Lecture 1 (Sep 10) - Review on Functions

Sections 1.1-1.4 of the book

Calculus is the study of (continuous) functions, through the concepts of limits, derivatives and integrals.

The rest of this course is about understanding what this means.

So... let's get started!

- $\boxed{\text{Sets and functions}}$

$\boxed{\text{Def:}}$ A <u>set</u> X is a collection of (mathematical) objects, called <u>elements</u> of X.

"$x \in X$" := "x is an element of X"

Slogan: We know a set when we know its elements.

Two main ways to describe sets:

① by listing its elements

$$X = \{\text{coffee}, \text{tea}\} \quad ;$$

Notation: the elements of the considered set (X) are all the objects between the brackets

② via a property that describes all its elements

$$X = \{x : x \text{ was born in Canada}\}$$

Notation: "such that"

$$Y = \{y : y \text{ is a number and } y^2 = 0\} =$$
$$= \{0\}$$

Important sets for us are the <u>number sets</u>:

$\mathbb{N} := \{0, 1, 2, 3, \dots\}$ <u>natural numbers</u>

$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$ <u>integers</u>

From Zahlen, "numbers" in German

$\mathbb{Q} := \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$ <u>rational numbers</u> (good ol' fractions)

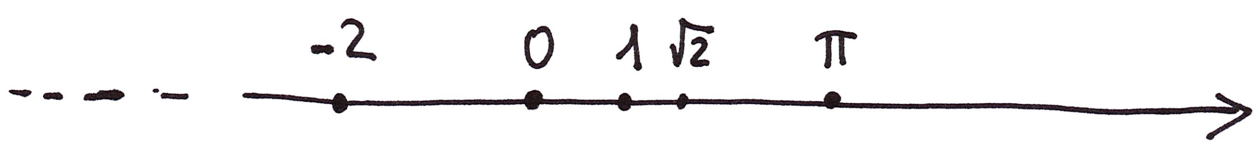
$\mathbb{R}$ is the set of <u>real numbers</u>.

It contains $\mathbb{Q}$ but also the irrational numbers like $\sqrt{2}$, π, e, ...

Google "Pi day"

We'll see this one later on

We can picture $\mathbb{R}$ as an oriented line



Other important sets are the <u>intervals</u>

Pick two numbers $a, b \in \mathbb{R}$ with $a < b$.

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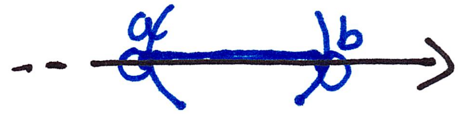
$(a,b) := \{x \in \mathbb{R} : a < x < b\}$

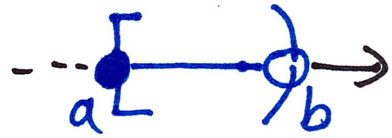


open interval with endpoints a and b

$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$

closed interval with endpoints a and b

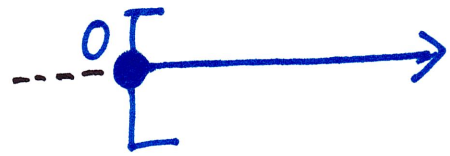
$[a,b) := \{x \in \mathbb{R} : a \leq x < b\}$



$(a,b] :=$ Figure this out :)

$\mathbb{R}_{\geq 0} := [0, +\infty) = \{x \in \mathbb{R} : x \geq 0\}$

Positive infinity



$\mathbb{R}_+ := (0, +\infty)$ $\quad$ $\mathbb{R}_- := (-\infty, 0)$

Negative infinity

Note: $+\infty$ (sometimes just written ∞) and $-\infty$ are symbols with the property that

$\mathbb{R} = (-\infty, +\infty)$ $\quad$ $-\infty < a < +\infty$ for any $a \in \mathbb{R}$

$-\infty, +\infty$ are NOT real numbers

Every set is just a collection of objects on their own, like a world in itself.

We would like to connect sets, move from one world to another.

We do this via functions.

$\boxed{\text{Def:}}$ Let X, Y be sets. A function f from X to Y is a rule that assigns to every element $x \in X$ a unique (that is, one and only one) element $f(x) \in Y$.

