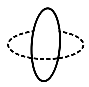
Gravitational Waves

Strain amplitude $h = \frac{\Delta L}{L}$ $\quad \Delta L \approx 10^{-20}$

Gravitational Waves · Spin 2 Structure - 90° Symmetry



Research Areas of GW:

→ Extreme Gravity
  → Strong and dynamical

→ Extreme Matter
  → Cold dense neuclear matter

→ Heavy Stars

→ Cosmology
  → Standard Siren

GW Data:

Time Series Data: $x_i(t)$ $\quad$ Where $i \in \{L1, H1, V1\}$ - detectors

Fourier Transforms (Frequency Domains).

Forward: $\tilde{x}(f) = \int_{-\infty}^{\infty} dt \; x(t)\, e^{-2\pi i f t}$

Inverse: $x(t) = \int_{-\infty}^{\infty} df \; \tilde{x}(f)\, e^{2\pi i f t}$

$$x(t) = \int df \left(\int dt' \; x(t')\, e^{-2\pi i f t'} \right) e^{2\pi i f t}$$

$$= \int dt' \; x(t') \int df \; e^{2\pi i f (t-t')} \longrightarrow \delta(t-t') = \int df \; e^{2\pi i f (t-t')}$$

Noise Can be assumed to be:

Gaussian + Stationary

Covariance $\langle x(t)\, x(t') \rangle = \rho(t-t')$ $\quad$ [Stationary]

$$\begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \quad \begin{pmatrix} \rho(t_1,t_1) & \cdots & \rho(t_1,t_4) \\ & \ddots & \vdots \\ & & \rho(t_4,t_4) \end{pmatrix}$$

In Fourier Domain:

$$\langle \tilde{x}(f), \tilde{x}^*(f) \rangle = \left\langle \int dt \; x(t)\, e^{-2\pi i f t} \int dt' \; x(t')\, e^{2\pi i f' t'} \right\rangle$$

$$= \int dt \int dt' \left\langle x(t)\, x(t') \right\rangle \times e^{2\pi i (f't' - ft)}$$

$$= \int dt \int dt' \; \rho(t-t') \; e^{2\pi i (f't' - ft')} \longrightarrow \text{Stationary property: } \left\langle x(t)\, x(t') \right\rangle = \rho(t-t')$$

Change of variables: $u = t - t'$ $\quad v = t + t'$

Overall: $\int du \int dv \; \frac{1}{2}\, \rho(u) \; e^{\pi i (f+f') u}\, e^{\pi i (f-f') v}$

$$= \int dv \; \frac{1}{2}\, \rho(u)\, e^{\pi i (f+f') v} \; \delta\left(\frac{f-f'}{2}\right)$$

$$= \int d\nu \, \rho(\nu)\, e^{2\pi i f \nu}\, \delta(f - f')$$

$$= \frac{1}{2} S_n(f)\, \delta(f-f')$$

$S_n(f)$ - PSD - Power Spectral density: Complete description of noise in Stationary / Gaussian time Series.

Stationary time Series with Gaussian noise:

Frequency domain is uncorrolated.

$$f_1 \; f_2 \; f_3 \; f_n \quad \propto \quad \begin{pmatrix} S_n(f_1) & & & \\ & S(f_2) & & \\ & & \ddots & \\ & & & S_n(f_n) \end{pmatrix}$$

$\rightarrow$ Diagonal matrix

$\Theta(n)$ - time + Space - Significantly reduced.

Matched Filtering (Frequentest Approach)

Good theoretical Prediction for Signal $h(t)$ $\longrightarrow$ Dependant on model parameters M_1, M_2, D

Noisy detector data, $S(t)$:

$$\boxed{S(t) = n(t) + h(t)}$$

$\hookrightarrow$ Stationary Gaussian Noise

Detection Statistic

$$\underbrace{(S * K)(T)}_{\text{Convolution}} = \int_{-\infty}^{\infty} dt \; S(t)\; K(t-T)$$

Let $K(t)$ - Weiner filter

$\hookrightarrow$ T - time offset , 'Sliding template forwards and backwards'

$$= \int_{-\infty}^{\infty} dt \; \big(n(t) + h(t)\big) \; K(t-T)$$

$$= \underbrace{\int dt \;\; n(t)\; K(t-T)}_{\substack{\text{Noise / Random Term} \\ N}} + \underbrace{\int dt \; h(t)\; K(t-T)}_{\substack{\text{Deterministic Part} \\ S}}$$

Using Parsevals theorem: (Time $\longleftrightarrow$ Frequency Domain)

Deterministic Term:

$$S = \int_{-\infty}^{\infty} dt\, h(t)\; K^*(t)$$

$$= \int_{-\infty}^{\infty} df \;\hat{h}(f)\;\hat{K}^*(f)$$

Noise / Random Term (N)

$$N^2 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \;\; K(t)\, K(t')\langle n(t)\, n(t')\rangle \qquad \left(N = \int_{-\infty}^{\infty} dt\, n(t)\;K^*(t)\right)$$

$\hookrightarrow$ RMS - as random property:

$$= \int dt \int dt' \;\; K(t)\,K(t') \times \int df \int df' \; \underbrace{\langle \tilde{n}(f)\;\tilde{n}^*(f')\rangle}_{= \frac{1}{2} S_n(f)\,\delta(f-f')} \; e^{2\pi i (ft - f't')}$$

$$= \int df \; \frac{1}{2}\; S_n(f) \int dt\; K(t)\; e^{2\pi i f t} \int dt' \;\; K^*(t')\; e^{-2\pi i f t'}$$

$$= \int df \;\; \frac{1}{2}\; S_n(f)\;\; \hat{K}(f)\;\; \hat{K}^*(f)$$

SNR, ρ

$$\boxed{\rho^2 = \frac{S^2}{N^2} = \frac{\langle \frac{1}{2} S_n(f)\,\hat{K}(f) \,|\, \hat{h}(f)\rangle}{\langle \frac{1}{2} S_n(f)\,\hat{K}(f) \,|\, \frac{1}{2} S_n(f)\,\hat{K}(f)\rangle}}$$

