Relativity Notes

Special Relativity

Newtonian Gravity

Space and time depenent gravitational potential, $\Phi(\vec{x}, t)$ — $\vec{x}$: Space, t: time

Force on test particle: $\boxed{\vec{f} = -m_G \vec{\nabla}\Phi}$ → Passive gravitational mass

Where potential determined by Poisson Equation:
$\boxed{\vec{\nabla}^2 \Phi = 4\pi G\rho}$ → ρ: poisson mass density 'Active gravitational mass'

Arising issues:
↳ 'Newtonion gravity is in consistent with special relativity'
Poisson equation implies: Φ responds instantly to changes in $\rho(t)$ (instanteous propogation)
But: no causal inference can travel faster than speed of light

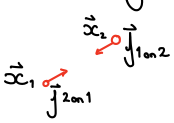
Equivelence of Passive and Active Masses: $\rho \longrightarrow m_G$

Passive Mass: M_G
Measures how strongly an object feels a gravitational field
Active mass: M_A
Measures how strongly an object creates a gravitational field

Mass density of point particle at $\vec{y}(t)$:
$$\boxed{\rho(\vec{x}, t) = M_A\, \delta^{(3)}(\vec{x} - \vec{y}(t))}$$

Potential: $$\boxed{\Phi(\vec{x}, t) = \frac{-G M_A}{|\vec{x} - \vec{y}(t)|}}$$ → M_A: Active Gravitational Mass.

So investigating force equation:



$$\vec{f}_{1on2} = -G\, m_{G,2}\, m_{A,1} \frac{(\vec{x}_2 - \vec{x}_1)}{|\vec{x}_2 - \vec{x}_1|^3}$$
$(\vec{x}_2 - \vec{x}_1)$: Direction; $|\vec{x}_2 - \vec{x}_1|^3$: Magnitude

$$\vec{f}_{2on1} = -G\, m_{G,1}\, m_{A,2} \frac{(\vec{x}_1 - \vec{x}_2)}{|x_1 - x_2|^3}$$

by Momentumn Conservation:
$$\vec{f}_{1on2} = -\vec{f}_{2on1} \longrightarrow m_{G,1} m_{A,2} = m_{G,2}, m_{A,2}$$

Thus $\boxed{\frac{M_{G,1}}{M_{A,1}} = \frac{M_{G,2}}{M_{A,2}}}$ } Universal ratio true for all particles. $\boxed{m_G = m_A}$

Equivelence of Inertial and Gravitational Masses:

Inertial Mass, m_I

Newtons 2^{nd} Law: $\vec{f} = m_I \frac{d^2 x}{dt^2}$

Motion under gravity $\vec{f} = -m_G \nabla \Phi$
$$\frac{d^2\vec{x}}{dt^2} = -\boxed{\frac{m_G}{m_I}}\vec{\nabla}\Phi$$
↳ equal for 1 part in 10^{13}.

Leads to Universal Acceleration:

Caveated: In EM q/m_I is universal

Weak Equivalence Principle:

All free falling particles follow the same path through space and time if they have the same initial position and velocity

↳ Implications:

For an observer in free fall within uniform gravitational field.

↳ Sees any free falling particle follow straight line path at constant velocity

Strong Equivelence Principle

In an arbitary gravitational field, all the laws of physics in a free-falling, non rotating laboratory occupying a sufficiently small region of space time look locally like special relativity.

→ Has to be local due to:

Tidal Effects: Observable manifestations of gravitational fields but undetectable for sufficiently local measurements.

So in Special Relativity:

↳ Without Gravity:

Physics looks simple in inertial reference frames.

With Gravity:

Physics looks simply locally in free falling reference frames.

→ Local Inertial Reference Frame

↳ Free falling, non rotating

→ Acceleration: Should be defined relative to frame

will be a result of non-gravitational forces

So next idea: Gravity as Spacetime Curvature (Geometry)

Where. → Trajectories of free-falling particles determine by local structure of 'space time'.

Problem: Local inertial frames do not mesh in general into one field

Gravity: Cannot extend a frame throughout all of space time

↳ Spacetime Curvature

General Relativity:

Abandons: Idea of gravity as a force defined on fixed spacetime of special relativity

Instead: Geometric theory → geometry of spacetime determines trajectories of free falling particles.

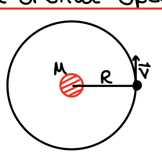
geometry itself is curved by presence of matter.

Further motivation: Extreme Gravity

Newtonian Gravity valid for low relative speeds: $v \ll c$.

weak fields: $|\Phi| \ll c$

Ie orbital speeds (Newtonian gravity)



$$\frac{v^2}{R} = \frac{GM}{R^2}$$

$$\frac{v^2}{c^2} = \frac{GM}{Rc^2} = \frac{|\Phi|}{c^2}$$

→ Many astrophysical systems where Newtonians limits fail:

↳ Gravitational Waves (Binary Mergers)

A Recap of Special Relativity:

Inertial Reference Frames:

↳ Event labelled by: 3 Space, 1 time } components.

Coordinates in inertial frame, S:

$(ct, \underbrace{x, y, z}_{\text{Cartesian Coords}})$

ct → Time in Sychronised Clocks

Inertial Frame:

Any free particle has: $\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0$ → At rest/Constant Speed in Straight line (Newton 1)

2 inertial frames (S, S') differ only by:

i) translation (choice of origin)

ii) rotation

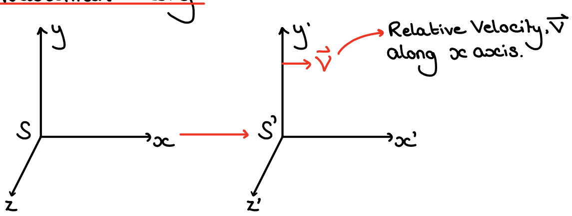
iii) relative motion (Constant Velocity).

Principle of Relativity

↳ All laws of physics take the Same form in every inertial frame

Differences between Newtonian Theory and Special Relativity

1) Newtonian Theory



↳ S, S' cords related by linear transform

Absolute time: $t' = t$ (fixed/equal)

Galilean Transformations

$t' = t$

$x' = x - vt$ → Velocity transformation Simply derivatives → $v_{x'} = v_x - V$

$y' = y$ $\qquad v_{y'} = v_y$

$z' = z$ $\qquad v_{z'} = v_z$

Results:

V constant → accl^n invariant

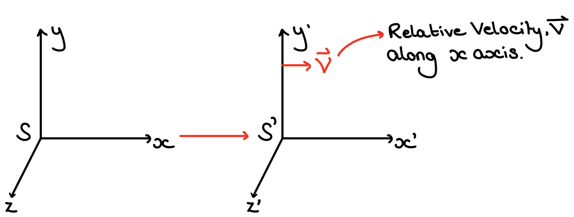
absolute time → Δt invariant

} $\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$ is invariant

↳ $\boxed{\text{Space and time are Seperate entities}}$ ⟹ Newtonian Result.

Spacetime Geometry of SR

↳ $\boxed{\text{Key point: Abandon absolute time → replace with Speed of light being Constant in all inertial frames}}$



Lorentz Transformation

$ct' = \gamma(ct - \beta x)$

$x' = \gamma(x - \beta ct)$

$y' = y$

$z' = z$

$\beta = V/c$

$\gamma = \frac{1}{\sqrt{1-\beta^2}} \geqslant 1$

For any two events:

$\boxed{\Delta S^2 = c\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2}$: Square Interval invariant under LT

↳ Space + Time United into Spacetime (4D) with invariant geometry Characterised by ΔS^2 } Minkowski Spacetime

Results of Special Relativity

↳ Lorentz Transform has 4D Rotations.

↳ Coordinate frame Systems ⟶ Relabel events in Minkowski Space.

$(ct, x, y, z) \longrightarrow (ct', x', y', z')$

Rapidity: a convinient transform definition

$\boxed{\beta = \tanh \psi}$ $\quad \beta = \frac{v}{c} \quad -\infty \leq \psi \leq \infty$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\tanh^2\psi}} = \cosh\psi \longrightarrow \boxed{\gamma\beta = \sinh\psi}$$

Rewriting Lorentz Transform

$$\boxed{\begin{aligned} ct' &= ct\cosh\psi - x\sinh\psi \\ x' &= -ct\sinh\psi + x\cosh\psi \end{aligned}}$$

→ Hyperbolic rotation in ct-x plane

→ Invariance of ΔS^2 follows: $c^2\Delta t^2 - \Delta x^2$

More Complicated lorentz transforms:

Previously we have explored a very simple configuration but there are lots of examples which would differ:

⟶ 4D origins may not coincide

ie $ct = x = y = z = 0$ may not be at $ct' = x' = y' = z'$

⟶ Relative velocity of two frames may be in arbitary directions

ie not along x axis

⟶ Spatial axis may not be aligned.

Can account for this by decomposing LT (between S and S').

→ i) Rotate axis in S s.t relative velocity is along new x axis

ii) Apply a standard lorentz boost to a frame comoving with S'

iii) Rotate spatial axis of boosted frame to align with S'

(+ overall spacetime translation - origins).

General LT Boost:

Homogenous LT between two inertial frames with:

→ Axis aligned

(but) relative velocity in arbitary direction.

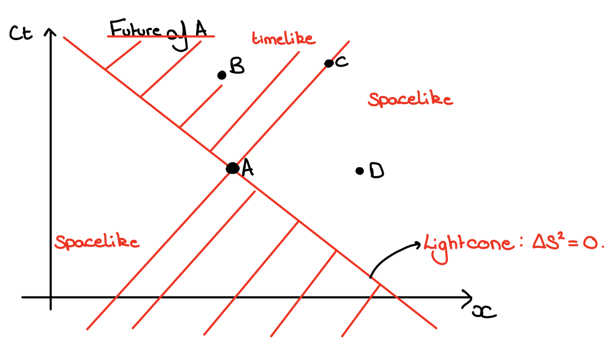
Interval and Light cone Structure

General LT: Invariant Interval

$$\boxed{\Delta S^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2}$$

- All Δ affected by shift in origin (S, S')
- $\Delta x^2, \Delta y^2, \Delta z^2$ unaffected by spatial variations.

Defines Invariant 'distance' between pairs of events:



$\Delta S^2 > 0$: Timelike

↳ Can find frame in which at same location

$\Delta S^2 = 0$: Null/Lightlike

$\Delta S^2 < 0$: Spacelike

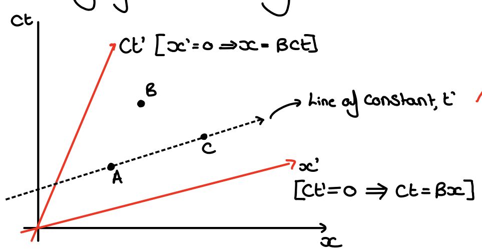
↳ Can find frame in which simultaneous.

Light cone: All events that can be connected to A by light signals.

$\boxed{A \to B: \text{ Causal Signals from A can reach and influence B.}}$ → Inside Cone ≈ Causally Connected.

D: Events outside light cone of A cannot influence / be influenced by A.

Relativity of Simultaneity



Simultaneity is relative: A and C: Simultaneous in S but not in S'

However:

Temporal Ordering: Invariant for Causally Connected (null/timelike) Seperated events

↳ Doesnt break Causality

Length Contraction and Time Dilation:

Length Contraction

Rod of Proper length: L_0 in S'

S.t. $L_0 = x_{B'} - x_{A'}$

Measuring in S:

Measuring positions of endpoints at time t

$x_A(t)$ and $x_B(t)$

Moving to S':

$x_{A'} = \gamma(x_A(t) - \beta ct)$ ⇒ $L_0 = \gamma(x_B(t) - x_A(t))$

$x_{B'} = \gamma(x_B(t) - \beta ct)$

$$\boxed{L = \frac{L_0}{\gamma} = L_0(1-\beta^2)^{1/2}}$$

Length of S is Contracted relative to rest frame of rod.

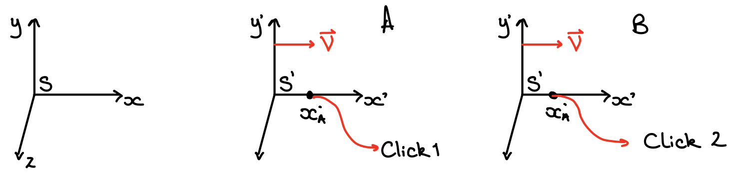
Volumes

↳ No contraction in perpendicular lengths: $V = V_0/\gamma$

Number Densities

↳ $n = \gamma n_0$

Time Dilation:



Clock at rest in S':

↳ Two Succesive 'clicks' of clock (events A, B) – Seperated by T_0

Times recorded in S are:

↳ $ct_A = \gamma(ct'_A + \beta x'_A)$

$ct_B = \gamma(ct'_A + cT_0 + \beta x'_B)$

$x'_A = x'_B$ as clock at rest.

↳ Time elapsed in S'

Thus: $\boxed{T = t_B - t_A = \gamma T_0 = \frac{T}{\sqrt{1-\beta^2}}}$

↳ Moving clock appears to run slower: Time Dilation

Paths in Spacetime:

Minkowski Spacetime Line Element

Invariant Interval: $\Delta S^2 = c\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$

'distance' in spacetime measured along straight line connecting two events.

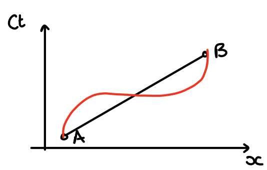
Minkowski Line Element

Infinitesimal Interval: $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ → For a 'wiggle' path connecting A and B:

$$\boxed{S = \int_A^B ds}$$

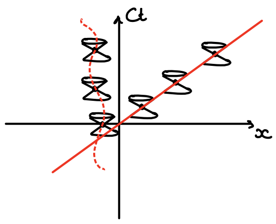
→ Path Dependent

→ Lorentz Invariant



Particle World Lines and Proper Time

A particle describes a world line path in Spacetime.



Massive Particle:

World line must lie inside light cone through every point on the path (Dashed Line)

Massless Particle (Photon):

at any point the path must be a tangent to the light cone (Solid Line)

Spacetime Path:

$x(t), y(t), z(t) \xrightarrow{\text{Parametised}} t(\lambda), x(\lambda), y(\lambda), z(\lambda)$

↳ Commonly parametised by

Proper time, τ:

Time on ideal clock carried by particle.

Increment in time, $d\tau$

↳ Increment in instanteous rest frame ($dx' = dy' = dz' = 0$)

↳ $\boxed{c^2 d\tau^2 = ds^2}$

Thus for two infitesimally close events on particles worldline seperated by dt, dx, dy, dz:

$$c^2 d\tau^2 = ds^2 = \underbrace{c^2dt^2 - dx^2 - dy^2 - dz^2}_{\text{General frame } S}$$

$$d\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt = \frac{dt}{\gamma_v}$$

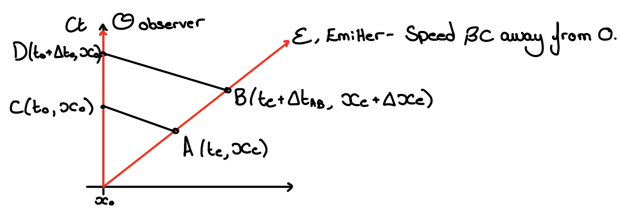
v - Speed of particle in frame S

γ_v - associated Lorentz factor

'Proper Time' elapsed between events A and B:

$$\Delta\tau = \int_A^B d\tau = \int_A^B \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt.$$

Doppler Effect



Consider two observers:

Observer O: at rest in position x_0

Emitter $\mathcal{E}$: moves at speed v along x axis

Both described in inertial frame S.

↳ Success waves emitted by $\mathcal{E}$ at events A, B

↳ Seperated by Proper time $\Delta\tau_{AB}$ (proper period of Source)

$$\boxed{\Delta t_{AB} = \Delta t_e \left(1 - \frac{v^2}{c^2}\right)^{1/2}}$$

Δt_e - time between emission events in S.

In time Δt_e, Source moves $\underline{\Delta x_e = v\Delta t_e}$

Thus observer time delay: $\boxed{\Delta t_{CD} = \Delta t_o = \left(1 + \frac{v}{c}\right)\Delta t_e}$

$$\text{Thus: } \frac{\Delta\tau_{AB}}{\Delta\tau_{CO}} = \frac{(1-\beta^2)^{1/2}\,\Delta t_e}{(1+\beta)\quad \Delta t_e} = \frac{(1-\beta)^{1/2}}{(1+\beta)^{1/2}}$$

→ Observed Freq.

→ Emitted Freq.

Velocity Transformations

For a particle on worldline: $x(t), y(t), z(t)$ in rest frame S:

3 Velocity Components in S: $u_x = \frac{dx}{dt}$, u_y, u_z

For Infinitesimally Seperated points on particle worldline:

Transforming $S \to S'$: (γ_v – Lorentz for Speed v)

$$dt' = \gamma_v \left(dt - v\frac{dx}{c^2}\right)$$
$$dx' = \gamma_v (dx - v\,dt)$$
$$dy' = dy$$
$$dz' = dz.$$

Velocity Components

Chain Rule

$$\boxed{u_x' = \frac{\partial x'}{\partial t'} = \frac{\partial x'}{\partial t} \cdot \frac{dt}{\partial t'} = \frac{u_x - v}{\left(1 - \frac{u_x v}{c^2}\right)}}$$

$$\boxed{u_y' = \frac{\partial y'}{\partial t'} = \frac{\partial y'}{\partial t} \cdot \frac{dt}{\partial t'} = \frac{u_y}{\gamma_v\left(1 - \frac{u_x v}{c^2}\right)}}$$

$$\boxed{u_z' = \frac{\partial z'}{\partial t'} = \frac{\partial z'}{\partial t} \cdot \frac{dt}{\partial t'} = \frac{u_z}{\gamma_v\left(1 - \frac{u_x v}{c^2}\right)}}$$

For $v \ll c$:

↳ Reduce to Galilean Transform

For multiple transforms (Co-Linear)

3 frames: S, S', S''

$S \to S'$: Boost along x by v

$S' \to S''$: Boost along x' by u'

Rapidities $\longrightarrow$ $\tanh \psi_v = v/c$ $(S \to S')$

$\longrightarrow$ $\tanh \psi_{u'} = u'/c$ $(S' \to S'')$

Successive Transformations

$$x'' = \cosh \psi_{u'}\, x' - \sinh \psi_{u'}\, ct'$$

↳ $x' = \cosh \psi_v\, x - \sinh \psi_v\, ct$

$ct' = \cosh \psi_v\, ct - \sinh \psi_v\, x$

$$= \cosh \psi_{u'} \left(\cosh \psi_v\, x - \sinh \psi_v\, ct\right) - \sinh \psi_{u'} \left(\cosh \psi_v\, ct - \sinh \psi_v\, x\right)$$
$$= \cosh(\psi_v + \psi_{u'})\, x - \sinh(\psi_v + \psi_{u'})\, ct$$

Equivelently: $ct'' = \cosh(\psi_v + \psi_{u'})\, ct - \sinh(\psi_v + \psi_{u'})\, ct$, $y'' = y$, $z'' = z$.

Colinear boost $\longrightarrow$ rapidities add

$S'' \longrightarrow S$: $c\tanh(\psi_{u'} + \psi_v)$

Acceleration in Special Relativity:

$$a_x = \frac{du_x}{dt}$$

In new reference frame: $t'(x, y, z, t)$

$$a_{x'} = \frac{du_{x'}}{dt'} = \left(\frac{\partial u_{x'}}{\partial u_x}\frac{\partial u_x}{\partial t} + \frac{\partial u_{y'}}{\partial u_y}\frac{\partial u_y}{\partial t} + \frac{\partial u_{z'}}{\partial u_z}\frac{\partial u_z}{\partial t}\right)\frac{dt'}{\partial t}$$

Chain rule

$$a_{x'} = \left(\frac{\partial u_{x'}}{\partial u_x} \cdot a_x + \cancel{\frac{\partial u_{y'}}{\partial u_y}} \cdot a_y + \cancel{\frac{\partial u_{z'}}{\partial u_z}}\, a_z\right)\frac{dt'}{dt} = \frac{\partial u_{x'}}{\partial u_x} \cdot a_x \cdot \frac{dt'}{dt}$$

Using Chain rule of $u_{x'}$, u_y, $u_{z'}$

$$du_x' = \frac{du_x}{\gamma_v^2 (1 - u_x v/c^2)^2}$$

$$dt' = \gamma\left(dt - v\frac{dx}{c^2}\right) = \gamma\left(1 - \frac{u_x v}{c^2}\right)dt.$$

$$du_y' = \frac{du_y}{\gamma_v^2 (1 - u_x v/c^2)^2} + \frac{u_y v\, du_x}{c^2 \gamma_v (1 - u_x v/c^2)^2}$$

$$du'_z = \frac{du_z}{\gamma_v^2 (1-u_x V/c^2)^2} + \frac{u_z V du_x}{c^2 \gamma_v (1-u_x V/c^2)^2}$$

Resultant Acceleration

$$\boxed{\begin{aligned} a'_x &= \frac{du'_x}{dt'} = \frac{1}{\gamma_v^3 (1-u_x V/c^2)^3} a_x \\ a'_y &= \frac{du'_y}{dt'} = \frac{1}{\gamma_v^2 (1-u_x V/c^2)^2} a_y + \frac{u_y V}{c^2 \gamma_v^2 (1-u_x V/c^2)^3} a_x \\ a'_z &= \frac{du'_z}{dt'} = \frac{1}{\gamma_v^2 (1-u_x V/c^2)^2} a_z + \frac{u_z V}{c^2 \gamma_v^2 (1-u_x V/c^2)^3} a_x \end{aligned}}$$

$\longrightarrow$ Acceleration in Special Relativity

$\longrightarrow$ not Lorentz invariant

is absolute – either is or not.

Rectilinear Acceleration Example

Particle moving along x axis with non uniform speed $u(\tau)$

↳ Proper acceleration in Instantaneous Rest Frame $g(\tau)$

↳ In IRF at time τ:

↳ $u'(\tau) = 0$, $\frac{du'}{dt'} = g(\tau)$,

Inverse into frame S:

$$\frac{du}{dt} = \left(1 - \frac{u^2}{c^2}\right)^{3/2} g(\tau) \qquad d\tau = \left(1 - \frac{u^2}{c^2}\right)^{1/2} dt$$

$$\boxed{\frac{du}{d\tau} = \left(1 - \frac{u^2}{c^2}\right) g(\tau) = \frac{g(\tau)}{\gamma_u^2}}$$

↳ In terms of rapidity: $u = c \tanh(\psi)$

$u(\tau) = c\tanh(\psi(\tau)) \longrightarrow c\frac{d\psi}{d\tau} = g(\tau)$

$$\boxed{c\psi(\tau) = \int_0^{\tau} g(\tau')\, d\tau'}$$ where $u=0$ at $\tau=0$

Parametising world line:

1) $\boxed{\frac{dt}{d\tau} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} = \cosh\psi(\tau)}$ 2) $\boxed{\frac{dx}{d\tau} = u\left(1 - \frac{u^2}{c^2}\right)^{-1/2} = c\sinh\psi(\tau)}$

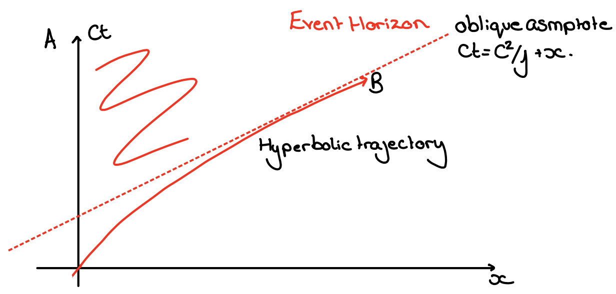
Special Case: $g(\tau)$ = Constant

Rapidity: $\psi(\tau) = \frac{1}{c}\int_0^{\tau'} g\, d\tau' = \frac{g\tau}{c}$ $\longrightarrow$ Rises linearly!

Making Worldline:

$$\frac{dt}{d\tau} = \cosh\left(\frac{g\tau}{c}\right) \longrightarrow ct(\tau) = \underbrace{ct_0}_{0} + \frac{c^2}{g}\sinh\left(\frac{g\tau}{c}\right)$$

$$\frac{dx}{d\tau} = c\sinh\left(\frac{g\tau}{c}\right) \longrightarrow x(\tau) = \underbrace{x_0}_{0} + \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right) - 1\right]$$



– Region of space time that can never influence ie communicate causally with the accelerated particle.

Ie. Light must be emitted by $t = c/g$ to ever reach object.

Manifolds and Coordinates

Manifolds

Informally: an N-dimensional manifold

A set of objects (events in Spacetime) that Locally resembles ND Euclidean Space, $\mathbb{R}^N$.

↳ time, 3 spatial

↳ There exists a map ϕ from the ND manifold M to an open subset of $\mathbb{R}^N$ (one-to-one and onto)

Invertible

In general: Can consider manifolds as Surfaces embedded in some higher-dimensional Euclidean Space.

Coordinates:

Points in an ND manifold

→ Labelled by N real-valued Coordinates: x^a (with $a = 1, \dots, N$).

Coordinates are not Unique → Change under coordinate transform/map ϕ.

Multiple Coordinate Systems:

↳ Generally not possible to cover a manifold with single non degenerate coord system.

↳ one-to-one Correspondence between Cordinates and labels.

Common Example: 2-Sphere S^2

Points on $\mathbb{R}^3$ with $x^2+y^2+z^2=1$

Polar Coords (θ, ϕ) → Degeneracy at poles $\theta = 0, \pi$

↳ At least 'two coordinate' patches needed to define manifold without degeneracy

Curves and Surfaces

↳ Defined by Subsets of manifolds.

Typically defined by:

→ Parametisation of Coordinate System.

Curve:

$x^a = x^a(u)$ $\quad a = 1, \dots, N$ → Single parameter.

Surface (Submanifold) - M dimensions

Require $M < N$ parameters:

$x^a = x^a(u^1, u^2, \dots, u^M)$ $\quad (a = 1, \dots, N)$

Alternatively can be defined using:

N-m independent constraints ie Intersection of N-M hypersurfaces.

$f_1(x^1, x^2, \dots, x^M) = 0, \dots, f_{N-M}(x^1, x^2, \dots, x^M) = 0.$

Hyper Surface

Special Case: $M = N-1$

Can eliminate N-1 params from N eqns to give 1 Constraint function: $f(x^1, x^2, \dots, x^N) = 0.$

Coordinate transformations:

Labelling of points in manifold is arbitary

Relabel by Coordinate transform (N eqns)

$\boxed{x'^a = x'^a(x^1, \dots, x^N)}$ → Passive Relabelling (ie in terms of previous Coords)

transform functions - Single valued, Continuous, differentiable.

Example

Using Chain rule to relate 'distance' before + after translation.

$P \circ x^a$ $\quad$ $Q \circ x^a + dx^a$ $\xrightarrow{\text{New, primed Coordinates}}$ $\boxed{dx'^a = \sum \frac{\partial x'^a}{\partial x^b}\, dx^b = \sum J^a{}_b\, dx^b}$

↳ Evaluate at P, Say.

Involves $N \times N$ transformation matrix, $J^a{}_b$

Transformation Matrix: $J^a{}_b = \frac{\partial x'^a}{\partial x^b} = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \cdots & \frac{\partial x'^1}{\partial x^N} \\ \vdots & & \vdots \\ \frac{\partial x'^N}{\partial x^1} & \cdots & \frac{\partial x'^N}{\partial x^N} \end{pmatrix}$ (columns: b, rows: a)

Jacobian of Transform:

$\boxed{J = \det(J^a{}_b)} \rightarrow$ Controls if Invertible: $J = \det(J^a{}_b) \neq 0$.

Inverse Transformation

$\boxed{x^a = x^a(x'^1, \ldots, x'^N)} \rightarrow$ Inverse of $J^a{}_b$; by Chain rule $\sum \frac{\partial x^a}{\partial x'^b} J^b{}_c = \sum \frac{\partial x^a}{\partial x'^b} \frac{\partial x'^b}{\partial x^c} = \frac{\partial x^a}{\partial x^c} = \delta^a{}_c$

Recap of Einstein Summation Convention:

If index occurs once in SubScript, one in Super Script $\longrightarrow$ Summation $1 \to N$ of index implied

Local Geometry of Riemannian Manifolds:

Geometry is Specified by additional Structure on the manifold.

Local Geometry: defined by invariant distance (interval) between neighbouring points

Reinman Manifold

↳ $\boxed{ds^2 = g_{ab}(x)\, dx^a dx^b}$ Coordinate dependent metric function

↳ Coordinates contain information about the local geometry but also depend on the particular Coordinate System.

