7. AVL Tree and 2-4 Tree

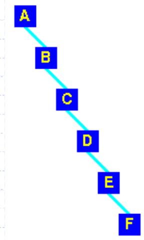
- AVL tree and 2-4 tree are balanced search trees.
- AVL tree, the definition, operations for re-balancing an AVL tree when it becomes unbalanced, the efficiency of rebalancing.
- 2-4 tree, the definition, operations of insertion and deletion, operations for rebalancing a 2-4 tree, the efficiency of rebalancing.

Problems of BST

- Binary Search Trees may be unbalanced.
- Insert this list of characters and form a tree

A B C D E F

- An unbalanced BST may also be the result of repeated insertions and deletions.



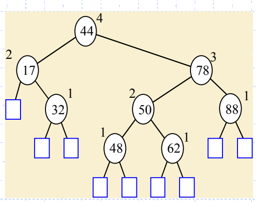
BST degenerates to a linked list

Binary Search tree, this is an unbalanced binary search tree, it has turned into a linked list O(n)

Balanced tree, balanced means for any node, the height of its left subtree, and the height of its right subtree, the difference between the heights is no more than one.

AVL Tree (Adelson-Velskii and Landis)

- **AVL trees are balanced.**
- An AVL tree is a ***binary search tree*** such that for every internal node v of T, the *heights of the children of v can differ by at most 1.*



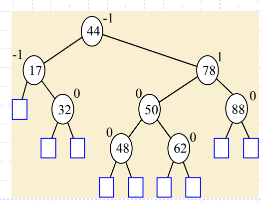
An example of an AVL tree where the heights are shown next to the nodes:

The height of the left minus the height of the right, is either -1, 0 or 1, the number of edges of the path is the heigh.

It is first a binary search tree, but the keys in the left subtree or less than the node, and the keys in the right subtree are greater than or equal to the node

Balance Factors in AVL Tree

- In an AVL tree, every internal node is associated with a ***balance factor***, which is calculated as the height of the left subtree minus the height of the right subtree.

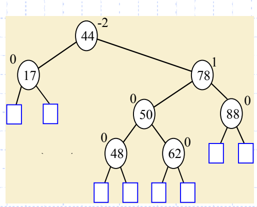


An example of an AVL tree where each internal node is associated with a balance factor.

Blanance factor for any internal node, is the difference between the right and left subtree, it's always left minus right

Balance Factors in AVL Tree

- In an AVL tree, if there is an internal node whose balance factor is less than -1 or greater than 1, the tree is said ***unbalanced***.



An example of an AVL tree which becomes unbalanced after a node is removed.

Height of an AVL Tree

- **Proposition**: The ***height*** of an AVL tree T storing n keys is O(log n).
- **Justification**: The easiest way to approach this problem is to find **n(h):** the *minimum number of internal nodes* of an AVL tree of height h.
- We see that n(1) = 1 and n(2) = 2
- For n ≥ 3, an AVL tree of height h contains the root node, one AVL subtree of height h-1 and the other AVL subtree of height at least h-2.
- i.e. n(h) = 1 + n(h-1) + n(h-2)

n(h) > 2^(h-2)/2 * N(1/(h-2^i)

n(h) > 2^(h-1)/2
log n(h) > (h-1)/2
2log(n(h)) > h -1
h < 2log(n(h))+1

Height of an AVL Tree (cont)

- Knowing n(h-1) > n(h-2), we get n(h) > 2n(h-2)
  - **n(h) > 2n(h-2)**
  - **n(h) > 4n(h-4)** (n(h-2) > 2n(h-4))
  - **n(h) > 8n(h-6)** (n(h-4) > 2n(h-6))
  - **...**
  - **n(h) > 2^in(h-2i)**
  - **For any integer i such that h-2i ≥ 1**
  - Let h – 2i = 1, then i = (h – 1)/2
- Solving the base case we get: n(h) ≥ 2 $^{(h-1)/2}$
- Taking logarithms: h < 2log n(h) + 1
- Thus the height of an AVL tree is O(log n)

