

Extragalactic Astrophysics and Cosmology

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Contents

1	The Smooth Universe	3
1.1	The Cosmological Principle; The Hubble Parameter; Scale Factor	3
1.1.1	The Cosmological Principle	3
1.1.2	Hubble Parameter	3
1.1.3	Scale Factor	3
1.2	The Friedmann Equation; The Equation of State; Radiation, Matter, and Dark Energy	4
1.2.1	The Friedmann Equation	4
1.2.2	Equation of State	6
1.2.3	Matter, Radiation, and Dark Energy	6
1.3	The Density Parameter; Open, flat, closed models; Redshift	7
1.3.1	Redshift	8
1.4	Time Evolution of Parameters: Hubble, Density, Scale Factor	8
1.4.1	Hubble Parameter	8
1.4.2	Density Parameter	8
1.4.3	Scale Factor (Flat Case)	9
1.4.4	Scale Factor (Open Case)	10
1.4.5	Scale Factor (Closed Case)	11
1.4.6	Deceleration Parameter	11
1.5	Robertson-Walker Metric	12
1.6	Basic Kinematic Properties of the Smooth Universe	14
1.6.1	Comoving, Radial Distance vs. Redshift (“The Hubble Diagram”)	14
A	Archival Lectures (C.P. Ma)	15
A.1	Lecture 1	15
A.1.1	The Cosmological Principle	15
A.1.2	Hubble’s Law (1929)	15
A.1.3	The Scale Factor	15
A.1.4	The Friedmann Equations	16
A.2	Lecture 2	17
A.2.1	The Friedmann Equations, continued	17
A.2.2	Critical Density	17
A.2.3	Density Parameter	18
A.2.4	Evolution of Hubble Parameter	18
A.2.5	Evolution of Density Parameter	18
A.2.6	Evolution of Scale Factor: Solving the Friedmann Equation	18
A.3	Lecture 3	19
A.3.1	Evolution of Scale Factor: Solving the Friedmann Equation, continued	19
A.3.2	Deceleration Parameter q	20

A.3.3	Redshift	21
A.3.4	Time-Redshift Relations and the Age of the Universe	21
A.4	Lecture 4	22
A.4.1	Time-Redshift Relations and the Age of the Universe	22
A.4.2	The Robertson-Walker Metric	23
A.5	Lecture 5	24
A.6	Lecture 6	24
A.7	Lecture 7	24
A.8	Lecture 8	24
A.9	Lecture 9	24
A.10	Lecture 10	24
A.11	Lecture 11	24

1 The Smooth Universe

1.1 The Cosmological Principle; The Hubble Parameter; Scale Factor

1.1.1 The Cosmological Principle

We begin with **The cosmological Principle**. It sounds simple, but incredibly well supported. It says that **the Universe (spatially) is homogeneous and isotropic on very large spatial scales**. Observationally, this is around 100 Mpc scales. Homogeneous means constant density (non-realistically, this is mass density; relativistically, this is energy density). Isotropic means the same in all directions.

Note: Isotropy about 2 points (or more) implies homogeneity. Isotropy about 1 point is not enough. Here is a quick illustration of that:

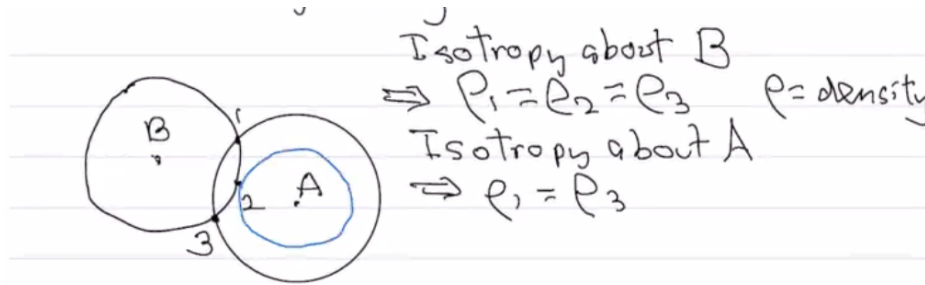


Figure 1: Isotropy about 2 points.

We can consider another example. A homogeneous universe can be anisotropic. Consider a homogeneous Universe that is expanding in different directions in a non-uniform way. This leads to different $H_0(x, y, z)$.

The reason we spend some time on the Cosmological Principle is the **Friedmann-Robertson-Walker metric**, which we will come to later on.

1.1.2 Hubble Parameter

Let's now talk about the **Hubble parameter**, which is not a constant! It, in fact, changes in time. An empirical linear relationship between the recession speed v and distance r can be seen, called the **Hubble's Law**:

$$v = Hr \quad (1)$$

Note that H_0 has units of 1/time. One convention to note is that:

$$H = 100 \underbrace{h}_{\text{to hide our ignorance}} \frac{\text{km}}{\text{s Mpc}} \quad (2)$$

One useful number to know is $H \approx \frac{h}{10^{10} \text{ yr}}$.

Sometimes, when we write H_0 , we mean the "present-day" value ($z = 0$). This is how we will use it now.

1.1.3 Scale Factor

We need a language to describe the expansion of the Universe. We will use $a(t)$, which describes the expansion (or contraction) of the Universe. It also relates two different coordinate systems: physical coordinates \vec{r} to comoving coordinates \vec{x} . The relation:

$$\vec{r} = a(t)\vec{x} \quad (3)$$

We typically use comoving coordinates in calculations in Cosmology. We can think of \vec{x} as the notches on a stretching ruler. Now consider:

$$\frac{d}{dt}\vec{r} = \vec{v} = \dot{a}\vec{x}a\vec{x} \quad (4)$$

$$\frac{d}{dt}\vec{r} = \vec{v} = \dot{a}\vec{x}a\vec{x} = \underbrace{\frac{\dot{a}}{a}\vec{r}}_{\text{Hubble}} + \underbrace{a\ddot{\vec{x}}}_{\text{motion relative to expansion ("peculiar velocity")}} \quad (5)$$

$$H(t) = \frac{\dot{a}(t)}{a(t)} \quad (6)$$

The name of the game for measuring H is to go far enough that the first term dominates. Otherwise, locally, the second term dominates since peculiar velocities are of order 100s of km/s.

One other convention we need to establish:

$$a(t_0) = 1 \rightarrow \text{comoving} = \text{today} \quad (7)$$

1.2 The Friedmann Equation; The Equation of State; Radiation, Matter, and Dark Energy

1.2.1 The Friedmann Equation

Below, we will use and derive these, but I am putting the equations at the top for convenience.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2} \quad (8)$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}(P + \rho) \quad (9)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \quad (10)$$

Let's motivate the origin with quasi-Newtonian physics. We can derive it from General Relativity, but that's overkill.

If we assume isotropy, we only need to worry about the radial coordinate r , not θ or ϕ . Homogeneity tells us that $\rho = \text{constant spatially}$, but *can* depend on time. We will model the Universe as an expanding, homogeneous medium that is adiabatically ($\Delta s = 0$) expanding. If it were not adiabatically expanding, we would have heat flow and thus no isotropy.

