GAMMA FUNCTION

SUMMARY OF THE GAMMA FUNCTION

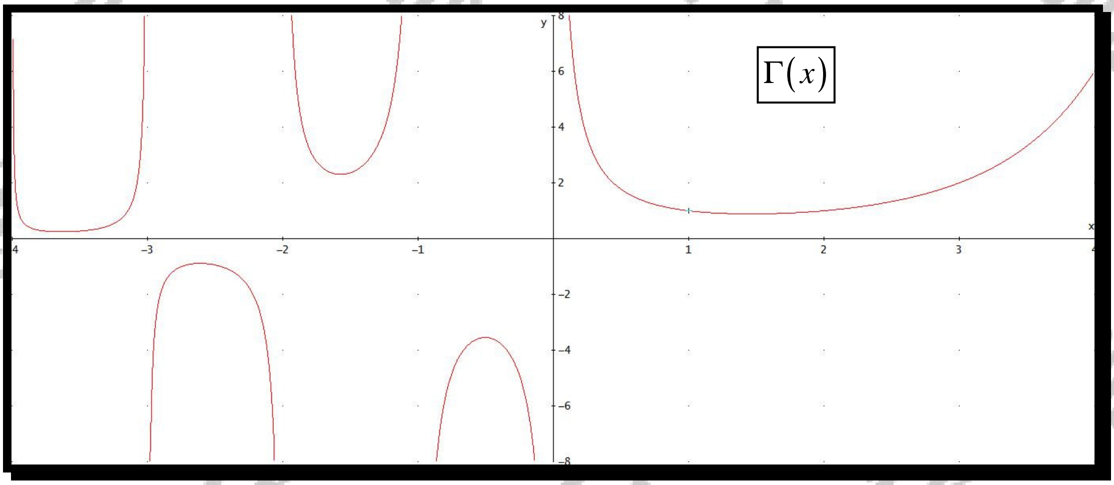
The Gamma function $\Gamma(x)$, is defined as

$$\Gamma(x) \equiv \int_0^{\infty} t^{x-1} \mathrm{e}^{-t} \, dt ,$$

where $x \neq \ldots -4, -3, -2, -1, 0$.

Gamma function common rules and facts

- $\Gamma(x+1) \equiv x\Gamma(x) \quad x \in \mathbb{R}$
- $\Gamma(n+1) \equiv n!$, $n \in \mathbb{N}$ or $\Gamma(n) \equiv (n-1)!$, $n \in \mathbb{N}$
- $\Gamma(1) = 1$, $\Gamma(2) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(0) = \pm\infty$, $\Gamma(-1) = \pm\infty$, $\Gamma(-2) = \pm\infty$, $\Gamma(-3) = \pm\infty$, etc
- $\Gamma(z)\Gamma(1-z) \equiv \dfrac{\pi}{\sin \pi z}$, $z \in \mathbb{C}$
- $\Gamma'(x) = \Gamma(x)\left[-\gamma + \dfrac{1}{x} + \displaystyle\sum_{k=1}^{\infty}\left(\dfrac{1}{k} - \dfrac{1}{x+k}\right)\right]$



Question 1

Evaluate each of the following expressions, leaving the final answer in exact simplified form.

a) $\dfrac{3\Gamma(6)}{\Gamma(4)}$.

b) $\Gamma\left(\frac{3}{2}\right)$.

c) $\displaystyle\int_0^{\infty} x^7 \, \mathrm{e}^{-x} \; dx$.

V, $\boxed{60}$, $\boxed{\frac{1}{2}\sqrt{\pi}}$, $\boxed{7!=5040}$

a) $\dfrac{3\Gamma(6)}{\Gamma(4)} = \dfrac{3\times 5\Gamma(5)}{\Gamma(4)} = \dfrac{15\times 4\cancel{\Gamma(4)}}{\cancel{\Gamma(4)}} = 60$

b) $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

c) $\displaystyle\int_0^{\infty} x^7 e^{-x}\,dx = \int_0^{\infty} x^{8-1} e^{-x}\,dx = \Gamma(8) = 7!$

Question 2

Evaluate each of the following expressions, leaving the final answer in exact simplified form.

a) $\dfrac{60\Gamma(5)}{\Gamma(7)}$.

b) $\Gamma\left(\frac{5}{2}\right)$.

c) $\displaystyle\int_0^{\infty} 2x^4 \, \mathrm{e}^{1-x} \; dx$.

V, $\boxed{2}$, $\boxed{\frac{3}{4}\sqrt{\pi}}$, $\boxed{48\mathrm{e}}$

a) $\dfrac{60\Gamma(5)}{\Gamma(7)} = \dfrac{60\Gamma(5)}{6\Gamma(6)} = \dfrac{60\cancel{\Gamma(5)}}{6\times 5\cancel{\Gamma(5)}} = 2$

b) $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

c) $\displaystyle\int_0^{\infty} 2x^4 e^{1-x}\,dx = \int_0^{\infty} 2x^{5-1} e^{1} e^{-x}\,dx = 2e\int_0^{\infty} x^{5-1}e^{-x}\,dx$

$= 2e\Gamma(5) = 2e\times 4! = 48e$

Question 3

Evaluate each of the following expressions, leaving the final answer in exact simplified form.

a) $\dfrac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$.

b) $\Gamma\left(-\frac{3}{2}\right)$.

c) $\displaystyle\int_0^\infty x^3 \mathrm{e}^{-4x} \ dx$.

V, $\boxed{-2}$, $\boxed{\frac{4}{3}\sqrt{\pi}}$, $\boxed{\frac{3}{128}}$

a) $\dfrac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \dfrac{-\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)}{-\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \dfrac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \dfrac{1}{-\frac{1}{2}} = -2$

OR $\dfrac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \dfrac{\cancel{\Gamma\left(-\frac{1}{2}\right)}}{-\frac{1}{2}\cancel{\Gamma\left(-\frac{1}{2}\right)}} = \dfrac{1}{-\frac{1}{2}} = -2$

b) $\Gamma\left(-\frac{3}{2}\right) \times \left(-\frac{3}{2}\right) = \Gamma\left(-\frac{1}{2}\right)$

$\Rightarrow \Gamma\left(-\frac{3}{2}\right) = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right)$

$\Rightarrow \Gamma\left(-\frac{3}{2}\right) = -\frac{2}{3} \times (-2) \times \left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)$

$\Rightarrow \Gamma\left(-\frac{3}{2}\right) = \frac{4}{3}\Gamma\left(\frac{1}{2}\right)$

$\Rightarrow \Gamma\left(-\frac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$

c) $\displaystyle\int_0^\infty x^3 e^{-4x}\,dx = \ldots \text{substitution} \ldots = \int_0^\infty \left(\tfrac{1}{4}t\right)^3 e^{-t}\,\frac{dt}{4}$

$\displaystyle = \frac{1}{256}\int_0^\infty t^3 e^{-t}\,dt = \frac{1}{256}\int_0^\infty t^{4-1} e^{-t}\,dt$

$= \frac{1}{256}\Gamma(4) = \frac{1}{256} \times 3! = \frac{6}{256}$

$= \frac{3}{128}$

$t = 4x$

$\frac{dt}{dx} = 4$

$dx = \frac{dt}{4}$

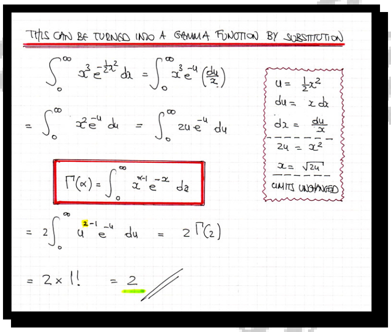
LIMITS UNCHANGED

Question 4

By using techniques involving the Gamma function, find the value of

$$\int_0^{\infty} x^3 \mathrm{e}^{-\frac{1}{2}x^2} \, dx .$$

V, 1M3, 2

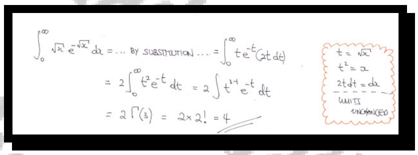


Question 5

By using techniques involving the Gamma function, find the value of

$$\int_0^{\infty} \sqrt{x}\, \mathrm{e}^{-\sqrt{x}} \, dx .$$

V, 4



Question 6

By using techniques involving the Gamma function, find the value of

$$\int_1^{\infty} \frac{(\ln x)^3}{x^2}\, dx .$$

V, $\boxed{6}$

$$\int_1^{\infty} \frac{(\ln x)^3}{x^2}\, dx = \ldots \text{ BY SUBSTITUTION } \ldots = \int_0^{\infty} \frac{u^3}{x^2}\,(x\,du)$$

$$= \int_0^{\infty} \frac{u^3}{x}\, du = \int_0^{\infty} \frac{u^3}{e^u}\, du = \int_0^{\infty} u^3 e^{-u}\, du$$

$$= \int_0^{\infty} u^{4-1} e^{-u}\, du = \Gamma(4) = 3! = 6$$

$u = \ln x$
$du = \frac{1}{x}\,dx$
$dx = x\,du$
$x = e^u$
$x = 1 \mapsto u = 0$
$x = \infty \mapsto u = \infty$

Question 7

By using techniques involving the Gamma function, find the exact value of

$$\int_0^{\infty} 2\sqrt{x}\, e^{-x^2}\, dx .$$

Give the answer in the form $\Gamma(k)$, where k is a rational constant.

V, $\boxed{\Gamma\left(\frac{3}{4}\right)}$

$$\int_0^{\infty} 2\sqrt{x}\, e^{-x^2}\, dx = \ldots \text{ BY SUBSTITUTION } \ldots = \int_0^{\infty} 2t^{\frac{1}{4}} e^{-t}\left(\frac{dt}{2x}\right)$$

$$= \int_0^{\infty} t^{\frac{1}{4}} e^{-t}\, \frac{dt}{x} = \int_0^{\infty} t^{\frac{1}{4}} e^{-t}\, \frac{dt}{t^{\frac{1}{2}}}$$

$$= \int_0^{\infty} t^{-\frac{1}{4}} e^{-t}\, dt = \int_0^{\infty} t^{\frac{3}{4}-1} e^{-t}\, dt$$

$$= \Gamma\left(\tfrac{3}{4}\right)$$

$t = x^2$
$t^{\frac{1}{2}} = x$
$t^{\frac{1}{4}} = x^{\frac{1}{2}}$
$dt = 2x\,dx$
LIMITS UNCHANGED

Question 8

By using techniques involving the Gamma function, find the exact value of

$$\int_0^{\infty} x^6 \, e^{-4x^2} \; dx .$$

Give the answer in the form $k\sqrt{\pi}$, where k is a rational constant.

V, $\boxed{\dfrac{15\sqrt{\pi}}{2048}}$

$$\int_0^{\infty} x^6 e^{-4x^2} dx = \dots \text{BY SUBSTITUTION} \dots = \int_0^{\infty} x^6 e^{-t} \frac{dt}{8x}$$

$$= \frac{1}{8}\int_0^{\infty} x^5 e^{-t} dt = \frac{1}{8}\int_0^{\infty} \left(\frac{t^{\frac{1}{2}}}{2}\right)^5 e^{-t} \, dt$$

$$= \frac{1}{256}\int_0^{\infty} t^{\frac{5}{2}} e^{-t} dt = \frac{1}{256}\int_0^{\infty} t^{\frac{7}{2}-1} e^{-t} dt$$

$$= \frac{1}{256}\Gamma\left(\frac{7}{2}\right) = \frac{1}{256} \times \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{1}{256} \times \frac{5}{2} \times \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{256} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{15}{2048}\Gamma\left(\frac{1}{2}\right) = \frac{15\sqrt{\pi}}{2048}$$

$t = 4x^2$

$dt = 8x \, dx$

$dx = \dfrac{dt}{8x}$

$x^2 = \dfrac{t}{4}$

$x = \dfrac{t^{1/2}}{2}$

LIMITS UNCHANGED

Question 9

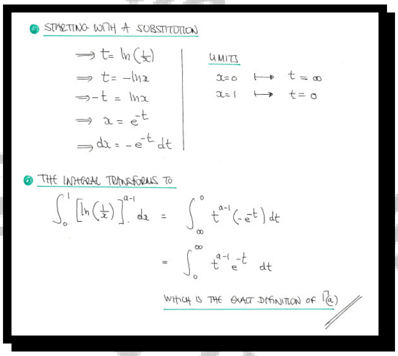
By using techniques involving the Gamma function, find the exact value of

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{a-1} \, dx ,$$

where $a \neq 1, \ 0, -1, -2, -3, \ldots$

Give the answer in terms of a Gamma function.

V , $\boxed{}$, $\boxed{\Gamma(a)}$



STARTING WITH A SUBSTITUTION

$\Rightarrow t = \ln\left(\frac{1}{x}\right)$

$\Rightarrow t = -\ln x$

$\Rightarrow -t = \ln x$

$\Rightarrow x = e^{-t}$

$\Rightarrow dx = -e^{-t}\,dt$

LIMITS

$x=0 \mapsto t=\infty$

$x=1 \mapsto t=0$

THE INTEGRAL TRANSFORMS TO

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{a-1} dx = \int_\infty^0 t^{a-1}\left(-e^{-t}\right) dt$$

$$= \int_0^\infty t^{a-1} e^{-t} \, dt$$

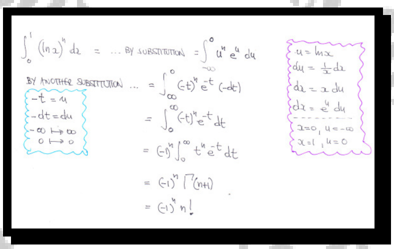
WHICH IS THE EXACT DEFINITION OF $\Gamma(a)$

Question 10

By using techniques involving the Gamma function, show that

$$\int_0^1 [\ln x]^n \, dx = (-1)^n \, n! , \quad n \in \mathbb{N} .$$

$\boxed{\text{proof}}$



$$\int_0^1 (\ln x)^n \, dx = \ldots \text{ BY SUBSTITUTION } = \int_{-\infty}^0 u^n e^u \, du$$

$u = \ln x$

$du = \frac{1}{x} dx$

$dx = x\,du$

$dx = e^u\,du$

$x=0, u=-\infty$

$x=1, u=0$

BY ANOTHER SUBSTITUTION ... $= \int_\infty^0 (-t)^n e^{-t} (-dt)$

$-t = u$

$-dt = du$

$-\infty \mapsto \infty$

$0 \mapsto 0$

$$= \int_0^\infty (-t)^n e^{-t} \, dt$$

$$= (-1)^n \int_0^\infty t^n e^{-t} \, dt$$

$$= (-1)^n \Gamma(n+1)$$

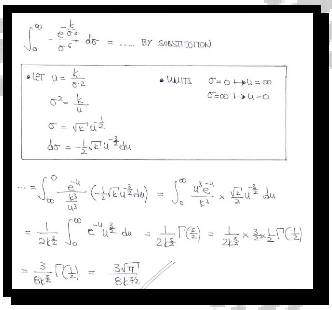
$$= (-1)^n n!$$

Question 11

By using techniques involving the Gamma function, show that

$$\int_0^\infty \frac{e^{-\frac{k}{\sigma^2}}}{\sigma^6}\, d\sigma = \frac{3\sqrt{\pi}}{8k^{\frac{5}{2}}}, \quad k \neq 0.$$

$\boxed{\text{proof}}$



$$\int_0^\infty \frac{e^{-\frac{k}{\sigma^2}}}{\sigma^6}\, d\sigma = \dots \text{ BY SUBSTITUTION}$$

- LET $u = \frac{k}{\sigma^2}$
  $\sigma^2 = \frac{k}{u}$
  $\sigma = \sqrt{k}\, u^{-\frac{1}{2}}$
  $d\sigma = -\frac{1}{2}\sqrt{k}\, u^{-\frac{3}{2}}\, du$
- LIMITS $\sigma = 0 \mapsto u = \infty$
  $\sigma = \infty \mapsto u = 0$

$$\dots = \int_\infty^0 \frac{e^{-u}}{\frac{k^3}{u^3}}\left(-\frac{1}{2}\sqrt{k}\, u^{-\frac{3}{2}}\, du\right) = \int_0^\infty \frac{u^3 e^{-u}}{k^3} \times \frac{\sqrt{k}}{2} u^{-\frac{3}{2}}\, du$$

$$= \frac{1}{2k^{\frac{5}{2}}} \int_0^\infty e^{-u} u^{\frac{3}{2}}\, du = \frac{1}{2k^{\frac{5}{2}}} \Gamma\left(\tfrac{5}{2}\right) = \frac{1}{2k^{\frac{5}{2}}} \times \frac{3}{2} \times \frac{1}{2} \Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{3}{8k^{\frac{5}{2}}} \Gamma\left(\tfrac{1}{2}\right) = \frac{3\sqrt{\pi}}{8k^{5/2}}$$

Question 12

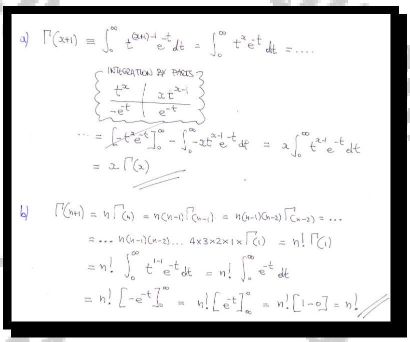
a) Show clearly that

$$\Gamma(x+1) \equiv x\,\Gamma(x), \; x \neq 0, \; x \notin -\mathbb{N}.$$

b) Hence show further that

$$\Gamma(n+1) \equiv n!\ , \; n \in \mathbb{N}.$$

$\boxed{\text{proof}}$



a) $\Gamma(x+1) \equiv \int_0^\infty t^{(x+1)-1} e^{-t}\, dt = \int_0^\infty t^x e^{-t}\, dt = \dots$

INTEGRATION BY PARTS

t^x	xt^{x-1}
$-e^{-t}$	e^{-t}

$$\dots = \cancel{\left[-t^x e^{-t}\right]_0^\infty} - \int_0^\infty -xt^{x-1}e^{-t}\, dt = x\int_0^\infty t^{x-1}e^{-t}\, dt$$

$$= x\,\Gamma(x)$$

b) $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) = \dots$

$$= \dots n(n-1)(n-2)\dots 4\times3\times2\times1\times\Gamma(1) = n!\,\Gamma(1)$$

$$= n!\int_0^\infty t^{1-1}e^{-t}\, dt = n!\int_0^\infty e^{-t}\, dt$$

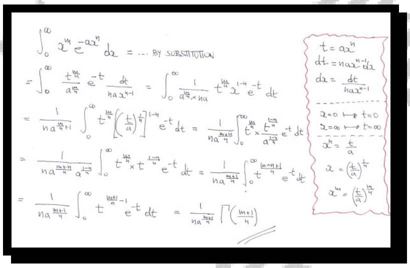
$$= n!\left[-e^{-t}\right]_0^\infty = n!\left[e^{-t}\right]_\infty^0 = n!\,[1-0] = n!$$

Question 13

By using techniques involving the Gamma function, show that

$$\int_0^\infty x^m \, e^{-ax^n} \; dx = \frac{\Gamma\left(\frac{m+1}{n}\right)}{na^{\frac{m+1}{n}}} \; .$$

proof

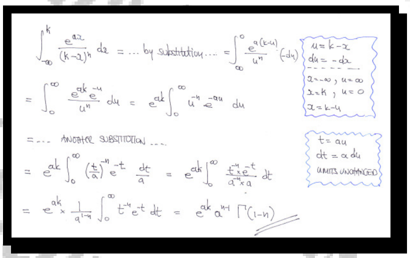


Question 14

By using techniques involving the Gamma function, show that

$$\int_{-\infty}^{k} \frac{e^{ax}}{(k-x)^n} \; dx = e^{ak} a^{n-1} \Gamma(1-n) \; , \; 0 < n < 1 .$$

proof



Question 15

The gamma function is defined as

$$\Gamma(x) = \int_0^{\infty} \mathrm{e}^{-t}\, t^{x-1}\ dt\,, \qquad x \neq 0, -1, -2, -3, \ldots$$

i. Show that for $x > 0$

$$\Gamma(x+1) = x\,\Gamma(x).$$

ii. Express the integral

$$I(s) = \int_0^{\infty} \mathrm{e}^{-p^s}\ dp\,, \qquad s > 0\,,$$

in terms of Gamma functions.

iii. Deduce that

$$\lim_{s\to\infty}\left[I(s)\right] = 1\,.$$

$$\boxed{I(s) = \frac{1}{s}\Gamma\left(\frac{1}{s}\right)}$$

I) BY DEFINITION $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt$

THUS $\Gamma(x+1) = \int_0^{\infty} e^{-t} t^{(x+1)-1}\, dt = \int_0^{\infty} e^{-t} t^{x}\, dt$

INTEGRATING BY PARTS W.R.T t

t^x	xt^{x-1}
$-e^{-t}$	e^{-t}

$\Gamma(x+1) = \cancel{\left[-t^x e^{-t}\right]_0^{\infty}} + x\int_0^{\infty} e^{-t} t^{x-1}\, dt$

$\Gamma(x+1) = x\int_0^{\infty} e^{-t} t^{x-1}\, dt$

$\Gamma(x+1) = x\Gamma(x)$ //

II) $I(s) = \int_0^{\infty} e^{-p^s}\, dp$

BY SUBSTITUTION

$t = p^s$

$\frac{dt}{dp} = sp^{s-1}$

$dp = \frac{dt}{sp^{s-1}}$

LIMITS UNCHANGED

$I(s) = \int_0^{\infty} e^{-t}\left(\frac{1}{s}p^{1-s}\right) dt$

$I(s) = \int_0^{\infty} \frac{1}{s} e^{-t}\left(t^{\frac{1}{s}}\right)^{1-s} dt$

$I(s) = \int_0^{\infty} \frac{1}{s} e^{-t} t^{\frac{1}{s}-1}\, dt$

$I(s) = \frac{1}{s}\int_0^{\infty} e^{-t} t^{\frac{1}{s}-1}\, dt$

$I(s) = \frac{1}{s}\Gamma\left(\frac{1}{s}\right)$ //

III) $\lim_{s\to\infty}\left[I(s)\right] = \lim_{s\to\infty}\left[\frac{1}{s}\Gamma\left(\frac{1}{s}\right)\right] = \lim_{s\to\infty}\left[\Gamma\left(\frac{1}{s}+1\right)\right]$

$= \Gamma(1) = 0! = 1$ //

Question 16

a) Write down the definition of the Gamma function $\Gamma(x)$, for $\text{Re}(x) > 0$.

b) Use the standard recurrence relation of the Gamma function to extent $\Gamma(x)$, for $\text{Re}(x) \le 0$, and hence find the residues at the simple poles at $x = -n$, $n = 0, 1, 2, 3 \ldots$

$$\boxed{\frac{(-1)^n}{n!}}$$

a) $\boxed{\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} \, dt}$

b) USING THE RECCURENCE $\Gamma(x+1) = x\Gamma(x)$

$$\boxed{\Gamma(x) = \frac{\Gamma(x+1)}{x}}$$

LET $x = u-n$, WHERE $n \in \mathbb{N}$ & $0 < u < 1$

$$\Gamma(u-n) = \frac{\Gamma(u-n+1)}{u-n}$$

$$\Gamma(u-n) = \frac{1}{u-n} \times \frac{\Gamma(u-n+2)}{u-n+1}$$

$$\Gamma(u-n) = \frac{1}{u-n} \times \frac{1}{u-n+1} \times \frac{\Gamma(u-n+3)}{u-n+2}$$

$$\Gamma(u-n) = \frac{1}{u-n} \times \frac{1}{u-n+1} \times \frac{1}{u-n+2} \times \cdots \times \frac{1}{u-2} \, \frac{\Gamma(u)}{u-1}$$

$$\Gamma(u-n) = \frac{1}{u-n} \times \frac{1}{u-n+1} \times \frac{1}{u-n+2} \times \cdots \times \frac{1}{u-2} \times \frac{1}{u-1} \times \frac{\Gamma(u+1)}{u}$$

$$\Gamma(u-n) = (-1)^n \frac{1}{(n-u)(n-u-1)(n-u-2)\ldots\times(2-u)(1-u)} \times \frac{\Gamma(u+1)}{u}$$