We write a function f from X to Y as:

$$f : X \longrightarrow Y$$

Terminology:

- X is the domain of the function f. It's the set of "inputs" of f.
- Y is the codomain of f. It's where all "outputs" of f live, but it might have other elements.

• The range of f is the set

$$R = \{y \in Y : y = f(x) \text{ for some } x \in X\}$$

It's the set made of all and only the outputs $f(x)$ of f.

All our functions will have subsets of $\mathbb{R}$ (that is, sets whose elements are real numbers) as domains and codomains.

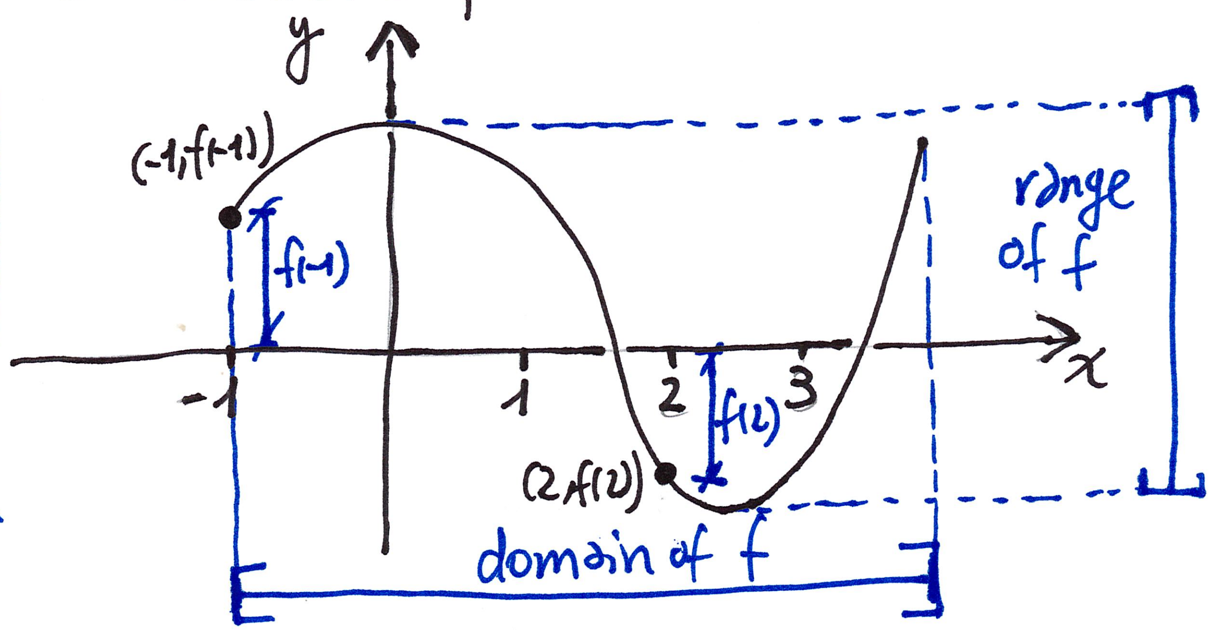
We can visualize a function via its graph.

The graph of a function $f: X \to Y$ is the set of all points

$$(x, f(x)) \qquad \text{for } x \in X$$

in the coordinate plane.

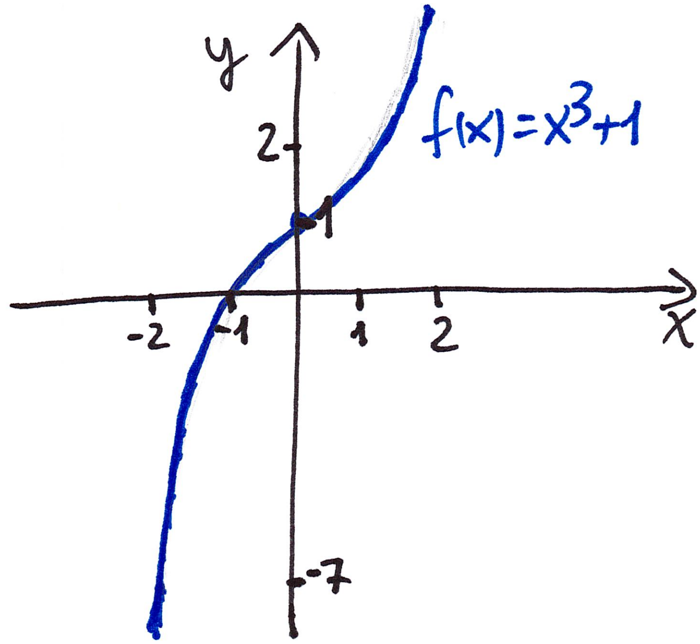
Check for "Vertical line test"



