# About the 3rd quiz

- It will cover:
  - Section 2.5 (Continuity)
  - Section 2.6 (Limits at $\infty$)

- The 4th quiz will be on

  Oct 17

  (Oct 8 - Oct 14 is READING WEEK ☺)

# Lecture 7: Limits at infinity & Derivatives. (Oct 1)

(Sections 2.6-2.7)

Limits at infinity are used to study the behaviour of $f(x)$ as $x$ becomes very large or very small.

**Def:** Let $f$ be a function defined on some interval $(b, +\infty)$.

① $$\lim_{x \to \infty} f(x) = L$$

means we can make $f(x) \to L$ by taking $x$ to be large enough.

② $$\lim_{x \to \infty} f(x) = \infty \qquad \left(\text{resp. } \lim_{x \to \infty} f(x) = -\infty\right)$$

means we can make $f(x)$ as large as we want (resp. as small as we want) by taking $x$ large enough.

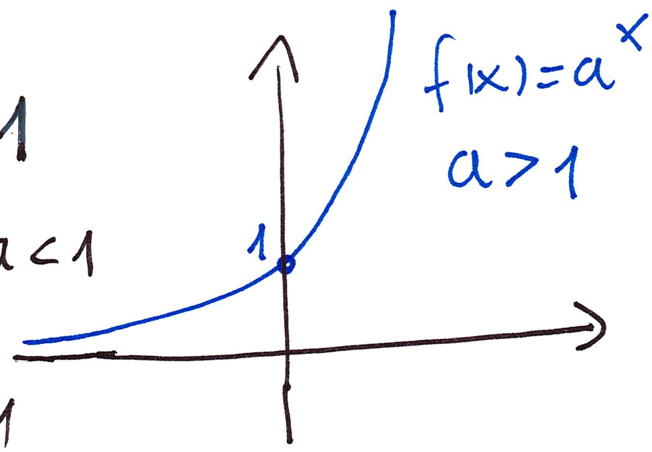
Similarly, we can get the definitions of

$$\lim_{x\to-\infty} f(x) = L \quad \& \quad \lim_{x\to-\infty} f(x) = \pm\infty$$

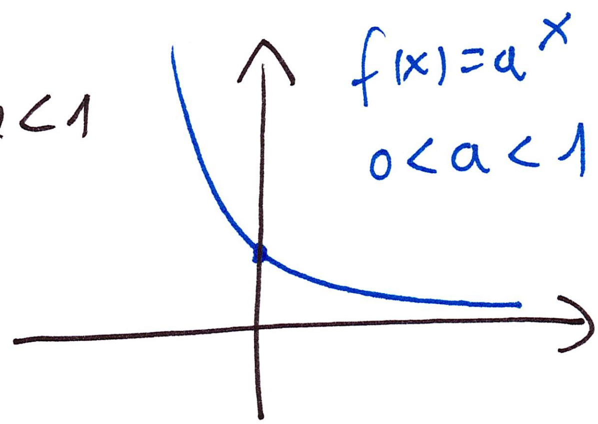
③ IF $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then we call the line $y = L$ a horizontal asymptote for f.

Facts / Examples :

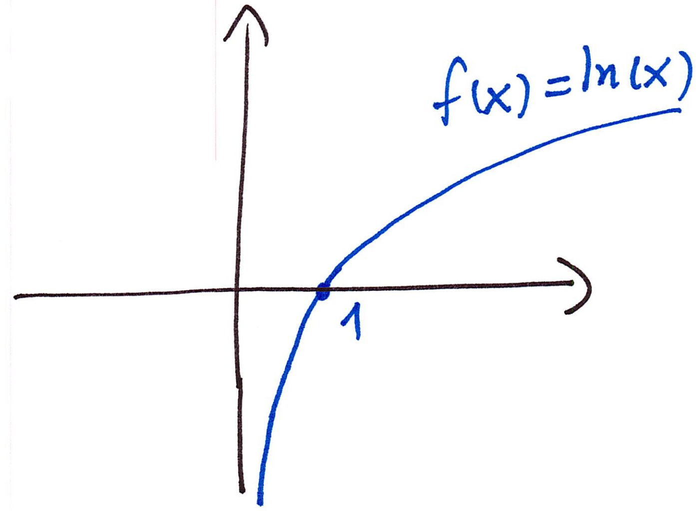
① $$\lim_{x\to\infty} a^x = \begin{cases} \infty & \text{if } a>1 \\ 0 & \text{if } 0<a<1 \end{cases}$$




$$\lim_{x\to-\infty} a^x = \begin{cases} 0 & \text{if } a>1 \\ \infty & \text{if } 0<a<1 \end{cases}$$

