MATH 315 CHAPTER 15

JACOB TERKEL

FEBRUARY 13, 2023

Problem 1, Chapter 16

Claim. *In the definition of an equivalence relation, it is enough to require that the relation be symmetric and transitive. That is, if the relation R on a set A is both symmetric and transitive, then it is also reflexive.*

Argument. Since R is symmetric, $a \sim b$ implies $b \sim a$. But R is also transitive, so $a \sim b$ and $b \sim a$ imply that $a \sim a$. Therefore, R is reflexive. $\boxed{J\tau}$

Claim. *The above is wrong.*

Proof. The argument subtly assumes that there exists a and b with the property that $a \sim b$.

To see a counterexample, define the relation $\sim_X$ on A to never hold for any $a, b \in A$. This relation satisfies the symmetric and transitive properties vacuously but clearly does not satisfy reflexivity. $\boxed{J\tau}$

Problem 2, Chapter 16

Claim (Part A). $R = \{(x, y) \in \mathbb{R}^2 \mid x \leq y + 1\}$ *is not an equivalence relation.*

Proof. Reflexivity is satisfied as $x \leq x + 1$ is always true.

We have that $1 \sim 3$ but $3 \nsim 1$ so symmetry is violated.

Transitivity is also seen to be violated as $1 \sim 0.5$, and $0.5 \sim -0.4$, but $1 \nsim -0.4$. $\boxed{J\tau}$

The Cartesian graph of this relation is exactly all the points above and on the line $x = y+1$.

Claim (Part B). $R = \{(x, y) \in \mathbb{R}^2 \mid xy > 0\}$ *is not an equivalence relation.*

Proof. Reflexivity is violated as $0 \nsim 0$.

Symmetry is satisfied as $xy = yx$.

Transitivity is satisfied as if $xy > 0$ and $yz > 0$ then we have that x, y, and z are all the same sign, and all are non-zero, and if that is the case then $xz > 0$. $\boxed{J\tau}$

The Cartesian graph of this relation is exactly all of the first and third quadrants.

Claim (Part C). $R = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}$ *is not an equivalence relation.*

Proof. Reflexivity is satisfied as for all non-zero x, we have that $x^2 > 0 \geq 0$ and if $x = 0$ then $x^2 = 0 \geq$

Symmetry is satisfied as $xy = yx$.

Transitivity is seen to be violated as $1 \sim 0$ and $0 \sim -1$, however, $1 \nsim -1$. $\boxed{J\tau}$

The Cartesian graph of this relation is exactly all of the first and third quadrants as well as the axes.

Claim (Part D). $R = \{(x, y) \in \mathbb{R}^2 \mid xy \neq 0\}$ *is not an equivalence relation*

Proof. Reflexivity is violated as $0 \nsim 0$.

Symmetry is satisfied as $xy = yx$.

Transitivity is satisfied as the $xy \neq 0 \iff 0 \notin \{x, y\}$, and so for any $a, b, c \in \mathbb{R}^2$ where $a \sim b$ and $b \sim c$ we have that

$$0 \notin \{a, b, c\} \implies 0 \notin \{a, c\} \iff ac \neq 0 \longleftrightarrow a \sim c.$$

$\boxed{J\tau}$

Claim (Part E). $R = \{(x, y) \in (\mathbb{R} \setminus \{0\})^2 \mid xy > 0\}$ *is an equivalence relation*

Proof. Reflexivity is satisfied as for all non-zero x, we have that $x^2 > 0 \geq 0$.

Symmetry is satisfied as $xy = yx$

Transitivity is satisfied as if $xy > 0$ and $yz > 0$ then we have that x, y, and z are all the same sign, and all are non-zero, and if that is the case then $xz > 0$. $\boxed{J\tau}$

The Cartesian graph of this relation is exactly all of the first and third quadrants.

Claim (Part F).

Proof. $R = \{(x, y) \in \mathbb{R}^2 \mid |x - y| \leq 1\}$ is not an equivalence relation $\boxed{J\tau}$

Proof. Reflexivity is satisfied as $|x - x| = 0 \leq 1$.

Symmetry is satisfied as $|x - y| = |y - x|$.

Transitivity is not satisfied as $1 \sim 2$ and $2 \sim 3$ but $1 \nsim 3$. $\boxed{J\tau}$

The Cartesian graph of this relation is the area between the lines $x = y + 1$ and $y = x + 1$ inclusive.

