Lecture 12: Log differentiation, Related Rates. (Oct 29)

(Sections 3.6 & 3.9)

Recall:

$$(\ln(x))' = \frac{1}{x}$$

If $y = y(x)$, then

$$(\ln(y))' = \frac{y'}{y} \quad (*)$$

or

$$y' = y \cdot (\ln(y))'$$

Log. differentiation is about using the log. laws and exploiting $(*)$ to find the derivative of functions y that involves products, quotients and powers.

Ex: Find the derivative of

$$y = \frac{x^5}{(1-10x)(\sqrt{x^2+2})}$$

Sol: ① Take ln on both sides.

$$\ln(y) = \ln\left(\frac{x^5}{(1-10x)(\sqrt{x^2+2})}\right)$$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

$$= \ln(x^5) - \ln((1-10x)(\sqrt{x^2+2}))$$

$\ln(ab) = \ln(a) + \ln(b)$

$$= \ln(x^5) - (\ln(1-10x) + \ln(\sqrt{x^2+2}))$$

$\sqrt{t} = t^{1/2}$

$$= \ln(x^5) - \ln(1-10x) - \ln((x^2+2)^{1/2})$$

$\ln(x^r) = r\ln(x)$

$$= 5\ln(x) - \ln(1-10x) - \frac{1}{2}\ln(x^2+2)$$

② Take derivatives on both sides (use CR for the LHS).

$$(\ln(y))' = \left(5\ln(x) - \ln(1-10x) - \frac{1}{2}\ln(x^2+2)\right)'$$

$$\frac{y'}{y} = (5\ln(x))' - (\ln(1-10x))' - \left(\frac{1}{2}\ln(x^2+2)\right)'$$

$$= 5\cdot\frac{1}{x} - \frac{1}{1-10x}\cdot(-10) - \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+2}\cdot\cancel{2}x$$

$$\frac{y'}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}$$

③ Solve for y':

$$y' = y\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$

check what y is!

$$= \frac{x^5}{(1-10x)(\sqrt{x^2+2})}\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$:)

$\boxed{\text{Ex:}}$ Differentiate

$$y = (\sin(x))^{x^3}.$$

Sol: The function

$$y = (\sin(x))^{x^3}$$

is different from

$$(\sin(x))^3 \quad \text{and} \quad (5)^{x^3}$$

because "there is an x both in the base and in the exponent".

In particular, we can NOT use the power rule. So

$$ y' \neq x^3 \sin(x)^{x^3-1} \cdot (+\cos(x)) \ldots $$

Apply log. differentiation instead!

① Apply ln to both sides:

$$ \ln(y) = \ln\left(\sin(x)^{x^3}\right) $$

$$ \overset{\ln(x^r)=r\ln(x)}{=} x^3 \ln(\sin(x)) $$

② Differentiate both sides:

$$ \frac{y'}{y} \overset{PR}{=} 3x^2 \ln(\sin(x)) + x^3 \cdot \underbrace{\frac{1 \cdot \cos(x)}{\sin(x)}}_{(\ln(\sin(x)))'} $$

$$ = \frac{3x^2 \ln(\sin(x))\sin(x) + x^3\cos(x)}{\sin(x)} $$

Factor x^2

$$ = \frac{x^2(3\ln(\sin(x))\cdot\sin(x) + x\cos(x))}{\sin(x)} $$

③ Solve for y':

$$ y' = y \cdot \left(\frac{x^2(3\ln(\sin(x))\cdot\sin(x) + x\cos(x))}{\sin(x)} \right) $$

$$ = \sin(x)^{x^3} \left(\frac{x^2(3\ln(\sin(x))\cdot\sin(x) + x\cos(x))}{\sin(x)} \right) $$

$= \sin(x)^{x^3-1}\left(x^2(3\ln(\sin(x))\cdot\sin(x) + x\cos(x)\right)$

$\frac{a^b}{a} = a^{b-1}$

If you want, you can do it this way.

