Lab 7
April 25, 2017
Introduction to Wavelets

INSTRUCTIONS:
All lab submissions include a written report and source code in the form of an m-file. The report contains all plots, images, and figures specified within the lab. All figures should be labeled appropriately. Answers to questions given in the lab document should be answered in the written report. ***The written report must be in PDF format***. Submissions are done electronically through **my.ECE**.

1 What are Wavelets?

Wavelets are basis functions with finite support. In the Fourier transform, sinusoidal basis functions are used to represent a signal. Sinusoids satisfy the orthogonality relationship defined by

$$\delta(\omega - \omega_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t}\, dt \tag{1}$$

Using this relationship, a signal $x(t)$ can be expressed as a continuum of sinusoids (the definition as CTFT) where a particular frequency $e^{i\omega_0 t}$ is weighted by a coefficient $X(\omega_0)$.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t}\, d\omega \tag{2}$$

Since the basis functions are orthogonal, one can easily determine $X(\omega_0)$ by performing an inner product between the $e^{-i\omega_0}t$ and $x(t)$. The orthogonality of the basis functions essentially permit us to count how many $e^{i\omega_0 t}$ are in $x(t)$.

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t) e^{-i\omega_0 t}\, dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\omega) e^{i(\omega-\omega_0)t}\, d\omega\, dt && (3)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t}\, dt\, d\omega && (4)\\
&= \int_{-\infty}^{\infty} X(\omega) \delta(\omega - \omega_0)\, d\omega = X(\omega_0) && (5)
\end{aligned}$$

Fourier transforms can capture what frequencies are contained in $x(t)$ but not *when* they occur. In order to obtain this information, we had to resort to short-time Fourier transform (STFT). Wavelet transforms are similar to STFT in that they capture both time *and* frequency information.

Suppose we have a basis function $f_{k,m}(t)$ with finite support ($f_{k,m}(t) \to 0$ as $|t| \to \infty$). Assuming that $f_{k,m}(t)$ is a complete basis (a math term meaning that the basis is suitable for representing square integrable signals), $x(t)$ can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \tilde{x}_{k,m} f_{k,m}(t) \tag{6}$$

$$\tilde{x}_{k,m} = \int_{-\infty}^{\infty} x(t) f_{k,m}(t)\, dt \tag{7}$$

Here, $f_{k,m}(t)$ can be thought of as a wavelet and $\tilde{x}_{k,m}$ is the associated wavelet coefficient. In a manner similar to STFT, an increase in k corresponds to an increase in the temporal rate of oscillation of the wavelet and m denotes a time shift of the wavelet. You are essentially moving a finite-support oscillating waveform across $x(t)$, performing an inner product to count the $x(t)$ which resemble the wavelet for a particular time shift. This is the same as moving a windowed sinusoid across $x(t)$ for STFT. Like the DFT, one can construct a matrix equation to determine the wavelet coefficients is:

$$\int_{-\infty}^{\infty} x(t) f_{k',m'}(t)\, dt = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \tilde{x}_{k,m} \int_{-\infty}^{\infty} f_{k',m'}(t) f_{k,m}(t)\, dt \tag{8}$$

If $\int_{-\infty}^{\infty} f_{k',m'}(t) f_{k,m}(t)\, dt = \delta_{kk'}\delta_{mm'}$, then

$$\tilde{x}_{k,m} = \int_{-\infty}^{\infty} x(t) f_{k,m}(t)\, dt \tag{9}$$

This is the simplest case. It says that wavelets with different oscillation rates or time shifts are orthogonal to one another.

