Data Generation

Extraneous covariate

Z is generated as the union of three disjoint intervals with nearly adjacent endpoints. That is, $Z = Z_1 \cup Z_2 \cup Z_3$ with $Z_1 = (a, b), Z_2 = (c, d), Z_3 = (m, n)$. Within each interval, the individuals' covariate values are equally spaced.

Precision matrix

All of the individuals in interval 1 have the same precision matrix, $\Omega^{(1)}$:

$$\Omega_{i,j}^{(1)} = \begin{cases} 2 & i = j \\ 1 & (i, j) \in \{(1, 2), (2, 1), (2, 3), (3, 2)\} \\ 0 & o.w. \end{cases}$$

Also, all of the individuals in interval 3 have the same precision matrix, $\Omega^{(3)}$:

$$\Omega_{i,j}^{(3)} = \begin{cases} 2 & i = j \\ 1 & (i, j) \in \{(1, 3), (3, 1), (2, 3), (3, 2)\} \\ 0 & o.w. \end{cases}$$

But, the individuals in interval 2 have a precision matrix that is dependent upon Z and (d, c). Let $\beta_0 = -c/(d - c)$ and $\beta_1 = 1/(d - c)$. Then:

$$\Omega_{i,j}^{(2)}(z) = \begin{cases} 2 & i = j \\ 1 & (i, j) \in \{(2, 3), (3, 2)\} \\ 1 - \beta_0 - \beta_1 z & (i, j) \in \{(1, 2), (2, 1)\} \\ \beta_0 + \beta_1 z & (i, j) \in \{(1, 3), (3, 1)\} \\ 0 & o.w. \end{cases}$$

Thus, $\Omega^{(2)}(c) = \Omega^{(1)}$ and $\Omega^{(2)}(d) = \Omega^{(3)}$. That is, the first individual in interval 2 has precision matrix $\Omega^{(1)}$ and the last individual in interval 2 has precision matrix $\Omega^{(3)}$.

Data matrix

Let z_l be the extraneous covariate for the l-th individual. To generate the data matrix for the l-th individual, a random sample is taken from $\mathcal{N}(0, \Omega_l^{-1})$, where:

$$\Omega_l(z_l) = \begin{cases} \Omega^{(1)} & z_l \in Z_1 \\ \Omega^{(2)}(z_l) & z_l \in Z_2 \\ \Omega^{(3)} & z_l \in Z_3 \end{cases}$$

Sensitivity Analysis

Overview

The goal of this analysis will be to examine the necessary signal strength in the interval 2 precision matrices to obtain nearly perfect structure recovery. To increase the signal strength, in each trial, I will further decrease the number of individuals in the second interval. Since the covariate values are uniformly spaced, this will result in larger jumps between the covariate values, which will result in greater signal strength.

That is, let $z_{2,l}^{(k)}$ be the value of the extraneous covariate for the l-th individual in interval 2 on the k-th trial. Then, $z_{2,l}^{(k)} \leq z_{2,l}^{(k+1)}$, with equality only for the first and last individuals in interval 2. Then, it follows that:

$$\beta_0 + \beta_1 z_{2,l}^{(k)} \leq \beta_0 + \beta_1 z_{2,l}^{(k+1)} \implies \Omega_{1,3}^{(2)}(z_{2,l}^{(k)}) \leq \Omega_{1,3}^{(2)}(z_{2,l}^{(k+1)})$$
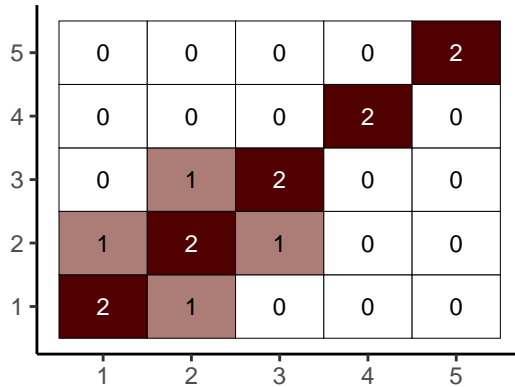
Trial 1

$$n_2 = 60$$

Interval	Individual Indices
1	$1, \ldots, 60$
2	$61, \ldots, 120$
3	$121, \ldots, 180$

First 4 Precision Matrices for Interval 2
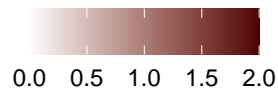
Individual 61
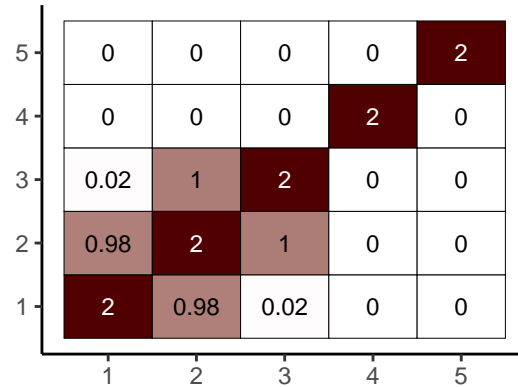
	1	2	3	4	5
5	0	0	0	0	2
4	0	0	0	2	0
3	0	1	2	0	0
2	1	2	1	0	0
1	2	1	0	0	0

