Extragalactic Astrophysics and Comsology

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1.1.1 The Cosmological Principle

We begin with **The cosmological Principle**. It sounds simple, but incredibly well supported. It says that **the Universe (spatially) is homogeneous and isotropic on very large spatial scales**. Observationally, this is around 100 Mpc scales. Homogeneous means constant density (non-realistically, this is mass density; relativistic ally, this is energy density). Isotropic means the same in all directions.

Note: Isotropy about 2 points (or more) implies homogeneity. Isotropy about 1 point is not enough. Here is a quick illustration of that:

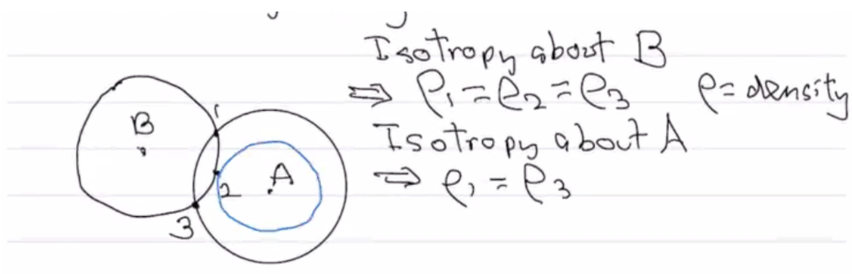

Figure 1: Istropy aobut 2 points.

We can consider another example. A homogeneous universe can anisotropic. Consider a homogeneous Universe that is expanding in different directions in a non-uniform way. This leads to different $H_0(x, y, z)$.

The reason we spend some time on the Cosmological Principle is the **Friedmann-Robertson-Walker metric**, which we will come to later on.

1.1.2 Hubble Parameter

Let's now talk about the **Hubble parameter**, which is not a constant! It, in fact, changes in time. An empirical linear relationship between the recession speed v and distance r can be seen, called the **Hubble's Law**:

$$v = Hr \tag{1}$$

Note that H_0 has units of 1/time. One convention to note is that:

$$H = 100 \underbrace{h}_{\text{to hide our ignorance}} \frac{\text{km}}{\text{s Mpc}} \tag{2}$$

One useful number to know is $H \approx \frac{h}{10^{10}\,\text{yr}}$.

Sometimes, when we write H_0, we mean the "present-day" value ($z = 0$). This is how we will use it now.

1.1.3 Scale Factor

We need a language to describe the expansion of the Universe. We will use $a(t)$, which describes the expansion (or contraction) of the Universe. It also relates two different coordinate systems: physical coordinates $\vec{r}$ to comoving coordinates $\vec{x}$. The relation:

$$\vec{r} = a(t)\vec{x} \tag{3}$$

We typically use comoving coordinates in calculations in Cosmology. We can think of $\vec{x}$ as the notches on a stretching ruler. Now consider:

$$\frac{d}{dt}\vec{r} = \vec{v} = \dot{a}\vec{x}a\dot{\vec{x}} \tag{4}$$

$$\frac{d}{dt}\vec{r} = \vec{v} = \dot{a}\vec{x}a\dot{\vec{x}} = \underbrace{\frac{\dot{a}}{a}\vec{r}}_{\text{Hubble}} + \underbrace{a\dot{\vec{x}}}_{\text{motion relative to expansion ("peculiar velocity")}} \tag{5}$$

$$\boxed{H(t) = \frac{\dot{a}(t)}{a(t)}} \tag{6}$$

The name of the game for measuring H is to go far enough that the first term dominates. Otherwise, locally, the second term dominates since peculiar velocities are of order 100s of km/s.

One other convention we need to establish:

$$a(t_0) = 1 \rightarrow \text{comoving} = \text{today} \tag{7}$$

1.2 The Friedmann Equation; The Equation of State; Radiation, Matter, and Dark Energy

1.2.1 The Friedmann Equation

Below, we will use and derive these, but I am putting the equations at the top for convenience.

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{c^2}} \tag{8}$$

$$\boxed{\dot{\rho} = -3\frac{\dot{a}}{a}\left(P + \rho\right)} \tag{9}$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right)} \tag{10}$$

Let's motivate the origin with quasi-Newtonian physics. We can derive it from General Relativity, but that's overkill.

If we assume isotropy, we only need to worry aobu the radial coordinate r, not θ or ϕ. Homogeneity tells us that $\rho =$ constant spatially, but *can* depend on time. We will model the Universe as an expanding, homogeneous medium that is adiabatically ($\Delta s = 0$) expanding. If it were not adiabatically expanding, we would have heat flow and thus no isotropy.

With these conditions, let's examine the motion of a thin, expanding, spherical shell of radius a. This depends **only** on the enclosed mass within a[1]:

$$M(< a) = \frac{4}{3}\pi a^3 \rho \tag{11}$$

Let's consider the energy:

$$E = \frac{1}{2}\dot{a}^2 - \frac{GM}{a} \tag{12}$$

$$E = \frac{1}{2}\dot{a}^2 - \frac{4}{3}\pi G\rho a^2 \tag{13}$$

Let's re-write E a bit: $kc^2 \equiv -2E$. Note that $k \propto 1/\text{length}^2$. There are three possibilities for kc^2:

- > 0, $E < 0$, bound
- $= 0$, $E = 0$, critical
- < 0, $E > 0$, unbound

Let's now evoke the First Law of Thermodynamics ($\Delta S = 0$):

$$\underbrace{\mathrm{d}U}_{\text{internal energy}} = -P\mathrm{d}V \tag{14}$$

We now equation the internal energy to the rest-mass energy:

$$\mathrm{d}\left(\rho c^2 a^3\right) = -P\mathrm{d}\left(a^3\right) \tag{15}$$

We will now set $c = 1$ and take a time derivative:

$$\dot{\rho}a^3 + 3\rho a^2\dot{a} = -3Pa^2\dot{a} \tag{16}$$

$$\dot{\rho} - 3\frac{\dot{a}}{a}\left(P + \rho\right) \tag{17}$$

Using this result and the energy equation from above

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k(c)^2}{a^2} \tag{18}$$

to get a new equation. If you stare at it hard enough and have divine intervention, take a derivative of the second equation and multiply by a^2. Doing so, you get:

$$2\dot{a}\ddot{a} = \frac{8\pi}{3}G\frac{d}{dt}(\rho a^2) = \frac{8\pi}{3}Ga^2\left(\dot{\rho} + 2\frac{\dot{a}a}{\rho}\right) \tag{19}$$

[1] see Birkhoff's Theorem for General Relativity proof

Simplifying with the other above equation, we get:

$$2\dot{a}\ddot{a} = -\frac{8\pi}{3}Ga^2\left(\frac{\dot{a}}{a}\rho + 3\frac{\dot{a}}{a}P\right) \tag{20}$$

Simplifying:

$$\frac{\ddot{a}}{\dot{a}} = -\frac{4\pi}{3}G\left(\rho + 3P\right) \tag{21}$$

Note that this is not independent from the other equations; rather it is massaged. Let's compare this to 1-D Newtonian forces:

$$\ddot{x} = -\frac{GM}{x^2} = -\frac{4}{3}\pi G\rho x \tag{22}$$

$$\frac{\ddot{x}}{x} = -\frac{4}{3}\pi G\rho \tag{23}$$

Had we done strictly Newtonian physics, we would have never gotten the $+3P$ term. The way to interpret this: we can think of ρ to have an extended meaning: $\rho_{eff} = \rho + 3P$.

The Grand Summary so far: two equations of motion for $a(t)$:

$$\frac{\ddot{a}}{\dot{a}} = -\frac{4\pi}{3}G\left(\rho + 3P\right) \tag{24}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(P + \rho\right) \tag{25}$$

Note this second equation tells us the acceleration! Very importantly, we have a minus sign. If ρ and P are positive, the Universe is **decelerating**! Conversely, if you have a *bizarre* P and could reverse the parenthetical term, we can have accelerated expansion!

Right now, we have three unknowns (P, ρ, a). How do we get that last piece – the equation of state ($P \iff \rho$ dependence)?

1.2.2 Equation of State

We choose to write:

$$P = w\rho(c^2) \tag{26}$$

We are in units where $c = 1$, but I threw it in for reference. Note – this means that pressure is the same thing as energy density! Think of the units.

With that definition of P, we can re-write the second boxed equation from above:

$$\dot{\rho} - 3\frac{\dot{a}}{a}\left(1 + w\right)\rho \tag{27}$$

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \rightarrow \rho \propto a^{-3(1+w)} \tag{28}$$

$$\boxed{\rho \propto a^{-3(1+w)}} \tag{29}$$

assuming that $\dot{w} = 0$ which might not be true!

1.2.3 Matter, Radiation, and Dark Energy

Let's look at a few special cases of the equation of state:

- Matter: Non-relativistic, pressure-less particles like cold dark matter. In this case, $w = 0, P = 0 \to \boxed{\rho \propto a^{-3}}$. This makes sense because it is units of 1/volume.
- Radiation: Relativistic particles, photons and neutrinos. In this case, $w = 1/3, P = \frac{1}{3}\rho \to \boxed{\rho \propto a^{-4}}$. This makes sense because it is units of (1/volume) (1/length) where the extra factor is from redshifting energy.
- The Cosmological Constant: In this case, $w = -1, P = -\rho \to \rho = \text{constant}$.
- Dark Energy: More general term, where $w < -\frac{1}{3}$ to make $\ddot{a} > 0$.

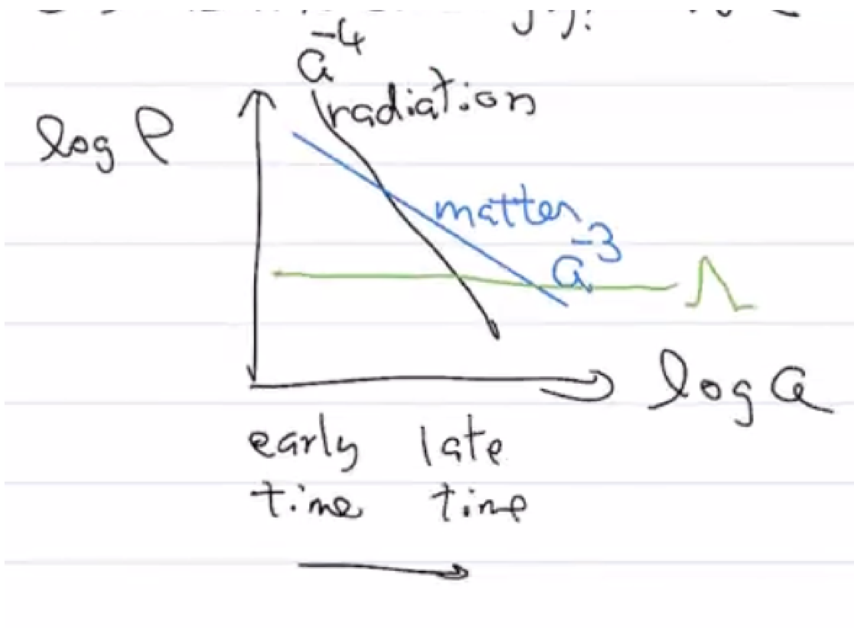

Figure 2: Sketch of density over cosmic time.