Cosmology

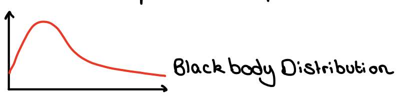
Very early universe:

Electron, Proton, Photon plasma

$e^- + p^+ \longleftrightarrow H + \gamma$

Cools under expansion.

Can observe primordial plasma.



5% Normal Matter

24% Dark matter

71% Dark Energy

Cosmological Principle:

On Large Scales the Universe is homogeneous and isotropic

Homogeneous: The Universe has the Same Composition and Structure everywhere when averaged over Sufficiently large distances.

Isotropic: Means the Universe looks the Same in every direction with no preferred orientation.

Therefore: treat our Universe as 'Small' perturbations on top of a completely Smooth background.

Statistical approach to fluctuations

Gaussian

initial amplitudes described by a power Spectrum. $P(\underline{k})$

$\underline{k} \rightarrow k$: Spatial homogeneity and isotropy.

Mathematically:

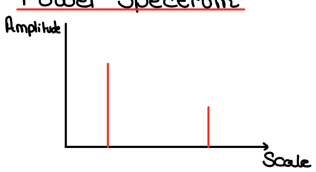
Observed CMB : field on the Sphere $T(\underline{i})$

Function drawn from a distribution with Kernel $C(\underline{i}, \underline{j})$

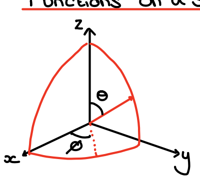
Constructed from 'angular power Spectrum' C_l of Cosmological model.

Cosmological parameters ≈ Hyper-parameters of the gaussian process.

Power Spectrum



Functions on a Sphere:



$z = \cos\theta$

$x = \sin\theta \cos\phi$

$y = \sin\theta \sin\phi$

Basic Functions

A Set of functions that one can describe any other functions as a Linear Combination of them

But can be used for value on a Spheres Surface.

Take the form:

$T(\underline{i}) = \sum_{lm} a_{lm} Y_{lm}(\underline{i})$

$\underline{i}$ - Vectors (θ_i, ϕ_i)

a_{lm} - Coefficients

Y_{lm} - basis functions

$a_{lm} = \int d^2\underline{i}\, T(\underline{i})\, Y^*_{lm}(\underline{i})$

Due to:

Orthogonality: $\int d^2\underline{i}\, Y_{lm}(i)\, Y^*_{l'm'}(i) = \delta_{ll'}\delta_{mm'}$

Completeness: $\sum_{lm} Y_{lm}(\underline{i})\, Y^*(\underline{j}) = \delta^{(2)}(\underline{i}-\underline{j})$

Y_{lm}'s - are eigenfunctions of the partial differential equations on the Surface of the Sphere

Form of Y_{lm}:

$Y_{lm}(\theta,\phi) \sim \underbrace{P_{lm}(\theta)}_{\theta}\, \underbrace{e^{im\phi}}_{\phi}$

P_{lm} - Associated Legendre Polynomials

$l = 0, 1, 2, \ldots$

$m = -l, -l+1, \ldots, l-1, l$ $(2l+1$ values$)$

The Addition Theorem:

$\sum_m Y_{lm}(\underline{i})\, Y^*_{lm}(\underline{j}) = \frac{(2l+1)}{4\pi} P_l(\underline{i}\cdot\underline{j})$

$P_l(\cos(\theta))$ is a Legendre Polynomial.

The Kernel is Stationary:

Statistical properties only dependent on seperations.

Statistically Isotropy for a Sphere

↳ No relations between points on the Sphere Should explicitly depend on the coordinates

But:

For two points: Any properties between two points can only depend on arc-length / angle between

For three points: three arc lengths between.

Two point function:

$\langle T(\underline{i})\, T(\underline{j})\rangle = f(\underbrace{\underline{i}\cdot\underline{j}}_{\text{angle between}})$

In harmonic space:

$\langle a_{lm}\, a^*_{l'm'}\rangle = C_l\, \delta_{ll'}\delta_{mm'}$

For a Statistically Isotropic Gaussian Field on the Sphere:

↳ all of its properties are determined by the angular power spectrum C_l

As Temprature is Real:

$T^* = T$

$\sum_{lm} a^*_{lm} Y^*_{lm}(\underline{i}) = \sum_{lm} a_{lm} Y_{lm}(\underline{i})$

