

Artificial neural networks and backpropagation

E. Decencière

MINES ParisTech
PSL Research University
Center for Mathematical Morphology



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- 2 Artificial neuron
- 3 Artificial neural networks
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Artificial neural networks and deep learning history

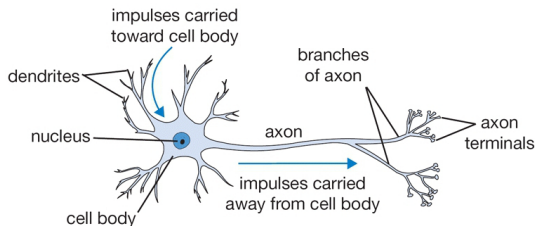
For a very complete state of the art on deep learning, see the overview by Schmidhuber [Schmidhuber, 2015].

- 1958: Rosenblatt's perceptron [Rosenblatt, 1958]
- 1980's: the backpropagation algorithm (see, for example, the work of Le Cun [LeCun, 1985])
- 2006-: CNN implementations using Graphical Processing Units (GPU): up to a 50 speed-up factor.
- 2011-: super-human performances [Cireşan et al., 2011]
- 2012: Imagenet image classification won by a CNN [Krizhevsky et al., 2012].

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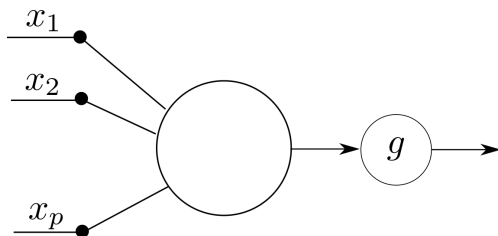
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Neuron



- The human brain contains 100 billion (10^{11}) neurons
- A human neuron can have several thousand dendrites
- The neuron sends a signal through its axon if during a given interval of time the net input signal (sum on excitatory and inhibitory signals received through its dendrites) is larger than a threshold.

Artificial neuron

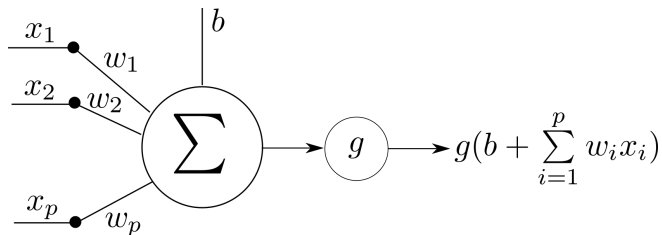


General principle

An artificial neuron takes p inputs $\{x_i\}_{1 \leq i \leq p}$, combines them to obtain a single value, and applies an **activation function** g to the result.

- The first artificial neuron model was proposed by [McCulloch and Pitts, 1943]
- Input and output signals were binary
- Input dendrites could be inhibitory or excitatory

Modern artificial neuron

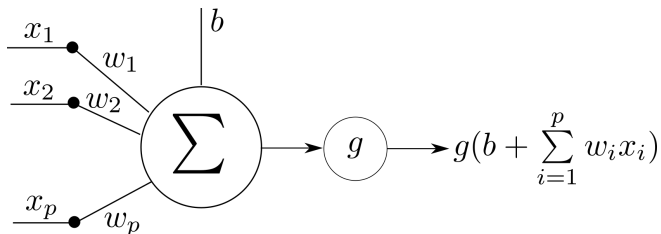


- The neuron computes a linear combination of the **inputs** x_i
 - The **weights** w_i are multiplied with the inputs
 - The **bias** b can be interpreted as a threshold on the sum
- The **activation function** g somehow decides, depending on its input, if a signal (the neuron's **activation**) is produced

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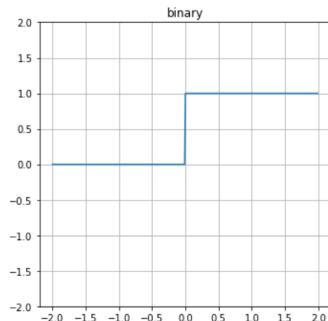
The role of the activation function



