A few more things about sets

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In class we saw a few things about sets. I will gather here some other useful concepts that might appear here and there as we move on, just in case you haven't seen them before. I will be using the number sets ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$) and the intervals we already introduced in class to give a few illustrative examples of the notions I will talk about.

Recall that, given a set X, we write:

- $x \in X$ to mean that x is an element of X;
- $x \notin X$ to mean that x is not an element of X.

There is a special set called the **empty set** and denoted by $\emptyset$. Its defining property is that it has no elements. In other words, given any x, we always have that $x \notin \emptyset$.

We also saw in class that there is a very handy way of defining sets. Namely, we can define a set by just specifying a property that is satisfied precisely by all the elements of that set. For example,

$$\mathbb{R}_{>0} := \{x : \ x \in \mathbb{R} \text{ and } x > 0\}$$

is the set defined by the property "$x \in \mathbb{R}$ and $x > 0$", so that an x is an element of $\mathbb{R}_{>0}$ precisely if it is a positive real number.

Fix now two sets X and Y.

1. We say that $Y$is a **subset** of X, written $Y \subseteq X$, if every element of Y is also an element of X.

   Examples

   - $\mathbb{Z} \subseteq \mathbb{Q}$ because every integer n can be written as a fraction (for example, as $\frac{n}{1}$).
   - $(1, 5) \subseteq (1, 5] \subseteq [1, 5] \subseteq (0, 7) \subseteq \mathbb{R}_{>0} \subseteq (-3, +\infty) \subseteq \mathbb{R}$. (Every time you see the symbol "$\subseteq$" here, think about why the left-hand side is a subset of the right-hand side).
   - $X \subseteq X$ and $\emptyset \subseteq X$.

2. We say that the sets X and Y are **equal** if $X \subseteq Y$ and $Y \subseteq X$. In other words, X and Y are equal sets if every element of X is an element of Y and, also, every element of Y is an element of X.

   Examples (Try to understand why the examples below are true!)

- $\{x: \ x \in \mathbb{R} \text{ and } x^2 < 0\} = \emptyset$
- $\{x: \ x \in \mathbb{R} \text{ and } x^3 = 1\} = \{1\}$
- $\{(x,y): \ x,y \in \mathbb{R} \text{ and } x+y=0\} = \{(x,y): \ x,y \in \mathbb{R} \text{ and } y=-x\}$.

3. The **union** of X and Y is the set $X \cup Y$ defined by

$$X \cup Y := \{z: \ z \in X \text{ or } z \in Y\}.$$

In plain words, z is an element of $X \cup Y$ if and only if z is an element of X or z is an element of Y (this includes the possibility that z is an element of *both* X and Y). To put it differently, $X \cup Y$ is made of all elements of X and all elements of Y.

Examples (Try to understand why the examples below are true!)

- $(-2,3) \cup (6,9]$ is the set of all real numbers x such that $-2 < x < 3$ or $6 < x \leq 9$.
- $(3,8) \cup \{8\} = (3,8]$, $(3,8) \cup \{3,8\} = [3,8]$.
- $[-4,7) \cup (2,9] = [-4,9]$, $(-\infty,4) \cup [-6,+\infty) = \mathbb{R}$.
- If $Y \subseteq X$, $X \cup Y = X$. In particular, $X \cup \emptyset = X$ and $X \cup X = X$.

4. The **intersection** of X and Y is the set $X \cap Y$ defined by

$$X \cap Y := \{z: \ z \in X \text{ and } z \in Y\}.$$

In other words, the elements of $X \cap Y$ are all and only the elements of X that are also elements of Y.

Examples (Try to understand why the examples below are true!)

- $(2,4) \cap (3,12] = (3,4)$, $[-3,8] \cap (-3,10] = (-3,8]$.
- $(1,2] \cap [2,4] = \{2\}$, $[3,6] \cap ([-4,3] \cup [6,9]) = \{3,6\}$, $\mathbb{Z} \cap (4,8) = \{5,6,7\}$, $\mathbb{R}_{\geq 0} \cap \mathbb{R}_{\leq 0} = \{0\}$.
- $(2,7) \cap (-3,0) = \emptyset$, $\mathbb{R}_{>0} \cap \mathbb{R}_{<0} = \emptyset$, $\mathbb{Z} \cap (4,5) = \emptyset$.
- If $Y \subseteq X$, $X \cap Y = Y$.
- If $f\colon X \to Y$ and $g\colon Z \to W$ are functions, with X, Y, Z, W subsets of $\mathbb{R}$, the functions $f+g$ and fg are defined on $X \cap Z$, whereas $\frac{f}{g}$ is defined on the subset of $X \cap Z$ consisting of all those x in $X \cap Z$ such that $g(x) \neq 0$.

5. The **complement** of Y in X is the set $X \setminus Y$ defined by

$$X \setminus Y := \{x: \ x \in X \text{ and } x \notin Y\}.$$

In other terms, $X \setminus Y$ has as elements exactly those elements of X that are not elements of Y.

Examples (Try to understand why the examples below are true!)

- $\mathbb{R} \setminus \{2,4,6\}$ is the set of all real numbers that are different from $2, 4$ and 6.
- $(2,4) \setminus [3,4) = (2,3)$, $(2,4) \setminus (3,4) = (2,3]$, $\mathbb{R} \setminus \mathbb{R}_{\geq 0} = \mathbb{R}_{<0}$.

- $[-2, 10) \setminus (0, 9) = [-2, 0] \cup [9, 10)$, $\mathbb{R} \setminus (-9, 9) = (-\infty, -9] \cup [9, +\infty)$.
- $(1, 2] \setminus (1, 2) = \{2\}$, $[1, 2] \setminus (1, 2) = \{1, 2\}$.
- For any set X, $X \setminus X = \emptyset$, $X \setminus \emptyset = X$.

Note. One can often use the concepts introduced above to describe the same set in many different ways. For example the sets

$$(2, 4) \cup (3, 6), \quad (-7, 6) \cap (2, 9), \quad (0, 6) \setminus (0, 2]$$

are all equal to the interval $(2, 6)$. (Why?)

EXERCISES

Explain in words what the elements of the following sets are and find another way of describing each of these sets. Exercise number 0 is already solved so as to give an example of what I am asking you to do.

0. $(2, 6) \cap (3, 9]$. By definition of intersection and of the displayed intervals, this is the set of all real numbers x such that both $2 < x < 6$ and $3 < x < 9$. But these two conditions together imply that $3 < x < 6$. Therefore, $(2, 6) \cap (3, 9)$ can also be described as the interval $(3, 6)$. In more symbolical terms, $(2, 6) \cap (3, 9) = (3, 6)$.
1. $(-7, 8) \cap [0, 9]$.
2. $\mathbb{N} \cap (-3, 8)$.
3. $(1, 7) \cup (-3, 4)$.
4. $(-\infty, 2) \cup (1, 5)$.
5. $\mathbb{N} \cup \{x \in \mathbb{R} : \ -x \in \mathbb{N}\}$.
6. $(4, +\infty) \setminus [6, +\infty)$.
7. $(4, +\infty) \setminus (-2, 5)$.
8. $\mathbb{R} \setminus \{-60, 34, 189\}$
9. $(-2, 5) \setminus (1, 2)$.
10. $(3, 12) \cap ((-2, 4) \cup (7, 35))$
11. $[-3, 1] \cup ((0, 7] \cap (-2, 9))$
12. $\mathbb{R}_{>0} \setminus ((9, 15) \cup (-4, 5))$