

## T.O.C.: The Math

The Irrelevant Phase

The Berry Phase

The Discrete Case

The Continuous Case

The Adiabatic Theorem

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## T.O.C.: Physical Systems

### Magnetic Moment in a Precessing Field

- Defining the System

- Calculating the Berry Phase

### The Aharonov-Bohm Effect

- The Electric Effect

- The Magnetic Effect

- The Berry Phase

### Outlook

## Section 1

### The Irrelevant Phase

#### The Irrelevant Phase

The Berry Phase

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## The Irrelevant Phase

1. All quantum mechanical observables correspond to Hermitian operators ( $\hat{O}$ ).
2. All Hermitian operators have a spectral decomposition ( $\hat{O} = \sum_n a_n |\Psi_n\rangle \langle \Psi_n|$ ).
3. The projector components ( $|\Psi_n\rangle \langle \Psi_n|$ ) of the spectrally decomposed operator  $\hat{O}$  are all invariant under the gauge transformation  $|\Psi_n\rangle \rightarrow \exp(i\alpha_n) |\Psi_n\rangle$ .

$\Rightarrow$  Phase is not physical!

$$\psi = \Psi \cdot \exp(i\phi)$$

$$\phi \in \{0, 2\pi\}$$

## Section 2

### The Berry Phase

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## The Relative Phase

- ▶ The relative phase between two states can be uniquely defined for a given gauge.
- ▶ The relative phase is not invariant under local gauge transformations (i.e. multiplication of the state by a phase factor).

Relative Phase:

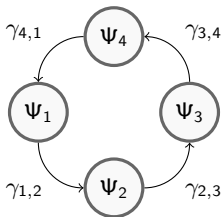
$$\gamma_{1,2} = -\arg \langle \Psi_1 | \Psi_2 \rangle \quad \gamma_{1,2} \in (-\pi, \pi]$$

Gauge Transformation:

$$\begin{aligned} |\Psi_j\rangle &\rightarrow \exp(i\alpha_j) |\Psi_j\rangle \\ \exp(-i\gamma_{1,2}) &\rightarrow \exp(-i\gamma_{1,2} + i(\alpha_2 - \alpha_1)) \end{aligned}$$

## The Berry Phase

- ▶ Summation of a non-gauge invariant quantity ( $\gamma_{i,j}$ ) along closed loops results in a gauge-invariant result!
- ▶ The Berry Phase can be expressed in terms of gauge-invariant operators.



The Berry Phase:

$$\gamma_L = \sum_{i=0}^{N-1} \gamma_{i \bmod N, i+1 \bmod N}$$

Invariance:

$$\gamma_L = \text{Tr} \left( \prod_{i=0}^{N-1} |\psi_i\rangle \langle \psi_i| \right)$$

## By Analogy

### Discrete:

- ▶ Parameter Space:  $\mathbb{P} = \mathbb{N}_0$
- ▶ Wave Function  
 $\Psi : \mathbb{P} \rightarrow \mathcal{H}, \quad n \mapsto |\Psi_n\rangle$
- ▶ Path:  $\mathcal{C} = [n_0, n_1, \dots, n_{N-1}]$
- ▶ Berry Phase:  
 $\gamma(\mathcal{C}) = \sum_{i=0}^{N-1} \gamma_{n_i \bmod N, n_{i+1} \bmod N}$

The sum of the relative phases of each wave function connected by the path  $\mathcal{C}$ .

### Continuous:

- ▶ Parameter Space:  $\mathbb{P} = \mathcal{M}$
- ▶ Wave Function  
 $\Psi : \mathbb{P} \rightarrow \mathcal{H}, \quad \mathbf{R} \mapsto |\Psi(\mathbf{R})\rangle$
- ▶ Path:  $\mathcal{C} : [0, 1) \rightarrow \mathcal{M}, \quad t \mapsto \mathbf{R}(t)$
- ▶ Berry Phase:  
 $\gamma(\mathcal{C}) = \lim_{dt \rightarrow 0} \oint \gamma_{\mathcal{C}(t), \mathcal{C}(t+dt)} d\mathbf{R}$

The integrated relative phase of all the wave functions connected by the path  $\mathcal{C}$ .



