MATH 315 CHAPTER 19

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Problem 1a i

1

1, 1

1, 2, 1

1, 3, 3, 1

1, 4, 6, 4, 1

1, 5, 10, 10, 5, 1

1, 6, **15**, **20**, 15, 6, 1

1, 7, 21, **35**, 35, 21, 7, 1

Problem 1a ii

1

1, 1

1, 2, 1

1, 3, 3, 1

1, **4**, 6, 4, 1

1, 5, **10**, 10, 5, 1

1, 6, 15, **20**, 15, 6, 1

1, 7, 21, **35**, 35, 21, 7, 1

Problem 1a iii

1

1, 1

1, 2, **1**

1, 3, **3**, 1

1, 4, **6**, 4, 1

1, 5, **10**, 10, 5, 1

1, 6, **15**, 20, 15, 6, 1

1, 7, 21, **35**, 35, 21, 7, 1

Problem 1b i

Claim. *For positive integers $m < n$*

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$$

Proof. We have that $\binom{n}{m}$ is the number of m-subsets of an n-set, A. Consider some specific element of the n-set, x. Note that every m-subset of the n-set falls into exactly one of the two disjoint cases

- x is in the m-subset
- x is not in the m-subset

Each m-subset with x in it corresponds to an $(m-1)$-subset of $A \setminus \{x\}$, of which there are $\binom{n-1}{m-1}$. The other case, when x is not in the m-subset means that these sets are exactly the m-subsets of $A \setminus \{x\}$ of which there are $\binom{n-1}{m}$ Thus the total number of m subsets is $\binom{n-1}{m} + \binom{n-1}{m-1}$, proving our claim.

Problem 1b ii

Claim. *For positive integers $m < n$*

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-2}{m-1} + \cdots + \binom{n-m-1}{0}$$

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$. First, note that every m-subset of A falls exactly into one of the following sets: "The set of m-subsets of A that do not contain a_i but do contain all a_j where $j < i$" We can describe the number of such sets as $\binom{n-i}{m-(i-1)}$ as we are selecting the remaining $m-(i-1)$ elements from the remaining $n-1$ elements of the set that have index greater than i. This gives us that

$$\binom{n}{m} = \sum_{i=1}^{n} \binom{n-i}{m-(i-1)}$$

which because $m < n$ is equivalent to

$$\binom{n}{m} = \sum_{i=1}^{m+1} \binom{n-i}{m-(i-1)} + \sum_{i=m+2}^{n} \binom{n-i}{m-(i-1)}$$

and because $m-(m+2) < 0$ we have that

$$\binom{n}{m} = \sum_{i=1}^{m+1} \binom{n-i}{m-(i-1)}$$

which is equivalent to our claim.

Remark. The partition described above can be proven to hold in many ways, my favorite of which is induction.

Problem 1b iii

Claim. *For positive integers $m < n$*

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-2}{m-1} + \cdots + \binom{m-1}{m-1}$$

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$. Note that we can partition the set of m-subsets of A into sets defined by the lowest value of i for which a_i is in the sets. For each such set, we are selecting a_i, and restricting the selection of the other $m-1$ elements to the $n-1-(i-1)$ elements with an index greater than i. Thus, we have that

$$\binom{n}{m} = \sum_{i=1}^{n} \binom{n-1-(i-1)}{m-1}$$

which is equal to

$$\binom{n}{m} = \sum_{i=1}^{n+1-m} \binom{n-1-(i-1)}{m-1} + \sum_{i=n+2-m}^{n} \binom{n-1-(i-1)}{m-1}$$

and because $n - 1 - (n + 2 - m) < m - 1$ we have that

$$\binom{n}{m} = \sum_{i=1}^{n+1-m} \binom{n-1-(i-1)}{m-1}$$

which is equivalent to our claim.

