Extragalactic Astrophysics and Comsology

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Spring 2021

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1 The Smooth Universe

1.1 The Cosmological Principle; The Hubble Parameter; Scale Factor

1.1.1 The Cosmological Principle

We begin with **The cosmological Principle**. It sounds simple, but incredibly well supported. It says that **the Universe (spatially) is homogeneous and isotropic on very large spatial scales**. Observationally, this is around 100 Mpc scales. Homogeneous means constant density (non-realistically, this is mass density; relativistic ally, this is energy density). Isotropic means the same in all directions.

Note: Isotropy about 2 points (or more) implies homogeneity. Isotropy about 1 point is not enough. Here is a quick illustration of that:

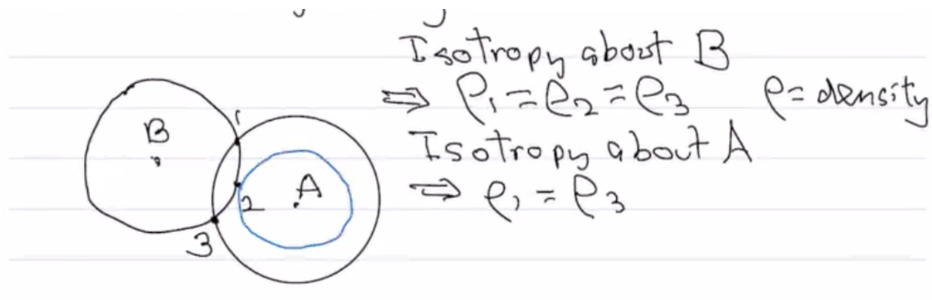

Figure 1: Istropy aobut 2 points.

We can consider another example. A homogeneous universe can anisotropic. Consider a homogeneous Universe that is expanding in different directions in a non-uniform way. This leads to different $H_0(x, y, z)$.

The reason we spend some time on the Cosmological Principle is the **Friedmann-Robertson-Walker metric**, which we will come to later on.

1.1.2 Hubble Parameter

Let's now talk about the **Hubble parameter**, which is not a constant! It, in fact, changes in time. An empirical linear relationship between the recession speed v and distance r can be seen, called the **Hubble's Law**:

$$v = Hr \tag{1}$$

Note that H_0 has units of 1/time. One convention to note is that:

$$H = 100 \underbrace{h}_{\text{to hide our ignorance}} \frac{\text{km}}{\text{s Mpc}} \tag{2}$$

One useful number to know is $H \approx \frac{h}{10^{10}\,\text{yr}}$.

Sometimes, when we write H_0, we mean the "present-day" value ($z = 0$). This is how we will use it now.

1.1.3 Scale Factor

We need a language to describe the expansion of the Universe. We will use $a(t)$, which describes the expansion (or contraction) of the Universe. It also relates two different coordinate systems: physical coordinates $\vec{r}$ to comoving coordinates $\vec{x}$. The relation:

$$\vec{r} = a(t)\vec{x} \tag{3}$$

We typically use comoving coordinates in calculations in Cosmology. We can think of $\vec{x}$ as the notches on a stretching ruler. Now consider:

$$\frac{d}{dt}\vec{r} = \vec{v} = \dot{a}\vec{x}a\dot{\vec{x}} \tag{4}$$

$$\frac{d}{dt}\vec{r} = \vec{v} = \dot{a}\vec{x}a\dot{\vec{x}} = \underbrace{\frac{\dot{a}}{a}\vec{r}}_{\text{Hubble}} + \underbrace{a\dot{\vec{x}}}_{\text{motion relative to expansion ("peculiar velocity")}} \tag{5}$$

$$\boxed{H(t) = \frac{\dot{a}(t)}{a(t)}} \tag{6}$$

The name of the game for measuring H is to go far enough that the first term dominates. Otherwise, locally, the second term dominates since peculiar velocities are of order 100s of km/s.

One other convention we need to establish:

$$a(t_0) = 1 \rightarrow \text{comoving} = \text{today} \tag{7}$$

1.2 The Friedmann Equation; The Equation of State; Radiation, Matter, and Dark Energy

1.2.1 The Friedmann Equation

Below, we will use and derive these, but I am putting the equations at the top for convenience.

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}} \tag{8}$$

$$\boxed{\dot{\rho} = -3\frac{\dot{a}}{a}(P+\rho)} \tag{9}$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)} \tag{10}$$

Let's motivate the origin with quasi-Newtonian physics. We can derive it from General Relativity, but that's overkill.

If we assume isotropy, we only need to worry aobu the radial coordinate r, not θ or ϕ. Homogeneity tells us that $\rho = \text{constant}$ spatially, but *can* depend on time. We will model the Universe as an expanding, homogeneous medium that is adiabatically ($\Delta s = 0$) expanding. If it were not adiabatically expanding, we would have heat flow and thus no isotropy.

With these conditions, let's examine the motion of a thin, expanding, spherical shell of radius a. This depends **only** on the enclosed mass within a[1]:

$$M(<a) = \frac{4}{3}\pi a^3 \rho \tag{11}$$

Let's consider the energy:

$$E = \frac{1}{2}\dot{a}^2 - \frac{GM}{a} \tag{12}$$

$$E = \frac{1}{2}\dot{a}^2 - \frac{4}{3}\pi G\rho a^2 \tag{13}$$

Let's re-write E a bit: $kc^2 \equiv -2E$. Note that $k \propto 1/\text{length}^2$. There are three possibilities for kc^2:

[1] see Birkhoff's Theorem for General Relativity proof

- > 0, $E < 0$, bound
- $= 0$, $E = 0$, critical
- < 0, $E > 0$, unbound

Let's now evoke the First Law of Thermodynamics ($\Delta S = 0$):

$$\underbrace{\mathrm{d}U}_{\text{internal energy}} = -P\mathrm{d}V \tag{14}$$

We now equation the internal energy to the rest-mass energy:

$$\mathrm{d}\left(\rho c^2 a^3\right) = -P\mathrm{d}\left(a^3\right) \tag{15}$$

We will now set $c = 1$ and take a time derivative:

$$\dot{\rho}a^3 + 3\rho a^2\dot{a} = -3Pa^2\dot{a} \tag{16}$$

$$\dot{\rho} - 3\frac{\dot{a}}{a}\left(P + \rho\right) \tag{17}$$

Using this result and the energy equation from above

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k(c)^2}{a^2} \tag{18}$$

to get a new equation. If you stare at it hard enough and have divine intervention, take a derivative of the second equation and multiply by a^2. Doing so, you get:

$$2\dot{a}\ddot{a} = \frac{8\pi}{3}G\frac{d}{dt}(\rho a^2) = \frac{8\pi}{3}Ga^2\left(\dot{\rho} + 2\frac{\dot{a}a}{\rho}\right) \tag{19}$$

Simplifying with the other above equation, we get:

$$2\dot{a}\ddot{a} = -\frac{8\pi}{3}Ga^2\left(\frac{\dot{a}}{a}\rho + 3\frac{\dot{a}}{a}P\right) \tag{20}$$

Simplifying:

$$\frac{\ddot{a}}{\dot{a}} = -\frac{4\pi}{3}G\left(\rho + 3P\right) \tag{21}$$

Note that this is not independent from the other equations; rather it is massaged. Let's compare this to 1-D Newtonian forces:

$$\ddot{x} = -\frac{GM}{x^2} = -\frac{4}{3}\pi G\rho x \tag{22}$$

$$\frac{\ddot{x}}{x} = -\frac{4}{3}\pi G\rho \tag{23}$$

Had we done strictly Newtonian physics, we would have never gotten the $+3P$ term. The way to interpret this: we can think of ρ to have an extended meaning: $\rho_{eff} = \rho + 3P$.

The Grand Summary so far: two equations of motion for $a(t)$:

$$\frac{\ddot{a}}{\dot{a}} = -\frac{4\pi}{3}G\left(\rho + 3P\right) \tag{24}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(P + \rho\right) \tag{25}$$

Note this second equation tells us the acceleration! Very importantly, we have a minus sign. If ρ and P are positive, the Universe is **decelerating**! Conversely, if you have a *bizarre* P and could reverse the parenthetical term, we can have accelerated expansion!

Right now, we have three unknowns (P, ρ, a). How do we get that last piece – the equation of state ($P \Leftrightarrow \rho$ dependence)?

1.2.2 Equation of State

We choose to write:

$$P = w\rho(c^2) \tag{26}$$

We are in units where $c = 1$, but I threw it in for reference. Note – this means that pressure is the same thing as energy density! Think of the units.

With that definition of P, we can re-write the second boxed equation from above:

$$\dot{\rho} - 3\frac{\dot{a}}{a}\left(1 + w\right)\rho \tag{27}$$

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \rightarrow \rho \propto a^{-3(1+w)} \tag{28}$$

$$\boxed{\rho \propto a^{-3(1+w)}} \tag{29}$$

assuming that $\dot{w} = 0$ which might not be true!

1.2.3 Matter, Radiation, and Dark Energy

Let's look at a few special cases of the equation of state:

- Matter: Non-relativistic, pressure-less particles like cold dark matter. In this case, $w = 0, P = 0 \rightarrow$ $\boxed{\rho \propto a^{-3}}$. This makes sense because it is units of 1/volume.
- Radiation: Relativistic particles, photons and neutrinos. In this case, $w = 1/3, P = \frac{1}{3}\rho \rightarrow \boxed{\rho \propto a^{-4}}$. This makes sense because it is units of (1/volume) (1/length) where the extra factor is from redshifting energy.
- The Cosmological Constant: In this case, $w = -1, P = -\rho \rightarrow \rho =$ constant.
- Dark Energy: More general term, where $w < -\frac{1}{3}$ to make $\ddot{a} > 0$.

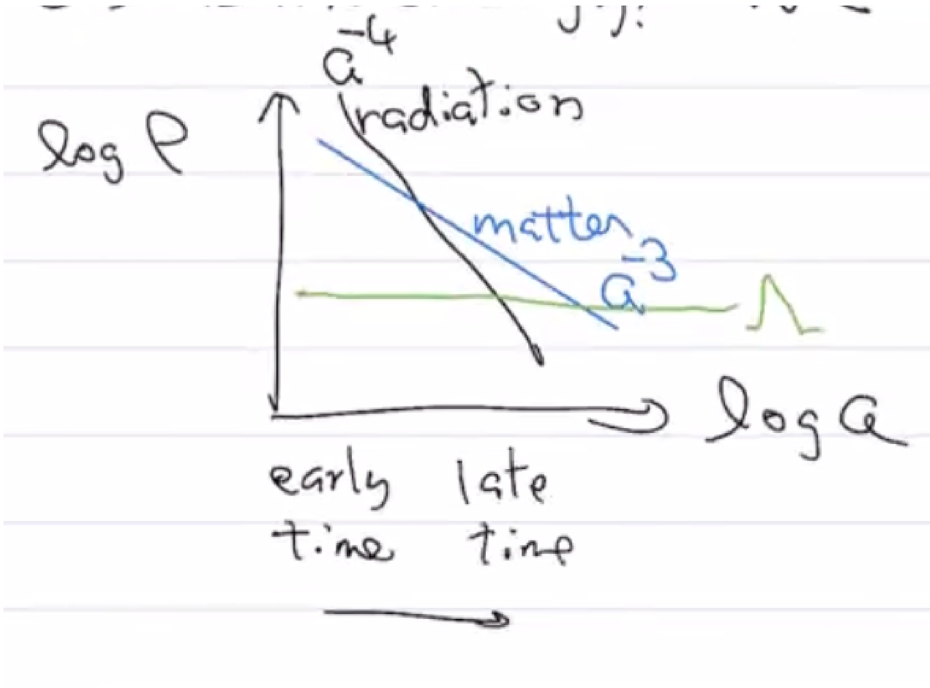

Figure 2: Sketch of density over cosmic time.

1.3 The Density Parameter; Open, flat, closed models; Redshift

Let's first define Ω in terms of the **critical density** ρ_c. This is defined to be the density needed to make the Universe flat. Recall the Friedmann Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2} \tag{30}$$

First, set $k = 0 \Leftrightarrow E = 0$, then solve for ρ:

$$\boxed{\rho_c = \frac{3H^2}{8\pi G}} \tag{31}$$

This critical density can be interpreted as well as the mean density of a flat Universe ($k = 0$). Importantly, ρ_c is a function of H, which itself is a function of time. Setting $H = H_0$, we can get Today's value of ρ_c:

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = 2.78 \times 10^{11} h^2 \frac{M_\odot}{\mathrm{Mpc}^3} = 1.88 \times 10^{-29} h^2 \frac{\mathrm{g}}{\mathrm{cm}^3} \tag{32}$$

This is about $10^{-5} h^2 m_{proton}\,\mathrm{cm}^{-3}$. Here are a few other useful numbers to carry around as well:

$$\rho_{thumb} \sim 1\frac{\mathrm{g}}{\mathrm{cm}^3} \tag{33}$$

$$\rho_{ISM} \sim 1\frac{m_p}{\mathrm{cm}^3} \tag{34}$$

$$\rho_{Universe} \sim 1\rho_c \tag{35}$$

Now with these definitions, we have the **density paramater**:

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\rho(t)}{3H^2(t)} \tag{36}$$

There are three obvious possibilities:

- $\Omega < 1 \rightarrow$ Open Universe ($k < 0$)
- $\Omega = 1 \rightarrow$ Flat Universe ($k = 0$)
- $\Omega > 1 \rightarrow$ Closed Universe ($k > 0$)

There are a few things to note. When we have *mutliple components*, we have:

$$\Omega(t) = \frac{\sum_i \rho_i}{\rho_c} \tag{37}$$

where i is often representing radiation, matter, Λ, baryonic component of matter, dark matter, neutrinos, etc. Another thing we want to stress is that ρ, ρ_c, Ω **all depend on time**. Note that if $\rho > \rho_c$, it will STAY that way. The same goes for Ω, and for other values. For example, these parameters will stay $>, <$, or $= 1$.

1.3.1 Redshift

Observationally, we have

$$1 + z \equiv \frac{\lambda_{obs}}{\lambda_{rest}} = \frac{a(t_0)}{a(t)} = \frac{1}{a(t)} \tag{38}$$

$$\boxed{\frac{1}{a(t)} = 1 + z} \tag{39}$$

1.4 Time Evolution of Paramaters: Hubble, Density, Scale Factor

1.4.1 Hubble Parameter

Worked out in Problem Set 1, but here is the solution:

$$H^2(a) = H_0^2\left(\frac{\Omega_0}{a^{3+3w}} + \frac{1-\Omega_0}{a^2}\right) \tag{40}$$

For multiple components (the common ones):

$$H^2(a) = H_0^2\left(\frac{\Omega_{0,m}}{a^3} + \Omega_{0,\Lambda} + \frac{\Omega_{0,r}}{a^4} + ... + \frac{1-\Omega_{0,m}-\Omega_{0,\Lambda}-\Omega_{0,r}}{a^2}\right) \tag{41}$$

1.4.2 Density Parameter

First, recall the critical density:

$$\rho_c = \frac{3H^2}{8\pi G} \rightarrow H^2\Omega = \frac{8\pi}{3}G\rho \tag{42}$$

We can also look at the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2} \tag{43}$$

We can replace the first term on the right hand side and set $a = 1$:

$$k = H_0^2 (\Omega_0 - 1) \tag{44}$$

All of these together, we get:

$$H^2 = H^2\Omega - \frac{H_0^2 (\Omega_0 - 1)}{a^2} \tag{45}$$

Rearrange:

$$1 - \Omega(a) = \frac{H_0^2}{H^2 a^2} (1 - \Omega_0) \tag{46}$$

We can use the time evolution of the Hubble Parameter to replace the H terms, leaving:

$$\boxed{1 - \Omega(a) = \frac{1 - \Omega_0}{\frac{\Omega_0}{a^{1+3w}} + 1 - \Omega_0}} \tag{47}$$

To be more explicit, let's write out the multi-component again:

$$1 - \Omega(a) = \frac{1 - \Omega_0}{1 - \Omega_0 + (\Omega_{m,0}a^{-1} + \Omega_{0,\Lambda}a^2 + \Omega_{r,0}a^{-2})} \tag{48}$$

where $\Omega_0 = \sum_i \Omega_{0,i}$. Note that the above equation essentially FORCES $\Omega(a) \sim 1$. This foreshadows the Flatness Problem.

1.4.3 Scale Factor (Flat Case)

This is essentially solving the Friedmann Equation for $a(t)$ for specific models (can, in general, be integrated). The most basic model is the Einstein-de Sitter model: a model that is flat $k = 0 \rightarrow \Omega = 1$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho \tag{49}$$

We have also see that $\rho \propto a^{-3(1+w)}$. Therefore:

$$\mathrm{d}a a^{\frac{3}{2}(1+w)-1} \propto \mathrm{d}t \tag{50}$$

$$\boxed{a(t) \propto t^{\frac{2}{3(1+w)}}} \tag{51}$$

Recall for the Einstein-de Sitter model ($k = 0, \Omega = 0$), we found:

$$a \propto t^{\frac{2}{3(1+w)}} \tag{52}$$

This is very useful since it works for each epoch (matter dominated, radiation dominated, dark energy dominated, etc;, but not in the transitions):

Matter Dominated Era ($w = 0, P \approx 0$)

$$\boxed{a_m \propto t^{\frac{2}{3}}} \tag{53}$$

Radiation Dominated Era ($w = \frac{1}{3}, P = \frac{1}{3}\rho$)

$$\boxed{a_r \propto t^{\frac{1}{2}}} \tag{54}$$

Λ **Dominated Era** ($w = -1$ (for example), $P = -\rho$)

$$a_r \propto t^{\infty} \rightarrow \text{ Wrong!} \tag{55}$$

This breaks down because H is a constant in this case! So here:

$$a \propto e^{Ht} \tag{56}$$

Isn't this reminiscent of inflation? That's because it is! It wasn't the Cosmological Constant but rather a scalar field called the inflaton.

1.4.4 Scale Factor (Open Case)

Let's consider $k < 0$ and $\Omega < 1$. Additionally, we assume $\Lambda = 0$ and that we are matter-dominated.

Start with the Friedmann Equation:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}G\rho - \frac{k}{a^2} > 0 \text{ always} \tag{57}$$

Immediately, we see that expansion will never stop! The right hand side is always positive, and therefore $\dot{a}$ is never equal to 0. Now, recall the equation for $H(a)$ and the fact that $H = \dot{a}/a$:

$$\dot{a}^2 = H_0^2\left(\frac{\Omega_0}{a^{1+3w}} + 1 - \Omega_0\right) \tag{58}$$

We now solve for $a(t)$ for this Universe:

$$\int_0^{a_f} \frac{\mathrm{d}a}{\sqrt{\frac{\Omega_0}{a} + 1 - \Omega_0}} = \int_0^{t_f} H_0 \mathrm{d}t \tag{59}$$

The right hand side is easy. The left hand side? Let's start with completing the squares to get rid of the $\frac{1}{a}$, so let's set that up.

$$\int_0^{a_f} \frac{a\mathrm{d}a}{(1-\Omega_0)\sqrt{a^2 + \frac{\Omega_0}{1-\Omega_0}a}} \tag{60}$$

Now we can actually complete the square in the denominator. Define $2\alpha \equiv \frac{\Omega_0}{1-\Omega_0}$:

$$\int_0^{a_f} \frac{a\mathrm{d}a}{\sqrt{1-\Omega_0}\sqrt{(a+\alpha)^2 - \alpha^2}} \tag{61}$$

We will now shift variables: $a\prime \rightarrow a + \alpha$:

$$\int_{\alpha}^{a_f+\alpha} \frac{(a-\alpha)\,\mathrm{d}a}{\sqrt{1-\Omega_0}\sqrt{a^2-\alpha^2}} \tag{62}$$

We can now do trigonometric substitution. Let $a \equiv \alpha\cosh\theta$. This gives $\mathrm{d}a = \alpha\sinh\theta\mathrm{d}\theta$. Lastly, $\sqrt{a^2\alpha^2} = \alpha\sinh\theta$. Properly adjusting limits, we get the expression:

$$\boxed{a(\theta) = \frac{\Omega_0}{2(1-\Omega_0)}(\cosh\theta - 1)} \tag{63}$$

$$\int_0^{\theta_f} \frac{\alpha^2(\cosh\theta - 1)\sinh\theta}{\sqrt{1-\Omega_0}\alpha\sinh\theta}\mathrm{d}\theta \tag{64}$$

$$\frac{\alpha}{\sqrt{1-\Omega_0}}\left(\sinh\theta-\theta\right)\Big|_0^{\theta_f} \tag{65}$$

$$H_0 t_f = \frac{\Omega_0}{2\left(1-\Omega_0\right)^{3/2}}\left(\sinh\theta_f-\theta_f\right) \tag{66}$$

This gives:

$$\boxed{t(\theta) = \frac{\Omega_0}{2H_0\left(1-\Omega_0\right)^{3/2}}\left(\sinh\theta-\theta\right)} \tag{67}$$

We can then use θ as some parameter, trace out a and t, and then find $a(t)$!

1.4.5 Scale Factor (Closed Case)

You can repeat the derivation above for the matter-dominated, closed case of $k > 0$, $\Omega > 1$. Here, there *will* be a maximum of a and the dynamics are different $\theta \to i\theta$):

$$\boxed{t(\theta) = \frac{\Omega_0}{2H_0\left(\Omega_0-1\right)^{3/2}}\left(\theta-\sin\theta\right)} \tag{68}$$

$$\boxed{a(\theta) = \frac{\Omega_0}{2\left(\Omega_0-1\right)}\left(1-\cos\theta\right)} \tag{69}$$

1.4.6 Deceleration Parameter

Let's define this – it was used historically since we now know the Universe is *accelerating*. This is a dimensionless parameter:

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} \to q = -\frac{1}{H^2}\frac{\ddot{a}}{a} \tag{70}$$

Recall one of the other results we had this semester:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\left(\rho+3P\right) \tag{71}$$

and

$$\rho_c = \frac{3H^2}{8\pi G} \to H^2\Omega = \frac{8\pi}{3}G\rho \tag{72}$$

Putting this together with q:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\rho\left(1+3w\right) = -\frac{H^2\Omega}{2}\left(1+3w\right) \tag{73}$$

This makes:

$$q = \frac{\Omega}{2}\left(1+3w\right) \tag{74}$$

For multiple components:

$$\boxed{q = \sum_i \frac{\Omega_i}{2}\left(1+3w_i\right)} \tag{75}$$

$$q = \frac{\Omega_m}{2} + \underbrace{\frac{1+3w}{2}\Omega_w}_{\text{Dark Energy}} + \Omega_r + ... \tag{76}$$

For a matter-dominated Universe and $\Omega_\Lambda \neq 0$:

$$\boxed{q = \frac{\Omega_m}{2} - \Omega_\Lambda} \tag{77}$$

The minus sign there is enormously important!

