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Abstract

We compare Pardon's framework of implicit atlases with Spivak's framework for an oo-category of derived manifolds.

1 Logistical stuff

In no particular order:

1. If you want to compile the file after adding new bibliography references, make sure you add the reference to the biblio.bib file . Also make sure to run BibTeX.

2. Hiro's comments are in blue, Jake's in red.

2 The structure sheaf

Consider the simplest case of an implicit atlas $\mathcal{A}$ with a single global chart, given by a smooth manifold Y, a smooth function $s : Y \to E$ into a finite dimensional vector space E, and the zero set $X = s^{-1}(0)$. Since the following diagram should be a pullback,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "s"] \\ * \arrow[r, "0"'] & E \end{tikzcd}$$

we would like to have that the C^∞-ring $\mathcal{O}(X)$ is the homotopy tensor product, $\mathcal{O}(X) = \mathbb{R} \otimes_{\mathcal{O}(E)} \mathcal{O}(Y)$.

Great simple example, and nice motivation. This could be cast as "Example x.y.z" after we finish writing a definition we know to be correct.

2.1 The cospan nerve of a category

Definition 2.1.1. Let $\mathcal{C}$ be a category. We construct a simplicial set $\mathrm{CoSpan}^\bullet(\mathcal{C})$, the **cospan nerve** of $\mathcal{C}$, as follows. Let $\mathcal{P}_+(n)$ denote the poset of nonempty subsets of the set $[n] = \{0, \ldots, n\}$. We define

$$\mathrm{CoSpan}^n(\mathcal{C}) = \mathrm{Fun}(\mathcal{P}_+(n), \mathcal{C}),$$

the set of functors from $\mathcal{P}_+(n)$ to $\mathcal{C}$. Given a map $f : [n] \to [m]$, we get an induced map $\mathrm{CoSpan}^m(\mathcal{C}) \to \mathrm{CoSpan}^n(\mathcal{C})$ given by pulling back along f, and so $\mathrm{CoSpan}^\bullet(\mathcal{C})$ is a simplicial set.

Proposition 2.1.2. Let $\mathcal{C}$ be a category with weak pushouts. Then $\mathrm{CoSpan}^\bullet(\mathcal{C})$ is a Kan complex.

Proof. [To be completed.] □

Speculation 2.1.3. Suppose $\mathcal{C}$ is a category with weak pushouts and a weak initial object. Then $\mathrm{CoSpan}^\bullet(\mathcal{C})$ is contractible.

2.2 Category of thickenings

Definition 2.2.1. A **thickening chart** $\mathcal{U} = (Y, E, f)$ consists of a smooth manifold Y, together with a smooth vector bundle E over Y and a smooth section $f : Y \to E$.

Given two thickening charts $\mathcal{U}_i = (Y_i, E_i, f_i)$ for $i = 0, 1$, a morphism $\mathcal{U}_0 \to \mathcal{U}_1$ is a smooth embedding $i : Y_0 \hookrightarrow Y_1$ together with an inclusion of vector bundles $j : E_0 \hookrightarrow i^*E_1$ such that $j \circ f_0 = i^*(f_1)$, and such that the induced diagram

$$\begin{tikzcd} Y_0 \arrow[r, hook, "i"] \arrow[d, "f_0"'] & Y_1 \arrow[d, "f_1"] \\ E_0 \arrow[r, hook, "\widetilde{j}"'] & E_1 \end{tikzcd}$$

is a transverse pullback of smooth manifolds. We let $\mathcal{T}$ denote the category whose objects are thickening charts, and whose morphisms are morphisms of thickening charts.

Wild Speculation 2.2.2. Consider the category whose objects are thickening charts, and whose morphisms are the same as morphisms of thickening charts but without the condition that i and $\widetilde{j}$ above be embeddings or the transverse pullback condition. I think there should be a model structure on this, where we have

- Cofibrations are maps with i and $\widetilde{j}$ embeddings.
- Weak equivalences are maps that are homeomorphisms on the zero set, and which are quasi-isomorphisms on the tangent complex [to be defined] around the zero set.
- Fibrations should have at least that the map $Y_0 \to Y_1$ be a submersion, plus probably the homotopy lifting property.

I think I can show that the acyclic cofibrations are just the things I'm calling morphisms of thickening charts above, with the extra condition that the zero sets be the same.

Remark 2.2.3. There are two functors out of $\mathcal{T}$ that should be very interesting. The first is a functor to Hausdorff spaces and open embeddings, and is called "take the zero set of the section f." The second is a functor to smooth manifolds, called "forget the vector bundle E and the section f."

Proposition 2.2.4. The category $\mathcal{T}$ has weak pushouts, and hence $\mathrm{CoSpan}^{\bullet}(\mathcal{T})$ is a Kan complex. [This is not true, only locally true.]

