

# VECTOR EXAM QUESTIONS

Part A

**Question 1    (\*\*)**

The straight line  $l_1$  passes through the points with coordinates  $(5,1,6)$  and  $(2,2,1)$ .

- a) Find a vector equation of  $l_1$ .

A different straight line  $l_2$  passes through the point  $C(6,6,-4)$  and is parallel to the vector  $4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ .

- b) Show clearly that  $l_1$  and  $l_2$  are skew.

$$\boxed{\quad}, \quad \mathbf{r} = \mathbf{5i} + \mathbf{j} + 6\mathbf{k} + \lambda(\mathbf{3i} - \mathbf{j} + 5\mathbf{k})$$

**a)** DETERMINE DIRECTION VECTOR FROM A(5,1,6) & B(2,2,1)

$$\vec{AB} = \mathbf{b} - \mathbf{a} = (2,2,1) - (5,1,6) = (-3,1,-5)$$

"SCALE IT<sup>2</sup> TO HAVE NEGATIVES"

$$\vec{AB} = (3,-1,5)$$

$$\Rightarrow \vec{s} = (5,1,6) + \gamma(3,-1,5)$$

$$\Rightarrow \vec{s} = (3\gamma+5, 1-\gamma, 5\gamma+6)$$

**b)** REPORT THE TWO LINES INCLUDING  $\vec{s}_2 = (6,6,-4) + \mu(4,-2,3)$

- $\vec{s}_1 = (3\gamma+5, 1-\gamma, 5\gamma+6)$
- $\vec{s}_2 = (4\mu+6, 6-2\mu, 3\mu+4)$

$$\begin{aligned} \frac{1}{3} &= \frac{3\gamma+5}{4\mu+6} \quad \gamma = 2\mu - 5 \\ 1-2 &= 1-2\mu \quad 2\mu = 1 \\ 2\mu &= 16 \quad \mu = 8 \end{aligned}$$

**CHECKING:** SUB THESE VALUES

$$\begin{aligned} 5\gamma+6 &= 5(8)-4 = 40-4 = 36 & \gamma = 20 \\ 3\gamma+4 &= 3(8)-1 = 24-1 = 23 & \mu = 11 \end{aligned}$$

$\gamma \neq 20$   
LINES NOT PARALLEL & NOT  
INTERSECTING SO SKEW

**Question 2 (\*\*)**

Relative to a fixed origin  $O$ , the respective position vectors of three points  $A$ ,  $B$  and  $C$  are

$$\begin{pmatrix} 3 \\ 2 \\ 9 \end{pmatrix}, \begin{pmatrix} -5 \\ 11 \\ 6 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix}.$$

- a) Determine, in component form, the vectors  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$ .
- b) Hence find, to the nearest degree, the angle  $BAC$ .
- c) Calculate the area of the triangle  $BAC$ .

$$\boxed{\overrightarrow{AB} = -8\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}}, \boxed{\overrightarrow{AC} = \mathbf{i} - 2\mathbf{j} - 17\mathbf{k}}, \boxed{\theta \approx 83^\circ}, \boxed{\text{area} \approx 106}$$

The diagram shows a triangle  $BAC$  with vertices  $B(-5, 11, 6)$ ,  $A(3, 2, 9)$ , and  $C(4, 0, -8)$ . The vectors  $\overrightarrow{AB} = \langle -8, 9, -3 \rangle$  and  $\overrightarrow{AC} = \langle 1, -2, -17 \rangle$  are calculated. The angle  $\theta$  between them is found using the dot product formula:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}| |\overrightarrow{AC}| \cos \theta$$

$$\langle -8, 9, -3 \rangle \cdot \langle 1, -2, -17 \rangle = \langle 8, -9, 3 \rangle \cdot \langle 1, -2, -17 \rangle \cos \theta$$

$$-8 - 18 + 51 = \sqrt{64 + 81 + 9} \sqrt{1 + 4 + 289} \cos \theta$$

$$25 = \sqrt{154} \sqrt{293} \cos \theta$$

$$\cos \theta = \frac{25}{\sqrt{154} \sqrt{293}}$$

$$\theta \approx 83^\circ$$

The area of triangle  $BAC$  is calculated using the formula:

$$\text{Area} = \frac{1}{2} |\overrightarrow{AB}| |\overrightarrow{AC}| \sin \theta$$

$$= \frac{1}{2} \sqrt{154} \sqrt{293} \sin(83^\circ)$$

$$\approx 105.6539$$

$$\approx 106$$

**Question 3 (\*\*)**

The straight line  $l_1$  passes through the points  $A(2,5,9)$  and  $B(6,0,10)$ .

- a) Find a vector equation for  $l_1$ .

The straight line  $l_2$  has vector equation

$$\mathbf{r} = \begin{pmatrix} 8 \\ 8 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix},$$

where  $\mu$  is a scalar parameter.

- b) Show that the point  $A$  is the intersection of  $l_1$  and  $l_2$ .  
 c) Show further that  $l_1$  and  $l_2$  are perpendicular to each other.

$$\boxed{\mathbf{r} = 2\mathbf{i} + 5\mathbf{j} + 9\mathbf{k} + \lambda(4\mathbf{i} - 5\mathbf{j} + \mathbf{k})}$$

(3)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (6,0,10) - (2,5,9) = (4, -5, 1)$   
 $\therefore l_1 \models (2,5,9) + \lambda(4, -5, 1)$   
 $\therefore l_1 = (2+4\lambda, 5-5\lambda, 9+\lambda)$

(4)  $C = (8,8,6) + \mu(2,1,3) = (2\mu+8, 1+\mu, 6-3\mu)$   
 • Point  $A(2,5,9)$  lies on  $l_1$  (we want it to find  $l_2$ )  
 • By inspection, if  $\mu = -3$ ,  $(8+(-6), 5+(-2), 6+9)$  gives  $(2,5,9)$   
 $\therefore A$  is the intersection of  $l_1, l_2$

(5) DOTTING DIRECTION VECTORS:  
 $(4, -5, 1) \cdot (2, 1, 3) = (4)(2) - 5(1) + (1)(3) = 8 - 5 - 3 = 0$   
 $\therefore l_1 \perp l_2$

**Question 4 (\*\*)**

Relative to a fixed origin  $O$ , the points  $A$  and  $B$  have respective position vectors

$$\mathbf{i} + 7\mathbf{j} + 5\mathbf{k} \text{ and } 5\mathbf{i} + \mathbf{j} - 5\mathbf{k}.$$

- a) Find a vector equation of the straight line  $l_1$  which passes through  $A$  and  $B$ .

The straight line  $l_2$  has vector equation

$$\mathbf{r}_2 = 5\mathbf{i} - 4\mathbf{j} + 4\mathbf{k} + \mu(\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}),$$

where  $\mu$  is a scalar parameter.

The point  $C$  is the point of intersection between  $l_1$  and  $l_2$ .

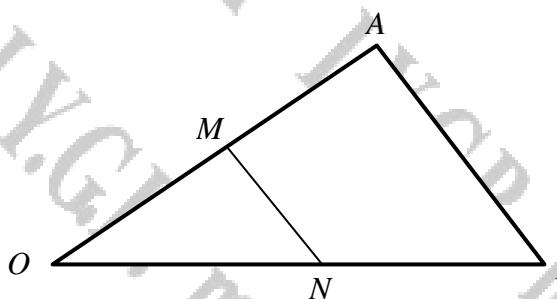
- b) Find the position vector of  $C$ .  
 c) Show that  $C$  is the midpoint of  $AB$ .

$$\boxed{\mathbf{r} = \mathbf{i} + 7\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k})}, \quad \boxed{\overrightarrow{OC} = 3\mathbf{i} + 4\mathbf{j}}$$

$$\begin{aligned}
 & \text{Q) } \vec{AB} = \mathbf{b} - \mathbf{a} = (5\mathbf{i} - \mathbf{j} - 5\mathbf{k}) - (\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}) = (4, -8, -10) \\
 & \text{SAG direction } (2, -3, -5) \\
 & \therefore \vec{r}_1 = (4\mathbf{i} - \mathbf{j} - 5\mathbf{k}) + \lambda(2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}) \\
 & \vec{r}_1 = (4\mathbf{i} + 1, -3\mathbf{j} - 5\mathbf{k}) \\
 & \text{(b) } \vec{r}_1 = (5, -4, 0) + \mu(1, -4, 2) \\
 & \vec{r}_2 = (5, -4, 0) + \mu(-1, 2, 4) \\
 & \text{equat. } \vec{r}_1 = \vec{r}_2 \\
 & \left\{ \begin{array}{l} 2\lambda + 1 = 5 \\ -3\lambda - 4 = -4 \\ -5\lambda = 4 \end{array} \right\} \Rightarrow \mu = 2, \lambda = -1 \\
 & \text{SOLVE} \\
 & \Rightarrow -7 - 3\lambda = -9(2) + 4 \Rightarrow -7 - 3(-1) = -9(2) + 4 \Rightarrow 2\lambda = 5 \Rightarrow \boxed{\lambda = 1} \\
 & \text{USING } \lambda = 1 \text{ in eqn. } (2\mathbf{i} + 1, -3\mathbf{j} - 5\mathbf{k}) = (3, -4, 0) \\
 & \therefore C(3, -4, 0) \\
 & \text{Q) MIDPOINT OF } AB = \left( \frac{4+5}{2}, \frac{-8+0}{2}, \frac{-10+0}{2} \right) = \left( \frac{14}{2}, \frac{74}{2}, \frac{50}{2} \right) = (3, -4, 0)
 \end{aligned}$$

**Question 5 (\*\*)**

The figure below shows the triangle  $OAB$ .

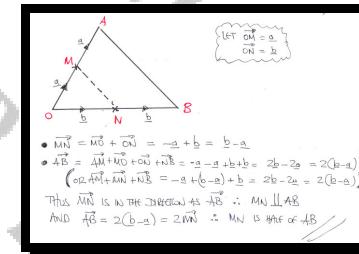


The point  $M$  is the midpoint of  $OA$  and the point  $N$  is the midpoint of  $OB$ .

Let  $\overrightarrow{OM} = \mathbf{a}$  and  $\overrightarrow{ON} = \mathbf{b}$ .

By finding simplified expressions for  $\overrightarrow{MN}$  and  $\overrightarrow{AB}$ , in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , show that  $MN$  is parallel to  $AB$ , and half its length.

proof



**Question 6 (\*\*+)**

The points  $A(2,4,4)$ ,  $B(6,8,4)$ ,  $C(6,4,0)$ ,  $D(2,0,0)$  and  $M(4,4,2)$  are given.

The straight line  $l_1$  has equation

$$\mathbf{r}_1 = 6\mathbf{i} + 4\mathbf{j} + \lambda(\mathbf{i} + \mathbf{j}),$$

where  $\lambda$  is a scalar parameter.

The straight line  $l_2$  passes through the points  $C$  and  $M$ .

- a) Find a vector equation of  $l_2$ .
- b) Show that  $\overrightarrow{AB}$  is parallel to  $l_1$ .
- c) Verify that  $D$  lies on  $l_1$ .
- d) Find the acute angle between  $\overrightarrow{AC}$  and  $l_1$ .

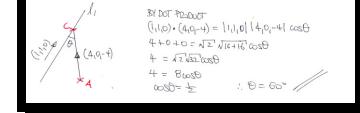
$$\boxed{\mathbf{r}_2 = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{k})}, [60^\circ]$$

(a)  $\overrightarrow{CM} = M - C = (4,4,2) - (6,4,0) = (-2,0,2)$   
 USE  $(1,0,-1)$  AS DIRECTION VECTOR  
 $\Sigma_1 = (4,4,2) + p(1,0,-1) = (4+4p, 4, 2-p)$

(b)  $\overrightarrow{AB} = B - A = (6,8,4) - (2,4,4) = (4,4,0) = C(1,0,0)$   
 DIRECTION OF  $l_1$  IS  $(1,1,0)$   $\therefore AB$  IS PARALLEL TO  $l_1$

(c)  $\Sigma_1 = (6,4,0) + q(1,0,0) = (6+q, 4, 0)$   
 BY INTERSECTION IF  $q=4$   $(6+4, 4, 0) = (2,0,0)$   $\therefore$  POINT D

(d)  $\overrightarrow{AC} = C - A = (6,4,0) - (2,4,4) = (4,0,-4)$



$\cos \theta = \frac{\overrightarrow{AC} \cdot \text{dir}(l_1)}{|\overrightarrow{AC}| |\text{dir}(l_1)|}$   
 $(4,0,-4) \cdot (1,1,0) = (4,0,-4) \cdot (4,4,2) \cos \theta$   
 $4+0+0 = \sqrt{4^2+0^2+(-4)^2} \cos \theta$   
 $4 = 8 \cos \theta$   
 $\cos \theta = \frac{1}{2} \therefore \theta = 60^\circ$

## Question 7 (\*\*+)

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$$

$$\mathbf{r}_2 = 8\mathbf{i} + 8\mathbf{j} + 13\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$$

where  $\lambda$  and  $\mu$  are scalar parameters

- a)** Show that  $l_1$  and  $l_2$  intersect at some point  $P$  and find its coordinates  
**b)** Calculate the acute angle between  $l_1$  and  $l_2$ .

$$P(6,11,7), \quad 77.0^\circ$$

(b)  $\Sigma_1 = \{B(3,1), B(1,3)\} = \{(3+4)(1+3)(1)\}$   
 $\Sigma_2 = \{B(1,3) + B(-2,-3), -B(0,5) - B(-2,-3)\}$

~~ANSWER 1 & 2~~

(1):  $2 + 3x - y = B$   
 $(2): 4x + 4y = B$

$\therefore 4x + 4y = B = -3x$   
 $y + 4 = B = -\frac{3}{4}x$

$\boxed{y = -1 + \frac{3}{4}x}$        $\boxed{B = -1 + \frac{3}{4}x}$

GIVE 1:  $B(1,1) = 8(2,2) + 1^2$   
 $6x + 3x = 8(2,2) + 1^2$

$\therefore$  USING ANSWERS AS THE THREE COMBINATIONS ABOVE  
 USING  $2x = 1$ ,  $1x = 0$ ,  $3x = 3$ , we can form  $P(6,1,1)$

(b) DOTTING DIRECTION VECTOR:  $\begin{pmatrix} 1,1,3 \\ -2,-3,1 \end{pmatrix} = \begin{pmatrix} 1,1,3 \\ -2,-3,1 \end{pmatrix} \cdot \begin{pmatrix} 1,1,3 \\ -2,-3,1 \end{pmatrix} = \frac{1}{\sqrt{14}} \cdot \frac{1}{\sqrt{14}} = \frac{1}{14}$

$\therefore \theta = \cos^{-1} \frac{1}{14} = 70^\circ$

**Question 8 (\*\*+)**

Relative to a fixed origin  $O$ , the points  $P$  and  $Q$  have respective position vectors

$$(-7\mathbf{j} + 4\mathbf{k}) \text{ and } (3\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}).$$

The straight line  $l_1$  passes through the points  $P$  and  $Q$ .

- a) Determine a vector equation for  $l_1$ .

The straight line  $l_2$  has vector equation

$$\mathbf{r} = (7\mathbf{i} + a\mathbf{j} + b\mathbf{k}) + \mu(\mathbf{i} + 4\mathbf{j} - \mathbf{k}),$$

where  $a$  and  $b$  are scalar constants, and  $\mu$  is a scalar parameter.

- b) Given that  $l_1$  and  $l_2$  intersect at  $Q$ , find the value of  $a$  and the value of  $b$ .  
 c) Calculate the acute angle between  $l_1$  and  $l_2$ .

,  $\mathbf{r} = -7\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$  ,  $a = 8$  ,  $b = -2$  ,  $86.4^\circ$

(a)  $\vec{PQ} = q - p = (3\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}) - (-7\mathbf{j} + 4\mathbf{k}) = (3\mathbf{i} - 1\mathbf{j} - 2)$   
 Thus  $\vec{s} = (3\mathbf{i} - 1\mathbf{j} - 2) + \lambda(3\mathbf{i} - 1\mathbf{j} - 2)$   
 $\vec{s} = (3\mathbf{i} - 1\mathbf{j} - 2) + \lambda(3\mathbf{i} - 1\mathbf{j} - 2)$

(b)  $l_2$  PASSES THROUGH Q(3, -1, 2)       $\vec{s}_1 = (7\mathbf{i} + a\mathbf{j} + b\mathbf{k})$   
 $(1): 7 + \mu = 3$       (2):  $a + 4\mu = -8$       (3):  $b + 4\mu = 2$   
 $\mu = -4$        $a - 16 = -8$        $a = 8$   
 $a = 8$        $b - 16 = -8$        $b = 8$   
 $b = 8$        $b + 4\mu = 2$   
 $b = -2$

(c) DOTTING THE JACOBIAN VECTORS OF THE TWO LINES  
 $(3\mathbf{i} - 1\mathbf{j} - 2) \cdot (1\mathbf{i} + a\mathbf{j} + b\mathbf{k}) = (3\mathbf{i} - 1\mathbf{j} - 2) \cdot (1\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$   
 $3 - 4 + 2 = \sqrt{9 + 1 + 4} \cdot \sqrt{1 + 16 + 4} \cos\theta$   
 $1 = \sqrt{14 + 18} \cos\theta$   
 $\cos\theta = \frac{1}{\sqrt{32}}$   
 $\theta \approx 86.4^\circ$

**Question 9 (\*\*+)**

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 2\mathbf{i} + 2\mathbf{j} + \lambda(\mathbf{i} + \mathbf{j})$$

$$\mathbf{r}_2 = 2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k}),$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- a) Show that  $l_1$  and  $l_2$  **do not** intersect.

The point  $P$  lies on  $l_1$  where  $\lambda = 4$  and the point  $Q$  lies on  $l_2$  where  $\mu = -1$ .

- b) Find the acute angle between  $PQ$  and  $l_1$ .

, 56.3°

(a)  $\mathbf{r}_1 = (2, 2, 0) + \lambda(1, 1, 0) = (2 + \lambda, 2 + \lambda, 0)$   
 $\mathbf{r}_1 = (2, 2, 0) + \mu(2, 1, -1) = (2 + 2\mu, 2 + \mu, -\mu)$

Equation of  $l_2$   
 $\left\{ \begin{array}{l} 2 + 2 = 2 + 2\mu \\ 2 + 2 = 2 + \mu \end{array} \right. \text{ subtract } 0 = \mu - 3 \Rightarrow \frac{\mu - 3}{\lambda - 2} = 1 \Rightarrow \lambda = 5$

Check:  $0 \neq 7 - 3 \Rightarrow$  LINES DO NOT INTERSECT

(b)

$\lambda = 4 \rightarrow \mathbf{P}(6, 6, 0)$   
 $\mu = -1 \rightarrow \mathbf{Q}(0, 1, -1)$

$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = (0, 1, -1) - (6, 6, 0) = (-6, -5, -1)$

- Substituting  $\overrightarrow{PQ}$  in direction of  $l_1$   
 $\rightarrow (-6, -5, -1) = k(1, 1, 0) \Rightarrow k = -6$   
 $\rightarrow -6 = \sqrt{6^2 + 5^2 + 0^2} \Rightarrow k = -6/\sqrt{6^2 + 5^2}$   
 $\rightarrow \cos \theta = \frac{-6}{\sqrt{6^2 + 5^2}}$   
 $\rightarrow \theta \approx 123.7^\circ$   
 $\therefore$  LINES ANGLE 56.3°

**Question 10 (\*\*\*)**

Relative to a fixed origin  $O$ , the points  $A$ ,  $B$  and  $C$  have respective position vectors

$$\mathbf{a} = 8\mathbf{i} + \mathbf{j}, \quad \mathbf{b} = 5\mathbf{j} + 8\mathbf{k} \quad \text{and} \quad \mathbf{c} = 14\mathbf{i} + \mathbf{j} + 15\mathbf{k}.$$

- a) Find a vector equation of the straight line which passes through  $A$  and  $B$ .

The point  $M$  is the midpoint of  $AB$ .

- b) Show that  $CM$  is perpendicular to  $AB$ .
- c) Determine the area of the triangle  $ABC$ .

$$\boxed{\mathbf{r} = 5\mathbf{j} + 8\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} + 2\mathbf{k})}, \quad [\text{area} = 90]$$

(a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (0, 5, 8) - (8, 1, 0) = (-8, 4, 8) \leadsto \text{SCALE TO } (-2, 1, 2)$

$$\begin{aligned}\Gamma &= (0, 5, 8) + t(-2, 1, 2) \\ \Gamma &= (2t, 5 + t, 8 + 2t)\end{aligned}$$

(b)  $M = \left( \frac{8+0}{2}, \frac{1+5}{2}, \frac{0+8}{2} \right) = (4, 3, 4)$

$$\vec{CM} = (4, 3, 4) - (14, 1, 0) = (-10, 2, 4)$$

Dot Product:

$$\vec{AB} \cdot \vec{CM} = (-2, 1, 2) \cdot (-10, 2, 4)$$

$$= 80 + 8 - 80$$

$$= 0$$

$\therefore CM \perp AB$

(c)  $\|\vec{AB}\| = \sqrt{(-8)^2 + 4^2 + 8^2} = \sqrt{160} = 4\sqrt{10}$

$$\|\vec{AC}\| = \sqrt{-10^2 + 2^2 + 4^2} = \sqrt{100 + 4 + 16} = 10$$

$$\|\vec{BC}\| = \sqrt{10^2 + 14^2} = \sqrt{100 + 196} = 14$$

$$\text{Area} = \frac{1}{2} \times 10 \times 14 = 70$$

**Question 11 (\*\*+)**

With respect to a fixed origin  $O$ , the respective position vectors of the points  $A$ ,  $B$  and  $C$  are

$$\begin{pmatrix} 2 \\ 9 \\ -1 \end{pmatrix}, \begin{pmatrix} 12 \\ 4 \\ 7 \end{pmatrix} \text{ and } \begin{pmatrix} 10 \\ -3 \\ 7 \end{pmatrix}.$$

- a) Find the position vector of the midpoint of  $AC$ .

The point  $D$  is such so that  $ABCD$  is a parallelogram.

- b) Determine the position vector of  $D$ .  
 c) Calculate, correct to one decimal place, the angle  $ABC$ .  
 d) Hence, calculate the area of the triangle  $ABC$ .

$$[6\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}], [2\mathbf{j} - \mathbf{k}], [\theta \approx 98.6^\circ], [\text{area} \approx 49.5]$$

c)  $\vec{BA} = a - b = (2, 9, -1) - (12, 4, 7) = (-10, 5, -6)$   
 $\vec{BC} = c - b = (10, -3, 7) - (12, 4, 7) = (-2, -7, 0)$

$\vec{BA} \cdot \vec{BC} = |BA| |BC| \cos \theta$   
 $(-10, 5, -6) \cdot (-2, -7, 0) = -10(-2) + 5(-7) = 130$   
 $20 - 35 = \sqrt{100 + 25 + 36} \sqrt{4 + 49} \cos \theta$   
 $-15 = \sqrt{145} \cos \theta$   
 $\cos \theta = -0.1496 \dots$   
 $\theta = 98.6194 \dots$   
 $\theta = 98.6^\circ$

d) Area of Triangle  $\triangle ABC = \frac{1}{2} |\vec{BA}| |\vec{BC}| \sin \theta$   
 $= \frac{1}{2} \sqrt{145} \sqrt{53} \sin(98.6^\circ)$   
 $= 49.477 \dots$   
 $\approx 49.5$

**Question 12 (\*\*\*)**

$OABC$  is a square.

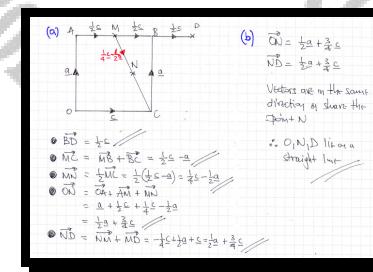
The point  $M$  is the midpoint of  $AB$  and the point  $N$  is the midpoint of  $MC$ .

The point  $D$  is such so that  $\overrightarrow{AD} = \frac{3}{2} \overrightarrow{AB}$ .

Let  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OC} = \mathbf{c}$ .

- Find simplified expressions, in terms of  $\mathbf{a}$  and  $\mathbf{c}$ , for each of the vectors  $\overrightarrow{BD}$ ,  $\overrightarrow{MC}$ ,  $\overrightarrow{MN}$ ,  $\overrightarrow{ON}$  and  $\overrightarrow{ND}$ .
- Deduce, showing your reasoning, that  $O$ ,  $N$  and  $D$  are collinear.

$$\boxed{\overrightarrow{BD} = \frac{1}{2}\mathbf{c}}, \boxed{\overrightarrow{MC} = \frac{1}{2}\mathbf{c} - \mathbf{a}}, \boxed{\overrightarrow{MN} = \frac{1}{4}\mathbf{c} - \frac{1}{2}\mathbf{a}}, \boxed{\overrightarrow{ON} = \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{c}}, \boxed{\overrightarrow{ND} = \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{c}}$$



**Question 13 (\*\*\*)**

Relative to a fixed origin  $O$ , the points  $P$  and  $Q$  have respective position vectors

$$5\mathbf{i} + 2\mathbf{k} \text{ and } 3\mathbf{i} + 3\mathbf{j}.$$

- a) Determine a vector equation of the straight line  $l$  which passes through the points  $P$  and  $Q$ .

The straight line  $m$  has a vector equation

$$\mathbf{r} = 4\mathbf{i} + 8\mathbf{j} - \mathbf{k} + \mu(5\mathbf{i} - \mathbf{j} + 3\mathbf{k}),$$

where  $\mu$  is a scalar parameter.

- b) Show that  $l$  and  $m$  intersect at some point  $A$  and find its position vector.  
 c) Find the size of the acute angle  $\theta$ , formed by  $l$  and  $m$ .

$$\boxed{\mathbf{r} = 5\mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})}, \boxed{\overrightarrow{OA} = -\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}}, \boxed{\theta \approx 38.8^\circ}$$

(a)  $\vec{PQ} = \vec{PQ} = \mathbf{q} - \mathbf{p} = (3, 3, 0) - (5, 0, 2) = (-2, 3, -2)$   
 $\therefore \mathbf{l}_1 = (\mathbf{q}_1, \mathbf{q}) + \lambda(2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$   
 $\mathbf{l}_1 = (2\lambda + 3, -3\lambda, 2\lambda + 2)$

(b)  $\mathbf{l}_2 = (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}, 2\mathbf{i} + 2\mathbf{j})$   
 $\mathbf{l}_2 = (5\lambda + 4, 8\lambda + 1, 3\lambda - 1)$

From  $\mathbf{l}_1$  &  $\mathbf{l}_2$   
 $\begin{cases} 2\lambda + 3 = 5\lambda + 4 \\ -3\lambda = 8\lambda + 1 \\ 2\lambda + 2 = 3\lambda - 1 \end{cases}$

SUBTRACT  
 $\begin{cases} 3 = 2\lambda + 5 \\ 4 = 11\lambda \\ 3 = \lambda - 3 \end{cases}$

Check  $\lambda = -3$ :  
 $-3(2) + 3 = 9$   
 $8(-3) + 1 = -23$   
 $-3 + 2 = -1$

Ans are the same, hence the lines intersect.

c) Using  $\mu = -1$  from  $(5\lambda + 4, 8\lambda + 1, 3\lambda - 1)$   
 $\mathbf{OA} = (5(-1) + 4, 8(-1) + 1, 3(-1) - 1)$   
 $\mathbf{OA} = (-1, -7, -4)$

Dot product  $\mathbf{OA} \cdot \mathbf{PQ} = -1 + 7 - 4 = 2$

$\cos \theta = \frac{2}{\sqrt{14} \sqrt{18}}$   
 $\cos \theta = \frac{1}{\sqrt{252}}$   
 $\cos \theta = \frac{1}{\sqrt{63}}$   
 $\cos \theta \approx 0.19$   
 $\theta \approx 79.8^\circ$

Rotating directions:  
 $(2, -3, 2), (5, 3, 0), (2, 3, 2), (5, 3, 0)$   
 $10 + 3 + 6 = \sqrt{49 + 9 + 4} \sqrt{25 + 9 + 4} \cos \theta$   
 $19 = \sqrt{58} \sqrt{38} \cos \theta$   
 $\cos \theta = \frac{19}{\sqrt{58} \sqrt{38}}$   
 $\cos \theta \approx 0.19$   
 $\theta \approx 79.8^\circ$

**Question 14 (\*\*\*)**

The points  $A(2,10,7)$  and  $B(0,15,12)$  are given.

- a) Determine a vector equation of the straight line  $l_1$  that passes through the points  $A$  and  $B$ .

The vector equation of the straight line  $l_2$  is

$$\mathbf{r}_2 = 4\mathbf{i} + \mathbf{j} - 6\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}),$$

where  $\mu$  is a scalar parameter.

- b) Show that  $l_1$  and  $l_2$  intersect at some point  $P$  and find its coordinates.  
 c) Calculate the acute angle between  $l_1$  and  $l_2$ .

 ,  $\mathbf{r}_1 = 2\mathbf{i} + 10\mathbf{j} + 7\mathbf{k} + \lambda(-2\mathbf{i} + 5\mathbf{j} + 5\mathbf{k})$ ,  $P(6,0,-3)$ ,  $77.4^\circ$

(a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (0,15,12) - (2,10,7) = (-2,5,5)$   
 Hence  $\mathbf{l}_1: (2,10,7) + \lambda(-2,5,5)$   
 $\mathbf{l}_1: (2-2\lambda, 10+5\lambda, 7+5\lambda)$

(b)  $\mathbf{l}_2: (2\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$   
 $\mathbf{l}_2: (2+2\mu, 10-\mu, 7+3\mu)$

Equating  $\mathbf{l}_1$  &  $\mathbf{l}_2$

(1)  $2-2\lambda = 2+2\mu \quad |+2\lambda$   
 $7+5\lambda = 10-\mu \quad |+5\lambda$   
 $4\lambda = 3 \quad |:4$   
 $\lambda = \frac{3}{4}$

(2)  $10+5\lambda = 1-\mu \quad |+5\lambda$   
 $10+25/4 = 1-\mu \quad |\cdot 4$   
 $50 = 4-4\mu \quad |+4\mu$   
 $54 = 4 \quad |:4$   
 $\mu = 13.5$

(3)  $7+5\lambda = 7+15\lambda \quad |+5\lambda$   
 $0 = 10\lambda \quad |:10$   
 $\lambda = 0$

Using  $\lambda = 1$  into  $(2)$ :  $(2+2\lambda)-1 = 10-\mu$  we obtain  $P(6,0,-3)$

Angle  $\theta$ :  $\cos \theta = \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}| |\mathbf{c}|}$   
 $\mathbf{b} \cdot \mathbf{c} = (-2,5,5) \cdot (2,-1,3) = -4+15+15 = 26$   
 $|\mathbf{b}| = \sqrt{4+25+25} = \sqrt{54} = 3\sqrt{6}$   
 $|\mathbf{c}| = \sqrt{4+1+9} = \sqrt{14}$   
 $\cos \theta = \frac{26}{3\sqrt{6} \cdot \sqrt{14}} = \frac{13}{3\sqrt{21}}$   
 $\theta \approx 77.4^\circ$

Dotting direction vectors  
 $(-2,5,5) \cdot (2,-1,3) = -2 \cdot 2 + 5 \cdot (-1) + 5 \cdot 3 = 26$   
 $|-2| \cdot |2| = \sqrt{4+25+25} \cdot \sqrt{4+1+9} = \sqrt{54} \cdot \sqrt{14}$   
 $\cos \theta = \frac{26}{\sqrt{54} \cdot \sqrt{14}} = \frac{13}{3\sqrt{21}}$   
 $\theta \approx 77.4^\circ$

**Question 15 (\*\*\*)**

Relative to a fixed origin  $O$  the following position vectors are given.

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 6 \\ 11 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{OD} = \begin{pmatrix} 2 \\ 8 \\ 9 \end{pmatrix}.$$

a) Show clearly that ...

- i. ...  $\overrightarrow{AD}$  is perpendicular to  $\overrightarrow{BD}$ .
- ii. ... the points  $A$ ,  $B$  and  $C$  are collinear and state the ratio  $AB : BC$ .

b) Determine the **exact** area of the triangle  $ABD$ .

$$AB : BC = 3 : 2, \quad \boxed{\text{area} = \frac{9}{2}\sqrt{5}}$$

(a)  $\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (4\hat{i} + 3\hat{j}) - (1\hat{i} + 6\hat{j}) = (3\hat{i} - 7\hat{j})$   
 $\overrightarrow{BC} = \overrightarrow{C} - \overrightarrow{B} = (6\hat{i} + \hat{j}) - (4\hat{i} + 3\hat{j}) = (2\hat{i} - 2\hat{j})$   
 $\overrightarrow{AB} \cdot \overrightarrow{BC} = (3\hat{i} - 7\hat{j}) \cdot (2\hat{i} - 2\hat{j}) = -6 + 10 - 14 = -10 \neq 0 \quad \therefore \text{THE POINTS ARE NOT COLLINEAR}$

(b)  $\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (4\hat{i} + 3\hat{j}) - (1\hat{i} + 6\hat{j}) = (3\hat{i} - 7\hat{j})$   
 $\overrightarrow{BD} = \overrightarrow{D} - \overrightarrow{B} = (2\hat{i} + 8\hat{j}) - (4\hat{i} + 3\hat{j}) = 2(1\hat{i} - 1\hat{j})$   
As  $\overrightarrow{BD}$  is in the same direction as  $\overrightarrow{BC}$ ,  $A, B, C$  are collinear.  
 $\therefore \overrightarrow{AB} \parallel \overrightarrow{BC}$



$$\text{AREA} = \frac{1}{2} |\overrightarrow{AB}| |\overrightarrow{BD}| \sin \theta$$

$$= \frac{1}{2} \sqrt{1+49} \sqrt{4+64} \sin 90^\circ$$

$$= \frac{1}{2} \sqrt{1+49} \times \sqrt{4+64}$$

$$= \frac{9}{2}\sqrt{5}$$

**Question 16** (\*\*\*)

Relative to a fixed origin  $O$ , the points  $A$  and  $B$  have respective position vectors

$$4\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} \text{ and } 2\mathbf{i} + 3\mathbf{k}.$$

- a) Determine a vector equation of the straight line  $l_1$  which passes through the points  $A$  and  $B$ .

The straight line  $l_2$  passes through the point  $C$  with position vector  $4\mathbf{i} - 6\mathbf{j}$  and is parallel to the vector  $3\mathbf{j} - \mathbf{k}$ .

- b) Write down a vector equation of  $l_2$ .
- c) Show that  $l_1$  and  $l_2$  intersect at the point  $A$ .
- d) Find the acute angle between  $l_1$  and  $l_2$ .

$$\boxed{\mathbf{r}_1 = 2\mathbf{i} + 3\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})}, \boxed{\mathbf{r}_2 = 4\mathbf{i} - 6\mathbf{j} + \mu(3\mathbf{j} - \mathbf{k})}, \boxed{47.3^\circ}$$

(a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (2, 0, 3) - (2, 3, 3) = (-2, -3, 6)$   
use scaled vector  $(2, 3, 3)$  & direction vector  
 $\mathbf{l}_1 = (2, 0, 3) + t(2, 3, -6) = (2+2t, 3t, 3-6t)$

(b)  $\mathbf{l}_2 = (4, 4, 0) + \mu(3, -1, -1) = (4, 3\mu+4, -\mu)$

(c) IT IS ENOUGH TO SHOW THAT  $A$  LIES ON  $l_2$   
BY INSPECTION IF  $\mu=3$   
 $(A, 3\mu+4, -\mu)$  becomes  $(A, 13, -3)$   
 i.e. point  $A$   
 i.e. both  $\mathbf{l}_1$  &  $\mathbf{l}_2$  meet at point  $A$

(d) DETERMINE DIRECTION VECTORS  
 $\Rightarrow (\mathbf{C}_1, \mathbf{C}_2) = (4, 3\mu+4, -\mu) - (2, 3, 3) = (2, 3\mu+1, -\mu-3)$   
 $\Rightarrow 4+4+6 = \sqrt{4+4+36} = \sqrt{44+36} = \sqrt{80}$   
 $\Rightarrow 15 = \sqrt{80}$   
 $\Rightarrow \cos \theta = \frac{-7}{\sqrt{80}}$   
 $\therefore \theta = 47.3^\circ$

**Question 17 (\*\*\*)**

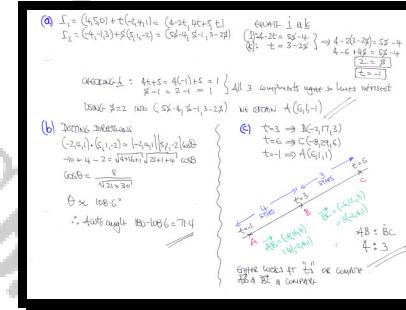
Relative to a fixed origin  $O$ , the straight lines  $L$  and  $M$  have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix} + s \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix},$$

where  $t$  and  $s$  are scalar parameters.

- Show that  $L$  and  $M$  intersect at some point  $A$  and find its coordinates.
- Find the size of the acute angle  $\theta$ , formed by  $L$  and  $M$ .
- Find the ratio  $AB : BC$ .

,  $[A(6,1,-1)]$  ,  $[\theta \approx 71.4]$  ,  $[AB : BC = 4 : 3]$



**Question 18 (\*\*\*)**

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 2\mathbf{i} + \mathbf{j} + \lambda(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

$$\mathbf{r}_2 = \mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{k})$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- Verify that both  $l_1$  and  $l_2$  pass through the point  $P$ , whose position vector is  $5\mathbf{j} + 6\mathbf{k}$ .
- Find the acute angle between  $l_1$  and  $l_2$ .

The point  $Q$  has position vector  $-\mathbf{i} + \mathbf{j} + \mathbf{k}$ .

- Find a vector equation of the straight line  $l_3$  that passes through the point  $Q$ , so that all three straight lines intersect.

$$[33.2^\circ], \quad \mathbf{r}_3 = -\mathbf{i} + \mathbf{j} + \mathbf{k} + t(\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$$

(a)  $\begin{aligned}\mathbf{r}_1 &= (2, 1, 0) + \lambda(-1, 2, 3) = (2, 1, 0) + \lambda(-1, 2, 3) \\ \mathbf{r}_2 &= (1, 5, 4) + \mu(1, -2, 0) = (1, 5, 4) + \mu(1, -2, 0)\end{aligned}$

BY INSPECTION: IF  $\lambda = 2$  &  $\mu = -1$ , lines produce  $(0, 5, 6)$  //

(b) DETERM. DIRECTIONS:  $(-1, 2, 3), (1, 0, 2) = (-1, 2, 3) \times (1, 0, 2) = \frac{1}{\sqrt{1+4+9}}(-1, 2, 3) \times (1, 0, 2)$   
 $\rightarrow \hat{\mathbf{n}} = \frac{1}{\sqrt{14}}(-1, 2, 3) \times (1, 0, 2)$   
 $\hat{\mathbf{n}} = \frac{1}{\sqrt{14}}(-1, 2, 3)$   
 $\theta \approx 14.6^\circ$        $\therefore$  Acute angle is  $33.2^\circ$

(c)  $\begin{aligned}\mathbf{r}_3 &= \mathbf{q} - \mathbf{p} = (-1, 1, 1) - (0, 5, 6) = (-1, -4, -5) \text{ at } (1, 0, 5) \text{ as direction} \\ \therefore \mathbf{r}_3 &= (-1, 1, 1) + t(1, 4, 5) = (t-1, 4t+1, 5t+1)\end{aligned}$

**Question 19 (\*\*\*)**

With respect to a fixed origin  $O$ , the point  $A$  has position vector  $8\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$  and the point  $B$  has position vector  $t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$ .

- a) Show clearly that

$$|AB|^2 = 6t^2 - 24t + 125.$$

Let  $f(t) = 6t^2 - 24t + 125$ .

- b) Find the value of  $t$  for which  $f(t)$  takes a minimum value.  
 c) Hence determine the closest distance between  $A$  and  $B$ .

$$t = 2, \sqrt{101}$$

(a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (t, t, 2t) - (8, -6, 5) = (t-8, t+6, 2t-5)$   
 $|AB| = \sqrt{(t-8)^2 + (t+6)^2 + (2t-5)^2}$   
 $|AB|^2 = \frac{t^2-16t+64}{6} + \frac{t^2+12t+36}{6} + \frac{4t^2-20t+25}{6}$   
 $|AB|^2 = 6t^2 - 24t + 125$

(b)  $f(t) = 6t^2 - 24t + 125$   
 $f'(t) = 12t - 24$   
 Since  $f'(t) = 0$   
 $0 = 12t - 24$   
 $t = 2$   
 Using  $t = 2$   
 $f(2) = 24 - 48 + 125 = 101$   
 $\therefore |AB|^2 = 101$   
 $|AB| = \sqrt{101}$

(c)  $|AB|^2 = 6t^2 - 24t + 125$   
 $|AB|^2 = 6\left[t^2 - 4t + \frac{125}{6}\right]$   
 $|AB|^2 = 6\left[\left(t - \frac{2}{2}\right)^2 - \frac{4}{6} + \frac{125}{6}\right]$   
 $|AB|^2 = 6(t-2)^2 - 24 + 125$   
 $|AB|^2 = 6(t-2)^2 + 101$   
 So min of 101 occurs when  
 $t = 2$   
 & min distance of  $\sqrt{101}$  when  $t = 2$

**Question 20 (\*\*\*)**

Relative to a fixed origin  $O$ , the points  $A$  and  $B$  have respective coordinates

$$(2, -3, 3) \quad \text{and} \quad (5, 1, b),$$

where  $b$  is a constant.

The point  $C$  is such so that  $OABC$  is a rectangle, where  $O$  is the origin.

- a) Show clearly that  $b = 5$ .
- b) Determine the position vector of  $C$ .
- c) Find the vector equation of the straight line  $l$  that passes through  $A$  and  $C$ .

$$C(3, 4, 2), \quad \boxed{\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 7\mathbf{j} - \mathbf{k})}$$

Diagram shows a 3D Cartesian coordinate system with axes i, j, k. Points A(2, -3, 3) and B(5, 1, b) are plotted. Point C is shown at (3, 4, 2). Vectors  $\vec{OA}$ ,  $\vec{OB}$ ,  $\vec{OC}$ , and  $\vec{AB}$  are drawn. Vector  $\vec{AC}$  is also shown.

(a)  $\vec{OA} = (2, -3, 3)$   
 $\vec{OB} = (5, 1, b)$   
 $\vec{OC} = (3, 4, 2)$   
 $\vec{AB} = (5-2, 1-(-3), b-3) = (3, 4, b-3)$   
 $\vec{AC} = (3-2, 4-(-3), 2-3) = (1, 7, -1)$

$\vec{OA} \times \vec{OB} = 0$   
 $(2, -3, 3) \times (5, 1, b-3) = 0$   
 $6-12+3b-9=0$   
 $3b=15$   
 $b=5$

(b)  $\vec{OC} = \vec{OA} + \vec{AC}$   
 $= \vec{OA} + \vec{AB}$   
 $= (3, 4, 2)$   
 $\therefore C(3, 4, 2)$

(c)  $\vec{AC} = \vec{C} - \vec{A} = (3, 4, 2) - (2, -3, 3) = (1, 7, -1)$   
 $\therefore \text{line } l:$   
 $\quad \vec{r} = (2, -3, 3) + \lambda(1, 7, -1)$   
 $\quad \vec{r} = (2+2\lambda, -3+7\lambda, 3-\lambda)$

**Question 21 (\*\*\*)**

The following vectors are given

$$\mathbf{a} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{b} = (4p+1)\mathbf{i} + (p-2)\mathbf{j} + \mathbf{k},$$

where  $p$  is a scalar constant.

Find the value of  $p$  if ...

- a) ...  $\mathbf{a}$  and  $\mathbf{b}$  are perpendicular.
- b) ...  $\mathbf{a}$  and  $\mathbf{b}$  are parallel.

$$p = -\frac{2}{3}, \quad p = \frac{1}{2}$$

**(a)**  $\mathbf{a} = (6, -3, 2)$   
 $\mathbf{b} = (4p+1, p-2, 1)$

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= 0 \\ (6, -3, 2) \cdot (4p+1, p-2, 1) &= 0 \\ 24p+6 - 3p + 6 - 2 &= 0 \\ 21p + 10 &= 0 \\ 21p &= -10 \\ p &= -\frac{10}{21} \\ p &= -\frac{2}{3}\end{aligned}$$

**(b)** IF PARALLEL, THEIR COMPONENTS MUST BE IN PROPORTION

$$\frac{4p+1}{6} = \frac{p-2}{-3} = \frac{1}{2}$$

Solving any pair:

$$\begin{aligned}\frac{p-2}{-3} &= \frac{1}{2} \\ p-2 &= -\frac{3}{2} \\ p-2 &= -1.5 \\ p &= -0.5\end{aligned}$$

**Question 22 (\*\*\*)**

$OABC$  is a parallelogram and the point  $M$  is the midpoint of  $AB$ .

The point  $N$  lies on the diagonal  $AC$  so that  $AN : NC = 1 : 2$ .

Let  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OC} = \mathbf{c}$ .

- Find simplified expressions, in terms of  $\mathbf{a}$  and  $\mathbf{c}$ , for each of the vectors  $\overrightarrow{AC}$ ,  $\overrightarrow{AN}$ ,  $\overrightarrow{ON}$  and  $\overrightarrow{NM}$ .
- Deduce, showing your reasoning, that  $O$ ,  $N$  and  $M$  are collinear.

$$\boxed{\quad}, \quad \boxed{\overrightarrow{AC} = \mathbf{c} - \mathbf{a}}, \quad \boxed{\overrightarrow{AN} = \frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a}}, \quad \boxed{\overrightarrow{ON} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c}}, \quad \boxed{\overrightarrow{NM} = \frac{1}{3}\mathbf{a} + \frac{1}{6}\mathbf{c}}$$

a) START BY DRAWING A DIAGRAM, AND LABEL VECTORS

$\bullet \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\mathbf{a} + \mathbf{c} = \underline{\underline{\mathbf{c} - \mathbf{a}}}$   
 $\bullet \overrightarrow{AN} = \frac{1}{3}\overrightarrow{AC} = \frac{1}{3}(\mathbf{c} - \mathbf{a}) = \underline{\underline{\frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a}}}$   
 $\bullet \overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \mathbf{a} + \frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a} = \underline{\underline{\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c}}}$   
 $\bullet \overrightarrow{NM} = \overrightarrow{NA} + \overrightarrow{AM} = -\overrightarrow{AN} + \overrightarrow{AM}$   
 $= -\left(\frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a}\right) + \frac{1}{2}\mathbf{a} = \underline{\underline{\frac{1}{3}\mathbf{a} + \frac{1}{6}\mathbf{c}}}$

b) ABOUT THE PROOF

$\overrightarrow{ON} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c} = \frac{1}{3}(2\mathbf{a} + \mathbf{c})$   
 $\overrightarrow{NM} = \frac{1}{3}\mathbf{a} + \frac{1}{6}\mathbf{c} = \frac{1}{6}(2\mathbf{a} + \mathbf{c})$

AS  $\overrightarrow{ON}$  &  $\overrightarrow{NM}$  ARE IN THE SAME DIRECTION & SINCE THE POINT  $N$ ,  $O$ ,  $N$  &  $M$  MUST BE COLLINEAR

**Question 23 (\*\*\*)**

The points with coordinates  $A(1, 4, 3)$ ,  $B(2, 2, 1)$  and  $C(5, 4, 0)$  are given.

- a) Find a vector equation of the straight line  $l$ , that passes through  $A$  and  $C$ .

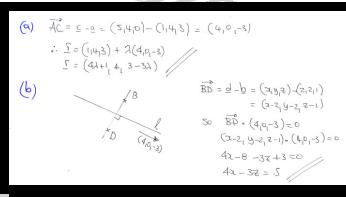
The point  $D(x, y, z)$  is such so that  $BD$  is perpendicular to  $l$ .

- b) Show clearly that

$$4x - 3z = 5.$$

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \lambda(4\mathbf{i} - 3\mathbf{k})$$

(a)  $\vec{AC} = C - A = (5, 4, 0) - (1, 4, 3) = (4, 0, -3)$   
 $\therefore \vec{l} = (1, 4, 3) + \lambda(4, 0, -3) \quad //$   
 $\vec{l} = (4\lambda + 1, 4, 3 - 3\lambda) \quad //$

(b) 

$$\begin{aligned}\vec{BD} &= d - b = (x, y, z) - (2, 2, 1) \\ &= (x-2, y-2, z-1) \\ \therefore \vec{BD} \cdot (\vec{l})_{\perp} &= 0 \\ (x-2, y-2, z-1) \cdot (4, 0, -3) &= 0 \\ 4x - 8 - 3z + 3 &= 0 \\ 4x - 3z &= 5 \quad //\end{aligned}$$

**Question 24 (\*\*\*)**

The straight line  $L_1$  passes through the points  $A(3,0,3)$  and  $B(5,5,2)$ .

The straight line  $L_2$  has a vector equation given by

$$\mathbf{r} = \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix},$$

where  $\mu$  is a scalar parameter.

- Write down the coordinates of the point of intersection of  $L_1$  and  $L_2$ .
- Find the size of the acute angle  $\theta$ , between  $L_1$  and  $L_2$ .
- Calculate the distance  $AB$ .

The point  $C$  lies on  $L_1$  so that the distance  $AB$  is equal to the distance  $AC$ .

- Determine the coordinates of  $C$ .

$$P(5,5,2), \quad \theta \approx 73.2^\circ, \quad |AB| = \sqrt{30}, \quad C(1, -5, 4)$$

④ INTERSECTION B OF POINT B(5,5,2)  
 ⑤  $\vec{AB} = \vec{b} - \vec{a} = (5,5,2) - (3,0,3) = (2,5,-1)$  → Job in case we need it  
 $\Sigma_1 = (3,0,3) + \lambda(2,5,-1) = (2\lambda+3, 5\lambda, 3-\lambda)$   
 Setting direction vectors  
 $(2,5,-1) \times (1,0,3) = [2,5,-1] \cdot [0,1,3] \text{ and}$   
 $2 + 0 + 3 = \sqrt{4+25+1} \cdot \sqrt{0+1+9}$   
 $= \frac{1}{\sqrt{30}} \text{ and}$   
 $\theta \approx 73.2^\circ$   
 ⑥  $|AB| = |2,5,-1| = \sqrt{4+25+1} = \sqrt{30}$   
 ⑦  $\vec{AC} = (1, -5, 4) - (3,0,3) = (-2, -5, 1)$   
 Using WORKING FORMULA  
 $\frac{-2-3}{\sqrt{30}} = 3$   
 $\frac{-5-0}{\sqrt{30}} = 0$   
 $\frac{1-3}{\sqrt{30}} = 3$   $\Rightarrow C(-1, -5, 4)$

**Question 25 (\*\*\*)**

A tunnel is to be dug through a mountain in order to link two cities.

Digging at one end of the tunnel begins at the point with coordinates  $(-3, -3, 9)$  and continues in the direction  $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ .

The digging at the other end of the tunnel starts at the point with coordinates  $(-19, -7, -3)$  and continues in the direction  $6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ .

Both sections are assumed to be straight lines.

The coordinates are measured relative to a fixed origin  $O$ , where one unit is 50 metres.

- Show that the two sections of the tunnel will eventually meet at a point  $P$ , and find the coordinates of this point.
- Find the total length of the tunnel.

|              |                      |
|--------------|----------------------|
| $P(5, 5, 5)$ | length = 2000 metres |
|--------------|----------------------|

**(a)**

$$\begin{aligned}\Gamma_1 &= (-3, -3, 9) + \lambda(2, 2, -1) = (2\lambda - 3, 2\lambda - 3, 9 - \lambda) \\ \Gamma_2 &= (-7, -7, 3) + \mu(6, 3, 2) = (6\mu - 7, 3\mu - 7, 2\mu + 3)\end{aligned}$$

Equation 1  $\perp$  Eqn 2

$$\begin{aligned}(1) : 2\lambda - 3 &= 6\mu - 7 \quad \left\{ \begin{array}{l} \text{Solve for } \lambda \\ \text{Solve for } \mu \end{array} \right. \\ (2) : 2\lambda - 3 &= 3\mu - 7 \end{aligned}$$

From (1):  $2\lambda - 3 = 6\mu - 7 \Rightarrow 2\lambda - 3 = 6\mu - 7 \Rightarrow 2\lambda = 6\mu + 4 \Rightarrow \lambda = 3\mu + 2$

Check (2):  $2\lambda - 3 = 3\mu - 7 \Rightarrow 2(3\mu + 2) - 3 = 3\mu - 7 \Rightarrow 6\mu + 4 - 3 = 3\mu - 7 \Rightarrow 3\mu + 1 = 3\mu - 7 \Rightarrow 1 = -7 \text{ (This is incorrect, so we need to recheck the components)} \Rightarrow \text{All three components agree so the lines intersect}$

Using  $\lambda = 4$  into  $(2\lambda - 3, 2\lambda - 3, 9 - \lambda)$  we get  $P(5, 5, 5)$  //

**(b)**

Diagram showing points A(-3, -3, 9), P(5, 5, 5), and B(-7, -7, 3). The distance AP is calculated as follows:

$$\begin{aligned}|AP| &= \sqrt{(5+3)^2 + (5+3)^2 + (5-9)^2} = \sqrt{64+64+16} \\ &= \sqrt{144} = 12\end{aligned}$$

$$\begin{aligned}|PB| &= \sqrt{(-7-5)^2 + (-7-5)^2 + (3-9)^2} \\ &= \sqrt{576+576+64} = \sqrt{1216} = 28\end{aligned}$$

$\therefore \text{TOTAL LENGTH} = (12 + 28) \times 50 = 2000 \text{ m}$

**Question 26 (\*\*\*)**

Relative to a fixed origin  $O$ , the straight lines  $l_1$  and  $l_2$  have respective vector equations given by

$$\mathbf{r}_1 = \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ p \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ -2 \\ 5 \end{pmatrix},$$

where  $\lambda$  and  $\mu$  are scalar parameters and  $p$  is a scalar constant.