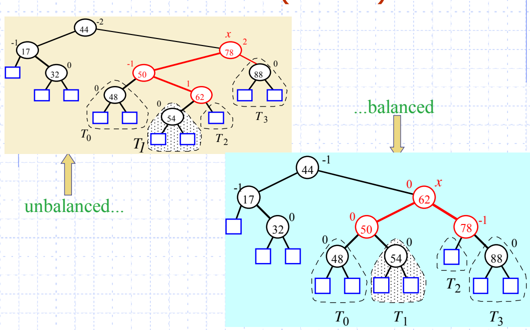
Insertion

- A binary search tree T is said to have ***AVL property*** if for every node v, the height of v's children differ by at most one, or the balance factor is -1, 0, 1.
- Inserting a node into an AVL tree may change the balance factors of some of the nodes in T.
- If an insertion causes AVL tree T to become unbalanced, we travel up the tree from the newly created node until we find the first node x such that its balance factor is -2 or 2.
- Node x is the root of the subtree to be rebalanced.
- Now to rebalance...

Two steps, first is to insert a new element as you would any other node. Check if the insertion violates the AVL property, if the insert has become an unbalanced an AVL tree, then you need to balance

Find the first ancestor such that the balance factor is 2 or -2, once we find the x, that is the unbalanced subtree, we do something in that subtree

Insertion (contd.)



Rotations

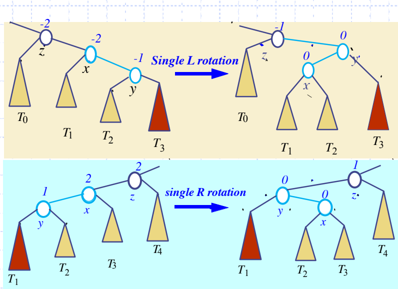
Rotations can be used to re-balance an AVL tree that becomes unbalanced after an insertion.

- There are four types of rotations: single left, single right, double right-left, double left-right.
- To re-balance an AVL tree, we travel up the tree from the newly inserted node until we find the first node x such that its balance factor is -2 or 2, then choose the type of rotation.

Rotations

Single Rotations:

L: $(T_1\, x\, (T_2\, y\, T_3)) \Rightarrow ((T_1 x\, T_2) y\, T_3)$ R: $((T_1\, y\, T_2) x\, T_3) \Rightarrow (T_1 y\, (T_2 x T_3))$

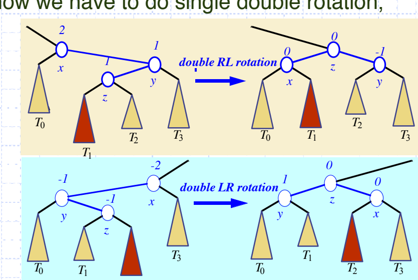


Left subtree has larger height, so right left, different from two cases, where we have right right or left left,
When we have right right, we can do single left rotation
when we have left left we can do single right rotation
now we have to do single double rotation,

Rotations

Double rotations:

RL: $(T_0 x((T_1 z T_2) y T_3)) \Rightarrow ((T_0 x T_1) z (T_2 y T_3))$ LR: $((T_0 y (T_1 z T_2)) x T_3) \Rightarrow ((T_0 x T_1) z (T_2 y T_3))$



Restructure Algorithm

function restructure(x):

Input: A node *x* of a binary search tree T that has both a child *y* and a grandchild *z*

Output: Tree T restructured involving nodes x, y, and z.

1: Let (*a, b, c*) be an inorder listing of the nodes *x, y, and z,* and let (T0, T1, T2, T3) be an inorder listing of the the four subtrees of *x, y, and z.*

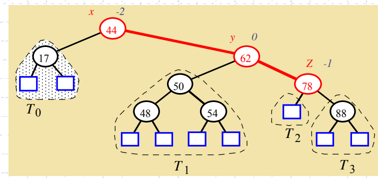
2. Replace the subtree rooted at x with a new subtree rooted at *b*

3. Let *a* be the left child of *b* and let T0, T1 be the left and right subtrees of *a*, respectively.

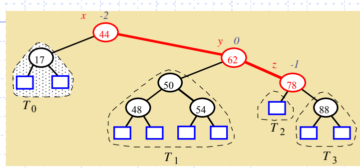
4. Let *c* be the right child of *b* and let T2, T3 be the left and right subtrees of *c*, respectively.