- The initial idea behind the activation function is that it works somehow as a gate
- If its input is “high enough”, then the neuron is activated, i.e. a signal (other than zero) is produced
- It can be interpreted as a source of abstraction: information considered as unimportant is ignored

Activation: binary

$$g(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

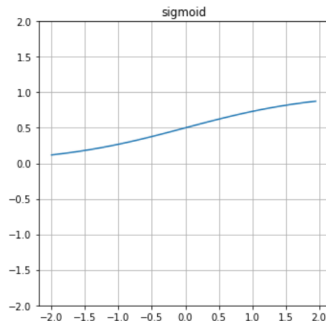


Remarks

- Biologically inspired
- + Simple to compute
- + High abstraction
 - Gradient nil except on one point
- In practice, almost never used

Activation: sigmoid

$$g(x) = \frac{1}{1 + e^{-x}}$$

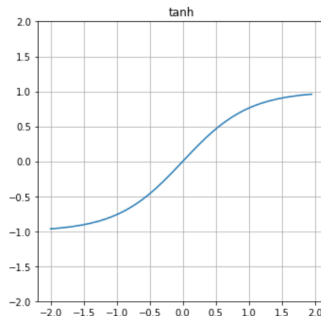


Remarks

- + Similar to binary activation, but with usable gradient
- However, gradient tends to zero when input is far from zero
- More computationally intensive

Activation: hyperbolic tangent

$$g(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

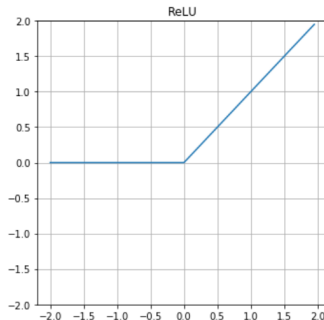


Remarks

- Similar to sigmoid

Activation: rectified linear unit

$$g(x) = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$



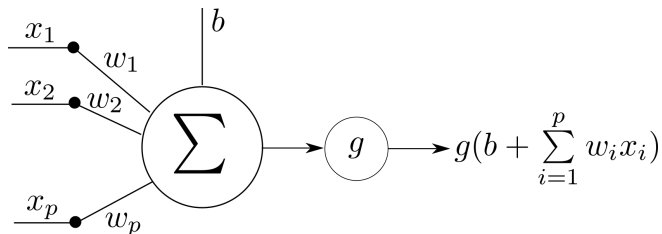
Remarks

- + Usable gradient when activated
- + Fast to compute
- + High abstraction

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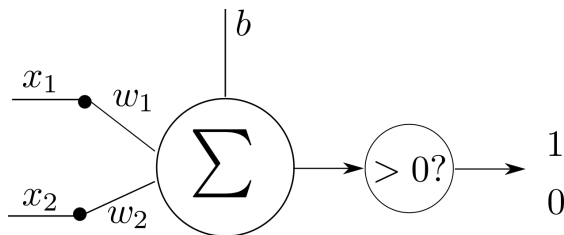
What can an artificial neuron compute?



In \mathbb{R}^p , $b + \sum_{i=1}^p w_i x_i = 0$ corresponds to a hyperplane. For a given point $\mathbf{x} = \{x_0, \dots, x_p\}$, decisions are made according to the side of the hyperplane it belongs to.