The point  $T$  is the point of intersection between  $l_1$  and  $l_2$ .

Find in any order ...

- a) ... the size of acute angle between  $l_1$  and  $l_2$ .
- b) ... the value of  $p$ .
- c) ... the coordinates of  $T$ .

$$\theta \approx 15.4^\circ, \quad p = 1, \quad T(-17, 5, -9)$$

**(a)**  $\theta = 15.4^\circ$

$$\begin{aligned} \mathbf{r}_1 - (2, 1, 2) &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -3 \\ 5 \end{pmatrix} \cos \theta \\ 72 + 2 + 40 &= \sqrt{40+1+1} \sqrt{81+9+25} \cos \theta \\ 84 &= \sqrt{65} \sqrt{106} \cos \theta \\ \cos \theta &= \frac{84}{\sqrt{65} \sqrt{106}} \\ \theta &\approx 15.4^\circ \end{aligned}$$

**(b)**  $\begin{cases} l_1 = (2, 1, 2) + t(9, -3, 5) \\ l_2 = (4, p+1, p-2) + s(9, -2, 5) \end{cases}$

Equating  $t$  to  $s$ :

$$\begin{cases} 2t + 1 = 4 + s \\ 9t + 7 = p + 1 + s \\ 2t - 3 = p - 2 + s \end{cases} \quad \begin{cases} 9t + 7 = p + 1 \\ -18 + 12 + 3s = 4 - 2s \\ 3s = -22 \end{cases} \quad \begin{cases} s = -2 \\ t = -2 \end{cases}$$

$$\begin{aligned} & 2t - 3 = s + 1 \\ & 2t - 3 = -20 + 1 \\ & 2t = -16 \\ & t = -8 \end{aligned}$$

Looking at  $s$ :

$$\begin{aligned} 2t - 3 &= p + 1 \\ -16 - 3 &= p + 1 \\ -19 &= p + 1 \\ p &= -20 \end{aligned}$$

**(c)** Using  $s = -2$ ,  
 $(8, -7, 2-2, 2-3) \rightarrow (-17, 5, -9)$

**Question 27 (\*\*\*)**

With respect to a fixed origin  $O$ , the points  $A(2, 6, 5)$  and  $B(5, 0, -4)$  are given.

- a) Find a vector equation of the straight line  $L_1$ , which passes through  $A$  and  $B$ .

The straight line  $L_2$  has a vector equation

$$\mathbf{r}_2 = \begin{pmatrix} -4 \\ 4 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

where  $\mu$  is a scalar parameter.

- b) Show clearly that  $L_1$  and  $L_2$  intersect, and find the coordinates of their point of intersection.

The straight line  $L_3$  is in the direction  $\begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix}$ .

- c) Given the acute angle between  $L_2$  and  $L_3$  is  $60^\circ$ , show clearly that  $k = \pm\sqrt{6}$

$$\boxed{\mathbf{r}_1 = 2\mathbf{i} + 6\mathbf{j} + 5\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})}, \quad \boxed{(3, 4, 2)}$$

Handwritten working for Question 27:

(a)  $\vec{AB} = B-A = (5-2, 0-6, -4-5) = (3, -6, -9)$  SOME DIRECTIONS TO  $(1, 1, 1)$   
 $\mathbf{l}_1 = (2, 6, 5) + \lambda(3, -6, -9)$

(b)  $\mathbf{l}_2 = (-4, 4, -5) + \mu(1, 0, 1) = (-4-\mu, 4, -5+\mu)$   
 $\text{SOLVE } \begin{cases} -4-\mu = 2 \\ 4 = 4 \\ -5+\mu = 2 \end{cases} \Rightarrow \frac{\mu=6}{\mu=6}$  ALL 3 components agree so lines intersect  
 $\mu=6 \Rightarrow (-4-6, 4, -5+6) = (-10, 4, 1) = (3, 4, 2)$

(c) BY DOT PRODUCT  
 $\mathbf{l}_1(1, 1, 1) \cdot \mathbf{l}_3(1, k, 1) = |(1, 1, 1)| |(1, k, 1)| \cos 60^\circ$   
 $1+1+k = \sqrt{1+1+1} \sqrt{1+1+k^2+1} \times \frac{1}{2}$   
 $2 = \sqrt{3} \sqrt{2+k^2} \times \frac{1}{2}$   
 $\frac{4}{\sqrt{3}} = \sqrt{2+k^2}$   
 $4 = 2+k^2$   
 $k^2 = 2$   
 $k = \pm\sqrt{2}$  M 2 marks

**Question 28 (\*\*\*)**

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 2\mathbf{i} + 3\mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

$$\mathbf{r}_2 = 5\mathbf{i} + 3\mathbf{j} + 9\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - \mathbf{k}),$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- a) Show that  $l_1$  and  $l_2$  intersect at some point  $A$ , and find its coordinates.
- b) Show further that  $l_1$  and  $l_2$  intersect at right angles.

The point  $B$  lies on  $l_1$  where  $\lambda = -1$  and the point  $C$  lies on  $l_2$  where  $\mu = 3$ .

- c) Find the exact area of the triangle  $BAC$ .

$$A(6,5,8), \quad \boxed{\text{area} = 9\sqrt{14}}$$

(a)  $\begin{aligned} l_1 &= (2\mathbf{i}, 3\mathbf{j}, 0) + \lambda(2\mathbf{i}, 1\mathbf{j}, 4\mathbf{k}) = (2\lambda+2, 3+\lambda, 4\lambda) \\ l_2 &= (5\mathbf{i}, 3\mathbf{j}, 9\mathbf{k}) + \mu(\mathbf{i}, 2\mathbf{j}, -\mathbf{k}) = (\mu+5, 2\mu+3, 9-\mu) \end{aligned}$

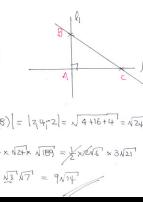
EQUATE  $\frac{l_1}{l_2}$  to  $\frac{1}{1}$

$\left\{ \begin{array}{l} 2\lambda+2 = \mu+5 \\ 3+\lambda = 2\mu+3 \\ 4\lambda = 9-\mu \end{array} \right. \quad \left. \begin{array}{l} \text{Add} \\ \text{Sub} \\ \text{Divide by } 4 \end{array} \right. \quad \left. \begin{array}{l} \lambda+2 = 16 \\ 6\lambda = 12 \\ \lambda = 2 \end{array} \right. \quad \left. \begin{array}{l} 4\lambda = 1-\mu \\ 8 = 9-\mu \\ \mu = 1 \end{array} \right.$

CHECK  $\lambda = 2$ :  $2+3 = 2\lambda+3$  ✓  
 $2\mu+3 = 2\lambda+3$  ✓ All 3 components agree so lines intersect

SOLVE  $\mu = 1$ :  $(4\mathbf{i}, 2\mathbf{j}, 3\mathbf{k}, 9-\mu)$  gives  $A(6,5,8)$  //

(b) DOTTING DIRECT VECTORS  $(2\mathbf{i}, 1\mathbf{j}, 4\mathbf{k}) \cdot (1\mathbf{i}, 2\mathbf{j}, -1\mathbf{k}) = 2+2-4 = 0 \Rightarrow \perp$

(c) 

$\bullet |AB| = \sqrt{(6+4)^2 + (5-2)^2 + (8+4)^2} = \sqrt{144+9+144} = \sqrt{307}$

$\bullet |AC| = \sqrt{(6-1)^2 + (5-7)^2 + (8-5)^2} = \sqrt{25+4+9} = \sqrt{38}$

$\bullet |BC| = \sqrt{(6-1)^2 + (5-7)^2 + (8-5)^2} = \sqrt{25+4+9} = \sqrt{38}$

$\therefore \text{Area} = \frac{1}{2} \times |BC| \times |AB| \sin 90^\circ = \frac{1}{2} \times \sqrt{38} \times \sqrt{307} = \sqrt{589 \times 307} = \sqrt{18271} = 134.67$

**Question 29 (\*\*\*)**

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 4\mathbf{i} + 7\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j})$$

$$\mathbf{r}_2 = 8\mathbf{i} + 5\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{k}),$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- Show that  $l_1$  and  $l_2$  intersect at some point  $A$  and find its coordinates.
- Calculate the acute angle between  $l_1$  and  $l_2$ .
- Find the distance  $AB$ .
- Show that the area of the triangle  $ABC$  is  $6\sqrt{3}$  square units.

$$A(6,5,4), [60^\circ], |AB| = 2\sqrt{2}$$

Q1

$$\begin{aligned} l_1 &= (4, 7, 4) + \lambda(1, -1, 0) = (4+4\lambda, 7-\lambda, 4) \\ l_2 &= (8, 5, 2) + \mu(1, 0, -1) = (8+\mu, 5, 2-\mu) \end{aligned}$$

• CIRCLE 1  
 $\begin{cases} 7-\lambda = 4 \\ 2-\mu = 4 \end{cases} \Rightarrow \begin{cases} \lambda = 3 \\ \mu = -2 \end{cases}$

• CIRCLE 2  
 $\begin{cases} 4+4\lambda = 8 \\ 2-\mu = 5 \end{cases} \Rightarrow \begin{cases} \lambda = 1 \\ \mu = -3 \end{cases}$

All 3 components agree in  $\lambda = 2$  &  $\mu = -2$   
So lines intersect

Using  $\lambda = 2$  we obtain  $A(6,5,4)$

(b)

Drawing direction vectors  
 $(1, -1, 0) \times (1, 0, -1) = (1, 1, 1) / |(1, 1, 1)| = \sqrt{3}$   
 $|1+0+0| = \sqrt{3} \text{ and } 0$   
 $\cos \theta = \frac{1}{\sqrt{3}}$   
 $\theta = 60^\circ$

(c)

$$|AB| = |\lambda - \mu| = |(6, 5, 4) - (8, 5, 2)| = |(-2, 0, 2)| = \sqrt{4+0+4} = \sqrt{8} = 2\sqrt{2}$$

(d)

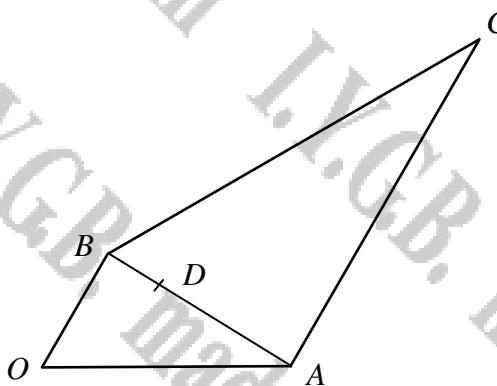
when  $\mu = -4$   $C(8, 5, -2)$   
 $|AC| = |3-8| = |(8, 5, 2) - (5, 5, 4)| = |(3, 0, -2)| = \sqrt{9+0+4} = \sqrt{13}$   
 $|BC| = \sqrt{(2-8)^2 + (5-5)^2 + (-2-2)^2} = \sqrt{36+0+16} = \sqrt{52} = 2\sqrt{13}$

$\text{Area} = \frac{1}{2}|AB||AC|\sin 60^\circ$   
 $= \frac{1}{2} \times 2\sqrt{2} \times \sqrt{13} \times \frac{\sqrt{3}}{2}$   
 $= 6\sqrt{3}$

ANSWER

**Question 30 (\*\*\*)**

The figure below shows a trapezium  $OBCA$  where  $OB$  is parallel to  $AC$ .



The point  $D$  lies on  $BA$  so that  $BD : DA = 1 : 2$ .

Let  $\overrightarrow{OA} = 4\mathbf{a}$ ,  $\overrightarrow{OB} = 3\mathbf{b}$  and  $\overrightarrow{AC} = 6\mathbf{b}$ .

- Find simplified expressions, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , for each of the vectors  $\overrightarrow{OC}$ ,  $\overrightarrow{AB}$ ,  $\overrightarrow{AD}$  and  $\overrightarrow{OD}$ .
- Deduce, showing your reasoning, that  $O, D$  and  $C$  are collinear and state the ratio of  $OD : DC$ .

$$\boxed{\overrightarrow{OC} = 4\mathbf{a} + 6\mathbf{b}}, \boxed{\overrightarrow{AB} = -4\mathbf{a} + 3\mathbf{b}}, \boxed{\overrightarrow{AD} = -\frac{8}{3}\mathbf{a} + 2\mathbf{b}}, \boxed{\overrightarrow{OD} = \frac{4}{3}\mathbf{a} + 2\mathbf{b}}, \\ \boxed{OD : DC = 1 : 2}$$

(a)

- $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = 4\mathbf{a} + 6\mathbf{b}$
- $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -4\mathbf{a} + 3\mathbf{b}$
- $\overrightarrow{AD} = \frac{2}{3}\overrightarrow{AB} = \frac{2}{3}(-4\mathbf{a} + 3\mathbf{b}) = -\frac{8}{3}\mathbf{a} + 2\mathbf{b}$
- $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = 4\mathbf{a} + (-\frac{8}{3}\mathbf{a} + 2\mathbf{b}) = \frac{4}{3}\mathbf{a} + 2\mathbf{b}$
- (b)  $\overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AC} = (\frac{8}{3}\mathbf{a} - 2\mathbf{b}) + 6\mathbf{b} = \frac{8}{3}\mathbf{a} + 4\mathbf{b} = 2(\frac{4}{3}\mathbf{a} + 2\mathbf{b})$

As vectors  $\overrightarrow{DC}$  &  $\overrightarrow{OD}$  ARE IN THE SAME DIRECTION AND SHARE D,  
THE POINTS O, D, C ARE COLLINEAR  
Hence,  $OD : DC = 1 : 2$

**Question 31 (\*\*\*)**

The points  $A(5,1,3)$ ,  $B(3,1,5)$ ,  $C(5,3,5)$  and  $D(4,0,3)$  are given.

- a) Show that the triangle  $ABC$  is equilateral and find its area.

- b) Show further that

$$\overrightarrow{AD} = \lambda \overrightarrow{AB} + \mu \overrightarrow{AC},$$

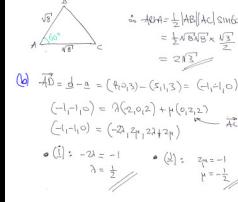
stating the exact values of the scalar constants  $\lambda$  and  $\mu$ .

- c) Find the size of the angle  $BAD$ .

$$\text{area} = 2\sqrt{3}, \quad \lambda = \frac{1}{2}, \mu = -\frac{1}{2}, \quad \angle BAD = 60^\circ$$

(a)  $|\overrightarrow{AB}| = |b - a| = |(3,1,5) - (5,1,3)| = |(-2,0,2)| = \sqrt{4+4} = \sqrt{8}$   
 $|\overrightarrow{BC}| = |c - b| = |(5,3,5) - (3,1,5)| = |(2,2,0)| = \sqrt{4+4} = \sqrt{8}$   
 $|\overrightarrow{CA}| = |a - c| = |(5,1,3) - (5,3,5)| = |(0,-2,-2)| = \sqrt{4+4} = \sqrt{8}$

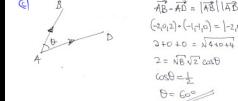
$\therefore$  EQUILATERAL AS ALL LENGTHS ARE EQUAL



$$\therefore \text{Area} = \frac{1}{2} |\overrightarrow{AB}| |\overrightarrow{AC}| \sin 60^\circ = \frac{1}{2} \sqrt{8} \sqrt{8} \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

(b)  $\overrightarrow{AB} = b - a = (3,1,5) - (5,1,3) = (-2,0,2)$   
 $(-1,1,0) = 2(0,1,0) + \mu(0,2,2)$   
 $(-1,1,0) = (-2,1,2) + 2\mu(0,2,2)$

$\bullet (1) \therefore -2\lambda = -1 \quad \lambda = \frac{1}{2}$        $\bullet (2) \therefore 2\mu = 1 \quad \mu = \frac{1}{2}$

(c) 

$$\overrightarrow{AB} \cdot \overrightarrow{AD} = |\overrightarrow{AB}| |\overrightarrow{AD}| \cos \theta$$

$$(2\sqrt{3})(1) \cos \theta = 1 \cdot 2\sqrt{3} \cos \theta$$

$$2\sqrt{3} \cos \theta = 2\sqrt{3} \cos \theta$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = 60^\circ$$

**Question 32 (\*\*\*)**

With respect to a fixed origin  $O$ , the straight lines  $l_1$  and  $l_2$  have respective vector equations given by

$$\mathbf{r}_1 = \begin{pmatrix} 9 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ p \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix},$$

where  $t$  and  $s$  are scalar parameters.

- a) If  $l_1$  and  $l_2$  are skew find the value that  $p$  **cannot** take.
- b) If  $l_1$  and  $l_2$  are not skew find the coordinates of their point of intersection.

$$p \neq 2, [3, -2, 0]$$

**(a)**  $\mathbf{r}_1 = (3, 0, 4) + t(3, 1, p) = (3+3t, t, tp+4)$   
 $\mathbf{r}_2 = (0, 4, 3) + s(1, -2, -1) = (s, 4-2s, 3-s)$

Given  $l_1 \not\parallel l_2$

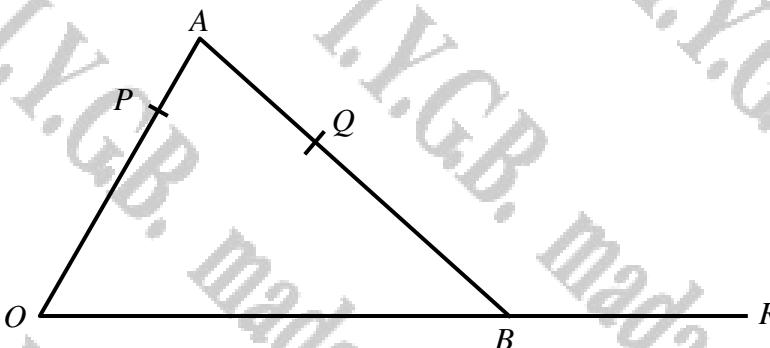
(i)  $3t+9 = s \Rightarrow 3(4-2s)+9 = s$   
(ii)  $t = 4-2s \Rightarrow 12-6s+9 = s$   
 $21 = 7s$   
 $s = 3$        $t = -2$

Now  $K$   
 $\frac{tp+4}{4-2s} \neq \frac{3-s}{3-2s}$   
 $-2p+8 \neq 3-2s$   
 $4 \neq 2p$   
 $p \neq 2$

**(b)** If lines do not skew they must intersect  $\Rightarrow p=2$   
using  $s=3$  into  $(s, 4-2s, 3-s)$  gives  $(3, -2, 0)$

**Question 33 (\*\*\*)**

The figure below shows a triangle  $OAB$ .

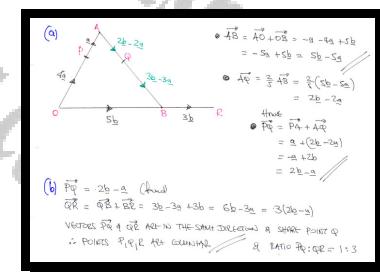


- The point  $P$  lies on  $OA$  so that  $OP:PA = 4:1$ .
- The point  $Q$  lies on  $AB$  so that  $AQ:QB = 2:3$ .
- The side  $OB$  is extended to the point  $R$  so that  $OB:BR = 5:3$ .

Let  $\overrightarrow{PA} = \mathbf{a}$  and  $\overrightarrow{OB} = 5\mathbf{b}$ .

- Find simplified expressions, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , for each of the vectors  $\overrightarrow{AB}$ ,  $\overrightarrow{AQ}$  and  $\overrightarrow{PQ}$ .
- Deduce, showing your reasoning, that  $P$ ,  $Q$  and  $R$  are collinear and state the ratio of  $PQ:QR$ .

$$\boxed{\overrightarrow{AB} = 5\mathbf{b} - 5\mathbf{a}}, \boxed{\overrightarrow{AQ} = 2\mathbf{b} - 2\mathbf{a}}, \boxed{\overrightarrow{PQ} = 2\mathbf{b} - \mathbf{a}}, \boxed{PQ:QR = 1:3}$$



**Question 34 (\*\*\*)**

The straight lines  $l_1$  and  $l_2$  have respective vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ a \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 5 \\ 8 \\ b \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix},$$

where  $\lambda$  and  $\mu$  are scalar parameters while  $a$  and  $b$  are positive constants.

Given that  $l_1$  and  $l_2$  intersect at some point  $P$ , forming an angle of  $60^\circ$ , determine in any order ...

- a) ... the value of  $a$ .
- b) ... the value of  $b$ .
- c) ... the coordinates of  $P$ .

$$a=5, b=10, P(12,9,10)$$

|  |
|--|
| <p>(a) <math>\mathbf{r}_1 = (6, 1, 0) + \lambda(3, 4, a)</math> <math>= (3\lambda+6, 4\lambda+1, a\lambda)</math><br/> <math>\mathbf{r}_2 = (5, 8, b) + \mu(7, 1, 0)</math> <math>= (7\mu+5, \mu+8, b)</math></p> <p>ANGLA MELAT <math>60^\circ</math> <math>\Rightarrow (3, 4, 0) \cdot (7, 1, 0) = \sqrt{9+16+a^2} \sqrt{49+1} \cos 60^\circ</math><br/> <math>21 \cdot 4 + 4 \cdot 0 = \sqrt{a^2+25} \cdot \sqrt{50} \cdot \frac{1}{2}</math><br/> <math>84 = \sqrt{a^2+25} \sqrt{50}</math><br/> <math>\sqrt{84} = \sqrt{a^2+25}</math><br/> <math>84 = a^2 + 25</math><br/> <math>a^2 = 59</math><br/> <math>a = \sqrt{59} \quad (a &gt; 0)</math></p> <p>(b) LINIS NONGERT (CONT'D)<br/> <math>\begin{aligned} 3\lambda+6 &amp;= 7\mu+5 \\ 3\lambda+1 &amp;= 7\mu+0 \end{aligned} \Rightarrow \boxed{\lambda=4\mu-7}</math> <math>\Rightarrow 3(4\mu-7) + 6 = 28\mu - 21 + 5</math><br/> <math>12\mu - 21 = 28\mu - 26</math><br/> <math>12\mu - 28\mu = -26 + 21</math><br/> <math>-16\mu = -5</math><br/> <math>\mu = \frac{5}{16}</math></p> <p><math>\therefore</math> INTERSECTION POINT IS <math>(3\lambda+6, 4\mu+1, a\mu) = (12, 9, 10)</math> //</p> <p>(c) <math>\therefore b=10</math> //</p> |
|--|

**Question 35 (\*\*\*)**

The points with coordinates  $A(8,0,12)$  and  $B(9,-2,14)$  are given.

- a) Find the vector equation of the straight line  $l_1$  that passes through  $A$  and  $B$ .

The straight line  $l_2$  has equation

$$\mathbf{r} = \mathbf{i} + 9\mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j}),$$

where  $\mu$  is a scalar parameter.

- b) Show that  $l_1$  and  $l_2$  are perpendicular.  
 c) Show further that  $l_1$  and  $l_2$  intersect at some point  $P$  and state the coordinates of  $P$ .

The point  $C(9,13,2)$  lies on  $l_2$  and the point  $D$  is the reflection of  $C$  about  $l_1$ .

- d) Determine the coordinates of  $D$ .

$$[ ] , \boxed{\mathbf{r}_1 = 8\mathbf{i} + 12\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})}, \boxed{P(3,10,2)}, \boxed{D(-3,7,2)}$$

Q)  $\vec{AB} = b - a = (9, -2, 14) - (8, 0, 12) = (1, -2, 2)$   
 $\vec{l}_1 = (\mathbf{i}, 0, 12) + \lambda(1, -2, 2)$   
 $\vec{l}_1 = (1 + \lambda, -2\lambda, 12 + 2\lambda)$

(L) Dot product method (check):  $(1, -2, 2) \cdot (2, 1, 0) = 2 - 2 + 0 = 0$   
 $\therefore \text{PERPENDICULAR}$

Q)  $\vec{l}_1 = (2 + \lambda, -2\lambda, 2 + 2\lambda)$   
 $\vec{l}_2 = (2\mu + 1, \mu + 9, 2)$   
 Equate  $\vec{l}_1 \parallel \vec{l}_2$   
 $\begin{cases} 2 + \lambda = 2\mu + 1 \\ -2\lambda = \mu + 9 \\ 2 + 2\lambda = 2 \end{cases}$   
 $\therefore \begin{cases} \lambda = -5 \\ -5 + 8 = 2\mu + 1 \\ 2 = 2 - 10 \\ \mu = 1 \end{cases}$   
 $\therefore \text{using } \mu = 1$   
 $\therefore \text{eqn } \vec{r} = (3, 10, 2)$

Q)  $P$  is the midpoint of  $DC$   
 $\frac{(9, 13, 2) + (-3, 7, 2)}{2} = (3, 10, 2)$   
 $\frac{2+9}{2} = 3 \Rightarrow \lambda = -3$   
 $\frac{-3+9}{2} = 3 \Rightarrow \mu = 1$   
 $\frac{2+2}{2} = 2 \Rightarrow z = 2$   
 $\therefore D(-3, 7, 2)$

**Question 36 (\*\*\*)**

Relative to a fixed origin  $O$ , the point  $A$  has position vector  $7\mathbf{i} + 4\mathbf{j}$  and the point  $B$  has position vector  $-3\mathbf{j} + 7\mathbf{k}$ . The straight line  $L_1$  passes through the points  $A$  and  $B$ .

- a) Find a vector equation for  $L_1$ .

The straight line  $L_2$  has a vector equation

$$\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} - 4\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$

where  $\mu$  is a scalar parameter.

- b) Show that  $L_1$  and  $L_2$  intersect at some point  $C$ , and find its position vector.  
 c) Show further that  $L_1$  and  $L_2$  are perpendicular.

The point  $D$  has position vector  $4\mathbf{i} - \mathbf{k}$ .

- d) Verify that  $D$  lies on  $L_2$ .

The point  $E$  is the image of  $D$  after reflection about  $L_1$ .

- e) Find the position vector of  $E$ .

$$[\quad], [\mathbf{r} = 7\mathbf{i} + 4\mathbf{j} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k})], [\overrightarrow{OC} = 5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}], [\overrightarrow{OE} = 6\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}]$$

(a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (0, -3, 7) - (7, 4, 0) = (-7, -7, 7)$   
 OR  $(1, 1, -1)$  is direction vector  
 $\mathbf{l}_1 = (7, 4, 0) + t(1, 1, -1) = (7+4t, 4+t, -t)$

(b)  $\mathbf{l}_2 = (3, -2, 4) + \mu(1, 2, 3) = (3+2\mu, -2+\mu, 3\mu+4)$   
 FOR  $\mathbf{l}_1 \perp \mathbf{l}_2$  components  
 $12+4\mu = -2+4 = 2$   $\Rightarrow \mu = -1$   
 $3+2\mu = 3-2 = 1$   $\Rightarrow \mu = 1$   
 $-2+\mu = -2+1 = -1$   $\Rightarrow \mu = 1$   
 Check  $\mu = 2+4 = -2+4 = 2$  As all three coordinates agree, the lines intersect  
 using  $\mu = 1$  in  $(3+2\mu, -2+\mu, 3\mu+4)$  we obtain:  $C(5, 1, 2)$

(c) Dotting direction vectors:  $(1, 1, -1) \cdot (1, 2, 3) = 1+2-3=0$   
 $\therefore L_1 \perp L_2$

(d) By inspection if  $\mu=1$   $(4+2\mu-2, 3\mu+4)$  becomes  $(4, 0, 1)$   
 $\therefore D$  lies on  $L_2$

(e)

C must be the midpoint of ED  
 $\left( \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right) = (5, 1, 2)$   
 $\left( \frac{4+5}{2}, \frac{0+1}{2}, \frac{-2+5}{2} \right) = (5, 1, 2)$   
 $\therefore z_2 = 6$   
 $y_2 = 4$   
 $x_2 = 5$   
 $\therefore E(5, 1, 6)$

**Question 37 (\*\*\*)+**

Relative to a fixed origin  $O$ , the points  $A$  and  $C$  have respective coordinates

$$(7, 2, 3) \text{ and } (3, -2, 1).$$

- a) Find the vector  $\overrightarrow{AC}$ .
- b) State the coordinates of the midpoint of  $AC$ .

The straight line  $l$  has vector equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix},$$

where  $\lambda$  is a scalar parameter.

- c) Show that  $\overrightarrow{AC}$  is perpendicular to  $l$ .

The point  $B$  lies on  $l$ , where  $\lambda = 1$ .

- d) Show further that the triangle  $ABC$  is isosceles but not equilateral.

The point  $D$  is such, so that  $ABCD$  is a rhombus.

- e) Show that the area of this rhombus is  $18\sqrt{2}$  square units.

,  $\boxed{\overrightarrow{AC} = -4\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}}$  ,  $\boxed{(5, 0, 2)}$

(a)  $\overrightarrow{AC} = S - B = (3, 2, 1) - (7, 1, 3) = (-4, 1, -2)$

(b) MIDPOINT  $M\left(\frac{3+2\lambda}{2}, \frac{1+1\lambda}{2}, \frac{2+2\lambda}{2}\right) \Rightarrow \left(\frac{3}{2}, 1, \frac{2+2\lambda}{2}\right) \Rightarrow M(5, 0, 2)$

(c)  $\overrightarrow{AC} \cdot ((1, 1, 1)) = (-4, 1, -2) \cdot (1, 1, 1) = -4 + 1 - 2 = -5 \therefore \text{PERPENDICULAR}$

(d)  $\overrightarrow{AB} = S - B = (3, 1, 1) - (7, 1, 3) = (-4, 0, -2)$   
 $\overrightarrow{BC} = b - a = (3, 1, 1) - (-7, 1, 3) = (10, 0, -2)$   
 $\overrightarrow{CA} = S - a = (3, 1, 1) - (3, 2, 1) = (0, -1, 0)$   
 $\overrightarrow{AB} = (10, 0, -2)$   
 $\overrightarrow{BC} = (10, 0, -2)$   
 $\overrightarrow{CA} = (0, -1, 0)$

$|\overrightarrow{AB}| = \sqrt{1+1+25} = \sqrt{27}$   
 $|\overrightarrow{BC}| = \sqrt{1+0+4} = \sqrt{5}$   
 $|\overrightarrow{CA}| = \sqrt{0+1+0} = 1$

$\therefore \text{isosceles but not equilateral}$

(e)

Area of rhombus  $ABCD$   
 $\frac{1}{2} |AC| |BD| = \frac{1}{2} \times 6\sqrt{3} \times 2\sqrt{5} = 6\sqrt{15}$   
 $\therefore \text{Area of } ABCD = 18\sqrt{2}$

$|\overrightarrow{BD}| = |b - d| = |\xi(0,2) - (\xi_1, \eta_1)|$   
 $= |(-1, -1, 4)| = \sqrt{18} = 3\sqrt{2}$

**Question 38 (\*\*\*)+**

Relative to a fixed origin  $O$ , the points  $A$  and  $B$  have respective position vectors

$$3\mathbf{i} - \mathbf{j} + 2\mathbf{k} \text{ and } -\mathbf{i} + \mathbf{j} + 9\mathbf{k}.$$

- Show that  $\overrightarrow{OA}$  and  $\overrightarrow{AB}$  are perpendicular.
- Find a vector equation of the straight line  $l$ , that passes through  $A$  and  $B$ .
- Determine the position vector of  $C$ .

$$\boxed{\quad}, \quad \boxed{\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(-4\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})}, \quad \boxed{\overrightarrow{OC} = -5\mathbf{i} + 3\mathbf{j} + 16\mathbf{k}}$$

(a)  $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (-1, 1, 9) - (3, -1, 2) = (-4, 2, 7)$   
 $\overrightarrow{OA} = \mathbf{a} = (3, -1, 2)$   
 $\therefore \overrightarrow{AB} \cdot \overrightarrow{OA} = (-4, 2, 7) \cdot (3, -1, 2) = -12 - 2 + 14 = 0$   
 $\therefore \overrightarrow{AB} \perp \overrightarrow{OA}$   $\therefore \overrightarrow{AB} \perp OA$

(b) Equation of line through  $A$  &  $B$   
 $\mathbf{l} = (3, -1, 2) + \lambda(-4, 2, 7) = (3 - 4\lambda, -1 + 2\lambda, 2 + 7\lambda)$

(c)   
 $|AB| = |BC|$   
 $B$  is the midpoint of  $AC$   
 $\therefore \frac{-1+3}{2}, \frac{1-1}{2} = (-1+\lambda, 1+2\lambda)$   
 $\therefore (-5, 3, 16)$

**Question 39 (\*\*\*)+**

Relative to a fixed origin  $O$ , the points  $A$ ,  $B$  and  $C$  have respective position vectors

$$\mathbf{i} + 10\mathbf{k}, \quad 4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k} \quad \text{and} \quad 8\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}.$$

- Show that  $A$ ,  $B$  and  $C$  are collinear, and find the ratio  $AB : BC$ .
- Find a vector equation for the straight line  $l$  that passes through  $A$ ,  $B$  and  $C$ .
- Show that  $OB$  is perpendicular to  $l$ .
- Calculate the area of the triangle  $OAC$ .

$$AB : BC = 3 : 4, \quad \mathbf{r} = \mathbf{i} + 10\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k}), \quad \text{area} = \frac{7}{2}\sqrt{222} \approx 52.15$$

(a)  $\vec{AB} = \mathbf{i} - 9\mathbf{k} = (1, 0, 7) - (1, 0, 10) = (0, 0, -3) = 3(0, 0, -1)$   
 $\vec{BC} = \mathbf{i} - \mathbf{j} - \mathbf{k} = (3, 7, 3) - (4, 3, 7) = (9, -4, -4) = 3(3, -4, -1)$   
 $\vec{AB}$  is in the same direction as  $\vec{BC}$ , so  $A, B, C$  are collinear  
AND RATIO IS  $3 : 4$

(b)  $\mathbf{l} = (1, 0, 10) + t(3, -4, -1)$

(c)  $\vec{OB} \cdot (\text{direction of } \omega_{OC}) = (4, 3, 7) \cdot (3, -4, -1) = 4 + 3 - 7 = 0$   
 $\therefore OB \perp \text{to line}$

(d)

$|\vec{OB}| = |(4, 3, 7)| = \sqrt{16 + 9 + 49} = \sqrt{74}$   
 $|\vec{AC}| = |\mathbf{i} - \mathbf{j} - \mathbf{k}| = \sqrt{(3, 7, 3) - (1, 0, 10)} = \sqrt{81 + 49} = \sqrt{130} = 3\sqrt{13}$   
 $\therefore \text{Area} = \frac{1}{2}|\vec{OB}| |\vec{AC}| = \frac{1}{2} \times \sqrt{74} \times 3\sqrt{13} = \frac{3}{2}\sqrt{958}$

**Question 40 (\*\*\*)+**

The points  $A$  and  $B$  have coordinates  $(11, 15, 4)$  and  $(13, 23, 7)$ , respectively.

- a) Find a vector equation for the straight line  $l$  that passes through  $A$  and  $B$ .

The point  $P$  lies on  $l$ , so that  $OP$  is perpendicular to  $l$ , where  $O$  is the origin.

- b) Show, without verification, that the coordinates of  $P$  are  $(7, -1, -2)$ .

- c) Calculate the area of the triangle  $OAB$ .

$$\boxed{\text{ }}, \boxed{\mathbf{r} = 11\mathbf{i} + 15\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} + 8\mathbf{j} + 3\mathbf{k})}, \boxed{\text{area} = \frac{3}{2}\sqrt{462} \approx 32.24}$$

①  $\vec{AB} = \vec{b} - \vec{a} = (13, 23, 7) - (11, 15, 4) = (2, 8, 3)$   
 $\therefore \vec{s} = (11, 15, 4) + t(2, 8, 3) \parallel l$   
 $\vec{s} = (2t+11, 8t+15, 3t+4)$

②  $\bullet \vec{v} \cdot \vec{p} = 0$   
 $\bullet \vec{OP} \perp l$   
 $\bullet (1, 2, 3) \cdot (7, -1, -2) = 0$   
 $\bullet \vec{OA} \times \vec{OB} = \vec{p}$   
 $\bullet (2, 8, 3) \cdot (2, 8, 3) = 462$   
 $\bullet \frac{3}{2}\sqrt{462} \approx 32.24$

SOLVING SIMULTANEOUS  $\Rightarrow$   $2(2t+1) + 8(8t+15) + 3(3t+4) = 0$   
 $4x + 2z + 64y + 120 + 9x + 12 = 0$   
 $11x + 64y + 12z = -136$   
 $t = -1.5$   
 $\vec{p} = (-2, -1, -2) \therefore P(-1, -1, -2)$

③  $A = \frac{1}{2} \times \text{base} \times \text{height}$   
 $= \frac{1}{2} |\vec{AB}| |\vec{OA}|$   
 $= \frac{1}{2} |2, 8, 3| |7, -1, -2|$   
 $= \frac{1}{2} \sqrt{462} \sqrt{58}$   
 $= \frac{1}{2} \sqrt{2774} \approx 32.24$

**Question 41 (\*\*\*)+**

Relative to a fixed origin  $O$ , the position vectors of the points  $A$ ,  $B$ ,  $C$  and  $D$  are

$$3\mathbf{i}, \quad 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \quad 4\mathbf{i} + \mathbf{k} \quad \text{and} \quad 4\mathbf{i} + \mathbf{j} + 4\mathbf{k}, \quad \text{respectively.}$$

- Show that  $\overrightarrow{AB}$  and  $\overrightarrow{BD}$  are perpendicular.
- Find the exact value of the cosine of the angle  $ABC$ .
- Determine the exact value of the area of triangle  $ABC$ .

$$\cos(\angle ABC) = \frac{8}{9}, \quad \text{area} = \frac{1}{2}\sqrt{17}$$

(a)  $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (2, 2, 2) - (3, 0, 0) = (-1, 2, 2)$   
 $\overrightarrow{BD} = \mathbf{d} - \mathbf{b} = (4, 1, 4) - (2, 2, 2) = (2, -1, 2)$   
 $\overrightarrow{AB} \cdot \overrightarrow{BD} = (-1, 2, 2) \cdot (2, -1, 2) = -2 - 2 + 4 = 0 \Rightarrow \perp$

(b)  $\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = (4, 0, 1) - (2, 2, 2) = (2, -2, -1)$   
 $\therefore \overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}| |\overrightarrow{BC}| \cos \theta$   
 $(1, -2, 1) \cdot (2, -2, -1) = |(1, -2, 1)| |(2, -2, -1)| \cos \theta$   
 $2 + 4 + 2 = \sqrt{1+4+1} \times \sqrt{4+4+1} \cos \theta$   
 $8 = 3\sqrt{2} \cos \theta$   
 $\cos \theta = \frac{8}{3\sqrt{2}}$

(c)  $\sqrt{\frac{8}{3}} \times \sqrt{\frac{8}{3}} \times \sin \theta$   
 $\text{BY PYTHAGOREAN}$   
 $\sin \theta = \frac{\sqrt{17}}{9}$

Area of  $ABC$  =  $\frac{1}{2} |\overrightarrow{AB}| |\overrightarrow{BC}| \sin \theta$   
 $= \frac{1}{2} \times 3\sqrt{2} \times \sqrt{\frac{17}{9}} \times \frac{\sqrt{17}}{9}$   
 $= \frac{1}{2} \times \sqrt{17}$

**Question 42 (\*\*\*)+**

Relative to a fixed origin  $O$ , the points  $A$ ,  $B$  and  $C$  have respective coordinates  $(5, 3, 1)$ ,  $(2, 2, 0)$  and  $(3, 4, -1)$ .

- Find the exact value of the cosine of the angle  $BAC$ .
- Show that the exact area of the triangle  $ABC$  is  $\frac{5}{2}\sqrt{2}$ .

$$\cos(\angle BAC) = \frac{7}{33}\sqrt{11}$$

(a)  $\overrightarrow{AB} = (2, 2, 0) - (5, 3, 1) = (-3, -1, -1)$   
 $\overrightarrow{AC} = (3, 4, -1) - (5, 3, 1) = (-2, 1, -2)$   
 $\Rightarrow (-3, -1, -1) \cdot (-2, 1, -2) = |(-3, -1, -1)| |(-2, 1, -2)| \cos \theta$   
 $6 + 1 + 2 = \sqrt{1+4+1} \times \sqrt{4+1+4} \cos \theta$   
 $\Rightarrow 7 = \sqrt{6} \times \sqrt{17} \cos \theta$   
 $\Rightarrow \cos \theta = \frac{7}{\sqrt{102}} \quad \text{or} \quad \frac{7}{33}\sqrt{11}$

(b)  $\sqrt{\frac{9}{2}} \times \sqrt{\frac{17}{2}} \times \sin \theta$   
 $\text{BY PYTHAGOREAN}$   
 $\sin \theta = \frac{\sqrt{11}}{33}$

Area of triangle  $ABC$  =  $\frac{1}{2} |\overrightarrow{AB}| |\overrightarrow{AC}| \sin \theta$   
 $= \frac{1}{2} \times \sqrt{6} \times \sqrt{17} \times \frac{\sqrt{11}}{33}$   
 $= \frac{3}{2} \times \sqrt{6} \times \sqrt{\frac{11}{17}}$   
 $= \frac{3}{2} \times \sqrt{6} \times \sqrt{\frac{1}{2}} = \frac{3}{2}\sqrt{3}$

**Question 43 (\*\*\*)+**

The straight line  $l$  passes through the points  $P(-5, 9, -9)$  and  $Q(a, b, 11)$ , where  $a$  and  $b$  are scalar constants.

The vector equation of  $l$  is given by

$$\mathbf{r} = \begin{pmatrix} 1 \\ 7 \\ c \end{pmatrix} + \lambda \begin{pmatrix} d \\ -1 \\ 2 \end{pmatrix},$$

where  $c$  and  $d$  are scalar constants and  $\lambda$  is a scalar parameter.

- a) Determine in any order the value of each the constants  $a$ ,  $b$ ,  $c$  and  $d$ .

The point  $T$  with  $x$  coordinate 4 lies on  $l$ .

- b) Show clearly that ...

i. ...  $OT$  is perpendicular to  $l$ , where  $O$  is the origin.

ii. ...  $PT:TQ = 3:7$ .

$$[a = 25], [b = -1], [c = -5], [d = 3]$$

(a)  $\mathbf{r} = \begin{pmatrix} 1 \\ 7 \\ c \end{pmatrix} + \lambda \begin{pmatrix} d \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+ad \\ 7-\lambda \\ c+2\lambda \end{pmatrix}$

Now  $P(3, 9, -9)$  lies on the line

$$\begin{aligned} 1+ad &= 3 & 7-\lambda &= 9 & c+2\lambda &= -9 \\ -2+a &= 2 & -\lambda &= 2 & c+4 &= -9 \\ \boxed{a=2} & & \boxed{\lambda=-2} & & \boxed{c=-5} & \\ d &= 3 & G &= 24 & & \\ \boxed{d=3} & & \boxed{G=24} & & & \end{aligned}$$

AND  $Q(4, b, 11)$  also lies on the line:  $\mathbf{r} = \begin{pmatrix} 1+ad \\ 7-\lambda \\ c+2\lambda \end{pmatrix}$

$$\begin{aligned} 4 &= 1+ad & 7-\lambda &= b & c+2\lambda &= 11 \\ 2a &= 3 & 7-a &= b & c+4 &= 11 \\ \boxed{a=2} & & \boxed{a=5} & & \boxed{c=7} & \\ \boxed{a=2} & & \boxed{a=5} & & \boxed{b=-1} & \end{aligned}$$

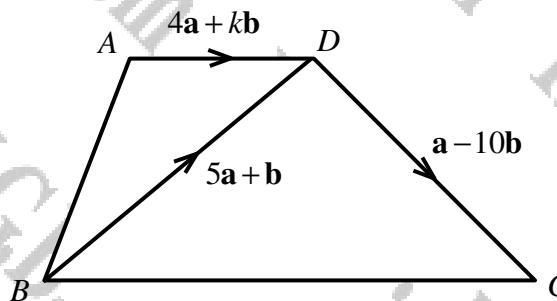
(b)  $T(4, y, z) = (1+3\lambda, 7-\lambda, 2\lambda-5)$

$$\begin{aligned} i: 1+3\lambda &= 4 & j: 7-\lambda &= 0 & k: 2\lambda-5 &= 0 \\ \boxed{\lambda=1} & & \boxed{\lambda=7} & & \boxed{\lambda=2.5} & \\ \overrightarrow{OT} &= (4, 6, -3) & \text{SLOPE OF LINE } l &= (1, -1, 2) & \overrightarrow{OT} &= (4, 6, -3) \\ \text{Hence } (4, 6, -3) \cdot (1, -1, 2) &= 12 - 6 - 6 = 0 & & & & \text{PERPENDICULAR INDICATED} \end{aligned}$$

(ii)  $\begin{aligned} ATP: & \lambda=2 \\ ATQ: & \lambda=8 \\ ATT: & \lambda=1 \end{aligned}$

$\therefore PT:TQ = 3:7$

Question 44 (\*\*\*)+



The figure above shows a trapezium  $ABCD$  where  $AD$  is parallel to  $BC$ .

The following information is given for this trapezium.

$$\overrightarrow{BD} = 5\mathbf{a} + \mathbf{b}, \quad \overrightarrow{DC} = \mathbf{a} - 10\mathbf{b} \quad \text{and} \quad \overrightarrow{AD} = 4\mathbf{a} + k\mathbf{b},$$

where  $k$  is an integer.

- Find the value of  $k$ .
- Find a simplified expression for  $\overrightarrow{AB}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

$$k = -6, \quad \overrightarrow{AB} = -\mathbf{a} - 7\mathbf{b}$$

(a)

$\overrightarrow{DC} = \overrightarrow{BD} + \overrightarrow{DC}$   
 $= 5\mathbf{a} + \mathbf{b} + \mathbf{a} - 10\mathbf{b}$   
 $= 6\mathbf{a} - 9\mathbf{b}$

$\overrightarrow{AD} = 4\mathbf{a} + k\mathbf{b}$   
 $\overrightarrow{DC} = \mathbf{a} - 10\mathbf{b}$  } these are equal  
 $\therefore \frac{1}{6}\mathbf{a} = \frac{k}{4}\mathbf{b}$   
 $\Rightarrow k = -6$

(b)  $\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} = 4\mathbf{a} - 6\mathbf{b} - 5\mathbf{a} - \mathbf{b} = -\mathbf{a} - 7\mathbf{b}$

**Question 45 (\*\*\*)+**

The straight lines  $L_1$  and  $L_2$  have respective vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix} + t \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 9 \\ 8 \\ -2 \end{pmatrix} + s \begin{pmatrix} -5 \\ -5 \\ 2 \end{pmatrix},$$

where  $t$  and  $s$  are scalar parameters.

- Show that  $L_1$  and  $L_2$  intersect at some point  $P$ , and find its coordinates.
- Find the exact value of the cosine of the acute angle  $\theta$ , between  $L_1$  and  $L_2$ .

The point  $A(9, -4, 4)$  lies on  $L_1$  and the point  $B(4, 3, 0)$  lies on  $L_2$ .

- Find the distance of  $AP$  and the distance of  $BP$ .
- Show the area of the triangle  $APB$  is  $9\sqrt{14}$ .

$$P(-1, -2, 2), \cos \theta = \frac{\sqrt{2}}{3}, |AP| = 6\sqrt{3}, |BP| = 3\sqrt{6}$$

**(a)** Given vector equations of lines:

$$\begin{aligned} \mathbf{r}_1 &= (4, -3, 3) + t(5, -1, 1) = (4+5t, -3-t, 3+t) \\ \mathbf{r}_2 &= (9, 8, -2) + s(-5, -5, 2) = (9-5s, 8-5s, -2+2s) \end{aligned}$$

Equating components:

$$\begin{cases} 4+5t = 9-5s \\ -3-t = 8-5s \\ 3+t = -2+2s \end{cases}$$

Solving the system of equations:

$$\begin{aligned} (1) &\quad 5t+5s = 5 \\ (2) &\quad -t = 11-5s \\ (3) &\quad t = 8-2s \end{aligned}$$

$$\begin{aligned} 5(8-2s) + 5s &= 5 \\ 40-10s + 5s &= 5 \\ -5s &= -35 \\ s &= 7 \end{aligned}$$

Check:  $t = 8-2(7) = -6$

$$\begin{aligned} t+3 &= -6+3 = -3 \\ -3 &= -3 \quad \text{All three components agree, lines intersect} \\ \text{Using } s=7 \text{ in } (9-5s, 8-5s, -2+2s) \\ \text{we obtain } P(-1, -2, 2) \end{aligned}$$

**(b)** Dotting direction vectors:

$$\begin{aligned} (5, -1, 1) \cdot (-5, -5, 2) &= (5, -1, 1) \cdot (-5, -5, 2) \cdot \cos \theta \\ -25 + 5 + 2 &= \sqrt{25+1+1} \cdot \sqrt{25+25+4} \cdot \cos \theta \\ -18 &= \sqrt{27} \cdot \sqrt{54} \cdot \cos \theta \\ \cos \theta &= -\frac{\sqrt{27}}{\sqrt{54}} \\ \cos \theta &= -\frac{\sqrt{3}}{2} \quad \therefore \text{acute angle is } \frac{\pi}{3} \end{aligned}$$

**(c)**  $|AP| = \sqrt{(9+1)^2 + (-4+2)^2 + (4-3)^2} = \sqrt{100+4+1} = \sqrt{105}$   
 $|BP| = \sqrt{(4+1)^2 + (3+2)^2 + (0-3)^2} = \sqrt{25+25+9} = 4\sqrt{6}$

**(d)** Diagram shows triangle  $APB$  with base  $AB$  and height from  $P$  perpendicular to  $AB$ .  
 $\text{Area of triangle} = \frac{1}{2}|AB| \sin \theta$   
 $= \frac{1}{2} \sqrt{105} \times \sqrt{54} \times \frac{\sqrt{3}}{2}$   
 $\therefore \text{Area} = \frac{1}{2} \times \sqrt{27} \times \sqrt{54} \times \frac{\sqrt{3}}{2}$   
 $= \frac{1}{2} \times 54 \times \frac{\sqrt{54}}{2} \times \frac{\sqrt{3}}{2}$   
 $= 9\sqrt{14}$

**Question 46 (\*\*\*)+**

With respect to a fixed origin  $O$ , the straight line  $l$  has vector equation

$$\mathbf{r} = \begin{pmatrix} a \\ b \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix},$$

where  $a$  and  $b$  are scalar constants and  $\lambda$  is a scalar parameter.

- a) If  $l$  passes through the point  $P(7, 3, 6)$ , find the value of  $a$  and the value of  $b$ .

The point  $Q$  lies on  $l$  so that  $OQ$  is perpendicular to  $l$ .

- b) Find the coordinates of  $Q$ .

The point  $T$  lies on  $l$  where  $\lambda = -7$ .

- c) Find the ratio  $PQ : QT$ .

,  $[a=7]$  ,  $[b=5]$  ,  $[Q(7,0,0)]$  ,  $[PQ:QT = 3:2]$

(a)  $\underline{s} = (a, b, 10) + \lambda(0, 1, 2) = (a, b+2\lambda, 10)$   
 $(7, 3, 6)$        $\therefore a=7$   
 $7 = a + 2\lambda$        $\lambda = 2$   
 $7 = 7 + 2\lambda$        $\lambda = 0$   
 $3 = b + 2\lambda$        $b = 3$   
 $3 = b + 2\lambda$        $b = 3$

(b) Let  $\underline{q} = (x, y, z)$   
 $OQ \perp l$   
 $(7, 3, 6) \cdot (0, 1, 2) = 0$   
 $7 + 2z = 0$   
 $z = -\frac{7}{2}$   
 $\bullet$   $Q$  lies on line  $l$   
 $(7, 3, 6) = (a, b+2\lambda, 10)$   
 $(7, 3, 6) = (7, 3, 6)$   
 $x = 7$   
 $y = 3$   
 $z = -\frac{7}{2}$   
 $\therefore Q(7, 3, -3.5)$

Hence:  $(7, 3, -3.5) = (a, b+2\lambda, 10)$   
 $7 = a + 2\lambda$   
 $3 = b + 2\lambda$   
 $-3.5 = 2\lambda$   
 $\lambda = -1.75$   
 $\therefore Q(7, 3, -3.5)$

(c) When  $\lambda = -7$ ,  $T(7, -7, -4)$  &  $P(7, 3, 6)$  &  $Q(7, 3, -3.5)$   
 $\overrightarrow{PQ} = q - p = (7, 3, -3.5) - (7, 3, 6) = (0, -3, -9.5) = 3(0, -1, -2.5)$   
 $\overrightarrow{QT} = t - q = (7, -7, -4) - (7, 3, -3.5) = (0, -10, -0.5) = 2(-1, -5, 0.25)$   
 $\therefore PQ : QT = 3 : 2$

**Question 47 (\*\*\*)+**

Relative to a fixed origin  $O$ , the straight lines  $l$  and  $m$  have respective vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix} + s \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix},$$

where  $t$  and  $s$  are scalar parameters, and  $a$  is a constant.