As $Y_{lm} - P_{lm}(\theta)\, e^{im\phi}$

But: $Y^*_{lm}(i) = (-1)^m\, Y_{l,-m}(\underline{i})$

$\sum_{lm} a^*_{lm} (-1)^m\, Y_{l,-m}(\underline{i}) = \sum_{lm} a_{lm} Y_{lm}(\underline{i})$

$\sum_{lm} a^*_{l,-m} (-1)^m\, Y_{l,m}(\underline{i}) = \sum_{lm} a_{lm} Y_{lm}(\underline{i})$

Thus $\boxed{a_{l\,-m} = (-1)^m a^*_{lm}}$

m=0:

$a_{l,0} = a^*_{l,0} \longrightarrow a_{l,0}$ - real. 1

m<0:

a_{lm}s are given in terms of $m>0$ which are complex.

$2l$

For a given l:

$2l+1$

Over all l:

$$\sum_{l=0}^{l_{max}} 2l+1 = (l_{max}+1)^2 \longrightarrow \langle a_{lm} a^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'}$$

C_l - Set of variance of effectively $2l+1$ quantaties

Multipole moments distributed:

$2l+1$ real variables, ie y_i, with the same variance C.

Sampling distribution

$$\Pi_i \frac{1}{2\pi\sqrt{C}} e^{\frac{-y_i^2}{2C}}$$

$\longrightarrow$ funtion of Σy_i^2

$\longrightarrow \frac{1}{2l+1} \sum_i^{2l+1} y_i^2$ Sufficient statistic for learning abount C:

Power Spectrum Estimator:

$$\bar{C}_l = \frac{1}{2l+1} \sum_m |a_{lm}|^2$$

$$\langle \vec{C}_l \rangle = C_l$$

$$\langle\langle \bar{C}_l^2 \rangle\rangle = \langle \bar{C}_l^2 \rangle - \langle \bar{C}_l \rangle^2$$

$$= \frac{2C_l^2}{2l+1}$$ (cosmic variance)

$\hookrightarrow$ Irreducible

Visualisation:

$l_{max} \approx 2\, n_{side}$

DoF: $(l_{max}+1)^2$ vs $N_{pix} = 12 n_{side}^2$

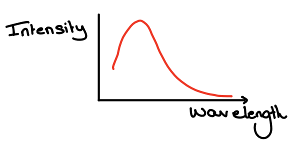
Data: Planck Legacy Archive

2018 'PR3' Legacy Archive

GHZ:

Maps are made from different detectors looking at the micro wave sky at different frequencies.

CMB: Almost perfect body



Foreground Contamination:

Galactic

Galactic Synchrotron emission (e^- in magnetic field) — Stronger low freq

Galactic Dust emission — Stronger high freq

Extragalactic

EG Point Sources

Cosmic Infrared Background (CIB) ('Continuous point sources')

Thermal and Kinetic Sunyaev Zeldovich effects.

Methods for Handeling This:

Remove (Subtraction)

Mask (Removal of regions)

Model (Dust maps + Syncotron + bayesian inference)

Correcting Power Spectrum to account for masking:

$$\langle \tilde{C}_l \rangle = M_{ll'} C_{l'} + N_l$$

Mask Deconvolved power Spectrum:

$$\hat{C}_l = M_{ll'}^{-1} (\tilde{C}_{l'} - N_{l'})$$

Unbiased: $\langle \tilde{C}_l \rangle = C_l$

$$\tilde{C}_l \approx \frac{\tilde{C}_l - N_l}{f_{sky}}$$

f_{sky} - Sky fraction: mean of the mask over the Sky $f_{sky} = \overline{W^2}$

Can model the Contribution of residual foregrounds at power spectrum level

$C_l^{fg} = A\, l^{\gamma}$ A, γ - parameters

Also must deal with:

Beam Response:

Each detector sees light in a general direction Convolved with telescopes beam response.

W_l - the Square of beam transfer function B_l

$$\hat{C}_l \longrightarrow \frac{\hat{C}_l}{W}$$

Pixel Window function Correction:

Finite Sized Pixels (tiny Smoothing Kernel) - especially at high monopoles l.

Can be corrected for using healp. pixwin

Binning power Spectrum

Smooths Cosmic Variance from raw power spectrum

Averaging adjacent multipoles values improving visual clarity

Implementation:

Straight average in D_l over an odd no. monopoles.

ΔL of 31 or 61 typical

Summary

Foreground

Masking

Beam (Instrumental)

Pixel window functions

More in detail on masking:

Investigating the effect of masking on the power Spectrum.