Geometry types:

Riemannian : $ds^2 > 0$ for all dx^a

Pseudo-Riemannian: Otherwise $\rightarrow$ eg. Spacetime

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

$$g_{ab} = \mathrm{diag}(1, -1, -1, -1)$$

Minkowski: $x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$

The metric functions:

↳ Relate infinitesimal changes in coordinates to invariantly defined distance in manifold ie $\boxed{\text{in GR - Proper distances and times.}}$

Showing $g_{ab}(x)$ can always be chosen as Symmetric: $g_{ab}(x) = g_{ba}(x)$

↳ $g_{ab}(x) = \frac{1}{2}[g_{ab}(x) + g_{ba}(x)] + \frac{1}{2}[g_{ab}(x) - g_{ba}(x)]$

Contribution of antisymmetric part to ds^2 Vanishes:

$$(g_{ab} - g_{ba})\, dx^a dx^b = g_{ab}\, dx^a dx^b - g_{ab}\, dx^b dx^a = (\cancel{g_{ab}} - \cancel{g_{ab}})\, dx^a dx^b = 0.$$

To Conserve Interval Invariance

Change of coords $\xrightarrow[\text{accompanied by}]{\text{must be}}$ Change of metric func.

$$ds^2 = g_{ab}(x)\, dx^a\, dx^b$$

$$= \underbrace{g_{ab}(x) \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d}}\, dx'^c\, dx'^d$$

$$= g'_{cd}(x')\, dx'^c\, dx'^d$$

$$\boxed{\text{Thus } g'_{cd}(x') = g_{ab}(x(x')) \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d}}$$

Example 1: Cartesian to Polar Coordinates in $\mathbb{R}^2$

Cartesian

$$ds^2 = (dx)^2 + (dy)^2$$

$\longrightarrow g_{ab} = \mathrm{diag}(1,1)$

Polar coordinates $(x'^1 = \rho,\ x'^2 = \phi)$

Transform: $x^1 = \rho\cos\phi$, $x^2 = \rho\sin\phi$

Differentials: $dx^1 = \cos\phi\, d\rho - \rho\sin\phi\, d\phi$

$dx^2 = \sin\phi\, d\rho + \rho\cos\phi\, d\phi$

So what is g'_{ab}?

Method 1:

$$(dx^1)^2 + (dx^2)^2 = (\sin^2\phi + \cos^2\phi)\, d\rho^2 + \rho^2(\sin^2\phi + \cos^2\phi)(d\phi)^2$$

$$= \partial\rho^2 + \rho^2 d\phi^2$$

$g'_{11} = g_{\rho\rho} = 1$ $\qquad$ $g'_{22} = g_{\phi\phi} = \rho^2$

Method 2 (Matrix transformation)

$$g'_{11} = \frac{\partial x^a}{\partial x'^1} \cdot \frac{\partial x^b}{\partial x'^1} g_{ab} = \frac{\partial x^1}{\partial\rho} \frac{\partial x^1}{\partial\rho} g_{11} + \frac{\partial x^2}{\partial\rho}\frac{\partial x^2}{\partial\rho} g_{22}$$

$$= \cos^2\phi \times 1 + \sin^2\phi \times 1 = 1$$

$$g'_{22} = \frac{\partial x^a}{\partial x'^2} \cdot \frac{\partial x^b}{\partial x'^2} g_{ab} = \frac{\partial x^1}{\partial\phi} \frac{\partial x^1}{\partial\phi} g_{11} + \frac{\partial x^2}{\partial\phi}\frac{\partial x^2}{\partial\phi} g_{22}$$

$$= \rho^2 \times \sin^2\phi \times 1 + \rho^2 \times \cos^2\phi \times 1 = \rho^2$$

Example 2: Sph. Polars in $\mathbb{R}^3$

Cartesian Coords: $ds^2 = dx^2 + dy^2 + dz^2$ $\qquad g_{ab} = \mathrm{diag}(1,1,1)$

$\downarrow$

Sph polar Coords $\quad x'^1 = r,\ x'^2 = \theta,\ x'^3 = \phi$

$x^1 = r\sin\theta\cos\phi,\ x^2 = r\sin\theta\sin\phi,\ x^3 = r\cos\theta$

$\downarrow$

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad g'_{ab} = (1, r^2, r^2\sin^2\theta)$$

Intrinsic VS Extrinsic Geometry $\longrightarrow$ GR - a theory of intrinsic geometry.

Intrinsic Geometry

↳ Independent of embedding in some higher dimensional space

ie ds^2-interval: Characterised by local geometry (curvature) only

} could by determined by bug confined by manifold

External Geometry (of Submanifold)

↳ Dependent on how it is embedded in higher dimension

} Could not be determined by bug requires external viewpoint.

Example: Induced metric of S^2

$x^2 + y^2 + z^2 = a^2$ - 2-Sphere embedded in $\mathbb{R}^3$

Constraint $z = \pm\sqrt{a^2 - x^2 - y^2}$

Taking derivative: $2x\,dx + 2y\,dy + 2z\,dz = 0 \longrightarrow dz = \dfrac{-(x\,dx + y\,dy)}{\sqrt{a^2 - x^2 - y^2}}$

Induced Line element:

$$ds^2 = dx^2 + dy^2 + dz^2 = dx^2 + dy^2 + \frac{(x\,dx + y\,dy)^2}{a^2 - (x^2+y^2)}$$

In Plane polars:

$x = \rho\cos\phi,\ y = \rho\sin\phi$

$$\left.\begin{aligned} dx &= \cos\phi\, d\rho - \rho\sin\phi\, d\phi \\ dy &= \sin\phi\, d\rho + \rho\cos\phi\, d\phi \end{aligned}\right\} \begin{aligned} dx^2 + dy^2 &= d\rho^2 + \rho^2 d\phi^2 \\ x\,dx + y\,dy &= \rho\, d\rho \end{aligned}$$

Overall: $$ds^2 = d\rho^2 + \rho^2 d\phi^2 + \frac{(\rho\, d\rho)^2}{a^2 - \rho^2}$$

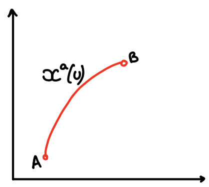
$$ds^2 = \frac{a^2}{a^2-\rho^2}\, d\rho^2 + \rho^2\, d\phi^2$$

Lengths and Volumes

Lengths along Curves:

Considering a curve $x^a(u)$ between points A and B on some manifold.

Path paramelised by u



Invariant Dist. between neighbouring points:

$$\boxed{ds^2 = g_{ab}(x)\, dx^a dx^b}$$

↳ einstein summation.

$$\boxed{ds = \left| g_{ab} \frac{dx^a}{du} \times \frac{dx^b}{du} \right|^{1/2} du}$$

Invariant Length along Curve

$$\boxed{L_{AB} = \int_{u_A}^{u_B} \left| g_{ab} \frac{dx^a}{du} \frac{dx^b}{du} \right|^{1/2} du}$$

Volumes of regions

Firstly: Considering diagonal metric

$g_{ab}(x) = 0$ for $a \neq b$.

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \dots + g_{NN}(dx^N)^2$$

Coordinate System is orthogonal:

tangents to coordinate curves are perpendicular

In 2D: Sides of invariant length $\sqrt{g_{11}}\, dx_1$, $\sqrt{g_{22}}\, dx_2$

Invariant Volume Element:

$$dV = \sqrt{|g_{11} g_{22}|}\, dx^1 dx^2$$

For ND manifold: (orthog coords)

$$\boxed{dV = \sqrt{|\underbrace{g_{11} \dots g_{NN}}|}\, dx^1 dx^2 \dots dx^N}$$

$\boxed{\det(g_{ab}) = g}$

Generalisation to Non-Orthog Coords

$$\boxed{dV = \sqrt{|g|}\, dx^1 \dots dx^N}$$

Proof of Invariant volume element:

$x^a \longrightarrow x'^a$ Diagonal $\longrightarrow$ Non-Diagonal.

$dx'^1 \dots dx'^N = J\, dx^1 dx^2 \dots dx^N$ where Jacobian, $J = \det\left(\frac{\partial x'^a}{\partial x^b}\right)$

Given metric transforms as:

$$g'_{ab} = \underbrace{\frac{\partial x^c}{\partial x'^a}}\ \underbrace{\frac{\partial x^d}{\partial x'^b}}_{\text{matrices.}}\, g_{cd}$$

$J = \det\left(\frac{\partial x'^a}{\partial x^b}\right)$ Then $-J^{-1} = \det\left(\frac{\partial x^a}{\partial x'^b}\right) = \frac{1}{J}$

Inverse transformation

Determinant Transform:

$$g' = \frac{g}{J^2}$$

Thus:

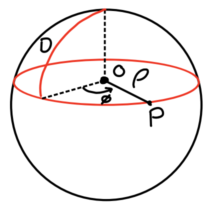
$$\underbrace{\sqrt{|g'|}}_{\sqrt{\frac{g}{J^2}}}\ \underbrace{dx'^1 \dots dx'^N}_{J\,dx^1 \dots dx^N} = \frac{\sqrt{|g|}}{J} J\, dx^1 \dots dx^N = \sqrt{|g|}\, dx^1 \dots dx^N \longrightarrow \text{QED Invariance}$$

Example: 2-Sphere in $\mathbb{R}^3$

Line Element (Shown previously)

$$\boxed{ds^2 = \frac{a^2\, d\rho^2}{(a^2-\rho^2)} + \rho^2\, d\phi^2}$$

↳ Diagonal with components: $\boxed{g_{11} = \frac{a^2}{a^2-\rho^2}}$ $\boxed{g_{22} = \rho^2}$



Taking the Circle drawn by $\rho = R$:

We will determine:

→ Length → Distance from Centre to Perimeter → Area enclosed.

Distance from Center O to perimeter along Curve ϕ = Const

$$\boxed{D = \int_0^R \frac{a}{(a^2-\rho^2)^{1/2}}\, d\rho = a \sin^{-1}\left(\frac{R}{a}\right)} \longrightarrow R \text{ at } R \ll A \text{, thus recover euclidean Space.}$$

Circumfrance of Circle

$$\boxed{C = \int_0^{2\pi} R\, d\phi = 2\pi R}$$

Area enclosed (volume)

$$g_{11} g_{22} = \frac{a^2\rho^2}{a^2-\rho^2} \qquad dv = \frac{a}{a^2-\rho^2}\, \rho\, d\rho\, d\phi$$

$$\boxed{A = \int_0^{2\pi}\int_0^R \frac{a}{(a^2-\rho^2)^{1/2}}\, \rho\, d\rho\, d\phi = 2\pi a^2\left[1 - \left(1 - \frac{R^2}{a^2}\right)^{1/2}\right]} \xrightarrow{R = a\sin\left(\frac{D}{a}\right)} A = 2\pi a^2\left[1 - \cos\left(\frac{D}{a}\right)\right] \longrightarrow \pi D^2 \quad R \ll a.$$

Can actually use these as Coordinates (not degenerate for each hemisphere).

$$D = a \sin^{-1}\left(\frac{\rho}{a}\right), \quad \rho = a \sin\left(\frac{D}{a}\right)$$

$$ds^2 = \frac{a^2\, d\rho^2}{a^2-\rho^2} + \rho^2\, d\phi^2 \longrightarrow ds^2 = dD^2 + a^2 \sin^2\left(\frac{D}{a}\right) d\phi^2$$

Local Cartesian Coordinates (Reinmann Manifolds $ds^2 > 0$)

Generally: not possible to choose coordinate Sys such that line element is Euclidean at every point.

Consider: $g_{ab} \longrightarrow \frac{N(N+1)}{2}$ independent functions but only N func in Coord transforms.

However

↳ Can always adopt coordinates Such that — In the neighbourhood of Some point P. the line element takes Euclidean form:

ie $g_{ab}(P) = \underset{\text{Cartesian metric.}}{\delta_{ab}}$ and $\left.\frac{\partial g_{ab}}{\partial x^c}\right|_P = 0 \longrightarrow g_{ab}(x) = \delta_{ab} + \mathcal{O}(|x - x_P|^2)$

→ In General Relativity: (Spacetime)

↳ Corrosponds to Coordinates Corrospond to locally-Inertial (free falling) observers.

↳ Equivelence Principle

Proof:

We will show existance of 'local Cartesian Coordinates' by:

→ Coordinate transform $x^a \to x'^a$ has enough degrees of freedom to create metric of form $\boxed{g_{ab}(x) = \delta_{ab} + O(|x-x_P|^2)}$

→ i.e. forcing → $g_{ab}(P) = \delta_{ab}$ → $\left.\dfrac{\partial g_{ab}}{\partial x^c}\right|_P = 0$

For a Coordinate transform:

$$\boxed{g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} g_{cd}} \qquad \boxed{\frac{\partial g'_{ab}}{\partial x'^e} = \frac{\partial}{\partial x'^e}\left(\frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\right) g_{cd} + \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\frac{\partial x^f}{\partial x'^e}\frac{\partial g_{cd}}{\partial x^f}}$$

Condition 1: $g'_{ab}(P) = \delta_{ab}$ → Requires $\frac{N(N+1)}{2}$ Constraints $\left(g'_{ab}(P)\right)$

→ We have N^2 dof in transformation matrix $\left(\left.\dfrac{\partial x^a}{\partial x'^b}\right|_P\right)$

More than enough:

Leaves $\frac{N(N-1)}{2}$ unused: N=4 Spacetime (6 dof)

→ 3 boosts / 3 rotations dof of definitions.

Condition 2: $\dfrac{\partial g'_{ab}}{\partial x'^c} = 0$ at P → Requires $\frac{N^2(N+1)}{2}$ Constraints $\dfrac{\partial^2 x^a}{\partial x'^b \partial x'^c}$

→ We have $\frac{N^2(N+1)}{2}$ dof in transformation matrix $\left(\left.\dfrac{\partial g'_{ab}}{\partial x'^c}\right|_P\right)$

Exact Sufficient no.

Could we go 1 step further: (2nd deriv → 0)

$\dfrac{\partial^2 g'_{ab}}{\partial x'^c \partial x'^d} = 0$ $\quad \frac{N^2(N+1)^2}{4}$ equations but only $\frac{N^2(N+1)(N+2)}{6}$ DoF

→ $\frac{N^2(N^2-1)}{12}$ Independent DoF

Space Time (N=4)

→ $\frac{16 \times 15}{12} = 20$ Indep. DoF

→ It is these that define the Curvature of the manifold

→ Physical DoF → Gravity in GR.

Pseudo-Riemannian Manifolds

True riemannian manifolds (eg. $\mathbb{R}^N$)

$ds^2 = g_{ab}\, dx^a dx^b > 0 \quad \forall\, dx^a$

↳ +ve definite - all eigenvalues +ve.

Pseudo-Riemannian Manifolds (eg Spacetime)

$ds^2 = g_{ab}\, dx^a dx^b \;\; = -\text{ve} / \text{zero} / +\text{ve}$

↳ -ve eigenvalues possible.

Can always find Coordinates for point P s.t.

$g_{ab}(P) = \eta_{ab} = \mathrm{diag}(\pm 1, \pm 1, \dots, \pm 1)$

$\left.\dfrac{\partial g_{ab}}{\partial x^c}\right|_P = 0$

↳ Singature of manifold : # +ve − # −ve

eg Minkowski Space : $ds^2 = d(ct)^2 - dx^2 - dy^2 - dz^2$

Topology of Manifolds.

Global geometry/Topology

↳ An instrinsic but not local property of a manifold.

GR - a local theory

↳ Do not constrain global topology of Spacetime manifold.

An overarching goal:

↳ Spacetime in GR is defined by (non-trivial) Pseudo-Riemannian manifold.

Must Satisfy: Equivelence principle

ie laws of physics locally reduce to SR in local-inertial Coordinates. → If you zoom in enough get Special relativity

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Vectors and Tensor Algebra for Manifolds:

Scalar Fields

Assign a real number to a point P independent of the Coordinate System.

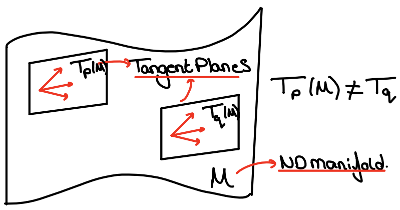
Coords x^a label P

↳ Field $\phi(x^a)$ of Coords ⟶ Independent of Coordinate System

⟶ $\phi'(x'^a) = \phi(x^a)$

Vector Fields and Tangent Spaces

↳ Can only define 'Local vectors' on general manifold.



$T_p(M) \neq T_q(M)$.

Tangent Vectors

↳ Tangent vectors to all curves in Surface lie in ND Subspace of embedding Space ⟶ Tangent Space, $T_p(M)$

$T_p(M)$: ND Vector Space at each point P whose elements are local vectors at P.

✗ Comparing Vectors (Can't add local vectors at different points)

↳ Tangent Spaces at different points are generally distinct.

Cannot compare vectors at different Points

⟶ (w/o Specifying additional Correction)

Vectors as differential Operators:

In Coordinate System x^a, Consider the differential operator at P:

$$\boxed{\vec{V} = V^a \frac{\partial}{\partial x^a}} \quad \text{for N real numbers } V^a.$$

Form ND vector Space at P with addition and multiplication by real numbers defined as:

Addition: $\vec{V} + \vec{W} = V_a \frac{\partial}{\partial x_a} + W_a \frac{\partial}{\partial x_a} = (V_a + W_a) \frac{\partial}{\partial x_a}$

Multiplication: $\lambda \vec{V} = (\lambda V^a) \frac{\partial}{\partial x^a}$

Effect under Change of Coordinates ($x^a \to x'^a$)

(V must not change but V^a can).

Coordinate Basis Vector Change:

$$\left.\frac{\partial}{\partial x'^a}\right|_p = \frac{\partial x^b}{\partial x'^a} \left.\frac{\partial}{\partial x^b}\right|_p$$

To allow $\vec{V}$ to be in Invariant:

$$\boxed{V'^a = \frac{\partial x'^a}{\partial x^b}\Big|_p V^b} \longrightarrow V'^a \frac{\partial}{\partial x'^a}\Big|_p = V^b \underbrace{\overbrace{\frac{\partial x'^a}{\partial x^b}\Big|_p \frac{\partial x^b}{\partial x'^a}}^{M\times M^{-1}=I}}_{\frac{\partial x^c}{\partial x^b} = \delta^c_b} \frac{\partial}{\partial x^b}\Big|_p = V^b \frac{\partial}{\partial x^b}\Big|_p = \vec{V}$$

Thus Invariant.

Dual Vector Fields:

↳ Gradient of Scalar field.

N-tuples associated with gradient of ϕ: $X_a = \frac{\partial \phi}{\partial x^a}$ → Downstairs Index: Not a Vector.

Behaviour Under Transformation: $(x^a \to x'^a)$

$$X'_a = \frac{\partial \phi'}{\partial x'^a} = \underbrace{\frac{\partial \phi}{\partial x^b}}_{X_b} \frac{\partial x^b}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a} X_b$$ → Not same relation

Dual Vector (Co-Vector) Transformation

$$\boxed{X'_a = \frac{\partial x^b}{\partial x'^a}\Big|_p X_b}$$ → Different Trans Rule to Vector.

$\boxed{\text{Define the Dual Vector Space } T_p^*(M)}$

Vector
$$V'^a = \frac{\partial x'^a}{\partial x^b}\Big|_p V^b$$
Dual Vector
$$X'_a = \frac{\partial x^b}{\partial x'^a}\Big|_p X_b$$

Contraction of a vector and dual vector $\boxed{X_a V^a}$ ⟶ Metric g_{ab} helps Vector ⟷ Dual Vector.

↳ A coordinate independent number.

$$X'_a V'^a = \underbrace{\frac{\partial x^b}{\partial x'^a} \frac{\partial x'^a}{\partial x^c}}_{\delta^b_c} X_b V^c = X_b V^b$$

$\boxed{\text{Formally: } T_p^*(M) \text{ consists of linear maps from } T_p(M) \text{ to numbers.}}$

Tensor Fields:

↳ Takes dual vectors and local vectors at point P → returns a number.

Geometric objects — multi linear maps from k copies of $T_p^*(M)$ and l copies of $T_p(M)$ ⟶ Real numbers.

Allow us to write down equations independent of any coordinate system.

Consider vector components U^a and V^a:

$$\boxed{T^{ab} = U^a V^b}$$

Behaviour under coordinate transformation:

$$T^{ab} \longrightarrow \left(\frac{\partial x'^a}{\partial x^c} U^c\right)\left(\frac{\partial x'^b}{\partial x^d} V^d\right) = \frac{\partial x'^a}{\partial x^c} \times \frac{\partial x'^b}{\partial x^d} U^c V^d$$

↳ Transforms as a Type-(2,0) tensor.

Generally: Type-(k,l) tensor:

Have k 'upstairs' indicies and l 'downstairs' indicies.

↳ Vector like Contravariant

↳ dual-vector-like covariant

Under transformation:

→ k factors Jacobian

→ l factors Inverse Jacobian.

T_{ab} → Type (0,2) tensor

$T^a{}_b$ → Type (1,1) tensor.

$$\boxed{T'^{a\ldots b}{}_{c\ldots d} = \underbrace{\frac{\partial x'^a}{\partial x^p} \cdots \frac{\partial x'^b}{\partial x^q}}\ \underbrace{\frac{\partial x^r}{\partial x'^c} \cdots \frac{\partial x^s}{\partial x'^d}} T^{p\ldots q}{}_{r\ldots s}}$$

Examples

Scalar fields $\longrightarrow$ Rank 0 tensors

Vectors $\longrightarrow$ Type $(1,0)$ tensors   Dual vectors $\longrightarrow$ Type $(0,1)$ tensors.

Tensor Fields

Assign a tensor of the same type to every point on $\mathcal{M}$.

Operations with tensors:

Addition and Multiplication by a Scalar

Addition

Tensors of the same type at a point P can be added to give a tensor of the same type.

$\underline{T}$, $\underline{S}$ ~ Tensors of type

$\underline{T} + \underline{S}$ with components $T_{ab} + S_{ab}$.

$$\boxed{T'_{ab} + S'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} T_{cd} + \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} S_{cd} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\left(T_{cd} + S_{cd}\right)}$$

Multiplication (by real number, c)

$c\underline{T}$ has components cT_{ab}

Outer / Tensor Product:

Considering two tensors: $\left.\begin{array}{l}\text{Type}(p,q): S^{a_1,\dots,a_p}{}_{b_1,\dots,b_q} \\ \text{Type}(r,s): T^{c_1,\dots,c_r}{}_{d_1,\dots,d_s}\end{array}\right\} \xrightarrow{\text{Outer / tensor product}} \boxed{S^{a_1,\dots,a_p}{}_{b_1,\dots,b_q} T^{c_1,\dots,c_r}{}_{d_1,\dots,d_s}} \sim \text{Type}(p+r, q+s).$

Generally not commutable: $\underline{T} \otimes \underline{S} \neq \underline{S} \otimes \underline{T}$

Contraction:

Operation: Type $(k,l) \longrightarrow$ Type $(k-1, l-1)$

Done by:

$\rightarrow$ Set 1 upstairs and downstairs index equal

Leads to implicit einstein notation.

eg Type ~ $(2,1)$ $T^{ab}{}_c \longrightarrow$ Type ~ $(1,0)$ $T^{ab}{}_b$ $\rightarrow$ Vector

Ensure $T^{ab}{}_b = S^a$ a Vector?

$$T'^{ab}{}_c = \frac{\partial x'^a}{\partial x^p}\frac{\partial x'^b}{\partial x^q}\frac{\partial x^r}{\partial x'^c} T^{pq}{}_r$$

$$S'^a = T'^{ab}{}_b = \frac{\partial x'^a}{\partial x^p}\underbrace{\frac{\partial x'^b}{\partial x^q}\frac{\partial x^r}{\partial x'^b}}_{\delta^r_q} T^{pq}{}_r = \frac{\partial x'^a}{\partial x^p} T^{pq}{}_r = \frac{\partial x'^a}{\partial x^p} S^p$$

Contraction of vector + dual no.

$\rightarrow$ more rigurously $V^a X_b$

Combining:

1) $\rightarrow$ Outer product $V^a X_b$

2) $\rightarrow$ Contraction $V^a X_a$

More generally

Inner Product: Outer + Contraction

Outer $T^{ab} S_{cd}$

Contraction $T^{ab} S_{bd}$ $\rightarrow$ Type $(1,1)$

Symmetrisation

For type ~ $(0,2)$ tensor:

Symmetric: $S_{ab} = S_{ba}$

Antisymmetric: $S_{ab} = -S_{ba}$

For type ~ $(2,0)$ tensor:

Symmetric: $T^{ab} = T^{ba}$

Antisymmetric: $T^{ab} = -T^{ba}$

Decomposition

$$S_{ab} = \underbrace{\tfrac{1}{2}(S_{ab}+S_{ba})}_{\text{Symmetric: } S_{(ab)}} + \underbrace{\tfrac{1}{2}(S_{ab}-S_{ab})}_{\text{AntiSym: } S_{[ba]}}$$

Coordinate independent.

Generalised for any number indices

$S_{(ab\ldots c)d}$, $S_{[ab\ldots c]d}$

Totally Symmetric: $S_{(ab\ldots c)}$

Totally AntiSymmetric: $S_{[ab\ldots c]}$

For general type ~ (0,k)

$S_{(a,b,\ldots c)} = \frac{1}{k!}$ (Sum over all permutations of indices)

$S_{[a,b,\ldots,c]} = \frac{1}{k!}$ (Alternating Sum over all permut a, bc)

Eg. S_{abc} $S_{[abc]} = \frac{1}{6}(S_{abc} - S_{acb} + S_{cab} - S_{cba} + S_{bca} - S_{bac})$

Even permutations +ve

odd permutations −ve

Quotient Theorem

A test if an object with indicies is a tensor.

→ Take the set of quantaties to be tested

→ Contract with arbitary tensor

→ See if outcome is another tensor → If so the original components were tensors.

Metric Tensor

$$\boxed{ds^2 = g_{ab}\,dx^a\,dx^b = g'_{ab}\,dx'^a\,dx'^b}$$

$g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} g_{cd}$ } g_{ab} are the components of a type-(0,2) tensor

The Metric Tensor

Takes in 2 vectors → Real no.

$T_p(M) \otimes T_p(M) \longrightarrow \mathbb{R}$

Coordinate Independent Scalar/Inner product.

$\boxed{g(\vec{u},\vec{v}) = g_{ab}\,u^a v^b}$ Contraction of metric tensor with two Vectors

Lower Indicies (using metric tensor)

Given a vector V^a contracted with metric tensor → dual vector $g_{ab}V^b$

Coordinate independent way of associating dual vector with vector (Mapping with metric of manifold)

Convention

Written with Same Kernel letter:

$\boxed{V_a = g_{ab}V^b}$ (V_a: Dual vector, V^b: vector)

$\boxed{V_a, V^b}$ → Mathematical - Distinct

Physically - Same object.

More general for all tensors:

$T_{ab} = g_{ac}T^c{}_b$ (Type (0,2), Type (1,1)) $\xrightarrow{\text{Chain together}}$ $g_{ap}\,g_{bq}\,T^{pq}$

Inverse Metric (Raising Indices)

By Convention: $\boxed{g^{ab} = (g^{-1})^{ab}}$ $g^{ab} g_{bc} = \delta^a{}_c$

Now: associates vectors with any dual vector x_a

$$X^a \equiv g^{ab} x_b$$

Lowering - Raising Loops

$$g_{ca} X^a = \underbrace{g_{ca} g^{ab}}_{\delta^b{}_c} x_b = \delta^b_c X_b = X_b$$

Mixed Components of the Metric:

Raising one index on g_{ab} gives mixed components of the metric:

Same as metric contracted with inverse metric.

$$\boxed{g^a{}_c \equiv g^{ab} g_{bc} = \delta^a{}_c}$$

Key Conclusion: Components of type (1,1) tensor. ≈ Kronecker delta in any coord system

$g^a{}_b$ - is always the identity: Shown with coord transformation:

$$g'^a{}_b = \underbrace{\frac{\partial x'^a}{\partial x^c}}_{J} \underbrace{\frac{\partial x^d}{\partial x'^b}}_{J^{-1}} g^c{}_d = \frac{\partial x'^a}{\partial x^c} \frac{\partial x^d}{\partial x'^b} \delta^c{}_d = \frac{\partial x'^a}{\partial x'^b} = \delta^a{}_b$$

Key Point: Geometrically a Linear Map

↳ $g^a{}_b$ is the identity map : $T_p(M) \rightarrow T_p(M)$

Scalar/Inner Product of Vectors

Recalling $\boxed{g(\vec{U},\vec{V}) = g_{ab} U^a V^b}$ equivelent to inner product.

Invariant length of a vector:

$\boxed{|\vec{V}| = \sqrt{|g_{ab} V^a V^b|}}$ → modulus for pseudo-Riemannian.

Lots of ways of writing the Inner Product:

↳ $g_{ab} U^a V^b = g^{ab} U_a V_b = U_a V^a = U^a V_a$

Orthogonal Vectors:

$\boxed{g(\vec{U},\vec{V}) = 0}$ ie $g_{ab} U^a V^b$

Vector and Tensor Calculus:

↳ Difficulties : Tangent spaces on manifold are different - need additional structure (connections) to link them to compare tensors.

How to connect tangent spaces at different points?

Covariant Derivatives:

Scalar fields:

Gradient of a scalar field → dual vectors $\vec{\nabla}\phi$ with components: by defn dual vector.

$\boxed{\frac{\partial \phi}{\partial x^a} \text{ (gradient of scalar field)}}$

↳ Can by written as a vector: $g^{ab} \frac{\partial \phi}{\partial x^a}$ however dual vector more natural description.

Contraction with infitesimal displacement vector:

$$\delta\phi = dx^a \frac{\partial \phi}{\partial x^a}$$ maps from vectors to real numbers.

Covariant Derivatives with tensor fields:

↳ Want derivatives that preserve the tensorial nature of the object being differentiated.

