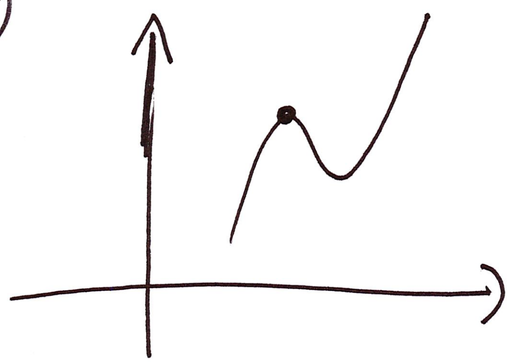
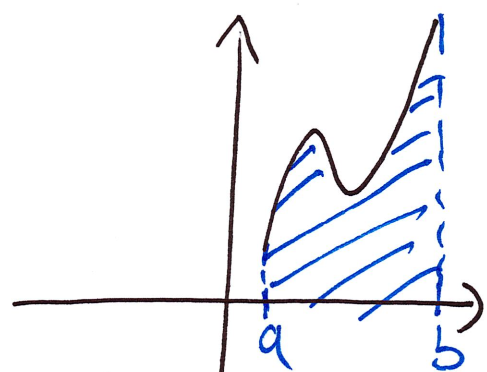
Lecture 4: Limits (Sep 14) (Section 2.2)

Two main goals:

① 

Find the slope of a curve at a point.

② 

Find the area under a curve

For both ① and ②, we can approximate very closely these quantities we are looking for. We would like to make an educated guess of those quantities based on these approximations.

That's limits' job.

Idea of limits.

Suppose a is a number and f is a function which is defined near $x=a$, that is, f is defined for all x close to a, ~~but~~ except possibly at $x=a$.

Ex: $f(x) = \dfrac{1}{x^2}$ is defined near $x=0$.

We'd like to predict the value that f should have at $x=a$, from the values around a.

This is what limits do!

Def: For a number L, we write

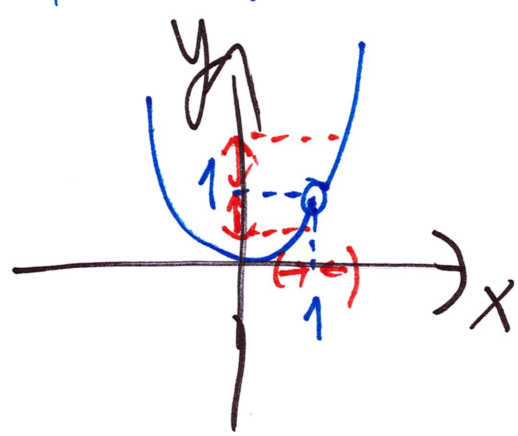
$$\lim_{x \to a} f(x) = L \quad \text{or} \quad f(x) \to L, \text{ as } x \to a$$

if we can make $f(x)$ as close to L as we want by taking x close enough to a, but not equal to a. (Here, f is defined near $x=a$)

Warning: Even if $\lim_{x \to a} f(x) = L$, this does NOT mean that $f(a) = L$. Why?

① f might not be defined at $x=a$.

Ex: $f(x) = x^2$ if $x \neq 1$, undefined at $x=1$

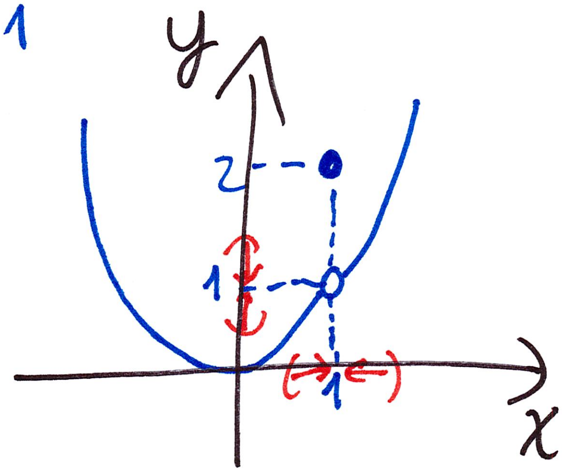


$\lim_{x \to 1} f(x) = 1$, but

f is not defined at $x=1$.

② Even if f is defined at $x=a$, $f(a)$ might still be $\neq L$.

