MATH 315 WEEK 2

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Problem 3, Chapter 14

Claim. *For any field $\mathbb{F}$, e_* and $e_\diamond$ are not equal*

Proof. In any field, e_* and $e_\diamond$ necessarily exist in $\mathbb{F}$. Furthermore, by the requirement that a field has at least two distinct elements there is some $a \in \mathbb{F}$ such that $a \neq e_*$. By Theorem 14.5 we have that $a \diamond e_* = e_*$, but this means that e_* is not $e_\diamond$ as if it were then we would have that $a \diamond e_\diamond = a$, and so $e_* \neq e_\diamond$. $\boxed{J\tau}$

Claim. For the real numbers 1 and 0 we have $0 = 1$.
According to part (a), this claim is false. Find the mistake(s) in the following argument.
Argument. Let a be an arbitrary real number. By axiom $(+5)$, we must have a real number x such that $a + x = 0$. Multiplying this equation by a yields $(a + x) \cdot a = 0 \cdot a$, or, after using distributivity, $a^2 + x \cdot a = 0 \cdot a$ (denoting $a \cdot a$ by a^2). Axiom $(\cdot 5)$ guarantees an inverse to $a^2 + x \cdot a$; that is, there exists a real number y for which $(a^2 + x \cdot a) \cdot y = 1$. Using this y to multiply our equation $a^2 + x \cdot a = 0 \cdot a$, we get $(a^2 + x \cdot a) \cdot y = (0 \cdot a) \cdot y$. Now the left-hand side equals 1, so we have $1 = (0 \cdot a) \cdot y$. According to Corollary 14.7, 0 times any real number is 0, so $(0 \cdot a) \cdot y = 0 \cdot y = 0$. Therefore, we proved that $0 = 1$.

Axiom $(\cdot 5)$ only guarantees inverse to non-zero values, and as we see $a^2 + x \cdot a = 0$.

Problem 4, Chapter 14

(a) Find the mistake(s) in the following argument.
Claim. We have $a * e_\diamond = e_\diamond$.
Argument. At each step we refer to a particular property under Definition 13.1.

$$\begin{aligned} a * e_\diamond &\overset{(\diamond 4)}{=} (a * e_\diamond) \diamond e_\diamond \\ &\overset{(\diamond 5')}{=} (a * e_\diamond) \diamond [(a * e_\diamond) \diamond -(a * e_\diamond)] \end{aligned}$$

$$\begin{aligned} &\overset{(\diamond 3)}{=} [(a * e_\diamond) \diamond (a * e_\diamond)] \diamond -(a * e_\diamond) \\ &\overset{(*D\diamond)}{=} [a * (e_\diamond \diamond e_\diamond)] \diamond -(a * e_\diamond) \\ &\overset{(\diamond 4)}{=} (a * e_\diamond) \diamond -(a * e_\diamond) \\ &\overset{(\diamond 5')}{=} e_\diamond \end{aligned}$$

Distributive Property was done backward, it is $\diamond$ that distributes over $*$, not vice versa.

Claim. *For a field $\mathbb{F}$ with $a \in \mathbb{F}$ if $a * e_\diamond = e_\diamond$ then $a = e_*$.*

Proof. Let us assume that $a * e_\diamond = e_\diamond$. We can do the following.

$$a * e_\diamond = e_\diamond$$

$$a * e_\diamond * (-e_\diamond) = e_\diamond * (-e_\diamond)$$

$$a * (e_\diamond * (-e_\diamond)) = e_*$$

$$a = e_*$$

$\boxed{J\tau}$

Problem 7, Chapter 14

Claim.

$$(A \cup \overline{B} \cup C) \cap \overline{(A \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}.$$

Proof. We do the following

$$(A \cup \overline{B} \cup C) \cap \overline{(A \cup C)}$$

Commutativity:

$$= ((A \cup C) \cup \overline{B}) \cap \overline{(A \cup C)}$$

Distributive:

$$= ((A \cup C) \cap \overline{(A \cup C)}) \cup (\overline{B} \cap \overline{(A \cup C)})$$