HENCE RESIDUE

$$\lim_{u \to 0} \left[u\,\Gamma(u-n) \right]$$

$$= \lim_{u \to 0} \left[\frac{\cancel{u}(-1)^n \Gamma(u+1)}{(n-u)(n-u-1)(n-u-2)\ldots(2-u)(1-u)\cancel{u}} \right]$$

$$= \frac{(-1)^n \Gamma(1)}{n(n-1)(n-2)\ldots\times 3\times 2\times 1}$$

$$= \frac{(-1)^n}{n!}$$

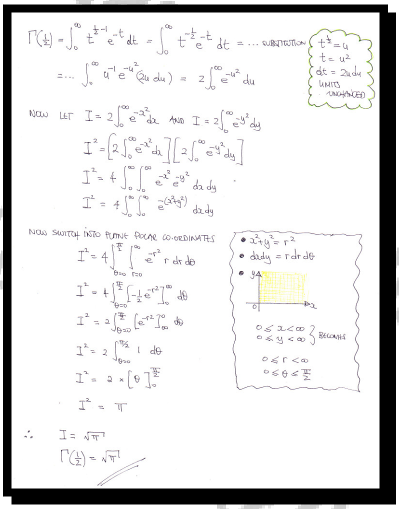
Question 17

$$I = 2\int_0^{\infty} \mathrm{e}^{-x^2}\ dx \quad \text{and} \quad I = 2\int_0^{\infty} \mathrm{e}^{-y^2}\ dy\ .$$

By finding an expression for I, show that

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\ .$$

$\boxed{\text{proof}}$



$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} t^{\frac{1}{2}-1} e^{-t}\,dt = \int_0^{\infty} t^{-\frac{1}{2}} e^{-t}\,dt = \dots$ SUBSTITUTION

$t^{\frac{1}{2}} = u$
$t = u^2$
$dt = 2u\,du$
LIMITS UNCHANGED

$= \dots \int_0^{\infty} u^{-1} e^{-u^2} (2u\,du) = 2\int_0^{\infty} e^{-u^2}\,du$

NOW LET $I = 2\int_0^{\infty} e^{-x^2}\,dx$ AND $I = 2\int_0^{\infty} e^{-y^2}\,dy$

$I^2 = \left[2\int_0^{\infty} e^{-x^2}\,dx\right]\left[2\int_0^{\infty} e^{-y^2}\,dy\right]$

$I^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-x^2} e^{-y^2}\,dx\,dy$

$I^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$

NOW SWITCH INTO PLANE POLAR CO-ORDINATES

- $x^2+y^2 = r^2$
- $dx\,dy = r\,dr\,d\theta$
- $0 \leq x < \infty$, $0 \leq y < \infty$ BECOMES $0 \leq r < \infty$, $0 \leq \theta \leq \frac{\pi}{2}$

$I^2 = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$

$I^2 = 4\int_{\theta=0}^{\frac{\pi}{2}}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty}\,d\theta$

$I^2 = 2\int_{\theta=0}^{\frac{\pi}{2}}\left[e^{-r^2}\right]_{\infty}^{0}\,d\theta$

$I^2 = 2\int_{\theta=0}^{\pi/2} 1\,d\theta$

$I^2 = 2 \times \left[\theta\right]_0^{\frac{\pi}{2}}$

$I^2 = \pi$

$\therefore\ I = \sqrt{\pi}$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Question 18

It is given that the following integral converges

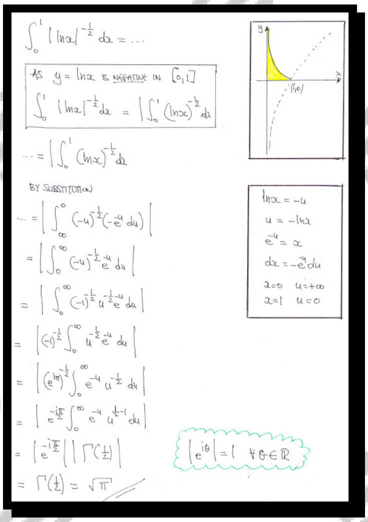
$$\int_0^1 |\ln x|^{-\frac{1}{2}} \ dx .$$

Evaluate the above integral by a suitable substitution introducing Gamma Functions.

You may assume that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} .$$

$\boxed{\sqrt{\pi}}$



$\int_0^1 |\ln x|^{-\frac{1}{2}} dx = \dots$

AS $y = \ln x$ IS NEGATIVE IN $[0,1]$

$\int_0^1 |\ln x|^{-\frac{1}{2}} dx = \left| \int_0^1 (\ln x)^{-\frac{1}{2}} dx \right|$

$\dots = \left| \int_0^1 (\ln x)^{-\frac{1}{2}} dx \right|$

BY SUBSTITUTION

$\ln x = -u$

$u = -\ln x$

$e^{-u} = x$

$dx = -e^{-u} du$

$x=0 \quad u=+\infty$

$x=1 \quad u=0$

$\dots = \left| \int_\infty^0 (-u)^{-\frac{1}{2}} (-e^{-u} du) \right|$

$= \left| \int_0^\infty (-u)^{-\frac{1}{2}} e^{-u} du \right|$

$= \left| \int_0^\infty (-1)^{-\frac{1}{2}} u^{-\frac{1}{2}} e^{-u} du \right|$

$= \left| (-1)^{-\frac{1}{2}} \int_0^\infty u^{-\frac{1}{2}} e^{-u} du \right|$

$= \left| (e^{i\pi})^{-\frac{1}{2}} \int_0^\infty e^{-u} u^{-\frac{1}{2}} du \right|$

$= \left| e^{-\frac{i\pi}{2}} \int_0^\infty e^{-u} u^{\frac{1}{2}-1} du \right|$

$= \left| e^{-\frac{i\pi}{2}} \right| \left| \Gamma(\tfrac{1}{2}) \right|$

$|e^{i\theta}| = 1 \quad \forall \theta \in \mathbb{R}$

$= \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

Question 19

It is given that the following integral converges

$$\int_0^1 (x \ln x)^n \, dx, \ n \in \mathbb{N}.$$

Evaluate the above integral by a suitable substitution introducing Gamma Functions.

$$\boxed{}, \quad \boxed{\frac{(-1)^n \, n!}{(n+1)^{n+1}}}$$

START BY A STANDARD SUBSTITUTION

$u = -\ln x$ ← IN ORDER TO REVERSE THE ZERO LIMIT TO $+\infty$

$\frac{du}{dx} = -\frac{1}{x}$

$dx = -x \, du$

$dx = -e^{-u} \, du$ ← SINCE $-u = \ln x$, $x = e^{-u}$

AND THE LIMITS TRANSFORM

$x = 0 \mapsto u = \infty$

$x = 1 \mapsto u = 0$

TRANSFORMING THE INTEGRAL

$$\int_0^1 (x \ln x)^n \, dx = \int_\infty^0 \left[e^{-u}(-u)\right]^n \left(-e^{-u} \, du\right) = \int_0^\infty e^{-nu}(-u)^n e^{-u} \, du$$

$$= (-1)^n \int_0^\infty e^{-nu} e^{-u} u^n \, du = (-1)^n \int_0^\infty e^{-(n+1)u} u^n \, du$$

ANOTHER SIMPLE LINEAR SUBSTITUTION TO TURN INTO A GAMMA FUNCTION

$t = (n+1)u \iff u = \frac{t}{n+1}$

$\implies du = \frac{1}{n+1} \, dt$

LIMITS UNCHANGED

$$\dots = (-1)^n \int_0^\infty e^{-t} \left(\frac{t}{n+1}\right)^n \left(\frac{1}{n+1} \, dt\right)$$

$$= (-1)^n \times \frac{1}{(n+1)^{n+1}} \int_0^\infty e^{-t} t^n \, dt$$

USING THE DEFINITION OF THE GAMMA FUNCTION

$$\dots = \frac{(-1)^n}{(n+1)^{n+1}} \times \Gamma(n+1)$$

$$= \frac{(-1)^n}{(n+1)^{n+1}} \times n!$$

$$\therefore \int_0^1 (x \ln x)^n \, dx = \frac{(-1)^n \, n!}{(n+1)^{n+1}}$$

Question 20

It is given that the following integral converges

$$\int_0^{\infty} e^{-\frac{1}{2}t} \ln t \ dt .$$

Evaluate the above integral by introducing a new parametric term in the integrand and carrying out a suitable differentiation under the integral sign.

You may assume that

$$\Gamma'(x) = \Gamma(x)\left[-\gamma + \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right)\right].$$

$\boxed{2(-\gamma + \ln 2)}$

$$I = \int_0^{\infty} e^{-\frac{1}{2}t} \ln t \, dt$$

- THIS HAS THE SIGNATURE OF A GAMMA FUNCTION

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} \, dt$$

- INTRODUCE A PARAMETER x, AS IN THE GAMMA FUNCTION

$\Longrightarrow$ LET $J = \int_0^{\infty} t^{x-1} e^{-\frac{1}{2}t} \ln t \, dt$ $\boxed{\text{WHERE } I \text{ IS } J \text{ WITH } x=1}$

- WE OBSERVE THAT $\frac{d}{dx}\left(t^{x-1}\right) = t^{x-1} \times \ln t \times 1 = t^{x-1} \ln t$

- THUS WE MAY WRITE J AS

$$\Longrightarrow J = \int_0^{\infty} \frac{\partial}{\partial x}\left[t^{x-1} e^{-\frac{1}{2}t}\right] dt = \frac{\partial}{\partial x}\left[\int_0^{\infty} t^{x-1} e^{-\frac{1}{2}t} \, dt\right]$$

WHICH IS ALMOST A GAMMA FUNCTION

LET $u = \frac{1}{2}t \Longleftrightarrow t = 2u$

$du = \frac{1}{2}dt$

$dt = 2du$

LIMITS UNCHANGED

$$\Longrightarrow J = \frac{\partial}{\partial x}\left[\int_0^{\infty} (2u)^{x-1} e^{-u} (2\,du)\right] = \frac{\partial}{\partial x}\left[\int_0^{\infty} 2^x u^{x-1} e^{-u} \, du\right]$$

$$\Longrightarrow J = \frac{\partial}{\partial x}\left[2^x \int_0^{\infty} e^{-u} u^{x-1} \, du\right] = \frac{\partial}{\partial x}\left[2^x \Gamma(x)\right]$$

- DIFFERENTIATING THE PRODUCT

$$\Longrightarrow J = 2^x \ln 2 \, \Gamma(x) + 2^x \Gamma'(x)$$

$$\Longrightarrow I = J(1) = 2\ln 2 \, \Gamma(1) + 2\Gamma'(1)$$

$$\Longrightarrow I = 2\ln 2 + 2\Gamma'(1) \qquad (\Gamma(1) = 0! = 1)$$

- NOW $\Gamma'(x) = \Gamma(x)\left[-\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left[\frac{1}{k} - \frac{1}{x+k}\right]\right]$

$$\Rightarrow \Gamma'(1) = \Gamma(1)\left[-\gamma - 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k} - \frac{1}{k+1}\right]\right]$$

$$\Rightarrow \Gamma'(1) = -\gamma - 1 + \left[\begin{array}{l} 1 + \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{4}} + \cancel{\tfrac{1}{5}} + \cdots \\ \quad - \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{5}} - \cdots \end{array}\right]$$

$$\Rightarrow \Gamma'(1) = -\gamma - 1 + 1$$

$$\Rightarrow \boxed{\Gamma'(1) = -\gamma}$$

$$\Longrightarrow I = 2\ln 2 + 2(-\gamma)$$

$$\Longrightarrow \int_0^{\infty} e^{-\frac{1}{2}t} \ln t \, dt = 2\left[-\gamma + \ln 2\right]$$

Question 21

Prove the validity of Legendre's duplication formula for the Gamma function, which states that

$$\Gamma\left(n+\tfrac{1}{2}\right) \equiv \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)}, \ n \in \mathbb{N}.$$

, proof

START BY THE RECURSIVE PROPERTY OF THE GAMMA FUNCTION

$$\Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})(n-\tfrac{3}{2})(n-\tfrac{5}{2})\ldots \tfrac{7}{2}\times\tfrac{5}{2}\times\tfrac{3}{2}\times\tfrac{1}{2}\Gamma(\tfrac{1}{2})$$

$$= \tfrac{1}{2}(2n-1)\times\tfrac{1}{2}(2n-3)\times\tfrac{1}{2}(2n-5)\ldots(\tfrac{1}{2}\times7)(\tfrac{1}{2}\times5)(\tfrac{1}{2}\times3)(\tfrac{1}{2}\times1)\times\sqrt{\pi}$$

$$= \left(\tfrac{1}{2}\right)^n(2n-1)(2n-3)(2n-5)\ldots7\times5\times3\times1\times\sqrt{\pi}$$

$$= \frac{1}{2^n}\,\frac{(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\ldots7\times6\times5\times4\times3\times2\times1\times\sqrt{\pi}}{(2n-2)(2n-4)(2n-6)\ldots6\times4\times2}$$

$$= \frac{1}{2^n}\times\frac{(2n-1)!\sqrt{\pi}}{2(n-1)\times2(n-2)\times2(n-3)\times\ldots\times(2\times3)\times(2\times2)\times(2\times1)}$$

$$= \frac{1}{2^n}\times\frac{\Gamma(2n)\sqrt{\pi}}{2^{n-1}\times(n-1)(n-2)(n-3)\ldots\times3\times2\times1}$$

$$= \frac{1}{2^n}\times\frac{\Gamma(2n)\sqrt{\pi}}{2^{n-1}(n-1)!}$$

$$= \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}(n-1)!}$$

AS REQUIRED

Question 22

Legendre's duplication formula for the Gamma function, states that

$$\Gamma\left(m+\tfrac{1}{2}\right) \equiv \frac{\Gamma(2m)\sqrt{\pi}}{2^{2m-1}\Gamma(m)}, \ m \in \mathbb{N}.$$

Show that

$$\Gamma\left(m+\tfrac{1}{2}\right) \equiv \frac{(2m)!\sqrt{\pi}}{2^{2m}\ m!}, \ m \in \mathbb{N}$$

proof

(BY LEGENDRE'S DUPLICATION FORMULA)

$$\Gamma(m+\tfrac{1}{2}) = \frac{\Gamma(2m)\sqrt{\pi}}{2^{2m-1}\Gamma(m)} = \frac{(2m-1)!\sqrt{\pi}}{2^{2m}\times\frac{1}{2}(m-1)!} = \frac{2(2m-1)!\sqrt{\pi}}{2^{2m}(m-1)!}$$

$$= \frac{2m(2m-1)!\sqrt{\pi}}{2^{2m}\,m(m-1)!} = \frac{(2m)!\sqrt{\pi}}{2^{2m}\,m!}$$

AS REQUIRED

Question 23

$$I(\lambda, x) = \int_0^\infty e^{-\lambda t} t^{x-1} \ln t \ dt .$$

a) By carrying a suitable differentiation over the integral sign, show that

$$I(\lambda, x) = \lambda^{-x}\left[\Gamma'(x) - \Gamma(x)\ln\lambda\right].$$

b) Find simplified expressions for $I(\lambda,1)$, $I(\lambda,2)$ and $I(\lambda,3)$.

You may assume that $\Gamma'(x) = \Gamma(x)\left[-\gamma + \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right)\right]$.

$\boxed{I(\lambda,1) = -\frac{1}{\lambda}[\gamma + \ln\lambda]}$, $\boxed{I(\lambda,2) = \frac{1}{\lambda^2}[1-\gamma-\ln\lambda]}$, $\boxed{I(\lambda,3) = \frac{1}{\lambda^3}[3-2\gamma-2\ln\lambda]}$

a) START BY NOTING THAT $\frac{d}{dx}(a^x) = a^x \ln a$

$$\int_0^\infty e^{-\lambda t} t^{x-1} \ln t \, dt = \int_0^\infty \frac{\partial}{\partial x}\left[e^{-\lambda t} t^{x-1}\right] dt$$
$$= \frac{\partial}{\partial x}\int_0^\infty e^{-\lambda t} t^{x-1} \, dt$$

AS THIS ALMOST LOOKS LIKE A GAMMA FUNCTION WE USE A SIMPLE LINEAR SUBSTITUTION

$u = \lambda t$
$\frac{du}{dt} = \lambda$
$dt = \frac{1}{\lambda} du$
LIMITS UNCHANGED

$$= \frac{\partial}{\partial x}\int_0^\infty e^{-u}\left(\frac{u}{\lambda}\right)^{x-1}\left(\frac{1}{\lambda}du\right)$$
$$= \frac{\partial}{\partial x}\int_0^\infty e^{-u} u^{x-1} \times \frac{1}{\lambda^x} du$$
$$= \frac{\partial}{\partial x}\left[\lambda^{-x}\int_0^\infty e^{-u} u^{x-1} du\right]$$
$$= \frac{\partial}{\partial x}\left[\lambda^{-x}\Gamma(x)\right]$$

DIFFERENTIATING W.R.T. x, USING PRODUCT RULE

$$= \lambda^{-x}(-1)\ln\lambda \, \Gamma(x) + \lambda^{-x}\Gamma'(x)$$
$$= \lambda^{-x}\left[\Gamma'(x) - \Gamma(x)\ln\lambda\right]$$ AS REQUIRED

b) START BY COMPUTING SIMPLIFIED EXPRESSIONS, IN TERMS OF γ, FOR THE DERIVATIVES OF GAMMA FOR $x = 1,2,3$

USING $\boxed{\Gamma'(x) = \Gamma(x)\left[-\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right)\right]}$

- $\Gamma'(1) = \Gamma(1)\left[-\gamma - 1 + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\right]$
$= 0!\left[-\gamma - 1 + 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots\right]$
$= 1\left[-\gamma - 1 + 1\right] = -\gamma$

- $\Gamma'(2) = \Gamma(2)\left[-\gamma - \frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+2}\right)\right]$
$= 1!\left[-\gamma - \frac{1}{2} + 1 + \frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{5}} \dots - \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} - \dots\right]$
$= 1\left[-\gamma - \cancel{\frac{1}{2}} + 1 + \cancel{\frac{1}{2}}\right] = 1 - \gamma$

- $\Gamma'(3) = \Gamma(3)\left[-\gamma - \frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+3}\right]$
$= 2!\left[-\gamma - \frac{1}{3} + 1 + \frac{1}{2} + \frac{1}{3} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{5}} + \cancel{\frac{1}{6}} \dots - \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} - \dots\right]$
$= 2\left[-\gamma - \cancel{\frac{1}{3}} + 1 + \frac{1}{2} + \cancel{\frac{1}{3}}\right] = 3 - 2\gamma$

FINALLY WE OBTAIN

$I(\lambda,1) = \lambda^{-1}\left[\Gamma'(1) - \Gamma(1)\ln\lambda\right] = \frac{1}{\lambda}\left[-\gamma - 0!\ln\lambda\right] = -\frac{1}{\lambda}(\gamma + \ln\lambda)$

$I(\lambda,2) = \lambda^{-2}\left[\Gamma'(2) - \Gamma(2)\ln\lambda\right] = \frac{1}{\lambda^2}\left[(1-\gamma) - 1!\ln\lambda\right] = \frac{1}{\lambda^2}(1 - \gamma - \ln\lambda)$

$I(\lambda,3) = \lambda^{-3}\left[\Gamma'(3) - \Gamma(3)\ln\lambda\right] = \frac{1}{\lambda^3}\left[(3-2\gamma) - 2!\ln\lambda\right] = \frac{1}{\lambda^3}\left[3 - 2\gamma - 2\ln\lambda\right]$

Question 24

By considering standard power series expansions and using Gamma functions, show

$$\int_0^1 x^x \, dx = \sum_{r=1}^{\infty}\left[\frac{(-1)^{r-1}}{r^r}\right].$$

You may assume that integration and summation commute.

$\boxed{}$, $\boxed{\text{proof}}$

REWRITING THE INTEGRAND

$$\int_0^1 x^x \, dx = \int_0^1 e^{\ln x^x} \, dx = \int_0^1 e^{x \ln x} \, dx$$

REWRITE AS THE EXPONENTIAL SERIES & INTERCHANGE INTEGRATION/SUMMATION

$$= \int_0^1 \sum_{r=0}^{\infty} \frac{(x \ln x)^r}{r!} \, dx = \sum_{r=0}^{\infty}\left[\frac{1}{r!}\int_0^1 (x \ln x)^r \, dx\right]$$

NOW USING A SUBSTITUTION (THE MINUS, TO TAKE THE LIMITS 0 TO ∞)

$t = -\ln x$
$-t = \ln x$
$x = e^{-t}$
$dx = -e^{-t}\,dt$
$x = 0 \mapsto t = \infty$
$x = 1 \mapsto t = 0$

$$\ldots = \sum_{r=0}^{\infty}\left[\frac{1}{r!}\int_{\infty}^{0}\left[e^{-t}(-t)\right]^r\left(-e^{-t}\,dt\right)\right]$$

$$= \sum_{r=0}^{\infty}\left[\frac{1}{r!}\int_0^{\infty}\left(-te^{-t}\right)^r e^{-t}\,dt\right]$$

$$= \sum_{r=0}^{\infty}\left[\frac{1}{r!}\int_0^{\infty}(-1)^r e^{-tr} e^{-t} t^r \, dt\right]$$

TIDY UP FURTHER & EMPLOY ANOTHER SUBSTITUTION

$u = t(r+1)$, $t = \frac{u}{r+1}$
$du = dt\,(r+1)$
$dt = \frac{du}{r+1}$
LIMITS UNCHANGED

$$= \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!}\int_0^{\infty} t^r e^{-t(r+1)}\,dt\right]$$

$$= \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!}\int_0^{\infty}\left(\frac{u}{r+1}\right)^r e^{-u}\left(\frac{du}{r+1}\right)\right]$$

$$= \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!}\int_0^{\infty}\frac{u^r}{(r+1)^r} e^{-u}\frac{1}{r+1}\,du\right]$$

$$= \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!}\times\frac{1}{(r+1)^{r+1}}\int_0^{\infty} u^r e^{-u}\,du\right]$$

NOW BY DEFINITION THIS IS A GAMMA FUNCTION

$$\Gamma(x) \equiv \int_0^{\infty} u^{x-1}e^{-u}\,dx \equiv (x-1)!$$

$$\Gamma(r+1) = \int_0^{\infty} u^r e^{-u}\,du = r!$$

FINALLY WE HAVE

$$\int_0^1 x^x\,dx = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!}\times\frac{1}{(r+1)^{r+1}}\int_0^{\infty}u^r e^{-u}\,du\right]$$

$$\int_0^1 x^x\,dx = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{\cancel{r!}}\times\frac{1}{(r+1)^{r+1}}\times\cancel{r!}\right]$$

$$\int_0^1 x^x\,dx = \sum_{r=0}^{\infty}\frac{(-1)^r}{(r+1)^{r+1}}$$

$$\therefore \int_0^1 x^x\,dx = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^r}$$

AS REQUIRED

$$1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \frac{1}{6^6} + \frac{1}{7^7} - \cdots$$

Question 25

By considering standard power series expansions and using Gamma functions, show

$$\int_0^1 \frac{1}{x^x}\,dx = 1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\frac{1}{5^5}+\ldots$$

You may assume that integration and summation commute.

