Results In Additive Combinatorics

Jacob Terkel

Summer 2021

Abstract

Below I have compiled my findings this summer with the exception of my results discussed extensively in my paper on Zero-h-sum-free sets.

Contents

1 $\tau(G, [s, t])$

Theorem F.17 from [1]

$$\tau(G, [1, t]) \leq \left\lfloor \frac{|G| - 1}{t} \right\rfloor$$

Proposition 1.1

Let G be an Abelian group with invariant factorization $n_1, n_2, \cdots, n_k$

$$\tau(G, [1, t]) \geq \sum_{i=1}^{k} \left(\tau(\mathbb{Z}_{n_i}, [1, t]) \cdot \prod_{j=1}^{i-1} n_j \right) = \sum_{i=1}^{k} \left(\left\lfloor \frac{n_i - 1}{t} \right\rfloor \cdot \prod_{j=1}^{i-1} n_j \right)$$

Proof. Let A be a $[1, t]$-zero sum-free set in G_1, and let B be a $[1, t]$-zero sum-free set in G_2 where both G_1 adn G_2 are finite Abelian groups. The set $G_1 \times B \cup A \times 0_{G_2}$ is $[1, t]$ zero sum-free in $G_1 \times G_2$ where 0_{G_2} is the zero element in G_2. Our proposition follows easily from this. $\square$

From Theorem F.17 in [1] and Proposition 1.1 we have

Corollary 1.2

If G factors into cyclic groups such that each of said factors has order 1 mod t

$$\tau(G, [1, t]) = \frac{|G| - 1}{t}$$

or more specifically

Corollary 1.3

$$\tau(\mathbb{Z}_k^r, [1, k - 1]) = \frac{k^r - 1}{k - 1}$$

Furthermore, as a result of the two theorems from [?] as well as Proposition 1.1

Corollary 1.4

For an integers t and w, if all divisors of w relatively prime to t are 1 mod t then

$$\tau(\mathbb{Z}_t \times \mathbb{Z}_{tw}, [1, t]) = tw - t$$

Problem 1.5

Find $\tau(\mathbb{Z}_3 \times \mathbb{Z}_{3k}, [1, 3])$ when k has factors 2 mod 3

Conjecture 1.6

If t does not divide the exponent of G, then equality holds in Proposition 1.1

Table 1: $\tau(\mathbb{Z}_k^2, [1, 3])$

k	4	5	6	7	8	9
$\tau(\mathbb{Z}_k^2, [1, 3])$	5	6	≥ 8	16	≥ 18	≥ 20

Proposition 1.7

For $t \geq 3$

$$\tau(G, [2, t]) = \tau(G, [1, t])$$

Proposition 1.8

For $t \geq 4$
$$\tau(G, [3, t]) = \tau(G, [1, t])$$

Proposition 1.9

For $t \geq 6$
$$\tau(G, [4, t]) = \tau(G, [1, t])$$

Proposition 1.10

For $t \geq P$ where P is the smallest number such that P is divisible by every number less than r, and is greater than r we have

$$\tau(G, [r, t]) = \tau(G, [1, t])$$

Recall that $\tau(\mathbb{Z}_n, h) = v_h(n, h)$ which gives us

Proposition 1.11

$$\left\lfloor \frac{n-1}{5} \right\rfloor \leq \tau(\mathbb{Z}_n, [4, 5]) \leq \min\{v_4(n, 4), v_5(n, 5)\}$$

It follows that

Proposition 1.12

If n has no factors 2,3,4 mod 5 or 2, 3 mod 4

$$\tau(\mathbb{Z}_n, [4, 5]) = \tau(\mathbb{Z}_n, [1, 5]) = \left\lfloor \frac{n-1}{5} \right\rfloor$$

and because with the congruence class $3\mathbb{Z}_n + 1$ we have $[4, 5](3\mathbb{Z}_n + 1) = (3\mathbb{Z}_n + 1) \cup (3\mathbb{Z}_n + 2)$ and $0 \notin (3\mathbb{Z}_n + 1) \cup (3\mathbb{Z}_n + 2)$.

Proposition 1.13

If n is divisible by 3
$$\tau(\mathbb{Z}_n, [4, 5]) = \frac{n}{3}$$

Problem 1.14

Find $\tau(\mathbb{Z}_n, [4, 5])$ for the remaining cases.

Proposition 1.15

If $\tau(\mathbb{Z}_n, [4,5]) > \left\lfloor \frac{n-1}{5} \right\rfloor$ then there is a set A such that $0 \in 3A$ and $0 \notin [4,5]A$.

Conjecture 1.16

$$\tau(\mathbb{Z}_n, [4,5]) = \begin{cases} \frac{n}{3} & n \text{ is divisible by } 3 \\ \left\lfloor \frac{n-1}{5} \right\rfloor & \text{Otherwise} \end{cases}$$

Proposition 1.17

Let H be a set of non-zero integers where the largest (not necessarily greatest) such integer has absolute value h, if for all $i \in H$, $i|h$ then

$$\tau(\mathbb{Z}_n, H) = \tau(\mathbb{Z}_n, h) = v_h(n, h)$$

2 $\mu_\pm(G, \{k, l\})$, $\mu\hat{_\pm}(G, \{k, l\})$, $\tau_\pm(G, h)$, and $\tau\hat{_\pm}(G, h)$

Definition 2.1

$\mu_\pm(G, \{k, l\})$ is equal to the maximum cardinality of a signed (k, l)-sum-free subset of G.

Proposition 2.2

If a (k, l)-sum-free set is symmetric, then it is signed (k, l)-sum-free as well.

Proposition 2.3

If a set s signed (k, l)-sum-free, then it is (k, l)-sum-free, this results in

$$\mu_\pm(G, \{k, l\}) \leq \mu(G, \{k, l\})$$

Proposition 2.4

$$\mu_\pm(G, \{2, 1\}) = \tau_\pm(G, 3) \geq v_3(\kappa, 3) \cdot \frac{n}{\kappa}$$

Equality holds for cyclic G.

Proof. A is $(2,1)$-signed sum-free if and only if $0 \notin 1_{\pm}A$ and $0 \notin 3_{\pm}A$, but the latter (zero-3-signed-sum-free) requires the former, meaning a set being $(2,1)$-signed sum-free is the same as being zero-3-signed-sum-free ☐

Proposition 2.5

For $k \neq l \mod 2$ and even $|G|$

$$\mu_{\pm}(G, \{k,l\}) = \frac{|G|}{2}$$

Problem 2.6

Find lower or upper bounds on $\mu_{\pm}(G, \{k,l\})$ or $\mu\hat{}_{\pm}(G, \{k,l\})$, specifically for $\{k,l\} = \{3,1\}$ and $\{k,l\} = \{4,1\}$

Via Theorem G.67 in [1], Proposition 2.4, and the fact that $\mu_{\pm}(G, \{2,1\}) \leq \mu\hat{}_{\pm}(G, \{2,1\})$:

Corollary 2.7

If G with order n has prime divisors congruent 2 to mod 3 and p is the smallest such divisor then

$$\mu\hat{}_{\pm}(G, \{2,1\}) = v_1(n,3) = \frac{n}{3}\left(1 + \frac{1}{p}\right)$$

As for the remaining cases:

$$v_3(n,3) \leq \mu\hat{}_{\pm}(G, \{2,1\}) \leq \mu\hat{}(G, \{2,1\})$$

Proposition 2.8

If n has no divisors congruent to 2 mod 3 then

$$\left\lfloor \frac{n}{3} \right\rfloor \leq \mu\hat{}_{\pm}(\mathbb{Z}_n, \{2,1\}) \leq \left\lfloor \frac{n}{3} \right\rfloor + 1$$

Proof. Let n be an odd number congruent to 0 mod 3. Let A be the subset of $\mathbb{Z}_n$ defined as

$$A = \left[\frac{n}{3} + 1, \frac{2n}{3}\right]$$

$1\hat{}_{\pm}A \cap 2\hat{}_{\pm}A = \emptyset$, thus we have proven our claim in combination with the above corollary ☐

Proposition 2.9

$$\mu\hat{\pm}(G,\{2,1\}) \le \mu(G,\{2,1\}) = v_1(\kappa,3)\cdot\frac{n}{\kappa}$$