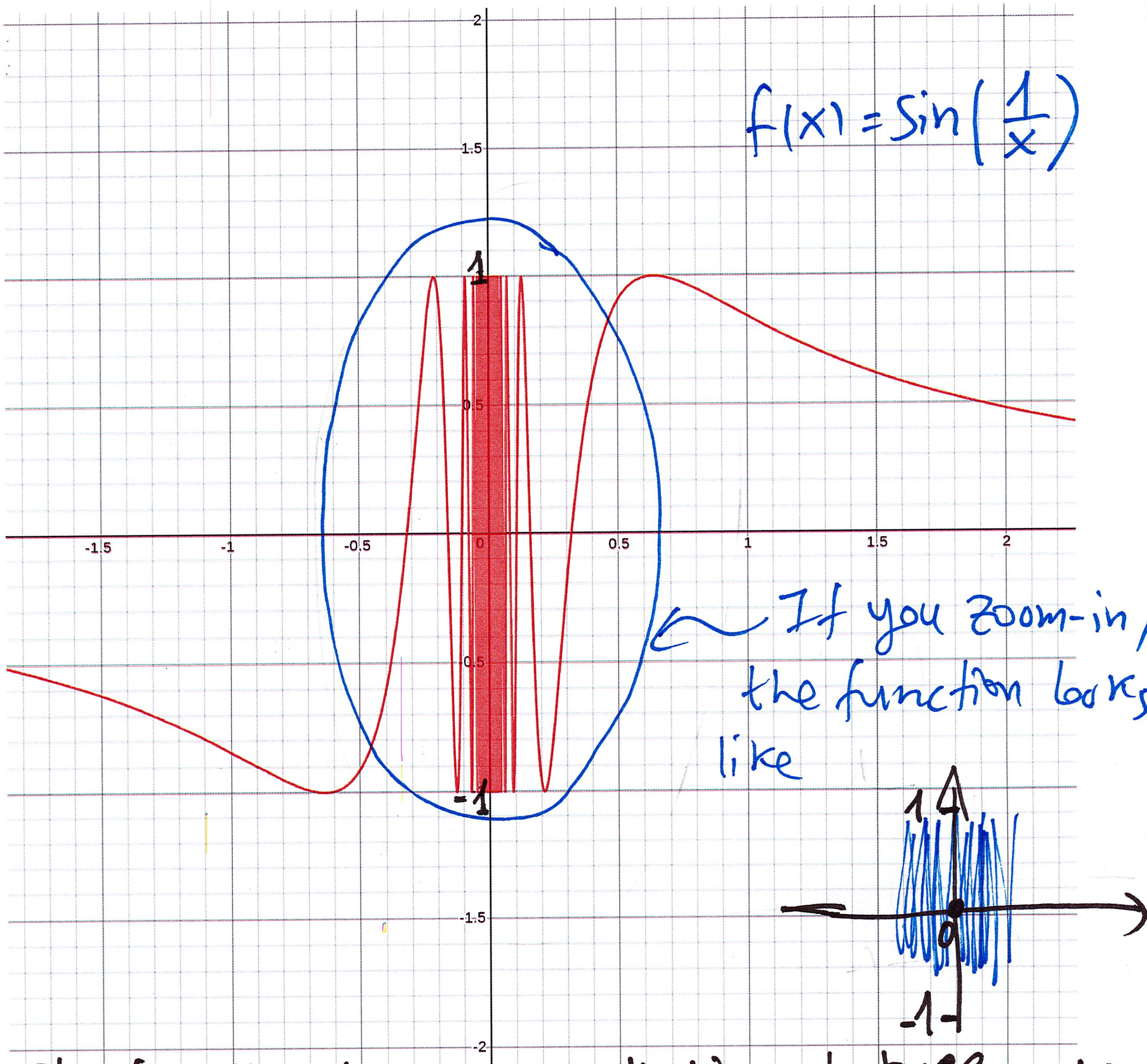
Ex:
$$f(x)=\begin{cases} x^2 & \text{if } x\neq 1 \\ 2 & \text{if } x=1 \end{cases}$$



We have that $f(1)=2$,
but $\lim_{x\to 1} f(x)=1$.

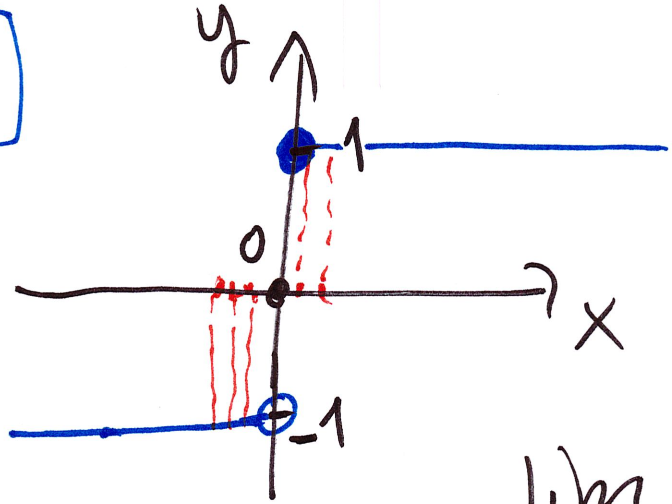
Slogan: "Limits don't care about what happens at a point $x=a$, they only care about what happens AROUND a".

