CodeChef Discussion

LCM - Editorial

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PROBLEM LINKS

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DIFFICULTY

HARD

EXPLANATION

SOLUTION TO LCM

The solution is to use Möbius inversion. See for example the Wikipedia for information on the Möbius function μ and related concepts.

Assume throughout that $A \le B$. We want to compute $\sum_{a=1}^{A}\sum_{b=1}^{B} abF(\gcd(a,b))$, where $F(n) = \frac{1}{n}$ if n is square-free and 0 otherwise. Adapting a standard approach, we do the following:

Let $f(n) = \sum_{d|n} \mu(d)F(\frac{n}{d})$. Then $F(n) = \sum_{d|n} f(d)$ by the Möbius inversion formula, and

$$
\begin{aligned}
&\sum_{a=1}^{A}\sum_{b=1}^{B} abF(\gcd(a,b)) \\
=& \sum_{a=1}^{A}\sum_{b=1}^{B} ab \sum_{d|\gcd(a,b)} f(d) \\
=& \sum_{d=1}^{A}\sum_{i=1}^{\lfloor A/d\rfloor}\sum_{j=1}^{\lfloor B/d\rfloor} (id)(jd)f(d) \\
=& \sum_{d=1}^{A} d^2 f(d) \sum_{i=1}^{\lfloor A/d\rfloor} i \sum_{j=1}^{\lfloor B/d\rfloor} j \\
=& \sum_{d=1}^{A} d^2 f(d) \binom{\lfloor A/d\rfloor+1}{2}\binom{\lfloor B/d\rfloor+1}{2} \\
=& \sum_{d=1}^{A} d^2 f(d) \lfloor A/d\rfloor \lfloor A/d+1\rfloor \lfloor B/d\rfloor \lfloor B/d+1\rfloor /4
\end{aligned}
$$

For the second equality: Since $a \le A \le B$ and d divides a, d can range from 1 to A. Since a and b are multiples of d, we can write them as id and jd, respectively, with $i \le \lfloor A/d\rfloor$ and $j \le \lfloor B/d\rfloor$.

For any $n \ge 1$, we have $f(n) = \sum_{d|n} \mu(n)F(\frac{n}{d}) = \frac{1}{n}\sum_{d|n} \mu(d)d\mu^2(\frac{n}{d})$. ($\mu^2(k) = 1$ iff k is square-free.) Since F is multiplicative, so is f. Using this it is easy to find f. If n is divisible by p^3 for some p, then $f(n) = 0$. Otherwise, $n = p_1^2 p_2^2 \cdots p_s^2 q_1 q_2 \cdots q_t$. Letting $k = p_1 p_2 \cdots p_s$ and $q = q_1 q_2 \cdots q_t$ we have $f(n) = \frac{1}{n}\prod_{p|k} -p \prod_{p|q}(1-p)$, where the products are taken over all prime divisors p.

Let $g(n) = \prod_{p|k} -p \prod_{p|q}(1-p)$. That is: Start with 1. For each prime p such that p divides n but p^2 does not divide n, multiply by $1-p$. For each prime p such that p^2 divides n, multiply by $-p$. If p^3 divides n for some prime p, $g(n) = 0$. We can precompute g as follows: First use a prime sieve to find a prime factor of n for each $n \le 4000000$. Then compute f[n] iteratively. The sieve of Eratosthenes is too slow. A wheel sieve should work, but a segmented sieve is simpler and faster in practise. In code:

```
   int MAX = 4000000;
   int g[4000000];

// Compute prime factors
//   .
//   .
// Now g[n] equals 0 if n is prime, otherwise g[n] is a prime factor of n.

   for(int n=2; n<=MAX; n++) {
      if(g[n]==0) {
         g[n]=1-n; // n is prime
         continue;
      }

      p=g[n];
      int m=n/p;

      if(m%p != 0) g[n]=f[m]*(1-p); // p divides i, but p^2 does not,
                                    // so g[n] = g[n/p]*(1-p)
      else { // p^2 divides i
         m/=p;
         if(m%p != 0) g[n] = g[m]*(-p); // p^3 does not divide i
         else g[n]=0; // p^3 divides i
     }
   }
```

It is possible to do better, using an all-in-one segmented approach that also avoids divisions, but this is good enough.

To compute $\sum_{d=1}^{A} dg(d)\binom{\lfloor A/d\rfloor+1}{2}\binom{\lfloor B/d\rfloor+1}{2}$, note that $\lfloor A/d\rfloor$ takes on at most $2\sqrt{A}+1$ values, and $\lfloor B/d\rfloor$ takes on at most $2\sqrt{B}+1$ values. So, we can precompute $\sum_{d=1}^{k} dg(d)$ for all $k \le 4000000$ and group terms that have the same values for $\lfloor A/d\rfloor$ and $\lfloor B/d\rfloor$.

Computing the result modulo 2^{30} allows using the "wrap-around" arithmetic of unsigned 32-bit integers, and clearing the two high bits of the final result. For languages that don't have wrap-around arithmetic, x mod 2^{30} can be computed very quickly as x AND $(2^{30}-1)$.

lcm hard editorial july10

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