RIYA Final Report (Summer 2021)

Dynamics of Vibration Absorber System with Focus on Nonlinear Paths

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Introduction

Motivation: Vibration absorber is used to control vibrations of a primary structure and prevent damage to structural components. But a nonlinear path adds complexity and allows larger variations in the operating frequency regime [1].

Potential applications of absorbers and tuned dampers:

- *Reduce response of structures subjected to seismic motions [2] or wind loads [3]*
- *Supress vibrations in the cutting tool during turning operations [4]*

Goal: Investigate the effects of nonlinear elastic path(s) on the dynamic response of a vibration absorber system using the multi-term harmonic balance method (MHBM)

[1] R. E. Roberson, Synthesis of a nonlinear dynamic vibration absorber, Journal of the Franklin Institute, 254(3), 205-220, 1952.

[2] F. Sadek, B. Mohraz, A.W. Taylor, and R.M. Chung, A method of estimating the parameters of tuned mass dampers for seismic applications, Earthquake Engineering & Structural Dynamics, 26(6), 1997.

[3] K.C.S. Kwok and B. Samali, Performance of tuned mass dampers under wind loads, Engineering structures, 17(9), 655-667, 1995.

[4] E.C. Lee, C.Y. Nian, and Y.S. Tarng, Design of a dynamic vibration absorber against vibrations in turning operations, Journal of Materials Processing Technology, 108(3), 278-285, 2001.

Tuned mass damper atop Taipei 101

Source: https://en.wikipedia.org/wiki/Tuned_mass_damper

Vibration absorbers installed on the Millenium Bridge, London

Source: https://en.wikipedia.org/wiki/Tuned_mass_damper

Literature Review

Author(s) (Journal, Year)	Topic	Comments
Roberson (JFI, 1952) [1]	Synthesis of a nonlinear dynamic vibration absorber	Undamped 2DOF model, Duffing iteration method
Sadek et al. (EESD, 1997) [2]	A method of estimating the parameters of tuned mass dampers for seismic applications	Multi-DOF structures, seismic loading, linear path with damping
Kwok & Samali (Engg. Struct., 1995) [3]	Performance of tuned mass dampers under wind loads	Linear undamped 2DOF model, passive and active vibration absorbers
Lee & Tarng (JMPT, 2001) [4]	Design of a dynamic vibration absorber against vibrations in turning operations	Linear damped 2DOF model, experimental verification
Sen & Singh (NOVEM Conf., 2018) [5]	Energy Exchange between Two Sub-systems Coupled with a Nonlinear Elastic Path	2DOF damped harmonic oscillator, nonlinear coupling, MHBM
Den Hartog (1985) [6]	Mechanical Vibrations	Dynamic vibration absorbers, forced vibrations with nonlinear springs

Problem Formulation

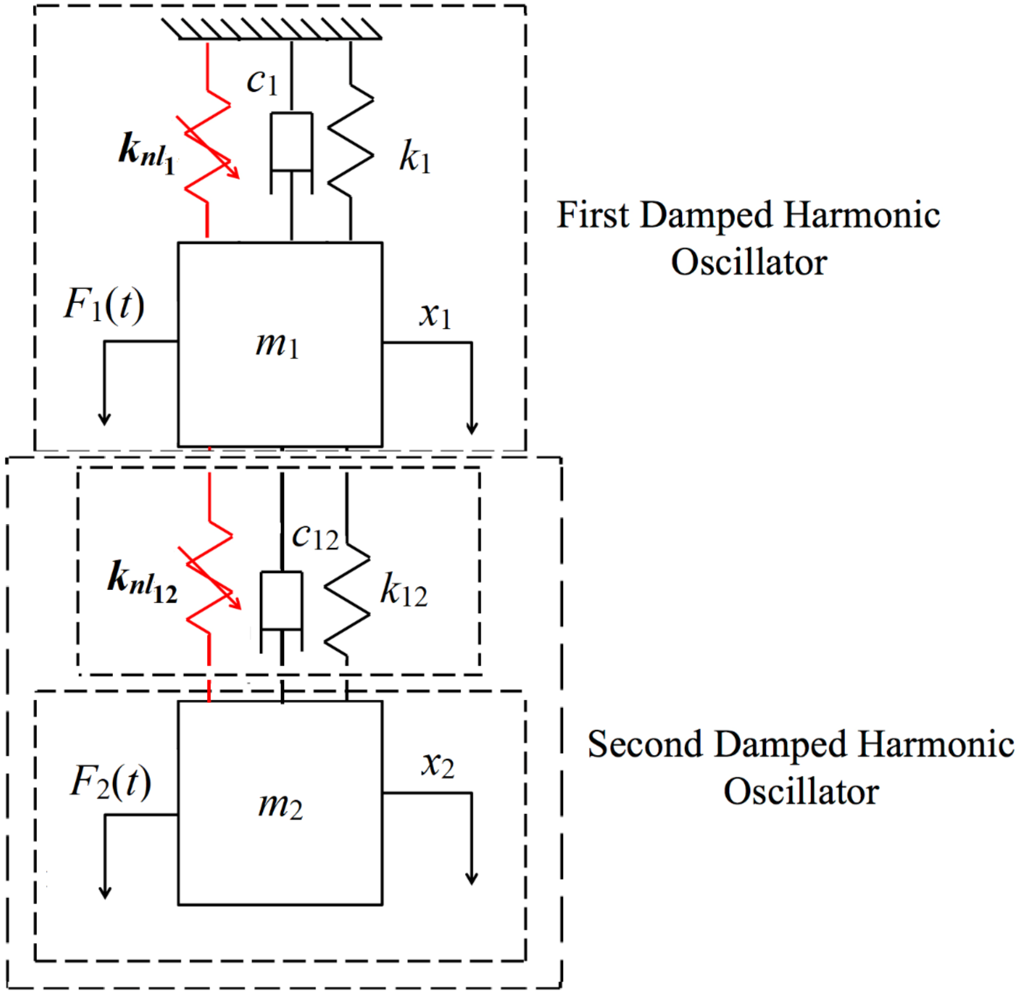
❖ Objective 1: Obtain nonlinear frequency responses of the example case using a semi-analytical multi-term harmonic balance method (MHBM), and verify results using Sen & Singh's approach [5]

❖ Objective 2: Investigate the dynamic effects of two nonlinear cubic springs on the nonlinear frequency responses

❖ Objective 3: Conduct parametric studies (relative linear stiffness, hardening/softening-type nonlinearity etc.) for the system

Scope and Assumptions

- 2DOF
- 2 springs with cubic nonlinearity
- Harmonic excitations on either/both masses
- Steady-state solutions and frequency-response curves
- Elastic coupling forces



Example Case: 2DOF nonlinear vibration absorber system

[5] O.T. Sen and R. Singh, Energy Exchange between Two Sub-systems Coupled with a Nonlinear Elastic Path, NOVEM 2018 Conference, Ibiza, May 2018.

Governing equations in the dimensionless form

Equations of motion

$$m_1\ddot{x}_1 + (c_1 + c_{12})\dot{x}_1 - c_{12}\dot{x}_2 + (k_1 + k_{12})x_1 - k_{12}x_2 + k_{nl_1}x_1^3 + k_{nl_{12}}(x_1 - x_2)^3 = F_1^S + F_1^D \sin(\omega t + \phi_1)$$
$$m_2\ddot{x}_2 + c_{12}\dot{x}_2 - c_{12}\dot{x}_1 + k_{12}x_2 - k_{12}x_1 + k_{nl_{12}}(x_2 - x_1)^3 = F_2^S + F_2^D \sin(\omega t + \phi_2)$$

Non-dimensional parameters

$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, \omega_n = \sqrt{\frac{k_1}{m_1}}, \zeta = \frac{c_1}{2m_1\omega_n}, \alpha_1 = \frac{k_{nl_1}x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}}x_0^2}{k_1}, \tau = \omega_n t$$
$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S}, \epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

where the operating point is defined as $x_0 = \frac{F_1^S}{k_1}$ **(Assumption: $\boldsymbol{F_1^S \neq 0}$)**.

Transforming the independent variable, $\theta = \omega\tau$, the final equations in the dimensionless form are:

$$\omega^2 \frac{d^2X_1}{d\theta^2} + [2\zeta(1+\gamma)]\omega\frac{dX_1}{d\theta} - 2\zeta\gamma\omega\frac{dX_2}{d\theta} + (1+\lambda)X_1 - \lambda X_2 + \alpha_1 X_1^3 + \alpha_{12}(X_1 - X_2)^3 = 1 + \epsilon_1 \sin\left(\frac{\theta}{\omega_n} + \phi_1\right)$$
$$\mu\omega^2 \frac{d^2X_2}{d\theta^2} + 2\zeta\gamma\omega\frac{dX_2}{d\theta} - 2\zeta\gamma\omega\frac{dX_1}{d\theta} + \lambda X_2 - \lambda X_1 + \alpha_{12}(X_2 - X_1)^3 = \sigma + \epsilon_2 \sin\left(\frac{\theta}{\omega_n} + \phi_2\right)$$

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Elastic coupling force: $\lambda(X_2 - X_1) + \alpha_{12}(X_2 - X_1)^3$

MHBM applied to 2DOF nonlinear system (Objective 1)

Assumed solution: Truncated Fourier series with N_h harmonics retained

$$X_1(\theta) = a_{0,1} + a_{1,1}\cos(\theta) + b_{1,1}\sin(\theta) + \ldots + a_{N_h,1}\cos(N_h\theta) + b_{N_h,1}\sin(N_h\theta)$$
$$X_2(\theta) = a_{0,2} + a_{1,2}\cos(\theta) + b_{1,2}\sin(\theta) + \ldots + a_{N_h,2}\cos(N_h\theta) + b_{N_h,2}\sin(N_h\theta)$$

Discretization of independent variable: $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_N]^T$, where $2N_h < N$ (Nyquist-Shannon sampling theorem)

Let $\begin{bmatrix} 1 & \cos(\theta_1) & \sin(\theta_1) & \cdots & \cos(N_h\theta_1) & \sin(N_h\theta_1) \\ & \vdots & & \ddots & \vdots & \\ 1 & \cos(\theta_N) & \sin(\theta_N) & \cdots & \cos(N_h\theta_N) & \sin(N_h\theta_N) \end{bmatrix} = \boldsymbol{F_r}$, and $\left[a_{0,k}\ a_{1,k}\ b_{1,k}\ \ldots a_{N_{h,k}}\ b_{N_{h,k}}\right]^T = \boldsymbol{A_k}\ (k = 1,2)$

$$\boldsymbol{R_1} = \omega^2 \boldsymbol{F_r D^2 A_1} + [2\zeta(1+\gamma)]\omega \boldsymbol{F_r D A_1} - 2\zeta\gamma\omega \boldsymbol{F_r D A_2} + (1+\lambda)\boldsymbol{F_r A_1} - \lambda \boldsymbol{F_r A_2} + \alpha_1(\boldsymbol{F_r A_1})^3 + \alpha_{12}(\boldsymbol{F_r A_1} - \boldsymbol{F_r A_2})^3 - 1 - \epsilon_1 \sin\left(\frac{\boldsymbol{\theta}}{\omega_n} + \phi_1\right)$$
$$\boldsymbol{R_2} = \mu\omega^2 \boldsymbol{F_r}\mathbf{D^2A_2} + 2\zeta\gamma\omega \boldsymbol{F_r D A_2} - 2\zeta\gamma\omega \boldsymbol{F_r D A_1} + \lambda \boldsymbol{F_r A_2} - \lambda \boldsymbol{F_r A_1} + \alpha_{12}(\boldsymbol{F_r A_2} - \boldsymbol{F_r A_1})^3 - \sigma - \epsilon_2 \sin\left(\frac{\boldsymbol{\theta}}{\omega_n} + \phi_2\right)$$