Complementation:

$$= \emptyset \cup (\overline{B} \cap \overline{(A \cup C)})$$

Identity:

$$= (\overline{B} \cap \overline{(A \cup C)})$$

DeMorgan's Laws (Problem 6):

$$= \overline{B} \cap (\overline{A} \cap \overline{C})$$

Commutative & Associative:

$$= \overline{A} \cap \overline{B} \cap \overline{C}.$$

$\boxed{J\tau}$

Problem 10, Chapter 14

Claim (Part A). *For all real a, we have that $-(-a) = a$.*

Proof. Definition 13.4 states that the negative of a number, a, is the number, $-a$, such that $a + (-a) = 0$. So, the negative of $-a$ is the number x such that $(-a) + x = 0$, however by the commutative property and the previous equation this gives us that $x = a$, and so $-(-a) = a$. $\boxed{J\tau}$

Lemma 1. *For any non-zero real number a and any real number c there exists a real number x such that $a \cdot x = c$, namely $x = c\dot{b}$ where b is the multiplicative inverse of a.*

Proof. Let a be a non-zero real number. Now, because $a \neq 0$ there exists some real number b such that $a \cdot b = 1$. Thus, for any real number c we have that $a \cdot b \cdot c = 1 \cdot c$ which by identity gives us that $a \cdot (b \cdot c) = c$. Furthermore, by closure, $b \cdot c$ is real and our claim is proven $\boxed{J\tau}$

Claim (Part B). *For all real a, we have that $(-1) \cdot a = -a$.*

Proof. Let a be a real number. If $a = 0$ we can see that $-a = 0$, as $0 + 0 = 0$. Furthermore, $(-1) \cdot 0 = 0$ by Theorem 14.5, and so we have that $(-1) \cdot 0 = 0 = -0$.

We know there is some $-a$ such that $-a + a = 0$. By Distributivity, Identity, and Lemma 1 we have that there exists a real number b such that $b \cdot a = -a$ and

$$a(b + 1) = 0.$$

However, by the non-zero product property and the fact that a is non-zero we can say that $b + 1 = 0$ which by the definition of negative gives us that $b = -1$, and by how b was defined we have that $a \cdot -1 = -a$.

$\boxed{J\tau}$

Claim (Part C). *For all real a and b, we have that $(-a) \cdot b = -(a \cdot b)$.*

Proof. Starting with the expression $(-a) \cdot b$, we can use Part B and the associative property to get that

$$(-a) \cdot b = (-1) \cdot a \cdot b = -(a \cdot b).$$

$\boxed{J\tau}$

Claim (Part D). *For all real a and b, we have that $(-a) \cdot (-b) = a \cdot b$.*

Proof. Starting with the expression $(-a) \cdot (-b)$, we can use Part B, the associative property, the commutative property, the identity as well as the fact that $-1 \cdot -1 = -(-1) = 1$ to get that

$$(-a) \cdot (-b) = (-1) \cdot a \cdot (-1) \cdot b = (-1 \cdot -1)(a \cdot b) = 1 \cdot (a \cdot b) = (a \cdot b).$$

$\boxed{J\tau}$

Lemma 2. *For non-zero real x and y we have that $\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{x \cdot y}$.*

Proof. Consider the quantity $\frac{1}{x \cdot y}$ for x and y described in the claim. We have that

$$\frac{1}{x \cdot y} \cdot (x \cdot y) = 1.$$

Now, we can do the following

$$\frac{1}{x \cdot y} \cdot (x \cdot y) \cdot \frac{1}{x} \cdot \frac{1}{y} = 1 \cdot \frac{1}{x} \cdot \frac{1}{y}$$

which by commutativity, associativity, and identity we get that

$$\frac{1}{x \cdot y} \cdot \left(x \cdot \left(y \cdot \frac{1}{y}\right) \cdot \frac{1}{x}\right) = \frac{1}{x} \cdot \frac{1}{y}$$

which then becomes

$$\frac{1}{x \cdot y} = \frac{1}{x} \cdot \frac{1}{y}$$

by identity and the definition of the reciprocal of non-zero reals, proving our claim. $\boxed{J\tau}$

Claim (Part E)**.** *For real a, b, c, and d with non-zero b and d we have that*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