$l$  and  $m$  intersect at the point  $P(8, 1, -3)$ .

a) Find the value of  $a$ .

b) Show that the vector  $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$  is perpendicular to both  $l$  and  $m$ .

c) Determine a vector equation of the straight line  $n$ , such that all three straight lines intersect, with the  $n$  being perpendicular to both  $l$  and  $m$ .

$$a = -2, \quad \mathbf{r} = 8\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \nu(4\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

|  |
|--|
| $\text{(a)} \quad \begin{aligned} \mathbf{r}_1 &= (6, 1, 1) + t(1, 0, a) = (6+t, 1, at+1) \\ \mathbf{r}_2 &= (4, -3, 7) + s(2, 2, -5) = (2s+4, 2s-3, 7-5s) \end{aligned}$  |
| LINE INTERSECT AT $(8, 1, -3)$<br>WORKING AT $\frac{1}{2}$ : $\begin{cases} 6+t = 8 \\ 2s+4 = 8 \\ 2s-3 = -3 \end{cases}$<br>$\begin{cases} t = 2 \\ s = 2 \\ s = 0 \end{cases}$   |
| $\begin{aligned} &\text{WORKING AT } \frac{1}{2}: \begin{cases} 6+t = 8 \\ 2s+4 = 8 \\ 2s-3 = -3 \end{cases} \Rightarrow \begin{cases} t = 2 \\ s = 2 \\ s = 0 \end{cases} \\ &\text{WORKING AT } \frac{1}{2}: \begin{cases} 6+t = 8 \\ 2s+4 = 8 \\ 2s-3 = -3 \end{cases} \Rightarrow \begin{cases} t = 2 \\ s = 2 \\ s = 0 \end{cases} \end{aligned}$ |
| $\begin{aligned} \text{(b)} \quad (4, 1, 2) \cdot (1, 0, -2) &= 4+0-4=0 \\ (4, 1, 2) \cdot (2, 2, -5) &= 8+2-10=0 \quad \therefore \text{PERPENDICULAR TO BOTH LINES} \end{aligned}$   |
| $\begin{aligned} \text{(c)} \quad \mathbf{r}_3 &= (4, -3, 7) + \lambda(4, 2, -5) \\ \mathbf{r}_3 &= (4\nu+4, 2\nu-3, 7-5\nu) \end{aligned}$  |

**Question 48 (\*\*\*)+**

With respect to a fixed origin  $O$ , the straight line  $l$  has vector equation

$$\mathbf{r} = \begin{pmatrix} 24 \\ 6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

where  $\lambda$  is a scalar parameter.

- a) If the point  $D(4, a, b)$  lies on  $l$ , find the value of  $a$  and the value of  $b$ .

The point  $P$  lies on  $l$  where  $\lambda = p$ , and the point  $C$  has coordinates  $(18, 6, 36)$ .

b) Show that  $\overrightarrow{CP} = \begin{pmatrix} p+6 \\ p \\ 2p-36 \end{pmatrix}$ .

- c) Given further that  $\overrightarrow{CP}$  is perpendicular to  $l$ , find the coordinates of  $P$ .

$$[a = -14], [b = -40], [P(35, 17, 22)]$$

**④**  $\Sigma = C(1, 6, 0) + \lambda C_1(1, 2) = (1+2\lambda, 1+\lambda, 2\lambda)$   
 By inspection of  $l$ :  $1+2\lambda=4$        $\therefore \lambda=2$        $a=1+6=7$        $b=2\lambda=4$   
 $\lambda=2$        $a=7$        $b=4$

**⑤** If  $\lambda=p$        $P(p+2, p+6, 2p)$   
 $C(18, 6, 36)$   
 $\therefore \overrightarrow{CP} = p-18 + C(p+2-18, p+6-6, 2p-36) = (p+6, p, 2p-36)$

**⑥**  $\overrightarrow{CP} = p-18 + C(p+2-18, p+6-6, 2p-36) = (p+6, p, 2p-36)$   
 If perpendicular:  $(p+6, p, 2p-36) \cdot (1, 1, 2) = 0$   
 $p+6+p+4p-72=0$   
 $6p=66$   
 $p=11$   
 $\therefore P(17, 17, 22)$

**Question 49** (\*\*\*)+

The straight line  $l$  has the following vector equation

$$\mathbf{r} = -2\mathbf{i} - 12\mathbf{j} - 9\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}),$$

where  $\lambda$  is a scalar parameter.

The point  $P(a, b, 3)$  lies on  $l$ .

- a) Find the value of each of the scalar constants  $a$  and  $b$ .

The point  $O$  represents a fixed origin.

The point  $Q$  lies on  $l$ , so that  $OQ$  is perpendicular to  $l$ .

- b) Show that the coordinates of  $Q$  are  $(2, 0, -1)$ .

You may not verify this fact by using the coordinates of  $Q$ .

- c) Find the exact area of the triangle  $OPQ$ .

$$[\square], [a=4], [b=6], [\text{area} = \sqrt{70}]$$

a) LOOKING AT THE LINE IN FULL PARAMETRIC FORM

$$\begin{aligned}\mathbf{r} &= \begin{pmatrix} -2 \\ -12 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2+\lambda \\ -12+3\lambda \\ -9+2\lambda \end{pmatrix} \\ \bullet 2-2=a & \bullet 3A-12=b & \bullet 2A-9=3 \\ 2A-12 & & 2A-9 \\ \therefore A=6 & & \therefore A=6 \\ \bullet 3A-12=b & & \\ \bullet 6-2=c & & \\ \therefore a=4 & & \end{aligned}$$

b) LOOKING AT THE DIAGRAM, LET  $Q = (2, 0, z)$

$$\begin{aligned}\mathbf{OQ} \perp l &\Rightarrow (\mathbf{Q}(2,0)) \cdot (1, 3, 2) = 0 \\ &\Rightarrow [2+3z+2z=0] \\ \text{BUT } \mathbf{Q}(2, 0, z) \text{ lies on } l & \\ &\Rightarrow \begin{cases} 2=2-2 \\ 0=3z \\ z=2z \end{cases} \\ &\Rightarrow (2-2)+3(3z-1)+2(2z-9)=0 \\ &\Rightarrow 2-2+9z-3+4z-18=0 \\ &\Rightarrow 13z=36 \\ &\Rightarrow z=4 \\ &\therefore Q(2, 0, -1) \end{aligned}$$

c) LOOKING AT THE DIAGRAM

$$\begin{aligned}|\mathbf{OQ}| = |Q| &= |(2, 0, -1)| = \sqrt{4+0+1} = \sqrt{5} \\ |\mathbf{PQ}| = |Q-P| &= |(2, 0, -1) - (4, 6, 3)| \\ &= \sqrt{4+36+16} \\ &= \sqrt{56} \\ \therefore \text{AREA} &= \frac{1}{2} |\mathbf{OQ}| |\mathbf{PQ}| = \frac{1}{2} \sqrt{5} \times \sqrt{56} = \frac{1}{2} \sqrt{280} \\ &= \sqrt{70} \end{aligned}$$

**Question 50 (\*\*\*)+**

The straight line  $L$  has the vector equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix},$$

where  $\lambda$  is a scalar parameter.

The straight line  $M$  passes through the points with coordinates  $A(10, 6, 6)$  and  $B(\alpha, \beta, 3)$ , where  $\alpha$  and  $\beta$  are scalar constants.

$L$  and  $M$  intersect at the point  $C(6, 8, 0)$ .

- a) Find the coordinates of  $B$ .
- b) Calculate the acute angle between  $L$  and  $M$ .

$$P(8, 7, 3), \quad [62.2^\circ]$$

(a)

EQUATION M:

$$\vec{AC} = \vec{C} - \vec{A} = (6, 8, 0) - (10, 6, 6) = (-4, 2, -6)$$

$$\text{and } C(6, 8, 0) \text{ is on } M \text{ so } \vec{AB} = \vec{B} - \vec{A} = (\alpha, \beta, 3) - (10, 6, 6) = (\alpha - 10, \beta - 6, 3 - 6)$$

$$\therefore M: \vec{r} = (10, 6, 6) + \lambda (6, 2, -6)$$

$$= (2\alpha + 10, 2\beta + 6, 3 - 6\lambda)$$

By inspection of E:  $\frac{\vec{r} - P}{\sqrt{1+1+1}} = \frac{3\vec{i} + 2\vec{j} + 3\vec{k}}{\sqrt{1+1+1}}$

$$\therefore \alpha = 2\alpha + 10 \quad \beta = 2\beta + 6 \quad \lambda = 3 - 6\lambda$$

$$\alpha = 6 \quad \beta = 6 \quad \lambda = 1$$

$$\therefore B(6, 6, 3)$$

(b) DOTTING DIRECTION VECTORS

$$(1, 4, -2) \cdot (3, -1, 3) = 1\sqrt{1+16+4} \sqrt{9+1+9} \cos \theta$$

$$\Rightarrow 2 + 4 - 6 = \sqrt{1+16+4} \sqrt{9+1+9} \cos \theta$$

$$\Rightarrow -2 = \sqrt{21} \sqrt{14} \cos \theta$$

$$\Rightarrow \cos \theta = \frac{-2}{\sqrt{21} \sqrt{14}}$$

$$\Rightarrow \theta \approx 117.8^\circ \quad \therefore \text{acute angle is } 62.2^\circ$$

**Question 51 (\*\*\*)+**

The straight line  $l_1$  passes through the points  $A(2,8,1)$  and  $B(2,4,3)$ .

- a) Find a vector equation for  $l_1$ , in terms of a scalar parameter  $\lambda$ .

The straight line  $l_2$  has a vector equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ a \end{pmatrix} + \mu \begin{pmatrix} 1 \\ b \\ -1 \end{pmatrix},$$

where  $a$  and  $b$  are scalar constants, and  $\mu$  is a scalar parameter.

The point  $C(2,-4,c)$ , where  $c$  is a scalar constant, is the point of intersection between  $l_1$  and  $l_2$ .

- b) Find the value of each of the scalar constants  $a$ ,  $b$  and  $c$ .  
 c) Determine the ratio  $AB : BC$ .

$$\boxed{\mathbf{r}_1 = 2\mathbf{i} + 8\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{j} - \mathbf{k})}, \quad \boxed{a=4}, \quad \boxed{b=2}, \quad \boxed{c=7}, \quad \boxed{AB : BC = 1 : 2}$$

**Q1**  $\vec{AB} = \mathbf{b} - \mathbf{a} = (2,4,3) - (2,8,1) = (0,-4,-2)$   
 USE  $(0,-4,-2) = 4k$  & DIVIDE BY 4  
 $\therefore l_1 = (2,8,1) + k(0,-4,-2)$   
 $\therefore l_1 = (2,8,1) + k(-1,2,1)$

**Q2**  $\vec{l}_2 = (5,2,a) + \mu(1,b,-1)$   
 $\vec{l}_2 = (5,2,7) + \mu(1,2,-1)$   
 EQUALISE COORDINATES

|   |                               |   |
|---|-------------------------------|---|
| $\begin{cases} 2 = 1 + \mu \\ 2 = 2 + \mu \\ 7 = a - \mu \end{cases}$ | $\Rightarrow \boxed{\mu = 3}$ | $\begin{cases} 2 + \mu = -3b + 2 = -4 \\ 1 + \mu = a + 3 = c \end{cases}$ |
|---|-------------------------------|---|

$\therefore \begin{cases} 2 + 3 = -4 \\ 2 + 3 = -4 \\ 7 = a + 3 \end{cases}$

$\therefore \begin{cases} 5 = -4 \\ 5 = -4 \\ 7 = a + 3 \end{cases}$

$\therefore \boxed{a = 4}, \boxed{b = 2}, \boxed{c = 7}$

**Q3**  $\vec{AB} = -\mathbf{b} = (2,-4,-2) - (2,8,1) = (0,-4,-3) = \boxed{(0,-4,-3)}$   
 $4B = (0,-4,-2) = 2(0,-2,1)$   
 $\therefore \text{RATIO } 0 : 2 : 4 \text{ i.e. } 1 : 2$

**Question 52 (\*\*\*)+**

The straight lines  $L_1$  and  $L_2$  have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 12 \\ 7 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 21 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix},$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- Show that  $L_1$  and  $L_2$  intersect at the point  $P$ , and find its coordinates .
- Show further that  $L_1$  and  $L_2$  are perpendicular to each other.
- Find the distance  $AB$ .
- Hence state the shortest distance of  $P$  from the line through  $A$  and  $B$ .

$$P(6,13,3) , |AB| = 12\sqrt{7} , 6\sqrt{7}$$

(a)  $\mathbf{r}_1 = (12, 7, 1) + \lambda(3, -3, -1) = (3\lambda + 12, 7 - 3\lambda, 1 - \lambda)$   
 $\mathbf{r}_2 = (0, 1, 21) + \mu(1, 2, -3) = (\mu, 1 + 2\mu, 21 - 3\mu)$

Given  $L_1$  &  $L_2$   
 $\therefore 3\lambda + 12 = \mu \quad (1)$   
 $7 - 3\lambda = 1 + 2\mu \quad (2)$   
 $1 - \lambda = 21 - 3\mu \quad (3)$

Multiply (1) by 3  
 $9\lambda + 36 = 3\mu$   
 $9\lambda + 36 = 3\mu$   
 $3\lambda + 12 = \mu$

Sub into (2)  
 $7 - 3(\mu - 12) = 1 + 2\mu$   
 $7 - 3\mu + 36 = 1 + 2\mu$   
 $43 - 3\mu = 1 + 2\mu$   
 $42 = 5\mu$   
 $\mu = 8.4$

Sub into (3)  
 $1 - (\mu - 12) = 21 - 3\mu$   
 $1 - 8.4 + 12 = 21 - 3(8.4)$   
 $13.6 = 21 - 25.2$   
 $13.6 = -3.6$   
 $17 = 6$  (WRONG)

Dot product  $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3(-3) + (-3)(-1) = -9 + 3 = -6$   
 $\therefore \mathbf{v}_1 \perp \mathbf{v}_2$

(b) Dot product  $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3(-3) + (-3)(-1) = -9 + 3 = -6$   
 $\therefore \mathbf{v}_1 \perp \mathbf{v}_2$

(c)

•  $|\overrightarrow{AP}| = \sqrt{(7-1)^2 + (1-1)^2 + (21-3)^2} = \sqrt{6^2 + 0^2 + 18^2} = \sqrt{36 + 0 + 324} = \sqrt{360} = 6\sqrt{10}$

•  $|\overrightarrow{BP}| = \sqrt{(6-6)^2 + (13-1)^2 + (3-21)^2} = \sqrt{0^2 + 12^2 + (-18)^2} = \sqrt{0 + 144 + 324} = \sqrt{468} = 2\sqrt{117}$

By Pythagoras  
 $|AB|^2 = |\overrightarrow{AP}|^2 + |\overrightarrow{BP}|^2$   
 $= 360 + 468 = 828$   
 $|AB| = \sqrt{828} = 6\sqrt{7}$

(d)  $C$  is the midpoint of  $AB \Rightarrow |CB| = |AC| = 6\sqrt{7}$   
 $\angle PCB = \angle PAC = 45^\circ \Rightarrow \triangle PCB \text{ is equilateral and isosceles}$   
 $\Rightarrow |PC| = |CB|$   
 $\therefore |PC| = 6\sqrt{7}$

**Question 53    (\*\*\*)+**

The straight lines  $l_1$  and  $l_2$  have the following Cartesian equations

$$l_1: \quad x-a = \frac{y+4}{-4} = \frac{z}{-2}$$

$$l_2: \quad \frac{x-a}{2} = \frac{y+1}{-5} = \frac{z-1}{-3}$$

where  $a$  is a scalar constant.

- a) Show that  $l_1$  and  $l_2$  intersect at for all values of  $a$ .

The intersection point of  $l_1$  and  $l_2$  has coordinates  $(b, b, b)$ , where  $b$  is a scalar constant.

- b) Find the value of  $a$  and the value of  $b$ .

- c) Calculate the acute angle formed by  $l_1$  and  $l_2$ .

\_\_\_\_\_ ,  $[a=6]$  ,  $[b=4]$  ,  $[\theta \approx 7.6^\circ]$

a) WRITE THE EQUATIONS IN PARAMETRIC FORM

$$\begin{aligned} l_1: \quad & \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ a+4 \\ a-2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -1 \\ -2 \end{pmatrix} \\ l_2: \quad & \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ a+1 \\ a-1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} \end{aligned}$$

EQUATE  $l_1$  &  $l_2$  COORDINATES

$$\begin{aligned} \begin{cases} -4\lambda = -1 - \mu \\ a + 4 = a + 1 \\ a - 2 = a - 1 \end{cases} \Rightarrow \begin{cases} \lambda = -\frac{1}{4} \\ \mu = 3 \\ \lambda = 1 \end{cases} \end{aligned}$$

SOLVE  $\lambda$  FOR UNKNOWN

$$\begin{aligned} a + \lambda &= a - 2 \\ a + 2\lambda &= a + 2(-\frac{1}{4}) = a - 2 \\ \lambda &= -2 \end{aligned}$$

IF  $2\lambda = -2$ ,  $\lambda = -1$  &  $l_1$  &  $l_2$  CONCURRENT  
HENCE, SO WHAT WOULD BE  $a$ ?

b) WORKSHEET

IF  $\lambda = -2$  &  $\mu = -1$  THE INTERSECTION POINT WILL BE

$$(a-2, a+4)$$

$$\therefore a = 6 \quad \text{&} \quad b = 4$$

c) DOTTING THE DIRECTION VECTORS OF  $l_1$  &  $l_2$

$$\begin{aligned} & (1, -4, -2) \cdot (2, -5, -3) = |(1, -4, -2)| |(2, -5, -3)| \cos \theta \\ & 2 + 20 + 6 = \sqrt{1+16+4} \sqrt{4+25+9} \cos \theta \\ & 28 = \sqrt{22} \sqrt{38} \cos \theta \\ & \cos \theta = \frac{28}{\sqrt{22} \sqrt{38}} \\ & \theta = 7.6^\circ \end{aligned}$$

**Question 54 (\*\*\*)+**

$OAB$  is a triangle with the point  $P$  being the midpoint of  $OB$  and the point  $Q$  being the midpoint of  $AB$ .

The point  $R$  is such so that  $\overrightarrow{AR} = \frac{2}{3}\overrightarrow{AP}$ .

Let  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OB} = \mathbf{b}$ .

- Find simplified expressions, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , for each of the vectors  $\overrightarrow{AB}$ ,  $\overrightarrow{AP}$ ,  $\overrightarrow{AQ}$  and  $\overrightarrow{AR}$ .
- By finding simplified expressions, in terms  $\mathbf{a}$  and  $\mathbf{b}$ , for two more suitable vectors, show that the points  $O$ ,  $R$  and  $Q$  are collinear.

$$\boxed{\overrightarrow{AB} = \mathbf{b} - \mathbf{a}}, \boxed{\overrightarrow{AP} = \frac{1}{2}\mathbf{b} - \mathbf{a}}, \boxed{\overrightarrow{AQ} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}}, \boxed{\overrightarrow{AR} = \frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{a}}$$

(a)  $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$   
 $\bullet \overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\mathbf{a} + \frac{1}{2}\mathbf{b}$   
 $\bullet \overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$   
 $\bullet \overrightarrow{AR} = \frac{2}{3}\overrightarrow{AP} = \frac{2}{3}(-\mathbf{a} + \frac{1}{2}\mathbf{b}) = -\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

(b)  $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR} = \mathbf{a} - \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$   
 $\overrightarrow{RQ} = \overrightarrow{RA} + \overrightarrow{AQ} = (\frac{2}{3}\mathbf{a} - \frac{1}{3}\mathbf{b}) + (\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}) = \frac{1}{6}\mathbf{a} + \frac{1}{6}\mathbf{b} = \frac{1}{6}(\mathbf{a} + \mathbf{b})$   
 $\overrightarrow{OR}$  is in the same direction as  $\overrightarrow{RQ}$  & share point  $R$   
 $\therefore O, R, Q$  are collinear

**Question 55 (\*\*\*)+**

The points  $A(1,1,2)$ ,  $B(2,1,5)$ ,  $C(4,0,1)$  and  $D$  form the parallelogram  $ABCD$ , where the above coordinates are measured relative to a fixed origin.

- a) Find the coordinates of  $D$ .

The points  $E$ ,  $B$  and  $D$  are collinear, so that  $B$  is the midpoint of  $ED$ .

- b) Determine the coordinates of  $E$ .

The point  $F$  is such so that  $ABEF$  is also a parallelogram.

- c) Find the coordinates of  $F$ .

- d) Show that  $B$  is the midpoint of  $FC$ .

- e) Prove that  $ADBF$  is another parallelogram.

,  $[D(3,0,-2)]$  ,  $[E(1,2,12)]$  ,  $[F(0,2,9)]$

a) LOOKING AT THE DIAGRAM

$$\begin{aligned} \rightarrow \vec{OD} &= \vec{OA} + \vec{AB} \\ \rightarrow \vec{OD} &= \vec{OA} + \vec{BC} \quad (\text{PARALLEL}) \\ \rightarrow d &= a + (c - b) \\ \rightarrow d &= a + c - b \\ \rightarrow d &= (1,1,2) + (4,0,1) - (2,1,5) \\ \rightarrow d &= (3,0,-2) \end{aligned}$$

$\therefore D(3,0,-2)$

b)  $E(1,2,12) \parallel B(2,1,5) \quad D(3,0,-2)$

BY SPLITTING / INSERTION / ANDROID FORMULA

|            |                |            |                |            |
|------------|----------------|------------|----------------|------------|
| $2:1$      | $\frac{-1}{2}$ | $2$        | $\frac{-1}{2}$ | $3$        |
| $y:2$      | $\frac{-1}{2}$ | $1$        | $\frac{-1}{2}$ | $0$        |
| $z:2$      | $\frac{-1}{2}$ | $5$        | $\frac{-1}{2}$ | $-2$       |
| $\uparrow$ | $\uparrow$     | $\uparrow$ | $\uparrow$     | $\uparrow$ |
| $E$        | $B$            | $D$        |                |            |

c) AS IN PART (a) OR INSERION

$$\begin{aligned} (3) &\leftarrow (E) \\ 2 &\leftarrow 1 \\ 1 &\leftarrow 2 \\ 5 &\leftarrow 2 \end{aligned}$$

(2)  $\leftarrow (F)$

$$\begin{aligned} 1 &\leftarrow 0 \\ 1 &\leftarrow 2 \\ 2 &\leftarrow 1 \end{aligned}$$

$\therefore F(0,2,9)$

d)  $F(0,2,9) \quad C(4,0,1) \quad B(2,1,5)$

MIDPOINT OF  $FC = \left(\frac{0+4}{2}, \frac{2+0}{2}, \frac{9+1}{2}\right) = (2,1,5)$  which is  $B$

e)  $\vec{AB} = \vec{d} - \vec{a} = (3,0,-2) - (1,1,2) = (2,-1,-7)$

$$\begin{aligned} \vec{FB} &= \vec{b} - \vec{f} = (2,1,5) - (0,2,9) = (2,-1,-7) \\ \vec{FD} &= \vec{a} - \vec{f} = (1,1,2) - (0,2,9) = (1,-1,-7) \\ \vec{BD} &= \vec{d} - \vec{b} = (3,0,-2) - (2,1,5) = (1,-1,-7) \end{aligned}$$

REFERRING TO THE CALCULATIONS & THE DIAGRAM

OPPOSITE SIDES ARE PARALLEL (AND EQUAL)

INDEED  $ADBF$  IS A PARALLELOGRAM

**Question 56** (\*\*\*)

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 2\mathbf{i} + \mathbf{j} + 8\mathbf{k} + \lambda(\mathbf{j} - 2\mathbf{k})$$

$$\mathbf{r}_2 = \mathbf{i} + \mu(a\mathbf{i} + b\mathbf{j} + 2\mathbf{k}),$$

where  $\lambda$  and  $\mu$  are scalar parameters, and  $a$  and  $b$  are scalar constants.

$l_1$  and  $l_2$  intersect at right angles at the point  $P$ .

- a) Find the value of  $a$  and the value of  $b$ .
- b) Determine the coordinates of  $P$ .

The straight line  $l_3$  passes through the point  $Q(1, -1, -1)$ .

- c) Find a vector equation for  $l_3$ , given that all three lines intersect at the same point.

$$a = 1, b = 4, P(2, 4, 2), \mathbf{r} = \mathbf{i} - \mathbf{j} - \mathbf{k} + \nu(\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$$

(a)

$$\begin{aligned} l_1 &= (2, 1, 8) + \lambda(0, 1, -2) = (2, 1+ \lambda, -2\lambda + 8) \\ l_2 &= (1, 0, 0) + \mu(a, b, 2) = (1+a\mu, b\mu, 2\mu) \end{aligned}$$

"MEET AT RIGHT ANGLES"  $\rightarrow (0, 1, -2) \cdot (a, b, 2) = 0$

$$0 + b - 4 = 0 \Rightarrow b = 4$$

"CROSS INTERSECT"  $\Rightarrow \begin{cases} 2 = 1+\mu+1 \\ 2 = a+4\mu \\ -2\lambda+8 = 2\mu \end{cases} \Rightarrow \begin{cases} 2 = 4\mu \\ -2\lambda+8 = 4\mu \\ a+1 = -2\lambda+4 \end{cases}$

$$\Rightarrow \begin{cases} \mu = 1/2 \\ \lambda = 1 \\ a = 1 \end{cases}$$

ANSWER:  $2 = 1+\mu+1$   
 $2 = a+4\mu$   
 $a = 1$

(b)

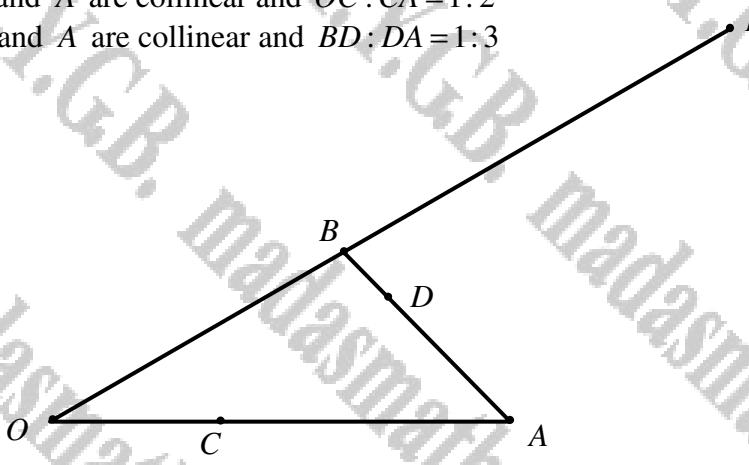
USING  $a=3$  into  $(2, 1+ \lambda, -2\lambda + 8)$  gives  $P(2, 4, 2)$

$\vec{PQ} = \mathbf{q} - \mathbf{p} = (1, -1, -1) - (2, 4, 2) = (-1, -3, -3)$   
 USE  $(1, -1, -1)$  AS DIRECTION  
 $\Rightarrow \mathbf{r} = (1, -1, -1) + \lambda(1, -1, -1)$   
 $\Gamma = (2, 4, 2) + \lambda(-1, -3, -3)$

**Question 57 (\*\*\*)+**

The figure below shows the points  $O$ ,  $C$ ,  $A$ ,  $D$ ,  $B$  and  $E$ , which are related as follows.

- $O$ ,  $B$  and  $E$  are collinear and  $OB : BE = 1 : 2$
- $O$ ,  $C$  and  $A$  are collinear and  $OC : CA = 1 : 2$
- $B$ ,  $D$  and  $A$  are collinear and  $BD : DA = 1 : 3$



Let  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OB} = \mathbf{b}$ .

- Find simplified expressions, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , for each of the vectors  $\overrightarrow{AB}$ ,  $\overrightarrow{DB}$ ,  $\overrightarrow{CD}$  and  $\overrightarrow{DE}$ .
- Show that the points  $C$ ,  $D$  and  $E$  are collinear, and find the ratio  $CD : DE$ .
- Show further that  $BC$  is parallel to  $EA$ , and find the ratio  $BC : EA$ .

$$\boxed{\overrightarrow{AB} = \mathbf{b} - \mathbf{a}}, \boxed{\overrightarrow{DB} = \frac{1}{4}\mathbf{b} - \frac{1}{4}\mathbf{a}}, \boxed{\overrightarrow{CD} = -\frac{1}{12}\mathbf{a} + \frac{3}{4}\mathbf{b}}, \boxed{\overrightarrow{DE} = -\frac{1}{4}\mathbf{a} + \frac{9}{4}\mathbf{b}}$$

$$\boxed{CD : DE = 1 : 3}, \boxed{BC : EA = 1 : 3}$$

(a)  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{DB} = \frac{1}{4}\overrightarrow{OB} - \frac{1}{4}\overrightarrow{OA} = \frac{1}{4}(\mathbf{b} - \mathbf{a}) = \frac{1}{4}\mathbf{b} - \frac{1}{4}\mathbf{a}$

$\overrightarrow{CD} = \overrightarrow{CA} - \overrightarrow{CD} = \overrightarrow{CA} + \frac{3}{4}\overrightarrow{AB}$

$= \frac{3}{4}\mathbf{a} + \frac{3}{4}(\mathbf{b} - \mathbf{a}) = \frac{3}{4}\mathbf{b} + \frac{3}{4}\mathbf{a} - \frac{3}{4}\mathbf{a} = \frac{3}{4}\mathbf{b}$

$\overrightarrow{DE} = \overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DB} + 2\overrightarrow{OB}$

$= \frac{1}{4}\mathbf{b} - \frac{1}{4}\mathbf{a} + 2\mathbf{b} = -\frac{1}{4}\mathbf{a} + \frac{9}{4}\mathbf{b}$

(b)  $\overrightarrow{CD} = -\frac{1}{12}\mathbf{a} + \frac{3}{4}\mathbf{b} \approx \frac{1}{2}(-\mathbf{a} + 3\mathbf{b})$

$\overrightarrow{DE} = -\frac{1}{4}\mathbf{a} + \frac{9}{4}\mathbf{b} \approx \frac{1}{2}(-\mathbf{a} + 9\mathbf{b})$

VECTORS ARE IN THE SAME DIRECTION AND SAME LENGTH  
 $\therefore C, D, E$  ARE COLLINEAR

$CD : DE = \frac{1}{2} : \frac{1}{2}$   
 $1 : 1$

(c)  $\overrightarrow{BC} = \overrightarrow{OB} - \overrightarrow{OC} = -\mathbf{b} + \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a} - 2\mathbf{b})$

$\overrightarrow{EA} = \overrightarrow{EB} + \overrightarrow{BA} = -2\mathbf{b} + \mathbf{a} = \mathbf{a} - 2\mathbf{b} = (\mathbf{a} - 2\mathbf{b})$

VECTORS ARE IN THE SAME DIRECTION, SO PARALLEL

$BC : EA = \frac{1}{2} : 1$   
 $1 : 2$

**Question 58 (\*\*\*)+**

With respect to a fixed origin  $O$ , the following points are given

$$A(2, 2, 5), B(12, 7, 0), C(0, 0, 1) \text{ and } D(9, k, 4),$$

where  $k$  is a scalar constant.

- a) Find the vector equation of the straight line  $l_1$  that passes through  $A$  and  $B$ .

The straight line  $l_2$  passes through  $C$  and  $D$ , and intersects  $l_1$  at the point  $P$ .

- b) Determine in any order ...

i. ... the coordinates of  $P$ .

ii. ... the value of  $k$ .

iii. ... the acute angle between  $l_1$  and  $l_2$ .

$$\boxed{\quad}, \boxed{\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k})}, \boxed{P(6, 4, 3)}, \boxed{k = 6}, \boxed{40.2^\circ}$$

(a)  $\overrightarrow{AB} = b - a = (12, 7, 0) - (2, 2, 5) = (10, 5, -5)$   
out  $(2, 1, -1)$  is direction vector

$$\begin{aligned} l_1 &= (2, 2, 5) + \lambda(2, 1, -1) \\ l_2 &= (2, 2, 5) + \mu(3, -k, 4) \end{aligned}$$

(b)  $\overrightarrow{CD} = d - c = (9, k, 4) - (0, 0, 1) = (9, k, 3)$

$$\begin{aligned} l_2 &= (0, 0, 1) + \nu(9, k, 3) \\ l_2 &= (0, 0, 1) + \nu(9, k, 3) \end{aligned}$$

\* EQUATE I & II

$$\begin{aligned} (1) : 2\lambda + 2 = 9\nu + 1 &\rightarrow [2 = 9\nu] \\ (2) : 5 - 5\nu = 3\nu + 1 &\rightarrow [5 = 8\nu] \end{aligned} \Rightarrow \begin{aligned} 2(4 - 3\nu) + 2 &= 9\nu \\ 8 - 6\nu + 2 &= 9\nu \\ 10 &= 15\nu \end{aligned} \Rightarrow \boxed{\nu = \frac{2}{3}}$$

$\boxed{\lambda = 2}$

\* looking at  $\lambda = 2$

$$\begin{aligned} 2\lambda + 2 &= 9\nu \\ 2\lambda + 2 &= \frac{3}{2}k \\ k &= 2 \end{aligned}$$

& since  $2 = 2\lambda$  we have  $(2\lambda, \lambda^2, 5 - 2)$   
 $W.C.B. P(6, 4, 3)$

DOTTING DIRECTION VECTORS

$$\begin{aligned} (2, 1, -1) \cdot (9, k, 3) &= 2(1)(9) + 1(k) - 1(3) = 18 + k - 3 = 15 + k = 0 \\ \Rightarrow 15 + k &= 0 \\ \Rightarrow k &= -15 \\ \Rightarrow \cos\theta &= \frac{21}{\sqrt{14} \sqrt{10}} \\ \Rightarrow \cos\theta &= \frac{21}{\sqrt{140}} \\ \Rightarrow \theta &\approx 40.2^\circ \end{aligned}$$

**Question 59 (\*\*\*\*)**

Two submarines  $S_1$  and  $S_2$ , are travelling through the ocean.

They both appeared on the radar screen of a tracking station at the same time. The distances are measured in hundreds of metres and the time  $t$ , in seconds, is measured from the instant they were both observed on the radar screen of the tracking station.

The coordinates of  $S_1$  and  $S_2$ , relative to a fixed origin  $O$ , are given by

$$S_1: \mathbf{r}_1 = (2t-4)\mathbf{i} + (t-15)\mathbf{j} + (t+5)\mathbf{k}$$

$$S_2: \mathbf{r}_2 = 10\mathbf{i} + (-2t+6)\mathbf{j} + (2t-2)\mathbf{k}$$

- a) Show that  $S_1$  and  $S_2$  are travelling in perpendicular directions to each other.

Suppose that  $S_1$  and  $S_2$  continue to travel according to the above vector equations.

- b) Show further that  $S_1$  and  $S_2$ , will eventually collide at some point  $P$ , and further determine the coordinates of  $P$ .
- c) Calculate, to the nearest metre, the distance between  $S_1$  and  $S_2$ , when they were first observed by the tracking station.

$$P(10, -8, 12), \text{ distance} = 700\sqrt{14} \approx 2619 \text{ m}$$

$$\begin{aligned} \text{(a)} \quad & S_1 = (2t-4, t-15, t+5) = (-4, -15, 5) + t(2, 1, 1) \\ & S_2 = (10, -2t+6, 2t-2) = (10, 6, -2) + t(0, -2, 2) \\ & \text{Dotting direction vectors: } (2, 1, 1) \cdot (0, -2, 2) = 0 - 2 + 2 = 0 \quad \therefore \text{PERPENDICULAR} \\ \text{(b)} \quad & \text{SOLVE } \begin{cases} 2t-4=10 \\ t-15=-2t+6 \\ t+5=2t-2 \end{cases} \Rightarrow \begin{cases} 2t=14 \\ 3t=21 \\ 7=t \end{cases} \Rightarrow t=7 \quad \therefore \text{THEY COLLIDE} \\ & \text{WHEN } t=7 \text{ AND } (2t-4, t-15, t+5) \text{ GIVES } P(10, -8, 12) \\ \text{(c)} \quad & \text{WHEN } t=0, \quad S_1 = (-4, -15, 5) \\ & \quad S_2 = (10, 6, -2) \\ d &= \left| \frac{\mathbf{S}_2 - \mathbf{S}_1}{\mathbf{S}_1} \right| = \sqrt{(10-6)^2 + (-4-15)^2 + (5-5)^2} = \sqrt{14^2 + 19^2} = \sqrt{530} = 7\sqrt{10} \\ & \text{BUT } 1 \text{ UNIT} = 100 \text{ metres} \quad \therefore \text{TRUE DISTANCE } 700\sqrt{14} \approx 2619 \end{aligned}$$

**Question 60 (\*\*\*)+**

The straight line  $l$  passes through the points with coordinates

$$A(-1, -4, 8) \quad \text{and} \quad B(1, -2, 5).$$

- a) Find a vector equation of  $l$ .

The origin is denoted by  $O$ .

The point  $P$  lies on  $l$ , so that  $OP$  is perpendicular to  $l$ .

- b) Determine the coordinates of  $P$ .

The point  $Q$  is the reflection of  $O$ , about  $l$ .

- c) State the coordinates of  $Q$ .

$$\boxed{\mathbf{r} = -\mathbf{i} - 4\mathbf{j} + 8\mathbf{k} + \lambda(2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})}, \boxed{P(3, 0, 2)}, \boxed{Q(6, 0, 4)}$$

(a)

$$\begin{aligned}\vec{AB} &= b - a = (1, -2, 5) - (-1, -4, 8) = (2, 2, -3) \\ \mathbf{l} &= (-1, -4, 8) + \lambda(2, 2, -3) \\ &= (2\lambda - 1, 2\lambda - 4, \lambda - 3)\end{aligned}$$

(b)

- Let  $\frac{OP}{OP} = (\lambda_1, \lambda_2, \lambda_3)$
- $\overrightarrow{OP} \perp \mathbf{l}$
- $(2, 2, -3) \cdot (2\lambda - 1, 2\lambda - 4, \lambda - 3) = 0$
- $2(2\lambda - 1) + 2(2\lambda - 4) - 3(\lambda - 3) = 0$
- $4\lambda - 2 + 4\lambda - 8 - 3\lambda + 9 = 0$
- $7\lambda = 3$
- $\lambda = \frac{3}{7}$

SOLVING SIMULTANEOUSLY:

$$\begin{aligned}2(2\lambda - 1) + 2(2\lambda - 4) - 3(\lambda - 3) &= 0 \\ 4\lambda - 2 + 4\lambda - 8 - 3\lambda + 9 &= 0 \\ 7\lambda &= 3 \\ \lambda &= \frac{3}{7}\end{aligned}$$

$\therefore P\left(\frac{3}{7}, \frac{6}{7}, -\frac{2}{7}\right)$

(c)

P IS THE MIDPOINT OF OQ

$\therefore Q(6, 0, 4)$

**Question 61 (\*\*\*)+**

The points  $A$  and  $B$  have coordinates  $(4, -7, 5)$  and  $(-2, 8, 17)$ , respectively.

- a) Find the equation of the straight line  $l$ , which passes through  $A$  and  $B$ .

The point  $C$  has coordinates  $(6, 6, 1)$ .

- b) Find the shortest distance from  $C$  to  $l$ .

$$\mathbf{r} = 4\mathbf{i} - 7\mathbf{j} + 5\mathbf{k} + \lambda(-2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}), \text{ shortest distance} = 12$$

(a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (-2, 8, 17) - (4, -7, 5) = (-6, 15, 12)$   
SCALE DIRECTION IS  $(-2, 5, 4)$

$$\therefore \mathbf{l} = (4, -7, 5) + \lambda(-2, 5, 4)$$

$$\mathbf{l} = (4-2\lambda, 5\lambda-7, 4\lambda+5)$$

(b)

•  $\mathbf{C} = (6, 6, 1)$   
•  $\mathbf{D} = (x, y, z)$   
•  $\mathbf{CD} = \mathbf{d} = \mathbf{C} - \mathbf{D} = (6-x, 6-y, 1-z)$   
 $\mathbf{CD} = (2-\lambda, 5\lambda-7, 4\lambda+5)$

•  $\mathbf{CD} \perp \mathbf{l}$

$$(2-\lambda, 5\lambda-7, 4\lambda+5) \cdot (-2, 5, 4) = 0$$

$$-2(2-\lambda) + 5(5\lambda-7) + 4(4\lambda+5) = 0$$

$$-4 + 2\lambda + 25\lambda - 35 + 16\lambda + 20 = 0$$

$$43\lambda - 19 = 0$$

$$43\lambda = 19$$

$$\lambda = \frac{19}{43}$$

POINT D LIES ON  $\mathbf{l}$

$$x = 4-2\lambda$$

$$y = 5\lambda-7$$

$$z = 4\lambda+5$$

SOLVING SIMULTANEOUSLY

$$-2(4-2\lambda) + 5(5\lambda-7) + 4(4\lambda+5) = 22$$

$$-8 + 4\lambda + 25\lambda - 35 + 16\lambda + 20 = 22$$

$$45\lambda = 45$$

$$\lambda = 1$$

$\therefore D(-2, 4)$  distance  $|CD| = \sqrt{(-2-6)^2 + (4-6)^2 + (1-1)^2} = \sqrt{16+4} = 12$

**Question 62 (\*\*\*)+**

Relative to a fixed origin  $O$ , the points  $A$ ,  $B$  and  $C$  have respective coordinates  $(-2, 5, 13)$ ,  $(1, 1, 1)$  and  $(3, 5, 5)$ .

- a) Determine the size of the angle  $ABC$ .

The point  $D$  has coordinates  $(9, -8, 6)$ .

- b) Show that  $BD$  is perpendicular to both  $AB$  and  $BC$ .
- c) Find the distance  $BD$ .
- d) Calculate the volume of the right triangular prism with base the triangle  $ABC$  and height  $BD$ .

$$[42.0^\circ], [BD] = \sqrt{170}, [340 \text{ cubic units}]$$

(a)

$$\vec{BA} = \mathbf{a} - \mathbf{b} = (-2, 5, 13) - (1, 1, 1) = (-3, 4, 12)$$

$$\vec{BC} = \mathbf{c} - \mathbf{b} = (3, 5, 5) - (1, 1, 1) = (2, 4, 4)$$

• DOT PRODUCT

$$\langle -3, 4, 12 \rangle \cdot \langle 2, 4, 4 \rangle = -3 \cdot 2 + 4 \cdot 4 + 12 \cdot 4 = -6 + 16 + 48 = 68$$

$$|\vec{AB}| = \sqrt{(-3)^2 + 4^2 + 12^2} = \sqrt{9 + 16 + 144} = \sqrt{170}$$

$$|\vec{BC}| = \sqrt{2^2 + 4^2 + 4^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

$$\cos \theta = \frac{68}{\sqrt{170} \cdot 6} \approx 0.633 \Rightarrow \theta \approx 42.0^\circ$$

(b)

$$\vec{BD} = \mathbf{d} - \mathbf{b} = (9, -8, 6) - (1, 1, 1) = (8, -9, 5)$$

$$\vec{BD} \cdot \vec{AB} = (8, -9, 5) \cdot (3, 4, 12) = 24 + 36 - 60 = 60 - 60 = 0$$

$$\vec{BD} \cdot \vec{BC} = (8, -9, 5) \cdot (2, 4, 4) = 16 - 36 + 20 = 36 - 36 = 0$$

∴  $BD$  is perpendicular to both  $\vec{AB}$  &  $\vec{BC}$

(c)

$$|BD| = \sqrt{8^2 + (-9)^2 + 5^2} = \sqrt{64 + 81 + 25} = \sqrt{170}$$

(d)

$$\text{Area } \triangle ABC = \frac{1}{2} \times 13 \times 6 \times \sin 60^\circ = \frac{1}{2} \times \sqrt{170} \times 6 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{510}}{2}$$

$$\text{Volume} = \frac{3\sqrt{510}}{2} \times 6 = 24\sqrt{510} \approx 340$$

**Question 63 (\*\*\*)+**

The points with coordinates  $A(3,0,3)$  and  $B(4,-1,5)$  are given.

- a) Find a vector equation of the straight line  $l_1$  that passes through  $A$  and  $B$ .

The straight line  $l_2$  has equation

$$\mathbf{r} = 5\mathbf{i} + 10\mathbf{j} + 4\mathbf{k} + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k}),$$

where  $\mu$  is a scalar parameter.

- b) Show that  $l_1$  and  $l_2$  are perpendicular.  
 c) Show further that  $l_1$  and  $l_2$  intersect at some point  $P$  and find its coordinates.

The point  $E$  is on the  $l_1$ .

A circle with centre at  $E$  is drawn so that it cuts  $l_2$  at the points  $C$  and  $D$ .

- d) Given that the coordinates of  $C$  are  $(0,-5,-1)$ , find the coordinates of  $D$ .

$$[\quad], [\mathbf{r}_1 = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})], [P(2,1,1)], [D(4,7,3)]$$

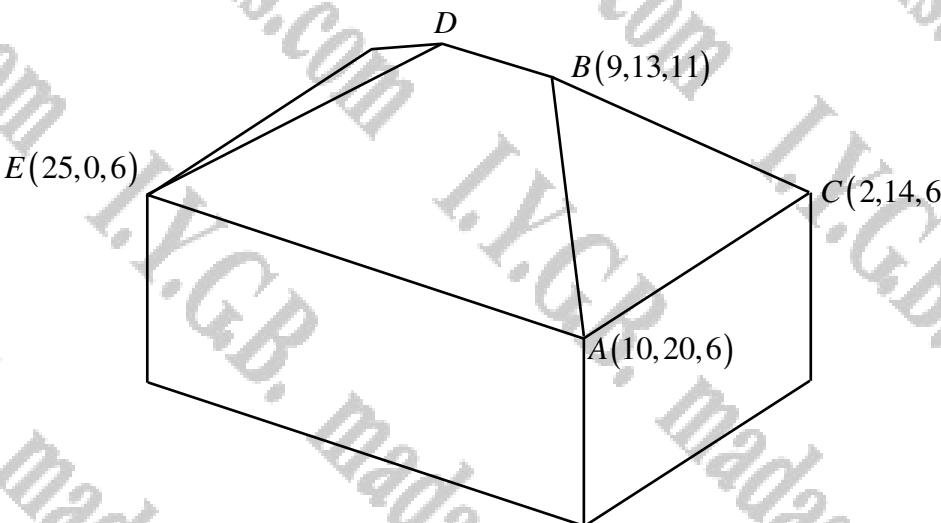
**Q1**  $\vec{AB} = \mathbf{b} - \mathbf{a} = (4, -1, 5) - (3, 0, 3) = (1, -1, 2)$   
 $\therefore l_1 = (3, 0, 3) + t(1, -1, 2) = (3t+3, -t+3, 3+t)$   
 $l_2 = (5, 10, 4) + \mu(1, 3, 1) = (\mu+5, 3\mu+10, \mu+4)$   
 Setting directions equal

**Q2** Given:  $\begin{cases} 1 \\ 2t+3 = \mu+5 \\ -t+3 = 3\mu+10 \end{cases}$   
 $\begin{aligned} (1) & \rightarrow 2t+3 = \mu+5 \\ (2) & \rightarrow -t+3 = 3\mu+10 \end{aligned}$   
 $\begin{aligned} & \text{Add } 3 \text{ to both sides} \\ & \begin{aligned} 2t+6 &= \mu+8 \\ -t+3 &= 3\mu+10 \end{aligned} \\ & \begin{aligned} 2t+6 &= \mu+8 \\ -t+3 &= 3\mu+10 \end{aligned} \\ & \begin{aligned} 2t+6 &= \mu+8 \\ -2t+6 &= -6 \\ 12 &= \mu+8 \\ 4 &= \mu \end{aligned} \end{aligned}$

Given  $t = 4$   
 $\begin{cases} 1 \\ 2(4)+3 = \mu+5 \\ -4+3 = 3\mu+10 \end{cases}$  As all 3 components agree the lines intersect

Solving  $t=1$  into  $(3t+3, -t+3, 3+t)$  we obtain  $P(2,1,1)$

**Q3** By circle theorem  $\angle CED = 90^\circ$  as the radius of  $CD$   
 $(0, -5, -1)$  and  $(2, 1, 1)$   
 $\therefore \frac{2-0}{2} = 1 \Rightarrow 2=4$   
 $\frac{-5-1}{2} = 1 \Rightarrow -6=-4$   
 $\frac{-1-1}{2} = 1 \Rightarrow -2=2$   
 $\therefore D(4,7,3)$

**Question 64 (\*\*\*\*\*)**

The figure above shows a solid, modelling a house with a standard slanted roof, where all the distances are measured in metres. With respect to a fixed origin, the coordinates of some of the vertices of the solid are marked in the diagram.

- Find a vector equation of  $AE$ .
- Show that  $AE$  is perpendicular to  $AC$ .
- Find the cosine of the angle  $ABC$ .
- Determine the coordinates of  $D$ .

The straight line  $BD$  is parallel to  $AE$ . The length of  $BD$  is 10 metres.

$$\boxed{\text{a) } \mathbf{r} = 10\mathbf{i} + 20\mathbf{j} + 6\mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{j})}, \boxed{\cos(\angle ABC) = \frac{1}{3}}, \boxed{D(15, 5, 11)}$$

(a)  $\vec{AE} = B-A = (25, 0, 6) - (10, 20, 6) = (15, -20, 0)$  • SCALE IT TO  $(3, -4, 0)$   
 $\therefore \vec{a} = (1, -2, 0)$  &  $\vec{AC} = (2, 14, 6) - (10, 20, 6) = (-8, -6, 0)$   
 $\vec{c} = (3, 1, 0)$

(b)  $\vec{AC} \times \vec{a} = (2, 14, 6) - (10, 20, 6) = (-8, -6, 0)$   
 $\vec{AE} \times \vec{AC} = (1, -2, 0) \times (-8, -6, 0) = -120 + 120 = 0$  INDEP PRACTICALLY

(c)  $\vec{AB} = B-A = (9, 13, 11) - (10, 20, 6) = (-1, -7, 5) \Rightarrow |\vec{AB}| = \sqrt{1+49+25 = \sqrt{75}}$   
 $\vec{CD} = D-C = (15, 5, 11) - (9, 13, 11) = (6, -8, 0) \Rightarrow |\vec{CD}| = \sqrt{36+64 = \sqrt{100}}$   
 BY DOT PRODUCT  $\vec{AB} \cdot \vec{CD} = |\vec{AB}| |\vec{CD}| \cos \theta$   
 $(-1, -7, 5) \cdot (6, -8, 0) = \sqrt{75} \sqrt{100} \cos \theta$   
 $25 = 75 \cos \theta$   
 $\cos \theta = \frac{1}{3}$

(d)  $BD$  IS PARALLEL TO  $AE$ , SO IN DIRECTION  $(3, -4, 0)$   
 $|B_1 - 4, 0| = \sqrt{1+16+0} = 5$   
 BUT  $|BD| = 10$   
 $\therefore \vec{BD} = 2(3, -4, 0) = (6, -8, 0)$   
 $d-B = (6, -8, 0)$   
 $d - (9, 13, 11) = (6, -8, 0)$   $\therefore d = (15, 5, 11)$

**Question 65 (\*\*\*\*)**

Relative to a fixed origin  $O$  the following position vectors are given.

$$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 13 \\ 1 \end{pmatrix}.$$

- a) Find a vector equation for the line straight  $l_1$  which passes through  $A$  and  $B$ .

The straight line  $l_2$  has vector equation

$$\mathbf{r} = \begin{pmatrix} 7 \\ 0 \\ 9 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},$$

where  $\mu$  is a scalar parameter.

- b) Show that  $l_1$  and  $l_2$  do not intersect.  
 c) Find the position vector of  $C$ , given it lies on  $l_2$  and  $\angle ABC = 90^\circ$ .

□,  $\boxed{\mathbf{r} = 8\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})}$ ,  $\boxed{C(-3, 15, 4)}$

**(a)**  $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = ((B_x)_1 - (A_x)_1, (B_y)_1 - (A_y)_1, (B_z)_1 - (A_z)_1) = (1-0, 13-8, 1-3) = (1, 5, -2)$   
 $\therefore \vec{l}_1 = (\mathbf{O}A)\vec{x} + \lambda((B-A)\vec{x})$   
 $\vec{l}_1 = (0, 8, 3)\vec{x} + \lambda(1, 5, -2)\vec{x}$

**(b)**  $\vec{l}_2 = (\mathbf{O}A)\vec{x} + \lambda(C-A)\vec{x} = (0, 8, 3) + \lambda(7, 0, 9) + \lambda(2, -3, 1)$

Given  $\vec{l}_1 \perp \vec{l}_2$   
 $\therefore \vec{l}_1 \cdot \vec{l}_2 = 0$   
 $\Rightarrow (1, 5, -2) \cdot (7, 0, 9) + \lambda(1, 5, -2) \cdot (2, -3, 1) = 0$   
 $\Rightarrow 7 + 45 - 18 + \lambda(2 - 25 + 2) = 0$   
 $\Rightarrow 32 - 18 + \lambda(-23) = 0$   
 $\Rightarrow 14 - 23\lambda = 0$   
 $\therefore \lambda = \frac{14}{23}$

Check if  $3 - 2\lambda = 3 - 2\left(\frac{14}{23}\right) = \frac{3}{23} \neq \frac{7}{13} \therefore$  lines do not intersect

**(c)**

- $\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = ((C_x)_1 - (A_x)_1, (C_y)_1 - (A_y)_1, (C_z)_1 - (A_z)_1) = (-3-0, 15-8, 4-3) = (-3, 7, 1)$
- $\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = ((C_x)_1 - (B_x)_1, (C_y)_1 - (B_y)_1, (C_z)_1 - (B_z)_1) = (-3-1, 15-13, 4-1) = (-4, 2, 3)$

$\therefore \overrightarrow{AB} \cdot \overrightarrow{BC} = 0$   
 $\Rightarrow (1, 5, -2) \cdot (-4, 2, 3) = 0$   
 $\Rightarrow -4 + 10 - 6 = 0$   
 $\therefore \vec{BC} \perp \vec{AB} \therefore l_2 \perp l_1$

$\therefore \lambda = \frac{14}{23}$

Since  $(C_x)_1 - 7 - \lambda(2) - 2(7) = 63$   
 $-15\lambda = 15$   
 $\therefore \lambda = -1$   
 $\therefore C(-3, 15, 4)$

**Question 66 (\*\*\*\*)**

The straight line  $L_1$  passes through the points  $A$  and  $B$ , whose respective position vectors relative to a fixed origin  $O$  are

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}.$$

- a) Find a vector equation for  $L_1$ .

