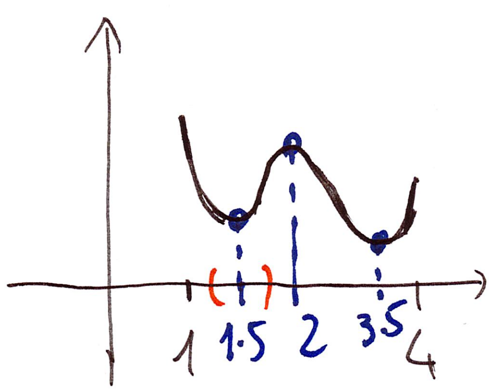
Some info

1. The quiz on Wednesday covers:
   - Section 3.6 (Derivative of log & log diff.)
   - Section 3.9 (Rel. rates)
   - Section 4.1 (Max and min values)

2. There was a change in Lecture 13 notes last Saturday. Check that out.
   The issue was the following. If f is defined on a closed interval $[a,b]$ the endpoint values $f(a)$ and $f(b)$ are never local min. or local max. They can only be ABSOLUTE min/max.



$f(1)$ ABS max

$f(1.5)$ loc min

$f(2)$ loc max

$f(3.5)$ ABS min

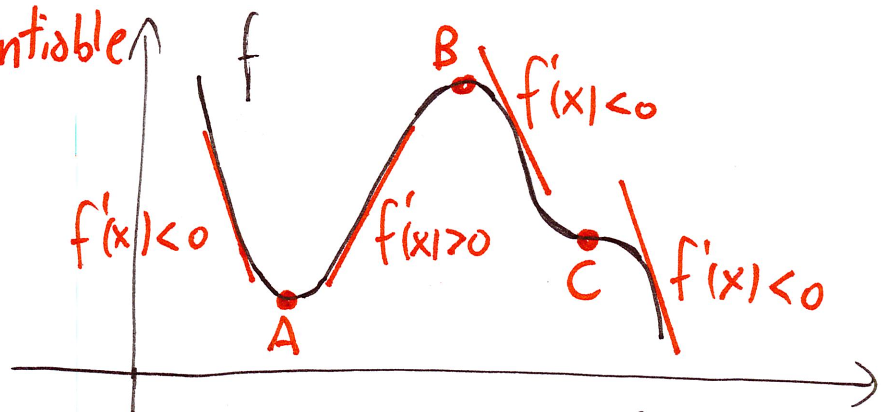
$f(4)$ is NOT a local max.

(Check Example 4 on page 277)

Lecture 14: Derivatives, shapes of graphs & local min/max. (Nov 5)

(Section 4.3)

Suppose f is a differentiable function on an open interval I.



A : local min (f' goes from $-$ to $+$)

B : local max (f' goes from $+$ to $-$)

C : f' goes from $-$ to $-$, no loc min/max

Increasing/Decreasing Test. (IT test).

(a) If $f'>0$ on I, then f is increasing on I.

(b) If $f'<0$ on I, then f is decreasing on I.

First Derivative Test (FDT). Suppose c is a critical number of f.

(a) If f changes from $+$ to $-$ at c, f has a loc. max at c

(b) If f changes from $-$ to $+$ at c, f has a loc. min at c

(c) If f does not change sign at c, f has no loc. min/max at c.

$\boxed{\text{Ex:}}$ For the function

$$f(x) = \frac{x^4}{2} - \frac{x^3}{3} - 2x^2 + 2x + 7$$

find:

(a) • the intervals where it is decreasing and where it is increasing;

(b) • the local min/max values.

Sol: We need to study where $f' > 0$ and where ~~it is~~ $f' < 0$, so that we can use the ID test and the FD test.

$$f'(x) = \frac{\cancel{4}^{\,2} x^3}{\cancel{2}_{\,1}} - \frac{\cancel{3}x^2}{\cancel{3}} - 4x + 2$$

$$= 2x^3 - x^2 - 4x + 2$$

$$= 2x(x^2 - 2) + (-1)(x^2 - 2)$$

$$= (2x - 1)(x^2 - 2) = (2x - 1)(x + \sqrt{2})(x - \sqrt{2})$$

$(a^2 - b^2) = (a+b)(a-b)$
with $a = x$, $b = \sqrt{2}$

To study where $f'>0$, and where $f'<0$, let's first look at $f'(x)=0$ (that is, let's find the critical numbers):

$$f'(x) = (2x-1)(x+\sqrt{2})(x-\sqrt{2}) = 0$$

$$\Leftrightarrow \begin{cases} x_1 = +\frac{1}{2} \\ x_2 = -\sqrt{2} \\ x_3 = \sqrt{2} \simeq 1.414\ldots \end{cases}$$