One thing to note: supernovae are sensitive to q, whereas CMB measurements are sensitive to the addition of Ω_m and Ω_Λ, so in some sense, these are orthogonal instead of parallel.

1.5 Robertson-Walker Metric

This is where you would start in General Relativity.

Let's set up the background. Recall Lorentz transformations in Special Relativity. There are two inertial observers, (x, y, z, t) and (x', y', z', t') with relative velocity $\vec{v} = v\hat{x}$:

$$x' = \gamma(x - vt) \tag{78}$$

$$y' = y \tag{79}$$

$$z' = z \tag{80}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{81}$$

We also talked about the Lorentz invariant interval:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) = \mathrm{d}s'^2 \tag{82}$$

Recall that the propagation of light follows the "null geodesic": $\mathrm{d}s^2 = 0$:

$$\mathrm{d}r = c\mathrm{d}t \tag{83}$$

We will quote the RW Metric here, and explore it next time. In short, Robertson and Walker showed that, for a **homogeneous and isotropic space, the most general metric is** (see Weinberg for a full proof):

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] \tag{84}$$

We are defining the coordinates as:

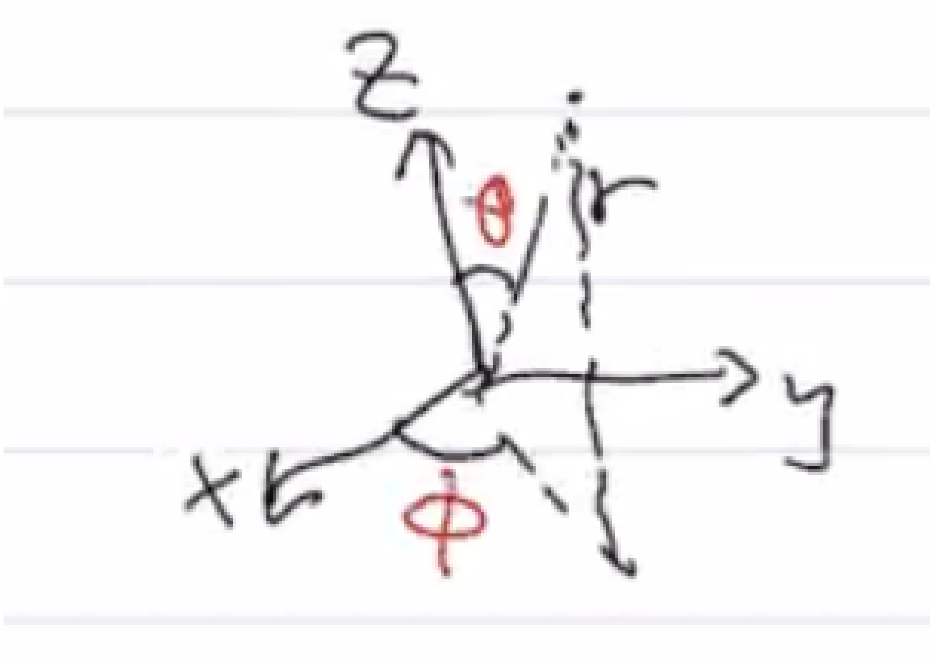

Figure 3: Spherical coordinate convention.

An immediate and obvious consequence is when $k = 0$, where we recover the Minkowski metric for flat space. Let's consider the more interesting cases of open ($k < 0$) and closed ($k > 0$). Immediately, we have a coordinate singularity at $r = \frac{1}{\sqrt{k}}$. Recall that $k = \frac{H_0^2}{c^2}(\Omega_0 - 1)$. We can thus define the **radius of curvature**:

$$R_0 \equiv \frac{1}{\sqrt{k}} = \frac{c}{H_0}\left(\frac{1}{\sqrt{\Omega_0 - 1}}\right) \tag{85}$$

Let's step back a second and look at the form of the FRW metric with some analogies:

3-sphere (in 4D)	2-sphere (in 3D)
$(x, y, z, w) \Leftrightarrow (R, \alpha, \beta, \gamma)$	$(x, y, z) \Leftrightarrow (R, \theta, \phi)$
$w = R\cos\alpha$	$z = R\cos\theta$
$z = R\sin\alpha\cos\beta$	$y = R\sin\theta\cos\phi$
$y = R\sin\alpha\sin\beta\cos\gamma$	$x = R\sin\theta\sin\phi$
$x = R\sin\alpha\sin\beta\sin\gamma$	
$x^2 + y^2 + z^2 + w^2 = R^2$	$x^2 + y^2 + z^2 = R^2$

Look at a line element of the 2-sphere embedded in 3D:

$$\mathrm{d}\ell^2 = R^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{86}$$

We will change variables to $u = \sin\theta \to \mathrm{d}u = \cos\theta\,\mathrm{d}\theta = \sqrt{1-u^2}\,\mathrm{d}\theta$. This makes the line element:

$$\mathrm{d}\ell^2 = R^2\left(\frac{\mathrm{d}u^2}{1-u^2} + u^2\mathrm{d}\phi^2\right) \tag{87}$$

Look at that! We're not quite there, but we have a similar form to the metric we want!
And the line element of the 3-sphere embedded in 4D:

$$\mathrm{d}\ell^2 = R^2\left(\mathrm{d}\alpha^2 + \sin^2\alpha\,\mathrm{d}\Omega^2\right) \tag{88}$$

where

$$\mathrm{d}\Omega^2 \equiv \mathrm{d}\beta^2 + \sin^2\beta \mathrm{d}\gamma^2 \tag{89}$$

Again, we can change variables, $u \equiv \sin\alpha \rightarrow \mathrm{d}u = \sqrt{1-u^2}\mathrm{d}\alpha$:

$$\mathrm{d}\ell^2 = R^2\left[\frac{\mathrm{d}u^2}{1-u^2} + u^2\mathrm{d}\Omega^2\right] \tag{90}$$

Again, where

$$\mathrm{d}\Omega^2 \equiv \mathrm{d}\beta^2 + \sin^2\beta \mathrm{d}\gamma^2 \tag{91}$$

Compared to the Friedmann-Robertson-Walker metric, these are identical!

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] \tag{92}$$

Thus, for a 3-sphere in 4-dimensions, we recover the FRW metric dependence in a hand-wavy way.

For the open model ($k < 0$), we don't have coordinate singularities. Instead of a spherical analogy, we use a saddle as an analogy (or, more tasty, Pringles!). This case has infinite volume and is called Lobachevsky space. Here, we let $u \equiv \sinh\theta$ which you can see will receover the proper FRW dependence.

There is an alternative form of the metric, too, we should know. We will see this in Problem Set 2. It is written as:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\mathrm{d}\chi^2 + \mathcal{S}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] \tag{93}$$

1.6 Basic Kinematic Properties of the Smooth Universe

Here, we have in mind:

- Comoving radial distance vs. Redshift
- Time vs. Redshift
- Age of the Universe
- Other useful quantities

1.6.1 Comoving, Radial Distance vs. Redshift ("The Hubble Diagram")

By the comoving radial distance, we mean the r in the RW metric.

First, we know that photons follow $\mathrm{d}s^2 = 0$, so let's take a radial path from the observer ($\mathrm{d}\theta = \mathrm{d}\phi = 0$. Now we have:

$$0 = c^2\mathrm{d}t^2 - \frac{a^2\mathrm{d}r^2}{1-kr^2} \tag{94}$$

$$\int_0^{r_1}\frac{\mathrm{d}r}{\sqrt{1-kr^2}} = c\int_{t_1}^{t_0}\frac{\mathrm{d}t}{a(t)} \tag{95}$$

where t_0 is the age of the Universe. Evaluating these integrals, starting with the flat, static Universe ($k = 0, a = 1$):

$$\mathrm{d}r = c\mathrm{d}t \rightarrow r = ct \tag{96}$$

What about other cases? First, let's replace $\mathrm{d}t$ by $\mathrm{d}z$ using $H = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}$ and $a = (1+z)^{-1}$:

$$H = -\frac{1}{1+z}\frac{\mathrm{d}z}{\mathrm{d}t} \tag{97}$$

Also recall:

$$H(z) = H_0\sqrt{\Omega_0(1+z)^{3+3w} + (1-\Omega_0)(1+z)^2} \tag{98}$$

Thus:

$$\mathrm{d}t = -\frac{\mathrm{d}z}{H(1+z)} = -\frac{\mathrm{d}z}{H_0(1+z)\sqrt{\text{stuff}}} \tag{99}$$

Going back to our integral expression:

$$\text{RHS} = \int_{t_1}^{t_0}\frac{c\mathrm{d}t}{a(t)} = \frac{c}{H_0}\int_0^{z_1}\frac{\mathrm{d}z}{\sqrt{\text{stuff}}} \tag{100}$$

We will look at two cases of the above.

- Einstein-de Sitter Model $k = 0, \Omega_0 = \Omega_{0,m} = 1$. No other components.
- Arbitrary $\Omega_{m,0}$ and $\Omega_{m,\Lambda}$ with negligible radiation.

Einstein-de Sitter

$$\text{RHS} = \frac{c}{H_0}\int_0^{z_1}\frac{\mathrm{d}z}{(1+z)^{3/2}} = \frac{2c}{H_0}\left[1-(1+z_1)^{-1/2}\right] \tag{101}$$

$$\text{LHS} = \int_0^{r_1}\mathrm{d}r = r \tag{102}$$

Thus, we have the comoving radial distance for the Einstein-de Sitter Universe:

$$\boxed{r = \frac{2c}{H_0}\left(1-(1+z)^{-1/2}\right]} \tag{103}$$

Let's examine a few limits. As $z \to \infty$, $r \to \frac{2c}{H_0} \approx 6h^{-1}\,\text{Gpc}$. When $z \ll 1$, $r \approx \frac{cz}{H_0}$. Note that this is just Hubble's Law! Since, for small z, $v \approx cz$.

Arbitrary Case

We will show this in PS2, but in this case:

$$\boxed{r = |k|^{-1/2}\,\text{sinn}\{\frac{c}{H_0}|k|^{1/2}\int_0^z\frac{\mathrm{d}z'}{\sqrt{\Omega_{0,m}(1+z')^3 + \Omega_{0,\Lambda} + (1-\Omega_{0,m}-\Omega_{0,\Lambda})(1+z')^2}}\}} \tag{104}$$

where

$$\boxed{\text{sinn} = \begin{cases}\sin \text{ when } k > 0\\ \text{absent when } k = 0\\ \sinh \text{ when } k < 0\end{cases}} \tag{105}$$

For $z \ll 1$, all three cases of k:

$$\boxed{r = \underbrace{\frac{c}{H_0}}_{\text{Hubble's Law}}\left(z - \frac{1}{2}\left[1+q_0\right]z^2\right) + \mathcal{O}(z^3)} \tag{106}$$

where

$$q_0 = \frac{\Omega_{0,m}}{2} - \Omega_{0,\Lambda} + \Omega_{0,r} \tag{107}$$

This is an enormously important equation, historically. Again, here we see that the Supernova experiments depend on q_0, thus the **difference** between $\Omega_{0,m}$ and $\Omega_{0,\Lambda}$, allowing good constraints.

1.6.2 A Note on Various Distances

There are two other commonly used cosmological "distances":

- Luminosity distance
- Angular diameter distance

Luminosity Distance D_L

This is based on standard candles. This is defined to be:

$$D_L \equiv \sqrt{\frac{L_{\text{emit}}}{4\pi S_{\text{obs}}}} \tag{108}$$

where the variables are emitted luminosity and observed flux. There are two factors for $1+z$ we have to look out for. For luminosity, we have energy per time. Energy is diluted by $1+z$ and time is dilated by $1+z$, so:

$$D_L = (1+z)\, r_{\text{comoving}} \tag{109}$$

Angular Diameter Distance D_A

This is based on standard rulers. This is defined to be:

$$D_A \equiv \frac{\ell_{\text{physical}}}{\underbrace{\theta}_{\text{angular size}}} \tag{110}$$

We can change this to comoving coordinates easily:

$$D_A = \frac{r_{\text{comoving}}}{(1+z)} \tag{111}$$

1.6.3 Time vs. Redshift

All we need here is $\mathrm{d}t$ vs. $\mathrm{d}z$, integrate, and we are okay. So how do we do that?

We actually did this earlier when we were deriving $H(a)$. Recall:

$$\mathrm{d}t = -\frac{1}{H_0}\frac{\mathrm{d}z}{(1+z)\sqrt{\Omega_0\,(1+z)^{3+3w} + (1-\Omega_0)\,(1+z)^2}} \tag{112}$$

Let's ask a few questions – what's the time since the Big Bang at redshift z?

$$t = \frac{1}{H_0}\int_z^\infty \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_0\,(1+z)^{3+3w} + (1-\Omega_0)\,(1+z)^2}} \tag{113}$$

A related question – what's the age of the Universe today? All we do is set $z = 0$ above!

$$t_0 = \frac{1}{H_0} \int_0^\infty \frac{\mathrm{d}z'}{(1+z')\sqrt{\Omega_0 (1+z)^{3+3w} + (1-\Omega_0)(1+z)^2}} \tag{114}$$

Again, as always, let's look at a few special cases, starting with EdS:

Einstein de Sitter

$$t_0 = \frac{1}{H_0} \int_)^\infty \frac{\mathrm{d}z}{(1+z)^{5/2}} = -\frac{2}{3H_0} (1+z)^{-3/2}|_0^\infty = \frac{2}{3H_0} \tag{115}$$

A different way to derive this – recall that $a(t) \propto \left(\frac{t}{t_0}\right)^{2/3}$. Thus, $H = \frac{\dot{a}}{a} = \frac{2}{3}\frac{1}{t}$. We can set this to today: $t_0 = \frac{2}{3H_0}$. Remind yourself that $H_0^{-1} = 10^{10} h^{-1}\,\mathrm{yr}$. This gives:

$$t_0 \approx 6.7 h^{-1}\,\mathrm{Gyr} \tag{116}$$

Now, another case, where $k = 0$ but allow for non-zero $\Omega_{0,\Lambda}$ (but $\Omega_m + \Omega_\Lambda = 1$. We can show this, but we won't, that:

$$t_0 = \frac{2}{3H_0} \frac{1}{\sqrt{\Omega_{0,\Lambda}}} \ln\left(\frac{1+\sqrt{\Omega_{0,\Lambda}}}{\sqrt{\Omega_{0,m}}}\right) \approx \frac{2}{3H_0} \Omega_{0,m}^{-0.3} \tag{117}$$

Open Case

Here, we will consider $k < 0$, $\Omega_0 < 1$ and matter only. We looked at this extensively when we did parametric solutions:

$$t(\theta) = \frac{\Omega_0}{2H_0 (1-\Omega_0)^{3/2}} (\sinh\theta - \theta) \tag{118}$$

$$a(\theta) = \frac{\Omega_0}{2(1-\Omega_0)} (\cosh\theta - 1) \tag{119}$$

Today, we have $a(\theta_0) = 1 \to \cosh\theta_0 = \frac{2-\Omega_0}{\Omega_0}$. After a little bit of algebra, we find:

$$\sinh\theta_0 = \frac{2}{\Omega_0}\sqrt{1-\Omega_0} \tag{120}$$

$$t_0 = \frac{1}{H_0}\left[(1-\Omega_0)^{-1} - \frac{1}{2}\Omega_0 (1-\Omega_0)^{-3/2} \cosh^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right)\right] \tag{121}$$

We can show that:

$$t_{0,\mathrm{open}} > \frac{2}{3H_0} (t_0 \text{ for flat}) \tag{122}$$

Additionally, if we take $\Omega_0 \to 0$ (a **Milne** Universe), then $t_0 \to \frac{1}{H_0}$.

Closed Case

This is when $\Omega_0 > 1$, with matter only, and $k > 0$. Correspondingly, we get:

$$t_0 = \frac{1}{H_0}\left[(1-\Omega_0)^{-1} + \frac{1}{2}\Omega_0 (\Omega_0 - 1)^{-3/2} \cos^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right)\right] \tag{123}$$

We can show that:

$$t_{0,\mathrm{closed}} < \frac{2}{3H_0} (t_0 \text{ for flat}) \tag{124}$$

Note that there is a nice identity: $\cosh^{-1}(x) = \ln\left(x + (x^2-1)^{1/2}\right)$.

2 The Bright Universe

We have discussed the smooth, Friedmann-Roberston-Walker Universe. What about fluctuations in density? Let's fill the Universe in, starting with the bright stuff (baryons). This climaxes with the Big Bang, and the prediction of mass ratios of the elements H, He, D, and ^{7}Li (with very minor tension). The thermodynamics that we learn here differs in that we have an expanding background with different particles with different statistics.

We must then compare the expansion rate (Hubble Rate) to the particle interactions that keep things in equilibrium.

2.1 The Planck Mass: The Ugliest Numbers in Physics

This might just look like unit conversion tricks in various branches physics:

- High Energy Physics
  - Energy $\sim \frac{1}{\text{length}}$, with $\hbar c \approx 200$ fermi-MeV where 1fermi $= 10^{-13}$ cm and $\hbar \equiv \frac{h}{2\pi}$. The convention is to instead say $\hbar c = 1$. Nominally, this says:
  $$1 \text{ MeV} \approx \frac{1}{200} \text{ fermi}$$
  - Energy $\sim$ mass, so $c = 1$. Thus,
  $$1 \text{ GeV} \approx m_p \approx 1.67 \times 10^{-24} \text{ g}$$
- Condensed Matter Physics
  - Energy $\sim$ Temperature, and the way we do this is setting $k_B T_{\text{room, 300 K}} = \frac{1}{40}$ eV. An easy example is the CMB, which has a temperature of around 3 K. Immediately, the characteristic photon energy is thus $\frac{1}{4000}$ eV, or around 2.5×10^{-4} eV. As a side note, if we want to say if something is "relativistic," we have to compare the particle's rest mass to the characteristic photon energy at a given temperature.
  $$k_B T_{\text{room, 300 K}} = \frac{1}{40} \text{ eV}$$
- Astrophysics
  - Here, we look to the Schwarzschild radius of the Sun:
  $$R_{\text{Sch}} = \frac{2GM}{c^2}$$
  Pinning this to the sun, we get $R_{\text{Sch}} \approx 3$ km.

So how does this relate to the Planck Mass? This is scale at which gravity and quantum mechanics are comparable in strength. We also call this the "unifying scale" of gravity and quantum mechanics. For gravity, this scale is R_{Sch} and for QM, this is $\lambda_c = \frac{hc}{mc^2} = \frac{2\pi\hbar c}{mc^2}$. When $R \sim \lambda$ and m_{pl}:

$$\frac{G m_{\text{pl}}}{c^2} = \frac{\hbar c}{m_{\text{pl}} c^2} \rightarrow \boxed{m_{\text{pl}} = \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{19} \text{ GeV}}$$

Schematically, we have:

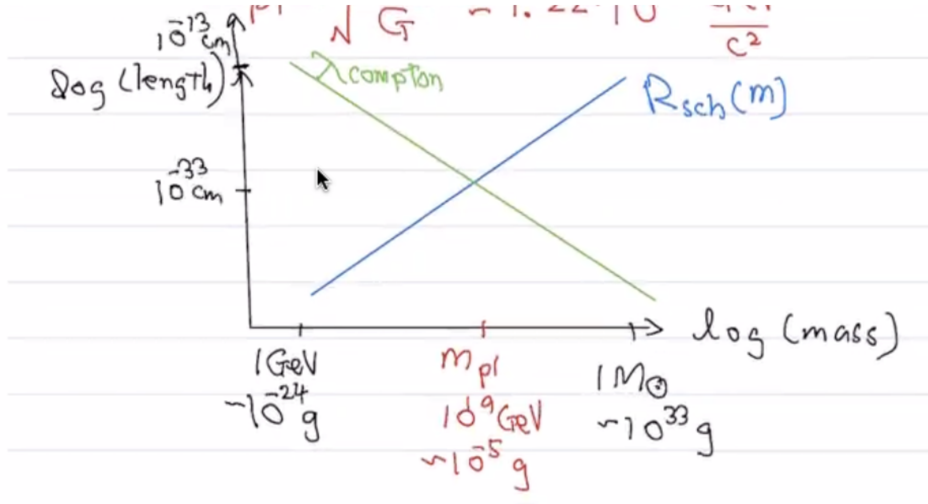

Figure 4: Length and mass scales, with the Planck mass being the intersection.

A very natural scale for Λ would be the Planck mass, right? Let's check that, looking at the Energy density in Λ to order of magnitude:

$$\Omega_\Lambda = 0.7 \approx 1 \to \rho_\Lambda \sim \rho_c \sim 1 \times 10^{-5} h^2 \frac{\text{GeV}}{\text{cm}^3} \tag{125}$$

Re-writing this with our conventions above (that $200\,\text{MeV} \approx \text{fm}^{-1} \sim 10^{13}\,\text{cm}^{-1}$.

$$\frac{1}{\text{cm}^3} \sim (0.2\,\text{GeV})^3 \times 10^{-39} \sim 10^{-41}\,\text{GeV} \tag{126}$$

Thus

$$\boxed{\rho_\Lambda \sim 10^{-46}\,\text{GeV}^4} \tag{127}$$

Here comes the comparsion that should shock you. What's the natural scale, again? The Planck Scale! So how does ρ_Λ compare with m_{pl}^4.

$$\boxed{m_{\text{pl}}^4 \sim 10^{76}\,\text{GeV}^4} \tag{128}$$

Notice!

$$\frac{\rho_\Lambda}{m_{pl}^4} \sim 10^{-122} \tag{129}$$

Why the heck is that so small? It's so ugly, unnatural, and not $\mathcal{O}(1)$.

2.2 Thermodynamics in an Expanding Universe

Here, we have lots of review of thermodynamics but applied to a non-static, fluid background. The conditions in the Early Universe (the first three minutes, or so). We had extremely high temperature, high pressure, and high density (either energy density ρ or number density n). To a good approximation, though not always true, we have thermal equilibrium. Additionally, we can treat all these particles as if they are ideal gases – gases where interactions are negligible.

