

State of the Art of Graph Visualization in non-Euclidean Spaces

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Abstract

Visualizing graphs and networks in non-Euclidean space can have benefits such as natural focus+context in hyperbolic space and the familiarity of interactions in spherical space. Despite work on these topics going back to the mid 1990s, there is no survey, or a part of a survey for this area of research. In this paper we review and categorize over 60 relevant papers and analyze them by geometry, (e.g., spherical, hyperbolic, torus), by contribution (e.g., technique, evaluation, proof, application), and by graph class (e.g., tree, planar, complex).

1. Introduction

When one begins to visualize a graph or network using a node-link diagram, one typically imagines placing the nodes and routing the edges in 2 dimensional Euclidean space; the space of the computer screen or a sheet of paper. However, representations in different geometries have distinct visualization benefits, such as natural focus+context, or the possibility to obtain a more faithful representation. There have been several works in applying graph visualization techniques beyond the standard 2 or 3 dimensional Euclidean space. Spherical [GSF97, WT06], hyperbolic [Mun97, LRP95], and toroidal [CDMB20] geometries have been considered in many visualization papers, dating back to the 1990s. Some user-studies have also been performed [DCL^{*}17, CDBM21] and there are many application-specific implementations [BKP^{*}12, Mun98, Glaa, Gü07]. In this paper we review relevant papers and analyze by geometry, (spherical, hyperbolic, torus), by contribution (technique, evaluation, proof, application), and by graph class (e.g., tree, small-world, large).

Several series of papers on non-Euclidean graph visualization appeared near the turn of the last century and recently this subject is of interest again to visualization researchers. While there are many surveys on graph visualization in general, none consider visualization in non-Euclidean spaces, leaving a notable gap in the literature. There are several graph visualization surveys for specific types of graph data, such as dynamic networks [BBDW17], multi-layer networks [MGM^{*}19], and multi-variate networks [NMSL19]. There are also more general surveys on graph visualization with topics on group structures [VBW17], and scientific visualization [WT17]. Von Landesburger et al. [VLKS^{*}11] survey visualizations of large networks, in which they point out that hyperbolic layouts scale well to large trees, though the survey itself does not overlap in content with ours.

We are guided by similar surveys on more narrow topics. Kale et al. review dynamic multivariate networks [KSP23] and use a sim-

ilar multi-dimensional tagging system to our methodology. Possibly the most related survey to our own is a survey paper analyzing the state of the art in geospatial network visualization [SYPB21]. While geospatial networks are (implicitly or explicitly) embedded on a sphere, these positions are typically given and have a very specific geographic interpretation. We also include graph visualization techniques that put arbitrary graphs on the sphere (not necessarily geographic), as well as in other geometric spaces.

We focus our attention on node-link diagram representations in non-Euclidean (or Riemannian) spaces. We initially generated a candidate set of papers by DBLP API, which includes papers from the proceedings of *IEEE VIS*, *IEEE PacificVis*, *EuroVis*, and the *Symposium on Graph Drawing*, along with the journals *TVCG* and *CGF*. Our search query was “graphnetwork+visualdrawlayout+hyperbollspherriemanntorus”, and returned an initial corpus of 43. We supplemented this with prior knowledge of the field, of which there were 13 additional papers that did not come up in the DBLP query. While reading and re-reading these papers, we paid close attention to the related work cited in them. This yielded an additional 8 papers, bringing the final total to 64.

2. Methodology

We consider three dimensions in our analysis of the related work, described below.

What is the geometry? We make note of the specific geometry studied for the graph visualization in the paper. We identify three non-standard geometries that are used for graph visualization:

- **Spherical geometry** is the surface of a 3-dimensional sphere, used to depict graphs with geographic or geographic-like information.
- **Hyperbolic geometry** is analogous to the sphere with ‘opposite’

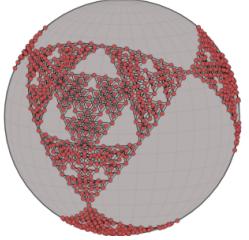
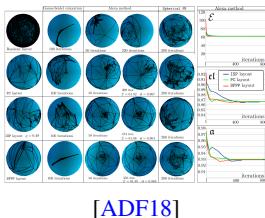
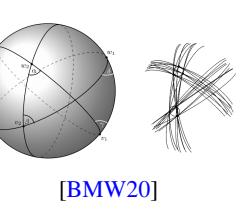
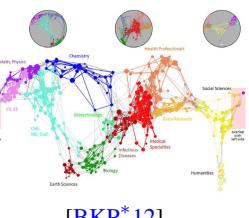
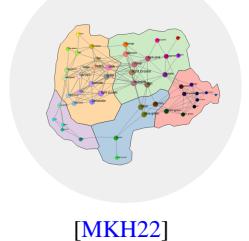
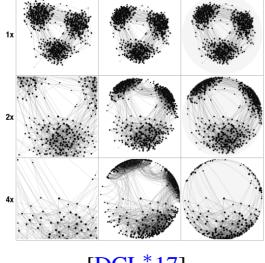
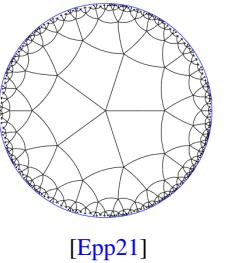
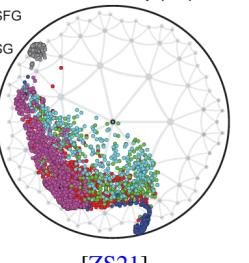
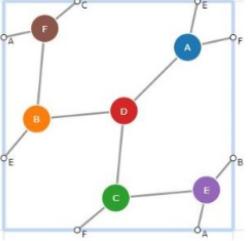
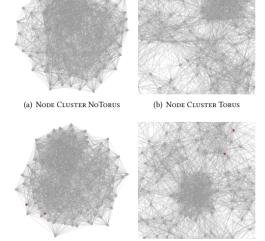
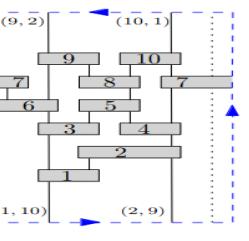
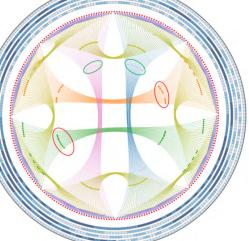
	Technique	Evaluation	Proof	Application
Spherical	 [MHK23]	 [ADF18]	 [BMW20]	 [BKP*12]
Hyperbolic	 [MKH22]	 [DCL*17]	 [Epp21]	 [ZS21]
Toroidal	 [CDMB20]	 [CDBM21]	 [Bie22]	 [CZIM18]

Table 1: Example works in a given geometry (row) and type of contribution (column).

curvature. Most notable for its exponentially expanding space allowing for ‘perfect’ drawings of trees and hierarchies.

- **Toroidal geometry** is the surface of a 3-dimensional ring or ‘doughnut’. Admits drawings of some non-planar graphs without crossings and can be embedded in the plane with no distortion.

What is the contribution? A cursory glance at the related work shows that there are many more theoretical and technique papers on this subject than human subject studies to show how such visualizations benefit us. We hypothesize that the contribution dimension of our analysis will be the most insightful in revealing what remains to be done in this subject area. We identify four type of contributions:

- **Technique** denotes papers whose main contribution is layout algorithm or drawing style.
- **Evaluation** are papers which include some experiment involving human participants to demonstrate a technique’s effectiveness.
- **Proof** style papers are theoretical papers that present results about embedding graphs in non-Euclidean space.
- **Application** papers provide a specific application in non-Euclidean geometry, such as showing how a given dataset can be analyzed.

What types of graphs? The third dimension of our analysis is

the type (or class) of graphs a paper considers in its scope. These include strict definitions such as trees and planar graphs, as well as more fuzzy definitions such as complex networks. To be consistent, we include a paper in a graph type category if the paper’s authors use that word to describe their graphs.

In summary, we review work on graph visualization in non-Euclidean spaces and provide a taxonomy and analysis. We believe this survey is timely, with recent interest on the subject in the visualization and human-computer interaction communities, and will provide interesting directions for future research where much has yet to be done.

