Study Notes of Principles of Mathematical Analysis

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1 The Real and Complex Number Systems

Ordered Sets

Definition 1.7. Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is an upper bound of E.

(ii) If $\gamma < \alpha$ the γ is not an upper bound of E.

Then α is called the *least upper bound of* E or the *supremum of* E, and we write

$$\alpha = \sup E$$

The *greatest lower bound*, or *infimum*, of a set E which is bounded below is defined in the same manner: The statement

$$\alpha = \inf E$$

means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E.

Note

- S 是有序集合的情况下，E 又是属于 S 的，并且 E 拥有上界。那么只会存在一个 α 是 E 的最小上界。同理如果是 E 拥有下界，只会存在一个 α 是 E 的最大下界。
- 上确界 Supremum [su:′pri:məm]；下确界 Infimum [′ɪnfaɪməm]。

Definition 1.10. An ordered set S is said to have the *least-upper-bound property* if the following is true: If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

Note S 中存在 $E \subset S$，且 E 具有最小上界，那么 S 就具有最小上界性，反之亦然。

Theorem 1.11. *Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then $\alpha = \sup L$ exists in S, and $\alpha = \inf B$. In particular, $\inf B$ exists in S.*

Proof.

因为 B 是有下界的，且 L 不为空。由于 L 包含了所有的 y（$y \in S$）且满足不等式 $y \leq x$（$x \in B$），那么所有的 $x \in B$ 都是 L 的上界。因此 L 是有上界的。关于 S 的假设意为在 S 中有一个 L 的最小上界，被称为 α。

如果 $\gamma < \alpha$ 那么（根据 Definition 1.8）γ 并不是 L 的一个上界，因此 $\gamma \notin B$。对于所有的 $x \in B$ 都有 $\alpha \leq x$。因此 $\alpha \in L$。

如果 $\alpha < \beta$ 那么 $\beta \notin L$，因为 α 是 L 的一个上界。

我们展示过了 $\alpha \in L$ 但是 $\beta \notin L$ 而 $\beta > \alpha$ 的情况。也就是说，α 是 B 的一个下界，但是当 $\beta > \alpha$ 时 β 却不是。这就意味着 $\alpha = \inf B$。 □

Fields

Definition 1.12. A field is a set F with two operations, called *addition* and *multiplication*, which satisfy the following so-called ”field axioms” (A), (M), and (D):

(A) Axioms for addition

(A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F.

(A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.

(A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.

(A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.

(A5) To every $x \in F$ corresponds an element $-x \in F$ such that $x + (-x) = 0$.

(M) Axioms for multiplication

(M1) If $x \in F$ and $y \in F$, then their product xy is in F.

(M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.

(M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.

(M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.

(M5) If $x \in F$ and $x \neq 0$ then there exists an element $\frac{1}{x} \in F$ such that $x \cdot (\frac{1}{x}) = 1$.

(D) The distributive law

$x(y + z) = xy + xz$ holds for all $x, y, z \in F$.

Note 域的定义：维基百科。

Definition 1.17. An *ordered field* is a field F which is also an ordered set, such that:

1. $x + y < x + z$ if $x, y, z \in F$ and $y < z$,
2. $xy > 0$ if $x \in F$, $y \in F$, $x > 0$, and $y > 0$.

如果 $x > 0$，我们称 x 为 *positive*；如果 $x < 0$，x 则为 *negative*。

The Real Field

Theorem 1.19. *There exists an ordered field R which has the least-upper-bound property. Moreover, R contains Q as a subfield.*

第二个声明意味着 $Q \subset R$ 以及加法与乘法在 R 上的运算，当应用于 Q 的成员时，与有理数的通常操作重合；同样的，正有理数成员是 R 的正元素。

R 的成员被称为 *real numbers*，即实数。

Theorem 1.20.

(a) *If $x \in R$, $y \in R$, and $x > 0$, then there is a positive integer n such that $nx > y$.*

(b) *If $x \in R$, $y \in R$, and $x < y$, then there exists a $p \in Q$ such that $x < p < y$.*

对于 (a) 部分通常认为是 R 具有 *archimedean property*，即阿基米德性质，详见维基百科。(b) 部分则表明 Q 是在 R 中 *dense*，即具有稠密性：在任意两个实数之间有一个有理数。

Proof.

(a) 令 A 作为所有 nx 的集合，其中 n 为所有的正整数。如果 (a) 是错误的，那么 y 则会是 A 的一个上界。但是接着 A 会在 R 中拥有一个最小上界，即 $\alpha = \sup A$。由于 $x > 0$，$\alpha - x < \alpha$，以及 $\alpha - x$ 不是 A 的上界，因此 $\alpha - x < mx$ 对于某些正整数 m 成立。但是这样就会有 $\alpha < (m+1)x \in A$，这是不可能的，因为 α 是 A 的上界。

(b) 因为 $x < y$，$y - x > 0$ 以及由 (a) 所知一个正整数 n 满足

$$n(y-x) > 1$$

再次应用 (a)，获取正整数 m_1 与 m_2 满足 $m_1 > nx$，$m_2 > -nx$，那么

$$-m_2 < nx < m_1$$

因此会有一个整数 m（$-m_2 \leq m \leq m_1$）满足

$$m - 1 \leq nx \leq m$$

如果我们结合这些不等式，则会得到

$$nx < m \leq 1 + nx < ny$$

因为 $n > 0$，它遵循

$$x < \frac{m}{n} < y$$

通过 $p = \frac{m}{n}$ 证明了 (b)。

□

Note (b) 的第一步将 $y-x$ 作为整体，将 (a) $nx>y$ 中的 x 替换为 $y-x$，y 替换为 1。同理对于整数 m_1 与 m_2 而言，可分别将 nx 与 $-nx$ 视为不等式右侧的 y，而不等式左侧的 x 视为 1，那么就有了 $m_1\cdot 1>nx$ 与 $m_2\cdot 1>-nx$。对于整数 m 的 $-m_2\leq m\leq m_1$ 最坏的情况可将 $-m_2$ 与 m_1 视为相邻的整数，比如说 1 和 2，那么当 m 取值为 2 时视为 $2-1\leq nx<2$，满足 $m-1\leq nx<m$。最后根据有理数的定义 $p=m/n\ m,n\in Q$ 可以得出 x 与 y 之间一定存在一个有理数。

Theorem 1.21. *For every real $x>0$ and every integer $n>0$ there is one and only one positive real y such that $y^n=x$.*

这个 y 数可以被写作 $\sqrt[n]{x}$ 或是 $x^{\frac{1}{n}}$。

Proof.