Problem 3a

Claim. *Let a_n be the number of ways to cover a 2 by n domino board with. We have that*

$$a_n = \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{\lceil n/2 \rceil}{\lfloor n/2 \rfloor}.$$

Proof. Note that the only way of placing dominoes is either vertically or horizontally, and if placed horizontally, another must be placed alongside it to cover a 2 by 2 square. So, the maximum number of such 2 by 2 squares is $\lfloor n/2 \rfloor$. Now, let b_i be the number of ways to cover the 2 by n board with exactly i horizontal pairs with the rest being vertical dominoes. With this, we have that

$$a_n = \sum_{i=0}^{\lfloor n/2 \rfloor} b_i.$$

Now, note that the covering of the boards with exactly i horizontal pairs consists of a total of $n - i$ horizontal pairs and vertical dominoes. And thus, each covering in b_i can be defined by an i-subset, B, of $\{1, 2, \ldots, n - i\}$ where you can construct the covering as follows:

- Start from the leftmost position and let $k = 1$.
- If $k \in B$ place two horizontal dominoes, otherwise place one vertical domino. Either way, place the domino(es) as far left as you can.
- If $k < n - i$: Increase k by one, and then repeat the previous step. If $k = n - i$, we have covered the board.

Thus b_i is the number of i subsets of $\binom{n-i}{i}$ and we have that

$$a_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}.$$

which is equivalent to our claim.

Problem 3b

Claim. $a_n = F_n$

Proof. See first that $a_1 = 1$ and $a_2 = 2$. Now see that $a_n = \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{\lceil n/2 \rceil}{\lfloor n/2 \rfloor}$ and so

$$a_n + a_{n+1} = \left(\binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{\lceil n/2 \rceil}{\lfloor n/2 \rfloor}\right) + \left(\binom{n+1}{0} + \binom{n}{1} + \cdots + \binom{\lceil n+1/2 \rceil}{\lfloor n+1/2 \rfloor}\right)$$

Or alternatively

$$a_n + a_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} + \sum_{i=0}^{\lfloor n+1/2 \rfloor} \binom{n+1-i}{i}$$

If n is even, then see that $\lfloor n + 1/2 \rfloor = \lfloor n/2 \rfloor$ giving us that

$$a_n + a_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} + \binom{n+1-i}{i}$$

which by problem 1i gives us that

$$a_n + a_{n+1} = \binom{n+1}{0} + \binom{\lceil n+1/2 \rceil}{\lceil n/2 \rceil} + \sum_{i=0}^{\lfloor n+2/2 \rfloor} \binom{n-i+1}{i+1}$$

which is equivalent to

$$a_n + a_{n+1} = \binom{n+2}{0} + \binom{\lceil n+2/2 \rceil}{\lfloor n/2 \rfloor} + \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n-i}{i}$$

which gives us that

$$a_n + a_{n+1} = \sum_{i=0}^{\lfloor n/2+2 \rfloor} \binom{n-i+1}{i+1} = a_{n+2}$$

If n is odd then we can use 1i to get that

$$a_n + a_{n+1} = \binom{n+1}{0} + \sum_{i=1}^{\lfloor n/2+1 \rfloor} \binom{n-i}{i}$$

which is the same as

$$a_n + a_{n+1} = \binom{n+2}{0} + \sum_{i=1}^{\lfloor n/2+1 \rfloor} \binom{n-i}{i}$$

which becomes

$$a_n + a_{n+1} = \sum_{i=0}^{\lfloor n/2+2 \rfloor} \binom{n-i}{i} = a_{n+2},$$

and so we have that $a_{n+2} = a_n + a_{n+1}$ regardless of parity, proving our claim in combination with $a_1 = 1$ and $a_2 = 2$.

Problem 5

Claim. *There are*

- $\binom{13}{4}\binom{4}{1}^4\binom{4}{2}$ *pair hands,*
- $\binom{13}{3}\binom{3}{2}\binom{4}{1}\binom{4}{2}$ *two-pair hands,*
- $\binom{13}{3}\binom{3}{1}\binom{4}{1}^2\binom{4}{3}$ *three-of-a-kind hands,*
- $10 \cdot (4^5 - 1)$ *straight hands,*
- $4 \cdot \binom{13}{5} - 40$ *flush hands,*
- 3744 *full house hands,*
- 156 *four-of-a-kind hands,*
- 36 *straight flush hands, and*
- 4 *Royal flush hands.*

A non-distinguished hand is slightly more likely.