Bra-ket notation for Inner product

$$\langle a(f) \,|\, b(f) \rangle = 4\, \mathrm{Re} \int_0^{\infty} df \; \frac{\tilde{a}^*(f)\;\tilde{b}(f)}{S_n(f)}$$

Cauchy Schwarz Inequality

$\frac{\langle a|b\rangle}{\langle a|a\rangle}$ maximised when $a \parallel b$ (parrallel ie proportional to).

We wish to pick:

$$\tfrac{1}{2} S_n(f)\,\tilde{K}(f) \propto \tilde{h}(f)$$

ie 'optimal' Wiener Filter:

$$\boxed{\tilde{K}(f) = \frac{\tilde{h}(f)}{S_n(f)}} \longrightarrow$$

Require templates, $h(f)$ from theoretical understanding

Makes peak as prominent as possible

Using the 'optimal' filter:

$$\rho^2_{\text{matched filter}} = \rho^2_{mf} = \langle s\,|\,h\rangle$$

$$\rho^2_{\text{optimal}} = \langle h\,|\,h\rangle$$

Random as $s = h + \underline{n}$ $\quad \rho_{mf} \sim N(\rho_{opt}, 1)$

Geometry of GW Detection (Attentena Function - Sensitivity to Signals).

Minkowski Space: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$= \Omega_{\mu\nu}\, dx^\mu dx^\nu \qquad \Omega_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} \qquad \mu,\nu \in \{0,1,2,3\}$$

$$= -dt^2 + \delta_{ij}\, dx^i dx^j \qquad ij \in \{1,2,3\}$$

Plane fronted monochromatic GW

$$\delta_{ij} \longrightarrow \delta_{ij} + h_{ij}(x)^{(t,x,y,z)} \qquad h_{ij}(x) \text{ - perturbation}$$

where $h_{ij}(x) = e^{ik_\mu x^\mu}\left(A_+ e^+_{ij} + A_\times e^\times_{ij}\right)$

$x^\mu = (t,x,y,z)$ Space-time

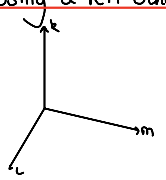
$k_\mu = (-1,0,0,1)$ - Considering wave propagating 'upwards' on Z axis

$\Big\}\ e^{i(z-t)\omega}$ $\qquad$ [Natural Units $c=1$]

$$e^+_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e^\times_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis tensors are transverse $\begin{pmatrix} k_i e^+_{ij} = 0 \\ e^+_{ii} = 0 \end{pmatrix}$

Choosing a RH othonormal set of vectors: $(\underline{l}, \underline{m}, \underline{k})$



$$\underline{\underline{e}}^+ = \underline{l}\,\underline{l}^T - \underline{m}\,\underline{m}^T \qquad \underline{\underline{e}}^\times = \underline{l} \otimes \underline{m} + \underline{m} \otimes \underline{l}$$

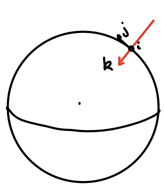
$$= \underline{l} \otimes \underline{l} - \underline{m} \times \underline{m}$$

$$e^+_{ij} = l_i l_j - m_i m_j$$

Consider a GW coming from a General Direction : $\hat{n}$

$$\underline{k} = -\hat{\underline{n}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

θ, ϕ - Spherical polar angles in the detection frame.



$\underline{k} = -\hat{n}$: direction to source

$\underline{j}$ - point to North pole

$\underline{i}$ - increasing ϕ (East/west).

Residual freedom: rotation about k : ψ - **polarisation angle**

$$\underline{l} = \underline{i}\cos\psi + \underline{j}\sin\psi$$

$$\underline{m} = -\underline{i}\sin\psi + \underline{j}\cos\psi$$

Overall: k, l, m in terms of θ, ϕ, ψ

Tensors:

$$\underline{\underline{\varepsilon}}^{+} = \underline{i}\otimes\underline{i} - \underline{j}\otimes\underline{j} \qquad \text{or} \qquad \underline{\underline{e}}^{+} = l\otimes l - m\otimes m = \underline{\underline{\varepsilon}}^{+}\cos 2\psi + \underline{\underline{\varepsilon}}^{\times}\sin 2\psi$$

$$\underline{\underline{\varepsilon}}^{\times} = \underline{i}\otimes j + \underline{j}\otimes i \qquad \underline{\underline{e}}^{\times} = l\otimes m + m\otimes l = \underbrace{-\underline{\underline{\varepsilon}}^{+}\sin 2\psi + \underline{\underline{\varepsilon}}^{\times}\cos 2\psi}_{\text{Similiarility to rotation but x2}}$$

$l, m \rightarrow ij$

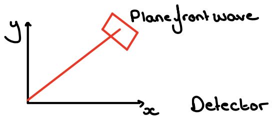
Ie x2 Spin / tensor field rather than vector field

Invariance under rotation of 180°.

