

# COMPLEX NUMBERS

(part 1)

# BASIC COMPLEX ALGEBRA

**Question 1**

Simplify the following complex number expressions, giving the final answer in the form  $a+bi$ , where  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ .

a)  $\frac{1}{1+2i} + \frac{1}{1-2i}$

b)  $5-4i + \frac{25}{3-4i}$

c)  $-1+3i + \frac{10}{-1+3i}$

d)  $\left(\frac{5+i}{2+3i}\right)^4$

$\boxed{\frac{2}{5}}$ ,  $\boxed{8}$ ,  $\boxed{-2}$ ,  $\boxed{-4}$

$$\begin{aligned} \text{(a)} \quad & \frac{1}{1+2i} + \frac{1}{1-2i} = \frac{(1-2i)+(1+2i)}{(1+2i)(1-2i)} = \frac{2}{1+2i-2i+4} = \frac{2}{5} \\ \text{(b)} \quad & 5-4i + \frac{25}{3-4i} = 5-4i + \frac{25(3+4i)}{(3-4i)(3+4i)} = 5-4i + \frac{25(3+4i)}{9+16} \\ & = 5-4i + 3+4i = 8 \\ \text{(c)} \quad & -1+3i + \frac{10}{-1+3i} = -1+3i + \frac{10(-1-3i)}{(-1+3i)(-1-3i)} = -1+3i + \frac{10(-1-3i)}{16} \\ & = -1+3i - \frac{10}{16} = -2 \\ \text{(d)} \quad & \left(\frac{5+i}{2+3i}\right)^4 = \left[\frac{(5+i)(2-3i)}{(2+3i)(2-3i)}\right]^4 = \left[\frac{10-15i+2i+3}{4+9}\right]^4 = \left[\frac{13-13i}{13}\right]^4 \\ & = (1-i)^4 = [(1-i)^2]^2 = [i-2i-1]^2 \\ & = [-2i]^2 = -4 \end{aligned}$$

**Question 2**

Find the value of  $x$  and the value of  $y$  in the following equation, given further that  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ .

$$(x+iy)(2+i) = 3-i.$$

$$(x, y) = (1, -1)$$

$$\begin{aligned}
 & \Rightarrow (x+iy)(2+i) = 3-i \\
 & \Rightarrow 2x+ix+iy^2-y = 3-i \\
 & \Rightarrow (2x-y)+i(x+iy) = 3-i \\
 & \text{EQUATE REAL AND IMAGINARY PARTS} \\
 & \left. \begin{aligned} 2x-y &= 3 \\ x+iy &= -1 \end{aligned} \right\} \Rightarrow \begin{aligned} 2x-3 &= y \\ x+2(-3-x) &= -1 \\ x+4x-6 &= -1 \\ 5x &= 5 \\ x &= 1 \end{aligned} \\
 & \text{Q. Thus if } x=2-3 \\
 & \Rightarrow y=-1 \\
 & \text{ACCURACY} \\
 & \Rightarrow (x+iy)(2+i) = 3-i \\
 & \Rightarrow x+iy = \frac{3-i}{2+i} \\
 & \Rightarrow x+iy = \frac{(3-i)(2-i)}{(2+i)(2-i)} \\
 & \Rightarrow x+iy = \frac{6-3i-2i+i^2}{4-2i+2i-i^2} \\
 & \Rightarrow x+iy = \frac{5-5i}{5} \\
 & \Rightarrow x+iy = 1-i \quad \therefore x=1 \text{ & } y=-1
 \end{aligned}$$

**Question 3**

Find the value of  $x$  and the value of  $y$  in the following equation, given further that  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ .

$$(x+iy)(3+4i) = 3-4i.$$

$$(x, y) = \left( -\frac{7}{25}, -\frac{24}{25} \right)$$

$$\begin{aligned}
 & \Rightarrow (x+iy)(3+4i) = 3-4i \\
 & \Rightarrow x+iy = \frac{3-4i}{3+4i} \\
 & \Rightarrow x+iy = \frac{(3-4i)(3-4i)}{(3+4i)(3-4i)} \\
 & \Rightarrow x+iy = \frac{9-12i-12i+16i^2}{9+12i-12i+16} \\
 & \Rightarrow x+iy = \frac{-7-24i}{25} \\
 & \Rightarrow x+iy = -\frac{7}{25} - \frac{24}{25}i \\
 & \therefore x = -\frac{7}{25} \quad \text{A} \neq 0 \quad y = -\frac{24}{25}
 \end{aligned}$$

**Question 4**

Find the value of  $x$  and the value of  $y$  in the following equation, given further that  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ .

$$\frac{1}{x+iy} - \frac{1}{1+i} = 2-3i.$$

$$(x, y) = \left( \frac{5}{37}, \frac{7}{37} \right)$$

METHOD: ALGEBRAIC

$$\begin{aligned}
 & \Rightarrow \frac{1}{x+iy} - \frac{1}{1+i} = 2-3i \\
 & \Rightarrow \frac{1}{x+iy} - \frac{(1-i)}{(1+i)(1-i)} = 2-3i \\
 & \Rightarrow \frac{1}{x+iy} - \frac{1-i}{2} = 2-3i \\
 & \Rightarrow \frac{1}{x+iy} - \frac{1}{2} + \frac{i}{2} = 2-3i \\
 & \Rightarrow \frac{2}{x+iy} - i + i = 4-6i \\
 & \Rightarrow \frac{2}{x+iy} = 5-7i \\
 & \Rightarrow \frac{x+iy}{2} = \frac{1}{5-7i} \\
 & \Rightarrow \frac{x+iy}{2} = \frac{5+7i}{(5-7i)(5+7i)} \\
 & \Rightarrow \frac{x+iy}{2} = \frac{5+7i}{25+49} \\
 & \Rightarrow \frac{x+iy}{2} = \frac{1}{74}(5+7i) \\
 & \Rightarrow \underline{\underline{x+iy}} = \frac{1}{74}(5+7i)
 \end{aligned}$$

∴  $x = \frac{5}{74}$ ,  $y = \frac{7}{74}$

**Question 5**

Find the value of  $x$  and the value of  $y$  in the following equation, given further that  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ .

$$\frac{1}{x+iy} + \frac{1}{1+2i} = 1.$$

$$(x, y) = \left(1, -\frac{1}{2}\right)$$

TRY UP TO FOCUS

$$\begin{aligned}
 & \Rightarrow \frac{1}{x+iy} + \frac{1}{1+2i} = 1 \\
 & \Rightarrow \frac{1}{x+iy} = 1 - \frac{1}{1+2i} \\
 & \Rightarrow \frac{1}{x+iy} = 1 - \frac{-i}{(1+2i)(1-2i)} \\
 & \Rightarrow \frac{1}{x+iy} = 1 - \frac{1-2i}{5} \\
 & \Rightarrow \frac{5}{x+iy} = 5 - (1-2i) \\
 & \Rightarrow \frac{5}{x+iy} = 4+2i \\
 & \Rightarrow \frac{x+iy}{5} = \frac{1}{4+2i} \\
 & \Rightarrow \frac{1}{5}(x+iy) = \frac{4-2i}{(4+2i)(4-2i)} \\
 & \Rightarrow \frac{1}{5}(x+iy) = \frac{4-2i}{16+16} \\
 & \Rightarrow \frac{1}{5}(x+iy) = \frac{4-2i}{32} \\
 & \Rightarrow \frac{1}{5}(x+iy) = \frac{1}{8} - \frac{1}{16}i \\
 & \Rightarrow x+iy = 1 - \frac{1}{2}i
 \end{aligned}$$

... 3 = 1  
 ...  $\frac{1}{2}i = \frac{1}{2}i$

**Question 6**

Find the value of  $x$  and the value of  $y$  in the following equation, given further that  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ .

$$\frac{x}{1+i} = \frac{1-5i}{3-2i} + \frac{y}{2-i}.$$

$$(x, y) = (2, 0)$$

Multiplying as follows

$$\begin{aligned} \Rightarrow \frac{x}{1+i} &= \frac{1-5i}{3-2i} + \frac{y}{2-i} \\ \Rightarrow \frac{x(1-i)}{(1+i)(1-i)} &= \frac{(1-5i)(3+2i)}{(3-2i)(3+2i)} + \frac{y(2+i)}{(2-i)(2+i)} \\ \Rightarrow \frac{x(1-i)}{2} &= \frac{3+2i-15i-10i^2}{9+4} + \frac{y(2+i)}{4+1} \\ \Rightarrow \frac{x(1-i)}{2} &= \frac{13-13i}{13} + \frac{y(2+i)}{5} \\ \Rightarrow \frac{x(1-i)}{2} &= -i + \frac{y(2+i)}{5} \\ \Rightarrow 5x(1-i) &= 10 - 10i + 2y(2+i) \\ \Rightarrow 5x - 5xi &= 10 - 10i + 4y + 2yi \\ \Rightarrow 5x - 5xi &= (10 - 4y) + (2y - 10)i \end{aligned}$$

Equating real & imaginary parts

$$\begin{aligned} 5x &= 10 + 4y \quad \text{and} \quad 0 = 2y \\ -5x &= 2y - 10 \quad \Rightarrow \quad 0 = 2y \\ &\Rightarrow \underline{\underline{y = 0}} \\ &\text{and hence } \underline{\underline{x = 2}} \end{aligned}$$

**Question 7**

Find the square roots of the following complex numbers.

a)  $15 + 8i$

b)  $16 + 30i$

Give the answers in the form  $a + bi$ , where  $a \in \mathbb{R}$  and  $b \in \mathbb{R}$ .

$\boxed{\pm(4+i)}, \boxed{\pm(5+3i)}$

<p>a) <math>z^2 = 15 + 8i</math></p> <p>• let <math>z = a+bi</math>, <math>a \in \mathbb{R}, b \in \mathbb{R}</math></p> $\Rightarrow (2a)^2 = 15 + 8i$ $\Rightarrow a^2 + 2abi - b^2 = 15 + 8i$ $\text{Then } \begin{cases} a^2 - b^2 = 15 \\ 2ab = 8 \end{cases} \Rightarrow \boxed{b = \frac{4}{a}}$ $\Rightarrow a^2 - \left(\frac{4}{a}\right)^2 = 15$ $\Rightarrow a^2 - \frac{16}{a^2} = 15$ $\Rightarrow a^4 - 15a^2 = 0$ $\Rightarrow a^2(a^2 - 15) = 0$ $\Rightarrow (a^2 - 16)(a^2 + 1) = 0$ $a^2 = \begin{cases} 16 \\ -1 \end{cases}$ $a = \begin{cases} 4 \\ -4 \end{cases} \Rightarrow b = \begin{cases} 1 \\ -1 \end{cases}$ $\therefore z_1 = 4+i$ $z_2 = -4-i$	<p>b) <math>z^2 = 16 + 30i</math></p> <p>• let <math>z = a+bi</math></p> $\Rightarrow (a+bi)^2 = 16 + 30i$ $\Rightarrow a^2 + 2abi - b^2 = 16 + 30i$ $\text{Then } \begin{cases} a^2 - b^2 = 16 \\ 2ab = 30 \end{cases} \Rightarrow \boxed{b = \frac{15}{a}}$ $\Rightarrow a^2 - \left(\frac{15}{a}\right)^2 = 16$ $\Rightarrow a^2 - \frac{225}{a^2} = 16$ $\Rightarrow a^4 - 16a^2 - 225 = 0$ $\Rightarrow (a^2 + 9)(a^2 - 25) = 0$ $a^2 = \begin{cases} 25 \\ -9 \end{cases}$ $a = \begin{cases} 5 \\ -5 \end{cases} \quad b = \begin{cases} 3 \\ -3 \end{cases}$ $\therefore z_1 = 5+3i$ $z_2 = 5-3i$
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**Question 8**

Solve the following equation.

$$z^2 = 21 - 20i, \quad z \in \mathbb{C}.$$

Give the answers in the form  $a + bi$ , where  $a \in \mathbb{R}$  and  $b \in \mathbb{R}$ .

