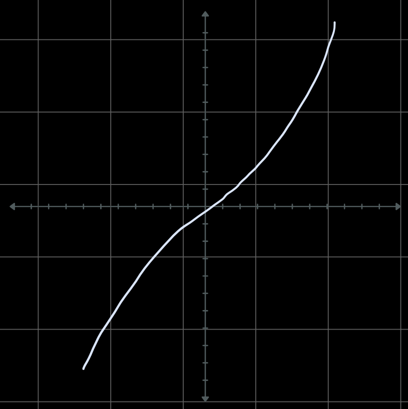
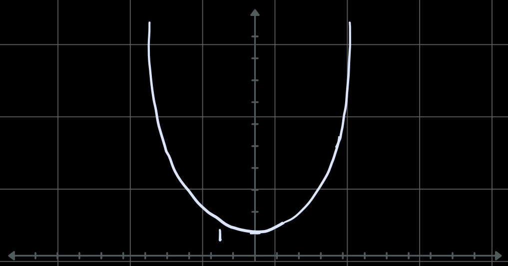

你相信光么

Ch-1 Hyperbolic Function

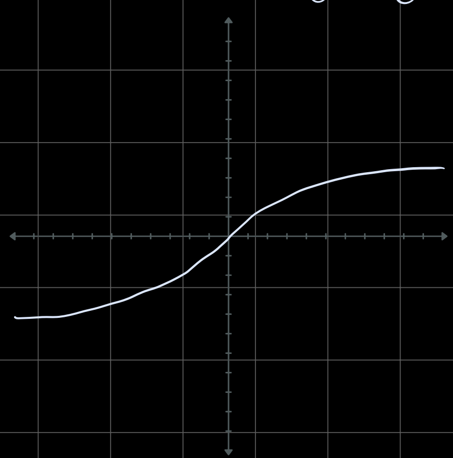
$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$



$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$



$$\tanh x = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$



Osborne's rule:

If a trigonometric identity involves the product of two sines, we change the sign to obtain corresponding hyperbolic identity.

Trigonometric identity	Hyperbolic identity
$\cos^2\theta + \sin^2\theta \equiv 1$	$\cosh^2 x - \sinh^2 x \equiv 1$
$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$	$\cosh 2x \equiv \cosh^2 x + \sinh^2 x$
$\sin 2\theta \equiv 2\sin\theta\cos\theta$	$\sinh 2x \equiv 2\sinh x\cosh x$

$1-\tanh^2 x = \operatorname{sech}^2 x$

$\coth^2 x - 1 = co\operatorname{sech}^2 x$

$\sinh(x \pm y) = \sinh x\cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x\cosh y \pm \sinh x \sinh y$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$\operatorname{ar}\sinh x = \ln(x + \sqrt{x^2+1})$

$\operatorname{ar}\cosh x = \ln(x + \sqrt{x^2-1})$ $(x \geq 1)$

$\operatorname{ar}\tanh x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $(|x| < 1)$

Ch-2 Differentiation

Hyperbolic & Inverse Function

$$\frac{d\sinh x}{dx} = \cosh x \qquad \frac{d\tanh x}{dx} = \operatorname{sech}^2 x$$

$$\frac{d\cosh x}{dx} = \sinh x$$

$$\frac{d\sinh^{-1}x}{dx} = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d\cosh^{-1}x}{dx} = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d\tanh^{-1}x}{dx} = \frac{1}{1-x^2}$$

proof eg.

$$y = \sinh^{-1}x$$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$\frac{dx}{dy} = \cosh y = \frac{e^y + e^{-y}}{2} = \sqrt{1+x^2}$$

Maclaurin series

Maclaurin's series:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{(all } x\text{)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 \leqslant x \leqslant 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 < x < 1)$$

Ch - 3 Integration

using Hyperbolic functions / trigonometric functions

$f(x)$	$f'(x)$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\sinh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2+x^2}}$
$\cosh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2-a^2}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

Reduction Formula

Arc length along a curve.

$$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

proof: $(ds)^2 = (dx)^2 + (dy)^2$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$S = \int_{x_1}^{x_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$(ds)^2 = (dx)^2 + (dy)^2$

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

proof: $(ds)^2 = (dr)^2 + (r\, d\theta)$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

surface & revolution

$S = 2\pi \int y\, ds$ (rotate about x-axis)

approximation using rectangles.