Proof. [To be done.] $\square$

Proposition 2.2.5. Let $\mathcal{U}$ be a thickening chart, and let $\mathcal{T}_{\mathcal{U}}$ denote the full subcategory of $\mathcal{T}$ on those thickening charts $\mathcal{V}$ such that $\mathcal{U}$ and $\mathcal{V}$ are in the same connected component of $\mathrm{CoSpan}^{\bullet}(\mathcal{T})$. Then $\mathcal{T}_{\mathcal{U}}$ has a weak initial object, and hence $\mathrm{CoSpan}^{\bullet}(\mathcal{T})$ is the disjoint union of contractible components. [This is not true, only locally true.]

Proof. [To be done.] □

2.3 Simplicial sheaf of n-thickenings

Definition 2.3.1. Let X be a Hausdorff space. An **n-thickening** is the data of a smooth manifold Y, together with vector spaces $E_0, \dots, E_n$ [maybe to avoid needing to sheafify, let E_i be a vector bundle over Y] and maps $\sigma_i : Y \to E_i$, and a homeomorphism $\psi : X \to \sigma^{-1}(0)$ (usually suppressed), where $\sigma = \sigma_0 \oplus \cdots \oplus \sigma_n$. We also demand a transversality requirement for $n \geq 1$. [Spell this out... it should be that all the simultaneous zero sets except the zero set of all the σ_i at once are smooth manifolds cut out transversely.] We certainly should spell this out; your transversality condition sounds like it only kicks in for k-simplices with $k \geq 2$.

Two n-thickenings $(Y^k, (E_i^k, \sigma_i^k))$, for $k = 1, 2$, are declared equivalent if they are isomorphic when restricted to open neighborhoods $X \subset U^k \subset Y^k$ (thus n-thickenings only depend on the germs of the functions σ_i around X). It might be nicer to talk about refinements. There's a filtered structure on these n-thickenings given by refinements—a refinement would be an (open?) embedding of the manifolds, together with embeddings of vector spaces that are all compatible with the s_i. There's probably a clean categorical way to say this—there's a category of n-thickenings that has a forgetful functor to Δ^{op}.

[Details here to be worked out and clarified. In particular, need to explain what "compatible with $\mathcal{A}$" means, as well as addressing the fact that the zero sets $f_i^{-1}(0)$ won't be smooth manifolds. Proving the simplicial identities would be good too.]

Let X be a Hausdorff space. There is a simplicial (pre?)sheaf $\mathcal{TH}_X^{\bullet}$ on X, called the **thickening sheaf** of X, given on an open $U \subset X$ by

$$\mathcal{TH}_X^n(U) = \{n\text{-thickenings of } U\}\,.$$

The face maps are given by

$$d_i : (Y, (E_0, \sigma_0; \dots; E_n, \sigma_n)) \mapsto \Big(\sigma_i^{-1}(0), (E_0, \sigma_0; \dots; \widehat{E}_i; \widehat{\sigma}_i, \dots; E_n, \sigma_n)\Big)\,,$$

and the degeneracies are given by

$$s_i : (Y, (E_0, \sigma_0; \dots ; E_n, \sigma_n)) \mapsto (Y \times E_i, (E_0, \sigma_0; \dots ; E_i, \pi_2 - \sigma_i \circ \pi_1; E_i, \pi_2; \dots ; E_n, \sigma_n)) \, .$$

Finally, if $V \subset U$ is open, then the restriction map is given by

$$r_{U,V} : (Y, (E_0, \sigma_0; \dots ; E_n, \sigma_n)) \mapsto (W, (E_0, \sigma_0|_W; \dots ; E_n, \sigma_n|_W)) \, ,$$

where $W \subset Y$ is any open subset with $W \cap U = V$. Any two choices of such a W are equivalent, since we are working locally around X.

Proposition 2.3.2. The simplicial set $\mathcal{TH}^\bullet(X)$ is a Kan complex.

Proof. [Proof only works locally on X really, but sheafifying should solve that.] By symmetry in the definition of $\mathcal{TH}^\bullet(X)$, it suffices to prove the claim for a 0-horn, given by the n-thickenings

$$(Y^k, (E_0, \sigma_0^k; \dots ; \widehat{E}_k, \widehat{\sigma}_k^k; \dots ; E_n, \sigma_n^k)),$$

for $1 \leq k \leq n$. Define

$$Z^k = (\sigma_1^k)^{-1}(0) \cap \dots \cap \widehat{(\sigma_k^k)^{-1}(0)} \cap \dots \cap (\sigma_n^k)^{-1}(0) \subset Y^k.$$

We have that Z^k is a smooth submanifold of Y^k, and further we have $Z^1 \cong \ldots \cong Z^n$ because the given data is a horn; we now suppress these isomorphisms and write Z for this space. Now, define

$$W^{k,j} = (\sigma_1^k)^{-1}(0) \cap \dots \cap \dots \cap \widehat{(\sigma_j^k)^{-1}(0)} \cap \dots \cap \widehat{(\sigma_k^k)^{-1}(0)} \cap \dots \cap (\sigma_n^k)^{-1}(0) \subset Y^k,$$

for $1 \leq j \neq k \leq n$. We have that $W^{k,j}$ is a smooth submanifold of Y^k, and further we have that $W^{1,j} \cong \ldots \cong \widehat{W^{j,j}} \cong \ldots W^{n,j}$, which we will suppress and denote by W^j.

