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Abstract

We compare Pardon's framework of implicit atlases with Spivak's framework for an oo-category of derived manifolds.

1 Logistical stuff

In no particular order:

1. If you want to compile the file after adding new bibliography references, make sure you add the reference to the biblio.bib file . Also make sure to run BibTeX.
2. Hiro's comments are in blue, Jake's in red.

2 The structure sheaf

Consider the simplest case of an implicit atlas \mathcal{A} with a single global chart, given by a smooth manifold Y , a smooth function $s : Y \rightarrow E$ into a finite dimensional vector space E , and the zero set $X = s^{-1}(0)$. Since the following diagram should be a pullback,

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow s \\ * & \xrightarrow{0} & E \end{array}$$

we would like to have that the C^∞ -ring $\mathcal{O}(X)$ is the homotopy tensor product, $\mathcal{O}(X) = \mathbb{R} \otimes_{\mathcal{O}(E)} \mathcal{O}(Y)$.

Great simple example, and nice motivation. This could be cast as "Example x.y.z" after we finish writing a definition we know to be correct.

2.1 Simplicial sheaf of n -thickenings

Definition 2.1.1. Let X be a Hausdorff space. An n -**thickening** is the data of a smooth manifold Y , together with vector spaces E_0, \dots, E_n and maps $\sigma_i : Y \rightarrow E_i$, and a homeomorphism $\psi : X \rightarrow \sigma^{-1}(0)$ (usually suppressed), where $\sigma = \sigma_0 \oplus \dots \oplus \sigma_n$. We also demand a transversality requirement for $n \geq 1$. [Spell this out... it should be that all the simultaneous zero sets except the zero set of all the σ_i at once are smooth manifolds cut out transversely.] We certainly should spell this out; your transversality condition sounds like it only kicks in for k -simplices with $k \geq 2$.

Two n -thickenings $(Y^k, (E_i^k, \sigma_i^k))$, for $k = 1, 2$, are declared equivalent if they are isomorphic when restricted to open neighborhoods $X \subset U^k \subset Y^k$ (thus n -thickenings only depend on the germs of the functions σ_i around X). It might be nicer to talk about refinements. There's a filtered structure on these n -thickenings given by refinements—a refinement would be an (open?) embedding of the manifolds, together with embeddings of vector spaces that are all compatible with the s_i . There's probably a clean categorical way to say this—there's a category of n -thickenings that has a forgetful functor to Δ^{op} .

Now, suppose (X, \mathcal{A}) is an implicit manifold. [Details here to be worked out and clarified. In particular, need to explain what "compatible with \mathcal{A} " means, as well as addressing the fact that the zero sets $f_i^{-1}(0)$ won't be smooth manifolds. Proving the simplicial identities would be good too.] There is a simplicial (pre?)sheaf $\mathcal{TH}_{\mathcal{A}}^\bullet$

on X , given by

$$\mathcal{TH}_{\mathcal{A}}^n(U) = \{n\text{-thickenings of } U \text{ compatible with } \mathcal{A}\}.$$

The face maps are given by

$$d_i : (Y, (E_0, \sigma_0, \dots, E_n, \sigma_n)) \mapsto (\sigma_i^{-1}(0), (E_0, \sigma_0, \dots, \widehat{E_i}, \widehat{\sigma_i}, \dots, E_n, \sigma_n)),$$

and the degeneracies are given by

$$s_i : (Y, (E_0, \sigma_0, \dots, E_n, \sigma_n)) \mapsto (Y \times E_i, (E_0, \sigma_0, \dots, E_i, \sigma_i \circ \pi_1 - \pi_2, E_i, \pi_2, \dots, E_n, \sigma_n)).$$

Proposition 2.1.2. The simplicial set $\mathcal{TH}_{\mathcal{A}}^\bullet(U)$ is a Kan complex. [I believe it is actually contractible, but I would have to use more detail about what “compatible with \mathcal{A} ” means to prove this].