In phase space, we can have a phase space volume: $\mathrm{d}^3x\mathrm{d}^3p$ which has units of h^3_{planck}. We use the term **phase space distribution function** $f(\vec{x}, \vec{p}, t)$, by the way. We thus have that:

$$\text{Particle No.: } \mathrm{d}N = f(\vec{x}, \vec{p}, t)\frac{\mathrm{d}^3x\mathrm{d}^3p}{h^3}g \tag{130}$$

When thermodynamics breaks down, we always can return to the evolution of f directly, which is goverened by the **Boltzmann equation**. On a good day, we don't have to solve the 6-dimensional Boltzmann equation and can return to fluid dynamics instead. On a bad day, we have to return to our distribution function analysis.

Now, recall from statistical mechanics: the equilibrium occupation number of a state of energy $\epsilon(p)$:

$$f = \frac{1}{e^{\frac{\epsilon-\mu}{kT}} \pm 1} \tag{131}$$

where $+$ is for fermions and $-$ is for bosons. Note as well that μ is the chemical potential:

$$\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N \tag{132}$$

The physical meaning of μ is thus the change in internal energy due to change in particle number! For a thermal radiation background, we have $\mu = 0$. This will be the case for most astrophysical contexts.

The most obvious consequence of the above Boltzmann statistics is that the temperature evolves in time, but thermal equilibrium gives us spatial homogeneity. We need to know the first order perturbations to f if we want to consider the non-linear Universe (the CMB!).

Here are a few macroscopic, useful quantities:

- Number density (where $\epsilon(p) = \sqrt{m^2c^4 + p^2c^2}$:

$$n = \frac{g}{h^3}\int_0^\infty \frac{4\pi p^2}{e^{\frac{\epsilon(p)}{kT}} \pm 1}\mathrm{d}p \tag{133}$$

- Energy density u:

$$u = \rho = \frac{g}{h^3}\int_0^\infty \epsilon(p)\frac{4\pi p^2}{e^{\frac{\epsilon(p)}{kT}} \pm 1}\mathrm{d}p \tag{134}$$

- Entropy density s:

$$s \equiv \frac{S}{V} = \frac{1}{V}\frac{1}{T}\left(U + PV - \mu N\right) \tag{135}$$

  when $\mu = 0$:

$$s = \frac{u+P}{T} \tag{136}$$

Two limits: We will see familiar expressions when we take some limits. Starting with the **ultrarelativistic limit** $kT \gg mc^2$.[2] Here, the particles are effectively massless and $\epsilon \to pc$. Computing these integrals:

- Number density:

$$n = \frac{g}{h^3}\int_0^\infty \frac{4\pi p^2}{e^{\frac{pc}{kT}} - 1}\mathrm{d}p = \frac{4\pi g}{(2\pi)^3}\left(\frac{kT}{\hbar c}\right)^3\int_0^\infty \frac{y^2}{e^y - 1}\mathrm{d}y \tag{137}$$

[2] Note, again, that T changes in time and therefore particles change from ultrarelativistic to non-relativistic throughout the evolution of the Universe.

- Energy density

$$u = \frac{4\pi g}{(2\pi)^3}\left(\frac{k^4T^4}{\hbar^3c^3}\right)\int_0^\infty \frac{y^3\mathrm{d}y}{e^y \pm 1} \tag{138}$$

  – Bosons:

$$n = \frac{1}{\Gamma(n)}\int_0^\infty \frac{y^{n-1}\mathrm{d}y}{e^y - 1} \tag{139}$$

  We need two results:

$$\int_0^\infty \frac{y^2\mathrm{d}y}{e^y - 1} = \Gamma(3)\zeta(3) \tag{140}$$

$$\int_0^\infty \frac{y^2\mathrm{d}y}{e^y - 1} = \Gamma(4)\zeta(4) = \frac{\pi^4}{15} \tag{141}$$

  Thus:

$$\boxed{n_{\text{bosons}} = \frac{g}{\pi^2}\zeta(3)\left(\frac{kT}{\hbar c}\right)^3} \tag{142}$$

$$\boxed{u_{\text{bosons}} = \frac{\pi^2}{30}g\left(\frac{(kT)^4}{(\hbar c)^3}\right)} \tag{143}$$

  For photons, this is blackbody radiation since $g = 2$ for two polarization states. This also gives us an energy flux of

$$\boxed{F = \frac{1}{4}uc = \frac{\pi^2}{60}\frac{k^4}{(\hbar c)^3}T^4 = \sigma T^4} \tag{144}$$

  where we introduce $\sigma = \frac{\pi^2 k^4}{60(\hbar c)^3} = 5.67 \times 10^{-8}\frac{\mathrm{W}}{\mathrm{m}^2\,\mathrm{K}^4}$. An obvious next question – what's the photon pressure (from equation of state)?

$$P = \frac{1}{3}u \tag{145}$$

  And lastly, the entropy density:

$$s = \frac{u + P}{T} = \frac{4}{3}\frac{U}{T} \tag{146}$$

  One thing that is important to note – recall that $\rho \propto a^{-4}$ for radiation. Since $u \propto T^4$, $\boxed{T \propto a^{-1}}$ assuming fixed g. **As the Universe cools, the effective g changes.**

  – Fermions:

  We can use one trick to make this easy! There's an identity that:

$$\frac{1}{e^y + 1} = \frac{1}{e^y - 1} - \frac{2}{e^{2y} - 1} \tag{147}$$

where $y = \frac{pc}{kT} \to 2y = \frac{pc}{k\frac{T}{2}}$.

$$n_{\text{fermions}}(T) = \left[n_B(T) - 2n_B\left(\frac{T}{2}\right)\right]\frac{g_{\text{fermions}}}{g_{\text{bosons}}} \tag{148}$$

$$n_{\text{fermions}}(T) = \left[1 - 2\left(\frac{1}{2}\right)^3\right] n_B(T)\frac{g_{\text{fermions}}}{g_{\text{bosons}}} \tag{149}$$

$$\boxed{n_{\text{fermions}}(T) = \frac{g_{\text{fermions}}}{g_{\text{bosons}}}\frac{3}{4}n_{\text{bosons}}(T)} \tag{150}$$

And thus the same thing for the energy density:

$$u_{\text{fermions}}\,(T) = [u_b(T) - 2u_B\,(T/2)]\,\frac{g_f}{g_b} \tag{151}$$

$$\boxed{u_{\text{fermions}}\,(T) = \frac{g_{\text{fermions}}}{g_{\text{bosons}}}\frac{7}{8}u_B\,(T)} \tag{152}$$

And similarly, for energy density:

$$\boxed{s_{\text{fermions}}\,(T) = \frac{g_{\text{fermions}}}{g_{\text{bosons}}}\frac{7}{8}s_B\,(T)} \tag{153}$$

For convenience, we can define an effective degeneracy:

$$\boxed{g^\star = \sum_{\text{boson types}} g_i + \frac{7}{8}\sum_{\text{fermion types}} g_i} \tag{154}$$

And thus:

$$\boxed{u_{\text{Tot}} \propto g^\star T^4} \tag{155}$$

Non-relativistic limit:

This is the case when $kT \ll mc^2$. In this limit, $\epsilon = \sqrt{m^2c^4 + p^2c^2} \to mc^2\left(1 + \frac{1}{2}\frac{p^2}{m^2c^2} + ...\right) \gg kT$. Our Fermi-Dirac or Bose-Einstein distribution thus become the same since the ± 1 becomes irrelevant! We drop that in the distribution function for f, and thus:

$$f \sim e^{-\epsilon/kT} \approx e^{-\frac{mc^2}{kT}}\underbrace{e^{-\frac{p^2}{2mkT}}}_{e^{-mv^2/2kT}}\ ... \tag{156}$$

Look at that! The second term is a Maxwellian distribution! This makes sense – if we talk about the distribution of stars in a cluster, we don't care if they are bosons!

What about number density?

$$n \approx \frac{g}{h^3}\int_0^\infty \mathrm{d}p 4\pi p^2 e^{-\frac{p^2}{2mkT}} \tag{157}$$

$$n \approx \frac{4\pi g}{h^3}\,(2mkT)^{3/2}\,e^{-\frac{mc^2}{kT}}\underbrace{\int_0^\infty \mathrm{d}y\, y^2 e^{-y^2}}_{\pi^{1/2}/2} \tag{158}$$

$$\boxed{n = \frac{g}{\hbar^3}\left(\frac{mkT}{2\pi}\right)^{3/2} e^{-\frac{\left(mc^2-\mu\right)}{kT}}} \tag{159}$$

where μ is the chemical potential, sometimes non-zero. The key feature here is that n is exponentially suppressed at $kT \ll mc^2$. For example, at $T \ll 1\,\mathrm{GeV}$ and in thermal equilibrium, neutrons and protons are non-relativistic:

$$\frac{n_n}{n_p} \approx e^{-\frac{(m_n-m_p)c^2}{kT}} \approx e^{-\frac{1.293\,\mathrm{MeV}}{kT}} \tag{160}$$

Note that neutrons are a bit heavier, and therefore a small under abundant! Also, as temperature drops, so does the ratio n_n to n_p. What matters here is at what value is this ratio frozen? This will give us the hydrogen to helium mass ratio!

2.3 Thermal History of the Universe: The Longest Three Minutes of Our Lives

Let's first agree to some rules or questions to ask:

- Rule 1: Thermal equilibrium (TE) vs. Decoupling or "Freeze-Out"[3]: Need to compare relevant interaction rate Γ with Hubble expansion rate H
  - $\Gamma \gg H \rightarrow$ many interactions per Hubble time, and thus can keep equilibrium.
  - $\Gamma \ll H \rightarrow$ interactions are negligible, and particles are decoupled
  - **Example**: Weak interactions (neutrinos) which freeze-out first since they are so weak:

$$\sigma_{\mathrm{weak}} \sim G_{\mathrm{Fermi}}^2 T^2 \sim 10^{-43}\left(\frac{kT}{1\,\mathrm{MeV}}\right)^2 \mathrm{cm}^2 \tag{161}$$

$$\sigma_{\mathrm{Thomson}} \sim 10^{-24}\,\mathrm{cm}^2 \text{ for reference} \tag{162}$$

    * This gives us an interaction rate dependent on T:

$$\Gamma_{\mathrm{weak}} \sim n\sigma_{\mathrm{weak}} v \propto G_F^2 T^5 \tag{163}$$

    * And we compare this to the Hubble rate:

$$H = \sqrt{\frac{8\pi}{3} G\rho} \propto \sqrt{Gg^\star} T^2 \text{ for radiation dom. era} \tag{164}$$

    * So when are neutrinos in equilibrium with the photons?

$$\Gamma_{\mathrm{weak}} < H \rightarrow G_F^2 T^5 < \sqrt{Gg^\star} T^2 \rightarrow \boxed{T \leq 1\,\mathrm{MeV}} \tag{165}$$

    * Thus, after about $T = 1\,\mathrm{MeV}$, the weak interaction decouples from the electromagnetic interaction. Thus neutrinos start to free-stream through the Universe! Neutrinos follow geodesics from General Relativity.

- Rule 2: Which particles contribute to thermal radiation/the radiation background?
  - Answer: All relativistic particles in equilibrium (high interaction rates with photons).

- Rule 3: How do we relate temperature to time in the radiation dominated era?

[3] Decoupling from photons! We might say something like "the photons and electrons froze out at..."

– Recall that $u \propto \rho c^2 = \frac{\pi^2}{30} g^\star \frac{(kT)^4}{(\hbar c)^3}$ for ultrarelativistic particles. Also recall from the Friedmann equation that $H^2 = \frac{8\pi G}{3}\rho = \frac{\dot{a}}{a}$ and $a \propto t^{1/2}$ in the radiation dominated era. Putting all of this together:

$$\left(\frac{1}{2t}\right)^2 = \frac{8\pi G}{3}\frac{\pi^2}{30} g^\star \frac{(kT)^4}{(\hbar c)^3} \tag{166}$$

$$\boxed{kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g^\star G}\right)^{1/4} t^{-1/2}} \tag{167}$$

– Usefully:

$$\boxed{kT = \frac{0.86\,\mathrm{MeV}}{\sqrt{t[\mathrm{s}]}}\left(\frac{10.75}{g^\star}\right)^{1/4}} \tag{168}$$

$$\boxed{T \sim \frac{10^{10}\,\mathrm{K}}{\sqrt{t\,[\mathrm{s}]}}} \tag{169}$$

– So, about 1 second after the Big Bang, the Universe is around 10^{10} Kelvin or 1MeV. We know from earlier that neutrinos decouple from the photons at this time! **So, 1 second after the Big Bang, neutrinos are already decoupled from the thermal background**.

With all of that out of the way, let's start with the first of 5 and a half frames!

2.3.1 Frame 1: t = 0.01 sec

Here, $kT = 8.6\,\mathrm{MeV}$ and $T \sim 10^{11}\,\mathrm{K}$. Additionally, this is a redshift of around $z \sim 3 \times 10^{10}$.

A first question – what's $g^\star$ for radiation? Well, the major players are:

- Photos: $g_\gamma = 2$
- Neutrinos: $g_\nu = 3^{\text{species}} \times 2^{\nu\bar{\nu}} \times 1^{\text{spin}} \times \left(\frac{7}{8}\right)^{\text{fermions}} = \frac{21}{4}$[4]
- Electrons, Positrons[5]: Highly relativistic from our rules above! Here, $g_e = 2^{e^+e^-} \times 2^{\text{spins}} \times \left(\frac{7}{8}\right)^{\text{fermions}} = \frac{7}{2}$

Together, we have:

$$g^\star = 2 + \frac{21}{4} + \frac{7}{2} = 10.75 \tag{170}$$

We also want to keep track of the baryons (just protons and neutrons for our purposes). They are highly non-relativistic since their rest mass is of order 1 GeV. However, we want to keep track of the neutron to proton ratio:

$$\frac{n_n}{n_p} = e^{-\frac{\Delta E}{kT}} \text{ where} \Delta E = (m_n - m_p)\,c^2 = 1.293\,\mathrm{MeV} \tag{171}$$

At this stage, we get:

$$\frac{n_n}{n_p} = 0.86 \rightarrow \boxed{\frac{n_n}{n_p + n_n} = 0.46} \tag{172}$$

We have a bit more protons than neutrons. Another interesting question – what is controlling this ratio? Well, a bunch of processes are, but the one with the highest branching ratios are:

[4] left handed neutrinos only!

[5] muons and taus are highly non-relativistic in this regime!

- $\mathrm{n} + \nu_e \Leftrightarrow \mathrm{p} + e^-$
- $\mathrm{n} + e^+ \Leftrightarrow \mathrm{p} + \bar{\nu_e}$
- $\mathrm{n} \Leftrightarrow \mathrm{p} + \bar{\nu_e} + e^+$: Neutron decay, which is of order 15 minutes! Right now, this is negligible, since we have only had $t = 0.01\,\mathrm{s}$[6].

2.3.2 Frame 2: t = 0.1 sec

Let's drop the energy by half a dex. This makes $kT \sim 2.72\,\mathrm{MeV} \to T \sim 10^{10.5}\,\mathrm{K}$. We know, too, that $\rho_{rad} \propto T^4$ drops by a factor of 100. We thus have:

$$\frac{n_n}{n_p} = 0.62 \tag{173}$$

and

$$\frac{n_n}{n_p + n_n} = \frac{n_n}{n_{\text{baryon}}} = 0.38 \tag{174}$$

2.3.3 Frame 3: t = 1 sec

Thus, $kT \sim 0.86\,\mathrm{MeV} \to T \sim 10^{10}\,\mathrm{K}$. Note that we crossed a threshold! The weak interaction has decoupled since we are lower than $kT \sim 1$ MeV. As a result, neutrinos decouple from thermal radiation.[7] At this time,

$$\frac{n_n}{n_B} = 0.24 \tag{175}$$

When the neutrinos decouple, remember, we have to change $g^\star$. Originally, $g^\star = 10.75$. Afterwards, neutrinos stop talking to photons, so:

$$g^\star = 2 + \frac{7}{2} = \frac{11}{2} \tag{176}$$

Now that they are free streaming, neutrinos are pretty similar to the CMB! This is actually termed the **comsic neutrino background**. Imagine if we had that information...

2.3.4 Frame 4: t = 14 sec

Because $g^\star$ changed, we have a different relationship between T and t. We will drop the energy to $kT = 0.272\,\mathrm{MeV} \to T = 10^{9.5}\,\mathrm{K} \to t \sim 14\,\mathrm{s}$.

An important threshold was crossed! We crossed the rest mass of the electrons (and positrons), and thus electrons are non-relativistic! The reaction $e^+e^- \to 2\gamma$ is favored compared to $2\gamma \to e^+e^-$. As electron-positron annihilations occur, entropy must be conserved. Thus, entropy is being transferred to photons!

Recall entropy density:

$$s \propto g^\star T^3 \tag{177}$$

Before the electrons and positrons disappear, $g^\star = 11/2$. Right after, $g^\star = 2$. s must be conserved. Bizarrely:

$$T_{\text{after}} = \left(\frac{11}{4}\right)^{1/3} T_{\text{before}} \tag{178}$$

[6] Only the very precocious neutrons have decayed by now.
[7] Not a sharp transition, but happens when $\Gamma_{\text{weak}} \leq H \to T \leq 1\,\mathrm{MeV}$

But neutrinos had decoupled in Frame 2, so T_ν is not affected. As a result:

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \tag{179}$$

This is a famous result! The photon temperatures are slightly higher because the entropy of the photon bath must increase. An incredibly powerful prediction – and it should hold even today! **We thus predicted the neutrino background temperature**.

One other thing to note – the photon temperature doesn't have a *kink*, but looks something like this:

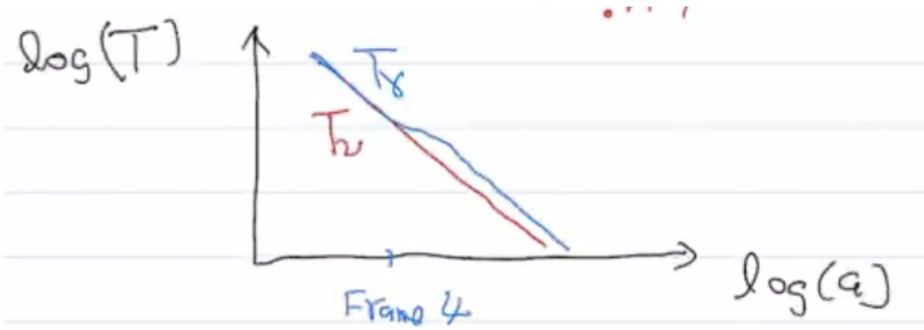

Figure 5: Caption

What about protons and neutrons? We need more detailed thermodynamics here, but the result:

$$\frac{n_n}{n_p} \sim 0.2 \tag{180}$$

The main reaction here to form nuceli is $\mathrm{p} + \mathrm{n} \Leftrightarrow \mathrm{D} + \underbrace{\gamma}_{2.2\text{ MeV}}$ since directly making He is incredibly unlikely. Are there enough reactions taking place? This is called the "deuterium bottleneck," and we **must** break this bottleneck. Thus, D is still unstable until T drops further.

2.3.5 Frame 5: t = 3 minutes

Here, we have $T_\gamma \sim 10^9\,\mathrm{K}$, giving $kT_\gamma = 0.086\,\mathrm{MeV}$. Remember that $g^\star = 2$. Most of the positrons and electrons have disappeared, and we still have the **deuterium bottleneck**, so there are no appreciable heavier nuclei yet. Free neutron decay starts to become more important, and the neutron-proton ratio is:

$$\frac{n_n}{n_p} \sim 0.16 \tag{181}$$

2.3.6 Frame 5.5: t = 3.75 minutes

Here, we have:

$$\frac{n_n}{n_p} \sim \frac{1}{7} \tag{182}$$

This is when deuterium becomes stable enough for: $\mathrm{n} + \mathrm{p} \to \mathrm{D} + \gamma$. **This is when nucleosynthesis begins**. Very quickly, nearly all neutrons turn into Helium-4. There are a few isotopes like tritium, He-3, and a bit of lithium. But, most of the neutrons are in Helium-4.

What is the helium abundance? Let Y be the mass fraction of Helium:

$$Y = \frac{4m_p n_{He-4}}{m_p n_{baryons}} \tag{183}$$

We also have $n_{He-4} = \frac{n_n}{2}$. Substituting that in:

$$Y = \frac{2\frac{n_n}{n_p}}{\frac{n_n}{n_p}} \tag{184}$$

For $n_n/n_p \sim 1/7$, we have $\boxed{Y = 1/4}$.

There is one more thing to note. The BBN He abundance Y depends on a number of other basic parameters in addition to temperature in the Universe. Some of these include:

- Baryon-to-photon ratio $\eta \equiv \frac{n_B}{n_\gamma} = \frac{n_n + n_p}{n_\gamma}$. We can show (in Problem Set 4): $\Omega_{0,B}h^2 = 0.00365\eta_{10}$ where $\eta_{10} = \frac{\eta}{10^{-10}}$. Thus, if we can measure η, we can measure $\Omega_{0,B}$! If we have a larger η, deuterium is more stable and the bottleneck is broken earlier. Thus, we have a higher Y for higher η.
- Neutron lifetime: $\tau_n = 880.2 \pm 1.0\,\mathrm{s}$ (2019) where $N(t) = N_0 e^{-t/\tau_n}$. If the neutron lifetime is longer, we have fewer neutron decays and thus higher Y.
- Number of species of neutrinos N_ν (or $g^\star$ generally). Larger N_ν gives us larger $g^\star$. The net result is that we have a larger Y.