$$z = \pm(5 - 2i)$$

LET  $z = a + bi$ , WHERE  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$

$$\Rightarrow z^2 = 21 - 20i$$
$$\Rightarrow (a+bi)^2 = 21 - 20i$$
$$\Rightarrow a^2 + 2ab i - b^2 = 21 - 20i$$

EQUATE REAL AND IMAGINARY PARTS

$$\begin{cases} a^2 - b^2 = 21 \\ 2ab = -20 \end{cases} \Rightarrow \boxed{b = -\frac{10}{a}}$$
$$\Rightarrow a^2 - \left(\frac{10}{a}\right)^2 = 21$$
$$\Rightarrow a^2 - \frac{100}{a^2} = 21$$
$$\Rightarrow a^4 - 100 = 21a^2$$
$$\Rightarrow a^4 - 21a^2 - 100 = 0$$
$$\Rightarrow (a^2 + 4)(a^2 - 25) = 0$$
$$\Rightarrow a^2 = 25 \quad a \in \mathbb{R}$$
$$\Rightarrow a = \sqrt{25} \quad a \in \mathbb{R}$$
$$\Rightarrow a = 5 \quad a \in \mathbb{R}$$
$$\Rightarrow b = -\frac{10}{5} = -2$$
$$\therefore z = \sqrt{25} - 2i$$
$$\therefore z = 5 - 2i$$

**Question 9**

Solve the following equation.

$$w^2 = 5 - 12i, \quad w \in \mathbb{C}.$$

Give the answers in the form  $a + bi$ , where  $a \in \mathbb{R}$  and  $b \in \mathbb{R}$ .

$$w = \pm(3 - 2i)$$

LET  $w = a+bi$ , where  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$

$$\Rightarrow w^2 = 5 - 12i$$
$$\Rightarrow (a+bi)^2 = 5 - 12i$$
$$\Rightarrow a^2 + 2ab - b^2 = 5 - 12i$$
$$\Rightarrow (a^2 - b^2) + i(2ab) = 5 - 12i$$

EQUATE REAL AND IMAGINARY PARTS

$$\begin{cases} a^2 - b^2 = 5 \\ 2ab = -12 \end{cases} \Rightarrow b = -\frac{5}{a}$$
$$\Rightarrow a^2 - (-\frac{5}{a})^2 = 5$$
$$\Rightarrow a^2 - \frac{25}{a^2} = 5$$
$$\Rightarrow a^4 - 36 = 5a^2$$
$$\Rightarrow a^4 - 5a^2 - 36 = 0$$
$$\Rightarrow (a^2 + 4)(a^2 - 9) = 0$$
$$\Rightarrow a^2 = 9 \quad a \in \mathbb{R}$$
$$\Rightarrow a = 3 \quad b = -2$$

$\therefore z = \sqrt{3-2i}$

**Question 10**

Find the square roots of  $1+i\sqrt{3}$ .

Give the answers in the form  $a+bi$ , where  $a \in \mathbb{R}$  and  $b \in \mathbb{R}$ .

$$\pm \frac{1}{2}(\sqrt{6} + i\sqrt{2})$$

Let  $z^2 = 1+i\sqrt{3}$ , where  $z = a+bi$ ,  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$

$$(a+bi)^2 = 1+i\sqrt{3}$$
$$a^2 + 2ab + b^2 = 1+i\sqrt{3}$$
$$(a^2 - b^2) + (2ab)i = 1+i\sqrt{3}$$

SOLVE REAL AND IMAGINARY PARTS

$$\begin{cases} a^2 - b^2 = 1 \\ 2ab = \sqrt{3} \end{cases} \Rightarrow b = \frac{\sqrt{3}}{2a}$$
$$\Rightarrow a^2 - \left(\frac{\sqrt{3}}{2a}\right)^2 = 1$$
$$\Rightarrow a^2 - \frac{3}{4a^2} = 1$$
$$\Rightarrow 4a^4 - 3 = 4a^2$$
$$\Rightarrow 4a^4 - 4a^2 - 3 = 0$$
$$\Rightarrow (2a^2 - 3)(2a^2 + 1) = 0$$
$$\Rightarrow a^2 = \frac{3}{2} \quad a \in \mathbb{R}$$
$$\Rightarrow a = \pm \sqrt{\frac{3}{2}} = \pm \sqrt{\frac{6}{4}} = \pm \frac{\sqrt{6}}{2}$$
$$\Rightarrow 2a = \pm \sqrt{6}$$
$$\Rightarrow \frac{1}{2a} = \pm \frac{1}{\sqrt{6}} = \pm \frac{\sqrt{6}}{6}$$
$$\Rightarrow b = \pm \frac{\sqrt{6} \times \sqrt{6}}{2} = \pm \frac{\sqrt{36}}{2} = \pm \frac{6}{2} = \pm 3$$
$$\therefore \frac{\sqrt{6}}{2} + i\frac{\sqrt{6}}{2} \quad \text{or} \quad -\frac{\sqrt{6}}{2} - i\frac{\sqrt{6}}{2}$$

**Question 11**

Solve the equation

$$2z^2 - 2iz - 5 = 0, \quad z \in \mathbb{C}.$$

$$z = \pm \frac{3}{2} + \frac{1}{2}i$$

$$\begin{aligned} 2z^2 - 2iz - 5 &= 0 \\ \text{By quadratic formula} \\ z &= \frac{2i \pm \sqrt{(-2i)^2 - 4 \times 2 \times (-5)}}{2 \times 2} = \frac{2i \pm \sqrt{-4 + 40}}{4} \\ z &= \frac{2i \pm 6}{4} = \frac{1}{2}i \pm \frac{3}{2} = \pm \frac{3}{2} + \frac{1}{2}i \end{aligned}$$

**Question 12**

$$z - 8 = i(7 - 2\bar{z}), \quad z \in \mathbb{C}.$$

The complex conjugate of  $z$  is denoted by  $\bar{z}$ .

Determine the value of  $z$  in the above equation, giving the answer in the form  $x+iy$ , where  $x$  and  $y$  are real numbers.

$$z = 2 + 3i$$

$$\begin{aligned} \text{Let } z &= x+iy, \quad \bar{z} = x-iy \\ \bullet \quad x+iy-8 &= i(7-2(x-iy)) \\ \Rightarrow (x-8)+iy &= i(7-2x+2iy) \\ \Rightarrow (x-8)+iy &= (7-2x)i-2y \end{aligned} \quad \left. \begin{array}{l} \text{This} \\ \{ \end{array} \right. \begin{aligned} x-8 &= -2y \\ y &= 7-2x \\ x-8 &= -2(7-2x) \\ x-8 &= -14+4x \\ 6 &= 3x \\ x &= 2 \\ y &= 3 \\ \therefore z &= 2+3i \end{aligned}$$

**Question 13**

The complex conjugate of  $z$  is denoted by  $\bar{z}$ .

Solve the equation

$$z - 12 = i(9 - 2\bar{z}),$$

giving the answer in the form  $x + iy$ , where  $x$  and  $y$  are real numbers.

$$\boxed{z = 2 + 5i}$$

$z - 12 = i(9 - 2\bar{z})$ <ul style="list-style-type: none"> <li>• Let <math>z = x+iy</math></li> <li><math>\Rightarrow x+iy - 12 = i((9 - 2(x-y))</math></li> <li><math>\Rightarrow x+iy - 12 = i(9 - 2x + 2y)</math></li> <li><math>\Rightarrow x+iy - 12 = 9i - 2xi + 2yi</math></li> <li><math>\Rightarrow (x-12) + iy = -2y + i(9-2x)</math></li> <li><math>\Rightarrow (x-12) + iy = -2y + i(9-2x)</math></li> </ul> $\begin{cases} x-12 = -2y \\ y = 9-2x \end{cases}$	<small>#METHOD</small> $\begin{aligned} z - 12 &= -2(9-2x) \\ \Rightarrow 12 &= -18 + 4x \\ 6 &= 3x \\ \boxed{x=2} \\ \boxed{y=5} \\ \therefore z &= 2 + 5i \end{aligned}$
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**Question 14**

The complex number  $z$  satisfies the equation

$$2z - i\bar{z} = 3(3 - 5i),$$

where  $\bar{z}$  denotes the complex conjugate of  $z$ .

Determine the value of  $z$ , giving the answer in the form  $x + iy$ , where  $x$  and  $y$  are real numbers.

$$\boxed{z = 1 - 7i}$$

$2z - i\bar{z} = 3(3 - 5i)$ <p style="text-align: center;"><math>\boxed{\text{LET } z = x+iy}</math></p> $2(x+iy) - i(x-y) = 9 - 15i$ $2x+2iy - ix + iy = 9 - 15i$ $(2x-y) + i(2y+x) = 9 - 15i$	<p style="text-align: center;"><small>SEPARATE REAL AND IMAGINARY</small></p> $\begin{aligned} 2x-y &= 9 \\ 2y+x &= 15 \end{aligned} \Rightarrow \boxed{\begin{aligned} 2x-y &= 9 \\ 2y-x &= 15 \end{aligned}}$ $\begin{aligned} \text{So, } 2(2x-y)-2x &= 15 \\ 4x-2x-2y &= 15 \\ 2x-2y &= 15 \\ 2x &= 3 \\ \text{So, } y &= -7 \\ \therefore z &= x+iy \\ z &= 1-7i \end{aligned}$
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**Question 15**

Find the value of  $z$  and the value of  $w$  in the following simultaneous equations

$$2z + 1 = -iw$$

$$z - 3 = w + 3i.$$

$$z = -1 + 2i, w = -4 - i$$

$$\begin{aligned}
 2z + 1 &= -iw \quad \Rightarrow \quad 2z = -1 - iw \\
 z - 3 &= w + 3i \quad \Rightarrow \quad z = 3 + w + 3i
 \end{aligned}$$

$$\begin{aligned}
 -1 - iw &= 2(3 + w + 3i) \\
 -1 - iw &= 6 + 2w + 6i \\
 -1 - 6i &= 2w + iw \\
 -7 - 6i &= w(2+i) \\
 w &= \frac{-7-6i}{2+i} \\
 w &= \frac{(-7-6i)(2-i)}{(2+i)(2-i)} \\
 w &= \frac{-14+7i-12i+6}{5} \\
 w &= \frac{-20-5i}{5} \\
 w &= -4 - i
 \end{aligned}$$

TMS  
 $z = 3 + w + 3i$   
 $z = 3 - 4 - i + 3i$   
 $z = -1 + 2i$

**Question 16**

Solve the equation

$$\frac{13z}{z+1} = 11 - 3i, \quad z \in \mathbb{C},$$

giving the answer in the form  $x + iy$ , where  $x$  and  $y$  are real numbers.

$$z = 1 - 3i$$

<u>METHOD A</u> $  \begin{aligned}  \Rightarrow \frac{13z}{z+1} &= 11 - 3i \\  \Rightarrow 13z &= (11-3i)(z+1) \\  \Rightarrow 13z &= (11-3i)z + (11-3i) \\  \Rightarrow 2z + 3i &= 11 - 3i \\  \Rightarrow 2(z+3i) &= 11 - 3i \\  \Rightarrow z &= \frac{11-3i}{2+3i} \\  \Rightarrow z &= \frac{(11-3i)(2-3i)}{(2+3i)(2-3i)} \\  \Rightarrow z &= \frac{22 - 35i - 6i - 9}{4+9} \\  \Rightarrow z &= \frac{13 - 39i}{13} \\  \Rightarrow z &= 1 - 3i  \end{aligned}  $	<u>METHOD B</u> $  \begin{aligned}  \Rightarrow \frac{13z}{z+1} &= 11 - 3i \\  \Rightarrow \frac{z+1}{13z} &= \frac{1}{11-3i} \\  \Rightarrow \frac{z+1}{z} &= \frac{1}{11-3i} \\  \Rightarrow 1 + \frac{1}{z} &= \frac{1}{11-3i} \\  \Rightarrow \frac{1}{z} &= \frac{1}{11-3i} - 1 \\  \Rightarrow z &= \frac{1}{\frac{1}{11-3i} - 1} \\  \text{MULTIPLY TOP & BOTTOM OF THE FRACTION BY } 11-3i \\  \Rightarrow z &= \frac{11-3i}{(11-3i)(\frac{1}{11-3i} - 1)} = \frac{11-3i}{2+3i} \\  \text{"CONJUGATE TO REMOVE i TO GET} \\  z &= 1 - 3i  \end{aligned}  $
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**Question 17**

$$z - 8 = i(7 - 2\bar{z}), z \in \mathbb{C}.$$

The complex conjugate of  $z$  is denoted by  $\bar{z}$ .