$\boxed{}$, $\boxed{\text{proof}}$

PROCEED AS FOLLOWS

$$\int_0^1 \frac{1}{x^x}\,dx = \int_0^1 x^{-x}\,dx = \int_0^1 e^{\ln x^{-x}}\,dx = \int_0^1 e^{-x\ln x}\,dx$$

USING THE EXPONENTIAL SERIES

$$\int_0^1 \sum_{r=0}^{\infty} \frac{(-x\ln x)^r}{r!}\,dx = \sum_{r=0}^{\infty}\left[\frac{1}{r!}\int_0^1 (-1)^r x^r (\ln x)^r\,dx\right]$$

$$= \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!}\int_0^1 x^r(\ln x)^r\,dx\right]$$

NOW USING A SUBSTITUTION TO CREATE A GAMMA FUNCTION (USE A MINUS TO GET THE "CORRECT" UNITS)

$t=-\ln x$
$-t=\ln x$
$x=e^{-t}$
$dx=-e^{-t}\,dt$
$x=0 \mapsto t=\infty$
$x=1 \mapsto t=0$

$$= \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!}\int_{\infty}^{0} e^{-tr}(-t)^r(-e^{-t})\,dt\right]$$

$$= \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!}\int_0^{\infty} t^r(-1)^r e^{-tr}e^{-t}\,dt\right]$$

$$= \sum_{r=0}^{\infty}\left[\frac{(-1)^r(-1)^r}{r!}\int_0^{\infty} t^r e^{-t(r+1)}\,dt\right]$$

ANOTHER SUBSTITUTION TO CREATE A "FULL GAMMA"

$u=t(r+1)$
$du=dt(r+1)$
$t=\frac{u}{r+1}$
$dt=\frac{du}{r+1}$
UNITS UNCHANGED

$$= \sum_{r=0}^{\infty}\left[\frac{\cancel{(-1)^{2r}}}{r!}\int_0^{\infty}\left(\frac{u}{r+1}\right)^r e^{-u}\left(\frac{du}{r+1}\right)\right]$$

$$= \sum_{r=0}^{\infty}\left[\frac{1}{r!}\int_0^{\infty}\frac{u^r}{(r+1)^r}e^{-u}\frac{1}{r+1}\,du\right]$$

$$= \sum_{r=0}^{\infty}\left[\frac{1}{r!}\frac{1}{(r+1)^{r+1}}\int_0^{\infty} u^r e^{-u}\,du\right]$$

NOW THIS IS A GAMMA FUNCTION

$$\Gamma(r) \equiv \int_0^{\infty} u^{r-1}e^{-u}\,du = (r-1)!$$

$$\therefore\ \Gamma(r+1) = \int_0^{\infty} u^r e^{-u}\,du = r!$$

THUS WE HAVE

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{r=0}^{\infty}\left[\frac{1}{r!\,(r+1)^{r+1}}\Gamma(r+1)\right]$$

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{r=0}^{\infty}\left[\frac{1}{r!\,(r+1)^{r+1}}(r!)\right]$$

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{r=0}^{\infty}\frac{1}{(r+1)^{r+1}} = \sum_{r=1}^{\infty}\frac{1}{r^r}$$

$$\therefore\ \int_0^1 \frac{1}{x^x}\,dx = 1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\frac{1}{5^5}+\frac{1}{6^6}+\ldots$$

Question 26

$$I = \int_1^{\infty} \frac{(\ln x)^3}{x^2(x-1)} \, dx \, .$$

Show by the means of partial fractions and a suitable substitution that

$$I = -6 + \int_0^{\infty} u^3 e^{-u} \left(1 - e^{-u}\right)^{-1} \, du \, ,$$

and hence show that $I = \frac{\pi^4}{15} - 6$.

You may assume without proof that

$$\zeta(4) = \sum_{r=1}^{\infty} \left[\frac{1}{r^4}\right] = \frac{\pi^4}{90} .$$

proof

$$\boxed{\int_1^{\infty} \frac{(\ln x)^3}{x^2(x-1)} \, dx = \frac{\pi^4}{15} - 6}$$

- START BY PARTIAL FRACTIONS

$$\frac{1}{x^2(x-1)} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1}$$
$$1 \equiv A(x-1) + Bx(x-1) + Cx^2$$
$$1 \equiv (B+C)x^2 + (A-B)x - A$$
$$\therefore A = -1, \; B = -1, \; C = 1$$

- REWRITE THE INTEGRAL

$$\Rightarrow I = \int_1^{\infty} \left[\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2}\right](\ln x)^3 \, dx$$

- BY SUBSTITUTION NEXT

$u = \ln x$, $e^u = x$, $dx = e^u \, du$; $x = 1 \mapsto u = 0$, $x = \infty \mapsto u = \infty$

$$\Rightarrow I = \int_0^{\infty} \left[\frac{1}{e^u - 1} - \frac{1}{e^u} - \frac{1}{e^{2u}}\right] u^3 \left(e^u \, du\right)$$
$$\Rightarrow I = \int_0^{\infty} \left[\frac{e^u}{e^u - 1} - 1 - \frac{1}{e^u}\right] u^3 \, du \quad \text{GAMMA}$$
$$\Rightarrow I = \int_0^{\infty} u^3 \left[\frac{e^u - e^u + 1}{e^u - 1}\right] du - \int_0^{\infty} u^3 e^{-u} \, du$$
$$\Rightarrow I = \int_0^{\infty} \frac{u^3}{e^u - 1} \, du - \Gamma(4)$$
$$\Rightarrow I = \int_0^{\infty} u^3 \left(\frac{e^{-u}}{1 - e^{-u}}\right) du - 3!$$
$$\Rightarrow I = \int_0^{\infty} u^3 e^{-u} (1 - e^{-u})^{-1} \, du - 6$$
$$\Rightarrow I = \int_0^{\infty} u^3 e^{-u} \left[1 + e^{-u} + e^{-2u} + e^{-3u} + \dots\right] du - 6$$

$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \dots \quad |r| < 1$

there $|e^{-u}| = \left|\frac{1}{e^u}\right| = \frac{1}{x} \quad x > 1$

$$\Rightarrow I = \int_0^{\infty} u^3 \left[e^{-u} + e^{-2u} + e^{-3u} + e^{-4u} + \dots\right] du - 6$$
$$\Rightarrow I = \int_0^{\infty} u^3 \left(\sum_{r=1}^{\infty} e^{-ru}\right) du - 6$$
$$\Rightarrow I = \sum_{r=1}^{\infty} \int_0^{\infty} u^3 e^{-ru} \, du - 6$$

- BY PARTS 3 TIMES, DIFFERENTIATION W.R.T r UNDER THE INTEGRAL SIGN, OR SWITCH INTO GAMMAS BY SUBSTITUTION

$v = ru$, $u = \frac{1}{r}v$, $du = \frac{1}{r} dv$, LIMITS UNCHANGED

$$\Rightarrow I = \sum_{r=1}^{\infty} \int_0^{\infty} \left(\frac{1}{r}v\right)^3 e^{-v} \frac{1}{r} \, dv - 6$$
$$\Rightarrow I = \sum_{r=1}^{\infty} \int_0^{\infty} \frac{1}{r^4} v^3 e^{-v} \, dv - 6$$
$$\Rightarrow I = \sum_{r=1}^{\infty} \frac{1}{r^4} \int_0^{\infty} v^3 e^{-v} \, dv - 6$$
$$\Rightarrow I = \sum_{r=1}^{\infty} \left[\frac{1}{r^4} \Gamma(4)\right] - 6$$
$$\Rightarrow I = \left[\sum_{r=1}^{\infty} \frac{3!}{r^4}\right] - 6$$
$$\Rightarrow I = 6\left[\sum_{r=1}^{\infty} \frac{1}{r^4}\right] - 6$$
$$\Rightarrow I = 6\zeta(4) - 6$$
$$\Rightarrow I = 6 \times \frac{\pi^4}{90} - 6$$
$$\Rightarrow I = \frac{\pi^4}{15} - 6$$

$\zeta(4) = \frac{\pi^4}{90}$ CAN EASILY BE DERIVED BY CONSIDERING THE FOURIER SERIES EXPANSION OF x^2 ON $(-\pi, \pi)$ & PARSEVAL'S IDENTITY

Question 27

By using the substitution $u = x + u\sqrt{x}$ in the integral definition of $\Gamma(x+1)$, show that for large values of n

$$n! \approx \sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}},\ n \in \mathbb{N}.$$

$\boxed{\text{proof}}$

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad \& \quad \Gamma(x+1) = \int_0^\infty t^{x}e^{-t}\,dt$$

- LET $t = x + u\sqrt{x}$ (NOTE x IS A CONSTANT IN THIS INTEGRATION)

$dt = \sqrt{x}\,du$

$t = 0 \quad u = -\sqrt{x}$

$t = \infty \quad u = \infty$

$$\Rightarrow \Gamma(x+1) = \int_{-\sqrt{x}}^{\infty} (x+u\sqrt{x})^x e^{-(x+u\sqrt{x})}\sqrt{x}\,du$$

$$\Rightarrow \Gamma(x+1) = \int_{-\sqrt{x}}^{\infty} x^x\left[1+\frac{\sqrt{x}}{x}u\right]^x e^{-x} e^{-u\sqrt{x}} x^{\frac{1}{2}}\,du$$

$$\Rightarrow \Gamma(x+1) = x^{x+\frac{1}{2}}e^{-x}\int_{-\sqrt{x}}^{0} e^{-u\sqrt{x}}\left[1+\frac{u}{\sqrt{x}}\right]^x du$$

$$\Rightarrow \frac{\Gamma(x+1)}{x^{x+\frac{1}{2}}e^{-x}} = \int_{-\sqrt{x}}^{0} e^{-u\sqrt{x}} \times e^{\ln\left[1+\frac{u}{\sqrt{x}}\right]^x} du$$

$$\Rightarrow \frac{\Gamma(x+1)}{x^{x+\frac{1}{2}}e^{-x}} = \int_{-\sqrt{x}}^{0} e^{-u\sqrt{x}} \times e^{x\ln\left(1+\frac{u}{\sqrt{x}}\right)} du$$

$$\Rightarrow \frac{\Gamma(x+1)}{x^{x+\frac{1}{2}}e^{-x}} = \int_{-\sqrt{x}}^{0} \exp\left[-u\sqrt{x} + x\ln\left(1+\frac{u}{\sqrt{x}}\right)\right] du$$

- NOW $\ln(1+w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \dots \quad -1 < w \leq 1$

SO IF $-\sqrt{x} < u \leq \sqrt{x}$

$$\Rightarrow \frac{\Gamma(x+1)}{x^{x+\frac{1}{2}}e^{-x}} \int_{-\sqrt{x}}^{0} \exp\left[-u\sqrt{x} + x\left[\frac{u}{\sqrt{x}} - \frac{u^2}{2x} + \frac{u^3}{3x^{3/2}} - \frac{u^4}{4x^2} + \dots\right]\right] du$$

$$\Rightarrow \frac{\Gamma(x+1)}{x^{x+\frac{1}{2}}e^{-x}} = \int_{-\sqrt{x}}^{\sqrt{x}} \exp\left[\cancel{-u\sqrt{x}} + x\left[\cancel{\frac{u}{\sqrt{x}}} - \frac{u^2}{2x} + \frac{u^3}{3x^{3/2}} - \frac{u^4}{4x^2} + \dots\right]\right] du$$

$$+$$

$$\int_{\sqrt{x}}^{\infty} \exp\left[\cancel{-u\sqrt{x}} + x\left[\cancel{\frac{u}{\sqrt{x}}} - \frac{u^2}{2x} + \frac{u^3}{3x^{3/2}} - \frac{u^4}{4x^2} + \dots\right]\right] du$$

- AS $x \to \infty$ THE SECOND INTEGRAL TENDS TO ZERO, GIVEN THE TOP LIMIT IS ∞

- AS $x \to \infty$ THE INTEGRAND IN THE FIRST INTEGRAL BECOMES $\exp(-u^2)$

THUS $\frac{\Gamma(x+1)}{x^{x+\frac{1}{2}}e^{-x}} \approx \int_{-\infty}^{\infty} e^{-u^2}\,du$ AS $x \to \infty$

$\Rightarrow \frac{\Gamma(x+1)}{x^{x+\frac{1}{2}}e^{-x}} \approx \sqrt{2\pi}$ AS $x \to \infty$

$\Rightarrow \Gamma(x+1) \approx \sqrt{2\pi}\, x^{x+\frac{1}{2}}e^{-x}$ AS $x \to \infty$

- LET $x = n$ = POSITIVE INTEGER.

$\Gamma(n+1) \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}}e^{-n}$

$n! \approx \sqrt{2\pi}\, e^{-n} n^{n+\frac{1}{2}}$

AS REQUIRED

BETA FUNCTION

SUMMARY OF THE BETA FUNCTION

The Beta function $\mathrm{B}(m,n)$, is defined as

$$\mathrm{B}(m,n) \equiv \int_0^1 x^{m-1}(1-x)^{n-1} \; dx .$$

Alternative definitions of $\mathrm{B}(m,n)$ are

$$\mathrm{B}(m,n) \equiv \int_0^{\frac{\pi}{2}} 2 \sin^{2m-1}\theta \; \cos^{2n-1}\theta \, d\theta .$$

$$\mathrm{B}(m,n) \equiv \int_0^{\infty} \frac{x^{m-1}}{(x+1)^{m+n}} \; dx$$

Beta function common rules and facts

- $\mathrm{B}(m,n) \equiv \mathrm{B}(n,m)$
- $\mathrm{B}(m,n) \equiv \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, or $\mathrm{B}(m,n) \equiv \dfrac{(m-1)!(n-1)!}{(m+n+1)!}$, $m \in \mathbb{N}$, $n \in \mathbb{N}$

  or $\mathrm{B}(m,n) \equiv \dfrac{(m-1)(n-1)}{(m+n-1)(m+n-2)} \mathrm{B}(m-1,n-1)$

Question 1

Show clearly that

$$\mathrm{B}(m,n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\sin\theta)^{2n-1} \ d\theta .$$

$\boxed{\text{proof}}$

$$\begin{aligned} \mathrm{B}(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \dots \text{ BY SUBSTITUTION} \\ &= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}(2\sin\theta\cos\theta\,d\theta) \\ &= 2\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\,(\cos^2\theta)^{n-1}\sin\theta\cos\theta\,d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{2n-2}\theta\,\sin\theta\cos\theta\,d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \end{aligned}$$

AS REQUIRED

SUBSTITUTION
$x = \sin^2\theta$
$dx = 2\sin\theta\cos\theta\,d\theta$
$x=1 \quad \theta = \frac{\pi}{2}$
$x=0 \quad \theta = 0$

Question 2

By using techniques involving the Beta function, find the exact value of

$$\int_0^1 7x^5(1-x)^4 \ dx .$$

$\boxed{\dfrac{1}{180}}$

$$\begin{aligned} \int_0^1 7x^5(1-x)^4\,dx &= 7\int_0^1 x^{6-1}(1-x)^{5-1}\,dx = 7\,\mathrm{B}(6,5) \\ &= 7\,\frac{\Gamma(6)\Gamma(5)}{\Gamma(11)} = \frac{7\times 5!\times 4!}{10!} \\ &= \frac{7\times 4!}{10\times 9\times 8\times 7\times 6} = \frac{\cancel{7}\times(4\times3\times2)}{10\times9\times8\times\cancel{7}\times6} \\ &= \frac{24}{10\times9\times48} = \frac{1}{10\times9\times2} = \frac{1}{180} \end{aligned}$$

Question 3

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^4\theta \; d\theta .$$

$$\boxed{\frac{8}{315}}$$

$$\int_0^{\frac{\pi}{2}} \sin^5\theta\cos^4\theta \, d\theta = \frac{1}{2}\times 2\int_0^{\frac{\pi}{2}} \sin^{2\times3-1}\theta \cos^{2\times\frac{5}{2}-1}\theta \, d\theta = \frac{1}{2}B\left(3,\frac{5}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma(3)\Gamma(\frac{5}{2})}{\Gamma(\frac{11}{2})} = \frac{1}{2}\left[\frac{2! \times \cancel{\Gamma(\frac{5}{2})}}{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\cancel{\Gamma(\frac{5}{2})}}\right]$$

$$= \frac{1}{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}} = \frac{8}{315}$$

Question 4

By using techniques involving the Beta function, find the exact value of

$$\int_0^1 x^4\sqrt{1-x^2} \; dx .$$

$$\boxed{\frac{\pi}{32}}$$

$$\int_0^1 x^4\sqrt{1-x^2}\,dx = \ldots \text{SUBSTITUTION} \ldots$$

LET $y = x^2$

$dy = 2x\,dx$

$x=0 \mapsto y=0$

$x=1 \mapsto y=1$

$$= \int_0^1 y^2(1-y)^{\frac{1}{2}}\frac{dy}{2x} = \int_0^1 \frac{1}{2}y^2(1-y)^{\frac{1}{2}} \times \frac{1}{y^{\frac{1}{2}}}\,dy$$

$$= \frac{1}{2}\int_0^1 y^{\frac{3}{2}}(1-y)^{\frac{1}{2}}\,dy = \frac{1}{2}\int_0^1 y^{\frac{5}{2}-1}(1-y)^{\frac{3}{2}-1}\,dy = \frac{1}{2}B\left(\frac{5}{2},\frac{3}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)} = \frac{1}{2}\frac{\frac{3}{2}\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{3!} = \frac{3}{4}\times\frac{1}{6}\left(\Gamma(\tfrac{3}{2})\right)^2$$

$$= \frac{1}{8}\left[\frac{1}{2}\Gamma(\tfrac{1}{2})\right]^2 = \frac{1}{8}\left[\frac{1}{2}\sqrt{\pi}\right]^2 = \frac{1}{8}\times\frac{1}{4}\pi = \frac{\pi}{32}$$

Question 5

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \ dx .$$

Express the value of the above integral in terms of Gamma functions, in their simplest form

$$\boxed{\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{2\Gamma\left(\frac{5}{4}\right)}}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\,(\sin x)^{\frac{1}{2}}(\cos x)^0\, dx = \frac{1}{2}\int 2(\sin x)^{2\times\frac{3}{4}-1}(\cos x)^{2\times\frac{1}{2}-1}\, dx$$

$$= \frac{1}{2}B\left(\frac{3}{4},\frac{1}{2}\right) = \frac{1}{2}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

Question 6

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^7\theta \ d\theta .$$

$$\boxed{\frac{8}{35}}$$

$$\int_0^{\frac{\pi}{2}} \sin^7\theta\, d\theta = \frac{1}{2}\times 2\int_0^{\frac{\pi}{2}} \sin^7\theta\cos^0\theta\, d\theta = \frac{1}{2}\times 2\int_0^{\frac{\pi}{2}} \sin^{2\times 4-1}\theta\cos^{2\times\frac{1}{2}-1}\theta\, d\theta$$

$$= \frac{1}{2}B\left(4,\frac{1}{2}\right) = \frac{1}{2}\,\frac{\Gamma(4)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}\right)} = \frac{1}{2}\times\frac{3!\,\cancel{\Gamma\left(\frac{1}{2}\right)}}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\cancel{\Gamma\left(\frac{1}{2}\right)}}$$

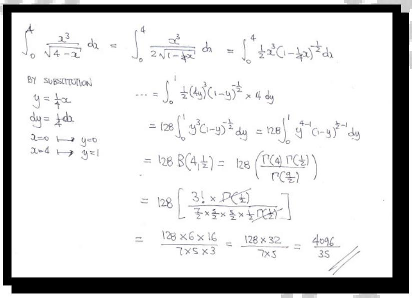
$$= \frac{3}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}} = \frac{\cancel{3}\times 8}{7\times 5\times\cancel{3}} = \frac{8}{35}$$

Question 7

By using techniques involving the Beta function, find the exact value of

$$\int_0^4 \frac{x^3}{\sqrt{4-x}}\ dx .$$

$$\boxed{\frac{4096}{35}}$$



$$\int_0^4 \frac{x^3}{\sqrt{4-x}}\,dx = \int_0^4 \frac{x^3}{2\sqrt{1-\frac{1}{4}x}}\,dx = \int_0^4 \frac{1}{2}x^3\left(1-\frac{1}{4}x\right)^{-\frac{1}{2}}dx$$

BY SUBSTITUTION

$y = \frac{1}{4}x$

$dy = \frac{1}{4}dx$

$x=0 \mapsto y=0$

$x=4 \mapsto y=1$

$$\dots = \int_0^1 \frac{1}{2}(4y)^3(1-y)^{-\frac{1}{2}} \times 4\,dy$$

$$= 128\int_0^1 y^3(1-y)^{-\frac{1}{2}}dy = 128\int_0^1 y^{4-1}(1-y)^{\frac{1}{2}-1}dy$$

$$= 128\,B\left(4,\tfrac{1}{2}\right) = 128\left(\frac{\Gamma(4)\,\Gamma(\frac{1}{2})}{\Gamma(\frac{9}{2})}\right)$$

$$= 128\left[\frac{3! \times \cancel{\Gamma(\frac{1}{2})}}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\cancel{\Gamma(\frac{1}{2})}}\right]$$

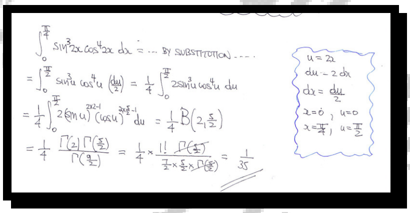
$$= \frac{128\times 6\times 16}{7\times 5\times 3} = \frac{128\times 32}{7\times 5} = \frac{4096}{35}$$

Question 8

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{4}} \sin^2 2x \cos^4 2x\ \ dx .$$

$$\boxed{\frac{1}{35}}$$



$$\int_0^{\frac{\pi}{4}} \sin^3 2x \cos^4 2x\,dx = \dots \text{ BY SUBSTITUTION } \dots$$

$$= \int_0^{\frac{\pi}{2}} \sin^3 u \cos^4 u \left(\frac{du}{2}\right) = \frac{1}{4}\int_0^{\frac{\pi}{2}} 2\sin^3 u\cos^4 u\,du$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 2(\sin u)^{2\times 2-1}(\cos u)^{2\times\frac{5}{2}-1}du = \frac{1}{4}B\left(2,\tfrac{5}{2}\right)$$

$$= \frac{1}{4}\,\frac{\Gamma(2)\,\Gamma(\frac{5}{2})}{\Gamma(\frac{9}{2})} = \frac{1}{4}\times\frac{1!\ \cancel{\Gamma(\frac{5}{2})}}{\frac{7}{2}\times\frac{5}{2}\times\cancel{\Gamma(\frac{5}{2})}} = \frac{1}{35}$$

$u = 2x$

$du = 2\,dx$

$dx = \frac{du}{2}$

$x = 0,\ u = 0$

$x = \frac{\pi}{4},\ u = \frac{\pi}{2}$

Question 9

By using techniques involving the Beta function, find the exact value of

$$\int_0^1 \frac{4}{\sqrt[4]{1-x^4}}\,dx\,.$$

$$\boxed{\pi\sqrt{2}}$$

$$\int_0^1 \frac{4}{\sqrt[4]{1-x^4}}\,dx = \dots \text{SUBSTITUTION} \dots$$

LET $u = x^4$

$\frac{du}{dx} = 4x^3$

$dx = \frac{du}{4x^3}$

LIMITS UNCHANGED

$x^3 = u^{\frac{3}{4}}$

$$= \int_0^1 \frac{4}{(1-u)^{\frac{1}{4}}}\,\frac{du}{4x^3} = \int_0^1 \frac{1}{x^3(1-u)^{\frac{1}{4}}}\,du$$

$$= \int_0^1 \frac{1}{u^{\frac{3}{4}}(1-u)^{\frac{1}{4}}}\,du = \int_0^1 u^{-\frac{3}{4}}(1-u)^{-\frac{1}{4}}\,du$$

$$= B\left(\tfrac{1}{4}, \tfrac{3}{4}\right) = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\underbrace{\cancel{\Gamma(1)}}_{=1}} = \Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right)$$

USING $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$

$$= \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\frac{1}{\sqrt{2}}} = \pi\sqrt{2}$$

Question 10

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\pi} \sin^5\left(\tfrac{1}{2}x\right)\cos^2\left(\tfrac{1}{2}x\right)\,dx\,.$$

$$\boxed{\frac{16}{105}}$$

$$\int_0^{\pi} \sin^5\left(\tfrac{1}{2}x\right)\cos^2\left(\tfrac{1}{2}x\right)\,dx = \frac{1}{2}\int_0^{\pi} 2\sin^5\left(\tfrac{1}{2}x\right)\cos^2\left(\tfrac{1}{2}x\right)\,dx \dots$$

