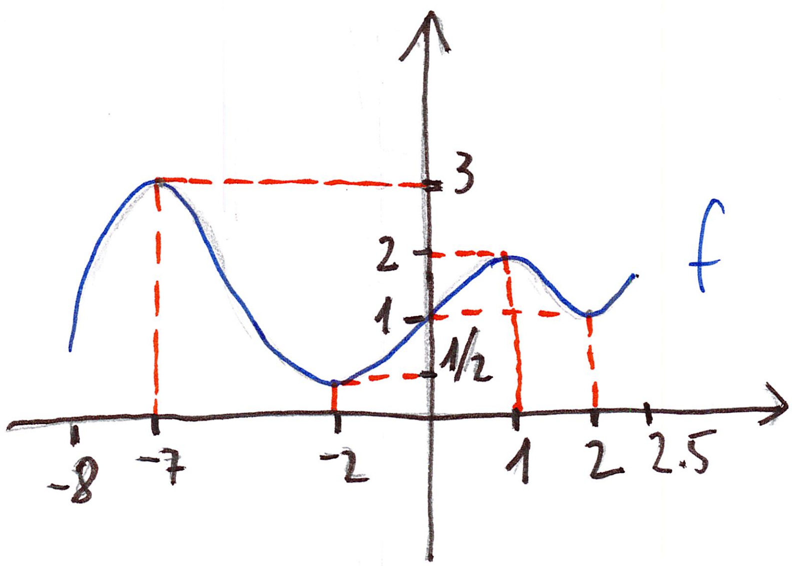
Lecture 13: Min & Max values
(Oct 31)

(Section 4.1)



The domain of f is $D=[-8,2.5]$

For f:

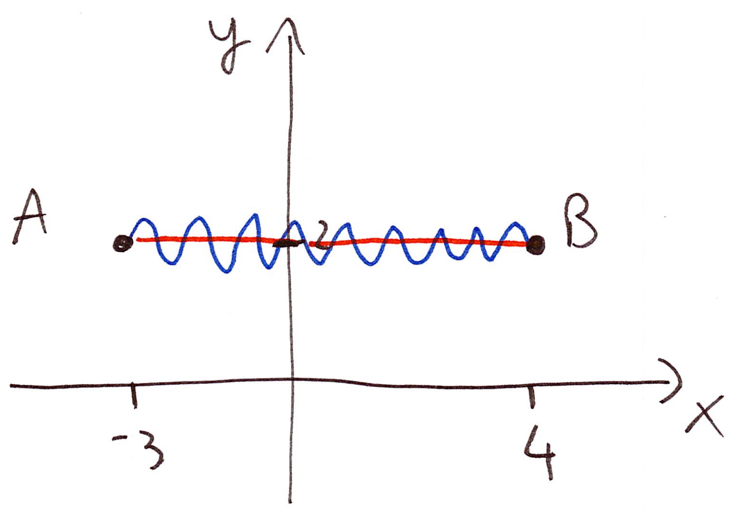
- $f(-7)=3$ is the **absolute maximum value**, since, for every x in D, $f(x)\leq 3=f(-7)$.
- $f(-2)=\frac{1}{2}$ is the **absolute minimum value**, since, for every x in D, $\frac{1}{2}=f(-2)\leq f(x)$.
- $f(1)=2$ is a **local maximum value**, since $2=f(1)\geq f(x)$, when x is close enough to $c=1$. This means: there is an open interval (a,b) contained in D such that $c=1$ is in (a,b) and $f(1)\geq f(x)$ for each $x\in(a,b)$. (For example, you can take the interval (a,b) to be $\left(\frac{1}{2},\frac{3}{2}\right)$ for f).
- $f(2)=1$ is a **local minimum value**, since $1=f(2)\leq f(x)$

when x is close enough to c (same meaning as above).

Note: For the book (so, also for us) and using the above definitions, if f is defined on a closed interval $[d_0, d_1]$ (with $d_0 < d_1$), $f(d_0)$ and $f(d_1)$ can only be abs. min/max values and not local min/max values, because there is no open interval (a,b) contained in $[d_0, d_1]$ such that $d_0 \in (a,b)$ or $d_1 \in (a,b)$.

However, if f is defined on $[d_0, d_1]$ and, for a c with $d_0 < c < d_1$, $f(c)$ is an abs. min/max value, then $f(c)$ is also a local min/max value.

Many functions have abs. min/max values.



For example, there are NO continuous functions defined on $[-3, 4]$ that have no abs min or abs max...

This is always the case because of the following result.