$\boxed{\text{Ex:}}$ ① $f(x) := x^3 + 1$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$



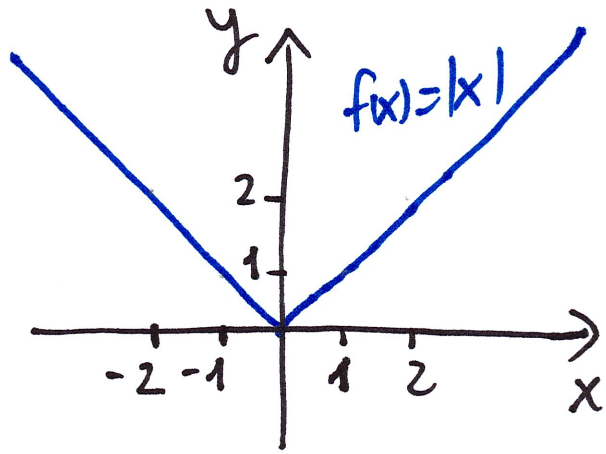
② The absolute value function.

This is the function

$$|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$$

defined by:

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



This is an example of a piecewise defined function, that is, a function that is defined by different formulas on different parts of its domain (the two parts are $[0,+\infty)$ and $(-\infty,0)$ in the example of $|x|$ above).

$\boxed{\text{Ex:}}$ Find the domain and the range of the function defined by the formula

$$f(x) = \frac{2}{x^2-3}.$$

Sol: ⓐ Domain. We need to find all $x \in \mathbb{R}$ for which $f(x)$ makes sense, that is, for which $f(x)$ is a real number.

$\frac{2}{x^2-3}$ is defined if and only if $x^2-3 \neq 0$.

But: $x^2-3=0 \iff x^2=3 \iff x=\sqrt{3}$ or $x=-\sqrt{3}$

Notation: "if and only if", "exactly when"

So, the domain of f is

$$X = \{x \in \mathbb{R} : x \neq \sqrt{3} \text{ and } x \neq -\sqrt{3}\}$$

ⓑ Range. We need to understand which numbers y in $\mathbb{R}$ can be written as $y = \frac{2}{x^2-3}$ for some x in the domain of f.

Here is a recipe to do this.

To find the range of $f(x)=\frac{2}{x^2-3}$ (or any other $f(x)$):

① Fix $y\in\mathbb{R}$ and suppose $y=\frac{2}{x^2-3}$ for some x. Now write x in terms of y (instead of y in terms of x).

$y=\frac{2}{x^2-3} \Rightarrow (x^2-3)y=2$

Multiply by (x^2-3)

$\boxed{\text{If } y\neq 0}$, we can divide by y:

$x^2-3=\frac{2}{y} \Rightarrow x^2=\frac{2}{y}+3$. To take the square root of the RHS, we need to make sure that $\frac{2}{y}+3\geq 0$

If $\frac{2}{y}+3\geq 0$, $x^2=\frac{2}{y}+3 \Rightarrow x=\pm\sqrt{\frac{2}{y}+3}$

Note: $\frac{2}{y}+3\geq 0 \Leftrightarrow \frac{2}{y}\geq -3$. This can happen in 2 ways:

① if $\boxed{y>0}$, then $\frac{2}{y}>0>-3$ ✓

② if $y<0$, $\frac{2}{y}\geq -3 \Leftrightarrow 2\leq -3y \Leftrightarrow \boxed{y\leq -\frac{2}{3}}$

y is negative!

② Make sure that, for each y found in ①, the corresponding x is in the domain of f.

