Lecture Notes for

Foundations in Representation Theory

With a View Towards Homological Algebra

Jendrik Stelzner

Winter Semester 2018/19*
University of Bonn

Available at https://github.com/cionx/foundations-in-representation-theory-notes-ws-18-19.
Please send comments and corrections at stelzner@uni-bonn.de.

*Last Changes: 2019-09-28 04:06:25 +0200 Commit: 2664340

Preface

The following text are my personal notes for the lecture course *Foundations in Representation Theory*, also known as *Homological Algebra*, that was held in the winter semester 2018/19 by Dr. Hans Franzen at the University of Bonn.

The recommended literature are [Mac78] and [Sch72] for category theory, [KS90, Chapter 1] and [Wei94] for homological algebra, and [ASS06] for representation theory. The author can also recommend [Lei14] und [Bra17] for category theory.

The numbering of these notes is consistent with the numbering from the lecture. Additional remarks and statements that were not present in the lecture itself are marked with the symbol * and use their own distinct numbering. Unnumbered remarks and statements were given the lecture, but outside of the usual numbering.

These notes are currently incomplete. The following key points are currently missing:

- The contents of the last two lectures.
- The proof of the snake lemma.
- The proof of the long exact cohomology sequence.
- The proof of the long exact sequence of the cone.
- A list of symbols.

Contents

1 Algebras and Modules

Convention. In this course we adhere to the following conventions:

i) All rings are unital, i.e. there exists for every ring A an element $1 \in A$ with both $1 \cdot a = a$ and $a \cdot 1 = a$ for every $a \in A$.

ii) If A and B are rings then every ring homomorphism $f \colon A \to B$ respects the unit, i.e. it holds that $f(1_A) = 1_B$.

Algebras

Convention. Throughout this lecture $\mathbf{k}$ denotes a commutative ring, that we will often additionally assume to be a field.

Definition 1.1. A $\mathbf{k}$-*algebra* is a ring A together with the structure of a $\mathbf{k}$-module on A, such that the ring multiplication and the scalar multiplication are compatible in the sense that

$$(\lambda a)b = \lambda(ab) = a(\lambda b) \tag{1.1}$$

for all $\lambda \in \mathbf{k}$ and all $a, b \in A$.

Definition 1.2. For $\mathbf{k}$-algebras A and B a map $f \colon A \to B$ is a *homomorphism of* $\mathbf{k}$-*algebras* if it is a ring homomorphism that is also $\mathbf{k}$-linear.

Remark 1.3. Let A be a ring. Then

$$\mathrm{Z}(A) \coloneqq \{a \in A \mid ab = ba \text{ for every } b \in A\}$$

is a commutative subring of A, the *center* of A.

Remark 1.4. Let A be a ring. To give a $\mathbf{k}$-algebra structure on A is the same as giving a ring homomorphism $\varphi \colon \mathbf{k} \to \mathrm{Z}(A)$. More precisely:

i) If A is a $\mathbf{k}$-algebra then we get a ring homomorphism $\tilde{\varphi} \colon \mathbf{k} \to A$ given by $\lambda \mapsto \lambda \cdot 1_A$. This ring homomorphism satisfies $\mathrm{im}(\tilde{\varphi}) \subseteq \mathrm{Z}(A)$ and therefore restricts to a ring homomorphism $\varphi \colon A \to \mathrm{Z}(A)$.

ii) Let $\varphi \colon \mathbf{k} \to \mathrm{Z}(A)$ be a ring homomorphism. Define $\lambda \cdot a = \varphi(\lambda) \cdot a$ for all $\lambda \in \mathbf{k}$ and all $a \in A$. This gives A the structure of a $\mathbf{k}$-module. The compatibility condition (1.1) is satisfied because $\mathrm{im}(\varphi) \subseteq \mathrm{Z}(A)$.

These two constructions are mutually inverse.

iii) Let A and B be $\mathbf{k}$-algebras and let $f\colon A \to B$ be a homomorphisms of rings. Then f is a homomorphism of $\mathbf{k}$-algebras if and only if it is compatible with the corresponding ring homomorphisms $\mathbf{k} \to \mathrm{Z}(A)$ and $\mathbf{k} \to \mathrm{Z}(B)$ in the sense that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "f"] & & B \\
\mathrm{Z}(A) \arrow[u, phantom, sloped, "\subseteq"] & & \mathrm{Z}(B) \arrow[u, phantom, sloped, "\subseteq"] \\
& \mathbf{k} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Example 1.5.

i) Let V be a $\mathbf{k}$-module and consider $\mathrm{End}_{\mathbf{k}}(V)$ with the multiplication

$$\mathrm{End}_{\mathbf{k}}(V) \times \mathrm{End}_{\mathbf{k}}(V) \to \mathrm{End}_{\mathbf{k}}(V)\,, \quad (f,g) \mapsto f \circ g\,.$$

Then $\mathrm{End}_{\mathbf{k}}(V)$ is a ring and becomes a $\mathbf{k}$-algebra via the ring homomorphim

$$\tilde{\varphi}\colon \mathbf{k} \to \mathrm{End}_{\mathbf{k}}(V)\,, \quad \lambda \mapsto \lambda \cdot \mathrm{id}_V\,,$$

for which $\mathrm{im}(\tilde{\varphi}) \subseteq \mathrm{End}_{\mathbf{k}}(V)$. (If $\mathbf{k}$ is a field then $\mathrm{Z}(\mathrm{End}_{\mathbf{k}}(V)) = \mathbf{k} \cdot \mathrm{id}_V \cong \mathbf{k}$.)

ii) Take $V = \mathbf{k}^n$ (the free $\mathbf{k}$-module of rank n). Then

$$\mathrm{End}_{\mathbf{k}}(V) \cong \mathrm{M}_{n\times n}(\mathbf{k})\,,$$

and

$$\mathrm{T}_n(\mathbf{k}) \coloneqq \{M \in \mathrm{M}_{n\times n}(\mathbf{k}) \mid M \text{ is upper triangular}\}$$

is a subalgebra of $\mathrm{M}_{n\times n}(\mathbf{k})$, i.e. it is both a subring and a $\mathbf{k}$-submodule of $\mathrm{M}_{n\times n}(\mathbf{k})$.

iii) Let G be a group. We define the *group algebra* $\mathbf{k}[G]$ as follows: As a $\mathbf{k}$-module we have that

$$\begin{aligned}
\mathbf{k}[G] \coloneqq \mathbf{k}^{(G)} &\coloneqq \text{free } \mathbf{k}\text{-module with basis } G \\
&= \left\{ \sum_{g\in G} a_g[g] \;\middle|\; \begin{array}{c} a_g \in \mathbf{k} \text{ for every } g \in G, \\ \text{all but finitely many } a_g \text{ vanish} \end{array} \right\}.
\end{aligned}$$

The multiplication of two elements $x, y \in \mathbf{k}[G]$ that are given by linear combinations $x = \sum_{g\in G} a_g[g]$ and $y = \sum_{g\in G} b_g[g]$ is given by

$$x \cdot y = \sum_{g,h\in G} a_g b_h [gh] = \sum_{g\in G} \left(\sum_{\substack{h,h'\in G \\ hh'=g}} a_h b_{h'} \right) [g]\,.$$

This multiplication is associative and $\mathbf{k}$-bilinear, and the unit of $\mathbf{k}[G]$ is given by $1_{\mathbf{k}[G]} = [e]$ (where e denotes the neutral element of G).

Definition 1.6. A *quiver* Q is a directed graph. Formally, Q is a quadruple (Q_0, Q_1, s, t) consisting of two sets Q_0, Q_1 and two functions $s, t\colon Q_1 \to Q_0$.

- The elements $i \in Q_0$ are the *vertices* of Q.
- The elements $\alpha \in Q_1$ are the *arrows* of Q.
- For an arrow $\alpha \in Q_1$ the vertex $s(\alpha)$ is the *source* of α.
- For an arrow $\alpha \in Q_1$ the vertex $t(\alpha)$ is the *target* of α.

An arrow $\alpha \in Q_1$ can pictorially be represented as follows:

$$s(\alpha) \xrightarrow{\alpha} t(\alpha)$$

Example 1.7.

i) The quiver $Q = (\{1\}, \emptyset, \emptyset, \emptyset)$ is given by a single vertex, labeled 1, and no arrows:[1]

$$1$$

ii) The quiver $Q = (\{1\}, \{\alpha\}, s, t)$ with (necessarily) $s(\alpha) = t(\alpha) = 1$ is given by a single vertex, labeled 1, together with a single arrow, labeled α:

$$\begin{tikzcd} 1 \arrow[loop above, "\alpha"] \end{tikzcd}$$

iii) The quiver $Q = (\{1, 2\}, \{\alpha, \beta\}, s, t)$ with $s(\alpha) = s(\beta) = 1$ and $t(\alpha) = t(\beta) = 2$ is given by two vertices, labeled 1 and 2, that are connected via two parallel arrows going from 1 to 2, and which are labeled α and β:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}$$

Definition⋆ 1.A. A quiver Q is *finite* if both Q_0 and Q_1 are finite.

Definition 1.8. Let Q be a quiver.

i) Let $\ell \in \mathbb{Z}_{\geq 1}$. A *path* in Q of length ℓ is a sequence $\alpha_\ell \cdots \alpha_1$ of arrows in Q such that $t(\alpha_i) = s(\alpha_{i+1})$ for every i. This may pictorially be represented as follows:

$$\bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_\ell} \bullet$$

The set of all paths of length ℓ in Q is denoted by Q_ℓ. For a path $p = \alpha_\ell \cdots \alpha_1 \in Q_\ell$ its *source* is given by $s(p) \coloneqq s(\alpha_1)$ and its *target* is given by $t(p) \coloneqq t(\alpha_\ell)$.

The set of paths of length 0 in Q is formally defined as Q_0, i.e. there exists for every $i \in Q_0$ a unique path ε_i of length 0, and every path of length 0 is of this form. The path ε_i is the 'lazy path' at i. We set $s(\varepsilon_i) \coloneqq i$ and $t(\varepsilon_i) \coloneqq i$.

[1] The third and fourth elements of the quadrupel $(\{1\}, \emptyset, \emptyset, \emptyset)$ refer to the empty function $\emptyset \to \{1\}$.

ii) Let $p = \alpha_\ell \cdots \alpha_1$ and $q = \beta_k \cdots \beta_1$ be paths in Q of lengths $\ell, k \geq 1$. If $t(p) = s(q)$ then the *composition* $q \circ p$ of p after q is defined as the path

$$q \circ p = \beta_k \cdots \beta_1 \alpha_\ell \cdots \alpha_1 \,.$$

This path $q \circ p$ can pictorially be represented as follows:

$$\bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_\ell} \bullet \xrightarrow{\beta_1} \bullet \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_k} \bullet$$

If p is a path in Q of length $\ell \geq 0$ and $i \in Q_0$ is a vertex then we set $\varepsilon_i \circ p = p$ if $t(p) = i$, as well as $p \circ \varepsilon_i = p$ if $s(p) = i$.

In all other cases the composition of paths is not defined.

iii) The set of all paths in Q is denoted by $Q_* \coloneqq \coprod_{\ell \geq 0} Q_\ell$. The *path algebra* $\mathbf{k}Q$ of Q is the following $\mathbf{k}$-algebra:

- The underlying $\mathbf{k}$-module of $\mathbf{k}Q$ is given by the free $\mathbf{k}$-module on the set Q_*, i.e. by

$$\mathbf{k}Q \coloneqq \mathbf{k}^{(Q_*)} = \left\{ \sum_{p \in Q_*} a_p p \;\middle|\; \begin{array}{c} a_p \in \mathbf{k} \text{ for every } p \in Q_*, \\ \text{all but finitely many } a_p \text{ vanish} \end{array} \right\} .$$

- The multiplication of two elements $x, y \in \mathbf{k}Q$ that are given by linear combinations $x = \sum_{p \in Q_*} a_p p$ and $y = \sum_{p \in Q_*} b_p p$ is given by

$$x \cdot y \coloneqq \sum_{p,q \in Q_*} a_p b_q (p \cdot q) \,,$$

where

$$p \cdot q \coloneqq \begin{cases} p \circ q & \text{if } s(p) = t(q) \,, \\ 0 & \text{else} \end{cases}$$

for all paths $p, q \in Q_*$. (This multiplication is associative because composition of paths is associative and $\mathbf{k}$-bilinear.)

- The unit of $\mathbf{k}Q$ is given by $1_{\mathbf{k}Q} = \sum_{i \in Q_0} \varepsilon_i$

Remark* 1.B. One can alternatively define for all vertices $i, j \in Q_0$ a path of length $\ell \geq 0$ from i to j in Q as a tuple

$$(j \mid \alpha_\ell \cdots \alpha_1 \mid i)$$

(consisting of $\ell + 2$ entries). We can then define the composatition of two such paths $(k \mid \beta_{\ell_2} \cdots \beta_1 \mid j)$ and $(j \mid \alpha_{\ell_1} \cdots \alpha_1 \mid i)$ in Q as the path

$$(k \mid \beta_{\ell_2} \cdots \beta_1 \mid j) \circ (j \mid \alpha_{\ell_1} \cdots \alpha_1 \mid i) \coloneqq (k \mid \beta_{\ell_2} \cdots \beta_1 \alpha_{\ell_1} \cdots \alpha_1 \mid i) \,.$$

This definition of a path already includes that there exists for every vertex $i \in Q_0$ a unique path ε_i of length 0 from i to i. The definition of composition of paths then also includes that for every path p from i to j we have that $\varepsilon_j \circ p = p$ and $p \circ \varepsilon_i = p$. We therefore wouldn't need to treat paths of length 0 separately when using this alternative definition.

Example 1.9. We determine the path algebras of the quivers from Example 1.7.

i) For the quiver
$$Q\colon\ 1$$
its path algebra is given by $\mathbf{k}Q = k$, as it is one-dimensional.

ii) For the quiver
$$Q\colon\ 1 \circlearrowleft \alpha$$
its set of paths is given by $Q_* = \{\alpha^n \mid n \geq 0\}$ with multiplication given by $\alpha^n \cdot \alpha^m = \alpha^n \circ \alpha^m = \alpha^{n+m}$ for all $n, m \geq 0$. It follows that $\mathbf{k}Q \cong \mathbf{k}[t]$ is the polynomial algebra over $\mathbf{k}$ in one indeterminate.

iii) For the quiver
$$Q\colon\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$
its set of paths is given by $Q_* = \{\varepsilon_1, \varepsilon_2, \alpha, \beta\}$. Its path algebra is given by
$$\mathbf{k}Q = \mathbf{k}\varepsilon_1 \oplus \mathbf{k}\varepsilon_2 \oplus \mathbf{k}\alpha \oplus \mathbf{k}\beta\,,$$
with multiplication being given on the basis elements given as follows:

row · column	ε_1	ε_2	α	β
ε_1	ε_1	0	0	0
ε_2	0	ε_2	α	β
α	α	0	0	0
β	β	0	0	0

Example* 1.C. Let Q be the following quiver:
$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} n$$
Then its path algebra $\mathbf{k}Q$ is isomorphic to the $\mathbf{k}$-algebra of lower triangular $(n \times n)$-matrices. (This is Exercise 2 of Exercise sheet 1.)

Lemma 1.10. Let $\mathbf{k}$ be a field and let A be a finite-dimensional $\mathbf{k}$-algebra. Then there exists an injective homomorphism of $\mathbf{k}$-algebras $\varphi\colon A \to \mathrm{M}_{n\times n}(\mathbf{k})$ for $n \coloneqq \dim_{\mathbf{k}}(A)$.

Proof. Consider the $\mathbf{k}$-algebra $\mathrm{End}_{\mathbf{k}}(A)$. It follows from choosing a $\mathbf{k}$-basis of A that $\mathrm{End}_{\mathbf{k}}(A) \cong \mathrm{M}_{n\times n}(\mathbf{k})$. It therefore sufficies to construct an injective homomorphism of $\mathbf{k}$-algebras $\varphi\colon A \to \mathrm{End}_{\mathbf{k}}(A)$. Let $a \in A$. Then the map
$$\varphi(a)\colon A \to A\,, \quad b \mapsto ab$$
is $\mathbf{k}$-linear by (1.1). The map $\varphi\colon A \to \mathrm{End}_{\mathbf{k}}(A)$ is the desired injective homomorphism of $\mathbf{k}$-algebras:

- The map φ is $\mathbf{k}$-linear by (1.1) and by the distributivity of the multiplication of A.

- It holds for all $a, a', b \in A$ that
$$\varphi(aa')(b) = (aa')b = a(a'b) = \varphi(a)(\varphi(a')(b)) = (\varphi(a)\varphi(a'))(b)\,,$$
which shows that $\varphi(aa') = \varphi(a)\varphi(a')$, i.e. that the map φ is multiplicative.
- It holds that $\varphi(1_A) = \mathrm{id}_A = 1_{\mathrm{End}_{\mathbf{k}}(A)}$.
- It holds for $a \in \ker(\varphi)$ that $\varphi(a) = 0$ and therefore that
$$0 = \varphi(a)(1) = a \cdot 1 = a\,,$$
which shows that φ is injective.

This altogether shows the claim. $\square$

End of lecture 1

Definition 1.11. Let A be a $\mathbf{k}$-algebra. The *opposite algebra* A^{op} of A has the same underlying $\mathbf{k}$-module as A but with multiplication given by
$$a \cdot_{A^{\mathrm{op}}} b \coloneqq b \cdot_A a$$
for all $a, b \in A^{\mathrm{op}}$.

Example 1.12. Let Q be a quiver. The *opposite quiver* Q^{op} of Q results from Q by reversing the direction of all its arrows. More formally, if $Q = (Q_0, Q_1, s, t)$ then the opposite quiver Q^{op} is given by $Q^{\mathrm{op}} = (Q_0, Q_1, s^{\mathrm{op}}, t^{\mathrm{op}})$ with
$$s^{\mathrm{op}}(\alpha) = t(\alpha) \quad \text{and} \quad t^{\mathrm{op}}(\alpha) = s(\alpha)$$
for every $\alpha \in Q_1$. Hence
$$(Q_0, Q_1, s, t)^{\mathrm{op}} = (Q_0, Q_1, t, s)\,.$$
It then holds that
$$(\mathbf{k}Q)^{\mathrm{op}} = \mathbf{k}(Q^{\mathrm{op}})\,.$$

Modules

Definition 1.13. Let A be a $\mathbf{k}$-algebra. A *left A-module* M is a $\mathbf{k}$-module M together with a multiplication
$$A \times M \to M\,, \quad (a, m) \mapsto am\,,$$
such that the following conditions are satisfied:

(L1) $a(m_1 + m_2) = am_1 + am_2$,

(L2) $(a_1 + a_2)m = a_1 m + a_2 m$,

(L3) $a_1(a_2 m) = (a_1 a_2)m$,

(L4) $1 \cdot m = m$,

(L5) $(\lambda a)m = \lambda(am) = a(\lambda m)$

for all $a, a_1, a_2 \in A$, all $m, m_1, m_2 \in M$ and all $\lambda \in \mathbf{k}$. That M is a left A-module is often signaled by the notation ${}_A M$.

The notion of a *right A-module* is defined analogous. That M is a right A-module is signaled by the notation M_A. Note that the axiom (R3) for a right A-module M reads

$$(ma_1)a_2 = m(a_1 a_2)$$

for all $a_1, a_2 \in A$ and all $m \in M$. Note also that a right A-module has the underlying structure of a right $\mathbf{k}$-module, and that scalars $\lambda \in k$ therefore also act from the right on M.

With an *A-module* we mean, if not mentioned otherwise, a left A-module.

Remark 1.14. For a $\mathbf{k}$-module M the data of a right A-module structure on M is equivalent to that of a left A^{op}-module structure on M. Hence right A-modules are 'the same' as left A^{op}-modules.

Definition 1.15. Let A be a $\mathbf{k}$-algebra and let M and N be two left A-modules. A map $f\colon M \to N$ is a *homomorphism of A-modules* if is it $\mathbf{k}$-linear and satisfies

$$f(am) = af(m)$$

for all $a \in A$ and all $m \in M$. The set of A-module homomorphisms $M \to N$ is denoted by $\mathrm{Hom}_A(M, N)$. A homomorphism of A-modules $f\colon M \to N$ is an *isomorphism* if it is bijective.

The notion of a homomorphism of right A-modules and that of an isomorphim of right A-modules is defined analogous.

Remark 1.16. Let A be a $\mathbf{k}$-algebra and let M and N be two left A-modules.

i) The set of homomorphisms $\mathrm{Hom}_A(M, N)$ becomes a $\mathbf{k}$-module when endowed with pointwise addition

$$(f + g)(m) \coloneqq f(m) + g(m)$$

and pointwise scalar multiplication

$$(\lambda \cdot f)(m) \coloneqq \lambda \cdot f(m)$$

for all $f, g \in \mathrm{Hom}_A(M, N)$, all $\lambda \in \mathbf{k}$ and all $m \in M$. That this scalar multiplication is well-defined follows from λ being central in A.

ii) The homomorphism space $\mathrm{Hom}_A(M, N)$ does in general carry neither the structure of a left A-module nor that of a right A-module.

iii) A map $f\colon M \to N$ is an isomorphism of A-modules if and only if there exists a homomorphism of left A-modules $g\colon N \to M$ with $g \circ f = \mathrm{id}_M$ and $f \circ g = \mathrm{id}_N$.

iv) Every homomorphism of left A-modules $f\colon M' \to M$ induces a $\mathbf{k}$-linear map

$$f^*\colon \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_A(M',N)\,, \quad h \mapsto h \circ f\,,$$

and every homomorphism of left A-modules $g\colon N \to N'$ induces a $\mathbf{k}$-linear map

$$g_*\colon \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_A(M,N')\,, \quad g \mapsto g \circ h\,.$$

Remark-Definition 1.17. Let A be a $\mathbf{k}$-algebra and let M, N be two left A-modules.

i) A subset $M' \subseteq M$ is a *left A-submodule* of M if
   (S1) $0 \in M'$,
   (S2) $m'_1 + m'_2 \in M'$ for all $m'_1, m'_2 \in M'$,
   (S3) $am' \in M'$ for all $a \in A$ and all $m' \in M'$.

ii) Let $M' \subseteq M$ be a left A-submodule. We can form the quotient $\mathbf{k}$-module M/M', which becomes a left A-module via the scalar multiplication

$$a \cdot (m + I) = (am) + I$$

for all $a \in A$ and all $m + I \in M/M'$. The canonical projection

$$\pi\colon M \to M/M'\,, \quad m \mapsto m + I$$

is a homomorphism of A-modules.

iii) A *left ideal* of A is a left A-submodule of ${}_AA$. The notion of a right ideal is defined analogous.

Note that if $I \subseteq A$ is a left ideal, then the quotient A/I does in general not inherit a $\mathbf{k}$-algebra structure from A.

A subset $I \subseteq A$ is a *two-sided ideal* if it is both a left ideal and a right ideal. The quotient A/I then inherits from A the structure of a $\mathbf{k}$-algebra with multiplication given by

$$(x + I) \cdot (y + I) \coloneqq xy + I$$

for all $x + I, y + I \in A/I$.

iv) Associated to every homomorphism of A-modules $f\colon M \to N$ are
- the *kernel* $\ker(f) \coloneqq \{m \in M \mid f(m) = 0\}$,
- the *image* $\operatorname{im}(f) \coloneqq \{f(m) \mid m \in M\}$,
- the *cokernel* $\operatorname{coker}(f) \coloneqq N/\operatorname{im}(f)$,
- the *coimage* $\operatorname{coim}(f) \coloneqq M/\ker(f)$,

all of which are again A-modules. The homomorphism f factors uniquely as a composition of the canonical projection $M \to \operatorname{coim}(f)$, followed by a homomorphism $\tilde{f}\colon \operatorname{coker}(f) \to \operatorname{im}(f)$ and then by the inclusion $\operatorname{im}(f) \to N$; the induced homomorphism $\tilde{f}$ is then an isomorphism. This results in the following commutative square:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, two heads] & N \\ \operatorname{coim}(f) \arrow[r, dashed, "\cong"] & \operatorname{im}(f) \arrow[u, hook] \end{tikzcd}$$

v) Let $(M_i)_{i\in I}$ be a family of left A-submodules M_i of M. Then the intersection $\bigcap_{i\in I} M_i$ and the sum $\sum_{i\in I} M_i$ are again A-submodules of M.

vi) For $x \in M$ the subset
$$Ax = \{ax \mid a \in A\}$$
is the *A-submodule of M generated by x*. It is the smallest A-submodule of M that contains the element x. For any subset $E \subseteq M$ the subset $\sum_{x\in E} Ax$ of M is the *A-submodule of M generated by E*. It is the smallest A-submodule of M that contains the subset E, i.e. it holds that
$$\sum_{x\in E} Ax = \bigcap_{\substack{\text{submodule } M' \subseteq M \\ \text{with } E \subseteq M'}} M' \,.$$
The A-module M is *finitely generated* if there exist finitely many $x_1, \dots, x_n \in M$ with $M = \sum_{i=1}^{n} Ax_i$.

vii) Let $(M_i)_{i\in I}$ be a family of A-modules. Then the product $\prod_{i\in I} M_i$ and the direct sum $\bigoplus_{i\in I} M_i$ are again left A-modules. For every $j \in I$ both the projection
$$\pi_j \colon \prod_{i\in I} M_i \to M_j \,, \quad (x_i)_{i\in I} \mapsto x_j$$
and the inclusion
$$\iota_j \colon M_j \to \bigoplus_{i\in I} M_i \,, \quad x \mapsto (\delta_{ij} x)_{i\in I}$$
are homomorphism of left A-modules.

viii) The left A-module M is finitely generated if and only if there exists for some $n \in \mathbb{N}$ a surjective homomorphism of left A-modules $A^n \to M$, where $A^n = \bigoplus_{i=1}^{n} A$.

The left A-module M is *finitely presented* if there exists for some $m, n \in \mathbb{N}$ an exact sequence of left A-modules
$$A^m \to A^n \to M \to 0 \,,$$
i.e. there exists a surjective homomorphism of A-modules $u \colon A^n \to M$ for which the kernel $\ker(u)$ is again finitely generated. (Finitely presented A-modules are in particular finitely generated.)

Definition⋆ 1.D.

i) A sequence

$$N \xrightarrow{f} M \xrightarrow{g} P$$

of A-modules and module homomorphisms is *exact* (*at* M) if $\operatorname{im}(f) = \ker(g)$.

ii) A (possibly infinite) sequence of A-modules and module homomorphisms

$$\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \to \cdots$$

is *exact* if it is exact at every M_i.

Proposition 1.18 (Left exactness of Hom)**.**

i) A sequence

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

of A-modules is exact if and only if for every A-module N the induced sequence

$$0 \to \operatorname{Hom}_A(M_3, N) \xrightarrow{f_2^*} \operatorname{Hom}_A(M_2, N) \xrightarrow{f_1^*} \operatorname{Hom}_A(M_1, N)$$

of $\mathbf{k}$-modules is exact.

ii) A sequence

$$0 \to N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3$$

of A-modules is exact if and only if for every A-module M the induced sequence

$$0 \to \operatorname{Hom}_A(M, N_1) \xrightarrow{(g_1)_*} \operatorname{Hom}_A(M, N_2) \xrightarrow{(g_2)_*} \operatorname{Hom}_A(M, N_3)$$

of $\mathbf{k}$-modules is exact.

Proof. We show only the implication ‘ $\Longrightarrow$ ’ for part ii), the rest will be done in the tutorials.

To show that $(g_1)_*$ is injective let $h \in \operatorname{Hom}_A(M, N_1)$ with $0 = (g_1)_*(h) = g_1 \circ h$. It then follows that $\operatorname{im}(h) \subseteq \ker(g_1) = 0$ and therefore that $h = 0$.

The inclusion $\operatorname{im}((g_1)_*) \subseteq \ker((g_2)_*)$ holds because

$$(g_2)_* \circ (g_1)_* = (g_2 \circ g_1)_* = 0_* = 0\,.$$

To show the inclusion $\ker((g_2)_*) \subseteq \operatorname{im}((g_1)_*)$ let $h \in \ker(g_2)_*$. It follows from the injectivity of g_1 that $g_1 = \tilde{g}_1 \circ i$ for a unique isomorphism $\tilde{g}_1 \colon N_1 \to \operatorname{im}(g_1)$ and the inclusion $i \colon \operatorname{im}(g_1) \to N_2$. This results in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[rr, "g_1"] \arrow[dr, dashed, "\tilde{g}_1"'] & & N_2 \arrow[rr, "g_2"] & & N_3 \\
& & \operatorname{im}(g_1) \arrow[ur, hook, "i"] \arrow[rr, equal] & & \ker(g_2) \arrow[ul, hook', "i"'] &
\end{tikzcd}$$

It follows from $0 = (g_2)_*(h) = g_2 \circ h$ that $\operatorname{im}(h) \subseteq \ker(g_2)$ and therefore that $h = i \circ \tilde{h}$ for a unique homomorphism $\tilde{h} \colon M \to \ker(g_2)$. This results in the following commutative digram:

$$\begin{tikzcd}
 & & & M \arrow[d, "h"] \arrow[drr, bend left, "0"] \arrow[ddr, dashed, "\tilde{h}"] & & \\
0 \arrow[r] & N_1 \arrow[rr, "g_1"] \arrow[dr, "\tilde{g}_1"'] & & N_2 \arrow[rr, "g_2"] & & N_3 \\
 & & \operatorname{im}(g_1) \arrow[ur, hook, "i"] \arrow[rr, equal] & & \ker(g_2) \arrow[ul, hook', "i"'] &
\end{tikzcd}$$

It follows for the homomorphism

$$h' \coloneqq \tilde{g}_1^{-1} \circ \tilde{h} \colon M \to N$$

that

$$(g_1)_*(h') = g_1 \circ h' = i \circ \tilde{g}_1 \circ \tilde{g}_1^{-1} \circ \tilde{h} = i \circ \tilde{h} = h\,.$$

This shows the claimed inclusion. □

Proposition 1.19. Let A be a $\mathbf{k}$-algebra.

i) To give a left A-module structure on a $\mathbf{k}$-module V is equivalent to giving a homomorphism of $\mathbf{k}$-algebras $A \to \operatorname{End}_k(V)$.

ii) To give a right A-module structure on a $\mathbf{k}$-module V is equivalent to giving a homomorphism of $\mathbf{k}$-algebras $A \to \operatorname{End}_k(V)^{\mathrm{op}}$.

Proof. This is Exercise 3 on Exercise sheet 1. □

Representations of Quivers

Remark-Definition 1.20. Let Q be a quiver.

i) A *representation* X of Q (over $\mathbf{k}$) consists of the following data:
   - A $\mathbf{k}$-module X_i for every vertex $i \in Q_0$.
   - For every arrow $\alpha \in Q_1$ a $\mathbf{k}$-linear map $X_\alpha \colon X_{s(\alpha)} \to X_{t(\alpha)}$.

Let X, Y and Z be representations of Q.

ii) A *homomorphism* $f \colon X \to Y$ is a tupel $(f_i)_{i \in Q_0}$ of $\mathbf{k}$-linear maps $f_i \colon X_i \to Y_i$ such that the square

$$\begin{tikzcd}
X_{s(\alpha)} \arrow[r, "X_\alpha"] \arrow[d, "f_{s(\alpha)}"'] & X_{t(\alpha)} \arrow[d, "f_{t(\alpha)}"] \\
Y_{s(\alpha)} \arrow[r, "Y_\alpha"] & Y_{t(\alpha)}
\end{tikzcd}$$

commutes for every arrow $\alpha \in Q_1$.

If $f\colon X \to Y$ and $g\colon Y \to Z$ are homomorphisms of representations then the *composition* $g \circ f$ is the homomorphism $X \to Z$ with component $(g \circ f)_i = g_i \circ f_i$ for every $i \in Q_0$. That this is indeed a homomorphism of representations follows from the commutativity of the following diagram for every arrow $\alpha \in Q_1$:

$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "X_\alpha"] \arrow[d, "f_{s(\alpha)}"'] \arrow[dd, bend right=60, "g_{s(\alpha)} \circ f_{s(\alpha)}"'] & X_{t(\alpha)} \arrow[d, "f_{t(\alpha)}"] \arrow[dd, bend left=60, "g_{t(\alpha)} \circ f_{t(\alpha}"] \\ Y_{s(\alpha)} \arrow[r, "Y_\alpha"] \arrow[d, "g_{s(\alpha)}"'] & Y_{t(\alpha)} \arrow[d, "g_{t(\alpha)}"] \\ Z_{s(\alpha)} \arrow[r, "Z_\alpha"] & Z_{t(\alpha)} \end{tikzcd}$$

The *identity homomorphism* of the representation X is the homomorphism of representations $\mathrm{id}_X : X \to X$ with $(\mathrm{id}_X)_i = \mathrm{id}_{X_i}$ for every $i \in Q_0$. That this is indeed a homomorphism of representations follows from the commutativity of the following diagram for every arrow $\alpha \in Q_1$:

$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "X_\alpha"] \arrow[d, "\mathrm{id}_{X_{s(\alpha)}}"'] & X_{t(\alpha)} \arrow[d, "\mathrm{id}_{X_{t(\alpha)}}"] \\ X_{s(\alpha)} \arrow[r, "X_\alpha"] & X_{t(\alpha)} \end{tikzcd}$$

It holds for every homomorphism of representations $f\colon X \to Y$ that both

$$\mathrm{id}_Y \circ f = f \quad \text{and} \quad f \circ \mathrm{id}_X = f\,.$$

iii) A homomorphism $f\colon X \to Y$ of representations is an *isomorphism* if there exists a homomorphism of representations $g\colon Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. The homomorphism f is an isomorphism if and only if at every vertex $i \in Q_0$ the component f_i is an isomorphism (of $\mathbf{k}$-modules).

Example 1.21.

i) For the quiver

$$Q: \ \bullet$$

(that consists of a single vertex without any arrows) a representation of Q is the same a $\mathbf{k}$-module V. For two such representations V and W, a homomorphism of representations $V \to W$ is just a $\mathbf{k}$-linear map.

ii) For the quiver

$$Q: \ \bullet \circlearrowleft$$

a representation of Q is the same as a pair (V, φ) consisting of a $\mathbf{k}$-module V together with a $\mathbf{k}$-linear endomorphism $\varphi\colon V \to V$.

Given two such representations (V, φ) and (W, ψ), a homomorphism of representations $f\colon (V, \varphi) \to (W, \psi)$ is the same as a $\mathbf{k}$-linear map $f\colon V \to W$ with $f \circ \varphi = \psi \circ f$.

iii) For the quiver

$$Q\colon\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

a representation of Q is the same as a quadruple (V_1, V_2, A_1, A_2) consisting of two $\mathbf{k}$-modules V_1 and V_2 and two $\mathbf{k}$-linear maps $A_1, A_2\colon V_1 \to V_2$.

Given two such representations (V_1, V_2, A_1, A_2) and (W_1, W_2, B_1, B_2), a homomorphism of representations $f\colon (V_1, V_2, A_1, A_2) \to (W_1, W_2, B_1, B_2)$ is the same as a pair (f_1, f_2) of $\mathbf{k}$-linear maps $f_1\colon V_1 \to W_1$ and $f\colon V_2 \to W_2$ with $f_2 A_1 = B_1 f_1$ and $f_2 A_2 = B_2 f_1$.

End of lecture 2

Remark 1.22. For a finite quiver Q we can consider its representations over $\mathbf{k}$ as well as modules over its path algebra $\mathbf{k}Q$. It turns out that both concepts are equivalent.

$$\begin{tikzcd}
 & \begin{array}{c} Q \text{ a quiver,} \\ \mathbf{k} \text{ a commutative ring} \end{array} \arrow[dl, rightsquigarrow] \arrow[dr, rightsquigarrow] & \\
\begin{array}{c} \text{left modules} \\ \text{over } \mathbf{k}Q \end{array} \arrow[rr, leftrightarrow] & & \begin{array}{c} \text{representations} \\ \text{of } Q \text{ over } \mathbf{k} \end{array}
\end{tikzcd}$$

i) Let X be a representation of Q over $\mathbf{k}$. We associate to X a left $\mathbf{k}Q$-module module $M = F(X)$ as follows:

As a $\mathbf{k}$-module let $M = \bigoplus_{i \in Q_0} X_i$. Define an action of $\mathbf{k}Q$ on M by actions of the paths $p \in Q_*$: Let p be a path of length ≥ 1 with $p = \alpha_\ell \cdots \alpha_1$ for arrows $\alpha_\ell, \ldots, \alpha_1 \in Q_1$. We define a $\mathbf{k}$-linear map $X_p\colon X_{s(p)} \to X_{t(p)}$ as

$$X_p \coloneqq X_{\alpha_\ell} \cdots X_{\alpha_1}\,.$$

We also define a $\mathbf{k}$-linear map $\tilde{X}_p\colon M \to M$ as the composition

$$\tilde{X}_p\colon M = \bigoplus_{i \in Q_0} X_i \xrightarrow{\pi_{s(p)}} X_{s(p)} \xrightarrow{X_p} X_{t(p)} \xrightarrow{\iota_{t(p)}} \bigoplus_{i \in Q_0} X_i\,.$$

By using these endomorphisms we define on M the structure of a $\mathbf{k}Q$-module via

$$\begin{gathered}
\mathbf{k}Q \times M \to M\,, \\
\left(a = \sum_{p \in Q_*} \lambda_p p, x = (x_i)_{i \in Q_0}\right) \mapsto ax = \sum_{p \in Q_*} \lambda_p \tilde{X}_p(x) = \sum_{p \in Q_*} \lambda_p \iota_{t(p)} X_p(x_{s(p)})\,.
\end{gathered}$$

We have to check that this action satisfies the module axioms. We will check the axiom (L3) as an example: We need to show that

$$a(bx) = (ab)x$$

for all $a, b \in \mathbf{k}Q$ and all $x \in M$. Both expressions are $\mathbf{k}$-bilinear in (a, b), so it sufficies to show this equality for the case that a and b are basis elements of $\mathbf{k}Q$, i.e. paths p and q in Q_*. It then holds that

$$p \cdot (q \cdot x) = \tilde{X}_p \tilde{X}_q(x) = \iota_{t(p)} X_p \underbrace{\pi_{s(p)} \iota_{t(q)}}_{= \begin{cases} \mathrm{id} & \text{if } s(p) = t(q), \\ 0 & \text{otherwise,} \end{cases}} X_q \pi_{s(q)}(x)$$

$$= \begin{cases} \iota_{t(p)} X_p X_q(x_{s(q)}) & \text{if } s(p) = t(q)\,, \\ 0 & \text{otherwise} \end{cases}$$

as well as

$$\underbrace{(p \cdot q)}_{= \begin{cases} p \circ q & \text{if } s(p) = t(q)\,, \\ 0 & \text{otherwise} \end{cases}} \cdot\, x = \begin{cases} \tilde{X}(p \circ q)(x) & \text{if } s(p) = t(q)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

It holds in the case $s(p) = t(q)$ that

$$\tilde{X}_{p \circ q} = \iota_{t(p \circ q)} X_{p \circ q} \pi_{s(p \circ q)} = \iota_{t(p)} X_p X_q \pi_{s(q)}\,,$$

which shows that the two expressions $p \cdot (q \cdot x)$ and $(p \cdot q) \cdot x$ do indeed coincide.

The construction F is functorial: If X and Y are representations of Q over $\mathbf{k}$ and $f \colon X \to Y$ is a homomorphism of representations then we get an induced homomorphism of left $\mathbf{k}Q$-modules $F(f) \colon F(X) \to F(Y)$ given by

$$F(f)\left((x_i)_{i \in Q_0}\right) = (f_i(x_i))_{i \in Q_0}$$

for every $(x_i)_{i \in Q_0} \in \bigoplus_{i \in Q_0} X_i = F(X)$.

ii) Let M be a left $\mathbf{k}Q$-module. We associate to M a representation $X = G(M)$ of Q over $\mathbf{k}$ as follows:

We set $X_i \coloneqq \varepsilon_i M$ for every $i \in Q_0$, which is a $\mathbf{k}$-submodule of M (but in general not a $\mathbf{k}Q$-submodule). For every arrow α of Q we define a $\mathbf{k}$-linear map $X_\alpha \colon X_{s(\alpha)} \to X_{t(\alpha)}$ by

$$X_\alpha(x) \coloneqq \alpha x\,.$$

for every $x \in X_{s(\alpha)}$. This map is well-defined because it holds for every $x \in X_{s(\alpha)}$ (and more generally every $x \in M$) that

$$\alpha x = (\varepsilon_{t(\alpha)} \alpha) x = \varepsilon_{t(\alpha)} \alpha x \in \varepsilon_{t(\alpha)} M = X_{t(\alpha)}\,.$$

This construction is again functorial: Let M and N be left $\mathbf{k}Q$-modules and let $g \colon M \to N$ be a homomorphism of left $\mathbf{k}Q$-modules. Let $X \coloneqq G(M)$ and $Y \coloneqq G(N)$. At every vertex $i \in Q_0$ the homomorphism g restricts to a $\mathbf{k}$-linear map

$$G(g)_i \colon X_i \to Y_i\,, \quad x \mapsto g(x)\,.$$

This restriction is well-defined because

$$g(X_i) = g(\varepsilon_i M) = \varepsilon_i g(M) \subseteq \varepsilon_i N = Y_i\,.$$

To show that $G(g)$ is a homomorphism of representations we need to show that for every arrow α in Q the following square commutes:

$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "X_\alpha"] \arrow[d, "G(g)_{s(\alpha)}"'] & X_{t(\alpha)} \arrow[d, "G(g)_{t(\alpha)}"] \\ Y_{s(\alpha)} \arrow[r, "Y_\alpha"] & Y_{t(\alpha)} \end{tikzcd}$$

It holds for every $x \in X_{s(\alpha)}$ that

$$G(g)_{t(\alpha)}(X_\alpha(x)) = g(\alpha x) = \alpha g(x) = Y_\alpha(G(g)_{s(\alpha)}(x)),$$

which shows the commutativity of the above square.

Theorem 1.23. Let Q be a finite quiver.

i) If M is a left $\mathbf{k}Q$-module then $FG(M) \cong M$ as left $\mathbf{k}Q$-modules.

ii) If X is a representation of Q over $\mathbf{k}$ then $GF(X) \cong X$ as representations of Q.

Proof.

i) Let $X := G(M)$. Then $F(X) = \bigoplus_{i \in Q_0} X_i = \bigoplus_{i \in Q_0} \varepsilon_i M$ as $\mathbf{k}$-modules. We start by showing that the $\mathbf{k}Q$-module M is actually the internal direct sum of its $\mathbf{k}$-submodules $\varepsilon_i M$:

- It follows from $1_{\mathbf{k}Q} = \sum_{i \in Q_0} \varepsilon_i$ that every $x \in M$ can be expressed as

$$x = 1 \cdot x = \sum_{i \in Q_0} \varepsilon_i x \in \sum_{i \in Q_0} \varepsilon_i M .$$

  This shows that $M = \sum_{i \in Q_0} \varepsilon_i M$.

- It holds for all $i, j \in Q_0$ with $i \neq j$ that $\varepsilon_i \varepsilon_j = 0$. It follows for $i \in Q_0$ and $x \in \varepsilon_i M \cap \sum_{j \neq i} \varepsilon_j M$ with $x = \varepsilon_i x_i$ for some $x_i \in M$ and $x = \sum_{j \neq i} \varepsilon_j x_j$ for some $x_j \in M$ that

$$x = \varepsilon_i x_i = \varepsilon_i \varepsilon_i x_i = \varepsilon_i x = \varepsilon_i \sum_{j \neq i} \varepsilon_j x_j = \sum_{j \neq i} \underbrace{\varepsilon_i \varepsilon_j}_{=0} x_j = 0 .$$

  This shows that the sum $\sum_{i \in Q_0} \varepsilon_i M$ is direct.

Together this shows that the $\mathbf{k}$-linear map

$$\varphi \colon F(X) = \bigoplus_{i \in Q_0} \varepsilon_i M \to M , \quad (x_i)_{i \in Q_0} \mapsto \sum_{i \in Q_0} x_i$$

is an isomorphism of $\mathbf{k}$-modules. We claim that φ is already an isomorphism of left $\mathbf{k}Q$-modules. For this we need to show that $\varphi(ax) = a\varphi(x)$ for all $a \in \mathbf{k}Q$

and all $x \in M$. It sufficies to show this equality in the cases that $a = p$ is a path $p \in Q_*$. It then holds for every element $x = (x_i)_{i \in Q_0} \in F(X)$ that

$$\varphi(px) = \varphi(\iota_{t(p)} X_p(x_{s(p)})) \underset{\text{def. } X_p}{=} \varphi(\iota_{t(p)}(p \cdot x_{s(p)})) = p \cdot x_{s(p)}\,.$$

as well as

$$p \cdot \varphi(x) = p \cdot \sum_{i \in Q_0} x_i = p \cdot \sum_{i \in Q_0} \varepsilon_i x_i = \sum_{i \in Q_0} p \varepsilon_i x_i = p \cdot x_{s(p)}\,.$$

because $p\varepsilon_i = 0$ for $i \neq s(p)$ and $p\varepsilon_{s(p)} = p$.

ii) Let $M \coloneqq F(X)$. Then

$$G(M)_i = \varepsilon_i G(M) = \varepsilon_i \bigoplus_{j \in Q_0} X_j \cong X_i$$

at every vertex $i \in Q_0$, and

$$G(M)_\alpha(x_{s(\alpha)}) = \alpha x_{s(\alpha)} = X_\alpha(x_{s(\alpha)})\,.$$

for every arrow $\alpha \in Q_1$. □

Definition⋆ 1.E. An *oriented cycle* in a quiver Q is a path p in Q of length ≥ 1 with $s(p) = t(p)$.

Remark 1.24. Let Q be a finite quiver.

i) If M is a left $\mathbf{k}Q$-module and $\mathbf{k}$ is a field then $\dim_k(M) = \sum_{i \in Q_0} \dim_k(X_i)$ for $X \coloneqq F(M)$.

ii) The path algebra $\mathbf{k}Q$ has finite rank as a $\mathbf{k}$-module if and only if Q contains no oriented cycles.

iii) If a $\mathbf{k}$-algebra A is finitely generated as a $\mathbf{k}$-module then an A-module M is finitely generated as an A-module if and only if it is finitely generated as a $\mathbf{k}$-module:

If M is finitely generated as a $\mathbf{k}$-module then every finite $\mathbf{k}$-generating set of M is also a finite A-generating set for M. If on the other hand M is finitely generated as a left A-module then there exists a surjective homomorphism of A-modules $A^n \to M$; the A-module A^n is again finitely generated as a $\mathbf{k}$-module, and so M is finitely generated as a $\mathbf{k}$-module.

It follows in particular that if Q has no oriented cycles then a $\mathbf{k}Q$-module M is finitely generated as an $\mathbf{k}Q$-module if and only if it is is finitely generated as a $\mathbf{k}$-module. If $\mathbf{k}$ is additionally a field, then this means that M is finitely generated as a $\mathbf{k}Q$-module if and only if it is finite-dimensional.

iv) If X is a representation of Q over $\mathbf{k}$ then a *subrepresentation* of X is a tupel $Y = (Y_i)_{i \in Q_0}$ of $\mathbf{k}$-submodules $Y_i \subseteq X_i$ such that $X_\alpha(Y_{s(\alpha)}) \subseteq Y_{t(\alpha)}$ for every arrow $\alpha \in Q_1$.

If $(Y_i)_{i \in Q_0}$ is a subrepresentation of X and $Y_\alpha \colon Y_{s(\alpha)} \to Y_{t(\alpha)}$ is for every arrow $\alpha \in Q_1$ the restriction of the $\mathbf{k}$-linear map X_α, then this defines a representation Y of Q of $\mathbf{k}$. The inclusion $\iota = (\iota_i)_{i \in Q_0} \colon Y \to X$ that it at every vertex $i \in Q_0$ given by the inclusion $\iota_i \colon Y_i \to X_i$ is then a homomorphism of representations.

The subrepresentations of X correspond under F bijectively to the left $\mathbf{k}Q$-submodules of $F(X)$.

v) If $(X^j)_{j \in J}$ is a family of representations X^j of Q over $\mathbf{k}$ then their *direct sum* is the representations $\bigoplus_{j \in J} X^j$ of Q over $\mathbf{k}$ that is given at vertices by

$$\left(\bigoplus_{j \in J} X^j \right)_i = \bigoplus_{j \in J} X^j_i$$

for every $i \in Q_0$, and on arrows by

$$\left(\bigoplus_{j \in J} X^j \right)_\alpha = \bigoplus_{j \in J} X^j_\alpha \colon \bigoplus_{j \in J} X^j_{s(\alpha)} \longrightarrow \bigoplus_{j \in J} X^j_{t(\alpha)}$$

for every $\alpha \in Q_1$. Under F the direct sum of representations corresponds to the direct sum of left $\mathbf{k}Q$-modules.

Bimodules and Tensor Products

Remark* 1.F. The author recommends [DF04, 10.4] for an introduction to tensor products.

Definition 1.25. Let A and B be $\mathbf{k}$-algebras. An A-B-bimodule is a $\mathbf{k}$-module M together with two $\mathbf{k}$-bilinear multiplications

$$A \times M \to M \,, \quad (a, m) \mapsto am \,,$$

and

$$M \times B \to M \,, \quad (m, b) \mapsto mb$$

such that

(B1) M becomes a left A-module,

(B2) M becomes a right B-module, and

(B3) $(am)b = a(mb)$ for all $a \in A$, $b \in B$ and all $m \in M$.

That M is an A-B-bimodule is denoted by ${}_A M_B$.

Remark 1.26.

i) If ${}_A M_B$ and ${}_A N_B$ are A-B-bimodules then a map $f\colon M \to N$ is a *homomorphisms of bimodules* if it is both a homomorphism of left A-modules and a homomorphism of right B-modules.
If ${}_A M_B$ is an A-B-bimodule then a subset $N \subseteq M$ is a bisubmodule if is both a left A-submodule and a right B-submodule. The A-B-bimodule structure of M then restricts to an A-B-bimodule module structure on N, which is the unique one that makes the inclusion $N \to M$ a homomorphis of A-B-bimodules.

ii) If A is a $\mathbf{k}$-algebra then A carries the structure of an A-A-bimodule by letting A act on itself both via left multiplication and right multiplication.

Lemma 1.27. Let A, B and C be $\mathbf{k}$-algebras, and let ${}_A M_B$ and ${}_A N_C$ be bimodules. Then $\operatorname{Hom}_A(M,N)$ becomes a B-C-bimodule via the multiplications

$$\begin{aligned} B \times \operatorname{Hom}_A(M,N) &\to \operatorname{Hom}_A(M,N)\,, \\ (b,f) &\mapsto \bigl[bf\colon M \to N,\ m \mapsto f(mb)\bigr]\,, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Hom}_A(M,N) \times C &\to \operatorname{Hom}_A(M,N)\,, \\ (g,c) &\mapsto \bigl[gc\colon M \to N,\ m \mapsto g(m)c\bigr]\,. \end{aligned}$$

Proof. We start by showing that for every $f \in \operatorname{Hom}_A(M,N)$ the $\mathbf{k}$-linear maps bf and fc are again homomorphisms of left A-module. This holds because

$$(bf)(am) = f(amb) = af(mb) = a((bf)(m))$$

for all $a \in A$ and all $m \in M$, and

$$(gc)(am) = g(am)c = ag(m)c = a((gc)(m))$$

for all $a \in A$ and all $m \in M$.

To show that $\operatorname{Hom}_A(M,N)$ becomes a left B-module and right C-module we need to verify the various module axioms. As an example, we check the axiom (L3) for the left B-module structure via the calculation

$$((bb')(f))(m) = f(m(bb')) = f((mb)b') = (b'f)(mb) = (b(b'f))(m)$$

for all $f \in \operatorname{Hom}_A(M,N)$, all $b, b' \in B$ and all $m \in M$.

The compatibility of the left B-module structure and right C-module structure of M follow from the calculation

$$((bf)c)(m) = (bf)(m)c = f(mb)c = (fc)(mb) = (b(fc))(m)$$

for all $f \in \operatorname{Hom}_A(M,N)$, all $b \in B$, all $c \in C$ and all $m \in M$. $\square$

End of lecture 3

Definition 1.28. Let A be a $\mathbf{k}$-algebra, let M_A be right A-module and let ${}_A N$ be a left A-module.

i) If P is a $\mathbf{k}$-module then a map $\varphi\colon M \times N \to P$ is *A-balanced* if it is $\mathbf{k}$-bilinear and satisfies

$$\varphi(xa, y) = \varphi(x, ay)$$

for all $x \in M$, all $y \in N$, and all $a \in A$.

ii) A pair (T, τ) consisting of a $\mathbf{k}$-module T and an A-balanced map $\tau\colon M \times N \to T$ is a *tensor product* of M with N over A if it satisifies the following universal property: There exists for every $\mathbf{k}$-module P and every A-balanced map $\varphi\colon M \times N \to P$ a unique $\mathbf{k}$-linear map $f\colon T \to P$ with $f \circ \tau = \varphi$, i.e. such that the following triangle commutes:

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[d, "\tau"'] & P \\ T \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

Lemma 1.29. Let A be a $\mathbf{k}$-algebra, let M_A be a right A-module and let ${}_A N$ be a left A-module.

i) There exists a tensor product (T, τ) of M with N over A.

ii) The tensor product of M with N over A is unique up to unique isomorphism: If (T', τ') is another such tensor product then there exists a unique homomorphism of $\mathbf{k}$-modules $f\colon T \to T'$ with $f \circ \tau = \tau'$, i.e. such that the triangle

$$\begin{tikzcd} & M \times N \arrow[dl, "\tau"'] \arrow[dr, "\tau'"] & \\ T \arrow[rr, dashed, "f"] & & T' \end{tikzcd}$$

commutes, and the homomorphism f is an isomorphism.

Proof. We start by showing the claimed uniqueness of the tensor product. It follows from the universal property of the tensor product (T, τ) applied to the A-balanced map $\tau'\colon M \times N \to T'$ that there exists a unique $\mathbf{k}$-linear map $f\colon T \to T'$ with $f \circ \tau = \tau'$. It also follows from the universal property of the tensor product (T', τ') applied to the A-balanced map $\tau\colon M \times N \to T$ that there exists a unique $\mathbf{k}$-linear map $g\colon T' \to T$ with $g \circ \tau' = \tau$. Together this means that the following two triangles commute:

$$\begin{tikzcd} & M \times N \arrow[dl, "\tau"'] \arrow[dr, "\tau'"] & \\ T \arrow[rr, dashed, "f"] & & T' \end{tikzcd} \qquad \begin{tikzcd} & M \times N \arrow[dl, "\tau'"'] \arrow[dr, "\tau"] & \\ T' \arrow[rr, dashed, "g"] & & T \end{tikzcd}$$

These two triangles team up to form the following commutative diagram:

$$\begin{tikzcd}
 & M \times N \arrow[dl, "\tau"'] \arrow[d, "\tau'"] \arrow[dr, "\tau"] & \\
T \arrow[r, "f"] \arrow[rr, bend right, dashed, "g \circ f"'] & T' \arrow[r, "g"] & T
\end{tikzcd}$$

It follows that $g \circ f$ is the unique homomorphism $T \to T$ that makes the triangle

$$\begin{tikzcd}
 & M \times N \arrow[dl, "\tau"'] \arrow[dr, "\tau"] & \\
T \arrow[rr, dashed] & & T
\end{tikzcd}$$

commute. This shows that $g \circ f = \mathrm{id}_T$ because the identity $\mathrm{id}_T \colon T \to T$ also makes this triangle commute. We find in the same way that also $f \circ g = \mathrm{id}_{T'}$. Together this shows that the homomorphism f is an isomorphism (with inverse g).

Now we show the existence of the tensor product: Let F be the free $\mathbf{k}$-module on the set $M \times N$ and let $U \subseteq F$ be the $\mathbf{k}$-submodule that is generated by the elements

$$(x_1 + x_2, y) - (x_1, y) - (x_2, y)\,, \qquad (x, y_1 + y_2) - (x, y_1) - (x, y_2)\,,$$
$$(\lambda x, y) - \lambda(x, y)\,, \qquad (x, \lambda y) - \lambda(x, y)\,, \qquad (xa, y) - (x, ay)$$

with $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$, $\lambda \in \mathbf{k}$, $a \in A$. We define $M \otimes_A N$ to be the $\mathbf{k}$-module

$$M \otimes_A N \coloneqq F/U$$

and define the required map $\tau \colon M \times N \to M \otimes_A N$ for all $x \in M$ and $y \in N$ by

$$\tau(x, y) \coloneqq x \otimes y \coloneqq [(x, y)]\,.$$

We need to check that $(M \otimes_A N, \tau)$ satisfies the universal property of the tensor product: Let P be a $\mathbf{k}$-module and let $\varphi \colon M \times N \to P$ be an A-balanced map. It follows from the universal property of the free $\mathbf{k}$-module F that there exists a unique $\mathbf{k}$-linear map $\widehat{\varphi} \colon F \to P$ that makes the triangle

$$\begin{tikzcd}
M \times N \arrow[r, "\varphi"] \arrow[d, "\text{incl.}"'] & P \\
F \arrow[ur, dashed, "\widehat{\varphi}"'] &
\end{tikzcd}$$

commute. That φ is A-balanced means precisely that $U \subseteq \ker(\widehat{\varphi})$. It follows that $\widehat{\varphi}$ factors uniquely through a $\mathbf{k}$-linear map $\overline{\varphi} \colon F/U \to P$, which then makes the following diagram commute:

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[d, "\text{incl.}"'] \arrow[dd, bend right, "\tau"'] & P \\ F \arrow[ur, "\widehat{\varphi}"'] \arrow[d, "\text{can.}"'] & \\ F/U \arrow[uur, dashed, bend right, "\overline{\varphi}"'] & \end{tikzcd}$$

This shows that there exists a unique **k**-linear map $f\colon F/U \to P$ with $\varphi = f \circ \otimes$, namely $f = \overline{\varphi}$. □

Notation. For a right A-module M and a left N-module we denote the (up to unique isomorphism unique) tensor product of M by N over A by $M \otimes_A N$, and the associated A-balanced map $M \times N \to M \otimes_A N$ by $\otimes$. For all $x \in M$ and $y \in N$ the resulting element of $M \otimes_A N$ is hence denoted by $x \otimes y$.

Warning⋆ 1.G. For A-modules M_A and ${}_AN$ not every element of their tensor product $M \otimes_A N$ has to be of the form $x \otimes y$ for some $x \in M$ and $y \in Y$. Such elements of $M \otimes_A N$ are *simple tensors*, and every elements of $M \otimes_A N$ is a **k**-linear combination of simple tensors.

It often sufficies to check things for simple tensors, since for every **k**-module P and every **k**-linear map $f\colon M \otimes_A N \to P$, the action of f is already uniquely determined by its action on the simple tensors. But to construct such a **k**-linear map one *has to* (in some instance) evoke the universal property of the tensor product; it does not sufficie to check well-definedness (or **k**-linearity) on simple tensors.

Lemma 1.30. Let A, B and C be **k**-algebras and let ${}_AM_B$ and ${}_BN_C$ be bimodules. Then $M \otimes_B N$ is an A-C-bimodule via

$$a(x \otimes y) = (ax) \otimes y \quad \text{and} \quad (x \otimes y)c = x \otimes (yc)$$

for all $x \in M$, $y \in N$, $a \in A$, $b \in C$.

Proof. The tensor product $M \otimes_A N$ is by definition a **k**-module, so it remains to show that the actions of A and C are well-defined, that they are module structures and that they are mutually compatible.

To show that the proposed action of A on $M \otimes_B N$ is well-defined let $a \in A$. Then the map

$$\tau_a\colon M \times N \to M \otimes_A N\,, \quad (x, y) \mapsto (ax) \otimes y$$

is B-balanced because

$$\begin{aligned}\tau_a(x_1+x_2,y) = (a(x_1+x_2))\otimes y = (ax_1+ax_2)\otimes y = (ax_1)\otimes y + (ax_2)\otimes y \\ = \tau_a(x_1,y)+\tau_a(x_2,y)\,,\\ \tau_a(x,y_1+y_2) = (ax)\otimes(y_1+y_2) = (ax)\otimes y_1 + (ax)\otimes y_2 = \tau_a(x,y_1)+\tau_a(x,y_2)\,,\\ \tau_a(\lambda x,y) = (a\lambda x)\otimes y = (\lambda ax)\otimes y = \lambda((ax)\otimes y) = \lambda\tau_a(x,y)\,,\\ \tau_a(x,\lambda y) = (ax)\otimes(\lambda y) = \lambda((ax)\otimes y) = \lambda\tau_a(x,y)\,,\\ \tau_a(xb,y) = (axb)\otimes y = (ax)\otimes(by) = \tau_a(x,by)\end{aligned}$$

for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$, $\lambda \in \mathbf{k}$, $b \in B$. It follows that that τ_a induces a well-defined $\mathbf{k}$-linear map

$$M \otimes_B N \to M \otimes_B N\,, \quad x \otimes y \mapsto (ax)\otimes y\,.$$

This shows that the proposed action of A on $M \otimes_B N$ is well-defined. It can be shown in the same way that the proposed action of C on $M \otimes_B N$ is well-defined.

It can be checked that the action of A on $M \otimes_B N$ satisfies the various left module axioms, and is therefore a left A-module structure on $M \otimes_B N$. The action of C on $M \otimes_B N$ is similarly a right C-module structure.

The left A-module structure of $M \otimes_B N$ is compatible with the right C-module structure because

$$(a(x\otimes y))c = ((ax)\otimes y)c = (ax)\otimes(yc) = a(x\otimes(yc)) = a((x\otimes y)c)$$

for all $x \in M$, $y \in N$, $a \in A$, $c \in C$. □

Remark 1.31. Let $\varphi\colon B \to A$ be a $\mathbf{k}$-algebra homomorphism and let ${}_BN$ be a left B-module. Then A carries the structure of an A-B-bimodule via

$$axb = ax\varphi(b)$$

for all $x \in A$ and all $a \in A$, $b \in B$. We can regard N as a B-$\mathbf{k}$-bimodule. It then follows that $A \otimes_B N$ carries the structure of an A-$\mathbf{k}$-bimodule, and hence the structure of a left A-module via

$$a' \cdot (a \otimes y) = (a'a)\otimes y$$

for all $a', a \in A$, $y \in N$. This construction is the *extension of scalars* or *induction* from B to A.

Lemma 1.32. Let A, B, C and D be $\mathbf{k}$-algebras.

i) Let $({}_A(M_i)_B)_{i\in I}$ be a family of bimodules and let ${}_BN_C$ be a bimodule. Then there exists a unique homomorphism of A-C-bimodules

$$\Phi_1\colon \left(\bigoplus_{i\in I} M_i\right)\otimes_B N \to \bigoplus_{i\in I}(M_i\otimes_B N)$$

with

$$\Phi_1\left((x_i)_{i\in I}\otimes y\right) = (x_i\otimes y)_{i\in I}$$

for all $(x_i)_{i\in I}\in\bigoplus_{i\in I} M_i$ and $y\in N$, and this homomorphism Φ_1 is an isomorphism.

ii) Let ${}_AM_B$ be a bimodule and let $({}_B(N_j)_C)_{j\in J}$ be a family of bimodules. Then there exists a unique homomorphism of A-C-bimodules

$$\Phi_2\colon M\otimes_B\left(\bigoplus_{j\in J} N_j\right)\to\bigoplus_{j\in J}(M\otimes_B N_j)$$

with

$$\Phi_2\left(x\otimes(y_j)_{j\in J}\right) = (x\otimes y_j)_{j\in J}$$

for all $x\in M$ and $(y_j)_{j\in J}\in\bigoplus_{j\in J} N_j$, and this homomorphism Φ_2 is an isomorphism.

iii) Let ${}_AM_B$, ${}_BN_C$ and ${}_CP_D$ be bimodules. There exists a unique homomorphism of A-D-bimodules

$$\Phi_3\colon (M\otimes_B N)\otimes_C P\to M\otimes_B(N\otimes_C P)$$

with

$$\Phi_3((x\otimes y)\otimes z) = x\otimes(y\otimes z)$$

for all $x\in M$, $y\in N$, $z\in P$, and this homomorphism Φ_3 is an isomorphism.

iv) For every bimodule ${}_AM_B$ there exist unique homomorphisms of A-B-bimodules

$$\Phi_4\colon A\otimes_A M\to M\quad\text{and}\quad\Phi_5\colon M\otimes_B B\to M$$

with

$$\Phi_4(a\otimes m) = am\quad\text{and}\quad\Phi_5(m\otimes b) = mb$$

for all $x\in M$, $a\in A$, $b\in B$, and the homomorphisms Φ_4 and Φ_5 are isomorphisms.

Proof. The proofs are the same as in the commutative case, see Proposition 2.27 in Algebra I. One only has to additionally check that the maps are already homomorphisms of bimodules. □

Proposition 1.33. Let A and B be $\mathbf{k}$-algebras and let M_A, ${}_AN_B$ and P_B be (bi)modules. Then the map

$$\Phi\colon\ \mathrm{Hom}_B(M\otimes_A N, P)\to\mathrm{Hom}_A(M,\mathrm{Hom}_B(N,P))$$

given by

$$\Phi(f)(x)(y) = f(x\otimes y)$$

is a well-defined isomorphism of $\mathbf{k}$-modules, that is natural in M, N and P.

Proof. For every $f \in \operatorname{Hom}_B(M \otimes_A N, P)$ and every $x \in M$ the map $\Phi(f)(x)\colon N \to P$ is a homomorphism of right B-modules because

$$\begin{aligned}\Phi(f)(x)(y_1 + y_2) &= f(x \otimes (y_1 + y_2)) = f(x \otimes y_1 + x \otimes y_2) \\ &= f(x \otimes y_1) + f(x \otimes y_2) = \Phi(f)(x)(y_1) + \Phi(f)(x)(y_2)\end{aligned}$$

and

$$\Phi(f)(x)(yb) = f(x \otimes (yb)) = f((x \otimes y)b) = f(x \otimes y)b = \Phi(f)(x)(y)b$$

for all $y, y_1, y_2 \in N$, $b \in B$. This shows that the map $\Phi(f)(x)$ is a well-defined element of $\operatorname{Hom}_B(N, P)$, which means that for every $f \in \operatorname{Hom}_B(M \otimes_A N, P)$ the map

$$\Phi(f)\colon M \to \operatorname{Hom}_B(N, P)$$

is well-defined. The map $\Phi(f)$ is then a homomorphism of right A-modules because

$$\begin{aligned}\Phi(f)(x_1 + x_2)(y) &= f((x_1 + x_2) \otimes y) = f(x_1 \otimes y + x_2 \otimes y) = f(x_1 \otimes y) + f(x_2 \otimes y) \\ &= \Phi(f)(x_1)(y) + \Phi(f)(x_2)(y) = (\Phi(f)(x_1) + \Phi(f)(x_2))(y)\end{aligned}$$

and

$$\Phi(f)(xa)(y) = f((xa) \otimes y) = f(x \otimes (ay)) = \Phi(f)(x)(ay) = (\Phi(f)(x)a)(y)\,.$$

for all $x, x_1, x_2 \in M$, $a \in A$, $y \in N$. This shows that the proposed map

$$\Phi\colon \operatorname{Hom}_B(M \otimes_A N, P) \to \operatorname{Hom}_A(M, \operatorname{Hom}_B(N, P))$$

is well-defined. The map Φ is a homomorphism of $\mathbf{k}$-modules because

$$\begin{aligned}\Phi(f_1 + f_2)(x)(y) &= (f_1 + f_2)(x \otimes y) = f_1(x \otimes y) + f_2(x \otimes y) \\ &= \Phi(f_1)(x)(y) + \Phi_2(f_2)(x)(y) = (\Phi(f_1) + \Phi(f_2))(x)(y)\end{aligned}$$

and

$$\Phi(\lambda f)(x)(y) = (\lambda f)(x \otimes y) = \lambda f(x \otimes y) = \lambda\Phi(f)(x)(y) = (\lambda\Phi)(f)(x)(y)$$

for all $f, f_1, f_2 \in \operatorname{Hom}_B(M \otimes_A N, P)$, $\lambda \in \mathbf{k}$, $x \in M$, $y \in N$.

To show that Φ is already an isomorphism of $\mathbf{k}$-modules we construct its inverse map: Let $g \in \operatorname{Hom}_A(M, \operatorname{Hom}_B(N, P))$. Then the map

$$\psi(g)\colon M \times N \to P\,, \quad (x, y) \mapsto g(x)(y)$$

is A-balanced because

$$\begin{aligned}\psi(g)(x_1 + x_2, y) &= g(x_1 + x_2)(y) = (g(x_1) + g(x_2))(y) \\ &= g(x_1)(y) + g(x_2)(y) = \psi(g)(x_1, y) + \psi(g)(x_2, y)\,, \\ \psi(g)(x, y_1 + y_2) &= g(x)(y_1 + y_2) = g(x)(y_1) + g(x)(y_2) = \psi(g)(x, y_1) + \psi(g)(x, y_2)\,, \\ \psi(g)(xa, y) &= g(xa)(y) = (g(x)a)(y) = g(x)(ay) = \psi(g)(x, ay)\,.\end{aligned}$$

for all $x, x_1, x_2 \in M$, $y, y_1, y_2 \in M$, $a \in A$. It follows from the universal property of the tensor product $M \otimes_A N$ that the A-balanced map $\psi(g)$ induces a well-defined $\mathbf{k}$-linear map $\Psi(g)\colon M \otimes_A N \to P$ with

$$\Psi(g)(x \otimes y) = \psi(g)(x, y) = g(x)(y)$$

for all $x \in M$, $y \in N$, i.e. such that the following triangle commutes:

$$\begin{tikzcd} M \times N \arrow[r, "\psi(g)"] \arrow[d, "\otimes"'] & P \\ M \otimes_A N \arrow[ur, dashed, "\Psi(g)"'] & \end{tikzcd}$$

The maps $\Psi(g)$ are already homomorphisms of right B-modules because

$$\Psi(g)((x \otimes y)b) = \Psi(g)(x \otimes (yb)) = g(x)(yb) = g(x)(y)b = \Psi(g)(x \otimes y)b$$

for all $x \in M$, $y \in N$, $b \in B$. This shows that $\Psi(g)$ is a well-defined element of the $\mathbf{k}$-module $\mathrm{Hom}_B(M \otimes_A N, P)$, and hence that the map

$$\begin{aligned} \Psi\colon \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P)) &\to \mathrm{Hom}_B(M \otimes_A N, P)\,, \\ g &\mapsto [\Phi(g)\colon x \otimes y \mapsto g(x)(y)] \end{aligned}$$

is well-defined.

The maps Φ and Ψ are mutually inverse: It holds for every $f \in \mathrm{Hom}_B(M \otimes_A N, P)$ that

$$\Psi(\Phi(f))(x \otimes y) = \Phi(f)(x)(y) = f(x \otimes y)$$

for all $x \in M$ and $y \in N$, which shows that $\Psi \circ \Phi = \mathrm{id}$. It similarly holds for every $g \in \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P))$ that

$$\Phi(\Psi(g))(x)(y) = \Psi(g)(x \otimes y) = g(x)(y)\,,$$

for all $x \in X$ and $y \in Y$, which shows that $\Phi \circ \Psi = \mathrm{id}$.

The naturality in M, N and P follows from a direct calculation. □

2 Categories and Functors

Categories

Definition 2.1. A *category* $\mathcal{C}$ consists of the following data:

- A class $\mathrm{Ob}(\mathcal{C})$, the elements of which are the *objects* of $\mathcal{C}$.
- For any two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ a set $\mathcal{C}(X, Y)$,[1] the elements of which are the *morphisms* from X to Y. That $f \in \mathcal{C}(X, Y)$ is denoted by $f\colon X \to Y$ or $X \xrightarrow{f} Y$.
- For any three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a map
$$\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)\,, \quad (g, f) \mapsto g \circ f\,.$$
For any two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$ the resulting morphism $g \circ f\colon X \to Z$ is the *composition* of g and f.

These data are subject to the following conditions:

(C1) The composition of morphisms is associative: It holds for all $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$ that
$$(h \circ g) \circ f = h \circ (g \circ f)\,.$$

(C2) There exists for every object $X \in \mathrm{Ob}(\mathcal{C})$ an *identity morphism* $\mathrm{id}_X\colon X \to X$ such that
$$f \circ \mathrm{id}_X = f \quad \text{and} \quad \mathrm{id}_X \circ g = g$$
for all morphisms $f\colon X \to Y$ and $g\colon Y \to X$ in $\mathcal{C}$.

Remark 2.2. Let $\mathcal{C}$ be a category.

i) It can happen that $\mathcal{C}(X, Y) = \emptyset$ for some objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ i.e. that there exists no morphism from X to Y in $\mathcal{C}$.

ii) For every object $X \in \mathrm{Ob}(\mathcal{C})$ the identity morphism id_X is unique. If id'_X is another identity morphism of X then
$$\mathrm{id}_X = \mathrm{id}_X \, \mathrm{id}'_X = \mathrm{id}'_X\,.$$

Remark⋆ 2.A. One often also requires an additional condition:

(C3) For all objects $X, X', Y, Y' \in \mathrm{Ob}(\mathcal{C})$ the sets $\mathcal{C}(X, Y)$ and $\mathcal{C}(X', Y')$ are disjoint.

[1] Other common notations are $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ or $\mathrm{Mor}_{\mathcal{C}}(X, Y)$.

This condition ensures that for every morphism f in $\mathcal{C}$ both its domain and codomain are unique. In other words, if $f\colon X \to Y$ but also $f\colon X' \to Y'$ (i.e. $f \in \mathcal{C}(X,Y) \cap \mathcal{C}(X',Y')$) then already $X = X'$ and $Y = Y'$.

One can always assume that this additional axiom (C3) holds: If $\mathcal{C}$ is any category then we can replace for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ the set $\mathcal{C}(X,Y)$ by the set of triples

$$\{(X, f, Y) \mid f \in \mathcal{C}(X,Y)\}\,.$$

These sets are then pairwise disjoint.

Remark 2.3. We sometimes want to consider categories whose objects are all sets (which satisfy certain conditions). This can lead to set theoretic problems (also known as *set theoretic difficulties*). One way out of this predicament are *universes*. (See [Mac78, I.6] and [Sch72, 3.2] for more details on this.) We will always fix a universe U and say that

- X is a set if $X \in U$, and that
- X is a class if $X \subseteq U$.

End of lecture 4

Example 2.4.

i) The category **Set** of sets: The objects of **Set** are given by

$$\mathrm{Ob}(\mathbf{Set}) = \{\text{sets (that are elements of the fixed universe)}\}\,,$$

and for any two sets X and Y the morphism set $\mathbf{Set}(X,Y)$ is given by

$$\mathbf{Set}(X,Y) = \{\text{maps } f\colon X \to Y\}\,.$$

The composition of morphisms in **Set** is the usual composition of maps.

ii) The category **Grp** of groups: The objects of **Grp** are given by

$$\mathrm{Ob}(\mathbf{Grp}) = \{\text{groups}\}\,,$$

and for any two groups G and H the morphism set $\mathbf{Grp}(G,H)$ is given by

$$\mathbf{Grp}(G,H) = \{\text{group homomorphisms } f\colon G \to H\}\,.$$

The composition of morphisms in **Grp** is the usual composition of group homomorphisms.

iii) The category **k-Alg** of **k**-algebras:[2] The objects of **k-Alg** are given by

$$\mathrm{Ob}(\mathbf{k\text{-}Alg}) = \{\mathbf{k}\text{-algebras}\}\,,$$

and for any two **k**-algebras A and B the morphism set $(\mathbf{k\text{-}Alg})(A,B)$ is given by

$$\mathbf{k\text{-}Alg}(A,B) = \{\mathbf{k}\text{-algebra homomorphisms } f\colon A \to B\}\,.$$

The composition of morphisms in **k-Alg** is the usual composition of **k**-algebra homomorphisms.

[2] This example was not given in the lecture, but will be used Example 2.9.

iv) For a **k**-algebra A the category A-**Mod** of left A-modules: The objects of A-**Mod** are given by

$$\mathrm{Ob}(A\text{-}\mathbf{Mod}) = \{\text{left } A\text{-modules}\}\,,$$

and for any two left A-modules M and N the morphism set $(A\text{-}\mathbf{Mod})(M, N)$ is given by

$$(A\text{-}\mathbf{Mod})(M, N) = \{\text{homomorphisms of left } A\text{-modules } f \colon M \to N\}\,.$$

The composition of morphisms in A-**Mod** is the usual composition of module homomorphisms.

The category **Mod**-A of right A-modules is defined analogous.

v) The category **Top** of topological spaces: The objects of **Top** are given by

$$\mathrm{Ob}(\mathbf{Top}) = \{\text{topological spaces}\}\,,$$

and for any two topological spaces X and Y the morphism set $\mathbf{Top}(X, Y)$ is given by

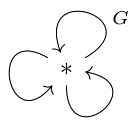
$$\mathbf{Top}(X, Y) = \{\text{continuous maps } f \colon X \to Y\}\,.$$

The composition of morphisms in **Top** is the usual composition of continuous maps.

vi) Let G be a group. We can consider a category $\mathcal{G}$ that consists of a single object $\mathrm{Ob}(\mathcal{G}) = \{*\}$, the single morphism set $\mathcal{G}(*, *) = G$, and for which the composition of morphisms is the multiplication of G, i.e. the composition is given by

$$g \circ h \coloneqq gh$$

for all $g, h \in \mathcal{G}(*, *) = G$. One may pictorially represent the category $\mathcal{G}$ as follows:



vii) Let Q be a quiver. Its *category of paths* $\mathbf{Path}(Q)$, is defined as follows:[3] The objects of $\mathbf{Path}(Q)$ are the vertices of Q, hence

$$\mathrm{Ob}(\mathbf{Path}(Q)) = Q_0\,.$$

For any two vertices i and j in Q the morphism set $\mathbf{Path}(Q)(i, j)$ is given by

$$\mathbf{Path}(Q)(i, j) = \{\text{paths from } i \text{ to } j \text{ in } Q\}\,.$$

The composition of morphisms in $\mathbf{Path}(Q)$ is given by concatenation of paths. This gives a category because concatenation of paths is associative, and because for every $i \in Q_0$ the lazy path $\varepsilon_i \in \mathbf{Path}(Q)(i, i)$ acts as the identity of the object i.

[3] In the lecture, the notation Q_* is used for the category of paths in Q.

Example⋆ 2.B. Let $\mathcal{C}$ be a category. The morphism category $\mathbf{Mor}(\mathcal{C})$ is defined as follows:

- The elements of $\mathbf{Mor}(\mathcal{C})$ are triples (X, f, Y) consisting of two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ and a morphism $f\colon X \to Y$; in other words, the objects of $\mathbf{Mor}(\mathcal{C})$ correspond to morphisms in $\mathcal{C}$.[4]
- A morphism $(X, f, Y) \to (X', f', Y')$ in $\mathbf{Mor}(\mathcal{C})$ is a pair (g, g') consisting of two morphisms $g\colon X \to X'$ and $g'\colon Y \to Y'$ that make the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

  commute.
- The composition of two such morphisms

$$(g, g')\colon (X, f', Y) \to (X', f, Y') \quad \text{and} \quad (h, h')\colon (X', f', Y') \to (X'', f'', Y'')$$

  is given by $(h, h') \circ (g, g') = (h \circ g, h', g')$. This is again a morphism in $\mathbf{Mor}(\mathcal{C})$ by the commutativity of the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dd, bend right=60, dashed, "h \circ g"'] & Y \arrow[d, "g'"] \arrow[dd, bend left=60, dashed, "h' \circ g'"] \\ X' \arrow[r, "f'"] \arrow[d, "h"'] & Y' \arrow[d, "h'"] \\ X'' \arrow[r, "f''"] & Y'' \end{tikzcd}$$

- The identity of an object $(X, f, Y) \in \mathrm{Ob}(\mathbf{Mor}(\mathcal{C}))$ is given by $\mathrm{id}_{(X,f,Y)} = (\mathrm{id}_X, \mathrm{id}_Y)$.

Definition 2.5. Let $\mathcal{C}$ be a category. The *opposite category* $\mathcal{C}^{\mathrm{op}}$ results from $\mathcal{C}$ by formally reversing the direction of all morphisms: The objects of $\mathcal{C}^{\mathrm{op}}$ are given by

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})\,,$$

for any two objects $X, Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$ the morphisms set $\mathcal{C}^{\mathrm{op}}(X, Y)$ is given by

$$\mathcal{C}^{\mathrm{op}}(X, Y) = \mathcal{C}(Y, X)\,,$$

and the composition of any two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}^{\mathrm{op}}$ is given by

$$g \circ_{\mathcal{C}^{\mathrm{op}}} f = f \circ_{\mathcal{C}} g\,.$$

Definition⋆ 2.C. Let $\mathcal{C}$ be a category.

[4] Or at least if we require the additional axiom (C3).

i) A *subcategory* $\mathcal{S}$ of $\mathcal{C}$ consists of
- a subclass $\mathrm{Ob}(\mathcal{S}) \subseteq \mathrm{Ob}(\mathcal{C})$, and
- for any two objects $X, Y \in \mathrm{Ob}(\mathcal{S})$ a subset $\mathcal{S}(X,Y) \subseteq \mathcal{C}(X,Y)$,

that are subject to the following conditions:

(S1) It holds that $\mathrm{id}_X \in \mathcal{S}(X,X)$ for every object $X \in \mathrm{Ob}(\mathcal{S})$.

(S2) It holds that $g \circ f \in \mathcal{S}(X,Z)$ for all objects $X, Y, Z \in \mathrm{Ob}(\mathcal{S})$ and all morphisms $f \in \mathcal{S}(X,Y)$ and $g \in \mathcal{S}(Y,Z)$.

ii) A subcategory $\mathcal{S}$ of $\mathcal{C}$ is *full* if $\mathcal{S}(X,Y) = \mathcal{C}(X,Y)$ for all objects $X, Y \in \mathrm{Ob}(\mathcal{S})$.

Remark⋆ 2.D. Let $\mathcal{C}$ be a category.

i) If $\mathcal{S}$ is a subcategory of $\mathcal{C}$ then $\mathcal{S}$ inherits from $\mathcal{C}$ the structure of a category.

ii) Any full subcategory $\mathcal{S}$ of $\mathcal{C}$ is uniquely determined by its class of objects $\mathrm{Ob}(\mathcal{S})$. Conversely, there exists for every subclass S of $\mathrm{Ob}(\mathcal{C})$ a unique full subcategory $\mathcal{S}$ of $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{S}) = S$.

Functors

Definition 2.6. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* $F \colon \mathcal{C} \to \mathcal{D}$ consists of the following data:

- A map (of classes) $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, $X \mapsto F(X)$.
- For any two objects X and Y of $\mathcal{C}$ a map

$$\mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y)), \quad f \mapsto F(f).$$

These data are subject to the following conditions:

(F1) It holds for every object X of $\mathcal{C}$ that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

(F2) It holds for any two composable morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$ that

$$F(g \circ f) = F(g) \circ F(f).$$

Notation⋆ 2.E. The application of a functor $F\colon \mathcal{C} \to \mathcal{D}$ to an object $X \in \mathrm{Ob}(\mathcal{C})$ or a morphisms $f\colon X \to Y$ in $\mathcal{C}$ is often written without parentheses as FX, resp. as Ff.