For a general coordinate transform:

$$\frac{\partial V'^b}{\partial x'^a} = \frac{\partial}{\partial x'^a}\left(\frac{\partial x'^b}{\partial x^c} V^c\right) = \frac{\partial x^d}{\partial x'^a}\frac{\partial}{\partial x^d}\left(\frac{\partial x'^b}{\partial x^c} V^c\right)$$

$$= \underbrace{\frac{\partial x^d}{\partial x'^a}\frac{\partial x'^b}{\partial x^c}\frac{\partial V^c}{\partial x^d}}_{\text{Usual tensor transform for type (1,1) tensor}} + \underbrace{\frac{\partial x^d}{\partial x'^a}\frac{\partial^2 x'^b}{\partial x^d \partial x^c} V^c}_{\text{extra.}}$$

overall doesn't transform as a tensor.

Solution: Covariant Derivative.

↳ In $\mathbb{R}^N$, define a tensor value derivative $\underline{T}$ of a vector by specifying its components to be $\frac{\partial V^b}{\partial x^a}$ in some Cartesian coordinates.

Needed for tensor valued – Plays another

$$T'^{\,b}_{a} = \frac{\partial x^d}{\partial x'^a}\frac{\partial x'^b}{\partial x^c}\frac{\partial V^c}{\partial x^d} = \underbrace{\frac{\partial V'^b}{\partial x'^a} - \frac{\partial x^d}{\partial x'^a}\frac{\partial^2}{\partial x^d \partial x^c}\frac{\partial x^c}{\partial x'^e}V'^e}_{\text{From above (+ extra transform } V^c \to V'^e)}$$

$$\frac{\partial}{\partial x^c}\left(\frac{\partial x'^b}{\partial x^d}\frac{\partial x^d}{\partial x'^a}\right) = \frac{\partial}{\partial x^c}(\delta^b{}_a) = 0$$

$$\frac{\partial x^d}{\partial x'^a}\frac{\partial^2 x'^b}{\partial x^c \partial x^d} + \underbrace{\frac{\partial x'^b}{\partial x^d}\frac{\partial x'^e}{\partial x^c}\frac{\partial^2 x^d}{\partial x'^a \partial x'^e}}_{\text{Sub this part in.}} = 0.$$

By Substitution:

$$T'^a{}_b = \frac{\partial V'^b}{\partial x'^a} + \underbrace{\frac{\partial^2 x^d}{\partial x'^a \partial x'^e}\frac{\partial x'^b}{\partial x^d}}_{\text{Connection: } \Gamma'^b_{ae}} V'^e$$

More on the Connection:

Since coordinate basis vectors (e.g. $\vec{e}_a = \frac{\partial}{\partial x^a}$) in cartesian ($x^a$) and general ($x'^a$) coords

are related by: $\boxed{\vec{e}_a = \frac{\partial x^b}{\partial x'} \vec{e}_b}$

The connection Γ'^a_{bc} arises from the variation of $\vec{e}'_a$ relative to $\vec{e}_a$

$$\boxed{\Gamma'^a_{bc} = \frac{\partial^2 x^d}{\partial x'^b \partial x'^c}\frac{\partial x'^a}{\partial x^d}}$$

⟶ Symmetric in lower indices (partial derivatives commute).

Since metric in Cartesian Coordinates $g_{ab} = \delta_{ab}$:

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\delta_{cd} \qquad g'^{ab} = \frac{\partial x'^a}{\partial x^c}\frac{\partial x'^b}{\partial x^d}\delta^{cd}$$

δ_{cd} ↳ g_{cd} $\qquad$ δ^{cd} ↳ g^{cd}

Thus (not full proof):

$$\boxed{\Gamma'^a_{bc} = \frac{1}{2} g'^{ad}\left(\frac{\partial g'_{cd}}{\partial x'^b} + \frac{\partial g'_{bd}}{\partial x'^c} - \frac{\partial g'_{bc}}{\partial x'^d}\right)}$$

⟶ Removed any reference of Cartesian coords.

ie If manifold has a metric ⟶ Can determine Connection

A more formal approach:

The basics:

Covariant Derivative

Type-(k,l) tensor : $T^{a_1 \dots a_k}{}_{b_1 \dots b_l} \longrightarrow$ Type $(k, l+1)$ $\nabla_c T^{a_1, \dots, a_k}{}_{b_1, \dots, b_l}$

↳ An extra downstairs index.

Properties:

1) Reduces to gradient on Scalar fields: $\nabla_a \phi = \dfrac{\partial \phi}{\partial x_a}$

2) Linearity: $\nabla_c \left(\alpha T^{a_1,\dots,a_k}{}_{b_1 \dots b_l} + \beta S^{a_1,\dots,a_k}{}_{b_1 \dots b_l}\right) = \alpha \nabla_c T^{a_1,\dots,a_k}{}_{b_1 \dots b_l} + \beta \nabla_c S^{a_1,\dots,a_k}{}_{b_1 \dots b_l}$

3) Leibnitz Product Rule: $\nabla_j \left(T^{a_1 \dots a_k}{}_{b_1 \dots b_l} S^{c_1 \dots c_m}{}_{d_1 \dots d_n}\right) = \left(\nabla_j T^{a_1 \dots a_k}{}_{b_1 \dots b_l}\right) S^{c_1 \dots c_m}{}_{d_1 \dots d_n} + T^{a_1 \dots a_k}{}_{b_1 \dots b_l} \left(\nabla_j S^{c_1 \dots c_m}{}_{d_1 \dots d_n}\right)$

The Connection:

$$\nabla_a V^b = \frac{\partial V^b}{\partial x^a} + \Gamma^b{}_{ac} V^c$$

→ Covariant Derivative wrt a of vector, V^b

→ Dummy Index c Summed over.

↳ The connection is not a tensor.

Must investigate how Γ transforms to ensure $\nabla'_a V'^b$ transforms as tensor:

$$\boxed{\nabla'_a V'^b = \frac{\partial x^d}{\partial x'^a} \frac{\partial x'^b}{\partial x^c} \nabla_d V^c}$$

Starting from: Covariate derivative in primed frame

$$\nabla'_a V'^b = \frac{\partial V'^b}{\partial x'^a} + \Gamma'^b{}_{ac} V'^c$$

See above

$$= \frac{\partial x^d}{\partial x'^a} \frac{\partial x'^b}{\partial x^c} \frac{\partial V^c}{\partial x^d} + \frac{\partial x^d}{\partial x'^a} \frac{\partial^2 x'^b}{\partial x^d \partial x^c} V^c + \Gamma'^b{}_{ac} \frac{\partial x'^c}{\partial x^d} V^d$$

$$= \frac{\partial x^d}{\partial x'^a} \frac{\partial x'^b}{\partial x^c} \nabla_d V^c - \frac{\partial x^d}{\partial x'^a} \frac{\partial x'^b}{\partial x^c} \Gamma^c_{de} V^e + \frac{\partial x^d}{\partial x'^a} \frac{\partial^2 x'^b}{\partial x^d \partial x^c} V^c + \Gamma'^b{}_{ac} \frac{\partial x'^c}{\partial x^d} V^d$$

Therefore to satisfy transformation above:

→ Note: dummy indicies rewritten.

$$\boxed{\Gamma'^a{}_{bc} = \underbrace{\frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \Gamma^d_{ef}}_{\text{Usual tensor transform law}} - \underbrace{\frac{\partial x^d}{\partial x'^b} \frac{\partial x^e}{\partial x'^c} \frac{\partial^2 x'^a}{\partial x^d \partial x^e}}_{\text{Inhomogenous part (Symmetric in b and c)}}}$$

→ It is this part which means Connection Coefficients do not transform as tensor components.

↳ The Connection is unique up to type-(1,2) tensor.

Extension to other tensor fields:

Type (2,0) tensor: $\boxed{T^{ab} = U^a V^b}$ tensor product of vectors.

Using Leibniz rule:

$$\nabla_a (U^b V^c) = (\nabla_a U^b) V^c + U^b (\nabla_a V^c)$$

$$= \left(\frac{\partial U^b}{\partial x^a} + \Gamma^b_{ad} U^d\right) V^c + U^b \left(\frac{\partial V^c}{\partial x^a} + \Gamma^c_{ad} V^d\right)$$

$$= \frac{\partial}{\partial x^a}(U^b V^c) + \Gamma^b_{ad} U^d V^c + \Gamma^c_{ad} U^b V^d$$

Generally by linearity:

Type (2,0) tensor: $$\boxed{\nabla_a T^{bc} = \frac{\partial T^{bc}}{\partial x^a} + \Gamma^b_{ad} T^{dc} + \Gamma^c_{ad} T^{bd}}$$

Application to dual vector fields

Extra Property

4) Covariant derivative Commutes with Contraction

$$\nabla_a (X_b V^c) = (\nabla_a X_b) V^c + X_b (\nabla_a V^c) \qquad \text{(Leibnitz)}$$

$\downarrow$ contract on b and c

$$\nabla_a (\underbrace{X_b V^b}_{\text{becomes scalar}}) = (\nabla_a X_b) V^b + X_b (\nabla_a V^b)$$

$\downarrow$ Simple partial derivitive

$$\frac{\partial (X_b V^b)}{\partial x^a} = \frac{\partial X_b}{\partial x^a} V^b + X_b \frac{\partial V^b}{\partial x^a} = (\nabla_a X_b) V^b + \underbrace{X_b (\nabla_a V^b)}_{\frac{\partial V^b}{\partial x^a} + \Gamma^b_{ac} V^c}$$

$$\frac{\partial X_b}{\partial x^a} \cancel{V^b} + X_b \cancel{\frac{\partial V^b}{\partial x^a}} = (\nabla_a X_b) \cancel{V^b} + X_b \cancel{\frac{\partial V^b}{\partial x^a}} + X_c \Gamma^c_{ac} \cancel{V^b}$$

$$\boxed{\boxed{\nabla_a X_b = \frac{\partial X_b}{\partial x^a} - \Gamma^c_{ab} X_c}} \rightarrow \text{Covariant derivative of dual vector}$$

Extensions $\boxed{\text{Shorthand } \partial_c = \partial / \partial x^c}$

$$\boxed{\begin{aligned} \nabla_c T_{ab} &= \partial_c T_{ab} - \Gamma^d_{ca} T_{db} - \Gamma^d_{cb} T_{ad} \\ \nabla_c T^a{}_b &= \partial_c T^a{}_b + \Gamma^a_{cd} T^d{}_b - \Gamma^d_{cb} T^a{}_d \end{aligned}}$$

Implication for mixed components of metric:

$\hookrightarrow$ $\nabla_c g^a{}_b = \partial_c \delta^a{}_b + \Gamma^a_{cd} \delta^d_b + \Gamma^d_{cb} \delta^a_d = 0$

The metric connection

For a manifold equipped with a metric $\longrightarrow$ Unique connection if demand:

5) Metric Compatibility:

$\boxed{\nabla_c g_{ab} = 0} \longrightarrow \boxed{\nabla_a g^{bc} = 0}$ *

$\hookrightarrow$ can show inverse metric also vanishes

$$0 = \nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d_{ca} g_{db} - \Gamma^d_{cb} g_{ad}$$

Other cyclic permutatons

$$0 = \partial_b g_{ca} - \Gamma^d_{bc} g_{da} - \Gamma^d_{ba} g_{cd}$$

$$0 = \partial_a g_{bc} - \Gamma^d_{ab} g_{dc} - \Gamma^d_{ac} g_{bd}$$

$$2 \Gamma^d_{ca} g_{db} = \partial_c g_{ab} + \partial_a g_{bc} - \partial_b g_{ca}$$

6) Commutative action on Scalar fields

$$\boxed{\nabla_a \nabla_b \phi = \nabla_b \nabla_a \phi}$$

$\hookrightarrow$ Symmetry in lower indices: $\boxed{\Gamma^c_{ab} = \Gamma^c_{ba}}$

Since: $$\nabla_a (\nabla_b \phi) = \partial_a (\nabla_b \phi) - \Gamma^c_{ab} \nabla_c \phi$$
$$= \underbrace{\partial_a \partial_b \phi}_{\text{Symmetric}} - \underbrace{\Gamma^c_{ab} \partial_c \phi}_{\text{Must by Sym.}}$$

We recover: $$\boxed{\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{db} + \partial_d g_{bc})}$$ Metric Connection

$$* \; g_{ab} g^{bc} = \delta^c_a$$
$$\nabla_d (g_{ab} g^{bc}) = 0$$
$$(\underbrace{\cancel{\nabla_d g_{ab}}}_{0}) g^{bc} + g_{ab} (\nabla_d g^{bc}) = 0$$
$$\therefore \text{ if } \nabla_d g_{ab} = 0 \longrightarrow \nabla_d g^{bc} = 0$$

Other Useful Properties of metric connection:

1) Since $\nabla_c g_{ab}=0$ Interchange order raising lowering Incides

$$\nabla_c T^{ab} = \nabla_c (g^{bd} T^a{}_d)$$
$$= (\underbrace{\nabla_c g^{bd}}_{0}) T^a{}_d + g^{bd} \nabla_c T^a{}_d = g^{bd} \nabla_c T^a{}_d$$

Relation to local Cartesian Coordinate:

Covariance derivative with metric connection $\longrightarrow$ partial differentiation in local cartesian coordinates.

@ any P, can find local CC such that

$$g_{ab}(P) = \mathrm{diag}(\pm 1, \dots, \pm 1) \quad \partial_c g_{ab}|_P = 0 \longrightarrow \Gamma^a_{bc}(P) = 0.$$

$$\boxed{\nabla_a V^b \longrightarrow \partial_a V^b}$$

Cov. derivative of tensor in local C.C. $\xrightarrow{\text{reduces}}$ partial derivatives.

Important Equations:

Vector Derivatives: $\nabla_a V^b = \partial_a V^b + \Gamma^b_{ac} V^c$

Dual Vector Derivatives: $\nabla_a X_b = \partial_a X_b - \Gamma^c_{ab} X_c$

Metric Connection: $\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})$

Example: Unit 2-Sphere in $\mathbb{R}^3$

line element: Coordinates θ, ϕ

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \longrightarrow g_{ab} = \mathrm{diag}(1, \sin^2\theta) \longrightarrow g_{\theta\theta} = 1 \quad g_{\phi\phi} = \sin^2\theta.$$

Computing Connection Coefficients:

For $a = \theta$:

$$2\Gamma^\theta_{bc} = g^{\theta\theta} (\partial_b g_{\theta c} + \partial_c g_{\theta b} + \partial_\theta g_{bc}) + \cancel{g^{\theta\phi}}^{\,0} (\dots)$$

$\downarrow g_{ab} = \mathrm{diag}(1, \sin^2\theta)$

$$2\Gamma^\theta_{bc} = 1 \; (\partial_b \delta_{\theta c} + \partial_c \delta_{\theta b} - \partial_\theta g_{bc})$$

$$\Gamma^\theta_{\phi\phi} = -\frac{1}{2} \partial_\theta \sin^2\theta = -\sin\theta\cos\theta.$$

For $a = \phi$:

no metric functions of ϕ

$$2\Gamma^\phi_{bc} = g^{\phi\phi} (\partial_b g_{\phi c} + \partial_c g_{\phi b} + \cancel{\partial_\phi g_{bc}}) + \cancel{g^{\phi\theta}} (\dots)$$

$$= \frac{1}{\sin^2\theta} \Big(\underbrace{\delta_{\phi c} \partial_b \sin^2\theta + \delta_{\phi b} \partial_c \sin^2\theta}_{\text{only } g_{\phi\phi} \text{ components.}} \Big)$$

$\Gamma^\phi_{\phi\phi} = 0, \; \Gamma^\phi_{\theta\theta} = 0.$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{1}{2} \frac{1}{\sin^2\theta} \partial_\theta \sin^2\theta = \frac{1}{2} \frac{1}{\sin^2\theta} 2\sin\theta\cos\theta = \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

Non Zero Connection Terms:

$\boxed{\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta}$ $\boxed{\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta.}$ $\boxed{\Gamma^{\theta}_{\theta\theta}=0 \quad \Gamma^{\theta}_{\theta\phi}=0 \quad \Gamma^{\phi}_{\phi\phi}=0 \quad \Gamma^{\phi}_{\theta\theta}=0}$

For a vector $\vec{V}$ with Coordinate Components V^θ and V^ϕ:

$$\boxed{\nabla_a V^b = \partial_a V^b + \Gamma^b_{ac} V^c}$$

Sum over the c

Drawn out for example.

$$\nabla_\theta V^\theta = \partial_\theta V^\theta + \Gamma^\theta_{\theta c} V^c = \partial_\theta V^\theta + \cancel{\Gamma^\theta_{\theta\theta}} V^\theta + \cancel{\Gamma^\theta_{\theta\phi}} V^\phi = \partial_\theta V^\theta$$

$$\nabla_\theta V^\phi = \partial_\theta V^\phi + \Gamma^\phi_{\theta c} V^c = \partial_\theta V^\phi + \cot\theta\, V^\phi$$

$$\nabla_\phi V^\theta = \partial_\phi V^\theta + \Gamma^\theta_{\phi c} V^c = \partial_\phi V^\theta - \sin\theta\cos V^\phi$$

$$\nabla_\phi V^\phi = \partial_\phi V^\phi + \Gamma^\phi_{\phi c} V^c = \partial_\phi V^\phi + \cot\theta\, V^\theta$$

Divergence, Curl and Laplacian

Divergence of a Vector V^a

Scalar Field: $\boxed{\operatorname{div}(\vec{V}) \equiv \nabla_a V^a}$

Curl of a dual-Vector field X_a

antisymmetrised Covariant derivative: $\boxed{\operatorname{Curl}(X)_{ab} = \nabla_a X_b - \nabla_b X_a}$ (type-(0,2) tensor).

Symmetry of Connection: equal by Symm

$$\nabla_a X_b - \nabla_b X_a = \partial_a X_b - \Gamma^c_{ab} X_c - (\partial_b X_a - \Gamma^c_{ba} X_c)$$

$$= \underbrace{\partial_a X_b - \partial_b X_a}$$

Anti Symmetrised usual Coordinate derivatives

Laplacian

Contracting two covariant derivatives.

1) Scalar: $\nabla^2\phi = \nabla^a\nabla_a\phi = g^{ab}\nabla_a\nabla_b\phi$

2) Tensor: $\nabla^2 T^{ab} = \nabla^c\nabla_c T^{ab} = g^{cd}\nabla_c\nabla_d T^{ab}$

Returning to example: 2-Sphere in $\mathbb{R}^3$

Recall:

$$g_{ab} = \operatorname{diag}(1, \sin^2\theta)$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta.$$

Gradient of Scalar Field, ψ

Cov deriv of Scalar

$$\nabla^a\psi = g^{ab}\nabla_b\psi = g^{ab}\partial_b\psi$$

$$= (\underbrace{g^{\theta\theta}}_{1}\partial_\theta\psi, \underbrace{g^{\phi\phi}}_{1/\sin^2\theta}\partial_\phi\psi) = (\underbrace{\partial_\theta\psi}_{\nabla^\theta\psi}, \underbrace{\frac{1}{\sin^2\theta}\partial_\phi\psi}_{\nabla^\phi\psi})$$

Divergence of a vector field, V^a

$$\operatorname{Div}(\vec{V}) = \nabla_a V^a = \partial_a V^a + \Gamma^a_{ac} V^c$$

$$= \partial_\theta V^\theta + \partial_\phi V^\phi + \underbrace{\Gamma^\phi_{\phi\theta}}_{\cot\theta} V^\theta$$

Laplacian of ψ

$$\nabla^2\psi = \nabla_a\nabla^a\psi = \partial_\theta(\underbrace{\nabla^\theta\psi}_{V^\theta}) + \partial_\phi(\underbrace{\nabla^\phi\psi}_{V^\phi}) + \cot\theta(\underbrace{\nabla^\theta\psi}_{V^\theta})$$

$$= \partial_\theta\partial_\theta\psi + \partial_\phi\left(\frac{1}{\sin^2\theta}\partial_\phi\psi\right) + \cot\theta\,\partial_\theta\psi$$

$$= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Intrinsic Derivatives Along Curves:

↳ Considering a vector $V^a(u)$ defined along a curve $x^a(u)$

↳ example parameter: proper time etc.

Intrinsic Derivative of $V^a(u)$ along the curve is the vector:

$$\boxed{\frac{DV^a}{Du} \equiv \underbrace{\frac{dx^b}{du}}_{\text{Tangent vector}} \underbrace{\nabla_b V^a}_{\text{Covariant derivative}} = \frac{dx^b}{du}\left(\frac{\partial V^a}{\partial x^b} + \Gamma^a_{bc}V^c\right)}$$

→ Contraction

→ only requires V^a to be defined along the curve:

(eg property of particle along world line).

Since:

$$\frac{dx^b}{du}\frac{\partial V^a}{\partial x^b} = \frac{dV^a}{du} \rightarrow \boxed{\frac{DV^a}{Du} = \frac{dV^a}{du} + \frac{dx^b}{du}\Gamma^a_{bc}V^c}$$

↳ form the components of a vector.

Intrinsic Derivative to general tensors:

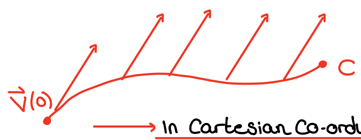
$$\frac{DT^a{}_b}{Du} \equiv \frac{dx^c}{du}\nabla_c T^a{}_b$$

$$= \frac{dT^a{}_b}{du} + \frac{dx^c}{du}\left(\Gamma^a_{cd}T^d{}_b - \Gamma^d_{cb}T^a{}_d\right)$$

Parallel Transport:

Goal: Generalise idea of keeping vector constant along curve.

Considering curve C defined in 2D Euclidean space in Cartesian coordinates by $x^a(u)$.



→ In Cartesian Co-ordinates:

↳ $\frac{dV^a}{du} = 0$

Writing this as a tensor eqn:

Parallel Transport defined on a general manifold:

$$\boxed{\frac{DV^a}{Du} = 0}$$

Generalised to other tensors. → $\boxed{\frac{DT^{ab}}{Du} = 0}$

Properties of Parallel Transport:

$$\boxed{\frac{DV^a}{Du} = \frac{dV^a}{du} + \frac{dx^b}{du}\Gamma^a_{bc}V^c}$$

Property 1:

$\frac{DV^a}{Du} = 0$ is an ODE for the components V^a → Unique solution if V^a is specified at some unique initial value.

Property 2:

Result of parallel transport from A to B independent of parametisation of curve:

→ Infitesimal Step: $\delta V^a + \delta u \frac{dx^b}{du}\Gamma^a_{bc}V^c = 0.$

$$\delta V^a = -\delta x^b\,\Gamma^a_{bc}V^c$$

↳ only change in coords – not dependent on parametisation of curve.

Property 3:

Length of vector preserved (+ Scalar product of two vectors preserved) under parallel transport.

$$\frac{d|v|^2}{du} = \frac{D}{Du}\Big(\underbrace{g_{ab}v^a v^b}_{\substack{\text{Scalar product} \\ |\vec{v}|^2}}\Big) = 2 g_{ab} v^a \frac{Dv^b}{Du} = 0.$$

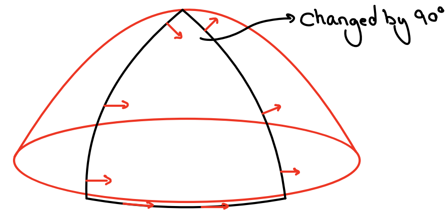
Property 4:

↳ For an infinitesimal step in local cartesian coordinates:

$$\boxed{\delta v^a = -\delta x^b \underbrace{\Gamma^a_{bc}}_{\text{Vanishes in Cartesian}} v^c = 0}$$

Property 5:

↳ Parallel transport IS path dependent between two points A and B as cannot find global cartesian coordinates.



Hence → Parallel transport around a closed loop, - result differs generally from original

$\boxed{\text{Path dependence: Measure of intrinsic Curvature}}$

Geodesic Curves

↳ The generalisation of straightlines in Euclidean Space. But... for a manifold (Curved space).

Path of free-falling particles in general relavistic manifolds

Two Definitions:

Geodisic Curve:

Curve of extremal distance between two points

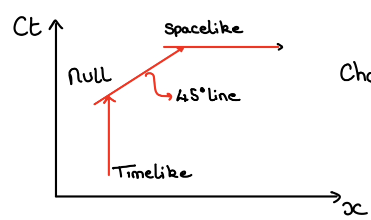
Auto-Parrallel Curve:

Curve $x^a(u)$ that parrallel transports tangent vector $t^a = \frac{dx^a}{du}$ in some suitable parametisation.

Tangent Vectors to Curves

Tangent vector to $x^a(u)$: $\boxed{t^a = \frac{dx^a}{du}}$ → Depends on choice of parametisation but parrallel for all choices.

$$\boxed{\begin{array}{l} \text{For any point along a curve in Spacetime:} \\ \hookrightarrow \text{The square of the tangent vector has character:} \\ \quad g_{ab}t^a t^b > 0 \ : \ \text{timelike} \\ \quad g_{ab}t^a t^b = 0 \ : \ \text{null / lightlike} \\ \quad g_{ab}t^a t^b < 0 \ : \ \text{Spacelike} \end{array}}$$



Character of t^a can change along curve

For a non-null curve:

'Length of tangent vector': $\frac{ds}{du}$

$$|\vec{t}| = |g_{ab} t^a t^b|^{1/2} = \left|g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}\right|^{1/2} = \frac{ds}{du}$$

Stationarity Properties of Non-null geodesics

↳ Invariant distance along the Curve $x^a(u)$

From $A(u=0)$ to $B(u=1)$:

↳ $$\boxed{\mathcal{L} = \int_A^B ds = \int_A^B \frac{ds}{du}\, du = \int_0^1 |g_{ab}\,\dot{x}^a \dot{x}^b|^{1/2}\, du}$$

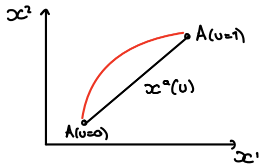
$\boxed{\dot{x}^a = dx^a/du}$

$\boxed{\text{Notation}: \frac{ds}{du} \equiv F}$

Extremal Curve: $\mathcal{L}$ - unchanged to first order for arbitary changes in the Path

$x^a(u) \longrightarrow x^a(u) + \delta x^a(u)$

↳ Fixed Endpoints: $\delta x^a(0) = 0 = \delta x^a(1)$



Extremal Curves Satisfy Euler Legrange Equations.

↳ $$\boxed{\frac{\partial F}{\partial x^a} = \frac{d}{du}\left(\frac{\partial F}{\partial \dot{x}^a}\right)}$$

$$F = \frac{ds}{du} = |g_{ab}\,\dot{x}^a \dot{x}^b|^{1/2} = \left|g_{ab}\,\frac{dx^a}{du}\frac{dx^b}{du}\right|^{1/2} \quad \begin{cases} + : \text{timelike} \\ - : \text{Spacelike} \end{cases}$$

Derivatives

$$\boxed{1)\ \frac{\partial F}{\partial x^a} = \pm \frac{1}{2F}(\partial_a g_{bc})\,\dot{x}^b \dot{x}^c} \quad \text{and} \quad \boxed{2)\ \frac{\partial F}{\partial \dot{x}^a} = \pm \frac{1}{F}\, g_{ab}\,\dot{x}^b}$$

Euler Legrange Equations Become:

$$\frac{1}{2F}(\partial_a g_{bc})\,\dot{x}^b \dot{x}^c = \frac{d}{du}\left(\frac{1}{F}\, g_{ab}\,\dot{x}^b\right)$$

$$= -\frac{1}{F^2}\frac{dF}{du}\, g_{ab}\,\dot{x}^b + \frac{1}{F}(\partial_c g_{ab})\,\dot{x}^b \dot{x}^c + \frac{1}{F}\, g_{ab}\,\ddot{x}^b$$

↳ By the chain rule: $\frac{dg_{ab}}{du} = \frac{\partial g_{ab}}{\partial x^c}\frac{dx^c}{du} = \partial_c\, g_{ab}\,\dot{x}^c$

$$\frac{1}{F^2}\frac{dF}{du}\, g_{ab}\,\dot{x}^b = \frac{1}{F}\, g_{ab}\,\ddot{x}^b + \frac{1}{F}(\partial_c g_{ab})\,\dot{x}^b \dot{x}^c - \frac{1}{2F}(\partial_a g_{bc})\,\dot{x}^b \dot{x}^c$$

Symmetry wrt bc: $\boxed{\frac{1}{2}(\partial_c g_{ab} + \partial_b g_{ac})}$

$\times F$

$$\frac{1}{F}\frac{dF}{du}\, g_{ab}\,\dot{x}^b = g_{ab}\,\ddot{x}^b + \underbrace{\frac{1}{2}(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc})}_{\boxed{g_{ad}\,\Gamma^d{}_{bc}}}\,\dot{x}^b \dot{x}^c$$

↳ By defn metric connection. $\boxed{\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})}$

Relabeling Dummy Indicies:

$\dot{S} = F = ds/du$

↳ $$\boxed{\boxed{\ddot{x}^a + \Gamma^a{}_{bc}\,\dot{x}^b \dot{x}^c = \left(\frac{\ddot{S}}{\dot{S}}\right)\dot{x}^a}}$$ Non-Null Geodesic

↳ Expressed in terms of tangent vectors: $t^a = \frac{dx^a}{du} \longrightarrow \boxed{\frac{Dt^a}{Du} = \frac{\ddot{S}}{\dot{S}}\, t^a}$

There exists a preferred Class of affine Parameters:

S.t $\ddot{S} = 0$

Leads to: Non-Null Geodesic in Affine Parametisation.

$$\boxed{\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0}$$

↳ Such parameters are linearly related to path length:

$u = as + b$ (a, b - constant).