With these conditions, let's examine the motion of a thin, expanding, spherical shell of radius a . This depends **only** on the enclosed mass within a ¹:

$$M(< a) = \frac{4}{3}\pi a^3 \rho \quad (11)$$

Let's consider the energy:

$$E = \frac{1}{2}\dot{a}^2 - \frac{GM}{a} \quad (12)$$

$$E = \frac{1}{2}\dot{a}^2 - \frac{4}{3}\pi G\rho a^2 \quad (13)$$

Let's re-write E a bit: $kc^2 \equiv -2E$. Note that $k \propto 1/\text{length}^2$. There are three possibilities for kc^2 :

¹see Birkhoff's Theorem for General Relativity proof

- > 0 , $E < 0$, bound
- $= 0$, $E = 0$, critical
- < 0 , $E > 0$, unbound

Let's now evoke the First Law of Thermodynamics ($\Delta S = 0$):

$$\underbrace{dU}_{\text{internal energy}} = -PdV \quad (14)$$

We now equation the internal energy to the rest-mass energy:

$$d(\rho c^2 a^3) = -Pd(a^3) \quad (15)$$

We will now set $c = 1$ and take a time derivative:

$$\dot{\rho}a^3 + 3\rho a^2\dot{a} = -3Pa^2\dot{a} \quad (16)$$

$$\dot{\rho} - 3\frac{\dot{a}}{a}(P + \rho) \quad (17)$$

Using this result and the energy equation from above

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k(c)^2}{a^2} \quad (18)$$

to get a new equation. If you stare at it hard enough and have divine intervention, take a derivative of the second equation and multiply by a^2 . Doing so, you get:

$$2\dot{a}\ddot{a} = \frac{8\pi}{3}G\frac{d}{dt}(\rho a^2) = \frac{8\pi}{3}Ga^2\left(\dot{\rho} + 2\frac{\dot{a}a}{\rho}\right) \quad (19)$$

Simplifying with the other above equation, we get:

$$2\dot{a}\ddot{a} = -\frac{8\pi}{3}Ga^2\left(\frac{\dot{a}}{a}\rho + 3\frac{\dot{a}}{a}P\right) \quad (20)$$

Simplifying:

$$\frac{\ddot{a}}{\dot{a}} = -\frac{4\pi}{3}G(\rho + 3P) \quad (21)$$

Note that this is not independent from the other equations; rather it is massaged. Let's compare this to 1-D Newtonian forces:

$$\ddot{x} = -\frac{GM}{x^2} = -\frac{4}{3}\pi G\rho x \quad (22)$$

$$\frac{\ddot{x}}{x} = -\frac{4}{3}\pi G\rho \quad (23)$$

Had we done strictly Newtonian physics, we would have never gotten the $+3P$ term. The way to interpret this: we can think of ρ to have an extended meaning: $\rho_{eff} = \rho + 3P$.

The Grand Summary so far: two equations of motion for $a(t)$:

$$\frac{\ddot{a}}{\dot{a}} = -\frac{4\pi}{3}G(\rho + 3P) \quad (24)$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}(P + \rho) \quad (25)$$

Note this second equation tells us the acceleration! Very importantly, we have a minus sign. If ρ and P are positive, the Universe is **decelerating!** Conversely, if you have a *bizarre* P and could reverse the parenthetical term, we can have accelerated expansion!

Right now, we have three unknowns (P, ρ, a) . How do we get that last piece – the equation of state ($P \Leftrightarrow \rho$ dependence)?

1.2.2 Equation of State

We choose to write:

$$P = w\rho(c^2) \quad (26)$$

We are in units where $c = 1$, but I threw it in for reference. Note – this means that pressure is the same thing as energy density! Think of the units.

With that definition of P , we can re-write the second boxed equation from above:

$$\dot{\rho} - 3\frac{\dot{a}}{a}(1 + w)\rho \quad (27)$$

$$\frac{\dot{\rho}}{\rho} = -3(1 + w)\frac{\dot{a}}{a} \rightarrow \rho \propto a^{-3(1+w)} \quad (28)$$

$$\boxed{\rho \propto a^{-3(1+w)}} \quad (29)$$

assuming that $\dot{w} = 0$ which might not be true!

1.2.3 Matter, Radiation, and Dark Energy

Let's look at a few special cases of the equation of state:

- **Matter:** Non-relativistic, pressure-less particles like cold dark matter. In this case, $w = 0, P = 0 \rightarrow \boxed{\rho \propto a^{-3}}$. This makes sense because it is units of 1/volume.
- **Radiation:** Relativistic particles, photons and neutrinos. In this case, $w = 1/3, P = \frac{1}{3}\rho \rightarrow \boxed{\rho \propto a^{-4}}$. This makes sense because it is units of (1/volume) (1/length) where the extra factor is from redshifting energy.
- **The Cosmological Constant:** In this case, $w = -1, P = -\rho \rightarrow \rho = \text{constant}$.
- **Dark Energy:** More general term, where $w < -\frac{1}{3}$ to make $\ddot{a} > 0$.

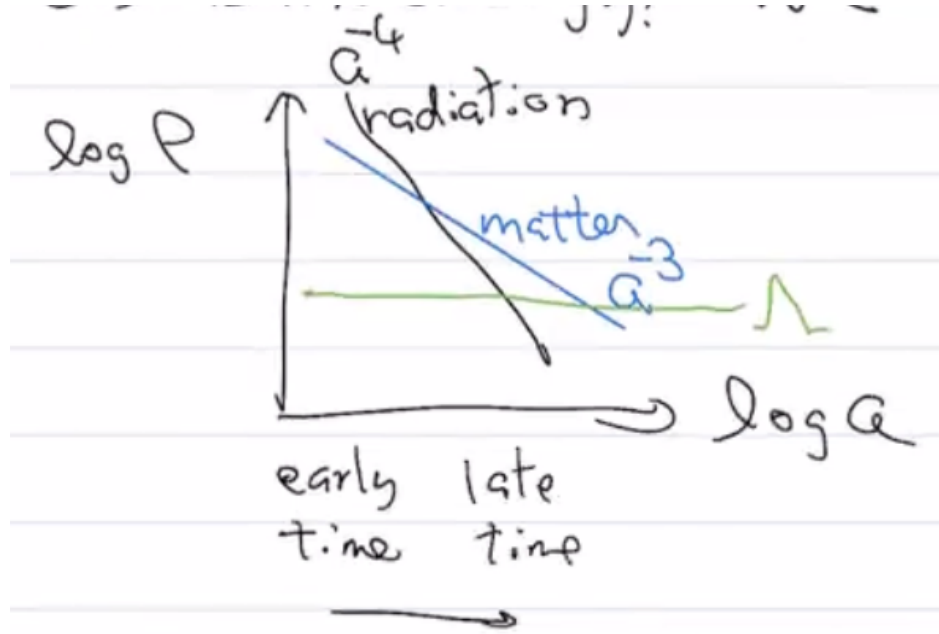


Figure 2: Sketch of density over cosmic time.

