About the 2nd Quiz

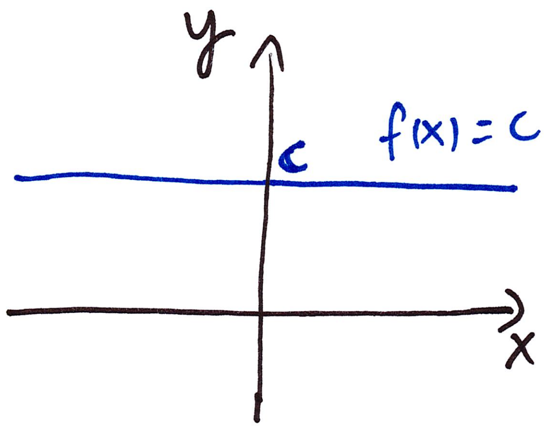
- It will cover:
  - inverse trigonometric functions (pages 63-66)
  - Section 2.2
  - Section 2.3 (Today's class)
- Even though you won't be tested on past material, you can NOT forget about it. You always need to remember basic things like exponentials, logarithms and their laws....
- No aids this time: no textbook, notes, calculators or other deviltries.

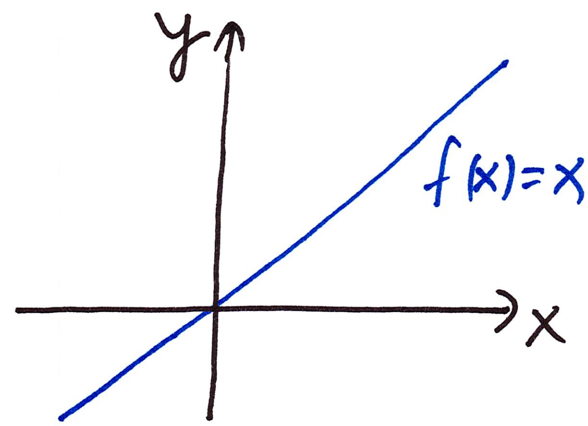
Lecture 5: Computing limits (Sep 24)

(Section 2.3)

There are two very simple limits. Suppose a, c are numbers. Then

$$\lim_{x \to a} c = c \qquad\qquad \lim_{x \to a} x = a$$





From these, we can compute more limits using the <u>limit laws</u>.

Limits' laws.

Suppose f, g are functions defined near $x=a$ and such that

$$\lim_{x\to a} f(x) \quad \& \quad \lim_{x\to a} g(x) \quad \text{both exist.}$$

Then

① $\lim\limits_{x\to a}(f(x)+g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

② $\lim\limits_{x\to a}(f(x)-g(x)) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

③ If c is any number, $\lim\limits_{x\to a}(c f(x)) = c \lim\limits_{x\to a} f(x)$

④ $\lim\limits_{x\to a}(f(x)\, g(x)) = \left(\lim\limits_{x\to a} f(x)\right)\left(\lim\limits_{x\to a} g(x)\right)$.

In particular, $\lim\limits_{x\to a}\left(f(x)^n\right) = \left(\lim\limits_{x\to a} f(x)\right)^n$.

⑤ $\lim\limits_{x\to a}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ IF $\lim\limits_{x\to a} g(x) \neq 0$

⑥ $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$.

$\boxed{\text{Ex}}$ Compute $\lim\limits_{x\to(-3)} e^2\left(\frac{\sqrt{(-x)}}{x+1}\right)$

Sol.: Apply the limit laws.