By the inverse function theorem, the transversality conditions in $\mathcal{TH}^\bullet(X)$, and the fact that we are identifying n-thickenings which agree in a neighborhood of X, we may assume WLOG, but not canonically, that $W^j \cong Z \times E_j$ [locally on X], and further that

$$Y^k = Z \times E_1 \times \dots \times \widehat{E}_k \times \dots \times E_n,$$

in such a way that $\sigma_i^k = \pi_{E_i}$. But we may then define an $(n+1)$-thickening that fills the horn, namely

$$(Z \times E_1 \times \dots \times E_n, (E_0, \sigma_0; E_1, \pi_{E_1}; \dots ; E_n, \pi_{E_n})) \, ,$$

where we have defined

$$\sigma_0(z, x_1, \dots, x_n) = \sum_{\varnothing \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|} \sigma_0^I(z, x_1, \dots, \widehat{x}_I, \dots, x_n),$$

where $\sigma_0^I : Z \times E_1 \times \cdots \times \widehat{E}_I \times \cdots \times E_n \to E_0$ is the restriction of σ_0^i for any $i \in I$. It is easy to check that we have

$$\sigma_0(z, x_1, \dots, x_i = 0, \dots, x_n) = \sigma_0^i(z, x_i, \dots, \widehat{x}_i, \dots, x_n),$$

as well as that the transversality condition is satisfied, and thus this is a filler for the given horn. □

Proposition 2.3.3. The simplicial set $\mathcal{TH}^\bullet(U)$ is equivalent to a discrete simplicial set.

Proof. [Proof again only works locally on X, some sheafy details to be worked out.]

We show that each component of $\mathcal{TH}^\bullet(U)$ is contractible. Let $p = (Y, (E, f)) \in \mathcal{TH}^0(U)$ be such that Y is of minimal dimension among all thickenings in the connected component of p. We claim that any n-simplex $(W, (E_0, f_0, \dots, E_n, f_n))$ with vertices at p is necessarily degenerate at p, which would imply that the component of p is contractible since X is a Kan complex.

We proceed by induction on n. For $n = 1$, suppose we have a 1-simplex $q = (W, (E_0, f_0, E_1, f_1))$ with vertices given by p. Explicitly, this means we are given embeddings $\iota_i : Y \to W$ such that $Y_i = \mathrm{im}(\iota_i)$ satisfies $Y_0 = f_1^{-1}(0)$ and vice versa, together with (not necessarily linear) isomorphisms of bundles $u_i : \overline{E} \to \iota_i^* \overline{E_i}$ such that $u_i \circ f = \iota_i^*(f_i)$. Choose $x \in U$, and consider the map

$$df_0 \oplus df_1 : T_x W \to E_0 \oplus E_1.$$

We claim that the rank of this map is equal to the dimension of $E \cong E_i$. Suppose not. Then we would have that $\ker(df_0 \oplus df_1)$ had dimension strictly lower than that of $\ker(df_i)$, which by Lemma 2.3.4 would construct a thickening of strictly lower dimension than Y in the same connected component, contradicting the minimality of Y.

Now, we have that $F = \mathrm{im}(df_0 \oplus df_1)$ has dimension equal to that of E_i, and further that $\pi_i : F \to E_i$ is an isomorphism since df_i is surjective. Thus, we may identify E_i with F by this given isomorphism, and we may assume that

$$q = (W, (F, f_0, F, f_1)),$$

such that $\mathrm{im}(df_0 \oplus df_1)$ at x is the diagonal $F \oplus F$, or equivalently, such that $df_0 = df_1$ as maps $T_xW \to F$. Now, choose an isomorphism $h : E \to F$ of vector spaces, and consider the map

□

Lemma 2.3.4. Let $(W, (E_0, f_0, E_1, f_1))$ be a 1-thickening of U with endpoints $p_i = (Y_i, (E_i, f_i))$, for $Y_0 = f_1^{-1}(0)$ and vice versa. Let $x \in U$, and let

$$d = \dim(T_xY_1 \cap T_xY_2).$$

Then there exists a 0-thickening $q = (Z, (H, h))$ of some neighborhood V of x in U with $\dim(Z) = d$, and such that q is in the same connected component of $\mathcal{TH}^\bullet(V)$ as $r_{U,V}(p_i)$.