SUBSTITUTION

$u = \frac{1}{2}x$

$x = 2u$

$dx = 2\,du$

$x = 0 \quad u = 0$

$x = \pi \quad u = \frac{\pi}{2}$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin^5 u\cos^2 u\,(2\,du) = \int_0^{\pi} 2(\sin u)^{2\times 3-1}(\cos u)^{2\times\frac{3}{2}-1}\,du$$

$$= B\left(3, \tfrac{3}{2}\right) = \frac{\Gamma(3)\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{9}{2}\right)} = \frac{2!\,\cancel{\Gamma\left(\frac{3}{2}\right)}}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\,\cancel{\Gamma\left(\frac{3}{2}\right)}}$$

$$= \frac{2}{\frac{7\times 5\times 3}{8}} = \frac{16}{105}$$

Question 11

a) Show clearly that

$$\int_0^1 x^{m-1}\left(1-x^n\right)^{p-1}\ dx = \frac{1}{n}\mathrm{B}\left(\frac{m}{n}, p\right),\ n \neq 0 .$$

b) Hence, find the exact value of

$$\int_0^1 x^5\left(1-x^3\right)^2\ dx .$$

$$\boxed{\frac{1}{24}}$$

a) $\int_0^1 x^{m-1}(1-x^n)^{p-1}\,dx = \ldots$ SUBSTITUTION $\ldots$

$= \int_0^1 t^{\frac{m-1}{n}}(1-t)^{p-1}\frac{dt}{n t^{\frac{n-1}{n}}} = \frac{1}{n}\int_0^1 t^{\frac{m-1}{n}}(1-t)^{p-1}t^{\frac{1-n}{n}}\,dt$

$= \frac{1}{n}\int_0^1 t^{\frac{m-n}{n}}(1-t)^{p-1}\,dt = \frac{1}{n}\int_0^1 t^{\frac{m}{n}-1}(1-t)^{p-1}\,dt$

$= \frac{1}{n}B\left(\frac{m}{n}, p\right)$

$t = x^n$
$x = t^{\frac{1}{n}}$
$dt = nx^{n-1}\,dx$
$dx = \frac{dt}{nx^{n-1}}$
$dx = \frac{dt}{nt^{\frac{n-1}{n}}}$
LIMITS UNCHANGED

b) $\int_0^1 x^5(1-x^3)^2\,dx = \int_0^1 x^{6-1}(1-x^3)^{3-1}\,dx = \ldots \frac{1}{3}B\left(\frac{6}{3}, 3\right) = \frac{1}{3}B(2,3)$

$= \frac{1}{3}\frac{\Gamma(2)\Gamma(3)}{\Gamma(5)} = \frac{1}{3}\frac{1!\times 2!}{4!} = \frac{1}{3}\times\frac{1}{12} = \frac{1}{24}$

Question 12

By using techniques involving the Beta function, find the exact value of

$$\int_0^1 \sqrt[11]{1-\sqrt[3]{x}}\ dx .$$

$$\boxed{\frac{1331}{1564}}$$

$\int_0^1 \left(1-x^{\frac{1}{3}}\right)^{\frac{1}{11}} dx = \frac{1331}{1564}$

THIS IS A STANDARD "BETA TYPE" INTEGRAL BY SUBSTITUTION

$u = x^{\frac{1}{3}}$
$u^3 = x$
$dx = 3u^2\,du$
LIMITS UNCHANGED

$\ldots = \int_0^1 (1-u)^{\frac{1}{11}} 3u^2\,du$

$= 3\int_0^1 (1-u)^{\frac{12}{11}-1}u^{3-1}\,du$

$= 3B\left(\frac{12}{11}, 3\right)$

$= 3\left[\frac{\Gamma\left(\frac{12}{11}\right)\Gamma(3)}{\Gamma\left(\frac{45}{11}\right)}\right]$

$= 3\left[\frac{\cancel{\Gamma\left(\frac{12}{11}\right)}\times 2!}{\frac{34}{11}\times\frac{23}{11}\times\frac{12}{11}\cancel{\Gamma\left(\frac{12}{11}\right)}}\right] =$

$= \frac{6}{\frac{34}{11}\times\frac{23}{11}\times\frac{12}{11}} = \frac{6\times 11^3}{34\times 23\times 12} = \frac{11^3}{34\times 46} = \frac{1331}{1564}$

Question 13

$$I_n = \int_0^1 \left(1-\sqrt{x}\right)^n \, dx .$$

Use Beta functions to show that

$$I_n = \frac{2}{(n+2)(n+1)} .$$

proof

$$I_n = \int_0^1 \left(1-\sqrt{x}\right)^n \, dx \quad \ldots \text{ BY SUBSTITUTION}$$

LET $u = \sqrt{x}$
$u^2 = x$
$dx = 2u\,du$
LIMITS UNCHANGED

$$\Rightarrow I_n = \int_0^1 (1-u)^n \, 2u \, du$$

$$\Rightarrow I_n = 2\int_0^1 (1-u)^{(n+1)-1} u^{2-1} \, du$$

$$\Rightarrow I_n = 2B[n+1, 2] = 2\left[\frac{\Gamma(n+1)\Gamma(2)}{\Gamma(n+3)}\right] = 2\left[\frac{\Gamma(n+1)\times 1!}{\Gamma(n+3)}\right]$$

$$\Rightarrow I_n = 2\left[\frac{\cancel{\Gamma(n+1)}}{(n+2)(n+1)\cancel{\Gamma(n+1)}}\right]$$

$$\Rightarrow I_n = \frac{2}{(n+2)(n+1)}$$

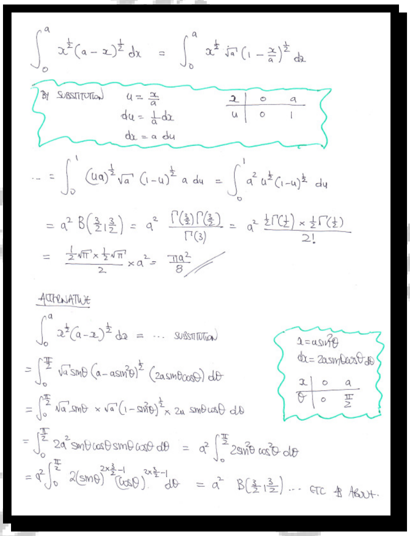
Question 14

$$\int_{0}^{a} x^{\frac{1}{2}}\sqrt{a-x}\ \, dx .$$

Find an exact simplified value for the integral above, by using a suitable substitution to transform into a Beta function.

You may assume that a is a positive constant.

$$\boxed{\frac{1}{8}\pi a^{2}}$$



$$\int_0^a x^{\frac{1}{2}}(a-x)^{\frac{1}{2}}\,dx = \int_0^a x^{\frac{1}{2}}\sqrt{a}\left(1-\frac{x}{a}\right)^{\frac{1}{2}}\,dx$$

BY SUBSTITUTION $u=\frac{x}{a}$, $du=\frac{1}{a}dx$, $dx = a\,du$

x	0	a
u	0	1

$$\dots = \int_0^1 (ua)^{\frac{1}{2}}\sqrt{a}\,(1-u)^{\frac{1}{2}}\,a\,du = \int_0^1 a^2 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\,du$$

$$= a^2 B\left(\tfrac{3}{2},\tfrac{3}{2}\right) = a^2\frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)} = a^2\frac{\frac{1}{2}\Gamma(\frac{1}{2})\times\frac{1}{2}\Gamma(\frac{1}{2})}{2!}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}\times\frac{1}{2}\sqrt{\pi}}{2}\times a^2 = \frac{\pi a^2}{8}$$

ALTERNATIVE

$$\int_0^a x^{\frac{1}{2}}(a-x)^{\frac{1}{2}}\,dx = \dots \text{ SUBSTITUTION}$$

$x = a\sin^2\theta$, $dx = 2a\sin\theta\cos\theta\,d\theta$

x	0	a
θ	0	$\frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a}\sin\theta\,(a-a\sin^2\theta)^{\frac{1}{2}}\,(2a\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a}\sin\theta\times\sqrt{a}(1-\sin^2\theta)^{\frac{1}{2}}\times 2a\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2a^2\sin\theta\cos\theta\sin\theta\cos\theta\,d\theta = a^2\int_0^{\frac{\pi}{2}} 2\sin^2\theta\cos^2\theta\,d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}} 2(\sin\theta)^{2\times\frac{3}{2}-1}(\cos\theta)^{2\times\frac{3}{2}-1}\,d\theta = a^2 B\left(\tfrac{3}{2},\tfrac{3}{2}\right)\dots \text{ ETC AS ABOUT.}$$

Question 15

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \ d\theta \ , \ m \in \mathbb{N} \, , \ n \in \mathbb{N} \, .$$

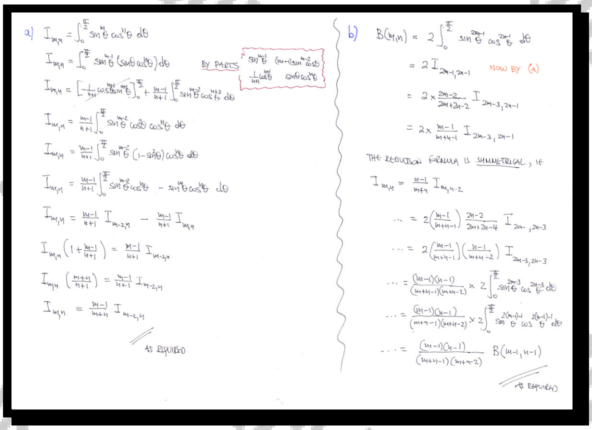
a) Show clearly that

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$$

b) Hence, show further that

$$\mathrm{B}(m,n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} \mathrm{B}(m-1,n-1) \, .$$

$\boxed{\text{proof}}$



a) $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta \, d\theta$

$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{m-1}\theta (\sin\theta \cos^n\theta) \, d\theta$

BY PARTS

$\sin^{m-1}\theta \quad (m-1)\sin^{m-2}\theta\cos\theta$

$-\frac{1}{n+1}\cos^{n+1}\theta \quad \sin\theta\cos^n\theta$

$I_{m,n} = \left[\cancel{-\frac{1}{n+1}\cos^{n+1}\theta\sin^{m-1}\theta}\right]_0^{\frac{\pi}{2}} + \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta \cos^{n+2}\theta \, d\theta$

$I_{m,n} = \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta \cos^2\theta \cos^n\theta \, d\theta$

$I_{m,n} = \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta (1-\sin^2\theta) \cos^n\theta \, d\theta$

$I_{m,n} = \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta \cos^n\theta - \sin^m\theta\cos^n\theta \, d\theta$

$I_{m,n} = \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$

$I_{m,n}\left(1 + \frac{m-1}{n+1}\right) = \frac{m-1}{n+1} I_{m-2,n}$

$I_{m,n}\left(\frac{m+n}{n+1}\right) = \frac{m-1}{n+1} I_{m-2,n}$

$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$

AS REQUIRED

b) $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$= 2 I_{2m-1,2n-1}$ NOW BY (a)

$= 2 \times \frac{2m-2}{2m+2n-2} I_{2m-3,2n-1}$

$= 2 \times \frac{m-1}{m+n-1} I_{2m-3,2n-1}$

THE REDUCTION FORMULA IS SYMMETRICAL, IE

$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$

$\dots = 2\left(\frac{m-1}{m+n-1}\right) \frac{2n-2}{2m+2n-4} I_{2m-,2n-3}$

$\dots = 2\left(\frac{m-1}{m+n-1}\right)\left(\frac{n-1}{m+n-2}\right) I_{2m-3,2n-3}$

$\dots = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} \times 2\int_0^{\frac{\pi}{2}} \sin^{2m-3}\theta \cos^{2n-3}\theta \, d\theta$

$\dots = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} \times 2\int_0^{\frac{\pi}{2}} \sin^{2(m-1)-1}\theta \cos^{2(n-1)-1}\theta \, d\theta$

$\dots = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} B(m-1,n-1)$

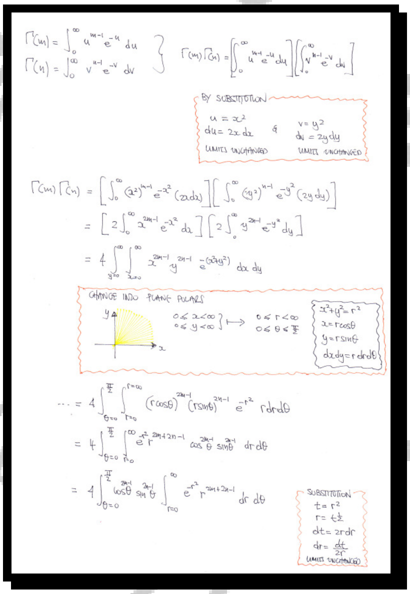
AS REQUIRED

Question 16

Show clearly that

$$\mathrm{B}(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

$\boxed{\text{proof}}$



$\Gamma(m)=\int_0^\infty u^{m-1}e^{-u}\,du$

$\Gamma(n)=\int_0^\infty v^{n-1}e^{-v}\,dv$

$\Gamma(m)\Gamma(n)=\left[\int_0^\infty u^{m-1}e^{-u}\,du\right]\left[\int_0^\infty v^{n-1}e^{-v}\,dv\right]$

BY SUBSTITUTION

$u=x^2$, $du=2x\,dx$, LIMITS UNCHANGED & $v=y^2$, $dv=2y\,dy$, LIMITS UNCHANGED

$\Gamma(m)\Gamma(n)=\left[\int_0^\infty (x^2)^{m-1}e^{-x^2}(2x\,dx)\right]\left[\int_0^\infty (y^2)^{n-1}e^{-y^2}(2y\,dy)\right]$

$=\left[2\int_0^\infty x^{2m-1}e^{-x^2}\,dx\right]\left[2\int_0^\infty y^{2n-1}e^{-y^2}\,dy\right]$

$=4\int_{y=0}^\infty\int_{x=0}^\infty x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}\,dx\,dy$

CHANGE INTO PLANE POLARS

$\left.\begin{array}{l}0\le x<\infty\\0\le y<\infty\end{array}\right\}\longmapsto\begin{array}{l}0\le r<\infty\\0\le\theta\le\frac{\pi}{2}\end{array}$

$x^2+y^2=r^2$, $x=r\cos\theta$, $y=r\sin\theta$, $dx\,dy=r\,dr\,d\theta$

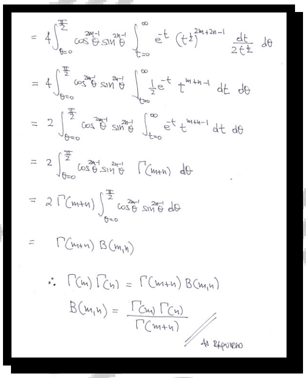
$\dots=4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{r=\infty}(r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1}e^{-r^2}\,r\,dr\,d\theta$

$=4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty}e^{-r^2}r^{2m+2n-1}\cos^{2m-1}\theta\sin^{2n-1}\theta\,dr\,d\theta$

$=4\int_{\theta=0}^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\int_{r=0}^{\infty}e^{-r^2}r^{2m+2n-1}\,dr\,d\theta$

SUBSTITUTION

$t=r^2$, $r=t^{\frac{1}{2}}$, $dt=2r\,dr$, $dr=\frac{dt}{2r}$, LIMITS UNCHANGED



$=4\int_{\theta=0}^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\int_{t=0}^{\infty}e^{-t}(t^{\frac{1}{2}})^{2m+2n-1}\frac{dt}{2t^{\frac{1}{2}}}\,d\theta$

$=4\int_{\theta=0}^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\int_{t=0}^{\infty}\frac{1}{2}e^{-t}t^{m+n-1}\,dt\,d\theta$

$=2\int_{\theta=0}^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\int_{t=0}^{\infty}e^{-t}t^{m+n-1}\,dt\,d\theta$

$=2\int_{\theta=0}^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,\Gamma(m+n)\,d\theta$

$=2\Gamma(m+n)\int_{\theta=0}^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta$

$=\Gamma(m+n)\,B(m,n)$

$\therefore\ \Gamma(m)\Gamma(n)=\Gamma(m+n)B(m,n)$

$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

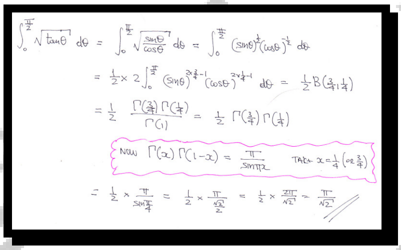
AS REQUIRED

Question 17

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}\ \ d\theta.$$

$\boxed{\frac{\pi}{\sqrt{2}}}$



$\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}\,d\theta=\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin\theta}{\cos\theta}}\,d\theta=\int_0^{\frac{\pi}{2}}(\sin\theta)^{\frac{1}{2}}(\cos\theta)^{-\frac{1}{2}}\,d\theta$

$=\frac{1}{2}\times2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2\times\frac{3}{4}-1}(\cos\theta)^{2\times\frac{1}{4}-1}\,d\theta=\frac{1}{2}B\left(\frac{3}{4},\frac{1}{4}\right)$

$=\frac{1}{2}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)}=\frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$

NOW $\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}$ TAKE $x=\frac{1}{4}$ (or $\frac{3}{4}$)

$=\frac{1}{2}\times\frac{\pi}{\sin\frac{\pi}{4}}=\frac{1}{2}\times\frac{\pi}{\frac{\sqrt{2}}{2}}=\frac{1}{2}\times\frac{2\pi}{\sqrt{2}}=\frac{\pi}{\sqrt{2}}$

Question 18

By using techniques involving the Beta function, find the exact value of

$$\int_0^4 x^2\sqrt{16-x^2}\ dx .$$

$\boxed{16\pi}$

$$\int_0^4 x^2\sqrt{16-x^2}\,dx = \int_0^4 x^2 \times 4\sqrt{1-\tfrac{1}{16}x^2}\,dx$$

USE THE SUBSTITUTION OPPOSITE

$$= \int_0^4 4x^2\left(1-\tfrac{1}{16}x^2\right)^{\frac{1}{2}}dx = \int_0^1 4(16u)(1-u)^{\frac{1}{2}}\frac{2}{u^{\frac{1}{2}}}\,du$$

$$= \int_0^1 128u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\,du = 128\int_0^1 u^{\frac{3}{2}-1}(1-u)^{\frac{3}{2}-1}\,du$$

$$= 128\,B\left(\tfrac{3}{2},\tfrac{3}{2}\right) = 128\times\frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)} = \frac{128\left[\Gamma(\frac{3}{2})\right]^2}{2!}$$

$$= 64\left[\tfrac{1}{2}\Gamma(\tfrac{1}{2})\right]^2 = 64\times\left(\tfrac{1}{2}\sqrt{\pi}\right)^2 = 16\pi$$

$u = \frac{1}{16}x^2$

$16u = x^2$

$16\,du = 2x\,dx$

$dx = \frac{8}{x}\,du$

$dx = \frac{8}{4u^{\frac{1}{2}}}$

$dx = \frac{2}{u^{\frac{1}{2}}}$

LIMITS

x	0	4
u	0	1

Question 19

By using techniques involving the Beta function, find the exact value of

$$\int_0^2 \sqrt{16x^2-x^6}\ dx .$$

$\boxed{2\pi}$

$$\int_0^2 \sqrt{16x^2-x^6}\,dx = \int_0^2 4x\sqrt{1-\tfrac{1}{16}x^4}\,dx$$

BY SUBSTITUTION ....

$$= \int_0^1 4\left(2u^{\frac{1}{4}}\right)\left[1-u\right]^{\frac{1}{2}}\times\tfrac{1}{2}u^{-\frac{3}{4}}\,du = \int_0^1 4u^{-\frac{1}{2}}(1-u)^{\frac{1}{2}}\,du$$

$$= 4\int_0^1 u^{\frac{1}{2}-1}(1-u)^{\frac{3}{2}-1}\,du = 4\,B\left(\tfrac{1}{2},\tfrac{3}{2}\right)$$

$$= 4\,\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)} = 4\times\frac{\Gamma(\frac{1}{2})\times\frac{1}{2}\Gamma(\frac{1}{2})}{1!} = 2\left[\Gamma(\tfrac{1}{2})\right]^2$$

$$= 2\pi$$

$u = \frac{1}{16}x^4$

$x^4 = 16u$

$x = 2u^{\frac{1}{4}}$

$dx = \frac{1}{2}u^{-\frac{3}{4}}\,du$

$x = 0 \mapsto u = 0$

$x = 2 \mapsto u = 1$

Question 20

By using techniques involving the Beta function, find the exact value of

$$\int_0^1 \frac{1}{\sqrt[3]{x^2 - x^3}} \, dx .$$

$$\boxed{\frac{2\pi}{\sqrt{3}}}$$

$$\int_0^1 \frac{1}{\sqrt[3]{x^2-x^3}}\,dx = \int_0^1 \frac{1}{x^{\frac{2}{3}}(1-x)^{\frac{1}{3}}}\,dx = \int_0^1 x^{-\frac{2}{3}}(1-x)^{-\frac{1}{3}}\,dx$$

$$= \int_0^1 x^{\frac{1}{3}-1}(1-x)^{\frac{2}{3}-1}\,dx = B\left(\tfrac{1}{3},\tfrac{2}{3}\right) = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)}$$

$$= \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{0!} = \Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{\sqrt{3}}$$

$$\Gamma(x)\Gamma(1-x) \equiv \frac{\pi}{\sin \pi x}$$

Question 21

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{1}{2}} (\cos\theta)^{\frac{7}{2}} \, dx .$$

$$\boxed{\frac{5\pi\sqrt{2}}{64}}$$

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{1}{2}}(\cos\theta)^{\frac{7}{2}}\,d\theta = \frac{1}{2}\int 2(\sin\theta)^{2\times\frac{3}{4}-1}(\cos\theta)^{2\times\frac{9}{4}-1}\,d\theta = \frac{1}{2}B\left(\tfrac{3}{4},\tfrac{9}{4}\right)$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma(3)} = \frac{1}{2}\,\frac{\Gamma\left(\frac{3}{4}\right)\times\frac{5}{4}\times\frac{1}{4}\times\Gamma\left(\frac{1}{4}\right)}{2!}$$

$$= \frac{5}{64}\,\Gamma\left(\tfrac{3}{4}\right)\Gamma\left(\tfrac{1}{4}\right) = \frac{5}{64}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{5\sqrt{2}\pi}{64}$$

$$\Gamma(x)\,\Gamma(1-x) \equiv \frac{\pi}{\sin \pi x}$$

Question 22

By using techniques involving the Beta function, find the exact value of

$$\int_0^2 \left(2x - x^2\right)^{\frac{5}{2}} \, dx .$$

$$\boxed{\frac{5}{16}\pi}$$

$$\int_0^2 (2x-x^2)^{\frac{5}{2}}\,dx = \int_0^2 x^{\frac{5}{2}}(2-x)^{\frac{5}{2}}\,dx = \int_0^2 x^{\frac{5}{2}}2^{\frac{5}{2}}\left(1-\tfrac{1}{2}x\right)^{\frac{5}{2}}\,dx$$

BY SUBSTITUTION

$u = \frac{1}{2}x$

$x = 2u$

$dx = 2\,du$

x	0	2
u	0	1

$$= \int_0^1 (2u)^{\frac{5}{2}} \times 2^{\frac{5}{2}} \times (1-u)^{\frac{5}{2}}\, 2\,du$$

$$= \int_0^1 2^{\frac{5}{2}} \times u^{\frac{5}{2}} \times 2^{\frac{5}{2}} \times (1-u)^{\frac{5}{2}} \times 2\,du$$

$$= \int_0^1 2^6 \times u^{\frac{5}{2}} \times (1-u)^{\frac{5}{2}}\,du$$

$$= 64\int_0^1 u^{\frac{7}{2}-1}(1-u)^{\frac{7}{2}-1}\,du$$

$$= 64\,B\left(\tfrac{7}{2},\tfrac{7}{2}\right) = 64 \times \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(7)} = 64 \times \frac{\left(\Gamma\left(\frac{7}{2}\right)\right)^2}{6!}$$

$$= \frac{64}{6!} \times \left[\tfrac{5}{2}\times\tfrac{3}{2}\times\tfrac{1}{2}\times\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \frac{64}{720}\times\left(\frac{15}{8}\sqrt{\pi}\right)^2 = \frac{\cancel{64}}{720}\times\frac{225}{\cancel{64}}\times\pi$$

$$= \frac{25}{80}\pi = \frac{5}{16}\pi$$

Question 23

By using techniques involving the Beta function, find the exact value of

$$\int_0^2 \left(4 - x^2\right)^{\frac{7}{2}} \, dx .$$

$$\boxed{35\pi}$$

$$\int_0^2 (4-x^2)^{\frac{7}{2}}\,dx = \int_0^2 4^{\frac{7}{2}}\left(1-\tfrac{1}{4}x^2\right)^{\frac{7}{2}}\,dx$$

BY SUBSTITUTION ....