Claim (Part G). $R = \{(x, y) \in \mathbb{Z}^2 \mid 2 \mid (a - b)\}$ *is an equivalence relation*

Proof. Reflexivity is seem to hold as $a - a = 0$ is divisible by 2.

Symmetry is seen to hold as the parity of a number is unchanged by taking its negative and $a - b = -(b - a)$.

Transitivity is seen to hold as if $a - b = 2k$ and $b - c = 2\ell$ then $2k + 2\ell = (a - b) + (b - c) = a - c$ gives us that $2(k + \ell) = a - c$, and thus $a - c$ is even if $a - b$ and $b - c$ are. $\boxed{J\tau}$

The Cartesian graph of this relation contains points at every (x, y) for which x and y have the same parity.

Claim (Part H). $R = \{(x, y) \in \mathbb{Z}^2 \mid 2 \mid (a + b)\}$ *is an equivalence relation*

Proof. Reflexivity is seem to hold as $a + a = 2a$ is divisible by 2.

Symmetry is seen to hold as $a + b = b + a$

Transitivity is seen to hold as if $a + b = 2k$ and $b + c = 2\ell$ then we have that $2k - 2\ell + 2c = (a + b) - (b + c) + 2c = a + c$ which means that $a + c = 2(k - \ell + c)$, and thus $a + c$ is even if $a + b$ and $b + c$ are even.

Alternatively, all three conditions must hold as $2 \mid (a + b) \iff 2 \mid (a - b)$, and thus R from this problem is equal to R from Part G. $\boxed{J\tau}$

The Cartesian graph of this relation contains points at every (x, y) for which x and y have the same parity.

Claim (Part I)**.** $R = \{(x, y) \in \mathbb{Z}^2 \mid 5 \mid (a - b)\}$ *is an equivalence relation*

Proof. Reflexivity is seem to hold as $a - a = 0$ is divisible by 5.

Symmetry is seen to hold as if x is divisible by 5, then $-x$ is as well, and $a - b = -(b - a)$.

Transitivity is seen to hold as if $a - b = 5k$ and $b - c = 5\ell$ then $5k + 5\ell = (a - b) + (b - c) = a - c$ gives us that $5(k + \ell) = a - c$, and thus $a - c$ is divisible 5 if $a - b$ and $b - c$ are. $\boxed{J\tau}$

The Cartesian graph of this relation contains every lattice point which lies on a line with integer y-intercepts and a slope of 5.

Claim (Part J)**.** $R = \{(x, y) \in \mathbb{Z}^2 \mid 5 \mid (a - b)\}$ *is not an equivalence relation*

Proof. Reflexivity is violated as $1 \nsim 1$.

Symmetry is seen to hold as $a + b = b + a$.

Transitivity is seen to not hold as $1 \sim 4$ and $4 \sim 6$ but $1 \nsim 6$. $\boxed{J\tau}$

The Cartesian graph of this relation contains every lattice point which lies on a line with integer y-intercepts and a slope of -5.

Problem 3, Chapter 16

Claim (Part A)**.** *i and iii are partitions of $\mathbb{Z}$, while ii and iv are not.*

Proof. ii does not form a partition of $\mathbb{Z}$ because 3 is not in any of the sets, and iv does not form a partition as 6 is in both $[0]_2$ and $[0]_3$.

We can see that i forms a partition as the set is equivalent to $\{[0]_4, [1]_4, [2]_4, [3]_4\}$.

iii forms a partition as $[0]_2 = [0]_8 \cup [2]_8 \cup [4]_8 \cup [6]_8$ and $[1]_4 = [1]_8 \cup [5]_8$. Thus together with $[3]_8$ and $[7]_8$ every integer is accomplished. $\boxed{J\tau}$

Claim (Part B)**.**

- $\{[1]_5, [2]_5, [3]_5, [4]_5, [5]_5\}$
- $\{[0]_3, [1]_6, [2]_6, [4]_6, [5]_6\}$
- $\{[0]_2, [1]_4, [3]_{12}, [7]_{12}, [11]_{12}\}$

Problem 5, Chapter 16

Claim (Part A). *For S with size four, there are* 65536 *relations on S.*

Proof. A relation on a set S is defined to be a subset of $S \times S$. We know that if $|S| = 4$ then $|S \times S| = 16$, and so the number of subsets of $S \times S$ is 2^{16} or 65536. $\boxed{J\tau}$

Claim (Part B). *For S with size four, there are* 4096 *reflexive relations on S.*