1.3 The Density Parameter; Open, flat, closed models; Redshift

Let's first define Ω in terms of the **critical density** ρ_c . This is defined to be the density needed to make the Universe flat. Recall the Friedmann Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2} \quad (30)$$

First, set $k = 0 \Leftrightarrow E = 0$, then solve for ρ :

$$\boxed{\rho_c = \frac{3H^2}{8\pi G}} \quad (31)$$

This critical density can be interpreted as well as the mean density of a flat Universe ($k = 0$). Importantly, ρ_c is a function of H , which itself is a function of time. Setting $H = H_0$, we can get Today's value of ρ_c :

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = 2.78 \times 10^{11} h^2 \frac{M_\odot}{\text{Mpc}^3} = 1.88 \times 10^{-29} h^2 \frac{\text{g}}{\text{cm}^3} \quad (32)$$

This is about $10^{-5} h^2 m_{\text{proton}} \text{cm}^{-3}$. Here are a few other useful numbers to carry around as well:

$$\rho_{\text{thumb}} \sim 1 \frac{\text{g}}{\text{cm}^3} \quad (33)$$

$$\rho_{\text{ISM}} \sim 1 \frac{m_p}{\text{cm}^3} \quad (34)$$

$$\rho_{\text{Universe}} \sim 1\rho_c \quad (35)$$

Now with these definitions, we have the **density parameter**:

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\rho(t)}{3H^2(t)} \quad (36)$$

There are three obvious possibilities:

- $\Omega < 1 \rightarrow$ Open Universe ($k < 0$)
- $\Omega = 1 \rightarrow$ Flat Universe ($k = 0$)
- $\Omega > 1 \rightarrow$ Closed Universe ($k > 0$)

There are a few things to note. When we have *multiple components*, we have:

$$\Omega(t) = \frac{\sum_i \rho_i}{\rho_c} \quad (37)$$

where i is often representing radiation, matter, Λ , baryonic component of matter, dark matter, neutrinos, etc. Another thing we want to stress is that ρ , ρ_c , Ω **all depend on time**. Note that if $\rho > \rho_c$, it will STAY that way. The same goes for Ω , and for other values. For example, these parameters will stay $>$, $<$, or $= 1$.

1.3.1 Redshift

Observationally, we have

$$1 + z \equiv \frac{\lambda_{obs}}{\lambda_{rest}} = \frac{a(t_0)}{a(t)} = \frac{1}{a(t)} \quad (38)$$

$$\boxed{\frac{1}{a(t)} = 1 + z} \quad (39)$$

1.4 Time Evolution of Parameters: Hubble, Density, Scale Factor

1.4.1 Hubble Parameter

Worked out in Problem Set 1, but here is the solution:

$$H^2(a) = H_0^2 \left(\frac{\Omega_0}{a^{3+3w}} + \frac{1 - \Omega_0}{a^2} \right) \quad (40)$$

For multiple components (the common ones):

$$H^2(a) = H_0^2 \left(\frac{\Omega_{0,m}}{a^3} + \Omega_{0,\Lambda} + \frac{\Omega_{0,r}}{a^4} + \dots + \frac{1 - \Omega_{0,m} - \Omega_{0,\Lambda} - \Omega_{0,r}}{a^2} \right) \quad (41)$$

1.4.2 Density Parameter

First, recall the critical density:

$$\rho_c = \frac{3H^2}{8\pi G} \rightarrow H^2\Omega = \frac{8\pi}{3}G\rho \quad (42)$$

We can also look at the Friedmann equation:

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2} \quad (43)$$

We can replace the first term on the right hand side and set $a = 1$:

$$k = H_0^2 (\Omega_0 - 1) \quad (44)$$

All of these together, we get:

$$H^2 = H^2 \Omega - \frac{H_0^2 (\Omega_0 - 1)}{a^2} \quad (45)$$

Rearrange:

$$1 - \Omega(a) = \frac{H_0^2}{H^2 a^2} (1 - \Omega_0) \quad (46)$$

We can use the time evolution of the Hubble Parameter to replace the H terms, leaving:

$$\boxed{1 - \Omega(a) = \frac{1 - \Omega_0}{\frac{\Omega_0}{a^{1+3w}} + 1 - \Omega_0}} \quad (47)$$

To be more explicit, let's write out the multi-component again:

$$1 - \Omega(a) = \frac{1 - \Omega_0}{1 - \Omega_0 + (\Omega_{m,0} a^{-1} + \Omega_{0,\Lambda} a^2 + \Omega_{r,0} a^{-2})} \quad (48)$$

where $\Omega_0 = \sum_i \Omega_{0,i}$. Note that the above equation essentially FORCES $\Omega(a) \sim 1$. This foreshadows the Flatness Problem.

1.4.3 Scale Factor (Flat Case)

This is essentially solving the Friedmann Equation for $a(t)$ for specific models (can, in general, be integrated). The most basic model is the Einstein-de Sitter model: a model that is flat $k = 0 \rightarrow \Omega = 1$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \rho \quad (49)$$

We have also see that $\rho \propto a^{-3(1+w)}$. Therefore:

$$da a^{\frac{3}{2}(1+w)-1} \propto dt \quad (50)$$

$$\boxed{a(t) \propto t^{\frac{2}{3(1+w)}}} \quad (51)$$

Recall for the Einstein-de Sitter model ($k = 0, \Omega = 0$), we found:

$$a \propto t^{\frac{2}{3(1+w)}} \quad (52)$$

This is very useful since it works for each epoch (matter dominated, radiation dominated, dark energy dominated, etc;, but not in the transitions):

Matter Dominated Era ($w = 0, P \approx 0$)

$$\boxed{a_m \propto t^{\frac{2}{3}}} \quad (53)$$

Radiation Dominated Era ($w = \frac{1}{3}, P = \frac{1}{3}\rho$)

$$\boxed{a_r \propto t^{\frac{1}{2}}} \quad (54)$$

Λ Dominated Era ($w = -1$ (for example), $P = -\rho$)

$$a_r \propto t^\infty \rightarrow \text{Wrong!} \quad (55)$$

This breaks down because H is a constant in this case! So here:

$$a \propto e^{Ht} \quad (56)$$

Isn't this reminiscent of inflation? That's because it is! It wasn't the Cosmological Constant but rather a scalar field called the inflaton.

1.4.4 Scale Factor (Open Case)

Let's consider $k < 0$ and $\Omega < 1$. Additionally, we assume $\Lambda = 0$ and that we are matter-dominated.

Start with the Friedmann Equation:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}G\rho - \frac{k}{a^2} > 0 \text{ always} \quad (57)$$

Immediately, we see that expansion will never stop! The right hand side is always positive, and therefore \dot{a} is never equal to 0. Now, recall the equation for $H(a)$ and the fact that $H = \dot{a}/a$:

$$\dot{a}^2 = H_0^2 \left(\frac{\Omega_0}{a^{1+3w}} + 1 - \Omega_0 \right) \quad (58)$$

We now solve for $a(t)$ for this Universe:

$$\int_0^{a_f} \frac{da}{\sqrt{\frac{\Omega_0}{a} + 1 - \Omega_0}} = \int_0^{t_f} H_0 dt \quad (59)$$

The right hand side is easy. The left hand side? Let's start with completing the squares to get rid of the $\frac{1}{a}$, so let's set that up.

$$\int_0^{a_f} \frac{ada}{(1 - \Omega_0) \sqrt{a^2 + \frac{\Omega_0}{1 - \Omega_0} a}} \quad (60)$$

Now we can actually complete the square in the denominator. Define $2\alpha \equiv \frac{\Omega_0}{1 - \Omega_0}$:

$$\int_0^{a_f} \frac{ada}{\sqrt{1 - \Omega_0} \sqrt{(a + \alpha)^2 - \alpha^2}} \quad (61)$$

We will now shift variables: $a' \rightarrow a + \alpha$:

$$\int_\alpha^{a_f + \alpha} \frac{(a - \alpha) da}{\sqrt{1 - \Omega_0} \sqrt{a^2 - \alpha^2}} \quad (62)$$

We can now do trigonometric substitution. Let $a \equiv \alpha \cosh \theta$. This gives $da = \alpha \sinh \theta d\theta$. Lastly, $\sqrt{a^2 - \alpha^2} = \alpha \sinh \theta$. Properly adjusting limits, we get the expression:

$$\boxed{a(\theta) = \frac{\Omega_0}{2(1 + \Omega_0)} (\cosh \theta - 1)} \quad (63)$$

$$\int_0^{\theta_f} \frac{\alpha^2 (\cosh \theta - 1) \sinh \theta}{\sqrt{1 - \Omega_0} \alpha \sinh \theta} d\theta \quad (64)$$

$$\frac{\alpha}{\sqrt{1-\Omega_0}} (\sinh \theta - \theta)|_0^{\theta_f} \quad (65)$$

$$H_0 t_f = \frac{\Omega_0}{2(1+\Omega_0)^{3/2}} (\sinh \theta_f - \theta_f) \quad (66)$$

This gives:

$$t(\theta) = \frac{\Omega_0}{2H_0(1+\Omega_0)^{3/2}} (\sinh \theta - \theta) \quad (67)$$

We can then use θ as some parameter, trace out a and t , and then find $a(t)$!

