Explain again, explain better?

① The business with $+\infty$ & $-\infty$.

Problem: There is no real number which is larger than any other real number nor any real number which is smaller than any other real number.

However, being able to have two objects with these properties turns out to be handy for Calculus.

So we ADD two objects/symbols to $\mathbb{R}$ — $(+\infty)$ & $(-\infty)$ — with the defining property that

$$(\heartsuit) \quad -\infty < x < +\infty \qquad \text{for every } x \in \mathbb{R}.$$

These are NOT numbers, they don't have any concrete value (they aren't equal to something really big like 10^{80} or very small like -10^{100}). They are there only to fix the

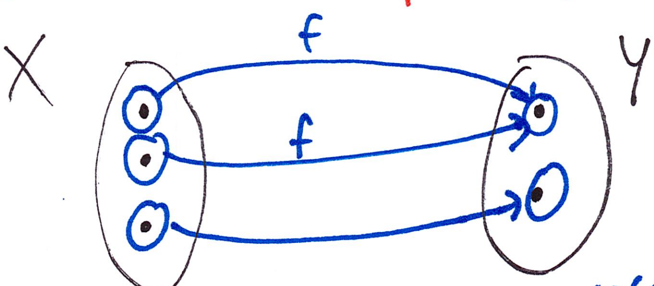
② The business with functions.

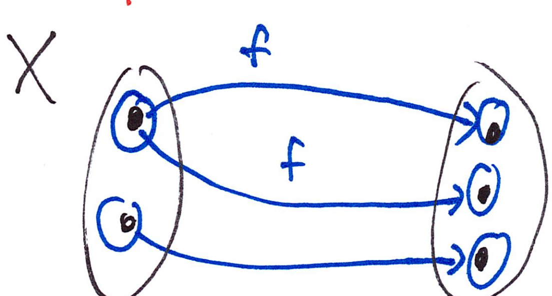
Remember, a function from a set X to a set Y is a rule that assigns to each $x \in X$ one and only one $f(x) \in Y$.

The blue statement means that the rule f has to be such that

"for one input x, there is only one associated output $f(x)$"

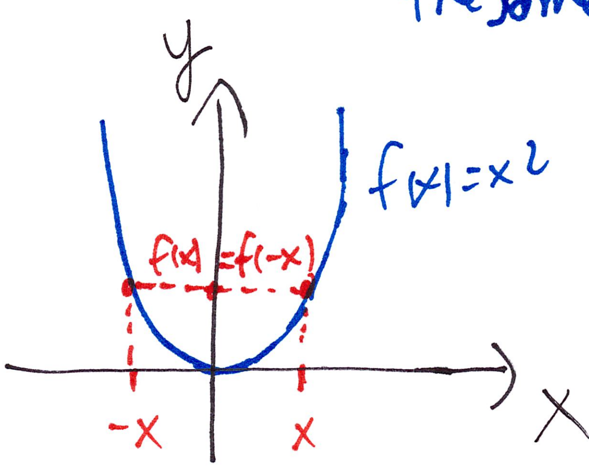
It is <u>fine</u> if "two distinct inputs X_1, X_2 have the same output $f(x_1) = f(x_2)$ in Y".

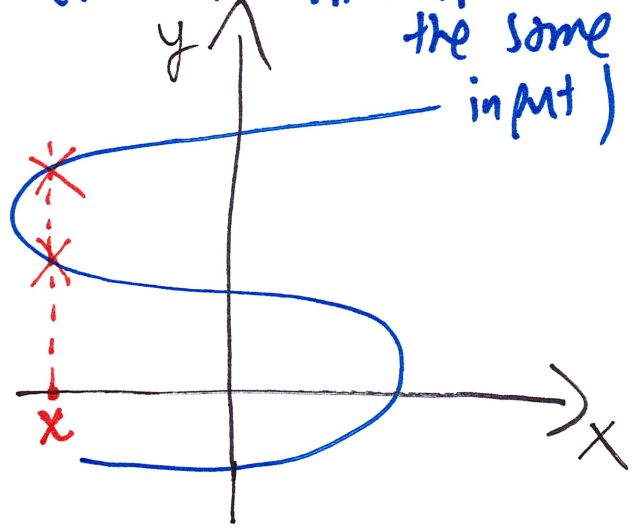




Functions (2 different inputs have the same output)

Not functions (two different outputs to the same input)





3) The business with $f(x) = \frac{2}{x^2-3}$ and ranges.