Proof. Let A be a weak signed sum free subset where $|A| > \mu(G,\{2,1\})$ With this, we would have that A cannot be sum free. This means there must exist a_0 and a_1 in A such that $a_0 = 2a_1$ (as A is certainly weakly sum free) . However, because $0 \notin A$, this would mean that $a_0 - a_1 = a_1 \in 2\hat{\pm}A$, and $1\hat{\pm}A$, meaning that such a set cannot exist, proving our claim. □

And when the previous three propositions/corollaries are combined, we arrive at

Theorem 2.10

For G of order n and exponent κ, we have the following

- For cyclic G
$$\mu\hat{\pm}(G,\{2,1\}) = v_1(n,3)$$
- If n is non-cyclic divisible by at least 1 number 2 mod 3
$$\mu\hat{\pm}(G,\{2,1\}) = v_1(n,3)$$
- If n is non-cyclic, and is divisible by 3 but no number 2 mod 3
$$\frac{n}{3}-\frac{n}{\kappa} \le \mu\hat{\pm}(G,\{2,1\}) \le \frac{n}{3}$$
- If n has only factors 1 mod 3
$$\mu\hat{\pm}(G,\{2,1\}) = \frac{\kappa-1}{3}\cdot\frac{n}{\kappa}$$

The only remaining case is non-cyclic G of Type II. However, we know that this differs from the remaining cases, as for the other cases, $\mu\hat{\pm}(G,\{2,1\}) = \mu(G,\{2,1\})$. However, in the non-cyclic Type II instance, we have $\mu\hat{\pm}(\mathbb{Z}_3^2,\{2,1\}) = \mu(G,\{2,1\}) - 1 = 2$. So, I ask the following.

Problem 2.11

Settle the Type II case for non-cyclic G, specifically $G = \mathbb{Z}_3^k$

There are some clues as to how this can be solved.

Theorem 2.12

If $|G| = n$ is divisible by 3 but not any not any number 2 mod 3

$$\mu\hat{\pm}(G,\{2,1\}) \le \min\left\{\frac{n}{3}-\frac{n}{\kappa}+\frac{|\mathrm{Ord}(G,3)|}{2},\frac{n}{3}\right\}$$

Proof. Let G be a group of type II. If $\mu\hat{_\pm}(G,\{2,1\}) > \tau(G,3)$, then there must exist a weak signed $(2,1)$-sum-free $A \subset G$ such that $0 \in 3A$. However, if it is 3 distinct elements in A that sum to 0 then $a+b=-c$, and if it is two then $a+b=-b$, both of which violate weak signed $(2,1)$-sum-freeness. Therefore, because $0 \notin A$, A must contain an element of order 3. Let $S = \mathrm{Ord}(G,3)$. If a has order 3, that means $2a = -a$, and because of this, both a and $2a$ cannot be in A, as if this is the case, $a - 2a = 2a \in 2\hat{_\pm}A$. This means that we can have no more than $\frac{|S|}{2}$ elements from S can be in A. This results in $|A| \le \tau(G,3) + \mathrm{Ord}(G,3)/2$ or

$$\mu\hat{_\pm}(G,\{2,1\}) \le \frac{n}{3} - \frac{n}{\kappa} + \frac{|\mathrm{Ord}(G,3)|}{2}$$

□

One example where this is useful is for $G = \mathbb{Z}_{21}^2$, by Theorem 2.12, we have that

$$\mu\hat{_\pm}(G,\{2,1\}) \le \frac{21^2}{3} - 13$$

Improving the previous upper bound by 13.

Corollary 2.13

$$\tau\hat{_\pm}(\mathbb{Z}_n,3) \ge v_1(n,3)$$

Proposition 2.14

$$\tau\hat{_\pm}(G,3) \le \mu\hat{\ }(G,\{2,1\}) + 1$$

Proof. If $\tau\hat{_\pm}(G,3) > \mu\hat{\ }(G,\{2,1\})$ then for a set A, if $2\hat{\ }A \cap A \neq \emptyset$ then one of the following must hold for distinct $a,b,c \in A$

1. $a+b-c=0$
2. $a=0$

If the first holds then clearly $0 \in 3\hat{_\pm}A$

So, if $\tau\hat{_\pm}(G,3) > \mu\hat{\ }(G,\{2,1\})$ then there must exist a set where $0 \in A$, $|A| > \mu\hat{\ }(G,\{2,1\})$ and $0 \notin 3\hat{_\pm}A$.

Now, consider A, a weak signed zero-3-sum free set of size $\mu\hat{\ }(G,\{2,1\})+2$. As we mentioned, it must contain 0, however there is a subset of A of size $\mu\hat{\ }(G,\{2,1\})+1$, A_0 without 0, because 0 is not in this set, and it is not weakly $(2,1)$-sum free we must have that $a+b-c=0$, contradicting the fact that A is weak signed zero-3-sum free, proving our claim. □

Corollary 2.15

In combination with Proposition 2.14 and Corollary 2.13, we can use Proposition F.156 from [1] to construct the following bounds:
If $\mathbb{Z}_n$ is type I (n has at least one factor that is 2 mod 3), then

$$v_1(n,3) \leq \tau\hat{_\pm}(\mathbb{Z}_n,3) \leq v_1(n,3)+1$$

If $\mathbb{Z}_n$ is type II ($\mathbb{Z}_n$ is not type I and n is divisible by 3) or III (Every factor of n is 1 mod 3) then

$$v_1(n,3)+1 \leq \tau\hat{_\pm}(\mathbb{Z}_n,3) \leq v_1(n,3)+2$$

Lemma 2.16

For odd n if

$$\tau\hat{}(G,3)/2 < \mu\hat{}(G,\{2,1\})$$

then

$$\tau\hat{_\pm}(G,3) \leq \mu\hat{}(G,\{2,1\})$$

Proof. Assume A is a weak signed zero-3-sum free set of size $\mu\hat{}(G,\{2,1\})+1$. We have that $0 \in A$ now, note that for odd n and $A_0 = A \setminus \{0\}$, $A_0 \cap (-A_0) = \emptyset$. Furthermore, $(A_0 \cup -A_0)$ is weakly zero-3-sum-free. This gives us that if $\tau\hat{_\pm}(G,3) = \mu\hat{}(G,\{2,1\})+1$ then

$$\tau\hat{}(G,3)/2+1 \geq \mu\hat{}(G,\{2,1\})+1$$

$$\tau\hat{}(G,3)/2 \geq \mu\hat{}(G,\{2,1\})$$

□

Theorem 2.17

For $n \geq 6$

$$\tau\hat{_\pm}(\mathbb{Z}_n,3) = \mu\hat{}(\mathbb{Z}_n,\{2,1\}) = \begin{cases} v_1(n,3) & n \text{ has at least one divisor 2 mod 3} \\ v_1(n,3)+1 & \text{Otherwise} \end{cases}$$

and $\tau\hat{_\pm}(\mathbb{Z}_5,3) = \tau\hat{_\pm}(\mathbb{Z}_4,3) = 3$, $\tau\hat{_\pm}(\mathbb{Z}_3,3) = \tau\hat{_\pm}(\mathbb{Z}_2,3) = 2$,

Proof. For odd n we have that

$$\chi\hat{}(G,2) = \frac{|G| + |\mathrm{Ord(G,2)}| + 3}{2}$$

Therefore, we have

$$\chi\hat{}(G,3) \leq \frac{|G| + |\mathrm{Ord(G,2)}| + 5}{2}$$

This gives us

$$\tau\hat{}(G,3) \leq \frac{|G| + |\mathrm{Ord(G,2)}| + 3}{2}$$

$$\tau\hat{}(G,3)/2 \leq \frac{|G| + |\mathrm{Ord(G,2)}| + 3}{4}$$

By Lemma 2.16, for cyclic $\mathbb{Z}_n$ of odd order greater than 5 we have

$$\tau\hat{_\pm}(\mathbb{Z}_n, 3) = \mu\hat{}(\mathbb{Z}_n, \{2,1\})$$

Even $n \geq 12$ comes from Theorem F.155 in [1], proving our claim. □

Theorem 2.18

For G, where $|G|$ has no factor 2 mod 3, and $|G|$ is divisible by 3

$$\frac{|G|}{\kappa} \cdot v_3(\kappa, 3) \leq \tau\hat{_\pm}(G,3) \leq \mu\hat{}(G, \{2,1\}) \leq \frac{|G|}{3} + 1$$

Proof. We have that

$$\chi\hat{}(G,2) = \frac{|G| + |\mathrm{Ord(G,2)}| + 3}{2}$$

Therefore, we have

$$\chi\hat{}(G,3) \leq \frac{|G| + |\mathrm{Ord(G,2)}| + 5}{2}$$

This gives us

$$\tau\hat{}(G,3) \leq \frac{|G| + |\mathrm{Ord(G,2)}| + 3}{2}$$

$$\tau\hat{}(G,3)/2 \leq \frac{|G| + |\mathrm{Ord(G,2)}| + 3}{4}$$

Let $|G|$ have no factors 2 mod 3, but $|G|$ divisible by 3.

$$\tau\hat{}(G,3)/2 \leq \frac{|G| + 3}{4}$$

Peter Francis showed that $\frac{|G|}{3} \leq \mu\hat{}(G, \{2,1\}) \leq \frac{|G|}{3} + 1$ This means by Lemma 2.16 as long as

$$\tau\hat{}(G,3)/2 < \mu\hat{}(G, \{2,1\})$$

then $\tau\hat{_\pm}(G,3) \leq \mu\hat{}(G, \{2,1\})$.

