MATH 315 CHAPTER 17

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Problem 9, Chapter 17

Claim. *The set of non-negative integers with partial order relation divisibility forms a complete lattice.*

Proof. By Theorem 17.13(b) it suffices to show that every subset of $P = \mathbb{N}_0$ has a supremum, that is: for every $S \subseteq P$ we have that $\sup S$ exists. First, consider when $S = \emptyset$. In this case, we have that $\sup \emptyset = \min P = 1$. Now, assume that S is non-empty. If S is finite and does not contain 0, then it follows that S has a supremum by Theorem 17.8, and induction.

If S does contain 0 then it follows that 0 is the supremum of S as the only number that 0 divides is zero, and every positive integer also divides 0.

If S is infinitely large then there is clearly no positive integer that is divisible by every element of S. However, because every non-negative integer divides 0, it follows that $0 = \sup S$. $\boxed{J\tau}$

Problem 10, Chapter 17

Claim (Part A). *Let $a, b, c \in \mathbb{R}$. If $a \leq b$, then $\min(a, c) \leq \min(b, c)$.*

Claim (Part B). *For a set S consider $A, B, C \in P(S)$. If $A \subseteq B$, then $A \cap C \subseteq B \cap C$.*

Claim (Part C). *Let $a, b, c \in \mathbb{N}$. If $a \mid b$, then $\gcd(a, c) \mid \gcd(b, c)$.*

Claim (Part D). *Consider a lattice P with partial order $\preceq$. Let $a, b, c \in P$. If $a \preceq b$, then $\inf(a, c) \preceq \inf(b, c)$.*

Proof. Since P is a lattice we have that $\inf(a, c) = i_1$ and $\inf(b, c) = i_2$ exist. Thus $i_1 \preceq a$, and so $i_1 \preceq b$ from the transitive property. And, because $i_1 \preceq c$, we have that $i_1 \in \{b, c\}^{\downarrow}$, but because $i_2 = \inf(b, c)$ we have that $i_1 \preceq i_2$, proving our claim.

$\boxed{J\tau}$

Claim (Part E). *Consider a lattice P with partial order $\preceq$. Let $a, b, c \in P$. If* $\inf(a, c) \preceq \inf(b, c)$ *then we don't necessarily have that $a \preceq b$.*

Proof. Consider the lattice $(\mathbb{N}, |)$ with $a = 2$, $b = 3$, and $c = 5$. We have that $\gcd(2, 5) \mid \gcd(3, 5)$ yet $2 \nmid 3$. $\boxed{J\tau}$

Problem 11, Chapter 17

Theorem 0.1 (17.15). *Suppose L is a lattice with partial order $\preceq$. For any $a, b, c \in L$ we have that*

$$\sup(\inf(a, b), \inf(a, c)) \preceq \inf(a, \sup(b, c))$$

Claim (Part A i). *For a set S and any $A, B, C \in P(S)$ we have that*

$$(A \cap B) \cup (A \cap C) \subseteq (A \cap (B \cup C)).$$

Claim (Part A ii). *The errors are listed below.*

Proof.

ii. What is wrong with the following "proof"? Wlog we can assume that $b \preceq c$; so $\sup\{b, c\} = c$. By Proposition 17.14, we also have $\inf\{a, b\} \preceq \inf\{a, c\}$, and thus $\sup\{\inf\{a, b\}, \inf\{a, c\}\} = \inf\{a, c\}$. Therefore, the claim of Theorem 17.15 simplifies to

$$\inf\{a, c\} \preceq \inf\{a, c\},$$

which obviously holds.

FIGURE 1. "Proof" of Theorem 17.15

The red highlight shows us the first error. We cannot assume that a relation holds between b and c.

The second error is highlighted in yellow. It does not prove that the theorem is true, it assumes the theorem and shows that a true statement follows. $\boxed{J\tau}$

Claim (Part A iii). *Theorem 17.15 Holds.*

Proof. Because $\inf(a,b)$ is a lower bound of a we have that $\inf(a,b) \leq a$. Similarly, $\inf(a,c) \leq a$.

Furthermore, we have that $\inf(a,b) \leq b$ and $b \leq \sup(b,c)$, and so by the transitive property we have that $\inf(a,b) \leq \sup(b,c)$. By a similar logic, we can arrive at the fact that $\inf(a,c) \leq \sup(b,c)$.

Because L is a lattice, we know that $\{a, \sup(b,c)\}$ has an infimum, let us call it i. Now, by the above, we have that $\inf(a,b)$ and $\inf(b,c)$ are lower bounds of the set $\{a, \sup(b,c)\}$, and so by the definition of i we have that $\inf(a,b) \leq i$ and $\inf(a,c) \leq i$. However, this gives us that i is an upper bound of the set $\{\inf(a,b), \inf(a,c)\}$ and thus, because the supremum of $\{\inf(a,b), \inf(a,c)\}$ exists we have that $\sup\{\inf(a,b), \inf(a,c)\} \leq i$. And because $i = \inf\{a, \sup(b,c)\}$ our claim is proven. $\boxed{J\tau}$

Remark. If I write that $x \leq y \leq z$ then that means that $(x \leq y)$ and $(y \leq z)$.

