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**Abstract**

We compare Pardon's framework of implicit atlases with Spivak's framework for an oo-category of derived manifolds.

# 1 Logistical stuff

In no particular order:

1. If you want to compile the file after adding new bibliography references, make sure you add the reference to the biblio.bib file . Also make sure to run BibTeX.
2. Hiro's comments are in blue, Jake's in red.

# 2 The structure sheaf

Consider the simplest case of an implicit atlas $\mathcal{A}$ with a single global chart, given by a smooth manifold $Y$, a smooth function $s : Y \to E$ into a finite dimensional vector space $E$, and the zero set $X = s^{-1}(0)$. Since the following diagram should be a pullback,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "s"] \\ * \arrow[r, "0"'] & E \end{tikzcd}$$

we would like to have that the $C^\infty$-ring $\mathcal{O}(X)$ is the homotopy tensor product, $\mathcal{O}(X) = \mathbb{R} \otimes_{\mathcal{O}(E)} \mathcal{O}(Y)$.

Great simple example, and nice motivation. This could be cast as "Example x.y.z" after we finish writing a definition we know to be correct.

## 2.1 Simplicial sheaf of $n$-thickenings

**Definition 2.1.1.** Let $X$ be a Hausdorff space. An **$n$-thickening** is the data of a smooth manifold $Y$, together with vector spaces $E_0, \ldots, E_n$ and maps $\sigma_i : Y \to E_i$, and a homeomorphism $\psi : X \to \sigma^{-1}(0)$ (usually suppressed), where $\sigma = \sigma_0 \oplus \cdots \oplus \sigma_n$. We also demand a transversality requirement for $n \geq 1$. [Spell this out... it should be that all the simultaneous zero sets except the zero set of all the $\sigma_i$ at once are smooth manifolds cut out transversely.] We certainly should spell this out; your transversality condition sounds like it only kicks in for $k$-simplices with $k \geq 2$.

Two $n$-thickenings $(Y^k, (E_i^k, \sigma_i^k))$, for $k = 1, 2$, are declared equivalent if they are isomorphic when restricted to open neighborhoods $X \subset U^k \subset Y^k$ (thus $n$-thickenings only depend on the germs of the functions $\sigma_i$ around $X$). It might be nicer to talk about refinements. There's a filtered structure on these $n$-thickenings given by refinements—a refinement would be an (open?) embedding of the manifolds, together with embeddings of vector spaces that are all compatible with the $s_i$. There's probably a clean categorical way to say this—there's a category of $n$-thickenings that has a forgetful functor to $\Delta^{\mathrm{op}}$.

Now, suppose $(X, \mathcal{A})$ is an implicit manifold. [Details here to be worked out and clarified. In particular, need to explain what "compatible with $\mathcal{A}$" means, as well as addressing the fact that the zero sets $f_i^{-1}(0)$ won't be smooth manifolds. Proving the simplicial identities would be good too.] There is a simplicial (pre?)sheaf $\mathcal{TH}_{\mathcal{A}}^{\bullet}$

on $X$, given by

$$\mathcal{TH}^n_{\mathcal{A}}(U) = \{n\text{-thickenings of } U \text{ compatible with } \mathcal{A}\}\,.$$

The face maps are given by

$$d_i : (Y, (E_0, \sigma_0, \dots, E_n, \sigma_n)) \mapsto \Big(\sigma_i^{-1}(0), (E_0, \sigma_0, \dots, \widehat{E}_i, \widehat{\sigma}_i, \dots, E_n, \sigma_n)\Big)\,,$$

and the degeneracies are given by

$$s_i : (Y, (E_0, \sigma_0, \dots, E_n, \sigma_n)) \mapsto (Y \times E_i, (E_0, \sigma_0, \dots, E_i, \sigma_i \circ \pi_1 - \pi_2, E_i, \pi_2, \dots, E_n, \sigma_n))\,.$$

**Proposition 2.1.2.** The simplicial set $\mathcal{TH}^\bullet_{\mathcal{A}}(U)$ is a Kan complex. [I believe it is actually contractible, but I would have to use more detail about what "compatible with $\mathcal{A}$" means to prove this].