Theoretical Power Spectrum: C_l

Expected $\tilde{C}_l$ post masking:

$$\tilde{C}_l = \frac{1}{2L+1} \sum_m \tilde{a}_{lm} \tilde{a}^*_{lm}$$

→ Spherical harmonic transforms

$$\tilde{a}_{lm} = \sum_i w_i \; T(\underline{i}) \; Y^*_{lm}(\underline{i})$$

T → Temperature; Y^*_{lm} → ang harmonic func; w_i → Mask function (0-1)

$$\tilde{a}^*_{lm} = \sum_j W_j \, T(\underline{j}) \; Y_{lm}(\underline{j})$$

$$\tilde{C}_l = \frac{1}{2L+1} \sum_m \sum_i w_i \; T(i) \, Y^*_{lm}(\underline{i}) \sum_j W_j \, T(\underline{j}) \; Y_{lm}(\underline{j})$$

$$\langle \tilde{C}_l \rangle = \frac{1}{2L+1} \sum_{imj} w_i w_j \; Y^*_{lm}(\underline{i}) \, Y_{lm}(\underline{j}) \, \underbrace{\langle T(i) T(j) \rangle}_{\sum_{l'} C_{l'} Y_{l'm'}(\underline{i}) Y^*_{l'm'}(\underline{j}) + N_{ij}}$$

→ Theoretical (Signal + Noise)

$$= \underbrace{\frac{1}{2L+1} \sum_{mm'ijl'} w_i w_j \; Y^*_{lm}(\underline{i}) \, Y_{lm}(\underline{j}) \, Y_{l'm'}(\underline{i}) \, Y^*_{l'm'}(\underline{j}) \; C_{l'}}_{\text{Signal}} + \underbrace{\frac{1}{2L+1} \sum_{mm'ijl'} w_i w_j \; Y^*_{lm}(\underline{i}) \, Y_{lm}(\underline{j}) \, N_{ij}}_{\text{Noise}}$$

$$= \frac{1}{2L+1} \sum_{l'} \left(\sum_{mm'ij} w_i w_j \; Y^*_{lm}(\underline{i}) \, Y_{lm}(\underline{j}) \; Y_{l'm'}(\underline{i}) \; Y^*_{l'm'}(\underline{j}) \right) C_{l'} \; + \text{Noise term}$$

$$\boxed{\boxed{\langle \tilde{C}_l \rangle = \sum_{l'} M_{ll'} C_{l'} \; + \langle \tilde{N}_l \rangle}}$$

'$M_{ll'}$' – Matrix Operation – Coupling matrix

→ Expectation masked power Spectrum – matrix × Theory power Spectrum

Deconvolved Power Spectrum, $\hat{C}_l$

$$\hat{C}_l = M^{-1}_{ll'} \left(\tilde{C}_{l'} - \langle N_{l'} \rangle \right)$$

$$\langle \hat{C}_l \rangle = M^{-1}_{ll'} \left(\langle \tilde{C}_{l'} \rangle - \langle N_{l'} \rangle \right)$$

$$= M^{-1}_{ll'} \left(M_{ll'} C_{l'} \; + \langle \tilde{N}_l \rangle - \langle \tilde{N}_{l'} \rangle \right)$$

$\langle C_l \rangle = C_l$ Unbiased estimator.

Further Approximations

$$\boxed{\hat{C}_l \approx \frac{\tilde{C}_l - \langle \tilde{N}_l \rangle}{f_{sky}}}$$

Sky fraction $f_{sky} = \overline{W^2}$ – mean of the Square of the mask over the sky.

Variance of masked $\tilde{C}_l$

→ The mask couples modes so the Covariance is no longer diagonal:

$$\boxed{\langle \langle \tilde{C}_l \, \tilde{C}_{l'} \rangle \rangle \neq 0 \text{ for } l \neq l'}$$

Each element of matrix is a multi dimensional quadratic function of the theory power spectrum - depending on all the C_ls

Can obtain $\langle\langle \hat{C}_l \hat{C}_{l'} \rangle\rangle$ from $\langle\langle \tilde{C}_l \tilde{C}_{l'} \rangle\rangle$ by matrix multiplying on one side with inverse of coupling matrix and on other side with the transpose.

Theory and Data Comparison

Likelihood:

A pixel based likelihood: (Impossible due to invertibility)

Pixel data (50 million)

$$\frac{1}{\sqrt{|2\pi (C+N)|}} e^{\left(-\frac{1}{2} \; \overline{} \; \boxed{C+N}^{-1} \; \| \right)}$$

Covariance matrix + noise from model

Too large to invert

Alternative: In an idealised circumstance

↳ Angular power spectrum is a sufficient statistic?