Remark 2.7. What we call a ‘functor’ is sometimes called a ‘covariant functor’. A *contravariant functor* $G\colon \mathcal{C} \to \mathcal{D}$ is then a (covariant) functor $G\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

This means in terms of the category $\mathcal{C}$ that G assigns to every object $X \in \mathrm{Ob}(\mathcal{C})$ an object $G(X) \in \mathcal{D}$ and to every morphism $f\colon X \to Y$ in $\mathcal{C}$ a morphism $G(f)\colon G(Y) \to G(X)$ in $\mathcal{D}$, in such a way that

- $G(\mathrm{id}_X) = \mathrm{id}_{G(X)}$ for every $X \in \mathcal{C}$, and
- $G(g \circ f) = G(f) \circ G(g)$ for every pair of composable morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$.

Remark 2.8.

i) Let $\mathcal{C}$ be a category. The *identity functor* $\mathrm{Id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is given by

$$\mathrm{Id}_{\mathcal{C}} \colon \begin{cases} X \mapsto X\,, \\ f \mapsto f\,. \end{cases}$$

ii) If $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are categories and $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ are functors, then they can be composed to a functor $G \circ F \colon \mathcal{C} \to \mathcal{E}$ that is given by

$$(F \circ G) \colon \begin{cases} X \mapsto G(F(X))\,, \\ f \mapsto G(F(f))\,. \end{cases}$$

Example 2.9.

i) We can define two functors that assign to each set X its power set $\mathrm{P}(X)$:
- We define a functor $P_* \colon \mathbf{Set} \to \mathbf{Set}$ that assigns to each set X its power set $\mathrm{P}(X)$, and to each map $f \colon X \to Y$ the induced map
$$P_*(f) \colon \mathrm{P}(X) \to \mathrm{P}(Y)\,, \quad A \mapsto f(A)\,.$$
- We can also define a (contravariant) functor $P^* \colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ that again assigns to each set X its power set $\mathrm{P}(X)$, but to each map $f \colon X \to Y$ the induced map
$$P^*(f) \colon \mathrm{P}(Y) \to \mathrm{P}(X)\,, \quad B \mapsto f^{-1}(B)\,.$$

ii) We have two functors between the categories **k-Alg** and **Grp**:
In one direction we have the functor

$$(-)^\times \colon \mathbf{k}\text{-}\mathbf{Alg} \to \mathbf{Grp}$$

that assigns to each **k**-algebra A its group of units $A^\times$ and to every homomorphism of **k**-algebras $f \colon A \to B$ its induced group homomorphism $f^\times \colon A^\times \to B^\times$ (that results from f by restriction).
In the other direction we have the functor

$$\mathbf{k}[-] \colon \mathbf{Grp} \to \mathbf{k}\text{-}\mathbf{Alg}$$

that assigns to each group G its group algebra $\mathbf{k}[G]$, and to each group homomorphism $\varphi \colon G \to H$ the induced homorphism of **k**-algebras $\mathbf{k}[\varphi] \colon \mathbf{k}[G] \to \mathbf{k}[H]$, i.e. the unique homomorphism of **k**-algebras $\mathbf{k}[G] \to \mathbf{k}[H]$ that makes the square

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, hook] & H \arrow[d, hook] \\ \mathbf{k}[G] \arrow[r, "\mathbf{k}[\varphi]"] & \mathbf{k}[H] \end{tikzcd}$$

commute. (Recall from Exercise sheet 1 that for every $\mathbf{k}$-algebra A, every group homomorphism $G \to A^\times$ extends uniquely to a $\mathbf{k}$-algebra homomorphism $\mathbf{k}[G] \to A$. By applying this to the composition $G \to H \hookrightarrow k[H]^\times$ it follows that there exists a unique $\mathbf{k}$-algebra homomorphism $\mathbf{k}[G] \to \mathbf{k}[H]$ that makes the above square commute.)

iii) The functor

$$V\colon \mathbf{Grp} \to \mathbf{Set}\,, \quad \begin{cases} G \mapsto (G \text{ as a set})\,, \\ f \mapsto f \end{cases}$$

is the *forgetful functor*. More generally, we call every functor *forgetful* if it forgets part of the structure of the objects. We have for example forgetful functors $\mathbf{Top} \to \mathbf{Set}$ and A-$\mathbf{Mod} \to \mathbf{k}$-$\mathbf{Mod}$, where A is a $\mathbf{k}$-algebra.

Example⋆ 2.F. Let $\mathcal{C}$ be a category and let $\mathbf{Mor}(\mathcal{C})$ be the morphism category of $\mathcal{C}$ from Example⋆ 2.B. Then there exist two functors $S, T\colon \mathbf{Mor}(\mathcal{C}) \to \mathcal{C}$:

- The functor S assigns to each object $(X, f, Y) \in \mathrm{Ob}(\mathbf{Mor}(\mathcal{C}))$, which represents a morphism $f\colon X \to Y$ in the category $\mathcal{C}$, its source

$$S(X, f, Y) \coloneqq X\,,$$

  and to each morphism $(h, h')\colon (X, f, Y) \to (X', f', Y')$ its first component

$$S(h, h') \coloneqq (h\colon X \to X')\,.$$

- The functor T assigns to each object $(X, f, Y) \in \mathrm{Ob}(\mathbf{Mor}(\mathcal{C}))$ its target $T(f)$, and to each morphism $(h, h')\colon (h, h')\colon (X, f, Y) \to (X', f', Y')$ its second component $T(h, h') \coloneqq (h'\colon Y \to Y')$.

Example 2.10. Let $\mathcal{C}$ be a category.

i) Every object $X \in \mathrm{Ob}(\mathcal{C})$ gives rise to a functor

$$h^X\colon \mathcal{C} \to \mathbf{Set}\,, \quad \begin{cases} Y \mapsto \mathcal{C}(X, Y)\,, \\ \Big(Y \xrightarrow{f} Y'\Big) \mapsto \Big(\mathcal{C}(X, Y) \xrightarrow{f_*} \mathcal{C}(X, Y')\Big)\,, \end{cases}$$

where the induced map $f_*\colon \mathcal{C}(X, Y) \to \mathcal{C}(X, Y')$ is given by $f_*(g) = f \circ g$ for every $g \in \mathcal{C}(X, Y)$. The functor h^X is also denoted by $\mathcal{C}(X, -)$.

ii) Every object $Y \in \mathrm{Ob}(\mathcal{C})$ gives rise to a (contravariant) functor

$$h_Y\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}\,, \quad \begin{cases} X \mapsto \mathcal{C}(X, Y)\,, \\ \Big(X \xrightarrow{f} X'\Big) \mapsto \Big(\mathcal{C}(X', Y) \xrightarrow{f^*} \mathcal{C}(X, Y)\Big)\,, \end{cases}$$

where the induced map $f^*\colon \mathcal{C}(X', Y) \to \mathcal{C}(X, Y)$ is given by $f^*(g) = g \circ f$ for every $g \in \mathcal{C}(X', Y)$. The functor h_Y is also denoted by $\mathcal{C}(-, Y)$.

Warning⋆ 2.G. Images of functors are not necessarily subcategories.

Definition 2.11. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

i) The functor F is *faithful* if the induced map $\mathcal{C}(X,Y) \to \mathcal{D}(F(X),F(Y))$, $f \mapsto F(f)$ is injective for all $X, Y \in \mathcal{C}$.

ii) The functor F is *full* if the induced map $\mathcal{C}(X,Y) \to \mathcal{D}(F(X),F(Y))$, $f \mapsto F(f)$ is surjective for all $X, Y \in \mathcal{C}$.

iii) The functor F is *fully faithful* if it is both full and faithful, i.e. if the induced map $\mathcal{C}(X,Y) \to \mathcal{D}(F(X),F(Y))$, $f \mapsto F(f)$ is bijective for all $X, Y \in \mathcal{C}$.

Example⋆ 2.H. For every subcategory $\mathcal{S}$ of a category $\mathcal{C}$ there exists an inclusion functor $I\colon \mathcal{S} \to \mathcal{C}$, that is given on objects by the inclusion $\mathrm{Ob}(\mathcal{S}) \hookrightarrow \mathrm{Ob}(\mathcal{C})$, and on morphisms by the inclusion $\mathcal{S}(X,Y) \hookrightarrow \mathcal{C}(X,Y)$ for any two objects $X, Y, \in \mathrm{Ob}(\mathcal{S})$. This inclusion functor I is faithful, and it is full (and hence fully faithful) if and only if $\mathcal{S}$ is full as a subcategory of $\mathcal{C}$.

Isomorphisms

Definition 2.12. A morphism $f\colon X \to Y$ in a category $\mathcal{C}$ is an *isomorphism* if there exist a morphism $g\colon Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Definition 2.11 (Continued)**.**

iv) The functor F is *dense* or *essentially surjective* if there exist for every object $Y \in \mathrm{Ob}(\mathcal{D})$ an object $X \in \mathrm{Ob}(\mathcal{C})$ with $Y \cong F(X)$.

Remark 2.13. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

i) If $f\colon X \to Y$ is an isomorphism in $\mathcal{C}$ then the morphism $g\colon Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ is uniquely determined. Indeed, if g' is another such morphism then
$$g = g \circ \mathrm{id}_Y = g \circ f \circ g' = \mathrm{id}_X \circ g' = g'\,.$$
The morphisms g is the *inverse* of f, and is denoted by f^{-1}.

ii) For every $X \in \mathrm{Ob}(\mathcal{C})$ its identity $\mathrm{id}_X\colon X \to X$ is an isomorphism.

iii) If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and f is an isomorphism in $\mathcal{C}$ then $F(f)$ is an isomorphism in $\mathcal{D}$, and it holds that $F(f)^{-1} = F(f^{-1})$.

Remark⋆ 2.I. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two composable morphisms in a category $\mathcal{C}$. If both f and g are isomorphisms then their composition $g \circ f$ is again an isomorphism. It then holds that
$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}\,.$$

Example 2.14.

i) In the categories **Set**, **Grp**, A-**Mod**, … a morphism f is an isomorphism if and only if it is bijective (as a set-theoretic map).

ii) In the category **Top** the isomorphisms are precisely the homeomorphisms, i.e. the continuous maps that are both bijective and open.

iii) In the path category $\mathbf{Path}(Q)$ of a quiver Q the isomorphisms are precisely the lazy paths $\varepsilon_i = \mathrm{id}_i$ for $i \in Q_0 = \mathrm{Ob}(\mathbf{Path}(Q))$.

Natural Transformations

Definition 2.15. Let $\mathcal{C}$, $\mathcal{D}$ be two categories and let $F, G \colon \mathcal{C} \to \mathcal{D}$ be two parallel functors between them. A *natural transformation* $\eta \colon F \to G$ is a family $\eta = (\eta_X)_{X \in \mathrm{Ob}(\mathcal{C})}$ of morphisms $\eta_X \colon F(X) \to G(X)$ such that for every morphism $f \colon X \to Y$ in $\mathcal{C}$ the following square (in $\mathcal{D}$) commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\eta_Y"] & G(Y)
\end{tikzcd}$$

Remark 2.16. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

i) Let $F, G, H \colon \mathcal{C} \to \mathcal{D}$ be functors and let $\eta \colon F \to G$ and $\zeta \colon G \to H$ be natural transformations. Their composition $\zeta \circ \eta$ is the natural transformation $F \to H$ that is given by

$$(\zeta \circ \eta)_X \coloneqq \zeta_X \circ \eta_X \colon F(X) \to H(X)$$

at every object $X \in \mathrm{Ob}(\mathcal{C})$. To see that this is indeed a natural transfomation we consider the following diagram:

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] \arrow[rr, bend left, dashed, "(\zeta\circ\eta)_X"] & G(X) \arrow[r, "\zeta_X"] \arrow[d, "G(f)"'] & H(X) \arrow[d, "H(f)"] \\
F(Y) \arrow[r, "\eta_Y"] \arrow[rr, bend right, dashed, "(\zeta\circ\eta)_Y"'] & G(Y) \arrow[r, "\zeta_Y"] & H(Y)
\end{tikzcd}$$

In this diagram the left and right squares commute because η and ζ are natural trasformations, and the upper and lower triangles commute by definition of $\zeta \circ \eta$. It follows that the outer square commutes, which shows that $\zeta \circ \eta$ is again a natural transformation.

ii) For any functor $F\colon \mathcal{C} \to \mathcal{D}$ its *identical natural transformation* $\mathrm{id}_F\colon F \to F$ is given by $(\mathrm{id}_F)_X = \mathrm{id}_{F(X)}$ for every $X \in \mathrm{Ob}(\mathcal{C})$. This is indeed a natural transformation because the square

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\mathrm{id}_X"'] & F(Y) \arrow[d, "\mathrm{id}_Y"] \\ F(X) \arrow[r, "F(f)"'] & F(Y) \end{tikzcd}$$

commutes for every morphism $f\colon X \to Y$ in $\mathcal{C}$. It holds for every other functor $G\colon \mathcal{C} \to \mathcal{D}$ that $\eta \circ \mathrm{id}_F = \eta$ for every natural transformation $\eta\colon F \to G$, and also $\mathrm{id}_F \circ \zeta = \zeta$ for every natural transformation $\zeta\colon G \to F$.

Example 2.17. Consider the two functors

$$G \coloneqq (-)^\times \colon \mathbf{k\text{-}Alg} \to \mathbf{Grp} \quad \text{and} \quad F \coloneqq k[-]\colon \mathbf{Grp} \to \mathbf{k\text{-}Alg}\,.$$

We examine how the compositions $F \circ G\colon \mathbf{Grp} \to \mathbf{Grp}$ and $G \circ F\colon \mathbf{k\text{-}Alg} \to \mathbf{k\text{-}Alg}$ relate to the identity functors $\mathrm{Id}_{\mathbf{Grp}}$ and $\mathrm{Id}_{\mathbf{k\text{-}Alg}}$.

Let us first examine the composition $G \circ F$: This functor is on objects given by

$$(G \circ F)(\Gamma) = \mathbf{k}[\Gamma]^\times$$

for every groups Γ. If $\varphi\colon \Gamma \to \Delta$ is a homomorphism of groups, then the group homomorphism

$$(G \circ F)(\varphi) = \mathbf{k}[\varphi]^\times \colon \mathbf{k}[\Gamma]^\times \to \mathbf{k}[\Delta]^\times$$

is the restriction of the induced homomorphism of $\mathbf{k}$-algebras $\mathbf{k}[\varphi]\colon \mathbf{k}[\Gamma] \to \mathbf{k}[\Delta]$ to the groups of units on both sides. We observe that we have every group Γ a group homomorphism

$$\eta_\Gamma\colon \Gamma \to \mathbf{k}[\Gamma]^\times\,, \quad \gamma \mapsto [\gamma]\,.$$

The resulting family $\eta \coloneqq (\eta_X)_{X \in \mathrm{Ob}(\mathbf{Grp})}$ is a natural transformation $\eta\colon \mathrm{Id}_{\mathbf{Grp}} \to G \circ F$, i.e. the square

$$\begin{tikzcd} \Gamma \arrow[r, "\varphi"] \arrow[d, "\eta_\Gamma"'] & \Delta \arrow[d, "\eta_\Delta"] \\ \mathbf{k}[\Gamma]^\times \arrow[r, "\mathbf{k}[\varphi]^\times"] & \mathbf{k}[\Delta]^\times \end{tikzcd}$$

commutes for every group homomorphism $f\colon \Gamma \to \Delta$. This holds because

$$\mathbf{k}[\varphi]^\times(\eta_\Gamma(\gamma)) = \mathbf{k}[\varphi]^\times([\gamma]) = [\varphi(\gamma)] = \eta_\Delta(\varphi(\gamma))$$

for every $\gamma \in \Gamma$.

The functor $F \circ G$ is given on objects by

$$(F \circ G)(A) = \mathbf{k}[A^\times]$$

for every $\mathbf{k}$-algebra A. If $f\colon A \to B$ is a homomorphism of $\mathbf{k}$-algebras then the induced $\mathbf{k}$-algebra homomorphism $(F \circ G)(f)$ is given by

$$(F \circ G)(f) = \mathbf{k}[f^\times]\colon \mathbf{k}[A^\times] \to \mathbf{k}[B^\times]\,, \qquad \sum_{a \in A^\times} \lambda_a [a] \mapsto \sum_{a \in A^\times} \lambda_a [f(a)]\,.$$

For every $\mathbf{k}$-algebra A the identity $A^\times \to A^\times$ corresponds (as seen on Exercise sheet 1) to a homomorphism of $\mathbf{k}$-algebra $\varepsilon_A\colon \mathbf{k}[A^\times] \to A$, that is given by

$$\varepsilon_A\colon \mathbf{k}[A^\times] \to A\,, \qquad \sum_{a \in A^\times} \lambda_a [a] \mapsto \sum_{a \in A^\times} \lambda_a a\,.$$

The resulting family $\varepsilon \coloneqq (\varepsilon_A)_{A \in \mathrm{Ob}(\mathbf{k}\text{-}\mathbf{Alg})}$ is a natural transformation $F \circ G \to \mathrm{Id}_{\mathbf{k}\text{-}\mathbf{Alg}}$, i.e. the square

$$\begin{tikzcd}
\mathbf{k}[A^\times] \arrow[r, "\mathbf{k}[f^\times]"] \arrow[d, "\varepsilon_A"'] & \mathbf{k}[B^\times] \arrow[d, "\varepsilon_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

commutes for every homomorphism of $\mathbf{k}$-algebras $f\colon A \to B$. This holds because

$$\begin{aligned}
\varepsilon_B\left(\mathbf{k}[f^\times]\left(\sum_{a \in A^\times} \lambda_a [a]\right)\right) &= \varepsilon_B\left(\sum_{a \in A^\times} \lambda_a [f(a)]\right) = \sum_{a \in A^\times} \lambda_a f(a) \\
&= f\left(\sum_{a \in A^\times} \lambda_a a\right) = f\left(\varepsilon_A\left(\sum_{a \in A^\times} \lambda_a [a]\right)\right)
\end{aligned}$$

for every $\sum_{a \in A^\times} \lambda_a [a] \in \mathbf{k}[A^\times]$.

We will see later on that the functors $F\colon \mathbf{Grp} \to \mathbf{k}\text{-}\mathbf{Alg}$ and $G\colon \mathbf{k}\text{-}\mathbf{Alg} \to \mathbf{Grp}$ are *adjoint* (with F left adjoint to G), and how this can be expressed via the natural transformations $\eta\colon \mathrm{Id}_{\mathbf{Grp}} \to G \circ F$ and $\varepsilon\colon F \circ G \to \mathrm{Id}_{\mathbf{k}\text{-}\mathbf{Alg}}$.

End of lecture 5

Remark-Definition 2.18. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\eta\colon F \to G$ is a *natural isomorphism* if at every object $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\eta_X\colon F(X) \to G(X)$ is an isomorphism.

The natural transformation η is a natural isomorphism if and only if there exists a natural transformation $\zeta\colon G \to F$ with $\zeta \circ \eta = \mathrm{id}_F$ and $\eta \circ \zeta = \mathrm{id}_G$:

If such a natural transformation exists then it holds for exery $X \in \mathrm{Ob}(\mathcal{C})$ that

$$\zeta_X \circ \eta_X = (\zeta \circ \eta)_X = (\mathrm{id}_F)_X = \mathrm{id}_{F(X)}$$

and similarly $\eta_X \circ \zeta_X = \mathrm{id}_{G(X)}$. This then shows that η_X is for every $X \in \mathrm{Ob}(\mathcal{C})$ an isomorphism, with inverse given by $\eta_X^{-1} = \zeta_X$.

If on the other hand $\eta_X \colon F(X) \to G(X)$ is an isomorphism for every $X \in \mathrm{Ob}(\mathcal{C})$, then it follows for every morphism $f \colon X \to Y$ in $\mathcal{C}$ from the commutativity of the square

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

that the square

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] & F(Y) \\ G(X) \arrow[r, "G(f)"] \arrow[u, "\eta_X^{-1}"] & G(Y) \arrow[u, "\eta_Y^{-1}"'] \end{tikzcd}$$

also commutes. This shows that the family $\zeta \coloneqq (\eta_X^{-1})_{X \in \mathrm{Ob}(\mathcal{C})}$ is a natural transformation $\zeta \colon G \to F$. It then holds by construction of ζ that $\zeta \circ \eta = \mathrm{id}_F$ and $\eta \circ \zeta = \mathrm{id}_G$.

That η is a natural isomorphism is denoted by

$$\eta \colon F \xrightarrow{\sim} G\,.$$

The two functors F and G are *isomorphic* if there exist a natural isomorphism $F \xrightarrow{\sim} G$. That the functors F and G are isomorphic is denoted by $F \cong G$.

Definition 2.19. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if there exists a functor $G \colon \mathcal{D} \to \mathcal{C}$ with $G \circ F \cong \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$. The categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exists an equivalence of categories between them. That $\mathcal{C}$ and $\mathcal{D}$ are equivalent is denoted by $\mathcal{C} \simeq \mathcal{D}$.

Example 2.20.

i) Let Q be a finite quiver. Theorem 1.23 shows that $\mathbf{Rep}_{\mathbf{k}}(Q) \simeq \mathbf{k}Q\text{-}\mathbf{Mod}$, where $\mathbf{Rep}_{\mathbf{k}}(Q)$ denotes the category of representations of Q over $\mathbf{k}$; the only missing ingredient is that the constructed isomorphisms

$$FG(M) \cong M \quad \text{and} \quad GF(X) \cong X$$

for $M \in \mathrm{Ob}(\mathbf{k}Q\text{-}\mathbf{Mod})$ and $X \in \mathrm{Ob}(\mathbf{Rep}_{\mathbf{k}}(Q))$ are natural, i.e. that for every homomorphism of left $\mathbf{k}Q$-modules $f \colon M \to N$ the square

$$\begin{tikzcd} FG(M) \arrow[r, "FG(f)"] \arrow[d, "\sim"'] & FG(N) \arrow[d, "\sim"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes, and that for every homomorphism of representations $f \colon X \to Y$ the square

$$\begin{tikzcd} GF(X) \arrow[r, "GF(f)"] \arrow[d, "\sim"'] & GF(Y) \arrow[d, "\sim"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

ii) Let G be a group. A *representation* of G over $\mathbf{k}$ is a pair (V, ρ) consisting of a $\mathbf{k}$-module V and a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$. A *homomorphism of representations* $f\colon (V, \rho) \to (W, \sigma)$ is a $\mathbf{k}$-linear map $f\colon V \to W$ such that the square

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ W \arrow[r, "\sigma(g)"] & W \end{tikzcd}$$

commutes for every $g \in G$. It holds for the category $\mathbf{Rep}_{\mathbf{k}}(G)$ of representations of G over $\mathbf{k}$ that $\mathbf{Rep}_{\mathbf{k}}(G) \simeq \mathbf{k}[G]\text{-}\mathbf{Mod}$.

iii) Let $\mathcal{C}$ be the category whose objects are pairs (A, φ) consisting of a ring A and a ring homomorphism $\varphi\colon \mathbf{k} \to \mathrm{Z}(A)$, and where a morphism $f\colon (A, \varphi) \to (B, \psi)$ is a ring homomorphism $f\colon A \to B$ that makes the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \\ \mathrm{Z}(A) \arrow[u, phantom, "\cup\!\mid"] & & \mathrm{Z}(B) \arrow[u, phantom, "\cup\!\mid"] \\ & \mathbf{k} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commute. Then Remark 1.4 shows that the categories $\mathbf{k}\text{-}\mathbf{Alg}$ and $\mathcal{C}$ are equivalent.

iv) Let A be a $\mathbf{k}$-algebra. Let $\mathcal{D}$ be the category whose objects are pairs (V, φ) consisting of a $\mathbf{k}$-module V and a homomorphism of $\mathbf{k}$-algebras $\varphi\colon A \to \mathrm{End}_{\mathbf{k}}(V)$, and where a morphism $f\colon (V, \varphi) \to (W, \psi)$ is a $\mathbf{k}$-linear map $f\colon V \to W$ that makes the squares

$$\begin{tikzcd} V \arrow[r, "\varphi(a)"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ W \arrow[r, "\psi(a)"] & W \end{tikzcd}$$

commute for every $a \in A$. Proposition 1.19 shows that the categories A-$\mathbf{Mod}$ and $\mathcal{D}$ are equivalent.

Functor Categories

Definition 2.21. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. The *functor category* $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ has as objects

$$\mathrm{Ob}(\mathbf{Fun}(\mathcal{C}, \mathcal{D})) \coloneqq \{\text{functors } F\colon \mathcal{C} \to \mathcal{D}\},$$

and for any two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ their morphism set $\mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, G)$ is given by

$$\mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, G) \coloneqq \{\text{natural transformations } \eta\colon F \to G\}.$$

The composition of morphisms in $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ is the composition of natural transformations.

Remark 2.22. We're running into set-theoretic issues again: If $\mathcal{C}$ and $\mathcal{D}$ are two categories for our fixed universe U (i.e. $\mathrm{Ob}(\mathcal{C}), \mathrm{Ob}(\mathcal{D}) \subseteq U$) then $\mathrm{Ob}(\mathbf{Fun}(\mathcal{C}, \mathcal{D}))$ might not be a subset of U. The solution to this problem is to choose another, larger universe V with $U \in V$. Then $\mathrm{Ob}(\mathbf{Fun}(\mathcal{C}, \mathcal{D})) \subseteq V$, so $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ becomes a category with respect to the universe V.

Example 2.23. Let Q be a quiver and consider the category $\mathbf{Fun}(\mathbf{Path}(Q), \mathbf{k}\text{-}\mathbf{Mod})$.

Every functor $V \in \mathrm{Ob}(\mathbf{Fun}(\mathbf{Path}(Q), \mathbf{k}\text{-}\mathbf{Mod}))$ gives rise to a representation $F(V)$ of Q over $\mathbf{k}$ with

$$F(V)_i \coloneqq V(i)$$

for every $i \in Q_0 = \mathrm{Ob}(\mathbf{Path}(Q))$ and

$$F(V)_\alpha \coloneqq V(\alpha)\colon F(V)_i \to F(V)_j$$

for every arrow α from i to j in Q_0 (and thus morphisms from i to j in $\mathbf{Path}(Q)$). In this way we obtain a functor

$$F\colon \mathbf{Fun}(\mathbf{Path}(Q), \mathbf{k}\text{-}\mathbf{Mod}) \to \mathbf{Rep}_{\mathbf{k}}(Q)\,.$$

Conversely, let X be a representation of Q over $\mathbf{k}$. We can use X to define a functor $G(X)\colon \mathbf{Path}(Q) \to \mathbf{k}\text{-}\mathbf{Mod}$ via

$$G(X)\colon \begin{cases} \qquad\qquad\quad i \mapsto X_i\,, \\ (p = \alpha_\ell \cdots \alpha_1) \mapsto X_{\alpha_\ell} \circ \cdots \circ X_{\alpha_1}\,. \end{cases}$$

This construction yields a functor

$$G\colon \mathbf{Rep}_{\mathbf{k}}(Q) \to \mathbf{Fun}(\mathbf{Path}(Q), \mathbf{k}\text{-}\mathbf{Mod})\,.$$

It can now be checked that $G \circ F = \mathrm{Id}_{\mathbf{Fun}(\mathbf{Path}(Q), \mathbf{k}\text{-}\mathbf{Mod})}$ and $F \circ G = \mathrm{Id}_{\mathbf{Rep}_{\mathbf{k}}(Q)}$, which shows that $\mathbf{Rep}_{\mathbf{k}}(Q) \simeq \mathbf{Fun}(\mathbf{Path}(Q), \mathbf{k}\text{-}\mathbf{Mod})$.

Example* 2.J. Let $\mathcal{C}$ be a category and let 2 denote the category that consists of two objects 0 and 1, and in which there exists precisely one non-idenity morphism, namely $\alpha\colon 0 \to 1$. The category 2 may be visualized as follows:

$$1 \xrightarrow{\alpha} 2$$

A functor $F\colon 2 \to \mathcal{C}$ is the same as a choice of objects $F(1), F(2) \in \mathrm{Ob}(\mathcal{C})$ together with a morphism $F(\alpha)\colon F(1) \to F(2)$. Given two such functors $F, G\colon 2 \to \mathcal{C}$, a morphism $\eta\colon F \to G$ consists of two morphisms $\eta_1\colon F(1) \to G(1)$ and $\eta_2\colon F(2) \to G(2)$ that make the square

$$\begin{tikzcd} F(1) \arrow[r, "\eta_1"] \arrow[d, dashed, "F(\alpha)"'] & F(2) \arrow[d, dashed, "G(\alpha)"] \\ G(1) \arrow[r, "\eta_2"] & G(2) \end{tikzcd}$$

commute.

We see from this explicit description of the functor category $\mathbf{Fun}(2, \mathcal{C})$ that it is equivalent to the morphism category $\mathbf{Mor}(\mathcal{C})$.

Definition 2.24. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. For every object $X \in \mathrm{Ob}(\mathcal{C})$ the *evaluation* at X is the functor $\mathrm{ev}_X \colon \mathbf{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$ given by

$$\mathrm{ev}_X \colon \begin{cases} F \mapsto F(X)\,, \\ (F \xrightarrow{\eta} G) \mapsto (\eta_X \colon F(X) \to G(X))\,. \end{cases}$$

Remark 2.25. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. If $f \colon X \to Y$ is a morphism in $\mathcal{C}$ then we get an induced natural transformation $\mathrm{ev}_f \colon \mathrm{ev}_X \to \mathrm{ev}_Y$ given by

$$(\mathrm{ev}_f)_F \coloneqq F(f) \colon F(X) \to F(Y)$$

for every functor $F \in \mathrm{Ob}(\mathbf{Fun}(\mathcal{C}, \mathcal{D}))$. This is indeed a natural transformation: Let $F, G \in \mathrm{Ob}(\mathbf{Fun}(\mathcal{C}, \mathcal{D}))$ be functors and let $\eta \colon F \to G$ be a natural transformation between them. We then have the following diagram:

$$\begin{tikzcd}
\mathrm{ev}_X(F) \arrow[rrr, "\mathrm{ev}_X(\eta)"] \arrow[ddd, "(\mathrm{ev}_f)_F"'] \arrow[dr, equal] & & & \mathrm{ev}_X(G) \arrow[ddd, "(\mathrm{ev}_f)_G"] \arrow[dl, equal] \\
& F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] & \\
& F(Y) \arrow[r, "\eta_Y"'] & G(Y) & \\
\mathrm{ev}_Y(F) \arrow[rrr, "\mathrm{ev}_Y(\eta)"'] \arrow[ur, equal] & & & \mathrm{ev}_Y(G) \arrow[ul, equal]
\end{tikzcd}$$

The inner square commutes because η is a natural transformation, and so it follows that the outer square commutes.

It also holds that $\mathrm{ev}_{\mathrm{id}_X} = \mathrm{id}_{\mathrm{ev}_X}$ and that $\mathrm{ev}_{f \circ g} = \mathrm{ev}_f \circ \mathrm{ev}_g$ for every two composable morphisms $f \colon X \to Y$ and $g \colon Y \to Z$ in $\mathcal{C}$. We hence obtain a functor

$$\mathrm{ev} \colon \mathcal{C} \to \mathbf{Fun}(\mathbf{Fun}(\mathcal{C}, \mathcal{D}), \mathcal{D})\,.$$

Representable Functors

Lemma 2.26 (Yoneda's lemma)**.** Let $\mathcal{C}$ be a category and let $X \in \mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$.

i) Let $F \colon \mathcal{C} \to \mathbf{Set}$ be a (covariant) functor. Then the map

$$Y^{F,X} \colon \mathbf{Fun}(\mathcal{C}, \mathbf{Set})(h^X, F) \to F(X)\,, \quad (\eta \colon h^X \to F) \mapsto \eta_X(\mathrm{id}_X)$$

is a bijection.

ii) Let $G\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a (contravariant) functor. Then the map

$$Y_{G,X} : \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})(h_X, G) \to G(X)\,, \quad (\eta\colon h_X \to G) \mapsto \eta_X(\mathrm{id}_X)$$

is a bijection.

Proof. Part i) follows from part ii) because $\mathcal{C} = (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ and

$$h^X_{(\mathcal{C})} = \mathcal{C}(X, -) = \mathcal{C}^{\mathrm{op}}(-, X) = h_X^{(\mathcal{C}^{\mathrm{op}})}$$

for every $X \in \mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})$. We therefore only prove part ii).

To show that the map $Y_{G,X}$ is injective we need to show that a natural transformation $\eta\colon h_X \to G$ is uniquely determined by its value $\eta_X(\mathrm{id}_X)$. This holds because it follows for every $Y \in \mathrm{Ob}(\mathcal{C})$ and every $f \in h_X(Y) = \mathcal{C}(Y, X)$ from the commutativity of the square

$$\begin{tikzcd} h_X(X) \arrow[r, "f^*"] \arrow[d, "\eta_X"'] & h_X(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

that

$$\eta_Y(f) = \eta_Y(\mathrm{id}_X \circ f) = \eta_Y(f^*(\mathrm{id}_X)) = G(f)(\eta_X(\mathrm{id}_X))\,.$$

To show the surjectivity of $Y_{G,X}$ let $z \in G(X)$. For every $Y \in \mathrm{Ob}(\mathcal{C})$ let

$$\zeta_Y : h_X(Y) = \mathcal{C}(Y, X) \to G(Y)\,, \quad f \mapsto G(f)(z)\,.$$

It then holds for every morphism $g\colon Y \to Y'$ in $\mathcal{C}$ that the square

$$\begin{tikzcd} h_X(Y') \arrow[r, "g^*"] \arrow[d, "\zeta_{Y'}"'] & h_X(Y) \arrow[d, "\zeta_Y"] \\ G(Y') \arrow[r, "G(g)"] & G(Y) \end{tikzcd}$$

commutes, because

$$G(g)(\zeta_{Y'}(f)) = G(g)(G(f)(z)) = G(f \circ g)(z) = \zeta_Y(f \circ g) = \zeta_Y(g^*(f))$$

for every $f \in h_X(Y')$. This shows that $\zeta \coloneqq (\zeta_X)_{X \in \mathrm{Ob}(\mathcal{C})}$ is a natural transformation $\zeta\colon h_X \to G$. It holds that

$$Y_{G,X}(\zeta) = \zeta_X(\mathrm{id}_X) = G(\mathrm{id}_X)(z) = \mathrm{id}_{G(X)}(z) = z\,,$$

which altogether shows that $Y_{G,X}$ is surjective. $\square$

Remark 2.27. The above proof shows that the inverse of the map $Y_{G,X}$ is given by

$$\begin{aligned} Y_{G,X}^{-1} : G(X) &\to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})(h_X, G)\,, \\ z &\mapsto \big(\eta_Y : h_X(Y) \to G(Y), f \mapsto G(f)(z)\big)_{Y \in \mathrm{Ob}(\mathcal{C})}\,. \end{aligned}$$

The inverse of the map $Y^{F,X}$ is similarly given by

$$(Y^{F,X})^{-1}\colon F(X) \to \mathbf{Fun}(\mathcal{C}, \mathbf{Set})(h^X, F)\,, \\ z \mapsto \big(\eta_Y\colon h^X(Y) \to F(Y), f \mapsto F(f)(z)\big)_{Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}})}\,.$$

$\boxed{\text{End of lecture 6}}$

Theorem 2.28 (Yoneda embedding)**.** Let $\mathcal{C}$ be a category.

i) The functor $h^{(-)}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Fun}(\mathcal{C}, \mathbf{Set})$ is fully faithful.

ii) The functor $h_{(-)}\colon \mathcal{C} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ is fully faithful.

Proof. It again sufficies to show part ii).

We need to show that for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ the map

$$\Phi\colon \mathcal{C}(X,Y) \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})(h_X, h_Y)\,, \quad f \mapsto f_*$$

is a bijection. We do so by exhibiting an inverse for Φ. For this we apply Yoneda's lemma to the functors $h_X, h_Y\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ to find that the map

$$\Psi\colon \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})(h_X, h_Y) \longrightarrow h_Y(X) = \mathcal{C}(X,Y)\,, \\ \zeta \longmapsto \zeta_X(\mathrm{id}_X)$$

is a bijection. We claim that this is the required inverse to Φ. Indeed, we have that

$$\Psi(\Phi(f)) = (\Phi(f))_X(\mathrm{id}_X) = f_*(\mathrm{id}_X) = f \circ \mathrm{id}_X = f$$

for all $f \in \mathcal{C}(X,Y)$, and hence $\Psi \circ \Phi = \mathrm{id}$. This shows that Φ is a right inverse to Ψ, and hence the (two-sided) inverse of Ψ (because Ψ is a bijection). $\square$

Definition 2.29. Let $\mathcal{C}$ be a category.

i) A (covariant) functor $F\colon \mathcal{C} \to \mathbf{Set}$ is *representable* if it is naturally isomorphic to a functor $h^X\colon \mathcal{C} \to \mathbf{Set}$ for some object $X \in \mathrm{Ob}(\mathcal{C})$.

ii) A (contravariant) functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is *representable* if it is naturally isomorphic to a functor $h_X\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ for some object $X \in \mathrm{Ob}(\mathcal{C})$.

The object X is then a *representing object* for F.

Remark. If a functor $F\colon \mathcal{C} \to \mathbf{Set}$ or $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ admits a representing object $X \in \mathrm{Ob}(\mathcal{C})$, then X is unique up to unique isomorphism.

Equivalence of Categories Revisited

Theorem 2.30. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence if and only if it is both fully faithful and dense.

Proof. Suppose first that F is an equivalence. Then let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor with $G \circ F \cong \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$, and let $\eta\colon G \circ F \to \mathrm{Id}_{\mathcal{C}}$ and $\zeta\colon F \circ G \to \mathrm{Id}_{\mathcal{D}}$ be natural isomorphisms.

The functor F is dense because it holds for every object $Y \in \mathrm{Ob}(\mathcal{D})$ that $F(X) \cong Y$ for the object $X \coloneqq G(Y) \in \mathrm{Ob}(\mathcal{C})$ via the isomorphism $\zeta_Y\colon F(G(Y)) \to Y$.

To show that F is fully faithful we first show that F is faithful, and then, by using that G is faithful, that F is full.

To see that F is faithful we note that for every morphism $f\colon X \to X'$ in $\mathcal{C}$ the square

$$\begin{tikzcd} GF(X) \arrow[r, "GF(f)"] \arrow[d, "\eta_X"'] & GF(X') \arrow[d, "\eta_{X'}"] \\ X \arrow[r, "f"] & X' \end{tikzcd} \tag{2.1}$$

commutes by the naturality of η. The morphism f is therefore uniquely determined by its image $GF(f)$ via the relation

$$f = \eta_{X'} \circ GF(f) \circ \eta_X^{-1} .$$

To show that F is full we first note that G is faithful, because we can switch the roles of F and G in the above discussion. Let $X, X' \in \mathrm{Ob}(\mathcal{C})$ be two objects and let $g\colon F(X) \to F(X')$ be a morphism in $\mathcal{D}$. To find a morphism $f\colon X \to X'$ with $F(f) = g$ we use the above calculation as a motivation, and define

$$f \coloneqq \eta_{X'} \circ G(g) \circ \eta_X^{-1}$$

This ensures that the square

$$\begin{tikzcd} GF(X) \arrow[r, "G(g)"] \arrow[d, "\eta_X"'] & GF(X') \arrow[d, "\eta_{X'}"] \\ X \arrow[r, "f"] & X' \end{tikzcd} \tag{2.2}$$

commutes. By applying the functor GF to this diagram we get the following commutative square:

$$\begin{tikzcd} GFGF(X) \arrow[r, "GFG(g)"] \arrow[d, "GF(\eta_X)"'] & GFGF(X') \arrow[d, "GF(\eta_{X'})"] \\ GF(X) \arrow[r, "GF(f)"] & GF(X') \end{tikzcd} \tag{2.3}$$

We can also get a similar commutative square by applying the naturality of $\eta\colon GF \to \mathrm{Id}_{\mathcal{C}}$ to the morphism $G(g)\colon GF(X) \to GF(X')$, resulting in the following commutative square:

$$\begin{tikzcd} GFGF(X) \arrow[r, "GFG(g)"] \arrow[d, "\eta_{GF(X)}"'] & GFGF(X') \arrow[d, "\eta_{GF(X')}"] \\ GF(X) \arrow[r, "G(g)"] & GF(X') \end{tikzcd} \tag{2.4}$$

We can also apply the naturality of η to the morphism $\eta_X \colon GF(X) \to X$ to get the following commutative square:

$$\begin{tikzcd} GFGF(X) \arrow[r, "GF(\eta_X)"] \arrow[d, "\eta_{GF(X)}"'] & GFG(X) \arrow[d, "\eta_X"] \\ GF(X) \arrow[r, "\eta_X"] & X \end{tikzcd}$$

It follows from the commutativity of this last square that

$$\eta_X \circ GF(\eta_X) = \eta_X \circ \eta_{GF(X)}$$

and hence

$$GF(\eta_X) = \eta_{GF(X)}$$

because η_X is an isomorphism. We find in the same way that also $GF(\eta_{X'}) = \eta_{GF(X')}$. With this we find that the commutative squares (2.3) and (2.4) coincide in both vertical morphisms and in the upper morphism. It hence follows that

$$\begin{aligned} GF(f) &= GF(\eta_{X'}) \circ GFG(g) \circ GF(\eta_X)^{-1} \\ &= \eta_{GF(X')} \circ GFG(g) \circ \eta_{GF(X)}^{-1} = G(g)\,. \end{aligned}$$

Because G is faithful it further follow that $g = F(f)$, which shows that F is fulll.

Suppose now on the other hand that the functor F is both fully faithful and dense. For every object $Y \in \mathrm{Ob}(\mathcal{D})$ let $G(Y) \in \mathrm{Ob}(\mathcal{C})$ be an object with $FG(Y) \cong Y$; we choose an isomorphism $\varepsilon_Y \colon FG(Y) \to Y$. If $g \colon Y \to Y'$ is a morphism in $\mathcal{D}$ then there exist for the conjugated morphism $\varepsilon_{Y'}^{-1} \circ g \circ \varepsilon_Y \colon FG(Y) \to FG(Y')$ a unique morphism $G(g) \colon G(Y) \to G(Y')$ in $\mathcal{D}$ with $FG(g) = \varepsilon_{Y'}^{-1} \circ g \circ \varepsilon_Y$, because F is fully faithful.

We claim that G is a functor $G \colon \mathcal{D} \to \mathcal{C}$ with both $G \circ F \cong \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$.

We first show that G is a functor: If $Y \in \mathrm{Ob}(\mathcal{D})$ then

$$\varepsilon_Y^{-1} \circ \mathrm{id}_Y \circ \varepsilon_Y = \mathrm{id}_{FG(Y)} = F(\mathrm{id}_{G(Y)})$$

and hence $\mathrm{id}_{G(Y)} = G(\mathrm{id}_Y)$. It holds for any two composable morphisms $g \colon Y \to Y'$ and $g' \colon Y' \to Y''$ in $\mathcal{D}$ that

$$\begin{aligned} &\varepsilon_{Y''}^{-1} \circ (g' \circ g) \circ \varepsilon_Y \\ &= \varepsilon_{Y''}^{-1} \circ g' \circ \varepsilon_{Y'} \circ \varepsilon_{Y'}^{-1} \circ g \circ \varepsilon_Y \\ &= FG(g') \circ FG(g) \\ &= F(G(g') \circ G(g))\,, \end{aligned}$$

which shows that $G(g' \circ g) = G(g') \circ G(g)$.

To show that $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$ we note that $\varepsilon \coloneqq (\varepsilon_Y)_{Y \in \mathrm{Ob}(\mathcal{D})}$ is a natural isomorphism $\varepsilon\colon F \circ G \to \mathrm{Id}_{\mathcal{D}}$. That ε is a natural transformation, i.e. that the square

$$\begin{tikzcd} FG(Y) \arrow[r, "FG(g)"] \arrow[d, "\varepsilon_Y"'] & FG(Y') \arrow[d, "\varepsilon_{Y'}"] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

commutes for every morphism $g\colon Y \to Y'$ in $\mathcal{D}$, holds by construction of $G(g)$. That ε_Y is an isomorphism for every $Y \in \mathcal{D}$ holds by choice of ε_Y.

To show that $G \circ F \cong \mathrm{Id}_{\mathcal{C}}$ we construct a natural isomorphism $\eta\colon G \circ F \to \mathrm{Id}_{\mathcal{C}}$:

There exist for every object $X \in \mathcal{C}$ for the morphisms $\varepsilon_{F(X)}\colon FGF(X) \to F(X)$ a unique morphisms $\eta_X\colon GF(X) \to X$ with $\varepsilon_{F(X)} = F(\eta_X)$ because F is fully faithful. We set $\eta \coloneqq (\eta_X)_{X \in \mathrm{Ob}(\mathcal{C})}$.

The family η is a natural transformation $\eta\colon G \circ F \to \mathrm{Id}_{\mathcal{C}}$: Let $f\colon X \to X'$ be a morphism in $\mathcal{C}$. Then the square

$$\begin{tikzcd} FGF(X) \arrow[r, "FGF(f)"] \arrow[d, "\varepsilon_{F(X)}"'] & FGF(X') \arrow[d, "\varepsilon_{F(X')}"] \\ F(X) \arrow[r, "F(f)"] & F(X') \end{tikzcd}$$

commutes because $\varepsilon\colon FG \to \mathrm{Id}_{\mathcal{D}}$ is a natural transformation. We may rewrite this diagram as

$$\begin{tikzcd} FGF(X) \arrow[r, "FGF(f)"] \arrow[d, "F(\eta_X)"'] & FGF(X') \arrow[d, "F(\eta_{X'})"] \\ F(X) \arrow[r, "F(f)"] & F(X') \end{tikzcd}$$

by construction of η. We thus find that

$$F(f \circ \eta_X) = F(f) \circ F(\eta_X) = F(\eta_{X'}) \circ FGF(f) = F(\eta_{X'} \circ GF(f)) .$$

It follows from F being faithful that already

$$\eta_{X'} \circ GF(f) = f \circ \eta_X ,$$

i.e. that the square

$$\begin{tikzcd} GF(X) \arrow[r, "GF(f)"] \arrow[d, "\eta_X"'] & GF(X') \arrow[d, "\eta_{X'}"] \\ X \arrow[r, "f"] & X' \end{tikzcd} \tag{2.5}$$

commutes. This shows that $\eta\colon GF \to \mathrm{Id}_{\mathcal{C}}$ is indeed a natural transformation.

It follows for every $X \in \mathrm{Ob}(\mathcal{C})$ from $\varepsilon_{F(X)} = F(\eta_X)$ being an isomorphism that η_X is again an isomorphism: Indeed, there exists for the inverse $\varepsilon_{F(X)}^{-1}\colon F(X) \to FGF(X)$ by the fully faithfulness of F a unique morphism $\eta'_X\colon X \to GF(X)$ with $\varepsilon_{F(X)}^{-1} = F(\eta'_X)$. Then

$$F(\eta_X \circ \eta'_X) = F(\eta_X) \circ F(\eta'_X) = \varepsilon_{F(X)} \circ \varepsilon_{F(X)}^{-1} = \mathrm{id}_{F(X)} = F(\mathrm{id}_X)$$

and hence $\eta_X \circ \eta'_X = \mathrm{id}_X$ because F is faithful. It can be shown similarly that also $\eta'_X \circ \eta_X = \mathrm{id}_{GF(X)}$. This shows that the morphism η_X is an isomorphism with $\eta_X^{-1} = \eta'_X$.

This shows altogether the claim that η is a natural isomorphism $\eta\colon G \circ F \to \mathrm{Id}_{\mathcal{C}}$. ☐

Remark⋆ 2.K. The above proof displays an important property that a faithful functor $F\colon \mathcal{C} \to \mathcal{D}$ possesses:

i) An identity between morphisms in $\mathcal{C}$ holds if and only if it holds after applying the functor F. In particular, a diagram in $\mathcal{C}$ commutes if and only if it does so after applying the functor F to it. We have used this observation in the above proof to show the commutativity of the square (2.5).

If F is not only faithful but also full, and hence fully faithful, then we can observe the following:

ii) A morphism $f\colon X \to X'$ in $\mathcal{C}$ is an isomorphism if and only if the morphism $F(f)\colon F(X) \to F(X')$ in $\mathcal{D}$ is an isomorphism. (This means that the functor F *reflects* isomorphisms.) We have used this observation in the above proof to show that η_X is again an isomorphism.

Remark⋆ 2.L.

i) To show in the first part of the proof that $G(g) = GF(f)$ (to show that F is full) one can also compare the squares (2.1) and (2.2) to find that

$$G(g) = \eta_{X'}^{-1} \circ f \circ \eta_X = GF(f)\,.$$

ii) The author prefers a slightly different argumentation for the fully faithfulness of the functor F in the first part of the proof:

It follows for any two objects $X, X' \in \mathrm{Ob}(\mathcal{C})$ from the commutativity of the square (2.1) that the composition

$$\mathcal{C}(X,X') \xrightarrow{F} \mathcal{D}(F(X),F(X')) \xrightarrow{G} \mathcal{C}(GF(X),GF(X')) \xrightarrow{\eta_{X'} \circ (-) \circ \eta_X^{-1}} \mathcal{C}(X,X')$$

is the identity $\mathrm{id}_{\mathcal{C}(X,X')}$. This tells us that F is fully faithful, and that G is full 'on the image of F', i.e. that the map

$$\mathcal{D}(Y,Y') \xrightarrow{G} \mathcal{C}(G(Y),G(Y'))$$

is surjective for any two objects $Y, Y' \in \mathrm{Ob}(\mathcal{D})$ that are of the form $Y = F(X)$ and $Y' = F(X')$ for suitable objects $X, X' \in \mathrm{Ob}(\mathcal{C})$.

We now use that the functor F is dense to extend this surjectivity to any two objects $Y, Y' \in \mathrm{Ob}(\mathcal{C})$: There exist objects $X, X' \in \mathrm{Ob}(\mathcal{C})$ for which there exist isomorphisms $\varepsilon\colon Y \to F(X)$ and $\varepsilon'\colon Y' \to F(X')$. It follows from the functoriality of G that $G(\varepsilon)\colon G(Y) \to GF(X)$ and $G(\varepsilon')\colon G(Y') \to GF(X')$ are again isomorphisms, and that the square

$$\begin{tikzcd}
\mathcal{D}(Y, Y') \arrow[r, "\varepsilon' \circ (-) \circ \varepsilon^{-1}"] \arrow[d, "G"'] & \mathcal{D}(F(X), F(X')) \arrow[d, "G"] \\
\mathcal{C}(G(Y), G(Y')) \arrow[r, "G(\varepsilon') \circ (-) \circ G(\varepsilon)^{-1}"] & \mathcal{C}(GF(X), GF(Y'))
\end{tikzcd}$$

commutes. The horizontal maps in this square are bijections because the morphisms ε, ε', $G(\varepsilon)$ and $G(\varepsilon')$ are isomorphisms; we have also seen above that the vertical map on the right is surjective, It follows that the vertical map on the left is surjective as well, which shows that G is full.

By switching the roles of F and G we also find that G is faithful and that F is full. This altogether shows that both F and G are fully faithful.

Adjunctions

Definition 2.31. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An *adjunction* (or *adjoint pair*) from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, G, φ) consisting of two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$, together with a family $(\varphi_{X,Y})_{X \in \mathrm{Ob}(\mathcal{C}), Y \in \mathrm{Ob}(\mathcal{D})}$ of bijections

$$\varphi_{X,Y} : \mathcal{D}(F(X), Y) \to \mathcal{C}(X, G(Y))$$

that are natural in both X and Y. The functor F is the *left adjoint* of the adjunction, and the functor G is the *right adjoint* of the adjunction.

Remark. The naturality in Definition 2.31 means that for every morphism $f\colon X \to X'$ in $\mathcal{C}$ and every object $Y \in \mathrm{Ob}(\mathcal{D})$ the square

$$\begin{tikzcd}
\mathcal{D}(F(X'), Y) \arrow[r, "\varphi_{X',Y}"] \arrow[d, "F(f)^*"'] & \mathcal{C}(X', G(Y)) \arrow[d, "f^*"] \\
\mathcal{D}(F(X), Y) \arrow[r, "\varphi_{X,Y}"] & \mathcal{C}(X, G(Y))
\end{tikzcd}$$

commutes, and that for every object $X \in \mathrm{Ob}(\mathcal{C})$ and every morphisms $g\colon Y \to Y'$ in $\mathcal{D}$

the square

$$\begin{tikzcd} \mathcal{D}(F(X), Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "g_*"'] & \mathcal{C}(X, G(Y)) \arrow[d, "G(g)_*"] \\ \mathcal{D}(F(X), Y') \arrow[r, "\varphi_{X,Y'}"] & \mathcal{C}(X, G(Y')) \end{tikzcd}$$

commutes.

Remark⋆ 2.M. The naturality of φ in Definition 2.31 amount to φ being a natural isomorphism

$$\varphi\colon \mathcal{D}(F(-), -) \to \mathcal{C}(-, G(-))$$

between the two (bi)functors

$$\mathcal{D}(F(-), -), \mathcal{C}(-, G(-))\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}\,.$$

Example 2.32. We give examples for adjoint pairs, where $F\colon \mathcal{C} \to \mathcal{D}$ is the left adjoint and $G\colon \mathcal{D} \to \mathcal{C}$ is the right adjoint.

i) We have an adjunction from the category $\mathcal{C} = \mathbf{Set}$ to the category $\mathcal{D} = \mathbf{Grp}$. The left adjoint functor $F\colon \mathbf{Set} \to \mathbf{Grp}$ assigns to every set X the free group on X, and to every map $f\colon X \to X'$ between sets X and X' the induced group homomorphisms

$$F(f)\colon F(X) \to F(X')\,, \quad F(x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}) = f(x_1)^{\varepsilon_1} \cdots f(x_n)^{\varepsilon_n}\,.$$

The right adjoint functor $G\colon \mathbf{Grp} \to \mathbf{Set}$ is the forgetful functor.

ii) Let A be a $\mathbf{k}$-algebra. We have an adjunction from the category $\mathcal{C} = \mathbf{Set}$ to the category $\mathcal{D} = A\text{-}\mathbf{Mod}$. The left adjoint functor $F\colon \mathbf{Set} \to A\text{-}\mathbf{Mod}$ assigns to each set X the free A-module on X, and to every map $f\colon X \to X'$ between sets X and X' the induced homomorphisms of A-modules

$$F(f)\colon F(X) \to F(X')\,, \quad F(f)\left(\sum_{x \in X} a_x[x]\right) = \sum_{x \in X} a_x[f(x)]\,.$$

The right adjoint functor $G\colon A\text{-}\mathbf{Mod} \to \mathbf{Set}$ is the forgetful functor.

iii) We have an adjunction from the category $\mathcal{C} = \mathbf{Grp}$ to the category $\mathcal{D} = \mathbf{k}\text{-}\mathbf{Alg}$. The left adjoint functor $F\colon \mathbf{Grp} \to \mathbf{k}\text{-}\mathbf{Alg}$ is given by the group algebra functor $F = \mathbf{k}[-]$ and the right functor $G\colon \mathbf{k}\text{-}\mathbf{Alg} \to \mathbf{Grp}$ is given by the unit group functor $G = (-)^{\times}$.

iv) We have an adjunctions from the category $\mathcal{C} = \mathbf{Set}$ to the category $\mathcal{D} = \mathbf{k}\text{-}\mathbf{CAlg}$ of commutative $\mathbf{k}$-algebras.[5] The left adjoint functor $F\colon \mathbf{Set} \to \mathbf{k}\text{-}\mathbf{CAlg}$ assigns

[5] In the lecture the notation **k-CommAlg** is used instead. The author prefers the shorter version **k-CAlg** as it helps him to avoid overfull hboxes.

to each set X the polynomial ring $k[T_x \mid x \in X]$ and to each map $f\colon X \to X'$ between sets X and X' the unique homomorphism of **k**-algebras

$$F(f)\colon k[T_x \mid x \in X] \to k[T_{x'} \mid x' \in X'] ,$$

that satisfies $F(f)(T_x) = T_{f(x)}$ for every $x \in X$. The right adjoint functor $G\colon \mathbf{k\text{-}CAlg} \to \mathbf{Set}$ is the forgetful functor.

v) Let A and B be two **k**-algebras and let ${}_A N_B$ be an A-B-bimodule. We then have an adjunction from the module category $\mathcal{C} = \mathbf{Mod}\text{-}A$ to the module category $\mathcal{D} = \mathbf{Mod}\text{-}B$. The left adjoint functor $F\colon \mathbf{Mod}\text{-}A \to \mathbf{Mod}\text{-}B$ assigns to each right A-module M_A the right B-module $F(M) = M \otimes_A N$, and the right adjoint functor $G\colon \mathbf{Mod}\text{-}B \to \mathbf{Mod}\text{-}A$ assigns to each right B-module P_B the right A-module $G(P) = \mathrm{Hom}_B(N, P)$. The required bijections $\varphi_{M,P}$ are given by Proposition 1.33.

vi) Let $\varphi\colon A \to B$ be a homomorphism of **k**-algebras. We have an adjunction from the module category $\mathcal{C} = A\text{-}\mathbf{Mod}$ to the module category $\mathcal{D} = B\text{-}\mathbf{Mod}$. The left adjoint functor F assigns to each left A-module ${}_A M$ the extension of scalars $F(M) = B \otimes_A N$, and the right adjoint functor G is the forgetful functor, i.e. the restriction of scalars.

vii) We define a category $\mathcal{C}$ as follows: The objects of $\mathcal{C}$ are pairs (A, S) consisting of a commutative ring A and a multiplicative subset $S \subseteq A$. For any two objects (A, S) and (B, T) in $\mathcal{C}$, a morphisms $f\colon (A, S) \to (B, T)$ is a ring homomorpism $f\colon A \to B$ with $f(S) \subseteq T$. Let $\mathcal{D} = \mathbf{CRing}$ be the category of commutative rings.[6] We have a functor $F\colon \mathcal{C} \to \mathbf{CRing}$ that assigns to each pair $(A, S) \in \mathrm{Ob}(\mathcal{C})$ the localization $F((A, S)) = S^{-1}A$, and assigns to each morphism $f\colon (A, S) \to (B, T)$ the induced ring homomorphism

$$F(f)\colon S^{-1}A \to T^{-1}B , \quad \frac{a}{s} \mapsto \frac{f(a)}{f(s)} .$$

The functor F is left adjoint to the functor $G\colon \mathbf{CRing} \to \mathcal{C}$ that assigns to each commutative ring B the pair $(B, B^\times)$, and to each ring homomorphism $g\colon B \to C$ between commutative rings B and C the morphism

$$G(g) = (g, g^\times)\colon (B, B^\times) \to (C, C^\times) .$$

The adjunction between F and G states that a ring homomorphism $S^{-1}A \to B$ is 'the same' as a ring homomorphism $f\colon A \to B$ with $f(S) \subseteq B^\times$, which is precisely the universal property of the localization $S^{-1}A$.

Example* 2.N.

i) Let A be a **k**-algebra. Then the forgetful functor $A\text{-}\mathbf{Mod} \to \mathbf{k\text{-}Mod}$ has both a left adjoint and a right adjoint. (This is part (i) of Exercise 2 of Exercise sheet 4.)

[6] Again, in the lecture the notation **CommRing** is used instead.

ii) The forgetful functor **Top** $\to$ **Set** has both a left adjoint a right adjoint. (This is part (ii) of Exercise 2 of Exercise sheet 4.)

Remark 2.33. Let (F, G, φ) be an adjunction from $\mathcal{C}$ to $\mathcal{D}$.

i) For any object $X \in \mathrm{Ob}(\mathcal{C})$ the identity $\mathrm{id}_{F(X)} \in \mathcal{D}(F(X), F(X))$ corresponds under the bijection

$$\varphi_{X,F(X)} \colon \mathcal{D}(F(X), F(X)) \xrightarrow{\cong} \mathcal{C}(X, GF(X))$$

to a morphism

$$\eta_X \coloneqq \varphi_{X,F(X)}(\mathrm{id}_{F(X)}) \colon X \to GF(X)\,.$$

The naturality of φ results in the naturality of the family $\eta \coloneqq (\eta_X)_{X \in \mathrm{Ob}(\mathcal{C})}$: If $f \colon X \to X'$ is a morphism in $\mathcal{C}$ then the diagram

$$\begin{tikzcd}
\mathcal{D}(F(X), F(X)) \arrow[r, "F(f)_*"] \arrow[d, "\varphi_{X,F(X)}"'] & \mathcal{D}(F(X), F(X')) \arrow[d, "\varphi_{X,F(X')}"] & \mathcal{D}(F(X'), F(X')) \arrow[l, "F(f)^*"'] \arrow[d, "\varphi_{X',F(X')}"] \\
\mathcal{C}(X, GF(X)) \arrow[r, "GF(f)_*"] & \mathcal{C}(X, GF(X')) & \mathcal{C}(X', GF(X')) \arrow[l, "f^*"']
\end{tikzcd} \tag{2.6}$$

commutes by the naturality of φ. Note that the elements $\mathrm{id}_{F(X)} \in \mathcal{D}(F(X), F(X))$ (in the top left corner of the diagram) and $\mathrm{id}_{X'} \in \mathcal{D}(F(X'), F(X'))$ (in the top right corner of the diagram) are assigned under the map $F(f)_*$, resp. $F(f)^*$, to the same element $F(f) \in \mathcal{D}(F(X), F(X'))$ (in the top middle of the diagram). It follows that the square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & X' \arrow[d, "\eta_{X'}"] \\
GF(X) \arrow[r, "GF(f)"] & GF(X')
\end{tikzcd}$$

commutes, because

$$\begin{aligned}
&GF(f) \circ \eta_X \\
&= GF(f)_*(\eta_X) \\
&= GF(f)_* \circ \varphi_{X,F(X)}(\mathrm{id}_{F(X)}) && (2.7)\\
&= \varphi_{X,F(X')} \circ F(f)_*(\mathrm{id}_{F(X)}) && (2.8)\\
&= \varphi_{X,F(X')}(F(f)) && (2.9)\\
&= \varphi_{X,F(X')} \circ F(f)^*(\mathrm{id}_{F(X')}) && (2.10)\\
&= f^* \circ \varphi_{X',F(X')}(\mathrm{id}_{F(X')}) && (2.11)\\
&= f^*(\eta_{X'}) && (2.12)\\
&= f \circ \eta_{X'}\,.
\end{aligned}$$

Here we use for (2.7) the definition of η_X, for (2.8) the commutativity of the left square in (2.6), for (2.9) and (2.10) the above observation about $F(f)$, for (2.11) the commutativity of the right square in (2.11), and for (2.12) the definition of $\eta_{X'}$.

We have thus constructed a natural transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \to G \circ F$. This natural transformation is the *unit* of the adjunction (F, G, φ).

ii) We similarly have for every $Y \in \mathrm{Ob}(\mathcal{D})$ that the identity $\mathrm{id}_{G(Y)} \in \mathcal{C}(G(Y), G(Y))$ corresponds under the bijection

$$\varphi_{G(Y),Y}\colon \mathcal{D}(FG(Y), Y) \xrightarrow{\cong} \mathcal{C}(G(Y), G(Y))\,,$$

to a morphism

$$\varepsilon_Y \coloneqq \varphi^{-1}_{G(Y),Y}(\mathrm{id}_{G(Y)})\colon FG(Y) \to Y\,.$$

The naturality of φ results (similarly as for η) in the naturality of the family $\varepsilon \coloneqq (\varepsilon_Y)_{Y \in \mathcal{D}}$: If $g\colon Y \to Y'$ is a morphisms in $\mathcal{D}$ then the diagram

$$\begin{tikzcd}
\mathcal{D}(FG(Y), Y) \arrow[r, "g_*"] & \mathcal{D}(FG(Y), Y') & \mathcal{D}(FG(Y'), Y') \arrow[l, "FG(g)^*"'] \\
\mathcal{C}(G(Y), G(Y)) \arrow[u, "\varphi^{-1}_{G(Y),Y}"] \arrow[r, "G(g)_*"] & \mathcal{C}(G(Y), G(Y')) \arrow[u, "\varphi^{-1}_{G(Y),Y'}"'] & \mathcal{C}(G(Y'), G(Y')) \arrow[l, "G(g)^*"'] \arrow[u, "\varphi^{-1}_{G(Y'),Y'}"']
\end{tikzcd} \tag{2.13}$$

commutes by the naturality of φ. Note that the elements $\mathrm{id}_{G(Y)} \in \mathcal{C}(G(Y), G(Y))$ (in the bottom left corner of the diagram) and $\mathrm{id}_{G(Y')} \in \mathcal{C}(G(Y), G(Y'))$ (in the bottom right corner of the diagram) are assigned under the map $G(g)_*$, resp. $G(g)^*$, to the same element $G(g) \in \mathcal{C}(G(Y), G(Y'))$ (in the bottom middle of the diagram). It follows that the square

$$\begin{tikzcd}
FG(Y) \arrow[r, "FG(g)"] \arrow[d, "\varepsilon_Y"'] & FG(Y') \arrow[d, "\varepsilon_{Y'}"] \\
Y \arrow[r, "g"] & Y'
\end{tikzcd}$$

commutes, because

$$
\begin{aligned}
& g \circ \varepsilon_Y \\
&= g_*(\varepsilon_Y) \\
&= g_* \circ \varphi^{-1}_{G(Y),Y}(\mathrm{id}_{G(Y)}) && (2.14) \\
&= \varphi^{-1}_{G(Y),Y'} \circ G(g)_*(\mathrm{id}_{G(Y)}) && (2.15) \\
&= \varphi^{-1}_{G(Y),Y'}(G(g)) && (2.16) \\
&= \varphi^{-1}_{G(Y),Y'} \circ G(g)^*(\mathrm{id}_{G(Y')}) && (2.17) \\
&= FG(g)^* \circ \varphi^{-1}_{G(Y'),Y'}(\mathrm{id}_{G(Y')}) && (2.18) \\
&= FG(g)^*(\varepsilon_{Y'}) && (2.19) \\
&= \varepsilon_{Y'} \circ FG(g)\,.
\end{aligned}
$$

Here we use for (2.14) the definition of ε_Y, for (2.15) the commutativity of the left square in (2.13), for (2.16) and (2.17) the above observation about $G(g)$, for (2.18) the commutativity of the right square in (2.13), and for (2.19) the definition of $\varepsilon_{Y'}$.

We have thus constructed a natural transformation $\varepsilon\colon F \circ G \to \mathrm{Id}_{\mathcal{D}}$. This natural transformation is the *counit* of the adjunction (F, G, φ).

Remark⋆ 2.O. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

i) A functor $G\colon \mathcal{D} \to \mathcal{C}$ admits a left adjoint functor $\mathcal{C} \to \mathcal{D}$ if and only if the functor $\mathcal{C}(X, G(-))\colon \mathcal{D} \to \mathbf{Set}$ is representable for every object $X \in \mathrm{Ob}(\mathcal{C})$. (This is Exercise 1 of Exercise sheet 4.)

ii) Dually, a functor $F\colon \mathcal{C} \to \mathcal{D}$ admits a right adjoint functor $\mathcal{D} \to \mathcal{C}$ if and only if the functor $\mathcal{D}(F(-), Y)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable for every object $Y \in \mathrm{Ob}(\mathcal{D})$.

End of lecture 7

Lemma⋆ 2.P. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors between two categories $\mathcal{C}$ and $\mathcal{D}$, and let $\zeta\colon F \to G$ be a natural transformation. Let $\mathcal{E}$ be another category.

i) If $H\colon \mathcal{E} \to \mathcal{C}$ is another functor then there exists a natural transformation

$$\zeta H\colon F \circ H \to G \circ H$$

defined by $(\zeta H)_Z = \zeta_{H(Z)}$ at every object $Z \in \mathrm{Ob}(\mathcal{E})$.

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, "H"] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\zeta"] & \mathcal{D}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "F\circ H"] \arrow[r, bend right, "G\circ H"'] \arrow[r, phantom, "\Downarrow\zeta H"] & \mathcal{D}
\end{tikzcd}
$$

ii) If $H\colon \mathcal{D} \to \mathcal{E}$ is another functor then there exists a natural transformation

$$H\zeta\colon H \circ F \to H \circ G$$

defined by $(H\zeta)_X = H(\zeta_X)$ at every object $X \in \mathrm{Ob}(\mathcal{C})$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\downarrow\zeta"] & \mathcal{D} \arrow[r, "H"] & \mathcal{E} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "H\circ F"] \arrow[rr, bend right, "H\circ G"'] \arrow[rr, phantom, "\downarrow H\zeta"] & & \mathcal{E} \end{tikzcd}$$

Remark 2.33 (Continued)**.**

iii) The constructed natural transformations η and ε make the triangles

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "\mathrm{id}_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "\mathrm{id}_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \tag{2.20}$$

commute:

To see the commutativity of the left triangle we need to show that at every object $Y \in \mathcal{D}$ the triangle

$$\begin{tikzcd} G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\ & G(Y) \end{tikzcd}$$

commutes, i.e. we need to show the equality

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)} \ .$$

This holds because it follows from the naturality of φ that the square

$$\begin{tikzcd} \mathcal{D}(FG(Y), FG(Y)) \arrow[r, "(\varepsilon_Y)_*"] \arrow[d, "\varphi_{G(Y),FG(Y)}"'] & \mathcal{D}(FG(Y), Y) \arrow[d, "\varphi_{G(Y),Y}"] \\ \mathcal{C}(G(Y), GFG(Y)) \arrow[r, "G(\varepsilon_Y)_*"] & \mathcal{C}(G(Y), G(Y)) \end{tikzcd}$$

commutes, which then gives

$$
\begin{aligned}
& G(\varepsilon_Y) \circ \eta_{G(Y)} \\
&= G(\varepsilon_Y)_*(\eta_{G(Y)}) \\
&= G(\varepsilon_Y)_* \circ \varphi_{G(Y),FG(Y)}(\mathrm{id}_{FG(Y)}) \\
&= \varphi_{G(Y),Y} \circ (\varepsilon_Y)_*(\mathrm{id}_{FG(Y)}) \\
&= \varphi_{G(Y),Y}(\varepsilon_Y) \\
&= \varphi_{G(Y),Y} \circ \varphi^{-1}_{G(Y),Y}(\mathrm{id}_{G(Y)}) \\
&= \mathrm{id}_{G(Y)} \,.
\end{aligned}
$$

The commutativity of the right triangle can be shown similarly: We need to show that at every object $X \in \mathrm{Ob}(\mathcal{C})$ the triangle

$$
\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
 & F(X)
\end{tikzcd}
$$

commutes, i.e. we need to show the equality

$$
\varepsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)} \,.
$$

We use the naturality of φ to find that the square

$$
\begin{tikzcd}
\mathcal{D}(FGF(X), F(X)) \arrow[r, "F(\eta_X)^*"] & \mathcal{D}(F(X), F(X)) \\
\mathcal{C}(GF(X), GF(X)) \arrow[r, "\eta_X^*"] \arrow[u, "\varphi^{-1}_{F(X),GF(X)}"] & \mathcal{C}(X, GF(X)) \arrow[u, "\varphi^{-1}_{X,F(X)}"']
\end{tikzcd}
$$

commutes, which allows us to calculate

$$
\begin{aligned}
& \varepsilon_{F(X)} \circ F(\eta_X) \\
&= F(\eta_X)^*(\varepsilon_{F(X)}) \\
&= F(\eta_X)^* \circ \varphi^{-1}_{GF(X),F(X)}(\mathrm{id}_{GF(X)}) \\
&= \varphi^{-1}_{X,F(X)} \circ \eta_X^*(\mathrm{id}_{GF(X)}) \\
&= \varphi^{-1}_{X,F(X)}(\eta_X) \\
&= \varphi^{-1}_{X,F(X)} \circ \varphi_{X,F(X)}(\mathrm{id}_{F(X)}) \\
&= \mathrm{id}_{F(X)} \,.
\end{aligned}
$$

The commutativity of the triangles (2.20) are the *triangle relations*.

Proposition 2.34. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors between categories $\mathcal{C}$ and $\mathcal{D}$, and let $\eta\colon \mathrm{Id}_{\mathcal{C}} \to G \circ F$ and $\varepsilon\colon F \circ G \to \mathrm{Id}_{\mathcal{D}}$ be natural transformation satisfying the triangle relations. Then for any two objects $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$ the map

$$\varphi_{X,Y}\colon \mathcal{D}(F(X), Y) \to \mathcal{C}(X, G(Y)), \quad h \mapsto G(h) \circ \eta_X$$

is a bijection with inverse given by

$$\varphi^{-1}_{X,Y}\colon \mathcal{C}(X, G(Y)) \to \mathcal{D}(F(X), Y), \quad k \mapsto \varepsilon_Y \circ F(k),$$

and (F, G, φ) is an adjunction from $\mathcal{C}$ to $\mathcal{D}$.

Proof. The family $\varphi \coloneqq (\varphi_{X,Y})_{X \in \mathrm{Ob}(\mathcal{C}), Y \in \mathrm{Ob}(\mathcal{D})}$ is natural: To show the naturality in X let $f\colon X \to X'$ be a morphism in $\mathcal{C}$ and let $Y \in \mathrm{Ob}(\mathcal{D})$ be any object in $\mathcal{D}$. We have to show that the square

$$\begin{tikzcd}
\mathcal{D}(F(X'), Y) \arrow[r, "F(f)^*"] \arrow[d, "\varphi_{X',Y}"'] & \mathcal{D}(F(X), Y) \arrow[d, "\varphi_{X,Y}"] \\
\mathcal{C}(X', G(Y)) \arrow[r, "f^*"] & \mathcal{C}(X, G(Y))
\end{tikzcd}$$

commutes, i.e. that

$$\varphi_{X,Y}(h \circ F(f)) = \varphi_{X',Y}(h) \circ f$$

for every $h \in \mathcal{D}(F(X'), Y)$. This holds because

$$\begin{aligned}
&\varphi_{X,Y}(h \circ F(f)) \\
&= G(h \circ F(f)) \circ \eta_X \\
&= G(h) \circ GF(f) \circ \eta_X \\
&= G(h) \circ \eta_{X'} \circ f \qquad (2.21) \\
&= \varphi_{X',Y}(h) \circ f,
\end{aligned}$$

where we use for the equality (2.21) that the square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & X' \arrow[d, "\eta_{X'}"] \\
GF(X) \arrow[r, "GF(f)"] & GF(X')
\end{tikzcd}$$

commutes by the naturality of η. The naturality in Y can be shown similarly.

It remains to show that the map $\varphi_{X,Y}\colon \mathcal{D}(F(X), Y) \to \mathcal{C}(X, G(Y))$ is for all objects $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$ a bijection with inverse as claimed. We denote the proposed inverse map by

$$\psi_{X,Y}\colon \mathcal{C}(X, G(Y)) \to \mathcal{D}(F(X), Y), \quad k \mapsto \varepsilon_Y \circ F(k).$$

We need to show that the map $\varphi_{X,Y}$ and $\psi_{X,Y}$ are mutually inverse. We have for every $k \in \mathcal{C}(X, G(Y))$ that

$$\begin{aligned}
&\varphi_{X,Y}(\psi_{X,Y}(k)) \\
&= G(\varepsilon_Y \circ F(k)) \circ \eta_X \\
&= G(\varepsilon_Y) \circ GF(k) \circ \eta_X \\
&= G(\varepsilon_Y) \circ \eta_{G(Y)} \circ k \qquad (2.22) \\
&= (G\varepsilon)_Y \circ (\eta G)_Y \circ k \\
&= (G\varepsilon \circ \eta G)_Y \circ k \\
&= (\mathrm{id}_G)_Y \circ k \qquad (2.23) \\
&= \mathrm{id}_{G(Y)} \circ k \\
&= k\,,
\end{aligned}$$

where we use for the equality (2.22) that the square

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "\eta_X"'] & G(Y) \arrow[d, "\eta_{G(Y)}"] \\
GF(X) \arrow[r, "GF(k)"] & GFG(Y)
\end{tikzcd}$$

commutes by the naturality of η, and where we use for the equality (2.23) a triangle relation. With this we have shown that $\varphi_{X,Y} \circ \psi_{X,Y} = \mathrm{id}$. We similarly compute for every $h \in \mathcal{D}(F(X), Y)$ that

$$\begin{aligned}
&\psi_{X,Y}(\varphi_{X,Y}(h)) \\
&= \varepsilon_Y \circ F(G(h) \circ \eta_X) \\
&= \varepsilon_Y \circ FG(h) \circ F(\eta_X) \\
&= h \circ \varepsilon_{F(X)} \circ F(\eta_X) \qquad (2.24) \\
&= h \circ (\varepsilon F)_X \circ (F\eta)_X \\
&= h \circ (\varepsilon F \circ F\eta)_X \\
&= h \circ (\mathrm{id}_F)_X \qquad (2.25) \\
&= h \circ \mathrm{id}_{F(X)} \\
&= h\,,
\end{aligned}$$

where we use for the equality (2.24) that the square

$$\begin{tikzcd}
FGF(X) \arrow[r, "FG(h)"] \arrow[d, "\varepsilon_{F(X)}"'] & FG(Y) \arrow[d, "\varepsilon_Y"] \\
F(X) \arrow[r, "h"] & Y
\end{tikzcd}$$

commutes by the naturality of η, and where we use for the equality (2.25) another triangle relation. $\square$

Remark 2.35. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence between two categories $\mathcal{C}$ and $\mathcal{D}$. Let $G\colon \mathcal{D} \to \mathcal{C}$ be a quasi-inverse to F, i.e. a functor for which there exist natural isomorphisms $\eta\colon \mathrm{Id}_{\mathcal{C}} \to G \circ F$ and $\zeta\colon \mathrm{Id}_{\mathcal{D}} \to F \circ G$. Then the maps

$$\varphi_{X,Y}\colon \mathcal{D}(F(X), Y) \to \mathcal{C}(X, G(Y))\,, \quad h \mapsto G(h) \circ \eta_X$$

and

$$\psi_{Y,X}\colon \mathcal{C}(G(Y), X) \to \mathcal{D}(Y, F(X))\,, \quad k \mapsto F(k) \circ \zeta_Y$$

are natural bijections, that make (F, G, φ) and (G, F, ψ) into adjoint pairs.

Lemma 2.36. Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor that is part of two adjoint pairs (F, G, φ) and (F', G, φ'). Then there exists a unique natural isomorphism $\zeta\colon F \to F'$ that makes the square

$$\begin{tikzcd}
\mathcal{D}(F(X), Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "(\zeta_X)^*"'] & \mathcal{C}(X, G(Y)) \arrow[d, equal] \\
\mathcal{D}(F'(X), Y) \arrow[r, "\varphi'_{X,Y}"] & \mathcal{C}(X, G(Y))
\end{tikzcd}$$

commute for all objects $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$.

Proof. The composition

$$\Phi_{X,Y}\colon \mathcal{D}(F'(X), Y) \xrightarrow{\varphi'_{X,Y}} \mathcal{C}(X, G(Y) \xrightarrow{\varphi^{-1}_{X,Y}} \mathcal{D}(F(X), Y)$$

is for any two objects $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$ a bijection. The bijection $\Phi_{X,Y}$ is natural in Y because both $\varphi_{X,Y}$ and $\varphi'_{X,Y}$ are natural in Y. This means for every object $X \in \mathrm{Ob}(\mathcal{C})$ that

$$\Phi_{X,(-)}\colon \mathcal{D}(F'(X), -) \to \mathcal{D}(F(X), -)$$

is a natural transformation, and hence a morphism in the functor category $\mathbf{Fun}(\mathcal{D}, \mathbf{Set})$. It follows from the fully faithfulness of the Yoneda embedding that there exists a unique morphism $\zeta_X\colon F(X) \to F'(X)$ in $\mathcal{D}$ with

$$(\zeta_X)_* = \Phi_{X,(-)} = \varphi^{-1}_{X,(-)} \circ \varphi'_{X,(-)}\,.$$

The bijections $\Phi_{X,Y}$ are also natural in X, because both $\varphi^{-1}_{X,Y}$ and $\varphi'_{X,Y}$ are natural in X; hence $\zeta_X\colon F(X) \to F'(X)$ is natural in X. We have therefore defined a natural transformation $\zeta\colon F \to F'$. It follows at every object $X \in \mathrm{Ob}(\mathcal{C})$ from $\Phi_{X,(-)}$ being an isomorphism that ζ_X is also an isomorphism, because the Yoneda embedding is fully faithful, and therefore reflects isomorphisms. This shows that ζ is a natural isomorphism.

Suppose that $\zeta' \colon F \to F'$ is another natural isomorphism that makes the square commute. It then holds at every object $X \in \mathcal{C}$ that

$$(\zeta'_X)^* = \Phi_{X,(-)} = (\zeta_X)^* ,$$

and therefore $\zeta'_X = \zeta_X$ by the faithfulness of the Yoneda embedding. This shows that $\zeta' = \zeta$. □

Remark 2.37. In the situation of Lemma 2.36, the natural isomorphism ζ and its inverse ζ^{-1} can be be constructed using the units η, η' and counits ε, ε' via

$$\zeta \colon F \xrightarrow{F\eta'} FGF' \xrightarrow{\varepsilon F'} F'$$

and

$$\zeta^{-1} \colon F' \xrightarrow{F'\eta} F'GF \xrightarrow{\varepsilon' F} F .$$

3 Additive and Abelian Categories

Monomorphisms and Epimorphisms

Definition 3.1. Let $f\colon X \to Y$ be a morphism in a category $\mathcal{C}$.

i) The morphism f is a *monomorphism* if it follows for every pair of parallel morphisms $u, v\colon W \to X$ in $\mathcal{C}$ from $f \circ u = f \circ v$ that already $u = v$.

ii) The morphism f is an *epimorphism* if it follows for every pair of parallel morphisms $u, v\colon Y \to Z$ in $\mathcal{C}$ from $u \circ f = v \circ f$ that already $u = v$.

Remark 3.2. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be composable morphisms in a category $\mathcal{C}$.

i) If f is an isomorphism then it is both a monomorphism and an epimorphism.

ii) If both f and g are monomorphisms then their composition $g \circ f$ is again a monomorphism. If both f and g are epimorphisms then their composition $g \circ f$ is again an epimorphism.

iii) If the composition $g \circ f$ is a monomorphism then f is a monomorphism. If the composition $g \circ f$ is an epimorphism then g is an epimorphism.

iv) The morphism f is a monomorphism (in $\mathcal{C}$) if and only if it is an epimorphism in $\mathcal{C}^{\mathrm{op}}$.

Example 3.3. We give examples of monomorphisms.

i) In the category **Set** the monomorphisms are precisely the injective maps. The same holds for the categories A-**Mod**, **Grp**, **Ring**, **CRing**, **k-Alg**, **k-CAlg**, **Top**.

ii) If Q is a quiver then in its path category $\mathbf{Path}(Q)$ every morphism is a monomorphism: Let $p = \alpha_\ell \cdots \alpha_1$ be a morphism in Q, i.e. a path in Q. If $u = u_r \cdots u_1$ and $v = v_s \cdots v_1$ are morphisms in $\mathbf{Path}(Q)$, i.e. paths in Q, with $s(u) = s(v)$ and $t(u) = t(v) = s(p)$ then the equality $p \circ u = p \circ v$ means that

$$\alpha_\ell \cdots \alpha_1 u_r \cdots u_1 = \alpha_\ell \cdots \alpha_1 v_s \cdots v_1 \,.$$

It then follows that $r = s$ and $u_i = v_i$ for all $i = 1, \dots, r$.

iii) Let $\mathbf{Conn}_*$ be the category of pointed, connected topological spaces: The objects of $\mathbf{Conn}_*$ are pairs (X, x_0) consisting of a connected topological space X and a base point $x_0 \in X$. A morphism $f\colon (X, x_0) \to (Y, y_0)$ is a continuous map $f\colon X \to Y$ with $f(x_0) = y_0$. The morphism $f\colon (\mathbb{R}, 0) \to (S^1, 1)$ with $f(x) = e^{2\pi i x}$ is then a monomorphism.

