

Study Notes of Principles of Mathematical Analysis

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1 The Real and Complex Number Systems

Ordered Sets

Definition 1.7. Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

- (i) α is an upper bound of E .
- (ii) If $\gamma < \alpha$ then γ is not an upper bound of E .

Then α is called the *least upper bound of E* or the *supremum of E* , and we write

$$\alpha = \sup E$$

The *greatest lower bound*, or *infimum*, of a set E which is bounded below is defined in the same manner: The statement

$$\alpha = \inf E$$

means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E .

Note S 是有序集合的情况下, E 又是属于 S 的, 并且 E 拥有上界。那么只会存在一个 α 是 E 的最小上界。同理如果是 E 拥有下界, 只会存在一个 α 是 E 的最大下界。发音: Supremum [su:'pri:məm]; Infimum ['ɪnfɪməm]。

Definition 1.10. An ordered set S is said to have the *least-upper-bound property* if the following is true: If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S .

Note S 中存在 $E \subset S$, 且 E 具有最小上界, 那么 S 就具有最小上界性, 反之亦然。

Theorem 1.11. Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B . Then $\alpha = \sup L$ exists in S , and $\alpha = \inf B$. In particular, $\inf B$ exists in S .

Proof.

因为 B 是有下界的, 且 L 不为空。由于 L 包含了所有的 y ($y \in S$) 且满足不等式 $y \leq x$ ($x \in B$), 那么所有的 $x \in B$ 都是 L 的上界。因此 L 是有上界的。关于 S 的假设意为在 S 中有一个 L 的最小上界, 被称为 α 。

如果 $\gamma < \alpha$ 那么 (根据定义 1.8) γ 并不是 L 的一个上界, 因此 $\gamma \notin L$ 。对于所有的 $x \in B$ 都有 $\alpha \leq x$ 。因此 $\alpha \in L$ 。

如果 $\alpha < \beta$ 那么 $\beta \notin L$, 因为 α 是 L 的一个上界。

我们展示过了 $\alpha \in L$ 但是 $\beta \notin L$ 而 $\beta > \alpha$ 的情况。也就是说, α 是 B 的一个下界, 但是当 $\beta > \alpha$ 时 β 却不是。这就意味着 $\alpha = \inf B$ 。 \square

Fields

Definition 1.12. A field is a set F with two operations, called *addition* and *multiplication*, which satisfy the following so-called "field axioms" (A), (M), and (D):

(A) Axioms for addition

- (A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F .
- (A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.
- (A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
- (A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
- (A5) To every $x \in F$ corresponds an element $-x \in F$ such that $x + (-x) = 0$.

(M) Axioms for multiplication

- (M1) If $x \in F$ and $y \in F$, then their product xy is in F .
- (M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.
- (M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
- (M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
- (M5) If $x \in F$ and $x \neq 0$ then there exists an element $\frac{1}{x} \in F$ such that $x \cdot (\frac{1}{x}) = 1$.

(D) The distributive law

$$x(y + z) = xy + xz \text{ holds for all } x, y, z \in F.$$

Note 域的定义：维基百科。

Definition 1.17. An *ordered field* is a field F which is also an ordered set, such that:

1. $x + y < x + z$ if $x, y, z \in F$ and $y < z$,
2. $xy > 0$ if $x \in F, y \in F, x > 0$, and $y > 0$.

如果 $x > 0$, 我们称 x 为 *positive*; 如果 $x < 0$, x 则为 *negative*.

The Real Field

Theorem 1.19. *There exists an ordered field R which has the least-upper-bound property. Moreover, R contains Q as a subfield.*

第二个声明意味着 $Q \subset R$ 以及加法与乘法在 R 上的运算，当应用于 Q 的成员时，与有理数的通常操作重合；同样的，正有理数成员是 R 的正元素。

R 的成员被称为 *real numbers*，即实数。

Theorem 1.20.

(a) *If $x \in R$, $y \in R$, and $x > 0$, then there is a positive integer n such that $nx > y$.*

(b) *If $x \in R$, $y \in R$, and $x < y$, then there exists a $p \in Q$ such that $x < p < y$.*

对于 (a) 部分通常认为是 R 具有 *archimedean property*，即阿基米德性质，详见维基百科。
(b) 部分则表明 Q 是在 R 中 *dense*，即具有稠密性：在任意两个实数之间有一个有理数。

Proof.