Determine the value of  $z$  in the above equation, giving the answer in the form  $x + iy$ , where  $x$  and  $y$  are real numbers.

$$\boxed{z = 2 + 3i}$$

$$\begin{aligned}
 & \text{Let } z = x + iy, \bar{z} = x - iy \\
 & \bullet \quad x + iy - 8 = i(7 - 2(x - iy)) \\
 & \Rightarrow (x - 8) + iy = i(7 - 2x + 2iy) \\
 & \Rightarrow (x - 8) + iy = (1 - 2i)x - 2y \\
 & \left. \begin{array}{l} \text{Thus } x - 8 = -2y \\ y = 7 - 2x \end{array} \right\} \\
 & \begin{array}{l} x - 8 = -2(7 - 2x) \\ x - 8 = -14 + 4x \\ 6 = 3x \\ x = 2 \\ y = 3 \end{array} \\
 & \therefore z = 2 + 3i
 \end{aligned}$$

**Question 18**

The complex conjugate of  $w$  is denoted by  $\bar{w}$ .

Given further that

$$w = 1 + 2i \text{ and } z = w - \frac{25\bar{w}}{w^2},$$

show clearly that  $z$  is a real number, stating its value.

[12]

$$\begin{aligned}
 z &= w - \frac{25\bar{w}}{w^2} = (1+2i) - \frac{25(1-2i)}{(1+2i)^2} = 1+2i - \frac{25(1-2i)}{(1+4i-4)} \\
 &= 1+2i - \frac{25(1-2i)}{-3+4i} = 1+2i - \frac{25(1-2i)(-3-4i)}{(-3+4i)(-3-4i)} \\
 &= 1+2i - \frac{25(-3+4i+6i-8)}{9+16} = 1+2i - \frac{25(-1+2i)}{25} \\
 &= 1+2i + 1-2i = 12. \quad \text{It R.M.C.}
 \end{aligned}$$

**Question 19**

The complex number  $z$  satisfies the equation

$$4z - 3\bar{z} = \frac{1-18i}{2-i},$$

where  $\bar{z}$  denotes the complex conjugate of  $z$ .

Solve the equation, giving the answer in the form  $x+iy$ , where  $x$  and  $y$  are real numbers.

$$\boxed{z = 4-i}$$

$$\begin{aligned} 4z - 3\bar{z} &= \frac{1-18i}{2-i} \\ \text{Let } z &= x+iy \\ \bar{z} &= x-iy \\ \Rightarrow 4(x+iy) - 3(x-iy) &= \frac{(1-18i)(2+i)}{(2-i)(2+i)} \\ \Rightarrow 4x + 4iy - 3x + 3iy &= \frac{2+4i - 36i + 18}{4+1} \\ &= 2x + 7iy = \frac{20-35i}{5} \\ &\Rightarrow x + 7iy = 4 - 7i \\ \therefore x &= 4 \\ y &= -1 \\ \therefore z &= 4-i \end{aligned}$$

**Question 20**

The complex conjugate of  $z$  is denoted by  $\bar{z}$ .

Solve the equation

$$\frac{2z + 3i(\bar{z} + 2)}{1+i} = 13 + 4i,$$

giving the answer in the form  $x+iy$ , where  $x$  and  $y$  are real numbers.

$$\boxed{z = 3+i}$$

$$\begin{aligned} \text{Let } z &= x+iy \\ \bar{z} &= x-iy \\ \Rightarrow \frac{2z + 3i(\bar{z} + 2)}{1+i} &= 13 + 4i \\ \Rightarrow 2z + 3i(\bar{z} + 2) &= (1+i)(13+4i) \\ \Rightarrow 2(x+iy) + 3i(x-iy+2) &= 13+4i+17i-4 \\ \Rightarrow 2x+2iy + 3ix+3y+6i &= 9+17i \\ \Rightarrow (2x+3y) + i(2y+3x+6) &= 9+17i \\ &= (2x+3y) + i(2y+3x) = 9+11i \end{aligned}$$

$$\begin{aligned} 2x+3y &= 9 && \times 3 \\ 2y+3x &= 11 && \times (-2) \\ 6x+9y &= 27 && \text{Add} \\ -6x-6y &= -22 && \\ 3y &= 5 && \\ y &= \frac{5}{3} && \boxed{y=\frac{5}{3}} \\ \text{And } 2x+3y &= 9 \\ 2x+3 &= 9 \\ 2x &= 6 \\ x &= 3 && \boxed{x=3} \\ \therefore z &= 3+i \end{aligned}$$

**Question 21**

The complex conjugate of  $z$  is denoted by  $\bar{z}$ .

Find the two solutions of the equation

$$(z-i)(\bar{z}-i) = 6z - 22i, \quad z \in \mathbb{C},$$

giving the answers in the form  $x+iy$ , where  $x$  and  $y$  are real numbers.

$$z_1 = 2 + 3i, \quad z_2 = \frac{28}{5} + \frac{9}{5}i$$

$$\begin{aligned} (z-i)(\bar{z}-i) &= 6z - 22i \\ 2\bar{z} - i\bar{z} - \bar{z}^2 + i - (6z - 22i) &= 0 \\ |\bar{z}|^2 - i(\bar{z}+2) - 1 &= 6(2+iy) - 22i \\ (\bar{z}^2+1) - i(\bar{z}+2) - 1 &= 6(2+iy) - 22i \\ (\bar{z}^2+1)^2 - i(\bar{z}+2) + i(\bar{z}+2) - 1 &= 6(2+iy) - 22i \\ \bar{z}^2 + 1 - 6z - 1 &= 0 \\ 2\bar{z} - 6y - 2z &= 0 \\ 2z - 6y &= 0 \\ z &= 3y \end{aligned}$$

$$\begin{aligned} \Rightarrow (5y-9)(y-3) &= 0 \\ \Rightarrow y &= \frac{9}{5} \\ \therefore 3z &< \frac{2}{\frac{9}{5}} \\ \therefore z &= 2+3i \\ \bar{z} &= \frac{28}{5} + \frac{9}{5}i \end{aligned}$$

**Question 22**

The complex conjugate of  $z$  is denoted by  $\bar{z}$ .

Solve the equation

$$2z - 3\bar{z} = \frac{-27 + 23i}{1+i},$$

giving the answer in the form  $x+iy$ , where  $x$  and  $y$  are real numbers.

$$z = 2 + 5i$$

$$\begin{aligned} 2z - 3\bar{z} &= \frac{-27 + 23i}{1+i} \\ \text{Let } z = x+iy & \\ \Rightarrow 2(x+iy) - 3(x-iy) &= \frac{(-27 + 23i)(1-i)}{(1+i)(1-i)} \\ \Rightarrow 2x + 2iy - 3x + 3iy &= \frac{-27 + 23i + 27i - 23}{1+i} \\ \Rightarrow -x + 5iy &= -2 + 25i \end{aligned}$$

$$\begin{aligned} \Rightarrow -x + 5iy &= \frac{-4 + 50i}{2} \\ \Rightarrow -x + 5iy &= -2 + 25i \\ \therefore x &= 2 \\ y &= 5 \\ \therefore z &= 2 + 5i \end{aligned}$$

**Question 23**

Find the three solutions of the equation

$$4z^2 + 4\bar{z} + 1 = 0, \quad z \in \mathbb{C},$$

where  $\bar{z}$  denotes the complex conjugate of  $z$ .

$$z = \frac{1}{2}, \frac{1}{2} + i, \frac{1}{2} - i$$

**Question 24**

Solve the following equations.

a)  $z^2 + 2iz + 8 = 0, \quad z \in \mathbb{C}.$

b)  $w^2 + 16 = 30i, \quad w \in \mathbb{C}.$

$$z_1 = 2i, \quad z_2 = -4i, \quad w = \pm(3 + 5i)$$

**Question 25**

It is given that

$$z + 2i = iz + k, \quad k \in \mathbb{R} \quad \text{and} \quad \frac{w}{z} = 2 + 2i, \quad \operatorname{Im} w = 8.$$

Determine the value of  $k$ .

$$\boxed{k = 4}$$

$$\begin{aligned} \bullet z+2i &= iz+k \\ z-i^2 &= k-2i \\ z(i-1) &= k-2i \\ \boxed{z = \frac{k-2i}{i-1}} & \\ \bullet w &= z(z+2i) \\ w &= \frac{k-2i}{i-1}(z+2i) \\ w &= (k-2i) \times \frac{2(i+1)}{i-1} \\ w &= (k-2i) \times \frac{2(i+1)(i+1)}{(i-1)(i+1)} \\ w &= (k-2i) \times \frac{2(i^2+2i+1)}{i^2-1} \\ w &= (k-2i) \times \frac{2i^2+4i+2}{i^2-1} \\ w &= (k-2i)2i \\ w &= 2ki+4 \\ |w|w = 8 &\Rightarrow 2k = 8 \\ k &= 4 \end{aligned}$$

**Question 26**

The complex number  $z$  satisfies the equation

$$z^2 = 3 + 4i.$$

- a) Find the possible values of ...

a....  $z$ .

b....  $z^3$ .

- b) Hence, by showing detailed workings, find a solution of the equation

$$w^6 - 4w^3 + 125 = 0, w \in \mathbb{C},$$

$$\boxed{z = \pm(2+i)}, \boxed{z^3 = 2 \pm 1i}, \boxed{w = \pm(2+i)}$$

(a) Let  $z = x+iy$

$$\Rightarrow (x+iy)^2 = 3+4i$$

$$\Rightarrow x^2 + 2xyi - y^2 = 3+4i$$

$$\Rightarrow (x^2 - y^2) + 2xyi = 3+4i$$

$$\Rightarrow (x^2 - y^2) + (2xy)i = 3+4i$$

$$\Rightarrow \begin{cases} x^2 - y^2 = 3 \\ 2xy = 4 \end{cases} \Rightarrow \begin{cases} y = \frac{2}{x} \\ x^2 - \left(\frac{2}{x}\right)^2 = 3 \end{cases}$$

$$\Rightarrow x^2 - \frac{4}{x^2} = 3 \Rightarrow x^4 - 4 = 3x^2 \Rightarrow x^4 - 3x^2 - 4 = 0 \Rightarrow (x^2 - 4)(x^2 + 1) = 0$$

$$\Rightarrow x^2 = 4 \quad x^2 = -1$$

$$\Rightarrow x = \sqrt[2]{4} \quad x = \sqrt[2]{-1}$$

$$\therefore z = \sqrt[2]{4} + i\sqrt[2]{-1}$$

$$\therefore z^3 = \sqrt[3]{4} \cdot \sqrt[3]{2+i} = (2+i)(3+4i) = 4+8i+3i-4 = 2+11i$$

$$= (-2-i)(3+4i) = -(2+i)(3+4i) = -2-11i$$

$$\therefore z^3 = \sqrt[3]{2+11i}$$

(b)  $w^6 - 4w^3 + 125 = 0$   
GIVE THE SOURCE IN  $w^3$

$$\Rightarrow (w^3)^2 - 4 + 125 = 0$$

$$\Rightarrow (w^3 - 2)^2 = -121$$

$$\Rightarrow w^3 - 2 = \pm 11i$$

$$\Rightarrow w^3 = 2 \pm 11i$$

WE ARE LOOKING "FOR A SOLUTION"

$$\Rightarrow w^3 = 2+11i \quad (\text{Gives } w)$$

$$\Rightarrow w = \sqrt[3]{2+11i}$$

**Question 27**

It is given that

$$z = -17 - 6i \text{ and } w = 3 + i.$$

Find the value of  $u$  given further that

$$\frac{1}{10u} = \frac{3}{z} + \frac{1}{2w}.$$

$$u = -9 - 7i$$

<b>METHOD A</b> $\begin{aligned} \Rightarrow \frac{1}{10u} &= \frac{3}{-17-6i} + \frac{1}{2(3+i)} \\ \Rightarrow \frac{1}{10u} &= \frac{3(-17+6i)}{(-17-6i)(-17+6i)} \\ \Rightarrow 10u &= \frac{-51+18i}{325} \\ \Rightarrow 10u &= \frac{-51+18i}{(5)(65)} \\ \Rightarrow 10u &= \frac{-51+18i}{325} \\ \Rightarrow 10u &= \frac{-51+18i}{325} \\ \Rightarrow u &= -\frac{51+18i}{325} \\ \Rightarrow u &= -9 - 7i \end{aligned}$	<b>METHOD B</b> $\begin{aligned} \frac{1}{10u} &= \frac{3}{-17-6i} + \frac{1}{2(3+i)} \\ \Rightarrow \frac{1}{10u} &= \frac{3}{-17-6i} + \frac{1}{6+4i} \\ \Rightarrow \frac{1}{10u} &= \frac{3(-17+6i)}{(-17-6i)(-17+6i)} + \frac{6+4i}{(6+4i)(6-4i)} \\ \Rightarrow \frac{1}{10u} &= \frac{-51+18i}{325} + \frac{6+4i}{36+16} \\ \Rightarrow \frac{1}{10u} &= \frac{-51+18i}{325} + \frac{6+4i}{52} \\ \Rightarrow u &= -9 - 7i \end{aligned}$
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**Question 28**

The complex conjugate of the complex number  $z$  is denoted by  $\bar{z}$ .