Proof. Consider

$$df_1 \oplus df_2 : T_xW \to E_1 \oplus E_2,$$

and let $F = \mathrm{im}(df_1 \oplus df_2)$. Choose a complement H for F such that $H \subset E_1$; such an F exists because df_2 is surjective by transversality. Let π_F, π_H be the projections induced by this decomposition, and consider the function

$$g = \pi_F \circ (f_1 \oplus f_2) : W \to F.$$

We have that dg is surjective at x, and so there is a neighborhood V of x where $Z = g^{-1}(0)$ is a smooth manifold. Consider now the 0-thickening q of V given by

$$q = (Z, (H, h = f_1)),$$

which is well defined, since on Z we have that f_1 takes values in H. This thickening has

$$\dim(Z) = \dim(\ker(df_2 \oplus df_2)) = d,$$

and further there is a 1-thickening

$$(f_1^{-1}(H), (H, f_1, E_2, f_2)),$$

with endpoints given by

$$(f_1^{-1}(0), (E_2, f_2)) = (Y, (E, f)),$$

and

$$(f_1^{-1}(H) \cap f_2^{-1}(0), (H, f_1)) = (Z, (H, h)),$$

and thus we have found the desired thickening.

□

Definition 2.3.5. An **implicit manifold** is a compact Hausdorff space X together with a choice of global section of $\mathcal{TH}^\bullet(X)$. This should agree with Pardon's definition of an implicit atlas.

We have the following candidate for the structure sheaf of $(X, \mathcal{A})$ as a derived manifold. Consider the simplicial sheaf $\mathcal{O}_X^\bullet$ on X, given by

$$\mathcal{O}_X^n(U) = \coprod_{(Y,(E_i,\sigma_i))\in\mathcal{TH}_{\mathcal{A}}^n(U)} C^\infty(Y).$$

I think this is a beautiful candidate. Do you want to try and work out the speculation? Spivak at some point must do something very similar in his thesis.

Speculation 2.3.6. If X is a smooth manifold, then $\mathcal{O}_X^\bullet(X)$ is equivalent to the discrete simplicial set $C^\infty(X)$. Further, if X is the intersection of the origin in $\mathbb{R}$ with itself, then $\pi_0(\mathcal{O}_X^\bullet(X)) \cong \mathbb{R}$ and $\pi_1(\mathcal{O}_X^\bullet(X)) \cong \mathbb{R}$. To see this last part, consider smooth functions on $\mathbb{R}^n$ that vanish on the coordinate hyperplanes, and see how strong a zero they must have when restricted to another generic plane.

3 Goals

In no particular order, but enumerated for sake of reference:

1. (The category of implicit manifolds) The pair $(X, \mathcal{A})$ of a Hausdorff X with an implicit atlas $\mathcal{A}$ (a la Pardon) is an object in some category. Define this category. Ideally, it should be a category enriched in Kan complexes.

   (a) Part of this should involve streamlining the definition of $\mathcal{A}$. Let's present it as categorically as possible.

   (b) One should do this when the implicit atlases are *smooth*, too.

   (c) So an ideal type of theorem would be something like:

   Theorem 3.0.1. (After defining some category.) The category of implicit manifolds is enriched over Kan complexes. The category of smooth manifolds (in the usual sense) embeds fully and faithfully.

   (d) If any of this makes sense, then there should be a close connection between defining the morphism spaces in this category and giving $(X, \mathcal{A})$ the structure of a derived manifold in the sense of [Spi07]. In particular, we should have that $\mathrm{Hom}(-, \mathbb{R}) \simeq \mathcal{O}_X$ as sheaves on X. This may help in figuring out the correct notion of morphism spaces, and in particular it gives an immediate candidate for $\mathrm{Hom}(X, Y)$ when Y is a smooth manifold (decompose Y into patches, map open subsets U of X into patches by tuples in $\mathcal{O}_X(U)^{\dim Y}$).

   (e) That's a great point.

2. (Comparing with Spivak) We should construct a functor from Pardon's framework (which I called implicit manifolds above—we can change the name) to Spivak's. This is where a lot of the logical meat is. Put another way: *How does a choice of $\mathcal{A}$ on X define a derived scheme?*

   (a) The first example of this to understand is for the zero locus of a section of a bundle. This is section 2.2.1 of [Par16].

   (b) An ideal type of theorem would be something like:

   Theorem 3.0.2. There is a functor F from implicit manifolds to derived manifolds. It is fully faithful on smooth manifolds (in the usual sense).

However, I am not sure to what extent this functor should be fully faithful on all implicit manifolds. This of course depends on the choice of homs, and it's not obvious to me that maps defined to be compatible with implicit atlases will recover the whole homotopy type of the hom spaces for derived manifolds.

3. (The virtual fundamental cycle and cobordisms) How should we think of the virtual fundamental cycle? Pardon presents it as an element of Cech cochains, but should it be thought of as an element of a cobordism group? See Remark 1.3.2 of [Par16]. I think Remark 1.3.3 is also helpful; but how is this an invariant of the derived manifold itself?