There are two types of wavelets: the mother wavelet $\psi(t)$ and father wavelet $\varphi(t)$. Other wavelets are derived from the mother wavelet. The coefficients associated with the mother wavelet are called *detail coefficients*. The coefficients associated with the father wavelet are called *approximation coefficients*. To obtain a wavelet of higher oscillation rate amounts to scrunching the mother wavelet in half. Mathematically,

$$\psi_{k,m}(t) = 2^{k/2}\psi_{0,0}(2^k t - m) \tag{10}$$

Report Item:

A mother wavelet is described by

$$\psi_{0,0}(t) = \begin{cases} \sin(2\pi(t + 1/2)), & |t| < 1/2 \\ 0, & \text{else} \end{cases} \tag{11}$$

Let $t = n\Delta t$ for $n = 0, \ldots, N-1$ where $N = 512$ and $\Delta t = 0.01$s. Plot $\psi_{1,0}(t), \psi_{0,1}(t), \psi_{1,1}(t)$, and $\psi_{2,1}(t)$ for $|t| < 5$. Additionally, plot the magnitude and phase response of $\psi_{1,0}(t), \psi_{0,1}(t), \psi_{1,1}(t)$, and $\psi_{2,1}(t)$. Based on the magnitude response, is this wavelet orthogonal (Hint: Do orthogonal waveforms have overlapping Fourier components?)? What is the value of the DC coefficient in each case?

The coefficients associated with $\psi_{k,m}(t)$ are called detail coefficients because they capture oscillations. Hence, they have no DC component. By continuously scrunching $\psi_{k,m}(t)$ by a factor of 2, high k mother wavelets oscillate faster with respect to t. However, without a DC component, $\psi_{k,m}(t)$ is incapable of representing $x(t)$ with DC components. This is where the father wavelet comes in. Sometimes called the corking function, $\varphi_{k,m}(t)$ has the responsibility of covering the frequency components from DC up to the minimum frequency component covered by $\psi_{k,m}(t)$. The relationship between mother and father wavelets is an inverse one. While higher k causes the mother wavelet to scrunch by a factor of 2, the father wavelet is stretched by a factor of 2.

$$\varphi_{k,m}(t) = 2^{-k/2}\varphi_{0,0}(2^{-k}t - m) \tag{12}$$

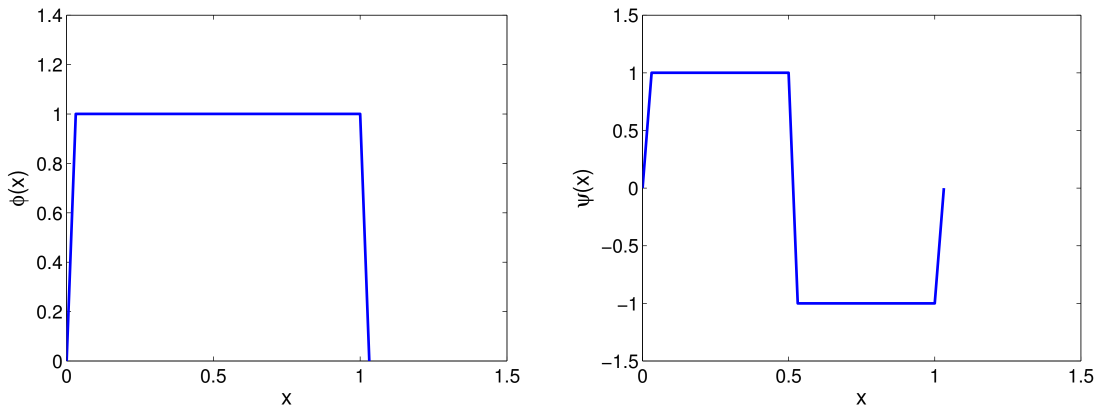
Report Item:
Why is the father wavelet sometimes called the corking function? What is the difference between wavelet coefficients and Fourier coefficients?

2 Types of Wavelets

As previously discussed, the mother wavelet is responsible for representing oscillations in a waveform while the father wavelet handles the low-frequency components of a waveform. To achieve different levels of detail, wavelets are scrunched and stretched. However, their general shape maintains the same. Fractals have similar behavior. There is a class of wavelets that are also fractals. Their description is beyond the scope of this course. However, one can gain intuition about wavelets by plotting their shape. A wavelet can be constructed iteratively using **wavefun**.