对于至多存在一个 y 的论证很简单，因为 $0<y_1<y_2$ 意味着 $y_1^n<y_2^n$。
令 E 为所有满足 $t^n<x$ 的正实数 t 的集合。
如果 $t=x/(1+x)$ 那么 $0\leq t<1$，那么 $t^n\leq t<x$。因此 $t\in E$，且 E 不为空。
如果 $t>1+x$ 那么 $t^n\geq t>x$，所以 $t\notin E$。因此 $1+x$ 是 E 的一个上界。
所以根据 Theorem 1.19 得出，存在一个

$$y=\sup E$$

而证明 $y^n=x$ 我们需要展示不等式 $y^n<x$ 与 $y^n>x$ 皆会导致矛盾。
当 $0<a<b$ 时，等式 $b^n-a^n=(b-a)(b^{n-1}+b^{n-2}a+\cdots+a^{n-1})$ 得出不等式

$$b^n-a^n<(b-a)nb^{n-1}$$

假设 $y^n<x$。选择 h 使得 $0<h<1$ 且

$$h<\frac{x-y^n}{n(y+1)^{n-1}}$$

令 $a=y$，$b=y+h$，那么就有

$$(y+h)^n-y^n<hn(y+h)^{n-1}<hn(y+1)^{n-1}<x-y^n$$

因此 $(y+h)^n<x$，且 $y+h\in E$。因为 $y+h>y$，这与 y 是 E 的一个上界相矛盾。
假设 $y^n>x$。令

$$k=\frac{y^n-x}{ny^{n-1}}$$

那么 $0<k<y$。如果 $t\geq y-k$，我们得出以下结论：

$$y^n-t^n\leq y^n-(y-k)^n\leq kny^{n-1}=y^n-x$$

因此 $t^n>x$，且 $t\notin E$。它遵循 $y-k$ 是 E 的一个上界。
但是因为 $y-k<y$，其与 y 是 E 的最小上界的事实相矛盾。
因此 $y^n=x$，证明完成。 □

Note 至多存在一个 y 换个角度也就是说但凡有第二个 y 使得 $y_1^n = y_2^n$，那么 $y_1 = y_2$。
等式

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$$

左侧可以视为

$$(b-a)(b^{n-1} + b^{n-1}\frac{a}{b} + \cdots + b^{n-1}\frac{a}{b^{n-1}})$$

提取 b^{n-1} 后得

$$(b-a)b^{n-1}(1 + \frac{a}{b} + \cdots + \frac{a^{n-1}}{b^{n-1}})$$

由于有 $0 < a < b$ 这么一个前提，可以将第三项变为

$$1 + \frac{a}{b} + \cdots + \frac{a^{n-1}}{b^{n-1}} < 1 + 1 + \cdots + 1 = n$$

因此可得不等式

$$b^n - a^n < (b-a)nb^{n-1}$$

至于在证明 $y^n < x$ 不成立时，选择 h 的 $h < \frac{x-y^n}{n(y+1)^{n-1}}$ 分母为什么是 $n(y+1)^{n-1}$，是因为这是为了之后处理不等式而特意设置的消除项（这里利用了函数 $f(x) = x^n$ 是连续的事实，也就是说分母一定也是实数，那么就可以将 h 视为小于某实数）；同样的在证明 $y^n > x$ 时的 k 也是如此。

Corollary. *If a and b are positive real numbers and n is a positive integer, then*

$$(ab)^{1/n} = a^{1/n}b^{1/n}$$

Proof.

令 $\alpha = a^{1/n}$，$\beta = b^{1/n}$，那么有

$$ab = \alpha^n\beta^n = (\alpha\beta)^n$$

而乘法是符合交换律的，因此

$$(ab)^{1/n} = \alpha\beta = a^{1/n}b^{1/n}$$

□

The Extended Real Number System

Definition 1.23. The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. We preserve the original order in R, and define

$$-\infty < x < +\infty$$

for every $x \in R$.

可以清楚的知道 $+\infty$ 是所有广义实数系子集的一个上界，且每个非空子集都有一个最小上界。如果 E 是一个实数的非空集合，且没有上界在 R 中，那么 $\sup E = +\infty$ 在广义实数系中。

下界同理。

广义实数系并不形成一个域，但它形成了一下惯例：

(a) 如果 x 是实数则

$$x + \infty = +\infty, \quad x - \infty = -\infty, \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0$$

(b) 如果 $x > 0$ 则 $x \cdot (+\infty) = +\infty$, $x \cdot (-\infty) = -\infty$

(c) 如果 $x < 0$ 则 $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = +\infty$

The Complex Field

Definition 1.24. A *complex number* is an ordered pair (a, b) of real numbers. "Ordered" means that (a, b) and (b, a) are regarded as distinct if $a \neq b$.