Example 3.4. We give examples for epimorphisms.

i) In the category **Set** a morphism is an epimorphism if and only if it surjective. The same holds for the category **Grp** of groups.[1]

ii) If Q is a quiver then in its path category **Path**(Q) every morphism in an epimorphism.

iii) Let **Haus** be the category of Hausdorff topological spaces (where morphisms are just continuous maps). A morphism $f\colon X \to Y$ in **Haus** is an epimorphism if and only if it has dense image.

iv) Let A be a commutative ring and let $S \subseteq A$ be a multiplicative subset. Then the canonical map $f\colon A \to S^{-1}A$, $a \mapsto a/1$ is an epimorphism: If $u, v\colon S^{-1}A \to B$ are two ring homomorphisms with $u \circ f = v \circ f$ then $u(a/1) = v(a/1)$ for every $a \in A$. It then follows for every fraction $a/s \in S^{-1}A$ that

$$u\left(\frac{a}{s}\right) = u\left(\frac{a}{1}\right) u\left(\frac{s}{1}\right)^{-1} = v\left(\frac{a}{1}\right) v\left(\frac{s}{1}\right)^{-1} = v\left(\frac{a}{s}\right) ,$$

which then shows that $u = v$.

Remark* 3.A. Let $f\colon X \to Y$ be a morphism in a category $\mathcal{C}$.

i) The morphism f is a monomorphism if and only if for every object $W \in \mathrm{Ob}(\mathcal{C})$ the induced map
$$f_*\colon \mathcal{C}(W, X) \to \mathcal{C}(W, Y)$$
is injective.

ii) The morphism f is an epimorphism if and only if for every object $Z \in \mathrm{Ob}(\mathcal{C})$ the induced map
$$f^*\colon \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$$
is injective.

Special Objects

Definition 3.5. Let X be an object in a category $\mathcal{C}$.

i) The object X is *initial* if there exists for every object $Y \in \mathrm{Ob}(\mathcal{C})$ a unique morphism $X \to Y$ in $\mathcal{C}$.

ii) The object X is *terminal* or *final* if there exists for every object $Y \in \mathrm{Ob}(\mathcal{C})$ a unique morphism $Y \to X$ in $\mathcal{C}$.

iii) The object X is a *zero object* if it is both initial and terminal.

Remark 3.6. Let $\mathcal{C}$ be a category.

i) An object X of $\mathcal{C}$ is initial (in $\mathcal{C}$) if and only if it is terminal in $\mathcal{C}^{\mathrm{op}}$.

[1] This is not as evident as one may suspect at first glance.

ii) Initial and terminal objects are unique up to unique isomorphisms (if they exist).

iii) If $\mathcal{C}$ admits a zero object then it is denoted by $0 = 0_{\mathcal{C}}$.

Example 3.7.

i) In the category **Set**, the empty set $\emptyset$ is the unique initial object, and every one-point set $\{*\}$ is a final object. The analogous statements hold for the category **Top**.

ii) In the category A-**Mod** the zero module 0 is the zero object.

iii) In the category **Grp** the trivial groups 1 is the zero object.

iv) In the category **k-Alg** the **k**-algebra **k** is initial, while the zero algebra 0 is final.

Example* 3.B. The category **Field** of fields has neither an intial nor a final object.

Remark-Definition 3.8. Let $\mathcal{C}$ be a category that admits a zero object 0. Then there exists for any two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ a unique morphism $0_{XY} \colon X \to Y$ that factors trough the zero object, i.e. that makes the triangle

$$\begin{tikzcd} X \arrow[rr, dashed, "0_{XY}"] \arrow[dr] & & Y \\ & 0 \arrow[ur] & \end{tikzcd}$$

commute. The morphism 0_{XY} is the *zero morphism* from X to Y.

Remark* 3.C. Let $\mathcal{C}$ be a category that admits a zero object 0. Then it holds for every morphism $f \colon X \to Y$ that

$$f \circ 0_{W,X} = 0_{W,Y} \qquad \text{and} \qquad 0_{Y,Z} \circ f = 0_{X,Z}$$

for all $W, Z \in \mathrm{Ob}(\mathcal{C})$. Indeed, we have the following commutative diagrams:

$$\begin{tikzcd} W \arrow[rr, "0_{W,X}"] \arrow[dr] \arrow[rrr, dashed, bend left, "f \circ 0_{W,X}"] & & X \arrow[r, "f"] & Y \\ & 0 \arrow[ur] \arrow[urr, dashed, bend right] & & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[drr, dashed, bend right] \arrow[rrr, dashed, bend left, "0_{Y,Z} \circ f"] & Y \arrow[rr, "0_{Y,Z}"] \arrow[dr] & & Z \\ & & 0 \arrow[ur] & \end{tikzcd}$$

The commutativity of the first diagram shows that the morphism $f \circ 0_{W,X}$ factors through the zero object 0, and is hence the zero morphism $W \to Y$. The commutativity of the second diagram shows similarly that $0_{Y,Z} \circ f$ factors through the zero objects 0 and is therefore the zero morphism $X \to Z$.

End of lecture 8

Products and Coproducts

Definition 3.9. Let $(X_i)_{i\in I}$ be a family of objects in a category $\mathcal{C}$.

i) A *product* of the family of objects $(X_i)_{i\in I}$ is a pair $(P,(p_i)_{i\in I})$ consisting of an object $P \in \mathrm{Ob}(\mathcal{C})$ and morphisms $p_i\colon P \to X_i$, such that for every other pair $(Q,(q_i)_{i\in I})$ consisting of an object $Q \in \mathrm{Ob}(\mathcal{C})$ and morphisms $q_i\colon Q \to X_i$ there exists a unique morphism $\lambda\colon Q \to P$ that makes the triangle

$$\begin{tikzcd} P \arrow[d, "p_i"'] & Q \arrow[l, dashed, "\lambda"'] \arrow[dl, "q_i"] \\ X_i & \end{tikzcd}$$

commute for every $i \in I$.

ii) A *coproduct* of the family of objects $(X_i)_{i\in I}$ is a pair $(C,(c_i)_{i\in I})$ consisting of an object $C \in \mathrm{Ob}(\mathcal{C})$ and morphisms $c_i\colon X_i \to C$, such that for every pair $(D,(d_i)_{i\in I})$ consisting of an object $D \in \mathrm{Ob}(\mathcal{C})$ and morphisms $d_i\colon X_i \to C$ there exists a unique morphism $\mu\colon C \to D$ that makes the triangle

$$\begin{tikzcd} C \arrow[r, dashed, "\mu"] & D \\ X_i \arrow[u, "c_i"] \arrow[ur, "d_i"'] & \end{tikzcd}$$

commute for every $i \in I$.

Remark 3.10. Let $(X_i)_{i\in I}$ be a family of objects X_i in a category $\mathcal{C}$.

i) A pair $(P,(p_i)_{i\in I})$ is a product of the family $(X_i)_{i\in I}$ in $\mathcal{C}$ if and only if it is a coproduct of this family in $\mathcal{C}^{\mathrm{op}}$.

ii) Products are unique up to unique isomorphism, i.e. if $(P,(p_i)_{i\in I})$ and $(P',(p'_i)_{i\in I})$ are two products of the family $(X_i)_{i\in I}$ in $\mathcal{C}$ then there exist a unique morphism $\lambda\colon P \to P'$ that makes the triangle

$$\begin{tikzcd} P \arrow[rr, dashed, "\lambda"] \arrow[dr, "p_i"'] & & P' \arrow[dl, "p'_i"] \\ & X_i & \end{tikzcd}$$

commute for every $i \in I$, and the morphism λ is already an isomorphism. Similarly, coproducts are unique up to unique isomorphism.

iii) The product of the family $(X_i)_{i\in I}$ is denoted by $\prod_{i\in I} X_i$, or by $X_1 \times \cdots \times X_n$ if $I = \{1,\dots,n\}$. The coproduct of the family $(X_i)_{i\in I}$ is denoted by $\coprod_{i\in I} X_i$, or by $X_1 \amalg \cdots \amalg X_n$ if $I = \{1,\dots,n\}$.[2]

[2]In the lecture the notations $\bigsqcap_{i\in I} X_i$, resp. $\bigsqcup_{i\in I} X_i$ and $X_1 \sqcup \cdots \sqcup X_n$ are used instead.

iv) If every family of objects in $\mathcal{C}$ has a product (resp. coproduct) in $\mathcal{C}$ then we say that that $\mathcal{C}$ *has products* (resp. *has coproducts*). If every finite family of objects in $\mathcal{C}$ has a product (resp. coproducts) in $\mathcal{C}$ then we say that $\mathcal{C}$ *has finite coproducts* (resp. *has finite coproducts*).

Example 3.11. In the following let $(X_i)_{i\in I}$ be a family objects in the given category $\mathcal{C}$.

i) Let $\mathcal{C} = \mathbf{Set}$. Then the (categorical) product $\prod_{\in I} X_i$ is the cartesian product, and the map $p_i \colon \prod_{j\in I} X_j \to X_i$ is for every $i \in I$ the usual projections onto the i-th factor. The coproduct $\coprod_{i\in I} X_i$ is the (formal) disjoint union of the sets X_i, and the map $c_i \colon X_i \to \coprod_{j\in I} X_j$ is for every $i \in I$ the inclusion into the j-th set.

ii) Let $\mathcal{C} = A\text{-}\mathbf{Mod}$. Then the (categorical) product $\prod_{i\in I} X_i$ is the cartesian product, and the morphism $p_i \colon \prod_{j\in I} X_j \to X_i$ is for every $i \in I$ the usual projection onto the i-th factor. The coproduct of the family $(X_i)_{i\in I}$ is the direct sum $\bigoplus_{i\in I} X_i$, and the morphisms $c_i \colon X_i \to \bigoplus_{j\in I} X_j$ is for every $i \in I$ the inclusion into the i-th summand.

iii) Let $\mathcal{C} = \mathbf{k}\text{-}\mathbf{CAlg}$. Then the (categorical) product $\prod_{i\in I} X_i$ is the cartesian product, and the morphism $p_i \colon \prod_{j\in I} X_j \to X_i$ is for every $i \in I$ the usual projection onto the i-th factor. The coproduct of finitely many commutative $\mathbf{k}$-algebra $A_1, \dots, A_n$ in the category $\mathbf{k}\text{-}\mathbf{CAlg}$ is their tensor product $A_1 \otimes \cdots \otimes A_n$, and the morphism $c_i \colon A_i \to A_1 \otimes \cdots \otimes A_n$ is for every $i \in I$ the inclusion into the i-th factor, i.e. the algebra homomorphism

$$A_i \to A_1 \otimes \cdots \otimes A_n \,, \quad x \mapsto 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \,.$$

(The coproduct of an arbitrary family $(A_i)_{i\in I}$ of commutative $\mathbf{k}$-algebras in the category $\mathbf{k}\text{-}\mathbf{CAlg}$ can be described similarly.)

Remark⋆ 3.D. Let $\mathcal{C}$ be a category. A product of an empty family of objects in $\mathcal{C}$ is the same a terminal object of $\mathcal{C}$. A coproduct of an empty family of objects in $\mathcal{C}$ is the same an initial object of $\mathcal{C}$.

Lemma 3.12. Let $(X_i)_{i\in I}$ be a family of objects in a category $\mathcal{C}$.

i) The following are equivalent:
   a) The product $\prod_{i\in I} X_i$ exists in $\mathcal{C}$.
   b) The functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ given by $Y \mapsto \prod_{i\in I} \mathcal{C}(Y, X_i)$ is representable.

ii) The following are equivalent:
   a) The coproduct $\coprod_{i\in I} X_i$ exists in $\mathcal{C}$.
   b) The functor $\mathcal{C} \to \mathbf{Set}$ given by $Y \mapsto \prod_{i\in I} \mathcal{C}(X_i, Y)$ is representable.

Proof. It sufficies to show part ii). We denote the given functor $\mathcal{C} \to \mathbf{Set}$ from condition b) by F.

Suppose first that the coproduct $\coprod_{i\in I} X_i$ exists, and denote the associated morphisms by $c_i \colon X_i \to \coprod_{j\in I} X_j$. We claim that the functor F is represented by the coproduct $\coprod_{i\in I} X_i$; we thus need to construct a natural isomorphism $\eta \colon h^{(\coprod_{i\in I} X_i)} \to F$.

We define the components of η as

$$\begin{aligned}\eta_Y : h^{(\coprod_{i\in I} X_i)}(Y) = \mathcal{C}\left(\coprod_{i\in I} X_i, Y\right) &\to \prod_{i\in I} \mathcal{C}(X_i, Y) = F(Y)\,,\\ f &\mapsto (f \circ c_i)_{i\in I}\,.\end{aligned}$$

Then the family $\eta \coloneqq (\eta_Y)_{Y\in\mathrm{Ob}(\mathcal{C})}$ defines a natural transformation $\eta\colon h^{(\coprod_{i\in I} X_i)} \to F$. Indeed, for every morphism $g\colon Y \to Y'$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd}
h^{(\coprod_{i\in I} X_i)}(Y) \arrow[rrr, "h^{(\coprod_{i\in I} X_i)}(g)"] \arrow[ddd, "\eta_Y"'] \arrow[dr, equal] & & & h^{(\coprod_{i\in I} X_i)}(Y') \arrow[ddd, "\eta_{Y'}"] \arrow[dl, equal] \\
& \mathcal{C}\left(\coprod_{i\in I} X_i, Y\right) \arrow[r, "g_*"] \arrow[d, "\eta_Y"'] & \mathcal{C}\left(\coprod_{i\in I} X_i, Y'\right) \arrow[d, "\eta_{Y'}"] & \\
& \prod_{i\in I} \mathcal{C}(X_i, Y) \arrow[r, "\prod_{i\in I} g_*"'] \arrow[dl, equal] & \prod_{i\in I} \mathcal{C}(X_i, Y') \arrow[dr, equal] & \\
F(Y) \arrow[rrr, "F(g)"'] & & & F(Y')
\end{tikzcd}$$

commutes, because the inner square is given on elements by

$$\begin{tikzcd}
f \arrow[r, mapsto, "g_*"] \arrow[d, mapsto, "\eta_Y"'] & g \circ f \arrow[d, mapsto, "\eta_{Y'}"] \\
(f \circ c_i)_{i\in I} \arrow[r, mapsto, "\prod_{i\in I} g_*"] & (g \circ f \circ c_i)_{i\in I}
\end{tikzcd}$$

and thus commutes.

The natural transformation η is already a natural isomorphism: There exist at every objects $Y \in \mathrm{Ob}(\mathcal{C})$ for every family of morphisms $(h_i)_{i\in I} \in \prod_{i\in I} \mathcal{C}(X_i, Y)$ by the definition of the coproduct $\coprod_{i\in I} X_i$ a unique morphism $g \in \mathcal{C}(\coprod_{i\in I} X_i, Y)$ with $g \circ c_i = h_i$ for every $i \in I$, i.e. with $\eta_Y(g) = (h_i)_{i\in I}$. This means that η_Y is bijective at every $Y \in \mathrm{Ob}(\mathcal{C})$.

Suppose now that the functor F is representable. Let C be a representing object for F and let $\eta\colon h^C \to F$ be a natural isomorphism. Then η_C is a map

$$\eta_C : h^C(C) = \mathcal{C}(C, C) \to F(C) = \prod_{i\in I} \mathcal{C}(X_i, C)\,.$$

By setting $(c_i)_{i\in I} \coloneqq \eta_C(\mathrm{id}_C)$ we therefore get for every $i \in I$ a morphism $c_i\colon X_i \to C$. We show that the pair $(C, (c_i)_{i\in I})$ is a coproduct of the family $(X_i)_{i\in I}$. So let $(D, (d_i)_{i\in I})$ be another pair consisting of an object $D \in \mathrm{Ob}(\mathcal{D})$ and a family $(d_i)_{i\in I}$ of morphisms $d_i\colon X_i \to D$. Then

$$(d_i)_{i\in I} \in \prod_{i\in I} \mathcal{C}(X_i, D) = F(D)$$

and it follows from $\eta_D \colon h^C(D) \to F(D)$ being a bijection that there exist a unique element $\lambda \in h^C(D) = \mathcal{C}(C, D)$, i.e. morphism $\lambda \colon C \to D$, with $(d_i)_{i \in I} = \eta_D(\lambda)$. It follows from the naturality of η that the diagram

$$\begin{tikzcd}
\mathcal{C}(C,C) \arrow[r, equal] \arrow[d, "\lambda_*"'] & h^C(C) \arrow[r, "\eta_C"] \arrow[d, "h^C(\lambda)"'] & F(C) \arrow[r, equal] \arrow[d, "F(\lambda)"] & \prod_{i \in I} \mathcal{C}(X_i, C) \arrow[d, "\prod_{i \in I} \lambda_*"] \\
\mathcal{C}(C,D) \arrow[r, equal] & h^C(D) \arrow[r, "\eta_D"'] & F(D) \arrow[r, equal] & \prod_{i \in I} \mathcal{C}(X_i, D)
\end{tikzcd}$$

commutes. It therefore follows for the element $\mathrm{id}_C \in \mathcal{C}(\mathcal{C}, \mathcal{C}) = h^C(C)$ that

$$\begin{aligned}
(d_i)_{i \in I} = \eta_D(\lambda) = \eta_D(\lambda_*(\mathrm{id}_C)) &= \eta_D(h^C(\lambda)(\mathrm{id}_C)) \\
&= F(\lambda)(\eta_C(\mathrm{id}_C))) = F(\lambda)((c_i)_{i \in I}) = (\lambda \circ c_i)_{i \in I} \,,
\end{aligned}$$

and hence that $\lambda \circ c_i = d_i$ for every $i \in I$. This shows the existence of the required morphism $\lambda \colon C \to D$. By reading the above argumentation from the bottom to the top we also find that the morphism λ is unique.

This shows that the object C together with the morphisms $c_i \colon X_i \to C$ is a coproduct of the family $(X_i)_{i \in I}$. □

Remark⋆ 3.E. The above proof actually shows that an object $C \in \mathrm{Ob}(\mathcal{C})$ is a coproduct of the family $(X_i)_{i \in I}$, with respect to some suitable morphisms $c_i \colon X_i \to C$, if and only if it represents the functor $\prod_{i \in I} \mathcal{C}(X_i, -) \colon \mathcal{C} \to \mathbf{Set}$. This statement is stronger than the formulation in Lemma 3.12.

One can also show a slightly stronger version of this: It follows for every object $C \in \mathrm{Ob}(\mathcal{C})$ from Yoneda's lemma that the map

$$\begin{aligned}
\{\text{natural transformations } \eta \colon h^C \to F\} &\to F(C) \,, \\
\eta &\mapsto \eta_C(\mathrm{id}_C)
\end{aligned}$$

is a bijection. An element of the right hand side is an element of $F(C)$, i.e. a family $(c_i)_{i \in I}$ of morphisms $c_i \colon X_i \to C$. It then holds that a natural transformation $\eta \colon h^C \to F$ is an isomorphism if and only if the corresponding family $(c_i)_{i \in I}$ makes the pair $(C, (c_i)_{i \in I})$ into a coproduct of the family $(X_i)_{i \in I}$.

Indeed, that the natural transformation η is a natural isomorphism means that at every object $D \in \mathrm{Ob}(\mathcal{C})$ the map

$$\eta_D \colon \mathcal{C}(C, D) = h^C(D) \to F(D)$$

is a bijection. The component η_D can be expressed via the element $(c_i)_{i \in I} \in F(C)$ as

$$\eta_D(\lambda) = F(\lambda)((c_i)_{i \in I}) = (\lambda \circ c_i)_{i \in I} \,.$$

The bijectivity of the component η_D therefore means that for every family of morphisms $(d_i)_{i \in I} \in F(D) = \prod_{i \in I} \mathcal{C}(X_i, D)$ there exists a unique element $\lambda \in \mathcal{C}(C, D)$

with $(\lambda \circ c_i)_{i \in I} = (d_i)_{i \in I}$. In other words, there exists for every object $D \in \mathrm{Ob}(\mathcal{C})$ and every family $(d_i)_{i \in I}$ of morphisms $d_i \colon X_i \to D$ a unique morphism $\lambda \colon C \to D$ with $d_i = \lambda \circ c_i$ for every $i \in I$. But this is precisely what it means for the pair $(C, (c_i)_{i \in I})$ to be a coproduct of the family $(X_i)_{i \in I}$.

This shows that for every object $C \in \mathrm{Ob}(\mathcal{C})$, a family $(c_i)_{i \in I}$ of morphisms $c_i \colon X_i \to C$ that makes $(C, (c_i)_{i \in I})$ into a coproduct of the family $(X_i)_{i \in I}$ is 'the same' as a natural isomorphism $\eta \colon h^C \to F$ (via Yoneda's lemma). It follows in particular that C is a coproduct of the family $(X_i)_{i \in I}$, with respect to a suitable choice of morphisms $c_i \colon X_i \to C$, if and only if there exist a natural isomorphism $h^C \to F$. (This is what was shown in the above proof.)

Remark⋆ 3.F (Functoriality of (co)products)**.** Let $\mathcal{C}$ be a category.

i) Suppose that $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ are two families of objects in $\mathcal{C}$ (over the same index set I) whose products $(\prod_{i \in I} X_i, (p_i)_{i \in I})$ and $(\prod_{i \in I} Y_i, (q_i)_{i \in I})$ exist.

If we are given for every $i \in I$ a morphism $f_i \colon X_i \to Y_i$, then there exists a unique morphism $\prod_{i \in I} f_i \colon \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ that makes for every $i \in I$ the following square commute:

$$\begin{tikzcd} \prod_{j \in I} X_j \arrow[r, "p_i"] \arrow[d, dashed, "\prod_{j \in I} f_j"'] & X_i \arrow[d, "f_i"] \\ \prod_{j \in I} Y_j \arrow[r, "q_i"] & Y_i \end{tikzcd}$$

The existence and uniqueness of the morphism $\prod_{i \in I} f_i$ follow from the universal property of the product $(\prod_{i \in I} Y_i, (q_i)_{i \in I})$, when applied to the diagonal morphisms $f_i \circ p_i \colon \prod_{j \in I} X_j \to Y_i$.

This induced morphism between products is functorial in the following sense:

- If $Y_i = X_i$, $p_i = q_i$ and $f_i = \mathrm{id}_{X_i}$ for every $i \in I$, then the induced morphisms $\prod_{i \in I} \mathrm{id}_{X_i}$ is the identity $\mathrm{id}_{(\prod_{i \in I} X_i)}$. Indeed, the identity $\mathrm{id}_{(\prod_{i \in I} X_i)}$ makes the square

$$\begin{tikzcd} \prod_{j \in I} X_j \arrow[r, "p_i"] \arrow[d, dashed, "\mathrm{id}_{(\prod_{j \in I} X_j)}"'] & X_i \arrow[d, "\mathrm{id}_{X_i}"] \\ \prod_{j \in I} X_j \arrow[r, "p_i"] & X_i \end{tikzcd}$$

  commute for every $i \in I$, and hence satisfies the defining property of the morphism $\prod_{i \in I} \mathrm{id}_{X_i}$.
- Let $(Z_i)_{i \in I}$ be another family of objects in $\mathcal{A}$ whose product $(\prod_{i \in I} Z_i, (r_i)_{i \in I})$ exists and let $g_i \colon Y_i \to Z_i$ with $i \in I$ be another collection of morphism. Then

$$\prod_{i \in I} (g_i \circ f_i) = \left(\prod_{i \in I} g_i \right) \circ \left(\prod_{i \in I} f_i \right) .$$

Indeed, we have for every $i \in I$ the following commutative diagram:

$$\begin{tikzcd} \prod_{j\in I} X_j \arrow[r, "p_i"] \arrow[d, "\prod_{j\in J} f_j"] \arrow[dd, dashed, bend right=60, "(\prod_{i\in I} g_i)\circ(\prod_{i\in I} f_i)"'] & X_i \arrow[d, "f_i"] \arrow[dd, dashed, bend left=60, "g_i\circ f_i"] \\ \prod_{j\in I} Y_j \arrow[r, "q_i"] \arrow[d, "\prod_{j\in J} g_j"] & Y_i \arrow[d, "g_i"] \\ \prod_{j\in I} Z_j \arrow[r, "r_i"] & Z_i \end{tikzcd}$$

That the outer square

$$\begin{tikzcd} \prod_{j\in J} X_j \arrow[r, "p_i"] \arrow[d, dashed, "(\prod_{i\in I} g_i)\circ(\prod_{i\in I} f_i)"'] & X_i \arrow[d, "g_i\circ f_i"] \\ \prod_{j\in J} Z_j \arrow[r, "r_i"] & Z_i \end{tikzcd}$$

commutes for every $i \in I$ shows that the composition $(\prod_{i\in I} g_i) \circ (\prod_{i\in I} f_i)$ satisfies the defining property of the morphism $\prod_{i\in I}(g_i \circ f_i)$.

ii) Suppose dually that $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ are two families of objects in $\mathcal{C}$ whose coproducts $(\coprod_{i\in I} X_i, (c_i)_{i\in I})$ and $(\coprod_{i\in I} Y_i, (d_i)_{i\in I})$ exist. Then for every family $(f_i)_{i\in I}$ of morphisms $f_i \colon X_i \to Y_i$ there exists a unique morphism

$$\coprod_{i\in I} f_i \colon \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$$

that makes for every $i \in I$ the following square commute:

$$\begin{tikzcd} X_i \arrow[r, "c_i"] \arrow[d, "f_i"'] & \coprod_{i\in I} X_i \arrow[d, dashed, "\coprod_{i\in I} f_i"] \\ Y_i \arrow[r, "d_i"] & \coprod_{i\in I} Y_i \end{tikzcd}$$

This induced morphism between coproducts is again functorial in the following sense:

- If $X_i = Y_i$ and $c_i = d_i$ for every $i \in I$, then $\coprod_{i\in I} \mathrm{id}_{X_i} = \mathrm{id}_{(\coprod_{i\in I} X_i)}$.
- If $(Z_i)_{i\in I}$ is another family of objects in $\mathcal{C}$ whose coproduct $(\coprod_{i\in I} Z_i, (e_i)_{i\in I})$ exists, and if $(g_i)_{i\in I}$ is a collection of morphisms $g_i \colon Y_i \to Z_i$, then

$$\coprod_{i\in I}(g_i \circ f_i) = \left(\coprod_{i\in I} g_i\right) \circ \left(\coprod_{i\in I} f_i\right) .$$

Additive Categories

Definition 3.13. A *preadditive category* is a category $\mathcal{A}$ together with the structure of an abelian group on $\mathcal{A}(X,Y)$ for all $X,Y \in \mathrm{Ob}(\mathcal{A})$ such that the composition in $\mathcal{A}$ is $\mathbb{Z}$-bilinear, i.e. such that

$$k \circ (g+h) = k \circ g + k \circ h \quad \text{and} \quad (g+h) \circ f = g \circ f + h \circ f$$

for all morphisms $f \colon W \to X$, $g,h \colon X \to Y$ and $k \colon Y \to Z$ in $\mathcal{A}$.

Remark 3.14.

i) Preddative categories are also known as **Ab**-*categories* (where **Ab** denotes the category of abelian groups).

ii) If $\mathbf{k}$ is a commutative ring then one can similarly define the notion of a *pre-*$\mathbf{k}$*-linear category* (also known as $\mathbf{k}$-**Mod**-*category*) $\mathcal{C}$. Every $\mathcal{C}(X,Y)$ is then endowed with the structure of a $\mathbf{k}$-module and the composition is $\mathbf{k}$-bilinear.

Remark* 3.G. In the language of enriched category theory, an **Ab**-category is precisely a category that is enriched over the monoidal category $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$, and a **k-Mod**-category is precisely a category that is enriched over the monoidal category $(\mathbf{k}\text{-}\mathbf{Mod}, \otimes_{\mathbf{k}})$.

Example 3.15.

i) The category $\mathbf{Ab} = \mathbb{Z}$-**Mod** is preadditive.

ii) If A is a $\mathbf{k}$-algebra then the categories A-**Mod** and **Mod**-A are pre-$\mathbf{k}$-linear.

iii) If R is a ring then we may think about R as a preadditive category $\mathcal{R}$ consisting of a single object $\mathrm{Ob}(\mathcal{R}) = \{*\}$ with $\mathcal{R}(*,*) = R$. The composition in $\mathcal{R}$ is given by the multiplication of R, i.e. by $g \circ f = gf$ for all $f, g \in R$, and the addition of morphisms is the addition in R.

One can similarly regard every $\mathbf{k}$-algebra A as a pre-$\mathbf{k}$-linear category $\mathcal{A}$ that consists of a single object $\mathrm{Ob}(\mathcal{A}) = \{*\}$ with $\mathcal{A}(*,*) = A$.

iv) Let $\mathcal{C}$ be any category and let $\mathcal{A}$ be a preadditive category. Then the functor category $\mathbf{Fun}(\mathcal{C},\mathcal{A})$ is again preadditive: For any two natural transformations $\eta, \zeta \colon F \to G$ between functors $F, G \in \mathrm{Ob}(\mathbf{Fun}(\mathcal{C},\mathcal{A}))$, their sum $\eta + \zeta$ is at an object $X \in \mathrm{Ob}(\mathcal{C})$ given by

$$(\eta + \zeta)_X = \eta_X + \zeta_X \,.$$

This defines again a natural transformation $\eta + \zeta \colon F \to G$. Indeed, for every morphism $f \colon X \to X'$ in $\mathcal{C}$ the square

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(X') \\ {\scriptstyle \eta_X + \zeta_X}\big\downarrow & & \big\downarrow{\scriptstyle \eta_{X'} + \zeta_{X'}} \\ G(X) & \xrightarrow{G(f)} & G(X') \end{array}$$

commutes because

$$\begin{aligned}(\eta_{X'} + \zeta_{X'}) \circ F(f) &= \eta_{X'} \circ F(f) + \zeta_{X'} \circ F(f) \\ &= G(f) \circ \eta_X + G(f) \circ \zeta_X = G(f) \circ (\eta_X + \zeta_X)\,.\end{aligned}$$

We find similarly that for every category $\mathcal{C}$ and every pre-$\mathbf{k}$-linear category $\mathcal{A}$ the functor category $\mathbf{Fun}(\mathcal{C}, \mathcal{A})$ is again pre-$\mathbf{k}$-linear.

Remark⋆ 3.H.

i) A preadditive category is the same as a pre-$\mathbb{Z}$-linear category.

ii) If $\mathcal{A}$ is a preadditive (resp. pre-$\mathbf{k}$-linear) category, then the opposite category $\mathcal{A}^{\mathrm{op}}$ is again preadditive (resp. pre-$\mathbf{k}$-linear) with the same addition (resp. $\mathbf{k}$-module structure) of morphisms.

Definition 3.16. Let $F \colon \mathcal{A} \to \mathcal{B}$ be a functor between categories $\mathcal{A}$ and $\mathcal{B}$.

i) If $\mathcal{A}$ and $\mathcal{B}$ are preaddive categories then the functor F is *additive* if

$$F(f+g) = F(f) + F(g)$$

for all morphisms $f, g \colon X \to Y$ in $\mathcal{A}$, i.e. if the map

$$\mathcal{A}(X,Y) \xrightarrow{F} \mathcal{B}(F(X), F(Y))$$

is a group homorphism for all $X, Y \in \mathrm{Ob}(\mathcal{A})$.

ii) If $\mathcal{A}$ and $\mathcal{B}$ are pre-$\mathbf{k}$-linear categories then the functor F is $\mathbf{k}$-*linear* if

$$F(f+g) = F(f) + F(g) \quad \text{and} \quad F(\lambda f) = \lambda F(f)$$

for all morphisms $f, g \colon X \to Y$ in $\mathcal{A}$ and scalars $\lambda \in \mathbf{k}$, i.e. if the map

$$\mathcal{A}(X,Y) \xrightarrow{F} \mathcal{B}(F(X), F(Y))$$

is $\mathbf{k}$-linear for all $X, Y \in \mathrm{Ob}(\mathcal{A})$.[3]

Lemma 3.17. Let $\mathcal{A}$ be a preadditive category.

i) For any object $X \in \mathrm{Ob}(\mathcal{A})$ the following conditions are equivalent:
   a) The object X is inital in $\mathcal{A}$.
   b) The object X is terminal in $\mathcal{A}$.
   c) The object X is a zero object for $\mathcal{A}$.
   d) It holds that $\mathrm{id}_X = 0_{\mathcal{A}(X,X)}$.
   e) The abelian group $\mathcal{A}(X,X)$ consists of only a single element.

ii) Suppose that the category $\mathcal{A}$ has a zero object. Then it holds for any two objects $X, Y \in \mathrm{Ob}(\mathcal{A})$ that $0_{X,Y} = 0_{\mathcal{A}(X,Y)}$.

[3] The notion of a $\mathbf{k}$-linear functor was not introduced in the lecture.

Proof. This is Exercise 3 on Exercise sheet 5. □

End of lecture 9

Definition 3.18. Let $\mathcal{A}$ be a preadditive category and let $X_1, \dots, X_n \in \mathrm{Ob}(\mathcal{A})$ be objects, where $n \in \mathbb{Z}_{\geq 0}$. A *biproduct* of $X_1, \dots, X_n$ is a triple $(X, (p_1, \dots, p_n), (c_1, \dots, c_n))$ consisting of an object $X \in \mathrm{Ob}(\mathcal{A})$ together with morphisms $p_i \colon X \to X_i$ and morphisms $c_i \colon X_i \to X$ in $\mathcal{A}$, such that

$$p_j c_i = \begin{cases} \mathrm{id}_{X_i} & \text{if } i = j\,, \\ 0_{X_i, X_j} & \text{if } i \neq j\,, \end{cases} \tag{3.1}$$

for all $i, j = 1, \dots, n$, and

$$\sum_{i=1}^{n} c_i p_i = \mathrm{id}_X\,.$$

Remark⋆ 3.I. In the lecture, the formula 3.1 was instead written as

$$p_j c_i = \delta_{ij} \, \mathrm{id}_{X_i}\,.$$

This is an abuse of notation: For $j \neq i$ the composition $p_j c_i$ is a morphism $X_i \to X_j$, whereas $\delta_{ij} \, \mathrm{id}_{X_i} = 0 \cdot \mathrm{id}_{X_i} = 0_{X_i, X_i}$ is the zero morphism $X_i \to X_i$. The author tries to avoid this abuse of notation, but will still sometimes write $p_j c_i = \delta_{ij}$ as an abbreviation for (3.1).

Remark. For a preadditive category $\mathcal{A}$, a biproduct of an empty family of objects in $\mathcal{A}$ is the same as a zero object of $\mathcal{A}$.

Lemma 3.19. Let $\mathcal{A}$ be a preadditive category, let $X_1, \dots, X_n \in \mathcal{A}$ where $n \in \mathbb{Z}_{\geq 0}$.

i) If $(X, (p_1, \dots, p_n), (c_1, \dots, c_n))$ is a biproduct of $X_1, \dots, X_n$ then $(X, (p_1, \dots, p_n))$ is a product of $X_1, \dots, X_n$ and $(X, (c_1, \dots, c_n))$ is a coproduct of $X_1, \dots, X_n$.

ii) Suppose that $(X, (p_1, \dots, p_n))$ is a product of $X_1, \dots, X_n$. Then there exist for every $i = 1, \dots, n$ a unique morphism $c_i \colon X_i \to X$ with $p_j c_i = \delta_{ij} \, \mathrm{id}_{X_i}$ for every $j = 1, \dots, n$. The triple $(X, (p_1, \dots, p_n), (c_1, \dots, c_n))$ is then a biproduct of $X_1, \dots, X_n$.

iii) Dually, suppose that $(X, (c_1, \dots, c_n))$ is a coproduct of $X_1, \dots, X_n$. Then there exist for every $i = 1, \dots, n$ a unique morphism $p_i \colon X \to X_i$ with $p_i c_j = \delta_{ij} \, \mathrm{id}_{X_i}$ for every $j = 1, \dots, n$. The triple $(X, (p_1, \dots, p_n), (c_1, \dots, c_n))$ is then a biproduct of $X_1, \dots, X_n$.

Proof. For security reasons we consider the case $n = 0$ separately: The product over the empty family is a final object of $\mathcal{A}$, the coproduct over the empty family is an initial object of $\mathcal{A}$, and the biproduct over the empty family is a zero object of $\mathcal{A}$. The statements therefore follow for $n = 0$ from Lemma 3.17. In the following we consider the case $n \geq 1$.

i) It sufficies by duality to show that $(X, (c_i)_i)$ is a coproduct for $X_1, \dots, X_n$. Let $(D, (d_i)_i)$ be another pair consisting of an object $D \in \mathrm{Ob}(\mathcal{A})$ and morphisms $d_i \colon X_i \to D$. We need to show that there exists a unique morphism $\mu \colon X \to D$ that makes the triangle

$$\begin{tikzcd} X_i \arrow[r, "c_i"] \arrow[dr, "d_i"'] & X \arrow[d, dashed, "\mu"] \\ & D \end{tikzcd}$$

commute for every $i = 1, \dots, n$. If such a morphism μ exists then

$$\mu = \mu \,\mathrm{id}_X = \mu \sum_{i=1}^n c_i p_i = \sum_{i=1}^n \mu c_i p_i = \sum_{i=1}^n d_i p_i \,,$$

which shows that μ is unique. If we define on the other hand $\mu \coloneqq \sum_{i=1}^n d_i p_i$ then

$$\mu c_i = \sum_{j=1}^n d_j \underbrace{p_j c_i}_{=\delta_{ij}} = d_i \,,$$

which shows the existence of μ.

ii) By the universal property of the product there exists for every $i = 1, \dots, n$ a unique morphism $c_i \colon X_i \to X$ that makes for all $j \neq i$ the triangles

$$\begin{tikzcd} X_i \arrow[d, dashed, "c_i"'] \arrow[dr, "\mathrm{id}_{X_i}"] & \\ X \arrow[r, "p_i"'] & X_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_i \arrow[d, dashed, "c_i"'] \arrow[dr, "0"] & \\ X \arrow[r, "p_j"'] & X_j \end{tikzcd}$$

commute. This means that $p_j c_i = \delta_{ij}$ for all $i, j = 1, \dots, n$.

We now show that $\sum_{i=1}^n c_i p_i = \mathrm{id}_X$. Indeed, we find for every $j = 1, \dots, n$ that

$$p_j \circ \sum_{i=1}^n c_i p_i = \sum_{i=1}^n \underbrace{p_j c_i}_{=\delta_{ij}} p_i = p_j \,.$$

That shows that for every $j = 1, \dots, n$ the triangle

$$\begin{tikzcd} X \arrow[rr, dashed, "\sum_{i=1}^n c_i p_i"] \arrow[dr, "p_j"'] & & X \arrow[dl, "p_j"] \\ & X_j & \end{tikzcd}$$

commutes. But it follows from the uniqueness of products up to unique isomorphism that there exist a *unique* morphism $X \to X$ that makes this triangle commute. The identity $\mathrm{id}_X \colon X \to X$ also makes the above triangle commute, and so it follows that $\sum_{i=1}^n c_i p_i = \mathrm{id}_X$.

iii) This can be shown dually to part ii). □

Remark 3.20. It follows from Lemma 3.19 that for a preadditive category $\mathcal{A}$ the following are equivalent:

i) $\mathcal{A}$ has finite products.

ii) $\mathcal{A}$ has finite coproducts.

iii) $\mathcal{A}$ has finite biproducts.

Definition 3.21. A preadditve (or pre-$\mathbf{k}$-linear) category $\mathcal{A}$ is *additive* (resp. $\mathbf{k}$-linear) if it has finite biproducts (and thus equivalently finite products, and equivalently finite coproducts).

Remark. Additive (and $\mathbf{k}$-linear) categories have zero objects, as these are the biproducts of empty family of objects.

Remark⋆ 3.J. A category $\mathcal{A}$ is additive (resp. $\mathbf{k}$-linear) if and only if its dual category $\mathcal{A}^{\mathrm{op}}$ is additive (resp. $\mathbf{k}$-linear).

Remark⋆ 3.K. In a preadditive category $\mathcal{A}$ one can express morphisms between biproducts as matrices: Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ be two families of objects in $\mathcal{A}$ whose biproducts $X_1 \oplus \cdots \oplus X_n$ and $Y_1 \oplus \cdots \oplus Y_m$ exist, and denote the associated morphisms by

$$c_i \colon X_i \to X_1 \oplus \cdots \oplus X_n \quad \text{and} \quad p_i \colon X_1 \oplus \cdots \oplus X_n \to X_i \,,$$

and

$$d_i \colon Y_i \to Y_1 \oplus \cdots \oplus Y_m \quad \text{and} \quad q_i \colon Y_1 \oplus \cdots \oplus Y_m \to Y_i \,.$$

i) Suppose first that we are given a morphism

$$f \colon X_1 \oplus \cdots \oplus X_n \to Y_1 \oplus \cdots \oplus Y_m$$

in $\mathcal{A}$. It then follows from the calculation

$$\begin{aligned} f &= \mathrm{id}_{Y_1 \oplus \cdots \oplus Y_m} \circ f \circ \mathrm{id}_{X_1 \oplus \cdots \oplus X_n} \\ &= \left(\sum_{i=1}^{m} d_i q_i \right) \circ f \circ \left(\sum_{j=1}^{n} c_j p_j \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} d_i (q_i \circ f \circ c_j) p_j \,. \end{aligned}$$

that the morphism f is unique determined by the compositions $q_i \circ f \circ c_j$. We will refer to the composition

$$[f]_{ij} \coloneqq q_i \circ f \circ c_j$$

as the (i,j)-th component of f. The above calculation shows that the morphism f can be retrieved from its components via the formula

$$f = \sum_{i=1}^{m} \sum_{j=1}^{n} d_i [f]_{ij} p_j \,.$$

To better visualize the relation between f and its components, we may arrange these components in the form of an $(m \times n)$-matrix

$$\begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix} .$$

This is the *representing matrix of* f and is denoted by $[f]$. (Note that the (i, j)-th entry of the matrix $[f]$ is precisely $[f]_{ij}$.)

ii) Let on the other hand $g_{ij} \colon X_j \to Y_i$ for $i = 1, \dots, m$ and $j = 1, \dots, n$ be a collection of morphisms. We can then define a morphism $g \colon X \to Y$ via

$$g \coloneqq \sum_{i=1}^{m} \sum_{j=1}^{n} d_i g_{ij} p_j .$$

The components $[g]_{ij}$ of the morphism g are for all $i = 1, \dots, m$ and $j = 1, \dots, n$ given by

$$\begin{aligned} [g]_{ij} = q_i \circ g \circ c_j &= q_i \circ \left(\sum_{i'=1}^{m} \sum_{j'=1}^{n} d_{i'} g_{i'j'} p_{j'} \right) \circ c_j \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{q_i d_{i'}}_{=\delta_{i,i'}} g_{i'j'} \underbrace{p_{j'} c_j}_{=\delta_{j',j}} = g_{ij} . \end{aligned}$$

The components $[g]_{ij}$ of g are hence the morphisms g_{ij} that we started with.

iii) This shows overall that we have constructed a bijection

$$\begin{gathered} \mathcal{A}(X_1 \oplus \cdots \oplus X_n, Y_1 \oplus \cdots \oplus Y_m) \longleftrightarrow \left\{ \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{m1} & \cdots & g_{mn} \end{bmatrix} \,\middle|\, g_{ij} \in \mathcal{A}(X_j, Y_i) \right\} , \\ f \longmapsto [f] , \\ \sum_{i=1}^{m} \sum_{j=1}^{n} q_i g_{ij} c_j = g \longleftarrow\!\shortmid \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{1n} & \cdots & g_{mn} \end{bmatrix} . \end{gathered}$$

This way of representing morphisms between biproducts as matrices is compatible with both sums and composition of morphisms, and if $\mathcal{A}$ is pre-**k**-linear then also with scalar multiplication of morphisms:

- Let $f_1, f_2 \colon X_1 \oplus \cdots \oplus X_n \to Y_1 \oplus \cdots \oplus Y_m$ be two parallel morphisms in $\mathcal{A}$. Then the morphism $f_1 + f_2$ has for all $i = 1, \dots, m$ and $j = 1, \dots, n$ the components

$$[f_1 + f_2]_{ij} = q_i \circ (f_1 + f_2) \circ c_j = q_i \circ f_1 \circ c_j + q_i \circ f_2 \circ c_j = [f_1]_{ij} + [f_2]_{ij} .$$

  This shows that indeed

$$[f_1 + f_2] = [f_1] + [f_2] .$$

- Let $Z_1, \dots, Z_l$ be objects in $\mathcal{A}$ whose biproduct $Z_1 \oplus \dots \oplus Z_l$ exists in $\mathcal{A}$, and let
$$e_i \colon Z_i \to Z_1 \oplus \dots \oplus Z_l \quad \text{and} \quad r_i \colon Z_1 \oplus \dots \oplus Z_l \to Z_i \,,$$
be the associated morphisms. It then holds for any two composable morphisms
$$X_1 \oplus \dots \oplus X_n \xrightarrow{f} Y_1 \oplus \dots \oplus Y_m \xrightarrow{g} Z_1 \oplus \dots \oplus Z_l$$
in $\mathcal{A}$ that
$$[g \circ f] = [g] \cdot [f]$$
where the product on the right hand side is taken in the naive way. Indeed the composition $g \circ f$ has for all $i = 1, \dots, l$ and $k = 1, \dots, n$ the components
$$\begin{aligned} [g \circ f]_{ik} &= r_i \circ (g \circ f) \circ c_k = r_i \circ g \circ \mathrm{id}_Y \circ f \circ c_k \\ &= r_i \circ g \circ \left(\sum_{j=1}^m d_j q_j \right) \circ f \circ c_k \\ &= \sum_{j=1}^m (r_i \circ g \circ d_j) \circ (q_j \circ f \circ c_k) = \sum_{j=1}^m [g]_{ij} [f]_{jk} \,. \end{aligned}$$
The resulting term $\sum_{j=1}^m [g]_{ij}[f]_{jk}$ is precisely the (i,k)-th entry of the matrix product $[g] \cdot [f]$.
- If $\mathcal{A}$ also pre-**k**-linear then let $f \colon X_1 \oplus \dots \oplus X_n \to Y_1 \oplus \dots \oplus Y_m$ be a morphism in $\mathcal{A}$ and let $\lambda \in \mathbf{k}$ be a scalar. Then the morphism λf has for all $i = 1, \dots, m$ and $j = 1, \dots, n$ the components
$$[\lambda f]_{ij} = q_i \circ (\lambda f) \circ c_j = \lambda (q_i \circ f \circ c_j) = \lambda [f]_{ij} \,.$$
This shows that indeed
$$[\lambda f] = \lambda [f] \,.$$

iv) In the following we will notationally often not distinguish between the morphism $f \colon X_1 \oplus \dots \oplus X_n \to Y_1 \oplus \dots \oplus Y_m$ and its matrix representation. So instead of
$$[f] = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix}$$
(where $f_{ij} = [f]_{ij}$ is the (i,j)-th component of f) we will just write
$$f = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix} .$$
If one of the morphisms f_{ij} is the identity id_Z of some object Z (that is then necessarily given by $X_j = Z = Y_i$) then we will often just write the corresponding matrix entry as 1 instead of id_Z.

v) We finish this remark by pointing out that the morphisms $c_i \colon X_i \to X_1 \oplus \cdots \oplus X_n$ and $p_i \colon X_1 \oplus \cdots \oplus X_n \to X_i$ are by these notional conventions given by the matrices

$$c_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{and} \qquad p_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} .$$

Remark 3.22. Let $\mathcal{A}$ be an additive category. For any objects $X \in \mathrm{Ob}(\mathcal{A})$ we can define the *diagonal* (*morphism*)

$$\Delta_X \colon X \to X \oplus X \,,$$

by using the universal property of the product for $X \oplus X$, as the unique morphism $X \to X \oplus X$ that makes the diagram

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}_X"'] \arrow[d, dashed, "\Delta_X"] \arrow[dr, "\mathrm{id}_X"] & \\ X & X \oplus X \arrow[l, "p_1"] \arrow[r, "p_2"'] & X \end{tikzcd}$$

commute. This means that

$$p_1 \circ \Delta_X = \mathrm{id}_X \quad \text{and} \quad p_2 \circ \Delta_X = \mathrm{id}_X \,,$$

so the morphism Δ_X can be written is matrix form as

$$\Delta_X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

We can dually define the *codiagonal* (*morphism*)

$$\nabla_X \colon X \oplus X \to X$$

by using the universal property of the coproduct for $X \oplus X$, as the unique morphism $X \oplus X \to X$ that makes the diagram

$$\begin{tikzcd} X \arrow[r, "c_1"] \arrow[dr, "\mathrm{id}_X"'] & X \oplus X \arrow[d, dashed, "\nabla_X"] & X \arrow[l, "c_2"'] \arrow[dl, "\mathrm{id}_X"] \\ & X & \end{tikzcd}$$

commute. This means that

$$\nabla_X \circ c_1 = \mathrm{id}_X \quad \text{and} \quad \nabla_X \circ c_2 = \mathrm{id}_X \,,$$

so the morphism ∇_X can be written in matrix form as

$$\nabla_X = \begin{bmatrix} 1 & 1 \end{bmatrix} .$$

(Note that for $\mathcal{A} = A$-**Mod**, where A is a $\mathbf{k}$-algebra, the diagonal Δ_X is the usual diagonal map $\Delta_X(x) = (x, x)$, and the codiagonal ∇_X is the addition $\nabla_X(x_1, x_2) = x_1 + x_2$.)

We can now describe the sum $f + g$ of two parallel morphisms $f, g \colon X \to Y$ in $\mathcal{A}$ as the compositions

$$X \xrightarrow{\Delta_X} X \oplus X \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right]} Y \oplus Y \xrightarrow{\nabla_Y} Y \,. \tag{3.2}$$

Indeed, we find by matrix multiplication that

$$\nabla_Y \circ \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} \circ \Delta_X = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = f + g \,.$$

By using that

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} = \begin{bmatrix} f & g \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

we can also rewrite the composition (3.2) as

$$X \xrightarrow{\Delta_X} X \oplus X \xrightarrow{[\, f \; g \,]} Y \qquad \text{or} \qquad X \xrightarrow{\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]} Y \oplus Y \xrightarrow{\nabla_Y} Y \,.$$

This shows that the addition of $\mathcal{A}$ can be retrieved from the categorical structure of $\mathcal{A}$. It follows that an arbitrary category $\mathcal{A}$ can be made into an additive category in at most one way. We can therefore regard 'being additive' as a property of a category.

Definition 3.23. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor between arbitrary categories $\mathcal{C}$ and $\mathcal{D}$.

i) The functor F *respects* (*finite*) *products* if it holds for every (finite) family $(X_i)_{i \in I}$ of objects $X_i \in \mathrm{Ob}(\mathcal{C})$ and every product $(P, (p_i)_{i \in I})$ of this family that the pair $(F(P), (F(p_i))_{i \in I})$ is a product of the family $(F(X_i))_{i \in I}$.

ii) The functor F *respects* (*finite*) *coproducts* if it holds for every (finite) family $(X_i)_{i \in I}$ of objects $X_i \in \mathrm{Ob}(\mathcal{C})$ and every coproduct $(C, (c_i)_{i \in I})$ of this family that the pair $(F(C), (F(c_i))_{i \in I})$ is a coproduct of the family $(F(X_i))_{i \in I}$.

Suppose now that $F \colon \mathcal{A} \to \mathcal{B}$ is a functor between preadditive categories $\mathcal{A}$ and $\mathcal{B}$.

iii) The functor F *respects biproducts* if it holds for all objects $X_1, \dots, X_n \in \mathrm{Ob}(\mathcal{A})$ (where $n \geq 0$) and every biproduct $(X, (p_1, \dots, p_n), (c_1, \dots, c_n))$ of these objects that the triple $(F(X), (F(p_1), \dots, F(p_n)), (F(c_1), \dots, F(c_n)))$ is a biproduct of the objects $F(X_1), \dots, F(X_n)$.

Theorem 3.24. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between preadditive categories $\mathcal{A}$ and $\mathcal{B}$.

i) If the functor F is additive then it respects biproducts (and hence also finite products and finite coproducts).

ii) If the categories $\mathcal{A}$ and $\mathcal{B}$ are already additive, then the following conditions on F are equivalent:

   a) F is additive.
   b) F respects biproducts.
   c) F respects finite products.
   d) F respects finite coproducts.

Proof.

i) Let $n \geq 0$, let $X_1, \dots, X_n \in \mathrm{Ob}(\mathcal{A})$ and let $(X, (p_i)_i, (c_i)_i)$ be a biproduct of $X_1, \dots, X_n$.

Once again we consider the case $n = 0$ separately: The biproduct X is then a zero object of $\mathcal{A}$. It follows that $\mathrm{id}_X = 0_{X,X}$ by Lemma 3.17, and hence

$$\mathrm{id}_{F(X)} = F(\mathrm{id}_X) = F(0_{X,X}) = 0_{F(X),F(X)}\,,$$

by the additivity of F. This shows that $F(X)$ is a zero object for $\mathcal{B}$.

Let now $n \geq 1$. We then calculate that

$$F(p_j)F(c_i) = F(p_j c_i) = F\left(\begin{cases} \mathrm{id}_{X_i} & \text{if } i = j \\ 0_{X_i,X_j} & \text{if } i \neq j \end{cases}\right) = \begin{cases} \mathrm{id}_{F(X_i)} & \text{if } i = j\,, \\ 0_{F(X_i),F(X_j)} & \text{if } i \neq j\,, \end{cases}$$

and

$$\sum_{i=1}^{n} F(c_i)F(p_i) = \sum_{i=1}^{n} F(c_i p_i) = F\left(\sum_{i=1}^{n} c_i p_i\right) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)}\ .$$

ii) a) $\implies$ b) This has been shown in part i).

b) $\implies$ c) Let $(X, (p_i)_i)$ be a product of the objects $X_1, \dots, X_n$. It follows from Lemma 3.19 that there exist unique morphisms $c_i\colon X_i \to X$ such that the triple $(X, (p_i)_i, (c_i)_i)$ is a biproduct for the objects $X_1, \dots, X_n$. It follows that the triple $(F(X), (F(p_i))_i, (F(c_i))_i)$ is a biproduct for the objects $F(X_1), \dots, F(X_n)$ as the functor F preserves biproducts. This entails that the tuple $(F(X), (F(p_i))_i)$ is a product of these objects.

c) $\implies$ b) It follows from F respecting products that F respects terminal objects, because a terminal object is the same as an empty product. Hence $F(0) = 0$. It follows that $F(0_{X,Y}) = 0_{F(X),F(Y)}$ for any two objects $X, Y \in \mathrm{Ob}(\mathcal{A})$. Indeed, by applying the functor F to the commutative triangle

$$\begin{tikzcd} X \arrow[rr, "0_{X,Y}"] \arrow[dr] & & Y \\ & 0 \arrow[ur] & \end{tikzcd}$$

we get the following commutative triangle:

$$\begin{tikzcd} F(X) \arrow[rr, "F(0_{X,Y})"] \arrow[dr] & & F(Y) \\ & 0 \arrow[ur] & \end{tikzcd}$$

The commutativity of this triangle shows that the morphism $F(0_{X,Y})$ factors through the zero object 0, which is precisly what is means for $F(0_{X,Y})$ to be the zero morphism.

Let now $(X, (p_i)_i, (c_i)_i)$ be a biproduct of some objects $X_1, \dots, X_n \in \mathrm{Ob}(\mathcal{A})$. Then $(X, (p_i)_i)$ is a product of $X_1, \dots, X_n$, and it follows that $(F(X), (F(p_i))_i)$ is a product of $F(X_1), \dots, F(X_n)$ because F respects finite products. We find for the morphisms $F(c_i)\colon F(X_i) \to F(X)$ that

$$F(p_i) \circ F(c_i) = F(p_i \circ c_i) = F(\mathrm{id}_{X_i}) = \mathrm{id}_{F(X_i)}$$

and we also find for $j \neq i$ that

$$F(p_j) \circ F(c_i) = F(p_j \circ c_i) = F(0_{X_i,X_j}) = 0_{F(X_i),F(X_j)} \,.$$

It follows from Lemma 3.19 that the triple $(F(X), (F(p_i))_i, (F(c_i))_i)$ is a biproduct of $F(X_1), \dots, F(X_n)$.

b) $\iff$ d) This can be shown dually to the equivalence of b) and c).

b) $\implies$ a) We find as in the implication c) $\implies$ b) that $F(0) = 0$ and that consequently $F(0_{X,Y}) = 0_{F(X),F(Y)}$ for all $X, Y \in \mathrm{Ob}(\mathcal{A})$. It also follows from the already proven implications b) $\implies$ c) and b) $\implies$ d) that F respects products and coproducts. We hence find that

$$F(\Delta_X) = \Delta_{F(X)} \quad \text{and} \quad F(\nabla_Y) = \nabla_{F(Y)}$$

for all $X, Y \in \mathrm{Ob}(\mathcal{A})$.

Let $f, g\colon X \to Y$ be two parallel morphisms in $\mathcal{A}$. We can then describe their sum $f + g$ as the composition

$$f + g\colon X \xrightarrow{\Delta_X} X \oplus X \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right]} Y \oplus Y \xrightarrow{\nabla_Y} Y \,. \tag{3.3}$$

It follows from F preserving coproducts and products, as well as identities and zero morphisms, that

$$F\left(\begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}\right) = \begin{bmatrix} F(f) & 0 \\ 0 & F(g) \end{bmatrix} \,.$$

By using that $F(\Delta_X) = \Delta_{F(X)}$ and $F(\nabla_Y) = \nabla_{F(Y)}$ we altogether find that applying the functor F to the compositon (3.3) exhibits the morphism $F(f+g)$ as the composition

$$F(f+g)\colon F(X) \xrightarrow{\Delta_{F(X)}} F(X) \oplus F(X) \xrightarrow{\left[\begin{smallmatrix} F(f) & 0 \\ 0 & F(g) \end{smallmatrix}\right]} F(Y) \oplus F(Y) \xrightarrow{\nabla_{F(Y)}} Y \,.$$

This composition is precisely $F(f) + F(g)$, so $F(f+g) = F(f) + F(g)$. □

Kernels and Cokernels

Definition 3.25. Let $\mathcal{C}$ be a category that has a zero object, or that is preaddive. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$.

i) A *kernel* of f is a pair (K, k) consisting of
- an object $K \in \mathrm{Ob}(\mathcal{C})$ and
- a morphism $k\colon K \to X$

such that

(K1) $f \circ k = 0$, and

(K2) the morphism k is universal with in property, in the sense that for every morphism $\ell\colon L \to X$ in $\mathcal{C}$ with $f \circ \ell = 0$ there exists a unique morphism $\lambda\colon L \to K$ that makes the following diagram commute:

$$\begin{tikzcd}
K \arrow[dr, "k"] \arrow[drr, bend left, "0"] & & \\
& X \arrow[r, "f"] & Y \\
L \arrow[uu, dashed, "\lambda"] \arrow[ur, "\ell"'] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

ii) A *cokernel* of f is a pair (C, c) consisting of
- an object $C \in \mathrm{Ob}(\mathcal{C})$ and
- a morphism $c\colon Y \to C$

such that

(C1) $c \circ f = 0$, and

(C2) the morphism c is universal with this property, in the sense that for every morphism $d\colon Y \to D$ in $\mathcal{C}$ with $d \circ f = 0$ there exists a unique morphism $\mu\colon C \to D$ that makes the following diagram commute:

$$\begin{tikzcd}
& & C \arrow[dd, dashed, "\mu"] \\
X \arrow[r, "f"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & Y \arrow[ur, "c"] \arrow[dr, "d"'] & \\
& & D
\end{tikzcd}$$

Remark 3.26. Let $\mathcal{C}$ be a category that has a zero object, or that is preaddive. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$.

i) A pair (K, k) is a kernel of f in $\mathcal{C}$ if and only if it is a cokernel of f in $\mathcal{C}^{\mathrm{op}}$.

ii) Kernels and cokernels are unique up to unique isomorphism.

iii) If every morphism in $\mathcal{C}$ has a kernel (resp. a cokernel) then the category $\mathcal{C}$ *has kernels* (resp. *has cokernels*).

iv) The kernel of f is denoted by $\ker(f) \to X$, and the cokernel of f is denoted by $Y \to \operatorname{coker}(f)$.

End of lecture 10

Notation⋆ 3.L.

i) Let $\mathcal{C}$ be a category that has a zero object and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. Then we write $\ker(f) = 0$ to mean that the zero morphism $0 \to X$ is a kernel of f. We dually write $\operatorname{coker}(f) = 0$ to mean that the zero morphism $Y \to 0$ is a cokernel of f.

ii) Let $\mathcal{C}$ be a category that is preadditive, or that has a zero object.
Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. If $g\colon Y \to Z$ is another morphism in $\mathcal{C}$ then we write that $\ker(f) = \ker(gf)$ to mean that a morphism $k\colon K \to X$ in $\mathcal{C}$ is a kernel of f if and only if it is a kernel of gf. (This entails in particular that a kernel for f exists if and only if a kernel for gf exist.)
Let $h\colon W \to X$ be another morphism in $\mathcal{C}$. We write that $\operatorname{coker}(f) = \operatorname{coker}(fh)$ to mean that a morphism $c\colon Y \to C$ in $\mathcal{C}$ is a cokernel of f if and only if it is a cokernel of fh. (This entails in particular that a cokernel for f exists if and only if a cokernel for fh exists.)

Remark⋆ 3.M. Let $\mathcal{C}$ be a category that has a zero object and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. Then $\ker(f) = 0$ if and only if it follows for every morphism $u\colon W \to X$ in $\mathcal{C}$ with $f \circ u = 0$ that already $u = 0$.

Indeed, the composition $0 \to X \xrightarrow{f} Y$ is the zero morphism. That $\ker(f) = 0$ therefore means that every morphism $u\colon W \to X$ with $f \circ u = 0$ factors uniquely trough the zero morphism $0 \to X$, i.e. that there exists a unique morphism $W \to 0$ that makes the triangle

$$\begin{tikzcd} W \arrow[d, dashed] \arrow[dr, "u"] \arrow[drr, bend left, "0"] & & \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \end{tikzcd}$$

commute. That u factors trough the zero morphism $0 \to X$ is equivalent to $u = 0$, and this factorization is necessarily unique because there exist only one morphism $W \to 0$.
It holds dually that $\operatorname{coker}(f) = 0$ if and only if it follows for every morphism $v\colon Y \to Z$ in $\mathcal{C}$ with $v \circ f = 0$ that already $v = 0$.

Lemma 3.27. Let $\mathcal{C}$ be a category that is preadditive, or that has a zero object. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$.

i) If $k\colon \ker(f) \to X$ is a kernel of f then the morphism k is a monomorphism. Dually, if $c\colon Y \to \operatorname{coker}(f)$ is a cokernel of f then the morphism c is an epimorphism.

ii) Suppose that $\mathcal{C}$ is both preadditive and has a zero object (e.g. $\mathcal{C}$ is additive). If $\ker(f) = 0$ then f is a monomorphism. Dually, if $\operatorname{coker}(f) = 0$ then f is an epimorphism.

iii) Suppose that $\mathcal{C}$ has a zero object. If f is a monomorphism then $\ker(f) = 0$, and if f is an epimorphism then $\operatorname{coker}(f) = 0$.

iv) If $u\colon Y \to Z$ is a monomorphism in $\mathcal{C}$ then $\ker(f) = \ker(uf)$. Dually, if $p\colon W \to X$ is an epimorphism in $\mathcal{C}$ then $\operatorname{coker}(f) = \operatorname{coker}(fp)$.

Proof.

i) Let $u, v\colon W \to \ker(f)$ be two parallel morphisms with $k \circ u = k \circ v$, and denote this composition by $w\colon W \to X$. Then

$$f \circ w = f \circ k \circ u = 0 \circ u = 0\,.$$

It follows from the universal property of the kernel $k\colon \ker(f) \to X$ that there exist a unique morphism $W \to \ker(f)$ that makes the triangle

$$\begin{tikzcd} W \arrow[d, dashed] \arrow[dr, "w"] & \\ \ker(f) \arrow[r, "k"] & X \end{tikzcd}$$

commute. Both u and v make this triangle commute, and so it follows that $u = v$.

That the cokernel $c\colon Y \to \operatorname{coker}(f)$ is an epimorphism can be shown dually.

ii) Let $u, v\colon W \to X$ be two morphisms with $f \circ u = f \circ v$. Then

$$f \circ (u - v) = f \circ u - f \circ v = 0\,,$$

and it follows from the universal property of the kernel $\ker(f)$ that the difference $u - v$ factors through $\ker(f) = 0$, which results in the following commutative triangle:

$$\begin{tikzcd} W \arrow[d, dashed] \arrow[dr, "u-v"] & \\ 0 \arrow[r] & X \end{tikzcd}$$

The morphisms $W \to 0$ and $0 \to X$ are necessarily the zero morphisms, so it follows that $u - v = 0 \circ 0 = 0$, and hence $u = v$.

That f is an epimorphism if $\operatorname{coker}(f) = 0$ can be shown dually.

iii) If $u\colon W \to X$ is any morphism with $f \circ u = 0$ then

$$f \circ u = 0 = f \circ 0$$

and hence $u = 0$.

That $\operatorname{coker}(f) = 0$ if f is an epimorphism can be shown dually.

iv) It holds for every morphism $v\colon W \to X$ that

$$f \circ v = 0 \iff uf \circ v = u \circ 0 \iff uf \circ v = 0\,,$$

where the first equivalence holds because u is a monomorphism. We thus find for every morphism $k\colon K \to X$ in $\mathcal{C}$ that

$$\begin{aligned} & k \text{ is a kernel of } f \\ \iff & \text{every morphism } v\colon W \to X \text{ with } f \circ v = 0 \text{ factors uniquely through } k \\ \iff & \text{every morphism } v\colon W \to X \text{ with } uf \circ v = 0 \text{ factors uniquely through } k \\ \iff & k \text{ is a kernel of } uf\,. \end{aligned}$$

That $\operatorname{coker}(f) = \operatorname{coker}(fp)$ can be shown dually. □

Notation* 3.N. Let $\mathcal{C}$ be a category that is preadditive, or that has a zero object. We say that a morphism $k\colon K \to X$ in $\mathcal{C}$ *is a kernel* if it is a kernel for some morphism $f\colon X \to Y$. Dually, we say that a morphism $c\colon Y \to C$ in $\mathcal{C}$ *is a cokernel* if it is a cokernel for some morphism $f\colon X \to Y$.

Example 3.28.

i) Let A be a $\mathbf{k}$-algebra and consider the module category A-**Mod**. Let $f\colon M \to N$ be a homomorphism of A-modules. Then the submodule $\ker(f) = f^{-1}(0)$ of M together with the inclusion $k\colon \ker(f) \to M$ is a kernel of f. The quotient module $\operatorname{coker}(f) = N/f(M)$ together with the canonical projection $c\colon N \to \operatorname{coker}(f)$ is a cokernel of f.

   Note that in the category A-**Mod**, a morphism k is a monomorphism if and only if k is a kernel, and similarly that a morphism c is an epimorphism if and only if c is a cokernel. (Recall that the monomorphisms in A-**Mod** are preciely the injective module homomorphisms, and that the epimorphisms in A-**Mod** are precisely the surjective module homomorphisms.)

ii) Consider the category **Grp** and let $f\colon G \to H$ be a group homorphism.

   Then the subgroup $\ker(f) = f^{-1}(1)$ together with the inclusion $k\colon \ker(f) \to G$ is a kernel of f. Note that $\ker(f)$ is always a normal subgroup of G, while for every subgroup $K' \subseteq G$ the inclusion $k'\colon K' \to G$ is a monomorphism. So in **Grp** not every monomorphism is a kernel.

   A cokernel of f is given by the quotient group $\operatorname{coker}(f) = H/\overline{f(G)}$ together with the canonical projection $c\colon H \to \operatorname{coker}(f)$, where

$$\overline{f(G)} \coloneqq \langle hf(g)h^{-1} \mid h \in H, g \in G\rangle$$

   is the normal subgroup of H generated by $f(G)$, i.e. the normal closure of $f(G)$ in H. If $p\colon H \to C$ is any epimorphism in **Grp**, then p is surjective, and hence

$$C \cong H/\ker(p) = \operatorname{coker}(\ker(p) \to H)$$

   This shows that in **Grp** every epimorphism is a cokernel.

iii) Let $\mathbf{Set}_*$ be the category of pointed sets:

The objects of $\mathbf{Set}_*$ are pairs (X, x_0) consisting of a set X and a fixed base point $x_0 \in X$. A morphism $f\colon (X, x_0) \to (Y, y_0)$ in $\mathbf{Set}_*$ is a map $f\colon X \to Y$ with $f(x_0) = y_0$. (Such maps are also known as *pointed maps*.) The category $\mathbf{Set}_*$ has the singleton $(\{*\}, *)$ as a zero object. The monomorphisms in $\mathbf{Set}_*$ are precisely the injective pointed maps, and the epimorphisms are the surjective pointed maps.

Let $f\colon (X, x_0) \to (Y, y_0)$ be a morphism in $\mathbf{Set}_*$. A kernel for f is given by $\ker(f) = (f^{-1}(y_0), x_0)$ together with the inclusion $k\colon \ker(f) \to (X, x_0)$. A cokernel for f is given by $\operatorname{coker}(f) = (Y/{\sim}, [y_0])$ together with the canonical projection $c\colon (Y, y_0) \to \operatorname{coker}(f)$, where $\sim$ is the equivalence relation on Y generated by $y \sim y'$ for all $y, y' \in f(X)$. More explicitely, we have for any two $y, y' \in Y$ that

$$y \sim y' \iff (y = y' \text{ or } y, y' \in f(X)) .$$

Every monomorphism in $\mathbf{Set}_*$ is a kernel (namely that of its cokernel). But not every epimorphism in $\mathbf{Set}_*$ is a cokernel, because it holds for every cokernel $c\colon (Y, y_0) \to (C, c_0)$ that every element $z \in C$ with $z \neq c_0$ has precisely one preimage under c. (This follows from the above explicit description of the cokernel.)

iv) The ring $A := \mathbb{Z}[t_1, t_2, t_3, \dots]$ is not noetherian because the ideal $I := (t_1, t_2, t_3, \dots)$ of A is not finitely generated. Consider the category $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ of finitely generated A-modules. This category is additive.

Every morphism $f\colon M \to N$ in $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ has a cokernel in $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ because the cokernel of f in $A\text{-}\mathbf{Mod}$ is already contained in $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ (and the zero morphisms in $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ coincides with the one in $A\text{-}\mathbf{Mod}$).

Let $f\colon A \to A/I$ be the canonical projection; note that A/I is finitely generated and hence contained in $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ (even though the A-modules I is not contained in $A\text{-}\mathbf{Mod}$). The morphism f has no kernel in $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$:

Morally speaking, the problem is that the kernel of f in $A\text{-}\mathbf{Mod}$, which is I, is not contained in $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$. But this is not yet a proper proof because a kernel of f in $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ does not necessarily have to also be a kernel of f in $A\text{-}\mathbf{Mod}$.

So instead, assume that there exists a kernel $k\colon K \to A$ of f in $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$. Then $f \circ k = 0$ and hence $k(K) \subseteq I$. Then for every $i \in I$ the morphism

$$\ell_i\colon A \to A\,, \quad a \mapsto t_i a$$

satisfies $\ell_i(A) \subseteq I$ and hence $f \circ \ell_i = 0$. It follows that the morphism ℓ_i factors uniquely trough the kernel k, i.e. that there exists a unique morphism $\lambda_i\colon A \to K$ that makes the triangle

$$\begin{tikzcd} A \arrow[d, dashed, "\lambda_i"'] \arrow[dr, "\ell_i"] & \\ K \arrow[r, "k"'] & A \end{tikzcd}$$

commute. It follows for every $i \in I$ that

$$t_i = \ell_i(1) = (k \circ \lambda_i)(1) = k(\lambda_i(1)) \in k(K)$$

and hence $t_i \in k(K)$. This shows that also $I \subseteq k(K)$, and hence $I = k(K)$. But it follows from K being finitely generated that $k(K)$ is also finitely generated, which contradicts I not being finitely generated.

Remark⋆ 3.O (Functoriality of the (co)kernel)**.** Let $\mathcal{C}$ be a category that is preadditive, or that has a zero object.

i) Let $f\colon X \to Y$ and $f'\colon X' \to Y'$ be two morphisms in the category $\mathcal{C}$ that admit kernels $k\colon K \to X$ and $k'\colon K' \to X'$. Then for any pair (φ, φ'') of morphisms $\varphi\colon X \to Y$ and $\varphi''\colon Y \to Y''$ that make the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\varphi"'] & Y \arrow[d, dashed, "\varphi''"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

commute, there exists a unique morphism $\varphi'\colon K \to K'$ that makes the diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, dashed, "\varphi'"'] & X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\varphi''"] \\ K' \arrow[r, "k'"] & X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

commute. Indeed, it holds for the composition $\varphi \circ k\colon K \to X'$ that

$$f' \circ \varphi \circ k = \varphi'' \circ f \circ k = \varphi'' \circ 0 = 0\,,$$

hence there exists a unique morphism $\varphi'\colon K \to K'$ with $k' \circ \varphi' = \varphi \circ k$ by the universal property of the kernel $k'\colon X' \to Y'$ of f'.

This induced morphism between kernels is functorial in the following sense:

- Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$ that admits a kernel $k\colon K \to X$. Then the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\mathrm{id}_X"'] & Y \arrow[d, dashed, "\mathrm{id}_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. The induced morphism $K \to K$ is given by the identity id_K because the diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, dashed, "\mathrm{id}_K"'] & X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[d, "\mathrm{id}_Y"] \\ K \arrow[r, "k"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

- Let
$$f \colon X \to Y\,, \quad f' \colon X' \to Y'\,, \quad f'' \colon X'' \to Y''$$
be morphisms in $\mathcal{C}$ that admit kernels
$$k \colon K \to X\,, \quad k' \colon K' \to X'\,, \quad k'' \colon K'' \to X''\,.$$
Let
$$\varphi \colon X \to Y,\, \varphi'' \colon X' \to Y' \quad \text{and} \quad \psi \colon X' \to X'',\, \psi'' \colon Y' \to Y''$$
be morphisms in $\mathcal{C}$ that makes the squares
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\varphi"'] & Y \arrow[d, dashed, "\varphi''"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[r, "f"] \arrow[d, dashed, "\psi"'] & Y' \arrow[d, dashed, "\psi''"] \\ X'' \arrow[r, "f'"] & Y'' \end{tikzcd}$$
commute. Then for
$$\theta \coloneqq \psi \circ \varphi \quad \text{and} \quad \theta'' \coloneqq \psi'' \circ \varphi''$$
the square
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\theta"'] & Y \arrow[d, dashed, "\theta''"] \\ X'' \arrow[r, "f'"] & Y'' \end{tikzcd}$$
again commutes by the commutativity of the following diagram:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dd, dashed, bend right=60, "\theta"'] & Y \arrow[d, "\varphi''"] \arrow[dd, dashed, bend left=60, "\theta''"] \\ X' \arrow[r, "f'"] \arrow[d, "\psi"'] & Y' \arrow[d, "\psi''"] \\ X'' \arrow[r, "f''"] & Y'' \end{tikzcd}$$
Let
$$\varphi' \colon K \to K'\,, \quad \psi' \colon K' \to K''\,, \quad \theta' \colon K \to K''$$
be the induced morphisms. Then
$$\theta' = \psi' \circ \varphi'\,.$$

Indeed, we have the following commutative diagram:

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "\varphi'"] \arrow[dd, dashed, bend right, "\psi' \circ \varphi'"'] & X \arrow[r, "f"] \arrow[d, "\varphi"] \arrow[dd, dashed, bend right, "\theta"'] & Y \arrow[d, "\varphi''"] \arrow[dd, dashed, bend right, "\theta''"'] \\
K' \arrow[r, "k'"] \arrow[d, "\psi'"] & X' \arrow[r, "f'"] \arrow[d, "\psi"] & Y' \arrow[d, "\psi''"] \\
K'' \arrow[r, "k''"] & X'' \arrow[r, "f''"] & Y''
\end{tikzcd}$$

The commutativity of the subdiagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, dashed, "\psi' \circ \varphi'"'] & X \arrow[r, "f"] \arrow[d, "\theta"'] & Y \arrow[d, dashed, "\theta''"] \\
K'' \arrow[r, "k''"] & X' \arrow[r, "f''"] & Y''
\end{tikzcd}$$

shows that $\psi' \circ \varphi'$ satisfies the defining property of θ', and hence $\theta' = \psi' \circ \varphi'$.

Suppose that the category $\mathcal{C}$ is preadditive. Then we also have the following additivity of the induced morphism between kernels:

- Let
$$f \colon X \to Y \quad \text{and} \quad f' \colon X' \to Y'$$
be morphisms in $\mathcal{C}$ that admit kernels
$$k \colon K \to X \quad \text{and} \quad k' \colon K' \to X' .$$
Let
$$\varphi \colon X \to X',\ \varphi'' \colon Y \to Y' \quad \text{and} \quad \psi \colon X \to X',\ \psi'' \colon Y \to Y'$$
be morphisms in $\mathcal{C}$ that make the squares
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "\varphi"'] & Y \arrow[d, dashed, "\varphi''"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "\psi"'] & Y \arrow[d, dashed, "\psi''"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$
commute. Then for the morphisms
$$\theta \coloneqq \psi + \varphi \quad \text{and} \quad \theta'' \coloneqq \psi'' + \varphi''$$
the square
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "\theta"'] & Y \arrow[d, dashed, "\theta''"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

again commutes because

$$\begin{aligned}\theta'' \circ f = (\varphi'' + \psi'') \circ f = \varphi'' \circ f + \psi'' \circ f \\ = f \circ \varphi + f \circ \psi = f \circ (\varphi + \psi) = f \circ \theta\,.\end{aligned}$$

It then holds for the induced morphisms between kernels

$$\varphi', \psi', \theta' \colon K \to K'$$

and

$$\theta' = \varphi' + \psi'\,.$$

Indeed, it follows from the commutativity of the squares

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, dashed, "\varphi'"'] & X \arrow[d, dashed, "\varphi"] \\ K' \arrow[r, "f'"] & X' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K \arrow[r, "f"] \arrow[d, dashed, "\psi'"'] & X \arrow[d, dashed, "\psi"] \\ K' \arrow[r, "f'"] & X' \end{tikzcd}$$

in the same way as above that the square

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, dashed, "\varphi'+\psi'"'] & X \arrow[d, dashed, "\varphi+\psi"] \\ K' \arrow[r, "f'"] & X' \end{tikzcd}$$

again commutes. Hence $\varphi' + \psi'$ satisfies the defining property of the induced morphism θ', and thus $\theta' = \varphi' + \psi'$.

ii) For cokernels the dual construction works:
Suppose that $f \colon X \to Y$ and $f' \colon X' \to Y'$ are morphisms in $\mathcal{C}$ that admit cokernels $c \colon Y \to C$ and $c' \colon Y' \to C'$. If $\varphi' \colon X \to X'$ and $\varphi \colon Y \to Y'$ are morphisms that makes the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\varphi'"'] & Y \arrow[d, dashed, "\varphi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

commute, then there exist a unique morphism $\varphi'' \colon C \to C'$ that makes the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi'"'] & Y \arrow[r, "c"] \arrow[d, "\varphi"] & C \arrow[d, dashed, "\varphi''"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "c'"] & C' \end{tikzcd}$$

commute. This induced morphism between cokernels is functorial and additive in the following way:

- If $f\colon X \to Y$ is a morphism in $\mathcal{C}$ that admits a cokernel $c\colon Y \to C$ then the square
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\mathrm{id}_X"'] & Y \arrow[d, dashed, "\mathrm{id}_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
commutes, and the induced morphism $C \to C$ is the identity id_C.
- Let
$$f\colon X \to Y\,, \quad f'\colon X' \to Y'\,, \quad f''\colon X'' \to Y''$$
be morphisms in $\mathcal{C}$ that admit cokernels
$$c\colon Y \to C\,, \quad c'\colon Y' \to C'\,, \quad c''\colon Y'' \to C''\,.$$
If
$$\varphi'\colon X \to X'\,, \varphi\colon Y \to Y' \quad \text{and} \quad \psi'\colon X' \to X''\,, \psi\colon Y' \to Y''$$
are morphisms that make the squares
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\varphi'"'] & Y \arrow[d, dashed, "\varphi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\psi'"'] & Y \arrow[d, dashed, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$
commute, then for
$$\theta' := \psi' \circ \varphi' \quad \text{and} \quad \theta := \psi \circ \varphi$$
the square
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\psi'\circ\varphi'"'] & Y \arrow[d, dashed, "\psi\circ\varphi"] \\ X'' \arrow[r, "f''"] & Y'' \end{tikzcd}$$
again commutes. It then holds for the induced morphisms
$$\varphi''\colon C \to C'\,, \quad \psi''\colon C' \to C''\,, \quad \theta''\colon C \to C''$$
that $\theta'' = \psi'' \circ \varphi''$.
- Suppose that $\mathcal{C}$ is preadditive, and let
$$f\colon X \to Y \quad \text{and} \quad f'\colon X' \to Y'$$
be morphisms in $\mathcal{C}$ that admit cokernels
$$c\colon Y \to C \quad \text{and} \quad c'\colon Y' \to C'\,.$$
Let
$$\varphi'\colon X \to X',\, \varphi\colon Y \to Y' \quad \text{and} \quad \psi'\colon X \to X',\, \psi\colon Y \to Y'$$

be morphisms in $\mathcal{C}$ that make the squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\varphi'"'] & Y \arrow[d, dashed, "\varphi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\psi'"'] & Y \arrow[d, dashed, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

commute. Then for the morphisms

$$\theta' \coloneqq \psi' + \varphi' \quad \text{and} \quad \theta \coloneqq \psi + \varphi$$

the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\theta'"'] & Y \arrow[d, dashed, "\theta"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

again commutes, and it holds for the induced morphisms

$$\varphi'', \psi'', \theta'' \colon C \to C'$$

that

$$\theta'' = \varphi'' + \psi'' .$$

iii) Suppose that the category $\mathcal{C}$ has kernels. The above 'functoriality' of the kernel can be used to construct a functor $\operatorname{Ker}\colon \mathbf{Mor}(\mathcal{C}) \to \mathbf{Mor}(\mathcal{C})$, where $\mathbf{Mor}(\mathcal{C})$ denotes the morphism category of $\mathcal{C}$ from Example⋆ 2.B:

For any object $(X, f, Y) \in \mathbf{Mor}(\mathcal{C})$ we choose[4] a kernel $k\colon K \to X$ of f and set $\operatorname{Ker}(f) \coloneqq (K, k, X)$. For a morphism $(g, g')\colon (X, f, Y) \to (X, f', Y)$ in $\mathbf{Mor}(\mathcal{C})$ we get the following commutative square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "g"'] & Y \arrow[d, dashed, "g'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

For the choosen kernels $\operatorname{Ker}(X, f, Y) = (K, k, X)$ and $\operatorname{Ker}(X', f, Y') = (K', k', X')$ we set $\operatorname{Ker}((h, h')) = (K, h'', K')$ where $h''\colon K \to K'$ is the unique morphism that makes the diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, dashed, "h''"'] & X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "h'"] \\ K' \arrow[r, "k'"] & X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

commute. The above 'functoriality' of the kernel shows that Ker is a functor $\operatorname{Ker}\colon \mathbf{Mor}(\mathcal{C}) \to \mathbf{Mor}(\mathcal{C})$.