Individual 62

	1	2	3	4	5
5	0	0	0	0	2
4	0	0	0	2	0
3	0.02	1	2	0	0
2	0.98	2	1	0	0
1	2	0.98	0.02	0	0

Individual 63

	1	2	3	4	5
5	0	0	0	0	2
4	0	0	0	2	0
3	0.04	1	2	0	0
2	0.96	2	1	0	0
1	2	0.96	0.04	0	0

Individual 64

	1	2	3	4	5
5	0	0	0	0	2
4	0	0	0	2	0
3	0.05	1	2	0	0
2	0.95	2	1	0	0
1	2	0.95	0.05	0	0

```r
out <- covdepGE(data_mat, Z, var_min = 1e-6, var_max = 1, n_sigma = 20, CS = T,
                parallel = T, num_workers = 7, stop_cluster = F)
```

```
## Warning in covdepGE(data_mat, Z, var_min = 1e-06, var_max = 1, n_sigma = 20, :
## No registered workers detected; registering doParallel with 7 workers
```

```
out
```

```
##                    Covariate Dependent Graphical Model
##
## Model ELBO: -80614.07              Unique conditional dependence structures: 5
## n: 180, variables: 5                        Hyperparameter grid size: 20 points
## CAVI converged for 5/5 variables
##
## Model fit completed in 5.967 secs
```

```
colors <- c("steelblue", "goldenrod", "forestgreen", "tomato2", "dodgerblue",
            "darkorchid", "powderblue", "seagreen", "turqoise", "springgreen2")
annotate_figure(ggarrange(plotlist = plot(out, colors)),
                text_grob("Unique Graphs, trial 1", size = 15))
```
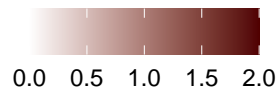

Results look slightly different than before since I am setting `CS = T`, *which uses Carbonetto-Stephens model to specify* σ^2 *and* π. *Also, I am using less grid points.*
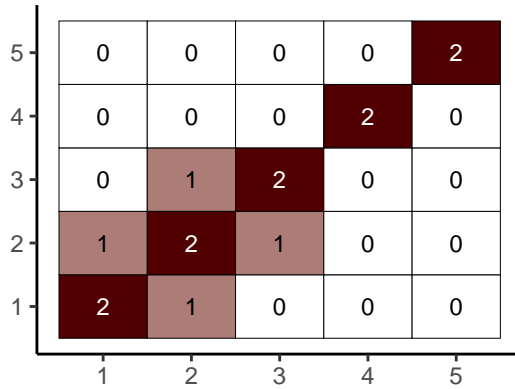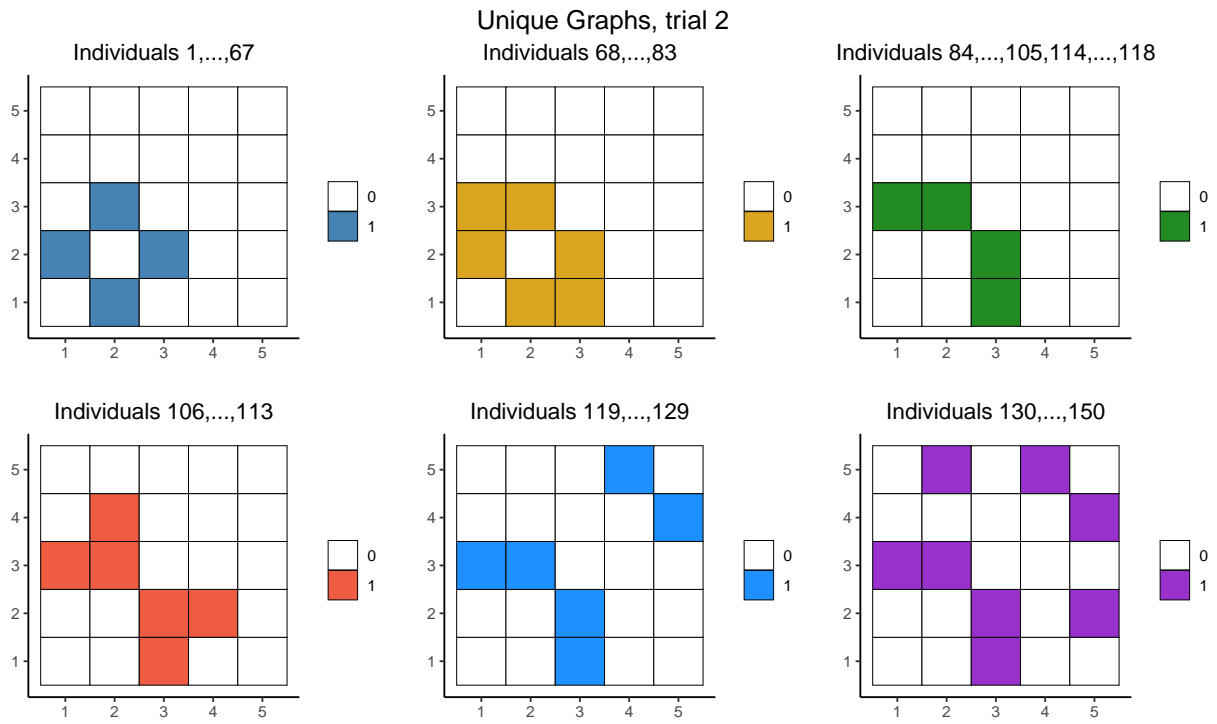
Trial 2