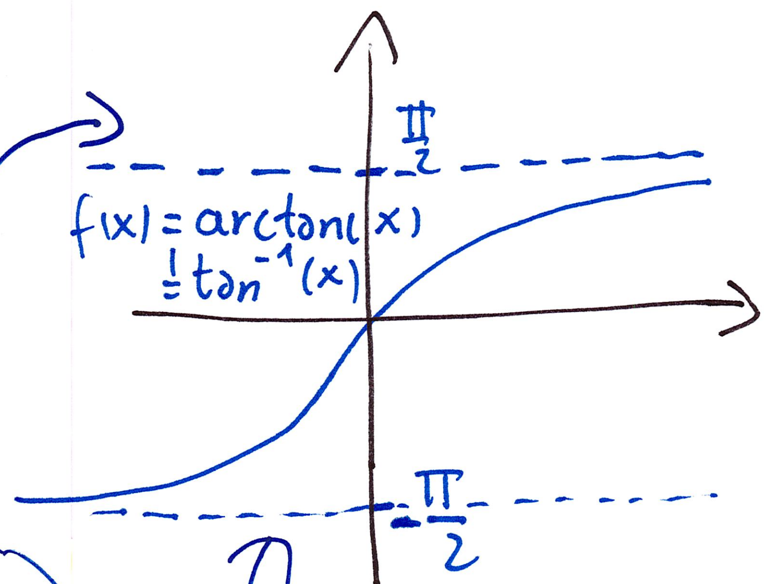


② $\lim_{x \to \infty} \ln(x) = \infty$




③ $\lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}$

$\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}$




So, $y = \frac{\pi}{2}$ & $y = -\frac{\pi}{2}$ are horizontal asymptotes for arctan(x).

4 Recall: $x^{\frac{p}{q}} = \sqrt[q]{x^p}$

$\boxed{\text{Thm:}}$ If $r = \frac{p}{q} > 0$ (important that $r > 0$),

then:

a $\lim\limits_{x \to \infty} \frac{1}{x^r} = 0$

b $\lim\limits_{x \to -\infty} \frac{1}{x^r} = 0$, IF $\frac{1}{x^r}$ is defined when $x < 0$

( $x^{\frac{1}{3}} = \sqrt[3]{x}$ is defined when $x < 0$

$x^{\frac{1}{2}} = \sqrt[2]{x}$ is not defined when $x < 0$ )

$\boxed{\text{Ex:}}$ Find $\lim\limits_{x \to \infty} \frac{x^3 + 3x^2}{-x^4 + 7}$.

Sol: As $x \to \infty$, $x^3 + 3x^2 \longrightarrow \infty$

$-x^4 + 7 \longrightarrow -\infty$

So $f(x) \to \frac{\infty}{-\infty}$ INDETERMINATE FORM

**Pro-tip:** Divide both the numerator and the denominator by the

highest power of x appearing in the denominator

and simplify ...

$$f(x) = \frac{x^3+3x^2}{-x^4+7}$$

Highest power of x in the denominator: $x^4$

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\boxed{\lim_{x\to\infty} \frac{x^3+3x^2}{-x^4+7}} = \lim_{x\to\infty} \frac{\frac{x^3+3x^2}{x^4}}{\frac{-x^4+7}{x^4}} =$$

$$= \lim_{x\to\infty} \frac{\frac{\cancel{x^3}}{\cancel{x^4}} + \frac{3\cancel{x^2}}{\cancel{x^4}2}}{-\frac{x^4}{x^4} + \frac{7}{x^4}} = \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{3}{x^2}}{-1 + \frac{7}{x^4}}$$

$\frac{1}{x} \to 0$, $\frac{3}{x^2} \to 0$, $\frac{7}{x^4} \to 0$ by Thm above

$$= \lim_{x\to\infty} \frac{0}{-1} = 0 .$$

☺

**Note:** We discovered, through the process, that $y=0$ is a horizontal asymptote for $f(x)$.

# Cheat sheet for $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$

DISCLAIMER: You should use what follows just as a sanity check for your computations. When asked to compute $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$ and to show ALL your work, you NEED to go through the process of dividing $P(x)$ and $Q(x)$ by the highest power of $x$ appearing in $Q(x)$ (the denominator).