↳ Satisfys $\ddot{s} = 0$.

Relation to Parallel Transport:

For a non-null geodesic - In affine param

→ Tangent Vector $t^a = \partial x^a / \partial u$ is parrallel transported.

$$\boxed{\frac{Dt^a}{Du} = 0} \qquad |t| = \frac{ds}{du} = \text{Constant}.$$

↳ Length of tangent Vector constant.

Null Geodesics

↳ Since Legrangian - $F = 0$ → Stationary property doesn't hold!

Affinely Parametised Null Geodisic:

↳ Curves with null tangent vectors $t^a = \frac{dx^a}{du}$ that are parrallel transported $\frac{Dt^a}{Du} \equiv 0$.

Timelike / Spacelike / null character preserved along geodisic: $\frac{Dt^a}{Du} = 0 \to g_{ab}t^a t^b = \text{Const.}$

Procedure to build up geodesics:

↳ 1) Pick vector at some starting point:

2) Solve $\frac{Dt^a}{Du} = 0$ and $t^a = \frac{dx^a}{du}$ → generates unique geodesic curve in affine parametisation

→ Everywhere Spacelike / timelike / null according to characterisation of initial vector.

↳ preservation $g_{ab}t_a t^b$

Alternative Procedure - Legrangian:

→ Generates equations for affinely parametised geodisic

↳ linearly related to path length.

For a geodisic in an affine Parametisation:

$$\frac{Dt^b}{Du} = 0 \xrightarrow[\text{index}]{\text{Lowering}} g_{ab}\frac{Dt^b}{Du} = 0$$

$$\to \frac{Dt_a}{Du} = \frac{dt_a}{du} - \Gamma^c_{ba}t^b t_c = 0.$$

$$= \frac{dt_a}{du} - \frac{1}{2}g^{cd}(\partial_b g_{ad} + \partial_a g_{bd} - \partial_d g_{ab})t^b t_c = 0$$

cancel by Symmetry property.

$$\boxed{\frac{dt_a}{du} = \frac{1}{2}\partial_a g_{bc}t^b t^c}$$ Affinely Parametised Geodisic

*

As $t_a = g_{ab}\frac{dx^b}{du}$ → $$\boxed{\frac{d}{du}\left(g_{ab}\frac{dx^b}{du}\right) - \frac{1}{2}\frac{\partial g_{bc}}{\partial x^a}\frac{dx^b}{du}\frac{dx^c}{du}}$$

This is the Euler-Legrange Eqn: $\boxed{\frac{\partial L}{\partial x^a} = \frac{d}{du}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right)}$ If Lagrangian is in form $\boxed{\mathcal{L} = g_{ab}\frac{dx^a}{du}\frac{dx^b}{du} = g_{ab}\dot{x}^a\dot{x}^b}$

Action: $\int L\,du$ not invariant under reparametisation.

As $\mathcal{L} = \left(\frac{ds}{du}\right)^2$: Eqn motions will give $\mathcal{L} = \text{Const}$

In terms of tangent vectors:

$(\partial_a g_{bc})\,t^b t^c = \frac{d}{du}(2g_{ab}t^b) = 2\frac{dt_a}{du}$

Conserved quantaties along affinely parametised geodisics:

$\frac{Dt^a}{Du} = 0$ or $\frac{dt^a}{du} = \frac{1}{2}(\partial_a g_{bc})\,t^b t^c$

1) $|\vec{t}| = \text{Const}$ Since $\frac{Dt^a}{Du} = 0$ (parralel-transported).

For non-null case: Can always take $|\vec{t}| = 1$ by $u = s$.

For null-case: $|\vec{t}| = 0$

2) If a manifold has symmetry s.t. g_{ab} is independent of some x^c

$\partial_c g_{ab} = 0 \longrightarrow \frac{dt_c}{du} = 0 \longrightarrow t_c = \text{Constant}.$

Or in Legrangian Approach: $(\mathcal{L} = g_{ab}\dot{x}^a\dot{x}^b)$

'Conservation of Conjugate momentum. $\pi_c = \frac{\partial \mathcal{L}}{\partial \dot{x}^c}$.

Minkowski Spacetime and Particle Dynamics.

Minkowski Spacetime in Cartesian Coordinates

4D Pseudo-Euclidean manifold:

over which we define global cartesian coordinates.

'Global Inertial Coordinates'

x^μ $(\mu = 0,1,2,3)$ $\longrightarrow$ $\boxed{x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z}$

$ds^2 = \eta_{\mu\nu}\,dx^\mu dx^\nu$, $\boxed{\eta_{\mu\nu} = \text{diag}(+1,-1,-1,-1)}$ $\longrightarrow$ Minkowski Metric, $\eta_{\mu\nu}$

Components of inverse metric: $\boxed{\eta^{\mu\nu} = \text{diag}(+1,-1,-1,-1)}$

Note: minkowski metric is a special case

In Global Inertial Coordinates:

$\Gamma^\mu_{\nu\rho} = 0 \longrightarrow$ Connection Vanishes everywhere.

Lorentz Transformations

Physically: Lorentz transformations are relabelling of events with coordinates assigned in difficult inertial frames

Mathematically: Represent the residual freedom in our choice of global inertial coordinates.

$x^\mu \rightarrow x'^\mu$

$\boxed{\eta_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma}}$ or $\boxed{\eta_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\eta_{\rho\sigma}}$

By differentiating: $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ with $\boxed{\eta_{\mu\nu} = \Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu\,\eta_{\rho\sigma}}$

Constant.

Change in spacetime origin.

If $a^\mu = 0$: Homogenous LT

$a^\mu \neq 0$: Poincare transformation.

Homogenous Lorentz Transform:

$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

↳ Constants $\Lambda^{\mu}{}_{\nu}$

→ depend on relative velocity and orientation of the two inertial frames.

For a standard Lorentz boost:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Speed: $V = \beta c$ $\quad \gamma = (1-\beta^2)^{-1/2}$ →

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$ct' = \gamma ct - \gamma\beta x$

etc

Inverse of transform.

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} = \eta^{\mu\rho}\eta_{\nu\sigma}\Lambda^{\sigma}{}_{\rho} \longrightarrow \text{Notation: } (\Lambda^{-1})^{\mu}{}_{\nu} \equiv \Lambda_{\nu}{}^{\mu}$$

Proper Lorentz Transformations

↳ $\boxed{\eta_{\mu\nu} = \Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\eta_{\rho\sigma}}$ transform between inertial frames with same handedness of spatial coordinates and excludes time reversal.

$\eta = \Lambda^{T}\eta\Lambda \quad \det(\eta) = (\det\Lambda)^2 \det(\eta)$

↳ $\det(\Lambda) = \pm 1$

and $\eta_{00} = \Lambda^{\rho}{}_{0}\Lambda^{\sigma}{}_{0}\eta_{\rho\sigma}$

$$1 = (\Lambda^{0}{}_{0})^2 - \sum_{i=1}^{3}(\Lambda^{i}{}_{0})^2 \longrightarrow (\Lambda^{0}{}_{0})^2 \geq 1$$

Defn: Proper Lorentz Transformation

↳ $\boxed{\det(\Lambda^{\mu}{}_{\nu}) = 1 \quad \Lambda^{0}{}_{0} \geq 1}$

↳ Time Reversal – Violates both conditions

↳ $\Lambda^{\mu}{}_{\nu} = \mathrm{diag}(-1,1,1,1) \longrightarrow \det(\Lambda) = -1, \ \Lambda^{0}{}_{0} = -1$

Cartesian Basis Vectors:

↳ On a general manifold, coordinate system x^a provides a set of basis vectors that span the tangent space $T_P(M)$ at given point P.

$$\boxed{\vec{e}_a = \left.\frac{\partial}{\partial x_a}\right|_P}$$

Basis vectors – differential operators corresponding to partial differentiation wrt coords.

Recalling Scalar Product between two vectors: $\underline{U}$ and $\underline{V}$

↳ $g(\underline{U}\cdot\underline{V}) \xrightarrow{\text{Components}} g_{ab}U^aV^b$

For basis vectors in a coordinate system:

$$g(\vec{e}_a, \vec{e}_b) = g_{cd}(e_a)^c(e_b)^d$$
$$= g_{cd}(\delta_a{}^c)(\delta_b{}^d) = g_{ab}$$

↳ Metric Components.

In Minkowski Space:

↳ Global Inertial Coordinates – basis vectors are orthonormal.

$\vec{e}_{\mu} = \frac{\partial}{\partial x^{\mu}}$ — $\boxed{g(\vec{e}_{\mu}, \vec{e}_{\nu}) = \eta_{\mu\nu}}$

Under a Lorentz transform, basis vectors transform with inverse transformation matrix:

$$\vec{e}\,'_\mu = \frac{\partial}{\partial x'^\mu} = \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{\text{Transformation matrix}} \underbrace{\frac{\partial}{\partial x^\nu}}_{\text{original basis vector.}} = (\Lambda^{-1})^\nu{}_\mu\, \vec{e}_\nu$$

$$(\Lambda^{-1})^\mu{}_\nu = \frac{\partial x^\mu}{\partial x'^\nu}$$

Since a LT – metric components unchanged:

$$\underline{g}(\vec{e}\,'_\mu, \vec{e}\,'_\nu) = \eta_{\mu\nu}$$

From: $\eta_{\mu\nu}\, \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu\, \eta_{\rho\sigma}$

From $\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu\, \eta_{\rho\sigma}$

$$\boxed{(\Lambda^{-1})^\nu{}_\mu = \eta^{\mu\rho}\eta_{\nu\sigma}\Lambda^\sigma{}_\rho}$$

not a tensor

$$\boxed{\begin{aligned}\vec{e}\,'_0 &= \gamma\vec{e}_0 + \gamma\beta\vec{e}_1 \\ \vec{e}\,'_1 &= \gamma\beta\vec{e}_0 + \gamma\vec{e}_1\end{aligned}}$$

4-Vectors and the lightcone:

Vectors in a 4D Spacetime

Vector at point P

$$\underline{V} = V^\mu \vec{e}_\mu$$

$\vec{e}_\mu$ → component relative to basis.

V^μ → Components of Vector.

Under LT:

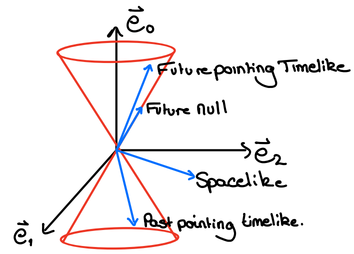
$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu = \Lambda^\mu{}_\nu V^\nu \quad \text{s.t.} \quad \boxed{\vec{V} = V^\mu \vec{e}_\mu = V'^\mu \vec{e}\,'_\mu \text{ is invariant.}}$$

Can invariantly classify vectors at any point using $g(\vec{V},\vec{V})$ (invariant)

$$\boxed{\begin{aligned}&\eta_{\mu\nu}V^\mu V^\nu > 0 && \text{timelike} \\ &\eta_{\mu\nu}V^\mu V^\nu = 0 && \text{lightlike/null} \\ &\eta_{\mu\nu}V^\mu V^\nu < 0 && \text{Spacelike.}\end{aligned}}$$

Follows that $\vec{e}_0$ – timelike

$\vec{e}_i$ – Spacelike
$i = 1,2,3$



Raising and lowering Indicies in Spacetime

Metric associates vectors – dual vectors

$$V_\mu = g_{\mu\nu} V^\nu$$

For Inertial Coordinates in Minkowski Space:

$$\boxed{V_\mu = \eta_{\mu\nu} V^\nu} \qquad V_0 = V^0 \text{ and } V_i = -V^i$$

→ timelike const
Spacelike flip.

Under LT:

Components of dual vectors transform w/ inverse transformation matrix:

$$\boxed{V'_\mu = (\Lambda^{-1})^\nu{}_\mu V_\nu}$$

Particle Dynamics

4-Velocity of a massive particle

trajectory given by worldline: $x^\mu(\tau)$ → 4-velocity: future pointing and timelike.

Parametised by Proper time: τ. (ideal clock).

an affine parameter as $ds^2 = c^2 d\tau^2$

ie linear relation to s.

Tangent Vector to worldline: $u^\mu = \dfrac{dx^\mu}{d\tau}$

dx^μ → 4-vector

$d\tau$ → Lorentz Scalar.

Length of 4-velocity Constant:

$$\eta_{\mu\nu} u^\mu v^\nu = \eta_{\mu\nu} \frac{\partial x^\mu}{d\tau}\frac{\partial x^\nu}{d\tau} = \left(\frac{ds}{d\tau}\right)^2 = c^2 > 0 \rightarrow \text{timelike.}$$

Writing out the Cartesian Coordinates u^μ:

$$u^\mu = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = \frac{dt}{d\tau}\Big(c, \underbrace{\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}}_{\text{usual three velocity.}}\Big)$$

$$\boxed{\vec{U} = (\vec{u}^1, \vec{u}^2, \vec{u}^3) \longrightarrow u^\mu = \frac{dt}{d\tau}(c, \vec{u})}$$

Notation: u' not same as $\vec{u}$

Relationship between Coordinate time and proper time:

↳ Normalisation of 4-velocity:

$$c^2 = \eta_{\mu\nu} U^\mu V^\nu$$

$$= \left(\frac{dt}{d\tau}\right)^2 (c^2 - |\vec{u}|^2) \qquad \frac{dt}{d\tau} = \left(1 - \frac{|\vec{U}|^2}{c^2}\right)^{-1/2} = \gamma_u$$

↳ Lorentz factor in particle frame.

Rewriting 4-Velocity:

$$\boxed{U^\mu = \gamma_u (c, \vec{u})}$$

Velocity Transformation Laws:

$$\boxed{u'^\mu = \Lambda^\mu{}_\nu u^\nu}$$

↳ Considering boost $S \rightarrow S'$

$$\begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} \vec{U}'_1 \\ \gamma_{u'} \vec{U}'_2 \\ \gamma_{u'} \vec{U}'_3 \end{pmatrix} = \begin{pmatrix} \gamma_v & -\beta\gamma_v & 0 & 0 \\ -\beta\gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ \gamma_u \vec{u}^1 \\ \gamma_u \vec{u}^2 \\ \gamma_u \vec{u}^3 \end{pmatrix}$$

0-component

$$\gamma_{u'} = \gamma_u \gamma_v \left(1 - \frac{\beta \vec{u}^1}{c}\right)$$

$$\boxed{\frac{\gamma_{u'}}{\gamma_u} = \gamma_v \left(1 - \frac{v\vec{u}^1}{c^2}\right)}$$

i^{th} Component

$$\gamma_{u'} \vec{u}'^1 = \gamma_v \gamma_u (\vec{u}^1 - \beta c)$$

$$\vec{u}'^1 = \frac{\vec{u}' - v}{\left(1 - \frac{\vec{u}' v}{c^2}\right)}$$

4-Acceleration

In SR, a free particle in inertial coordinates has:

$\vec{u} = \text{const} \longrightarrow \gamma_u = \text{const.}$

Using $U^\mu = \gamma_u (c, \vec{U})$

$$\frac{\partial u^\mu}{d\tau} = 0 \quad (\text{not a tensor eqn})$$

Tensor Eqn version (in Cartesian $\Gamma = 0$: $d/d\tau \rightarrow D/D\tau$ flat space)

$$\frac{D u^\mu}{D\tau} = 0. \quad (\text{parralel transported})$$

Since u^μ is tangent vector to affine parametised world-line:

↳ Free particles move on timelike geodisic in minkowski spacetime.

↳ **Equivelence Principle:** Same will be true in curved spacetime.

Turning on external (non gravitational) force:

Non zero acceleration:

$$\boxed{a^\mu = \frac{D u^\mu}{D\tau}} \xrightarrow{\text{inertial coords}} a^\mu = \frac{du^\mu}{d\tau}$$

In inertial coords: $u^\mu = \gamma_u (c, \vec{U})$

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma_u \frac{\partial}{\partial t}(\gamma_u c, \gamma_u \vec{U}) = \gamma_u \left(c\frac{d\gamma_u}{dt}, \frac{d\gamma_u}{dt}\vec{u} + \gamma_u \vec{a}\right) \quad \vec{a} = \frac{d\vec{u}}{dt} \text{ - 3 acceleration}$$

$$\frac{dt}{d\tau} = \gamma_u$$

Derivative of Lorentz factor:

$$\frac{1}{\gamma_u^2} = 1 - \frac{\vec{u}\cdot\vec{u}}{c^2} \longrightarrow \frac{-2}{\gamma_u^3}\frac{d\gamma_u}{dt} = -\frac{2}{c^2}\frac{d\vec{u}}{dt}\cdot u = -\frac{2}{c^2}\,\vec{a}\cdot\vec{u}$$

$$\boxed{\frac{d\gamma_u}{dt} = \frac{\gamma_u^3}{c^2}\,\vec{u}\cdot\vec{a}}$$

Follows that:

$$\boxed{a^\mu = \gamma_u^2\left(\frac{\gamma_u^2}{c}\,\vec{u}\cdot\vec{a},\ \vec{a} + \frac{\gamma_u^2}{c^2}(\vec{u}\cdot\vec{a})\,\vec{u}\right)}$$

Properties of acceleration 4-Vector

1) Acceleration 4-Vector $\left(a^\mu = \frac{Du^\mu}{D\tau}\right)$ is orthogonal to u^μ : $g(a,u)=0$.

Since $g_{\mu\nu}u^\nu u^\mu = c^2$ $\qquad 0 = \frac{d}{d\tau}\left(g_{\mu\nu}u^\nu u^\mu\right) = \frac{D}{D\tau}\left(g_{\mu\nu}u^\nu u^\mu\right) = 2g_{\mu\nu}u^\mu\frac{Du^\nu}{D\tau}$

2) In inst rest frame $\vec{u}=0$:

$$a^\mu = (0, \vec{a}_{IRF})$$

↳ The magnitude of $\vec{a}_{IRF}$ determines the invariant magnitude of 4-acceleration.

$$\eta_{\mu\nu}\, a^\mu a^\nu = -|\vec{a}_{IRF}|^2 < 0$$

↳ Space like Vector.

Relativistic Mechanics of massive Particles:

Momentum 4-Vector:

↳ Massive particle of rest mass m:

$$\boxed{\underline{p} = m\,\underline{u}} \longrightarrow$$

↳ 4 velocity

Recalling components of $\underline{u}$ in inertial Coordinates $\longrightarrow$ $\boxed{p^\mu = (\gamma_u mc, \vec{p})}$

$$u^\mu = \gamma_u(c, \vec{u})$$

↳ Relativistic 3D Momentum $\vec{p} = \gamma_u m\vec{u}$

Recalling $g(\underline{u},\underline{u}) = c^2$:

$$\boxed{|\underline{p}|^2 = g(\underline{p},\underline{p}) = m^2c^2}$$

Properties:

1) Reduces to newtonian case: $\vec{p} = m\vec{u}$ for $|\vec{u}| \ll c$.
2) Constant for free particles (Since $\vec{u}$ is Constant)
3) Conserved for Systems of particles interacting with Short-range Interaction.
4) Newtons 2^{nd} Law:

$$\vec{f} = \frac{d\vec{p}}{dt} \quad \text{where } \vec{f} \text{ is the 3D force}$$

Writing down $p^\mu = mu^\mu = m\gamma_u(c,\vec{u})$ in an Inertial frame:

The zero-component p^0 is the total energy of the particle:

$$\boxed{p^0 = \gamma_u mc = \frac{E}{c}}$$

Considering rate of working on the 3D force:

$$\vec{u}\cdot\vec{f} = \vec{u}\cdot\frac{d\vec{p}}{dt} = \vec{u}\cdot\frac{d}{dt}(\gamma_u m\vec{u}) = m\left(|\vec{u}|^2\frac{d\gamma_u}{dt} + \gamma_u\,\vec{u}\cdot\vec{a}\right)$$

using $\vec{u}\cdot\vec{a} = \frac{c^2}{\gamma_u^3}\frac{d\gamma_u}{dt}$

$$= mc^2\frac{d\gamma_u}{dt}\underbrace{\left(\frac{|\vec{u}|^2}{c^2} + \frac{1}{\gamma_u^2}\right)}_{1}$$

Hence: Rate of working $= \frac{d}{dt}(\text{energy}) = \frac{d}{dt}(\gamma_u m c^2)$

$\hookrightarrow E = \gamma_u m c^2 \longrightarrow \boxed{p^\mu = \left(\frac{E}{c}, \vec{p}\right)}$

Energy-Momentum Invariant:

$$\boxed{|\underline{p}|^2 = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2 = m^2c^2}$$

→ 1) For a free particle in inertial coordinates:

$\frac{dp^\mu}{dt} = 0$ Tensor Eqn: $\frac{Dp^\mu}{D\tau}$

2) For isolated system of point particles undergoing short range interactions:

$\hookrightarrow \boxed{\sum_{\text{particles}} p^\mu = \text{Constant}}$ → only for Minkowski Space (Global Inertial).

Force 4-Vector:

For a particle acted on by a force, p^μ is not constant

Define the force 4-vector by:

$$\boxed{f^\mu = \frac{Dp^\mu}{D\tau} = ma^\mu}$$

$\hookrightarrow$ Intrinsic proper time derivative.

→ 1) f^μ is orthogonal to u^μ since:

$m^2c^2 = g_{\mu\nu}p^\mu p^\nu \longrightarrow \frac{D}{D\tau}(g_{\mu\nu}p^\mu p^\nu) = 0$

$g_{\mu\nu}p^\mu \frac{Dp^\nu}{D\tau} = 0 \longrightarrow g_{\mu\nu}u^\mu f^\mu = 0 \longrightarrow g(\underline{f}, \underline{u}) = 0$

2) Components of f^μ in inertial coordinates relate to the 3D force:

$f^\mu = \frac{dp^\mu}{d\tau} = \gamma_u \frac{d}{dt}\left(\frac{E}{c}, \vec{p}\right) = \gamma_u\left(\frac{\vec{f}\cdot\vec{u}}{c}, \vec{f}\right)$

$\frac{dt}{d\tau} = \gamma_u$.

$\hookrightarrow$ used $\frac{dE}{dt} = \vec{f}\cdot\vec{u}$

3) 4-force and 4-acceleration relation:

$\underline{f} = m\,\underline{a}$

Momentum 4-Vector of a Photon:

$\hookrightarrow$ To conserve energy and momentum in interactions involving photons.

$$\boxed{p^\mu \equiv \left(\frac{E}{c}, \vec{p}\right)}$$

For zero rest mass:

$\hookrightarrow \boxed{E = |\vec{p}|c}$ and $\boxed{g(\underline{p}\cdot\underline{p}) = 0.}$

$\hookrightarrow$ Future pointing null vector.

For a free photon in Minkowski Space:

In inertial coords E and $\vec{p}$ constant.

$\boxed{\frac{Dp^\mu}{D\lambda} = 0}$ $\lambda \longrightarrow$ arbitrary parameter along path of photon: $x^\mu(\lambda)$.

Cannot use τ as path is null

Relationship of p^μ to photon worldline:

$\hookrightarrow$ For a massive particle: $p^\mu = mu^\mu = m\frac{dx^\mu}{d\tau}$

For a photon, we can always choose a (dimensional) λ to parametise the worldline s.t.:

$$\boxed{p^\mu = \frac{dx^\mu}{d\lambda}}$$ → 4 momentum parallel to tangent vector to path

→ In inertial coordinates:

↳ $p^\mu = \frac{E}{c}\left(1, \frac{\vec{p}}{|\vec{p}|}\right) = \frac{E}{c^2}\left(c\frac{dt}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$

↳ unit vector in direction of propagation.

$= \frac{E}{c^2}\frac{dx^\mu}{dt}$

↳ Follows p^μ parallel to tangent vector to the worldline

→ E/dt is Lorentz invariant.

By $d\lambda = \frac{c^2 dt}{E}$ → $p^\mu = \frac{dx^\mu}{d\lambda}$ → As $\frac{Dp^\mu}{D\lambda} = 0$ we see $x^\mu(\lambda)$ is affinely parametised null geodisic.

Key Point:

Free massless particles move in Minkowski Space on null geodisics, with $p^\mu = \frac{dx^\mu}{d\lambda}$ for some affine parametisation, λ

Example: Compton Scattering

↳ Considering scattering of a photon from an electron in the rest frame of the electron.

Compton Scattering:

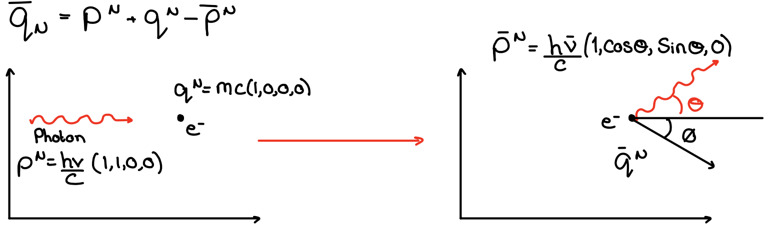
↳ Consider scattering of a photon from an electron in the rest frame of the electron.

Total 4-momentum Conserved

$p^\mu + q^\mu = \bar{p}^\mu + \bar{q}^\mu$

$\bar{q}_\nu = p^\mu + q^\mu - \bar{p}^\mu$

h - planks constant.



Eliminate final state properties of electron by squaring and using invariance $|\bar{q}|^2$

$\underbrace{|\bar{q}|^2}_{m_e^2c^2} = |p^\mu + q^\mu - \bar{p}^\mu|$

$\underbrace{|p|^2}_{0} + \underbrace{|q|^2}_{m_e^2c^2} + \underbrace{|\bar{p}|^2}_{0} + 2g(p, q) - 2g(p, \bar{p}) - 2g(q, \bar{p})$

$0 = \eta_{\mu\nu} p^\mu q^\nu - \eta_{\mu\nu} p^\mu \bar{p}^\nu - \eta_{\mu\nu} q^\mu \bar{p}^\nu$

$0 = h\nu m_e - h\bar{\nu} m_e - \frac{h\nu}{c}\frac{h\bar{\nu}}{c}(1-\cos\theta)$ → $$\boxed{\bar{\nu} = \frac{\nu}{1 + \left(\frac{h\nu}{m_e c^2}\right)(1-\cos\theta)}}$$ → Photon freq reduced - quantum effect.

Local Reference Frame of a General Observer:

Consider a general observer following a worldline, O:

$$u^\mu = \frac{dx^\mu}{d\tau} \qquad a^\mu = \frac{D u^\mu}{d\tau}$$

At any proper time τ (event P) Coordinate basis Vectors of an IRF there $e_\mu(\tau)$ $(\mu = 0,1,2,3)$ form an orthonormal basis

With: $\underline{U}(\tau) = c\,\underline{e}_0(\tau)$ (particle at rest).

↳ Vectors $\vec{e}_i(\tau)$ $(i=1,2,3)$ span the Instantaneous rest space of the observer.

Maxwell's Equations in Mikowski Spacetime

Original Equations:

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$

ρ → Charge density

ε_0 → Permitivity of free Space

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \overbrace{\mu_0\varepsilon_0}^{1/c^2}\frac{\partial\vec{E}}{\partial t}}$$

μ_0 → Permitivity of Free S

$\vec{J}$ → Current density

Lorentz force Law:

↳ $\boxed{\vec{f} = q(\vec{E} + \vec{u}\times\vec{B})}$ Electromagnetic 3-force on a particle of Charge q and 3-velocity $\vec{u}$.

Charge Conservation built into Maxwell's Eqn:

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot\left(\mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) = 0$. and $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ ⟶ $\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$ Continuity Equation.

Lorentz Force Law:

Recalling for force 4-vector: $f^\mu = \gamma_u\left(\frac{\vec{f}\cdot\vec{u}}{c}, \vec{f}\right)$ with $\vec{f} = q(\vec{E} + \vec{u}\times\vec{B})$

As force vector is linear in fields and Velocity:

Postulate: $f_\mu = q F_{\mu\nu} u^\nu$

↳ $F_{\mu\nu}$ – Maxwell file tensor.