$$n_2 = 30$$

Interval	Individual Indices
1	$1, \ldots, 60$
2	$61, \ldots, 90$
3	$91, \ldots, 150$

First 4 Precision Matrices for Interval 2

out <- covdepGE(data_mat, Z, var_min = 1e-6, var_max = 1, n_sigma = 20, CS = T,
                parallel = T, stop_cluster = F)

Detected 7 workers

out

Covariate Dependent Graphical Model
##
Model ELBO: -56059.04   Unique conditional dependence structures: 6
n: 150, variables: 5   Hyperparameter grid size: 20 points
CAVI converged for 5/5 variables
##
Model fit completed in 2.77 secs

```
annotate_figure(ggarrange(plotlist = plot(out, colors)),
                text_grob("Unique Graphs, trial 2", size = 15))
```

Unique Graphs, trial 2

Trial 3

$$n_2 = 10$$

Interval	Individual Indices
1	$1, \ldots, 60$
2	$61, \ldots, 70$
3	$71, \ldots, 130$

First 4 Precision Matrices for Interval 2
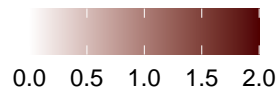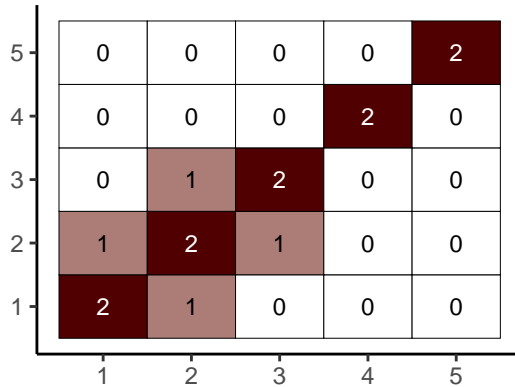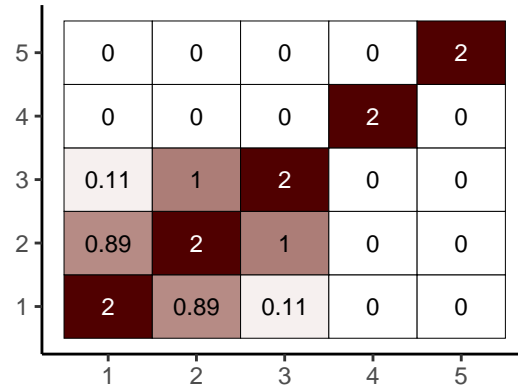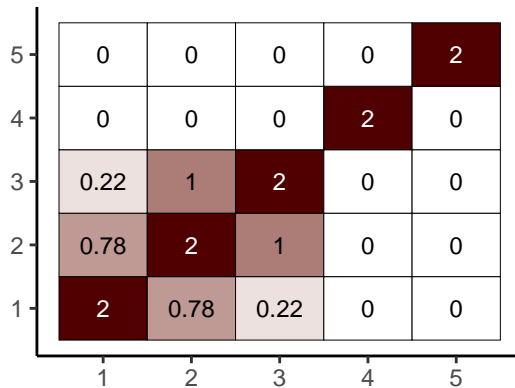

```r
