

# **SERIES EXPANSIONS 59 QUESTIONS**

# **MACLAURIN EXPANSIONS**

## **6 BASIC QUESTIONS**

**Question 1** (\*\*)

$$f(x) = (1-x)^2 \ln(1-x), -1 \leq x < 1.$$

Find the Maclaurin expansion of  $f(x)$  up and including the term in  $x^3$ .

$$\boxed{\quad}, \quad f(x) = -x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + O(x^4)$$

USING STANDARD RESULTS, OTHER THAN DIFFERENTIATION

$$\begin{aligned} \Rightarrow f(x) &= (1-x)^2 \ln(1-x) \\ \Rightarrow f(x) &= (1-2x+2x^2) \left[ -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + O(x^4) \right] \\ \ln(1+x) &\equiv x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5) \\ \ln(1-x) &\equiv -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + O(x^4) \\ \Rightarrow f(x) &= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + O(x^4) \\ &\quad + 2x^2 + O(x^4) \\ &\quad - x^3 + O(x^4) \\ \Rightarrow f(x) &= -x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + O(x^4) \end{aligned}$$

**Question 2** (\*\*+)

$$f(x) = e^{-2x} \cos 4x.$$

Find the Maclaurin expansion of  $f(x)$  up and including the term in  $x^4$ .

$$\boxed{\quad}, \quad e^{-2x} \cos 4x = 1 - 2x - 6x^2 + \frac{44}{3}x^3 - \frac{14}{3}x^4 + O(x^5)$$

USING STANDARD EXPANSIONS

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^5)$
- $e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + O(x^5)$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^5)$
- $\cos 4x = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} + O(x^5)$
- $\cos 4x = 1 - 8x^2 + \frac{32}{3}x^4 + O(x^5)$

COMBINING THESE RESULTS

$$\begin{aligned} f(x) &= e^{-2x} \cos 4x = (\cos 4x)(e^{-2x}) \\ f(x) &= \left[ 1 - 8x^2 + \frac{32}{3}x^4 + O(x^5) \right] \left[ 1 - 2x - 2x^2 - \frac{4}{3}x^3 + \frac{3}{2}x^4 + O(x^5) \right] \\ f(x) &= 1 - 2x - 2x^2 - \frac{4}{3}x^3 + \frac{3}{2}x^4 + O(x^5) \\ &\quad - 8x^2 + (16x^3 - 16x^4) + O(x^5) \\ &\quad + \frac{32}{3}x^4 + O(x^5) \\ f(x) &= 1 - 2x - 6x^2 + \frac{44}{3}x^3 - \frac{14}{3}x^4 + O(x^5) \end{aligned}$$

**Question 3** (\*\*\*)

$$y = e^{2x} \sin 3x.$$

- a) Use standard results to find the series expansion of  $y$ , up and including the term in  $x^4$ .
- b) Hence find an approximate value for

$$\int_0^{0.1} e^{2x} \sin 3x \, dx.$$

<input type="text"/>	$e^{2x} \sin 3x = 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4 + O(x^5)$	$\approx 0.0170275$
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a) USING STANDARD EXPANSIONS

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + O(x^4)$
- $e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + O(x^4)$
- $e^{2x} = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + O(x^4)$
- $\sin 3x = 3x - \frac{3x^3}{3!} + O(x^4)$
- $\sin 3x = (3x) - \frac{(3x)^3}{3!} + O(x^4)$
- $\sin 3x = 3x - \frac{3}{2}x^3 + O(x^4)$

COMBINING RESULTS

$$\Rightarrow y = e^{2x} \sin 3x = \left[ 1 + 2x + 2x^2 + \frac{8}{3}x^3 + O(x^4) \right] \left[ 3x - \frac{3}{2}x^3 + O(x^4) \right]$$

$$\Rightarrow y = \begin{aligned} & 3x & - \frac{3}{2}x^3 & + O(x^4) \\ & 6x^2 & - 12x^4 & + O(x^4) \\ & 6x^3 & + O(x^4) \\ & 4x^4 & + O(x^4) \end{aligned}$$

$$\Rightarrow y = 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4 + O(x^5)$$

b) USING PART (a)

$$\int_0^{0.1} e^{2x} \sin 3x \, dx \approx \int_0^{0.1} 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4 \, dx$$

$$\approx \left[ \frac{3}{2}x^2 + 2x^3 + \frac{3}{8}x^4 - x^5 \right]_0^{0.1}$$

$$\approx \left( \frac{3}{200} + \frac{1}{500} + \frac{3}{8000} - \frac{1}{100000} \right) - (0)^5$$

$$\approx 0.0170275 \quad //$$

**Question 4** (\*\*\*)

Find the Maclaurin's expansion of  $\ln\left[\sqrt[3]{\frac{1+2x}{1-2x}}\right]$ , up and including the term in  $x^3$ .

$$\boxed{\text{Answer: } \ln\left[\sqrt[3]{\frac{1+2x}{1-2x}}\right] = \frac{4}{3}x + \frac{16}{9}x^3 + O(x^5)}$$

USING STANDARD EXPANSIONS FOR  $\ln(1+x)$

- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$
- $\ln(1-x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$
- $\ln(1+2x) = (2x) - \frac{4}{3}(2x)^2 + \frac{1}{3}(2x)^3 - \frac{1}{4}(2x)^4 + O(x^5)$

Then we have

$$\begin{aligned}\ln\left[\sqrt[3]{\frac{1+2x}{1-2x}}\right] &= \ln\left[\left(\frac{1+2x}{1-2x}\right)^{\frac{1}{3}}\right] = \frac{1}{3}\ln\left(\frac{1+2x}{1-2x}\right) \\ &= \frac{1}{3}\left[\ln(1+2x) - \ln(1-2x)\right] \\ &= \frac{1}{3}\left[2x - 2x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + O(x^5)\right] \\ &= \frac{1}{3}\left[\frac{2x}{2x} - \frac{-2x^2}{2x} - \frac{\frac{2}{3}x^3}{2x} - \frac{-\frac{1}{3}x^4}{2x} + O(x^5)\right] \\ &= \frac{1}{3}\left[1 - x - \frac{x^2}{3} + \frac{2}{9}x^3 + O(x^4)\right] \\ &= \frac{1}{3}\left(1 + \frac{4}{3}x + \frac{16}{9}x^3 + O(x^5)\right) \\ &= \underline{\underline{\frac{4}{3}x + \frac{16}{9}x^3 + O(x^5)}}\end{aligned}$$

**Question 5** (\*\*\*)

$$f(x) = \ln(1 + \sin x), \sin x \neq -1.$$

a) Find the Maclaurin expansion of  $f(x)$  up and including the term in  $x^3$ .

b) Hence show that

$$\int_0^{\frac{1}{4}} \ln(1 + \sin x) dx \approx 0.028809.$$

$$\boxed{\text{[ ]}}, \quad \boxed{\ln(1 + \sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)}$$

a) By direct differentiation

$$\begin{aligned} f(0) &= \ln(1 + \sin 0) & f'(0) &= \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} \\ f'(0) &= \frac{\cos x}{1 + \sin x} & f'(0) &= 1 \\ f''(0) &= \frac{(1 + \sin x)(\cos x) - \cos x \cdot \cos x}{(1 + \sin x)^2} & f''(0) &= -\frac{1 + \sin x}{(1 + \sin x)^2} \\ &= \frac{-\sin x - \sin^2 x - \cos x}{(1 + \sin x)^2} & &= -\frac{1}{(1 + \sin x)^2} \\ &\approx -\frac{1}{1 + \sin x} & f''(0) &\approx -1 \\ f'''(0) &= \frac{(1 + \sin x)^2 \cos x}{(1 + \sin x)^3} & f'''(0) &= 1. \end{aligned}$$

By Maclaurin's theorem

$$\begin{aligned} f(x) &= f(0) + x(f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)) \\ \ln(1 + \sin x) &= 0 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4) \\ \ln(1 + \sin x) &= x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4) \end{aligned}$$

Approximate using sigma notation

$$\begin{aligned} \ln(1 + x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4) \\ \sin x &= x - \frac{1}{6}x^3 + O(x^5) \\ \therefore \ln(1 + \sin x) &= \ln(1 + \sin x) = x - \frac{1}{2}x^2 - \frac{1}{2}(x - \frac{1}{6}x^3)^2 + \frac{1}{3}(x - \frac{1}{6}x^3)^3 + O(x^4) \\ &= x - \frac{1}{2}x^2 - \frac{1}{2}(x^2 - \frac{1}{3}x^4) + \frac{1}{3}(x^3 - \frac{1}{6}x^5) + O(x^4) \\ &= x - \frac{1}{2}x^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4) \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4) \end{aligned}$$

b) As  $\alpha$  is small in the curve

$$\begin{aligned} \int_0^{\frac{1}{4}} \ln(1 + \sin x) dx &\approx \int_0^{\frac{1}{4}} x - \frac{1}{2}x^2 + \frac{1}{3}x^3 dx \\ &\approx \left[ \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 \right]_0^{\frac{1}{4}} \\ &\approx \left( \frac{1}{32} - \frac{1}{384} + \frac{1}{6144} \right) - (0) \\ &\approx \frac{21}{384} \\ &\approx 0.028809 \end{aligned}$$

As required

**Question 6** (\*\*\*)

$$f(x) \equiv \frac{e^x + 1}{2e^{\frac{1}{2}x}}, \quad x \in \mathbb{R}.$$

Use standard results to determine the Maclaurin series expansion of  $f(x)$ , up and including the term in  $x^6$ .

□, 
$$f(x) = 1 + \frac{1}{8}x^2 + \frac{1}{384}x^4 + \frac{1}{7680}x^6 + O(x^8)$$

Start by "splitting the fraction"

$$\begin{aligned} f(x) &= \frac{e^x + 1}{2e^{\frac{1}{2}x}} = \frac{e^x}{2e^{\frac{1}{2}x}} + \frac{1}{2e^{\frac{1}{2}x}} = \frac{1}{2}e^{\frac{1}{2}x} + \frac{1}{2}e^{-\frac{1}{2}x} \\ &= \cosh(\frac{x}{2}) \end{aligned}$$

Now  $\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + O(u^8)$

$$\begin{aligned} f(x) &= 1 + \frac{(\frac{x}{2})^2}{2!} + \frac{(\frac{x}{2})^4}{4!} + \frac{(\frac{x}{2})^6}{6!} + O(x^8) \\ f(x) &\sim 1 + \frac{1}{2}x^2 + \frac{1}{384}x^4 + \frac{1}{46080}x^6 + O(x^8) \end{aligned}$$

Alternative (using exponentials)

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + O(x^4)$
- $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + O(x^4)$

$$\begin{aligned} f(x) &= \frac{1}{2}e^{\frac{1}{2}x} + \frac{1}{2}e^{-\frac{1}{2}x} = \frac{1}{2}(e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}) \\ &= \frac{1}{2} \left[ 1 + \frac{1}{2}x + \frac{1}{2}\frac{x^2}{2!} + \frac{1}{2}\frac{x^3}{3!} + \frac{1}{2}\frac{x^4}{4!} + \frac{1}{2}\frac{x^5}{5!} + \frac{1}{2}\frac{x^6}{6!} + O(x^7) \right] \\ &= \frac{1}{2} \left[ 1 + \frac{1}{2}x + \frac{1}{2}\frac{x^2}{2!} + \frac{1}{2}\frac{x^3}{3!} + \frac{1}{2}\frac{x^4}{4!} - \frac{1}{2}\frac{x^5}{5!} + \frac{1}{2}\frac{x^6}{6!} + O(x^7) \right] \\ &= 1 + \frac{1}{2}x^2 + \frac{1}{384}x^4 + \frac{1}{46080}x^6 + O(x^8) \end{aligned}$$

As above

# MACLAURIN EXPANSIONS

## 20 STANDARD QUESTIONS

**Question 1** (\*\*\*)+

$$y = (1+x)^2 \cos x.$$

Show clearly that ...

a) ...  $\frac{d^3y}{dx^3} = (x^2 + 2x - 5) \sin x - 6(x+1) \cos x.$

b) ...  $y \approx 1 + Ax + Bx^2 + Cx^3$ , where  $A$ ,  $B$  and  $C$  are constants to be found.

, proof

<p>a) <u>Differentiate 3 times by the product rule</u></p> <ul style="list-style-type: none"> <li><math>y = (1+x)^2 \cos x</math></li> <li><math>\frac{dy}{dx} = 2(1+x) \cos x - (1+x)^2 \sin x</math></li> <li><math>\frac{d^2y}{dx^2} = 2(2x+2) \cos x - 2(1+x) \sin x - (1+x)^2 \cos x</math>  <math>= [2 - (1+x)^2] \cos x - 4(1+x) \sin x</math>  <math>\approx (2 - 1 - 2x - x^2) \cos x - 4(1+x) \sin x</math>  <math>= (1 - 2x - x^2) \cos x - 4(1+x) \sin x</math></li> <li><math>\frac{d^3y}{dx^3} = (-2-2x) \cos x - (1-2-x^2) \sin x - 4 \sin x - 4(1+x) \cos x</math>  <math>\frac{d^4y}{dx^4} = (-2-2x-4x) \cos x + (-1+2x^2-4) \sin x</math>  <math>\frac{d^5y}{dx^5} = (-6-6) \cos x + (x^2+2x-5) \sin x</math>  <math>\frac{d^6y}{dx^6} = (x^2+2x-5) \sin x - 6(1+x) \cos x</math> // As required         </li> </ul>	<p>By the MacLaurin theorem</p> $y = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + \dots$ $(1+x)^2 \cos x = 1 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$ $(1+x)^3 \cos x = 1 + 2x + \frac{x^2}{2} - x^3 + \dots$
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**Question 2** (\*\*\*)+

Find the Maclaurin expansion of  $\ln(2 - e^x)$ , up and including the term in  $x^3$ .

$$\boxed{\quad}, \quad \ln(2 - e^x) = -x - x^2 - x^3 + O(x^4)$$

BY DIRECT DIFFERENTIATION

$$\begin{aligned} f(x) &= \ln(2 - e^x) \\ f'(x) &= \frac{1}{2 - e^x} \times (-e^x) = \frac{-e^x}{2 - e^x} = \frac{e^x}{e^x - 2} = \frac{e^x(2 - 2)}{e^x(2 - 2)} \\ &= \frac{2e^x - 2}{e^x - 2} = 1 + 2(\frac{e^x - 1}{e^x - 2})^{-1} \\ f''(x) &= -2(e^x - 2)^{-2}e^x = -\frac{2e^x}{(e^x - 2)^2} \\ f'''(x) &= -\frac{(e^x)^2(e^x - 2)^2 - 2e^x(2)(e^x - 2)e^x}{(e^x - 2)^4} = -\frac{2e^x(e^x - 2) - 4e^x(e^x - 2)}{(e^x - 2)^3} \\ &= \frac{4e^x(e^x - 4e^x + 4e^x)}{(e^x - 2)^3} = \frac{2e^x + 4e^x}{(e^x - 2)^3} \end{aligned}$$

NOW EVALUATING AT  $x=0$

$$f(0) = 0, \quad f'(0) = -1, \quad f''(0) = -2, \quad f'''(0) = -6$$

BY THE MACLAURIN THEOREM

$$\begin{aligned} f(x) &= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + O(x^4) \\ \ln(2 - e^x) &= 0 - x - x^2 - x^3 + O(x^4) \end{aligned}$$

**Question 3** (\*\*\*)+

$$f(x) = \ln(1 + \cos 2x), \quad 0 \leq x < \frac{\pi}{2}.$$

a) Find an expression for  $f'(x)$ .

b) Show clearly that

$$f''(x) = -2 - \frac{1}{2}(f'(x))^2.$$

c) Show further that the series expansion of the first three non zero terms of  $f(x)$  is given by

$$\ln 2 - x^2 - \frac{1}{6}x^4.$$

$$\boxed{\phantom{000}}, \quad f'(x) = \frac{2 \sin 2x}{1 + \cos 2x}$$

a)  $f(x) = \ln(1 + \cos 2x)$

$$f'(x) = \frac{1}{1 + \cos 2x} \times (-2\sin 2x)$$

$$f'(x) = -\frac{2\sin 2x}{1 + \cos 2x}$$

b) MANIPULATE ABOVE FIRST

$$f'(x) = -\frac{2(\cos x + \cos x)}{1 + (\cos x + \cos x - 1)} = \frac{-4\sin x \cos x}{2\cos x} = -2\sin x$$

NOW WE HAVE

$$\Rightarrow f''(x) = -2\cos x = -2(1 + \cos^2 x)$$

$$\Rightarrow f''(x) = -2 - 2\cos^2 x$$

$$\Rightarrow 2f''(x) = -4 - 4\cos^2 x$$

$$\Rightarrow 2f''(x) = -4 - (-2\sin x)^2$$

$$\Rightarrow 2f''(x) = -4 - (f'(x))^2$$

$$\Rightarrow f''(x) = -2 - \frac{1}{2}(f'(x))^2$$

c) USING PART (b)

$$f'''(x) = 0 - (f'(x)) \times f''(x) = -f'(x)f''(x)$$

$$f'''(x) = -f'(x)f''(x) - f''(x)f''(x)$$

EVALUATE AT  $x=0$

$$f(x) = \ln(1 + \cos 0) = \ln 2$$

$$f'(x) = -2\sin 0 = 0$$

$$f''(x) = -2 - \frac{1}{2}(f'(0))^2 = -2$$

$$f'''(x) = -f'(x)f''(x) = 0$$

$$f^{(4)}(x) = -f''(x)f''(x) - f'(x)f'''(x) = -(x)(-2) - 0 = -4$$

Finally we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + O(x^5)$$

$$\ln(1 + \cos x) = \ln 2 + 0 + \frac{1}{2}x^2 + 0 + \frac{x^3}{24}(-4) + O(x^5)$$

$$\ln(1 + \cos x) = \ln 2 - x^2 - \frac{1}{6}x^4 + O(x^5)$$

AS Required

**Question 4    (\*\*\*)+**

Find the Maclaurin expansion of  $\ln(1 + \sinh x)$  up and including the term in  $x^3$ .