We have that $\tau\hat{}(G,3)/2 < \mu\hat{}(G,\{2,1\})$ holds certainly when

$$\frac{|G|+3}{4} < \frac{|G|}{3}$$
$$|G| > 9$$

meaning as long as $|G| > 9$ we have that

$$\mu\hat{}(G,\{2,1\}) \leq \frac{|G|}{3} + 1$$

Furthermore, by Corollary F.25 in [1] we have our lower bound

□

It was also found that $\mu\hat{}(\mathbb{Z}_3^3,\{2,1\}) = 9$, which gives us $\tau\hat{\pm}(\mathbb{Z}_3^r,3) \leq 9$.

Theorem 2.19

If $|G|$ is odd, $G \neq \mathbb{Z}_5$, and $|G|$ has at least one factor 2 mod 3, where p is the smallest such factor

$$\tau\hat{\pm}(G,3) = \frac{|G|}{3}\left(\frac{1}{p}+1\right)$$

and $\tau\hat{\pm}(\mathbb{Z}_5,3) = 3$

Proof. We have that

$$\chi\hat{}(G,2) = \frac{|G| + |\mathrm{Ord(G,2)}| + 3}{2}$$

Therefore, we have

$$\chi\hat{}(G,3) \leq \frac{|G| + |\mathrm{Ord(G,2)}| + 5}{2}$$

This gives us

$$\tau\hat{}(G,3) \leq \frac{|G| + |\mathrm{Ord(G,2)}| + 3}{2}$$
$$\tau\hat{}(G,3)/2 \leq \frac{|G| + |\mathrm{Ord(G,2)}| + 3}{4}$$

For G of order n where n is odd, and has at least 1 factor 2 mod 3, and p is the smallest such factor

$$\tau\hat{}(G,3)/2 \leq \frac{n+3}{4}$$

Peter Francis showed that

$$\mu\hat{}(G,\{2,1\}) = v_1(n,3) = \frac{n}{3}\left(\frac{1}{p}+1\right)$$

$\mu\hat{}(G,\{2,1\}) > \tau\hat{}(G,3)/2$ holds when

$$\frac{n}{3}\left(\frac{1}{p}+1\right) > \frac{n+3}{4}$$

$$\frac{4n}{p} + n > 9$$

Because p is at most n the above becomes

$$n > 5$$

$$\tau\hat{\pm}(G, h) \leq \mu\hat{}(G, \{2, 1\})$$

this holds for all $n > 5$, or all appropriate G with the exception of $\mathbb{Z}_5$.

Let κ be the exponent of G, and let A be the zero-3-sum-free construction used in the proof of Theorem F.4 for $\mathbb{Z}_\kappa$. See that this set is symmetric, and thus $A \times (G/\mathbb{Z}_\kappa)$ is symmetric, and also zero-3-sum-free. This means it is also weakly signed zero-3-sum-free, and it has order $\frac{n}{3}\left(\frac{1}{p} + 1\right)$, proving our claim.

$\tau\hat{\pm}(\mathbb{Z}_5, 3) = 3$ is verified via computer. ☐

Theorem 2.20

If $|G|$ has only factors 1 mod 3 and $G \neq \mathbb{Z}_1$ then

$$\frac{\kappa - 1}{3} \cdot \frac{|G|}{k} \tau\hat{\pm}(G, 3) \leq \mu\hat{}(G, \{2, 1\})$$

Proof. We have that

$$\chi\hat{}(G, 2) = \frac{|G| + |\mathrm{Ord(G, 2)}| + 3}{2}$$

Therefore, we have

$$\chi\hat{}(G, 3) \leq \frac{|G| + |\mathrm{Ord(G, 2)}| + 5}{2}$$

This gives us

$$\tau\hat{}(G, 3) \leq \frac{|G| + |\mathrm{Ord(G, 2)}| + 3}{2}$$

$$\tau\hat{}(G, 3)/2 \leq \frac{|G| + |\mathrm{Ord(G, 2)}| + 3}{4}$$

If $|G|$ has only factors 1 mod 3, then we have

$$\tau\hat{}(G, 3)/2 \leq \frac{|G| + 3}{4}$$

Peter Francis showed that for κ exponent of G

$$\mu(G, \{1, 2\}) \geq \frac{\kappa - 1}{3} \frac{|G|}{\kappa} + 1$$

Meaning by lemma 2.16 we have $\tau\hat{\pm}(G, 3) \leq \mu\hat{}(G, \{2, 1\})$ When

$$\frac{|G| + 3}{4} < \frac{\kappa - 1}{3} \frac{|G|}{\kappa} + 1$$

$$3\kappa|G| - 3\kappa < 4\kappa|G| - 4|G|$$
$$4|G| < \kappa|G| + 3\kappa$$
Because κ is 7 at the very least(unless $G \cong \mathbb{Z}_1$)
$$0 < 3|G| + 21$$
holds, so does $\tau\hat{\pm}(G,3) \leq \mu\hat{}(G,\{2,1\})$ Moving forward we now have as long as
$$1 < |G|$$
or $G \neq \mathbb{Z}_1$, then
$$\tau\hat{\pm}(G,3) \leq \mu\hat{}(G,\{2,1\})$$
Furthermore, because $\tau\hat{\pm}(G,3) \geq \tau_{\pm}(G,3)$ □

Corollary 2.21

For all G where $|G|$ is odd, if $G \notin \{\mathbb{Z}_1, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_3^2\}$ then
$$\tau\hat{\pm}(G,3) \leq \mu\hat{}(G,\{2,1\})$$

Proposition 2.22

If $|G| \geq 12$ is even, and $G \notin \{\mathbb{Z}_2^r, \mathbb{Z}_2^r \times \mathbb{Z}_4\}$ we have
$$\tau\hat{\pm}(G,3) = \frac{|G|}{2}$$
For all positive integers r
$$\tau\hat{\pm}(\mathbb{Z}_2^r,3) = \tau\hat{}(\mathbb{Z}_2^r,3) = 2^{r-1} + 1$$
and for $r \geq 2$
$$2^{r+1} \leq \tau\hat{\pm}(\mathbb{Z}_2^r \times \mathbb{Z}_4,3) \leq 2^{r+1} + 1$$

Proof. By Theorem E.60 from [1], we know that for even $|G|$, and $|G| \geq 12$ unless $G = \mathbb{Z}_2^r$ or $G = \mathbb{Z}_2^r \times \mathbb{Z}_4$ that $\tau\hat{\pm}(G,3) \leq \frac{|G|}{2}$. Furthermore, it is clear that if $\mathbb{Z}_\kappa$ is the exponent of G, it is clear that $(2 \cdot \mathbb{Z}_\kappa + 1) \times (G/\mathbb{Z}_\kappa)$ is weakly signed zero-3-sum-free, giving us $\tau\hat{\pm}(G,3) = \frac{|G|}{2}$ For even $|G|$ where $G \notin \{\mathbb{Z}_2^r, \mathbb{Z}_2^r \times \mathbb{Z}_4\}$ and $|G| \geq 12$.

