Using Neural Networks and Other Machine Learning Algorithms with Non-Standard Loss Functions in Realized Volatility Forecasting

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Abstract

Neural networks have recently become widespread since they allow approximating a nonlinear dependence of any complexity. And many researchers began to use them for modeling and forecasting various time series. In addition, a random forest machine learning method is also used to predict time series. This paper compares the predictive power of various machine learning algorithms, including neural networks and random forest, with standard econometric approaches (such as MEM and HAR models). A distinctive feature of the work is the use of non-standard (asymmetric) loss functions such as the LinEx function and the asymmetric least squares function. The result of the work is that the considered machine learning methods are not able to outperform econometric approaches in predictive power, both with symmetric and with asymmetric loss functions.

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Contents

1 Introduction

Volatility forecasting is important not only in academia, but also in business, investment banking, trading and even government regulation. Moreover, option prices, Value at Risk, Sharpe ratio and other indicators are highly dependent on volatility, which reveals the necessity of its accurate forecasts. For example, volatility is used by option traders as an input for an option pricing using Black–Scholes' (Black & Scholes (1973)) and Heston's (Heston (1993)) models.

When high-frequency data appeared, the realized volatility (RV) measure was introduced by Andersen et al. (2003). Unlike GARCH-like measures, the RV is model-free. In this paper we compare recently developed machine learning algorithms (such as neural networks and random forests) and classical econometric approaches (such as the HAR and the MEM models) in forecasting RV. Moreover, we will consider using non-standard loss functions — the LinEx loss function (see references in Khatun & Matin (2020)) and the asymmetric least squares (ALS) loss function (as defined in Tian (2009)).

Luong & Dokuchaev (2018) researched the possibility to forecast the RV using the random forest algorithm. Vortelinos (2017) compared the HAR model against the GARCH, feedforward neural network and Principal Components Combining (which will not be considered in this paper). Bucci (2020) compared different recurrent neural network (RNN) architectures (applied to the RV) in terms of predictability performance.

The purposes of this paper can be formulated as follows:

1. Compare the HAR and the MEM models with the random forest algorithm and the LSTM model, using symmetric MSE loss function and criteria.

2. Compare the HAR and the LSTM models, using the LinEx loss function and criteria.

3. Compare the HAR and the LSTM models, using the ALS loss function and criteria.

This paper is structured as follows. In Section 2 we provide a literature review on all topics we cover in this paper: volatility, models (for loss functions see references in already presented papers). In Section 3 we give a definition of RV. In Section 4 we introduce description of all models we used to forecast realized volatility. In Section 5 we will describe all loss functions we used. In Section 6 we provide a data description used in the paper. In Section 7 there are results of our empirical research and in Section 8 there is a conclusion of the paper.

2 Literature Review

Volatility has been estimated and forecast a lot (you can see the references in Andersen & Bollerslev (1997), Dokuchaev (2014)).

ARCH and GARCH models were proposed by Engle (1982) and Bollerslev (1986), respectively. These models were extended in many directions: EGARCH (Nelson (1991)), GJR-GARCH (Glosten et al. (1993)), AGARCH (Engle (1990)), and TGARCH (Zakoian (1994)). But these models cannot be appropriate estimators for the RV since they were developed for low-frequency data and cannot capture intraday information well. And Vortelinos (2017) showed that GARCH models cannot indeed contest with HAR model in terms of forecasting. So, we won't consider GARCH-like models in this paper.

As we already mentioned in introduction, Andersen et al. (2003) proposed the RV measure. The HAR model, which was introduced by Corsi (2003), was developed specifically for estimating and predicting the RV. The MEM model, presented by Engle (2002), was not developed specifically for the RV estimation, but it is widely used for it. We will use the MEM and the HAR models in this paper.

The algorithm of random forests was introduced by Breiman (2001) and is basically an ensemble of some number of decision tree algorithms, which were presented in Breiman et al. (1984). The random forest algorithm will be used in the paper.

Feedforward neural network, according to Vortelinos (2017), is not a perfect choice for the RV forecasting. On the contrary, recurrent neural network (RNN) architectures are applied for time series estimation and prediction not only in academia, but also in business. The LSTM neural network architecture (presented in Hochreiter (1997)) is one of the most popular RNNs, and it will be used in this paper.

3 Realized Volatility

As we mentioned, Andersen et al. (2003) proposed the RV measure, which is based on high-frequency data.