   (a) I can't find where Pardon actually sets up a theory of cobordisms between implicit manifolds. It'd be nice to prove a statement like

   Theorem 3.0.3. (After defining a notion of cobordism between implicit manifolds.) If s_t is a homotopy between two sections s_0, s_1 of a vector bundle, then the implicit manifolds $(X_i, \mathcal{A}_i)$ associated to the s_i are cobordant. (i=0,1.)

   (b) Then it would be nice to show that Borel-Moorse cochains on X are actually just sections of some sort of stabilized "normal bundle" on X. (Roughly, there should be some notion of a normal bundle for an "embedding" of X into $\mathbb{R}^N$ for large N.) Then, the same way characteristic classes are preserved via cobordism, these cochains may be preserved under cobordism (however we define this), and we can try to show that the VFCs defined on X_i are compatible.

   (c) Finally, an ideal theorem would be to prove that

   Theorem 3.0.4. The functor F from above preserves cobordisms. That is, $X_0 \sim X_1$ cobordant $\implies F(X_0) \sim F(X_1)$, where $\sim$ is the cobordism relation. Further, F also preserves normal bundle classes and sections thereof. (This last sentence is intentionally vague.)

   See Section 6.2 of [Spi07] and 3.1 of [Spi10] for the derived manifolds definition of cobordism.

4. (Examples) We should write out the examples of Morse theory, and of holomorphic curves, as presented in [Par16].

5. (Intersections of virtual fundamental cycles) The Kunneth formula is much harder; I think we'll actually need to deal with derived smooth stacks to do that bit, because negative-dimensional things will show up.

4 Derived manifolds

Let's keep a running document here of what we're learning about Spivak's work.

Here we summarize the portions of [Spi10] and [Spi07] salient to our work. The notation follows [Spi07] closely, and nothing original is in this section of our paper.

4.1 Local models and C^∞ rings

I might eventually remove this section; it's a non-homotopical version of the "smooth rings" we review in the next section.

Let $\mathcal{R}$ be a small Grothendieck site.

1. Let $U : \mathcal{R} \to \mathsf{Top}$ be a morphism of Grothendieck sites. We say U is a basis for its image if and only if, for every $R \in \mathcal{R}$ and every covering of $U(R)$ in Top, there is a refinement of the covering by open sets in the image of U.

2. Given a collection of diagrams, $\mathcal{L} = \{L : \mathcal{C}_L \to \mathcal{R}\}$, we say a full subcategory $\mathcal{S} \subset \mathcal{R}$ is closed under $\mathcal{L}$-limits if any L factoring through $\mathcal{S}$ has a limit in $\mathcal{S}$.

3. $\mathcal{S}$ is closed under gluing if—whenever an open cover of $T \in \mathcal{R}$ is fully contained in $\mathcal{S}$, then T is in $\mathcal{S}$ as well.

4. We say a collection of objects $G \subset \mathrm{Ob}\,\mathcal{R}$ generates $\mathcal{S}$ if $\mathcal{S}$ is the smallest full subcategory of $\mathcal{R}$ containing G, and closed under $\mathcal{L}$-limits and gluing.

Example 4.1.1. Let $\mathcal{R} = \mathcal{E}$ denote the category of finite-dimensional Euclidean spaces. We let its morphisms be all smooth maps. The forgetful map $U : \mathcal{E} \to \mathsf{Top}$ is a basis for its image.

We let $\mathcal{L}$ denote the collection of all functors from a finite, discrete category to $\mathcal{E}$. (I.e., any functor $I \to \mathcal{E}$ where I is a possibly empty finite set.)

Then $G = \{\mathbb{R}\}$ generates $\mathcal{E}$ under $\mathcal{L}$-limits. We refer to

$$(\mathcal{E}, U, \mathcal{L}, \mathbb{R})$$

as the Euclidean category of local models. We call $\mathbb{R}$ the affine line for $\mathcal{E}$.

Note that any functor $\mathcal{R} \to \mathsf{Sets}$ preserving $\mathcal{L}$-limits is determined by what it does on any full subcategory containing a G which generates $\mathcal{R}$. In particular, any functor $\mathcal{E} \to \mathsf{Sets}$ is determined by what it does to $\mathbb{R}$ and to its smooth endomorphisms.

Definition 4.1.2. A C^∞-ring is a covariant functor

$$F : \mathcal{E} \to \mathsf{Sets}$$

preserving $\mathcal{L}$-limits. We refer to $|F| := F(\mathbb{R})$ as the underlying set of F.

As an example, the corepresentable functor $\hom(\mathbb{R}^n, -) : \mathcal{E} \to \mathsf{Sets}$ is a C^∞ ring. Its value on $\mathbb{R}$ is the set of smooth functions on $\mathbb{R}^n$.

Remark 4.1.3. Since $\mathbb{R}$ is a commutative ring object in the category $\mathcal{E}$, F induces a commutative ring structure on $|F|$ by virtue of preserving finite limits.

Remark 4.1.4. Other geometries, notably the geometry of affine $\mathbb{Z}$-schemes, also fit into this framework of local models.