$$\lim_{x\to(-3)} e^2\left(\frac{\sqrt{(-x)}}{x+1}\right) \overset{③}{=} e^2\left(\lim_{x\to(-3)} \frac{\sqrt{(-x)}}{x+1}\right) =$$

$$\overset{⑤}{=} e^2\left(\frac{\lim\limits_{x\to(-3)}\sqrt{(-x)}}{\lim\limits_{x\to(-3)} x+1}\right) \overset{⑥}{\underset{①}{=}} e^2\left(\frac{\sqrt{\lim\limits_{x\to(-3)} -x}}{\lim\limits_{x\to(-3)} x + \lim\limits_{x\to(-3)} 1}\right) =$$

$$\overset{③}{=} e^2\left(\frac{\sqrt{-\lim\limits_{x\to(-3)} x}}{\lim\limits_{x\to(-3)} x + \lim\limits_{x\to(-3)} 1}\right) \underset{\text{substitution}}{=} e^2\left(\frac{\sqrt{-(-3)}}{-3+1}\right) =$$

$$= e^2\left(\frac{\sqrt{3}}{-2}\right) = -\frac{\sqrt{3}\,e^2}{2}$$

Fact (Substitution property). If f is a polynomial ($f(x) = a_n x^n + \dots + a_1 x + a_0$) or f is a rational function ($f(x) = \frac{P(x)}{Q(x)}$ with P, Q polynomials) AND f is defined at $x = a$, THEN

$$\lim_{x \to a} f(x) = f(a)$$

Ex: Compute $\lim_{x \to 2} \frac{x^3 + 2x - 3}{4x + 2}$

Sol: $f(x) = \frac{x^3 + 2x - 3}{4x + 2}$ is a rational function AND $x = 2$ is in the domain of f.

$\Rightarrow$ Use subst. $\boxed{\lim_{x \to 2} f(x)} = \frac{2^3 + 2 \cdot 2 - 3}{4 \cdot 2 + 2}$

Substitute x with 2 everywhere

$$= \frac{8 + 4 - 3}{8 + 2} = \frac{12 - 3}{10} = \boxed{\frac{9}{10}}$$

It is important that a is in the domain of f!

Ex: Compute $\lim_{x\to 2} \frac{x^2-4x+4}{x^2-2x}$

Sol: Gut instinct: $\lim_{x\to 2} f(x) = \frac{2^2-4\cdot 2+4}{2^2-2\cdot 2} =$

$$= \frac{4-8+4}{4-4} = \frac{0}{0} \;???$$

$\frac{0}{0}$: INDETERMINATE FORM

This happened because 2 is not in the domain of f(x)!

If we get $\frac{0}{0}$, we try to rewrite the numerator and the denominator and see if we can simplify something.

$$\frac{x^2-4x+4}{x^2-2x} = \frac{(x-2)^2}{x(x-2)}$$

$$\frac{(x-2)^{\cancel{2}}}{x\cancel{(x-2)}} = \frac{x-2}{x}$$

$x^2-4x+4 =$
$= (x)^2 - 4x + (-2)^2$
$a^2 - 2ab + (-b)^2$
$= (a-b)^2 = (x-2)^2$

So $\boxed{\lim_{x \to 2} \frac{x^2-4x+4}{x^2-2x}} = \lim_{x \to 2} \frac{x-2}{x} \underset{\text{substitution}}{=}$

$= \frac{2-2}{2} = \frac{0}{2} = \boxed{0}.$

Note: $\frac{x^2-4x+4}{x^2-2x} \neq \frac{x-2}{x}$ in general

BUT $\frac{x^2-4x+4}{x^2-2x} = \frac{x-2}{x}$ if $x \neq 2$

This is what we care about when we compute $\lim_{x \to 2} f(x)$.

In general,

If $f(x) = g(x)$ near $x = a$, but $f(a) \neq g(a)$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ (if the limits exist)

$\boxed{\text{Ex:}}$ Compute $\lim\limits_{x\to(-3)} \dfrac{\sqrt{x^2+16}-5}{x+3}$.

Sol: If you substitutes you get $\frac{0}{0}$ again.

