Wavelet Analysis and Its Application in CFD

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Prologue

The concept of wavelet is originated from Fourier analysis. The classical Fourier transform

$$\hat{f}(\gamma) = \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-2\pi\mathrm{i}\gamma x}\mathrm{d}x \tag{1}$$

converts information to the *frequency space*. However, the Fourier transform is not localized in both physical space and frequency space. One method of obtain local information on physical space of a signal is the Fourier transform with a windowed function $g(t-b)$:

$$(G_{a,b}f)(\gamma) := \int_{-\infty}^{+\infty} f(t)e^{-2\pi\mathrm{i}\gamma x}g_a(t-b)\mathrm{d}t$$

where

$$g_a(x) := \mathrm{e}^{-2\pi a\,x^2}.$$

The above transform is also called *Gabor transform*. The shortcoming of Gabor transform is that it has only fixed window, so it is not suitable for signals with singularity or severe oscillation. Inspired by the above methods, wavelet transform provides a systematical tool of analyzing unstable signals.

Part I

Mathematical Foundations of Wavelet Analysis

Chapter 1
Basic Concepts of Wavelet

1.1 An Abstract Way of Introducing Discrete Wavelet Transform

1.1.1 Wavelet

In general, the Hilbert space has the property described in the following theorem:[2]

Theorem 1.1. (Existence of orthonormal bases) *Every separable Hilbert space $\mathcal{H}$ has an orthonormal basis.*

Discrete wavelet transform provides a way of constructing orthonormal bases in $L^2(\mathbb{R})$ with a special structure: all of them are scaled and translated version of a fixed function. The character that distinguishes bases with wavelet structure from other orthonormal bases in $L^2(\mathbb{R})$ is that the relevant function can be approximated well by finite partial sums, and even with just a few nonzero coefficients.

Definition 1.2. (Wavelet) *Let $\psi \in L^2(\mathbb{R})$.*

i. For $j, k \in \mathbb{Z}$, define the function $\psi_{j,k}$ by

$$\psi_{j,k} := 2^{j/2}\psi(2^j x - k) \quad x \in \mathbb{R}.$$

ii. The function ψ is called a wavelet if the functions $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ form an orthonormal basis for $L^2(\mathbb{R})$.

In terms of the translation operator T_k and the dilation operator D, we have

$$\psi_{j,k} = D^j T_k \psi \quad j, k \in \mathbb{Z}.$$

The systematical theory of constructing wavelet began in middle 1980s, but the first wavelet is constructed much earlier and it is proved by Haar in 1910 that the following function consistutes an example of wavelet.

Example 1.3. (Haar Function) The *Haar function* is defined by $\psi(x) = \begin{cases} 1 & \text{if} \quad 0 \leqslant x < 1/2 \\ -1 & \text{if} \quad 1/2 \leqslant x < 1 \\ 0 & \text{otherwise.} \end{cases}$

It is very complicated to verify that Haar function is wavelet, and it is mentioned here just to point out the existence of wavelet, and an illustration on how Haar function forms a wavelet is in Section 1.2.1.

1.1.2 Multiresolution Analysis

Multiresolution analysis is a general tool to construct wavelet orthonormal bases. A multiresolution analysis consists of a collection of conditions on certain subspaces of $L^2(\mathbb{R})$, and an associated function $\phi \in L^2(\mathbb{R})$.

Definition 1.4. (Multiresolution Analysis) *A multiresolution analysis for $L^2(\mathbb{R})$ consists of a squence of closed subspaces $\{V_i\}_{i\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and a function $\phi\in V_0$, such that the following conditions hold:*

i. The space V_j are nested, i.e.,

$$\cdots\subset V_{-1}\subset V_0\subset V_1\subset\cdots.$$

ii. $\overline{\bigcup_{j\in\mathbb{Z}}V_j}=L^2(\mathbb{R})$ *and* $\bigcap_{j\in\mathbb{Z}}V_j=\{0\}$.

iii. $\forall j\in\mathbb{Z},\ V_{j+1}=D(V_j)$.

iv. $\forall k\in\mathbb{Z},\ f\in V_0\Rightarrow T_kf\in V_0$.

v. $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is an orthonormal basis for* V_0.

A closer look at the condition the condition in Definition 1.4 reveals that the choice of the function ϕ in a multiresolution analysis acturally determines the space V_j uniquely:

Lemma 1.5. (The Space V_j) *Assume that the conditions (iii) and (iv) in Definition 1.4 are satisfied. Then the following hold:*

i. $V_j=D^j(V_0)$ *for all* $j\in\mathbb{Z}$.

ii. $V_j=\overline{\operatorname{span}\{D^jT_k\phi\}_{k\in\mathbb{Z}}}$ *for all* $j\in\mathbb{Z}$.

Proof. For $j\in\mathbb{N}$:

$$V_j=D(V_{j-1})=DD(V_{j-2})=\cdots=D^j(V_0),$$

and for $j<0$, (i) holds similarly.

$$V_j=D^j(V_0)=D^j(\overline{\operatorname{span}\{T_k\phi\}_{k\in\mathbb{Z}}})=\overline{\operatorname{span}\{D^jT_k\phi\}_{k\in\mathbb{Z}}}. \tag{1.1}$$

□

Lemma 1.5(ii) shows that the space V_j in a multiresolution analysis are uniquely determined by the function ϕ, so we say that the function ϕ *generates the multiresolution analysis*. Later in this Chapter, it will shown that only very special function ϕ can generate multiresolution analysis.

Example 1.6. (Haar Multiresolution Analysis) We can define a multiresolution analysis by

$$\begin{cases}\phi & := \chi_{[0,1)};\\ V_j & := \{f\in L^2(\mathbb{R}): f \text{ is constant on } [2^{-j}k,2^{-j}(k+1)), \forall k\in\mathbb{Z}\}.\end{cases}$$

Note that the Haar wavelet can be written as

$$\begin{aligned}\psi(x) &= \chi_{[0,1/2)}(x)-\chi_{[1/2,1)}(x)\\ &= \chi_{[0,1)}(2x)-\chi_{[0,1)}(2x-1)\\ &= \frac{1}{\sqrt{2}}(D\chi_{[0,1)}(x)-DT_1\chi_{[0,1)}(x))\\ &= \frac{1}{\sqrt{2}}(D\phi(x)-DT_1\phi(x))\end{aligned}$$

In order to construct an orthonormal basis for $L^2(\mathbb{R})$ with multiresolution analysis, we need to consider a class of vector space W_j associated with $\{V_j\}_{j\in\mathbb{Z}}$:

Definition 1.7. (The Space W_j) *Assume that V_j is a sequence of closed subspace of $L^2(\mathbb{R})$ and that the condition (i) in Definition 1.4 is satisfied. For any $j\in\mathbb{Z}$, let W_j denote the orthogonal complement of V_j with respect to V_{j+1}, i.e.,*

$$W_j:=\{f\in V_{j+1}|\langle f,g\rangle=0,\forall g\in V_j\}.$$

We denote the orthogonal projection of $L^2(\mathbb{R})$ onto W_j by Q_j.

The above construction of space W_j is very similar to Gram-Schmidt process in finite-dimensional linear algebra. However, in infinite-dimensional vector spaces, the Gram-Schmidt process does not make sense in general. The multiresolution analysis provides a special condition with which a process similar to Gram-Schmidt process is feasible.

It turns out that the space W_0 plays a very special role in wavelet analysis. In fact, the next result shows that in order to obtain an orthonormal basis $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$, it is enough to find a function $\psi\in W_0$ such that $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_0.

Proposition 1.8. *Assume that the function $\phi\in L^2(\mathbb{R})$ generates a multiresolution analysis. Let $\psi\in L^2(\mathbb{R})$ and suppose that $\{T_k\psi\}_{k\in\mathbb{Z}}$ form an orthonormal basis for W_0. Then the following holds:*

- *i. For each $j\in\mathbb{Z}$, the functions $\{D^jT_k\psi\}_{k\in\mathbb{Z}}$ form an orthonormal basis for W_j.*
- *ii. The functions $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ form an orthonormal basis for $L^2(\mathbb{R})$, i.e., ψ is a wavelet.*
- *iii. The functions $\{T_k\phi\}_{k\in\mathbb{Z}}\cup\{D^jT_k\psi\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ form an orthonormal basis for $L^2(\mathbb{R})$.*

Since function ψ is a wavelet and function ϕ generates the corresponding multiresolution analysis, it is important to point out the relationship between the two functions. The following in this section also shows the general method of constructing a multiresolution analysis.

Proposition 1.9. (Scaling Function) *Assume that the function $\phi\in L^2(\mathbb{R})$ generates a multiresolution analysis. Then there exists a 1-periodic function $H_0\in L^2(0,1)$ such that*

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma) \tag{1.2}$$

Solution to Equation (1.2) is called a *scaling function*, or said to be *refinable*. Formulated in this language, a necessary condition for a function ϕ to generate a multiresolution analysis is that ϕ is a scaling function. Later it will be shown that that condition added up with two extra conditions can become sufficient. The following theorem provides a concrete statement on how to generate a wavelet with Equation (1.2).