Proof. If M is the matrix representation of a reflexive relation on S with size four then the four main diagonal entries must all be 1's, while the other 12 entries can be either 0 or 1, of which there are $2^{12} = 4096$ ways to select. $\boxed{J\tau}$

Claim (Part C). *For S with size four, there are* 4096 *irreflexive relations on S.*

Proof. Like in Part B, we can use the matrix representation of the equivalence relation, but this time the main diagonal must all be 0's, and like before the other 12 entries can be either 0 or 1 resulting in $2^{12} = 4096$ such matrices, and the same number of irreflexive relations. $\boxed{J\tau}$

Claim (Part D). *For S with size four, there are* 1024 *symmetric relations on S.*

Proof. If an equivalence relation is symmetric, then its matrix representation is seen to be as well, and in a symmetric 4x4 matrix 6 of the 16 entries are equal to their mirrors across the diagonal, and so the number of different possible matrices is the number of ways to pick $16 - 6 = 10$ selections of either 1 or 0, $2^{10} = 1024$. $\boxed{J\tau}$

Claim (Part E). *For S with size four, there are* 729 *symmetric relations on S.*

Proof. Note first that we cannot have any such $a \sim a$ in an asymmetric relation as that would imply that $a \nsim a$, a contradiction, and so in the matrix representation of an asymmetric relation the main diagonal must be all zeroes, and as for the other 12, we can split them into symmetric pairs eg. $(a, b), (b, a)$. For each of these pairs, asymmetry restricts us to the following possibilities.

- $a \sim b, b \nsim a$
- $a \nsim b, b \sim a$
- $a \nsim b, b \nsim a$

Thus, for each of these 6 pairs, we have three options, and thus a total of $3^6 = 729$ possibilities. $\boxed{J\tau}$

Claim (Part F). *For S with size four, there are* 11664 *antisymmetric relations on S.*

Proof. First, we see that any of the relations on the diagonal can be either 1 or 0, and there are $2^4 = 16$ ways to select these four.

As for the other 12 entries, we can do the same thing as we did in the asymmetric case on the symmetric pairs to get that there are $3^6 = 729$ options for the other 12 entries. Together, there are $729 * 16 = 11664$ $\boxed{J\tau}$

Claim. *For S with size four, there are* 64 *graph relations on S.*

Proof. Recall that a graph relation is defined to be irreflexive and symmetric. We know that reflexivity forces the diagonal of the matrix representation to contain all zeroes, and symmetry forces the matrix to be symmetric, and so a symmetric relation can be defined by the upper triangle of the matrix alone. The four entries on the main diagonal are already decided and the other six can be either 1 or 0 and so there are $2^6 = 64$ possibilities $\boxed{J\tau}$

Claim. *For S with size four, there are* 15 *equivalence relations on S.*

Proof. By the fundamental theorem of equivalence relations, the number of relations on a set of size four is the same as the number of partitions on a set of four elements. In the chapter, this number was determined to be 15. $\boxed{J\tau}$

Problem 7, Chapter 16

Claim (Part A). *The relation* $R = \{((a, b), (c, d)) \in (\mathbb{N}^2)^2 \mid a + d = b + c\}$ *is an equivalence relation.*

Proof. Let a and b be natural numbers. We see that $(a, b) \sim (a, b)$ is equivalent to $a + b = a + b$, which is clearly true and so reflexivity of R is satisfied.

Now we will prove the symmetric property. Consider $a, b, c, d \in \mathbb{N}$ for which $(a, b) \sim (c, d)$. This implies that $a+d = b+c$, which is equivalent to $c+b = d+a$ which is equivalent to $(c, d) \sim (a, b)$, proving symmetry.

Now, let $a, b, c, d, e, f \in \mathbb{N}$ such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. This gives us that $a + d = b + c$ and $c + f = d + e$ doing a bit of algebra on both of these equations gives us that $a - b = c - d$ and $c - d = e - f$, and because equality is transitive we have that $a - b = e - f$ which is equivalent to $a + f = b + e$ which is the same as $(a, b) \sim (e, f)$, and thus we have that $(a, b) \sim (c, d) \wedge (c, d) \sim (e, f) \implies (a, b) \sim (e, f)$ which is transitivity.

$\boxed{J\tau}$

Claim (Part B). *The relation* $R = \{((a, b), (c, d)) \in (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})) \mid ad = bc\}$ *is an equivalence relation.*

Proof. Let a and b be integers with $b \neq 0$. We see that $(a, b) \sim (a, b)$ is equivalent to $ab = ba$, which is clearly true and so reflexivity of R is satisfied.