If the category $\mathcal{C}$ has cokernels then one can similarly use the above 'functoriality' of the cokernel to construct a functor $\operatorname{Coker}\colon \mathbf{Mor}(\mathcal{C}) \to \mathbf{Mor}(\mathcal{C})$.

[4]This choice can be avoided by working with *anafunctors* instead of just functors, as explained in [Mak96].

iv) Suppose that the category $\mathcal{C}$ has kernels. The data of the constructed kernel functor $\mathrm{Ker}\colon \mathbf{Mor}(\mathcal{C}) \to \mathbf{Mor}(\mathcal{C})$ can also arranged in a different way. Namely that of a kernel functor $\ker\colon \mathbf{Mor}(\mathcal{C}) \to \mathcal{C}$ together with a natural transformation $k\colon S \to \ker$.

Let $S\colon \mathbf{Mor}(\mathcal{C}) \to \mathcal{C}$ be the source-functor from Example⋆ 2.F. The above functor $\mathrm{Ker}\colon \mathbf{Mor}(\mathcal{C}) \to \mathbf{Mor}(\mathcal{C})$ assigns to each morphism $f\colon X \to Y$ in $\mathcal{C}$ a triple $(\ker(f), k_f, X)$ such that

$$k_f\colon \ker(f) \to X$$

is a kernel of f. We furthermore get for every commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\varphi"'] & Y \arrow[d, dashed, "\varphi''"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

in $\mathcal{C}$ that $\mathrm{Ker}((\varphi, \varphi''))$ is the triple $(\ker(f), \ker(\varphi, \varphi''), \ker(f'))$ where

$$\ker(\varphi, \varphi'')\colon \ker(f) \to \ker(f')$$

is the unique morphism that makes the square

$$\begin{tikzcd} \ker(f) \arrow[r, "k_f"] \arrow[d, dashed, "\ker(\varphi,\varphi'')"'] & X \arrow[r, "f"] \arrow[d, "\varphi"] & Y \arrow[d, "\varphi''"] \\ \ker(f') \arrow[r, "k_{f'}"] & X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

commute. In this way we have constructed $\ker\colon \mathbf{Mor}(\mathcal{C}) \to \mathcal{C}$ together with a natural transformation $k\colon \ker \to S$.

If the category $\mathcal{C}$ has cokernels, then we can dually encode the data of the cokernel functor $\mathrm{Coker}\colon \mathbf{Mor}(\mathcal{C}) \to \mathbf{Mor}(\mathcal{C})$ in form of a functor $\mathrm{coker}\colon \mathbf{Mor}(\mathcal{C}) \to \mathcal{C}$ together with a natural transformation $c\colon T \to \mathrm{coker}$, where $T\colon \mathbf{Mor}(\mathcal{C}) \to \mathcal{C}$ denotes the target-functor from Example⋆ 2.F.

Definition 3.29. Let $\mathcal{C}$ be a category that is preadditive, or that has a zero object, and that has kernels and cokernels. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$.

i) An *image* of f is a kernel of a cokernel of f, and is denotedy by $\mathrm{im}(f) \to Y$.

ii) A *coimage* of f is a cokernel of a kernel of f, and is denoted by $X \to \mathrm{coim}(f)$.

Example⋆ 3.P. Let A be a $\mathbf{k}$-algebra and let $f\colon M \to N$ be a morphism in A-**Mod** (or **Mod**-A). Then an image of f is given by the submodule $\mathrm{im}(f) = f(M)$ of N together with the inclusion $\mathrm{im}(f) \to N$. A coimage of f is given by the quotient module $\mathrm{coim}(f) = M/\ker(f)$ together with the canonical projection $M \to \mathrm{coim}(f)$.

Remark. Images and coimages are unique up to unique isomorphisms.

Remark⋆ 3.Q. Let $\mathcal{C}$ be a category that is preadditive, or that admits a zero object.

i) It follows from Lemma 3.27 that images are monomorphisms and coimages are epimorphisms.

ii) If $f\colon X \to Y$ is a morphism in $\mathcal{C}$ and $h\colon W \to X$ is an epimorphism in $\mathcal{C}$ then $\operatorname{im}(fh) = \operatorname{im}(f)$ (in the sense of Notation⋆ 3.L). Indeed, it follows from part iv) of Lemma 3.27 that $\operatorname{coker}(f) = \operatorname{coker}(fh)$, i.e. that for the morphisms f and fh the same cokernel can be choosen (and that a cokernel exists for one of them if and only if a cokernel exists for the other one). We can then also choose the kernels of $\operatorname{coker}(f)$ and $\operatorname{coker}(fh)$ to be the same.
   Similarly, if $g\colon Y \to Z$ is a monomorphism then $\operatorname{coim}(f) = \operatorname{coim}(gf)$.

Lemma 3.30 (Canonical factorization)**.** Let $\mathcal{C}$ be a category that is preadditive, or that has a zero object, and that has kernels and cokernels. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$.

i) There exists a unique morphism $f'\colon X \to \operatorname{im}(f)$ in $\mathcal{C}$ that makes the following triangle commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "f'"'] & Y \\ & \operatorname{im}(f) \arrow[u] \end{tikzcd}$$

ii) There exists a unique morphism $\bar{f}\colon \operatorname{coim}(f) \to Y$ in $\mathcal{C}$ that makes the following triangle commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ \operatorname{coim}(f) \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

iii) There exists a unique morphism $\tilde{f}\colon \operatorname{coim}(f) \to \operatorname{im}(f)$ in $\mathcal{C}$ that makes the following square commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ \operatorname{coim}(f) \arrow[r, dashed, "\tilde{f}"] & \operatorname{im}(f) \arrow[u] \end{tikzcd} \tag{3.4}$$

iv) The morphism $\tilde{f}$ is compatible with the morphisms f' and $\bar{f}$ in the sense that the diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, "f'"] & Y \\ \operatorname{coim}(f) \arrow[r, "\tilde{f}"] & \operatorname{im}(f) \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ \operatorname{coim}(f) \arrow[r, "\tilde{f}"] \arrow[ur, "\bar{f}"] & \operatorname{im}(f) \arrow[u] \end{tikzcd} \tag{3.5}$$

commute.

Proof. We denote the various kernels and cokernels by

$$k\colon \ker(f) \to X\,, \quad c\colon Y \to \operatorname{coker}(f)\,, \quad i\colon \operatorname{im}(f) \to Y\,, \quad p\colon X \to \operatorname{coim}(f)\,.$$

i) The morphism $i\colon \operatorname{im}(f) \to Y$ is a kernel of the cokernel $c\colon Y \to \operatorname{coker}(f)$, hence it follows from $c \circ f = 0$ that there exist a unique morphism $f'\colon X \to \operatorname{im}(f)$ that makes the following diagram commute:

$$\begin{tikzcd}
 & \operatorname{coker}(f) \\
X \arrow[ur, "0"] \arrow[r, "f"] \arrow[dr, dashed, "f'"'] & Y \arrow[u, "c"'] \\
 & \operatorname{im}(f) \arrow[u, "i"']
\end{tikzcd}$$

ii) This can be shown dually to part i).

iii) We construct $\tilde{f}$ by using the already constructed morphism $f'\colon X \to \operatorname{im}(f)$: The morphism $p\colon X \to \operatorname{coim}(f)$ is a cokernel of the the kernel $k\colon \ker(f) \to X$. It holds that

$$i \circ f' \circ k = f \circ k = 0 = i \circ 0\,,$$

and hence $f' \circ k = 0$ because i is a monomorphism. It follows from the universal property of the cokernel $p\colon X \to \operatorname{coim}(f)$ that there exist a unique morphism $\tilde{f}\colon \operatorname{coim}(f) \to \operatorname{im}(f)$ that makes the following diagram commute:

$$\begin{tikzcd}
\ker(f) \arrow[d, "k"'] \arrow[dr, bend left, "0"] \arrow[ddr, dashed, "0"] & \\
X \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, "f'"] & Y \\
\operatorname{coim}(f) \arrow[r, dashed, "\tilde{f}"'] & \operatorname{im}(f) \arrow[u, "i"']
\end{tikzcd}$$

This shows the existence of $\tilde{f}$. Suppose that $\tilde{\tilde{f}}\colon \operatorname{coim}(f) \to \operatorname{im}(f)$ is another morphism that makes the diagram (3.4) commute. Then

$$i \circ \tilde{\tilde{f}} \circ p = f = i \circ \tilde{f} \circ p\,.$$

It follows from i being a monomorphism that

$$\tilde{\tilde{f}} \circ p = \tilde{f} \circ p\,,$$

and it then further follows from p being a epimorphism that

$$\tilde{\tilde{f}} = \tilde{f}\,.$$

This shows the desired uniqueness of the morphism $\tilde{f}$.

iv) It follows from the above construction of $\tilde{f}$ that of the two diagrams in (3.5) the left one commutes. We could have dually constructed $\tilde{f}$ by using the morphism $\bar{f}$ instead of f', which would then give us that the right diagram commutes. We can alternatively check the commutativity of the right hand diagram by hand: It holds that

$$i \circ \tilde{f} \circ p = f = \bar{f} \circ p$$

and hence $i \circ \tilde{f} = \bar{f}$ because p is an epimorphism. □

Example⋆ 3.R. Let A be a $\mathbf{k}$-algebra and consider the module category A-**Mod**, which is additive. Let $f\colon M \to N$ be a homomorphism of left A-modules, i.e. a morphism in A-**Mod**. Then the canonical morphism $f'\colon \operatorname{im}(f) \to N$ is the inclusion $n \mapsto n$, the canonical morphism $\tilde{f}\colon M \to \operatorname{coker}(f) = M/\ker(f)$ is the canonical projection $m \mapsto [m]$, and the canonical morphism $\bar{f}\colon M/\ker(f) \to \operatorname{im}(f)$ is the isomorphism $[m] \mapsto f(m)$.

Remark⋆ 3.S (Functoriality of the (co)image)**.** Let $\mathcal{C}$ be a category that is preadditive, or that has a zero object, and that has kernels and cokernels. It follows from the functoriality of the (co)kernel (as discussed in Remark⋆ 3.O) that images and coimages are also functorial: Let $f\colon X \to Y$ and $f'\colon X' \to Y'$ be two morphisms in $\mathcal{C}$.

i) Let $i\colon \operatorname{im}(f) \to Y$ and $i'\colon \operatorname{im}(f') \to Y'$ be images of f and f'. If f and f' fit into a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "h"'] & Y \arrow[d, dashed, "h''"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

then there exists a unique morphism $h'\colon \operatorname{im}(f) \to \operatorname{im}(f')$ that makes the following square commute:

$$\begin{tikzcd} \operatorname{im}(f) \arrow[r, "i"] \arrow[d, dashed, "h'"'] & Y \arrow[d, "h''"] \\ \operatorname{im}(f') \arrow[r, "i'"] & Y' \end{tikzcd}$$

To show the existence of h' let $c\colon Y \to \operatorname{coker}(f)$ and $c'\colon Y' \to \operatorname{coker}(f')$ be cokernels of f, resp. f', so that i is a kernel of c and i' is a kernel of c'. Then there exists by the functoriality of the cokernel a unique morphism $\bar{h}\colon \operatorname{coker}(f) \to \operatorname{coker}(f')$ that makes the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[r, "c"] \arrow[d, "h''"] & \operatorname{coker}(f) \arrow[d, dashed, "\bar{h}"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "c'"] & \operatorname{coker}(f') \end{tikzcd}$$

commute. By applying the functoriality of the kernel to the right square, it follows that there exists a unique morphism $h' \colon \operatorname{im}(f) \to \operatorname{im}(f')$ that makes the diagram

$$\begin{tikzcd}
\operatorname{im}(f) \arrow[r, "i"] \arrow[d, dashed, "h'"'] & Y \arrow[r, "c"] \arrow[d, "h''"] & \operatorname{coker}(f) \arrow[d, "\bar{h}"] \\
\operatorname{im}(f') \arrow[r, "i'"] & Y' \arrow[r, "c'"] & \operatorname{coker}(f')
\end{tikzcd}$$

commute. This shows the existence of h'. The uniqueness of h' follows from the uniqueness of h' in the above construction; it also follows from h' being uniquely determined by the composition $i'h' = h''i$ because i' is a monomorphism.

This induced morphism between images is functorial and additive in the following sense:

- The morphism $\operatorname{im}(f) \to \operatorname{im}(f)$ that results from the commutative square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "\mathrm{id}_X"'] & Y \arrow[d, dashed, "\mathrm{id}_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

  is the identity $\mathrm{id}_{\operatorname{im}(f)}$.
- Suppose that we are given another morphism $f'' \colon X'' \to Y''$, and that the morphisms f, f' and f'' fits into a commutative diagram of the following form:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "h"'] & Y \arrow[d, dashed, "h''"] \\
X' \arrow[r, "f'"] \arrow[d, dashed, "g"'] & Y' \arrow[d, dashed, "g''"] \\
X'' \arrow[r, "f''"] & Y''
\end{tikzcd}$$

  Let $i'' \colon \operatorname{im}(f'') \to Y''$ be an image of f'' and let $g' \colon \operatorname{im}(f') \to \operatorname{im}(f'')$ be the morphism induced by lower commutative square of the above diagram. Then the composition $g' \circ h' \colon \operatorname{im}(f) \to \operatorname{im}(f'')$ is the morphism that is induced by the outer commutative square, i.e. by the following commutative square:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "g \circ h"'] & Y \arrow[d, dashed, "g'' \circ h''"] \\
X'' \arrow[r, "f''"] & Y''
\end{tikzcd}$$

- Suppose that the category $\mathcal{C}$ is preadditive and that $g \colon X \to X'$ and $g'' \colon Y \to Y'$ is another pair of morphisms in $\mathcal{C}$ that make the square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "g"'] & Y \arrow[d, dashed, "g''"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

commute, and let $g'\colon \operatorname{im}(f) \to \operatorname{im}(f')$ be the resulting induced morphism between images. Then the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "h+g"'] & Y \arrow[d, dashed, "h''+g''"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

again commutes, and resulting induced morphism $\operatorname{im}(f) \to \operatorname{im}(f')$ is precisely the sum $h' + g'$.

This claimed functoriality and additivity of the induced morphism follow via the above explicit construction of the induced morphism from functoriality of the (co)kernel, as explained in Remark⋆ 3.O.

ii) Dually, if $p\colon X \to \operatorname{coim}(f)$ and $p'\colon X' \to \operatorname{coim}(f')$ are coimages of f and f' then there exists a unique morphism $\bar{h}\colon \operatorname{coim}(f) \to \operatorname{coim}(f')$ that makes the following square commute:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "h"'] & \operatorname{coim}(f) \arrow[d, dashed, "\bar{h}"] \\ X' \arrow[r, "p'"] & \operatorname{coim}(f') \end{tikzcd}$$

This induced morphism between coimages is functorial and additive in the following sense:

- The morphism $\operatorname{coim}(f) \to \operatorname{coim}(f)$ that results from the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\mathrm{id}_X"'] & Y \arrow[d, dashed, "\mathrm{id}_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is the identity $\mathrm{id}_{\operatorname{coim}(f)}$.

- Suppose that we are given another morphism $f''\colon X'' \to Y''$, and that the morphisms f, f' and f'' fits into a commutative diagram of the following form:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "h"'] & Y \arrow[d, dashed, "h''"] \\ X' \arrow[r, "f'"] \arrow[d, dashed, "g"'] & Y' \arrow[d, dashed, "g''"] \\ X'' \arrow[r, "f''"] & Y'' \end{tikzcd}$$

Let $p''\colon X'' \to \operatorname{coim}(f'')$ be a coimage of f'' and let $g'\colon \operatorname{coim}(f') \to \operatorname{coim}(f'')$ be the morphism induced by lower commutative square of the above diagram. Then the composition $g' \circ h'\colon \operatorname{coim}(f) \to \operatorname{coim}(f'')$ is the morphism that is induced by the outer commutative square, i.e. by the following commutative square:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "g\circ h"'] & Y \arrow[d, dashed, "g''\circ h''"] \\
X'' \arrow[r, "f''"] & Y''
\end{tikzcd}$$

- Suppose that the category $\mathcal{C}$ is preadditive and that $g\colon X \to X'$ and $g''\colon Y \to Y'$ is another pair of morphisms in $\mathcal{C}$ that make the square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "g"'] & Y \arrow[d, dashed, "g''"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

commute, and let $g'\colon \operatorname{coim}(f) \to \operatorname{coim}(f')$ be the resulting induced morphism. Then the square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "h+g"'] & Y \arrow[d, dashed, "h''+g''"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

again commutes, and resulting induced morphism $\operatorname{coim}(f) \to \operatorname{coim}(f')$ is precisely the sum $h' + g'$.

iii) These induced morphisms between images and coimages are compatible with the canonical factorization from Lemma 3.30 in the sense that for every commutative sqare

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "h"'] & Y \arrow[d, dashed, "h''"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rrr, "f"] \arrow[dr, "p"'] \arrow[dd, dashed, "h"'] & & & Y \arrow[dd, dashed, "h''"] \\
& \operatorname{coim}(f) \arrow[r, "\tilde{f}"] \arrow[dd, dashed, "\bar{h}"'] & \operatorname{im}(f) \arrow[ur, "i"'] \arrow[dd, dashed, "h'"] & \\
X' \arrow[rrr, "f'"] \arrow[dr, "p'"'] & & & Y' \\
& \operatorname{coim}(f') \arrow[r, "\tilde{f}'"] & \operatorname{im}(f') \arrow[ur, "i'"'] &
\end{tikzcd}$$

Indeed, the trapezoid on the top commutes by definition of $\tilde{f}$, the trapezoid on the bottom commutes by definition of $\tilde{f}'$, the square on the left commutes by definition of $\bar{h}$, the square on the right commutes by definition of h', and the rectangle in the back commutes by assumption. It follows that the front square commutes. Indeed, it follows that

$$i'h'\tilde{f}p = h''i\tilde{f}p = h''f = f'h = i'\tilde{f}'p'h = i'\tilde{f}'\bar{h}p\,,$$

and hence $h'\tilde{f} = \tilde{f}'\bar{h}$ because p is an epimorphism and i' is a monomorphism.

It follows that also the following diagrams commute, where the unlabeled arrows are the ones from the canonical factorization lemma.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] \arrow[dd, "h"'] & & Y \arrow[dd, "h''"] \\
& \mathrm{coim}(f) \arrow[ur] \arrow[dd, "\bar{h}", near start] & \\
X' \arrow[rr, "f'", near end, crossing over] \arrow[dr, "p'"'] & & Y' \\
& \mathrm{coim}(f') \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] \arrow[dd, "h"'] & & Y \arrow[dd, "h''"] \\
& \mathrm{im}(f) \arrow[ur, "i"'] \arrow[dd, "h'", near start] & \\
X' \arrow[rr, "f'", near start, crossing over] \arrow[dr] & & Y' \\
& \mathrm{im}(f') \arrow[ur, "i'"'] &
\end{tikzcd}$$

Example⋆ 3.T. Let A be a $\mathbf{k}$-algebra and let $f\colon X \to Y$ and $f'\colon X' \to Y'$ be two homomorphisms of A-modules that fit into a commutative square:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "h''"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

Then $h''(\mathrm{im}(f)) \subseteq \mathrm{im}(f')$, and hence the homomorphism h'' restricts to a homomorphism $h'\colon \mathrm{im}(f) \to \mathrm{im}(f')$. It similarly holds that the homomorphism h induces a well-defined homomorphism $\bar{h}\colon X/\ker(f) \to X'/\ker(f')$ that is given on residue classes by by $\bar{h}([x]) = [h(x)]$ for every $[x] \in X/\ker(f)$.

Abelian Categories

Definition 3.31. An *abelian category* is an additive category $\mathcal{A}$ that has kernels and cokernels and in which for every morphism $f\colon X \to Y$ the induced morphism $\tilde{f}\colon \mathrm{coim}(f) \to \mathrm{im}(f)$ from the canonical factorization lemma is an isomorphism.

Remark⋆ 3.U. Let $\mathcal{A}$ be a category. We have previously seen that 'being additive' is a property of $\mathcal{A}$. It follows that 'being abelian' is also a property of $\mathcal{A}$.

Remark 3.32.

i) A category $\mathcal{A}$ is abelian if and only if its dual category $\mathcal{A}^{\mathrm{op}}$ is abelian.

ii) If $\mathcal{A}$ is an abelian category then we will for $X, Y \in \mathrm{Ob}(\mathcal{A})$ often write $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ instead of $\mathcal{A}(X,Y)$.

iii) If $F: \mathcal{A} \to \mathcal{B}$ is an equivalence of categories and $\mathcal{A}$ is abelian, then $\mathcal{B}$ is also abelian.[5]

Example 3.33.

i) The category **Ab** of abelian groups is an abelian category.

ii) If more generally A is a **k**-algebra then the module categories A-**Mod** and **Mod**-A are abelian.

iii) If G is a group then the category $\mathbf{Rep}_k(G)$ is equivalent to the the module category $\mathbf{k}[G]$-**Mod** and hence abelian.

iv) If A is a left noetherian ring then the additive category $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ of finitely generated left A-modules is again abelian:
   Let $f: M \to N$ be a homomorphism of A-modules where both M and N are finitely generated. Then the submodule $\ker(f) \subseteq M$ is again finitely generated because M is noetherian (since finitely generated modules over noetherian rings are noetherian themselves), and the quotient module $N/f(M)$ is also again finitely generated. We can therefore utilize for $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ the usual kernels and cokernels.
   The analogous statement and argumentation also holds for $\mathbf{Mod}^{\mathrm{fg}}\text{-}A$, the category of finitely generated right A-modules.

v) Let A be a finite-dimensional **k**-algebra over a field **k**. Then the category A-**mod** of finite-dimensional left A-modules coincides with the category $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ of finitely generated left A-modules, and is hence abelian. The analogous observation holds for **mod**-A, the category of finite-dimensional right A-modules.

vi) If $\mathcal{C}$ is a category and $\mathcal{A}$ is an abelian category then the functor category $\mathbf{Fun}(\mathcal{C}, \mathcal{A})$ is again abelian.

Example⋆ 3.V. If $\mathcal{A}$ is any abelian category then the morphism category $\mathbf{Mor}(\mathcal{A})$ is again abelian since it is a functor category, as seen in Example⋆ 2.J.

Remark⋆ 3.W. If $\mathcal{A}$ is an abelian category and $f: X \to Y$ is a morphism in $\mathcal{A}$ that is both a monomorphism and an epimorphism, then f is already an isomorphism. This will be shown in Proposition 3.38.

Example 3.34 (Non-example)**.** Consider the category $F^*\mathbf{Ab}$ of filtered abelian groups: The objects of $F^*\mathbf{Ab}$ are pairs (A, F^*) consisting of an abelian group A and an increasing $\mathbb{Z}$-filtration F^* on A, i.e. an increasing sequence

$$\cdots \subseteq F^{i-1}(A) \subseteq F^i(A) \subseteq F^{i+1}(A) \subseteq \cdots$$

of subgroups $F^i(A) \subseteq A$ with $i \in \mathbb{Z}$. A morphism $f: (A, F^*) \to (B, G^*)$ in $F^*\mathbf{Ab}$ is a group homomorphism $f: A \to B$ with $f(F^i(A)) \subseteq G^i(B)$ for every $i \in \mathbb{Z}$. The

[5] In the lecture it was instead stated that one can 'use F to make $\mathcal{B}$ into an abelian category'. It was then sketched how one can use F to transfer the group structure on the morphism sets of $\mathcal{A}$ to a group structure of the morphism sets of $\mathcal{B}$; one can then check that F respects biproducts, kernels and cokernels (this was not proven in the lecture). The author finds this presentation somewhat misleading, as 'being abelian' is really a property of a category, and not an additional structure. Hence we don't 'make' a category abelian, it *is* abelian.

category $F^*\mathbf{Ab}$ is additive and has kernels and cokernels: If $f\colon (A, F^*) \to (B, G^*)$ is a morphism in $F^*\mathbf{Ab}$ then a kernel of f is given by

$$\ker_{F^*\mathbf{Ab}}(f) = (\ker_{\mathbf{Ab}}(f), (F^i(A) \cap \ker_{\mathbf{Ab}}(f))_{i\in\mathbb{Z}})\,,$$

and a cokernel of f is given by

$$\operatorname{coker}_{F^*\mathbf{Ab}}(f) = (\operatorname{coker}_{\mathbf{Ab}}(f), ((F^i(B) + f(A))/f(A))_{i\in\mathbb{Z}})\,.$$

Let now A be an abelian group and suppose that F^* and $\tilde{F}^*$ are two filtrations on A such that $F_i(A) \subseteq \tilde{F}_i(A)$ for all $i \in \mathbb{Z}$, but $F_j(A) \subsetneq \tilde{F}_j(A)$ for some $j \in \mathbb{Z}$. (One can for example choose any nontrivial abelian group A and define such filtrations via

$$F_i(A) = \begin{cases} 0 & \text{if } i < 1\,, \\ A & \text{if } i \geq 1\,, \end{cases} \qquad \text{and} \qquad \tilde{F}_i(A) = \begin{cases} 0 & \text{if } i < 0\,, \\ A & \text{if } i \geq 0\,. \end{cases}$$

Then both filtrations agree except for $F_0(A) = 0 \subsetneq A = F_1(A)$.) We can now regard the identity id_A as a morphism $f\colon (A, F^*) \to (A, \tilde{F}^*)$. It then follows from $\ker_{\mathbf{Ab}}(f) = 0$ and $\operatorname{coker}_{\mathbf{Ab}}(f) = 0$ that $\ker_{F^*\mathbf{Ab}}(f) = 0$ and $\operatorname{coker}_{F^*\mathbf{Ab}}(f) = 0$ by the above explicit construction of kernels and cokernels in $F^*\mathbf{Ab}$. Hence f is both a monomorphism and an epimorphism. But f is not an isomorphism as the filtrations of F^* and $\tilde{F}^*$ differ, so $F^*\mathbf{Ab}$ cannot be abelian.

End of lecture 11

Example 3.35. Let X be a topological space. We will sketch how to show that the category of sheaves of abelian groups over X is abelian.

i) We start with the notion of a presheaf on X:

Definition. A *presheaf* $\mathscr{F}$ (of abelian groups over X) consists of

- an abelian group $\mathscr{F}(U)$ for every open subset $U \subseteq X$, and
- a group homomorphism $\rho_{V,U}\colon \mathscr{F}(V) \to \mathscr{F}(U)$ for all open subsets $U \subseteq V \subseteq X$,

such that

(P1) $\rho_{U,U} = \mathrm{id}_{\mathscr{F}(U)}$ for every open subset $U \subseteq X$, and
(P2) $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$ for all open subsets $U \subseteq V \subseteq W \subseteq X$.

For an open subset $U \subseteq X$ one may think about the abelian group $\mathscr{F}(U)$ as consisting of certain functions on U that can be added together. One can consequently think about the homomorphism $\rho_{V,U}\colon \mathscr{F}(V) \to \mathscr{F}(U)$ associated to open subsets $U \subseteq V \subseteq X$ as restricting the functions on V to the functions on U.

For an open subset $U \subseteq X$, the elements of $\mathscr{F}(U)$ are called *sections* of $\mathscr{F}$ on U. For open subsets $U \subseteq V \subseteq X$, the homomorphism $\rho_{V,U}\colon \mathscr{F}(V) \to \mathscr{F}(U)$ is the *restriction homomorphism* from V to U, and for a section $s \in \mathscr{F}(V)$ one calls $\rho_{V,U}(s) \in \mathscr{F}(U)$ the *restriction* of s to U. This restriction is also denoted by $s|_U$ instead of $\rho_{V,U}(s)$. We then have that

$$s|_U = s$$

for every open subset $U \subseteq X$ and every section $s \in \mathscr{F}(U)$, and

$$(s|_V)|_U = s|_U$$

for all open subsets $U \subseteq V \subseteq W \subseteq X$ and every section $s \in \mathscr{F}(W)$. It also holds that

$$(s+t)|_U = s|_U + t|_U$$

for all open subsets $U \subseteq V \subseteq X$ and all sections $s, t \in \mathscr{F}(V)$ because the restriction homomorphism $\rho_{V,U}$ is a group homomorphism.

Example.

a) For every open subset $U \subseteq X$ let

$$\mathscr{C}_X(U) \coloneqq \{f \colon U \to \mathbb{R} \mid f \text{ is continuous}\},$$

and for all open subsets $U \subseteq V \subseteq X$ let

$$\rho_{V,U} \colon \mathscr{C}_X(V) \to \mathscr{C}_X(U), \quad f \mapsto f|_U$$

be the (literal) restriction homomorphism. This defines a presheaf $\mathscr{C}_X$ on X, the *presheaf of continuous functions*.

b) For every abelian group A we can consider the *constant A-valued presheaf* on X, which is denoted by $\widetilde{\mathscr{C}}_{X,A}$ and given by $\widetilde{\mathscr{C}}_{X,A}(U) = A$ for every open subset $U \subseteq X$, and $\rho_{V,U} = \mathrm{id}_A$ for all open subsets $U \subseteq V \subseteq X$.

Let $\mathscr{F}, \mathscr{G}$ be two presheaves on X. A *homomorphism* of presheaves $f \colon \mathscr{F} \to \mathscr{G}$ is a tuple $(f_U)_{U \subseteq X}$ of group homomorphisms $f_U \colon \mathscr{F}(U) \to \mathscr{G}(U)$, where $U \subseteq X$ ranges through the open subsets of X, such that for all open subsets $U \subseteq V \subseteq X$ the square

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r, "f_V"] \arrow[d, "\rho^{\mathscr{F}}_{V,U}"'] & \mathscr{G}(V) \arrow[d, "\rho^{\mathscr{G}}_{V,U}"] \\
\mathscr{F}(U) \arrow[r, "f_U"] & \mathscr{G}(U)
\end{tikzcd}$$

commutes. Let $\mathcal{P} \coloneqq \mathbf{PSh}_X(\mathbf{Ab})$ be the category of presheaves on X.

ii) We can interpret the presheaf category $\mathcal{P}$ as a functor category: Let $\mathcal{X}$ be the category defined by objects

$$\mathrm{Ob}(\mathcal{X}) \coloneqq \{U \mid U \subseteq X \text{ is open}\}$$

and morphism sets

$$\mathcal{X}(U,V) \coloneqq \begin{cases} \{i_{U,V}\} & \text{if } U \subseteq V, \\ \emptyset & \text{otherwise}, \end{cases}$$

where $i_{U,V} \colon U \to V$ is the inclusion. The composition of morphisms in $\mathcal{X}$ is defined in the only possible way. The presheaf category $\mathcal{P}$ is then equivalent to the functor category $\mathbf{Fun}(\mathcal{X}^{\mathrm{op}}, \mathbf{Ab})$. We see from this alternative description of the presheaf category $\mathcal{P}$ that it is abelian.

Remark⋆ 3.X. A *preorder* on a set P is a relation $\leq$ that is reflexive and transitive, i.e. it holds that $x \leq x$ for every $x \in P$, and it holds for all $x, y, z \in P$ with $x \leq y$ and $y \leq z$ that also $x \leq z$. (But in contrast to a partial order, a preorder does not have to be antisymmetric. So there may exist $x, y \in P$ with both $x \leq y$ and $y \leq x$ but $x \neq y$.) A *preordered set* is a pair $(P, \leq)$ consisting of a set P and a preorder $\leq$ on P.

If $(P, \leq)$ is a preordered set then one can define a category $\mathcal{P}$ whose objects are given by the elements of P, and in which there exists for every two elements $x, y \in P$ a morphism $x \to y$ in $\mathcal{P}$ if and only if $x \leq y$, and this morphism is then unique. More formally speaking, we have that

$$\mathrm{Ob}(\mathcal{P}) = P\,,$$

and the morphisms sets of $\mathcal{P}$ are for any two objects $x, y \in P$ given by

$$\mathcal{P}(x, y) = \begin{cases} \{*\} & \text{if } x = y\,, \\ \emptyset & \text{if } x \neq y\,. \end{cases}$$

The composition of morphisms in defined in the only possible way. In the resulting category $\mathcal{P}$ there exists between any two objects at most one morphism. Such categories are called *thin*.

If on the other hand $\mathcal{T}$ is any thin category whose class of objects $T \coloneqq \mathrm{Ob}(\mathcal{T})$ is a set (and not a proper class), then one can define a preorder $\leq$ on T via

$$s \leq t \iff \mathcal{T}(s, t) \neq \emptyset$$

for all $s, t \in T$. This then results in a preordered set $(T, \leq)$.

These two constructions are mutually inverse, and show that preordered sets $(P, \leq)$ are 'the same' as thin categories $\mathcal{T}$ whose class of objects form a set.

In the above example, the preordered set (that is already an ordered set) is given by $P = \{U \subseteq X \mid U \text{ is open}\}$ together with the inclusion $\subseteq$ as a preorder. The category $\mathcal{X}$ then results from the preordered set $(P, \subseteq)$ via the above construction.

iii) We are now ready to introduce sheaves:

Definition. A presheaf $\mathscr{F}$ on X is a *sheaf* if for every open subset $U \subseteq X$ and every open cover $\{U_i\}_{i \in I}$ of U the following two conditions are satisfied:

(S1) If $s \in \mathscr{F}(U)$ is a section with $s|_{U_i} = 0$ for every $i \in I$ then already $s = 0$.

(S2) Suppose that $s_i \in \mathscr{F}(U_i)$ is a section for every $i \in I$, so that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for all $i, j \in I$. Then there exists a section $s \in \mathscr{F}(U)$ with $s|_{U_i} = s_i$ for every $i \in I$.

Condition (S1) is the *separation axiom* and condition (S2) is the *glueing axiom*.

We denote by $\mathcal{S} = \mathbf{Sh}_X(\mathbf{Ab})$ the category of sheaves on X, which is the full subcategory of the presheaf category $\mathcal{P} = \mathbf{PSh}_X(\mathbf{Ab})$ whose objects are the sheaves on X.

Note that if $\mathscr{F}$ is any sheaf on X then $\mathscr{F}(\emptyset) = 0$: We may choose for the open subset $U = \emptyset$ the empty covering $U = \bigcup_{i \in \emptyset} U_i$. It then follows for any two sections $s, t \in \mathscr{F}(\emptyset)$ from the separation axiom that $s = t$, which shows that the abelian group $\mathscr{F}(\emptyset)$ consists of only a single element.

Example.

a) The presheaf of continuous functions $\mathscr{C}_X$ is already a sheaf.

b) If A is a nonzero abelian group then the constant presheaf $\widetilde{\mathscr{C}}_{X,A}$ is not a sheaf, because $\widetilde{\mathscr{C}}_{X,A}(\emptyset) = A \neq 0$.

iv) The sheaf category $\mathcal{S}$ is additive:

Preadditive: The sheaf category $\mathcal{S}$ is a full subcategory of the presheaf category $\mathcal{P}$, which is a preadditive category. It therefore inherits the structure of a preadditive category from $\mathcal{P}$. Note that the inclusion functor $I \colon \mathcal{S} \to \mathcal{P}$ is additive.

Zero object: The zero presheaf 0 is already a sheaf, and hence contained in $\mathcal{S}$. It is also a zero object in $\mathcal{S}$ because $\mathcal{S}$ is a full subcategory of $\mathcal{P}$.

Biproducts: The biproduct $\mathscr{F} \oplus \mathscr{G}$ of two presheaves $\mathscr{F}$ and $\mathscr{G}$ in the presheaf category $\mathcal{P}$ is given by

$$(\mathscr{F} \oplus \mathscr{G})(U) = \mathscr{F}(U) \oplus \mathscr{G}(U)$$

for every open subset $U \subseteq X$, together with the restriction homomorphisms

$$\rho^{\mathscr{F} \oplus \mathscr{G}}_{V,U} = \rho^{\mathscr{F}}_{V,U} \oplus \rho^{\mathscr{G}}_{V,U}$$

for all open subsets $U \subseteq V \subseteq X$, i.e.

$$(s,t)|_U = (s|_U, t|_U)$$

for every section $(s,t) \in \mathscr{F}(U) \oplus \mathscr{G}(U)$.

If both $\mathscr{F}$ an $\mathscr{G}$ are sheaves then their biproduct $\mathscr{F} \oplus \mathscr{G}$ is again a sheaf: Let $U \subseteq X$ be an open subset and let $\{U_i\}_{i \in I}$ be an open cover of U.

(S1) Let $s \in \mathscr{F}(U) \oplus \mathscr{G}(U)$ be a section with $s|_{U_i} = 0$ for every $i \in I$. We may write $s = (t,u)$ for some sections $t \in \mathscr{F}(U)$ and $u \in \mathscr{G}(U)$, and it holds for every $i \in I$ that

$$0 = s|_{U_i} = (t,u)|_{U_i} = (t|_{U_i}, u|_{U_i})\,.$$

It therefore holds for every $i \in I$ that $t|_{U_i} = 0$ and $u|_{U_i} = 0$. It follows that $t = 0$ and $u = 0$ because both $\mathscr{F}$ and $\mathscr{G}$ are sheaves (and hence satisify the separation axiom). This shows that $s = 0$, and hence that $\mathscr{F} \oplus \mathscr{G}$ satisfies the separation axiom.

(S2) For every $i \in I$ let $s_i \in U_i$ be a section, such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for all $i, j \in I$. Then every s_i can be written as $s_i = (t_i, u_i)$ for some sections $t_i \in \mathscr{F}(U_i)$ and $u_i \in \mathscr{G}(U_i)$, and it holds for all $i, j \in I$ that

$$\begin{aligned}(t_i|_{U_i \cap U_j}, u_i|_{U_i \cap U_j}) &= (t_i, u_i)|_{U_i \cap U_j} = s_i|_{U_i \cap U_j} \\ &= s_j|_{U_i \cap U_j} = (t_j, u_j)|_{U_i \cap U_j} = (t_j|_{U_i \cap U_j}, u_j|_{U_i \cap U_j}) \,.\end{aligned}$$

It therefore holds for all $i, j \in I$ that

$$t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j} \quad \text{and} \quad u_i|_{U_i \cap U_j} = u_j|_{U_i \cap U_j} \,.$$

It follows that there exist sections $t \in \mathscr{F}(U)$ and $u \in \mathscr{G}(U)$ with $t|_{U_i} = t_i$ and $u|_{U_i} = u_i$ for every $i \in I$ because both $\mathscr{F}$ and $\mathscr{G}$ are sheaves (and hence satisfiy the glueing axiom). It follows for the section $s \coloneqq (t, u) \in (\mathscr{F} \oplus \mathscr{G})(U)$ that

$$s|_{U_i} = (t, u)|_{U_i} = (t|_{U_i}, u|_{U_i}) = (t_i, u_i) = s_i$$

for every $i \in I$. This shows that $\mathscr{F} \oplus \mathscr{G}$ satisfies the glueing axiom.

This shows together that the biproduct $\mathscr{F} \oplus \mathscr{G}$ is contained in $\mathcal{S}$. The sheaf $\mathscr{F} \oplus \mathscr{G}$ is also a biproduct of $\mathscr{F}$ and $\mathscr{G}$ in $\mathcal{S}$ because $\mathcal{S}$ is a full subcategory of $\mathcal{P}$.

This shows that $\mathcal{S}$ admits binary biproducts; it follows inductively that $\mathcal{S}$ admits biproducts $\mathscr{F}_1 \oplus \cdots \oplus \mathscr{F}_n$ for any collections of sheaves $\mathscr{F}_1, \dots, \mathscr{F}_n$ with $n \geq 1$. For $n = 0$ this is also true because $\mathcal{S}$ contains a zero object.

This shows alltogether that the sheaf category $\mathcal{S}$ is indeed additive.

v) The sheaf category $\mathcal{S}$ admits kernels: To be more precise, let $\mathscr{F}$ and $\mathscr{G}$ be two sheaves and let $f \colon \mathscr{F} \to \mathscr{G}$ be a homomorphism of sheaves. We may forget that $\mathscr{F}$ and $\mathscr{G}$ are sheaves and regard them as just presheaves; then f is a homomorphisms of presheaves, and hence a homomorphism in the abelian category $\mathcal{P}$. We can therefore consider its kernel $\ker_{\mathcal{P}}(f)$ in the category $\mathcal{P}$.

This kernel $\ker_{\mathcal{P}}(f)$ of f in the presheaf category $\mathcal{P}$ is already a sheaf: The kernel $\ker_{\mathcal{P}}(f)$ is given by

$$(\ker_{\mathcal{P}}(f))(U) = \ker(f_U)$$

for every open subset $U \subseteq X$, together with the restriction homomorphisms

$$\rho^{\ker_{\mathcal{P}}(f)}_{V,U} \colon \ker(f_V) \to \ker(f_U)$$

for all open subsets $U \subseteq V \subseteq X$ that are given by restriction of the restriction homomorphisems $\rho^{\mathscr{F}}_{V,U} \colon \mathscr{F}(V) \to \mathscr{F}(U)$. Let $U \subseteq X$ be open and let $\{U_i\}_{i \in I}$ be an open cover of U.

(S1) Let $s \in (\ker_{\mathcal{P}}(f))(U) = \ker(f_U)$ be a section with $s|_{U_i} = 0$ for every $i \in I$. This section s is an element of $\mathscr{F}(U)$ (because $\ker(f_U)$ is a subgroup of $\mathscr{F}(U)$), and the relation $s|_{U_i} = 0$ also holds in the sheaf $\mathscr{F}$ (i.e. it follows from $\rho^{\ker_{\mathcal{P}}(f)}_{U,U_i}(s) = 0$ that also $\rho^{\mathscr{F}}_{U,U_i}(s) = 0$) because the square

$$\begin{tikzcd}
\ker(f_U) \arrow[r, hook] \arrow[d, "\rho_{U,U_i}"'] & \mathscr{F}(U) \arrow[d, "\rho_{U,U_i}"] \\
\ker(f_{U_i}) \arrow[r, hook] & \mathscr{F}(U_i)
\end{tikzcd}$$

commutes. It follows that $s = 0$ in $\mathscr{F}(U)$, and hence also in $\ker(f_U)$, because $\mathscr{F}$ is a sheaf and hence satisfies the separation axiom. This shows that the presheaf $\ker_{\mathcal{P}}(f)$ satisfies the separation axiom.

(S2) For every $i \in I$ let $s_i \in (\ker_{\mathcal{P}}(f))(U_i) = \ker(f_{U_i})$ be a section such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$. We can then regard every s_i as an element of $\mathscr{F}(U_i)$, and the relation $s_i|_{U_i \cap U_j} = s_i|_{U_i \cap U_j}$ also holds in the sheaf $\mathscr{F}$ because the diagram

$$\begin{tikzcd}
\ker(f_{U_i}) \arrow[dr, "\rho_{U_i,U_i\cap U_j}"] \arrow[dd, hook] & & \ker(f_{U_j}) \arrow[dl, "\rho_{U_j,U_i\cap U_j}"'] \arrow[dd, hook] \\
 & \ker(f_{U_i \cap U_j}) \arrow[dd, hook] & \\
\mathscr{F}(U_i) \arrow[dr, "\rho_{U_i,U_i\cap U_j}"'] & & \mathscr{F}(U_j) \arrow[dl, "\rho_{U_j,U_i\cap U_j}"] \\
 & \mathscr{F}(U_i \cap U_j) &
\end{tikzcd}$$

commutes. It follows that there exists a section $s \in \mathscr{F}(U)$ with $s|_{U_i} = s_i$ for every $i \in I$ because $\mathscr{F}$ is a sheaf and hence satisfies the glueing axiom. We need to show that already $s \in \ker(f_U)$, i.e. that $f_U(s) = 0$. We use that f is a homomorphism of sheaves to calculate

$$f_U(s)|_{U_i} = f_{U_i}(s|_{U_i}) = f_{U_i}(s_i) = 0$$

for every $i \in I$. It follows that $f_U(s) = 0$ because $\mathscr{G}$ is a sheaf and hence satisfies the separation axiom. This shows that $\ker(f)$ satisfies the glueing axiom.

We can similarly consider the cokernel $\operatorname{coker}_{\mathcal{P}}(f)$. This presheaf is given by

$$(\operatorname{coker}_{\mathcal{P}}(f))(U) = \operatorname{coker}(f_U)$$

for every open subset $U \subseteq X$, and the restriction homomorphism

$$\rho^{\operatorname{coker}_{\mathcal{P}}(f)}_{V,U} \colon \mathscr{G}(V) \to \mathscr{G}(U)$$

is for all open subsets $U \subseteq V \subseteq X$ induced by the restriction homomorphism $\rho^{\mathscr{G}}_{V,U} \colon \mathscr{G}(V) \to \mathscr{G}(U)$ in the sense that the following square commutes:

$$\begin{tikzcd}
\mathscr{G}(V) \arrow[r, two heads] \arrow[d, "\rho_{V,U}"'] & \operatorname{coker}^{\mathcal{P}}(f_V) \arrow[d, "\rho_{V,U}"] \\
\mathscr{G}(U) \arrow[r, two heads] & \operatorname{coker}^{\mathcal{P}}(f_U)
\end{tikzcd}$$

In constrast to kernels it does not necessarily hold that the cokernel $\operatorname{coker}_{\mathcal{P}}(f)$ is already a sheaf.

Example. Let $X \coloneqq \mathbb{S}^1$ and let $\mathscr{F} \coloneqq \mathscr{G} \coloneqq \mathscr{C}^\infty$ be the sheaf of smooth real-valued functions on $\mathbb{S}^1$, that is given by

$$\mathscr{C}^\infty(U) = \{f \colon U \to \mathbb{R} \mid f \text{ is smooth}\}$$

for every open subset $U \subseteq \mathbb{S}^1$ and has for all open subsets $U \subseteq V \subseteq \mathbb{S}^1$ as restriction homomorphisms

$$\rho_{V,U} \colon \mathscr{C}^\infty(V) \to \mathscr{C}^\infty(U)\,, \quad f \mapsto f|_U$$

the (literal) restriction maps.

Let us consider the homomorphism $d \colon \mathscr{C}^\infty \to \mathscr{C}^\infty$ that is given by the derivative, i.e. by

$$d_U \colon \mathscr{C}^\infty(U) \to \mathscr{C}^\infty(U)\,, \quad f \mapsto f'$$

for every open subset $U \subseteq \mathbb{S}^1$. If $U \subsetneq \mathbb{S}^1$ is a proper open subset then U corresponds to an open subset of $(0,1)$, hence there exists for every $f \in \mathscr{C}^\infty(U)$ an antiderivative for f on U. The homomorphism $d_U \colon \mathscr{C}^\infty(U) \to \mathscr{C}^\infty(U)$ is hence surjective, and has therefore the cokernel

$$\operatorname{coker}(d_U) = 0\,.$$

Consider on the other hand the open subset $U = S^1$. On this open subset, the constant 1-function $1 \in \mathscr{C}^\infty(\mathbb{S}^1)$ has no antiderivative because it lifts to the constant 1-function $1 \in \mathscr{C}^\infty(\mathbb{R})$, whose antiderivatives $x \mapsto x + c$ with $c \in \mathbb{R}$ are not periodic. The homomorphism $d_{\mathbb{S}^1}$ does therefore have a nonvanishing cokernel

$$\operatorname{coker}(f_{\mathbb{S}^1}) \neq 0\,.$$

(One can actually identify the cokernel $\operatorname{coker}(f_{\mathbb{S}^1})$ with the constant functions on $\mathbb{S}^1$, so that $\operatorname{coker}(f_{\mathbb{S}^1}) \cong \mathbb{R}$.) Let $\mathbb{S}^1 = U_1 \cup U_2$ be an open cover by proper open subsets $U_1, U_2 \subsetneq \mathbb{S}^1$. If $\operatorname{coker}(d)$ were a sheaf then it would follows from $\operatorname{coker}(d_{U_1}) = 0$ and $\operatorname{coker}(d_{U_2}) = 0$ by the separation axiom that also $\operatorname{coker}(d_{\mathbb{S}^1}) = 0$. Hence $\operatorname{coker}(d)$ is not a sheaf.

vi) Let $I \colon \mathcal{S} \to \mathcal{P}$ be the inclusion functor, which is both fully faithful and additive.

Fact 1. The inclusion functor I has a left adjoint $S\colon \mathcal{P} \to \mathcal{S}$. The functor S is again additive and the adjunction is also additive, in the sense that the natural bijections

$$\varphi_{\mathscr{F},\mathscr{G}} \colon \mathcal{S}(S(\mathscr{F}),\mathscr{G}) \to \mathcal{P}(\mathscr{F}, I(\mathscr{G}))$$

are isomorphisms of abelian groups.

Definition. For a presheaf $\mathscr{F}$ on X, the sheaf $S(\mathscr{F})$ is the *sheafification* of $\mathscr{F}$.

Let $\eta\colon \mathrm{id}_{\mathcal{P}} \to I \circ S$ be the unit of the adjunction (S, I, φ). This adjunction then states the the sheafification has the following universal property: There exists for every sheaf $\mathscr{G}$ and every homomorphism of presheaves $f\colon \mathscr{F} \to I(\mathscr{G})$ a unique homomorphism of sheaves $\lambda\colon S(\mathscr{F}) \to \mathscr{G}$ that makes the following triangle commute:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\eta_{I(\mathscr{F})}"] \arrow[dr, "f"'] & IS(\mathscr{F}) \arrow[d, dashed, "I(\lambda)"] \\
 & I(\mathscr{G})
\end{tikzcd}$$

If a presheaf $\mathscr{F}$ is already a sheaf, then it follows from the fully faithfulness of the inclusion functor I that the sheafification $SI(\mathscr{F})$ of $\mathscr{F}$ is just $\mathscr{F}$ itself.

To be more precise, the sheaf $\mathscr{F}$ is also a sheafification of the presheaf $I(\mathscr{F})$, in the sense that the identity homomorphism $\mathrm{id}_{I(\mathscr{F})}\colon I(\mathscr{F}) \to I(\mathscr{F})$ fullfills the same universal property as the counit $\eta_{I(\mathscr{F})}\colon I(\mathscr{F}) \to ISI(\mathscr{F})$: Let $\mathscr{G}$ be another sheaf and let $f\colon I(\mathscr{F}) \to I(\mathscr{G})$ be a homomorphism of presheaves. Then, as I is fully faithful, there exist a unique homomorphism of sheaves $\lambda\colon \mathscr{F} \to \mathscr{G}$ that makes the desired triangle

$$\begin{tikzcd}
I(\mathscr{F}) \arrow[r, "\mathrm{id}_{I(\mathscr{F})}"] \arrow[dr, "f"'] & I(\mathscr{F}) \arrow[d, "I(\lambda)"] \\
 & I(\mathscr{G})
\end{tikzcd}$$

commute.

As both $\eta\colon I(\mathscr{F}) \to ISI(\mathscr{F})$ and $\mathrm{id}_{I(\mathscr{F})}\colon I(\mathscr{F}) \to I(\mathscr{F})$ have the same universal property, we can conclude that there exist a unique homomorphism of presheaves $ISI(\mathscr{F}) \to I(\mathscr{F})$ that makes the triangle

$$\begin{tikzcd}
 & I(\mathscr{F}) \arrow[dl, "\eta_{I(\mathscr{F})}"'] \arrow[dr, "\mathrm{id}_{I(\mathscr{F})}"] & \\
ISI(\mathscr{F}) \arrow[rr, dashed] & & I(\mathscr{F})
\end{tikzcd}$$

commute, and that this homomorphism is already an isomorphism. This homomorphism is just $\eta_{I(\mathscr{F})}$ itself (as $\eta_{I(\mathscr{F})}$ makes the above triangle commute) and so $\eta_{I(\mathscr{F})}$ is an isomorphism (of presheaves, and hence also of sheaves).

Remark⋆ 3.Y.

a) One can also express the above argumentation in the language of representable functors: It holds that by the naturality of φ and the fully faithfulness of I that
$$\mathcal{S}(S(I(\mathscr{F})), -) \cong \mathcal{P}(I(\mathscr{F}), I(-)) \cong \mathcal{S}(\mathscr{F}, -)\,,$$
and hence $SI(\mathscr{F}) \cong \mathscr{F}$ (because representing objects are unique up to isomorphism).

b) The constructed isomorphism $SI(\mathscr{F}) \cong \mathscr{F}$ is furthermore natural (in $\mathscr{F}$), and hence gives a natural isomorphism $S \circ I \cong \mathrm{Id}_{\mathcal{S}}$.

c) Let more generally (F, G, φ) be an adjunction from a category $\mathcal{C}$ to a category $\mathcal{D}$, with unit $\eta\colon \mathrm{Id}_{\mathcal{C}} \to G \circ F$ and counit $\varepsilon\colon F \circ G \to \mathrm{Id}_{\mathcal{D}}$. Then
   - F is fully faithful if and only if the unit η is an isomorphism, and dually
   - G is fully faithful if and only if the counit ε is an isomorphism.

   Proofs of this can be found in [St075B], [Bra17, Lemma 7.6.6] and [Mac78, IV, 3, Theorem 1] (where it more generally explained under what conditions F an G are faithful, and under what conditions they are full).

Example (Sheafificaton)**.**

a) Let A be an abelian group. The *constant sheaf* $\mathscr{C}_{X,A}$ is defined to be the sheafification of the constant presheaf $\widetilde{\mathscr{C}}_{X,A}$. It is given by
$$\mathscr{C}_{X,A}(U) = \{ f\colon U \to A \mid f \text{ is locally constant} \}\,.$$
for every open subset $U \subseteq X$, and for all open subsets $U \subseteq V \subseteq X$ the restriction homomorphism
$$\rho_{V,U}\colon \mathscr{C}_{X,A}(V) \to \mathscr{C}_{X,A}(U)\,, \quad f \mapsto f|_U$$
is the (literal) restriction map.

b) Let $\mathscr{F}$ be a sheaf and let $\mathscr{F}' \subseteq \mathscr{F}$ be a subpresheaf, i.e. $\mathscr{F}$ is a presheaf such that $\mathscr{F}'(U) \subseteq \mathscr{F}(U)$ is a subgroup for every open subset $U \subseteq X$, and such that the square
$$\begin{tikzcd} \mathscr{F}'(V) \arrow[r, hook] \arrow[d, "\rho^{\mathscr{F}'}_{V,U}"'] & \mathscr{F}(V) \arrow[d, "\rho^{\mathscr{F}}_{V,U}"] \\ \mathscr{F}'(U) \arrow[r, hook] & \mathscr{F}(U) \end{tikzcd}$$
commutes for all open subsets $U \subseteq V \subseteq X$. Then the sheafification $S(\mathscr{F}')$ is given by
$$S(\mathscr{F}')(U) = \left\{ s \in \mathscr{F}(U) \;\middle|\; \begin{array}{c} \text{there exists an open cover } \{U_i\}_{i \in I} \text{ of } U \\ \text{with } s|_{U_i} \in \mathscr{F}'(U_i) \text{ for every } i \in I \end{array} \right\},$$

and the restriction homomorphism

$$\rho^{S(\mathscr{F}')}_{V,U} : S(\mathscr{F}')(V) \to S(\mathscr{F}')(U)$$

is for all open subsets $U \subseteq V \subseteq X$ the restriction of the restriction homomorphism $\rho^{\mathscr{F}}_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$.

(This is Exercise 1 of Exercise sheet 7.)

For the general construction of the sheafification functor one uses stalks. We will not do this here.

vii) Let $f : \mathscr{F} \to \mathscr{G}$ be a homomorphism of sheaves. That the cokernel $\operatorname{coker}_{\mathcal{P}}(I(f))$ is in general not a sheaf can be fixed by applying the sheafification functor S:

Claim. The sheafification $S(\operatorname{coker}_{\mathcal{P}}(I(f)))$ is a cokernel of f in $\mathcal{S}$.

Proof. Let η: $\operatorname{coker}_{\mathcal{P}}(I(f)) \to IS(\operatorname{coker}_{\mathcal{P}}(I(f)))$ be the canonical homomorphism, and let $c' : I(\mathscr{G}) \to \operatorname{coker}(I(f))$ be the homomorphism belonging to the cokernel $\operatorname{coker}(I(f))$. The functor I is fully faithful, and so there exists for the composition

$$I(\mathscr{G}) \xrightarrow{c'} \operatorname{coker}_{\mathcal{P}}(I(f)) \xrightarrow{\eta} IS(\operatorname{coker}_{\mathcal{P}}(I(f)))$$

a unique homomorphism of sheaves $c : \mathscr{G} \to S(\operatorname{coker}_{\mathcal{P}}(I(f)))$ such that the homomorphism $I(c)$ is the above composition. Then c is a cokernel of f:

It holds that

$$I(c \circ f) = I(c) \circ I(f) = c' \circ I(f) = 0$$

because c' is a cokernel of $I(f)$ and I is additive. Suppose that $h : \mathscr{G} \to \mathscr{H}$ is a homomorphism of sheaves with $h \circ f = 0$. Then

$$I(h) \circ I(f) = I(h \circ f) = I(0) = 0$$

and it follows from the universal property of the cokernel c' that there exists a unique homomorphism of presheaves $\lambda' : I(\mathscr{H}) \to \operatorname{coker}_{\mathcal{P}}(I(f))$ that makes the triangle

$$\begin{tikzcd} & I(\mathscr{H}) \\ I(\mathscr{G}) \arrow[ur, "I(h)"] \arrow[r, "c'"'] & \operatorname{coker}_{\mathcal{P}}(I(f)) \arrow[u, dashed, "\lambda'"'] \end{tikzcd}$$

commute. It follows from the universal property of the sheafification that there exists a unique homomorphism of sheaves $\lambda' : \mathscr{H} \to S(\operatorname{coker}_{\mathcal{P}}(I(f)))$ that makes the triangle

$$\begin{tikzcd} \operatorname{coker}_{\mathcal{P}}(I(f)) \arrow[r, "\eta"] \arrow[dr, "\lambda'"'] & IS(\operatorname{coker}_{\mathcal{P}}(I(f))) \arrow[d, "I(\lambda)"] \\ & I(\mathscr{H}) \end{tikzcd}$$

commute. It holds that

$$I(h) = \lambda' \circ c' = I(\lambda) \circ \eta \circ c' = I(\lambda) \circ I(c) = I(\lambda \circ c)$$

and hence $h = \lambda \circ c$ because f is fully faithful.

The uniqueness of λ can be shown similarly. □

Remark⋆ 3.Z. One can express the above proof in the language of representable functors: It follows from the naturality of the adjunction φ, the universal property of the cokernel $\operatorname{coker}_{\mathcal{P}}(f)$ and the fully faithfulness of the inclusion functor I that

$$\begin{aligned}
&\mathcal{S}(S(\operatorname{coker}_{\mathcal{P}}(I(f))), -) \\
&\cong \mathcal{P}(\operatorname{coker}_{\mathcal{P}}(I(f)), I(-)) \\
&\cong \{g \in \mathcal{P}(I(\mathscr{G}), I(-)) \mid g \circ I(f) = 0\} \\
&\cong \{h \in \mathcal{S}(\mathscr{G}, -) \mid I(h) \circ I(f) = 0\} \\
&\cong \{h \in \mathcal{S}(\mathscr{G}, -) \mid I(h \circ f) = 0\} \\
&\cong \{h \in \mathcal{S}(\mathscr{G}, -) \mid h \circ f = 0\}\,.
\end{aligned}$$

This shows that $S(\operatorname{coker}_{\mathcal{P}}(I(f)))$ represents the right kind of functor to make it a cokernel of f.

End of lecture 12

viii) Let $f\colon \mathscr{F} \to \mathscr{G}$ be a homomorphism of presheaves. Then there exists a unique homomorphism of sheaves $\lambda\colon S(\ker_{\mathcal{P}}(f)) \to \ker_{\mathcal{S}}(S(f))$ that makes the triangle

$$\begin{tikzcd}
S(\ker_{\mathcal{P}}(f)) \arrow[d, dashed, "\lambda"'] \arrow[dr] & \\
\ker_{\mathcal{S}}(S(f)) \arrow[r] & S(\mathscr{F})
\end{tikzcd}$$

commute; here $\ker_{\mathcal{S}}(S(f)) \to S(\mathscr{F})$ denotes the canonical homomorphism of the kernel, and $S(\ker_{\mathcal{P}}(f)) \to S(\mathscr{F})$ denotes the homomorphism induced by the canonical homomorphism $\ker_{\mathcal{P}}(f) \to \mathscr{F}$. Indeed, the composition

$$\ker_{\mathcal{P}}(f) \to \mathscr{F} \xrightarrow{f} \mathscr{G}$$

is the zero homomorphism. It follows that the composition

$$\ker_{\mathcal{P}}(f) \to S(\mathscr{F}) \xrightarrow{S(f)} S(\mathscr{G})$$

is again the zero homomorphism because the sheafification functor S is additive. The existence and uniqueness of the homomorphism λ thus follows from the universal property of the kernel $\ker_{\mathcal{S}}(S(f)) \to S(\mathscr{F})$.

Fact 2. The homomorphism $\lambda\colon S(\ker_{\mathcal{P}}(f)) \to \ker_{\mathcal{S}}(S(f))$ is an isomorphism.

ix) If $f\colon \mathscr{F} \to \mathscr{G}$ is a homomorphism of sheaves then the canonical homomorphism $\tilde{f}\colon \operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism:

The resulting homomorphism of presheaves $I(f)\colon I(\mathscr{F}) \to I(\mathscr{G})$ has the following canonical factorization:

$$\begin{tikzcd}
I(\mathscr{F}) \arrow[r, "I(f)"] \arrow[d] & I(\mathscr{G}) \\
\operatorname{coim}(I(f)) \arrow[r, "\widetilde{I(f)}"] & \operatorname{im}(I(f)) \arrow[u]
\end{tikzcd}$$

We note that the homomorphism $\widetilde{I(f)}\colon \operatorname{coim}(I(f)) \to \operatorname{im}(I(f))$ is an isomorphism because the presheaf category $\mathcal{P}$ is abelian. By applying the sheafification functor S to this square diagram we get the following commutative square:

$$\begin{tikzcd}
SI(\mathscr{F}) \arrow[r, "SI(f)"] \arrow[d] & SI(\mathscr{G}) \\
S(\operatorname{coim}(I(f))) \arrow[r, "S(\widetilde{I(f)})"] & S(\operatorname{im}(I(f))) \arrow[u]
\end{tikzcd}$$

The homomorphisms of sheaves $S(\widetilde{I(f)})$ is again an isomorphism. We now get the following diagram:

$$\begin{tikzcd}
\mathscr{F} \arrow[rrr, "f"] \arrow[dr, "\cong"] \arrow[ddd] & & & \mathscr{G} \arrow[dl, "\cong"'] \\
& SI(\mathscr{F}) \arrow[r, "SI(f)"] \arrow[d] & SI(\mathscr{G}) & \\
& S(\operatorname{coim}_{\mathcal{P}}(I(f))) \arrow[r, "S(\widetilde{I(f)})"] & S(\operatorname{im}_{\mathcal{P}}(I(f))) \arrow[u] & \\
\operatorname{coim}_{\mathcal{S}}(f) \arrow[ur, "\cong"] \arrow[rrr, "\tilde{f}"] & & & \operatorname{im}_{\mathcal{S}}(f) \arrow[ul, "\cong"'] \arrow[uuu]
\end{tikzcd}$$

This diagram commutes:

- We have seen above that the inner square commutes.
- The outer square commutes by definition of $\tilde{f}$.
- The upper trapezoid commutes by the naturality of $\mathrm{Id}_{\mathcal{S}} \cong S \circ I$.
- The author finds the argument that was given in the lecture for the commutativtiy of the left trapezoid not sufficient. This argument is therefore currently missing from these notes.

- The commutativity of the right trapezoid was not explained in the lecture.
- The commutativity of the lower trapezoid follows from the commutativity of the rest of the diagram: It holds that

$$\begin{aligned} &(\mathscr{F} \to \operatorname{coim}_{\mathcal{S}}(f) \to S(\operatorname{coim}_{\mathcal{P}}(I(f))) \to S(\operatorname{im}_{\mathcal{P}}(I(f))) \to \operatorname{im}_{\mathcal{S}}(f) \to \mathscr{G}) \\ &= (\mathscr{F} \to SI(\mathscr{F}) \to S(\operatorname{coim}_{\mathcal{P}}(I(f))) \to S(\operatorname{im}_{\mathcal{P}}(I(f))) \to SI(\mathscr{G}) \to \mathscr{G}) \\ &= (\mathscr{F} \to SI(\mathscr{F}) \to SI(\mathscr{G}) \to \mathscr{G}) \\ &= (\mathscr{F} \to \mathscr{G}) \\ &= (\mathscr{F} \to \operatorname{coim}_{\mathcal{S}}(f) \to \operatorname{im}_{\mathcal{S}}(f) \to \mathscr{G}) \,. \end{aligned}$$

  The homomorphism $\mathscr{F} \to \operatorname{coim}_{\mathcal{S}}(f)$ is an epimorphism and the homomorphism $\operatorname{im}_{\mathcal{S}}(f) \to \mathscr{G}$ is a monomorphism. We hence find that already

$$\begin{aligned} &(\operatorname{coim}_{\mathcal{S}}(f) \to S(\operatorname{coim}_{\mathcal{P}}(I(f))) \to S(\operatorname{im}_{\mathcal{P}}(I(f))) \to \operatorname{im}_{\mathcal{S}}(f)) \\ &= (\operatorname{coim}_{\mathcal{S}}(f) \to \operatorname{im}_{\mathcal{S}}(f)) \,. \end{aligned}$$

The commutativity of the lower trapezoid shows that the canonical homomorphism $\tilde{f}$ is a composition of three isomorphisms, and therefore an isomorphism itself.

This altogether shows that the sheaf category $\mathcal{S} = \mathbf{Sh}_X(\mathbf{Ab})$ is abelian.

Lemma 3.36. Let $\mathcal{A}$ be an abelian category.

i) Every monomorphism in $\mathcal{A}$ is the kernel of its cokernel. It is in particular a kernel.

ii) Every epimorphism in $\mathcal{A}$ is the cokernel of its kernel. It is in particular a cokernel.

Proof.

i) Let $f\colon X \to Y$ be a monomorphism in $\mathcal{A}$. Then $\ker(f) = 0$ and hence the identity $\mathrm{id}_Y\colon X \to X$ is a cokernel of $\ker(f) \to X$. The canonical factorization of f is therefore given as follows:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \\ \operatorname{coim}(f) \arrow[r, "\cong", "\tilde{f}"'] & \operatorname{im}(f) \arrow[u] \end{tikzcd}$$

This commutative square can also be compressed into the following commutative triangle:

$$\begin{tikzcd} X \arrow[rr, "\cong", "\tilde{f}"'] \arrow[dr, "f"'] & & \operatorname{im}(f) \arrow[dl] \\ & Y & \end{tikzcd}$$

The canonical morphism $\operatorname{im}(f) \to Y$ is a kernel of the cokernel $Y \to \operatorname{coker}(f)$. It follows from the above triangle that $f\colon X \to Y$ is also a kernel of $Y \to \operatorname{coker}(f)$.

ii) This can be shown dually. □

Remark 3.37. Let $f\colon X \to Y$ be a morphism in an abelian category $\mathcal{A}$. Then the canonical morphism $f'\colon X \to \operatorname{im}(f)$ is an epimorphism and the canonical morphism $\bar{f}\colon \operatorname{coim}(f) \to Y$ is a monomorphism. Indeed, the morphism f' is given by the composition of epimorphisms

$$X \to \operatorname{coim}(f) \xrightarrow{\tilde{f}} \operatorname{im}(f)\,,$$

while $\bar{f}$ is given by the composition of monomorphisms

$$\operatorname{coim}(f) \xrightarrow{\tilde{f}} \operatorname{im}(f) \to Y\,.$$

Proposition 3.38. A morphism f in an abelian category $\mathcal{A}$ is an isomorphism if and only if it is both an epimorphism and a monomorphism.

Proof. If f is an isomorphism then it is both an epimorphism and a monomorphism, as this holds in every category.

Suppose on the other hand that f is both an epimorphism and a monomorphism. Then the zero morphism $0 \to X$ is a kernel of f and the zero morphism $Y \to 0$ is a cokernel of f. The identity $\mathrm{id}_X\colon X \to X$ is therefore a coimage of f, while the identity $\mathrm{id}_Y\colon Y \to Y$ is an image of f. The canonical factorization of f is therefore given as follows:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, equal] \\ \operatorname{coim}(f) \arrow[r, "\tilde{f}"] & \operatorname{im}(f) \end{tikzcd}$$

The isomorphism $\tilde{f}$ is therefore just f itself. □

Pullbacks and Pushouts

Definition⋆ 3.AA. Let $\mathcal{C}$ be a category.

i) Let (f, g) be a pair of morphisms

$$X \xrightarrow{f} Y \xleftarrow{g} Y'$$

in $\mathcal{C}$. A *pullback* of the pair (f, g) is a triple (X', f', g') consisting of

- an object $X' \in \operatorname{Ob}(\mathcal{C})$ and
- two morphisms

$$Y' \xleftarrow{f'} X' \xrightarrow{g'} X\,,$$

such that

(PB1) the square diagram

$$\begin{tikzcd} X' \arrow[r, dashed, "g'"] \arrow[d, dashed, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

commutes, and

(PB2) the triple (X', f', g') is universal with this property, in the following sense: If (X'', f'', g'') is any other triple consisting of an object $X'' \in \mathrm{Ob}(\mathcal{C})$ and two morphisms

$$Y'' \xleftarrow{f''} X'' \xrightarrow{g''} X$$

such that the square

$$\begin{tikzcd} X'' \arrow[r, dashed, "g''"] \arrow[d, dashed, "f''"'] & X \arrow[d, "f"] \\ Y'' \arrow[r, "g"] & Y \end{tikzcd}$$

commutes, then there exists a unique morphism $\lambda\colon X'' \to X'$ that makes the following diagram commute:

$$\begin{tikzcd} X'' \arrow[dr, dashed, "\lambda"] \arrow[drr, bend left, "g''"] \arrow[ddr, bend right, "f''"'] & & \\ & X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

ii) Let (f, g) be a pair of morphisms

$$Y \xleftarrow{f} X \xrightarrow{g} X'$$

in $\mathcal{C}$. A *pushout* of the pair (f, g) is a triple (Y', f', g') consisting of

- an object $Y' \in \mathrm{Ob}(\mathcal{A})$ and
- two morphisms

$$X' \xrightarrow{f'} Y' \xleftarrow{g'} Y\,,$$

such that

(PO1) the square diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[d, dashed, "f'"] \\ Y \arrow[r, dashed, "g'"] & Y' \end{tikzcd}$$

commutes, and

(PO2) the triple (Y', f', g') is universal with this property, in the following sense: If (Y'', f'', g'') is another triple consisting of an object $Y'' \in \mathrm{Ob}(\mathcal{C})$ and two morphisms

$$X'' \xrightarrow{f''} Y'' \xleftarrow{g''} Y$$

such that the square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X'' \arrow[d, dashed, "f''"] \\ Y \arrow[r, dashed, "g''"] & Y'' \end{tikzcd}$$

commutes, then there exists a unique morphism $\mu\colon Y' \to Y''$ in $\mathcal{C}$ that makes the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"'] \arrow[ddr, bend left, "f''"] & \\ Y \arrow[r, "g'"] \arrow[drr, bend right, "g''"'] & Y' \arrow[dr, dashed, "\mu"] & \\ & & Y'' \end{tikzcd}$$

Remark⋆ 3.AB.

i) The notions of pullback and pushout are dual to each other: A triple (X', f', g') is a pullback of morphisms $X \xrightarrow{f} Y \xleftarrow{g} Y'$ in $\mathcal{C}$ if and only if (X', f', g') is a pushout of $X \xleftarrow{f} Y \xrightarrow{g} Y'$ in $\mathcal{C}^{\mathrm{op}}$.

ii) Pullbacks and pushouts are unique up to unique isomorphism: If (X', f', g') and (X'', f'', g'') are two pullbacks of morphisms $X \xrightarrow{f} Y \xleftarrow{g} Y'$, then the unique morphism $\lambda\colon X'' \to X'$ that makes the diagram

$$\begin{tikzcd} X'' \arrow[dr, dashed, "\lambda"] \arrow[drr, bend left, "g''"] \arrow[ddr, bend right, "f''"'] & & \\ & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

commute is already an isomorphism. For pushouts the dual statement holds.

iii) Pullbacks are also know as *fibre products*, and the fibre product of two morphisms $X \xrightarrow{f} Y \xleftarrow{g} Y'$ is then denoted by $X \times_Y Y'$. Pushouts are knows as *fibre coproducts* and *amalgamated sums*, and the amalgamated sum of two morphisms $Y \xleftarrow{f} X \xrightarrow{g} X'$ is then denoted by $X' \amalg_X Y$.

iv) A square diagram

$$\begin{tikzcd} X' \arrow[r, dashed, "g'"] \arrow[d, dashed, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is a *pullback square* if the triple (X', f', g') is a pullback of the two morphisms $X \xrightarrow{f} Y \xleftarrow{g} Y'$. This is often denoted by adding the symbol $\lrcorner$ inside of the square:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Dually, a square diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[d, dashed, "f'"] \\ Y \arrow[r, dashed, "g'"] & Y' \end{tikzcd}$$

is a *pushout square* if the triple (X', f', g') is a pushout of the two morphisms $Y \xleftarrow{f} X \xrightarrow{g} X'$. This is often denoted by adding the symbol $\ulcorner$ inside of the square:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "g'"] & Y' \end{tikzcd}$$

v) We say that the category $\mathcal{C}$ *has pullbacks* if every pair of morphisms $X \xrightarrow{f} Y \xleftarrow{g} Y'$ in $\mathcal{C}$ admits a pullback in $\mathcal{C}$. We say dually that $\mathcal{C}$ *has pushouts* if every pair of morphisms $Y \xleftarrow{f} X \xrightarrow{g} X'$ in $\mathcal{C}$ admits a pushout in $\mathcal{C}$.

Example⋆ 3.AC.

i) In the category **Set** any pair of maps $X \xrightarrow{f} Y \xleftarrow{g} Y'$ admits a pullback. This pullback is given by the set

$$X' := \{(x, y') \in X \times Y' \mid f(x) = g(y')\}$$

together with the maps $f' \colon X' \to Y'$ and $g' \colon X' \to X$ that are the restrictions of the canonical projections $X \times Y' \to X$ and $X \times Y' \to Y'$.

We observe that the set X' is given by

$$X' = \coprod_{y' \in Y'} \left(\{y'\} \times f^{-1}(g(y'))\right),$$

which explains the term *fibre product* and the notation $X \times_Y Y'$

The existence and above construction of pullbacks also holds for the categories

$$\mathbf{Top}\,,\ \mathbf{Grp}\,,\ \mathbf{Ring}\,,\ \mathbf{CRing}\,,\ \mathbf{k\text{-}Alg}\,,\ \mathbf{k\text{-}CAlg}\,,\ A\text{-}\mathbf{Mod}\,,\ \mathbf{Mod}\text{-}A\,,$$

where A is a **k**-algebra.

ii) In the category **Set** any every pair of maps $Y \xleftarrow{f} X \xrightarrow{g} X'$ admits a pushout. This pushout is given by the set

$$Y' := (X' \amalg Y)/{\sim}$$

where $\sim$ is the equivalence relation generated by $i(f(x)) \sim j(g(x))$ for $x \in X$, where $i\colon X' \to X' \amalg Y$ and $j\colon Y \to X' \amalg Y$ are the canonical inclusions. The map $f'\colon X' \to Y'$ is induced by the inclusion i and the map $g'\colon Y \to Y'$ is induced by the inclusion j.

This construction also explains the notation $X' \amalg_X Y$.

Remark* 3.AD. Let $\mathcal{C}$ be a category and let $Y \in \mathrm{Ob}(\mathcal{C})$ be an object in $\mathcal{C}$. The *over-Y-category* $\mathcal{C}/Y$ is defined as follows:

- The objects of $\mathcal{C}/Y$ are pairs (X, x) consisting of an an object $X \in \mathrm{Ob}(\mathcal{C})$ and a morphism $x\colon X \to Y$. The objects of $\mathcal{C}/Y$ may be visualized as follows:

$$\begin{tikzcd} X \arrow[d, "x"'] \\ Y \end{tikzcd}$$

- A morphism $f\colon (X, x) \to (X', x')$ in $\mathcal{C}/Y$ is a morphism $f\colon X \to X'$ that makes the triangle

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "x"'] & & X' \arrow[dl, "x'"] \\ & Y & \end{tikzcd}$$

commute.

- The composition of two morphism $f\colon (X, x) \to (X, x')$ and $f'\colon (X', x') \to (X'', x'')$ is given by $f' \circ f$, the composition of f and f' in $\mathcal{C}$. This is again a morphism in $\mathcal{C}/Y$ by the commutativity of the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, dashed, bend left, "f' \circ f"] \arrow[dr, "x"'] & X' \arrow[r, "f'"] \arrow[d, "x'"] & X'' \arrow[dl, "x''"] \\ & Y & \end{tikzcd}$$

- The identity morphism of an object $(X, x) \in \mathrm{Ob}(\mathcal{C}/Y)$ is given by $\mathrm{id}_{(X,x_0)} = \mathrm{id}_X$.

A pair of morphisms $X \xrightarrow{f} Y \xleftarrow{g} Y'$ can be seen as two objects (X, f) and (Y', g) of the category $\mathcal{C}/Y$. Let (X', f', g') be a triple consisting of an object $X \in \mathrm{Ob}(\mathcal{C})$ and two morphisms $Y' \xleftarrow{f'} X' \xrightarrow{g'} X$ such that the square

$$\begin{tikzcd} X' \arrow[r, dashed, "g'"] \arrow[d, dashed, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \tag{3.6}$$

commutes. For $h \coloneqq f \circ g' = g \circ f'$ we then have the following commutative diagram:

$$\begin{tikzcd} Y' \arrow[dr, "g"'] & X' \arrow[l, dashed, "f'"'] \arrow[r, dashed, "g'"] \arrow[d, "h"] & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

The two morphisms f' and g' are therefore also morphisms $f' \colon (X', h) \to (Y', g)$ and $g' \colon (X', h) \to (X, f)$ in $\mathcal{C}/Y$. That the diagram (3.6) is a pullback square means precisely that the triple $((X', h), f', g')$ is a product of the two objects (X, f) and (Y, g) in the category $\mathcal{C}/Y$.

Lemma* 3.AE (Transitivity of pullback and pushout)**.** Let $\mathcal{C}$ be a category.

i) Let

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "h"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "g'"] & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$.

a) If the left square and the right square are both pullback squares, then the outer reactangle is again a pullback square.

b) If the right square and the outer reactangle are both pullback squares, then the left square is also a pullback square.

ii) Let

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[r, "g'"] \arrow[d, "f'"'] & X'' \arrow[d, "f''"'] \\ Y \arrow[r, "h"] & Y' \arrow[r, "h'"] & Y'' \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$.

a) If the right square and the left square are both pushout diagrams, then the outer rectangle is again a pushout square.

b) If the left square and outer reactangle are both pushout squares, then the right square is also a pushout square.

Proof.

i) This is Exercise 1 on Exercise sheet 6.

ii) This follows from part ii) by duality. □

Lemma⋆ 3.AF. Let $\mathcal{A}$ be an additive category that has kernels and cokernels. Then the category $\mathcal{A}$ has pullbacks and pushouts.

Proof. This is Exercise 3 on the Exercise sheet 6. □

Proposition 3.AG. Let $\mathcal{C}$ be a category.

i) Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a pullback square in $\mathcal{C}$. If f is a monomorphism then f' is again a monomorphism.

ii) Let

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow \\ Y & \xrightarrow{g'} & Y' \end{array}$$

be a pushout square in $\mathcal{C}$. If f is an epimorphism then f' is again an epimorphism.

In an abelian category $\mathcal{A}$ the same holds when we switch 'monomorphism' and 'epimorphism' in the above statements:

iii) Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a pullback square in $\mathcal{A}$. If f is an epimorphism then f' is again a epimorphism.

iv) Let

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow \\ Y & \xrightarrow{g'} & Y' \end{array}$$

is a pushout square in $\mathcal{A}$. If f is a monomorphism then f' is again a monomorphism.