3. Background on non-Euclidean and Riemannian Geometry

Geometry is among the oldest and most studied areas of mathematics, with roots in ancient times. Around 300 BC, Euclid wrote his *Elements* which employed the notion of an axiomatic system: a mathematical system in which all statements should be proved from a small number of indisputably true axioms (postulates). Euclid gave five axioms which make up what we now call Euclidean geometry.

1. Any two points describe a (unique) line.

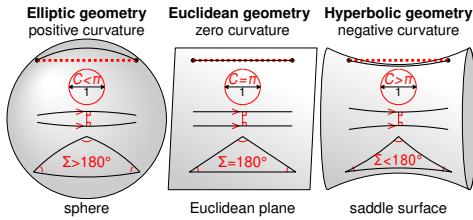


Figure 1: Illustration of non-Euclidean geometries and Playfair's axiom (middle two lines). Image credit <https://commons.wikimedia.org/w/index.php?curid=94781281>

2. Any line segment can be extended to a line.
3. A circle is described by a center and radius.
4. All right angles are equivalent.
5. "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles."

The fifth postulate (which states that any two non-parallel lines will intersect exactly once) is noticeably more involved than the previous four. Mathematicians tried for centuries to prove the fifth postulate using the first four, but invariably failed. Despite its perceived clunky-ness, the fifth postulate seemed necessary to prove even simple theorems such as the Pythagorean theorem about right angle triangles.

3.1. Hyperbolic geometry

As it would turn out, the Euclidean notion of parallel lines is not necessary for a consistent geometric system. It was independently discovered by mathematicians Nikolai Lobachevsky and János Bolyai around 1830, that by allowing arbitrarily many parallel lines to exist, a distinct geometry arises. This non-Euclidean geometry came to be known as *hyperbolic geometry*.

The fifth postulate is equivalent to the following statement, known as Playfair's axiom: "Given a line, L , and a point, p , not on L , **exactly one** line parallel to L can be drawn through p "; see Fig 1 for an illustration. Then, hyperbolic space is obtained by replacing the fifth postulate by the statement: "Given a line, L , and a point, p , not on L , **infinitely many** lines parallel to L can be drawn through p ."

Most properties of Euclidean geometry hold in hyperbolic geometry; for instance, any two lines can intersect at most once. Things differ when parallel lines or more than two lines are involved. Hyperbolic triangles have many properties distinct from traditional trigonometry, notably that the sum of their internal angles is strictly less than 180.

Another notable difference is the absence of relative scale. In Euclidean geometry, affine transformations are linear maps of the plane to itself that can be nicely represented as a function with real valued matrix A and vector b , $f(x) = Ax + b$. This includes translations, reflections, rotations, and resizing (scale). Once we have a drawing, we can shrink it to a tablet screen or expand it to a billboard and shapes (i.e. angles) are unaltered. In hyperbolic space

however, there is a relationship between distance and angles. Translations, reflections, and rotations can all preserve angles, but resizing will not.

3.2. Spherical (elliptical) geometry

Although the geometry of the surface of a sphere had been studied for many centuries (e.g., work by Theodosius around 200 B.C. [Ros12]), it was considered a separate system with its own rules. Even today, the sphere is often thought of as a surface embedded in 3 dimensions of Euclidean space, but its geometry can be studied intrinsically only using the 2 dimensions of its surface.

It wasn't until 1859 that the classical understanding of spherical geometry was unified with the new ideas produced from the study of non-Euclidean geometry [Cay59]. Cayley introduced *elliptic geometry*, a sphere with antipodal points identified to satisfy the result (present in both Euclidean and hyperbolic) that lines can intersect at most once (note that on a full sphere, lines always intersect twice). Then elliptic geometry is obtained by the following modification of Playfair's axiom: "Given a line, L , and a point, p , not on L , **no** lines parallel to L can be drawn through p ." Or in other words, all lines must intersect.

Although elliptic geometry removes the trouble of antipodal points, we generally refer to spherical geometry (also called doubly elliptic geometry) throughout this paper, as it is more widely used for visualization.

While the first four of Euclid's postulates hold in spherical geometry, the third must be slightly modified; a circle of arbitrary radius cannot exist. Since the sphere is finite, a circle can only have radius equal to the radius of the sphere it exists on.

Many of the unique properties of hyperbolic geometry have the opposite implication for spherical geometry. For instance, the sum of internal angles of a spherical triangle is strictly greater than 180 as seen in Fig. 2. Spherical space also has an absolute scale though since the surface is finite there is a strict upper limit on how much things can be stretched.

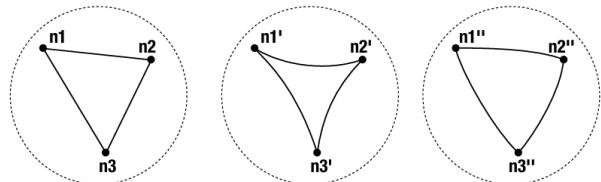


Figure 2: Comparison of visualization of a triangle in Euclidean plane, hyperbolic plane, and sphere [DCL*17]. Note how the hyperbolic triangle has smaller angles while the spherical triangle has larger.

3.3. Riemannian and Toroidal geometry

Riemannian geometry is the study of manifolds (spaces which locally resemble Euclidean space) together with a Riemannian metric (an inner product on the tangent space at each point). This includes the Euclidean and non-Euclidean geometries with the standard dot

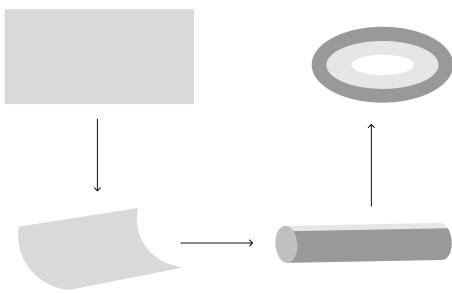


Figure 3: Creating a torus from a piece of paper. Note how first two opposite ends of the rectangle are connected, then the top/bottom of the cylinder is connected to create the torus. This type of torus is generally called a flat torus.

product, along with many other surfaces. The torus has been used for graph visualization and is worth exploring.

A torus is a product of two circles that is topologically closed, which produces the typically doughnut-shape. The torus can be described as the surface of an axis of revolution of a circle in 3 dimensional space. It is an example of a surface with genus 1, where genus is the maximum number of closed, circular cuts that can be made and the surface remain connected. Note that the spaces discussed before all have genus 0.

A property of interest in graph drawing is the crossing number of the graph [Sch12], which is the minimum number of crossings required to draw a graph on a 2 dimensional surface of genus 0. A surface with higher genus admits smaller crossing numbers [Hea90]. Notably, the two canonical non-planar graphs (K_5 and $K_{3,3}$) can be realized without crossings on the torus.

Although the surface of a torus is often depicted in 3 dimensions with curvature, a flat torus actually has curvature of 0, just like the Euclidean plane. Similar to how one can bend a sheet of paper into a cylinder in 3 dimensions with no distortion, the torus could be created from a sheet of paper with no distortions if we had a 4th dimension to work with. While no perfect 3 dimensional embedding is possible, a nearly perfect embedding of the flat torus is obtained in 2 dimensions by a rectangle with the left/right and top/bottom edges identified as displayed in Fig. 3.

3.4. Terminology

A **graph** (network) is a finite set of objects V (called *vertices*) together with a set of relationships on those objects $E \subseteq V \times V$ (called *edges*). Edges may be directed or undirected, and potentially weighted. Graphs arise in many application areas, making their visualization of particular research interest. While some fields make a distinction between graphs and networks, we treat them interchangeably in this survey.

A common visualization idiom for graphs is the **node-link diagram**, with vertices drawn as nodes (simple shapes) and edges

drawn as links (curves between the corresponding vertices). Node-link diagrams are constructed from an **embedding** (layout) of the graph, an assignment of positions to the vertices and a routing of edges. The resulting picture is called a **drawing** of the graph. In a typical straight-line drawing, the routing of the edges is totally determined by the placement of the incident vertices, as the line segment between them.