$u = \frac{1}{4}x^2$

$x^2 = 4u$

$x = 2u^{\frac{1}{2}}$

$dx = u^{-\frac{1}{2}}\,du$

$x=0 \mapsto u=0$

$x=2 \mapsto u=1$

$$\int_0^1 128\,(1-u)^{\frac{7}{2}}u^{-\frac{1}{2}}\,du = 128\int_0^1 u^{\frac{1}{2}-1}(1-u)^{\frac{9}{2}-1}\,du$$

$$= 128\,B\left(\tfrac{1}{2},\tfrac{9}{2}\right) = 128\left[\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{9}{2}\right)}{\Gamma(5)}\right] = 128 \times \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{9}{2}\right)}{4!}$$

$$= \frac{128}{24}\times\left[\Gamma\left(\tfrac{1}{2}\right)\times\tfrac{7}{2}\times\tfrac{5}{2}\times\tfrac{3}{2}\times\tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)\right] = \frac{16}{3}\times\left[\frac{7\times5\times3}{16}\,\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right)\right]$$

$$= 35\sqrt{\pi}\times\sqrt{\pi} = 35\pi$$

Question 24

By using techniques involving the Beta function, find the exact value of

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\ dx .$$

$\boxed{\pi}$

$$\int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} dx = \dots \text{SUBSTITUTION} \dots = \int_{0}^{1} \left[\frac{1+(2t-1)}{1-(2t-1)}\right]^{\frac{1}{2}} 2\,dt = \int_{0}^{1} \left(\frac{2t}{2-2t}\right)^{\frac{1}{2}} 2\,dt$$

WE NEED TO CONSTRUCT A SUBSTITUTION IN ORDER TO CREATE A BETA FUNCTION

WE WILL MATCH THE LIMITS $[-1,1] \longmapsto [0,1]$

LET $t = Ax + B$

- $0 = A(-1) + B$

  $A = B$

- $1 = A \times 1 + B$

  $A + B = 1$

  $2B = 1$

  $B = \frac{1}{2}$

$\therefore A = B = \frac{1}{2}$

LET

$t = \frac{1}{2}x + \frac{1}{2}$

$2t = x + 1$

$2\,dt = dx$

$x = 2t - 1$

$$= 2\int_{0}^{1} \left(\frac{t}{1-t}\right)^{\frac{1}{2}} dt$$

$$= 2\int_{0}^{1} t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}} dt$$

$$= 2B\left(\tfrac{3}{2}, \tfrac{1}{2}\right)$$

$$= \frac{2\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}$$

$$= \frac{2 \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1!}$$

$$= \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2$$

$$= \left(\sqrt{\pi}\right)^2$$

$$= \pi$$

Question 25

By using techniques involving the Beta function, find the exact value of

$$\int_{0}^{\infty} \frac{1}{1+x^2}\ dx .$$

$\boxed{\frac{\pi}{2}}$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \dots \text{BY SUBSTITUTION} = \int_{0}^{\infty} \frac{1}{1+u}\left(\tfrac{1}{2}u^{-\frac{1}{2}}\,du\right)$$

$u = x^2$

$x = u^{\frac{1}{2}}$

$dx = \frac{1}{2}u^{-\frac{1}{2}}\,du$

LIMITS UNCHANGED

$$= \frac{1}{2}\int_{0}^{\infty} \frac{u^{-\frac{1}{2}}}{(1+u)^1}\,du = \frac{1}{2}\int_{0}^{\infty} \frac{u^{\frac{1}{2}-1}}{(1+u)^{\frac{1}{2}+\frac{1}{2}}}\,du = \frac{1}{2}B\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$\boxed{B(m,n) \equiv \int_{0}^{\infty} \frac{u^{m-1}}{(1+u)^{m+n}}\,du}$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \frac{1}{2}\,\frac{\sqrt{\pi}\sqrt{\pi}}{0!} = \frac{\pi}{2}$$

Question 26

Use the substitution $x^2 = \sin\theta$ to show that

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx = \frac{\left[\Gamma\left(\tfrac{1}{4}\right)\right]^2}{\sqrt{32\pi}}.$$

$\boxed{\text{proof}}$

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx \;\dots\; \text{START BY A SUBSTITUTION}$$

$x^2 = \sin\theta$
$2x\,dx = \cos\theta\,d\theta$
$dx = \frac{\cos\theta\,d\theta}{2x}$
LIMITS
$x=1 \quad \theta=\frac{\pi}{2}$
$x=0 \quad \theta=0$

$$\dots = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2\theta}}\left(\frac{\cos\theta}{2x}\,d\theta\right)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\cos\theta} \times \frac{1}{2x}\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{x}\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{(\sin\theta)^{\frac{1}{2}}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin\theta)^{-\frac{1}{2}}\,d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 2(\sin\theta)^{2\times\frac{1}{4}-1}(\cos\theta)^{2\times\frac{1}{2}-1}\,d\theta$$

$$= \frac{1}{4}B\left(\tfrac{1}{4},\tfrac{1}{2}\right) = \frac{1}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}$$

$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}$
$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}}$
$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi$
$\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{2}\pi}{\Gamma\left(\frac{1}{4}\right)}$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{\sqrt{2}\pi}$$

$$= \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\sqrt{32\pi}}$$

Question 27

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{3}} \sin\left(\tfrac{3}{2}x\right)\sqrt{8\sin 3x}\,dx.$$

$$\boxed{\frac{\sqrt{2}}{3}\pi}$$

$$\int_0^{\frac{\pi}{3}} \sin\tfrac{3}{2}x\,\sqrt{8\sin 3x}\,dx = \dots \text{SUBSTITUTION}$$

$u = \frac{3}{2}x$
$x = \frac{2}{3}u$
$dx = \frac{2}{3}du$
$x=0 \quad u=0$
$x=\frac{\pi}{3} \quad u=\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} \sin u\sqrt{8\sin 2u}\left(\tfrac{2}{3}du\right) = \int_0^{\frac{\pi}{2}} \sin u\sqrt{16\sin u\cos u}\left(\tfrac{2}{3}du\right)$$

$$= \int_0^{\frac{\pi}{2}} \frac{8}{3}\sin u\sqrt{\sin u}\sqrt{\cos u}\,du = \frac{4}{3}\int_0^{\frac{\pi}{2}} 2(\sin u)^{\frac{3}{2}}\cos u^{\frac{1}{2}}\,du$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} 2(\sin u)^{2\times\frac{5}{4}-1}(\cos u)^{2\times\frac{3}{4}-1}\,du = \frac{4}{3}B\left(\tfrac{5}{4},\tfrac{3}{4}\right) = \frac{4}{3}\,\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(2)}$$

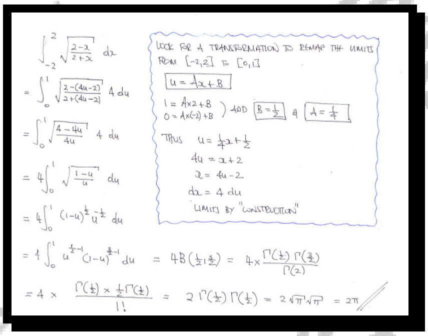
$$= \frac{4}{3}\times\frac{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{1!} = \frac{1}{3}\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right) = \frac{1}{3}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\sqrt{2}}{3}\pi$$

Question 28

By using techniques involving the Beta function, find the exact value of

$$\int_{-2}^{2} \sqrt{\frac{2-x}{2+x}}\ dx .$$

$\boxed{2\pi}$



$$\int_{-2}^{2} \sqrt{\frac{2-x}{2+x}}\ dx$$

LOOK FOR A TRANSFORMATION TO REMAP THE LIMITS FROM $[-2,2]$ TO $[0,1]$

$\boxed{u = Ax + B}$

$1 = A\times 2 + B$
$0 = A\times(-2) + B$) AND $\boxed{B=\frac{1}{2}}$ & $\boxed{A=\frac{1}{4}}$

THUS $u = \frac{1}{4}x + \frac{1}{2}$
$4u = x + 2$
$x = 4u - 2$
$dx = 4\,du$
LIMITS BY "CONSTRUCTION"

$$= \int_{0}^{1} \sqrt{\frac{2-(4u-2)}{2+(4u-2)}}\ 4\,du$$

$$= \int_{0}^{1} \sqrt{\frac{4-4u}{4u}}\ 4\,du$$

$$= 4\int_{0}^{1} \sqrt{\frac{1-u}{u}}\ du$$

$$= 4\int_{0}^{1} (1-u)^{\frac{1}{2}} u^{-\frac{1}{2}}\ du$$

$$= 4\int_{0}^{1} u^{\frac{1}{2}-1}(1-u)^{\frac{3}{2}-1}\ du = 4B\left(\tfrac{1}{2},\tfrac{3}{2}\right) = 4\times\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)}$$

$$= 4\times\frac{\Gamma(\frac{1}{2})\times\frac{1}{2}\Gamma(\frac{1}{2})}{1!} = 2\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}) = 2\sqrt{\pi}\sqrt{\pi} = 2\pi$$

Question 29

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\infty} \frac{x^3}{\left(1+8x^3\right)^2}\, dx .$$

$\boxed{}$, $\boxed{\dfrac{\pi}{72\sqrt{3}}}$

- START BY A SELF SUGGESTING SUBSTITUTION

$$\Rightarrow u = 8x^3$$
$$\Rightarrow x^3 = \tfrac{1}{8}u$$
$$\Rightarrow x = \tfrac{1}{2}u^{\frac{1}{3}}$$
$$\Rightarrow dx = \tfrac{1}{6}u^{-\frac{2}{3}}\,du$$

& LIMITS ARE UNCHANGED

- THE INTEGRAL TRANSFORMS TO

$$\int_0^{\infty} \frac{x^3}{(1+8x^3)^2}\,dx = \int_0^{\infty} \frac{\frac{1}{8}u}{(1+u)^2}\left(\tfrac{1}{6}u^{-\frac{2}{3}}\,du\right) = \frac{1}{48}\int_0^{\infty} \frac{u^{\frac{1}{3}}}{(1+u)^2}\,du$$

- NOW USING THE ALTERNATIVE FORM OF A BETA FUNCTION

$$\boxed{B(m,n) \equiv \int_0^{\infty} \frac{u^{m-1}}{(1+u)^{m+n}}\,du}$$

- THE INTEGRAL BECOMES

$$\ldots = \frac{1}{48}\int_0^{\infty} \frac{u^{\frac{4}{3}-1}}{(1+u)^{\frac{4}{3}+\frac{2}{3}}}\,du = \frac{1}{48}B\left(\tfrac{4}{3},\tfrac{2}{3}\right) = \frac{1}{48}\,\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}+\frac{2}{3}\right)}$$

$$= \frac{1}{48}\,\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(2)} = \frac{1}{48}\,\frac{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{1!} = \frac{1}{144}\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right)$$

- FINALLY USING $\boxed{\Gamma(x)\Gamma(1-x) \equiv \dfrac{\pi}{\sin \pi x}}$

$$= \frac{1}{144}\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(1-\tfrac{1}{3}\right) = \frac{1}{144}\,\frac{\pi}{\sin\frac{\pi}{3}} = \frac{1}{144}\,\frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{\pi}{72\sqrt{3}}$$

Question 30

By using techniques involving the Beta function, find the exact value of

$$\int_2^4 \frac{1}{\sqrt{(x-2)(4-x)}}\,dx\,.$$

$$\boxed{\pi}$$

$$\boxed{\int_2^4 \frac{1}{\sqrt{(x-2)(4-x)}}\,dx = \pi}$$

TRY TO FIND A LINEAR SUBSTITUTION TO TRANSFORM THE LIMITS TO THAT OF A BETA FUNCTION, I.E FROM 0 TO 1

$x = At+B \qquad 2 \mapsto 0 \implies 2 = 0\times A + B \implies B=2$

$\qquad\qquad\qquad 4 \mapsto 1 \implies 4 = 1\times A + B \implies A=2$

THUS WE USE THE SUBSTITUTION $\boxed{\begin{array}{l} x = 2t+2 \\ dx = 2\,dt \\ \text{WITH LIMITS 0 TO 1} \end{array}}$

$$\ldots = \int_0^1 \frac{1}{\sqrt{(2t+2-2)(4-(2t+2))}}\,(2\,dt)$$

$$= \int_0^1 \frac{2}{\sqrt{2t(2-2t)}}\,dt = \int_0^1 \frac{2}{2\sqrt{t(1-t)}}\,dt$$

$$= \int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt = B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}$$

$$= \frac{\sqrt{\pi}\sqrt{\pi}}{0!} = \frac{\pi}{1} = \pi$$

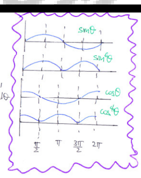
Question 31

By using techniques involving the Beta function, find the exact value of

$$\int_0^{2\pi} \cos^4\theta + \sin^6\theta \;d\theta\,.$$

$$\boxed{\frac{11\pi}{4}}$$

$$\int_0^{2\pi} \cos^4\theta + \sin^6\theta\,d\theta = \int_0^{2\pi} \cos^4\theta\,d\theta + \int_0^{2\pi} \sin^6\theta\,d\theta = \ldots \text{ BY SYMMETRIES}$$



$$= \int_0^{\frac{\pi}{2}} 4\cos^4\theta\,d\theta + \int_0^{\frac{\pi}{2}} 4\sin^6\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} 2(\cos\theta)^{2\times\frac{5}{2}-1}(\sin\theta)^{2\times\frac{1}{2}-1}\,d\theta + 2\int_0^{\frac{\pi}{2}} 2(\sin\theta)^{2\times\frac{7}{2}-1}(\cos\theta)^{2\times\frac{1}{2}-1}\,d\theta$$

$$= 2B\left(\tfrac{5}{2},\tfrac{1}{2}\right) + 2B\left(\tfrac{7}{2},\tfrac{1}{2}\right) = 2\left[\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)}\right] + 2\times\left[\frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{\Gamma(4)}\right]$$

$$= \cancel{2}\times\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\cancel{2!}} + 2\times\frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{3!} = \Gamma(\tfrac{5}{2})\Gamma(\tfrac{1}{2}) + \tfrac{1}{3}\times\boxed{\tfrac{5}{2}\Gamma(\tfrac{5}{2})}\Gamma(\tfrac{1}{2})$$

$$= \Gamma(\tfrac{5}{2})\Gamma(\tfrac{1}{2}) + \tfrac{5}{6}\Gamma(\tfrac{5}{2})\Gamma(\tfrac{1}{2}) = \tfrac{11}{6}\Gamma(\tfrac{5}{2})\Gamma(\tfrac{1}{2}) = \tfrac{11}{6}\times\boxed{\tfrac{3}{2}\Gamma(\tfrac{1}{2})}\Gamma(\tfrac{1}{2})$$

$$= \tfrac{11}{4}\left[\Gamma(\tfrac{1}{2})\right]^2 = \tfrac{11}{4}\left(\sqrt{\pi}\right)^2 = \frac{11\pi}{4}$$

Question 32

By using techniques involving the Beta function, show that

$$\int_{1}^{5} \sqrt[4]{(5-x)(x-1)}\ dx = \frac{2\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{3\sqrt{\pi}}.$$

proof

$$\boxed{\int_{1}^{5} \sqrt[4]{(5-x)(x-1)}\ dx}$$

- SEEK FOR SUBSTITUTIONS INTO BETA IN STEPS

$$\dots = \int_{4}^{0} \sqrt[4]{u(5-u-1)}\ (-du)$$

$$= \int_{0}^{4} u^{\frac{1}{4}}(4-u)^{\frac{1}{4}}\ du$$

$$\boxed{\begin{array}{l} u = 5-x \\ x = 5-u \\ dx = -du \\ x=1 \mapsto u=4 \\ x=5 \mapsto u=0 \end{array}}$$

- ANOTHER SUBSTITUTION TO "SCALE" THE LIMITS BY $\frac{1}{4}$

$$\dots = \int_{0}^{1} (4t)^{\frac{1}{4}}(4-4t)^{\frac{1}{4}}\ 4\,dt$$

$$= \int_{0}^{1} 4^{\frac{1}{4}}t^{\frac{1}{4}}4^{\frac{1}{4}}(1-t)^{\frac{1}{4}}\times 4\ dt$$

$$= \int_{0}^{1} 4^{\frac{3}{2}}t^{\frac{1}{4}}(1-t)^{\frac{1}{4}}\ dt = 8\,B\left(\tfrac{5}{4},\tfrac{5}{4}\right)$$

$$\boxed{\begin{array}{l} t = \frac{1}{4}u \\ u = 4t \\ du = 4\,dt \\ \hline u=4 \mapsto t=1 \\ u=0 \mapsto t=0 \end{array}}$$

- SWITCHING INTO "GAMMAS"

$$= \frac{8\,\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{5}{2}\right)} = \frac{8\times\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\times\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}{\frac{3}{2}\times\frac{1}{2}\ \Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{2}}{\frac{3}{4}}\times\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\Gamma\left(\frac{1}{2}\right)} = \frac{2}{3}\,\frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\sqrt{\pi}}$$

$$\therefore \int_{1}^{5} \sqrt[4]{(5-x)(x-1)}\ dx = \frac{2\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{3\sqrt{\pi}}$$

AN ALTERNATIVE SUBSTITUTION CAN BY FOUND IF WE LOOK FOR A LINEAR TRANSFORMATION TO MATCH THE LIMITS OF A BETA FUNCTION

E.G

$x = At+B \qquad x=5 \mapsto t=1$

$\qquad\qquad\qquad\ \ x=1 \mapsto t=0$

$5 = A+B$

$1 = B$

$A = 4$

$$\therefore \boxed{\begin{array}{l} \text{LET } x = 4t+1 \\ \quad dx = 4dt \\ \quad \text{\& LIMITS NOW MATCH} \end{array}}$$

$$\therefore \int_{1}^{5} \sqrt[4]{(5-x)(x-1)}\ dx = \int_{0}^{1} \sqrt[4]{[5-(4t+1)][4t+1-1]}\ (4dt)$$

$$= \int_{0}^{1} \sqrt[4]{(4-4t)(4t)}\ 4dt = \int_{0}^{1} (4-4t)^{\frac{1}{4}}(4t)^{\frac{1}{4}}\ dt$$

WHICH NOW MERGES WITH THE ORIGINAL WORKINGS

Question 33

$$\mathrm{B}(m,n) \equiv \int_0^1 x^{m-1}(1-x)^{n-1}\ dx\ .$$

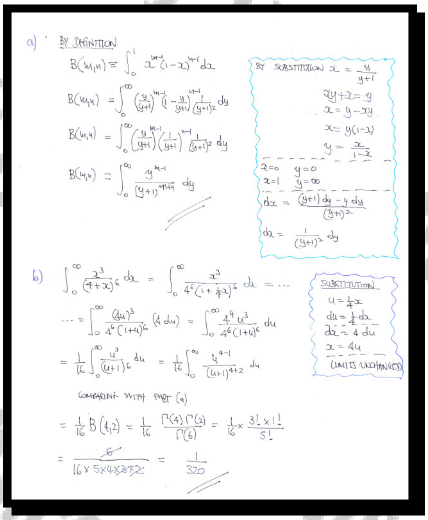
a) Use the substitution $x = \dfrac{y}{y+1}$ to show that

$$\mathrm{B}(m,n) \equiv \int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}}\ dy\ .$$

b) Hence find the exact value of

$$\int_0^\infty \frac{x^3}{(4+x)^6}\ dx\ .$$

$$\boxed{\frac{1}{320}}$$



a) BY DEFINITION

$$B(m,n) \equiv \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$B(m,n) = \int_0^\infty \left(\frac{y}{y+1}\right)^{m-1}\left(1-\frac{y}{y+1}\right)^{n-1}\frac{1}{(y+1)^2}\,dy$$

$$B(m,n) = \int_0^\infty \left(\frac{y}{y+1}\right)^{m-1}\left(\frac{1}{y+1}\right)^{n-1}\frac{1}{(y+1)^2}\,dy$$

$$B(m,n) = \int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}}\,dy$$

BY SUBSTITUTION $x = \frac{y}{y+1}$

$xy + x = y$

$x = y - xy$

$x = y(1-x)$

$y = \frac{x}{1-x}$

$x=0 \quad y=0$

$x=1 \quad y=\infty$

$dx = \frac{(y+1)\,dy - y\,dy}{(y+1)^2}$

$dx = \frac{1}{(y+1)^2}\,dy$

b)

$$\int_0^\infty \frac{x^3}{(4+x)^6}\,dx = \int_0^\infty \frac{x^3}{4^6(1+\frac{1}{4}x)^6}\,dx = \dots$$

$$\dots = \int_0^\infty \frac{(4u)^3}{4^6(1+u)^6}(4\,du) = \int_0^\infty \frac{4^4u^3}{4^6(1+u)^6}\,du$$

$$= \frac{1}{16}\int_0^\infty \frac{u^3}{(u+1)^6}\,du = \frac{1}{16}\int_0^\infty \frac{u^{4-1}}{(u+1)^{4+2}}\,du$$

SUBSTITUTION

$u = \frac{1}{4}x$

$du = \frac{1}{4}dx$

$dx = 4\,du$

$x = 4u$

LIMITS UNCHANGED

COMPARING WITH PART (a)

$$= \frac{1}{16}B(4,2) = \frac{1}{16}\frac{\Gamma(4)\Gamma(2)}{\Gamma(6)} = \frac{1}{16}\times\frac{3!\times 1!}{5!}$$

$$= \frac{\cancel{6}}{16\times 5\times 4\times \cancel{3\times 2}} = \frac{1}{320}$$

Question 34

a) Show clearly that

$$\int_a^b (b-x)^{m-1}(x-a)^{n-1}\ dx = (b-a)^{m+n-1}\,\mathrm{B}(m,n).$$

b) Hence find the exact value of

$$\int_3^5 \sqrt{\frac{5-x}{x-3}}\ dx.$$

$\boxed{\pi}$

a) $\int_a^b (b-x)^{m-1}(x-a)^{n-1}\,dx \ldots$

FIRSTLY MATCH THE LIMITS $a \mapsto 0$, $b \mapsto 1$

$\boxed{t = mx + c}$

$\left.\begin{array}{l} 0 = ma + c \\ 1 = mb + c \end{array}\right\} \Rightarrow m(b-a) = 1$

$m = \frac{1}{b-a}$

$c = -ma = -\frac{a}{b-a}$

$\boxed{t = \frac{1}{b-a}x - \frac{a}{b-a}}$

$(b-a)t = x - a$

$x = a + (b-a)t$

$dx = (b-a)\,dt$

$= \int_a^b \left[b-a-(b-a)t\right]^{m-1}\left[a+(b-a)t-a\right]^{n-1}(b-a)\,dt$

$= \int_0^1 \left[(b-a)-(b-a)t\right]^{m-1}\left[(b-a)t\right]^{n-1}(b-a)\,dt$

$= (b-a)^{m-1}(b-a)^{n-1}(b-a)\int_0^1 (1-t)^{m-1}t^{n-1}\,dt$

$= (b-a)^{m+n-1}\int_0^1 t^{n-1}(1-t)^{m-1}\,dt$

$= (b-a)^{m+n-1}\,B(n,m)$

AS REQUIRED

b) $\int_3^5 \sqrt{\frac{5-x}{x-3}}\,dx = \int_3^5 (5-x)^{\frac{1}{2}}(x-3)^{-\frac{1}{2}}\,dx = \int_3^5 (5-x)^{\frac{3}{2}-1}(x-3)^{\frac{1}{2}-1}\,dx$