1.4.5 Scale Factor (Closed Case)

You can repeat the derivation above for the matter-dominated, closed case of $k > 0$, $\Omega > 1$. Here, there *will* be a maximum of a and the dynamics are different $\theta \rightarrow i\theta$:

$$t(\theta) = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}} (\theta - \sin \theta) \quad (68)$$

$$a(\theta) = \frac{\Omega_0}{2(\Omega_0 - 1)} (1 - \cos \theta) \quad (69)$$

1.4.6 Deceleration Parameter

Let's define this – it was used historically since we now know the Universe is *accelerating*. This is a dimensionless parameter:

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} \rightarrow q = -\frac{1}{H^2} \frac{\ddot{a}}{a} \quad (70)$$

Recall one of the other results we had this semester:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G(\rho + 3P) \quad (71)$$

and

$$\rho_c = \frac{3H^2}{8\pi G} \rightarrow H^2 \Omega = \frac{8\pi}{3} G\rho \quad (72)$$

Putting this together with q :

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G\rho(1+3w) = -\frac{H^2\Omega}{2} (1+3w) \quad (73)$$

This makes:

$$q = \frac{\Omega}{2} (1+3w) \quad (74)$$

For multiple components:

$$q = \sum_i \frac{\Omega_i}{2} (1+3w_i) \quad (75)$$

$$q = \frac{\Omega_m}{2} + \underbrace{\frac{1+3w}{2}\Omega_w}_{\text{Dark Energy}} + \Omega_r + \dots \quad (76)$$

For a matter-dominated Universe and $\Omega_\Lambda \neq 0$:

$$\boxed{q = \frac{\Omega_m}{2} - \Omega_\Lambda} \quad (77)$$

The minus sign there is enormously important!

One thing to note: supernovae are sensitive to q , whereas CMB measurements are sensitive to the addition of Ω_m and Ω_Λ , so in some sense, these are orthogonal instead of parallel.

1.5 Robertson-Walker Metric

This is where you would start in General Relativity.

Let's set up the background. Recall Lorentz transformations in Special Relativity. There are two inertial observers, (x, y, z, t) and (x', y', z', t') with relative velocity $\vec{v} = v\hat{x}$:

$$x' = \gamma(x - vt) \quad (78)$$

$$y' = y \quad (79)$$

$$z' = z \quad (80)$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad (81)$$

We also talked about the Lorentz invariant interval:

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) = ds'^2 \quad (82)$$

Recall that the propagation of light follows the “null geodesic”: $ds^2 = 0$:

$$dr = c dt \quad (83)$$

We will quote the RW Metric here, and explore it next time. In short, Robertson and Walker showed that, for a **homogeneous and isotropic space, the most general metric is** (see Weinberg for a full proof):

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (84)$$

We are defining the coordinates as:

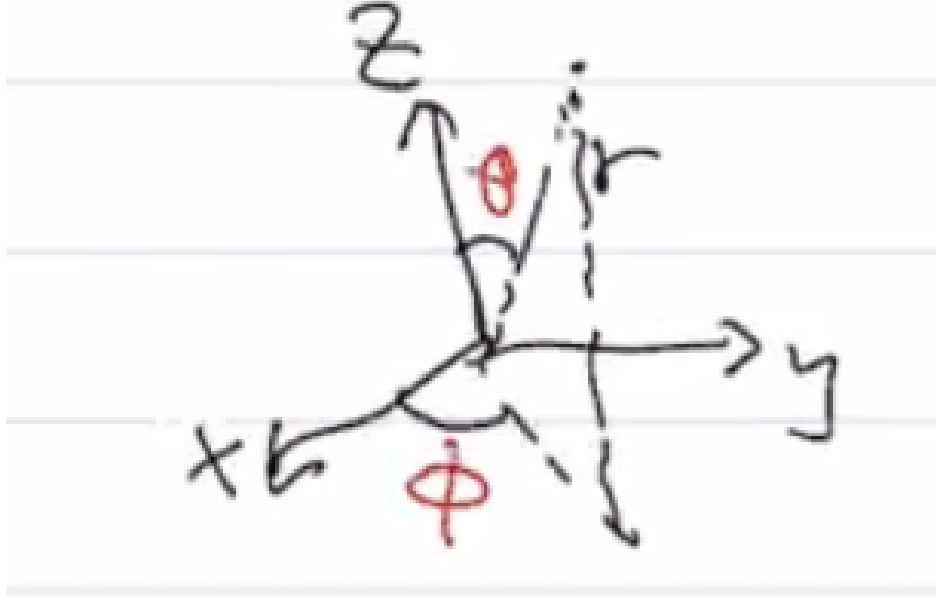


Figure 3: Spherical coordinate convention.

An immediate and obvious consequence is when $k = 0$, where we recover the Minkowski metric for flat space. Let's consider the more interesting cases of open ($k < 0$) and closed ($k > 0$). Immediately, we have a coordinate singularity at $r = \frac{1}{\sqrt{k}}$. Recall that $k = \frac{H_0^2}{c^2} (\Omega_0 - 1)$. We can thus define the **radius of curvature**:

$$R_0 \equiv \frac{1}{\sqrt{k}} = \frac{c}{H_0} \left(\frac{1}{\sqrt{\Omega_0 - 1}} \right) \quad (85)$$

Let's step back a second and look at the form of the FRW metric with some analogies:

3-sphere (in 4D) $(x, y, z, w) \Leftrightarrow (R, \alpha, \beta, \gamma)$ $w = R \cos \alpha$ $z = R \sin \alpha \cos \beta$ $y = R \sin \alpha \sin \beta \cos \gamma$ $x = R \sin \alpha \sin \beta \sin \gamma$ $x^2 + y^2 + z^2 + w^2 = R^2$	2-sphere (in 3D) $(x, y, z) \Leftrightarrow (R, \theta, \phi)$ $z = R \cos \theta$ $y = R \sin \theta \cos \phi$ $x = R \sin \theta \sin \phi$ $x^2 + y^2 + z^2 = R^2$
---	---

Look at a line element of the 2-sphere embedded in 3D:

$$d\ell^2 = R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (86)$$

We will change variables to $u = \sin \theta \rightarrow du = \cos \theta d\theta = \sqrt{1 - u^2} d\theta$. This makes the line element:

$$d\ell^2 = R^2 \left(\frac{du^2}{1 - u^2} + u^2 d\phi^2 \right) \quad (87)$$

Look at that! We're not quite there, but we have a similar form to the metric we want! And the line element of the 3-sphere embedded in 4D:

$$d\ell^2 = R^2 (d\alpha^2 + \sin^2 \alpha d\Omega^2) \quad (88)$$

where

$$d\Omega^2 \equiv d\beta^2 + \sin^2 \beta d\gamma^2 \quad (89)$$

Again, we can change variables, $u \equiv \sin \alpha \rightarrow du = \sqrt{1-u^2} d\alpha$:

$$d\ell^2 = R^2 \left[\frac{du^2}{1-u^2} + u^2 d\Omega^2 \right] \quad (90)$$

Again, where

$$d\Omega^2 \equiv d\beta^2 + \sin^2 \beta d\gamma^2 \quad (91)$$

Compared to the Friedmann-Robertson-Walker metric, these are identical!