Theorem 1.10. (Wavelet Orthonormal Basis) *Assume that $\phi\in L^2(\mathbb{R})$ generates a multiresolution analysis, and let $H_0\in L^2(0,1)$ be a 1-periodic function satisfying the scaling equation (1.2). Define the 1-periodic function H_1 by*

$$H_1 := \overline{H_0\left(\gamma + \frac{1}{2}\right)}\mathrm{e}^{-2\pi\mathrm{i}\gamma} \tag{1.3}$$

Also define the function ψ via

$$\hat{\psi}(2\gamma) := H_1(\gamma)\hat{\phi}(\gamma). \tag{1.4}$$

Then the following holds:

- *i. $\{T_k\psi\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_0.*
- *ii. $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ is orthonormal basis for $L^2(\mathbb{R})$, i.e., ψ is a wavelet.*

The direct way of defining the wavelet ψ is applying inverse Fourier transform on $\hat{\psi}(\gamma)$:

Theorem 1.11. (Explicit Expression for the Wavelet) *Assume that (1.4) holds for a 1-periodic function $H_1\in L^2(0,1)$,*

$$H_1(\gamma) = \sum_{k\in\mathbb{Z}} d_k \mathrm{e}^{2\pi\mathrm{i}k\gamma}.$$

Then

$$\psi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} d_k DT_{-k}\phi(x) = 2\sum_{k\in\mathbb{Z}} d_k\phi(2x+k) \quad x\in\mathbb{R}. \tag{1.5}$$

The Equation (1.5) provides the method of finding the wavelet ψ whenever the function H_0 has been calculated. In most cases of practical interest, H_0 is actually a trigonometric polynomial:

$$H_0(\gamma) = \sum_{k=-N}^{N} c_k \mathrm{e}^{2\pi\mathrm{i}k\gamma}.$$

The explicit expression of the wavelet in (1.5) immediately leads to a criterion for how to obtain a compactly supported wavelet:

Corollary 1.12. (Compactly Supported Wavelet) *Assume that the function $\phi \in L^2(\mathbb{R})$ is compactly supported and generates a multiresolution analysis. Assume further that the function H_0 in the scaling equation (1.2) is a trigonometric polynomial. Then the wavelet ψ in (1.5) is compactly supported.*

Example 1.13. (Haar Wavelet) The previous conclusions can all be applied to the case of Haar wavelet.

Since a multiresolution analysis is uniquely characterized by the scaling function ϕ, it is natural to examine how to formulate the multiresolution analysis conditions directly in terms of conditions on the function ϕ. Such conditions are:

Theorem 1.14. (Construction of Multiresolution Analysis) *Let $\phi \in L^2(\mathbb{R})$. Define the spaces V_j by (1.1), and assume that the following conditions holds:*

i. $\inf_{\gamma \in (-\varepsilon, \varepsilon)} |\hat{\phi}(\gamma)| > 0$ *for some* $\varepsilon > 0$*;*

ii. the scaling equation

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma)$$

is satisfied with a bounded 1-periodic function H_0;

iii. $\{T_k\phi\}_{k\in\mathbb{Z}}$ *form an orthonormal system.*

Theorem 1.15. (Characterization of Orthonormal System $\{T_k\phi\}_{k\in\mathbb{Z}}$) *Let $\phi \in L^2(\mathbb{R})$. Then $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthornormal system if and only if*

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(\gamma + k)|^2 = 1 \quad \gamma \in \mathbb{R}$$

1.1.3 Vanishing Moments and Daubechies' Wavelet

This section mainly discusses the properties that make wavelet analysis useful in signal processing, and the construction by Ingrid Daubechies.

Since the functions $\{T_k\phi\}_{k\in\mathbb{Z}} \cup \{D^jT_k\psi\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ form an orthonormal basis for $L^2(\mathbb{R})$, we know that any $f \in L^2(\mathbb{R})$ has the representation

$$f = \sum_{k\in\mathbb{Z}} \langle f, T_k\phi\rangle T_k\phi + \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}. \tag{1.6}$$

All information about the function f is stored in the coefficients

$$\{\langle f, T_k\phi\rangle\}_{k\in\mathbb{Z}} \cup \{\langle f, \psi_{j,k}\rangle\}_{j\in\mathbb{N},k\in\mathbb{Z}}.$$

However, in practice computers can only process finite sequence, so one has to select a finite number of coefficients to keep. In most situation of practical interest, such as the wavelet generated by a compactly supported scaling function $\phi \in L^2(\mathbb{R})$, the sequence $\{\langle f, T_k\phi\rangle\}_{k\in\mathbb{Z}}$ is actually finite, thus the problems is how to deal with the infinite sequence $\{\langle f, \psi_{j,k}\rangle\}_{j\in\mathbb{N},k\in\mathbb{Z}}$. This is usually done by *thresholding*[5]: we choose a certain $\varepsilon > 0$ and keep only the coefficients in (1.6) under condition:

$$|\langle f, \psi_{j,k}\rangle| \geqslant \varepsilon. \tag{1.7}$$

By convergence of (1.6) the coefficients $\{\langle f, \psi_{j,k}\rangle\}_{j,k\in\mathbb{Z}} \subset \ell^2$, so only finite number of indices $(j,k) \in \mathbb{Z} \times \mathbb{Z}$ satisfy (1.7). A key feature of wavelet theory is that it is often posible to choose the wavelet ψ such that many of the coefficients $\{\langle f, \psi_{j,k}\rangle\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ are small.

Example 1.16. (Haar Wavelet) For Haar wavelet, we have (1.6) as

$$\langle f, \psi_{j,k}\rangle \psi_{j,k}(x) = 2^{j/2}\langle f, \psi_{j,k}\rangle \psi(2^jx - k),$$

and it can be shown that

$$\begin{aligned} d_{j,k} &:= 2^{j/2}\langle f, \psi_{j,k}\rangle \\ &= \frac{1}{2}(\text{average of } f \text{ over}[2^{-j}k, 2^{-j}(k+1/2)) - \text{average of } f \text{ over } [2^{-j}(k+1/2), 2^{-j}(k+1)). \end{aligned}$$

Since

$$\langle f, \psi_{j,k}\rangle = 2^{-j/2} d_{j,k} \quad j \geqslant 1 \quad k \in \mathbb{Z},$$

most coefficients in (1.6) are very small if f is continuous or oscillates slightly.

In order to find wavelets that are better than Haar wavelet, we need to introduce *vanishing moment* first.

Definition 1.17. (Vanishing Moment) *Let $N \in \mathbb{N}$. A function ψ has N vanishing moments if*

$$\int_{-\infty}^{+\infty} x^\ell \psi(x) \mathrm{d}x = 0 \quad \text{for } \ell = 0, 1, \ldots, N-1.$$

The Haar wavelet has only one vanishing moment. We have the following result showing that only relatively few coefficients $\langle f, \psi_{j,k}\rangle$ will be large if ψ has a large number of vanishing moments.

Theorem 1.18. (Decay of Wavelet Coefficients) *Assume that the function $\psi \in L^2(\mathbb{R})$ is compactly supported and has N vanishing moments. Then, for any Ntimes differentiable function $f \in L^2(\mathbb{R})$ for which the Nth derivative $f^{(N)}$ is bounded, there exists a constant $C > 0$ such that*

$$|\langle f, \psi_{j,k}\rangle| \leqslant C 2^{-jN} 2^{-j/2} \quad \forall j \geqslant 1 \quad k \in \mathbb{Z} \tag{1.8}$$

The proof of Theorem (1.18) is complicated [8], but the result shows clearly that: the higher number of vanishing moments a wavelet has, the fewer coefficients $\{\langle f, \psi_{j,k}\rangle\}_{j\in\mathbb{N}, k\in\mathbb{Z}}$ remain after thresholding. Another import theorem is provided by [8]:

Theorem 1.19. (Vanishing Moment) *Let ϕ be a compactly supported scaling function associated with a multiresolution analysis, and let ψ be the associated wavelet as in (1.2). Then the following are equivalent:*

i. ψ has N vanishing moments.

ii. The function H_0 can be factorized

$$H_0(\gamma) = \left(\frac{1 + \mathrm{e}^{-2\pi \mathrm{i}\gamma}}{2}\right)^N L(\gamma) \tag{1.9}$$

for some 1-periodic trigonometric polynomial L.

Combining Theorem (1.14) and Theorem (1.19) produces one approach to construct a wavelet ψ having N vanishing moments by searching for a proper 1-periodic function $H_0 \in L^2(0,1)$. Then

$$\hat{\phi}(\gamma) = H_0(\gamma/2)\hat{\phi}(\gamma/2) = H_0(\gamma/2)H_0(\gamma/4)\hat{\phi}(\gamma/4) = H_0(\gamma/2)H_0(\gamma/4)H_0(\gamma/8)\hat{\phi}(\gamma/8),$$

and finally for any $K \in \mathbb{N}$

$$\hat{\phi}(\gamma) = H_0(\gamma/2)H_0(\gamma/4)H_0(\gamma/8)\cdots H_0(\gamma/2^K)\hat{\phi}(\gamma/2^K) = \hat{\phi}(\gamma/2^K)\prod_{j=1}^{K} H_0(\gamma/2^j).$$

It can be proved that the following limit process is feasible

$$\begin{aligned}\hat{\phi}(\gamma) &= \lim_{K\to\infty}\left(\hat{\phi}(\gamma/2^K)\prod_{j=1}^{K} H_0(\gamma/2^j)\right)\\ &= \hat{\phi}(0)\prod_{j=1}^{\infty} H_0(\gamma/2^j).\end{aligned}$$

This also ensures the uniqueness (up to a scalar multiplication) of scaling function ϕ since once $\hat{\phi}(0)$ is given, all other values are determined.

