Lecture 6: Continuity - Sep 26 (Section 2.5)

Def: A function f is continuous at $x=a$ if:

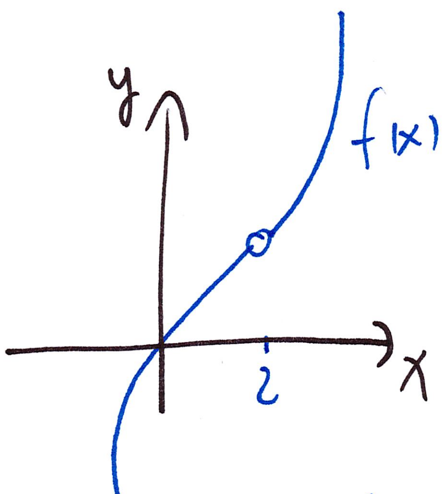
① a is in the domain of f.

② $\lim_{x\to a} f(x)$ exists (it has to be a number, not $\pm\infty$)

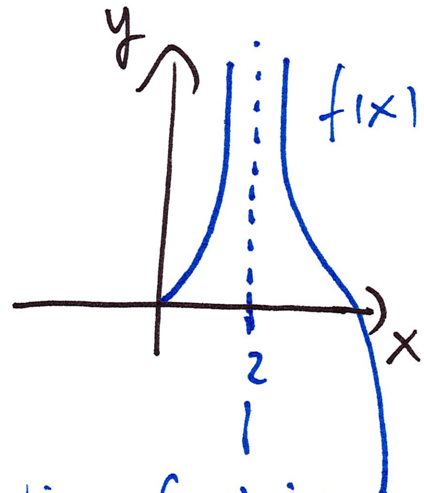
AND

③ $\lim_{x\to a} f(x) = f(a)$ (This means that the substitution property works for $\lim_{x\to a} f(x)$).

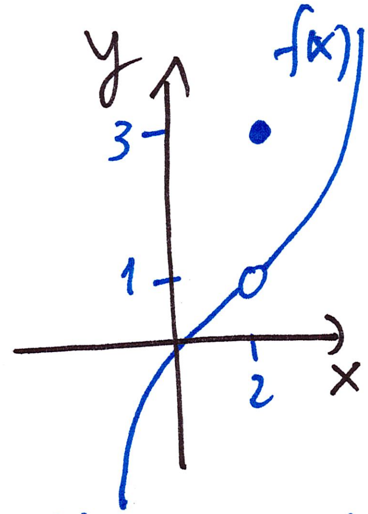
Ex: The following functions are NOT continuous at $a=2$.



f is not defined at 2



$\lim_{x\to 2} f(x)$ is not a number (it's ∞)



$\lim_{x\to 2} f(x)$ exists but $f(2) \neq \lim_{x\to 2} f(x)$

Thm: The following functions are continuous at every point a in their domain:

- polynomials
- rational functions
- root functions ($\sqrt[n]{x}$)
- trig. functions and their inverses ($\sin, \cos, \tan, \sin^{-1}, \cos^{-1}, \dots$)
- exponential and logarithmic functions.

More examples:

Thm: Suppose f, g are continuous at $x=a$.
Then all the following functions are continuous at $x=a$:

- $f+g$
- $f-g$
- cf $(c \in \mathbb{R})$
- $f \cdot g$
- $\frac{f}{g}$ if $g(a) \neq 0$

$\boxed{\text{Ex:}}$ Find all the numbers for which the following function is continuous

$$f(x) = \begin{cases} \sin^{-1}(x) - \frac{\pi}{2}x^2 & \text{if } x \in (0,1) \\ 0 & \text{if } x = 1 \\ \frac{x^2+2x+5}{x^3+1} - \frac{3}{x} & \text{if } x \in (1,+\infty) \end{cases}$$

Sol.: ① If $x \in (0,1)$

$$f(x) = \underbrace{\sin^{-1}(x)}_{\text{continuous at } x \in (0,1)} - \underbrace{\frac{\pi}{2}x^2}_{\text{continuous at } x \in (0,1)}$$

So f is continuous at $x \in (0,1)$ because $f = \text{cont} - \text{cont}$.

