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June 9, 2016

Abstract

We compare Pardon's framework of implicit atlases with Spivak's framework for an oo-category of derived manifolds.

1 Logistical stuff

In no particular order:

1. If you want to compile the file after adding new bibliography references, make sure you add the reference to the biblio.bib file . Also make sure to run BibTeX.

2. Hiro's comments are in blue, Jake's in red.

2 The structure sheaf

Consider the simplest case of an implicit atlas $\mathcal{A}$ with a single global chart, given by a smooth manifold Y, a smooth function $s : Y \to E$ into a finite dimensional vector space E, and the zero set $X = s^{-1}(0)$. Since the following diagram should be a pullback,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "s"] \\ * \arrow[r, "0"'] & E \end{tikzcd}$$

we would like to have that the C^∞-ring $\mathcal{O}(X)$ is the homotopy tensor product, $\mathcal{O}(X) = \mathbb{R} \otimes_{\mathcal{O}(E)} \mathcal{O}(Y)$.

2.1 Simplicial sheaf of n-thickenings

Definition 2.1.1. Let X be a Hausdorff space. An **n-thickening** is the data of a smooth manifold Y, together with vector spaces $E_0, \ldots, E_n$ and maps $\sigma_i : Y \to E_i$, and a homeomorphism $\psi : X \to \sigma^{-1}(0)$ (usually suppressed), where $\sigma = \sigma_0 \oplus \cdots \oplus \sigma_n$. We also demand a transversality requirement for $n \geq 1$. [Spell this out... it should be that all the simultaneous zero sets except the zero set of all the σ_i at once are smooth manifolds cut out transversely.]

Two n-thickenings $(Y^k, (E_i^k, \sigma_i^k))$, for $k = 1, 2$, are declared equivalent if they are isomorphic when restricted to open neighborhoods $X \subset U^k \subset Y^k$ (thus n-thickenings only depend on the germs of the functions σ_i around X).

Now, suppose $(X, \mathcal{A})$ is an implicit manifold. [Details here to be worked out and clarified. In particular, need to explain what "compatible with $\mathcal{A}$" means, as well as addressing the fact that the zero sets $f_i^{-1}(0)$ won't be smooth manifolds. Proving the simplicial identities would be good too.] There is a simplicial (pre?)sheaf $\mathcal{TH}_{\mathcal{A}}^\bullet$ on X, given by

$$\mathcal{TH}_{\mathcal{A}}^n(U) = \{n\text{-thickenings of } U \text{ compatible with } \mathcal{A}\}\,.$$

The face maps are given by

$$d_i : (Y, (E_0, \sigma_0, \ldots, E_n, \sigma_n)) \mapsto \Big(\sigma_i^{-1}(0), (E_0, \sigma_0, \ldots, \widehat{E}_i, \widehat{\sigma}_i, \ldots, E_n, \sigma_n)\Big)\,,$$

and the degeneracies are given by

$$s_i : (Y, (E_0, \sigma_0, \ldots, E_n, \sigma_n)) \mapsto (Y \times E_i, (E_0, \sigma_0, \ldots, E_i, \sigma_i \circ \pi_1 - \pi_2, E_i, \pi_2, \ldots, E_n, \sigma_n))\,.$$

Proposition 2.1.2. The simplicial set $\mathcal{TH}^\bullet_{\mathcal{A}}(U)$ is a Kan complex, and is in fact contractible.

We have the following candidate for the structure sheaf of $(X, \mathcal{A})$ as a derived manifold. Consider the simplicial sheaf $\mathcal{O}^\bullet_X$ on X, given by

$$\mathcal{O}^n_X(U) = \coprod_{(Y,(E_i,\sigma_i)) \in \mathcal{TH}^n_{\mathcal{A}}(U)} C^\infty(Y).$$

[A slight modification of this should also give a notion of morphism spaces for implicit manifolds. For X' another implicit manifold, simply replace $C^\infty(Y)$ above with $\mathrm{Hom}(Y, X')$; because Y is a manifold, it already knows how to map into what is morally a limit of smooth manifolds.]

Speculation 2.1.3. If X is a smooth manifold, then $\mathcal{O}^\bullet_X(X)$ is equivalent to the discrete simplicial set $C^\infty(X)$. Further, if X is the intersection of the origin in $\mathbb{R}$ with itself, then $\pi_0(\mathcal{O}^\bullet_X(X)) \cong \mathbb{R}$ and $\pi_1(\mathcal{O}^\bullet_X(X)) \cong \mathbb{R}$. To see this last part, consider smooth functions on $\mathbb{R}^n$ that vanish on the coordinate hyperplanes, and see how strong a zero they must have when restricted to another generic plane.