Can $x = \pm\sqrt{\frac{2}{y}+3}$ ever be $\pm\sqrt{3}$?

No! $\pm\sqrt{\frac{2}{y}+3} = \pm\sqrt{3} \underset{\text{taking squares}}{\Longrightarrow} \frac{2}{y}\cancel{+3} = \cancel{3}$

$\Downarrow$

$\frac{2}{y} = 0$ which can't be

③ Check whether specific values of y excluded when performing the computations in ① could actually belong to the range.

Is there an x in the domain of f such that $\frac{2}{x^2-3} = 0$? (Remember, we had to assume $y \neq 0$ at some point). No! (If $\frac{2}{x^2-3}$ is defined, it's never zero - because $2 \neq 0$).

$\Rightarrow$ Range of $f = \{y \in \mathbb{R} : y > 0 \text{ or } y \leq -\frac{2}{3}\} =$

$= (-\infty, -\frac{2}{3}] \cup (0, +\infty)$ union set

• New functions from old

We can combine two or more functions together to obtain new ones in many different ways.

For example, given two functions $f, g : X \to Y$, we can define:

If f and g have different domains, the domains of $f+g$ and fg are the intersection of the domains of f and g

$$f+g : X \to Y, \quad (f+g)(x) := f(x) + g(x)$$

$$f \cdot g : X \to Y, \quad (fg)(x) := f(x)g(x)$$

Here is another VERY IMPORTANT way of combining two functions together.

$\boxed{\text{Def:}}$ Given two functions

$$f : X \to Y \text{ and } g : Y \to Z,$$

the composition of f and g (or the composite function of f and g) is the function

$$g \circ f : X \to Z, \quad (g \circ f)(x) := g(f(x))$$

Note that:

① $g \circ f$ is defined only when the codomain of f is the same as the domain of g $(f: X \to \underline{Y},\ g: \underline{Y} \to Z)$ or if the range of f is contained in the domain of g.

② $g(f(x))$ means: take $x \in X$, apply f to get $f(x)$ and then apply g to $y = f(x)$ to get $g(f(x))$.

$\boxed{\text{Ex:}}$ Consider the functions

$$f: \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) := x^2$$

$$g: \mathbb{R} \longrightarrow \mathbb{R}, \quad g(x) := x + 3.$$

Compute $g \circ f$ and $f \circ g$.

<u>Sol:</u> Note that f and g have both the same domain and the same codomain, so we can consider both $g \circ f$ and $f \circ g$.

Take the formula for g and substitute x with x^2

$$g \circ f: \mathbb{R} \to \mathbb{R}, \quad (g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 3$$

$$f \circ g: \mathbb{R} \to \mathbb{R}, \quad (f \circ g)(x) = f(g(x)) = f(x+3) = (x+3)^2.$$

$$\boxed{(g \circ f)(x) \neq (f \circ g)(x)}$$

Some special composite functions have an easy-to-describe effect on the graph of $f(x)$.

Exercise: Interpret all the functions in the table below as $g\circ f$ or as $f\circ g$ for a suitable function g.

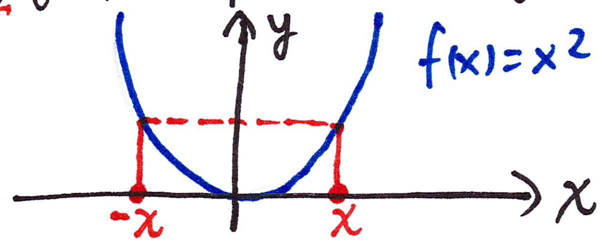
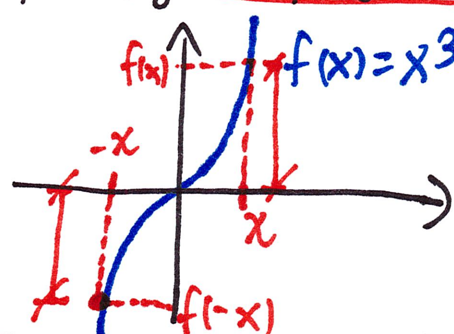
Choose your favorite f and plot the composite functions below with Geogebra/Desmos!