Cut/Link Restructure Algorithm

- Let's go into a little more detail on this algorithm...
- Any tree that needs to be balanced can be grouped into 7 parts: x, y, z, and the 4 trees anchored at the children of those nodes (T0-T3)



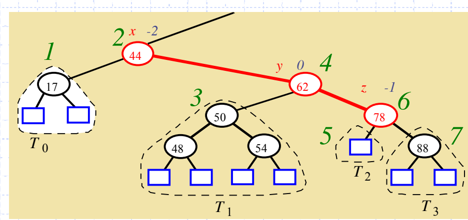
Cut/Link Restructure Algorithm



- Make a new tree which is balanced and put the 7 parts from the old tree into the new tree so that the numbering is still correct when we do an in-order-traversal of the new tree.
- This works regardless of how the tree is originally unbalanced.
- Let's see how it works!

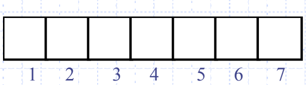
Cut/Link Restructure Algorithm

- Number the 7 parts by doing an in-order-traversal.

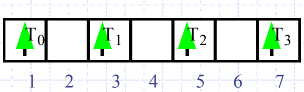


Cut/Link Restructure Algorithm

- Now create an Array, numbered 1 to 7 (the 0th element can be ignored with minimal waste of space)

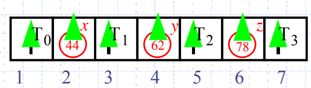


•Cut() the 4 T trees and place them in their inorder rank in the array



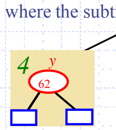
Cut/Link Restructure Algorithm

- Now cut x,y, and z in that order (child,parent,grandparent) and place them in their inorder rank in the array.



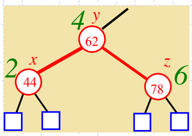
•Now we can re-link these subtrees to the main tree.

•Link in rank 4 (y) where the subtree's root formerly



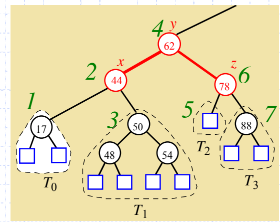
Cut/Link Restructure Algorithm

- Link in ranks 2 (x) and 6 (z) as 4's children.



Cut/Link Restructure Algorithm

- Finally, link in ranks 1,3,5, and 7 as the children of 2 and 6.



- Now you have a balanced tree!

Cut/Link Restructure algorithm

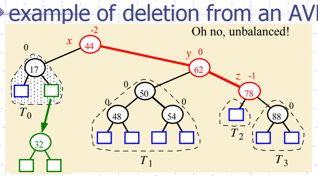
- This algorithm for restructuring has the same effect as using the four rotation cases discussed earlier.
- Advantages: no case analysis, more elegant
- Disadvantage: can be more code to write
- Same time complexity

Removal

- We can easily see that performing a remove(w) can cause T to become unbalanced.
- Let *x* be the first unbalanced node encountered while traveling up the tree from w. Also, let y be the child of x with the larger height, and let z be the child of y with the larger height.
- We can perform operation restructure(x) to restore balance at the subtree rooted at x.
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached

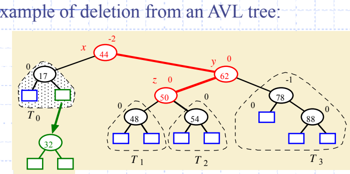
Removal (contd.)

- example of deletion from an AVL tree:



Removal (contd.)

- example of deletion from an AVL tree:



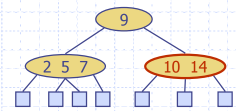
Oh no, unbalanced!

