A few things

- Please, complete the feedback form at feedback.uwo.ca
- Please, check the announcement "Midterm rooms: Important Information".
- No more quizzes until Nov, 7
- I'll be out of town Fri - Sun.

Lecture 10: Implicit Differentiation

(Oct 17, Section 3.5)

It is an APPLICATION OF THE CHAIN RULE.

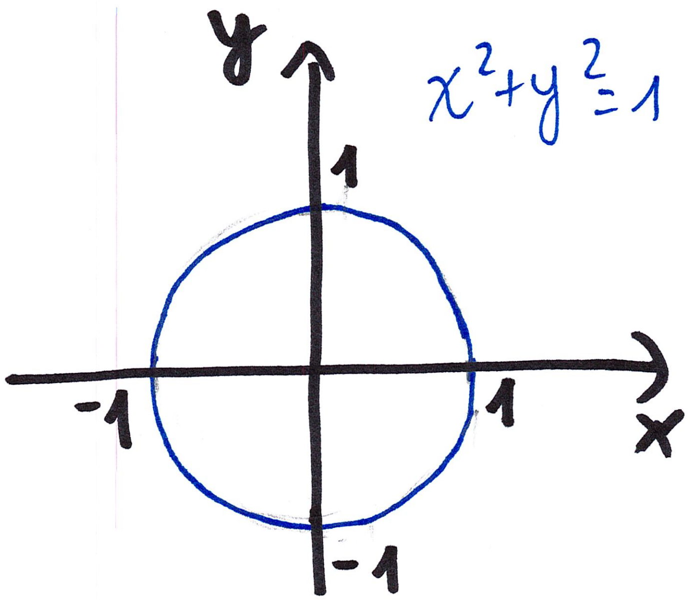
Q: Do we have a tool to find the tangent line to a curve at a point?

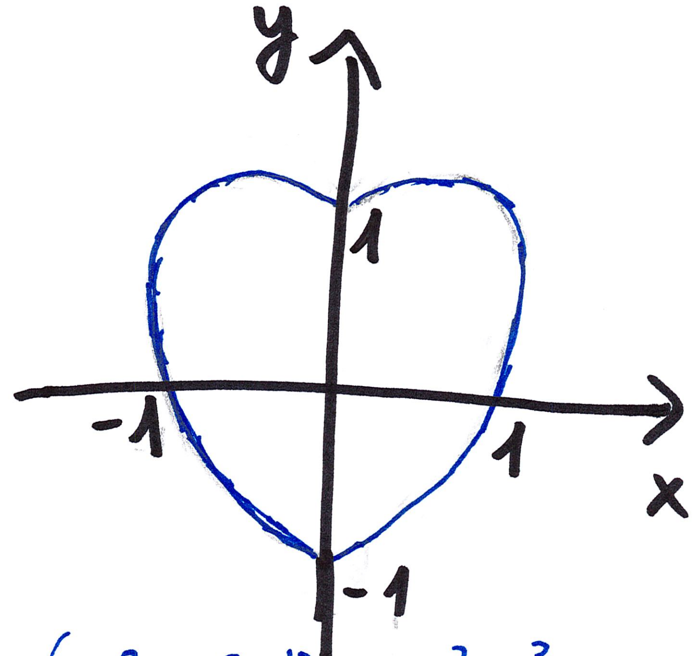
A: Yes, derivatives!

Q: Can we use derivatives to find the tangent line to ANY curve at a point?

A: Not quite. For example, we need the curve to be the graph of a function.

Here are two curves





$(x^2+y^2-1)^3 - x^2y^3 = 0$

Neither of these curves is the graph of a function and for the heart, it is hard to write y in terms of x.

Implicit differentiation helps you to find the derivative of a function $y(x)$ even when we donot have a formula for y in terms of x.

