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GEAR AND POWER TRANSMISSION RESEARCH LABORATORY

Problem Formulation Report (Summer 2022)

Numerical Solutions and Statistical Analysis of Fatigue Data using Maximum Likelihood Estimates

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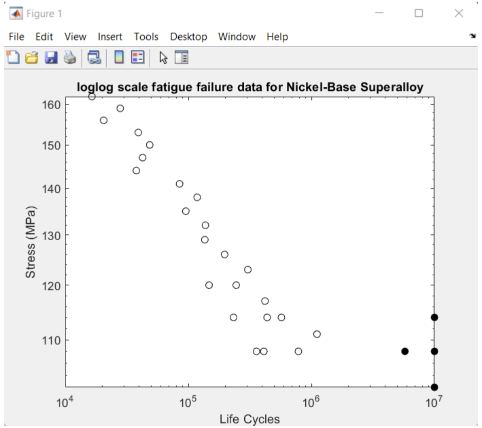
Introduction

Motivation:

1. The relationship between the fatigue life of gear metal and applied stress is an important input to design-for-reliability processes.
2. Experimental Gear Tooth Bending Fatigue Data needs to be analyzed statistically because of the natural occurrence of variance.
3. Curve-fitting on the data helps in estimating the fatigue limit and quantitatively comparing different curve-fitting models lets us know which model provides a better fit.

Goals:

1. Develop Numerical Solutions for various curve-fitting statistical models on Fatigue Data using Maximum Likelihood Estimates.
2. Evaluate statistical goodness of fit of MLE estimates of various distributions to gear tooth bending fatigue data.



A typical SN plot showing fatigue limit characteristic

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Problem Formulation

Objectives:

1. Develop Maximum Likelihood solutions to fatigue data using Log-Normal and Weibull Distributions [1],[2]
2. Develop a robust numerical solver for the Random Fatigue Limit Model (RFL) as applied to gear tooth bending and pitting data [3],[4]
3. Investigate goodness-of-fit of gear fatigue data to log-normal and Weibull distributions [1],[2]
4. Investigate regression differences between subsurface and surface failure data

Literature Review

Author(s) (Journal, Year)	Topic	Comments
W. Nelson (JTEVA, 1984)	Fatigue Curves Fitting with Nonconstant Standard Deviation	Introduction to Fatigue Curve Fitting; Maximum Likelihood Estimates (MLE)
F. G. Pascual and W. Q. Meeker (JTEVA, 1997)	Model with Nonconstant Standard Deviation and a Fatigue Limit Parameter	Incorporated fatigue limit parameter to the previous model
F. Pascual and W. Q. Meeker (Technometric, 1999)	Estimating Fatigue Curves with the Random Fatigue-Limit Model	Development and Likelihood calculation of Random Fatigue Limit (RFL) Model
R. D. Pollak and A. N. Palazotto (Probabilistic Eng., 2009)	Comparison of maximum likelihood models for fatigue strength characterization	RFL; Bilinear and Hyperbolic Model

MLE formulations with the Weibull distribution are sparse, unlike the Log-normal Distribution. In this project, along with developing numerical solutions for curve-fitting models, we will compare the results of both the Normal and Weibull Distributions using goodness-of-fit techniques to find which is better.

Objective 1: Likelihood Calculation

	Normal Distribution - [1] W. Nelson (JTEVA, 1984)	**Weibull Distribution**
L	$= f = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(Y-\mu)^2}{2\sigma^2}}$	$= f = \begin{cases} \frac{b}{a} \cdot \left(\frac{Y-c}{a}\right)^{b-1} \cdot e^{-\left(\frac{Y-c}{a}\right)^b}, Y \geq c \\ 0, \quad Y < 0 \end{cases}$
L'	$= 1 - F = 1 - \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{Y} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$	$= 1 - F = 1 - e^{-\left(\frac{Y-c}{a}\right)^b}$

$\mathrm{Y} = log(N)$;
μ = mean;
σ = standard deviation;

a, b, c =Weibull scale, shape and location parameters respectively;

Goal: Maximize Likelihood which is the same as maximizing **log(Likelihood)**

4 Cases for Varying Mu and Sigma

Case 1: Linear mu and constant sigma

$\mu = c_1 + c_2[log(Stress) - mean(log(Stress)]$;