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (92)$$

Thus, for a 3-sphere in 4-dimensions, we recover the FRW metric dependence in a hand-wavy way.

For the open model ($k < 0$), we don't have coordinate singularities. Instead of a spherical analogy, we use a saddle as an analogy (or, more tasty, Pringles!). This case has infinite volume and is called Lobachevsky space. Here, we let $u \equiv \sinh \theta$ which you can see will recover the proper FRW dependence.

There is an alternative form of the metric, too, we should know. We will see this in Problem Set 2. It is written as:

$$ds^2 = c^2 dt^2 - a^2(t) [d\chi^2 + \mathcal{S}^2 (d\theta^2 + \sin^2 \theta d\phi^2)] \quad (93)$$

1.6 Basic Kinematic Properties of the Smooth Universe

Here, we have in mind:

- Distance vs. Redshift
- Time vs. Redshift
- Age of the Universe
- Other useful quantities

1.6.1 Comoving, Radial Distance vs. Redshift (“The Hubble Diagram”)

First, we know that photons follow $ds^2 = 0$, so let's take a radial path from the observer ($d\theta = d\phi = 0$). Now we have:

$$0 = c^2 dt^2 - \frac{a^2 dr^2}{1-kr^2} \quad (94)$$

$$\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = c \int_{t_1}^{t_0} \frac{dt}{a(t)} \quad (95)$$

where t_0 is the age of the Universe. Evaluating these integrals, starting with the flat, static Universe ($k = 0, a = 1$):

$$dr = c dt \rightarrow r = ct \quad (96)$$

What about other cases? First, let's replace dt by dz using $H = \frac{1}{a} \frac{da}{dt}$ and $a = (1+z)^{-1}$:

$$H = -\frac{1}{1+z} \frac{dz}{dt} \quad (97)$$

Also recall:

$$H(z) = H_0 \sqrt{\Omega_0 (1+z)^{3+3w} + (1-\Omega_0)(1+z)^2} \quad (98)$$

Thus:

$$dt = -\frac{dz}{H(1+z)} = -\frac{dz}{H_0(1+z)\sqrt{\text{stuff}}} \quad (99)$$

Going back to our integral expression:

$$\text{RHS} = \int_{t_1}^{t_0} \frac{cdt}{a(t)} = \frac{c}{H_0} \int_0^{z_1} \frac{dz}{\sqrt{\text{stuff}}} \quad (100)$$

A Archival Lectures (C.P. Ma)

A.1 Lecture 1

A.1.1 The Cosmological Principle

The **Cosmological Principle** states that the universe is spatially *isotropic* (looks the same in all directions) and *homogeneous* (has constant density everywhere) on large scales. The **Perfect Cosmological Principle** states that the universe is also *temporally* isotropic and homogeneous (a steady state universe). This is unlikely because it doesn't describe the Cosmic Microwave Background (CMB). The CMB and Hubble's Law are both provide evidence for isotropy and homogeneity.

A.1.2 Hubble's Law (1929)

Hubble's Law is an empirical law stating that, on large scales, recessional velocity is proportional to distance from observer.

$$\boxed{v = Hr}$$

where H , the Hubble parameter, is not constant, but can vary slowly with time. By convention, H is often expressed as $H = 100 \cdot h \frac{km}{s \cdot Mpc}$, where 1 parsec (pc) $\approx 3 \cdot 10^{18} cm = 3.26 ly$, is the distance at which 1 AU appears as 1 arcsec on the sky. The Hubble Space Telescope Key Project (Freedman et al. ApJ 553, 47, 2001) measured the present day value of Hubble Constant $H_0 = 72 \pm 8 \frac{km}{s \cdot Mpc}$, giving us that the current timescale for the expansion of the universe is $H_0^{-1} \approx \frac{h}{10^{11}} yrs \approx 9.778 h^{-1} Gyr$.

A.1.3 The Scale Factor

$a(t)$ relates physical (r) and *comoving* (x) coordinates in an expanding universe:

$$r = a(t)x \quad (101)$$

$$\dot{r} = \dot{a}x + a\dot{x} = \underbrace{\frac{\dot{a}}{a}}_{\equiv H} r + \underbrace{a\dot{x}}_{\equiv v_p} \quad (102)$$

$$(103)$$

Thus, the two components of physical velocity are H (the Hubble expansion parameter) and v_p (the peculiar velocity, or motion relative to expansion) By convention, $t_0 \equiv$ today and $a(t_0) = 1$.

A.1.4 The Friedmann Equations

The Friedmann Equation is an equation of motion for $a(t)$. A rigorous derivation requires General Relativity, but we can fake it with a quasi-Newtonian derivation. We will model the universe as an *adiabatically* ($\Delta S = 0$) expanding, isotropic, homogeneous medium. Isotropy allows us to use r as a scalar. Consider a thin, expanding spherical shell of radius a . Birkhoff's Theorem states that even in GR, the motion of such a shell depends only on the enclosed mass $M = 4\pi 3a^3\rho$. Thus, the energy per unit mass per unit length is:

$$E = \overbrace{\frac{1}{2}\dot{a}^2}^{\text{Kinetic}} - \overbrace{\frac{G \cdot M}{a}}^{\text{Potential}} = \frac{1}{2}\dot{a}^2 - \frac{4\pi}{3}G\rho a^2.$$

We define $k \equiv -\frac{2E}{c^2}$, and we will show later that k is a measure of the curvature of the universe:

$$k \begin{cases} > 0 & \text{for } E < 0 \text{ (bound)} \\ = 0 & \text{for } E = 0 \text{ (critical)} \\ < 0 & \text{for } E > 0 \text{ (unbound)} \end{cases}$$

where k has units of $\frac{1}{\text{length}^2}$ if a is dimensionless. Substituting k into the above energy equation, and solving for $\frac{\dot{a}}{a}$, we get:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}$$

(1st Friedmann Equation)

This is a statement of conservation of E . The first law of thermodynamics ($\Delta S = 0$) requires that any system with positive pressure must lose energy as the volume enclosing it expands. Thus, if U is our internal energy and P is our pressure:

$$\frac{dU}{dt} = -P\frac{dV}{dt}.$$

In an expanding universe, $U = EV \cdot V = \rho a^3$, where ρ is the energy density of the universe. P is the pressure of the photon gas, so:

$$\dot{\rho}a^3 + 3\rho a^2\dot{a} = -P3a^2\dot{a},$$

which simplifies to:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P)$$

(2nd Friedmann Equation)

This is a statement of the temperature loss of the universe due to adiabatic expansion.

Finally, $\frac{d}{dt}$ (1st Friedmann Equation) gives us:

$$2\dot{a}\ddot{a} = \frac{8\pi}{3}G\frac{d}{dt}(\rho a^2) = \frac{8\pi}{3}Ga^2(\dot{\rho} + 2\frac{\dot{a}}{a}\rho).$$

Substituting the 2nd Friedmann Equation for $\dot{\rho}$:

$$2\dot{a}\ddot{a} = \frac{8\pi}{3}Ga^2\left(-\frac{\dot{a}}{a}\rho - 3\frac{\dot{a}}{a}P\right) = -\frac{8\pi}{3}G\frac{\dot{a}}{a}(\rho + 3P).$$

Now we have our 3rd Equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

(3rd Friedmann Equation)

The 3rd Friedmann Equation relates the acceleration of the expansion of the universe to the pressure of photon gas and the density of the universe. Note that if $3P \leq -\rho$, we have an accelerating universe.

