5.

Reasoning with Horn Clauses

Horn clauses

Clauses are used two ways:

- as disjunctions: $(\text{rain} \vee \text{sleet})$
- as implications: $(\neg\text{child} \vee \neg\text{male} \vee \text{boy})$

Here focus on 2nd use

Horn clause = at most one +ve literal in clause

- positive / definite clause = exactly one +ve literal

  e.g. $[\neg p_1, \neg p_2, ..., \neg p_n, q]$

- negative clause = no +ve literals

  e.g. $[\neg p_1, \neg p_2, ..., \neg p_n]$ and also [ ]

Note: $[\neg p_1, \neg p_2, ..., \neg p_n, q]$ is a representation for
$(\neg p_1 \vee \neg p_2 \vee ... \vee \neg p_n \vee q)$ or $[(p_1 \wedge p_2 \wedge ... \wedge p_n) \supset q]$

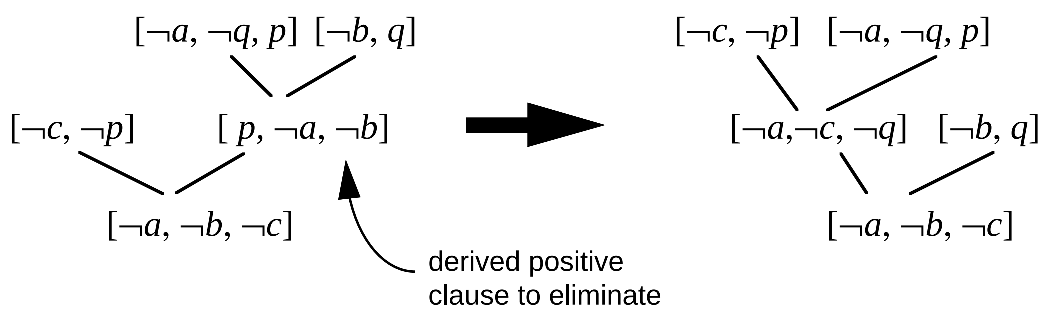
so can read as: If p_1 and p_2 and ... and p_n then q

and write as: $p_1 \wedge p_2 \wedge ... \wedge p_n \Rightarrow q$ or $q \Leftarrow p_1 \wedge p_2 \wedge ... \wedge p_n$

Resolution with Horn clauses

Only two possibilities:

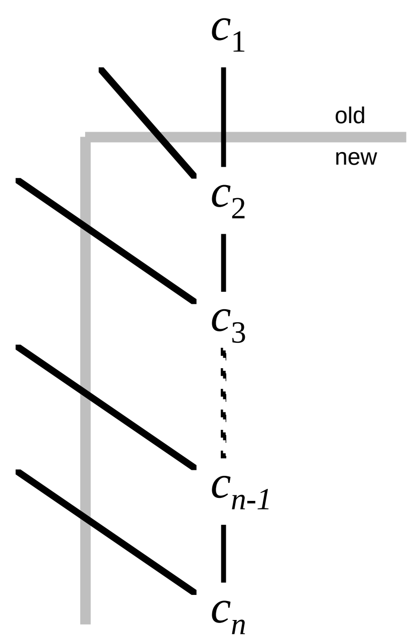
It is possible to rearrange derivations of negative clauses so that all new derived clauses are negative



Further restricting resolution

Can also change derivations such that each derived clause is a resolvent of the previous derived one (negative) and some positive clause in the original set of clauses

- Since each derived clause is negative, one parent must be positive (and so from original set) and one parent must be negative.
- Chain backwards from the final negative clause until both parents are from the original set of clauses
- Eliminate all other clauses not on this direct path



This is a recurring pattern in derivations

- See previously:
  - example 1, example 3, arithmetic example
- But not:
  - example 2, the 3 block example

SLD Resolution

An <u>SLD-derivation</u> of a clause c from a set of clauses S is a sequence of clause $c_1, c_2, \ldots c_n$ such that $c_n = c$, and

1. $c_1 \in S$
2. c_{i+1} is a resolvent of c_i and a clause in S

Write: $S \overset{\text{SLD}}{\rightarrow} c$

SLD means S(elected) literals
L(inear) form
D(efinite) clauses

Note: SLD derivation is just a special form of derivation and where we leave out the elements of S (except c_1)

In general, cannot restrict ourselves to just using SLD-Resolution

Proof: $S = \{[p, q], [p, \neg q], [\neg p, q] [\neg p, \neg q]\}$. Then $S \rightarrow []$.

Need to resolve some $[\ \rho\ ]$ and $[\ \overline{\rho}\ ]$ to get [].
But S does not contain any unit clauses.

So will need to derive both $[\ \rho\ ]$ and $[\ \overline{\rho}\ ]$ and then resolve them together.

Completeness of SLD

However, for Horn clauses, we can restrict ourselves to SLD-Resolution

Theorem: SLD-Resolution is refutation complete for Horn clauses: $H \rightarrow []$ iff $H \overset{\text{SLD}}{\rightarrow} []$

So: H is unsatisfiable iff $H \overset{\text{SLD}}{\rightarrow} []$

This will considerably simplify the search for derivations

Note: in Horn version of SLD-Resolution, each clause in the $c_1, c_2, ..., c_n$, will be negative

So clauses H must contain at least one negative clause, c_1 and this will be the only negative clause of H used.