Ex:



The function keeps oscillating between -1 and 1 as $x \to 0$, so it doesn't get close to any specific L. $\lim_{x \to 0} f(x)$ DNE

Ex:



$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$\lim_{x \to 0} f(x)$ DNE

Note: We can make f(x) as close as we want to a number L if:

(1) EITHER x gets close to 0, while ALWAYS being smaller than 0. (In this case $L = -1$)

(2) OR x gets close to 0, while ALWAYS being larger than 0. (In this case, $L = 1$).

But we can NOT make $f(x) \to L$ for a number L, if we allow x to get close to 0 while being sometimes < 0, sometimes > 0 (because f(x) "jumps" back and forth between $y = -1$ and $y = 1$)

Def: Suppose a is a number and f is a function that is defined for all x near AND to the right of a, excluding perhaps a. ("To the right of a" = "Larger than a").

We write

$$\lim_{x \to a^+} f(x) = L \quad \text{or} \quad f(x) \to L, \text{ as } x \to a^+$$

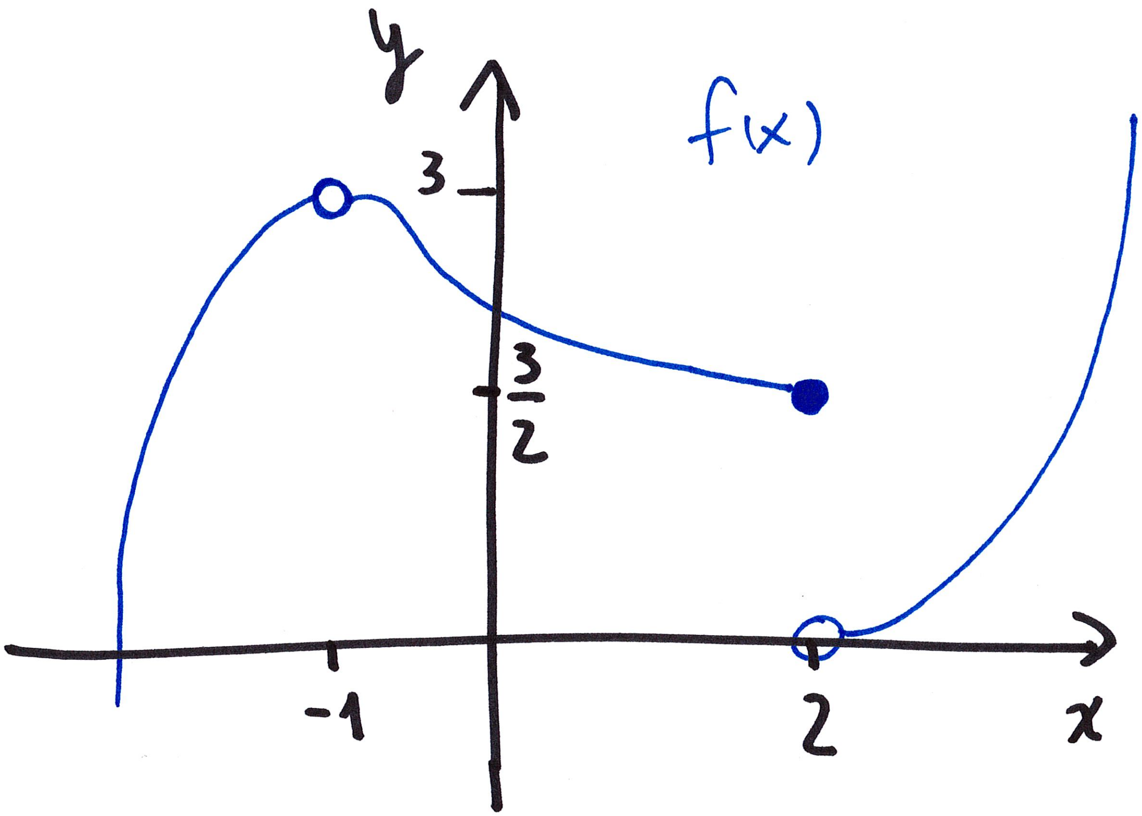
if we can make $f(x)$ as close to L as we want by taking x close enough to a, while being to the right of a.