$\sigma = e^{c_3}$;

$\boldsymbol{x} = [\boldsymbol{c_1}, \boldsymbol{c_2}, \boldsymbol{c_3}]$

Case 2: Linear mu and log linear sigma

$\mu = c_1 + c_2[log(Stress) - mean(log(Stress)]$;

$\sigma = e^{c_3 + c_4[log(Stress) - mean(log(Stress)]}$

$\boldsymbol{x} = [\boldsymbol{c_1}, \boldsymbol{c_2}, \boldsymbol{c_3}, \boldsymbol{c_4}]$

Case 3: Quadratic mu and constant sigma

$\mu = c_1 + [log(Stress) - mean(log(Stress)] \cdot (c_2 + c_3[log(Stress) - mean(log(Stress)])$;

$\sigma = e^{c_4}$;

$\boldsymbol{x} = [\boldsymbol{c_1}, \boldsymbol{c_2}, \boldsymbol{c_3}, \boldsymbol{c_4}]$

Case 4: Quadratic mu and log linear sigma

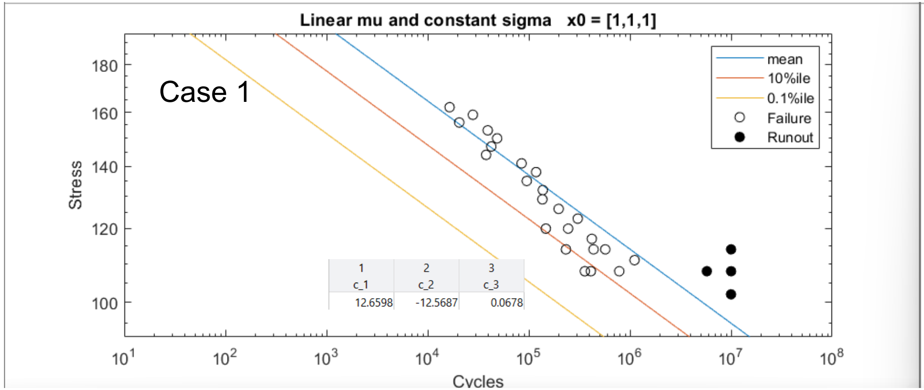
$\mu = c_1 + [log(Stress) - mean(log(Stress)] \cdot (c_2 + c_3[log(Stress) - mean(log(Stress)])$

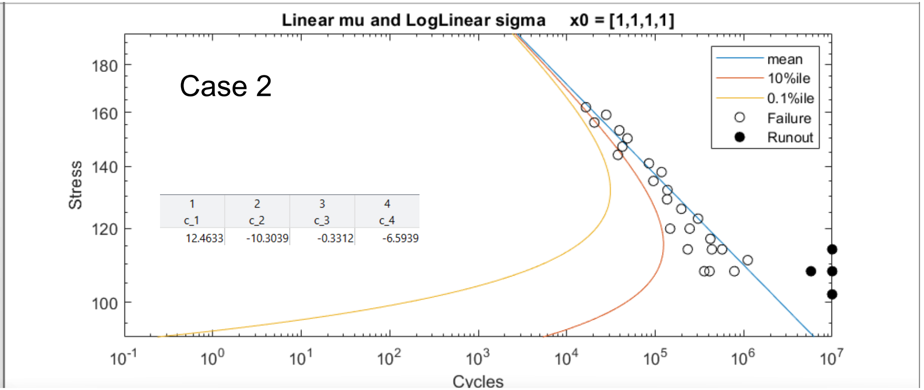
$\sigma = e^{(c_4 + c_5[log(Stress) - mean(log(Stress)])}$

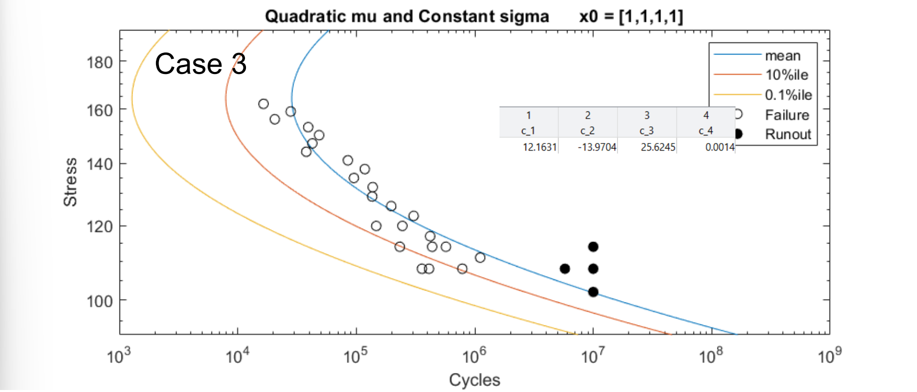
$\boldsymbol{x} = [\boldsymbol{c_1}, \boldsymbol{c_2}, \boldsymbol{c_3}, \boldsymbol{c_4}, \boldsymbol{c_5}]$

