News from the world of Calculus

- Handouts on: (a) continuity & asymptotes
  (b) Important limits

  available on "OWL" ---> "Resources".

- Next week's office hours:

  Tuesday, Oct 9, 2.30-4.00 pm
  in MC 104

- Review session for the midterm:

  Tuesday, Oct 23, 2.30-4.00 pm
  in MC 204.

Lecture 8: More on Derivatives (Oct 3)

(Sections 2.8, 3.1, 3.2)

Recall:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If we let $x = a$ vary, we get the derivative as a function of x:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Sometimes people write:

- $f'(x) = \frac{dy}{dx}(x)$ if $y = f(x)$
- $f''(x) = (f'(x))'$ (This is the derivative of the derivative of f)

  $f'''(x) = (f''(x))'$

  $f^{IV}(x)$ ....

Def: We say that f is <u>differentiable at $x=a$</u> if $f'(a)$ exists (that is, a is in the domain of the function $f'(x)$).

Fact: IF f is differentiable at $x=a$, THEN f is continuous at $x=a$.

(Equivalently, if f is NOT continuous at $x=a$, then f is NOT differentiable at $x=a$)

Ex: Consider

$$f(x) = \sqrt{3x+4}$$

Find a formula for $f'(x)$ and all the points $x=a$ at which f is differentiable.

Sol:

$$f'(x) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3(x+h)+4} - \sqrt{3x+4}}{h}$$

$$\underset{\text{Rationalize}}{=} \lim_{h\to 0} \frac{\sqrt{3(x+h)+4}-\sqrt{3x+4}}{h} \cdot \frac{\sqrt{3(x+h)+4}+\sqrt{3x+4}}{\sqrt{3(x+h)+4}+\sqrt{3x+4}}$$

$$= \lim_{h\to 0} \frac{3(x+h)+4-(3x+4)}{h\left(\sqrt{3(x+h)+4}+\sqrt{3x+4}\right)}$$

$$3(x+h)+4-(3x+4) = \cancel{3x}+3h+\cancel{4}\cancel{-3x}\cancel{-4}$$

$(a-b)(a+b)=a^2-b^2$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}\left(\sqrt{3(x+h)+4}+\sqrt{3x+4}\right)}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)+4}+\sqrt{3x+4}} =$$

$$\lim_{h\to 0} \sqrt{3(x+h)+4} = \sqrt{3x+4}$$

$$= \frac{3}{\sqrt{3x+4}+\sqrt{3x+4}} = \boxed{\frac{3}{2\sqrt{3x+4}}} = f'(x)$$

f is differentiable at $x=a \iff a$ is in the domain of $f'(x)$

Domain of $f'(x)$: $\left(\sqrt{3x+4} \neq 0\right)$

$$3x+4>0 \iff x>-\frac{4}{3}$$

Domain of $f'(x)$ is $\left(-\frac{4}{3}, +\infty\right)$, so

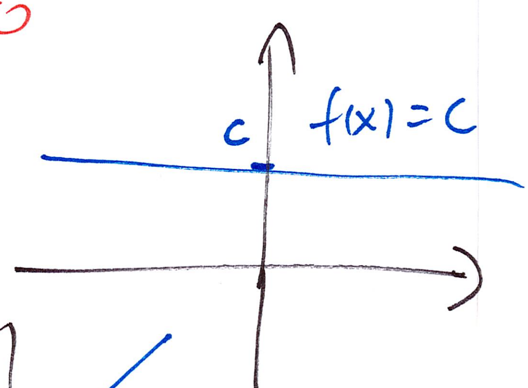
f is differentiable at every $a \in \left(-\frac{4}{3}, +\infty\right)$

- <u>Computing derivatives</u> (Section 3.1 & 3.2)

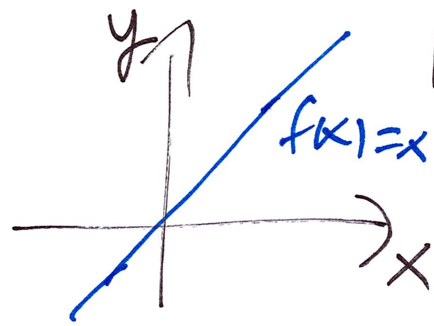
$\boxed{\text{Basic derivatives:}}$

- If $c \in \mathbb{R}$, $\frac{d}{dx}(c) = 0$

  $\left(\overset{\shortparallel}{(c)'} \right)$



- $\frac{d}{dx}(x) = 1$



- For any <u>REAL</u> number $r \in \mathbb{R}$,

  $\frac{d}{dx}(x^r) = r(x^{r-1})$

$\boxed{\text{Ex:}}$ If $f(x) = \sqrt[5]{x^2} = x^{\frac{2}{5}}$

$f'(x) = \left(x^{\frac{2}{5}}\right)' = \frac{2}{5} x^{\frac{2}{5}-1} = \frac{2}{5} x^{-\frac{3}{5}} =$

$-\frac{3}{5} = (-1)\cdot\frac{3}{5}$

$$= \frac{2}{5} \cdot \left(x^{-1}\right)^{\frac{3}{5}} = \frac{2}{5} \cdot \left(\frac{1}{x}\right)^{3/5} = \frac{2}{5\sqrt[5]{x^3}}$$

$$x^{a \cdot b} = \left(x^{a}\right)^{b}$$

- If $a > 0$, $\frac{d}{dx}(a^x) = a^x \ln(a)$

In particular, $\frac{d}{dx}(e^x) = e^x$.