Denote P_t the asset price at time $t \in [0, T]$, $p_t = \log P_t$. The asset return is $r_{t,\Delta t} = s_{t+\Delta t} - s_t$ for some small Δt such that $t + \Delta t \leq T$. The assumption is that p_t can be represented by the following diffusion Ito equation:

$$dp_t = \mu_t dt + \sigma_t dW_t, \quad 0 \leq t \leq T,$$

where W_t is Brownian motion, μ_t and σ_t are predictable processes with σ_t being the standard deviation of dp_t and independent of dW_t. From the diffusion equation it follows that

$$r_{t,\delta t} = p_{t+\Delta t} - p_t = \int\limits_0^{\Delta t} \mu_{t+\tau} d\tau + \int\limits_0^{\Delta t} \sigma_{t+\tau} dW(t+\tau).$$

The integrated variance (IV) is defined as follows:

$$IV_t = \int\limits_0^{\Delta t} \sigma^2_{t+\tau} d\tau.$$

Consider even splitting $\left\{ t_i = t + \frac{\Delta t}{m} \times i \right\}_{i=1}^{m}$ of the segment $[t, t+\Delta t]$. Then the realized variance can be calculated as follows:

$$RV_t = \sum_{i=1}^{m} r^2_{t_i},$$

and from the stochastic calculus it follows that RV_t is a consistent estimator for IV_t (we assuming the absence of jumps, according to the diffusion equation).

Finally, the realized volatility (RV) is simply a square root of realized variance (and from now on, we denote it with RV_t). From now on let us assume that $\Delta t = 1$ day (so that each day we calculate RV based on intradaily data).

4 Models

4.1 HAR Model

Let $RV_t^n = \frac{\sum_{i=0}^{n-1} RV_{t-i}}{n}$. Then, the daily HAR model can be expressed as follows:

$$RV_t = \beta_0 + \beta_d RV^1_{t-1} + \beta_w RV^5_{t-1} + \beta_m RV^{22}_{t-1} + \epsilon_t,$$

where subscripts "d", "w", and "m" stand for "day", "week", and "month", respectively. There are some extensions of this model (i.e. taking into account jumps or asymmetry in the relationship between returns and volatility), but we won't consider them in this paper.

We will estimate the HAR model with three different loss functions:

1. Mean Squared Error Loss Function (MSE);

2. Linear Exponential Loss Function (LinEx);

3. Asymmetric Least Squares Loss Function (ASL).

All of them will be considered in Section 5.

In the HAR model we will use logs of RV to deal with negative predictions.

4.2 MEM Model

The MEM model is used only for positive time series, and the RV is an example of it. The model has the following expression:

$$RV_t = \psi_t \eta_t,$$

where $\phi_t = \mathbb{E}_{t-1} RV_t$, and η_t is a i.i.d. process with a mean 1 (and, ideally, positive elements). We will use exp(1) distribution. Thus, while considering the MEM model, we will use exponential quasi-maximum likelihood estimate (QMLE). We also assume that ψ_t can be expressed as follows:

$$\psi_t = \beta_0 + \beta_1 RV_{t-1} + \beta_2 \psi_{t-1}.$$

To guarantee the positivity of forecasts, we should impose $\beta_0 > 0$.

Since the exponential density function is $f(x) = e^{-x}$, log-likelihood function of the whole sample is

$$L = -\sum_{i=1}^{T} \left(\frac{RV_t}{\phi_t} + \log \psi_t \right),$$

where $\psi_t = \beta_0 + \beta_1 RV_{t-1} + \beta_2 \psi_{t-1}$.

4.3 Random Forests

For this algorithm, as well as for the next LSTM algorithm, we will build a dependency of RV_t on some number k of previous observations $RV_{t-1}, \ldots, RV_{t-k}$, where k is some hyperparameter. In this paper we take $k = 10$. So, let us call RV_t a target and $RV_{t-1}, \ldots, RV_{t-k}$ features. For each $t \geq k$ we have a separate object with a target and features. Denote our sample as X (consisting of vectors $((RV_{t-1}, \ldots, RV_{t-k}), RV_t)$, $t \in [k, T]$).

To fully describe the random forest algorithm, we first should define the decision tree method:

1. Let N be the root node of the tree with all available data.

2. Check whether some stop condition holds. If so, make N a leaf node and assign to it the average target of the objects in this node (meaning that while forecasting, if an out-of-sample object ends up being in this node, than we predict this value as a target). Otherwise, go ahead to the next step.

3. Select the feature F and a threshold T, such that all data is separated into two subsets: X_{TRUE} and X_{FALSE}, where data with $F < T$ is going to X_{TRUE} and otherwise in X_{FALSE}. And we select F and T maximizing some criteria $Q(X, F, T)$.

4. Assign a pair (F, T) to the node N, make two new nodes N_{TRUE} and N_{FALSE}, passing to them X_{TRUE} and X_{FALSE}, respectively.