Let's rationalize!

$$\frac{\sqrt{x^2+16}-5}{x+3}\cdot\frac{\sqrt{x^2+16}+5}{\sqrt{x^2+16}+5}=$$

$(a-b)(a+b)=$
$=a^2-b^2$

$$=\frac{(\underbrace{\sqrt{x^2+16}}_{=a}-\underbrace{5}_{=b})(\sqrt{x^2+16}+5)}{(x+3)(\sqrt{x^2+16}+5)}=$$

$$=\frac{(\sqrt{x^2+16})^2-(5)^2}{(x+3)(\sqrt{x^2+16}+5)}\overset{(\sqrt{x})^2=x}{=}\frac{x^2+16-25}{(x+3)(\sqrt{x^2+16}+5)}=$$

$(a^2-b^2)=(a+b)(a-b)$

$$=\frac{x^2-\underbrace{9}_{=3^2}}{(x+3)(\sqrt{x^2+16}+5)}=\frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(\sqrt{x^2+16}+5)}=$$

$$=\frac{x-3}{\sqrt{x^2+16}+5}$$

$$\boxed{\lim_{x\to(-3)} \frac{\sqrt{x^2+16}-5}{x+3}} = \lim_{x\to(-3)} \frac{x-3}{\sqrt{x^2+16}+5} =$$

(5)
$$= \frac{\lim_{x\to(-3)} x-3}{\lim_{x\to(-3)} \sqrt{x^2+16}+5} = \dots \overset{\text{Subst.}}{=} \frac{(-3)-3}{\sqrt{(-3)^2+16}+5} =$$

$$= \frac{-6}{\sqrt{25}+5} = \frac{-6}{5+5} = -\frac{\cancel{6}^{3}}{\cancel{10}_{5}} = \boxed{-\frac{3}{5}}$$

- Other techniques to compute limits.

(1) Compare $\lim_{x\to a^+}$ & $\lim_{x\to a^-}$

Recall: $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L$ AND $\lim_{x\to a^+} f(x) = L$

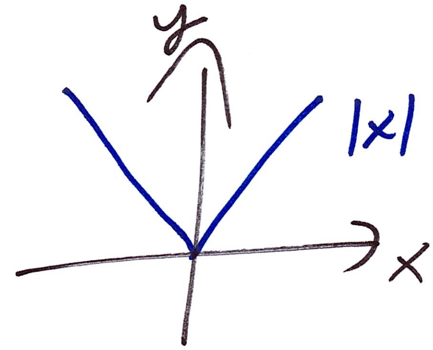
Pro-tip: This fact is useful to compute limits involving $|\cdot|$ (absolute value) or other piecewise defined functions.

$\boxed{\text{Ex:}}$ Compute, if it exists,

$$\lim_{x\to(-1)} \frac{-3x-3}{|x+1|}$$

Sol: Recall

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



$$|x+1| = \begin{cases} x+1 & \text{if } x+1 \geq 0 \iff x \geq -1 \\ -(x+1) & \text{if } x+1 < 0 \iff x < -1 \end{cases}$$

There are 2 formulas for $|x+1|$, which suggests us to study $\lim\limits_{x\to(-1)^+}$ & $\lim\limits_{x\to(-1)^-}$.

- $\lim\limits_{x\to(-1)^+} \dfrac{-3x-3}{|x+1|} = \lim\limits_{x\to(-1)^+} \dfrac{-3x-3}{x+1} = \lim\limits_{x\to(-1)^+} \dfrac{(-3)\cancel{(x+1)}}{\cancel{x+1}}$

As $x\to(-1)^+$, $x>(-1)$

$$= \lim_{x\to(-1)^+} (-3) = -3$$

$$\lim_{x\to(-1)^-} \frac{-3x-3}{|x+1|} \overset{}{=} \lim_{x\to(-1)^-} \frac{-3x-3}{-(x+1)} = \lim_{x\to(-1)^-} \frac{(-3)\cancel{(x+1)}}{(-1)\cancel{(x+1)}} =$$

As $x \to (-1)^-$, $x < -1$

$$= \lim_{x\to(-1)^-} 3 = 3$$

$$\lim_{x\to(-1)^+} f(x) = -3$$

$$\neq$$

$$\lim_{x\to(-1)^-} f(x) = 3$$

So, $\lim_{x\to(-1)} f(x)$ DNE.

Note: We had to do this because the formula for $f(x)$ changes exactly at $x=-1$.

