MATH 315 CHAPTER 17

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Problem 1, Chapter 17

Claim (Part A)**.** *Strict order relations being irreflexive, asymmetric, and transitive follow from the definition of partial order relations.*

Proof. Consider some partial order relation $\preceq$ on a set P. Let $\prec$ be the relation defined to be all $(a, b) \in P^2$ for which $a \preceq b$ and $a \neq b$.

It easily follows from the definition that $\prec$ is irreflexive.

Now, observe that since $\preceq$ is antisymmetric, and so for any $(a, b) \in P^2$ we have that $a \neq b \wedge a \preceq b \implies b \npreceq a$ However, by the definition of $\prec$ we have that

$$a \neq b \wedge a \preceq b \iff a \prec b,$$

and this gives rise to the implication chain:

$$a \prec b \iff (a \neq b \wedge a \preceq b) \implies b \npreceq a \implies b \nprec a.$$

This now gives us that $\prec$ is indeed asymmetric.

Now, to prove transitivity we first consider some $a, b, c \in P$ such that $a \prec b$ and $b \prec c$. Both of these imply that $a \preceq b$ and $b \preceq c$ which by the transitivity of $\preceq$ gives us that $a \preceq c$. Now, I will show that $a \neq c$. In fact, if we assume the contrary then we get that that $a \preceq b$ and $b \preceq a$ which by $\preceq$'s antisymmetry means that $a = b$, but this cannot be as $a \prec b$, and so we cannot have that $a = c$. Thus, $a \neq c$ and $a \preceq c$ giving us that $a \prec c$. With this, transitivity is proven. $\boxed{J\tau}$

Claim (Part B)**.** *Partial order relations being reflexive, anti-symmetric, and transitive follow from the definition of strict order relations (as well as the fact that* $\preceq$ *is the union of* $=$ *and* $\prec$*)*

Proof. Consider some strict order relation $\prec$ on a set P.

Because $\preceq$ is the union of $=$ and $\prec$ we have that

$$a \preceq b \iff (a \prec b) \lor (a = b),$$

Thus, we have that

$$(a \preceq b) \land (b \preceq a) \iff [(a \prec b) \lor (a = b)] \land [(b \prec a) \lor (a = b)],$$

and because $\prec$ is asymmetric we can never have both $a \prec b$ and $b \prec a$, and from thus it follows that the right side of our logical equivalence can be true only when $a = b$ and so we have that

$$(a \preceq b) \land (b \preceq a) \iff a = b.$$

This simultaneously gives us that $(a \preceq b) \land (b \preceq a) \implies a = b$ and $(a \preceq b) \land (b \preceq a) \impliedby a = b$ which are antisymmetry and reflexivity respectively.

Now, to show that $\preceq$ is transitive. Consider some $a, b, c \in P$ for which $a \preceq b$ and $b \preceq c$. If all of a, b, c are different or the same then the transitivity of $\prec$ follows from the transitivity of $\prec$ or $=$ respectively.

So, we only need now to consider the case where two of a, b, and c are equal.

If $a = c$ then reflexivity gives us that $a \preceq c$, and thus $\preceq$ is transitive.

If $a = b$ then we can substitute b for a in $b \preceq c$ to get that $a \preceq c$, and if $b = c$ then we can substitute b for c on $a \preceq b$ to get that $a \preceq c$.

With this, transitivity is proven.

$\boxed{J\tau}$

Problem 4, Chapter 17

Let P be a poset with partial order relation $\leq$. Let S be some subset of P.

Claim (Part A)**.** *If the minimum element of S exists, then it must be unique.*

Proof. Assume that there are multiple minimum elements of S. Let a and b be two such minimum elements. By the definition of minimum, we have that $a \leq b$ and $b \leq a$, and by antisymmetry, this implies that $a = b$, and from this we have that there can be no two distinct minimum elements of S. $\boxed{J\tau}$

Claim (Part B)**.** *If S has more than one minimal element then S has no minimum element.*

Proof. Assume our claim is false, that S has its minimum element a (which is unique by Part A), as well as a non-minimum minimal element b. By the definition of minimum, we have that $a \leq b$, but because b is a minimal element this implies that $a = b$ which cannot be as a is a minimum, and b is not, and thus we have a contradiction. $\boxed{J\tau}$

Claim (Part C)**.** *If S has exactly one minimal element in it, then it is not necessarily the minimum.*