The *Daubechies' wavelets* are the best known constructions based on the above idea. First denote $P_{N-1}, N \in \mathbb{N}$ by

$$P_{N-1}(y) = \sum_{k=0}^{N-1} \frac{(2N-1)!}{k!(2N-1-k)!} y^k (1-y)^{N-1-k},$$

then [3] provides another important theorem:

Theorem 1.20. (Daubechies' Wavelets) *For any $N \in \mathbb{N}$, there exists a trigonometric polynomial L such that*

$$|L(\gamma)|^2 = P_{N-1}(\sin^2 \pi\gamma). \tag{1.10}$$

With such a choice for L, the following hold:

i. The function H_0 given by (1.9) is associated with a multiresolution analysis.

ii. With H_0 as in (i), the wavelet ψ given in Theorem 1.10 has N vanishing moments, and support in $[0, 2N-1]$.

Example 1.21. (The Polynomial P_{N-1}) It can be calculated that:

$$\begin{aligned} P_0(y) &= 1 \\ P_1(y) &= 1+2y \\ P_2(y) &= 1+3y+6y^2 \end{aligned}$$

Daubechies further provides a procedure called *spectral factorization* to find the trigonometric polynomial L satisfying (1.10).

Example 1.22. (Spectral Factorization with $N=1$) For $N=1$, $|L(\gamma)|^2=1$, which is satisfied for $L(\gamma)=-1$. Via (1.9), this leads to

$$H_0(\gamma) = \frac{1+\mathrm{e}^{-2\pi\mathrm{i}\gamma}}{2} L(\gamma) = \frac{-1-\mathrm{e}^{-2\pi\mathrm{i}\gamma}}{2}.$$

We can calculate H_1:

$$\begin{aligned} H_1(\gamma) &= \overline{H_0\left(\gamma+\frac{1}{2}\right)} \mathrm{e}^{-2\pi\mathrm{i}\gamma} \\ &= \overline{\frac{-1-\mathrm{e}^{-2\pi\mathrm{i}(\gamma+1/2)}}{2}} \mathrm{e}^{-2\pi\mathrm{i}\gamma} \\ &= \frac{1-\mathrm{e}^{-2\pi\mathrm{i}\gamma}}{2} \\ &= \sum_{k=-1}^{0} d_k \mathrm{e}^{2\pi\mathrm{i}kx}, \end{aligned}$$

where $d_0 = 1/2, d_{-1} = -1/2$. By (1.5), the associated wavelet is

$$\psi(x) = \sqrt{2} \sum_{k=-1}^{0} d_k \, DT_{-k}\phi(x) = \phi(2x) - \phi(2x-1)\,,$$

which is exactly the expression for the Haar wavelet in Example 1.6.

1.2 An Elementary Way of Introducing Wavelet

This section is mainly according to [1, 7], and provides another way of defining (orthogonal) wavelet which is better for programming, but lacks mathematical rigorousness.

1.2.1 Example: Haar Wavelet

This section is an demonstration on how to approximate an arbitrary function f in $L^2(\mathbb{R})$ by linear combination of Haar wavelet [3]. First, any function $f \in L^2(\mathbb{R})$ can be arbitrarily well approximated by a function with compact support which is piecewise constant on the interval $[\ell 2^{-j}, (\ell+1)2^{-j})$ (it suffices to take the support and j large enough). As a result, we consider the following piecewise constant function: assume f to be supported on $[-2^{J_1}, 2^{J_1}]$, and to be piecewise constant on $[\ell 2^{-J_0}, (\ell+1)2^{-J_0})$, as is shown in Figure 1.1. Let us define the function $f^0 = f$, and denote the constant value of f^0 on $[\ell 2^{-J_0}, (\ell+1)2^{-J_0})$ by f^0_ℓ. We now represent function f^0 as a sum of two parts, $f^0 = f^1 + \delta^1$, where the function f^1 is an approximation to f^0 which is piecewise constant over intervals twice as large as the originally, i.e.,

$$f^1|_{[k2^{-J_0+1}, (k+1)2^{-J_0+1})} := \text{constant} = f^1_k\,.$$

And the values of f^1 is given by the average of function f^0. Now we can write down that:

$$f_k^1 = \frac{1}{2}(f_{2k}^0 + f_{2k+1}^0)\,. \tag{1.11}$$

The function δ^1 is piecewise constant with the same stepwidth as f^0, and we have:

$$\delta_{2\ell}^1 = f_{2\ell}^0 - f_\ell^1 = \frac{1}{2}(f_{2\ell}^0 - f_{2\ell+1}^0) \tag{1.12}$$

and

$$\delta_{2\ell+1}^1 = f_{2\ell+1}^0 - f_\ell^1 = \frac{1}{2}(f_{2\ell+1}^0 - f_{2\ell}^0) = -\delta_{2\ell}^1\,. \tag{1.13}$$

So on interval $[k2^{-J_0+1}, (k+1)2^{-J_0+1})$, we have that

$$\delta^1(x) = \delta_{2l}^1\,\psi(2^{J_0-1}x - \ell)\,,$$

hence

$$\delta^1(x) = \sum_{\ell=-2^{J_1+J_0+1}}^{2^{J_1+J_0-1}} \delta_{2\ell}^1\,\psi(2^{J_0-1} - \ell)\,. \tag{1.14}$$

Now we have written f as

$$f = f^0 = f^1 + \sum_l c_{-J_0+1,\ell}\,\psi_{-J_0+1,\ell}\,,$$

where f^1 is the same type of function as f^0 but with stepwidth twice as large.

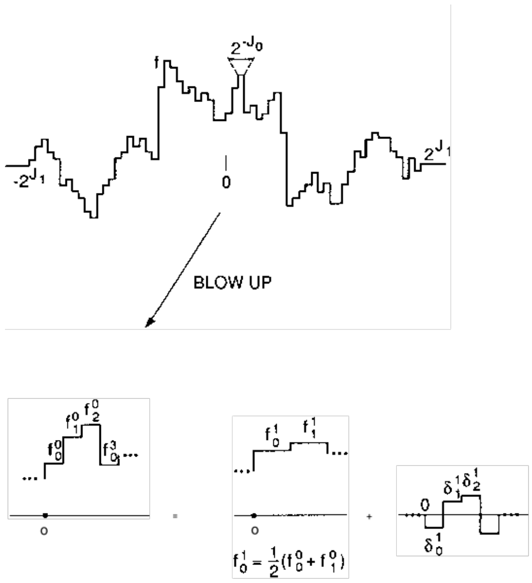

Figure 1.1. (upper) A function f with support $[-2^{J_1}, 2^{J_1}]$, piecewise constant on the $[\ell 2^{-J_0}, (\ell+1)2^{-J_0})$. (lower) A blowup of a portion of f.

We can apply the same trick to f^1, so that

$$f^1 = f^2 + \sum_l c_{-J_0+2,\ell}\,\psi_{-J_0+2,\ell}\,.$$

We keep going the trick until we have

$$f = f^{J_0+J_1} + \sum_{m=-J_0+1}^{J_1} \sum_\ell c_{m,\ell}\,\psi_{m,\ell}\,.$$

Here $f^{J_0+J_1}$ consists of two constant pieces (see Figure 1.2).

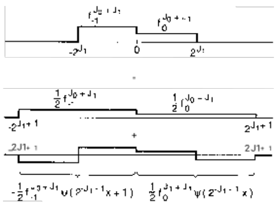

Figure 1.2. Extending interval $[-2^{J_1}, 2^{J_1}]$ to $[-2^{J_1+1}, 2^{J_1+1}]$.

As is shown in Figure 1.2, the interval can be extended to twice larger, which means the support of function f is extended:

$$f = f^{J_0+J_1+K} + \sum_{m=-J_0+1}^{J_1+K} \sum_{\ell} c_{m,\ell}\, \psi_{m,\ell}\,.$$

It follows immediately that

$$\begin{aligned} \left\| f - \sum_{m=-J_0+1}^{J_1+K} \sum_{\ell} c_{m,\ell}\, \psi_{m,\ell} \right\|_{L^2}^2 &= \| f^{J_0+J_1+K} \|_{L^2}^2 \\ &= 2^{-K/2} \cdot 2^{J_1/2} (|f_0^{J_0+J_1}|^2 + |f_{-1}^{J_0+J_1}|^2)^{1/2}, \end{aligned}$$

which can be arbitrarily small by taking sufficiently large K. Finally, this illustrates that f can be approximated to arbitrary precision by a finite linear combination of Haar wavelets.

1.2.2 Scaling Function

The core part of wavelet analysis is the wavelet transform which is a tool that split data, function or operator into components with different frequency, and then studies the above componet with a resolution matched to its scale[7, 3]. The spliting of data, function or operator is described mathematically with multiresolution of a functional space defined below.

