MATH 315 CHAPTER 20

JACOB TERKEL

2023

Problem 2a

Claim.

$$g(m,n) = \begin{cases} (m-1)^2 + n & m \geq n \\ m^2 - n + 1 & m < n \end{cases},$$

Problem 2b

Claim.

$$h(m,n) = \frac{m+n}{2} \cdot \frac{m+n-2}{2} - n + 1$$

Problem 3

Claim. *For $A = \{0, 1, 2\}$ let*

$$\mathcal{A} = \prod_{n=1}^{\infty} A$$

and let $\mathcal{B}$ be the sequences in $\mathcal{A}$ such that consecutive terms are always distinct. $\mathcal{B}$ is uncountable.

Proof. Define C as

$$C = \prod_{n=1}^{\infty} \{0, 1\} \times \{2\}.$$

That is, C consists of the sequences with 2 at the even positions, and 0 or 1 at the odd positions. See that $C \subseteq \mathcal{B}$, and also see that there is a bijection between C and the uncountable set $\prod_{n=1}^{\infty}\{0, 1\}$ defined by removing all the 2's from the sequence. Giving us that C is uncountable, and because $C \subseteq \mathcal{B}$ by Proposition 20.4 our claim is proven.

Problem 5a

Claim. *If X is a set of pairwise disjoint intervals then it is countable*

Proof. From the density of $\mathbb{Q}$ in $\mathbb{R}$, each interval contains at least one rational number, and because the elements of X are pairwise disjoint the function $f : X \longrightarrow \mathbb{Q}$ that takes an interval to some rational number in the interval is an injection. Thus, there is a surjection g from $\mathbb{Q}$ to X by Theorem 20.13. Furthermore, because $\mathbb{Q}$ is countable, there is a surjection from $\mathbb{N}$ to $\mathbb{Q}$ called h. Because both g and h are surjective the function $g \circ h : \mathbb{N} \longrightarrow X$ is a surjection, and so X is countable.

Problem 5b

Claim. *If no pair of intervals in X are disjoint then X is not necessarily countable.*

Proof. Note that the set

$$X = \{(0, r) \mid r \in \mathbb{R}\}$$

satisfies the above property as for any $(0, a), (0, b) \in X$ we have that $\frac{\min\{a,b\}}{2} \in (0, a) \cap (0, b)$.

However, this set is clearly uncountable as there is a bijection from $\mathbb{R}$ to X.

Problem 5c

Claim. *If no interval in X contains another then X is not necessarily countable*

Proof. See that the set

$$X = \{(r, r + 1) \mid r \in \mathbb{R}\}$$

satisfies the above property. Furthermore, this set is clearly uncountable as there is a bijection from $\mathbb{R}$ to X.

Problem 6

Claim. *The set of algebraic numbers, $\mathcal{A}$, is countable.*

Proof. Let S_n be the set of solutions to integer coefficient polynomials with degree n. Note that there exists a function $f : \mathbb{Z}^{n+2} \setminus \{0\}^n \longrightarrow S_n \cup \{\infty\}$ such that

$f(a_1, a_2, \cdots, a_{n+1}, a_{n+2}) =$ the $a_{n+2}th$ largest real solution to $a_1 + a_2 x + \cdots + a_{n+1} x^n$ if it exists

and

$$f(a_1, a_2, \cdots, a_{n+1}, a_{n+2}) = \infty$$

if the $a_{n+2}th$ largest real solution to $a_1 + a_2 x + \cdots + a_n x^n$ doesn't exist.

Now, let T_n the subset of $\mathbb{Z}^{n+2} \setminus \{0\}^n$ consisting of all elements of $\mathbb{Z}^{n+2} \setminus \{0\}^n$ that do not get taken to ∞ by f, and with the property that that $a_{n+1} \neq 0$. Now, note that there is a surjection $g : T_n \longrightarrow S_n$ that can be defined solely by

$$g(a_1, a_2, \cdots, a_{n+1}, a_{n+2}) = \text{ the } a_{n+1}th \text{ largest real solution to } a_1 + a_2 x + \cdots + a_{n+1} x^n.$$

Note that by Now, note that $\mathbb{Z}^{n+2}$ is countable by Lemma 20.7, and so by Proposition 20.4 we have that T_n is countable, and because g is a surjection, S_n is countable. Now, note that

$$\mathcal{A} = \bigcup_{i=1}^{\infty} S_i,$$

and so by Lemma 20.9 our claim is proven.

Claim. *The set of transcendental numbers $\mathcal{T}$ is uncountable.*

Proof. By definition, $\mathbb{R} = \mathcal{T} \cup \mathcal{A}$. Now, because $\mathcal{A}$ is countable, if $\mathcal{T}$ were to be countable, that would imply that $\mathbb{R}$ is countable, which is false. Thus, $\mathcal{T}$ is uncountable.