Proof. Consider the poset $P = \mathbb{Z} \cup \{\emptyset\}$ with the partial order relation $\leq$ on P defined as

$$R = \{(a, b) \in P^2 \mid (a, b \in \mathbb{Z} \land a < b) \lor a = b\}.$$

In this relation, $\emptyset$ is the only minimal element. However, there is no minimum element of P. $\boxed{J\tau}$

Claim (Part D)**.** *If S has an infimum then it must be unique.*

Proof. The infimum of S can clearly be seen to be the maximum of the set of lower bounds of S, and as the set of lower bounds of S is a subset of P if it has a maximum, it is unique. (This can be proven nearly identically to how we proved that if a set has a minimum it must be unique). $\boxed{J\tau}$

Claim (Part E)**.** *If S has multiple lower bounds then it can still have an infimum*

Proof. For $P = \mathbb{Z}$ with partial order relation, $\leq$, the set $S = \{1, 2, 3\}$ has infinitely many lower bounds, but 1 is clearly the infimum. $\boxed{J\tau}$

Claim (Part F)**.** *If S has one lower bound it is the infimum.*

Proof. If S has one lower bound then $S^{\downarrow}$ is a singleton and by reflexivity, it follows that its maximum exists and that it is the sole element contained within it. Our claim follows from the fact that the infimum of S is the maximum of $S^{\downarrow}$. $\boxed{J\tau}$

Claim (Part G)**.** *If S has a minimum element, then this element is the infimum of S in P.*

Proof. Let a be the minimum of S. For all $b \in S^{\downarrow}$, we have by the definition of lower bound that $b \preceq a$ as $a \in S$. Furthermore, because $a \in S^{\downarrow}$ (follows easily from the definitions) we can see that a is the maximum of $S^{\downarrow}$, and thus the infimum of S. $\boxed{J\tau}$

Claim (Part H)**.** *If S has an infimum, then it is not necessarily the minimum of S.*

Proof. Consider $P = \mathbb{R}$ with relation $\leq$, and $S = \{a \in \mathbb{R} \mid 0 < a < 1\}$. The infimum of S is 0, but it is not the minimum of S, as it is not even contained in the set. $\boxed{J\tau}$

Claim (Part I)**.** *If S has an infimum, then it is the maximum of $S^{\downarrow}$.*

Proof. The infimum of S, a, is defined to be an element of $S^{\downarrow}$ such that $c \preceq a$ for $c \in S^{\downarrow}$. But this is simply the definition of maximum. $\boxed{J\tau}$

Claim (Part J)**.** *If S has a minimum, then it is the supremum of $S^{\downarrow}$.*

Proof. The minimum of S, a, is certainly an upper bound of $S^{\downarrow}$, and furthermore, it is also the maximum of $S^{\downarrow}$ by Part G and Part I, and from this, it is the supremum of $S^{\downarrow}$ (This follows from the fact that if the maximum of a set exists then it is the supremum, proved in a similar way to Part G).

$\boxed{J\tau}$

Problem 6, Chapter 17

Claim (Part A). *Figure 17.4 is a complete lattice.*

Proof. Every subset of points has an infimum and a supremum.

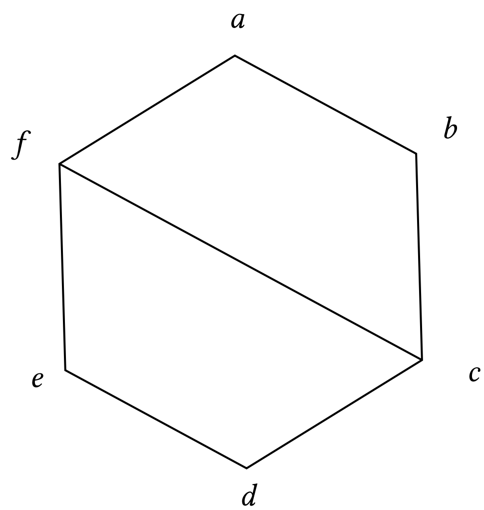

FIGURE 1. Figure 17.4

$\boxed{J\tau}$

Claim (Part B). *Figure 17.5 is not a lattice.*

Proof. There is no supremum for $\{e, c\}$.