In the plane, one can draw a straight line segment between two points and measure its length – this is known as the Euclidean distance or L_2 norm. This intuitive concept can be generalized to arbitrary surfaces with the **geodesic**: the length of the shortest curve between two points. In non-Euclidean geometries (such as 2D sphere and 2D hyperbolic space), this is a circular arc defined by a closed-form function. In Euclidean and hyperbolic space, there is a unique ‘straight line’ between any two points. For the sphere there may be many possible straight lines if the points are antipodal, but the length is the same for each. On the torus, however, there are several possible lines so it is required to measure the length of each of them to compute the geodesic length

Non-Euclidean geometries cannot be perfectly embedded into the 2 dimensional plane. The study of cartography is a field dedicated to this problem [Sny87] as it has long been known for the sphere. A **projection** is a linear map that preserves some combination of angles, areas, geodesics, or distances but a projection (from spherical or hyperbolic space) that preserves all these aspects simultaneously is not possible.

There are many classes of graphs that are subject of particular interest. **Trees** are graphs that contain no cycles. If a root of the tree is specified, this encodes a hierarchy, where each non-root vertex has exactly one parent it descends from. **Planar graphs** are graphs which can be drawn as a node-link diagram without any two links intersecting. **Complex networks** are graphs that arise naturally in real-world networks, defined by phenomena such as small average path length and high clustering coefficient.

Many dimension reduction (DR) techniques can be applied to graph embedding. These techniques take as input a distance matrix, D , on a set of high dimensional Euclidean coordinates where $D_{i,j}$ is the distance between objects i and j . Defining distance with the graph theoretic (i.e. shortest path) distance allows one to apply these DR techniques to graph embedding. A particular technique used in graph embedding is **multi-dimensional scaling (MDS)** (stress minimization), which tries to match the pairwise embedding distances to the original input distances.

4. Geometry

Our first dimension of analysis is geometry. We give a brief overview of the chronology of how each geometry has been used in graph visualization.

4.1. Sphere

Spherical geometry has been used in visualization since antiquity. Much of the visualization work on the sphere has been inspired from the Earth’s globe, with orthographic projections and map metaphors being common.

Gross et al. [GGSF97] describe an approach for graph layout on the sphere. This is a force-directed method which places nodes initially with an initialization scheme that attempts to keep nodes close to the surface of the sphere. Edges are not routed along geodesics, but as straight-line segments in 3D Euclidean space.

Sangole and Knopf [SK03] apply a self-organizing map (SOM) that creates closed spherical surfaces from the input data, transforming the discrete input into a continuous distribution on the sphere. For a globe-like feel, they doubly encode the density of the input data with a rainbow color map and the height of the point on the globe. Low density areas are blue and the lowest points to resemble oceans, while high density areas are red and high, resembling mountains.

Wu and Takatsuka [WT06] adapt a SOM to the sphere for multivariate network visualization. This requires adapting the grid used in a SOM to a triangulation of the sphere. They also implement a visual interface to view the embedding with either an orthographic or cylindrical projection. The spherical SOM has also been used in the field of networking [TKY11], to visualize file sharing activities on a computer network with the help of an orthographic projection.

Traditional force-directed algorithms were generalized to the sphere [KW05] and the tree visualization scheme SphereTree was created in [BM12]. MDS has also been generalized to the sphere with a few different approaches. Osinska et al. [OB08] perform 3D Euclidean MDS and normalize the vectors to lie on a unit sphere. Another approach is to perform Euclidean MDS in 3D, but constrain the optimization so that the points never leave the surface of a sphere during the algorithm [DLM09, PYGK20]. Finally, [MHK23, EKK05] describe methods to perform MDS using latitude and longitude coordinates, constraining the computations and movements to the surface of the sphere.

Spherical layouts have been investigated in an immersive setting such as virtual reality [KMLM16, YJD^{*}18], discussed further in section 5.4. Several other examples of spherical graph visualization include the Map of Science [BKP^{*}12], as well as the “Places and Spaces” [Bö12] and “Worldprocessor” [Gü07] exhibitions.

4.2. Hyperbolic

Hyperbolic space first received attention in the visualization community due to its exponentially expanding structure, and eye catching representations such as Fig. 5. This particular Escher work is inspired from the Poincaré disk, a model of hyperbolic space which maps lines to circular arcs in Euclidean space and admits many aesthetically pleasing tessellations. Hyperbolic embeddings are also of interest in the field of machine learning, as a desirable latent space for embeddings [PVM^{*}21]. Here we cover works which deal primarily with hyperbolic geometry in graph visualization.

One of the earliest approaches of graph drawing in the hyperbolic plane is by Lampert et al [LRP95], dubbed the hyperbolic browser; see Fig. 6(a). They embed trees in hyperbolic space using a linear time algorithm and construct the embedding top-down from parent to children, evenly spacing children on the half-plane opposite the parent all at unit length. This is always possible thanks to the exponentially expanding hyperbolic geometry. To represent this

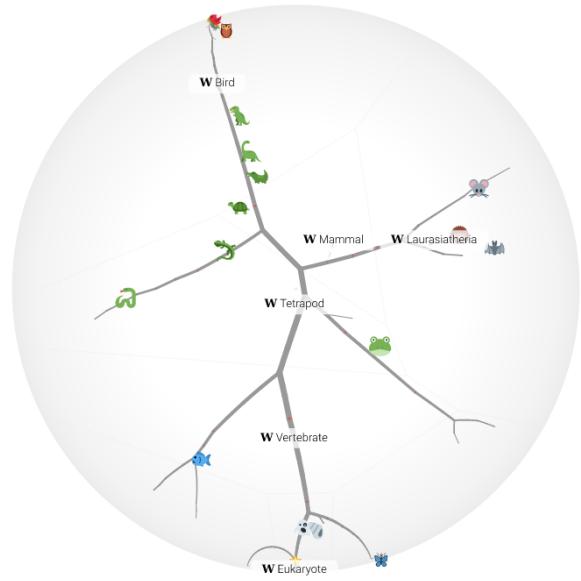


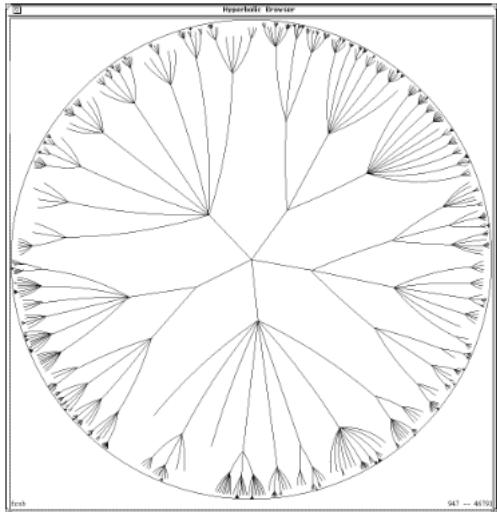
Figure 4: Hyperbolic Tree of Life for browser based hyperbolic visualization. Source code and demo found from [[Glaa](#)]



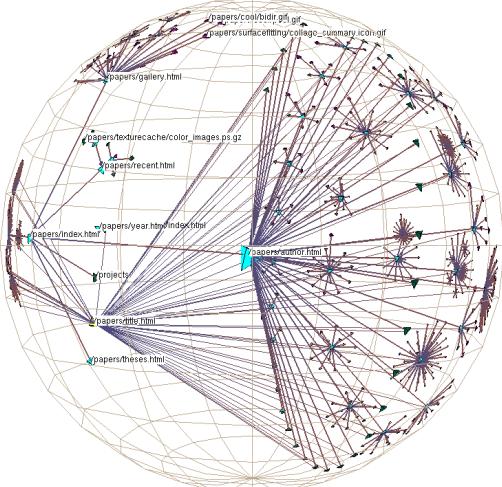
Figure 5: Circle Limit III; M.C. Escher [[EV89](#)]

embedding, they use the Poincaré disk which maps the entirety of hyperbolic space into a Euclidean disk. Since the entire tree is visible (up to pixel resolution) this provides a powerful focus+context effect, allowing one to see the parent node and its children in high detail, but also get a sense of how deep or wide each of the subtrees are. Clicking on a node places it in the center of the projection, via a smooth transition in the drawing. This approach was generalized to arbitrary graphs by [HDK00] by first computing a spanning tree and filling in the removed edges after layout (or keeping them hidden).