For the case of $G = \mathbb{Z}_2^r$, due to the fact that for any $a \in \mathbb{Z}_2^r$ $a = -a$, we can use Theorem F.75 from [1] and we get
$$\tau\hat{\pm}(\mathbb{Z}_2^r,3) = \tau\hat{}(\mathbb{Z}_2^r,3) = 2^{r-1} + 1$$
For remaining case of $G = \mathbb{Z}_2^r \times \mathbb{Z}_4$ Theorem E.60 gives us for
$$\tau\hat{\pm}(\mathbb{Z}_2^r \times \mathbb{Z}_4,3) \leq 2^{r+1} + 1$$
We have that $0 \notin 3\hat{\pm}(\{1\} \times \mathbb{Z}_2^{r-1} \times \mathbb{Z}_4)$, this leads us to
$$2^{r+1} \leq \tau\hat{\pm}(\mathbb{Z}_2^r \times \mathbb{Z}_4,3) \leq 2^{r+1} + 1$$
□

Below are all the results for $\tau\hat{_\pm}(G,3)$ summarized

Theorem 2.23

For an Abelian group G of order n:

- If G is cyclic and $n \geq 6$ then
$$\tau\hat{_\pm}(G,3) = \mu\hat{}(G,\{2,1\}) = \begin{cases} v_1(n,3) & n \text{ has at least one divisor 2 mod 3} \\ v_1(n,3)+1 & \text{Otherwise} \end{cases}$$
- If n is even, $n \geq 12$ and
$$G \notin \{\mathbb{Z}_2^r, \mathbb{Z}_2^r \times \mathbb{Z}_4\}$$
then
$$\tau\hat{_\pm}(G,3) = \frac{n}{2}$$
- If $G \cong \mathbb{Z}_2^r$
$$\tau\hat{_\pm}(G,3) = \frac{n}{2} + 1$$
- If $G \cong \mathbb{Z}_2^r \times \mathbb{Z}_4$
$$\frac{n}{2} \leq \tau\hat{_\pm}(G,3) \leq \frac{n}{2} + 1$$
- If $G \ncong \mathbb{Z}_5$ and n has at least one factor congruent to 2 mod 3 and p is the smallest such factor then
$$\tau\hat{_\pm}(G,3) = \frac{n}{3}\left(\frac{1}{p} + 1\right)$$
- For $G \notin \{\mathbb{Z}_3, \mathbb{Z}_3^2, \mathbb{Z}_9\}$, where n has no factor 2 mod 3, and n is divisible by 3
$$\frac{n}{3} - \frac{n}{\kappa} \leq \tau\hat{_\pm}(G,3) \leq \mu\hat{}(G,\{2,1\}) \leq \frac{n}{3} + 1$$
- If n has only factors 1 mod 3 and $G \neq \mathbb{Z}_1$ then
$$\frac{\kappa - 1}{3} \cdot \frac{n}{\kappa} \leq \tau\hat{_\pm}(G,3) \leq \mu\hat{}(G,\{2,1\})$$

Problem 2.24

Find or improve the bounds on $\tau\hat{_\pm}(G,3)$ for G of type II, III, and $G \cong \mathbb{Z}_2^r \times \mathbb{Z}_4$.

Proposition 2.25

If $\tau\hat{_\pm}(\mathbb{Z}_n, h) > \mu\hat{}(\mathbb{Z}_n, \{h-1, 1\})$ then one of the following must hold for $|A| = \tau\hat{_\pm}(\mathbb{Z}_n, h)$ and $a_i \in A$

1. $a_1 + a_2 + \cdots + a_{h-1} - a_h = 0$
2. $a_1 + a_2 + \cdots + a_{h-2} = 0$

Because the first one obviously cannot hold, the second must be true, meaning $0 \in (h-2)\hat{}A$

Corollary 2.26

$$\tau\hat{_\pm}(\mathbb{Z}_n, 4) \leq \mu\hat{}(\mathbb{Z}_n, \{3, 1\}) + 2$$

Proof. For $h = 4$ in Proposition 2.25, this would mean that there exists $\{a, -a\} \subset A$. Note that if more than one such pair exists then $0 \in 4\hat{_\pm}A$. So, only one of these symmetric pairs can be in A, thus we have that if $|A| \geq \mu\hat{}(\mathbb{Z}_n, \{3, 1\}) + 3$ then there exists $A_0 = A \setminus \{a, -a\}$ where $|A_0| \geq \mu\hat{}(\mathbb{Z}_n, \{3, 1\}) + 1$ with no symmetric pairs, meaning $0 \in 4\hat{_\pm}A$. Giving us

$$\tau\hat{_\pm}\ (\mathbb{Z}_n, 4) \leq \mu(\mathbb{Z}_n, \{3, 1\}) + 2$$

□

Proposition 2.27

If $\tau\hat{_\pm}(\mathbb{Z}_n, h) > \mu\hat{}(\mathbb{Z}_n, \{h-2, 2\})$ then one of the following must hold for $|A| = \tau\hat{_\pm}(\mathbb{Z}_n, h)$ and $a_i \in A$

1. $a_1 + a_2 + \cdots + a_{h-2} - a_{h-1} - a_h = 0$
2. $a_1 + a_2 + \cdots + a_{h-3} - a_{h-2} = 0$
3. $a_1 + a_2 + \cdots + a_{h-4} = 0$

Because the first one obviously cannot hold, meaning either $0 \in (h-4)\hat{}A$ or there exists some $a \in A$ such that $a \in (h-3)\hat{}(A \setminus a)$

Proposition 2.28

For an Abelian group G and $A \subset G$, the following are equivalent

- $0 \notin 3\hat{_\pm}A$ and $0 \notin A$
- $0 \notin 3\hat{}A$ and $0 \notin 2\hat{}A - A$

Conjecture 2.29

For odd h and prime $p > h$

$$\tau_\pm(\mathbb{Z}_p, h) = 2\left\lfloor \frac{p+h-2}{2h} \right\rfloor$$

Or equivalently

$$\tau_\pm(\mathbb{Z}_p, h) = \begin{cases} \left\lfloor \frac{p-2}{h} \right\rfloor + 1 & p = 1 \text{ or even mod h} \\ \\ \left\lfloor \frac{p-2}{h} \right\rfloor & \text{otherwise} \end{cases}$$

Problem* 2.30

Prove or disprove Conjecture 2.29

Problem 2.31

Investigate $\tau_\pm(\mathbb{Z}_p, h)$ for even h.

3 $\phi_\pm(\mathbb{Z}_n, h) = 2$

Definition 3.1

$\phi_\pm(G, h)$ is equal to the minimum cardinality of a subset of G such that the h fold signed sumset of said subset spans G.

Proposition 3.2

If $\phi_\pm(\mathbb{Z}_n, h) = 2$ then $n \leq 4h - 1$

Known Cases When $\phi_\pm(\mathbb{Z}_n, h) = 2$

- If $2 \leq n \leq 2h + 1$ then $h_\pm\{0, 1\} = \mathbb{Z}_n$.
- If both n and h are odd and $2h + 3 \leq n \leq 3h$ then $h_\pm\{1, 3\} = \mathbb{Z}_n$.
- For odd n and h is $2^{k-1} \mod 2^k$ and $2h + 1 \leq n \leq \frac{2^k+1}{2^{k-1}}h$ then $h_\pm\{2^k - 1, 2^k + 1\} = \mathbb{Z}_n$.

Proposition From Torrence

For even $n \geq 2h + 1$: $\phi_\pm(\mathbb{Z}_n, h) \neq 2$

Remark 3.3

For the set $S = h_{\pm}\{a, b\}$ we have

$$S = h\{a, b\} \cup h\{-a, b\} \cup h\{a, -b\} \cup \{-a, -b\}$$

Or

$$T = h\{a, b\} \cup h\{-a, b\}$$
$$S = T \cup -T$$

Proposition 3.4

If for a subset of $\mathbb{Z}_n$: $A = \{a, b\}$ and $\gcd(a, b, n) \neq 1$ then $h_{\pm}A \neq \mathbb{Z}_n$

Proposition 3.5

If $\gcd(a - b, n) = 1$ and $\phi_{\pm}(\mathbb{Z}_n, h) = 2$ then there exists a set A such that $A = \{a, a + 1\}$ and $h_{\pm}A = \mathbb{Z}_n$

The above is true for all $\phi_{\pm}(\mathbb{Z}_p, h) = 2$ for prime p.