$= (5-3)^{\frac{3}{2}+\frac{1}{2}-1} \times B\left(\frac{3}{2},\frac{1}{2}\right) = 2^1\,B\left(\frac{3}{2},\frac{1}{2}\right)$

$= 2 \times \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = 2 \times \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1!}$

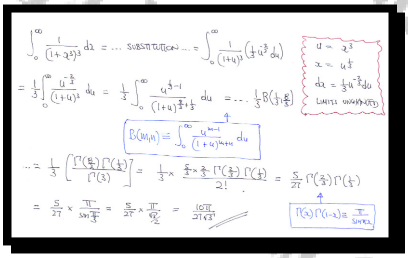
$= \pi$

Question 35

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\infty} \frac{1}{\left(1+x^3\right)^3}\, dx .$$

$$\boxed{\frac{10\pi}{27\sqrt{3}}}$$

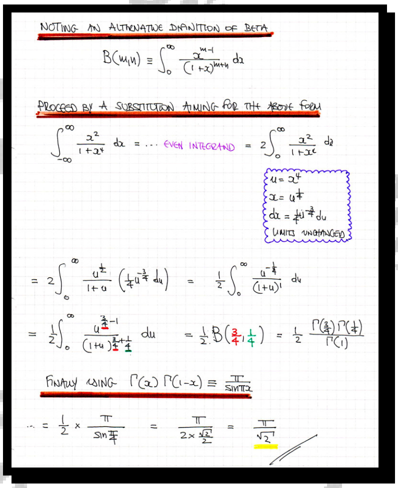


Question 36

By using techniques involving the Beta function, find the exact value of

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx .$$

$$\boxed{}, \quad \boxed{\frac{\pi}{\sqrt{2}}}$$



Question 37

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\infty} \frac{1}{1+x^6}\,dx\,.$$

$$\boxed{\frac{\pi}{3}}$$

$$\int_0^{\infty} \frac{1}{1+x^6}\,dx = \dots \int_0^{\infty} \frac{1}{1+u}\left(\frac{1}{6}u^{-\frac{5}{6}}\right) = \int_0^{\infty} \frac{1}{6}\,\frac{u^{-\frac{5}{6}}}{(1+u)^1}\,du$$

$$= \int_0^{\infty} \frac{1}{6}\,\frac{u^{\frac{1}{6}-1}}{(1+u)^{\frac{5}{6}+\frac{1}{6}}}\,du = \frac{1}{6}B\left(\frac{1}{6},\frac{5}{6}\right) = \frac{1}{6}\,\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma(1)}$$

$$= \frac{1}{6}\,\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{0!} = \frac{1}{6}\times\frac{\pi}{\sin\frac{\pi}{6}} = \frac{1}{6}\times\frac{\pi}{\frac{1}{2}} = \frac{\pi}{3}$$

$u = x^6$
$x = u^{\frac{1}{6}}$
$dx = \frac{1}{6}u^{-\frac{5}{6}}$
UNITS UNCHANGED

$$\boxed{B(m,n) \equiv \int_0^{\infty} \frac{u^{m-1}}{(1+u)^{m+n}}\,du}$$

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}}$$

Question 38

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\infty} \frac{1}{\left(2+x^2\right)^4}\,dx\,.$$

$$\boxed{\frac{5\pi\sqrt{2}}{512}}$$

$$\int_0^{\infty} \frac{1}{(2+x^2)^4}\,dx = \dots \text{SUBSTITUTION} \dots \int_0^{\infty} \frac{1}{(2+2u)^4}\left(\frac{1}{2}\sqrt{2}\,u^{-\frac{1}{2}}\,du\right)$$

$$= \int_0^{\infty} \frac{\frac{1}{2}\sqrt{2}\,u^{-\frac{1}{2}}}{16(1+u)^4}\,du = \frac{\sqrt{2}}{32}\int_0^{\infty} \frac{u^{-\frac{1}{2}}}{(1+u)^4}\,du = \frac{\sqrt{2}}{32}\int_0^{\infty} \frac{u^{\frac{1}{2}-1}}{(1+u)^{\frac{7}{2}+\frac{1}{2}}}\,du$$

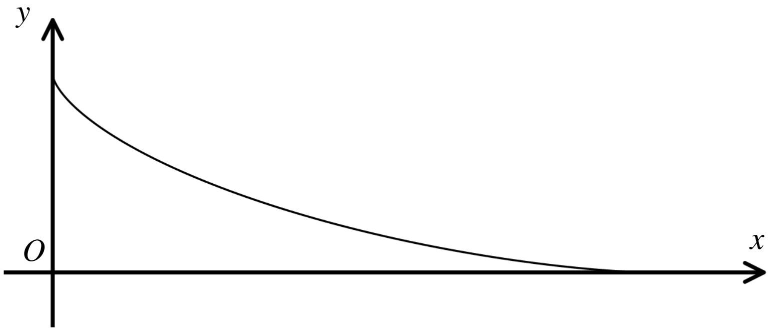
$x^2 = 2u$
$x = \sqrt{2}\,u^{\frac{1}{2}}$
$dx = \frac{1}{2}\sqrt{2}\,u^{-\frac{1}{2}}\,du$
UNITS UNCHANGED

$$B(m,n) \equiv \int_0^{\infty} \frac{u^{m}}{(1+u)^{m+n}}\,du$$

$$= \frac{\sqrt{2}}{32}B\left(\frac{1}{2},\frac{7}{2}\right) = \frac{\sqrt{2}}{32}\times\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(4)} = \frac{\sqrt{2}}{32}\,\frac{\Gamma\left(\frac{1}{2}\right)\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{3!}$$

$$= \frac{\sqrt{2}}{32}\times\frac{1}{\cancel{6}_{2}}\times\frac{\cancel{15}^{5}}{8}\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \frac{5\sqrt{2}}{512}\pi$$

Question 39



The figure above shows the curve with parametric equations

$$x = 8\cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le \tfrac{1}{2}\pi.$$

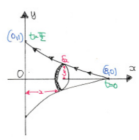
The finite region bounded by the curve and the coordinate axes is revolved fully about the x axis, forming a solid of revolution S.

Determine the x coordinate of the centre of mass of S.

$\boxed{}$, $\boxed{\bar{x} = \frac{21}{16}}$

- START WITH THE DIAGRAM OPPOSITE

$$x = 8\cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le \tfrac{\pi}{2}$$



- THE MASS OF THE INFINITESIMAL DISC OF RADIUS y & THICKNESS δx IS GIVEN BY

$$\delta m = \pi\rho y^2\,\delta x$$

(ρ = DENSITY)

- THE "MOMENT" OF THIS INFINITESIMAL MASS, ABOUT THE y AXIS IS GIVEN BY

$$x\,\delta m = x(\pi\rho y^2\,\delta x) = \pi\rho y^2 x\,\delta x$$

- SUMMING UP, TAKING LIMITS, WE OBTAIN

$$\Rightarrow M\bar{x} = \int_{x=0}^{8} \pi\rho y^2 x\,dx$$

$$\Rightarrow \bar{x}\int_{x=0}^{8} \cancel{\pi\rho}\, y^2\,dx = \int_{x=0}^{8} \cancel{\pi\rho}\, y^2 x\,dx$$

$$\Rightarrow \bar{x}\int_{t=\frac{\pi}{2}}^{0} \underbrace{(\sin^3 t)^2}_{y^2}\underbrace{(-24\cos^2 t\sin t\,dt)}_{dx} = \int_{t=\frac{\pi}{2}}^{0} \underbrace{(\sin^3 t)^2}_{y^2}\underbrace{(-24\cos^2 t\sin t\,dt)}_{dx}\underbrace{(8\cos^3 t)}_{x}$$

$$\Rightarrow \bar{x}\int_{0}^{\frac{\pi}{2}} \sin^7 t\cos^2 t\,dt = \int_{0}^{\pi/2} 8\sin^7 t\cos^5 t\,dt$$

- EVALUATE USING BETA FUNCTIONS

$$\Rightarrow \bar{x}\int_{0}^{\frac{\pi}{2}} 2(\sin t)^{2\times 4-1}(\cos t)^{2\times\frac{3}{2}-1}\,dt = 8\int_{0}^{\frac{\pi}{2}} 2(\sin t)^{2\times 4-1}(\cos t)^{2\times 3-1}\,dt$$

$$\Rightarrow \bar{x}\,B(4,\tfrac{3}{2}) = 8\,B(4,3)$$

$$\Rightarrow \bar{x}\,\frac{\cancel{\Gamma(4)}\,\Gamma(\frac{3}{2})}{\Gamma(\frac{11}{2})} = \frac{8\,\cancel{\Gamma(4)}\,\Gamma(3)}{\Gamma(7)}$$

$$\Rightarrow \bar{x} = \frac{8\,\Gamma(3)\,\Gamma(\frac{11}{2})}{\Gamma(7)\,\Gamma(\frac{3}{2})}$$

$$\Rightarrow \bar{x} = \frac{8\times 2!}{6!} \times \frac{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\cancel{\Gamma(\frac{3}{2})}}{\cancel{\Gamma(\frac{3}{2})}}$$

$$\Rightarrow \bar{x} = \frac{\cancel{16}}{720} \times \frac{9\times 7\times 5\times 3}{\cancel{16}}$$

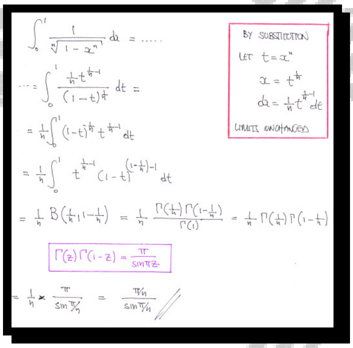
$$\Rightarrow \bar{x} = \frac{21}{16}$$

Question 40

By using techniques involving the Beta function, show that

$$\int_0^1 \frac{1}{\sqrt[n]{1-x^n}}\, dx = \frac{\frac{\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}.$$

proof



$$\int_0^1 \frac{1}{\sqrt[n]{1-x^n}}\, dx = \dots\dots$$

BY SUBSTITUTION
LET $t = x^n$
$x = t^{\frac{1}{n}}$
$dx = \frac{1}{n}t^{\frac{1}{n}-1}\,dt$
LIMITS UNCHANGED

$$\dots = \int_0^1 \frac{\frac{1}{n}t^{\frac{1}{n}-1}}{(1-t)^{\frac{1}{n}}}\,dt =$$

$$= \frac{1}{n}\int_0^1 (1-t)^{-\frac{1}{n}}\, t^{\frac{1}{n}-1}\,dt$$

$$= \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1}(1-t)^{\left(1-\frac{1}{n}\right)-1}\,dt$$

$$= \frac{1}{n}B\left(\frac{1}{n}, 1-\frac{1}{n}\right) = \frac{1}{n}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{\Gamma(1)} = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)$$

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}}$$

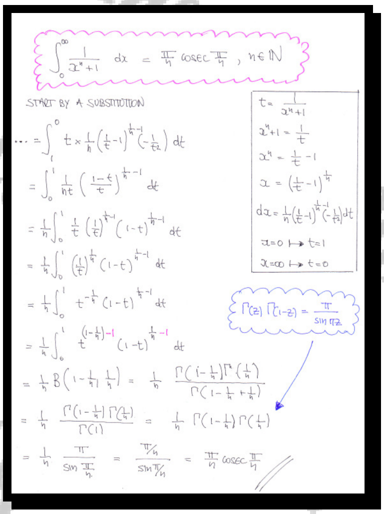
$$= \frac{1}{n} \times \frac{\pi}{\sin\frac{\pi}{n}} = \frac{\pi/n}{\sin\pi/n}$$

Question 41

By using techniques involving the Beta function, show that

$$\int_0^1 \frac{1}{x^n+1}\, dx \;=\; \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right).$$

proof



$$\int_0^\infty \frac{1}{x^n+1}\,dx = \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}, \quad n \in \mathbb{N}$$

START BY A SUBSTITUTION

$t = \frac{1}{x^n+1}$
$x^n + 1 = \frac{1}{t}$
$x^n = \frac{1}{t} - 1$
$x = \left(\frac{1}{t}-1\right)^{\frac{1}{n}}$
$dx = \frac{1}{n}\left(\frac{1}{t}-1\right)^{\frac{1}{n}-1}\left(-\frac{1}{t^2}\right)dt$
$x = 0 \mapsto t = 1$
$x = \infty \mapsto t = 0$

$$\dots = \int_1^0 t \times \frac{1}{n}\left(\frac{1}{t}-1\right)^{\frac{1}{n}-1}\left(-\frac{1}{t^2}\right)dt$$

$$= \int_0^1 \frac{1}{nt}\left(\frac{1-t}{t}\right)^{\frac{1}{n}-1}dt$$

$$= \frac{1}{n}\int_0^1 \frac{1}{t}\left(\frac{1}{t}\right)^{\frac{1}{n}-1}(1-t)^{\frac{1}{n}-1}dt$$

$$= \frac{1}{n}\int_0^1 \left(\frac{1}{t}\right)^{\frac{1}{n}}(1-t)^{\frac{1}{n}-1}dt$$

$$= \frac{1}{n}\int_0^1 t^{-\frac{1}{n}}(1-t)^{\frac{1}{n}-1}dt$$

$$= \frac{1}{n}\int_0^1 t^{\left(1-\frac{1}{n}\right)-1}(1-t)^{\frac{1}{n}-1}dt$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$= \frac{1}{n}B\left(1-\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{n}\frac{\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(1-\frac{1}{n}+\frac{1}{n}\right)}$$

$$= \frac{1}{n}\frac{\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)}{\Gamma(1)} = \frac{1}{n}\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\frac{\pi}{\sin\frac{\pi}{n}} = \frac{\pi/n}{\sin\pi/n} = \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}$$

Question 42

$$\mathrm{B}(m,n) \equiv \int_0^1 x^{m-1}(1-x)^{n-1}\ dx.$$

a) Use the substitution $x = \dfrac{y}{y+1}$ to show that

$$\mathrm{B}(m,n) \equiv \int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}}\ dy.$$

b) Hence find the exact value of

$$\int_0^\infty \frac{\sqrt{x}}{16+x^2}\ dx.$$

$$\boxed{\frac{\pi\sqrt{2}}{4}}$$

a) BY DEFINITION

$$B(m,n) \equiv \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$B(m,n) = \int_0^\infty \left(\frac{y}{y+1}\right)^{m-1}\left[1-\frac{y}{y+1}\right]^{n-1}\frac{1}{(y+1)^2}\,dy$$

$$B(m,n) = \int_0^\infty \left(\frac{y}{y+1}\right)^{m-1}\left(\frac{1}{y+1}\right)^{n-1}\frac{1}{(y+1)^2}\,dy$$

$$B(m,n) = \int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}}\,dy$$

AS REQUIRED

BY SUBSTITUTION

- $x = \frac{y}{y+1}$, $xy + x = y$, $x = y - xy$, $y(1-x) = x$, $y = \frac{x}{1-x}$

LIMITS: $x=0, y=0$; $x=1, y=\infty$

- $dx = \frac{(y+1)dy - y\,dy}{(y+1)^2}$, $dx = \frac{1}{(y+1)^2}\,dy$

b)

$$\int_0^\infty \frac{\sqrt{x}}{16+x^2}\,dx = \int_0^\infty \frac{x^{\frac{1}{2}}}{16(1+\frac{1}{16}x^2)}\,dx$$

$$\dots = \int_0^\infty \frac{(4u^{\frac{1}{2}})^{\frac{1}{2}}}{16(1+u)}\left(2u^{-\frac{1}{2}}du\right) = \int_0^\infty \frac{2u^{\frac{1}{4}}}{16(1+u)}\left[2u^{-\frac{1}{2}}du\right]$$

$$= \int_0^\infty \frac{u^{-\frac{1}{4}}}{4(1+u)}\,du = \dots \text{PART (a)}$$

$$= \frac{1}{4}\int_0^\infty \frac{u^{\frac{3}{4}-1}}{(u+1)^{\frac{3}{4}+\frac{1}{4}}}\,du = \frac{1}{4}B\left(\tfrac{3}{4},\tfrac{1}{4}\right)$$

$$= \frac{1}{4}\,\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)} = \frac{1}{4}\Gamma\left(\tfrac{3}{4}\right)\Gamma\left(\tfrac{1}{4}\right)$$

$$= \frac{1}{4}\,\frac{\pi}{\sin(\frac{1}{4}\pi)} = \frac{1}{4}\times\frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}\pi}{4}$$

OR $\frac{\pi}{\sin(\frac{3}{4}\pi)}$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}$$

SUBSTITUTION

$u = \frac{1}{16}x^2$, $16u = x^2$, $x = 4u^{\frac{1}{2}}$

$dx = 2u^{-\frac{1}{2}}du$, $dx = \frac{2}{u^{\frac{1}{2}}}du$

LIMITS UNCHANGED

Question 43

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{(\cos x+\sin x)^2}\,dx .$$

$\boxed{}$, $\boxed{\frac{\pi}{2}}$

START WITH A SUBSTITUTION

$t = \tan x$

$dt = \sec^2 x\,dx$

$dx = \frac{dt}{\sec^2 x}$

$dx = \cos^2 x\,dt$

& THE LIMITS

$x=0 \longmapsto t=0$

$x=\frac{\pi}{2} \longmapsto t=\infty$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{(\cos x+\sin x)^2}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\cos^2 x\left(1+\frac{\sin x}{\cos x}\right)^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\cos^3 x(1+\tan x)^2}\,dx$$

$$= \int_0^{\infty} \frac{t^{\frac{1}{2}}}{\cancel{\cos^3 x}(1+t)^2}\left(\cancel{\cos^2 x}\,dt\right)$$

$$= \int_0^{\infty} \frac{t^{\frac{1}{2}}}{(1+t)^2}\,dt$$

NOW USING AN ALTERNATIVE DEFINITION OF THE BETA FUNCTION

$$B(m,n) = \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

AND BY REWRITING THE ABOVE INTEGRAL TO "FIT" THIS DEFINITION

$$= \int_0^{\infty} \frac{t^{\frac{3}{2}-1}}{(1+t)^{\frac{3}{2}+\frac{1}{2}}}\,dt = B\left(\tfrac{3}{2},\tfrac{1}{2}\right)$$

SWITCHING INTO GAMMA FUNCTIONS

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1!} = \frac{\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{1} = \frac{\pi}{2}$$

Question 44

By using techniques involving the Beta function, find the exact value of

$$\int_{1}^{\infty} \frac{1}{x^2\sqrt{x-1}}\, dx .$$

$$\boxed{\frac{\pi}{2}}$$

$\int_1^\infty \frac{1}{x^2\sqrt{x-1}}\,dx = \ldots$ LIMITS SUGGEST RECIPROCATION

$u = \frac{1}{x}$

$x = \frac{1}{u}$

$dx = -\frac{1}{u^2}\,du$

$x=1 \mapsto u=1$

$x=\infty \mapsto u=0$

$= \int_1^0 \frac{1}{\frac{1}{u^2}\sqrt{\frac{1}{u}-1}}\left(-\frac{1}{u^2}\,du\right) = \int_1^0 \frac{\cancel{u^2}}{\sqrt{\frac{1}{u}-1}}\left(-\frac{1}{\cancel{u^2}}\right)du$

$= \int_0^1 \frac{1}{\sqrt{\frac{1}{u}-1}}\,du = \int_0^1 \frac{1}{\left(\frac{1-u}{u}\right)^{\frac{1}{2}}}\,du$

$= \int_0^1 \left(\frac{u}{1-u}\right)^{\frac{1}{2}} du = \int_0^1 u^{\frac{1}{2}}(1-u)^{-\frac{1}{2}}\,du$

$= \int_0^1 u^{\frac{3}{2}-1}(1-u)^{\frac{1}{2}-1}\,du = B\left(\frac{3}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}$

$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\times\Gamma\left(\frac{1}{2}\right)}{1!} = \frac{1}{2}\sqrt{\pi}\times\sqrt{\pi} = \frac{\pi}{2}$

Question 45

$$I=\int_{-1}^{1}\left(1-x^2\right)^n\ dx\ ,\ n\in\mathbb{N}.$$

Use techniques involving the Beta function to show that

$$I=\frac{2^{2n+1}(n!)^2}{(2n+1)!}.$$

$\boxed{}$, $\boxed{\text{proof}}$

START BY A SUBSTITUTION, NOTING FIRST, THAT THE INTEGRAND IS EVEN IN A SYMMETRICAL DOMAIN.

$$\int_{-1}^{1}(1-x^2)^n\,dx = 2\int_{0}^{1}(1-x^2)^n\,dx$$

$$= 2\int_0^1 (1-y)^n\left(\frac{1}{2\sqrt{y}}\,dy\right) = \int_0^1 (1-y)^n y^{-\frac{1}{2}}\,dy$$

$$\boxed{\begin{aligned}\sqrt{y}&=x\\ y&=x^2\\ dy&=2x\,dx\\ dx&=\frac{dy}{2x}\\ dx&=\frac{dy}{2\sqrt{y}}\end{aligned}\quad \text{LIMITS ARE UNCHANGED}}$$

BY BETA & GAMMA FUNCTIONS

$$\ldots = B\left(n+1,\tfrac{1}{2}\right) = \frac{\Gamma(n+1)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)}$$

$$= \frac{n!\,\sqrt{\pi}}{\Gamma\left(n+\frac{3}{2}\right)}$$

MANIPULATE FURTHER AS FOLLOWS

$$= \frac{n!\,\sqrt{\pi}}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{n!\,\cancel{\sqrt{\pi}}}{\frac{1}{2}(2n+1)\times\frac{1}{2}(2n-1)\times\frac{1}{2}(2n-3)\cdots\left(\frac{1}{2}\times5\right)+\left(\frac{1}{2}\times3\right)+\left(\frac{1}{2}\times1\right)\times\cancel{\sqrt{\pi}}}$$

$$= \frac{n!}{\left(\frac{1}{2}\right)^{n+1}(2n+1)(2n-1)(2n-3)\cdots\times5\times3\times1}$$

$$= \frac{2^{n+1}\,n!\,(2n)(2n-2)(2n-4)\ldots\times6\times4\times2}{(2n+1)(2n)(2n-1)(2n-2)(2n-3)\ldots\times6\times5\times4\times3\times2\times1}$$

$$= \frac{2^{n+1}\,n!\times2n\times2(n-1)\times2(n-2)\times2(n-3)\times\cdots\times(2\times3)\times(2\times2)\times(2\times1)}{(2n+1)!}$$

$$= \frac{2^{n+1}\times n!\times2^n\ n(n-1)(n-2)(n-3)\cdots\times3\times2\times1}{(2n+1)!}$$

$$= \frac{2^{2n+1}\times n!\times n!}{(2n+1)!}$$

$$= \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

AS REQUIRED

Question 46

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\infty} \frac{1}{(x+1)\sqrt{x}}\,dx.$$

$\boxed{\quad}$, $\boxed{\pi}$

START WITH A SUBSTITUTION FIRST

$$\boxed{\begin{array}{l} u = \frac{1}{x+1} \iff x+1 = \frac{1}{u} \quad \left(x = \frac{1}{u} - 1\right) \\ \qquad \Rightarrow dx = -\frac{1}{u^2}\,du \\ x = 0 \longmapsto u = 1 \\ x = \infty \longmapsto u = 0 \end{array}}$$