$$\ln(1 + \sinh x) = x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + O(x^4)$$

$$\begin{aligned}
 & \bullet f(x) = \ln(1 + \sinh x) \\
 & \bullet f'(x) = \frac{\cosh x}{1 + \sinh x} \\
 & \bullet f''(x) = \frac{(1 + \sinh x)\sinh x - \cosh x \cosh x}{(1 + \sinh x)^2} = \frac{\sinh^2 x + \sinh x - \cosh^2 x}{(1 + \sinh x)^2} \\
 & = \frac{\sinh x - (\cosh^2 x - \sinh^2 x)}{(1 + \sinh x)^2} = \frac{\sinh x - (\cosh x + \sinh x)(\cosh x - \sinh x)}{(1 + \sinh x)^2} \\
 & \bullet f'''(x) = \frac{(\cosh x + \sinh x)^2 - (\cosh x - \sinh x) \cdot 2(\cosh x + \sinh x)}{(\sinh x + 1)^3} \\
 & \bullet f''''(x) = \frac{\cosh x (\sinh x + 1) - 2\cosh x (\cosh x + 1)}{(\sinh x + 1)^4} \\
 & \bullet f^{(5)}(x) = \frac{3\sinh x - (\cosh x \sinh x)}{(\sinh x + 1)^5} \\
 & f(0) = \ln 1 = 0 \quad f''(0) = \frac{2e^2}{2e^2 - 1} + \frac{2e^2}{2!} f''(0) + \frac{2e^2}{3!} f'''(0) + O(x^4) \\
 & f'(0) = \frac{0-1}{1+0} = 1 \quad f''(0) = 0 + 2x \left( -\frac{1}{2!} e^2 + \frac{2}{3!} e^3 + O(x^3) \right) \\
 & f''(0) = \frac{0-1}{(2e^2)^2} = -1 \quad f''(0) = 2x - \frac{1}{2}e^2 + \frac{2}{3}e^3 + O(x^3) \\
 & f'''(0) = \frac{3-0}{(2e^2)^3} = 3 \quad f'''(0) = 0 \\
 & f^{(5)}(0) = \frac{3-0}{(2e^2)^5} = 3 \quad f^{(5)}(0) = 0
 \end{aligned}$$

**Question 5    (\*\*\*)+**

$$f(x) \equiv \ln(2e^x - 1), \quad x \in \mathbb{R}.$$

Find the Maclaurin expansion of  $f(x)$ , up and including the term in  $x^3$ .

$$f(x) \equiv 2x - x^2 + x^3 + O(x^4)$$

$$\begin{aligned}
 & f(x) = \ln(2e^x - 1) \\
 & f'(x) = \frac{2e^x}{2e^x - 1} \\
 & f''(x) = \frac{(2e^x - 1)(2e^x)^2 - 2e^x(2e^x)^2}{(2e^x - 1)^2} = \frac{4e^{2x} - 2e^{2x} - 4e^{2x}}{(2e^x - 1)^2} = -\frac{2e^{2x}}{(2e^x - 1)^2} \\
 & f'''(x) = -\frac{(2e^{2x})^2(2e^x) - 2e^{2x} \cdot 2(2e^x - 1)(2e^x)^2}{(2e^x - 1)^3} = \frac{-2e^{2x}(2e^x - 1) + 8e^{2x}}{(2e^x - 1)^3} \\
 & = \frac{4e^{2x} + 2e^{2x}}{(2e^x - 1)^3} \\
 & \text{Now } f(0) = \ln 1 = 0 \quad \text{Then } f''(0) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + O(x^4) \\
 & f'(0) = 2 \quad f''(0) = 2x + \frac{2^2}{2!} (2e^2) + \frac{2^3}{3!} (2e^3) + O(x^3) \\
 & f''(0) = -2 \quad f'''(0) = 2x - x^2 + x^3 + O(x^4) \\
 & f(0) = 0 \quad f'''(0) = 0
 \end{aligned}$$

**Question 6** (\*\*\*)+

$$y = e^{\tan x}, \quad x \in \mathbb{R}.$$

a) Show clearly that

$$\frac{d^2y}{dx^2} = (1 + \tan x)^2 \frac{dy}{dx}.$$

b) Find a series expansion for  $e^{\tan x}$ , up and including the term in  $x^3$ .

$$e^{\tan x} = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + O(x^4)$$

<p>(a) <math>y = e^{\tan x}</math></p> $\frac{dy}{dx} = e^{\tan x} \sec^2 x = y \sec^2 x$ $\frac{d^2y}{dx^2} = \frac{dy}{dx} \sec^2 x + 2y \sec^2 \tan x = \frac{dy}{dx} \sec^2 x + 2y \sec^2 x$ $= \frac{dy}{dx} [2\sec^2 x + 2\tan^2 x] = \frac{dy}{dx} [1 + \tan^2 x + 2\tan^2 x]$ $\therefore \frac{d^2y}{dx^2} = (1 + \tan^2 x) \frac{dy}{dx} \quad // \quad \text{to } \sin x \neq 0$
<p>(b) <math>\frac{dy}{dx} = 2(1 + \tan x) \sec^2 x \frac{dy}{dx} + (1 + \tan x)^2 \frac{dy}{dx}</math></p> $\text{At } x=0, \quad y=1, \quad \frac{dy}{dx}=1, \quad \frac{d^2y}{dx^2}=1, \quad \frac{d^3y}{dx^3}=2+1=3$ $\therefore y = y_0 + x y'_0 + \frac{x^2}{2!} y''_0 + \frac{x^3}{3!} y'''_0 + O(x^4)$ $\therefore y = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + O(x^4)$

**Question 7** (\*\*\*)+

$$y = \tanh x, \quad x \in \mathbb{R}.$$

By expressing the derivatives of  $\tanh x$  in terms of  $y$ , or otherwise find the first 2 non zero terms of a series expansion for  $\tanh x$ .

$$y \approx x - \frac{1}{3}x^3 + O(x^5)$$

①  $y = \tanh x$

$$\begin{aligned} \frac{dy}{dx} &= \text{sech}^2 x = \frac{1 - \tanh^2 x}{\cosh^2 x} = \frac{1 - y^2}{1 + y^2} \\ \frac{dy}{dx} &= 1 - 2y \frac{dy}{dx} = -2y(1-y^2) = 2y^3 - 2y \\ \frac{d^2y}{dx^2} &= (2y^2 - 2)\frac{dy}{dx} = (2y^2 - 2)(1-y^2) = -2y^4 + 4y^2 - 2 \end{aligned}$$

② THIS EVALUATING THESE AT  $x=0$

$$\begin{aligned} y_0 &= 0 \\ \frac{dy}{dx}|_{x=0} &= 1 - 0^2 = 1 \\ \frac{d^2y}{dx^2}|_{x=0} &= -2(0^2 + 2) = 0 \\ \frac{d^3y}{dx^3}|_{x=0} &= (6(0^2 - 2)(1 - 0^2)) = -12 \end{aligned}$$

③ HENCE  $y = y_0 + x \frac{dy}{dx}|_0 + \frac{1}{2!} \frac{d^2y}{dx^2}|_0 + \frac{1}{3!} \frac{d^3y}{dx^3}|_0 + O(x^4)$

$$\begin{aligned} \tanh x &= 0 + 2x1 + 0 + \frac{1}{3!}(-12) + O(x^4) \\ \tanh x &= x - \frac{1}{3}x^3 + O(x^5) \end{aligned}$$

**Question 8** (\*\*\*)+

By using results for series expansions of standard functions, find the series expansion of  $\ln(1-x-2x^2)$  up and including the term in  $x^4$ .

$$\boxed{\quad}, \quad \ln(1-x-2x^2) = -x - \frac{5}{2}x^2 - \frac{7}{3}x^3 - \frac{17}{4}x^4 + O(x^5)$$

SIMPLIFYING:

$$1-x-2x^2 = -(x^2+x-1) = -(2x-1)(x+1) = (-2x)\ln(x+1)$$

LOGIC OF EXPANSION:

$$\ln(1-x-2x^2) = \ln[(1-2x)(1+x)] = \ln(1-2x) + \ln(1+x)$$

NOW USE STANDARD EXPANSIONS:

$$\begin{aligned}\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \\ \ln(1-x) &= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\end{aligned}$$

THIS WE HAVE

$$\begin{aligned}\ln(1-2x) &= -2x - \frac{1}{2}(2x)^2 - \frac{1}{3}(2x)^3 - \frac{1}{4}(2x)^4 - \dots \\ \ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \\ \ln(1-x-2x^2) &= -x - \frac{5}{2}x^2 - \frac{7}{3}x^3 - \frac{17}{4}x^4 + \dots\end{aligned}$$

**Question 9** (\*\*\*)+

By using results for series expansions of standard functions, or otherwise, find the series expansion of  $\ln(x^2+4x+4)$  up and including the term in  $x^4$ .

$$\boxed{V}, \quad \boxed{\quad}, \quad \ln(x^2+4x+4) = 2\ln 2 + x - \frac{1}{4}x^2 + \frac{1}{12}x^3 - \frac{1}{32}x^4 + O(x^5)$$

WORKING AS FOLLOWS:

$$\begin{aligned}\ln(x^2+4x+4) &= \ln[(2x+2)^2] = 2\ln(2x+2) = 2\ln(2+\frac{1}{2}x) \\ &= 2\ln[2(1+\frac{1}{2}x)] \\ &= 2\ln 2 + 2\ln(1+\frac{1}{2}x)\end{aligned}$$

NOW USE STANDARD EXPANSION:

$$\begin{aligned}\ln(1+\frac{1}{2}x) &= \frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}x\right)^2+\frac{1}{3}\left(\frac{1}{2}x\right)^3-\frac{1}{4}\left(\frac{1}{2}x\right)^4+\dots \\ \ln(1+\frac{1}{2}x) &= \frac{1}{2}-\frac{1}{2}\cdot\frac{1}{4}x^2+\frac{1}{3}\cdot\frac{1}{8}x^3-\frac{1}{4}\cdot\frac{1}{16}x^4+\dots \\ \therefore \ln(x^2+4x+4) &= 2\ln 2 + 2\left[\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{4}x^2+\frac{1}{3}\cdot\frac{1}{8}x^3-\frac{1}{4}\cdot\frac{1}{16}x^4+\dots\right] \\ &= 2\ln 2 + x - \frac{1}{4}x^2 + \frac{1}{12}x^3 - \frac{1}{32}x^4 + O(x^5)\end{aligned}$$

**Question 10** (\*\*\*)

$$f(x) \equiv \cos x + \cosh x, \quad x \in \mathbb{R}.$$

Use the first 3 non zero terms of the Maclaurin expansion of  $f(x)$  to approximate the solutions of the equation

$$f(x) = 2.1.$$

$$\boxed{\phantom{00}}, \quad x \approx \pm 1.046$$

SOLVE BY DIFFERENTIATION OF STANDARD EXPANSION

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + O(x^8)$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + O(x^8)$$

$$f(x) = 2 + \frac{x^2}{4!} + \frac{x^4}{8!} + O(x^6)$$

SUBSTITUTE AND SOLVE

$$\Rightarrow f(x) = 2.1$$

$$\Rightarrow 2 + \frac{1}{12}x^2 + \frac{1}{20160}x^4 = 2.1$$

$$\Rightarrow \frac{1}{20160}x^4 + \frac{1}{12}x^2 - \frac{1}{10} = 0$$

$$\Rightarrow x^8 + 1680x^4 - 2016 = 0$$

THIS IS A QUADRATIC IN  $x^4$

$$\Rightarrow x^4 = \frac{-1680 \pm \sqrt{1680^2 - 4(1)(-2016)}}{2}$$

$$\Rightarrow x^4 = \frac{-1680 \pm 72\sqrt{5144}}{2} = -840 \pm 36\sqrt{5144}$$

$$\Rightarrow x^4 = -840 + 36\sqrt{5144} \quad [-840-36\sqrt{5144} < 0]$$

$$\Rightarrow x = \pm \sqrt{-840 + 36\sqrt{5144}}$$

$$\Rightarrow x \approx \pm 1.046$$

**Question 11** (\*\*\*\*)

$$f(x) \equiv \sin[\ln(1+x)], \quad x \in \mathbb{R}, \quad x > -1.$$

a) Show that

$$(1+x)^2 f''(x) + (1+x) f'(x) + f(x) = 0$$

b) Hence find first 3 non zero terms of the Maclaurin expansion of  $f(x)$ .

c) Use the result of part (b) to find first 2 non zero terms of the Maclaurin expansion of  $\sin[\ln(1+x)]$ .

$$\boxed{(1+x)^2 f''(x) + (1+x) f'(x) + f(x) = 0}, \quad \boxed{\sin[\ln(1+x)] \approx x - \frac{1}{2}x^2 + \frac{1}{6}x^3}, \quad \boxed{\cos[\ln(1+x)] \approx 1 - \frac{1}{2}x^2}$$

a) DIFFERENTIATE & THEN SET UP

$$f(x) = \sin[\ln(1+x)]$$

$$f'(x) = \cos[\ln(1+x)] \cdot \frac{1}{1+x}$$

$$(1+x)f'(x) = \cos[\ln(1+x)]$$

Differentiate again:

$$f''(x) + \cos[\ln(1+x)] = -\sin[\ln(1+x)] \cdot \frac{1}{(1+x)^2}$$

$$(1+x)^2 f''(x) + (1+x) f'(x) = -\sin[\ln(1+x)]$$

$$(1+x)^2 f''(x) + (1+x)^2 f'(x) = -f(x)$$

$$(1+x)^2 f''(x) + (1+x) f'(x) + f(x) = 0$$

b) DIFFERENTIATE TWO MORE TIMES (NOT SURE IF NEEDED)

$$2(1+x)^2 f''(x) + (1+x)^2 f'''(x) + (1+x)^2 f''(x) + f''(x) = 0$$

$$(1+x)^2 f'''(x) + 2(1+x)^2 f''(x) + 2f''(x) = 0$$

$$2(1+x)^2 f''(x) + (1+x)^2 f'''(x) + 3f''(x) + 3(1+x)^2 f''(x) + 2f'''(x) = 0$$

$$(1+x)^2 f'''(x) + 5f''(x) + f'''(x) = 0$$

SIMPLIFY INDIVIDUALLY

- $f(0) = \sin(0) = 0$
- $f'(0) = \cos(0) \times 1 = \cos 0 = 1$
- $f''(0) + f'(0) + f(0) = 0$
- $f''(0) + 1 + 0 = 0$
- $f''(0) = -1$

$\bullet \quad f''(0) + 3f''(0) + 2f''(0) = 0$   
 $\bullet \quad f''(0) + 3(-1) + 2(-1) = 0$   
 $\bullet \quad f''(0) = 1$   
 $\bullet \quad f''(0) + 5x + 5(-1) = 0$

HENCE WE HAVE

$$f(x) - f(0) + x(f'(0) + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!}) + \frac{f''(0)}{2!}x^2 + o(x^2)$$

$$\sin[\ln(1+x)] = 0 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$$

$$\sin[\ln(1+x)] = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

By differentiation with respect to  $x$ ,

$$\sin[\ln(1+x)] = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

$$\frac{d}{dx} [\sin[\ln(1+x)]] = \frac{1}{1+x} [x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots]$$

$$\cos[\ln(1+x)] \times \frac{1}{1+x} = 1 - x + \frac{1}{2}x^2 + \dots$$

$$\cos[\ln(1+x)] = (1+x)(1 - x + \frac{1}{2}x^2 + \dots)$$

$$\cos[\ln(1+x)] = 1 - x + \frac{1}{2}x^2 + \dots$$

$$\cos[\ln(1+x)] = 1 - \frac{1}{2}x^2 + \dots$$

**Question 12** (\*\*\*)

By using results for series expansions of standard functions, or otherwise, find the series expansion of  $\ln(x^2 + 2x + 1) - (x-2)(e^x - 2)$  up and including the term in  $x^3$ .

<input type="text"/>	$\ln(x^2 + 2x + 1) - (x-2)(e^x - 2) = -2 + 5x - x^2 + \frac{1}{2}x^3 + O(x^4)$
----------------------	--

Workings with Standard Expansions

$$\begin{aligned}
 f(x) &= \ln(x^2 + 2x + 1) - (x-2)(e^x - 2) \\
 &= \ln((x+1)^2) - (x-2)(e^x - 2) \\
 &= 2\ln(x+1) - (x-2)x^2 - 2(x-2) \\
 &= 2\left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4)\right] - (x-2)\left[1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)\right] - 4 + 2x \\
 &= 2x - x^2 + \frac{2}{3}x^3 + O(x^4) + 2 + 2x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4) - 4 + 2x \\
 &= 2x - x^2 + \frac{5}{3}x^3 + O(x^4) + 2 + 2x - \frac{1}{2}x^2 + O(x^4) - 4 + 2x \\
 &= \boxed{2x - x^2 + \frac{5}{3}x^3 + O(x^4) + 2 + 2x - \frac{1}{2}x^2 + O(x^4) - 4 + 2x}
 \end{aligned}$$

## Question 13 (\*\*\*\*)

$$f(x) = e^x \cos x, \quad x \in \mathbb{R}.$$

a) Show clearly that

$$f''(x) = f'(x) - f(x) - e^x \sin x.$$

b) Find a series expansion for  $f(x)$ , up and including the term in  $x^5$ .

c) Hence find a series expansion for  $e^x \sin x$ , up and including the term in  $x^4$ , showing further that the coefficient of  $x^4$  is zero.

$$f(x) = 1 + x + \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{30}x^5 + O(x^6), \quad e^x \sin x = x + x^2 + \frac{1}{3}x^3 + O(x^5)$$

(a)  $f(x) = e^x \cos x$   
 $f'(x) = e^x \cos x - e^x \sin x = f(x) - e^x \sin x$   
 $f''(x) = f'(x) - e^x \sin x - e^x \cos x = f'(x) - f(x) - e^x \sin x$  *As required*

(b)  $f'''(x) = f''(x) - f'(x) - e^x \sin x = f''(x) - f'(x) - f(x) - e^x \sin x$   
*Similarly*  
 $f^{(4)}(x) = f'''(x) - f''(x) - f'(x) - f(x) - e^x \sin x$   
 $\text{Thus } f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -2, \quad f^{(4)}(0) = -4$   
 $\therefore f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + O(x^4)$   
 $e^x \cos x = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{30}x^4 + O(x^5)$

(c) Differentiate w.r.t.  $x$   
 $e^x \cos x - e^x \sin x = 1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + O(x^4)$   
 $-e^x \sin x + (1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3) = 1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + O(x^4)$   
 $-e^x \sin x = -x^2 - \frac{1}{3}x^3 + O(x^4)$   
 $e^x \sin x = x^2 + \frac{1}{3}x^3 + O(x^4)$   
*Q.E.D. This is no true in  $\mathbb{C}$*

**Question 14    (\*\*\*)**

The functions  $f$  and  $g$  are given below.