But: $$\lim_{x\to 5} \frac{-3x-3}{|x+1|} = \lim_{x\to 5} \frac{-3x-3}{x+1} =$$

As $x \to 5$, $x \geq 1$

$$\overset{\text{substitute}}{=} \frac{-3\cdot 5-3}{5+1} = \frac{-18}{6} = -3.$$

② Use the Squeeze theorem.

Thm: (Squeeze or Sandwich theorem). Suppose that

$$f(x) \le g(x) \le h(x)$$

for all x near a, except possibly AT $x=a$.

Suppose also that

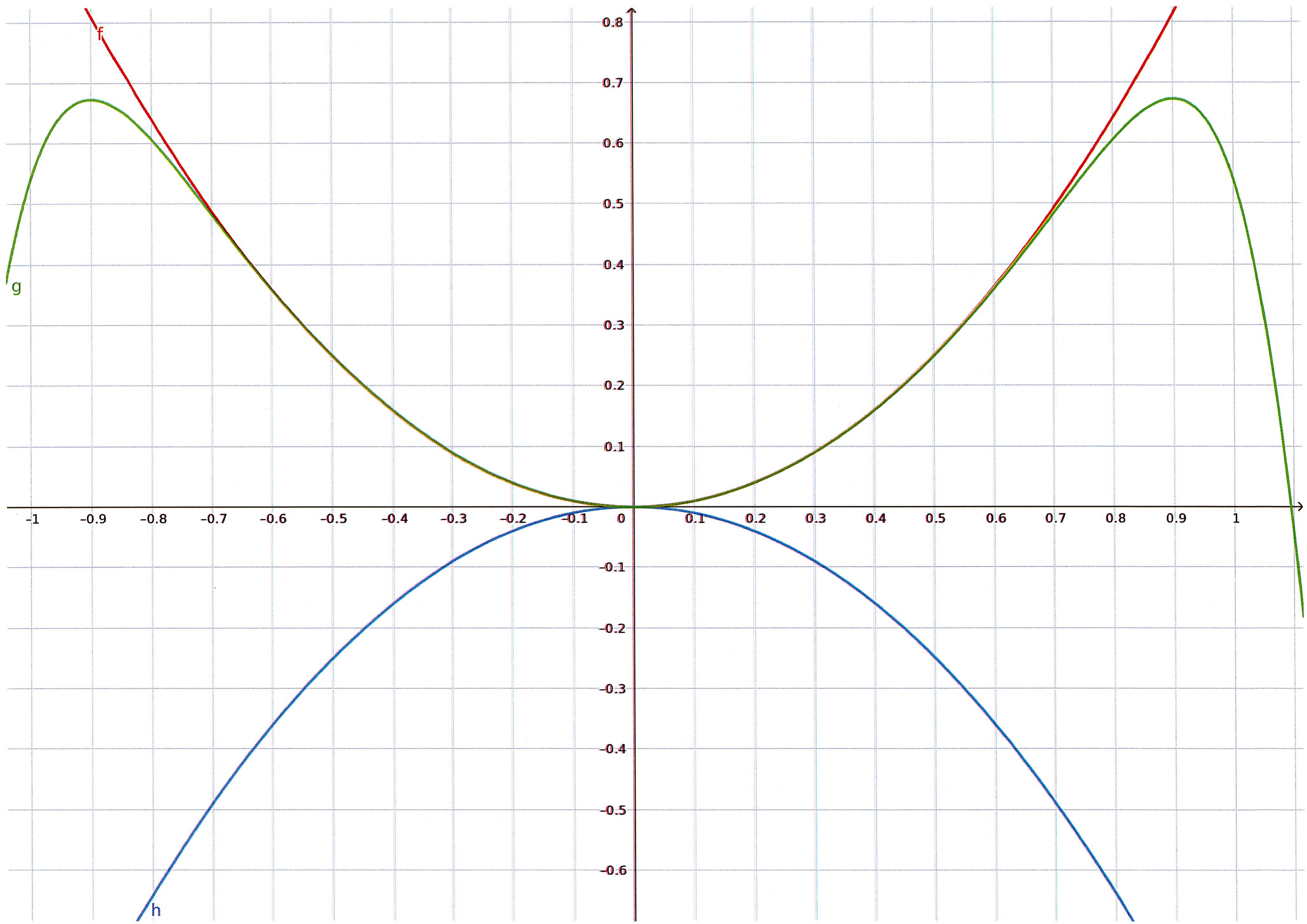
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = \begin{cases} L & \text{a number} \\ +\infty \\ -\infty \end{cases}$$