Proof.

i) This is Exercise 2 on the Exercise sheet 2.

ii) This is the dual statement to part i).

iii) This is Exercise 4 on the Exercise sheet 6.

iv) This is the dual statement to part iii). □

Remark⋆ 3.AH (Functoriality of pullback and pushout)**.** Let $\mathcal{C}$ be a category.

i) Suppose that we are given two diagrams

$$\begin{tikzcd} & X_1 \arrow[d] \\ Y_1' \arrow[r] & Y_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X_2 \arrow[d] \\ Y_2' \arrow[r] & Y_2 \end{tikzcd}$$

in $\mathcal{C}$ and morphisms $X_1 \to X_2$, $Y_1 \to Y_2$ and $Y_1' \to Y_2'$ that make the resulting diagram

$$\begin{tikzcd} & X_1 \arrow[dd] \arrow[dr, dashed] & \\ & & X_2 \arrow[dd] \\ Y_1' \arrow[r] \arrow[dr, dashed] & Y_1 \arrow[dr, dashed] & \\ & Y_2' \arrow[r] & Y_2 \end{tikzcd} \tag{3.7}$$

commute. If the pullbacks

$$\begin{tikzcd} X_1' \arrow[r, dashed] \arrow[d, dashed] & X_1 \arrow[d] \\ Y_1' \arrow[r] & Y_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_2' \arrow[r, dashed] \arrow[d, dashed] & X_2 \arrow[d] \\ Y_2' \arrow[r] & Y_2 \end{tikzcd}$$

exist, then there exists a unique morphism $X_1' \to X_2'$ that makes the resulting cube

$$\begin{tikzcd} X_1' \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & X_1 \arrow[dr] \arrow[dd] & \\ & X_2' \arrow[rr] \arrow[dd] & & X_2 \arrow[dd] \\ Y_1' \arrow[rr] \arrow[dr] & & Y_1 \arrow[dr] & \\ & Y_2' \arrow[rr] & & Y_2 \end{tikzcd}$$

commute. Both the existence and uniqueness follow by the universal property of the pullback (applied to X_2') from the commutativity of the diagram (3.7). This induced morphism is functorial in the following sense:

- Let
$$\begin{tikzcd}
 & X \arrow[d] \\
Y' \arrow[r] & Y
\end{tikzcd}$$
be a diagram in $\mathcal{C}$ whose pushout
$$\begin{tikzcd}
X' \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d] \\
Y' \arrow[r] & Y
\end{tikzcd}$$
exists. Then the diagram
$$\begin{tikzcd}
 & & X \arrow[dr, dashed, "\mathrm{id}_X"] \arrow[dd] & \\
 & & & X \arrow[dd] \\
Y' \arrow[rr] \arrow[dr, dashed, "\mathrm{id}_{Y'}"'] & & Y \arrow[dr, dashed, "\mathrm{id}_Y"] & \\
 & Y' \arrow[rr] & & Y
\end{tikzcd}$$
commutes, and the induced morphism $X' \to X'$ that makes the cube
$$\begin{tikzcd}
X' \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & X \arrow[dr, "\mathrm{id}_X"] \arrow[dd] & \\
 & X' \arrow[rr] \arrow[dd] & & X \arrow[dd] \\
Y' \arrow[rr] \arrow[dr, "\mathrm{id}_{Y'}"'] & & Y \arrow[dr, "\mathrm{id}_Y"] & \\
 & Y' \arrow[rr] & & Y
\end{tikzcd}$$
commute is the identity $\mathrm{id}_{X'}$.
- Let
$$\begin{tikzcd}
 & X_1 \arrow[d] \\
Y_1' \arrow[r] & Y_1
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & X_2 \arrow[d] \\
Y_2' \arrow[r] & Y_2
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & X_3 \arrow[d] \\
Y_3' \arrow[r] & Y_3
\end{tikzcd}$$

be diagram in $\mathcal{C}$ whose pullbacks

$$\begin{tikzcd} X_1' \arrow[r, dashed] \arrow[d, dashed] & X_1 \arrow[d] \\ Y_1' \arrow[r] & Y_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_2' \arrow[r, dashed] \arrow[d, dashed] & X_2 \arrow[d] \\ Y_2' \arrow[r] & Y_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_3' \arrow[r, dashed] \arrow[d, dashed] & X_3 \arrow[d] \\ Y_3' \arrow[r] & Y_3 \end{tikzcd}$$

exist. Suppose that we are given morphisms

$$X_i \to X_{i+1}\,, \quad Y_i \to Y_{i+1}\,, \quad Y_i' \to Y_{i+1}'$$

that make the resulting diagram

$$\begin{tikzcd}
 & & & X_1 \arrow[dr, dashed] \arrow[ddd] & & \\
 & & & & X_2 \arrow[dr, dashed] \arrow[ddd] & \\
 & & & & & X_3 \arrow[ddd] \\
Y_1' \arrow[rrr] \arrow[dr, dashed] & & & Y_1 \arrow[dr, dashed] & & \\
 & Y_2' \arrow[rrr] \arrow[dr, dashed] & & & Y_2 \arrow[dr, dashed] & \\
 & & Y_3' \arrow[rrr] & & & Y_3
\end{tikzcd}$$

commutes. Let $f\colon X_1' \to X_2'$ and $g\colon X_2' \to X_3'$ be the induced morphisms that make the diagram

$$\begin{tikzcd}
X_1' \arrow[rrr] \arrow[dr, dashed, "f"] \arrow[ddd] & & & X_1 \arrow[dr] \arrow[ddd] & & \\
 & X_2' \arrow[rrr] \arrow[dr, dashed, "g"] \arrow[ddd] & & & X_2 \arrow[dr] \arrow[ddd] & \\
 & & X_3' \arrow[rrr] \arrow[ddd] & & & X_3 \arrow[ddd] \\
Y_1' \arrow[rrr] \arrow[dr] & & & Y_1 \arrow[dr] & & \\
 & Y_2' \arrow[rrr] \arrow[dr] & & & Y_2 \arrow[dr] & \\
 & & Y_3' \arrow[rrr] & & & Y_3
\end{tikzcd}$$

commute. It then follows from the commutativity of the subcube

$$\begin{tikzcd}
X_1' \arrow[rr] \arrow[dr, dashed, "gf"] \arrow[dd] & & X_1 \arrow[dr] \arrow[dd] & \\
& X_3' \arrow[rr] \arrow[dd] & & X_3 \arrow[dd] \\
Y_1' \arrow[rr] \arrow[dr] & & Y_1 \arrow[dr] & \\
& Y_3' \arrow[rr] & & Y_3
\end{tikzcd}$$

that gf is the morphism $X_1' \to X_3'$ induced by the commutativity of the following diagram:

$$\begin{tikzcd}
& & X_1 \arrow[dr, dashed] \arrow[dd] & \\
& & & X_3 \arrow[dd] \\
Y_1' \arrow[rr] \arrow[dr, dashed] & & Y_1 \arrow[dr, dashed] & \\
& Y_3' \arrow[rr] & & Y_3
\end{tikzcd}$$

ii) We observe similarly the functoriality of the pushout: Suppose that

$$\begin{tikzcd}
X_1' \arrow[r] \arrow[d] & Y_1' \\
X_1 &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X_2' \arrow[r] \arrow[d] & Y_2' \\
X_2 &
\end{tikzcd}$$

are diagrams in $\mathcal{C}$ whose pushouts

$$\begin{tikzcd}
X_1' \arrow[r] \arrow[d] & Y_1' \arrow[d, dashed] \\
X_1 \arrow[r, dashed] & Y_1
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X_2' \arrow[r] \arrow[d] & Y_2' \arrow[d, dashed] \\
X_2 \arrow[r, dashed] & Y_2
\end{tikzcd}$$

exist. If $X_1 \to X_2$, $Y_1 \to Y_2$ and $X_1' \to X_2'$ are morphisms that make the diagram

$$\begin{tikzcd}
X_1' \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & X_1 \arrow[dr, dashed] & \\
& X_3' \arrow[rr] \arrow[dd] & & X_3 \\
Y_1' \arrow[dr, dashed] & & & \\
& Y_3' & &
\end{tikzcd}$$

commute, then there exists a unique morphism $Y_1' \to Y_1'$ that makes the resulting cube

$$\begin{tikzcd}
X_1' \arrow[rr] \arrow[dr] \arrow[dd] & & X_1 \arrow[dr] \arrow[dd] & \\
& X_3' \arrow[rr] \arrow[dd] & & X_3 \arrow[dd] \\
Y_1' \arrow[rr] \arrow[dr] & & Y_1 \arrow[dr, dashed] & \\
& Y_3' \arrow[rr] & & Y_3
\end{tikzcd}$$

commute. We also find that this induced morphism is functorial in the same sense as for the pullback.

Exact Sequences and Diagram Lemmata

Remark. (Orally) We will (at least for now) neither use nor state the Freyd–Mitchell embedding theorem. We will instead explain how one can 'diagram chase' in abelian categories.

Remark-Definition 3.39.

i) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two composable morphisms in $\mathcal{A}$. If $gf = 0$ then there exists a unique morphism $\lambda\colon \operatorname{im}(f) \to \ker(g)$ that makes the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "f'"'] & & Y \arrow[rr, "g"] & & Z \\
& \operatorname{im}(f) \arrow[ur, "i"] \arrow[rr, dashed, "\lambda"] & & \ker(g) \arrow[ul, "k"'] &
\end{tikzcd}$$

commute, and this morphism is a monomorphism. Indeed, it holds that

$$gif' = gf = 0$$

and hence $gi = 0$ because f' is an epimorphism. The existence and uniqueness of λ thus follow from the universal property of the kernel $k\colon \ker(g) \to Y$.

The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is *exact* if

(E1) $gf = 0$, and

(E2) the canonical morphism $\lambda\colon \operatorname{im}(f) \to \ker(g)$ is an isomorphism.

ii) A (possibly infinite) sequence

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

of composable morphisms in $\mathcal{A}$ is *exact* if at every position i for which both an incoming morphism $f_{i-1}\colon X_{i-1} \to X_i$ and an outgoing morphism $f_i\colon X_i \to X_{i+1}$ exist, the resulting sequence

$$X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1}$$

is exact.

iii) We have the following special cases of exact sequences:

- A sequence of the form
$$0 \to X \xrightarrow{f} Y$$
is exact if and only if the morphism f is a monomorphism.
- A sequence of the form
$$Y \xrightarrow{g} Z \to 0$$
is exact if and only if the morphism g is an epimorphism.
- A sequence of the form
$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$
is exact if and only if the morphism f is a monomorphism, the morphism g is an epimorphism, and the canonical morphism $\operatorname{im}(f) \to \ker(g)$ is an isomorphism. This is furthermore equivalent to f being a kernel of g and g being a cokernel of f (at the same time).

Remark* 3.AI. Let $\mathcal{A}$ be an abelian category and let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence in $\mathcal{A}$. If $h\colon W \to X$ is an epimorphism in $\mathcal{A}$ then the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if and only if the sequence $W \xrightarrow{fh} Y \xrightarrow{g} Z$ is exact:

Indeed, we have that $gf = 0$ if and only if $gfh = 0$ because h is an epimorphism. We hence need to show that if $gf = 0$ (and then also $gfh = 0$) then the induced morphism $\mathrm{im}(f) \to \ker(g)$ is an isomorphism if and only if the induced morphism $\mathrm{im}(fh) \to \ker(g)$ is an isomorphism.

We have seen in Remark* 3.Q that $\mathrm{im}(f) = \mathrm{im}(fh)$, in the sense that a morphism $Y' \to Y$ is an image of f if and only if it is an image of fh. We hence get the following commutative diagram:

$$\begin{tikzcd}
W \arrow[rr, "h"] \arrow[dr] & & X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[rr, "g"] & & Z \\
& \mathrm{im}(fh) \arrow[rr, equal] & & \mathrm{im}(f) \arrow[ur] \arrow[rr, dashed] & & \ker(g) \arrow[ul] &
\end{tikzcd}$$

The commutativity of this diagram shows that the induced morphism $\mathrm{im}(fh) \to \ker(g)$ is the same as the induced morphism $\mathrm{im}(f) \to \ker(g)$. So the morphism $\mathrm{im}(f) \to \ker(g)$ is an isomorphism if and only if the morphism $\mathrm{im}(fh) \to \ker(g)$ is an isomorphism, as both are the same morphism.

If dually $h' \colon Z \to Z'$ is a monomorphism in $\mathcal{A}$, then the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if and only if the sequence $X \xrightarrow{f} Y \xrightarrow{h'g} Z'$ is exact.

Definition* 3.AJ. Let $\mathcal{A}$ be an abelian category.

i) A *short exact sequence* in $\mathcal{A}$ is an exact sequence of the form
$$0 \to X' \to X \to X'' \to 0 .$$

ii) A *left exact sequence* in $\mathcal{A}$ is an exact sequence of the form
$$0 \to X' \to X \to X'' .$$

iii) A *right exact sequence* in $\mathcal{A}$ is an exact sequence of the form
$$X' \to X \to X'' \to 0 .$$

Remark* 3.AK. Let $\mathcal{A}$ be an abelian category.

i) Sometimes a sequence $X' \to X \to X''$ in $\mathcal{A}$ is called *short exact* if the corresponding sequence $0 \to X' \to X \to X'' \to 0$ is short exact.

Similarly, a sequence $X' \to X \to X''$ is sometimes called *left exact* (resp. *right exact*) if the sequence $0 \to X' \to X \to X''$ is (left) exact (resp. if the sequence $X' \to X \to X'' \to 0$ is (right) exact).

ii) A sequence $0 \to X' \to X \to X''$ in $\mathcal{A}$ is (left) exact if and only if the morphism $X' \to X$ is a kernel of the morphism $X \to X''$.

Dually, a sequence $X' \to X \to X'' \to 0$ is (right) exact if and only if the morphism $X \to X''$ is a cokernel of the morphism $X' \to X$.

Proposition 3.40. Let $\mathcal{A}$ be an abelian category and let

$$\begin{tikzcd}
 & & Y' \arrow[r, "g'"] \arrow[d, "h'"] \arrow[dr, phantom, "\lrcorner"] & Z' \arrow[d, "h"] & \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram in $\mathcal{A}$ where the bottom row is exact and the square is a pullback square. Then there exists a unique morphism $f' \colon X \to Y'$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, dashed, "f'"] \arrow[d, equal] & Y' \arrow[r, "g'"] \arrow[d, "h'"] \arrow[dr, phantom, "\lrcorner"] & Z' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd} \tag{3.8}$$

commutes and such that the upper row

$$0 \to X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \to 0 \tag{3.9}$$

is exact.

Proof. It follows from Proposition 3.AG that g' is an epimorphism because g is one.

To construct the desired morphism $f' \colon X \to Y$ we use that the given square is a pullback square: The diagram

$$\begin{tikzcd}
X \arrow[drr, dashed, bend left, "0"] \arrow[ddr, bend right, "f"'] & & \\
 & Y' \arrow[r, "g'"] \arrow[d, "h'"] \arrow[dr, phantom, "\lrcorner"] & Z' \arrow[d, "h"] \\
 & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

commutes because $gf = 0$. So it follows that there exists a unique morphism $f' \colon X \to Y'$ that makes the diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "f"'] \arrow[dr, dashed, "f'"] & & \\
 & Y' \arrow[r, "g'"] \arrow[d, "h'"] \arrow[dr, phantom, "\lrcorner"] & Z' \arrow[d, "h"] \\
 & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

commute. This means precisely that the morphism f' makes the diagram (3.8) commute and satisfies $g'f' = 0$. The morphism f is a monomorphism because the composition $h'f' = f$ is a monomorphism.

To show the exactness of the row (3.9) it remains to show that f' is already a kernel of g'. So let $u\colon W \to Y'$ be a morphism with $g'u = 0$.

$$\begin{tikzcd}
 & & W \arrow[d, dashed, "u"] \arrow[dr, dashed, bend left, "0"] & & \\
0 \arrow[r] & X \arrow[r, "f'"] \arrow[d, equal] & Y' \arrow[r, "g'"] \arrow[d, "h'"] \arrow[dr, phantom, "\lrcorner"] & Z' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

Then
$$gh'u = hg'u = h \circ 0 = 0$$
and hence $h'u$ factors uniquely over the kernel of g, which is f. So there exists a unique morphism $\lambda\colon W \to X$ with $f\lambda = h'u$.

$$\begin{tikzcd}
 & & W \arrow[dl, dashed, bend right, "\lambda"'] \arrow[ddl, dashed, "\lambda"] \arrow[d, "u"] \arrow[dr, bend left, "0"] \arrow[ddr, dashed, "0"] & & \\
0 \arrow[r] & X \arrow[r, dashed, "f'"] \arrow[d, equal] & Y' \arrow[r, "g'"] \arrow[d, "h'"] \arrow[dr, phantom, "\lrcorner"] & Z' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

To show that $u = f'\lambda$, i.e. that the above diagram commutes, we again use that the right-hand square is a pullback square: The two parallel morphisms $u, f'\lambda\colon W \to Y'$ coincide if and only if they coincide after composition with both g' and h'. We have that
$$h'u = f\lambda = f \operatorname{id}_X \lambda = h'f'\lambda$$
and also
$$g'u = 0 = g'f'\lambda$$
because $g'f' = 0$. Hence $u = f'\lambda$, which shows the existence of the desired morphism λ. The morphism λ is uniquely determined by the composition $f'\lambda = u$ because f' is a monomorphism. This shows the uniqueness of λ. $\square$

Remark-Definition 3.41. Let $\mathcal{A}$ be an abelian category and let X be an object in $\mathcal{A}$.

i) For every object $A \in \operatorname{Ob}(\mathcal{A})$ let
$$X(A) := \operatorname{Hom}_{\mathcal{A}}(A, X)\,.$$
The elements of $X(A)$ are the *A-valued points* of X.

ii) We denote by $x \in_{\mathcal{A}} X$ that x is an A-valued points of X for some $A \in \mathrm{Ob}(\mathcal{A})$.

Remark⋆ 3.AL. Let $\mathcal{A}$ be an abelian category and let $f\colon X \to Y$ be a morphism in $\mathcal{A}$.

i) We can reformulate the universal property of the kernel in terms of points of objects: A morphism $k\colon K \to X$ is a kernel of f if and only if for every point $x \in X$ with $fx = 0$ there exist a unique point $\tilde{x} \in_{\mathcal{A}} K$ with $k\tilde{x} = x$. (If x is an A-valued point then it automatically follows that $\tilde{x}$ is again A-valued, because $f\tilde{x}$ is A-valued.)

ii) However, another similar rule from diagram chasing does not hold (or at least not up to equality, as we will see below): If the morphism f is an epimorphism and $y \in_{\mathcal{A}} Y$ is a point, say $y \in Y(A)$, then there does not have to exist a point $x \in X$ with $fx = y$. In other words, there does not have to exist a lift $x\colon A \to X$ of the morphism $y\colon A \to Y$ along f; here we mean by 'lift' a morphism that makes the triangle

$$\begin{tikzcd} & X \arrow[d, "f"] \\ A \arrow[ur, dashed, "x"] \arrow[r, "y"'] & Y \end{tikzcd}$$

commute. Indeed, we may consider as a (counter)example the abelian category $\mathcal{A} = \mathbf{Ab}$ and choose $f\colon \mathbb{Z} \to \mathbb{Z}/2$ to be the canonical projection. Then for the morphism $\mathrm{id}_{\mathbb{Z}}\colon \mathbb{Z}/2 \to \mathbb{Z}/2$ no such lift $\mathbb{Z}/2 \to \mathbb{Z}$ exists.

We will in the following circumvent this problem by relaxing under what conditions we consider two points of X to be 'the same': Instead of equality of points we will work with equivalence of points, as we will now explain.

Remark-Definition 3.41 (Continued)**.**

iii) Two points $x, y \in_{\mathcal{A}} X$, say $x \in X(A)$ and $y \in X(B)$, are *equivalent* if there exists for some object $C \in \mathrm{Ob}(\mathcal{A})$ epimorphisms $u\colon C \to A$ and $v\colon C \to B$ that make the square

$$\begin{tikzcd} C \arrow[r, dashed, "u"] \arrow[d, dashed, "v"'] & A \arrow[d, "x"] \\ B \arrow[r, "y"] & X \end{tikzcd}$$

commute. That the points x and y are equivalent is denoted by $x \equiv y$.

This concept of equivalence $\equiv$ defines an equivalence relation on the class of points of X: Every point $x \in_{\mathcal{A}} X$, say $x \in X(A)$, is equivalent to itself because square

$$\begin{tikzcd} A \arrow[r, dashed, "\mathrm{id}_A"] \arrow[d, dashed, "\mathrm{id}_A"'] & A \arrow[d, "x"] \\ A \arrow[r, "x"] & X \end{tikzcd}$$

commutes. The relation $\equiv$ is symmetric because the definition of $x \equiv y$ is symmetric in x and y. Let $x, y, z \in_{\mathcal{A}} X$, say

$$x \in X(A)\,, \quad y \in X(B)\,, \quad z \in X(C)\,,$$

with $x \equiv y$ and $y \equiv z$. Let

$$u\colon D \to A\,, \quad v\colon D \to B\,, \quad r\colon E \to B\,, \quad s\colon E \to C\,,$$

be epimorphism that make the squares

$$\begin{tikzcd} D \arrow[r, dashed, "u"] \arrow[d, dashed, "v"'] & A \arrow[d, "x"] \\ B \arrow[r, "y"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \arrow[r, dashed, "r"] \arrow[d, dashed, "s"'] & B \arrow[d, "y"] \\ C \arrow[r, "z"] & X \end{tikzcd}$$

commute. Together with the pullback square

$$\begin{tikzcd} F \arrow[r, dashed, "r'"] \arrow[d, dashed, "v'"'] \arrow[dr, phantom, "\lrcorner"] & D \arrow[d, "v"] \\ E \arrow[r, "r"] & B \end{tikzcd}$$

we get the following commutative diagram:

$$\begin{tikzcd} F \arrow[r, dashed, "r'"] \arrow[d, dashed, "v'"'] \arrow[dr, phantom, "\lrcorner"] & D \arrow[r, dashed, "u"] \arrow[d, dashed, "v"] & A \arrow[d, "x"] \\ E \arrow[r, dashed, "r"] \arrow[d, dashed, "s"'] & B \arrow[r, "y"] \arrow[d, "y"] & X \arrow[d, dashed, "\mathrm{id}_X"] \\ C \arrow[r, "z"] & X \arrow[r, dashed, "\mathrm{id}_X"] & X \end{tikzcd}$$

The two morphisms v' and r' are epimorphisms by Proposition 3.AG because the morphisms v and r are epimorphisms. The compositions $ur'\colon F \to A$ and $sv'\colon F \to C$ are therefore epimorphisms, and they make the outer square

$$\begin{tikzcd} F \arrow[r, dashed, "ur'"] \arrow[d, dashed, "sv'"'] & A \arrow[d, "x"] \\ C \arrow[r, "z"] & X \end{tikzcd}$$

commute. This shows that also $x \equiv z$.

iv) Let 0 be the zero morphism $0_{\mathcal{A}} \to X$. It holds for every point $x \in_{\mathcal{A}} X$ that $x \equiv 0$ if and only if there exists an epimorphism $u\colon B \to A$ that makes the square

$$\begin{tikzcd} B \arrow[r, dashed, "u"] \arrow[d, dashed] & A \arrow[d, "x"] \\ 0_{\mathcal{A}} \arrow[r] & X \end{tikzcd}$$

commute (because the zero morphism $B \to 0$ is automatically an epimorphism), i.e. such that $xu = 0_{B,X}$. It then follows that $x = 0_{A,X}$ because u is an epimorphism; and if on the other hand $x = 0_{A,X}$ then we can choose $B = A$ and $u = \mathrm{id}_A$ for the above commutative square. This shows that $x \equiv 0$ if and only if x is the zero morphism $A \to X$.

This shows in particular that the various zero morphisms $A \to X$ for $A \in \mathrm{Ob}(\mathcal{A})$ all give the same point of X, namely $0 \in_{\mathcal{A}} X$.

v) Let $Y \in \mathrm{Ob}(\mathcal{A})$ be another object and let $f\colon X \to Y$ be a morphism. Then f induces for every object $A \in \mathrm{Ob}(\mathcal{A})$ a map

$$X(A) \to Y(A)\,, \quad x \mapsto fx\,.$$

If $x, y \in_{\mathcal{A}} X$ with $x \equiv y$ then also $fx \equiv fy$: If $x \in X(A)$ and $y \in X(B)$ then there exist epimorphisms $u\colon C\colon A$ and $v\colon C \to B$ that make the square

$$\begin{tikzcd} C \arrow[r, dashed, "u"] \arrow[d, dashed, "v"'] & A \arrow[d, "x"] \\ B \arrow[r, "y"] & X \end{tikzcd}$$

commute. We then get the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "v"'] & A \arrow[d, "x"'] \arrow[ddr, dashed, bend left, "fx"] & \\ B \arrow[r, "y"] \arrow[drr, dashed, bend right, "fy"'] & X \arrow[dr, "f"] & \\ & & Y \end{tikzcd}$$

The commutativity of the outer square

$$\begin{tikzcd} C \arrow[r, dashed, "u"] \arrow[d, dashed, "v"'] & A \arrow[d, "fx"] \\ B \arrow[r, "fy"] & Y \end{tikzcd}$$

shows that $fx \equiv fy$.

Remark* 3.AM. Let $\mathcal{A}$ be an abelian category and let X be an object in $\mathcal{A}$.

If $x \in_{\mathcal{A}} X$, say $x \in X(A)$, and $u\colon A' \to A$ is an epimorphism, then $x \equiv xu$. This follows from the commutativity of the following square:

$$\begin{tikzcd} A' \arrow[r, dashed, "u"] \arrow[d, dashed, "\mathrm{id}_{A'}"'] & A \arrow[d, "x"] \\ A' \arrow[r, "xu"] & X \end{tikzcd}$$

The equivalence relation $\equiv$ is already determined by this property: Let $\equiv'$ be the equivalence relation on the class of points of X that is generated by $x \equiv' y$ for all $x, y \in_{\mathcal{A}} X$ for which there exists an epimorphism u with $y = xu$. Then the equivalence relations $\equiv$ and $\equiv'$ coincide.

Indeed, we have seen above that the equivalence relation $\equiv$ is finer than the equivalence relation $\equiv'$. Suppose on the other hand that $x, y \in_{\mathcal{A}} X$ are points with $x \equiv y$, say $x \in X(A)$ and $y \in X(B)$. Then let $u \colon C \to A$ and $v \colon C \to B$ be epimorphisms that make the square

$$\begin{tikzcd} C \arrow[r, dashed, "u"] \arrow[d, dashed, "v"'] & A \arrow[d, "x"] \\ B \arrow[r, "y"] & X \end{tikzcd}$$

commute. Then

$$x \equiv' xu = yv \equiv' y\,.$$

This shows that the equivalence relation $\equiv'$ is finer than the equivalence relation $\equiv$.

Theorem 3.42 (Rules for diagram chase)**.** Let $\mathcal{A}$ be an abelian category.

i) For a morphism $f \colon X \to Y$ in $\mathcal{A}$ the following three conditions are equivalent:
   a) The morphism f is a monomorphism.
   b) It follows for all $x, x' \in_{\mathcal{A}} X$ from $fx \equiv fx'$ that $x \equiv x'$.
   c) It follows for every $z \in_{\mathcal{A}} X$ from $fz \equiv 0$ that $z \equiv 0$.

ii) For a morphism $g \colon Y \to Z$ in $\mathcal{A}$ the following two conditions are equivalent:
   a) The morphism g is an epimorphism.
   b) There exists for every $z \in_{\mathcal{A}} Z$ some $y \in_{\mathcal{A}} Y$ with $gy \equiv z$.

iii) For a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ the following two conditions are equivalent:
   a) The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact.
   b) It holds that $gf = 0$ and there exists for every $y \in_{\mathcal{A}} Y$ with $gy \equiv 0$ some $x \in_{\mathcal{A}} X$ with $fx \equiv y$.

iv) Let $f \colon X \to Y$ be a morphism in $\mathcal{A}$ and let $x, x' \in_{\mathcal{A}} X$ be two points with $fx \equiv fx'$. Then there exists a point $\tilde{x} \in_{\mathcal{A}} X$ such that
   a) $f\tilde{x} \equiv 0$, and
   b) it holds for every morphism $h \colon X \to W$ that
      - if $hx' \equiv 0$ then $h\tilde{x} \equiv hx$, and
      - if $hx \equiv 0$ then $h\tilde{x} \equiv -hx'$.

Notation. The morphism $\tilde{x}$ from part iv) of Theorem 3.42 is denoted by $x - x'$.

Proof.

i) Suppose that f is a monomorphism and let $x, x' \in_{\mathcal{A}} X$ be two points with $fx \equiv fx'$, say $x \in X(A)$ and $x' \in X(A')$. Then there exist epimorphisms $u\colon B \to A$ and $v\colon B \to A'$ that make the square

$$\begin{tikzcd} B \arrow[rr, "u"] \arrow[dd, "v"'] & & A \arrow[d, "x"] \\ & & X \arrow[d, "f"] \\ A' \arrow[r, "x'"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

commute, i.e. such that $fxu = fx'v$. It follows from f being a monomorphism that already $xu = x'v$, i.e. that the square

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[d, "v"'] & A \arrow[d, "x"] \\ A' \arrow[r, "x'"] & X \end{tikzcd}$$

commutes. This shows that $x \equiv x'$.

Suppose that it follows for all points $x, x' \in_{\mathcal{A}} X$ from $fx \equiv fx'$ that $x \equiv x'$, and let $z \in_{\mathcal{A}} X$ with $fz \equiv 0$. Then $fz \equiv 0 = f \circ 0$ and hence $z \equiv 0$.

Suppose lastly that $z \equiv 0$ for every $z \in_{\mathcal{A}} X$ with $fz \equiv 0$. It then follows for every $x \in_{\mathcal{A}} X$ that

$$fx = 0 \implies fx \equiv 0 \implies x \equiv 0 \implies x = 0\,.$$

This shows that $\ker(f) = 0$ and hence that f is a monomorphism.

ii) Suppose that g is an epimorphism and let $z \in_{\mathcal{A}} Z$. We consider the following pullback square:

$$\begin{tikzcd} A' \arrow[r, "g'"] \arrow[d, "y"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "z"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

The morphism g' is by Proposition 3.AG again an epimorphism because g is an epimorphism. It therefore holds that

$$z \equiv zg' = gy\,.$$

Suppose on the other hand that for every point $z \in_{\mathcal{A}} Z$ there exists some $y \in_{\mathcal{A}} Y$ with $gy \equiv z$. By choosing $z = \mathrm{id}_Z$ we find that there exists some $y \in_{\mathcal{A}} Y$ with $gy \equiv \mathrm{id}_Z$, say $y \in X(A)$. There then exist epimorphisms $u\colon B \to Z$ and $v\colon B \to A$ that make the square

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[d, "v"'] & Z \arrow[d, "\mathrm{id}_Z"] \\ A \arrow[r, "gy"] & Z \end{tikzcd}$$

commute. The composition $gyv = u$ is therefore an epimorphism, hence the morphism g is an epimorphism.

End of lecture 13

iii) If $gf = 0$ then let $\lambda\colon \operatorname{im}(f) \to \ker(g)$ be the canonical morphism, i.e. the unique morphism $\lambda\colon \operatorname{im}(f) \to \ker(g)$ that makes the diagram

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "f'"'] & & Y \arrow[rr, "g"] & & Z \\
& \operatorname{im}(f) \arrow[ur, "i"] \arrow[rr, dashed, "\lambda"] & & \ker(g) \arrow[ul, "k"'] &
\end{tikzcd}
$$

commute. If $y \in_{\mathcal{A}} Y$ with $gy \equiv 0$ then $gy = 0$ and it follows from the universal property of the kernel $k\colon \ker(g) \to Y$ that there exist a unique morphism $\tilde{y}\colon A \to \ker(g)$ that makes the following diagram commute:

$$
\begin{tikzcd}
& & A \arrow[d, "y"'] \arrow[ddr, dashed, bend left, "\tilde{y}"] & & \\
X \arrow[rr, "f"] \arrow[dr, "f'"'] & & Y \arrow[rr, "g"] & & Z \\
& \operatorname{im}(f) \arrow[ur, "i"] \arrow[rr, "\lambda"] & & \ker(g) \arrow[ul, "k"'] &
\end{tikzcd}
$$

Suppose now that the sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact. Then the canonical morphism $\lambda\colon \operatorname{im}(f) \to \ker(g)$ is an isomorphism. Let $y \in_{\mathcal{A}} Y$ such that $gy \equiv 0$, hence $gy = 0$. Then

$$\lambda^{-1}\tilde{y} \in_A \operatorname{im}(f)\,.$$

The morphism $f'\colon X \to \operatorname{im}(f)$ is an epimorphism, hence it follows from part ii) that there exists some point $x \in_{\mathcal{A}} X$ with $f'x \equiv \lambda^{-1}\tilde{y}$. It holds for this point that

$$fx = if'x \equiv i\lambda^{-1}\tilde{y} = k\tilde{y} = y\,.$$

Suppose on the other hand that $gf = 0$ and that for every $y \in_{\mathcal{A}} Y$ with $gy \equiv 0$ there exist some $x \in_{\mathcal{A}} X$ with $fx = y$. We need to show that the morphism $\lambda\colon \operatorname{im}(f) \to \ker(g)$ is an isomorphism. We already know that λ is a monomorphism (as this is always the case), so it remains to show that λ is an epimorphism. For this we use part ii): Let $\tilde{y} \in_{\mathcal{A}} \ker(g)$ and set $y := k\tilde{y} \in_{\mathcal{A}} Y$. Then

$$gy = gk\tilde{y} = 0 \circ \tilde{y} = 0\,.$$

It follows by assumption that there exist some point $x \in_{\mathcal{A}} X$ with $fx \equiv y$. For the point
$$\tilde{x} := f'x \in_{\mathcal{A}} \operatorname{im}(f)$$
we have that $\lambda\tilde{x} = \tilde{y}$; indeed, we have that
$$k\lambda\tilde{x} = i\tilde{x} = if'x = fx \equiv y = k\tilde{y}\,,$$
and hence $\lambda\tilde{x} \equiv \tilde{y}$ because k is a monomorphism (by part i)). This shows by part ii) that λ is an epimorphism.

iv) Let $x, x' \in_{\mathcal{A}} X$ with $fx \equiv fx'$, say $x \in X(A)$ and $x' \in X(A')$. It follows from $fx \equiv fx'$ that there exist epimorphisms $u \colon B \to A$ and $v \colon B \to A'$ that make the diagram
$$\begin{tikzcd} B \arrow[r, "u"] \arrow[dd, "v"'] & A \arrow[d, "x"] \\ & X \arrow[d, "f"] \\ A' \arrow[r, "x'"] & X \arrow[r, "f"] & Y \end{tikzcd}$$
commute. It follows for the element
$$\tilde{x} := xv - x'u \in X(B)$$
that
$$f\tilde{x} = f(xv - x'u) = fxv - fx'u = 0$$
and hence $f\tilde{x} \equiv 0$. If $h \colon X \to W$ is a morphism with $hx' \equiv 0$ then $hx' = 0$, hence
$$h\tilde{x} = h(xv - x'u) = hxv - hx'u = hxv \equiv hx$$
because v is an epimorphism. We similarly find that if $hx \equiv 0$ then $h\tilde{x} \equiv -hx'$. $\square$

Lemma 3.43 (Snake lemma)**.** Let $\mathcal{A}$ be an abelian category. Let
$$\begin{tikzcd} & X' \arrow[r, "i"] \arrow[d, "f'"] & X \arrow[r, "p"] \arrow[d, "f"] & X'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & Y' \arrow[r, "j"'] & Y \arrow[r, "q"'] & Y'' \end{tikzcd} \tag{3.10}$$
be a commutative diagram in $\mathcal{A}$ with exact rows. Let
$$k' \colon \ker(f') \to X'\,, \quad k \colon \ker(f) \to X\,, \quad k'' \colon \ker(f'') \to X''$$
be kernels of f', f and f'', and let
$$\tilde{i} \colon \ker(f') \to \ker(f) \quad \text{and} \quad \tilde{p} \colon \ker(f) \to \ker(f'')$$

be the unique morphisms that make the following diagram commute:

$$
\begin{tikzcd}
\ker(f') \arrow[r, dashed, "\tilde{i}"] \arrow[d, "k'"] & \ker(f) \arrow[r, dashed, "\tilde{p}"] \arrow[d, "k"] & \ker(f'') \arrow[d, "k''"] \\
X' \arrow[r, "i"] & X \arrow[r, "p"] & X''
\end{tikzcd}
$$

Dually, let

$$
c' \colon Y' \to \operatorname{coker}(f') \,, \quad c \colon Y \to \operatorname{coker}(f) \,, \quad c'' \colon Y'' \to \operatorname{coker}(f'')
$$

be cokernels of f', f and f'', and let

$$
\bar{j} \colon \operatorname{coker}(f') \to \operatorname{coker}(f) \quad \text{and} \quad \bar{q} \colon \operatorname{coker}(f) \to \operatorname{coker}(f'')
$$

be the unique morphisms that make the following diagram commute:

$$
\begin{tikzcd}
Y' \arrow[r, "j"] \arrow[d, "c'"] & Y \arrow[r, "q"] \arrow[d, "c"] & Y'' \arrow[d, "c''"] \\
\operatorname{coker}(f') \arrow[r, dashed, "\bar{j}"] & \operatorname{coker}(f) \arrow[r, dashed, "\bar{q}"] & \operatorname{coker}(f'')
\end{tikzcd}
$$

(See Remark⋆ 3.O for a more thorough explanation on these induced morphisms.)

i) There exists a morphism

$$
\delta \colon \ker(f'') \to \operatorname{coker}(f')
$$

such that the sequence

$$
\begin{tikzcd}
\ker(f') \arrow[r, "\tilde{i}"] & \ker(f) \arrow[r, "\tilde{p}"] & \ker(f'') \arrow[dll, "\delta"] \\
\operatorname{coker}(f') \arrow[r, "\bar{j}"] & \operatorname{coker}(f) \arrow[r, "\bar{q}"] & \operatorname{coker}(f'')
\end{tikzcd}
\tag{3.11}
$$

is exact.

ii) If i is a monomorphism then $\tilde{i} \colon \ker(f') \to \ker(f)$ is again a monomorphism, and if q is an epimorphism then $\bar{q} \colon \operatorname{coker}(f) \to \operatorname{coker}(f'')$ is again an epimorphism.

Proof. The proof of the snake lemma is currently missing from these notes, but will be added in the future. ☐

Remark⋆ 3.AN.

i) One can also draw the exact sequence (3.11) into the diagram (3.10). This then results in the following diagram:

$$
\begin{tikzcd}
 & \ker(f') \arrow[r, dashed] \arrow[d] & \ker(f) \arrow[r, dashed] \arrow[d] & \ker(f'') \arrow[d] \arrow[dddll, dashed, rounded corners, "\delta"] & \\
 & X' \arrow[r, "i"] \arrow[d, "f'"'] & X \arrow[r, "p"] \arrow[d, "f"'] & X'' \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & Y' \arrow[r, "j"] \arrow[d] & Y \arrow[r, "q"] \arrow[d] & Y'' \arrow[d] & \\
 & \operatorname{coker}(f') \arrow[r, dashed] & \operatorname{coker}(f) \arrow[r, dashed] & \operatorname{coker}(f'') &
\end{tikzcd}
$$

The name ‘snake lemma’ stems from the form of this dashed sequence.

ii) Part ii) of the snake lemma states that the sequence

$$
\begin{tikzcd}
0 \arrow[r, dashed] & \ker(f') \arrow[r] & \ker(f) \arrow[r] & \ker(f'') \arrow[dll, rounded corners, "\delta"] & \\
 & \operatorname{coker}(f') \arrow[r] & \operatorname{coker}(f) \arrow[r] & \operatorname{coker}(f'') \arrow[r, dotted] & 0
\end{tikzcd}
$$

inherits on the left and right the same additional exactness conditions as the original diagram:

$$
\begin{tikzcd}
0 \arrow[r, dashed] & X' \arrow[r, "i"] \arrow[d, "f'"] & X \arrow[r, "p"] \arrow[d, "f"] & X'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & Y' \arrow[r, "j"'] & Y \arrow[r, "q"'] & Y'' \arrow[r, dotted] & 0
\end{tikzcd}
$$

iii) The morphism δ from the snake lemma is known as the ‘connecting morphism’.

iv) The connecting morphism in natural in the following sense: Suppose that we are given a commutative diagram of the following form, where the four horizontal rows are exact.

$$
\begin{tikzcd}
& & & X' \arrow[rr, "i"] \arrow[dl, dashed, "\alpha'"'] \arrow[dd, "f'"] & & X \arrow[rr, "p"] \arrow[dl, dashed, "\alpha"'] \arrow[dd, "f"] & & X'' \arrow[rr] \arrow[dl, dashed, "\alpha''"'] \arrow[dd, "f''"] & & 0 \\
& & W' \arrow[rr, "i'"] \arrow[dd, "g'"'] & & W \arrow[rr, "p'"] \arrow[dd, "g"'] & & W'' \arrow[rr] \arrow[dd, "g''"'] & & 0 & \\
& 0 \arrow[rr] & & Y' \arrow[rr, "j"'] \arrow[dl, dashed, "\beta'"] & & Y \arrow[rr, "q"'] \arrow[dl, dashed, "\beta"] & & Y'' \arrow[dl, dashed, "\beta''"] & & \\
0 \arrow[rr] & & Z' \arrow[rr, "j'"'] & & Z \arrow[rr, "q'"'] & & Z'' & & &
\end{tikzcd}
$$

We can then apply the snake lemma to both the back and the front of this diagram. It then follows from the commutativity of the square

$$
\begin{tikzcd}
X'' \arrow[r, dashed, "\alpha''"] \arrow[d, "f''"'] & W'' \arrow[d, "g''"] \\
Y'' \arrow[r, dashed, "\beta''"] & Z''
\end{tikzcd}
$$

that α'' induces a unique morphism $\ker(f'') \to \ker(g'')$ that make the square

$$
\begin{tikzcd}
\ker(f'') \arrow[r, dashed] \arrow[d] & \ker(g'') \arrow[d] \\
X'' \arrow[r, dashed, "\alpha''"] & W''
\end{tikzcd}
$$

commute. Similarly, the commutativity of the square

$$
\begin{tikzcd}
X' \arrow[r, dashed, "\alpha'"] \arrow[d, "f'"'] & W' \arrow[d, "g'"] \\
Y' \arrow[r, dashed, "\beta'"] & Z'
\end{tikzcd}
$$

shows that β' induces a unique morphism $\operatorname{coker}(f') \to \operatorname{coker}(g')$ that makes the square

$$
\begin{tikzcd}
Y' \arrow[r, dashed, "\beta'"] \arrow[d] & Z' \arrow[d] \\
\operatorname{coker}(f') \arrow[r, dashed] & \operatorname{coker}(g')
\end{tikzcd}
$$

commute. (See Remark* 3.O for a more detailed explanation on this.) The connecting morphisms $\delta\colon \ker(f'') \to \operatorname{coker}(f')$ and $\delta' \ker(g'') \to \operatorname{coker}(g')$ then make the square

$$\begin{tikzcd} \ker(f'') \arrow[r, "\delta"] \arrow[d, dashed] & \operatorname{coker}(f') \arrow[d, dashed] \\ \ker(g'') \arrow[r, "\delta'"] & \operatorname{coker}(g') \end{tikzcd}$$

commute.

It then follows more more generally that the diagram

$$\begin{tikzcd} \ker(f') \arrow[r] \arrow[d, dashed] & \ker(f) \arrow[r] \arrow[d, dashed] & \ker(f'') \arrow[r, "\delta"] \arrow[d, dashed] & \operatorname{coker}(f') \arrow[r] \arrow[d, dashed] & \operatorname{coker}(f) \arrow[r] \arrow[d, dashed] & \operatorname{coker}(f'') \arrow[d, dashed] \\ \ker(g') \arrow[r] & \ker(g) \arrow[r] & \ker(g'') \arrow[r, "\delta'"] & \operatorname{coker}(g') \arrow[r] & \operatorname{coker}(g) \arrow[r] & \operatorname{coker}(g'') \end{tikzcd}$$

commutes, where the rows are the exact sequences from the snake lemma. The vertical morphisms in this diagram are induced in the same way as $\ker(f'') \to \ker(g'')$ and $\operatorname{coker}(f') \to \operatorname{coker}(g')$, as explained above.

One may also think about this naturality of the connecting morphism as the functoriality of the exact squences from the snake lemma.

Definition 3.44. Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$. The functor F is

i) exact, resp.

ii) left exact, resp.

iii) right exact,

if for every short exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$

i) the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is again (short) exact, resp.

ii) the sequence $0 \to F(X') \to F(X) \to F(X'')$ is again (left) exact, resp.

iii) the sequence $F(X') \to F(X) \to F(X'') \to 0$ is again (right) exact.

Remark 3.45. Let $F: \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$.

i) Left exactness can be detected via left exact sequences: The functor F is left exact if and only if for every (left) exact sequence $0 \to X' \to X \to X''$ in $\mathcal{A}$ the sequence $0 \to F(X') \to F(X) \to F(X'')$ in $\mathcal{B}$ is again (left) exact.

ii) Right exactness can similarly be detected via right exact sequences: The functor F is right exact if and only if for every (right) exact sequence $X' \to X \to X'' \to 0$ in $\mathcal{A}$ the sequence $F(X') \to F(X) \to F(X'') \to 0$ in $\mathcal{B}$ is again (right) exact.

Remark⋆ 3.AO.

i) In other words, a functor $F: \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}$ and $\mathcal{B}$ is left exact (resp. right exact) if and only if it respects kernels (resp. cokernels); this follows from the characterization of left exact sequences via kernels and right exact sequences via cokernels, as explained in Remark⋆ 3.AK.

ii) If a functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}$ and $\mathcal{B}$ is exact then for every exact sequence
$$\cdots \to X_{i-1} \to X_i \to X_{i+1} \to \cdots$$
in $\mathcal{A}$ the sequence
$$\cdots \to F(X_{i-1}) \to F(X_i) \to F(X_{i+1}) \to \cdots$$
in $\mathcal{B}$ is again exact.

Example 3.46. Let $\mathcal{A}$ be an abelian category.

i) For every object $X \in \mathrm{Ob}(\mathcal{A})$ the covariant Hom-functor
$$\mathrm{Hom}_{\mathcal{A}}(X, -)\colon \mathcal{A} \to \mathbf{Ab}$$
is left exact.

ii) For every object $X \in \mathrm{Ob}(\mathcal{A})$ the contravariant Hom-functor
$$\mathrm{Hom}_{\mathcal{A}}(-, X)\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$$
is left exact.

iii) Let A be a $\mathbf{k}$-algebra. Then for every left A-module ${}_AN$ the functor
$$- \otimes_A N\colon \mathbf{Mod}\text{-}A \to \mathbf{k}\text{-}\mathbf{Mod}$$
is right exact. Similarly, for every right A-module M_A the functor
$$M \otimes_A -\colon A\text{-}\mathbf{Mod} \to \mathbf{k}\text{-}\mathbf{Mod}$$
is right exact.

iv) Let X be a topological space. The inclusion functor $I\colon \mathbf{Sh}(X) \to \mathbf{PSh}(X)$ is left exact. But it is not right exact because it does not respect cokernels. The sheafification functor $S\colon \mathbf{PSh}(X) \to \mathbf{Sh}(X)$ is exact.

v) Let again X be a topological space. For every presheaf $\mathscr{F}$ on X let
$$\Gamma(X, \mathscr{F}) \coloneqq \mathscr{F}(X).^{6}$$
Then $\Gamma(X, -)\colon \mathbf{PSh}(X) \to \mathbf{Ab}$ is the *global section functor*, and this functor is exact. The restriction $\Gamma(X, -)\colon \mathbf{Sh}(X) \to \mathbf{Ab}$ is again left exact, but not right exact.

End of lecture 14

[6] A more general notation is $\Gamma(U, \mathscr{F}) \coloneqq \mathscr{F}(U)$ for every open subseteq $U \subseteq X$ and every presheaf $\mathscr{F}$ on X.

Lemma* 3.AP (5-Lemma)**.** Let $\mathcal{A}$ be an abelian category

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & X_3 \arrow[r] \arrow[d, "f_3"] & X_4 \arrow[r] \arrow[d, "f_4"] & X_5 \arrow[d, "f_5"] \\ Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & Y_4 \arrow[r] & Y_5 \end{tikzcd}$$

be a commutative diagram in $\mathcal{A}$ with exact rows.

i) If f_2 and f_4 are monomorphisms and f_1 is an epimorphism then f_3 is again a monomorphism.

ii) If f_2 and f_4 are epimorphisms and f_5 is a monomorphism then f_3 is again an epimorphism.

iii) If f_2 and f_4 are isomorphisms, f_1 is an epimorphism and f_5 is a monomorphism, then f_3 is again an isomorphism.

Proof. This is Exercise 3 of Exercise sheet 7. □

Lemma 3.47 (9-Lemma)**.** Let $\mathcal{A}$ be an abelian category and let

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

be a commutative diagram in $\mathcal{A}$ with exact columns.

i) If the upper two rows are exact then the lower row is also exact.

ii) If the lower two rows are exact then the upper row is also exact.

iii) If the upper and lower rows are exact and the composition $Y' \to Y \to Y''$ is zero then the middle row is also exact.

Proof. This is Exercise 4 of Exercise sheet 7. □

4 Complexes

Convention. In the following $\mathcal{A}$ denotes an abelian category.

Chain and Cochain Complexes

Definition 4.1.

i) A *chain complex* in $\mathcal{A}$ is a pair $C_\bullet = ((C_n)_{n\in\mathbb{Z}}, (d_n)_{n\in\mathbb{Z}})$ consisting of
   - a family of objects $C_n \in \mathrm{Ob}(\mathcal{A})$, where $n \in \mathbb{Z}$, and
   - a family of morphisms $d_n \colon C_n \to C_{n-1}$, where again $n \in \mathbb{Z}$,

   subject to the condition $d_{n-1}d_n = 0$ for every $n \in \mathbb{Z}$. The family $d = (d_n)_{n\in\mathbb{Z}}$ is the *differential* of $C_\bullet$.

ii) Dually, a *cochain complex* in $\mathcal{A}$ is a par $C^\bullet = ((C^n)_{n\in\mathbb{Z}}, (d^n)_{n\in\mathbb{Z}})$ consisting of
   - a family of objects $C^n \in \mathrm{Ob}(\mathcal{A})$, where $n \in \mathbb{Z}$, and
   - a family of morphisms $d^n \colon C^n \to C^{n+1}$, where again $n \in \mathbb{Z}$,

   subject to the condition $d^{n+1}d^n = 0$ for every $n \in \mathbb{Z}$. The family $d \coloneqq (d^n)_{n\in\mathbb{Z}}$ is the *differential* of $C^\bullet$.[1]

iii) Let $C_\bullet$ and $D_\bullet$ be two chain complexes in $\mathcal{A}$. A *morphism* of chain complexes $f \colon C_\bullet \to D_\bullet$ is a family $f = (f_n)_{n\in\mathbb{Z}}$ of morphisms $f_n \colon C_n \to D_n$ such that '$df = fd$', i.e. such that for every $n \in \mathbb{Z}$ the following diagram commutes:

$$\begin{tikzcd} C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "d_n"] & D_{n-1} \end{tikzcd}$$

   We obtain with this notion of morphism a category $\mathbf{Ch}_\bullet(\mathcal{A})$ of chain complexes in $\mathcal{A}$.

   The notion of a morphism of cochain complexes is defined similarly, and we obtain a category $\mathbf{Ch}^\bullet(\mathcal{A})$ of cochain complexes in $\mathcal{A}$.

Remark* 4.A. If $f = (f_n)_n \colon C_\bullet \to D_\bullet$ is a morphism of chain complexes such that f_n is for every $n \in \mathbb{Z}$ an isomorphism then $(f_n^{-1})_{n\in\mathbb{Z}}$ is a morphism of chain complexes $D_\bullet \to C_\bullet$. The morphism f is therefore an isomorphism if and only if f_n is for every $n \in \mathbb{Z}$ an isomorphism.

[1] The author thinks that d ought to be called *codifferential* instead.

Remark 4.2.

i) A cochain complexes in $\mathcal{A}$ is the same as a chain complex in $\mathcal{A}^{\mathrm{op}}$ shifted by 1: If $C^\bullet$ is a cochain complex in $\mathcal{A}$ then $\widetilde{C}_\bullet = ((\widetilde{C}_n)_{n\in\mathbb{Z}}, (\tilde{d}_n)_{n\in\mathbb{Z}})$ given by $\widetilde{C}_n := C^n$ and $\tilde{d}_n := d^{n-1}$ for every $n \in \mathbb{Z}$ is a chain complex in $\mathcal{A}^{\mathrm{op}}$.
   To every chain complex $C_\bullet$ in $\mathcal{A}$ we can also associate a cochain complex $\widetilde{C}^\bullet$ in $\mathcal{A}$ by setting $\widetilde{C}^n := C_{-n}$ for every $n \in \mathbb{Z}$, as well as $\tilde{d}^n := d_{-n}$ for every $n \in \mathbb{Z}$.
   These constructions can be extended to equivalences of categories

$$\mathbf{Ch}^\bullet(\mathcal{A}) \simeq \mathbf{Ch}_\bullet(\mathcal{A}^{\mathrm{op}}) \quad \text{and} \quad \mathbf{Ch}_\bullet(\mathcal{A}) \simeq \mathbf{Ch}^\bullet(\mathcal{A})$$

ii) One can also describe chain complexes as *differential graded objects*:
   Suppose that the category $\mathcal{A}$ has countable coproducts, which will be denoted by $\bigoplus$. A $\mathbb{Z}$-graded object of $\mathcal{A}$ is an object C together with a decomposition $C = \bigoplus_{n\in\mathbb{Z}} C_n$ for some countable family $(C_n)_{n\in\mathbb{Z}}$ of objects $C_n \in \mathrm{Ob}(\mathcal{A})$.[2]
   If $C = \bigoplus_{n\in\mathbb{Z}}$ and $D = \bigoplus_{n\in\mathbb{Z}} D_n$ are $\mathbb{Z}$-graded objects in $\mathcal{A}$, then a morphism $C \to D$ (in $\mathcal{A}$) is *homogeneous* of degree $d \in \mathbb{Z}$ if for every $n \in \mathbb{Z}$ the morphism f 'restricts to a morphism $C_n \to D_{n+d}$', in the sense that there exist a morphism $f_n \colon C_n \to D_{n+d}$ that makes the following square commute:

$$\begin{tikzcd} C \arrow[r, "f"] & D \\ C_n \arrow[u] \arrow[r, "f_n"] & D_{n+d} \arrow[u] \end{tikzcd}$$

   A *morphism of* $\mathbb{Z}$*-graded objects* $f \colon C \to D$ is a homogeneous morphism of degree 0.
   If $C_\bullet$ is a chain complex in $\mathcal{A}$ then $C := \bigoplus_{n\in\mathbb{Z}} C_n$ is (together with this decomposition) a $\mathbb{Z}$-graded object in $\mathcal{A}$, and the differential $(d_n)_{n\in\mathbb{N}}$ induces a morphism $d \colon C \to C$ of degree -1 with $d^2 = 0$; this morphism is the unique one that makes the following square commute:

$$\begin{tikzcd} C \arrow[r, "d"] & C \\ C_n \arrow[u] \arrow[r, "d_{n-1}"] & C_{n-1} \arrow[u] \end{tikzcd}$$

   Every morphism of chain complexes $(f_n)_{n\in\mathbb{Z}} \colon C_\bullet \to D_\bullet$ results in a morphism of $\mathbb{Z}$-graded objects $f \colon C \to D$ in $\mathcal{A}$, namely the unique morphism $C \to D$ such that the following square commutes for every $n \in \mathbb{Z}$:

$$\begin{tikzcd} C \arrow[r, dashed, "f"] & D \\ C_n \arrow[u] \arrow[r, "f_n"] & D_n \arrow[u] \end{tikzcd}$$

[2] The author thinks that it is more appropiate to define the described $\mathbb{Z}$-graded object as simply the family $(C_n)_{n\in\mathbb{Z}}$ itself.

(See Remark⋆ 3.F for more details on this induced morphism and its functoriality.) This morphism then satisfies $df = fd$. Indeed, we have for every $n \in \mathbb{Z}$ the following diagram:

$$\begin{tikzcd}
C_n \arrow[rrr, "d_n"] \arrow[dr, "i_n"] \arrow[ddd, "f_n"'] & & & C_{n-1} \arrow[dl, "i_{n-1}"'] \arrow[ddd, "f_{n-1}"] \\
& C \arrow[r, "d"] \arrow[d, "f"'] & C \arrow[d, "f"] & \\
& D \arrow[r, "d"] & D & \\
D_n \arrow[rrr, "d_n"] \arrow[ur, "j_n"'] & & & D_{n-1} \arrow[ul, "j_{n-1}"]
\end{tikzcd}$$

The four trapezoids commute by the definitions of d,d and f, and the outer sqare commutes because $(f_n)_{n \in \mathbb{Z}}$ is a morphism of chain complexes. It follows that the inner square also commutes; indeed, it follows that for every $n \in \mathbb{Z}$ that

$$df i_n = dj_n f_n = j_{n-1} d_n f_n = j_{n-1} f_{n-1} d_n = f i_{n-1} d_n = f d i_n \,,$$

and hence overall that $df = fd$ by the universal property of the coproduct C

This leads us to consider the category $\mathcal{C}$ where

- an object of $\mathcal{C}$ is a pair (C, d) consisting of a $\mathbb{Z}$-graded object C and a homogeneous morphism $d\colon C \to C$ of degree -1 with $d^2 = 0$, and
- a morphism $f\colon (C, d) \to (D, d)$ is a morphism $f\colon C \to D$ of $\mathbb{Z}$-graded objects such that $df = fd$.

We have above constructed a functor $F\colon \mathbf{Ch}_\bullet(\mathcal{A}) \to \mathcal{C}$, and this functor is an equivalence of categories. The objects of $\mathcal{C}$ are called *differential graded objects* in $\mathcal{A}$.

Definition 4.3.

i) Let $C_\bullet$ be a chain complex. For every $n \in \mathbb{Z}$, the *n-th cycle object* of $C_\bullet$ is

$$\mathrm{Z}_n(C_\bullet) \coloneqq \ker(d_n) \,,$$

and the *n-th boundary object* of $C_\bullet$ is

$$\mathrm{B}_n(C_\bullet) \coloneqq \mathrm{im}(d_{n+1}) \,.$$

It follows for every $n \in \mathbb{Z}$ from $d_n d_{n+1} = 0$ that there exist a unique morphism $\mathrm{B}_n(C_\bullet) \to \mathrm{Z}_n(C_\bullet)$ that makes the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots \\
& \mathrm{B}_n(C_\bullet) \arrow[ur] \arrow[r, dashed] & \mathrm{Z}_n(C_\bullet) \arrow[u] & &
\end{tikzcd}$$

commute, and this morphism is a monomorphism. The *n-th homology object* of the chain complex $C_\bullet$ is

$$\mathrm{H}_n(C_\bullet) \coloneqq \operatorname{coker}(\mathrm{B}_n(C_\bullet) \to \mathrm{Z}_n(C_\bullet)) .$$

(One may think about $\mathrm{H}_n(C_\bullet)$ as a quotient $\mathrm{H}_n(C_\bullet) = \mathrm{Z}_n(C_\bullet)/\mathrm{B}_n(C_\bullet)$.)

ii) Dually, let $C^\bullet$ be a cochain complex in $\mathcal{A}$. For every $n \in \mathbb{Z}$, the *n-th cocycle object* of $C^\bullet$ is

$$\mathrm{Z}^n(C^\bullet) \coloneqq \ker(d^n) ,$$

and the *n-th coboundary object* of $C^\bullet$ is

$$\mathrm{B}^n(C^\bullet) \coloneqq \operatorname{im}(d^{n-1}) .$$

The *n-th cohomology object* of $C^\bullet$ is

$$\mathrm{H}^n(C^\bullet) \coloneqq \operatorname{coker}(\mathrm{B}^n(C^\bullet) \to \mathrm{Z}^n(C^\bullet))$$

where $\mathrm{B}^n(C^\bullet) \to \mathrm{Z}^n(C^\bullet)$ is the unique morphism that makes the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1} \arrow[rr, "d^{n-1}"] \arrow[dr] & & C^n \arrow[rr, "d^n"] & & C^{n+1} \arrow[r] & \cdots \\
& & \mathrm{B}^n(C^\bullet) \arrow[ur] \arrow[rr, dashed] & & \mathrm{Z}^n(C^\bullet) \arrow[ul] & &
\end{tikzcd}$$

Example⋆ 4.B. Let A be a $\mathbf{k}$-algebra and let $C_\bullet$ be a chain complex of A-modules, i.e. a chain complex in the abelian category A-**Mod**. Then $\mathrm{B}_n(C_\bullet) \subseteq \mathrm{Z}_n(C_\bullet) \subseteq C_n$ are submodules for every $n \in \mathbb{Z}$, whose quotient $\mathrm{H}_n(C_\bullet) = \mathrm{Z}_n(C_\bullet)/\mathrm{B}_n(C_\bullet)$ is the n-th homology of $C_\bullet$. The elements of $\mathrm{Z}_n(C_\bullet)$ are the *n-cycles* of $C_\bullet$, and the elements of $\mathrm{B}_n(C_\bullet)$ are the *n-boundaries* of $C_\bullet$.

Remark⋆ 4.C. Let $C_\bullet$ be a chain complex in $\mathcal{A}$ and let $n \in \mathbb{Z}$. Then the canonical morphism $C_{n+1} \to \mathrm{B}_n(C_\bullet)$ is an epimorphism, and hence

$$\mathrm{H}_n(C_\bullet) = \operatorname{coker}(\mathrm{B}_n(C_\bullet) \to \mathrm{Z}_n(C_\bullet)) = \operatorname{coker}(C_{n+1} \to \mathrm{B}_n(C_\bullet) \to \mathrm{Z}_n(C_\bullet)) .$$

The morphism $C_{n+1} \to \mathrm{Z}_n(C_\bullet)$ is induced by the differential $d_{n+1} \colon C_{n+1} \to C_n$ via the universal property of the kernel, which can be used because $d_{n+1}d_n = 0$.

Remark 4.4 (Functoriality of homology).

i) Let $f = (f_n)_n \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. It follows for every $n \in \mathbb{Z}$ from the commutativity of the square

$$\begin{tikzcd}
C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_n \arrow[r, "d_n"] & D_{n-1}
\end{tikzcd}$$

that there exists a unique induced morphism $\mathrm{Z}_n(f)\colon \mathrm{Z}_n(C_\bullet) \to \mathrm{Z}_n(D_\bullet)$ that makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{Z}_n(C_\bullet) \arrow[r] \arrow[d, dashed, "\mathrm{Z}_n(f)"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\
\mathrm{Z}_n(D_\bullet) \arrow[r] & D_n \arrow[r, "d_n"] & D_{n-1}
\end{tikzcd}$$

This induced morphism is functorial in the following sense:

- If $D_\bullet = C_\bullet$ and $f = \mathrm{id}_{C_\bullet}$ then $\mathrm{Z}_n(\mathrm{id}_{C_\bullet}) = \mathrm{id}_{\mathrm{Z}_n(C_\bullet)}$ for every $n \in \mathbb{Z}$.
- If $E_\bullet$ is another chain complex and $g\colon D_\bullet \to E_\bullet$ is another morphism of chain complexes, then
$$\mathrm{Z}_n(g \circ f) = \mathrm{Z}_n(g) \circ \mathrm{Z}_n(f)$$
for every $n \in \mathbb{Z}$.
- If $g\colon C_\bullet \to D_\bullet$ is another morphisms of chain complexes that is parallel to the morphism f, then
$$\mathrm{Z}_n(f + g) = \mathrm{Z}_n(f) + \mathrm{Z}_n(g)\,.$$
for every $n \in \mathbb{Z}$.

(See Remark⋆ 3.O for more details on this induced morphism and its functoriality and additivity.)

ii) It also follows for every $n \in \mathbb{Z}$ from the commutativity of the square

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[d, "f_n"] \\
D_{n+1} \arrow[r, "d_{n+1}"] & D_n
\end{tikzcd}$$

that there exists a unique induced morphism $\mathrm{B}_n(f)\colon \mathrm{B}_n(C_\bullet) \to \mathrm{B}_n(D_\bullet)$ that makes the following diagram commute:

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r] \arrow[d, "f_n"] & \mathrm{B}_n(C_\bullet) \arrow[d, dashed, "\mathrm{B}_n(f)"] \\
D_{n+1} \arrow[r, "d_{n+1}"] & D_n \arrow[r] & \mathrm{B}_n(D_\bullet)
\end{tikzcd}$$

This induced morphism is functorial in the following sense:

- If $D_\bullet = C_\bullet$ and $f = \mathrm{id}_{C_\bullet}$ then $\mathrm{B}_n(\mathrm{id}_{C_\bullet}) = \mathrm{id}_{\mathrm{B}_n(C_\bullet)}$ for every $n \in \mathbb{Z}$.
- If $E_\bullet$ is another chain complex and $g\colon D_\bullet \to E_\bullet$ is another morphism of chain complexes, then
$$\mathrm{B}_n(g \circ f) = \mathrm{B}_n(g) \circ \mathrm{B}_n(f)$$
for every $n \in \mathbb{Z}$.

- If $g\colon C_\bullet \to D_\bullet$ is another morphism of chain complexes that is parallel to the morphism g then
$$\mathrm{B}_n(f+g) = \mathrm{B}_n(f) + \mathrm{B}_n(g)$$
for every $n \in \mathbb{Z}$.

(See Remark* 3.S for more details on this induced morphism and its functoriality and additivity.)

iii) We get from the above for every $n \in \mathbb{Z}$ the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "p"] \arrow[dd, "f_{n+1}"'] & & C_n \arrow[rr, "d_n"] \arrow[dd, "f_n"] & & C_{n-1} \arrow[r] \arrow[dd, "f_{n-1}"] & \cdots \\
& & \mathrm{B}_n(C_\bullet) \arrow[ur, "i"] \arrow[rr, dashed, "\lambda"] \arrow[dd, dashed, "\mathrm{B}_n(f)"] & & \mathrm{Z}_n(C_\bullet) \arrow[ul, "j"'] \arrow[dd, dashed, "\mathrm{Z}_n(f)"'] & & \\
\cdots \arrow[r] & D_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "p'"'] & & D_n \arrow[rr, "d_n"] & & D_{n-1} \arrow[r] & \cdots \\
& & \mathrm{B}_n(D_\bullet) \arrow[ur, "i'"] \arrow[rr, dashed, "\lambda'"] & & \mathrm{Z}_n(D_\bullet) \arrow[ul, "j'"'] & &
\end{tikzcd}$$

This diagram commutes: It remains to show that the dashed square in the front commutes. This holds because

$$\begin{aligned} j'\,\mathrm{Z}_n(f)\lambda p = f_n j \lambda p = f_n i p = f_n d_{n+1} = d_{n+1} f_{n+1} = i' p' f_{n+1} &= j' \lambda' p' f_{n+1} \\ &= j' \lambda'\,\mathrm{B}_n(f) p\,, \end{aligned}$$

and hence $\mathrm{Z}_n(f)\lambda = \lambda'\,\mathrm{B}_n(f)$ because p is an epimorphism and j' is a monomorphism.

It follows from this commutativity of the frontal square

$$\begin{tikzcd}
\mathrm{B}_n(C_\bullet) \arrow[r, "\lambda"] \arrow[d, "\mathrm{B}_n(f)"'] & \mathrm{Z}_n(C_\bullet) \arrow[d, "\mathrm{Z}_n(f)"] \\
\mathrm{B}_n(D_\bullet) \arrow[r, "\lambda'"] & \mathrm{Z}_n(D_\bullet)
\end{tikzcd}$$

that there exists a unique induced morphism $\mathrm{H}_n(f)\colon \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(D_\bullet)$ that makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{B}_n(C_\bullet) \arrow[r, "\lambda"] \arrow[d, "\mathrm{B}_n(f)"'] & \mathrm{Z}_n(C_\bullet) \arrow[r] \arrow[d, "\mathrm{Z}_n(f)"] & \mathrm{H}_n(C_\bullet) \arrow[d, dashed, "\mathrm{H}_n(f)"] \\
\mathrm{B}_n(D_\bullet) \arrow[r, "\lambda'"] & \mathrm{Z}_n(D_\bullet) \arrow[r] & \mathrm{H}_n(D_\bullet)
\end{tikzcd}$$

It follows from the functorialty of both B_n and Z_n, together with the functoriality of the cokernel (as explained in Remark* 3.O), that this induced morphism is both functorial and additive in the following sense:

- If $D_\bullet = C_\bullet$ and $f = \mathrm{id}_{C_\bullet}$ then $\mathrm{H}_n(\mathrm{id}_{C_\bullet}) = \mathrm{id}_{\mathrm{H}_n(C_\bullet)}$ for every $n \in \mathbb{Z}$.
- If $E_\bullet$ is another chain complex and $g\colon D_\bullet \to E_\bullet$ is another morphism of chain complexes, then
$$\mathrm{H}_n(g \circ f) = \mathrm{H}_n(g) \circ \mathrm{H}_n(f)$$
for every $n \in \mathbb{Z}$.
- If $g\colon C_\bullet \to D_\bullet$ is another morphism of chain complexes that is parallel to the morphism f, then
$$\mathrm{H}_n(f + g) = \mathrm{H}_n(f) + \mathrm{H}_n(g)$$
for every $n \in \mathbb{Z}$.

Remark* 4.D. We have seen in Remark* 4.C that the n-th homology $\mathrm{H}_n(C_\bullet)$ of a chain complex $C_\bullet$ can alse be described as the cokernel of the morphism $C_{n+1} \to \mathrm{Z}_n(C_\bullet)$ that is induced by the morphism $d_{n+1}\colon C_{n+1} \to C_n$ via the universal propery of the kernel $\mathrm{Z}_n(C_\bullet) = \ker(d_n)$. This description of the n-th homology leads to the same induced morphism between homology objects:

Let $f\colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. We have the following commutative diagram in $\mathcal{A}$:

$$\begin{tikzcd}
C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr] \arrow[drrr, dashed] \arrow[dd, dashed, "f_{n+1}"'] & & C_n \arrow[rr, "d_n"] \arrow[dd, "f_n"'] & & C_{n-1} \arrow[dd, "f_{n-1}"] & \\
& \mathrm{B}_n(C_\bullet) \arrow[rr] \arrow[dd, "\mathrm{B}_n(f)"] & & \mathrm{Z}_n(C_\bullet) \arrow[ul] \arrow[rr, dashed] \arrow[dd, dashed, "\mathrm{Z}_n(f)"] & & \mathrm{H}_n(C_\bullet) \arrow[dd, dashed, "\mathrm{H}_n(f)"] \\
D_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr] \arrow[drrr, dashed] & & D_n \arrow[rr, "d_n"] & & D_{n-1} & \\
& \mathrm{B}_n(D_\bullet) \arrow[rr] & & \mathrm{Z}_n(D_\bullet) \arrow[ul] \arrow[rr, dashed] & & \mathrm{H}_n(D_\bullet)
\end{tikzcd}$$

The commutativity of the dashed subdiagram

$$\begin{tikzcd}
C_{n+1} \arrow[r] \arrow[d, "f_{n+1}"'] & \mathrm{Z}_n(C_\bullet) \arrow[r] \arrow[d, "\mathrm{Z}_n(f)"] & \mathrm{H}_n(C_\bullet) \arrow[d, dashed, "\mathrm{H}_n(f)"] \\
D_{n+1} \arrow[r] & \mathrm{Z}_n(D_\bullet) \arrow[r] & \mathrm{H}_n(D_\bullet)
\end{tikzcd}$$

shows the claim.

Example* 4.E. Let A be a $\mathbf{k}$-algebra. Let $C_\bullet$ and $D_\bullet$ be chain complexes of A-modules, i.e. chain complexes in the abelian category A-**Mod**, and let $f\colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. Then $f(\mathrm{Z}_n(C_\bullet)) \subseteq \mathrm{Z}_n(D_\bullet)$ and $f(\mathrm{B}_n(C_\bullet)) \subseteq \mathrm{B}_n(D_\bullet)$ for every $n \in \mathbb{Z}$, and the homomorphisms $\mathrm{Z}_n(f)$ and $\mathrm{B}_n(f)$ are the resulting restrictions of f. The induced morphism $\mathrm{H}_n(f)\colon\ \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(D_\bullet)$ is on elements given by

$$\mathrm{H}_n(f)([x]) = [f(x)] \in \mathrm{H}_n(D_\bullet)$$

for every $[x] \in \mathrm{H}_n(C_\bullet)$.

Definition 4.5. A morphism $f\colon C_\bullet \to D_\bullet$ of chain complexes is a *quasi-isomorphism* if for every $n \in \mathbb{Z}$ the morphism $\mathrm{H}_n(f)\colon\ \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(D_\bullet)$ is an isomorphism. Dually, a morphism $f\colon C^\bullet \to D^\bullet$ of cochain complexes is a *quasi-isomorphism* if for every $n \in \mathbb{Z}$ the morphism $\mathrm{H}^n(f)\colon\ \mathrm{H}^n(C^\bullet) \to \mathrm{H}^n(D^\bullet)$ is an isomorphism.

Remark-Definition 4.6. For a chain complex $C_\bullet$ in $\mathcal{A}$ the following conditions are equivalent:

i) The sequence
$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$
is exact.

ii) It holds that $\mathrm{H}_n(C_\bullet) = 0$ for every $n \in \mathbb{Z}$.

iii) The morphism $0 \to C_\bullet$ is a quasi-isomorphism.

iii') The morphism $C_\bullet \to 0$ is a quasi-isomorphism.

iv) The chain complex $C_\bullet$ is quasi-isomorphic to the zero complex.

If the chain complex $C_\bullet$ satisfies these equivalent conditions then $C_\bullet$ is *acyclic*.

Example 4.7. Consider the chain complex $C_\bullet$ of abelian groups, i.e. in the abelian category **Ab**, given by

$$C_n \coloneqq \begin{cases} \mathbb{Z}/8 & \text{if } n \geq 0\,, \\ 0 & \text{if } n < 0\,, \end{cases}$$

together with the differential morphisms $d_n\colon C_n \to C_{n-1}$ given by

$$d_n \coloneqq \begin{cases} \text{multiplication by } 4 & \text{if } n > 0\,, \\ 0 & \text{if } n \leq 0\,. \end{cases}$$

This is indeed a chain complex, and its homology is given by

$$\mathrm{H}_n(C_\bullet) \cong \begin{cases} 0 & \text{if } n < 0\,, \\ \mathbb{Z}/4 & \text{if } n = 0\,, \\ \mathbb{Z}/2 & \text{if } n \geq 1\,. \end{cases}$$

Indeed, we have for every $n < 0$ that $C_n = 0$, hence $\mathrm{Z}_n(C_\bullet) = 0$ and therefore also $\mathrm{H}_n(C_\bullet) = 0$. For $n = 0$ we have that $\mathrm{Z}_0(C_\bullet) = \mathbb{Z}/8$ and $\mathrm{B}_0(C_\bullet) = 4\mathbb{Z}/8$, and hence that

$$\mathrm{H}_0(C_\bullet) = \mathrm{Z}_0(C_\bullet)/\mathrm{B}_0(C_\bullet) = (\mathbb{Z}/8)/(4\mathbb{Z}/8) \cong \mathbb{Z}/4\,.$$

For $n \geq 1$ we have that $\mathrm{Z}_n(C_\bullet) = 2\mathbb{Z}/8$ and $\mathrm{B}_n(C_\bullet) = 4\mathbb{Z}/8$, and hence that

$$\mathrm{H}_n(C_\bullet) = \mathrm{Z}_n(C_\bullet)/\mathrm{B}_n(C_\bullet) = (2\mathbb{Z}/8)/(4\mathbb{Z}/8) \cong 2\mathbb{Z}/4 \cong \mathbb{Z}/2\,.$$

Example 4.8. For every $n \geq 0$ let

$$\Delta^n \coloneqq \mathrm{conv}(e_0, \dots, e_n) = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, t_0, \dots, t_n \geq 0, \sum_{i=0}^{n} t_i = 1 \right\}$$

be the *standard n-simplex*, together with the usual topology. For every $k = 0, \dots, n$ let

$$f_k^{(n)} \colon \Delta^{n-1} \to \Delta^n\,, \quad (t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

be the inclusion of Δ^{n-1} into Δ^n as the k-th face.

Let now X be a topological space. A n-simplex in X is a continuous map $\sigma \colon \Delta^n \to X$. For every $n \geq 0$ let

$$C_n^{\mathrm{sing}}(X) \coloneqq \text{free abelian group on the set } \{n\text{-simplices } \sigma \colon \Delta^n \to X\}\,,$$

and for every $n < 0$ let $C_n^{\mathrm{sing}}(X) \coloneqq 0$. We define for every $n \in \mathbb{Z}$ a differential

$$d_n^{\mathrm{sing}} \colon C_n^{\mathrm{sing}}(X) \to C_{n-1}^{\mathrm{sing}}(X)$$

by $d_n^{\mathrm{sing}} = 0$ for $n \leq 0$, and for $n > 0$ on basis elements by

$$d_n^{\mathrm{sing}}(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ f_i^{(n)}$$

for every n-simplex σ in X. This resulting chain complex

$$C_\bullet^{\mathrm{sing}} \coloneqq ((C_n^{\mathrm{sing}}(X))_{n \in \mathbb{Z}}, (d_n^{\mathrm{sing}})_{n \in \mathbb{Z}})$$

is the *singular chain complex* of X. (That this really defines a chain complex of abelian groups follows from Exercise 4 of Exercise sheet 8.) Its n-th homology

$$\mathrm{H}_n^{\mathrm{sing}}(X) \coloneqq \mathrm{H}_n(C_\bullet^{\mathrm{sing}}(X))$$

is the n-th *singular homology* of X.

The *singular cochain complex* of X is denoted by $C^\bullet_{\mathrm{sing}}(X)$; it is given by

$$C^n_{\mathrm{sing}}(X) \coloneqq \mathrm{Hom}_{\mathbb{Z}}(C_n^{\mathrm{sing}}(X), \mathbb{Z})$$

for every $n \in \mathbb{Z}$, and the differential $d^n_{\mathrm{sing}} \colon C^n_{\mathrm{sing}}(X) \to C^{n+1}_{\mathrm{sing}}(X)$ is for every $n \in \mathbb{Z}$ the dual map to $d_{n+1}^{\mathrm{sing}} \colon C_{n+1}^{\mathrm{sing}}(X) \to C_n^{\mathrm{sing}}(X)$. The n-th cohomology of this cochain complex $C^\bullet_{\mathrm{sing}}(X)$,

$$\mathrm{H}^n_{\mathrm{sing}}(X) \coloneqq \mathrm{H}^n(C^\bullet_{\mathrm{sing}}(X))\,,$$

is the n-th *singular cohomology* of X.

Lemma 4.9. The categories $\mathbf{Ch}_\bullet(\mathcal{A})$ and $\mathbf{Ch}^\bullet(\mathcal{A})$ are additive. Moreover, both sums of morphisms and biproducts can be computed componentwise.

Proof. It sufficies to consider the category $\mathbf{Ch}_\bullet(\mathcal{A})$. For any two parallel morphisms of chain complexes $f, g\colon C_\bullet \to D_\bullet$ their sum is given by

$$(f+g)_n \coloneqq f_n + g_n$$

for every $n \in \mathbb{Z}$. This makes $\mathbf{Ch}_\bullet(\mathcal{A})(C_\bullet, D_\bullet)$ into an abelian group.

The category $\mathbf{Ch}_\bullet(\mathcal{A})$ has biproducts: Let $C_\bullet^{(1)}, \dots, C_\bullet^{(k)}$ be chain complexes in $\mathcal{A}$. Their biproduct $C_\bullet$ is given by

$$C_n \coloneqq C_n^{(1)} \oplus \dots \oplus C_n^{(k)}$$

for every $n \in \mathbb{Z}$, together with the differentials $d_n\colon C_n \to C_{n-1}$ given by

$$d_n \coloneqq \begin{bmatrix} d_n^{(1)} & & \\ & \ddots & \\ & & d_n^{(k)} \end{bmatrix} : C_n \to C_{n-1}$$

for every $n \in \mathbb{Z}$. The canonical morphisms $c_i\colon C_\bullet^{(i)} \to C_\bullet$ and $p_i\colon C_\bullet \to C_\bullet^{(i)}$ are given in components by

$$c_i \coloneqq (c_{i,n})_{n\in\mathbb{Z}} \quad \text{and} \quad p_i \coloneqq (p_{i,n})_{n\in\mathbb{Z}}\,,$$

where

$$c_{i,n}\colon C_n^{(i)} \to C_n^{(1)} \oplus \dots \oplus C_n^{(k)} \quad \text{and} \quad p_{i,n}\colon C_n^{(1)} \oplus \dots \oplus C_n^{(k)} \to C_n^{(i)}$$

are for every $n \in \mathbb{Z}$ the canonical morphisms belonging to the biproduct in $\mathcal{A}$. These are indeed morphisms of chain complexes. It remains check that $p_j c_i = 0$ for all $i \neq j$, and that $\sum_{i=1}^k c_i p_i = \mathrm{id}_{C_\bullet}$. This holds true because it holds componentwise. ☐

Lemma 4.10. Let $f\colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. It follows for every $n \in \mathbb{Z}$ from the commutativity of the square

$$\begin{tikzcd} C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "d_n"] & D_{n-1} \end{tikzcd}$$

that there exist unique morphisms

$$d_n^{\ker}\colon \ker(f_n) \to \ker(f_{n-1})$$

and

$$d_n^{\mathrm{coker}}\colon \mathrm{coker}(f_n) \to \mathrm{coker}(f_{n-1})$$

that make the following two squares commute:

$$\begin{tikzcd} \ker(f_n) \arrow[r, dashed, "d_n^{\ker}"] \arrow[d, "k_n"'] & \ker(f_{n-1}) \arrow[d, "k_{n-1}"] \\ C_n \arrow[r, "d_n"] & C_{n-1} \end{tikzcd} \qquad \begin{tikzcd} D_n \arrow[r, "d_n"] \arrow[d, "c_n"'] & D_{n-1} \arrow[d, "c_{n-1}"] \\ \operatorname{coker}(f_n) \arrow[r, dashed, "d_n^{\operatorname{coker}}"] & \operatorname{coker}(f_{n-1}) \end{tikzcd}$$

Then $\ker(f) := ((\ker(f_n))_{n\in\mathbb{Z}}, (d_n^{\ker})_{n\in\mathbb{Z}})$ is again a chain complex, $k := (k_n)_{n\in\mathbb{Z}}$ is a morphism of chain complexes $k\colon \ker(f) \to C_\bullet$, and k is a kernel of f. Dually, $\operatorname{coker}(f) := ((\operatorname{coker}(f_n))_{n\in\mathbb{Z}}, (d_n^{\operatorname{coker}})_{n\in\mathbb{Z}})$ is again a chain complex, $c := (c_n)_{n\in\mathbb{Z}}$ is a morphism of chain complexes $c\colon D_\bullet \to \operatorname{coker}(f)$, and c is a cokernel of f.

Proof. This is part of Exercise 1 of Exercise sheet 8. □

End of lecture 15

Theorem 4.11. The categories $\mathbf{Ch}_\bullet(\mathcal{A})$ and $\mathbf{Ch}^\bullet(\mathcal{A})$ are again abelian.

Proof. It sufficies to show that $\mathbf{Ch}_\bullet(\mathcal{A})$ is abelian. It follows from Lemma 4.9 and Lemma 4.10 that the category $\mathbf{Ch}_\bullet(\mathcal{A})$ is additive and has kernels and cokernels. It remains to show that for a morphism $f = (f_n)_{n\in\mathbb{Z}}\colon C_\bullet \to D_\bullet$ of chain complexes the induced morphism $\tilde{f}\colon \operatorname{coim}(f) \to \operatorname{im}(f)$, i.e. the unique morphism that makes the square

$$\begin{tikzcd} C_\bullet \arrow[r, "f"] \arrow[d] & D_\bullet \arrow[d] \\ \operatorname{coim}(f) \arrow[r, dashed, "\tilde{f}"] & \operatorname{im}(f) \end{tikzcd}$$

commute, is an isomorphism. Let $E_\bullet := \operatorname{coim}(f)$ and $F_\bullet := \operatorname{im}(f)$. It follows from Lemma 4.10 that $C_n \to E_n$ is for every $n \in \mathbb{Z}$ a coimage of f_n and that $F_n \to D_n$ is for every $n \in \mathbb{Z}$ an image of f_n. It holds for $\tilde{f} = (\tilde{f}_n)_{n\in\mathbb{Z}}$ that $\tilde{f}_n\colon E_n \to F_n$ is for every $n \in \mathbb{Z}$ the canonical morphism from the canonical factorization lemma induced by f_n, because the square

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d] & D_n \arrow[d] \\ \operatorname{coim}(f_n) \arrow[r, dashed, "\tilde{f}_n"] & \operatorname{im}(f_n) \end{tikzcd}$$

commutes. It follows that $\tilde{f}_n$ is an isomorphism for every $n \in \mathbb{Z}$ because $\mathcal{A}$ is abelian. Hence $\tilde{f}$ is an isomorphism. □

Remark⋆ 4.F. Kernels and cokernels in the abelian categories $\mathbf{Ch}_\bullet(\mathcal{A})$ and $\mathbf{Ch}^\bullet(\mathcal{A})$ are computed degreewise, so the same goes for images and coimages. It follows that a sequence $C'_\bullet \to C_\bullet \to C''_\bullet$ of chain complexes in $\mathcal{A}$ (resp. a sequence ${}'C^\bullet \to C^\bullet \to {}''C^\bullet$ of cochain complexes in $\mathcal{A}$) is exact if and only if it is exact in each degree, i.e. if and only if the sequence $C'_n \to C_n \to C''_n$ (resp. the sequence ${}'C^n \to C^n \to {}''C^n$) is exact for every $n \in \mathbb{Z}$.