In order to develop a multilevel representation of a function in $L^2(\mathbb{R})$, we seek a squence of embedded subspaces V_i such that

$$\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2(\mathbb{R}) \tag{1.15}$$

with the following properties:

- $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$.
- $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$.
- The embedded subspaces are related by a scaling law

$$g(x) \in V_j \quad \Leftrightarrow \quad g(2x) \in V_{j+1}$$

- Each subspace is spanned by integer translates of a single function $g(x)$ such that

$$g(x) \in V_0 \quad \Leftrightarrow \quad g(x+1) \in V_0$$

Definition 1.23. (Multiresolution Analysis) *If there exists function $\phi(x) \in V_0$ such that its integer translates $\{\phi(x-k)\}_{k \in \mathbb{Z}}$ form a Riesz basis*[1.1] *for the space V_0, then the sequence of subspaces $\{V_j\}_{j \in \mathbb{Z}}$ forms a multiresolution analysis of $L^2(\mathbb{R})$ space.*

1.1. Sequence $\{x_k\}_{k \in \mathbb{Z}}$ forms a Riesz squence of a Hilbert space $\mathcal{H}$ if there exists constants $0 < c < C < +\infty$ such that

$$c\Big(\sum_n |a_n|^2\Big) \leqslant \Big\| \sum_n a_n x_n \Big\|^2 \leqslant C\Big(\sum_n |a_n|^2\Big)$$

for all sequences $\{a_k\}_{k \in \mathbb{Z}} \subset l^2$, and a Riesz squence $\{x_k\}_{k \in \mathbb{Z}}$ forms a Riesz basis if

$$\overline{\mathrm{span}(x_k)} = \mathcal{H}.$$

Definition 1.24. (Scaling Function) *The function $\phi(x)$ is called scaling function if it produces a multiresolution of $L^2(\mathbb{R})$.*

It can be proved that $\{\phi(2x-k\}_{k\in\mathbb{Z}}$ form a basis for the space V_1[1.2]. Thus,

$$\begin{aligned} V_0 &= \overline{\text{span}\{\phi(x-k)\}_{k\in\mathbb{Z}}} \\ V_1 &= \overline{\text{span}\{\phi(2x-k)\}_{k\in\mathbb{Z}}} \end{aligned}.$$

Since $V_0 \subset V_1$, we have the following *dilation equation*:

$$\phi(x) = \sum_{k=-\infty}^{+\infty} a_k \phi(2x-k), \quad \{a_k\}_{k\in\mathbb{Z}} \subset l^2(\mathbb{R}). \tag{1.16}$$

Similarly we can define

$$\phi_{m,k}(x) = 2^{m/2}\phi(2^m x - k) \tag{1.17}$$

then it can be deduced that $\{\phi_{m,k}(x)\}_{k\in\mathbb{Z}}$ form a Riesz basis for the space V_m.

If the integer translates of scaling function $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is further constrained as being orthonormal basis of space V_0, then we have $\{\phi_{m,k}(x)\}_{k\in\mathbb{Z}}$ forming an orthonormal basis of space V_m. The dilation parameter m is referred to as the *scale*.

1.2.3 Wavelet

The concept *wavelet* comes from the difference between subspace V_{m-1} and V_m. The subspace W_{m-1} of $L^2(\mathbb{R})$ is defined as the orthogonal complement of V_{m-1} in V_m:

$$V_m = V_{m-1} \oplus W_{m-1}. \tag{1.18}$$

It follows that the spaces W_j are orthogonal and that

$$\bigoplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{R}). \tag{1.19}$$

Definition 1.25. (Wavelet) *Similar to the definition of scaling function to subspace V_0, a wavelet function $\psi(x)$ can be introduced such that $\{\psi(x-k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis for subspace W_0. Then*

$$\psi_{m,k}(x) = 2^{m/2}\psi(2^m x - k) \quad m,k\in\mathbb{Z} \tag{1.20}$$

forming a Riesz basis for subspace W_m is defined as a wavelet in $L^2(\mathbb{R})$.

After defining the multiresolution analysis and wavelet, two projection operators can be introduced, assuming that $\{\phi_{m,k}(x)\}_{k\in\mathbb{Z}}$ and $\{\psi_{m,k}(x)\}_{k\in\mathbb{Z}}$ are both orthonormal basis. We may now approximate a function $f\in L^2(\mathbb{R})$ by its projection

$$P_m f(x) = \sum_{k=-\infty}^{+\infty} c_{m,k}\phi_{m,k}(x), \tag{1.21}$$

we may also define projection of function f on subspace W_m as Q_m and it follows that

$$P_m = P_{m-1} + Q_{m-1}. \tag{1.22}$$

This implies that $Q_m f$ is represents the details that was lost from one level of approximation to a coarser level.

The multiresolution analysis defined in a functional way can be explained as follows. If we have the expansion coefficients $c_{m,k}$ in equation (1.21), then we can decomposes them into two parts with equation (1.22):

1. the expansion coefficients $c_{m-1,k}$ of the approximation $P_{m-1}f$,
2. the expansion coefficients $d_{m-1,k}$ of the detail component $Q_{m-1}f$.

1.2. Scaling by power other than 2 is also feasible.

Chapter 2

Construction and Properties of Wavelet System

This chapter is mainly according to [7].

2.1 The Construction of Daubechies Wavelet System

2.1.1 Quadrature Mirror Filters

In general, a scaling function $\phi(x)$ is solution to dilation equation (1.16), and the constants a_k are called filter coefficients and it is often the case that only a finite number of these are non-zero. Given the filter coefficients, the scaling function can be deduced, and the corresponding wavelet transform is determined; if certain conditions are imposed on the scaling functions, the filter coefficients can be fully derived, and this is the most important contribution of modern wavelet theory (along with frame theory).

One of the most important condition on scaling function is orthogonality:

$$\int_{-\infty}^{\infty} \phi(x)\phi(x+l)\,\mathrm{d}x = \delta_{0,l} \quad l \in \mathbb{Z},$$

so a wavelet being orthogonal to the scaling function can defined by:

$$\psi(x) = \sum_{k=-\infty}^{+\infty} (-1)^k a_{N-1-k}\phi(2x-k)$$

where N is an even integer[2.1]. The orthogonality is verified as follows:

$$\begin{aligned}\langle \phi(x), \psi(x)\rangle &= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{+\infty} a_k\phi(2x-k) \sum_{l=-\infty}^{+\infty} (-1)^l a_{N-1-l}\phi(2x-l) \\ &= \frac{1}{2}\sum_{k=-\infty}^{+\infty} (-1)^k a_{N-1-k}a_k \\ &= 0.\end{aligned}$$

The set of coefficients $\{a_k\}$ and $\{(-1)^k a_{N-1-k}\}$ are said to form a pair of *quadrature mirror filters*.

2.1.2 Derivation of Filter Coefficients

There are several properties that are essential to a useful basis for functional analysis which lead to the corresponding conditions constraining the filter coefficients. They are listed below:

1. Normalization to unity, i.e.

$$\int_{-\infty}^{+\infty} \phi(x)\mathrm{d}x = 1,$$

2.1. It will then be shown that the wavelet has support over $[0, N-1]$.

which leads to

$$\sum_{k=-\infty}^{+\infty} a_k = 2. \tag{2.1}$$

2. Orthogonality of scaling function, i.e.

$$\int_{-\infty}^{+\infty} \phi(x)\phi(x+l)\mathrm{d}x = \delta_{0,l} \quad l \in \mathbb{Z},$$

which yields

$$\sum_{k=-\infty}^{+\infty} a_k a_{k+2l} = 2\delta_{0,l} \quad l \in \mathbb{Z}. \tag{2.2}$$

3. Vanishing moment, i.e.

$$\int_{-\infty}^{+\infty} \psi(x)x^l\mathrm{d}x = 0 \quad l = 0, 1, 2, ..., p-1,$$

which yields (by Daubechies [3])

$$\sum_{k=-\infty}^{+\infty} (-1)^k a_k k^l = 0 \quad l = 0, 1, 2, ..., N/2 - 1, \tag{2.3}$$

and a more detailed discussion on equation (2.3) can be found in [7].

The filter coefficients $\{a_k\}_{k=0,1,...,N-1}$ for an N coefficient system are uniquely defined by equation (2.1, 2.2, and 2.3).

2.1.3 Construction of Scaling Function

In general, there is no closed-form solution of scaling function, and they have to be attained recursively from the dilation equation (1.16) instead. In the case of quadrature mirror filter:

$$\phi(x) = a_0\phi(2x) + a_1\phi(2x-1) + a_2\phi(2x-2) + \cdots + a_{N-1}\phi(2x-N+1),$$

and it can be proved that[7] all integer points outside $[0, N-1]$ have zero value, we have

$$\begin{aligned}
\phi(0) &= a_0\phi(0) \\
\phi(1) &= a_0\phi(2) + a_1\phi(1) + a_2\phi(0) \\
\phi(2) &= a_0\phi(4) + a_1\phi(3) + a_2\phi(2) + a_3\phi(1) + a_4\phi(0) \\
&\vdots \\
\phi(N-2) &= a_{N-3}\phi(N-1) + a_{N-2}\phi(N-2) + a_{N-1}\phi(N-3) \\
\phi(N-1) &= a_{N-1}\phi(N-1).
\end{aligned}$$

or

$$M\Phi = \Phi$$

This linear equation system is actually finding the eigenvector of matrix M corresponding to the eigenvalue 1, so the normalizing condition is necessary:

$$\sum_{k=-\infty}^{+\infty} \phi(i) = 1 \quad i \in \mathbb{Z}.$$

The above process gives the scaling function on integer points, and the scaling function on all real points can be calculated by bisection method:

$$\phi\left(\frac{x}{2}\right) = \sum_{k=-\infty}^{+\infty} a_k\phi(x-k).$$

2.1.4 Example: The Daubechies 4 Coefficient Wavelet System

Here we demonstrate the construction of the so called D4 Wavelet. According to equation (2.1, 2.2, 2.3), we have

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 2 \\ a_0^2 + a_1^2 + a_2^2 + a_3^2 &= 2 \\ a_0 - a_1 + a_2 - a_3 &= 0 \\ -a_1 + 2a_2 - 3a_3 &= 0 \end{aligned}$$

One set of solution is

$$\begin{aligned} a_0 &= \frac{1+\sqrt{3}}{4} \\ a_1 &= \frac{3+\sqrt{3}}{4} \\ a_2 &= \frac{3-\sqrt{3}}{4} \\ a_3 &= \frac{1-\sqrt{3}}{4} \end{aligned}$$

and the other set of solution is the antithesis of this set leading to $\phi(-x)$ instead of $\phi(x)$.

