Challenge Problems Set 1 Solutions

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Key:

- NT: Number Theory
- AL: Algebra
- AN: Analysis
- CO: Combinatorics
- GM: Geometry

Problems

1. Prove that the equation
$$\binom{m}{3} + \binom{m}{2} + \binom{m}{1} + 1 = 2^n$$
only has finitely many solutions for positive integers m and n.

2. Given some Gaussian integer $a \in \mathbb{Z}[i]$ show that the group of Gaussian integers mod a is cyclic under addition if a is a Gaussian prime with both real and imaginary components non-zero.

3. Show that the quantity
$$a = \sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \sqrt{\cdots}}}}}$$
exists, ie. the above nested radical converges onto a real number.

4. Prove that in any set A of 7 numbers with different remainders when divided by 15, there exists $a, b, c \in A$ such that $a + b + c$ is divisible by 15. (a, b, c need not be distinct).

5. Find all positive integer solutions to
$$a^2 - b^2 = 2^k$$
where both a and b are odd.

6. Prove that the square is the only regular polygon with side length 1 that has integer area.

7. Prove that
$$1^3 + 2^3 + \cdots + n^3$$
is never prime

8. Determine when the polynomial $f(x) = x^3 + 2abx^2 + ax$ has only real rational roots for $a, b \in \mathbb{Z}$.

9. Equations of the form
$$x^3 + y^3 + ax^2 + by^2 + cxy = 0$$
sometimes have two self intersections, find the conditions for this occurring (in terms of an equation or inequality containing only a, b and c as variables) and the points of self intersection in terms of a, b and c).

10. Let p be some prime not equal to 2 or 5. Prove that $\frac{1}{p}$ cannot be represented as a terminating decimal in base 10, and prove that ℓ_{10}, the period of the decimal's repeating cycle, divides $p - 1$.

Solutions

1. Simplify the equation, showing that $m+1$ can only take the form $3 \cdot 2^k$ or 2^k, then show that k is bounded from above via parity and/or rationality.

2. Demonstrate that for any $z = x + yi$, the group Gaussian integers is cyclic if $\gcd(x, y) = 1$, and explain that all Gaussian primes with non-zero components have relatively prime x and y.

3. Show that a sequence defined by iterations of $a = \sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \sqrt{\cdots}}}}}$ is monotone and bounded from above. (Many ways to do this)

4. First, one may note that if A has no three elements who have a sum divisible by 15, then no element in A is divisible by 5. Therefore, every element of A must be 1,2,3,4 mod 5. Now, we examine the sets mod 15 $A_2 = \{2a \; a \in A\}$ and $A_1 = \{15 - a \; a \in A\}$. Now, note that for A to not contain a 15 divisible triplet, $A_1 \cap A_2 = \emptyset$. Furthermore, neither set can contain 0, 5 or 10, giving us that $A_1, A_2, \{0, 5, 10\}$ are pairwise disjoint. However, $|A_1| = 7$ and $|A_2| = 7$, however $|A_1 \cup A_2 \cup \{0, 5, 10\}| \leq 15$ but $7 + 7 + 3 \leq 15$ is obviously untrue.

5. We have that $(a + b)(a - b) = 2^k$ Meaning $a + b = 2^m$ and $a - b = 2^n$, this means that $\frac{a+b}{a-b} = 2^{m-n}$. Because they are both odd, $-2b$ is only divisble by 2 once, and we would have that $a - b = 2$, and $a + b = 2^{k-1}$.

6. Many ways to show this, I prefer showing that the area of such a regular n-gon with side length 1 is $\frac{n}{4} \cot\left(\frac{\pi}{n}\right)$, and that for this to be an integer, it is required that $\frac{n}{4}$ is an integer, thus n is divisible by 4. Furthermore, $\cot\left(\frac{\pi}{n}\right)$ is only rational for $n = 4$.

7. One way to show this is that the quantity is always a square.

8. First, show that the roots for $f(x)$ are 0 and $\pm\sqrt{a^2b^2 - a} - ab$. Now, the question has been simplified to when $a^2b^2 - a$ is square. We have that
$$a^2b^2 - a = a(ab^2 - 1)$$
must also be square, however this means that either a is square or $ab^2 - 1$ is divisible by a. The former is clearly never valid as this would make ab^2 a square, and a square minus 1 is never a square. The latter implies that $a = \pm 1$ or $b = 0$. However, if $a = 1$, then that would also mean that $b^2 - 1$ is a square, giving us that b is either 1, 0 or -1. If $a = -1$, then we must have that $b^2 + 1$ is square, which is only possible when $b = 0$. Now, The last case, when $b = 0$. This is easily seen to result in a rational real solution when a is a negative square. Thus, we have the full list of solutions
$$(1, 1), (1, -1), (1, 0), (A, 0)$$
where A is -1 times any integer square.

9. Condition for double intersection: $a + b - c = 0$

   Points of self-intersection when two self intersection points are present:
$$(0, 0), \left(\frac{b-a}{3}, \frac{a-b}{3}\right)$$

10. First part follows relatively easily from the fact that p is prime, second follows from Fermat's little theorem.