Solve the equation

$$\frac{2\bar{z}(1-2i)}{5z} + \frac{i}{1+2i} = \frac{2-3i}{z},$$

giving the answer in the form  $x+iy$ .

$$z = 5 + 2i$$

Working shown in a box:

$$\begin{aligned} \frac{2\bar{z}(1-2i)}{5z} + \frac{i}{1+2i} &= \frac{2-3i}{z} \\ \Rightarrow \frac{2\bar{z}(1-2i)(1+2i)}{5z(1+2i)} + i &= \frac{(2-3i)(1+2i)}{z} \\ \Rightarrow \frac{2\bar{z}(1-2i)(1+2i)}{5z} + i &= \frac{2+4i-3i+6}{z} \\ \Rightarrow \frac{2\bar{z}}{5z} + i &= \frac{8-i}{z} \quad (\text{cancel } 1+2i) \\ \Rightarrow 2\bar{z} + 5iz &= 8-i \\ \Rightarrow 2(\bar{z} + iz) &= 8+i \\ \Rightarrow 2(-i\bar{z}) + 2z &= 8+i \\ 2x - y &= 8 \\ -2y - x &= 1 \end{aligned}$$

(cancel  $\bar{z}$ )  
So by substitution  
 $\Rightarrow 2(2y+1) - y = 8$   
 $\Rightarrow 3y = 6$   
 $\Rightarrow y = 2$  &  $x = 5$   $\therefore z = 5+2i$

**Question 29**

It is given that

$$z = \cos \theta + i \sin \theta, \quad 0 \leq \theta < 2\pi.$$

Show clearly that

$$\frac{2}{1+z} = 1 - i \tan\left(\frac{\theta}{2}\right).$$

proof

$$\begin{aligned}
 \frac{2}{1+z} &= \frac{2}{1+\cos\theta+i\sin\theta} = \frac{2[(1+\cos\theta)-i\sin\theta]}{[(1+\cos\theta)+i\sin\theta][(1+\cos\theta)-i\sin\theta]} \\
 &= \frac{2[(1+\cos\theta)-i\sin\theta]}{(1+\cos\theta)^2+\sin^2\theta} = \frac{2[(1+\cos\theta)-i\sin\theta]}{1+2\cos\theta+(\cos^2\theta+\sin^2\theta)} \\
 &= \frac{2[(1+\cos\theta)-i\sin\theta]}{2+2\cos\theta} = \frac{1+\cos\theta-i\sin\theta}{1+\cos\theta} = \frac{1-\sin\theta}{1-\cos\theta} \\
 &= 1 - i \frac{2\sin\theta \cos\frac{\theta}{2}}{1+2\cos^2\frac{\theta}{2}-1} \\
 &= 1 - i \frac{2\sin\theta \cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} \\
 &= 1 - i \tan\frac{\theta}{2}.
 \end{aligned}$$

$\sin 2A = 2\sin A \cos A$   
 $\sin\left(\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2} \cos\frac{\theta}{2}$   
 $\sin\theta = 2\sin\frac{\theta}{2} \cos\frac{\theta}{2}$   
 $\cos 2A = 2\cos^2 A - 1$   
 $\cos\left(\frac{\theta}{2}\right) = 2\cos^2\frac{\theta}{2} - 1$   
 $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

**Question 30**

By considering the solutions of the equation

$$z^4 = 16,$$

find the solutions of the equation

$$w^4 = 16(1-w)^4,$$

giving the answers in the form  $x+iy$ , where  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ .

$$z_1 = 2, \quad z_2 = \frac{2}{3}, \quad z_3 = \frac{4}{5} + i\frac{2}{5}, \quad z_4 = \frac{4}{5} - i\frac{2}{5}$$

Given  $\frac{z^4 - 16}{z^2 - 4} = 0 \Rightarrow z = \begin{cases} \text{real} \\ \text{non-real} \end{cases}$

Now  $w^4 = 16(1-w)^4 \Rightarrow w^4 = 16$

$$\Rightarrow \frac{w^4}{(1-w)^4} = 16 \Rightarrow \left(\frac{w}{1-w}\right)^4 = 16$$

$$\Rightarrow \left|\frac{w}{1-w}\right|^4 = 16 \Rightarrow \left|\frac{w}{1-w}\right| = \sqrt[4]{16} = 2$$

$$\Rightarrow \frac{|w|}{\sqrt{1-w^2}} = 2 \Rightarrow \frac{1}{\sqrt{1-w^2}} = \frac{1}{|w|} \Rightarrow |w| = \sqrt{1-w^2}$$

$$\Rightarrow \frac{1-w}{w} = \frac{1}{|w|} \Rightarrow \frac{1}{w} - 1 = \frac{1}{|w|} \Rightarrow \frac{1}{w} = \frac{1}{|w|} + 1 \Rightarrow \frac{1}{w} = \frac{|w|+1}{|w|} \Rightarrow w = \frac{|w|}{|w|+1}$$

•  $w_1 = \frac{2}{2+1} = \frac{2}{3}$

•  $w_2 = \frac{-2}{-2+1} = \frac{-2}{-1} = 2$

•  $w_3 = \frac{2i}{2i+1} = \frac{(6-2)(2i)}{(14i-2)(2i-1)} = \frac{2i+4}{1+4} = \frac{4+2i}{5} = \frac{\frac{4}{5}+\frac{2}{5}i}{1+4}$

•  $w_4 = \frac{-2i}{-2i+1} = \frac{-2(1+2i)}{(-2-2)(2i+1)} = \frac{-2i+4}{1+4} = \frac{4-2i}{5} = \frac{\frac{4}{5}-\frac{2}{5}i}{1+4}$

**Question 31**

The complex number  $z$  is given by

$$z = \frac{a+bi}{a-bi}, \quad a \in \mathbb{R}, \quad b \in \mathbb{R}.$$

Show clearly that

$$\frac{z^2 + 1}{2z} = \frac{a^2 - b^2}{a^2 + b^2}.$$

proof

$$\begin{aligned}
 z &= \frac{a+bi}{a-bi} = \frac{(a+bi)(a+bi)}{(a-bi)(a+bi)} = \frac{a^2 + 2abi - b^2}{a^2 + b^2} \\
 \frac{z^2 + 1}{2z} &= \frac{\left(\frac{a+bi}{a-bi}\right)^2 + 1}{2\left(\frac{a+bi}{a-bi}\right)} = \frac{\frac{(a+bi)^2}{(a-bi)^2} + 1}{2(a+bi)} = \frac{(a+bi)^2 + 1}{2(a+bi)^2} = \frac{\text{MULTIPLY TOP & BOTTOM BY } (a-bi)^2}{(a-bi)^4} \\
 &= \frac{(a+bi)^2 + (a-bi)^2}{2(a+bi)(a-bi)} = \frac{a^2 + 2abi - b^2 + a^2 - 2ab + b^2}{2(a^2 + b^2)} = \frac{2a^2 - 2b^2}{2(a^2 + b^2)} = \frac{2(a^2 - b^2)}{2(a^2 + b^2)} = \frac{a^2 - b^2}{a^2 + b^2} // 
 \end{aligned}$$

**Question 32**

Solve the following equations.

a)  $z^3 - 27 = 0$ .

b)  $w^2 - i(w-2) = (w-2)$ .

$$z_1 = 3, \quad z_2 = \frac{3}{2}(-1 \pm \sqrt{3}), \quad w_1 = 2i, \quad w_2 = 1-i$$

(a)  $z^3 - 27 = 0$

$$\Rightarrow z^3 - 3^3 = (a-b)(a^2+ab+b^2)$$

$$\Rightarrow (z-3)(z^2 + 3z + 9) = 0$$

$$\text{From } z=3 \quad \text{or} \quad z^2 + 3z + 9 = 0$$

$$(z + \frac{3}{2})^2 - \frac{9}{4} + 9 = 0$$

$$(z + \frac{3}{2})^2 = -\frac{27}{4}$$

$$z + \frac{3}{2} = \pm \sqrt{-\frac{27}{4}}$$

$$z = -\frac{3}{2} \pm \frac{\sqrt{27}}{2}$$

(b)  $w^2 - i(w-2) = w-2$

$$\Rightarrow w^2 - iw + 2i - w + 2 = 0$$

$$\Rightarrow w^2 + iw(-1+i) + (2i)^2 = 0$$

By quadratic formula

$$w = \frac{-(-1+i) \pm \sqrt{(-1+i)^2 - 4w(2i^2)}}{2w}$$

$$w = \frac{1+i \pm \sqrt{1+2i-8-8i}}{2}$$

$$w = \frac{1+i \pm \sqrt{-7-6i}}{2}$$

Method

$$z = -8-6i$$

$$(a+bi) = -8-6i$$

$$a^2+2ab+b^2 = -8-6i$$

$$a^2+b^2 = -8$$

$$2ab = -6i$$

$$\left(b = \frac{3}{a}\right)$$

$$a^2 - \frac{9}{a^2} = -8$$

$$a^4 - 9 = -8a^2$$

$$a^4 + 8a^2 - 9 = 0$$

$$(a^2+9)(a^2-1) = 0$$

$$a^2 < 1 \quad a < 1$$

$$a^2 > 1 \quad a > -1$$

**Question 33**

Solve the quadratic equation

$$z^2 - 7z + 16 = i(z-11), \quad z \in \mathbb{C}.$$

$$\boxed{z = 2 + 3i, \quad z = 5 - 2i}$$

$$\begin{aligned}
 & z^2 - 7z + 16 = i(z-11) \\
 & z^2 - 7z + 16 = iz - 11i \\
 & z^2 - 7z - iz + 16 + 11i = 0 \\
 & z^2 - (7+i)z + (16+11i) = 0 \\
 & \text{By quadratic formula} \\
 & z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 & z = \frac{-7-i \pm \sqrt{(7+i)^2 - 4(1)(16+11i)}}{2} \\
 & z = \frac{-7-i \pm \sqrt{49+14i-1-64-44i}}{2} \\
 & z = \frac{-7-i \pm \sqrt{-16-30i}}{2} \\
 & \text{Now } w^2 = -16-30i \\
 & \Rightarrow (u+v)^2 = -16-30i \\
 & \Rightarrow u^2+2uv+v^2 = -16-30i \\
 & (u^2-v^2) + 2uv = -16-30i \\
 & u^2 = -16 \quad \text{so } v = -\frac{15}{u}
 \end{aligned}$$

$$\begin{aligned}
 & \text{From } u^2 = -16 \\
 & \Rightarrow u^2 - 225 = -16u^2 \\
 & \Rightarrow u^4 + 16u^2 - 225 = 0 \\
 & \Rightarrow (u^2 - 9)(u^2 + 25) = 0 \\
 & \Rightarrow u^2 = 9 \\
 & \Rightarrow u = \pm 3 \\
 & \text{Thus} \\
 & z = \frac{(7+i) \pm (3-5i)}{2} \\
 & z = \frac{(7+3) + (1-5)i}{2} \\
 & z = \frac{10 - 4i}{2} \\
 & z = 5 - 2i
 \end{aligned}$$