Proposition 3.6

If r is even and $h = 2^r$ let $n = 2h + 5$, then

$$A = \{2^{r-1}, 2^{r-1} + 1\} \subset \mathbb{Z}_n$$

has the property that $h_{\pm}A = \mathbb{Z}_n$

Proof. If r is even and $h = 2^r$ let $n = 2h + 5$, then consider the set

$$A = \{2^{r-1}, 2^{r-1} + 1\} \subset \mathbb{Z}_n$$

We have that for $h_1 + h_2 = h$

$$h_2(2^{r-1} + 1) - h_1(2^{r-1}) \in h_{\pm}A$$

If we let $h_1 = 5\frac{\frac{h}{2}+1}{3}$, then the above simplifies to

$$-\frac{(h+2)(2h+5)}{6}$$

Because r is even, $h = 4 \mod 6$, making $\frac{h+2}{6}$ and integer and $-\frac{(h+2)(2h+5)}{6} = 0 \mod n$. Thus, $0 \in h_{\pm}A$.

We also have that

$$\left[\frac{h^2}{2}, \frac{h^2}{2} + h\right] \in h_{\pm}A$$

and

$$\left[\frac{h(2h+5)}{2} - \frac{h^2}{2} - h, \frac{h(2h+5)}{2} - \frac{h^2}{2}\right] = \left[\frac{h^2+3h}{2}, \frac{h^2+h}{2} - 5\right] \in h_{\pm}A$$

This gives us

$$\left[\frac{h^2}{2}, \frac{h^2}{2} + h\right] \cup \left[\frac{h^2+3h}{2}, \frac{h^2+h}{2} - 5\right] = \left[\frac{h}{4} - \frac{2h^2}{4} - \frac{5h}{4}, \frac{h^2}{2} + h\right] = \left[\frac{h}{4}, n - \frac{h}{4}\right]$$

Now, all that needs to be proven is that $[1, \frac{h}{4} - 1] \subset h_{\pm}A$.

Consider
$h\{\frac{h}{2}, \frac{3h}{2} + 4\} \subset h_{\pm}A$ and $h\{\frac{3h}{2} + 5, \frac{h}{2} + 1\} \subset h_{\pm}A$
Now see that

$$\frac{h}{2}\left(2\frac{h-1}{3}\right) - \left(\frac{3h}{2} + 4\right)\left(h - 2\frac{h-1}{3}\right) \in h\{\frac{h}{2}, \frac{3h}{2} + 4\}$$

We have that for any $k \in [0, h]$

$$k\frac{h}{2} + (h-k)\left(\frac{3h}{2} + 4\right) \in h_{\pm}A$$

when $k = 2\frac{\frac{h}{4}-1}{3}$, the above simplifies to

$$\frac{(2h+1)(2h+5)}{3} + 1$$

and because $h = 4 \mod 6$, $\frac{(2h+1)(2h+5)}{3}$ is an integer, meaning $1 \in h_{\pm}A$.

Now, we show that for $x \in h\{\frac{h}{2}, \frac{3h}{2} + 4\}$, $x + 3 \in h\{\frac{h}{2}, \frac{3h}{2} + 4\}$ as long as $x = k\frac{h}{2} + (h-k)\left(\frac{3h}{2} + 4\right)$ and $k \geq 2$. Furthermore, because $k = 2\frac{\frac{h}{4}-1}{3}$, which gives us $\frac{\frac{h}{4}-1}{3}$ numbers 1 mod 3, which is the exact number of numbers congruent to 1 mod 3 less than or equal to $\frac{h}{4}$, we have have shown that $[1, \frac{h}{4} - 1] \subset h_{\pm}A$ holds for numbers 1 mod 3 in $[1, \frac{h}{4} - 1]$.

The same as above can be done for the 0 mod 3 case, as $k = 5\frac{\frac{h}{2}-1}{3}$ to get $x = 0$, which is much greater than our value of x that found 1, and thus $[1, \frac{h}{4} - 1] \subset h_{\pm}A$ holds for elements in $[1, \frac{h}{4} - 1]$ that are 0 mod 3.

Now, instead consider $h\{\frac{3h}{2} + 5, \frac{h}{2} + 1\}$, and note that the $k + 2 \mapsto x + 3$ holds here as well. In

$$k(\frac{h}{2} + 1) + (h-k)\left(\frac{3h}{2} + 5\right) \in h\{\frac{3h}{2} + 5, \frac{h}{2} + 1\}$$

If $k = \frac{5h-8}{6}$ then the above simplifies to

$$\left(\frac{(h+2)(2h+5)}{3}\right) + 2$$

One final time, because $h = 4 \mod 6$, $\left(\frac{(h+2)(2h+5)}{3}\right)$ is an integer and we have that $2 \in h_{\pm}A$. Clearly $\frac{5h-8}{6}$ is also much larger than $\frac{\frac{h}{4}-1}{3}$ as well, thus we have that $[1, \frac{h}{4} - 1] \subset h_{\pm}A$ holds for elements in $[1, \frac{h}{4} - 1]$ that are 2 mod 3, and thus in general. $\square$

Proposition 3.7

If r is odd and $h = 2^r$ let $n = 2h + 7$, then

$$A = \left\{ \frac{h}{2} + 2, \frac{h}{2} + 3 \right\} \subset \mathbb{Z}_n$$

has the property that $h_{\pm}A = \mathbb{Z}_n$

Proof. Let $h = 2^r$ where r is an odd integer and let $n = 2h + 7$. Given the subset of $\mathbb{Z}_n$,

$$A = \left\{ \frac{h}{2} + 2, \frac{h}{2} + 3 \right\}$$

we have that

$$h_{\pm}A = hA_1 \cup hA_2 \cup hA_3 \cup hA_4$$

where

$$A_1 = A \quad A_2 = \left\{ \frac{3h}{2} + 5, \frac{h}{2} + 3 \right\} \quad A_3 = \left\{ \frac{h}{2} + 2, \frac{3h}{2} + 4 \right\} \quad A_4 = -A$$

Starting with A_1 and A_4, we have

$$hA_1 = \left[\frac{h^2}{2} + 2h, \frac{h^2}{2} + 3h \right] = \left[\frac{h^2 + 4h}{2}, \frac{h^2 + 6h}{2} \right]$$

$$hA_4 = \left[\frac{3h^2}{2} + 4h, \frac{3h^2}{2} + 5h \right] = \left[\frac{h^2 + h + h(2h+7)}{2}, \frac{h^2 + 3h + h(2h+7)}{2} \right] = \left[\frac{h^2 + h}{2}, \frac{h^2 + 3h}{2} \right]$$

$$hA_1 \cup hA_4 = \left[\frac{h^2 + h}{2}, \frac{h^2 + 3h}{2} \right] \cup \left[\frac{h^2 + 4h}{2}, \frac{h^2 + 6h}{2} \right]$$

$$hA_1 \cup hA_4 = \left[\frac{2h^2 + 2h}{4}, \frac{2h^2 + 6h}{4} \right] \cup \left[\frac{2h^2 + 8h}{4}, \frac{2h^2 + 12h}{4} \right]$$

$$hA_1 \cup hA_4 = \cup \left[\frac{h}{4}, \frac{5h}{4} \right] \cup \left[\frac{3h}{4} + 7, \frac{7h}{4} + 7 \right]$$

As long as $\frac{h}{2} \geq 7$ then

$$hA_1 \cup hA_4 = \cup \left[\frac{h}{4}, \frac{7h}{4} + 7 \right]$$

Now all that needs to be proven is that $\left[0, \frac{h}{4} - 1\right] \subset h_{\pm}A$