Proof. First, see that

$$\frac{a}{b} \cdot \frac{c}{d} = a \cdot \frac{1}{b} \cdot c \cdot \frac{1}{d},$$

and by the commutative property we have that this can become

$$(a \cdot c) \cdot \left(\frac{1}{b} \cdot \frac{1}{d}\right),$$

and by Lemma 2 we have that this is equal to

$$(a \cdot b) \cdot \left(\frac{1}{c \cdot d}\right)$$

which by the definition of the quotient is equal to $\frac{a \cdot c}{b \cdot d}$, proving our claim. $\boxed{J\tau}$

Lemma 3. *For any real a, b, and c where $c \neq 0$ we have that $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$.*

Proof. By the distributive property and the definition of a quotient we have that

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} = \frac{1}{c}(a+b) = \frac{a+b}{c}.$$

$\boxed{J\tau}$

Claim (Part F)**.** *For real a, b, c, and d with non-zero b and d we have that*

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}.$$

Proof. We start with the quantity $\frac{a}{b} + \frac{c}{d}$. It is clear that $\frac{b}{b} = \frac{d}{d} = 1$ and thus we have that

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b} = \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d}.$$

Finally, by Lemma 3 we have that

$$\frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{a \cdot d + b \cdot c}{b \cdot d},$$

and with this our claim is proven. $\boxed{J\tau}$

Problem 11, Chapter 14

Problem 9a, Chapter 13

Definition 1. The negative elements of a ring, R, with positive elements P is defined to be $N = \{-x \mid x \in P\}$.

Alternatively, $N = R \setminus (P \cup \{0\})$.

Problem 9b, Chapter 13

Claim. *For a pair of elements a and b in an ordered ring R with positive set P and negative set N we define the following order relations.*

($\geq$) $a \geq b$ *if* $a - b \in P \cup \{0\}$.
($\leq$) $a \leq b$ *if* $a - b \in N \cup \{0\}$.
($>$) $a > b$ *if* $a - b \in P$.
($<$) $a < b$ *if* $a - b \in N$

Claim (Part A)**.** *For every real a and b we have that exactly one of $a = b$, $a < b$, and $b > a$ holds.*

Proof. First see that is claim is equivalent to P, N, and $\{0\}$ are pairwise disjoint but their union equals R as $a = b$ is equivalent to $a - b = 0$. This easily is seen to follow from the definitions as $N = R \setminus \{P \cup \{0\}\}$, and P does not contain 0. $\boxed{J\tau}$

Claim (Part B)**.** *For every real a, b, and c for which $a > b$ and $b > c$ we have that $a > c$.*

Proof. We have that $a - b \in P$ and $b - c \in P$, and by $O+$ this means that $(a - b) + (b - c) = a - c \in P$, proving our claim as this is equivalent to $a > c$ by our definition. $\boxed{J\tau}$

Claim (Part C)**.** *For every real a, b for which $a > b$ we have that $a + c > b + c$ for all real c.*

Proof. We have that $a - b \in P$. Note that $(a + c) - (b + c)$, and thus $(a + c) - (b + c) \in P$, which by our definition from Problem 13.9b gives us that $a + c > b + c$, our claim. $\boxed{J\tau}$

Claim (Part D)**.** *For every real a, b for which $a > b$ we have that $ac > bc$ for all real $c > 0$.*

Proof. We have that $a - b \in P$. If $c > 0$, that gives us that $c \in P$, and by axiom $(O\cdot)$ this means that $c(a - b) = ca - cb \in P$ which then gives us that $ac > bc$. $\boxed{J\tau}$

Claim (Part E)**.** *For every real a, b for which $a > b$ we have that $ac < bc$ for all real $c < 0$.*

Proof. By Part D, we have that $-ca > -cb$, and thus $-ca-(-cb) = -c(a-b) \in P$. However, by axiom O, this means that $c(a - b) \notin P$, and from the fact that $a \neq b$ and $c \neq 0$ we know that $c(a-b) \neq 0$ and thus by Part A we must have that $c(a-b) \in N$ and thus $ac < bc$ by definition. $\boxed{J\tau}$