令 $x = (a, b)$，$y = (c, d)$ 为两个复数。当且仅当 $a = c$ 以及 $b = d$ 时有 $x = y$。（注意该定义并非是完全不必要的；考虑有理数的等式，表现为整数的商。）我们定义：

$$x + y = (a + c, b + d)$$

$$xy = (ac - bd, ad + bc)$$

Note 作为补充（详见该篇文章），对于任意两个复数 $x = (a, b)$, $y = (c, d)$ 的四则运算：

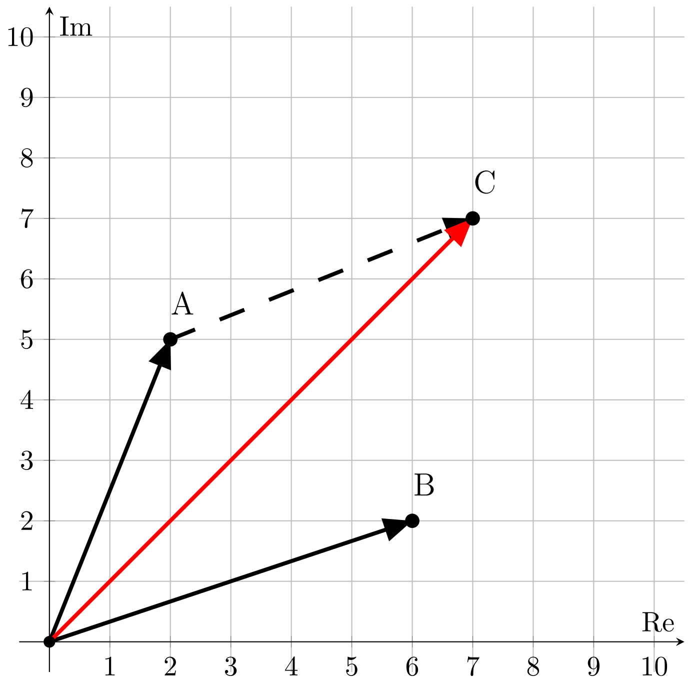
(1) $x + y = (a + c) + i(b + d)$

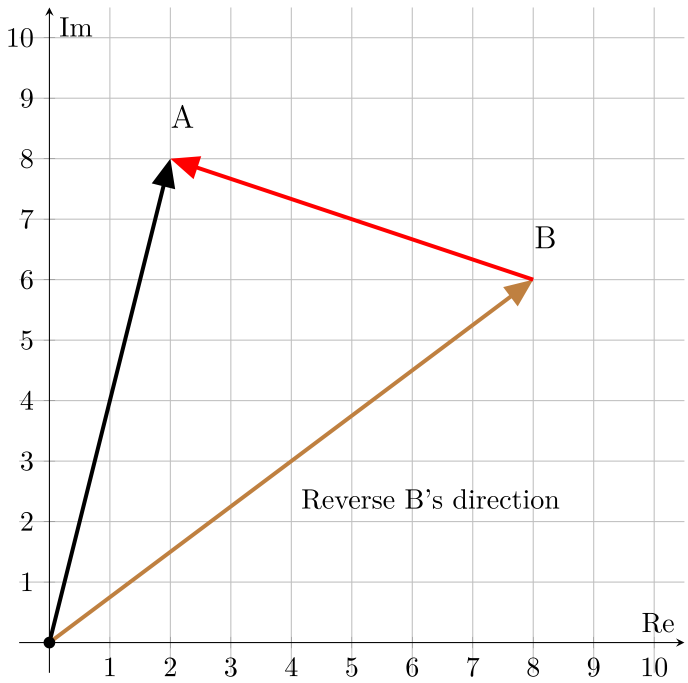
(2) $x - y = (a - c) + i(b - d)$

(3) $xy = (a + ib)(c + id) = ac - bd + i(ad + bc)$

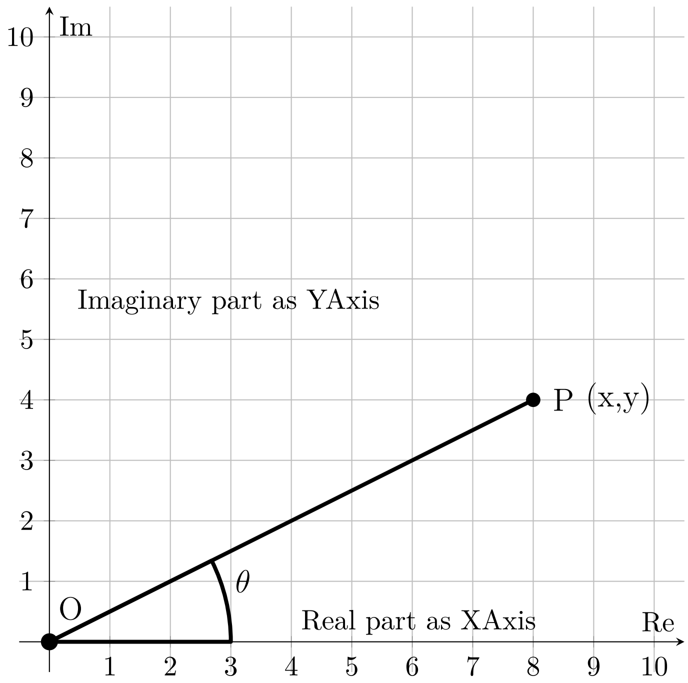
(4) $\frac{x}{y} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