Several application papers made use of the on the Lamp-



(a) Hyperbolic Browser from [LRP95]



(b) H3 from [MB95]

Figure 6: Hyperbolic tree visualizations either took the 2 dimensional (a) or 3 dimensional (b) approach.

ing et al. [LRP95] technique. Phylogenetic trees [BS00], word trees [BW02], the tree of life [Glaa] all made natural candidates, as large trees with interest to domain experts. The hyperbolic browsing system also made its way into several systems and tools, such as the Hierarchy Visualization System (HVS) [APN07], the Java InfoVis Toolkit [Bel], and TreeBolic [Bou].

More recently, Lamping et al.'s [LRP95] hyperbolic browser has been re-implemented using d3.js [Gla18, Gla18] for web-based visualization of large hierarchies such as the tree of life [Glaa] as shown in Fig. 4.

General graphs can be drawn in hyperbolic space as well, [MKH22] provide several different techniques to do so, building on earlier approaches from [KW05, Wal04, PYGK20].

While most work considers the 2-dimensional hyperbolic plane,

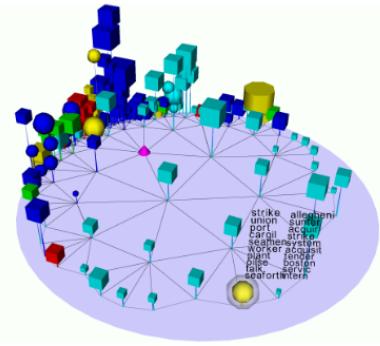


Figure 7: Hyperbolic self-organizing map visualization from [WOWR03]. The shown flat disk is a Poincaré projection, while the data are drawn as 3 dimensional shapes coming out of the plane.

Munzner used 3D hyperbolic space to visualize trees [Mun97, Mun98, MB95], one of the examples shown in Fig. 6(b). Just like 2 dimensional hyperbolic space can be projected into the 2 dimensional plane, 3 dimensional hyperbolic space can be projected into the 3 dimensional Euclidean space. While the Poincaré projection is preferred in two dimensions due to its simple construction and conformal properties, it maps geodesics to curved arcs which are more difficult for a reader to follow in 3 dimensions. For this reason, Munzner chose to use the Beltrami-Klein projection in spite of the larger distortion, as it maps geodesics to Euclidean lines. The ball itself can be rotated when viewed from outside, or entered by zooming to get a closer look. The node in the center of the ball can be changed by navigating the tree. This technique was later extended to general graphs by first computing a spanning tree [Mun00], laying out with the above technique, and filling in the cycles afterwards. Munzner's work has been re-implemented in two subsequent systems: Walrus [Hyu00] and h3py [ZK16].

In the early 2000s, hyperbolic space received attention as a potential target space for self-organizing maps (SOMs) with the hyperbolic SOM (HSOM) [WR02]. Although restricted to a grid, the HSOM algorithm is only linear in complexity, requires only input distances, and can be used as a visualization. This algorithm was integrated into a visualization tool [WOWR03] where the SOM lattice was drawn on a Poincaré disk and the data points drawn as 3 dimensional objects coming up out of the plane; see Fig. 7.

4.3. Torus

Torus-based graph visualizations are relatively recent in comparison to the other non-Euclidean geometries above. Early on, torus embeddings of graphs was primarily of theoretical interest. Since the torus has genus 1, it admits crossing-free drawings of many non-planar graphs. An example of a non-planar graph (K_5) drawn without crossings is shown in Fig. 9 [Hea90].

Kocay et al. [KNS01] first describe an algorithm to embed any 2-connected toroidal graph on the torus, where a toroidal graph is one that can be embedded without crossings on the torus. They do this

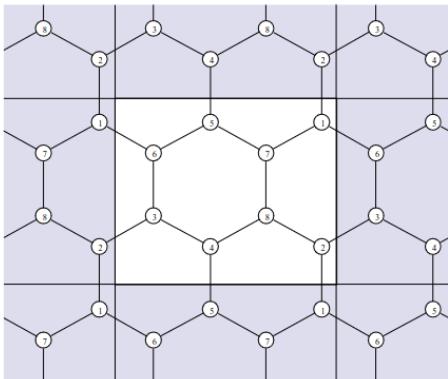


Figure 8: Torus graph embedding depiction from [KNS01]. This tiling approach is often taken for small graphs on the torus.

by generalizing Read’s algorithm for Euclidean planar graphs, and go on to show that this will always produce a crossing free drawing on the torus. These drawings are depicted as rectangles, with the surrounding area duplicated to show the ‘wrapping around’ of the torus; see Fig. 8.

This rectangular embedding of the flat torus has been used to create planar drawings of non-planar graphs, by duplicating vertices and edges like the sides of the square in Fig. 8. Biedl [Bie22] shows that any toroidal graph has a visibility representation on a rectangular flat torus, where a visibility representation is an embedding in which vertices are mapped to horizontal line segments and edges are mapped to vertical line segments.

More recently, a series of papers has introduced a practical technique for graph layout and interactive visualization of graphs on the torus. Starting with [CDBM20], Chen et al. identify 3 research questions that they aim to address:

- **RQ1:** develop layout algorithms that take advantage of the extra flexibility of graph layout on the torus;
- **RQ2:** determine how we can best visualize the layout of a node-link diagram on the surface of a torus on a piece of paper or 2D computer monitor; and
- **RQ3:** determine what, if any, perceptual benefits graph layout on a torus has over standard layout on a 2D plane.

This initial paper begins to address these challenges by proposing a stress-based algorithm for torus layout, providing some interaction styles, and a small but thorough user study to evaluate the different interactions and how they compare to traditional drawings. A follow up paper [CDBM21] improved the previous optimization, and took a closer look at cluster based tasks on the torus, performing a user study and found some advantages over traditional drawings (e.g., when identifying clusters).

Most recently, Chen et al. [CNK*23] use eye tracking data to further understand how readers use the toroidal graph drawings to make inferences. They use the tracking data to identify patterns and compare different representations of the torus. They found that duplicating the torus at each of the 8 neighboring grid squares was the most effective, but poorly utilizes space. This leaves a space-efficient full-context torus visualization as future work.

4.4. Other

Focus+context effects can also be achieved with lens effects [TGK*17, TGK*14]. In particular, at first glance the Poincaré disk appears to resemble a fisheye lens, which in some cases is preferable to a flat drawing [Fur86]. However, a lens effect generally applies only to a subset of the visible data, scaling or warping it to bring it into focus. The focus+context view is applied across all of the data in the Poincaré disk, with an exponential decrease in data size away from the center, but the entirety of the object remaining in view.

5. Contribution

Our second dimension of analysis is contribution. We discuss papers based on what we believe is their primary contribution. The four broad categories are *techniques*, *evaluation*, *theory*, and *application*. It is common for any visualization research paper to have some mix of all four of these, in this case we select which contribution we believe to be the most novel.

5.1. Technique

The classical straight-line graph layout problem is to assign 2 dimensional Euclidean coordinates to the vertices of the graph so that the resulting drawing captures well the underlying graph properties (e.g. preserving distances, minimizing crossings, preserving symmetries, etc.). This assignment is known as an *embedding*, and it totally determines a straight line drawing. Nodes are drawn as circles centered at the embedded position, and edges are straight-line segments connecting the corresponding endpoints. There are many different approaches for producing Euclidean graph embeddings; the survey by Gibson et al. includes a good selection [GFV13].

The straight-line graph layout problem naturally generalizes to Riemannian spaces, by instead finding an embedding in the desired space with the straight line requirement becoming geodesic curves. We identify four distinct approaches to solving this problem algorithmically.