$$f(x) = \arctan\left(\frac{2}{3}x\right), \quad x \in \mathbb{R}.$$

$$g(y) = \frac{1}{1+y}, \quad y \in \mathbb{R}, \quad -1 < y < 1.$$

- a) Expand  $g(y)$  as a binomial series, up and including the term in  $y^3$ .
- b) Use  $f'(x)$  and the answer to part (a) to show clearly that

$$\arctan\left(\frac{2}{3}x\right) \approx \frac{2}{3}x - \frac{8}{81}x^3 + \frac{32}{1215}x^5 - \frac{128}{15309}x^7.$$

$$g(y) = 1 - y + y^2 - y^3 + O(y^4)$$

$$\begin{aligned}
 \text{(a)} \quad (1+y)^{-1} &= 1 + \frac{-1}{1}(y) + \frac{-1(-2)}{1 \times 2} y^2 + \frac{(-1)(-2)(-3)}{1 \times 2 \times 3} y^3 + O(y^4) \\
 (1+y)^{-1} &= 1 - y + y^2 - y^3 + O(y^4)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad f(x) &= \arctan\left(\frac{2}{3}x\right) \\
 f'(x) &= \frac{\frac{2}{3}}{1 + (\frac{2}{3}x)^2} = \frac{\frac{2}{3}}{1 + \frac{4}{9}x^2} = \frac{\frac{2}{3}}{1 + \frac{4}{9}x^2} \\
 \text{Now } f'(x) &= \frac{2}{3} \left(1 + \frac{4}{9}x^2\right)^{-1} \\
 y &\mapsto \frac{4}{9}x^2 \text{ in } \arctan(u) \\
 f'(x) &= \frac{2}{3} \left[1 - \left(\frac{4}{9}x^2\right) + \left(\frac{4}{9}x^2\right)^2 - \left(\frac{4}{9}x^2\right)^3 + O(x^8)\right] \\
 f'(x) &= \frac{2}{3} \left[1 - \frac{4}{9}x^2 + \frac{32}{81}x^4 - \frac{128}{729}x^6 + O(x^8)\right] \\
 f'(x) &= \frac{2}{3} - \frac{8}{27}x^2 + \frac{32}{243}x^4 - \frac{128}{2187}x^6 + O(x^8) \\
 \therefore f(x) &= \int \left( \frac{2}{3} - \frac{8}{27}x^2 + \frac{32}{243}x^4 - \frac{128}{2187}x^6 + O(x^8) \right) dx \\
 \text{and } f(0) &= \frac{2}{3}x - \frac{8}{27}x^3 + \frac{32}{1215}x^5 - \frac{128}{15309}x^7 + O(x^9) + C \\
 \text{When } x=0 \Rightarrow 0 = C \\
 \therefore \arctan\left(\frac{2}{3}x\right) &\approx \frac{2}{3}x - \frac{8}{27}x^3 + \frac{32}{1215}x^5 - \frac{128}{15309}x^7. // \text{ As required}
 \end{aligned}$$

## Question 15 (\*\*\*\*)

$$y = \sqrt{9 + 2 \sin 3x}.$$

- a) Find a simplified expression for  $y \frac{dy}{dx}$ .
- b) Hence show that if  $x$  is numerically small

$$y \approx 3 + x - \frac{1}{6}x^2 - \frac{13}{9}x^3.$$

$$\boxed{3 \cos 3x}$$

**a)**  $y = (9 + 2 \sin 3x)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(9 + 2 \sin 3x)^{-\frac{1}{2}} \times 3 \cos 3x = \frac{3 \cos 3x}{(9 + 2 \sin 3x)^{\frac{1}{2}}}$$

$$y \frac{dy}{dx} = 3 \cos 3x (9 + 2 \sin 3x)^{\frac{1}{2}} (9 + 2 \sin 3x)^{-\frac{1}{2}}$$

$$y \frac{dy}{dx} = 3 \cos 3x$$

**b)** Now

- $y = (9 + 2 \sin 3x)^{\frac{1}{2}} \Rightarrow \boxed{y_0 = 3}$
- $3y' = 3 \cos 3x \Rightarrow y_0 y'_0 = 3$
- $\frac{3y'}{y_0} = \cos 3x \Rightarrow \boxed{\frac{y'_0}{y_0} = 1}$
- $y''_0 + 2y'_0 = -9 \sin 3x$
- $(y')^2 + y y'' = -18 \sin 3x \Rightarrow (y'_0)^2 + y_0 y''_0 = 0$
- $1 + \frac{3y''_0}{y_0} = 0 \Rightarrow \boxed{\frac{y''_0}{y_0} = -\frac{1}{3}}$
- $2y''_0 + y'_0 + y y''' = -27 \cos 3x$
- $3y''_0 y'_0 + y y''' = -27 \cos 3x \Rightarrow 3y_0 y''_0 + y_0 y'''_0 = -27$
- $3x \frac{1}{3} + 3 \frac{1}{3} = -27 \Rightarrow \boxed{\frac{y'''_0}{y_0} = -26}$

Therefore

$$y \approx y_0 + x y'_0 + \frac{x^2}{2!} y''_0 + \frac{x^3}{3!} y'''_0 + O(x^4)$$

$$(9 + 2 \sin 3x)^{\frac{1}{2}} = 3 + x - \frac{1}{6}x^2 - \frac{13}{9}x^3 + O(x^4)$$

**Question 16** (\*\*\*)

$$f(x) = \operatorname{arsinh}(x+1), \quad x \in \mathbb{R}.$$

Show clearly that ...

a) ...  $f''(x) + (x+1)[f'(x)]^3 = 0$ .

b) ...  $\operatorname{arsinh}(x+1) \approx \ln(1+\sqrt{2}) + \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{8}x^2 + \frac{\sqrt{2}}{48}x^3$ .

proof

**a)**  $\bullet \hat{f}'(x) = \operatorname{arsinh}(x+1)$   
 $\bullet \hat{f}'(x) = \frac{1}{\sqrt{(x+1)^2+1}} = \frac{1}{\sqrt{x^2+2x+2}} = (x^2+2x+2)^{-\frac{1}{2}}$   
 $\bullet \hat{f}''(x) = -\frac{1}{2}(x^2+2x+2)^{-\frac{3}{2}} \times (2x+2) = -(2x)(x^2+2x+2)^{-\frac{3}{2}} = -\frac{x+1}{(x^2+2x+2)^{\frac{3}{2}}}$   
 Then  $\hat{f}''(x) + (x+1)[\hat{f}'(x)]^3 = -\frac{2x+1}{(x^2+2x+2)^{\frac{3}{2}}} + (x+1)\left(\frac{1}{(x^2+2x+2)^{\frac{1}{2}}}\right)^3 = -\frac{x+1}{(x^2+2x+2)^{\frac{5}{2}}} + \frac{x+1}{(x^2+2x+2)^{\frac{3}{2}}} = 0$   
 $\Rightarrow \operatorname{arsinh}(x+1)'' + (x+1)[\operatorname{arsinh}(x+1)']^3 = 0$

**b)**  $\hat{f}''(x) = -(x+1)\left[\frac{\hat{f}'(x)}{\hat{f}'(x)}\right]^2$   
 $\hat{f}''(x) = -\left[\hat{f}'(x)\right]^2 - 3(x+1)\left[\frac{\hat{f}'(x)}{\hat{f}'(x)}\right]^2 \times \hat{f}''(x)$   
 $\hat{f}''(x) = \operatorname{arsinh} 1 = \ln(1+\sqrt{1+1}) - \ln(1+\sqrt{2})$   
 $\hat{f}''(x) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$   
 $\hat{f}''(x) = -\left(\frac{\sqrt{2}}{2}\right)^2 = -\frac{\sqrt{2}}{2}$   
 $\hat{f}''(x) = -\frac{\sqrt{2}}{2} - 3\left(\frac{\sqrt{2}}{2}\right)^2 \left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} + \frac{3}{8}\sqrt{2} = -\frac{1}{8}\sqrt{2}$   
 $\hat{f}(x) = \hat{f}(0) + 2\hat{f}'(0)x + \frac{2^2}{2!}\hat{f}''(0)x^2 + \frac{2^3}{3!}\hat{f}'''(0)x^3 + O(x^4)$   
 $\operatorname{arsinh}(x+1) = \ln(1+\sqrt{2}) + \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{8}x^2 - \frac{4\sqrt{2}}{3!}x^3 + O(x^4)$   
 $\operatorname{arsinh}(x+1) = \ln(1+\sqrt{2}) + \frac{1}{2}\sqrt{2}x - \frac{1}{8}\sqrt{2}x^2 - \frac{1}{24}\sqrt{2}x^3 + O(x^4)$

**Question 17** (\*\*\*\*)

$$y = \tan x, \quad 0 \leq x < \frac{\pi}{2}.$$

a) Show clearly that ...

i. ...  $\frac{d^2y}{dx^2} = 2y \frac{dy}{dx}$ .

ii. ...  $\frac{d^5y}{dx^5} = 6\left(\frac{d^2y}{dx^2}\right)^2 + 8\frac{dy}{dx} \frac{d^3y}{dx^3} + 2y \frac{d^4y}{dx^4}$ .

b) Use these results to find the first 3 non zero terms of a series expansion for  $y$ .

$$y \approx x + \frac{1}{3}x^3 + \frac{2}{15}x^5$$

a) (i)

$$\begin{aligned} y &= \tan x \\ \frac{dy}{dx} &= \sec^2 x = 1 + \tan^2 x = 1 + y^2 \\ y' &= 1 + y^2 \\ \text{Diff. wrt } x \\ y'' &= 0 + 2yy' \\ y'' &= 2yy' \quad \text{or} \quad \frac{d^2y}{dx^2} = 2y \frac{dy}{dx} \end{aligned}$$

(ii)

$$\begin{aligned} y''' &= 2y'y' + 2y^2y'' \\ y''' &= 2y^2y' + 2y^2y'' \\ \text{Diff. wrt } x \\ y'''' &= 6y^2y' + 6y^2y'' + 2y^3y''' \\ y'''' &= 6y^2y' + 6y^2y'' + 2y^3y''' \\ \text{or} \quad \frac{d^4y}{dx^4} &= 6\left(\frac{dy}{dx}\right)^3 + 8\frac{dy}{dx} \frac{d^3y}{dx^3} + 2y \frac{d^4y}{dx^4} \end{aligned}$$

b)

$$\begin{aligned} y_0 &= 0 \\ y_1 &= 1 + y_0^2 = 1 \\ y_2 &= 2y_1y' = 0 \\ y_3 &= 2y_1y'^2 + \frac{d}{dx}(y_1^2) = 2 \\ y_4 &= 2y_1y'^2 + 2y_2y'' = 0 \\ y_5 &= 6y_1y'^3 + 8y_1y'^2y'' + 2y_2y'^2y''' = 16 \end{aligned}$$

Thus

$$\begin{aligned} y &= y_0 + y_1y' + \frac{y_2}{2!}y'' + \frac{y_3}{3!}y''' + \frac{y_4}{4!}y'''' + O(x^5) \\ y &= x + \frac{2}{3}x^3 + \frac{16}{24}x^5 + O(x^5) \\ y &= x + \frac{1}{3}x^3 + \frac{2}{3}x^5 + O(x^5) \end{aligned}$$

**Question 18** (\*\*\*\*)

$$y = \ln(4+3x), \quad x > -\frac{4}{3}.$$

- a) Find simplified expressions for  $\frac{dy}{dx}$ ,  $\frac{d^2y}{dx^2}$  and  $\frac{d^3y}{dx^3}$ .
- b) Hence, find the first 4 terms in the Maclaurin expansion of  $y = \ln(4+3x)$ .
- c) State the range of values of  $x$  for which the expansion is valid.
- d) Show that for small values of  $x$ ,

$$\ln\left(\frac{4+3x}{4-3x}\right) \approx \frac{3}{2}x + \frac{9}{32}x^3.$$

$$\boxed{\frac{dy}{dx} = \frac{3}{3x+4}}, \quad \boxed{\frac{d^2y}{dx^2} = -\frac{9}{(3x+4)^2}}, \quad \boxed{\frac{d^3y}{dx^3} = \frac{54}{(3x+4)^3}}, \quad \boxed{-\frac{4}{3} < x \leq \frac{4}{3}},$$

$$\boxed{\ln(4+3x) = \ln 4 + \frac{3}{4}x - \frac{9}{32}x^2 + \frac{9}{64}x^3 + O(x^4)}$$

a)  $y = \ln(4+3x)$

$$\frac{dy}{dx} = \frac{3}{4+3x} = 3(4+3x)^{-1}$$

$$\frac{d^2y}{dx^2} = -9(4+3x)^{-2} = -\frac{9}{(4+3x)^2}$$

$$\frac{d^3y}{dx^3} = 54(4+3x)^{-3} = \frac{54}{(4+3x)^3}$$

b)  $y_{x=0} = \ln 4$

$$\frac{dy}{dx}|_{x=0} = \frac{3}{4}$$

$$\frac{d^2y}{dx^2}|_{x=0} = -\frac{9}{16}$$

$$\frac{d^3y}{dx^3}|_{x=0} = \frac{54}{32}$$

$$y = y_0 + 2y'_0x + \frac{2^2}{2!}y''_0x^2 + \frac{2^3}{3!}y'''_0x^3 + O(x^4)$$

$$\Rightarrow \ln(4+3x) = \ln 4 + \frac{3}{4}x - \frac{9}{32}x^2 + \frac{9}{64}x^3 + O(x^4)$$

$$\Rightarrow \ln(4+3x) = \ln 4 + \frac{3}{4}x - \frac{9}{32}x^2 + \frac{9}{64}x^3 + O(x^4)$$

c) Looking at  $3(4+3x)^{-1} = 3x^{-1}(1+\frac{3}{4x})^{-1}$  : valid for  $|\frac{3}{4x}| < 1$   
 $|x| < \frac{4}{3}$   
 $-\frac{4}{3} < x < \frac{4}{3}$

d)  $\ln(4-3x) = \ln 4 - \frac{3}{4}x - \frac{9}{32}x^2 - \frac{9}{64}x^3 + O(x^4)$

$$\ln\left[\frac{4+3x}{4-3x}\right] = \ln(4+3x) - \ln(4-3x) = \ln 4 + \frac{3}{4}x - \frac{9}{32}x^2 + \frac{9}{64}x^3 + O(x^4)$$

$$- \left[ \ln 4 + \frac{3}{4}x - \frac{9}{32}x^2 - \frac{9}{64}x^3 + O(x^4) \right]$$

$$= \frac{9}{32}x^2 + \frac{9}{64}x^3 + O(x^4)$$

**Question 19** (\*\*\*\*)

If  $m$  and  $n$  are non zero constants, then the first non zero term in the Maclaurin expansion of  $e^{mx} - (1+4x)^n$  is  $-4x^2$ .

Find the coefficient of  $x^3$  in this expansion.

You may NOT use standard series expansions in this question.

$$\boxed{\phantom{000}}, \boxed{x^3} = \frac{56}{3}$$

EXPAND UP TO  $x^2$

$$y = e^{mx} - (1+4x)^n$$

$$\frac{dy}{dx} = me^{mx} - n(1+4x)^{n-1}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx} - n(n-1)(1+4x)^{n-2}$$

$$\frac{d^3y}{dx^3} = m^3e^{mx} - n(n-1)(n-2)(1+4x)^{n-3}$$

$$\frac{d^4y}{dx^4} = m^4e^{mx} - n(n-1)(n-2)(n-3)(1+4x)^{n-4}$$

BY MACLAURIN THEOREM

$$y = y_0 + 2y_1 x + \frac{2}{3!}y_2 x^2 + \frac{2}{4!}y_3 x^3 + O(x^4)$$

$$y = 0 + (2m+4n)x + \frac{1}{2}[m^2 - 4n(n-1)]x^2 + \frac{1}{3!}[m^3 - 6n(n-1)(n-2)]x^3 + O(x^4)$$

SUMMING COEFFICIENTS FOR  $x^2$  &  $x^3$

$$\begin{aligned} m+4n &= 0 \\ \frac{1}{2}[m^2 - 4n(n-1)] &= -4 \end{aligned} \quad \left\{ \begin{array}{l} m = -4n \\ \frac{1}{2}[m^2 - 4n(n-1)] = -4 \end{array} \right.$$

$$\begin{aligned} \rightarrow \frac{1}{2}[(-4n)^2 - 4n(n-1)] &= -4 \\ \rightarrow 8n^2 - 4n^2 + 4n &= -8 \\ \rightarrow 4n^2 + 4n &= -8 \\ \rightarrow 4n(n+1) &= -8 \\ \rightarrow n(n+1) &= -2 \end{aligned}$$

THE COEFFICIENT OF  $x^3$  WILL BE

$$\begin{aligned} \frac{1}{3!}[m^3 - 6n(n-1)(n-2)] &= \frac{1}{3!}[m^3 - 6n(n-1)(n-2)] \\ &= \frac{1}{3!}[-8(-2)] \\ &= \frac{16}{3} \end{aligned}$$

ANSWER

**Question 20** (\*\*\*\*)

Determine the first 3 non zero terms in the Maclaurin expansion of

$$y = e^{\sin^2 x}.$$

,  $y = 1 + x^2 + \frac{1}{2}x^4 + O(x^6)$

DEFINING THE FIRST FEW DERIVATIVES - NOTE THAT FUNCTION IS EVEN

- $y = e^{\sin^2 x}$
- $\frac{dy}{dx} = e^{\sin^2 x} \cdot 2\sin x \cos x = e^{\sin^2 x} \sin 2x = y \sin 2x$
- $\frac{d^2y}{dx^2} = \frac{dy}{dx} \sin 2x + 2y \cos 2x$
- $\frac{d^3y}{dx^3} = \frac{d}{dx}(\sin 2x) + 2\frac{dy}{dx} \cos 2x - 4y \sin 2x$   
 $= \frac{d}{dx}(\sin 2x) + 4\frac{dy}{dx} \cos 2x - 4y \sin 2x$
- $\frac{d^4y}{dx^4} = \frac{d}{dx}(\sin 2x) + 2\frac{d}{dx}(\cos 2x) + 4\frac{d^2y}{dx^2} \cos 2x - 8y \sin 2x - 4y = 0$   
 $= \frac{d}{dx}(\sin 2x) + 6\frac{dy}{dx} \cos 2x - 16y \sin 2x - 8y = 0$

EVALUATE AT  $x=0$  & WRITE THE EXPANSION

• $y=1$	$\Rightarrow y = y_0 + 2y_1 x + \frac{2^2}{2!}y_2 x^2 + \frac{2^3}{3!}y_3 x^3 + \frac{2^5}{5!}y_5 x^5 + O(x^7)$
• $y_0=0$	$\Rightarrow y = 0 + 0 + \frac{2^2}{2!}y_2 x^2 + 0 + \frac{2^5}{5!}y_5 x^5 + O(x^7)$
• $y_2=2$	$\Rightarrow e^{2x^2} = 0 + x^2 + \frac{2^2}{2!}x^4 + O(x^6)$
• $y_5=0$	$\Rightarrow e^{2x^2} = 1 + x^2 + \frac{2^2}{2!}x^4 + O(x^6)$
• $y^6=4$	