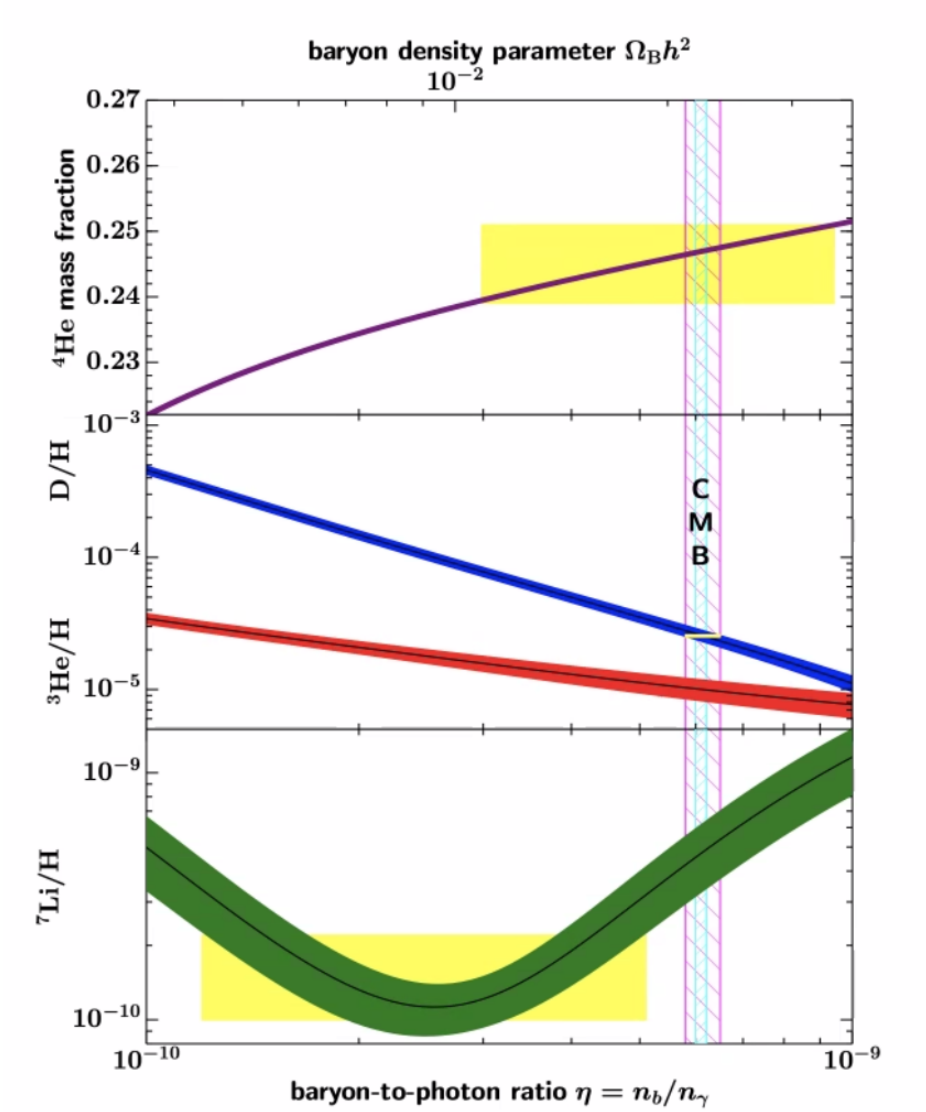

Figure 6: The master plot for discussions of BBN. This is from https://pdg.lbl.gov/2020/reviews/rpp2020-rev-bbang-nucleosynthesis.pdf.

3 The Mildly Lumpy Universe

3.1 Gravitational Instability in a Static Medium

How does a *nearly* smooth Universe become some lumpy? The basic principle is **gravitational instability**: tiny deviations from equilibrium become greatly enhanced. In practice, this means that **slightly overdense regions become denser**, whereas **slightly underdense regions become less dense**. How do we quantify this process?

In the static case, we have the **Jeans Instability**. For a static Universe filled with non-relativistic fluid with mass density ρ, pressure P, (fluid) velocity $\vec{v}$, and gravitational potential Φ. In this classical limit, what are the equations that govern these quantities?

Here are the fluid equations:

Continuity Equation/Mass Conservation:

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{185}$$

Equation of Motion (Euler Equation):

$$\underbrace{\partial_t \vec{v} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v}}_{\text{acceleration}} = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \Phi \tag{186}$$

Note that above, we are ignoring viscosity, magnetic fields, etc. We can include those on the right side of the Euler equation.

Poisson Equation:

$$\nabla^2 \Phi = 4\pi G \rho \tag{187}$$

We have three equations with four unknowns, so we need the equation of state for a unique solution.

The unperturbed piece (with subscript 0): static uniform fluid:

$$\rho_0 = P_0 = \vec{v}_0 = \text{const.} \tag{188}$$

And we introduce small perturbations (like $\rho_1 \ll \rho_0$ with a subscript 1 (which are not necessarily constant):

$$\rho = \rho_0 + \rho_1 \tag{189}$$

$$P = P_0 + P_1 \tag{190}$$

$$\vec{v} = \vec{v}_1 \tag{191}$$

$$\Phi = \Phi_0 + \Phi_1 \tag{192}$$

Let's introduce one other concept: sound speed. Specifically, we are talking about **adiabatic sound speed**:

$$v_s^2 \equiv \left(\frac{\partial P}{\partial \rho} \right)_{\text{const. entropy}} = \frac{P_1}{\rho_1} \tag{193}$$

Note! If we don't have perturbations, we don't have a sound speed! We return to the master equations and we get two sets of equations – perturbed and unperturbed. Starting with the zeroth-order fluid equations:

Unperturbed Fluid Equations:

$$0 + 0 = 0 \tag{194}$$

$$\partial_t \vec{v}_0 + \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_0 = -\frac{1}{\rho_0} \vec{\nabla} P_0 - \vec{\nabla} \Phi_0 \to \Phi_0 = \text{const.} \tag{195}$$

$$\nabla^2 \Phi_0 = 4\pi G \rho_0 \to \vec{\nabla} \Phi_0 = \frac{4\pi}{3} G \rho_0 \vec{r} \tag{196}$$

But we just said that the term had to be 0! Historically, this is called **Jean's Swindle**: a basic inconsistency between Poisson and Euler equation. It's not worth doing, there is no reconciliation! Mathematically this checks out. What doesn't check out – there is no such thing as a **static, uniform, fluid**. We are not allowing a pressure gradient to balance gravity, but we **NEED A PRESSURE GRADIENT TO BALANCE GRAVITY**. So, what's the swindle? We worry about this if it were relevant at all to physics; this is not the case since we never worry about it!

Any system with rotation, or any system that is not static, we don't have a Jean's swindle. Let's look at the **perturbed equations** to first order:

$$\partial_t \rho_1 + \rho_0 \vec{\nabla} \cdot \vec{v}_1 = 0 \tag{197}$$

$$\partial_t \vec{v}_1 = -\frac{v_s^2}{\rho_0} \vec{\nabla} P_1 - \vec{\nabla} \Phi_1 \tag{198}$$

$$\nabla^2 \Phi_1 = 4\pi G \rho_1 \tag{199}$$

We can combine these since we have three equations with three unknowns (treating v_s as a parameter): Let's combine the first two perturbed equations (by taking the derivative of the first equation with respect to time and exchanging order of partials):

$$\partial_t^2 \rho_1 + \rho_0 \left(-\frac{v_s^2}{\rho_0} \nabla^2 \rho_1 - \nabla^2 \Phi_1 \right) \tag{200}$$

Using equation 3...

$$\boxed{\partial_t^2 - v_s^2 \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0} \tag{201}$$

This is the linearized fluid equation. It is functionally similar to the wave equation with a non-zero driving term. The standard trick is to go to Fourier space with an ansatz:

$$\rho_1(\vec{r}, t) \propto \int e^{i(\vec{k}\vec{r} - \omega t)} c\left(\vec{k}\right) \mathrm{d}^3 k \tag{202}$$

This gives:

$$-\omega^2 + v_s^2 k^2 - 4\pi G \rho_0 = 0 \tag{203}$$

$$\boxed{\omega^2 = v_s^2 k^2 - 4\pi G \rho_0} \tag{204}$$

This is the **dispersion relation for a linearized gravitational instability**. What is gravity doing here? The minus sign is crucial. Contrast this with the plasma frequency (which comes from electromagnetism which has *two* signs):

$$\omega^2 = v_s^2 k^2 + \frac{4\pi n_e e^2}{m_e} > 0 \tag{205}$$

What happens when $\omega^2 < 0$ in the gravitational case? Let's rewrite ω^2:

$$\omega^2 = v_s^2 \left(k^2 - k_J^2\right) \tag{206}$$

where we define the **Jeans wavenumber** and correspondingly a **Jeans' length** and a **Jeans' mass**:

$$k_J \equiv \sqrt{\frac{4\pi G \rho_0}{v_s^2}} \tag{207}$$

$$\lambda_J = \frac{2\pi}{k_J} = 2 \times \text{ Jean's Radius} \tag{208}$$

$$M_J = \frac{4\pi}{3} \lambda_J^3 \rho_0 \tag{209}$$

This last quantity gives us a scale for gravitational collapse.

Anyways, let's continue!

Naturally, we have two regimes:

- $k > k_J$ or $\lambda < \lambda_J$. In this case, we have $\omega^2 > 0$ and ω is thus real. As a consequence, ρ_1 oscillates like sound waves! This is stable.

- $k > k_J$ or $\lambda > \lambda_J$. In this case, $\omega^2 < 0$, and $\omega \in \mathbb{C}$, giving us an exponential damping in time for the solution: $\rho_1 \propto e^{\pm|\omega|t}$. This grows or decays **exponentially**. This is **Jean's Instability**, or more generally, a **gravitational instability**. There are other instaiblities – Kelvin-Helmholtz, Rayleigh-Jeans, etc.

What are the underlying physics? Balance between pressure (outward) and gravity (inward). When pressure is larger, you get oscillations. When gravity is larger, we have an instability.

By the way, let's do the SAME THING but much easier. This is called a **desert-island** analysis – we are stuck on an island, and we need Jean's instability to get our boat moving. Let's start by looking at the dimensions of the forces and then the timescales:

1. Instability occurs if gravity is larger than pressure, so consider the forces on a fluid element at the boundary of a star of density ρ and radius R.

$$\frac{F_G}{V_\star} \sim \frac{GM\rho}{R^2} \sim GR\rho^2 \tag{210}$$

$$\frac{F_P}{V_\star} \sim \nabla P \sim v_s^2 \nabla\rho \sim v_s^2 \frac{\rho}{R} \tag{211}$$

   - Gravity wins if $F_G > F_P$

$$G\rho^2 R > \frac{v_s^2 \rho}{R} \to \boxed{R > \sqrt{\frac{v_s^2}{G\rho}} \sim \lambda_J} \tag{212}$$

2. Instability occurs if free-fall timescale (gravity) is smaller than sound crossing time:

$$R \sim a_{\text{accel.}} t_g^2 \sim \frac{GM}{R^2} t_g^2 \sim G\rho R t_g^2 \tag{213}$$

$$\boxed{t_g \sim \frac{1}{G\rho}} \tag{214}$$

$$t_p \sim \frac{R}{v_s} \tag{215}$$

$$t_g < t_p \to \frac{1}{\sqrt{G\rho}} < \frac{R}{v_s} \to R > \sqrt{\frac{v_s^2}{G\rho}} \tag{216}$$

3.2 Gravitational Instability in an Expanding Medium

James Jeans definitely did not know about this!

Recall our fluid equations:

$$\rho_0 = \rho_0(t) \text{ and } \rho_1 \equiv \delta\rho_0 \text{ where } \delta \equiv \frac{\rho - \rho_0}{\rho_0} \tag{217}$$

We introduced the "fractional deviation" δ from uniform background. The smallest δ can be is -1, but it has no bound! Just keep that in mind. We also have to take Hubble expansion into account:

$$\vec{v}_0 = H\vec{r} = \frac{\dot{a}}{a}\vec{r} \tag{218}$$

And $\vec{v}_1$ is the peculiar velocity. One thing to note for later that we will need: $\vec{\nabla} \cdot \vec{v}_0 = \frac{\dot{a}}{a} \vec{\nabla} \cdot \vec{r} = 3 \frac{\dot{a}}{a}$. Returning to our fluid equations:

$$\partial_t \rho + \vec{\nabla} (\rho \vec{v}) = 0 \tag{219}$$

To 0th order,

$$\frac{\partial \rho_0}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{v}_0 = \dot{\vec{\rho}}_0 + 3 \frac{\dot{a}}{a} \rho_0 = 0 \tag{220}$$

Solving in our head...we get what we had seen earlier in class!

$$\rho_0 a^{-3} \tag{221}$$

The linearised (first order piece):

$$\dot{\rho}_1 + \rho_0 \vec{\nabla} \cdot \vec{v}_1 + \vec{\nabla} (\rho_1 \vec{v}_0) \tag{222}$$

We can now switch to δ notation...

$$\underbrace{\dot{\rho}_0 \delta}_{-3\dot{a}/a\rho_0} + \rho_0 \dot{\delta} + \rho_0 \vec{\nabla} \cdot \vec{v}_1 + \rho_0 \vec{v}_0 \cdot \vec{\nabla} \delta + \underbrace{\rho_0 \delta \vec{\nabla} \cdot \vec{v}_0}_{3\dot{a}/a\rho_0} = 0 \tag{223}$$

Simplifying....

$$\boxed{\dot{\delta} + \vec{\nabla} \cdot \vec{v}_1 + \left(\frac{\dot{a}}{a} \vec{r} \cdot \vec{\nabla} \right) \delta} \tag{224}$$

The first term is the density change, the second term is a velocity flux, and the third term reminds us that we are in an expanding fluid.

Now take a look at the equation of motion:

To 0th order,

$$\frac{\partial \vec{v}_0}{\partial t} + \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_0 = -\vec{\nabla} \Phi_0 \neq 0 \text{ unlike the static case} \tag{225}$$

There is no Jeans Swindle here! Now to first order, we get...

$$\partial_t \vec{v}_1 + \left(\vec{v}_0 \cdot \vec{\nabla} \right) + \left(\vec{v}_1 \cdot \vec{\nabla} \right) \vec{v}_0 = -\frac{v_s^2 \vec{\nabla} \rho}{\rho_0} - \vec{\nabla} \Phi_1 \tag{226}$$

Note:

$$\left(\vec{v}_1 \cdot \vec{\nabla} \right) \vec{v}_0 = H \left(v_{1x} \partial_x + v_{1y} \partial_y + v_{1z} \partial_z \right) \vec{r} = H \vec{v}_1 = \frac{\dot{a}}{a} \vec{v}_1 \tag{227}$$

This makes:

$$\boxed{\partial_t \vec{v}_1 + \frac{\dot{a}}{a} \left(\vec{r} \cdot \vec{\nabla} \right) \vec{v}_1 + \frac{\dot{a}}{a} \vec{v}_1 = -v_s^2 \vec{\nabla} \delta - \vec{\nabla} \Phi_1} \tag{228}$$

The last is trivial:

$$\boxed{\nabla^2 \Phi_0 = 4\pi G \rho_0 \text{ and } \nabla^2 \Phi_1 = 4\pi G \rho_0 \delta} \tag{229}$$

Just so we are clear, here are the **unperturbed fluid equations**:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} (\rho \vec{v}) = 0 \tag{230}$$

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{\nabla} = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \Phi \tag{231}$$

$$\nabla^2 \Phi = 4\pi G \rho \tag{232}$$

The **perturbed, 1st order** equations:

$$\frac{\partial \rho_1}{\partial t} + \vec{\nabla} \left(\rho_0 \vec{v}_1\right) = 0 \tag{233}$$

$$\frac{\partial \vec{v}_1}{\partial t} = -\frac{v_s^2}{\rho} \vec{\nabla} P - \vec{\nabla} \Phi_1 \tag{234}$$

$$\nabla^2 \Phi_1 = 4\pi G \rho_1 \tag{235}$$

Above is for **a static medium**. When we have an expanding medium, we have $\vec{v}_0 = H\vec{r} = \frac{\dot{a}}{a}\vec{r}$. Also $\vec{\nabla} \cdot \vec{v}_0 = \frac{\dot{a}}{a} \vec{\nabla} \cdot \vec{r} = 3\frac{\dot{a}}{a}$.

Note:

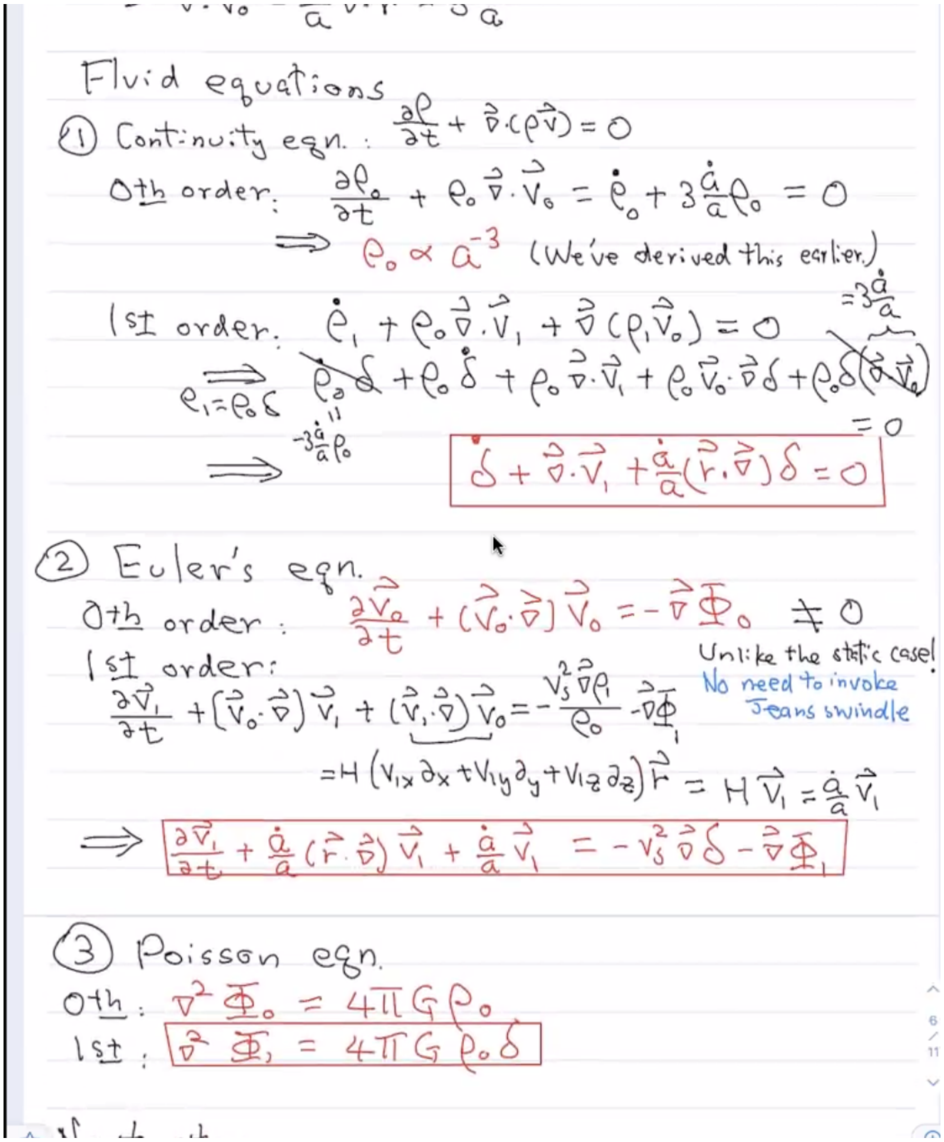

Figure 7: Summary of perturbed and unperturbed equations in an expanding Universe.

Let's work with some of these equations. The best way to proceed is to go to Fourier Space. Also, let's move to co-moving coordinates. The key variables are:

$$\delta(\vec{r},t) = \int d^3k\, e^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}}\delta_k(\vec{r},t) \tag{236}$$

$$v_1(\vec{r},t) = \int d^3k\, e^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}}\vec{v}_k(\vec{r},t) \tag{237}$$

$$\Phi_1(\vec{r},t) \int d^3k\, e^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}}\Phi_k(\vec{r},t) \tag{238}$$

We also have $\vec{r} = a\vec{x}$ where x is our co-moving coordinate and $\vec{k}$ is the co-moving wavenumber. There are a few pieces of math we should do, first:

$$\dot{\delta}(\vec{r},t) = \int d^3k\, e^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}}\,\delta_k(\vec{r},t)\left(\dot{\delta}_k - \frac{\dot{a}}{a}\frac{i\vec{k}\cdot\vec{r}}{a}\delta_k\right) \tag{239}$$

$$\vec{\nabla}\delta(\vec{r},t) = \frac{i\vec{k}}{a}\int d^3k\, e^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}}\,\delta_k(\vec{r},t) \tag{240}$$

$$\vec{\nabla}\cdot\vec{v}_1 = \frac{i\vec{k}}{a}\int d^3k\, e^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}}\,\vec{v}_k(\vec{r},t) \tag{241}$$

Knowing this, we turn to the continuity equation:

$$\left(\dot{\delta}_k - \frac{\dot{a}}{a}\frac{i\vec{k}\cdot\vec{r}}{a}\delta_k\right) + i\vec{k}\cdot\frac{\vec{v}_k}{a} + \frac{\dot{a}}{a}\vec{r}\cdot\frac{i\vec{k}}{a}\delta_k = 0 \tag{242}$$

Simplifying:

$$\boxed{\dot{\delta}_k + \frac{i\vec{k}\cdot\vec{v_k}}{a} = 0} \tag{243}$$

And now, going to the Euler equation:

$$\vec{v}_k - \frac{\dot{a}}{a}\frac{i\vec{k}\cdot\vec{r}}{a}\vec{v}_k + \frac{\dot{a}}{a}\frac{i\vec{r}\cdot\vec{k}}{a}\vec{v}_k + \frac{\dot{a}}{a}\vec{v}_k = -v_s^2\frac{i\vec{k}}{a}\delta_k - i\frac{\vec{k}}{a}\Phi_k \tag{244}$$

Again, simplifying and re-writing a bit (multiplying by a^2), we get:

$$\boxed{\partial_t\left(a\vec{v}_k\right) + iv_s^2\vec{k}\delta_k + i\vec{k}\Phi_k = 0} \tag{245}$$

And lastly, Poisson's equation:

$$\boxed{\frac{k^2}{a^2}\Phi_k = -4\pi G\rho_0\delta_k} \tag{246}$$

Let's use the middle and top equation to get rid of v, giving a single equation for δ. Stare at the top equation, and convince yourself that you should multiply by a^2 and take the time derivative. Doing that, we get:

$$a^2\ddot{\delta}_k + 2a\dot{a} + i\vec{k}\cdot\left(-iv_s^2\vec{k}\delta_k - i\vec{k}\Phi_k\right) = 0 \tag{247}$$

Simplifying and using the Poisson equation:

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k + \frac{v_s^2k^2}{a^2}\delta_k - 4\pi G\rho_0\delta_k = 0 \tag{248}$$

This is suspiciously similar to what we had earlier! Let's make some new definitions:

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k + \frac{v_s^2}{a^2}\left(k^2 - k_J^2\right)\delta_k = 0 \text{ where } k_J \equiv \sqrt{\frac{4\pi G\rho_0 a^2}{v_s^2}} \tag{249}$$

The last definition we had there is nothing but the **co-moving Jeans wavenumber**. We can look at two regimes, starting with the unstable case of $k \ll k_J$:

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k - 4\pi G\rho_0\delta_k = 0 \tag{250}$$

The time dependence here is complicated because of our dependence on a.