The values of the scaling function on integer points are given by

$$\begin{bmatrix} a_0-1 & 0 & 0 & 0 \\ a_2 & a_1-1 & a_0 & 0 \\ 0 & a_3 & a_2-1 & a_1 \\ 0 & 0 & 0 & a_3-1 \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and solved as:

$$\phi(0) = 0 \quad \phi(1) = \frac{1+\sqrt{3}}{2} \quad \phi(2) = \frac{1-\sqrt{3}}{2} \quad \phi(3) = 0.$$

Additionally we have the half integers:

$$\begin{aligned} \phi\left(\frac{1}{2}\right) &= a_0\phi(1) = \frac{2+\sqrt{3}}{4} \\ \phi\left(\frac{3}{2}\right) &= a_1\phi(2) + a_2\phi(1) = 0 \\ \phi\left(\frac{5}{2}\right) &= a_3\phi(2) = \frac{2-\sqrt{3}}{4}. \end{aligned}$$

Any real number point on $[0,3]$ can be calculated similarly, and it can be verified easily that the scaling function always has null value outside $[0,3]$, which means it has compact support.

2.2 Classification of Wavelet Bases

There are many families of orthogonal wavelets that have been constructed in $L^2(\mathbb{R})$. We can classify them with the following criteria: localization in physical space (owing to their fast decay or even compact support[5]), localization in frequency space (owing to their vanishing moments and smoothness[5]), continuity, and differentiablity.

2.3 Mallat Transform

The Mallat Transform provides a simple means of transforming data from one level of resolution m to the next coarser level of resolution $m-1$. The inverse Mallat transform is a transform from the coarser level $m-1$ back to the finer level m.

2.3.1 Multiresolution Decomposition

As is mentioned before, the multiresolution decomposition consists of two parts:

1. the expansion coefficients $c_{m-1,k}$ of the approximation $P_{m-1}f$,
2. the expansion coefficients $d_{m-1,k}$ of the detail component $Q_{m-1}f$.

Consider a function f:

$$\begin{aligned} P_m f &= \sum_{k=-\infty}^{+\infty} c_{m,k}\phi_{m,k}(x) \quad c_{m,k} = \langle f, \phi_{m,k}\rangle \\ Q_m f &= \sum_{k=-\infty}^{+\infty} d_{m,k}\psi_{m,k}(x) \quad d_{m,k} = \langle f, \psi_{m,k}\rangle \\ P_{m-1}f &= P_m f - Q_{m-1}f. \end{aligned}$$

Substituting the above in

$$c_{m-1,k} = \langle P_{m-1}f, \phi_{m-1,k}\rangle$$

leads to the following result

$$c_{m-1,k} = \frac{1}{\sqrt{2}}\sum_{j=-\infty}^{+\infty} c_{m,j}a_{j-2k}. \tag{2.4}$$

Similarly there is:

$$d_{m-1,k} = \frac{1}{\sqrt{2}}\sum_{j=-\infty}^{+\infty} c_{m,j}(-1)^j a_{N-1-j+2k}. \tag{2.5}$$

Equation (2.4, 2.5) form the basis of the Mallat transform algorithm.

2.3.2 Multiresolution Reconstruction

Multiresolution construction use $c_{m-1,k}$ and $d_{m-1,k}$ to reconstruct $c_{m,k}$. Considering the following relationship:

$$\begin{aligned} P_m f &= P_{m-1}f + Q_{m-1}f \\ c_{m,k} &= \langle P_m f, \phi_{m,k}\rangle \end{aligned}$$

leads to

$$c_{m,k} = \frac{1}{\sqrt{2}}\sum_{j=-\infty}^{+\infty} c_{m-1,j}a_{k-2j} + \frac{1}{\sqrt{2}}\sum_{j=-\infty}^{+\infty} d_{m-1,k}(-1)^k a_{N-1-k+2j}. \tag{2.6}$$

Equation (2.6) forms the basis of the inverse Mallat transform algorithm.

2.3.3 The Mallat Transform and Inverse Transform Algorithm

The equations (2.4, 2.5, and 2.6) is expressed by the following algorithm. Consider a string of data $c_{m,k}$ of finite length n which represents the approximation $P_m f$ to a function. For convenience, suppose that this data is periodic with period n. The matrix form of equation (2.4) is

$$\begin{bmatrix} c_{m-1,0} \\ \times \\ c_{m-1,1} \\ \times \\ c_{m-1,2} \\ \times \\ \cdots \\ \cdots \\ c_{m-1,n/2-1} \\ \times \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} & \cdots & 0 \\ 0 & a_0 & a_1 & a_2 & \cdots & a_{N-2} & \cdots & 0 \\ 0 & 0 & a_0 & a_1 & \cdots & a_{N-3} & \cdots & 0 \\ 0 & 0 & 0 & a_0 & \cdots & a_{N-4} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{N-1} \\ a_{N-1} & 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots & 0 & \cdots & a_1 \\ a_1 & a_2 & a_3 & a_4 & \cdots & 0 & \cdots & a_0 \end{bmatrix} \begin{bmatrix} c_{m,0} \\ c_{m,1} \\ c_{m,2} \\ c_{m,3} \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ c_{m,n-2} \\ c_{m,n-1} \end{bmatrix}$$

in which $\times$ represents information of no value. The matrix of filter coefficients is a circulant matrix, so that the right-hand side of the equation is effectively a convolution of the discrete filter

$$\tilde{h} = \frac{1}{\sqrt{2}}[a_0, 0, 0, 0, \ldots, 0, a_{N-1}, \ldots, a_2, a_1]^{\mathrm{T}}$$

with the data $c_{m,k}$. Only every other sample of the result needs be kept; this process is called decimation. Similarly, the $d_{m-1,k}$ is effectively a convolution of the discrete filter

$$\tilde{g} = \frac{1}{\sqrt{2}}[a_{N-1}, 0, 0, 0, \ldots, 0, -a_0, \ldots, a_{N-3}, -a_{N-2}]^{\mathrm{T}}$$

with the data $c_{m,k}$. The above two steps constitutes the Mallat transform (decomposition). The reconstruction algorithm can allso be obtained by the matrix form of equation (2.6):

1. insert a zero between every sample in $c_{m-1,k}$ and $d_{m-1,k}$,
2. convolve $c_{m-1,k}$ with the filter h,
3. convolve $d_{m-1,k}$ with the filter g,
4. add results in 2. and 3. to get the original data, $c_{m,k}$,

where the filters h and g are

$$h = \frac{1}{\sqrt{2}}[a_0, a_1, a_2, a_3, \ldots, a_{N-2}, a_{N-1}, \ldots, 0, 0]^{\mathrm{T}}$$

and

$$g = \frac{1}{\sqrt{2}}[a_{N-1}, -a_{N-2}, a_{N-3}, -a_{N-4}, \ldots, a_1, -a_0, \ldots, 0, 0]^{\mathrm{T}}$$

2.3.4 Notes on Wavelet Transform Algorithm

In practice [7], the scaling function decomposition does not usually need to be carried out since, at the finest scale, the scaling function approaches the delta function, i.e.

$$\phi_{m,k}(x) \to 2^{m/2}\delta(2^m x - k) \quad \text{as } m \to \infty,$$

and this is the basis for a discrete sampling of $f(x)$. The projection of f onto the space of delta function is

$$P_m f(x) = \sum_{k=-\infty}^{+\infty} f_k\, 2^m \delta(2^m x - k) \tag{2.7}$$

where

$$f_k = f(2^{-m}k).$$

A comparision between equation (1.21) and (2.7) leads to

$$c_{m,k} \approx 2^m f_k = 2^m f(2^{-m}k) \quad \text{as } m \text{ is very large.}$$

But if m is not large enough, the sampling process described above may introduce extra error in which case the integral according to definition of $c_{m,k}$ is required.

Chapter 3

Wavelet in Numerical Analysis

3.1 Some Useful Orthogonal Wavelets

3.1.1 Meyer Wavelet

The Mayer Wavelet is defined by the following formula according to [4], [3], and [6][3.1],

$$\hat{\phi}(\xi) = \begin{cases} 1 & ; |\xi| \leqslant \frac{1}{3} \\ \cos\left(\frac{\pi}{2}\nu(3|\xi|-1)\right) & ; \frac{1}{3} \leqslant |\xi| \leqslant \frac{2}{3} \\ 0 & ; \text{elsewhere} \end{cases}$$

and

$$\hat{\psi}(\xi) = \begin{cases} e^{i\pi\xi}\cos\left(\frac{\pi}{2}\nu(-3|\xi|+2)\right) & ; \frac{1}{3} \leqslant |\xi| \leqslant \frac{2}{3} \\ e^{i\pi\xi}\cos\left(\frac{\pi}{2}\nu\left(\frac{3}{2}|\xi|-1\right)\right) & ; \frac{2}{3} \leqslant |\xi| \leqslant \frac{4}{3} \\ 0 & ; \text{elsewhere} \end{cases}.$$

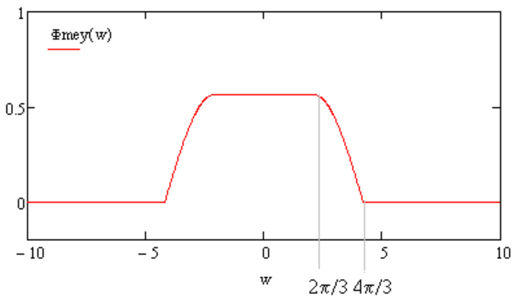

Figure 3.1. The Meyer scaling function in frequency space, it should be pointed out that the definition of Fourier transform in the figure is different from Equation (1) in Prologue, and the relation is $\Phi_{\text{meyer}}(\xi) = \hat{\phi}(2\pi\xi)/\sqrt{2\pi}$.