Listing 1: An example of **wavefun** for haar wavelets

```
[phi,psi,t] = wavefun('haar',5);
figure;
plot(t,phi,'linewidth',2);
xlabel('t','fontsize',16);
ylabel('\phi(t)','fontsize',16);
set(gca,'fontsize',14);
figure;
plot(t,psi,'linewidth',2);
xlabel('t','fontsize',16);
ylabel('\psi(t)','fontsize',16);
set(gca,'fontsize',14);
```



(a) Father Wavelet (b) Mother Wavelet

Figure 1: Haar wavelet

Report Item:
Plot $\psi_{0,0}(k,t)$ and $\varphi_{0,0}(k,t)$ for the wavelet types: *coif1*, *db1*, and *sym4* for *iter*=5. Zero pad each signal to $N = 512$ and plot the magnitude spectrum of each wavelet from $-f_s/2$ to $f_s/2$.

Different wavelet types have different properties and shapes. Depending on the application, the use of a particular wavelet may be advantageous. For example, in compression, one should choose a wavelet that compactly represents the original signal in as few wavelet coefficients as possible with minimal error.

3 Wavelet Decomposition

Despite the vast amount of new information we have just covered, the discrete wavelet transform (DWT) can be implemented using Fourier theory we are already familiar with. Using our knowledge of filters and downsampling, a DWT can be implemented in accordance with Figure 2.

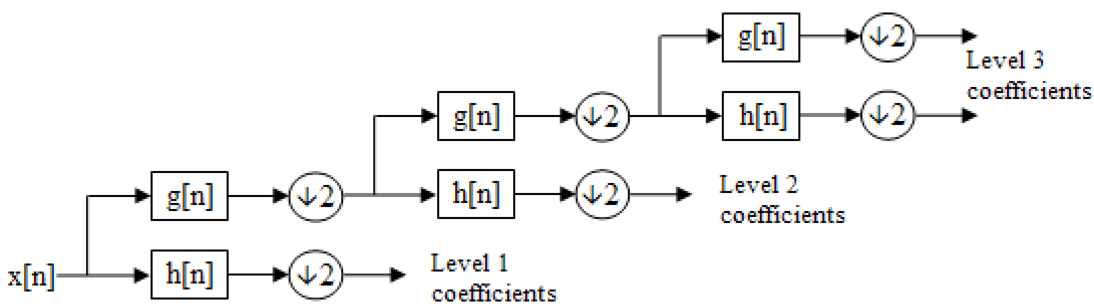


Figure 2: A 3-level filter bank.

Here, $g(n)$ is a lowpass filter and $h(n)$ is a highpass filter but they may also be thought of as the father and mother wavelets, respectively. Both filters are orthogonal, ensuring that redundant frequency content is not captured. Since each filter captures a half of the overall bandwidth, a decimation by 2 can be used to remove extra information. The level 1 coefficients are a result of convolution between the father wavelet and the input. The coefficients at level 1 can be written as

$$a_{1,m} = \sum_{n=-\infty}^{\infty} x(n)g(m-n) \equiv \sum_{n=-\infty}^{\infty} x(n)\varphi_{1,m}(n) \tag{13}$$

$$d_{1,m} = \sum_{n=-\infty}^{\infty} x(n)h(m-n) \equiv \sum_{n=-\infty}^{\infty} x(n)\psi_{1,m}(n) \tag{14}$$

where a_m denotes the father wavelet coefficients for a particular shift m. The output of the lowpass filter are the mother wavelet coefficients, d_m. This is essentially the portion of the signal represented by the lower half of the original signal band.

Decimation by 2 lets us re-use $g(n)$ and $h(n)$ on a_m, the father wavelet coefficients. Decimation by 2 operation can be viewed as stretching the mother and father wavelets by a factor of 2. Stretching the mother wavelet allows detection of lower frequency oscillations while stretching of the father wavelet filters out more frequency oscillations (think about the inverse relationship between the width of a rect function and the mainlobe width of its Fourier transform).