5. Repeat steps 2-4 with both nodes each considered as a new root of a respective subtree.

To finish the definition, we should set particular stop condition and a criteria $Q(X, F, T)$. The latter is defined as follows:

$$Q(X, F, T) = H(X) - \frac{|X_{TRUE}|}{|X|} H(X_{TRUE}) - \frac{|X_{FALSE}|}{|X|} H(X_{FALSE}),$$

where $H(X) = \min\limits_{c \in \mathbb{Y}} \frac{1}{|X|} \sum\limits_{((x_1^i, \ldots, x_k^i), y^i) \in X} L(y^i, c)$, and $L(y, c)$ is some loss function we choose. In this paper the standard MSE loss function was used for random forests: $L(y, c) = (y - c)^2$ (note that we use non-standard loss function only for the LSTM model; using it in random forests is a future plan for extension). The stop condition is the following: stop whenever the best splitting into two subsets yields negative criteria $Q(X, F, T)$. When it is positive, then the process of subtree construction continues. Actually, for checking this condition we need to calculate $Q(X, F, T)$ for each pair of (F, T) to take the best, so we can use these calculations to select the optimal pair on step 3 (of course, if stop condition does not hold).

Now we can define an algorithm of random forest. Hyperparameters (besides those for decision trees) are n_{tree} (number of trees in the ensemble) and m (the number of features in each tree). The algorithm is the following:

1. Use bootstrap to generate n_{tree} samples from the original one (and use each of them while training a separate decision tree).

2. For i_{tree} from 1 to n_{tree}: train a decision tree on i_{tree}'s generated bootstrap sample, but when selecting the best splitting of each node (i.e. on step 3 of tree construction) use only m features, whose numbers are randomly chosen before each splitting.

3. Return the average of the predictions of all trees (i.e. while forecasting, we get the predictions from each tree and after that the averaged value will be a prediction from a random forest).

As we can see, all predictions in a random forest lie between the smallest and the largest values of sample targets, so there is no need to worry about negative predictions of the RV, since all targets in the sample are positive.

Random forest makes the variance of the overall algorithm smaller, which prevents overfitting. However, it does not change the bias, so it cannot help with underfitting (if it exists). Therefore, we should use decision trees with large depth (so that they are complex enough not to be underfitted). In fact, in this paper we do not restrict the depth of trees at all (using the only stop condition described above).

For the Random Forest algorithm we will use only MSE loss function.

4.4 LSTM Model

The general neural network consists of the following elements:

- neurons (or nodes), grouped in
- layers, connected to each other;
- weights of neurons and
- (non-linear) activation functions for each layer.

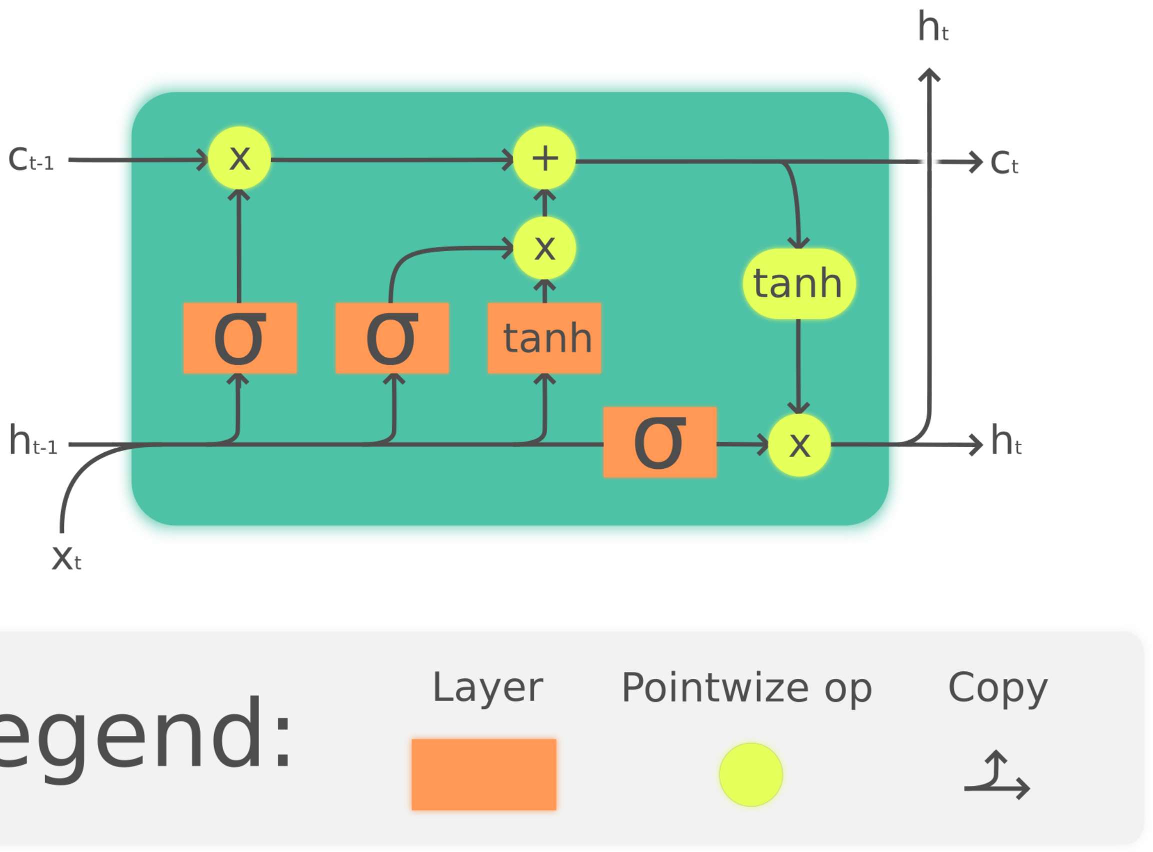
Layers are numerated from 0 (the input of the neural network) to L (the output of the neural network), all layers except the input and the output are called hidden layers. Each node in layer $l-1$ is connected to each node in layer l via arrows and weights under arrows. Each layer l is specified by an activation function $a_l(\cdot)$, so that the value in the node of layer $l+1$ is an activation function applied to the linear combination of values of nodes of layer l with weights under arrows entering this node (including a constant term). Activation function are necessary (and they are to be non-linear) to be able to approximate non-linear dependencies. We will use the only hidden layer in this paper, since Donaldson & Kamstra (1996) showed that a single hidden neural network can approximate any non-linear function given enough number of neurons in this hidden layer. Overall, the output function of one hidden layer neural network is