*Proof.* By symmetry in the definition of $\mathcal{TH}^\bullet_{\mathcal{A}}(U)$, it suffices to prove the claim for a 0-horn, given by the $n$-thickenings

$$(Y^k, (E_0, \sigma_0^k, \dots, \widehat{E}_k, \widehat{\sigma}_k^k, \dots, E_n, \sigma_n^k)),$$

for $1 \le k \le n$. Define

$$Z^k = (\sigma_1^k)^{-1}(0) \cap \dots \cap \widehat{(\sigma_k^k)^{-1}(0)} \cap \dots \cap (\sigma_n^k)^{-1}(0) \subset Y^k.$$

We have that $Z^k$ is a smooth submanifold of $Y^k$, and further we have $Z^1 \cong \ldots \cong Z^n$ because the given data is a horn; we now suppress these isomorphisms and write $Z$ for this space. Now, define

$$W^{k,j} = (\sigma_1^k)^{-1}(0) \cap \dots \cap \dots \cap \widehat{(\sigma_j^k)^{-1}(0)} \cap \dots \cap \widehat{(\sigma_k^k)^{-1}(0)} \cap \dots \cap (\sigma_n^k)^{-1}(0) \subset Y^k,$$

for $1 \le j \ne k \le n$. We have that $W^{k,j}$ is a smooth submanifold of $Y^k$, and further we have that $W^{1,j} \cong \ldots \cong \widehat{W^{j,j}} \cong \ldots W^{n,j}$, which we will suppress and denote by $W^j$.

By the tubular neighborhood theorem and the fact that we are identifying $n$-thickenings which agree in a neighborhood of $U$, we may assume WLOG, but not canonically, that $W^j$ has the structure of a smooth vector bundle over $Z$, and further that

$$Y^k = W^1 \oplus_Z \dots \oplus_Z \widehat{W}^k \oplus_Z \dots \oplus_Z W^n.$$

But we may then define an $(n+1)$-thickening that fills the horn, namely

$$()$$

$\square$

**Proposition 2.1.3.** The simplicial set $\mathcal{TH}^\bullet(U)$ is equivalent to a discrete simplicial set.

**Definition 2.1.4.** An **implicit manifold** is a compact Hausdorff space $X$ together with a choice of global section $\mathcal{TH}^\bullet(X)$. This should agree with Pardon's definition of an implicit atlas.

We have the following candidate for the structure sheaf of $(X, \mathcal{A})$ as a derived manifold. Consider the simplicial sheaf $\mathcal{O}_X^\bullet$ on $X$, given by

$$\mathcal{O}_X^n(U) = \coprod_{(Y,(E_i,\sigma_i))\in\mathcal{TH}_{\mathcal{A}}^n(U)} C^\infty(Y).$$

I think this is a beautiful candidate. Do you want to try and work out the speculation? Spivak at some point must do something very similar in his thesis.

**Speculation 2.1.5.** If $X$ is a smooth manifold, then $\mathcal{O}_X^\bullet(X)$ is equivalent to the discrete simplicial set $C^\infty(X)$. Further, if $X$ is the intersection of the origin in $\mathbb{R}$ with itself, then $\pi_0(\mathcal{O}_X^\bullet(X)) \cong \mathbb{R}$ and $\pi_1(\mathcal{O}_X^\bullet(X)) \cong \mathbb{R}$. To see this last part, consider smooth functions on $\mathbb{R}^n$ that vanish on the coordinate hyperplanes, and see how strong a zero they must have when restricted to another generic plane.

# 3 Goals

In no particular order, but enumerated for sake of reference:

1. (The category of implicit manifolds) The pair $(X, \mathcal{A})$ of a Hausdorff $X$ with an implicit atlas $\mathcal{A}$ (a la Pardon) is an object in some category. Define this category. Ideally, it should be a category enriched in Kan complexes.

   (a) Part of this should involve streamlining the definition of $\mathcal{A}$. Let's present it as categorically as possible.

   (b) One should do this when the implicit atlases are *smooth*, too.

   (c) So an ideal type of theorem would be something like:

   **Theorem 3.0.1.** (After defining some category.) The category of implicit manifolds is enriched over Kan complexes. The category of smooth manifolds (in the usual sense) embeds fully and faithfully.