If the instrument noise is gaussian and isotropic, all that changes is that

$C_l \longrightarrow C_l + N_l$

$$p(\hat{C}_l \mid C_l) \propto \frac{1}{\sqrt{2\pi (C_l + N_l)}^{2l+1}} e^{-\frac{\frac{1}{2}(2l+1)\hat{C}_l}{C_l + N_l}}$$

In the real world case:

Gaussian likelihood: full power spectra

$\langle \hat{C}_l \rangle = C_l + N_l$

$\langle\langle \hat{C}_l \hat{C}_{l'} \rangle\rangle = \dots$

$$\frac{1}{\sqrt{|2\pi \langle\langle C_l C_{l'} \rangle\rangle|}} e^{-\frac{1}{2}\left\{ \left(\tilde{C}_l - C_l - N_l \right) \left(\langle\langle C_l C_{l'} \rangle\rangle \right) \left(\tilde{C}_l - C_l - N_l \right)^T \right\}}$$

⟶ Every model must recompute

Fudicial Gaussian

⟶ Scatter / Covariance matrix kept constant - does not adjust for each model adjustment

$\langle\langle C_l C_{l'} \rangle\rangle_F$

Advantage of 'unbiased'.

$$\frac{1}{\sqrt{2\pi |\langle C_l, C_{l'} \rangle|_F}} e^{-\frac{1}{2}\left\{ \left(\tilde{C}_l - C_l - N_l \right) \left(\langle\langle C_l C_{l'} \rangle\rangle_F \right) \left(\tilde{C}_l - C_l - N_l \right)^T \right\}}$$

Must better understand noise modelling:

↳ In the 'numerator' - we have $\hat{C}_l - C_l - C_l^{fg} - N_l$

N_l factor subtracts the noise off the model

Not known perfectly ⟶ residual trend will be fitted by the cosmology 'biasing' the inference.

Cross Spectra:

↳ Rather than power spectrum of one map - $\boxed{\text{Cross power spectrum of two maps}}$

If noise in the maps is independent ⟶ Noise contribution to cross spectrum is on average zero.

$$\tilde{C}_L = \frac{1}{2L+1}\sum_m \tilde{a}_{lm}\,\tilde{a}^*_{lm} \longrightarrow \tilde{C}_L^{IJ} = \frac{1}{2L+1}\sum_m \tilde{a}^I_{lm}\,\tilde{a}^{J*}_{lm} \longrightarrow$$ I, J two different maps

Methods of Splitting by:

Frequency (143 × 217)

Detector (143-ds1 × 143-ds2)

Time (143hm1 × 143hm2)

Result: Can make a likelihood out of multiple Spectra.

Hybrid Approach

Exact pixel base likelihood at low L L = 2-29

Approx power spectrum based likelihood at high L L = 30-2500.

Gravitational Lensing: Statistical Challenges

Cosmology background:

Distances / Redshift:

The Universe is expanding:

$Z = \frac{\lambda^{obs} - \lambda^{emitted}}{\lambda^{emitted}}$ (redshift) tells us how much the universe has expanded

Since light was emitted before reaching us.

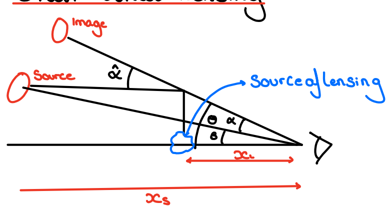
Comoving Distance $X(z)$ – denotes distance.

Standard cosmological Model: (~6 free params)

Ω_m: Matter density Ω_Λ: Dark Energy density A_s: Strength of initial perturbations

Ω_b: Baryon density H_0: Hubble constant n_s: Tilt of initial power spectrum

Gravitational Lensing



Simple Case: Deflection Angle $\hat{\alpha} = \frac{4GM}{c^2 b}$ *
(as percieved by object)

* In a flat universe

Change in angular position $\alpha = \frac{X_s - X_L}{X_s}\frac{4GM}{c^2 b}$ *
(Seen by observer)

Evidence for GR:

Deflection is twice that in Newtonian gravity

Generalised to total deflection from gradient of grav. potential along photon path

$$\alpha = \frac{2}{c^2}\int_0^{X_s} dx_L\, \frac{X_s - X_L}{X_s}\nabla_\perp \phi(x, \theta, X_L)$$

$$= -\frac{2}{c^2}\nabla\psi(\theta, X_L)$$

Image transformed according to lens equation.

$$\vec{\beta} = \vec{\theta} - \vec{\alpha} = \vec{\theta} - \frac{2}{c^2}\nabla\psi$$

Source and Image Coordinates mapping

$$\boxed{I(\vec{\theta}) = I^{(s)}[\vec{\beta}(\vec{\theta})]}$$ Coordinate transformation.