3.1. The formula given in appendix of [4] is not correct.

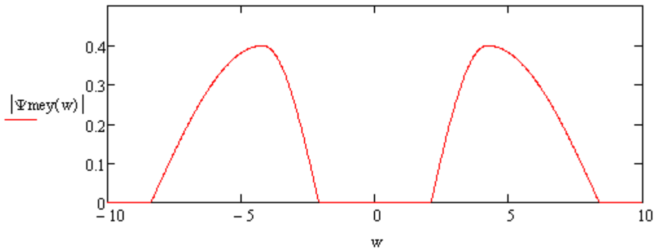

Figure 3.2. The Meyer wavelet function in frequency space, it should be pointed out that the definition of Fourier transform in the figure is different from Equation (1) in Prologue, and the relation is $\Psi_{\text{meyer}}(\xi) = \hat{\psi}(2\pi\xi)/\sqrt{2\pi}$.

For the smooth "step function" ν, two properties are required:

$$\nu(x) = \begin{cases} 0 & ; x \leqslant 0 \\ 1 & ; x \geqslant 1 \end{cases} \quad \text{and} \quad \nu(x) + \nu(1-x) = 1,$$

and the choice given by [3] with best localisation in physical domain is

$$\nu(x) = x^4(35 - 84x + 70x^2 - 20x^3) \quad 0 \leqslant x \leqslant 1,$$

leading to the 3rd order of Mayer wavelet as the exponential index in term $-20x^3$ is 3. If the zeroth order of Mayer wavelet is chosen:

$$\nu(x) = \begin{cases} 0 & ; x < 0 \\ x & ; 0 < x < 1 \\ 1 & ; x > 1 \end{cases},$$

analytical expressions of scaling function and wavelet on time domain can be attained [6]:

$$\phi(t) = \begin{cases} \dfrac{2}{3} + \dfrac{4}{3\pi} & ; t = 0 \\ \dfrac{\sin\frac{2\pi t}{3} + \frac{4}{3} t \cos\frac{4\pi t}{3}}{\pi t - \frac{16}{9}\pi t^3} & ; \text{elsewhere} \end{cases}$$

and

$$\psi(t) = \psi_1(t) + \psi_2(t)$$

where

$$\psi_1(t) = \frac{\frac{4}{3\pi}(t-0.5)\cos\left(\frac{2\pi}{3}(t-0.5)\right) - \frac{1}{\pi}\sin\left(\frac{4\pi}{3}(t-0.5)\right)}{(t-0.5) - \frac{16}{9}(t-0.5)^3}$$

and

$$\psi_2(t) = \frac{\frac{8}{3\pi}(t-0.5)\cos\left(\frac{8\pi}{3}(t-0.5)\right) + \frac{1}{\pi}\sin\left(\frac{4\pi}{3}(t-0.5)\right)}{(t-0.5) - \frac{64}{9}(t-0.5)^3}$$

Mayer wavelet, as one of the most important orthogonal wavelets is constructed earlier than the invention of MRA. It is defined in *frequency space* and the purpose of this construction is to realize localization in frequency space (shown in Figure 3.1 and 3.2). The 3rd order Meyer wavelet system in time domain is sketched in Figure 3.3 and 3.4.

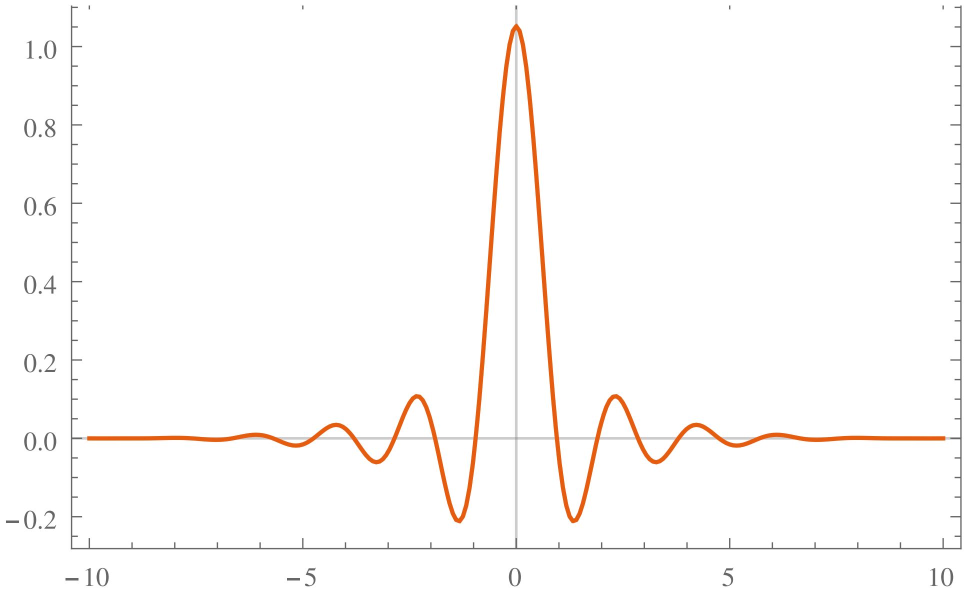

Figure 3.3. Scaling function of Meyer wavelet

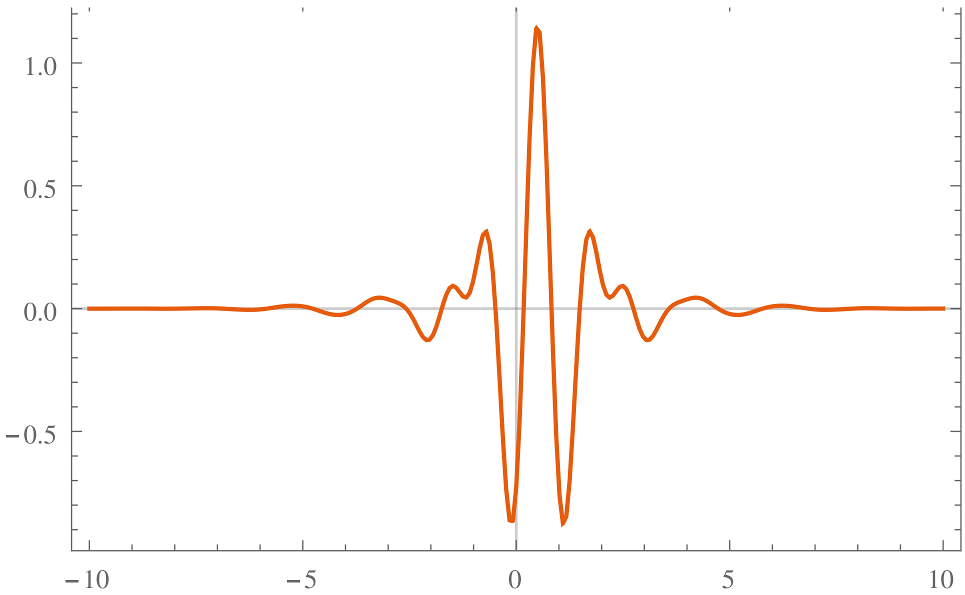

Figure 3.4. Meyer wavelet function

3.1.2 Shannon Wavelet

The Shanonn wavelet is very similar to Meyer wavelet while it is also defined on frequency space with normalised gate function. As a result, it has very simple analytical expression (though poor convergence property):

$$\phi(t) = \mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t}$$

and

$$\psi(t) = \mathrm{sinc}\left(\frac{t}{2}\right) \cdot \cos\left(\frac{3\pi t}{2}\right) = 2 \cdot \mathrm{sinc}(2t-1) - \mathrm{sinc}(t).$$