Residue $\boldsymbol{R} = [\boldsymbol{R_1}\ \boldsymbol{R_2}]^T$, Jacobian $\boldsymbol{J} = \begin{bmatrix} \frac{\partial \boldsymbol{R_1}}{\partial \boldsymbol{A_1}} & \frac{\partial \boldsymbol{R_1}}{\partial \boldsymbol{A_2}} \\ \frac{\partial \boldsymbol{R_2}}{\partial \boldsymbol{A_1}} & \frac{\partial \boldsymbol{R_2}}{\partial \boldsymbol{A_2}} \end{bmatrix}$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & N_h \\ 0 & 0 & 0 & & 0 & -N_h & 0 \end{bmatrix}$$

The Fourier coefficients vector $\boldsymbol{A} = [\boldsymbol{A_1}\ \boldsymbol{A_2}]^T$ is computed using Newton Raphson method

At each iteration i, $\boldsymbol{A}_{i+1} = \boldsymbol{A}_i - (\boldsymbol{J}_i^T \boldsymbol{J}_i)^{-1} \boldsymbol{J}_i^T \boldsymbol{R}_i$

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Verification of frequency responses using Sen & Singh [5]

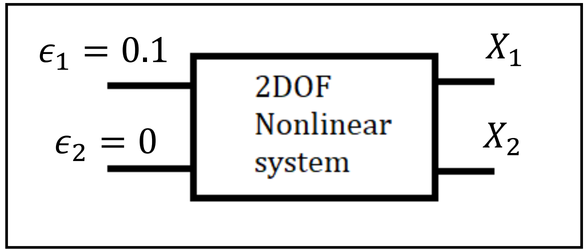
System parameters: $\mu = 0.2, \gamma = 0.2, \lambda = 0.2, \zeta = 0.005, \sigma = \frac{\mu}{1+\mu}$

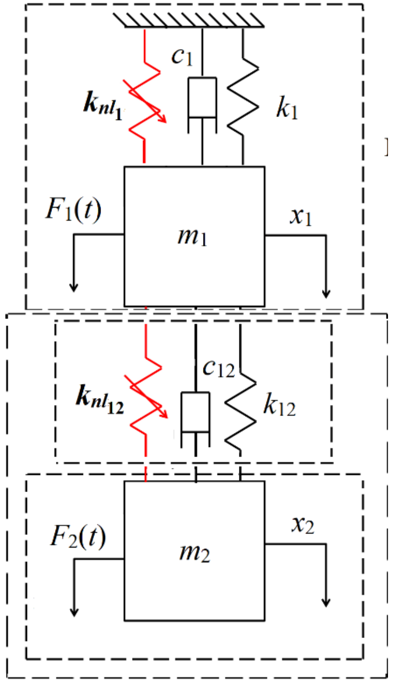
$\boldsymbol{\alpha_1 = 0.01, \alpha_{12} = 0.01}$

Natural frequencies (for linear system): $0.8\omega_n, 1.25\omega_n$

Solution parameters: $N_h = 80, N = 200$

Excitation





$$F_1(t) = F_1^S + F_1^D \sin(\omega t)$$
$$F_2(t) = F_2^S + F_2^D \sin(\omega t)$$

$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
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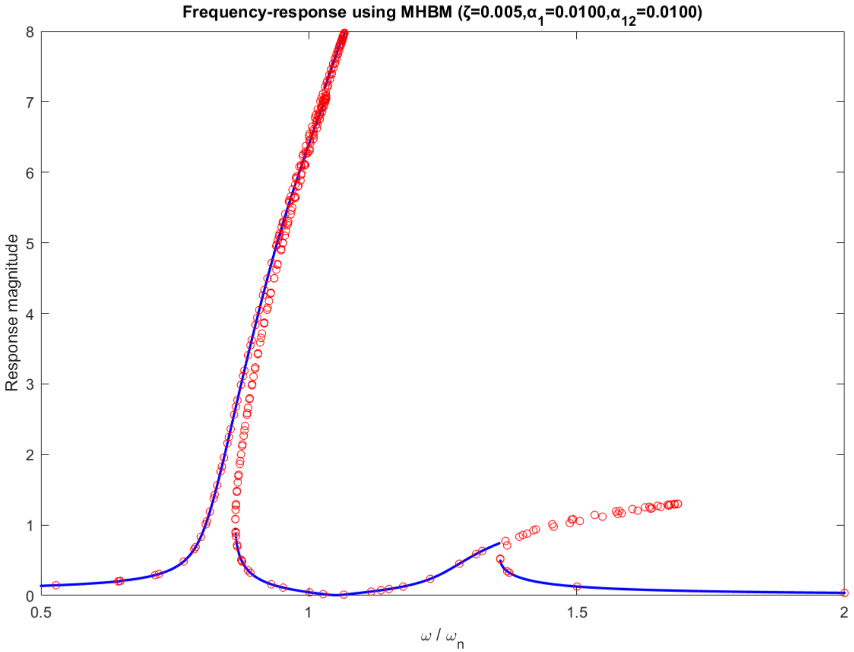
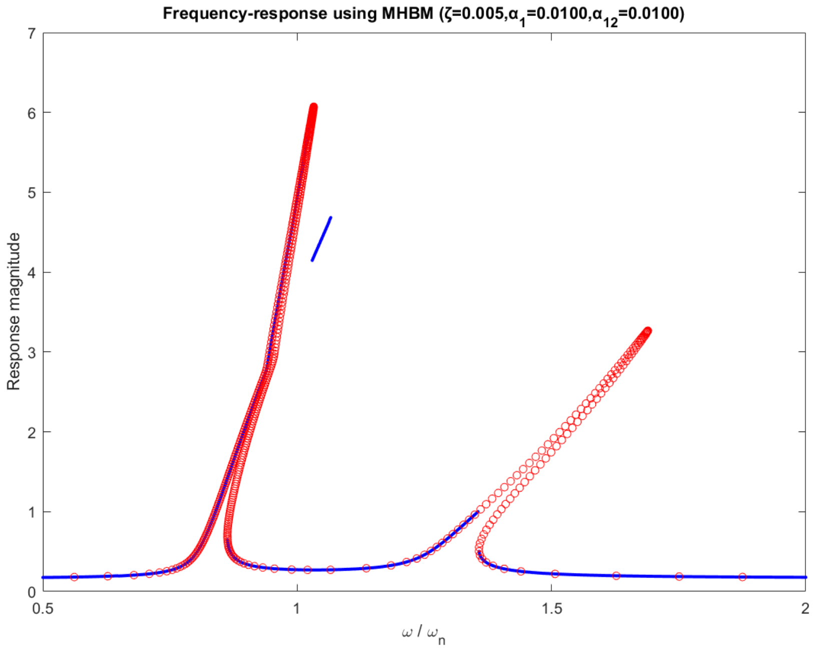


Frequency response curve: X_1



Elastic coupling force vs ω

Legend
MHBM without continuation method
MHBM with continuation method [5]

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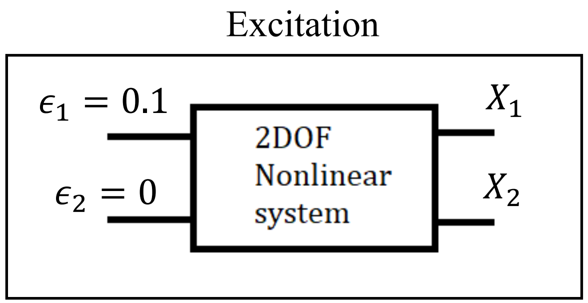
Jump phenomenon

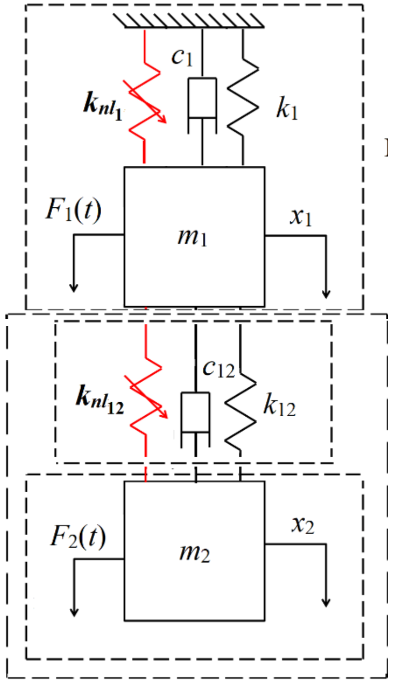
System parameters: $\mu = 0.2, \gamma = 0.2, \lambda = 0.2, \zeta = 0.005, \sigma = \frac{\mu}{1+\mu}$

$\boldsymbol{\alpha_1 = 0.01, \alpha_{12} = 0.01}$

Natural frequencies (for linear system): $0.8\omega_n, 1.25\omega_n$

Solution parameters: $N_h = 80, N = 200$





$F_1(t) = F_1^S + F_1^D \sin(\omega t)$
$F_2(t) = F_2^S + F_2^D \sin(\omega t)$

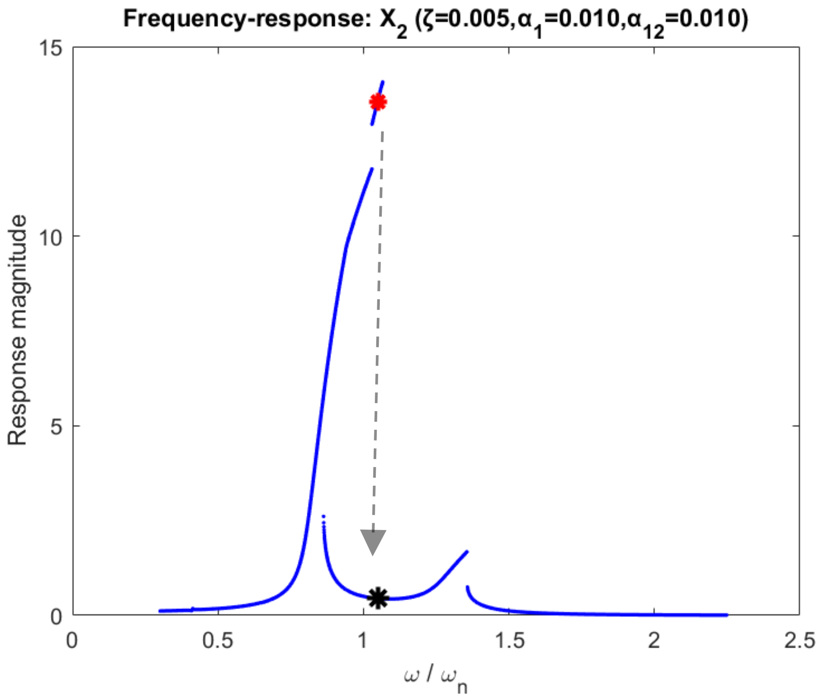
$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$

$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$

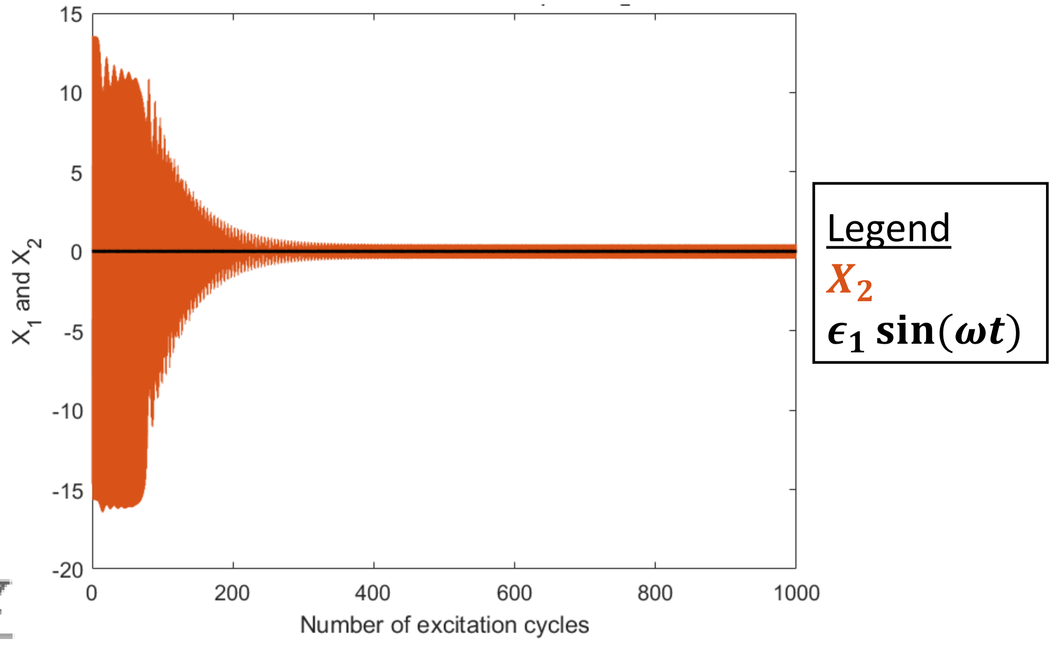
$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$