General Metric Perturbation:

$$\underline{\underline{h}} = A_{+}\underline{\underline{e}}^{+} + A_{\times}\underline{\underline{e}}^{\times}$$

Interaction with detector:



Detector in x-y plane

Stretch 1 arm, Shrink another

Interference (change in Phase) * Intuition (not full theory)

↳ $\propto$ Difference in arm length

Squared Length arm 1: $= x_i x_j (\delta_{ij} + h_{ij})$

Change in length : $x_i x_j h_{ij}$

Squared Length arm 2: $= y_i y_j (\delta_{ij} + h_{ij})$

Change in length : $y_i y_j h_{ij}$

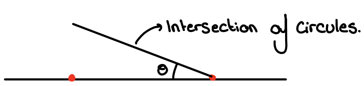
Measured Output:

$$h(t) = D_{ij} h_{ij}$$

Where $D_{ij} = \dfrac{x_i x_j - y_i y_j}{2}$ detector tensor, Projection like operator.

Coursework:

→ Time Delay between arrival at detectors



Useful methods:

→ time_domain_strain

→ frequency domain strain

Given a noisy time Series, $S(t)$

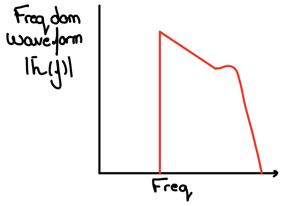
Whitened version of this as:

$$\tilde{S}_w(f) = \frac{\tilde{S}(f)}{\sqrt{S_n(f)}}$$

$$S_w(t) = FT^{-1}\{S_w(f)\}(t)$$

Ie. Take fourier transform - divide by Sqrt PSD - Inverse fourier transform } Rescale noise in each freq component. Rebalance amplitudes of fourier components (Standardised your data)

Graph
Log-log Scale



'Characteristic $f^{-7/3}$ Slope' - Corresponding to inspiral (Period of Signal where binary orbit Steadily Shrinks)

On infall Frequency ↑, Signal ↑

Despite graph looking like it is getting quieter quieter

(Reason - Signal Spends longer at Slower frequencies)

Match Filter (Squared) SNR:

$$\rho_{mf}^2 = 4\,\mathrm{Re}\int_0^\infty df\, \frac{\tilde{h}(f)\,\tilde{s}^*}{S_n(f)}$$

template

noisy data

Wish to determine value for lots of templates (each shifted by time slide):

FT of Shifted version:

$$\boxed{FT\{h(t+t_c)\}(f) = \tilde{h}(f)\, e^{+2\pi i f t_c}}$$

$$\boxed{\begin{aligned}\rho_{mf}^2(t_c) &= 4\,\mathrm{Re}\int_0^\infty df\, \frac{\tilde{h}(f)\,\tilde{s}^*(f)}{S_n(f)}\, e^{2\pi i f t_c} \\ \rho_{mf}^2(t_c) &= 4\,\mathrm{Re}\; FT^{-1}\left\{\frac{\tilde{h}(f)\,\tilde{s}^*(f)}{S_n(f)}\right\}(t_c)\end{aligned}}$$

Take: your fourier domain Signal $\tilde{h}(f)$

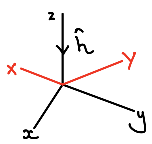
multiply fourier transform of data

Divide by $S_n(f)$ - PSD

Inverse Fourier transform ⟶ Time function: SNR at every time shift, t_c

Recap:

Decomposition of GW into '+', '×' polarisation.



$\hat{n}$ - wave

$\hat{n}_i$ - Detector arms.

$$h_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 \\ 0 & h_{12} & -h_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = h_+(t)\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\varepsilon^+_{\mu\nu}} + h_\times(t)\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\varepsilon^\times_{\mu\nu}}$$

$$h_{\mu\nu} = h_+ \varepsilon^+_{\mu\nu} + h_\times \varepsilon^\times_{\mu\nu}$$

Strain of antenna: $h(t) = F_+ h_t + F_\times h_\times$

Gravitational Wave Sources:

Generation of GWs need Strong accelerating quadrupoles

Example: Compact binaries in quasi-circular orbits

$$\boxed{h_{ij}(t,\vec{x}) = \frac{4G}{rc^4}\, \ddot{Q}_{ij}\left(t - \frac{r}{c}\right)}$$

Wave Detection

Each detection arm stretches/squeezes in a different way.

Detection response depends on direction of propogation:

$$h(t) = F_+(\hat{n})\, h_+(\hat{n}) + F_\times(\hat{n})\, h_\times(t)$$

$$F_p(\hat{n}) = D^{ij}\, \varepsilon^p_{ij}(\hat{n}) \qquad p = +, \times$$

Detector output $h(t)$ (Strain):

measures differential deformation of arms:

$$\boxed{D^{ij} = \frac{1}{2}\left(\hat{x}_i \hat{x}_j - \hat{y}_i \hat{y}_j\right)}$$

Wave form Modelling

Parameter Space, Θ $\longrightarrow$ Signal Space

$$\underbrace{d(t)}_{\text{data}} = \underbrace{n(t)}_{\text{noise}} + \underbrace{s(t)}_{\text{signal}} \longrightarrow \tilde{d}(f) = \tilde{n}(f) + \tilde{h}(f)$$

Quadropole Formula

$$h_{ij} = \frac{1}{r}\frac{4G}{c^4}\ddot{Q}_{ij}^{TT}\left(t - \frac{r}{c}\right)$$

Prescription: Calculate Quadropole of Source

Calculate radiated power

$$\frac{d\mathcal{E}}{dt} = P_{gw} \simeq \frac{G}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle$$

$$\frac{dJ^i}{dt} = \mathcal{E}^{ikl}\langle \ddot{Q}_{kj}\dddot{Q}_{lj}\rangle$$

Calculate orbital evolution due to loss of $\mathcal{E}, J^i$

Compact binaries (circularised)