Let's check again what we did with the simpler example $f(x) = \frac{2}{x^2}$. We want to find its range.

We need to find those $y \in \mathbb{R}$ for which $y = \frac{2}{x^2}$ for some x in the domain of f.

① Consider $y = f(x)$ and write x in terms of y.

<u>IF</u> $y = \frac{2}{x^2}$, then $x^2 y = 2$.

If $y \neq 0$, $x^2 y = 2 \Rightarrow x^2 = \frac{2}{y}$. Since $x^2 > 0$, $y > 0$ and $x^2 = \frac{2}{y} \Rightarrow x = \pm\sqrt{\frac{2}{y}}$.

So, if $y > 0$, $x = \pm\sqrt{\frac{2}{y}}$ is such that $f(x) = y$. But.... we need to make sure we can apply f to $x = \pm\sqrt{\frac{2}{y}}$!

② Check if all x's found in ① belong to the domain of f.

We ask: if $y > 0$, is $x = \pm\sqrt{\frac{2}{y}}$ <u>always</u> in the domain of f?

Domain of $f = \mathbb{R} \setminus \{0\}$. Can $x = \pm\sqrt{\frac{2}{y}}$ ever be 0, when $y > 0$? No! (If $y > 0$, $\frac{2}{y} > 0$, so $\sqrt{\frac{2}{y}} > 0$ and $-\sqrt{\frac{2}{y}} < 0$).

When deriving the formula, we had to assume $y \neq 0$, in order to divide by y.

But maybe $y=0$ actually belongs to the range of f... Does it?

Is there x in the domain of f s.t.

$$\frac{2}{x^2} = 0 \, ?$$

No! $\left(\frac{2}{x^2} = 0 \iff 2=0\right.$, but $\left.2 \neq 0\right)$.

③ Check whether any specific value y you had to exclude during the computations in ① might actually belong to the range of f.

So, we discovered that:

(a) if $y \in \mathbb{R}$ is such that $y = \overbrace{\frac{2}{x^2}}^{=f(x)}$ (that is, if y is an element of the range of f), then y has to be >0 $\left(\frac{2}{x^2}\right.$ is always $\left.>0\right)$. So:

$$\text{Range of } f \subseteq \{y \in \mathbb{R} : y > 0\} = \mathbb{R}_+$$

"is a subset of" / "is contained in"

(b) on the other hand, if $y>0$, $x = \pm\sqrt{\frac{2}{y}}$ is in the domain of f and $f\left(\pm\sqrt{\frac{2}{y}}\right) = y$. So:

$\mathbb{R}_+ = \{y \in \mathbb{R} : y > 0\} \subseteq \text{Range of } f$.

These (a) and (b) together mean:

$$\text{Range of } f = \mathbb{R}_+ \; (= (0, +\infty))$$

④ The business with $g \circ f$.

Recall that, given $f: X \to Y$, $g: Y \to Z$, we had defined

$$g \circ f : X \to Z \quad , \quad (g \circ f)(x) := g(f(x))$$

For this to work, we need to know:

- (a) the formulas for f and g
- (b) the domain of f and the domain of g
- (c) the fact that the range of f is contained in the domain of g.

If you only know the formulas for f and g, you need to find appropriate domains of f and g so that (c) holds and you can actually write down $g \circ f$.

Let's understand this a bit better with an example.

Ex: Consider the functions given by

$$f(x) = x^2 \quad \text{and} \quad g(x) = \sqrt{x-2}$$

Find an appropriate domain for $g \circ f$ to be defined.

__Sol:__ Let's try to understand what the problem is first. Usually, we would say that:

- Domain of $f = \mathbb{R}$
  Range of $f = \mathbb{R}_{\geq 0}$
- Domain of $g = [2, +\infty) = \{x \in \mathbb{R} : 2 \leq x < +\infty\}$
  $(x - 2 \geq 0 \iff x \geq 2)$

  Range of $g = \mathbb{R}_{\geq 0}$ (think about this)

So we would have:

$$f : \mathbb{R} \to \mathbb{R}_{\geq 0} \qquad g : [2, +\infty) \to \mathbb{R}_{\geq 0}$$

The range of of f ($\mathbb{R}_{\geq 0}$) is __NOT__ contained in the domain of g (why?)

So, we can't do $g \circ f$ if we pick these domains for f and g. So... what do we do?