Case 1 and 2: 0 and 1 mod 3 Consider hA_2, we have

$$\left(\frac{h}{2} + 3 \right) \frac{\frac{5h}{2} + 7}{3} + \left(\frac{3h}{2} + 5 \right) \left(h - \frac{\frac{5h}{2} + 7}{3} \right) \in hA_2$$

$$\frac{(h-2)(2h+7)}{3} \in hA_2$$

because h is an odd power of 2 this becomes

$$0 \in hA_2$$

Consider hA_2, we have

$$\left(\frac{h}{2}+2\right)\frac{\frac{h}{2}+2}{3}+\left(\frac{3h}{2}+5\right)\left(h-\frac{\frac{h}{2}+2}{3}\right) \in hA_2$$

$$\frac{4h^2+12h-4}{3} \in hA_2$$

$$\frac{4h^2+12h-7}{3}+1 \in hA_2$$

$$\frac{(2h-1)(2h+7)}{3}+1 \in hA_2$$

Because h is an odd power of 2, $2h-1$ is divisible by 3, giving us

$$1 \in hA_2$$

Note that for any

$$\left(\frac{h}{2}+3\right)k+\left(\frac{3h}{2}+5\right)(h-k) \in hA_2$$

If $k \leq h-2$

$$\left(\frac{h}{2}+3\right)(k+2)+\left(\frac{3h}{2}+5\right)(h-k-2) \in hA_2$$

$$\left(\frac{h}{2}+3\right)(k)+\left(\frac{3h}{2}+5\right)(h-k)+h+6-3h-10 \in hA_2$$

$$\left(\frac{h}{2}+3\right)(k)+\left(\frac{3h}{2}+5\right)(h-k)+h+6-3h-10 \in hA_2$$

$$\left(\frac{h}{2}+3\right)(k)+\left(\frac{3h}{2}+5\right)(h-k)-2h-4 \in hA_2$$

$$\left(\frac{h}{2}+3\right)(k)+\left(\frac{3h}{2}+5\right)(h-k)+3 \in hA_2$$

Furthermore, because

$$\left\lfloor h-\frac{1}{2}\frac{\frac{h}{2}+2}{3}+1\right\rfloor \cdot 3 \geq \frac{h}{4}-1$$

for the 1 mod 3 case and Furthermore, because

$$\left\lfloor h-\frac{1}{2}\frac{\frac{3h}{2}+7}{3}+1\right\rfloor \cdot 3 \geq \frac{h}{4}-1$$

for the 0 mod 3 case we have that $\left[0,\frac{h}{4}-1\right] \subset h_{\pm}A$ holds for the numbers 0 and 1 mod 3 in the set

Case 3: 2 mod 3
Now, consider hA_3.

$$\left(\frac{h}{2}+2\right)\frac{\frac{5h}{2}+4}{3}+\left(\frac{3h}{2}+4\right)\left(h-\frac{\frac{5h}{2}+4}{3}\right)\in hA_3$$

$$\frac{2h^2+3h-8}{3}\in hA_3$$

$$\frac{(h-2)(2h+7)}{3}+2\in hA_3$$

$$2\in hA_3$$

Finally, because for $k\leq h-2$

$$\left(\frac{h}{2}+2\right)(k+2)+\left(\frac{3h}{2}+4\right)(h-k-2)\in hA_2$$

$$\left(\frac{h}{2}+2\right)(k)+\left(\frac{3h}{2}+4\right)(h-k)+3\in hA_2$$

we have by the same logic as the first two cases that $\left[0,\frac{h}{4}-1\right]\subset h_{\pm}A$ holds for the numbers 2 mod 3 in the set, completing or proof.

$\square$

Corollary 3.8

For $r\geq 2$

$$\phi_{\pm}(\mathbb{Z}_{2\cdot 4^r+5},4^r)=2$$

and

$$\phi_{\pm}(\mathbb{Z}_{4^{r+1}+7},2\cdot 4^r)=2$$

Problem* 3.9

Classify more cases where $\phi_{\pm}(\mathbb{Z}_n, h) = 2$.

Below is a list of unclassified cases for which $\phi_{\pm}(\mathbb{Z}_n, h) = 2$. This is **by no means** comprehensive.

Table 2: $\phi(\mathbb{Z}_n, h) = 2$ Unclassified Examples

n	h	Element 1	Element 2
67	32	24	25
71	32	18	19
77	32	18	19
131	64	47	48
135	64	56	57
137	64	32	33
139	64	35	36
145	64	35	36
151	64	38	39
157	64	38	39
261	128	44	45
259	128	95	96
265	128	61	62
267	128	95	96
269	128	66	67
273	128	33	34
275	128	69	70
279	128	47	48
281	128	69	70
283	128	105	106
287	128	72	73
293	128	72	73
299	128	75	76
305	128	75	76
311	128	78	79
317	128	78	79
519	256	225	226
521	256	64	65
523	256	131	132
525	256	126	127
527	256	37	38
529	256	131	132
531	256	198	199
533	256	127	128
535	256	134	135
1037	512	258	259

Remark 3.10

I believe that this problem is **extremely** challenging, as one would expect. While the brute force proofs are valid for this topic, they are extremely ugly and a pain to write. The above data suggests that $\phi_{\pm}(\mathbb{Z}_n, h) = 2$ is far from predictable, and far from complete classification. While improvements can be made by slowly checking off cases from this list via general proofs I would still be reluctant to conjecture, even if all of the above are classified and prove, that we have found all cases where $\phi_{\pm}(\mathbb{Z}_n, h) = 2$. To solve this problem properly I believe it will require a very clever approach, one that I cannot think of as of now. Of the problems in [1] I believe that classifying $\phi_{\pm}(\mathbb{Z}_n, h) = 2$ is among the most difficult questions in the entire book

4 Dissociativity

Proposition 4.1

If $|A| = 1$ and $0 \in \Sigma_{\pm}A$ iff $A = \{0\}$

Proof. If $A = \{a\}$ then $\Sigma_{\pm}A = \{a, -a\}$, neither of which are 0 unless $a = 0$. $\square$

Proposition 4.2

If $|A| = 2$ and $0 \in \Sigma_{\pm}A$ iff $0 \in A$ or $A = \{a, -a\}$

Proof. If $A = \{a_1, a_2\}$, then $\Sigma_{\pm}A = A_1 \cup A_2$ where $A_1 = \{a_1, -a_1, a_2, -a_2\}$ and $A_2 = \{a_1 + a_2, a_1 - a_2, -a_1 - a_2, -a_1 + a_2\}$. Note that $0 \in A_1$ iff $0 \in A$, and $0 \in A_2$ iff $a_1 - a_2 = 0$ ($a_1 - a_2$ cannot be 0 as they are distinct). $\square$

Corollary 4.3

If $|A| \geq 4$ then $\dim(A) \geq 2$, and $|A| = 3$ and $\dim(A) = 1$ iff $A = \{0, a_1, a_1\}$ for some $a_1 \in G$.

Lemma 4.4

If $|G| > m_2 \geq m_1$ then

$$\dim(G, m_2) \geq \dim(G, m_1)$$

Lemma 4.5

If H is a subgroup of G then

$$\dim(H, m) \geq \dim(G, m)$$

Proposition 4.6

If $|A| = 3$ and $0 \in \Sigma_{\pm} A$ then either $0 \in \Sigma_{\pm}(A \setminus \{a\})$ for some $a \in A$ or $A = \{a_1, a_2, a_3\}$ (labeling is done without loss of generality) and one of $a_1 + a_2 + a_3 = 0$ or $a_1 + a_2 = a_3$ hold.

Proposition 4.7

For G of order $n \geq m$ and exponent κ we have for $m = 6$

$$\dim(G, 6) = \begin{cases} 2 & \kappa \geq 3 \\ 3 & \kappa = 2 \end{cases}$$

and $m = 7$

$$\dim(G, 7) = \begin{cases} 2 & \kappa \geq 3 \\ 3 & \kappa = 2 \text{ or } G \cong \mathbb{Z}_2^r \times \mathbb{Z}_4 \end{cases}$$

Proof. The case for $\kappa = 2$ for both $m = 6$ and $m = 7$ and follows from Ben Shepard's paper. Corollary 5.3 gives us that $\dim(G, 6) \geq 2$. Now, for groups with exponent greater than 2, we can do the following. Let κ be the exponent of G, and

$$A = (0, a, b, -a, -b, a + b)$$

Granted every element in A is distinct, which as long as $\kappa \geq 6$ will surely occur.

If $\kappa = 5$ then we can consult

$$(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (4, 0) \subset \mathbb{Z}_5^2$$

If $\kappa = 4$ and G has a subgroup isomorphic to $\mathbb{Z}_4^2$ then consult

$$(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (3, 0) \subset \mathbb{Z}_4^2$$

' If $\kappa = 4$ and G has no subgroup isomorphic to $\mathbb{Z}_4^2$ then consult

$$(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (1, 3) \subset \mathbb{Z}_2 \times \mathbb{Z}_4$$

Furthermore, Corollary 5.3 also gives us $\dim(G, 7) \geq 2$, in combination with the sets $A \cup I$ where $I = \{-(a + b)\}$, $(4, 4)$, and $(3, 3)$ for the $\kappa \geq 6$, $\kappa = 5$, and the first $\kappa = 4$ case respectively having dimension 2.