② If $x \in (1,+\infty)$,

$$f(x) = \frac{x^2+2x+5}{\underbrace{x^3+1}_{\text{continuous at } x \in (1,+\infty)}} - \underbrace{\frac{3}{x}}_{\text{continuous at } x \in (1,+\infty)}$$

$\Rightarrow$ f is continuous on $(1,+\infty)$.

We need to check if f is continuous at $x=1$.

① Is f defined at $x=1$?

Yes, $f(1)=0$.

② Does $\lim_{x\to 1} f(x)$ exist?

Check $\lim_{x\to 1^-}$ and $\lim_{x\to 1^+}$!

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sin^{-1}(x) - \frac{\pi}{2}x^2 =$$

$$= \sin^{-1}(1) - \frac{\pi}{2}(1)^2 = \frac{\pi}{2} - \frac{\pi}{2} = 0$$

f IS continuous to the left of 1, that is when $x<1$.

$$\boxed{\lim_{x\to 1^+} f(x)} = \lim_{x\to 1^+} \frac{x^2+2x+5}{x^3+1} - \frac{3}{x} =$$

$$= \frac{1^2+2\cdot 1+5}{1^3+1} - \frac{3}{1} = \frac{8}{2} - 3 = 4-3 = \boxed{1}$$

f IS continuous to the right of 1

$$\Rightarrow \lim_{x\to 1} f(x) \text{ DNE.}$$

So, f is NOT continuous at $x=1$.

f is continuous on

$$(0,1) \cup (1,+\infty).$$

<u>Note</u>: $\lim_{x\to 1^-} f(x) = f(1)$

we call functions for which $\lim_{x\to a^-} f(x) = f(a)$ continuous ~~to~~ at the left of a.

— o — o — o —

If f is continuous at $x=a$, then

$$f(a) = \lim_{x\to a} f(x).$$

But $$a = \lim_{x\to a} x$$

So, if f is continuous, we can write

$$f\left(\lim_{x\to a} x\right) = \lim_{x\to a} f(x)$$

More generally:

$\boxed{\text{Thm:}}$ If f is continuous at b and $\lim\limits_{x \to a} g(x) = b$, then

$$f(b) = f\left(\lim_{x \to a} g(x)\right) = \lim_{x \to a} f(g(x))$$

$\boxed{\text{Cor:}}$ (How you use the thm). If g is continuous at a and f is continuous at $b = g(a)$, then

$f \circ g$ is continuous at a

(Recall $(f \circ g)(x) = f(g(x))$)

$\boxed{\text{Ex:}}$ $\lim\limits_{x \to \frac{1}{2}} \sqrt{e^{\sin\left(\frac{\pi}{x}\right)}}$

So!: $h(x) = \sin\left(\frac{\pi}{x}\right)$ is continuous on $\mathbb{R} \setminus \{0\}$, so also at $x = 1/2$
domain of h $\parallel$

$g(y) = e^y$ is continuous on $\mathbb{R}$, so also at $x = 1/2$

$f(z) = \sqrt{z}$ " " on $\mathbb{R}_{\geq 0} = [0, +\infty)$, so at $x = 1/2$

$\sqrt{e^{\sin\left(\frac{\pi}{x}\right)}} = f(g(h(x)))$ is continuous at $\frac{1}{2}$ by the Thm.

So $\lim_{x \to \frac{1}{2}} \sqrt{e^{\sin\left(\frac{\pi}{x}\right)}} =$ ← substitution

$$= \sqrt{e^{\sin\left(\frac{\pi}{\left(\frac{1}{2}\right)}\right)}} = \sqrt{e^{\sin(2\pi)}} = \sqrt{e^0} =$$

$$= \sqrt{1} = 1.$$

There is one nice consequence of continuity which allows you to find solutions to equations....