$f(x) = x^2$

$g(x) = \sqrt{x-2}$

We want to find for which $x \in \mathbb{R}$, we have that

$$f(x) = x^2 \in [2, +\infty) \text{ (the domain of } g)$$

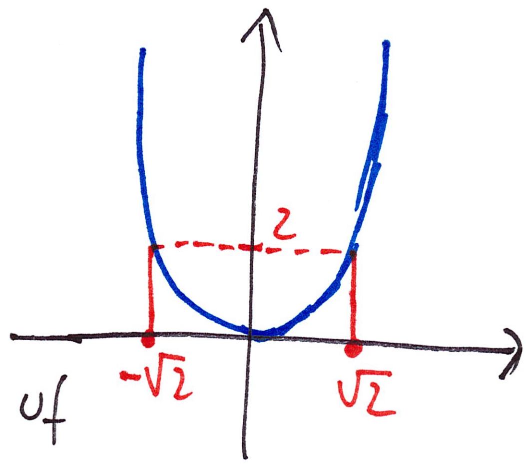
That is, we need to find all $x \in \mathbb{R}$ for which

$$2 \leq x^2 \; (< +\infty \text{, but this is always true}).$$

This isn't too hard...

We have that

$$x^2 = 0 \iff x = \pm\sqrt{2}$$



and so - because the graph of x^2 is a convex (upward) parabola - we have that

$$x^2 \geq 2 \iff x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2}$$

So, the domain of $g \circ f$ is

$$\{x \in \mathbb{R} : x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2}\} =$$
$$= (-\infty, -\sqrt{2}] \cup [\sqrt{2}, +\infty)$$

(Try to explain to yourself why this is the domain of $g \circ f$).

Lecture 2 (Sep 12) - Inverse functions with examples

Sections 1.4–1.5 & Appendix D.

<u>Recall</u>: A function f is <u>one-to-one</u> if

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

We are interested in <u>one-to-one functions</u> because they can be <u>reverted/inversed</u>.

Def: Suppose f is a one-to-one function with

X = domain of f $\qquad$ Y = range of f.

The <u>inverse function</u> of f is the function

$$f^{-1}: Y \to X$$

(Reversed domain and range.)

defined by:

$$\boxed{f^{-1}(y) = x \Leftrightarrow f(x) = y} \quad \text{for } y \in Y.$$

What does this mean?

It means that, for every $y \in Y$ and $x \in X$,

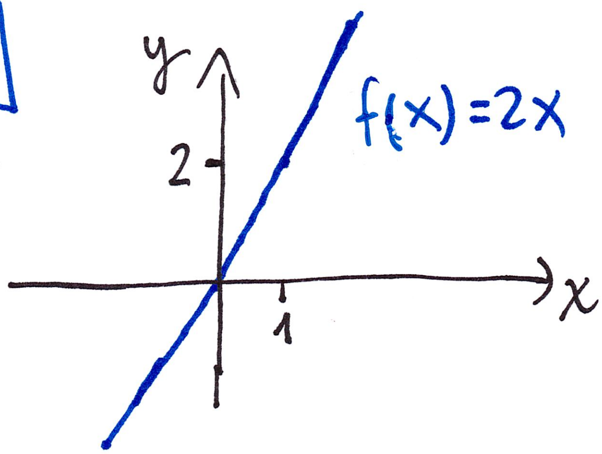
$f^{-1}(y)$ is the <u>only</u> (because f is 1-to-1) number in X such that

$$\boxed{f(f^{-1}(y)) = y}$$

and

$$\boxed{f^{-1}(f(x)) = x}$$

$\boxed{\text{Ex:}}$



$f: \mathbb{R} \to \mathbb{R}$

$f(x) = 2x$

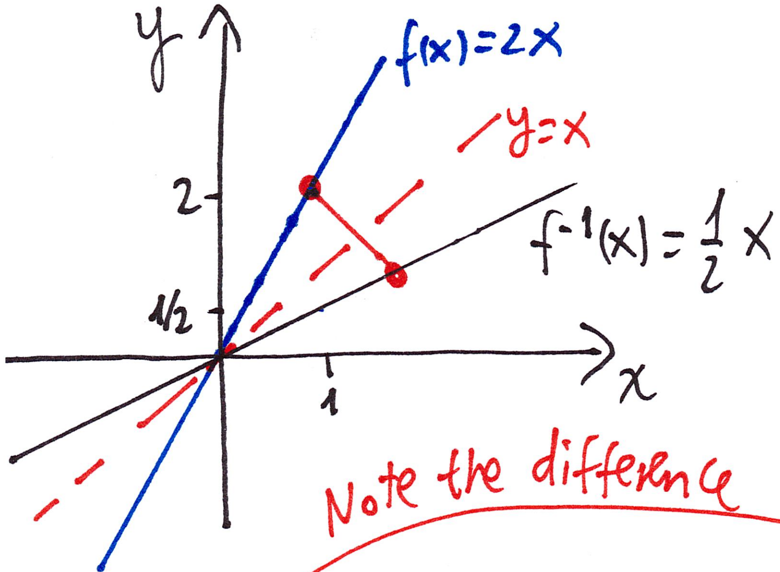
What is f^{-1}?