out <- covdepGE(data_mat, Z, var_min = 1e-6, var_max = 1, n_sigma = 20, CS = T,
                parallel = T, stop_cluster = F)
```

```
## Detected 7 workers

## Warning in covdepGE(data_mat, Z, var_min = 1e-06, var_max = 1, n_sigma = 20, :
## For 1/5 variables, the selected value of sigmabeta_sq was on the grid boundary.
## See return value CAVI_details
```

```
out
```
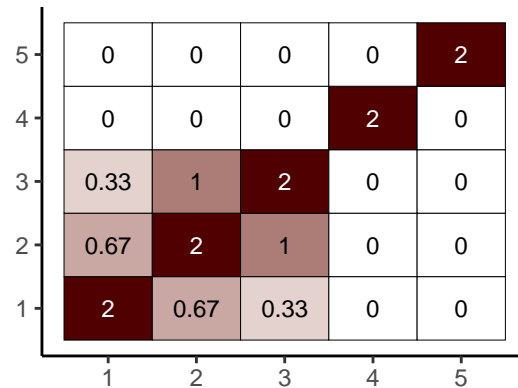
```
##                      Covariate Dependent Graphical Model
##
## Model ELBO: -41549.58           Unique conditional dependence structures: 4
## n: 130, variables: 5                    Hyperparameter grid size: 20 points
```

```
## CAVI converged for 5/5 variables
##
## Model fit completed in 0.879 secs
```

```
annotate_figure(ggarrange(plotlist = plot(out, colors)),
                text_grob("Unique Graphs, trial 3", size = 15))
```
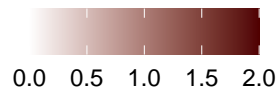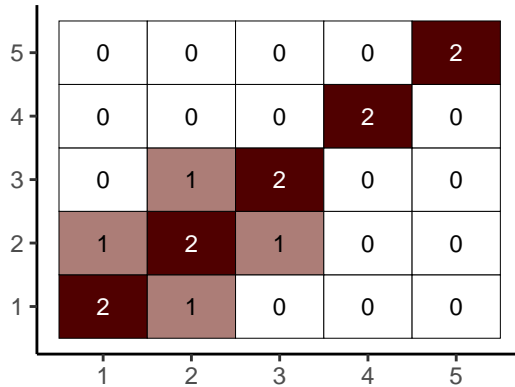
Unique Graphs, trial 3
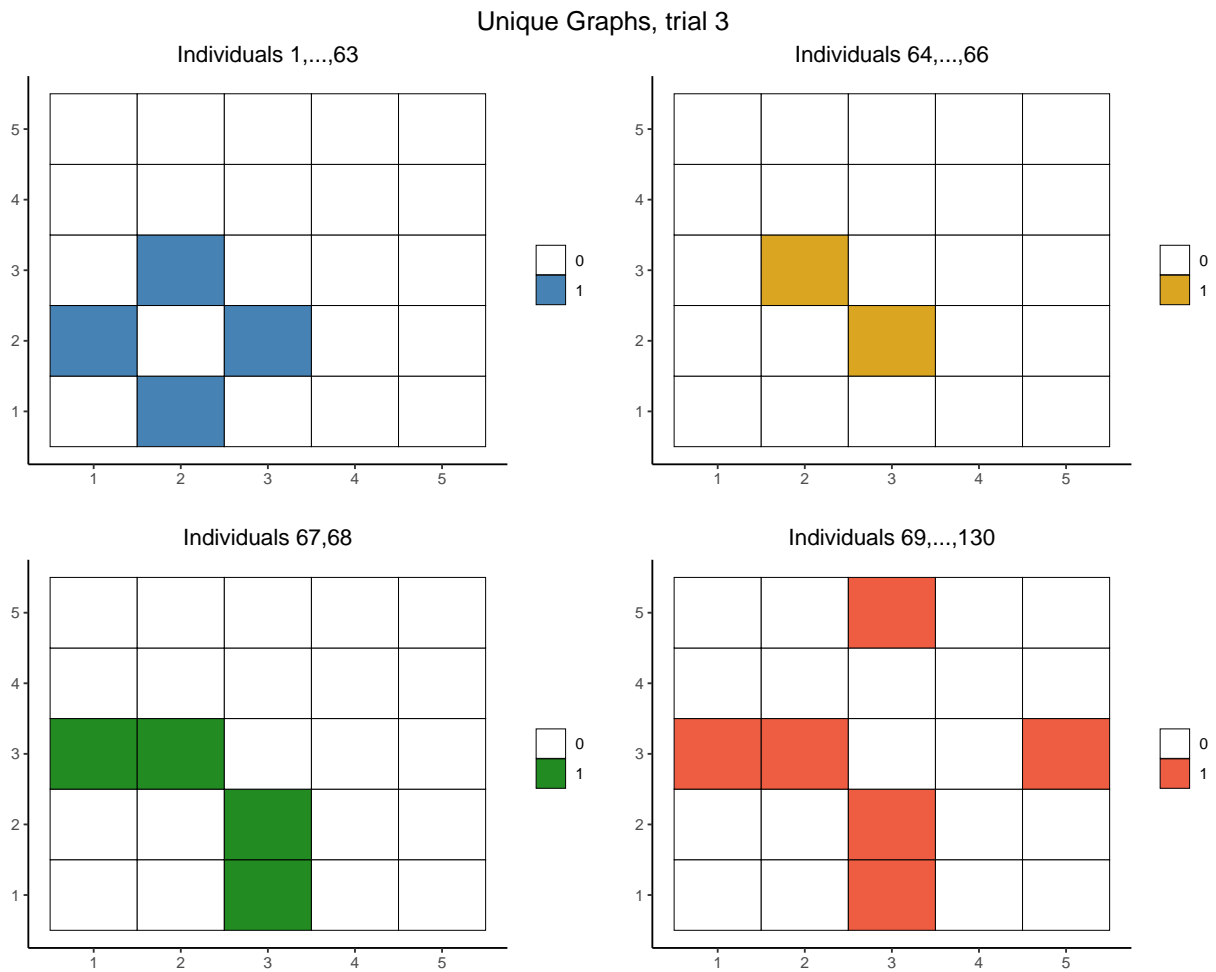
Trial 4