$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$

Numerical solution (using Runge-Kutta method) is computed at the **red** marker; the response attains a steady-state value represented by the **black** marker (on the stable branch)



Frequency response curve: X_2



Time-domain solution (Runge-Kutta method)

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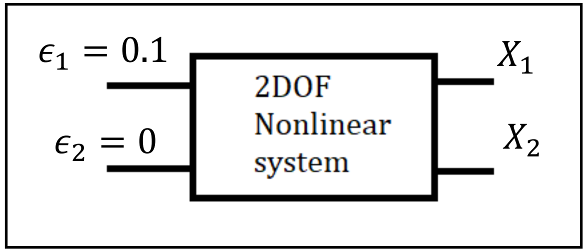
Effect of relative linear stiffness (λ)

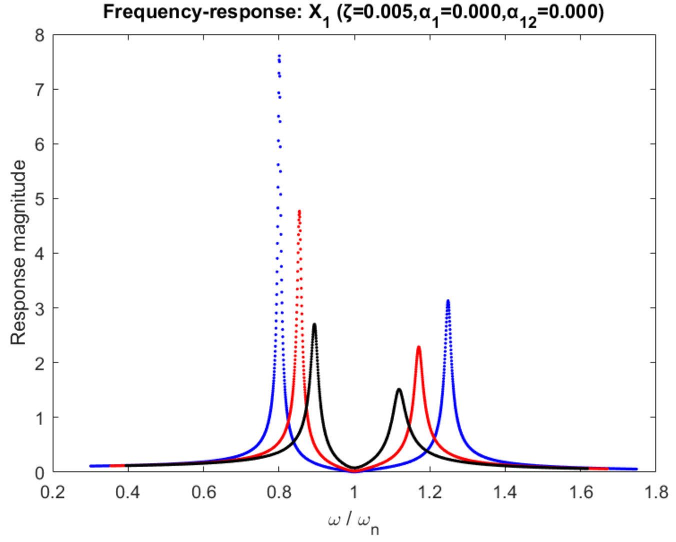
System parameters: $\gamma = 0.2, \boldsymbol{\mu} = \boldsymbol{\lambda}, \zeta = 0.005, \sigma = \frac{\mu}{1+\mu}$

$\boldsymbol{\alpha_1 = 0.01, \alpha_{12} = 0.01}$

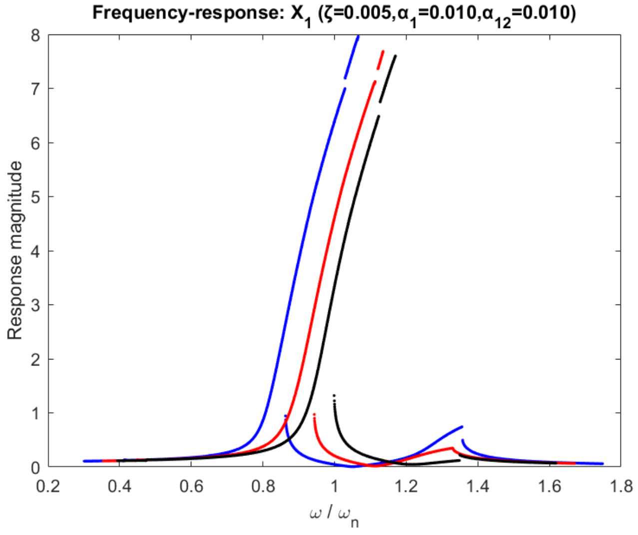
Solution parameters: $N_h = 80, N = 200$

Excitation





Frequency response curve: X_1
(Linear system)



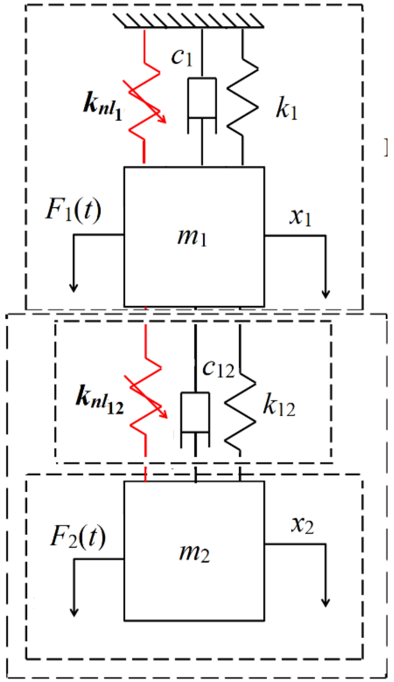
Frequency response curve: X_1
(Nonlinear system)

Legend

$\boldsymbol{\lambda = 0.2}$

$\boldsymbol{\lambda = 0.1}$

$\boldsymbol{\lambda = 0.05}$



$$F_1(t) = F_1^S + F_1^D \sin(\omega t)$$
$$F_2(t) = F_2^S + F_2^D \sin(\omega t)$$

$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
$$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

The second natural frequency decreases (due to $\lambda \downarrow$), but the hardening-type paths ($\alpha_1 > 0, \alpha_{12} > 0$) prevent the left-shift in the response curves near the second natural frequencies

Linear natural frequencies

$\boldsymbol{0.8\omega_n, 1.25\omega_n}$

$\boldsymbol{0.85\omega_n, 1.17\omega_n}$

$\boldsymbol{0.9\omega_n, 1.1\omega_n}$

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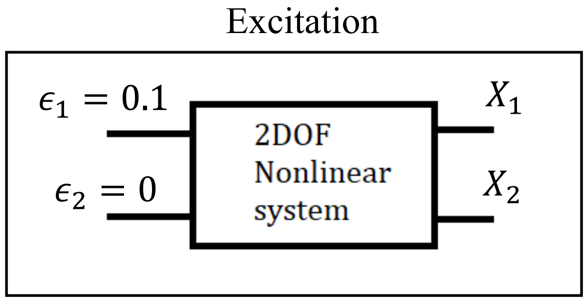
Vibration absorber for linear primary system

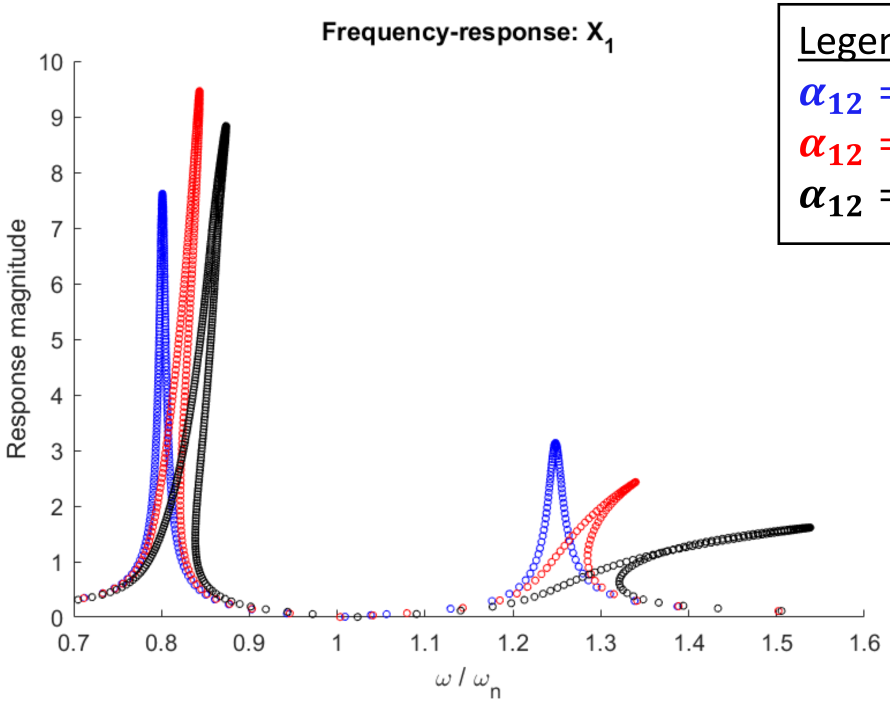
System parameters: $\mu = 0.2, \gamma = 0.2, \lambda = 0.2, \zeta = 0.005, \sigma = \frac{\mu}{1+\mu}$

$\boldsymbol{\alpha_1 = 0}$

Natural frequencies (for linear system): $0.8\omega_n, 1.25\omega_n$

Solution parameters: $N_h = 80, N = 200$





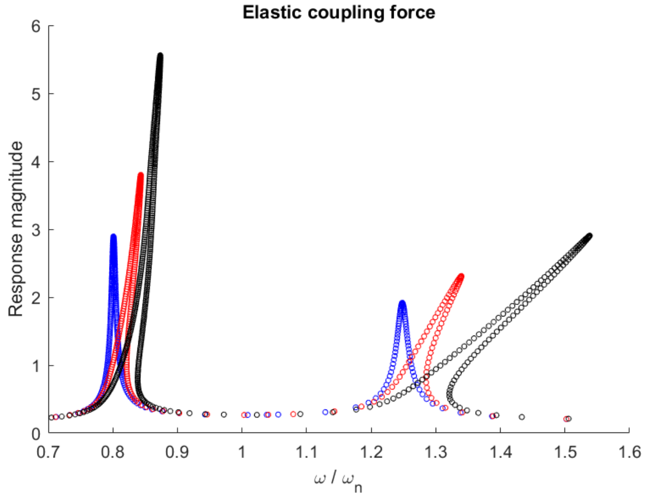
Frequency response curve: X_1

Legend

$\boldsymbol{\alpha_{12} = 0}$

$\boldsymbol{\alpha_{12} = 0.001}$

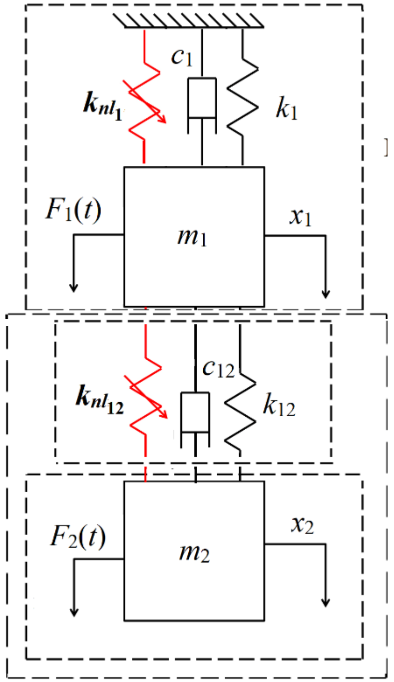
$\boldsymbol{\alpha_{12} = 0.005}$



Elastic coupling force vs ω

Vibration amplitudes of the primary system reduce at a given frequency (ω) with increasing path nonlinearity levels

Drawback: The maximum elastic force also increases with increasing nonlinearity levels



$$F_1(t) = F_1^S + F_1^D \sin(\omega t)$$
$$F_2(t) = F_2^S + F_2^D \sin(\omega t)$$