Proof. By symmetry in the definition of $\mathcal{TH}_{\mathcal{A}}^\bullet(U)$, it suffices to prove the claim for a 0-horn, given by the n -thickenings

$$(Y^k, (E_0, \sigma_0^k, \dots, \widehat{E_k}, \widehat{\sigma_k^k}, \dots, E_n, \sigma_n^k)),$$

for $1 \leq k \leq n$. Define

$$Z^k = (\sigma_1^k)^{-1}(0) \cap \dots \cap (\widehat{(\sigma_k^k)^{-1}(0)}) \cap \dots \cap (\sigma_n^k)^{-1}(0) \subset Y^k.$$

We have that Z^k is a smooth submanifold of Y^k , and further we have $Z^1 \cong \dots \cong Z^n$ because the given data is a horn; we now suppress these isomorphisms and write Z for this space. Now, define

$$W^{k,j} = (\sigma_1^k)^{-1}(0) \cap \dots \cap \dots \cap (\widehat{(\sigma_j^k)^{-1}(0)}) \cap \dots \cap (\widehat{(\sigma_k^k)^{-1}(0)}) \cap \dots \cap (\sigma_n^k)^{-1}(0) \subset Y^k,$$

for $1 \leq j \neq k \leq n$. We have that $W^{k,j}$ is a smooth submanifold of Y^k , and further we have that $W^{1,j} \cong \dots \cong \widehat{W^{j,j}} \cong \dots \cong W^{n,j}$, which we will suppress and denote by W^j .

By the tubular neighborhood theorem and the fact that we are identifying n -thickenings which agree in a neighborhood of U , we may assume WLOG, but not canonically, that W^j has the structure of a smooth vector bundle over Z , and further that

$$Y^k = W^1 \oplus_Z \dots \oplus_Z \widehat{W^k} \oplus_Z \dots \oplus_Z W^n.$$

But we may then define an $(n+1)$ -thickening that fills the horn, namely

$$()$$

□

We have the following candidate for the structure sheaf of (X, \mathcal{A}) as a derived manifold. Consider the simplicial sheaf \mathcal{O}_X^\bullet on X , given by

$$\mathcal{O}_X^n(U) = \coprod_{(Y, (E_i, \sigma_i)) \in \mathcal{TH}_{\mathcal{A}}^n(U)} C^\infty(Y).$$

I think this is a beautiful candidate. Do you want to try and work out the speculation? Spivak at some point must do something very similar in his thesis.

Speculation 2.1.3. If X is a smooth manifold, then $\mathcal{O}_X^\bullet(X)$ is equivalent to the discrete simplicial set $C^\infty(X)$. Further, if X is the intersection of the origin in \mathbb{R} with itself, then $\pi_0(\mathcal{O}_X^\bullet(X)) \cong \mathbb{R}$ and $\pi_1(\mathcal{O}_X^\bullet(X)) \cong \mathbb{R}$. To see this last part, consider smooth functions on \mathbb{R}^n that vanish on the coordinate hyperplanes, and see how strong a zero they must have when restricted to another generic plane.

3 Goals

In no particular order, but enumerated for sake of reference:

1. (The category of implicit manifolds) The pair (X, \mathcal{A}) of a Hausdorff X with an implicit atlas \mathcal{A} (a la Pardon) is an object in some category. Define this category. Ideally, it should be a category enriched in Kan complexes.

- (a) Part of this should involve streamlining the definition of \mathcal{A} . Let's present it as categorically as possible.
- (b) One should do this when the implicit atlases are *smooth*, too.
- (c) So an ideal type of theorem would be something like:

Theorem 3.0.1. (After defining some category.) The category of implicit manifolds is enriched over Kan complexes. The category of smooth manifolds (in the usual sense) embeds fully and faithfully.

- (d) If any of this makes sense, then there should be a close connection between defining the morphism spaces in this category and giving (X, \mathcal{A}) the structure of a derived manifold in the sense of [Spi07]. In particular, we should have that $\mathrm{Hom}(-, \mathbb{R}) \simeq \mathcal{O}_X$ as sheaves on X . This may help in figuring out the correct notion of morphism spaces, and in particular it gives an immediate candidate for $\mathrm{Hom}(X, Y)$ when Y is a smooth manifold (decompose Y into patches, map open subsets U of X into patches by tuples in $\mathcal{O}_X(U)^{\dim Y}$).
- (e) That's a great point.