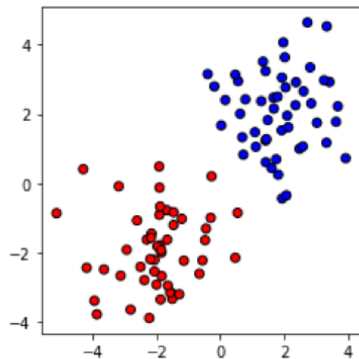
When the activation function is binary, we obtain a **perceptron**

Example of what we can do with a neuron

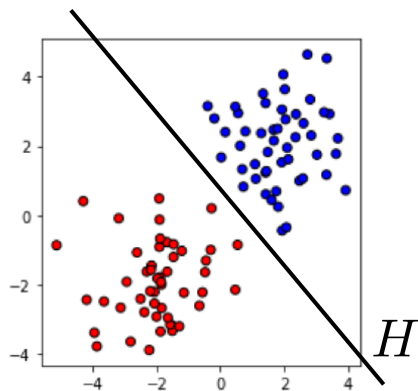


- $p = 2$: 2 dimensional inputs (can be represented on a screen!)
- Activation: binary
- Classification problem

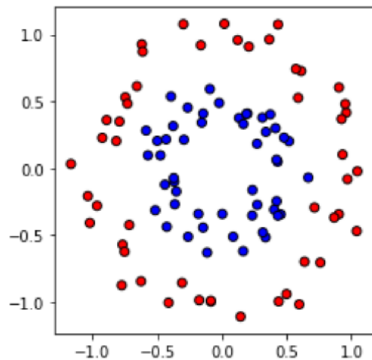
Gaussian clouds



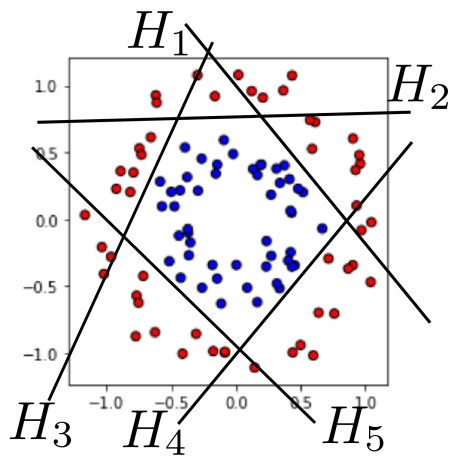
Gaussian clouds



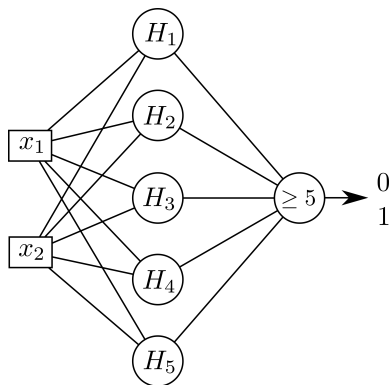
Circles



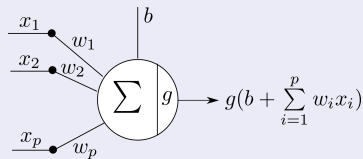
Circles



Solution



Artificial neuron compact representation



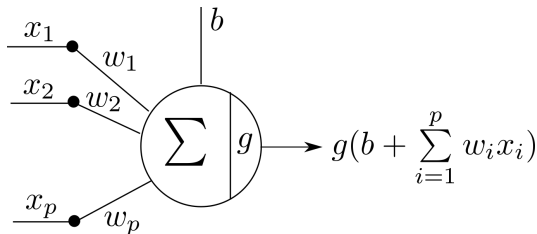
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Notations



With

$$\mathbf{w} = (w_1, \dots, w_p)^T$$

$$\mathbf{x} = (x_1, \dots, x_p)^T$$

We can simply write:

$$g(b + \sum_{i=1}^p w_i x_i) = g(b + \mathbf{w}^T \mathbf{x})$$

Neural network (NN)

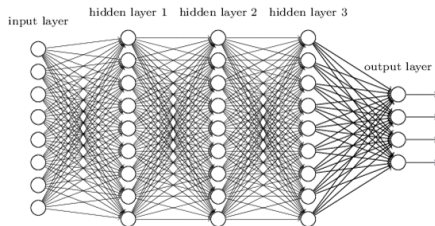
Definitions

- An (artificial) neural network is a directed graph, where:
 - the nodes are artificial neurons and
 - the edges are connections between the neurons.
- The **input layer** is the set of neurons without incoming edges.
- The **output layer** is the set of neurons without outgoing edges.