Now we will prove the symmetric property. Consider $a, b, c, d \in \mathbb{Z}$ with b, c non-zero for which $(a, b) \sim (c, d)$. This implies that $ad = bc$, which is equivalent to $cb = da$ which is equivalent to $(c, d) \sim (a, b)$, which proves symmetry.

Now, let $a, b, c, d, e, f \in \mathbb{Z}$ with non-zero b, d, and f such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. This gives us that $ad = bc$ and $cf = de$ doing a bit of algebra on both of these equations gives us that $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$, and because equality is transitive we have that $\frac{a}{b} = \frac{e}{f}$ which is equivalent to $af = be$ which is the same as $(a, b) \sim (e, f)$, and thus we have that $(a, b) \sim (c, d) \wedge (c, d) \sim (e, f) \implies (a, b) \sim (e, f)$ which is transitivity.

$\boxed{J\tau}$

Arguement. The equivalence class $[(3,1)]_{R_A}$ is the set of (a,b) that satisfy $3+b=1+a$, or equivalently $a-b=2$ Thus, the members of $[(3,1)]_{R_A}$ take the form $(b+2,b)$. Similarly, the members of $[(2,2)]_{R_A}$ all take the form (b,b), and the members of $[(1,3)]_{R_A}$ take the form $(b-2,b)$.

Now, consider the function the function $f : R[A] \longrightarrow \mathbb{Z}$ defined by $f([(a,b)]_{R_A}) = a-b$ where $R[A]$ is the set of equivalence classes formed by the relation R in Part A. We can see that this is indeed a well-defined function as for all $(a,b),(c,d) \in [(a,b)]_{R_A}$ we have that $a+d=c+b$ which gives us that $a-b=c-d$ and thus regardless of the representative selected from $[(a,b)]_{R_A}$ f will output the same result.

To prove this is a bijection, I must prove both injectivity and surjectivity hold.

First, I prove that injectivity holds by considering two equivalence classes $[(a,b)]_{R_A}$ and $[(c,d)]_{R_A}$ such that $f([(a,b)]_{R_A}) = f([(c,d)]_{R_A})$. Which is equivalent to $a-b=c-d$ or equivalently $a+d=b+c$, which gives us that $(a,b) \sim (c,d)$ which is true if and only if $[(a,b)]_{R_A} = [(c,d)]_{R_A}$. And so, f is injective.

Now, to prove that f is surjective. We can see that for $a \in \mathbb{N}$ if $z \geq 0$ that $[(a+z,a)]_{R_A}$ is a valid equivalence class and $f([(a+z,a)]_{R_A}) = z$ and if $z \leq 0$ then $(a-z,a)]_{R_A}$ is a valid equivalence class and $f([(a-z,a)]_{R_A}) = -z$, and so every element of the codomain gets mapped to by f, and thus f is surjective.

Because f is both injective and surjective it is bijective and we are done. $\boxed{J\tau}$

Argument. The equivalence class $[(3,1)]_{R_B}$ is the set of (a,b) that satisfy $3b=1a$. Thus, the members of $[(3,1)]_{R_B}$ take the form $(3b,b)$. Similarly, the members of $[(2,2)]_{R_B}$ all take the form (b,b), and the members of $[(1,3)]_{R_B}$ take the form $(b,3b)$.

Now, consider the function the function $f : R[B] \longrightarrow \mathbb{Q}$ defined by $f([(a,b)]_{R_A}) = \frac{a}{b}$ where $R[B]$ is the set of equivalence classes formed by the relation R in Part B

We can see that this is indeed a well-defined function as for all $(a,b),(c,d) \in [(a,b)]_{R_B}$ as we have that $ad=cb$ which because $b,d \neq 0$ gives us that $\frac{a}{b} = \frac{c}{d}$ and thus regardless of the representative selected from $[(a,b)]_{R_B}$, f will output the same result.

To prove this is a bijection, I must prove both injectivity and surjectivity hold.