Properties of $F_{\mu\nu}$

↳ $F_{\mu\nu}$ is antisymmetric:

↳ force 4 vector is orthogonal to particles 4 vector velocity:

$$0 = f_\mu U^\nu = q\,F_{\mu\nu}\underbrace{U^\mu U^\nu}_{\text{Symmetric}}$$

↳ Vanishes for ∀ future pointing timelike u^μ

⟶ $\boxed{F_{\mu\nu} = -F^{\mu\nu}}$ Also holds for $\boxed{F^{\mu\nu} = -F^{\mu\nu}}$

3D Lorentz force law: $f_\mu = \gamma_u\left(\frac{\vec{f}\cdot\vec{u}}{c}, -\vec{f}\right) = q\left(\frac{\vec{E}\cdot\vec{u}}{c}, -\vec{E} - \vec{u}\times\vec{B}\right)$

↳ $\vec{u}\times\vec{B}\cdot u = 0$

Comparing with $f_\mu = q F_{\mu\nu} U^\nu$

For f_0:

$$f_0 = q F_{0\nu} U^\nu = q \sum_i F_{0i} \gamma \vec{U}_i = \frac{q}{c} \gamma_u \vec{E} \cdot u \qquad F_{0i} = \frac{\vec{E}^i}{c}$$

For f_i:

$$\boxed{f_i = q F_{i0} U^0 + q \sum_j F_{ij} U^j} \qquad U^\mu = \gamma_u (c, \vec{u})$$

$$= -q \cdot \frac{\vec{E}^i}{c} \cdot \underbrace{c\gamma_u}_{U^0} + q \gamma_u \sum_j F_{ij} \vec{U}^j$$

$$= \boxed{-\gamma_u q \left[\vec{E}^i + (\vec{u} \times B)^i \right]} \longrightarrow \boxed{F_{12} = -\vec{B}^3,\ F_{13} = \vec{B}^2,\ F_{23} = -\vec{B}^1}$$

Putting it all together: $F_{\mu\nu} = \begin{pmatrix} 0 & \vec{E}^1/c & \vec{E}^2/c & \vec{E}^3/c \\ -\vec{E}^1/c & 0 & -\vec{B}^3 & \vec{B}^2 \\ -\vec{E}^2/c & \vec{B}^3 & 0 & -\vec{B}^1 \\ -\vec{E}^3/c & -\vec{B}^2 & \vec{B}^1 & 0 \end{pmatrix}$

Under a Lorentz transformation:

Under a Lorentz transformation:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Goal: Search for a field strength tensor $T^{\mu\nu}$ such that the form of the Lorentz force law is preserved.

Note: for a type (2,0) Tensor:

Components transform as: $F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$

For a standard Lorentz boost:

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

$$\Lambda^0{}_0 = \Lambda^1{}_1 = \gamma, \quad \Lambda^1{}_0 = \Lambda^0{}_1 = -\beta\gamma$$

$$F'^{01} = \Lambda^0{}_\rho \Lambda^1{}_\sigma F^{\rho\sigma} = \underbrace{\Lambda^0{}_0}_{\gamma} \underbrace{\Lambda^1{}_1}_{\gamma} F^{01} + \underbrace{\Lambda^0{}_1}_{-\beta c} \underbrace{\Lambda^1{}_0}_{-\beta c} F^{10}$$

$$\vec{E}'^1 = \underbrace{(\gamma^2 - \gamma^2 \beta^2)}_{1} \vec{E} = \vec{E}.$$

Overall:

$$\boxed{\vec{E}' = \begin{pmatrix} \vec{E}^1 \\ \gamma(\vec{E}^2 - v\vec{B}^3) \\ \gamma(\vec{E}^3 - v\vec{B}^2) \end{pmatrix} \qquad \vec{B} = \begin{pmatrix} \vec{B}^1 \\ \gamma\left(\vec{B}^2 + v\frac{\vec{E}^3}{c^2}\right) \\ \gamma\left(\vec{B}^3 - v\frac{\vec{E}^2}{c^2}\right) \end{pmatrix}}$$

Charged Cylinder:

↳ Cylinder at rest in S with charge per length, λ:

Radial electric field (at distance r):

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}} \quad \text{(Gauss' Law)}$$

In frame S':

Cylinder has velocity $-v$ along x' axis:

↳ Charge per length (contraction): $\gamma_v \lambda$ → Increase in charge density.

Current along x' axis: $-\gamma_v \lambda v$

Sources Radial and Electric Fields:

↳ $E' = \frac{\gamma_v \lambda}{2\pi \varepsilon_0 r'}$ $\quad B' = \frac{\mu_0 \gamma_v \lambda v}{2\pi r'}$ → (ampere's law).

↳ Larger e field

Recreate these with transform laws:

↳ Fields in S at $\vec{x} = (0, 0, r)$

$$\vec{E} = \left(0, 0, \frac{\lambda}{2\pi \varepsilon_0 r}\right) \qquad \vec{B} = 0 \text{ (at rest).}$$

Transform the fields at this event with:

$$\vec{E}' = \begin{pmatrix} \vec{E}' \\ \gamma_v(\vec{E}^2 - v\vec{B}^3) \\ \gamma_v(\vec{E}^3 + v\vec{B}^2) \end{pmatrix} \qquad \vec{B}' = \begin{pmatrix} \vec{B} \\ \gamma_v(\vec{B}^2 + v\vec{E}^3/c^2) \\ \gamma_v(\vec{B}^3 - v\vec{E}^2/c^2) \end{pmatrix}$$

No length contraction (perpendicular to motion) $(r' = r)$:

$$\vec{E}' = \left(0, 0, \frac{\gamma_v \lambda}{2\pi \varepsilon_0 r'}\right) \text{ and } \vec{B}' = \left(0, \frac{\gamma_v v}{c^2} \frac{\lambda}{2\pi \varepsilon_0 r'}, 0\right) = \left(0, \frac{\mu_0 \gamma_v \lambda v}{2\pi r'}, 0\right)$$

→ Same as ampere's law.

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

Maxwell's Equations

Writing them as tensor equations in Minkowski Spacetime. (i.e. Lorentz-Covariant).

Maxwell's Equations are linear in fields and charge and current densities, and their derivatives (both time and space).

$$\boxed{\text{Suggests a tensor equation: } \nabla F \sim j}$$

Current 4-Vector:

Charge density ρ and current density $\vec{J}$ are the components of the current 4-vector (in inertial coordinates).

$$\boxed{j^\mu = (\rho c, \vec{J})}$$

→ Considering charges at rest in S' with (proper) charge density $\rho_0(t', x')$:

$$\boxed{j'^\mu = (\rho_0 c, \vec{0})}$$

Length contraction increases charge density to:

↳ $\rho(t, \vec{x}) = \gamma_v \rho_0(t', \vec{x}')$

Current density for these moving charges is:

↳ $\vec{J}(t, \vec{x}) = \rho(t, \vec{x})\vec{V} = \gamma_v \rho_0(t', \vec{x}')\vec{V}$

Follows that:

$\hookrightarrow j^\mu = \gamma_v \rho_0 (c, \vec{v}) = \gamma_v \rho_0 (c, v, 0, 0)$

Recalling:

$j'^\mu = (\rho_0 c, \vec{0})$

$$\boxed{\begin{aligned} j^0 &= \gamma_v (j'^0 + \beta j'^1) = \gamma_v c \rho_0 = c\rho \\ j^1 &= \gamma_v (j'^1 + \beta j'^0) = \gamma_v \beta c \rho_0 = \rho v = \vec{J}^1 \\ j^2 &= j'^2 = 0 \\ j^3 &= j'^3 = 0 \end{aligned}}$$

Relativistic Field Equations:

Trying: $\boxed{\nabla_\mu F^{\mu\nu} = k j^\nu}$

In global inertial coordinate:

Assymetry of $F^{\mu\nu}$ implies: $\partial_\mu F^{\mu\nu} = k j^\nu \longrightarrow \partial_\nu \partial_\mu \overset{\text{antisymmetry}}{F^{\mu\nu}} = 0 = k \partial_\nu j^\nu$

$\partial_\mu j^\mu = 0.$

Can show that generally: tensor continuity equation

$\hookrightarrow \boxed{\nabla_\mu j^\mu = 0}$ continuity equation

We now know:

$\nabla_\mu F^{\mu\nu} = k j^\nu$ and
$$F^{\mu\nu} = \begin{pmatrix} 0 & -\vec{E}^1/c & -\vec{E}^2/c & -\vec{E}^3/c \\ \vec{E}^1/c & 0 & -\vec{B}^3 & \vec{B}^2 \\ \vec{E}^2/c & \vec{B}^3 & 0 & -\vec{B}^1 \\ \vec{E}^3/c & -\vec{B}^2 & \vec{B}^1 & 0 \end{pmatrix}$$

Can we recover (sourced) maxwell's equations?

$\hookrightarrow$ working in (global) Inertial Coordinates ($\nabla_\mu \to \partial_\mu$)

For $\nu = 0$

$$\underbrace{\frac{\partial F^{00}}{\partial (ct)}}_{0} + \underbrace{\sum_i \frac{\partial F^{i0}}{\partial x^i}}_{\frac{\vec{E}^i}{c}} = k \overset{\rho c}{j^0} \longrightarrow \vec{\nabla}\cdot\vec{E} = kc^2\rho$$

Recovered Maxwell's first equation:

$\hookrightarrow \vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$ where $\mu_0 \varepsilon_0 = \dfrac{1}{c^2}$

$\hookrightarrow$ $\boxed{\text{Thus } k = \mu_0}$

For $\nu = i$:

$$\frac{\partial F^{0i}}{\partial (ct)} + \sum_j \frac{\partial^{ji}}{\partial x^j} = \mu_0 j^i = \mu_0 \vec{J}^i$$

$\hookrightarrow -\dfrac{\vec{E}_i}{c}$

For $i = 1$ ..

$$-1/c^2\, \partial \vec{E}^1/\partial t + \underbrace{\partial F^{21}/\partial x^2 + \partial F^{31}/\partial x^3}_{\partial_2 \vec{B}^3 - \partial_3 \vec{B}^2 = (\vec{\nabla}\wedge\vec{B})^1} = \mu_0 \vec{J}^1 \longrightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}}$$

We have shown we can write Maxwell's Eqns:

$$\underbrace{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}_{\text{3 equations}} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$\hookrightarrow$ 3equations. $\qquad \hookrightarrow$ one equation.

All contained within:

$$\nabla_{[\mu} F_{\nu\rho]} = 0 \xrightarrow{\text{Since } F_{\mu\nu} \text{ antisym}} \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0$$

↳ Antisymmetric Parts

The four independent choices of indicies are:

$\mu = (\mu, \nu, \rho) = (0,1,2), (0,1,3), (0,2,3), (1,2,3)$

In global inertial coords:

↳ $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0.$

For $(\mu,\nu,\rho) = (0,1,2)$

↳ $$\underbrace{\frac{\partial F_{12}}{\partial(ct)}}_{-\frac{1}{c}\frac{\partial \vec{B}^3}{\partial t}} + \underbrace{\frac{\partial F_{20}}{\partial x^1} + \frac{\partial F_{01}}{\partial x^2}}_{-\frac{1}{c}\left(\frac{\partial \vec{E}^2}{\partial x^1} - \frac{\partial \vec{E}^1}{\partial x^2}\right)} = 0 \longrightarrow (\vec{\nabla} \times \vec{E})^3 = -\frac{\partial \vec{B}^3}{\partial t}.$$

⟶ Other two cases $\mu = 0$ gives $(\vec{\nabla} \times \vec{E})^i$ equations.

Electromagnetism in SpaceTime:

$$\boxed{\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu}$$

and

$$\boxed{\nabla_{[\mu} F_{\nu\rho]} = 0}$$

Reduce to maxwells equations on minkowskis Spacetime.

Spacetime Curvature:

Overarching aim: Formulate laws of physics as tensor equations that reduce in local inertial coordinates to SR form:

Requires Spacetime to be pseudo-Riemannian:

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}$$

↳ Key point: must be able to construct local inertial coordinates X^μ at any point P s.t.

$$\boxed{ds^2 \simeq \underbrace{\eta_{\mu\nu}}_{\text{flat space}} dx^\mu dx^\nu}$$ in the vicinity of P.

In the presence of gravity:

↳ must not be able to extend local inertial coordinates to all of Spacetime (otherwise gravity wouldn't be observed).

↳ gravity ⟹ Spacetime Curvature.

Example: Free massive particle with 4-Velocity

↳ Tangent vector to world line $\quad U^\mu = \frac{dx^\mu}{d\tau}$

In general spacetime is parrallel transported $\quad \frac{DU^\mu}{D\tau} = 0$

As in local inertial coordinates:

$$\boxed{\frac{d^2 X}{d\tau^2} = 0}$$ ⟶ usual equation of motion for a free particle in Cartesian inertial coordinates in SR.

a geodisic

Reminder: Local Inertial Coordinates

Mathematically: X^μ in the vicinity of P s.t.

$$\boxed{g_{\mu\nu}(P) = \eta_{\mu\nu}} \qquad \boxed{(\partial_\rho g_{\mu\nu})_P = 0}$$

↳ Minkowski at point, P

↳ Derivative of metric vanishes at point, P.

What this means:

Coordinates assigned by free falling non rotating Cartesian frame over limited spacetime region around P.

Results: Basis Vectors $\vec{e}_N = \frac{\partial}{\partial x^N}$ are orthonormal @ P.

↳ Differential operators.

$$\boxed{g(\vec{e}_N, \vec{e}_N) = \eta_{NV}}$$ (Local inertial coordinates at P).

x^N are only unique up to Lorentz transformations

↳ Freedom to specify velocity and orientation of free falling frames at point P.

In the Vicinity of P:

$$g_{NV} = \eta_{NV} + 0 + \frac{1}{2}\left(\frac{\partial^2 g_{NV}}{\partial x^\rho \partial x^\sigma}\right)_P [x^\rho - x^\rho(P)][x^\sigma - x^\sigma(P)] + \dots$$

↳ Shown earlier insufficient DoF

↳ Derivative of $g_{NV} = 0$ @ P (DoF ✓)

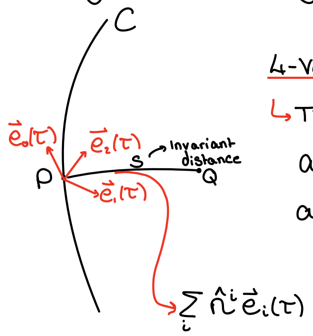
→ When second derivative large → non Minkowski

Fermi Normal Coordinates

A free falling observer parallel-transports an orthonormal frame along their world line C

↳ C $\vec{e}_0(\tau) = \underline{u}(\tau)$

Connect any point Q in the vicinity of C to some unique point with a spacelike geodesic with tangent vector orthogonal to $\vec{e}_0$ at P.



4-Velocity: C $\vec{e}_0(\tau) = u(\tau)$

↳ The proper time τ at P, invariant distance S and direction cosines $\hat{n}^i$ define local inertial coordinates all along C

C: $x^N(Q) = (c\tau, S\hat{n}^i)$

Coordinates extend the idea of local inertial point → vicinity of geodisic

Newtonian Limit for a free falling particle:

↳ Free particles move according to the geodisic equation

$$\boxed{\frac{d^2 x^N}{d\tau^2} + \Gamma^N_{\rho\sigma} \frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0}$$

↳ Does this reduce to newtonian form for weak fields and slow moving particles.

↳ For weak fields: Choose global coordinates where metric close to Minkowski

$g_{NV} = \eta_{NV} + h_{NV}$ with $|h_{NV}| << 1$ → Also metric is stationary in these coordinates (not time varying)

↳ $\frac{\partial h_{NV}}{\partial x^0} = 0$

Thus for slow moving particles: $\left|\frac{\partial x^i}{\partial t}\right| << c \rightarrow \left|\frac{\partial x^i}{\partial \tau}\right| << \frac{dx^0}{d\tau}$ $\quad x^0 = ct$.

Thus can ignore $dx^i/d\tau$ in geodisic eqn compared to $\frac{dx^0}{d\tau}$

$$\boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\, c^2 \left(\frac{dt}{d\tau}\right)^2 \approx 0}$$

The required connection coefficients are:

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\nu}\left(2\,\partial_0 \cancel{g_{\nu 0}} - \partial_\nu g_{00}\right) = -\frac{1}{2}\sum_i g^{\mu i}\partial_i h_{00} \approx -\frac{1}{2}\sum_i \eta^{\mu i}\partial_i h_{00}$$

no off diags.

Follows that $\Gamma^0_{00} \approx 0$, $\Gamma^i_{00} \approx \frac{1}{2}\partial_i h_{00}$

The zero component of the geodisic equation:

$$\frac{d^2t}{d\tau^2} = 0 \qquad \frac{dt}{d\tau} \approx \text{const.}$$

i^{th} component.

$$\frac{d^2x^i}{d\tau^2} \approx -\frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i}\left(\frac{dt}{d\tau}\right)^2 \rightarrow \boxed{\frac{d^2x^i}{dt^2} \approx -\frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i}}$$

weak field → slows speeds.

Newtonian result: $\frac{d^2x^i}{dt^2} = -\frac{\partial\Phi}{\partial x^i} \longrightarrow h_{00} \approx \frac{2\Phi}{c^2}$

It follows that in the weak field limit, to recover the correct Newtonian Result

$$\boxed{g_{00} \approx \left(1 + \frac{2\Phi^2}{c^2}\right)}$$

Weak-Field Limit: $|\Phi|^2 << c^2$

Holds: in nearly all situations

eg. the surface of a white dwarf star: $\left(\frac{\Phi}{c^2} \sim 10^{-4}\right)$

Fails: Near horizon of a black hole.

Intrinsic Curvature of a Manifold:

A flat manifold

If there exists global Cartesian Coordinates X^a

St $ds^2 = \epsilon_1 (dx^1)^2 + \dots + \epsilon_N (dx^N)^2$, $\epsilon_a = \pm 1$

How to Check:

Look for tensor valued objects, involving second derivatives of the metric which can test for curvature (non-flatness) in arbitary coordinates

$\Big\}$ Riemann Curvator Tensor

In General Relativity:

Flat Spacetime: Can find global inertial coordinates where $g_{\mu\nu} = \eta_{\mu\nu}$

SR globally: no gravity.

Riemann Curvature Tensor:

Covariant Derivative acting on Scalar fields:

$\boxed{\nabla_a \nabla_b \phi = \nabla_b \nabla_a \phi}$ Commutativity

This isn't the case for tensors:

Considering action on dual vector V_a:

$$\nabla_a \nabla_b V_c = \partial_a (\nabla_b V_c) - \Gamma^d_{ab}\nabla_d V_c - \Gamma^d_{ac}\nabla_b V_d$$

$$= -\partial_a\left(\partial_b V_c - \Gamma^d_{bc} V_d\right) - \Gamma^d_{ab}\left(\partial_d V_c - \Gamma^e_{dc} V_e\right) - \Gamma^d_{ac}\left(\partial_b V_d - \Gamma^e_{bd} V_e\right)$$

$$= \underbrace{\left[\partial_a\partial_b V_c - \Gamma^d_{bc}\partial_a V_d - \Gamma^d_{ac}\partial_b V_d - \Gamma^d_{ab}\left(\partial_d V_c - \Gamma^e_{bd} V_e\right)\right]} - \left(\partial_a \Gamma^d_{bc}\right) V_d + \Gamma^d_{ac}\Gamma^e_{bd} V_e$$

Symmetric in a and b

Subtracting same expression $a \leftrightarrow b$

$$\nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c = -(\partial_a \Gamma^d_{bc})V_d + (\partial_b \Gamma^d_{ac})V_d + \Gamma^e_{ac}\Gamma^d_{bd}V_d - \Gamma^e_{ac}\Gamma^d_{bd}V_d$$

$$\boxed{\nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c = R_{abc}{}^d V_d}$$

$R_{abc}{}^d$ - Riemann Curvature Tensor ~ Type (1,3)

$$\boxed{R_{abc}{}^d = -\partial_a \Gamma^d_{bc} + \partial_b \Gamma^d_{ac} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{bc}\Gamma^d_{ae}}$$

↳ Derivatives of Connection (2^{nd} Derivatives of metric):

↳ If manifold is flat ⟷ Can construct global cartesian Coordinates ⟷ $R_{abc}{}^d = 0$

Symmetries of the Riemann Tensor:

Following from $\nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c = V_{abc}{}^d V_d$

1) $\boxed{R_{abc}{}^d = -R_{bac}{}^d}$ →

↳ Anti-symmetric in first two indicies.

Explicit expression for $R_{abc}{}^d$:

Implies:

2) $\boxed{R_{abc}{}^d + R_{cab}{}^d + R_{bca}{}^d = 0}$

↳ Cyclic Symmetry

Note: rewriting Contraction type (1,3) → Type (0,4)

$$\boxed{R_{abcd} = g_{de} R_{abc}{}^e}$$

3) $\boxed{R_{abcd} = -R_{abdc}}$

↳ Antisymmetry in last two incidies

4) $\boxed{R_{abcd} = R_{cdab}}$

↳ Symmetry in swapping first and last pair.

Bianchi Identity:

The Reimann tensor satisfies:

$$\boxed{\nabla_a R_{bcd}{}^e + \nabla_b R_{acd}{}^e + \nabla_c R_{abd}{}^e = 0} \xleftrightarrow[\text{written}]{\text{Can also be}} \boxed{\nabla_{[a} R_{bc]d}{}^e = 0}$$

using anti-symmetry

↳ Cyclic Sum of indicies a, b, c.

Origin of the Bianchi Identity:

↳ Since a tensor equation → Verified by working in local Cartesian Coordinates at an arbitary point p (S.t. $\Gamma^a_{bc} = 0$):

1 Permutation:

$\partial_a \leftrightarrow \nabla_a$

$$(\nabla_a R_{bcd}{}^e)_p = (\partial_a[-\partial_b \Gamma^e_{cd} + \partial_c \Gamma^e_{bd} + \Gamma^f_{bd}\Gamma^e_{cf} - \Gamma^f_{cd}\Gamma^e_{bf}])_p$$

$$= (-\partial_a\partial_b \Gamma^e_{cd} + \partial_a\partial_c \Gamma^e_{bd})_p$$

$$\boxed{\nabla_a R_{bcd}{}^{e} + \nabla_b R_{cad}{}^{e} + \nabla_c R_{abd}{}^{e} = 0} \rightarrow$$ Local Cartesian Coordinates at P.
$\downarrow$
P arbitrary $\rightarrow$ thus holds for general coords

Ricci Tensor and Ricci Scalar:

Riemmann tensor to form Lower-ranked tensors.

$$\boxed{R_{ab} = R_{cab}{}^{c}}$$

$\hookrightarrow$ Symmetric: $R_{ba} = R_{cba}{}^{c} = -R_{abc}{}^{c} - \underbrace{R_{bac}{}^{c}}_{0 \text{ by antisym of second pair of indicies}}$ (Cyclic Symmetry).
$= R_{cab}{}^{c} = R_{ab}$

Ricci Scalar:

$\hookrightarrow$ $\boxed{R \equiv g^{ab} R_{ab}}$

$\rightarrow$ Flat manifold: $R_{abc}{}^{d} = 0 \rightarrow R_{ab} = 0$. $\rightarrow$ When does this arrise?

GR vaccum regions: $R_{\mu\nu} = 0$ but tidal effects remain $R_{\mu\nu\rho}{}^{\sigma} \neq 0$.

$R_{ab} = 0 \not\rightarrow \underbrace{R_{abc}{}^{d} = 0}_{\text{So not neccesarily flat.}}$

Contracted Bianchi Identity:

$\hookrightarrow$ $$\boxed{\nabla^a \big(\underbrace{R_{ab} - \tfrac{1}{2} g_{ab} R}_{\text{Einstein Tensor, } G_{ab}}\big) = 0}$$

Overall Importance:

Einstein tensor is divergence free.

Bianchi Identity:

$$\nabla_a R_{bcd}{}^{e} + \nabla_b R_{acd}{}^{e} + \nabla_c R_{abd}{}^{e} = 0$$

$\downarrow$ Contracting on b and e.

$$\nabla_a R_{cd} + \nabla^b R_{cadb} - \nabla_{c} R_{ad} = 0$$

$\downarrow$ further contraction with g^{ad}:

$$g^{ad}\left(\nabla_a R_{cd} + \nabla^b R_{cadb} - \nabla_c R_{ad}\right) = 0$$

$$\nabla^a R_{ca} + \nabla^b R_{cb} + \nabla_c R = 0$$

$$2\nabla^d R_{cd} - \nabla_c R = 0.$$

Physical Manifestations of Curvature.

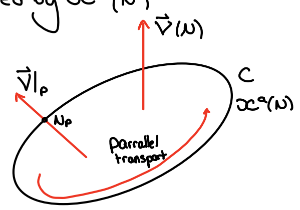
Curvature and Parallel Transport:

$\hookrightarrow$ Parallel transport is path dependent when the manifold has intrinsic curvature.

$\hookrightarrow$ A vector is rotated after parallel transport around a closed loop when the manifold has intrinsic curvature.

Considering parallel transporting a vector $V^a(N)$ around a closed infitesimal loop, C described by $x^a(N)$

$$\frac{DV^a}{DN} = 0 \rightarrow \frac{dV^a}{dN} = -\Gamma^a_{bc} \underbrace{\frac{dx^b}{dN}}_{\text{tangent vector.}} V^c$$



Integrate formally from point P where $N = N_P$:

$\hookrightarrow$ $$V^a(N) = V^a|_P - \int_{N_P}^{N} \Gamma^a_{bc}[x(N')] \frac{dx^b}{dN'} V^c(N')\, dN'$$

Infinitesimal loop: (taylor expansion)

$\hookrightarrow$ $$\Gamma^a_{bc}(N) = \Gamma^a_{bc}|_P + \partial_d \Gamma^a_{bc}|_P \left(x^d(N) - x^d|_P\right) + \ldots$$

$$V^c(N) = V^c|_P - \Gamma^c_{ef}|_P V^f|_P \left(x^e(N) - x^e|_P\right) + \ldots$$

On Integrating back to P, change in V^a is:

$$\Delta V^a = -(\partial_d \Gamma^a_{bc} - \Gamma^a_{be}\Gamma^e_{dc})|_P V^c|_P \underbrace{\oint (x^d - x^d|_P)\, dx^b}$$

Infinitesimal Loop Integral:

Anti symmetric tensor (at P) encoding the planar area of the loop.

$$\oint (x^d - x^d|_P)\, dx^b = \oint x^d\, dx^b = \oint x^b dx^d$$
$$= -\oint x^d\, dx^b \qquad \text{since } \oint d(x^b x^d) = 0$$
$$= \oint x^{[d} dx^{b]}$$

Can anti Symmetrise:

$\boxed{\partial_d \Gamma^a_{bc} - \Gamma^a_{be}\Gamma^e_{dc}}$ in b and d.

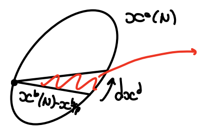
Overall:

$$\boxed{\Delta V_a = \frac{1}{2} R^a_{bcd}|_P V^d|_P \oint x^{[b} dx^{c]}}$$

If manifold flat (Riemann Curvature tensor vanishes):

No change in the vector around a closed loop.

Symbolically:



$$ds^{bd} = \frac{1}{2}(x^{[b}(N) - x^b|_P)\, dx^{d]}$$

Curvature and geodesic deviation:

Considering two geodesics which are initially parrallel:

- in R^N – They remain parrallel
- on S^2 – Converge (eg at the North Pole) as a result of intrinsic curvature.

Mathematically:

Let $x^a(N)$ and $\bar{x}^a(N)$ be affinely-parametised neighbouring geodisics, C and $\bar{C}$ respectively.

- Infinitesimal Connecting Vector:
  - $\xi^a(N) \equiv \bar{x}^a(N) - x^a(N)$
    - Moniter Convergence by: $\dfrac{D^2\xi^a}{DN^2}$

tangent vector

Geodesic Equations: $\dfrac{d^2x^a}{dU^2} + \Gamma^a_{bc}\dfrac{dx^b}{dN}\dfrac{dx^c}{dN} = 0$

$$\frac{d^2\bar{x}^a}{dU^2} + \bar{\Gamma}^a_{bc}\frac{d\bar{x}^b}{dN}\frac{d\bar{x}^c}{dN} = 0$$

Taking difference and expanding: we can write this as the tensor-valued equations of geodisic deviation:

$$\boxed{\frac{D}{DN}\left(\frac{D\xi^a}{DN}\right) - R^a_{dbc}\, \dot{x}^b \dot{x}^c \xi^d = 0}$$

$\dfrac{D^2\xi^a}{DN^2}$: Riemann Curvature sources relative acceleration of neighbouring geodesics.

If the manifold is flat $\dfrac{d^2\xi^a}{dN^2} = 0$ in global cartesian coordinates.

Tidal Acceleration:

↳ For timelike geodesics in Spacetime with $u = \tau$ so

$\dot{x}^a = u^a$

$$\frac{D}{D\tau}\left(\frac{D\xi^\mu}{D\tau}\right) = \underbrace{R_{\nu\alpha\beta}{}^{\mu}\, u^\alpha u^\beta}_{S_\nu{}^\mu \,\rightarrow\, \text{Type (1,1) tensor.}}\, \xi^\nu$$

4-velocity

Relative acceleration of neighbouring free-falling particles due to tidal effects described by Symmetric tidal tensor:

$$\boxed{S_{\mu\nu} = R_{\mu\alpha\beta\nu}\, u^\alpha u^\beta}$$

Newtonian tidal theory:

$$\frac{d^2 x^i}{dt^2} = -\frac{\partial \Phi}{\partial x^i}\bigg|_{x(t)} \quad \text{and} \quad \frac{d^2 \bar{x}^i}{dt^2} = -\frac{\partial \Phi}{\partial x^i}\bigg|_{\bar{x}(t)}$$

Connecting Vector gives:

$$\boxed{\frac{d^2\xi^i}{dt^2} = -\left(\frac{\partial \Phi}{\partial x^i}\bigg|_{\bar{x}(t)} - \frac{\partial \Phi}{\partial x^i}\bigg|_{x(t)}\right) = -\frac{\partial^2 \Phi}{\partial x^i \partial x^j}\,\xi^j}$$

Spacetime Curvature:

'how matter tells Space time how to curve'.