For $a>0$, $b>1$

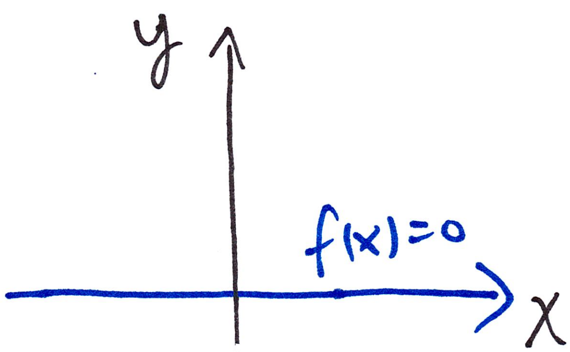
$f(x)+a$	shift the graph of f a units upward
$f(x)-a$	" " " " " " " downward
$f(x+a)$	" " " " " " " to the left
$f(x-a)$	" " " " " " " to the right
$f(-x)$	reflect the graph of f about the y-axis
$-f(x)$	" " " " " " the x-axis
$f(bx)$	shrink the graph of f horizontally by a factor of b
$b\,f(x)$	stretch the graph of f vertically by a factor of b
$f\left(\frac{x}{b}\right)$	stretch the graph horizontally by a factor of b
$\frac{f(x)}{b}$	shrink the graph of f vertically by a factor of b

• (Some) properties of a function f

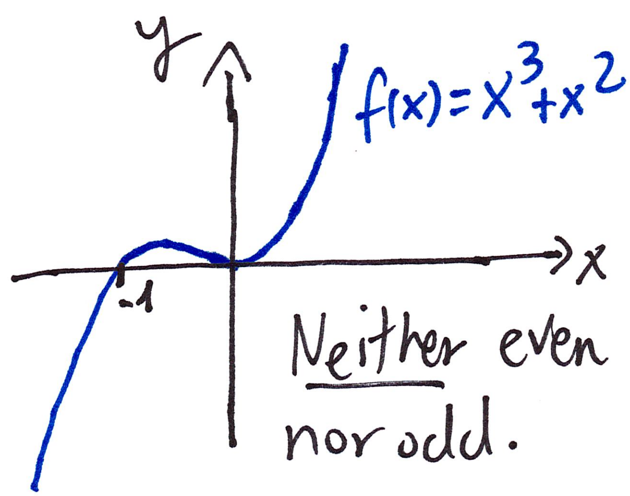
Suppose $f: X \to Y$ is a function.

Property	Graph (or Example)
• f is **even** if $f(x) = f(-x)$ for all $x \in X$	The graph of f is symmetric wrt (with respect to) the y-axis 
• f is **odd** if $f(-x) = -f(x)$ for all $x \in X$	The graph of f is symmetric about the origin 

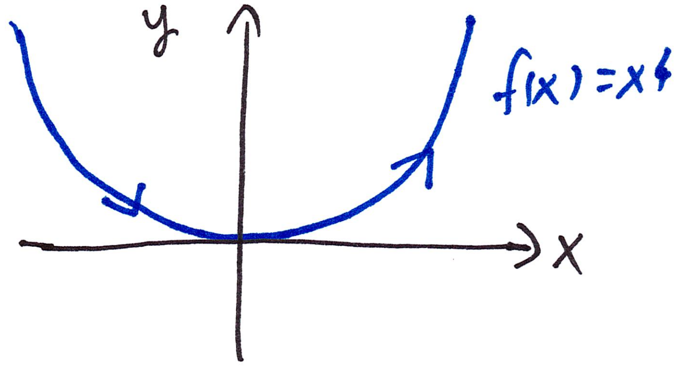
Note: Even and odd are NOT contraries for functions.