# MODULUS AND ARGUMENT

**Question 1**

$$w = \frac{-9+3i}{1-2i}.$$

Find the modulus and the argument of the complex number  $w$ .

$$|w| = 3\sqrt{2}, \quad \arg w = -\frac{3\pi}{4}$$

**METHOD A**

$$\begin{aligned} w &= \frac{-9+3i}{1-2i} = \frac{(-9+3i)(1+2i)}{(1-2i)(1+2i)} = \frac{-9-18i+3i-6}{1+2i-2i-4} \\ &= \frac{-15-15i}{5} = -3-3i \\ \bullet |w| &= |-3-3i| = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2} \\ \bullet \arg w &= \arg(-3-3i) = \arctan\left(\frac{-3}{-3}\right) - \pi \\ &= \frac{\pi}{4} - \pi = -\frac{3\pi}{4} \end{aligned}$$

**METHOD B**

$$\begin{aligned} \bullet |w| &= \left| \frac{-9+3i}{1-2i} \right| = \frac{|-9+3i|}{|1-2i|} = \frac{\sqrt{81+9}}{\sqrt{1+4}} = \frac{\sqrt{90}}{\sqrt{5}} \\ &= \frac{\sqrt{18} \times \sqrt{5}}{\sqrt{18}} = 3\sqrt{2} \\ \bullet \arg w &= \arg\left[\frac{-9+3i}{1-2i}\right] = \arg(-9+3i) - \arg(1-2i) \\ &= \left[ \arctan\left(\frac{3}{-9}\right) + \pi \right] - \left[ \arctan\left(\frac{-2}{1}\right) \right] \quad (\text{see sketch diagram}) \\ &= \pi - \arctan\frac{1}{3} + \arctan 2 \\ &\approx \frac{5}{4}\pi \quad \rightarrow -2\pi \text{ TO GET IN RANGE} \\ &= -\frac{3\pi}{4} \end{aligned}$$

**Question 2**

$$z = -3 + 4i \quad \text{and} \quad zw = -14 + 2i.$$

By showing clear workings, find ...

- a) ...  $w$  in the form  $a+bi$ , where  $a$  and  $b$  are real numbers.
- b) ... the modulus and the argument of  $w$ .

$$w = 2 + 2i, |w| = 2\sqrt{2}, \arg w = \frac{\pi}{4}$$

$  \begin{aligned}  @) \quad zw &= -14 + 2i \\  &\Rightarrow (-3 + 4i)w = -14 + 2i \\  &\Rightarrow w = \frac{-14 + 2i}{-3 + 4i} \\  &\Rightarrow w = \frac{(-14 + 2i)(-3 - 4i)}{(-3 + 4i)(-3 - 4i)} \\  &\Rightarrow w = \frac{42 + 50i - 6i + 8}{25} \\  &\Rightarrow w = \frac{50 + 44i}{25} \\  &\Rightarrow w = 2 + 2i  \end{aligned}  $	$  \begin{aligned}  (b) \quad  w  &=  2 + 2i  = \sqrt{2^2 + 2^2} \\  &= \sqrt{8} = 2\sqrt{2} \\  \bullet \arg(w) &= \arg(2 + 2i) \\  &= \arg\left(\frac{2}{\sqrt{2}}\right) \\  &= \arg\left(\sqrt{2}\right) \\  &= \text{constant} \\  &= \frac{\pi}{4}  \end{aligned}  $
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**Question 3**

$$z = 22 + 4i \quad \text{and} \quad \frac{z}{w} = 6 - 8i.$$

By showing clear workings, find ...

- a) ...  $w$  in the form  $a+bi$ , where  $a$  and  $b$  are real numbers.
- b) ... the modulus and the argument of  $w$ .

$$w = 1 + 2i, |w| = \sqrt{5}, \arg w \approx 1.11^\circ$$

$  \begin{aligned}  @) \quad \frac{z}{w} &= 6 - 8i \\  \frac{22 + 4i}{w} &= 6 - 8i \\  w &= \frac{22 + 4i}{6 - 8i} \\  w &= \frac{11 + 2i}{3 - 4i} \\  w &= \frac{(11 + 2i)(3 + 4i)}{(3 - 4i)(3 + 4i)} \\  w &= \frac{33 + 44i + 6i - 8}{9 + 16} \\  w &= \frac{25 + 50i}{25} \\  w &= 1 + 2i  \end{aligned}  $	$  \begin{aligned}  (b) \quad  w  &=  1 + 2i  \\  &= \sqrt{1^2 + 2^2} \\  &= \sqrt{5} \\  \bullet \arg w &= \arg(1 + 2i) \\  &= \arg\left(\frac{1}{\sqrt{5}}\right) \\  &= \arg\sqrt{5} \\  &= \text{constant} \\  &= 1.107^\circ  \end{aligned}  $
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**Question 4**

$$z = 1 + \sqrt{3}i \quad \text{and} \quad \frac{w}{z} = 2 + 2i.$$

Find the exact value of the modulus of  $w$  and the exact value of the argument of  $w$ .

$$|w| = 4\sqrt{2}, \quad \arg w = \frac{7\pi}{12}$$

<p><b>METHOD A:</b></p> $\frac{w}{z} = \frac{(1 + \sqrt{3}i)}{2 + 2i} \Rightarrow \frac{w}{2 + 2i} = 1 + \sqrt{3}i \Rightarrow w = (2 + 2i)(1 + \sqrt{3}i)$ <p>• <math>W = 2 + 2\sqrt{3}i + 2i - 2\sqrt{3}</math>  <math>W = (2 - 2\sqrt{3}) + (2 + 2\sqrt{3})i</math></p> <p>• <math> W  = \sqrt{(2 - 2\sqrt{3})^2 + (2 + 2\sqrt{3})^2}</math>  <math> W  = \sqrt{4 - 8\sqrt{3}i + 12 + 4 + 8\sqrt{3}i + 12}</math>  <math>\Rightarrow  W  = \sqrt{32} = 4\sqrt{2}</math>  <math>\Rightarrow  w  = 4\sqrt{2}</math></p> <p>• <math>\arg W = \arg((2 - 2\sqrt{3}) + (2 + 2\sqrt{3})i)</math>  <math>\Rightarrow \arg W = \arctan\left(\frac{2 + 2\sqrt{3}}{2 - 2\sqrt{3}}\right) + \pi</math>  <math>\Rightarrow \arg W = \arctan\left(\frac{\frac{1 + \sqrt{3}}{1 - \sqrt{3}}}{1}\right) + \pi</math>  <math>\Rightarrow \arg W = \arctan\left[\frac{(1 + \sqrt{3})(1 + \sqrt{3})}{(1 - \sqrt{3})(1 + \sqrt{3})}\right] + \pi</math>  <math>\Rightarrow \arg W = \arctan\left[\frac{1 + 2\sqrt{3} + 3}{1 - 3}\right] + \pi</math>  <math>\Rightarrow \arg W = \arctan(-2 - \sqrt{3}) + \pi</math>  <math>\Rightarrow \arg W = -\arctan(2 + \sqrt{3}) + \pi</math>  <math>\Rightarrow \arg W = -\frac{3\pi}{12} + \pi</math>  <math>\Rightarrow \arg W = \frac{7\pi}{12}</math></p>	<p><b>METHOD B:</b></p> <p>• <math>W = (2 + 2i)(1 + \sqrt{3}i)</math>  <math>\Rightarrow  W  = \sqrt{(2 + 2i)(1 + \sqrt{3}i)}</math>  <math>\Rightarrow  W  = \sqrt{(2 + 2i)^2(1 + \sqrt{3}i)}</math>  <math>\Rightarrow  W  = \sqrt{2 + 2i} \times \sqrt{1^2 + (\sqrt{3})^2}</math>  <math>\Rightarrow  W  = \sqrt{8} \times \sqrt{4}</math>  <math>\Rightarrow  W  = 2\sqrt{2} \times 2</math>  <math>\Rightarrow  W  = 4\sqrt{2}</math></p> <p>• <math>\arg W = \arg[(2 + 2i)(1 + \sqrt{3}i)]</math>  <math>\Rightarrow \arg W = \arg(2 + 2i) + \arg(1 + \sqrt{3}i)</math>  <math>\Rightarrow \arg W = \arctan\left(\frac{2}{2}\right) + \arctan\left(\frac{\sqrt{3}}{1}\right)</math>  <math>\Rightarrow \arg W = \arctan 1 + \arctan\sqrt{3}</math>  <math>\Rightarrow \arg W = \frac{\pi}{4} + \frac{\pi}{3}</math>  <math>\Rightarrow \arg W = \frac{7\pi}{12}</math></p>
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**Question 5**

The following complex numbers are given.

$$z_1 = 2 - 2i, \quad z_2 = \sqrt{3} + i \quad \text{and} \quad z_3 = a + bi \quad \text{where } a \in \mathbb{R}, b \in \mathbb{R}.$$

- a) If  $|z_1 z_3| = 16$ , find the modulus of  $z_3$ .
- b) Given further that  $\arg\left(\frac{z_3}{z_2}\right) = \frac{7\pi}{12}$ , determine the argument of  $z_3$ .
- c) Find the values of  $a$  and  $b$ , and hence show  $\frac{z_3}{z_1} = -2$ .

$$|z_3| = 4\sqrt{2}, \quad \arg z_3 = \frac{3\pi}{4}, \quad [a = -4], \quad [b = 4]$$

a)  $|z_1 z_3| = |z_1||z_3|$

$$\begin{aligned} &\rightarrow |z_1||z_3| = 16 \\ &\rightarrow |z_1| |z_3| = 16 \\ &\rightarrow |2-2i| |z_3| = 16 \\ &\rightarrow \sqrt{4+4} |z_3| = 16 \\ &\rightarrow \sqrt{8} |z_3| = 16 \\ &\rightarrow \sqrt{8} |z_3| = 16\sqrt{2} \\ &\rightarrow |z_3| = 16\sqrt{2} \\ &\rightarrow |z_3| = 4\sqrt{2} \end{aligned}$$

b)  $\arg\left(\frac{z_3}{z_2}\right) = \arg z_3 - \arg z_2$

$$\begin{aligned} &\rightarrow \arg\left(\frac{z_3}{z_2}\right) = \frac{7\pi}{12} \\ &\rightarrow \arg z_3 - \arg z_2 = \frac{7\pi}{12} \\ &\rightarrow \arg z_3 - \arg(\sqrt{3}+i) = \frac{7\pi}{12} \\ &\rightarrow \arg z_3 - \arg\left(\frac{1}{\sqrt{3}}+i\right) = \frac{7\pi}{12} \\ &\rightarrow \arg z_3 - \frac{\pi}{6} = \frac{7\pi}{12} \\ &\rightarrow \arg z_3 = \frac{3\pi}{4} \end{aligned}$$

c) Given if  $z_3 = a+bi$ ,  $|z_3| = 4\sqrt{2}$ ,  $\arg z_3 = \frac{3\pi}{4}$

$$\begin{aligned} &\rightarrow |a+bi| = 4\sqrt{2} \\ &\rightarrow \sqrt{a^2+b^2} = 4\sqrt{2} \\ &\rightarrow a^2+b^2 = 32 \end{aligned}$$

(use sum of squares)

$$\begin{aligned} &\arg z_3 = \frac{3\pi}{4} \\ &\arctan \frac{b}{a} + \pi = \frac{3\pi}{4} \\ &\arctan \frac{b}{a} = -\frac{\pi}{4} \\ &\frac{b}{a} = -\tan\left(-\frac{\pi}{4}\right) \\ &\frac{b}{a} = -1 \\ &b = -a \end{aligned}$$

$a^2 + a^2 = 32$

$$\begin{aligned} &2a^2 = 32 \\ &a^2 = 16 \\ &a = -4 \quad (\text{as } z_3 \text{ lies in the 2nd quadrant}) \\ &b = +4 \end{aligned}$$