- Einstein-de Sitter: Flat, $\Omega_m = 1$. Here, we have $4\pi G\rho = \frac{3}{2}H^2 = \frac{3}{2}\left(\frac{2}{3t}\right)^2 = \frac{2}{3t^2} \rightarrow a \propto t^{2/3}$. We assume a power-law ansatz t^α. Thus:

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta_k = 0 \tag{251}$$

$$\alpha\left(\alpha - 1\right) + \frac{4}{3}\alpha - \frac{2}{3} = 0 \rightarrow \alpha = -1, \frac{2}{3} \tag{252}$$

$$\boxed{\delta\left(t\right) \propto a(t) \propto t^{2/3}} \text{ and } \boxed{\delta(t) \propto t^{-1}} \tag{253}$$

  - We have a growing and decaying part! Compare this with the static gravitational instability, in which $\delta \propto e^{\pm\omega t}$. The origin of this slowing is the friction term in the 2nd order differential equation for δ. **The expansion of the Universe slows down the instability growth to a power law**. We can use this simple calculation to show that the perturbations MUST GROW faster in order for galaxies to form. With dark matter, δ can grow much, much faster since they are not coupled to photons.

- Open Universe Model, $\Omega_m < 1$, $\Omega_\lambda = 0$. The only thing that is different is our a. Recall:

$$a(\theta) \propto \cosh\theta - 1 \tag{254}$$

$$t(\theta) \propto \sinh\theta - \theta \tag{255}$$

  - Our key equation becomes:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\delta - 4\pi G\rho\delta = 0 \tag{256}$$

  - Through some simple, but tedious math:

$$\left(\cosh\theta - 1\right)\frac{d^2\delta}{d\theta^2} + \sinh\theta\frac{d\delta}{d\theta} - 3\delta = 0 \tag{257}$$

$$\text{Growing: } \delta_+ \propto \frac{3\sinh\theta\left(\sinh\theta - \theta\right)}{\left(\cosh\theta - 1\right)^2} - 2 \tag{258}$$

$$\text{Decaying: } \delta_- \propto \frac{\sinh\theta}{\left(\cosh\theta - 1\right)^2} \tag{259}$$

- Closed Case: $\Omega_m > 1$
  - The key equation becomes:

$$\left(1 - \cos\theta\right)\frac{d^2\delta}{d\theta^2} + \sin\theta\frac{d\delta}{d\theta} - 3\delta = 0 \tag{260}$$

  - Which has solutions:

$$\text{Growing: } \delta_+ \propto 2 - \frac{3\sin\theta\left(\theta - \sin\theta\right)}{\left(1 - \cos\theta\right)^2} \tag{261}$$

$$\text{Decaying: } \delta_- \propto \frac{\sin\theta}{\left(1 - \cos\theta\right)^2} \tag{262}$$

3.3 Time Evolution of V and Phi

Let's return to the Einstein-de Sitter model and solve for the other two parameters, $\vec{v}$ and Φ; that is, the perturbed $\vec{v}_k$ and Φ_k.

A standard technique for solving these systems is to decompose our vectors into parallel and perpendicular components:

$$\vec{v} = \vec{v}_\perp + \vec{v}_\parallel \text{ where } \vec{\nabla} \cdot \vec{v}_\perp = 0 \text{ and } \vec{\nabla} \times \vec{v}_\parallel = 0 \tag{263}$$

In Fourier-space, we have:

$$\vec{\nabla} \cdot \vec{v}_\perp \to i\vec{k} \cdot \vec{v}_\perp \left(\vec{k}\right) = 0 \to \vec{v}_\perp \perp \vec{k} \tag{264}$$

$$\vec{\nabla} \times \vec{v}_\parallel = 0 \to \vec{k} \times \vec{v}_\parallel \left(\vec{k}\right) = 0 \to \vec{v}_\parallel \parallel \vec{k} \tag{265}$$

We often call the perpendicular piece the **transverse piece** (divergence-free) and the parallel (curl-free) piece is the **longitudinal piece**.

Now returning to our Euler equation with the perpendicular piece:

$$\partial_t \left(a\vec{v}_\perp\right) = 0 \to \vec{v}_\perp \propto \frac{1}{a(t)} \tag{266}$$

Angular momentum conservation falls out! We care thus about the parallel piece of the velocity, since that is affecting the perturbed equations. For this parallel component:

$$\partial_t \left(av_\parallel\right) + iv_s^2 k\delta_k + ik\Phi_k = 0 \tag{267}$$

We can use the continuity equation to derive the time evolution since we know how $\dot{\delta}$ behaves. The continuty equation tells us:

$$\dot{\delta}_k + \frac{ikv_\parallel}{a} = 0 \to v_\parallel = i\frac{a}{k}\dot{\delta}_k \tag{268}$$

$$v_\parallel \propto t^{1/3} \propto a^{1/2} \tag{269}$$

What about the perturbed Φ_k. From the third equation, we have:

$$\Phi_k \propto \text{ constant in t} \tag{270}$$

3.4 Full Linear Perturbation Theory

The framework in which we were working breaks down if we consider non-fluid things in the Universe! We can move to kinetic theory and work in phase space to overcome this.

Let's start by extending the fluid equations to the "Boltzmann equation," broadly called **kinetic theory**. Additionally, our Newtonian potential Φ needs to become the Einstein equations of General Relativity.

Kinetic Theory

We start with the phase space: a 6-dimensional space of positions $\vec{r}$ and momenta $\vec{p}$. Note that this is a relativistic momentum (conjugate of position). Additionally, we have the phase-space distribution function $f(\vec{x}, \vec{p}, t)$. Integrating this over momentum, you get number density, etc. The collisionless Boltzmann equation (conservation of phase-space density):

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i} = 0 \tag{271}$$

In the absence of collisions, phase-space density should be conserved. In the case when collisions are important (i.e., Thomson scattering at the surface of last scattering), we have the **collisional** Boltzmann equation:

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \left(\frac{\partial f}{\partial t}\right)_c \tag{272}$$

Note that here, f is totally generic and we need to keep track/evolve them simultaneously for each component's f. In general, one has to solve the Boltzmann equation, and that's a total mess in 6 + 1-dimensions.

A standard way to move forward: take velocity (momentum) moments. For example, the 0th order velocity moment is the number density $n = \frac{N}{V} = \int d^3p\, f$ since, recall: $\mathrm{d}N = \mathrm{d}^3x\,\mathrm{d}^3p\, f$.

The first velocity moment of the distribution function f is the **mean** velocity:

$$\langle v \rangle = \frac{\int f\vec{v}\,\mathrm{d}^3p}{\int f\,\mathrm{d}^3p} = \frac{1}{n}\int f\vec{v}\,\mathrm{d}^3p \tag{273}$$

The second velocity moment:

$$\langle v_i v_j \rangle = \frac{1}{n}\int \mathrm{d}^3p\, v_i v_j f \tag{274}$$

Often, we use $\sigma_{ij}^2 = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle$.

The ℓth velocity moment:

$$\langle v_1 v_2 \cdots v_\ell \rangle = \cdots \tag{275}$$

Now that we have our bearings, let's take velocity moments of the Boltzmann equation to see what we find. The **zeroth order velocity moment of the collisionless Boltzmann equation is**:

$$\int \mathrm{d}^3p\ (\text{Boltzmann equation}) = \frac{\partial}{\partial t}\int \mathrm{d}^3p\, f + \int \mathrm{d}^3p\, \vec{v}\cdot\vec{\nabla} f - \underbrace{m\partial_i\Phi}_{\dot{p}_i}\int \mathrm{d}^3p\, \frac{\partial f}{\partial p_i} = 0 \tag{276}$$

The last term is trivial since we can use integration by parts, leaving only the surface terms. The last term is thus 0 since f has to drop off. We could also do this last term by the divergence theorem, assuming that $f \to 0$ at large p_i. This gives:

$$\frac{\partial}{\partial t}n + \vec{\nabla}\underbrace{\int \mathrm{d}^3p\, \vec{v}f}_{n\langle v \rangle} - \underbrace{m\partial_i\Phi}_{\dot{p}_i}\int \mathrm{d}^3p\, \frac{\partial f}{\partial p_i} = 0 \tag{277}$$

$$\boxed{\frac{\partial}{\partial t}n + \vec{\nabla}\left(n\langle \vec{v} \rangle\right) = 0} \tag{278}$$

This looks just look the continuity equation for fluids!

Let's look at the **first velocity moment of the Boltzmann equation**

$$\int \mathrm{d}^3p\, v_j\ (\text{Boltzmann equation}) = \frac{\partial}{\partial t}\int \mathrm{d}^3p\, v_j\, f + \int \mathrm{d}^3p\, v_j v_j \partial_i f + -m\partial_i\Phi \int \mathrm{d}^3p\, v_j \frac{\partial f}{\partial p_i} = 0 \tag{279}$$

This looks messy, but look at each term (first, second, and third) separately:

- Term 1: $\partial_t\left(n\langle v_j \rangle\right) = n\frac{\partial \langle v_j \rangle}{\partial t} + \langle v_j \rangle \frac{\partial n}{\partial t} = n\frac{\partial \langle v_j \rangle}{\partial t} - \underbrace{\langle v_j \rangle \partial_i\left(n\langle v_i \rangle\right)}_{\text{Note!}}$

- Term 2: $\partial_i \left(n\langle v_i v_j\rangle\right) = \partial_i \left(n\sigma_{ij}^2\right) +_i \left(n\langle v_i\rangle\langle v_j\rangle\right) = \partial_i \left(n\sigma_{ij}^2\right) + \underbrace{\langle v_j\rangle\partial_i \left(n\langle v_i\rangle\right)}_{\text{Note!}} + n\langle v_i\rangle\partial_i\langle v_j\rangle$

- Term 3: $-\partial_i\Phi \int \mathrm{d}^3p\, p_j \frac{\partial f}{\partial p_i} = -\partial_i\Phi \int \mathrm{d}^3p \left(\underbrace{\frac{\partial}{\partial p_i}\left[p_j f\right]}_{\text{surface}} - f\frac{\partial p_j}{\partial p_i}\right) = \partial_j\Phi n$

Doing these messy simplifications and collecting our prizes:

$$\frac{\partial\langle v_j\rangle}{\partial t} - \langle v_i\rangle\partial_i\langle v_j\rangle = -\partial_j\Phi - \frac{1}{n}\partial_i\left(n\sigma_{ij}^2\right) \tag{280}$$

This looks familiar! If we drop the ensemble averages $\langle\rangle$, and compare with our Euler equation:

$$\frac{\partial\vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}\Phi - \frac{1}{\rho} - \vec{\nabla}P \tag{281}$$

The key difference between the two are the pressure term versus the second velocity moment term σ_{ij}^2. The right side is the stress tensor.

A Archival Lectures (C.P. Ma)

A.1 Lecture 1

A.1.1 The Cosmological Principle

The **Cosmological Principle** states that the universe is spatially *isotropic* (looks the same in all directions) and *homogeneous* (has constant density everywhere) on large scales. The **Perfect Cosmological Principle** states that the universe is also *temporally* isotropic and homogeneous (a steady state universe). This is unlikely because it doesn't describe the Cosmic Microwave Background (CMB). The CMB and Hubble's Law are both provide evidence for isotropy and homogeneity.

A.1.2 Hubble's Law (1929)

Hubble's Law is an empirical law stating that, on large scales, recessional velocity is proportional to distance from observer.

$$\boxed{v = Hr}$$

where H, the Hubble parameter, is not constant, but can vary slowly with time. By convention, H is often expressed as $H = 100 \cdot h\frac{km}{s\cdot Mpc}$, where 1 parsec (pc) $\approx 3\cdot 10^{18}cm = 3.26ly$, is the distance at which 1 AU appears as 1 arcsec on the sky. The Hubble Space Telescope Key Project (Freedman et al. ApJ 553, 47, 2001) measured the present day value of Hubble Constant $H_0 = 72 \pm 8\frac{km}{s\cdot Mpc}$, giving us that the current timescale for the expansion of the universe is $H_0^{-1} \approx \frac{h}{10^{11}}yrs \approx 9.778h^{-1}Gyrs$.

A.1.3 The Scale Factor

$a(t)$ relates physical (r) and *comoving* (x) coordinates in an expanding universe:

$$r = a(t)x \tag{282}$$

$$\dot{r} = \dot{a}x + a\dot{x} = \underbrace{\frac{\dot{a}}{a}}_{\equiv H} r + \underbrace{a\dot{x}}_{\equiv v_p} \tag{283}$$

$$\tag{284}$$

Thus, the two components of physical velocity are H (the Hubble expansion parameter) and v_p (the peculiar velocity, or motion relative to expansion) By convention, $t_0 \equiv$ today and $a(t_0) = 1$.

A.1.4 The Friedmann Equations

The Friedmann Equation is an equation of motion for $a(t)$. A rigorous derivation requires General Relativity, but we can fake it with a quasi-Newtonian derivation. We will model the universe as an *adiabatically* ($\Delta S = 0$) expanding, isotropic, homogeneous medium. Isotropy allows us to use r as a scalar. Consider a thin, expanding spherical shell of radius a. Birkhoff's Theorum states that even in GR, the motion of such a shell depends only on the enclosed mass $M = 4\pi 3 a^3 \rho$. Thus, the energy per unit mass per unit length is:

$$E = \overbrace{\frac{1}{2}\dot{a}^2}^{Kinetic} \overbrace{-\frac{G \cdot M}{a}}^{Potential} = \frac{1}{2}\dot{a}^2 - \frac{4\pi}{3} G \rho a^2.$$

We define $k \equiv -\frac{2E}{c^2}$, and we will show later that k is a measure of the curvature of the universe:

$$k \begin{cases} > 0 & for\ E < 0\ (bound) \\ = 0 & for\ E = 0\ (critical) \\ < 0 & for\ E > 0\ (unbound) \end{cases}$$

where k has units of $\frac{1}{length^2}$ if a is dimensionless. Substituting k into the above energy equation, and solving for $\frac{\dot{a}}{a}$, we get:

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2}}$$

(1^{st} Friedmann Equation)

This is a statement of conservation of E. The first law of thermodynamics ($\Delta S = 0$) requires that any system with positive pressure must lose energy as the volume enclosing it expands. Thus, if U is our internal energy and P is our pressure:

$$\frac{dU}{dt} = -P\frac{dV}{dt}.$$

In an expanding universe, $U = EV \cdot V = \rho a^3$, where ρ is the energy density of the universe. P is the pressure of the photon gas, so:

$$\dot{\rho} a^3 + 3\rho a^2 \dot{a} = -P 3 a^2 \dot{a},$$

which simplifies to:

$$\boxed{\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P)}$$

(2^{nd} Friedmann Equation)

This is a a statement of the temperature loss of the universe due to adiabatic expansion.

Finally, $\frac{d}{dt}$(1^{st} Friedmann Equation) gives us:

$$2\dot{a}\ddot{a} = \frac{8\pi}{3} G \frac{d}{dt}(\rho a^2) = \frac{8\pi}{3} G a^2 (\dot{\rho} + 2\frac{\dot{a}}{a}\rho).$$

Substituting the 2^{nd} Friedmann Equation for $\dot{\rho}$:

$$2\dot{a}\ddot{a} = \frac{8\pi}{3} G a^2 (-\frac{\dot{a}}{a}\rho - 3\frac{\dot{a}}{a}P) = -\frac{8\pi}{3} G \frac{\dot{a}}{a}(\rho + 3P).$$

Now we have our 3^{rd} Equation:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}$$

$$(3^{rd} \text{ Friedmann Equation})$$

The 3^{rd} Friedmann Equation relates the acceleration of the expansion of the universe to the pressure of photon gas and the density of the universe. Note that if $3P \leq -\rho$, we have an accelerating universe.

Compare the 3^{rd} Friedmann Equation to the Newtonian equation for gravity, with $M = \frac{4\pi}{3}\rho x^3$ and $\rho_{eff} = \rho + 3P$:

$$\ddot{x} = \frac{-G \cdot M}{x^2} = -4\pi 3G\rho x$$

A.2 Lecture 2

A.2.1 The Friedmann Equations, continued

Recall we had the following equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = 8\pi 3G\rho - \frac{k}{a^2}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G(\rho + 3P)$$

To close the equations, we need to relate P and ρ with an equation of state:

$$\boxed{P = w\rho(c^2)}$$

Note that we will generally set $c = 1$ in this class. Combined with (2), this gives us:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(1 + w)\rho \tag{285}$$

$$\dot{\rho}\rho = -3(1 + w)\frac{\dot{a}}{a} \tag{286}$$

$$\rho \propto a^{-3(1+w)} \tag{287}$$

$$\tag{288}$$

Note that we've assumed $\dot{w} = 0$, which is okay most of the time. Some special cases of interest are:

- Pressure-less "dust" $P = 0$, $w = 0 \Rightarrow \rho \propto a^{-3}$ because volume goes as $V \propto \frac{1}{a^3}$.
- Relativistic particles (photons, bosons): $w = \frac{1}{3}$, $P = \frac{\rho}{3} \Rightarrow \rho \propto a^{-4}$ because $VV \propto \frac{1}{a^3}$, and energy is given by $E \propto \frac{1}{a}$.
- (Λ)/Dark Energy: $w = -1$, $P = -\rho \Rightarrow \rho =$ constant in time.

To get density (ρ) as a function of time, want to solve for w.

A.2.2 Critical Density

We define ρ_{crit} to be the critical density at which $k = 0$ (and $E = 0$):

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

Today, we measure: $\rho_{crit,0} = \frac{3H_0^2}{8\pi G} = \frac{3}{8\pi}\frac{\left(100h\frac{km}{sMpc}\right)^2}{G} = 2.78 \cdot 10^{11}h^2\frac{M_\odot}{Mpc^3} = 1.88 \cdot 10^{-29}h^2\frac{g}{cm^3}$. Note that the mass of the sun is $M_\odot = 2 \cdot 10^{33}g$, and the mass of the proton is $M_p = 1.67 \cdot 10^{-24}g$.

A.2.3 Density Parameter

Ω measures the ratio of the density of the universe to the critical density:

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\rho(t)}{3H^2(t)}$$

$$\Omega \begin{cases} \leq 1 & \Rightarrow open\ (k \leq 0) \\ = 1 & \Rightarrow flat\ (k = 0) \\ \geq 1 & \Rightarrow closed\ (k \geq 0) \end{cases}$$

In general, Ω can consist of multiple components: $\Omega = \sum_i \Omega_i$ e.g.

$$\Omega \begin{cases} r & = radiation \\ m & = matter\ (dark\ and\ luminous) \\ b & = baryons\ (dark\ and\ luminous) \\ \nu & = neutrinos \\ \Lambda & = dark\ energy \end{cases}$$

$\Omega = 1$ is an unstable equilibrium; any perturbation from $\Omega = 1$ in the early universe ensures Ω is far from 1 today. That we measure $\Omega_m \approx 0.3$ today implies that the early universe must have been extremely finely tuned.

A.2.4 Evolution of Hubble Parameter

Today (at $a = 1$): $k = \frac{8\pi}{3}G\rho_0 - H_0^2 = H_0^2(\Omega_0 - 1)$. Using the 1^{st} Friedmann equation (1), we have:

$$\boxed{H^2 = H_0^2(\frac{\Omega_0}{a^{3+3w}} + \frac{1-\Omega_0}{a^2})}$$

(THE Friedmann Equation)

where $H_0 \equiv \sqrt{\frac{8\pi G\rho}{3\rho_{crit}}}$, and it is understood that $\Omega_0 = \sum_i \Omega_{i,0}$. Each Ω_i has it's own w_i, so really $\frac{\Omega_0}{a^{3+3w}} = \sum_i \frac{\Omega_{i,0}}{a^{3+3w_i}}$.

A.2.5 Evolution of Density Parameter

We'll show in PS#1 that for any single component:

$$\frac{1-\Omega(a)}{\Omega(a)} = \frac{1-\Omega_0}{\Omega_0}a^{1+3w}$$

Plotting $\Omega(a)$, we will find that for early a, Ω is *extremely* close to 1.

A.2.6 Evolution of Scale Factor: Solving the Friedmann Equation

The evolution of the expansion of the universe is governed by:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2}$$

We can apply this to several models of the universe:

- The Einstein-deSitter (flat) Model: $k = 0$, $\Omega(a) = \Omega_0 = 1$. Using that $\rho \propto a^{-3(1+w)}$, we have:

$$\left(\frac{\dot{a}}{a}\right)^2 \propto a^{-3(1+w)} \quad (289)$$

$$a^{-1}a^{\frac{3}{2}(1+w)}da \propto dt \quad (290)$$

$$a^{-\frac{3}{2}(1+w)} \propto t \quad (291)$$

$$(292)$$

$\boxed{a(t) \propto t^{\frac{2}{3(1+w)}}}$ Thus, the rate of expansion of the universe depends on w:

- (a) The matter-dominated era: $\Omega \approx \Omega_m \Rightarrow w = 0, P = 0, a \propto t^{\frac{2}{3}}$.
- (b) The radiation-dominated era: $\Omega \approx \Omega_r \Rightarrow w = \frac{1}{3} \Rightarrow a(t) \propto t^{\frac{1}{2}}$.
- (c) The Λ-dominated era: $\Omega \approx \Omega_\Lambda \Rightarrow w = -1, P = -\rho, \rho$ constant in time $\Rightarrow a(t) \propto e^{Ht}$, where H is now actually a constant. This is exponential inflation. We used to think that this only happened early on (like 10^{-34} seconds), but now we think that this has also been happening recently. Next time, we will do the harder two cases: open and closed.

A.3 Lecture 3

A.3.1 Evolution of Scale Factor: Solving the Friedmann Equation, continued

Last time we did (1), the flat universe:

Einstein-deSitter, $k = 0$ (flat), $\Omega = 1, \Rightarrow a \propto t^{\frac{2}{3(1+w)}}$. In general, we consider the evolution of the universe to be *matter dominated* if $a \propto t^{\frac{2}{3}}$, *radiation dominated* if $a \propto t^{\frac{1}{2}}$, and Λ *dominated* if $a \propto e^{1+t}$ (that is, H is constant).