AVL Trees - Data Structures

- AVL trees can be implemented with a flag to indicate the balance state

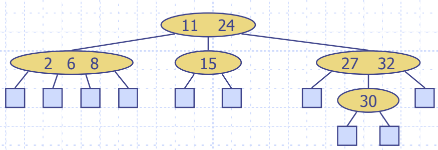
```
typedef enum {RightTooHeavy, RightHeavy,
  Balanced, LeftHeavy, LeftTooHeavy
         } BalanceFactor;

typedef struct node {
      BalanceFactor bf;
      void *item;
      struct node *left, *right;
} AVL_node;
```


(2,4) Trees



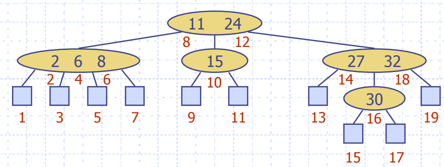
Multi-Way Search Tree

- A multi-way search tree is an ordered tree such that
  - Each internal node has at least two children and stores $d-1$ key-element items (k_i, o_i), where d is the number of children
  - For a node with children $v_1\ v_2 \ldots v_d$ storing keys $k_1\ k_2 \ldots k_{d-1}$
    - keys in the subtree of v_1 are less than k_1
    - keys in the subtree of v_i are between k_{i-1} and k_i $(i = 2, \ldots, d-1)$
    - keys in the subtree of v_d are greater than k_{d-1}



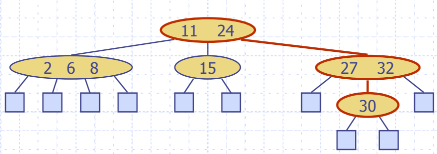
Multi-Way Inorder Traversal

- We can extend the notion of inorder traversal from binary trees to multi-way search trees
- Namely, we visit item (k_i, o_i) of node v between the recursive traversals of the subtrees of v rooted at children v_i and v_{i+1}
- An inorder traversal of a multi-way search tree visits the keys in increasing order



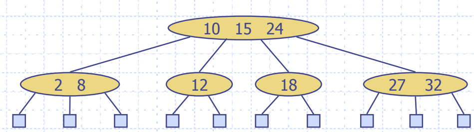
Multi-Way Searching

- Similar to search in a binary search tree
- A each internal node with children $v_1\ v_2 \ldots v_d$ and keys $k_1\ k_2 \ldots k_{d-1}$
  - $k = k_i$ $(i = 1, \ldots, d-1)$: the search terminates successfully
  - $k < k_1$: we continue the search in child v_1
  - $k_{i-1} < k < k_i$ $(i = 2, \ldots, d-1)$: we continue the search in child v_i
  - $k > k_{d-1}$: we continue the search in child v_d
- Reaching an external node terminates the search unsuccessfully
- Example: search for 30



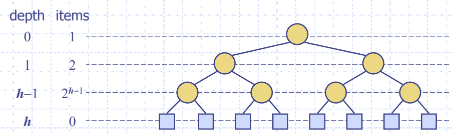
(2,4) Tree

- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search with the following properties
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node



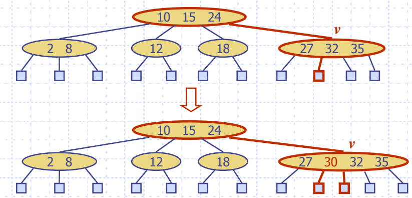
Height of a (2,4) Tree

- Theorem: A (2,4) tree storing n items has height $O(\log n)$
  Proof:
  - Let h be the height of a (2,4) tree with n items
  - Since there are at least 2^i items at depth $i = 0, \ldots, h-1$ and no items at depth h, we have
    $n \geq 1 + 2 + 4 + \ldots + 2^{h-1} = 2^h - 1$
  - Thus, $h \leq \log(n+1)$
- Searching in a (2,4) tree with n items takes $O(\log n)$ time