   (d) If any of this makes sense, then there should be a close connection between defining the morphism spaces in this category and giving $(X, \mathcal{A})$ the structure of a derived manifold in the sense of [Spi07]. In particular, we should have that $\mathrm{Hom}(-, \mathbb{R}) \simeq \mathcal{O}_X$ as sheaves on $X$. This may help in figuring out the correct notion of morphism spaces, and in particular it gives an immediate candidate for $\mathrm{Hom}(X, Y)$ when $Y$ is a smooth manifold (decompose $Y$ into patches, map open subsets $U$ of $X$ into patches by tuples in $\mathcal{O}_X(U)^{\dim Y}$).

   (e) That's a great point.

2. (Comparing with Spivak) We should construct a functor from Pardon's framework (which I called implicit manifolds above—we can change the name) to Spivak's. This is where a lot of the logical meat is. Put another way: *How does a choice of $\mathcal{A}$ on $X$ define a derived scheme?*

   (a) The first example of this to understand is for the zero locus of a section of a bundle. This is section 2.2.1 of [Par16].

   (b) An ideal type of theorem would be something like:

   **Theorem 3.0.2.** There is a functor $F$ from implicit manifolds to derived manifolds. It is fully faithful on smooth manifolds (in the usual sense).

However, I am not sure to what extent this functor should be fully faithful on all implicit manifolds. This of course depends on the choice of homs, and it's not obvious to me that maps defined to be compatible with implicit atlases will recover the whole homotopy type of the hom spaces for derived manifolds.

3. (The virtual fundamental cycle and cobordisms) How should we think of the virtual fundamental cycle? Pardon presents it as an element of Cech cochains, but should it be thought of as an element of a cobordism group? See Remark 1.3.2 of [Par16]. I think Remark 1.3.3 is also helpful; but how is this an invariant of the derived manifold itself?

   (a) I can't find where Pardon actually sets up a theory of cobordisms between implicit manifolds. It'd be nice to prove a statement like

   **Theorem 3.0.3.** (After defining a notion of cobordism between implicit manifolds.) If $s_t$ is a homotopy between two sections $s_0, s_1$ of a vector bundle, then the implicit manifolds $(X_i, \mathcal{A}_i)$ associated to the $s_i$ are cobordant. (i=0,1.)

   (b) Then it would be nice to show that Borel-Moorse cochains on $X$ are actually just sections of some sort of stabilized "normal bundle" on $X$. (Roughly, there should be some notion of a normal bundle for an "embedding" of $X$ into $\mathbb{R}^N$ for large $N$.) Then, the same way characteristic classes are preserved via cobordism, these cochains may be preserved under cobordism (however we define this), and we can try to show that the VFCs defined on $X_i$ are compatible.

   (c) Finally, an ideal theorem would be to prove that

   **Theorem 3.0.4.** The functor $F$ from above preserves cobordisms. That is, $X_0 \sim X_1$ cobordant $\implies F(X_0) \sim F(X_1)$, where $\sim$ is the cobordism relation. Further, $F$ also preserves normal bundle classes and sections thereof. (This last sentence is intentionally vague.)

   See Section 6.2 of [Spi07] and 3.1 of [Spi10] for the derived manifolds definition of cobordism.

4. (Examples) We should write out the examples of Morse theory, and of holomorphic curves, as presented in [Par16].

5. (Intersections of virtual fundamental cycles) The Kunneth formula is much harder; I think we'll actually need to deal with derived smooth stacks to do that bit, because negative-dimensional things will show up.

# 4 Derived manifolds

Let's keep a running document here of what we're learning about Spivak's work.

Here we summarize the portions of [Spi10] and [Spi07] salient to our work.

# 5 Implicit atlases

Let's keep a running document here of what we're learning about Pardon's work.

Here we summarize the portions of [Par16] salient to our work.

# References

[Par16] John Pardon, *An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves*, Geom. Topol. **20** (2016), no. 2, 779–1034. MR 3493097

[Spi07] David Isaac Spivak, *Quasi-smooth derived manifolds*, ProQuest LLC, Ann Arbor, MI, 2007, Thesis (Ph.D.)–University of California, Berkeley. MR 2710585

[Spi10] David I. Spivak, *Derived smooth manifolds*, Duke Math. J. **153** (2010), no. 1, 55–128. MR 2641940