The energy-momentum tensor

In newtonian gravity, Poisson equation involves mass density ρ:

$$\boxed{\vec{\nabla}^2 \Phi = 4\pi G\rho}$$

Wish to generalise mass density to some tensor 'Energy-Momentum Tensor'

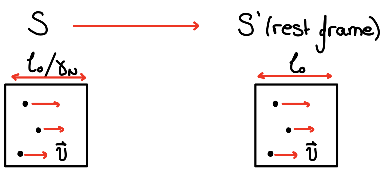
↳ encodes energy density, momentum density and their fluxes at each point.

First considering the 'Dust' case:

↳ non-interacting point particles of rest mass, m.

→ No velocity dispersion (no random components)

→ All particles at event x have the same 4-velocity: $u^\mu(x)$.



$\gamma_u n_0$ Number Density n_0

$\rho c^2 = (\gamma_u n_0)(\gamma m c^2)$ Energy Density $\rho_0 c^2 = m n_0 c^2$ → Energy per particle × No. density

$= \gamma_u^2 \rho_0 c^2$

Remember Relativistic Energy Expression:

↳ $\gamma_u m c^2$.

$$\boxed{\rho_0 c^2 \longrightarrow \rho c^2 \equiv \gamma_u^2 \rho_0 c^2}$$

↳ Rest frame

n_0, ρ_0 are lorentz Scalars (defined in rest frame)

Energy Density:

↳ Not a lorentz Scalar (0,0 component of type (2,0).)

Energy Momentum Tensor for Dust:

$$\boxed{T^{\mu\nu} = \rho_0\, u^\mu u^\nu}$$

In the Inertial Rest Frame (with local inertial Coords)

↳ $\boxed{T'^{00} = \rho_0 c^2}$ since $u'^{\mu} = c\,\delta_0$

In frame S (moving dust)

↳ $u^{\mu} = \gamma_u (c, \vec{u})$

$T^{00} = \rho_0 (\gamma_u c)^2 = \rho c^2$ (energy density)

$\mu = i$

$T^{i0} = m n_0 (\vec{\gamma_u}\, \vec{u}_i)(\gamma_u c)$ or $cT^{i0} = (\gamma_u^2 n_0 mc^2)\vec{u}^i$

$= c(\gamma_u n_0)\,(\underbrace{m\gamma_u \vec{u}^i}_{p_i})$ $=$ energy density $\times \vec{u}_i$

$= c \times$ no. density $\times$ 3-momentum. $=$ energy flux.

$= c \times$ 3-momentum density.

T^{ij} (ij^{th} components in general frame s).

$T_{ij} = m n_0 (\gamma_u \vec{u}^i)(\gamma_u \vec{u}^j)$

$= (\gamma_u m n_0 \vec{u})\,\vec{u}^j$

$= [3\text{-momentum density}]^i \times \vec{u}^j$

$=$ flux of 3-momentum density

Summary:

Summarising in local inertial coordinates

T^{00}: Energy Density

$T^{0i} = T^{i0}$: i^{th} component of 3 momentum density

T^{ij}: flux of i-component of 3-momentum in j direction.

The energy-momentum tensor

↳ Generalises to other Sources eg. EM Fields with Components as above.

EM: T^{00}: Energy Density (field)^2

T^{i0}, T^{0i}: Poynting Vector

T^{ij}: Maxwell Stress

An ideal Fluid:

↳ More general we must include All Sources of Energy/Momentum:

↳ - Kinectic energy (due to velocity dispersion).

- Energy/Momentum due to heat Conduction

- Shear stresses and velocity dispersion for T^{ij}

$$\boxed{\begin{array}{l}\text{An ideal fluid is: isotropic in its IRF (no preffered spatial direction).}\\ \hookrightarrow \text{In IRF: } T^{i0} = 0\ ,\ T^{ij} \propto \delta^{ij}\end{array}}$$

Requires the mean free path << Scale of any thermal/velocity gradients.

In IRF - only have energy density ρc^2 and isotropic pressure p.

$$\boxed{T^{\mu\nu} = \text{diag}(\rho c, p, p, p).}$$

If the fluid 4-velocity is $u^{\mu}(x)$:

$$\boxed{T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^{\mu} u^{\nu} - p g^{\mu\nu}}$$

ρ and p - rest frame defined - Lorentz scalars.

↳ Components in rest frames in local inertial Coords ($u^{\mu} = c\delta_0^{\mu}$ and $g^{\mu\nu} = \eta^{\mu\nu}$)

$T^{00} = c^2\left(\rho + \frac{p}{c^2}\right) - p = \rho c^2$

$T^{i0} = 0$

$T^{ij} = -p\eta_{ij} = p\,\delta^{ij}$

$\xrightarrow{\text{Ideal fluid has 3-velocity } \vec{u}}$

$T^{00} = c^2\left(\rho + \frac{p}{c^2}\right)\gamma_u^2 - p$

$T^{i0} = \gamma_u^2 \rho c^2 + \gamma_u^2 p \left(1 - \frac{1}{\gamma_u^2}\right)$

$T^{ij} = \gamma_u^2 \left(\rho c^2 + p\,|\vec{u}|^2/c^2\right)$

For a non-relativistic fluid: $p << \rho c^2$ and recover dust like $T^{\mu\nu} \rightarrow \rho u^{\mu} u^{\nu}$

Conservation of energy and momentum:

Local Conservation of energy and momentum:

$$\boxed{\nabla_\mu T^{\mu\nu} = 0}$$

Shown by:

In local IRF: $\nabla \to \partial$

$$\frac{\partial T^{0\nu}}{\partial (ct)} + \frac{\sum_i \partial T^{i\nu}}{\partial x^i} = 0 \longrightarrow \text{For } \nu = 0: \quad \frac{\partial T^{00}}{\partial t} + c\sum_i \frac{\partial T^{i0}}{\partial x^i} = 0$$

$$\frac{\partial}{\partial t}(\text{energy density}) + \vec{\nabla}\cdot(\text{energy flux}) = 0 \longrightarrow \text{True by local energy conservation:}$$

Energy density in a region result of flow in/out

For $\nu = j$, have: $\frac{\partial T^{0j}}{\partial (ct)} + \sum_i \frac{\partial T^{ij}}{\partial x^i} = 0$

$$\frac{\partial}{\partial t}(\text{3-momentum density}) + \vec{\nabla}\cdot(\text{momentum flux}) = 0$$

Einstein Equations:

Recall:

Weak field limit: $g_{00} = 1 + h_{00} \approx \left(1 + \frac{2\Phi}{c^2}\right)$ weak field static limit.

Considering dust at rest (in weak field coordinates).

$T^{\mu\nu} = \rho u^\mu u^\nu$ with $u^\mu = A\,\delta^\mu_0$ where $c^2 = g_{\mu\nu} U^\mu U^\nu = A^2(1 + h_{00})$

$$A^2 \approx c^2(1 - h_{00})$$

Follows:

$$\boxed{T^{\mu\nu} \approx \rho c^2 (1 - h_{00})\,\delta^\mu_0\,\delta^\nu_0}$$

$$\begin{aligned} T_{00} &= g_{0\mu}\, g_{0\nu}\, T^{\mu\nu} \\ &\approx \rho c^2 (1 - h_{00})(g_{00})^2 \\ &= \rho c^2 (1 + h_{00}) \\ &\approx \rho c^2 \end{aligned}$$

Thus poisson equation equivelent to:

$$\vec{\nabla}^2 g_{00} \sim \frac{8\pi G}{c^4} T_{00} \qquad \left(\vec{\nabla}^2 g_{00} = \frac{2}{c^2}\vec{\nabla}^2 \Phi = \frac{8\pi G}{c^2}\rho = \frac{8\pi G}{c^4} T_{00} \right)$$

Tensor valued field equation:

Candidate: $\boxed{K_{\mu\nu} = \kappa T_{\mu\nu}}$ $\quad \kappa = \frac{8\pi G}{c^4}$

Properties: Type (2,0) Symmetric tensor ($T_{\mu\nu}$ is Symmetric).

Related to Spacetime Curvature $\sim \partial^2 g$

Conserved such that $T_{\mu\nu}$ is conserved:

$$\nabla^\mu K_{\mu\nu} = 0 \longrightarrow \nabla^\mu T_{\mu\nu} = 0.$$

Contracted Bianchi Identity: $\nabla^\mu G_{\mu\nu} = 0$ $\quad G_{\mu\nu}$ - Covariantly Conserved.

Arrive at: Einstein Field Equations

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}} \longrightarrow \text{This describes 10, coupled nonlinear partial differential equations}$$

Equivelently written: $\boxed{g^{\mu\nu}\left(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R\right) = -\frac{8\pi G}{c^4}\underbrace{g^{\mu\nu}T^{\mu\nu}}_{\text{Trace of } T_{\mu\nu}=T.} = -\frac{8\pi G}{c^4}T}$

It follows: $R = \frac{8\pi G}{c^4}T \longrightarrow \boxed{R_{\mu\nu} = -\left(\frac{8\pi G}{c^4}\right)\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)}$

Einstein Equations in Empty Space:

↳ If a region of Spacetime is a vacumm (no matter and fields).

$\boxed{T_{\mu\nu}=0}$ (vavumm)

Thus:

$\boxed{R_{\mu\nu}=0}$ (Einstein field equations in vacuum).

↳ This does not mean that: $R_{\mu\nu\sigma}{}^{\sigma}=0$ (ie 'gravitational' tidal effects - in a vaccumm region).

Weak field limits of Einstein equations:

↳ Can we recover Poisson equation of Newtonian theory in the weak field limit:

↳ $\vec{\nabla}^2\Phi = 4\pi G\rho$

For the Source, Consider a non-relativistic fluid ($p \ll \rho c^2$) at rest in coordinates where:

$\boxed{g_{\mu\nu} = \eta_{\mu\nu}+h_{\mu\nu}}$ and metric Stationary $\boxed{\partial_0 = 0}$

Previously shown this leads to: $\boxed{T^{\mu\nu} = \rho u^{\mu}u^{\nu}}$ and $\boxed{u^{\nu} = c\left(\frac{1-h_{00}}{2}\right)\delta^{\mu}_{0}}$

$\boxed{T^{\mu\nu} \approx \rho c^2(1-h_{00})\,\delta^{\mu}_{0}\delta^{\nu}_{0} \approx \rho c^2\,\delta^{\mu}_{0}\delta^{\nu}_{0}}$

So

$\boxed{T^{00} \approx \rho c^2,\ T = g_{\mu\nu}T^{\mu\nu} = \rho c^2}$ neglecting $h_{00} \ll 1$.

It follows that: $T_{00} - \frac{1}{2}g_{00} \approx \frac{1}{2}\rho c^2$

Subbing in EFE $\boxed{R_{00} \approx -\frac{4\pi G}{c^2}\rho}$ (weak field, slows sources).

Recalling:

↳ Riemann curvature tensor:

$R_{\mu\nu\rho}{}^{\sigma} = -\partial_{\mu}\Gamma^{\sigma}_{\nu\rho} + \partial_{\nu}\Gamma^{\sigma}_{\mu\rho} + \Gamma^{\tau}_{\mu\rho}\Gamma^{\sigma}_{\nu\tau} - \Gamma^{\tau}_{\nu\rho}\Gamma^{\sigma}_{\mu\tau}$

↳ Since $g_{\mu\nu} = \eta_{\mu\nu}+h_{\mu\nu}$ (metric)

↳ $\Gamma \sim \partial h$ (connection)

Thus Ricci tensor (Contract on ν, σ)

↳ $R_{\mu\rho} \approx -\partial_{\mu}\Gamma^{\nu}_{\nu\rho} + \partial_{\nu}\Gamma^{\nu}_{\mu\rho}$

↳ $\boxed{R_{00} \approx -\sum_i \partial_i \Gamma^{i}_{00}}$ (Since $\partial_0 = 0$).

We previously were shown (Newtonian limit of geodesic equation):

↳ $\Gamma^{i}_{00} \approx \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}$ and $h_{00} \approx \frac{2\Phi}{c^2} \longrightarrow R_{00} \approx -\frac{4\pi G\rho}{c^2} \longrightarrow -\frac{1}{2}\vec{\nabla}^2 h_{00} \approx -\frac{4\pi G}{c^2}\rho$

Therefore: $\boxed{\vec{\nabla}^2\Phi \approx 4\pi G\rho}$ Justifies proportionality constant

The Cosmological Constant:

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}}$$

Einstein Tensor $G_{\mu\nu}$ satisfies following properties:

1) Divergence-free ($\nabla^\mu G_{\mu\nu} = 0$).
2) Constructed from metric and its first and second derivatives.
3) Linear in the second derivative of the metric.

Lovelock's Theorem:

In 4D spacetime, the only tensor satisfying these properties is:

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu}}$$

Λ → Constant.

Including this 'cosmological constant' term:

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}}$$

$\Lambda \sim (3.04 \text{ Gpc})^{-2} \sim (\text{Size of observable universe})^{-2}$

A very small correction to EFE (unless on a cosmological scale!). If considering singular object: Star/BH - does not need considering.

Alternative form: Contraction $g^{\mu\nu}$.

$$\boxed{-R + 4\Lambda = -\frac{8\pi G}{c^4} T}$$

R: Ricci Scalar. 4 from 'trace'.

Treating ρ and Λ as small perturbations to Minkowski spacetime:

$$R_{00} \approx \frac{-1}{c^2} \vec{\nabla}^2 \Phi \approx \frac{-4\pi G}{c^2} \rho + \Lambda$$

$$\boxed{\vec{\nabla}^2 \Phi = 4\pi G \rho - \Lambda c^2}$$

For a point mass M at the origin, have additional contribution to Newtonian potential $\Delta\Phi$ with:

$$\boxed{\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Delta\Phi}{\partial r}\right) = -\Lambda c^2}$$

i) $\times r^2$
ii) integrate.
iii) $\div r^2$

$$-\frac{\partial \Delta\Phi}{\partial r} = \frac{1}{3} \Lambda c^2 r$$

The gravitational acceleration →

$$\boxed{-\vec{\nabla}\Phi(\vec{x}) = \underbrace{-GM\frac{\vec{x}}{|\vec{x}|^3}}_{\text{Usual Coloumb term}} + \underbrace{\frac{\Lambda c^2}{3}\vec{x}}_{\text{new term}}}$$

Λ provides universal gravitational repulsion:

- Increasing linearly with distance
- Accelerating expansion of the universe.

Cosmological constant as Vacuum Energy:

Consider an ideal fluid with energy-momentum tensor:

$$\boxed{T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U^\mu U^\nu + P g^{\mu\nu}}$$

But suppose it had some weird constant tension → $P = -\rho c^2$ → $\boxed{T_{\mu\nu} = \rho c^2 g_{\mu\nu}}$

All observers agree that energy density in their frame: ρc^2

Pressure: $-\rho c^2$

$\therefore$ must be a 'fundemental property' of the vaccum.

If we include such a fluid with $T_{\mu\nu}^{vac} = \rho_{vac} c^2 g_{\mu\nu}$ in the EFE:

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(\underbrace{T_{\mu\nu}}_{\rho_{vac}c^2 g_{\mu\nu}} - \frac{1}{2} g_{\mu\nu} \underbrace{T}_{4\rho_{vac}c^2}\right)$$

$$\boxed{T_{\mu\nu}^{vac} - \frac{1}{2} g_{\mu\nu} T^{vac} = -\rho_{vac} c^2 g_{\mu\nu}}$$

effectively like a cosmological constant: $\rho_{vac} c^2 = \dfrac{\Lambda c^4}{8\pi G}$

Origins of vaccum energy:

1) Zero-point quantum fluctuations of standard model quantum fields.

2) empty space is not empty.

Characteristic feature of Gravitational Field:

Source for gravitational field is the entire energy-momentumn tensor.

Opens up possibility of vaccum Energy:

Energy density of empty space.

Example considering EM field:

Zero point energy of EM-field

Zero point energy of 1D SHO's is $\frac{\hbar\omega}{2}$ (Black body radiation).

$$\boxed{\rho_{vac,em} c^2 = \frac{2}{(2\pi)^3} \int \frac{1}{2} \hbar \omega(k)\, d^3\vec{k}}$$

Density of space in k-space.

$\omega(k) = ck$: Integral is divergent, cut off at Planck scale. $k_{max} \sim 1/l_{Pl}$

$$\boxed{l_{Pl} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35}\, m}$$

Then: $\rho_{vac,EM}\, c^2 \sim \dfrac{\hbar c}{l_{Pl}^4} \rightarrow \boxed{\Lambda_{vac,em} \sim \dfrac{G}{c^4} \dfrac{\hbar c}{l_{Pl}^4} \sim \dfrac{1}{l_{Pl}^2} \sim 10^{70} m^{-2}}$

Terrible in comparison to $\Lambda_{obs} \approx 1.1 \times 10^{-52} m^{-2}$.

Schwarzschild Solution

Non trivial solution of EFE

'Describes the vacuum spacetime outside a spherically-symmetric non-rotating mass distribution.'

$$\boxed{ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\Omega^2}$$

Spherically Symmetric Spacetimes:

↳ Solutions more simple with high degrees of symmetry

Passive vs Active Symmetries:

↳ **Active:** Transforming a system results in identical solution.

Passive: Change our coordinate system such that the functional form of components of fields are the same in the new coordinates as in the old.

Example:

↳ Coulomb field in $\mathbb{R}^3$:

↳ In Cartesian Coordinates, with $\vec{x} = (x_1, x_2, x_3)$

$$\boxed{E^i(x) = \frac{x^i}{|\vec{x}|^3}}$$

Radial field with magnitude falling like $1/(\text{distance})^2$

Rotate coordinates with an orthogonal transformation:

↳ $x'^i = \sum_j (O^{-1})^i{}_j x^j$

then:

$$\boxed{E'^i = \sum_j \frac{\partial x'^i}{\partial x^j} E^j = \sum_j (O^{-1})^i{}_j \frac{x^j}{|\vec{x}|^3} = \frac{x'^i}{|\vec{x}'|^3}}$$

↳ $E'^i(\vec{x}')$ are the same function of x'^i coordinates as $E^i(\vec{x})$ are of x^i coordinates

Spherical Symmetry:

Functional form of the Coulomb law remaining the same after an orthogonal transformation.

Spacetime possesses a symmetry if under some:

↳ $\boxed{x^N \longrightarrow x'^N}$

→ The functional form of the metric components $g'_{\mu\nu}(x')$ is the same as in the original coordinates $g_{\mu\nu}(x)$.

→ Functional form of the line element on x'^N and dx'^N same as on x^N and dx^N.

For the Specific Case of Spherical Symmetry:

Cartesian like coordinates x^i $(i = 1,2,3)$ must exist such that under the constant coordinate transformation:

$$\boxed{x'^i = \sum_{j=1}^{3} (O^{-1})^i{}_j x^j}$$

where $O^i{}_j$ is an orthogonal matrix → where the functional form of the line element is unchanged.

Using the Coordinates t and $\vec{x} = (x_1, x_2, x_3)$.

↳ $ds^2 = g_{00}(t,\vec{x})\,dt^2 + 2g_{0i}(t,\vec{x})\,dt\,dx^i + g_{ij}(t,\vec{x})\,dx^i dx^j$

Under orthogonal transformation of $\vec{x}$:

↳ $\boxed{ds^2 = g_{00}(t, O\vec{x}')\,dt^2 + 2g_{0i}(t, \underline{O}\,\vec{x}')\,dt\,O^i{}_j\,dx'^j + g_{ij}(t, \underline{O}\,\vec{x}')\,O^i{}_k O^j{}_l\,dx'^k dx'^l}$

↳ If the spacetime has Spherical Symmetry:

↳ This transformed line element must be equivelent to:

$$\boxed{ds^2 = g_{00}(t,\vec{x}')\,dt^2 + 2g_{0i}(t,\vec{x}')\,dt\,dx'^i + g_{ij}(t,\vec{x}')\,dx'^i dx'^j}$$

$\xrightarrow{\text{ie}}$

$$\boxed{\begin{aligned} g_{00}(t,\vec{x}) &= g_{00}(t, O\vec{x}) \\ g_{0j}(t,\vec{x})\,dx^j &= g_{0i}(t,\underline{O}\,\vec{x})\,O^i{}_j\,dx^j \\ g_{kl}(t,\vec{x})\,dx^k dx^l &= g_{ij}(t,\underline{O}\,\vec{x})\,O^i{}_k\,dx^k O^j{}_l\,dx^l \end{aligned}}$$

The general rotational invariants of the spacelike coords / differentials:

$\vec{x}\cdot\vec{x} = r^2$

$d\vec{x}\cdot d\vec{x}$

$x \cdot d\vec{x}$

Spherical Symmetry therefore constrain results to:

$g_{00}(t,\vec{x})$ can only depend on $\vec{x}$ through rotational invariant $r=\sqrt{\vec{x}\cdot\vec{x}}$

$$\boxed{g_{00}(t,\vec{x}) = A(t,r)}$$

$g_{0i}(t,\vec{x})dx^i$ can only involve invariant $\vec{x}\cdot d\vec{x}$ multiplying a function of t and r

$$\boxed{g_{0i}(t,\vec{x})\,dx^i = -B(t,r)\,\vec{x}\cdot d\vec{x}}$$

$g_{ij}(t,\vec{x})\,dx^i dx^j$ must be of form:

$$\boxed{g_{ij}(t,\vec{x})\,dx^i dx^j = -C(t,r)(\vec{x}\cdot d\vec{x})^2 - D(t,r)\,d\vec{x}\cdot d\vec{x}}$$

Generally Spherically-Symmetric line-element.

$$\boxed{ds^2 = A(t,r)\,dt^2 - 2B(t,r)\,dt\,(\vec{x}\cdot d\vec{x}) - C(t,r)(\vec{x}\cdot d\vec{x})^2 - D(t,r)\,d\vec{x}\cdot d\vec{x}}$$

Switch to Polar Coordinates:

$x^1 = r\sin\theta\cos\phi$, $x^2 = r\sin\theta\sin\phi$, $x^3 = r\cos\theta$.

Differentials:

$$\vec{x}\cdot d\vec{x} = \frac{1}{2}\,d(\vec{x}\cdot\vec{x}) = r\,dr$$

$$d\vec{x}\cdot d\vec{x} = dr^2 + r^2\underbrace{(d\theta^2 + \sin^2\theta\, d\phi^2)}_{d\Omega^2}$$

Line element becomes:

$$\boxed{ds^2 = A(t,r)\,dt - 2B(t,r)\,dt\,dr - C(t,r)\,dr^2 - D(t,r)\,d\Omega^2.}$$

Introduce new radial coordinate with: $\bar{r}^2 = D(t,r)$

$$\boxed{ds^2 = A(t,\bar{r})\,dt - 2B(t,\bar{r})\,dt\,d\bar{r} - C(t,\bar{r})\,d\bar{r}^2 - \bar{r}^2 d\Omega^2.}$$

Aim to remove the $dt\,d\bar{r}$ term with another transformation

$\bar{t} = f(t,\bar{r})$ for some function f

$$A(t,\bar{r})\,dt^2 - 2B(t,r)\,dt\,d\bar{r} = \frac{1}{A(t,\bar{r})}\left[A(t,\bar{r})\,dt - B(t,\bar{r})\,d\bar{r}\right]^2 - \frac{B^2(t,\bar{r})}{A(t,\bar{r})}\,d\bar{r}^2.$$

Introducing an integrating factor so that:

$\Phi(t,r)\left[A(t,\bar{r})\,dt - B(t,\bar{r})\,d\bar{r}\right]$ is an exact differential.

$$d\bar{t} = \Phi(t,\bar{r})\left[A(t,\bar{r})\,dt - B(t,\bar{r})\,d\bar{r}\right]$$

$$\boxed{ds^2 = \frac{1}{A\Phi^2}\,d\bar{t}^2 - \left(\frac{B^2}{A} + C\right)d\bar{r}^2 - \bar{r}^2 d\Omega^2}$$

Functions of $\bar{t}$ and $\bar{r}$

Solution of the field equation in Vaccum

$$\boxed{ds^2 = A(t,r)\,dt^2 - B(t,r)\,dr^2 - r^2 d\Omega^2}$$

Considering Static Spacetime outside a Spherically-Symmetric mass distribution.

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - r^2 d\Omega^2$$

Vaccum (and ignoring cosmological constant):

$R_{\mu\nu} = 0$

Where:

$$\boxed{R_{\mu\nu} = -\partial_\rho\Gamma^\rho_{\mu\nu} + \partial_\mu\Gamma^\rho_{\rho\nu} + \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\mu\rho} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\sigma\rho}}$$

Definitions

Stationary Spacetime:

Symmetry under $t \to t+T$ for any constant T.

$g_{\mu\nu}$ independent of t (time translation invariance)

Static:

In addition to Stationary, Symmetry under:

$t \to -t \longrightarrow g_{0i} = 0$ (unchanged under time reversal).

$g_{tt} = A(r) \longrightarrow g^{tt} = 1/A(r)$

$g_{rr} = -B(r) \longrightarrow g^{rr} = -1/B(r)$

$g_{\theta\theta} = -r^2 \longrightarrow g^{\theta\theta} = -1/r^2$

$g_{\phi\phi} = -r^2\sin^2\theta \longrightarrow g^{\phi\phi} = -1/(r^2\sin^2\theta)$

$\xrightarrow{\text{Non zero, Independent Connection Coefficients}}$

$\Gamma^t_{tr} = A'/(2A)$ $\quad \Gamma^r_{tt} = A'/2B$

$\Gamma^r_{rr} = B'/(2B)$ $\quad \Gamma^r_{\theta\theta} = -r/B$

$\Gamma^r_{\phi\phi} = -r\sin^2\theta/B$ $\quad \Gamma^\theta_{r\theta} = 1/r$

$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ $\quad \Gamma^\phi_{r\phi} = 1/r$

$\Gamma^\phi_{\theta\phi} = \cot\theta$

Non zero components of Ricci tensor:

$$R_{tt} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}$$

$$R_{\theta\theta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right)$$

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

Primes: Derivatives w.r.t r.

Given $R_{\mu\nu} = 0$: $\longrightarrow AR_{\nu\nu} + BR_{tt} = 0 \longrightarrow \frac{A'}{A} + \frac{B'}{B} = 0$, $AB = \alpha = \text{const}$.

$R_{\theta\theta} = 0$, $1/B = A/\alpha$

$$\frac{A}{\alpha} - 1 + \frac{Ar}{2\alpha}\left(2\frac{A'}{A}\right) = 0 \longrightarrow (rA)' = \alpha \longrightarrow rA = \alpha(r+k).$$

Follows:

$$A(r) = \alpha\left(1 + \frac{k}{r}\right) \text{ and } B(r) = \left(1 + \frac{k}{r}\right)^{-1}$$

$$\boxed{ds^2 = \alpha\left(1 + \frac{k}{r}\right)dt^2 - \left(1 + \frac{k}{r}\right)^{-1}dr^2 - r^2 d\Omega^2}$$

Comparison with weak field limit

$$ds^2 \approx \left(1 + \frac{2\Phi}{c^2}\right)d(ct)^2 + \ldots \quad \text{with } \Phi = -\frac{GM}{r}$$

Follows that: $\alpha = c^2$, $K = -2\frac{GM}{c^2}$

$$\boxed{ds^2 = c^2\left(1 - \frac{2N}{r}\right)dt^2 - \left(1 - \frac{2N}{r}\right)^{-1}dr^2 - r^2 d\Omega^2} \quad N = \frac{GM}{c^2}$$

Metric is singular at $r = 2N$ (coordinate singularity)

As $r \longrightarrow \infty$: Spacetime tends to Minkowski (Solution - asymptotically flat).

Birkoff's Theorem:

Static solution of EFEs so far:

Dropping this condition:

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2 d\Omega^2$$

Any Spherically Symmetric Solution of the EFE's in vacuum is the Schwarzchild Solution and therefore asymptotically flat.