## The Berry Connection

Relative Phase:  $\lim_{r \rightarrow 0} \gamma_{R, R+r} = \lim_{r \rightarrow 0} -\arg \langle \Psi(R) | \Psi(R+r) \rangle$

$$\lim_{r \rightarrow 0} \exp(-i\gamma_{R, R+r}) = \lim_{r \rightarrow 0} \frac{\langle \Psi(R) | \Psi(R+r) \rangle}{|\langle \Psi(R) | \Psi(R+r) \rangle|}$$

$$1 - i\gamma_{R, R+r} = \langle \Psi(R) | (|\Psi(R)\rangle + \nabla_R |\Psi(R)\rangle)$$

$$\Rightarrow \gamma_{R, R+dr} = i \langle \Psi(R) | \nabla_R | \Psi(R) \rangle$$

$$\gamma(C) = \oint i \langle \Psi(R) | \nabla_R | \Psi(R) \rangle dR$$

## The Berry Connection and Curvature

- ▶ Discrete relative phase quantities in the discrete case are replaced by the Berry Connection in the continuous case.

- ▶ The Berry Connection is non-gauge invariant under gauge transformations.

- ▶ Stokes' theorem can be utilized to reformulate the path integral to a surface integral.

- ▶ The resulting integrand defines what is known as the Berry Curvature.

- ▶ The Berry Curvature is gauge-invariant under gauge transformations.

### Berry Connection:

$$\mathcal{A} = i \langle \Psi(\mathbf{R}) | \nabla | \Psi(\mathbf{R}) \rangle$$

### Stokes' Theorem:

$$\gamma(\mathcal{C}) = \sum_{\mu, \nu} \int_{\mathcal{S}(\mathcal{C})} \frac{1}{2} \partial_{\mu} \mathcal{A}_{\nu} - \partial_{\nu} \mathcal{A}_{\mu} dR^{\mu} \wedge dR^{\nu}$$

### Berry Curvature:

$$\Omega_{\mu, \nu}(\mathbf{R}) = \partial_{\mu} \mathcal{A}_{\nu} - \partial_{\nu} \mathcal{A}_{\mu}$$

## Magnetic Analogies

- The relationship between the Berry Phase ( $\gamma$ ), Berry Connection ( $\mathcal{A}$ ) and Berry Curvature ( $\Omega$ ) is analogous to the relationship between the magnetic flux ( $\Phi$ ), magnetic vector potential ( $\mathbf{A}$ ) and magnetic flux density ( $\mathbf{B}$ ).

**Phase:**

$$\begin{aligned}\gamma(C) &= \oint \mathcal{A} \cdot d\mathbf{R} \\ &= \int_{S(C)} \nabla \times \mathcal{A} \cdot d\mathbf{s} \\ &= \int_{S(C)} \Omega \cdot d\mathbf{s}\end{aligned}$$

**Flux:**

$$\begin{aligned}\Phi(C) &= \oint \mathbf{A} \cdot d\mathbf{R} \\ &= \int_{S(C)} \nabla \times \mathbf{A} \cdot d\mathbf{s} \\ &= \int_{S(C)} \mathbf{B} \cdot d\mathbf{s}\end{aligned}$$

## Section 3

### The Adiabatic Theorem

The Irrelevant Phase

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## Statement

### Theorem

*Under a slowly changing Hamiltonian with instantaneous eigenstates  $|n(t)\rangle$  and energies  $E_n(t)$ , the state of a quantum system evolves ( $\Psi(t) = \sum_n c_n(t) |n(t)\rangle$ ) such that the amplitudes of the instantaneous eigenstates composing the wave function are given by:*

$$c_n(t) = c_n(0) e^{i\theta_n(t)} e^{i\gamma_n(t)}$$

$$\theta_n(t) = -\hbar^{-1} \int_0^t E_n(t') dt'$$

$$\gamma_n(t) = i \int_0^t \langle m(t') | \dot{m}(t') \rangle dt'$$

### Assumptions

1.  $i\hbar |\dot{\Psi}(t)\rangle = \mathbf{H} |\Psi(t)\rangle$
2.  $\frac{\langle m(t) | \dot{\mathbf{H}}(t) | n(t) \rangle}{E_m(t) - E_n(t)} = 0$

## Proof: Part I

1. Begin with the time-dependent Schrödinger equation.
2. Expand  $|\Psi(t)\rangle$  and apply the Hamiltonian  $\mathbf{H}$  to the eigenstates  $|n(t)\rangle$ .
3. Take the inner product of both sides of the equation with  $|m(t)\rangle$ .

$$i\hbar |\dot{\Psi}(t)\rangle = \mathbf{H}(t) |\Psi(t)\rangle$$

$$\mathbf{L.H.S} = \sum_n c_n(t) E_n(t) |n(t)\rangle$$

$$i\hbar \sum_n [\dot{c}_n(t) |n(t)\rangle + c_n(t) |\dot{n}(t)\rangle] = \mathbf{R.H.S}$$

$$i\hbar \dot{c}_m(t) + i\hbar \sum_n \langle m(t) | \dot{n}(t) \rangle c_n(t) = c_m(t) E_m(t)$$