$$f(X, W, b) = a_1\left(b^1 + \sum_{j=1}^{q} a_0(b_j^0 + XW^0)W_j^1\right),$$

where $k, q, 1$ are sizes of layers $0, 1, 2$ respectively, W^0 (of size (q, k)) is a matrix consisting of weights connecting the input layer and hidden layer, and W^1 (of size $(1, q)$) is a row vector consisting of weights connecting the hidden layer and the output layer, b^0 (of size $(q, 1)$) is a column vector consisting of bias terms in the hidden layer, and b^1 (scalar) is a bias term in the output layer. X (of size (m, k)) is an input matrix consisting of m examples, so that $f(X, W, b)$ has size $(m, 1)$ approximating the target values on these examples. Remember (from the previous point) that k is the number of features (regressors). In this paper we have $k = 10$ and $q = 100$.

As already mentioned in Section 2, feedforward neural networks are not ideal for estimating and forecasting time series. And we will use the LSTM model for it.The hidden layer in this model (which is the only as we already discussed) is represented by the so-called memory block. Each block includes one or more self-connected memory cells and is equipped with three multiplicative units called input, forget and output gates.

Figure 1: LSTM memory cell



Define the following activation functions:

$$\sigma(z) = \frac{1}{1 + e^{-z}},$$

$$\tanh(z) = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}.$$

The illustration of the LSTM memory cell is given on Figure 1 on page 8. x_t denotes the input vector at time t, c_t is a cell state at time t, h_t is a hidden state or an output state at time t. The input gate is i_t, whereas f_t is the forget gate and o_t is the output gate. The entire process can be described by the following equations:

$$f_t = \sigma\left(W_f h_{t-1} + U_f x_t + b_f\right),$$

$$i_t = \sigma\left(W_i h_{t-1} + U_i x_t + b_i\right),$$

$$\bar{c}_t = \tanh\left(W_c h_{t-1} + U_c x_t + b_c\right),$$

$$c_t = f_t \circ c_{t-1} + i_t \circ \bar{c}_t,$$

$$o_t = \sigma\left(W_o h_{t-1} + U_o x_t + V_o c_t + b_o\right),$$

$$h_t = o_t \circ \tanh(c_t),$$

$$\hat{y}_t = h_t.$$

As well as for the HAR model, we will use three loss functions (MSE, LinEx and ASL). And similarly to the HAR model, we will take logs of RV to deal with negative predictions.

5 Loss Functions

Each model we consider (except for the MEM model, where we use QMLE) while being estimated optimize some cost function:

$$C(y, \hat{y}) = \frac{1}{m}\sum_{i=1}^{m} L(y^i, \hat{y}^i),$$

where m is a number of examples, and $L(y^i, \hat{y}^i)$ is a loss function, which estimates an error of a prediction on i'th example (from the true target value). Let us consider some of them.