$$n_2 = 5$$

Interval	Individual Indices
1	$1, \ldots, 60$
2	$61, \ldots, 65$
3	$66, \ldots, 125$

First 4 Precision Matrices for Interval 2
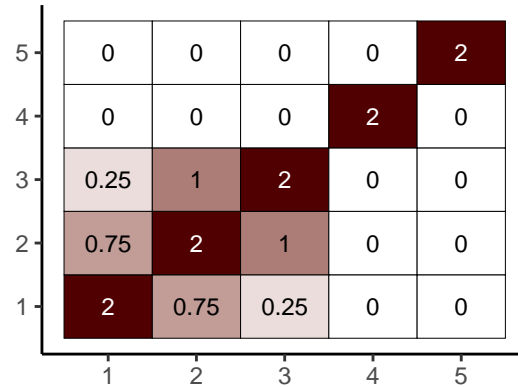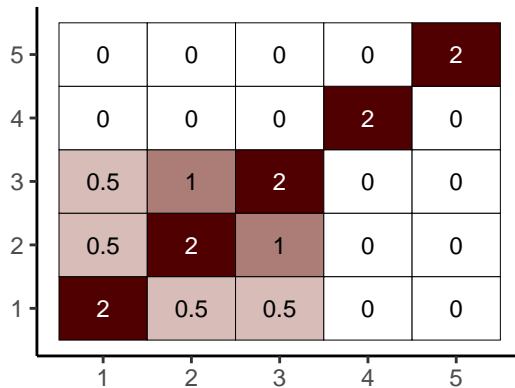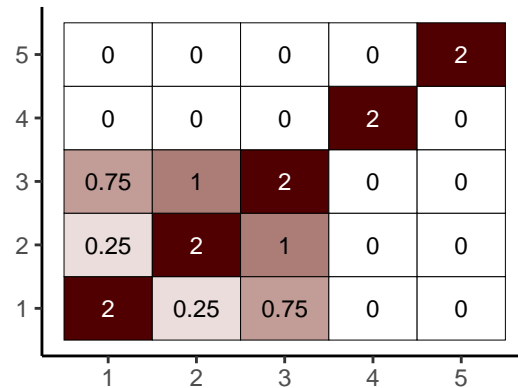


```
out <- covdepGE(data_mat, Z, var_min = 1e-6, var_max = 1, n_sigma = 20, CS = T,
                parallel = T, stop_cluster = F)
```
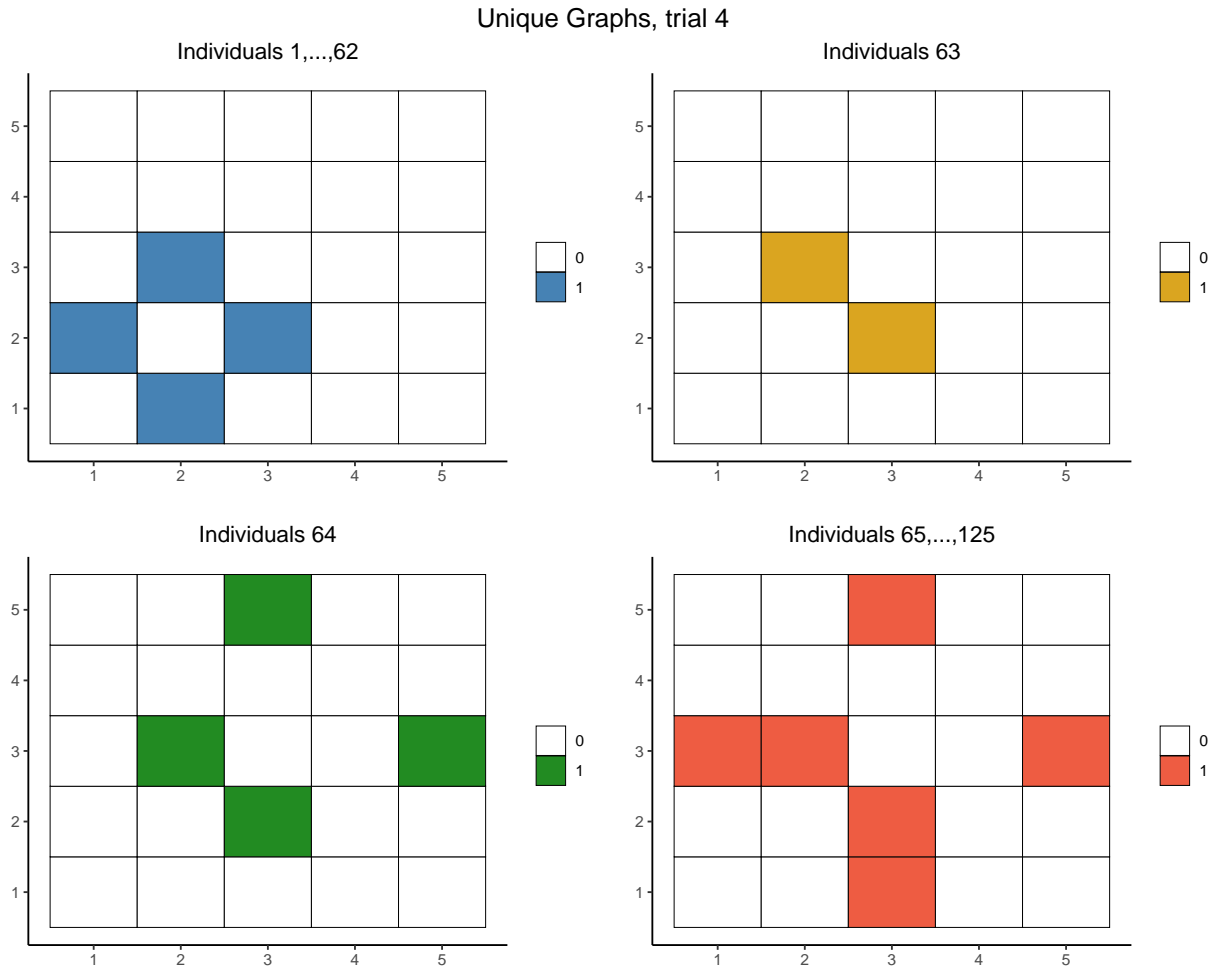
```
## Detected 7 workers
```

```
## Warning in covdepGE(data_mat, Z, var_min = 1e-06, var_max = 1, n_sigma = 20, :
## Variable 5: CAVI did not converge in 10000 iterations for 1/20 grid search
## candidates
```

```