首先，对于任意一个复数 $z = a + ib$，可以用复平面上的一个点来表示，那么复数加减法可以通过向量来理解：

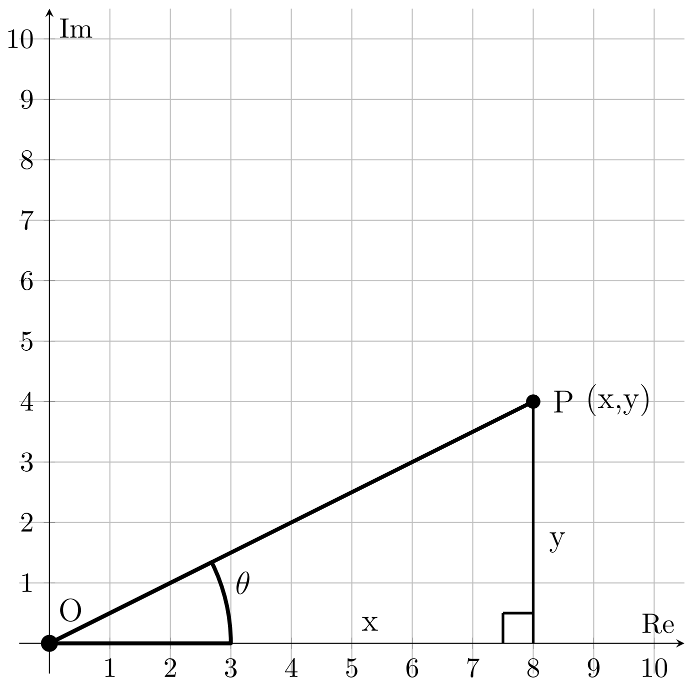




而对于复数的乘除法，需要先引入复数在极坐标上的几何意义 - 对于任意一个复数 $z = a + ib$ 用极坐标来表示：



那么复数的三角表示为：



这里将 OP 的长度作为复数 z 的**模（Modulus）**，用 $|z|$ 表示；而角 θ 为复数 z 的**幅角（Ar-**

gument)，用 $\arg(z)$ 表示。那么复数的三角表示为：

$$z = x + iy = r(\cos\theta + i\sin\theta), \quad r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}$$

接下来是复数乘法的几何意义，使用复数的三角形式计算下列两个复数的乘积：

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

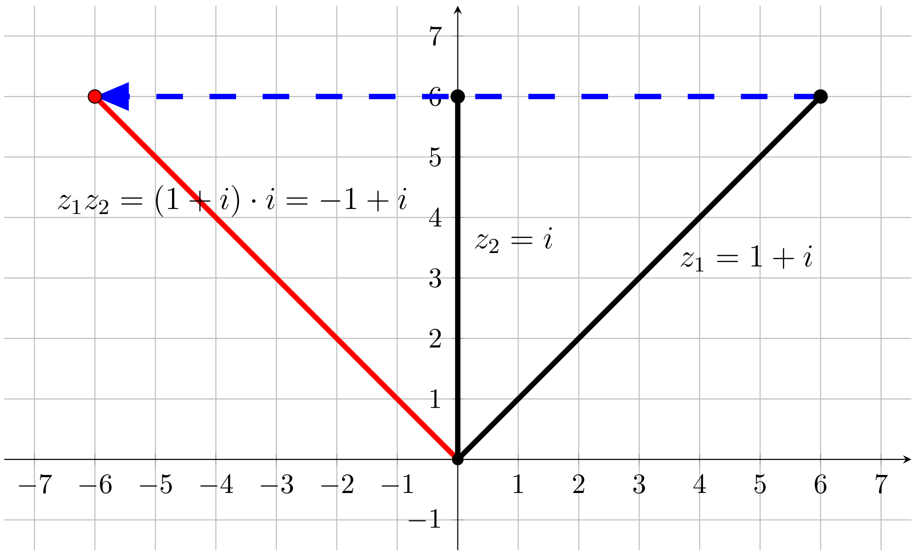
那么有：

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big) \end{aligned}$$

那么根据三角和差公式：

$$z_1 z_2 = r_1 r_2\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big)$$

可以发现 $z_1 \cdot z_2$ 计算后模为两个复数模的乘积 $|z_1||z_2|$，幅角为两个复数幅角之和 $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$。因此复数的乘积可以理解为**拉伸与旋转**。例如：



因为

$$z_1 = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}), \quad z_2 = 1(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$$

所以 z_1z_2 的模长为 $\sqrt{2}$ 且幅角为 $\frac{3\pi}{4}$。而复数的除法只需要将除法写成乘法形式即可

$$\begin{aligned} z^{-1} &= \big(r(\cos\theta + i\sin\theta)\big)^{-1} \\ &= r^{-1}\frac{1}{\cos\theta + i\sin\theta} \\ &= r^{-1}\frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)} \\ &= r^{-1}(\cos\theta - i\sin\theta) \end{aligned}$$

那么两个复数

$$\begin{aligned} z_1 &= r_1(\cos\theta_1 + i\sin\theta_1) \\ z_2 &= r_2(\cos\theta_2 + i\sin\theta_2) \end{aligned}$$

的除法便是

$$\begin{aligned} \frac{z_1}{z_2} &= z_1{z_2}^{-1} \\ &= \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_2)(\cos\theta_1 - i\sin\theta_2) \\ &= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)) \end{aligned}$$

因此，$\frac{z_1}{z_2}$ 的模长为两个模相除 $\frac{|z_1|}{|z_2|}$，幅角为 $\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$。所以复数的除法也可以理解为**拉伸与旋转**。

综上所述，复数的加减法就是向量的加减法，乘除法就是拉伸与旋转变换。

Theorem 1.25. *These definitions of addition and multiplication turn the set of all complex numbers into a field, with (0, 0) and (1, 0) in the role of 0 and 1.*

Proof.