$\boxed{\text{Thm:}}$ (Extreme value thm). IF

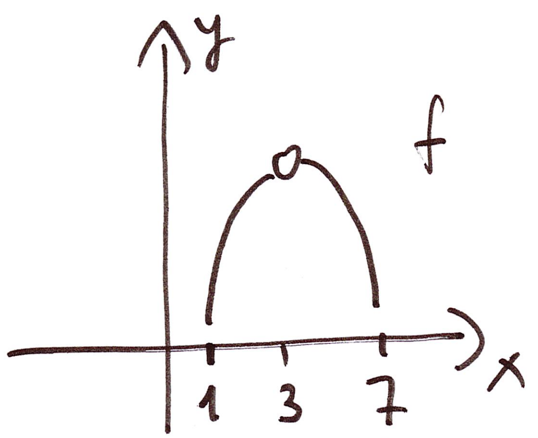
f is continuous on $[a,b]$

THEN

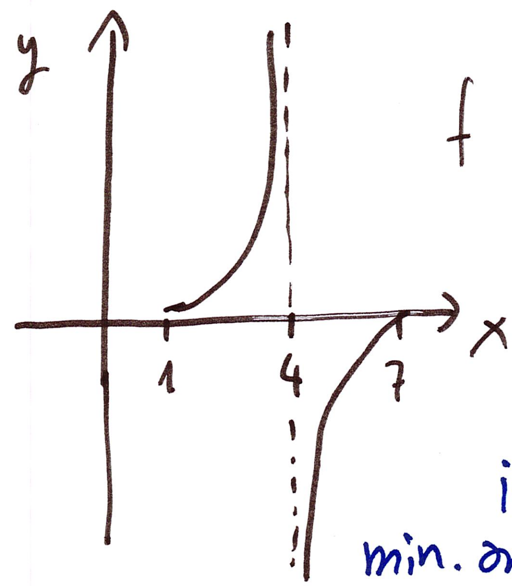
f has an abs. max. value $f(c)$ AND an abs. min. value $f(d)$ for some points $c, d \in [a,b]$.

Both the hypotheses that f is continuous and the interval is closed are needed for this theorem.

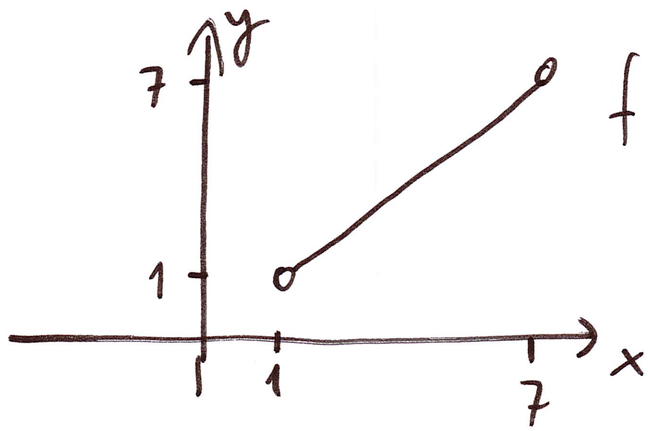
Rmk: If f is not continuous on $[a,b]$ or the interval is not closed (we take (a,b) or $[a,b)$, $(a,b]$ instead of $[a,b]$), then f might not have abs. min value or abs. max. value.



f is NOT continuous on $[a,b]$ and there is no abs. max. value.

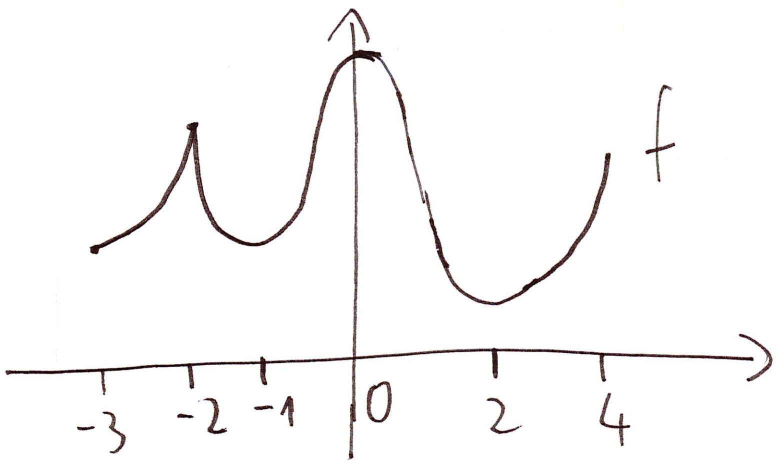


f is NOT continuous (vertical asymptote) and there is no abs. min. and no abs max. values.



f is continuous, but on $(1,7)$, not $[1,7]$ and there is no abs. min or max value.

Let's look at



$f(0)$ is a loc. max value & $f'(0)=0$
(even an abs. max)

$f(-1)$ & $f(2)$ are loc. min values & $f'(-1)=0=f'(2)$

(horizontal tangent lines)

$f(-2)$ is a loc. max value & $f'(-2)$ DNE (vertical tangent line).