## Warning in covdepGE(data_mat, Z, var_min = 1e-06, var_max = 1, n_sigma = 20, :
## For 1/5 variables, the selected value of sigmabeta_sq was on the grid boundary.
## See return value CAVI_details
```

```
out
```

```
##                      Covariate Dependent Graphical Model
##
## Model ELBO: -38231.61              Unique conditional dependence structures: 4
## n: 125, variables: 5                     Hyperparameter grid size: 20 points
## CAVI converged for 5/5 variables
##
## Model fit completed in 5.941 secs
```

```
annotate_figure(ggarrange(plotlist = plot(out, colors)),
                text_grob("Unique Graphs, trial 4", size = 15))
```
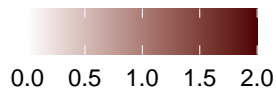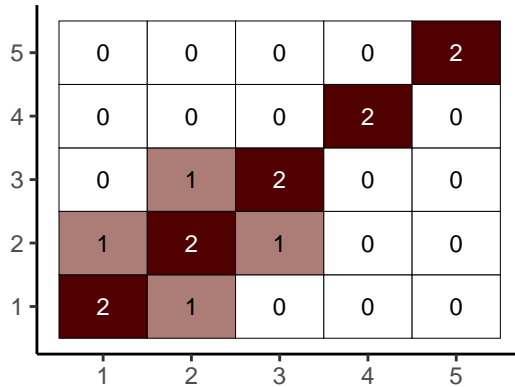
Unique Graphs, trial 4