f^{-1} should be such that, for $x \in \mathbb{R}$,

$$x = f^{-1}(f(x)) = f^{-1}(2x)$$

How to pass from $2x$ to x? Divide by 2!

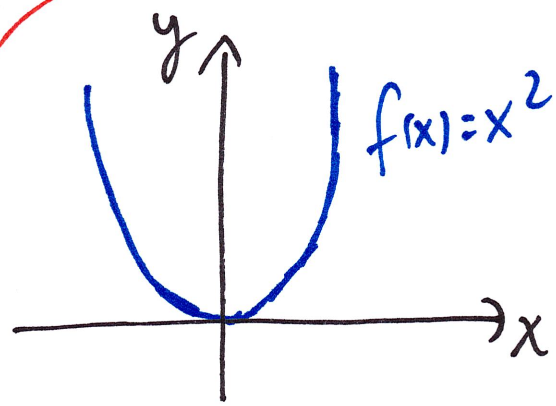
$$f^{-1}(x) = \frac{1}{2}x$$



Fact: The graph of f^{-1} is obtained by reflecting the graph of f about the line $y=x$.

Note the difference

Ex:



$f: \mathbb{R} \to \mathbb{R}_{\geq 0}$

$f(x)=x^2$

What is f^{-1}?

The question DOES NOT MAKE SENSE. Why?

Because f is NOT 1-TO-1: $f(2)=4=f(-2)$

(An even function is never 1-to-1).

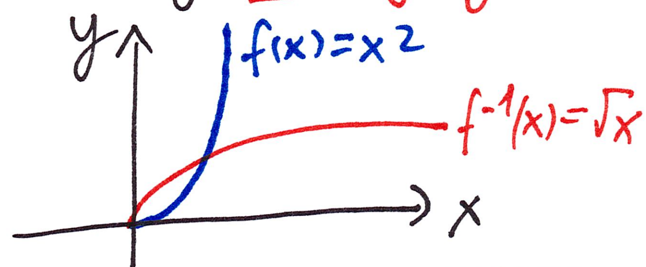
However, we can make it one-to-one by changing its domain

$f: \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$

$f(x)=x^2$

$f^{-1}: \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$

$f^{-1}(x)=\sqrt{x}$



$\sqrt{x}$ only makes sense when $x \geq 0$

Recipe: How to find f^{-1}.

① Make sure f is one-to-one. (Optional: don't do this step if they don't ask you to).

② Write $y = f(x)$.

③ Solve this equation for x in terms of y.
(Recall: this is exactly what we did to find the range of $f(x) = \frac{2}{x^2-3}$ last time...)

④ In order to write f^{-1} as a function of x, interchange x and y in the formula you found in ③. Then $y = f^{-1}(x)$.

⑤ If X = domain of f and Y = range of f,

$$f^{-1} : Y \longrightarrow X$$

domain of f^{-1} = range of f

range of f^{-1} = domain of f

Let's see an example right away.

$\boxed{\text{Ex:}}$ Given the function

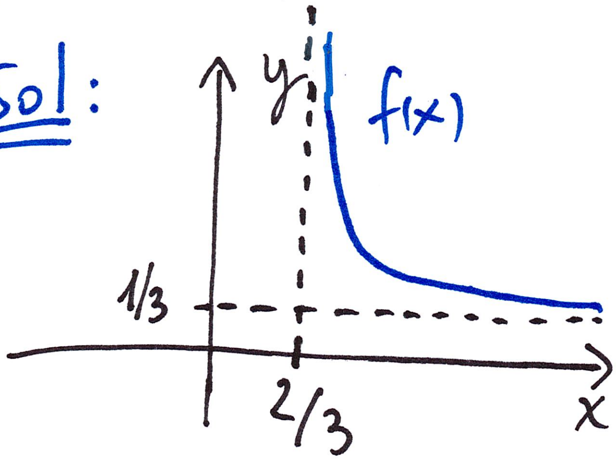
$$f : \left(\tfrac{2}{3}, +\infty\right) \longrightarrow \left(\tfrac{1}{3}, +\infty\right),$$

Range off

$$f(x) = \frac{x+2}{3x-2}$$

find its inverse.