<u>Recall</u>: $t = e^{\ln(t)}$

Let's pick $t = (\sin(x))^{x^3}$. Then

$$y = (\sin(x))^{x^3} = e^{\ln(\sin(x)^{x^3})}$$

So computing the derivative of y is the same as finding $\left(e^{\ln(\sin(x)^{x^3})}\right)'$:

$$\left(e^{\ln(\sin(x)^{x^3})}\right)' \overset{\ln(x^r) = r\ln(x)}{=} \left(e^{x^3\ln(\sin(x))}\right)'$$

$$\overset{CR}{=} e^{x^3\ln(\sin(x))} \cdot (x^3\ln(\sin(x)))' =$$

$(e^{g(t)})' = e^{g(t)} \cdot g(t)'$

$$= e^{x^3 \ln(\sin(x))} \left(3x^2 \ln(\sin(x)) + \frac{x^3 \cos(x)}{\sin(x)} \right)$$

$r\ln(x) = \ln(x^r)$

$$= e^{\ln(\sin(x)^{x^3})} \left(3x^2 \ln(\sin(x)) + x^3 \cot\text{an}(x) \right)$$

$$= \sin(x)^{x^3} \left(3x^2 \ln(\sin(x)) + x^3 \cot\text{an}(x) \right).$$

- **Related Rates** (Section 3.9).

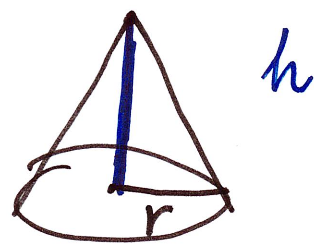
What is this about?

1. They deal with "rates of change", aka derivatives.

2. It is an application of the chain rule to "real-life" problems.

Ex: At the North Pole, an iceberg, in the shape of a right circular cone, is melting in such a way that both its radius and its height are decreasing at a rate of 0.005 ft/hr. How fast is the volume of the iceberg decreasing when radius = height = 0.5 ft?

__Sol:__ The iceberg at time t looks like:



$h = h(t)$ is the height of the cone at time t

$r = r(t)$ is the radius of the base of the cone at time t.

What we know		What we want to find	
In words	In Math	In words	In Math
"The radius r is decreasing at 0.005 ft/hr" (velocity)	$r'(t) = -0.005$ ft/hr (derivative) $\doteq -\frac{5}{1000}$ ft/hr	"How fast is the volume decreasing when $r = h = 0.5$ ft" or "What is the rate of change of the volume when $r = h = 0.5$ ft"	Let $V = V(t)$ be the volume of the iceberg at time t. WTF: $V'(t) = \frac{dV}{dt}$ when $r = h = 0.5$ ft $\doteq \frac{1}{2}$ ft
"The height h is decreasing at 0.005 ft/hr"	$h'(t) = \frac{-5}{1000}$ ft/hr (OR $\frac{dh}{dt}(t) = \frac{-5}{1000}$ ft/hr)		

Note: We want to find $V'(t)$ but we are given info about $h(t)$ and $r(t)$.

Strategy: Find a way to connect what you want to find ($V'(t)$) to what you know (info about $h(t)$, $r(t)$) by writing $V(t)$ in terms of $h(t)$ and $r(t)$.

Use Geometry: In this case we need to know the formula for the volume of a cone.

A cone with height h and radius r has volume:

$$V = \frac{\pi}{3} \cdot r^2 \cdot h$$

So for us:

$$V(t) = \frac{\pi}{3} r^2(t)\, h(t) \qquad (*)$$

But, remember, we need $V'(t)$.

So, let's just differentiate $(*)$ using the CR:

$$V'(t) = \frac{\pi}{3} \left(r^2(t)\, h(t) \right)'$$

$$\overset{PR}{\underset{!}{=}} \frac{\pi}{3} \left(2r(t) r'(t) \cdot h(t) + r^2(t)\, h'(t) \right)$$

$$\underset{!}{=} \frac{\pi}{3} \left(2 r \cdot r' \cdot h + r^2 h' \right)$$

Now, the problem tell's us that

$$r'(t) = h'(t) = -\frac{\cancel{5}\,1}{\cancel{1000}\,200} = -\frac{1}{200} \text{ ft/hr}$$