(a) 令 A 作为所有 nx 的集合，其中 n 为所有的正整数。如果 (a) 是错误的，那么 y 则会是 A 的一个上界。但是接着 A 会在 R 中拥有一个最小上界，即 $\alpha = \sup A$ 。由于 $x > 0$ ， $\alpha - x < \alpha$ ，以及 $\alpha - x$ 不是 A 的上界，因此 $\alpha - x < mx$ 对于某些正整数 m 成立。但是这样就会有 $\alpha < (m+1)x \in A$ ，这是不可能的，因为 α 是 A 的上界。

(b) 因为 $x < y$ ， $y - x > 0$ 以及由 (a) 所知一个正整数 n 满足

$$n(y - x) > 1$$

再次应用 (a)，获取正整数 m_1 与 m_2 满足 $m_1 > nx$ ， $m_2 > -nx$ ，那么

$$-m_2 < nx < m_1$$

因此会有一个整数 m ($-m_2 \leq m \leq m_1$) 满足

$$m - 1 \leq nx \leq m$$

如果我们结合这些不等式，则会得到

$$nx < m \leq 1 + nx < ny$$

因为 $n > 0$ ，它遵循

$$x < \frac{m}{n} < y$$

通过 $p = \frac{m}{n}$ 证明了 (b)。

□

Note (b) 的第一步将 $y - x$ 作为整体, 将 (a) $nx > y$ 中的 x 替换为 $y - x$, y 替换为 1。同理对于整数 m_1 与 m_2 而言, 可分别将 nx 与 $-nx$ 视为不等式右侧的 y , 而不等式左侧的 x 视为 1, 那么就有了 $m_1 \cdot 1 > nx$ 与 $m_2 \cdot 1 > -nx$ 。对于整数 m 的 $-m_2 \leq m \leq m_1$ 最坏的情况可将 $-m_2$ 与 m_1 视为相邻的整数, 比如说 1 和 2, 那么当 m 取值为 2 时视为 $2 - 1 \leq nx < 2$, 满足 $m - 1 \leq nx < m$ 。最后根据有理数的定义 $p = m/n$ $m, n \in \mathbb{Q}$ 可以得出 x 与 y 之间一定存在一个有理数。

Theorem 1.21. For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.

这个 y 数可以被写作 $\sqrt[n]{x}$ 或是 $x^{\frac{1}{n}}$ 。

Proof.

对于至多存在一个 y 的论证很简单, 因为 $0 < y_1 < y_2$ 意味着 $y_1^n < y_2^n$ 。

令 E 为所有满足 $t^n < x$ 的正实数 t 的集合。

如果 $t = x/(1+x)$ 那么 $0 \leq t < 1$, 那么 $t^n \leq t < x$ 。因此 $t \in E$, 且 E 不为空。

如果 $t > 1+x$ 那么 $t^n \geq t > x$, 所以 $t \notin E$ 。因此 $1+x$ 是 E 的一个上界。

所以根据 Theorem 1.19 得出, 存在一个

$$y = \sup E$$

而证明 $y^n = x$ 我们需要展示不等式 $y^n < x$ 与 $y^n > x$ 皆会导致矛盾。

当 $0 < a < b$ 时, 等式 $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ 得出不等式

$$b^n - a^n < (b-a)nb^{n-1}$$

假设 $y^n < x$ 。选择 h 使得 $0 < h < 1$ 且

$$h < \frac{x - y^n}{n(y+1)^{n-1}}$$

令 $a = y$, $b = y + h$, 那么就有

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$

因此 $(y+h)^n < x$, 且 $y+h \in E$ 。因为 $y+h > y$, 这与 y 是 E 的一个上界相矛盾。

假设 $y^n > x$ 。令

$$k = \frac{y^n - x}{ny^{n-1}}$$

那么 $0 < k < y$ 。如果 $t \geq y - k$, 我们得出以下结论:

$$y^n - t^n \leq y^n - (y-k)^n \leq kny^{n-1} = y^n - x$$

因此 $t^n > x$, 且 $t \notin E$ 。它遵循 $y-k$ 是 E 的一个上界。

但是因为 $y-k < y$, 其与 y 是 E 的最小上界的事实相矛盾。

因此 $y^n = x$, 证明完成。 □

Note 至多存在一个 y 换个角度也就是说但凡有第二个 y 使得 $y_1^n = y_2^n$, 那么 $y_1 = y_2$ 。
等式

$$b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$$

左侧可以视为

$$(b - a)(b^{n-1} + b^{n-1}\frac{a}{b} + \cdots + b^{n-1}\frac{a^{n-1}}{b^{n-1}})$$

提取 b^{n-1} 后得

$$(b - a)b^{n-1}(1 + \frac{a}{b} + \cdots + \frac{a^{n-1}}{b^{n-1}})$$

由于有 $0 < a < b$ 这么一个前提, 可以将第三项变为

$$1 + \frac{a}{b} + \cdots + \frac{a^{n-1}}{b^{n-1}} < 1 + 1 + \cdots + 1 = n$$

因此可得不等式

$$b^n - a^n < (b - a)nb^{n-1}$$

至于在证明 $y^n < x$ 不成立时, 选择 h 的 $h < \frac{x - y^n}{n(y+1)^{n-1}}$ 分母为什么是 $n(y+1)^{n-1}$, 是因为这是为了之后处理不等式而特意设置的消除项 (这里利用了函数 $f(x) = x^n$ 是连续的事实, 也就是说分母一定也是实数, 那么就可以将 h 视为小于某实数); 同样的在证明 $y^n > x$ 时的 k 也是如此。

Corollary. *If a and b are positive real numbers and n is a positive integer, then*

$$(ab)^{1/n} = a^{1/n}b^{1/n}$$

Proof.

令 $\alpha = a^{1/n}$, $\beta = b^{1/n}$, 那么有

$$ab = \alpha^n \beta^n = (\alpha\beta)^n$$

而乘法是符合交换律的, 因此

$$(ab)^{1/n} = \alpha\beta = a^{1/n}b^{1/n}$$

□

The Extended Real Number System

Definition 1.23. The extended real number system consists of the real field \mathbb{R} and two symbols, $+\infty$ and $-\infty$. We preserve the original order in \mathbb{R} , and define

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

可以清楚的知道 $+\infty$ 是所有衍生的实数系统子集的一个上界, 且每个非空子集都有一个最小上界。如果 E 是一个实数的非空集合, 且没有上界在 \mathbb{R} 中, 那么 $\sup E = +\infty$ 在衍生实数系统中。

下界同理。

衍生实数系统并不形成一个域, 但它形成了一下惯例:

(a) 如果 x 是实数则

$$x + \infty = +\infty, \quad x - \infty = -\infty, \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0$$

(b) 如果 $x > 0$ 则 $x \cdot (+\infty) = +\infty, x \cdot (-\infty) = -\infty$

(c) 如果 $x < 0$ 则 $x \cdot (+\infty) = -\infty, x \cdot (-\infty) = +\infty$

The Complex Field

Definition 1.24. A *complex number* is an ordered pair (a, b) of real numbers. "Ordered" means that (a, b) and (b, a) are regarded as distinct if $a \neq b$.

令 $x = (a, b)$, $y = (c, d)$ 为两个复数。当且仅当 $a = c$ 以及 $b = d$ 时有 $x = y$ 。(注意该定义并非是完全不必要的; 考虑有理数的等式, 表现为整数的商。) 我们定义:

$$x + y = (a + c, b + d)$$

$$xy = (ac - bd, ad + bc)$$

Note 作为补充 (详见该篇文章), 对于任意两个复数 $x = (a, b)$, $y = (c, d)$ 的四则运算:

$$(1) \quad x + y = (a + c) + i(b + d)$$

$$(2) \quad x - y = (a - c) + i(b - d)$$

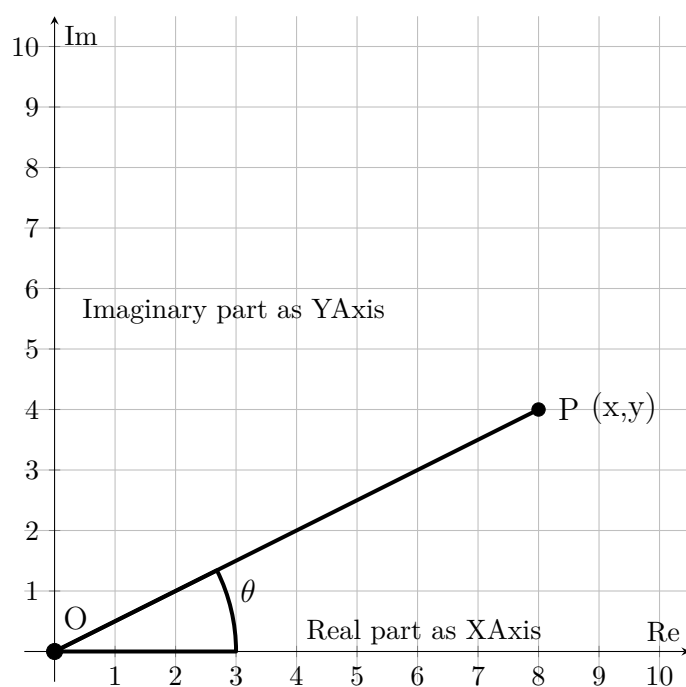
$$(3) \quad xy = (a + ib)(c + id) = ac - bd + i(ad + bc)$$

$$(4) \quad \frac{x}{y} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

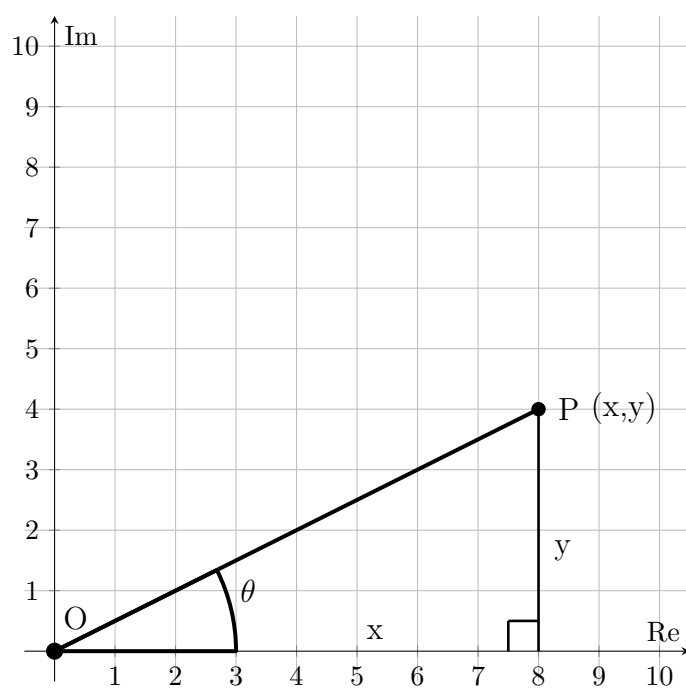
Note 首先, 对于任意一个复数 $z = a + ib$, 可以用复平面上的一点来表示, 那么复数加减法可以通过向量来理解:



Note 而对于复数的乘除法,需要先引入复数在极坐标上的几何意义 – 对于任意一个复数 $z = a + ib$ 用极坐标来表示:



那么复数的三角表示为:



这里将 OP 的长度作为复数 z 的模 (Modulus), 用 $|z|$ 表示; 而角 θ 为复数 z 的幅角 (Argument), 用 $\arg(z)$ 表示。那么复数的三角表示为:

$$z = x + iy = r(\cos \theta + i \sin \theta), \quad r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}$$

接下来是复数乘法的几何意义, 使用复数的三角形式计算下列两个复数的乘积:

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

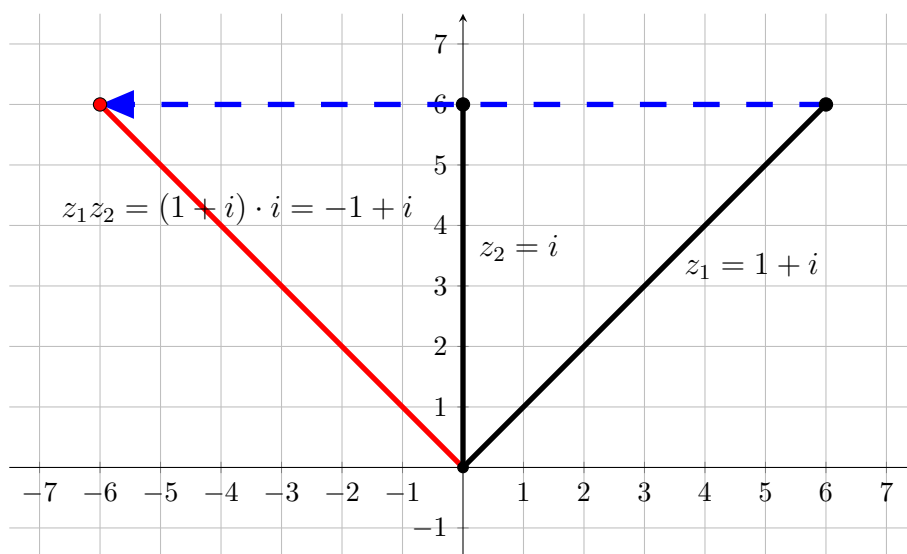
那么有:

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)) \end{aligned}$$

那么根据三角和差公式:

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

可以发现 $z_1 \cdot z_2$ 计算后模为两个复数模的乘积 $|z_1||z_2|$, 幅角为两个复数幅角之和 $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ 。因此复数的乘积可以理解为**拉伸与旋转**。例如:



因为

$$z_1 = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}), \quad z_2 = 1(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$$

所以 $z_1 z_2$ 的模长为 $\sqrt{2}$ 且幅角为 $\frac{3\pi}{4}$ 。而复数的除法只需要将除法写成乘法形式即可

$$\begin{aligned} z^{-1} &= (r(\cos \theta + i \sin \theta))^{-1} \\ &= r^{-1} \frac{1}{\cos \theta + i \sin \theta} \\ &= r^{-1} \frac{\cos \theta - i \sin \theta}{(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)} \\ &= r^{-1}(\cos \theta - i \sin \theta) \end{aligned}$$

那么两个复数

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

的除法便是

$$\begin{aligned} \frac{z_1}{z_2} &= z_1 z_2^{-1} \\ &= \frac{r_1}{r_2} (\cos \theta_1 + i \sin \theta_1)(\cos \theta_1 - i \sin \theta_1) \\ &= \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)) \end{aligned}$$

因此, $\frac{z_1}{z_2}$ 的模长为两个模相除 $\frac{|z_1|}{|z_2|}$, 幅角为 $\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$ 。所以复数的除法也可以理解为**拉伸与旋转**。

Note 综上所述, 复数的加减法就是向量的加减法, 乘除法就是拉伸与旋转变换。

Theorem 1.25. *These definitions of addition and multiplication turn the set of all complex numbers into a field, with $(0, 0)$ and $(1, 0)$ in the role of 0 and 1.*

Proof.

我们简单的验证一下域的公理 (Definition 1.12), 使用 \mathbb{R} 的域结构。令 $x = (a, b), y = (c, d), z = (e, f)$ 。

(A1) 很清楚。

$$(A2) \quad x + y = (a + c, b + d) = (c + a, d + b) = y + x$$

$$\begin{aligned} (A3) \quad (x + y) + z &= (a + c, b + d) + (e, f) \\ &= (a + c + e, b + d + f) \\ &= (a, b) + (c + e, d + f) \\ &= x + (y + z) \end{aligned}$$