2. (Comparing with Spivak) We should construct a functor from Pardon's framework (which I called implicit manifolds above—we can change the name) to Spivak's. This is where a lot of the logical meat is. Put another way: *How does a choice of \mathcal{A} on X define a derived scheme?*

- (a) The first example of this to understand is for the zero locus of a section of a bundle. This is section 2.2.1 of [Par16].
- (b) An ideal type of theorem would be something like:

Theorem 3.0.2. There is a functor F from implicit manifolds to derived manifolds. It is fully faithful on smooth manifolds (in the usual sense).

However, I am not sure to what extent this functor should be fully faithful on all implicit manifolds. This of course depends on the choice of homs, and it's not obvious to me that maps defined to be compatible with implicit atlases will recover the whole homotopy type of the hom spaces for derived manifolds.

3. (The virtual fundamental cycle and cobordisms) How should we think of the virtual fundamental cycle? Pardon presents it as an element of Cech cochains, but should it be thought of as an element of a cobordism group? See Remark 1.3.2 of [Par16]. I think Remark 1.3.3 is also helpful; but how is this an invariant of the derived manifold itself?

- (a) I can't find where Pardon actually sets up a theory of cobordisms between implicit manifolds. It'd be nice to prove a statement like

Theorem 3.0.3. (After defining a notion of cobordism between implicit manifolds.) If s_t is a homotopy between two sections s_0, s_1 of a vector bundle, then the implicit manifolds (X_i, \mathcal{A}_i) associated to the s_i are cobordant. (i=0,1.)

- (b) Then it would be nice to show that Borel-Moorse cochains on X are actually just sections of some sort of stabilized “normal bundle” on X . (Roughly, there should be some notion of a normal bundle for an “embedding” of X into \mathbb{R}^N for large N .) Then, the same way characteristic classes are preserved via cobordism, these cochains may be preserved under cobordism (however we define this), and we can try to show that the VFCs defined on X_i are compatible.
 - (c) Finally, an ideal theorem would be to prove that

Theorem 3.0.4. The functor F from above preserves cobordisms. That is, $X_0 \sim X_1$ cobordant $\implies F(X_0) \sim F(X_1)$, where \sim is the cobordism relation. Further, F also preserves normal bundle classes and sections thereof. (This last sentence is intentionally vague.)

See Section 6.2 of [Spi07] and 3.1 of [Spi10] for the derived manifolds definition of cobordism.

4. (Examples) We should write out the examples of Morse theory, and of holomorphic curves, as presented in [Par16].

5. (Intersections of virtual fundamental cycles) The Kunneth formula is much harder; I think we'll actually need to deal with derived smooth stacks to do that bit, because negative-dimensional things will show up.

4 Derived manifolds

Let's keep a running document here of what we're learning about Spivak's work.

Here we summarize the portions of [Spi10] and [Spi07] salient to our work. The notation follows [Spi07] closely, and nothing original is in this section of our paper.

4.1 Local models and C^∞ rings

I might eventually remove this section; it's a non-homotopical version of the “smooth rings” we review in the next section.

Let \mathcal{R} be a small Grothendieck site.

1. Let $U : \mathcal{R} \rightarrow \mathbf{Top}$ be a morphism of Grothendieck sites. We say U is a basis for its image if and only if, for every $R \in \mathcal{R}$ and every covering of $U(R)$ in \mathbf{Top} , there is a refinement of the covering by open sets in the image of U .
2. Given a collection of diagrams, $\mathcal{L} = \{L : \mathcal{C}_L \rightarrow \mathcal{R}\}$, we say a full subcategory $\mathcal{S} \subset \mathcal{R}$ is closed under \mathcal{L} -limits if any L factoring through \mathcal{S} has a limit in \mathcal{S} .
3. \mathcal{S} is closed under gluing if—whenever an open cover of $T \in \mathcal{R}$ is fully contained in \mathcal{S} , then T is in \mathcal{S} as well.
4. We say a collection of objects $G \subset \mathbf{Ob} \mathcal{R}$ generates \mathcal{S} if \mathcal{S} is the smallest full subcategory of \mathcal{R} containing G , and closed under \mathcal{L} -limits and gluing.