5.1.1. Projection-Reprojection

The first approach is to have the algorithm take as input a Euclidean drawing of the graph, and project the positions into the desired space. This is particularly intuitive in spherical space, where geographic map projections from the sphere to the Euclidean plane have been studied since classical times. This is one of the approaches presented by Perry et al. [PYGK20], referred to as the “Projection–Reprojection” method. Here, any Euclidean visualization (e.g. node-link diagram) can be drawn on the sphere, by treating the input as a sheet of paper, and ‘folding’ it around the curved sphere surface; see Fig. 10. A similar technique for the sphere is presented in [DCL*17] and another is adapted to hyperbolic space in [MKH22]. Any conceivable node-link based graph visualization idiom can be adapted to a non-Euclidean space using this technique, including map-style drawing [GHK10] or BubbleSets [CPC09], as it only requires a set of coordinates and lines to draw between them. Additionally, this method requires only linear time (given a pre-computed input) and is straightforward to implement. However, this method does not use any unique aspect of the

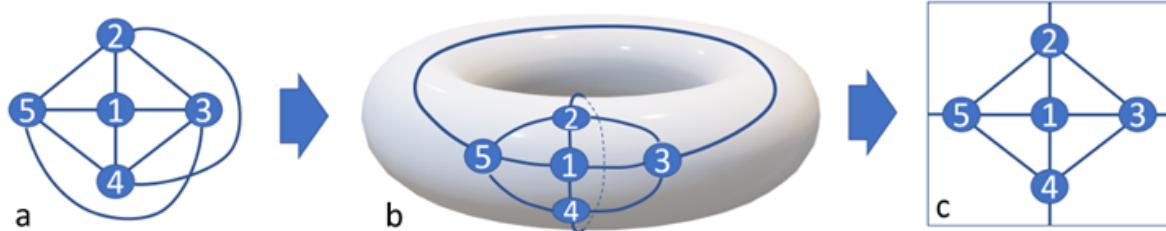


Figure 9: The canonical non-planar graph K_5 can be drawn without crossings on the torus, as shown in this example from [CDMB20].

target geometry (such as the ‘wrap-around’ of the sphere, or exponential expansion of hyperbolic space), as shown in [PYGK20]. This method works for any space which admits a projection to Euclidean (since a projection is linear, it has an inverse), but we are unaware of any work which attempts such a method for the torus.

5.1.2. Tangent planes

The second technique is to exploit the fact that Riemannian geometries are locally flat, and perform a force-directed style layout algorithm in the tangent spaces about the vertices. This is the approach in [KW05] and generalizes layout algorithms driven by forces. When computing the forces on a vertex, first compute the tangent plane about that point (a plane which, in 3 dimensions, intersects the surface at exactly one point). Project the other vertices into that plane, and compute forces and update the central vertex position as normal. Finally, project the central vertex back into the target space. Repeatedly apply until the quality of the drawing is sufficient. The tangent plane technique has been reapplied in [DCL^{*}17, MKH22]. While this approach makes use of some unique properties of the geometry, this method is computationally quite expensive and does not achieve as high of quality as the later approaches [MKH22]. While the formulation is for a general Riemannian geometry, we are unaware of any implementation for the torus.

5.1.3. Constrained optimization

The third technique requires us to view our target space as the 2 dimensional surface of a 3 dimensional object. Then, this surface is defined by a set of 3 dimensional Euclidean coordinates that satisfy some property or equation (e.g. a unit sphere is all points with magnitude exactly one). One can perform a force-based graph layout technique in 3 dimensions, requiring that any movement of the vertices still satisfies the properties of the desired space.

This particular technique has been applied only to the sphere, first described by [DLM09] for dimension reduction and adapted to graph layout by [PYGK20]. This is also the approach taken by [SKOM12], who show this is related to information diffusion models, such as independent cascade. Rodighiero [Rod20] describes this constraint as ‘gravity’ attracting the nodes to a spherical surface. The Fruchterman-Reingold algorithm was also adapted in this fashion [GF22]. Although this is easy to describe at a high level, the technical details are quite difficult. One must ensure the step size is sufficiently small, errors will quickly accumulate in the 3 dimensional representation, and the underlying mathematics is quite complicated.

This approach could be extended to hyperbolic space and the torus, but has not been done so. For instance, just like the equation $x^2 + y^2 + z^2 = 1$ describes a sphere in 3 dimensions with components x, y, z the equation $x^2 + y^2 - z^2 = 1$ describes a hyperbolic surface.

5.1.4. Native formulation

Something common to the previous three methods is that they all require the use of the friendly properties of Euclidean space. It is desirable to solve the problem entirely within the target geometry, for fewer computations, simplicity, and in order to fully benefit from the advantages of the particular geometry (e.g., more space in hyperbolic geometry).

The tree embedding algorithms discussed for hyperbolic geometry [MB95, LRP95, SSGR18] are native formulations but they are limited to trees or spanning trees of graphs. Similarly, there are algorithms to generate crossing free embeddings of toroidal graphs on the torus [KNS01], but again these are limited to toroidal graphs.

For general graphs, a native formulation typically requires generalizing some Euclidean graph layout approach. The most popular choice for this has been stress minimization, also known as Multi-dimensional Scaling (MDS) [Kru64]. Stress-based approaches aim to match the graph theoretic distance between all pairs of vertices to the geodesic distance between the corresponding pairs of nodes in the embedding.

This was first done in the sphere for dimension reduction by [EKK05], then adapted to graph embedding with an improved optimization by [MHK23]. The basic idea is straightforward: replace the L_2 norm computation in Euclidean MDS with the spherical geodesic distance function. Since positions on the sphere are completely defined by a pair of angles, and this function is differentiable, one can find a minimum without considering cumulative error, or points ending up outside the sphere.

Classical MDS has been explored in hyperbolic space, by replacing the conversion to similarities with an appropriate hyperbolic scaling function [CE17, SSGR18]. Using a similar idea to the spherical MDS, metric and non-metric MDS have been generalized to hyperbolic space by incorporating hyperbolic geodesic distance into the cost function [WOWR03, Wal04, WR02, ZS21, MKH22].

While [CDMB20] use a stress minimization scheme to compute their toroidal layouts, the torus is missing a closed-form distance function. Chen et al. overcome this by checking which of the

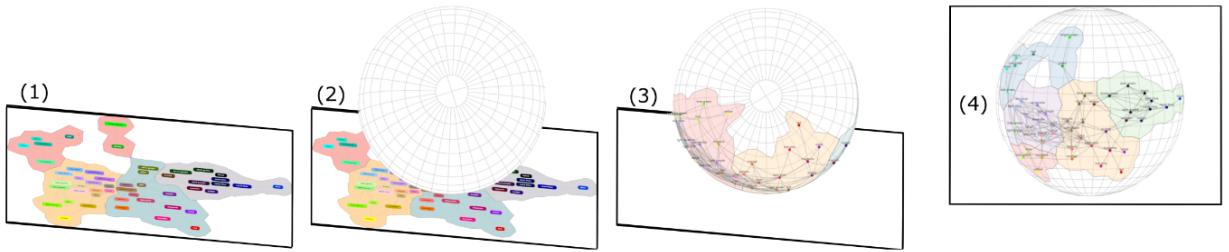


Figure 10: Example from [PYGK20], illustrating the Projection-Reprojection drawing method for the sphere. By treating the input Euclidean drawing as a sheet of paper, the spherical drawing can be obtained by ‘wrapping’ the input around the ball.

straight-line distances is closest to the ideal for each pair of vertices, but a toroidal layout that uses closed form functions is an interesting direction for future work.

5.2. Evaluation

Here we focus on papers whose primary contribution is an evaluation of non-Euclidean graph layouts, and also, more generally, typical ways in which such layouts are evaluated.

5.2.1. Quality Measures

The evaluation in most technique and application papers are quantitative metrics: metrics that evaluate a layout or embedding with a single number. This may be the time in seconds to produce the drawing, or a faithfulness or aesthetic metric which captures how well the drawing reflects the data or how readable the drawing is. Such quality metrics were originally defined and studied in the Euclidean setting.

Aesthetic metrics: Properties such as number of crossings, crossing angle, angular resolution, vertex resolution, symmetry, and average edge length are all desirable to optimize for readability, with number of crossings being particularly impactful [Pur02]. These concepts can extend to non-Euclidean geometries, but there is not much work in this area. Eppstein shows that the vertex and angular resolution is bounded in hyperbolic space, meaning one cannot achieve a good resolution in general [Epp21]. The torus admits smaller crossing numbers in theory [KNS01], and this has been shown to be achievable for some real world graphs [CDBM21]. However, much remains to be explored in this area. For example, hyperbolic spaces can have symmetries not possible in Euclidean space (see Fig. 5), but measuring hyperbolic symmetry cannot yet be done numerically.