Insertion

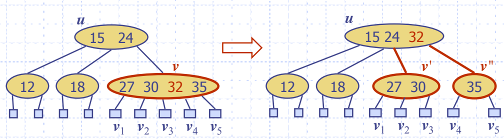
- We insert a new item (k, o) at the parent v of the leaf reached by searching for k
  - We preserve the depth property but
  - We may cause an overflow (i.e., node v may become a 5-node)
- Example: inserting key 30 causes an overflow



We first insert and store the key into the parent of the leaf and then we check if the insertion caused the problem of overflow, We need to split the overflow node into two nodes, we have four keys and five pointers, after the split we put three pointers to the left node, and two pointers into the right node, and two keys in the left node, and the last key in the right node, and then we push the third key up to the parent of the overflow node. And you can check the nodes involved and they are not overflowed. if we reach the root and the root is overflowed we create a new root above it.

Overflow and Split

- We handle an overflow at a 5-node v with a split operation:
  - let $v_1 \ldots v_5$ be the children of v and $k_1 \ldots k_4$ be the keys of v
  - node v is replaced by nodes v' and v''
    - v' is a 3-node with keys $k_1\ k_2$ and children $v_1\ v_2\ v_3$
    - v'' is a 2-node with key k_4 and children $v_4\ v_5$
  - key k_3 is inserted into the parent u of v (a new root may be created)
- The overflow may propagate to the parent node u



Analysis of Insertion

function *insertItem*(k, o)

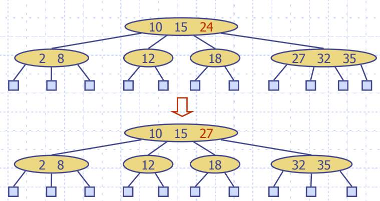
1. We search for key k to locate the insertion node v
2. We add the new item (k, o) at node v
3. **while** (*overflow*(v)) {
   if (*isRoot*(v))
   create a new empty root above v;
   $v \leftarrow$ *split*(v) // return parent of v;
   }

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
  - Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
  - Step 2 takes $O(1)$ time
  - Step 3 takes $O(\log n)$ time because each split takes $O(1)$ time and we perform $O(\log n)$ splits
- Thus, an insertion in a (2,4) tree takes $O(\log n)$ time

Text

Deletion

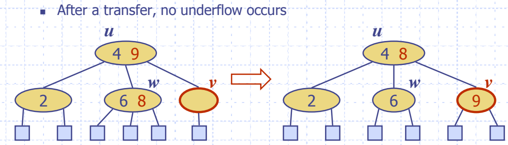
- We reduce deletion of an item to the case where the item is at the node with leaf children
- Otherwise, we replace the item with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter item
- Example: to delete key 24, we replace it with 27 (inorder successor)



One method to handle underflow, see if we can move something from sibling, check the siblings

Underflow and Transfer

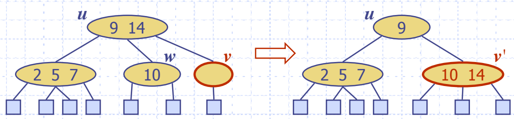
- Deleting an item from a node v may cause an underflow, where node v becomes a 1-node with one child and no keys
- To handle an underflow at node v with parent u, we consider two cases
- Case 1: an adjacent sibling w of v is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of w to v
    2. we move an item from u to v
    3. we move an item from w to u
  - After a transfer, no underflow occurs



If we don't have a sibling that can give a value, you can fuse items

Underflow and Fusion

- Case 2: the adjacent siblings of v are 2-nodes
  - Fusion operation: we merge v with an adjacent sibling w and move an item from u to the merged node v'
  - After a fusion, the underflow may propagate to the parent u



Analysis of Deletion

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
- In a deletion operation
  - We visit $O(\log n)$ nodes to locate the node from which to delete the item
  - We handle an underflow with a series of $O(\log n)$ fusions, followed by at most one transfer
  - Each fusion and transfer takes $O(1)$ time
- Thus, deleting an item from a (2,4) tree takes $O(\log n)$ time

Text