Then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x)$$

Pro-tip: Try using the Squeeze thm when you want to compute $\lim_{x\to 0} g(x)$ and in the formula for $g(x)$ you see a function like $\cos\left(\frac{1}{x}\right)$ or $\sin\left(\frac{4}{x}\right)$ (or $\sin\left(\frac{1}{x^3}\right)$ ...).

(The limits as $x\to 0$ of these functions DNE).


f
g
h
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Ex: Compute
$$\lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)+6}{x^4}.$$

Sol: We CAN'T use the limit laws because

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ DNE.

For every $\theta \in \mathbb{R}$, $\sin(\theta) \in [-1, 1]$.

So $\left(\theta = \frac{1}{x}\right)$:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

for x near 0 (but not at x=0)

Add 6 to both sides:

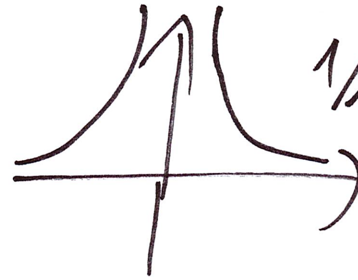
$$5 \leq \sin\left(\frac{1}{x}\right)+6 \leq 7 \quad (*)$$

Divide by x^4:

$$\underbrace{\frac{5}{x^4}}_{f(x)} \leq \underbrace{\frac{\sin\left(\frac{1}{x}\right)+6}{x^4}}_{g(x)} \leq \underbrace{\frac{7}{x^4}}_{h(x)}$$

Note: $x^4 > 0$ when $x \neq 0$, so if you divide by x^4 in (*), the order of the inequalities does not change.

As $x \to 0$, $\frac{1}{x^4} \to +\infty$,



so also $\frac{5}{x^4} \to +\infty$, $\frac{7}{x^4} \to +\infty$

So $\lim\limits_{x\to 0} f(x) = \lim\limits_{x \to 0} h(x) = +\infty$

Then by the squeeze thm:

$$\lim_{x\to 0} g(x) = +\infty.$$

$\boxed{\text{Ex:}}$ Try to compute $\lim\limits_{x\to 0} x^2 \sin\left(\frac{1}{x^4}\right)$.

We used the following fact:

$\boxed{\text{Thm:}}$ 1) If n is a positive, even integer,

$$\lim_{x\to a} \frac{1}{(x-a)^n} = +\infty.$$

Ex: $\lim\limits_{x\to 0} \frac{1}{x^4} = \infty$

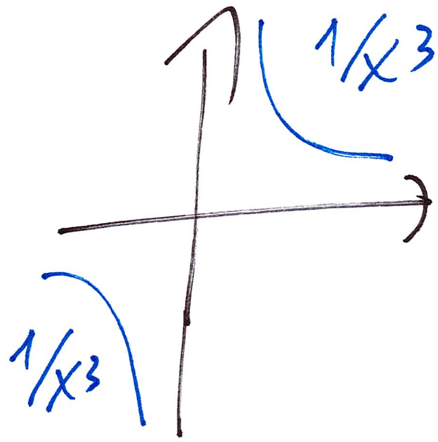
② If n is a positive, odd integer, then

(i) $\displaystyle\lim_{x\to a^+} \frac{1}{(x-a)^n} = +\infty$

(ii) $\displaystyle\lim_{x\to a^-} \frac{1}{(x-a)^n} = -\infty$

Ex: $\displaystyle\lim_{x\to 0^+} \frac{1}{x^3} = +\infty$

$\displaystyle\lim_{x\to 0^-} \frac{1}{x^3} = -\infty$



$\boxed{\text{Ex:}}$ Compute $\displaystyle\lim_{x\to 0} \frac{1}{x^2} - \frac{1}{x^4}$.