我们简单的验证一下域的公理（Definition 1.12），使用 R 的域结构。令 $x = (a, b), y = (c, d), z = (e, f)$。

(A1) 很清楚。

(A2) $x + y = (a + c, b + d) = (c + a, d + b) = y + x$

(A3)
$$\begin{aligned} (x + y) + z &= (a + c, b + d) + (e, f) \\ &= (a + c + e, b + d + f) \\ &= (a, b) + (c + e, d + f) \\ &= x + (y + z) \end{aligned}$$

(A4) $x + 0 = (a, b) + (0, 0) = (a, b) = x$

(A5) 令 $-x = (-a, -b)$，那么 $x + (-x) = (0, 0) = 0$

(M1) 很清楚。

(M2) $xy = (ac - bd, ad + bc) = (ca - db, da + cb) = yx$

(M3)
$$\begin{aligned}(xy)z &= (ac - db, ad + bc)(e, f)\\ &= (ace - bde - adf - bef, acf - bdf + ade + bce)\\ &= (a, b)(ce - df, cf + de)\\ &= x(yz)\end{aligned}$$

(M4) $1x = (1, 0)(a, b) = (a, b) = x$

(M5) 如果 $x \neq 0$ 那么 $(a, b) \neq (0, 0)$，也就是说 a 和 b 至少有一个实数不等于 0。因此 $a^2 + b^2 > 0$，根据 Proposition 1.18(d)，我们可以定义

$$\frac{1}{x} = (\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2})$$

那么

$$x \cdot \frac{1}{x} = (a, b)(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}) = (1, 0) = 1$$

(D)
$$\begin{aligned}x(y + z) &= (a, b)(c + e, d + f)\\ &= (ac + ae - bd - bf, ad + af + bc + be)\\ &= (ac - bd, ad + bc) + (ae - bf, af + be)\\ &= xy + yz\end{aligned}$$

□

Theorem 1.26. *For any real numbers a and b we have*

$$(a, 0) + (b, 0) = (a + b, 0), \quad (a, 0)(b, 0) = (ab, 0)$$

Definition 1.27. $i = (0, 1)$

Theorem 1.28. $i^2 = -1$

Proof.
$i^2 = (0, 1)(0, 1) = (-1, 0) = -1$ □

Theorem 1.29. *If a and b are real, then $(a, b) = a + bi$*

Proof.

$$\begin{aligned} a + bi &= (a, 0) + (b, 0)(0, 1) \\ &= (a, 0) + (0, b) \\ &= (a, b) \end{aligned}$$

□

Definition 1.30. If a, b are real and $z = a + bi$, then the complex number $\overline{z} = a - bi$ is called the *conjugate* of z. The numbers a and b are the *real part* and the *imaginary part* of z, respectively. We shall occasionally write

$$a = Re(z), \quad b = Im(z)$$

Theorem 1.31. *If z and w are complex, then*

(a) $\overline{z + w} = \overline{z} + \overline{w}$

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$

(c) $z + \overline{z} = 2\,Re(z)$, $z - \overline{z} = 2i\,Im(z)$

(d) $z\overline{z}$ *is real and positive (except when* $z = 0$*)*

Definition 1.32. If z is a complex number, its absolute value $|z|$ is the non-negative square root of $z\overline{z}$; that is, $|z| = (z\overline{z})^{1/2}$.

$|z|$ 的存在（以及唯一性）遵循 Theorem 1.12 以及 Theorem 1.31 (d)。

注意当 x 为实数时，那么 $\overline{x} = x$，因此 $|x| = \sqrt{x^2}$。所以如果 $x \geq 0$ 时 $|x| = x$，如果 $x < 0$ 时 $|x| = -x$。

Theorem 1.33. *Let z and w be complex numbers. Then*

(a) $|z| > 0$ *unless* $z = 0$, $|0| = 0$

(b) $|\overline{z}| = |z|$

(c) $|zw| = |z||w|$

(d) $|Re\, z| \leq |z|$

(e) $|z + w| \leq |z| + |w|$

Proof.

(a) 与 (b) 不足为道。令 $z = a + bi$, $w = c + di$，其 a, b, c, d 皆为实数。那么

$$|zw|^2 = (ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2) = |z|^2|w|^2$$

即 $|zw|^2 = (|z||w|)^2$。(c) 遵循 Theorem 1.21 所声明的唯一性。

证明 (d)，有 $a^2 \leq a^2 + b^2$，因此

$$|a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}$$

证明 (e)，有 $\overline{z}w$ 与 $z\overline{w}$ 是共轭的，因此 $\overline{z}w + z\overline{w} = 2\,Re\,(z\overline{w})$。因此

$$\begin{aligned}
|z + w|^2 &= (z + w)(\overline{z} + \overline{w}) \\
&= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w} \\
&= |z|^2 + 2\,Re\,(z\overline{w}) + |w|^2 \\
&\leq |z|^2 + 2|z\overline{w}| + |w|^2 \\
&= |z|^2 + 2|z||w| + |w|^2 \\
&= (|z| + |w|)^2
\end{aligned}$$

两边开根号后即可得 (e)。 □

Note 计算中第一步的 $|z + w|^2 = (z + w)(\overline{z} + \overline{w})$ 是将 $z + w$ 视为整体，并使用了 Definition 1.32 中的 $|z| = (z\overline{z})^{1/2}$ 转换为 $(z + w)(\overline{z + w})$，而又因为 Theorem 1.31 (a) 可将第二项变为 $(\overline{z} + \overline{w})$；而第三步到第四步的不等式则利用了 Theorem 1.33 (d)，即 $|Rez| \leq |z|$；第四步到第五步则使用了 Theorem 1.33 (c)，即 $|zw| = |z||w|$。

Notation 1.34. If $x_1, \ldots, x_n$ are complex numbers, we write

$$x_1 + x_2 + \cdots + x_n = \sum_{j=1}^{n} x_j$$

我们用一个重要的不等式来结束本节，它通常被称为 *Schwarz inequality*。

Theorem 1.35. *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then*

$$\left|\sum_{j=1}^{n} a_j\overline{b}_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Proof.