4.2 Smooth rings

One of the most powerful principles in algebraic geometry is that affine schemes are the same thing as commutative rings. This is somewhat tautological, but is not so tautological that its transportation into the world of smooth geometry is self-evident. (For instance, it takes care to define the appropriate tensor product for which $C^\infty(M) \otimes C^\infty(N) \cong C^\infty(M \times N)$.) At the same time, we also seek a notion of "rings" which is sufficiently homotopical. So the C^∞ rings described in the previous sections should be thought of as the tip of a homotopical iceberg, in a way similar to how cdgas are a homotopical iceberg whose tip comprises the usual notion of commutative rings.

Let $\mathsf{Fun}(\mathsf{Man}, \mathsf{sSets})$ be the category of all functors from the category of manifolds to the category of simplicial sets. We call $F : \mathsf{Man} \to \mathsf{sSets}$ discrete if $F(M)$ if a discrete simplicial set for all manifolds M.

Here are some facts we will take for granted:

1. $\mathsf{Fun}(\mathsf{Man}, \mathsf{sSets})$ has an injective model structure. Its weak equivalences are object wise—this means $F \simeq G$ if and only if $F(M) \simeq G(M)$ for every manifold M. Its cofibrations are object-wise as well, meaning $F \to G$ is a cofibration if and only if $F(M) \to G(M)$ is a cofibration of simplicial sets for each M. This model structure is proper, and cofibrantly generated.

2. Let S be a simplicial set. We let $\underline{S} : \mathsf{Man} \to \mathsf{sSets}$ denote the constant functor. This makes $\mathsf{Fun}(\mathsf{Man}, \mathsf{sSets})$ tensored over simplicial sets.

3. All discrete F are fibrant. In particular, if M is a manifold, the functor

$$H_M := \mathsf{Man}(M, -)$$

is fibrant.

Definition 4.2.1. The model category of smooth rings is the localization of $\mathsf{Fun}(\mathsf{Man}, \mathsf{sSets})$ along Ψ.

We define Ψ now. Consider a diagram

$$\begin{tikzcd} & P \arrow[d, "a"] \\ M \arrow[r, "s"] & N \end{tikzcd}$$

where s is a submersion. Then the pullback $Q \cong M \times_N P$ exists in the category of smooth manifolds. On the other hand, one could also take the homotopy pushout

$$\begin{tikzcd} H_N \arrow[r, "s^*"] \arrow[d, "a^*"'] & H_M \arrow[d] \\ H_P \arrow[r] & G \end{tikzcd}$$

in the model category $\mathsf{Fun}(\mathsf{Man}, \mathsf{sSets})$. We let

$$\psi_{s,a} : G \to H_Q$$

denote the induced map. Note that localizing with respect to these $\psi_{s,a}$ means that the assignment "corepresenting functor" would preserve the one good operation there is in smooth manifolds: pulling back along submersions. That is, pullbacks obtained from submersions will be sent to homotopy pushouts.

Finally, note that the 0-manifold $pt \cong \mathbb{R}^0$ is terminal in Man. We'd like this to be reflected in the category of smooth rings. (This reflects as principle from algebraic geometry: if $\operatorname{Spec} k$ is terminal in k-schemes, we'd like the structure ring k to be initial in k-algebras.) However, in $\mathsf{Fun}(\mathsf{Man}, \mathsf{sSets})$, the initial object is not H_{pt}, but rather the functor sending $M \mapsto \emptyset$. So we would like to further localize with respect to the unique map

$$(M \mapsto \emptyset) \to H_{pt}.$$

Definition 4.2.2. We let Ψ denote the collection of all morphisms of the form $\psi_{s,a}$ (for s a submersion and a a smooth map) together with the unique map $(M \mapsto \emptyset) \to H_{pt}$.

It would be nice to just think of this as an ∞-category from the outset, without having to resort to model categories, and then defining an ∞-category. But so it is.

Example 4.2.3 (Example 2.1.10 in [Spi07].)**.** If M, N are smooth manifolds, the object-wise coproduct $H_M \coprod H_N$ is in $\mathsf{Fun}(\mathsf{Man}, \mathsf{sSets})$. However, it is not a fibrant object in the model category of smooth rings. For instance, apply it to the pullback diagram realizing $M \times_{pt} N \cong M \times N$. However, the map

$$H_M \coprod H_N \to H_{M \times N}$$

is a fibrant replacement map.