Sol:



① f is decreasing

$\Rightarrow$ f is one-to-one

(Horizontal line test)

② Write $y = f(x) = \dfrac{x+2}{3x-2}$

③ Solve for x in terms of y.

$$y = \frac{x+2}{3x-2} \xRightarrow[\text{by } (3x-2)]{\text{Multiply}} (3x-2)y = x+2 \Rightarrow$$

$$\Rightarrow 3xy - 2y = x + 2$$

$$\Rightarrow 3xy - x = 2y + 2 \xRightarrow[\text{x on the LHS}]{\text{Factor}} x(3y-1) = 2(y+1)$$

$$\xRightarrow[\text{by } (3y-1)]{\text{Divide}} x = \frac{2(y+1)}{3y-1}$$

④ Interchange x and y to get $f^{-1}(x)$.

$$\boxed{f^{-1}(x) = y = \frac{2(x+1)}{3x-1}}$$

⑤ Get $f^{-1}: \left(\frac{1}{3}, +\infty\right) \longrightarrow \left(\frac{2}{3}, +\infty\right)$

$$f^{-1}(x) = \frac{2(x+1)}{3x-1}$$

⑥ As a sanity check that you got the right answer, you can actually check whether $f^{-1}(f(x)) = x$ and/or $f(f^{-1}(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}\left(\frac{x+2}{3x-2}\right) \overset{}{=}$$

⟵ Sub. $\frac{x+2}{3x-2}$ for x in the formula for f^{-1}

$$= \frac{2\left(\frac{x+2}{3x-2} + 1\right)}{3\left(\frac{x+2}{3x-2}\right) - 1} = \frac{2\left(\frac{x+\cancel{2}+3x-\cancel{2}}{3x-2}\right)}{\frac{\cancel{3x}+6-\cancel{3x}+2}{3x-2}} =$$

$$= \frac{\left(\frac{\cancel{8x}}{\cancel{3x-2}}\right)}{\left(\frac{\cancel{8}}{\cancel{3x-2}}\right)} = x.$$

Ex: Suppose f is a decreasing function and suppose that $f^{-1}(5)=8$. Which one of the following inequalities follows from this information?

ⓐ $f(7)>5$ ⓑ $f(8)>5$ ⓒ $f(7)<5$

ⓓ $f(8)<5$

Sol: Since f is decreasing, we have that

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

Therefore, $7<8 \Rightarrow f(7)>f(8)$.

But $f^{-1}(5)=8$ and $f(f^{-1}(5))=5$. So:

$$f(7)>f(8) \underset{f^{-1}(5)=8}{\Longrightarrow} f(7)>f(f^{-1}(5)) \underset{f(f^{-1}(5))=5}{\Longrightarrow} f(7)>5$$

Thus, the right answer is given by ⓐ.

Note: You could have excluded ⓑ and ⓓ right off the bat because

$$f(8)=f(f^{-1}(5))=5 \quad (\text{and } 5 \not> 5, \; 5 \not< 5).$$

We will next study very important examples of one-to-one functions and their inverses:

① Exponential and logarithmic functions.

② Trigonometric functions and their inverses.

• Exponentials and logarithms

Def: An exponential function is a function of the form

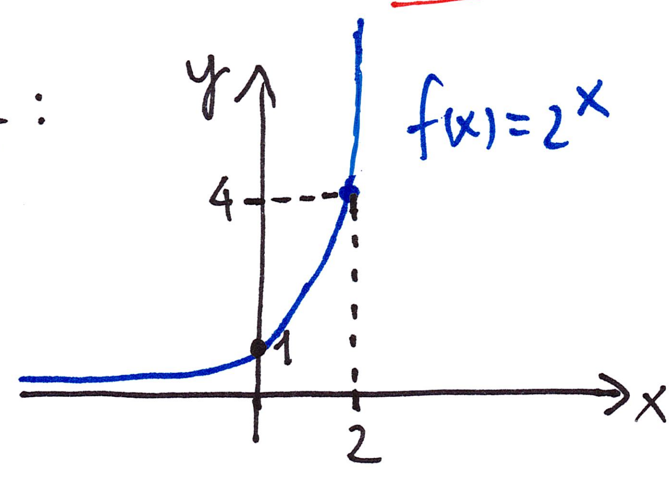
$$f: \mathbb{R} \longrightarrow \mathbb{R}_{>0}$$
$$f(x) = b^x$$

Domain of $f = \mathbb{R}$

Range of $f = (0, +\infty)$

where b is a fixed positive ($b>0$) number.