Fact: Suppose $f(x) = \frac{P(x)}{Q(x)}$ with

$$P(x) = a_n x^n + \dots + a_1 x + a_0$$
$$Q(x) = b_m x^m + \dots + b_1 x + b_0$$

polynomials ($a_n, b_m \neq 0$)

THEN:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{P(x)}{Q(x)} = \begin{cases} \infty & \text{if } n>m \text{ and } a_n, b_m \text{ have the SAME sign} \\ -\infty & \text{if } n>m \text{ and } a_n, b_m \text{ have DIFFERENT signs} \\ \frac{a_n}{b_m} & \text{if } n=m \\ 0 & \text{if } n<m \end{cases}$$

Ex:

① $\lim\limits_{x\to\infty} \dfrac{2x^5+3x^2}{-7x^3+2} = -\infty.$

Indeed: $P(x) = 2x^5+3x^2$ $n=5$, $a_n=2$

$Q(x) = -7x^3+2$ $m=3$, $b_m=-7$

Since $n>m$ & $a_n, b_m$ have different signs ($a_n>0$ and $b_m<0$), $\lim\limits_{x\to\infty} \dfrac{P(x)}{Q(x)} = -\infty.$

② $\lim\limits_{x\to\infty} \dfrac{-3x^3+4}{-2x^2+5x} = \infty$

Indeed: $P(x) = -3x^3+4$ $n=3$, $a_n=-3$

$Q(x) = -2x^2+5x$ $m=2$, $b_m=-2$

Since $n>m$ & $a_n, b_m$ have the same sign ($a_n>0$ & $b_m>0$), $\lim\limits_{x\to\infty} \dfrac{P(x)}{Q(x)} = \infty.$

③ $\lim\limits_{x\to\infty} \dfrac{4x^4+7x^3}{5x^4-3x^2+4} = \dfrac{4}{5}$

Indeed: $P(x) = 4x^4+7x^3$ $n=4$, $a_n=4$

$Q(x) = 5x^4-3x^2+4$ $m=4$, $b_m=5$

Since $n=m$, $\lim\limits_{x\to\infty} \dfrac{P(x)}{Q(x)} = \dfrac{a_n}{b_n} = \dfrac{4}{5}.$

④ $\lim\limits_{x\to\infty} \dfrac{3x^2}{-7x^3+4} = 0.$

Indeed: $P(x) = 3x^2$ $n=2$, $a_n=3$

$Q(x) = -7x^3+4$ $m=3$, $b_m=-7.$

Since $n<m$, $\lim\limits_{x\to\infty} \dfrac{P(x)}{Q(x)} = 0.$

## • Limits by "variable substitution".

**General idea/framework**. Suppose you want to compute

$$\lim_{x\to a^{\pm},\pm\infty} f(g(x))$$

and $f$ is one of the nice functions (exponential, log, $\sqrt[n]{\ }$, sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, ...).

set

$$t = g(x)$$

and suppose you know that

$$\text{as } x \to a^{\pm}, \pm\infty, \quad t \to L^{\pm}, \pm\infty$$

Then

$$\lim_{x\to a^{\pm},\pm\infty} f(g(x)) = \lim_{t\to L^{\pm},\pm\infty} f(t)$$

(if this last limit on the right exists)

$\boxed{\text{Ex:}}$ Find $\displaystyle\lim_{x\to \frac{\pi}{2}^+} \ln\left(\sin\left(x-\frac{\pi}{2}\right)\right)$.