5.1 Mean Squared Error (MSE)

The most famous and widely used in regression problems loss function is the MSE function:

$$L(y, \hat{y}) = (y - \hat{y})^2.$$

It is continuously differentiable, and it is symmetric (meaning that errors of underestimation and overestimation are symmetric and depend only on the distance between the predicted value and the true value). However, its use often contradicts the reality, e.g. in situations where overprediction is much less preferred than underprediction (or vice versa). Therefore, using symmetric loss functions (e.g. MSE) in these situations is undesirable. And the next two loss functions are asymmetric (in general).

5.2 Linear Exponential Loss Function (LinEx)

LinEx loss function is defined as follows:

$$L(y, \hat{y}) = \exp(\alpha(y - \hat{y})) - \alpha(y - \hat{y}) - 1,$$

where α is a parameter which represents a degree of asymmetry. If $\alpha > 0$, than the loss is approximately exponential for positive errors, and approximately linear for negative errors. Therefore, it is smaller for overprediction than for underprediction. We assume that this is the case with the RV: it is much better to overestimate it than to underestimate (e.g. for risk management purposes). Thus, we will use $\alpha = 0.5$ which estimating models. We will use LinEx loss with two models: the HAR and the LSTM models. For the HAR model we need to calculate a derivative of the LinEx loss function with respect to the weights of the model:

$$\frac{\partial L(y, Xw)}{\partial w} = -\alpha X^T e^{y - Xw} + X^T \alpha \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(the size of the vector of ones is $(m, 1)$; we calculate a simple sum with it). Here and in the next point X is a matrix of regressors with a size $(T - 22, k)$. For the LSTM model we do not need a derivative, since it is calculated automatically in the backpropagation algorithm.

5.3 Asymmetric Least Squares Loss Function (ALS)

ALS loss function is defined as follows:

$$L(y, \hat{y}) = \left| \alpha - \mathbb{1}_{(y - \hat{y} < 0)} \right| \times (y - \hat{y})^2.$$

As we can see, when $y < \hat{y}$ (i.e. when there is an overprediction), the coefficient is $1 - \alpha$, otherwise α. We assumed that underprediction is less preferred than overprediction, thus, we need to have $\alpha > 1 - \alpha \Leftrightarrow \alpha > 0.5$. In this paper we use $\alpha = 0.7$.

Note that this function is continuously differentiable with respect to $y - \hat{h}$ (in $y - \hat{y} = 0$ it is differentiable by definition and the derivative is 0, and both left and right derivatives converge to 0 as well). Again, we will use ALS loss with two models: the HAR and the LSTM models. And again, for the HAR model we need to calculate a derivative of the ALS loss function with respect to the weights of the model:

$$\frac{\partial L(y, Xw)}{\partial w} = -2 \left| \alpha - \mathbb{1}_{(y-Xw<0)} \right| \times (y - Xw).$$

6 Data

We used the RV of S&P 500 Index, calculated by Oxford-Man Institute of Quantitative Finance (ticks of 5 minutes were used to calculate the RV of each day). The period is from 1/3/2000 to 1/14/2020, which is splitted into an in-sample and an out-of-sample periods with a proportion of 7 : 3.

All the code and data can be found here.

7 Results

The following models were estimated:

1. the MEM model with the exponential QMLE;
2. the Random Forest model with the MSE loss function;
3. the HAR model with the MSE loss function;
4. the LSTM model with the MSE loss function;
5. the HAR model with the LinEx loss function;
6. the LSTM model with the LinEx loss function;
7. the HAR model with the ALS loss function;
8. the LSTM model with the ALS loss function.

Compare the first four models using MSE for RV_t and for $\log(RV_t)$ on out-of-sample data (we used one-step-ahead predictions):

Model	MSE for RV_t	MSE for $\log(RV_t)$
MEM	4.67×10^{-6}	0.057
Random Forest	1.11×10^{-5}	0.136
HAR	1.21×10^{-5}	0.11
LSTM	3.96×10^{-5}	0.453

As we can see, the MEM model has the best result in both criteria, although RV_t were predicted, not their logs. The HAR and the random forest models are very close, and each model has lower criteria corresponding to their loss function (the HAR was estimated using the MSE for logs and the Random Forest model used the MSE for RV_t themselves). The LSTM model shows the worst performance. On Figures 2–5 one-step-ahead predictions (on the first 200 days of out-of-sample data) of all four models are represented.