However, if we let $G \cong \mathbb{Z}_2^r \times \mathbb{Z}_4$ and $m = 7$ then something different occurs. This becomes apparent when we note that for some $A \subset \mathbb{Z}_n$ of size 7, if A has less than 3 elements of order 4, then by Ben Shepard's Proposition 7, $\dim(A) \geq 3$. Thus, if $\dim(G, 7) = 2$, we only need to look at A with 3 or more elements of order 4. Let $\{a, b, c\}$ be three such elements. Because $\dim(G, 7) = 2$, by Proposition 5.5, we have without loss of generality

$$a + b = \pm c$$

. However, this cannot happen as $a + b$ will have order 2, but c will have order 4. This is a contradiction, meaning $\dim(G, 7) \neq 2$, and instead we have $\dim(G, 7) \geq 3$, furthermore we can consult

$$\{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2)\} \subset \mathbb{Z}_2 \times \mathbb{Z}_4$$

to get $\dim(G, 7) \leq 3$

□

Proof. If $A = \{a_1, a_2, a_3\}$, and $0 \in \Sigma_{\pm}A$, then either for some $a \in A$ we have $0 \in \Sigma_{\pm}(A \setminus \{a\})$ or $0 \in 3_{\pm}A$. If the latter is true but not the former, then by that assumption $0 \notin A$. From this, we have that if $0 \in 3_{\pm}A$ then either $a_1 + a_2 + a_3 = 0$, or $a_1 + a_2 = a_3$ □

Theorem 4.8

For any G of order 9 or greater

$$\dim(G, 8) = \begin{cases} 2 & |\mathrm{Ord}(G, 3)| \geq 4 \\ 3 & \text{Otherwise} \end{cases}$$

and for G of order 10 or greater

$$\dim(G, 9) = \begin{cases} 2 & |\mathrm{Ord}(G, 3)| \geq 4 \\ 3 & |\mathrm{Ord}(G, 3)| < 4, \text{ and } \kappa \geq 5 \text{ or } G \cong \mathbb{Z}_4^r \\ 4 & G \cong \mathbb{Z}_2^r \times \mathbb{Z}_4 \text{ or } G \cong \mathbb{Z}_2^r \end{cases}$$

Proof. Assume $\dim(G, 8) < 3$. This means that the must exist some subset of G, S, with $|S| = 8$ such that every 3-subset of S (A) has one of the following properties.

1. $0 \in A$.
2. $\{-a, a\} \subset A$ for some $a \in G$ such that $a \neq -a$.
3. $\{a, b, c\} = A$ for some $\{a, b, c\} \subset G$ and $a + b + c = 0$,
4. $\{a, b, c\} = A$ for some $\{a, b, c\} \subset G$ and $a + b = c$ (WLOG).

We cannot only rely on the first two properties for such a set as $|S| > 7$, and thus we will need three disjoint symmetric pairs, 0 and at least one other element, meaning there will always exist 4 elements in S which are guaranteed to avoid three first two properties. We will call these elements a, b, c and d.

Let

$$W = \{a, b, c\}, \qquad X = \{a, b, d\}, \qquad Y = \{a, c, d\}, \qquad \text{and} \qquad Z = \{b, c, d\}$$

Note that no more than one of the above sets can have their elements sum to 0, and they all share exactly 2 elements with each other set. Furthermore, all the sets which do not have their elements sum to 0 will have two of their elements sum to their third.

We will first bring attention to the possibility that no three of a, b, c, d sum to 0. If this is the case, then without loss of generality we can assume $a + b = c$.

Thus, when it comes to set X we have that $a + d = b$ without loss of generality.

So, for set Y our options are $a + c = d$ or $d + c = a$.

When we come to set Z we can only have $b + c = d$, which means for set Y only $d + c = a$ is possible.

In all, we have that

- $a + b = c$
- $a + d = b$
- $d + c = a$
- $b + c = d$

Substitution b in equation 1 for its value in equation 2 gives us

$$2a + d = c$$

and because $d = b + c$ and $a = d + c$

$$a + b + c + d = 0$$

$$a + c = 0$$

which is a contradiction, meaning exactly one of W, X, Y, and Z have have elements which sum to 0.

Next, we come to the second case: when one of the sets does have three elements which sum to 0.

Without loss of generality, we will assume that if any of the sets have three elements that sum to 0, that it is W. If this is the case, we have

$$a + b + c = 0.$$

As for set X, we cannot have that $a + b = d$, as that would imply d and c are symmetric pairs. Thus we can assume without loss of generality that

$$a + d = b.$$

Moving to Y, for the same reason as in X, $a + c \neq d$. This means either $a + d = c$ or $c + d = a$. The former cannot be true as $a + d = b$ giving us

$$c + d = a.$$

Last, we examine Z. Which by similar log to above, we arrive at

$$b + d = c.$$

If we add these equations in various ways we get that for all $x \in \{a, b, c, d\}$ that $3x = 0$ In other words, a, b, c, d all have order 3. However, of G does not have 4 or more elements of order 3, then

For this to occur, G must have 4 or more elements of order 3 itself, Or equivalently, $\mathbb{Z}_3^2 \leq G$. Because we have for

$$B = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1)\} \subset \mathbb{Z}_3^2$$

and $\dim(B) = 2$. So, for any G with $\mathbb{Z}_3^2$ as a subgroup $\dim(G, 8) \leq 2$, and because $\dim(G, 8) \geq \dim(G, 7) = 2$ as $\exp(G) \geq 3$, we have have that if $|\mathrm{Ord}(G, 3)| \geq 4$ then $\dim(G, 8) = 2$.

If $\kappa = 2$, then Ben Shepard's Theorem 14 gives us $\dim(G, 8) = 3$.

For other G with exponent $\kappa \geq 4$ we do the following:

$$A = \{0, 1, 2, 3, 4, -3, -2, -1\}$$

This set clearly has dimension 3, and we can guarantee that all of these elements are distinct in cyclic G of order 9 (and thus exponent at least 9).

If $\kappa = 8$, then $\mathbb{Z}_2 \times \mathbb{Z}_8$ is a subgroup of G which has the dimension 3 subset

$$\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3)\}$$

If $\kappa = 7$ then $\mathbb{Z}_7 \times \mathbb{Z}_7$ is a subgroup of G which has the dimension 3 subset

$$\{(0,0),(0,1),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0)\}$$

If $\kappa = 6$ and $|\mathrm{Ord}(G, 3)| < 4$ then $\mathbb{Z}_2 \times \mathbb{Z}_6$ has the following dimension 3 subset

$$\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3)\}$$

If $\kappa = 5$ then $\mathbb{Z}_5 \times \mathbb{Z}_5$ has the following dimension 3 subset

$$\{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1),(3,0),(4,0)\}$$

If $\kappa = 4$ then at least one of $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ or $\mathbb{Z}_4^2$ will be subgroups of G, and thus will mean they will have dimension 3 subsets corresponding to the ones of those groups which are listed below

$$\{((0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\} \subset \mathbb{Z}_2^2 \times \mathbb{Z}_4$$

$$\{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1),(3,0),(3,1)\} \subset \mathbb{Z}_4^2$$

This concludes the long list of examples. However, we must address the case of $m = 9$. We have that $\dim(G, 9) \geq \dim(G, 8)$, and for most scenarios we can do the following example subsets of size 9