$$(A4) \quad x + 0 = (a, b) + (0, 0) = (a, b) = x$$

$$(A5) \quad \text{令 } -x = (-a, -b), \text{ 那么 } x + (-x) = (0, 0) = 0$$

(M1) 很清楚。

$$(M2) \quad xy = (ac - bd, ad + bc) = (ca - db, da + cb) = yx$$

$$\begin{aligned} (M3) \quad (xy)z &= (ac - db, ad + bc)(e, f) \\ &= (ace - bde - adf - bef, acf - bdf + ade + bce) \\ &= (a, b)(ce - df, cf + de) \\ &= x(yz) \end{aligned}$$

$$(M4) \quad 1x = (1, 0)(a, b) = (a, b) = x$$

(M5) 如果 $x \neq 0$ 那么 $(a, b) \neq (0, 0)$, 也就是说 a 和 b 至少有一个实数不等于 0。因此 $a^2 + b^2 > 0$, 根据 Proposition 1.18(d), 我们可以定义

$$\frac{1}{x} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right)$$

那么

$$x \cdot \frac{1}{x} = (a, b) \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) = (1, 0) = 1$$

$$\begin{aligned} (D) \quad x(y + z) &= (a, b)(c + e, d + f) \\ &= (ac + ae - bd - bf, ad + af + bc + be) \\ &= (ac - bd, ad + bc) + (ae - bf, af + be) \\ &= xy + yz \end{aligned}$$

□

Theorem 1.26. *For any real numbers a and b we have*

$$(a, 0) + (b, 0) = (a + b, 0), \quad (a, 0)(b, 0) = (ab, 0)$$

Definition 1.27. $i = (0, 1)$

Theorem 1.28. $i^2 = -1$

Proof.

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1$$

□

Theorem 1.29. *If a and b are real, then $(a, b) = a + bi$*

Proof.

$$\begin{aligned} a + bi &= (a, 0) + (b, 0)(0, 1) \\ &= (a, 0) + (0, b) \\ &= (a, b) \end{aligned}$$

□

Definition 1.30. If a, b are real and $z = a + bi$, then the complex number $\bar{z} = a - bi$ is called the *conjugate* of z . The numbers a and b are the *real part* and the *imaginary part* of z , respectively. We shall occasionally write

$$a = \operatorname{Re}(z), \quad b = \operatorname{Im}(z)$$

Theorem 1.31. *If z and w are complex, then*

- (a) $\overline{z + w} = \bar{z} + \bar{w}$
- (b) $\overline{zw} = \bar{z} \cdot \bar{w}$
- (c) $z + \bar{z} = 2 \operatorname{Re}(z), \quad z - \bar{z} = 2i \operatorname{Im}(z)$
- (d) $z\bar{z}$ is real and positive (except when $z = 0$)

Definition 1.32. If z is a complex number, its absolute value $|z|$ is the non-negative square root of $z\bar{z}$; that is, $|z| = (z\bar{z})^{1/2}$.

$|z|$ 的存在 (以及唯一性) 遵循 Theorem 1.12 以及 Theorem 1.31 (d)。

注意当 x 为实数时, 那么 $\bar{x} = x$, 因此 $|x| = \sqrt{x^2}$ 。所以如果 $x \geq 0$ 时 $|x| = x$, 如果 $x < 0$ 时 $|x| = -x$ 。

Theorem 1.33. *Let z and w be complex numbers. Then*

- (a) $|z| > 0$ unless $z = 0, |0| = 0$
- (b) $|\bar{z}| = |z|$
- (c) $|zw| = |z||w|$
- (d) $|\operatorname{Re} z| \leq |z|$
- (e) $|z + w| \leq |z| + |w|$

Proof.

(a) 与 (b) 不足为道。令 $z = a + bi$, $w = c + di$, 其 a, b, c, d 皆为实数。那么

$$|zw|^2 = (ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2) = |z|^2|w|^2$$

即 $|zw|^2 = (|z||w|)^2$ 。(c) 遵循 Theorem 1.21 所声明的唯一性。

证明 (d), 有 $a^2 \leq a^2 + b^2$, 因此

$$|a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}$$

证明 (e), 有 $\bar{z}w$ 与 $z\bar{w}$ 是共轭的, 因此 $\bar{z}w + z\bar{w} = 2 \operatorname{Re}(z\bar{w})$ 。因此

