Name: ____________________

Time Limit: **Student ID:** ____________________

This exam contains 5 pages (including this cover page) and 4 questions. Total number of points is 24.

Question	Points	Score
1	6	
2	6	
3	6	
4	6	
Total:	24	

No aiding materials are allowed. You may use a calculator.

1. (a) (3 points) Determine whether $(p \vee q) \wedge (p \to r) \wedge (q \to r) \to r$ is a tautology, **without using the truth table**.

   (b) (3 points) Determine whether $\forall x(P(x) \vee Q(x))$ is logically equivalent to $(\forall x\, P(x)) \vee (\forall x\, Q(x))$.

(a) Assume $p \vee q = t$, $p \to r = t$, $q \to r = t$

If $p = t$, since $p \to r = t$, then $r = t$.

If $q = t$, since $q \to r = t$, then $r = t$.

Tautology.

(b) $U = \mathbb{R}$, $P(x)$: $x \geq 0$, $Q(x)$: $x < 0$

$\forall x [P(x) \vee Q(x)) = \forall x \in \mathbb{R} (x \geq 0$ or $x < 0)$ t.

$\forall x P(x) \vee \forall x Q(x) = (\forall x \in \mathbb{R}, x \geq 0)$ or $(\forall x \in \mathbb{R}, x < 0)$

$= f$.

Not logically equivalent.

$a_{n+1} = a_n + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 + \frac{-(n+2)+n+1}{(n+2)!}$

$= 1 - \frac{1}{(n+2)!}$ ✓

2. (a) (3 points) Determine whether this argument is valid.
"If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist."

(b) (3 points) Evaluate the sum

$$\sum_{i=1}^{n} \frac{i}{(i+1)!}$$

for $n = 1, 2$ and 3. Make a conjecture about the formula for this sum for general n, and prove your conjecture by mathematical induction.

(a)
1. $A \wedge W \to P$
2. $\sim A \to I.$
3. $\sim W \to M$
4. $\sim P$
5. $E \to \sim I \wedge \sim M$

$\therefore \sim E.$

Valid.

6. $\sim P \to \sim A \vee \sim W$ 1
7. $\sim A \vee \sim W$ 6, 4
8. If $\sim A = t$
9. $I = t$ 2, 8
10. $I \vee M \to \sim E.$ 5
11. $\sim E = t.$ 9, 10
12. $\sim W = t$
13. $M = t$ 3, 12
14. $\sim E = t$ 10, 13

(b) $a_n = \sum_{i=1}^{n} \frac{i}{(i+1)!}$ $a_1 = \frac{1}{2!} = \frac{1}{2}$

$a_2 = \frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{6} = \frac{5}{6}.$

$a_3 = a_2 + \frac{3}{(3+1)!} = \frac{5}{6} + \frac{3}{24} = \frac{20}{24} + \frac{3}{24}$

$= \frac{23}{24} = 1 - \frac{1}{24} = 1 - \frac{1}{4!}$

$a_n = a_{n-1} + \frac{n}{(n+1)!}$

$a_4 = a_3 + \frac{4}{5!} = \frac{23}{24} + \frac{4}{120} = \frac{115+4}{120} = \frac{119}{120} = 1 - \frac{1}{120}$

$= 1 - \frac{1}{5!}$

$a_n = 1 - \frac{1}{(n+1)!}$

3. (a) (3 points) Let R and S be both antisymmetric relations on X. Is $R \cap S$ antisymmetric? Is $R \cup S$ antisymmetric? Justify your answer.

(b) (3 points) Is there a function $f : \mathbb{R} \to \mathbb{R}$, which is injective but not surjective? Justify your answer

(a) Antisymmetric: $x \neq y$, $(x, y) \in R$

$\Rightarrow (y, x) \notin R$. (definition).

let $x \neq y$, $(x, y) \in R \cap S \overset{?}{\Rightarrow} (y, x) \notin R \cap S$. ✓

$(x, y) \in R \cap S \Rightarrow (x, y) \in R$, $x \neq y$, R anti sym $\Rightarrow (y, x) \notin R$

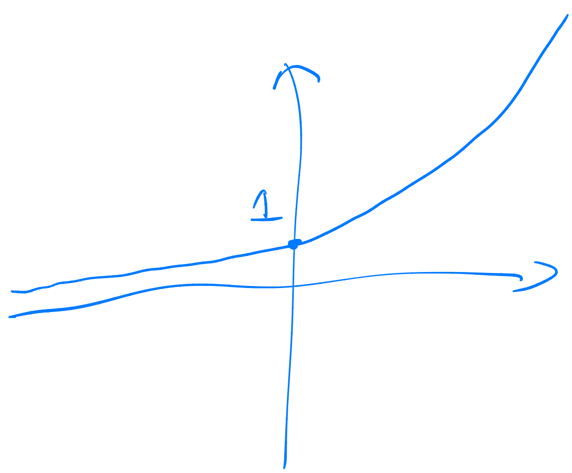
$\Rightarrow (y, x) \notin R \cap S$ $R \cap S$ is Antisymmetric

---

$X = \{1, 2\}$ $R = \{(1, 2)\}$, $S = \{(2, 1)\}$

$R \cup S = \{(1, 2), (2, 1)\}$ Not anti-sym.

(b) $f(x) = e^x > 0$ $\forall x \in \mathbb{R}$.



4. (a) (3 points) How many ways to sit 6 men and 4 women in a row, where all women must sit next to one another?

(b) (3 points) There are 42 students who are to share 12 computers. Each student uses exactly one computer and no computer is used by more than six students. Show that at least five computers are used by three or more students.

(a) $4! \times (6+1)!$ ←

(b) Suppose n computers are used by 3 or more studens.

$3 \le S_1, \dots, S_n \le 6$

$S_{n+1}, \dots, S_{12} \le 2$ ←

$S_1, S_2, \dots, S_n, \underline{S_{n+1}, \dots, S_{12}}$

$\uparrow$

# of studens use the first computer.

$42 = \underbrace{S_1 + \dots + S_n}_{} + S_{n+1} + \dots + S_{12}$

$\le n \times 6 + (12-n) \times 2 = 24 + 4n$

$4n \ge 18 \qquad n \ge 4.5 \quad \Rightarrow \quad n \ge 5.$