Open, $k < 0$, $\Omega_0 < 1$, $\Omega_{0,\Lambda} = 0$, $\Omega_{0,M} < 1 \Rightarrow$ our Friedmann Equation $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2}$ becomes:

$$\dot{a}^2 = H_0^2\left(\frac{\Omega_0}{a^{1+3w}} + 1 - \Omega_0\right)$$

Since we are matter dominated, $w = 0$. Solving for $a(t)$:

$$\int_0^{a_f} \frac{da}{\sqrt{\Omega_0 a + 1 - \Omega_0}} = \int_0^{t_f} H_0 \cdot dt \quad (293)$$

$$\int_0^{a_f} \frac{a \cdot da}{\sqrt{1-\Omega_0}\sqrt{a^2 + \frac{\Omega_0}{1-\Omega_0}a}} = H_0 t_f \quad (294)$$

$$(295)$$

Let's define $2\alpha \equiv \frac{\Omega_0}{1-\Omega_0}$. Then:

$$\int_0^{a_f} \frac{a \cdot da}{\sqrt{1-\Omega_0}\sqrt{(a+\alpha)^2 - \alpha^2}} = H_0 t_f$$

Defining $a' = a + \alpha$, we get:

$$\int_0^{a_f+\alpha} \frac{(a'-\alpha)da'}{\sqrt{1-\Omega_0}\sqrt{a'^2 - \alpha^2}} = H_0 t_f \quad (296)$$

$$\int_0^{a_f+\alpha} \frac{(a-\alpha)da}{\sqrt{1-\Omega_0}\sqrt{a^2 - \alpha^2}} = H_0 t_f \quad (297)$$

$$(298)$$

Now we define $\alpha\cosh\theta \equiv a'$, $\alpha\sinh\theta\, d\theta = da$, so that $\sqrt{a'^2-\alpha^2} = \alpha\sinh\theta$. We'll be lazy and just say that $a'_f \to \frac{\Omega_0}{2(1-\Omega_0)(\cosh\theta-1)}$ because there was nothing special about time a'_f. Then we have:

$$\int_0^{\theta_f} \frac{\alpha^2(\cosh\theta-1)\sinh\theta\, d\theta}{\sqrt{1-\Omega_0}\alpha\sinh\theta} = H_0 t_f \tag{299}$$

$$\frac{\alpha}{1-\Omega_0}(\sinh\theta-\theta)\Big|_0^{\theta_f} = H_0 t_f \tag{300}$$

$$\tag{301}$$

Plugging α back in:

$$H_0 t_f = \frac{\Omega_0}{2(1-\Omega_0)^{\frac{3}{2}}}(\sinh\theta-\theta) \tag{302}$$

$$t(\theta) = \frac{\Omega_0}{2H_0(1-\Omega_0)^{\frac{3}{2}}}(\sinh\theta-\theta) \tag{303}$$

$$\tag{304}$$

Thus we have a parametric relationship between a and t with a dummy variable θ:

$$a(\theta) = \frac{\Omega_0}{2(1-\Omega_0)}(\cosh\theta-1) \tag{305}$$

$$t(\theta) = \frac{\Omega_0}{2\,H_0(1-\Omega_0)^3 2}(\sinh\theta-\theta) \tag{306}$$

$$\tag{307}$$

Note that a has no maximum and expands forever.

Closed, $k>0$, $\Omega_0>1$, $\Omega_\Lambda=0$. As usual, we start with the Friedmann Equation:

$$(\frac{\dot{a}}{a})^2 = \underbrace{\overbrace{\frac{8\pi}{3}}^{positive}G\rho - \overbrace{\frac{k}{a^2}}^{positive}}_{can\ be\ 0}$$

Therefore, in this model it is possible for $\dot{a}=0$. We'll again solve for a matter-dominated era, $w=0$. The solution is similar, to the open model, but $(1-\Omega_0<0)$, so $\theta\to i\,\theta$. Thus our solution:

$$a(\theta) = \frac{\Omega_0}{2(1-\Omega_0)}(\cos\theta-1) \tag{308}$$

$$t(\theta) = \frac{\Omega_0}{2H_0(1-\Omega_0)^{\frac{3}{2}}}(\sin\theta-\theta) \tag{309}$$

$$\tag{310}$$

is a cycloid solution. That is, a oscillates, with a maximum at $\theta=\pi$.

A.3.2 Deceleration Parameter q

$$q \equiv -\ddot{a}a\dot{a}^2$$

Note that q is *dimensionless*, and has a negative sign. The minus is there because historically, the universe was thought to be decelerating. Thus, $q<0 \Rightarrow$ *acceleration*, and $q>0 \Rightarrow$ *deceleration*. We can find out what q does from the 2^{nd} Friedmann Equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G(\rho+3P)$$

Recall that $P = w\rho$, $\Omega = \frac{\rho}{\rho_{crit}}$, $\rho_{crit} = \frac{3H^2}{8\pi G}$, $H^2\Omega = \frac{8\pi}{3}G\rho$. Thus, the above equation becomes:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\rho(1+3w) = -\frac{H^2\Omega}{2}(1+3w)$$

Substituting in our friend, q:

$$q = \frac{\Omega}{2}(1+3w) = \sum_i \frac{\Omega_i}{2}(1+3w_i)$$

$$\boxed{q = \frac{\Omega_m}{2} - \Omega_\Lambda + \Omega_r}.$$

Note that today, $\Omega_r \ll 1$. That we are accelerating $\Rightarrow q < 0 \Rightarrow \Omega_{0,\Lambda} \neq 0$.

A.3.3 Redshift

$$\frac{\lambda_{obs}}{\lambda_{emit}} \equiv 1 + z$$

Or alternately,

$$\boxed{1 + z = \frac{1}{a}}$$

z is what we call the *redshift*. Reionization happened somewhere around $z = 17 \pm 6$, and recombination around $z = 1091$.

A.3.4 Time-Redshift Relations and the Age of the Universe

We seek a relation between t and z. Beginning with the Friedmann Equation:

$$H^2 = H_0^2\left[\frac{\Omega_0}{a^{3+3w}} + \frac{1-\Omega_0}{a^2}\right] \quad (311)$$

$$\frac{1}{a}\frac{da}{dt} = H_0\sqrt{\frac{\Omega_0}{a^{3+3w}} + \frac{1-\Omega_0}{a^2}} \quad (312)$$

$$dt = \frac{1}{H_0}\left(\frac{1}{a}\right)\frac{da}{\sqrt{\frac{\Omega_0}{a^{3+3w}} + \frac{1-\Omega_0}{a^2}}} \quad (313)$$

$$dt = -\frac{1}{H_0}\frac{dz}{(1+z)\sqrt{\Omega_{0,m}(1+z)^3 + \Omega_{0,\Lambda} + (1-\Omega_{0,m}-\Omega_{0,\Lambda})(1+z)^2}} \quad (314)$$

$$(315)$$

We can calculate the time since the Big Bang at redshift z_1 by solving the integral:

$$t_1 = -\frac{1}{H_0}\int_{z_1}^{\infty}\frac{dz}{1+z}\frac{1}{\sqrt{\Omega_{0,m}(1+z)^3 + \Omega_{0,\Lambda} + (1-\Omega_{0,m}-\Omega_{0,\Lambda})(1+z)^2}}$$

For the Age of the Universe, set $z_1 = 0$. In the special case of a flat (Einstein-deSitter) universe:

$$t_0 = \frac{1}{H_0}\int_0^{\infty}\frac{dz}{(1+z)^{\frac{5}{2}}} = 23H_0$$

If $H_0^{-1} \approx 10Gyrs$, then $t_0 \approx 6.7h^{-1}Gyrs$, so the Hubble constant needs to be pretty small to get reasonable estimates of the age of the universe.

A.4 Lecture 4

A.4.1 Time-Redshift Relations and the Age of the Universe

Last time we found the age of a flat universe. in a flat (Einstein-deSitter) universe: $k = 0$, $\Omega_m = 1$, $\Omega_\Lambda = 0$, so:

$$t_0 = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)^{\frac{5}{2}}} = \frac{2}{3H_0} \approx 6.7h^{-1}Gyr$$

Alternatively, recall that for a matter-dominated era, $a(t) = (tt_0)^{\frac{2}{3}}$. Thus, $H = \frac{\dot{a}}{a} = \frac{2}{3t} \Rightarrow t_0 = \frac{2}{3H_0}$.

If we have $\Lambda \neq 0$: $k = 0$, $\Omega_{0,M} + \Omega_{0,\Lambda} = 1$, then:

$$t_0 = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{0,M}(1+z)^3 + \Omega_{0,\Lambda}}}$$

Assuming $0.1 \leq \Omega_{0,M} \leq 1$, this integral is solvable:

$$t_0 = \frac{2}{3H_0}\frac{1}{\sqrt{1-\Omega_{0,M}}}\ln\left(\frac{1+\sqrt{1-\Omega_{0,M}}}{\sqrt{\Omega_{0,M}}}\right) \approx \frac{2}{3H_0}(0.7\Omega_{0,M} + 0.3 - 0.3\Omega_{0,\Lambda})^{-0.3} \approx \frac{2}{3H_0}\Omega_{0,M}^{-0.3}$$

Generally, in a flat universe, $t_0 \propto \frac{1}{H_0}$. If $\Omega_0 \leq 1$, it will be longer.

In an open universe: $k < 0$, $\Omega_0 < 1(\Omega_{0,\Lambda} = 0)$. Recall:

$$t(\theta) = \tfrac{\Omega_0}{2H_0(1-\Omega_0)^{\frac{3}{2}}}(\sinh\theta - \theta), \quad a(\theta) = \tfrac{\Omega_0}{2(1-\Omega_0)}(\cosh\theta - 1)$$

So today:

$$a(\theta_0) = \Omega_0 2(1-\Omega_0)(\cosh\theta_0 - 1) = 1 \tag{316}$$

$$\cosh\theta_0 = \frac{2(1-\Omega_0)}{\Omega_0} + 1 = \frac{2-\Omega_0}{\Omega_0} \tag{317}$$

$$\sinh\theta_0 = \sqrt{\cosh^2\theta_0 - 1} = \frac{2}{\Omega_0}\sqrt{1-\Omega_0} \tag{318}$$

$$t_0 = t(\theta_0) - \frac{\Omega_0}{2H_0(1-\Omega_0)^{\frac{3}{2}}}\left[\frac{2}{\Omega_0}\sqrt{1-\Omega_0} - \cosh^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right)\right] \tag{319}$$

$$\tag{320}$$

$$\boxed{t_0 = \frac{1}{H_0}\left[(1-\Omega_0)^{-1} - \frac{1}{2}(1-\Omega_0)^{-\frac{3}{2}}\cosh^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right)\right]}$$

Thus $t_0 > \frac{2}{3H_0}$ for $\Omega_0 \leq 1$, and $t_0 = \frac{1}{H_0}$ for $\Omega_0 = 0$ (an empty universe).

In a closed universe: $k > 0$, $\Omega_0 > 1$, $(\Omega_{0,\Lambda} = 0)$. Recall:

$$t(\theta) = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{\frac{3}{2}}}(\theta - sin\theta) \tag{321}$$

$$a(\theta) = \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\theta) \tag{322}$$

$$\tag{323}$$

Thus, today:

$$a(\theta_0) = \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\theta) = 1$$

$$\boxed{t_0 = \frac{1}{H_0}\left[(1-\Omega_0)^{-1} + \frac{1}{2}\Omega_0(\Omega_0 - 1)^{-\frac{3}{2}}\cos^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right)\right]}$$

A.4.2 The Robertson-Walker Metric

Lorentz invariance dictates that two inertial frame (x, y, z, t) and (x', y', z', t'), with one moving with respect to the other at velocity $\hat{v} = v\hat{x}$, are related by:

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

where $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Note, to give a taste of tensor forms, this all may be written as $x'^{\alpha} = \Lambda^{\alpha}_{\beta} x^{\beta} + I^{\alpha}_{0}$.

Remember the Lorentz invariant interval, which is conserved between frames:

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

Light travels a $ds^2 = 0$ path. In tensor form, this equation looks like:

$$ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

where $g_{\alpha\beta}$, the metric tensor, is given by:

$$g_{\alpha\beta} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Look at Weinberg, Ch. 13 for full proof, but for a homogeneous, γ-isotropic space, the metric looks like:

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega\right]$$

where r is a radial direction (in comoving coordinates), and $d\Omega = d\theta^2 + \sin^2\theta d^2\phi$ is the differential angle seperation of two points in space. As usual, k is the measure of curvature.

The $k = 0$ Model:

$$dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) = dx^2 + dy^2 + dz^2$$

so we recover the Minkowski metric for flat space, using comoving coordinates.

The $k > 0$ (closed) Model:

We get a coordinate singularity at $r = \frac{1}{\sqrt{k}}$, so this universe has a finite volume. For $k > 0$, we need to define "Polar Coordinates" in 4-D (to describe a 3-sphere embedded in 4-D). Here is a comparison of how we define polar coordinates for a 3-sphere in 4-D versus for a 2-sphere in 3-D:

$$\begin{array}{cc} 3-sphere & 2-sphere \\ (x, y, z, w) \Leftrightarrow (R, \alpha, \beta, \gamma) & (x, y, z) \Leftrightarrow (R, \theta, \phi) \\ w = R\cos\alpha & z = R\cos\theta \\ z = R\sin\alpha\cos\beta & y = R\sin\theta\cos\phi \\ y = R\sin\alpha\sin\beta\cos\gamma & x = R\sin\theta\sin\phi \\ x = R\sin\alpha\sin\beta\sin\gamma & \\ x^2 + y^2 + z^2 + w^2 = R^2 & x^2 + y^2 + z^2 = R^2 \end{array}$$

Take a line element on a 2-sphere:

$$d\gamma = R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Changing variables for $v \equiv \sin\theta$:

$$dv = \cos\theta d\theta = \sqrt{1 - v^2} d\theta$$

Then $d\theta^2 = \frac{dv^2}{1-v^2}$, so rewriting our line element, we get:

$$d\gamma^2 = R^2 \left(\frac{dv^2}{1-v^2} + v^2 d\phi^2 \right)$$

For a 3-sphere,

$$d\gamma^2 = R^2(d\alpha^2 + \sin^2 \alpha d\Omega^2)$$

where $d\Omega^2 \equiv d\beta^2 + \sin^2 \beta d\gamma^2$. Again, using a change of variables so that $v \equiv \sin\alpha$, $d\alpha = \frac{dv}{\sqrt{1-v^2}}$, we get that:

$$\boxed{d\gamma^2 = R^2 \left(\frac{dv^2}{1-v^2} + v^2 d\Omega^2 \right)}$$

This is what Robertson-Walker showed.

A.5 Lecture 5

Finishing the Robertson-Walker Metric

We've been following Weinberg's derivation to show there are discrete metrics. We'll start with:

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

Recall that k is the *curvature constant* and r is in *comoving* coordinates. If we define $U \equiv \sinh\theta$, then:

$$dU = \cosh\theta \cdot d\theta \tag{324}$$

$$d\theta = \frac{dU}{\sqrt{1+U^2}} \tag{325}$$

$$\tag{326}$$

The beauty of this metric is that we derived it only using symmetry (no dynamics). There is an alternate way of writing the metric above:

$$ds^2 = c^2 dt^2 - a^2 \left[d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2 \right]$$

where:

$$S(\chi) \equiv \begin{cases} \chi & (for\ k = 0,\ \chi = r) \\ k^{-\frac{1}{2}} \sin\left(k^{\frac{1}{2}}\chi\right) & (for\ k > 0) \\ (-k)^{-\frac{1}{2}} \sinh\left(\sqrt{-k}\chi\right) & (for\ k < 0) \end{cases}$$

Comoving Radial Distance vs. Redshift: the Hubble Diagram

This is the fundamental diagram behind using a standard candle (supernova) to infer the curvature of the universe. What we want here is an algebraic expression relating r and z. We'll look at light propagation ($ds^2 = 0$), and take a radial path ($d\theta = d\phi = 0$) to know from the Robertson-Walker metric that:

$$c^2 dt^2 = a^2 \frac{dr^2}{1-kr^2}$$

Separating out our r dependencies and our t dependencies and integrating, we get:

$$\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_1}^{t_0} \frac{c \cdot dt}{a(t)}$$

- For the flat, $k = 0$, matter-dominated model ($w = 0$, $\Omega_{0,\Lambda} = 0$, $\Omega_{0,M} = 1$), we'll start with the Friedmann Equation:

$$H = \frac{\dot{a}}{a} = H_0 \sqrt{\frac{\Omega_0}{a^{3+3w}} + \frac{1-\Omega_0}{a^2}}$$

Recognizing that $a = \frac{1}{1+z}$, we have:

$$H_0 dt = -\frac{dz}{(1+z)\sqrt{\Omega_0(1+z)^{3+3w} + (1-\Omega_0)(1+z)^2}}$$

These integrals evaluate to (in comoving r):

$$r_1 = \int_{t_1}^{t_0} \frac{c \cdot dt}{a(t)}$$

$$\boxed{r_1 = \frac{2c}{H_0}\left[1 - (1+z)^{-\frac{1}{2}}\right]}$$

Here, $\frac{2c}{H_0}$ is called the Hubble distance and is the definition of how far away we can possibly see–how far light could have traveled since the beginning of time. Notice that as $z \to \infty$, $r = \frac{2c}{H_0} \approx 10000h^{-1}Mpc$.

- For the open, $k < 0$, $\Omega_0 < 1$, $w = 0$ model, we'll substitute $\sqrt{-k}r = \sinh\chi$, $dr = \frac{\cosh\chi d\chi}{\sqrt{-k}}$ for r in the integral above:

$$r = \int_{-}^{\chi_1} \frac{\cosh\chi d\chi}{\sqrt{-k}\sqrt{1+\sinh^2\chi}} \tag{327}$$

$$\frac{\chi_1}{\sqrt{-k}} = \sinh^{-1}(\sqrt{-k}r_1)\sqrt{-k} \tag{328}$$

$$\tag{329}$$

From this, we can use *Mattig's Formula*, which states for $w = 0$, arbitrary $\Omega_{0,M} = \Omega_0$, that:

$$r = \frac{2c}{H_0\Omega_0^2}\frac{1}{1+z}\left[\Omega_0 z + (\Omega_0 - 2)(\sqrt{1+\Omega_0 z} - 1)\right]$$

In general, for arbitrary $\Omega_{0,M}, \Omega_{0,\Lambda}$ (we'll derive this in PS#3), one can show that, in comoving r:

$$\boxed{r = \frac{1}{\sqrt{|k|}} sinn\left(\frac{c}{H_0}\sqrt{|k|}\int_0^z \frac{dz'}{\sqrt{\Omega_{0,M}(1+z)^3 + \Omega_{0,\Lambda} + (1-\Omega_{0,M}-\Omega_{0,\Lambda})(1+z)^2}}\right)}$$

where *sinn* is a funny function:

$$sinn \equiv \begin{cases} \sin & for\ k > 0 \\ absent & for\ k = 0 \\ \sinh & for\ k < 0 \end{cases}$$

Note that when $w = 0$, for arbitrary $\Omega_{0,M}$, we recover Mattig's Formula.

Angular Diameter Distance

The angular diameter distance is a useful quantity which relates the physical size or separation of objects to the angular size on the sky. For normal, Euclidean geometries, this is trivial trigonometry. For a curved universe, this is not trivial. For example, in some universes, an object pulled far enough away may actually start looking larger (have a larger angular diameter) than a closer object!

This brings us to the end of the smooth universe. We've seen $a(t)$, but we have not seen any perturbations off of that. Similarly, we've seen $\rho(t)$, but no spatial components of density. We will begin to talk about perturbations off of the Smooth Universe, and we will call this:

The Bright Side of the Universe

Let's do a quick tour of the the particles out there to give some context to what we're talking about. Take a look at *Review of Particle Physics*, which is also at *http://pgd.lbl.gov*, for some more detailed information.

First, we'll talk about *fermions*. Fermions come in two varieties: Leptons and Quarks. Quarks are hadrons and group together to from baryons (made of 3 quarks) and mesons (made of quark-antiquark pairs).

Elementary Particles: Fermions (spin $\frac{1}{2}$)

$Leptons$	$Charge$	$Mass$	$Mean\ Lifetime$
e	-1	$0.51099892 \pm 0.00000004 MeV$	∞
ν_e	0	$< 3eV$	∞
μ	-1	$106MeV$	$2.2\mu s,\ \mu^- \to e^- \bar{\nu}_e \nu_p$
ν_μ	0	$< 190keV$	∞
τ	-1	$1.78GeV$	$2.9 \cdot 10^{-13}$
ν_τ	0	$< 18.2MeV$	∞

$Quarks$ (3 colors each)	$Charge$	$Mass$
U	$\frac{2}{3}$	$1.5 \to 4MeV$
d	$-\frac{1}{3}$	$4 \to 8MeV$
s	$-\frac{1}{3}$	$80 \to 130MeV$
c	$\frac{2}{3}$	$1.15 \to 1.35GeV$
b	$-\frac{1}{3}$	$4.1 \to 4.4GeV$
t	$\frac{2}{3}$	$174.3 \pm 5.1GeV$

A.6 Lecture 6

The Bright Side, continued

$Gauge\ Bosons$	$spin$	$charge$	$mass$	
γ	1	0	0	$(carries\ electroweak\ force)$
$w^\pm$	1	± 1	$80.4GeV$	$(carries\ electroweak\ force)$
Z^0	1	0	$91.2GeV$	$(carries\ electroweak\ force)$
$gluons$	1	0	$don't\ know$	$(carries\ strong\ force)$
$Higgs$	0	0	$> 114.4GeV$	
$graviton$	2	0	0	

$Baryons$	$quark\ content$	$charge$	$mass$	$lifetime$
$proton$	uud	$+1$	$938.07003MeV$	$\geq 10^{32} yrs$
$neutron$	udd	0	$939.56536MeV$	$885.7sec$
Λ	uds	0	$1.11GeV$	$10^{-10} sec$
$\Sigma^{+,0,-}$	uus, uds, dds	$+1, 0, 1$	$1.2GeV$	$10^{-10}, 10^{-19}$
Ξ^0	uss	0	$1.3GeV$	10^{-10}
Ξ^-	dss	-1	$1.3GeV$	10^{-10}
Ω^-	sss	-1	$1.67GeV$	?

$Mesons(q\bar{q},\ Spin\ 0)$	$charge$	$quark$	$mass$	$lifetime$
$\Pi^\pm$	± 1	$u\bar{d}, d\bar{u}$	$140MeV$	$10^{-8}s$
Π^0	0	$u\bar{u}, d\bar{d}$	$135MeV$	$10^{-16}s$
$K^\pm$	± 1	$u\bar{s}, \bar{u}s$	$494MeV$	$10^{-8}s$
$(Spin\ 1)$				
J, Ψ	0	$c\bar{c}$	$3.1GeV$	10^{-20}
Υ	0	$b\bar{b}$	$9.5GeV$	10^{-20}

Fundamental Interactions	*Gravity*	*Weak*	*EM*	*Strong*
Classical	*Newton/Einstein*	$-$	*Maxwell*	$-$
Quantum	?	$V-A$ *Theory(flawed)*	*QED, Gauge*	$QCD[SU(3)]$

Useful Scales/Conversion tricks

In particle physics, $E \sim \frac{1}{length}$, because $\hbar c = 1$ (which had units $\frac{E}{cm}$). In condensed matter physics, $E \sim T$.