TRANSFORMING THE INTEGRAL INCLUDING LIMITS

$$\int_1^0 \frac{1}{\underbrace{\sqrt{\frac{1}{u}-1}}_{\sqrt{x}}\ \underbrace{\frac{1}{u}}_{x+1}} \underbrace{\left(-\frac{1}{u^2}\,du\right)}_{dx} = \int_1^0 \frac{-1}{\sqrt{\frac{1-u}{u}}\ \frac{1}{u}\,u^2}\,du$$

$$= \int_0^1 \frac{+1}{\frac{\sqrt{1-u}}{\sqrt{u}} \times u}\,du$$

$$= \int_0^1 \frac{1}{(1-u)^{\frac{1}{2}}\left(u^{\frac{1}{2}}\right)}\,du$$

$$= \int_0^1 (1-u)^{-\frac{1}{2}} u^{-\frac{1}{2}}\,du$$

NOW USING THE BETA FUNCTION DEFINITION

$$\dots = \int_0^1 (1-u)^{\frac{1}{2}-1} u^{\frac{1}{2}-1}\,du = B\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

SWITCHING INTO GAMMA FUNCTIONS FOR THE FINAL EVALUATION

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \frac{\sqrt{\pi}\,\sqrt{\pi}}{0!} = \frac{\pi}{1} = \pi$$

Question 47

By using techniques involving the Beta function, find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\cos^2 x + 4\sin^2 x}\, dx .$$

$$\boxed{\frac{\pi}{4}}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\cos^2 x+4\sin^2 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\cos^2 x\left[1+4\tan^2 x\right]}\,dx = \dots$$

$$\boxed{\begin{aligned} u &= \tan x \\ du &= \sec^2 x\,dx \\ dx &= \frac{du}{\sec^2 x} \\ dx &= du\cos^2 x \\ x=\tfrac{\pi}{2} &\mapsto u=\infty \\ x=0 &\mapsto u=0 \end{aligned}}$$

$$\dots = \int_0^{\infty} \frac{u^{\frac{1}{2}}}{\cancel{\cos^2 x}(1+4u^2)}\left(\cancel{\cos^2 x}\,du\right) = \int_0^{\infty} \frac{u^{\frac{1}{2}}}{1+4u^2}\,du$$

$\dots =$ ANOTHER SUBSTITUTION

$$\boxed{\begin{aligned} t &= 4u^2 \\ u^2 &= \frac{t}{4} \\ u &= \frac{t^{\frac{1}{2}}}{2} = \frac{1}{2}t^{\frac{1}{2}} \\ du &= \frac{1}{4}t^{-\frac{1}{2}}\,dt \\ &\text{LIMITS UNCHANGED} \end{aligned}}$$

$$= \dots \int_0^{\infty} \frac{\left(\frac{t^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}}{1+t}\,\frac{1}{4}t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{4}\times\frac{1}{\sqrt{2}}\int_0^{\infty} \frac{t^{-\frac{1}{4}}}{1+t}\,dt = \frac{\sqrt{2}}{8}\int_0^{\infty} \frac{t^{\frac{3}{4}-1}}{(1+t)^{\frac{1}{4}+\frac{3}{4}}}\,dt = \frac{\sqrt{2}}{8}B\left(\tfrac{3}{4},\tfrac{1}{4}\right)$$

$$\boxed{B(m,n) = \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}}\,dt}$$

$$= \frac{\sqrt{2}}{8}\times\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \frac{\sqrt{2}}{8}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{0!} = \frac{\sqrt{2}}{8}\Gamma\left(\tfrac{3}{4}\right)\Gamma\left(\tfrac{1}{4}\right)$$

$$\boxed{\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}}$$

$$= \frac{\sqrt{2}}{8}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\sqrt{2}}{8}\times\frac{\pi}{\frac{\sqrt{2}}{2}} = \frac{\pi}{4}$$

Question 48

$$\int_0^{\infty} e^{-u^2} u^{2x-1} \ du \, .$$

a) Determine the value of the above integral in terms of Gamma functions.

b) Show that

$$\int_0^{\frac{1}{2}\pi} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} \ d\theta \ \equiv \tfrac{1}{2}\mathrm{B}(x,y) \, .$$

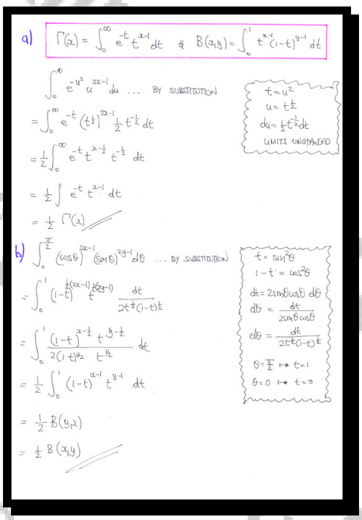
c) Use the results of part **(a)** and **(b)** to show further that

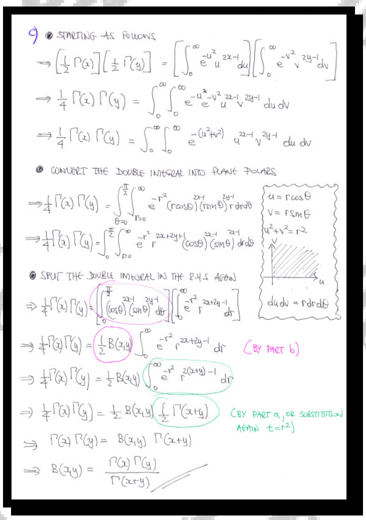
$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \, .$$

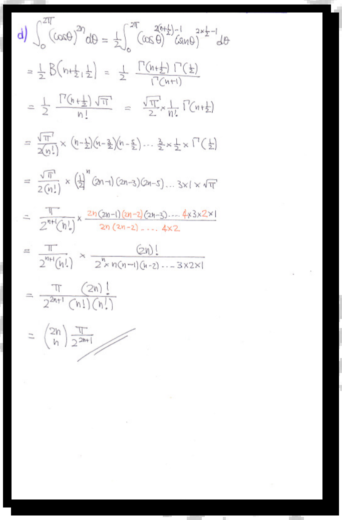
d) Hence deduce that

$$\int_0^{\frac{1}{2}\pi} (\cos\theta)^{2n} \ d\theta \ = \binom{2n}{n} \frac{\pi}{2^{2n+1}} \, .$$

$$\boxed{\tfrac{1}{2}\Gamma(x)}$$







Question 49

a) By using techniques involving the Beta function and the Gamma function, show that

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2k+1}\, d\theta = \frac{(k!)^2\, 2^{2k}}{(2k+1)!}.$$

b) Use a suitable substitution in the above integral to deduce an expression for

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{2k+1}\, d\theta.$$

$$\boxed{\int_0^{\frac{\pi}{2}} (\sin\theta)^{2k+1}\, d\theta = \int_0^{\frac{\pi}{2}} (\cos\theta)^{2k+1}\, d\theta = \frac{(k!)^2\, 2^{2k}}{(2k+1)!}}$$

a) $\int_0^{\frac{\pi}{2}} (\cos\theta)^{2k+1}\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 2(\cos\theta)^{\overset{2(k+1)-1}{2k+1}} (\sin\theta)^{2\times\frac{1}{2}-1}\, d\theta$

$= \frac{1}{2}B\left(k+1,\frac{1}{2}\right) = \frac{1}{2}\,\frac{\Gamma(k+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}$

$= \frac{1}{2} \times \frac{k!\,\cancel{\Gamma\left(\frac{1}{2}\right)}}{\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\dots\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\cancel{\Gamma\left(\frac{1}{2}\right)}}$

$= \frac{k!}{2} \times \frac{1}{\frac{1}{2}(2k+1)\,\frac{1}{2}(2k-1)\,\frac{1}{2}(2k-3)\dots\frac{1}{2}\times5\times\frac{1}{2}\times3\times\frac{1}{2}\times1}$

$= \frac{k!}{2} \times \frac{1}{\left(\frac{1}{2}\right)^{k+1}\times(2k+1)(2k-1)(2k-3)\dots5\times3\times1}$

$\begin{matrix} 1 & 3 & 5 & 7 & \dots & 2k+1 \\ 1 & 2 & 3 & 4 & & k+1 \end{matrix}$

$= \frac{k!\,2^{k+1}}{2} \times \frac{2k(2k-2)\dots6\times4\times2}{(2k+1)\,2k\,(2k-1)(2k-2)(2k-3)\dots6\times5\times4\times3\times2\times1}$

$= \frac{k!\,2^{k}}{(2k+1)!} \times 2^{k}\left[k(k-1)(k-2)\dots3\times2\times1\right]$

$= \frac{k!\,2^{2k}\,k!}{(2k+1)!} = \frac{(k!)^2\,2^{2k}}{(2k+1)!}$ // AS REQUIRED

b) $\int_0^{\frac{\pi}{2}} (\cos\theta)^{2k+1}\, d\theta = \frac{(k!)^2\,2^{2k}}{(2k+1)!}$

LET $\theta = \frac{\pi}{2} - \Theta$

$d\theta = -d\Theta$

$\theta = 0,\ \Theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{2},\ \Theta = 0$

$\int_{\frac{\pi}{2}}^{0} \left[\cos\left(\frac{\pi}{2}-\Theta\right)\right]^{2k+1}(-d\Theta) = \frac{(k!)^2\,2^{2k}}{(2k+1)!}$

$\int_0^{\frac{\pi}{2}} (\sin\Theta)^{2k+1}\, d\Theta = \frac{(k!)^2\,2^{2k}}{(2k+1)!}$ //

Question 50

$$f(z) = \frac{z^{p-1}}{z+1}, \; z \in \mathbb{C}, \; p \in \mathbb{R}, \; 0 < p < 1.$$

a) By integrating $f(z)$ over a keyhole contour with a branch cut along the positive x axis, show that

$$\int_0^{\infty} \frac{x^{p-1}}{1+x} \, dx = \frac{\pi}{\sin(p\pi)}.$$

b) Hence show further that

$$\Gamma(x)\Gamma(1-x) = \int_0^1 u^{x-1}(1-u)^{-x} \, du.$$

c) Use the substitution $u = \frac{t}{t+1}$ to deduce that

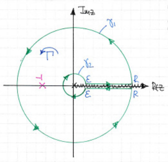
$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \; 0 < x < 1.$$

V, MM3, proof

[solution overleaf]

a) $f(z)=\frac{z^{p-1}}{z+1}$ CONTAINS FRACTIONAL POWERS AS $0<p<1$, SO IT HAS A BRANCH POINT AT $z=0$, SO WE MUST MAKE A BRANCH CUT (~~~~) FROM $z=0$, "OUTWARDS" — IT IS SENSIBLE HERE TO DO SO ON THE POSITIVE x AXIS AS THERE IS A POLE AT $z=-1$ AND IT IS EASY TO PARAMETRIZE ALONG THE x AXIS

CONSIDER $\int_{\Gamma} f(z)\,dz$ OVER THE "KEYHOLE CONTOUR" SHOWN BELOW



NOTE THAT $0 \le \arg z < 2\pi$ BECAUSE OF THE POSITION OF THE BRANCH CUT

$f(z)$ HAS SIMPLE POLE AT $z=-1$

$$\operatorname{Res}[f;-1]=\lim_{z\to-1}\left[(z+1)f(z)\right]=\lim_{z\to-1}\left[(z+1)\frac{z^{p-1}}{z+1}\right]=(-1)^{p-1}=(e^{i\pi})^{p-1}=e^{i\pi(p-1)}$$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z)\,dz = 2\pi i \times \sum(\text{RESIDUES OF } f \text{ INSIDE } \Gamma)$$

$$\left\{\int_{\gamma_1}+\int_R^{\varepsilon}+\int_{\gamma_2}+\int_{\varepsilon}^{R}\right\}f(z)\,dz = 2\pi i \times e^{i\pi(p-1)}$$

NEXT PARAMETRIZE z AROUND γ_1

$z=Re^{i\theta}$
$dz=iRe^{i\theta}$
θ RUNS FROM 0 TO 2π

$$\left|\int_{\gamma_1} f(z)\,dz\right| = \left|\int_0^{2\pi}\frac{(Re^{i\theta})^{p-1}}{Re^{i\theta}+1}(iRe^{i\theta}d\theta)\right| = \left|\int_0^{2\pi}\frac{iR^{p-1}Re^{i\theta}e^{i\theta(p-1)}}{Re^{i\theta}+1}\,d\theta\right|$$

$$\le \int_0^{2\pi}\frac{|iR^p e^{i\theta}e^{i\theta(p-1)}|}{|Re^{i\theta}+1|}\,d\theta = \int_0^{2\pi}\frac{\cancel{|i|}|R^p|\cancel{|e^{i\theta}|}\cancel{|e^{i\theta(p-1)}|}}{|Re^{i\theta}+1|}\,d\theta$$

$$= \int_0^{2\pi}\frac{R^p}{|Re^{i\theta}+1|}\,d\theta$$

$|z+w| \ge |z|-|w| \quad |z|>|w|$

$\frac{1}{|z+w|} \le \frac{1}{|z|-|w|}$

$$\le \int_0^{2\pi}\frac{R^p}{|Re^{i\theta}|-|1|}\,d\theta$$

$$= \int_0^{2\pi}\frac{R^p}{|R|\cancel{|e^{i\theta}|}-1}\,d\theta = \int_0^{2\pi}\frac{R^p}{R-1}\,d\theta = \frac{R^p}{R-1}\int_0^{2\pi}1\,d\theta$$

$$= \frac{2\pi R^p}{R-1} = O(R^{p-1}) \to 0 \text{ AS } R\to\infty \quad [p-1<0]$$

NEXT PARAMETRIZE z AROUND γ_2

$z=\varepsilon e^{i\theta}$
$dz=i\varepsilon e^{i\theta}$
θ RUNS FROM 2π TO 0

$$\left|\int_{\gamma_2} f(z)\,dz\right| = \left|\int_{2\pi}^{0}\frac{(\varepsilon e^{i\theta})^{p-1}}{\varepsilon e^{i\theta}+1}(i\varepsilon e^{i\theta}d\theta)\right| = \left|\int_0^{2\pi}\frac{i\varepsilon^{p-1}\varepsilon e^{i\theta}e^{i\theta(p-1)}}{\varepsilon e^{i\theta}+1}\,d\theta\right|$$

$$\le \int_0^{2\pi}\left|\frac{i\varepsilon^p e^{i\theta}e^{i\theta(p-1)}}{\varepsilon e^{i\theta}+1}\right|d\theta = \int_0^{2\pi}\frac{\cancel{|i|}|\varepsilon^p|\cancel{|e^{i\theta}|}\cancel{|e^{i\theta(p-1)}|}}{|\varepsilon e^{i\theta}+1|}\,d\theta$$

SAME INEQUALITY APPLIED AS IN THE PREVIOUS ESTIMATION AROUND γ_1

$$\le \int_0^{2\pi}\frac{\varepsilon^p}{|\varepsilon e^{i\theta}|-|1|}\,d\theta = \int_0^{2\pi}\frac{\varepsilon^p}{|\varepsilon|\cancel{|e^{i\theta}|}-1}\,d\theta = \int_0^{2\pi}\frac{\varepsilon^p}{\varepsilon-1}\,d\theta$$

$$= \frac{\varepsilon^p}{\varepsilon-1}\int_0^{2\pi}1\,d\theta = \frac{\varepsilon^p}{\varepsilon-1}\times 2\pi \to 0 \text{ AS } \varepsilon\to 0$$

(NUMERATOR $\to 0$, DENOMINATOR $\to -1$)

FINALLY WE HAVE AS $R\to\infty$ & $\varepsilon\to 0$

$$\underbrace{\int_{\infty}^{0} f(z)\,dz}_{\text{BOTTOM SEGMENT}} + \underbrace{\int_{0}^{\infty} f(z)\,dz}_{\text{TOP SEGMENT}} = 2\pi i \times e^{i\pi(p-1)}$$

- $z=xe^{2i\pi}$
- $dz=dx\,e^{2i\pi}$

- $z=xe^{0i}$
- $dz=dx\,e^{0i}$

EVERYTHING IS OK, IE $z=x$ & $dz=dx$ EXCEPT z^{p-1} ON THE BOTTOM SEGMENT SINCE $p-1$ IS NOT AN INTEGER
$z^p=(xe^{2i\pi})^{p-1}=x^{p-1}e^{2\pi(p-1)i}$

THUS WE OBTAIN

$$\int_{\infty}^{0}\frac{x^{p-1}e^{2\pi(p-1)i}}{x+1}\,dx + \int_0^{\infty}\frac{x^{p-1}}{x+1}\,dx = 2\pi i \times e^{i\pi(p-1)}$$

$$-e^{2\pi(p-1)i}\int_0^{\infty}\frac{x^{p-1}}{x+1}\,dx + \int_0^{\infty}\frac{x^{p-1}}{x+1}\,dx = 2\pi i \times e^{i\pi(p-1)}$$

$$\left[1-e^{2\pi(p-1)i}\right]\int_0^{\infty}\frac{x^{p-1}}{x+1}\,dx = 2\pi i \times e^{i\pi(p-1)}$$

FINAL TIDY UP

$$\int_0^{\infty}\frac{x^{p-1}}{x+1}\,dx = \frac{2\pi i \times e^{i\pi(p-1)}}{1-e^{2\pi(p-1)i}} = \frac{2\pi i \times \cancel{e^{i\pi(p-1)}}}{\cancel{e^{i\pi(p-1)}}\left[e^{-i\pi(p-1)}-e^{i\pi(p-1)}\right]}$$

$$= \frac{2\pi i}{-\left[e^{i\pi(p-1)}-e^{-i\pi(p-1)}\right]} = \frac{2\pi i}{-2\sinh\left[i\pi(p-1)\right]}$$

$$= \frac{2\pi i}{-2i\sin(\pi(p-1))} = \frac{-\pi}{\sin(\pi p-\pi)}$$

$$= \frac{-\pi}{\sin\pi p\cos\pi - \cancel{\cos\pi p\cos\pi}} = \frac{-\pi}{-\sin\pi p} = \frac{\pi}{\sin\pi p}$$

OR $\pi\operatorname{cosec}\pi p$

b) NOW WE HAVE

$$\Gamma(x)\Gamma(1-x) = \frac{\Gamma(x)\Gamma(1-x)}{1} = \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(x+(1-x))} = B(x,1-x)$$

$$= \int_0^1 u^{x-1}(1-u)^{(1-x)-1}\,du = \int_0^1 u^{x-1}(1-u)^{-x}\,du$$

AS REQUIRED

c) USING A SUBSTITUTION FOR THE ABOVE INTEGRAL

- $u=\frac{t}{t+1}$
  $ut+u=t$
  $u=t-ut$
  $u=t(1-u)$
  $t=\frac{u}{1-u}$
- $\frac{du}{dt}=\frac{(t+1)\times1-t\times1}{(t+1)^2}$
  $\frac{du}{dt}=\frac{t+1-t}{(t+1)^2}$
  $du=\frac{1}{(t+1)^2}\,dt$
- $u=0 \mapsto t=0$
- $u=1 \mapsto t=\infty$

RETURNING TO OUR INTEGRAL

$$\Gamma(x)\Gamma(1-x) = \dots = \int_0^1 u^{x-1}(1-u)^{-x}\,du = \int_0^{\infty}\left(\frac{t}{t+1}\right)^{x-1}\left(1-\frac{t}{t+1}\right)^{-x}\frac{1}{(t+1)^2}\,dt$$

$$= \int_0^{\infty}\left(\frac{t}{t+1}\right)^{x-1}\times\left(\frac{t+1-t}{t+1}\right)^{-x}\times\frac{1}{(t+1)^2}\,dt$$

$$= \int_0^{\infty}\left(\frac{t}{t+1}\right)^{x-1}\times\frac{1}{(t+1)^{-x}}\times\frac{1}{(t+1)^2}\,dt = \int_0^{\infty}\frac{t^{x-1}}{(t+1)^{x-1-x+2}}\,dt$$

$$= \int_0^{\infty}\frac{t^{x-1}}{t+1}\,dt$$

& FROM PART (a) IF $0<x<1$

$$= \frac{\pi}{\sin\pi x}$$

AS REQUIRED

Question 51

$$\mathrm{B}(m,n) \equiv \int_0^1 x^{m-1}(1-x)^{n-1}\ dx .$$

a) Use a suitable substitution to show that

$$\mathrm{B}(k,k) = \frac{1}{2^{2k-1}} \int_{-1}^{1} (1+u)^{k-1}(1-u)^{k-1}\ du .$$

b) Hence show further

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{\Gamma(2k)\sqrt{\pi}}{2^{2k-1}\Gamma(k)} .$$

$\boxed{\text{proof}}$

a) $B(m,n) \equiv \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$B(k,k) = \int_0^1 x^{k-1}(1-x)^{k-1}\,dx$

$B(k,k) = \int_{-1}^{1} \left(\frac{u+1}{2}\right)^{k-1}\left(1-\frac{u+1}{2}\right)^{k-1} \times \frac{1}{2}\,du$

$B(k,k) = \int_{-1}^{1} \left(\frac{u+1}{2}\right)^{k-1}\left(\frac{1-u}{2}\right)^{k-1} \times \frac{1}{2}\,du$

$B(k,k) = \frac{1}{2^{2k-1}}\int_{-1}^{1} (1+u)^{k-1}(1-u)^{k-1}\,du$

SUBSTITUTION

$u = Ax + B$ SO THAT

$0 \mapsto -1$, $1 \mapsto 1$; $-1 = B$, $1 = A + B$

$\therefore A = 2 \quad B = -1$

$\therefore \boxed{u = 2x - 1}$

$du = 2\,dx$

$dx = \frac{1}{2}\,du$

LIMITS BY "CONSTRUCTION"

b) $B(k,k) = \frac{1}{2^{2k-1}}\int_{-1}^{1}(1-u^2)^{k-1}\,du$

$= \frac{1}{2^{2k-1}} \times 2\int_0^1 (1-u^2)^{k-1}\,du$ (EVEN INTEGRAND)

$= \frac{1}{2^{2k-1}} \times 2\int_0^1 (1-y)^{k-1} \times \frac{1}{2}y^{-\frac{1}{2}}\,dy$

$= \frac{1}{2^{2k-1}}\int_0^1 y^{-\frac{1}{2}}(1-y)^{k-1}\,dy$

$= \frac{1}{2^{2k-1}} B\left(\frac{1}{2},k\right)$

ANOTHER SUBSTITUTION

$y = u^2$

$u = y^{\frac{1}{2}}$

$du = \frac{1}{2}y^{-\frac{1}{2}}\,dy$

LIMITS UNCHANGED

Thus $B(k,k) = \frac{1}{2^{2k-1}} B\left(k,\frac{1}{2}\right)$

$\frac{\cancel{\Gamma(k)}\Gamma(k)}{\Gamma(2k)} = \frac{1}{2^{2k-1}} \frac{\cancel{\Gamma(k)} \times \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}$

$\frac{\Gamma(k)}{\Gamma(2k)} = \frac{1}{2^{2k-1}} \frac{\sqrt{\pi}}{\Gamma\left(k+\frac{1}{2}\right)}$

$\Gamma(2k) = \frac{2^{2k-1}}{\sqrt{\pi}} \Gamma(k)\Gamma\left(k+\frac{1}{2}\right)$

AS REQUIRED

$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{1}{2}(2n-1)\frac{1}{2}(2n-3)\frac{1}{2}(2n-5)\ldots\left(\frac{1}{2}\times 7\right)\times\left(\frac{1}{2}\times 5\right)\times\left(\frac{1}{2}\times 3\right)\times\left(\frac{1}{2}\times 1\right)\times\sqrt{\pi}$

$= \left(\frac{1}{2}\right)^n (2n-1)(2n-3)(2n-5)\ldots 7\times 5\times 3\times 1 \times \sqrt{\pi}$

$= \frac{1}{2^n} \times \frac{(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\ldots 7\times 6\times 5\times 4\times 3\times 2\times 1}{(2n-2)(2n-4)(2n-6)\ldots 6\times 4\times 2} \times \sqrt{\pi}$

$= \frac{1}{2^n} \times \frac{(2n-1)!}{2(n-1)\,2(n-2)\,2(n-3)\ldots(2\times 3)\times(2\times 2)\times(2\times 1)} \times \sqrt{\pi}$

$= \frac{1}{2^n} \times \frac{\Gamma(2n)}{2^{n-1}(n-1)(n-2)(n-3)\ldots 3\times 2\times 1} \times \sqrt{\pi}$