Feed-forward neural networks

Definition

- A feed-forward neural networks is a NN without cycles
- Neurons are organized in **layers**
 - A neuron belongs to layer q if the longest path in the graph between the input layer and the neuron is of length q .
- Any layers other than input and output layers are called **hidden layers**



(from <http://www.jtoy.net>)

Feed-forward neural networks

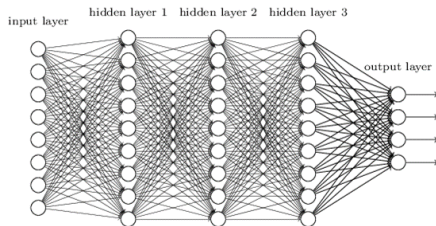
In the following of this course, except when otherwise specified, all NNs will be feed-forward. Indeed, this is the preferred type of NN for image processing.

What about other architectures?

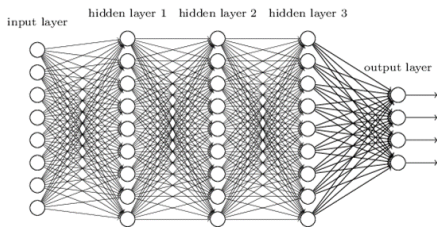
- Recurrent neural networks (RNN)
 - Long short-term memory networks (LSTM)
- + More powerful than feed-forward NNs
- Complex dynamics; more difficult to train
 - Mainly used for processing temporal data

Fully-connected network

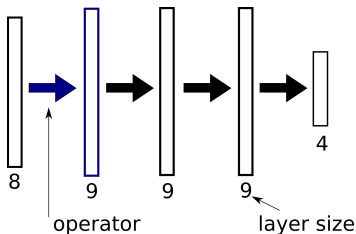
- A layer is said to be fully-connected (FC) if each of its neurons is connected to all the neurons of the previous and following layers
- If a FC layer contains r neurons, and the previous layer q , then its weights are 2D dimensional array (a matrix) of size $q \times r$
- A NN is said to be fully connected if all its hidden layers are fully connected



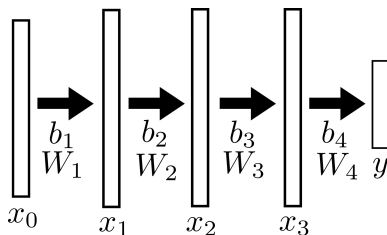
Graphical representation of NNs



- Data is organized into arrays, linked with operators
- A layer corresponds to an operator between arrays (and often an activation) as well as the resulting array.



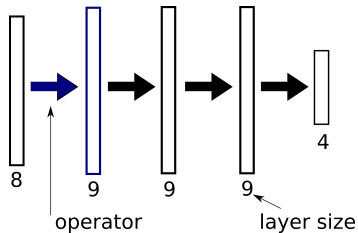
The equations of a fully connected neural network



$$\mathbf{x}_i = \mathbf{g}_i(\mathbf{W}_i \mathbf{x}_i + \mathbf{b}_i), \quad i = 1, 2, 3$$

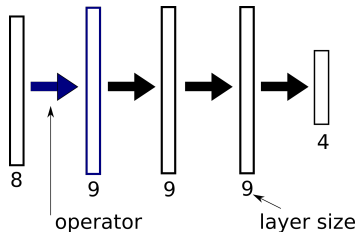
$$\mathbf{y} = \mathbf{g}_4(\mathbf{W}_4 \mathbf{x}_4 + \mathbf{b}_4)$$

Number of parameters



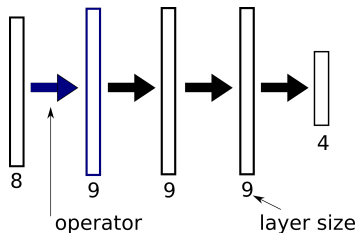
- How many parameters does the above network contain?

