Lecture 9: More derivatives. Chain Rule.

Recall:

$$(\sin(x))' = \cos(x) \qquad (\cos(x))' = -\sin(x)$$

What about $(\tan(x))'$?

Quotient rule

$$(\tan(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)'$$

$$= \frac{(\sin(x))'\cos(x) - \sin(x)(\cos(x))'}{\cos^2(x)}$$

$$= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} =$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

$$= \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} = 1 + \tan^2(x)$$

$\boxed{\text{Ex:}}$ Let $\quad f(x) = \dfrac{\sin^2(x)+1}{\cos(x)}$

① Find $f'(x)$.

<u>Note</u>: $f(x) = \dfrac{\sin^2(x)}{\cos(x)} + \dfrac{1}{\cos(x)}$

$\overset{!}{=} \sin(x) \cdot \left(\dfrac{\sin(x)}{\cos(x)} \right) + \dfrac{1}{\cos(x)}$

$\overset{!}{=} \sin(x) \cdot \tan(x) + \dfrac{1}{\cos(x)}$

$f'(x) = \left(\sin(x)\tan(x) + \dfrac{1}{\cos(x)} \right)' =$

$= \left(\sin(x)\tan(x) \right)' + \left(\dfrac{1}{\cos(x)} \right)' =$

$= \sin'(x)\tan(x) + \sin(x)\tan'(x) + \dfrac{\cancelto{0}{(1)'\cos(x)} - 1 \cdot \cos(x)'}{\cos^2(x)}$

$\overset{!}{=} \cos(x)\tan(x) + \sin(x)\sec^2(x) + \dfrac{\sin(x)}{\cos^2(x)} = \sin(x) \cdot \dfrac{1}{\cos^2(x)}$

$\overset{!}{=} \cancel{\cos(x)} \dfrac{\sin(x)}{\cancel{\cos(x)}} + \sin(x)\sec^2(x) + \sin(x)\sec^2(x)$

$= \sin(x)\left(1 + 2\sec^2(x)\right)$

② Find the values of x for which the graph of f has a horizontal tangent line at $(x, f(x))$.

Sol: "Horizontal tangent line at $(x, f(x))$"

$$\Updownarrow$$

"$f'(x) = 0$"

We need to solve

$$f'(x) = \sin(x)\left(1 + 2\sec^2(x)\right) = 0$$

This happens if:

ⓐ $\sin(x) = 0$: $x = 0, \pi, 2\pi, -\pi, -277\pi$
$= k\pi, \quad k \in \mathbb{Z}$.
$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

ⓑ $1 + 2\sec^2(x) = 0 \iff \underbrace{\sec^2(x)}_{\geq 0} = -\frac{1}{2}$ IMPOSS.
$(\sec^2(x) = (\sec(x))^2)$

The x's we want are $k\pi$, $k \in \mathbb{Z}$.

$\boxed{\text{Ex:}}$ Find the 17^{th} derivative of

$$f(x) = e^x \sin(x)$$

Sol: **DEFINITELY** DO NOT compute 17 derivatives.

Pro-tip: Compute higher derivatives until you see a "pattern" or a "cycle" and then use the properties of derivatives to find $f^{(17)}(x)$.

$$f'(x) = (e^x)'\sin(x) + (e^x)\sin'(x) =$$
$$= e^x \sin(x) + e^x \cos(x)$$
$$= e^x(\sin(x) + \cos(x))$$

$$f''(x) = (f'(x))' = (e^x(\sin(x)+\cos(x)))' =$$ Product rule

$$= e^x(\sin(x)+\cos(x)) + e^x(\cos(x) - \sin(x))$$
$$= e^x(\cancel{\sin(x)} + \cos(x) + \cos(x) - \cancel{\sin(x)})$$
$$= 2e^x\cos(x)$$

$$f'''(x) = \left(2e^x \cos(x)\right)'$$
$$= 2\left(e^x \cos(x) + e^x(-\sin(x)\right)$$
$$\overset{!}{=} 2e^x\left(\cos(x) - \sin(x)\right)$$

$$f^{(4)}(x) = \left(2e^x\left(\cos(x) - \sin(x)\right)\right)' =$$
$$= 2\left(e^x\left(\cos(x) - \sin(x)\right) + e^x\left(-\sin(x) - \cos(x)\right)\right)$$
$$= 2e^x\left(\cancel{\cos(x)} - \sin(x) - \sin(x) - \cancel{\cos(x)}\right)$$
$$= -4e^x \sin(x) = -4f(x)$$

This gives us a "cycle":

$$f^{(5)}(x) = \left(f^{(4)}(x)\right)' = \left(-4f(x)\right)' = -4f'(x)$$
$$f^{(6)}(x) = \left(-4f'(x)\right)' = -4f''(x)$$
$$f^{(7)}(x) = \left(-4f''(x)\right)' = -4f'''(x)$$
$$f^{(8)}(x) = \left(-4f'''(x)\right)' = -4f^{(4)}(x) =$$
$$= (-4)\left(-4f(x)\right) = 16 f(x)$$

$8 = 4 \cdot 2 \longleftrightarrow 16 = 4^2$

$$f^{(16)}(x) = (-4)(-4)(-4)(-4)\,f(x)$$

$16 = 4 \cdot 4$

$$= 256\,f(x)$$

$$\boxed{f^{(17)}(x)} = 256\,f'(x) = \boxed{256\left(e^{x}\left(\sin(x)+\cos(x)\right)\right)}$$

• The (dreaded) Chain rule

Let's fix this

<u>Recall</u>: $(f \circ g)(x) = f(g(x))$

The chain rule tells you how to compute the derivative of a composite function $f \circ g$.