Ch-4 Differential Equations

First Order:

$$\frac{dy}{dx} + yP(x) = Q(x)$$

$$f(x)\frac{dy}{dx} + yf(x)P(x) = f(x)Q(x)$$

$$\frac{d}{dx}\, yf(x) = f(x)\frac{dy}{dx} + yf'(x)$$

$$yf(x)P(x) = yf'(x)$$

$$P(x) = \frac{f'(x)}{f(x)}$$

$$\int P(x)\,dx = \int \frac{f'(x)}{f(x)}\,dx = \ln(f(x))$$

$$f(x) = e^{\int P(x)\,dx}$$ → Integrating Factor

$$y = \frac{\int f(x)\cdot Q(x)\,dx}{f(x)}$$

Second Order

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$a\lambda^2 + b\lambda + c = 0 \rightarrow$ auxiliary equation

General Solution:

1. Complementary function y_c

$\Delta > 0 \qquad y = Ce^{\lambda_1 x} + De^{\lambda_2 x}$

$\Delta = 0 \qquad y = (Cx + D)e^{\lambda_1 x}$

$\Delta < 0 \qquad y = e^{\alpha x}(C\cos\beta x + D\sin\beta x)$

$\lambda_1 = \alpha + \beta i \quad \lambda_2 = \alpha - \beta i$

2. Particular Integral y_p

$f(x)$	Assumed Form $F(x)$
$a_n x^n + \dots + a_1 x + a_0$	$b_n x^n + \dots + b_1 x + b_0$
ae^{kx}	be^{kx}
$a_1 \sin kx + a_2 \cos kx$	$b_1 \sin kx + b_2 \cos kx$

Notice: if the assumption is part of y_c and does not work, $y_p = x F(x)$

Ch-5 Complex Numbers

Cartesian Form: $z = x + iy$

Modulus-argument form: $z = r(\cos\theta + i\sin\theta)$

$$r_1(\cos\alpha + i\sin\alpha) \cdot r_2(\cos\beta + i\sin\beta) =$$
$$r_1 r_2[\cos(\alpha+\beta) + i\sin(\alpha+\beta)]$$

$$\frac{r_1(\cos\alpha + i\sin\alpha)}{r_2(\cos\beta + i\sin\beta)} = \frac{r_1}{r_2}[\cos(\alpha-\beta) + i\sin(\alpha-\beta)]$$

Exponential Form: $z = re^{i\theta}$

De Moivre's Theorem:

$$z^n = \cos n\theta + i\sin n\theta$$

1. Express $\cos n\theta$ and $\sin n\theta$ as polynomials of $\cos\theta$ or $\sin\theta$ (using trigonometric identities)

a. express $\cos^n\theta$ and $\sin^n\theta$

$$z+\frac{1}{z}=2\cos\theta \qquad z^n+\frac{1}{z^n}=2\cos n\theta$$

$$z-\frac{1}{z}=2i\sin\theta \qquad z^n-\frac{1}{z^n}=2i\sin n\theta$$

Summation using complex numbers

Roots of Unity

The nth roots of unity are

$1, w, w^2, w^3, \dots, w^{n-1}$

where $w=e^{\frac{2\pi}{n}i}$

$$1+w+w^2+\dots+w^{n-1}=0$$

nth root of any complex number

$$z=r(\cos\theta+i\sin\theta)$$

$$z^{\frac{1}{n}}=r^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n}\right)$$

$k=0,1,\dots,n-1$

Ch-6 Matrix

Intersection of three planes:

1. Do not intersect
2. intersect in two parallel lines
3. intersect in a single point
4. sheaf: share a common line
5. triangle prism: each pair intersect a line
   three lines are parallel

$\det M \neq 0$: one unique solution ③

$\det M = 0$: inconsistent equations
① ② ⑤

consistent equations
④

Eigenvalues & Eigenvectors

$$AX = \lambda X$$

characteristic equation:

$$|A - \lambda I| = 0$$

each λ is associated with eigenvector X.

Solve $(A - \lambda_k I)X = 0$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$$

$$\lambda_1 \lambda_2 \cdots \lambda_n = |A|$$

Diagonalization

$$A(e_1, e_2, \cdots e_n) = (Ae_1, Ae_2 \cdots, Ae_n)$$

$$= (\lambda e_1, \lambda e_2, \cdots, \lambda e_n)$$

$$= \underbrace{(e_1, e_2, \cdots, e_n)}_{P} \underbrace{\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}}_{D}$$

$$AP = PD$$

$$A = PDP^{-1}$$

Cayley-Hamilton theorem:

assume the characteristic equation is $P_A(\lambda) = \det(A-\lambda I)$

$$P_A(A) = 0$$

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注：配色仅根据个人喜好，
不统一请见谅

仅有标题无内容的topic
说明没有key points要标注，
直接做题即可。

本notes本质上仅是一篇
高配版List of Formulas，
（甚至不全的那种）

在下深度大字爱好者，
对近视选手十分友好。
能认出来的字就是好字