↳ $T^{0,0} \longrightarrow \rho(t,\vec{x}) = m_1\delta^{(3)}(\vec{x}-\vec{x}_1(t)) + m_2\delta^{(3)}(\vec{x}-\vec{x}_2(t))$

Keplers Law: $\omega = \sqrt{\frac{GM}{R^3}}$

Chirp mass $\mathcal{M}_c = \frac{(M_1M_2)^{3/5}}{(M_1+M_2)^{1/5}}$

Orbital Energy: $E_{orb} = -\left(\frac{G^2}{32}\mathcal{M}_c^3\omega_{gw}^2\right)^{1/3}$

$$\boxed{P_{GW} = \frac{32}{5}\frac{c^5}{G}\left(\frac{G}{2c^3}\mathcal{M}_c\,\omega_{gw}\right)^{10/3}} \longrightarrow \text{Power}$$

$$\boxed{\frac{dE_{orb}}{dt} = -P_{GW}} \longrightarrow \boxed{\dot{\omega} = F(\omega)}$$

Ways of describing binary systems:

$(m_1, m_2) \longrightarrow (M, q) \longrightarrow (\mathcal{M}_c, \eta)$

→ two masses; → Masses, Mass ratio; Chirp mass; ↳ $\frac{m_1m_2}{(m_1+m_2)^2}$

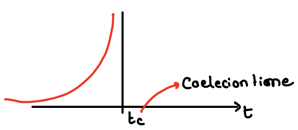
$$\dot{\mathcal{E}}(\omega) = -P_{gw}(\omega)$$

Freq of GW $f_{gw} = \frac{\omega_{gw}}{2\pi} = \frac{\omega}{\pi}$

$$\dot{f}_{gw} = \frac{96}{5}\pi^{8/3}\left(\frac{G}{c^3}\mathcal{M}_c\right)^{5/3}\cdot f_{gw}^{11/3}$$

$$\dot{f}_{gw} \propto \mathcal{M}_c^{5/3} f_{gw}^{11/3} \xrightarrow{\int dt} f_{gw}(t) = \frac{1}{\tau^{3/8}}$$

$Z = t_c - t$



Cut off f: f_{ISCO}

GW Phase:

$$\Phi(t) = \int_{t_{ref}}^{t} dt'\, 2\pi f_{gw}(t) \propto \mathcal{M}_c^{-5/8}(t_c - t)^{5/8} + \phi_0$$

Simple frequency-domain waveform (SPA):

$$\tilde{h}(f) = A f^{-7/6} \cos(2\phi(f: m_1, m_2) + \phi_0) \qquad v = (\pi M f)^{1/3}$$

$$\Phi = \left(\frac{v}{c}\right)^{-5} \sum_{i=0}^{N} \left[\psi_i + \psi_i^{(l)} \ln\frac{v}{c}\right]\left(\frac{v}{c}\right)^i \qquad \text{PN phase coefficients } \psi_i(m_1, m_2, S_1, S_2)\ \psi_i^{(l)}(M_1, m_2, S_1, S_2)$$

Theory of Blackholes:

Schwarzchild BHs:

Coordinates (t, r, θ, ϕ):

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & & & \\ & g_{rr} & & \\ & & g_{\theta\theta} & \\ & & & g_{\phi\phi} \end{pmatrix}$$

or

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -\underbrace{\left(1 - \frac{2M}{r}\right) dt^2}_{\text{'time like'}} + \underbrace{\left(1 - \frac{2M}{r}\right) ds^2 + r^2 d\Omega^2}_{\text{'space like'}}$$

where $d\Omega^2 = d\theta^2 + \sin\theta\, d\phi$

Which is a solution of Vacumm field eqns: $R_{\mu\nu} = 0$

↳ 1-parameter family of solutions: can vary M

↳ can be thought of as Mass: (from Kepler III). $\Omega^2 = \frac{M}{r^3}$

This Schwarzchild radius only has a clear interpretation for $r > 2M$:

Birkoff's Theorem:

This is the only spherically symmetric vaccum solution of the field equations.

↳ Leads to key properties of Schwarzchild metric:

→ Exterior spherical (non rotating body): Exterior Schwarzchild metric

Thus can be used as approx of gravitational sphere

Experimental test of GR:

Classical test of GR:

↳ 1) Postdiction: Orbit of mercury eccentric orbit presesion.

2) Deflection of light (einstein: Sun)

3) Gravitational Redshift

Additional test:

↳ Event Horizon Telescope

Gravity probe b (Kerr metric)

Comments on natural units:

$G = c = 1$ Units of everything ≈ 1 quantabity (arbitary considered mass).

Mass: M^1

time: $M^1 = \frac{GM}{c^3}$

Velocity M^0 (dimensionless)

ang momentum: M^2

energy: $M^1 = Mc^2$

Length: $M^1 = \frac{GM}{c^2}$

Power: $M^0 = \frac{c^5}{G}$

This Surface $r=2M$: (event horizon)

Radial null Geodesics: (Geodesic - Curve repsenting the Locally Shortest path between two points in a Surface.)