Geodesics in Schwarzschild Spacetime:

↳ Dynamics of massive and massless particles in the Schwarzschild Spacetime:

Recalling:

$$\boxed{ds^2 = c^2\left(1-\frac{2\mu}{r}\right)dt^2 - \left(1-\frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\Omega^2}$$

Alternative 'Lagrangian Procedure':

↳ $$L = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \left(\frac{ds}{d\lambda}\right)^2$$

$$= c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2 \quad \text{eg } \dot{t} = \frac{dt}{d\lambda}$$

Recalling Euler Legrange Equations:

↳ With $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$

$$\frac{\partial L}{\partial x^\mu} = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right)$$

For θ-gives:

$$-2r^2\sin\theta\cos\theta\,\dot{\phi}^2 = -2\frac{d(r^2\dot{\theta})}{d\lambda}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

Possible Solution: $\theta = \pi/2 \rightarrow$ Planar motion in equatorial plane

$$L = c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$

Euler Legrange equation for t gives:

↳ $\frac{\partial L}{\partial \dot{t}} = \text{Const} \longrightarrow \boxed{\left(1-\frac{2\mu}{r}\right)\dot{t} = k}$

→ Conserved Quantities

Euler Legrange equation for ϕ gives:

↳ $\frac{\partial L}{\partial \dot{\phi}} = \text{Const} \longrightarrow \boxed{r^2\dot{\phi} = h}$

For r:

↳ Simpler to use constraint $\begin{cases} L = c^2 & \text{massive particle} \\ L = 0 & \text{massless particle}\end{cases}$

$$\boxed{\left(1-\frac{2\mu}{r^2}\right)c^2\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = \begin{cases} c^2 & \text{massive.} \\ 0 & \text{massless.}\end{cases}}$$

Interpretation of k and h:

↳ $\left(1-\frac{2\mu}{r}\right)\dot{t} = k$ and $r\dot{\phi} = h$

k - Related to energy of a particle measured by a stationary observer at spatial infinity.

h - arrises from Symmetry around the z axis and can be interpreted as the Specific angular momentum

A stationary observer is at rest in (r, θ, ϕ) coordinates and so has 4-velocity

$\hookrightarrow$ $\boxed{u^\mu = A\,\delta^\mu_0}$ with $c^2 = g_{\mu\nu} u^\mu u^\nu \longrightarrow c^2 = c^2 A^2 \left(1 - \frac{2\mu}{r}\right)$

$$\longrightarrow A = \left(1 - \frac{2\mu}{r}\right)^{-1/2}.$$

Generally – the energy of a particle with 4-momentum $\underline{p}$ measured by an observer with 4-velocity $\underline{u}$ is

$$E = g(\underline{p}, \underline{u})$$

Since in IRF of observer in local inertial coordinates:

$\hookrightarrow$ $u'^\mu = c\,\delta^\mu_0$ and $p'^\mu = \left(\frac{E}{c}, \vec{p}\right)$

$$\eta_{\mu\nu} p'^\mu u'^\nu = \frac{E}{c} \cdot c = E \quad \text{as required.}$$

For a massive particle:

$$p^\mu = m \frac{dx^\mu}{d\tau} \longrightarrow E = g(\underline{p}, \underline{u}) = g_{00} p^0 A = c^2\left(1 - \frac{2\mu}{r}\right) \times m\dot{t} \times \left(1 - \frac{2\mu}{r}\right)^{-1/2}$$

$$= kmc^2 \left(1 - \frac{2\mu}{r}\right)^{-1/2} \qquad k = \left(1 - \frac{2\mu}{r}\right)\dot{t}$$

Thus

kmc^2 is the energy measured at $r \to \infty$

$\hookrightarrow$ requires $k \geq 1$ to get there

For a massless particle:

$\hookrightarrow$ $p^\mu = \frac{dx^\mu}{d\lambda} \longrightarrow E = g(\underline{p}, \underline{u}) = g_{00} p^0 A = c^2\left(1 - \frac{2\mu}{r}\right) \times \dot{t} \times \left(1 - \frac{2\mu}{r}\right)^{-1/2}$

$$E = kc^2 \left(1 - \frac{2\mu}{r}\right)^{-1/2}.$$

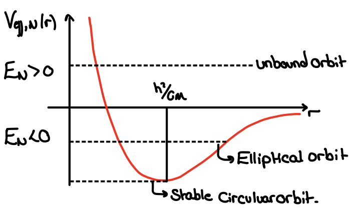
Thus

$\hookrightarrow$ k is energy measured at $r \to \infty$

$\hookrightarrow$ requires $k > 0$ to get there

Energy Equation and Effective Potential:

Newtonian Effective Potential.



$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V_{eff,N}(r) = \frac{E_N}{m}$$

Massive Particles in GR:

$$\boxed{\mathcal{L} = c^2\left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = c^2}$$

Massless Particles:

$$\boxed{\mathcal{L} = c^2\left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = c^2}$$

Eliminate $\dot{t}$ and $\dot{\phi}$ with:

$\boxed{\left(1-\frac{2N}{r}\right)\dot{t}=k}$ and $\boxed{r^2\dot{\phi}=h.}$

Gives:

$$\boxed{\frac{1}{2}\dot{r}^2\underbrace{-\frac{GM}{r}+\frac{h^2}{2r^2}\left(1-\frac{2N}{r}\right)}_{V_{eff,N}(r)}=\frac{1}{2}c^2(k^2-1)}$$

GR correction: Dominates at small r

– reverses sign of centrifugal barrier.

Stationary Points of V_{eff}:

$$\boxed{V_{eff}(r)=\frac{-Nc^2}{r}+\frac{h}{2r^2}\left(1-\underbrace{\frac{2N}{r}}_{\text{non-Newtonian Correction.}}\right)} \qquad N=\frac{GM}{c^2}$$

Derivatives: $\frac{dV_{eff}}{dr}=\frac{Nc^2}{r^2}+\frac{h^2}{r^3}\left(\frac{3N}{r}-1\right)$

Stationary Points – $h>\sqrt{12}\,Nc$: $\boxed{\frac{r_\pm}{N}=\frac{1}{2}\left(\frac{h}{Nc}\right)^2\left[1\pm\sqrt{1-12\left(\frac{Nc}{h}\right)^2}\right]}$

r_+ (Stable) – local minimum | r_- (unstable) – local maximum.

As $h/Nc\to\infty$, Locations of Stationary Points tend to:

$r_+\to\infty$, $r_-\to 3N$. While $h\to\sqrt{12}\,Nc$ merge : $r_\pm=6N$

Innermost Stable Circular Orbit (ISCO):

↳ $r=6N$ (and particle $h=\sqrt{12}Nc$)

Circular obits: Exist up to $3N\to 6N$

↳ Unstable $3N\to 6N$.

Eliminate $\dot{t}$ and $\dot{\phi}$ with:

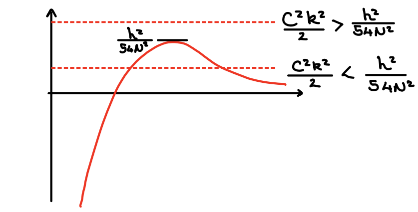
$\boxed{\left(1-\frac{2N}{r}\right)\dot{t}=k}$ and $\boxed{r^2\dot{\phi}=h.}$

Gives:

$$\boxed{\frac{1}{2}\dot{r}^2\underbrace{+\frac{h^2}{2r^2}\left(1-\frac{2N}{r}\right)}_{V_{eff,N}(r)}=\frac{1}{2}c^2k^2}$$

$$\boxed{V_{eff}(r)=\frac{h^2}{2r^2}\left(1-\frac{2N}{r}\right)}\longrightarrow\frac{dV_{eff}}{dr}=-\frac{h^2}{r^3}\left(1-\frac{3N}{r}\right)$$

Stationary Point (unstable): $r=3N$ $\quad V_{eff}(3N)=\frac{h^2}{54N^2}$



If $\frac{h}{ck}<\sqrt{27}N$

Photon coming in from infinity is captured

If $\frac{h}{ck}>\sqrt{27}N$

Photon turns around / is deflected.

Singular turning at closest a

Impact parameter (interpretation of h/ck)

Shape Eqn (Shape of the orbit):

Combing energy eqn with $r^2\dot{\phi}=h$

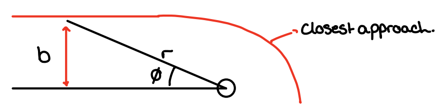
$$\frac{1}{2}\left(\frac{dr}{d\phi}\right)^2\frac{h^2}{r^4}+\frac{h^2}{2r^2}\left(1-\frac{2N}{r}\right)=\frac{1}{2}c^2k^2$$

$$\frac{dr}{d\phi}=\pm r\left[\frac{c^2k^2}{h^2}r^2-\left(1-\frac{2N}{r}\right)\right]^{1/2}$$

At $r\to\infty$: $r\sin\phi=b$ (Straight line – Minkowski)

↳ Follows that: $\frac{dr}{d\phi}\sin\phi+r\cos\phi=0 \qquad \frac{dr}{d\phi}=\pm r\left(\frac{r^2}{b^2}-1\right)^{1/2}$

$\boxed{b=\frac{h}{ck}}$



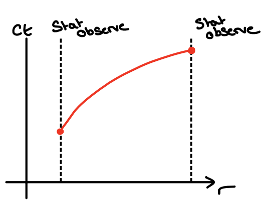
Gravitational Redshift:

Energy of photon measured by stationary observer - Constant (r,θ,ϕ)

Changes along photons path.

$$E=g(\underline{p},\underline{u})=c^2\dot{t}\left(1-\frac{2N}{r}\right)^{1/2}=kc^2\left(1-\frac{2N}{r}\right)^{-1/2}$$

$$\frac{\nu_R}{\nu_E}=\frac{E_R}{E_n}=\left(\frac{1-2N/r_E}{1-2N/r_R}\right)^{1/2}=\frac{1}{1+z}$$



Note: $z \to \infty$ as $r_E \to 2\mu$

↳ In the weak field limit ($r \gg 2\mu$)

$$1+z \approx 1 - \frac{\mu}{r_R} + \frac{\mu}{r_E} = 1 + \frac{1}{c^2}\left[\Phi(r_R) - \Phi(r_E)\right]$$

Key observation: Photon looses

Classical Tests of General Relativity (weak field limit):

Shapes of orbits for massive and massless particles:

Energy Equations:

Massive: $\boxed{\frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{h^2}{2r^2}\left(1 - \frac{2M}{r}\right) = \frac{1}{2}c^2(k^2-1)}$

Massless: $\boxed{\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2}\left(1 - \frac{2M}{r}\right) = \frac{1}{2}c^2k^2}$

Shape Eqns:

$\dot{r} = \dot{\phi}\frac{dr}{d\phi}$

Massive: $\boxed{\frac{d^2u}{d\phi^2} + u - 3\mu u^2 = \frac{GM}{h^2}}$

Massless: $\boxed{\frac{d^2u}{d\phi^2} + u - 3\mu u^2 = 0}$

Newtonian Orbits of Massive Particles:

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2}$$

↳ Taking the turning point $du/d\phi = 0$ at $\phi = 0$

Conic Section:

$$u = \frac{GM}{h^2}(1 + e\cos\phi) \qquad \text{(eccentricity } e\text{)}$$

Relating to Newtonian Energy:

↳ $\frac{E_N}{m} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r}$

$\boxed{\dot{r} = -h\frac{du}{d\phi}}$ $\boxed{\dot{\phi} = hu^2}$

$$\frac{E_N}{mh^2} = \frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}u^2 - \left(\frac{GM}{h^2}\right)u$$

$\boxed{u = \frac{GM}{h^2}(1 + e\cos\phi)}$

$$\boxed{\frac{E_N}{mh^2} = \frac{1}{2}\left(\frac{GM}{h}\right)^2 (e^2 - 1)}$$

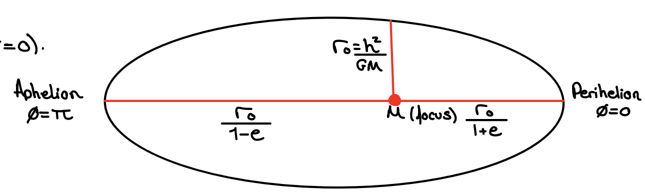
Newtonian bound orbits:

↳ $\boxed{\frac{1}{r} = \frac{GM}{h^2}(1 + e\cos\phi)}$ ⟶ $e = 0$ ⟶ Circular Orbits at $r_0 = h^2/GM$ $(\Phi = \pi/2)$

$0 < e < 1$ ⟶ elliptical orbits with mass M at focus $(r = 0)$.

Semi-Major axis ($\phi = 0 - \pi$)

$$2a = \frac{r_0}{1+e} + \frac{r_0}{1-e} = \frac{2r_0}{1-e^2} \qquad \boxed{a = \frac{h^2}{GM(1-e^2)}}$$



Precession of Planetary Orbits in GR:

$$\boxed{\frac{d^2U}{d\phi^2} + U = \frac{GM}{h^2} + \frac{3GM}{c^2}U^2} \qquad U = \frac{1}{r}$$

In weak field limit: ($GM \ll rc^2$: for all points on orbit).

GR treated as perturbation.

$$U(\phi) = \frac{GM}{h^2}u(\phi) \qquad \frac{d^2u}{d\phi^2} + u = 1 + \underbrace{\frac{3(GM)^2}{c^2h^2}}_{\alpha}u^2$$

$$\boxed{\frac{d^2u}{d\phi^2} + u = 1 + \alpha u^2} \qquad \alpha \equiv 3\left(\frac{GM}{hc}\right)^2$$

Expand $u(\phi)$ in small parameter α:

$$u(\phi) = 1 + e\cos(\phi) + \alpha u_1(\phi) + \alpha^2 u_2(\phi) + \ldots$$

At first order in α:

$$\frac{d^2u_1}{d\phi^2} + u_1 = (1 + e\cos(\phi))^2 = 1 + 2e\cos(\phi) + e^2\cos^2\phi$$

$$= \left(1 + \frac{1}{2}e^2\right) + 2e\cos\phi + \frac{1}{2}e^2\cos(2\phi)$$

$$u_1(\phi) = \left(1 + \frac{1}{2}e^2\right) + e\phi\sin\phi - \frac{1}{6}e^2\cos 2\phi$$

$$U(\phi) = \frac{GM}{h^2}\left[1 + e\cos\phi + \alpha\left(1 + \frac{1}{2}e^2 + \underbrace{e\phi\sin\phi}_{\substack{\text{only significant} \\ \text{(accumulating) term}}} - \frac{1}{6}e^2\cos 2\phi\right)\right.$$

$$U(\phi) \approx \frac{GM}{h^2}\left(1 + e\cos(\phi(1-\alpha))\right)$$

Precessing ellipse, with angle at perihelion ($dr/d\phi = 0$) increasing each orbit by:

$$\Delta\phi = 2\pi\left(\frac{1}{1-\alpha} - 1\right) \approx 2\pi\alpha$$

Using $\alpha = 3\left(\frac{GM}{hc}\right)^2$ $\quad a = \frac{h^2}{GM(1-e^2)}$

$$\boxed{\Delta\phi = \frac{6\pi GM}{a(1-e^2)c^2}} \rightarrow \text{Excellent agreement with mercury!}$$

Bending of Light (Gravitational Lensing)

$$\boxed{\frac{d^2U}{d\phi^2} + U = \frac{3GM}{c^2}U^2}$$

For Minkowski Space time ($M=0$):

Straight line with impact parameter, b:

$$U = \frac{\sin\phi}{b}$$

For non-zero M:

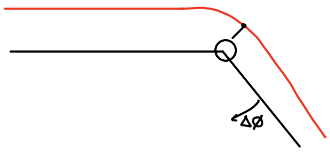
$$bU(\phi) = u(\phi) \quad \text{where} \quad \frac{d^2u}{d\phi^2} + u = \underbrace{\frac{3GM}{c^2b}}_{\equiv\beta}u^2$$

Assume dimensionless $\beta = \frac{3M}{b} \ll 1$

$$U(\phi) = \sin\phi + \beta U_1(\phi) + \beta^2 U_2(\phi) + \dots$$

the $O(\beta)$ correction satisfies:

$$\frac{d^2 U_1}{d\phi^2} + U_1 = \sin^2\phi = \frac{1}{2}(1-\cos 2\phi).$$



Solved by:

$$U_1(\phi) = C_1 \sin\phi + C_2\cos\phi + \frac{1}{2}\left(1 + \frac{1}{2}\cos 2\phi\right)$$

$$U(\phi) = \frac{\sin\phi}{b} + \frac{3GM}{c^2 b^2}\left[\frac{2}{3}\cos\phi + \frac{1}{2}\left(1 + \frac{1}{3}\cos 2\phi\right)\right]$$

Overall: $\boxed{\Delta\phi \approx \frac{4GM}{c^2 b}}$ → verified by effects of Sun by eddington in 1919.

Schwarzchild Black Holes

Singularities in the Schwarzchild Metric:

$$\boxed{ds^2 = c^2\left(1 - \frac{2\mu}{r}\right)dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\Omega^2} \rightarrow \mu = \frac{GM}{c^2}$$

The Schwarzchild metric is singular at $r=0$, $r=2\mu$ $(g_{rr}=\infty)$

→ Schwarzchild Radius

$$r_s \equiv 2\mu = \frac{2GM}{c^2}$$

→ Only important where $r_s > R$:

→ As Schwarzchild Solution is a vaccum soln - if $r_s < R$: no consequence

Nature of Singularities:

→ Investigating nature of singularities at $r=0$, $r=2\mu$

Since $R_{\mu\nu} = 0$ for $r > 0$: Vacumm Spacetime

→ $\boxed{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \propto \frac{\mu^2}{r^6}}$ Kreschman Scalar

→ tells us about magnitude of tidal effects

Regular at $r=2\mu$, Singular at $r=0$. → $r=2\mu$: Simply a coordinate singularity.

$r=0$: genuine intrinsic singularity.

Example: 2 sphere in cylindrical coords.

$$ds^2 = \frac{a^2 d\rho^2}{(a^2-\rho^2)} + \rho^2 d\phi^2 \qquad (0 \le \rho \le a).$$

$\mathbb{R}^3$ in Cylindrical Polar Coordinates:

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

Staying on surface requires:

$$\rho^2 + z^2 = a^2 \longrightarrow \rho\, d\rho = -z\, dz$$

Induced line element:

$$\boxed{ds^2 = \frac{a^2 d\rho^2}{a^2 - \rho^2} + \rho^2 d\phi^2} \longrightarrow$$ Metric is singular at $\rho = a$.

Curvature is regular.

Analagous Behaviour in Schwarzchild Metric:

$$\boxed{ds^2 = C^2\left(1-\frac{2N}{r}\right)dt^2 - \left(1-\frac{2N}{r}\right)^{-1}dr^2 - r^2 d\Omega^2}$$

Region I ($r=2N$)

$\underline{\underline{e}}_0 = \frac{\partial}{\partial t}$ is timelike

$$\boxed{g(\underline{e}_0,\underline{e}_0) = g_{00} = C^2\left(1-\frac{2N}{r}\right) > 0}$$

Curve at constant (r,θ,ϕ) is timelike

$$\boxed{g(\underline{e}_1,\underline{e}_1) = g_{rr} < 0}$$

Spatial basis vectors are spacelike

Region II ($r<2N$)

$\underline{\underline{e}}_0$ is spacelike $(g_{00}<0)$

Basis vector $\frac{\partial}{\partial r}$ is timelike.

A particle cannot stay fixed (r,θ,ϕ) – no stationary observers

Switched roles for r and t: $r<2N$

Causal Structure:

By plotting timelike and null curves:

light cones display causal structure of spacetime

Radial Null Geodesic:

$$\boxed{ds^2 = C^2\left(1-\frac{2N}{r}\right)dt^2 - \left(1-\frac{2N}{r}\right)^{-1}dr^2 = 0} \qquad d\theta, d\phi = 0$$

$$\frac{d(ct)}{dr} = \pm\left(1-\frac{2N}{r}\right)^{-1}$$

Plus(+) Case:

Outgoing in region I $(t\uparrow, r\uparrow)$

$$\boxed{Ct = r + 2N\ln\left|\frac{r}{2N}-1\right| + \text{Const.}}$$

Where the absolute value means we can consider both $r>2N$ and $r<2N$

Minus (–) Case:

Ingoing in region I $(t\uparrow r\downarrow)$

$$\boxed{Ct = -r - 2N\ln\left|\frac{r}{2N}-1\right| + \text{Const}}$$

Ingoing and Outgoing Solns:

Related by time reversal: $t \to -t$

Outgoing (+ Sign) → asymptotically, $Ct = r + \text{Const}$ as $r\to\infty$

(usual Minkowski light cones).

$Ct \to -\infty$ as $r\to 2N$ (lightcone squashed radially).

Ingoing: (– Sign) asympotically, $Ct \to -r + \text{Const}$ as $r\to\infty$

(usual Minkowski light cones).

$Ct \to \infty$ as $r\to 2N$ (light cone squashed radially).

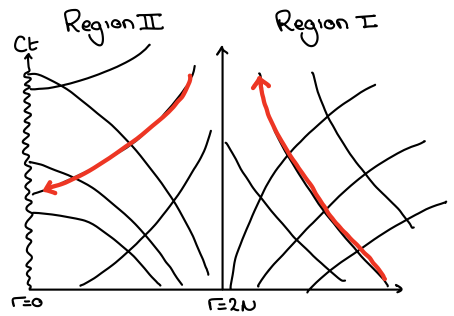
Confusion: It seems to take an infinite coordinate time for an ingoing null geodesic to reach $r=2N$ but a finite change in affine parameter.

Shown by: $\frac{dt}{d\lambda} = k\left(1-\frac{2N}{r}\right)^{-1}$ where λ is an affine parameter. $\longrightarrow$ $\frac{dr}{d\lambda} = \pm C\left(1-\frac{2N}{r}\right)k\left(1-\frac{2N}{r}\right)^{-1} = \pm kC$

Thus: $\boxed{r = \pm ck\lambda + \text{Const.}}$

↳ finite λ to reach $r=2N$

Ingoing photon passes through $r=2N$ Region I and II.



Extending (affine param) geodesic into region II

However $\frac{dt}{d\lambda} < 0$ in region 2: $\frac{dt}{d\lambda} = k\left(1 - \frac{2N}{\lambda}\right)^{-1}$

> The Causal feature of Region I includes a type-II region in which the forward light cone is directed towards $r=0$.

What this means:

→ Any particle that falls through $r=2N$ will eventually reach the Singularity at $r=0$, no matter how they accelerate.

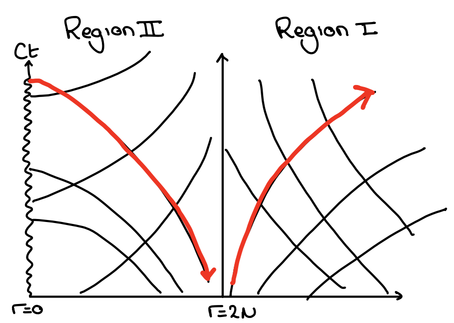
→ The hypersurface $r=2N$ is an event horizon

↳ Out most boundary of a region of Spacetime from which particals cannot escape in the future to $r=\infty$: (Black Hole).

→ Black holes are regions from which no particle can escape in the future to spatial infinity.

Causal Past of Region I:

Considering outgoing null geodesics instead in region I:



Seem to: Emerge from $r=2N$ at $t=-\infty$

In fact: finite change in affine parameter

$$\frac{dr}{d\lambda} = \pm kc \quad \text{and} \quad \frac{dt}{d\lambda} = k\left(1 - \frac{2N}{r}\right)^{-1}$$

Extend outgoing rays in region I into region II.

$$\frac{dr}{d\lambda} > 0 \quad \text{and} \quad \frac{dt}{d\lambda} < 0$$

Defines a different type-II region in spacetime:

↳ A white hole from which all particles are expelled.

↳ Not believed to exist in nature - require singularity from past.

Radially Infalling Massive Particles:

Setting $h=0$ in the energy equation for a massive particle gives:

$$\boxed{\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = \frac{1}{2}c^2(k^2-1)}$$

Differentiating w.r.t proper time τ gives:

$$\ddot{r} = -\frac{GM}{r^2}$$

↳ Newtonian like - But in proper time, τ.

For a particle Starting at rest, $r=\infty$: $K=1$.

$$\dot{r} = -\sqrt{\frac{2GM}{r}} = -\sqrt{\frac{2Nc^2}{r}} \qquad \boxed{c(\tau-\tau_0) = \frac{2}{3}\left[\left(\frac{r_0^3}{2N}\right)^{1/2} - \left(\frac{r^3}{2N}\right)^{1/2}\right]} \qquad \text{where } r(\tau_0) = r_0$$

> It takes finite proper time to reach $r=2N$ from $r_0>2N$ and the particle passes through Smoothly

$$\dot{r} = -\sqrt{\frac{2Nc^2}{r}} \quad \text{and} \quad \dot{t} = \left(1-\frac{2N}{r}\right)^{-1}$$

Path $ct(r)$ from:

$$\frac{d(ct)}{dr_0} = \frac{c\dot{t}}{\dot{r}} = -c\,\frac{(1-2N/r)^{-1}}{\sqrt{2Nc^2/r}} = -\sqrt{\frac{r}{2N}}\left(1-\frac{2N}{r}\right)^{-1}$$

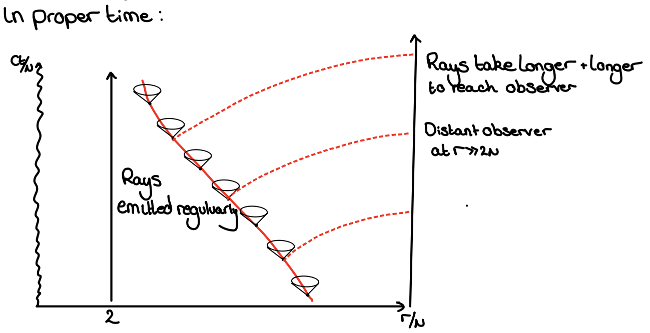
Solution with $r=r_0$ and $ct=ct_0$ is

$$c(t-t_0) = 2N\int_{r_0/2N}^{r/2N} \frac{x^{3/2}}{x-1}\,dx = -2N\left[\frac{2}{3}x^{3/2} + 2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right|\right]_{r_0/2N}^{r/2}$$

Takes Infinite Coordinate time to fall into $r\to 2N$
↳ as integral diverges logarithmically as $r\to 2N$.

From Perspective of distant Stationary observer:

Considering photons emitted by particle at equal intervals in proper time:



Infalling particle appears to take infinite τ_{obs} to reach $r=2N$
↳ Infinitely redshifted and fades away.

Eddington Finkelstein Coordinates:

We wish to adopt Coordinates that cover region I and the type II region in its causal future without coordinate Singularity

Ie Coordinates Such that radially Infalling Photons are Continuous through $r=2N$.

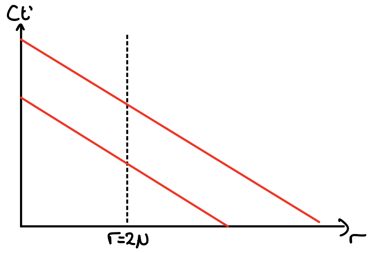
Recalling: Radial Null Geodesics

Outgoing in region I. $\boxed{ct = r + 2N\ln\left|\frac{r}{2N}-1\right| + \text{Const}}$

Ingoing in region II $ct = -r - 2N\ln\left|\frac{r}{2N}-1\right| + \text{Const.}$

Define a new time coordinate:

$$\boxed{ct' \equiv ct + 2N \ln\left|\frac{r}{2N} - 1\right|}$$



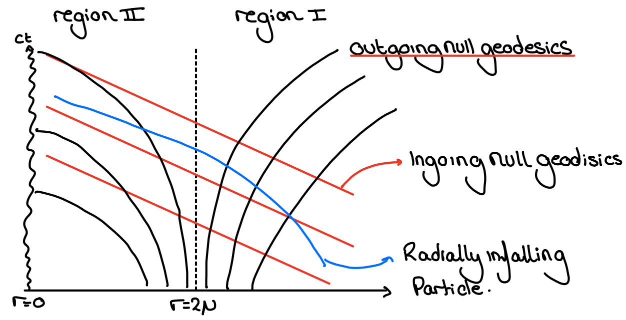
In going radial null geodesic becomes:

$$\boxed{ct' = -r + \text{Const}}$$

Outgoing null geodesics:

$$ct' = r + 4N \ln\left|\frac{r}{2N} - 1\right| + \text{Const.}$$

In Region I Still extend back to $r = 2N$ at finite λ in the past.



Ingoing: Smooth

Outgoing: Discontinous.

Extend past $r = 2N$ at finite affine coordinates in past

Note: Type II region in Casaul past of Region I not included.

$\boxed{(t', r, \theta, \phi) \text{ are Ingoing Eddington-Finkelstein Coordinates.}}$

Finding the line element:

Using $c\,dt' = c\,dt + \left(\frac{r}{2N} - 1\right)^{-1} dr$

Schwarzchild line element: $ds^2 = c^2\left(1 - \frac{2N}{r}\right)dt^2 - \left(1 - \frac{2N}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$

In IEF: $$\boxed{ds^2 = c^2\left(1 - \frac{2N}{r}\right)dt'^2 - \frac{4Nc}{r}\,dt'\,dr - \left(1 + \frac{2N}{r}\right)dr^2 - r^2 d\Omega^2}$$

No Longer Singular at $r = 2N$:

$g_{tt} = 0$ but non-diagonal metric so Inverse exists.

At $r \to \infty$: Metric $\longrightarrow$ Minkowsi

Outgoing Eddington Finkel Stein Coordinates:

We wish to adopt Coordinates that Cover region I and the type II region in its Causal past without Coordinate Singularity

Ie Coordinates Such that radially Outgoing Photons are Continuous through $r = 2N$.