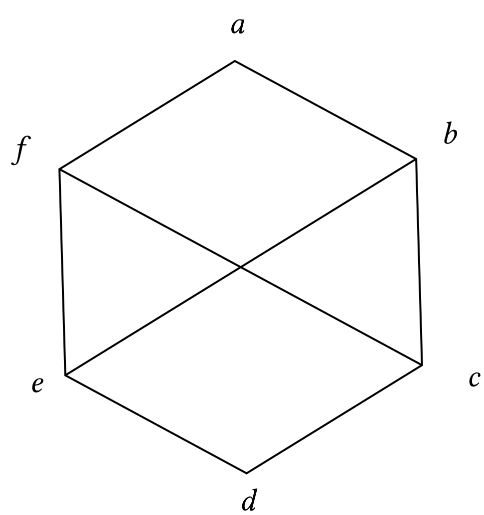

FIGURE 2. Figure 17.5

$\boxed{J\tau}$

Claim (Part C)**.** *Let $P = \{1, 2, 4, 5, 6, 12, 20, 30, 60\}$ with the partial order being divisibility. P is not a lattice*

Proof. There is no supremum of $\{2, 5\}$. $\boxed{J\tau}$

Claim (Part D)**.** *Let $P = \{1, 2, 5, 15, 20, 60\}$ with the partial order being divisibility. P is a complete lattice.*

Proof. For any subset of P, the supremum and infimum exist. $\boxed{J\tau}$

Claim (Part E)**.** *Let $P = (6, 7]$ with the partial order being $\leq$. P is a lattice, but not a complete one.*

Proof. P as a whole has no infimum, but each pair has both an infimum and a supremum: the lesser and greater of the two elements respectively. $\boxed{J\tau}$

Claim (Part F)**.** *Let $P = [6, 7]$ with the partial order being $\leq$. P is a complete lattice,*

Proof. Let $S \subseteq P$. By Completeness, $i = \inf S$ and $s = \sup S$ exist in $\mathbb{R}$. If $S = \emptyset$ then $i = 6$ and $s = 7$, and S has a supremum and infimum. Now, assume that S is non-empty. Every element of S is greater than or equal to 6, and so the infimum of S is greater than or equal to 6, but less than or equal to 7, as it is less or equal to some number less than or equal to 7, and so the i lies in $[6, 7]$. The supremum can be argued similarly to lie in $[6, 7]$ giving us that the supremum and infimum of every subset of P are in P making P complete. $\boxed{J\tau}$

Lemma 1. *Let $f : P_1 \longrightarrow P_2$ be a bijection between two posets such that $a \leq b \iff f(a) \leq f(b)$. If P_1 is a complete lattice, then P_2 is a complete lattice.*

Proof. From the conditions, we would have that $\inf(f(S)) = f(\inf(S))$ and $\sup(f(S)) = f(\sup(S))$, and so our claim follows. $\boxed{J\tau}$

Claim (Part G)**.** *Let $P = \{5\} \cup (6, 7]$ with the partial order being $\leq$. P is a complete lattice,*

Proof. Let $f : [6, 7] \longrightarrow \{5\} \cup (6, 7]$ be the bijection defined as $f(x) = x$ for $x \in (6, 7]$ and $f(6) = 5$. This function clearly preserves $\leq$, and so by Lemma 1 our claim follows. $\boxed{J\tau}$

Problem 8, Chapter 17

> *Argument.* According to Theorem 17.13, it is enough to verify that every set of positive integers has an infimum for $\leq$. But, by Theorem 10.6, every subset of $\mathbb{N}$ has a minimum element; this element then is clearly the infimum of the set.

Claim. *The above argument is wrong*

Proof. Theorem 10.6 only accounts for non-empty subsets, and the empty subsets do not have a minimum element. $\boxed{J\tau}$

> *Argument.* We need to prove that for all pairs of positive integers a and b, there are integers $i = \inf\{a, b\}$ and $s = \sup\{a, b\}$. We will show that $i = \gcd(a, b)$ and $s = \operatorname{lcm}(a, b)$ satisfy the definition of infimum and supremum, respectively. We will only do this here for i; the argument for s is similar.
> Note that the partial order $\preceq$ here is divisibility; therefore, we need to prove that i is a common divisor of a and b and that, if c is any common divisor of a and b, then i is greater than or equal to c. But both of these claims follow trivially from the definition of the greatest common divisor. This proves that $\mathbb{N}$ is a lattice for the divisibility relation.

Claim. *The above argument is wrong*

Proof. I think there are two errors in this proof.

The first error is that the author assumes the claim in the proof, that i and s exist, and that if i and s exist that they are gcd and lcm respectively

The second error made is with the sentence: "If c is any common divisor of a and b then i is greater than or equal to c."

You actually need to prove that if c is a common divisor of a and b then $\gcd(a, b)$ divides c, it is not enough to say that it is the largest one. $\boxed{J\tau}$