Long Exact Sequence

Theorem 4.12 (Long exact (co)homology sequence)**.**

i) For every short exact sequence of chain complexes

$$\xi\colon 0 \to C'_\bullet \to C_\bullet \to C''_\bullet \to 0$$

and every $n \in \mathbb{Z}$ there exists a connecting morphism

$$\partial_n = \partial_n^\xi\colon\ \mathrm{H}_n(C''_\bullet) \to \mathrm{H}_{n-1}(C'_\bullet)$$

with the following two properties:

a) The resulting sequence

$$\cdots \longrightarrow \mathrm{H}_{n+1}(C''_\bullet) \xrightarrow{\partial_{n+1}} \mathrm{H}_n(C'_\bullet) \longrightarrow \mathrm{H}_n(C_\bullet) \longrightarrow \mathrm{H}_n(C''_\bullet) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(C'_\bullet) \longrightarrow \cdots$$

is exact.

b) If

$$\begin{tikzcd} 0 \arrow[r] & C'_\bullet \arrow[r] \arrow[d] & C_\bullet \arrow[r] \arrow[d] & C''_\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D'_\bullet \arrow[r] & D_\bullet \arrow[r] & D''_\bullet \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of chain complexes with (short) exact rows ξ and ζ then the following square commutes for every $n \in \mathbb{Z}$:

$$\begin{tikzcd} \mathrm{H}_n(C''_\bullet) \arrow[r, "\partial_n^\xi"] \arrow[d] & \mathrm{H}_{n-1}(C'_\bullet) \arrow[d] \\ \mathrm{H}_n(D''_\bullet) \arrow[r, "\partial_n^\zeta"] & \mathrm{H}_{n-1}(D'_\bullet) \end{tikzcd}$$

ii) Dually, there exists for every short exact sequence of cochain complexes

$$\xi\colon 0 \to {}'C^\bullet \to C^\bullet \to {}''C^\bullet \to 0$$

and every $n \in \mathbb{Z}$ a connecting morphism

$$\partial^n = \partial^n_\xi\colon\ \mathrm{H}^n({}''C^\bullet) \to \mathrm{H}^{n+1}({}'C^\bullet)$$

with the following two properties:

a) The resulting sequence

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}^{n-1}({}''C^{\bullet}) \arrow[dll, rounded corners, "\partial^{n-1}"] \\ \mathrm{H}^{n}({}'C^{\bullet}) \arrow[r] & \mathrm{H}^{n}(C^{\bullet}) \arrow[r] & \mathrm{H}^{n}({}''C^{\bullet}) \arrow[dll, rounded corners, "\partial^{n}"] \\ \mathrm{H}^{n+1}({}'C_{\bullet}) \arrow[r] & \cdots & \end{tikzcd}$$

is exact.

b) If

$$\begin{tikzcd} 0 \arrow[r] & {}'C^{\bullet} \arrow[r] \arrow[d] & C^{\bullet} \arrow[r] \arrow[d] & {}''C^{\bullet} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & {}'D^{\bullet} \arrow[r] & D^{\bullet} \arrow[r] & {}''D^{\bullet} \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of cochain complexes with (short) exact rows ξ and ζ then the following square commutes for every $n \in \mathbb{Z}$:

$$\begin{tikzcd} \mathrm{H}^{n}({}''C^{\bullet}) \arrow[r, "\partial^{n}_{\xi}"] \arrow[d] & \mathrm{H}^{n+1}({}'C^{\bullet}) \arrow[d] \\ \mathrm{H}^{n}({}''D^{\bullet}) \arrow[r, "\partial^{n}_{\zeta}"] & \mathrm{H}^{n+1}({}'D^{\bullet}) \end{tikzcd}$$

Proof. This proof is currently missing from these notes and will be added later. □

Remark⋆ 4.G. If

$$\begin{tikzcd} 0 \arrow[r] & C'_{\bullet} \arrow[r] \arrow[d] & C_{\bullet} \arrow[r] \arrow[d] & C''_{\bullet} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D'_{\bullet} \arrow[r] & D_{\bullet} \arrow[r] & D''_{\bullet} \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence of chain complexes in $\mathcal{A}$ then property b) states that the induced ladder diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}_{n+1}(C''_{\bullet}) \arrow[r, "\partial^{\xi}_{n+1}"] \arrow[d] & \mathrm{H}_{n}(C'_{\bullet}) \arrow[r] \arrow[d] & \mathrm{H}_{n}(C_{\bullet}) \arrow[r] \arrow[d] & \mathrm{H}_{n}(C''_{\bullet}) \arrow[r, "\partial^{\xi}_{n}"] \arrow[d] & \mathrm{H}_{n-1}(C'_{\bullet}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \mathrm{H}_{n+1}(D''_{\bullet}) \arrow[r, "\partial^{\zeta}_{n+1}"] & \mathrm{H}_{n}(D'_{\bullet}) \arrow[r] & \mathrm{H}_{n}(D_{\bullet}) \arrow[r] & \mathrm{H}_{n}(D''_{\bullet}) \arrow[r, "\partial^{\zeta}_{n}"] & \mathrm{H}_{n-1}(D'_{\bullet}) \arrow[r] & \cdots \end{tikzcd}$$

with (long) exact rows is again commutative. For cochain complexes the analogous statement holds.

Lemma⋆ 4.H.

i) Let $0 \to C'_\bullet \to C_\bullet \to C''_\bullet \to 0$ be a short exact sequence of chain complexes in $\mathcal{A}$. If any two of the chain complexes, $C_\bullet$, $C_\bullet$, $C''_\bullet$ are acyclic then the third one is also acyclic.

ii) Let $f\colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. If $\ker(f)$ and $\operatorname{coker}(f)$ are acyclic then f is a quasi-isomorphism.

Proof. This is Exercise 2 of Exercise sheet 8. □

Example 4.13. Let X be a topological space.

i) The singular chain complex $C^{\mathrm{sing}}_\bullet(X)$ is covariantly functorial in X, in the following sense: If Y is another topological space and $f\colon X \to Y$ is a continuous map, then for every n-simplex $\sigma\colon \Delta^n \to X$ in X we get an n-simplex $f\sigma\colon \Delta^n \to Y$ in Y. We hence get for every $n \geq 0$ an induced group homomorphism

$$f_n\colon C^{\mathrm{sing}}_n(X) \to C^{\mathrm{sing}}_n(Y)\,,$$

that is on basis elements given by $f_n(\sigma) = f\sigma$. By also setting $f_n = 0$ for every $n < 0$ we get a family of group homomorphisms $f_* \coloneqq (f_n)_{n\in\mathbb{Z}}$ that is a morphism of chain complexes

$$f_*\colon C^{\mathrm{sing}}_\bullet(X) \to C^{\mathrm{sing}}_\bullet(Y)\,.$$

If $A \subseteq X$ is a subspace then the chain complex $C^{\mathrm{sing}}_\bullet(A)$ is a subcomplex of the chain complex $C^{\mathrm{sing}}_\bullet(X)$, i.e. it holds that $C^{\mathrm{sing}}_n(A) \subseteq C^{\mathrm{sing}}_n(X)$ is a subgroup for every $n \in \mathbb{Z}$, and the differential of $C^{\mathrm{sing}}_\bullet(A)$ is the (well-defined) restriction of the differential of $C^{\mathrm{sing}}_\bullet(X)$. We can hence form the quotient chain complex

$$C^{\mathrm{sing}}_\bullet(X, A) \coloneqq C^{\mathrm{sing}}_\bullet(X)/C^{\mathrm{sing}}_\bullet(A)\,,$$

that is given in components by

$$C^{\mathrm{sing}}_n(X, A) = C^{\mathrm{sing}}_n(X)/C^{\mathrm{sing}}_n(A)$$

for every $n \in \mathbb{Z}$, and where the differential $C^{\mathrm{sing}}_n(X, A) \to C^{\mathrm{sing}}_{n-1}(X, A)$ is for every $n \in \mathbb{Z}$ induced by the differential $C^{\mathrm{sing}}_n(X) \to C^{\mathrm{sing}}_{n-1}(X)$. The homology

$$\mathrm{H}^{\mathrm{sing}}_n(X, A) \coloneqq \mathrm{H}_n(C^{\mathrm{sing}}_\bullet(X, A))$$

is the n-th *relative singular homology* of X with respect to A.

The inclusion $i\colon A \to X$ is an injective continuous map. It follows at every position $n \in \mathbb{Z}$ that the induced group homomorphism $i_n\colon C^{\mathrm{sing}}_n(A) \to C^{\mathrm{sing}}_n(X)$ is a monomorphism, and hence overall that the induced morphism of chain complexes $i_*\colon C^{\mathrm{sing}}_\bullet(A) \to C^{\mathrm{sing}}_\bullet(X)$ is a monomorphism. The above chain complex $C_\bullet(X, A)$ is precisely the cokernel of this morphism i_*, and we therefore have a short exact sequence of chain complexes

$$0 \to C^{\mathrm{sing}}_\bullet(A) \xrightarrow{i_*} C^{\mathrm{sing}}_\bullet(X) \to C^{\mathrm{sing}}_\bullet(X, A) \to 0\,. \tag{4.1}$$

We hence get the folloing long exact sequence:

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{H}^{\mathrm{sing}}_{n+1}(X,A) \arrow[dll, "\partial_{n+1}"] \\
\mathrm{H}^{\mathrm{sing}}_{n}(A) \arrow[r] & \mathrm{H}^{\mathrm{sing}}_{n}(X) \arrow[r] & \mathrm{H}^{\mathrm{sing}}_{n}(X,A) \arrow[dll, "\partial_{n}"] \\
\mathrm{H}^{\mathrm{sing}}_{n-1}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

ii) The singular cochain complex $C^{\bullet}_{\mathrm{sing}}(X)$ is the dual of the singular chain complex $C^{\bullet}_{\mathrm{sing}}(X)$, and is hence contravariantly functorial X: If Y is another topological space and $f\colon X \to Y$ is a continuous map then we get an induced morphism of cochain complexes $f^*\colon C^{\bullet}_{\mathrm{sing}}(Y) \to C^{\bullet}_{\mathrm{sing}}(X)$, where for every $n \in \mathbb{Z}$ the group homomorphism $f^n\colon C^n_{\mathrm{sing}}(Y) \to C^n_{\mathrm{sing}}(X)$ is the dual homomorphism to the homomorphism $f_n\colon C^{\mathrm{sing}}_n(X) \to C^{\mathrm{sing}}_n(Y)$ as constructed above.

If $A \subseteq X$ is a subspace with inclusion $i\colon A \to X$ then the three chain complexes $C^{\mathrm{sing}}_{\bullet}(X)$, $C^{\mathrm{sing}}_{\bullet}(A)$ and $C^{\mathrm{sing}}_{\bullet}(X,A)$ consist of free abelian groups. We hence get by dualizing the short exact sequence (4.1) again a short exact sequence[3]

$$0 \to C^{\bullet}_{\mathrm{sing}}(X,A) \to C^{\bullet}_{\mathrm{sing}}(X) \xrightarrow{i^*} C^{\bullet}_{\mathrm{sing}}(A) \to 0\,. \tag{4.2}$$

The cochain complex $C^{\bullet}_{\mathrm{sing}}(X,A)$ is the dual of the chain complex $C^{\mathrm{sing}}_{\bullet}(X,A)$, i.e. it is given in components by

$$C^n_{\mathrm{sing}}(X,A) \coloneqq \mathrm{Hom}_{\mathbb{Z}}(C^{\mathrm{sing}}_n(X,A), \mathbb{Z})$$

for every $n \in \mathbb{Z}$, and its differential $C^n_{\mathrm{sing}}(X,A) \to C^{n+1}_{\mathrm{sing}}(X,A)$ is the dual homomorphism to the differential $C^{\mathrm{sing}}_{n+1}(X,A) \to C^{\mathrm{sing}}_n(X,A)$.[4]

We now get from the short exact sequence (4.2) the following long exact sequence:

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{H}^{n-1}_{\mathrm{sing}}(A) \arrow[dll, "\partial^{n-1}"] \\
\mathrm{H}^{n}_{\mathrm{sing}}(X,A) \arrow[r] & \mathrm{H}^{n}_{\mathrm{sing}}(X) \arrow[r] & \mathrm{H}^{n}_{\mathrm{sing}}(A) \arrow[dll, "\partial^{n}"] \\
\mathrm{H}^{n+1}_{\mathrm{sing}}(X,A) \arrow[r] & \cdots &
\end{tikzcd}$$

Constructions with Complexes

Definition 4.14.

[3] Every short exact sequence $0 \to X' \to X \to X'' \to 0$ of free abelian groups is already a split short exact sequence. Its dual is then again a split short exact sequence.

[4] It follows from the exactness of the sequence (4.2) that $C^{\bullet}_{\mathrm{sing}}(X,A)$ is a kernel of the morphism f^*, hence we could have also defined $C^{\bullet}_{\mathrm{sing}}(X,A)$ in this way.

i) Let $C_\bullet$ be a chain complex in $\mathcal{A}$. Then for every $p \in \mathbb{Z}$ the *shifted chain complex* $C_\bullet[p]$ is given by

$$(C_\bullet[p])_n \coloneqq C_{n+p} \quad \text{and} \quad d_n^{C_\bullet[p]} \coloneqq (-1)^p d_{n+p}^{C_\bullet}$$

for every $n \in \mathbb{Z}$.

ii) Let $C^\bullet$ be a chain complex in $\mathcal{A}$. Then for every $p \in \mathbb{Z}$ the *shifted cochain complex* $C^\bullet[p]$ is given by

$$(C^\bullet[p])^n \coloneqq C^{n-p} \quad \text{and} \quad d^n_{C^\bullet[p]} \coloneqq (-1)^p d^{n-p}_{C^\bullet}$$

for every $n \in \mathbb{Z}$.

Remark 4.15.

i) Shifting complexes is covariantly functorial (for both chain complexes and cochain complexes):

If $f = (f_n)_{n\in\mathbb{Z}} \colon C_\bullet \to D_\bullet$ is a morphism of chain complexes then for every $p \in \mathbb{Z}$ the family $f[p] \coloneqq (f_{n+p})_{n\in\mathbb{Z}}$ is a morphism of chain complexes $f[p] \colon C_\bullet[p] \to D_\bullet[p]$. Dually, if $f = (f^n)_{n\in\mathbb{Z}} \colon C^\bullet \to D^\bullet$ is a morphism of cochain complexes then for every $p \in \mathbb{Z}$ the family $f[p] \coloneqq (f^{n-p})_{n\in\mathbb{Z}}$ is a morphism of cochain complexes $f[p] \colon C^\bullet[p] \to D^\bullet[p]$.

The shift operator $[1]$ is also denoted by Σ (for both chain complexes and cochain complexes).

ii) Let $p \in \mathbb{Z}$. If $C_\bullet$ is a chain complex in $\mathcal{A}$ then

$$\mathrm{H}_n(C_\bullet[p]) = \mathrm{H}_{n+p}(C_\bullet)$$

for every $n \in \mathbb{Z}$, and if $C^\bullet$ is a cochain complex in $\mathcal{A}$ then

$$\mathrm{H}^n(C^\bullet[p]) = \mathrm{H}^{n-p}(C^\bullet)$$

for every $n \in \mathbb{Z}$.

Remark 4.16. Let $C_\bullet$ be a chain complex in $\mathcal{A}$. We can consider the subcomplexes $Z_\bullet$ and $B_\bullet$ of $C_\bullet$ that are given in components by

$$Z_n = \mathrm{Z}_n(C_\bullet) \quad \text{and} \quad B_n = \mathrm{B}_n(C_\bullet)$$

for every $n \in \mathbb{Z}$, together with the zero morphisms as differentials. We then get a short exact sequence of chain complexes

$$0 \to Z_\bullet \to C_\bullet \to B_\bullet[-1] \to 0\,.$$

We get dually for every cochain complex $C^\bullet$ in $\mathcal{A}$ a short exact sequence of cochain complexes

$$0 \to Z^\bullet \to C^\bullet \to B^\bullet[-1] \to 0\,.$$

End of lecture 16

Remark-Definition 4.17.

i) Let $C_\bullet$ be a chain complex in $\mathcal{A}$ and let $p \in \mathbb{Z}$.

- The differential $d_{p+1} \colon C_{p+1} \to C_p$ induces for $\widetilde{C}_p := \mathrm{Z}_p(C_\bullet) = \ker(d_p)$ a morphism $\tilde{d}_{p+1} \colon C_{p+1} \to \widetilde{C}_p$. This results in a chain complex
$$\tau_{\geq p}(C_\bullet) := \left(\cdots \to C_{p+2} \xrightarrow{d_{p+2}} C_{p+1} \xrightarrow{\tilde{d}_{p+1}} \widetilde{C}_p \to 0 \to 0 \to \cdots \right) .$$
- The differential $d_p \colon C_p \to C_{p-1}$ induces for $\widetilde{C}_p := \operatorname{coker}(d_{p+1})$ a morphism $\tilde{d}_p \colon \tilde{C}_p \to C_{p-1}$. This results in a chain complex
$$\tau_{\leq p}(C_\bullet) := \left(\cdots \to 0 \to 0 \to \widetilde{C}_p \xrightarrow{\tilde{d}_p} C_{p-1} \xrightarrow{d_{p+1}} C_{p-2} \to \cdots \right) .$$

The chain complexes $\tau_{\geq p}(C_\bullet)$ and $\tau_{\leq p}(C_\bullet)$ are the *truncated complexes* of $C_\bullet$.

ii) Dually, let $C^\bullet$ be a cochain complex in $\mathcal{A}$ and let $p \in \mathbb{Z}$.

- The differential $d^{p-1} \colon C^{p-1} \to C^p$ induces for $\widetilde{C}^p := \mathrm{Z}^p(C^\bullet) = \ker(d^p)$ a morphism $\tilde{d}^{p-1} \colon C^{p-1} \to \widetilde{C}^p$. This results in a cochain complex
$$\tau^{\leq p}(C^\bullet) := \left(\cdots \to C^{p-2} \xrightarrow{d^{p-2}} C^{p-1} \xrightarrow{\tilde{d}^{p-1}} \widetilde{C}^p \to 0 \to 0 \to \cdots \right) .$$
- The differential $d^p \colon C^p \to C^{p+1}$ induces for $\widetilde{C}^p := \operatorname{coker}(\tilde{d}^{p+1})$ a morphism $\tilde{d}^p \colon \widetilde{C}^p \to C^{p+1}$. This results in a cochain complex
$$\tau^{\geq p}(C^\bullet) := \left(\cdots \to 0 \to 0 \to \widetilde{C}^p \xrightarrow{\tilde{d}^p} C^{p+1} \xrightarrow{d^{p+1}} C^{p+2} \to \cdots \right) .$$

The chain complexes $\tau^{\leq p}(C^\bullet)$ and $\tau^{\geq p}(C^\bullet)$ are the *truncated complexes* of $C^\bullet$.

Remark 4.18.

i) Truncation of complexes is functorial (for both chain complexes and cochain complexes):

Let $f = (f_n)_{n \in \mathbb{Z}} \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes and let $p \in \mathbb{Z}$. Then the morphism $f_p \colon C_p \to D_p$ restrict to a morphism $\tilde{f}_p \colon \ker(d_p^C) \to \ker(d_p^D)$. We get from this a morphism of chain complexes
$$\tau_{\geq p}(f) \colon \tau_{\geq p}(C_\bullet) \to \tau_{\geq p}(D_\bullet)$$
that is given in components given by
$$\tau_{\geq p}(f)_n := \begin{cases} f_n & \text{if } n > p \,, \\ \tilde{f}_n & \text{if } n = p \,, \\ 0 & \text{if } n < p \,. \end{cases}$$

We similarly get an induced morphism $\tilde{f}_p\colon \operatorname{coker}(d^C_{p+1}) \to \operatorname{coker}(d^D_{p+1})$, from which we get a morphism of chain complexes

$$\tau_{\leq p}(f)\colon \tau_{\leq p}(C_\bullet) \to \tau_{\leq p}(D_\bullet)$$

that is given in components by

$$\tau_{\leq p}(f)_n \coloneqq \begin{cases} f_n & \text{if } n < p\,, \\ \tilde{f}_n & \text{if } n = p\,, \\ 0 & \text{if } n > p\,. \end{cases}$$

These induced morphisms are compatible with identity morphisms and composition of morphisms by the functoriality of (co)kernels.

For truncation of cochain complexes the analogous construction works.

ii) If $C^\bullet$ is a chain complex in $\mathcal{A}$ and $p \in \mathbb{Z}$ then

$$\mathrm{H}_n(\tau_{\geq p}(C_\bullet)) \cong \begin{cases} \mathrm{H}_n(C_\bullet) & \text{if } n \geq p\,, \\ 0 & \text{if } n < p\,, \end{cases}$$

and

$$\mathrm{H}_n(\tau_{\leq p}(C_\bullet)) \cong \begin{cases} \mathrm{H}_n(C_\bullet) & \text{if } n \leq p\,, \\ 0 & \text{if } n > p\,. \end{cases}$$

Similarly, if $C^\bullet$ is a cochain complex in $\mathcal{A}$ and $p \in \mathbb{Z}$ then

$$\mathrm{H}^n(\tau^{\leq p}(C^\bullet)) \cong \begin{cases} \mathrm{H}^n(C^\bullet) & \text{if } n \leq p\,, \\ 0 & \text{if } n > p\,, \end{cases}$$

and

$$\mathrm{H}^n(\tau_{\geq p}(C^\bullet)) \cong \begin{cases} \mathrm{H}^n(C^\bullet) & \text{if } n \geq p\,, \\ 0 & \text{if } n < p\,. \end{cases}$$

Remark* 4.I. If $C_\bullet$ as a chain complex in $\mathcal{A}$ and $p \in \mathbb{Z}$ then one can also define the 'stupid truncations'

$$\sigma_{\geq p}(C_\bullet) \coloneqq (\cdots \to C_{p+2} \xrightarrow{d_{p+2}} C_{p+1} \xrightarrow{d_{p+1}} C_p \to 0 \to 0 \to \cdots)$$

and

$$\sigma_{\leq p}(C_\bullet) \coloneqq (\cdots \to 0 \to 0 \to C_p \xrightarrow{d_p} C_{p-1} \xrightarrow{d_{p-1}} C_{p-2} \to \cdots)\,.$$

For a cochain complex $C^\bullet$ in $\mathcal{A}$ and $p \in \mathbb{Z}$ one can define the 'stupid truncations'

$$\sigma^{\leq p}(C^\bullet) \coloneqq (\cdots \to C^{p-2} \xrightarrow{d^{p-2}} C^{p-1} \xrightarrow{d^{p-1}} C_p \to 0 \to 0 \to \cdots)$$

and

$$\sigma^{\geq p}(C^\bullet) \coloneqq (\cdots \to 0 \to 0 \to C^p \xrightarrow{d^p} C^{p+1} \xrightarrow{d^{p+1}} C^{p+2} \to \cdots)\,.$$

But then

$$\mathrm{H}_p(\sigma_{\geq p}(C_\bullet)) \cong \operatorname{coker}(d_{p+1})\,, \quad \mathrm{H}_p(\sigma_{\leq p}(C_\bullet)) = \mathrm{Z}_p(C_\bullet)\,,$$

and

$$\mathrm{H}^p(\sigma^{\leq p}(C^\bullet)) \cong \operatorname{coker}(d^{p-1})\,, \quad \mathrm{H}^p(\sigma^{\geq p}(C^\bullet)) = \mathrm{Z}^p(C^\bullet)\,.$$

Hence the p-th (co)homology is (in general) not preserved by the stupid truncations.

Chain Homotopies

Remark⋆ 4.J (Quotient categories)**.** Let $\mathcal{C}$ be a category and let $\mathcal{A}$ be a preadditve category.

i) Suppose that we are given for any two objects $X, Y \in \mathcal{C}$ an equivalence relation $\sim_{X,Y}$ of $\mathcal{C}(X,Y)$. Then we can try to define a *quotient category* $\widetilde{\mathcal{C}}$, that is supposed to be given by

- by the class of objects $\mathrm{Ob}(\widetilde{\mathcal{C}}) \coloneqq \mathrm{Ob}(\mathcal{C})$, and
- for any two objects $X, Y \in \mathrm{Ob}(\widetilde{\mathcal{C}}) = \mathrm{Ob}(\mathcal{C})$ the set of morphisms

$$\widetilde{\mathcal{C}}(X,Y) \coloneqq \mathcal{C}(X,Y)/{\sim_{X,Y}}\,,$$

- with the composition of $[f] \in \widetilde{\mathcal{C}}(X,Y)$ and $g \in \widetilde{\mathcal{C}}(Y,Z)$ being given by

$$[g] \circ [f] \coloneqq [g \circ f]\,.$$

However, the composition in $\widetilde{\mathcal{C}}$ will not be well-defined in general. More specifically, this composition is well-defined if and only if it holds for all morphisms $f, f' \colon X \to Y$ and $g, g' \colon Y \to Z$ in $\mathcal{C}$ that

$$f \sim_{X,Y} f' \text{ and } g \sim_{Y,Z} g' \implies g \circ f \sim_{X,Z} g' \circ f'\,. \tag{4.3}$$

If this condition is satisfied then $\widetilde{\mathcal{C}}$ is indeed a category: The associativity of compositions of morphisms is inherited from $\mathcal{C}$, and for every obect $X \in \mathrm{Ob}(\widetilde{\mathcal{C}}) = \mathrm{Ob}(\mathcal{C})$ the identity $\mathrm{id}_X^{\widetilde{\mathcal{C}}}$ is given by

$$\mathrm{id}_X^{\widetilde{\mathcal{C}}} = \left[\mathrm{id}_X^{\mathcal{C}}\right]\,.$$

The collection of equivalence relations $\sim \coloneqq (\sim_{X,Y})_{X,Y \in \mathrm{Ob}(\mathcal{C})}$ is then a *congruence relations* on $\mathcal{C}$, and the quotient category $\widetilde{\mathcal{C}}$ as described above is denoted by $\mathcal{C}/{\sim}$. The equivalencen relations $\sim_{X,Y}$ are for simplicity just denoted by $\sim$.

One may also think about a congruence relation on $\mathcal{C}$ as an equivalence relations on the class $\coprod_{X,Y \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(X,Y)$ of all morphisms in $\mathcal{C}$, that is compatible with domains, codomains and composition of morphisms.

It is also worthwile to notice that the single condition (4.3) can equivalently be replaced by the two conditions

$$f \sim f' \implies g \circ f \sim g \circ f'$$

for all morphisms $f, f' \colon X \to Y$ and $g \colon Y \to Z$ in $\mathcal{C}$, and

$$f \sim f' \implies f \circ h \sim f' \circ h$$

for all morphisms $f, f' \colon X \to Y$ and $h \colon W \to X$ in $\mathcal{C}$.

ii) If $\sim$ is a congruence relation on $\mathcal{C}$ then we get a *projection functor*

$$P \colon \mathcal{C} \to \mathcal{C}/{\sim}$$

that is given on objects by $P(X) = X$ and on morphisms by $P(f) = [f]$. This projection functor has the property that $P(f) = P(f')$ for any two morphisms f and f' in $\mathcal{C}$ with $f \sim f'$, and the functor P is universal with this property: Whenever $F \colon \mathcal{C} \to \mathcal{D}$ is another functor such that $F(f) = F(f')$ for all morphisms f and f' in $\mathcal{C}$ with $f \sim f'$, then there exists a unique induced functor $F' \colon (\mathcal{C}/{\sim}) \to \mathcal{D}$ that makes the following triangle commute:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "P"'] & \mathcal{D} \\ \mathcal{C}/{\sim} \arrow[ur, dashed, "F'"'] & \end{tikzcd}$$

iii) Let $\sim$ be a congruence relation on $\mathcal{A}$. Then we would like the preadditive structure of $\mathcal{A}$ to descend to a preadditive structure on $\mathcal{A}/{\sim}$ via

$$[f] + [g] \coloneqq [f + g]$$

for any two parallel morphisms $[f], [g] \colon X \to Y$ in $\mathcal{A}/{\sim}$. This addition of morphisms in the quotient category $\mathcal{A}/{\sim}$ is well-defined if and only if it holds for all morphisms $f, f', g, g' \colon X \to Y$ in $\mathcal{A}$ that

$$f \sim f' \text{ and } g \sim g' \implies f + g \sim f' + g' .$$

The congruence relation $\sim$ is then called *additive*.

If $\sim$ is additive then $\mathcal{A}/{\sim}$ is together with the above addition of morphisms indeed a preadditive category: The associativity of the addition on $(\mathcal{A}/{\sim})(X, Y)$ is inherited from $\mathcal{A}(X, Y)$. The neutral element of $(\mathcal{A}/{\sim})(X, Y)$ is given by the morphism $[0]$, and the additive inverse of a morphism $[f] \in (\mathcal{A}/{\sim})(X, Y)$ is given by the morphism $[-f]$. The bilinearity of the composition of morphisms in $\mathcal{A}/{\sim}$ follows from the bilinearity of the composition of morphisms in $\mathcal{A}$.

Note that if $\sim$ is additive then the projection functor $P \colon \mathcal{A} \to \mathcal{A}/{\sim}$ is additive.

iv) An *ideal* $\mathcal{I}$ in $\mathcal{A}$ consists of a subgroup $\mathcal{I}(X, Y) \subseteq \mathcal{A}(X, Y)$ for any two objects $X, Y \in \mathrm{Ob}(\mathcal{A})$ such that

$$\mathcal{A}(Y, Z) \circ \mathcal{I}(X, Y) \subseteq \mathcal{I}(X, Z) \quad \text{and} \quad \mathcal{I}(X, Y) \circ \mathcal{A}(W, X) \subseteq \mathcal{I}(W, Y)$$

for all objects $W, X, Y, Z \in \mathrm{Ob}(\mathcal{A})$.

v) Additive congruence relations on $\mathcal{A}$ are in one-to-one correspondence with ideals in $\mathcal{A}$:

If $F\colon \mathcal{A} \to \mathcal{B}$ is any additive functor, where $\mathcal{B}$ is another preadditive category, then

$$\begin{aligned}\ker(F)(X,Y) &\coloneqq \{f \in \mathcal{A}(X,Y) \mid F(f) = 0\} \\ &= \ker\Bigl(\mathcal{A}(X,Y) \xrightarrow{F} \mathcal{B}(F(X),F(Y))\Bigr)\end{aligned}$$

is for any two objects $X, Y \in \mathrm{Ob}(\mathcal{A})$ a subgroup of $\mathcal{A}(X,Y)$. This collection of subgroups defines an ideal $\ker(F)$ in $\mathcal{A}$, the *kernel* of F If $\sim$ is an additive congruence relation on $\mathcal{A}$ then the projection functor $P\colon \mathcal{A} \to (\mathcal{A}/{\sim})$ is additive, and hence $\mathcal{I} \coloneqq \ker(P)$ is an ideal in $\mathcal{A}$. This ideal $\mathcal{I}$ in $\mathcal{A}$ is given by

$$\mathcal{I}(X,Y) = \{f \in \mathcal{A}(X,Y) \mid P(f) = 0\} = \{f \in \mathcal{A}(X,Y) \mid f \sim 0\}$$

for all $X, Y \in \mathrm{Ob}(\mathcal{A})$.

If on the other hand $\mathcal{I}$ is an ideal in $\mathcal{A}$ then we can define an additive congruence relation $\sim$ on $\mathcal{A}$ that is given by

$$f \sim g \iff f - g \in \mathcal{I}(X,Y)$$

for any two parallel morphisms $f, g\colon X \to Y$ in $\mathcal{A}$. Indeed, it is known from basic algebra that $\sim$ defines for all $X, Y \in \mathrm{Ob}(\mathcal{A})$ an equivalence relation on $\mathcal{A}(X,Y)$ that is compatible with the addition on $\mathcal{A}(X,Y)$, in the sense that

$$f \sim f' \text{ and } g \sim g' \implies f + g \sim f' + g'$$

for all morphisms $f, f', g, g' \in \mathcal{A}(X,Y)$. (This equivalence relation on $\mathcal{A}(X,Y)$ is the one used to construct the quotient group $\mathcal{A}(X,Y)/\mathcal{I}(X,Y)$.) If $f, f'\colon X \to Y$ and $g, g'\colon Y \to Z$ are morphisms in $\mathcal{A}$ with $f \sim f'$ and $g \sim g'$ then

$$gf - g'f' = gf - gf' + gf' - g'f' = g(f - f') + (g - g')f' \in \mathcal{I}(X,Z)$$

because $f - f' \in \mathcal{I}(X,Y)$ and $g - g' \in \mathcal{I}(Y,Z)$ and because $\mathcal{I}$ is an ideal. Hence also $gf \sim g'f'$. This shows altogether that $\sim$ is indeed an additive congruence relation on $\mathcal{A}$.

These two constructions are mutually inverse, and give a one-to-one correspondence between additive congruence relations in $\mathcal{A}$ and ideals in $\mathcal{A}$.

If $\mathcal{I}$ is an ideal in $\mathcal{A}$ with corresponding additive congruence relation $\sim$ then we define the quotient of $\mathcal{A}$ by $\mathcal{I}$ as

$$\mathcal{A}/\mathcal{I} \coloneqq \mathcal{A}/{\sim}\,.$$

This is by the above discussion again a preadditive category, and we have that the projection functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$ is additive (by construction of the preadditive structure on $\mathcal{A}/\mathcal{I}$). We note that the morphisms groups of $\mathcal{A}/\mathcal{I}$ are for any two objects $X, Y \in \mathrm{Ob}(\mathcal{A}/\mathcal{I}) = \mathrm{Ob}(\mathcal{A})$ given by

$$(\mathcal{A}/\mathcal{I})(X,Y) = \mathcal{A}(X,Y)/\mathcal{I}(X,Y)\,.$$

vi) If $\mathcal{I}$ and $\mathcal{J}$ are two ideal in $\mathcal{A}$ then the ideal $\mathcal{I}$ is a *subideal* of the ideal $\mathcal{J}$, denoted by $\mathcal{I} \subseteq \mathcal{J}$, if $\mathcal{I}(X,Y) \subseteq \mathcal{J}(X,Y)$ for all $X, Y \in \mathrm{Ob}(\mathcal{A})$.

If $\mathcal{I}$ is an ideal in $\mathcal{A}$ with corresponding additive equivalence relation $\sim$ on $\mathcal{A}$, then the aforementioned universal propery of the quotient category $\mathcal{A}/{\sim}$ leads to the following universal property of the quotient category $\mathcal{A}/\mathcal{I}$: If $F\colon \mathcal{A} \to \mathcal{B}$ is an additive functor, where $\mathcal{B}$ is another preadditive category, then F decends to a functor $F'\colon \mathcal{A}/\mathcal{I} \to \mathcal{B}$ that makes the triangle

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "P"'] & \mathcal{B} \\ \mathcal{A}/\mathcal{I} \arrow[ur, dashed, "F'"'] & \end{tikzcd}$$

commute if and only if $\mathcal{I} \subseteq \ker(F)$. The induced functor F' is then unique with this property and again additive.

vii) If $\mathcal{A}$ is an additive category and $\mathcal{I}$ is an ideal in $\mathcal{A}$ then the resulting quotient category $\mathcal{A}/\mathcal{I}$ is again additive: Let $X_1, \dots, X_n$ be finitely many objects in $\mathrm{Ob}(\mathcal{A}/\mathcal{I})$ and let $(X, (c_i)_{i=1}^n, (p_i)_{i=1}^n)$ be a biprodct of $X_1, \dots, X_n$ in $\mathcal{A}$. Then $(X, ([c_i])_{i=1}^n, ([p_i])_{i=1}^n)$ is a biproduct of $X_1, \dots, X_n$ in $\mathcal{A}/\mathcal{I}$ because the projection functor $P\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ respects biproducts by part i) of Theorem 3.24.

Example⋆ 4.K.

i) In the category **Top** of topological spaces a congruence relation $\sim$ is given by

$$f \sim g \iff f \text{ and } g \text{ are homotopic}$$

for any two parallel continuous maps $f, g\colon X \to Y$. The resulting quotient category $\mathbf{Top}/{\sim}$ is precisely the homotopy category **hTop**.

ii) We may regard a ring R as a preadditive category $\mathcal{R}$ that consists of a single object $*$, for which $\mathcal{R}(*,*) = R$. Then an ideal $\mathcal{I}$ in the category $\mathcal{R}$ is the same as an additive subgroup $I = \mathcal{I}(X,X)$ of R that satisfies $RI = I$ and $IR = R$. In other words, ideals in $\mathcal{R}$ are the same as two-sided ideals in R.

We also note that the quotient category $\mathcal{R}/\mathcal{I}$ is then precisely the quotient ring R/I regarded as a one-object preadditive category.

iii) If G is a group, then we can similarly regard G as a category $\mathcal{G}$ consisting of a single object $*$ for which $\mathcal{G}(*,*) = G$. Then a congruence relations $\sim$ in $\mathcal{G}$ is the same as normal subgroup N of G via

$$N := \{g \in G \mid g \sim 1\}.$$

The quotient category $\mathcal{G}/{\sim}$ is then precisely the quotient group G/N regarded as a single-object category.

Warning⋆ 4.L. If $\mathcal{A}$ is an abelian category and $\mathcal{I}$ is an ideal in $\mathcal{A}$ then the quotient category $\mathcal{A}/\mathcal{I}$, which is again additive, won't necessarily be abelian again.

Definition 4.19.

i) Let $C_\bullet$ and $D_\bullet$ be two chain complexes in $\mathcal{A}$, and let $f, g\colon C_\bullet \to D_\bullet$ be two parallel morphisms of chain complexes.

   a) The morphism $f = (f_n)_{n \in \mathbb{Z}}$ is *null homotopic* if there exists a family $(s_n)_{n \in \mathbb{Z}}$ of morphisms $s_n\colon C_n \to D_{n+1}$ such that $f = ds + sd$, i.e. such that

   $$f_n = d_{n+1} s_n + s_{n-1} d_n$$

   for every $n \in \mathbb{Z}$.

   b) The morphisms f and g are *homotopic* if their difference $f - g$ is null homotopic. That the morphisms f and g are homotopic is denoted by $f \sim g$.

   c) The morphism f is a *homotopy equivalence* if there exist a morphism of chain complexes $h\colon D_\bullet \to C_\bullet$ such that $fh \sim \mathrm{id}_{D_\bullet}$ and $hf \sim \mathrm{id}_{C_\bullet}$.

ii) Let $C^\bullet$ and $D^\bullet$ be two cochain complexes in $\mathcal{A}$, and let $f, g\colon C^\bullet \to D^\bullet$ be two parallel morphisms of cochain complexes.

   a) The morphism $f = (f^n)_{n \in \mathbb{Z}}$ is *null homotopic* if there exists a family $(s^n)_{n \in \mathbb{Z}}$ of morphisms $s^n\colon C^n \to D^{n-1}$ such that $f = ds + sd$, i.e. such that

   $$f^n = d^{n-1} s^n + s^{n+1} d^n$$

   for every $n \in \mathbb{Z}$.

   b) The morphisms f and g are *homotopic* if their difference $f - g$ is null homotopic. That the morphisms f and g are homotopic is denoted by $f \sim g$.

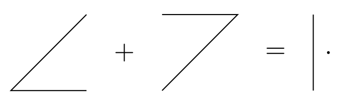
   c) The morphisms f is a *homotopy equivalence* if there exist a morphism of cochain complexes $h\colon D^\bullet \to C^\bullet$ such that $fh \sim \mathrm{id}_{D^\bullet}$ and $hf \sim \mathrm{id}_{C^\bullet}$.

Remark⋆ 4.M. Let $C_\bullet$ and $D_\bullet$ be two chain complexes in $\mathcal{A}$ and let $f, g\colon C_\bullet \to D_\bullet$ be two parallel morphisms of chain complexes.

i) A family $s = (s_n)_{n \in \mathbb{Z}}$ of morphisms $s_n\colon C_n \to D_{n+1}$ with $f = ds + sd$ is a *null homotopy* for f. Such a null homotopy s for f may be visualized as the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] \arrow[dl, dashed] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dl, dashed, "s_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, dashed, "s_{n-1}"'] & \cdots \arrow[dl, dashed] \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d_{n+1}"] & D_n \arrow[r, "d_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

together with the formula



ii) A family $s = (s_n)_{n\in\mathbb{Z}}$ of morphisms $s_n\colon C_n \to D_{n+1}$ with $f - g = ds + sd$, i.e. a null homotopy for $f - g$, is a *homotopy* between f and g.

Remark 4.20. Let $C_\bullet$ and $D_\bullet$ be morphisms of chain complexes in $\mathcal{A}$. If $(s_n)_{n\in\mathbb{N}}$ is any family of morphisms $s_n\colon C_n \to D_{n+1}$ then $f \coloneqq (f_n)_{n\in\mathbb{Z}}$ with $f_n \coloneqq d_{n+1}s_n + s_{n-1}d_n$ is a morphism of chain complexes $f\colon C_\bullet \to D_\bullet$ because

$$df = d(ds + sd) = d^2 s + dsd = dsd = dsd + sd^2 = (ds + sd)d = fd\,.$$

Lemma 4.21. Let $C_\bullet$ and $D_\bullet$ be chain complexes and let $f, g\colon C_\bullet \to D_\bullet$ be two parallel morphisms.

i) If the morphism f is null homotopic then $\mathrm{H}_n(f) = 0$ for every $n \in \mathbb{Z}$.

ii) If the morphisms f and g are homotopic then $\mathrm{H}_n(f) = \mathrm{H}_n(g)$ for every $n \in \mathbb{Z}$.

iii) If f is a homotopy equivalence then f is a quasi-isomorphism.

The same holds for cochain complexes.

Proof.

i) This is Exercise 1 of Exercise sheet 9.

ii) This follows from part i) by the additivity of H_n.

iii) This follows from part ii) by the functoriality of H_n. □

Remark 4.22. At this point in the lecture it was remarked that the homology functors $\mathrm{H}_n\colon \mathbf{Ch}_\bullet(\mathcal{A}) \to \mathcal{A}$ are additive. We have (thanks to some personal modifications) already seen this (in a slightly different way than in the lecture) in Remark 4.4.

Example 4.23. Let $f, g\colon X \to Y$ be two continuous maps between topological spaces X and Y. If the maps f and g are homotopic then their induced morphisms of chain complexes $f_*, g_*\colon C_\bullet^{\mathrm{sing}}(X) \to C_\bullet^{\mathrm{sing}}(Y)$ are again homotopic.

Remark-Definition 4.24. Let $B_\bullet$, $C_\bullet$, $D_\bullet$ and $E_\bullet$ be chain complexes in $\mathcal{A}$.

i) The set

$$N = \{f \in \mathrm{Hom}_{\mathbf{Ch}_\bullet(\mathcal{A})}(C_\bullet, D_\bullet) \mid f \text{ is null homotopic}\}$$

is a subgroup of $\mathrm{Hom}_{\mathbf{Ch}_\bullet(\mathcal{A})}(C_\bullet, D_\bullet)$:

- It holds that $0 \in \mathrm{Hom}_{\mathbf{Ch}_\bullet(\mathcal{A})}(C_\bullet, D_\bullet)$, i.e. the zero morphism is null homotopic. This can be seen by choosing for the required nullhomotpy $s = (s_n)_{n\in\mathbb{Z}}$ the zero morphism $s_n = 0$ for every $n \in \mathbb{Z}$.
- If two morphisms of chain complexes $f, g\colon C_\bullet \to D_\bullet$ are null homotopic then there exist null homotopies $s = (s_n)_{n\in\mathbb{Z}}$ and $t = (t_n)_{n\in\mathbb{Z}}$ for f and g. Then $u = (u_n)_{n\in\mathbb{Z}}$ with $u_n \coloneqq s_n - t_n$ is a null homotopy for $f - g$ because

$$\begin{aligned} du + ud = d(s - t) + (s - t)d &= ds - dt + sd - td \\ &= ds + sd - (dt + td) = f - g\,. \end{aligned}$$

  This show that for all $f, g \in N$ also $f - g \in N$.

It follows that homotopy of morphisms of chain complexes is an equivalence relation on $\mathrm{Hom}_{\mathbf{Ch}_\bullet(\mathcal{A})}(C_\bullet, D_\bullet)$.

ii) If $f, f', g, g' \colon C_\bullet \to D_\bullet$ are morphisms of chain complexes with both $f \sim g$ and $f' \sim g'$ then also $f + g \sim f' + g'$: We find for N as above that it follows from $f - f', g - g' \in N$ that also

$$(f+g) - (f'+g') = (f - f') + (g - g') \in N\,,$$

because N is a subgroup of $\mathrm{Hom}_{\mathbf{Ch}_\bullet(\mathcal{A})}(X, Y)$.

iii) Let $f \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes that is null homotopic, say via a null homotopy $(s_n)_{n\in\mathbb{Z}}$. Then for every morphism of chain complexes $g \colon D_\bullet \to E_\bullet$ the composition gf is again null homotopic: The family $t = (t_n)_{n\in\mathbb{Z}}$ given by $t_n \coloneqq g_{n+1} s_n$ is a null homotopy for gf because

$$gf = g(ds + sd) = gds + gsd = dgs + gsd = dt + td\,.$$

We find similarly that for every morphism of chain complexes $h \colon B_\bullet \to C_\bullet$ the composition fh is again null homotopic: The family $u = (u_n)_{n\in\mathbb{Z}}$ given by $u_n \coloneqq s_n h_n$ is a null homotopy for fh because

$$fh = (ds + sd)h = dsh + sdh = dsh + shd = du + ud\,.$$

It follows for all morphisms of chain complexes $f, f' \colon C_\bullet \to D_\bullet$ and $g, g' \colon D_\bullet \to E_\bullet$ with $f \sim f'$ and $g \sim g'$ that also $gf \sim g'f'$. Indeed, the difference

$$gf - g'f' = gf - gf' + gf' - g'f' = g(f - f') + (g - g')f'$$

is nullhomotpic because both $f - f'$ and $g - g'$ are null homotopic.

iv) This shows that the composition of morphisms in $\mathbf{Ch}_\bullet(\mathcal{A})$ descends to a composition of equivalence classes with respect to homotopy. We hence obtain a category $\mathbf{K}_\bullet(\mathcal{A})$ with

- objects $\mathrm{Ob}(\mathbf{K}_\bullet(\mathcal{A})) \coloneqq \mathrm{Ob}(\mathbf{Ch}_\bullet(\mathcal{A}))$,
- morphism sets

$$\begin{aligned}\mathbf{K}_\bullet(\mathcal{A})(C_\bullet, D_\bullet) &\coloneqq \mathrm{Hom}_{\mathbf{Ch}_\bullet(\mathcal{A})}(C_\bullet, D_\bullet)/\!\sim \\ &= \mathrm{Hom}_{\mathbf{Ch}_\bullet(\mathcal{A})}(C_\bullet, D_\bullet)/\text{homotopy}\end{aligned}$$

  for every two chain complexes $C_\bullet$ and $D_\bullet$ in $\mathcal{A}$,
- and composition of morphisms $[f] \colon C_\bullet \to D_\bullet$ and $[g] \colon D_\bullet \to E_\bullet$ given by

$$[g] \circ [f] \coloneqq [g \circ f]\,.$$

The category $\mathbf{K}_\bullet(\mathcal{A})$ is the *homotopy category* of $\mathcal{A}$.

v) The homotopy category $\mathbf{K}_\bullet(\mathcal{A})$ is again additve: The preadditive structure of $\mathbf{K}_\bullet(\mathcal{A})$ is inherited from the category $\mathbf{Ch}_\bullet(\mathcal{A})$ via

$$[f] + [g] \coloneqq [f + g]$$

for any two parallel morphisms $[f], [g] \colon C_\bullet \to D_\bullet$ in $\mathbf{K}_\bullet(\mathcal{A})$. Part ii) shows that this addition is well-defined, and that $\mathbf{K}_\bullet(\mathcal{A})$ satisfies the axioms of a preadditive category follows from $\mathbf{Ch}_\bullet(\mathcal{A})$ satisfying these axioms.

If $C^{(1)}_\bullet, \dots, C^{(n)}_\bullet$ are finitely many chain complexes in $\mathcal{A}$, i.e. objects of the homotopy category $\mathbf{K}_\bullet(\mathcal{A})$, then a biproduct of $C^{(1)}_\bullet, \dots, C^{(n)}_\bullet$ in $\mathbf{K}_\bullet(\mathcal{A})$ is given by $(C_\bullet, ([c_i])_{i=1}^n, ([p_i])_{i=1}^n)$ if $(C_\bullet, (c_i)_{i=1}^n, (p_i)_{i=1}^n)$ is a biproduct of $C^{(1)}_\bullet, \dots, C^{(n)}_\bullet$ in $\mathbf{Ch}_\bullet(\mathcal{A})$.

Warning. Even though $\mathcal{A}$ and $\mathbf{Ch}_\bullet(\mathcal{A})$ are abelian, the homotopy category $\mathbf{K}_\bullet(\mathcal{A})$ will in general not be abelian again. (A counterexample can be found in Exercise 4 of Exercise sheet 9, where is shown that $\mathbf{K}_\bullet(\mathbf{Ab})$ is not abelian.)

Remark⋆ 4.N. We may reformulate Remark-Definition 4.24 using the language introduced in Remark⋆ 4.J: The null homotopic morphisms of chain complexes form an ideal $\mathcal{N}$ in the additive category $\mathbf{Ch}_\bullet(\mathcal{A})$, whose corresponding additive congruence relation $\sim$ is given by homotopy of morphisms of chain complex. The homotopy category $\mathbf{K}_\bullet(\mathcal{A})$ is precisely the quotient category $\mathbf{Ch}_\bullet(\mathcal{A})/\mathcal{N} = \mathbf{Ch}_\bullet(\mathcal{A})/{\sim}$.

Mapping Cones

Definition 4.25.

i) Let $f \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. The *mapping cone* of f is the chain complex $\operatorname{cone}(f)$ that is given by
   - the components $\operatorname{cone}(f)_n \coloneqq C_{n-1} \oplus D_n$ for every $n \in \mathbb{Z}$, together with
   - the differential

   $$d_n^{\operatorname{cone}(f)} \coloneqq \begin{bmatrix} -d^C_{n-1} & 0 \\ -f_{n-1} & d^D_n \end{bmatrix} \colon C_{n-1} \oplus D_n \to C_{n-2} \oplus D_{n-1}$$

   for every $n \in \mathbb{Z}$.

ii) Let $f \colon C^\bullet \to D^\bullet$ be a morphism of cochain complexes. The *mapping cone* of f is the cochain complex $\operatorname{cone}(f)$ that is given by
   - the components $\operatorname{cone}(f)^n \coloneqq C^{n+1} \oplus D^n$ for every $n \in \mathbb{Z}$, together with
   - the differential

   $$d^n_{\operatorname{cone}(f)} \coloneqq \begin{bmatrix} -d^{n+1}_C & 0 \\ -f^{n+1} & d^n_D \end{bmatrix} \colon C^{n+1} \oplus D^n \to C^{n+2} \oplus D^{n+1}$$

   for every $n \in \mathbb{Z}$.

Remark 4.26. If $f\colon C_\bullet \to D_\bullet$ is a morphism of chain complexes, or $f\colon C^\bullet \to D^\bullet$ a morphism of cochain complexes, then cone(f) is indeed a (co)chain complex because

$$\begin{bmatrix} -d & 0 \\ -f & d \end{bmatrix} \begin{bmatrix} -d & 0 \\ -f & d \end{bmatrix} = \begin{bmatrix} d^2 & 0 \\ fd - df & d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

Proposition 4.27. Let $f\colon C_\bullet \to D_\bullet$ be a morphism of chain complexes.

i) The family $\alpha = (\alpha_n)_{n\in\mathbb{Z}}$ of morphisms

$$\alpha_n \coloneqq \begin{bmatrix} 0 \\ \mathrm{id}_{D_n} \end{bmatrix} : D_n \to C_{n-1} \oplus D_n$$

is a morphism of chain complexes $\alpha\colon D_\bullet \to \mathrm{cone}(f)$.

ii) The family $\beta = (\beta_n)_{n\in\mathbb{Z}}$ of morphisms

$$\beta_n \coloneqq \begin{bmatrix} -\mathrm{id}_{C_{n-1}} & 0 \end{bmatrix} : C_{n-1} \oplus D_n \to C_{n-1}$$

is a morphism of chain complexes $\beta\colon \mathrm{cone}(f) \to C_\bullet[-1]$.

iii) The morphisms α and β fit into a short exact sequence

$$0 \to D_\bullet \xrightarrow{\alpha} \mathrm{cone}(f) \xrightarrow{\beta} C_\bullet[-1] \to 0 .$$

iv) In the induced long exact homology sequence

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}_{n+1}(C_\bullet[-1]) \arrow[dll, "\partial_{n+1}"] \\ \mathrm{H}_n(D_\bullet) \arrow[r, "\mathrm{H}_n(\alpha)"] & \mathrm{H}_n(\mathrm{cone}(f)) \arrow[r, "\mathrm{H}_n(\beta)"] & \mathrm{H}_n(C_\bullet[-1]) \arrow[dll, "\partial_n"] \\ \mathrm{H}_{n-1}(D_\bullet) \arrow[r] & \cdots & \end{tikzcd}$$

the connecting morphism

$$\partial_{n+1}\colon \mathrm{H}_n(C_\bullet) = \mathrm{H}_{n+1}(C_\bullet[-1]) \to \mathrm{H}_n(D_\bullet)$$

is given by $\mathrm{H}_n(f)$ for every $n \in \mathbb{Z}$.

Proof. This proof is currently missing from these notes, and will be added later. (The proof relies on the explicit construction of the connecting homomorphism in the snake lemma.) □

Corollary 4.28. A morphism of chain complexes $f\colon C_\bullet \to D_\bullet$ is a quasi-isomorphism if and only if its cone is acyclic.

Proof. We get by Proposition 4.27 a long exact sequence

$$\cdots \to \mathrm{H}_{n+1}(\mathrm{cone}(f)) \to \mathrm{H}_n(C_\bullet) \xrightarrow{\mathrm{H}_n(f)} \mathrm{H}_n(D_\bullet) \to \mathrm{H}_n(\mathrm{cone}(f)) \to \cdots$$

The morphism f is a quasi-isomorphism if and only if $\mathrm{H}_n(f)$ is for every $n \in \mathbb{Z}$ an isomorphism. This holds by the exactness of the above sequence if and only if $\mathrm{H}_n(\mathrm{cone}(f)) = 0$ for every $n \in \mathbb{Z}$. This is precisely what it means for $\mathrm{cone}(f)$ to be acyclic. ☐

Remark⋆ 4.O. Let $C_\bullet$, $D_\bullet$ and $E_\bullet$ be chain complexes in $\mathcal{A}$.

i) Let $f\colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. Then for a family $g = (g_n)_{n\in\mathbb{Z}}$ of morphisms

$$g_n \colon C_{n-1} \oplus D_n \to E_n$$

we have that g is a morphism of chain complexes $g\colon \mathrm{cone}(f) \to E_\bullet$ if and only if

$$d^E_n g_n = g_{n-1} d^{\mathrm{cone}(f)}_n \tag{4.4}$$

for every $n \in \mathbb{Z}$. We may write for every $n \in \mathbb{Z}$ the morphism g_n as

$$g_n = \begin{bmatrix} s_{n-1} & -h_n \end{bmatrix}$$

for unique morphisms $s_{n-1}\colon C_{n-1} \to E_n$ and $h_n\colon D_n \to E_n$. We can then rewrite the condition (4.4) for every $n \in \mathbb{Z}$ as

$$\begin{aligned} & d^E_n g_n = g_{n-1} d^{\mathrm{cone}(f)}_n \\ \iff & d_n \begin{bmatrix} s_{n-1} & -h_n \end{bmatrix} = \begin{bmatrix} s_{n-2} & -h_{n-1} \end{bmatrix} \begin{bmatrix} -d_{n-1} & 0 \\ -f_{n-1} & d_n \end{bmatrix} \\ \iff & \begin{bmatrix} d_n s_{n-1} & -d_n h_n \end{bmatrix} = \begin{bmatrix} -s_{n-2} d_{n-1} + h_{n-1} f_{n-1} & -h_{n-1} d_n \end{bmatrix} \\ \iff & \left\{ \begin{array}{rcl} h_{n-1} f_{n-1} & = & d_n s_{n-1} + s_{n-2} d_{n-1} \,, \\ d_n h_n & = & h_{n-1} d_n \,. \end{array} \right. \end{aligned}$$

That the second condition holds for every $n \in \mathbb{Z}$ means precisely that $h \coloneqq (h_n)_{n\in\mathbb{Z}}$ is a morphism of chain complexes $h\colon D_\bullet \to E_\bullet$, and the first condition means that $s \coloneqq (s_n)_{n\in\mathbb{Z}}$ is a null homotopy for the composition hf.

We hence find that morphisms of chain complexes $g\colon \mathrm{cone}(f) \to E_\bullet$ are in one-to-one correspondence to pairs (h, s) consisting of

- a morphism of chain complexes $h\colon D_\bullet \to E_\bullet$ such that the composition hf is null homotopic, and
- a choosen null homotopy s for hf.

ii) The *cone* of the chain complex $C_\bullet$ it the mapping cone of its identity, i.e.

$$\mathrm{cone}(C_\bullet) \coloneqq \mathrm{cone}(\mathrm{id}_{C_\bullet}) \,.$$

We have the canonical morphism of chain complexes $C_\bullet \to \operatorname{cone}(C_\bullet)$, whose n-th component is for every $n \in \mathbb{Z}$ given by

$$\begin{bmatrix} 0 & \mathrm{id} \end{bmatrix} : C_n \to C_{n-1} \oplus C_n \,.$$

We get from the above discussion that a morphism of chain complexes $f \colon C_\bullet \to D_\bullet$ extends to a morphism of chain complexes $\hat{f} \colon \operatorname{cone}(f) \to D_\bullet$, in the sense that the diagram

$$\begin{tikzcd} \operatorname{cone}(f) \arrow[dr, dashed, "\hat{f}"] & \\ C_\bullet \arrow[u] \arrow[r, "f"'] & D_\bullet \end{tikzcd}$$

commutes, if and only if the morphism f is null homotopic. (This is also Exercise 3 of Exercise Sheet 9.) To be more precise, such extensions $\hat{f}$ of f are in one-to-one correspondence to null homotopies of f.

$\boxed{\text{End of lecture 17}}$

5 Derived Functors

δ-Functors

Convention. In the following, $\mathcal{A}$ and $\mathcal{B}$ denote abelian categories.

Definition 5.1 ((Co)homological δ-functors)**.**

i) A *homological δ-functor* from $\mathcal{A}$ to $\mathcal{B}$ is a pair $((T_n)_{n\geq 0}, (\delta_n^\xi)_{n\geq 1,\xi})$ consisting of
   - a family $(T_n)_{n\geq 0}$ of additive functors $T_n \colon \mathcal{A} \to \mathcal{B}$, and
   - a family $(\delta_n^\xi)_{n\geq 1,\xi}$ of morphisms $\delta_n^\xi \colon T_n(X'') \to T_{n-1}(X')$, where ξ ranges through the class of short exact sequences in $\mathcal{A}$.

   These data are subject to the following conditions:

   (Hδ1) For every short exact sequence

   $$\xi \colon 0 \to X' \to X \to X'' \to 0$$

   in $\mathcal{A}$, the resulting sequence

   $$\cdots \longrightarrow T_{n+1}(X'') \xrightarrow{\delta_{n+1}^\xi} T_n(X') \longrightarrow T_n(X) \longrightarrow T_n(X'') \xrightarrow{\delta_n^\xi} T_{n-1}(X') \longrightarrow \cdots \longrightarrow T_1(X'') \xrightarrow{\delta_1^\xi} T_0(X') \longrightarrow T_0(X) \longrightarrow T_0(X'') \longrightarrow 0$$

   in $\mathcal{B}$ is exact.

   (Hδ2) If

   $$\begin{tikzcd}
   \xi \colon & 0 \arrow[r] & X' \arrow[r] \arrow[d, dashed, "f'"] & X \arrow[r] \arrow[d, dashed, "f"] & X'' \arrow[r] \arrow[d, dashed, "f''"] & 0 \\
   \zeta \colon & 0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 0
   \end{tikzcd}$$

is a commutative diagram in $\mathcal{A}$ with (short) exact rows ξ and ζ, then the induced square

$$\begin{tikzcd} T_n(X'') \arrow[r, "\delta_n^\xi"] \arrow[d, dashed, "T_n(f'')"'] & T_{n-1}(X') \arrow[d, dashed, "T_{n-1}(f')"] \\ T_n(Y'') \arrow[r, "\delta_n^\zeta"] & T_{n-1}(Y') \end{tikzcd}$$

commutes for every $n \geq 1$.

ii) A *cohomological δ-functor* from $\mathcal{A}$ to $\mathcal{B}$ is a pair $((T^n)_{n\geq 0}, (\delta_\xi^n)_{n\geq 0,\xi})$ consisting of

- a family $(T^n)_{n\geq 0}$ of additive functors $T^n \colon \mathcal{A} \to \mathcal{B}$, and
- a family $(\delta_\xi^n)_{n\geq 0,\xi}$ of morphisms $\delta_\xi^n \colon T^n(X'') \to T^{n+1}(X')$, where ξ ranges through the class of short exact sequences in $\mathcal{A}$.

These data are subject to the following conditions:

(Cδ1) For every short exact sequence

$$\xi \colon 0 \to X' \to X \to X'' \to 0$$

in $\mathcal{A}$, the resulting sequence

$$\begin{tikzcd} 0 \arrow[r] & T^0(X') \arrow[r] & T^0(X) \arrow[r] & T^0(X'') \arrow[dll, "\delta_\xi^0"] \\ & T^1(X') \arrow[r] & \cdots \arrow[r] & T^n(X'') \arrow[dll, "\delta_\xi^n"] \\ & T^{n+1}(X') \arrow[r] & T^{n+1}(X) \arrow[r] & T^{n+1}(X'') \arrow[dll, "\delta_\xi^{n+1}"] \\ & T^{n+2}(X') \arrow[r] & \cdots \end{tikzcd}$$

in $\mathcal{B}$ is exact.

(Cδ2) If

$$\begin{tikzcd} \xi\colon & 0 \arrow[r] & X' \arrow[r] \arrow[d, dashed, "f'"] & X \arrow[r] \arrow[d, dashed, "f"] & X'' \arrow[r] \arrow[d, dashed, "f''"] & 0 \\ \zeta\colon & 0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram in $\mathcal{A}$ with (short) exact rows ξ and ζ, then the induced square

$$\begin{tikzcd} T^n(X'') \arrow[r, "\delta_\xi^n"] \arrow[d, dashed, "T^n(f'')"'] & T^{n+1}(X') \arrow[d, dashed, "T^{n+1}(f')"] \\ T^n(Y'') \arrow[r, "\delta_\zeta^n"] & T^{n+1}(Y') \end{tikzcd}$$

commutes for every $n \geq 0$.

Remark 5.2.

i) If $T_\bullet = ((T_n)_{n\geq 0}, (\delta_n^\xi)_{n\geq 0,\xi})$ is a homological δ-functor from $\mathcal{A}$ to $\mathcal{B}$ then we get a cohomological δ-functor $T^\bullet = ((T^n)_{n\geq 0}, (\delta_\xi^n)_{n\geq 0,\xi})$ from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}^{\mathrm{op}}$ by setting
$$T^n \coloneqq T_n \colon \mathcal{A} \to \mathcal{B}$$
for every $n \in \mathbb{Z}$, and
$$\delta_\xi^n = \delta_{n+1}^\xi \colon T^n(X'') \to T^{n+1}(X')$$
for all $n \geq 0$ and every short exact sequence
$$\xi \colon 0 \to X' \to X \to X'' \to 0$$
in $\mathcal{A}^{\mathrm{op}}$. (Note that the above short exact sequence in $\mathcal{A}^{\mathrm{op}}$ is the same as a short exact sequence
$$0 \leftarrow X' \leftarrow X \leftarrow X'' \leftarrow 0 : \xi$$
in $\mathcal{A}$, and that the resulting connecting morphism $\delta_{n+1}^\xi \colon T_{n+1}(X') \to T_n(X'')$ in $\mathcal{B}$ is a morphism $T^n(X'') \to T^{n+1}(X')$ in $\mathcal{B}^{\mathrm{op}}$.) The converse construction also works, and hence homological δ-functors $\mathcal{A} \to \mathcal{B}$ are the same as cohomological δ-functors $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$.

ii) If $T_\bullet$ is a homological δ-functor then its zeroeth component T_0 is right exact.

iii) If $T^\bullet$ is a cohomological δ-functor then its zeroeth component T^0 is left exact.

Example 5.3. Let $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ be the full subcategories of $\mathbf{Ch}_\bullet(\mathcal{A})$ whose objects are given by those chain complexes $C_\bullet$ in $\mathcal{A}$ with $C_n = 0$ for every $n < 0$. This subcategory of $\mathbf{Ch}_\bullet(\mathcal{A})$ is closed under taking biproducts, kernels and cokernels. Hence $\mathbf{Ch}_\bullet(\mathcal{A})$ is again abelian (and the inclusion functor $\mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{Ch}_\bullet(\mathcal{A})$ is exact). The long exact sequence of homology, together with the connecting morphisms ∂_n^ξ, result in a δ-functor $\mathrm{H}_\bullet = ((\mathrm{H}_n)_{n\geq 0}, (\partial_n^\xi)_{n\geq 1,\xi})$ from $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ to $\mathcal{A}$.

Dually, the full subcategory $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ of $\mathbf{Ch}^\bullet(\mathcal{A})$, whose objects are given by those cochain complexes $C^\bullet$ in $\mathcal{A}$ with $C^n = 0$ for every $n < 0$, is again abelian, and $\mathrm{H}^\bullet = ((\mathrm{H}^n)_{n\geq 0}, (\partial_\xi^n)_{n\geq 0})$ is a δ-functor from $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ to $\mathcal{A}$.

Example 5.4. Let A be a $\mathbf{k}$-algebra and let $a \in A$. The *a-torsion* of an A-module M is
$${}_{(a)}M \coloneqq \{m \in M \mid am = 0\},$$
which is a $\mathbf{k}$-submodule of M. For every short exact sequence of A-modules
$$0 \to M' \to M \to M'' \to 0$$
we get the following commutative diagram of $\mathbf{k}$-modules with (short) exact rows:
$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, dashed, "a\cdot(-)"] & M \arrow[r] \arrow[d, dashed, "a\cdot(-)"] & M'' \arrow[r] \arrow[d, dashed, "a\cdot(-)"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

It follows from the snake lemma that we get a long exact sequence of $\mathbf{k}$-modules

$$0 \to {}_{(a)}M' \to {}_{(a)}M \to {}_{(a)}M'' \xrightarrow{\delta} M'/aM' \to M/aM \to M''/aM'' \to 0\,.$$

For the additive functors $T^n \colon A\text{-}\mathbf{Mod} \to \mathbf{k}\text{-}\mathbf{Mod}$ with

$$T^0(M) \coloneqq {}_{(a)}M \quad \text{and} \quad T^1(M) \coloneqq M/aM$$

for every A-module M, and $T^n = 0$ for every $n \geq 2$, we get from the above a cohomological δ-functor $T^\bullet \colon A\text{-}\mathbf{Mod} \to \mathbf{k}\text{-}\mathbf{Mod}$.

Definition 5.5.

i) Let $S_\bullet, T_\bullet \colon \mathcal{A} \to \mathcal{B}$ be two parallel homological δ-functors. A *morphism* of homological δ-functor $\eta \colon S_\bullet \to T_\bullet$ is a family $\eta = (\eta_n)_{n \geq 0}$ of natural transformations $\eta_n \colon S_n \to T_n$ such that for every short exact sequence

$$\xi \colon 0 \to X' \to X \to X'' \to 0$$

in $\mathcal{A}$ the induced square

$$\begin{tikzcd} S_n(X'') \arrow[r, "\delta_n^\xi"] \arrow[d, dashed, "(\eta_n)_{X''}"'] & S_{n-1}(X') \arrow[d, dashed, "(\eta_{n-1})_{X'}"] \\ T_n(X'') \arrow[r, "\delta_n^\xi"] & T_{n-1}(X'') \end{tikzcd}$$

in $\mathcal{B}$ commutes for every $n \geq 1$.

ii) Let $S^\bullet, T^\bullet \colon \mathcal{A} \to \mathcal{B}$ be two parallel cohomological δ-functors. A *morphism* of cohomological δ-functor $\eta \colon S^\bullet \to T^\bullet$ is a family $\eta = (\eta^n)_{n \geq 0}$ of natural transformations $\eta^n \colon S^n \to T^n$ such that for every short exact sequence

$$\xi \colon 0 \to X' \to X \to X'' \to 0$$

in $\mathcal{A}$ the induced square

$$\begin{tikzcd} S^n(X'') \arrow[r, "\delta_\xi^n"] \arrow[d, dashed, "\eta^n_{X''}"'] & S^{n+1}(X') \arrow[d, dashed, "\eta^{n+1}_{X'}"] \\ T^n(X'') \arrow[r, "\delta_\xi^n"] & T^{n+1}(X'') \end{tikzcd}$$

in $\mathcal{B}$ commutes for every $n \geq 0$.

iii) A homological δ-functor $U_\bullet \colon \mathcal{A} \to \mathcal{B}$ is *universal* if for every homological δ-functor $T_\bullet \colon \mathcal{A} \to \mathcal{B}$, every natural transformation $\eta_0 \colon T_0 \to U_0$ can be uniquely extended to a morphism of homological δ-functors $\eta \colon T_\bullet \to U_\bullet$.

iv) Dually, a cohomological δ-functor $U^\bullet \colon \mathcal{A} \to \mathcal{B}$ is *universal* if for every cohomological δ-functor $T^\bullet \colon \mathcal{A} \to \mathcal{B}$, every natural transformation $\eta^0 \colon U^0 \to T^0$ can be extended uniquely to a morphism of cohomological δ-functors $\eta \colon U^\bullet \to T^\bullet$.

End of lecture 18

Example 5.6. We will later see that the δ-functors

$$\mathrm{H}_\bullet \colon \mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{A} \qquad \text{and} \qquad \mathrm{H}^\bullet \colon \mathbf{Ch}^{\geq 0}(\mathcal{A}) \to \mathcal{A}$$

are universal.

Definition 5.7. Let $F \colon \mathcal{A} \to \mathcal{B}$ be an additive functor.

i) A *left derived functor* of F is a universal homological δ-functor $\mathrm{L}_\bullet F \colon \mathcal{A} \to \mathcal{B}$ with $\mathrm{L}_0 F \cong F$.

ii) A *right derived functor* of F is a universal cohomological δ-functor $\mathrm{R}^\bullet F \colon \mathcal{A} \to \mathcal{B}$ with $\mathrm{R}^0 F \cong F$.

Remark 5.8. Left derived functors are, if they exist, unique up to isomorphism in the following sense: If $F \colon \mathcal{A} \to \mathcal{B}$ is an additive functor and $U_\bullet, V_\bullet \colon \mathcal{A} \to \mathcal{B}$ are two left derived functors of F then $U_0 \cong V_0$, and every isomorphism $U_0 \to V_0$ extends uniquely to a morphism of δ-functors $U_\bullet \to V_\bullet$:

Indeed, it holds that $U_0 \cong F \cong V_0$. If $\eta_0 \colon U_0 \to V_0$ is such an isomorphism then η_0 extends uniquely to a morphism of δ-functors $\eta \colon U_\bullet \to V_\bullet$, because the δ-functor $V_\bullet$ is universal. Similarly, we can extend $\eta_0^{-1} \colon V_0 \to U_0$ uniquely to a morphism of δ-functors $\zeta \colon V_\bullet \to U_\bullet$. Then $\zeta\eta \colon U_\bullet \to U_\bullet$ is a morphism of δ-functors that extends the natural transformation

$$(\zeta\eta)_0 = \zeta_0\eta_0 = \eta_0^{-1}\eta_0 = \mathrm{id}_{U_0} \ .$$

The identity $\mathrm{id}_{U_\bullet} \colon U_\bullet \to U_\bullet$ is another morphism of δ-functors that extends id_{U_0}, and hence $\zeta\eta = \mathrm{id}_{U_\bullet}$ by the uniqueness of this extension (which holds because the δ-functor $U_\bullet$ is universal). We find similarly that $\eta\zeta = \mathrm{id}_{V_\bullet}$, and hence together that η is an isomorphism of δ-functors with $\eta^{-1} = \zeta$.

For right derived functors the analogous uniqueness result holds.

Notation⋆ 5.A. Because of this uniqueness of left (resp. right) derived functors, we will often talk about *the* left (resp. right) derived functor of an additive functor F.

Remark 5.9. Let $F \colon \mathcal{A} \to \mathcal{B}$ be an additive functor.

i) If F has a left derived functor then F is right exact.

ii) If F has a right derived functor then F is left exact.

Lemma 5.10. Let $F \colon \mathcal{A} \to \mathcal{B}$ be an additive functor and suppose that F is exact.

i) The left derived functor of F exists and is given by $\mathrm{L}_0 F = F$ and $\mathrm{L}_n F = 0$ for every $n > 0$.

ii) The right derived functor of F exists and is given by $\mathrm{R}^0 F = F$ and $\mathrm{R}^n F = 0$ for all $n > 0$.

Proof. We only prove part ii), as part i) follows from duality.

It follows from F being exact that $R^\bullet := ((F, 0, 0, \dots), (0, 0, 0, \dots))$ is a cohomological δ-functor, since for every short exact sequence

$$\xi \colon 0 \to X' \to X \to X'' \to 0$$

in $\mathcal{A}$ the induced sequence

$$0 \to F(X') \to F(X) \to F(X'') \to 0 \to 0 \to \cdots$$

in $\mathcal{B}$ is exact.

Let $T^\bullet \colon \mathcal{A} \to \mathcal{B}$ is a cohomological δ-functor and let $\eta^0 \colon F \to T^0$ be a natural transformation. Then by setting $\eta^n = 0$ for every $n > 0$ we get a morphism of δ-functor $\eta \colon R^\bullet \to T^\bullet$. Indeed, for every short exact sequence

$$\xi \colon 0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

in $\mathcal{A}$ the diagram

$$\begin{tikzcd}
0 \arrow[r] & F(X') \arrow[r] \arrow[d, "\eta^0_{X'}"] & F(X) \arrow[r, "F(g)"] \arrow[d, "\eta^0_X"] & F(X'') \arrow[r] \arrow[d, "\eta^0_{X''}"] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & T^0(X') \arrow[r] & T^0(X) \arrow[r, "T^0(g)"] & T^0(X'') \arrow[r, "\delta^0_\xi"] & T^1(X') \arrow[r] & \cdots
\end{tikzcd}$$

commutes: The first two squares (from the left) commute because η^0 is a natural transformation. For the commutativity of the third square we note that

$$\delta^0_\xi \eta^0_{X''} F(g) = \delta^0_\xi T^0(g) \eta^0_X = 0 \circ \eta^0_X = 0$$

and hence $\delta^0_\xi \eta^0_{X''} = 0$ because $F(g)$ is an epimorphism. That η is already the unique morphism of δ-functor $R^\bullet \to T^\bullet$ that extends the natural transformation η^0 follows from $R^n = 0$ for every $n > 0$. $\square$

Injective and Projective Resolutions

Definition 5.11. Let $\mathcal{C}$ be a category.

i) An object $P \in \mathrm{Ob}(\mathcal{C})$ is *projective* if for every epimorphism $f \colon X \to Y$ in $\mathcal{C}$ and every morphism $g \colon P \to Y$ in $\mathcal{C}$ there exists a lift of g along f, i.e. a morphism $g' \colon P \to X$ that makes the following diagram commute:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "g'"'] \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

ii) An object $I \in \mathrm{Ob}(\mathcal{C})$ is *injective* if for every monomorphism $f\colon X \to Y$ and every morphism $g\colon X \to I$ there exist a morphism $g'\colon Y \to I$ that makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[dl, dashed, "g'"] \\ I & \end{tikzcd}$$

Remark 5.12. Let $\mathcal{C}$ be a category.

i) An object $X \in \mathrm{Ob}(\mathcal{C})$ is projective in $\mathcal{C}$ if and only if it is injective in $\mathcal{C}^{\mathrm{op}}$.

ii) The induced morphisms g' in Definition 5.11 are in general not unique.

Lemma 5.13.

i) For an object $P \in \mathrm{Ob}(\mathcal{A})$ the following conditions are equivalent:
   a) The object P is projective.
   b) The functor $\mathrm{Hom}_{\mathcal{A}}(P, -)\colon \mathcal{A} \to \mathbf{Ab}$ is exact.
   c) The functor $\mathrm{Hom}_{\mathcal{A}}(P, -)\colon \mathcal{A} \to \mathbf{Ab}$ maps epimorphisms to epimorphisms.

ii) For an object $I \in \mathrm{Ob}(\mathcal{A})$ the following conditions are equivalent:
   a) The object I is injective.
   b) The functor $\mathrm{Hom}_{\mathcal{A}}(-, I)\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is exact.
   c) The functor $\mathrm{Hom}_{\mathcal{A}}(-, I)\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ maps epimorphisms to epimorphisms.[1]

Proof. It sufficies to show part i) because part ii) then follows by duality.

a) $\iff$ c) That the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ maps epimorphisms to surjections is just a reformulation of the definition of a projective object.

c) $\implies$ b) This holds because the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is already left exact.

b) $\implies$ c) Every epimorphism $f\colon X \to Y$ can be extended to a short exact sequence

$$0 \to \ker(f) \to X \to Y \to 0$$

in $\mathcal{A}$, and the exactness of the induced sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(P, \ker(f)) \to \mathrm{Hom}_{\mathcal{A}}(P, X) \to \mathrm{Hom}_{\mathcal{A}}(P, Y) \to 0$$

entails that the group homomorphism $\mathrm{Hom}_{\mathcal{A}}(P, X) \to \mathrm{Hom}_{\mathcal{A}}(P, Y)$ is surjective. $\square$

Definition⋆ 5.B. Let $\mathcal{C}$ be a category and let $s\colon X \to Y$ and $r\colon Y \to X$ be two morphisms in $\mathcal{C}$ with $rs = \mathrm{id}_X$. Then the morphism s is a *section* (*for* r) and the morphism r is a *retraction* (*for* s). (This definition is given in Exercise 2 of Exercise Sheet 9.)

[1] If one thinks about $\mathrm{Hom}_{\mathcal{A}}(-, I)$ not as a covariant functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ but instead as a contravariant functor $\mathcal{A} \to \mathbf{Ab}$, then this means that $\mathrm{Hom}_{\mathcal{A}}(-, I)$ maps monomorphisms (in $\mathcal{A}$) to epimorphisms.

Remark⋆ 5.C. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

i) Sections are also called *split monomorphisms*, and retractions are also called *split epimorphisms*.

Split monomorphisms are monomorphisms, and split epimorphisms are epimorphisms (as the names indicate). (This observation is also made in Exercise 2 of Exercise Sheet 9.) Indeed, if $s\colon X \to Y$ and $r\colon Y \to X$ are morphisms in $\mathcal{C}$ with $rs = \mathrm{id}_X$ then it follows for all morphisms $t, t'\colon W \to X$ that

$$st = st' \implies rst = rst \implies t = t' \,,$$

and simiarly for all morphisms $u, u'\colon Y \to Z$ that

$$ur = u'r \implies urs = u'rs \implies u = u' \,.$$

A monomorphism is said to be *split* if it is a split monomorphism. Similary, an epimorphism is said to be *split* if it is a split epimorphism.

ii) A morphism f in $\mathcal{C}$ is a section in $\mathcal{C}$ if and only if it is a retraction in $\mathcal{C}^{\mathrm{op}}$.

iii) A morphism f in $\mathcal{C}$ is both a section and a rectration if and only it is an isomorphism.

iv) Functors respect sections and retractions: Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, let s is a section and $\mathcal{C}$ and r is a retraction in $\mathcal{C}$. Then $F(s)$ is a section in $\mathcal{D}$ and $F(r)$ is a retraction in $\mathcal{D}$.

Lemma⋆ 5.D. For a short exact sequence

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

in $\mathcal{A}$, the following conditions are equivalent:

i) The morphism f is a section.

ii) The morphism g is a retraction.

iii) There exists an isomorphism $\alpha\colon X \to X' \oplus X''$ that makes the diagram

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "f"] \arrow[d, equal] & X \arrow[r, "g"] \arrow[d, dashed, "\alpha"] & X'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X' \arrow[r] & X' \oplus X'' \arrow[r] & X'' \arrow[r] & 0 \end{tikzcd}$$

commute, where $X' \to X' \oplus X''$ and $X' \oplus X'' \to X''$ are the canonical morphisms that are part of the biproduct structure of $X' \oplus X''$.

iv) There exists morphisms $r\colon X \to X'$ and $s\colon X'' \to X$ such that $(X, (f, s), (r, g))$ is a biproduct of X' and X''.

Proof⋆. The equivalence of the parts i), ii) and iv) is Exercise 2 of Exercise sheet 9.

The implication iii) $\implies$ iv) can be seen by pulling back the X'-X''-biproduct structure of $X' \oplus X''$ along α to a X'-X''-biproduct structure on X. It then follows from the commutativity of the given diagram that the canonical morphism $X' \to X' \oplus X''$ corresponds to the morphism $f\colon X' \to X$, and that the canonical morphism $X' \oplus X'' \to X''$ corresponds to the morphism $g\colon X \to X''$.

For the implications i) $\implies$ iii) one chooses a retraction $r\colon X \to X'$ of f. Then the morphism

$$\alpha \coloneqq \begin{bmatrix} r \\ g \end{bmatrix} : X \to X' \oplus X''$$

makes the given diagram commute, and it follows from the 5-lemma that α is an isomorphism. □

Definition⋆ 5.E. A short exact sequence in $\mathcal{A}$ is *split* if it satisfies the equivalent conditions from Lemma⋆ 5.D.

Warning⋆ 5.F. While it still makes sense to talk about short exact sequences in the category **Grp** of groups, it is for such a short exact sequence of groups

$$1 \to K \xrightarrow{f} G \xrightarrow{g} H \to 1$$

not equivalent that f is a section and that g is a retraction. Indeed, that f is a section is—roughly speaking—equivalent to G being a direct product of the groups K and H; whereas g being a retract is equivalent to G being a semidirect product of the groups K and H.

Remark⋆ 5.G (Split chain complexes)**.**

i) If more generally $C_\bullet$ is a chain complex in $\mathcal{A}$, then $C_\bullet$ *splits* if there exists a family $s = (s_n)_{n\in\mathbb{Z}}$ of morphisms $s_n\colon C_n \to C_{n+1}$ with $dsd = d$, i.e. such that $d_n s_n d_n = d_n$ for every $n \in \mathbb{Z}$. Such a family s is a *split* for $C_\bullet$, and may be visualized as follows:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[l, dashed, bend right] & C_n \arrow[r, "d_n"] \arrow[l, dashed, bend right, "s_n"'] & C_{n-1} \arrow[r] \arrow[l, dashed, bend right, "s_{n-1}"'] & \cdots \arrow[l, dashed, bend right] \end{tikzcd}$$

ii) A more intuitive but equivalent definition of $C_\bullet$ being split is the following: There exist families $(B_n)_{n\in\mathbb{Z}}$ and $(H_n)_{n\in\mathbb{Z}}$ of objects $B_n, H_n \in \mathrm{Ob}(\mathcal{A})$, and isomorphisms

$$\alpha_n\colon C_n \to B_n \oplus H_n \oplus B_{n-1}$$

that make the following diagram commute for every $n \in \mathbb{Z}$:

$$\begin{tikzcd} \cdots \arrow[r] & C_n \arrow[r, "d_n"] \arrow[d, "\alpha_n"] & C_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & \cdots \\ \cdots \arrow[r] & B_n \oplus H_n \oplus B_{n-1} \arrow[r, "\left[\begin{smallmatrix} 0&0&1\\0&0&0\\0&0&0 \end{smallmatrix}\right]"'] & B_{n-1} \oplus H_{n-1} \oplus B_{n-2} \arrow[r] & \cdots \end{tikzcd}$$

(This characterization of split chain complexes should be compared with the characterization iii) of a split short exact sequence from Lemma* 5.D.) The split morphisms $s_n\colon C_n \to C_{n+1}$ correspond to the morphisms

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} : B_n \oplus H_n \oplus B_{n-1} \to B_{n+1} \oplus H_{n+1} \oplus B_n \,.$$

We furthermore have that $\mathrm{B}_n(C_\bullet) \cong B_n$, $\mathrm{Z}_n(C_\bullet) \cong B_n \oplus H_n$ and $\mathrm{H}_n(C_\bullet) \cong H_n$ for every $n \in \mathbb{Z}$. Moverover, under these identifications the canonical (mono)morphism $\mathrm{B}_n(C_\bullet) \to \mathrm{Z}_n(C_\bullet)$ corresponds to the canonical morphism $B_n \to B_n \oplus H_n$, and the canonical (epi)morphism $\mathrm{Z}_n(C_\bullet) \to \mathrm{H}_n(C_\bullet)$ corresponds to the canonical morphism $B_n \oplus H_n \to H_n$.

iii) A short exact sequence

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

in $\mathcal{A}$ can be regarded as a chain complex

$$\cdots \to 0 \to 0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0 \to 0 \to \cdots$$

in $\mathcal{A}$. Then both notions of 'being split' coincide:

The chain complex is split (according to the above definition) if and only if there exists morphisms $r\colon X \to X'$ and $s\colon X'' \to X$ such that $frf = f$ and $gsg = g$.