Example 4.1.1. Let $\mathcal{R} = \mathcal{E}$ denote the category of finite-dimensional Euclidean spaces. We let its morphisms be all smooth maps. The forgetful map $U : \mathcal{E} \rightarrow \mathbf{Top}$ is a basis for its image.

We let \mathcal{L} denote the collection of all functors from a finite, discrete category to \mathcal{E} . (I.e., any functor $I \rightarrow \mathcal{E}$ where I is a possibly empty finite set.)

Then $G = \{\mathbb{R}\}$ generates \mathcal{E} under \mathcal{L} -limits. We refer to

$$(\mathcal{E}, U, \mathcal{L}, \mathbb{R})$$

as the Euclidean category of local models. We call \mathbb{R} the affine line for \mathcal{E} .

Note that any functor $\mathcal{R} \rightarrow \mathbf{Sets}$ preserving \mathcal{L} -limits is determined by what it does on any full subcategory containing a G which generates \mathcal{R} . In particular, any functor $\mathcal{E} \rightarrow \mathbf{Sets}$ is determined by what it does to \mathbb{R} and to its smooth endomorphisms.

Definition 4.1.2. A C^∞ -ring is a covariant functor

$$F : \mathcal{E} \rightarrow \mathbf{Sets}$$

preserving \mathcal{L} -limits. We refer to $|F| := F(\mathbb{R})$ as the underlying set of F .

As an example, the corepresentable functor $\mathrm{hom}(\mathbb{R}^n, -) : \mathcal{E} \rightarrow \mathbf{Sets}$ is a C^∞ ring. Its value on \mathbb{R} is the set of smooth functions on \mathbb{R}^n .

Remark 4.1.3. Since \mathbb{R} is a commutative ring object in the category \mathcal{E} , F induces a commutative ring structure on $|F|$ by virtue of preserving finite limits.

Remark 4.1.4. Other geometries, notably the geometry of affine \mathbb{Z} -schemes, also fit into this framework of local models.

4.2 Smooth rings

One of the most powerful principles in algebraic geometry is that affine schemes are the same thing as commutative rings. This is somewhat tautological, but is not so tautological that its transportation into the world of smooth geometry is self-evident. (For instance, it takes care to define the appropriate tensor product for which $C^\infty(M) \otimes C^\infty(N) \cong C^\infty(M \times N)$.) At the same time, we also seek a notion of “rings” which is sufficiently homotopical. So the C^∞ rings described in the previous sections should be thought of as the tip of a homotopical iceberg, in a way similar to how cdgas are a homotopical iceberg whose tip comprises the usual notion of commutative rings.

Let $\mathbf{Fun}(\mathbf{Man}, \mathbf{sSets})$ be the category of all functors from the category of manifolds to the category of simplicial sets. We call $F : \mathbf{Man} \rightarrow \mathbf{sSets}$ discrete if $F(M)$ is a discrete simplicial set for all manifolds M .

Here are some facts we will take for granted:

1. $\mathbf{Fun}(\mathbf{Man}, \mathbf{sSets})$ has an injective model structure. Its weak equivalences are object wise—this means $F \simeq G$ if and only if $F(M) \simeq G(M)$ for every manifold M . Its cofibrations are object-wise as well, meaning $F \rightarrow G$ is a cofibration if and only if $F(M) \rightarrow G(M)$ is a cofibration of simplicial sets for each M . This model structure is proper, and cofibrantly generated.
2. Let S be a simplicial set. We let $\underline{S} : \mathbf{Man} \rightarrow \mathbf{sSets}$ denote the constant functor. This makes $\mathbf{Fun}(\mathbf{Man}, \mathbf{sSets})$ tensored over simplicial sets.