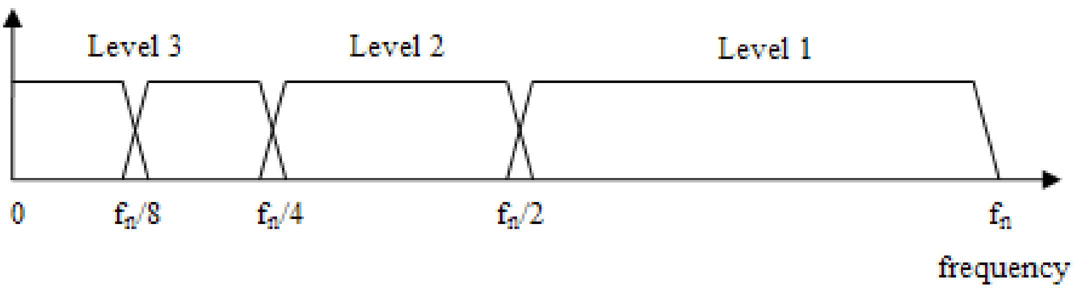


Figure 3: Frequencies captured at each level.

Note that the number of coefficients at a particular level is half the number of coefficients of the previous level. Unlike STFT where the window size is fixed, DWT has the advantage of having a short window for high frequencies and a long window for low frequencies (via stretching through decimation). This gives DWT good high and low frequency resolution. DWT is also less prone to error compared to DFT due to the temporal locality of wavelets.

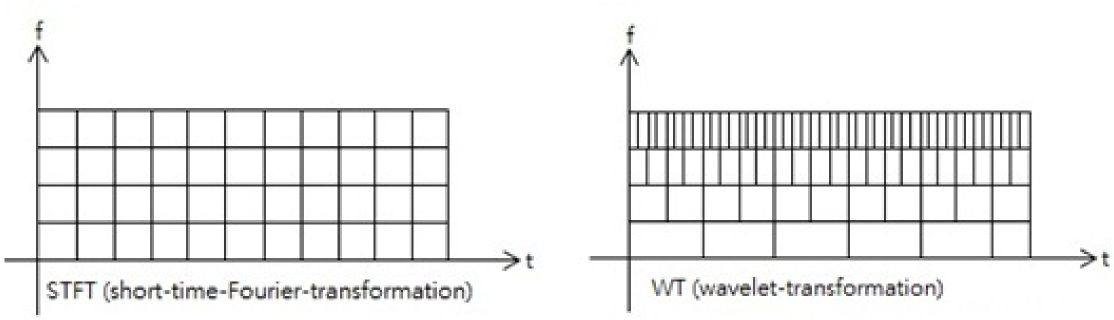


Figure 4: A comparison of window sizes for STFT and DWT.

Report Item:
Use **dwt** to find the single level DWT of *signal.mat* using *sym4* and *coif1* wavelets. For each case, plot both set of coefficients using stem.

To perform the inverse DWT, one simply has to undo the downsampling operation on both d_m and a_m. This can be done via upsampling. To upsample by U, simply insert U zeros between every sample and then apply a lowpass filter with a cutoff of $\omega_c = \pi/U$. Since upsampling effectively scrunches the signal in the ω domain by a factor of U, one must also multiply the output of the lowpass filter by U in order to preserve the energy in the signal. Once we obtain the upsampled versions of a_m and d_m, we simply add the results together since these coefficients represent two orthogonal bands of the original signal.

Report Item:
Use **dwt** to find the single level DWT of *signal.mat* using *sym4* and *coif1* wavelets. Use **idwt** to perform the inverse DWT to verify that it works as expected. For each wavelet, set the coefficients whose absolute values are less than the average of the absolute coefficient values. Take the IDWT of this new set of coefficients. Use stem to plot the results for both cases and both coefficients.