Problem 7a

(a) **Claim.** Every nonempty set is countable.
Argument. Since every nonempty finite set is clearly countable, it suffices to prove our claim for infinite sets.
Let A be an infinite set. Then, by definition, there is an injection $f : \mathbb{N} \to A$. Consider the set

$$\operatorname{Im}(f) = \{f(1), f(2), \ldots\}.$$

If $\operatorname{Im}(f) = A$, then f is a surjection, and A is countable. Assume then that $\operatorname{Im}(f)$ is a proper subset of A, and let $a \in A \setminus \operatorname{Im}(f)$. Define a function g as follows:

$$g : \mathbb{N} \to A$$

$$n \mapsto \begin{cases} a & \text{if } n = 1 \\ f(n-1) & \text{if } n \geq 2 \end{cases}$$

Note that

$$\operatorname{Im}(g) = \{g(1), g(2), \ldots\} = \{a, f(1), f(2), \ldots\}.$$

If $\operatorname{Im}(g) = A$, then g is a surjection, so A is countable. If $\operatorname{Im}(g)$ is a proper subset of A, then we proceed as above; continuing the process until we eliminate all of A, we finally arrive at a function h from $\mathbb{N}$ to A whose image is all of A, and thus A is countable.

This argument uses induction, and induction only works on countable sets. Thus, this argument subtly assumes its claim.

Problem 7c

) **Claim.** The set $P(\mathbb{N})$ is countable.

Argument. Let $P = \{2, 3, 5, 7, 11, \dots\}$ be the set of positive primes. Note that, by the Fundamental Theorem of Arithmetic, every positive integer n can be written in the form

$$n = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdots$$

with nonnegative integers $\alpha_1, \alpha_2, \alpha_3, \dots$, and the expression of n in this form is unique.

Using the expression of n above, we can define the function

$$f : \mathbb{N} \to P(\mathbb{N})$$

$$n \mapsto \{\alpha_i \mid \alpha_i \geq 1\}$$

(For example, we have $f(1) = \emptyset$, $f(2) = f(6) = \{1\}$, $f(63) = \{1, 2\}$, and $f(63,000,000) = \{1, 2, 6\}$.)

It is easy to see that f is a surjection (though clearly not an injection), thus $P(\mathbb{N})$ is countable.

The function f is not a surjection as it there are no infinite sets in the image of f.

Problem 7d

Claim. The set of Plutonian words is uncountable.

Argument. Let $S = \{A, B, C, D, \smile\}$ be the set consisting of the four letters in the Plutonian alphabet and the symbol $\smile$. Each Plutonian word can be thought of as an infinite sequence of elements of S where a finite string of the four letters is followed by infinitely many $\smile$s. For example, the word $AABCDA$ can be identified with the infinite sequence $AABCDA \smile\smile\smile \cdots$. Therefore, the set of Plutonian words is essentially the same as the Cartesian product $S \times S \times \cdots$, which, by Lemma 20.10, is uncountable.

The equivalence established is false.

Problem 7e

Claim. The set of real numbers in the interval $[0, 1)$ is countable.
Argument. Write each real number $x \in [0, 1)$ in its binary representation (cf. Theorem 18.24):

$$x = 0.d_1 d_2 \ldots$$

where the binary digits (bits) $d_1, d_2, \ldots$ all equal 0 or 1. Note that certain numbers have two such representations; namely, if in the representation of x above, there is a $k \in \mathbb{N}$ for which the k-th bit is 0 and it is followed by infinitely many 1 bits, then x is unchanged if we replace the k-th bit by a 1 and each successive bit by 0. Therefore, we may assume that each real number between 0 and 1 has a binary representation with only finitely many 1 bits.
We can now create a list of all real numbers between 0 and 1, as follows. The list will start with the only real number in $[0, 1)$ with no 1 bit: $0 = 0.00000\ldots$. It is followed by the other number that has no 1 bits beyond the first bit, $\frac{1}{2} = 0.100000\ldots$. Then, we list the two numbers in $[0, 1)$ that have no 1 bits beyond the second bit (and that have not been listed before): $\frac{1}{4} = 0.010000\ldots$ and $\frac{3}{4} = 0.110000\ldots$, and so on. Note that, for each positive integer n, there is only a finite number of binary representations that have no 1 bits beyond the n-th bit; these can obviously be arranged in a finite list. Therefore, proceeding like this for successive values of n creates a list that contains all real numbers with only finitely many 1 bits and, therefore, all real numbers in the interval $[0, 1)$.

The listing given is not comprehensive.