$\vec{\beta}$ - unlensed / source plane coords

$\vec{\theta}$ - Coords in lensed / 'image' plane.

In weak lensing regime (linearise mapping approx)

$$I(\vec{\theta}) = I^{(s)}\left[\vec{\beta}_0 + \frac{\partial\vec{\beta}}{\partial\vec{\theta}}\cdot(\vec{\theta} - \vec{\theta}_0)\right]$$

$$\frac{\partial\beta_i}{\partial\theta_j} = \delta_{ij} - \partial_i\partial_j\psi \qquad \text{(from lens equation)}$$

$$\equiv \underbrace{\begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa-\gamma_1 \end{pmatrix}}_{\text{Jacobian Lensing Equation}}$$

γ = 'Shear' γ_1, γ_2

κ = Convergence

Effects on a Circular isophotes: $I^s(\beta_1,\beta_2)$ for $(\beta_1,\beta_2) = (R\cos\phi, R\sin\phi)$

$$I(\vec{\theta}) = I^s[\vec{\beta}(\vec{\theta})] = I^s[\vec{\beta}_0 + A\cdot(\vec{\theta} - \vec{\theta}_0)]$$

$= 1$ when $A\cdot\vec{\theta} = (R\cos\phi, R\sin\phi)$

$\vec{\theta} = A^{-1}\cdot(R\cos\phi, R\sin\phi)$

Special Cases:

$\gamma_1 = \gamma_2 = 0$

$$(\theta_1, \theta_2) \simeq (1+\kappa)(R\cos t, R\sin t)$$

$\gamma_2 = 0$

$$(\theta_1, \theta_2) = ((1-\kappa-\gamma_1)^{-1} R\cos\phi,\ (1-\kappa+\gamma_1)^{-1} R\sin\phi)$$

Ellipticity

$$|\varepsilon| = \frac{a-b}{a+b} \qquad \varepsilon = |\varepsilon| e^{2i\phi}$$

$$\boxed{\begin{aligned} \varepsilon_1 &= |\varepsilon|\cos 2\phi \\ \varepsilon_2 &= |\varepsilon|\sin 2\phi \end{aligned}}$$

In weak lensing regime: $\kappa, \gamma \ll 1$: Initial shape of galaxy.

$$\varepsilon^{obs}_{1,2} = \varepsilon^{intrinsic}_{1,2} + \gamma_{1,2}$$

Orientation of ellipse globally = 0 (although may be elipses - no preferred orientation).

$$\langle \varepsilon^{intrinsic}_{1,2} \rangle = 0 \longrightarrow \langle \varepsilon^{obs}_{1,2} \rangle = \gamma_{1,2}$$

Allows it $\gamma_{1,2}$ to be inferred on large scales.

Shear of an image:

Quantify ellipticity via quadropole moments:

$$\varepsilon_1 = \frac{Q_{xx} - Q_{yy}}{Q_{xx} + Q_{yy} + 2(Q_{xx}Q_{yy} - Q_{xy}^2)^{1/2}} \qquad \varepsilon_2 = \frac{Q_{xy}}{Q_{xx} + Q_{yy} + 2(Q_{xx}Q_{yy} - Q_{xy}^2)^{1/2}}$$

CMB Lensing (Blake Sherwin)

Basics:

Galaxies: Gas: <20%

Dark Matter: > 80%

Cosmic structure grows from gravitational collapse of dark matter and gas. (Run away collapse).

Sensitive to initial conditions – theory of gravity etc.

Large Scale Dark Matter Structure:

Gravitational Lensing on CMB Radiation $\Theta(\hat{n})$

CMB: Oldest source – most distance source of light – passes through all of universe.

Distribution of Dark Matter deflect CMB light that it passes through (Preserves Surface Brightness.

↳ Original, unlensed, CMB fluctuations

↳ Very well understood statistical properties, eg isotropy

Add lensing

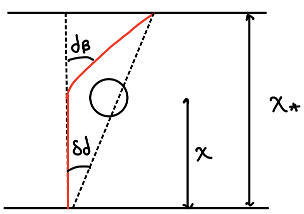
↳ Causes magnification / Shearing.