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\ &= |z|^2 + 2 \operatorname{Re}(z\bar{w}) + |w|^2 \\ &\leq |z|^2 + 2|z\bar{w}| + |w|^2 \\ &= |z|^2 + 2|z||w| + |w|^2 \\ &= (|z| + |w|)^2 \end{aligned}$$

两边开根号后即可得 (e)。 □

Note 计算中第一步的 $|z + w|^2 = (z + w)(\bar{z} + \bar{w})$ 是将 $z + w$ 视为整体, 并使用了 Definition 1.32 中的 $|z| = (z\bar{z})^{1/2}$ 转换为 $(z + w)(\bar{z} + \bar{w})$, 而又因为 Theorem 1.31 (a) 可将第二项变为 $(\bar{z} + \bar{w})$; 而第三步到第四步的不等式则利用了 Theorem 1.33 (d), 即 $|\operatorname{Re} z| \leq |z|$; 第四步到第五步则使用了 Theorem 1.33 (c), 即 $|zw| = |z||w|$ 。

Notation 1.34. If x_1, \dots, x_n are complex numbers, we write

$$x_1 + x_2 + \cdots + x_n = \sum_{j=1}^n x_j$$

我们用一个重要的不等式来结束本节, 它通常被称为 *Schwarz inequality*。

Theorem 1.35. If a_1, \dots, a_n and b_1, \dots, b_n are complex numbers, then

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2$$

Proof.

令 $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \bar{b}_j$, 本证明中 j 取值 $1, \dots, n$ 。如果 $B = 0$, 那么就有 $b_1 = \dots = b_n = 0$, 那么结论很清楚。因此假设 $B > 0$ 。根据 Theorem 1.31 有

$$\begin{aligned} \sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\bar{a}_j - \overline{Cb_j}) \\ &= B^2 \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B|C|^2 \\ &= B(AB - |C|^2) \end{aligned}$$

因为在首次求和的每个项都是非负的, 可以知道

$$B(AB - |C|^2) \geq 0$$

又因为 $B > 0$, 它遵循 $AB - |C|^2 \geq 0$ 。它便是预期的不等式。 \square

Note $|Ba_j - Cb_j|^2$ 为构造项, 将其中 $Ba_j - Cb_j$ 视为整体根据 Definition 1.32, 可转换为 $(Ba_j - Cb_j)(\overline{Ba_j - Cb_j})$, 而后者根据 Theorem 1.31 即可写为 $B\bar{a}_j - \overline{Cb_j}$ (这里 $B = \sum |b_j|^2$ 的共轭还是其本身); 根据乘法分配律得出第二步后, 将之前设定的 A, B, C 带入即可得出 $B(AB - |C|^2)$; 最后根据起始的构造项 $|Ba_j - Cb_j|^2$ 必然非负以及之前假设的 $B > 0$ 可以得出 $AB - |C|^2 \geq 0$; 将 A, B, C 原本代表的值带入, 即 $\sum |a_j|^2 \sum |b_j|^2 - |\sum a_j \bar{b}_j| \geq 0$ 。

Euclidean Spaces

Definition 1.36. For each positive integer k , let R^k be the set of all ordered k -tuples

$$\mathbf{x} = (x_1, x_2, \dots, x_k)$$

where x_1, \dots, x_k are real numbers, called the *coordinatates* of \mathbf{x} . The elements of R^k are called points, or vectors, especially when $k > 1$. We shall denote vectors by boldfaced letters. If $\mathbf{y} = (y_1, \dots, y_k)$ and if α is a real number, put

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_k + y_k)$$

$$\alpha \mathbf{x} = (\alpha x_1, \dots, \alpha x_k)$$

so that $\mathbf{x} + \mathbf{y} \in R^k$ and $\alpha \mathbf{x} \in R^k$. This defines addition of vectors, as well as multiplication of a vector by a real number (a scalar). These two operations satisfy the commutative, associative, and distributive laws (the proof is trivial, in view of the analogous for the real numbers) and make R^k into a *vector space over the real field*. The zero element of R^k (sometimes called the *origin* or the *null vector*) is the point $\mathbf{0}$, all of whose coordinatates are 0.