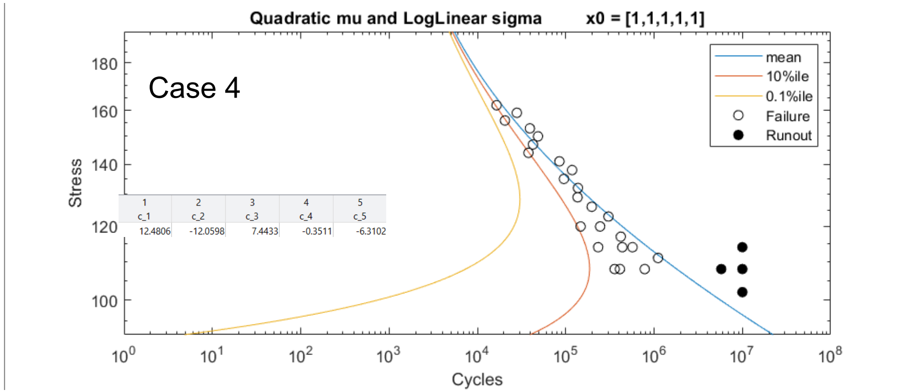
$\boldsymbol{x}$ = the vector that is evaluated by the optimization function

Normal Distribution Results for all 4 Cases



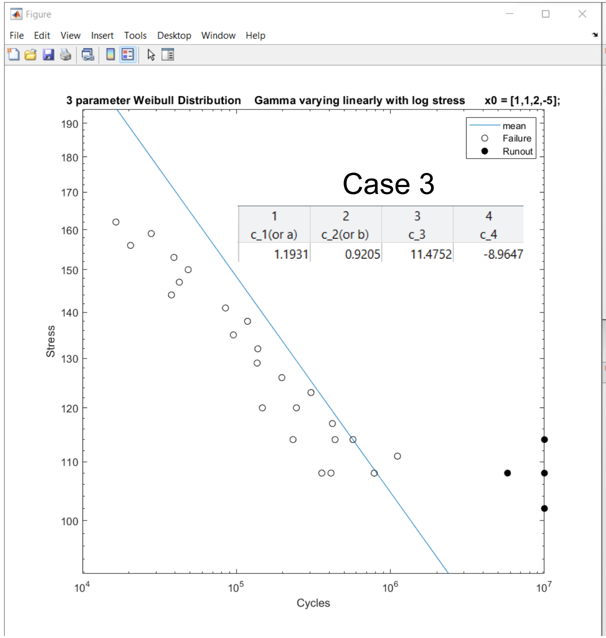


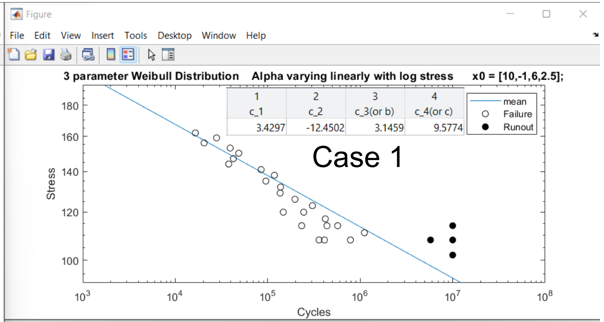


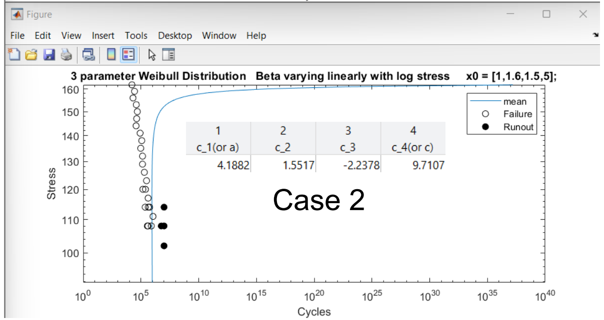


Weibull Distribution Result

Similar to Normal Distribution, the Weibull Distribution was fitted onto the data