Faithfulness metrics: Faithfulness metrics instead measure how well the embedding captures the information present in the data [NEH13]. These include metrics such as stress [GKN04], neighborhood preservation [KRM*17], and distortion [KPK*10].

In the graph drawing literature, the normalized stress of a layout is a standard quality measure [GHN13, KRM*17, ZCH*21], defined as:

$$\sum_{i < j} \frac{(||X_i - X_j|| - d_{i,j})^2}{(d_{i,j})^2} \quad (1)$$

where X_i is the embedded position of vertex i , and $d_{i,j}$ is the desired distance between vertex i and vertex j .

This is perfectly acceptable in Euclidean space where a layout is not meaningfully changed when the layout is resized. For non-Euclidean graph layouts there is a possible issue of *dilation* or *resizing*. Formally, a dilation is a function on a metric space M , $f : M \rightarrow M$ that satisfies $d(f(x), f(y)) = rd(x, y)$ for $x, y \in M$, $r > 0 \in \mathbb{R}$ and $d(x, y)$ being the distance between x and y .

In non-Euclidean spaces the size of a layout can have drastic effects since they have an absolute scale [MHK23]. For this reason, *distortion* [SSGR18, MKH22] should be used as a quality metric in place of the normalized stress, defined as

$$\binom{V}{2}^{-1} \sum_{i < j} \frac{|||X_i - X_j|| - d_{i,j}|}{d_{i,j}} \quad (2)$$

Distortion is less sensitive to the dilation of the drawing, but note that selecting the correct size is still important to finding a good embedding. Stress has also been generalized to the torus in [CDMB20]. Distortion can be generalized to the torus, but no work so far makes use of it for toroidal embeddings.

Other quality metrics such as neighborhood preservation have yet to be studied in non-Euclidean spaces.

5.2.2. User studies

While less common, an important form of evaluation for non-Euclidean graph visualization is the user study. Several papers incorporate some form of user study as part of the evaluation/justification for a technique or application [CDMB20, SG12] but there are fewer papers dedicated to user evaluation.

Du et al. [DCL*17] compare Euclidean, spherical, and hyperbolic graph drawings by performing a user study. They first present their iSphere system for large graph visualization, which takes any Euclidean, straight line drawing, and performs an inverse Stereographic (conformal) projection on it to map it to the sphere. They present participants with layouts of stochastic block model random graphs, and ask them questions about some nodes or groups of nodes. The experiment design was within-subjects, each participant saw examples of Euclidean, spherical, and hyperbolic displays, supporting zooming and panning. The layout was driven by MDS for all 3 displays: standard for Euclidean, inverse stereographic projection for sphere, and [KW05] for hyperbolic. All the

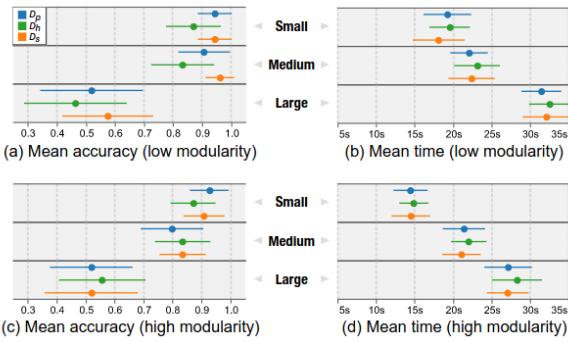


Figure 11: Plots from [DCL^{*}17], time (right) and error (left) of participant results. The top row are results for low modularity graphs (weak clustering) and the bottom for high modularity graphs (strong clustering). The colors indicate the display geometry: blue for Euclidean, green for hyperbolic, orange for spherical.

tasks are about node degrees in the drawings. In particular, they ask for: (1) the neighbor of highest degree for a highlighted node; (2) the common neighbor of highest degree for two highlighted nodes; and (3) the highest degree node along a highlighted path. The sizes of graphs were also controlled: $|V|$ ranged from {128, 512, 2048} and $|E|$ from {1024, 4096, 16384}. The number of clusters generated was 3 for each graph. For each size of graph, they used two different modularities, low and high (a graph statistic which expresses how dense the clusters in a graph are). They conclude with statistical analysis on their collected participant data. The results seemed to indicate that the sphere and Euclidean visualizations had comparable time and accuracy, while hyperbolic performed worse (see Fig. 11).

Meanwhile, for the torus a thorough user study was conducted in [CDBM21]. This study was primarily aimed at asking whether toroidal visualizations were more beneficial for cluster identification. The graph sizes used in the study ranged from $|V| \in (68, 134)$ with $|E| \in (710, 2590)$. Similar to [DCL^{*}17], they divide their graphs into small and large classes, and into low and high modularity. Graphs are randomly generated from stochastic block model type generators.

The authors registered several hypotheses on the open science forum, such as that the torus-based visualization would be beneficial over Euclidean drawings in identifying the number of clusters (in time and error) when the graph modularity is low (i.e. when clusters are not very dense). They performed a within-subjects study so that all participants see all variations; namely both the torus and Euclidean layouts with small/large, low/high modularity, and 2 tasks. The two tasks were: (1) What is the number of clusters; and (2) Do the two highlighted nodes belong to the same cluster?

The statistical analysis of the participant data provides several insights. They found statistically significant results that torus-based layouts increased participant accuracy over Euclidean layout in some cases; we include a figure showing the mean time and accuracy from their study in Fig. 12.

5.3. Proof/theory

Planar graphs are of particular interest in spherical graph visualization. Aleardi et al. [ADF18] develop an algorithm that embeds a planar graph on the sphere without crossings. The algorithm is efficient, but can generate very short edges, resulting in occasionally undesirable layouts. The authors present several use cases, including as an initialization for a Euclidean layout.

Kryven et al. introduce a measure of visual complexity, *spherical cover number* of a drawing, which is the minimum number of spheres needed to cover all edges of a 3 dimensional graph drawing (sim. circles for 2 dimensions) [KRW19].

Kang and Lin [KL21] show that planar Cayley graphs (graphs that describe a finite group and operations on that group, and is also planar) admit symmetrical spherical drawings. These drawings reside in \mathbb{S}^2 , are rotationally symmetric, and have uniform edge lengths. They provide a construction for each drawing and enumerate all planar Cayley graphs.

A generalization of the straight line drawing for the Euclidean plane is the geodesic drawing for the sphere, where all edges must be geodesics of the endpoints. This type of drawing has been studied by [BMW20].

Mohar first showed how one can draw general graphs in the hyperbolic plane [Moh99]. Note also that there are theoretical limits on the effectiveness of hyperbolic embeddings for general graphs. Some graphs can be embedded trivially with a low, constant embedding error (e.g., as cycles and square lattices) but have non-trivial embedding error in the hyperbolic plane [Epp21, VS16]. However, other graphs such as trees and hyperbolic tilings can be embedded better in hyperbolic space than in Euclidean space. For example, while Euclidean geometry only admits 3 regular tessellations (triangles, squares, hexagons), the hyperbolic plane admits infinitely many.

Although embedding of arbitrary graphs have theoretical limitations, there are still many graph structures that cannot be well captured in Euclidean space but can be well captured in hyperbolic space. Ouyang et al. [OCCZ12] construct arbitrary hyperbolic tessellations, and provide an algorithm to color and distort edges in aesthetically pleasing and interesting patterns.

It has been shown that some graphs can be embedded with lower error in hyperbolic space than in Euclidean space [BFKL18]. Zhou and Sharpee [ZS21] show that hyperbolic MDS (H-MDS) can be used to detect the underlying geometry of a dataset, when comparing its embedding error to Euclidean metric MDS. They go further to show that the underlying space of genomes is hyperbolic. Krioukov et al's [KPK^{*}10] work indicates that hyperbolic geometry may underlie complex networks and hierarchical networks, such as phylogenetic trees and the internet.