Trial 5

$$n_2 = 4$$

Interval	Individual Indices
1	$1, \ldots, 60$
2	$61, \ldots, 64$
3	$65, \ldots, 124$

Interval 2 Precision Matrices
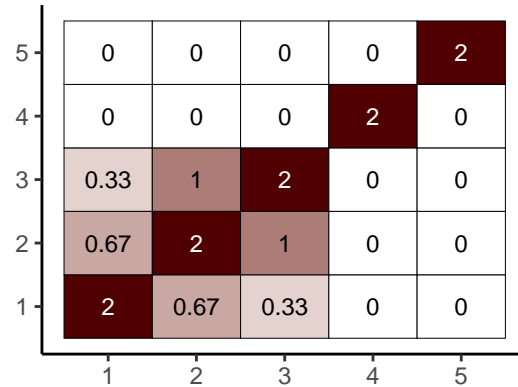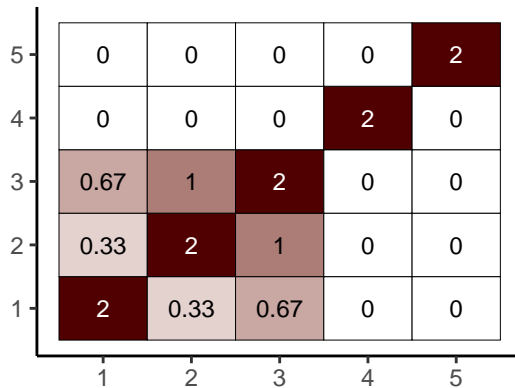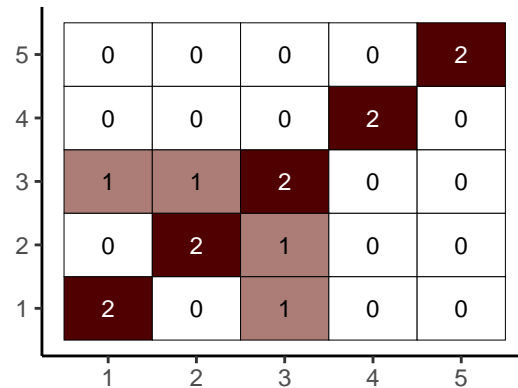

```r
out <- covdepGE(data_mat, Z, var_min = 1e-6, var_max = 1, n_sigma = 20, CS = T,
                parallel = T, stop_cluster = F)
```
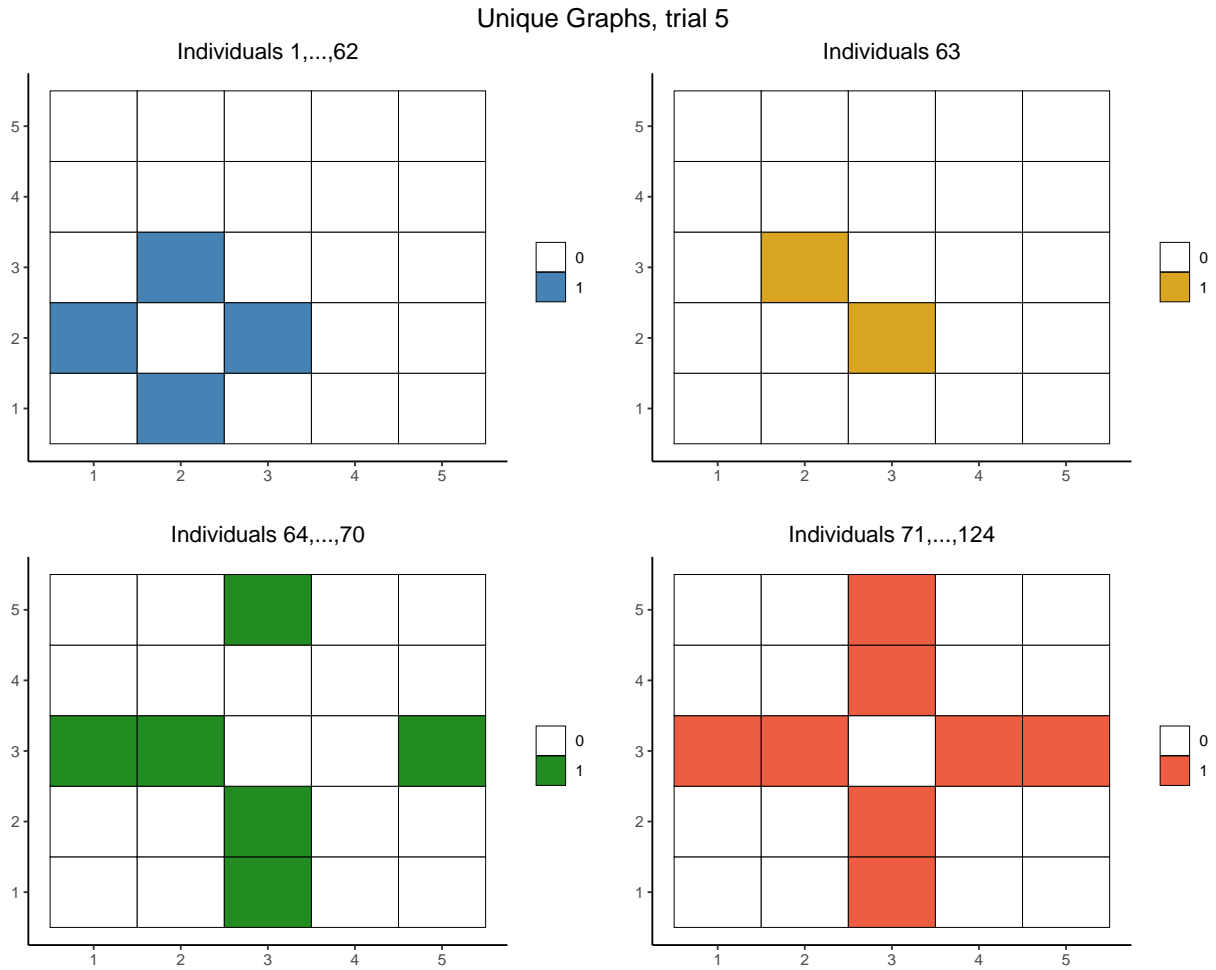
```
## Detected 7 workers
```

```
## Warning in covdepGE(data_mat, Z, var_min = 1e-06, var_max = 1, n_sigma = 20, :
## Variable 5: CAVI did not converge in 10000 iterations for 3/20 grid search
## candidates
```

```
## Warning in covdepGE(data_mat, Z, var_min = 1e-06, var_max = 1, n_sigma = 20, :
## For 1/5 variables, the selected value of sigmabeta_sq was on the grid boundary.
## See return value CAVI_details
```

```
out
```

```
##                    Covariate Dependent Graphical Model
##
## Model ELBO: -37538.41          Unique conditional dependence structures: 4
## n: 124, variables: 5                   Hyperparameter grid size: 20 points
## CAVI converged for 5/5 variables
##
## Model fit completed in 14.714 secs
```

```
annotate_figure(ggarrange(plotlist = plot(out, colors)),
                text_grob("Unique Graphs, trial 5", size = 15))
```
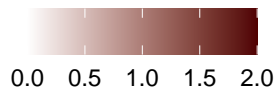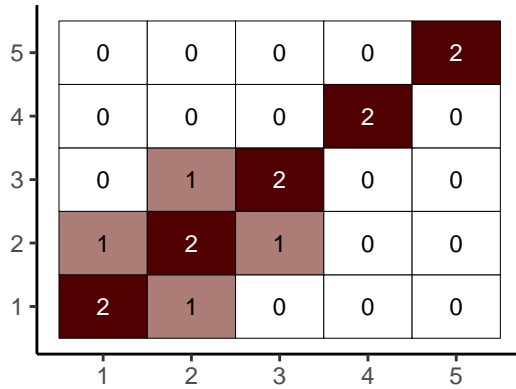