Case 1: a varying linearly wrt log(Stress), b and c are constant
Case 2: b varying linearly wrt log(Stress), a and c are constant
Case 3: c varying linearly wrt log(Stress), a and b are constant

a = Scale parameter
b = Shape parameter
c = Location parameter

Need for Random Fatigue Limit (RFL) Model

So far, we didn't incorporate the fatigue limit characteristic of the data to the curve-fitting models. There are fixed fatigue-limit models already developed by [2] Pascual and Meeker, 1997

However, fixed fatigue-limit models do not address the possible variability of the fatigue limit. This variability could be expected to be due to the dependence of the fatigue limit on material structural properties that may vary from specimen to specimen. The random fatigue-limit model provides a description of the commonly observed increase in variability in log fatigue life at low levels of stress/strain and suggests a possible physical explanation for this behavior. These issues are the main motivations for the random fatigue-limit model.

Source- [3] Pascual, Meeker - 1999 - Estimating Fatigue Curves with the Random Fatigue-Limit Model

Objective 2: Likelihood Calculation for RFL Model

The difference between other models and RFL model is that the latter models the fatigue limit as a random variable. $V = \log(\gamma)$

Let, $x = \log(s)$ $and\ W = \log(Y)$ Where s = Stress and Y = Life Cycles

Marginal pdf of W would be: $\boldsymbol{f_W} = \int_{-\infty}^{x} \frac{1}{\sigma\sigma_\gamma} \emptyset_{W|V}\left(\frac{w-\mu}{\sigma}\right) \emptyset_V\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right) dv$

Marginal cdf of W would be: $\boldsymbol{F_W} = \int_{-\infty}^{x} \frac{1}{\sigma_\gamma} \Phi_{W|V}\left(\frac{w-\mu}{\sigma}\right) \emptyset_V\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right) dv$

Where,

$\mu = [\beta_0 + \beta_1(\log(e^x - e^v))$ = mean for W | V μ_γ = mean for γ

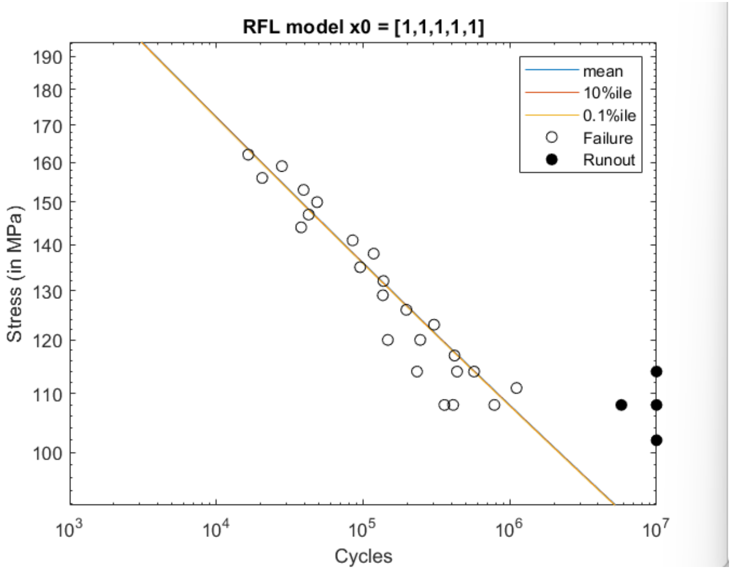
σ_γ, σ = standard deviation for γ and W | V $\emptyset(\cdot)$ = Normal pdf $\boldsymbol{\Phi}(\cdot)$ = Normal cdf

- Likelihood was evaluated using $\boldsymbol{f_W}$ & $\boldsymbol{F_W}$ in the same way as done for Objective 1

Source- [3] Pascual, Meeker - 1999 - Estimating Fatigue Curves with the Random Fatigue-Limit Model