Hyperbolic geometry is of interest in networking and routing, in the form of greedy embeddings. It has been shown that any connected, finite graph admits a greedy embedding in hyperbolic space, which is not generally true in Euclidean geometry [Kle07]. Greedy embeddings of graphs allow for greedy routing, which is particularly useful when a node may not know the global topology, but only its own position and that of its neighbors such as in social networks and the internet [EG11, EG08].

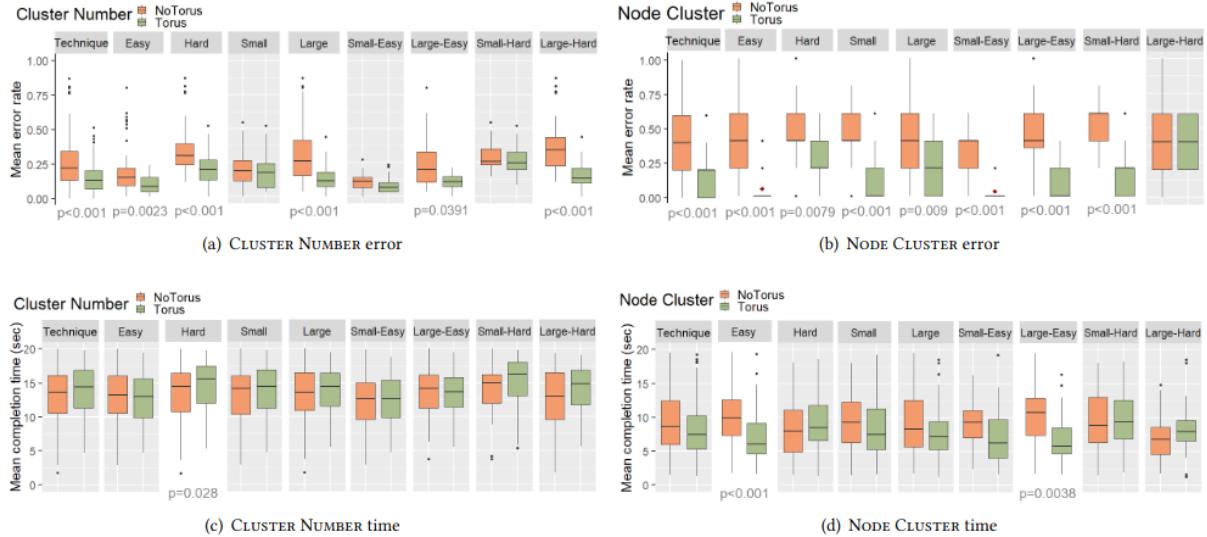


Figure 12: Box plots of results from [CDBM21], showing that Torus based visualization was often more accurate than NoTorus (Euclidean) visualization of graphs for cluster counting task.



Figure 13: Using 3D edge routing around the globe to visualize international air interconnections network from [LBA10].

Cabello et al. [CMS15] study toroidal embeddings: given a disconnected graph $G \cup G'$, does there exist a toroidal embedding of the graph with optimal crossings such that no edge of G intersects an edge of G' . Some results are presented, but the problem appears to remain open. Norine [Nor09] shows that any 4-Pfaffian graph can be drawn on the torus such that every perfect matching intersects itself an even number of times.

5.4. Application

Applications of network visualization on the sphere are primarily of the form of a map metaphor, or for geographic data.

Shelley et al. [SG12] present a software titled GerbilSphere, which places a user in the center of a sphere with a network drawn

on the sphere surface. The target is dynamic, large-scale networks so the tasks are primarily high level movement of groups. The paper includes a small user study to test the effectiveness, reporting that navigation is more efficient with GerbilSphere than traditional network visualization.

Lambert et al. [LBA10] use a 3D edge bundling technique based on edge routing to support visualization of geographical networks as shown in Fig. 13 where they visualize international air traffic network. The width of the bundles is based on the number of edges within it, and the data is overlaid on the globe since the data has geographic coordinates.

Efforts to visualize the difficult to conceptualize processes in non-Euclidean geometry has had some interest. Francis and Sullivan [FS04] create a visualization algorithm and rendering to visualize a sphere eversion (turning it inside-out). Further, hyperbolic tessellations have been the subject of study for education and aesthetics [OCCZ12]. An open-source hyperbolic visualization tool, RogueViz [CK17], includes different projections and educational tools, although its restriction to tessellations of the hyperbolic plane makes it less than ideal for general graphs.

Non-Euclidean geometry is of particular interest in virtual reality. This has been explored for the sphere by Kwon et al. [KMLM15, KMLM16], who present a technique for edge routing on the sphere with VR in mind, running the edges outside of the sphere. A user is placed inside the sphere with a head-mounted display, and is free to look around to gain understanding of the network data. In [KDB23], a user is placed in the center of a spherical drawing dubbed an ‘egocentric’ view. This also allows areas to be ‘retracted’ and brought closer to the user.

A class of graph known as a ‘torus graph’ is common in networking and HPC, in fact some years ago high-dimensional torus networks were used in four of the top ten supercomputers and eight

of the top ten on the Graph500 list [MIB^{*}14]. These are graphs which describe the topology of the torus in n dimensions. They are created by starting with an n -dimensional rectangular lattice, then connecting the outside vertices in the same dimension; left to right and top to bottom for a two dimensional torus graph. There has been interest in visualizing these types of networks, and we have classified these works as application papers since they are a method of visualizing data from a torus.

TorusVis [CDJM14] is a visualization design study attempting to visualize the topology of torus networks from super computing clusters. They use a circular layout for the graph visualization part of their dashboard, with an ordering algorithm based on Hilbert curves. They use edge bundling to decrease clutter within the circle. This circular layout was later improved and extended into a technique and analytics tool [CZIM18]. Also [MIB^{*}14] look specifically at the IBM Blue Gene machine's torus network using four connected views depicting the network at different levels of detail. The approach is demonstrated by analyzing network traffic for a simulation running on the IBM Blue architecture. Small multiples with links between them are used by [TSW14] to visualize torus networks.

6. Types of Graphs

Our final dimension for analysis is the types of graphs under consideration, as the choice of graphs for visualization in different geometries can affect the visualization methods and goals.

6.1. Trees and Planar graphs

Hyperbolic tree browsers had a large effect on the research landscape of tree and hierarchy visualization. Schneiderman et al. [SDSW12] perform a thorough investigation of early tree visualization techniques, including hyperbolic trees as well as treemaps and cone trees. They show that at the time of writing, the Lamping et al. [LRP95] paper for the hyperbolic browser was one of the most cited tree visualization papers.

Hyperbolic tree visualizations are typically in 2D or 3D. Most 2D approaches employ node-link diagrams drawn in the Poincaré disk [LRP95, HHDK00, BS00]. Meanwhile, most 3D approaches [MB95, Mun97, Mun98, Mun00, Hyu00, ZK16] make use of the Beltrami-Klein projection.

[Treevis.net](#) provides several examples of hyperbolic and sphere-based tree visualizations [Sch11]. For example, Sphere-Tree [BM12] puts a treemap onto the surface of a sphere, encoding vertices as nested rectangles. The sphere appears hollow, allowing a reader to see the ‘backside’ of the surface. The hyperbolic wheel [LD12] scheme places a sunburst chart in the Poincaré disk, so that areas of vertices further from the center decrease exponentially.

Planar and non-planar graphs remain planar and non-planar respectively in spherical and hyperbolic geometries. Spherical planar embeddings of graphs are studied by both [BMW20] who aim to eliminate geodesic crossings and [KL21] who exhaustively show a small class of graphs (Planar Cayley graphs) have symmetrical planar spherical embeddings. Planarity can be generalized to

higher-genus surfaces. A graph which can be drawn without crossings on the torus is called a toroidal graph. Algorithms to construct crossing-free embeddings of toroidal graphs are known [KNS01], but there are no known efforts to use these algorithms for visualization.

6.2. Complex networks

Krioukov et al. [KPK^{*}10] show that complex networks have an underlying hyperbolic geometry. What this means is that the graph theoretic distance between vertices in a complex network is roughly equivalent to the hyperbolic geodesic distance of a hyperbolic embedding of that graph. One of the ways they show this is by defining a random graph model, a hyperbolic random graph, which uniformly distributes points in the hyperbolic plane and connects points with an edge if they are within some radius. They show this random graph model exhibits complex network properties where many other random graph models (e.g. Euclidean geometric random graphs) do not.