# MACLAURIN EXPANSIONS

## 7 HARD QUESTIONS

**Question 1** (\*\*\*\*+)

$$y = \ln(1 + \sin x), \sin x \neq -1.$$

a) Show clearly that

$$\frac{dy}{dx} = f(y),$$

where  $f(y)$  is a function to be found.

b) Hence show further that

$$\ln(1 + \sin x) \approx x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{24}x^5.$$

$$y = -e^{-y}$$

(a)  $y = \ln(1 + \sin x)$

$$\frac{dy}{dx} = \frac{\cos x}{1 + \sin x}$$

$$\frac{d^2y}{dx^2} = \frac{(1 + \sin x)(\cos x) - \cos x(\cos x)}{(1 + \sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2}$$

$$= \frac{-\sin x - (\cos^2 x + \sin^2 x)}{(1 + \sin x)^2} = \frac{-\sin x - 1}{(1 + \sin x)^2} = \frac{-1 - \sin x}{(1 + \sin x)^2}$$

$$= \frac{1}{1 + \sin x}$$

But  $y = \ln(1 + \sin x)$

$$e^y = 1 + \sin x$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{e^y} = -e^{-y}$$

$$\text{LHS } f(y) = -e^{-y}$$

(b)  $\frac{d^3y}{dx^3} = -e^{-y}$

$$\frac{d^4y}{dx^4} = -e^{-y} \frac{dy}{dx} = -\frac{dy}{dx} e^{-y}$$

$$\frac{d^5y}{dx^5} = -\frac{d}{dx} \left( \frac{dy}{dx} e^{-y} \right) = -\frac{dy}{dx} \frac{d}{dx} e^{-y} - e^{-y} \frac{d^2y}{dx^2}$$

$$\frac{d^6y}{dx^6} = -\frac{d}{dx} \left( -\frac{dy}{dx} e^{-y} - e^{-y} \frac{d^2y}{dx^2} \right) = -\frac{d^2y}{dx^2} e^{-y} - \frac{d}{dx} \left( e^{-y} \frac{d^2y}{dx^2} \right)$$

Now  $\frac{dy}{dx} \ln 1 = 0$

$$\frac{dy}{dx} = 1$$

$$\frac{d^2y}{dx^2} = -1$$

$$\frac{d^3y}{dx^3} = -[x(-1)] = -x$$

$$\frac{d^4y}{dx^4} = -[x(-1)]^2 = -x^2$$

$$\frac{d^5y}{dx^5} = -[x(-1)] \cdot -2x = 2x^2$$

$$\frac{d^6y}{dx^6} = 5$$

Therefore  $y = y_0 + a_1 y_1' + \frac{a_2}{2!} y_2'' + \frac{a_3}{3!} y_3''' + \frac{a_4}{4!} y_4'''' + \dots$

$$\Rightarrow y = 0 + 1x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \dots$$

$$\Rightarrow y = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{24}x^5 - \dots$$

**Question 2** (\*\*\*\*+)

$$y = \tan\left(x + \frac{\pi}{4}\right), -\frac{3\pi}{4} < x < \frac{\pi}{4}.$$

Use the Maclaurin theorem to show that

$$y = \tan\left(x + \frac{\pi}{4}\right) \approx 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \frac{64}{15}x^5.$$

proof

$$\begin{aligned}
 & y = \tan\left(2x + \frac{\pi}{4}\right) \\
 \Rightarrow & y = \tan\left(2x + \frac{\pi}{4}\right) = 1 + \tan^2\left(2x + \frac{\pi}{4}\right) = 1 + y^2 \Rightarrow y \leq 1+y^2 \\
 \Rightarrow & y' = 2yy' \\
 \Rightarrow & y' = 2y^2 + 2yy'' \\
 \Rightarrow & y'' = 2y^2y' + 2yy''' \\
 \Rightarrow & y''' = 4y^2y' + 2y'y' + 2yy'' \\
 \Rightarrow & y'''' = 6y^2y' + 2yy'' \\
 \Rightarrow & y'''' = 6y^3 + 8y^4 + 2y^5 \\
 & \begin{array}{l} y_0=1 \\ y_1=1 \\ y_2=2 \\ y_3=4 \\ y_4=16 \\ y_5=80 \end{array} \quad \begin{array}{l} y_0'=1 \\ y_1'=2 \\ y_2'=4 \\ y_3'=16 \\ y_4'=80 \end{array} \quad \begin{array}{l} y_0''=2 \\ y_1''=4 \\ y_2''=16 \\ y_3''=80 \end{array} \quad \begin{array}{l} y_0'''=2 \\ y_1'''=4 \\ y_2'''=16 \\ y_3'''=80 \end{array} \quad \begin{array}{l} y_0''''=8 \\ y_1''''=16 \\ y_2''''=80 \\ y_3''''=80 \end{array} \\
 & \begin{array}{l} y_0=1 \\ y_1=1 \\ y_2=2 \\ y_3=4 \\ y_4=16 \\ y_5=80 \end{array} \quad \begin{array}{l} y_0'=2 \\ y_1'=4 \\ y_2'=8 \\ y_3'=16 \\ y_4'=32 \\ y_5'=64 \end{array} \quad \begin{array}{l} y_0''=2 \\ y_1''=4 \\ y_2''=8 \\ y_3''=16 \\ y_4''=32 \\ y_5''=64 \end{array} \quad \begin{array}{l} y_0'''=2 \\ y_1'''=4 \\ y_2'''=8 \\ y_3'''=16 \\ y_4'''=32 \\ y_5'''=64 \end{array} \quad \begin{array}{l} y_0''''=8 \\ y_1''''=16 \\ y_2''''=32 \\ y_3''''=64 \\ y_4''''=128 \\ y_5''''=256 \end{array} \\
 & \text{Thus } y = y_0 + x y_1' + \frac{x^2}{2!} y_2'' + \frac{x^3}{3!} y_3''' + \frac{x^4}{4!} y_4'''' + O(x^5) \\
 & \tan\left(2x + \frac{\pi}{4}\right) = 1 + 2x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 + \frac{2^5}{5!}x^5 + O(x^6) \\
 & \tan\left(2x + \frac{\pi}{4}\right) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \frac{64}{15}x^5 + O(x^6)
 \end{aligned}$$

**Question 3** (\*\*\*\*+)

Find the Maclaurin expansion, up and including the term in  $x^4$ , for  $y = e^{\sin 2x}$ .

$$e^{\sin 2x} = 1 + x + 2x^2 - 2x^4 + O(x^5)$$

$y = e^{\sin 2x}$

$$\begin{aligned} \boxed{y_0 = e^0 = 1} \\ \Rightarrow y' = e^{\sin 2x} (2\cos 2x) = 2y \cos 2x \\ \boxed{y'_0 = 2y_0 = 2} \\ \Rightarrow y'' = 2y' \cos 2x - 4y \sin 2x \\ \boxed{y''_0 = 2y'_0 = 4} \\ \Rightarrow y''' = 2y'' \cos 2x - 4y' \sin 2x - 4y \sin 2x - 8y \cos 2x \\ \boxed{y'''_0 = 2y''_0 - 8y_0 = 8 - 8 = 0} \\ \Rightarrow y^{(4)} = (2y'' - 8y') \cos 2x - 8y \sin 2x \\ \Rightarrow y^{(4)} = (2y'' - 8y') \cos 2x - 2(2y'' - 8y) \sin 2x - 8y \sin 2x - 16y \cos 2x \\ \boxed{y^{(4)}_0 = -8x^2 - 16x^2 = -48} \\ \therefore y = y_0 + y'_0 x + \frac{y''_0}{2!} x^2 + \frac{y'''_0}{3!} x^3 + \frac{y^{(4)}_0}{4!} x^4 + O(x^5) \\ e^{\sin 2x} = 1 + 2x + 2x^2 + O(x^3) \\ e^{\sin 2x} = 1 + 2x + 2x^2 - 2x^4 + O(x^5) \end{aligned}$$

ALTERNATIVE BY SUBSTITUTED SERIES

$$\begin{aligned} \boxed{y = e^t} \\ \boxed{t = \sin 2x} \\ * \sin 2x = 2x - \frac{x^3}{3!} + O(x^5) \\ * \sin 2x = 2x - \frac{x^3}{3!} + O(x^5) \\ \therefore y = e^t \quad \text{where } t = 2x - \frac{x^3}{3!} + O(x^5) \\ \Rightarrow y = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + O(t^5) \\ \Rightarrow y = 1 + [2x - \frac{x^3}{3!} + O(x^5)] + \frac{1}{2}[2x - \frac{x^3}{3!} + O(x^5)]^2 + \frac{1}{3!}[2x - \frac{x^3}{3!} + O(x^5)]^3 \\ + \frac{1}{4!}[2x - \frac{x^3}{3!} + O(x^5)]^4 + O\left[\left(2x - \frac{x^3}{3!} + O(x^5)\right)^5\right] \\ \dots \text{EXPAND THE BOTTLED DAY UP TO TERMS OF } O(x^5) \dots \\ \Rightarrow y = 1 + [2x - \frac{x^3}{3!}] + \frac{1}{2}[4x^2 - \frac{16x^4}{3!}] + \frac{1}{3!}[8x^3] + \frac{1}{4!}[16x^4] + O(x^5) \\ \Rightarrow y = 1 + 2x - \frac{4}{3}x^3 + 2x^2 - \frac{8}{3}x^4 + \frac{2}{3}x^3 + O(x^5) \\ \Rightarrow y = 1 + 2x + 2x^2 - 2x^4 + O(x^5) \end{aligned}$$

**Question 4** (\*\*\*\*+)

Consider the following infinite convergent series.

$$\frac{3}{1 \times 2} - \frac{5}{2 \times 3} + \frac{7}{3 \times 4} - \frac{9}{4 \times 5} + \frac{11}{5 \times 6} - \dots$$

- a) Use the method of differences, to find the sum of this series.
- b) Verify the answer of part (a) by using a method based on the Maclaurin expansion of  $\ln(1+x)$ .

**V**,  , **1**

a) Start by obtaining the general term in sigma notation

$$\frac{3}{1 \times 2} - \frac{5}{2 \times 3} + \frac{7}{3 \times 4} - \frac{9}{4 \times 5} + \frac{11}{5 \times 6} - \dots = \sum_{n=1}^{\infty} [(-1)^{n+1} \frac{(2n+1)}{(n)(n+1)}]$$

Factorising  $(-1)^{n+1}$  separates the last two partial fractions by cancellation

$$\frac{2n+1}{n(n+1)} = \frac{1}{n} + \frac{1}{n+1}$$

Now, we have

$P_{n1} = \frac{-3}{1 \times 2} = -\frac{1}{1} + \frac{1}{2}$
$P_{n2} = -\frac{5}{2 \times 3} = -\frac{1}{2} + \frac{1}{3}$
$P_{n3} = \frac{7}{3 \times 4} = \frac{1}{3} + \frac{1}{4}$
$P_{n4} = -\frac{9}{4 \times 5} = -\frac{1}{4} + \frac{1}{5}$
$\vdots$
$P_{n-n} = (-1)^{n+1} \frac{(2n+1)}{n(n+1)}$

$$\sum_{n=1}^{\infty} [(-1)^{n+1} \frac{(2n+1)}{n(n+1)}] = 1 + (-1)^{\infty} \frac{1}{n+1}$$

As  $n \rightarrow \infty$  the sum to infinity is  $\boxed{1}$

b) Working at the expansion of  $\ln(1+x)$ , valid for  $-1 < x \leq 1$

- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$
- Let  $x = 1$
- $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Using the partial fractions from part (a)

$$\sum_{n=1}^{\infty} \left[ \frac{(-1)^{n+1}}{n(n+1)} \right] = \sum_{n=1}^{\infty} \left[ (-1)^{n+1} \left( \frac{1}{n} + \frac{1}{n+1} \right) \right]$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

$$= \ln 2 + \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} \right)$$

Re-indenting and manipulating

$$\begin{aligned} &= \ln 2 + \left[ -\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \right] \\ &= \ln 2 + \left[ 1 - 1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \right] \\ &= \ln 2 + \left[ 1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \right] \\ &= \ln 2 + (1 - \ln 2) \\ &= 1 \end{aligned}$$

Alternative to re-indenting & manipulating

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \\ -S &= -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \\ 1 - \frac{S}{2} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \\ 1 - \frac{S}{2} &= \ln 2 \\ S &= 1 - \ln 2 \quad \text{as required} \end{aligned}$$

**Question 5** (\*\*\*\*+)

$$y = \ln(2 - e^x), \quad x < \ln 2.$$

Show clearly that

$$e^y \left[ \frac{d^3 y}{dx^3} + 3 \frac{dy}{dx} \frac{d^2 y}{dx^2} + \left( \frac{dy}{dx} \right)^3 \right] + e^x = 0,$$

and hence find the first 3 non zero terms in the Maclaurin expansion of

$$y = \ln(2 - e^x), \quad x < \ln 2.$$

$$\boxed{\quad}, \quad \boxed{y = \ln(2 - e^x) = -x - x^2 - x^3 + O(x^4)}$$

START THE DIFFERENTIATION AFTER REMOVING THE LOGS.

$$\begin{aligned} \rightarrow y &= \ln(2 - e^x) \\ \rightarrow e^y &= 2 - e^x \\ \rightarrow \frac{d}{dx}(e^y) &= \frac{d}{dx}(2 - e^x) \\ \rightarrow e^y \frac{dy}{dx} &= -e^x \\ \rightarrow e^y \frac{dy}{dx} + e^x &= 0 \end{aligned}$$

WRITE THE DERIVATIVE IN THE EXPRESSION MORE COMPACTLY AND DIFFERENTIATE AGAIN.

$$\begin{aligned} \rightarrow e^y y' + e^x &= 0 \\ \rightarrow \frac{d}{dx}(e^y y' + e^x) &= \frac{d}{dx}(0) \\ \rightarrow e^y y'' + e^y y' + e^x &= 0 \\ \rightarrow e^y [y'' + y'] + e^x &= 0 \end{aligned}$$

DIFFERENTIATE ONE MORE TIME WITH RESPECT TO x.

$$\begin{aligned} \rightarrow e^y [(y')^2 + y''] + e^x [2y'y' + y''] + e^x &= 0 \\ \rightarrow e^y [y'^2 + 2y'y' + y''] + e^x &= 0 \\ \rightarrow e^y [ \underline{y'' \left( \frac{dy}{dx} \right)^2 + 3 \frac{dy}{dx} \frac{d^2 y}{dx^2} + \frac{d^3 y}{dx^3}} ] + e^x &= 0 \end{aligned}$$

NOW EVALUATE THESE AT x=0

$$\begin{aligned} \bullet y_0 &= \ln(2 - e^0) = \ln 1 = 0 \quad \therefore y_0 = 0 \\ \bullet e^{y_0} &= e^0 = 1 \\ \bullet e^{y_0} y'_0 + e^{y_0} &= 0 \\ 1 \cdot y'_0 + 1 &= 0 \quad \therefore y'_0 = -1 \\ \bullet e^{y_0} [ (y')^2 + y''_0 ] + e^x &= 0 \\ e^{y_0} [ (y'_0)^2 + y''_0 ] + e^0 &= 0 \\ 1 [ (-1)^2 + y''_0 ] + 1 &= 0 \\ 1 + y''_0 + 1 &= 0 \quad \therefore y''_0 = -2 \\ \bullet e^{y_0} [ (y')^2 + 3y'y' + y'''] + e^x &= 0 \\ e^{y_0} [ (y'_0)^2 + 3(y'_0)(y''_0) + y'''_0 ] + e^0 &= 0 \\ 1 \times [ (-1)^2 + 3(-1)(-2) + y'''_0 ] + 1 &= 0 \\ -1 + 6 + y'''_0 + 1 &= 0 \quad \therefore y'''_0 = -6 \end{aligned}$$

FINALLY WE HAVE

$$\begin{aligned} y &= y_0 + y'_0 x + \frac{y''_0}{2!} x^2 + \frac{y'''_0}{3!} x^3 + O(x^4) \\ \ln(2-x) &= 0 - 1(x) + \frac{(-2)}{2!} x^2 + \frac{-6}{3!} x^3 + O(x^4) \\ \ln(2-x) &= -x - x^2 - x^3 + O(x^4) \end{aligned}$$

**Question 6** (\*\*\*\*+)

Find the Maclaurin expansion, up and including the term in  $x^4$ , for  $y = \sin(\cos x)$ .

$$\boxed{\quad}, \quad \sin(\cos x) = \sin 1 - \frac{1}{2}x^2 \cos 1 + x^4 \left( \frac{1}{24} \cos 1 - \frac{1}{8} \sin 1 \right) + O(x^6)$$

DETERMINE THE EXPANSION BY DIRECT DIFFERENTIATION

$$\begin{aligned}
 y &= \sin(\cos x) & y_0 &= \sin 1 \\
 \frac{dy}{dx} &= \cos(\cos x) (-\sin x) = -\sin x \cos(\cos x) & y'_0 &= 0 \\
 \frac{d^2y}{dx^2} &= -\cos x \cos(\cos x) - \sin x [-\sin(\cos x)] (-\sin x) \\
 &= -\cos x \cos(\cos x) - \sin^2 x \sin(\cos x) \\
 &= -y_0 \cos x - \cos x \sin(x) & y''_0 &= -\cos x \\
 \frac{d^3y}{dx^3} &= -\frac{d}{dx} \sin^2 x - 2y_0 \sin x + \sin x \cos x - \cos x [-\sin(\cos x)] (-\sin x) \\
 &= -\frac{d}{dx} \sin^2 x - y_0 \sin x - \frac{d}{dx} \cos x \sin x \\
 &= -\frac{d}{dx} \sin^2 x - y_0 \sin x - \frac{1}{2} \sin x \cos x \\
 &= -\frac{1}{2} \sin x \cos x - (1+y_0) \frac{dy}{dx} & y'''_0 &= 0 \\
 \frac{d^4y}{dx^4} &= -\frac{1}{2} \frac{d}{dx} \sin x \cos x - 2y_0 \cos x - 2\cos x \frac{dy}{dx} - (1+y_0) \frac{d^2y}{dx^2} \\
 &= -\frac{1}{2} \frac{d}{dx} \sin x \cos x - \frac{d}{dx} \cos x - (1+y_0) \frac{d^2y}{dx^2} \\
 &= -\frac{1}{2} \sin x \cos x - \frac{1}{2} \cos x \sin x - (1+y_0) \frac{d^2y}{dx^2} & y^{(4)}_0 &= 0
 \end{aligned}$$

BY THE MACLAURIN THEOREM

$$\begin{aligned}
 \Rightarrow y &= y_0 + y'_0 x + \frac{y''_0}{2!} x^2 + \frac{y'''_0}{3!} x^3 + \frac{y^{(4)}_0}{4!} x^4 + O(x^5) \\
 \Rightarrow \sin(\cos x) &= \sin 1 - \frac{1}{2}x^2 \cos 1 + \frac{1}{24}x^4 (\cos 1 - \sin 1) + O(x^5)
 \end{aligned}$$