So we can Now plug-in $r' = h' = -\frac{1}{200}$:

$$V' = \frac{\pi}{3}\left(2r\left(-\frac{1}{200}\right)h + r^2\left(-\frac{1}{200}\right)\right)$$

$$= \frac{\pi}{3}\left(-\frac{1}{200}\right)\left(2rh + r^2\right)$$

$$= -\frac{\pi}{600}\left(2rh + r^2\right)$$

We want to find V' when $r = h = \frac{1}{2}$ ft.
So let's plug-in these values:

$$V' = -\frac{\pi}{600}\left(\cancel{2} \cdot \frac{1}{\cancel{2}} \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2\right) =$$

$$= -\frac{\pi}{600}\left(\frac{1}{2} + \frac{1}{4}\right) = -\frac{\pi}{\cancel{600}\,200}\left(\frac{\cancel{2+1}}{4}\right) =$$

$$= -\frac{\pi}{800} \text{ ft/hr}.$$

In general:

1. Decide what the variables are in the problem (r, h, V).
2. Find an equation that relates them, using geometry $\left(V = \frac{\pi}{3} r^2 h\right)$
3. Derive both sides using the chain rule
4. plug in the values you know about
5. Solve for the unknown derivative $(V'(t))$.

You should FIRST derive (do 3) and THEN plug in the specific values (do 4).

For 1 it is also helpful to draw a picture of the situation.

The pages following this one solve this problem. You should attempt each part first, then look at the solution for that part, then move to the next part.

A tank of water in the shape of an inverted cone is leaking water at a constant rate of 2 ft^3/hr. The base radius of the tank is 5 ft. and the height of the tank is 14 ft.

Question: At what rate is the depth of the water in the tank changing when the depth is 6 ft?

(A) Sketch a 2D picture of the situation at time t and introduce variables for the radius, the height and the volume of the cone of water inside the tank.

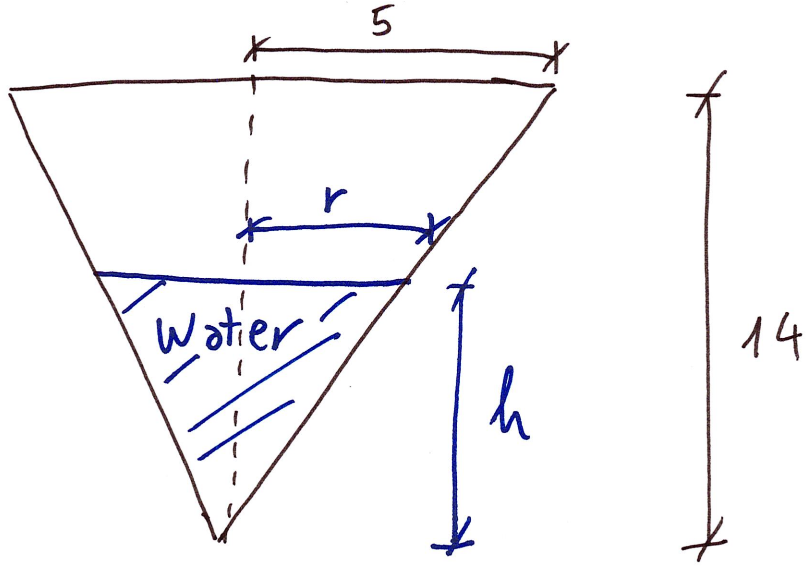
(B) Write what you know and what you need to find.

(C) Write a formula for the volume of water in the tank at any time t, *as a function only of the height of the cone of water.*

(D) Use the chain rule to solve the problem.

(A)

Situation at time t:



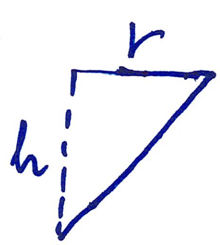
r = radius of water cone $(r = r(t))$
h = height of water cone $(h = h(t))$
V = volume of water cone $(V = V(t))$

(B) We Know: $\dfrac{dV}{dt} = V'(t) = -2 \ ft^3/hr$

Want to find: $h'(t)$ when $h = 6\,ft$.

(C) $V(t) = \dfrac{\pi}{3} r(t)^2 h(t)$ Volume of the water cone at time t

The triangles



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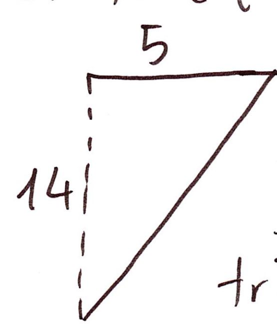


are similar triangles.