The Planck mass is the mass scale at which the effect of gravity and quantum effects are comparable. We can get an estimate of the Planck mass by setting the Schwarzschild radius ($R_{sch} = \frac{2Gm}{c^2}$) equal to the Compton wavelength of a particle ($\lambda = \frac{hc}{mc^2} = \frac{2\pi\hbar c}{mc^2}$). Dropping our constants (2's, π's), we get:

$$\frac{Gm_{planck}}{c^2} \sim \frac{\hbar c}{m_{planck}c^2}$$

$$m_{planck} = \sqrt{\frac{\hbar c}{G}} \approx 1.2 \cdot 10^{19} \frac{GeV}{c^2}$$

Another number of interest is the energy density in Λ. We know

$$\Omega_{0,\Lambda} \equiv \frac{\rho_\Lambda}{\rho_{crit}} \approx 0.7 \sim 1$$

$$\rho_\Lambda \sim \rho_{crit} = 1.88 \cdot 10^{-29} h^2 \frac{g}{cm^3} \tag{330}$$

$$= 1.054 \cdot 10^{-5} h^2 \frac{GeV}{cm^3} \tag{331}$$

Converting $\frac{GeV}{cm^3}$ to GeV^4, we get:

$$\rho_\Lambda \sim \rho_{crit} \sim 10^{-46} GeV^4$$

$$\frac{\rho_\Lambda}{m_{planck}^4} \approx 10^{-122} = worst\ number\ in\ physics$$

A.7 Lecture 7

The lower the temperature of the universe, the easier life gets: once the temperature of the universe drops below the rest energy of a particle, it is no longer often created in particle/antiparticle pairs.

Thermodynamics of a Fermi/Bose Gas

Phase Space is a 6-dimensional space of directions and linear momenta. It has volume:

$$V = \frac{d^3x d^3p}{h^3}$$

The *Distribution Function* $f(\hat{x}, \hat{p}, t)$ describes the number of particles in a particular $\hat{x}, \hat{p}$ state at a given time. Thus, the number of particles in a phase space volume is:

$$dN = f(\hat{x}, \hat{p}, t) \frac{d^3x d^3p}{h} \cdot g$$

where g is used to account for *degeneracy.* Reminder of Fermi/Bose statistics:

$$f(\hat{x}, \hat{p}, t) = \frac{1}{e^{\frac{E-\mu}{kT}} \pm 1}$$

Where μ is the chemical potential (usually 0 for us), + is for fermions, and - is for bosons. The chemical potential is usually 0 because we are talking about photons, and it can be shown that since photon number is not conserved, then $\mu = 0$.

- # density in thermal equilibrium (integrating dN above, dividing by volume, recall $E = \sqrt{p^2c^2 + m^2c^4}$):

$$n = \frac{g}{h^3} \int \frac{4\pi p^2 dp}{e^{\frac{E(p)}{kT}} \pm 1}$$

- Energy density (dN, weighted by energy):

$$u = \frac{g}{hA^3} \int_0^\infty E(p) \frac{4\pi p^2 dp}{e^{\frac{E(p)}{kT}} \pm 1}$$

- Entropy density $s = \frac{S}{V} = \frac{1}{V}\frac{1}{T}(U + PV - \mu N)$. P is pressure. For $\mu = 0$:

$$s = \frac{u + P}{T}$$

  Without proving, we'll state $P = \langle \frac{p^2c^2}{3E} \rangle$. The proof comes from kinetic theory. Thus:

$$s = \frac{g}{h^3} \int_o^\infty \frac{4\pi p^2 dp}{e^{\frac{E(p)}{kT}} \pm 1} \left(\frac{E(p)}{T} + \frac{p^2c^2}{3ET} \right)$$

So we have ways to calculate 3 very useful quantities for the early universe. We can take two useful limits of these quantities:

Ultra-relativistic ($kT \gg mc^2$) Particles in the Early Universe

In this limit, particles are effectively massless (their energy is dominated by their motion, $E = \sqrt{m^2c^4 + p^2c^2} \approx pc$).

- "A." Bosons
- (1) The # density of relativistic bosons is:

$$n = \frac{g}{h^3} \int_0^\infty \frac{4\pi p^2 dp}{e^{\frac{pc}{kT}} \pm 1} \tag{332}$$
$$= \frac{4\pi g}{(2\pi)^2} \left(\frac{kT}{\hbar c} \right)^3 \int_0^\infty \frac{y^2 dy}{e^y \pm 1} \tag{333}$$
$$\tag{334}$$

  where $y \equiv \frac{pc}{kT}$. This integral is calculable, and is the definition of the Reimann-Zeta function:

$$\zeta(n) \equiv \frac{1}{\Gamma(n)} \int_0^\infty \frac{y^{n-1} dy}{e^y - 1} \tag{335}$$
$$\int_0^\infty \frac{y^2 dy}{e^y - 1} = \Gamma(3)\zeta(3) \approx 2 \cdot 1.202 \tag{336}$$
$$\int_0^\infty \frac{y^3 dy}{e^y - 1} = \Gamma(4)\zeta(4) = \frac{\pi^4}{15} \tag{337}$$
$$\tag{338}$$

  So getting back to the # density of bosons:

$$\boxed{n_B = \frac{g}{\pi^2}\zeta(3) \left(\frac{kT}{\hbar c} \right)^3}$$

- (2) The energy density of relativistic bosons is:

$$u = \frac{g}{h^3}\int_0^\infty pc\frac{4\pi p^2 dp}{e^{\frac{pc}{kT}} \pm 1} \tag{339}$$

$$= \frac{4\pi g}{(2\pi)^3}\frac{(kT)^4}{(\hbar c)^3}\int_0^\infty \frac{y^3 dy}{e^y \pm 1} \tag{340}$$

$$\tag{341}$$

$\boxed{u_B = \frac{\pi^2}{30}g\frac{(kT)^4}{(\hbar c)^3}}$ For photons, $g = 2$. We can calculate a flux related to their energy density:

$$F = \frac{1}{4}u_\gamma c = \frac{\pi^2}{60}\frac{K^4}{\hbar^3 c^3}T^4 = \sigma = 5.67 \cdot 10^{-8}\frac{W}{m^2K^4}$$

- (3) The entropy density of relativistic bosons is:

$$\boxed{s_B = \frac{u_B + P_B}{T} = \frac{4}{3}\frac{u_B}{T} = \frac{2\pi^2}{45}\left(\frac{kT}{\hbar c}\right)^3 K = 3.602 n_B K}$$

Recall that for radiation, the energy density $\rho \propto a^{-3(1+w)} \propto a^{-4}$. Since $u \propto \rho \propto T^4 \Rightarrow T \propto \frac{1}{a}$. This is only true if g is fixed.

- "B." Fermions: here's a trick:

$$\frac{1}{e^y+1} = \frac{1}{e^y-1} - \frac{2}{e^{2y}-1}$$

This is an identity.

- (1) # density. Since $y \equiv \frac{pc}{kT}$, then $2y = \frac{pc}{K(\frac{T}{2})}$, then using our identity:

$$n_F(T) = \frac{g_F}{g_B}[n_B(T) - 2n_B(\frac{T}{2})] \tag{342}$$

$$= \frac{g_F}{g_B}n_B(T)[1 - 2(\frac{1}{2})^3] \tag{343}$$

$$\tag{344}$$

$$\boxed{n_F(T) = \frac{g_F}{g_B}\frac{3}{4}n_B(T)}$$

- (2) Energy density:

$$u_F(T) = \frac{g_F}{g_B}[u_B(t) - 2u_B(\frac{T}{2})] \tag{345}$$

$$= \frac{g_F}{g_B}u_B(t)[1 - 2(\frac{1}{2})^4] \tag{346}$$

$$\tag{347}$$

$$\boxed{u_F(T) = \frac{g_F}{g_B}\frac{7}{8}u_B(T)}$$

- (3) Entropy density:

$$s_F(T) = \frac{4}{3}\frac{u_F}{T}$$

$$\boxed{s_F(T) = \frac{g_F}{g_B}\frac{7}{8}s_B(T)}$$

When doing computations like this for the universe, we'll have to keep track of the g's of each particle's contribution. To aid us in doing this, we'll define an *effective degeneracy*:

$$\boxed{g^* = \sum_{i=bosons} g_i + \frac{7}{8}\sum_{i=fermions} g_i}$$

Using this definition, the total energy density $u_{tot} \propto g^* T^4$.

Non-Relativistic ($kT \ll mc^2$) Particles In the Early Universe

In this limit, a particle's energy is dominated by its rest mass: $E = \sqrt{m^2c^4 + p^2c^2} \approx mc^2(1 + \frac{1}{2}\frac{p^2c^2}{m^2c^4} + \dots)$. Then in general, the denominator inside the integral over momentum simplifies:

$$e^{\frac{E}{kT}} \pm 1 \approx e^{\frac{E}{kT}} \approx e^{\frac{mc^2}{kT} + \frac{p^2}{2kTm}}$$

So dropping the "± 1" in the denominator:

$$n \approx \frac{g}{h^3}\int_0^\infty 4\pi p^2 dp e^{-\frac{mc^2}{kT}} e^{-\frac{p^2}{2mkT}} \tag{348}$$

$$\approx \frac{4\pi g}{h^3} e^{-\frac{mc^2}{kT}} (2mkT)^{\frac{3}{2}} \int_0^\infty dy \cdot y^2 e^{-y^2} \tag{349}$$

$$\tag{350}$$

$$\boxed{n \approx \frac{g}{h^3}\left(\frac{mkT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{mc^2}{kT}}}$$

This the the Maxwell-Boltzmann distribution if we put μ, the chemical potential, back in the exponent with the energy (mc^2). Thus, the # density is exponentially suppressed for $kT \ll mc^2$. For example, at $kT \ll 1GeV \sim m_{proton}$, if we are in thermal equilibrium, then the neutron-to-proton ratio is:

$$\frac{n_n}{n_p} \approx e^{-\frac{(m_n - m_p)c^2}{kT}} \approx e^{\frac{-1.293MeV}{kT}}$$

This determines the Hydrogen/Helium ratio!

A.8 Lecture 8

Thermal Equilibrium vs. Decoupling ("Freeze-Out")

The rule of thumb here is to compare the interaction rate (Γ) of the particle we are interested in to the expansion rate of the universe. We'll examine two extremes: $\Gamma \gg H \Rightarrow TE$, and $\Gamma \ll H \Rightarrow$ *decoupled*.

- "Ex:" Weak Interactions:
  The cross-section for weak interaction goes as $\sigma \sim G_F^2 T^2$, where G_F^2 is the Fermi constant, $\frac{G_F m_o^2}{(\hbar c)^2} \sim 10^{-5}$. Thus, for $v \sim c$ (recall that $n \propto T^3$ in the relativistic limit):

$$\Gamma_{weak} \sim n\sigma v \sim G_F^2 T^5$$

Let's compare this to the Hubble expansion rate of early universe, when $\Omega \approx 1$, so curvature is negligible. Therefore:

$$H = \sqrt{\frac{8\pi}{3} G\rho} \sim \sqrt{G g_* T^4}$$

where G is Newton's constant and g_* is the effective degeneracy. In order for $\Gamma_{weak} \ll H$, we need:

$$G_F^2 T^5 \ll \sqrt{g_* G} T^2$$

$$T^3 \ll \frac{\sqrt{g_* \hbar c}}{G_F^2 m_{planck}}$$

We know $m_{planck} \sim \sqrt{\frac{\hbar c}{G}} \sim 10^{19} GeV$, so the temperature requirement for decoupling of the weak interaction is:

$$\Gamma_{weak} < 1MeV$$

In general we can say that particles decouple after the rest mass stops being much more that kT. We can compute the threshold temperature for particles based on their rest mass:

$Particles$	mc^2	$T_{thresh} = \frac{mc^2}{K}$
τ	$1.78 GeV$	$2.1 \cdot 10^{13} K$
n, p	$.94 GeV$	$1.1 \cdot 10^{13}$
π	$140 MeV$	$1.6 \cdot 10^{12}$
μ	$106 MeV$	$1.2 \cdot 10^{12}$
e	$.511 MeV$	$6 \cdot 10^{9}$

Relating Temperature and Time in the Radiation-Dominated Era

Recall that the energy density of relativistic bosons is given by:

$$u = \rho c^2 = \frac{\pi^2}{30} g_* \frac{(kT)^4}{(\hbar c)^3}$$

We also have shown that in the radiation-dominated era, $a \propto t^{\frac{1}{2}}$, so:

$$H = \frac{\dot{a}}{a} = \frac{1}{2t}$$

Now since $H^2 = \frac{8\pi}{3} G\rho$:

$$\left(\frac{1}{2t}\right)^2 = \frac{8\pi}{3} G \frac{\pi^2}{30} \frac{g_* (kT)^4}{\hbar^3 c^5}$$

$$\boxed{kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g_* G}\right)^{\frac{1}{4}} t^{-\frac{1}{2}} = \frac{0.86 MeV}{\sqrt{t(sec)}} \left(\frac{10\frac{3}{4}}{g_*}\right)^{\frac{1}{4}}}$$

$$\boxed{T \approx \frac{10^{10} Kelvin}{\sqrt{t(sec)}} \left(\frac{10.75}{g_*}\right)^{\frac{1}{4}}}$$

A useful relation is that $1 sec \sim 10^{10} Kelvin \sim 1 MeV$.

The First 30 Minutes (in 6 frames)

- "Frame 1:" $T = 10^{11} Kelvin$, $t = 0.01 sec$, $kT = 8.6 MeV$, $z \sim 1 + z = \frac{1}{a} = \frac{T}{T_0} \sim 3 \cdot 10^{10}$. To put things in perspective, $m_e < T < m_\mu$. The major players at this point in time are photons (γ), electrons and positrons (e^-, e^+), neutrinos ($\nu_i \bar{\nu}_i$), and protons and neutrons (p, n). For p, n, we're assuming baryon asymmetry has occurred, and we should note that they only show up in small numbers. γ, e, ν particles are all ultra-relativistic. Let's figure out our g_*:

$$g_* \; composed \; of \begin{cases} \gamma & g_\gamma = 2 \\ \nu\bar{\nu} & g_\nu = 3 \cdot 2 \cdot \frac{7}{8} = \frac{21}{4} \\ e^+ e^- & g_{ee} 2 \cdot 2 \cdot \frac{7}{8} = \frac{7}{2} \end{cases}$$

$$g_* = 10\frac{3}{4}$$

Note that g_ν was computed as [3 species $\cdot$ 2 particle/antiparticle pairs with 1 spin state each $\cdot$ fermion factor]. We ignored p, n because they were not relativistic. Their reactions are:

$$n + \nu_e \to p + e^-$$

$$n + e^+ \to p + \bar{\nu}_e$$

$$[n \to p + e^- + \bar{\nu}_e]$$

The last reaction is negligible because it has timescale ~ 15 minutes. Remember that we derived the neutron-to-proton ratio:

$$\frac{n_n}{n_p} = e^{\frac{-\Delta E}{kT}}$$

where $\Delta E \approx 1.293 MeV$. Thus:

$$\frac{n_n}{n_p} \approx 0.86$$

The neutron-to-baryon ratio is:

$$\frac{n_n}{n_B} = \frac{n_n}{n_p + n_n} = \frac{0.86}{0.86 + 1} = 0.46$$

- "Frame 2:" $T = 10^{10.5} Kelvin$,$t = 0.1 sec$, $kT = 2.72 MeV$, ρ drops by a factor of 100. Our major player are the same, so:

$$\frac{n_n}{n_p} \approx 0.62, \quad \frac{n_n}{n_B} \approx 0.38$$

- "Frame 3:" $T = 10^{10} Kelvin$, $t = 1 sec$, $kT = 0.86 MeV$. Now the weak interaction rate falls below the Hubble time ($< 1 MeV$). Therefore, neutrinos are decoupling from the thermal bath. Before this decoupling occurred, recall that $\gamma, e^+, e^-, \nu, \bar{\nu}$ were all in thermal equilibrium, $g_* = 10\frac{3}{4}$. After decoupling, γ, e^+, e^- are in thermal equilibrium, so $g_* = 2 + \frac{7}{2} = \frac{11}{2}$. For the neutrinos, their Fermi-Dirac distribution is "frozen" in place:

$$dn_\nu = g_\nu \frac{d^3 p}{h^3} \frac{1}{e^{\frac{pc}{kT}} + 1}$$

- "Frame 4:" $T = 10^{9.5} Kelvin$, $t = 14 s$, $kT = .272 MeV$. Because $T < 0.511 MeV$, it's hard to make new e^+, e^-. That is, $e^+ e^- \to 2\gamma$ is a favored interaction over the reverse, so $e^+ e^-$ start to disappear. As they annihilate, entropy is transferred to photons, so the entropy density $S = const \cdot g_* \cdot T^3$ is conserved (see Kolb and Turner, 66).

$$\frac{n_n}{n_p} = 0.2, \quad g_* = 2$$

A.9 Lecture 9

- "Frame 5:" $T = 10^9 K$, $t = 3min$, $kT = 0.086MeV$. As was started in Frame 4, the major player is γ, with only small contributions from (p,n,e^-). Therefore:

$$g_* = 2$$

At this point, neutron decay becomes important ($\frac{n_n}{n_p} \sim 0.16$):

$$p + n \to D + \overbrace{\gamma}^{2.2MeV}$$

There isn't much deuterium in the universe because n_p, n_n are small and n_γ is large, so deuterium gets broken up as soon as it's made.

Digression: The primary channel for energy generation in the center of stars is the *pp chain*, whereby H is fused into He. The pp chain generates about 85% of the energy in the sun. This chain goes as follows:

$$(1) p + p \to D + e^+ + \nu_e \tag{351}$$
$$(2) p + D \to^3 He + \gamma \tag{352}$$
$$(3)^3 He +^3 He \to^4 He + 2p \tag{353}$$
$$(net) 4p \to^4 He + 2e^+ + 2\nu_e + energy \tag{354}$$
$$\tag{355}$$

This generates a temperature of about $1.5 \cdot 10^7 K$ in the core of the sun. The early universe, this temperature occurred about 9 days in. Back then, the dominant energy-releasing reaction was:

$$p + n \to D + \gamma$$

Why are the primary reactions different for the center of a star and the early universe? Well, in the center of a star, there aren't so many free neutrons floating around, so the early universe reaction doesn't work there. Also, the density of the core of a star ($\sim 100 \frac{g}{cm^3}$) is much greater than the density of the early universe ($\sim 10^{-8} \frac{g}{cm^3}$). Notice that one of the reactions for the sun generates a neutrino, so these reactions involve weak interactions. We already discussed that the universe's density was out of the realm of weak interactions after about 1 sec. Now back to our regularly scheduled program.

- "Frame 5.5:" $t = 3.75min$. Here T finally gets low enough for deuterium to persist. Nucleosynthesis begins. $\frac{n_n}{n_p} \sim 0.15$.

- "Frame 6:" $T = 10^{8.5} K$, $t \sim 30min$, $kT = .027MeV$. The major players are still γ, with a little more p, e^-. Free neutrons are now mostly in He.

After Frame 6, the next important epoch is matter-radiation equality. After that, it's the last scattering of e^- off γ. The CMB epoch is when recombination occurs.

Frame 4 Revisited

This is the epoch of e^-, e^+ annihilation. Energy conservation dictates that S (the entropy density) be conserved. Thus, the entropy is transferred into γ as e^-, e^+ annihilate. Since $S = constant \cdot g_* T^3$, we can calculate how much the photon gas was heated up in this epoch:

$$\overbrace{e^+ + e^-}^{g_* = \frac{11}{2}} \to \overbrace{2\gamma}^{g_* = 2}$$

$$\tfrac{11}{2}T^3_{before} = 2T^3_{after} \quad \Rightarrow \quad \tfrac{T_{after}}{T_{before}} = \left(\tfrac{11}{4}\right)^{\frac{1}{3}}$$

Since the neutrinos have already decoupled, they are unaffected by this heating. Therefore:

$$\boxed{T_\nu = \left(\frac{4}{11}\right)^{\frac{1}{3}} T_\gamma}$$

This is true even today.

Frame 5.5 Revisited

Recall that this was the epoch of nucleosynthesis. In Frame 5, there was a bottleneck where elements heavier that H could not be made because D kept being destroyed by radiation. In 5.5, that bottleneck has been lifted, and heavier elements are synthesized:

$$D + D \to^3 He + n \tag{356}$$
$$D + D \to^4 He + \gamma \tag{357}$$
$$n + D \to^3 H + \gamma \tag{358}$$
$$D +^3 H \to^4 He + n \tag{359}$$
$$^4He +^3 H \to^7 Li + \gamma \tag{360}$$
$$^4He +^3 He \to^7 Be + \gamma \tag{361}$$
$$^7Be + e^- \to^7 Li + \nu_e \tag{362}$$
$$\tag{363}$$

This tree continues ad infinitum. The most abundant element in the universe is H, then He. The abundance of ^4He can be calculated:

$$Y \equiv \frac{mass\ in\ ^4He}{mass\ of\ all\ baryons} = \frac{4m_p \cdot n_{^4He}}{m_p(n_p + n_n)}$$

We can estimate that $n_{^4He} \approx \frac{n_n}{2}$, since all other n-containing elements are rare. Thus:

$$\boxed{Y = \frac{2\frac{n_n}{n_p}}{1 + \frac{n_n}{n_p}}}$$

We can use an estimate of $\frac{n_n}{n_p} \sim \frac{1}{7}$.