Approximate Quantification:

$\tilde{\Theta}(\hat{n}) = \Theta(\hat{n} + d(\hat{n}))$ Lensing deflection field: d.

↳ Described by remapping by vector field: d.

CMB Last Scattering Surface:



CMB is deflected by mass:

Described by i component of geodesic equation:

$$\frac{dp^{\nu}}{d\lambda} + \Gamma^{\nu}_{\nu} p^{\nu} p^{\rho} = 0$$

Derive by $\boxed{\delta \beta = -2\, \delta \chi \nabla_{\perp} \Phi}$

Deflection Angle

Distance

Grav. potential

$$\chi_* \delta \underline{d} = (\chi_* - \chi)\, \delta \underline{\beta}, \quad \delta d = \frac{\chi_* - \chi}{\chi_*} \delta \beta$$

Hence:

Observed deflection angle

$$\boxed{\delta \underline{d} = -2\, \frac{\chi_* - \chi}{\chi_*}\, \delta_{\chi} \nabla_{\perp} \Phi}$$

Perpendicular Gradient.

The lensing is related to an integral of the potential:

Total Deflection

$$\boxed{d(\hat{n}) = -\int_0^{\chi_*} 2\, d\chi\, \frac{\chi_* - \chi}{\chi_*}\, \underbrace{\nabla_{\perp} \Phi(\chi\hat{n}, \eta_0 - \chi)}}$$

↳ Small angle approx ∇ / χ

$$d(\hat{n}) = \nabla \phi$$

$$\phi(\hat{\underline{n}}) = -2 \int_0^{\chi} d\chi\, \frac{\chi_0 - \chi}{\chi \chi_0}\, \Phi$$

Lensing Convergence:

$$\boxed{\kappa = -\frac{1}{2}\nabla^2 \phi = \int_0^{\chi_0} \frac{\chi_0-\chi}{\chi_0\chi}\, \chi^2 \underbrace{\nabla_\perp^2 \Phi}\, d\chi}$$

$$\nabla^2_{3D}\, \Phi = \frac{3}{2a}\, \Omega_{m,0} H_0^2\, \delta$$

$$\delta \to \frac{\delta\rho}{\rho}$$

Overall: $\kappa = \Omega_{m,0} \int_0^{\chi} d\chi\, \frac{3H_0^2}{2a(\chi)}\, \frac{(\chi_0-\chi)\chi}{\chi_0}\, \delta$

Overall point:

↳ Lensing probes the projected total mass density in each direction (of which most is dark matter)

The power spectrum of a lensing map: $C_L^{\phi\phi}$

$$\langle \phi(L)\, \phi^*(L')\rangle = (2\pi)^2\, C_L^{\phi\phi}\, \delta(L-L')$$

Flat Sky Approximation:

Neglect the curvature of the sky:

$$\langle a^\phi_{LM}\, (a^\phi_{L'M'})^*\rangle = C_L^{\phi\phi}\, \delta_{LL'}\, \delta_{MM'}$$

$$\boxed{\langle \phi(L)\, \phi^*(L)\rangle = (2\pi)^2\, C_L^{\phi\phi}\, \delta(\underline{L}-\underline{L}')}$$

What does $C_L^{\phi\phi}$ tell us:

δ - density fluctuation

Lensing power spectrum probes square density fluctuation $\langle \delta^2\rangle$

↳ $\sim \sigma_8 \sim \sqrt{\langle \delta^2\rangle} \approx$ 'Clumpiness of parameter'

Must also consider neutrino mass.

Lensing Reconstruction:

2point Correlation function (describes Gaussian Random Field).

↳ $\xi(x,x') = \langle \Theta(x)\, \Theta(x')\rangle = \int D\Theta\, \Theta(x)\, \Theta(x')\, \Pr[\Theta]$

Require Statistical Isotropy/translation Invariance:

$$\xi(x,x') = \xi(x+a,\, x'+a)$$

Fourier Transform: $\boxed{\Theta(x) = \int \frac{d^2L}{(2\pi)^2}\, \Theta(L)\, e^{iL\cdot x}}$

Translational Invariance effect on Fourier Coefficients

$$\xi(x,x') = \langle \Theta(x), \Theta(x')\rangle = \int \frac{dL}{(2\pi)^2} \int \frac{dL'}{(2\pi)^2}\, e^{iL\cdot x + iL'\cdot x'}\, \langle \Theta(L)\, \Theta(L')\rangle$$

$$= \xi(\underline{x}+\underline{a},\, \underline{x}'+\underline{a}) \qquad = \int \frac{dL}{(2\pi)^2} \int \frac{dL'}{(2\pi)^2}\, e^{iL\cdot x + iL'\cdot x' + i(L+L')\underline{a}}\, \langle \Theta(L)\, \Theta(L')\rangle$$