Ex: $b=2$:



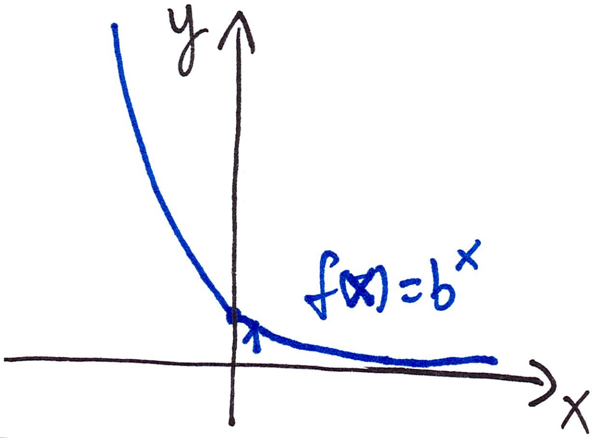
Beware: $b^x \neq x^b$

The functions 2^x and x^2 are NOT the same

(Plot the two functions on Geogebra)

The graph of b^x depends on the value of b.

ⓐ $f(x) = b^x$ for $0 < b < 1$

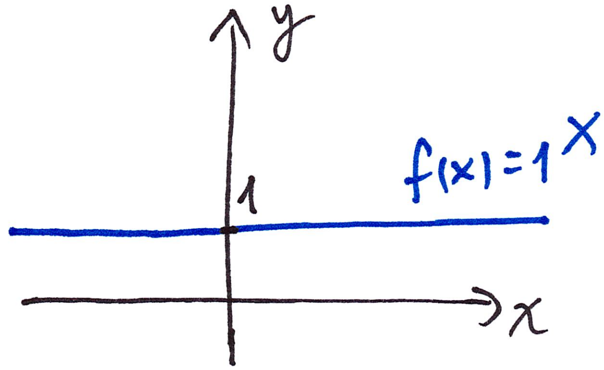


Decreasing function

Domain: $\mathbb{R}$

Range: $\mathbb{R}_{>0} = (0, +\infty)$

ⓑ $f(x) = 1^x$

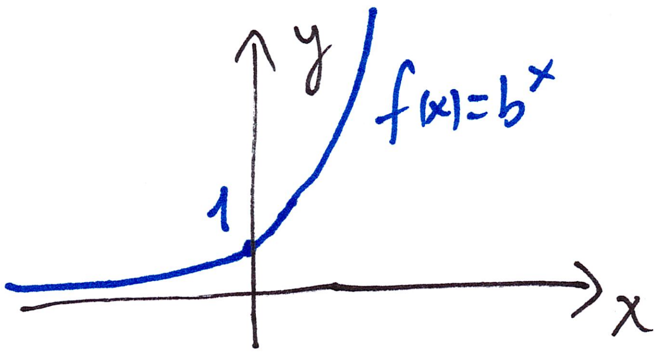


Constant function at 1

Domain: $\mathbb{R}$

Range: $\{1\}$

ⓒ $f(x) = b^x$ for $b > 1$



Increasing function

Domain: $\mathbb{R}$

Range: $\mathbb{R}_{>0} = (0, +\infty)$

In real life:

- bacteria and mold on food left out of the fridge grow according to an increasing exponential function;
- atoms in radioactive material decay (that is, they die - sort of) according to a decreasing exponential function.