Note: $-\sqrt{2} < \frac{1}{2} < \sqrt{2}$

We need to understand the sign of f' on the intervals:

$$(-\infty, -\sqrt{2}), \ (-\sqrt{2}, \tfrac{1}{2}), \ (\tfrac{1}{2}, \sqrt{2}), \ (\sqrt{2}, +\infty)$$

$$\left[\mathbb{R} = (-\infty, -\sqrt{2}) \cup \{-\sqrt{2}\} \cup (-\sqrt{2}, \tfrac{1}{2}) \cup \{\tfrac{1}{2}\} \cup (\tfrac{1}{2}, \sqrt{2}) \cup \{\sqrt{2}\} \cup (\sqrt{2}, +\infty)\right]$$

Sign of:	$(-\infty, -\sqrt{2})$	$(-\sqrt{2}, 1/2)$	$(1/2, \sqrt{2})$	$(\sqrt{2}, +\infty)$	
$2x-1$	$-$	$-$	$+$	$+$	$(2x-1>0$ $2x>1$ $x>1/2)$
$x+\sqrt{2}$	$-$	$+$	$+$	$+$	$x+\sqrt{2}>0$ $x>-\sqrt{2}$
$x-\sqrt{2}$	$-$	$-$	$-$	$+$	$x-\sqrt{2}>0$ $x>\sqrt{2}$
$f'(x)$	$((-)\cdot(-)\cdot(-))$ $-$	$((-)\cdot(+)\cdot(-))$ $+$	$-$	$+$	
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	

USING I/D test

(a) f is decreasing on $(-\infty, -\sqrt{2}]$, $(1/2, \sqrt{2}]$

f is increasing on $(-\sqrt{2}, 1/2]$, $[\sqrt{2}, +\infty)$

(b) • At $x=-\sqrt{2}$, f' is changing from $-$ to $+$, so $f(-\sqrt{2})$ is a local min.

• At $x=\frac{1}{2}$, f' is changing from $+$ to $-$, so $f(\frac{1}{2})$ is a local max

• At $x=\sqrt{2}$, f' is changing from $-$ to $+$, so $f(\sqrt{2})$ is a loc. min value.

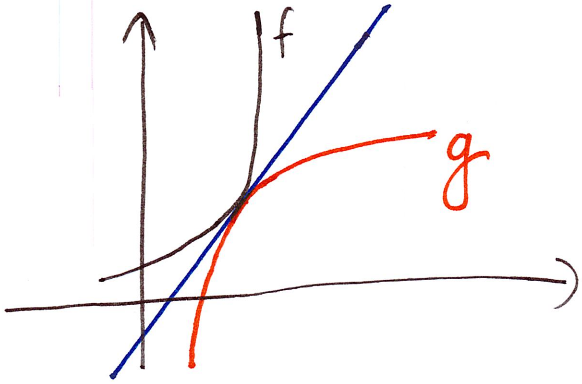
Compute the values $f(-\sqrt{2})$, $f(\sqrt{2})$, $f(\frac{1}{2})$ at home.

Bonus: Does f have any ABSOLUTE min or max? If not, say why; if yes, find them.

(Hint: Will write the hint at the bottom of the page.

Remember: except for endpoint values, abs min/max are loc. min/max...

What happens if you take $\lim_{x\to\infty} f(x)$ & $\lim_{x\to-\infty} f(x)$?)



Given an interval (a,b), a differentiable function is:

- **Concave Up(ward)** on (a,b) if, like for f, its graph always lies above its tangent lines on (a,b).
- **Concave down(ward)** on (a,b) if, like for g, its graph always lies below its tangent lines on (a,b).

Concavity Test. Suppose a function f is differentiable on (a,b) and suppose $f''(x)$ exists for $x \in (a,b)$.

(a) If $f''(x) > 0$ for all x in (a,b), then f is concave up on (a,b).

(b) If $f''(x) < 0$ for all x in (a,b), then f is concave down on (a,b).