Thus $\langle \Theta(L)\, \Theta(L')\rangle \propto \delta^{(D)}(L+L')$

Implies:

$$\langle \Theta(l)\,\Theta(l')\rangle = (2\pi)^2 C_{|l|}\,\delta^{(D)}(l+l')$$

$\hookrightarrow C$ power Spectrum of CMB

$\Theta^*(l) = \Theta(-l)$

Complex Conjugate:

$$\langle \Theta(l)\,\Theta^*(l')\rangle = (2\pi)^2 C_{|l|}\,\delta^{(D)}(\underline{l}-\underline{l}')$$

$\boxed{\text{Different modes are uncorrolated}}$

$\boxed{\langle \Theta(l)\,\Theta^*(l-L)\rangle = 0}$ without lensing

Lensing breaks this translational invariance:

$$\boxed{\langle \tilde{\Theta}(l)\,\tilde{\Theta}^*(l-L)\rangle \sim \phi(L)}$$

measure lensing by looking for these new Correlations in the CMB two-point functions

Proof: $\tilde{\Theta}(x) = \tilde{\Theta}(x+d) = \Theta(x) + \nabla \underbrace{\phi}_{\text{lens potential}} \cdot \nabla\Theta + \ldots$

$$\tilde{\Theta}(l) = \Theta(l) - \int \frac{d^2 l'}{(2\pi)^2}\,[l'\,\Theta(l')]\cdot(l-l')\,\phi(l-l')$$

$$\langle \tilde{\Theta}(\underline{l})\,\tilde{\Theta}(\underline{l}-\underline{L})\rangle = -\int \frac{d^2 l'}{(2\pi)^2}\,[\,l'\overbrace{\langle \Theta(l)\,\Theta(l')\rangle}^{(2\pi)^2 C_l\,\delta(l+l')}\,]\,(\underline{L}-\underline{l}-\underline{l}')\,\phi(\underline{L}-\underline{l}-\underline{l}')$$

$$= [\,l\,C_l\,]\,\underline{L}\,\phi(\underline{L}) \equiv \underbrace{K(\underline{l},\underline{L})}_{\text{Response Function.}}\,\phi(L)$$

Quadratic Lensing Estimator:

$$\hat{\phi}(\underline{L}) = \int \frac{d^2 l}{(2\pi)^2}\,f(\underline{l},\underline{L})\,\tilde{\Theta}(l)\,\tilde{\Theta}(\underline{L}-\underline{l})$$

$$\hat{\phi}(\underline{L}) \sim \frac{\tilde{\Theta}(\underline{l})\,\tilde{\Theta}(\underline{L}-\underline{l})}{K(l,L)}$$

Say have i measurements of quantatity x, $\hat{x}$

$\langle \hat{x}_i\rangle = x$

different $\sigma(x_i) = \sigma_i$

Best estimate of x? $\hat{x} = \dfrac{\sum w_i \hat{x}_i}{\sum w_i}$ $\quad w = \dfrac{1}{\sigma_i^2}$

Combine with Inverse-Variance Weight:

"Optimal" Quadratic Estimator:

$$\boxed{\hat{\phi}(L) = N(\underline{L})\,\underline{L}\cdot\int \frac{d^2 l}{(2\pi)^2}\,\frac{\underline{l}\,C_l}{C_l^{\text{noisy}}}\,\Theta(l)\,\frac{1}{C^{\text{noisy}}_{|L-l|}}\,\Theta^*(l-L)}$$

Sum over all pairs of modes

A fast approach:

Using Convolution theorem $|FT(A*B)| = |FT(A)| \times |FT(B)|$

(IFT - Inverse Fourier Transform)

Thus:

$A*B = FT(IFT(A) \times IFT(B))$ $\longrightarrow$ Scales $N\log N$ vs N^2

$$\hat{\phi}(L) = N(\underline{L})\,\underline{L}\cdot\int \frac{d^2 l}{(2\pi)^2}\underbrace{\left[\frac{\underline{l}\,C_l}{C_l^{\text{noisy}}}\,\Theta(l)\right]}_{\underline{l}F_2(\underline{l})}\underbrace{\left[\frac{1}{C^{\text{noisy}}_{|L-l|}}\,\Theta^*(l-L)\right]}_{F_1(l-\underline{L})}$$

$$= N(\underline{L})\,\underline{L}\cdot FT\left[IFT(F_1) \times IFT(\underline{l}F_2)\right] = \boxed{-N(\underline{L})\,i\underline{L}\int d^2x\, e^{-i\underline{L}\cdot x}\,F_1(x)\,\nabla F_2(x)}$$