令 $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \bar{b}_j$，本证明中 j 取值 $1, \ldots, n$。如果 $B = 0$，那么就有 $b_1 = \cdots = b_n = 0$，那么结论很清楚。因此假设 $B > 0$。根据 Theorem 1.31 有

$$\begin{aligned}\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\bar{a}_j - \overline{CB_j}) \\ &= B^2 \sum |a_j|^2 - B\overline{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B|C|^2 \\ &= B(AB - |C|^2)\end{aligned}$$

因为在首次求和的每个项都是非负的，可以知道

$$B(AB - |C|^2) \geq 0$$

又因为 $B > 0$，它遵循 $AB - |C|^2 \geq 0$。它便是预期的不等式。 □

Note $|Ba_j - Cb_j|^2$ 为构造项，将其中 $Ba_j - Cb_j$ 视为整体根据 Definition 1.32，可转换为 $(Ba_j - Cb_j)(\overline{Ba_j - Cb_j})$，而后者根据 Theorem 1.31 即可写为 $B\bar{a}_j - \overline{Cb_j}$（这里 $B = \sum |b_j|^2$ 的共轭还是其本身）；根据乘法分配律得出第二步后，将之前设定的 A, B, C 带入即可得出 $B(AB - |C|^2)$；最后根据起始的构造项 $|Ba_j - Cb_j|^2$ 必然非负以及之前假设的 $B > 0$ 可以得出 $AB - |C|^2 \geq 0$；将 A, B, C 原本代表的值带入，即 $\sum |a_j|^2 \sum |b_j|^2 - |\sum a_j \bar{b}_j| \geq 0$。

Euclidean Spaces

Definition 1.36. For each positive integer k, let R^k be the set of all ordered k-tuples

$$\mathbf{x} = (x_1, x_2, \ldots, x_k)$$

where $x_1, \ldots, x_k$ are real numbers, called the *coordintates* of $\mathbf{x}$. The elements of R^k are called points, or vectors, especially when $k > 1$. We shall denote vectors by boldfaced letters. If $\mathbf{y} = (y_1, \ldots, y_k)$ and if α is a real number, put

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_k + y_k)$$

$$\alpha\mathbf{x} = (\alpha x_1, \ldots, \alpha x_k)$$

so that $\mathbf{x} + \mathbf{y} \in R^k$ and $\alpha\mathbf{x} \in R^k$. This defines addition of vectors, as well as multiplication of a vector by a real number (a scalar). These two operations satisfy the commutative, associative, and distributive laws (the proof is trivial, in view of the analogous for the real numbers) and make R^k into a *vector space over the real field*. The zero element of R^k (sometimes called the *origin* or the *null vector*) is the point $\mathbf{0}$, all of whose coordintates are 0.

We also define the so-called "inner product" (or scalar product) of $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{k} x_i y_i$$

and the *norm* of $\mathbf{x}$ by

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{1}^{k} x_i^2 \right)^{1/2}$$

The structure now defined (the vector space R^k with the above inner product and norm) is called euclidean k-space.

Theorem 1.37. *Suppose* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^k$, *and* α *is real. Then*

(a) $|\mathbf{x}| \geq 0$;

(b) $|\mathbf{x}| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$;

(c) $|\alpha \mathbf{x}| = |\alpha||\mathbf{x}|$;

(d) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$;

(e) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;

(f) $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$

Proof.

前三项不必赘述，而 (d) 是 Schwarz 不等式的间接结论。通过 (d) 可以有

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} \\ &\leq |\mathbf{x}|^2 + 2|\mathbf{x}||\mathbf{y}| + |\mathbf{y}|^2 \\ &= (|\mathbf{x}| + |\mathbf{y}|)^2 \end{aligned}$$

这样 (e) 就被证明了。最后替换 (e) 中的 $\mathbf{x}$ 为 $\mathbf{x} - \mathbf{y}$ 以及 $\mathbf{y}$ 为 $\mathbf{y} - \mathbf{z}$ (f) 可以得出 (f)。 □

Remark 1.38. *Theorem 1.37 (a), (b), and (f) will allow us (see Chap. 2) to regard* R^k *as a metric space.*

R^1 *(the set of all real numbers) is usually called the line, or the real line. Likewise,* R^2 *is called the plane, or the complex plane (compare Definitions 1.24 and 1.36). In these two cases the norm is just the absolute value of the corresponding real or complex number.*

2 Basic Topology

Finite, Countable, and Uncoutable Sets

本节由函数概念的定义开始。

Definition 2.1. Consider two sets A and B, whose elements may be any objects whatsover, and suppose that with each element x of A there is associated, in some manner, an element of B, which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* of A into B). The set A is called the *domain* of f (we also say f is defined on A), and the elements $f(x)$ are called the *values* of f. The set of all values of f is called the *range* of f.

Definition 2.2. Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$. We call $f(E)$ the *image* of E under f. In this notation, $f(A)$ is the range of f. It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A *onto* B. (Note that, according to this usage, *onto* is more specific than *into*.)

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the *inverse image* of E under f. If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$. If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A, then f is said to be a 1-1 (*one-to-one*) mapping of A into B. This may also be expressed as follows: f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2,\ x_1 \in A,\ x_2 \in A$.

(The notation $x_1 \neq x_2$ means that x_1 and x_2 are distinct elements; otherwise we write $x_1 = x_2$.)

Note 简单来说：

1. $f(E)$ 是集合 E 通过 f 得到的像（image）。
2. $f(A)$ 是 f 的范围。
3. 如果 $f(A) = B$，那么称 f 将 A 完全映射至（onto）B 。
4. 而当 $E \subset B$ 且 $x \in A$ 时，反函数 $f^{-1}(E)$ 是集合 E 通过 f 得到的反像（inverse image）。
5. $\forall x \in A$ 通过 f 映射后且满足 $\forall f(x) \in B$ 被称为一一映射（1-1 mapping）。

Definition 2.3. If there exists a 1-1 mapping of A *onto* B, we say that A and B can be put in 1-1 *correspondence*, or that A and B have the same *cardinal number*, or, briefly, that A and B are *equivalent*, and we write $A \sim B$. This relation clearly has the following properties:

It is reflexive: $A \sim A$.