3. All discrete F are fibrant. In particular, if M is a manifold, the functor

$$H_M := \mathbf{Man}(M, -)$$

is fibrant.

Definition 4.2.1. The model category of smooth rings is the localization of $\mathbf{Fun}(\mathbf{Man}, \mathbf{sSets})$ along Ψ .

We define Ψ now. Consider a diagram

$$\begin{array}{ccc} & P & \\ & \downarrow a & \\ M & \xrightarrow{s} & N \end{array}$$

where s is a submersion. Then the pullback $Q \cong M \times_N P$ exists in the category of smooth manifolds. On the other hand, one could also take the homotopy pushout

$$\begin{array}{ccc} H_N & \xrightarrow{s^*} & H_M \\ a^* \downarrow & & \downarrow \\ H_P & \longrightarrow & G \end{array}$$

in the model category $\mathbf{Fun}(\mathbf{Man}, \mathbf{sSets})$. We let

$$\psi_{s,a} : G \rightarrow H_Q$$

denote the induced map. Note that localizing with respect to these $\psi_{s,a}$ means that the assignment "corepresenting functor" would preserve the one good operation there is in smooth manifolds: pulling back along submersions. That is, pullbacks obtained from submersions will be sent to homotopy pushouts.

Finally, note that the 0-manifold $pt \cong \mathbb{R}^0$ is terminal in \mathbf{Man} . We'd like this to be reflect in the category of smooth rings. (This reflects a principle from algebraic geometry: if $\mathrm{Spec} k$ is terminal in k -schemes, we'd like the structure ring k to be initial in k -algebras.) However, in $\mathbf{Fun}(\mathbf{Man}, \mathbf{sSets})$, the initial object is not H_{pt} , but rather the functor sending $M \mapsto \emptyset$. So we would like to further localize with respect to the unique map

$$(M \mapsto \emptyset) \rightarrow H_{pt}.$$

Definition 4.2.2. We let Ψ denote the collection of all morphisms of the form $\psi_{s,a}$ (for s a submersion and a a smooth map) together with the unique map $(M \mapsto \emptyset) \rightarrow H_{pt}$.

It would be nice to just think of this as an ∞ -category from the outset, without having to resort to model categories, and then defining an ∞ -category. But so it is.

Example 4.2.3 (Example 2.1.10 in [Spi07].). If M, N are smooth manifolds, the object-wise coproduct $H_M \amalg H_N$ is in $\mathbf{Fun}(\mathbf{Man}, \mathbf{sSets})$. However, it is not a fibrant object in the model category of smooth rings. For instance, apply it to the pullback diagram realizing $M \times_{pt} N \cong M \times N$. However, the map

$$H_M \amalg H_N \rightarrow H_{M \times N}$$

is a fibrant replacement map.

Example 4.2.4 (The fat point, 2.1.13 in [Spi07].). Let us compute the homotopy pushout

$$\begin{array}{ccc} H_{\mathbb{R}} & \longrightarrow & H_{pt} \\ \downarrow & & \downarrow \\ H_{pt} & \longrightarrow & G \end{array}$$

where the arrows $H_{\mathbb{R}} \rightarrow H_{pt}$ are induced by the inclusion of the origin in \mathbb{R} . As usual, to compute this homotopy pushout, we just replace either the top or lefthand arrow by a cofibration. (The localization is left proper because $\mathbf{Fun}(\mathbf{Man}, \mathbf{sSets})$ is; see [?] 4.1.1.) Consider the composite map

$$g : \mathbb{R} \rightarrow pt \xrightarrow{0} \mathbb{R}.$$

Then we have a functor $C : \mathbf{Man} \rightarrow \mathbf{sSets}$ whose only non-degenerate simplices are in dimensions 1 and 0, given by

$$d_0, d_1 : H_{\mathbb{R}} \rightarrow H_{\mathbb{R}}$$

where d_0 is the identity and d_1 is g^* . Obviously, the inclusion $H_{\mathbb{R}} \rightarrow C$ is a cofibration because it is a levelwise injection for any manifold M . One can check straightforwardly that $H_{pt}(M) \simeq C(M)$ for every manifold M . Hence the homotopy pushout G can be computed as the honest pushout of the diagram

$$\begin{array}{ccc} H_{\mathbb{R}} & \longrightarrow & C \\ \downarrow & & \downarrow \\ H_{pt} & \longrightarrow & G. \end{array}$$