$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
$$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

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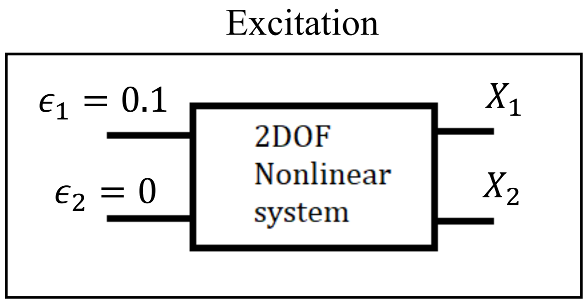
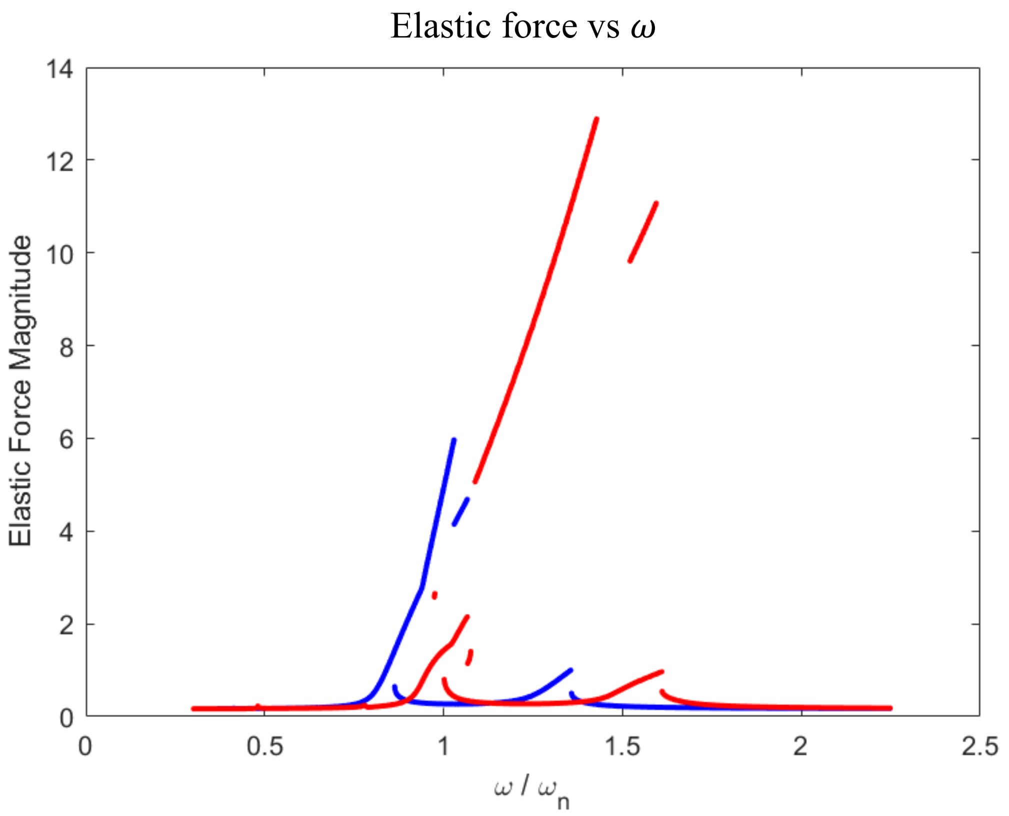
Effect of α on the elastic force

System parameters: $\mu = 0.2, \gamma = 0.2, \lambda = 0.2, \zeta = 0.005, \sigma = \frac{\mu}{1+\mu}$

Natural frequencies (for linear system): $0.8\omega_n, 1.25\omega_n$

Solution parameters: $N_h = 80, N = 200$





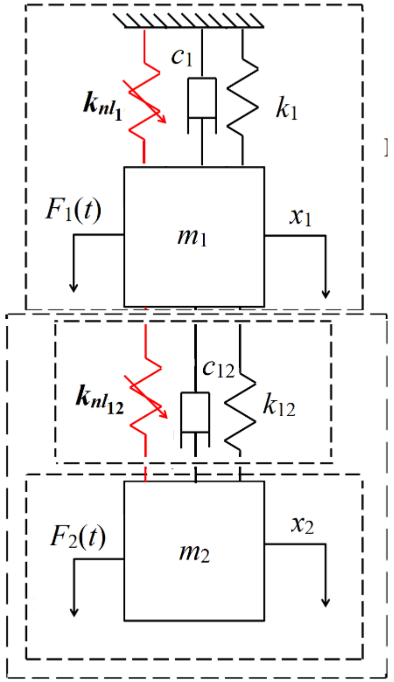
Legend

$\alpha_1 = \alpha_{12} = 0.01$

$\alpha_1 = \alpha_{12} = 0.1$

The magnitude of elastic coupling force increases with an increase in path nonlinearity level

High elastic forces may cause mechanical fatigue in the springs



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$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
$$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

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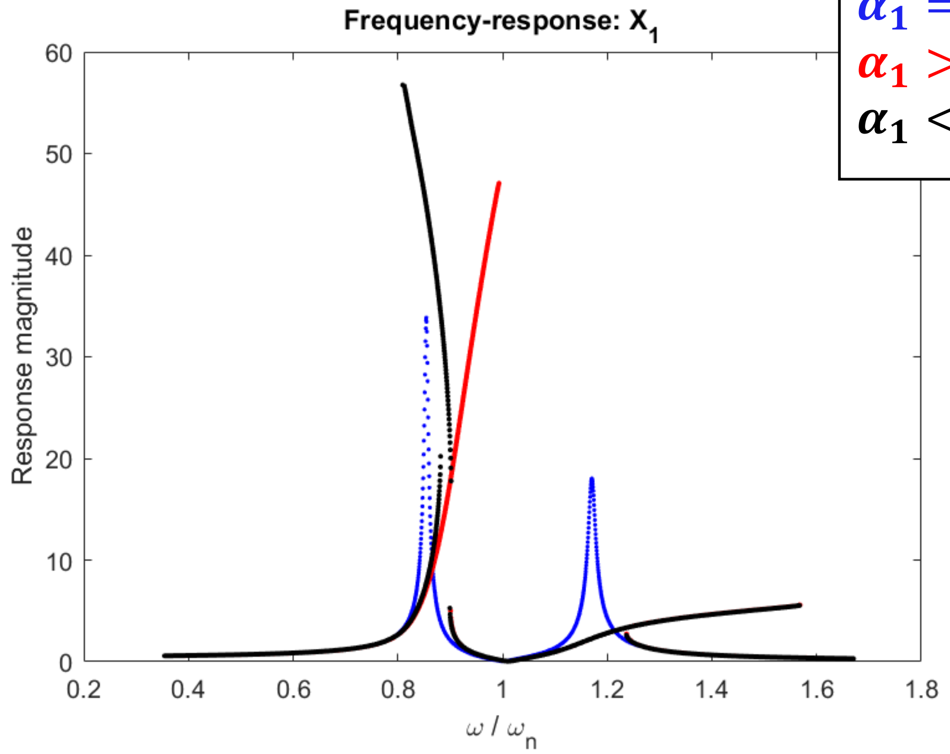
Mixed nonlinearities

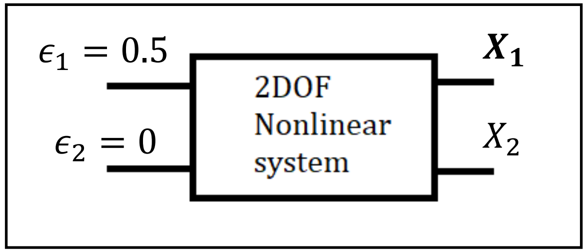
System parameters: $\mu = 0.1, \gamma = 0.1, \lambda = 0.1, \zeta = 0.005, |\boldsymbol{\alpha_1}| = |\boldsymbol{\alpha_{12}}| = \mathbf{0.00001}$
Natural frequencies (for linear system): $0.85\omega_n, 1.7\omega_n$
Solution parameters: $N_h = 80, N = 200$

Legend
$\boldsymbol{\alpha_1 = 0, \alpha_{12} = 0}$
$\boldsymbol{\alpha_1 > 0, \alpha_{12} > 0}$
$\boldsymbol{\alpha_1 < 0, \alpha_{12} > 0}$



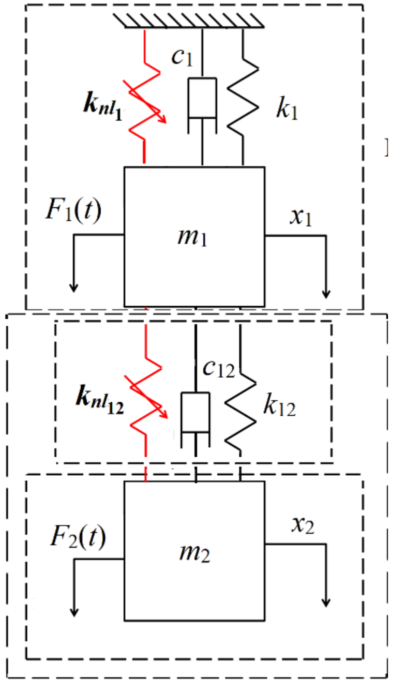
Frequency response curve: X_1

Excitation



Overlap between black and red curves near the second natural frequency

The response near the second natural frequency is significantly affected only by the path nonlinearity ($k_{nl_{12}}$)



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$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
$$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

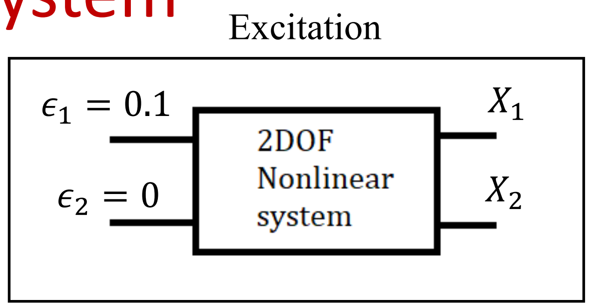
Vibration absorber for nonlinear primary system

System parameters: $\mu = 0.2, \gamma = 0.2, \lambda = 0.2, \zeta = 0.005, \sigma = \frac{\mu}{1+\mu}$