<u>Chain Rule</u>: Suppose g is differentiable at x (that is, $g'(x)$ exists) and f is differentiable at $g(x)$. Then $f \circ g$ is differentiable at x and

$$(f(g(x)))' = (f \circ g(x))' = \underbrace{f'}_{\text{derivative of the outer function}} \underbrace{(g(x))}_{\text{evaluated at the inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of the inner function}}$$

$\boxed{\text{Ex:}}$ Derive $F(x) = \sin(\cos(x))$

Sol: $F(x) = \sin(\cos(x)) = f(g(x))$

where: $f(t) = \sin(t)$ $\qquad f'(t) = \cos(t)$

$\qquad\quad g(x) = \cos(x)$ $\qquad g'(x) = -\sin(x)$

$$x \overset{g}{\longmapsto} \cos(x) \overset{f}{\longmapsto} \sin(\cos(x)) = F(x)$$

$$(\sin(\cos(x)))' = \underbrace{\sin'(\cos(x))}_{f'(g(x))} \cdot \underbrace{(\cos(x))'}_{g'(x)}$$

$\sin'(t) = \cos(t) \rightarrow$

$t = \cos(x)$

$$= \cos(\cos(x)) \cdot (-\sin(x))$$

$$= -\sin(x) \cdot \cos(\cos(x))$$

- Another approach to the chain rule

Recall: We also write

$$\frac{d}{dx}(f(x))$$

to mean $f'(x)$.

Let's compute $h'(x) = \frac{d}{dx}(h(x))$, where

$$h(x) = (\sin(x))^2.$$

We have: $h(x) = f(g(x))$

with $f(x) = x^2$

$g(x) = \sin(x)$

Now:

$$\frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dt}(t^2) = 2t$$

$$\frac{d}{d(\sin(x))}((\sin(x))^2) = 2\sin(x)$$

Chain rule:

$$\frac{d}{dx}\left((\sin(x))^2\right) = \frac{d}{d(\sin(x))}\left((\sin(x))^2\right) \cdot \frac{d}{dx}(\sin(x))$$

$$= \frac{d((\sin(x))^2)}{d(\sin(x))} \cdot \frac{d(\sin(x))}{dx}$$

$$= 2\sin(x) \cdot \cos(x)$$

$$= h'(x)$$

In general

Chain rule:

$$\frac{d(f(g(x)))}{dx} = \underbrace{\frac{d(f(g(x)))}{dg(x)}}_{\text{derivative of the outer function with respect to the inner function}} \cdot \underbrace{\frac{dg(x)}{dx}}_{\text{derivative of the inner funct. with respect to x.}}$$

$\boxed{\text{Ex:}}$ Derive

$$F(x) = (x^7 + e^x)^8$$

<u>Sol:</u> $F(x) = f(g(x))$

$$f(t) = t^8 \qquad x \xrightarrow{g} x^7 + e^x \xrightarrow{f} F(x)$$

$$g(x) = x^7 + e^x$$

$$f'(t) = 8t^7$$

$$g'(x) = 7x^6 + e^x$$

$$(F(x))' = ((x^7 + e^x)^8)'$$

$$\overset{!}{=} f'(g(x)) \cdot g'(x)$$

$(t^8)' = 8t^7$
$t = x^7 + e^x$

$$\overset{!}{=} 8(x^7 + e^x)^7 \cdot (7x^6 + e^x)$$

In general, for n any number

$$\left(g(x)^n\right)' = n\left(g(x)\right)^{n-1} \cdot g'(x)$$

$\boxed{\text{Ex:}}$ If $F = f\left(h(x)^2\right)$ and $h(1) = 2$, $h'(0) = 7$, $h'(1) = -1$, $f(1)=3$, $f'(4) = 3$, find $F'(1)$.

Sol: Use the Chain rule.

$$F'(x) = f'\left(\underbrace{h(x)^2}_{g(x)}\right) \cdot \left(h(x)^2\right)'$$

Chain rule $f(g(x))' = f'(g(x)) \cdot g'(x)$ with $f(t) = t^2$, $g(x) = h(x)$

$$= f'\left(h(x)^2\right) \cdot 2\, h(x)\, h'(x)$$

Now, take $x = 1$:

$$F'(1) = f'\left(h(1)^2\right) \cdot 2\, h(1)\, h'(1)$$

$$= f'\left(2^2\right) \cdot 2 \cdot 2 \cdot (-1)$$

$= f'(4)\cdot(-4) = 3\cdot(-4) = -12$

$\boxed{\text{Ex:}}$ Compute $F'(x)$ if

$$F(x) = \frac{\sin\left(e^{\sqrt{3x^2+1}}\right)}{\cos(x)}$$

For next class:

- Review sections 2.7–3.3 for the quiz (Check "Fourth Quiz info" on OWL - Announcements)
- Complete the Midterm feedback questionnaire at feedback.uwo.ca
- Do the exercises of Section 3.4.
- We will do Section 3.5.