We have

$$(*) \quad (x^2+y^2-1)^3 - x^2y^3 = 0$$

We think of y as a function of x ($y = y(x)$) and apply $(-)'$ on both sides of $(*)$:

$$\left((x^2+y^2-1)^3 - x^2y^3\right)' = 0'$$

$$\left((x^2+y^2-1)^3\right)' - (x^2y^3)' = 0$$

① $\bullet$ $\left((x^2+y^2-1)^3\right)' \overset{\text{Chain rule}}{=}$: $f(t) = t^3$ $(t = x^2+y^2-1)$ $f'(t) = 3t^2$

$$= 3(x^2+y^2-1)^2 \cdot (x^2+y^2-1)'$$

$$\overset{CR}{=} 3(x^2+y^2-1)^2 \cdot (\underbrace{2x}_{(x^2)'} + \underbrace{2yy'}_{((y(x))^2)'})$$

② • $(x^2y^3)' \underset{\text{rule}}{\overset{\text{product}}{=}} (x^2)'y^3 + (x^2)(y^3)'$

$= 2xy^3 + x^2\,3y^2y'$

$\uparrow\ (y^3)' = (y(x)^3)' \overset{CR}{=} 3\,y(x)^2 \cdot y'(x)$

From $\left((x^2+y^2-1)^3\right)' - (x^2y^3)' = 0$

we get

$3(x^2+y^2-1)^2 \cdot (2x+2yy') - (2xy^3 + 3x^2y^2y') = 0$

You now want to factor y' and write it in terms of x and y.

$6x(x^2+y^2-1)^2 + \underline{6yy'(x^2+y^2-1)^2} - 2xy^3 -$

$- \underline{3x^2y^2y'} = 0$

$6yy'(x^2+y^2-1)^2 - 3x^2y^2y' = 2xy^3 - 6x(x^2+y^2-1)^2$

$y'(6y(x^2+y^2-1)^2 - 3x^2y^2) = 2xy^3 - 6x(x^2+y^2-1)^2$

$$\boxed{y' = \frac{2xy^3 - 6x(x^2+y^2-1)^2}{6y(x^2+y^2-1)^2 - 3x^2y^2}} \quad (**)$$

Note: implicit differentiation gives you y' in terms of both x AND y.

Q: What is the tangent line at the point $P=(1,1)$?

The general formula is:

$$y - y_P = \underbrace{m}_{=f'(x_P)}(x - x_P)$$

$x_P = 1$, $y_P = 1$, $m = y'(1)$

Find $y'(1)$ by plugging in $x=1$, $y=1$ in $(**)$

$y'(1) = 2 \cdot 1 \cdot 1 - 6 \cdot 1$

$\boxed{\text{Ex:}}$ If $xe^{(x^2+y^2)} = e^2$,

find y' and y'' at the point $P = (1, 1)$.

Sol: Use impl. diff. to find y':

$$\left(xe^{(x^2+y^2)}\right)' = (e^2)'$$

Product rule

$$1 \cdot \left(e^{(x^2+y^2)}\right) + x\left(e^{(x^2+y^2)}\right)' = 0$$

Chain rule

$$e^{x^2+y^2} + x\left(e^{x^2+y^2} \cdot (2x + 2yy')\right) = 0$$

Isolate y':

$$2x^2\left(\cancel{e^{(x^2+y^2)}}\right) + 2xyy'\left(\cancel{e^{(x^2+y^2)}}\right) = -\cancel{e^{(x^2+y^2)}}$$

$$2xyy' = -1 - 2x^2$$

$$\boxed{y' = -\frac{1+2x^2}{2xy}}.$$

So: $\boxed{y'(1)} = \frac{-1-2\cdot 1}{2\cdot 1\cdot 1} = \boxed{-\frac{3}{2}}$

Now, for $y'' = (y')'$:

$$y'' = (y')' = \left(-\frac{2x^2+1}{2xy}\right)' =$$

Quotient rule

$$= -\frac{4x(2xy) - (2x^2+1)(2xy)'}{4x^2y^2} =$$

$$= -\frac{8x^2y - (2x^2+1)(2(y+xy'))}{4x^2y^2}$$

Now, plug-in $x=1, y=1, y'=-\frac{3}{2}$, since we only care about y'' AT $x=1, y=1$:

$$\boxed{y''(1)} = -\frac{8\cdot 1 - (2\cdot 1+1)(2(1-\frac{3}{2}))}{4} =$$

$$= -\frac{8-(3)(-1)}{4} = \boxed{-\frac{11}{4}}$$

For the next class:

- Start doing the exercises of section 3.5 (do not do the ones with inverse trigonometric functions - we will do those on the next class).
- If you need help for midterm preparation:
  - I have office hours M 5-6.30 pm in MC 104
  - There is a review session Tuesday, 2.30-4 pm in MC 204. I am available after that for questions
- Next Wednesday's class will be review (no quiz) →

→ • The Math Help-Centre still runs 2.30-6.30 pm every day (M-F) in MC 106.