Compare the 3rd Friedmann Equation to the Newtonian equation for gravity, with $M = \frac{4\pi}{3}\rho x^3$ and $\rho_{eff} = \rho + 3P$:

$$\ddot{x} = \frac{-G \cdot M}{x^2} = -4\pi 3G\rho x$$

A.2 Lecture 2

A.2.1 The Friedmann Equations, continued

Recall we had the following equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = 8\pi 3G\rho - \frac{k}{a^2}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G(\rho + 3P)$$

To close the equations, we need to relate P and ρ with an equation of state:

$$\boxed{P = w\rho(c^2)}$$

Note that we will generally set $c = 1$ in this class. Combined with (2), this gives us:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(1 + w)\rho \tag{104}$$

$$\dot{\rho}\rho = -3(1 + w)\frac{\dot{a}}{a} \tag{105}$$

$$\rho \propto a^{-3(1+w)} \tag{106}$$

$$\tag{107}$$

Note that we've assumed $\dot{w} = 0$, which is okay most of the time. Some special cases of interest are:

- Pressure-less “dust” $P = 0$, $w = 0 \Rightarrow \rho \propto a^{-3}$ because volume goes as $V \propto \frac{1}{a^3}$.
- Relativistic particles (photons, bosons): $w = \frac{1}{3}$, $P = \frac{\rho}{3} \Rightarrow \rho \propto a^{-4}$ because $VV \propto \frac{1}{a^3}$, and energy is given by $E \propto \frac{1}{a}$.
- (Λ)/Dark Energy: $w = -1$, $P = -\rho \Rightarrow \rho = \text{constant}$ in time.

To get density (ρ) as a function of time, want to solve for w .

A.2.2 Critical Density

We define ρ_{crit} to be the critical density at which $k = 0$ (and $E = 0$):

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

Today, we measure: $\rho_{crit,0} = \frac{3H_0^2}{8\pi G} = \frac{3}{8\pi} \frac{(100h \frac{km}{sMpc})^2}{G} = 2.78 \cdot 10^{11} h^2 \frac{M_\odot}{Mpc^3} = 1.88 \cdot 10^{-29} h^2 \frac{g}{cm^3}$. Note that the mass of the sun is $M_\odot = 2 \cdot 10^{33} g$, and the mass of the proton is $M_p = 1.67 \cdot 10^{-24} g$.

A.2.3 Density Parameter

Ω measures the ratio of the density of the universe to the critical density:

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\rho(t)}{3H^2(t)}$$

$$\Omega \begin{cases} \leq 1 & \Rightarrow \text{open } (k \leq 0) \\ = 1 & \Rightarrow \text{flat } (k = 0) \\ \geq 1 & \Rightarrow \text{closed } (k \geq 0) \end{cases}$$

In general, Ω can consist of multiple components: $\Omega = \sum_i \Omega_i$ e.g.

$$\Omega \begin{cases} r & = \text{radiation} \\ m & = \text{matter (dark and luminous)} \\ b & = \text{baryons (dark and luminous)} \\ \nu & = \text{neutrinos} \\ \Lambda & = \text{dark energy} \end{cases}$$

$\Omega = 1$ is an unstable equilibrium; any perturbation from $\Omega = 1$ in the early universe ensures Ω is far from 1 today. That we measure $\Omega_m \approx 0.3$ today implies that the early universe must have been extremely finely tuned.

A.2.4 Evolution of Hubble Parameter

Today (at $a = 1$): $k = \frac{8\pi}{3}G\rho_0 - H_0^2 = H_0^2(\Omega_0 - 1)$. Using the 1st Friedmann equation (1), we have:

$$H^2 = H_0^2 \left(\frac{\Omega_0}{a^{3+3w}} + \frac{1 - \Omega_0}{a^2} \right)$$

(THE Friedmann Equation)

where $H_0 \equiv \sqrt{\frac{8\pi G\rho}{3\rho_{crit}}}$, and it is understood that $\Omega_0 = \sum_i \Omega_{i,0}$. Each Ω_i has it's own w_i , so really $\frac{\Omega_0}{a^{3+3w}} = \sum_i \frac{\Omega_{i,0}}{a^{3+3w_i}}$.

A.2.5 Evolution of Density Parameter

We'll show in PS#1 that for any single component:

$$\frac{1 - \Omega(a)}{\Omega(a)} = \frac{1 - \Omega_0}{\Omega_0} a^{1+3w}$$

Plotting $\Omega(a)$, we will find that for early a , Ω is *extremely* close to 1.

A.2.6 Evolution of Scale Factor: Solving the Friedmann Equation

The evolution of the expansion of the universe is governed by:

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2}$$

We can apply this to several models of the universe:

- The Einstein-deSitter (flat) Model: $k = 0$, $\Omega(a) = \Omega_0 = 1$. Using that $\rho \propto a^{-3(1+w)}$, we have:

$$\left(\frac{\dot{a}}{a}\right)^2 \propto a^{-3(1+w)} \quad (108)$$

$$a^{-1} a^{\frac{3}{2}(1+w)} da \propto dt \quad (109)$$

$$a^{-\frac{3}{2}(1+w)} \propto t \quad (110)$$

$$(111)$$

$a(t) \propto t^{\frac{2}{3(1+w)}}$ Thus, the rate of expansion of the universe depends on w :

- (a) The matter-dominated era: $\Omega \approx \Omega_m \Rightarrow w = 0, P = 0, a \propto t^{\frac{2}{3}}$.
- (b) The radiation-dominated era: $\Omega \approx \Omega_r \Rightarrow w = \frac{1}{3} \Rightarrow a(t) \propto t^{\frac{1}{2}}$.
- (c) The Λ -dominated era: $\Omega \approx \Omega_\Lambda \Rightarrow w = -1, P = -\rho, \rho$ constant in time $\Rightarrow a(t) \propto e^{Ht}$, where H is now actually a constant. This is exponential inflation. We used to think that this only happened early on (like 10^{-34} seconds), but now we think that this has also been happening recently. Next time, we will do the harder two cases: open and closed.

A.3 Lecture 3

A.3.1 Evolution of Scale Factor: Solving the Friedmann Equation, continued

Last time we did (1), the flat universe:

Einstein-deSitter, $k = 0$ (flat), $\Omega = 1, \Rightarrow a \propto t^{\frac{2}{3(1+w)}}$. In general, we consider the evolution of the universe to be *matter dominated* if $a \propto t^{\frac{2}{3}}$, *radiation dominated* if $a \propto t^{\frac{1}{2}}$, and Λ *dominated* if $a \propto e^{1+t}$ (that is, H is constant).