Claim (Part B i). $(\mathbb{R}, \leq)$ *is a distributive lattice.*

Proof. We know already that $(\mathbb{R}, \leq)$ is a lattice. The distributivity condition for $\mathbb{R}$ is equivalent to $\max(\min(a,b), \min(a,c)) = \min(a, \max(b,c))$, and so it suffices to prove this statement.

Because $\mathbb{R}$ is totally ordered, we can show that this holds in each of the six different possible orderings of these three elements. However, we can cut this number in half by assuming that $b \leq c$ without loss of generality.

- If $a \leq b \leq c$ then both sides of the equation equal a.
- If $b \leq a \leq c$ then both sides of the equation will equal a.
- If $b \leq c \leq a$ then both sides of the equation will equal c.

$\boxed{J\tau}$

Claim (Part B ii). $(P(U), \subseteq)$ *is a distributive lattice.*

Proof. The claim is equivalent to

$$(A \cap B) \cup (A \cap C) = (A \cap (B \cup C)).$$

which we know to be true as the power set of U forms a boolean algebra with $\cap$ and $\cup$ and the above condition is equivalent to distributive property in boolean algebras. $\boxed{J\tau}$

Claim (Part B iii). $(\mathbb{N}, |)$ *is a distributive lattice.*

Proof. Consider three natural numbers a, b, c. By the fundamental theorem of arithmetic, they all have unique prime factorizations. Thus, if we let P_a, P_b, and P_c be the set of prime numbers that divide a, b, and c respectively we have that

$$a = \prod_{p \in P} p^{\alpha_p}$$

where α_p is the number of times p appears in the prime factorization of a. Similarly define β_p for b, and γ_p for c. Let $P = P_a \cup P_b \cup P_c$. From this and Theorem 17.8 we have that

$$\operatorname{lcm}(\gcd(a, b), \gcd(a, c)) = \prod_{p \in P} p^{\max(\min(\alpha_p, \beta_p), \min(\alpha_p, \gamma_p))},$$

as well as

$$\gcd(a, \operatorname{lcm}(b, c)) = \prod_{p \in P} p^{\min(\alpha_p, \max(\beta_p, \gamma_p))},$$

and because the natural numbers are a subset of $\mathbb{R}$, Part B i gives us that

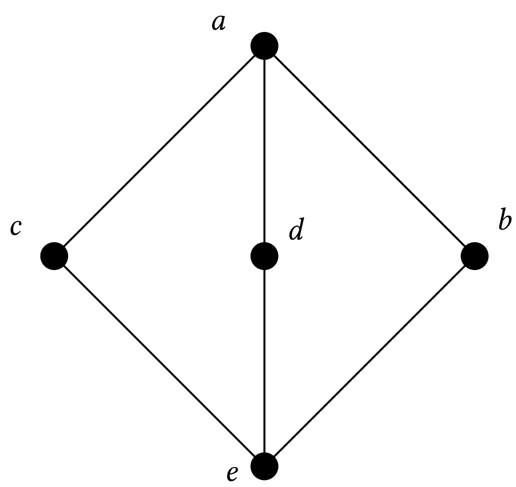
$$\min(\alpha_p, \max(\beta_p, \gamma_p)) = \max(\min(\alpha_p, \beta_p), \min(\alpha_p, \gamma_p)),$$

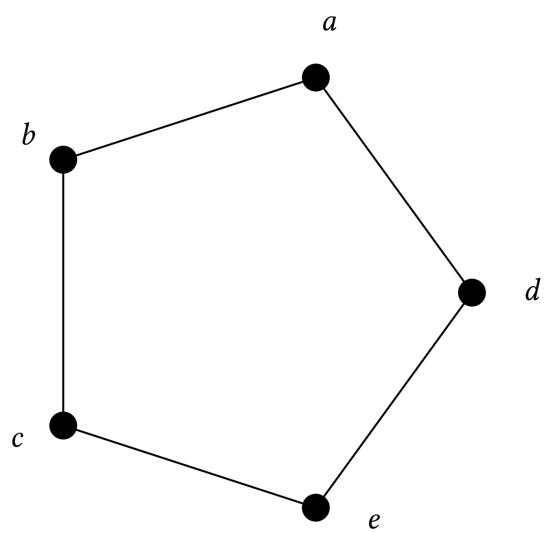
and so we have that

$$\gcd(a, \operatorname{lcm}(b, c)) = \operatorname{lcm}(\gcd(a, b), \gcd(a, c))$$

which is equivalent to our claim by Theorem 17.8. $\boxed{J\tau}$

Claim. *M_3 and N_5 are not distributive.*





Proof. M_3 can be seen to be non-distributive as $\sup(\inf(b,c), \inf(b,d)) = \sup(e,e) = e$ and $\inf(b, \sup(c,d)) = \inf(b,a) = b$.

N_5 can be seen to be non-distributive as $\sup(\inf(b,c), \inf(b,d)) = \sup(c,e) = c$ and $\inf(b, \sup(c,d)) = \inf(b,a) = b$. $\boxed{J\tau}$