$d\theta = d\phi = 0$ (radial)

$g_{\mu\nu} v^{\mu} v^{\nu} = ds^2 = 0$ (null)

↳ 'norm of 4 velocity' = 0

$x^{\mu}(\lambda) \longrightarrow$ Curve in these Coords (geodesic Curve - World Line)

↳ t, r, θ, ϕ

Velocity: $v^{\mu} = \frac{dx^{\mu}}{d\lambda} = \Big(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \overbrace{0, 0}^{\text{Radial condition}} \Big)$

$$0 = -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2$$

$$\boxed{\frac{dr}{dt} = \pm\left(1-\frac{2M}{r}\right)}$$

2 familys of radial null geodesic: + outgoing

− ingoing

$$\int \frac{\pm 1}{1-\frac{2M}{r}}\, dr = \int dt \longrightarrow t_{\pm}(r) = \pm\left(r + 2M\log|r-2M|\right) + \text{Const}$$
$$= \pm r_*$$

where $r_* = r + 2M\log\left|\frac{r}{2M} - 1\right| + \text{Const}^*$ (tortoise coordinate)

$$\frac{dr_*}{dr} = -\left(1-\frac{2M}{r}\right)^{-1}$$

New Coordinate Systems adapted to radial geodesic:

Ingoing EF Coordinates: (v, r, θ, ϕ) where $v = t + r_*$

$$dv = \frac{dv}{dt}dt + \frac{dv}{dr}dr$$
$$= dt + \frac{dr_*}{dr}dr = dt - \left(1-\frac{2M}{r}\right)^{-1} dr$$

$dr = dr$

$d\theta = d\theta$

$d\phi = d\phi$.

Schwarzchild metric:

Ingoing EF Coords (v, r, θ, ϕ): $v = t + r_*$

$$\boxed{ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dv\,dr + r^2 d\Omega^2} \longrightarrow \text{Curves of constant } v \text{ (and } \theta,\phi) \longrightarrow \text{Radial null geodesics (Ingoing)}$$

as

$$\begin{pmatrix} * & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

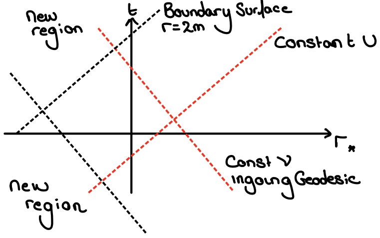
$r = 2m$: is not a problematic point.

$r = 0$: Problematic

Outgoing EF Coords (u, r, θ, ϕ) : $u = t - r_*$

$$\boxed{ds^2 = -\left(1-\frac{2M}{r}\right)du^2 - 2du\,dr + r^2 d\Omega^2} \longrightarrow \text{Curves of constant } u \text{ (and } \theta,\phi) \longrightarrow \text{Radial null geodesics (outgoing)}$$

as $u = t + r_*$



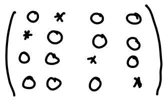
A set of coordinates that covers the maximally extended Schwarzschild Radius.

$$U' = -e^{-U/2m}$$

$$V' = e^{V/2m}$$

In coords $(U', V', \theta, \varphi)$ the metric becomes: Double Null Coordinates

$$\boxed{ds^2 = -\frac{32M^3}{r} e^{-r/2m} \, dU' dV' + r^2 d\Omega^2}$$



where r is defined implicitly

$$\frac{1}{2}(V-U) = r + 2M \log|r-2M|$$

Krusal Coordinates:

$$T = \tfrac{1}{2}(V' + U')$$

$$R = \tfrac{1}{2}(V' - U')$$

$$\boxed{ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dT^2 + d\Omega^2\right) + r^2 d\Omega^2}$$

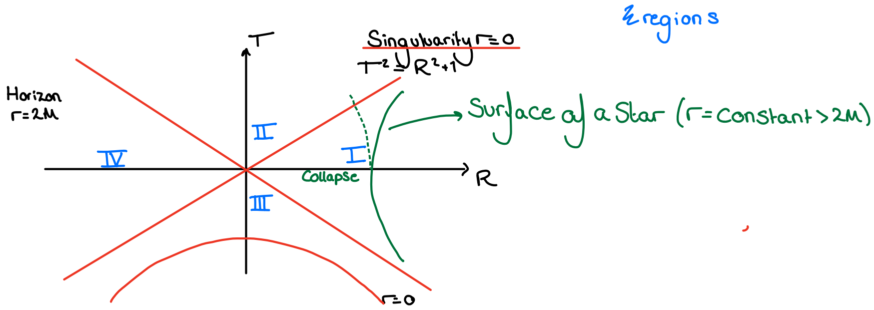
r defined implicitly by T, R.

$$\boxed{T^2 - R^2 = \left(1 - \frac{2M}{r}\right) e^{r/2M}} \longrightarrow$$

When $r=0$: $T^2 - R^2 = 1$

When $r = 2m$: $T^2 = R^2$

4 regions

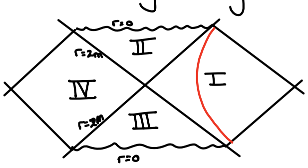


Parts of region I and II are thought to describe the gravitational field exterior to a collapsing star

This arguement relies heavily on spherical symmetry (Birkoff) but qualitative picture holds in general

Alternative Diagram:

Penrose Diagram of Schwarzchild



Reduced Order Models I

Predominantely - Binary Models

Model Composition:

$$q = \frac{m_1}{m_2} \qquad \Theta = \{ \underbrace{q}_{\text{Mass Ratio}} , \underbrace{\chi_{1,x}, \chi_{1,y}, \chi_{1,z}, \chi_{2,x}, \chi_{2,y}, \chi_{2,z}}_{\text{Spin parameters}} \}$$

Full parameter Space.

Θ = [
- α — right ascension
- β — declination
- r — distance
- t_α — arrival time at geocenter
- i — inclination angle
- ψ — Polarization angle
- ϕ_c — Coalensence angle
- D — luminosity distance
- m_1 — first mass component
- m_2 — Second mass component
- $\vec{S}_1$ — first spin component
- $\vec{S}_2$ — Second Spin component.