There is also complex continuous Shanno wavelet:

$$\psi^{(C\,\mathrm{Sha})}(t) = \mathrm{sinc}(t) \cdot \mathrm{e}^{-2\pi i t}$$

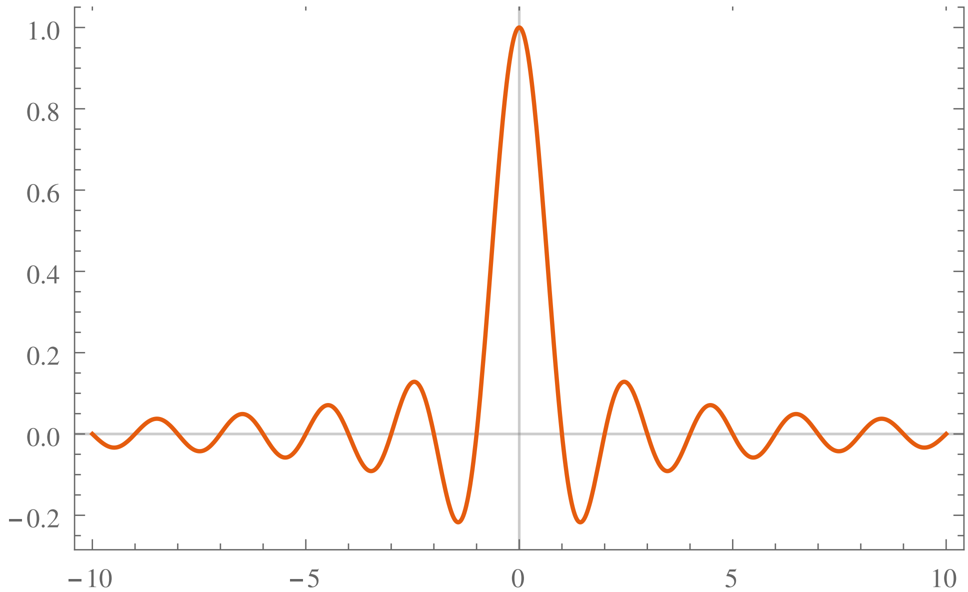

Figure 3.5. sinc function

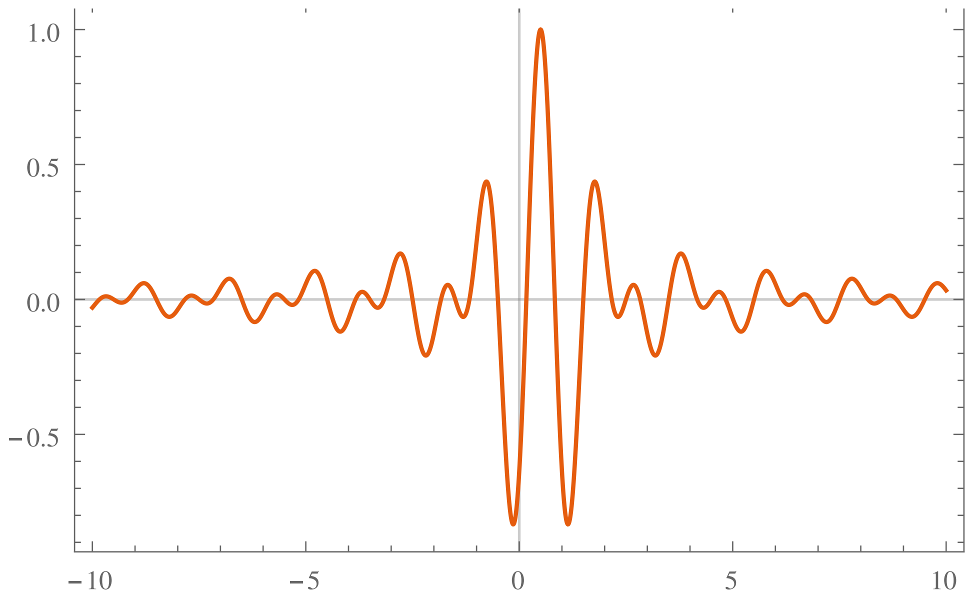

Figure 3.6. Shanonn wavelet (real)

3.1.3 Daubechies Wavelet

The construction of Daubechies wavelet has been discussed in great detail in Chapter 2. The difference between different orders of Daubechies wavelet are quite large, so they are sketched out in detail. D1 means the first order[3.2] of Daubechies wavelet, i.e. Haar wavelet, D2 means the second order of Daubechies wavelet, and so on.