$\boldsymbol{\alpha_1 = 0.001}$

Natural frequencies (for linear system): $0.8\omega_n, 1.25\omega_n$

Solution parameters: $N_h = 80, N = 200$

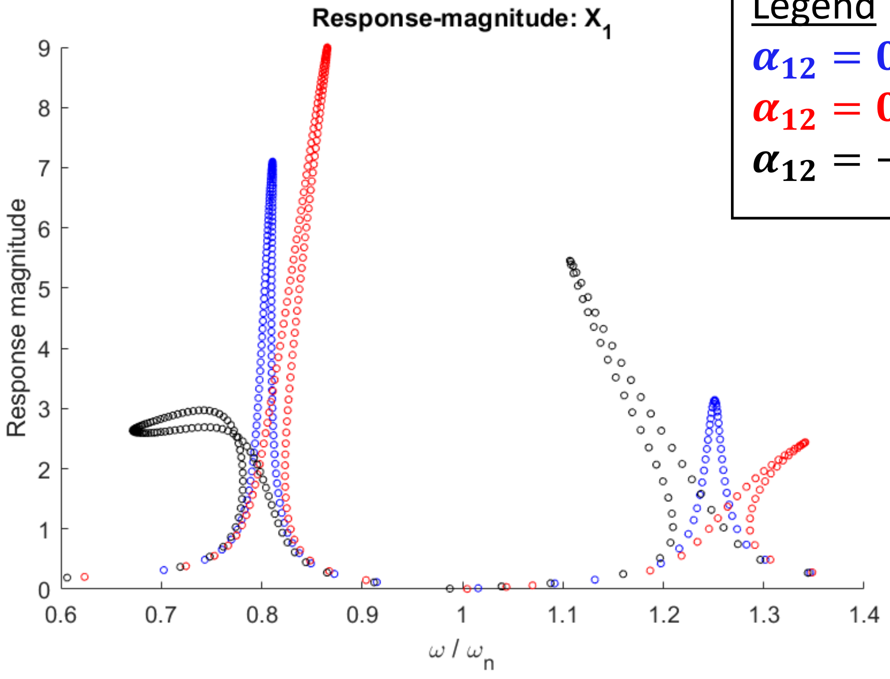


Legend

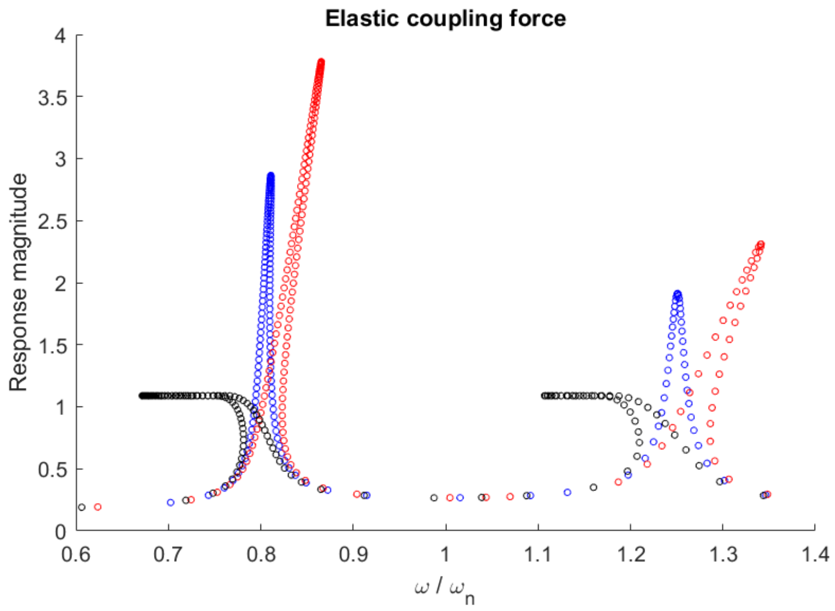
$\boldsymbol{\alpha_{12} = 0}$

$\boldsymbol{\alpha_{12} = 0.001}$

$\boldsymbol{\alpha_{12} = -0.001}$

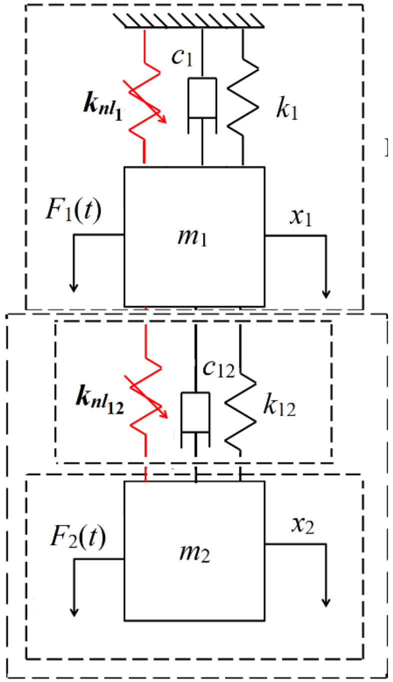


Frequency response curve: X_1



Elastic coupling force vs ω

Elastic coupling force corresponding to the softening-type path (black curve) appears to saturate near both natural frequencies (effect of $k_{12} > 0$ & $\alpha_{12} < 0$)



$$F_1(t) = F_1^S + F_1^D \sin(\omega t)$$
$$F_2(t) = F_2^S + F_2^D \sin(\omega t)$$

$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
$$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

Softening-type path

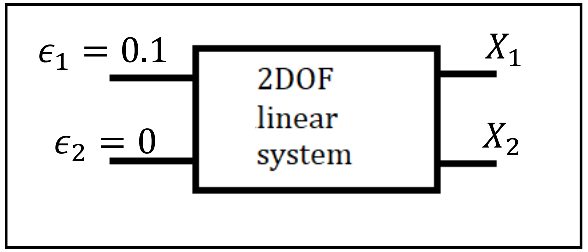
System parameters: $\mu = 0.2, \gamma = 0.2, \lambda = 0.2, \zeta = 0.005, \sigma = \frac{\mu}{1+\mu}$

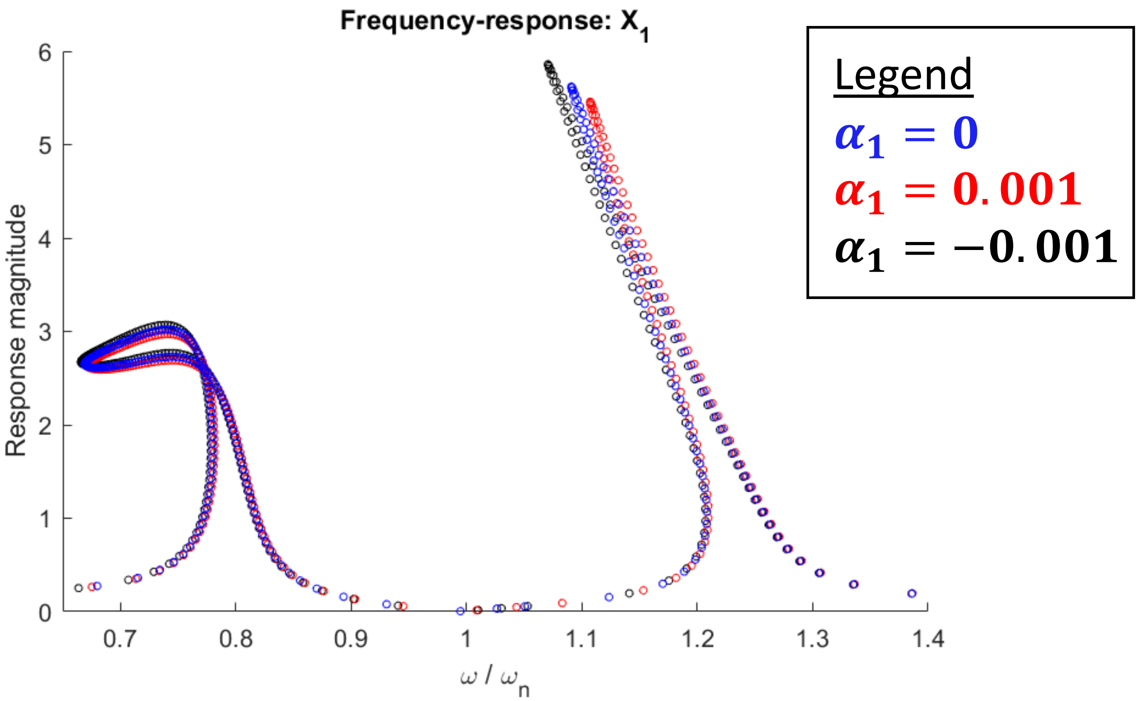
$\boldsymbol{\alpha_{12} = -0.001}$

Natural frequencies (for linear system): $0.8\omega_n, 1.25\omega_n$

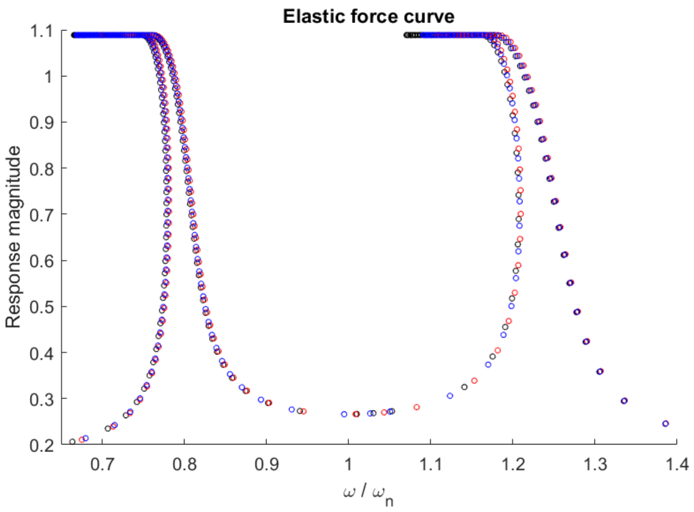
Solution parameters: $N_h = 80, N = 200$

Excitation

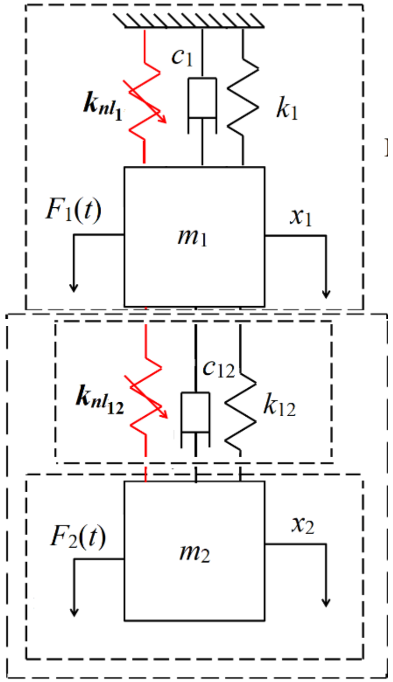




Frequency response curve: X_1



Elastic coupling force vs ω



$$F_1(t) = F_1^S + F_1^D \sin(\omega t)$$
$$F_2(t) = F_2^S + F_2^D \sin(\omega t)$$

$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
$$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

In this case, the type of nonlinearity (α_1) in the primary system does not significantly affect the nature of the response curves

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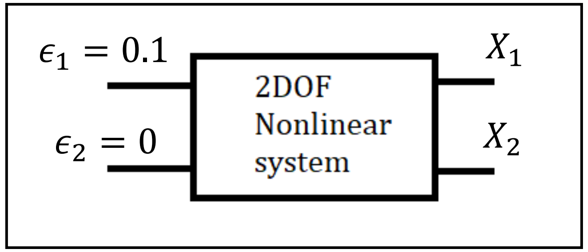
Effects of individual nonlinearity

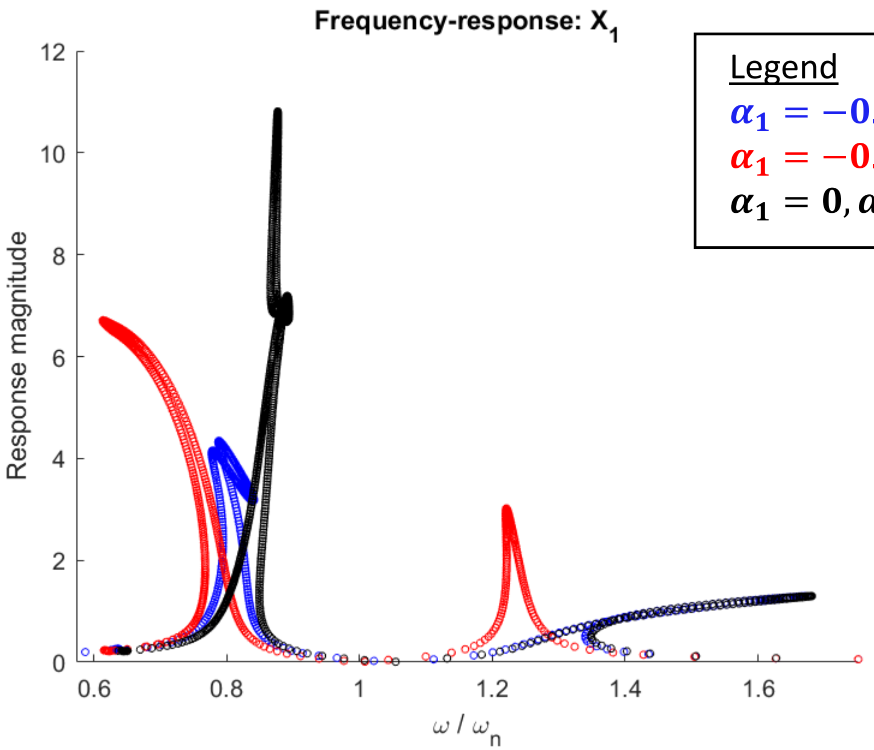
System parameters: $\mu = 0.2, \gamma = 0.2, \lambda = 0.2, \zeta = 0.005, \sigma = \frac{\mu}{1+\mu}$