- If $|\mathrm{Ord}(G, 3)| \geq 4$, then either $\mathbb{Z}_3^3 \leq G$ and we have

  $$\{(0,0,0),(0,0,1),(0,1,0),(0,2,0),(1,0,0),(1,1,0),(1,2,0),(2,0,0),(2,1,0),(2,2,0)\}$$

  is a dimension 2-subset of $\mathbb{Z}_3^3$. Or $\mathbb{Z}_3 \times \mathbb{Z}_{3w} \leq G$ for some integer $w > 1$ and we have $\mathbb{Z}_3 \times \mathbb{Z}_{3w}$'s dimension-2 9-subset:

  $$\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0),(2,1)\}$$

- $\kappa \geq 10$ we have
$$\{0, 1, 2, 3, 4, -4, -3, -2, -1\}$$
has dimension 3 in $\mathbb{Z}_\kappa \leq G$.
- The $\kappa = 9$ case is covered by $|\text{Ord}(G, 3)| \geq 4$
- If $\kappa = 8$, then $\mathbb{Z}_2 \times \mathbb{Z}_8$ is a subgroup of G which has the dimension 3 subset
$$\{(0, 0), (0, 1), (0, 2), (0, 3), (0, 5), (1, 0), (1, 1), (1, 2), (1, 3)\}$$
- If $\kappa = 7$, then $\mathbb{Z}_7 \times \mathbb{Z}_7$ is a subgroup of G which has the dimension 3 subset
$$\{(0, 0), (0, 1), (0, 6), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0)\}$$
- If $\kappa = 6$ and $|\text{Ord}(G, 3)| < 4$ then $\mathbb{Z}_2 \times \mathbb{Z}_6 \leq G$ and it has the following dimension 3 subset
$$\{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 0), (1, 1), (1, 2), (1, 3)\}$$
- If $\kappa = 5$ then $\mathbb{Z}_5 \times \mathbb{Z}_5 \leq G$ and it has the following dimension 3 subset
$$\{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1), (4, 0)\}$$
- If $\kappa = 4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4 \leq G$, $\mathbb{Z}_4 \times \mathbb{Z}_4$ has dimension 3 subset
$$\{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1)\}$$
- If $\kappa = 3$ then $|\text{Ord}(G, 3)| \geq 4$ gives us $\dim(G, 9) = 2$
- If $\kappa = 2$ then Ben Shepard's Theorem 14 gives us that $\dim(G, 9) = 4$

The only case left is when $\kappa = 4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ and is not a subgroup of G, ie. $G \cong \mathbb{Z}_2^r \times \mathbb{Z}_4$ for some $r \geq 2$. Pressing further into this case, assume $\dim(G, 9) = 3$. we note that for any $A \subseteq G$ of size 9, if A contains only elements of order 2 or 1 then by Ben Shepard's Theorem 14, $\dim(A) = 4$, so if $\dim(G, 9) = 3$, then there must exists some set $S \subset G$ with dimension 3, and at least one element $a \in S$ of order 4. However, because $\dim(G, 8) = 3$, ther must exist a 3-subset $S_1 \subset (S \setminus \{a\})$ where S_1 is dissociated. From this, one can see that $S_1 \cup \{a\}$ is clearly dissociated, and thus we have that $\dim(G, 9) > 3$. In combination with Lemma 5.4 and Ben Shepard's Theorem 14 to produce an upper bound, we have proven our claim. □

Below are the summarized results for $\dim(G, m)$ when m is constant $m \in \{1, 2, 3\}$ credited to Bajnok, and $m \in \{4, 5\}$ credited to Shepard.

- $$\dim(G, 1) = 0$$
- $$\dim(G, 2) = 1$$

- $$\dim(G,3) = \begin{cases} 1 & \kappa \geq 3 \\ 2 & \kappa = 2 \end{cases}$$
- $$\dim(G,4) = 2$$
- $$\dim(G,5) = \begin{cases} 2 & \kappa \geq 3 \\ 3 & \kappa = 2 \end{cases}$$
- $$\dim(G,6) = \begin{cases} 2 & \kappa \geq 3 \\ 3 & \kappa = 2 \end{cases}$$
- $$\dim(G,7) = \begin{cases} 2 & \kappa \geq 3 \text{ and } G \ncong \mathbb{Z}_2^r \times \mathbb{Z}_4 \\ 3 & \kappa = 2 \text{ or } G \cong \mathbb{Z}_2^r \times \mathbb{Z}_4 \end{cases}$$
- $$\dim(G,8) = \begin{cases} 2 & |\mathrm{Ord}(G,3)| \geq 4 \\ 3 & \text{Otherwise} \end{cases}$$
- $$\dim(G,9) = \begin{cases} 2 & |\mathrm{Ord}(G,3)| \geq 4 \\ 3 & |\mathrm{Ord}(G,3)| < 4, \text{ and either } \kappa \geq 5 \text{ or } G \cong \mathbb{Z}_4^r \\ 4 & G \cong \mathbb{Z}_2^r \times \mathbb{Z}_4 \text{ or } G \cong \mathbb{Z}_2^r \end{cases}$$

Finding $\dim(G,m)$ for constant m is a fun yet challenging exercise. As m gets larger, $\dim(G,m)$ will almost certainly get more complex, especially once m starts approaching 14 and 15, as that is the earliest point known where $\dim(\mathbb{Z}_n,m)$ begins to deviate from $\lfloor \log_2 m \rfloor$. This is a definite contrast to the cases of $m \in [1,9]$, which we know $\dim(\mathbb{Z}_n,m) = \lfloor \log_2 m \rfloor$ holds for each m. I cannot help but be curious about the following questions

Problem 4.9

Find $\dim(G,m)$ for all G and a constant $m \geq 10$

The above can be done by finding all cases where $\dim(G,10) = \dim(G,9)$, and then finding a size 10 subset of those G with dimension $\dim(G,9)$. For the groups where no such subset is found, determine what could be preventing that group from having such a set.

I have not yet observed any difference between $\dim(G,9)$ and $\dim(G,10)$

Problem 4.10

Find $\dim(G,15)$ and $\dim(G,14)$ for all G, or at least cyclic G.

Problem 4.11

Find $\dim(G, m)$ for all m and G

Problem 4.12

Does every m-subset of $\mathbb{Z}_n$ have a dissociated sequence of length $\lfloor \log_2 m \rfloor$?

Proposition 4.13

If $|A| = 4$ and $0 \in \Sigma_{\pm} A$ then either $0 \in \Sigma_{\pm}(A \setminus \{a\})$ for some $a \in A$ or $A = \{a_1, a_2, a_3, a_4\}$ (labeling is done without loss of generality) and one of the following is true

- $a + b + c + d = 0$
- $a + b + c = d$
- $a + b = c + d$

By Theorem 4.8, we have that for any $n \geq 15$ that

$$\dim(\mathbb{Z}_n, 15) \geq 3.$$

Assume equality holds. That is, for all $A \subset \mathbb{Z}_n$ with $|A| = 15$ all $S \subset A$ such that $|S| = 4$: $0 \in \Sigma_{\pm} S$. We will write A as follows

$$A = \{a_1, a_2, a_3, \cdots, a_{15}\}$$

By the pigeonhole principle, there must be at least 7 non-zero elements in A such that no two of them form a symmetric pair. A_1 will denote the set of 7 such elements, so without loss of generality we have

$$A = \{a_1, a_2, a_3, \cdots, a_7\}$$

In Theorem 4.8, we showed that unless $|\text{Ord}(G, 3)| \geq 4$ then for any set, B of 4 or more non-zero elements that contain no symmetric pairs, that $\dim(B) \geq 3$. Because we have that $\dim(\mathbb{Z}_n, 15) = 3$, we must have one of the following for all combinations of four distinct $i, j, k, l \in [1, 7]$

- $a_i + a_j + a_k + a_l = 0$
- $a_i + a_j + a_k = a_l$
- $a_i + a_j = a_k + a_l$

In Theorem 4.8, there were 8 total configurations that the equations could possibly be in, whereas here, there are a whopping 105. Going further will likely require a clever trick, or significant brute force.

Conjecture 4.14

For m and n with $\lfloor \log_2 m \rfloor < \lfloor \log_2 n \rfloor$and the subset of $\mathbb{Z}_n$:

$$A = [0, m-1]$$

we have

$$\dim(A) = \lfloor \log_2 m \rfloor + 1$$

This would imply that if $\lfloor \log_2 m \rfloor < \lfloor \log_2 n \rfloor$, then $\dim(\mathbb{Z}_n, m) \leq \lfloor \log_2 m \rfloor + 1$

Clearly

$$\dim(A) \geq \lfloor \log_2 m \rfloor + 1$$

as $\{1, 2, 4, \cdots, 2^{\lfloor \log_2 m \rfloor}\}$ is dissociated.

Table 3: Some known values of $\dim(G, m)$

G	m	$\dim(G, m)$
$\mathbb{Z}_4^2$	13	3
$\mathbb{Z}_4^2$	14	4
$\mathbb{Z}_2^2 \times \mathbb{Z}_6$	11	3
G	m	
G	m	
G	m	

Proposition 4.15

References

[1] B. Bajnok Additive Combinatorics A Menu of Research Problems *CRC Press, Boca Raton*, 2018, p.284,