Thus G is given by the functor $\mathbf{Man} \rightarrow \mathbf{sSets}$ whose non-degenerate bit we represent by

$$d_0, d_1 : H_{\mathbb{R}} \rightarrow H_{pt}.$$

Here, both d_i are induced by the inclusion $0 : pt \rightarrow \mathbb{R}$.

Remark 4.2.5. We remind the reader of what localization of \mathcal{C} to $\mathcal{C}[\Psi^{-1}]$ does, at least at a superficial level. Heuristically, it turns any morphism in $\Psi \subset \mathcal{C}$ into an equivalence in the localization. In terms of model categories:

1. \mathcal{C} and $\mathcal{C}[\Psi^{-1}]$ are the same simplicial category.
2. The cofibrations of $\mathcal{C}[\Psi^{-1}]$ are the cofibrations of \mathcal{C} .
3. A fibrant object X of $\mathcal{C}[\Psi^{-1}]$ is one which is both fibrant in \mathcal{C} , and Ψ -local. That is, for any $\psi : A \rightarrow B \in \Psi$, the map

$$\mathrm{Map}(B, X) \rightarrow \mathrm{Map}(A, X)$$

is a weak equivalence.

4. The weak equivalences in $\mathcal{C}[\Psi^{-1}]$ are those $f : X \rightarrow Y$ such that

$$\mathrm{Map}(Y, -) \rightarrow \mathrm{Map}(X, -)$$

is a weak equivalence whenever $-$ is fibrant in $\mathcal{C}[\Psi^{-1}]$.

4.3 Derived manifolds

Let X be a topological space. Let \mathcal{F} be a sheaf of C^∞ rings. This means that \mathcal{F} is a contravariant functor from $\mathbf{Open}(X)$ to $\mathbf{Fun}(\mathbf{Man}, \mathbf{sSets})$, satisfying the following property: [Continue here](#).

We say that \mathcal{F} is local if, for every open cover $\{U_\alpha$ of M , the natural map of sheaves of sets

$$\pi_0 \left(\coprod_{\alpha} \mathcal{F}(-, M_\alpha) \right) \rightarrow \pi_0 \mathcal{F}(-, M)$$

is a surjection. If \mathcal{F} is local, we say that the pair (X, \mathcal{F}) is a local smooth-ringed space.

We will show how to compute homotopy pullbacks of C^∞ -ringed spaces momentarily. But for now:

Definition 4.3.1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a smooth map. Then the C^∞ -ringed space given by the homotopy fiber product

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathbb{R}^0 \\ \downarrow & & \downarrow 0 \\ \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m \end{array}$$

which we write $\mathcal{U} = (U, \mathcal{O}_U)$, is called a principal derived manifold. Any Hausdorff smooth-ring space (X, \mathcal{O}_X) is called a derived manifold if it can be covered by countably many principal derived manifolds.

In other words, derived manifolds are locally modeled on zero locuses of functions. The salient point here is that the homotopy fiber product is what produces a robust interpretation of the zero locus—one that is independent of perturbations, and which can shrug off non-transversality.

4.4 Simplicial commutative algebras

[Section 3.5 of \[Spi07\]](#).

Smooth-ring spaces define derived manifolds, but they rarely allow us to compute. Thankfully, replacing a smooth ring with its underlying simplicial commutative \mathbb{R} -algebra preserves many homotopy colimits.

5 Implicit atlases

Let's keep a running document here of what we're learning about Pardon's work.

Here we summarize the portions of [Par16] salient to our work.

References

- [Par16] John Pardon, *An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves*, *Geom. Topol.* **20** (2016), no. 2, 779–1034. MR 3493097
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