Number of parameters



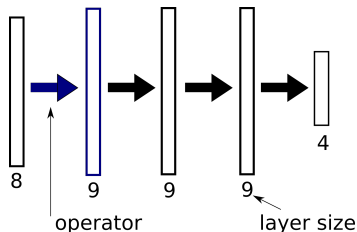
- How many parameters does the above network contain?
- First hidden layer:

Number of parameters



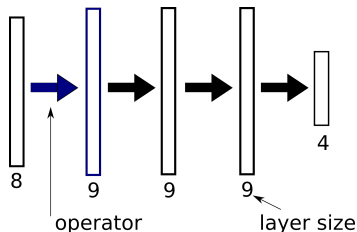
- How many parameters does the above network contain?
- First hidden layer:
 - $9 \text{ neurons} \times 8 \text{ neurons in the previous layer} + 9 \text{ biases} = 81$

Number of parameters



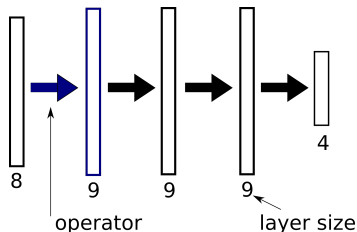
- How many parameters does the above network contain?
- First hidden layer:
 - $9 \text{ neurons} \times 8 \text{ neurons in the previous layer} + 9 \text{ biases} = 81$
- Second and third layers:

Number of parameters



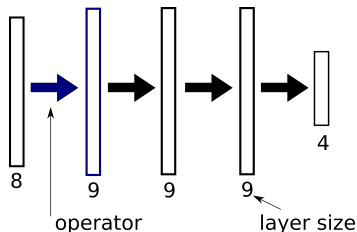
- How many parameters does the above network contain?
- First hidden layer:
 - $9 \text{ neurons} \times 8 \text{ neurons in the previous layer} + 9 \text{ biases} = 81$
- Second and third layers: $9 \times 9 + 9 = 90$

Number of parameters



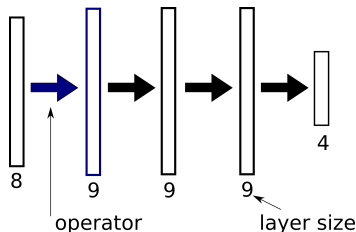
- How many parameters does the above network contain?
- First hidden layer:
 - $9 \text{ neurons} \times 8 \text{ neurons in the previous layer} + 9 \text{ biases} = 81$
- Second and third layers: $9 \times 9 + 9 = 90$
- Output layer:

Number of parameters



- How many parameters does the above network contain?
- First hidden layer:
 - $9 \text{ neurons} \times 8 \text{ neurons in the previous layer} + 9 \text{ biases} = 81$
- Second and third layers: $9 \times 9 + 9 = 90$
- Output layer: $4 \times 9 + 4$

Number of parameters



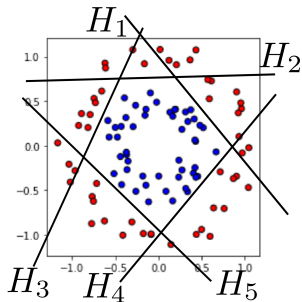
- How many parameters does the above network contain?
- First hidden layer:
 - $9 \text{ neurons} \times 8 \text{ neurons in the previous layer} + 9 \text{ biases} = 81$
- Second and third layers: $9 \times 9 + 9 = 90$
- Output layer: $4 \times 9 + 4$
- Total: 305 parameters

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Universal approximation theorem

- We have previously seen that a neuron can be used as a linear classifier and that combining several of them one can build complex classifiers
- We will see that this observation can be generalized



Universal approximation theorem

Let f be a **continuous** real-valued function of $[0, 1]^p$ ($p \in \mathbb{N}^*$) and ϵ a strictly positive real. Let g be a non-constant, increasing, bounded real function (*the activation function*).