## Proof: Part II

4. Differentiate the time-independent Schrödinger equation with respect to time.
5. Take the inner product of both sides of the equation with  $|m(t)\rangle$ .
6. Rearrange in terms of the inner product  $\langle m(t)|\dot{n}(t)\rangle$ .

$$\mathbf{L.H.S} = \dot{E}_n(t) |n(t)\rangle + E_n(t) |\dot{n}(t)\rangle$$

$$\dot{\mathbf{H}}(t) |n(t)\rangle + \mathbf{H}(t) |\dot{n}(t)\rangle = \mathbf{R.H.S}$$

$$\mathbf{L.H.S} = E_n(t) \langle m(t)|\dot{n}(t)\rangle$$

$$\langle m(t)|\dot{\mathbf{H}}(t)|n(t)\rangle + E_m \langle m(t)|\dot{n}(t)\rangle = \mathbf{R.H.S}$$

$$\langle m(t)|\dot{n}(t)\rangle = \frac{\langle m(t)|\dot{\mathbf{H}}(t)|n(t)\rangle}{E_n(t) - E_m(t)} \quad (m \neq n)$$

## Proof: Conclusion

7. Insert the result from Part II into the result from Part I.

8. For  $\langle m(t) | \dot{\mathbf{H}}(t) | n(t) \rangle \ll E_m(t) - E_n(t)$   
 and  $\forall n \quad \forall t : E_m(t) - E_n(t) \neq 0$ .

9. Solving the resulting differential equation for  $c_m(t)$ .

$$L.H.S = \sum_{\substack{n \\ n \neq m}} \frac{\langle m(t) | \dot{\mathbf{H}}(t) | n(t) \rangle}{E_m(t) - E_n(t)}$$

$$\dot{c}_m(t) + \left( \frac{i}{\hbar} E_m(t) + \langle m(t) | \dot{m}(t) \rangle \right) c_m(t) = R.H.S$$

$$\dot{c}_m(t) = i \left( -\frac{1}{\hbar} E_m(t) + i \langle m(t) | \dot{m}(t) \rangle \right) c_m(t)$$

$$c_m(t) = c_m(0) e^{-i \frac{1}{\hbar} \int_0^t E_m(t') dt'} e^{i \int_0^t \langle m(t') | \dot{m}(t') \rangle dt'}$$



## Proof: The Berry Phase

Time evolution given by the variation of parameters  $\mathbf{R} \in \mathbb{P}$ :

$$\mathbf{H}(t) = \mathbf{H}(\mathbf{R}(t)) \quad \Psi(t) = \Psi(\mathbf{R}(t))$$

Therefore:

$$\begin{aligned}
 \gamma &= \int_0^t i \langle m(\mathbf{R}(t)) | \dot{m}(\mathbf{R}(t)) \rangle dt \\
 &= \int_0^t i \langle m(\mathbf{R}(t)) | \nabla_{\mathbf{R}} | m(\mathbf{R}(t)) \rangle \dot{\mathbf{R}}(t) dt \\
 &= \oint i \langle m(\mathbf{R}) | \nabla_{\mathbf{R}} | m(\mathbf{R}) \rangle d\mathbf{R}
 \end{aligned}$$

## Section 4

### Magnetic Moment in a Precessing Field

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**Magnetic Moment in a Precessing Field**

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## Defining the System

### System:

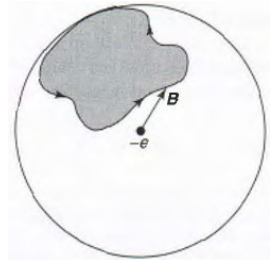
Magnetic moment at the origin in the presence of a magnetic field  $B$  of fixed magnitude, moving along some closed path  $\mathcal{C}$ .

$$\mathbf{B}(t) = B_0 \begin{pmatrix} \sin \theta(t) \cos \phi(t) \\ \sin \theta(t) \sin \phi(t) \\ \cos \theta(t) \end{pmatrix}$$

$$\mathbf{H}(t) = \frac{\hbar e}{2m} \mathbf{B} \cdot \boldsymbol{\sigma}$$

$$\chi_+(t) = \begin{pmatrix} \cos \frac{\theta(t)}{2} \\ e^{i\phi(t)} \sin \frac{\theta(t)}{2} \end{pmatrix} \quad E_+ = \frac{\hbar \omega_1}{2}$$

$$\chi_-(t) = \begin{pmatrix} e^{-i\phi(t)} \sin \frac{\theta(t)}{2} \\ -\cos \frac{\theta(t)}{2} \end{pmatrix} \quad E_- = -\frac{\hbar \omega_1}{2}$$



