

Challenge Problems 4

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April 13, 2022

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Problem	1	2a	2b-2d	3	4	5	6a	6b-6d
Difficulty/10	7	5	8	6.75	4.5	4.5	6	8
Category	NT	GM	GM	CO	CO	NT	CO	CO

Key:

- NT: Number Theory
- CO: Combinatorics
- GM: Geometry

1. Prove that

$$\lim_{n \rightarrow \infty} \frac{\varphi(n) \sum_{i=1}^n \frac{1}{i^2}}{n} = 1$$

where $\varphi(m)$ is the number of positive integers less than m relatively prime to m .

2. (a) What is the maximum number of regions a plane can be divided into with m lines?
 (b) What is the maximum number of regions three-dimensional space can be divided into with m planes?
 (c) What is the maximum number of regions four-dimensional space can be divided into with m hyper-planes?
 (d) What is the maximum number of regions n -dimensional euclidean space can be divided into with m hyper-planes?
3. Prove that in any set A of 10 positive integers less than or equal to some positive n there exists some $B \subset A$ such that $|B| = 3$, and the sum the the elements in each subset of B is distinct mod n .
4. Determine the number of 2-subsets, S , of \mathbb{Z}_n (the group of integers mod n) with the property that $S = \{a, b\}$ and $0 \notin \{a, b, a + b, a - b, -a - b, -a - b\}$.
5. Let f_a, f_b, f_c , and f_d be distinct positive Fibonacci numbers with the property that

$$f_a + f_b = f_c + f_d.$$

Prove that $\{f_a, f_b\} = \{f_c, f_d\}$.

6. (a) What is the number of paths you can take from the point $(0, 0)$ to $(2n, 0)$ without going below the x axis with the options $(1, 1)$, $(-1, 1)$, and $(0, 1)$ for steps.
 (b) What if you remove the x axis clause?
 (c) What if any point with x value $2n$ suffices as the endpoint?
 (d) What's the answer to part c if the x axis clause is reinstated?