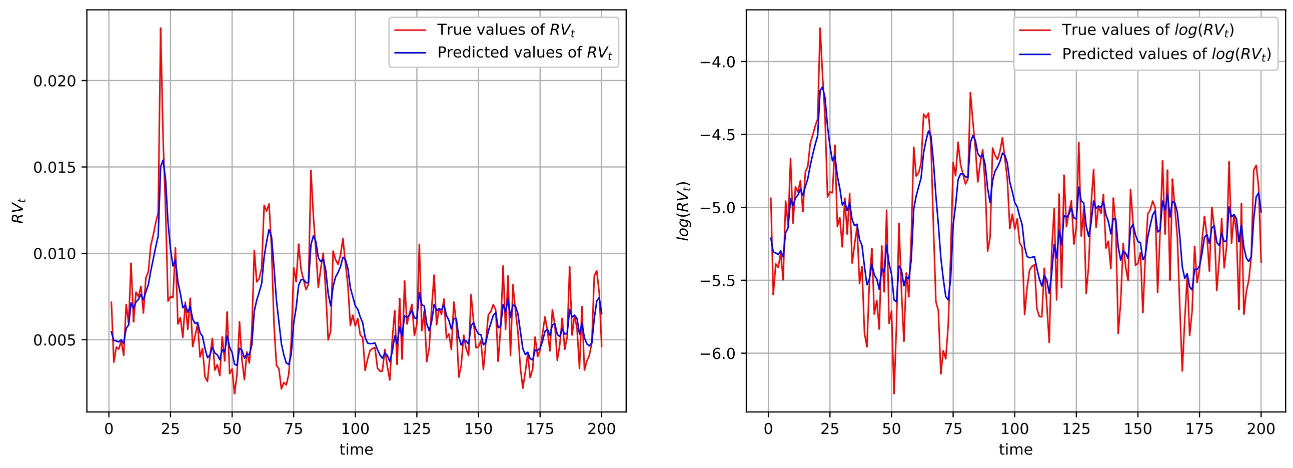


Figure 2: MEM model with QMLE, one-step-ahead predictions

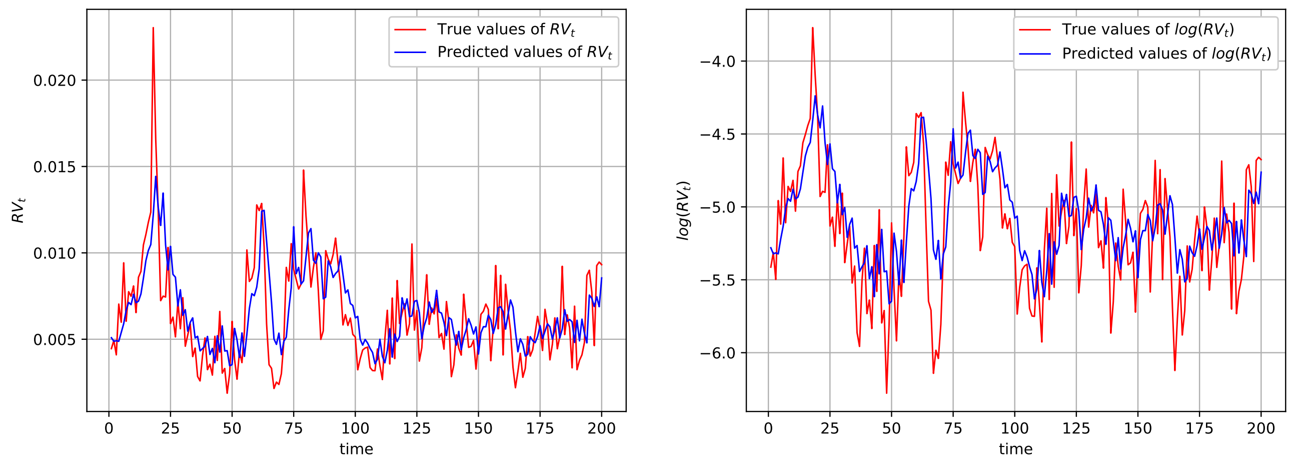


Figure 3: Random Forest model with the MSE loss, one-step-ahead predictions

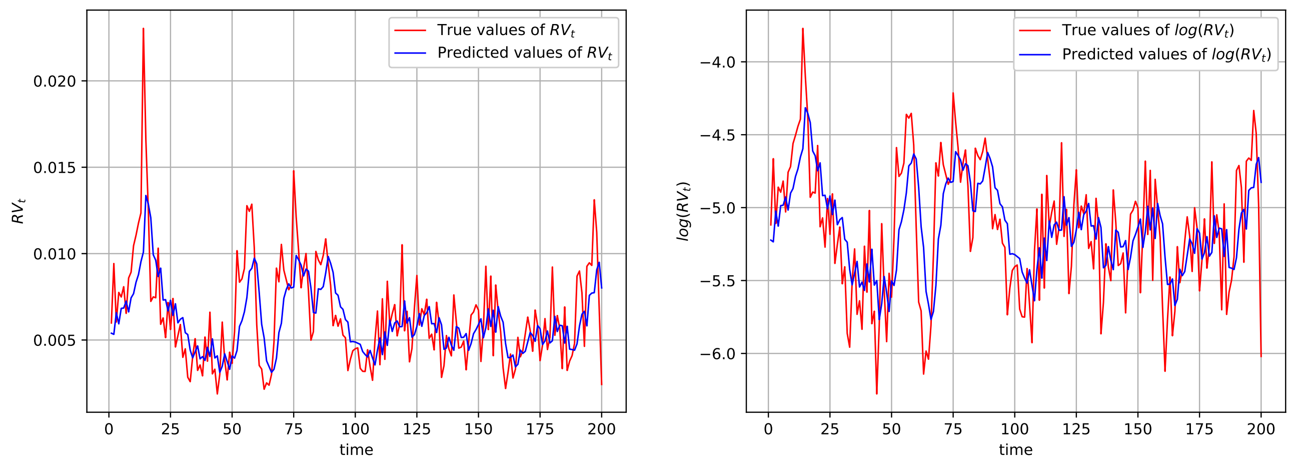


Figure 4: HAR model with the MSE loss, one-step-ahead predictions

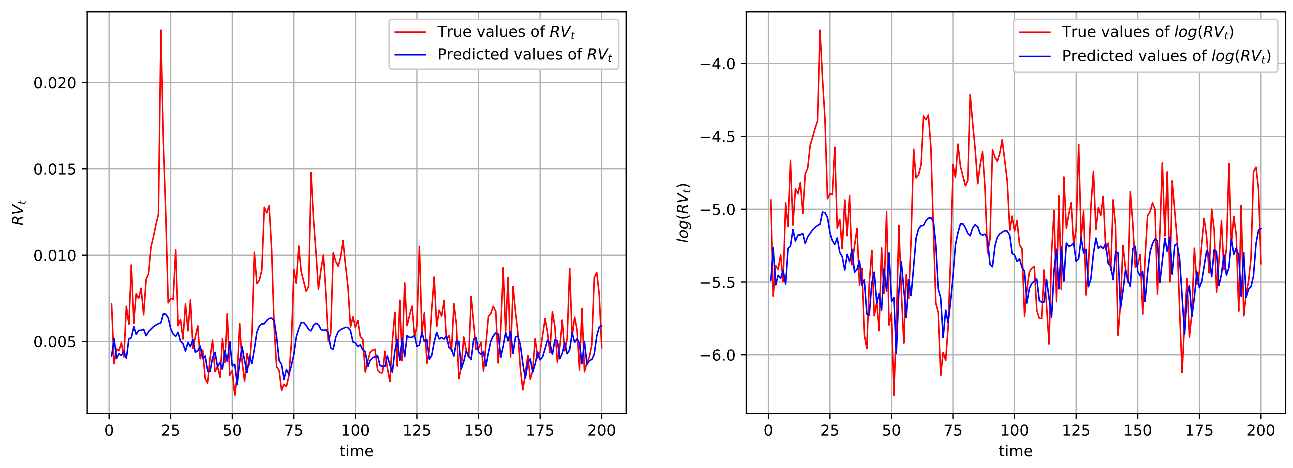


Figure 5: LSTM model with the MSE loss, one-step-ahead predictions

But we can see that often there is an underestimation of the RV, which is undesirable. Move to asymmetric loss. Compare the HAR and the LSTM models, using the LinEx loss function/criteria and the ALS loss function/criteria (again, we used one-step-ahead predictions):

Model	LinEx for $\log(RV_t)$	ALS for $\log(RV_t)$
HAR	0.0142	0.0518
LSTM	0.0666	0.246

As we can see, again the LSTM model predicts worse than the HAR model, even with an asymmetric criteria. On Figures 6–7 one-step-ahead predictions (on the first 200 days of out-of-sample data) of these two models (estimated using asymmetric loss functions) are represented.