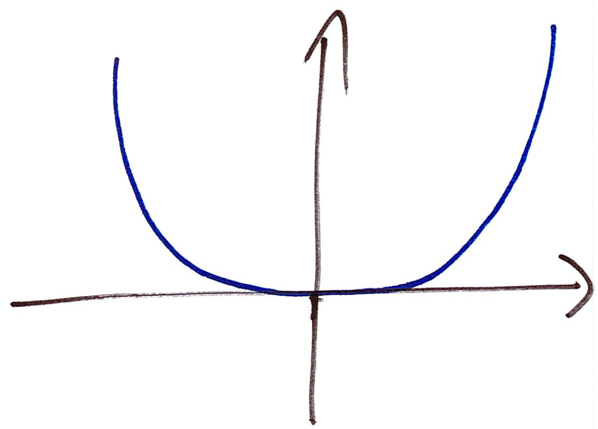
A point $(x, f(x))$ is an **inflection point** if f changes concavity at x. If $f''(x)$ exists, then an inflection point is one where f'' changes sign (from + to − or from − to +).

Beware: IF $(x, f(x))$ is an inflection point, THEN

$$f''(x) = 0.$$

But $f''(x)$ might be zero even if $(x, f(x))$ is not an inflection point

(Ex: take $f(x) = x^4$; $x=0$ is s.t. $f''(x) = 0$ but it's not an inflection point



$y = x^4$
)

$\boxed{\text{Ex:}}$ For

$$f(x) = x^4 - 2x^2 + 3$$

find:
- where the function is concave up and where it is concave down;
- the inflection points.

Sol: We need to study $f''(x)$.

$f'(x) = 4x^3 - 4x$

$f''(x) = (f'(x))' = 12x^2 - 4$

. We need to understand where $f''(x) > 0$ and where $f''(x) < 0$.

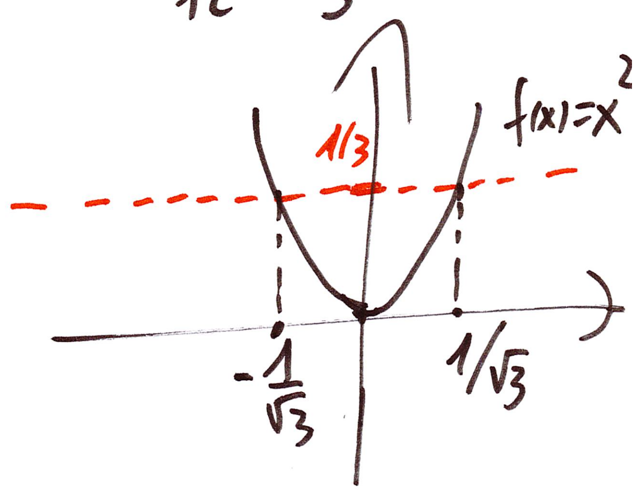
Let's first solve: $12x^2 - 4 = 0 \iff$

$\iff 12x^2 = 4 \iff x^2 = \frac{4}{12} = \frac{1}{3} \iff$

$\iff x = \pm \frac{1}{\sqrt{3}}$.

Now: • $12x^2 - 4 > 0 \iff x^2 > \frac{4}{12} = \frac{1}{3}$

$\iff x < -\frac{1}{\sqrt{3}}$ or $x > \frac{1}{\sqrt{3}}$



• $12x^2 - 4 < 0 \iff x^2 < \frac{1}{3}$

$\iff -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

So

		$-\frac{1}{\sqrt{3}}$	$1/\sqrt{3}$
$f''(x)$	$+$	$-$	$+$

Thus, by CT (Concavity Test):

- f is concave up on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, +\infty\right)$.

- f is concave down on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

The function f is s.t. $f''(x)$ is changing signs both at $-\frac{1}{\sqrt{3}}$ and at $\frac{1}{\sqrt{3}}$, so

$$\left(-\frac{1}{\sqrt{3}}, f\left(-\frac{1}{\sqrt{3}}\right)\right) \;\&\; \left(\frac{1}{\sqrt{3}}, f\left(\frac{1}{\sqrt{3}}\right)\right)$$

are both inflection points.

(Compute $f\left(-\frac{1}{\sqrt{3}}\right)$ & $f\left(\frac{1}{\sqrt{3}}\right)$).

We can use $f''(x)$ to find local min/max.