The angle  $ABC$  is denoted by  $\theta$ , where  $C$  is the point with position vector  $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ .

- b) Show clearly that  $\cos \theta = \frac{1}{3}$ .

The straight line  $L_2$  passes through  $C$  and is parallel to  $L_1$ .

The points  $P$  and  $Q$  both lie on  $L_2$  so that  $|AB| = |CP| = |CQ|$ .

- c) Determine the position vector of  $P$  and the position vector of  $Q$ , given that  $P$  is furthest away from  $O$ .
- d) Show further that the area of the quadrilateral  $ABPQ$  is  $9\sqrt{2}$ .

$$\boxed{\mathbf{r}_1 = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})}, \boxed{\overrightarrow{OP} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}}, \boxed{\overrightarrow{OQ} = 2\mathbf{i} + \mathbf{j}}$$

a)  $\vec{AB}^2 = b - a = (2, 1, 6) - (1, 2, 5) = (1, -1, 1)$

$\vec{L}_1 = (1, 2, 5) + \lambda(1, -1, 1)$

$\vec{L}_1 = (1+1\lambda, 2-1\lambda, 5+1\lambda)$

b)  $\vec{CB} = b - c = (2, 1, 6) - (3, 0, 1) = (-1, 1, 5)$

$\vec{CB} = (-1, 1, 5)$

c)  $\vec{BC} = c - b = (3, 0, 1) - (2, 1, 6) = (1, -1, -5)$

$\vec{BC} = (1, -1, -5)$

using  $G = (3, 0, 1)$

$\vec{L}_2 = (3, 0, 1) + \mu(1, -1, -5)$

$\vec{L}_2 = (3+1\mu, 0-1\mu, 1-5\mu)$

$\vec{L}_2 = (3+1\mu, -1\mu, 1-5\mu)$

$\vec{L}_2 = (3+1\mu, -1\mu, 1-5\mu)$

$\vec{L}_2 = (3+1\mu, -1\mu, 1-5\mu)$

d) RESOLVING THE PARALLELOGRAM

• 3 CONGRUENT TRIANGLES

• IF  $\cos \theta = \frac{1}{3}$

So for  $\frac{1}{3} = \frac{1}{3}$

•  $\text{Area of } \triangle ABC = \frac{1}{2}|AB||BC|\sin \theta$

$= \frac{1}{2}\sqrt{14}\sqrt{27}\frac{1}{3}$

$= \frac{1}{2}\sqrt{14}\sqrt{27}\frac{1}{3}$

$= 9\sqrt{2}$

• REQUIRED  $\text{Area } ABC = 2 \times \triangle ABC$

$= 9\sqrt{2}$

As Required

**Question 67 (\*\*\*\*)**

The points  $A$  and  $B$  have position vectors  $9\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$  and  $9\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ , respectively.

- a) Find a vector equation of the straight line  $l_1$  that passes through  $A$  and  $B$ .

The straight line  $l_2$  has the vector equation

$$\mathbf{r}_2 = 6\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k}),$$

where  $\mu$  is a scalar parameter.

- b) Show that  $l_1$  and  $l_2$  intersect and find the position vector of their point of intersection.
- c) Find the acute angle between  $l_1$  and  $l_2$ .

The point  $C$  lies on  $l_2$  in such a position so that is closest to  $A$ .

- d) Show that the position vector of  $C$  is given by

$$\mathbf{c} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

$$\boxed{\mathbf{r}_1 = 9\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{j} - 4\mathbf{k})}, \quad \boxed{9\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}}, \quad \boxed{45.6^\circ}$$

(a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (9, 4, 1) - (9, 3, 5) = (0, 1, -4)$   
 $\mathbf{f}_1 = (9, 3, 5) + \lambda(0, 1, -4) = (9, 3+3\lambda, 5-\lambda)$

(b)  $\mathbf{f}_2 = (6, 3, -4) + \mu(1, 1, -1) = (6+4\mu, 3+\mu, -4-\mu)$

- Equate  $x$  &  $y$ 

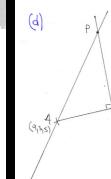
$$\begin{cases} (1): 9+4\mu = 6+4\mu \\ (2): 3+\mu = 3+\mu \end{cases} \Rightarrow \begin{cases} y=3 \\ \lambda=3 \end{cases}$$

check  $\mathbf{z}$   
 $5-4 = 5-4 = 1 = 1$

As all three components agree the lines intersect.

Hence  $\mu = 3$ . Nearest point is  $P(9, 6, -7)$

(c)   
 DOTTING PRODUCT METHOD  
 $\Rightarrow (\mathbf{c}, \mathbf{v}_{l_1}) \cdot (\mathbf{v}_{l_1}, \mathbf{c}) = |\mathbf{c}| |\mathbf{v}_{l_1}| \cos \theta$   
 $\Rightarrow 1 \cdot 4 = \sqrt{1+16} \cdot \sqrt{1+16} \cos \theta$   
 $\Rightarrow S = \sqrt{17} \sqrt{17} \cos \theta$   
 $\Rightarrow \cos \theta = \frac{S}{\sqrt{17}}$   
 $\therefore \theta \approx 45.6^\circ$

(d)   
 • Let  $\mathbf{z} = (0, 0, 0)$   
 •  $\vec{AC} = \mathbf{z} - \mathbf{a} = (9, 3, 5) - (9, 3, 5) = (0, 0, 0)$   
 •  $\vec{AC} \cdot (1, 1, -1) = 0$   

$$(3-1, 4-3, 2-5) \cdot (1, 1, -1) = 0$$

$$2-4+4-3-2+5=0$$

$$\therefore \mathbf{z} \text{ is on } l_2$$

But  $C$  is on  $l_2$   
 $(3, 6, -7) = (9, 3, 5) + \mu(4, 1, -4)$   
 $\therefore 3 = 9 + 4\mu \Rightarrow \mu = -1$   
 $6 = 3 + \mu \Rightarrow \mu = 3$   
 $-7 = 5 - 4\mu \Rightarrow \mu = -1$

$\therefore C(-2, 1, -2)$

**Question 68 (\*\*\*\*)**

With respect to a fixed origin  $O$ , the points with coordinates  $A(2,3,5)$ ,  $B(6,-1,5)$ ,  $C(9,2,2)$  and  $D(5,6,2)$  are given.

Prove that  $ABCD$  is a rectangle and show that its area is  $12\sqrt{6}$  square units.

The diagonals of the rectangle intersect at the point  $E$ .

- Find the coordinates of  $E$ .
- Find the size of the angle  $BEA$ .
- State the exact area of the triangle  $BEA$ .

$$E\left(\frac{11}{2}, \frac{5}{2}, \frac{7}{2}\right), \angle BEA = 94.9^\circ, \text{area}(\triangle BEA) = 3\sqrt{6}$$

(a)

$$\begin{aligned}|AB| &= |b-a| = |(6,-1,5)-(2,3,5)| = |4,-4,0| = \sqrt{4+16} = 2\sqrt{5}\\|BC| &= |c-b| = |(9,2,2)-(6,-1,5)| = |3,3,3| = \sqrt{9+9+9} = 3\sqrt{3}\\|AD| &= |d-a| = |(5,6,2)-(2,3,5)| = |3,3,-3| = \sqrt{9+9+9} = 3\sqrt{3}\\|DC| &= |c-d| = |(9,2,2)-(5,6,2)| = |4,-4,0| = \sqrt{16+16} = 4\sqrt{2}\end{aligned}$$

- OPPOSITE SIDES ARE EQUAL  $\Rightarrow$  PARALLELOGRAM
- $\vec{AB} \cdot \vec{AD} = (4,-4,0) \cdot (3,3,-3) = 12 - 12 + 0 = 0 \Rightarrow \hat{A} = 90^\circ$
- A PARALLELOGRAM WITH A DIHEDRAL ANGLE OF  $90^\circ$  IS A RECTANGLE

TO find  $\angle BEA = |\angle AEB| = \sqrt{\frac{1}{2} \times 32} = 2\sqrt{2} = 4\sqrt{2} = 12\sqrt{6}$

By Pythag

(b)

$E$  IS THE MIDPOINT OF  $AC$  (OR  $BD$ )

$$\therefore E\left(\frac{2+9}{2}, \frac{3+2}{2}, \frac{5+2}{2}\right) = E\left(\frac{11}{2}, \frac{5}{2}, \frac{7}{2}\right)$$

(c)

$$\begin{aligned}\vec{BA} \times \vec{EA} &= (4,-4,0) \times (4,-2,-2) = (2,-2,2) \\|\vec{BA}| &= \sqrt{16+16} = 4\sqrt{2} \\|\vec{EA}| &= \sqrt{1+1+1} = \sqrt{3}\end{aligned}$$

$$\begin{aligned}\vec{BA} \cdot \vec{EA} &= (4,-4,0) \cdot (4,-2,-2) = 16 + 8 + 0 = 24 \\|\vec{BA}| \cdot |\vec{EA}| &= 4\sqrt{2} \times \sqrt{3} = 4\sqrt{6} \\&\therefore \cos \theta = \frac{24}{4\sqrt{6}} = \frac{6}{\sqrt{6}} = \sqrt{6} \\&\therefore \theta = \cos^{-1}(\sqrt{6}) \\&\therefore \theta \approx 94.9^\circ\end{aligned}$$

(d)

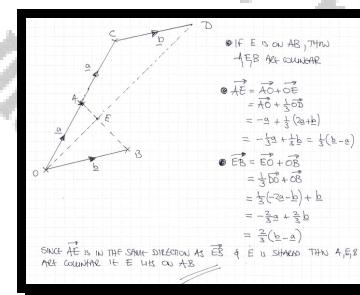
$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 12\sqrt{6}^2 = 36\sqrt{6}$$

**Question 69    (\*\*\*\*\*)**

Let  $\overrightarrow{OA} = \mathbf{a}$ ,  $\overrightarrow{OB} = \mathbf{b}$ ,  $\overrightarrow{OC} = 2\mathbf{a}$  and  $\overrightarrow{OD} = 2\mathbf{a} + \mathbf{b}$ .

If  $\overrightarrow{OE} = \frac{1}{3}\overrightarrow{OD}$  prove that the point  $E$  lies on the straight line  $AB$ .

proof



**Question 70 (\*\*\*\*)**

The point  $A$  has position vector  $-5\mathbf{j} + 7\mathbf{k}$ .

- a) Find a vector equation of the straight line  $l$  that passes through  $A$  and is parallel to the vector  $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ .

The point  $P$  lies on  $l$  so that  $OP$  is perpendicular to  $l$ , where  $O$  is the origin.

- b) Determine the coordinates of  $P$ .  
 c) Show that the point  $B(5, 10, 2)$  lies on  $l$ .

The point  $C$  is on  $l$  so that  $|OB| = |OC|$ .

- d) Find the coordinates of  $C$ .

$$\mathbf{r} = -5\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} - \mathbf{k}), [P(2, 1, 5)], [C(-1, -8, 8)]$$

**(a)**  $\mathbf{r} = (0, -5, 7) + \lambda(1, 3, -1) = (0, 3\lambda - 5, 7 - \lambda)$

Let  $\mathbf{r} = (x, y, z)$   
 $OP \perp \text{line } l$   
 $(0, 0, 0), (x, y, z) = 0$   
 $3\lambda - 5 = 0$

$\mathbf{r}$  is on the line  $l$   
 $(x, y, z) = (0, 3\lambda - 5, 7 - \lambda)$

|                    |
|--------------------|
| $x = 2$            |
| $y = 1$            |
| $z = 5$            |
| $3\lambda - 5 = 1$ |

Solving simultaneously  $\Rightarrow$

$$\begin{aligned} 2 &= 3\lambda - 5 \quad |+5 \\ 7 &= 3\lambda \quad ||:3 \\ 7 &= \lambda \end{aligned}$$

$\therefore P(2, 1, 5)$

**(c)** By inspection if  $3\lambda = 5$   
 $(0, 3\lambda - 5, 7 - \lambda)$  yields  $B(5, 10, 2)$

**(d)** Point  $B$  is the midpoint of  $BC$   
 $\frac{2+5}{2} = 2 \Rightarrow x = -1$   
 $\frac{1+8}{2} = 4.5 \Rightarrow y = -8$   
 $\frac{5+2}{2} = 3.5 \Rightarrow z = 8$   
 $\therefore C(-1, -8, 8)$

**Question 71 (\*\*\*\*)**

Relative to a fixed origin  $O$ , the straight lines  $l_1$  and  $l_2$  have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 2 \\ a \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} b \\ 2 \\ 14 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ -4 \\ 6 \end{pmatrix},$$

where  $\lambda$  and  $\mu$  are scalar parameters, and  $a$  and  $b$  are constants.

$l_1$  and  $l_2$  intersect at the point  $P$ , whose  $z$  coordinate is 8.

- a) Find the coordinates of the point  $P$ .
- b) Show that the value of both  $a$  and  $b$ , is zero.

The point  $A$ , whose  $z$  coordinate is zero, lies on  $l_1$ .

The point  $C$  lies on  $l_2$ , so that  $AC$  is perpendicular to  $l_1$ .

- c) Determine the coordinates of  $C$ .

$$P(-7, 6, 8), \quad C(21, -10, 32)$$

(a)  $\mathbf{r}_1 = (2, a, 2) + \lambda(3, -2, -2) = (3\lambda+2, -2\lambda+a, 2-2\lambda)$   
 $\mathbf{r}_2 = (b, 2, 14) + \mu(7, -4, 6) = (7\mu+b, 2-4\mu, 14+6\mu)$

As intersection is  $(2, 6, 8)$   $\Rightarrow 2\lambda+2=2 \Rightarrow \lambda=0$   
 $\Rightarrow 2-2\lambda+a=6 \Rightarrow a=4$   
 $\Rightarrow 14+6\mu=8 \Rightarrow \mu=-1$

Hence  
if  $\lambda=0$  &  $\mu=-1$  (1)  $\Rightarrow 3(-3)+2=2(-1)+b$   
 $\Rightarrow -7=b$   
 $\Rightarrow b=-7$   
(2)  $\Rightarrow -2(-3)+a=2-4(-1)$   
 $\Rightarrow a=6$

∴ Hence using  $\mu=-1$  (with b=-7) we obtain  $P(-7, 6, 8)$

(c)

$\bullet A(2, 6, 8)$  on  $l_1$ , i.e.  $2\lambda+2=2$   
 $\therefore A(2, 6, 8)$   
 $\bullet$  let  $S = (2, 6, 8)$   
 $\vec{AS} = S-A = (2, 6, 8)-(2, 0, 0) = (0, 6, 8)$   
 $\vec{AC} = C-A = (0, 0, 14)-(2, 0, 0) = (-2, 0, 14)$   
 $\vec{AC} \perp \vec{AS}$   
 $(-2, 0, 14) \cdot (0, 6, 8) = (3, -2, -2) \cdot (0, 6, 8) = 0$   
 $3(-2)-2(0)-2(8)=0$   
 $3(-2)-2(0)-2(8)=0$   
 $\bullet$  Hence  $C$  lies on  $l_2$   
 $(2, 6, 8) = (2, 0, 0) + t(7, -4, 6) \quad t=2$   
 $\begin{cases} 2=2+7t \\ 6=0-4t \\ 8=0+6t \end{cases}$   
 $t=2$   
 $\therefore C(21, -10, 32)$

**Question 72 (\*\*\*\*)**

The straight line  $l$  passes through the points  $A$  and  $C$  whose respective coordinates are  $(-2, 7, 9)$  and  $(8, -3, -1)$ .

- a) Find a vector equation for  $l$ .

The point  $E(2, p, q)$  lies on  $l$  and the point  $B$  has coordinates  $(-4, 1, 1)$ .

- b) Determine the value of  $p$  and the value of  $q$ .  
 c) Show that  $BE$  is perpendicular to  $l$ .

The point  $D$  is such, so that  $ABCD$  is a kite with  $\angle ABC = \angle ADC$ .

Determine ...

- d) ... the coordinates of  $D$ .  
 e) ... the area of the kite  $ABCD$ .

 ,  $\mathbf{r} = -2\mathbf{i} + 7\mathbf{j} + 9\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - \mathbf{k})$ ,  $p = 3$  and  $q = 5$ ,  $D(8, 5, 9)$ ,  $20\sqrt{42}$

**Q1**  $\vec{AC} = 3 - 2 = (8, -3, -1) - (-2, 7, 9) = (10, -10, -10) \leftarrow \text{SCALE TO } (1, -1, -1)$   
 $\vec{l} = (2, 7, 9) + \lambda(1, -1, -1) = (2 + \lambda, 7 - \lambda, 9 - \lambda)$

**Q2**  $(2, p, q) = (2 + \lambda, 7 - \lambda, 9 - \lambda)$   $\therefore \lambda = 4$  since  $\vec{l}$   
 $\therefore p = 7 - \lambda = 3$   
 $\therefore q = 9 - \lambda = 5$   $\therefore \lambda = 3$   $\therefore q = 5$

**Q3**  $\vec{BE} = \vec{s} - \vec{b} = (2, 7, 5) - (-4, 1, 1) = (6, 6, 4)$   
 DISTANCE FROM LINE  $\vec{l}$  DIRECTION VECTOR:  $(1, -1, -1)$   
 $= \sqrt{6^2 + 6^2 + 4^2} = \sqrt{76}$   
 $= \sqrt{4 \times 19}$   
 $= 2\sqrt{19}$  Perpendicular indeed

**Q4**

- $E$  MUST BE THE MIDPOINT OF  $BD$
- $\frac{2+4}{2} = 3 \quad x = 8$   
 $\frac{-4+2}{2} = -1 \quad y = 5$   
 $\frac{5+9}{2} = 7 \quad z = 9$   
 $\therefore D(8, 5, 9)$

**Q5**

- $\vec{AC} = (10, -10, -10)$   
 $|\vec{AC}| = \sqrt{100 + 100 + 100} = \sqrt{300} = 10\sqrt{3}$
- $\vec{BD} = (6, 6, 4)$   
 $|\vec{BD}| = \sqrt{36 + 36 + 16} = \sqrt{88} = 2\sqrt{22}$

 $\text{Area of kite} = 2 \times \frac{1}{2} |\vec{AC}| |\vec{BD}|$   
 $= 10\sqrt{3} \times 2\sqrt{22}$   
 $= 20\sqrt{66}$

**Question 73 (\*\*\*\*)**

The straight line  $l$  passes through the points with coordinates  $(4, -1, 1)$  and  $(-1, 4, 6)$ .

- a) Determine a vector equation of  $l$ .

The points  $C$  and  $D$  have coordinates  $(4, -2, -3)$  and  $(p, q, -1)$ , respectively.

The midpoint of  $CD$  is the point  $M$ , where  $M$  lies on  $l$ .

Find in any order ...

- b) ... the coordinates of  $M$ .  
 c) ... the value of  $p$  and the value of  $q$ .  
 d) ... the size of the acute angle  $\theta$ , between  $CD$  and  $l$ .

$[C]$ ,  $\boxed{\mathbf{r} = 4\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + \mathbf{k})}$ ,  $\boxed{M(7, -4, -2)}$ ,  $\boxed{p=10, q=-6}$ ,  $\boxed{\theta \approx 51.9^\circ}$

**(a)**  $\vec{AB} = \mathbf{b} - \mathbf{a} = (-1, 4, 6) - (4, -1, 1) = (-5, 5, 5)$   
 USE  $(-1, 1)$  AS DIRECTION  $\Rightarrow \mathbf{d} = (4, -1, 1) + t(-1, 1)$   
 $\mathbf{r} = (4, -1, 1) + t(-1, 1)$

**(b)**  $M\left(\frac{3x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = M\left(\frac{7+1}{2}, \frac{-4-2}{2}, \frac{-2-3}{2}\right)$   
 M IS ON THE LINE  $\Rightarrow$  BY SUBSTITUTION  
 $\begin{cases} 3t+1 = 2 \\ -3+t = -4 \\ 2t = -5 \end{cases} \Rightarrow \begin{cases} t = 1 \\ t = -1 \\ t = -2.5 \end{cases} \Rightarrow M(7, -4, -2)$

**(c)**  $4m + \frac{p+4}{2} = 7 \Rightarrow p = 10$   
 $\frac{q-2}{2} = -4 \Rightarrow q = -6$

**(d)**  $\vec{CD} = \mathbf{d} - \mathbf{c} = (-1, 4, 6) - (4, -1, 1) = (-5, 5, 5)$   
 BUT  $\vec{CD}$  A DIRECTION VECTOR OF UNIT E(Y),  
 $\Rightarrow (5, 5, 5) \cdot (1, 1, 1) = |(5, 5, 5)| \cdot |(1, 1, 1)|$   
 $\Rightarrow -6 - 4 + 2 = \sqrt{5^2 + 5^2 + 5^2} \cdot \sqrt{1^2 + 1^2 + 1^2}$   
 $\Rightarrow -8 = \sqrt{45} \cos \theta$   
 $\Rightarrow \cos \theta = \frac{-8}{\sqrt{45}}$   
 $\Rightarrow \theta \approx 127.1^\circ$   $\therefore$  Acute Angle:  $51.9^\circ$

**Question 74 (\*\*\*\*)**

Relative to a fixed origin  $O$ , the straight lines  $L$  and  $M$  have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 4 \\ 10 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 0 \\ 14 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix},$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- a) Show that  $L$  and  $M$  represent the same straight line and find a linear relationship between  $\lambda$  and  $\mu$ , giving the answer in the form  $\lambda = f(\mu)$ .

The points  $A$ ,  $B$  and  $C$  lie on  $L$ , where  $\lambda = 3$ ,  $\lambda = 5$  and  $\lambda = 8$  respectively.

- b) State the ratio  $AB : BC$ .

,  $\lambda = 4 - 2\mu$  ,  $AB : BC = 2 : 3$

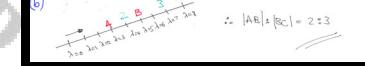
(a)  $\mathbf{r}_1 = (4, 10, 1) + \lambda(-1, 1, -2) = (4-\lambda, 10+\lambda, 1-2\lambda)$   
 $\mathbf{r}_2 = (0, 14, -7) + \mu(2, -2, 4) = (2\mu, 14-2\mu, -7)$

(DIRECTION OF  $L_1$ )  $\Rightarrow$   $(-1, 1, -2) = -1(1, -1, 2)$  (L-SIMILAR DIRECTION)  
(DIRECTION OF  $L_2$ )  $\Rightarrow$   $(2, -2, 4) = 2(1, -1, 2)$  (PARALLEL LINE POSSIBLY DIFFERENT)

( $L_1$  GIVES POINT  $(4, 10, 1)$ )  
IF  $\mu = 2$ ,  $(2\mu, 14-2\mu, -7) = (4, 10, 1)$  LIES ON  $L_2$  AT THE SAME POINT

∴ LINE  $L_1$  IS THE SAME LINE AS  $L_2$

NOTE:  $4-\lambda = 2\mu \Rightarrow 4-2\mu = \lambda$   
 $2\mu + 10 = 14-2\mu \Rightarrow 2\mu + 2\mu = 14-10 \Rightarrow 4\mu = 4 \Rightarrow \mu = 1$   
 $(-2) = 4\mu - 7 \Rightarrow 8-4\mu = 21 \Rightarrow \lambda = 4-2\mu \Rightarrow \lambda = 4-2(1) = 2$

(b) 

$\therefore |AB| : |BC| = 2 : 3$

**Question 75 (\*\*\*\*)**

Relative to a fixed origin  $O$ , the points  $A$  and  $B$  have respective position vectors

$$\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} \quad \text{and} \quad \mathbf{b} = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}.$$

- a) Find the position vector of the point  $C$ , given that  $\mathbf{c} = \mathbf{a} + \mathbf{b}$ .

- b) Show that  $OACB$  is a rectangle, and calculate its area.

The diagonals of the rectangle  $OACB$ ,  $OC$  and  $AB$ , meet at the point  $D$ .

- c) State the position vector of  $D$ .

- d) Calculate the size of the angle  $BDC$ .

$$\boxed{\mathbf{c} = 6\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}}, \quad \boxed{\text{area} = 8\sqrt{42}}, \quad \boxed{\mathbf{d} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}}, \quad \boxed{94.4^\circ}$$

(a)  $\mathbf{c} = \mathbf{a} + \mathbf{b} = (2, 4, 6) + (4, 4, -4) = (6, 8, 2)$

(b)

$\bullet |OC| = |\mathbf{c}| = \sqrt{2^2 + 4^2 + 6^2} = \sqrt{4 + 16 + 36} = \sqrt{56}$   
 $\bullet |BC| = |\mathbf{b}| = \sqrt{(6,8,2) - (4,4,-4)}$   
 $= \sqrt{2^2 + 4^2 + 6^2} = \sqrt{56}$   
 $\bullet |AC| = |\mathbf{c} - \mathbf{a}| = [(6,8,2) - (2,4,6)]$   
 $= [4,4,-4] = \sqrt{16 + 16 + 16} = \sqrt{48}$   
 $\bullet |OB| = [4,4,-4] = \sqrt{48}$   
 $\bullet \text{OPPOSITE SIDES ARE EQUAL} \Rightarrow \text{PARALLELGRAM}$   
 $\bullet \text{CHECK FOR RIGHT ANGLE}$   
 $\overrightarrow{OA} \cdot \overrightarrow{OB} = (2,4,6) \cdot (4,4,-4) = 8 + 16 - 24 = 0$   
 $\therefore \text{Hence } A \text{ rectangle}$   
 $\text{AND AREA IS } \sqrt{56} \times \sqrt{48} = \sqrt{56 \times 48} = \sqrt{56 \times 4 \times 12} = 8\sqrt{42}$

(c)  $D$  is the midpoint of  $OC$  (or  $AB$ )  
 $\therefore D(3,4,1)$

(d)  $\text{Dot } \overrightarrow{OC} \text{ and } \overrightarrow{AB} \Rightarrow \mathbf{b} - \mathbf{a} = (4,4,-4) - (2,4,6) = (2,0,-10)$   
 $\Rightarrow (6,8,2) \cdot (2,0,-10) = |6\mathbf{i}| |2\mathbf{i}| \cos \theta$   
 $\Rightarrow 12 + 0 - 20 = \sqrt{36+64+4} \sqrt{4+0+0} \cos \theta$   
 $\Rightarrow -8 = 10\sqrt{13} \cos \theta$   
 $\Rightarrow \cos \theta = -\frac{8}{10\sqrt{13}}$   
 $\theta = 94.4^\circ$

**Question 76 (\*\*\*\*)**

With respect to a fixed origin  $O$ , the points  $A$  and  $B$  have coordinates  $(5, -1, -1)$  and  $(1, -5, 7)$ , respectively.

- a) Find a vector equation of the straight line  $l$  which passes through  $A$  and  $B$ .

The point  $C$  has coordinates  $(4, -2, 1)$ .

- b) Show that  $C$  lies on  $l$ .

- c) Show further that  $\overrightarrow{OC}$  is perpendicular to  $l$ .

The point  $D$  lie on  $l$  so that  $|\overrightarrow{CD}| = 2|\overrightarrow{CA}|$ .

- d) Find the two possible sets for the coordinates of  $D$ .

$$\boxed{\text{[ ]}}, \boxed{\mathbf{r} = 5\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - 2\mathbf{k})}, \boxed{D(2, -4, 5) \text{ or } D(6, 0, -3)}$$

**(a)**  $\frac{\mathbf{a}}{|a|} = \frac{(5, -1, -1)}{\sqrt{27}} = \frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$

$\frac{\mathbf{b}}{|b|} = \frac{(1, -5, 7)}{\sqrt{75}} = \frac{1}{5}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{7}{5}\mathbf{k}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \left( \frac{1}{5}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{7}{5}\mathbf{k} \right) - \left( \frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \right) = \left( -\frac{2}{15}\mathbf{i} + \frac{14}{15}\mathbf{k} \right)$

Since direction to  $C(4, -2, 1)$

If  $\vec{\Gamma} = (4, -2, 1) = \lambda(1, -1, 2)$

$\vec{\Gamma} = (4, -2, 1) - (1, -1, 2) = (3, -1, -1)$

**(b)**  $C = (4, -2, 1)$  If  $A = (-1, 1, -1)$  then  $\vec{\Gamma} = (4, -2, 1) - (-1, 1, -1) = (5, -3, 2)$

$\therefore C$  lies on  $l$

**(c)**  $\overrightarrow{OC} \perp \overrightarrow{AB}$  Now  $2\vec{\Gamma} = (6, -6, 2)$ ,  $\vec{\Gamma} \cdot \overrightarrow{AB} = (4, -2, 1) \cdot (1, -1, 2) = 4 - 2 - 2 = 0$

$\therefore \overrightarrow{OC}$  is perpendicular to  $l$

**(d)**  $|\overrightarrow{CD}| = 2|\overrightarrow{CA}|$

AT  $A$   $\lambda = 0$  (by instruction)

AT  $C$   $\lambda = 1$  (from part c))

AT  $D$   $\lambda = -3$  or  $\lambda = 1$

$\therefore D(2, -4, 5) \text{ or } D(6, 0, -3)$

**Question 77 (\*\*\*\*)**

$OABC$  is a rectangle, with  $A(2, 2, 0)$ ,  $B(3, a, b)$ , where  $a$  and  $b$  are positive constants and  $O$  is a fixed origin.

- Given that the area of  $OABC$  is 12 square units determine the value of  $a$  and the value of  $b$ .
- Find a vector equation of the straight line  $l$  that passes through  $A$  and  $C$ .

$$[a=1], [b=4], [\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + \lambda(-\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})]$$

(a)  $\vec{OA} \cdot \vec{AB} = 0$   
 $(2, 2, 0) \cdot (1, a-2, b) = 0$   
 $2 + 2a - 4 = 0$   
 $2a = 2$   
 $a = 1$

(b)  $|\vec{AB}| = |\vec{AB}| = \sqrt{2^2 + 2^2} = \sqrt{8}$   
 $\vec{AB} = (3, a, b) - (2, 2, 0) = (1, a-2, b)$   
 $|\vec{AB}| = \sqrt{1^2 + (a-2)^2 + b^2} = \sqrt{1 + a^2 - 4a + 4 + b^2} = \sqrt{5 + a^2 - 4a + b^2} = \sqrt{16} = 4$   
 $5 + a^2 - 4a + b^2 = 16$   
 $a^2 - 4a + b^2 = 11$   
 $a^2 - 4a + 4 + b^2 = 15$   
 $(a-2)^2 + b^2 = 15$   
 $b^2 = 15 - (a-2)^2$   
 $b^2 = 15 - 1$   
 $b^2 = 14$   
 $b = \sqrt{14}$  (b > 0)

(c)  $\vec{OC} = \vec{AB} = (1, a-2, b) = (1, -1, 4) \Rightarrow C(1, -1, 4)$   
 $\vec{OC} = \vec{OB} = (1, a, b) = (1, 1, 4) \Rightarrow C(1, 1, 4)$   
 $\text{Hence } \vec{C} = (2, 2, 0) + \lambda(-1, -3, 4)$   
 $\vec{C} = (2-1, 2-3, 4)$

**Question 78 (\*\*\*\*)**

With respect to a fixed origin  $O$ , the points  $A$  and  $B$  have coordinates  $(1, 5, 4)$  and  $(3, 4, 5)$ , respectively.

- a) Find a vector equation of the straight line  $l$  that passes through  $A$  and  $B$ .

The point  $C$  lie on  $l$  so that  $\overrightarrow{AC} = \frac{1}{2}\overrightarrow{CB}$ .

- b) Determine the coordinates of  $C$ .  
 c) Calculate the size of the angle  $OAC$ .

$$\boxed{\mathbf{r} = \mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})}, \boxed{C\left(\frac{11}{5}, \frac{22}{5}, \frac{19}{5}\right)}, \boxed{93.6^\circ}$$

a)  $A(1, 5, 4)$   
 $B(3, 4, 5)$

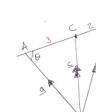
$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (3, 4, 5) - (1, 5, 4) = (2, -1, 1)$$

From  $CE$

$$\mathbf{f} = (1, 5, 4) + \lambda(2, -1, 1)$$

$$\mathbf{f} = (2\lambda + 1, 5 - \lambda, 4 + \lambda)$$

b)  $\overrightarrow{AC} = \frac{3}{2}\overrightarrow{CB}$        $\Rightarrow |\lambda| : |CB| : \frac{3}{2}$        $\therefore \overrightarrow{AC} = \frac{3}{2}\overrightarrow{AB}$



Thus  $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$$\overrightarrow{OC} = \overrightarrow{OA} + \frac{3}{2}\overrightarrow{AB}$$

$$\overrightarrow{OC} = \mathbf{a} + \frac{3}{2}\mathbf{b} - \frac{3}{2}\mathbf{a}$$

$$\overrightarrow{OC} = \frac{3}{2}\mathbf{b} + \frac{1}{2}\mathbf{a}$$

$$\overrightarrow{OC} = \frac{1}{2}(3\mathbf{b} + \mathbf{a})$$

$$\overrightarrow{OC} = \frac{1}{2}(1, 15, 13)$$

$$\therefore C\left(\frac{11}{5}, \frac{22}{5}, \frac{19}{5}\right)$$

c)  $\overrightarrow{AB} \cdot \overrightarrow{AO} = |\overrightarrow{AB}| |\overrightarrow{AO}| \cos \theta$

$$\Rightarrow (2, -1, 1) \cdot (-2, -5, -4) = (2, -1, 1) \cdot (-1, -5, -4) \cos \theta$$

$$\Rightarrow -2 + 5 + 4 = \sqrt{4 + 1 + 1} \sqrt{1 + 25 + 16} \cos \theta$$

$$\Rightarrow -1 = 4\sqrt{2} \cos \theta$$

$$\Rightarrow -1 = 4\sqrt{2} \cos \theta$$

$$\Rightarrow \cos \theta = -\frac{\sqrt{2}}{4}$$

$$\Rightarrow \theta \approx 93.6^\circ$$

**Question 79 (\*\*\*\*)**

Relative to a fixed origin  $O$ , the points  $A$  and  $B$  have position vectors  $8\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$  and  $8\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$ , respectively.

- a) Find a vector equation of the straight line  $l_1$  which passes through  $A$  and  $B$ .

The straight line  $l_2$  has the vector equation

$$\mathbf{r}_2 = 5\mathbf{i} + 5\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k}),$$

where  $\mu$  is a scalar parameter.

- b) Show that  $l_1$  and  $l_2$  intersect and find the position vector of their point of intersection.  
 c) Calculate the acute angle between  $l_1$  and  $l_2$ .

The point  $C$  lies on  $l_2$  so that  $C$  is as close as possible to  $A$ .

- d) Find the position vector of  $C$ .

$$\boxed{\mathbf{r}_1 = 8\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{j} - 4\mathbf{k})}, \boxed{[8\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}]}, \boxed{[45.6^\circ]}, \boxed{[\overrightarrow{OC} = 3\mathbf{i} + 3\mathbf{j}]}$$

(a)  $\vec{AB} = b - a = (8, 6, 2) - (8, 5, 7) = (0, 1, -5)$   
 $\therefore l_1 = (8, 5, 7) + \lambda(0, 1, -5) = (8, 5 + \lambda, 7 - 5\lambda)$

(b)  $\vec{l}_2 = (5, 5, -2) + \mu(1, 1, -1) = (5 + \mu, 5 + \mu, -2 - \mu)$

Given  $\vec{l}_1 \perp \vec{l}_2$   
 (1):  $0 \cdot 1 + 5 \cdot 1 + (-5) \cdot (-1) = 0 \Rightarrow \mu = 3$   
 (2):  $1 \cdot 1 + 5 \cdot 1 + (-1) \cdot (-1) = 0 \Rightarrow \lambda = 3$

Check:  $k$   
 $-7 + 1 = 7 - 4 \times 3 = -5$   
 $-7 + 1 = -1 - 2 = -5$   
 At the three components agree that lines intersect

Using  $\lambda = 3$  into  $(8, 5 + \lambda, 7 - 5\lambda)$  gives  $(8, 8, -5)$

(c) Defining Direction Vectors  
 $(v_1, v_2, v_3) = (0, 1, -5)$  and  $c = 3$   
 $\sqrt{1^2 + (-5)^2} = \sqrt{26}$   
 $\|cv_1\| = \sqrt{26}$   
 $\theta \approx 45.6^\circ$

(d)  $\vec{AC} = S - a = (8, 5, 7) - (8, 3, 0) = (0, 2, 7)$   
 $\vec{AC} \perp \vec{l}_2 \Rightarrow (0, 2, 7) \cdot (1, 1, -1) = 0$   
 $0 + 2 + 7 - 7 = 0$   
 Point  $C$  lies on  $\vec{l}_2 \Rightarrow (x, y, z) = (8 + \mu, 5 + \mu, -2 - \mu)$   
 $x = 8 + \mu, y = 5 + \mu, z = -2 - \mu$   
 Solving simultaneously:  
 $(y+5) + (z+2) - (x+8) = 0$   
 $3\mu = -6$   
 $\mu = -2$   
 Using  $\mu = -2$  into  $(8 + \mu, 5 + \mu, -2 - \mu)$  gives  
 $\boxed{(6, 3, 0)}$

**Question 80 (\*\*\*\*)**

Relative to a fixed origin  $O$ , the position vectors of three points  $A$ ,  $B$  and  $C$  are

$$\overrightarrow{OA} = \mathbf{i} - 2\mathbf{k}, \quad \overrightarrow{AB} = 2\mathbf{i} + 10\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \overrightarrow{BC} = 6\mathbf{i} - 12\mathbf{j}.$$

- Show that  $\overrightarrow{AC}$  is perpendicular to  $\overrightarrow{AB}$ .
- Show further that the area of the triangle  $ABC$  is  $18\sqrt{6}$ .
- Hence, or otherwise, determine the shortest distance of  $A$  from the straight line through  $B$  and  $C$ .

[ ] , distance =  $\frac{6}{5}\sqrt{30}$

①  $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = (2, 10, 2) + (6, -12, 0) = (8, -2, 2)$   
 $\therefore \overrightarrow{AC} \cdot \overrightarrow{AB} = (8, -2, 2) \cdot (2, 10, 2) = 16 - 20 + 4 = 0 \therefore \overrightarrow{AC} \perp \overrightarrow{AB}$

②  $|\overrightarrow{AB}| = \sqrt{2^2 + 10^2 + 0^2} = \sqrt{4 + 100} = \sqrt{104} = 2\sqrt{26}$   
 $|\overrightarrow{AC}| = \sqrt{8^2 + (-2)^2 + 2^2} = \sqrt{64 + 4 + 4} = \sqrt{72} = 6\sqrt{2}$   
 $\therefore \text{Area} = \frac{1}{2} |\overrightarrow{AB}| |\overrightarrow{AC}| = \frac{1}{2} \times 2\sqrt{26} \times 6\sqrt{2} = \frac{12\sqrt{52}}{2} = \frac{12\sqrt{4 \times 13}}{2} = \frac{24\sqrt{13}}{2} = 12\sqrt{13}$

③  $|\overrightarrow{BC}| = \sqrt{6^2 + (-12)^2 + 0^2} = \sqrt{36 + 144} = \sqrt{180} = 6\sqrt{5}$   
 $\Rightarrow \frac{1}{2} |\overrightarrow{BC}| \times d = 18\sqrt{6}$   
 $\Rightarrow 3\sqrt{5}d = 18\sqrt{6}$   
 $\Rightarrow d = 18\sqrt{6}/3\sqrt{5}$   
 $\Rightarrow d = \frac{6}{5}\sqrt{30}$

**Question 81 (\*\*\*\*)**

The straight line  $l_1$  passes through the points  $A(6,2,0)$  and  $B(5,0,5)$ .

- a) Find a vector equation of  $l_1$ .

The straight line  $l_2$  has vector equation

$$\mathbf{r}_2 = \begin{pmatrix} -7 \\ 6 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 0 \\ 2 \end{pmatrix},$$

where  $\mu$  is a scalar parameter.

- b) Show that  $l_1$  and  $l_2$  intersect at some point  $C$ , and find its coordinates.

The point  $D$  lies on  $l_2$  so that  $DBC = 90^\circ$ .

- c) Determine the coordinates of  $D$ .

,  $\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + \lambda(-5\mathbf{i} + 2\mathbf{k})$  ,  $[C(8,6,-10)]$  ,  $[D(-22,6,2)]$

**(a)**  $A(2,4,0) \equiv B(5,0,5)$   $\rightarrow \overrightarrow{AB} = (5,0,5) - (2,4,0) = (-3,4,5)$   
 $\therefore l_1 \equiv (5,0,5) + \lambda(-3,4,5) = (5-3\lambda, 4\lambda, 5)$

**(b)**  $\mathbf{r}_2 = (-7, 6, -4) + \mu(-5, 0, 2) = (-7-5\mu, 6, -4+2\mu)$

\* EQUATE  $l_1$  &  $l_2$   
 $\left\{ \begin{array}{l} 5-3\lambda = -7-5\mu \\ 4\lambda = 6 \\ -4+2\mu = -4 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \lambda = 2 \\ \mu = 0 \\ \mu = 0 \end{array} \right.$

\* CHECK  $l_1$   
 $\left\{ \begin{array}{l} 5-3\lambda = -7-5\mu \\ 4\lambda = 6 \\ -4+2\mu = -4 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \lambda = 2 \\ \mu = 0 \\ \mu = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} 5-3(2) = -7-5(0) \\ 4(2) = 6 \\ -4+2(0) = -4 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} -1 = -7 \\ 8 = 6 \\ -4 = -4 \end{array} \right. \Rightarrow \text{OK}$

As all three components agree, this value is correct.  
using  $\lambda = 2$  gives  $(5-3(2), 4(2), -4+2(0)) = (5,8,-4) \in C(\mathbf{r}_1 = \mathbf{r})$

**(c)** 

Let  $\mathbf{d} = (2,4,2)$   
 $\overrightarrow{BD} = \mathbf{d} - \mathbf{r} = (2,4,2) - (5,0,5) = (-3,4,-3)$

\* USE  $\overrightarrow{BD} \cdot \mathbf{d} = 0$   
 $(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = 0$   
 $-3(-3) + 4(4) - 3(-3) = 0$   
 $27 + 16 + 9 = 52 \neq 0$

Also D lies on the line  $l_2 \Rightarrow$   
 $\left\{ \begin{array}{l} x = -7 + 5\mu \\ y = 6 \\ z = -4 + 2\mu \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x = -7 + 5\mu \\ y = 6 \\ z = -4 + 2\mu \end{array} \right.$

SOLVING SIMULTANEOUSLY :  
 $\begin{aligned} -(-7 + 5\mu) - 2(4) + 5(-4 + 2\mu) &= 20 \\ 7 + 5\mu - 8 + 10\mu - 20 &= 20 \\ 15\mu - 11 &= 20 \\ 15\mu &= 31 \\ \mu &= \frac{31}{15} \end{aligned}$

$\therefore D(-22,6,2)$

**Question 82 (\*\*\*\*)**

The straight lines  $L_1$  and  $L_2$  have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 2 \\ 10 \\ 14 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} a \\ 8 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ b \\ 1 \end{pmatrix},$$

where  $\lambda$  and  $\mu$  are scalar parameters, and  $a$  and  $b$  are scalar constants.

$L_1$  and  $L_2$  intersect at the point  $P$  whose  $z$  coordinate is 6, and the acute angle between  $L_1$  and  $L_2$ , is  $\theta$ .

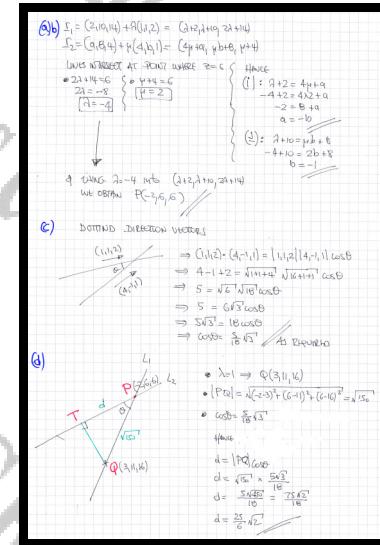
- a) Determine the coordinates of  $P$ .
- b) Find the value of  $a$  and the value of  $b$ .
- c) Show that  $\cos \theta = \frac{5}{18}\sqrt{3}$ .

The point  $Q$  lies on  $L_1$  where  $\lambda=1$ .

The point  $T$  lies on  $L_2$  so that  $\overrightarrow{QT}$  is perpendicular to  $L_2$ .

- d) Determine the exact distance  $PT$ .

$$[ ] , P(-2, 6, 6) , [a = -10, b = -1] , [ |PT| = \frac{25}{6}\sqrt{2} ]$$



**Question 83 (\*\*\*\*)**

The points  $A$  and  $B$  have coordinates  $(3, -1, 2)$  and  $(2, 0, 2)$ , respectively.

- a) Find a vector equation of the straight line  $l_1$  that passes through  $A$  and  $B$ .

The straight line  $l_2$  has equation

$$\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \boldsymbol{\mu}(\mathbf{i} - \mathbf{k}),$$

where  $\boldsymbol{\mu}$  is a scalar parameter.

- b) Show that  $l_1$  and  $l_2$  intersect at some point  $P$  and find its coordinates.  
 c) Verify that the point  $C(9, 1, -6)$  lies on  $l_2$ .

The point  $D$  lies on  $l_1$  so that  $CD$  is perpendicular to  $l_1$ .

- d) Determine the coordinates of  $D$ .  
 e) Calculate the area of the triangle  $PDC$ .  
 f) Deduce the acute angle between  $l_1$  and  $l_2$ .

$$\boxed{\mathbf{r}_1 = 2\mathbf{i} + 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j})}, \boxed{P(1, 1, 2)}, \boxed{D(5, -3, 2)}, \boxed{\text{area} = 16\sqrt{3}}, \boxed{60^\circ}$$

Handwritten working for Question 83:

- a)**  $\vec{AB} = \mathbf{b} - \mathbf{a} = (2, 0, 2) - (3, -1, 2) = (-1, 1, 0)$   
 $\therefore l_1 = \mathbf{a} + t\vec{AB} = (3, -1, 2) + t(-1, 1, 0)$   
 $\therefore l_1 = (3-t, -1+t, 2)$
- b)**  $\mathbf{r}_2 = (1+2t, 1, 1-t)$   
 Given  $\mathbf{r}_1 \perp \mathbf{r}_2$   
 $\therefore 3-t + 1 - t = 0$   
 $\therefore t = 2$   
 $\therefore P(1, 1, 2)$
- c)** By inspection if  $t=7$   
 $(4+2, 1, 1-7) = (9, 1, -6)$   $\therefore C$  lies on  $l_2$
- d)** Using  $\mathbf{r}_1$   
 $\vec{CD} = \mathbf{d} - \mathbf{c} = (5, -3, 2) - (9, 1, -6) = (-4, -4, 8)$   
 $\therefore \vec{CD} \perp l_1$   
 $\therefore \vec{CD} \cdot \vec{AB} = 0$   
 $-2(-4) + (-4)(-1) + 8(0) = 0$   
 $8 - 4 = 4$   
 $4 = 4$   
 $\therefore D(5, -3, 2)$
- e)**  $\text{Area } \triangle PDC = \frac{1}{2} |\vec{PD}| |\vec{CD}| \sin \theta$   
 $|\vec{PD}| = |P - D| = |(1, 1, 2) - (5, -3, 2)| = |(-4, 4, 0)| = \sqrt{16+16} = \sqrt{32}$   
 $|\vec{CD}| = |C - D| = |(9, 1, -6) - (5, -3, 2)| = |(4, 4, -8)| = \sqrt{16+16+64} = \sqrt{96}$   
 $\sin \theta = \frac{1}{2} \sqrt{32} \sqrt{96} = \frac{1}{2} \sqrt{32 \cdot 96} = \frac{1}{2} \sqrt{3072} = 16\sqrt{3}$
- f)** Looking at diagram  
 $\tan \theta = \frac{|\vec{CD}|}{|\vec{PD}|} = \frac{\sqrt{96}}{\sqrt{32}} = \sqrt{3}$   
 $\theta = 60^\circ$

**Question 84 (\*\*\*\*)**

Relative to a fixed origin  $O$ , the points  $A$  and  $B$  have respective position vectors

$$2\mathbf{i} + 10\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

The angle  $AOB$  is  $\theta$ .

a) Show that  $\sin \theta = \frac{\sqrt{6}}{3}$ .

b) Calculate the exact area of the triangle  $AOB$ .

c) Show further that the shortest distance of ...

i. ...  $A$  from the straight line  $OB$  is  $6\sqrt{2}$ ,

ii. ... the straight line  $AB$  from  $O$  is  $2\sqrt{2}$ .

$\boxed{\quad}$ ,  $\boxed{\text{area} = 9\sqrt{2}}$

Q)  $O = (2, 10, 2)$        $B = (-2, 1, 2)$

By  $\cos \theta = \frac{\mathbf{OA} \cdot \mathbf{OB}}{|\mathbf{OA}| |\mathbf{OB}|}$

$$\Rightarrow (2\mathbf{i} + 10\mathbf{j} + 2\mathbf{k}) \cdot (-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = (2, 10, 2) \cdot (-2, 1, 2) \cos \theta$$

$$\Rightarrow 4 + 10 + 4 = \sqrt{4 + 100 + 4} \sqrt{4 + 1 + 4} \cos \theta$$

$$\Rightarrow 18 = \sqrt{108} \sqrt{9} \cos \theta$$

$$\Rightarrow \cos \theta = \frac{18}{\sqrt{108}}$$

$$\Rightarrow \cos \theta = \frac{\sqrt{6}}{3}$$

By Pythagoras

$$\sin^2 \theta = 1 - \cos^2 \theta = 1 - \left(\frac{\sqrt{6}}{3}\right)^2 = \frac{5}{9}$$

$$\sin \theta = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

(i)  $\text{Area} = \frac{1}{2} |\mathbf{OA}| |\mathbf{OB}| \sin \theta = \frac{1}{2} \times \sqrt{108} \times 3 \times \frac{\sqrt{5}}{3} = 9\sqrt{2}$   $\checkmark$   
 Found in diagram below

(ii)  $AM = \frac{1}{2} |\mathbf{OB}| |\mathbf{AM}|$   
 $9\sqrt{2} = \frac{1}{2} \times 3 \times |\mathbf{AM}|$   
 $|\mathbf{AM}| = 6\sqrt{2} \checkmark$  AS REQUIRED

(iii)  $|\mathbf{AB}| = |b - a|$   
 $= |(-2, 1, 2) - (2, 10, 2)|$   
 $= |(0, -9, 0)|$   
 $= \sqrt{0 + 81 + 0} = 9$   
 Now  
 $\Delta AM = \frac{1}{2} |\mathbf{AB}| |\mathbf{on}|$   
 $9\sqrt{2} = \frac{1}{2} \times 9 \times |\mathbf{on}|$   
 $2\sqrt{2} = |\mathbf{on}| \checkmark$  AS REQUIRED

**Question 85 (\*\*\*\*)**

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} - \mathbf{k})$$

$$\mathbf{r}_2 = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k} + \mu(\mathbf{i} + 4\mathbf{j} - \mathbf{k})$$

where  $\lambda$  and  $\mu$  are scalar parameters.

The point  $A$  is the intersection of  $l_1$  and  $l_2$ .

The point  $B(b, 1, -1)$ , where  $b$  is a scalar constant, lies on  $l_1$ .

The point  $D(4, d, 3)$ , where  $d$  is a scalar constant, lies on  $l_2$ .

- a) Find the value of  $b$  and the value of  $d$ .
- b) Calculate the cosine of  $\theta$ , where  $\theta$  is the acute angle formed by  $l_1$  and  $l_2$ .

The point  $C$  is such so that  $ABCD$  is a parallelogram.

- c) Determine the coordinates of  $C$ .
- d) Show that the area of the parallelogram  $ABCD$  is  $48\sqrt{2}$  square units.

$$[ ] , [b=8, d=9] , [\cos \theta = \frac{1}{3}] , [C(10,9,-3)]$$

④  $\mathbf{l}_1 = (2, 1, 5) + \lambda(1, 0, -1) = (2+2\lambda, 1, -2+\lambda)$   
 $\mathbf{l}_2 = (2, 1, 5) + \mu(1, 4, -1) = (2+2\mu, 1+4\mu, -2+\mu)$

BY INSPECTION IF  $\lambda=6$  THEN  $(2+2\lambda, 1, -2+\lambda) = (14, 1, -1)$  i.e.  $b=14$   
 BY INSPECTION IF  $\mu=2$  THEN  $(2+2\mu, 1+4\mu, -2+\mu) = (6, 9, -3)$  i.e.  $d=9$

⑤ Drawing Direction Vectors  
 $(1, 0, -1) \times (1, 4, -1) = |1, 0, -1| |1, 4, -1| \cos \theta$   
 $|1, 0, -1| = \sqrt{1+0+1} = \sqrt{2}$   
 $|1, 4, -1| = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$   
 $\cos \theta = 2 / (\sqrt{2} \cdot 3\sqrt{2}) = 1/3$

⑥

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{O} + \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OA}$   
 $\overrightarrow{AB} = (2, 1, 5) - (2, 1, 5) = (0, 0, 0)$   
 $\overrightarrow{AB} = (0, 0, 0)$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \overrightarrow{OD} - \overrightarrow{O} + \overrightarrow{OA} = \overrightarrow{OD} + \overrightarrow{OA}$   
 $\overrightarrow{AD} = (4, d, 3) - (2, 1, 5) = (2, d-1, -2)$   
 $\overrightarrow{AD} = (2, d-1, -2)$

$\overrightarrow{AB} \times \overrightarrow{AD} = [(0, 0, 0) \times (2, d-1, -2)] = [0, 0, 0]$   
 $\text{Magnitude} = \sqrt{0+0+0} = 0$

AREA OF PARALLELOGRAM =  $2 \times \text{CONTRASTING TRIANGLES}$   
 $= 2 \times \frac{1}{2} \times |\overrightarrow{AB}| |\overrightarrow{AD}| \sin \theta$   
 $= 2 \times \frac{1}{2} \times \sqrt{2} \times \sqrt{18} \times \frac{1}{3}$   
 $= 6\sqrt{2} \times \sqrt{2} \times \frac{1}{3}$   
 $= 48\sqrt{2}$

**Question 86 (\*\*\*\*)**

The straight line  $L_1$  passes through the point  $A(5, -2, 1)$  and is parallel to the vector

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

- a) Find a vector equation for  $L_1$ , in terms of a scalar parameter  $\lambda$ .