## Calculating the Berry Phase

The adiabatic approximation applies:

$$\gamma = \int_C i \langle \chi_+ | \nabla_{r,\theta,\phi} | \chi_+ \rangle \cdot d\mathbf{B}$$

$$\langle \chi_+ | \nabla_{r,\theta,\phi} | \chi_+ \rangle = i \frac{\sin^2(\theta/2)}{r \sin(\theta)} \hat{\phi}$$

Applying Stokes' theorem:

$$\gamma = \int_{S(C)} i \nabla_{r,\theta,\phi} \times \langle \chi_+ | \nabla_{r,\theta,\phi} | \chi_+ \rangle \cdot d\mathbf{s}$$

$$\nabla_{r,\theta,\phi} \times \langle \chi_+ | \nabla_{r,\theta,\phi} | \chi_+ \rangle = \frac{i}{2r^2} \hat{r}$$

$$\gamma = -\frac{\Omega}{2}$$

## Section 5

### The Aharonov-Bohm Effect

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The Electric Effect

The Magnetic Effect

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## The Electric Effect

1. Electron beam is split at point A.
2. Dual beams pass through *Faraday Cages* to which a voltage is applied.
3. Electron beams are recombined at point F.

$$H(t) = H_0 + V(t)$$

$$\Psi(t) = \Psi_0 \exp(-iS)$$

$$S = \int V(t)dt$$

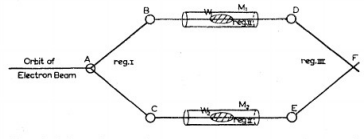


Figure: Electric effect schematic.

## The Magnetic Effect

1. Electron beam is split at point A.
2. Dual beams pass over and under a solenoid coil producing a constant magnetic field in a infinitesimally small region at the origin.
3. Electron beams are recombined at point F.

$$H = \frac{[\mathbf{P} - \frac{e}{c}\mathbf{A}]^2}{2m}$$

$$\Psi(t) = \Psi_0 \exp(-iS)$$

$$S = \int \mathbf{A} \cdot d\mathbf{x}$$

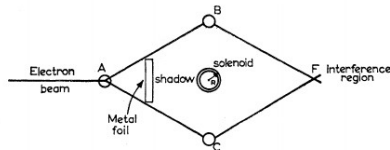


Figure: Magnetic effect schematic.

## The Berry Phase

The General Case:

$$H = \frac{[\mathbf{P} - \frac{e}{c}\mathbf{A}]^2}{2m} + V(t)$$

$$\Psi(t) = \Psi_0 \exp(-iS)$$

$$S = \oint V(t)dt + \mathbf{A} \cdot d\mathbf{x}$$

The Berry Phase:

$$\gamma = \oint i \langle \Psi(t) | \nabla_r | \Psi(t) \rangle d\mathbf{r} = S$$



## Section 6

### Outlook

The Irrelevant Phase

The Berry Phase

The Adiabatic Theorem

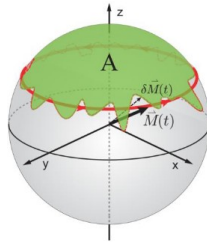
Magnetic Moment in a Precessing Field

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## Quantum Information

- Use of the geometric phase in a universal set of quantum gates
- Fault tolerance because of geometric nature



$$\hat{H} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

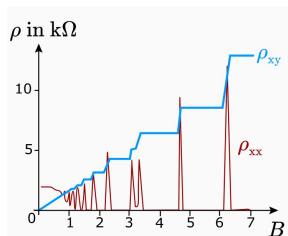
$$\hat{\phi} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

$$\hat{U}_{cphase} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{bmatrix}$$

## Quantum Hall Effect

- Quantized conductivity closely related to the Berry curvature ( $\Omega_n$ ) in the Thoules, Kohomoto, Nightingale and den Nijs (TKNN) formalism.

$$\sigma_{xy} = \frac{e^2}{\hbar} \sum_n \int \frac{dk}{2\pi} \Omega_z^n$$



## Conclusion

- ▶ The Berry Phase is a gauge-invariant property of the manifold describing the parameter space of the system.
- ▶ It is a property of a parametrized loop through parameter space.
- ▶ It is given as the integral of the berry connection over the parametrized path ( $\oint i \langle \Psi(\mathbf{R}) | \nabla | \Psi(\mathbf{R}) \rangle$ ) which can be considered as the relative phase of two infinitesimally close states.
- ▶ The Berry Phase arises naturally in the adiabatic theorem describing the quantum evolution of a state in a system undergoing infinitesimally small changes to its parameters.
- ▶ The Berry Phase is just one example of the much wider phenomena of holonomy.