Some special values of b^x:

- when $x=0$, $b^0=1$;
- when $x=n=1,2,3\cdots$, $b^n = \underbrace{b\cdot b\cdots b}_{n \text{ times}}$;
- when $x=-n$, for $n=1,2,3,\cdots$, $b^{-n} = \frac{1}{b^n}$;
- when $x=\frac{p}{q}$, $q\neq 0$, $b^{\frac{p}{q}} = \sqrt[q]{b^p}$ $\quad b^{1/2} = \sqrt[2]{b}$

Laws of Exponents. (VERY important)

For $a,b \in \mathbb{R}_{>0} = (0,+\infty)$ and $x,y \in \mathbb{R}$:

① $b^{x+y} = b^x \cdot b^y$, ② $b^{x-y} = \frac{b^x}{b^y}$, ③ $(b^x)^y = b^{xy}$

④ $(ab)^x = a^x b^x$

$\boxed{\text{Ex:}}$ Compute $\left(b^{2x}\right)^{(7y+\frac{1}{x})}$.

Sol: $\left(b^{2x}\right)^{(7y+\frac{1}{x})} \underset{③}{=} b^{2x(7y+\frac{1}{x})} = b^{14xy+2}$

$\underset{①}{=} b^2 \cdot b^{14xy}$.

When considering

$$f(x) = b^x,$$

there is a special choice for the number $b>0$ that is often used in Math and comes up in real life.

This number is called e.

"e" has to do with growth in real life. The bacteria's growth ex. of page 9 increases according to an exponential function with base e.

$$e \approx 2.71828\ldots$$

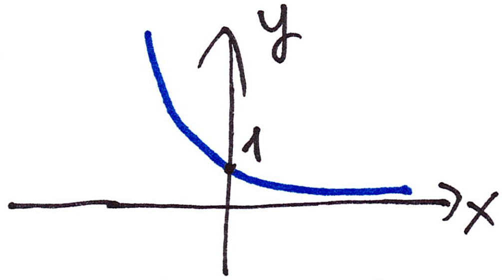
e is <u>not</u> a rational number

We will get tools to specify better what number e is later on:

- $e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$
- if $f(x) = a^x$ and $f'(x) = f(x)$, then $a = e$.

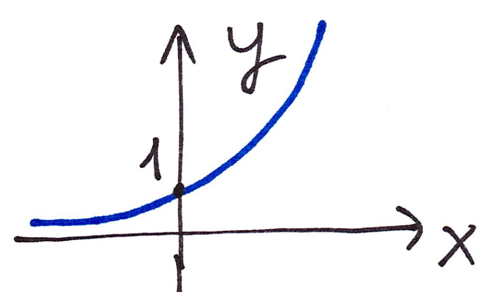
— o — o — o —

As we saw earlier on, if $b>0$ and $b \neq 1$, $f(x) = b^x$ looks like



(if $0<b<1$)

or like



(if $b>1$)

In either case, when $b>0$ and $b \neq 1$, $f(x)=b^x$ is one-to-one, hence we can invert it!

$\boxed{\text{Def:}}$ For $b>0$ and $b\neq 1$, the logarithm with base b is the inverse function of $f(x)=b^x$ and is denoted by

$$f(x)=\log_b(x).$$

$\ln(x)=\log_e(x)$ has to do with time needed for a certain growth.
e^x = amount of growth at time x
$\ln(x)$ = time needed to grow of x

By definition:

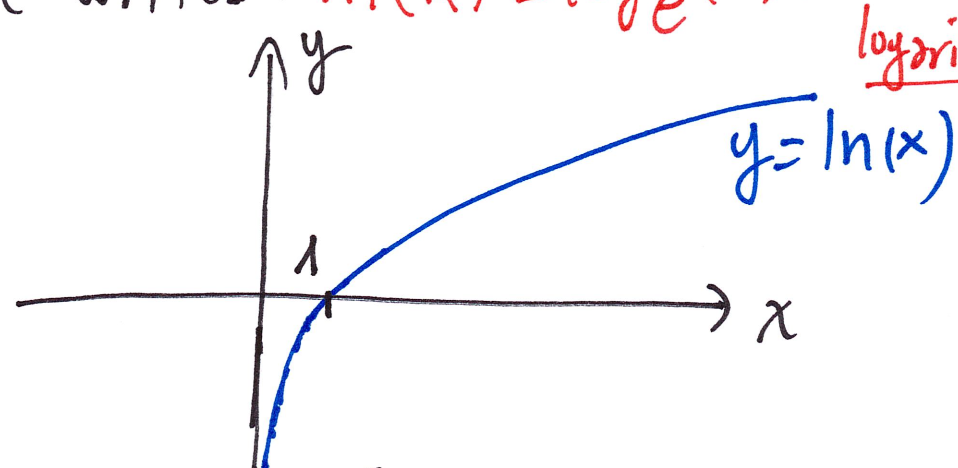
(1) the domain of $\log_b$ is $\mathbb{R}_{>0}$, the range is $\mathbb{R}$;

(2) $\log_b(b^x)=x$, for $x\in\mathbb{R}$.
$b^{\log_b(x)}=x$, for $x>0$

In particular, $\log_b(1)=0$, since $b^0=1$.