- Derivative rules

① $(c\,f(x))' = c\,f'(x)$ ($c \in \mathbb{R}$ constant)

② $(f(x) \pm g(x))' = f'(x) \pm g'(x)$

③ (Product rule)

$$(f(x)\,g(x))' = f'(x)\,g(x) + f(x)\,g'(x)$$

④ (Quotient rule)

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

$\boxed{\text{Ex:}}$ for

$$h(x) = \frac{\sqrt{x} - 7}{\sqrt{x} + 7}$$

find $h'(x)$.

Sol: $h'(x) = \left(\frac{\overbrace{\sqrt{x} - 7}^{= f(x)}}{\underbrace{\sqrt{x} + 7}_{= g(x)}}\right)' \overset{④}{=}$

$$= \frac{(\sqrt{x}-7)'(\sqrt{x}+7) - (\sqrt{x}-7)(\sqrt{x}+7)'}{(\sqrt{x}+7)^2} =$$

$$\overset{②}{=} \frac{((\sqrt{x})' - (7)')(\sqrt{x}+7) - (\sqrt{x}-7)((\sqrt{x})' + (7)')}{(\sqrt{x}+7)^2}$$

$$= \frac{\frac{1}{2\sqrt{x}}(\sqrt{x}+7) - (\sqrt{x}-7)\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x}+7)^2} =$$

$$(\sqrt{x})' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$(7)' = 0$$

$$= \frac{\frac{1}{2\sqrt{x}}(\cancel{\sqrt{x}}+7-\cancel{\sqrt{x}}+7)}{(\sqrt{x}+7)^2} =$$

$$= \frac{\frac{\cancel{14}^{\,7}}{\cancel{2}\sqrt{x}}}{(\sqrt{x}+7)^2} = \boxed{\frac{7}{\sqrt{x}(\sqrt{x}+7)^2}}$$

$$\| $$

$$\frac{7\sqrt{x}}{x(\sqrt{x}+7)^2}$$

:)

• Derivatives of trig. functions (Section 3.3)

We have:

$$\boxed{\begin{aligned} \frac{d}{dx}(\sin(x)) &= \cos(x) \\ \frac{d}{dx}(\cos(x)) &= -\sin(x) \end{aligned}}$$

Proof that $\frac{d}{dx}(\cos(x)) = -\sin(x)$:

$$(\cos(x))' \overset{def}{=} \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

Addition formula for cos

$$\overset{\text{Addition formula for cos}}{=} \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} =$$

$$\overset{\text{shuffle things}}{=} \lim_{h\to 0} \left(\frac{\cos(x)\cos(h) - \cos(x)}{h} - \frac{\sin(x)\sin(h)}{h} \right)$$

$$= \lim_{h\to 0} \cos(x)\left(\frac{\cos(h)-1}{h}\right) - \lim_{h\to 0} \frac{\sin(x)\sin(h)}{h}$$

$$= \underbrace{\left(\lim_{h\to 0} \cos(x)\right)}_{= \cos(x)} \cdot \underbrace{\left(\lim_{h\to 0} \frac{\cos(h)-1}{h}\right)}_{= 0 \text{ Fact}} - \underbrace{\left(\lim_{h\to 0} \sin(x)\right)}_{= \sin(x)} \underbrace{\left(\lim_{h\to 0} \frac{\sin(h)}{h}\right)}_{= 1 \text{ Fact}}$$

$$= -\sin(x).$$

We used:

($\heartsuit$) $\boxed{\lim_{h\to 0} \frac{\sin(h)}{h} = 1}$ $\qquad$ $\boxed{\lim_{h\to 0} \frac{\cos(h)-1}{h}}$ ($\diamond$)

We can use ($\heartsuit$) & ($\diamond$) to compute some fancy limits.

Ex: Find $\displaystyle\lim_{x\to 0} \frac{\sin(4x)}{\sin(3x)}$

Sol: <u>Trick</u>: find a way to use ($\heartsuit$)

$$\boxed{\lim_{x\to 0} \frac{\sin(4x)}{\sin(3x)}} = \lim_{x\to 0} \frac{\sin(4x)}{4x} \cdot \frac{4x}{\sin(3x)} =$$

$$= \lim_{x\to 0} \frac{\sin(4x)}{4x} \cdot \frac{4x}{3x} \cdot \frac{3x}{\sin(3x)} \quad (*)$$

If $t = 4x$, then $\lim_{x\to 0} t = \lim_{x\to 0} 4x = 0$,

So $\lim_{x\to 0} \frac{\sin(4x)}{4x} = \lim_{t\to 0} \frac{\sin(t)}{t} = 1$

Similarly, $\lim_{x\to 0} \frac{3x}{\sin(3x)} \underset{u=3x}{=} \lim_{u\to 0} \frac{u}{\sin(u)} = 1$

(If $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$, then, if $f(h) = \frac{\sin(h)}{h}$,

$$\lim_{h\to 0} \frac{1}{f(h)} = \lim_{h\to 0} \frac{h}{\sin(h)}$$

$$\overset{\text{Because } \lim_{h\to 0} f(h) \text{ exists}}{=} \frac{1}{\lim_{h\to 0} f(h)} = \frac{1}{1} = 1 \Bigg).$$

$$(*) = \lim_{x\to 0} \frac{4\cancel{x}}{3\cancel{x}} = \lim_{x\to 0} \frac{4}{3} = \boxed{\frac{4}{3}}$$

Exercise: Find $\lim_{x\to 0} \frac{\sin(6x)}{\sin(3x)}$ WITHOUT using $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$.

For next class: (Oct 15)

- Enjoy your reading week
- Do the exercises of section 2.8
- Do as many suggested exercises of sections 3.1 & 3.2 as you feel are needed for you to be comfortable in using the product & the quotient rule.
- Enjoy your reading week.
- If you want a break from your reading week's enjoyment, I have office hours Tuesday Oct 9, 2.30 - 4 pm in MC 104.
  Regular office hours will also be on Oct 15 15-6.30 pm in MC104).