Open, $k < 0$, $\Omega_0 < 1$, $\Omega_{0,\Lambda} = 0$, $\Omega_{0,M} < 1 \Rightarrow$ our Friedmann Equation $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2}$ becomes:

$$\dot{a}^2 = H_0^2 \left(\frac{\Omega_0}{a^{1+3w}} + 1 - \Omega_0 \right)$$

Since we are matter dominated, $w = 0$. Solving for $a(t)$:

$$\int_0^{a_f} \frac{da}{\sqrt{\Omega_0 a + 1 - \Omega_0}} = \int_0^{t_f} H_0 \cdot dt \quad (112)$$

$$\int_0^{a_f} \frac{a \cdot da}{\sqrt{1 - \Omega_0} \sqrt{a^2 + \frac{\Omega_0}{1 - \Omega_0} a}} = H_0 t_f \quad (113)$$

$$(114)$$

Let's define $2\alpha \equiv \frac{\Omega_0}{1 - \Omega_0}$. Then:

$$\int_0^{a_f} \frac{a \cdot da}{\sqrt{1 - \Omega_0} \sqrt{(a + \alpha)^2 - \alpha^2}} = H_0 t_f$$

Defining $a' = a + \alpha$, we get:

$$\int_0^{a_f + \alpha} \frac{(a' - \alpha) da'}{\sqrt{1 - \Omega_0} \sqrt{a'^2 - \alpha^2}} = H_0 t_f \quad (115)$$

$$\int_0^{a_f + \alpha} \frac{(a - \alpha) da}{\sqrt{1 - \Omega_0} \sqrt{a^2 - \alpha^2}} = H_0 t_f \quad (116)$$

$$(117)$$

Now we define $\alpha \cosh \theta \equiv a'$, $\alpha \sinh \theta d\theta = da$, so that $\sqrt{a'^2 - \alpha^2} = \alpha \sinh \theta$. We'll be lazy and just say that $a'_f \rightarrow \frac{\Omega_0}{2(1-\Omega_0)(\cosh \theta - 1)}$ because there was nothing special about time a'_f . Then we have:

$$\int_0^{\theta_f} \frac{\alpha^2 (\cosh \theta - 1) \sinh \theta d\theta}{\sqrt{1 - \Omega_0} \alpha \sinh \theta} = H_0 t_f \quad (118)$$

$$\frac{\alpha}{1 - \Omega_0} (\sinh \theta - \theta) \Big|_0^{\theta_f} = H_0 t_f \quad (119)$$

$$(120)$$

Plugging α back in:

$$H_0 t_f = \frac{\Omega_0}{2(1 - \Omega_0)^{\frac{3}{2}}} (\sinh \theta - \theta) \quad (121)$$

$$t(\theta) = \frac{\Omega_0}{2H_0(1 - \Omega_0)^{\frac{3}{2}}} (\sinh \theta - \theta) \quad (122)$$

$$(123)$$

Thus we have a parametric relationship between a and t with a dummy variable θ :

$$a(\theta) = \frac{\Omega_0}{2(1 - \Omega_0)} (\cosh \theta - 1) \quad (124)$$

$$t(\theta) = \frac{\Omega_0}{2H_0(1 - \Omega_0)^{\frac{3}{2}}} (\sinh \theta - \theta) \quad (125)$$

$$(126)$$

Note that a has no maximum and expands forever.

Closed, $k > 0$, $\Omega_0 > 1$, $\Omega_\Lambda = 0$. As usual, we start with the Friedmann Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \underbrace{\frac{8\pi}{3}G\rho}_{\text{can be 0}} - \underbrace{\frac{k}{a^2}}_{\text{positive}}$$

Therefore, in this model it is possible for $\dot{a} = 0$. We'll again solve for a matter-dominated era, $w = 0$. The solution is similar, to the open model, but $(1 - \Omega_0 < 0)$, so $\theta \rightarrow i\theta$. Thus our solution:

$$a(\theta) = \frac{\Omega_0}{2(1 - \Omega_0)} (\cos \theta - 1) \quad (127)$$

$$t(\theta) = \frac{\Omega_0}{2H_0(1 - \Omega_0)^{\frac{3}{2}}} (\sin \theta - \theta) \quad (128)$$

$$(129)$$

is a cycloid solution. That is, a oscillates, with a maximum at $\theta = \pi$.

A.3.2 Deceleration Parameter q

$$q \equiv -\ddot{a}a\dot{a}^2$$

Note that q is *dimensionless*, and has a negative sign. The minus is there because historically, the universe was thought to be decelerating. Thus, $q < 0 \Rightarrow \text{acceleration}$, and $q > 0 \Rightarrow \text{deceleration}$. We can find out what q does from the 2nd Friedmann Equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G(\rho + 3P)$$

Recall that $P = w\rho$, $\Omega = \frac{\rho}{\rho_{crit}}$, $\rho_{crit} = \frac{3H^2}{8\pi G}$, $H^2\Omega = \frac{8\pi}{3}G\rho$. Thus, the above equation becomes:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\rho(1+3w) = -\frac{H^2\Omega}{2}(1+3w)$$

Substituting in our friend, q :

$$q = \frac{\Omega}{2}(1+3w) = \sum_i \frac{\Omega_i}{2}(1+3w_i)$$

$$q = \frac{\Omega_m}{2} - \Omega_\Lambda + \Omega_r.$$

Note that today, $\Omega_r \ll 1$. That we are accelerating $\Rightarrow q < 0 \Rightarrow \Omega_{0,\Lambda} \neq 0$.

A.3.3 Redshift

$$\frac{\lambda_{obs}}{\lambda_{emit}} \equiv 1 + z$$

Or alternately,

$$1 + z = \frac{1}{a}$$

z is what we call the *redshift*. Reionization happened somewhere around $z = 17 \pm 6$, and recombination around $z = 1091$.

A.3.4 Time-Redshift Relations and the Age of the Universe

We seek a relation between t and z . Beginning with the Friedmann Equation:

$$H^2 = H_0^2 \left[\frac{\Omega_0}{a^{3+3w}} + \frac{1-\Omega_0}{a^2} \right] \quad (130)$$

$$\frac{1}{a} \frac{da}{dt} = H_0 \sqrt{\frac{\Omega_0}{a^{3+3w}} + \frac{1-\Omega_0}{a^2}} \quad (131)$$

$$dt = \frac{1}{H_0} \left(\frac{1}{a} \right) \frac{da}{\sqrt{\frac{\Omega_0}{a^{3+3w}} + \frac{1-\Omega_0}{a^2}}} \quad (132)$$

$$dt = -\frac{1}{H_0} \frac{dz}{(1+z) \sqrt{\Omega_{0,m}(1+z)^3 + \Omega_{0,\Lambda} + (1-\Omega_{0,m} - \Omega_{0,\Lambda})(1+z)^2}} \quad (133)$$

$$(134)$$

We can calculate the time since the Big Bang at redshift z_1 by solving the integral:

$$t_1 = -\frac{1}{H_0} \int_{z_1}^{\infty} \frac{dz}{1+z} \frac{1}{\sqrt{\Omega_{0,m}(1+z)^3 + \Omega_{0,\Lambda} + (1-\Omega_{0,m} - \Omega_{0,\Lambda})(1+z)^2}}$$

For the Age of the Universe, set $z_1 = 0$. In the special case of a flat (Einstein-deSitter) universe:

$$t_0 = \frac{1}{H_0} \int_0^{\infty} \frac{dz}{(1+z)^{\frac{5}{2}}} = 23H_0$$

If $H_0^{-1} \approx 10Gyrs$, then $t_0 \approx 6.7h^{-1}Gyrs$, so the Hubble constant needs to be pretty small to get reasonable estimates of the age of the universe.