Then there exist an integer n , real vectors $\{\mathbf{w}_i\}_{1 \leq n}$ of \mathbb{R}^p , and reals $\{b_i\}_{1 \leq n}$ and $\{v_i\}_{1 \leq n}$ such that for all \mathbf{x} in $[0, 1]^p$:

$$\left| f(\mathbf{x}) - \sum_{i=1}^n v_i g(\mathbf{w}_i^T \mathbf{x} + b_i) \right| < \epsilon$$

A first version of this theorem, using sigmoidal activation functions, was proposed by [Cybenko, 1989]. The version above was demonstrated by [Hornik, 1991].

Universal approximation theorem: what does it mean?

$$\left| f(\mathbf{x}) - \sum_{i=1}^n v_i g(\mathbf{w}_i^T \mathbf{x} + b_i) \right| < \epsilon$$

This means that function f can be approximated with a neural network containing:

- an input layer of size p ;
- a hidden layer containing n neurons with activation function g , weights \mathbf{w}_i and biases b_i ;
- an output layer containing a single neuron, with weights v_i (and an identity activation function).

Universal approximation theorem in practice

- The number of neurons increases very rapidly with the complexity of the function
- Empirical evidence has shown that multi-layer architectures give better results

Universal approximation theorem in practice

- The number of neurons increases very rapidly with the complexity of the function
- Empirical evidence has shown that multi-layer architectures give better results

A NN can potentially have a lot of parameters. How can we set them?

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Introduction

- We have seen that NNs have a lot of potential. However, how can the parameters $\theta = (\mathbf{W}_i, \mathbf{b}_i)$ be set?
- What is our objective ?
- A very general solution, that is also the mostly used, is **gradient descent**

Learning problem

We recall that our training set contains n samples:

$$(\mathbf{x}_i, y_i) \in \mathbb{R}^p \times \mathbb{R}$$

We **choose** a family f_{θ} of functions from \mathbb{R}^p into \mathbb{R} , depending on our set of parameters θ , and **find** the value of θ that minimizes a **chosen** loss function L :

$$\theta^* = \arg \min_{\theta} (L(\theta) + \mathcal{R}(\theta))$$

where $\mathcal{R}(\theta)$ is a regularization term.

For the time being, for the sake of simplicity, we will drop the regularization term until further notice

Loss function

A general form of the loss function is:

$$L(\boldsymbol{\theta}) = \sum_{i=1}^n d(y_i, f(\mathbf{x}_i, \theta))$$

where d is some disparity function (the more similar its parameters, the smaller its value).

Loss function: examples

Squared error

$$L(\boldsymbol{\theta}) = \sum_{i=1}^n (y_i - f(\mathbf{x}_i, \theta))^2$$

This loss function is mainly used in regression problems. However, it has also been used for binary classification problems.

Cross-entropy

In this case, $y_i \in \{0, 1\}$:

$$L(\boldsymbol{\theta}) = - \sum_{i=1}^n y_i \ln(f(\mathbf{x}_i, \theta))$$

This loss function is used in binary classification problems, where the network's output can be interpreted as a probability of belonging to a class.

Gradient descent

Definition

Gradient descent is an optimization algorithm. For a derivable function L , a positive real γ (the **learning rate**) and a starting point θ_0 , it computes a sequence of values:

$$\forall i \in \mathbb{N} : \theta_{i+1} = \theta_i - \gamma \nabla L(\theta_i)$$

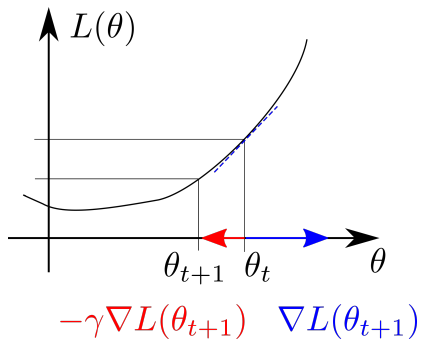
Property

If γ is small enough, then:

$$L(\theta_{i+1}) \leq L(\theta_i)$$

Gradient descent is an essential tool in optimization.