Finally  $\frac{z_3}{z_1} = \frac{-4+4i}{2-2i} = \frac{-2(2-2i)}{2-2i} = -2$  (as required)

ANALYSIS ON GRAPH

$$\begin{aligned} z_3 &= r(\cos\theta + i\sin\theta) \\ z_3 &= 4\sqrt{2} \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \\ z_3 &= 4\sqrt{2} \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \\ z_3 &= -4 + 4i \end{aligned}$$

**Question 6**

$$z = \sqrt{3} + i \quad \text{and} \quad w = 3i.$$

- a) Find, in exact form where appropriate, the modulus and argument of  $z$  and the modulus and argument of  $w$ .
- b) Determine simplified expressions for  $zw$  and  $\frac{w}{z}$ , giving the answers in the form  $x+iy$ , where  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ .
- c) Find, in exact form where appropriate, the modulus and argument of  $zw$  and the modulus and argument of  $\frac{w}{z}$ .

$$|z| = 2, |w| = 3, \arg z = \frac{\pi}{6}, \arg w = \frac{\pi}{2}, zw = -3 + 3\sqrt{3}i, \frac{w}{z} = \frac{3}{4} + \frac{3}{4}\sqrt{3}i,$$

$$|zw| = 6, \left| \frac{w}{z} \right| = \frac{3}{2}, \arg(zw) = \frac{2\pi}{3}, \arg\left(\frac{w}{z}\right) = \frac{\pi}{3}$$

(a)  $|z| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$   
 $|w| = |3i| = 3$   
 $\arg z = \arg(\sqrt{3}+i) = \arg \frac{(\sqrt{3}+i)}{|z|} = \frac{\pi}{6}$   
 $\arg w = \arg(3i) = \frac{\pi}{2}$

(b)  $zw = (\sqrt{3}+i)(3i) = 3\sqrt{3}i - 3 = -3 + 3\sqrt{3}i$   
 $\frac{w}{z} = \frac{3i}{\sqrt{3}+i} = \frac{3i(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)} = \frac{3\sqrt{3}i + 3}{3+1} = \frac{3}{4} + \frac{3}{4}\sqrt{3}i$

(c)  $|zw| = |z||w| = 2 \times 3 = 6$   
 $\left| \frac{w}{z} \right| = \frac{|w|}{|z|} = \frac{3}{2}$   
 $\arg(zw) = \arg z + \arg w = \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}$   
 $\arg\left(\frac{w}{z}\right) = \arg w - \arg z = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

**Question 7**

The following complex numbers are given

$$z = \frac{1+i}{1-i} \text{ and } w = \frac{\sqrt{2}}{1-i}.$$

- Calculate the modulus of  $z$  and the modulus of  $w$ .
- Find the argument of  $z$  and the argument of  $w$ .
- By considering the quadrilateral  $OABC$  and the argument of  $z+w$ , show that

$$\tan\left(\frac{3\pi}{8}\right) = 1 + \sqrt{2}.$$

$$|z| = 1, |w| = 1, \arg z = \frac{\pi}{2}, \arg w = \frac{\pi}{4}$$

**(a)**

$$|z| = \sqrt{\frac{1+i}{1-i} \cdot \frac{1-i}{1-i}} = \sqrt{\frac{(1+i)(1-i)}{1-i^2}} = \sqrt{\frac{1-i^2}{1+1}} = \sqrt{\frac{2}{2}} = 1$$

$$|w| = \sqrt{\frac{2}{1-i} \cdot \frac{1-i}{1-i}} = \sqrt{\frac{2(1-i)}{1-i^2}} = \sqrt{\frac{2(1-i)}{2}} = \sqrt{2}$$

**(b)**

$$\arg(z) = \arg\left(\frac{1+i}{1-i}\right) = \arg(1+i) - \arg(1-i) = \operatorname{arctan}\left(\frac{1}{1}\right) - \operatorname{arctan}\left(\frac{-1}{1}\right) = \operatorname{arctan}(1) - \operatorname{arctan}(-1) = \frac{\pi}{4} - (-\frac{\pi}{4}) = \frac{\pi}{2}$$

$$\arg w = \arg\left(\frac{\sqrt{2}}{1-i}\right) = \arg(1+i) - \arg(1-i) = 0 - \operatorname{arctan}\left(\frac{1}{1}\right) = -\operatorname{arctan}\left(\frac{1}{1}\right) = -\left(\frac{\pi}{4}\right) = -\frac{\pi}{4}$$

**(c)**

- THE PARALLELOGRAM OABC IS A PARALLELLOGRAM WITH EQUAL SIDES, SO IT IS A PARAL.
- OB IS THE ANGLE BISECTOR OF AC, SO THAT  $\angle COB = \frac{\pi}{2}$
- SO THE ARGUMENT OF NUMBER REPRESENTED BY B IS  $\frac{\pi}{2}$

Method A:

$$z+w = \frac{1+i}{1-i} + \frac{\sqrt{2}}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} + \frac{\sqrt{2}(1-i)}{(1-i)(1+i)} = \frac{(1+i)^2 + i\sqrt{2}}{2} = \frac{1+2i+i^2 + i\sqrt{2}}{2} = \frac{2i + i\sqrt{2}}{2} = i(2+\sqrt{2})$$

$$\tan \frac{\theta}{2} = \frac{2+\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\tan \frac{\theta}{2} = \frac{2}{2} + 1$$

$$\tan \frac{\theta}{2} = 1 + \sqrt{2}$$

**(d)**

$$\tan \frac{\theta}{2} = \frac{1+i}{1-i} + \frac{\sqrt{2}}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} + \frac{\sqrt{2}(1-i)}{(1-i)(1+i)} = \frac{(1+i)^2 + i\sqrt{2}}{2} = \frac{1+2i+i^2 + i\sqrt{2}}{2} = \frac{2i + i\sqrt{2}}{2} = i(2+\sqrt{2})$$

$$\Rightarrow \arg(2+i) = \arg\left(\frac{1+i}{1-i}\right) + \arg\left(\frac{\sqrt{2}}{1-i}\right)$$

$$\Rightarrow \frac{\theta}{2} = \arg\left(\frac{1}{1-i}\right) - \arg(1-i)$$

$$\Rightarrow \frac{\theta}{2} = \operatorname{arctan}\left(\frac{1}{1-i}\right) - \operatorname{arctan}\left(\frac{-1}{1}\right)$$

$$\Rightarrow \frac{\theta}{2} = \operatorname{arctan}\left(\frac{1}{1-i}\right) - \left(-\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\theta}{2} = \operatorname{arctan}\left(\frac{1}{1-i}\right) + \frac{\pi}{4}$$

$$\Rightarrow \tan \frac{\theta}{2} = \tan\left[\operatorname{arctan}\left(\frac{1}{1-i}\right) + \frac{\pi}{4}\right]$$

$$\Rightarrow \tan \frac{\theta}{2} = \tan(\operatorname{arctan}\left(\frac{1}{1-i}\right)) + \tan \frac{\pi}{4}$$

$$\Rightarrow \tan \frac{\theta}{2} = \frac{1 - \tan(\operatorname{arctan}\left(\frac{1}{1-i}\right))}{1 + \tan(\operatorname{arctan}\left(\frac{1}{1-i}\right))}$$

$$\Rightarrow \tan \frac{\theta}{2} = \frac{1 - \frac{1}{1+i}}{1 + \frac{1}{1+i}} = \frac{1 - \frac{1}{1+i}}{1 + \frac{1}{1+i}} = \frac{1 - \frac{1}{1+i}}{1 + \frac{1}{1+i}} = \frac{1 - \frac{1}{1+i}}{1 + \frac{1}{1+i}}$$

$$\Rightarrow \tan \frac{\theta}{2} = \frac{1 - \frac{1}{1+i}}{1 + \frac{1}{1+i}} = \frac{1 - \frac{1}{1+i}}{1 + \frac{1}{1+i}} = \frac{1 - \frac{1}{1+i}}{1 + \frac{1}{1+i}}$$

NOW  $\tan(\theta) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\Rightarrow \tan \frac{\theta}{2} = \frac{(1+\sqrt{2}) + 1}{(1+\sqrt{2})(1)}$$

$$\Rightarrow \tan \frac{\theta}{2} = \frac{2+\sqrt{2}}{\sqrt{2}}$$

$$\Rightarrow \tan \frac{\theta}{2} = 1 + \sqrt{2}$$

**Question 8**

$$\frac{(3+4i)(1+2i)}{1+3i} = q(1+i), \quad q \in \mathbb{R}.$$

- a) Find the value of  $q$ .  
 b) Hence simplify

$$\arctan \frac{4}{3} + \arctan 2 - \arctan 3,$$

giving the answer in terms of  $\pi$ .

$$q = \frac{5}{2}, \quad \boxed{\frac{1}{4}\pi}$$

$$\begin{aligned}
 \text{(a)} \quad & \frac{(3+4i)(1+2i)}{1+3i} = \frac{3+6i+4i-8}{1+3i} = \frac{-5+10i}{1+3i} = \frac{(-5+10i)(1-3i)}{(1+3i)(1-3i)} \\
 &= \frac{-5+15i+10i-30}{1+9} = \frac{25+25i}{10} = \frac{5(5+5i)}{10} = \frac{5(1+i)}{2} \\
 & \Rightarrow |q| = \frac{5}{2}.
 \end{aligned}$$
  

$$\begin{aligned}
 \text{(b)} \quad & \frac{(3+4i)(1+2i)}{1+3i} = \frac{5(1+i)}{2} \\
 & \Rightarrow \arg\left[\frac{(3+4i)(1+2i)}{1+3i}\right] = \arg\left[\frac{5(1+i)}{2}\right] \\
 & \Rightarrow \arg(3+4i) + \arg(1+2i) - \arg(1+3i) = \arg\frac{5}{2} + \arg(1+i) \\
 & \Rightarrow \arg\frac{4}{3} + \arg\frac{2}{1} - \arg\left(\frac{3}{1}\right) = 0 + \arg\frac{1}{2} \\
 & \Rightarrow \arg\frac{4}{3} + \arg\frac{2}{1} - \arg\frac{3}{1} = \frac{\pi}{4}
 \end{aligned}$$

**Question 9**

It is given that

$$z = \frac{1+8i}{1-2i}.$$

- a) Express  $z$  in the form  $x+iy$ .
- b) Find the modulus and argument of  $z$ .
- c) Show clearly that

$$\arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \pi.$$

$$z = -3 + 2i, |z| = \sqrt{13}, \arg z \approx 2.55^\circ$$

$$\begin{aligned}
 \text{(a)} \quad z &= \frac{1+8i}{1-2i} = \frac{(1+8i)(1+2i)}{(1-2i)(1+2i)} = \frac{1+2i+8i-16}{1+4} = \frac{-15+10i}{5} = -3+2i \\
 \text{(b)} \quad |z| &= |-3+2i| = \sqrt{(-3)^2+2^2} = \sqrt{13} \\
 \arg(z) &= \pi + \arctan\left(\frac{2}{3}\right) = \pi - \arctan\frac{2}{3} \approx 2.55^\circ \\
 \text{(c)} \quad \frac{1+8i}{1-2i} &= -3+2i \\
 \Rightarrow \arg\left(\frac{1+8i}{1-2i}\right) &= \arg(-3+2i) \\
 \Rightarrow \arg(1+8i) - \arg(1-2i) &= \arg(-3+2i) \\
 \Rightarrow \arctan\left(\frac{8}{1}\right) - \arctan\left(\frac{2}{1}\right) &= \pi - \arctan\frac{2}{3} \xrightarrow{\text{Part (b)}} \\
 \Rightarrow \arctan 8 + \arctan 2 &= \pi - \arctan\frac{2}{3} \\
 \Rightarrow \arctan 8 + \arctan 2 + \arctan\frac{2}{3} &= \pi \quad \text{Part (c)}
 \end{aligned}$$

# COMPLEX POLYNOMIAL QUESTIONS

**Question 1**

The cubic equation

$$2z^3 - 5z^2 + cz - 5 = 0, \quad c \in \mathbb{R},$$

has a solution  $z = 1 - 2i$ .

Find in any order ...