4 Wavelet De-noise

We learned that Fourier transforms can denoise signals when the noise and the signal occupy separate bands. However, if the noise and signal occupy the same band, then Fourier denoising techniques become ineffective. When we looked at spectrograms, we noticed from the time-frequency separation of the signal that there are certain times when the signal is no longer present in a particular frequency band but the noise is. STFT let us observe this phenomena but we could not perform any denoising techniques because we were not given an inverse STFT. Fortunately, the DWT does have an inverse.

A simple approach to denoising using DWT is hard thresholding. In this case, you set coefficients below a certain threshold equal to zero. A combination of Fourier and wavelet denoising techniques could prove much more effective than Fourier denoising alone.

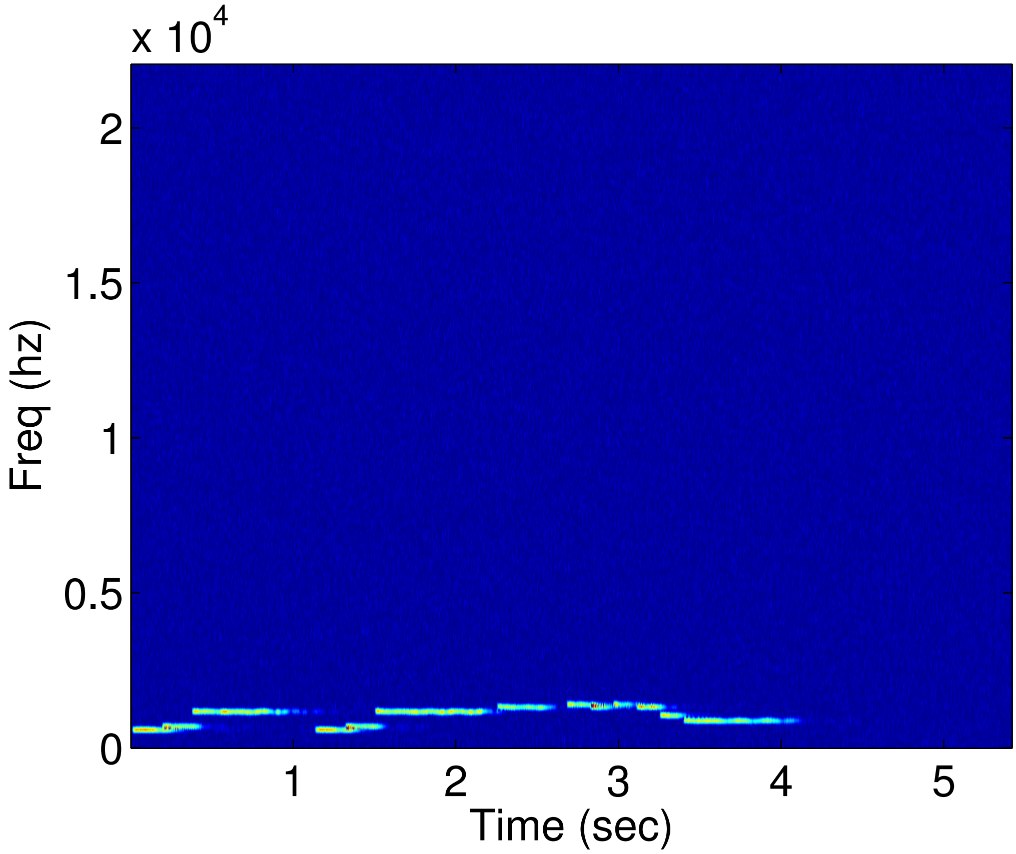


Figure 5: Spectrogram of *noisy.wav*.

Report Item:

A code for performing wavelet denoise called *waveletdenoise.m* is provided. Using this code, choose a combination of thresholds that minimizes the noise and the relative percent error. List the values you used for each threshold and the percent error you obtained in your report.