It is symmetric: If $A \sim B$, then $B \sim A$.

It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Any relation with these three properties is called an *equivalence relation*.

Note Reflexive 自反性，维基百科。

Definition 2.4. For any positive integer n, let J_n be the set whose elements are the integers $1, 2, \ldots, n$; let J be the set consisting of all positive integers. For any set A, we say:

(a) A is *finite* if $A \sim J_n$ for some n (the empty set is also considered to be finnite).

(b) A is *infinite* if A is not finite.

(c) A is *countable* if $A \sim J$.

(d) A is *uncountable* if A is neither finite nor countable.

(e) A is *at most countable* if A is finite or countable.

Countable sets are sometimes called *enumerable*, or *denumerable*.

For two finite sets A and B, we evidently have $A \sim B$ if and only if A and B contain the same number of elements. For infinite sets, however, the idea of "having the same number of elements" becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

Note 可数集（countable set），是指每个元素都能与自然数集 N 的每个元素之间能建立一一对应的集合；不可数集顾名思义就是无法与自然数集 N 建立一一对应的集合；至多可数集（at most coutable）是有限集（finite）与可数集（coutable）的统称。

Definition 2.7. By a *sequence*, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, x_3, \ldots$. The values of f, that is, the elements x_n, are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a *sequence* in A, or a *sequnce of elements* of A.

注意一个数列的 $x_1, x_2, x_3, \ldots$ 项不需要是独特的。

由于每个可数集合是一个定义在 J 上一一映射的范围，可将每个可数集合视为一系列不同项的范围。更宽泛来说，任何可数集合中的原始可以被"排列在一个数列上"。

有时可以将定义中的 J 替换为所有非负整数集合，这样可能会更加的方便，例如开始于 0 而不是 1。

Theorem 2.8. *Every infinite subset of a countable set A is coutable.*

Proof.

假设 $E \subset A$，且 E 为无限的。排列 A 中的元素 x 构建 $\{x_n\}$ 独特数列。构建一个满足如下的数列 $\{n_k\}$：

令 n_1 为最小的正整数使得 $x_{n_1} \in E$。选择 $n_1, \dots, n_{k-1}$ $(k = 2, 3, 4, \dots)$，令 n_k 为最小的大于 n_{k-1} 的整数使得 $x_{n_k} \in E$。

令 $f(k) = x_{n_k}$ $(k = 1, 2, 3, \dots)$，我们获取一个 E 与 J 的一一映射关系。

根据定理，粗略的说可数集合表示了 "最小的" 无限性：没有不可数集合可以成为一个可数集合的子集。 □

Note 一个可数集合 A 的任意无限子集都是可数的。

Definition 2.9. Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α.

The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$. Instead of speaking of sets of sets, we shall sometimes speak of a collection of sets, or a family of sets.

The *union* of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

(1) $$S = \bigcup_{\alpha \in A} E_\alpha.$$

If A consists of the integers $1, 2, \dots, n$, one usually writes

(2) $$S = \bigcup_{m=1}^{n} E_m$$

or

(3) $$S = E_1 \cup E_2 \cup \cdots \cup E_n.$$

If A is the set of all positive integers, the usual notation is

(4) $$S = \bigcup_{m=1}^{\infty} E_m.$$

The symbol ∞ in (4) merely indicates that the union of a *countable* collection of sets is taken, and should not be confused with the symbols $+\infty$, $-\infty$, introduced in Definition 1.23.

The *intersection* of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. We use the notation

(5) $$P = \bigcap_{\alpha \in A} E_\alpha,$$

or

$$P = \bigcap_{m=1}^{n} E_m = E_1 \cap E_2 \cap \cdots \cap E_n, \tag{6}$$

or

$$P = \bigcap_{m=1}^{\infty} E_m, \tag{7}$$

as for unions. If $A \cap B$ is not empty, we say that A and B *intersect*; otherwise they are *disjoint*.

Note S 代表所有 E_α 集合的并集；P 代表所有 E_α 集合的交集。

Remark 2.11. *Many properties of unions and intersections are quite similar to those of sums and products; in fact, the words sum and product were sometimes used in this connection, and the symbols Σ and Π were written in place of $\bigcup$ and $\bigcap$.*

The commutative and associative laws are trivial:

$$A \cup B = B \cup A; \quad A \cap B = B \cap A. \tag{8}$$

$$(A \cup B) \cup C = A \cup (B \cup C); \quad (A \cap B) \cap C = A \cap (B \cap C). \tag{9}$$

Thus the omission of parenthese in (3) and (6) is justified.

The distributive law also holds:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{10}$$

To prove this, let the left and right members of (10) be denoted by E and F, respectively.

Suppose $x \in E$. Then $x \in A$ and $x \in B \cup C$, that is, $x \in B$ or $x \in C$ (possibly both). Hence $x \in A \cap B$ or $x \in A \cap C$, so that $x \in F$. Thus $E \subset F$.

Next, suppose $x \in F$. Then $x \in A \cap B$ or $x \in A \cap C$. That is, $x \in A$, and $x \in B \cup C$. Hence $x \in A \cap (B \cup C)$, so that $F \subset E$.

It follows that $E = F$.