**Question 7** (\*\*\*\*+)

Find the first four non zero terms in the Maclaurin expansion of

$$y = \ln(1 + \cosh x).$$

$$\boxed{\quad}, \quad \ln(1 + \cosh x) = \ln 2 + \frac{1}{4}x^2 - \frac{1}{96}x^4 + \frac{1}{1440}x^6 + O(x^8)$$

SIMPLY IN DIRECT DIFFERENTIATION — NOTE THE FUNCTION IS EVEN  
SO WE NEED DERIVATIVES UP TO  $x^4$

$$\begin{aligned}
 \rightarrow y &= \ln(1 + \cosh x) \\
 \rightarrow \frac{dy}{dx} &= \frac{\sinh x}{1 + \cosh x} \\
 \rightarrow \frac{d^2y}{dx^2} &= \frac{(1+\cosh x)\cosh x - \sinh x(\sinh x)}{(1+\cosh x)^2} = \frac{\cosh x + \cosh^2 x - \sinh^2 x}{(1+\cosh x)^2} \\
 &= \frac{\cosh x + 1}{(1+\cosh x)^2} = \frac{1}{1+\cosh x}
 \end{aligned}$$

DETERMINE 4 MORE DERIVATIVES DIRECTLY IS DIFFICULT TO DO.  
WE MAY PREFER TO USE FOCUSES

$$\begin{aligned}
 y &= \ln(1 + \cosh x) = -\ln\left(\frac{1}{1 + \cosh x}\right) = -\ln\left(\frac{dx}{dx}\right) \\
 \Rightarrow -y' &= \ln\left(\frac{dx}{dx}\right) \\
 \Rightarrow e^{-y} &= \frac{dx}{dx} \\
 \Rightarrow \frac{dy}{dx} &= e^{-y}
 \end{aligned}$$

CONSTITUTE THE DIFFERENTIATIONS W.R.T.  $x$

$$\begin{aligned}
 \rightarrow \frac{d^2y}{dx^2} &= -e^{-y} \frac{dy}{dx} \\
 \rightarrow \frac{d^3y}{dx^3} &= e^{-y} \left( \frac{dy}{dx} \right)^2 - e^{-y} \frac{d^2y}{dx^2} = e^{-y} \left( \frac{dy}{dx} \right)^2 - e^{-y} \\
 \rightarrow \frac{d^4y}{dx^4} &= -e^{-y} \left( \frac{dy}{dx} \right)^3 + 2e^{-y} \frac{dy}{dx} \frac{d^2y}{dx^2} + 2e^{-2y} \frac{d^3y}{dx^3}
 \end{aligned}$$

TIDY BEFORE THE FINAL DIFFERENTIATION

$$\begin{aligned}
 \Rightarrow \frac{dy}{dx} &= -e^{-y} \left( \frac{dy}{dx} \right)^3 + 2e^{-2y} \left( \frac{dy}{dx} \right) \\
 \Rightarrow \frac{dy}{dx} &= \frac{4e^{-2y} \frac{dy}{dx} - e^{-y} \left( \frac{dy}{dx} \right)^3}{4e^{-2y} \left( \frac{dy}{dx} \right)^2 + e^{-2y} \left( \frac{dy}{dx} \right)^3 - 3e^{-2y} \left( \frac{dy}{dx} \right)^2 \frac{dy}{dx}} \\
 \Rightarrow \frac{dy}{dx} &= -8e^{-2y} \left( \frac{dy}{dx} \right)^2 + 4e^{-2y} + e^{-2y} \left( \frac{dy}{dx} \right)^4 - 3e^{-2y} \left( \frac{dy}{dx} \right)^2 \\
 \Rightarrow \frac{dy}{dx} &= \frac{4e^{-2y} + e^{-2y} \left( \frac{dy}{dx} \right)^4 - 16e^{-2y} \left( \frac{dy}{dx} \right)^2}{4e^{-2y} \left( \frac{dy}{dx} \right)^2 + e^{-2y} \left( \frac{dy}{dx} \right)^3}
 \end{aligned}$$

EVALUATING THESE DERIVATIVES AT  $x=0$

$$\begin{aligned}
 y_0 &= \ln 2, \quad \frac{dy}{dx}|_{x=0} = 0, \quad \frac{d^2y}{dx^2}|_{x=0} = e^{-\ln 2} = \frac{1}{2} \\
 \frac{d^3y}{dx^3}|_{x=0} &= 0, \quad \frac{d^4y}{dx^4}|_{x=0} = -e^{-2\ln 2} = -\frac{1}{4} \\
 \frac{d^5y}{dx^5}|_{x=0} &= 0, \quad \frac{d^6y}{dx^6}|_{x=0} = 4e^{-3\ln 2} = \frac{1}{2}
 \end{aligned}$$

HENCE WE CAN OBTAIN THE MACLAURIN, IGNORING ODD TERMS

$$\begin{aligned}
 y &= y_0 + \frac{2e^{-2y}}{2!} y''|_{x=0} + \frac{2e^{-4y}}{4!} y''''|_{x=0} + \frac{2e^{-6y}}{6!} y^{(6)}|_{x=0} + O(x^6) \\
 y &= \ln 2 + \frac{1}{2}x^2 + \left( \frac{1}{2} \left( \frac{1}{2} \right)^2 x^4 + \frac{1}{24} \frac{1}{2} x^6 \right) + O(x^8) \\
 y &= \ln 2 + \frac{1}{4}x^2 - \frac{1}{96}x^4 + \frac{1}{1440}x^6 + O(x^8)
 \end{aligned}$$

# MACLAURIN EXPANSIONS

## 9 ENRICHMENT QUESTIONS

**Question 1** (\*\*\*\*\*)

The curve with equation  $y = f(x)$  is the solution of the differential equation

$$f(x) \equiv \ln\left(\frac{1-x+x^2}{1+x+x^2}\right).$$

Determine, in its simplest form, the coefficient of  $x^{6n-3}$ ,  $n \in \mathbb{N}$ , in the Maclaurin series expansion of  $f(x)$ .

,  $\frac{4}{6n-3}$

USING THE DIFFERENCE OF CUBES IDENTITY & THE SUM OF CUBES IDENTITY

$$\begin{aligned} 1+x^2 &\equiv (1+x)(1-x+x^2) \\ 1-x^3 &\equiv (1-x)(1+x+x^2) \\ \ln\left(\frac{1-x+x^2}{1+x+x^2}\right) &= \ln\left(\frac{(1-x)(1+x+x^2)}{(1+x)(1-x+x^2)}\right) \\ &= \ln\left(\frac{1+x^2}{1-2x+2x^2}\right) \end{aligned}$$

THIS, WE HAVE

- $f(x) = \ln\left(\frac{1-x+x^2}{1+x+x^2}\right) + (\ln(1+x) - \ln(1-x)) = \ln(1+x^2) - \ln(1-2x)$
- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots$
- $\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$
- ∴  $\ln(1+x) - \ln(1-x) = 2x + \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{4}x^4 - \dots + \frac{2n-1}{2n-2}x^{2n-3}$
- $\ln(1+x^2) - \ln(1-2x) = 2(2x) + \frac{3}{2}(2x)^2 + \frac{5}{3}(2x)^3 + \frac{7}{4}(2x)^4 + \dots + \frac{2n-1}{2n-2}(2x)^{2n-3}$

Now, multiply the  $\frac{2^{2n-3}}{2n-2}$  times in  $\ln(1+x^2) - \ln(1-2x)$

$$\begin{aligned} &\dots + \frac{3}{2}2^3 + \dots + \frac{5}{3}2^4 + \dots + \frac{7}{4}2^5 + \dots + \frac{2n-1}{2n-2}2^{2n-3} \\ \text{Thus } f(x) &= [\ln(1+x^2) - \ln(1-2x)] \\ \therefore \text{coeff of } x^{2n-3} \text{ will be} &= \frac{2}{2n-2} - \frac{2}{2n-3} = \frac{2(2n-3)-4(2n-2)}{(2n-3)(2n-2)} \\ &= \frac{4(2n-11)}{(2n-3)(2n-2)} = \frac{4}{2n-3} \end{aligned}$$

**Question 2** (\*\*\*\*\*)

Find the Maclaurin expansion of  $\arctan x$ , and use it to show that

$$\pi = \sum_{n=0}^{\infty} f(n),$$

for some suitable function  $f$ .

$$\boxed{\quad}, \quad \boxed{\pi = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}}$$

START WITH DIFFERENTIATION & INTEGRATION

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

$$\frac{d}{dx}(\arctan x) = \sum_{n=0}^{\infty} [(-1)^n x^{2n}]$$

INTEGRATE WITH RESPECT TO x, ASSUMING INTEGRATION/SUMMATION COINCIDE

$$\arctan x = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx.$$

$$\arctan x = \sum_{n=0}^{\infty} \left[ (-1)^n \frac{x^{2n+1}}{2n+1} \right] + C$$

Using  $x=0 \Rightarrow 0 = 0 + C$

$$\arctan x = \sum_{n=0}^{\infty} \left[ (-1)^n \frac{x^{2n+1}}{2n+1} \right]$$

FINALLY SUBSTITUTE x=1

$$\arctan 1 = \sum_{n=0}^{\infty} \left[ (-1)^n \frac{1^{2n+1}}{2n+1} \right]$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$$

if  $f(x) = \frac{4(-1)^x}{2x+1}$

**Question 3** (\*\*\*\*\*)

- a) Use an appropriate integration method to evaluate the following integral.

$$\int_0^1 x^3 \arctan x \, dx.$$

- b) Obtain an infinite series expansion for  $\arctan x$  and use this series expansion to verify the answer obtained for the above integral in part (a).

[you may assume that integration and summation commute]

,  $\frac{1}{6}$

a) START BY INTEGRATION BY PARTS

$$\begin{aligned} \int_0^1 x^3 \arctan x \, dx &= [\arctan x]_0^1 - \int_0^1 \frac{x^3}{1+x^2} \, dx \\ &= \frac{1}{2}\pi - 0 - \frac{1}{4} \int_0^1 \frac{x^3}{x^2+1} \, dx \\ &= \frac{\pi}{4} - \frac{1}{4} \int_0^1 \frac{x(x^2+1)-(x^2+1)+1}{x^2+1} \, dx \\ &= \frac{\pi}{4} - \frac{1}{4} \int_0^1 (x^2-1 + \frac{1}{x^2+1}) \, dx \\ &= \frac{\pi}{4} - \frac{1}{4} \left[ \frac{1}{3}x^3 - x + \arctan x \right]_0^1 \\ &= \frac{\pi}{4} - \frac{1}{4} \left[ \left( \frac{1}{3} - 1 + \frac{\pi}{4} \right) - 0 \right] \\ &= \frac{\pi}{4} - \frac{1}{4} \left( -\frac{2}{3} + \frac{\pi}{4} \right) \\ &= \frac{\pi}{4} + \frac{1}{6} - \frac{\pi}{16} \\ &= \frac{1}{16}(4\pi + 6) \end{aligned}$$

b) NEED THE EXPANSION OF  $\arctan x$ .

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1-x^2+x^4-x^6+x^8-\dots$$

INTEGRATE BOTH SIDES GIVES

$$\begin{aligned} \arctan x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots + C \\ \arctan 0 = 0 &\Rightarrow C = 0 \end{aligned}$$

- $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots$
- $\arctan x = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{2n+1} x^{2n+1} \right]$

THIS WE KNOW HAVE

$$\begin{aligned} \int_0^1 x^3 \arctan x \, dx &= \int_0^1 x^3 \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{2n+1} x^{2n+1} \right] \, dx \\ &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{2n+1} \int_0^1 x^{2n+4} \, dx \right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left[ \frac{1}{2n+5} \right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+5)} \left[ \frac{1}{2n+5} \right] \end{aligned}$$

NEED TO SUM THIS SERIES BY PARTIAL FRACTION

$$\frac{1}{(2n+1)(2n+5)} = \frac{A}{2n+1} - \frac{B}{2n+5} \quad (\text{BY ASPECTR})$$

THIS WE NOW HAVE

$$\int_0^1 x^3 \arctan x \, dx = \frac{1}{4} \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n+5} \right]$$

INVERSE THE PARENTHESIS

- $n=0$   $\frac{1}{1} - \frac{1}{5}$
- $n=1$   $\frac{1}{3} - \frac{1}{7}$
- $n=2$   $\frac{1}{5} - \frac{1}{9}$
- $n=3$   $\frac{1}{7} - \frac{1}{11}$
- $\vdots$
- $n=4$   $\frac{1}{9} - \frac{1}{13}$
- $n=5$   $\frac{1}{11} - \frac{1}{15}$

FINALLY WE HAVE THE RESULT

$$\begin{aligned} \int_0^1 x^3 \arctan x \, dx &= \frac{1}{4} \lim_{n \rightarrow \infty} \sum_{k=0}^n \left[ \frac{(-1)^k}{2k+1} - \frac{(-1)^k}{2k+5} \right] \\ &= \frac{1}{4} \lim_{n \rightarrow \infty} \left[ 1 - \frac{1}{3} - \frac{(-1)^n}{2n+3} - \frac{(-1)^n}{2n+5} \right] \\ &\quad \uparrow \quad \uparrow \\ &\quad \text{These tend to zero} \\ &= \frac{1}{4} \times \left( 1 - \frac{1}{3} \right) \\ &= \frac{1}{4} \times \frac{2}{3} \\ &= \frac{1}{6} \quad \text{As Expected} \end{aligned}$$

**Question 4** (\*\*\*\*\*)

It is given that

- ♦  $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{1}{4}\pi$
- ♦  $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots = \frac{1}{12}\pi^2$
- ♦  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln 2$

Assuming the following integral converges find its exact value.

$$\int_0^1 (\ln x)(\arctan x) dx.$$

[you may assume that integration and summation commute]

,  $\frac{1}{48}[\pi^2 - 12\pi + 24\ln 2]$

IT IS UNLIKELY THAT THIS INTEGRAL HAS A CLOSED FORM IN TERMS OF ELEMENTARY FUNCTIONS. NO INDEFINITE FORM---USE PARTS INSTEAD

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

INTEGRATING WITH RESPECT TO  $x$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + C$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad x > 0, C=0$$

YOU RETURNING TO THE INTEGRAL, A SMART APPROXIMATION AND SUMMATION

$$\begin{aligned} \int_0^1 (\arctan x)(\ln x) dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \times \ln x dx \\ &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{2n+1} \int_0^1 x^{2n+1} \ln x dx \right] \end{aligned}$$

INTEGRATION BY PARTS INSIDE THE SUM

$$\begin{aligned} \int_0^1 \ln x dx &\Big| \begin{array}{l} \frac{1}{x} \\ -x^2 \\ 2x^2 \\ x^4 \\ \vdots \\ 2^{2n} \\ 2^{2n+1} \end{array} &= \int_0^1 \frac{1}{x} \left[ \frac{(-1)^n x^{2n+1}}{2n+1} - \int_0^1 \frac{1}{x} x^{2n+1} dx \right] \\ &= \int_0^1 \frac{1}{x} \ln x \left[ \frac{(-1)^n x^{2n+1}}{2n+1} \right] dx \\ &= \int_0^1 \frac{1}{x} \ln x \left[ \frac{(-1)^n}{2n+1} x^{2n+1} \right] dx \end{aligned}$$

THIS IS A RECURSIVE SUM

$$\begin{aligned} &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{2n+1} \int_0^1 x^{2n+1} dx \right] \\ &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{2n+1} \cdot \frac{1}{2n+2} \right] \end{aligned}$$

SUMMING UP SO FAR

$$\int_0^1 (\arctan x)(\ln x) dx = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(2n+1)(2n+2)} \right] = -\frac{1}{2} \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(2n+1)(2n+2)} \right]$$

CERTAINLY SOME PARTIAL FRACTION

$$\frac{1}{(2n+1)(2n+2)} = \frac{A}{2n+1} + \frac{B}{2n+2}$$

$$1 = A(2n+2) + B(2n+1) + C(2n+1)^2$$

$$\bullet 1(2n+1) \quad \bullet 1(2n+2) \quad \bullet N(n+1)$$

$$1 = A + B \quad 1 = 2A + 3C \quad 1 = -2B + 4C$$

$$A = -1 \quad B = -\frac{1}{2} \quad C = \frac{1}{4}$$

$$B = -2$$

THIS WE KNOW HAVE

$$\begin{aligned} \int_0^1 (\arctan x)(\ln x) dx &= -\frac{1}{2} \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(2n+1)(2n+2)} + \frac{-2(-1)^n}{(2n+1)^2} + \frac{(-1)^n}{2n+1} \right] \\ &= -\frac{1}{2} \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(2n+1)^2} \right] + \frac{1}{2} \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{2n+1} \right] - \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{2n+1} \right] \end{aligned}$$

LOOKING AT THE RESULT (PNT)

$$\begin{aligned} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{1}{4}\pi \\ 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \frac{1}{12}\pi^2 \\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \ln 2 \end{aligned}$$

FINALLY WE HAVE

$$\begin{aligned} \int_0^1 (\arctan x)(\ln x) dx &= -\frac{1}{2} \left( \frac{\pi^2}{12} \right) + \frac{1}{2} \ln 2 - \frac{1}{4}\pi \\ &= \frac{\pi^2}{24} - \frac{1}{4}\pi + \frac{1}{2}\ln 2 \\ &= \frac{1}{48}[\pi^2 - 12\pi + 24\ln 2] \end{aligned}$$

**Question 5** (\*\*\*\*\*)

Show with detailed workings that

$$\sum_{r=1}^{\infty} \left[ \frac{2r+3}{(r+1)3^r} \right] = 3\ln\left(\frac{3}{2}\right).$$

V,  proof

SIMPLY MANIPULATING THE SUMMAND

$$\frac{2r+3}{3(r+1)} = \frac{2(r+1)+1}{3(r+1)} = \left(\frac{1}{3}\right)^r + \frac{1}{r+1} \left(\frac{1}{3}\right)^r$$

SPLIT THE SUM INTO TWO, AND CARRY THE SUM TO INFINITY OF THE G.P.

$$\sum_{r=1}^{\infty} \left[ \frac{2r+3}{3(r+1)} \right] = \sum_{r=1}^{\infty} \left[ \left(\frac{1}{3}\right)^r \right] + \sum_{r=1}^{\infty} \left[ \frac{1}{r+1} \left(\frac{1}{3}\right)^r \right]$$

$\uparrow$

G.P. with  $a = \frac{1}{3}$  }  $\Rightarrow S_\infty = \frac{\frac{1}{3}}{1-\frac{1}{3}} = 1$