So: $\dfrac{r}{h} = \dfrac{5}{14} \Rightarrow r = \dfrac{5}{14} h$.

Then:

$$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}\left(\frac{5h}{14}\right)^2 h = \frac{25\pi}{588} h^3 .$$

(D) Remember: $V = V(t)$ and $h = h(t)$. We derive wrt time:

$$V'(t) = \frac{25\pi}{588} \cdot 3h^2 h' = \frac{25\pi}{196} h^2 h'$$

$$\left(\text{or } \frac{dV}{dt}(t) = \frac{dV}{dh}(h(t)) \cdot \frac{dh}{dt}(t) \right.$$

$$\left. \overset{!}{=} \left(\frac{25\pi}{588} \cdot 3h^2\right) \frac{dh}{dt} = \frac{25\pi}{196} h^2 \frac{dh}{dt} \right).$$

Plug-in $h = 6$ & $V' = -2$ and solve for h':

$$(-2) = \frac{25\pi}{196} \cdot 6^2 h' \Rightarrow h' = \cancel{-2} \cdot \frac{\overset{98}{\cancel{196}}}{25\pi} \cdot \frac{1}{\underset{9}{\cancel{\underset{18}{\cancel{36}}}}}$$

$$= \frac{-98}{225\pi} \text{ ft/hr.}$$

Summary of this class.

1. We saw how to use the log laws and the rule

$$(\ln(y))' = \frac{y'}{y}$$

to find y' when the formula for y involves products, quotients, powers. (Logarithmic Differentiation).

2. We also saw another method to compute y', using $t = e^{\ln(t)}$ (e.g. to compute the derivative of $y = \sqrt{x}^{\sqrt{x}} e^{x^2}$, we can write $y = e^{\ln(y)} = e^{\ln(\sqrt{x}^{\sqrt{x}} e^{x^2})}$ and then derive $e^{\ln(\sqrt{x}^{\sqrt{x}} e^{x^2})}$).

3. We talked about related rates, aka problems with real-life situations where we have two (or more) quantities $x(t)$, $y(t)$ depending on a variable t (time). We know what $x'(t)$ is at a certain $t = t_0$ and we want to find $y'(t)$ at $t = t_0$ by

relating (with geometry) y' to x'.
We try to write y in terms of x, that is, we find a formula for $y = y(x)$ and then use the chain rule

$$y'(t) = y'(x(t)) \cdot x'(t)$$

or $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

For the next class/next week:

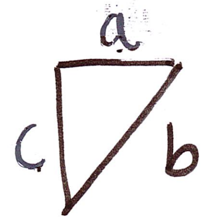
① Do the exercises of Section 3.6 & 3.9.
For related rates, the kind of geometry you need to use is usually in one of these forms:

(a) Volumes of spheres, cylinders, cones (Check Reference page 1 in the book) or areas of triangles, circles, rectangles etc.

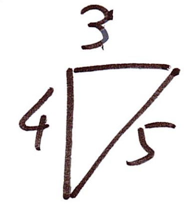
(b) Pythagora's Thm ($Hyp^2 = Adj^2 + Opp^2$)

(c) Similar triangles (the lengths of corresponding sides are proportionals).

E.g: If you know that



is similar to



then you know:

$$\frac{a}{b} = \frac{3}{5}, \quad \frac{a}{c} = \frac{3}{4}, \quad \frac{b}{c} = \frac{5}{4}$$

etc.

(d) How to write the sides of a right-angle triangle using cos, sin, tan, i.e.

$$\cos(\theta) = \frac{Adj}{Hyp} \qquad \sin(\theta) = \frac{Opp}{Hyp}$$

$$\tan(\theta) = \frac{Opp}{Adj}$$

(2) There is no Quiz on W, Oct 31.