Example 4.2.4 (The fat point, 2.1.13 in [Spi07].)**.** Let us compute the homotopy pushout

$$\begin{tikzcd} H_{\mathbb{R}} \arrow[r] \arrow[d] & H_{pt} \arrow[d] \\ H_{pt} \arrow[r] & G \end{tikzcd}$$

where the arrows $H_{\mathbb{R}} \to H_{pt}$ are induced by the inclusion of the origin in $\mathbb{R}$. As usual, to compute this homotopy pushout, we just replace either the top or lefthand arrow by a cofibration. (The localization is left proper because $\mathsf{Fun}(\mathsf{Man}, \mathsf{sSets})$ is; see [**?**] 4.1.1.) Consider the composite map

$$g : \mathbb{R} \to pt \xrightarrow{0} \mathbb{R}.$$

Then we have a functor $C : \mathsf{Man} \to \mathsf{sSets}$ whose only non-degenerate simplices are in dimensions 1 and 0, given by

$$d_0, d_1 : H_{\mathbb{R}} \to H_{\mathbb{R}}$$

where d_0 is the identity and d_1 is g^*. Obviously, the inclusion $H_{\mathbb{R}} \to C$ is a cofibration because it is a levelwise injection for any manifold M. One can check straightforwardly that $H_{pt}(M) \simeq C(M)$ for every manifold M. Hence the homotopy pushout G can be computed as the honest pushout of the diagram

$$\begin{tikzcd} H_{\mathbb{R}} \arrow[r] \arrow[d] & C \arrow[d] \\ H_{pt} \arrow[r] & G. \end{tikzcd}$$

Thus G is given by the functor $\mathsf{Man} \to \mathsf{sSets}$ whose non-degenerate bit we represent by

$$d_0, d_1 : H_{\mathbb{R}} \to H_{pt}.$$

Here, both d_i are induced by the inclusion $0 : pt \to \mathbb{R}$.

Added after phone discussion. Note that this sheaf of simplicial sets is not fibrant. Even when evaluated on $\mathbb{R}$, or on pt, it is not a Kan complex. What we can try to prove is that, on each manifold, the map

$$G \to Implicit - stuff$$

is left/right anodyne. Then by 4.1.1.3 of HTT, this map is initial/final, hence an homotopy equivalence in the Quillen/Kan sense. Aside from the definition (4.1.1.1 of HTT) of being final, Joyal's characterization (Theorem 4.1.3.1 of HTT) may also be helpful.

Remark 4.2.5. We remind the reader of what localization of $\mathcal{C}$ to $\mathcal{C}[\Psi^{-1}]$ does, at least at a superficial level. Heuristically, it turns any morphism in $\Psi \subset \mathcal{C}$ into an equivalence in the localization. In terms of model categories:

1. $\mathcal{C}$ and $\mathcal{C}[\Psi^{-1}]$ are the same simplicial category.

2. The cofibrations of $\mathcal{C}[\Psi^{-1}]$ are the cofibrations of $\mathcal{C}$.

3. A fibrant object X of $\mathcal{C}[\Psi^{-1}]$ is one which is both fibrant in $\mathcal{C}$, and Ψ-local. That is, for any $\psi : A \to B \in \Psi$, the map

$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$$

   is a weak equivalence.

4. The weak equivalences in $\mathcal{C}[\Psi^{-1}]$ are those $f : X \to Y$ such that

$$\mathrm{Map}(Y, -) \to \mathrm{Map}(X, -)$$

   is a weak equivalence whenever $-$ is fibrant in $\mathcal{C}[\Psi^{-1}]$.

4.3 Sheaves, local sheaves, derived manifolds

Let X be a topological space. Let $\mathcal{F}$ be a sheaf of C^∞ rings. This means that $\mathcal{F}$ is a contravariant functor from $\mathsf{Open}(X)$ to $\mathsf{Fun}(\mathsf{Man}, \mathsf{sSets})$, satisfying descent for hypercovers.

4.3.1 Local sheaves

Definition 4.3.1. We say that $\mathcal{F}$ is local if, for every open cover $\{U_\alpha$ of M, the natural map of sheaves of sets

$$\pi_0 \left(\coprod_\alpha \mathcal{F}(-, M_\alpha) \right) \to \pi_0 \mathcal{F}(-, M)$$

is a surjection.

Definition 4.3.2. If $\mathcal{F}$ is local, we say that the pair $(X, \mathcal{F})$ is a local smooth-ringed space.

Definition 4.3.3. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, we say that $\phi : \mathcal{F} \to \mathcal{G}$ is a local morphism of sheaves if and only if: For any open $U \subset M$, the natural diagram

$$\begin{tikzcd} \pi_0 \mathcal{F}(-, U) \arrow[r, "\phi"] \arrow[d, "res"] & \pi_0 \mathcal{F}(-, U) \arrow[d, "res"] \\ \pi_0 \mathcal{F}(-, M) \arrow[r, "\phi"] & \pi_0 \mathcal{F}(-, M) \end{tikzcd}$$

exhibits $\pi_0 \mathcal{F}(-, U)$ as a pullback of sheaves of sets.

We let

$$\mathrm{Map}_{\mathrm{loc}}(\mathcal{F}, \mathcal{G}) \subset \mathrm{Map}(\mathcal{F}, \mathcal{G})$$

denote the full simplicial set spanned by local morphisms. That is, every simplex on the righthand side has vertices given by local morphisms.