FOR THE SECOND PART OF THE SUM CONSIDER  $\ln(1-x)$  AS A POWER SERIES (NOTE THAT  $\ln(1-x)$  HAS ASCENDING TERMS)

$$\begin{aligned} \ln(-x) &\approx -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \dots \\ -\frac{1}{3}\ln(-x) &= 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \frac{1}{5}x^4 \\ -\frac{1}{3}\ln(-x) &= 1 + \sum_{n=1}^{\infty} \left[ \frac{1}{n+1} x^n \right] \end{aligned}$$

LET  $x = -\frac{1}{3}$  AND NOTE THAT IS WITHIN THE RANGE OF CONVERGENCE

$$\begin{aligned} -\frac{1}{3}\ln\left(-\frac{1}{3}\right) &= 1 + \sum_{n=1}^{\infty} \left[ \frac{1}{n+1} \left(\frac{1}{3}\right)^n \right] \\ -1 - 3\ln\frac{2}{3} &= \sum_{n=1}^{\infty} \left[ \frac{1}{n+1} \left(\frac{1}{3}\right)^n \right] \end{aligned}$$

PUTTING IT ALL TOGETHER

$$\sum_{r=1}^{\infty} \left[ \frac{2r+3}{3(r+1)} \right] = \sum_{r=1}^{\infty} \left[ \left(\frac{1}{3}\right)^r \right] + \sum_{r=1}^{\infty} \left[ \frac{1}{r+1} \left(\frac{1}{3}\right)^r \right] = 1 + \left( -1 - 3\ln\frac{2}{3} \right) = 3\ln\frac{2}{3}$$

**Question 6 (\*\*\*\*\*)**

By considering the series expansions of  $\ln(1-x^2)$  and  $\ln\left(\frac{1+x}{1-x}\right)$ , or otherwise, find the exact value of the following series.

$$\sum_{r=1}^{\infty} \left[ \left( \frac{1}{2r} + \frac{1}{2r+1} \right) \left( \frac{1}{4} \right)^r \right].$$

,   $-1 + \frac{1}{2} \ln 12$

SIMPLY WITH SUGGESTION GIVEN

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad |x| < 1$   
 $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \quad |x| < 1$

$\ln(1-x^2) = -x^2 - \frac{x^3}{2} - \frac{x^4}{3} - \frac{x^5}{4} - \dots$

$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$   
 $= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$   
 $\frac{2}{2} + \frac{3}{3}x^3 + \frac{4}{4}x^5 + \dots$

$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{3}{2}x^3 + \frac{5}{4}x^5 + \frac{7}{8}x^7 + \dots$

NOW LOOKING AT THE FIRST FEW TERMS OF OUR SERIES

$(\frac{1}{2} + \frac{1}{3})(\frac{1}{2}) + (\frac{1}{3} + \frac{1}{4})(\frac{1}{3}) + (\frac{1}{4} + \frac{1}{5})(\frac{1}{4}) + \dots$   
 $= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{4} + \dots$   
 $= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{3} + \dots \quad \leftarrow \text{looks like } \ln(1-x)$   
 $= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{3} + \dots \quad \leftarrow \text{looks like } \ln\left(\frac{1+x}{1-x}\right)$

PROCEEDED AS FOLLOWS LOOKING AT THE EXPANSION OF  $\ln(1-x)$

$-\frac{1}{2} \ln(1-x) = -\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$   
 $-\frac{1}{2} \ln\left[1 - (-\frac{1}{2})\right] = -\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{3} + \dots$   
 $-\frac{1}{2} \ln\left(\frac{3}{2}\right) = \frac{1}{2} \times \frac{1}{2} + \frac{3}{2} \times \frac{1}{3} + \frac{5}{4} \times \frac{1}{2} + \frac{7}{8} \times \frac{1}{3} + \dots$

LOOKING AT THE REST OF THE SERIES - COMPARE WITH  $\ln\left(\frac{1+x}{1-x}\right)$

$\Rightarrow \frac{1}{2x} \ln\left(\frac{1+x}{1-x}\right) = 1 + \frac{1}{3}x^2 + \frac{5}{9}x^4 + \frac{35}{81}x^6 + \frac{315}{729}x^8$   
 $\Rightarrow \frac{1}{2x} \ln\left(\frac{1+x}{1-x}\right) = 1 + \underbrace{\frac{1}{3}x^2 + \frac{5}{9}x^4 + \frac{35}{81}x^6}_{\text{"our series"}}$   
 $\Rightarrow \ln\left(\frac{3}{2}\right) - 1 = \text{"our series"}$

COLLECTING THE RESULTS

$\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{3} + \dots = -\frac{1}{2} \ln\left(\frac{3}{2}\right)$   
 $\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{3} + \dots = 1 \ln\left(\frac{3}{2}\right) - 1$   
 $(\frac{1}{2} + \frac{1}{3})\frac{1}{2} + (\frac{1}{3} + \frac{1}{4})\frac{1}{3} + (\frac{1}{4} + \frac{1}{5})\frac{1}{4} + \dots = -\frac{1}{2} \ln\left(\frac{3}{2}\right) + \ln 3 - 1$   
 $\sum_{n=1}^{\infty} \left[ \left( \frac{1}{2n} + \frac{1}{2n+1} \right) \left( \frac{1}{2n} \right) \right] = \frac{1}{2} [2 \ln 3 - \ln\left(\frac{3}{2}\right)] - 1$   
 $\sum_{n=1}^{\infty} \left[ \left( \frac{1}{2n} + \frac{1}{2n+1} \right) \left( \frac{1}{2n} \right)^2 \right] = \frac{1}{2} [\ln 3 + \ln\left(\frac{3}{2}\right)] - 1$   
 $\sum_{n=1}^{\infty} \left[ \left( \frac{1}{2n} + \frac{1}{2n+1} \right) \left( \frac{1}{2n} \right)^3 \right] = \frac{1}{2} \ln 2 - 1$

**Question 7** (\*\*\*\*\*)

Find the sum to infinity of the following series.

$$\frac{1}{1} - \frac{1}{1+4} + \frac{1}{1+4+9} - \frac{1}{1+4+9+16} + \frac{1}{1+4+9+16+25} + \dots$$

You may find the series expansion of  $\arctan x$  useful in this question.

,  $6(\pi - 3)$

WRITE THE SERIES IN COMPACT<sup>3</sup> NOTATION

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(1+4+9+\dots+n^2)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\frac{1}{4}n(2n+1)}$$

IGNORE THE  $\frac{1}{4}$  TERM & SPLIT THE REST INTO PARTIAL FRACTIONS BY INSPECT

$$\frac{1}{n(2n+1)} = \frac{A}{n} + \frac{B(2n+1)}{2n+1} = \frac{1}{n} + \frac{1}{2n+1} - \frac{1}{2n+1}$$

HENCE WE HAVE

$$\dots = \sum_{n=1}^{\infty} (-1)^{n+1} \left[ \frac{1}{n} + \frac{1}{2n+1} - \frac{1}{2n+1} \right]$$

$$= 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} - 24 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

NEXT CONSIDER EACH TERM OF THE SUMMATION SEPARATELY

- $6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 6 \left[ -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right] = 6G_2$  (from fact)
- $6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} = 6 \left[ \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \dots \right]$ 
 $= -6 \left[ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots \right]$ 
 $= -6 \left[ \frac{1}{2}(1+2+3+4+5+6+7) \right]$ 
 $= 6 - 6G_2$

NEXT CONSIDER THE SERIES EXPANSION OF ARCTAN

$$\Rightarrow \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

$$\Rightarrow \int \frac{1}{1+x^2} dx = x - \frac{1}{2}x^3 + \frac{1}{4}x^5 - \frac{1}{6}x^7 + \dots + C$$

$$\Rightarrow \arctan x = C + x - \frac{1}{2}x^3 + \frac{1}{4}x^5 - \frac{1}{6}x^7 + \dots$$
 $\text{let } x=0 \rightarrow C=0$ 
 $\Rightarrow \arctan x = x - \frac{1}{2}x^3 + \frac{1}{4}x^5 - \frac{1}{6}x^7 + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$ 
 $\Rightarrow \arctan 1 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ 
 $\Rightarrow \frac{1}{\pi} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ 
 $\Rightarrow \pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ 
 $\Rightarrow G_1 = 2 \pi \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ 
 $\Rightarrow G_1 = 2\pi \left[ 1 + \sum_{n=2}^{\infty} \frac{(-1)^n}{2n-1} \right]$ 
 $\Rightarrow G_1 = 2\pi + 24 \sum_{n=2}^{\infty} \frac{(-1)^n}{2n-1}$ 
 $\Rightarrow 24 \sum_{n=2}^{\infty} \frac{(-1)^n}{2n-1} = G_1 - 2\pi$ 

FINALLY COLLECTING ALL THE PARTS

$$\frac{1}{1} - \frac{1}{1+4} + \frac{1}{1+4+9} - \frac{1}{1+4+9+16} + \dots$$
 $= 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} - 24 \sum_{n=2}^{\infty} \frac{(-1)^n}{2n-1}$ 
 $= 6G_2 + (G - 4\pi) + (G_1 - 2\pi)$ 
 $= G_1 - 16 = G(\pi - 3)$

**Question 8** (\*\*\*\*\*)

Find the sum to infinity of the following series.

$$1 + \frac{1}{3 \times 4} + \frac{1}{5 \times 4^2} + \frac{1}{7 \times 4^3} + \frac{1}{9 \times 4^4} + \dots$$

 ,  $\ln 3$

Method A — Using Series Expansions

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^4)$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^4)$$

SUBTRACTING THE EXPANSIONS (WE OBTAIN)

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + O(x^7)$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2 \sum_{k=0}^{\infty} \left( \frac{x^{2k+1}}{(2k+1)} \right)$$

Now (from the radius of convergence), let  $x = \frac{1}{2}$

$$\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = 2 \sum_{k=0}^{\infty} \left( \frac{(\frac{1}{2})^{2k+1}}{(2k+1)} \right)$$

$$\ln\left(\frac{3}{2}\right) = 2 \sum_{k=0}^{\infty} \left[ \frac{1}{(2k+1)2^{2k+1}} \right]$$

$$\ln 3 = \sum_{k=0}^{\infty} \frac{2}{(2k+1)2^{2k+1}}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)2^{2k+1}} = \ln 3$$

$$\therefore \sum_{k=0}^{\infty} \frac{1}{(2k+1)2^{2k+1}} = \ln 3$$

$$\therefore 1 + \frac{1}{3 \times 4} + \frac{1}{5 \times 4^2} + \frac{1}{7 \times 4^3} + \frac{1}{9 \times 4^4} + \dots = \ln 3$$

Method B — Algebraic Techniques

$$\int_0^{\frac{1}{2}} x^{\infty} dx = \left[ \frac{1}{2k+1} x^{2k+1} \right]_0^{\frac{1}{2}} = \frac{1}{2k+1} \left[ \left( \frac{1}{2} \right)^{2k+1} - 0 \right] = \frac{1}{(2k+1)2^{2k+1}}$$

$$= \frac{1}{(2k+1)2^{2k+1}} = \frac{1}{2} \frac{1}{(2k+1)4^k}$$

Now consider the infinite sum (from A)

$$1 + \frac{1}{3 \times 4} + \frac{1}{5 \times 4^2} + \frac{1}{7 \times 4^3} + \dots = \sum_{k=0}^{\infty} \left[ \frac{1}{(2k+1)4^k} \right]$$

$$= 2 \sum_{k=0}^{\infty} \left[ \frac{1}{(2k+1)4^k} \right] = 2 \sum_{k=0}^{\infty} \left[ \frac{1}{2(2k+1)4^k} \right] = 2 \sum_{k=0}^{\infty} \left[ \frac{1}{2} \frac{1}{(2k+1)2^{2k+1}} \right]$$

INTEGRATE BY SUBSTITUTION & INTEGRATION BY PARTS

$$\dots = 2 \int_0^{\frac{1}{2}} \left[ \sum_{k=0}^{\infty} 2^{-2k} \right] dx = 2 \int_0^{\frac{1}{2}} \left[ 1 + 2^2 + 2^4 + 2^6 + \dots \right] dx$$

$$= 2 \int_0^{\frac{1}{2}} \frac{1}{1-2^2} dx = \int_0^{\frac{1}{2}} \frac{2}{(1-2^2)(1+2)} dx$$

$$= \int_0^{\frac{1}{2}} \frac{1}{1-2^2} + \frac{1}{1+2} dx = \left[ \ln|1+2| - \ln|1-2| \right]_0^{\frac{1}{2}}$$

$$= \left( \ln\frac{3}{2} - \ln\frac{1}{2} \right) - \left( \ln 1 - \ln 1 \right) = \ln\frac{3}{2} = \ln 3$$

**Question 9** (\*\*\*\*\*)

Given that  $p$  and  $q$  are positive, show that the natural logarithm of their arithmetic mean exceeds the arithmetic mean of their natural logarithms by

$$\sum_{r=1}^{\infty} \left[ \frac{2}{2r-1} \left( \frac{\sqrt{p} - \sqrt{q}}{\sqrt{p} + \sqrt{q}} \right)^{4r-2} \right].$$

You may find the series expansion of  $\operatorname{artanh}(x^2)$  useful in this question.

[ ] , proof

**SUMMING FROM THE SERIES EXPANSION OF  $\operatorname{artanh}x$  IN LOG FORM**

$$\Rightarrow \operatorname{artanh}x > \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2} [\ln(1+x) - \ln(1-x)]$$

$$\Rightarrow \operatorname{artanh}x = \frac{1}{2} \left[ \left( x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \dots \right) - \left( x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \dots \right) \right]$$

$$\Rightarrow \operatorname{artanh}x = \frac{1}{2} \left[ 2x - \frac{2x^3}{3} + \frac{2x^5}{5} - \frac{2x^7}{7} + \dots \right]$$

$$\Rightarrow \operatorname{artanh}x = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \dots$$

$$\Rightarrow \operatorname{artanh}(x^2) = x^2 + \frac{1}{3}x^6 + \frac{1}{5}x^8 + \frac{1}{7}x^{10} + \dots$$

$$\therefore \operatorname{artanh}(x^2) = \sum_{r=1}^{\infty} \left[ \frac{2}{2r-1} \left( \frac{\sqrt{p} - \sqrt{q}}{\sqrt{p} + \sqrt{q}} \right)^{4r-2} \right] = \frac{1}{2} \ln\left(\frac{1+\sqrt{pq}}{1-\sqrt{pq}}\right)$$

**NEXT LET**  $x = \frac{\sqrt{p} - \sqrt{q}}{\sqrt{p} + \sqrt{q}}$  **IN THE ARGUMENT OF THE LOG ABOVE.**

$$\Rightarrow \frac{1+\sqrt{pq}}{1-\sqrt{pq}} = 1 + \frac{(\sqrt{p} + \sqrt{q})^2}{1 - (\sqrt{p} - \sqrt{q})^2} \quad \text{REDUCE TOP & BOTTOM OF THE FRACTION BY }$$

$$(1-x^2) = \frac{(\sqrt{p} + \sqrt{q})^2 + (\sqrt{p} - \sqrt{q})^2}{(\sqrt{p} + \sqrt{q})^2 - (\sqrt{p} - \sqrt{q})^2}$$

$$1-x^2 = \frac{p+2\sqrt{pq}+q + p-2\sqrt{pq}+q}{p+2\sqrt{pq}-q - p+2\sqrt{pq}+q}$$

$$\frac{1-x^2}{1-x^2} = \frac{2p+2q}{4\sqrt{pq}} = \frac{p+q}{2\sqrt{pq}}$$

**POTTING ALL THE RESULTS TOGETHER**

$$\sum_{r=1}^{\infty} \left[ \frac{2}{2r-1} \right] = \frac{1}{2} \ln\left(\frac{1+\sqrt{pq}}{1-\sqrt{pq}}\right)$$

$$\sum_{r=1}^{\infty} \left[ \frac{1}{2r-1} \left( \frac{\sqrt{p} - \sqrt{q}}{\sqrt{p} + \sqrt{q}} \right)^{4r-2} \right] = \frac{1}{2} \ln\left(\frac{p+q}{2\sqrt{pq}}\right)$$

$$2 \sum_{r=1}^{\infty} \left[ \frac{1}{2r-1} \left( \frac{\sqrt{p} - \sqrt{q}}{\sqrt{p} + \sqrt{q}} \right)^{4r-2} \right] = \ln\left[\frac{p+q}{2\sqrt{pq}}\right]$$

$$\sum_{r=1}^{\infty} \left[ \frac{2}{2r-1} \left( \frac{\sqrt{p} - \sqrt{q}}{\sqrt{p} + \sqrt{q}} \right)^{4r-2} \right] = \ln\left(\frac{p+q}{2}\right) - \ln\sqrt{pq}^2$$

$$\sum_{r=1}^{\infty} \left[ \frac{2}{2r-1} \left( \frac{\sqrt{p} - \sqrt{q}}{\sqrt{p} + \sqrt{q}} \right)^{4r-2} \right] = \ln\left(\frac{p+q}{2}\right) - \frac{1}{2} \ln(pq)$$

THUS WE FINALLY HAVE THE DESIRED RESULT

$$\ln\left(\frac{p+q}{2}\right) - \frac{\ln p + \ln q}{2} = \sum_{r=1}^{\infty} \left[ \frac{2}{2r-1} \left( \frac{\sqrt{p} - \sqrt{q}}{\sqrt{p} + \sqrt{q}} \right)^{4r-2} \right]$$

# **TAYLOR SERIES EXPANSIONS 4 BASIC QUESTIONS**

**Question 1** (\*\*\*)

$$y = \frac{1}{\sqrt{x}}, x > 0$$

- a) Find the first four terms in the Taylor expansion of  $y$  about  $x=1$ .
- b) Use the first three terms of the expansion found in part (a), with  $x = \frac{8}{9}$  to show that  $\sqrt{2} \approx \frac{229}{162}$ .