This example suggests that if $f(c)$ is a loc. min or max value, then $f'(c)=0$ (this is the case for $f(0), f(-1), f(2)$) or $f'(c)$ DNE (this is the case for $f'(-2)$).

Def: A **critical number** (or critical point) of a function f is a number c in the domain of f such that

$$f'(c) = 0 \quad \text{OR} \quad f'(c) \text{ DNE.}$$

Thm: (Fermat's theorem). If f has a local min. or a local max value at c, then c is a critical number of f.

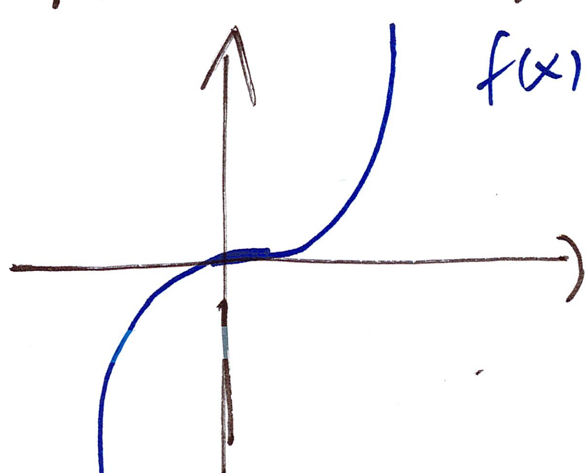
This means:

Every loc. min or max. point is a critical point.

It <u>DOES NOT</u> MEAN:

Every crit. point is a loc. min or max point.

Ex:



$f'(x) = 3x^2$

So, $f'(0) = 0$, (0 is a critical number)

but $f(0)$ is neither loc. min nor loc. max value.

We can use the Extreme value Thm (EVT) & Fermat Thm to find abs. min. and max values of functions.

We use the CLOSED INTERVAL METHOD (CIM).

General Recipe	Example
This method applies to a <u>continuous</u> <u>function</u> f on a <u>closed</u> interval $[a,b]$. It will tell you the abs min & max values of f.	Find the abs. min & max value of $f(x) = 2x^3 + 3x^2 - 12x + 4$ on $[-4, 2]$.
⓪ Compute $f'(x)$	$f'(x) = 2 \cdot 3x^2 + 3 \cdot 2x - 12$ $\overset{!}{=} 6x^2 + 6x - 12$ Factor 6! $\overset{!}{=} 6(x^2 \boxed{+1}x - 2)$ We need: $\begin{cases} c\,d = -2 \\ c+d = \boxed{1} \end{cases}$ $\overset{!}{=} 6(x+2)(x-1)$ Thus, $(x^2+x-2) = (x+2)(x-1)$ So: $\begin{cases} c = 2 \\ d = -1 \end{cases}$

1. Find the critical numbers of f, that is the c's in the interval $[a,b]$ f s.t. $f'(c)=0$ or $f'(c)$ DNE.

$f'(x)$ exists everywhere

$f'(x)=0 \iff$

$\iff 6(x+2)(x-1)=0$

$$\iff \begin{cases} x+2=0 \iff x=-2 \\ \text{or} \\ x-1=0 \iff x=1 \end{cases}$$

Both -2 and 1 are in the interval $[-4,2]$, so the critical numbers are

$x=-2$, $x=1$.

2. Compute the value of f at the critical numbers AND at a and b (the endpoint of the interval $[a,b]$).

x	$\overset{a}{\overset{\shortparallel}{-4}}$	$\overset{\text{crit. number}}{\overset{\shortparallel}{-2}}$	$\overset{\text{crit. number}}{\overset{\shortparallel}{1}}$	$2 = b$
$f(x)$	-28	24	-3	8

$$f(-4) = 2\cdot(-4)^3 + 3(-4)^2 + $$
$$-12(-4) + 4 =$$
$$= 2\cdot(-64) + 3(16) + 48 + 4$$
$$= -128 + \underbrace{48 + 48 + 4}_{=100}$$
$$= -28$$

③ Compare the values you found on Step ②.

The largest of those values is the abs. max value.

The smallest of those values is the abs. min. value.

$$\underset{\text{Smallest}}{\overset{f(-4)}{-28}} < \overset{f(1)}{-3} < \overset{f(2)}{8} < \underset{\text{largest}}{\overset{f(-2)}{24}}$$

So: • $f(-4)=-28$ is the abs. min. value.

• $f(-2)=24$ is the abs. max. value.

$\boxed{\text{Ex:}}$ Find the abs. min & max values of

$$f(x) = xe^{-\frac{x^2}{8}}$$

on $[-1,4]$.

<u>Sol:</u> We use the CIM (f is continuous on $\mathbb{R}$, so also on $[-1,4]$).