Proof. The number of hands that are a pair can be determined as follows: Every pair hand consists of four different numbers, and so there $\binom{13}{4}$ ways to select those. Furthermore, one of those four will be the paired number, and there are $\binom{4}{1}$ ways to select that number. Lastly, when selecting suits, there are $\binom{4}{1}$ options for each the three unpaired cards, and $\binom{4}{2}$ for the paired cards giving us a total of $\binom{13}{4}\binom{4}{1}^4\binom{4}{2}$ pair hands

Similarly, for two-pair, we select three different numbers, and there are $\binom{13}{3}$ ways to select this. Then, two of the three selected numbers are selected to be paired. There are $\binom{3}{2}$ ways to select this. Lastly, when selecting the suit of the unpaired there are $\binom{4}{1}$ options, and $\binom{4}{2}$ options for each of the pairs' suits. Giving us $\binom{13}{3}\binom{3}{2}\binom{4}{1}\binom{4}{2}$

For three-of-a-kind, it is very similar to two-pair there are still three different numbers, and so there are $\binom{13}{3}$ ways to select those numbers. Then there are $\binom{3}{1}$ options for the card that there are three of. Then there are $\binom{4}{1}$ different options for the suits of the un-"tripled" cards, and $\binom{4}{3}$ options for the suit of the tripled cards, and so there are $\binom{13}{3}\binom{3}{1}\binom{4}{1}^2\binom{4}{3}$ three-of-a-kind hands.

For straight, we must have a smallest card, and this can be any card except for a jack, queen, or king. Thus, there are 10 options for the first card's rank. Each card's rank thereafter is guaranteed, and thus the only thing that changes about the cards after are the suits, and so there $10 \cdot 5^4$ total straights, but since this includes the straight flushes (royal flushes too) we must remove the cases of which there are $10 \cdot 4$ as there are 10 valid starting points, and 4 valid suits. giving us that there are $10 \cdot (5^4 - 1)$ straights.

For flushes, there are 4 different suits, and $\binom{13}{5}$ different 5-number combinations, but as mentioned above this includes the 40 straight flushes and thus there are $4 \cdot \binom{13}{5} - 40$ flushes.

For full house, all five numbers are made up of two different numbers there are 13 different options for the triple and 12 options for the double. To determine the triple, one must select the suit of the predetermined number to not be in the triple, of which there are 4 options. Doing the same for the double, there are $\frac{4 \cdot 3}{2} = 6$ options. In total there are $13 \cdot 12 \cdot 4 \cdot 6 = 3744$ full house hands

For four-of-a-kind hands, there are a total of 13 options for the four cards that make up the four-of-a-kind itself and 12 options for the fifth card giving us 156 four-of-a-kind hands.

As mentioned before there are 40 straight flush hands, but four of them are royal flushes. This completes our proof.

Upon adding the quantities together and dividing by $\binom{52}{5}$ we get a value just below 0.5, and so a non-distinguished hand is ever so slightly more likely.

Problem 6a

Claim. *If the books are all different and the order of the books on each shelf matters there are* $\binom{n}{m} k^{\overline{m}}$ *arrangements.*

Proof. There are $\binom{n}{m}$ options of books for Alvin to buy, and once he buys those books he orders the $m + k$ books and shelves on his shelves, but the order of the k shelves must be divided out giving us $(m + k)!/k! = k^{\overline{m}}$, and so there are $\binom{n}{m} k^{\overline{m}}$ arrangements.

Problem 6b

Claim. *If the books are all different and the order of the books on each shelf doesn't matter there are* $\binom{n}{m} k^m$ *arrangements.*

Proof. There are $\binom{n}{m}$ options of books for Alvin to buy. Each book can go on one of k shelves, and thus there are k^m ways to place those books on the shelves. Thus, there are $\binom{n}{m} k^m$ arrangements.