We also define the so-called “inner product” (or scalar product) of \mathbf{x} and \mathbf{y} by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^k x_i y_i$$

and the *norm* of \mathbf{x} by

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{i=1}^k x_i^2 \right)^{1/2}$$

The structure now defined (the vector space R^k with the above inner product and norm) is called euclidean k -space.

Theorem 1.37. *Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^k$, and α is real. Then*

- (a) $|\mathbf{x}| \geq 0$;
- (b) $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;
- (c) $|\alpha \mathbf{x}| = |\alpha| |\mathbf{x}|$;
- (d) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| |\mathbf{y}|$;
- (e) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;
- (f) $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$

Proof.

前三项不必赘述，而 (d) 是 Schwarz 不等式的间接结论。通过 (d) 可以有

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} \\ &\leq |\mathbf{x}|^2 + 2|\mathbf{x}| |\mathbf{y}| + |\mathbf{y}|^2 \\ &= (|\mathbf{x}| + |\mathbf{y}|)^2 \end{aligned}$$

这样 (e) 就被证明了。最后替换 (e) 中的 \mathbf{x} 为 $\mathbf{x} - \mathbf{y}$ 以及 \mathbf{y} 为 $\mathbf{y} - \mathbf{z}$ (f) 可以得出 (f)。 \square

Remark 1.38. Theorem 1.37 (a), (b), and (f) will allow us (see Chap. 2) to regard R^k as a metric space.

R^1 (the set of all real numbers) is usually called the line, or the real line. Likewise, R^2 is called the plane, or the complex plane (compare Definitions 1.24 and 1.36). In these two cases the norm is just the absolute value of the corresponding real or complex number.

2 Basic Topology

Finite, Countable, and Uncountable Sets

本节由函数概念的定义开始。

Definition 2.1. Consider two sets A and B , whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B , which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* of A into B). The set A is called the *domain* of f (we also say f is defined on A), and the elements $f(x)$ are called the *values* of f . The set of all values of f is called the *range* of f .

Definition 2.2. Let A and B be two sets and let f be a mapping of A into B . If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$. We call $f(E)$ the *image* of E under f . In this notation, $f(A)$ is the range of f . It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A *onto* B . (Note that, according to this usage, *onto* is more specific than *into*.)

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the *inverse image* of E under f . If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$. If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A , then f is said to be a 1-1 (*one-to-one*) mapping of A into B . This may also be expressed as follows: f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1 \in A$, $x_2 \in A$.

(The notation $x_1 \neq x_2$ means that x_1 and x_2 are distinct elements; otherwise we write $x_1 = x_2$.)

Definition 2.3. If there exists a 1-1 mapping of A *onto* B , we say that A and B can be put in 1-1 *correspondence*, or that A and B have the same *cardinal number*, or, briefly, that A and B are *equivalent*, and we write $A \sim B$. This relation clearly has the following properties:

It is reflexive: $A \sim A$.

It is symmetric: If $A \sim B$, then $B \sim A$.

It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Any relation with these three properties is called an *equivalence relation*.

Definition 2.4. For any positive integer n , let J_n be the set whose elements are the integers $1, 2, \dots, n$; let J be the set consisting of all positive integers. For any set A , we say:

- (a) A is *finite* if $A \sim J_n$ for some n (the empty set is also considered to be finite).
- (b) A is *infinite* if A is not finite.

- (c) A is if $A \sim J$.
- (d) A is if A is neither finite nor countable.
- (e) A is if A is finite or countable.

Countable sets are sometimes called *enumerable*, or *denumerable*.

For two finite sets A and B , we evidently have $A \sim B$ if and only if A and B contain the same number of elements. For infinite sets, however, the idea of “having the same number of elements” becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

Metric Spaces

WIP

Compact Sets

WIP

Perfect Sets

WIP

Connected Sets

WIP

3 Numerical Sequences and Series

4 Continuity

5 Differentiation

6 The Riemann-Stieltjes Integral

7 Sequences and Series of Functions

8 Some Special Functions

9 Functions of Several Variables

10 Integration of Differential Forms

11 The Lebesgue Theory