Y dependencies

- Y depends on the baryon-to-photon ratio$=\eta \propto \Omega_b h^2$. As η increases, $p + n \to D + \gamma$ is favored over photo-dissociation, so D is more stable and the bottleneck breaks earlier, so $\frac{n_n}{n_p}$ is higher when He forms. Thus Y would be larger.
- Y depends on the lifetime of neutrons: τ_n.
- Y depends on the # of species of neutrinos, N_ν. This parameter affects the energy density of the universe. A higher N_ν causes the expansion of the universe to increase, since:

  $$H \propto \sqrt{\rho} \propto \sqrt{g_*}T^2$$

  This affects how many neutrons have decayed. We can establish a fitting formula for Y:

  $$Y = 0.23 + 0.025 \log_{10}(\frac{\eta}{10^{10}}) + 0.0075(g_* - 10.75) + 0.014(\tau - 10.6min)$$

  This shows, for example, that an increase in N_ν of 1 would change g_*:

  $$\Delta g_* = 2 \cdot \tfrac{7}{8} = \tfrac{7}{4} \quad \Rightarrow \quad \Delta Y \approx 0.013$$

A.10 Lecture 10

The Dark Side of the Universe

There is evidence that dark matter exists:

- Zwicky (1933): measurements of σ_ν (the total mass) of galaxies in the Coma Cluster showed:

  $$\frac{M}{L} \approx 300h\frac{M_\odot}{L_\odot}$$

  This means that there is lots of mass that doesn't appear in luminosity (the mass is 300 times what you'd expect if all the stars were sun-like).

- Dynamics (70's, Rubin and others): the rotation curves of galaxies is *flat*. That is, $v^2 \propto \frac{M(r)}{r}$, and this is constant, so $M(r) \propto r$.

- Structure formation: A baryonic universe can't grow structure until decoupling (CMB era) has occurred, since rapid collisions with photons prevents gravitational collapse. But if that's when gravitational collapse started, there wouldn't have been time to form the structure we see today. Since dark matter does not couple, it could start collapsing earlier and suck the baryonic matter down with it.

- CMB fluctuations give us a measure of Ω_m.

- Gravitational lensing probes luminous *and* dark matter.

Nature of Dark Matter

If we calculate the mass of luminous matter we get $\Omega_{stars} + \Omega_{gas} \approx .003$ (Fukugita et al, 98). For baryons (p,n), $\Omega_b h^2 \approx .024$, as calculated from Big Bang Nucleosynthesis. Overall, matter is estimated to constitute $\Omega_m \sim .2 \to .4$ (baryons and non-baryons). This brings us to the two levels of the "dark matter problem".

- $\Omega_b \gg \Omega_{luminous} \Rightarrow$ some baryons must be "dark". This is the "Baryon Dark Matter problem". What are these "dark baryons"? Candidates are:

- (a) MACHOs (compact stellar-mass objects), searched for by micro-lensing. These could be faded white dwarfs ($\sim \frac{1}{2}M_{sun}$) or brown dwarfs ("failed stars" with $M < .08M_{sun}$). So far it seems that these cannot account for the mass density difference between baryonic and luminous matter.

- (b) Black holes, neutron stars: too few.

- (c) Planets: $M_{Jupiter} \sim \frac{1}{300}M_{sun}$, too little mass.

- (d) Dust: radiates, bounded by metallicity constraints.

- $\to$(e) Diffuse warm gas: $T \leq 10^6 K$. Can't detect this very well. It will emit in the soft x-ray range. This one is still a viable option.

- The "Non-Baryonic Dark Matter Problem": $\Omega_m \gg \Omega_{baryon}$ tells us that there is mass coming from non-baryons. Some options here are hot dark matter (massive neutrinos, for example), and cold dark matter (SUSY=SUper SYmmetry particles). See PS#5 for why neutrinos are "hot". In order for neutrinos to work out as hot dark matter, they need to have mass.

Massive Neutrinos (as Hot Dark Matter)

- 1. Current lab limits (based on weak-decay kinematics): $m_{\nu_\tau} < 18.2MeV$ based on τ decays: $\tau^- \to \nu_\tau + pions$. $m_{\nu_\mu} < 190keV$ based on Π^+ decays: $\Pi^+ \to \mu^+ + \nu_\mu$. $m_{\nu_e} < 3eV$ based on tritium decays.
- 2. Neutrino oscillations (in vacuum) measures $\Delta m^2 = m_1^2 + m_2^2$. When solving the interaction Lagrangian, there are mass eigenstates and weak-interaction eigenstates, which do not have to be identical. Considering ν_1, ν_2 as the mass eigenstates of a 2-generation interaction, and ν_e, ν_μ as the weak eigenstates, then:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

The Weak interaction Lagrangian is $S = \int d^4x \mathfrak{L}$, where $\mathfrak{L}$ is the Lagrangian density. Then:

$$\mathfrak{L}_{weak} \propto \bar{\nu}_e \gamma^\alpha (1-\gamma_5) e W_\alpha + \bar{\nu}_\mu \gamma^\alpha (1-\gamma_5) \mu W_\alpha + \overbrace{m_1 \bar{\nu}_1 \nu_1 + m_2 \bar{\nu}_2 \nu_2}^{mass\ terms}$$

Where the "mass terms" are what's relevant for determining neutrino oscillations.

A.11 Lecture 11

Massive ν (Hot Dark Matter) continued

Last time we started talking about (1) and (2) below:

- Lab upper bounds on m_ν.
- Neutrino oscillations: $\Delta m^2 = m_1^2 = m_2^2$:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

Now suppose at time $t = 0$ we have a pure ν_e produced. Then:

$$|\nu(0)\rangle = |\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle$$

At time t:

$$|\nu(t)\rangle = \cos\theta e^{-iE_1 t}|\nu_1\rangle + \sin\theta e^{-iE_2 t}|\nu_2\rangle$$

Where $E_1 = \sqrt{p^2 + m_1^2}$, and $E_2 = \sqrt{p^2 + m_2^2}$, assuming a fixed momentum state. We assume this for simplicity, but a full wave-packet-based derivation is done in (Kayser (81) PRD 24,110). Anyway, the probability of finding a ν_μ state at time t is:

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= |\langle \nu_\mu | \nu(t)\rangle|^2 && (364) \\ &= |(-\sin\theta\langle\nu_1| + \cos\theta\langle\nu_2|)(\cos\theta e^{-iE_1 t}|\nu_1\rangle + \sin\theta e^{-iE_2 t}|\nu_2\rangle)|^2 && (365) \\ &= |-\cos\theta\sin\theta e^{-iE_1 t} + \cos\theta\sin\theta e^{-iE_2 t}|^2 && (366) \\ &= \cos^2\theta\sin^2\theta|e^{-iE_1 t} - e^{-iE_2 t}|^2 && (367) \\ &= \frac{\sin^2 2\theta}{4}(1-\cos(E_1 - E_2)t) && (368) \\ &= \sin^2 2\theta \sin^2\left(\frac{(E_1 - E_2)t}{2}\right) && (369) \\ & && (370) \end{aligned}$$

For relativistic ν (we don't really need to make this assumption, it just makes our lives easier), then:

$$E_1 - E_2 \approx \frac{1}{2}\frac{m_1^2 - m_2^2}{E} \sim \frac{1}{2}\frac{\Delta m^2}{E}$$

Thus:

$$\frac{1}{2}(E_1 - E_2)t \sim \frac{1}{4}\frac{\Delta m^2 L c^3}{e} \tag{371}$$
$$= 1.27\frac{\Delta m^2 L}{E} \tag{372}$$
$$\tag{373}$$

Where $L \sim ct$ is the distance ν travels in meters, Δm^2 is in eV^2, and E is in MeV Thus we have our expression for the probability of an e^- neutrino turning into a μ neutrino in a vacuum:

$$\boxed{P(\nu_e \to \nu_\mu) = \sin^2 2\theta \sin^2\left(\frac{1.27\Delta m^2 L}{E}\right)}$$

- Supernova ν: (SN1987A: Feb 23 1987)
- "(a)" Supernovae produce ν via: $e^- + p \to n + \nu_e$, which happens within the first (0.1 sec) of final star collapse (note that the n is what makes neutron stars.
- "(b)" They also produce ν via: $e^- + e^+ \to \nu_i + \bar{\nu}_i$, where $i = e, \mu, \tau$.
- "(c)" Time for ν to travel from 1987A: If neutrinos are massless, $\mu_\nu = 0$, $T_0 = \frac{d}{c} \sim 169,000 lyrs$. However if $\mu_\mu \neq 0$, then $E = \gamma mc^2$, $p = \gamma mv$, so:

$$\frac{v}{c} = \frac{pc}{E} = \frac{\sqrt{E^2 - m^2c^4}}{E} = \sqrt{1 - \frac{m_\nu^2 c^4}{E^2}}$$

$$T = \frac{d}{v} = \frac{d}{c}\left(1 - \frac{m_\nu^2 c^4}{E^2}\right)^{-\frac{1}{2}}$$

The difference in times of arrival for if neutrinos are massive or not is:

$$\Delta t \equiv T - T_0 \approx \frac{d}{c}\frac{m_\nu^2 c^4}{2E^2} \tag{374}$$
$$= 0.0257sec\left(\frac{d}{50kpc}\right)\left(\frac{m_\nu}{1eV}\right)\left(\frac{10MeV}{E}\right)^2 \tag{375}$$
$$\tag{376}$$

A.12 Lecture 12

Wrapping Up Massive ν

So far we've discussed (1), (2), and (3) below:

- Direct Lab upper limits.
- ν oscillations
- Supernova ν
- Cosmology (PS #5): There is a relic ν background from 1 second after the Big Bang,

$$T_{\nu,0} = 1.945K \sim \left(\frac{4}{11}\right)^{\frac{1}{3}} T_{\gamma,0}$$

We calculated that the neutrino mass required to close the universe is:

$$\Omega_\nu h^2 = \frac{\sum_{i=e,\mu,\tau} m_i}{93.5eV}$$

(for $m_i \ll 1MeV$). This puts a pretty tight limit on the mass of neutrinos, given that we observe an open universe. The assumption we made about mass allowed us to make the assumption that ν's were relativistic when they decoupled from the rest of the universe.

If $m_\nu > 1MeV$, then the # density of neutrinos will have a $e^{-\frac{m_\nu c^2}{kT}}$ suppression, since $\nu\bar{\nu}$ will annihilate to Z^0. Lee-Weinberg (1977) showed that $\Omega_\nu \propto m_\nu^{-2}$ for $m_\nu \gg 1MeV$. Since this is a square function, there are two values for which $\Omega_\nu = 1$. If we want $\Omega_\nu < 1$, we have that either $m_\nu > 2GeV$ or $m_\nu < 100eV$.

Cold Dark Matter (CDM)

The two best known candidates for CDM are:

- WIMPs (Weakly Interacting Massive Particles): The lightest super-symmetric particle is in the range of 10-100 GeV.
- Axions: These were "introduced" to resolve the *strong CP* problem in QCD. The problem was that non-perturbative effects in QCD led to a CP,T,P violation. Which would predict an excessively large electric dipole moment for the neutron. Axions were invented to suppress this effect. In terms of the Lagrangian density:

$$L_{QCD} = L_{pert} + \underbrace{\bar{\Theta}\frac{g^2}{32\pi^2}\overbrace{G^{a}_{\mu\nu}}^{\substack{gluon\\fields}}\tilde{G}^a_{\mu\nu}}_{violates\ CP}$$

The CP violating term predicted an electric dipole moment of the neutron of $d_n \sim 10^{-15}\bar{\Theta}\ e\cdot cm$. Experimental results show that $d_n < .63\cdot 10^{-25}e\cdot cm \Rightarrow \bar{\Theta} < 10^{-10}$. Thus, the axion mass required to suppress this is $10^{-5}eV$. Why are we calling these cold? Clearly, these are still relativistic particles even today. However, since axions couple with photons only very weakly, they were never really in thermal equilibrium.

The Lumpy Universe: Structure Formation

The universe has a dichotomy between being initially nearly smooth (as per the CMB), and quite lumpy (as in galaxy clusters). The general belief is that the lumpiness of the universe is caused by small perturbations amplified by gravitational instability. We will concern ourselves with the time-evolution of small perturbations to *uniform* (Robertson-Walker-Friedmann mode) parameters such as ρ (density), P (pressure), $\vec{v}$ (fluid velocity), and ϕ (gravitational potential).

A.13 Lecture 13

Jeans (Gravitational) Instability (in a Static Universe)

In a non-relativistic, non-dissipative, static fluid described by ρ, P, the fluid velocity $\vec{v}$, and gravitational potential ϕ, we can write three equations to describe motion:

- The Continuity equation (describing mass conservation):

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0$$

(Continuity Equation)

- Euler's Equation:

$$\underbrace{\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}}_{\frac{d\vec{v}}{dt}=(\frac{\partial}{\partial t}+\dot{x}_i\frac{d}{dx_i})\vec{v}} = \underbrace{-\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\phi}_{\frac{force}{mass}} + other\ terms$$

(Euler's Fluid Equation)

where "$other/terms$" could come from other forces like magnetic fields or viscous force. The full version of the above equation is called the Wavier-Stokes equation. This equation describes how the fluid velocity changes with force.

- Poisson's equation:

$$\nabla^2\phi = 4\pi G\rho$$

(Poisson's Equation)

This is just a Gauss' Law for a gravitational field.

At this point, we would like to write an equation of state which relates P to ρ. To add a little bit of generality to this derivation so that we can perturb the equation for a lumpy universe, we'll start using $\rho_0 = const$ to describe the "zeroth order" of mass density. We'll also say $P_0 = const$, and $\vec{v}_0 = 0$. We have to be careful about ϕ_0, though: Euler's equation (2) tells us that $\boldsymbol{\nabla}\cdot\phi_0 = 0$, but if $\phi_0 = const = 0$, then Poisson's Equation (3) says:

$$\boldsymbol{\nabla}\cdot\phi_0 = \frac{4\pi}{3}G\rho_0\vec{x} \neq 0$$

(Jeans Swindle)

There is really no way to fix this. On PS#6 we will show that although this fails for the completely static case, when we start doing first-order perturbations to this equation, things work out better.

For small perturbations to the static uniform solution $(\rho_1, P_1, \vec{v}_1, \phi_1)$ such that:

$$\rho = \rho_0 + \rho_1 \qquad P = P_0 + P_1$$
$$\vec{v} = \vec{v}_0 + \vec{v}_1 = \vec{v}_1 \qquad \phi = \phi_0 + \phi_1$$

We will define the **Adiabatic Sound Speed** as $v_s^2 \equiv \frac{\partial P}{\partial \rho}\big|_s = \frac{P_1}{\rho_1}\big|_s$, so our equations become:

$$\frac{\partial P}{\partial t} + \rho_0(\boldsymbol{\nabla}\cdot\cdot\vec{v}) = 0$$

$$\frac{\partial \vec{v}_1}{\partial t} = \frac{-v_s^2}{\rho_0}\boldsymbol{\nabla}\cdot\rho_1 - \boldsymbol{\nabla}\cdot\phi_1$$

$$\nabla^2\phi_1 = 4\pi G\rho_1$$

We can combine (1) and (2) to get:

$$\frac{\partial P_1}{\partial t^2} + \rho_0\left(-\frac{v_s^2}{\rho_0}\nabla^2\rho_1 - \nabla^2\phi_1\right) = 0$$

Then using (3) we get:

$$\boxed{\frac{\partial^2 P_1}{\partial t^2} - v_s^2\nabla^2\rho_1 - 4\pi G\rho_0\rho_1 = 0}$$

Given that this equation looks like a wave equation, we can guess there is a solution where $\rho_1(\vec{r},t) = \int d^3k a_k e^{i(\vec{k}\vec{r}-\omega t)}$. This gives us a dispersion relation:

$$\boxed{\omega^2 = v_s^2k^2 - 4\pi G\rho_0}$$

This looks similar to the dispersion relation of plasma physics. However, in a plasma, $\omega^2 = v_s^2 k^2 + \frac{4\pi n_e e^2}{m_e}$. Therefore, in plasma, $\omega^2 > 0$, so it is always oscillating. In our case, we can rewrite ω as:

$$\omega^2 = v_s^2(k^2 - k_J^2)$$

where k_J is:

$$\boxed{k_J \equiv \sqrt{\frac{4\pi G\rho_0}{v_s^2}}}$$

(Jeans Wavenumber)

There is also a Jeans Wavelength: $\lambda_J = \frac{2\pi}{k_J}$, and a Jeans Mass: $M_J = \frac{4\pi}{3}(\frac{\pi}{k_J})^3\rho_0$.

We have two regimes for the Jeans Wavenumber:

- $k < k_J$ (or $\lambda > \lambda_J$):

  $$\omega^2 = v_s^2(k^2 - k_J^2) < 0$$

  So ω is imaginary, and thus:

  $$\rho_1 \propto e^{\pm|\omega|t}$$

  (Jeans Instability)

  So the density of waves can grow or decay exponentially.

- $k > k_J$ (or $\lambda < \lambda_J$):

  ω is real, so ρ_1 oscillates stably.

Thus, in a static universe, small scale density waves are stable, but for larger scale waves, we have runaway growth.

A.14 Lecture 14

Jeans Instability Continued

So far we've written down 3 equations: the Continuity Equation, Euler's Equation, and Poisson's Equation. From these we derived a dispersion relation for a static medium. We evaluated two wavelength regimes for this relation and found that for wavelengths longer than the Jeans wavelength, we had exponential collapse due to self-gravity, and for wavelengths shorter than the Jeans wavelength, we have normal, stable oscillation.

Basically, we are seeing a balance between pressure outward and gravity inward. Collapse occurs when gravity wins out over pressure. Consider a sphere of density ρ, and a piece of volume V at distance R. Then the force of gravity on that piece is:

$$\frac{F_{grav}}{V} \sim \frac{GM\rho}{R^2} \sim G\rho^2 R$$

since $M \sim \rho R^3$. The pressure force on that volume is:

$$\frac{F_{pres}}{V} \sim \nabla P \sim v_s^2 \nabla\rho \sim v_s^2\frac{\rho}{R}$$

Comparing these forces, we find that gravity wins if:

$$G\rho^2 R > v_s^2\frac{\rho}{R}$$

Solving for R:

$$R > \sqrt{\frac{v_s^2}{G\rho}} \sim \lambda_J$$

So this is a quick-and-dirty way of deriving the Jeans length scale. The timescale for collapse is also a balance between gravity and pressure. The time for gravitational free fall is given by:

$$R \sim a_{grav} t_{ff}^2 \sim \frac{GM}{R^2} t_{ff}^2 \sim G\rho R t_{ff}^2 \tag{377}$$

$$t_{ff} \sim \frac{1}{\sqrt{G\rho}} \tag{378}$$

$$\tag{379}$$

Thus, the free fall time is independent of distance. The pressure timescale is the sound crossing time, given by:

$$t_{sound} \sim \frac{R}{v_s}$$

We will have gravitational collapse if $t_{ff} < t_{sound}$, which means:

$$\tfrac{1}{\sqrt{G\rho}} < \tfrac{R}{v_s} \quad \Rightarrow \quad R > \sqrt{\tfrac{v_s^2}{G\rho}}$$

So again we get out the Jeans wavelength.

Gravitational Instability in an Expanding Fluid

Density is obviously affected by expansion. To zeroth order:

$$\rho_0 = \rho_0(t) = \frac{\rho_0(t_{today})}{a^3} \tag{380}$$

$$\frac{\partial \rho_0}{\partial t} = -3\frac{\dot{a}}{a}\rho_0 \tag{381}$$

$$\tag{382}$$

where a is our familiar scale factor. To examine first order perturbations, we define the **density field** δ, where $\rho_1 \equiv \rho_0 \delta$, so that:

$$\delta = \frac{\rho_1}{\rho_0} = \frac{\rho - \rho_0}{\rho_0}$$

$$\boxed{\delta = \frac{\partial \rho}{\rho_0}}$$

Our velocities are also changed by expansion:

$$\vec{v}_0 = H\vec{r} = \frac{\dot{a}}{a}\vec{r}$$

where $\vec{r}$ is in physical coordinates. This velocity is purely from Hubble expansion. Note that:

$$\nabla \cdot \vec{v}_0 = \frac{\dot{a}}{a}(\nabla \cdot \vec{r}) = 3\frac{\dot{a}}{a}$$

We'll use $\vec{v}_1$ to describe the peculiar velocity (motion with respect to the expansion). Now we need to solve our fluid equations using these parameters. The continuity equation, to zeroth order, says:

$$\frac{\partial \rho_0}{\partial t} + \rho_0(\nabla \cdot \vec{v}_0) = 0 \tag{383}$$

$$\frac{\partial \rho_0}{\partial t} + 3\frac{\dot{a}}{a}\rho_0 = 0 \tag{384}$$

$$\rho_0(t) \propto a^{-3} \tag{385}$$

$$\tag{386}$$

To first order:

$$\frac{\partial \rho_1}{\partial t} + \rho_0(\nabla \cdot \vec{v}_1) + \nabla(\rho_1 \vec{v}_1) = 0 \tag{387}$$

$$\rho_0\dot{\delta} + \dot{\rho}_0\delta + \rho_0(\nabla \cdot \vec{v}_1) + \rho_0\vec{v}_0(\nabla \cdot \delta) + \rho_0\delta(\nabla \cdot \vec{v}_0) = 0 \tag{388}$$

(389)

Using that $\dot{\rho}_0\delta = -3\frac{\dot{a}}{a}\rho_0\delta$, and $\nabla \cdot \vec{v}_0 = 3\frac{\dot{a}}{a}$:

$$\boxed{\dot{\delta} + \nabla \cdot \vec{v}_1 + \frac{\dot{a}}{a}(\vec{r} \cdot \nabla)\delta = 0}$$

Euler's equation gives us, to zeroth order:

$$\frac{\partial \vec{v}_0}{\partial t} + (\vec{v}_0 \cdot \nabla)\vec{v}_0 = -\nabla \cdot \phi_0 \neq 0$$

So there is no Jeans swindle in this case. Expanding this to first order, we get:

$$\frac{\partial \vec{v}_1}{\partial t} + (\vec{v}_0 \cdot \nabla)\vec{v}_1 + (\vec{v}_1 \cdot \nabla)\vec{v}_0 = -\frac{v_s^2}{\rho_0}\nabla\rho_1 - \nabla\phi_1$$

To figure out what $(\vec{v}_1 \cdot \nabla)\vec{v}_0$ is, let's expand it:

$$(\vec{v}_1 \cdot \nabla)\vec{v}_0 = H(v_{1x}\hat{x} + v_{1y}\hat{y} + v_{1z}\hat{z})(\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z)\vec{r} \tag{390}$$

$$= H\vec{v}_1 = \frac{\dot{a}}{a}\vec{v}_1 \tag{391}$$

(392)

Thus our equation above becomes:

$$\boxed{\frac{\partial \vec{v}_1}{\partial t}\frac{\dot{a}}{a}(\vec{r} \cdot \nabla)\vec{v}_1 + \frac{\dot{a}}{a}\vec{v}_1 = -v_s^2\nabla\rho - \nabla\pi_1}$$

Finally, we have Poisson's Equation:

$$\boxed{\nabla^2\phi = 4\pi G\rho_1 = 4\pi G\rho_0\delta}$$

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