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Created, developed, and nurtured by Eric Weisstein at Wolfram Research

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Permutation

A permutation, also called an "arrangement number" or "order," is a rearrangement of the elements of an ordered list S into a one-to-one correspondence with S itself. The number of permutations on a set of n elements is given by $n!$ (n factorial; Uspensky 1937, p. 18). For example, there are $2! = 2 \cdot 1 = 2$ permutations of $\{1, 2\}$, namely $\{1, 2\}$ and $\{2, 1\}$, and $3! = 3 \cdot 2 \cdot 1 = 6$ permutations of $\{1, 2, 3\}$, namely $\{1, 2, 3\}$, $\{1, 3, 2\}$, $\{2, 1, 3\}$, $\{2, 3, 1\}$, $\{3, 1, 2\}$, and $\{3, 2, 1\}$. The permutations of a list can be found in the Wolfram Language using the command `Permutations`[*list*]. A list of length n can be tested to see if it is a permutation of 1, ..., n in the Wolfram Language using the command `PermutationListQ`[*list*].

Sedgewick (1977) summarizes a number of algorithms for generating permutations, and identifies the minimum change permutation algorithm of Heap (1963) to be generally the fastest (Skiena 1990, p. 10). Another method of enumerating permutations was given by Johnson (1963; Séroul 2000, pp. 213-218).

The number of ways of obtaining an *ordered* subset of k elements from a set of n elements is given by

$$ {}_nP_k \equiv \frac{n!}{(n-k)!} \tag{1}$$

(Uspensky 1937, p. 18), where $n!$ is a factorial. For example, there are $4!/2! = 12$ 2-subsets of $\{1, 2, 3, 4\}$, namely $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 1\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 1\}$, $\{3, 2\}$, $\{3, 4\}$, $\{4, 1\}$, $\{4, 2\}$, and $\{4, 3\}$. The *unordered* subsets containing k elements are known as the k-subsets of a given set.

A representation of a permutation as a product of permutation cycles is unique (up to the ordering of the cycles). An example of a cyclic decomposition is the permutation $\{4, 2, 1, 3\}$ of $\{1, 2, 3, 4\}$. This is denoted $(2)(143)$, corresponding to the disjoint permutation cycles (2) and (143). There is a great deal of freedom in picking the representation of a cyclic decomposition since (1) the cycles are disjoint and can therefore be specified in any order, and (2) any rotation of a given cycle specifies the same cycle (Skiena 1990, p. 20). Therefore, (431)(2), (314)(2), (143)(2), (2)(431), (2)(314), and (2)(143) all describe the same permutation.

Another notation that explicitly identifies the positions occupied by elements before and after application of a permutation on n elements uses a $2 \times n$ matrix, where the first row is $(123 \ldots n)$ and the second row is the new arrangement. For example, the permutation which switches elements 1 and 2 and fixes 3 would be written as

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}. \tag{2}$$

Any permutation is also a product of transpositions. Permutations are commonly denoted in lexicographic or transposition order. There is a correspondence between a permutation and a pair of Young tableaux known as the Schensted correspondence.

The number of wrong permutations of n objects is $[n!/e]$ where $[x]$ is the nearest integer function. A permutation of n ordered objects in which no object is in its natural place is called a derangement (or sometimes, a complete permutation) and the number of such permutations is given by the subfactorial $!n$.

Using

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \tag{3}$$

with $x = y = 1$ gives

$$2^n = \sum_{r=0}^{n} \binom{n}{r}, \tag{4}$$

so the number of ways of choosing 0, 1, ..., or n at a time is 2^n.

The set of all permutations of a set of elements 1, ..., n can be obtained using the following recursive procedure

```
  1 2
  /                 (5)
2 1
  1   2 3
      /
  1 3 2
  /
3 1   2
|                   (6)
3 2   1
  \
  2 3 1
      \
  2   1 3
```

Consider permutations in which no pair of consecutive elements (i.e., rising or falling successions) occur. For $n = 1$, 2, ... elements, the numbers of such permutations are 1, 0, 0, 2, 14, 90, 646, 5242, 47622, ... (OEIS A002464).

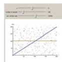
Let the set of integers 1, 2, ..., N be permuted and the resulting sequence be divided into increasing runs. Denote the average length of the nth run as N approaches infinity, L_n. The first few values are summarized in the following table, where e is the base of the natural logarithm (Le Lionnais 1983, pp. 41-42; Knuth 1998).

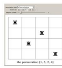
n	L_n	Sloane	approximate
1	$e-1$	A091131	1.7182818...
2	e^2-2e	A091132	1.9524...
3		A091133	1.9957...

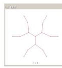
binomial theorem

THINGS TO TRY:
binomial theorem
symmetric group
permutation (1 3 5)(2 4)(6


Generating Permutations b Sequence of Transpositions
George Beck


Random Permu of a Given Leng
Neil Chriss


Permutations, Derangements, Other Forbidde Position Proble Using Non-Atta Rooks
Marc Brodie


Permutation Tr
Michael Schreibe

$e^3 - 3e^2 + \frac{3}{2}e$		

SEE ALSO:
Alternating Permutation, Ball Picking, Binomial Coefficient, Choose, Circular Permutation, Combination, Derangement, Eulerian Number, Even Permutation, *k*-Subset, Linear Extension, Married Couples Problem, Multichoose, Multinomial Coefficient, Multiset, Odd Permutation, Permutation Ascent, Permutation Cycle, Permutation Inversion, Permutation Matrix, Permutation Pattern, Permutation Run, Permutation Symbol, Random Permutation, String, Subfactorial, Transposition

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Referenced on Wolfram|Alpha: Permutation

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Weisstein, Eric W. "Permutation." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Permutation.html