Second Derivative Test (SDT). Suppose that:

(1) c is a critical number of f s.t. $f'(c)=0$;

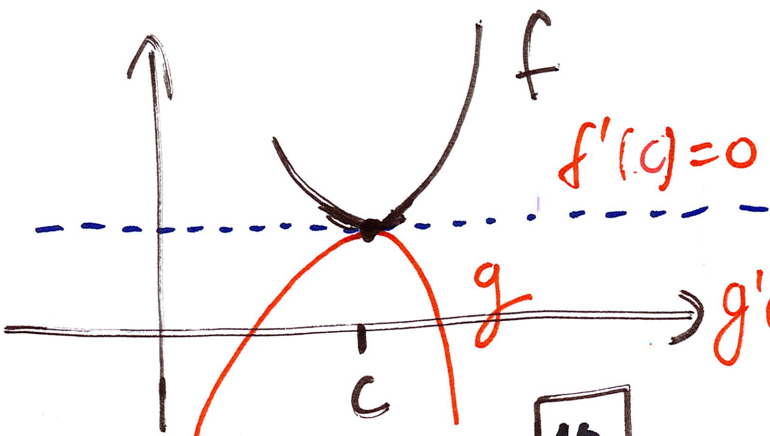
AND

(2) f'' is continuous on an open interval containing c.

THEN

(a) If $f''(c)>0$, then f has a loc. min at c;

(b) if $f''(c)<0$, then f has a local max at c.



$f'(c)=0$ and $f''(c)>0$ (f is concave up)

$g'(c)=0$ and $g''(c)<0$ (g is concave down)

WARNING: This test CAN'T BE USED (it doesn't say anything) if:

(1) $f'(c)=0$ & $f''(c)=0$

(2) $f''(c)$ DNE (f'' can't be continuous at c);

(3) $f'(c)$ DNE ($f'(c) \neq 0$).

In the cases above, use FDT (First Derivative Test).

$\boxed{\text{Ex:}}$ Find the local min/max of

$$f(x) = x^3 - 9x^2 + 15x - 7$$

Sol: Let's try to use the SDT.

$f'(x) = 3x^2 - 18x + 15$

$f''(x) = 6x - 18$

Let's find those c's for which $f'(c)=0$.

$$f'(x) = 3x^2 - 18x + 15 = 0 \Leftrightarrow x^2 - 6x + 5 = 0$$

Divide by 3

$$x_{1,2} = \frac{6 \pm \sqrt{36-20}}{2} = \frac{6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2} = \begin{cases} x_1 = 5 \\ x_2 = 1 \end{cases}$$

So $c_1 = 5$, $c_2 = 1$ are s.t $f'(c_1) = 0 = f'(c_2)$.

- $f''(5) = 6 \cdot 5 - 18 = 30 - 18 = 12 > 0$, so f has a loc. min at $x = 5$.
- $f''(1) = 6 \cdot 1 - 18 = 6 - 18 = -12 < 0$, so f has a loc. max at $x = 1$.

$\boxed{\text{Ex:}}$ For the function

$$f(x) = x - \sqrt{2}\sin(x)$$

defined on $\left[-\frac{\pi}{4}, \pi\right]$, find:

(a) the critical numbers;

(b) the intervals where it's increasing/decreasing;

(c) the local min & max and the abs min & max;

(d) the intervals where it's concave up & down;

(e) the inflection points.

Sol:

(a) $-\frac{\pi}{4}, \frac{\pi}{4}$

(b) Increasing: $[-\frac{\pi}{2}, -\frac{\pi}{4}]$ & $[\frac{\pi}{2}, \pi]$

Decreasing: $[-\frac{\pi}{4}, \frac{\pi}{2}]$.

(c) Local min at $x = \frac{\pi}{4}$

Local max at $-\frac{\pi}{4} = x$

Abs min at $x = -\frac{\pi}{2}$

Abs max at $x = \pi$

(d) Concave up: $[0, \frac{\pi}{2}]$

Concave down: $[-\frac{\pi}{2}, 0]$ & $[\frac{\pi}{2}, \pi]$

(e) Inflection points: $(0, f(0))$
$= (0, 0)$.

Summary of this class

1. We saw how the first derivative, $f'(x)$, of a differentiable function $f(x)$ can tell us:
   - where f is increasing & decreasing (ID test)
   - which critical points of f are loc. min, loc. max or neither (FDT)

(2) We saw how the second derivative, $f''(x)$, of a function f (for which $f''(x)$ exists) can tell us:

- where f is concave up & down (CT);
- what critical points of f are loc. min or local max (SDT).

Key-words for this class:

- Increasing/Decreasing Test (ID test).
- First Derivative Test (FDT).
- Concave up, Concave down.
- Inflection points.
- Concavity Test (CT).
- Second Derivative Test (SDT).

For next class.

1. Revise for the Quiz (Sections 3.6, 3.9, 4.1)

2. Start doing the exercises for section 4.3.

3. We will do Section 4.7 and, maybe, start section 4.4.