(3) On W, Oct 31 we will do 4.1 & maybe start 4.3. (More on next page)

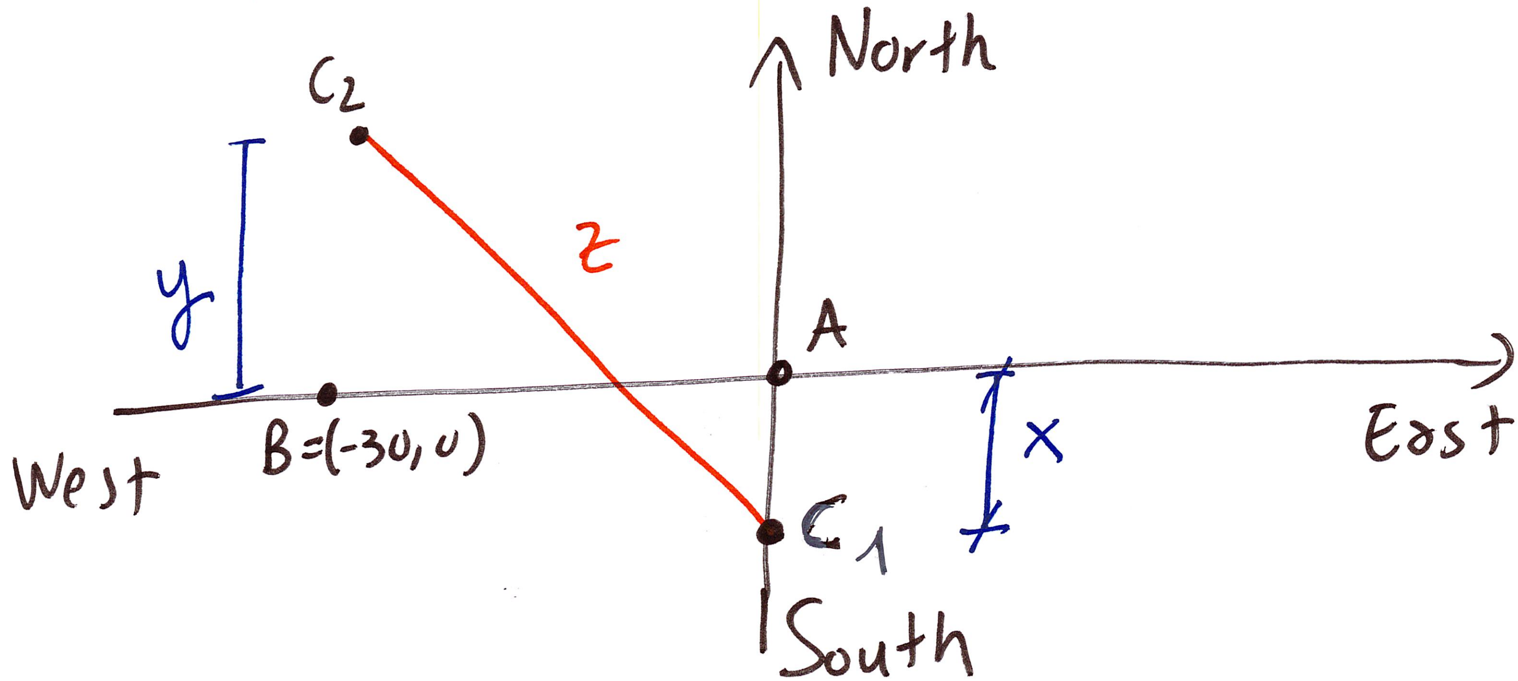
Here is another exercise on Related Rates.

$\boxed{\text{Ex:}}$ A bike starts moving south at 25 km/h from a point A. At the same time another bike starts moving north at 15 km/h from a point 30 km due west of A. At what rate is the distance between the two bikes changing 1 hr after they began travelling?

It is VERY HELPFUL to sketch a picture of the situation here. I will draw one on the next page. Try first to draw yours and see if they look similar.

You should draw a picture of the situation at a time t after both cyclists have started to move.

At a time t after both cyclists have started to move, the situation looks like this:



A = starting point of the first cyclist (C_1)
B = starting point of the second cyclist (C_2)
$x = x(t)$ = distance of C_1 from A
$y = y(t)$ = distance of C_2 from B
$z = z(t)$ = distance btwn C_1 & C_2

We know:

$$x'(t) = \frac{dx}{dt} = 25 \ \frac{Km}{h}$$

$$y'(t) = \frac{dy}{dt} = 15 \ Km/h$$

We want $z'(t)$ for $t = 1$ (hr).