- a) ... the other two solutions of the equations.
- b) ... the value of  $c$ .

$$z_2 = 1 + 2i, \quad z_3 = \frac{1}{2}, \quad c = 12$$

**(a) Method A:**

$$\begin{aligned} z_1 &= 1-2i \\ z_2 &= 1+2i \\ z_3 &= \frac{1}{2} \end{aligned}$$

using  $x+y+z = -\frac{b}{a}$

$$\begin{aligned} (1-2i)+(1+2i)+z &= \frac{-b}{a} \\ 2+z &= \frac{-b}{a} \\ z &= \frac{-b}{a}-2 \end{aligned}$$

$$\begin{aligned} z_1 &= 1-2i \\ z_2 &= 1+2i \\ z_3 &= \frac{1}{2} \end{aligned}$$

**(b)**

$$\begin{aligned} ax^3+bx^2+cx+d &= 0 \\ ((-2)(1+2i))+\frac{1}{2}(1-2i)+c(1+2i)+d &= 0 \\ 1+4+\frac{1}{2}i-2i+\frac{1}{2}+2i+c &= 0 \\ \frac{5}{2}+c &= 0 \\ c &= -\frac{5}{2} \\ c &= 12 \end{aligned}$$

**(b) Method B:**

- $2z^3 - 5z^2 + cz - 5 = 0$
- $\begin{cases} z_1 = 1-2i \\ z_2 = 1+2i \\ z_3 = \frac{1}{2} \end{cases}$
- $\begin{aligned} &[(z-1-2i)][(z-1+2i)][z-\frac{1}{2}] \\ &= [(z-i)-1][[(z-i)+1]-1] \\ &= (z-i)^2 - (\frac{1}{2})^2 \\ &= z^2 - 2z + 1 + \frac{1}{4} \\ &= z^2 - 2z + \frac{5}{4} \end{aligned}$
- $2z^3 - 5z^2 + cz - 5 = (z-1)(z^2 - 2z + \frac{5}{4})$  BY NRFC (or)
- $\begin{cases} z_1 = 1+2i \\ z_2 = \frac{1}{2} \end{cases}$
- AND BY MULTIPLYING,**
- $\begin{aligned} (2z-1)(z^2 - 2z + \frac{5}{4}) &= 2z^3 - 4z^2 + 10z - 5 \\ &= 2z^3 - 5z^2 + Dz - 5 \\ \therefore D &= 12 \end{aligned}$

**Question 2**

The following cubic equation is given

$$z^3 + az^2 + bz - 5 = 0,$$

where  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ .

One of the roots of the above cubic equation is  $2+i$ .

- a) Find the other two roots.
- b) Determine the value of  $a$  and the value of  $b$ .

$$z_2 = 2-i, z_3 = 1, a = -5, b = 9$$

<p><b>METHOD A</b></p> <p>(a) <math>\alpha = 2+i</math>  <math>\beta = 2-i</math></p> $\Rightarrow \alpha\beta\gamma = -5$ $\Rightarrow (2+i)(2-i)\gamma = 5$ $\Rightarrow 5\gamma = 5$ $\Rightarrow \gamma = 1$ <p><math>\therefore z_1 = 2+i</math>  <math>z_2 = 2-i</math>  <math>z_3 = 1</math></p> <p>(b)</p> $\begin{aligned} -a &= \alpha + \beta + \gamma \\ &= (2+i) + (2-i) + 1 \\ &\Rightarrow -a = 5 \\ &\Rightarrow a = -5 \end{aligned}$ $\begin{aligned} b &= ab + \beta\gamma + \alpha\gamma \\ &= (2+i)(2-i) + (2+i)(1) + (2-i)(1) \\ &\Rightarrow b = 4 + 1 + 2i + 2 - i \\ &\Rightarrow b = 9 \end{aligned}$	<p><b>METHOD B</b></p> <p>(c) <math>z_1 = 2+i</math>  <math>z_2 = 2-i</math></p> $\begin{aligned} [z - (2+i)][z - (2-i)] \\ = [(z-2)-i][(z-2)+i] \\ = (z-2)^2 - i^2 \\ = z^2 - 4z + 5 \end{aligned}$ <p>BY INSPECTION</p> $z^3 + az^2 + bz - 5 \equiv (2-i)(z^2 - 4z + 5)$ $\therefore z_1 = 2+i$ $z_2 = 2-i$ $z_3 = 1$	<p>(d) METHOD OF  <math>(2-i)(z^2 - 4z + 5) = z^3 - 4z^2 + 5z - 2i + 4z - 5</math>  <math>= z^3 - 5z^2 + 9z - 5</math></p> $\therefore a = -5$ $b = 9$
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**Question 3**

The following cubic equation is given

$$z^3 + pz^2 + 6z + q = 0,$$

where  $p \in \mathbb{R}$ ,  $q \in \mathbb{R}$ .

One of the three solutions of the above cubic equation is  $5 - i$ .

- a) Find the other two solutions of the equation.
- b) Determine the value of  $p$  and the value of  $q$ .

$$z_2 = 5+i, z_3 = 2, p = -8, q = 52$$

<p><u>Method A:</u></p> <p>(a) <math>z = 5-i</math>  <math>\bar{z} = 5+i</math></p> $\Rightarrow z\bar{z} + \bar{z}z + 6 = 0$ $\Rightarrow (5-i)(5+i) + (5+i)(5-i) + 6 = 0$ $\Rightarrow 25+1 + 5i+5i + 5i-5i = 6$ $\Rightarrow 10i = -20$ $\Rightarrow i = -2$ $\therefore z_2 = 5-i$ $\bar{z}_2 = 5+i$ $\bar{z}_3 = -2$ <p>(b) <math>\frac{-q}{z_1} = \alpha + bi</math>  <math>\Rightarrow -i = (5-i)(5+i) - 2</math> <math>\Rightarrow -i = 25+1 - 2</math> <math>\Rightarrow -i = 24</math> <math>\Rightarrow i = -24</math> <p>AND</p> <math>\Rightarrow \frac{-q}{i} = \alpha + bi</math> <math>\Rightarrow -q = (5-i)(5+i)(-i)</math> <math>\Rightarrow -q = (25+1)(-i)</math> <math>\Rightarrow -q = -52</math> <math>\Rightarrow q = 52</math></p>	<p><u>Method B:</u></p> <p>(a) <math>z_1 = 5-i</math>  <math>\bar{z}_1 = 5+i</math></p> $\text{THUS}$ $\begin{aligned} & [z - (5-i)][z - (5+i)] \\ &= [(z-5) + i][(z-5) - i] \\ &= (z-5)^2 - i^2 \\ &= z^2 - 10z + 25 + 1 \\ &= z^2 - 10z + 26 \end{aligned}$ <p><math>\rightarrow \text{系数}</math></p> $\begin{aligned} z^2 + z^2 + 6z + q &= (2+c)(z^2 - 10z + 26) \\ &\equiv z^2 - 10z^2 + 26z \\ &\quad - c^2 - 10c^2 + 26c \\ &\equiv z^2 + (-c-2)^2 + (26-10c)z + 26c \end{aligned}$ <p>EQUATE COEFFICIENTS</p> $\begin{aligned} z^2 &= 10c \\ -10c &= -20 \\ \therefore c &= 2 \end{aligned}$ $\begin{aligned} z^2 + c &= 0 \\ z^2 &= 0 \\ z &= 0 \end{aligned}$ $\begin{aligned} -c-2 &= b \\ -c &= b \\ \therefore c &= -b \\ \therefore b &= -2 \end{aligned}$ $\begin{aligned} 26c &= q \\ 26 &= q \\ \therefore q &= 52 \end{aligned}$ <p>THUS</p> $\begin{aligned} z_1 &= 5-i \\ \bar{z}_1 &= 5+i \\ z_3 &= -2 \\ p &= -8 \\ q &= 52 \end{aligned}$
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**Question 4**

The following cubic equation is given

$$z^3 + 2z^2 + az + b = 0,$$

where  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}$ .

One of the roots of the above cubic equation is  $1+i$ .

- a) Find the real root of the equation.
- b) Find the value of  $a$  and the value of  $b$ .

$$z = -4, [a = -6], [b = 8]$$

**METHOD A:**

If  $z_1 = 1+i$   
 $\bar{z}_1 = 1-i$

Then  $(z - (1+i))(z - (1-i)) = [(z-1)-i][(z-1)+i]$   
 $= (z-1)^2 - i^2 = z^2 - 2z + 1 + 1 = z^2 - 2z + 2$

Hence  $z^2 + 2z^2 + az + b \equiv (z+2)(z^2 - 2z + 2)$   
 $\equiv z^3 + 2z^2 + 2z + 2z^2 - 4z + 4c$   
 $\equiv z^3 + (c-2)z^2 + (2-2c)z + 4c$

Thus  $c-2=2$   
 $\frac{c}{2}=2$   
 $c=4$

$\frac{a}{2}=-2$   
 $a=-4$

$4c=b$   
 $b=16$

$\therefore a=-4$   
 $b=16$   
 $\bar{z}_3 = -4$

**METHOD B:**

SUM OF THE 3 ROOTS IS  $\frac{-b}{a} = -\frac{2}{1} = -2$

THUS  $(1+i) + (1-i) + \bar{z}_3 = -2$   
 $2 + \bar{z}_3 = -2$   
 $\bar{z}_3 = -4$

$\frac{c}{a} = -2$   
 $c = -2a$

$\frac{a}{2} = (1+i)(1-i) + (1-i)(-4) + (1+i)(-4)$   
 $a = 2 - 4 + 4i - 4 - 4i$   
 $a = -6$

$\frac{b}{a} = (1+i)(1-i)(-4)$   
 $b = (1+i)(1-i)(-4)$   
 $b = 2 \times 4$   
 $b = 8$

**Question 5**

The following cubic equation is given

$$z^3 + Az^2 + Bz + 26 = 0,$$

where  $A \in \mathbb{R}$ ,  $B \in \mathbb{R}$

One of the roots of the above cubic equation is  $1+i$ .

- a) Find the real root of the equation.
- b) Determine the value of  $A$  and the value of  $B$ .

$$z = -13, A = 11, B = -24$$

(a)  $z^3 + Az^2 + Bz + 26 = 0$

Given  $z = 1+i$  and solutions  
 $\therefore z - (1+i)$   
 $\therefore z - (1-i)$   
 $\therefore z - (-1)$

Thus by inspection of  $z^3 + 2z^2 + Az^2 + Bz + 26 = 0$   
 $(z^3 - 2z^2 + z)(z + 13) = 0$   
 $\therefore$  Real Root is  $z = -13$

(b) Expanding  $(z^3 - 2z^2 + z)(z + 13) = \frac{z^3 + Bz^2}{-2z^2 - 26z} + \frac{26 + 26}{z^3 + 11z - 24z + 26}$   
 $\therefore A = 11, B = -24$

**Question 6**

The cubic equation

$$2z^3 - z^2 + 4z + p = 0, \quad p \in \mathbb{R},$$

is satisfied by  $z = 1 + 2i$ .