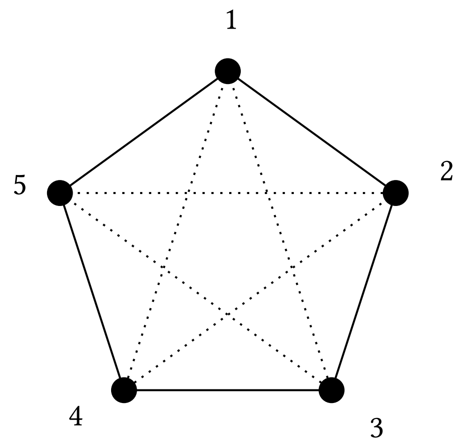
First, I prove that injectivity holds by considering two equivalence classes $[(a,b)]_{R_B}$ and $[(c,d)]_{R_B}$ such that $f([(a,b)]_{R_B}) = f([(c,d)]_{R_B})$. This gives us that $\frac{a}{b} = \frac{c}{d}$ or equivalently $ad=bc$, which means that $(a,b) \sim (c,d)$ and thus $[(a,b)]_{R_B} = [(c,d)]_{R_B}$. And so, f is injective.

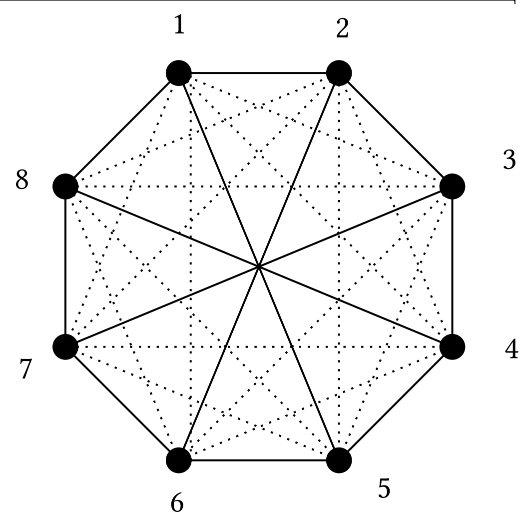
Now, to prove that f is surjective. This can be easily done as we simply need to prove that for an arbitrary $q \in \mathbb{Q}$ we have that q is in the image of f. This is done by considering q's fractional representation of $q = \frac{a}{b}$ where $a,b \in \mathbb{Z}$ and $b \neq 0$, and because a,b have these properties we have that $[(a, b)]_{R_B}$ exists, and it can be seen that $f([(a, b)]_{R_B}) = \frac{a}{b}$, which proves surjectivity.

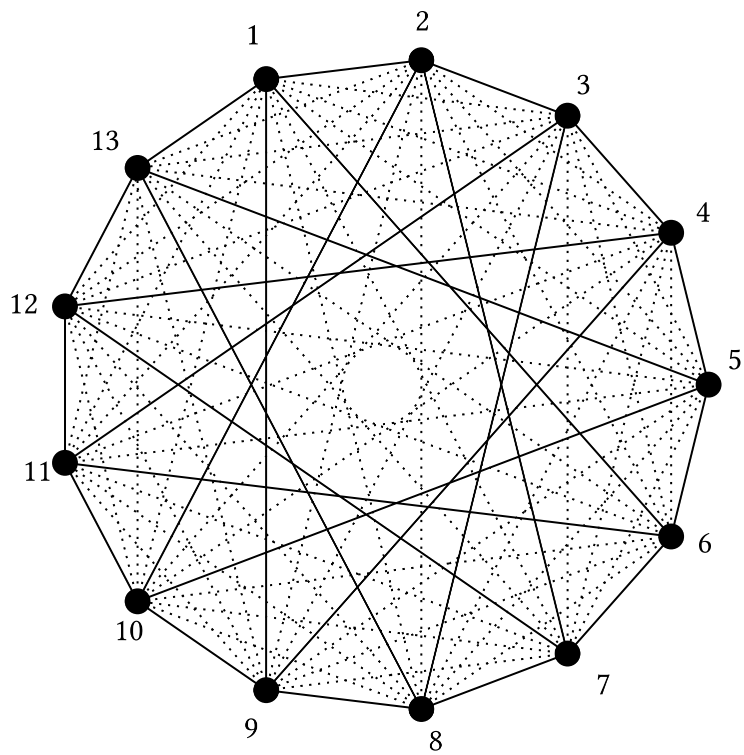
Because f is both injective and surjective it is bijective and we are done.

$\boxed{\boldsymbol{J\tau}}$

Problem 9, Chapter 16







All of these graphs produce a situation where no three people, know each other, while no 3, 4, or 5 people don't know each other in groups of 5, 8, and 13 people respectively. Thus, they give lower bounds of 6, 9, and 14 on $R(3,3)$, $R(4,3)$, and $R(5,3)$.