Typical case:

- KB is a collection of positive Horn clauses
- Negation of query is the negative clause

Example 1 (again)

KB

FirstGrade

FirstGrade $\supset$ Child

Child $\wedge$ Male $\supset$ Boy

Kindergarten $\supset$ Child

Child $\wedge$ Female $\supset$ Girl

Female

SLD derivation

[¬Girl]

|

[¬Child, ¬Female]

|

[¬Child]

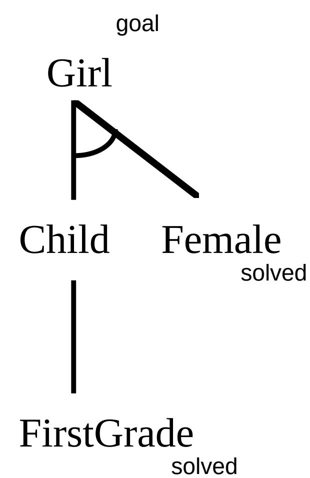
|

[¬FirstGrade]

|

[]

alternate representation



Show KB $\cup$ {¬Girl} unsatisfiable

A goal tree whose nodes are atoms, whose root is the atom to prove, and whose leaves are in the KB

Back-chaining procedure

Solve[$q_1, q_2, ..., q_n$] = /* to establish conjunction of q_i */

If n=0 then return **YES**; /* empty clause detected */

For each $d \in$ KB do

If d = [$q_1, \neg p_1, \neg p_2, ..., \neg p_m$] /* match first q */

and /* replace q by -ve lits */

Solve[$p_1, p_2, ..., p_m, q_2, ..., q_n$] /* recursively */

then return **YES**

end for; /* can't find a clause to eliminate q */

Return **NO**

Depth-first, left-right, back-chaining

- depth-first because attempt p_i before trying q_i
- left-right because try q_i in order, 1,2, 3, ...
- back-chaining because search from goal q to facts in KB p

This is the execution strategy of Prolog

First-order case requires unification *etc.*

Problems with back-chaining

Can go into infinite loop

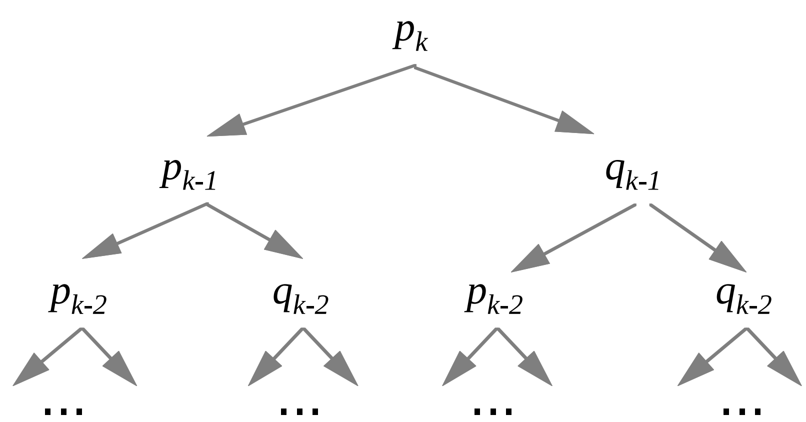
tautologous clause: $[p\ ,\ \neg p]$ (corresponds to Prolog program with `p :- p`).

Previous back-chaining algorithm is inefficient

Example: Consider $2n$ atoms, $p_0, \ldots, p_{n-1}, q_0, \ldots, q_{n-1}$ and $4n$-4 clauses

$$(p_{i-1} \Rightarrow p_i),\ (q_{i-1} \Rightarrow p_i),\ (p_{i-1} \Rightarrow q_i),\ (q_{i-1} \Rightarrow q_i).$$

With goal p_k the execution tree is like this



Solve[p_k] eventually fails after 2^k steps!

Is this problem inherent in Horn clauses?

Forward-chaining

Simple procedure to determine if Horn KB |= q.

main idea: mark atoms as solved

1. If q is marked as solved, then return **YES**
2. Is there a $\{p_1, \neg p_2, ..., \neg p_n\} \in$ KB such that $p_2, ..., p_n$ are marked as solved, but the positive lit p_1 is not marked as solved?

   no: return **NO**

   yes: mark p_1 as solved, and go to 1.

FirstGrade example:

Marks: FirstGrade, Child, Female, Girl then done!

Note: FirstGrade gets marked since all the negative atoms in the clause (none) are marked

Observe:

- only letters in KB can be marked, so at most a linear number of iterations
- not goal-directed, so not always desirable
- a similar procedure with better data structures will run in *linear* time overall

First-order undecidability

Even with just Horn clauses, in the first-order case we still have the possibility of generating an infinite branch of resolvents.

KB:

$\text{LessThan}(\text{succ}(x),y) \Rightarrow \text{LessThan}(x,y)$

Query:

$\text{LessThan}(\text{zero},\text{zero})$

As with full Resolution, there is no way to detect when this will happen

$$[\neg\text{LessThan}(0,0)] \xrightarrow{x/0,\ y/0} [\neg\text{LessThan}(1,0)] \xrightarrow{x/1,\ y/0} [\neg\text{LessThan}(2,0)] \xrightarrow{x/2,\ y/0} \ldots$$

There is no procedure that will test for the satisfiability of first-order Horn clauses

the question is undecidable

As with non-Horn clauses, the best that we can do is to give control of the deduction to the *user*

to some extent this is what is done in Prolog, but we will see more in “Procedural Control”