Claim (Part F). *For every non-zero real a we have that $a^2 > 0$.*

Proof. The case for positive a follows easily from axiom $O\cdot$. If $a < 0$ then part D implies that $a \cdot a > 0 \cdot a$ which is equivalent to $a^2 > 0$, proving our claim as every non-zero a is either positive or negative. $\boxed{J\tau}$

Claim (Part G). *For positive real a and b with $a > b$ we have that $a^2 > b^2$.*

Proof. From Part D we have that $a^2 > ab$ and $ab > b^2$, which from Part B gives us $a^2 > b^2$, proving our claim. $\boxed{J\tau}$

Lemma 4. *Let z be a positive real number. For real x and y that are solutions to $xy = z$ we have that either $x, y \in P$ or $x, y \in N$.*

Proof. First, note that neither of x or y are zero by the non-zero product property and so we need only show that we cannot have exactly one of x and y in P, and the other in N.

Let us assume, without loss of generality that $x \in P$ and $y \in N$. If this is this case we have that $x > 0$ and $y < 0$, which by part E would give us that $x \cdot y < 0 \cdot y = 0$, a contradiction as $x \cdot y = z$, which is positive and thus greater than 0. This contradicts the assumption that we need to prove our claim, and so it is proven.

$\boxed{J\tau}$

Lemma 5. *For every pair of real numbers, a and b, $a^2 + ab + b^2$ must be positive with the sole exception of $a = b = 0$, in which case the expression equals zero.*

Proof. First, we examine when $a^2 + ab + b^2 = 0$. Note that if one of a or b are zero, then the other must be zero, this can be seen as if $a = 0$ then the equation becomes $b^2 = 0$ which implies that $b = 0$ by the non-zero product property.

From now on, we will assume that $a, b \neq 0$ in attempt to determine the existence of any remaining cases. The original equation can be rearranged to $a^2 = -(b^2 + ab)$ which would then imply that $-(b^2 + ab)$ is positive as a is non-zero, making a^2 positive.

With this, we have that $b^2 + ab$ is negative, which axiom $O+$ means that ab must be negative, but this would give that $(a^2 + 2ab + b^2)$ is negative, which is equivalent to $(a+b)^2 < 0$.

However, this cannot be as if $a + b \neq 0$ then $(a+b)^2 > 0$ by Part F and if $a + b = 0$ then $(a+b)^2 = 0$ by the zero-product property. Either way, $(a+b)^2 < 0$ is not satisfied and we have reached a contradiction to our assumption that there are non-zero solutions to $a^2 + ab + b^2 = 0$.

Let us assume the contrary, that $(a^2+ab+b^2) \in N$. This would mean that $a^2+ab+b^2 < 0$, however note that a^2 and b^2 are positive and so by Part F and $O+$ we must have that $ab < 0$. However, this would then imply that $a^2 + 2ab + b^2 < a^2 + ab + b^2$ by Part C, and so we have that $a^2 + 2ab + b^2 < 0$ by Part A, but this gives us that $(a + b)^2 < 0$ which results in a contradiction identically to the above. $\boxed{J\tau}$

Claim (Part H). *For real a and b we have that $a > b$ if and only if $a^3 > b^3$.*

Proof. First, see that for either of $a > b$ or $a^3 > b^3$ to be true true, we cannot have $a = b$, so for this proof assume that $a \neq b$

We know that $a^3 > b^3$ if and only if $a^3 - b^3 \in P$, however this is equivalent to $(a - b)(a^2 + ab + b^2) \in P$. By Lemma 4 this is equivalent to one of $(a - b), (a^2 + ab + b^2) \in P$ or $(a-b), (a^2+ab+b^2) \in N$ being true. By Lemma 5 we must have that a^2+ab+b^2 is positive as it can only be non-positive if $a = b = 0$, and $a \neq b$, so we have that $(a-b), (a^2+ab+b^2) \in N$ is never true, and $(a^2 + ab + b^2) \in P$ is always true meaning $(a - b)(a^2 + ab + b^2) \in P$ is equivalent to $a - b \in P$, and our claim is proven. $\boxed{J\tau}$