A.4 Lecture 4

A.4.1 Time-Redshift Relations and the Age of the Universe

Last time we found the age of a flat universe. in a flat (Einstein-deSitter) universe: $k = 0$, $\Omega_m = 1$, $\Omega_\Lambda = 0$, so:

$$t_0 = \frac{1}{H_0} \int_0^\infty \frac{dz}{(1+z)^{\frac{5}{2}}} = \frac{2}{3H_0} \approx 6.7h^{-1}Gyr$$

Alternatively, recall that for a matter-dominated era, $a(t) = (tt_0)^{\frac{2}{3}}$. Thus, $H = \frac{\dot{a}}{a} = \frac{2}{3t} \Rightarrow t_0 = \frac{2}{3H_0}$.

If we have $\Lambda \neq 0$: $k = 0$, $\Omega_{0,M} + \Omega_{0,\Lambda} = 1$, then:

$$t_0 = \frac{1}{H_0} \int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_{0,M}(1+z)^3 + \Omega_{0,\Lambda}}}$$

Assuming $0.1 \leq \Omega_{0,M} \leq 1$, this integral is solvable:

$$t_0 = \frac{2}{3H_0} \frac{1}{\sqrt{1-\Omega_{0,M}}} \ln \left(\frac{1 + \sqrt{1-\Omega_{0,M}}}{\sqrt{\Omega_{0,M}}} \right) \approx \frac{2}{3H_0} (0.7\Omega_{0,M} + 0.3 - 0.3\Omega_{0,\Lambda})^{-0.3} \approx \frac{2}{3H_0} \Omega_{0,M}^{-0.3}$$

Generally, in a flat universe, $t_0 \propto \frac{1}{H_0}$. If $\Omega_0 \leq 1$, it will be longer.

In an open universe: $k < 0$, $\Omega_0 < 1$ ($\Omega_{0,\Lambda} = 0$). Recall:

$$t(\theta) = \frac{\Omega_0}{2H_0(1-\Omega_0)^{\frac{3}{2}}} (\sinh \theta - \theta), \quad a(\theta) = \frac{\Omega_0}{2(1-\Omega_0)} (\cosh \theta - 1)$$

So today:

$$a(\theta_0) = \Omega_0 2(1-\Omega_0) (\cosh \theta_0 - 1) = 1 \quad (135)$$

$$\cosh \theta_0 = \frac{2(1-\Omega_0)}{\Omega_0} + 1 = \frac{2-\Omega_0}{\Omega_0} \quad (136)$$

$$\sinh \theta_0 = \sqrt{\cosh^2 \theta_0 - 1} = \frac{2}{\Omega_0} \sqrt{1-\Omega_0} \quad (137)$$

$$t_0 = t(\theta_0) - \frac{\Omega_0}{2H_0(1-\Omega_0)^{\frac{3}{2}}} \left[\frac{2}{\Omega_0} \sqrt{1-\Omega_0} - \cosh^{-1} \left(\frac{2-\Omega_0}{\Omega_0} \right) \right] \quad (138)$$

$$(139)$$

$$\boxed{t_0 = \frac{1}{H_0} \left[(1-\Omega_0)^{-1} - \frac{1}{2} (1-\Omega_0)^{-\frac{3}{2}} \cosh^{-1} \left(\frac{2-\Omega_0}{\Omega_0} \right) \right]}$$

Thus $t_0 > \frac{2}{3H_0}$ for $\Omega_0 \leq 1$, and $t_0 = \frac{1}{H_0}$ for $\Omega_0 = 0$ (an empty universe).

In a closed universe: $k > 0$, $\Omega_0 > 1$, ($\Omega_{0,\Lambda} = 0$). Recall:

$$t(\theta) = \frac{\Omega_0}{2H_0(\Omega_0-1)^{\frac{3}{2}}} (\theta - \sin \theta) \quad (140)$$

$$a(\theta) = \frac{\Omega_0}{2(\Omega_0-1)} (1 - \cos \theta) \quad (141)$$

$$(142)$$

Thus, today:

$$a(\theta_0) = \frac{\Omega_0}{2(\Omega_0-1)} (1 - \cos \theta) = 1$$

$$\boxed{t_0 = \frac{1}{H_0} \left[(1-\Omega_0)^{-1} + \frac{1}{2} \Omega_0 (\Omega_0-1)^{-\frac{3}{2}} \cos^{-1} \left(\frac{2-\Omega_0}{\Omega_0} \right) \right]}$$

A.4.2 The Robertson-Walker Metric

Lorentz invariance dictates that two inertial frame (x, y, z, t) and (x', y', z', t') , with one moving with respect to the other at velocity $\hat{v} = v\hat{x}$, are related by:

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

where $\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$. Note, to give a taste of tensor forms, this all may be written as $x'^\alpha = \Lambda^\alpha_\beta x^\beta + I_0^\alpha$.

Remember the Lorentz invariant interval, which is conserved between frames:

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

Light travels a $ds^2 = 0$ path. In tensor form, this equation looks like:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

where $g_{\alpha\beta}$, the metric tensor, is given by:

$$g_{\alpha\beta} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Look at Weinberg, Ch. 13 for full proof, but for a homogeneous, γ -isotropic space, the metric looks like:

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega \right]$$

where r is a radial direction (in comoving coordinates), and $d\Omega = d\theta^2 + \sin^2 \theta d\phi^2$ is the differential angle separation of two points in space. As usual, k is the measure of curvature.

The $k = 0$ Model:

$$dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) = dx^2 + dy^2 + dz^2$$

so we recover the Minkowski metric for flat space, using comoving coordinates.

The $k > 0$ (closed) Model:

We get a coordinate singularity at $r = \frac{1}{\sqrt{k}}$, so this universe has a finite volume. For $k > 0$, we need to define ‘‘Polar Coordinates’’ in 4-D (to describe a 3-sphere embedded in 4-D). Here is a comparison of how we define polar coordinates for a 3-sphere in 4-D versus for a 2-sphere in 3-D:

<i>3 - sphere</i>	<i>2 - sphere</i>
$(x, y, z, w) \Leftrightarrow (R, \alpha, \beta, \gamma)$	$(x, y, z) \Leftrightarrow (R, \theta, \phi)$
$w = R \cos \alpha$	$z = R \cos \theta$
$z = R \sin \alpha \cos \beta$	$y = R \sin \theta \cos \phi$
$y = R \sin \alpha \sin \beta \cos \gamma$	$x = R \sin \theta \sin \phi$
$x = R \sin \alpha \sin \beta \sin \gamma$	
$x^2 + y^2 + z^2 + w^2 = R^2$	$x^2 + y^2 + z^2 = R^2$

Take a line element on a 2-sphere:

$$d\gamma = R^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

Changing variables for $v \equiv \sin \theta$:

$$dv = \cos \theta d\theta = \sqrt{1 - v^2} d\theta$$

Then $d\theta^2 = \frac{dv^2}{1-v^2}$, so rewriting our line element, we get:

$$d\gamma^2 = R^2 \left(\frac{dv^2}{1-v^2} + v^2 d\phi^2 \right)$$

For a 3-sphere,

$$d\gamma^2 = R^2 (d\alpha^2 + \sin^2 \alpha d\Omega^2)$$

where $d\Omega^2 \equiv d\beta^2 + \sin^2 \beta d\gamma^2$. Again, using a change of variables so that $v \equiv \sin \alpha$, $d\alpha = \frac{dv}{\sqrt{1-v^2}}$, we get that:

$$\boxed{d\gamma^2 = R^2 \left(\frac{dv^2}{1-v^2} + v^2 d\Omega^2 \right)}$$

This is what Robertson-Walker showed.

A.5 Lecture 5

A.6 Lecture 6

A.7 Lecture 7

A.8 Lecture 8

A.9 Lecture 9

A.10 Lecture 10

A.11 Lecture 11