RFL Results

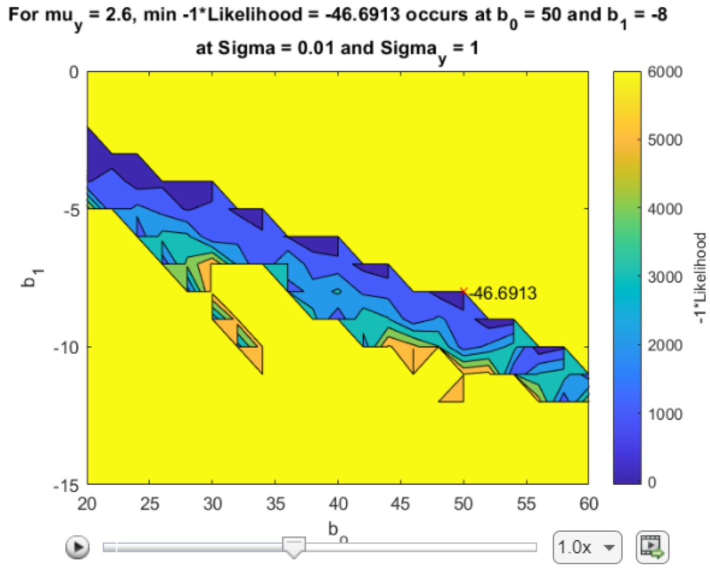
RFL curve-fitting



1 b_0	2 b_1	3 sigma	4 mu_y	5 sigma_y
54.5396	-8.9372	0.0057	2.5427	1.0321

Solution fatigue limit is not consistent with the physics. Not a global minima

RFL Likelihood values visualization



1 b_0	2 b_1	3 sigma	4 mu_y	5 sigma_y
50	-8	0.0100	2.6000	1

Likelihood solution space shows multiple local minima

Future Research

- **Next Steps:**

1. Study characteristics of the solution space of the RFL model to aid in development of a numerical solution method

2. Explore statistical goodness of fit metrics to quantify the accuracy of MLE Normal and Weibull Distribution fits with respect to Gear Tooth Bending Fatigue Data

3. Evaluate sets of gear tooth bending fatigue data to provide insight into proper statistical assumptions on distribution fit.

Conclusion

- **Successful Problem Formulation:**

  Clear definition of the objectives.

- **Progress made thus far (with significant mentor guidance)**

1. Literature Review
2. Log-normal and Weibull Distribution models without fatigue limit has been developed (Objective 1)
3. Random Fatigue Limit model has been developed which needs some refining (Partially Completed Objective 2)

- **Difficulties Encountered**

1. Global Optimization is complex compared to Local Optimization
2. Visualization of models with more than 3 dimensions is challenging

Appendix I: Lessons Learned (Expected)

- **Subjects:**

1. Statistical Analysis
2. Numerical Solutions
3. Optimization (Global and Local)
4. MALTAB Programming for the aforementioned topics

- **Research Methods:**

1. Experimental data analysis
2. Interpretation of results and underlying physics
3. “Best practice” in technical presentation

Appendix II: Resources

- Weibull Distribution:
  - Weibull Distribution: Uses, Parameters & Examples - Statistics By Jim
  - Weibull Distribution (Definition, Properties, Plot, Reliability & Examples) (byjus.com)
  - 3 Parameter Weibull Distribution (Formulae for pdf and cdf)
- Journal Papers

1. W. Nelson, “Fitting of Fatigue Curves with Nonconstant Standard Deviation to Data with Runouts,” *J. Test. Eval.*, vol. 12, no. 2, pp. 69–77, 1984.
2. F. G. Pascual and W. Q. Meeker, “Analysis of Fatigue Data with Runouts Based on a Model with Nonconstant Standard Deviation and a Fatigue Limit Parameter,” *J. Test. Eval.*, vol. 25, no. 3, pp. 292-301, 1997.
3. F. Pascual and W. Q. Meeker, “Estimating Fatigue Curves with the Random Fatigue-Limit Model,” *Technometrics*, vol. 41, no. 4, pp. 277–289, 1999.
4. R. D. Pollak and A. N. Palazotto, “A comparison of maximum likelihood models for fatigue strength characterization in materials exhibiting a fatigue limit,” *Probabilistic Eng. Mech.*, vol. 24, no. 2, pp. 236–241, 2009.