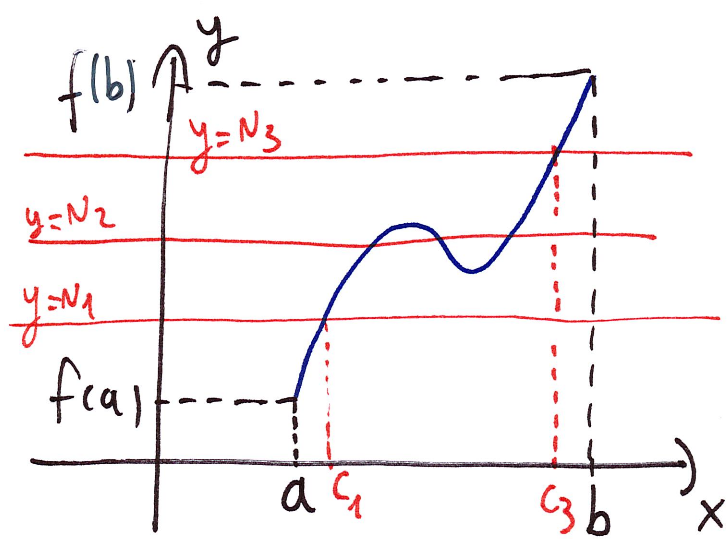
$\boxed{\text{Thm}}$ (Intermediate Value Theorem - IVT).

Suppose that:

1. f is continuous on an interval $[a,b]$ $(a<b)$. This means:
   - (a) $\lim_{x \to a^+} f(x) = f(a)$
   - (b) $\lim_{x \to b^-} f(x) = f(b)$
   - (c) f is continuous at $x=d$, for <u>every</u> $d \in (a,b)$
2. $f(a) \neq f(b)$
3. N is a number with $\begin{cases} f(a) < N < f(b), & \text{if } f(a)<f(b) \\ f(b) < N < f(a), & \text{if } f(b)<f(a) \end{cases}$

THEN there is a number c such that:

- $a < c < b$;
- $f(c) = N$.



<u>Note</u>: N is a number on the y-axis, whereas c is a number on the x-axis.

Pro-tip: Use IVT when a question asks "show that the equation ——— has at least one solution on an interval (a,b)".

Ex: Show that the equation
$$\sqrt[3]{x} = 1 - x^2$$
has a solution in $(0,1)$.

Sol: $\sqrt[3]{x} = 1 - x^2 \iff \sqrt[3]{x} - 1 + x^2 = 0$.
We set $f(x) = \sqrt[3]{x} - 1 + x^2$. The question is asking to ~~find~~ show that there is a $c \in (\underset{a}{0}, \underset{b}{1})$ such that $f(c) = \underset{N}{0}$.

Let's try to apply the IVT. "Applying a theorem" means "check its hypotheses", that is check that ①, ②, ③ of IVT are true for this f.

① f is continuous on $[0,1]$?

$$f = \underbrace{\sqrt[3]{x}}\ -\underbrace{1}\ +\underbrace{x^2}$$

continuous on $[0,1]$, because they are continuous at EVERY $x \in \mathbb{R}$.

$\Rightarrow$ f is continuous on $[0,1]$ because it is cont − cont + cont.

② $f(0) \neq f(1)$?

$f(0) = \sqrt[3]{0} - 1 + 0^2 = -1$ AND $-1 \neq 1$.

$f(1) = \sqrt[3]{1} - 1 + 1 = 1$

③ $N=0$ is a number between $f(0)$ and $f(1)$?

$$-1 = f(0) < 0 < f(1) = 1$$

Victory! By IVT, there is a $c \in (0,1)$ s.t. $f(c) = 0$. ☺

For next class:

- Do the exercises of Section 2.5
- Take a look at section 2.6.