Some info.

1. Review sessions for the Midterm:
   (a) Tu, Oct 23, 2.30-4 pm in MC 204
   (b) W, Oct 24, 7-9 pm in class.
2. Midterm is F, 7-9pm (Oct 26). Check where you are supposed to go on OWL.
3. Make-up is W, Oct 31, 7-9 pm in NS 1.
4. Check the "Midterm Summary" on OWL → "Resources".

Lecture 11: Derivatives of $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, log. (Oct 22).

(Sections 3.5-3.6)

Q: Suppose f is a one-to-one function, with inverse f^{-1}. Using that

$$f(f^{-1}(x)) = x \qquad (x \in \text{domain of } f^{-1}),$$

what is $(f^{-1}(x))'$?

(a) $\dfrac{f(x)}{f'(f^{-1}(x))}$

(b) $\dfrac{f'(x)}{f'(f^{-1}(x))}$

(c) $\dfrac{f'(x)}{f^{-1}(x)}$

(d) It depends on f

(e) None of the above

What is the answer then?

$$\left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))},$$

if f is differentiable at $a = f^{-1}(x)$.

Why?

$$f(f^{-1}(x)) = x$$

Apply $(-)'$ on both sides:

$$\left(f(f^{-1}(x))\right)' = 1$$

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$$f'(f^{-1}(x)) \cdot \left(f^{-1}(x)\right)' = 1$$

Dividing by $f'(f^{-1}(x))$, we get

$$\left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))}$$

Let's use this formula to find a bunch of new derivatives.

- $f(x) = \cos(x)$ (on $[0,\pi]$)

If $f(x) = \cos(x)$, $f^{-1}(x) \overset{!}{=} \arccos(x) \overset{!}{=} \cos^{-1}(x)$

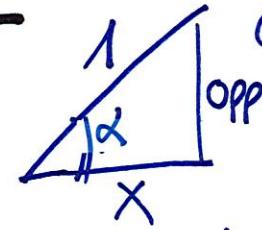
$$\left(\cos^{-1}(x)\right)' = \frac{1}{\cos'\left(\cos^{-1}(x)\right)}$$

$$\cos'\left(\cos^{-1}(x)\right) = -\sin\left(\cos^{-1}(x)\right) =$$

$(\cos(t))' = -\sin(t) \qquad t = \cos^{-1}(x)$

$$= -\sqrt{1 - \left(\cos\left(\cos^{-1}(x)\right)\right)^2}$$

$$\overset{!}{=} -\sqrt{1-x^2}$$



$\cos^{-1}(x) =: \alpha$

$\text{opp} = \sqrt{1-x^2}$

See extra page 3-bis for an explanation of why $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$.

$\sin(\cos^{-1}(x)) = ?$

Let's call $\alpha = \cos^{-1}(x)$.

Then:

$\alpha \in [0, \pi]$ = range of $\cos^{-1}(x)$

$(0 \leq \alpha \leq \pi)$

$\cos(\alpha) = x$

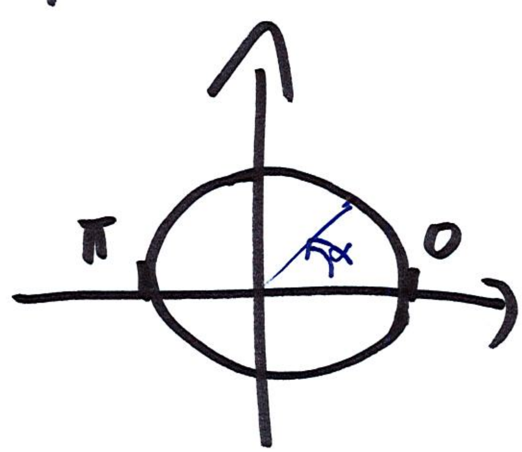
Now: $\sin^2(\alpha) + \cos^2(\alpha) = 1$

$\Rightarrow \sin^2(\alpha) = 1 - \cos^2(\alpha)$

$\sin(\alpha) = \pm\sqrt{1 - \cos^2(\alpha)}$

$\cos(\alpha) = x$

$= \pm\sqrt{1 - x^2}$



Because $\alpha \in [0, \pi]$,

$\sin(\alpha) = +\sqrt{1 - x^2}$

3-bis

All in all,

$$\boxed{\left(\cos^{-1}(x)\right)' = -\frac{1}{\sqrt{1-x^2}}}$$

Similarly:

$$\boxed{\begin{aligned} \left(\sin^{-1}(x)\right)' &= \frac{1}{\sqrt{1-x^2}} \\ \left(\tan^{-1}(x)\right)' &= \frac{1}{1+x^2} \end{aligned}}$$

Remember:

$\sin^{-1}(x)$ can also be written $\arcsin(x)$. $\tan^{-1}(x)$ is $\arctan(x)$.