Ex: $\lim_{x \to 1^+} f(x) = 1$ where $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

Replacing:
- <u>right</u> w/ <u>left</u> ("larger" w/ "smaller")
- $+$ with $-$,

we get the definition of $\lim_{x \to a^-} f(x) = L$.

Consider the function with the following graph.



Which of the following is true?

(a) $\lim_{x \to -1} f(x)$ exists & $\lim_{x \to 2} f(x)$ DNE

(b) $\lim_{x \to -1} f(x)$ DNE & $\lim_{x \to 2} f(x)$ exists

(c) $\lim_{x \to -1} f(x)$ DNE & $\lim_{x \to 2} f(x)$ DNE

(d) $\lim_{x \to -1} f(x)$ exists & $\lim_{x \to 2} f(x)$ exists

Thm: (Very important)

If f is defined near $x=a$ and L is a number, then

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L \text{ AND } \lim_{x\to a^+} f(x) = L.$$

Here the RHS means 3 things:

① that $\lim_{x\to a^-} f(x)$ exists

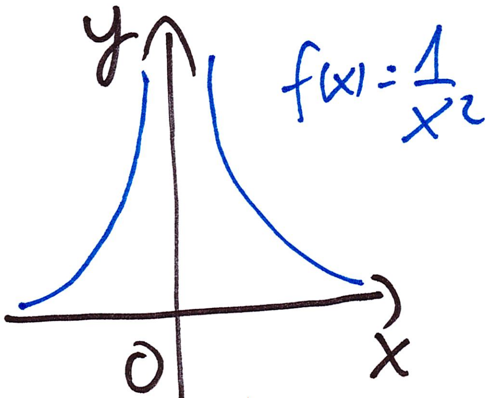
② that $\lim_{x\to a^+} f(x)$ exists

③ that $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, we say that the limit DOES NOT EXIST (DNE).

Let's see a different example.

Ex:



As $x\to 0^-$ or $x \to 0^+$, $f(x)$ does not get close to a single number, but it gets bigger and bigger.

We write $\lim_{x\to 0^-} f(x) = (+)\infty = \lim_{x\to 0^+} f(x)$.

In general, if f is defined near $x=a$, we write

$$\lim_{x \to a} f(x) = (+)\infty \quad \left(\text{resp. } \lim_{x \to a} f(x) = -\infty\right)$$

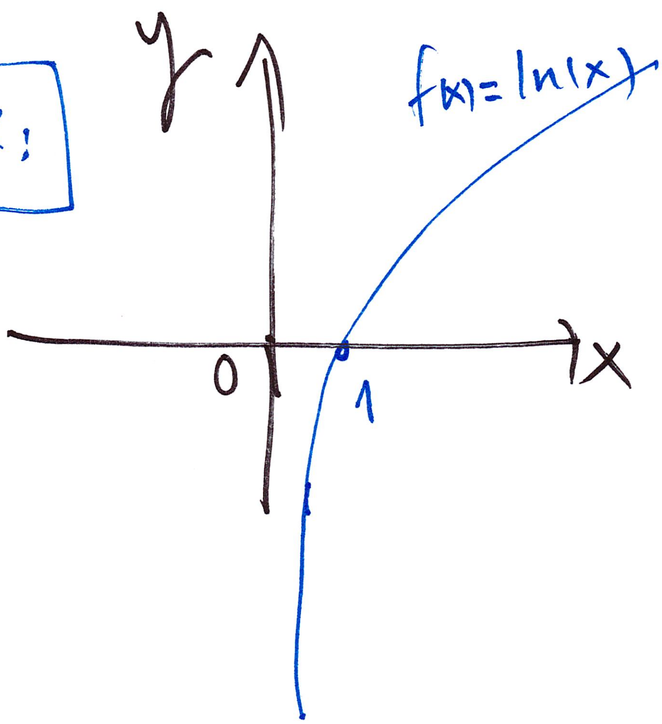
if $f(x)$ gets larger and larger (resp. smaller and smaller) as x approaches a (but $x \neq a$).

We can also define

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty$$

$$\left(\lim_{x \to a^{+}} f(x) = +\infty, \quad \lim_{x \to a^{+}} f(x) = -\infty \ldots\right)$$

$\boxed{\text{Ex:}}$



$$\boxed{\lim_{x \to 0^{+}} \ln(x) = -\infty}$$

For next class:

- Do the exercises in Section 2.2. If some of them don't make much sense and you don't understand how to compute a limit, do not worry, we will get much better tools to compute limits that make this operation almost routine.

- We will cover Section 2.3 and maybe start section 2.5 next time.

- Section 2.1 & 2.4 are not needed.

- You can get your quiz back on MONDAY 5-6 pm in MC104.