Unique Graphs, trial 5

Trial 6

$$n_2 = 3$$

Interval	Individual Indices
1	$1, \ldots, 60$
2	$61, \ldots, 63$
3	$64, \ldots, 123$

Interval 2 Precision Matrices
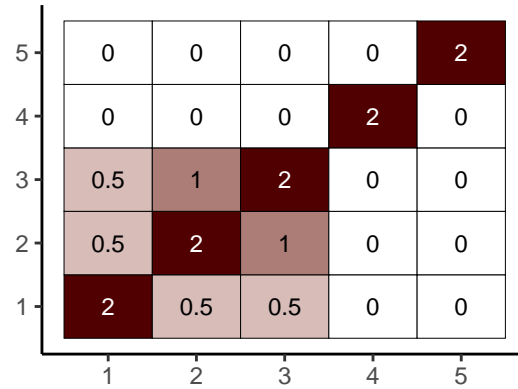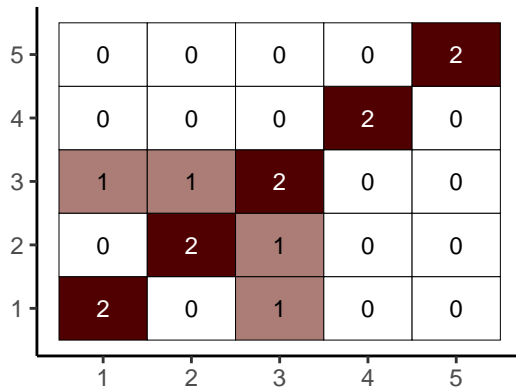

```
out <- covdepGE(data_mat, Z, var_min = 1e-6, var_max = 1, n_sigma = 20, CS = T,
                parallel = T, stop_cluster = F)
```

```
## Detected 7 workers
```

```
## Warning in covdepGE(data_mat, Z, var_min = 1e-06, var_max = 1, n_sigma = 20, :
## Variable 5: CAVI did not converge in 10000 iterations for 3/20 grid search
## candidates
```

```
## Warning in covdepGE(data_mat, Z, var_min = 1e-06, var_max = 1, n_sigma = 20, :
## For 1/5 variables, the selected value of sigmabeta_sq was on the grid boundary.
## See return value CAVI_details
```

```
out
```

```
##                       Covariate Dependent Graphical Model
##
## Model ELBO: -36945.24                Unique conditional dependence structures: 4
## n: 123, variables: 5                        Hyperparameter grid size: 20 points
## CAVI converged for 5/5 variables
##
## Model fit completed in 14.792 secs
```

```
annotate_figure(ggarrange(plotlist = plot(out, colors)),
                text_grob("Unique Graphs, trial 6", size = 15))
```


Unique Graphs, trial 6