We list a few more relations which are easily verified:

$$A \subset A \cup B, \tag{11}$$

$$A \cap B \subset A. \tag{12}$$

If 0 denotes the empty set, then

$$A \cup 0 = A, \quad A \cap 0 = 0. \tag{13}$$

If $A \subset B$, then

$$A \cup B = B, \quad A \cap B = A. \tag{14}$$

Theorem 2.12. *Let $\{E_n\}$, $n = 1, 2, 3, \dots$, be a sequence of countable sets, and put*

$$S = \bigcup_{n=1}^{\infty} E_n. \tag{15}$$

Then S is countable.

Proof.

Let every set E_n be arranged in a sequence $\{x_{nk}\}$, $k = 1, 2, 3, \dots$, and consider the infinite array

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \tag{16}$$

in which the elements of E_n form the nth row. The array contains all elements of S. As indicated by the arrows, these elements can be arranged in a sequence

$$x_1 1; x_2 1, x_1 2; x_3 2, x_2 2, x_1 3; x_4 1, x_3 2, x_2 3, x_1 4; \dots \tag{17}$$

If any two of the sets E_n have elements in common, these will appear more than once in (17). Hnce there is a subset T of the set of all positive integers such that $S \sim T$, which shows that S is at most coutable (Theorem 2.8). Since $E_1 \subset S$, and E_1 is infinite, S is infinite, and thus countable. □

Corollary. *Suppose A is at most coutable, and, for every $\alpha \in A$, B_α is at most countable. Put*

$$T = \bigcup_{\aleph \in A} B_\alpha \ .$$

Then T is at most countable.

T 相当于 (15) 的子集。

Theorem 2.13. *Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \dots, a_n)$, where $a_k \in A$ $(k = 1, \dots, n)$, and the elements $a_1, \dots, a_n$ need not be distinct. Then B_n is coutable.*

Proof.

B_1 可数是显而易见的，因为 $B_1 = A$。假设 B_{n-1} 是可数的（$n = 2, 3, 4, \ldots$）。B_n 的元素形式是

$$(b, a) \quad (b \in B_{n-1}, \alpha \in A). \tag{18}$$

对于每个固定的 b，成对集合（set of pairs）b, a 等同于 A，即是可数的。因此 B_n 是若干可数集合的并集构成的可数集合。根据 Theorem 2.12，B_n 是可数的。 □

Corollary. *The set of all rational numbers is countable.*

Proof.

我们应用 Theorem 2.13 同时 $n = 2$，所有有理数 r 都可以表示为 b/a，其中 a 与 b 都是整数。那么成对集合 (a, b) 就是分数 b/a 的集合，即是可数的。 □

实际上，所有代数集合都是可数的。

然而并不是所有的无限集合是可数的，详见下个定理。

Theorem 2.14. *Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncoutable.*

A 集合中的元素数列类似于 $1, 0, 0, 1, 0, 1, 1, 1, \ldots$。

Proof.

令 E 为集合 A 中的一个可数子集，且令 E 由数列 $s_1, s_2, s_3, \ldots$ 构成。再构建一个满足以下条件的数列 s。如果在 s_n 中的第 n 个小数是 1，令 s 的第 n 个小数为 0，以此类推。那么数列 s 至少有一处是有别于所有 E 中的成员；因此 $s \notin E$。但是陷入 $s \in A$，因此 E 是 A 的一个合理子集。

我们证明了所有 A 集合的可数子集是合理的子集。对于 A 是不可数的也同理（否则 A 将会是 A 合理的子集，这是荒谬的）。 □

Metric Spaces

Definition 2.15. A set X, whose elements we shall call *points*, is said to be a *metirc space* if with any two points p and q of X there is associated a real number $d(p, q)$, called the *distance* from p to q, such that

(a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;

(b) $d(p, q) = d(q, p)$;

(c) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

任何拥有上述三个性质的函数都被称为距离函数 *distance function*，或者度规 *metric*。

Example 2.16. *The most important examples of metric spaces, from our standpoint, are the euclidean spaces R^k, especially R^1 (the real line) and R^2 (the complex plane); the distance in R^k is defined by*

$$d(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}| \quad (\mathbf{x},\mathbf{y} \in R^k). \tag{19}$$

Note

- 欧几里得空间的距离概念抽象化后，即度量空间（Metric Space）。
- (a) 正定性，(b) 对称性，(c) 三角不等式。

Definition 2.17. By the *segment* (a, b) we mean the set of all real numbers x such that $a < x < b$.

By the *interval* $[a, b]$ we mean the set of all real numbers x such that $a \le x \le b$.

Occasionally we shall also encounter "half-open intervals" $[a, b)$ and $(a, b]$; the first consists of all x such that $a \le x < b$, the second of all x such that $a < x \le b$.

If $a_i < b_i$ for $i = 1, \ldots, k$, the set of all points $\mathbf{x} = (x_1, \ldots, x_k)$ in R^k whose coordinates satisfy the inequalities $a_i \le x_i \le b_i$ $(1 \le i \le k)$ is called a *k-cell*. Thus a 1-cell is an interval, a 2-cell is a rectangle, etc.

If $\mathbf{x} \in R^k$ and $r > 0$, the *open (or closed) ball* B with center at $\mathbf{x}$ and radius r is defined to be the set of all $\mathbf{y} \in R^k$ such that $|\mathbf{y}-\mathbf{x}| < r$ (or $|\mathbf{y}-\mathbf{x}| \le r$).

We call a set $E \subset R^k$ *convex* if

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in E$$

whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $0 < \lambda < 1$.

Note

- k-方格（k-cell）
- R^k 空间定义**开/闭球（open/closed ball**
- ball 和 k-cell 都是凸的（**convex**）

WIP

Compact Sets

WIP

Perfect Sets

WIP

Connected Sets

WIP

3 Numerical Sequences and Series

4 Continuity

5 Differentiation

6 The Riemann-Stiltjes Integral

7 Sequences and Series of Functions

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10 Integration of Differential Forms

11 The Lebesgue Theory