Definition 4.3.4. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are local smooth-ringed spaces, we let

$$\mathrm{Map}(X, Y) := \coprod_{\phi : X \to Y} \mathrm{Map}_{\mathrm{loc}}(\phi^* \mathcal{O}_Y, \mathcal{O}_X).$$

where ϕ runs over all continuous maps.

The following gives some intuition for why we insist on the adjective local:

Theorem 4.3.5 (Theorem 3.3.6 of [Spi07].)**.** Let $\mathcal{F}$ be a sheaf of smooth ring son X. The following are equivalent:

1. $\mathcal{F}$ is local.

2. For every open cover $U_\bullet \to M$ of a smooth manifold M, the natural map
$$\mathrm{hocolim}(\mathcal{F}(-, U_\bullet)) \to \mathcal{F}(-, M)$$
is a weak equivalence of simplicial sheaves on X. Here, the hocolim is over the Cech nerve of the cover.

3. The same holds above if each element of the open cover $U_\bullet$ is by Euclidean spaces.

4. If $f : \mathcal{F} \to \mathcal{G}$ is a local morphism of sheaves of smooth rings, and if f induces a weak equivalence of sheaves
$$\mathcal{F}(-, \mathbb{R}) \simeq \mathcal{G}(-, \mathbb{R})$$
then f is a weak equivalence of sheaves of local smooth rings. (This is Corollary 3.3.7 of [Spi07].)

4.3.2 Derived manifolds

We will show how to compute homotopy pullbacks of C^∞-ringed spaces momentarily. But for now:

Definition 4.3.6. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. Then the C^∞-ringed space given by the homotopy fiber product
$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] & \mathbb{R}^0 \arrow[d, "0"'] \\ \mathbb{R}^n \arrow[r, "f"] & \mathbb{R}^m \end{tikzcd}$$
which we write $\mathcal{U} = (U, \mathcal{O}_U)$, is called a principal derived manifold.

Definition 4.3.7. Any Hausdorff smooth-ringed space $(X, \mathcal{O}_X)$ is called a derived manifold if it can be covered by countably many principal derived manifolds.

In other words, derived manifolds are locally modeled on zero locuses of functions. The salient point here is that the homotopy fiber product is what produces a robust interpretation of the zero locus—one that is independent of perturbations, and which can shrug off non-transversality.

Definition 4.3.8. The category of derived manifolds is the full simplicial subcategory of the category of locally smooth-ringed spaces.

Here are some basic properties to get the reader acclimated:

Theorem 4.3.9. 1. Any smooth manifold M defines a structure sheaf $\mathcal{O}_M$ which sends any manifold N to the set of smooth functions from M to N. This is a locally smooth-ringed space. (Proposition 3.2.2 of [Spi07].)

2. The inclusion of smooth manifolds (with a discrete set of smooth maps) into the simplicial category of locally ringed spaces is fully faithful. (Proposition 3.2.3 of [Spi07].)

3. If $(X, \mathcal{O}_X)$ is a locally smooth-ringed space, then for any smooth manifold M, the simplicial set of (local) maps from $(X, \mathcal{O}_X)$ to $(M, \mathcal{O}_M)$ is homotopy equivalent to $\mathcal{O}_X(X, M)$. In other words, the global sections of $\mathcal{O}_X$, evaluated on the manifold M, recovers the space of maps from X to M. (Theorem 3.3.3 of [Spi07].)

4.3.3 Computing with derived manifolds

First we discuss how to compute fiber products. The algorithm is: Take the fiber product in the usual category of topological spaces, then take the homotopy pushout in the category of sheaves of simplicial rings.

4.4 Simplicial commutative algebras

Section 3.5 of [Spi07].

Smooth-ringed spaces define derived manifolds, but they rarely allow us to compute. Thankfully, replacing a smooth ring with its underlying simplicial commutative $\mathbb{R}$-algebra preserves many homotopy colimits. This buys us mileage in local computations: While taking global sections is a limit, taking stalks is a colimit.

For instance, we have:

Lemma 4.4.1 (Lemma 3.5.7 of [Spi07])**.** Let $\mathcal{F}$ be a sheaf of smooth rings on X.

5 Implicit atlases

Let's keep a running document here of what we're learning about Pardon's work.

Here we summarize the portions of [Par16] salient to our work.

Example 5.0.1 (Zero loci, Section 2.2.1 of [Par16]). Let $p : E \to B$ be a smooth vector bundle an $s : B \to E$ a section with $s^{-1}(0)$ compact.

A thickening datum α is a triple of

1. $V_\alpha \subset B$ open
2. E_α finite-dimensional vector space
3. A smooth map $\lambda_\alpha : V_\alpha \times E_\alpha \to p^{-1}(V_\alpha)$.

Then a thickening is given by pairs $(x, (e_\alpha))$ where

1. $x \in \cap V_\alpha$
2. continue...

References

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