$= \frac{1}{2^n} \times \frac{\Gamma(2n)}{2^{n-1}(n-1)!} \times \sqrt{\pi}$

$= \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)}$

AS REQUIRED

Question 52

$$I(x) \equiv \int_0^1 \left[\frac{1}{4} - \left(t - \frac{1}{2}\right)^2\right]^{x-1} dt .$$

a) Express $I(x)$ in terms of Gamma functions.

b) By considering the symmetry of the integrand about the line $t = \frac{1}{2}$ and using the substitution $y = 4\left(t - \frac{1}{2}\right)^2$, show that

$$\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) = 2^{1-2x}\sqrt{\pi}\ \Gamma(2k).$$

$$\boxed{I(x) = \frac{\Gamma(x)\Gamma(x)}{\Gamma(2x)}}$$

a) $\boxed{I(x) = \int_0^1 \left[\frac{1}{4} - (t-\frac{1}{2})^2\right]^{x-1} dt.}$

$I(x) = \int_0^1 \left[\left[\frac{1}{2} - (t-\frac{1}{2})\right]\left[\frac{1}{2} + (t-\frac{1}{2})\right]\right]^{x-1} dt$

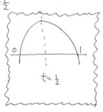
$= \int_0^1 \left[(1-t)t\right]^{x-1} dt = \int_0^1 (1-t)^{x-1} t^{x-1}\, dt$

$= B(x,x) = \frac{\Gamma(x)\Gamma(x)}{\Gamma(x+x)} = \frac{[\Gamma(x)]^2}{\Gamma(2x)}$

b) NOW THE INTEGRAND OF $I(x)$ IS EVEN ABOUT $t=\frac{1}{2}$

HENCE WE CAN WRITE

$I(x) = 2\int_0^{\frac{1}{2}} \left[\frac{1}{4} - (t-\frac{1}{2})^2\right]^{x-1} dt = 2\int_{\frac{1}{2}}^1 \left[\frac{1}{4} - (t-\frac{1}{2})^2\right]^{x-1} dt$



THUS

$I(x) = 2\int_{\frac{1}{2}}^1 \left[\frac{1}{4} - (t-\frac{1}{2})^2\right]^{x-1} dt$

$= 2\int_{\frac{1}{2}}^1 \left(\frac{1}{4}\right)^{x-1}\left(1 - \underbrace{4(t-\frac{1}{2})^2}_{y}\right)^{x-1} dt$

$= 2\int_0^1 \left(\frac{1}{4}\right)^{x-1}(1-y)^{x-1}\left(\frac{1}{4}y^{-\frac{1}{2}}\,dy\right)$

$= 2\int_0^1 \left(\frac{1}{4}\right)^{x}(1-y)^{x-1}y^{-\frac{1}{2}}\,dy$

$= 2\left(\frac{1}{4}\right)^{x}\int_0^1 (1-y)^{x-1}y^{-\frac{1}{2}}\,dy$

$= 2\left(\frac{1}{2}\right)^{2x}\int_0^1 (1-y)^{x-1}y^{\frac{1}{2}-1}\,dy$

BY SUBSTITUTION

$y = 4(t-\frac{1}{2})^2$

$\frac{y}{4} = (t-\frac{1}{2})^2$

$t-\frac{1}{2} = \frac{1}{2}y^{\frac{1}{2}}$

$dt = \frac{1}{4}y^{-\frac{1}{2}}\,dy$

$t=\frac{1}{2} \mapsto y=0$

$t=1 \mapsto y=1$

$= \left(\frac{1}{2}\right)^{2x-1} B(x,\frac{1}{2})$

$= 2^{1-2x} B(x,\frac{1}{2})$

COMBINING RESULTS

$$\left.\begin{aligned} I(x) &= \frac{[\Gamma(x)]^2}{\Gamma(2x)} \\ I(x) &= 2^{1-2x}B(x,\tfrac{1}{2}) \end{aligned}\right\} \Rightarrow \frac{\Gamma(x)\Gamma(x)}{\Gamma(2x)} = 2^{1-2x}B(x,\tfrac{1}{2})$$

$\Rightarrow \frac{\Gamma(x)\cancel{\Gamma(x)}}{\Gamma(2x)} = 2^{1-2x}\frac{\cancel{\Gamma(x)}\Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})}$

$\Rightarrow \frac{\Gamma(x)}{\Gamma(2x)} = \frac{2^{1-2x}\sqrt{\pi}}{\Gamma(x+\frac{1}{2})}$

$\Rightarrow \Gamma(x)\Gamma(x+\frac{1}{2}) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x)$

Question 53

$$I(\alpha) = \int_0^{\infty} \arctan\left(x^{\alpha}\right)\, dx, \quad \alpha < -1.$$

By using integration by parts, and justifying all steps, show that

$$I(\alpha) = \frac{1}{2}\pi \sec\left(\frac{\pi}{2\alpha}\right) = \frac{1}{2}\mathrm{B}\left(\frac{1}{2}-\frac{1}{2\alpha}, \frac{1}{2}+\frac{1}{2\alpha}\right).$$

You may assume that $\displaystyle\int_0^{\infty} \frac{x^{p-1}}{(x+1)}\, dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.

proof

$$\boxed{\int_0^{\infty} \arctan\left(x^{\alpha}\right) dx \qquad \alpha < -1}$$

STARTING BY INTEGRATION BY PARTS

$$\int_0^{\infty} 1 \times \arctan\left(x^{\alpha}\right) dx = \dots$$

$\arctan(x^{\alpha})$	$\frac{\alpha x^{\alpha-1}}{1+x^{2\alpha}}$
x	1

$$\dots = \Big[\cancel{x\arctan(x^{\alpha})}\Big]_0^{\infty} - \int_0^{\infty} \frac{\alpha x^{\alpha}}{1+x^{2\alpha}}\, dx$$

$$\frac{d}{dy}(\arctan y) = \frac{1}{1+y^2} = 1 - y^2 + y^4 - y^5 + \dots$$

$$\arctan y = y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \frac{1}{7}y^7 + \dots + \cancel{C}$$

$$x\arctan(x^{\alpha}) = x\left(x^{\alpha} - \frac{1}{3}x^{3\alpha} + \frac{1}{5}x^{5\alpha} - \dots\right)$$

$$= x^{\alpha+1} - \frac{1}{3}x^{3\alpha+1} + \frac{1}{5}x^{5\alpha+1} - \dots \qquad \alpha < -1$$

NOW THE "$\arctan(\ )$" FUNCTION IS BOUNDED $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

SO AT $x = 0$, $x\arctan(x^{\alpha}) \to 0$

AND FROM THE POWER SERIES ABOVE IF $\alpha < -1$

$x\arctan(x^{\alpha}) \to 0$ AS $x \to +\infty$

$$\dots = -\alpha\int_0^{\infty} \frac{x^{\alpha}}{1+x^{2\alpha}}\, dx$$

WHICH RESEMBLES A STANDARD CONTOUR INTEGRAL $\displaystyle\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$

SO WE REQUIRE SOME SUBSTITUTIONS | MANIPULATIONS

LET $y = x^{2\alpha}$

$\Rightarrow x = y^{\frac{1}{2\alpha}}$

$\Rightarrow \frac{dx}{dy} = \frac{1}{2\alpha}y^{\frac{1}{2\alpha}-1}$

$\Rightarrow dx = \frac{1}{2\alpha}y^{\frac{1}{2\alpha}-1}\, dy$

NOTE THAT LIMITS SWAP AS $\alpha < -1$

$$\dots = -\alpha\int_{\infty}^{0} \frac{\left(y^{\frac{1}{2\alpha}}\right)^{\alpha}}{1+y}\left[\frac{1}{2\alpha}y^{\frac{1}{2\alpha}-1}\, dy\right]$$

$$\dots = \frac{1}{2}\int_0^{\infty} \frac{y^{\frac{1}{2}}y^{\frac{1}{2\alpha}-1}}{1+y}\, dy$$

$$\dots = \frac{1}{2}\int_0^{\infty} \frac{y^{\frac{1}{2\alpha}-\frac{1}{2}}}{1+y}\, dy$$

MANIPULATE $\frac{1}{2\alpha}-\frac{1}{2}$ IN THE FORM $p-1$

$$\dots = \frac{1}{2}\int_0^{\infty} \frac{y^{\left(\frac{1}{2\alpha}+\frac{1}{2}\right)-1}}{1+y}\, dy$$

IE $p = \frac{1}{2\alpha} + \frac{1}{2}$

[NOTE IF $\alpha < -1$, $0 < p < \frac{1}{2}$]

USING STANDARD RESULT NOW

$$\dots = \frac{1}{2}\,\frac{\pi}{\sin\left[\pi\left(\frac{1}{2\alpha}+\frac{1}{2}\right)\right]} = \frac{\pi}{2\sin\left[\frac{\pi}{2\alpha}+\frac{\pi}{2}\right]}$$

$$= \frac{\pi}{2} \times \frac{1}{\sin\frac{\pi}{2\alpha}\cos\frac{\pi}{2} + \cos\frac{\pi}{2\alpha}\sin\frac{\pi}{2}} = \frac{\pi}{2\cos\frac{\pi}{2\alpha}}$$

// AS REQUIRED

NOTE THIS CAN ALSO BE WRITTEN AS

$$\dots = \frac{1}{2}\,\frac{\pi}{\sin\left[\pi\left(\frac{1}{2\alpha}+\frac{1}{2}\right)\right]} = \frac{1}{2}\Gamma\left[\frac{1}{2\alpha}+\frac{1}{2}\right]\Gamma\left(1-\frac{1}{2\alpha}-\frac{1}{2}\right)$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$B\left(\frac{1}{2}-\frac{1}{2\alpha}, \frac{1}{2}+\frac{1}{2\alpha}\right) = \frac{\Gamma\left(\frac{1}{2}-\frac{1}{2\alpha}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2\alpha}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2\alpha}+\frac{1}{2}+\frac{1}{2\alpha}\right)} \leftarrow 1$$

$$\dots = \frac{1}{2}B\left(\frac{1}{2}-\frac{1}{2\alpha}, \frac{1}{2}+\frac{1}{2\alpha}\right)$$

Question 54

It is given that for any real constants x and y

$$\mathrm{B}(x+1, y+1) = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}.$$

a) Use the integral definition of $\mathrm{B}(x+1, y+1)$, with a suitable substitution to derive Gauss' definition of the gamma function

$$\Gamma(x) = \lim_{n\to\infty}\left[\frac{n^x\, n!}{x(x+1)(x+2)\ldots(x+n-1)(x+n)}\right].$$

b) Hence show

$$\Gamma'(x) = \Gamma(x)\left[-\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\frac{x}{k(x+k)}\right].$$

proof

a) $\boxed{\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}}$

- $\mathrm{B}(x+1,y+1) = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}$

$$\int_0^1 t^x(1-t)^y\,dt = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}$$

- BY SUBSTITUTION, NOTING y IS NOT A VARIABLE IN THE INTEGRAL

$$t = \frac{v}{y} \Rightarrow dt = \frac{1}{y}\,dv$$
$$t=1 \mapsto v=y$$
$$t=0 \mapsto v=0$$

$$\Rightarrow \int_0^y \left(\frac{v}{y}\right)^x\left(1-\frac{v}{y}\right)^y \frac{1}{y}\,dv = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}$$

$$\Rightarrow \frac{1}{y^{x+1}}\int_0^y v^x\left(1-\frac{v}{y}\right)^y dv = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}$$

- NOW

$$\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y = e^1 \qquad \lim_{y\to\infty}\left(1-\frac{1}{y}\right)^y = e^{-1}$$
$$\lim_{y\to\infty}\left(1+\frac{a}{y}\right)^y = e^a \qquad \lim_{y\to\infty}\left(1-\frac{a}{y}\right)^y = e^{-a}$$

- SO TAKING THE LIMIT AS $y \to \infty$ ON BOTH SIDES

$$\Rightarrow \lim_{y\to\infty}\left[\frac{1}{y^{x+1}}\int_0^y v^x\left(1-\frac{v}{y}\right)^y dv\right] = \lim_{y\to\infty}\left[\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}\right]$$

$$\Rightarrow \lim_{y\to\infty}\left[\frac{1}{y^{x+1}}\right] \times \int_0^\infty v^x e^{-v}\,dv = \lim_{y\to\infty}\left[\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}\right]$$

$$\Rightarrow \lim_{y\to\infty}\left[\frac{1}{y^{x+1}}\right]\cancel{\Gamma(x+1)} = \cancel{\Gamma(x+1)}\lim_{y\to\infty}\left[\frac{\Gamma(y+1)}{\Gamma(x+y+2)}\right]$$

$$\Rightarrow \frac{\lim_{y\to\infty}\left[\frac{\Gamma(y+1)}{\Gamma(x+y+2)}\right]}{\lim_{y\to\infty}\left[\frac{1}{y^{x+1}}\right]} = 1$$

$$\Rightarrow \lim_{y\to\infty}\left[\frac{\Gamma(y+1)\,y^{x+1}}{\Gamma(x+y+2)}\right] = 1$$

- NOW USING $\Gamma(n) = (n-1)\Gamma(n-1)$

$$\Rightarrow \lim_{y\to\infty}\left[\frac{y\cdot y^x\cdot\Gamma(y+1)}{(x+y+1)\Gamma(x+y+1)}\right] = 1$$

$$\Rightarrow \cancel{\lim_{y\to\infty}\left[\frac{y}{x+y+1}\right]} \times \lim_{y\to\infty}\left[\frac{y^x\Gamma(y+1)}{\Gamma(x+y+1)}\right] = 1$$

$$\Rightarrow \lim_{y\to\infty}\left[\frac{y^x\Gamma(y+1)}{\Gamma(x+y+1)}\right] = 1$$

$$\Rightarrow \lim_{y\to\infty}\left[\frac{y^x\Gamma(y+1)}{(x+y)(x+y-1)(x+y-2)\ldots(x+1)x\,\Gamma(x)}\right] = 1$$

LET $y = n$ = INTEGER

$$\Rightarrow \lim_{n\to\infty}\left[\frac{n^x\Gamma(n+1)}{(x+n)(x+n-1)(x+n-2)\ldots(x+1)x\,\Gamma(x)}\right] = 1$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{n^x\, n!}{(x+n)(x+n-1)(x+n-2)\ldots(x+1)x}\right] = \Gamma(x)$$

OR

$$\Gamma(x) = \lim_{n\to\infty}\left[\frac{n!\, n^x}{x(x+1)(x+2)\ldots(x+n-1)(x+n)}\right]$$

AS REQUIRED.

b)
- USING GAUSS' DEFINITION OF $\Gamma(x)$

$$\Gamma(x) = \lim_{n\to\infty}\left[\frac{n!\, n^x}{x(x+1)(x+2)\ldots(x+n-1)(x+n)}\right]$$

- TAKING LOGARITHMS

$$\ln[\Gamma(x)] = \lim_{n\to\infty}\left[\ln(n!) + \ln(n^x) - \ln x - \ln(x+1) - \ln(x+2) - \ldots - \ln(x+n-1) - \ln(x+n)\right]$$

- DIFF WITH RESPECT TO x

$$\Rightarrow \frac{\Gamma'(x)}{\Gamma(x)} = \lim_{n\to\infty}\left[0 + \ln(n) - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} - \ldots - \frac{1}{x+n-1} - \frac{1}{x+n}\right]$$

$$\Rightarrow \frac{\Gamma'(x)}{\Gamma(x)} = \lim_{n\to\infty}\left[\ln(n) - \sum_{k=0}^{n}\frac{1}{x+k}\right]$$

- BUT $\boxed{\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right]}$

$$\Rightarrow \frac{\Gamma'(x)}{\Gamma(x)} = \lim_{n\to\infty}\left[\ln(n) - \frac{1}{x} - \sum_{k=1}^{n}\frac{1}{x+k}\right]$$

$$\Rightarrow \frac{\Gamma'(x)}{\Gamma(x)} = \lim_{n\to\infty}\left[\ln(n) - \frac{1}{x} - \sum_{k=1}^{n}\frac{1}{x+k} + \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k}\right]$$

$$\Rightarrow \frac{\Gamma'(x)}{\Gamma(x)} = \lim_{n\to\infty}\left[\underbrace{\ln(n) - \sum_{k=1}^{n}\frac{1}{k}}_{-\gamma} - \frac{1}{x} + \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{x+k}\right]$$

$$\Rightarrow \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{x+k}\right)\right]$$

$$\Rightarrow \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{x+k-k}{k(x+k)}\right]$$

$$\Rightarrow \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\frac{x}{k(x+k)}$$

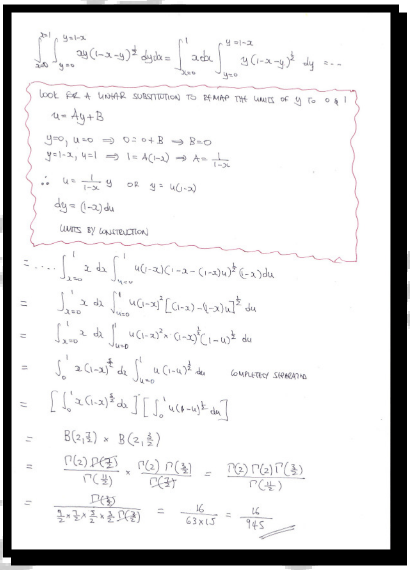
$$\Rightarrow \Gamma'(x) = \Gamma(x)\left[-\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\frac{x}{k(x+k)}\right]$$

Question 55

$$I = \int_0^1 \int_0^{1-x} xy\left(1-x-y\right)^{\frac{1}{2}}\ dydx\,.$$

Determine the exact value of I by transforming it into an expression involving Beta functions.

$$\boxed{\frac{16}{945}}$$



$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x} xy(1-x-y)^{\frac{1}{2}}\,dy\,dx = \int_{x=0}^{1} x\,dx \int_{y=0}^{y=1-x} y(1-x-y)^{\frac{1}{2}}\,dy = \dots$$

LOOK FOR A LINEAR SUBSTITUTION TO REMAP THE LIMITS OF y TO 0 & 1

$u = Ay + B$

$y=0,\ u=0 \implies 0 = 0 + B \implies B = 0$

$y = 1-x,\ u=1 \implies 1 = A(1-x) \implies A = \frac{1}{1-x}$

$\therefore\ u = \frac{1}{1-x}y$ OR $y = u(1-x)$

$dy = (1-x)\,du$

LIMITS BY CONSTRUCTION

$$= \dots \int_{x=0}^{1} x\,dx \int_{u=0}^{1} u(1-x)(1-x-(1-x)u)^{\frac{1}{2}}(1-x)\,du$$

$$= \int_{x=0}^{1} x\,dx \int_{u=0}^{1} u(1-x)^2\left[(1-x)-(1-x)u\right]^{\frac{1}{2}}\,du$$

$$= \int_{x=0}^{1} x\,dx \int_{u=0}^{1} u(1-x)^2 \times (1-x)^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\,du$$

$$= \int_0^1 x(1-x)^{\frac{5}{2}}\,dx \int_{u=0}^{1} u(1-u)^{\frac{1}{2}}\,du \qquad \text{COMPLETELY SEPARATED}$$

$$= \left[\int_0^1 x(1-x)^{\frac{5}{2}}\,dx\right]\left[\int_0^1 u(1-u)^{\frac{1}{2}}\,du\right]$$

$$= B(2,\tfrac{7}{2}) \times B(2,\tfrac{3}{2})$$

$$= \frac{\Gamma(2)\cancel{\Gamma(\frac{7}{2})}}{\Gamma(\frac{11}{2})} \times \frac{\Gamma(2)\Gamma(\frac{3}{2})}{\cancel{\Gamma(\frac{7}{2})}} = \frac{\Gamma(2)\Gamma(2)\Gamma(\frac{3}{2})}{\Gamma(\frac{11}{2})}$$

$$= \frac{\cancel{\Gamma(\frac{3}{2})}}{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\cancel{\Gamma(\frac{3}{2})}} = \frac{16}{63\times 15} = \frac{16}{945}$$

Question 56

A finite region R defined by the inequalities

$$x^3 + y^3 + z^3 \le 1, \quad x \ge 0, \quad y \ge 0, \quad z \ge 0.$$

Show that the volume of R is

$$\left[\frac{1}{k}\Gamma\left(\frac{1}{k}\right)\right]^k,$$

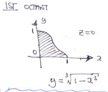
where k is a positive integer to be found.

$\boxed{k=3}$

FIRSTLY REWRITE THE SURFACE $z^3 = 1 - x^3 - y^3$

$z = \sqrt[3]{1-x^3-y^3}$

& WE REQUIRE THE VOLUME UNDER THE SURFACE IN THE <u>1ST</u> OCTANT



$$V = \int_{x=0}^{1} \int_{y=0}^{y=\sqrt[3]{1-x^3}} \sqrt{1-x^3-y^3}\, dy\, dx$$

$$V = \int_{x=0}^{1} \int_{y=0}^{y=(1-x^3)^{\frac{1}{2}}} (1-x^3-y^3)^{\frac{1}{3}}\, dy\, dx$$

SUBSTITUTE IN y — SEE OPPOSITE

$$V = \int_{x=0}^{1} \int_{u=0}^{1} \left[1-x^3-u^3(1-x^3)\right]^{\frac{1}{3}} (1-x^3)^{\frac{1}{3}}\, du\, dx$$

$$V = \int_{x=0}^{1} \int_{u=0}^{1} (1-x^3)^{\frac{1}{3}} \left[(1-x^3)(1-u^3)\right]^{\frac{1}{3}}\, du\, dx$$

SEPARATE THE INTEGRAL AS VARIABLES HAVE SEPARATED

$$V = \left[\int_{x=0}^{1} (1-x^3)^{\frac{2}{3}}\, dx\right]\left[\int_{0}^{1} (1-u^3)^{\frac{1}{3}}\, du\right]$$

$$V = \left[\int_{0}^{1} (1-x^3)^{\frac{2}{3}}\, dx\right]\left[\int_{0}^{1} (1-x^3)^{\frac{1}{3}}\, dx\right] \quad \text{FOR SIMPLICITY!}$$

RE-MAP LIMITS IN y FROM $y=0$ TO $y=1$

$$u = \frac{1}{(1-x^3)^{\frac{1}{3}}}\, y$$

(BY INSPECTION OR SEEK A SUBSTITUTION OF THE FORM $u = Ay + B$ & TRY TO WORK A & B)

$$y = u(1-x^3)^{\frac{1}{3}}$$

$$dy = du(1-x^3)^{\frac{1}{3}}$$

(x IS CONSTANT IN THE y INTEGRATION)

ANOTHER FINAL SUBSTITUTION

$$V = \int_{0}^{1} (1-t)^{\frac{2}{3}} \times \tfrac{1}{3} t^{-\frac{2}{3}}\, dt \int_{0}^{1} (1-t)^{\frac{1}{3}} \times \tfrac{1}{3} t^{-\frac{2}{3}}\, dt$$

$t = x^3$

$x = t^{\frac{1}{3}}$

$dx = \frac{1}{3} t^{-\frac{2}{3}}\, dt$

LIMITS UNCHANGED

$$V = \tfrac{1}{3} B\left(\tfrac{5}{3}, \tfrac{1}{3}\right) \times \tfrac{1}{3} B\left(\tfrac{4}{3}, \tfrac{1}{3}\right)$$

$$V = \frac{1}{9}\, \frac{\cancel{\Gamma\left(\frac{5}{3}\right)}\Gamma\left(\frac{1}{3}\right)}{\cancel{\Gamma(2)}} \times \frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\cancel{\Gamma\left(\frac{5}{3}\right)}}$$

$$V = \frac{1}{9} \Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{4}{3}\right)$$

$$V = \frac{1}{9} \Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{1}{3}\right) \times \frac{1}{3}\Gamma\left(\tfrac{1}{3}\right)$$

$$V = \frac{1}{27}\left[\Gamma\left(\tfrac{1}{3}\right)\right]^3 = \left[\tfrac{1}{3}\Gamma\left(\tfrac{1}{3}\right)\right]^3 \quad \text{I.E } k=3$$