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & X' \arrow[r, "f"] & X \arrow[r, "g"] \arrow[l, dashed, bend right, "r"'] & X'' \arrow[r] \arrow[l, dashed, bend right, "s"'] & 0 \arrow[r] & \cdots \end{tikzcd}$$

That $frf = f$ is equivalent to $rf = \mathrm{id}_{X'}$ because f is a monomorphism, and that $gsg = g$ is equivalent to $gs = \mathrm{id}_{X''}$ because g is an epimorphism.

Remark* 5.H.

i) Additive functors respect split short exact sequences: Suppose that $F\colon \mathcal{A} \to \mathcal{B}$ is an an additive functor and let

$$0 \to X' \to X \to X'' \to 0$$

be a split short exact sequence in $\mathcal{A}$. Then the resulting sequence

$$0 \to F(X') \to F(X) \to F(X'') \to 0$$

in $\mathcal{B}$ is again split short exact. Indeed, there exists in $\mathcal{A}$ an isomorphism

$$\alpha\colon X \to X' \oplus X''$$

that makes the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\alpha"] & X'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X' \arrow[r, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & X' \oplus X'' \arrow[r, "{[\,0\ 1\,]}"] & X'' \arrow[r] & 0
\end{tikzcd}$$

By applying the functor F to this diagram, and using that F is additive, we get the following commutative diagram in $\mathcal{B}$:

$$\begin{tikzcd}
0 \arrow[r] & F(X') \arrow[r] \arrow[d, equal] & F(X) \arrow[r] \arrow[d, "F(\alpha)"] & F(X'') \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F(X') \arrow[r, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & F(X') \oplus F(X'') \arrow[r, "{[\,0\ 1\,]}"] & F(X'') \arrow[r] & 0
\end{tikzcd}$$

The lower row is again split short exact and the vertical arrow are isomorphisms, thus the upper row is split short exact.

ii) We find in the same way that additive functors respect split chain complexes and split acyclic chain complexes.

iii) That additive functors respect split chain complexes can also be seen by exhibiting a split for the resulting chain complex: Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor and let $C_\bullet$ be a split chain complex in $\mathcal{A}$, and denote by $F(C_\bullet)$ the resulting chain complex in $\mathcal{B}$, namely

$$\cdots \to F(C_{n+1}) \xrightarrow{F(d_{n+1})} F(C_n) \xrightarrow{F(d_n)} F(C_{n-1}) \to \cdots$$

If $s = (s_n)_{n\in\mathbb{Z}}$ is a split for the chain complex $C_\bullet$, then the resulting family $F(s) \coloneqq (F(s_n))_{n\in\mathbb{Z}}$ is a split for the chain complex $F(C_\bullet)$ because

$$F(d)F(s)F(d) = F(dsd) = F(d)\,.$$

Therefore, the chain complex $F(C_\bullet)$ again splits.

Definition⋆ 5.I. Let $X \in \mathrm{Ob}(\mathcal{A})$ be an object in $\mathcal{A}$. An object $Y \in \mathrm{Ob}(\mathcal{A})$ is a *direct summand* of X if there exists another object $Y' \in \mathrm{Ob}(\mathcal{A})$ with $X \cong Y \oplus Y'$.

Lemma 5.14. Let $P, I \in \mathrm{Ob}(\mathcal{A})$ be a objects in $\mathcal{A}$.

i) The object P is projective if and only if every short exact sequence of the form $0 \to X' \to X \to P \to 0$ in $\mathcal{A}$ splits.

ii) Dually, the object I is injective if and only if every short exact sequence of the form $0 \to I \to X \to X'' \to 0$ in $\mathcal{A}$ splits.

iii) If $P \cong P_1 \oplus P_2$ for objects $P_1, P_2 \in \mathrm{Ob}(\mathcal{A})$ then P is projective if and only if both P_1 and P_2 are projective.

iv) If $I \cong I_1 \oplus I_2$ for objects $I_1, I_2 \in \mathrm{Ob}(\mathcal{A})$ then I is injective if and only if both I_1 and I_2 are projective.

Proof.

i) Suppose that P is projective and let $0 \to X' \to X \to P \to 0$ be a short exact sequence in $\mathcal{A}$ that ends in P. Then there exists a lift of the identity $\mathrm{id}_P \colon P \to P$ along the epimorphism $X \to P$, that is then a section for $X \to P$. Hence $X \to P$ is a retraction, which shows that the given short exact sequence splits.

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] & X \arrow[r] & P \arrow[r] & 0 \\
& & & P \arrow[u, "\mathrm{id}_P"'] \arrow[ul, dashed] &
\end{tikzcd}$$

Suppose now on the other hand that every short exact sequence that ends in P splits. Let $f \colon X \to Y$ be an epimorphism in $\mathcal{A}$ and let $g \colon P \to Y$ be a morphism in $\mathcal{A}$. We may extend the epimorphism f to a short exact sequence

$$0 \to \ker(f) \to X \xrightarrow{f} Y \to 0$$

in $\mathcal{A}$. By using Proposition 3.40 we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r] \arrow[d, equal] & X' \arrow[r, "f'"] \arrow[d, "g'"] & P \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \ker(f) \arrow[r] & X \arrow[r, "f"] & Y \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The upper row splits by assumption, hence there exists a section $s \colon P \to X'$ for f'. For the morphism $g'' \coloneqq g's \colon P \to X$ we then have

$$fg'' = fg's = gf's = g\,\mathrm{id}_P = g\,.$$

ii) This is dual to part i).

iii) This is Exercise 2 on Exercise sheet 10.

iv) This is dual to part iii). □

Remark⋆ 5.J. Let $\mathcal{C}$ be a category.

i) If $(P_j)_{j \in J}$ is a family of projective objects P_j in $\mathcal{C}$ that admit a coproduct $\coprod_{j \in J} P_j$ then $\coprod_{j \in J} P_j$ is again projective.

If there exists in the category $\mathcal{C}$ for any two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ a morphism $X \to Y$ (e.g. if $\mathcal{C}$ is preadditive or has a zero object) then the converse also holds: If $\coprod_{j \in J} P_j$ is projective then P_j is projective for every $j \in J$.

ii) If $(I_j)_{j \in J}$ is a family of injective objects I_j in $\mathcal{C}$ that admit a product $\prod_{j \in J} I_j$ then $\prod_{j \in J} I_j$ is again injective.

If there exists in the category $\mathcal{C}$ for any two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ a morphism $X \to Y$ then the converse also holds: If $\prod_{j \in J} I_j$ is injective then I_j is injective for every $j \in J$.

Definition 5.15.

i) The category $\mathcal{A}$ *has enough projectives* if for every object $X \in \mathrm{Ob}(\mathcal{A})$ there exists an epimorphism $P \to X$ coming from some projective object $P \in \mathrm{Ob}(\mathcal{A})$.

ii) The category $\mathcal{A}$ *has enough injectives* if for every object $X \in \mathrm{Ob}(\mathcal{A})$ there exists a monomorphism $X \to I$ into some injective object $I \in \mathrm{Ob}(\mathcal{A})$.

Definition 5.16. Let $X \in \mathrm{Ob}(\mathcal{A})$ be an object in $\mathcal{A}$.

i) A *chain resolution* of X is a pair $(C_\bullet, p_0)$ consisting of
   - a chain complex $C_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$ bounded below by degree 0, together with
   - a morphism $p_0 \colon C_0 \to X$,

   such that the resulting sequence
$$\cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{p_0} X \to 0$$
   is exact.

ii) A *projective resolution* of X is a chain resolution $(P_\bullet, p_0)$ of X in which P_n is projective for every $n \geq 0$.

iii) A *cochain resolution* of X is a pair $(C^\bullet, i^0)$ consisting of
   - a cochain complex $C^\bullet \in \mathbf{Ch}^{\geq 0}(\mathcal{A})$ bounded below by degree 0, together with
   - a morphism $i^0 \colon X \to C^0$,

   such that the resulting sequence
$$0 \to X \xrightarrow{i^0} C_0 \xrightarrow{d^0} C_1 \xrightarrow{d^1} C_2 \to \cdots$$
   is exact.

iv) An *injective resolution* of X is a cochain resolution $(I^\bullet, i^0)$ of X in which I_n is injective for every $n \geq 0$.

Remark 5.17. Let $X \in \mathrm{Ob}(\mathcal{A})$ be an object in $\mathcal{A}$.

i) We can consider the object X as a chain (resp. cochain) complex that is concentrated in degree 0. Then $\mathrm{H}_0(X) = X$ (resp. $\mathrm{H}^0(X) = X$) and $\mathrm{H}_n(X) = 0$ (resp. $\mathrm{H}^n(X) = 0$) for every $n \neq 0$.

ii) For a chain complex $C_\bullet \in \mathbf{Ch}_\bullet(\mathcal{A})$, a morphism $f \colon C_\bullet \to X$ is uniquely determined by the single morphism $C_0 \to X$ (for all $n \neq 0$ it holds that $f_n = 0$), and the morphism f_0 is subject to the single conditions $f_0 d_1 = 0$. Indeed, this is precisely what it means for the following diagram to commute:
$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r, "d_2"] \arrow[d] & C_1 \arrow[r, "d_1"] \arrow[d] & C_0 \arrow[r, "d_0"] \arrow[d, "f_0"] & C_{-1} \arrow[r, "d_{-1}"] \arrow[d] & C_{-2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

iii) For a cochain complex $C^\bullet \in \mathbf{Ch}^\bullet(\mathcal{A})$, a morphism $f\colon X \to C^\bullet$ is uniquely determined by the single morphism $f^0\colon X \to C^0$ (for all $n \neq 0$ it holds that $f^n = 0$), and the morphism f^0 is subject to the single condition $d^0 f^0 = 0$. Indeed, this is precisely what it means for the following diagram to commute:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f^0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^{-2} \arrow[r, "d^{-2}"] & C^{-1} \arrow[r, "d^{-1}"] & C^0 \arrow[r, "d^0"] & C^1 \arrow[r, "d^1"] & C^2 \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 5.18. Let $X \in \mathrm{Ob}(\mathcal{A})$ be an object in $\mathcal{A}$.

i) Let $C_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$ be a chain complex that is bounded below by degree 0, and let $p_0\colon C_0 \to X$ be a morphism with $p_0 d_1 = 0$. Let $p\colon C_\bullet \to X$ be the corresponding morphism of chain complexes. Then the pair $(C_\bullet, p_0)$ is a chain resolution of X if and only if the morphism of chain complexes p is a quasi-isomorphism.

ii) Let $C^\bullet \in \mathbf{Ch}^{\geq 0}(\mathcal{A})$ be a cochain complex this is bounded below by degree 0, and let $i^0\colon X \to C^0$ be a morphism with $d^0 i^0 = 0$. Let $i\colon X \to C^\bullet$ be the corresponding morphism of cochain complexes. Then the pair $(C^\bullet, i^0)$ is a cochain resolution of X if and only if the morphism of cochain complexes i is a quasi-isomorphism.

Proof. It sufficies to prove part i), as part ii) follows by duality.

That $(C_\bullet, p_0)$ is a chain resolution of X, i.e. that the sequence

$$\cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{p_0} X \to 0$$

is exact, means that $\mathrm{H}_n(C_\bullet) = 0$ for every $n \neq 0$, that $C_1 \xrightarrow{d_1} C_0 \xrightarrow{p_0} X \to 0$ is exact. That the morphism $p\colon C_\bullet \to X$ given by the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r, "d_2"] \arrow[d] & C_1 \arrow[r, "d_1"] \arrow[d] & C_0 \arrow[r] \arrow[d, "p_0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism means that $\mathrm{H}_n(C_\bullet) = 0$ for every $n \neq 0$ and that the induced morphism $\mathrm{coker}(d_1) \to X$ is an isomorphism. This induced morphism is the unique morphism that makes the triangle

$$\begin{tikzcd}
C_0 \arrow[r] \arrow[dr, "p_0"'] & \mathrm{coker}(d_1) \arrow[d, dashed] \\
 & X
\end{tikzcd}$$

commute. The commutativity of this triangle together with $\mathrm{coker}(d_1) \to X$ being an isomorphism is equivalent to p_0 being a cokernel of d_1, which is in turn equivalent to the exactness of the sequence $C_1 \xrightarrow{d_1} C_0 \xrightarrow{p_0} X \to 0$. ☐

Lemma 5.19.

i) If the category $\mathcal{A}$ has enough projectives, then every object $X \in \mathrm{Ob}(\mathcal{A})$ admits a projective resolution.

ii) If the category $\mathcal{A}$ has enough injectives, then every object $X \in \mathrm{Ob}(\mathcal{A})$ admits an injective resolution.

Proof. It sufficies to prove part i), as part ii) follows from part i) by duality. So suppose that the abelian category $\mathcal{A}$ has enough enough projectives.

For any object $X \in \mathrm{Ob}(\mathcal{A})$ there exists by assumption an epimorphism $p_0 \colon P_0 \to X$ for some projective object P_0 in $\mathcal{A}$. We have thus an exact sequence.

$$P_0 \xrightarrow{p_0} X \to 0 \,.$$

Suppose now that we have already constructed an exact sequence

$$P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{p_0} X \to 0 \,.$$

for some $n \geq 0$. For the kernel $\ker(d_n)$ there exists an epimorphism

$$\tilde{d}_{n+1} \colon P_{n+1} \to \ker(d_n)$$

for some projective object P_{n+1} in $\mathcal{A}$. It follows for the composition

$$d_{n+1} \colon P_{n+1} \xrightarrow{\tilde{d}_{n+1}} \ker(d_n) \to P_n$$

from the exactness of the sequence

$$\ker(d_n) \to P_n \xrightarrow{d_n} P_{n-1}$$

that the sequence

$$P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1}$$

is again exact, as seen in Remark* 3.AI. We hence arrive at the exact sequence

$$P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{p_0} X \to 0 \,.$$

By induction we get the desired projective resolution of X. □

Example 5.20. We give a short overview about which abelian categories have enough projectives or enough injectives. (Details on this will be given in the upcoming Chapter 6.)

category	**enough projectives**	**enough injectives**
A-**Mod** and **Mod**-A where A is a **k**-algebra	Yes	Yes
$\mathbf{Sh}_X(\mathbf{Ab})$	No (in general)	Yes
$\mathbf{PSh}_X(\mathbf{Ab})$	Yes	Yes
$\mathbb{Z}$-$\mathbf{Mod}^{\mathrm{fg}}$	Yes	No
A-**mod** and **mod**-A where **k** is a field and A is a f.d. **k**-algebra	Yes	Yes

End of lecture 19

Theorem 5.21 (Comparison theorem)**.** Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$.

i) Let $P_\bullet \xrightarrow{p_0} X$ be a projective resolution of X and let $C_\bullet \xrightarrow{c_0} Y$ be a chain resolution of Y. Then there exists a morphism of chain complexes $\hat{f}\colon P_\bullet \to C_\bullet$ that makes the square

$$\begin{tikzcd} P_0 \arrow[r, "p_0"] \arrow[d, dashed, "\hat{f}_0"'] & X \arrow[d, "f"] \\ C_0 \arrow[r, "c_0"] & Y \end{tikzcd} \tag{5.1}$$

commute. The morphism $\hat{f}$ is up to homotopy uniquely determined by the commutativity of the above square.

ii) Let $Y \xrightarrow{i^0} I^\bullet$ be an injective resolution of Y and let $X \xrightarrow{c^0} C^\bullet$ be a cochain resolution of X. Then there exists a morphism of cochain complexes $\hat{f}\colon C^\bullet \to I^\bullet$ that makes the square

$$\begin{tikzcd} X \arrow[r, "c^0"] \arrow[d, "f"'] & C^0 \arrow[d, dashed, "\hat{f}^0"] \\ Y \arrow[r, "i^0"] & I^0 \end{tikzcd} \tag{5.2}$$

commute. The morphism $\hat{f}$ is up to homotopy uniquely determined by the commutativity of the above square.

Proof. It sufficies to prove part i), as part ii) follows from part i) by duality. We first construct the morphism $\hat{f}$, and then show that it is unique up to homotopy.

We set for $P_{-1} := X$ and $P_{-2} := 0$ with $d_0^P := p_0$ and $d_{-1}^P = 0$, and we similarly set $C_{-1} := Y$ and $C_{-2} := 0$ with $d_0^C = c_0$ and $d_{-1}^C = 0$. We also set $\hat{f}_{-1} := f$

and $\hat{f}_{-2} = 0$. The given commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "p_0"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
\cdots \arrow[r] & C_2 \arrow[r, "d_2"'] & C_1 \arrow[r, "d_1"'] & C_0 \arrow[r, "c_0"'] & Y \arrow[r] & 0
\end{tikzcd}$$

is hence the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & P_{-1} \arrow[r, "d_{-1}"] \arrow[d, "\hat{f}_{-1}"] & P_{-2} \arrow[d, "\hat{f}_{-2}"] \\
\cdots \arrow[r] & C_2 \arrow[r, "d_2"'] & C_1 \arrow[r, "d_1"'] & C_0 \arrow[r, "d_0"'] & C_{-1} \arrow[r, "d_{-1}"'] & C_{-2}
\end{tikzcd}$$

This commutative diagram has exact rows and the object P_n is projective for every $n \geq 0$.

We construct the required morphisms $\hat{f}_n \colon P_n \to C_n$ by induction over $n \geq -2$, with $\hat{f}_{-1}$ and $\hat{f}_{-2}$ being already given. So suppose more generally that we have already constructed morphisms $\hat{f}_n, \dots, \hat{f}_0, \hat{f}_{-1}, \hat{f}_{-2}$ for some $n \geq -1$ that make the diagram

$$\begin{tikzcd}
P_n \arrow[r, "d_n"] \arrow[d, "\hat{f}_n"] & P_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\hat{f}_{n-1}"] & \cdots \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "\hat{f}_0"] & P_{-1} \arrow[r] \arrow[d, "\hat{f}_{-1}"] & P_{-2} \arrow[d, "\hat{f}_{-2}"] \\
C_n \arrow[r, "d_n"'] & C_{n-1} \arrow[r, "d_{n-1}"'] & \cdots \arrow[r, "d_1"'] & C_0 \arrow[r, "d_0"'] & C_{-1} \arrow[r] & C_{-2}
\end{tikzcd}$$

commute. It follows from $d_n^P d_{n+1}^P = 0$ and $d_n^C d_{n+1}^C = 0$ that the differentials d^P and d^C induce morphisms $p \colon P_{n+1} \to \ker(d_n^P)$ and $q \colon C_{n+1} \to \ker(d_n^C)$ that make the triangles

$$\begin{tikzcd}
P_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, dashed, "p"'] & & P_n \\
& \ker(d_n) \arrow[ur, "i"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, dashed, "q"'] & & C_n \\
& \ker(d_n) \arrow[ur, "j"'] &
\end{tikzcd}$$

commute. It follows from the commutativity of the square

$$\begin{tikzcd}
P_n \arrow[r, "d_n"] \arrow[d, "\hat{f}_n"'] & P_{n-1} \arrow[d, "\hat{f}_{n-1}"] \\
C_n \arrow[r, "d_n"] & C_{n-1}
\end{tikzcd}$$

that the morphism $\hat{f}_n \colon P_n \to C_n$ induces a morphism $\tilde{f}_n \colon \ker(d_n^P) \to \ker(d_n^C)$ that makes the following diagram commute:

$$\begin{tikzcd}
P_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "p"'] & & P_n \arrow[dd, "\hat{f}_n"] \\
& \ker(d_n) \arrow[ur, "i"'] \arrow[dd, dashed, "\tilde{f}_n"] & \\
C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "q"'] & & C_n \\
& \ker(d_n) \arrow[ur, "j"'] &
\end{tikzcd}$$

The morphism $q\colon C_{n+1} \to \ker(d_n^C)$ is an epimorphism because it is the composition

$$q\colon C_{n+1} \to \operatorname{im}(d_{n+1}^C) \to \ker(d_n^C)$$

with $C_{n+1} \to \operatorname{im}(d_{n+1})$ being an epimorphism, and $\operatorname{im}(d_{n+1}^C) \to \ker(d_n^C)$ being an isomorphism by the exactness of the sequence $C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$. It follows from P_{n+1} being projective that there exist a morphism $\hat{f}_{n+1}\colon P_{n+1} \to C_{n+1}$ that makes the square

$$\begin{tikzcd}
P_{n+1} \arrow[r, "p"] \arrow[d, dashed, "\hat{f}_{n+1}"'] & \ker(d_n) \arrow[d, "\tilde{f}_n"] \\
C_{n+1} \arrow[r, "q"'] & \ker(d_n)
\end{tikzcd}$$

commute. Then also the square

$$\begin{tikzcd}
P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, dashed, "\hat{f}_{n+1}"'] & P_n \arrow[d, "\hat{f}_n"] \\
C_{n+1} \arrow[r, "d_n"] & C_n
\end{tikzcd}$$

commute, i.e. the complete diagram

$$\begin{tikzcd}
P_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "p"'] \arrow[dd, dashed, "\hat{f}_{n+1}"'] & & P_n \arrow[dd, "\hat{f}_n"] \\
& \ker(d_n) \arrow[ur, "i"'] \arrow[dd, "\tilde{f}_n"] & \\
C_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "q"'] & & C_n \\
& \ker(d_n) \arrow[ur, "j"'] &
\end{tikzcd}$$

commutes. Indeed, we have that

$$d_{n+1}\hat{f}_{n+1} = jq\hat{f}_{n+1} = j\tilde{f}_n p = \hat{f}_n ip = \hat{f}_n d_{n+1}\,.$$

To show that the morphism $\hat{f}$ is unique up to homotopy we may assume that $f = 0$: Indeed, if $\hat{f}' \colon P_\bullet \to C_\bullet$ is another lift of $f \colon X \to Y$ then the difference $\hat{f}' - \hat{f}$ is a lift of $f - f = 0$. And the morphisms $\hat{f}'$ and $\hat{f}$ are homotopic if and only if this difference $\hat{f}' - \hat{f}$ is null homotopic, i.e. is homotopic to the zero morphism $P_\bullet \to C_\bullet$, which is another lift of the zero morphism $X \to Y$.

So let $f = 0$, and let us shows that any lift $\hat{f} \colon P_\bullet \to C_\bullet$ is null homotopic. For this we need to construct morphisms $s_n \colon P_n \to C_{n+1}$ for $n \geq 0$ such that

$$d_1 s_0 = \hat{f}_0 \qquad \text{and} \qquad d_{n+1} s_n + s_{n-1} d_n = \hat{f}_n \text{ for every } n \geq 1\,.$$

We may visualize the role of the morphisms s_n as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "\hat{f}_2"] \arrow[dl, dashed] & P_1 \arrow[r, "d_1"] \arrow[d, "\hat{f}_1"] \arrow[dl, dashed, "s_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "\hat{f}_0"] \arrow[dl, dashed, "s_0"] & P_{-1} \arrow[r, "d_{-1}"] \arrow[d, "0"] & P_{-2} \arrow[d, "0"] \\
\cdots \arrow[r] & C_2 \arrow[r, "d_2"'] & C_1 \arrow[r, "d_1"'] & C_0 \arrow[r, "d_0"'] & C_{-1} \arrow[r, "d_{-1}"'] & C_{-2}
\end{tikzcd}$$

We define $s_{-1} \colon P_{-1} \to C_0$ and $s_{-2} \colon P_{-2} \to C_{-1}$ as the respective zero morphisms, and hence need to show that

$$\hat{f}_n = d_{n+1} s_n + s_{n-1} d_n$$

for every $n \geq -1$, with this relation already holding for $n = -1$.

Suppose that $s_{n-1}, \dots, s_{-1}, s_{-2}$ are already constructed for some $n \geq 0$. Then the morphism

$$\hat{f}_n - s_{n-1} d_n \colon P_n \to C_n$$

satisfies

$$\begin{aligned}
d_n(\hat{f}_n - s_{n-1} d_n) &= d_n \hat{f}_n - d_n s_{n-1} d_n \\
&= d_n \hat{f}_n - (\hat{f}_{n-1} - s_{n-2} d_{n-1}) d_n \\
&= d_n \hat{f}_n - \hat{f}_{n-1} d_n \\
&= 0
\end{aligned}$$

because $\hat{f}$ is a morphism of chain complexes. It follows from the universal property of the kernel $\ker(d_n) \to C_n$ that there exists a (unique) morphism $\tilde{s}_n \colon P_n \to \ker(d_n)$ that makes the diagram

$$\begin{tikzcd}
 & P_n \arrow[dl, dashed, "\tilde{s}_n"'] \arrow[d, "\hat{f}_n - s_{n-1} d_n"] \arrow[dr, bend left, "0"] & \\
\ker(d_n) \arrow[r] & C_n \arrow[r, "d_n"'] & C_{n+1}
\end{tikzcd}$$

commute. The canonical morphism $C_{n+1} \to \ker(d_n)$ that makes the diagram

$$\begin{tikzcd}
C_{n+1} \arrow[r, dashed] \arrow[rr, bend right, "d_{n+1}"'] & \ker(d_n) \arrow[r] & C_n
\end{tikzcd}$$

commute is an epimorphism by the exactness of the sequence $C_{n+1} \to C_n \to C_{n-1}$, as seen in the previous part of the proof. It follows from P_n being projective that there exists a morphism $s_n \colon P_n \to C_{n+1}$ that makes the diagram

$$\begin{tikzcd}
 & & P_n \arrow[dll, dashed, bend right, "s_n"'] \arrow[dl, "\tilde{s}_n"'] \arrow[d, "\hat{f}_n - s_{n-1} d_n"] \\
C_{n+1} \arrow[r] \arrow[rr, bend right, "d_{n+1}"'] & \ker(d_n) \arrow[r] & C_n
\end{tikzcd}$$

commute. Then

$$d_{n+1} s_n + s_{n-1} d_n = \hat{f}_n - s_{n-1} d_n + s_{n-1} d_n = \hat{f}_n \,,$$

as desired. □

Remark⋆ 5.K. One may rewrite the commutative square (5.1) as

$$\begin{tikzcd}
P_\bullet \arrow[r, "p_0"] \arrow[d, dashed, "\hat{f}"'] & X \arrow[d, "f"] \\
C_\bullet \arrow[r, "c_0"] & Y
\end{tikzcd}$$

and the commutative square (5.2) as follows:

$$\begin{tikzcd}
X \arrow[r, "c^0"] \arrow[d, "f"'] & C^\bullet \arrow[d, dashed, "\hat{f}"] \\
Y \arrow[r, "i^0"] & I^\bullet
\end{tikzcd}$$

Corollary⋆ 5.L.

i) Any two projections resolutions $(P_\bullet, p_0)$ and $(P'_\bullet, p'_0)$ of an object $X \in \mathrm{Ob}(\mathcal{A})$ are homotopy equivalent.

ii) Any two injective resolutions $(I^\bullet, i^0)$ and $({}^{\backprime}I^\bullet, {}^{\backprime}i^0)$ of an object $X \in \mathrm{Ob}(\mathcal{A})$ are homotopy equivalent.

Proof. It sufficies to prove part i), as part ii) follows from part i) by duality. It follows from the comparison theorem that there exists lifts $f \colon P_\bullet \to P'_\bullet$ and $g \colon P'_\bullet \to P_\bullet$ of the identity $\mathrm{id}_X \colon X \to X$. The composition $g \circ f \colon P_\bullet \to P_\bullet$ is then a lift of the composition $\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X$. The identity $\mathrm{id}_{P_\bullet} \colon P_\bullet \to P_\bullet$ is also a lift of id_X, hence $g \circ f$ and $\mathrm{id}_{P_\bullet}$ are homotopic by the comparison theorem. We find similarly that the composition $f \circ g$ is homotopic to the identity $\mathrm{id}_{P'_\bullet}$. □

Lemma 5.22 (Horseshoe lemma)**.**

i) Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence in $\mathcal{A}$. If $P'_\bullet \xrightarrow{p'_0} X'$ and $P''_\bullet \xrightarrow{p''_0} X''$ are projective resolutions then there exists a projective resolution $P_\bullet \xrightarrow{p_0} X$ of X such that the short exact sequence $0 \to X' \to X \to X'' \to 0$ lifts to a short exact sequence of chain complexes

$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0\,,$$

in the sense that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & P'_\bullet \arrow[r] \arrow[d, "p'_0"] & P_\bullet \arrow[r] \arrow[d, "p_0"] & P''_\bullet \arrow[r] \arrow[d, "p''_0"] & 0 \\
0 \arrow[r] & X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & 0
\end{tikzcd}$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & P'_0 \arrow[r] \arrow[d, "p'_0"] & P_0 \arrow[r] \arrow[d, "p_0"] & P''_0 \arrow[r] \arrow[d, "p''_0"] & 0 \\
0 \arrow[r] & X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & 0
\end{tikzcd}$$

The short exact sequence $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ can moverover be choosen such that in every degree n, the short exact sequence $0 \to P'_n \to P_n \to P''_n \to 0$ splits.

ii) Let $0 \to {}^{\backprime}X \to X \to {}^{\backprime\backprime}X \to 0$ be a short exact sequence in $\mathcal{A}$. If ${}^{\backprime}X \xrightarrow{{}^{\backprime}i^0} {}^{\backprime}I^\bullet$ and ${}^{\backprime\backprime}X \xrightarrow{{}^{\backprime\backprime}i^0} {}^{\backprime\backprime}I^\bullet$ are injective resolutions then there exists an injective resolution $X \xrightarrow{i^0} I^\bullet$ of X such that the short exact sequence $0 \to {}^{\backprime}X \to X \to {}^{\backprime\backprime}X \to 0$ extends to a short exact sequence of cochain complexes

$$0 \to {}^{\backprime}I^\bullet \to I^\bullet \to {}^{\backprime\backprime}I^\bullet \to 0\,,$$

in the sense that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & {}^{\backprime}X \arrow[r] \arrow[d, "{}^{\backprime}i^0"] & X \arrow[r] \arrow[d, "i^0"] & {}^{\backprime\backprime}X \arrow[r] \arrow[d, "{}^{\backprime\backprime}i^0"] & 0 \\
0 \arrow[r] & {}^{\backprime}I^\bullet \arrow[r] & I^\bullet \arrow[r] & {}^{\backprime\backprime}I^\bullet \arrow[r] & 0
\end{tikzcd}$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & {}^{\backprime}X \arrow[r] \arrow[d, "{}^{\backprime}i^0"] & X \arrow[r] \arrow[d, "i^0"] & {}^{\backprime\backprime}X \arrow[r] \arrow[d, "{}^{\backprime\backprime}i^0"] & 0 \\
0 \arrow[r] & {}^{\backprime}I^0 \arrow[r] & I^0 \arrow[r] & {}^{\backprime\backprime}I^0 \arrow[r] & 0
\end{tikzcd}$$

The short exact sequence $0 \to {}^{\backprime}I^\bullet \to I^\bullet \to {}^{\backprime\backprime}I^\bullet \to 0$ can moreover be choosen such that in every degree n, the short exact sequence $0 \to {}^{\backprime}I^n \to I^n \to {}^{\backprime\backprime}I^n \to 0$ splits.

Proof. It sufficies to prove part i), as part ii) follows from part i) by duality. We denote the given short exact sequence by

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0 \,.$$

We set $P_n \coloneqq P'_n \oplus P''_n$ for every $n \geq 0$ and consider for every $n \geq 0$ the short exact sequence

$$0 \to P'_n \xrightarrow{\left[\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right]} P_n \xrightarrow{[\, 0 \ \mathrm{id} \,]} P''_n \to 0$$

We need to construct morphisms $d_n \colon P_n \to P_{n-1}$ and $p_0 \colon P_0 \to X$ that make the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d, dashed] & \vdots \arrow[d] & \\
0 \arrow[r] & P'_2 \arrow[r, "\left[\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right]"] \arrow[d, "d'_2"'] & P_2 \arrow[r, "{[\, 0 \ \mathrm{id} \,]}"] \arrow[d, dashed, "d_2"] & P''_2 \arrow[r] \arrow[d, "d''_2"] & 0 \\
0 \arrow[r] & P'_1 \arrow[r, "\left[\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right]"] \arrow[d, "d'_1"'] & P_1 \arrow[r, "{[\, 0 \ \mathrm{id} \,]}"] \arrow[d, dashed, "d_1"] & P''_1 \arrow[r] \arrow[d, "d''_1"] & 0 \\
0 \arrow[r] & P'_0 \arrow[r, "\left[\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right]"] \arrow[d, "p'_0"'] & P_0 \arrow[r, "{[\, 0 \ \mathrm{id} \,]}"] \arrow[d, dashed, "p_0"] & P''_0 \arrow[r] \arrow[d, "p''_0"] & 0 \\
0 \arrow[r] & X' \arrow[r, "f"] \arrow[d] & X \arrow[r, "g"] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

commute and that make the middle column into a projective resolution of X. That the sequence $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ of chain complexes is then already (short) exact follows because exactness in $\mathbf{Ch}_\bullet(\mathcal{A})$ is component componentwise (as explained in Remark* 4.F).

We start by contructing the morphism p_0: We have the following diagram in which the morphism g is an epimorphism and the object P''_0 is projective:

$$\begin{tikzcd}
 & P'_0 \arrow[d, "p'_0"] & & P''_0 \arrow[d, "p''_0"] & \\
0 \arrow[r] & X' \arrow[r, "f"] \arrow[d] & X \arrow[r, "g"] & X'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

It follows that there exist a morphism $g' \colon P_0'' \to X$ that makes the triangle

$$\begin{tikzcd} & P_0'' \arrow[dl, dashed, "g'"'] \arrow[d, "p_0''"] \\ X \arrow[r, "g"'] & X'' \end{tikzcd}$$

commute. We define

$$p_0 \coloneqq \begin{bmatrix} fp_0' & g' \end{bmatrix} : P_0 = P_0' \oplus P_0'' \to X \,.$$

Then the diagram

$$\begin{tikzcd} 0 \arrow[r] & P_0' \arrow[r, "\left[\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right]"] \arrow[d, "p_0'"'] & P_0 \arrow[r, "{[0 \ \mathrm{id}]}"] \arrow[d, dashed, "p_0"] & P_0'' \arrow[r] \arrow[d, "p_0''"] & 0 \\ 0 \arrow[r] & X' \arrow[r, "f"] & X \arrow[r, "g"] & X'' \arrow[r] & 0 \end{tikzcd}$$

commutes because

$$p_0 \begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix} = \begin{bmatrix} fp_0' & g' \end{bmatrix} \begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix} = fp_0'$$

and

$$gp_0 = g \begin{bmatrix} fp_0' & g' \end{bmatrix} = \begin{bmatrix} gfp_0' & gg' \end{bmatrix} = \begin{bmatrix} 0 & p_0'' \end{bmatrix} = p_0'' \begin{bmatrix} 0 & \mathrm{id} \end{bmatrix} \,.$$

It follows from (part ii) of) the 5-lemma that the morphisms p_0 is an epimorphism, because both p_0' and p_0'' are epimorphism. We have hence the following commutative diagram with exact columns:

$$\begin{tikzcd} & \vdots \arrow[d] & & \vdots \arrow[d] & \\ & P_2' \arrow[d, "d_2'"] & & P_2'' \arrow[d, "d_2''"] & \\ & P_1' \arrow[d, "d_1'"] & & P_1'' \arrow[d, "d_1''"] & \\ 0 \arrow[r] & P_0' \arrow[r, "\left[\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right]"] \arrow[d, "p_0'"] & P_0 \arrow[r, "{[0 \ \mathrm{id}]}"] \arrow[d, "p_0"] & P_0'' \arrow[r] \arrow[d, "p_0''"] & 0 \\ 0 \arrow[r] & X' \arrow[r, "f"'] \arrow[d] & X \arrow[r, "g"'] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

To construct the morphism $d_1 \colon P_1 \to P_0$ we proceed similar as before: We get the following commutative diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(p_0') \arrow[r] \arrow[d, "\tilde{d}_0'"] & \ker(p_0) \arrow[r] \arrow[d, "\tilde{d}_0"] & \ker(p_0'') \arrow[r] \arrow[d, "\tilde{d}_0''"] & 0 \\
0 \arrow[r] & P_0' \arrow[r] \arrow[d, "p_0'"] & P_0 \arrow[r] \arrow[d, "p_0"] & P_0'' \arrow[r] \arrow[d, "p_0''"] & 0 \\
0 \arrow[r] & X' \arrow[r, "f"'] \arrow[d] & X \arrow[r, "g"'] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The lower two rows are exact, hence the upper row is also exact by the 9-lemma. We can therefore consider the following subdiagram, in which the morphism $\ker(p_0) \to \ker(p_0'')$ is an epimorphism and the object P_1'' is projective.

$$\begin{tikzcd}
 & P_1' \arrow[d, "\tilde{d}_1'"] & & P_1'' \arrow[d, "\tilde{d}_1''"] & \\
0 \arrow[r] & \ker(p_0') \arrow[r] \arrow[d] & \ker(p_0) \arrow[r] & \ker(p_0'') \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

We find as before that there exists an epimorphism $d_1 \colon P_1 \to \ker(p_0)$ that makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & P_1' \arrow[r, "\begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix}"] \arrow[d, "\tilde{d}_1'"] & P_1 \arrow[r, "{[\,0 \ \mathrm{id}\,]}"] \arrow[d, dashed, "\tilde{d}_1"] & P_1'' \arrow[r] \arrow[d, "\tilde{d}_1''"] & 0 \\
0 \arrow[r] & \ker(p_0') \arrow[r] & \ker(p_0) \arrow[r] & \ker(p_0'') \arrow[r] & 0
\end{tikzcd}$$

commute. It follows from the exactness of the sequence

$$\ker(p_0) \xrightarrow{\tilde{d}_0} P_0 \xrightarrow{p_0} X$$

that the sequence

$$P_1 \xrightarrow{\tilde{d}_0 \tilde{d}_1} P_0 \xrightarrow{p_0} X$$

is again exact because $\tilde{d}_1$ is an epimorphism. We have hence arrived at the following commutatitve diagram with exact columns:

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & P_2' \arrow[d, "d_2'"] & & P_2'' \arrow[d, "d_2''"] & \\
0 \arrow[r] & P_1' \arrow[r, "\begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix}"] \arrow[d, "d_1'"] & P_1 \arrow[r, "\begin{bmatrix} 0 & \mathrm{id} \end{bmatrix}"] \arrow[d, "d_0"] & P_1'' \arrow[r] \arrow[d, "d_1''"] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "\begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix}"] \arrow[d, "p_0'"] & P_0 \arrow[r, "\begin{bmatrix} 0 & \mathrm{id} \end{bmatrix}"] \arrow[d, "p_0"] & P_0'' \arrow[r] \arrow[d, "p_0''"] & 0 \\
0 \arrow[r] & X' \arrow[r, "f"'] \arrow[d] & X \arrow[r, "g"'] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We can now continue by induction. ☐

Remark⋆ 5.M. The term 'horse shoe lemma' comes from the form of the following diagram, that we are filling in with an exact column in the middle:

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & P_2' \arrow[d, "d_2'"'] & & P_2'' \arrow[d, "d_2''"] & \\
 & P_1' \arrow[d, "d_1'"'] & & P_1'' \arrow[d, "d_1''"] & \\
 & P_0' \arrow[d, "p_0'"'] & & P_0'' \arrow[d, "p_0''"] & \\
0 \arrow[r] & X' \arrow[r, "f"] \arrow[d] & X \arrow[r, "g"] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Existence of Derived Functors

Goal. We want to construct for every left exact (resp. right exact) functor a right (resp. left) derived functor.

Convention. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$, where the category $\mathcal{A}$ has enough injectives.

Notation⋆ 5.N. If $C^\bullet$ is a cochain complex in $\mathcal{A}$ then we denote by $F(C^\bullet)$ the cochain complex

$$\cdots \to F(C^{n+1}) \xrightarrow{F(d^{n+1})} F(C^n) \xrightarrow{F(d^n)} F(C^{n-1}) \to \cdots$$

in $\mathcal{B}$. If $f\colon C^\bullet \to D^\bullet$ is a morphism of cochain complexes in $\mathbf{Ch}^\bullet(\mathcal{A})$ then we denote by $F(f)$ the morphism of cochain complexes $F(f) \to F(C^\bullet) \to F(D^\bullet)$ that is given by $F(f) \coloneqq (F(f^n))_{n\in\mathbb{Z}}$.

Lemma 5.23. Let $X \in \mathrm{Ob}(\mathcal{A})$ be an object in $\mathcal{A}$. Let $(I^\bullet, i^0)$ be an injective resolution of X, and set

$$(\mathrm{R}^n_{(I^\bullet,i^0)} F)(X) \coloneqq \mathrm{H}^n(F(I^\bullet))$$

for every $n \geq 0$.

i) Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$ and let $(J^\bullet, j^0)$ be an injective resolution of Y. Then for any two morphisms of cochain complexes $\hat{f}, \hat{f}'\colon I^\bullet \to J^\bullet$ that extend the morphism f, it holds that $\mathrm{H}^n(F(\hat{f})) = \mathrm{H}^n(F(\hat{f}'))$ for every $n \geq 0$.

Let $(\mathrm{R}^n_{(I^\bullet,i^0),(J^\bullet,j^0)} F)(f) \coloneqq \mathrm{H}^n(F(\hat{f}))$ in the above situation.

ii) For any two composable morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{A}$ and injective resolutions $(J^\bullet, j^0)$ of Y and $(K^\bullet, k^0)$ of Z it holds for every $n \geq 0$ that

$$(\mathrm{R}^n_{(J^\bullet,j^0),(K^\bullet,k^0)} F)(g) \circ (\mathrm{R}^n_{(I^\bullet,i^0),(J^\bullet,j^0)} F)(f) = (\mathrm{R}^n_{(I^\bullet,i^0),(K^\bullet,k^0)} F)(g \circ f)\,.$$

iii) If $({}^{\backprime}I^\bullet, {}^{\backprime}i^0)$ is another injective resolution of X then $(\mathrm{R}^n_{(I^\bullet,i^0),({}^{\backprime}I^\bullet,{}^{\backprime}i^0)} F)(\mathrm{id}_X)$ is an isomorphism for every $n \geq 0$.

iv) If furthemore $f\colon X \to Y$ is a morphism in $\mathcal{A}$ and $(J^\bullet, j^0)$ and $({}^{\backprime}J^\bullet, {}^{\backprime}j^0)$ are two injective resolutions of Y, then the following square commutes:

$$\begin{tikzcd}
(\mathrm{R}^n_{(I^\bullet,i^0)} F)(X) \arrow[r, "(\mathrm{R}^n_{(I^\bullet,i^0),(J^\bullet,j^0)} F)(f)"] \arrow[d, "(\mathrm{R}^n_{(I^\bullet,i^0),({}^{\backprime}I^\bullet,{}^{\backprime}i^0)} F)(\mathrm{id}_X)"] & (\mathrm{R}^n_{(J^\bullet,j^0)} F)(Y) \arrow[d, "(\mathrm{R}^n_{(J^\bullet,j^0),({}^{\backprime}J^\bullet,{}^{\backprime}j^0)} F)(\mathrm{id}_Y)"'] \\
(\mathrm{R}^n_{({}^{\backprime}I^\bullet,{}^{\backprime}i^0)} F)(X) \arrow[r, "(\mathrm{R}^n_{({}^{\backprime}I^\bullet,{}^{\backprime}i^0),({}^{\backprime}J^\bullet,{}^{\backprime}j^0)} F)(f)"'] & (\mathrm{R}^n_{({}^{\backprime}J^\bullet,{}^{\backprime}j^0)} F)(Y)
\end{tikzcd}$$

Proof.

i) It follows from the comparison theorem that any two such extensions $\hat{f}$ and $\hat{f}'$ are homotopic. Then the morphisms $F(\hat{f})$ and $F(\hat{f}')$ is are also homotopic, and hence induce the same morphism in homology.

ii) If $\hat{f}\colon I^\bullet \to J^\bullet$ is a lift of f and $\hat{g}\colon J^\bullet \to K^\bullet$ is a lift of g, then the composition $\hat{g} \circ \hat{f}\colon I^\bullet \to K^\bullet$ is a lift of the composition $g \circ f\colon X \to Z$. Hence

$$\begin{aligned} &(\mathrm{R}^n_{(I^\bullet,i^0),(K^\bullet,k^0)} F)(g \circ f) \\ =&\,\mathrm{H}^n(F(\hat{g} \circ \hat{f})) \\ =&\,\mathrm{H}^n(F(\hat{g}) \circ F(\hat{f})) \\ =&\,\mathrm{H}^n(F(\hat{g})) \circ \mathrm{H}^n(F(\hat{f})) \\ =&(\mathrm{R}^n_{(J^\bullet,j^0),(K^\bullet,k^0)} F)(g) \circ (\mathrm{R}^n_{(I^\bullet,i^0),(J^\bullet,j^0)} F)(f)\,. \end{aligned}$$

iii) It holds that

$$(\mathrm{R}^n_{(I^\bullet,i^0),(I^\bullet,i^0)} F)(\mathrm{id}_X) = \mathrm{id}_{(\mathrm{R}^n_{(I^\bullet,i^0)} F)(X)}\ .$$

It hence follows from part ii) that the morphism $(\mathrm{R}^n_{(I^\bullet,i^0),({}^{\backprime}I^\bullet,{}^{\backprime}i^0)} F)(\mathrm{id}_X)$ is an isomorphism with

$$(\mathrm{R}^n_{(I^\bullet,i^0),({}^{\backprime}I^\bullet,{}^{\backprime}i^0)} F)(\mathrm{id}_X)^{-1} = (\mathrm{R}^n_{({}^{\backprime}I^\bullet,{}^{\backprime}i^0),(I^\bullet,i^0)} F)(\mathrm{id}_X)\,.$$

iv) This follows from part ii). $\square$

End of lecture 20

Remark-Definition 5.24. For every object $X \in \mathrm{Ob}(\mathcal{A})$ we fix an injective resolution $(I^\bullet_X, i^0_X)$ of X. We define for every $n \geq 0$ a functor $\mathrm{R}^n F\colon \mathcal{A} \to \mathcal{B}$ by

$$(\mathrm{R}^n F)(X) \coloneqq \mathrm{H}^n(F(I^\bullet_X))$$

for every object $X \in \mathrm{Ob}(\mathcal{A})$, and

$$(\mathrm{R}^n F)(f) \coloneqq \mathrm{H}^n(F(\hat{f}))$$

and every morphism $f\colon X \to Y$ in $\mathcal{A}$, where $\hat{f}\colon I^\bullet_X \to I^\bullet_Y$ is an extension of f. This is a well-defined functor by parts i) and ii) of Lemma 5.23. The functor $\mathrm{R}^n F$ is up to isomorphism independent of the choice of injective resolutions, again by Lemma 5.23.

Lemma 5.25. The functors $\mathrm{R}^n F\colon \mathcal{A} \to \mathcal{B}$ with $n \geq 0$ are additive.

Proof. Let $f, g\colon X \to Y$ be two parallel morphisms in $\mathcal{A}$ and let $\hat{f}, \hat{g}\colon I^\bullet_X \to I^\bullet_Y$ be lifts of f and g to morphisms of chain complexes. Then $\hat{f} + \hat{g}$ is a lift of the morphism $f + g$. It follows that

$$\begin{aligned} (\mathrm{R}^n F)(f+g) &= \mathrm{H}^n(F(\hat{f} + \hat{g})) \\ &= \mathrm{H}^n(F(\hat{f})) + \mathrm{H}^n(F(\hat{g})) \\ &= (\mathrm{R}^n F)(f) + (\mathrm{R}^n F)(g)\,, \end{aligned}$$

as desired. $\square$

Lemma 5.26. It holds that $\mathrm{R}^0 F \cong F$.

Proof. It follows for every object $X \in \mathrm{Ob}(\mathcal{A})$ from the exactness of the sequence

$$0 \to X \to I_X^0 \to I_X^1 \to I_X^2 \to \cdots$$

that the sequence

$$0 \to F(X) \to F(I_X^0) \to F(I_X^1)$$

is again exact, because the functor F is left exact. This exactness means that the morphism $F(X) \to F(I_X^0)$ is a kernel of the morphism $F(I_X^0) \to F(I_X^1)$. The chain complex $F(I_X^\bullet)$ is given by

$$\cdots \to 0 \to F(I_X^0) \to F(I_X^1)l \to F(I_X^2) \to \cdots$$

The zeroeth cohomology of this cochain complex, which is $(\mathrm{R}^0 F)(X)$, is also a kernel of the morphism $F(I_X^0) \to F(I_X^1)$. It follows that $F(X) \cong (\mathrm{R}^0 F)(X)$. More precisely, there exists a unique morphism $\lambda_X \colon (\mathrm{R}^0 F)(X) \to F(X)$ that makes the square

$$\begin{tikzcd}
(\mathrm{R}^0 F)(X) \arrow[r, "j_X"] \arrow[d, "\lambda_X"'] & F(I_X^0) \arrow[d, equal] \\
F(X) \arrow[r, "F(i_X^0)"'] & F(I_X^0)
\end{tikzcd}$$

commute, and the morphism λ_X is an isomorphism.

We need to show that this isomorphism is natural in X. For this, we consider for a morphism $f \colon X \to Y$ in $\mathcal{A}$ and an extension $\hat{f} \colon I_X^\bullet \to I_Y^\bullet$ of f the following cube:

$$\begin{tikzcd}
(\mathrm{R}^0 F)(X) \arrow[rr, "j_X"] \arrow[dr, "(\mathrm{R}^0 F)(f)"] \arrow[dd, "\lambda_X"'] & & F(I_X^0) \arrow[dr, "F(\hat{f}^0)"] \arrow[dd, equal] & \\
& (\mathrm{R}^0 F)(Y) \arrow[rr, "j_Y"] \arrow[dd, "\lambda_Y"'] & & F(I_Y^0) \arrow[dd, equal] \\
F(X) \arrow[rr, "F(i_X^0)"] \arrow[dr, "F(f)"'] & & F(I_X^0) \arrow[dr, "F(\hat{f}^0)"'] & \\
& F(Y) \arrow[rr, "F(i_Y^0)"'] & & F(I_Y^0)
\end{tikzcd}$$

This cube commutes: The front and back squares commute by choice of the isomorphisms λ_X and λ_Y. The top square commutes by construction of $(\mathrm{R}^0 F)(f)$. The bottom square results by applying the functor F to the following commutative square:

$$\begin{tikzcd}
X \arrow[r, "i_X^0"] \arrow[d, "f"'] & I_X^0 \arrow[d, "\hat{f}"] \\
Y \arrow[r, "i_Y^0"'] & I_Y^0
\end{tikzcd}$$

The commutativity of the left square follows from the commutativity of the other sides and $F(i_Y^0)$ being a monomorphism (which holds because it is a kernel): It holds that

$$\begin{aligned}
F(i_Y^0)\lambda_Y(\mathrm{R}^0 F)(f) &= \mathrm{id}_{F(I_Y^0)}\, j_Y(\mathrm{R}^0 F)(f) \\
&= \mathrm{id}_{F(I_Y^0)}\, F(\hat{f}^0) j_X \\
&= F(\hat{f}^0)\, \mathrm{id}_{F(I_X^0)}\, j_X \\
&= F(\hat{f}^0) F(i_X^0)\lambda_X \\
&= F(i_Y^0) F(f)\lambda_X \,,
\end{aligned}$$

and hence

$$\lambda_Y(\mathrm{R}^0 F)(f) = F(f)\lambda_X \,.$$

This commutativity of the left square

$$\begin{tikzcd}
(\mathrm{R}^0 F)(X) \arrow[r, "(\mathrm{R}^0 F)(f)"] \arrow[d, "\lambda_X"'] & (\mathrm{R}^0 F)(Y) \arrow[d, "\lambda_Y"] \\
F(X) \arrow[r, "F(f)"'] & F(Y)
\end{tikzcd}$$

shows that the family $\lambda = (\lambda_X)_{X \in \mathrm{Ob}(\mathcal{A})}$ is a natural isomorphism $\mathrm{R}^0 F \to F$. $\square$

Goal. We want to show that the family $(\mathrm{R}^n F)_{n \geq 0}$ of functors $\mathrm{R}^n F \colon \mathcal{A} \to \mathcal{B}$ can be made into a universal cohomological δ-functor.

Lemma. Let

$$\begin{tikzcd}
 & & X' \arrow[rr, "a"] \arrow[dl, "i'"'] \arrow[dd, "f'"] & & X \arrow[rr, "b"] \arrow[dl, "i"'] \arrow[dd, "f"] & & X'' \arrow[rr] \arrow[dl, "i''"'] \arrow[dd, "f''"] & & 0 \\
0 \arrow[r] & I' \arrow[rr, "\alpha"] \arrow[dd, "g'"'] & & I \arrow[rr, "\beta"] & & I'' \arrow[rr] \arrow[dd, "g''"] & & 0 & \\
 & & Y' \arrow[rr, "c"] \arrow[dl, "j'"] & & Y \arrow[rr, "d"] \arrow[dl, "j"] & & Y'' \arrow[dl, "j''"] & & \\
0 \arrow[r] & J' \arrow[rr, "\gamma"] & & J \arrow[rr, "\delta"] & & J'' \arrow[rr] & & 0 &
\end{tikzcd}$$

be a commutative diagram in $\mathcal{A}$ with exact rows, such that the objects I' and J' are injective, and the diagonal morphism i'' is a monomorphism.[2] Then there exists a morphism $g''\colon I \to J$ that makes the resulting diagram commute.

Proof. The frontal short exact sequences

$$0 \to I' \to I \to I'' \to 0 \qquad \text{and} \qquad 0 \to J' \to J \to J'' \to 0$$

split because the objects I' and J' are injective. We may therefore assume that the given diagram is of the form

$$\begin{tikzcd}
 & & X' \arrow[rr, "a"] \arrow[dl, "i'"'] \arrow[dd, "f'"] & & X \arrow[rr, "b"] \arrow[dl, "i"'] \arrow[dd, "f"] & & X'' \arrow[rr] \arrow[dl, "i''"'] \arrow[dd, "f''"] & & 0 \\
0 \arrow[r] & I' \arrow[rr, "\alpha"] \arrow[dd, "g'"] & & I' \oplus I'' \arrow[rr, "\beta"] & & I'' \arrow[rr] \arrow[dd, "g''"] & & 0 & \\
 & & Y' \arrow[rr, "c"] \arrow[dl, "j'"] & & Y \arrow[rr, "d"] \arrow[dl, "j"] & & Y'' \arrow[dl, "j''"] & & \\
0 \arrow[r] & J' \arrow[rr, "\gamma"] & & J' \oplus J'' \arrow[rr, "\delta"] & & J'' \arrow[rr] & & 0 &
\end{tikzcd}$$

with

$$\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad \gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \delta = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

We will define the desired morphism $g\colon I' \oplus I'' \to J' \oplus J''$ as

$$g = \begin{bmatrix} g' & \varepsilon \\ & g'' \end{bmatrix}$$

for a suitable morphism $\varepsilon\colon I'' \to J'$. To contruct ε, we start by defining a morphism $X \to J'$, that we then extend to a morphism $X'' \to J'$, and then further extend to the desired morphism $I'' \to J'$.

Let $\alpha'\colon I' \oplus I'' \to I'$ and $\gamma'\colon J' \oplus J'' \to J'$ be the splits of α and γ given by given by

$$\alpha' \coloneqq \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad \gamma' \coloneqq \begin{bmatrix} 1 & 0 \end{bmatrix}$$

We start off with the morphism $\varepsilon''\colon X \to J'$ given by

$$\varepsilon'' \coloneqq \gamma' j f - g' \alpha' i \,.$$

Then

$$\begin{aligned} \varepsilon'' a = (\gamma' j f - g' \alpha' i) a = \gamma' j f a - g' \alpha i a &= \gamma' j c f' - g' \alpha i a \\ &= \gamma' \gamma j' f' - g' \alpha' \alpha i' = j' f' - g' i' = 0 \,. \end{aligned}$$

[2]One should think about the diagonal morphisms in the diagram as starting points of injective resolutions.

It follows that the morphism ε'' factors through the morphism b, because b is a cokernel of a by the exactness of the sequence

$$X' \xrightarrow{a} X \xrightarrow{b} X'' \to 0 \,.$$

There hence exists a (unique) morphism $\varepsilon' \colon X'' \to J'$ that makes the triangle

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "\varepsilon''"'] & X'' \arrow[dl, dashed, "\varepsilon'"] \\ J' & \end{tikzcd}$$

commute. It follows from the injectivity of the object J' and i'' being a monomorphism that the morphism ε' further extends to a morphism $I'' \to J'$, i.e. that there exists a morphism $\varepsilon \colon I'' \to J'$ that makes the following triangle commute:

$$\begin{tikzcd} X'' \arrow[r, "i''"] \arrow[d, "\varepsilon'"'] & I'' \arrow[dl, dashed, "\varepsilon''"] \\ J' & \end{tikzcd}$$

We now find for the morphism

$$g \coloneqq \begin{bmatrix} g' & \varepsilon \\ & g'' \end{bmatrix} : I' \oplus I'' \to J' \oplus J''$$

that the diagram

$$\begin{tikzcd}
 & & X' \arrow[rr, "a"] \arrow[dl, "i'"'] \arrow[dd, "f'" near start] & & X \arrow[rr, "b"] \arrow[dl, "i"'] \arrow[dd, "f" near start] & & X'' \arrow[rr] \arrow[dl, "i''"'] \arrow[dd, "f''" near start] & & 0 \\
0 \arrow[r] & I' \arrow[rr, "\alpha" near end] \arrow[dd, "g'"] & & I' \oplus I'' \arrow[rr, "\beta" near end] \arrow[dd, dashed, "g" near start] & & I'' \arrow[rr] \arrow[dd, "g''" near start] & & 0 & \\
 & & Y' \arrow[rr, "c" near start] \arrow[dl, "j'"] & & Y \arrow[rr, "d" near start] \arrow[dl, "j"] & & Y'' \arrow[dl, "j''"] & & \\
0 \arrow[r] & J' \arrow[rr, "\gamma"] & & J' \oplus J'' \arrow[rr, "\delta"] & & J'' \arrow[rr] & & 0 &
\end{tikzcd}$$

commutes. Indeed, it holds that

$$g\alpha = \begin{bmatrix} g' & \varepsilon \\ & g'' \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} g' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} g' = \gamma g'$$

and

$$\delta g = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} g' & \varepsilon \\ & g'' \end{bmatrix} = \begin{bmatrix} 0 \\ g'' \end{bmatrix} = g'' \begin{bmatrix} 0 \\ 1 \end{bmatrix} = g''\beta \,,$$

which shows the commutativity of the frontal two squares. (Note that this commutativity only needs that g has an upper triangular form with diagonal entries g' and g''.) To check that $gi = jf$ we use the universal property of the product $(J' \oplus J'', \gamma', \delta)$, and show that $\delta gi = \delta jf$ and $\gamma' gi = \gamma' jf$. We have that

$$\delta gi = g''\beta i = g'' i'' b = j'' f'' b = j'' df = \delta jfr$$

which shows the first equality. We also have that

$$\gamma' g = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} g' & \varepsilon \\ & g'' \end{bmatrix} = \begin{bmatrix} g' & \varepsilon \end{bmatrix} = g'\alpha' + \varepsilon\beta \,,$$

and therefore

$$\begin{aligned} \gamma' gi = (g'\alpha' + \varepsilon\beta)i = g'\alpha' i + \varepsilon\beta i = g'\alpha' i + \varepsilon i'' b \\ = g'\alpha' i + \varepsilon' b = g'\alpha' i + \varepsilon'' = g'\alpha' i + \gamma' jf - g'\alpha' i = \gamma' jf \,. \end{aligned}$$

This shows the second equality. □

Corollary⋆ 5.O. Let

$$\begin{tikzcd} 0 \arrow[r] & {}^{\backprime}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & {}^{\backprime\backprime}X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & {}^{\backprime}Y \arrow[r] & Y \arrow[r] & {}^{\backprime\backprime}Y \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram in $\mathcal{A}$ with (short) exact rows. Let

$$\begin{tikzcd} & 0 \arrow[rr] & & {}^{\backprime}X \arrow[rr] \arrow[dl] \arrow[dd] & & X \arrow[rr] \arrow[dl] \arrow[dd] & & {}^{\backprime\backprime}X \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \\ 0 \arrow[rr] & & {}^{\backprime}I^\bullet \arrow[rr] \arrow[dd, "{}^{\backprime}f"'] & & I^\bullet \arrow[rr] & & {}^{\backprime\backprime}I^\bullet \arrow[rr] \arrow[dd, "{}^{\backprime\backprime}f"'] & & 0 & \\ & 0 \arrow[rr] & & {}^{\backprime}Y \arrow[rr] \arrow[dl] & & Y \arrow[rr] \arrow[dl] & & {}^{\backprime\backprime}Y \arrow[rr] \arrow[dl] & & 0 \\ 0 \arrow[rr] & & {}^{\backprime}J^\bullet \arrow[rr] & & J^\bullet \arrow[rr] & & {}^{\backprime\backprime}J^\bullet \arrow[rr] & & 0 & \end{tikzcd}$$

be a resulting commutative diagram of injective resolutions with short exact rows. Then there exists a morphism of chain complexes $f \colon I^\bullet \to J^\bullet$ that makes the resulting diagram commute.

Proof. We construct the required morphisms $f^n \colon I^n \to J^n$ inductively. For $n = 0$ we apply the previous lemma to the commutative diagram

$$\begin{tikzcd} & 0 \arrow[rr] & & {}^{\backprime}X \arrow[rr] \arrow[dl] \arrow[dd] & & X \arrow[rr] \arrow[dl] \arrow[dd] & & {}^{\backprime\backprime}X \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \\ 0 \arrow[rr] & & {}^{\backprime}I^0 \arrow[rr] \arrow[dd, "{}^{\backprime}f^0"'] & & I^0 \arrow[rr] & & {}^{\backprime\backprime}I^0 \arrow[rr] \arrow[dd, "{}^{\backprime\backprime}f^0"'] & & 0 & \\ & 0 \arrow[rr] & & {}^{\backprime}Y \arrow[rr] \arrow[dl] & & Y \arrow[rr] \arrow[dl] & & {}^{\backprime\backprime}Y \arrow[rr] \arrow[dl] & & 0 \\ 0 \arrow[rr] & & {}^{\backprime}J^0 \arrow[rr] & & J^0 \arrow[rr] & & {}^{\backprime\backprime}J^0 \arrow[rr] & & 0 & \end{tikzcd}$$

to get the desired morphism $f^0 \colon I^0 \to J^0$. To construct the morphism $f^1 \colon I^1 \to J^0$ we use that the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[rr] & & {`X} \arrow[rr] \arrow[dl, "{`i^0}"'] \arrow[dd] & & X \arrow[rr] \arrow[dl, "i^0"'] \arrow[dd] & & {``X} \arrow[rr] \arrow[dl, "{``i^0}"'] \arrow[dd] & & 0 \\
0 \arrow[rr] & & {`I^0} \arrow[rr] \arrow[dd, "{`f^0}"] & & I^0 \arrow[rr] \arrow[dd, "f^0"] & & {``I^0} \arrow[rr] \arrow[dd, "{``f^0}"] & & 0 & \\
 & 0 \arrow[rr] & & {`Y} \arrow[rr] \arrow[dl, "{`j^0}"] & & Y \arrow[rr] \arrow[dl, "j^0"] & & {``Y} \arrow[rr] \arrow[dl, "{``j^0}"] & & 0 \\
0 \arrow[rr] & & {`J^0} \arrow[rr] & & J^0 \arrow[rr] & & {``J^0} \arrow[rr] & & 0 &
\end{tikzcd}$$

induces the following commuative diagram:

$$\begin{tikzcd}
\operatorname{coker}(`i^0) \arrow[r] \arrow[d] & \operatorname{coker}(i^0) \arrow[r] \arrow[d] & \operatorname{coker}(``i^0) \arrow[d] \\
\operatorname{coker}(`j^0) \arrow[r] & \operatorname{coker}(j^0) \arrow[r] & \operatorname{coker}(``j^0)
\end{tikzcd}$$

We also find from the snake lemma that the two sequences

$$\operatorname{coker}(`i^0) \to \operatorname{coker}(i^0) \to \operatorname{coker}(``i^0) \to 0$$

and

$$\operatorname{coker}(`j^0) \to \operatorname{coker}(j^0) \to \operatorname{coker}(``j^0) \to 0$$

are (right) exact. We hence get the following commutative diagram with (right) exact rows:

$$\begin{tikzcd}
\operatorname{coker}(`i^0) \arrow[r] \arrow[d] & \operatorname{coker}(i^0) \arrow[r] \arrow[d] & \operatorname{coker}(``i^0) \arrow[r] \arrow[d] & 0 \\
\operatorname{coker}(`j^0) \arrow[r] & \operatorname{coker}(j^0) \arrow[r] & \operatorname{coker}(``j^0) \arrow[r] & 0
\end{tikzcd}$$

The commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[rr] & & {`I^0} \arrow[rr] \arrow[dl] \arrow[dd] & & I^0 \arrow[rr] \arrow[dl] \arrow[dd] & & {``I^0} \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \\
0 \arrow[rr] & & {`I^1} \arrow[rr] \arrow[dd] & & I^1 \arrow[rr] & & {``I^1} \arrow[rr] \arrow[dd] & & 0 & \\
 & 0 \arrow[rr] & & {`J^0} \arrow[rr] \arrow[dl] & & J^0 \arrow[rr] \arrow[dl] & & {``J^0} \arrow[rr] \arrow[dl] & & 0 \\
0 \arrow[rr] & & {`J^1} \arrow[rr] & & J^1 \arrow[rr] & & {``J^1} \arrow[rr] & & 0 &
\end{tikzcd}$$

factors through the following commutative diagram:

$$
\begin{tikzcd}
 & & \operatorname{coker}({}^{\backprime}i^0) \arrow[rr] \arrow[dl] \arrow[dd] & & \operatorname{coker}(i^0) \arrow[rr] \arrow[dl] \arrow[dd] & & \operatorname{coker}({}^{\backprime\backprime}i^0) \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \\
0 \arrow[r] & {}^{\backprime}I^1 \arrow[rr] \arrow[dd] & & I^1 \arrow[rr] & & {}^{\backprime\backprime}I^1 \arrow[rr] \arrow[dd] & & 0 & \\
 & & \operatorname{coker}({}^{\backprime}j^0) \arrow[rr] \arrow[dl] & & \operatorname{coker}(j^0) \arrow[rr] \arrow[dl] & & \operatorname{coker}({}^{\backprime\backprime}j^0) \arrow[rr] \arrow[dl] & & 0 \\
0 \arrow[r] & {}^{\backprime}J^1 \arrow[rr] & & J^1 \arrow[rr] & & {}^{\backprime\backprime}J^1 \arrow[rr] & & 0 &
\end{tikzcd}
$$

We can now apply the previous lemma to get the desired morphism $f^1 \colon I^1 \to J^1$. To construct the morphisms f^n with $n \geq 2$ we can proceed inductively in the same way. $\square$

Theorem 5.27. There exist morphisms

$$\delta_\xi^n \colon (\mathrm{R}^n F)(X'') \to (\mathrm{R}^{n+1} F)(X'),$$

where $n \geq 0$ and where

$$\xi \colon 0 \to X' \to X \to X'' \to 0$$

is any short exact sequence in $\mathcal{A}$, that make $\mathrm{R}^\bullet F := ((\mathrm{R}^n F)_{n\geq 0}, (\delta_\xi^n)_{n\geq 0,\xi})$ into a cohomological δ-functor.

Proof. Let

$$\xi \colon 0 \to X' \xrightarrow{g} X \xrightarrow{h} X'' \to 0$$

be a short exact sequence in $\mathcal{A}$. We get by the horseshoe lemma an injective resolution $(I^\bullet, i^0)$ of X such that the short exact sequence ξ lifts to a short exact sequence of chain complexes

$$0 \to I^\bullet_{X'} \to I^\bullet \to I^\bullet_{X''} \to 0,$$

such that the following diagram commutes:

$$
\begin{tikzcd}
0 \arrow[r] & X'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^\bullet_{X'} \arrow[r] & I^\bullet \arrow[r] & I^\bullet_{X''} \arrow[r] & 0
\end{tikzcd}
$$

The injective resolution $(I^\bullet, i^0)$ can moreover be choosen such that the short sequence $0 \to I^\bullet_{X'} \to I^\bullet \to I^\bullet_{X''} \to 0$ is split in each degree, i.e. such that the sequence $0 \to I^n_{X'} \to I^n \to I^n_{X''} \to 0$ splits for every $n \geq 0$ (again by the horseshoe lemma). It follows with the additivity of F that for every $n \geq 0$ the resulting sequence

$$0 \to F(I^n_{X'}) \to F(I^n) \to F(I^n_{X''}) \to 0$$

is again exact (as explained in Remark* 5.H). This shows that the sequence

$$0 \to F(I^\bullet_{X'}) \to F(I^\bullet) \to F(I^\bullet_{X''}) \to 0$$

is again (short) exact. We can therefore consider the resulting long exact cohomology sequence

$$\cdots \to \mathrm{H}^n(F(I^\bullet_{X'})) \to \mathrm{H}^n(F(I^\bullet)) \to \mathrm{H}^n(F(I^\bullet_{X''})) \xrightarrow{\delta^n_\xi} \mathrm{H}^{n+1}(F(I^\bullet_{X''})) \to \cdots$$

It follows from Lemma 5.23 that we get the following commutative diagram, in which the vertical arrows are isomorphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^n(F(I^\bullet_{X'})) \arrow[r] \arrow[d, equal] & \mathrm{H}^n(F(I^\bullet)) \arrow[r] \arrow[d, "\mathrm{R}^n_{(I^\bullet, i^0),(I^\bullet_X, i^0_X)}(\mathrm{id}_X)"] & \mathrm{H}^n(F(I^\bullet_{X''})) \arrow[r, "\delta^n_\xi"] \arrow[d, equal] & \mathrm{H}^{n+1}(F(I^\bullet_{X'})) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & (\mathrm{R}^n F)(X') \arrow[r] & (\mathrm{R}^n F)(X) \arrow[r] & (\mathrm{R}^n F)(X'') \arrow[r, "\delta^n_\xi"] & (\mathrm{R}^{n+1} F)(X') \arrow[r] & \cdots
\end{tikzcd}$$

The lower row of this diagram is therefore a long exact sequence:

$$\cdots \to (\mathrm{R}^n F)(X') \to (\mathrm{R}^n F)(X) \to (\mathrm{R}^n F)(X'') \xrightarrow{\delta^n_\xi} (\mathrm{R}^{n+1} F)(X') \to \cdots$$

We need to show that this induced long exact sequence in natural in ξ: We hence consider a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{A}$ with (short) exact rows. We get by the horseshoe lemma injective resolutions $(I^\bullet, i^0)$ and $(J^\bullet, j^0)$ of X and Y, and short exact sequences

$$0 \to I^\bullet_{X'} \to I^\bullet \to I^\bullet_{X''} \to 0 \qquad \text{and} \qquad 0 \to J^\bullet_{X'} \to J^\bullet \to J^\bullet_{X''} \to 0\,,$$

that fit into the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[rr] & & X' \arrow[rr] \arrow[dl] \arrow[dd] & & X \arrow[rr] \arrow[dl] \arrow[dd] & & X'' \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \\
0 \arrow[rr] & & I^\bullet_{X'} \arrow[rr] & & I^\bullet \arrow[rr] & & I^\bullet_{X''} \arrow[rr] & & 0 & \\
 & 0 \arrow[rr] & & Y' \arrow[rr] \arrow[dl] & & Y \arrow[rr] \arrow[dl] & & Y'' \arrow[rr] \arrow[dl] & & 0 \\
0 \arrow[rr] & & J^\bullet_{X'} \arrow[rr] & & J^\bullet \arrow[rr] & & J^\bullet_{X''} \arrow[rr] & & 0 &
\end{tikzcd}$$

It follows from the comparison theorem that there exists morphisms of chain complexes $I^\bullet_{X'} \to J^\bullet_{X'}$ and $I^\bullet_{X''} \to J^\bullet_{X''}$ that make the resulting diagram

$$
\begin{tikzcd}
& 0 \arrow[rr] & & X' \arrow[rr] \arrow[dl] \arrow[dd] & & X \arrow[rr] \arrow[dl] \arrow[dd] & & X'' \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \\
0 \arrow[rr] & & I^\bullet_{X'} \arrow[rr] \arrow[dd, dashed, "{}'f"'] & & I^\bullet \arrow[rr] & & I^\bullet_{X''} \arrow[rr] \arrow[dd, dashed, "{}''f"'] & & 0 & \\
& 0 \arrow[rr] & & Y' \arrow[rr] \arrow[dl] & & Y \arrow[rr] \arrow[dl] & & Y'' \arrow[rr] \arrow[dl] & & 0 \\
0 \arrow[rr] & & J^\bullet_{X'} \arrow[rr] & & J^\bullet \arrow[rr] & & J^\bullet_{X''} \arrow[rr] & & 0 &
\end{tikzcd}
$$

commute. It follows from Corollary⋆ 5.O that there exists a morphism of chain complexes $I^\bullet \to J^\bullet$ that makes the diagram

$$
\begin{tikzcd}
& 0 \arrow[rr] & & X' \arrow[rr] \arrow[dl] \arrow[dd] & & X \arrow[rr] \arrow[dl] \arrow[dd] & & X'' \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \\
0 \arrow[rr] & & I^\bullet_{X'} \arrow[rr] \arrow[dd, "{}'f"'] & & I^\bullet \arrow[rr] \arrow[dd, dashed, "f"'] & & I^\bullet_{X''} \arrow[rr] \arrow[dd, "{}''f"'] & & 0 & \\
& 0 \arrow[rr] & & Y' \arrow[rr] \arrow[dl] & & Y \arrow[rr] \arrow[dl] & & Y'' \arrow[rr] \arrow[dl] & & 0 \\
0 \arrow[rr] & & J^\bullet_{X'} \arrow[rr] & & J^\bullet \arrow[rr] & & J^\bullet_{X''} \arrow[rr] & & 0 &
\end{tikzcd}
$$

commute. We get in cohomology the following diagram with long exact rows:

$$
\begin{tikzcd}
& \cdots \arrow[rr] & & \mathrm{H}^n(F(I^\bullet_{X'})) \arrow[rr] \arrow[dd] & & \mathrm{H}^n(F(I^\bullet)) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{H}^n(F(I^\bullet_{X''})) \arrow[r] \arrow[dd] & \cdots \\
\cdots \arrow[r] & (\mathrm{R}^n F)(X') \arrow[rr] \arrow[ur, equal] \arrow[dd] & & (\mathrm{R}^n F)(X) \arrow[rr] \arrow[dd] & & (\mathrm{R}^n F)(X'') \arrow[rr] \arrow[ur, equal] \arrow[dd] & & \cdots & \\
& \cdots \arrow[rr] & & \mathrm{H}^n(F(I^\bullet_{Y'})) \arrow[rr] & & \mathrm{H}^n(F(J^\bullet)) \arrow[rr] \arrow[dl] & & \mathrm{H}^n(F(I^\bullet_{Y''})) \arrow[r] & \cdots \\
\cdots \arrow[r] & (\mathrm{R}^n F)(Y') \arrow[rr] \arrow[ur, equal] & & (\mathrm{R}^n F)(Y) \arrow[rr] & & (\mathrm{R}^n F)(Y'') \arrow[rr] \arrow[ur, equal] & & \cdots &
\end{tikzcd}
$$

We have seen above that the top and bottom squares commute in this diagram, and the back squares commute by the naturality of the long exact cohomology sequence. The commutativity of the front squares folllows because the diagonal morphisms are isomorphisms. □

End of lecture 21

Definition 5.28. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$.

i) The functor F is *effaceable* if for every object $X \in \mathrm{Ob}(\mathcal{A})$ there exists a monomorphism $u : X \to M$ in $\mathcal{A}$ with $F(u) = 0$.

ii) The functor $F : \mathcal{A} \to \mathcal{B}$ is *coeffaceable* if there exists for every object $X \in \mathrm{Ob}(\mathcal{A})$ an epimorphism $p : N \to X$ with $F(p) = 0$.

Remark 5.29. In the given situation (the category $\mathcal{A}$ has enough injectives, the functor $F\colon \mathcal{A} \to \mathcal{B}$ is left exact) we have for the constructed functors $\mathrm{R}^n F\colon \mathcal{A} \to \mathcal{B}$ that they are effaceable for every $n \geq 1$:

Indeed, we have for every injective object $I \in \mathrm{Ob}(\mathcal{A})$ the injective resolution $(I^\bullet, i^0)$ given by

$$0 \to I \xrightarrow{\mathrm{id}_I} I \to 0 \to 0 \to \cdots$$

It follows that

$$(\mathrm{R}^n F)(I) \cong (\mathrm{R}^n_{(I^\bullet, i^0)} F)(I) = 0$$

for every $n \geq 1$. For every object $X \in \mathrm{Ob}(\mathcal{A})$ there exists a monomorphism $u\colon X \to I$ with I being injective (because $\mathcal{A}$ has enough injectives) and we find for every $n \geq 1$ that $(\mathrm{R}^n F)(u) = 0$ for the morphim

$$(\mathrm{R}^n F)(u)\colon (\mathrm{R}^n F)(X) \to (\mathrm{R}^n F)(I)$$

because $(\mathrm{R}^n F)(I) = 0$.

Theorem 5.30. Let $T^\bullet\colon \mathcal{A} \to \mathcal{B}$ be a cohomological δ-functor. If the functors T^n are effaceable for every $n \geq 1$, then the δ-functor $T^\bullet$ is universal.

Proof. Let $S^\bullet\colon \mathcal{A} \to \mathcal{B}$ be another cohomological δ-functor and let $\eta\colon T^0 \to S^0$ be a natural transformation.

For every object $X \in \mathrm{Ob}(\mathcal{A})$ there exists a monomorphism $u\colon X \to M$ in $\mathcal{A}$ with $T^1(u) = 0$ because the functor T^1 is effaceable. Let $N := \mathrm{coker}(u)$ and let $v\colon M \to N$ be the canonical morphism. The short exact sequence

$$\xi\colon 0 \to X \xrightarrow{u} M \xrightarrow{v} N \to 0$$

induces the exact sequence

$$T^0(X) \xrightarrow{u} T^0(M) \xrightarrow{v} T^0(N) \xrightarrow{\delta^0_\xi} T^1(X) \xrightarrow{T^1(u)} T^1(M) \to \cdots$$

It follows with $T^1(u) = 0$ that the sequence

$$T^0(X) \xrightarrow{u} T^0(M) \xrightarrow{v} T^0(N) \xrightarrow{\delta^0_\xi} T^1(X) \to 0$$

is exact. We arrive at the following diagram with exact rows:

$$\begin{tikzcd}
T^0(X) \arrow[r, "T^0(u)"] \arrow[d, "\eta_X"] & T^0(M) \arrow[r, "T^0(v)"] \arrow[d, "\eta_M"] & T^0(N) \arrow[r, "\delta^0_{T,\xi}"] \arrow[d, "\eta_N"] & T^1(X) \arrow[r] & 0 \\
S^0(X) \arrow[r, "S^0(u)"] & S^0(M) \arrow[r, "S^0(v)"] & S^0(N) \arrow[r, "\delta^0_{S,\xi}"] & S^1(X) &
\end{tikzcd}$$

We show that

i) there exists a unique morphism $\eta^1_X \colon T^1(X) \to S^1(X)$ that makes the above diagram commute,

and that this morphism η^1_X is

ii) independent of the choice of the morphism u,

iii) natural in X, and

iv) compatible with the long exact sequence.

Indeed:

i) The exactness of the sequence

$$T^0(M) \xrightarrow{T^0(v)} T^0(N) \xrightarrow{\delta^0_{T,\xi}} T^1(X) \to 0$$

means that $\delta^0_{T,\xi}$ is a cokernel of the morphism $T^0(v)$. It holds that

$$\delta^0_{S,\xi} \circ \eta_N \circ T^0(v) = \underbrace{\delta^0_{S,\xi} \circ S^0(v)}_{=0} \circ \eta_M = 0\,,$$

whence there exists by the universal property of the cokernel a unique morphism $\eta^1_{X,u} \colon T^1(X) \to S^1(X)$ that makes the following resulting diagram commute:

$$\begin{tikzcd}
T^0(X) \arrow[r, "T^0(u)"] \arrow[d, "\eta_X"] & T^0(M) \arrow[r, "T^0(v)"] \arrow[d, "\eta_M"] & T^0(N) \arrow[r, "\delta^0_{T,\xi}"] \arrow[d, "\eta_N"] & T^1(X) \arrow[r] \arrow[d, dashed, "\eta^1_{X,u}"] & 0 \\
S^0(X) \arrow[r, "S^0(u)"] & S^0(M) \arrow[r, "S^0(v)"] & S^0(N) \arrow[r, "\delta^0_{S,\xi}"] & S^1(X) &
\end{tikzcd}$$

ii) Let

$$\xi' \colon 0 \to X \xrightarrow{u'} M' \xrightarrow{v'} N' \to 0$$

and

$$\xi'' \colon 0 \to X \xrightarrow{u''} M'' \xrightarrow{v''} N'' \to 0$$

be two short exact sequences with $T^1(u') = 0$ and $T^1(u'') = 0$. We consider the pushout

$$\begin{tikzcd}
X \arrow[r, "u'"] \arrow[d, "u''"'] \arrow[dr, phantom, "\ulcorner"] & M' \arrow[d] \\
M'' \arrow[r] & M
\end{tikzcd}$$

The morphism $M' \to M$ is a monomorphism by part iv) of Proposition 3.AG. Hence the composition $u \colon X \to M' \to M$ (which coincides with the composition $X \to M'' \to M$) is again a monomorphism. We note that $T^1(u) = 0$ because $T^1(u') = 0$. We get for $N \coloneqq \operatorname{coker}(u)$ a short exact sequence

$$0 \to X \to M \to N \to 0$$

that fits into the following commutative diagram with exact rows ξ' and ζ:

$$\begin{tikzcd}
\xi'\colon & 0 \arrow[r] & X \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d] & N' \arrow[r] & 0 \\
\zeta\colon & 0 \arrow[r] & X \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

It follows from the functoriality of the cokernel that there exists a unique morphism $N' \to N$ that makes the resulting diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & X \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

commute. We get the following diagram:

$$\begin{tikzcd}
 & T^0(N') \arrow[rr] \arrow[dl] \arrow[dd] & & T^1(X) \arrow[dl, equal] \\
T^0(N) \arrow[rr] \arrow[dd] & & T^1(X) & \\
 & S^0(N') \arrow[rr] \arrow[dl] & & S^1(X) \arrow[dl, equal] \\
S^0(N) \arrow[rr] & & S^1(X) &
\end{tikzcd}$$

The left square commutes by the naturality of η, the commutativity of the top and bottom squares are part of the axioms of a δ-functor. The morphism

$$\eta^1_{X,u'} \colon T^1(X) \to S^1(X)$$

is the unique morphism $T^1(X) \to S^1(X)$ that makes the resulting back square

$$\begin{tikzcd}
T^0(N') \arrow[r] \arrow[d] & T^1(X) \arrow[d, dashed, "\eta^1_{X,u'}"] \\
S^0(N') \arrow[r] & S^1(X)
\end{tikzcd}$$

commute, while the morphism $\eta^1_{X,u} \colon T^1(X) \to S^1(X)$ is the unique morphism $T^1(X) \to S^1(X)$ that makes the resulting front square

$$\begin{tikzcd}
T^0(N) \arrow[r] \arrow[d] & T^1(X) \arrow[d, dashed, "\eta^1_{X,u}"] \\
S^0(N) \arrow[r] & S^1(X)
\end{tikzcd}$$

commute. It follows that the cube

$$\begin{tikzcd}[row sep=small, column sep=small]
 & T^0(N') \arrow[rr] \arrow[dl] \arrow[dd] & & T^1(X) \arrow[dl, equal] \arrow[dd, dashed, "\eta^1_{X,u'}"] \\
T^0(N) \arrow[rr] \arrow[dd] & & T^1(X) \arrow[dd, dashed, "\eta^1_{X,u}" near start] & \\
 & S^0(N') \arrow[rr] \arrow[dl] & & S^1(X) \arrow[dl, equal] \\
S^0(N) \arrow[rr] & & S^1(X) &
\end{tikzcd}$$

commutes (because the identity $S^1(X) \to S^1(X)$ going from the back to the front is a monomorphism), and hence that $\eta^1_{X,u'} = \eta^1_{X,u}$. We find similarly that $\eta^1_{X,u''} = \eta^1_{X,u}$ and hence $\eta^1_{X,u'} = \eta^1_{X,u''}$.