Gradient descent in the scalar case



$$\theta_{t+1} = \theta_t - \gamma \nabla L(\theta_t)$$

Gradient descent applied to neural networks

In the case of neural networks, the loss L depends on each parameter θ_i via the composition of several simple functions. In order to compute the gradient $\nabla_{\theta} L$ we will make extensive use of the chain rule theorem.

Chain rule theorem

Let f_1 and f_2 be two derivable real functions ($\mathbb{R} \rightarrow \mathbb{R}$). Then for all x in \mathbb{R} :

$$(f_2 \circ f_1)'(x) = f_2'(f_1(x)) \cdot f_1'(x)$$

Leibniz notation

Let us introduce variables x , y and z :

$$x \xrightarrow{f_1} y \xrightarrow{f_2} z$$

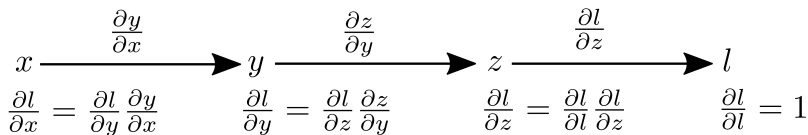
Then:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

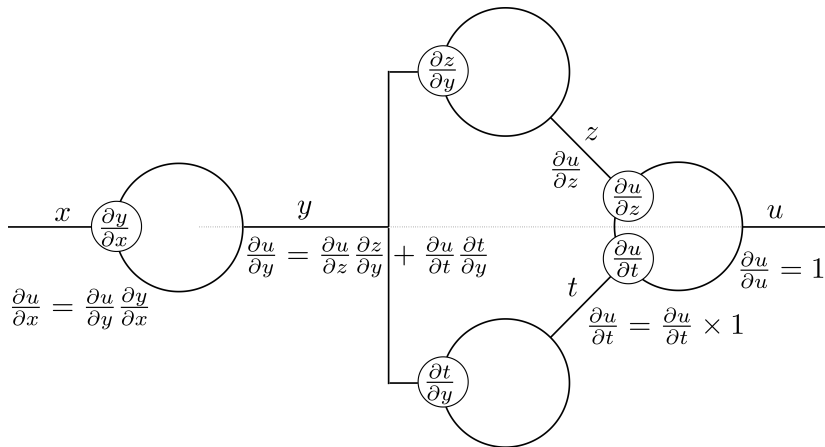
The backpropagation algorithm

- The backpropagation algorithm is used in a neural network to efficiently compute the partial derivative of the loss with respect to each parameter of the network.
- After a forward propagation, one can compute
- One can trace the origins of the method to the sixties
- It was first applied to NN in the eighties
[Werbos, 1982, LeCun, 1985]

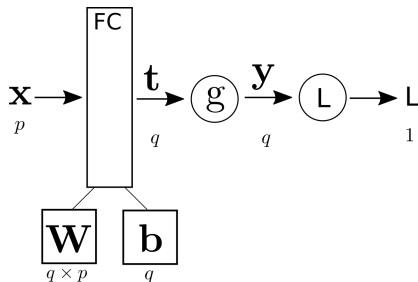
Simple backpropagation example



Simple backpropagation example



Backpropagation through a fully connected layer



Setup:

$$n, q \in \mathbb{N}^*$$

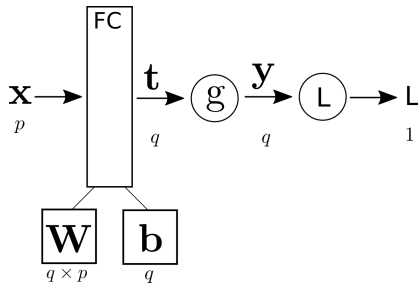
$$\mathbf{x} \in \mathbb{R}^n$$

$$\mathbf{W} \in \mathbb{R}^q \times \mathbb{R}^n$$

$$\mathbf{b}, \mathbf{t}, \mathbf{y} \in \mathbb{R}^q$$

$$L \in \mathbb{R}$$

Backpropagation through a fully connected layer



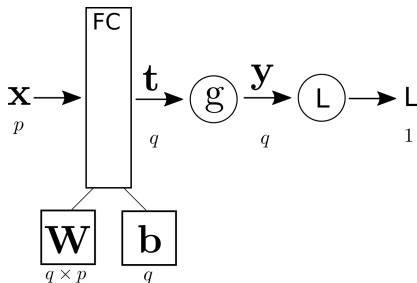
Local gradients:

Forward pass:

$$\begin{aligned}\mathbf{t} &= \mathbf{W}\mathbf{x} + \mathbf{b} \\ \mathbf{y} &= g(\mathbf{W}\mathbf{x} + \mathbf{b}) \\ L &= L(\mathbf{y})\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathbf{t}}{\partial \mathbf{W}} &= \mathbf{x}^t \\ \frac{\partial \mathbf{t}}{\partial \mathbf{b}} &= \mathbf{1} \\ \frac{\partial \mathbf{y}}{\partial \mathbf{t}} &= g'\end{aligned}$$

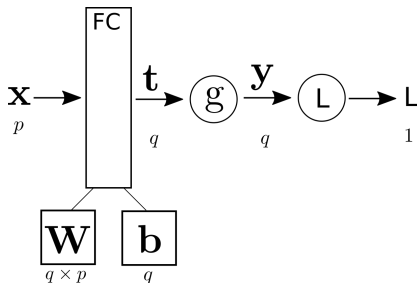
Backpropagation through a fully connected layer



Backpropagation:

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{t}} &= \frac{\partial L}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial \mathbf{t}} \\ &= \frac{\partial L}{\partial \mathbf{y}} \odot g'(\mathbf{t})\end{aligned}$$

Backpropagation through a fully connected layer



Backpropagation:

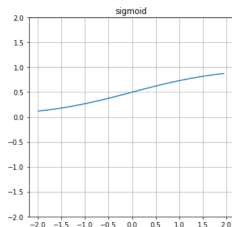
$$\begin{aligned}\frac{\partial L}{\partial \mathbf{W}} &= \frac{\partial L}{\partial \mathbf{t}} \cdot \frac{\partial \mathbf{t}}{\partial \mathbf{W}} \\ &= \frac{\partial L}{\partial \mathbf{y}} \odot \mathbf{g}'(\mathbf{t}) \cdot \mathbf{x}^t\end{aligned}$$

$$\frac{\partial L}{\partial \mathbf{b}} = \frac{\partial L}{\partial \mathbf{y}} \odot \mathbf{g}'(\mathbf{t})$$

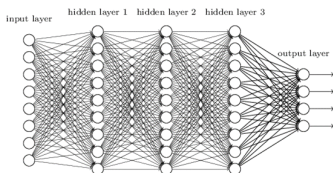
Network parameters initialization

General idea

Inputs of activation functions should be in an appropriate range (high gradient)



- If all parameters are initialized to zero, then in each layer the activations will remain equal – symmetry will never be broken
- Simple solution: random values from a normal or uniform distribution
- More advanced solutions exist: [LeCun et al., 2012, Glorot and Bengio, 2010, He et al., 2015]



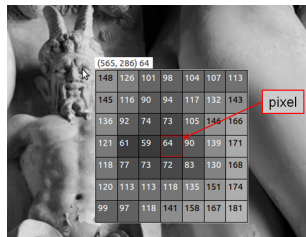
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Images

Definition

- Classically, an image is a matrix of values belonging to $[0, \dots, 255]$ (grey level images) or to $[0, \dots, 255]^3$ (color images).
- More generally, an image is a q -dimensional array of values belonging to R^d .



Conclusion

We have seen:

- What is an artificial neuron and an artificial neural network (NN)
- The (potential) power of a NN
- The backpropagation algorithm
- NN learning basics

In the following, we will see how to process images using NNs.

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