]

(m_1, m_2, $\vec{S}_1$, $\vec{S}_2$): Intrinsic Parameters (Properties of Source).

Model Searches

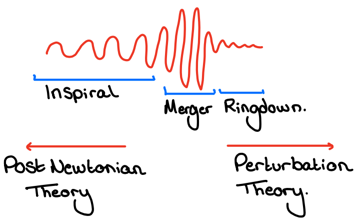
From frequency evolution we infer the masses of the System:

$$f_{GW} = \frac{\omega}{\pi} \qquad \omega = \sqrt{\frac{GM}{r^3}}$$

→ Once we have the masses together with the Amplitude we can infer the distance

→ From the arrival time, t_a amplitude and phase at the detectors → Sky localisation

→ modulations of amplitude and phase, we infer Spin and eccentricity



GWs modes:

The GWs strain, h, on a sphere can be decomposed as:

$$h(t, r, \theta, \phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} {}_{-2}Y_{lm}(\theta,\phi)\, h_{lm}(t,r)$$

→ ${}_{-2}Y_{lm}(\theta,\phi)$ are the $S=-2$ Spin-weighted Spherical harmonics.

→ h_{lm} are the GW modes.

} Quadruple modes ($l=2$, $m=\pm 2$) typically dominate.

Surrogates / ROMs

Simulations → Waveform + Resultant Blackholes

→ Learn ROM (Reduced order Models / Approximation) from Data

→ Use ROM for inference (Data Driven Approximation).

→ Surragate allows Rapid Waveform Generation

Reduced-Order Modeling (ROM)

$\boxed{\text{Reduced Basis}}$ Compression in parameter Space (remove redundant Information)

Lower Dimensional Embedding to represent waveform (greedy algorithm)

↓

$\boxed{\text{Empirical Interpolation}}$

↙ ↘

$\boxed{\text{Surrogate}}$ $\boxed{\text{Reduced order Quadratures}}$

Mathematically:

GW Waveform - function of frequency (or time) with parameters $\theta \in \{\theta_1, \dots, \theta_p\}$: $h(t, \theta)$ or $h(f, \theta)$

Set of Waveforms, $\mathcal{F}$

For a given GW Source, can define a catalog, C_N on the Space of all normalized GWs: $\mathcal{F}$

C_N is a Linear Space made of the "best linear combinations of waveforms":

$\longrightarrow$ Reduced Basis (RBs) Space $C_N = \{h(t, \theta_j)\}_{j=1}^{N}$

Idea: Seek "greedy parameter values" $\{\theta_i\}_{j=1}^{N}$ Such that:

$$h(t, \underline{\theta}) \approx \sum C_{ij} h(t, \underline{\theta}_i) \quad \text{or} \quad h(f, \underline{\theta}) \approx \sum_{j=1}^{m} C_{ij}(\underline{\theta}) h(f, \underline{\theta}_i)$$

We Seek θ_i iteratively: add the one that gives the largest projection error

The Greedy Algorithm:

Build 'Reduced Basis' iteratively:

i) from initial bank of waveforms, Choose θ_i and $h(\cdot, \theta_i) \equiv h_{\theta_i}$

This is the first basis vector: $e_i \equiv h_{\theta_i}$: Catalogue, $C_1 = \{h_{\theta_i}\}$

ii) Add another basis waveform e_2, We seek θ_2 Such that:

$\boxed{\underset{\theta_2}{\arg\max} \|h_\theta - P_1(h_\theta)\|}$ **Greedy Sweep**

$$h_1 - \frac{\langle h_\theta, h_1\rangle}{\langle h_1, h_1\rangle} h_1 \longrightarrow e_1$$

Where $P_1(h_\theta) = e_1 \langle e_1, h_\theta\rangle$ is the orthogonal projection h_θ onto C_1

- The waveform corresponding to θ_2 is added back to the catalogue so that $C_2 = \{h_{\theta_1}, h_{\theta_2}\}$
- The new basis vector e_2 constructed via Gram-Schmidt orthonormalisation

$$e_i = \frac{h_i}{\|h_i\|} = \frac{h_i}{\langle h_i, h_i\rangle}$$

End up with a set of vectors $\{e_1, e_2, \dots e_m\}$

Remarks:

We have a bank of waveforms (training set) known as a discrete set of M training points:

$\boxed{\Theta_M = \{\theta_i\}_{i=1}^{M}}$

Greedy algorithm: Identifies a set of parameter values $\{\theta_1, \dots, \theta_m\} \subset \Theta_M$ and associated set of waveforms (the reduced basis): $\{h_{\theta_1}, \dots, h_{\theta_m}\}$

This basis is hierarchical: if $\{h_i\}_{i=1}^{m'}$ is the basis is for $m' < m$: $\{h_i\}_{i=1}^{m'} \subset \{h_i\}_{i=1}^{m}$

$$\boxed{\begin{array}{l}\text{Waveform in the training set are well approximated by expansion:} \\ \quad h(t, \theta) \approx \sum_{i=1}^{m} C_i(\theta)\, e_i(t) \quad \text{where } \theta \in \Theta\end{array}}$$

$\longrightarrow$ Due to Similiar Structure of GW: $m \ll M$

Greedy Error, σ_m:

Bounded by a specified tolerance ε:

$$\sigma_m \equiv \max_{\theta} \min_{c_i \in \mathbb{C}} \left\| h(\cdot,\theta) - \sum_{i=1}^{m} c_i(\theta)\, e_i(\cdot) \right\|^2 \leq \varepsilon$$

We see: $c_i(\theta) = \langle h(\cdot,\theta), e_i(\cdot) \rangle$

Empirical Interpolation:

We wish to approximate a given $h(t;\theta)$ with high accuracy for an arbitrary θ, given:

- A reduced basis set: $\{e_i\}_{i=1}^{m}$
- find specific points in time at which we can perform parameter space interpolation.

i) Sampling $h(t,\theta)$ at these selected times, T_i (same for frequency)
→ Ensures interpolation problem is well conditioned or even has a solution.

ii) Interpolation error is minimised with nearly optical convergence rate.