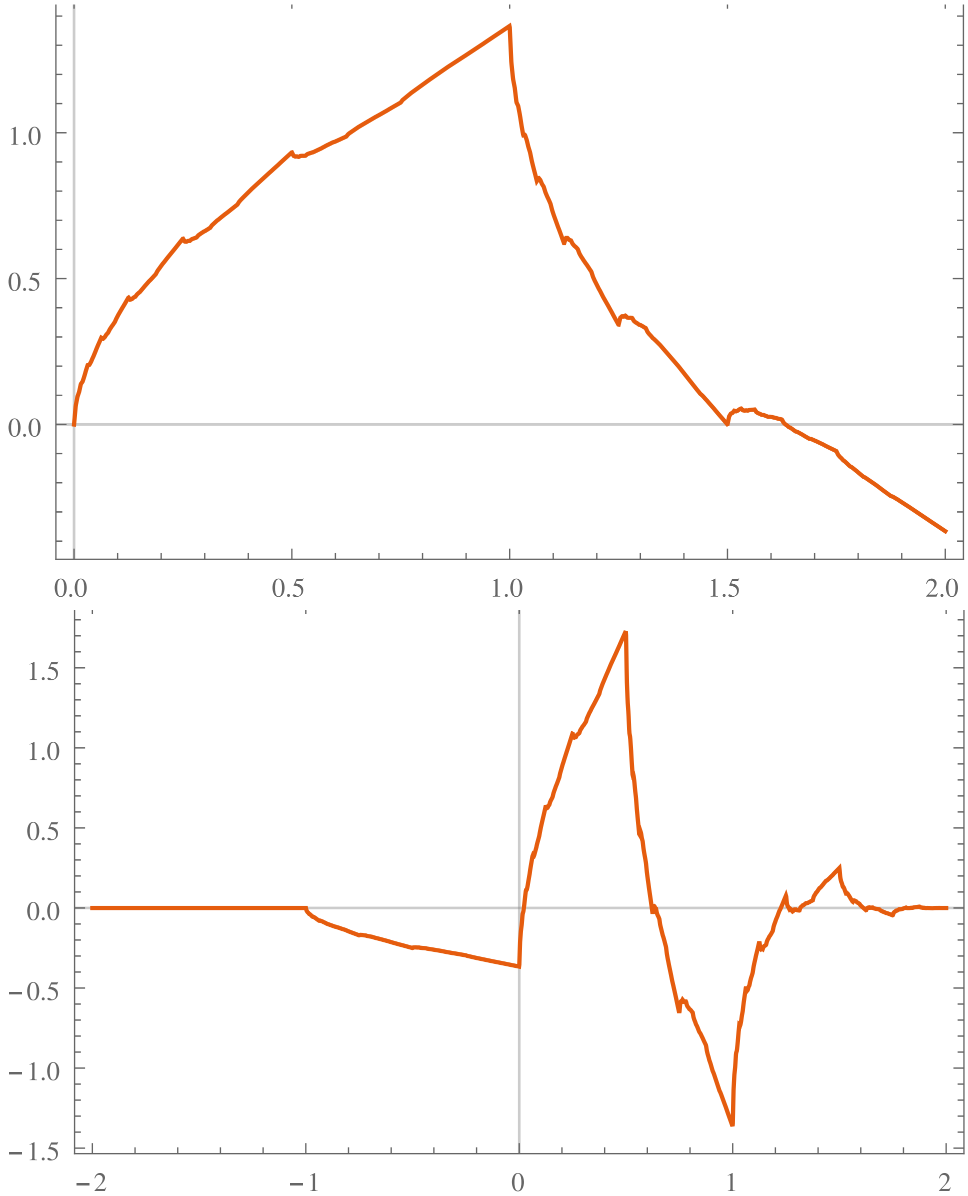

Figure 3.7. The scaling function (upper) and wavelet function (lower) of D2

3.2. The order of Daubechies wavelet is defined as the order of its vanishming moment.

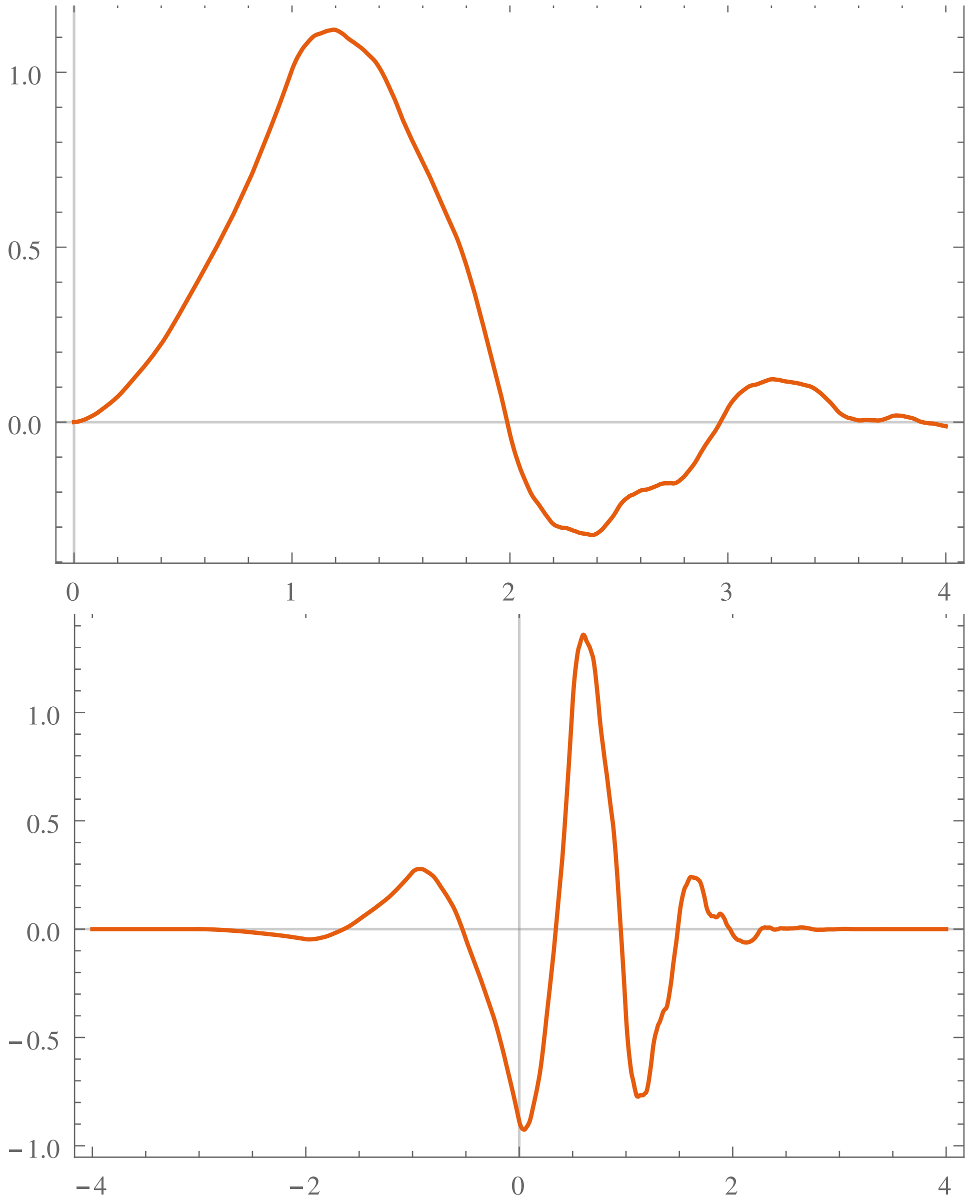

Figure 3.8. The scaling function (upper) and wavelet function (lower) of D4

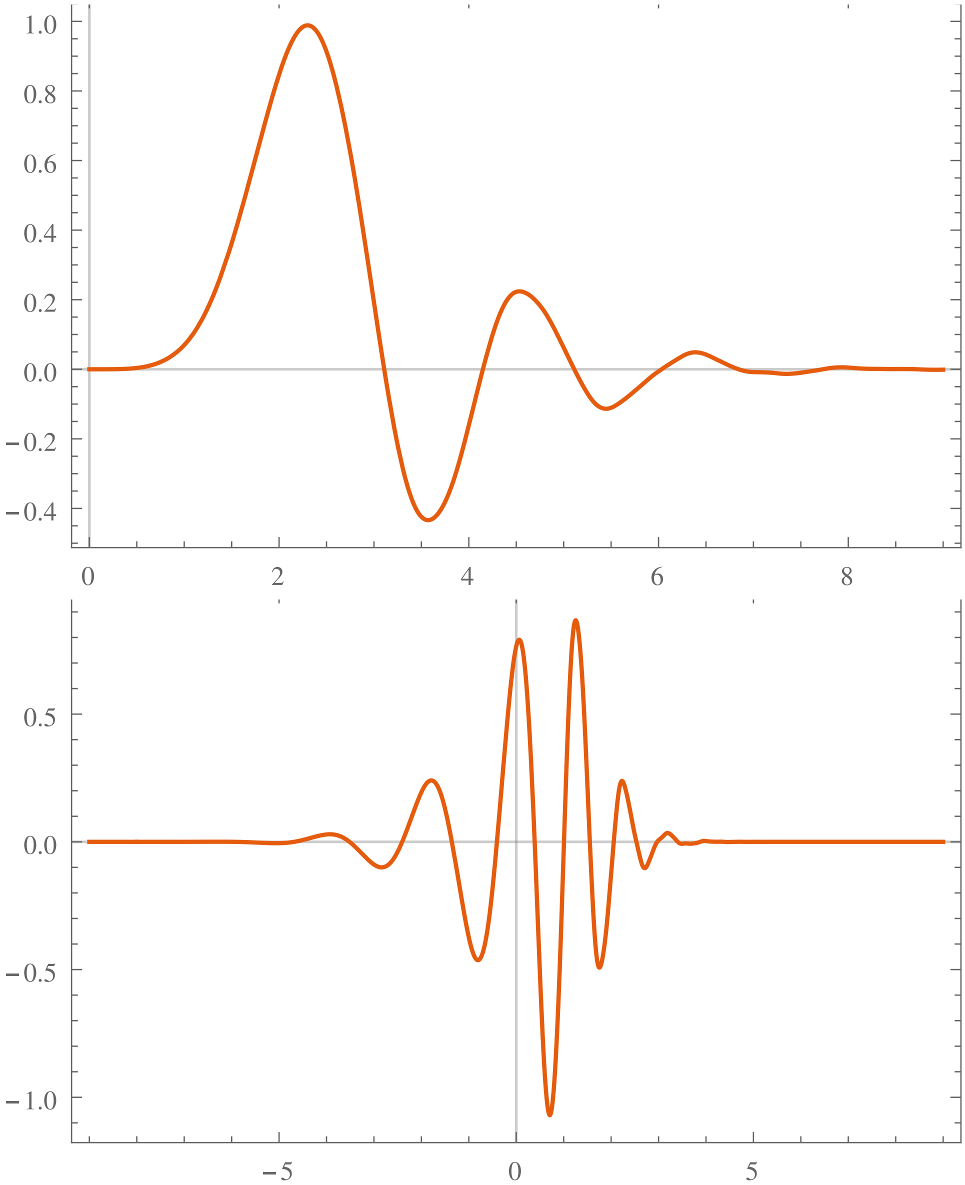

Figure 3.9.

3.1.4 Symlet Wavelet

3.1.5 Coiflet Wavelet

3.1.6 Battle-Lemarie Wavelet

3.1.7 Biorthogonal spline Wavelet

3.1.8 Reverse B-spline Wavelet

3.2 Miscellaneous Staff

3.2.1 Learning Resources

- All references are good reading materials, especially [3].
- Mathematica's *Wolfram Documentation* is the best material as a starting point of learning how to apply wavelet.

Part II

Application of Wavelet Analysis in CFD

Chapter 4
Miscellaneous Staff

4.1 Orthogonal Wavelet with Gaussian Distribution as Scaling Function?

The answer is that this is not feasible. Suppose we have

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(x-x_0)^2}{2}},$$

then it can be calculated that

$$\hat{\phi}(\gamma) = \int_{-\infty}^{+\infty} \phi(x) \mathrm{e}^{-2\pi \mathrm{i} \gamma x} \mathrm{d}x = \mathrm{e}^{-2\pi\gamma(\mathrm{i}x_0 + 2\pi\gamma)}.$$

However we have also

$$H_0(\gamma) = \frac{\hat{\phi}(2\gamma)}{\hat{\phi}(\gamma)} = \mathrm{e}^{-6\pi^2\gamma^2} \mathrm{e}^{-2\pi \mathrm{i} \gamma x_0}$$

which is unfortunately not 1-periodic: this violates theorem 1.14. So it is imposible to construct an orthogonal wavelet with Gaussian distriubtion as its scaling function.

However, it is still posible to construct a biorthogonal wavelet from wavelet frame generated by Gaussian distribution.

4.2 Approximation of Gaussian Distribution with Orthogonal Wavelets

In this section, a series of numerical test is carried out to compress the data representing a Gaussian distribution. The Gaussian distribution is modeled here as:

$$f(x) = \exp\left(-\frac{x^2}{2}\right).$$

Wavelet transform with four levels of refinement is applied on sampled data points, and a threshold of $\epsilon = 5 \times 10^{-5}$ is applied to both the original data and the transformed data. The error caused by the compression is measured with a numerical integral by rectangle formula

$$\mathrm{Err} = \frac{1}{10} \int_{-5}^{5} |f(x) - \tilde{f}(x)| \mathrm{d}x = \frac{1}{10} \sum_n |f(x_n) - \tilde{f}(x_n)| \Delta x.$$

The comparision is shown in table 4.1 and 4.2.

Two typical cases are examined. The sampling interval are both $[-5, 5]$ since $\mathrm{e}^{-12.5} = 3.7 \times 10^{-6}$ is small enough. In the first case, the sampling rate is one sample point per 0.1 unit of length; in the second case, the sampling rate is one sample point per 0.01 unit of length. Meyer wavelet and Shannon wavelet do not work well on compressing the data[4.1], and the B-spline wavelet is the only one that performs good compressing ratio among all spline-based wavelets. The result of Haar wavelet is also provides for comparison.

4.1. The compression ratio is less than 2 even in the second case.

	N_{ori}	N_{trans}	α	Err
Haar	89	88	1.011	6.12×10^{-7}
Daubechies 9	89	36	2.472	1.24×10^{-5}
Coiflet 5	89	41	2.171	1.73×10^{-5}
Symlet 10	89	39	2.282	1.04×10^{-5}
B-spline 6 2	89	25	3.560	5.75×10^{-7}

Table 4.1. Case 1 where there is one sample point per 0.1 unit of length has totally 101 sampling points. α is compression ration, N_{ori} is the number of sampling points in original data that is lager than the threshold, and N_{trans} is the number of data points in tranformed data that is lager than the threshold.

	N_{ori}	N_{trans}	α	Err
Haar	891	760	1.172	4.60×10^{-7}
Daubechies 8	891	59	15.10	4.58×10^{-7}
Coiflet 5	891	59	15.10	4.43×10^{-7}
Symlet 5	891	59	15.10	4.45×10^{-7}
B-spline 4 2	891	59	15.10	3.68×10^{-7}

Table 4.2. Case 1 where there is one sample point per 0.01 unit of length has totally 1001 sampling points. One thing to mention is that Daubechies 5 has already reduce the amout of meaningfu data to 60 samples. α is compression ration, N_{ori} is the number of sampling points in original data that is lager than the threshold, and N_{trans} is the number of data points in tranformed data that is lager than the threshold.

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