3 Goals

In no particular order, but enumerated for sake of reference:

1. (The category of implicit manifolds) The pair $(X, \mathcal{A})$ of a Hausdorff X with an implicit atlas $\mathcal{A}$ (a la Pardon) is an object in some category. Define this category. Ideally, it should be a category enriched in Kan complexes.

   (a) Part of this should involve streamlining the definition of $\mathcal{A}$. Let's present it as categorically as possible.

   (b) One should do this when the implicit atlases are *smooth*, too.

   (c) So an ideal type of theorem would be something like:

   Theorem 3.0.1. (After defining some category.) The category of implicit manifolds is enriched over Kan complexes. The category of smooth manifolds (in the usual sense) embeds fully and faithfully.

   (d) If any of this makes sense, then there should be a close connection between defining the morphism spaces in this category and giving $(X, \mathcal{A})$ the structure of a derived manifold in the sense of [Spi07]. In particular, we should have that $\mathrm{Hom}(-, \mathbb{R}) \simeq \mathcal{O}_X$ as sheaves on X. This may help in figuring out the correct notion of morphism spaces, and in particular it gives an immediate candidate for $\mathrm{Hom}(X, Y)$ when Y is a smooth manifold (decompose Y into patches, map open subsets U of X into patches by tuples in $\mathcal{O}_X(U)^{\dim Y}$).

2. (Comparing with Spivak) We should construct a functor from Pardon's framework (which I called implicit manifolds above—we can change the name) to Spivak's. This is where a lot of the logical meat is. Put another way: *How does a choice of $\mathcal{A}$ on X define a derived scheme?*

   (a) The first example of this to understand is for the zero locus of a section of a bundle. This is section 2.2.1 of [Par16].

   (b) An ideal type of theorem would be something like:

   Theorem 3.0.2. There is a functor F from implicit manifolds to derived manifolds. It is fully faithful on smooth manifolds (in the usual sense).

   However, I am not sure to what extent this functor should be fully faithful on all implicit manifolds. This of course depends on the choice of homs, and it's not obvious to me that maps defined to be compatible with implicit atlases will recover the whole homotopy type of the hom spaces for derived manifolds.

3. (The virtual fundamental cycle and cobordisms) How should we think of the virtual fundamental cycle? Pardon presents it as an element of Cech cochains, but should it be thought of as an element of a cobordism group? See Remark 1.3.2 of [Par16]. I think Remark 1.3.3 is also helpful; but how is this an invariant of the derived manifold itself?

   (a) I can't find where Pardon actually sets up a theory of cobordisms between implicit manifolds. It'd be nice to prove a statement like

      Theorem 3.0.3. (After defining a notion of cobordism between implicit manifolds.) If s_t is a homotopy between two sections s_0, s_1 of a vector bundle, then the implicit manifolds $(X_i, \mathcal{A}_i)$ associated to the s_i are cobordant. (i=0,1.)

   (b) Then it would be nice to show that Borel-Moorse cochains on X are actually just sections of some sort of stabilized "normal bundle" on X. (Roughly, there should be some notion of a normal bundle for an "embedding" of X into $\mathbb{R}^N$ for large N.) Then, the same way characteristic classes are preserved via cobordism, these cochains may be preserved under cobordism (however we define this), and we can try to show that the VFCs defined on X_i are compatible.

   (c) Finally, an ideal theorem would be to prove that

      Theorem 3.0.4. The functor F from above preserves cobordisms. That is, $X_0 \sim X_1$ cobordant $\implies F(X_0) \sim F(X_1)$, where $\sim$ is the cobordism relation. Further, F also preserves normal bundle classes and sections thereof. (This last sentence is intentionally vague.)

      See Section 6.2 of [Spi07] and 3.1 of [Spi10] for the derived manifolds definition of cobordism.

4. (Examples) We should write out the examples of Morse theory, and of holomorphic curves, as presented in [Par16].

5. (Intersections of virtual fundamental cycles) The Kunneth formula is much harder; I think we'll actually need to deal with derived smooth stacks to do that bit, because negative-dimensional things will show up.

4 Derived manifolds

Let's keep a running document here of what we're learning about Spivak's work.

Here we summarize the portions of [Spi10] and [Spi07] salient to our work.

5 Implicit atlases

Let's keep a running document here of what we're learning about Pardon's work.

Here we summarize the portions of [Par16] salient to our work.

References

[Par16] John Pardon, *An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves*, Geom. Topol. **20** (2016), no. 2, 779–1034. MR 3493097

[Spi07] David Isaac Spivak, *Quasi-smooth derived manifolds*, ProQuest LLC, Ann Arbor, MI, 2007, Thesis (Ph.D.)–University of California, Berkeley. MR 2710585

[Spi10] David I. Spivak, *Derived smooth manifolds*, Duke Math. J. **153** (2010), no. 1, 55–128. MR 2641940