The straight line  $L_2$  has a vector equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix},$$

where  $\mu$  is scalar parameter.

- b) Show that the lines intersect at some point  $P$ , and find its coordinates.

The point  $B$  lies on  $L_2$  where  $\mu = -2$ .

The point  $C$  lies on a straight line which is parallel to  $L_1$  and passes through  $B$ .

The points  $A$ ,  $B$ ,  $C$  and  $P$  are vertices of a parallelogram.

- c) Show that one of the possible positions for  $C$  is the origin  $O$  and find the coordinates of the other possible position for  $C$ .

|   |   |               |              |
|---|---|---------------|--------------|
| □ | $\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{k})$ | $P(3, -2, 0)$ | $C(4, 0, 2)$ |
|---|---|---------------|--------------|

(a)  $L_1 = (5, -2, 1) + \lambda(2\mathbf{v}_1)$   
 $L_1 = (2\lambda + 5, -2, \lambda + 1)$

(b)  $L_2 = (4, 2, 3) + \mu(-2, 1, 1)$   
 $L_2 = (4, 2\mu + 2, \mu + 3)$

FOR LINE  $L_2$  AND  $L_1$  TO MEET  
 $(1) : 2\lambda + 5 = 4$   
 $(2) : -2\mu + 2 = -2$   
 $\therefore \begin{cases} \lambda = -1 \\ \mu = 1 \end{cases}$

∴  $L_1 = (3, -2, 0)$  AND  $L_2 = (2, 3, 2)$

∴  $P(3, -2, 0)$  AND  $C(4, 0, 2)$

(c)  $\vec{PA} = \vec{PB} = (3, -2, 0) - (2, 0, 1) = (1, -2, -1)$   
 $\vec{PQ} = (2, 0, 1)$

HENCE  
 $S = \vec{BQ} = \vec{B} + \vec{PQ} = \vec{B} + \vec{PA}$   
 $S = (2, 0, 1) + (2, 0, 1) = (4, 0, 2)$

$\vec{CQ} = \vec{B} + \vec{PQ} = \vec{B} + \vec{AP}$   
 $\vec{CQ} = (2, 0, 1) + (-2, 0, -1) = (0, 0, 0)$

$\therefore C(0, 0, 0)$  OR  $C(4, 0, 2)$

**Question 87 (\*\*\*\*)**

The straight line  $L$  has vector equation

$$\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k}),$$

where  $\lambda$  is a scalar parameter.

The point  $A$  has position vector  $\mathbf{i} + \mathbf{j} - \mathbf{k}$ .

The point  $P$  lies on  $L$  so that  $AP$  is perpendicular to  $L$ .

- a) Find the position vector of  $P$ .

The point  $B$  is the reflection of  $A$  about  $L$ .

- b) Determine the position vector of  $B$ .

$$\boxed{\quad}, \boxed{\overrightarrow{OP} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}}, \boxed{\overrightarrow{OB} = 7\mathbf{i} + \mathbf{j} + 5\mathbf{k}}$$

**a)**  $\Gamma = (\mathbf{i}, -2\mathbf{j}, 5) + \lambda(\mathbf{i}, \mathbf{j}, -\mathbf{k}) = (x+1, \lambda-2, 5-\lambda)$

LOOKING AT THE DIAGRAM

Let  $P(x, y, z)$   
i.e.  $P = (x, y, z)$

$\vec{AP}$  IS PERPENDICULAR TO  $L$

$$\begin{aligned} \Rightarrow \vec{AP} \cdot (\mathbf{i}, \mathbf{j}, -\mathbf{k}) &= 0 \\ \Rightarrow (x-1, y+1, z-1) \cdot (\mathbf{i}, \mathbf{j}, -\mathbf{k}) &= 0 \\ \Rightarrow [(x-1)\mathbf{i} + (y+1)\mathbf{j} + (z-1)\mathbf{k}] \cdot (\mathbf{i}, \mathbf{j}, -\mathbf{k}) &= 0 \\ \Rightarrow (x-1)\mathbf{i} \cdot \mathbf{i} + (y+1)\mathbf{j} \cdot \mathbf{i} + (z-1)\mathbf{k} \cdot \mathbf{i} &= 0 \\ \Rightarrow x-1 + y+1 - z-1 &= 0 \\ \Rightarrow x+y-z &= 3 \end{aligned}$$

SOLVING SIMULTANEOUSLY WE OBTAIN

$$\begin{aligned} \Rightarrow (x+1) + (x-2) - (z-2) &= 3 \\ \Rightarrow 3x - 6 &= 3 \\ \Rightarrow 3x &= 9 \\ \Rightarrow x &= 3 \end{aligned}$$

$\therefore P(3, 1, 2)$

LOOKING AT THE DIAGRAM

P MUST BE THE MIDPOINT OF AB

BY INSPECTION

|      |                    |     |                    |     |
|------|--------------------|-----|--------------------|-----|
| A    | $\xrightarrow{+3}$ | P   | $\xrightarrow{+3}$ | B   |
| (1)  |                    | (4) |                    | (7) |
| (1)  | $\xrightarrow{+0}$ | (1) | $\xrightarrow{+6}$ | (5) |
| (-1) | $\xrightarrow{+3}$ | (2) | $\xrightarrow{+3}$ |     |

$\therefore B(7, 1, 5)$

**Question 88 (\*\*\*\*)**

All the position vectors and coordinates in this question are measured from a fixed origin  $O$ .

The point  $P$  lies on the straight line  $l$  with vector equation

$$\mathbf{r} = \mathbf{i} - 3\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}),$$

where  $\lambda$  is a scalar parameter.

The point  $Q$  has position vector  $3\mathbf{i} + 9\mathbf{j} + 6\mathbf{k}$ .

- a) Determine, in terms of  $\lambda$ , an expression for the vector  $\overrightarrow{QP}$ .
- b) By considering  $|\overrightarrow{QP}|^2$ , find the value of  $\lambda$  which makes  $|\overrightarrow{QP}|$  minimum.
- c) Hence, or otherwise, find the shortest distance of  $Q$  from  $l$ .

$\boxed{\quad}$ ,  $\boxed{\overrightarrow{QP} = (2\lambda - 2)\mathbf{i} + (3\lambda - 9)\mathbf{j} + (5\lambda - 9)\mathbf{k}}$ ,  $\boxed{\lambda = 2}$ ,  $\boxed{\sqrt{14}}$

$\Gamma = (1, 0, -3) + \lambda(2, 3, 5) = (2\lambda + 1, 3\lambda, 5\lambda - 3)$

a) AS  $P$  IS ON THIS LINE  
 $\mathbf{P} = (2\lambda + 1, 3\lambda, 5\lambda - 3)$ , FOR SOME  $\lambda$

$$\overrightarrow{QP} = \mathbf{P} - \mathbf{Q} = (2\lambda + 1, 3\lambda, 5\lambda - 3) - (3, 9, 6)$$

$$\overrightarrow{QP} = (2\lambda - 2, 3\lambda - 9, 5\lambda - 9)$$

b) LOOKING AT THE MODULUS

$$\begin{aligned} \Rightarrow |\overrightarrow{QP}| &= |(2\lambda - 2, 3\lambda - 9, 5\lambda - 9)| \\ \Rightarrow |\overrightarrow{QP}| &= \sqrt{(2\lambda - 2)^2 + (3\lambda - 9)^2 + (5\lambda - 9)^2} \\ \Rightarrow |\overrightarrow{QP}|^2 &= \frac{40\lambda^2 - 80\lambda + 14}{13} \\ &\quad - 27\lambda + 81 + 25\lambda^2 - 90\lambda + 81 \\ \Rightarrow |\overrightarrow{QP}|^2 &= 38\lambda^2 - 152\lambda + 166 \end{aligned}$$

BY COMPLETING THE SQUARE (OR CALCULUS)

$$\begin{aligned} \Rightarrow |\overrightarrow{QP}|^2 &= 38\left[\lambda^2 - 4\lambda + \frac{89}{38}\right] \\ \Rightarrow |\overrightarrow{QP}|^2 &= 38(\lambda - 2)^2 - 152 + 166 \\ \Rightarrow |\overrightarrow{QP}|^2 &= 38(\lambda - 2)^2 + 14 \\ \Rightarrow |\overrightarrow{QP}|^2 &\approx 38(\lambda - 2)^2 + 14 \end{aligned}$$

$\therefore \lambda = 2$

a) LOOKING AT THE DIAGONAL

THE SHORTEST DISTANCE OF  $Q$  FROM  $l$  IS  $\sqrt{14}$

BY LOOKING AT  $|\overrightarrow{QP}| = \sqrt{38(\lambda - 2)^2 + 14}$

CALCULATE BY CALCULUS

LOOKING AT  $|\overrightarrow{QP}|^2 = 38\lambda^2 - 152\lambda + 166$

$$\begin{aligned} \Rightarrow f(\lambda) &= 38\lambda^2 - 152\lambda + 166 && \curvearrowleft \text{Minimum} \\ \Rightarrow f'(\lambda) &= 76\lambda - 152 && \text{Look for stationary points} \\ \Rightarrow 152 &= 76\lambda \\ \Rightarrow 2 &= \lambda \\ \Rightarrow f(2) &= 38(2)^2 - 152(2) + 166 \\ \Rightarrow f(2) &= 14 \\ \Rightarrow |\overrightarrow{QP}|^2 &= 14 \\ \Rightarrow |\overrightarrow{QP}|_{\text{min}} &= \sqrt{14} \end{aligned}$$

**Question 89 (\*\*\*\*)**

The points  $A(-1,4)$ ,  $B(2,3)$  and  $C(8,1)$  lie on the  $x$ - $y$  plane, where  $O$  is the origin.

- a) Show that  $A$ ,  $B$  and  $C$  are collinear.

The point  $D$  lies on  $BC$  so that  $\overrightarrow{BD} : \overrightarrow{BC} = 2:3$ .

- b) Find the coordinates of  $D$ .

The straight line  $OB$  is extended to the point  $P$ , so that  $\overrightarrow{AP}$  is parallel to  $\overrightarrow{OC}$ .

- c) Determine the coordinates of  $P$ .

,  $D\left(6, \frac{5}{3}\right)$  ,  $P\left(3, \frac{9}{2}\right)$

**a) FIND THE VECTORS  $\overrightarrow{AB}$  &  $\overrightarrow{BC}$**

$$\overrightarrow{AB} = b - a = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\overrightarrow{BC} = c - b = \begin{pmatrix} 8 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

AS  $\overrightarrow{AB}$  &  $\overrightarrow{BC}$  ARE IN THE SAME DIRECTION & SHARE THE POINT  $B$ ,  $A, B, C$  MUST BE COLLINEAR.

**b) LOOKING AT THE DIAGRAM BELOW**

**LOOKING AT THE DIAGRAM**

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$

$$\overrightarrow{OD} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{BC}$$

$$\overrightarrow{d} = b + \frac{2}{3}(c - b)$$

$$3\overrightarrow{d} = 3b + 2c - 2b$$

$$3\overrightarrow{d} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

$$3\overrightarrow{d} = \begin{pmatrix} 15 \\ -5 \end{pmatrix}$$

$$\overrightarrow{d} = \begin{pmatrix} 5 \\ -\frac{5}{3} \end{pmatrix}$$

$$\therefore D\left(5, \frac{5}{3}\right)$$

**b) LOOKING AT THE DIAGRAM BELOW**

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$$2\overrightarrow{OP} = \overrightarrow{OA} + 2\overrightarrow{AP}$$

$$2\overrightarrow{b} = \overrightarrow{a} + 2\overrightarrow{c}$$

$$2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 16 \\ 2 \end{pmatrix}$$

$$-2\lambda = -1 - 16$$

$$-2\lambda = -33$$

$$\lambda = \frac{33}{2}$$

HENCE AS  $\overrightarrow{OP} = 2\overrightarrow{OB}$

$$\overrightarrow{OP} = 2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\therefore P\left(3, \frac{9}{2}\right)$$

**Question 90 (\*\*\*\*)**

Relative to a fixed origin  $O$ , the points  $A$ ,  $B$  and  $C$  have respective position vectors

$$-6\mathbf{i} - 5\mathbf{j} - 21\mathbf{k}, \quad 8\mathbf{i} + 9\mathbf{j} \quad \text{and} \quad u\mathbf{i} - 3\mathbf{j} + v\mathbf{k},$$

where  $u$  and  $v$  are scalar constants.

$A$ ,  $B$  and  $C$  lie on the straight line  $l$ .

- Find a vector equation of  $l$ .
- Determine the value of  $u$  and the value of  $v$ .
- Calculate the distance  $AB$ .

The point  $D$  lies on  $l$  so that  $\overrightarrow{OD}$  is perpendicular to  $l$ .

- Determine the position vector of  $D$ .
- Calculate, correct to three significant figures, the area of the triangle  $OAB$ .

$$\boxed{\quad}, \quad \mathbf{r} = 8\mathbf{i} + 9\mathbf{j} + \lambda(2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}), \quad u = -4, \quad v = -18, \quad |AB| = 7\sqrt{17},$$

$$\mathbf{d} = 4\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}, \quad \text{area} \approx 127$$

a) USING  $A(-6, -5, -21)$  &  $B(8, 9, 0)$

- $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (8\mathbf{i} + 9\mathbf{j}) - (-6\mathbf{i} - 5\mathbf{j} - 21\mathbf{k}) = (14\mathbf{i} + 14\mathbf{j} + 21\mathbf{k})$
- SCALE THE DIRECTION TO  $(2, 2, 3)$
- $\Sigma = (\text{fixed point}) + (\text{direction vector})$
- $\Sigma = (8, 9, 0) + 2(2, 2, 3)$
- $\Sigma = (20, 13, 21)$

b) USING THE EQUATIONS ABOVE & THE FACT  $C(4, -3, v)$  MUST SATISFY IT

$$(4, -3, v) = (20, 13, 21) + \lambda(2, 2, 3)$$

$$\begin{aligned} (i) \quad 2\lambda + 20 &= 4 \\ 2\lambda &= -16 \\ \lambda &= -8 \end{aligned} \quad \begin{aligned} (j) \quad 2\lambda + 13 &= -3 \\ 2\lambda &= -16 \\ \lambda &= -8 \end{aligned} \quad \begin{aligned} (k) \quad 3\lambda + 21 &= v \\ v &= 3(-8) \\ v &= -18 \end{aligned}$$

c) USING PART (a)

$$|AB| = \sqrt{(14)^2 + (14)^2 + (21)^2} = \sqrt{2(14^2 + 21^2)} = \sqrt{7 \cdot 2 \cdot 14^2} = 7\sqrt{17}$$

d) LOOKING AT THE DIAGRAM, LET  $\mathbf{D}(2, 2, 2)$ , i.e.  $\mathbf{d} = (2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

$\overrightarrow{OB}$  IS PERPENDICULAR TO  $l$

$$\overrightarrow{OB} \cdot (2, 2, 3) = 0$$

$$(2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \cdot (2, 2, 3) = 0$$

$$2x + 2y + 3z = 0$$

$$2(2) + 2(2) + 3(2) = 0$$

$$4 + 4 + 6 = 0$$

$$14 = 0$$

$$\therefore \lambda = -2$$

$D$  IS ON THE LINE  $l$

$$(x, y, z) = (20, 13, 21) + \lambda(2, 2, 3)$$

$$\begin{aligned} x &= 20 + 2\lambda \\ y &= 13 + 2\lambda \\ z &= 21 + 3\lambda \end{aligned}$$

SETTING SIMULTANEOUSLY

$$\begin{aligned} \Rightarrow 2(20 + 2\lambda) + 2(13 + 2\lambda) + 3(21 + 3\lambda) &= 0 \\ \Rightarrow 4\lambda + 16 + 4\lambda + 18 + 9\lambda &= 0 \\ \Rightarrow 17\lambda &= -34 \\ \Rightarrow \lambda &= -2 \end{aligned}$$

$$\therefore D(4, 5, 6)$$

LOOKING AT THE DIAGRAM

- THREE SIDES =  $|OB|$
- $h_1 = \sqrt{14^2 + 21^2} = \sqrt{14^2 + 25 + 36} = \sqrt{77}$
- $h_2 = \sqrt{14^2 + 21^2} = \sqrt{77}$
- AREA =  $\frac{1}{2}|AB|(h_1)$
- $= \frac{1}{2}(7\sqrt{17})(\sqrt{77})$
- $\approx 127$

**Question 91 (\*\*\*\*\*)**

The position vectors and coordinates in this question are relative to a fixed origin  $O$ .

The straight lines  $l_1$ ,  $l_2$  and  $l_3$  have the following vector equations

$$\mathbf{r}_1 = 10\mathbf{i} + 6\mathbf{j} + 9\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

$$\mathbf{r}_2 = -4\mathbf{j} + 13\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

$$\mathbf{r}_3 = -3\mathbf{i} - 4\mathbf{k} + \nu(4\mathbf{i} + 3\mathbf{j} + \mathbf{k}),$$

where  $\lambda$ ,  $\mu$  and  $\nu$  are scalar parameters.

- a) Show that  $l_1$  and  $l_2$  intersect at some point  $A$ , and find its coordinates.

- b) Verify that  $B(5, 6, -2)$  lies on both  $l_2$  and  $l_3$ .

The point  $C$  is the intersection of  $l_1$  and  $l_3$ .

- c) Find the coordinates of  $C$ .

- d) Show that  $|CA| = |CB|$ .

- e) Hence calculate the shortest distance of  $C$  from  $l_2$ .

|              |               |  |
|--------------|---------------|--|
| $A(4, 4, 1)$ | $C(1, 3, -3)$ | $\text{distance} = \frac{3}{2}\sqrt{10}$ |
|--------------|---------------|--|

(a)  $\mathbf{r}_1 = (0, 6, 9) + \lambda(3, 1, 4) = (3\lambda + 0, 6 + \lambda, 9 + 4\lambda)$   
 $\mathbf{r}_2 = (0, 4, 0) + \mu(1, 2, -3) = (0, 4 + 2\mu, -3\mu)$

• Equate  $\mathbf{r}_1$  &  $\mathbf{r}_2$   
 $\begin{cases} 0 = 3\lambda + 0 \\ 6 + \lambda = 4 + 2\mu \\ 9 + 4\lambda = -3\mu \end{cases} \Rightarrow \begin{cases} \lambda + \mu = 2(3\lambda + 0) - 4 \\ 2\lambda + 1 = 6 + 2\mu - 4 \\ 10 = 5\lambda \end{cases}$   
 $\begin{cases} \lambda = 2 \\ \mu = 1 \\ \lambda = 2 \end{cases}$

• Check  $\lambda = 2$ :  $4 + 9 = 4(2 + 1) = 13$  ✓ UNITS IN PARENTHESIS  
 $13 - 3\mu = 13 - 3(2) = 1$

Check  $\mu = 1$ :  $4(1, 2\mu - 4, 13 - 2\lambda) \Rightarrow A(4, 4, 1)$

(b)  $\mathbf{r}_2 = (0, 4, 0) + \mu(1, 2, -3) \Rightarrow \text{BY INSPECTION } \mu = 5$   
 $\mathbf{r}_3 = (40 - 5, 30, 0 - 5) = (35, 30, -5) = (5, 6, -1)$

• BY INSPECTION  $\lambda = 2$ :  
 $(40 - 5, 30, 0 - 5) + (3, 1, 4) = (35, 31, -1) = (5, 6, -1)$

∴  $B(5, 6, -1)$  lies on both  $l_2$  &  $l_3$

(c)  $\mathbf{r}_1 = (0, 6, 9) + \lambda(3, 1, 4)$   
 $\mathbf{r}_3 = (40 - 5, 30, 0 - 5) = (35, 30, -5) = (5, 6, -1)$

• Equate  $\mathbf{r}_1$  &  $\mathbf{r}_3$   
 $\begin{cases} 0 = 3\lambda + 0 \\ 6 + \lambda = 5 \\ 9 + 4\lambda = -5 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ \lambda = 1 \\ \lambda = -3 \end{cases}$

Hence  $3\lambda + 0 = 3(1) + 0 = 3$   
 $6 + \lambda = 6 + 1 = 7$   
 $9 + 4\lambda = 9 + 4(-3) = -33 = 10$

∴  $C(1, 3, -3)$  (NO NEED TO FIND  $\lambda$ )

(d)  $|\overrightarrow{CA}| = |\mathbf{a} - \mathbf{c}| = |(4, 4, 1) - (1, 3, -3)| = |(3, 1, 4)| = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}$   
 $|\overrightarrow{CB}| = |\mathbf{b} - \mathbf{c}| = |(5, 6, -1) - (1, 3, -3)| = |(4, 3, 2)| = \sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}$   
 $\therefore |\overrightarrow{CA}| = |\overrightarrow{CB}|$

By Pythagoras  
 $d = \sqrt{(CM)^2 + (MA)^2} = \sqrt{12.5 + 12.5} = \sqrt{25} = 5$   
 $d = \sqrt{\frac{25}{2}} = \frac{5}{2}\sqrt{10}$

**Question 92 (\*\*\*\*)**

The position vectors and coordinates in this question are relative to a fixed origin  $O$ .

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$$

$$\mathbf{r}_2 = 9\mathbf{i} + \mu(\mathbf{i} - 3\mathbf{j} + a\mathbf{k})$$

where  $\lambda$  and  $\mu$  are scalar parameters, and  $a$  is a scalar constant.

The point  $A$  is the intersection between  $l_1$  and  $l_2$ , and the acute angle between them is denoted by  $\theta$ .

a) Find in any order ...

- i. ... the value of  $a$ .
- ii. ... the coordinates of  $A$ .
- iii. ... the value of  $\theta$ .

The point  $B$  has coordinates  $(5, 13, 11)$ .

The point  $P$  lies on  $l_1$  so that the angle  $APB$  is  $90^\circ$ .

b) Calculate the distance  $BP$ .

$$a = -2, A(7, 6, 4), \theta = 54.2^\circ, |BP| = \sqrt{61}$$

**Part (a) Working:**

Q)  $\mathbf{r}_1 = (3, 2, 1) + \lambda(4, 4, 3) = (4\lambda+3, 4\lambda+2, 3\lambda+1)$   
 $\mathbf{r}_2 = (9, 0, 0) + \mu(1, -3, a) = (\mu+9, -3\mu, a\mu)$

\* Equate  $\mathbf{r}_1$  &  $\mathbf{r}_2$   
(1):  $4\lambda+3 = \mu+9 \Rightarrow \mu = 4\lambda+6$   
(2):  $4\lambda+2 = -3\mu \Rightarrow -3\mu = 4\lambda+2 \Rightarrow \mu = -\frac{4\lambda+2}{3}$   
From (1) & (2):  $4\lambda+6 = -\frac{4\lambda+2}{3} \Rightarrow 12\lambda+18 = -4\lambda-2 \Rightarrow 16\lambda = -20 \Rightarrow \lambda = -\frac{5}{4}$

\* Calculate  $a$ :  
 $3\lambda+1 = a\mu \Rightarrow 3(-\frac{5}{4})+1 = a(-\frac{4(-\frac{5}{4})+2}{3}) \Rightarrow a = -2$

(i) Using  $\lambda = 1$  into  $(4\lambda+3, 4\lambda+2, 3\lambda+1)$  gives  $A(7, 6, 4)$

(ii) Using  $\lambda = -\frac{5}{4}$  into  $(4\lambda+3, 4\lambda+2, 3\lambda+1)$  gives  $P(1, 10, 7)$

(iii) Using  $\lambda = -\frac{5}{4}$  into  $(4, 4, 3)$  gives  $\cos \theta$   
 $\cos \theta = \frac{-4 - 1 - 6}{\sqrt{41} \sqrt{17}} = \frac{-11}{\sqrt{41} \sqrt{17}} \approx -0.55 \Rightarrow \theta = 125.75^\circ \Rightarrow 90^\circ - \theta = 54.25^\circ$

**Part (b) Working:**

(b)  $\angle APB = 90^\circ$   
 $(7-5, 6-13, 4-11) \cdot (1, 4, 3) = 0$   
 $4-10 + 24 - 52 + 12 - 33 = 0$   
 $24 - 44 = 0$

Point  $P$  lies on  $l_1$ :  
 $x = 4\lambda+3$   
 $y = 4\lambda+2$   
 $z = 3\lambda+1$

Scalene triangle:  
 $|PA|^2 + |PB|^2 = |AB|^2$   
 $(4^2 + 1^2 + 1^2) + (1^2 + 4^2 + 3^2) = 10^2$   
 $16 + 1 + 1 + 1 + 16 + 9 = 100$   
 $41 = 100$

$|AB| = \sqrt{100} = 10$

So, distance  $|PO| = \sqrt{6^2 + 3^2 + 4^2} = \sqrt{36 + 9 + 16} = \sqrt{61}$

**Question 93 (\*\*\*\*)**

The position vectors and coordinates in this question are relative to a fixed origin  $O$ .

The points  $A$  and  $B$  have respective position vectors

$$2\mathbf{i} + 3\mathbf{j} \quad \text{and} \quad 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}.$$

- a) Find a vector equation of the straight line  $l_1$  that passes through  $A$  and  $B$ .

The straight line  $l_2$  has vector equation

$$\mathbf{r} = 5\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}),$$

where  $\mu$  is a scalar parameter.

- b) Show that  $l_1$  and  $l_2$  intersect at the point  $A$ .  
 c) Find the exact value of  $\cos \theta$ , where  $\theta$  is the acute angle between  $l_1$  and  $l_2$ .

The point  $C$  with position vector  $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$  lies on  $l_2$ .

- d) Show that the shortest distance from  $C$  to  $l_1$  is exactly one unit.

$$\boxed{\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k})}, \quad \boxed{\cos \theta = \frac{2}{3}\sqrt{2}}$$

(a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (5, -2, 3) - (2, 3, 0) = (3, -5, 3)$   
 $\Gamma_1 = (2, 3, 0) + t(4, -5, 3)$   
 $(4t+2, 3-5t, 3t)$

$\Gamma_2 = (5, -2, 3) + \mu(1, -2, 2)$   
 $(5+\mu, -2-2\mu, 3\mu+6)$

By inspection if  $t = -3$  and  $\mu = 0$   
 $(4(-3)+2, 3-5(-3), 3(-3)) = (2, 3, 0)$   
 $\therefore l_1 \text{ and } l_2 \text{ intersect at } A(2, 3, 0)$

(b) Drawing direction vectors of the two lines  
 $\Rightarrow (4, -5, 3), (1, -2, 2) = (4, -5, 3) \parallel (1, -2, 2) \text{ since}$   
 $\Rightarrow 4 \cdot 1 + 5 \cdot (-2) = \sqrt{1^2 + 2^2} \sqrt{4^2 + 5^2} \cos \theta$   
 $\Rightarrow 20 = \sqrt{5} \sqrt{41} \cos \theta$   
 $\Rightarrow 20 = 15\sqrt{2} \cos \theta$   
 $\Rightarrow \cos \theta = \frac{2}{3}\sqrt{2}$

(c)  $|\vec{AC}| = \sqrt{(5-2)^2 + (3-1)^2 + (0-2)^2} = \sqrt{1+4+4} = 3$   
 $\text{If } \cos \theta = \frac{2}{3}\sqrt{2} \quad \sin \theta = \sqrt{1 - (\frac{2}{3}\sqrt{2})^2} = \sqrt{1 - \frac{8}{9}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$   
 $\therefore \sin \theta = \frac{1}{3}$   
 $\text{Hence } d = 3 \sin \theta$   
 $d = 3 \cdot \frac{1}{3} = 1 \text{ unit}$

**Question 94 (\*\*\*\*)**

The position vectors and coordinates in this question are relative to a fixed origin  $O$ .

The straight line  $l$  has vector equation

$$\mathbf{r} = \begin{pmatrix} 10 \\ 7 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},$$

where  $\lambda$  is a scalar parameter.

The point  $P(14,15,15)$  lies on  $l$  and the point  $A$  has coordinates  $(5,1,2)$ .

- a) Calculate the size of the acute angle that  $AP$  makes with  $l$ .

The point  $B$  lies on  $l$  so that  $ABP = 90^\circ$ .

- b) Determine the coordinates of  $B$ .

The point  $C$  is such so that  $l$  is the angle bisector of  $APC$ .

- c) Find a set of the possible coordinates of  $C$ .

$$\theta \approx 6.1^\circ, [B(7,1,1)], [C(9,1,0)]$$

(a)  $\Sigma = (0,7,7) + \lambda(1,2,2) = (\lambda+7, 2\lambda+7, 2\lambda+7)$   $\vec{AP} = (5,1,2)$   
 $\vec{AB} = \vec{B} - \vec{A} = (0,15,15) - (5,1,2) = (5,14,13)$   $\vec{P}(14,15,15) \leftarrow \text{not used}$

$\vec{AP} \cdot \vec{AB} = 0 + 14 + 14 = 28$   $\vec{AP} \cdot \vec{AB} = (1,2,2) \cdot (5,14,13) \cos \theta$   
 $28 = 28 \cos \theta = \sqrt{1+4+4} \sqrt{25+196+169} \cos \theta$   
 $\cos \theta = \frac{28}{\sqrt{288}} \cos \theta$   
 $\theta = 6.1^\circ$

(b)  $\vec{AB} = (0,14,13)$   $\vec{AB} = b = (0,14,13)$   
 $\vec{AB} = b = (0,14,13) - (5,1,2) = (-5,13,11)$

$\vec{AC} = (x-5, y-1, z-2)$   $\vec{AC} \cdot \vec{AB} = 0$   
 $(x-5)(-5) + (y-1)(13) + (z-2)(11) = 0$   
 $-5x + 25 + 13y - 13 + 11z - 22 = 0$   
 $-5x + 13y + 11z - 2 = 0$

But  $B$  lies on  $l \Rightarrow$   $\begin{cases} 3x + 2y + 2z = 20 \\ 2x + 4y + 2z = 24 \\ x + 2y + 2z = 11 \end{cases}$   $\Rightarrow 3x + 10 + 2(2x+7) + 2(2x+7) = 11$   
 $7x + 27 = 11 \Rightarrow x = -2$   
 $y = -3 \Rightarrow B(-2, 11, 1)$

(c)  $\vec{AC} = (x-5, y-1, z-2)$   $\vec{AC} = (x-5, y-1, z-2)$   
 $\vec{AP} = (5,1,2)$   $\vec{AP} = (5,1,2)$   
 $B$  IS THE MIDPOINT OF  $AC$   $\frac{x+5}{2} = 7 \Rightarrow x = 9$   
 $\frac{y+1}{2} = 1 \Rightarrow y = 1$   $\therefore C(9,1,0)$   
 $\frac{z+2}{2} = 1 \Rightarrow z = 0$

**Question 95 (\*\*\*\*)**

The position vectors and coordinates in this question are relative to a fixed origin  $O$ .

The points  $A$ ,  $B$  and  $C$  have coordinates  $(0,0,8)$ ,  $(2,6,4)$  and  $(8,8,0)$ , respectively.

The point  $D$  is such so that  $ABCD$  is a parallelogram.

The angle  $ABC$  is  $\theta$ .

- Determine the coordinates of  $D$ .
- Use the scalar product to find an exact value for  $\cos \theta$  and hence show

$$\sin \theta = \frac{2}{7}\sqrt{6}.$$

- Explain, with reference to the calculations of part (b), why  $AC$  must be perpendicular to  $BD$ .
- Show that the area of the parallelogram is  $16\sqrt{6}$ .

|  |   |            |   |                              |
|--|---|------------|---|------------------------------|
|  | , | $D(6,2,4)$ | , | $\cos \theta = -\frac{5}{7}$ |
|--|---|------------|---|------------------------------|

a) LOOK AT THE DIAGRAM

WORK BY INSPECTION

$$\begin{aligned} \overrightarrow{BD} &= \overrightarrow{CD} + \overrightarrow{CB} \\ \overrightarrow{BD} &= \overrightarrow{CA} + \overrightarrow{CB} \\ \overrightarrow{BD} &= \overrightarrow{A} + \overrightarrow{C} - \overrightarrow{B} \\ \overrightarrow{BD} &= (0,0,8) + (8,8,0) - (2,6,4) \\ \overrightarrow{d} &= (6,2,4) \end{aligned}$$

AS  $ABCD$

b) LOOK AT THE DIAGRAM

$\bullet \overrightarrow{BC} = \overrightarrow{a} - \overrightarrow{b}$

$$\begin{aligned} \bullet \overrightarrow{BC} &= ((8,8,0) - (2,6,4)) \\ &= (6,2,4) \end{aligned}$$

$\bullet \overrightarrow{AB} = \overrightarrow{c} - \overrightarrow{b}$

$$\begin{aligned} \bullet \overrightarrow{AB} &= ((0,0,8) - (2,6,4)) \\ &= (-2,-6,4) \end{aligned}$$

$\bullet \cos \theta = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}| |\overrightarrow{b}|}$

$$\begin{aligned} \bullet \cos \theta &= \frac{(6,2,4) \cdot (-2,-6,4)}{\sqrt{6^2+2^2+4^2} \sqrt{(-2)^2+(-6)^2+4^2}} \\ &= \frac{-12-24}{\sqrt{60} \sqrt{60}} = \frac{-36}{\sqrt{3600}} = \frac{-36}{60} = -\frac{3}{5} \end{aligned}$$

$\bullet \sin \theta = \sqrt{1 - \cos^2 \theta}$

$$\begin{aligned} \sin \theta &= \sqrt{1 - \left(\frac{3}{5}\right)^2} \\ \sin \theta &= \sqrt{\frac{16}{25}} \\ \sin \theta &= \frac{4}{5} \end{aligned}$$

AS  $\sin \theta = \frac{2}{7}\sqrt{6}$

c) BOTH  $|\overrightarrow{a}|$  &  $|\overrightarrow{b}|$  ARE  $\sqrt{60}$  SO THE PARALLELOGRAM IS IN FACT A RHOMBUS, SO ITS DIAGONALS MUST BE PERPENDICULAR

CONSIDERING THE "RHOMBUS" AS TWO TRIANGLES

$\text{Area} = \frac{1}{2} \times |\overrightarrow{a}| |\overrightarrow{b}| \sin \theta$

$$\begin{aligned} \text{Area} &= \frac{1}{2} \times \sqrt{60} \times \sqrt{60} \times \frac{2}{7}\sqrt{6} \\ \text{Area} &= \sqrt{60} \times \sqrt{60} \times \frac{2}{7}\sqrt{6} \\ \text{Area} &= 36\sqrt{6} \\ \text{Area} &= 16\sqrt{6} \end{aligned}$$

**Question 96 (\*\*\*\*)**

The position vectors and coordinates in this question are relative to a fixed origin  $O$ .

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 5\mathbf{i} + 3\mathbf{j} + 6\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

$$\mathbf{r}_2 = -\mathbf{i} + 5\mathbf{j} + a\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}),$$

where  $\lambda$  and  $\mu$  are scalar parameters, and  $a$  is a scalar constant.

The point  $A$  lies on both  $l_1$  and  $l_2$ .

- a) Find the value of  $a$  and the coordinates of  $A$ .

The point  $P(11, p, 12)$ , where  $p$  is a scalar constant, lies on  $l_1$ .

The point  $Q(q, -9, -8)$ , where  $q$  is a scalar constant, lies on  $l_2$ .

- b) Find the value of  $p$  and the value of  $q$ .
- c) Determine the coordinates of the midpoint of  $PQ$ .
- d) Show that  $|AP| = |AQ|$ .
- e) Hence, or otherwise, find a vector equation of the angle bisector of  $\angle PAQ$ .

,  $[a=6]$ ,  $[A(1,1,2)]$ ,  $[p=6, q=6]$ ,  $[M\left(\frac{17}{2}, -\frac{3}{2}, 2\right)]$ ,  $[\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j})]$

(a)

$$\begin{aligned} l_1: & 5\mathbf{i} + 3\mathbf{j} + 6\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = (2\lambda+5)\mathbf{i} + (\lambda+3)\mathbf{j} + (2\lambda+6)\mathbf{k} \\ l_2: & (-1)\mathbf{i} + 5\mathbf{j} + a\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = (\mu-1)\mathbf{i} + (5-2\mu)\mathbf{j} + (a-2\mu)\mathbf{k} \end{aligned}$$

From  $\mathbf{i}$  coefficients:

$$2\lambda+5 = \mu-1 \quad \Rightarrow \quad \mu = 2\lambda+6$$

From  $\mathbf{j}$  coefficients:

$$\lambda+3 = 5-2\mu \quad \Rightarrow \quad \lambda+3 = 5-2(2\lambda+6) \quad \Rightarrow \quad 3\lambda+7 = 0 \quad \Rightarrow \quad \lambda = -\frac{7}{3}$$

From  $\mathbf{k}$  coefficients:

$$2\lambda+6 = a-2\mu \quad \Rightarrow \quad a = 2\lambda+6+2\mu = 2\lambda+6+2(2\lambda+6) = 6\lambda+18 = 6\left(-\frac{7}{3}\right)+18 = 6$$

∴  $A(1, 1, 2)$

Using  $\mu = 2$  into  $(\mathbf{i}-\mathbf{j}-2\mathbf{k})$  we get  $Q(1, -9, -8)$

By inspection  $(11, p, 12) = (2\lambda+5, 2\lambda+3, 2\lambda+6) \Rightarrow 2\lambda+5 = 11 \Rightarrow \lambda = 3$

By inspection  $(q, -9, -8) = (\mu-1, 5-2\mu, a-2\mu) \Rightarrow \mu-1 = q \Rightarrow \mu = q+1$

MIDPOINT OF  $(1, 1, 2)$  &  $(q, -9, -8)$  is  $\left(\frac{q+1}{2}, \frac{-8-1}{2}, \frac{2-8}{2}\right) \Rightarrow M\left(\frac{q+1}{2}, -\frac{9}{2}, -3\right)$

(d)

$$\begin{aligned} |\overrightarrow{AP}|^2 &= [p-9]^2 = [(11, p, 12) - (1, 1, 2)]^2 = |10\mathbf{j}|^2 = \sqrt{100+25+100} = 15 \\ |\overrightarrow{AQ}|^2 &= [q-9]^2 = [(q, -9, -8) - (1, 1, 2)]^2 = |s_1\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k}|^2 = \sqrt{25+81+100} = 17 \end{aligned}$$

∴  $|\overrightarrow{AP}| = |\overrightarrow{AQ}|$

(e)

Angle bisector of  $\angle PAQ$

$\overrightarrow{AM} = \overrightarrow{AP} - \overrightarrow{AQ} = \left(\frac{11}{2}, \frac{p-9}{2}, \frac{10}{2}\right) - \left(\frac{q+1}{2}, \frac{-9}{2}, -3\right)$

CALC. AS DIFFERENCE  $(3, \frac{p-9}{2}, 13)$

$\therefore \overrightarrow{M} = \left(\frac{11}{2}, \frac{p-9}{2}, 13\right)$

**Question 97 (\*\*\*\*)**

The points with coordinates  $A(4, 0, -4)$  and  $B(5, -1, -6)$  lie on the line  $L$ , where the point  $O$  is a fixed origin.

- Find a vector equation of the line  $L$ .
- Find the distance between the points  $A$  and  $B$ .

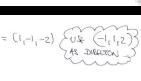
The point  $D$  lies on the line  $L$ , so that  $OD$  is perpendicular to  $L$ .

- Find the coordinates of the point  $D$ , and hence show that  $|OD| = \sqrt{8}$

The point  $C$  is such so that  $OABC$  is parallelogram.

- Find the exact area of the parallelogram  $OABC$ .

$$\boxed{\mathbf{r} = 4\mathbf{i} - 4\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})}, \boxed{|AB| = \sqrt{6}}, \boxed{D(2, 2, 0)}, \boxed{\text{area} = 4\sqrt{3}}$$

(a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (5, -1, -6) - (4, 0, -4) = (1, -1, -2)$  

$$\begin{aligned}\mathbf{l} &= (1, -1, -2) + \lambda(-1, 1, 2) \\ \mathbf{l} &= (4 - \lambda, \lambda, 2\lambda - 4)\end{aligned}$$

(b)  $|AB| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6}$

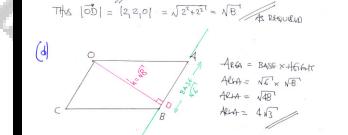
(c) Let  $\mathbf{d} = (2, 2, 0)$

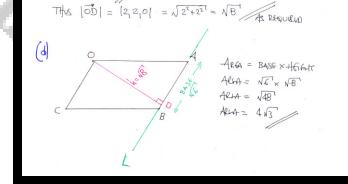
- $OD \perp L$ :  $(2, 2, 0) \times (-1, 1, 2) = 0$   
 $\rightarrow 2(-1) + 2(1) = 0$
- Point  $D$  lies on  $L$ :  
 $(2, 2, 0) = (4 - \lambda, \lambda, 2\lambda - 4)$   
 $2 = 4 - \lambda$   
 $2 = \lambda$   
 $2 = 2\lambda - 4$

Solve simultaneously:

$$\begin{aligned}-4 + \lambda + \lambda + 2\lambda - 4 &= 0 \\ -4 + 2\lambda + 2\lambda - 4 &= 0 \\ 6\lambda &= 12 \\ \lambda &= 2\end{aligned}$$

$\therefore D(2, 2, 0)$

Thus,  $|OD| = \sqrt{2^2 + 2^2} = \sqrt{8}$  

(d) 

$$\begin{aligned}A_{\text{par}} &= \text{base} \times \text{height} \\ A_{\text{par}} &= \sqrt{6} \times \sqrt{8} \\ A_{\text{par}} &= 4\sqrt{3}\end{aligned}$$

**Question 98 (\*\*\*\*)**

The lines  $l_1$  passes through the point  $A(2, 2, -2)$  and has direction vector  $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ .

- a) Find a vector equation for  $l_1$ .

The line  $l_2$  has the vector equation

$$\mathbf{r}_2 = 7\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - \mathbf{k}),$$

where  $\mu$  is a scalar parameter.

The lines intersect at the point  $B$  and the acute angle between the two lines is  $\theta$ .

- b) Find the coordinates of  $B$ .

- c) Show that  $\cos \theta = \frac{1}{6}$ .

The point  $P(15, -3, -6)$  lies on  $l_2$  and the point  $Q$  lies on  $l_1$  so that  $\angle PQB = 90^\circ$ .

- d) Find  $|\overrightarrow{BP}|$  and show that  $|\overrightarrow{BQ}| = \sqrt{6}$ .

- e) Show that  $|\overrightarrow{PQ}| = \sqrt{210}$ .

- f) Verify that the point  $Q$  is in fact the same the point as  $A$ .

|   |              |  |
|---|--------------|--|
| $\mathbf{r}_1 = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ | $B(3, 3, 0)$ | $ \overrightarrow{BP}  = \sqrt{216} = 6\sqrt{6}$ |
|---|--------------|--|

(a)  $\Sigma_1 = (2, 2, -2) + \lambda(1, 1, 2) = (2+2\lambda, 2+\lambda, -2+2\lambda)$

(b)  $\Sigma_1 = (2, 2, -2) + \mu(2, -1, -1) = (2+2\mu, 2-\mu, -2-\mu)$

\* EQUATE 1 & 2

(i):  $2+2\lambda = 2+2\mu \quad \text{subtract } 0 = 3\mu + 6$

(ii):  $2+\lambda = 2-\mu \quad \therefore \lambda = -\mu - 2$

(iii):  $2-2\lambda = 2-\mu \quad \therefore \lambda = 1-\mu$

DOTTING DIRECTION VECTORS

$(1, 1, 2) \cdot (2, -1, -1) = 1(1)(2) + (-1)(-1) = 3$

$2-2\lambda = 2-\mu \quad \therefore \sqrt{1^2 + 1^2 + 2^2} \sqrt{(-1)^2 + (-1)^2 + (-1)^2} = \sqrt{6}$

$-1 = -\mu \quad \therefore \cos \theta = \frac{3}{\sqrt{6}}$

$\cos \theta = \frac{1}{2}$

(d)  $|\overrightarrow{BP}| = \sqrt{(3-15)^2 + (3+3)^2 + (0+6)^2} = \sqrt{144 + 36 + 36} = \sqrt{216} = 6\sqrt{6}$

BY TRIGONOMETRY, LET  $d = |\overrightarrow{BQ}|$

$\frac{d}{\sqrt{6}} = \cos \theta$

$d = \sqrt{6}\cos \theta = 6\sqrt{6} \times \frac{1}{6} = \sqrt{6}$

(e) BY PYTHAGORES  $|\overrightarrow{PQ}| = \sqrt{(|\overrightarrow{PB}|^2 - d^2)} = \sqrt{216 - 6} = \sqrt{210} \neq |\overrightarrow{BQ}|$

(f)  $A(2, 2, -2)$   
 $B(3, 3, 0)$   
 $P(15, -3, -6)$

$|\overrightarrow{PA}| = \sqrt{(2-15)^2 + (2+3)^2 + (-2+6)^2} = \sqrt{144 + 25 + 16} = \sqrt{185} \neq |\overrightarrow{BQ}|$

$|\overrightarrow{PB}| = \sqrt{(15-2)^2 + (-3+2)^2 + (-6+2)^2} = \sqrt{169 + 1 + 16} = \sqrt{186} \neq |\overrightarrow{BQ}|$

$\therefore Q \text{ is in fact } A$

**Question 99 (\*\*\*\*)**

With respect to a fixed origin, the points  $A$ ,  $B$  and  $C$  have coordinates  $(-2, -4, 6)$ ,  $(-16, 1, 4)$  and  $(4, 8, -6)$ , respectively.

- Find a vector equation for the line  $L_1$ , through the points  $A$  and  $B$ .
- Find a vector equation for the line  $L_2$ , that passes through the point  $C$  and is parallel to the vector  $p\mathbf{i} + q\mathbf{j} - 4\mathbf{k}$ , where  $p$  and  $q$  are scalar constants.

The line  $L_2$  passes through the  $z$  axis, and is perpendicular to  $L_1$ .

- Find the values of  $p$  and  $q$ .
- Verify that  $L_1$  and  $L_2$  lines intersect at the point  $A$ .

$$\boxed{\mathbf{r}_1 = -2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k} + \lambda(14\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})}, \quad \boxed{\mathbf{r}_2 = 4\mathbf{i} + 8\mathbf{j} - 6\mathbf{k} + \mu(p\mathbf{i} + q\mathbf{j} - 4\mathbf{k})},$$

$$\boxed{p = 2, \quad q = 4}$$

**Q. 99** Given  $\vec{AB} = \mathbf{b} - \mathbf{a} = (-16, 1, 4) - (-2, -4, 6) = (-14, 5, -2)$ .  
 $\vec{a} = (-2, -4, 6)$ ,  $\vec{b} = (4, 8, -6)$ .  
 $\vec{L}_1: \vec{r}_1 = (-2, -4, 6) + \lambda(-14, 5, -2)$ .  
 $\vec{L}_2: \vec{r}_2 = (4, 8, -6) + \mu(p\mathbf{i} + q\mathbf{j} - 4\mathbf{k})$ .  
 $\vec{L}_2$  passes through the  $z$ -axis, so  $\vec{r}_2 = (0, 0, z)$ .  
 $\vec{r}_1 = \vec{r}_2 \Rightarrow (-2, -4, 6) + \lambda(-14, 5, -2) = (0, 0, z)$ .  
 $\begin{cases} -2 = 0 \\ -4 = 0 \\ 6 - \lambda(-14) = z \end{cases} \Rightarrow \lambda = 0, z = 6$ .  
 $\vec{r}_1 = (-2, -4, 6) + 0(-14, 5, -2) = (-2, -4, 6)$ .  
 $\vec{r}_2 = (0, 0, 6) + \mu(2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k})$ .  
 $\vec{r}_1 = \vec{r}_2 \Rightarrow (-2, -4, 6) = (0, 0, 6) + \mu(2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k})$ .  
 $\begin{cases} -2 = 0 \\ -4 = 0 \\ 6 = 6 + \mu(2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}) \end{cases} \Rightarrow \mu = -1$ .  
 $\vec{r}_2 = (0, 0, 6) + -1(2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}) = (-2, -4, 6)$ .  
 $\therefore L_1 \text{ and } L_2 \text{ intersect at } (-2, -4, 6).$

**Question 100 (\*\*\*\*)**

The straight lines  $L$  and  $M$  have the following vector equations

$$L: \mathbf{r}_1 = (3\lambda + 3)\mathbf{i} + (4 - 4\lambda)\mathbf{j} + 2\lambda\mathbf{k}$$

$$M: \mathbf{r}_2 = (3\mu + 12)\mathbf{i} + (20 - 4\mu)\mathbf{j} + (2\mu + 4)\mathbf{k},$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- a) Show that  $L$  and  $M$  are parallel.
- b) Show further that  $A(6, 0, 2)$  lies on  $L$ .

The point  $B$  lies on  $M$  so that  $AB$  is perpendicular to  $M$ .

- c) Find the coordinates of  $B$ .
- d) Hence determine the shortest distance between  $L$  and  $M$ .

$B(18, 12, 8)$ , distance = 18 units

a)  $\mathbf{r}_1 = (3\lambda + 3, 4 - 4\lambda, 2\lambda) = (3, 4, 0) + \lambda(3, -4, 2)$   
 $\mathbf{r}_2 = (3\mu + 12, 20 - 4\mu, 2\mu + 4) = (12, 20, 4) + \mu(3, -4, 2)$

SAME DIRECTION VECTOR, SO PARALLEL

(b) BY INSPECTION IF  $\mathbf{r}_1 = (3\lambda + 3, 4 - 4\lambda, 2\lambda)$  GIVES  $(6, 0, 2)$   
 $\therefore A(6, 0, 2)$  IS ON  $L$

(c)



Let  $\mathbf{d} = (3, 0, 2)$   
 $\mathbf{AB} = (3, 12, 8) - (6, 0, 2) = (3, 12, 6) = (3, 4, 2)$

$\mathbf{AB} \perp M$   
 $(3, 4, 2) \cdot (3, -4, 2) = 0$   
 $3x - 12y + 2z = 4 = 0$   
 $3x - 4y + 2z = 22$

ALSO  $B$  LIES ON LINE  $M$ :  
 $(3\mu + 12, 20 - 4\mu, 2\mu + 4) = (3, 12, 8)$

SOLVING SIMULTANEOUSLY:  
 $3(3\mu + 12) - 4(20 - 4\mu) = 22$   
 $9\mu + 36 - 80 + 16\mu + 8 = 22$   
 $25\mu = 56$   
 $\mu = 2$

$\therefore B(18, 12, 8)$

(d)  $|AB| = |(3, 4, 2)| = \sqrt{(3^2 + 4^2 + 2^2)} = \sqrt{34 + 16 + 4} = \sqrt{54} = 3\sqrt{6}$

**Question 101 (\*\*\*\*)**

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = -2\mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$$

$$\mathbf{r}_2 = 8\mathbf{i} - 5\mathbf{j} + 26\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- Given  $l_1$  and  $l_2$  intersect at the point  $P$ , find the coordinates of  $P$ .
- Show  $l_1$  and  $l_2$  are perpendicular.

The points  $A(-8, a, -2)$  and  $C(c, 11, -2)$  lie on  $l_1$ .

- Find the value of each of the constants  $a$  and  $c$ , and show further that  $P$  is the midpoint of  $AC$ .

The quadrilateral  $ABCD$  is a square.

- Determine the coordinates of the points  $B$  and  $D$ .

$P(1, 2, -2)$ ,  $a = -7$ ,  $c = 10$ ,  $B(4, -1, 10)$  &  $D(-2, 5, -14)$  in any order

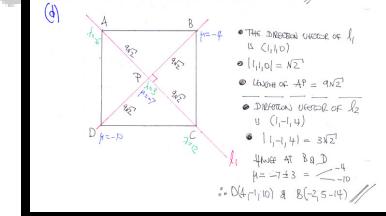
(a)  $\begin{aligned}\mathbf{r}_1 &= (-2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) + \lambda(\mathbf{i} + \mathbf{j}) = (2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ \mathbf{r}_2 &= (8\mathbf{i} - 5\mathbf{j} + 26\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = (9\mathbf{i} - 6\mathbf{j} + 26\mathbf{k})\end{aligned}$

Given  $\mathbf{r}_2 - \mathbf{r}_1 = 2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k} \perp \mathbf{r}_1$  (i.e.  $\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$ )  
 $\therefore \text{using } \mathbf{r}_1 \cdot \mathbf{r}_2 = 0 \Rightarrow 2(-2) + 4(-1) + 2(-2) = 0 \Rightarrow \lambda = -7$

(b) DOTTING (DIRECTION) VECTORS  
 $\langle 1, 1, 0 \rangle \cdot \langle 1, -1, 4 \rangle = |1| + |4| + 0 = 1 - 1 = 0 \Rightarrow \text{LINES } l_1 \text{ & } l_2 \text{ ARE PERPENDICULAR}$

(c)  $\begin{aligned}A(-8, a, -2) &= (2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \Rightarrow a = -6 \\ C(c, 11, -2) &= (2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \Rightarrow c = 10\end{aligned}$

MIDPOINT OF  $AC$ :  $\frac{A(-8, -7, -2) + C(10, 11, -2)}{2} = M\left(\frac{-8+10}{2}, \frac{-7+11}{2}, \frac{-2-2}{2}\right) = M(1, 2, -2)$   
 $\therefore \text{MIDPOINT IS } P$

(d) 

- THE DIRECTION VECTOR OF  $l_1$  IS  $\langle 1, 1, 0 \rangle$
- $|1, 1, 0| = \sqrt{3}$
- LENGTH OF  $AP = \sqrt{3^2}$
- DIRECTION VECTOR OF  $l_2$  IS  $\langle 1, -1, 4 \rangle$
- $|1, -1, 4| = \sqrt{18}$
- Length of  $BD = \sqrt{18}$
- $\therefore D(-2, 5, -14) \& B(4, -1, 10)$

**Question 102 (\*\*\*\*)**

Relative to a fixed origin, the points  $P$  and  $Q$  have position vectors  $9\mathbf{j} - 2\mathbf{k}$  and  $7\mathbf{i} - 8\mathbf{j} + 1\mathbf{l}\mathbf{k}$ , respectively.