Sol: "Try to substitute": "$\displaystyle\lim_{x\to 0} \frac{1}{x^2} - \frac{1}{x^4} =$

$\displaystyle= \lim_{x\to 0} \frac{1}{x^2} - \lim_{x\to 0} \frac{1}{x^4} = \infty - \infty$"

No sense...

Now, $\displaystyle\frac{1}{x^2} - \frac{1}{x^4} = \frac{x^2-1}{x^4}$. As $x\to 0$, we have:

- $x^2-1 \longrightarrow -1$ (substitution)
- $\displaystyle\frac{1}{x^4} \to \infty$ (by the Thm).

So $\frac{x^2-1}{x^4}$ is going to approach ∞ while being negative as $x \to 0$, that is, $\lim_{x\to 0} \frac{x^2-1}{x^4} = -\infty$.

Morally "$\lim_{x\to 0} \frac{x^2-1}{x^4} = \left(\lim_{x\to 0} x^2-1\right)\left(\lim_{x\to 0} \frac{1}{x^4}\right) =$

$= (-1)\cdot(+\infty) = -\infty$".

You can also think about this in this way.

$$\lim_{x\to 0} \frac{x^2-1}{x^4} = \lim_{x\to 0} \frac{1}{\left(\frac{x^4}{x^2-1}\right)}.$$

Now, $\lim_{x\to 0} \frac{x^4}{x^2-1} = \frac{0}{-1} = 0^{-}$

substitution

This "minus" is just saying that we are approaching 0 while taking negative values. Since we are going to look at $\lim_{x\to 0} \frac{1}{\left(\frac{x^4}{x^2-1}\right)}$ we want to keep track of whether we approach 0 with positive or negative values, to understand if $\frac{1}{\left(\frac{x^4}{x^2-1}\right)} \to \infty$ or $-\infty$.

Since $\frac{x^4}{x^2-1} \to 0$ while being negative, as $x \to 0$, $\frac{1}{\left(\frac{x^4}{x^2-1}\right)} \to -\infty$ (if you divide by something that is negative and gets smaller and smaller as you approach 0, the limit is $-\infty$, by definition).

I put together a few "infinite limit" laws in the next pages. We will talk more about these when we do limits at infinity.

"Infinite limit" laws

1. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = L$ or ∞, then $\lim_{x\to a} (f(x) + g(x)) = \infty$. (This also works if we substitute ∞ with $-\infty$).

Morally: "$\infty + L = \infty$" & "$\infty + \infty = \infty$)

2. If $\lim_{x\to a} f(x) = L$ or $-\infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} (f(x) - g(x)) = -\infty$.

Morally: "$L - (\infty) = -\infty$" & "$-\infty - (\infty) = -\infty$"

3. If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = L > 0$ or $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x) g(x) = \pm\infty$.

If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = L < 0$ or $\lim_{x\to a} g(x) = -\infty$, then $\lim_{x\to a} f(x) g(x) = \mp\infty$.

Morally: "
$$\begin{aligned}
c \cdot \infty &= \infty \quad \text{if } c > 0 \\
c \cdot (-\infty) &= -\infty \quad \text{if } c > 0 \\
\infty \cdot (\pm\infty) &= \pm\infty \\
c \cdot \infty &= -\infty \quad \text{if } c < 0 \\
c \cdot (-\infty) &= \infty \quad \text{if } c < 0 \\
-\infty \cdot (\pm\infty) &= \mp\infty
\end{aligned}$$
"

In general, anything else does not hold true.

For example, ALL of the following expressions are indeterminate forms.

"$\infty - \infty$" "$0 \cdot (\pm\infty)$"

"$\frac{\infty}{\infty}$" "$\frac{0}{\infty}$"

"$\frac{\infty}{0}$" ("$\frac{0}{0}$")

(We will encounter more later on).

In general, if you find some of these, you need to try and rewrite the formula or expression you are trying to find the limit of, so as to get rid of the indeterminate form.

For the next class:

- Do the exercises of section 2.3
- Take a look at section 2.5.