(Note: These 2 equations are just the equations $f^{-1}(f(x))=x$ and $f(f^{-1}(x))=x$, with $f(x)=b^x$ and $f^{-1}(x)=\log_b(x)$).

When $b=e$, one writes $\ln(x)=\log_e(x)$. (ln for natural logarithm)



From the laws of exponentials, we can derive some important laws for logarithms.

Laws of logarithms (VERY important).

For $b>0$ and $b\neq 1$ and for $x, y \in \mathbb{R}$, we have:

(1) $\log_b(xy) = \log_b(x) + \log_b(y)$

(2) $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$ (for $y\neq 0$)

(3) $\log_b(x^r) = r\log_b(x)$ (for $r\in\mathbb{R}$).

We also have another rule that allows us to express every logarithm with base b in terms of the natural logarithm.

(4) (Change of base rule) $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$

Let's try to use the laws of exponentials to derive the first law of logarithms.

Let's do it!

Recall: b^x is a <u>1-to-1 function</u>.

This means that, if $x_1, x_2 \in \mathbb{R}$,

$$b^{x_1} = b^{x_2} \implies x_1 = x_2.$$

So if we want to show that $x_1 = x_2$, we can instead show that $b^{x_1} = b^{x_2}$....

... But isn't b^{x_1} "more complicated" than x_1?

Not if $x_1 = \log_b(x)$, because of the rule

$$b^{\log_b(x)} = x \; !$$

So now to show that

$$\underbrace{\log_b(xy)}_{= x_1} = \underbrace{\log_b(x) + \log_b(y)}_{= x_2}$$

We can show that $b^{\log_b(xy)} = b^{\log_b(x) + \log_b(y)}$.

But: $b^{\log_b(xy)} = xy$

$$b^{\log_b(x)+\log_b(y)} = \underbrace{b^{\log_b(x)}}_{= x} \cdot \underbrace{b^{\log_b(y)}}_{= y} = xy$$

They are the same! :)

First rule of exponentials

Hence, since:

(1) $b^{\log_b(xy)} = b^{\log_b(x)+\log_b(y)}$

AND

(2) b^z is a one-to-one function,

we can conclude that

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

as we wanted.

Congratulations! You just saw your first mathematical proof in this course!

Take some time later on to look back at it and think through what was done. In particular, notice how we used properties we already knew about exponentials to derive something new about logarithms.

$\boxed{\text{Ex:}}$ Solve: $2e^{x^2-\ln(2)} = \sqrt{8}$

Sol: Divide by 2 both sides:

$$e^{x^2-\ln(2)} = \frac{\sqrt{8}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\sqrt{8} = \sqrt{2^3} = \sqrt{2\cdot 4}$$

We know: $\ln(e^t) = t$, so we can apply ln to both sides and "kill e".

$$\underbrace{\ln(e^{x^2-\ln(2)})}_{= x^2-\ln(2)} = \ln(\sqrt{2}) = \ln(2^{1/2}) = \frac{1}{2}\ln(2)$$

③ Laws of log

$$\Rightarrow x^2 - \ln(2) = \frac{1}{2}\ln(2)$$

$$\Rightarrow x^2 = \frac{1}{2}\ln(2) + \ln(2) = \frac{3}{2}\ln(2)$$

$$\Rightarrow x = \pm\sqrt{\frac{3}{2}\ln(2)}$$

$$\left(x = \pm\sqrt{\ln(2)+\ln(\sqrt{2})}\right)$$

Remember:

$$\ln(x) \text{ is } \begin{cases} <0 & \text{if } 0<x<1 \\ =0 & \text{if } x=1 \\ >0 & \text{if } x>1 \end{cases}$$

For next class

- Do the suggested exercises for section 1.4.
- Start doing the suggested exercises for section 1.5 - but only those about functions we saw already (no sin, cos, tan just yet).
- Next time, we will look at Appendix D and finish off section 1.5.

First quiz is on Wednesday (20'-long, at the beginning of class, open book just for this time).
The quiz will be on Sections 1.1-1.5 and Appendix D.