- Determine the distance between the points  $P$  and  $Q$ .
- Find the position vector of the point  $M$ , where  $M$  is the midpoint of  $PQ$ .
- Show that the length of one of the sides of the cube is 13 units.
- Show that the origin  $O$  lies inside the cube.

$$\boxed{\quad}, \quad \boxed{PQ = \sqrt{507}}, \quad \boxed{OM = \frac{7}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{9}{2}\mathbf{k}}$$

(a)  $|PQ| = |\mathbf{q} - \mathbf{p}| = |(7, -8, 1) - (0, 9, -2)| = |7, -17, 13| = \sqrt{49 + 289 + 169} = \sqrt{507}$

(b)  $M\left(\frac{7+0}{2}, \frac{-8+9}{2}, \frac{1+(-2)}{2}\right) = M\left(\frac{7}{2}, \frac{1}{2}, \frac{-1}{2}\right) \Leftrightarrow \mathbf{m} = \left(\frac{7}{2}, \frac{1}{2}, \frac{-1}{2}\right)$

(c) 

- $|PR| = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$  by Pythagoras on  $\triangle PSR$
- $|PS| = \sqrt{(3\sqrt{2})^2 + 3^2} = \sqrt{33}$  by Pythagoras on  $\triangle PQR$

therefore  $\sqrt{33^2} = \sqrt{507}$   
 $33^2 = 507$   
 $a^2 = 169$   
 $a = 13$  After rationalising

(d)  $|OM| = |y| = |7, \frac{1}{2}, \frac{9}{2}| = \sqrt{49 + \frac{1}{4} + \frac{81}{4}} = \sqrt{\frac{251}{4}} = \frac{\sqrt{251}}{2}$

(e) 

LOOKING AT FIGURE OF A SQUARE INSIDE A CIRCLE OF SIDE LENGTH 13  
If  $|OM| < \frac{13}{2}$ , THEN THE CIRCLE MUST BE INSIDE THE SQUARE AND HENCE INSIDE THE CUBE.  
 $\frac{1}{2}|OM|^2 < 507 \Rightarrow < \frac{13}{2}$   
 $\therefore$  CIRCLE IS INSIDE THE CUBE

**Question 103 (\*\*\*\*)**

The points  $A(7, a, 5)$  and  $B(b, 1, 12)$  lie on the straight line  $L$ , with vector equation

$$\mathbf{r} = 19\mathbf{i} - 2\mathbf{j} - 9\mathbf{k} + \lambda(6\mathbf{i} - \mathbf{j} - 7\mathbf{k}),$$

where  $\lambda$  is a scalar parameter.

- a) Find the value of  $a$  and the value of  $b$ .

The point  $C$  has coordinates  $(-3, 19, 10)$  and  $M$  is the midpoint of  $BC$ .

- b) Determine the coordinates of  $M$ .

The point  $D$  is such so that  $ABMD$  is a parallelogram.

- c) Find the coordinates of  $D$ .

- d) Show that  $|AB| = |BM|$ .

- e) Find the exact area of the parallelogram  $ABMD$ .

$$a = 0, b = 1, A(-1, 10, 11), D(5, 9, 4), \text{area} = 60\sqrt{2}$$

**④**  $\vec{s} = (a, -2, -9) + 2(6, -1, -7) = (6a+19, -1-2, -7a-9)$   
 BY INSPECTION OR  $\vec{s}$ :  $6a+19=7 \Rightarrow a=-2$   
 $6a+12 \Rightarrow a=-2$   
 $-2=-2$   
 BY INSPECTION OR  $\vec{s}$ :  $2=-3 \Rightarrow b=6a+19$   
 $b=1$

**⑤**  $B(1, 1, 12)$   
 $C(-3, 1, 10)$   $\therefore$  MIDPOINT M  $\left(\frac{-3+1}{2}, \frac{1+1}{2}, \frac{12+10}{2}\right)$   
 $M(-1, 1, 11)$

**⑥**  $\vec{AB} = \vec{b} - \vec{a} = (1, 1, 12) - (-3, 1, 10) = (4, 0, 2) = \sqrt{4^2+0^2+2^2} = \sqrt{20}$   
 $\vec{BD} = \vec{d} - \vec{b} = (5, 9, 4) - (1, 1, 12) = (4, 8, -8) = \sqrt{4^2+8^2+(-8)^2} = \sqrt{144} = 12$   
 $\vec{AD} = \vec{d} - \vec{a} = (5, 9, 4) - (-3, 1, 10) = (8, 8, -6) = \sqrt{8^2+8^2+(-6)^2} = \sqrt{224}$   
 $\therefore |AB| = |BD|$

**⑦**  $ABMD$  IS A PARALLELOGRAM  
 $|AB| = |BD| = \sqrt{20} = \sqrt{4^2+0^2+2^2} = \sqrt{16+0+4} = \sqrt{20} = 2\sqrt{5}$   
 $* |BD| = |\vec{d} - \vec{b}| = |(5, 9, 4) - (1, 1, 12)| = |4, 8, -8| = \sqrt{4^2+8^2+(-8)^2} = 12$   
 $\therefore \text{Area} = 4 \times \left(\frac{1}{2} \times 6 \times 5\sqrt{2}\right) = 60\sqrt{2}$

**Question 104 (\*\*\*\*)**

The straight line  $L_1$  passes through the points  $A(1, -2, 5)$  and  $B(4, -3, 3)$ .

- a) Find a vector equation for  $L_1$ .

The straight line  $L_2$  has a vector equation

$$\mathbf{r} = \begin{pmatrix} 8 \\ p \\ q \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix},$$

where  $\mu$  is a scalar parameter, and  $p$  and  $q$  are scalar constants.

- b) Given that  $L_1$  and  $L_2$  intersect at  $B$ , find the value of  $p$  and the value of  $q$ .
- c) Find the cosine of the acute angle  $\theta$ , between  $L_1$  and  $L_2$ .

The point  $C$  is on  $L_2$ , so that  $AC$  is perpendicular to  $L_2$ .

- d) Show that the length of  $AC$  is  $\frac{\sqrt{502}}{6}$ .

$$\boxed{\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})}, \quad \boxed{p = 13, q = -1}, \quad \boxed{\cos \theta = \frac{1}{42}\sqrt{7}}$$

(a)  $\vec{AB} = b - a = (4-1, -3-(-2), 3-5) = (3, -1, -2)$   
 $\therefore L_1 = (1, -2, 5) + \lambda(3, -1, -2) = (3\lambda+1, -1-\lambda, 5-2\lambda)$

(b)  $L_2 = (8, p, q) + \mu(1, 4, -1) = (8+\mu, 4\mu+p, q-\mu)$   
Given they intersect at  $(4, -3, 3)$   
 $8+\mu=4 \Rightarrow \mu=-4$   
 $4\mu+p=-3 \Rightarrow -16+p=-3 \Rightarrow p=13$   
 $q-\mu=3 \Rightarrow q+4=3 \Rightarrow q=-1$

(c) Dotting direction vectors  
 $(3, -1, -2) \cdot (1, 4, -1) = [3, -1, -2] \cdot [1, 4, -1] \text{ dot product}$   
 $3+4+2 = \sqrt{9+1+4} \sqrt{1+16+1} \text{ lengths}$   
 $1 = \sqrt{26} \sqrt{18} \text{ lengths}$   
 $\cos \theta = \frac{1}{\sqrt{26}\sqrt{18}} = \frac{1}{42}$

(d)

- $\|\vec{AB}\| = \sqrt{(3-1)^2 + (-1-(-2))^2 + (5-3)^2} = \sqrt{9+1+4} = \sqrt{14}$
- If  $\cos \theta = \frac{1}{42} \rightarrow \sqrt{\frac{1}{14}} \frac{42}{\sqrt{14}} = \frac{42}{\sqrt{14}}$  (By Pythagoras)

Since  $d = |\vec{AB}| \sin \theta$   
 $d = \sqrt{\frac{1}{14}} \times \frac{\sqrt{14}}{42}$   
 $d = \frac{\sqrt{14}}{42} \sqrt{281}$   
 $d = \frac{\sqrt{394}}{42} \times \sqrt{281}$   
 $d = \frac{1}{6}\sqrt{394}$  AS REQUIRED

**Question 105 (\*\*\*\*)**

With respect to a fixed origin  $O$ , the variable points  $A$  and  $B$  have the following position vectors

$$\overrightarrow{OA} = \begin{pmatrix} t-1 \\ t^2-6t+14 \\ 28-27t+9t^2-t^3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 2t^2-12t+18 \\ 3-t \\ 1 \end{pmatrix},$$

where  $t$  is a scalar parameter.

- Calculate the angle  $AOB$  when  $t = 5$
- Find the values of  $t$  for which the angle  $AOB$  is a right angle.

$$\theta \approx 86.0^\circ, \quad t = 4 \text{ or } t = \frac{13}{4}$$

(a) When  $t=5$

$$\overrightarrow{OA} = (4, 9, 7)$$

$$\overrightarrow{OB} = (8, -2, 1)$$

BY THE DOT PRODUCT

$$(4, 9, 7) \cdot (8, -2, 1) = |(4, 9, 7)| |(8, -2, 1)| \cos \theta$$

$$32 - 18 - 7 = \sqrt{16 + 81 + 49} \sqrt{64 + 4 + 1} \cos \theta$$

$$7 = \sqrt{144} \sqrt{66} \cos \theta$$

$$\cos \theta = \frac{7}{\sqrt{144} \sqrt{66}}$$

$$\theta \approx 86.0^\circ$$
  

(b)  $\overrightarrow{OA} \cdot \overrightarrow{OB} = 0$

$$\Rightarrow (-1)(2t^2-12t+18) + (3-t)(t^2-6t+14) + (28-27t+9t^2-t^3) = 0$$

$$\Rightarrow 2t^2-12t+18-3t^2+18t-54-3t+t^2+42 = 0$$

$$-t^2+9t-27+28 = 0$$

$$4t^2-30t+52 = 0$$

By quadratic formula or  $(t-4)(4t-13)=0$

$$t = 4 \quad \text{or} \quad t = \frac{13}{4}$$

**Question 106 (\*\*\*\*)**

The points  $A$  and  $B$  have coordinates  $(7, 13, 14)$  and  $(15, 19, 15)$ , respectively.

- a) Find a vector equation of the straight line  $l_1$  which passes through  $A$  and  $B$ .

The line  $l_2$  has vector equation

$$\mathbf{r} = 5\mathbf{i} - 8\mathbf{j} - 9\mathbf{k} + \mu(-2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}),$$

where  $\mu$  is a scalar parameter.

- b) Show that  $l_1$  and  $l_2$  do not intersect.  
 c) Find a vector with integer components in their simplest proportions, which is perpendicular to both lines.

[you may not use the cross product for this part]

$$\boxed{\mathbf{r} = 7\mathbf{i} + 13\mathbf{j} + 14\mathbf{k} + \lambda(8\mathbf{i} + 6\mathbf{j} + \mathbf{k})}, \boxed{\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}}$$

**(a)**  $\vec{AB} = b - a = (15, 19, 15) - (7, 13, 14) = (8, 6, 1)$   
 $\vec{a}_1 = (7, 13, 14) + t(8, 6, 1)$   
 $\vec{L}_1 \sim (8t+7, 13t+13, 14+t)$

**(b)**  $\vec{L}_2 = (5, 12, 9) + p(-2, 5, 3) = (5-2p, 12+5p, 9+3p)$

(1):  $8t+7 = 5-2p \Rightarrow 2t = 3p-12$       (4)  $8(3p-2) = 7 = 5-2p$   
 (2):  $13t+13 = 3p-9 \Rightarrow 13t = 3p-28$        $24p = 16t+17 = 5-2p$   
 $\frac{13}{2}p = 18t$   
 $\frac{13}{2}p = 18$   
 $p = 2\left(\frac{13}{2}\right) - 23$   
 $p = -2$

CHECK 1:  
 $5-15 = 6(-2)+13 = 1$   
 $5-9 = 5(-2)-9 = 27$  ✓ so ends don't agree, so lines don't intersect

**(c)** DOTTING DIRECTION VECTORS TO ZERO WITH A THIRD CROSS.  $(3, 1, 2)$   
 $(8, 6, 1) \cdot (2, 5, 3) = 0 \rightarrow 8x+30=0 \rightarrow x=-\frac{15}{4}$   
 $(5, 12, 9) \cdot (2, 5, 3) = 0 \rightarrow 10x+60=0 \rightarrow x=-6$   
 LET  $y=1$   
 $8x+6+2=0 \rightarrow \frac{8x+6+2=0}{-8x-20+20=0} \rightarrow \frac{-8x=0}{24+12=0} \rightarrow x=-3$   
 $2x+3y+3z=0 \rightarrow -6+3y+3z=0 \rightarrow 3y+3z=6 \rightarrow y+z=2$   
 IF  $x=-2$        $8x+6+2=0 \rightarrow 8(-2)+6+2=0 \rightarrow -16+6+2=0 \rightarrow -8=0$   
 $x=-2 \rightarrow y=-4 \rightarrow z=6$   
 $\therefore \left(-\frac{15}{4}, 1, -2\right)$  SORRY TO  
 $x \in \mathbb{Z}$        $(1, 1, 2)$

**Question 107 (\*\*\*\*)**

The straight lines  $l_1$  and  $l_2$  have respective vector equations

$$\mathbf{r}_1 = 5\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = -3\mathbf{i} + 4\mathbf{j} + 8\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - 3\mathbf{k}),$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- a) Show that  $l_1$  and  $l_2$  intersect at some point  $P$ , further finding its coordinates.

- b) Calculate the acute angle between  $l_1$  and  $l_2$ .

The point  $A(7,5,3)$  lies on  $l_1$  and the point  $B$  lies on  $l_2$ , such that the straight line  $AB$  is perpendicular to  $l_2$ .

- c) Determine the area of the triangle  $ABP$ .

|                        |                     |                      |  |
|------------------------|---------------------|----------------------|--|
| $\boxed{\phantom{00}}$ | $\boxed{P(3,1,-1)}$ | $\boxed{72.0^\circ}$ | $\boxed{\text{area}(ABC) = \frac{8}{7}\sqrt{38} \approx 7.05}$ |
|------------------------|---------------------|----------------------|--|

**a) WRITING THE EQUATIONS OF THE LINES IN PARAMETRIC**

$$l_1: (3, 5, 3) + \lambda(1, 1, 1) = (3+5\lambda, 5+\lambda, 3+\lambda)$$

$$l_2: (-3, 4, 8) + \mu(2, -1, -3) = (2\mu-3, 4-\mu, 8-3\mu)$$

EQUATE  $\underline{l}_1$  &  $\underline{l}_2$  COMPONENTS

$$\begin{aligned} \underline{l}_1: 2+5 &= 2\mu-3 \\ \underline{l}_2: \lambda+3 &= 4-\mu \end{aligned} \quad \text{SUBTRACTING} \Rightarrow 2 = 3\mu-7$$

$$\begin{aligned} \Rightarrow \mu &= 3\lambda+7 \\ \Rightarrow \underline{l}_2: \mu &= 3\lambda+7 \\ \Rightarrow \lambda+3 &= 4-3\lambda \\ \Rightarrow \lambda &= -2 \end{aligned}$$

CHECKING  $\lambda$

- $2+5 = -2+(-2-1)$
- $8-3\mu = 8-3(-2-1)$

AS ALL 3 COMPONENTS ARE 0 WHEN  $\lambda=-2$  &  $\mu=3$ , THE LINES INTERSECT AT SOME POINT

USING  $\lambda=-2$  OR  $\mu=3$  WE OBTAIN  $P(3, 1, -1)$

**b) DOTTING THE DIRECTION VECTORS**

$$\begin{aligned} \rightarrow (1, 1, 1) \cdot (2, -1, -3) &= (1)(2) + (1)(-1) + (1)(-3) \\ \Rightarrow 2 - 1 - 3 &= \sqrt{1+1+1} \sqrt{4+1+9} \cos\theta \\ \Rightarrow -2 &= \sqrt{3} \sqrt{14} \cos\theta \end{aligned}$$

$\Rightarrow -\frac{\sqrt{3}}{\sqrt{14}} = \cos\theta$

$$\Rightarrow \theta \approx 107.75293\dots$$

$\therefore$  ACUTE ANGLE  $\approx 72.0^\circ$

**c) START WITH A DIAGRAM**

$\bullet |\vec{AP}| = |P-A| = |(3, 1, -1) - (7, 5, 3)| = |-4, -4, -4| = \sqrt{16+16+16} = \sqrt{48} \dots$   
 $\bullet |AB| = |AP| \sin\theta$   
 $\bullet |PB| = |AP| \cos\theta$   
 $\bullet \text{Area} = \frac{1}{2} |AB||PB| = \frac{1}{2} |AB| \sin\theta \times |AP| \cos\theta$   
 $= \frac{1}{2} \cdot 4\sqrt{3} \cdot 4\sqrt{3} \sin(2\pi - \theta)$   
 $= \frac{1}{2} \cdot 48 \cdot \sin(2\pi - \theta)$   
 $= 7.0495157\dots$   
 $\approx 7.05$

**Question 108 (\*\*\*\*)**

The straight line  $L$  passes through the points  $B(1, 4, 0)$  and  $D(2, 2, 6)$ .

- a) Find a vector equation of  $L$ .

The point  $A(1, 0, p)$ , where  $p$  is a scalar constant, is such so that  $\angle BAD = 90^\circ$ .

- b) Find the possible values of  $p$ .

The rectangle  $ABCD$  has an area of  $12\sqrt{2}$  square units.

- c) Find the coordinates of  $C$ .

$$\boxed{\quad}, \boxed{\mathbf{r} = \mathbf{i} + 4\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})}, \boxed{p = 2, 4}, \boxed{C(2, 6, 2)}$$

**a) FIND A DIRECTION VECTOR FOR  $L$**

$$\vec{BD} = d - b = (2, 2, 6) - (1, 4, 0) = (1, -2, 6)$$

$$\mathbf{f} = (1, -2, 6) + \lambda(1, 2, -4)$$

$$(3, 0, 2) = (2+1, 4-2\lambda, 6\lambda) \quad \cancel{\text{This is wrong}}$$

**b) LOOKING AT THE DIAGONAL**

- $\vec{AB} = b - a = (1, 4, 0) - (1, 0, p) = (0, 4, -p)$
- $\vec{AD} = d - a = (2, 2, 6) - (1, 0, p) = (1, 2, 6-p)$

$$\Rightarrow \vec{AB} \cdot \vec{AD} = 0$$

$$\Rightarrow (0, 4, -p) \cdot (1, 2, 6-p) = 0$$

$$\Rightarrow 0 + 8 - p(6-p) = 0$$

$$\Rightarrow 8 - 6p + p^2 = 0$$

$$\Rightarrow p^2 - 6p + 8 = 0$$

$$\Rightarrow (p-4)(p-2) = 0$$

$$p = \underline{\underline{-2, 4}}$$

**c) LOOKING AT THE RECTANGLE**

- $\text{AREA } ABCD = 12\sqrt{2}$   
 $(\text{Area } \triangle ABD = 6\sqrt{2})$
- If  $p=2$ ,  $|\vec{AB}| = |\vec{a}-\vec{b}| = \sqrt{(1-1)^2 + (4-0)^2 + (0-2)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

• If  $p=4$

$$|\vec{AB}| = |\vec{a}-\vec{b}| = \sqrt{(1-1)^2 + (4-0)^2 + (0-4)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$$

$$\therefore \triangle A(1,0,4)$$

HENCE BY INSPECTION

**Question 109** (\*\*\*\*)

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 12\mathbf{i} + 7\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\mathbf{r}_2 = \mathbf{i} + 3\mathbf{j} + \mu(3\mathbf{i} - \mathbf{k}),$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- Show that  $l_1$  and  $l_2$  intersect at some point  $A$ , further finding its coordinates.
- Calculate the acute angle between  $l_1$  and  $l_2$ .
- If  $BD$  is perpendicular to  $l_2$  find the coordinates of  $D$ .
- Find the coordinates of a point  $C$  so that the triangle  $ABC$  is isosceles.

,  $A(4,3,-1)$  ,  $49.8^\circ$  ,  $D(13,3,-4)$  ,  $C(22,3,-7)$

a) WRITE THE EQUATIONS IN PARAMETRIC FORM

$$\begin{aligned}\mathbf{r}_1 &= (12, 7, 3) + \lambda(2, 1, 1) = (2\lambda+12, 1+\lambda, \lambda+3) \\ \mathbf{r}_2 &= (1, 3, 0) + \mu(3, 0, -1) = (3\mu+1, 3, -\mu)\end{aligned}$$

SPOT THE  $\perp$  &  $\parallel$

$$\begin{aligned}\mathbf{i}: 2\lambda+12 &= 3\mu+1 \\ \mathbf{j}: 1+\lambda &= 3 \\ \mathbf{k}: \lambda+3 &= -\mu\end{aligned} \Rightarrow \begin{aligned}2\lambda+12 &= 3\mu+1 \\ 2\lambda+11 &= 3\mu \\ 2\lambda-3\mu &= -11\end{aligned} \quad \begin{aligned}1+\lambda &= 3 \\ \lambda &= 2\end{aligned} \quad \begin{aligned}\lambda+3 &= -\mu \\ 2+3 &= -\mu \\ 5 &= -\mu \\ \mu &= -5\end{aligned}$$

AT ALL 3 COORDINATES AGREE WHEN  $\lambda=2$  &  $\mu=-5$  THE LINES INTERSECT AT SAME POINT

USING  $\lambda=2$  OR  $\mu=-5$  FIND  $A(4,3,-1)$

b) DOTTING THE  $\perp$  (INTERCEPT) VECTORS

$$\begin{aligned}(2, 1, 1) \cdot (3, 0, -1) &= |2, 1, 1| |3, 0, -1| \cos \theta \\ 6+0-1 &= \sqrt{4+1+1} \sqrt{9+0+1} \cos \theta \\ 5 &= \sqrt{18} \sqrt{10} \cos \theta \\ \cos \theta &= \frac{5}{\sqrt{180}} \\ \theta &\approx 49.8^\circ\end{aligned}$$

c) WORKING AT THE DIAGONAL

$$\begin{aligned}\overrightarrow{BD} &= \frac{1}{2} \mathbf{i} - \mathbf{j} = (2, 0, 2) - (16, 9, 5) \\ &= (2-16, 0-9, 2-5) \\ &= (-14, -9, -3) \\ (2-14, 0-9, 2-5) \cdot (3, 0, -1) &= 0 \\ (2-14)(3) + (-9)(0) + (2-5)(-1) &= 0 \\ -42 - 9 + 5 &= 0 \\ -46 &= 0\end{aligned}$$

OR  $(2, 0, 2) = (3\mu+1, 3, -\mu)$

$$\begin{aligned}2 = 3\mu+1 &\Rightarrow 3\mu+1 = 2 \\ 3\mu &= 1 \\ \mu &= \frac{1}{3}\end{aligned}$$

$\therefore D(13,3,-4)$

d) THE POINT C IS NOT UNIQUE  
HENCE C COULD BE SUCH SO THAT D IS THE MIDDLE OF AC

|    |    |    |
|----|----|----|
| A  | D  | C  |
| 4  | 13 | 22 |
| 3  | 3  | 22 |
| -1 | -4 | -7 |

$\therefore C(22,3,-7)$

**Question 110 (\*\*\*\*)**

The vectors  $\mathbf{p}$  and  $\mathbf{q}$  are defined as

$$\mathbf{p} = \mathbf{a} + 2\mathbf{b} \quad \text{and} \quad \mathbf{q} = 5\mathbf{a} - 4\mathbf{b}$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are unit vectors.

Given that  $\mathbf{p}$  and  $\mathbf{q}$  are perpendicular, determine the acute angle between  $\mathbf{a}$  and  $\mathbf{b}$ .

$$\boxed{\quad}, \quad \theta = 60^\circ$$

We are given that  $|\mathbf{a}| = |\mathbf{b}| = 1$ .  
Then we have  $\mathbf{p} \perp \mathbf{q}$ , i.e.  $\mathbf{p} \cdot \mathbf{q} = 0$   
 $\Rightarrow (\mathbf{a} + 2\mathbf{b}) \cdot (5\mathbf{a} - 4\mathbf{b}) = 0$   
 $\Rightarrow 5\mathbf{a} \cdot \mathbf{a} - 4\mathbf{b} \cdot \mathbf{a} + 10\mathbf{a} \cdot \mathbf{b} - 8\mathbf{b} \cdot \mathbf{b} = 0$   
 $\Rightarrow 5\|\mathbf{a}\| \|\mathbf{a}\| \cos 0 + 10\mathbf{a} \cdot \mathbf{b} - 8\|\mathbf{b}\| \|\mathbf{b}\| \cos 0 = 0$   
 $\Rightarrow 5 \times 1 \times 1 \times 1 + 10\mathbf{a} \cdot \mathbf{b} - 8 \times 1 \times 1 \times 1 = 0$   
 $\Rightarrow 10\mathbf{a} \cdot \mathbf{b} = 3$   
 $\Rightarrow \mathbf{a} \cdot \mathbf{b} = \frac{3}{10}$   
 $\Rightarrow |\mathbf{a}| |\mathbf{b}| \cos \theta = \frac{3}{10}$   
 $\Rightarrow 1 \times 1 \times \cos \theta = \frac{3}{10}$   
 $\Rightarrow \cos \theta = \frac{3}{10}$   
 $\Rightarrow \theta = 60^\circ$

**Question 111 (\*\*\*\*)**

The straight lines  $l_1$  and  $l_2$  have the following Cartesian equations

$$l_1: \frac{x-8}{1} = \frac{y+1}{-1} = \frac{z-2}{1}.$$

$$l_2: \frac{x-3}{-1} = \frac{y-4}{1} = \frac{z-1}{1}.$$

- a) Show that  $l_1$  and  $l_2$  intersect at some point  $P$ , and find its coordinates.  
 b) Find the exact value of  $\cos\theta$ , where  $\theta$  is the acute angle formed by  $l_1$  and  $l_2$ .

The point  $A(6,1,0)$  lies on  $l_1$  and the point  $B(4,3,0)$  lies on  $l_2$ .

- c) By considering  $|AP|$  and  $|BP|$  show further that the angle bisector of  $\angle APB$  is parallel to the vector  $\mathbf{k}$ .

 ,  $P(5,2,-1)$ ,  $\cos\theta = \frac{1}{3}$

**a) WRITE THE EQUATIONS IN PARAMETRIC**

$$\begin{aligned} \frac{x-8}{1} + \frac{y+1}{-1} - \frac{z-2}{1} &= \lambda \Rightarrow l_1: (8, -1, 2) + \lambda(-1, 1, 1) \\ \frac{x-3}{-1} - \frac{y-4}{1} - \frac{z-1}{1} &= \mu \Rightarrow l_2: (3, 4, 1) + \mu(-1, 1, 1) \end{aligned}$$

$$\begin{aligned} l_1 &= (\lambda+8, -\lambda-1, \lambda+2) \\ l_2 &= (-\mu+3, \mu+4, \mu+1) \end{aligned}$$

**EQUATE  $\lambda + \mu = k$**  (assuming  $\mathbf{j} \perp \mathbf{k}$ , which is true)

$$\begin{aligned} \frac{\partial}{\partial \lambda} & \rightarrow -1 = \mu+2 \quad \left\{ \text{ADDS} \right. \\ \frac{\partial}{\partial \mu} & \rightarrow \lambda+2 = \mu+1 \quad \left. \begin{array}{l} \Rightarrow 1 = 2\mu + 3 \\ \Rightarrow \mu = -\frac{1}{2} \\ \Rightarrow \lambda = -\frac{3}{2} \end{array} \right\} \end{aligned}$$

**POINT P**

$$\begin{aligned} \lambda+8 &= -\frac{3}{2}+8 = 5 \\ -\mu+3 &= -\left(-\frac{1}{2}\right)+3 = 3 \end{aligned}$$

**INTERSECT THE LINES AT**  $P(5,2,-1)$

**b) DOTTING THE DIRECTION VECTORS**

$$\begin{aligned} (1, -1, 1) \cdot (-1, 1, 1) &= |(1, -1, 1)| |(-1, 1, 1)| \cos\theta \\ -1 - 1 + 1 &= \sqrt{1+1+1} \sqrt{1+1+1} \cos\theta \\ -1 &= \sqrt{3}\sqrt{3} \cos\theta \\ \cos\theta &= -\frac{1}{3} \end{aligned}$$

**SO  $\cos\theta = -\frac{1}{3}$**

**9 LOOKING AT THE DIAGRAM BELOW**

$|\vec{AP}| = |\vec{P}-\vec{A}| = |(5,2,-1) - (6,1,0)| = |(-1,1,-1)| = \sqrt{3}$

$|\vec{BP}| = |\vec{P}-\vec{B}| = |(5,2,-1) - (4,3,0)| = |(1,-1,-1)| = \sqrt{3}$

HENCE  $\triangle APB$  IS ISOSCELES SO THE PERPENDICULAR BISECTOR OF  $\angle APB$  MUST GO THROUGH THE MIDPOINT OF  $AB$ .

$M\left(\frac{6+4}{2}, \frac{1+3}{2}, \frac{0+0}{2}\right) \rightarrow M(5,2,0)$

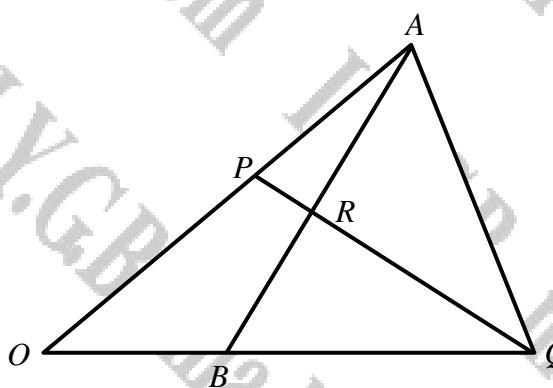
$\vec{MP} = \vec{P}-\vec{M} = (5,2,-1) - (5,2,0) = (0,0,-1)$

**K MIDPOINT OF B**

**PARALLEL TO K**

**Question 112 (\*\*\*)+**

The figure below shows a triangle  $OAQ$ .



- The point  $P$  lies on  $OA$  so that  $OP:OA = 3:5$ .
- The point  $B$  lies on  $OQ$  so that  $OB:BQ = 1:2$ .

Let  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OB} = \mathbf{b}$ .

- a) Given that  $\overrightarrow{AR} = h\overrightarrow{AB}$ , where  $h$  is a scalar parameter with  $0 < h < 1$ , show that

$$\overrightarrow{OR} = (1-h)\mathbf{a} + h\mathbf{b}.$$

- b) Given further that  $\overrightarrow{PR} = k\overrightarrow{PQ}$ , where  $k$  is a scalar parameter with  $0 < k < 1$ , find a similar expression for  $\overrightarrow{OR}$  in terms of  $k, \mathbf{a}, \mathbf{b}$ .
- c) Determine ...
  - ... the value of  $k$  and the value of  $h$ .
  - ... the ratio of  $\overrightarrow{PR}:\overrightarrow{PQ}$ .

$$[ ] , \boxed{\overrightarrow{OR} = \frac{3}{5}(1-k)\mathbf{a} + k\mathbf{b}} , \boxed{k = \frac{1}{6}} , \boxed{h = \frac{1}{2}} , \boxed{PR:PQ = 1:6}$$

|   |
|---|
| <p>a) <u>WORKING AT THE DIAGRAM</u></p> $\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OP} + k(\overrightarrow{PQ} + \overrightarrow{QR}) \\ &= \frac{3}{5}\mathbf{a} + k(\frac{2}{3}\mathbf{a} + \frac{3}{2}\mathbf{b}) \\ &= \frac{3}{5}\mathbf{a} - \frac{3}{5}\mathbf{a} + \frac{3}{2}k\mathbf{b} \\ &= \frac{3}{2}k\mathbf{b} + \frac{3}{2}\mathbf{b}\end{aligned}$ <p>b) <u>WORKING SIMPLY ALGEBRAIC</u></p> $\begin{aligned}\overrightarrow{PR} &= (-h)\mathbf{a} + \mathbf{b} \\ &= \frac{2}{3}(1-k)\mathbf{a} + \frac{3}{2}\mathbf{b} \\ \therefore 1-k &= \frac{2}{3}(1-k) \\ 1-3k &= \frac{2}{3}(1-k) \\ 3-9k &= 2-2k \\ 7-7k &= 0 \\ k &= \frac{1}{6}\end{aligned}$ <p>c) <u>BY INSPECTION</u></p> $\begin{aligned}\overrightarrow{PQ} &= \mathbf{b} - \mathbf{a} \\ \overrightarrow{PR} &= \frac{1}{6}\overrightarrow{PQ} \\ \therefore \frac{PR}{PQ} &= \frac{1}{6}\end{aligned}$ |
|---|

**Question 113 (\*\*\*)+**

The points  $A$  and  $B$  have coordinates  $(4, 0, 2)$  and  $(7, 0, -1)$ , respectively.

- a) Find the vector  $\overrightarrow{AB}$ .

The straight line  $l$  has vector equation

$$\mathbf{r} = -3\mathbf{i} - 4\mathbf{j} + \mathbf{k} + \lambda(7\mathbf{i} + 4\mathbf{j} + \mathbf{k}),$$

where  $\lambda$  is a scalar parameter.

- b) Show that  $A$  lies on  $l$ .  
 c) Calculate the acute angle between  $\overrightarrow{AB}$  and  $l$ .

The point  $C$  lies on  $l$  so that  $ABCD$  is a rectangle.

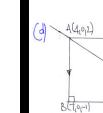
- d) Find the coordinates of  $D$ .

$$\boxed{\overrightarrow{AB} = 3\mathbf{i} - 3\mathbf{k}}, \quad \boxed{\theta \approx 58.5^\circ}, \quad \boxed{D(8, 4, 6)}$$

**(a)**  $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (7, 0, -1) - (4, 0, 2) = (3, 0, -3)$

**(b)**  $\mathbf{l} = (-3, -4, 1) + \lambda(7, 4, 1) = (14\lambda - 3, 4\lambda - 4, \lambda + 1)$   
 By inspection if  $\lambda = 1$ :  $(14 - 3, 4 - 4, 1 + 1) = (7, 0, -1) = (4, 0, 2)$   
 $\therefore A$  lies on  $l$

**(c)**   
 By dot product  
 $(7\mathbf{i} + 4\mathbf{j} + \mathbf{k}) \cdot (3\mathbf{i} - 3\mathbf{k}) = |7\mathbf{i}| |3\mathbf{i} - 3\mathbf{k}| \cos\theta$   
 $21 + 0 - 3 = \sqrt{49 + 16 + 1} \sqrt{9 + 0 + 9} \cos\theta$   
 $18 = \sqrt{66} \sqrt{18} \cos\theta$   
 $\cos\theta = 0.5222\dots$   
 $\theta \approx 58.5^\circ$

**(d)**   
 Since  $C \in l$   
 $\bullet \overrightarrow{BC} = \mathbf{c} - \mathbf{b} = (4, 0, 2) - (7, 0, -1) = (-3, 0, 3)$   
 $\bullet \overrightarrow{AB} \perp \overrightarrow{BC}$   
 $(-3, 0, 3) \cdot (3, 0, -3) = (3, 0, -3) \cdot (3, 0, -3) = 0$   
 $3(-3) - 3(-3) = 0$   
 $-9 + 9 = 0$   
 $\therefore C \in BC$   
 Since  $C \in l$   
 $\bullet \overrightarrow{BC} \perp \overrightarrow{CD}$   
 $(-3, 0, 3) \cdot (1, 4, -1) = 0$   
 $-3 + 0 - 3 = 0$   
 $-6 = 0$   
 $\therefore D \in BC$   
 To find  $D$ :  $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = \overrightarrow{OC} + \overrightarrow{BA} = \mathbf{c} + \mathbf{a} - \mathbf{b}$   
 $= (4, 0, 2) + (4, 0, 2) - (7, 0, -1)$   
 $= (8, 4, 6)$   
 $\therefore D(8, 4, 6)$

**Question 114 (\*\*\*\*+)**

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 3\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

$$\mathbf{r}_2 = 2\mathbf{i} + 7\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 2\mathbf{k}),$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- a) Show that  $l_1$  and  $l_2$  intersect at some point  $P$  and find its position vector.

The points  $A$  and  $C$  lie on  $l_1$  and the points  $B$  and  $D$  lie on  $l_2$ .

The point  $A$  has position vector  $4\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ .

The quadrilateral  $ABCD$  is a parallelogram with an area of 54 square units.

- b) State the position vector of the point  $C$ .  
 c) Show that the distance of the point  $B$  from  $l_1$  is 3 units.

$$[\mathbf{p} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}], [\mathbf{c} = -8\mathbf{i} + 7\mathbf{j} - 7\mathbf{k}]$$

(a)  $\begin{aligned}\mathbf{r}_1 &= (0, 3, 1) + \lambda(2, -1, 2) = (2\lambda, 3-\lambda, 2\lambda+1) \\ \mathbf{r}_2 &= (2, 0, 7) + \mu(1, -1, 2) = (2+\mu, -\mu, 7+2\mu)\end{aligned}$

Equating  $\frac{1}{2}$  &  $\frac{1}{2}$ :

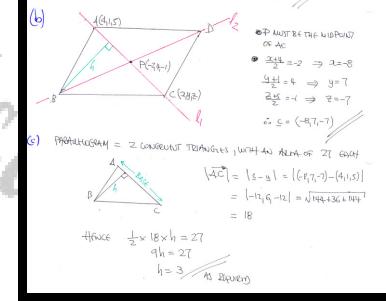
(1):  $2\lambda = 2 + \mu \quad \text{Subtract } -1 = -\mu - \lambda \quad \frac{-1 = -\mu - \lambda}{\lambda = -1}$

(2):  $3 - \lambda = -\mu \quad \frac{2\lambda = 4 + 2\mu}{\lambda = -1}$

(3):  $2\lambda + 1 = 7 + 2\mu \quad \frac{2\lambda = 6 + 2\mu}{\lambda = -1}$

Solve  $\frac{1}{2}$ :  $3 - \lambda = 3 - (-1) = 4 \quad \text{all three components against the base reflected}$

Using  $\lambda = -1$  into  $(2\lambda, 3-\lambda, 2\lambda+1)$  gives pos  $(-2, 4, -1)$

(b) 

• P must be THE MIDPOINT OF AC  
 $\bullet \frac{x+4}{2} = -2 \Rightarrow x = -8$   
 $\bullet \frac{y+1}{2} = 1 \Rightarrow y = 1$   
 $\bullet \frac{z+5}{2} = -1 \Rightarrow z = -7$   
 $\therefore P = (-4, 1, -7)$

(c) PARALLELOGRAM = 2 CONGRUENT TRIANGLES, WITH AN AREA OF 54 EACH

$|\Delta ABC| = |A-C| = |(6, 7, -1) - (4, 1, 5)|$   
 $= |(-1, 6, -12)| = \sqrt{144 + 36 + 144} = 18$

Hence  $\frac{1}{2} \times 18 \times h = 27$   
 $9h = 27$   
 $h = 3$  ✓ AS 21 MARKS

**Question 115 (\*\*\*)+**

With respect to a fixed origin  $O$ , the points  $A$  and  $B$  have respective position vectors

$$\mathbf{i} + 11\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad 11\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}.$$

- a) Find a vector equation for the straight line  $l$ , which passes through  $A$  and  $B$ .

The point  $C$  is the point on  $l$  closest to the origin  $O$ .

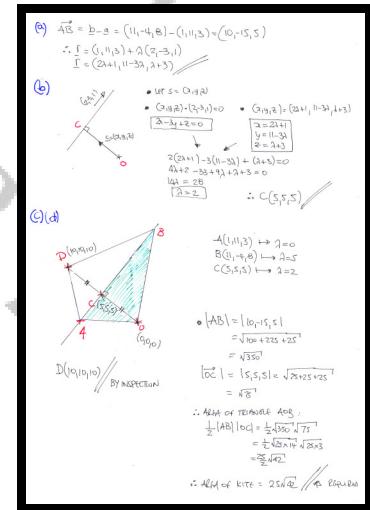
- b) Determine the position vector of  $C$ .

The point  $D$  is the reflection of  $O$  about  $l$ .

- c) State the position vector of  $D$ .

- d) Show that the area of the kite  $OADB$  is  $25\sqrt{42}$  square units.

$$\boxed{\mathbf{r} = \mathbf{i} + 11\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})}, \boxed{\overrightarrow{OC} = 5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}}, \boxed{\overrightarrow{OD} = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}}$$



**Question 116 (\*\*\*\*+)**

Relative to a fixed origin  $O$ , the points  $A(3t-19, 2t-14, 28-t)$ ,  $B(t+1, t-2, 5t)$  and  $C(2t-11, 10-t, 2t+4)$ , represent the coordinates of the paths of three helicopters, where  $t$  represents the time in minutes after a certain instant.

All distances are in kilometres with the coordinates axes  $Ox, Oy, Oz$  oriented due east, due north and vertically upwards, respectively.

- Show that all three helicopters pass through a point  $P$  and find its coordinates.
- Explain why only two of the helicopters will collide at the point  $P$  if they maintain their courses as described in this problem.
- Show that the paths of  $A$  and  $B$  are perpendicular.

$$P(5, 2, 20)$$

(a)  $\Sigma_A = (3t-19, 2t-14, 28-t)$        $\Sigma_B = (3t-19, 2t-14, 28-t)$   
 $\Sigma_B = (t+1, t-2, 5t)$   
 $\Sigma_C = (2t-11, 10-t, 2t+4)$

From  $t-2$  &  $t-4$  or  $A \neq B$   
 $(1): 2t-14 = t-2$   
 $(2): 28-t = 5t$        $\Rightarrow [t=2t-12]$   
 $28-t = 5t$   
 $28 = 6t$   
 $t = 12$   
 $t = 12$  &  $t=4$

Check 1:  $3t-19 = 3 \times 12 - 19 = 5$   
 $t+1 = 12 + 1 = 13$

• Helicopters A & B paths will cross  
using  $t=4$   $(t+1, t-2, 5t)$  we obtain  $P(5, 2, 20)$

Now looking at  $\Sigma_C = (2t-11, 10-t, 2t+4)$  &  $P(5, 2, 20)$   
By inspection if  $t=8$  helicopter C also passes through  $P(5, 2, 20)$   
 $\therefore$  THE PATHS OF THE THREE PASS THROUGH  $P(5, 2, 20)$

(b) Helicopter A, passes through  $P$  when  $t=8$   
B, passes through  $P$  when  $t=4$   
C, passes through  $P$  when  $t=8$   
Hence only A & C will collide

(c) Direction vector of "A" is  $(3, 2, -1)$   
Direction vector of "B" is  $(1, 1, 5)$   
 $(3, 2, -1) \cdot (1, 1, 5) = 3+2-5 \Rightarrow \therefore$  PATHS ARE PERPENDICULAR

**Question 117 (\*\*\*\*+)**

The straight lines  $l_1$  and  $l_2$ , where  $\lambda$  and  $\mu$  are scalar parameters, have the following vector equations

$$\mathbf{r}_1 = (\lambda + 2)\mathbf{i} + (2\lambda + 6)\mathbf{j} + (-\lambda - 1)\mathbf{k}$$

$$\mathbf{r}_2 = (2\mu - 4)\mathbf{i} + (4 - \mu)\mathbf{j} + (3 - \mu)\mathbf{k}.$$

$l_1$  and  $l_2$  intersect at the point  $A$  and the acute angle between  $l_1$  and  $l_2$  is  $\theta$ .

a) Find in any order...

i. .... the coordinates of  $A$ .

ii. .... the exact value of  $\cos \theta$ .

The point  $B$  lies on  $l_1$  and the point  $C$  lies on  $l_2$ . The triangle  $ABC$  is isosceles with  $|AB| = |AC| = 6\sqrt{6}$ .

b) Find the two possible sets of coordinates for the points  $B$  and  $C$ .

c) Show that either  $|BC| = 6\sqrt{14}$  or  $|BC| = 6\sqrt{10}$

In the triangle  $ABC$  the angle  $BAC$  is acute.

a) In the triangle  $ABC$ , determine the two possible pairings for the coordinates of the point  $B$  and the corresponding coordinates of the point  $C$ .

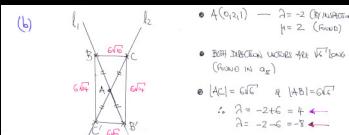
$$A(0, 2, 1), \cos \theta = \frac{1}{6}, B_1(-6, -10, 7) \text{ or } B_2(6, 14, -5),$$

$$C_1(-12, 8, 7) \text{ or } C_2(12, -4, -5), B_1 \text{ & } C_1 \text{ or } B_2 \text{ & } C_2$$

(a) (i)  $\mathbf{r}_1 = (\lambda + 2)\mathbf{i} + (2\lambda + 6)\mathbf{j} + (-\lambda - 1)\mathbf{k}$   
 $\mathbf{r}_2 = (2\mu - 4)\mathbf{i} + (4 - \mu)\mathbf{j} + (3 - \mu)\mathbf{k}$

Equate  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$   
 $\lambda + 2 = 2\mu - 4$  Add  $\lambda = 2\mu - 6$   
 $-\lambda - 1 = 3 - \mu$  Add  $\mu = 2 - \lambda$   
 $\lambda = 2 - \mu$

(ii) Dot product method  
 $(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{r}_2 - \mathbf{r}_1) = |\mathbf{r}_1 - \mathbf{r}_2| |\mathbf{r}_2 - \mathbf{r}_1| \cos \theta$   
 $2 - 2 + 1 = \sqrt{(4 + 4 + 1)} \sqrt{4 + 1 + 1} \cos \theta$   
 $1 = 6\sqrt{6} \cos \theta$   
 $\cos \theta = \frac{1}{6}$

(b) 

- $A(0, 2, 1) \rightarrow \lambda = 2 \quad (\text{from } l_1)$   
 $\mu = 2 \quad (\text{from } l_2)$
- Both methods give the same angle (acute in  $\theta$ )
- $|AC| = 6\sqrt{6} \quad \& \quad |AB| = 6\sqrt{6}$   
 $\therefore \lambda = -2 + 6 = 4 \leftarrow$   
 $\lambda = -2 - 6 = -8 \leftarrow$   
 $\mu = 2 + 6 = 8 \leftarrow$   
 $\mu = 2 - 6 = -4 \leftarrow$
- $\therefore C_1(2, -4, -5) \cup (-2, 8, 7)$   
 $\& \quad B(-6, -10, 7) \cup (6, 14, -5)$
- DOT PRODUCT  $BC = \sqrt{(12 - 4)^2 + (-4 + 10)^2 + (-5 - 7)^2} = \sqrt{144 + 36 + 144} = 6\sqrt{14}$   
 OR  $= \sqrt{(12 - 4)^2 + (-4 + 10)^2 + (5 - 7)^2} = \sqrt{144 + 36 + 16} = 6\sqrt{10}$
- $C(12, -4, -5) \text{ with } B(6, 14, -5) \quad \& \quad C(-2, 8, 7) \text{ with } B(-6, -10, 7)$

**Question 118    (\*\*\*)+**

Relative to a fixed origin  $O$ , the points  $A$ ,  $B$ ,  $C$  and  $D$  have coordinates  $(7, 6, 2)$ ,  $(12, 10, 5)$ ,  $(1, -4, -8)$  and  $(11, 4, -2)$ , respectively.

- Find the vector equation of the straight line  $l_1$  which passes through the point  $A$  and  $B$  and the vector equation of the straight line  $l_2$  which passes through the point  $C$  and  $D$ .
- Explain why  $l_1$  and  $l_2$  do not intersect.

The point  $P$  lies on  $l_2$ .

- Find an expression for  $|\overrightarrow{AP}|^2$ , in terms of  $\mu$ .
- Calculate the distance between  $l_1$  and  $l_2$ .

$$\boxed{\mathbf{r}_1 = 7\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} + \lambda(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})}, \quad \boxed{\mathbf{r}_2 = \mathbf{i} - 4\mathbf{j} - 8\mathbf{k} + \mu(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})}$$

$$\boxed{|\overrightarrow{PA}|^2 = 50\mu^2 - 200\mu + 236}, \quad \boxed{6 \text{ units}}$$

④  $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (12, 10, 5) - (7, 6, 2) = (5, 4, 3)$   
 $\overrightarrow{CD} = \mathbf{d} - \mathbf{c} = (11, 4, -2) - (1, -4, -8) = (10, 8, 6) \leftarrow \text{SCALAR TO } (1, 1, 3)$

$\mathbf{l}_1: \quad \mathbf{r}_1 = (7, 6, 2) + \lambda(5, 4, 3) = (5\lambda + 7, 4\lambda + 6, 3\lambda + 2)$

$\mathbf{l}_2: \quad \mathbf{r}_2 = (1, -4, -8) + \mu(10, 8, 6) = (10\mu + 1, 8\mu - 4, 6\mu - 8)$

(b) LINES HAVE THE SAME DIRECTION  $\Rightarrow$  PARALLEL  $\Rightarrow$  DO NOT INTERSECT

(c)



$\bullet \overrightarrow{AP} = \mathbf{p} - \mathbf{a}$   
 $= (x, y, z) - (7, 6, 2)$   
 $= (x-7, y-6, z-2)$   
 $\bullet |\overrightarrow{AP}|^2 = |(x-7, y-6, z-2)|^2$   
 $= \sqrt{(x-7)^2 + (y-6)^2 + (z-2)^2}$

But  $P(x,y,z)$  lies on  $l_2 \Rightarrow$

$$|\overrightarrow{AP}|^2 = \sqrt{(x-7)^2 + (4\mu + 6 - 6)^2 + (3\mu + 2 - 2)^2}$$

$$\Rightarrow |\overrightarrow{AP}|^2 = \sqrt{(x-7)^2 + (4\mu)^2 + (3\mu)^2}$$

$$\Rightarrow |\overrightarrow{AP}|^2 = \sqrt{30\mu^2 - 140\mu + 32}$$

$$\frac{d}{d\mu} (30\mu^2 - 140\mu + 32) = 60\mu - 140$$

$$\therefore \mu = 7 \Rightarrow |\overrightarrow{AP}|^2 = 50 \times 7^2 - 200 \times 7 + 236$$

(d) Let  $f(\mu) = 50\mu^2 - 200\mu + 236$   
 $f'(\mu) = 100\mu - 200$

Solve for zero  
 $100\mu - 200 = 0$   
 $\mu = 2$

MINIMUM  
 $f''(\mu) = 100 > 0$   
 $f''(2) = 36 > 0$   
 $|f''(2)| = 6$

**Question 119 (\*\*\*)+**

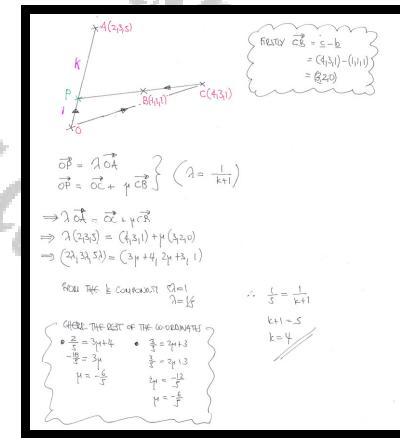
Relative to a fixed origin  $O$ , the points  $A$ ,  $B$  and  $C$  have coordinates  $(2,3,5)$ ,  $(1,1,1)$  and  $(4,3,1)$ , respectively.

The line segment  $CB$  is extended to the point  $P$ .

It is further given that  $P$  lies on the line segment  $OA$  so that  $|OP| : |PA| = 1 : k$ .

Determine the value of  $k$ .

$$\boxed{k = 4}$$



**Question 120 (\*\*\*)+**

The straight line  $l_1$  passes through the points with coordinates  $A(-2, 3, 4)$  and  $B(8, -1, 14)$ .

- a) Find a vector equation for  $l_1$ .

The straight line  $l_2$  has vector equation

$$\mathbf{r} = \mathbf{i} + 5\mathbf{j} - 5\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}),$$

where  $\mu$  is a scalar parameter.

The point  $C$  lies on  $l_2$  so that  $AC$  is perpendicular to  $BC$ .