⓪ $$f'(x) = e^{-\frac{x^2}{8}} + xe^{-\frac{x^2}{8}} \cdot \left(-\frac{\cancel{2}^{1}x}{\cancel{8}_{4}}\right)$$

PR+CR

$= e^{-\frac{x^2}{8}} - \frac{x^2}{4} e^{-\frac{x^2}{8}}$

| Factor $e^{-\frac{x^2}{8}}$

$= e^{-\frac{x^2}{8}}\left(1 - \frac{x^2}{4}\right).$

(1) Critical numbers?

$f'(x) = 0$?

$f'(x) = 0 \iff \begin{cases} e^{-\frac{x^2}{8}} = 0 \\ \text{or} \\ 1 - \frac{x^2}{4} = 0 \iff 1 = \frac{x^2}{4} \end{cases}$

IMPOSSIBLE: $e^{\text{something}} > 0$

$\iff 4 = x^2$

$\iff x = \pm 2$

Now, -2 is not in the interval $[-1, 4]$, so we only keep $x = 2$ as a critical number.

(2)

x	-1 $= a$	2 $=$ crit. number	4 $= b$
$f(x)$	$-e^{-\frac{1}{8}}$	$2e^{-\frac{4}{8}} \overset{!}{=} 2e^{-\frac{1}{2}} \overset{!}{=} \frac{2}{\sqrt{e}}$	$4e^{-\frac{16}{8}} = 4e^{-2} = \frac{4}{e^2}$

③ Compare the values.

$f(-1) < 0$, whereas $f(2), f(4) > 0$,

so $f(-1) = -e^{-\frac{1}{8}}$ is the smallest value among the three we found.

Hence $-e^{-\frac{1}{8}}$ is the abs. min. value.

Which one is larger between $\frac{2}{\sqrt{e}}$ & $\frac{4}{e^2}$?

Remember: $2 < e < 3$.

Therefore: $(7 < 9 \Rightarrow \frac{1}{7} > \frac{1}{9})$

① $2^2 < e^2$ (since $2 < e$) $\Rightarrow \frac{1}{2^2} > \frac{1}{e^2}$

Multiply by 4 $\Rightarrow \underbrace{\frac{4}{2^2}}_{=1} > \frac{4}{e^2}$. So $1 > \frac{4}{e^2}$.

$(a < b \Rightarrow \frac{1}{a} > \frac{1}{b})$

② $\sqrt{e} < \sqrt{3} < \sqrt{4} = 2 \Rightarrow \frac{1}{\sqrt{e}} > \frac{1}{2}$

(since $e < 3$)

Multiply by 2 $\Rightarrow \frac{2}{\sqrt{e}} > \frac{2}{2} = 1$

So, all in all,

$$\frac{2}{\sqrt{e}} > 1 > \frac{4}{e^2}$$

Hence, $\frac{2}{\sqrt{e}} = f(2)$ is the abs. max value.

Extra exercise: Find the abs. min & max values of

$$f(x) = x\sqrt{6-x^2} \quad \text{on } [-1,2].$$

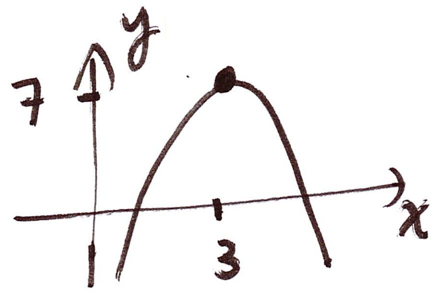
[Abs. min value: $f(-1) = -\sqrt{5}$
Abs. max value: $f(\sqrt{3}) = 3$]

Summary of this class.

1. We introduced the concepts of abs. min/max values and rel. min/max values and we remarked that abs. min/max $\Rightarrow$ rel. min/max, but not the other way round.

<u>Note</u>: If c is in the domain of f and $f(c)$ is an abs (or rel.) min/max

value, then c is called an abs. (or rel.) min/max point of f. (Min/Max points are "on the x axis", while Min/max values are "on the y axis".



3 is an abs. max POINT, $f(3)=7$ is an abs. max VALUE).

(2) We saw that if f is continuous on $[a,b]$, then f has both a min. and a max. value on $[a,b]$. (Extreme Value Thm). We discussed how both the hypotheses that f is continuous and $[a,b]$ is closed are needed.

(3) We introduced critical points and talked about Fermat theorem.

(4) We used the Closed Interval Method to find abs. min/max values.

For next class:

1. Do the exercises of section 4.1.
2. Take a look at the first Derivative test and at functions that are "Concave up" & "Concave down" in section 4.3. We will try to go through that section on Monday.
3. The Quiz on next Wednesday will cover:
   - Log. differentiation & derivative of log (section 3.6)
   - Related rates problems (Section 3.9)
   - What we did today (Section 4.1).