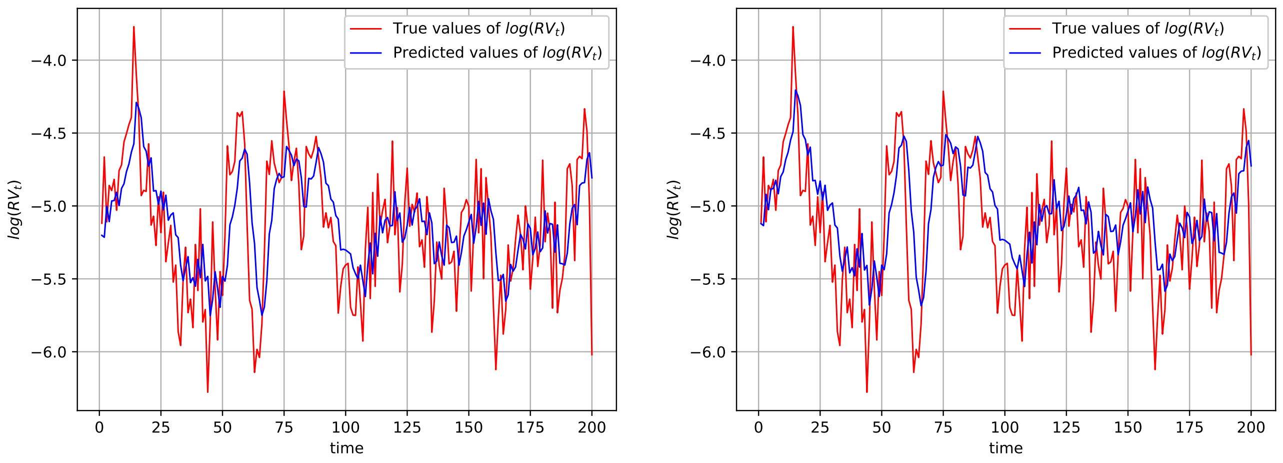


Figure 6: HAR model with the LinEx loss (left) and the ALS loss (right), one-step-ahead predictions

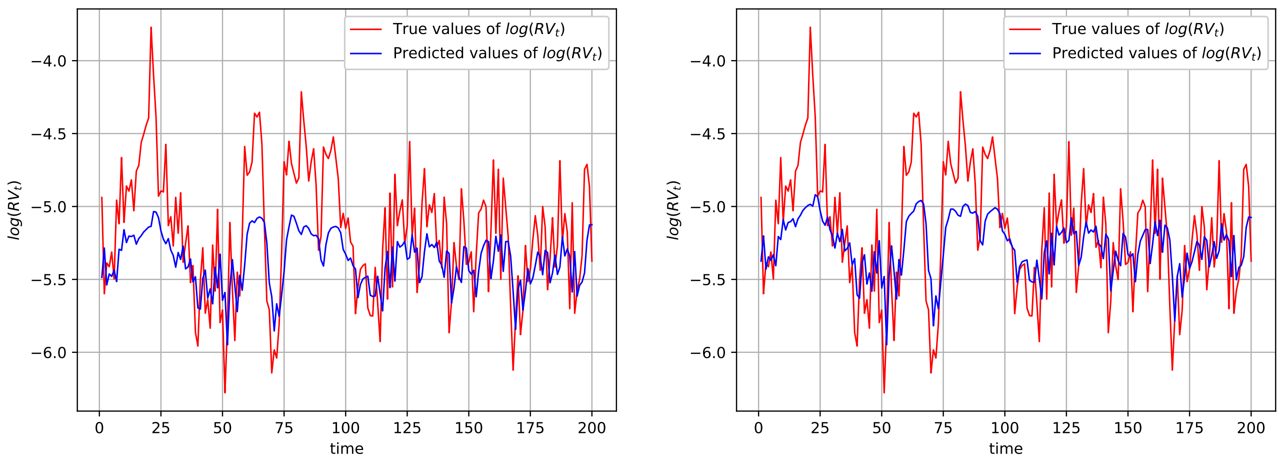


Figure 7: LSTM model with the LinEx loss (left) and the ALS loss (right), one-step-ahead predictions

We can see on the plots that the predictions of the models estimated with asymmetric loss functions are slightly higher (in general) than with symmetric. This is because we assumed that underprediction is less desirable than overprediction. If we want to highlight the asymmetry more, we should increase α (in both asymmetric losses).

8 Conclusion

Firstly, we compared the predictive power of four models: the MEM model with QMLE, the HAR model with the MSE loss, the Random Forest models with the MSE loss and the LSTM model with the MSE loss — with the MSE criteria for RV_t and their logs. And it turned out that the MEM model was the best (by margin). So, the considered machine learning algorithms were not able to outperform the MEM model, an econometric approach.

Secondly, we compared the predictive power of two models: the HAR model (with the LinEx loss and ALS loss) and the LSTM model (with the same losses) — using criteria corresponding to the loss functions of the models. And it turned out, again, that the considered machine learning algorithm (the LSTM model) cannot outperform the LSTM model, an econometric approach (with both asymmetric loss functions).

In the future new criteria and loss functions can be added for the comparison. Also it can be interesting to add jumps in the diffusion equation (and in the HAR model).

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