Empirical Interpolation Method:

Provides a sparse subset of empirical time (or freq) nodes given the RBs function: $\{e_i\}_{i=1}^{m}$

Output: A set of m empirical nodes $\{T_i\}_{i=1}^{m} \subset \{t_i\}_{i=1}^{L}$

$$I_0[h](t,\theta) = 0$$

$$T_m = \text{argmax} \| h(t,\sigma_m) - I_{m-1}[h](t,\theta_m) \|$$

$$J_m[h](t;\theta) = \sum_{i=1}^{m} c_i(\theta)\, e_i(t)$$

→ $\{c_i\}_{i=1}^{m}$ defined by: $\sum_{i=1}^{m} c_i(\theta)\, e_i(T_j) = h(T_j;\theta) \quad j = 1,\dots,m$

Equivalent to Solving:

m-by-m System $\sum_{i=1}^{m} V_{ji}\, c_i(\theta) = h(T_j;\theta)$

Where interpolation matrix $V \equiv \begin{pmatrix} e_1(T_1) & \cdots & e_m(T_1) \\ \vdots & \ddots & \vdots \\ e_1(T_m) & \cdots & e_m(T_m) \end{pmatrix}$ then $c_i = \sum_{j=1}^{m} (V^{-1})_{ij}\, h(T_j;\theta)$

Sub in. $\boxed{I_m[h](t;\theta) = \sum_{j=1}^{m} B_j(t)\, h(T_j;\theta)}$

where: $B_j(t) = \sum_{i=1}^{m} e_i(t)(V^{-1})_{ij}$ ⟶ independent of θ

Evaluating any waveform at these $\{T_i\}_{i=1}^{m}$ nodes one can reconstruct the full time series of the waveform with high accuracy:

$$\boxed{\max_{\theta_i} \| h(t,\theta) - J_m[h](t;\theta) \|^2 < \Lambda_m \sigma_m}$$

Fitting the EI:

We want to estimate the values of $h(T_i;\theta)_{i=1}^{m}$ for arbitary θ.

Most GWs can be modeled as:

$$\boxed{h(t;\theta) = A(t;\theta)\, e^{-i\phi(t,\theta)}}$$

It is easier to fit the phase and amplitude than the complex waveform itself (due to highly oscillatory behaviour).

Find $2m$ functions $\{A_i(\theta)\}_{i=1}^m$ and $\{\phi_i(\theta)\}_{i=1}^m$ approximating the amplitude and phase at each $\{T_i\}_{i=1}^m$:

$$\boxed{h(T_i;\theta) \approx A_i(\theta)\, e^{-i\phi_i(\theta)}}$$

Building the GW Surrogate Model:

Surrogate model: $h_s(t;\theta)$

$$\boxed{h_s(t;\theta) \equiv \sum_{i=1}^{m} B_i(t)\, A_i(\theta)\, e^{-i\phi_i(\theta)}} \longrightarrow$$

$\{B_i(t)\}_{i=1}^m$ terms - Computed offline

Fitting functions $A_i(\theta), \phi_i(\theta)$ - evaluated online when θ specified

Important: Only m RB waveforms evaluated at m empirical times $\{T_i\}_{i=1}^m$ are needed to build the surrogate model and thus predict fiducial waveform at any time and parameter value

Given a fiducial waveform $h(t_i;\theta)$ and its surrogate $h_s(t;\theta)$:

The surrogate error is given by:

(for equally spaced time samples)

$$\Delta t \sum_{i=1}^{L} |h(t_i;\theta) - h_s(t_i;\theta)|^2 \qquad \Delta t = \frac{t_{max} - t_{min}}{L-1}$$

Data Analysis for GW

Detect presence of signal: match signal.

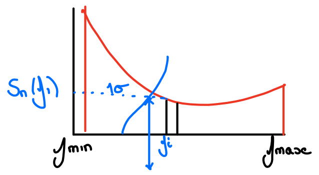
Next Stage: Signal $\longrightarrow$ Parameter Space. (Bayesian Inference - Parameter Estimation).

CBC Parameter Space:

15 dim Param Space $\boxed{\vec{\theta} = \left((m_1, m_2), \vec{S}_1, \vec{S}_2, r, \alpha, \delta, \epsilon, \psi, \phi_c, t_c\right)}$

Stationary Gaussian Noise - p

Noise Curve



Detector Data:

TD: $d(t) = n(t) + h(t)$

FD: $\tilde{d}(f) = \tilde{n}(f) + \tilde{h}(f)$

Noise: stochastic process that assume to be stationary:

$\longrightarrow$ Stationary

$\longrightarrow$ Gaussian

Without loss of generality $\langle n(t)\rangle = 0$. $\quad R(\tau) = \langle n(t+\tau)\, n(t)\rangle = \frac{1}{2}\int_{-\infty}^{\infty} df\, S_n(f)\, e^{-i2\pi f\tau}$

In the frequency domain:

for some freq bin f_i: $\qquad (H_{noise}: d = n)$

$$\boxed{P(\tilde{n} \mid H_{noise}) = \frac{1}{\sqrt{2\pi \frac{1}{2} S_n(f)}}\, e^{-\frac{1}{2}\frac{|\tilde{n}_i|^2}{\frac{1}{2}S_n}}} = \frac{1}{\sqrt{\pi S_n(f_i)}}\, e^{-\frac{|\tilde{n}_i|^2}{S_n(f_i)}}$$

Glitches: $d = n + s + g$ only 1 detector - no coherence across them.