Sol: Set $t \overset{s}{=} \sin\left(x-\frac{\pi}{2}\right)$.

$$\lim_{x\to\frac{\pi}{2}^+} t = \lim_{x\to\frac{\pi}{2}^+} \sin\left(x-\frac{\pi}{2}\right) \overset{\leftarrow}{=}$$

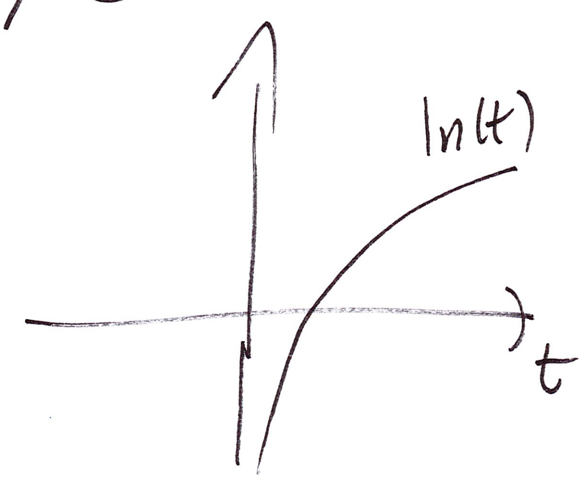
sin is continuous at $\frac{\pi}{2}$

$$= \sin\left(\frac{\pi}{2}^+ - \frac{\pi}{2}\right) = \sin(0^+) = 0^+$$

↑ This means that $\sin\left(x-\frac{\pi}{2}\right)$ is getting to 0 while being $>0$, as $x\to\frac{\pi}{2}^+$.

Then: $\displaystyle\lim_{\boxed{x\to\frac{\pi}{2}^+}} \ln\left(\sin\left(x-\frac{\pi}{2}\right)\right) =$

$$= \lim_{\boxed{t\to 0^+}} \ln(t) = -\infty$$



$\boxed{\text{Ex:}}$ Find $\lim_{x \to -\infty} \arctan\left(\frac{x^2+1}{x}\right)$

Sol: Set $t \overset{s}{\equiv} \frac{x^2+1}{x}$.

$$\lim_{x \to -\infty} t = \lim_{x \to -\infty} \frac{x^2+1}{x} = \lim_{x \to -\infty} \frac{\frac{x^2+1}{x}}{\frac{x}{x}} =$$

$$= \lim_{x \to -\infty} \frac{\frac{x^{\cancel{2}}}{\cancel{x}} + \frac{1}{x}}{1} = \lim_{x \to -\infty} \cancel{x} + \frac{1}{x} \xrightarrow{} 0$$

as $x \to -\infty$ by Ex 4

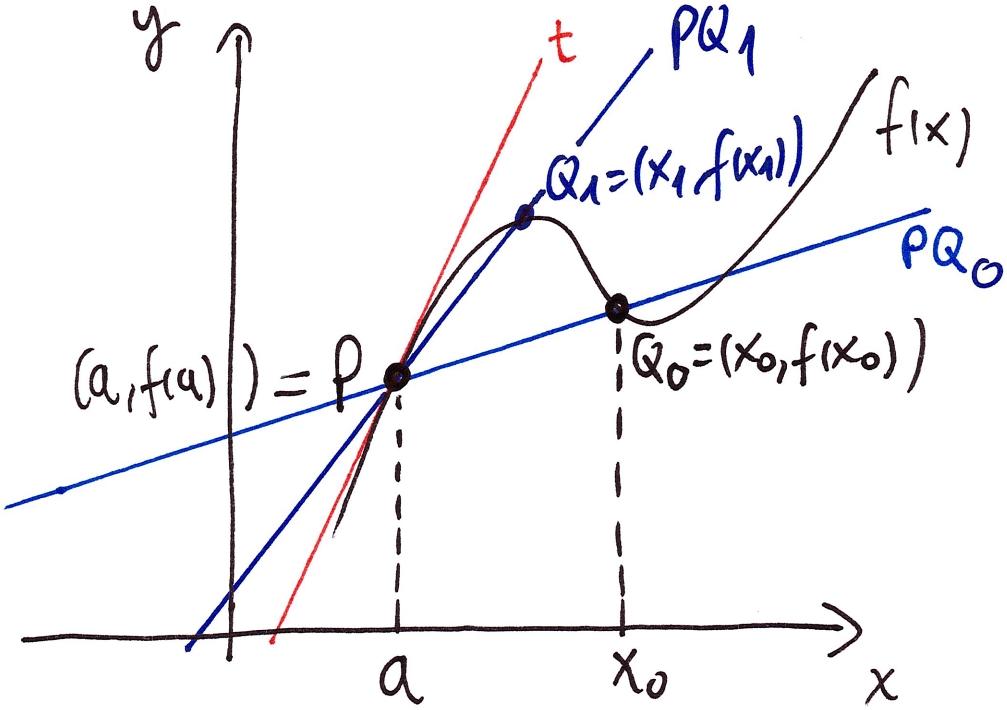
$$= \lim_{x \to -\infty} x = -\infty.$$

So, $\lim_{x \to -\infty} \arctan\left(\frac{x^2+1}{x}\right) = \lim_{t \to -\infty} \arctan(t) = -\frac{\pi}{2}$

from the graph

(Look at page 132 for more examples.
Ex 35, 40, 41 in Section 2.6 can also be done in this way)