- b) Show that one possible position for the point  $C$  has coordinates  $(2, 3, 2)$  and find the other.
- c) Assuming further that  $C$  has coordinates  $(2, 3, 2)$ , show that the area of the triangle  $ABC$  is  $14\sqrt{5}$  square units.

$$\boxed{\mathbf{r}_1 = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} + \lambda(5\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})}, \quad \boxed{(4, -1, 16)}$$

**Q**  $\vec{AB} = \frac{1}{6}\mathbf{i} - \mathbf{j} - \mathbf{k} = (8, -1, 14) - (-2, 3, 4) = (6, -4, 10)$   
 Scale it to  $(3, -2, 5)$

$$\vec{l}_1 = (3, -2, 5) + \lambda(5, -2, 5) = (3+5\lambda, -2-2\lambda, 5+5\lambda)$$

**Q**  $\vec{l}_2 = (1, 3, 2) + \mu(1, -2, 1) = (1+\mu, 3-2\mu, 2+\mu)$

$\vec{AC} = (2, 3, 2) - (-2, 3, 4) = (4, 0, -2)$   
 $\vec{BC} = (2, 3, 2) - (8, -1, 14) = (-6, 4, -12)$   
 $\vec{CA} = \vec{CB} = 0$   
 $(2, 3, 2) \cdot (4, 0, -2) = 0$   
 $(2, 3, 2) \cdot (-6, 4, -12) = 0$   
 $(2, 3, 2) \cdot (4, 0, -2) + (2, 3, 2) \cdot (-6, 4, -12) = 0$

But  $C$  lies on  $l_2 \Rightarrow \begin{cases} x = 1+\mu \\ y = 3-2\mu \\ z = 2+\mu \end{cases}$

$$\therefore (-2-4\mu)(8-(1+\mu)) + (3-(1-2\mu))(1-(2\mu-2)) + (4-(2\mu-2))(14-(1+\mu-2)) = 0$$

$$(-3-\mu)(7-\mu) + (2\mu-2)(2\mu-2) + (-7\mu)(2\mu-2) = 0$$

$$1^2 - 4\mu + 2\mu^2 - 12\mu + 12 = 0$$

$$2\mu^2 - 16\mu + 12 = 0$$

$$\frac{2\mu^2 - 16\mu + 12}{2\mu^2 - 16\mu + 12} = 0$$

$$\mu^2 - 8\mu + 6 = 0$$

$$(\mu-3)(\mu-1) = 0$$

$$\therefore \mu = 1 \rightarrow C(3, 1, 2)$$

**Q**  $\vec{AC} = (1, 3, 2) - (-2, 3, 4) = (3, 0, -2) = \sqrt{13}\mathbf{j}$   
 $\vec{BC} = (1, 3, 2) - (8, -1, 14) = (-7, 4, -12) = \sqrt{13}\mathbf{j} + 4\mathbf{k}$

$\bullet |\vec{AC}| = \sqrt{13^2 + 2^2} = \sqrt{165}$

$\bullet |\vec{BC}| = \sqrt{7^2 + 4^2} = \sqrt{65}$

$\therefore \text{Area} = \frac{1}{2}\sqrt{13}\sqrt{65} = 14\sqrt{5}$

**Question 121 (\*\*\*\*+)**

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 13\mathbf{i} - 5\mathbf{j} + 8\mathbf{k} + \lambda(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

$$\mathbf{r}_2 = -5\mathbf{i} - 4\mathbf{j} + 8\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}),$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- Show that  $l_1$  and  $l_2$  intersect at some point  $C$  and find its coordinates.
- Find the cosine of the acute angle between  $l_1$  and  $l_2$ .

The point  $A$  lies on  $l_1$  where  $\lambda = -1$  and the point  $B$  lies on  $l_2$  where  $\mu = 4$ .

- Determine a vector equation of the angle bisector of  $\angle ACB$ .

$$[ ] , [C(1, -1, 2)] , [\cos \theta = \frac{4}{21}] , [\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + t(32\mathbf{i} + \mathbf{j} - 5\mathbf{k})]$$

a)  $\begin{aligned}\Gamma_1 &= (13, -5, 8) + \lambda(6, -2, 3) = (13+6\lambda, -5-2\lambda, 8+3\lambda) \\ \Gamma_2 &= (-5, -4, 8) + \mu(2, 1, -2) = (-5+2\mu, -4+\mu, 8-2\mu)\end{aligned}$

• Separate  $\mathbf{i}, \mathbf{j}, \mathbf{k}$

$\begin{cases} 13+6\lambda = 2\mu-5 \\ -5-2\lambda = 8-2\mu \\ 8+3\lambda = -4+\mu \end{cases}$  Solving  $\Rightarrow \begin{cases} \lambda = -2 \\ \mu = 3 \end{cases}$

$\begin{aligned} &\text{Hence } C = (-5, -4, 8) + 3(2, 1, -2) \\ &= (-5+6, -4+3, 8-6) \\ &= (1, -1, 2) \end{aligned}$

• Check  $\lambda$   
 $3A+B = 3(-2)+3 = 2$   
 $B-2A = 3-2(-2) = 2$

• As all three component affinities  $\lambda = -2$  &  $\mu = 3$ , the lines intersect

• Using  $\mu = 3$  we obtain  $C(1, -1, 2)$

b) Dotting the direction vectors  
 $(6, -2, 3) \cdot (2, 1, -2) = |(6, -2, 3)| |(2, 1, -2)| \cos \theta$   
 $|2-2-6| = \sqrt{36+4+9} \sqrt{4+1+4} \cos \theta$   
 $4 = 7\sqrt{3} \cos \theta$   
 $\cos \theta = \frac{4}{21}$

• The direction vector of  $l_1$  is 7 units long & the direction vector of  $l_2$  is 3 units long.  
 Orthogonality  $\Rightarrow 21$  units

• Hence  $|CP| = |CQ| = 21$

$P(1, -1, 1) \Leftarrow \lambda = 1$   
 $Q(5, 6, -2) \Leftarrow \mu = 6$

• By inspection  
 $M\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$   
 $\overrightarrow{MC} = \underline{s} - \underline{m} = (1, -1, 2) - \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{5}{2}\right)$   
 Scale it to 4 direction vector  $(-3, 1, 1, 5)$   
 or better  $(32, 1, -5)$

• Using  $C(1, -1, 2)$  as a fixed point

$$\begin{aligned}\underline{r} &= (1, -1, 2) + t(32, 1, -5) \\ \underline{r} &= (32t+1, t-1, 2-5t)\end{aligned}$$

**Question 122 (\*\*\*\*\*)**

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 9\mathbf{i} + 7\mathbf{j} + 11\mathbf{k} + \lambda(4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})$$

$$\mathbf{r}_2 = -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k} + \mu(3\mathbf{i} - 4\mathbf{j} + a\mathbf{k}),$$

where  $\lambda$  and  $\mu$  are scalar parameters and  $a$  is a scalar constant.

The point  $A$  is the intersection of  $l_1$  and  $l_2$ .

b) Find in any order ...

i. ... the value of  $a$ .

ii. ... the coordinates of  $A$ .

The acute angle between  $l_1$  and  $l_2$  is  $\theta$ .

c) Show that  $\theta = 60^\circ$ .

The point  $B$  lies on  $l_1$  and the point  $C$  lies on  $l_2$ .

The triangle  $ABC$  is equilateral with sides of length  $15\sqrt{2}$ .

d) Find the two possible pairings for the coordinates of  $B$  and  $C$ .

$\boxed{\quad}, \boxed{a=5}, \boxed{A(1,1,1)},$

$\boxed{B(13,10,16) \text{ & } C(10,-11,16) \text{ or } B(-11,-8,-14) \text{ & } C(-8,13,-14)}$

**Q**  $\begin{aligned} l_1 &= (3,7,11) + \lambda(4,3,5) = (4\lambda+9, 3\lambda+7, 5\lambda+11) \\ l_2 &= (-2,5,-4) + \mu(3,-4,a) = (3\mu-2, 5-4\mu, a\mu-4) \end{aligned}$

DOTTING THE DIRECTION VECTORS

$$\begin{aligned} \mathbf{d} \cdot \mathbf{e} &= 0 \Rightarrow 4\lambda+9 + 3\mu-2 = 0 \Rightarrow 4\lambda+3\mu = -7 \\ \rightarrow 12+10+25 &= 4\lambda+3\mu+23 \Rightarrow 4\lambda+3\mu = -28 \end{aligned}$$

ADDITION GIVES  $25\lambda+57 = 7$

$$\begin{aligned} 25\lambda &= -50 \\ \lambda &= -2 \end{aligned}$$

AND  $4\lambda+9 = 3\mu-2$

$$\begin{aligned} -8+9 &= 3\mu-2 \\ 1 &= 3\mu \\ \mu &= \frac{1}{3} \end{aligned}$$

EQUATE  $\mathbf{d}$  with  $\lambda=-2$ ,  $\mu=\frac{1}{3}$

$$\begin{aligned} 3\lambda+11 &= 3(-2)+11 \\ -6+11 &= -6+11 \\ \alpha &= 5 \end{aligned}$$

ANSWER:  $\lambda=-2$ ,  $\mu=\frac{1}{3}$ ,  $\alpha=5$ ,  $(4(-2)+9, 3(-2)+7, 5(-2)+11)$  or  $A(1,1,1)$

WE FOUND  $\lambda=-2$ ,  $\mu=\frac{1}{3}$ ,  $\alpha=5$

$$\begin{aligned} |\mathbf{d}| &= \sqrt{4^2+3^2+5^2} = \sqrt{50} = 5\sqrt{2} \\ |\mathbf{e}| &= \sqrt{3^2+(-4)^2+a^2} = \sqrt{16+25+a^2} = \sqrt{41+a^2} \end{aligned}$$

THIS MEANS THAT INCREASING  $\mu$  BY 1 UNIT ADVANCES UP BY A DISTANCE OF  $|\mathbf{e}|^2$  ON EITHER LINE.

SINCE THE JOURNEY IS  $15\sqrt{2}$  THIS CORRESPONDS TO 3 UNITS IN  $\lambda$  OR  $\mu$ .

AT THE INTERSECTION  $A(1,1,1)$  WE FOUND  $\lambda=-2$ ,  $\mu=\frac{1}{3}$

$\therefore B$  MUST BE AT  $\lambda=-2+3$  OR  $C$  MUST BE AT  $\mu=\frac{1}{3}+3$

$$\begin{aligned} \lambda=1 &: B(13,10,16) & \mu=4 &: C(10,-11,16) \\ \lambda=3 &: B(-11,-8,-14) & \mu=2 &: C(-8,13,-14) \end{aligned}$$

FINALLY WE NEED TO FIND THE DISTANCE  $|BC|$  FOR ANY PAIR TO SEE WHICH WAY THEY TAKE UP. (MOREOVER WE MAY PICK THE OTHER TRIANGLES BETWEEN THE 2 COORDINATES)

SAY USE THE THIRD PAIR:  $|BC|^2 = \sqrt{(13-10)^2 + (10+8)^2 + (16+14)^2}$

$$\begin{aligned} &= \sqrt{9+144+484} = \sqrt{637} = 25\sqrt{2} \\ &= 15\sqrt{2} \end{aligned}$$

SO THIS IS ONE OF THE CORRECT ANSWERS:

$$\begin{aligned} \therefore B(13,10,16) & \quad C(10,-11,16) \\ C(10,-11,16) & \quad B(13,10,16) \end{aligned}$$

**Question 123 (\*\*\*)+**

The points  $A(3,2,14)$ ,  $B(0,1,13)$  and  $C(5,6,8)$  are defined with respect to a fixed origin  $O$ .

- a) Show that the cosine of the angle  $ABC$  is  $\frac{3}{\sqrt{33}}$ .

The straight line  $L$  passes through  $A$  and it is parallel to the vector  $\overrightarrow{BC}$ .

- b) Find a vector equation of  $L$ .

The point  $D$  lies on  $L$  so that  $ABCD$  is a parallelogram.

- c) Find the coordinates of  $D$ .

- d) If instead  $ABCD$  is an isosceles trapezium and the point  $D$  still lies on  $L$ , determine the new coordinates of  $D$ .

$$[\quad], [\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 14\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k})], [D(8,7,9)], [D(6,5,11)]$$

**a)**

$$\begin{aligned}\overrightarrow{BA} &= \mathbf{a} - \mathbf{b} = (3,2,14) - (0,1,13) = (3,1,1) \\ \overrightarrow{BC} &= \mathbf{c} - \mathbf{b} = (5,6,8) - (0,1,13) = (5,5,-5)\end{aligned}$$

BY THE DOT PRODUCT

$$\begin{aligned}\overrightarrow{BA} \cdot \overrightarrow{BC} &= (3,1,1) \cdot (5,5,-5) = 3 \cdot 5 + 1 \cdot 5 - 1 \cdot 5 = 15 \\ |\overrightarrow{BA}| &= \sqrt{3^2 + 1^2 + 1^2} = \sqrt{11} \\ |\overrightarrow{BC}| &= \sqrt{5^2 + 5^2 + (-5)^2} = \sqrt{55} \\ \cos \theta &= \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| |\overrightarrow{BC}|} = \frac{15}{\sqrt{11} \sqrt{55}} = \frac{15}{\sqrt{605}} = \frac{3}{\sqrt{33}}\end{aligned}$$

**b)** PARALLEL TO  $BC \dots$  i.e.  $(1,1,-1)$  AS DIRECTION VECTOR, SCALAR MULT OF  $(5,5,-5)$

$$\begin{aligned}\mathbf{d} &= (3,2,14) + \lambda(1,1,-1) \\ \mathbf{d} &= (3+1\lambda+2, 2+1\lambda, 14-\lambda)\end{aligned}$$

**c)**

BY INSPECTION

|         |        |    |
|---------|--------|----|
| B to A  | 0 → 3  | +3 |
| 1 → 2   | 1 → 1  |    |
| 13 → 14 | 13 → 1 | -1 |

|        |       |    |
|--------|-------|----|
| C to D | 5 → 2 | -3 |
| 5 → 1  | 5 → 1 |    |
| 8 → 9  | 8 → 9 |    |

MIDPOINT OF  $AC \dots (4,4,11)$

|         |         |        |   |
|---------|---------|--------|---|
| B       | 0 → 4   | 4 → 8  | 8 |
| 1 → 4   | 1 → 4   | 4 → 7  | 7 |
| 13 → 11 | 13 → 11 | 11 → 9 | 9 |

**d)**

$|AB| = |CD| = \sqrt{11}$  (FOUND EARLIER IN THE DOT PRODUCT)

LET THE COORDINATES OF D BE  $(x,y,z)$

$$|CD| = |\mathbf{d} - \mathbf{c}| = |(x,y,z) - (5,6,8)| = |(x-5, y-6, z-8)| = \sqrt{(x-5)^2 + (y-6)^2 + (z-8)^2}$$

AS WE HAVE AN ISOSCELES TRAPEZIUM  $|AB| = |CD| = \sqrt{11}$

$$\begin{aligned}\sqrt{(x-5)^2 + (y-6)^2 + (z-8)^2} &= \sqrt{11} \\ (x-5)^2 + (y-6)^2 + (z-8)^2 &= 11\end{aligned}$$

LET D LIES ON THE LINE  $\mathbf{l} = (3,1,1) + \lambda(5,5,-5) = (3+5\lambda, 1+5\lambda, 1-5\lambda)$

$$\begin{aligned}(2+5\lambda)^2 + (4+5\lambda)^2 + (2-5\lambda)^2 &= 11 \\ (2+5\lambda)^2 + (1+5\lambda)^2 + (1-5\lambda)^2 &= 11 \\ 2^2 + 4 \times 2 \times 4 + 4 &= 11 \\ 2^2 + 20\lambda + 16 &= 11 \\ 20\lambda + 17 &= 0 \\ 20\lambda &= -17 \\ \lambda &= -\frac{17}{20}\end{aligned}$$

$\therefore D = \left(3 + 5 \left(-\frac{17}{20}\right), 1 + 5 \left(-\frac{17}{20}\right), 1 - 5 \left(-\frac{17}{20}\right)\right) = \left(\frac{1}{4}, -\frac{1}{4}, \frac{51}{4}\right)$

$\therefore \angle D \leftarrow \text{PARALLELGRAM}$

**Question 124 (\*\*\*\*+)**

With respect to a fixed origin  $O$ , the point  $A$  and the point  $B$  have position vectors  $\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$  and  $-9\mathbf{j} + 6\mathbf{k}$ , respectively.

- a) Find a vector equation of the straight line  $l$  which passes through  $A$  and  $B$ .

A variable vector is defined as

$$\mathbf{p} = (p+6)\mathbf{i} + (2p+3)\mathbf{j} - p\mathbf{k},$$

where  $p$  is a scalar parameter.

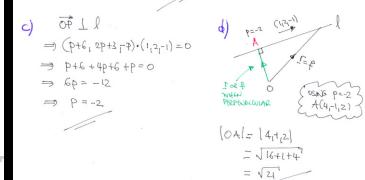
- b) Show that for all values of  $p$ , the point  $P$  with position vector  $\mathbf{p}$ , lies on  $l$ .  
 c) Determine the value of  $p$  for which  $\overrightarrow{OP}$  is perpendicular to  $l$ .  
 d) Hence, or otherwise, find the shortest distance of  $l$  from the origin  $O$ .

 ,  $\mathbf{r} = \mathbf{i} - 7\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ ,  $p = -2$ ,  $\text{shortest distance} = \sqrt{21}$

**a)**  $\begin{aligned} \mathbf{a} &= (\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}) \\ \mathbf{b} &= (\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}) \end{aligned}$   $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}) - (\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}) = (-2, -3, 1)$   
 USING  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$  AS A DIRECTION VECTOR  
 $\begin{aligned} \mathbf{l} &= (\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}) + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\ \mathbf{l} &= (\mathbf{i} + \lambda, 2\lambda - 7, \lambda + 5) // \end{aligned}$

**b)** Rewrite THE EQUATION  
 $\mathbf{p} = (p+6)\mathbf{i} + (2p+3)\mathbf{j} - p\mathbf{k}$  DIRECTION VECTOR  
SO EQUAL DIRECTION  
OR COINCIDENT  
 USING  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$   
 $\begin{aligned} \mathbf{p} + \mathbf{a} &= \mathbf{l} & 2\mathbf{j} + \mathbf{b} &= -7 & \mathbf{s} &= -\mathbf{p} \\ p + 6 &= 1 & 2p + 3 &= -7 & p &= -s \\ p &= -5 & 2p &= -10 & p &= -s \\ p &= -5 & p &= -5 & p &= -s \end{aligned}$   
 $\therefore$  THE LINE ALSO PASSES THROUGH A  
 $\therefore$  LINES ARE COINCIDENT

**c)**  $\overrightarrow{OP} \perp l$   
 $\Rightarrow (\mathbf{i} + 6, 2p + 3, -p) \cdot (\mathbf{i}, 2\mathbf{j}, -\mathbf{k}) = 0$   
 $\Rightarrow p + 6 + 4p + 6 - p = 0$   
 $\Rightarrow 6p = -12$   
 $\Rightarrow p = -2$

**d)**   
 $\begin{aligned} |\mathbf{OA}| &= [4, 1, 2] \\ &= \sqrt{16 + 1 + 4} \\ &= \sqrt{21} \end{aligned}$

**Question 125 (\*\*\*)+**

The points with coordinates  $A(7, 6, 10)$ ,  $B(6, 5, 6)$  and  $C(1, 0, 4)$  are the vertices of the parallelogram  $ABCD$ .

a) Find ...

i. ... the coordinates of  $D$ .

ii. ... a vector equation of the straight line  $l$  which passes through the points  $A$  and  $C$ .

iii. ... the distance  $AC$ .

b) Show that the shortest distance of  $l$  from  $B$  is  $\sqrt{6}$  units.

c) Hence find the exact area of the parallelogram  $ABCD$ .

$$\boxed{\quad}, \boxed{D(2,1,8)}, \boxed{\mathbf{r} = 7\mathbf{i} + 6\mathbf{j} + 10\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})}, \boxed{|AC| = 6\sqrt{3}}, \boxed{18\sqrt{2}}$$

**Diagram:** A 3D coordinate system showing points  $A(7, 6, 10)$ ,  $B(6, 5, 6)$ , and  $C(1, 0, 4)$ . A parallelogram  $ABCD$  is drawn with  $A$  at the top-left,  $B$  at the top-right,  $C$  at the bottom-right, and  $D$  at the bottom-left.

**By Vector Calculations:**

- Find  $\vec{B}C$  to  $C$ :
 
$$\begin{pmatrix} 6 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix}$$
- Find  $\vec{AC}$  to  $C$ :
 
$$\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 7 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} -6 \\ -6 \\ -6 \end{pmatrix}$$
- Since  $\vec{B}C$  and  $\vec{AC}$  are parallel, they lie in the same plane.
- Let  $\vec{DC} = \vec{a} + \vec{b}$  where  $\vec{a} = \vec{AC}$  and  $\vec{b} = \vec{BC}$ .
- Then  $\vec{DC} = \begin{pmatrix} -6 \\ -6 \\ -6 \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix}$
- Therefore,  $D(2, 1, 8)$ .

**Alternative by Vectors:**

- The midpoint of  $AC$  is  $(4, 3, 7)$ .
- This must also be the midpoint of  $BD$ :
- $\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ 6 \\ 10 \end{pmatrix} = 2 \times \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 14 \end{pmatrix}$
- Therefore,  $D(2, 1, 8)$ .

**Alternative by Successive Vectors:**

- $\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 6 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 7 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix}$
- $\vec{AD} = \vec{OD} - \vec{OA} = \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix} - \begin{pmatrix} 7 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$
- $\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 7 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} -6 \\ -6 \\ -6 \end{pmatrix}$
- Using  $\vec{AC} = \vec{AD} + \vec{DC}$ , we obtain:
 
$$\begin{pmatrix} -6 \\ -6 \\ -6 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix} + \vec{DC}$$

$$\vec{DC} = \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix}$$
- Therefore,  $D(2, 1, 8)$ .

**(i)**  $\vec{AC} = \begin{pmatrix} -6 \\ -6 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times (-6) = 6\sqrt{3}$

**(ii)**  $|\vec{AC}| = \sqrt{(-6)^2 + (-6)^2 + (-6)^2} = 6\sqrt{3}$

**Diagram:** A 3D coordinate system showing points  $A(7, 6, 10)$ ,  $B(6, 5, 6)$ , and  $C(1, 0, 4)$ . A line  $l$  passes through  $A$  and  $C$ .

**By Vector Calculations:**

- Let  $\vec{AC}(=6\sqrt{3})$ , i.e.  $\vec{p} = (6\sqrt{3})\mathbf{i}$
- $\vec{BP} = \vec{p} - \frac{1}{2}\vec{b} = (6\sqrt{3})\mathbf{i} - (6\sqrt{3})\mathbf{k}$
- $\vec{BP} = (2, -3, -6)$

**By Vector Calculations:**

- $\vec{BP}$  is perpendicular to  $l$ :
 
$$(2-1)(x-6) + (1)(y-1) = 0$$

$$x-6-y+5+2=0$$

$$x+y+2=17$$
- $\vec{BP}$  must satisfy the parametric equations of  $l$ :
 
$$\begin{cases} x=2t+7 \\ y=1+6t \\ z=2t+10 \end{cases}$$
- Solving simultaneously:
 
$$(2t+7) + (1+6t) + (2t+10) = 17$$

$$3t = -6$$

$$t = -2$$
- ∴  $\vec{P}(5, -1, 6)$
- $\therefore |\vec{BP}| = \sqrt{(5-6)^2 + (-1-1)^2 + (6-10)^2} = \sqrt{1+4+16} = \sqrt{21}$

**Alternative by Calculus of Guarding the Square:**

- $\Rightarrow |\vec{BP}| = \sqrt{(6-4)^2 + (5-3)^2 + (6-4)^2} = \sqrt{8}$
- $\Rightarrow |\vec{BP}| = \sqrt{(x-4)^2 + (y-3)^2 + (z-4)^2}$
- $\Rightarrow |\vec{BP}| = \sqrt{(4t-4)^2 + (1+6t-3)^2 + (2t+10-4)^2}$
- $\Rightarrow f(t) = (4t-4)^2 + (1+6t-3)^2 + (2t+10-4)^2$
- $\Rightarrow f'(t) = 32t^2 + 144t + 112$
- $\Rightarrow f'(t) = 64t + 144$

**(iii)**  $f'(t) = 0 \Rightarrow t = -\frac{18}{8} = -2$

**Scaling for 2D:**  $t = -2$ .

So  $\lambda = -2$ .  $\lambda$  is negative.  $f''(t) = 64 > 0$  (NB)

$f''(-2) = 12 - 24 + 16 = 6$

$|BP| = \sqrt{f(-2)} = \sqrt{12} = \sqrt{6}$  As before

**Alternative by Vector Products:**

$\vec{BP} = (2, -3, -6)$

- $\vec{BP} = (2, -3, -6)$
- $d = |\vec{BP}| \sin \theta$
- $d = \frac{1}{\sqrt{3}}(|\vec{BP}|)|\vec{AC}| \sin \theta$
- $\text{Sd} = \frac{1}{\sqrt{3}}(|\vec{BP}|)|\vec{AC}| \sin \theta \sqrt{2}$
- $|\vec{A}B| = \sqrt{\frac{1}{3}(|\vec{BP}|)^2 + |\vec{AC}|^2 - \frac{1}{2}|\vec{BP}|^2|\vec{AC}|^2 \cos^2 \theta}$
- $d = \sqrt{\frac{1}{3}(|\vec{BP}|)^2 + |\vec{AC}|^2 - \frac{1}{2}|\vec{BP}|^2|\vec{AC}|^2 \cos^2 \theta}$
- $d = \sqrt{\frac{1}{3}(1+4+16) + 36 - \frac{1}{2}(1+4+16)(36) \cos^2 \theta}$
- $d = \sqrt{\frac{1}{3}(21) + 36 - \frac{1}{2}(21)(36) \cos^2 \theta}$

**(iv)**  $\text{U} \text{ Using the Area of } ABCD = 2 \times \text{Area of } ABC$

- $= 2 \times \frac{1}{2} \times |\vec{BP}| \times |\vec{AC}|$
- $= \sqrt{6} \times 6\sqrt{3}$
- $= 18\sqrt{2}$
- $= 6\sqrt{3} \times \sqrt{2}$
- $= 18\sqrt{2}$

**Question 126    (\*\*\*)+**

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = \mathbf{i} - 5\mathbf{j} + \lambda(4\mathbf{j} - \mathbf{k})$$

$$\mathbf{r}_2 = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \mu(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- a) Given that  $l_1$  and  $l_2$  intersect at some point  $Q$ , find the position vector of  $Q$ .
- b) Given further that the point  $P$  lies on  $l_1$  and has position vector  $\mathbf{i} + p\mathbf{j} - 3\mathbf{k}$ , find the value of  $p$ .

The point  $T$  lies on  $l_2$  so that  $|\overrightarrow{PQ}| = |\overrightarrow{QT}|$ .

- c) Determine the two possible position vectors for  $T$ .

,  $\boxed{\mathbf{q} = \mathbf{i} - \mathbf{j} - \mathbf{k}}$  ,  $\boxed{p = 7}$  ,  $\boxed{\mathbf{t} = -5\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}}$  or  $\boxed{\mathbf{t} = 7\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}}$

a)

$$\begin{aligned} l_1: & \mathbf{r}_1 = (1, -5, 0) + \lambda(4, 1, -1) = (1, 4\lambda - 5, -\lambda) \\ l_2: & \mathbf{r}_2 = (4, -3, 1) + \mu(3, -2, 2) = (4 + 3\mu, -3 - 2\mu, 2\mu) \end{aligned}$$

EQUATE I COMPONENTS

$$\begin{aligned} 3\mu + 4 &= 1 \\ 3\mu &= -3 \\ \mu &= -1 \end{aligned}$$

USING  $\mu = -1$  IN  $l_2$  WE OBTAIN  $Q(l_1 \cap l_2)$

(NO NEED TO DO ANYTHING ELSE, OR GIVE OTHER COMPONENTS AS THE INTERSECTION IS A GIVEN)

b)

$$(1, 4\lambda - 5) = (1, 4\lambda - 5, -\lambda) \text{ BY INSPECTION. } \lambda = 3$$

$$\therefore p = 4\lambda - 5$$

$$p = 7$$

c)

THE DIRECTION VECTOR OF  $l_1$  IS  $(0, 4, -1)$  & ITS MAGNITUDE IS  $\sqrt{17}$ .  
THE DIRECTION VECTOR OF  $l_2$  IS  $(3, -2, 2)$  & ITS MAGNITUDE IS ALSO  $\sqrt{17}$ .  
FROM  $P(1, -3, 2)$  TO  $Q(1, -1, -1)$  THERE IS  $\lambda^2$  DIFFERENCE IN  $\lambda^2$  OF 2.  
HENCE AS BOTH DIRECTION VECTORS ARE IDENTICAL WE ADD AND REMOVE A DIFFERENCE IN  $\lambda^2$  BY 2.  
 $\therefore \lambda = \pm 1$   
 $\therefore$  AT THE TWO POSSIBLE POSITIONS  $\begin{cases} \lambda = 1 & \Rightarrow T(-5, 3, -5) \\ \lambda = -1 & \Rightarrow T(7, -5, 3) \end{cases}$

ALTERNATIVE FOR PART C

- $P(1, 7, -3)$   $|\overrightarrow{PQ}| = |4 - 1| = |(1, 4\lambda - 5) - (1, 7, -3)| = |(0, 4\lambda - 12)| = \sqrt{16\lambda^2 + 144} = \sqrt{16}$
- $Q(1, 4\lambda - 5)$
- $\overrightarrow{QT} = \frac{1}{2}\mathbf{q} = (2, 1, 2) - (1, 4\lambda - 5)$
- $|\overrightarrow{QT}| = \sqrt{16\lambda^2 + 144}$
- $|\overrightarrow{QT}| = \sqrt{16\lambda^2 + 144} = \sqrt{16}$
- $\sqrt{16\lambda^2 + 144} = \sqrt{16}$
- $16\lambda^2 + 144 = 16$
- $16\lambda^2 = -128$
- $\lambda^2 = -8$
- $\lambda = \pm 2i$
- $\lambda = 2i$  LIES ON  $l_2$
- $\lambda = -2i$  LIES ON  $l_2$
- $\lambda = 2i$   $\Rightarrow$   $\mathbf{q} = (2, 1, 2)$
- $\lambda = -2i$   $\Rightarrow$   $\mathbf{q} = (-2, 1, 2)$
- $\therefore T(-5, 3, -5)$
- $\therefore T(7, -5, 3)$

**Question 127 (\*\*\*\*+)**

The point  $P$  has position vector  $2\mathbf{i} + 2\mathbf{j} + 21\mathbf{k}$ .

- a) Find the vector equation of the straight line  $l$  which passes through  $P$  and is parallel to the vector  $\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ .

The points  $A$  and  $B$  have coordinates  $(-1, 2, 3)$  and  $(2, 5, 3)$ , respectively.

The point  $C$  lies on  $l$  so that the triangle  $ABC$  is equilateral.

- b) Find the two possible position vectors for  $C$ .

$$\boxed{\text{[ ]}}, \boxed{\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + 21\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 5\mathbf{k})}, \boxed{\mathbf{c} = -\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} \text{ or } \mathbf{c} = -\frac{17}{9}\mathbf{i} + \frac{53}{9}\mathbf{j} + \frac{14}{9}\mathbf{k}}$$

a)  $\mathbf{r} = (2, 2, 3) + \lambda(1, -1, 5)$   
 $\underline{\mathbf{r}} = \lambda\mathbf{i} + 2\mathbf{j} - \lambda\mathbf{j} + 2\mathbf{k}$

b)

CALCULATE  
 $|AB| = \sqrt{(2+1)^2 + (5-2)^2 + (3-3)^2} = \sqrt{10}$   
 $|AC| = \sqrt{(x+1)^2 + (y-2)^2 + (z-3)^2} = \sqrt{10}$   
 $|BC| = \sqrt{(x-2)^2 + (y-5)^2 + (z-3)^2} = \sqrt{10}$

NEXT FIND THE POSITION VECTOR ON  $l$  OF THE POINT  $C$ , SO THAT  $|AC| = \sqrt{10}$ . THEN CHECK IT AGAINST  $|BC|$

LET  $C(x, y, z)$   
 $|AC| = \sqrt{(x+1)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x+1)^2 + (y-2)^2 + (z-3)^2} = 10$   
 $\Rightarrow \sqrt{10^2} = \sqrt{(x+1)^2 + (y-2)^2 + (z-3)^2}$   
 $\Rightarrow (x+1)^2 + (y-2)^2 + (z-3)^2 = 10$

BUT  $C(x, y, z)$  LIES ON  $l$   
 $\Rightarrow (2+1)^2 + (x-2)^2 + (z-3)^2 + (5-2)^2 + (y-5)^2 + (z-3)^2 = 10$   
 $\Rightarrow (x+3)^2 + (z-1)^2 + (z+1)^2 + (y-3)^2 = 10$   
 $\Rightarrow x^2 + 6x + 9 + z^2 - 2z + 1 + z^2 + 2z + 1 + y^2 - 6y + 9 = 10$   
 $\Rightarrow 2x^2 + 2y^2 + 18x - 12y + 32 = 10$   
 $\Rightarrow 2x^2 + 2y^2 + 18x - 12y + 22 = 0$

$\Rightarrow 2x^2 + 2y^2 + 18x - 12y + 22 = 0$ 

FACTORISING OR USING THE QUADRATIC FORMULA

 $\Rightarrow (x+3)(x+1) = 0$ 
 $\Rightarrow x = -3 \quad \text{or} \quad x = -1$ 

CHECK  $|BC|$  WITH EACH OF THESE  $x$ 's

- $|BC| = \sqrt{(2+1)^2 + (5-2)^2 + (3-3)^2} = \sqrt{10}$
- $|BC| = \sqrt{(2+1)^2 + (5-2)^2 + (3-3)^2} = \sqrt{(2+1)^2 + (5-2)^2 + (3-3)^2} = \sqrt{10}$

SO BOTH WORK SO  $C(-1, 5, 3)$   
OR  $C(-3, 2, 3)$

**Question 128 (\*\*\*\*+)**

The quadrilateral  $ABCD$  is a rectangle with the vertex  $A$  having coordinates  $(2, 1, 2)$ .

The diagonals of the rectangle intersect at the point with coordinates  $(7, 0, 4)$ .

- a) Find the coordinates of the point  $C$ .

The points  $B$  and  $D$  both lie on the straight line with vector equation

$$\mathbf{r} = 4\mathbf{i} + 15\mathbf{j} + 10\mathbf{k} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}),$$

where  $\lambda$  is a scalar parameter.

- b) Determine the coordinates of  $B$  and  $D$ .

,   $C(12, -1, 6)$ ,   $B(6, 5, 6)$ ,   $D(8, -5, 2)$  in any order

a)

By inspection (slicing-out counting) as  $E$  is the midpoint of  $AC$ .

$$\begin{pmatrix} 2 & \xrightarrow{\text{+5}} & 7 \\ 1 & \xrightarrow{\text{+5}} & 6 \\ 2 & \xrightarrow{\text{+5}} & 4 \end{pmatrix} \quad \therefore C(12, -1, 6)$$

b)

- $|AE| = |e_1 - a| = |(7,0,4) - (2,1,2)| = |5, -1, 2| = \sqrt{25 + 1 + 4} = \sqrt{30}$
- THE DIRECTION VECTOR of  $\ell$  is  $(1, -5, -2) + (1, -5, -2) = \sqrt{30}$
- THE EQUATION of  $\ell$  is

$$\begin{aligned} \Gamma_1 &= (4, 1, 2) + 2(1, -5, -2) \\ \Gamma_2 &= (4 + 2t, 1 - 10t, 2 - 2t) \end{aligned}$$

BY INSPECTION THE VALUE OF  $t$  AT  $E(7,0,4)$  IS 3

- AS  $|AE| = \text{LENGTH OF THE DIRECTION VECTOR OF } \ell$  THEN
- AT THE POINTS  $B$  &  $D$   $t = 2 \pm 1$

$$\begin{aligned} t &= \frac{1}{2} \\ \therefore B(6, 5, 6) \text{ & } D(8, -5, 2) & \text{ IN EITHER ORDER} \end{aligned}$$

ALTERNATIVE FOR PART (b)

$\vec{BC} = \vec{C} - \vec{B} = (2, -7, 4) - (2, 1, 2) = (2, -8, 2)$

$\vec{BD} = \vec{D} - \vec{B} = (2, -7, 4) - (2, 5, 6) = (2, -12, -2)$

$\vec{CB} = \vec{B} - \vec{C} = (2, 5, 6) - (2, -7, 4) = (2, 12, 2)$

$\bullet$  let  $\vec{BC} = \lambda \vec{BD} \Rightarrow \vec{B} = (2, 5, 6)$

$\bullet$   $\vec{BD} = \vec{B} - \vec{D} = (2, 5, 6) - (2, -7, 4) = (2, 12, 2)$

$\bullet$   $\vec{CB} = \vec{B} - \vec{C} = (2, 5, 6) - (2, -7, 4) = (2, 12, 2)$

$\bullet$  BUT POINT  $B$  LIES ON  $\ell$

$$\begin{aligned} \vec{r} &= (4, 1, 2) + \lambda(1, -5, -2) \\ \vec{r} &= (4 + \lambda, 1 - 5\lambda, 2 - 2\lambda) \\ \vec{r} &= (1 + 4\lambda, 1 - 5\lambda, 2 - 2\lambda) \end{aligned}$$

$\begin{cases} x = 1 + 4\lambda \\ y = 1 - 5\lambda \\ z = 2 - 2\lambda \end{cases}$

$\bullet$  SOLVING SIMULTANEOUSLY THE TWO "BOXED" EQUATIONS,

$$(2, 5, 6) = (1 + 4\lambda, 1 - 5\lambda, 2 - 2\lambda) \Rightarrow (1 + 4\lambda, 1 - 5\lambda, 2 - 2\lambda) = (2, 5, 6)$$

$$(2 + 4\lambda)^2 - 100\lambda^2 + 24\lambda^2 + (2 - 2\lambda)^2 - 100\lambda^2 + 24\lambda^2 + (2 - 2\lambda)^2 - 100\lambda^2 + 24\lambda^2 = 0$$

$$32\lambda^2 - 16\lambda + 2 = 0$$

$$16\lambda^2 - 8\lambda + 1 = 0$$

$$(\lambda - 1)(\lambda - 1) = 0$$

$$\lambda = 1 \quad \rightarrow \quad \begin{cases} B(6, 5, 6) \\ D(8, -5, 2) \end{cases}$$

As required

**Question 129** (\*\*\*\*+)

The vectors  $\mathbf{a}$  and  $\mathbf{b}$  are such so that

$$|\mathbf{a}| = 3, |\mathbf{b}| = 12 \text{ and } \mathbf{a} \cdot \mathbf{b} = 18.$$

Show clearly that

$$|\mathbf{a} - \mathbf{b}| = 3\sqrt{13}.$$

□, proof

**METHOD A**

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a} - \mathbf{b}|(|\mathbf{a} - \mathbf{b}|) \cos 0^\circ$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = |\mathbf{a} - \mathbf{b}| |\mathbf{a} - \mathbf{b}|$$

$$\Rightarrow |\mathbf{a}| |\mathbf{a}| \cos 0^\circ - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}| |\mathbf{b}| \cos 0^\circ = |\mathbf{a} - \mathbf{b}|^2$$

$$\Rightarrow |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$$

$$\Rightarrow 3^2 - 2 \times 18 + 12^2 = |\mathbf{a} - \mathbf{b}|^2$$

$$\Rightarrow 9 - 36 + 144 = |\mathbf{a} - \mathbf{b}|^2$$

$$\Rightarrow |\mathbf{a} - \mathbf{b}|^2 = 117$$

$$\Rightarrow |\mathbf{a} - \mathbf{b}| = \sqrt{117} \quad (\because |\mathbf{a} - \mathbf{b}| > 0)$$

$$\Rightarrow |\mathbf{a} - \mathbf{b}| = 3\sqrt{13} \quad \text{As required}$$
  

**METHOD B**

DRAW A DIAGRAM

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \cos 0^\circ \\ |\mathbf{b}| &= 3 \times 12 \times \cos 0^\circ \\ |\mathbf{b}| &= 36 \cos 0^\circ \\ \cos 0^\circ &= \frac{1}{2} \\ \theta &= 60^\circ \end{aligned}$$

BY THE COSINE RULE

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a}^2 + \mathbf{b}^2 - 2|\mathbf{a}||\mathbf{b}|\cos 60^\circ) \\ |\mathbf{a} - \mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}||\mathbf{b}| \\ |\mathbf{a} - \mathbf{b}|^2 &= 9 + 144 - 36 \\ |\mathbf{a} - \mathbf{b}|^2 &= 117 \\ |\mathbf{a} - \mathbf{b}| &= 3\sqrt{13} \quad \text{As required} \end{aligned}$$

**Question 130** (\*\*\*\*+)

It is given that

$$\mathbf{w} = \mathbf{u} + \mathbf{v},$$

where  $\mathbf{w} = 2\mathbf{i} + 8\mathbf{j} - \mathbf{k}$ .

Given further that  $\mathbf{u}$  is in the direction  $\mathbf{i} + \mathbf{j} + \mathbf{k}$ , and the vectors  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular to one another, determine  $\mathbf{u}$  and  $\mathbf{v}$  in component form.

$$[\quad], [\mathbf{u} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}], [\mathbf{v} = -\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}]$$

**METHOD A**

$\mathbf{w} = \mathbf{u} + \mathbf{v}$

REARRANGING THE EQUATION BY  $\mathbf{u}$

$\mathbf{w} - \mathbf{v} = \mathbf{u} + \mathbf{u} + \mathbf{v} - \mathbf{v}$  (PERPENICULAR)

LET  $\mathbf{u} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}), \lambda \neq 0$

$\Rightarrow \begin{pmatrix} 2 \\ 8 \\ -1 \end{pmatrix} - \mathbf{v} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow \lambda(2 + \lambda - 1) = \lambda^2(1 + 1 + 1)$

$\Rightarrow 9\lambda = 3\lambda^2$

$\Rightarrow 3 = \lambda$  ( $\lambda \neq 0$ )

Finally as  $\mathbf{w} = \mathbf{u} + \mathbf{v}$

$\Rightarrow \begin{pmatrix} 2 \\ 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \mathbf{v}$

$\therefore \mathbf{v} = \begin{pmatrix} -1 \\ 5 \\ -4 \end{pmatrix}$

**METHOD B**

START WITH A DIAGRAM WITH  $\mathbf{u} \neq \mathbf{v}$  PERPENDICULAR AND  $\mathbf{w}$  THERE "PROJECT"

PROJECT  $\mathbf{w}$  onto THE DIRECTION OF  $\mathbf{u}$

$d = \mathbf{w} \cdot \hat{\mathbf{u}}$

THIS WE HAVE

$\Rightarrow \mathbf{u} = d\hat{\mathbf{u}}$

$\Rightarrow \mathbf{u} = (\lambda\hat{\mathbf{u}})\hat{\mathbf{u}}$

$\Rightarrow \mathbf{u} = \left[ \begin{pmatrix} 2 \\ 8 \\ -1 \end{pmatrix} \cdot \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{3}} \right] \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{3}}$

$\Rightarrow \mathbf{u} = \frac{1}{\sqrt{3}}(2 + 8 - 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow \mathbf{u} = \frac{1}{3}(2 + 8 - 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow \mathbf{u} = 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow \mathbf{u} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

4. Finally since  $\mathbf{w} = \mathbf{u} + \mathbf{v}$

$(2, 8, -1) = (3, 3, 3) + \mathbf{v}$

$\mathbf{v} = (-1, 5, -4)$

**METHOD C**

LET  $\mathbf{v} = (x, y, z), \mathbf{u} = \lambda(1, 1, 1), \lambda \neq 0$

$\mathbf{v} \perp \mathbf{u} \Rightarrow (\mathbf{v} \cdot \mathbf{u}) \cdot \mathbf{u} = 0$

$\Rightarrow \lambda(x + y + z) = 0$

$\Rightarrow x + y + z = 0$

Now we have  $\mathbf{w} = \mathbf{u} + \mathbf{v}$

$\Rightarrow (2, 8, -1) = (x, y, z) + (\lambda, \lambda, \lambda)$

$\Rightarrow \begin{cases} x + \lambda = 2 \\ y + \lambda = 8 \\ z + \lambda = -1 \end{cases}$

ADDING THESE EQUATIONS

$2\lambda + 2 = 9$

$0 + 3\lambda = 9$

$\lambda = 3$

$\therefore \mathbf{u} = (3, 3, 3) \quad \mathbf{v} = (3, 3, 3) = (-1, 5, -4)$

**Question 131 (\*\*\*\*+)**

The points  $A(3,3,2)$ ,  $B(6,4,3)$  and  $C(5,1,4)$  are referred with respect to a fixed origin  $O$ . The point  $M$  is the midpoint of  $AC$ .

- a) Show that  $\overrightarrow{BM}$  is perpendicular to  $\overrightarrow{AC}$ .

The point  $D$  is such so that  $ABCD$  is a kite with an area of  $6\sqrt{6}$ .

The straight line  $BD$  is a line of symmetry for the kite  $ABCD$ .

- b) Find the coordinates of  $D$ .

$$\boxed{\text{SOL}}, \boxed{D(0,-2,3)}$$

a)

$A(3,3,2)$     $B(6,4,3)$     $C(5,1,4)$

- $M(4,2,3)$  BY INSPECTION
- $\overrightarrow{BM} = m - b = (4,2,3) - (6,4,3) = (-2,-2,0)$
- $\overrightarrow{AC} = c - a = (5,1,4) - (3,3,2) = (2,-2,2)$
- $\overrightarrow{BM} \cdot \overrightarrow{AC} = (-2,-2,0) \cdot (2,-2,2) = -4 + 4 + 0 = 0$  INDEED PERPENDICULAR

b)

FIRSTLY  $|AC| = |(2,-2,2)| = \sqrt{4+4+4^2} = \sqrt{12} = 2\sqrt{3}$   
 $|AM| = |MC| = \sqrt{3}$

AREA OF KITE = AREA OF 2 TRIANGLES  
 $6\sqrt{6} = 2 \times \frac{1}{2} |\overrightarrow{BD}| |\overrightarrow{AM}|$   
 $6\sqrt{6} = |\overrightarrow{BD}| |\overrightarrow{AM}|$   
 $6\sqrt{6} = |\overrightarrow{BD}| \sqrt{3}$   
 $6\sqrt{2}\sqrt{3} = |\overrightarrow{BD}| \sqrt{3}$   
 $|\overrightarrow{BD}| = 6\sqrt{2}$

SET AND EQUATION OF LINE STARTING AT  $B(2,0,0)$  AND "HEADING" IN THE DIRECTION  $\overrightarrow{BM}$ . (IE AT  $M(2,1)$ )

$$\begin{aligned} \Gamma &= b + t \overrightarrow{BM} \\ \Gamma &= (6,4,3) + t(-2,-2,0) \\ \Gamma &= (6-2t, 4-2t, 3) \end{aligned}$$

BTW  $|\overrightarrow{BM}| = |(-2,-2,0)| = \sqrt{4+4+0^2} = \sqrt{8} = 2\sqrt{2}$   
 $|\overrightarrow{BD}| = 6\sqrt{2}$

△ BY INSPECTION  $t=3$  AT D  
 $\therefore D(0,-2,3)$

**Question 132    (\*\*\*)+**

Relative to a fixed origin  $O$ , the straight lines  $l$  and  $m$  have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} p \\ 4 \\ 5 \end{pmatrix} + t \begin{pmatrix} q \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 9 \\ 0 \\ 16 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix},$$

where  $t$  and  $s$  are scalar parameters, and  $p$  and  $q$  are scalar constants.

The point  $A$  is the intersection of  $l$  and  $m$ , and the cosine of acute angle  $\theta$  between  $l$  and  $m$  is  $\frac{1}{3}\sqrt{6}$ .

- a) Find the value of  $p$  and the value of  $q$ , given that  $q$  is a positive integer.
- b) Determine the coordinates of  $A$ .
- c) Find the cosine of the acute angle  $\varphi$  between  $AB$  and  $l$ .
- d) Hence show, without the use of any calculating aid, that

$$\varphi = 2\theta.$$

$$p = 0, q = 2, A(8, 2, 9), \cos \varphi = \frac{1}{3}$$

(a)  $\mathbf{r}_1 = (p, 4, 5) + t(4, -1, 2) = (p+4t, 4-t, 2t+5)$   
 $\mathbf{r}_2 = (9, 0, 16) + s(1, -2, 7) = (9+s, -2s, 7s+16)$

BY DOT PRODUCT:  $(q+4t, 4-t, 2t+5) \cdot (1, -2, 7) = |(q+4t, 4-t, 2t+5)| |(1, -2, 7)| \cos \theta$   
 $\Rightarrow q+4t+4-t = \sqrt{q^2+32t^2+40t+16} \sqrt{1+4s+49s^2} \frac{\sqrt{6}}{3}$   
 $\Rightarrow q+16 = \sqrt{q^2+32t^2+40t+16} \sqrt{1+4s+49s^2}$   
 $\Rightarrow (q+16)^2 = 32(q^2+5)$   
 $\Rightarrow (q^2+20q+256) = 32(q^2+5)$   
 $\Rightarrow 0 = 30q^2 - 32q - 76$   
 By quadratic formula or otherwise  
 $\Rightarrow 0 = (q-2)(30q+38)$   
 $\therefore q = \begin{cases} 2 \\ -\frac{38}{30} \end{cases}$

EQUATE COORDINATES IN THE LINES, FROM THEY INTERSECT

(1):  $p+4t = 9 \quad \text{①}$   
 $4-t = -2s \quad \text{②} \Rightarrow t = 2s+4 \quad \text{so into ①: } p = 9-8s \quad \text{③}$   
 $2t+5 = 7s+16 \quad \text{④}$

$\begin{cases} p+2(2s+4) = 9+8s \\ 2(2s+4)+5 = 7s+16 \end{cases} \Rightarrow \begin{cases} p+4s+8 = 9+8s \\ 4s+8+5 = 7s+16 \end{cases} \Rightarrow \begin{cases} p = -8s+1 \\ s = -1 \end{cases}$   
 $\therefore \begin{cases} s = -1 \\ t = 2 \\ q = 4 \\ p = 16 \end{cases}$

using  $t=2$  (as  $s=-1$ ) we obtain  $A(8, 2, 9)$

(b)  $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (12, 5, 9) - (8, 2, 9) = (4, 3, 0)$

$\cos \theta = \frac{(4, 3, 0) \cdot (1, -2, 7)}{|(4, 3, 0)| |(1, -2, 7)|} = \frac{4-6+0}{\sqrt{25} \sqrt{6}} = \frac{-2}{5\sqrt{6}} = \frac{1}{3}\sqrt{6}$

$\cos \varphi = \frac{(\overrightarrow{AB}) \cdot (\mathbf{d})}{|\overrightarrow{AB}| |\mathbf{d}|} = \frac{(4, 3, 0) \cdot (1, -2, 7)}{\sqrt{25} \sqrt{1+4+49}} = \frac{4-6+0}{5\sqrt{55}} = \frac{-2}{5\sqrt{55}} = \frac{2}{5\sqrt{55}} = \frac{2}{5\sqrt{11}} = \frac{2}{5\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}} = \frac{2\sqrt{11}}{55} = \frac{2}{5\sqrt{5}}$

**Question 133** (\*\*\*)+

The straight lines  $l_1$  and  $l_2$  have the vector equations given below.

$$\mathbf{r}_1 = 3\mathbf{i} + \mathbf{j} + 7\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$$

$$\mathbf{r}_2 = 3\mathbf{i} + 3\mathbf{k} + \mu(2\mathbf{i} - 2\mathbf{j} - \mathbf{k}).$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- a) Show that  $l_1$  and  $l_2$  intersect at some point  $P$  and find its coordinates

The points  $A$  and  $C$  lie on  $l_1$  and the points  $B$  and  $D$  lie on  $l_2$ , such that  $ABCD$  forms a parallelogram.

The point  $A$  has coordinates  $(7, -1, 13)$ .

- b) Find ...

  - ... the coordinates of  $C$ .
  - ... the coordinates of  $B$  and  $D$ , given further that  $|BD| = 12$
  - ... the angle  $\angle BAD$ .

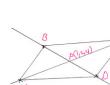
c) Show that the exact area of the parallelogram  $ABCD$  is  $36\sqrt{13}$ .

$$|P(1,2,4)|, |C(-5,5,-5)|, |B(5,-2,2), D(-3,6,6) \text{ in any order}|, \angle BAD \approx 55.3^\circ$$

(3)  $I_1 = (3,1,7) + 2(5,-1,3) = (24+3, -2+14, 21+7)$   
 $I_2 = (3,1,7) + 4(-2,-7,1) = (24+12, -4+28, 7-4)$

• GRAPH  $\begin{cases} x+y=1 \\ x-y=3 \end{cases}$   
 $\begin{array}{l} 1: 2x+3 = 2y+7 \text{ Add} \\ 2: 1-x = -2y \\ \hline 2x+4 = 5 \\ x = -1 \end{array}$   $\left\{ \begin{array}{l} x=-1 \\ y=1 \end{array} \right.$

\* GFG E.  
 $\begin{array}{l} 2x+3 = 3(y+7) = 4 \\ 3-y = 3-(y+4) = 4 \end{array}$   $\rightarrow$  IN COMPONENTS TREAT THE LINES AS EQUATIONS  
 $x = -1$   $\rightarrow$  obt.  $P(1,2,4)$

(4) 

(2) DIRECTION VECTOR OF  $l_1$ :  $B(2,2,-1)$   
 $|B| = \sqrt{2^2+2^2+(-1)^2} = \sqrt{9} = 3$   
 $\therefore |BP| = |PD| = 6$   
 NOW AT P  $\begin{cases} x=-1 \\ y=1 \\ z=-1 \end{cases}$

(1) UR C( $A(0,4,9)$ )  
 $\frac{x-0}{2} = \frac{y-4}{1} = \frac{z-9}{-5}$   
 $\frac{y-4}{1} = 2 \Rightarrow y = 5$   
 $\frac{z-9}{-5} = 2 \Rightarrow z = -5$   
 $\therefore C(5,5,-5)$

Hence At B:  $\begin{cases} x=1 \\ y=5 \\ z=10 \end{cases}$   
 $\begin{array}{l} \mu = -1 \\ \nu = 1 \\ \omega = -2 \end{array}$   
 $\therefore B(5,-2,2)$   
 $D(-6,6)$   
 (IN ANY ORDER)

(II)  $\begin{array}{l} \overrightarrow{AB} = b-a = (-5,-2,1) - (7,-1,0) = (-12,-1,1) \\ \overrightarrow{AD} = d-a = (-5,-2,1) - (7,-1,0) = (-12,-1,1) \end{array}$

Hence  $(-12,-1,1) = (4,-7,-2) = -2(-6,-1,1) = -2(\overrightarrow{AT_1T_2})$   $\therefore \omega = 6$   
 $20+7+77 = \sqrt{4+1+121} \cdot \sqrt{100+49+64} = 105$   
 $90 = \sqrt{105} \cdot 13 \approx 105$   
 $\cos \theta = \frac{20}{\sqrt{105} \cdot \sqrt{13}}$   $\therefore \theta \approx 55.3^\circ$

G) Area of parallelogram =  $2 \times \frac{1}{2} |AB| |AD| \sin B$

$$= |AB| |AD| \sin B$$

$$= \sqrt{10^2 + 8^2} \times \sqrt{10^2 + 2^2} \times \frac{\sqrt{3}}{2}$$


$$= 3\sqrt{10} \times 3\sqrt{2} \times \frac{\sqrt{3}}{2}$$

$$= 18\sqrt{10} \sqrt{2} \sqrt{3} / 4\sqrt{11}$$

$$= \frac{18\sqrt{10} \sqrt{2} \sqrt{3} \sqrt{2} \sqrt{2}}{4\sqrt{11} \sqrt{11}}$$

$$= 36\sqrt{10}$$

As  $\sin 60^\circ = \frac{\sqrt{3}}{2}$

**Question 134 (\*\*\*\*+)**

The straight lines  $l_1$  and  $l_2$  have the following vector equations

$$\mathbf{r}_1 = 7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + a\mathbf{k})$$

$$\mathbf{r}_2 = 3\mathbf{i} + b\mathbf{j} + 5\mathbf{k} + \mu(\mathbf{i} - \mathbf{k}),$$

where  $\lambda$  and  $\mu$  are scalar parameters, and  $a$  and  $b$  are scalar constants.