Problem 6c

Claim. *If the books are not necessarily different and the order of the books on each shelf matters there are* $n^m \begin{bmatrix} k \\ m \end{bmatrix}$ *total arrangements.*

Proof. We can define every shelf arrangement as a sequence of books in the order they are bought, and a placement of shelves in between those books. See that there are n^m ways to do this. Then, k shelves can be placed at any of the $m + 1$ points in the sequence (1 at the start $m - 1$ between consecutive pair, 1 at the end) with repetition allowed. The books on any given shelf are the books that come before that shelf, but no earlier shelf. This means that one of our shelves must go at the very bottom, since we cannot have any books left out. Accounting for the other $k - 1$ shelves, there are $\begin{bmatrix} m+1 \\ k-1 \end{bmatrix}$ possible shelf placements. Giving us

$$n^m \begin{bmatrix} m+1 \\ k-1 \end{bmatrix} = n^m \begin{bmatrix} k \\ m \end{bmatrix}$$

total arrangements

Problem 6d

Claim. *If the books are not necessarily different and the order on each shelf doesn't matter there are* $\begin{bmatrix} nk \\ m \end{bmatrix}$ *total arrangements.*

Proof. There are $\begin{bmatrix} nk \\ m \end{bmatrix}$ ways to buy m books if Alvin decides the shelf the book will reside on upon purchase, and since the order of the books does not matter, there are $\begin{bmatrix} nk \\ m \end{bmatrix}$ total arrangements

Problem 8a

Claim. *There are*
$$\frac{(a+b+c+d)!}{a!b!c!d!}$$
Plutonian words with a A's, b B's, c C's, and d D's.

Proof. Note that all such Plutonian words have $a+b+c+d$ letters, and so the number of possible placements of the a A's is easily seen to be $\binom{a+b+c+d}{a}$. The placements of the B's can be described as a b-subset of the $b+c+d$ remaining non-A letters, and so there are $\binom{b+c+d}{b}$. By the same reasoning, there are $\binom{c+d}{c}$ ways to place our c C's, and our d D's are placed in the remaining spots. Thus the number of all such Plutonian words is equal to
$$\begin{aligned}\binom{a+b+c+d}{a}\binom{b+c+d}{b}\binom{c+d}{c} &= \left(\frac{(a+b+c+d)!}{a!(b+c+d)!}\right)\left(\frac{(b+c+d)!}{b!(c+d)!}\right)\left(\frac{(c+d)!}{c!d!}\right) \\ &= \frac{(a+b+c+d)!}{a!b!c!d!}\end{aligned}$$
as claimed.

Problem 8b

Claim. *There are*
$$\frac{(a+b+c+d+1)!}{(a+b+c+1)a!b!c!d!}$$
Plutonian words with a A's, b B's, c C's, and at most d D's.

Proof. By 8a, we have that there are
$$\sum_{x=0}^{d}\frac{(a+b+c+x)!}{a!b!c!x!}$$
Plutonian words with a A's, b B's, c C's, and at most d D's. See that
$$\sum_{x=0}^{d}\frac{(a+b+c+x)!}{a!b!c!x!} = \frac{1}{a!b!c!}\sum_{x=0}^{d}\frac{(a+b+c+x)!}{x!}.$$
And because $\binom{a+b+c+x}{x} = \frac{(a+b+c+x)!}{(a+b+c)!x!}$ we have that $(a+b+c)!\binom{a+b+c+x}{x} = \frac{(a+b+c+x)!}{x!}$ and so our expression from earlier is equal to
$$\frac{(a+b+c)!}{a!b!c!}\sum_{x=0}^{d}\binom{a+b+c+x}{x}$$

which by identity 1ii is equal to

$$\frac{(a+b+c)!}{a!b!c!}\binom{a+b+c+d+1}{d} = \frac{(a+b+c)!}{a!b!c!}\frac{(a+b+c+d+1)!}{(a+b+c+1)!d!} = \frac{(a+b+c+d+1)!}{(a+b+c+1)a!b!c!d!},$$

and with this our claim is proven

Problem 8c

Claim. *There are*

$$\frac{(2a+2b+2c+1)!}{(a+b+c+1)a!b!c!(a+b+c)!}$$

Plutonian words with a A's, b B's, c C's, with at most half of its letters being D's.

Proof. Easily follows from Problem 8b