This result lead to a series of papers in hyperbolic graph drawing. The main question is how to generate a hyperbolic graph embedding which realizes the properties above. Bläsius et al. [BFKL18] propose a maximum likelihood estimation formulation for hyperbolic graph embedding, and show computationally that they achieve lower distortion than the force-directed approach of [KW05]. Meanwhile, Sala et al. [SSGR18] develop a hyperbolic classical MDS approach that achieves lower distortion than Euclidean embeddings on example datasets. For metric MDS, Miller et al. [MKH22] apply and stochastic gradient descent optimization to also achieve lower distortion than Euclidean embeddings. Bläsius et al. [BFK21] recently adapted a force-directed approach with two types of forces: popularity force, which operates on the radial component (i.e. its distance from the origin) and similarity force, which operates on the angle of a node. By treating these forces independently, they achieve comparable distortion error to competing methods.

7. Discussion

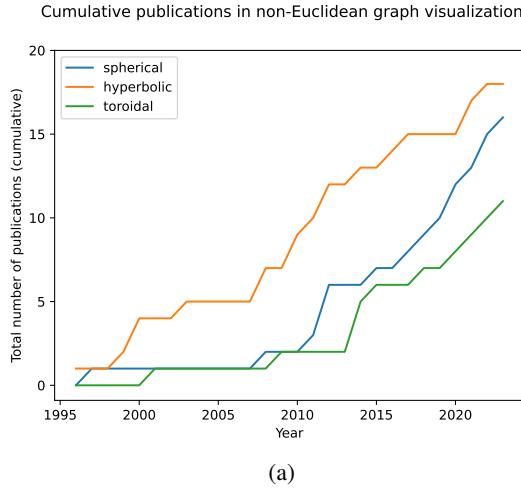
In this survey we attempt to summarize and analyze work on visualizing graphs in non-Euclidean spaces. We have tagged all papers under consideration and categorized them in a json file with paper titles, authors, DOIs, and other attributes, which we provide as supplemental material. Plots and tables referred to in this section are based on this data.

Geometry	Papers
spherical	16
hyperbolic	18
torus	11

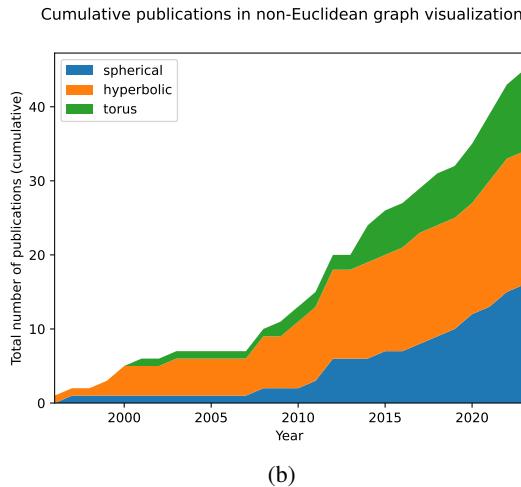
Table 2: Count of papers in survey that fall into each geometry.

Looking at the total number of papers over time, grouped by geometry, we see that the three categories have roughly an equal number of papers currently. Most of the papers in hyperbolic geometry are from before 2010, when spherical papers become more

popular. Torus papers have been more frequent in recent years and seem to be catching up.



(a)



(b)

Figure 14: Number of publications in each geometry over time (cumulative). (a) depicts a simple line chart with total number of publications increasing cumulatively each year. (b) depicts the same data as a stacked area chart, with the height of the chart being the total number of publications of all categories and the colors within the distribution of spherical, hyperbolic, and toroidal geometries.

The count of papers by geometry can be found in Table 2. As to be expected, the number of spherical and hyperbolic papers is roughly equal, while there fewer torus papers. We investigate how this trend evolved over time in Fig. 14 (a). From 1995–2009, hyperbolic publications were more numerous with their applications to tree visualization, and received another round of papers in the early 2010s for their relationship to complex networks. Spherical visualization papers became more prevalent in the 2010s as well, perhaps due to the development of map-based network drawings and the release of open source web tools like d3.geo and THREE.js. We see a similar spike in torus papers, when visualization of torus networks for HPC systems became of interest.

A count of papers instead by contribution is found in Table 3.

Contribution	Papers
technique	15
evaluation	4
theory	13
application	19

Table 3: Number of Papers based on contribution.

Technique, theory, and application papers are roughly equally common. Notably, there are far fewer evaluation papers than one might expect for a field of research originating 30 years ago.

	technique	evaluation	proof	application
spherical	6	2	3	5
hyperbolic	4	1	6	6
torus	1	2	4	4

Table 4: Papers in each geometry based on contribution.

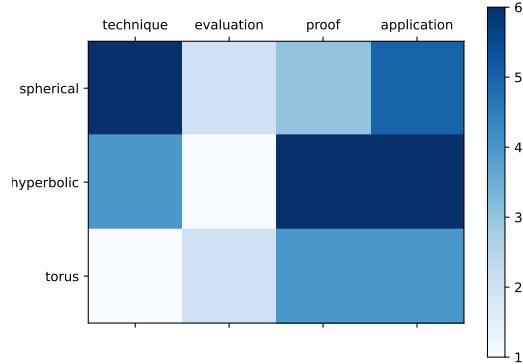


Figure 15: A matrix based visualization of Table 4. Darker squares indicate more papers published in that category, while lighter squares indicate fewer papers.

We can observe the intersection of two of the dimensions of our categorization; geometry and contribution. A table that counts the occurrences of these intersections is found in Table 4. A more striking visual is the matrix visualization Fig. 15. We can clearly see where there is potential for future research. Toroidal techniques and evaluations for all geometries have much room to explore (the white spaces in the table).

8. Conclusions and Future work

While there exists a large body of interesting research in this subject area, there is still much to be done. We give an overview of potential future work.

We are unaware of any human-subject studies evaluating graph embedding techniques in spherical and hyperbolic space. For

example, is there qualitative difference between the Projection-Reprojection methods (which are very fast, but do not take advantage of the underlying geometry) and the more computationally expensive native formulations?

A large advantage of the sphere as pointed out by [GF22, MHK23] is that there is no defined ‘center’ or ‘periphery’. Any node can be placed in the center of the drawing, something not possible in Euclidean space. Can this be used to prevent anchoring or similar biases?

There are many potentially interesting questions in quantitative measures for non-Euclidean geometries. Are there bounds for other aesthetic criteria such as crossing angle like there are for angular resolution in hyperbolic space [Epp21]? Do similar bounds exist for the sphere and torus? Can we characterize the graphs that ‘live naturally’ (i.e. have low distortion) in hyperbolic/spherical/toroidal space? Do other faithfulness metrics like Neighborhood Preservation need modifying to make sense in non-Euclidean space?

Like the spherical and hyperbolic space, the torus has an absolute scale. However, the effect of dilation/resizing of a drawing on the torus has not yet been studied.

We do not know how people read, understand, and utilize non-Euclidean embeddings. More human-subject studies can provide guidelines for the design of other techniques and applications.

Rodighiero [Rod20] posits that the choice of projection of the sphere effects the reader perception of the network data and argue that a projection that preserves the continuity of the network is desirable. Can we: (1) show that this is true; and (2) select projections more intelligently?

Although there have been some attempts to utilize non-Euclidean geometry in the virtual reality [KMLM15, KMLM16], this field seems ripe for exploration of immersive visualization. Of course, data can be put on a globe, but the focus+context effect of hyperbolic space is exaggerated in 3D.

We have reviewed over 60 papers in the non-Euclidean and Riemannian graph visualization area, as well as summarized and categorized the literature. This subject has been of interest to the visualization community in the last three decades and we hope that visualization researchers might use this survey to get an overview of what is known and what remains to be explored..

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We have obtained permissions to include most of the images in this survey, and we are waiting to hear from the authors on a few more. We will only include images with explicit authorization by the authors in a camera-ready version.

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