Natural frequencies (for linear system): $0.8\omega_n, 1.25\omega_n$

Solution parameters: $N_h = 80, N = 200$

Excitation

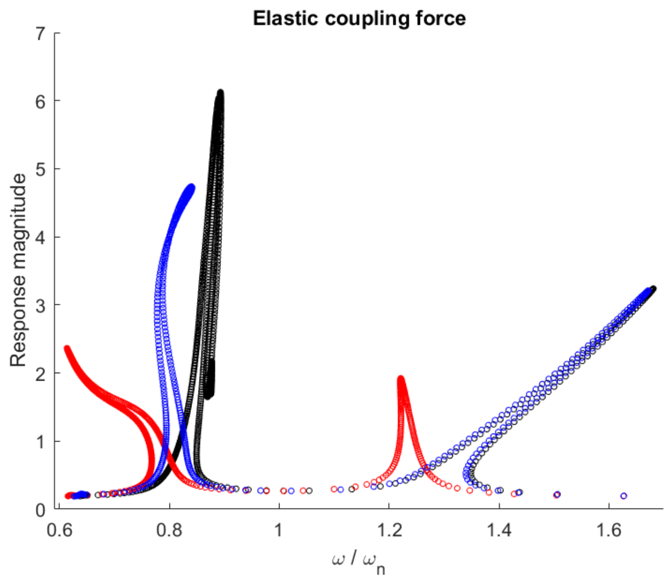




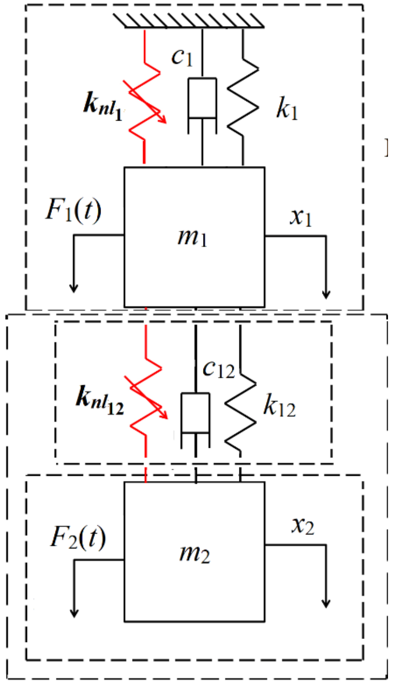
Frequency response curve: X_1

Legend

- $\boldsymbol{\alpha_1 = -0.01, \alpha_{12} = 0.01}$
- $\boldsymbol{\alpha_1 = -0.01, \alpha_{12} = 0}$
- $\boldsymbol{\alpha_1 = 0, \alpha_{12} = 0.01}$



Elastic coupling force



$$F_1(t) = F_1^S + F_1^D \sin(\omega t)$$
$$F_2(t) = F_2^S + F_2^D \sin(\omega t)$$

$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
$$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

Addition of a hardening-type path to the softening-type primary system limits the response amplitude (blue curve) in this case, in spite of the individual nonlinearities (red and black curves) having higher responses

Conclusion

- MHBM is successfully applied to a 2DOF nonlinear system
- Nonlinear paths affect the dynamics of the system; frequency response curves have multiple branches
- The MHBM solutions using both methods match for the most stable branches; some set of solutions are mutually exclusive
- The response near the second natural frequency is governed primarily by the path nonlinearity ($k_{nl_{12}}$) for all cases
- Choice of nonlinear path (hardening/softening-type) depends on the primary system, and the response amplitude and elastic force considerations

Methods of verifying results

- Sen & Singh's approach [5]
- Numerical solutions (Runge-Kutta method)

Difficulties/Challenges

- Obtaining unstable branches in the frequency response curves using MHBM
- Numerical errors in MHBM solutions
- Physical interpretation of results in some cases of softening-type nonlinear path (Appendix IV)

Future scope

- Determining system parameters for a specific application
- Excitation applied to secondary system (e.g. excitation due to wind loads)
- Experimental validation
- Presentation of work in a conference proceeding

[5] O.T. Sen and R. Singh, Energy Exchange between Two Sub-systems Coupled with a Nonlinear Elastic Path, NOVEM 2018 Conference, Ibiza, May 2018.

Appendix I: Lessons Learnt

- Basic characteristics of Duffing oscillators
- Vibration isolation
- Multi-term Harmonic Balance Method
- Numerical methods to solve nonlinear equations and the associated errors
- Basics of signal processing
- Importance and some techniques of non-dimensionalization
- Judgement of real-life parameters and approximations
- Practices in technical presentation

Appendix II: List of Selected References

[1] R. E. Roberson, Synthesis of a nonlinear dynamic vibration absorber, Journal of the Franklin Institute, 254(3), 205-220, 1952.

[2] F. Sadek, B. Mohraz, A.W. Taylor, and R.M. Chung, A method of estimating the parameters of tuned mass dampers for seismic applications, Earthquake Engineering & Structural Dynamics, 26(6), 1997.

[3] K.C.S. Kwok and B. Samali, Performance of tuned mass dampers under wind loads, Engineering structures, 17(9), 655-667, 1995.

[4] E.C. Lee, C.Y. Nian, and Y.S. Tarng, Design of a dynamic vibration absorber against vibrations in turning operations, Journal of Materials Processing Technology, 108(3), 278-285, 2001.

[5] O.T. Sen and R. Singh, Energy exchange between two sub-systems coupled with a nonlinear elastic path, NOVEM 2018 Conference, Ibiza, May 2018. *(Also, the matlab code used to obtain nonlinear frequency responses with both stable and unstable solutions.)*

[6] J.P. Den Hartog, Mechanical Vibrations, Dover, 79-105 and 370-377, 1985.

Appendix III: Pendulum – SDOF system with cubic nonlinearity

Governing equation of SDOF pendulum with

harmonic excitation: $\ddot{\phi} + \frac{g}{l}\sin(\phi) = \theta \sin(\omega t)$

Using series approximation, $\sin\phi \approx \phi - \frac{\phi^3}{3!}$,

$$\ddot{\phi} + \frac{g}{l}\phi - \frac{g}{(3!)l}\phi^3 = \theta \sin(\omega t)$$

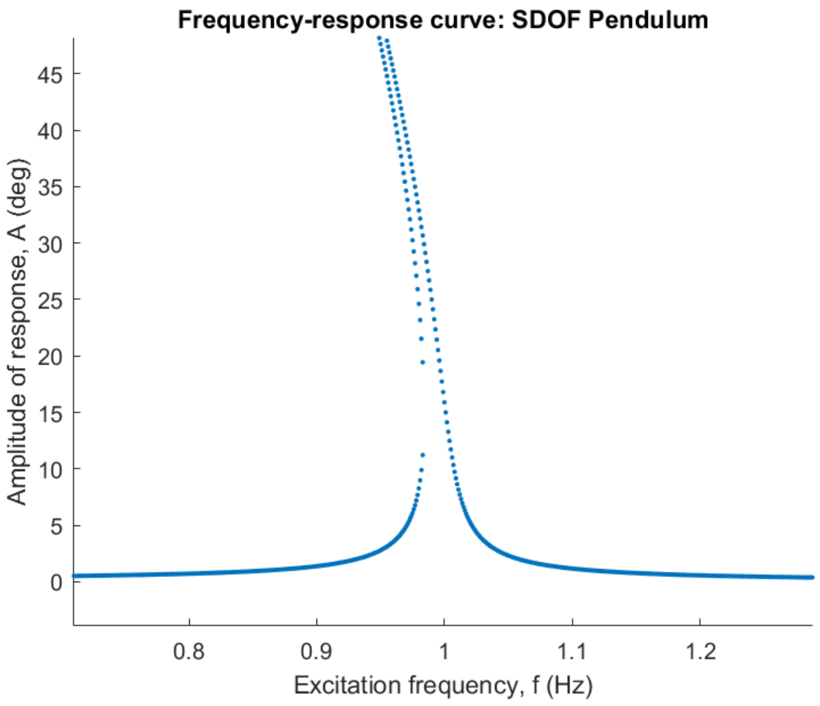
Harmonic Balance Method (HBM) for Duffing Oscillator

Assumed solution: $\phi(t) = A\sin(\omega t)$, $\omega = 2\pi f$

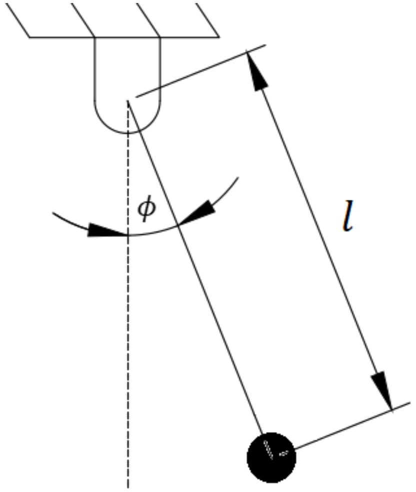
Substituting the assumed solution and balancing $\sin(\omega t)$ on both sides,

$$\left(\frac{g}{8l}\right)A^3 + \left(\omega^2 - \frac{g}{l}\right)A + \theta = 0$$

Real roots of the above equation are plotted against f to get the characteristic backbone curve



Backbone curve: $A\ vs\ f\ (l = 250mm, \theta = \frac{\pi}{18} N/kgm)$



Schematic: SDOF Pendulum

Pendulum is often used as a vibration absorber

Limitations of Single-term HBM:

- Cannot capture higher harmonics of the solution
- Cannot accommodate the effect of damping

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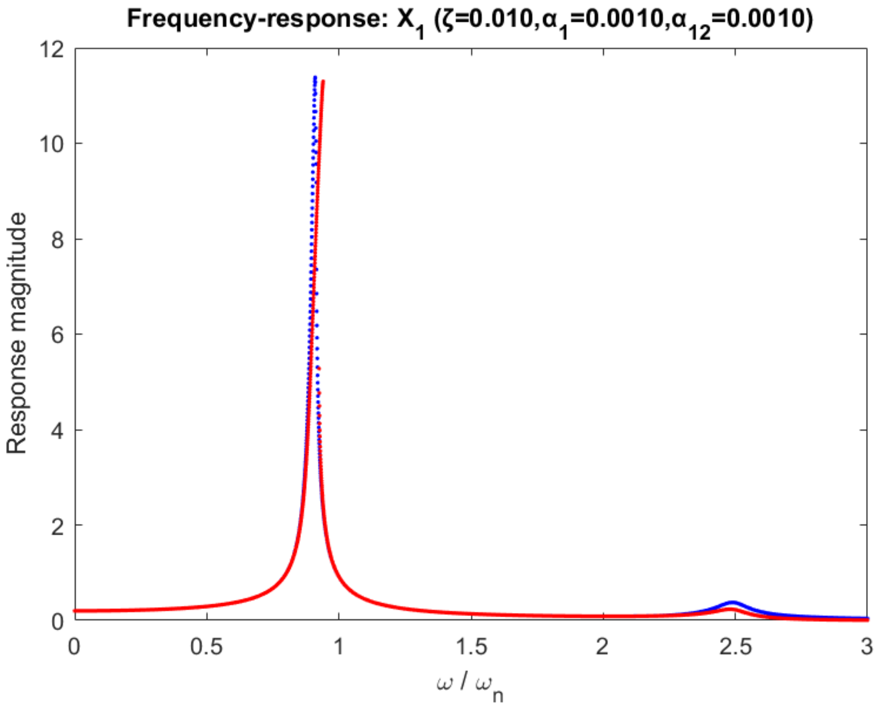
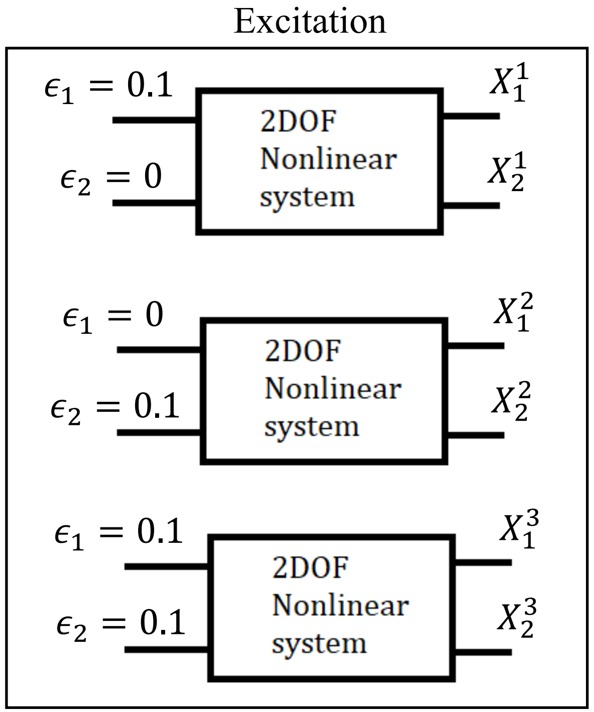
Appendix III: Principle of superposition