Reconstruction Noise:

$$\hat{C}_L^{\phi\phi} = \langle \hat{\phi}(\underline{L}) \hat{\phi}^*(\underline{L}) \rangle \sim \langle TTTT \rangle \sim C_L^{\phi\phi} + \langle TT \rangle \langle TT \rangle$$

Cross Corrolation:

$$\langle T_1 T_2 \rangle = \langle TT \rangle + \langle n_1 \rangle \langle T \rangle + \langle n_2 \rangle \langle T \rangle + \langle n_1 \rangle \langle n_2 \rangle$$

Challenge 2: Fore grounds

$$T(l) = T^{CMB}(l) + T^{fg}(l)$$

Effect on 4 point function:

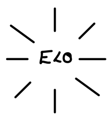
↳ $C_L^{\phi\phi} \sim \langle TTTT \rangle \sim \langle T^{CMB} T^{CMB} T^{CMB} T^{CMB} \rangle$

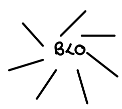
$$\left.\begin{aligned} &+ \langle T^{fg} T^{fg} T^{fg} T^{fg} \rangle \\ &+ \langle T^{CMB} T^{CMB} T^{fg} T^{fg} \rangle + \dots \end{aligned}\right\} \text{foreground bias terms.}$$

Correct for: frequency based Seperation.

Delensing (ie extracting original CMB Map before lensing effect.

CMB Polarization Basics

E mode map: 

B mode map: 

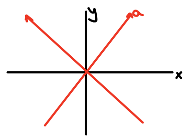
B-mode: Contains signals from inflation.

Describe with Q and U fields defined via electric fields:

Stokes Q,U parameters:

$$Q \propto E_x^2 - E_y^2$$

$$U \propto E_a^2 - E_b^2$$



CMB Polarization:

$$Q(x) = \int \frac{d^2L}{(2\pi)^2} \left(E_1 \cos 2\phi_1 - B_1 \sin 2\phi_1 \right) e^{iL\cdot x}$$

alligned/orientated with wavevector.

$$U(\underline{x}) = \int \frac{d^2L}{(2\pi)^2} \left(E_1 \sin 2\phi_1 + B_1 \cos 2\phi_1 \right) e^{iL\cdot x}$$

B mode polarization Channel ⟶ Primodial GW from Inflation.

However:

↳ Lensing also produced B-mode polarization by Converting E to B modes.

For CMB Polarisations:

$$\tilde{B}(\underline{L}) = \int dL\, W(\underline{L}, \underline{L}') E(\underline{L}') \phi(L - L') \qquad W(L, L') = L' \cdot (L - L') \sin(2\varphi_{L,L'})$$

If I have a lensing estimate:

↳ Delensing: Construct $B_{Lensing} \sim E\phi$ map from measured ϕ and E and Subtract B-$B_{Lensing}$

Control Variance:

Say have an estimator m with $\langle m \rangle = \mu$, unknown.

Wish to lower variance:

Have another Statistic t with $\langle t \rangle = \tau$

New unbiased estimator : $\boxed{m^* = m - c(t - \tau)}$

Minimise Variance in m^* for $c = \frac{\mathrm{Cov}(m,t)}{\mathrm{Var}(t)}$

$$\mathrm{Variance}(m^*) = (1 - \rho^2_{m,t})\,\mathrm{Var}(m)$$

→ Coerrelation Coefficient.

Posterior Analysis:

The evidence $P(d) = \int d\theta\, P(d|\theta)\, P(\theta)$.

Affine Invariant Sampler

Affine transformation:

$$x' = A \cdot x + b$$

Markov Process:

$x_{t+1} = R(x_t, \xi, P(x))$ → update algorithm

Affine Invariant Sampler:

$$R(ax + b, \xi, P) = AR(x, \xi, P) + b$$

Example from Cosmology

CMB lensing distorts the CMB:

$$T_{obs}(\vec{n}) = T_{unlens}(\vec{n} + \nabla \phi(\vec{n}))$$

Estimate of ϕ $\quad \hat{\phi} \sim \sum T_{obs} T'_{obs}$.