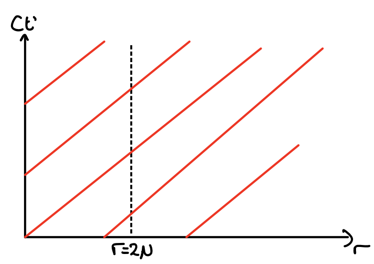
Recalling: Radial Null Geodesics

Outgoing in region I. $$\boxed{ct = r + 2N \ln\left|\frac{r}{2N} - 1\right| + \text{Const}}$$

Ingoing in region II $$ct = -r - 2N \ln\left|\frac{r}{2N} - 1\right| + \text{Const.}$$

Define a new time coordinate:

$$\boxed{ct^* \equiv ct - 2N \ln\left|\frac{r}{2N} - 1\right|}$$



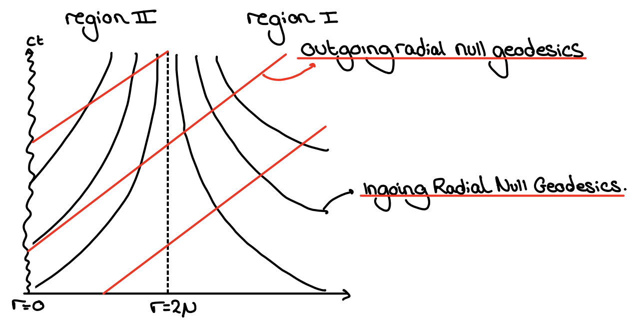
In going radial null geodesic becomes:

$$\boxed{ct^* = r + \text{Const}}$$

Ingoing null geodesics:

$$ct^* = -r - 4N \ln\left|\frac{r}{2N} - 1\right| + \text{Const.}$$

↳ In Region I - Still extend back to $r = 2N$ at finite λ in the past:



↳ Ingoing: Smooth

Outgoing: Discontinous.

Extend past $r = 2N$ at finite affine Coordinates in pas

$\boxed{(t^*, r, \theta, \phi) \text{ are outgoing Eddington-Finkelstein Coordinates.}}$

Line element in (t^*, r, θ, ϕ)

$$\boxed{ds^2 = c^2\left(1 - \frac{2N}{r}\right)dt^{*2} + \frac{4Nc}{\gamma}\,dt^* dr - \left(1 + \frac{2N}{r}\right)dr^2 - r^2 d\Omega^2}$$

Kruskal - Szekeres Coordinates: → Combines them all in a non singular way

Formation of Black Holes:

↳ expected to form as the endpoint of stellar evolution of sufficiently massive stars

Star Contracts to radius $< \frac{2GM}{c^2}$ → Collapse to Singularity

Stellar Mass	Endpoint	Radius
$M < 1.4 M_\odot$	White dwarf (Electron Degeneracy Pressure)	$\sim R_{earth}$
$1.4 M_\odot < M < 4 M_\odot$	Neutron Star (Neutron Degeneracy Pressure)	~ 10 km
$M > 4 M_\odot$	Black Hole	

Spherically Symmetric Collapse of Dust:

↳ Modelling a star collapsing to a black hole.

For this a star is considered:

↳ Symmetric/isotropic ball of dust collapsing under gravity

No pressure Support: → dust falls on geodesic

By Birkhoff's Theorem: the Spacetime outside the dust cloud is described by the Schwarzschild Solution

↳ Can treat outside edge of Cloud as particle free-falling radially in Schwarzchild Spacetime.

Model Set up:

↳ Collapse Starts at rest at infinity - $k = 1$

Stationary observer at rest at large radius $r \gg N$.

↳ Will never see Surface of the dust cloud pass through $r = 2N$.

Light emitted within $r = 2N$ will instead end up at $r = 0$.

↳ Light increasingly redshifted as $r \to 2N$.

Frequency (and arrival rate) of Photons falls to zero.

Considering light ray sent from edge of cloud at (ct'_E, r_E) and recieved by a distant stationary observer at (ct'_R, r_R)

Radial Null Geodesics:

Schwarzschild: $ct = r + 2N\ln\left|\frac{r}{2N} - 1\right| + \text{Const.}$

$\downarrow$

Ingoing Eddingston Finkelstein: $ct' = r + 4N\ln\left|\frac{r}{2N} - 1\right| + \text{Const.}$

$$ct_R' - r_R - 4N\ln\left|\frac{r_R}{2N} - 1\right| = ct_E' - r_E - 4N\ln\left|\frac{r_E}{2N} - 1\right|$$

$\rightarrow$ t'_R is related to proper time of distant observer by:

$$ds^2 = c^2 d\tau_R^2 = c^2\left(1 - \frac{2N}{R}\right) dt'^2_R$$

$$\tau_R \approx t'_R + \text{Const} \quad \text{as } r \rightarrow \infty$$

In IEF – worldline of the edge of the cloud is regular through $r = 2N$

$$ct_R' - r_R - 4N\ln\left|\frac{r_R}{2N} - 1\right| = ct_E' - r_E - 4N\ln\left|\frac{r_E}{2N} - 1\right|$$

$$c\tau_R \approx ct'_R + \text{Const} \approx \text{Const} - 4N\ln\left|\frac{r_E}{2N} - 1\right|$$

It follows that as the edge of the cloud is observed to approach $r_E = 2N$.

$$\boxed{r_E(\tau_R) \approx 2N(1 + ae^{-c\tau_R/4N})}$$

$\rightarrow$ Observer see's cloud approach Schwarzchild radius exponentially with characteristic time $\frac{4N}{c}$.

Asymtotic Redshift:

$$\boxed{\frac{\nu_R}{\nu_E} = \frac{d\tau_E}{d\tau_R} = \frac{d\tau_E}{dr_E}\frac{dr_E}{d\tau_E} \sim \left.\frac{d\tau_E}{dr_E}\right|_{r_E = 2N} e^{-\frac{c\tau_R}{4N}}}$$

$\rightarrow$ Frequency becomes exponentially small

Cosmology

If we smooth the universe over sufficiently large scales ($\sim$Mpc) – very symmetric

Isotropic:

↳ A universe is isotropic if it looks the same in all directions from any give point. (no preferred direction in the large-scale structure).

Homogenous

↳ Its properties are the same at every point in space (on large scales) roughly uniform matter and energy density

Cosmological Principle:

↳ The universe is both homogenous and isotropic on large scales

Fundamental Observers

↳ Defined by: Observe the universe to be isotropic (look the same in all directions).
At rest relative to average motion of matter in the universe.

Have the following properties:

↳ 1. Must comove with matter (rest in CMB frame - no dipole in CMB)

2. Must be free-falling (non-accelerating) - must follow geodesics → Note nice definition: Natural straightest possible paths through curved spacetime.

3. Hyper surfaces of constant proper density must be orthogonal to world lines

Synchronous Coordinates:

Assign fixed spatial coordinates x^i $(i=1,2,3)$ to each fundemental observer.

Label surface of homogenity with proper time as measured by one of the fundemental observers

↳ Synchronous (cosmic) time : $x^0 = t$.

Line element in Synchronous Coordinates:

$$\boxed{ds^2 = c^2 dt^2 + g_{ij}(t, \vec{x})\, dx^i dx^j}$$ (explicit sum over i,j)

Proof:

i) Consider displacement along world line of fundemental observer:

$dx^\mu = (dt, \vec{0})$

↳ $\boxed{ds^2 = g_{00} dt^2 = c^2 dt^2 = c^2 d\tau^2}$ → $dt = d\tau$ (so can take $t = \tau$)

ii) 4-velocity of fundemental observers has to be orthogonal to any displacement $dx^\mu (0, dx^i)$ in the surfaces of homogeneity.

$$U^\mu = \frac{dx^\mu}{d\tau} = \delta^\mu{}_0$$

$$g_{\mu\nu} U^\mu dx^\nu = 0 \quad \forall\, dx_i$$

$$g_{0\nu} U^0 dx^\nu = 0 \quad \forall\, dx_i$$

$$g_{0i} = 0$$

iii) Must verify world lines $x^\mu = (t, x^i)$ $\forall x^i$ are geodesics:

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} = 0 \longrightarrow \Gamma^\mu_{00} = 0$$

This is satisfied for the metric in Eq (1):

$$\boxed{\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\nu} (2\partial_0 g_{\nu 0} - \partial_\nu g_{00}) = 0}$$ as $g_{00} = c^2$ and $g_{0i} = 0$.

Robertson-Walker Metric:

Intrinsic Geometry of $t=$ const hypersurfaces must be homogenous and isotropic for all t.

Induced line element in $t=$ const. $\boxed{ds^2 = g_{ij}(t, \vec{x})\, dx^i dx^j}$ → Requiring homogeneity and isotropy for all t requires $\boxed{g_{ij}(t, \vec{x}) = -a^2(t)\, \gamma_{ij}(\vec{x})}$

↳ Every component evolves in same way with t.

Where: $a(t)$ - Scale factor

$\gamma_{ij}(\vec{x})$ - 3D metric upto Scaling

↳ Transforms as a type $(0,2)$ - 3D tensor under time independent $x^i \to x^{i'}$

Isotropy Requires:

↳ γ_{ij} must be Spherically Symmetric

$$\boxed{d\sigma^2 \equiv \gamma_{ij}\, dx^i dx^j = B(r)\, dr^2 + r^2\, d\Omega^2} \longrightarrow \text{Spherical polar coords.}$$

It follows: $\gamma_{rr} = B(r)$ $\quad \gamma^{rr} = 1/B(r)$

$\gamma_{\theta\theta} = r^2$ $\quad \gamma^{\theta\theta} = 1/r^2$

$\gamma_{\phi\phi} = r^2 \sin^2$ $\quad \gamma^{\phi\phi} = \dfrac{1}{r^2 \sin^2\theta}.$

Constructing 3D metric Connection:

↳ ${}^{(3)}\Gamma^i_{jk}$ from γ_{ij} with (independent)

$${}^{(3)}\Gamma^r_{rr} = \frac{1}{2B}\frac{dB}{dr} \qquad {}^{(3)}\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad {}^{(3)}\Gamma^r_{\theta\theta} = -\frac{r}{B} \qquad {}^{(3)}\Gamma^\phi_{r\phi} = \frac{1}{r}$$

$${}^{(3)}\Gamma^r_{\phi\phi} = \frac{-r\sin^2\theta}{B} \qquad {}^{(3)}\Gamma^\phi_{\theta\phi} = \cot\theta \qquad {}^{(3)}\Gamma^\theta_{r\theta} = \frac{1}{r}$$

The 3D Riemann Curvature tensor is given by:

$$\boxed{{}^{(3)}R_{ijk}{}^{l} = -\partial_i {}^{(3)}\Gamma^l_{jk} + \partial_j {}^{(3)}\Gamma^l_{ik} + {}^{(3)}\Gamma^m_{ik}\, {}^{(3)}\Gamma^l_{jm} - {}^{(3)}\Gamma^m_{jk}\, {}^{(3)}\Gamma^l_{im}}$$

Only 3 non-zero components for Spherically Symmetric metric:

$${}^{(3)}R_{r\theta r\theta} = \frac{r}{2B}\frac{dB}{dr}, \quad {}^{(3)}R_{r\phi r\phi} = \frac{r}{2B}\frac{dB}{dr}\sin^2\theta, \quad {}^{(3)}R_{\theta\phi\theta\phi} = \left(1 - \frac{1}{B}\right) r^2 \sin^2\theta.$$

3D Ricci Tensor:

↳ $\boxed{{}^{(3)}R_{ij} = \gamma^{kl\,(3)} R_{kijl}}$ $\longrightarrow$ Non zero (independent) Components: ${}^{(3)}R_{rr} = -\dfrac{1}{rB}\dfrac{dB}{dr}$ | ${}^{(3)}R_{\theta\theta} = -1 + \dfrac{1}{B} - \dfrac{r}{2B^2}\dfrac{dB}{dr}$ | $R_{\phi\phi} = \sin^2\theta\, {}^{(3)}R_{\theta\theta}.$

3D Ricci Scalar:

$${}^{(3)}R = \gamma^{ij\,(3)} R_{ij} = -\frac{2}{r^2}\left(1 - \frac{1}{B} + \frac{r}{B^2}\frac{dB}{dr}\right)$$

$${}^{(3)}R = -\frac{2}{r^2}\left[1 - \frac{d}{dr}\left(\frac{r}{B}\right)\right]$$

By homogeneity – Ricci Scalar cannot depend on position:

↳ $\boxed{{}^{(3)}R = -6K}$ where K constant

Thus $1 - \dfrac{d}{dr}\left(\dfrac{r}{B}\right) = 3Kr^2$, $\quad \dfrac{r}{B} = A + r - Kr^3$, $\quad \boxed{B = \dfrac{1}{A/r + (1 - Kr^2)}}$

By homogenity (also), Curvature Invariants must be independent of r:

↳ $R_{ij}R^{ij} = 12K^2 + \dfrac{3A^2}{2r^6} \longrightarrow \boxed{A = 0}$ $\quad \boxed{B = \dfrac{1}{1 - Kr^2}}$

Thus results:

$$\boxed{d\sigma^2 = \gamma_{ij}\, dx^i dx^j = \frac{dr^2}{(1-Kr^2)} + r^2 d\Omega^2}$$

Robertson-Walker Line Element

$$\boxed{ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]}$$

Maximally Symmetric Spaces

$$\boxed{R_{abcd} = K\left(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}\right)}$$

Subing in $B(r) = (1-K^2 r)^{-1}$:

$$^{(3)}R_{rr} = \frac{-2K}{(1-Kr^2)} = -2K\gamma_{rr} \quad ; \quad ^{(3)}R_{\theta\theta} = -2Kr^2 = -2K\gamma_{\theta\theta} \quad ; \quad ^{(3)}R_{\phi\phi} = -2Kr^2 \sin^2\theta = -2K\gamma_{\phi\phi}$$

Generally: $\boxed{^{(3)}R_{ij} = -2K\gamma_{ij}}$

Geometry of 3D Spaces:

$$\boxed{d\sigma^2 = \gamma_{ij}\, dx^i dx^j = \frac{dr^2}{(1-Kr^2)} + r^2 d\Omega^2}$$

For $K=0$: $\boxed{d\sigma^2 = dr^2 + r^2 d\Omega^2}$ → Euclidean Space in Spherical Polar Coords

↳ Our Universe appears to be Spatially flat.

For $K>0$:

Introducing new comoving radial coordinate χ with:

$$\boxed{r = \frac{1}{\sqrt{K}} \sin(\sqrt{K}\chi) \equiv S_K(\chi)}$$

$$dr = \cos(\sqrt{K}\chi)\, d\chi \longrightarrow d\sigma^2 = \frac{\cos^2(\sqrt{K}\chi)}{(1-\sin^2(\sqrt{K}\chi))}\, d\chi^2 + \overbrace{S_K^2(\chi)}^{r^2}\, d\Omega^2$$

$$\boxed{d\sigma^2 = d\chi^2 + S_K^2(\chi)\, d\Omega^2}$$

→ ≡ Line element on the Surface of 3-Sphere of radius $1/\sqrt{K}$ embedded in 4D Euclidean Space

Shown by: Cartesian Coordinates in $\mathbb{R}^4$: $ds^2 = dw^2 + dx^2 + dy^2 + dz^2$

3-Sphere $w^2 + x^2 + y^2 + z^2 = 1/K$

Parametrised by: $w = 1/\sqrt{K}\, \cos(\sqrt{K}\chi)$, $\quad x = S_K(\chi)\sin\theta\cos\phi$

$y = S_K(\chi)\sin\theta\sin\phi \quad z = S_K(\chi)\cos\theta$.

θ, ϕ - usual ang coords; $0 \le \sqrt{K}\chi \le \pi$

Induced metric on 3D Sphere:

$$dx^2 + dy^2 + dz^2 = \cos^2(\sqrt{K}\chi)\, d\chi^2 + S_K^2(\chi)\, d\Omega^2$$

$$dw^2 = \sin^2(\sqrt{K}\chi)\, d\chi^2.$$

It follows for a Maximally-Symmetric Space with $K>0$:

↳ Compact/Closed cosmological model

Area of 2-Sphere ($\chi = \text{const}$)

$$\boxed{A(\chi) = 4\pi S_k(\chi)}$$

Volume of 3D Space:

$$V = \int \sqrt{\det \gamma}\, d^3x = \int_0^{\pi/\sqrt{k}} d\chi \int d\Omega\, S_k^2(\chi)$$

$$= \int_0^{\pi/\sqrt{k}} 4\pi S_k^2(\chi)\, d\chi = \frac{2\pi^2}{k^{3/2}}$$

For $k < 0$:

Introducing new comoving radial coordinate χ with:

$$\boxed{r = \frac{1}{\sqrt{k}} \sinh(\sqrt{|k|}\chi) \equiv S_k(\chi)}$$

$$dr = \cosh(\sqrt{|k|}\chi)\, d\chi \longrightarrow d\sigma^2 = \frac{\cosh^2(\sqrt{k}\chi)}{(1 - \sinh^2(\sqrt{k}\chi))} d\chi^2 + \overbrace{S_k^2(\chi)}^{r^2} d\Omega^2$$

$$\boxed{d\sigma^2 = d\chi^2 + S_k^2(\chi)\, d\Omega^2}$$

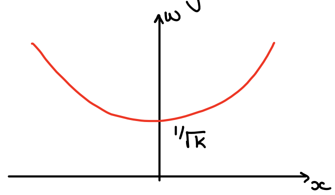
Open Space line element induced on the hyperboloid (open → infinite volume)

$$w^2 - x^2 - y^2 - z^2 = \frac{1}{|k|} \quad \text{with} \quad w = \frac{1}{\sqrt{|k|}} \cosh \sqrt{|k|}\,\chi \quad (0 \leq \sqrt{|k|}\,\chi \leq \infty)$$

$$x = S_k(\chi) \sin\theta \cos\phi \quad \text{etc}$$

Have:

$$ds^2 = dw^2 - dx^2 - dy^2 - dz^2 \longrightarrow ds^2 = -d\chi^2 - S_k^2(\chi)\, d\Omega^2$$



Overall: Robertson-Walker Line Element:

$$\boxed{ds^2 = c^2 dt^2 - a^2(t)\left[d\chi^2 + S_k^2(\chi)\, d\Omega^2\right]}$$

↳ Area function is defined as:

$$\boxed{S_k(\chi) = \begin{cases} \sin(\sqrt{k}\,\chi)/\sqrt{k} & \text{for } k > 0 \text{ (closed)} \\ \chi & \text{for } k = 0 \text{ (flat)} \\ \sinh(\sqrt{|k|}\,\chi)/\sqrt{k} & \text{for } k < 0 \text{ (open)} \end{cases}}$$

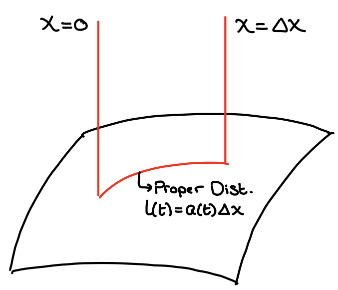
Expanding Universe:

Important:

Worldlines of Fundemental Observer: — χ and ang. coords Ω constant

Proper time measured by observers: t

Overall Scale of Spatial Sections: Scale factor, $a(t)$.



Proper distance changes at fractional rate:

$$\boxed{\frac{1}{l}\frac{dl}{dt} = \frac{1}{a}\frac{da}{dt} \equiv H(t)}$$

Expanding Distance: $H > 0$

↳ Every fundemental observer sees every other moving away at a fractional rate - $H(t)$.

Cosmological Redshift:

↳ In an expanding universe - light is redshifted during propogation.

Photon emitted by a fundemental observer at coordinates $(t_E, 0, 0, 0)$

Which is recieved later by a fundemental observer $(t_R, \chi_R, \theta_R, \phi_R)$:

Considering radial null Geodesics:

↳ $x^\mu(\lambda) = (t(\lambda), \chi(\lambda), \theta_R, \phi_R)$

4-momentum of photon:

↳ $p^\mu = \frac{dx^\mu}{d\lambda} = (p^0, p^1, 0, 0)$, $p_\mu = (p_0, p_1, 0, 0)$ where $p_0 = c^2 p^0$ $p_1 = -a^2 p^1$

↳ Photon travels on a null geodesic, parallel transporting p^μ.

Energy of photon as measured by a fundemental observer (recall $u^\mu = \delta_0{}^\mu$)

$$E = g(p_\mu, u) = g_{\mu\nu} p^\mu u^\nu = p_\mu u^\mu = p_0$$
$$= p_0 = c^2 p^0$$

Need to determine relation between p^0 at emission and reception to determine affect of propogation on Energy.

'Lagrangian' for Radial Motion:

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$
$$= c^2 \dot{t}^2 - a^2 \dot{\chi}^2$$

Euler-Lagrange Equation for χ gives:

$$\frac{\partial L}{\partial \chi} = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\chi}}\right) \longrightarrow a^2 \dot{\chi} = \text{Const.}$$

$$\boxed{p^1 \propto \frac{1}{a^2}}$$

4 momentum: Null vector:

↳ $0 = c^2 (p^0)^2 - a^2 (p^1)^2$ $E = c^2 p^0 \propto a p^1 \propto \frac{1}{a}$.

Cosmological Redshift in expanding Universe:

$$\boxed{1 + z = \frac{a(t_R)}{a(t_E)}}$$

Cosmological Field Equations

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}} \qquad \kappa \equiv \frac{8\pi G}{c^4}$$

↳ Cosmological Constant – Important on Cosmological Scales.

Isotropy → Ideal fluid form:

$$\boxed{T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu}}$$

with u^μ - 4 velocity of fundamental observers

$$u^\mu = \delta^\mu_0$$

Homogeneity → Proper energy density ρc^2 and pressure p → time dependent only.

Robertson-Walker Line Element:

$$ds^2 = c^2 dt^2 - a^2(t) \gamma_{ij} dx^i dx^j$$

$$g_{00} = c^2 \longrightarrow g^{00} = 1/c^2$$

$$g_{ij} = -a^2 \gamma_{ij} \longrightarrow g^{ij} = -1/a^2 \gamma^{ij}$$

γ^{ij} (inverse of γ_{ij})

Connection Coefficients for the Robertson-Walker Metric:

↳ $$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho})$$

Thus $$\Gamma^0_{ij} = \frac{1}{2} g^{0\sigma} (\partial_i g_{\sigma j} + \partial_j g_{\sigma i} - \partial_\sigma g_{ij})$$

$$= \frac{1}{2} g^{00} (\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij})$$

$$= \frac{1}{2c^2} \left(\frac{da^2}{dt} \gamma_{ij} \right) = \frac{\dot{a} a}{c^2} \gamma_{ij}$$

Similarly: $\Gamma^i_{0j} = \frac{\dot{a}}{a} \delta^i{}_j$; $\Gamma^i_{jk} = {}^{(3)}\Gamma^i_{jk}$

Ricci tensor:

$$R_{\mu\nu} = -\partial_\rho \Gamma^\rho_{\mu\nu} + \partial_\mu \Gamma^\rho_{\rho\nu} + \Gamma^\sigma_{\rho\nu} \Gamma^\rho_{\mu\sigma} - \Gamma^\sigma_{\mu\nu} \Gamma^\rho_{\rho\sigma}$$

By isotropy: $R_{0\alpha} = 0$ $\quad \boxed{R_{00} = 3\frac{\ddot{a}}{a}} \quad \boxed{R_{ij} = -\frac{1}{c^2} (\ddot{a} a + 2\dot{a}^2 + 2Kc^2) \gamma_{ij}}$

Rewriting Einstein Field Equations:

↳ $$\boxed{R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + \Lambda g_{\mu\nu}}$$ → where $T \equiv g_{\mu\nu} T^{\mu\nu}$ (trace of energy momentum tensor).

For the ideal fluid we have:

↳ $$T = g_{\mu\nu} \left(\rho + \frac{p}{c^2} \right) u^\mu u^\nu - p g_{\mu\nu} g^{\mu\nu}$$

$$\boxed{= \rho c^2 - 3p}$$

So that:

$$T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T = \left(\rho + \frac{p}{c^2} \right) u_\mu u_\nu - \frac{1}{2} (\rho c^2 - p) g_{\mu\nu} \qquad u_\mu = g_{\mu 0} = c^2 \delta_{\mu 0}.$$

Non-Zero Einstein Equations reduce to:

$$\boxed{R_{00} = -4\pi G \left(\rho + \frac{3p}{c^2} \right) + \Lambda c^2} + R_{00} = 3\frac{\ddot{a}}{a} \longrightarrow$$

$$\boxed{R_{ij} = -\frac{4\pi G}{c^2} \left(\rho - \frac{p}{c^2} \right) a^2 \gamma_{ij}} + R_{ij} = -\frac{1}{c^2} (\ddot{a} a + 2\dot{a}^2 + 2Kc^2) \gamma_{ij} \longrightarrow$$

The Friedman Equations:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) + \Lambda c^2}$$

$$\boxed{\left(\frac{\dot{a}}{a} \right)^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3} \rho + \frac{1}{3} \Lambda c^2}$$

Conservation of Energy-Momentum Tensor

Noted by $\nabla_\mu T^{\mu\nu} = 0$.

$$\boxed{T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu}}$$

$\longrightarrow$ We have: $0 = \nabla_\mu \left[\left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu}\right]$

$$= u^\nu u^\mu \nabla_\mu \left(\rho + \frac{p}{c^2}\right) + \left(\rho + \frac{p}{c^2}\right)\left(u^\nu \nabla_\mu u^\mu + u^\mu \nabla_\mu u^\nu\right) - \nabla^\nu p$$

$u^\mu \nabla_\mu u^\nu$

Since $u^\mu = \delta^\mu{}_0$, we have $u^\mu \nabla_\mu \rho = \dot{\rho}$ and $u^\mu = dx^\mu/d\tau$

$$u^\mu \nabla_\mu u^\nu = \frac{Du^\nu}{D\tau} = 0 \rightarrow \text{Moving along geodesic/ free falling.}$$

$$\nabla_\mu u^\mu = \partial_\mu u^\mu + \Gamma^\mu_{\mu\nu} u^\nu$$
$$= \Gamma^\mu_{\mu 0} = 3\dot{a}/a$$

$$u^\nu \left[\dot{\rho} + \frac{\dot{p}}{c^2} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)\right] - \nabla^\nu p = 0.$$

Contracting with u_ν

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0}$$ Conservation of Energy Equation

Dust:

$p = 0$ - no pressure

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \longrightarrow \boxed{\rho a^3 = \text{Const}}$$

$\hookrightarrow$ Number density falls like $1/a^3$ so does (rest mass) energy.

Radiation:

$$p = \frac{\rho c^2}{3}$$

$$\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a} \longrightarrow \boxed{\rho a^4 = \text{Const}}$$

$\hookrightarrow$ Falls more quickly as universe expands.
(Due to work done by fluid pV).

Cosmological Models:

$$\boxed{H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2}$$

H - Hubble parameter $H = \frac{\dot{a}}{a}$

Defining Critical density: $\boxed{\rho_{crit} = \frac{3H^2}{8\pi G}}$

For $\Lambda = 0$

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$

$\rho > \rho_{crit} \longrightarrow k > 0$ (closed)

$\rho = \rho_{crit} \longrightarrow k = 0$ (flat)

$\rho < \rho_{crit} \longrightarrow k < 0$ (open)

While Using $\Lambda=0$:

Considering ordinary matter with $\rho>0$ and $P>0$:

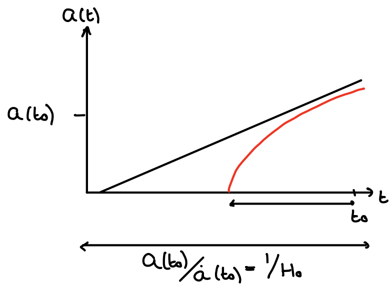
$$\boxed{\frac{\ddot{a}}{a}<0}$$ deceleration

In an expanding Universe, Implies that:

$a=0$ at finite time in past $\longrightarrow$ Past Singularity (big bang).

Bounding the age of the Universe:

↳ $\text{age} < \dfrac{a(t_0)}{\dot{a}(t_0)} = \dfrac{1}{H_0}$



$H_0 \approx 68\ \text{kms}^{-1}$ $\longrightarrow$ In a Universe Ignoring Λ: $\boxed{\text{age} < 14\ \text{Gyr}}$

For evolution for $\Lambda=0$:

$$\boxed{H^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3}\rho} \quad (\Lambda=0)$$

In a flat or open, $\Lambda=0$, expanding Universe $(K\leq 0)$

↳ $\boxed{H^2 = \dfrac{8\pi G}{3}\rho - \dfrac{Kc^2}{a^2} > 0}$ Expands Forever

In a Closed, $\Lambda=0$, expanding Universe $(K>0)$:

$H=0$ when $\dfrac{Kc^2}{a^2_{max}} = \dfrac{8\pi G}{3}\rho(a_{max})$

Expands to a maximum Size a_{max} then Contracts to $a=0$.

Special Case: $\Lambda=0$ and $K=0$

Dust: $p=0$ (Einstein-de Sitter Universe).

$$\rho \propto a^{-3} \longrightarrow \left(\frac{\dot{a}}{a}\right)^2 \propto a^{-3} \longrightarrow a \propto t^{2/3} \longrightarrow H = \frac{2}{3t}$$

Radiation: $p = \rho c^2/3$

$$\rho \propto a^{-4} \longrightarrow \left(\frac{\dot{a}}{a}\right)^2 \propto a^{-4} \longrightarrow a \propto t^{1/2} \longrightarrow H = \frac{1}{2t}$$

When $\Lambda>0$:

$$H^2 + \frac{Kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda c^2$$

For models with Sufficiently large Cosmological Constants

↳ Can undergo accelerated expansion — Evidence our Universe is!

Considering flat $(K=0)$, $\Lambda>0$:

$H^2 \longrightarrow \dfrac{1}{3}\Lambda c^2$ $\quad \boxed{a \propto \exp\left(\sqrt{\dfrac{\Lambda c^2}{3}}\,t\right)}$ $\longrightarrow$ De-Sitter universe!