Instead of $\eta^1_{X,u}$ we will use the notation η^1_X from now on.

iii) Let $f\colon X \to Y$ be a morphism in $\mathcal{A}$. Let

$$\xi\colon 0 \to X \xrightarrow{u_X} M_X \to N_X \to 0$$

be a short exact sequence in $\mathcal{A}$ with $T^1(u_X) = 0$. In the pushout

$$\begin{tikzcd}
X \arrow[r, "u_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & M_X \arrow[d] \\
Y \arrow[r, "u'"'] & P
\end{tikzcd}$$

the morphism u' is again a monomorphism by part iv) of Proposition 3.AG. Let $u\colon P \to M_Y$ be a monomorphism with $T^1(u) = 0$ and let $u_Y\colon Y \to M_Y$ be the composition

$$u_Y\colon Y \xrightarrow{u'} P \xrightarrow{u} M_Y\,.$$

With $N_Y \coloneqq \operatorname{coker}(u_Y)$ and the composition $M_X \to P \xrightarrow{u} M_Y$ we get the following commutative diagram with (short) exact rows ξ and ζ:

$$\begin{tikzcd}
\xi\colon & 0 \arrow[r] & X \arrow[r, "u_X"] \arrow[d, "f"] & M_X \arrow[r] \arrow[d] & N_X \arrow[r] & 0 \\
\zeta\colon & 0 \arrow[r] & Y \arrow[r, "u_Y"'] & M_Y \arrow[r] & N_Y \arrow[r] & 0
\end{tikzcd}$$

There exists a unique morphism $N_X \to N_Y$ that makes the resulting diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u_X"] \arrow[d, "f"] & M_X \arrow[r] \arrow[d] & N_X \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & Y \arrow[r, "u_Y"'] & M_Y \arrow[r] & N_Y \arrow[r] & 0
\end{tikzcd}$$

commute. Together with $T^1(u_X) = 0$ we get the following diagram in which the upper row is exact:

$$\begin{tikzcd}
 & T^0(N_X) \arrow[rr, "\delta^0_{T,\xi}"] \arrow[dl] \arrow[dd, "\eta_{N_X}"] & & T^1(X) \arrow[r] \arrow[dl] \arrow[dd, "\eta^1_X"] & 0 \\
T^0(N_Y) \arrow[rr] \arrow[dd, "\eta_{N_Y}"'] & & T^1(Y) \arrow[dd, "\eta^1_Y"] & & \\
 & S^0(N_X) \arrow[rr] \arrow[dl] & & S^1(X) \arrow[dl] & \\
S^0(N_Y) \arrow[rr] & & S^1(Y) & &
\end{tikzcd}$$

The left side of this cube commutes by the naturality of η, the back and front sides commute by construction of η^1_X and η^1_Y, and the commutativity of the upper and lower sides are part of the axioms of a cohomological δ-functor. The commutativity of the right side follows from the commutativity of the other sides because the morphism $\delta^0_{T,\xi}$ is an epimorphism. The commutativity of this right side shows the desired naturality of η^1.

iv) Let

$$\xi \colon 0 \to X' \to X \to X'' \to 0$$

be a short exact sequence in $\mathcal{A}$. We need to show the commutativity of the following square:

$$\begin{tikzcd}
T^0(X'') \arrow[r, "\delta^0_{T,\xi}"] \arrow[d, "\eta_{X''}"'] & T^1(X') \arrow[d, "\eta^1_{X'}"] \\
S^0(X'') \arrow[r, "\delta^0_{S,\xi}"'] & S^1(X')
\end{tikzcd}$$

Let $u \colon X \to M$ be a monomorphism with $T^1(u) = 0$ and let $u' \colon X' \to M$ be the composition

$$u' \colon X' \to X \xrightarrow{u} M\,.$$

Note that this morphism u' is again a monomorphism and that $T^1(u') = 0$. We get for $N'' \coloneqq \operatorname{coker}(u')$ the following commutative diagram with (short) exact rows ξ and ζ:

$$\begin{tikzcd}
\xi\colon & 0 \arrow[r] & X' \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "u"] & X'' \arrow[r] & 0 \\
\zeta\colon & 0 \arrow[r] & X' \arrow[r, "u'"] & M \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

We find (as above) that there exists a unique morphism $X'' \to N''$ that makes the resulting diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "u"] & X'' \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & X' \arrow[r, "u'"] & M \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

commute. We get the following diagram:

$$\begin{tikzcd}
& T^0(X'') \arrow[rr, "\delta^0_{T,\xi}"] \arrow[dl] \arrow[dd, "\eta_{X''}"] & & T^1(X') \arrow[dl, equal] \arrow[dd, "\eta^1_{X'}"] \\
T^0(N'') \arrow[rr, "\delta^0_{T,\zeta}"'] \arrow[dd, "\eta_{N''}"'] & & T^1(X') \arrow[dd, "\eta^1_{X'}"] & \\
& S^0(X'') \arrow[rr, "\delta^0_{S,\xi}"] \arrow[dl] & & S^1(X') \arrow[dl, equal] \\
S^0(N'') \arrow[rr, "\delta^0_{S,\zeta}"'] & & S^1(X') &
\end{tikzcd}$$

The left side commute by the naturality of η, the right side commutes, the front site commutes by construction of $\eta^1_{X'}$, and the commutativity of the top and bottom sides are part of the axioms of a cohomological δ-functor. It follows that the back side also commutes (because the identity $S^1(X') \to S^1(X')$ going from the back to the front is a monomorphism).

For the construction and uniqueness of the components η^n with $n \geq 2$ we can proceed indutively in the same way. ☐

Theorem 5.31. The constructed δ-functor $\mathrm{R}^\bullet F$ (from Theorem 5.27) is a right derived functor of F.

Proof. By Lemma 5.26 and Theorem 5.27 it remains to show that the δ-functor $\mathrm{R}^\bullet F$ is universal. This follows from Theorem 5.30 because the functors $\mathrm{R}^n F$ are for every $n \geq 1$ effaceable by Remark 5.29. ☐

Corollary 5.32. Let $T^\bullet \colon \mathcal{A} \to \mathcal{B}$ be a cohomological δ-functor and let $\mathcal{A}$ have enough injectives. Then the following conditions on $T^\bullet$ are equivalent:

i) The δ-functor $T^\bullet$ is universal.

ii) The functors T^n are effaceable for every $n \geq 1$.

iii) It holds for every injective object $I \in \mathrm{Ob}(\mathcal{A})$ and every $n \geq 1$ that $T^n(I) = 0$.

iv) It holds that $T^\bullet \cong \mathrm{R}^\bullet(T^0)$ as cohomological δ-functor.

Proof.

i) $\implies$ iv) The δ-functors $T^\bullet$ and $\mathrm{R}^\bullet(T^0)$ are both universal and $T^0 \cong \mathrm{R}^0(T^0)$. Hence $T^\bullet \cong \mathrm{R}^\bullet(T^0)$ by the uniqueness of right derived functors.

iv) $\implies$ iii) This was shown in Remark 5.29.

iii) $\implies$ ii) This follows from $\mathcal{A}$ having enough injectives, as seen in Remark 5.29.

ii) $\implies$ i) This is Theorem 5.30. □

Lemma⋆ 5.P. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor and let $G\colon \mathcal{B} \to \mathcal{C}$ be an exact functor. Then the composition $G \circ F\colon \mathcal{A} \to \mathcal{C}$ is again left exact, and

$$\mathrm{R}^n(G \circ F) \cong G \circ \mathrm{R}^n F$$

for every $n \geq 0$.

Proof. This is Exercise 1 on Exercise sheet 11. □

End of lecture 22

6 Projectives and Injectives in Interesting Categories

Projectives in Module Categories

Convention. Let $\mathbf{k}$ be a commutative ring and let A be a $\mathbf{k}$-algebra.

Lemma 6.1. A left A-module P is projective if and only if it is a direct summand of a free A-module.

Proof. This is Exercise 4 on Exercise sheet 10. □

Corollary 6.2. The module category A-**Mod** has enough projectives.

Proof. If M is any A-module with generating set $\{x_i\}_{i\in I}$ then there exists a surjective homomorphism of A-modules $A^{\oplus I} \to M$, with $A^{\oplus I}$ being free and hence projective by Lemma 6.1 □

Remark 6.3. Lemma 6.1 also holds for right modules, whence **Mod**-A has enough projectives as well.

Remark 6.4. Let M_A and ${}_AN$ be A-modules.

i) The functors

$$- \otimes_A N \colon \mathbf{Mod}\text{-}A \to \mathbf{k}\text{-}\mathbf{Mod} \quad \text{and} \quad M \otimes_A - \colon A\text{-}\mathbf{Mod} \to \mathbf{k}\text{-}\mathbf{Mod}$$

are right exact and hence admit left derived functors

$$\mathrm{L}_\bullet(- \otimes_A N) \colon \mathbf{Mod}\text{-}A \to \mathbf{k}\text{-}\mathbf{Mod} \quad \text{and} \quad \mathrm{L}_\bullet(M \otimes_A -) \colon A\text{-}\mathbf{Mod} \to \mathbf{k}\text{-}\mathbf{Mod}\,.$$

ii) The category $(A\text{-}\mathbf{Mod})^{\mathrm{op}}$ has enough injectives because the category A-**Mod** has enough projectives. The functor

$$\mathrm{Hom}_A(-, N) \colon (A\text{-}\mathbf{Mod})^{\mathrm{op}} \to \mathbf{k}\text{-}\mathbf{Mod}$$

is left exact and does therefore admit a right derived functor

$$\mathrm{R}^\bullet \mathrm{Hom}_A(-, N) \colon (A\text{-}\mathbf{Mod})^{\mathrm{op}} \to \mathbf{k}\text{-}\mathbf{Mod}\,.$$

Remark 6.5. Let A be a left noetherian $\mathbf{k}$-algebra. Then the category $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ of finitely generated left A-modules is abelian and the forgetful functor

$$U\colon A\text{-}\mathbf{Mod}^{\mathrm{fg}} \to A\text{-}\mathbf{Mod}$$

is exact. The exact functor U is in particular right exact and hence respects cokernels. Therefore a morphism in $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ is an epimorphism if and only if it is an epimorphism in $A\text{-}\mathbf{Mod}$. It follows that an object P of $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ is projective if and only if it is a direct summand of a finitely generated free A-module.

For finitely generated A-module M_A and ${}_AN$ the functors

$$- \otimes_A N\colon A\text{-}\mathbf{Mod}^{\mathrm{fg}} \to \mathbf{k}\text{-}\mathbf{Mod} \quad \text{and} \quad M \otimes_A -\colon A\text{-}\mathbf{Mod}^{\mathrm{fg}} \to \mathbf{k}\text{-}\mathbf{Mod}$$

are right exact, and the functor

$$\mathrm{Hom}_A(-, N)\colon \left(A\text{-}\mathbf{Mod}^{\mathrm{fg}}\right)^{\mathrm{op}} \to \mathbf{k}\text{-}\mathbf{Mod}$$

is left exact. It follows from the exactness of the forgetful functor U by Lemma⋆ 5.P that their right, resp. left derived functor agrees with the right, resp. left derived functor for $A\text{-}\mathbf{Mod}$, resp. $(A\text{-}\mathbf{Mod}^{\mathrm{op}})$.

End of lecture 23

Example 6.6. In the category $\mathbf{Ab}$ (resp. $\mathbf{Ab}^{\mathrm{fg}}$) of (finitely generated) abelian groups we find for every object P that

$$\begin{aligned} & P \text{ is projective} \\ \iff\ & P \text{ is a direct summand of a free (finitely generated) abelian group} \\ \iff\ & P \text{ is a free (finitely generated) abelian group} \end{aligned}$$

because subgroups of free abelian groups are again free abelian (of smaller rank). The same argument works for $A\text{-}\mathbf{Mod}$ and $A\text{-}\mathbf{Mod}^{\mathrm{fg}}$ where A is any PID.

Projectives in the Category of Quiver Representations

Convention. Let $\mathbf{k}$ be a field, Q a finite quiver (i.e. Q_0 and Q_1 are finite), and abbreviate $A \coloneqq \mathbf{k}Q$.

Definition 6.7.

i) For all vertices $i, j \in Q_0$ let

$$Q_*(i, j) = \{p \in Q_* \mid s(p) = i, t(p) = j\}$$

be the set of paths from i to j in Q.

ii) For every vertex $i \in Q_0$ let $P(i)$ be the representation of Q over $\mathbf{k}$ that is given by the following data:

- For every vertex $j \in Q_0$ the $\mathbf{k}$-vector space $P(i)_j$ is the free $\mathbf{k}$-vector space with basis $Q_*(i,j)$.
- For every arrow $\alpha \in Q_1$ with $\alpha\colon j \to k$ the $\mathbf{k}$-linear map $P(i)_\alpha\colon P(i)_j \to P(i)_k$ is given on the basis $Q(i,j)$ of $P(i)_j$ by the concatenation of paths. We hence have for every $p \in Q(i,j)$ that

$$P(i)_\alpha(p) = \alpha \circ p\,.$$

Remark 6.8. Recall the equivalence of categories $F\colon \mathbf{Rep}_{\mathbf{k}}(Q) \to A\text{-}\mathbf{Mod}$ from part i) of Example 2.20: For every representation X of Q over $\mathbf{k}$ the A-module $M \coloneqq F(X)$ has the underlying $\mathbf{k}$-vector space

$$M = \bigoplus_{j \in Q_0} X_j\,,$$

and the action of an arrow $\alpha\colon j \to k$ (i.e. an element of the basis Q_* of A) on M is given by the composition

$$M \xrightarrow{\text{projection}} X_j \xrightarrow{X_\alpha} X_k \xrightarrow{\text{inclusion}} M\,.$$

Under this equivalence of categories the representation $P(i)$ corresponds to

$$F(P(i)) \cong A\varepsilon_i\,.$$

Indeed, the $\mathbf{k}$-vector space $F(P(i)) = \bigoplus_{j \in Q_0} P(i)_j$ has as a basis the set of paths

$$\{p \in Q_* \mid s(p) = i\}\,. \tag{6.1}$$

This is also a basis of $A\varepsilon_i$: We have for every $p \in Q_*$ that

$$p\varepsilon_i = \begin{cases} p & \text{if } s(p) = i\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and hence find that the linearly independent set (6.1) is a generating set, and therefore basis, for $A\varepsilon_i$. The action of an arrow $\alpha \in Q_1$ on a basis element p from (6.1) is for $\alpha\colon j \to k$ given by

$$\alpha p = \begin{cases} \alpha \circ p & \text{if } t(\alpha) = s(p)\,, \\ 0 & \text{otherwise}\,, \end{cases} = \begin{cases} \alpha \circ p & \text{if } s(p) = j\,, \\ 0 & \text{otherwise}\,, \end{cases} = P(i)_\alpha(p)\,.$$

This shows that the actions of A on M and $A\varepsilon_i$ coincide.

Corollary 6.9. The representations $P(i)$ of Q are projective objects of $\mathbf{Rep}_{\mathbf{k}}(Q)$.

Proof. It sufficies to show that the A-modules $A\varepsilon_i$ are projective. We observe that

$$A = \bigoplus_{i \in Q_0} A\varepsilon_i$$

because on the level of bases

$$Q_* = \coprod_{i \in Q_0} \{p \in Q_* \mid s(p) = i\}\,.$$

The $A\varepsilon_i$ are therefore direct summands of the free A-module A, whence projective. ☐

Lemma⋆ 6.A. For every representation X of Q over $\mathbf{k}$ the map

$$\mathrm{Hom}(P(i), X) \to X_i\,, \quad f \mapsto f_i(\varepsilon_i)$$

is an isomorphism of $\mathbf{k}$-vector space.

Proof. This is part (i) of Exercise 4 of Exercise sheet 11. ☐

Remark⋆ 6.B. That the representations $P(i)$ are projective can also be seen with the help of Lemma⋆ 6.A: For every representation X of Q over $\mathbf{k}$ let

$$\varphi_X \colon \mathrm{Hom}(P(i), X) \to X_i$$

be the above isomorphism. Let X and Y be two such representations and let $f \colon X \to Y$ be an epimorphism of representations. We note the commutativity of the following square:

$$\begin{array}{ccc} \mathrm{Hom}(P(i), X) & \xrightarrow{f_*} & \mathrm{Hom}(P(i), Y) \\ {\scriptstyle\varphi_X}\downarrow & & \downarrow{\scriptstyle\varphi_Y} \\ X_i & \xrightarrow[f_i]{} & Y_i \end{array}$$

The $\mathbf{k}$-linear map f_i is surjective because f is an epimorphism. It follows that f_* is also surjective. This shows that the functor $\mathrm{Hom}(P(i), -)$ maps epimorphisms to surjections, which means that $P(i)$ is projective.

Definition⋆ 6.C. A quiver Q is *acyclic* if it contains no oriented circles (of length ≥ 1).

Corollary⋆ 6.D. If the quiver Q is acyclic then every $P(i)$ is finite-dimensional with $\mathrm{End}_k(P(i)) = k$.

Proof. This is part (iii) of Exercise 4 of Exercise sheet 11. ☐

Remark 6.10. If the quiver Q is acyclic then A is finite-dimensional and every $P(i)$ is indecomposable. (This is part (iv) of Exercise 4 of Exercise sheet 11.)

Theorem (Krull–Remak–Schmidt)**.** Let X be a finite-dimensional representation of an acyclic quiver Q.

i) There exists a decomposition $X \cong X_1^{\oplus a_1} \oplus \cdots \oplus X_r^{\oplus a_r}$ with X_i indecomposable, $X_i \ncong X_j$ for $i \neq j$ and $a_i > 0$ for every i.

ii) If $X \cong Y_1^{\oplus b_1} \oplus \cdots \oplus Y_s^{\oplus b_s}$ is another such decomposition then $r = s$ and, up to reordering, $X_i \cong Y_i$ and $a_i = b_i$ for every i.

Remark. We will not prove the Krull–Remak–Schmidt theorem. A proof can be found in [ASS06].

Remark⋆ 6.E. The Krull–Remak–Schmidt theorem holds for every **k**-algebra A and every finite-dimensional A-module M; it holds even more generally for every ring R and every R-module M of finite length. We will use the Krull–Remak–Schmidt theorem to show that every finite-dimensional projective representation of Q over **k** is isomorphic to a direct sum of copies of $P(i)$.

Theorem 6.11 (The standard projective resolution)**.** Let M be an A-module.[1] Then the sequence

$$0 \to \bigoplus_{\alpha \in Q_1} A\varepsilon_{t(\alpha)} \otimes_{\mathbf{k}} \varepsilon_{s(\alpha)} M \xrightarrow{f} \bigoplus_{i \in Q_0} A\varepsilon_i \otimes_{\mathbf{k}} \varepsilon_i M \xrightarrow{g} M \to 0$$

given by the **k**-linear maps

$$g((a_i \otimes x_i)_i) \coloneqq \sum_{i \in Q_0} a_i x_i$$

and

$$f((a_\alpha \otimes x_\alpha)_\alpha) \coloneqq \sum_{\alpha \in Q_1} \Big(\iota_{s(\alpha)}(a_\alpha \alpha \otimes x_\alpha) - \iota_{t(\alpha)}(a_\alpha \otimes \alpha x_\alpha) \Big)$$

is exact, and the appearing A-modules

$$P_0 \coloneqq \bigoplus_{i \in Q_0} A\varepsilon_i \otimes_{\mathbf{k}} \varepsilon_i M \qquad \text{and} \qquad P_1 \coloneqq \bigoplus_{\alpha \in Q_1} A\varepsilon_{t(\alpha)} \otimes_{\mathbf{k}} \varepsilon_{s(\alpha)} M$$

are projective.

Proof. The **k**-linear maps f and g are well-defined, and $f \circ g = 0$. We have for every $m \in M$ that

$$m = 1 \cdot m = \sum_{i \in Q_0} \varepsilon_i m = \sum_{i \in Q_0} \varepsilon_i^2 m = g\left(\sum_{i \in Q_0} \varepsilon_i \otimes \varepsilon_i m \right)$$

which shows that g is surjective. The A-modules P_0 and P_1 are projective because

$$P_0 \cong \bigoplus_{i \in Q_0} (A\varepsilon_i)^{\oplus \dim \varepsilon_i M} \quad \text{and} \quad P_1 \cong \bigoplus_{\alpha \in Q_1} (A\varepsilon_{t(\alpha)})^{\oplus \dim \varepsilon_{s(\alpha)} M}$$

are direct sums of the A-modules $A\varepsilon_i$.

To show the injectivity of f and the exactness at P_0 we observe that every element $\xi \in P_0$ can be uniquely written as

$$\xi = \left(\sum_{\substack{p \in Q_* \\ s(p) = i}} p \otimes \xi_p \right)_{i \in Q_0}$$

[1] We do not require M to be finite-dimensional, nor Q to be acyclic.

with $\xi_p \in \varepsilon_i A$ for $i = s(p)$, because the set $\{p \in Q_* \mid s(p) = i\}$ is a basis for $A\varepsilon_i$. The *length* of $\xi \neq 0$ is given by

$$\ell(\xi) = \max\{\ell(p) \mid \xi_p \neq 0\} .$$

Claim. For every $\xi \in P_0$ the residue class $\xi + \operatorname{im}(f)$ contains either 0 or an element of length 0.

Proof. Suppose that $\xi_p \neq 0$ for a path $p \in Q_*$ with starting vertex $i := s(p)$ such that $\ell(p) \geq 1$. If α denotes the starting arrow of p then there exists a unique path $p' \in Q_*$ with $p = p'\alpha$. We find for $\zeta \in P_1$ with components $\zeta_\alpha = p' \otimes \xi_p$ and $\zeta_{\alpha'} = 0$ otherwise, i.e.

$$(\delta_{\alpha,\alpha'} p' \otimes \xi_p)_{\alpha' \in Q_1}$$

that

$$f(\zeta) = \iota_{s(\alpha)}(p'\alpha \otimes \zeta_p) - \iota_{t(\alpha)}(p' \otimes \alpha\zeta_p) = \iota_{s(\alpha)}(p \otimes \zeta_p) - \iota_{t(\alpha)}(p' \otimes \alpha\zeta_p)$$

We have that $s(\alpha) = s(p) = i$ because the path p starts with α. We hence find that the element

$$\xi' := \xi - f(\zeta)$$

differs from ξ in (at most) two coordinates: In the i-th coordinate we're losing the summand $p \otimes \zeta_p$, while gaining the summand $p' \otimes \alpha\zeta_p$ in the $s(p')$-th coordinate. We note that both ξ' and ξ have the same residue class modulo $\operatorname{im}(f)$, and that $\ell(p') < \ell(p)$.

We have thus shows that by changing the representative of the residue class $\xi + \operatorname{im}(f)$ from ξ to ξ' we can replace a summand of length d in one coordinate by a summand of smaller length in another coordinate. By repeating this process finitely many times we arrive at a representative of the residue class $\xi + \operatorname{im}(f)$ that is either of length 0 or just 0 itself. □

We now show that $\ker(g) \subseteq \operatorname{im}(f)$: Suppose that there exists some $\xi \in \ker(g)$ with $\xi \notin \operatorname{im}(f)$. Then the residue class $\xi + \operatorname{im}(f)$ does not contain 0, and so we may assume by the above claim that $\ell(\xi) = 0$. Then

$$\xi = (\varepsilon_i \otimes \xi_{\varepsilon_i})_{i \in Q_0}$$

with $\xi_{\varepsilon_i} \in \varepsilon_i M$ for every $i \in Q_0$. We have that

$$0 = g(\xi) = \sum_{i \in Q_0} \varepsilon_i \xi_{\varepsilon_i} = \sum_{i \in Q_0} \xi_{\varepsilon_i}$$

because it follows from $\xi_{\varepsilon_i} \in \varepsilon_i M$ that $\varepsilon_i \xi_{\varepsilon_i} = \xi_{\varepsilon_i}$. We find from the directness of the sum $M = \bigoplus_{i \in Q_0} \varepsilon_i M$ that $\xi_{\varepsilon_i} = 0$ for every $i \in Q_0$, and hence $\xi = 0$. But this contradicts $\xi \notin \operatorname{im}(f)$. We find that indeed $\ker(g) \subseteq \operatorname{im}(f)$.

All that is left to show is the injectivity of f: We may write $\eta \in P_1$ uniquely as

$$\eta = \left(\sum_{\substack{p \in Q_* \\ s(p) = t(\alpha)}} p \otimes \eta_{\alpha,p} \right)_{\alpha \in Q_1}$$

with $\eta_{\alpha,p} \in \varepsilon_{s(\alpha)} A$ for every $\alpha \in Q_1$ and path $p \in Q_*$ with $s(p) = t(\alpha)$. Suppose that $\eta \neq 0$. Let $p_0 \in Q_*$ for which there exists an arrow $\alpha \in Q_1$ with $\eta_{\alpha,p_0} \neq 0$, and suppose that p_0 is of maximal length with this property. We may write the element

$$\xi \coloneqq f(\eta) = \sum_{\alpha \in Q_1} \left(\iota_{s(\alpha)} \left(\sum_{\substack{p \in Q_* \\ s(p)=t(\alpha)}} p\alpha \otimes \eta_{\alpha,p} \right) - \iota_{t(\alpha)} \left(\sum_{\substack{p \in Q_* \\ s(p)=t(\alpha)}} p \otimes \alpha\eta_{\alpha,p} \right) \right)$$

as

$$\xi = \left(\sum_{\substack{p \in Q_* \\ s(p)=i}} p \otimes \xi_p \right)_{i \in Q_0}$$

in the same way as before. We then find that

$$\xi_{p_0\alpha} = \eta_{\alpha,p_0} \neq 0\,.$$

and hence that $f(\eta) \neq 0$. This shows that $\ker(f) = 0$. ☐

Remark 6.12. Under the equivalence of categories $F\colon \mathbf{Rep}_{\mathbf{k}}(Q) \to \mathbf{k}Q\text{-}\mathbf{Mod}$ the standard projective resolution of a representation X is given by

$$0 \to \bigoplus_{\alpha \in Q_1} P(t(\alpha)) \otimes_{\mathbf{k}} X_{s(\alpha)} \to \bigoplus_{i \in Q_0} P(i) \otimes_{\mathbf{k}} X_i \to X \to 0\,.$$

Corollary 6.13. Let X be a representation Q over $\mathbf{k}$. Then $\mathrm{R}^n \operatorname{Hom}(-, X) = 0$ for every $n \geq 2$.

Definition 6.14. The $\mathbb{Z}$-bilinear form

$$\langle -, - \rangle \colon \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$$

given by

$$\langle d, e \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} d_{s(\alpha)} e_{t(\alpha)}$$

is the *Euler form* of Q.

Definition* 6.F. The *dimension vector* of a finite-dimensional representation X of Q over $\mathbf{k}$ is the tupel

$$\mathbf{dim}\, X \coloneqq (\dim X_i)_{i \in Q_0}$$

Corollary 6.15. Let X and Y be finite-dimensional representations of Q over $\mathbf{k}$. Then

$$\dim \operatorname{Hom}(X, Y) - \dim(\mathrm{R}^1 \operatorname{Hom}(-, Y))(X) = \langle \mathbf{dim}\, X, \mathbf{dim}\, Y \rangle\,.$$

Proof. Let

$$\underbrace{\cdots \to 0 \to 0 \to P_1 \to P_0}_{P_\bullet} \to X \to 0$$

be the standard projective resolution of X. By applying the functor $\operatorname{Hom}(-, Y)$ to this resolution we arrive at the cochain complex

$$\operatorname{Hom}(P_\bullet, Y) = \big(\cdots \to 0 \to 0 \to \operatorname{Hom}(P_0, Y) \to \operatorname{Hom}(P_1, Y)) \to 0 \to 0 \to \cdots\big) .$$

We find with the universal property of the coproduct and Lemma* 6.A that

$$\begin{aligned} \operatorname{Hom}(P_0, Y) &= \operatorname{Hom}\left(\bigoplus_{i \in Q_0} P(i) \otimes_{\mathbf{k}} X_i, Y\right) \\ &\cong \left(\bigoplus_{i \in Q_0} P(i)^{\oplus \dim X_i}, Y\right) \\ &\cong \bigoplus_{i \in Q_0} \operatorname{Hom}(P(i), Y)^{\oplus \dim X_i} \\ &\cong \bigoplus_{i \in Q_0} Y_i^{\oplus \dim X_i} \end{aligned}$$

and hence

$$\dim \operatorname{Hom}(P_0, Y) = \sum_{i \in Q_0} \dim X_i \dim Y_i .$$

We similarly find that

$$\dim \operatorname{Hom}(P_1, Y) = \sum_{\alpha \in Q_1} \dim X_{s(\alpha)} \dim Y_{t(\alpha)} .$$

We get from the short exact sequence

$$0 \to P_1 \to P_0 \to X \to 0$$

the induced (long) exact equence

$$0 \to \operatorname{Hom}(X, Y) \to \operatorname{Hom}(P_0, Y) \to \operatorname{Hom}(P_1, Y) \to (\mathrm{R}^1 \operatorname{Hom}(-, Y))(X) \to 0 \to \cdots$$

where we use that $(\mathrm{R}^1 \operatorname{Hom}(-, Y))(P_0) = 0$ because P_0 is projective. It follows that

$$\begin{aligned} &\dim \operatorname{Hom}(X, Y) - \dim(\mathrm{R}^1 \operatorname{Hom}(-, Y))(X) \\ =\ &\dim \operatorname{Hom}(P_0, Y) - \operatorname{Hom} \operatorname{Hom}(P_1, Y) \\ =\ &\sum_{i \in Q_0} \dim X_i \dim Y_i - \sum_{\alpha \in Q_1} \dim X_{s(\alpha)} \dim Y_{t(\alpha)} \\ =\ &\langle \mathbf{dim}\, X, \mathbf{dim}\, Y \rangle \end{aligned}$$

as claimed. □

End of lecture 24

Categories Without Enough Projectives

Example 6.16. The category $\mathcal{A}$ of finite abelian groups does not have any projective object aside from 0: For every $n \neq 1$ the short exact sequence

$$0 \to \mathbb{Z}/n \to \mathbb{Z}/n^2 \to \mathbb{Z}/n \to 0$$

does not split, which shows that $\mathbb{Z}/n$ is not projective. Every finite abelian group is (up to isomorphism) of the form

$$(\mathbb{Z}/n_1)^{\oplus a_1} \oplus \cdots \oplus (\mathbb{Z}/n_r)^{\oplus a_r}$$

for some $n_i \neq 0$ and $a_i \geq 0$, and hence also not projective by part iii) of Lemma 5.14 (aside for the trivial group 0 which admit zero such summand).

Example 6.17. If X is a topological space then the category $\mathbf{Sh}(X)$ of sheaves over X has enough projectives if and only if the topological space X is Alexandrov (i.e. arbitrary intersections of open subsets are again open). (See [MO5378] for more details.)

Injectives in Module Categories

Convention. Let $\mathbf{k}$ be a commutative ring and let A be a $\mathbf{k}$-algebra.

Theorem 6.18 (Baer's criterion)**.** For an A-module M the following conditions are equivalent:

i) The module M is injective.

ii) For every left ideal $J \subseteq A$, every module homomorphism $J \to M$ extends to a module homomorphism $A \to M$.

$$\begin{tikzcd} J \arrow[r, hook] \arrow[d] & A \arrow[dl, dashed] \\ M & \end{tikzcd}$$

Remark⋆ 6.G. Baer's criterion states that it sufficies to prove the defining condition of an injective module for inclusions $J \to A$ of left ideals $J \subseteq A$.

Proof of Baer's criterion. If M is injective then condition ii) holds because the inclusion $J \to A$ is a monomorphism.

Suppose on the other hand that condition ii) holds. Let N be any A-module, $N' \subseteq N$ a submodule and $f' \colon N' \to M$ a module homomorphism. We need to show that there exists a module homomorphism $f \colon N \to M$ with $f|_{N'} = f'$.

It follows from Zorn's lemma that there exists a maximal extension $\overline{f}$ of f. More explicitely, there exists a submodule $\overline{N} \subseteq M$ and a module homomorphism $\overline{f} \colon \overline{N} \to M$ with $N' \subseteq \overline{N}$ and $\overline{f}|_{N'} = f'$, such that $\overline{N}$ is maximal with this property (with respect to the inclusion of submodules).

Suppose that $\overline{N} \neq N$. Then let $y \in N$ with $y \notin \overline{N}$ and consider the left ideal

$$J \coloneqq \{a \in A \mid ay \in \overline{N}\} .$$

The map

$$g \colon J \to M , \quad a \mapsto \overline{f}(ay)$$

is a module homomorphism and can therefore (by assumption) be extended to a module homomorphism

$$\overline{g} \colon A \to M .$$

For $N'' \coloneqq \overline{N} + Ay$ the map

$$f'' \colon N'' \to M , \quad x + ay \mapsto \overline{f}(x) + \overline{g}(a)$$

where $x \in \overline{N}$ and $a \in A$ is a well-defined module homomorphism that extends f. To see that f'' is well-defined suppose that $x + ay = x' + a'y$ for some $x, x' \in \overline{N}$ and $a, a' \in A$. Then

$$\overline{N} \ni x - x' = (a' - a)y$$

and therefore $a' - a \in J$; then

$$\overline{f}(x) - \overline{f}(x') = \overline{f}(x - x') = \overline{f}((a' - a)y) = \overline{g}(a' - a) = \overline{g}(a') - \overline{g}(a)$$

and therefore

$$\overline{f}(x) + \overline{g}(a) = \overline{f}(x') + \overline{g}(a') .$$

But $\overline{N}$ is a proper submodule of N'', which contradicts the maximality of $\overline{N}$. $\square$

Definition* 6.H. An A-module M is *divisible* if there exists for every $m \in M$ and $a \in A$ with $a \neq 0$ some $m' \in M$ such that $m = am'$.

Corollary 6.19. An abelian group M is injective in **Ab** if and only if it is divisible.

Proof. If M is injective then for every $n \in \mathbb{Z}$ with $n \neq 0$ the linear map

$$f \colon n\mathbb{Z} \to M , \quad k \mapsto \frac{k}{n}a$$

extends to a linear map $g \colon \mathbb{Z} \to M$. Then

$$ng(1) = g(n) = f(n) = a .$$

Suppose on the other hand that M is disivible. We use Baer's criterion to show that M is is injective: Let $J \subseteq \mathbb{Z}$ be an ideal and let $f \colon J \to M$ be a group homomorphism; we need to show that f extends to a group homomorphism $\mathbb{Z} \to M$. The ideal J is of the form $J = (n)$ for some $n \in \mathbb{Z}$. If $J = 0$ then the zero morphism $\mathbb{Z} \to M$ is such an extension. Otherwise $n \neq 0$ and it follows for the element $f(n) \in M$ from the divisibility of M that there exists some $x \in M$ with $f(n) = nx$. The group homomorphism

$$g \colon \mathbb{Z} \to M , \quad k \mapsto kx$$

then satisfies

$$g(n) = nx = f(n)$$

and is therefore an extension on f. $\square$

Remark 6.20. Corollary 6.19 holds more generally when $\mathbb{Z}$ is replaced by any PID.

Example 6.21.

i) The abelian group $\mathbb{Z}$ is no injective.
ii) The abelian group $\mathbb{Q}$ is injective.
iii) The abelian group $\mathbb{Q}/\mathbb{Z}$ is injective.

Lemma 6.22. If $(I_\beta)_{\beta \in B}$ is a family of injective objects in a category $\mathcal{C}$ then the product $\prod_{\beta \in B} I_\beta$ (if it exists) is again injective.

Corollary 6.23. The category **Ab** has enough injectives.

Proof. Let M be an abelian group and let

$$I \coloneqq \prod_{f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}\,.$$

The abelian group I again injective by Lemma 6.22. Let $i\colon M \to I$ be the group homomorphism that is given in the f-th coordinate (for $f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$) by f, i.e.

$$i(x) = (f(x))_{f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})}\,.$$

There exists by the upcomming Lemma 6.24 for every $x \in M$ with $x \neq 0$ a group homomorphism $f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ with $f(x) \neq 0$. This shows that $i(x) \neq 0$, which shows that i is injective. □

Lemma 6.24. Let M be an abelian group. There exists for every $x \in M$ with $x \neq 0$ a group homomorphism $f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ with $f(x) \neq 0$.

Proof. We may assume that $M = \mathbb{Z}x$ because $\mathbb{Q}/\mathbb{Z}$ is injective, and every such homomorphism $\mathbb{Z}x \to \mathbb{Q}/\mathbb{Z}$ therefore extends to a homomorphism $M \to \mathbb{Q}/\mathbb{Z}$. The group $\mathbb{Z}x$ is cyclic and so we may assume that $M = \mathbb{Z}/n$ for some $n \geq 0$ and that $x = [1]$. In the case $n = 0$ we can use the projection $\mathbb{Z} \to \mathbb{Z}/2$ to assume that $n > 0$. We can now use the embedding $\mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$ given by $[1] \to 1/n$ as a suitable morphism. □

Theorem 6.25. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let (F, G, φ) be an adjunction from $\mathcal{A}$ to $\mathcal{B}$ (i.e. $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ and F is left adjoint to G).

i) The functors F and G are additive, and the map $\varphi_{X,Y}$ is for all object $X \in \mathrm{Ob}(\mathcal{A})$ and $Y \in \mathrm{Ob}(\mathcal{B})$ an isomorphism of abelian groups.
ii) The functor F is right exact, whereas the functor G is left exact.
iii) If F is exact and I is an injective object of $\mathcal{B}$ then $G(I)$ is an injective object of $\mathcal{A}$. If G is exact and P is a projective object of $\mathcal{A}$ then $F(P)$ is a projective object of $\mathcal{B}$.

Proof.

i) This is part (ii) of Exercise 4 of Exercise sheet 13.

ii) This is part (iii) of Exercise 4 of Exercise sheet 13.

iii) We show the first assertion, the second assertion then follows by duality. We show that $\mathrm{Hom}_{\mathcal{A}}(-, G(I))$ maps monomorphisms to surjections. So let $f\colon X' \to X$ be a monomorphism in $\mathcal{A}$. We get the following commutative square:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(X, G(I)) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathcal{A}}(X', G(I)) \\ \mathrm{Hom}_{\mathcal{B}}(F(X), I) \arrow[u, "\varphi_{X,Y}"] \arrow[r, "F(f)^*"'] & \mathrm{Hom}_{\mathcal{B}}(F(X'), I) \arrow[u, "\varphi_{X',Y}"'] \end{tikzcd}$$

The functor F respects kernels because it is left exact, and hence respect monomorphisms. We therefore find that the morphism $F(f)$ is again a monomorphism. It follows from the injectivity of the object I that $F(f)^*$ is surjective. Therefore $f^* = \varphi_{X',Y} \circ F(f)^* \circ \varphi_{X,Y}^{-1}$ is surjective. □

Corollary 6.26. If I is an injective abelian group then $\mathrm{Hom}_{\mathbb{Z}}(A, I)$ is an injective left A-module, where A acts on $\mathrm{Hom}_{\mathbb{Z}}(A, I)$ via

$$(a.f)(a') = f(a'a)$$

for all $a \in A$ and all $f \in \mathrm{Hom}_{\mathbb{Z}}(A, I)$ and $a' \in A$. Similarly for right A-modules.

Proof. The forgetful functor $F\colon A\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is exact and has a right adjoint

$$G\colon \mathbf{Ab} \to A\text{-}\mathbf{Mod}\,, \quad M \mapsto \mathrm{Hom}_{\mathbb{Z}}(A, M)\,.$$

It follows from part iii) of Theorem 6.25 that G respects injectives. It therefore follows from the injectivity of I that $G(I) = \mathrm{Hom}_{\mathbb{Z}}(A, I)$ is again injective. □

Example* 6.I. The A-module $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is injective.

Lemma* 6.J. Let M be an A-module. Then there exists for every $x \in M$ with $x \neq 0$ a module homomorphism $f\colon M \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ with $f(x) \neq 0$.

Proof. We may assume that $M = Ax$ because by the injectivity of $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ every such module homomorphism $Ax \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ can be extended to a module homomorphism $M \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. We may further assume that $M = A/J$ for a left ideal $J \subseteq A$ and that $x = [1]$. It follows from $x \neq 0$ by Lemma 6.24 that there exists some $h \in \mathrm{Hom}_{\mathbb{Z}}(A/J, \mathbb{Q}/\mathbb{Z})$ with $h([1]) \neq 0$. Let $g \in \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ be the composition

$$g\colon A \xrightarrow{\pi} A/J \xrightarrow{h}$$

where $\pi\colon A \to A/J$ is the canonical projection. Let $\tilde{f}\colon A \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ be the unique module homomorphism with $\tilde{f}(1) = g$. It holds for every $y \in J$ that

$$\tilde{f}(y)(a) = (yg)(a) = g(ay) = h([ay]) = h(0) = 0$$

for every $a \in A$ because $ay \in J$; hence $\tilde{f}(y) = 0$. This shows that $\tilde{f}$ factors through a well-defined module homomorphism $f\colon A/J \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. This homomorphism satisfies

$$f([1]) = \tilde{f}(1) = g$$

with $g \neq 0$ (because $h \neq 0$). $\square$

Corollary 6.27. The category A-**Mod** has enough injectives (and so has **Mod**-A).

Proof. The proof proceed analogous to Corollary 6.23, with the role of $\mathbb{Q}/\mathbb{Z}$ replaced by $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$,. $\square$

End of lecture 25

Remark 6.28. We can for every A-module ${}_AM$ form the right derived functors of the left exact functor $\mathrm{Hom}_A(M, -)\colon A\text{-}\mathbf{Mod} \to \mathbf{k}\text{-}\mathbf{Mod}$.

Injectives in the Category of Quiver Representations

Remark 6.29. Let Q be a finite quiver and suppose that $\mathbf{k}$ is a field. For every $i \in Q_0$ we define a representation $I(i)$ of Q by the following data:

- For every vertex $j \in Q_0$ the $\mathbf{k}$-vector space $I(i)_j$ is the free $\mathbf{k}$-vector space with basis $Q_*(j, i)$.
- For every arrow $\alpha \in Q_1$ with $\alpha\colon j \to k$ the $\mathbf{k}$-linear map $I(i)_\alpha\colon I(i)_j \to I(i)_k$ is given on the basis elements $p \in Q_*(j, i)$ of $P(i)_j$ by

$$I(i)_\alpha(p) := \begin{cases} p' & \text{if } p = p'\alpha\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We get for every representation X of Q an isomorphism of $\mathbf{k}$-vector space

$$\varphi\colon\ \mathrm{Hom}(X, I(i)) \to X_i^*\,,$$

where for every $f \in \mathrm{Hom}(X, I(i))$ the element $\varphi(f) \in X_i^*$ is given by the composition

$$\varphi(f)\colon X_i \xrightarrow{f_i} I(i)_i \xrightarrow{\pi_{\varepsilon_i}} k\,.$$

It follows that $I(i)$ is injective.

Remark⋆ 6.K.

i) The injectivity of the $I(i)$ can be shown similarly to the projectivity of the $P(i)$ in Remark⋆ 6.B: For every representation X of Q over $\mathbf{k}$ let

$$\varphi_X\colon\ \mathrm{Hom}(X, I(i)) \to X_i^*$$

be the above isomorphism of **k**-vector spaces. Let X and Y be two representations of Q over **k** and let $f\colon X \to Y$ be a monomorphism of representations We then have the following commutative rectangle:

$$\begin{tikzcd} \mathrm{Hom}(Y, I(i)) \arrow[r, "f^*"] \arrow[d, "\varphi_X"'] & \mathrm{Hom}(X, I(i)) \arrow[d, "\varphi_Y"] \\ Y_i^* \arrow[r, "f_i^*"'] & X_i^* \end{tikzcd}$$

Indeed, we have for $g \in \mathrm{Hom}(Y, I(i))$ and $x \in X_i$ that

$$\varphi_Y(f^*(g))(x) = \varphi_Y(g \circ f)(x) = \pi_{\varepsilon_i}((g \circ f)_i(x)) = \pi_{\varepsilon_i}(g_i(f_i(x)))$$

and similarly

$$(f_i^* \circ \varphi_X)(g)(x) = f_i^*(\varphi_X(g))(x) = \varphi_X(g)(f_i(x)) = \pi_{\varepsilon_i}(g_i(f_i(x)))\,.$$

The **k**-linear map $f_i\colon X_i \to Y_i$ is injective, so its dual map $f_i^*\colon Y_i^* \to X_i^*$ is surjective. It follows that $f^* = \varphi_Y^{-1} \circ f_i^* \circ \varphi_X$ is surjective. This shows that the functor $\mathrm{Hom}(-, I(i))$ maps monomorphisms (in $\mathcal{A}$) to surjections, which means that the representation $I(i)$ is injective.

ii) For any right A-module M its dual space M^* becomes a left A-module via

$$(a\varphi)(m) = \varphi(ma)$$

for all $a \in A$, $\varphi \in M^*$ and $m \in M$. We find for the (projective) right A-module $\varepsilon_i A$ that its dual $(\varepsilon_i A)^*$ becomes a left A-module. The A-module $I(i)$ can be constructed as a submodule of $(\varepsilon_i A)^*$:

The right A-module $\varepsilon_i A$ has as a basis the set

$$\{p \in Q_* \mid t(p) = i\}\,, \tag{6.2}$$

hence its dual $(\varepsilon_i A)^*$ contains the linear independent family

$$\{p^* \mid p \in Q_*, t(p) = i\}\,. \tag{6.3}$$

We have for every arrow $\alpha \in Q_1$ with $\alpha\colon j \to k$ that

$$\alpha \cdot p^* = \begin{cases} (p')^* & \text{if } p = p'\alpha\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{6.4}$$

Indeed, it holds for every basis element q of $\varepsilon_i A$, i.e. $q \in Q_*$ with $t(\alpha) = i$, that

$$(\alpha \cdot p^*)(q) = p^*(q\alpha) = \begin{cases} 1 & \text{if } p = q\alpha\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

If $p = p'\alpha$ for some path $p' \in Q_*$ (with necessarily $t(p') = t(p) = i$) then therefore

$$(\alpha \cdot p^*)(q) = \begin{cases} 1 & \text{if } q = p' , \\ 0 & \text{otherwise} \end{cases} = (p')^*(q)$$

and hence $\alpha \cdot p^* = (p')^*$. If no such p' exists then $(\alpha \cdot p^*)(q) = 0$ for every q and hence $\alpha \cdot p^* = 0$.

We can now identify the elements of (6.3) with the elements of the original basis (6.2) via the bijection $p^* \mapsto p$. We then get a left A-module I with basis (6.2), and on which the multiplication rule (6.4) becomes

$$\alpha \cdot p = \begin{cases} p' & \text{if } p = p'\alpha , \\ 0 & \text{otherwise} . \end{cases}$$

We have in particular for every $j \in Q_0$ that $\varepsilon_j I$ has as a basis the set

$$\begin{aligned} &\{p' \in Q_* \mid p = p'\varepsilon_j \text{ for some } p \in Q_* \text{ with } t(p) = i\} \\ = &\{p' \in Q_* \mid p = p' \text{ for some } p \in Q_* \text{ with } t(p) = i \text{ and } s(p) = j\} \\ = &\{p \in Q_* \mid s(p) = j, t(p) = i\} = Q_*(j, i) . \end{aligned}$$

This shows that $I_j = I(i)_j$ for every $j \in Q_0$ and that the actions of Q on I and $I(i)$ coincide. Hence $I(i) = I \hookrightarrow (\varepsilon_i A)^*$.

Injectives in Other Categories

Example 6.30. Let X be a topological space. Then the catgory $\mathbf{Sh}_X(\mathbf{Ab})$ has enough injectives. Let us give a sketch of the poof:

If Y is another topological space and $f\colon X \to Y$ is a continuous map, then f induces a functor

$$f_*\colon \mathbf{Sh}_Y(\mathbf{Ab}) \to \mathbf{Sh}_X(\mathbf{Ab}) ,$$

which is given on objects by

$$f_*(\mathscr{G})(U) := \mathscr{G}(f^{-1}(U))$$

for every sheaf $\mathscr{G}$ on Y and every open subset $U \subseteq X$. For the special case $X = \{*\}$ the sheaf $f_*(\mathscr{G})$ consists of the single abelian group $\mathscr{G}(Y) = \Gamma(Y, \mathscr{G})$.

The functor f_* admits a left adjoint $f^*\colon \mathbf{Sh}_X(\mathbf{Ab}) \to \mathbf{Sh}_Y(\mathbf{Ab})$, that can be shown to be exact. It follows from Theorem 6.25 that f_* respects injectives, i.e. that $f_*(\mathscr{I})$ is injective whenever $\mathscr{I}$ is injective.

We can for every $x \in X$ consider the inclusion $i_x\colon \{x\} \to X$ to get for every sheaf $\mathscr{F}$ on X an abelian group $\mathscr{F}_x := i_x^*(\mathscr{F})$. The category $\mathbf{Sh}_{\{x\}}(\mathbf{Ab}) = \mathbf{Ab}$ has enough injectives, whence we have for some injective abelian group I_x an embedding $\mathscr{F}_x \to I_x$. The induced morphism of sheaves $(i_x)_*(\mathscr{F}_x) \to (i_x)_*(I_x)$ is again a monomorphism because $(i_x)_*$ is left exact (since it is right adjoint) and hence respects kernels. Together with the adjunction unit

$$\mathscr{F} \to (i_x)_*(i_x^*(\mathscr{F})) = (i_x)_*(\mathscr{F}_x)$$

we get a natural morphism

$$\mathscr{F} \to (i_x)_*(\mathscr{F}_x) \to (i_x)_*(I_x)\,.$$

These natural morphisms fit together to form a monomorphism

$$\mathscr{F} \to \prod_{x \in X} (i_x)_*(\mathscr{F}_x)\,.$$

The composition

$$\mathscr{F} \to \prod_{x \in X} (i_x)_*(\mathscr{F}_x) \to \prod_{x \in X} (i_x)_*(I_x)$$

is a monomorphism into the injective object $\prod_{x \in X} (i_x)_*(I_x)$. This shows that $\mathbf{Sh}_X(\mathbf{Ab})$ has enough injectives.

We can therefore consider for every open subset $U \subseteq X$ the right derived functors of the section functor $\Gamma(U, -) \colon \mathbf{Sh}_X(\mathbf{Ab}) \to \mathbf{Ab}$, or more generally the right derived functors of the aforementioned functor $f_* \colon \mathbf{Sh}_Y(\mathbf{Ab}) \to \mathbf{Sh}_X(\mathbf{Ab})$. The right derived functors $(\mathrm{R}^n \, \Gamma(U, -))(\mathscr{F}) =: \mathrm{H}^i(U, \mathscr{F})$ are known[2] as *sheaf cohomology*, whereas the derived functors $(\mathrm{R}^n \, f_*)(\mathscr{F})$ are known as the *higher direct image*.

Categories Without Enough Injectives

Example 6.31.

i) The category of finitely generated abelian groups has no injective objects aside from the trivial group 0:

Every finitely generated abelian groups M is a finite direct sum

$$M \cong \bigoplus_{n \geq 0} (\mathbb{Z}/n\mathbb{Z})^{\oplus a_n}$$

with $a_n \geq 0$, and $a_n = 0$ for almost all n. The abelian group M is injective if and only if each direct summand is injective, by part iv) of Lemma 5.14. But for $n \geq 1$ none of the $\mathbb{Z}/n\mathbb{Z}$ is injective because the short exact sequence

$$0 \to \mathbb{Z}/n \to \mathbb{Z}/n^2 \to \mathbb{Z}/n \to 0$$

does not split, and $\mathbb{Z}$ (for $n = 0$) is not injective because the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

does not split.

ii) The same argument shows that the category of finite abelian groups has no injective objects aside from the trivial group 0.

[2] and feared

7 Extensions

Yoneda extensions and Ext^1

Convention. Let $\mathcal{A}$ be an abelian category for which the class of isomorphism classes $\operatorname{Ob}(\mathcal{A})/\cong$ is actually a set.

Remark-Definition 7.1. Let X and Y be two objects in $\mathcal{A}$.

i) We denote by $\mathcal{E}(X,Y)$ the class

$$\mathcal{E}(X,Y) := \left\{ \xi = (a,E,b) \;\middle|\; \begin{array}{c} E \in \operatorname{Ob}(\mathcal{A}), \\ a\colon Y \to E,\ b\colon E \to X, \\ 0 \to Y \xrightarrow{a} E \xrightarrow{b} X \to 0 \text{ is exact} \end{array} \right\}.$$

ii) Two such sequences $\xi, \xi' \in \mathcal{E}(X,Y)$ given by $\xi = (a,E,b)$ and $\xi' = (a',E',b')$ are *equivalent* if there exists a morphism $\varphi\colon E \to E'$ that makes the resulting diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "a"] \arrow[d, equal] & E \arrow[r, "b"] \arrow[d, dashed, "\varphi"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "a'"] & E' \arrow[r, "b'"] & X \arrow[r] & 0 \end{tikzcd}$$

commute. That ξ is equivalent to ξ' is denoted by $\xi \sim \xi'$. Note that it follows from the 5-lemma that φ is an isomorphism, which shows tells us that $\sim$ is symmetric. We also observe that $\sim$ is reflexive and transitive. We thus find that $\sim$ is an equivalence relation on the class $\mathcal{E}(X,Y)$.

iii) The quotient $\operatorname{Ext}^1_{\mathcal{A}}(X,Y) := \mathcal{E}(X,Y)/\!\sim$ is by assumption a set. An equivalence class $[\xi] \in \operatorname{Ext}^1(X,Y)$ is a *Yoneda extension*.

Remark⋆ 7.A. If there exists a short exact sequence $0 \to Y \to E \to X \to 0$ then E is an *extension* of X by Y. The class $\mathcal{E}(X,Y)$ can therefore be though of as the class of extensions of Y by X. (Hence the letter $\mathcal{E}$.)

Remark 7.2. Let X and Y be objects in $\mathcal{A}$ and let $\xi = (a,E,b) \in \mathcal{E}(X,Y)$.

i) Every morphism $f\colon X' \to X'$ in $\mathcal{A}$ induces a map $f^*\colon \operatorname{Ext}^1(X,Y) \to \operatorname{Ext}^1(X',Y)$ as follows:

We start with the following pullback square:

$$\begin{tikzcd} E' \arrow[r, "b'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[d, "f"] \\ E \arrow[r, "b"'] & X \end{tikzcd}$$

We know from Proposition 3.40 that there exists a unique morphism $a' \colon Y \to E'$ that makes the resulting diagram

$$
\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "a'"] \arrow[d, equals] & E' \arrow[r, "b'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & Y \arrow[r, "a"'] & E \arrow[r, "b"'] & X \arrow[r] & 0
\end{tikzcd}
$$

commute, and such that the rows of this diagram are (short) exact. Observe that $(a', E', b') \in \mathcal{E}(X', Y)$.

We claim that this construction is compatible with equivalence. More explicitely, let $\xi_1, \xi_2 \in \mathcal{E}(X, Y)$ with $\xi_1 \sim \xi_2$ Then for $\xi_1', \xi_2' \in \mathcal{E}(X', Y)$ resulting from the above construction, also $\xi_1' \sim \xi_2'$.

Indeed, let $\xi_i = (a_i, E_i, b_i)$ and $\xi_i' = (a_i', E_i', b_i')$. Let $\varphi \colon E_1 \to E_2$ be a morphism that makes the resulting diagram

$$
\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "a_1"] \arrow[d, equals] & E_1 \arrow[r, "b_1"] \arrow[d, dashed, "\varphi"] & X \arrow[r] \arrow[d, equals] & 0 \\
0 \arrow[r] & Y \arrow[r, "a_2"'] & E_2 \arrow[r, "b_2"'] & X \arrow[r] & 0
\end{tikzcd}
$$

commute. We get the following commutative diagram:

$$
\begin{tikzcd}
 & 0 \arrow[rr] & & Y \arrow[rr, "a_1'"] \arrow[dl, equals] \arrow[dd, equals] & & E_1' \arrow[rr, "b_1'"] \arrow[dd, "f_1'"] & & X' \arrow[rr] \arrow[dl, equals] \arrow[dd, "f"] & & 0 \\
0 \arrow[rr] & & Y \arrow[rr, "a_2'"] \arrow[dd, equals] & & E_2' \arrow[rr, "b_2'"] \arrow[dd, "f_2'"] & & X' \arrow[rr] \arrow[dd, "f"] & & 0 & \\
 & 0 \arrow[rr] & & Y \arrow[rr, "a_1"] \arrow[dl, equals] & & E_1 \arrow[rr, "b_1"] \arrow[dl, "\varphi"] & & X \arrow[rr] \arrow[dl, equals] & & 0 \\
0 \arrow[rr] & & Y \arrow[rr, "a_2"'] & & E_2 \arrow[rr, "b_2"'] & & X \arrow[rr] & & 0 &
\end{tikzcd}
$$

It follows from the functoriality of the pullback that there exists a unique morphism $\varphi' \colon E_1' \to E_2'$ that makes the resulting cube

$$
\begin{tikzcd}
 & E_1' \arrow[rr, "b_1'"] \arrow[dl, dashed, "\varphi'"'] \arrow[dd, "f_1'"] & & X' \arrow[dl, equals] \arrow[dd, "f"] \\
E_2' \arrow[rr, "b_2'"] \arrow[dd, "f_2'"] & & X' \arrow[dd, "f"] & \\
 & E_1 \arrow[rr, "b_1"] \arrow[dl, "\varphi"] & & X \arrow[dl, equals] \\
E_2 \arrow[rr, "b_2"'] & & X &
\end{tikzcd}
$$

commute. It then follows that the complete diagram

$$\begin{tikzcd}
& 0 \arrow[rr] & & Y \arrow[rr, "a_1'"] \arrow[dl, equal] \arrow[dd, equal] & & E_1' \arrow[rr, "b_1'"] \arrow[dl, dashed, "\varphi'"'] \arrow[dd, "f_1'"] & & X' \arrow[rr] \arrow[dl, equal] \arrow[dd, "f"] & & 0 \\
0 \arrow[rr] & & Y \arrow[rr, "a_2'"] \arrow[dd, equal] & & E_2' \arrow[rr, "b_2'"] \arrow[dd, "f_2'"] & & X' \arrow[rr] \arrow[dd, "f"] & & 0 & \\
& 0 \arrow[rr] & & Y \arrow[rr, "a_1"'] \arrow[dl, equal] & & E_1 \arrow[rr, "b_1"'] \arrow[dl, "\varphi"] & & X \arrow[rr] \arrow[dl, equal] & & 0 \\
0 \arrow[rr] & & Y \arrow[rr, "a_2"'] & & E_2 \arrow[rr, "b_2"'] & & X \arrow[rr] & & 0 &
\end{tikzcd}$$

commutes: It remains to show that the square

$$\begin{tikzcd}
Y \arrow[r, "a_1'"] \arrow[d, equal] & E_1' \arrow[d, "\varphi'"] \\
Y \arrow[r, "a_2'"'] & E_2'
\end{tikzcd}$$

commutes. We have that

$$f_2'\varphi' a_1' = \varphi f_1' a_1' = \varphi a_1 \operatorname{id}_Y = a_2 \operatorname{id}_Y \operatorname{id}_Y = a_2 \operatorname{id}_Y \operatorname{id}_Y = f_2' a_2' \operatorname{id}_Y \ .$$

and also that

$$b_2'\varphi' a_1' = \operatorname{id}_{X'} \underbrace{b_1' a_1'}_{=0} = 0 = \underbrace{b_2' a_2'}_{=0} \operatorname{id}_Y \ .$$

It follows from the universal property of the pullback (applied to E_2') that indeed $\varphi' a_1' = a_2' \operatorname{id}_Y$.

This shows that ξ_1' and ξ_2' are again equivalent. We hence get a well-defined[1] map

$$f^*\colon \ \mathrm{Ext}^1(X, Y) \to \mathrm{Ext}^1(X', Y)\,.$$

For $[\xi] \in \mathrm{Ext}^1(X, Y)$ we also write

$$[\xi] \cdot f \coloneqq f^*([\xi])\,.$$

ii) Let $g\colon Y \to Y'$ be a morphism in $\mathcal{A}$. We find dually to above discussion that the morphism g induces a well-defined map

$$g_*\colon \ \mathrm{Ext}^1(X, Y) \to \mathrm{Ext}^1(X, Y')\,,$$

[1] We use, without proof, that different choices of pullback give equivalent sequences.

and we denote for $[\xi] \in \mathrm{Ext}^1(X,Y)$ the Yoneda extension $g_*([\xi]) \in \mathrm{Ext}^1(X,Y')$ by $g \cdot [\xi]$. If $\xi = (a,E,b)$ and $g \cdot [\xi] = [\xi']$ then one such a representative $\xi' = (a',E',b')$ is given by the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "a"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[r, "b"] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y' \arrow[r, "a'"'] & E' \arrow[r, "b'"'] & X \arrow[r] & 0 \end{tikzcd}$$

where the left square is a pushout.

iii) If $f_2 \colon X' \to X$ and $f_1 \colon X'' \to X'$ are morphisms in $\mathcal{A}$ then

$$(f_2 \circ f_1)^* = f_1^* \circ f_2^*$$

for the induced maps

$$\begin{aligned} f_2^* &\colon \mathrm{Ext}^1(X,Y) \to \mathrm{Ext}^1(X',Y)\,, \\ f_1^* &\colon \mathrm{Ext}^1(X',Y) \to \mathrm{Ext}^1(X'',Y)\,. \end{aligned}$$

Indeed, let $[\xi] \in \mathrm{Ext}^1(X,Y)$ with $\xi = (a,E,b)$. Then $f_2^*([\xi]) = [\xi']$ for a sequence $\xi' = (a',E',b') \in \mathcal{E}(X',Y)$ such that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "a'"] \arrow[d, equal] & E' \arrow[r, "b'"] \arrow[d, "f_2'"] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[r] \arrow[d, "f_2"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "a"'] & E \arrow[r, "b"'] & X \arrow[r] & 0 \end{tikzcd}$$

in which the right square is a pullback. We similarly have that $f_1^*([\xi']) = [\xi'']$ for a sequence $\xi'' = (a'',E'',b'') \in \mathcal{E}(X'',Y)$ such that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "a''"] \arrow[d, equal] & E'' \arrow[r, "b''"] \arrow[d, "f_1'"] \arrow[dr, phantom, "\lrcorner"] & X'' \arrow[r] \arrow[d, "f_1"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "a'"'] & E' \arrow[r, "b'"'] & X' \arrow[r] & 0 \end{tikzcd}$$

in which the right square is a pullback. By glueing the above two diagrams together we get the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "a''"] \arrow[d, equal] & E'' \arrow[r, "b''"] \arrow[d, "f_1'"] \arrow[dr, phantom, "\lrcorner"] & X'' \arrow[r] \arrow[d, "f_1"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "a'"] \arrow[d, equal] & E' \arrow[r, "b'"] \arrow[d, "f_2'"] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[r] \arrow[d, "f_2"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "a"'] & E \arrow[r, "b"'] & X \arrow[r] & 0 \end{tikzcd}$$

It follow from the transitivity of pullbacks that in the subdiagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "a''"] \arrow[d, equal] & E'' \arrow[r, "b''"] \arrow[d, "f_2' f_1'"] \arrow[dr, phantom, "\lrcorner"] & X'' \arrow[r] \arrow[d, "f_2 f_1"] & 0 \\
0 \arrow[r] & Y \arrow[r, "a"'] & E \arrow[r, "b"'] & X \arrow[r] & 0
\end{tikzcd}$$

the right square is again a pullback. We find from this diagram that

$$(f_2 \circ f_1)^*([\xi] = [\xi'']$$

and therefore

$$(f_2 \circ f_1)^*([\xi]) = [\xi''] = f_1^*([\xi']) = f_1^*(f_2^*([\xi])) = (f_1^* \circ f_2^*)([x_i]) ,$$

as desired. We note this this equality can also be expressed as

$$([\xi] \cdot f_2) \cdot f_1 = [\xi] \cdot (f_2 \cdot f_1) .$$

We find similarly for all morphisms $g_1 \colon Y \to Y'$ and $g_2 \colon Y' \to Y''$ in $\mathcal{A}$ that

$$(g_2 \circ g_1)_* = (g_2)_* \circ (g_1)_*$$

for the induced maps

$$\begin{aligned}
(g_1)_* &\colon \mathrm{Ext}^1(X, Y) \to \mathrm{Ext}^1(X, Y') , \\
(g_2)_* &\colon \mathrm{Ext}^1(X, Y') \to \mathrm{Ext}^1(X, Y'') .
\end{aligned}$$

This equality can also be expressed as

$$g_1 \cdot (g_2 \cdot [\xi]) = (g_1 \circ g_2) \cdot [\xi]$$

for every $[\xi] \in \mathrm{Ext}^1(X, Y)$.

We hence have functors

$$\begin{aligned}
\mathrm{Ext}^1_{\mathcal{A}}(X, -) &\colon \mathcal{A} \to \mathbf{Ab} , \\
\mathrm{Ext}^1_{\mathcal{A}}(-, Y) &\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab} .
\end{aligned}$$

iv) We have for all morphisms $f \colon X' \to X$ and $g \colon Y \to Y'$ in $\mathcal{A}$ that

$$g_* \circ f^* = f^* \circ g_* ,$$

which an also be expressed as

$$g \cdot ([\xi] \cdot f) = (g \cdot [\xi]) \cdot f$$

for every $[\xi] \in \operatorname{Ext}^1(X, Y)$. To prove this we consider the following commutative diagram:

$$\begin{tikzcd}
g_*f^*\xi : & 0 \arrow[r] & Y' \arrow[r, "a^\circ"] & E^\circ \arrow[r, "b^\circ"] & X' \arrow[r] & 0 \\
f^*\xi : & 0 \arrow[r] & Y \arrow[r, "a'"] \arrow[u, "g"] \arrow[d, equal] \arrow[ur, phantom, "\llcorner"] & E' \arrow[r, "b'"] \arrow[u, "g^\circ"'] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[r] \arrow[u, equal] \arrow[d, "f"] & 0 \\
\xi : & 0 \arrow[r] & Y \arrow[r, "a"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[r, "b"] \arrow[d, "g''"'] & X \arrow[r] \arrow[d, equal] & 0 \\
g^*\xi : & 0 \arrow[r] & Y' \arrow[r, "a''"] \arrow[d, equal] & E'' \arrow[r, "b''"] \arrow[dr, phantom, "\urcorner"] & X \arrow[r] & 0 \\
f^*g_*\xi : & 0 \arrow[r] & Y' \arrow[r, "a^{\circ\circ}"] & E^{\circ\circ} \arrow[r, "b^{\circ\circ}"] \arrow[u, "f^{\circ\circ}"'] & X' \arrow[r] \arrow[u, "f"'] & 0
\end{tikzcd}$$

It follows from the universal property of the pushout (applied to E°) that there exists a unique morphism $\lambda \colon E^\circ \to E''$ that makes the diagram

$$\begin{tikzcd}
Y \arrow[r, "a'"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & E' \arrow[d, "g^\circ"'] \arrow[ddr, bend left, "g''f'"] & \\
Y' \arrow[r, "a^\circ"] \arrow[drr, bend right, "a''"'] & E^\circ \arrow[dr, dashed, "\lambda"] & \\
 & & E''
\end{tikzcd}$$

commute. Then the square

$$\begin{tikzcd}
E^\circ \arrow[r, "b^\circ"] \arrow[d, "\lambda"'] & X' \arrow[d, "f"] \\
E'' \arrow[r, "b''"'] & X
\end{tikzcd}$$

commutes. This follows from the universal property of the pushout (applied to E°) because

$$b''\lambda a^\circ = b''a'' = f \circ 0 = fb^\circ a^\circ$$

and

$$b''\lambda g^\circ = b''g''f' = bf' = fb' = f\,\mathrm{id}_{X'}\, b^\circ g^\circ = fb^\circ g^\circ\,.$$

It now follows from the universal property of the pullback (applied to $E^{\circ\circ}$) that there exists a unique morphism $\varphi\colon E^\circ \to E^{\circ\circ}$ that makes the diagram

$$\begin{tikzcd}
E^\circ \arrow[drr, bend left, "b^\circ"] \arrow[dr, dashed, "\varphi"] \arrow[ddr, bend right, "\lambda"'] & & \\
& E^{\circ\circ} \arrow[r, "b^{\circ\circ}"] \arrow[d, "f^{\circ\circ}"'] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[d, "f"] \\
& E'' \arrow[r, "b''"'] & X
\end{tikzcd}$$

commute. We claim that the morphism φ makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y' \arrow[r, "a^\circ"] \arrow[d, equal] & E^\circ \arrow[r, "b^\circ"] \arrow[d, dashed, "\varphi"] & X' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y' \arrow[r, "a^{\circ\circ}"'] & E^{\circ\circ} \arrow[r, "b^{\circ\circ}"'] & X' \arrow[r] & 0
\end{tikzcd} \tag{7.1}$$

commute. The right square commutes by the construction of φ. The commutativity of the left square follows from the universal property of the pullback (applied to $E^{\circ\circ}$) because

$$b^{\circ\circ}\varphi a^\circ = b^\circ a^\circ = 0 = b^{\circ\circ}a^{\circ\circ}$$

and

$$f^{\circ\circ}\varphi a^\circ = \lambda a^\circ = a'' = f^{\circ\circ}a^{\circ\circ}\,.$$

The commutativity of the diagram (7.1) proves the claim.

End of lecture 26

Remark-Definition 7.3. Let $[\xi], [\xi'] \in \operatorname{Ext}^1(X, Y)$ be two extensions given by representatives $\xi = (a, E, b)$ and $\xi' = (a', E', b')$. Then the sequence

$$\xi \oplus \xi' \colon 0 \to Y \oplus Y \xrightarrow{\left[\begin{smallmatrix} a & 0 \\ 0 & a' \end{smallmatrix}\right]} E \oplus E \xrightarrow{\left[\begin{smallmatrix} b & 0 \\ 0 & b' \end{smallmatrix}\right]} X \oplus X \to 0$$

is again exact, and hence $[\xi \oplus \xi'] \in \operatorname{Ext}^1(X \oplus X, Y \oplus Y)$.

The extension $[\xi \oplus \xi']$ is independent of the choice of representatives ξ and ξ': Let $\xi_1, \xi_2, \xi_1', \xi_2' \in \mathcal{E}(X, Y)$ with $\xi_1 \sim \xi_2$ and $\xi_1' \sim \xi_2'$. Suppose that $\xi_i = (a_i, E_i, b_i)$ and $\xi_i' = (a_i', E_i', b_i')$ and let $\varphi\colon E_1 \to E_2$ and $\varphi'\colon E_1' \to E_2'$ be morphisms that make the diagrams

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "a_1"] \arrow[d, equal] & E_1 \arrow[r, "b_1"] \arrow[d, dashed, "\varphi"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "a_1'"] & E_1' \arrow[r, "b_1'"] & X \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "a_2"] \arrow[d, equal] & E_2 \arrow[r, "b_2"] \arrow[d, dashed, "\varphi"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "a_2'"] & E_2' \arrow[r, "b_2'"] & X \arrow[r] & 0
\end{tikzcd}$$

commute. Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \oplus X \arrow[r, "\begin{bmatrix} a_1 & 0 \\ 0 & a_1' \end{bmatrix}"] \arrow[d, equal] & E_1 \oplus E_1' \arrow[r, "\begin{bmatrix} b_1 & 0 \\ 0 & b_1' \end{bmatrix}"] \arrow[d, "\begin{bmatrix} \varphi & 0 \\ 0 & \varphi' \end{bmatrix}"] & Y \oplus Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \oplus X \arrow[r, swap, "\begin{bmatrix} a_2 & 0 \\ 0 & a_2' \end{bmatrix}"] & E_2 \oplus E_2' \arrow[r, swap, "\begin{bmatrix} b_2 & 0 \\ 0 & b_2' \end{bmatrix}"] & Y \oplus Y \arrow[r] & 0
\end{tikzcd}$$

commutes, which shows that $\xi_1 \oplus \xi_1' \sim \xi_2 \oplus \xi_2'$. This proves the claimed independence, and consequently allows us to write

$$[\xi] \oplus [\xi'] \coloneqq [\xi \oplus \xi'] \in \mathrm{Ext}^1(X \oplus X, Y \oplus Y)\,.$$

for all $[\xi], [\xi'] \in \mathrm{Ext}^1(X, Y)$.

Together with the diagonal morphism $\Delta_X \colon X \to X \oplus X$ and the codiagonal morphism $\nabla_Y \colon Y \oplus Y \to Y$ we get for all extensions $[\xi], [\xi'] \in \mathrm{Ext}^1(X, Y)$ an extension

$$[\xi] + [\xi'] \coloneqq \nabla_Y \cdot ([\xi] \oplus [\xi']) \cdot \Delta_X\,.$$

The extension $[\xi] + [\xi']$ is the *Baer sum* of $[\xi]$ and $[\xi']$.

Remark 7.4. Let $\mathbf{k}$ be commutative ring and let A be a $\mathbf{k}$-algebra. For all A-modules M and N and all short exact sequences

$$\xi \colon 0 \to N \xrightarrow{a} E \xrightarrow{b} M \to 0\,,$$
$$\xi' \colon 0 \to N' \xrightarrow{a'} E' \xrightarrow{b'} M' \to 0$$

the Baer sum $[\xi] + [\xi']$ is given by the short exact sequence

$$0 \to N \xrightarrow{a''} E'' \xrightarrow{b''} M \to 0$$

where

$$E'' \coloneqq \{(e, e') \in E \oplus E' \mid b(e) = b'(e')\} \big/ \{(a(y), -a'(y)) \mid y \in N\}$$

and the homomorphisms a'' and b'' are given by

$$a''(y) \coloneqq [(a(y), a'(y))]\,, \quad b''([(e, e')]) \coloneqq b(e) = b'(e')\,.$$

Lemma 7.5. Let

$$\begin{tikzcd}
0 \arrow[r] & Y' \arrow[r, "a'"] \arrow[d, "g"] & E' \arrow[r, "b'"] \arrow[d, "h"] & X' \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & Y \arrow[r, swap, "a"] & E \arrow[r, swap, "b"] & X \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram with (short) exact rows in $\mathcal{A}$.

i) If the morphism g is an isomorphism then the right square is a pullback.

ii) If the morphism f is an isomorphism then the left square is a pushout.

Proof. We prove part i). Part ii) then follows from part i) by duality. We consider for part i) the pullback:

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[d, "f"] \\ E \arrow[r, "b"'] & X' \end{tikzcd}$$

It follows from the universal property of the pullback (P, p, q) that there exists a unique morphism $\varphi\colon E' \to P$ that makes the diagram

$$\begin{tikzcd} E' \arrow[drr, bend left, "b'"] \arrow[dr, dashed, "\varphi"] \arrow[ddr, bend right, "h"'] & & \\ & P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[d, "f"] \\ & E \arrow[r, "b"'] & X \end{tikzcd}$$

commute. We show that φ is an isomorphism. For this it is enough to show that φ is both a monomorphism and an epimorphism, because the category $\mathcal{A}$ is abelian.

The morphism φ is a monomorphism: Let $e' \in_{\mathcal{A}} E'$ with $\varphi e' = 0$. Then $p\varphi e' = 0$ and $q\varphi e' = 0$ (and by the universal property of the pullback P this really is equivalent to $\varphi e' = 0$). Hence $k\varphi e' = 0$ for the morphism

$$k := \begin{bmatrix} p \\ q \end{bmatrix} : P \to E \oplus X' \,.$$

We know that

$$k\varphi = \begin{bmatrix} p \\ q \end{bmatrix} \varphi = \begin{bmatrix} p\varphi \\ q\varphi \end{bmatrix} = \begin{bmatrix} h \\ b' \end{bmatrix}$$

and hence conclude from $k\varphi e' = 0$ that $b'e' = 0$ and $he' = 0$. It follows from the exactness of the upper row that there exists some $y' \in_{\mathcal{A}} Y'$ with $a'y' \equiv e'$. Then

$$0 = he' \equiv ha'y' = agy' \,.$$

Both a and g are monomorphisms, hence $y' = 0$. Therefore also $e' \equiv a'y' = 0$.

The morphism φ is an epimorphism: Let $z \in_{\mathcal{A}} P$ and set $e := pz$ and $x' := qz$. Then $be = fx'$. As b' is an epimorphism, there exists some $\tilde{e}' \in_{\mathcal{A}} E'$ with $b'\tilde{e}' \equiv x'$. Then

$$bh\tilde{e}' = fb'\tilde{e}' \equiv fx' = be \,.$$

We can therefore consider the point $e - h\tilde{e}' \in_{\mathcal{A}} E$ (from part iv) of Theorem 3.42); then in particular $b(e - h\tilde{e}') = 0$. It follows from the exactness of the lower row that there exists some $y \in_{\mathcal{A}} Y$ with $ay \equiv e - h\tilde{e}'$. As g is an epimorphism, there further exists some $y' \in_{\mathcal{A}} Y'$ wih $gy' \equiv y$. Then

$$e - h\tilde{e}' \equiv ay \equiv agy' = ha'y'$$

Therefore $e \equiv h(a'y' + \tilde{e}')$.[2] Let $e' := a'y' + \tilde{e}' \in_{\mathcal{A}} E'$. Then

$$he' = h(a'y' + \tilde{e}') \equiv e$$

and

$$b'e' = b'(a'y' + \tilde{e}') = b'a'y' + b'\tilde{e}' = 0 + b'\tilde{e}' = b'\tilde{e}' \equiv x\,.$$

We note that the morphism $k\colon P \to E \oplus X'$ (as above) is a monomorphism by the universal property of the pullback (P, p, q). The above calculation shows that

$$k\varphi e' = \begin{bmatrix} p \\ q \end{bmatrix} \varphi e' = \begin{bmatrix} p\varphi e' \\ q\varphi e' \end{bmatrix} = \begin{bmatrix} he' \\ b'e' \end{bmatrix} \equiv \begin{bmatrix} e \\ x' \end{bmatrix} = \begin{bmatrix} pz \\ qz \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} z = kz\,.$$

It follows that $\varphi e' \equiv z$ because k is a monomorphism. This shows that φ is an epimorphism. $\square$

Remark. Lemma 7.5 is a converse to Proposition 3.40.

Theorem 7.6. Let X and Y be objects in $\mathcal{A}$.

i) The Baer sum makes $\mathrm{Ext}^1(X, Y)$ into an abelian group.

ii) The neutral element of $\mathrm{Ext}^1(X, Y)$ is given by the split exact sequence

$$0 \to Y \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} X \oplus Y \xrightarrow{[\,1\ 0\,]} X \to 0\,,$$

and for every extension $[\xi] \in \mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ with $\xi = (a, E, b)$ the inverse to $[\xi]$ is given by the extension $[\xi']$ with $\xi' = (-a, E, b)$.

iii) The functors $\mathrm{Ext}^1_{\mathcal{A}}(X, -)\colon \mathcal{A} \to \mathbf{Ab}$ and $\mathrm{Ext}^1_{\mathcal{A}}(-, Y)\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ are additive.

End of lecture 27

[2]The author doesn't understand why this holds.

Index

Bibliography

[ASS06] Ibrahim Assem, Daniel Simson and Andrzej Skowroński. *Elements of the Representation Theory of Associative Algebras.* Vol. 1. London Mathematical Society Student Texts 65. Cambridge University Press, Cambridge, 2006, pp. x+458. ISBN: 978-0-521-58423-4. DOI: 10.1017/CBO9780511614309.

[Bra17] Martin Brandenburg. *Einführung in die Kategorientheorie. Mit ausführlichen Erklärungen und zahlreichen Beispielen.* 2nd ed. Springer Spektrum, 2017, pp. x+345. DOI: 10.1007/978-3-662-53521-9.

[DF04] David S. Dummit and Richard M. Foote. *Abstract Algebra.* Wiley, 2004. ISBN: 9780471433347.

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds. With a Short History. «Les débuts de la théorie des faisceaux». By Christian Houzel.* Grundlehren der mathematischen Wissenschaften 292. Springer-Verlag, Berlin Heidelberg, 1990, pp. x+512. ISBN: 978-3-540-51861-7. DOI: 10.1007/978-3-662-02661-8.

[Lei14] Tom Leinster. *Basic Category Theory.* Cambridge Studies in Advanced Mathematics 143. Cambridge University Press, 2014. arXiv: 1612.09375 [math.CT].

[Mac78] Saunders Mac Lane. *Categories for the Working Mathematician.* 2nd ed. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1978, pp. xii+317. ISBN: 978-0-387-98403-2. DOI: 10.1007/978-1-4757-4721-8.

[Mak96] Michael Makkai. 'Avoiding the axiom of choice in general category theory'. In: *Journal of Pure and Applied Algebra* 108.2 (1996), pp. 109–173. ISSN: 0022-4049. DOI: https://doi.org/10.1016/0022-4049(95)00029-1. URL: http://www.sciencedirect.com/science/article/pii/0022404995000291 (visited on 25th November 2018).

[MO5378] Pete L. Clark (https://mathoverflow.net/users/1149/pete-l-clark). *When are there enough projective sheaves on a space X?* URL: https://mathoverflow.net/q/5378 (visited on 22nd November 2018).

[Sch72] Horst Schubert. *Categories.* Trans. from the German by Eva Gray. Springer-Verlag Berlin Heidelberg, 1972, pp. xiii+385. ISBN: 978-3-642-65366-7. DOI: 10.1007/978-3-642-65364-3.

[St075B] The Stacks Project Authors. *Stacks Project.* URL: https://stacks.math.columbia.edu/tag/07RB (visited on 22nd November 2018).

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics 38. Cambridge University Press, Cambridge, 1994, pp. xiv+450. ISBN: 978-0-521-55987-4. DOI: 10.1017/CBO9781139644136.