System parameters: $\mu = 0.2, \gamma = 1, \lambda = 1, \zeta = 0.01, \boldsymbol{\alpha_1 = \alpha_{12} = 0.001}$
Natural frequencies (for linear system): $0.9\omega_n, 2.49\omega_n$
Solution parameters: $N_h = 80, N = 200$

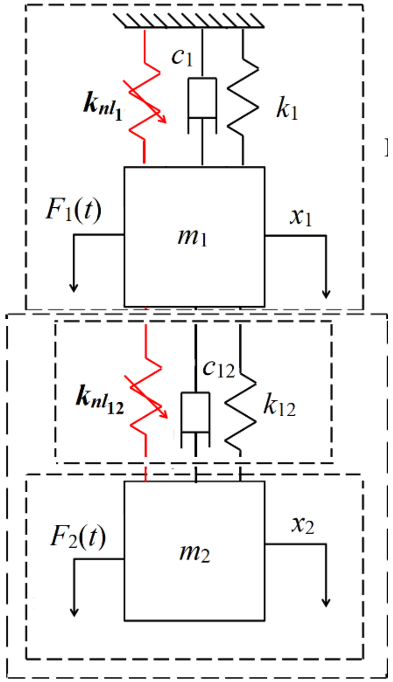


Frequency response curve, X_1

Legend
$(X_1^1 + X_1^2)$
X_1^3



Superposition principle doesn't apply to the given nonlinear system



$$F_1(t) = F_1^S + F_1^D \sin(\omega t)$$
$$F_2(t) = F_2^S + F_2^D \sin(\omega t)$$

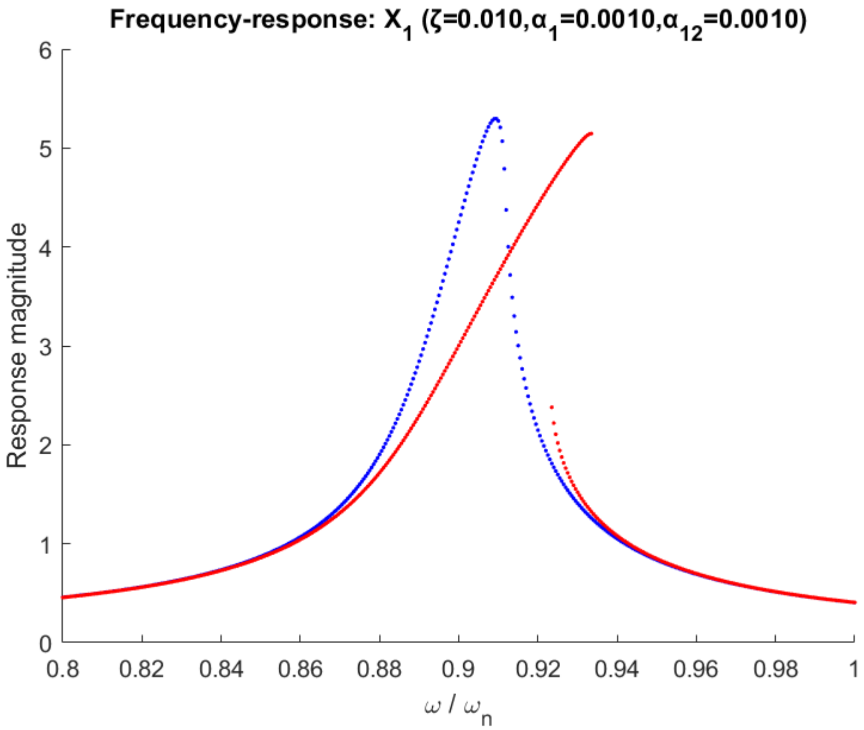
$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
$$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

Appendix III: Amplitude effect

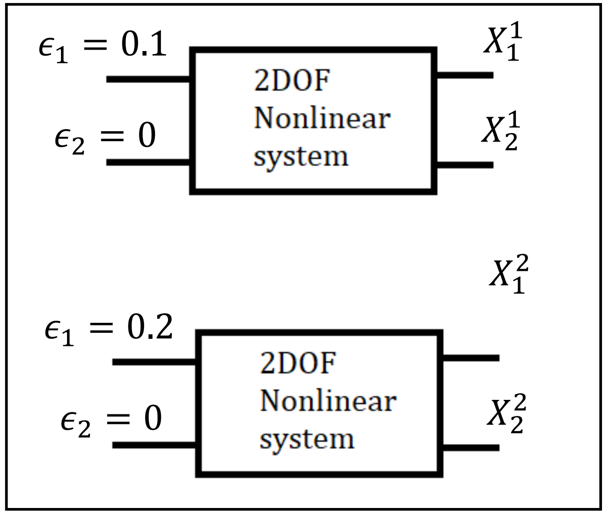
System parameters: $\mu = 0.2, \gamma = 1, \lambda = 1, \zeta = 0.01, \boldsymbol{\alpha_1 = \alpha_{12} = 0.001}$
Natural frequencies (for linear system): $0.9\omega_n, 2.49\omega_n$
Solution parameters: $N_h = 80, N = 200$



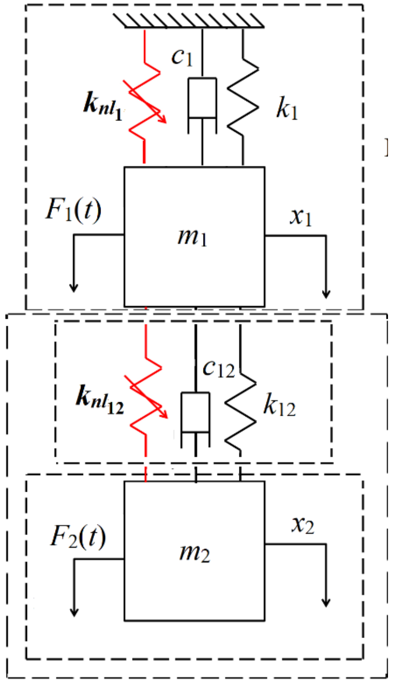
Frequency response curve: X_1
(near first natural frequency)

Legend
$\boldsymbol{X_1^1}$
$\boldsymbol{(0.5)X_1^2}$

Excitation



Unlike a linear system, the response of the given nonlinear system cannot be scaled



$$F_1(t) = F_1^S + F_1^D \sin(\omega t)$$
$$F_2(t) = F_2^S + F_2^D \sin(\omega t)$$

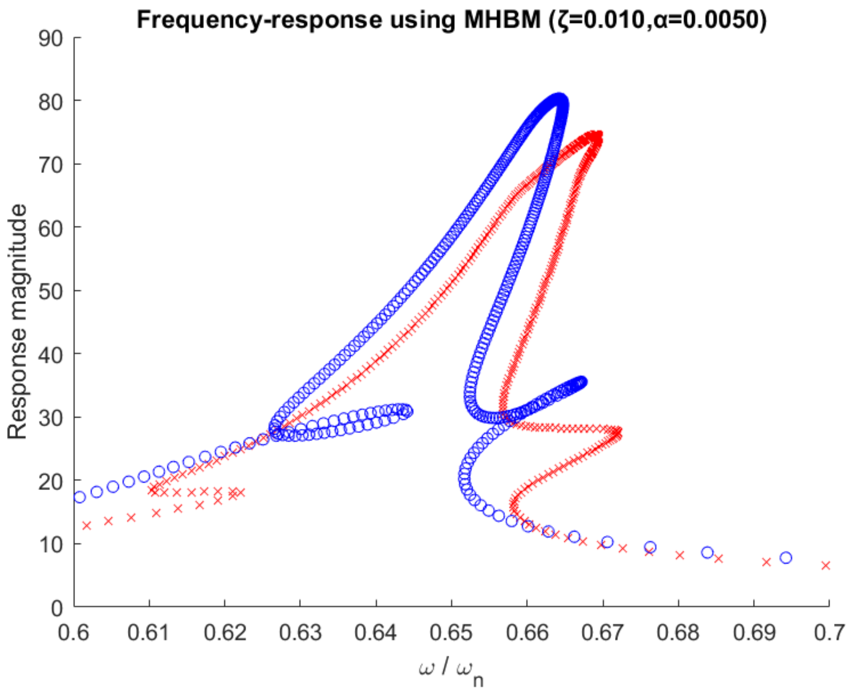
$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
$$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

Appendix III: Reciprocity

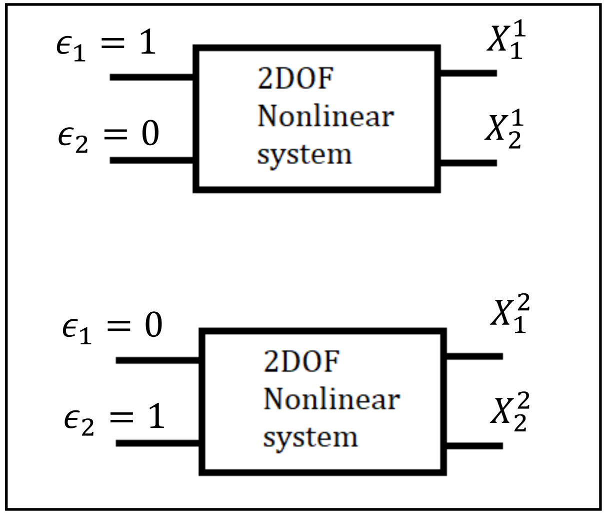
System parameters: $\mu = 1, \gamma = 1, \lambda = 1, \zeta = 0.01, \boldsymbol{\alpha_1 = 0, \alpha_{12} = 0.005}$
Natural frequencies (for linear system): $0.62\omega_n, 1.62\omega_n$
Solution parameters: $N_h = 80, N = 200$



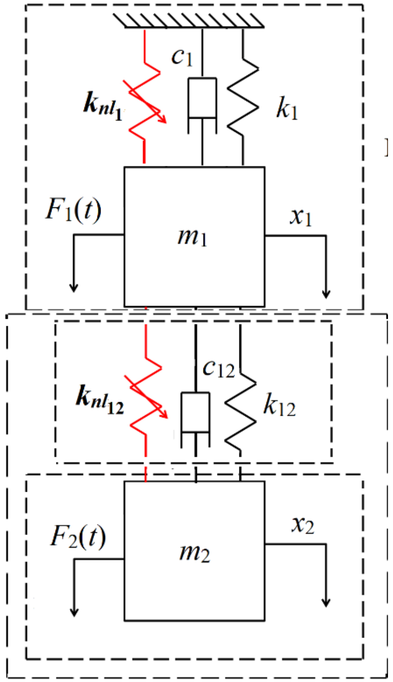
Frequency response curve
(near first natural frequency)