Gravitational Wave Likelihood.

$$P(n|H_{noise}) = \prod_{i=0}^{n} p(\tilde{n}(f_i)|H_{noise})$$

$$= \boxed{\mathcal{N} e^{-\frac{1}{2}\sum_{i=0}^{N}\frac{|\tilde{n}(f_i)|^2}{S_n(f_i)}}} = \mathcal{N} e^{-\frac{(n|n)}{2}}$$ Noise Likelihood.

$$\mathcal{L}(d,\vec{\theta}) = P(d|\vec{\theta}, H_{signal}) \qquad H_{signal}: d = n + h_{\vec{\theta}}$$

$$= P(d - h_{\vec{\theta}} | H_{noise}) = \mathcal{N} e^{-\frac{\langle d-h_{\vec{\theta}} | d-h_{\vec{\theta}}\rangle}{2}}$$

Posterior: $$P(\vec{\theta}|d) = \frac{\overbrace{P(d|\vec{\theta})}^{\mathcal{L}(\vec{\theta})}\overbrace{P(\vec{\theta})}^{\pi(\vec{\theta})}}{\underbrace{P(d)}_{Z}}$$

Combining Information from multiple detectors: $d \longrightarrow \{d_i\}_{I=1}^{N_{det}}$: $P(\{d_i\}|\vec{\theta}) = \prod_I p(d_I|\vec{\theta}) \propto e^{-\sum_I \frac{(d_i - h_{\vec{\theta}}|d-h_{\vec{\theta}})}{2}}$

$$P(\vec{\theta}|\{d\}_I) \propto \pi(\vec{\theta})\, e^{-\sum_I \frac{(d_i - h\vec{\theta}|\dots)}{2}}$$

Gaussian Noise Likelihood

$$\langle a|b\rangle = \mathbb{R}\left\{\int_{-\infty}^{\infty} \frac{\tilde{a}^*(f)\tilde{b}(f)}{S_n(f)}\right\} = 4\mathbb{R}\left\{\int_0^{\infty} \frac{\tilde{a}^*(f)\tilde{b}(f)}{S_n(f)}\right\}$$

$$\mathcal{L}(d;\vec{\theta}) = p(d - h_{\vec{\theta}}|H_{noise}, I) = \mathcal{N}\, e^{-\frac{(d-h_\theta|d-h_\theta)}{2}}$$

$$\langle d - h_{\vec{\theta}} | d - h_{\vec{\theta}}\rangle = 4\mathbb{R}\left\{\int_0^{\infty} \frac{(\widetilde{d - h_{\vec{\theta}}})^* (\widetilde{d - h_{\vec{\theta}}})}{S_n(f)}\,df\right\}$$

Fourier Domain / Time Domain $\longrightarrow$ Shift by $e^{-2\pi i f t_c}$

$$\mathcal{N}\exp e^{-\frac{(n|n)}{2}} = \mathcal{N} e^{-\frac{1}{2}\sum\frac{|\tilde{d} - \tilde{h}_{\vec{\theta}}|^2}{S_n(f_i)}}$$

SBI Theory:

Probabilistic Model: $\theta \xrightarrow[P(x|\theta)]{} x$

Simulator$(\theta) = x$

$x \sim p(x|\theta)$

Alternative to Classical: $\theta, x \quad p(x|\theta)$

Bayes Theorem: $$\underbrace{p(\theta|x)}_{\text{Neural Posterior Estimation}} = \underbrace{\frac{\overbrace{p(x|\theta)}^{\text{Neural Likelihood estimation}}\, p(\theta)}{p(x)}}_{\text{Neural Ratio Estimation.}}$$

NPE

↳ Density Estimation

Normalising Flow

Very fast to Sample /evaluate densities

Struggles with dimensionality

NLE

Need hand crafted Summarys

Algorithms

NPE

Prior $p(\theta)$, $x \sim p(x|\theta)$

Initialising some density estimator $q_\phi(\theta|x)$

↳ Network works

$D: \{(\theta_i, x_i)\}$ — θ from prior, $p(x|\theta)$ from posterior. $P(x,\theta) = P(x|\theta)P(\theta)$

$$\mathcal{L}_{NPE}[\phi, B] = -\frac{1}{|B|} \sum_{\theta, x \in B} \log q_\phi(\theta|x)$$

Result: $q_\phi(\theta|x) \simeq p(\theta|x)$

Proof of Validity

Assume Infinite Data Limit: $N \to \infty$

$$\mathcal{L}_{NPE} \longrightarrow \mathbb{E}_{p(\theta,x)}[-q_\phi(\theta|x)] = \int d\theta dx \, p(\theta,x)\left[-\log q_\phi(\theta|x)\right]$$

$$= \int d\theta dx \, p(\theta,x)\left[-\log q_\phi(\theta|x)\right] + p(\theta|x)\log p(\theta|x) - p(\theta|x)\log p(\theta|x)$$

$$= \int d\theta dx \, p(\theta,x)\left[\log \frac{p(\theta|x)}{q_\phi(\theta|x)}\right] - \int d\theta dx \, p(\theta,x) \log p(\theta|x)$$

$$= \mathbb{E}_{p(x)}\left[D_{KL}\left(p(\theta|x) \| q_\phi(\theta|x)\right)\right.$$

Know $D_{KL} \geq 0$ Minimisation. When $p(\theta|x) = q_\phi(\theta|x)$