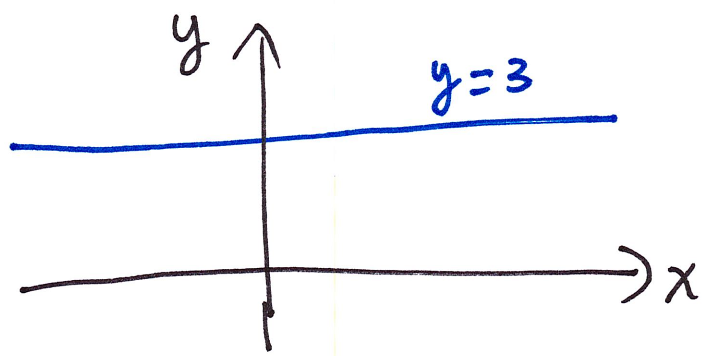
Both even and odd



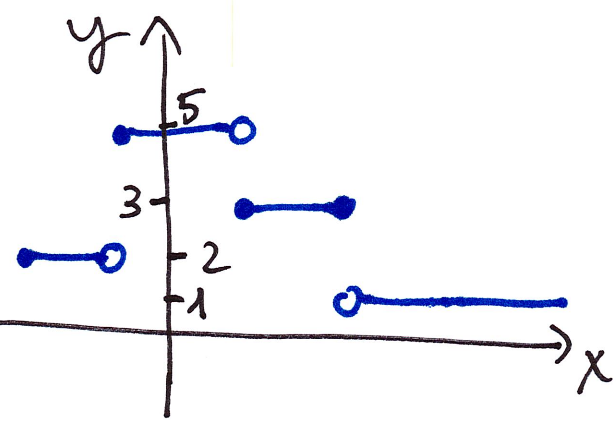
Neither even nor odd.

Property	Graph (or Example)
• f is <u>increasing</u> on an interval [a,b] contained in X if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in [a,b]	
• f is <u>decreasing</u> on [a,b] if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in [a,b]	f is: • decreasing on $(-\infty, 0]$ • increasing on $[0, +\infty)$

<u>Note</u>: Decreasing is <u>NOT</u> the contrary of increasing for a function on an interval [a,b] of its domain.



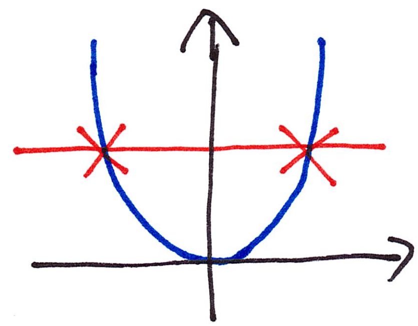
Neither increasing nor decreasing



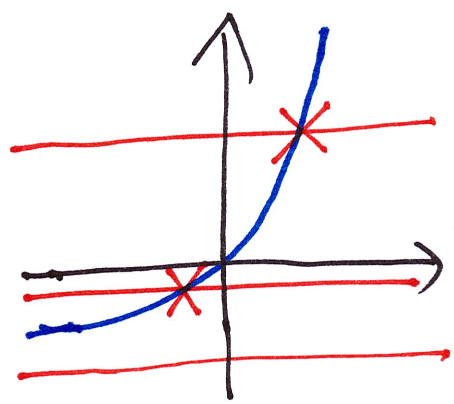
Neither increasing nor decreasing

Property	Graph (and Example)
• f is one-to-one (or injective) if $x_1 \neq x_2$ in $X \Rightarrow f(x_1) \neq f(x_2)$ (Different elements in the domain remain distinct when f is applied) Equivalently, f is one-to-one if $f(x_1) = f(x_2)$ in $Y \Rightarrow x_1 = x_2$ in X (If two elements in the domain become equal when f is applied, they were equal to start with)	The graph of f satisfies the horizontal line test: every horizontal line crosses the graph of f AT MOST once.

Ex:



Not one-to-one



One-to-one

$\boxed{\text{Slogan:}}$ One-to-one functions can be reversed/undone!

We'll see this next time. For now....

$\boxed{\text{Facts / Exercise}}$ (1) If $f: X \to Y$ is increasing or decreasing, then f is one-to-one.

(2) If $f: X \to Y$ and $g: Y \to Z$ are one-to-one, then the composite function $g \circ f$ is also 1-to-1.

- Check the set-theory handout online for some more stuff about sets.

- Look at the suggested exercises for Sections 1.1 & 1.3 and start doing them.

- On the next class, we will be looking at material from Section 1.4, 1.5 & Appendix D of the book.