- a) Find the other two roots of the equation.
- b) Determine the value of  $p$ .

$$\boxed{1-2i, -\frac{3}{2}}, \quad \boxed{p=15}$$

**a)** AS THE COEFFICIENTS OF THE POLYNOMIAL EQUATION ARE REAL, ANY COMPLEX ROOTS MUST APPEAR AS CONJUGATE PAIRS. — SO WE HAVE:

$$\begin{aligned} z_1 &= 1+2i, \quad \text{say } \alpha \\ z_2 &= 1-2i, \quad \text{say } \beta \end{aligned}$$

NOW  $\alpha(\beta+\gamma) = -\frac{1}{2}$

$$(1+2i)(1-2i) + \gamma = -\frac{1}{2}$$

$$2 + \gamma = -\frac{1}{2}$$

$$\gamma = -\frac{5}{2}$$

∴ SOLUTIONS ARE:  $1+2i, 1-2i, -\frac{5}{2}$

**b)** NOW  $\alpha\beta\gamma = -\frac{1}{2}$

$$(1+2i)(1-2i)\left(-\frac{5}{2}\right) = -\frac{5}{2}$$

$$3(1+2i)(1-2i) = p$$

$$p = 3(1^2 + 2^2)$$

$$p = 15$$

ALTERNATIVE WITHOUT USING ROOT RELATIONSHIPS

$$\begin{aligned} (1+2i)^2 &= 1+4i+(2i)^2 = 1+4i-4 = -3+4i \\ (1+2i)^3 &= (-3+4i)(1+2i) = -3-6i+4i-8 = -11-2i \end{aligned}$$

SUB INTO THE ABC TO FIND  $p$  FIRST

$$\begin{aligned} 2(-11-2i) - 11-2i + 4(-11-2i) + p &= 0 \\ 2(-11-2i) - (-3+4i) + 4(-11-2i) + p &= 0 \\ -20-4i + 3-4i - 44-8i + p &= 0 \\ p &= 15 \end{aligned}$$

NON-SOLUTIONS MUST APPEAR IN CONJUGATE PAIRS IF COMPLEX

$$\begin{aligned} (z-1-2i)(z-1+2i) &= [(z-1)-2i][(z-1)+2i] \\ &= (z-1)^2 - (2i)^2 \\ &= z^2 - 2z + 1 + 4 \\ &= z^2 - 2z + 5 \end{aligned}$$

BY INSPECTION

$$2z^2 - z^2 + 4z + 15 = (2z+5)(z^2 - 2z + 5)$$

∴  $z = \boxed{\begin{array}{c} 1+2i \\ 1-2i \\ -\frac{5}{2} \end{array}}$

**Question 7**

Consider the cubic equation

$$z^3 + z + 10 = 0, \quad z \in \mathbb{C}.$$

- Verify that  $1+2i$  is a root of this equation.
- Find the other two roots.

$$z_1 = 1 - 2i, \quad z_2 = -2$$

(a)  $(1+2i)^3 + (1+2i) + 10 = (1+2i)(1+2i)^2 + 1 + 2i$   
 $= (1+2i)(1+4i-4) + 1 + 2i$   
 $= (1+2i)(-3+4i) + 1 + 2i$   
 $= -3 - 8i + 6i + 8 + 1 + 2i$   
 $= 0$   
 $\therefore z_1 = 1+2i$  IS indeed a solution

(b)  $\bullet z_2 = -2$  ✓ (AS EQUATION HAS REAL COEFFICIENTS, COMPLEX ROOTS WILL EXIST IN CONJUGATE PAIRS)

$\bullet [z - (1-2i)][z - (1+2i)] = [(z-1)+2i][(z-1)-2i] = (z-1)^2 - (2i)^2$   
 $= z^2 - 2z + 1 + 4 = z^2 - 2z + 5$

THUS BY INSPECTION  $z^2 - 2z + 10 = 0$   
 $(z+2)(z-2+5) = 0$   
 $\therefore z_3 = -2$

ALTERNATIVE USING ROOTS OF POLYNOMIALS THEORY

 $\alpha + \beta + \gamma = -\frac{b}{a}$   
 $\alpha\beta + \gamma = 0$   
 $(1+2i) + (1-2i) + \gamma = 0$   
 $2 + \gamma = 0$   
 $\gamma = -2$

**Question 8**

Solve the equation

$$2z^4 - 14z^3 + 33z^2 - 26z + 10 = 0, \quad z \in \mathbb{C}$$

given that one of its roots is  $3+i$ .

$$z = 3+i, \quad z = 3-i, \quad z = \frac{1}{2} + \frac{1}{2}i, \quad z = \frac{1}{2} - \frac{1}{2}i$$

As the polynomial equation has real coefficients any solutions  
MUST appear in conjugate pairs, so  $z = 3-i$  is a solution.

$$\begin{aligned} [z - (3+i)][z - (3-i)] &= [(z-3) - i][(z-3) + i] \\ &= (z-3)^2 - i^2 \\ &= z^2 - 6z + 9 + 1 \\ &= z^2 - 6z + 10 \end{aligned}$$

By long division

$$\begin{array}{r} 2z^3 - 2z^2 + 10 \\ 2z^4 - 14z^3 + 33z^2 - 26z + 10 \\ \hline -2z^4 + 14z^3 - 33z^2 \\ \hline -2z^3 + 28z^2 - 26z + 10 \\ +2z^3 - 12z^2 - 10z \\ \hline z^2 - 6z + 10 \\ -z^2 + 6z - 10 \\ \hline 0 \end{array}$$

Hence  $2z^2 - 2z + 1 = 0$

$$\begin{aligned} 4z^2 - 4z + 2 &= 0 \\ 4z^2 - 4z + 1 &= -1 \\ (2z-1)^2 &= -1 \\ 2z-1 &= \pm i \\ 2z &= 1 \pm i \\ z &= \frac{1}{2} \pm \frac{1}{2}i \end{aligned}$$

The full solution set is  $3+i, 3-i, \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} - \frac{1}{2}i$

**Question 9**

$$2z^3 + pz^2 + qz + 16 = 0, \quad p \in \mathbb{R}, \quad q \in \mathbb{R}.$$

The above cubic equation has roots  $\alpha$ ,  $\beta$  and  $\gamma$ , where  $\gamma$  is real.

It is given that  $\alpha = 2(1+i\sqrt{3})$ .

- Find the other two roots,  $\beta$  and  $\gamma$ .
- Determine the values of  $p$  and  $q$ .

$$\boxed{\beta = 2(1-i\sqrt{3})}, \boxed{\gamma = -\frac{1}{2}}, \boxed{p = -7}, \boxed{q = 28}$$

a) As coefficients are real  $R = 2(-i\sqrt{3})$

- $\alpha\beta\gamma = -\frac{p}{2}$   
 $\Rightarrow 2(-i\sqrt{3}) \times 2(-i\sqrt{3}) \times \gamma = -8$   
 $\Rightarrow 4\gamma(-i^2 + \sqrt{3}i) = -8$   
 $\Rightarrow 4\gamma = -8$   
 $\Rightarrow \gamma = -2$   
 $\Rightarrow \gamma = -\frac{1}{2} \cancel{/}$
- b)  $\alpha + \beta + \gamma = -\frac{p}{2}$   
 $\Rightarrow 2(-i\sqrt{3}) + 2(-i\sqrt{3}) - \frac{1}{2} = -\frac{p}{2}$   
 $\Rightarrow 4 - \frac{1}{2} = -\frac{p}{2}$   
 $\Rightarrow \frac{7}{2} = -\frac{p}{2}$   
 $\Rightarrow p = -7 \cancel{/}$
- $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{q}{2}$   
 $\Rightarrow 2(-i\sqrt{3}) \times 2(-i\sqrt{3}) + 2(-i\sqrt{3})(-\frac{1}{2}) - \frac{1}{2} \times 2(-i\sqrt{3}) = \frac{q}{2}$   
 $\Rightarrow 4(-i^2) - (-i\sqrt{3}) - (i\sqrt{3}) = \frac{q}{2}$   
 $\Rightarrow 4 + 2 = \frac{q}{2}$   
 $\Rightarrow q = 28 \cancel{/}$

**Question 10**

$$z^4 - 8z^3 + 33z^2 - 68z + 52 = 0, z \in \mathbb{C}.$$

One of the roots of the above quartic equation, is  $2 + 3i$ .

Find the other roots of the equation.

$$z = 2 - 3i, z = 2$$

$z^4 - 8z^3 + 33z^2 - 68z + 52 = 0, z \in \mathbb{C}$

AS THE EQUATION HAS REAL COEFFICIENTS, ANY ROOTS IF COMPLEX MUST EXIST AS CONJUGATE PAIRS

$\therefore z_1 = 2 + 3i \implies z_2 = 2 - 3i$

PROCEEDED AS FOLLOWS:

$$\begin{aligned} (z - z_1)(z - z_2) &= [z - (2+3i)][z - (2-3i)] \\ &= [(z-2) - 3i][(z-2) + 3i] \\ &= (z-2)^2 - (3i)^2 \\ &= z^2 - 4z + 4 \\ &= z^2 - 4z + 13 \end{aligned}$$

BY "LONG DIVISION" OR "SYNTHETIC"

$z^2 - 4z + 13$	$\overline{z^4 - 8z^3 + 33z^2 - 68z + 52}$
$+ z^2 + 4z + 13z^2$	$- 1z^3 + 26z^2 - 68z + 52$
$+ 4z^2 + 16z^2 + 52z$	$+ 4z^3 - 16z^2 + 52$
$+ 16z^2 + 16z^2 - 52$	$- 4z^3 + 16z^2 - 52$
	$0$

HENCE WE HAVE

$$z^4 - 8z^3 + 33z^2 - 68z + 52 = (z^2 - 4z + 13)(z^2 - 4z + 4)$$

$$= (z - 4z + 13)(z - 2)^2$$

HENCE THE FULL SET OF SOLUTIONS IS

$z =$

2+3i (green)  
2-3i  
2 (yellow)

**Question 11**

It is given that  $z = 2$  and  $z = 1 + 2i$  are solutions of the equation

$$z^4 - 3z^3 + az^2 + bz + c = 0.$$

where  $a$ ,  $b$  and  $c$  are real constants.

Determine the values of  $a$ ,  $b$  and  $c$ .

$$\boxed{a = 5}, \boxed{b = -1}, \boxed{c = -10}$$

Plotted to follow – As quartic has real coefficients any complex roots will appear to conjugate pairs

$$\begin{aligned} z_1 &= 2 & z_2 &= 1+2i & z_3 &= 1-2i \\ \text{Now, the sum of all 4 roots satisfy} \\ z_1 + z_2 + z_3 + z_4 &= -\frac{b}{a} \\ 2 + (1+2i) + (1-2i) + z_4 &= -\frac{-3}{1} \\ 4 + z_4 &= 3 \\ z_4 &= -1 \end{aligned}$$

This we find

$$\begin{aligned} \rightarrow [z - (1+2i)][z - (1-2i)][z + 1](z - 2) &= 0 \\ \rightarrow [(z-1)-2i][(z-1)+2i](z^2-2z-2) &= 0 \\ \rightarrow [(z-1)^2 - (2i)^2](z^2-2z-2) &= 0 \\ \rightarrow [z^2-2z+1 - (-4)](z^2-2z-2) &= 0 \\ \rightarrow (z^2-2z+5)(z^2-2z-2) &= 0 \\ \rightarrow z^4 - z^3 - 2z^2 - 2z^3 + 2z^2 + 4z &= 0 \\ z^4 - 3z^3 + 2z^2 - 2z - 10 &= 0 \\ \rightarrow z^4 - 3z^3 + 2z^2 - z - 10 &= 0 \end{aligned}$$

$\therefore a = 5$   
 $b = -1$   
 $c = -10$

**Question 12**

If  $1 - 2i$  is a root of the quartic equation

$$z^4 - 6z^3 + 18z^2 - 30z + 25 = 0$$

find the other three roots.

$$z_2 = 1 + 2i, \quad z_3 = 2 - i, \quad z_4 = 2 + i$$

IF  $z_1 = 1 - 2i$  IS A ROOT, THEN  $z_2 = 1 + 2i$  MUST ALSO BE A SOLUTION AS THE COEFFICIENTS OF THE QUADRATIC ARE REAL.

$$\begin{aligned} [z - (1-2i)][z - (1+2i)][(z-1) - 2i][z - 1] &= (z-1)^2 - (2i)^2 \\ &= z^2 - 2z + 1 + 4 = z^2 - 2z + 5 \end{aligned}$$

LONG DIVIDE TO REDUCE THE QUADRATIC.

$$\begin{array}{r} z^2 - 4z + 5 \\ \overline{z^2 - 2z + 5} \\ \underline{-z^2 + 2z^2} \\ \hline 4z^2 - 30z + 25 \\ \underline{4z^2 + 20z} \\ \hline -50z + 25 \\ \underline{-50z - 10z} \\ \hline 0 \end{array}$$

SOLVE THE REMAINING QUADRATIC EQUATION

$$\begin{aligned} z^2 - 4z + 5 &= 0 \\ (z-2)^2 + 4i^2 &= 0 \\ (z-2)^2 &= -4 \\ z-2 &= \pm 2i \\ z &= 2 \pm 2i \end{aligned}$$