Challenge Problems 4

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Difficulty/10	7	5	8	6.75	4.5	4.5	6	8
Category	NT	GM	GM	CO	CO	NT	CO	CO

Key:

- NT: Number Theory
- CO: Combinatorics
- GM: Geometry

1. Prove that
$$\lim_{n\to\infty} \frac{n\sum_{i=1}^{n} \frac{1}{i^2}}{\varphi(n)} = 1$$
where $\varphi(m)$ is the number of positive integers less than m relatively prime to m.

2. (a) What is the maximum number of regions a plane can be divided into with m lines?
   (b) What is the maximum number of regions three-dimensional space can be divided into with m planes?
   (c) What is the maximum number of regions four-dimensional space can be divided into with m hyper-planes?
   (d) What is the maximum number of regions n-dimensional euclidean space can be divided into with m hyper-planes?

3. Prove that in any set A of 10 positive integers less than or equal to some positive n there exists some $B \subset A$ such that $|B| = 3$, and the sum the the elements in each subset of B is distinct mod n.

4. Determine the number of 2-subsets, S, of $\mathbb{Z}_n$ (the group of integers mod n) with the property that $S = \{a, b\}$ and $0 \notin \{a, b, a+b, a-b, -a-b, -a-b\}$.

5. Let f_a, f_b, f_c, and f_d be distinct positive Fibonacci numbers with the property that
$$f_a + f_b = f_c + f_d.$$
Prove that $\{f_a, f_b\} = \{f_c, f_d\}$.

6. (a) What is the number of paths you can take from the point $(0, 0)$ to $(2n, 0)$ without going below the x axis with the options $(1, 1)$, $(-1, 1)$, and $(0, 1)$ for steps.
   (b) What if you remove the x axis clause?
   (c) What if any point with x value $2n$ suffices as the endpoint?
   (d) What's the answer to part c if the x axis clause is reinstated?