	$\boxed{\quad}, \quad y = 1 - \frac{1}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{5}{16}(x-1)^3 + O((x-1)^4)$
--	---

**a) Obtain the first three derivatives of  $y = x^{-\frac{1}{2}}$**

$$y' = -\frac{1}{2}x^{-\frac{3}{2}}, \quad y'' = \frac{3}{4}x^{-\frac{5}{2}}, \quad y''' = -\frac{15}{8}x^{-\frac{7}{2}}$$

EVALUATE AT  $x=1$

$$y|_{x=1}, \quad y'|_{x=1} = -\frac{1}{2}, \quad y''|_{x=1} = \frac{3}{4}, \quad y'''|_{x=1} = -\frac{15}{8}$$

BY THE TAYLOR FORMULA

$$\begin{aligned} y &= y_0 + (x-a)y'_0 + \frac{(x-a)^2}{2!}y''_0 + \frac{(x-a)^3}{3!}y'''_0 + O[(x-a)^4] \\ \frac{1}{\sqrt{x}} &= 1 - \frac{1}{2}(x-1) + \frac{1}{2}(x-1)^2 \left(\frac{3}{4}\right) + \frac{1}{6}(x-1)^3 \left(-\frac{15}{8}\right) + O[(x-1)^4] \\ \frac{1}{\sqrt{x}} &= 1 - \frac{1}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{5}{16}(x-1)^3 + O[(x-1)^4] \end{aligned}$$

**b) Now using the first three terms with  $a = \frac{8}{9}$**

$$\begin{aligned} \rightarrow \frac{1}{\sqrt{\frac{8}{9}}} &= 1 - \frac{1}{2}\left(\frac{8}{9}-1\right) + \frac{3}{8}\left(\frac{8}{9}-1\right)^2 + \dots \\ \rightarrow \frac{3}{4\sqrt{2}} &= 1 - \frac{1}{18} + \frac{3}{8}\left(\frac{1}{81}\right) + \dots \\ \rightarrow \frac{3\sqrt{2}}{162} &= 1 + \frac{1}{16} + \frac{1}{216} + \dots \\ \rightarrow \frac{3\sqrt{2}}{162} &= \frac{229}{162} + \dots \\ \Rightarrow \sqrt{2} &= \frac{229}{162} + \dots \quad \therefore \sqrt{2} \approx \frac{229}{162} \end{aligned}$$

**Question 2** (\*\*\*)

$$f(x) = x^2 \ln x, \quad x > 0$$

- a) Find the first three non zero terms in the Taylor expansion of  $f(x)$ , in powers of  $(x-1)$ .
- b) Use the first three terms of the expansion to show  $\ln 1.1 \approx 0.095$ .

$$\boxed{f(x) = (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + O((x-1)^4)}$$

a) START BY COMPUTING DERIVATIVES & THEIR EVALUATIONS AT  $x=1$ , AS THE EXPANSION IS IN POWERS OF  $(x-1)$

- $f(0) = x^2 \ln x$   
 $\frac{df}{dx} = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x$   
 $f'(0) = 2x \ln 1 + 1 = 1$
- $\frac{d^2f}{dx^2} = 2\ln x + 2x \frac{1}{x} + 1 = 2\ln x + 2 + 1 = 2\ln x + 3$   
 $\frac{f''(0)}{2!} = 2\ln 1 + 3 = 3$
- $\frac{d^3f}{dx^3} = \frac{2}{x}$   
 $f'''(0) = 2$

HENCE WE CAN DEFINE AN EXPANSION

$$\Rightarrow f(x) = f(1) + (x-1)\frac{f'(1)}{1!} + \frac{(x-1)^2}{2!}\frac{f''(1)}{2!} + \frac{(x-1)^3}{3!}\frac{f'''(1)}{3!} + \dots$$

$$\Rightarrow x^2 \ln x = 1 + (x-1) \times 1 + \frac{(x-1)^2}{2} \times 3 + \frac{(x-1)^3}{6} \times 2 + \dots$$

$$\Rightarrow x^2 \ln x = (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \dots$$

b) LET  $x=1.1$  IN THE ABOVE EXPANSION GIVES

$$\Rightarrow (1.1)^2 \ln(1.1) \approx (0.1) + \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3$$

$$\Rightarrow 1.21 \ln(1.1) \approx \frac{1.13}{1500}$$

$$\Rightarrow \ln(1.1) \approx \frac{1.13}{1500} \text{ or } 0.095$$

**Question 3    (\*\*\*)**

$$f(x) = \cos 2x.$$

- a) Find the first three non zero terms in the Taylor expansion of  $f(x)$ , in powers

$$\text{of } \left(x - \frac{\pi}{4}\right).$$

- b) Use the first three terms of the expansion to show  $\cos 2 \approx -0.416$ .

$$\boxed{\quad}, \quad f(x) = -2\left(x - \frac{\pi}{4}\right) + \frac{4}{3}\left(x - \frac{\pi}{4}\right)^3 - \frac{4}{15}\left(x - \frac{\pi}{4}\right)^5 + O\left(\left(x - \frac{\pi}{4}\right)^7\right)$$

**a) DIFFERENTIATE A GRADUATE INTEGRAL AT  $x = \frac{\pi}{4}$**

$f(x) = \cos 2x$	$f(x) \approx 0$
$f'(x) = -2\sin 2x$	$f'(\frac{\pi}{4}) = -2$
$f''(x) = -4\cos 2x$	$f''(\frac{\pi}{4}) = 0$
$f'''(x) = 8\sin 2x$	$f'''(\frac{\pi}{4}) = 8$
$f^{(4)}(x) = -16\cos 2x$	$f^{(4)}(\frac{\pi}{4}) = 0$
$f^{(5)}(x) = 32\sin 2x$	$f^{(5)}(\frac{\pi}{4}) = -32$

**USING TAYLOR THEOREM**

$$f(x) = f(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})}{1!} f'(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^2}{2!} f''(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^3}{3!} f'''(\frac{\pi}{4}) + \dots$$

$$\cos 2x = -2(x - \frac{\pi}{4}) + \frac{8}{3}(x - \frac{\pi}{4})^3 - \frac{32}{3!}(x - \frac{\pi}{4})^5 + O((x - \frac{\pi}{4})^7)$$

$$\cos 2x = -2(x - \frac{\pi}{4}) + \frac{8}{3}(x - \frac{\pi}{4})^3 - \frac{32}{3!}(x - \frac{\pi}{4})^5 + O((x - \frac{\pi}{4})^7)$$

**b) LETTING  $x=1$  IN THE ABOVE EXPANSION AND OBTAIN**

$$\Rightarrow \cos 2 \approx -2(1 - \frac{\pi}{4}) + \frac{8}{3}(1 - \frac{\pi}{4})^3 - \frac{32}{3!}(1 - \frac{\pi}{4})^5$$

$$\Rightarrow \cos 2 \approx -0.41647368\dots$$

$$\Rightarrow \cos 2 \approx \underline{-0.416}$$

As required

**Question 4**    (\*\*\*)

$$f(x) = \cos x.$$

- a) Find the first four terms in the Taylor expansion of  $f(x)$ , in ascending powers

$$\text{of } \left( x - \frac{\pi}{6} \right).$$

- b) Use the expansion of part (a) to show that

$$\cos \frac{\pi}{4} \approx \frac{\sqrt{3}}{2} - \frac{\pi}{24} - \frac{\sqrt{3}\pi^2}{576} - \frac{\pi^3}{20736}.$$

$$\boxed{\quad}, \quad f(x) = \frac{\sqrt{3}}{2} - \frac{1}{2} \left( x - \frac{\pi}{6} \right) - \frac{\sqrt{3}}{4} \left( x - \frac{\pi}{6} \right)^2 + \frac{1}{12} \left( x - \frac{\pi}{6} \right)^3 + O\left( \left( x - \frac{\pi}{6} \right)^4 \right)$$

a) FIND THE FIRST 3 DERIVATIVES OF  $\cos x$

$$f(x) = -\sin x$$

$$f'(x) = -\cos x$$

$$f''(x) = \sin x$$

SUMMARISE THE FUNCTION AND ITS DERIVATIVES AT  $x = \frac{\pi}{4}$

$$f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \quad f'(\frac{\pi}{4}) = -\frac{1}{2}, \quad f''(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \quad f'''(\frac{\pi}{4}) = \frac{1}{2}$$

BY THE TAYLOR THEOREM

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + O[(x-a)^4]$$

$$\cos x = \frac{\sqrt{2}}{2} + (x-\frac{\pi}{4}) \left( -\frac{1}{2} \right) + \frac{(x-\frac{\pi}{4})^2}{2} \left( \frac{\sqrt{2}}{2} \right) + \frac{(x-\frac{\pi}{4})^3}{6} \left( \frac{1}{2} \right) + O[(x-\frac{\pi}{4})^4]$$

$$\cos x = \frac{\sqrt{2}}{2} - \frac{1}{2}(x-\frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2 + \frac{1}{12}(x-\frac{\pi}{4})^3 + O[(x-\frac{\pi}{4})^4]$$

Let  $a = \frac{\pi}{4}$ , so THAT  $(x - \frac{\pi}{4}) = \frac{x\pi}{4}$

$$\Rightarrow \cos \frac{x\pi}{4} = \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{x\pi}{4} - \frac{\sqrt{2}}{4} \left( \frac{x\pi}{4} \right)^2 + \frac{1}{12} \left( \frac{x\pi}{4} \right)^3 + \dots$$

$$\Rightarrow \cos \frac{x\pi}{4} = \frac{\sqrt{2}}{2} - \frac{x\pi}{8} - \frac{\sqrt{2}\pi^2}{32} + \frac{x^3\pi^3}{3072} + \dots$$

\* APPROX

# **TAYLOR SERIES EXPANSIONS 3 STANDARD QUESTIONS**

**Question 1** (\*\*\*)+

$$f(x) \equiv \sin 2x, \quad x \in \mathbb{R}.$$

- a) Determine, in exact simplified form, the first 3 non zero terms, in the Taylor expansion of  $f(x)$ , centred at  $x = \frac{1}{4}\pi$ .
- b) Write the entire expansion of  $f(x)$ , as a simplified expression in  $\Sigma$  notation.

$$\boxed{\quad}, \quad f(x) = 1 - 2\left(x - \frac{1}{4}\pi\right)^2 + \frac{2}{3}\left(x - \frac{1}{4}\pi\right)^4 + \dots$$

$$f(x) = \sum_{r=0}^{\infty} \left[ \frac{(-4)^r}{(2r)!} \left(x - \frac{1}{4}\pi\right)^{2r} \right]$$

**a) START BY DIFFERENTIATION & EVALUATION AT  $x = \frac{1}{4}\pi$**

$f(x) = \sin 2x$	$f'(x) \approx 1$
$f'(x) = 2\cos 2x$	$f''(x) = 0$
$f''(x) = -4\sin 2x$	$f'''(x) \approx -4$
$f'''(x) = -8\cos 2x$	$f^{(4)}(x) = 0$
$f^{(4)}(x) = 16\sin 2x$	$f^{(5)}(x) \approx 16$

**Taylor MC  $\rightarrow$   $x = \frac{1}{4}\pi$**

$$f(x) = f\left(\frac{1}{4}\pi\right) + (x - \frac{1}{4}\pi)f'\left(\frac{1}{4}\pi\right) + (x - \frac{1}{4}\pi)^2 \frac{f''\left(\frac{1}{4}\pi\right)}{2!} + (x - \frac{1}{4}\pi)^3 \frac{f'''\left(\frac{1}{4}\pi\right)}{3!} + \dots$$

$$\sin 2x = 1 - \frac{4}{2!}(x - \frac{1}{4}\pi)^2 + \frac{16}{4!}(x - \frac{1}{4}\pi)^4 + \dots$$

$$\sin 2x = 1 - 2(x - \frac{1}{4}\pi)^2 + 8(x - \frac{1}{4}\pi)^4 + \dots$$

**b) LOOKING AT THE PATTERN OF THE INDIVIDUALS ONE SIMPLIFIED**

$$\begin{array}{ccccccc} 1, & 0, & -4, & 0, & 16, & 0, & -64, \\ (x-\frac{1}{4}\pi), & x(\frac{1}{4}\pi), & (x-\frac{1}{4}\pi)^2, & x(\frac{1}{4}\pi)^3, & (x-\frac{1}{4}\pi)^4, & x(\frac{1}{4}\pi)^5, & (x-\frac{1}{4}\pi)^6 \end{array}$$

$$\therefore \sin 2x = \sum_{r=0}^{\infty} \left[ \frac{(-4)^r}{(2r)!} (x - \frac{1}{4}\pi)^{2r} \right]$$

**Question 2** (\*\*\*\*)

$$y = \tan x .$$

a) Show that

$$\frac{d^3 y}{dx^3} = 2y \frac{d^2 y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2.$$

- b) Determine the first four terms in the Taylor expansion of  $\tan x$ , in ascending powers of  $\left(x - \frac{\pi}{4}\right)$ .
- c) Hence deduce that

$$\tan \frac{5\pi}{18} \approx 1 + \frac{\pi}{18} + \frac{\pi^2}{648} + \frac{\pi^3}{17496} .$$

$$\boxed{\quad}, \quad y = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + O\left(\left(x - \frac{\pi}{4}\right)^4\right)$$

a) NOTING THAT  $1 + \tan^2 x = \sec^2 x$  we have

$$\begin{aligned} y &= \tan x \\ \frac{dy}{dx} &= \sec^2 x \\ \frac{dy}{dx} &= 1 + \tan^2 x \\ \frac{dy}{dx} &= 1 + y^2 \end{aligned}$$

DIFFERENTIATE AGAIN WITH RESPECT TO x

$$\begin{aligned} \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}\left(1 + y^2\right) \\ \frac{dy^2}{dx^2} &= 0 + 2y \frac{dy}{dx} \end{aligned}$$

DIFFERENTIATE WITH RESPECT TO x ONCE MORE

$$\begin{aligned} \frac{d}{dx}\left(\frac{dy^2}{dx^2}\right) &= \frac{d}{dx}\left(2y \frac{dy}{dx}\right) \leftarrow \text{PRODUCT RULE} \\ \frac{dy^3}{dx^3} &= 2y \times \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{d}{dx}(2y) \times \frac{dy}{dx} \\ \frac{dy^3}{dx^3} &= 2y \frac{dy^2}{dx^2} + 2 \frac{dy}{dx} \cdot \frac{dy}{dx} \\ \frac{dy^3}{dx^3} &= 2y \frac{dy^2}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 \end{aligned}$$

AS REQUIRED

b) SIMPLY AT  $x = \frac{\pi}{4}$

$$\begin{aligned} \frac{dy}{dx} &= 1 + \tan^2 \frac{\pi}{4} = 1 + 1 = 2 \\ \frac{dy^2}{dx^2} &= 2y \frac{dy}{dx} = 2 \times 1 \times 2 = 4 \\ \frac{dy^3}{dx^3} &= 2y \frac{dy^2}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 2 \times 1 \times 4 + 2 \times 2^2 = 8 + 8 = 16 \end{aligned}$$

FROM WE KNOW HAVE

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots \\ \tan x &= 1 + (x-\frac{\pi}{4}) \times 2 + \frac{(x-\frac{\pi}{4})^2}{2!} \times 4 + \frac{(x-\frac{\pi}{4})^3}{3!} \times 16 + \dots \\ \tan x &= 1 + 2(x-\frac{\pi}{4}) + 2(x-\frac{\pi}{4})^2 + \frac{8}{3}(x-\frac{\pi}{4})^3 + \dots \end{aligned}$$

LET  $x = \frac{5\pi}{18}$  IN THE ABOVE EXPANSION

$$\begin{aligned} \text{FIRST } \frac{dy}{dx} &= \frac{dy}{dx} \\ \therefore \frac{dy}{dx} &\approx 1 + 2 \times \frac{\pi}{18} + 2 \times \left(\frac{\pi}{18}\right)^2 + \frac{8}{3} \left(\frac{\pi}{18}\right)^3 \\ \frac{dy}{dx} &\approx 1 + \frac{\pi}{9} + \frac{\pi^2}{162} + \frac{\pi^3}{17496} \end{aligned}$$

AS REQUIRED

**Question 3** (\*\*\*\*)

$$y = \tan^2 x.$$

a) Show that

$$\frac{d^4 y}{dx^4} = 120 \sec^6 x - 120 \sec^4 x + 16 \sec^2 x.$$

- b) Determine the first 5 terms in the Taylor expansion of  $\tan^2 x$ , in ascending powers of  $\left(x - \frac{\pi}{3}\right)$ .

**V** ,  ,

$$y = 3 + 8\sqrt{3}\left(x - \frac{\pi}{3}\right) + 40\left(x - \frac{\pi}{3}\right)^2 + \frac{176}{3}\left(x - \frac{\pi}{3}\right)^3 + \frac{728}{3}\left(x - \frac{\pi}{3}\right)^4 + O\left(\left(x - \frac{\pi}{3}\right)^5\right)$$

**a) START WITH DIFFERENTIATIONS**

•  $y = \tan^2 x = \sec^2 x - 1$

•  $\frac{dy}{dx} = 2 \sec x (\sec x \tan x) = 2 \sec^2 x \tan x$

•  $\frac{d^2 y}{dx^2} = 4 \sec x (\sec x + \tan x) \tan x + 2 \sec^2 x \sec^2 x$   
 $= 4 \sec^2 x \tan^2 x + 2 \sec^3 x$   
 $= 4 \sec^2 x (\sec^2 x - 1) + 2 \sec^3 x$   
 $= 4 \sec^2 x - 4 \sec^4 x + 2 \sec^3 x$   
 $= 2 \sec^3 x - 4 \sec^4 x$

•  $\frac{d^3 y}{dx^3} = 2 \sec x (2 \sec^2 x) \tan x - \sec x (3 \sec^2 x) \sec x$   
 $= 2 \sec^3 x \tan x - \sec^4 x \tan x$

•  $\frac{d^4 y}{dx^4} = 6 \sec x (\sec x + \tan x) \tan x + 4 \sec^2 x (-\sec x (\sec x + \tan x)) \sec x$   
 $= 6 \sec^3 x \tan x + 2 \sec^3 x - 4 \sec^3 x \tan x - \sec^4 x$   
 $= 10 \sec^3 x (\sec^2 x - 1) + 2 \sec^3 x - 4 \sec^3 x (\sec x - 1) - \sec^4 x$   
 $= 10 \sec^5 x - 10 \sec^3 x + 2 \sec^4 x - 4 \sec^3 x - \sec^4 x$   
 $= 12 \sec^5 x - (12 \sec^3 x + 6 \sec^2 x)$  AS REQUIRED

SUMMARISE THESE AT  $x = 3$ , SO  $\tan^2 3 = \frac{\pi^2}{9} + \sec^2 3 = 2$

$\bullet y = 3 + 8\sqrt{3}(x - \frac{\pi}{3}) + \frac{f''(3)}{2!}(x - \frac{\pi}{3})^2 + \frac{f'''(3)}{3!}(x - \frac{\pi}{3})^3 + \frac{f''''(3)}{4!}(x - \frac{\pi}{3})^4 + \dots$

$\tan^2 3 = 3 + 8\sqrt{3}(x - \frac{\pi}{3}) + 40(x - \frac{\pi}{3})^2 + \frac{176}{3}(x - \frac{\pi}{3})^3 + \frac{728}{3}(x - \frac{\pi}{3})^4 + \dots$

•  $\frac{d^4 y}{dx^4} = 120 \sec^2 x - 120 \sec^4 x + 16 \sec^2 x$   
 $= 7680 - 1440 + 64$   
 $= 6244$

**APPLYING TAYLOR'S THEOREM**

$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$

$\tan^2 x = 3 + 8\sqrt{3}(x - \frac{\pi}{3}) + 40(x - \frac{\pi}{3})^2 + \frac{176}{3}(x - \frac{\pi}{3})^3 + \frac{728}{3}(x - \frac{\pi}{3})^4 + \dots$

**O.D.E.**

**TAYLOR SERIES  
EXPANSIONS**

**3 BASIC**

**QUESTIONS**

**Question 1** (\*\*+)

A curve has equation  $y = f(x)$  which satisfies the differential equation

$$\frac{dy}{dx} = x^2 - y^2,$$

subject to the condition  $x = 0, y = 2$ .