Ex: Find $f'(x)$ for

$$f(x) = \frac{\sin^{-1}(x)}{1+x}$$

Sol: $$f'(x) \overset{QR}{=} \frac{\left(\sin^{-1}(x)\right)'(1+x) - \sin^{-1}(x)(1+x)'}{(1+x)^2} =$$

$$= \frac{\frac{1}{\sqrt{1-x^2}}(1+x) - \sin^{-1}(x)\cdot 1}{(1+x)^2} =$$

$$= \frac{\dfrac{(1+x) - \sin^{-1}(x)\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}}{(1+x)^2} =$$

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$$

$$= \frac{1+x - \sin^{-1}(x)\sqrt{1-x^2}}{\left(\sqrt{1-x^2}\right)(1+x)^2}$$

$\boxed{\text{Ex:}}$ Derive

$$f(x) = e^{3x} \cdot \left(\arctan\left(\frac{1}{1+3e^x}\right)\right)$$

Sol: $f'(x) = \left(e^{3x}\right)' \cdot \left(\arctan\left(\frac{1}{1+3e^x}\right)\right) +$

$$+ \left(e^{3x}\right) \cdot \left(\arctan\left(\frac{1}{1+3e^x}\right)\right)'$$

$$= \underbrace{e^{3x}}_{} \cdot 3 \cdot \arctan\left(\frac{1}{1+3e^x}\right) +$$

"First do $e'(t)$ and then put $t = 3x$"

$$+ e^{3x} \cdot \left(\frac{1}{1+\left(\frac{1}{1+3e^x}\right)^2}\right) \cdot \left(\frac{1}{1+3e^x}\right)'$$

$$= 3e^{3x}\arctan\left(\frac{1}{1+3e^{x}}\right) +$$

$$+ e^{3x}\left(\frac{1}{1+\left(\frac{1}{1+3e^{x}}\right)^{2}}\right)\cdot\left(\frac{0\cdot(1+3e^{x}) - 1\cdot\overbrace{(1+3e^{x})'}^{3e^{x}}}{(1+3e^{x})^{2}}\right)$$

$$= 3e^{3x}\arctan\left(\frac{1}{1+3e^{x}}\right) + e^{3x}\left(\frac{1}{1+\left(\frac{1}{1+3e^{x}}\right)^{2}}\right)\cdot\left(\frac{-3e^{x}}{(1+3e^{x})^{2}}\right)$$

$$= 3e^{3x}\left(\arctan\left(\frac{1}{1+3e^{x}}\right) - \frac{e^{x}}{\left(1+\left(\frac{1}{1+3e^{x}}\right)^{2}\right)(1+3e^{x})^{2}}\right)$$

$$= 3e^{3x}\left(\arctan\left(\frac{1}{1+3e^{x}}\right) - \frac{e^{x}}{(1+3e^{x})^{2}+1}\right)$$

END OF MIDTERM'S TOPICS

• Derivative of logarithmic functions. (Section 3.6)

<u>Recall:</u> If $f(x) = b^x$, $\left(\begin{array}{l} b>0, \\ b\neq 1 \end{array}\right)$

$$f^{-1}(x) = \log_b(x)$$

Then
$$f(f^{-1}(x)) = x$$

becomes
$$b^{\log_b(x)} = x$$

Let's differentiate both sides

$\boxed{(b^t)' = b^t \ln(b)}$
$$\left(b^{\log_b(x)}\right)' = x'$$

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$$b^{\log_b(x)} \ln(b) \cdot (\log_b(x))' = 1$$

$\boxed{b^{\log_b(x)} = x}$
$$x \ln(b) \cdot (\log_b(x))' = 1$$

$$(\log_b(x))' = \frac{1}{x \ln(b)}$$

So $$\boxed{\left(\log_b(x)\right)' = \frac{1}{x\ln(b)}}$$

In particular (if $b=e$):

$$\boxed{\left(\ln(x)\right)' = \frac{1}{x}}$$

We also have

$$\boxed{\left(\ln(|x|)\right)' = \frac{1}{x}}$$

$\boxed{\text{Ex:}}$ Find $f'(x)$ if

$$f(x) = \ln\left(\frac{e^x}{x^2+1}\right)$$

<u>Sol:</u> Use the logarithmic laws:

$$\ln\left(\frac{e^x}{x^2+1}\right) = \ln(e^x) - \ln(x^2+1)$$
$$\overset{!}{=} x - \ln(x^2+1)$$

So:

$$\left(\ln\left(\frac{e^x}{x^2+1}\right)\right)' = \left(x - \ln(x^2+1)\right)'$$

$$= (x)' - \left(\ln(x^2+1)\right)'$$

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$$= 1 - \ln'(x^2+1) \cdot (x^2+1)' =$$

$$= 1 - \frac{1}{x^2+1} \cdot 2x =$$

$(a^2 - 2ab + b^2)$
$= (a-b)^2$

$$= 1 - \frac{2x}{x^2+1} = \frac{x^2+1-2x}{x^2+1} =$$

$$= \frac{(x-1)^2}{x^2+1}.$$

In general:

$$\boxed{\left(\ln(g(x))\right)'} = \frac{1}{g(x)} \cdot g'(x) = \boxed{\frac{g'(x)}{g(x)}}$$