# • Tangent lines & derivatives (Section 2.7)




Consider a curve C which is the graph of a function f(x).

$P=(a,f(a))$, a point on the graph

We want to find:

$t$ = tangent line at P with slope m

Let $Q_0=(x_0,f(x_0))$ be another point on the graph of f(x) with $x_0 \neq a$.
Let $PQ_0$ the line that passes through P and $Q_0$. The slope of $PQ_0$ is

$$m_{PQ_0} = \frac{y_{Q_0}-y_P}{x_{Q_0}-x_P} = \frac{f(x_0)-f(a)}{x_0-a}$$

$m_{PQ_0}$ is not that great of an approximation for $m$.

Idea: Take $Q_1 = (x_1, f(x_1))$ "closer to $P$" (but not equal to $P$).

$$m_{PQ_1} = \frac{y_{Q_1} - y_P}{x_{Q_1} - x_P} = \frac{f(x_1) - f(a)}{x_1 - a}$$

This is a better approx, but not quite what we wanted...

We want: "Predict" the value of $m$ based on the various approximations we can do of $m$ by taking points $Q_0, Q_1, Q_2, \ldots$ closer and closer to $P$ and computing $m_{PQ_0}, m_{PQ_1}, m_{PQ_2}, \ldots$

Limits are predictions' best friends!

$\boxed{\text{Def:}}$ The tangent line to the curve $y=f(x)$ at $P=(a,f(a))$ is the line with equation

$$y-f(a)=m(x-a)$$

where

$$m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

is the slope of the tangent line.

Note: We can take

$$x=a+h \qquad \text{with } h\to 0$$

Then $\lim_{h\to 0} x = \lim_{h\to 0} a+h = a$

Then:

subst. $x=a+h$

$$m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

We write

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

and call it the derivative of $f$ at $x=a$

So the tangent line to $y=f(x)$ at $P=(a,f(a))$ is

$$y-f(a)=f'(a)(x-a)$$

$\boxed{\text{Ex:}}$ Consider $f(x)=\ln(x)+2$

1. Using the def. of derivative, find $f'(1)$.

2. Find the tangent line to $y=f(x)$ at the point $P=(1,2)$.

Sol: By def:

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$\stackrel{!}{=}\lim_{h\to 0}\frac{\ln(1+h)+2-(\ln(1)+2)}{h}$$

$\ln(1)=0$

$$\stackrel{!}{=}\lim_{h\to 0}\frac{\ln(1+h)+\cancel{2}-0-\cancel{2}}{h}$$

$$= \lim_{h\to 0} \frac{\ln(1+h)}{h} = \lim_{h\to 0} \frac{1}{h} \ln(1+h)$$

$$= \lim_{h\to 0} \ln\left((1+h)^{\frac{1}{h}}\right)$$

Property of logarithm

Fact: $\lim_{h\to 0} (1+h)^{\frac{1}{h}} = e = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

Then, set $t = (1+h)^{\frac{1}{h}}$:

$t \to e$ as $h \to 0$. $\left(\lim_{h\to 0} t = e\right)$

So $\lim_{h\to 0} \ln\left((1+h)^{\frac{1}{h}}\right) = \lim_{t\to e} \ln(t)$

$$= \ln(e) \left(= \ln(e^1)\right) = \boxed{1 = f'(1)}$$

ln is continuous at $t=e$

(2) The eq. for the tangent line at $P=(1,2)$ is

$$y - f(1) = f'(1)(x-1)$$

But: $f(1) = 2$ , $f'(1) = 1$

So, the needed equation is

$$y - 2 = 1(x-1)$$

$$\boxed{y} = x - 1 + 2 \boxed{= x + 1}$$

For next class:

- Do the exercises of Section 2.6
- Take a look at Section 2.8.