Legend
X_1^2
X_2^1

Excitation



Reciprocity doesn't apply to the given nonlinear system



$$F_1(t) = F_1^S + F_1^D \sin(\omega t)$$
$$F_2(t) = F_2^S + F_2^D \sin(\omega t)$$

$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
$$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

Appendix III: Numerical solution ($\alpha = 0.1$)

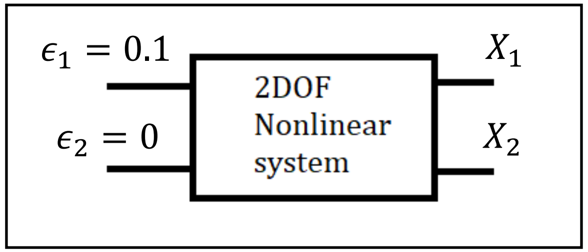
System parameters: $\mu = 0.2, \gamma = 0.2, \lambda = 0.2, \zeta = 0.005, \sigma = \frac{\mu}{1+\mu}$

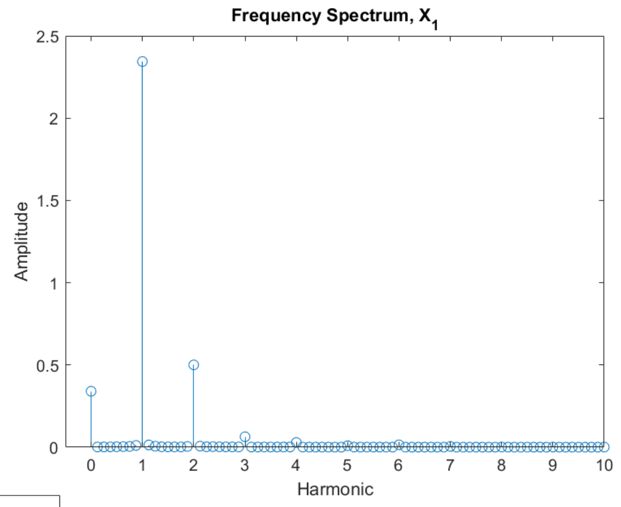
$\boldsymbol{\alpha_1 = 0.1, \alpha_{12} = 0.1}$

Natural frequencies (for linear system): $0.8\omega_n, 1.25\omega_n$

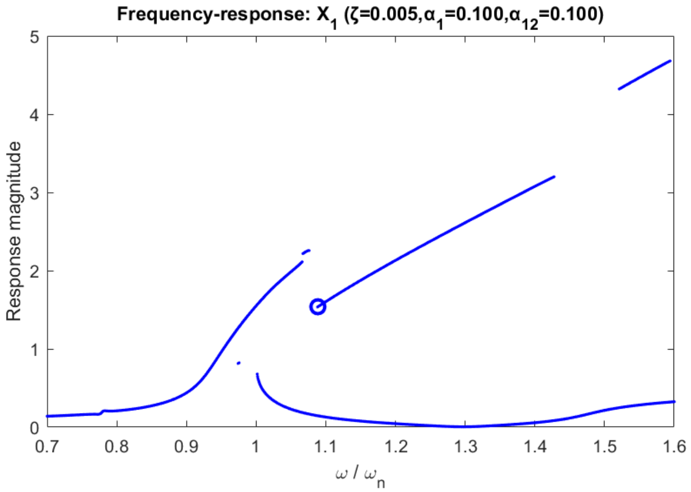
Solution parameters: $N_h = 80, N = 200$

Excitation

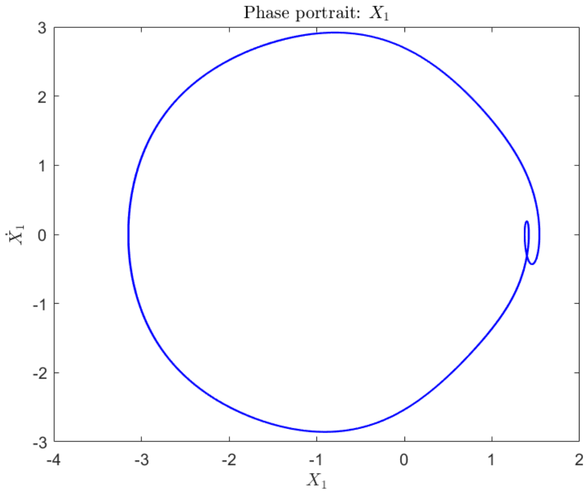




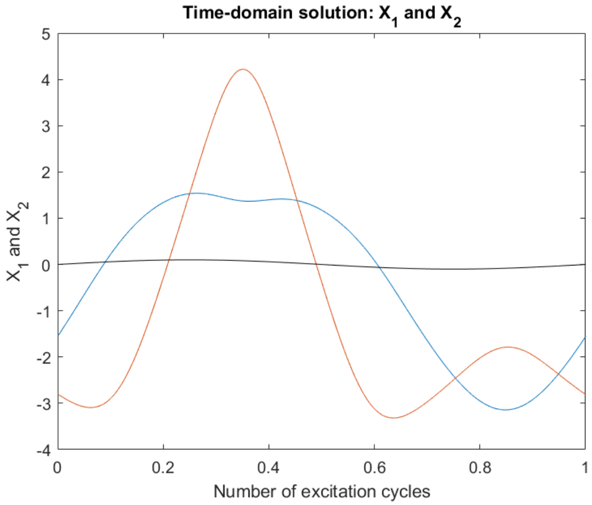
Frequency spectrum: X_1



Frequency response curve: X_1
(near first natural frequency)



Phase portrait: X_1



Time-domain solution

Legend
$\boldsymbol{X_1}$
$\boldsymbol{X_2}$
$\boldsymbol{\epsilon_1 \sin(\omega t)}$

Numerical solution (using Runge-Kutta method) is computed at $\omega = 1.09\omega_n$ (blue marker)

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At higher nonlinearity levels, the response contains multiple harmonics

Appendix IV: Softening-type path (α=0.01)

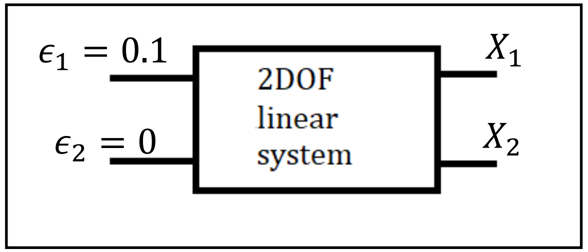
System parameters: $\mu = 0.2, \gamma = 0.2, \lambda = 0.2, \zeta = 0.005, \sigma = \frac{\mu}{1+\mu}$

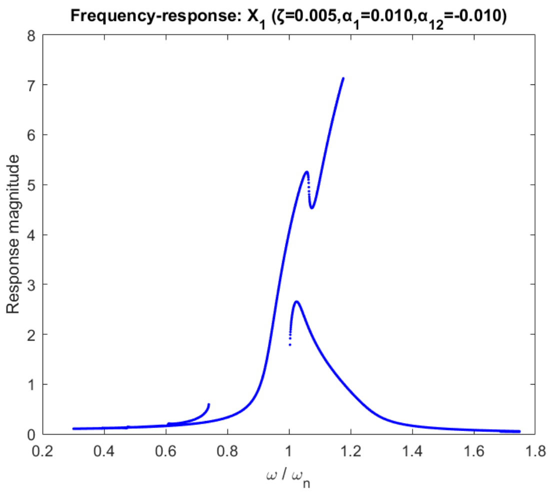
$\boldsymbol{\alpha_1 = 0.01, \alpha_{12} = -0.01}$

Natural frequencies (for linear system): $0.8\omega_n, 1.25\omega_n$

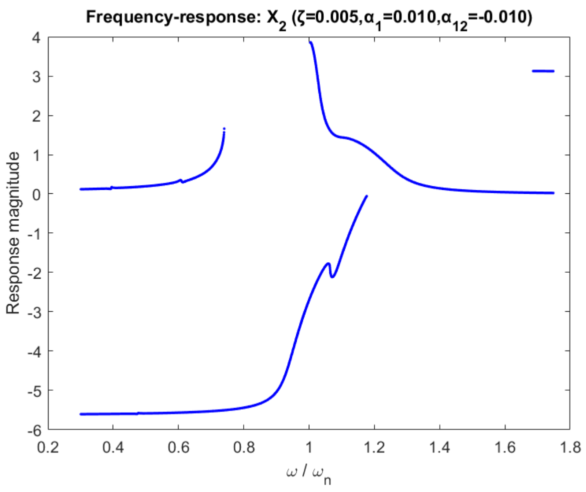
Solution parameters: $N_h = 80, N = 200$

Excitation

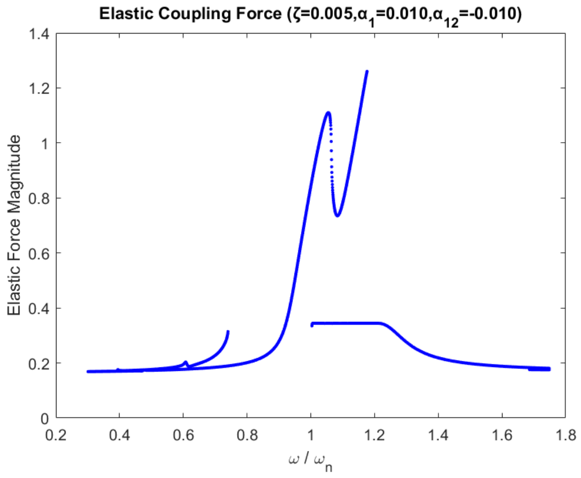




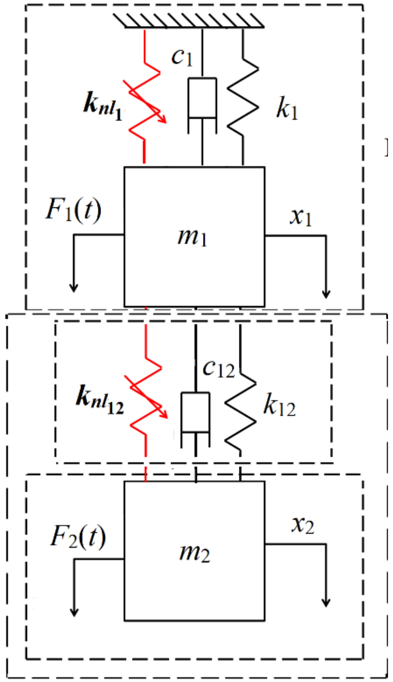
Frequency response curve: X_1



Frequency response curve: X_2



Elastic coupling force vs ω



$$F_1(t) = F_1^S + F_1^D \sin(\omega t)$$
$$F_2(t) = F_2^S + F_2^D \sin(\omega t)$$

$$\mu = \frac{m_2}{m_1}, \gamma = \frac{c_{12}}{c_1}, \lambda = \frac{k_{12}}{k_1}, x_0 = \frac{F_1^S}{k_1}$$

$$\omega_n = \sqrt{\frac{k_1}{m_1}}, \alpha_1 = \frac{k_{nl_1} x_0^2}{k_1}, \alpha_{12} = \frac{k_{nl_{12}} x_0^2}{k_1}$$

$$X_1 = \frac{x_1}{x_0}, X_2 = \frac{x_2}{x_0}, \epsilon_1 = \frac{F_1^D}{F_1^S},$$
$$\epsilon_2 = \frac{F_2^D}{F_1^S}, \sigma = \frac{F_2^S}{F_1^S}$$

The backbone curves (X_1) appear to collapse into each other

Challenge: Isolating nonlinear effects from numerical errors in the MHBM solution

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