Determine the first 4 terms in the infinite series expansion of  $y = f(x)$  in ascending powers of  $x$ .

,  $y = 2 - 4x + 8x^2 - \frac{47}{3}x^3 + O(x^4)$

DIFFERENTIATE THE O.D.E IN SUCCESSION AND SUBSTITUTE THE DERIVATIVES AT $x=0$	
DIFFERENTIATIONS	EVALUATIONS
$y' = x^2 - y^2$	$y_0 = 2$ (Given)
$y'' = 2x - 2yy'$	$y_1' = x^2 - y_0^2$ $y_1'' = 0 - 2y_0^2$ $y_1 = f$
$y''' = 2 - 2yy' - 2y'y''$	$y_2' = 2x - 2y_1y_1'$ $y_2'' = 2x_0 - 2x_0y_1^2$ $y_2 = 16$
$y^{(4)} = 2 - 2y_1y_1' - 2y_1^2y_1''$	$y_3' = 2 - 2y_2y_2' - 2y_2^2y_2''$ $y_3'' = 2 - 2y_3y_3' - 2y_3^2y_3''$ $y_3 = 24$

  

EXPANDING AS A POWER SERIES	
$y = y_0 + xy_1' + \frac{x^2}{2!}y_2'' + \frac{x^3}{3!}y_3''' + O(x^4)$	
$y = 2 + x(-4) + \frac{x^2}{2}(16) + \frac{x^3}{3}(-48) + O(x^4)$	
$y = 2 - 4x + 8x^2 - \frac{47}{3}x^3 + O(x^4)$	

**Question 2 (\*\*\*)**

A curve has an equation  $y = f(x)$  that satisfies the differential equation

$$y \frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^2 + xy = 0,$$

subject to the conditions  $x = 0, y = 1, \frac{dy}{dx} = 1$ .

By using the first four terms in the expansion of  $y = f(x)$  in ascending powers of  $x$ , show that  $y \approx 1.08$  at  $x = \frac{1}{12}$ .

proof

WEIT RELATIONSHIP IN "COMPACT" NOTATION

$\bullet$   $yy'' + (y')^2 + xy = 0$

$x=0$
$y=1$
$y'=1$

$yy'' + (y')^2 + 0xy = 0$

$1xy' + 1^2 = 0$

$y= -1$
---------

$\bullet$  DIFFERENTIATE O.D.E w.r.t  $x$

$yy'' + yy''' + 2y(y')^2 + y + xy' = 0$

$y'y'' + y_0y'' + 2y_0y' + y_0 + 0xy' = 0$

$1x(-1) + 1xy' + 2x(-1) + 1 = 0$

$-1 + y'' - 2 + 1 = 0$

$y_0 = 2$
-----------

$y = y_0 + xy'_0 + \frac{2^2}{2!} y''_0 + \frac{3^2}{3!} y'''_0 + \dots$

$y = 1 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$

MATCH  $2 = \frac{1}{12}$

$y = 1 + \frac{1}{12} - \frac{1}{288} + \frac{1}{5184} \dots$

$y \approx \frac{5593}{5184}$

$y \approx 1.08$

**Question 3    (\*\*\*)**

A curve has an equation  $y = f(x)$  that satisfies the differential equation

$$x \frac{dy}{dx} - y^2 = 3, \quad x \neq 0,$$

subject to the condition  $y = 2$  at  $x = 1$ .

Find the first four terms in the expansion of  $y = f(x)$  as powers of  $(x-1)$ .

$$y = 2 - 7(x-1) + \frac{21}{2}(x-1)^2 + \frac{70}{3}(x-1)^3 O((x-1)^4)$$

THIS  
 $y = y_1 + (x-1)y'_1 + \frac{(x-1)^2}{2!}y''_1 + \frac{(x-1)^3}{3!}y'''_1 + \dots$   
 $y = 2 + 7(x-1) + \frac{21}{2}(x-1)^2 + \frac{70}{3}(x-1)^3 + \dots$

THIS  
 $\bullet \quad 2y' - y^2 = 3$   
 $2y'_1 - y_1^2 = 3$   
 $y'_1 - 4 = 3$   
 $\boxed{y'_1 = 7}$

$\bullet \quad y' + xy'' - 2yy' = 0$   
 $y'_1 + 7y''_1 - 2y_1y'_1 = 0$   
 $7 + y''_1 = 2x \times 7 = 0$   
 $\boxed{y''_1 = 21}$

$\bullet \quad y' + y'' + xy'' - 2yy' - 2yy'' = 0$   
 $y'_1 + y''_1 + 2y''_1 - 2y_1y'_1 - 2y_1y''_1 = 0$   
 $21 + 21 + y''_1 - 2x \times 7 - 2x \times 21 = 0$   
 $42 + y''_1 - 98 = 0 \Rightarrow y''_1 = 56$

**O.D.E.**

**TAYLOR SERIES**

**EXPANSIONS**

**3 STANDARD**

**QUESTIONS**

**Question 1** (\*\*\*)+

$$\frac{dy}{dx} = \frac{3x + y^2}{x}, \quad x \neq 0.$$

Given that  $y=1$  at  $x=1$ , find a series solution for the above differential equation in ascending powers of  $(x-1)$ , up and including the terms in  $(x-1)^3$ .

$$y = 1 + 4(x-1) + \frac{7}{2}(x-1)^2 + \frac{16}{3}(x-1)^3 + O[(x-1)^4]$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{3x + y^2}{x} & 2y' - 3 + 2y_0'y' - y' \\
 2 \frac{dy}{dx} &= 3x + y^2 & \text{Diff w.r.t. } x \text{ gives} \\
 2y' &= 3x + y^2 & (y_0'y')' = 0 + 2xy' + 2y'y'' - y''' \\
 2y' &= 3x + y^2 & y' + 2y'' = 2(y_0'y')^2 + 2yy'' - y''' \\
 \text{Diff w.r.t. } x & & 2y' = 2(y_0'y')^2 + 2yy'' - 2y''' \\
 2y'' + y''' &= 3 + 2y_0'y' & 2y'' = 2(y_0'y')^2 + 2y_0''(y_0'-1) \\
 2y'' &= 3 + 2y_0'y' - y' & \\
 2y'' &= 3 + 2y_0'y' - y' & \bullet \text{ Now w.r.t. } x \text{ again, } y_0 = 1 \\
 & & \bullet 2y' = 3 + 2y_0'y' \Rightarrow (y_0'y')' = 3x + 1 \\
 & & \bullet 2y_0'y' = 3 + 2y_0'y' - y' \Rightarrow (y_0'y')' = 3 + 2x + 1 = 4 \\
 & & \bullet y_0'' = 2(y_0'y')^2 + 2y_0''(y_0'-1) \\
 & & \bullet y_0'' = 2(y_0'y')^2 + 2y_0''(1-1) \\
 & & \bullet y_0'' = 32 \\
 y &= y_0 + \frac{y_0'(x-1)}{1!} + \frac{y_0''(x-1)^2}{2!}(x-1)^2 + \frac{y_0'''(x-1)^3}{3!}(x-1)^3 + O[(x-1)^4] \\
 y &= 1 + 4(x-1) + \frac{7}{2}(x-1)^2 + \frac{16}{3}(x-1)^3 + O[(x-1)^4]
 \end{aligned}$$

**Question 2** (\*\*\*)+

A curve has an equation  $y = f(x)$  that satisfies the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\sin 2x + 4y \cos 2x = 0,$$

subject to the conditions  $y = 3$ ,  $\frac{dy}{dx} = 0$  at  $x = 0$ .

Find a series solution for  $f(x)$  up and including the term in  $x^4$ .

$$y = 3 - 6x^2 + 8x^4 + O(x^6)$$

WITH O.D.E IN COMPACT NOTATION

$\bullet y'' + 2y' \sin 2x + 4y \cos 2x = 0$

$y_0 = 0$
$y_1 = 3$
$y_2 = 0$

$\Rightarrow y''_0 + 2y'_0 \sin 2x + 4y_0 \cos 2x = 0$   
 $\Rightarrow y''_0 + 4x^2 \times 1 = 0$   
 $\Rightarrow [y''_0 = -12]$

$\bullet y'' + 2y' \sin 2x + 2y''(2x) + 4y' \cos 2x + 4y(-\sin 2x) = 0$   
 $\Rightarrow y''_0 + 2y''_1 \sin 2x + 2y''(2x) + 4y'_0 \cos 2x + 4y_1(-\sin 2x) = 0$   
 $\Rightarrow [y''_1 = 0]$

$\bullet$  Tidy up first  
 $y'' + 2y' \sin 2x + By''(2x) - By \sin 2x = 0$   
 $y'' + 2y' \sin 2x + 2y''(2x) + By' \cos 2x + By(-\sin 2x) = -By \sin 2x - By(-\sin 2x) = 0$   
 $y'' + 2y' \sin 2x + 2y''(2x) + By' \cos 2x + By(-\sin 2x) = -By' \sin 2x - By_0(2x \sin 2x) = 0$   
 $y'' + 2(-12)x^2 + 8x(-12) \times 1 - 8x^3 \times 2 = 0$   
 $y'' = 48 - 16x^2 - 48x^3 = 192$

THIS

$y = y_0 + 2y'_0 + \frac{2^2}{2!}y''_0 + \frac{2^3}{3!}y'''_0 + \dots$   
 $y = 3 + 0 + \frac{2^2}{2}(-12) + 0 + \frac{2^3}{3}(192) + \dots$   
 $y = 3 - 6x^2 + 8x^4 + \dots$

**Question 3** (\*\*\*)+

A curve has an equation  $y = f(x)$  that satisfies the differential equation

$$e^{-x} \frac{d^2y}{dx^2} = 2y \frac{dy}{dx} + y^2 + 1$$

with  $y = 1$ ,  $\frac{dy}{dx} = 2$  at  $x = 0$ .

- a) Show clearly that

$$e^{-x} \frac{d^3y}{dx^3} = (2y + e^{-x}) \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} \left( y + \frac{dy}{dx} \right).$$

- b) Find a series solution for  $f(x)$ , up and including the term in  $x^3$ .

$$\boxed{\quad}, \quad y = 1 + 2x + 3x^2 + 5x^3 + O(x^4)$$

**a) DIFFERENTIATE THE EQUATION WITH RESPECT TO  $x$**

$$\begin{aligned} \Rightarrow \frac{d}{dx} \left[ e^{-x} \frac{dy}{dx} \right] &= \frac{d}{dx} \left[ 2y \frac{dy}{dx} \right] + \frac{d}{dx} [y^2 + 1] \\ \Rightarrow -e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2y}{dx^2} &= 2 \frac{dy}{dx} \frac{dy}{dx} + 2y \frac{d^2y}{dx^2} + 2y \frac{dy}{dx} \\ \Rightarrow e^{-x} \frac{d^2y}{dx^2} &= (e^{-x} + 2y) \frac{d^2y}{dx^2} + 2y \frac{dy}{dx} + 2y \frac{dy}{dx} \\ \Rightarrow e^{-x} \frac{d^2y}{dx^2} &= (e^{-x} + 2y) \frac{d^2y}{dx^2} + 2y \frac{dy}{dx} + 2y \frac{dy}{dx} \end{aligned}$$

As Required

**b) SUBSTITUTE AT  $x=0$**

$$\begin{aligned} x=0, \quad y &= 1 \\ \frac{dy}{dx} &= 2 \\ \frac{dy}{dx} &= 6 \quad \rightarrow \quad e^{-x} \frac{d^2y}{dx^2} \Big|_{x=0} = 2x(1^2) + 1^2 + 1 \\ \frac{d^2y}{dx^2} &= 30 \quad \rightarrow \quad e^{-x} \frac{d^2y}{dx^2} \Big|_{x=0} = (e^{-x} + 2x)6 + 2x^2[2+1] \end{aligned}$$

**THUS WE HAVE**

$$\begin{aligned} y &= y_0 + 2xy' + \frac{2x^2}{2!}y'' + \frac{2x^3}{3!}y''' + O(x^4) \\ y &= 1 + 2x + \frac{2x^2}{2} \times 6 + \frac{2x^3}{6} \times 30 + O(x^4) \\ y &= 1 + 2x + 3x^2 + 5x^3 + O(x^4) \end{aligned}$$

# MIXED SERIES EXPANSIONS 3 QUESTIONS

**Question 1** (\*\*\*)+

$$f(x) = \frac{\cos 3x}{\sqrt{1-x^2}}, |x| < 1.$$

Show clearly that

$$f(x) \approx 1 - 4x^2 + \frac{3}{2}x^4.$$

**proof**

$$\begin{aligned} f(x) &= \frac{\cos 3x}{\sqrt{1-x^2}} = \frac{\cos 3x \times (1-x^2)^{-\frac{1}{2}}}{1} \\ &\approx \left[ 1 - \frac{(3x)^2}{2!} + O(x^4) \right] \left[ 1 + \frac{(-x^2)}{1} + \frac{(-x^2)(2x^2)}{2!} + O(x^4) \right]^{-\frac{1}{2}} \\ &= \left[ 1 - \frac{9x^2}{2} + \frac{27x^4}{8} + O(x^4) \right] \left[ 1 + 2x^2 + \frac{1}{2}x^4 + O(x^4) \right]^{-\frac{1}{2}} \\ &= \frac{1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + O(x^4)}{1 - \frac{9x^2}{2} + \frac{27x^4}{8} + O(x^4)} \\ &= \frac{\frac{2}{3}x^2 + O(x^4)}{1 - 4x^2 + \frac{3}{2}x^4 + O(x^4)} \end{aligned}$$

As required

**Question 2** (\*\*\*)+

- a) Find the first four terms in the series expansion of  $\left(1 - \frac{1}{2}y\right)^{\frac{1}{2}}$ .

- b) By considering the first two non zero terms in the expansion of  $\sin 3x$  and the answer from part (a), show that

$$\sqrt{1 - \frac{1}{2}\sin 3x} \approx 1 - \frac{3}{4}x - \frac{9}{32}x^2 + \frac{117}{128}x^3.$$

$$\boxed{1 - \frac{1}{4}y - \frac{1}{32}y^2 - \frac{1}{128}y^3 + O(y^4)}$$

$$\begin{aligned} a) \quad \left(1 - \frac{1}{2}y\right)^{\frac{1}{2}} &= 1 + \frac{1}{2}\left(\frac{1}{2}y\right) + \frac{\frac{1}{2}(\frac{1}{2})}{1 \times 2} \left(\frac{1}{2}y\right)^2 + \frac{\frac{1}{2}(\frac{1}{2})(\frac{1}{2})}{1 \times 2 \times 3} \left(\frac{1}{2}y\right)^3 + O(y^4) \\ &= 1 - \frac{1}{4}y - \frac{1}{16}y^2 - \frac{1}{128}y^3 + O(y^4) \end{aligned}$$
  

$$\begin{aligned} b) \quad \sqrt{1 - \frac{1}{2}\sin 3x} &= \left[ 1 - \frac{1}{2}(3x) - \frac{(3x)^2}{2!} \right]^{\frac{1}{2}} = \left[ 1 - \frac{1}{2}(3x - \frac{9}{2}x^2) \right]^{\frac{1}{2}} \\ &= 1 - \frac{1}{4}(3x - \frac{9}{2}x^2) - \frac{1}{2}(\frac{3x}{2} - \frac{9}{4}x^2)^2 - \frac{1}{16}(2x^3 - \dots) \\ &= 1 - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{1}{2}x^3 - \frac{27}{128}x^4 \\ &= 1 - \frac{3}{4}x - \frac{9}{32}x^2 - \frac{177}{128}x^4 \end{aligned}$$

As required

**Question 3** (\*\*\*\*\*)

By considering a suitable binomial expansion, show that

$$\arcsin x = \sum_{r=0}^{\infty} \left[ \binom{2r}{r} \frac{2}{2r+1} \left( \frac{x}{2} \right)^{2r+1} \right].$$

 , proof

Starting from the binomial expansion of  $(1-x)^{-\frac{1}{2}}$

$$\frac{1}{\sqrt{1-x^2}} = (1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}x^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}x^3 + \dots$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{15}{32}x^3 + \dots + O(x^4)$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{15}{32}x^3 + \frac{105}{64}x^4 + O(x^5)$$

Multiplying through by  $x$

$$\frac{x}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3x^3}{8} + \frac{15x^4}{32} + \frac{105x^5}{64} + O(x^6)$$

$$\frac{x}{\sqrt{1-x^2}} = 1 + \frac{21}{12}x^2 + \frac{41}{24}x^3 + \frac{61}{48}x^4 + \frac{81}{64}x^5 + O(x^6)$$

$$\frac{x}{\sqrt{1-x^2}} = 1 + \frac{21}{12}x^2 + \frac{41}{24}x^3 + \frac{61}{48}x^4 + \frac{81}{64}x^5 + O(x^6)$$

$$\frac{x}{\sqrt{1-x^2}} = 1 + \frac{21}{12}x^2 + \frac{41}{24}x^3 + \frac{61}{48}x^4 + \frac{81}{64}x^5 + O(x^6)$$

$$\frac{x}{\sqrt{1-x^2}} = 1 + \sum_{n=2}^{\infty} \left[ \frac{(2n)!}{(n!)^2} \left( \frac{3}{2} \right)^n \right]$$

Integrating both sides, within the radius of convergence

$$\int \frac{1}{1-x^2} dx = \int \sum_{n=2}^{\infty} \left[ \frac{(2n)!}{(n!)^2} \left( \frac{3}{2} \right)^n \right] dx$$

$$\text{TERM}_2 = \frac{21}{12} \left[ \frac{(2n)!}{(n!)^2} \cdot \frac{x^{2n+1}}{2n+1} \right] + C \quad \text{as } n=0, \text{ since } C=0$$

$$\text{TERM}_3 = \frac{41}{24} \left[ \left( \frac{2n)!}{(n!)^2} \cdot \frac{x^{2n+1}}{2n+1} \right) \times \frac{3}{2} \right]$$

$$\text{TERM}_4 = \frac{61}{48} \left[ \left( \frac{2n)!}{(n!)^2} \cdot \frac{x^{2n+1}}{2n+1} \right) \times \left( \frac{3}{2} \right)^2 \right]$$