It is further given that the point  $A$  is the intersection of  $l_1$  and  $l_2$ , and the acute angle between  $l_1$  and  $l_2$  is  $60^\circ$ .

Find in any order ...

... the two possible pairings for the value of  $a$  and the value  $b$ .

... the possible coordinates of  $A$  for each possible pair of  $a$  and  $b$ .

|           |                                       |   |
|-----------|---------------------------------------|---|
| $\square$ | $a = 0$ with $b = 4$ and $A(5, 4, 3)$ | $a = 4$ with $b = \frac{12}{5}$ and $A\left(\frac{33}{5}, \frac{12}{5}, \frac{7}{5}\right)$ |
|-----------|---------------------------------------|---|

$\begin{aligned} l_1 &= (7, 2, 3) + \lambda(1, -1, a) = (3\lambda, 2 - \lambda, 2\lambda + 3) \\ l_2 &= (3, 1, 5) + \mu(1, 0, -1) = (\mu + 3, 1, 5 - \mu) \end{aligned}$

- By the dot product using the fact that the lines intersect at  $60^\circ$ 

$$\Rightarrow (1, -1, a) \cdot (1, 0, -1) = |1, -1, a| |1, 0, -1| \cos 60^\circ$$

$$\Rightarrow 1 + a = \sqrt{1 + a^2} \sqrt{1 + 1} \times \frac{1}{2}$$

$$\Rightarrow 1 + a = \sqrt{1 + a^2} \times \frac{1}{2}$$

$$\Rightarrow 2(1 + a) = \sqrt{1 + a^2} \times 2$$

$$\Rightarrow 4(1 + a)^2 = (1 + a^2) \times 2$$

$$\Rightarrow 4(1 + 2a + a^2) = 2(1 + a^2)$$

$$\Rightarrow 2(a^2 + 2a + 1) = a^2 + 2$$

$$\Rightarrow 2a^2 + 4a + 2 = a^2 + 2$$

$$\Rightarrow 2a^2 + 4a = 0$$

$$\Rightarrow a(a + 4) = 0$$

$$\Rightarrow a < 0$$
- Next we need to find the value of  $b$  & the corresponding point of intersection for each value of  $a$

|   |  |
|---|--|
| $\text{IF } a = 0$<br>$\begin{aligned} l_1 &= (7, 2, 3) \\ l_2 &= (4, 1, 5) \end{aligned}$<br>EQUATING COMPONENTS<br>$\begin{aligned} 3\lambda &= 4 \\ 2 - \lambda &= 1 \\ 2\lambda + 3 &= 5 \end{aligned}$<br>$\begin{cases} \lambda = 1 \\ b = 4 \end{cases}$ | $\begin{aligned} 1 &= 2 - \lambda \\ 1 &= 2 - 1 \\ 1 &= 1 \end{aligned}$<br>A. INTERSECTION IS AT $A(5, 4, 3)$ |
|---|--|

IF  $a = 4$

$\begin{aligned} l_1 &= (3\lambda, 2 - \lambda, 2\lambda + 3) \\ l_2 &= (\mu + 3, 1, 5 - \mu) \end{aligned}$

EQUATING COMPONENTS

$\begin{aligned} \lambda + 3 &= \mu + 3 \\ 2 - \lambda &= 1 \\ 2\lambda + 3 &= 5 - \mu \end{aligned}$

$\begin{cases} \lambda = \mu \\ b = 2 - \lambda \\ 2\lambda + 3 = 5 - \mu \end{cases}$

$\begin{aligned} \lambda &= \mu \\ b &= 2 - \mu \\ 2\mu + 3 &= 5 - \mu \end{aligned}$

$\begin{aligned} 3\mu &= 2 \\ \mu &= \frac{2}{3} \\ b &= 2 - \frac{2}{3} \\ b &= \frac{4}{3} \end{aligned}$

INSPECTION AT  $A\left(\frac{33}{5}, \frac{12}{5}, \frac{7}{5}\right)$

$\therefore A\left(\frac{33}{5}, \frac{12}{5}, \frac{7}{5}\right)$

• THIS WE OBTAIN

$\begin{cases} a < 0 \\ b = \frac{4}{3} \end{cases}$

INSPECTION  $\leftarrow A\left(\frac{33}{5}, \frac{12}{5}, \frac{7}{5}\right)$

**Question 135 (\*\*\*\*+)**

The point  $A$  has position vector  $-\mathbf{i} + 7\mathbf{j} - \mathbf{k}$ .

- a) Find the vector equation of the straight line  $l_1$  which passes through  $A$  and is parallel to the vector  $3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ .

The straight line  $l_2$  has equation

$$\mathbf{r}_2 = 9\mathbf{i} - 9\mathbf{j} + 8\mathbf{k} + \mu(3\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}),$$

where  $\mu$  is a scalar parameter.

- b) Show that ...

- i. ...  $l_1$  and  $l_2$  do not intersect.
- ii. ... the vector  $2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$  is perpendicular to both  $l_1$  and  $l_2$ .

The point  $P$  lies on  $l_1$  and the point  $Q$  lies on  $l_2$  so that the distance  $PQ$  is least.

- c) Find the coordinates of  $P$  and  $Q$ .

$$\boxed{\mathbf{r}_1 = -\mathbf{i} + 7\mathbf{j} - \mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})}, \quad \boxed{P(5, 3, 3) \text{ & } Q(3, -3, 0)}$$

(a)  $\mathbf{l}_1: (-1, 7, -1) + \frac{1}{3}(3, -2, 2) = (3\lambda - 1, 7 - 2\lambda, 2\lambda - 1)$

(b)  $\mathbf{l}_2: (1, -1, 8) + \mu(3, -3, 4) = (3\mu + 1, -3\mu - 1, 4\mu + 8)$

(c) Gradient  $\mathbf{A} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

(d)  $(3\lambda - 1, 7 - 2\lambda, 2\lambda - 1) \perp (3\mu + 1, -3\mu - 1, 4\mu + 8) \Rightarrow 3\lambda + 2\mu + 1 = 0 \Rightarrow \boxed{\lambda = -\frac{1}{2}}$

Hence  $3\lambda + 2\mu + 1 = 0$   
 $3\lambda + 2(-\frac{1}{2}) + 1 = 0$   
 $3\lambda - 1 + 1 = 0$   
 $3\lambda = 0$   
 $\lambda = 0$

Check  $\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$   
 $\bullet 2\lambda - 1 = -1$   
 $\bullet 4\mu + 8 = -8$

$\therefore$  Lines do NOT intersect

(e)  $(3, -2, 2) \cdot (2, 6, 3) = 6 - 12 + 6 = 0$   
 $(3, -3, 4) \cdot (2, 6, 3) = 6 - 18 + 12 = 0$   
 $\therefore (2, 6, 3)$  is perpendicular to both lines

$\bullet \mathbf{P} = (3\lambda - 1, 7 - 2\lambda, 2\lambda - 1) \Rightarrow \lambda = 0$  since  $\mathbf{A} \perp \mathbf{OP}$

$\bullet \mathbf{Q} = (3\mu + 1, -3\mu - 1, 4\mu + 8) \Rightarrow \mu = 0$  since  $\mathbf{A} \perp \mathbf{OQ}$

$\bullet \overrightarrow{PQ} = \mathbf{P} - \mathbf{Q} = (3\lambda - 1, 7 - 2\lambda, 2\lambda - 1) - (3\mu + 1, -3\mu - 1, 4\mu + 8)$

Now  $\overrightarrow{PQ} = \lambda(2, 6, 3)$

$\therefore 3\lambda - 3\mu - 10 = 2\lambda \quad (\times 3)$   
 $-21 + 18 + 16 = 6\lambda \quad (\times 3)$   
 $21 - 4\mu - 9 = 3\lambda \quad (\times 2)$

$3\lambda - 3\mu - 10 = 2\lambda \quad (\times 3)$   
 $-21 + 18 + 16 = 6\lambda \quad (\times 3)$   
 $21 - 4\mu - 9 = 3\lambda \quad (\times 2)$

$9\lambda - 9\mu - 30 = 6\lambda \quad (\times 2)$   
 $-21 + 18 + 16 = 9\lambda \quad (\times 3)$   
 $4\lambda - 8\mu - 18 = 6\lambda \quad (\times 3)$

$9\lambda = 54 - 12$   
 $\therefore -12 + 18 = 6\lambda - 9\lambda$   
 $6 = -3\lambda$   
 $\lambda = -2$

$\therefore \overrightarrow{PQ} = -2(2, 6, 3)$   
 $\overrightarrow{PQ} = (-4, -12, -6)$   
 $\boxed{\overrightarrow{PQ} = \frac{1}{3}(2, 6, 3)}$

$\therefore P(5, 3, 3)$   
 $Q(3, -3, 0)$

**Question 136 (\*\*\*\*\*)**

With respect to a fixed origin  $O$ , the straight lines  $L_1$  and  $L_2$  have respective vector equations

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

where  $t$  and  $s$  are scalar parameters.

The points  $A$  and  $C$  lie on  $L_1$  and  $L_2$ , where  $t=0$  and  $s=0$ , respectively.

- a) Find  $|\overrightarrow{AC}|$ , in exact surd form.
- b) Show that  $L_1$  and  $L_2$  intersect at some point  $B$  and find its coordinates.
- c) Find the size of the angle  $\theta$ , between  $L_1$  and  $L_2$ .

The point  $D$  is such so that  $ABCD$  is a kite.

- d) Show further that ...
  - i. ... the area of the kite is  $16\sqrt{3}$  square units.
  - ii. ... the length of  $BD$  is  $\frac{8}{7}\sqrt{42}$ .

$$|\overrightarrow{AC}| = 2\sqrt{14}, \quad [B(1,1,1)], \quad [\theta = 90^\circ]$$

(a)

$$\begin{aligned} L_1 &= (C_1, S_1) + t(C_2 - C_1) = (C_1, S_1 - 2t, t) \\ L_2 &= (C_2, S_2) + s(C_3 - C_2) = (C_2, S_2 - t, S_2) \\ t=0 &\Rightarrow \underline{\underline{S_1 = C_1, S_1 = 1}} \quad |\overrightarrow{AC}| = |S_1 - S_2| = |(C_2 - C_1, S_2 - S_1)| \\ s=0 &\Rightarrow \underline{\underline{S_2 = C_2, S_2 = 5}} \quad = |(-2, -4, 4)| \\ &= \sqrt{4^2 + (-4)^2 + 4^2} = \sqrt{32} \\ &= 2\sqrt{8} \end{aligned}$$

(b)  $\text{equat } 1 \text{ & } 2$   
 $\begin{cases} 1: t-1 = \frac{s-3}{2} \\ 2: 5-2t = 1 \end{cases} \Rightarrow -2t = -4 \quad \boxed{t=2}$   
 $\begin{cases} \text{check } 3 \\ t-1 = \frac{s-3}{2} \\ 2-1 = \frac{s-3}{2} \\ 2-1 = \frac{s-3}{2} \end{cases}$   
 $\therefore \text{all three components agree}$   
 $\therefore \text{the lines intersect}$   
 $\text{using } t=2 \text{ into } (C_1, S_1 - 2t, t)$   
 $\therefore \text{Gives } B(1,1,1)$

(c)

Determine direction vectors:

$$\begin{aligned} C_1, C_2, 1 &= (1, -2, 1) \\ C_2, C_3, 1 &= (1, 0, -1) \\ \therefore \theta &= \dots \quad \text{Hence dot to zero!} \\ \therefore \theta &= 90^\circ \end{aligned}$$

(d)

Area of kite

$$\begin{aligned} &= 2 \times \text{triangles} = 2 \times \frac{1}{2} |\overrightarrow{AC}| |\overrightarrow{BC}| \\ &= \frac{1}{2} \sqrt{32} \times \sqrt{48} \\ &= 8\sqrt{12} \\ &= 16\sqrt{3} \end{aligned}$$

Final soln

$$\begin{aligned} \frac{1}{2} |\overrightarrow{AC}| |\overrightarrow{MB}| &= 8\sqrt{3} \\ \frac{1}{2} \times \sqrt{64} \times |\overrightarrow{MB}| &= 8\sqrt{3} \\ |\overrightarrow{MB}| &= \frac{8\sqrt{3}}{4} \\ \therefore |\overrightarrow{BD}| &= 2|\overrightarrow{MB}| = \frac{16\sqrt{3}}{4} \\ \therefore |\overrightarrow{BD}| &= \frac{8}{7}\sqrt{42} \quad \text{as required} \end{aligned}$$

**Question 137 (\*\*\*\*+)**

A person standing at a fixed origin  $O$  observes an insect taking off from a point  $A$  on level horizontal ground. The position vector of the insect  $\mathbf{r}$  metres,  $t$  seconds after taking off, is given by

$$\mathbf{r} = (t+1)\mathbf{i} + \left(2t + \frac{1}{2}\right)\mathbf{j} + 2t\mathbf{k}.$$

All distances are in metres and the coordinates axes  $Ox, Oy, Oz$  are oriented due east, due north and vertically upwards, respectively.

a) Find ...

i. ... the bearing of the insect's flight path.

ii. ... the angle between the flight path and the horizontal ground.

The roof top of a garden shed is located at  $B\left(5, \frac{9}{2}, 3\right)$ .

b) Calculate the shortest distance between the insect's path and the point  $B$ .

When the insect reaches a height of 20 metres above the ground, at the point  $C$ , the insect gets eaten by a bird.

c) Determine the coordinates of  $C$ .

$\boxed{\quad}$ , bearing  $\approx 027^\circ$ ,  $\theta \approx 42^\circ$ ,  $\sqrt{5}$ ,  $\boxed{C\left(11, \frac{41}{2}, 20\right)}$

**a)** START BY REWRITING THE EQUATION OF THE LINE

$$\begin{aligned} \mathbf{r} &= [t\mathbf{i} + 1\mathbf{i}, 2t + \frac{1}{2}\mathbf{j}, 2t\mathbf{k}] \\ \mathbf{r} &= [1, 2t + \frac{1}{2}, 2t] \end{aligned}$$

WHEN  $t = 0$ ,  $A(1, \frac{1}{2}, 0)$

LOOKING AT THE MOTION OF THE INSECT FROM  $t = 0$  TO  $t = 1$

View Realistic  
Angle  $\theta = 12^\circ$   
 $\theta = 46^\circ$   
 $\theta = 41.8^\circ$   
 $\theta = 42^\circ$

SIMILARLY  
 $t = 0$  &  $t = 1$

**b)**  $\bullet$  let  $\mathbf{d} = (2t, 0, 0)$

$$\begin{aligned} \mathbf{B}\mathbf{d} &= \mathbf{1} - \mathbf{b} \\ &= (2t, 0, 0) - (5, \frac{9}{2}, 3) \\ &= (2t-5, 0, -\frac{5}{2}) \end{aligned}$$

$\bullet$   $\mathbf{B}\mathbf{d} \perp$  TO THE FLIGHT PATH

$$(2t-5, 0, -\frac{5}{2}) \cdot (2, 0, 2) = 0$$

$$2t-5 + 0 - 5 = 0$$

$$2t-10 = 0$$

$$2t = 10$$

$$t = 5$$

$\bullet$   $\mathbf{B}\mathbf{d}$  LIES ON THE FLIGHT PATH

$$\mathbf{d} = (2t, 0, 0) = (10, 0, 0)$$

$$\begin{aligned} x &= 10 \\ y &= 0 \\ z &= 0 \end{aligned}$$

$\bullet$  SCALING SIMULTANEOUSLY

$$\begin{aligned} \rightarrow (10, 0, 0) \cdot 2(2t+1) + 2(2t) &= 20 \\ \Rightarrow t+1 + 4t+1 + 4t &= 20 \\ \Rightarrow 9t+2 &= 20 \\ \Rightarrow 9t &= 18 \\ \Rightarrow t &= 2 \end{aligned}$$

$\therefore (2, 0, 0) + (5, \frac{9}{2}, 3) = (7, \frac{9}{2}, 3)$

$$\begin{aligned} |\mathbf{B}\mathbf{d}| &= \sqrt{(2-7)^2 + (\frac{9}{2}-0)^2 + (3-0)^2} \\ |\mathbf{B}\mathbf{d}| &= \sqrt{25 + \frac{81}{4}} = \sqrt{56.25} \\ \therefore \text{SHORTEST DISTANCE} &\approx 2.34 \text{ m} \end{aligned}$$

**c)** FINISHED HEIGHT OF 20 METRES  $\Rightarrow z = 20$

$$\begin{aligned} \mathbf{r} &= (1, 0, 0) + (t, 2t + \frac{1}{2}, 2t) \\ (1, 0, 0) + (t, 2t + \frac{1}{2}, 2t) &= (11, 10.5, 20) \\ \begin{matrix} t = 10 \\ t = 10 \end{matrix} & \\ \therefore 2t+1 &= 20.5 = \frac{41}{2} \\ t+1 &= 11 \\ \therefore C(11, \frac{41}{2}, 20) & \end{aligned}$$

**Question 138 (\*\*\*\*+)**

With respect to a fixed origin  $O$ , the points with coordinates  $A(4,3,-1)$ ,  $B(5,1,2)$ ,  $C(2,0,3)$  and  $D(4,2,-1)$  are given.

- Find the vector equation of the line  $l_1$  which passes through  $A$  and  $B$ , and the vector equation of the line  $l_2$  which passes through  $C$  and  $D$ .
- Show that  $l_1$  and  $l_2$  do not intersect.

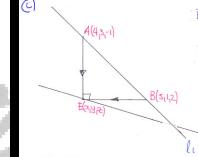
The point  $E$  is on  $l_2$  so that  $\angle AEB = 90^\circ$ .

- Show that one possible position for  $E$  has coordinates  $(\frac{25}{6}, \frac{13}{6}, -\frac{4}{3})$  and find the coordinates of the other possible position.

$$\boxed{\mathbf{r}_1 = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})}, \boxed{\mathbf{r}_2 = 2\mathbf{i} + 3\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - 2\mathbf{k})}, \boxed{E(3,1,1)}$$

(a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (5,1,2) - (4,3,-1) = (1,-2,3)$   
 $\vec{CD} = \mathbf{d} - \mathbf{c} = (4,2,-1) - (2,0,3) = (2,2,-4) \leftarrow \text{SCALAR } 2 \text{ TO } (1,1,-2)$   
 $\mathbf{l}_1 = (4,3,-1) + \lambda(1,-2,3) = (4+\lambda, 3-2\lambda, -1+3\lambda)$   
 $\mathbf{l}_2 = (2,0,3) + \mu(1,1,-2) = (2+\mu, 1+\mu, -3\mu)$

(b) **SOLVE**  $\mathbf{l}_1 \& \mathbf{l}_2$   
 $\begin{cases} 1: & 1+\lambda=1 \\ 2: & 1-2\lambda=1 \\ 3: & -1+3\lambda=-2 \end{cases}$  **SIMPLIFY**  
 $\begin{cases} 1: & \lambda=0 \\ 2: & \lambda=0 \\ 3: & \lambda=\frac{1}{3} \end{cases}$   
 $\lambda = \frac{1}{3}$  **CHECK**  
 $\bullet 2\lambda-1 = 2(\frac{1}{3})-1 = 0$   
 $\bullet -2\lambda+3 = -2(\frac{1}{3})+3 = \frac{7}{3}$   
 $\bullet 0 \neq \frac{7}{3}$   
 $\therefore \text{LINES DO NOT INTERSECT}$

(c)   
 $\vec{AE} = \mathbf{e} - \mathbf{a} = (2,0,3) - (4,3,-1) = (-2,-3,4)$   
 $\vec{BE} = \mathbf{e} - \mathbf{b} = (2,0,3) - (5,1,2) = (-3,-1,-1)$   
 $\vec{AB} = \mathbf{b} - \mathbf{a} = (1,-2,3)$   
 $\vec{AC} = \mathbf{c} - \mathbf{a} = (2,0,3)$   
 $\vec{AD} = \mathbf{d} - \mathbf{a} = (0,2,-4)$   
 $\vec{AE} \cdot \vec{BE} = 0$   
 $(-2,-3,4) \cdot (-3,-1,-1) = (2)(-3) + (-3)(-1) + (4)(-1) = 0$   
 $(2)(2) + (-3)(0) + (4)(-2) = 0$   
 $(2)(0) + (-3)(2) + (4)(-4) = 0$   
 $\therefore \text{E lies on } l_2$

$$\therefore \begin{cases} 6\mu^2 - 19\mu + 13 = 0 \\ (3\mu - 13)(\mu - 1) = 0 \end{cases}$$
  
 $\mu = \begin{cases} 1 \\ \frac{13}{6} \end{cases} \rightarrow E(3,1,1)$   
 $\rightarrow E(\frac{25}{6}, \frac{13}{6}, -\frac{4}{3})$

**Question 139 (\*\*\*\*+)**

Relative to a fixed origin  $O$ , the points  $A$  and  $B$  have respectively position vectors  $\mathbf{i} - 3\mathbf{j} - 9\mathbf{k}$  and  $-4\mathbf{j} - 10\mathbf{k}$ .

- a) Find the vector equation of the straight line  $l_1$  which passes through  $A$  and  $B$ .

The straight line  $l_2$  has the vector equation

$$\mathbf{r}_2 = 6\mathbf{i} + \mathbf{k} + \mu(-\mathbf{i} + p\mathbf{j} + q\mathbf{k}),$$

where  $\mu$  is a scalar parameter, and  $p$  and  $q$  are scalar constants.

- b) Given that  $l_1$  and  $l_2$  are perpendicular, write an equation in terms of  $p$  and  $q$ .
- c) Given further that  $l_1$  and  $l_2$  intersect, find the value of  $p$  and the value of  $q$ .
- d) Determine the position vector of the point of intersection of  $l_1$  and  $l_2$ .

$$\boxed{\mathbf{r}_1 = \mathbf{i} - 3\mathbf{j} - 9\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})}, \boxed{p + q = 1}, \boxed{p = -3, q = 4}, \boxed{7\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}}$$

(a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (\mathbf{c}_1 - \mathbf{c}_2, \mathbf{c}_2 - \mathbf{c}_1) = (-1, -1, -1)$   
 i.e.  $(1, 1, 1) \perp \text{to } l_1, l_2$

$$\begin{aligned} l_1 &= (\mathbf{c}_1 - \mathbf{c}_2) + \lambda(\mathbf{c}_1 - \mathbf{c}_2) = (\lambda + 1, \lambda + 3, \lambda + 9) \\ (\mathbf{c}_1 - \mathbf{c}_2) \cdot (-1, p, q) &= 0 \\ -1 + p + q &= 0 \\ p + q &= 1 \end{aligned}$$

(b)  $\circ$  calculate  $l_1 \perp l_2$

$$\left. \begin{aligned} (1) \lambda + 1 &= 6 - p \\ (2) \lambda + 3 &= p + 9 \\ (3) \lambda + 9 &= pq + 1 \end{aligned} \right\} \Rightarrow \boxed{p + 1 = q} \quad \begin{aligned} \lambda + 1 &= 6 - p \\ \lambda - 3 &= p + 9 \\ \lambda - 9 &= pq + 1 \end{aligned} \quad \begin{aligned} \lambda + 1 &= 6 - p \\ \lambda - 3 &= p + 9 \\ \lambda - 9 &= pq + 1 \end{aligned}$$

ADD THE LAST TWO

$$\begin{aligned} 3\lambda - 11 &= 7 \\ 3\lambda &= 18 \\ \lambda &= 6 \end{aligned} \quad \begin{aligned} \therefore \lambda + 1 &= 6 - p \\ 7 &= 6 - p \\ p &= -1 \end{aligned} \quad \begin{aligned} \therefore \lambda - 9 &= pq + 1 \\ 6 - 9 &= pq + 1 \\ q &= 4 \end{aligned}$$

$$\therefore p = 1 - q \quad \boxed{p = -3}$$

(d) using  $2\lambda + 6$  in  $(\lambda + 1, \lambda + 3, \lambda + 9)$  we get  $(2, 3, 3)$

**Question 140 (\*\*\*)+**

The straight lines  $l_1$  and  $l_2$  have respective vector equations

$$\mathbf{r}_1 = 3\lambda \mathbf{i} + (6 - 2\lambda) \mathbf{j} + (2\lambda + 1) \mathbf{k}$$

$$\mathbf{r}_2 = (3\mu + 10) \mathbf{i} + (-3\mu - 10) \mathbf{j} + (4\mu + 10) \mathbf{k}$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- a) Show that  $l_1$  and  $l_2$  do not intersect.

The point  $P$  lies on  $l_1$  and the point  $Q$  lies on  $l_2$  so that the distance  $PQ$  is least.

- b) Find the coordinates of  $P$  and the coordinates of  $Q$ .

|              |   |               |
|--------------|---|---------------|
| $P(6, 2, 5)$ | & | $Q(4, -4, 2)$ |
|--------------|---|---------------|

(a)

$$\begin{aligned} l_1 &= (3\lambda - 2, 2\lambda + 1) = (3, 0, 1) + \lambda(3, -2, 2) \\ l_2 &= (3\mu + 10, -3\mu - 10, 4\mu + 10) = (3, -3, 4) + \mu(3, -3, 4) \end{aligned}$$

• Equate 2. 4.1

$$\begin{cases} (1) : 3\lambda - 2 = 3\mu + 10 \\ (2) : 2\lambda + 1 = -3\mu - 10 \end{cases} \quad \begin{array}{c} \text{Add} \\ (1) + (2) \end{array} \quad \begin{array}{c} 3\lambda + 2\lambda = 12 \\ 5\lambda = 12 \end{array} \quad \begin{array}{c} \text{Divide by } 5 \\ \lambda = \frac{12}{5} \end{array}$$

• Check E:

$$\begin{cases} 2\lambda + 1 = 2(\frac{12}{5}) + 1 = -11 \\ 4\mu + 10 = 4(\frac{12}{5}) + 10 = -\frac{38}{5} \end{cases} \quad \therefore \text{LINES DO NOT INTERSECT.}$$

(b)

Let  $P = (3\lambda - 2, 2\lambda + 1)$   
 $Q = (3\mu + 10, -3\mu - 10, 4\mu + 10)$

Find  $\vec{PQ} = \vec{P} - \vec{Q}$   
 $\vec{PQ} = (3\lambda - 3\mu - 12, 2\lambda + 3\mu + 9, 2\lambda - 4\mu - 1)$

$\vec{PQ} \cdot (3, -3, 4) = 0$   
 $\vec{PQ} \cdot (3, -2, 2) = 0$

Hence  
 $(3\lambda - 3\mu - 12, 2\lambda + 3\mu + 9, 2\lambda - 4\mu - 1) \cdot (3, -3, 4) = 0$   
 $9\lambda - 9\mu - 36 - 9\lambda + 6\lambda + 36 + 8\lambda - 12\mu - 4 = 0$   
 $[2\lambda - 3\mu] = 14$   
 $(3\lambda - 3\mu - 10, 2\lambda + 3\mu + 9, 2\lambda - 4\mu - 1) \cdot (3, -2, 2) = 0$   
 $9\lambda - 9\mu - 30 - 6\lambda - 18 + 12 + 4\lambda - 8\mu - 2 = 0$   
 $[2\lambda - 2\mu] = 80$

Hence  
 $(2\lambda - 3\mu) = 14 \quad (x 3) \quad 36\lambda - 57\mu = 192$   
 $(2\lambda - 2\mu) = 80 \quad (x 3) \quad 36\lambda - 57\mu = 1920$   
 $57\mu = 1920 - 192 \quad \therefore \mu = 32$   
 $\lambda = \frac{14}{3} = 2$

Hence  $P(6, 2, 5)$  &  $Q(4, -4, 2)$

**Question 141 (\*\*\*)+**

Relative to a fixed origin  $O$ , the straight lines  $l_1$  and  $l_2$  have the following respective vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 8 \\ q \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ p \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where  $\lambda$  and  $\mu$  are scalar parameters, and  $p$  and  $q$  are scalar constants.

- a) Given that  $l_1$  and  $l_2$  are perpendicular, determine the value of  $p$ .

The point  $D$  is the intersection of  $l_1$  and  $l_2$ .

- b) Find the value of  $q$  and the coordinates of  $D$ .

Another straight line  $l_3$  intersects **both**  $l_1$  and  $l_2$ , and is also perpendicular to **both**  $l_1$  and  $l_2$ .

- c) Find a vector equation for  $l_3$ .

*You may not use the vector (cross) product in this part*

The points  $A(8,1,-3)$ ,  $B(8,1,0)$  and  $C(8,-1,-1)$  lie on  $l_1$ ,  $l_2$  and  $l_3$ , respectively.

- d) Show that the volume of the triangle based pyramid with vertices at  $A$ ,  $B$ ,  $C$  and  $D$  is 1 cubic unit.

,  $p = -2$  ,  $q = 1$  ,  $D(7,0,-1)$  ,  $\mathbf{r} = 7\mathbf{i} - \mathbf{k} + v(\mathbf{i} - \mathbf{j})$

a) FINDING THE DIRECTION VECTORS

$$\begin{aligned} & (1,1,p) \cdot (1,1,1) = 0 \\ & \Rightarrow 1+1+p=0 \\ & \Rightarrow p=-2 \end{aligned}$$

b)  $\begin{aligned} l_1 &= (8,q,-3) + \lambda(1,1,-2) = (8+q, 1+q, -3-2\lambda) \\ l_2 &= (3,-4,-5) + \mu(1,1,1) = (3+\mu, -4+\mu, -5+\mu) \end{aligned}$

equate i & j

$$\begin{aligned} 1 &= 8+q+4\lambda \quad \text{SUBTRACT} \quad \Rightarrow 5\lambda+11=8 \\ 1 &= -3+2\mu-4-\mu \quad \Rightarrow 3\lambda+1=3 \\ & \Rightarrow \frac{5\lambda}{5}=4 \quad \Rightarrow \lambda=\frac{4}{5} \end{aligned}$$

equate k

$$\begin{aligned} 1 &= q+4 \\ -1 &= 4+4 \\ q &= 4 \end{aligned}$$

c) NEED THE DIRECTION OF  $l_3$  WHICH IS PERPENDICULAR TO  $l_1$  &  $l_2$

$$\begin{aligned} (3,q,-3) \cdot (1,1,1) &= 0 \\ (3,q,-3) \cdot (1,1,-2) &= 0 \end{aligned}$$

$$\begin{aligned} & \Rightarrow 3+q+3=0 \\ & \Rightarrow 3+q+2\lambda=0 \\ & \cancel{3+2\lambda=0} \quad \Rightarrow 3+q+2\lambda=0 \\ & \Rightarrow \frac{3+q}{2}=0 \quad \Rightarrow \frac{3+q}{2}+2\lambda=0 \\ & \Rightarrow \frac{3+q}{2}+2\lambda=0 \quad \Rightarrow \frac{3+q}{2}+2\lambda=0 \end{aligned}$$

c) FINDING THE MUTUAL PERPENDICULAR DIRECTION  $(1,1,0)$  AND THE POINT  $D(7,0,-1)$

$$\begin{aligned} l_1 &= (7,0,-1) + t(1,1,0) \\ l_2 &= (7,0,-1) + t(1,1,1) \end{aligned}$$

d) CALCULATING THE RELATIVE LENGTHS

$$\begin{aligned} |AD| &= \sqrt{1^2+1^2+2^2} = \sqrt{6} \\ |BD| &= \sqrt{1^2+1^2+1^2} = \sqrt{3} \\ |DC| &= \sqrt{1^2+1^2+1^2} = \sqrt{3} \end{aligned}$$

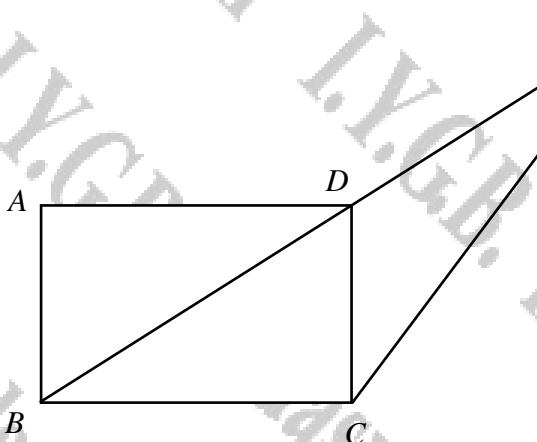
AREA OF A TRIANGLE =  $\frac{1}{2} \times \text{base} \times \text{height}$

$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2} \times \sqrt{3} \times \sqrt{6} \\ &= \frac{1}{2} \times \sqrt{3} \times \sqrt{2} \times \sqrt{3} \\ &= \frac{1}{2} \times \sqrt{6} \\ &= \frac{1}{2} \times \sqrt{6} \end{aligned}$$

VOLUME =  $\frac{1}{3} \times \text{base area} \times \text{height}$

$$\begin{aligned} \text{Volume} &= \frac{1}{3} \times \frac{1}{2} \times \sqrt{6} \times \sqrt{3} \\ &= \frac{1}{6} \times \sqrt{18} \\ &= \frac{1}{6} \times 3\sqrt{2} \\ &= \frac{1}{2} \times \sqrt{2} \\ &= 1 \text{ cubic unit} \end{aligned}$$

**Question 142** (\*\*\*\*\*)



The figure above shows the rectangle  $ABCD$ , where  $C(3,7,12)$  and  $D(5,1,4)$ .

The point  $E(2,1,0)$  is such so that  $BDE$  and  $EC$  are straight lines.

Use vector methods to determine the coordinates of  $A$ .

,  $A(19, -5, 12)$

SOLVING BY THE DIAGRAM

DETERMINE THE COORDINATES OF  $B - D - E$

$$\vec{DE} = \underline{s - d} = (2, 1, 0) - (5, 1, 4) = (-3, 0, -4)$$

[ $\underline{CON} = (3, 0, 4)$ ]

$$\underline{s} = (2, 1, 0) + 2(3, 0, 4)$$

$$(2, 1, 0) = (3, 4, 8)$$

NOW LET  $B(3, 4, 8)$

$$\vec{CE} = \underline{b - c} = (3, 4, 8) - (3, 7, 12) = (2, -3, 4, -4, -2)$$

$$\vec{CB} = \underline{d - c} = (5, 1, 4) - (3, 7, 12) = (2, -6, -8)$$

? Scale to  $(1, -3, -4)$

BY THE DOT PRODUCT

$$(2, -3, 4, -4, -2) \cdot (1, -3, -4) = 0$$

$$2 \cdot 1 - 3 \cdot (-3) - 4 \cdot (-4) - 4 \cdot (-2) = 0$$

$$2 - 3 + 12 - 16 + 8 = 0$$

$$2 - 3 + 12 = 6$$

BY  $(2, 1, 0) = (3, 4, 8)$

$$\Rightarrow \underline{\alpha - 3j - 4k = -2i}$$

$$\Rightarrow (3, 4, 8) - 3j - 4k = -2i$$

$$\Rightarrow 3i + 2 - 3 - 4k = -2i$$

$$\Rightarrow -3i + 1 = -2i$$

$$\Rightarrow 3i = 1$$

$$\Rightarrow i = \frac{1}{3}$$

THUS,  $B(3, 4, 8) = B(7, 1, 12)$  AND  $A$  CAN BE FOUND BY INSPECTION

$\therefore A(19, -5, 12)$

**Question 143 (\*\*\*\*\*)**

The coordinates in this question are relative to a fixed origin  $O$  at  $(0,0,0)$ .

The straight line  $l_1$  has vector equation

$$\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} + \mathbf{k}),$$

where  $\lambda$  is a scalar parameter.

The straight line  $l_2$  passes through the point with coordinates  $(6,0,6)$  and is in the direction  $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ .

- a) Verify that  $A(4,3,5)$  is the intersection of  $l_1$  and  $l_2$ , and show further that  $B(12,-9,9)$  lies on  $l_2$ .

The point  $C(6,-3,3)$  lies on  $l_1$ .

The straight line  $l_3$  passes through  $B$  and  $C$ .

The straight line  $l_4$  is parallel to  $l_2$  and passes through  $C$ .

The straight line  $l_5$  is perpendicular to  $l_3$  and passes through  $A$ .

- b) Given that  $l_4$  and  $l_5$  intersect at the point  $D$ , find the coordinates of  $D$ .

,  $D(4,0,2)$

a)  $\begin{cases} l_1: \mathbf{r}_1 = (3,0,6) + \lambda(-1,3,1) = (3-\lambda, 3+3\lambda, 7+\lambda) \\ l_2: \mathbf{r}_2 = (6,0,6) + \mu(2,-3,1) = (2\mu+6, -3\mu+6, 6+\mu) \end{cases}$

• CHECKING THE POINT  $(4,3,5)$  BY INSPECTION

- IF  $\lambda = -1$   $[3 - (-1), 3(-1) + 6, -1 + 6] = (4,3,5)$
- IF  $\mu = -1$   $[2(-1) + 6, -3(-1) + 6, 6 - 1] = (4,3,5)$

$\therefore A(4,3,5)$  IS THE INTERSECTION OF  $l_1$  AND  $l_2$

• CHECKING  $B(12,-9,9)$

- $2\mu + 6 = 12 \quad \mu = 3$
- $-3\mu + 6 = -9 \quad \mu = 3$
- $6 + \mu = 9 \quad \mu = 3$

$\therefore B(12,-9,9)$  LIES ON  $l_2$

b) START BY DRAWING A GOOD DIAGRAM

FIND AN EQUATION OF  $l_4$  (PARALLEL TO  $l_2$  & PASSING THROUGH C)  
 $\mathbf{r}_4 = (6,-3,3) + t(2,-3,1) = (6+2t, -3-3t, 3+t)$

NEXT FIND THE DIRECTION VECTOR OF  $l_3$  & IT FINDS  $\overrightarrow{BC}$   
 $\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = (6,-3,3) - (12,-9,9) = (-6,6,-6)$   
 $\downarrow$   
 $\therefore \mathbf{d} = (1,-1,1)$

NEXT LET THE POINT  $D(x,y,z)$  HAVE POSITION VECTOR  $\mathbf{d} = (x,y,z)$

 $\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = (x,y,z) - (4,3,5) = (x-4, y-3, z-5)$ 

BOT  $\overrightarrow{AD}$  IS PERPENDICULAR TO  $l_3$

 $\Rightarrow \overrightarrow{AD} \cdot (\text{direction vector of } l_3) = 0$ 
 $\Rightarrow (x-4)(-1) + (y-3)(3) + (z-5)(1) = 0$ 
 $\Rightarrow x-4-y+3+z-5 = 0$ 
 $\Rightarrow x-y+z = 6$ 

BUT  $D(4,0,2)$  LIES ON  $l_4$ , SO THE COORDINATES MUST SATISFY THE PARAMETRIC EQUATIONS OF  $l_4$

$$\begin{cases} x = 6+2t \\ y = -3-3t \\ z = t+2 \end{cases}$$

SOLVING SIMULTANEOUSLY THE LAST TWO SETS OF EQUATIONS

$$\begin{aligned} &\Rightarrow (2t+6) - (-3t-3) + (t+2) = 6 \\ &\Rightarrow 6t+6+3t+3 = 6 \\ &\Rightarrow 9t+9 = 6 \\ &\Rightarrow 9t = -3 \\ &\Rightarrow t = -\frac{1}{3} \end{aligned}$$

$\therefore D\left(\frac{11}{3}, -\frac{10}{3}, \frac{1}{3}\right)$

**Question 144 (\*\*\*\*\*)**

Relative to a fixed origin  $O$  at  $(0,0,0)$  the points  $A$ ,  $B$  and  $C$  have coordinates  $(0,4,6)$ ,  $(3,5,4)$  and  $(2,0,0)$ , respectively.

- The straight line  $l_1$  passes through  $A$  and  $B$ .
- The straight line  $l_2$  passes through  $C$  and is parallel to  $l_1$ .
- The point  $D$  lies on  $l_1$  so that  $\angle ACD = 90^\circ$ .
- The point  $E$  lies on  $l_2$  so that  $\angle CDE = 90^\circ$ .
- The point  $F$  lies on  $l_2$  so that  $|EC| = 2|EF|$ .

Determine the coordinates of the possible positions of  $F$ .

$$\boxed{\quad}, \boxed{F(8,2,-4) \cup F(20,6,-12)}$$

• CONST WITH A GOOD Diagram - NOTE THAT "ACED" IS A PARALLELOGRAM

• FIND EQUATION OF  $l_1$

- $\vec{AB} = b-a = (3,5,4)-(0,4,6) = (3,1,-2)$
- $\vec{l}_1 = (6,4,4) + \lambda(3,1,-2)$
- $\vec{l}_1 = (3\lambda, 4+4\lambda, 4-2\lambda)$

• FIND VECTOR/DIRECTION OF AC

$$\vec{AC} = c-a = (2,0,0)-(0,4,6) = (2,-4,-6)$$

SCALE IT TO  $(-4,2,3)$

• NEXT FIND THE COORDS OF D( $2\lambda, 4+4\lambda, 4-2\lambda$ )

- $\vec{CD} = d-c = (2\lambda, 4+4\lambda, 4-2\lambda) - (2,0,0) = (2-2\lambda, 4+4\lambda, 4-2\lambda)$
- $\vec{CB} \perp \vec{AC} \Rightarrow (2-2\lambda, 4+4\lambda, 4-2\lambda) \cdot (-4,2,3) = 0$
- $-4(2-2\lambda) + 2(4+4\lambda) + 3(4-2\lambda) = 0$
- $-8 + 8\lambda + 8 + 8\lambda + 12 - 6\lambda = 0$
- $2 - 2\lambda = 0 \Rightarrow \lambda = 1$

• BUT  $(2,4,2)$  MUST SATISFY  $\vec{l}_1$

|                  |                  |
|------------------|------------------|
| $x = 2\lambda$   | $y = 4+4\lambda$ |
| $z = 4-2\lambda$ |                  |

$$\Rightarrow 2\lambda = 2(1)=2$$

$$\Rightarrow 6\lambda = 2\lambda - 8 = 10 + 6\lambda = 2$$

$$\Rightarrow 7\lambda = 28$$

$$\Rightarrow \lambda = 4$$

$\therefore D(8,2,-4)$

• NEXT FIND THE COORDINATES OF E

|                        |                          |
|------------------------|--------------------------|
| $0 \xrightarrow{+2} 2$ | $12 \xrightarrow{-2} 10$ |
| $4 \xrightarrow{-2} 0$ | $8 \xrightarrow{-2} 6$   |
| $6 \xrightarrow{-2} 0$ | $-2 \xrightarrow{-2} -4$ |

$\therefore E(14,4,-8)$

• FIND THE MIDPOINT OF C( $2,0,0$ ) & E( $14,4,-8$ ) BY INSPECTION

$$(8,2,-4)$$

• FINALLY

|                         |                          |                           |                            |
|-------------------------|--------------------------|---------------------------|----------------------------|
| $2 \xrightarrow{+2} 4$  | $8 \xrightarrow{-2} 6$   | $14 \xrightarrow{-2} 12$  | $20 \xrightarrow{-2} 18$   |
| $0 \xrightarrow{+2} 2$  | $12 \xrightarrow{-2} 10$ | $4 \xrightarrow{-2} 2$    | $6 \xrightarrow{-2} 4$     |
| $0 \xrightarrow{-2} -4$ | $6 \xrightarrow{-2} -2$  | $-8 \xrightarrow{-2} -10$ | $-12 \xrightarrow{-2} -14$ |

$\therefore$  EITHER  $(8,2,-4)$  OR  $(20,6,-12)$

**Question 145 (\*\*\*\*\*)**

With respect to a fixed origin  $O$ , the points  $A$ ,  $B$  and  $C$  have position vectors

$$\mathbf{a} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 8 \\ 2 \\ 7 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 11 \\ 0 \\ 1 \end{pmatrix}.$$

- a) Determine the volume of the cube, with vertices the points  $A$ ,  $B$  and  $C$ .

The points  $P$ ,  $Q$  and  $R$  are vertices of a different cube, so that

$$\overrightarrow{PQ} = \begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix} \text{ and } \overrightarrow{PR} = \begin{pmatrix} k \\ 4 \\ 3 \end{pmatrix},$$

where  $k$  is a positive constant.

- b) Given that  $\angle QPR = 60^\circ$ , determine ...

- i. ... the value of  $k$ .  
 ii. ... the length of the diagonal of the second cube.

,  $\boxed{\text{volume} = 343}$  ,  $\boxed{k = 5}$  ,  $\boxed{\sqrt{75} = 5\sqrt{3}}$

|   |  |   |
|---|--|---|
| <p>a) AS WE DO NOT KNOW THE LOCATION OF THE CUBES, WE NEED TO FIND ALL THE POSSIBLE LENGTHS BETWEEN <math>A</math>, <math>B</math> &amp; <math>C</math></p> $ \overrightarrow{AB}  =  \mathbf{b} - \mathbf{a}  =  (8, 2, 7) - (0, 5, 2)  =  8, 3, 5  = \sqrt{64 + 9 + 25} = \sqrt{98}$ $ \overrightarrow{AC}  =  c - a  =  (11, 0, 1) - (0, 5, 2)  =  11, -5, -1  = \sqrt{21 + 25 + 1} = \sqrt{47}$ $ \overrightarrow{BC}  =  c - b  =  (11, 0, 1) - (8, 2, 7)  =  3, -2, -6  = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$ <p>THUS THE CONFIGURATION IS AS OPPOSITE<br/>     ∵ SIDE LENGTH IS 7 UNITS<br/> <math>\therefore</math> VOLUME = <math>7 \times 7 \times 7</math><br/> <math>= 49 \times 7</math><br/> <math>= 343</math> ✓</p> <p>b) i) DRAWING THE SECOND CUBE</p> <p>BY THE DOT PRODUCT</p> $\Rightarrow \overrightarrow{PQ} \cdot \overrightarrow{PR} =  \overrightarrow{PQ}   \overrightarrow{PR}  \cos 60^\circ$ $\Rightarrow (0, 1, 7) \cdot (k, 4, 3) = \sqrt{1+49} \sqrt{k^2+16} \cos 60^\circ$ $\Rightarrow 0+4+21 = \sqrt{50} \sqrt{k^2+16} \frac{1}{2}$ $\Rightarrow 25 = \sqrt{50} \sqrt{k^2+16} \frac{1}{2}$ $\Rightarrow 50 = \sqrt{50} \sqrt{k^2+25}$ $\Rightarrow \frac{50}{\sqrt{50}} = \sqrt{k^2+25}$ $\Rightarrow \frac{50\sqrt{50}}{50} = k^2+25$ | $\Rightarrow k^2 + 25 = 50$ $\Rightarrow k^2 = 25$ $\Rightarrow k = 5$ ✓ | <p>ii) LOOKING AT THE PENTAGONAL PRISM ON THE "FLOOR"</p> <p><math> \overrightarrow{PT}  =  \overrightarrow{QT}  =  \overrightarrow{RT}  = \sqrt{50}</math></p> <p>NOW LOOKING AT <math>\triangle PRT</math> ON THE "ROOF"</p> $\begin{aligned} \text{PT}^2 + \text{RT}^2 &= (\sqrt{50})^2 \\ 25 + 25 &= 50 \\ x^2 &= 50 \\ x &= \sqrt{50} \\ \text{SIDE LENGTH} &= \sqrt{50} \end{aligned}$ <p>∴ LENGTH OF THE LONGEST DIAGONAL IS <math>\sqrt{5^2 + 5^2 + 5^2}</math><br/> <math>= \sqrt{75}</math><br/> <math>= 5\sqrt{3}</math> ✓</p> |
|---|--|---|

**Question 146 (\*\*\*\*\*)**

Relative to a fixed origin  $O$  at  $(0,0,0)$  the points  $A$ ,  $B$  and  $C$  have coordinates  $(1,2,5)$ ,  $(-1,0,7)$  and  $(4,-2,8)$ , respectively.

The point  $D$  is such so that  $ABCD$  is an isosceles trapezium with  $|BC|=|AD|$ .

Determine the coordinates of  $D$ .

$$\boxed{\quad}, \boxed{D\left(\frac{14}{3}, -\frac{4}{3}, \frac{22}{3}\right)}$$

- $\vec{AB} = b - a = (-1,0,7) - (1,2,5) = (-2,-2,2)$
- USE QM  $C(1,-1)$  AS A DIRECTION, WE CAN FIND AN EQUATION OF  $L_2$ 

$$L_2 = (4,-2,8) + \lambda(1,-1,1)$$

$$L_2 = (4+1\lambda, -2-\lambda, 8+\lambda)$$
- FIND THE LENGTH OF  $BC$ 

$$|BC| = |s - k| = |(4,-2,8) - (-1,0,7)| = |5,-2,1|$$

$$= \sqrt{25+4+1} = \sqrt{30}$$
- NOW  $|AD| = \sqrt{30}$  WITH  $D(x,y,z)$  LYING ON  $L_2$ 

$$\rightarrow |\vec{AD}| = \sqrt{30}$$

$$\rightarrow |d - a| = \sqrt{30}$$

$$\rightarrow |(x,y,z) - (1,2,5)| = \sqrt{30}$$

$$\Rightarrow |x-1, y-2, z-5| = \sqrt{30}$$

$$\Rightarrow \sqrt{(x-1)^2 + (y-2)^2 + (z-5)^2} = \sqrt{30}$$

$$\Rightarrow (x-1)^2 + (y-2)^2 + (z-5)^2 = 30$$

$$\Rightarrow [(3+4)-1]^2 + [(1-2)-2]^2 + [(0-5)-5]^2 = 30$$

$$\Rightarrow (2+5)^2 + (-1-1)^2 + (-3-2)^2 = 30$$

$$\Rightarrow \begin{pmatrix} 7^2 + 6 & 1 & 9 \\ 7^2 - 2 & 1 & 6 \end{pmatrix} = 30$$

$$\Rightarrow 49 + 36 + 9 = 30$$

$$\Rightarrow 94 = 30$$

$$\Rightarrow 94 - 30 = 0$$

$$\Rightarrow (3+2)(\lambda-2) = 0$$

$$\Rightarrow \lambda = \frac{2}{3}$$

- NOW  $D(6,6)$  OR  $D\left(\frac{14}{3}, -\frac{4}{3}, \frac{22}{3}\right)$
- $|\vec{AB}| = |b - a| = |2,-2,2| = \sqrt{4+4+4} = \sqrt{12}$
- $|CD| = |d - a| = |(6,6) - (4,-2,8)| = |2,2,-2| = \sqrt{12}$
- $\therefore D$  LIES ON  $L_2$  AS THIS VECTORS ARE PARALLEL/SEGMENT
- $\therefore D\left(\frac{14}{3}, -\frac{4}{3}, \frac{22}{3}\right)$

**Question 147 (\*\*\*\*\*)**

Relative to a fixed origin  $O$ , the points  $A$  and  $B$  have position vectors  $4\mathbf{i} + 5\mathbf{j} + 8\mathbf{k}$  and  $6\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$ , respectively. The straight line  $l_1$  passes through  $A$  and  $B$  and crosses the  $y-z$  plane at the point  $C$ . The straight line  $l_2$  passes through the point  $D$  with position vector  $p\mathbf{j} + (2p+2)\mathbf{k}$ , where  $p$  is a scalar constant.

Given that  $l_1$  and  $l_2$  are perpendicular, and intersect at  $C$ , find the value of  $p$ .

$$\boxed{\phantom{00}}, \boxed{p=5}$$

• FIND THE VECTOR EQUATION OF  $\ell_1$

$$\vec{r}_1 = \mathbf{b} - \mathbf{a} = (6, 6, 7) - (4, 5, 8) = (2, 1, -1)$$

$$\vec{r}_1 = (4, 5, 8) + \lambda(2, 1, -1)$$

$$\ell_1 = (2\lambda + 4, 5\lambda + 5, 8 - \lambda)$$

• NEXT FIND THE CO-ORDINATES OF THE POINT C "CROSSES THE y-z PLANE"  $\Rightarrow z=0$

$$\begin{aligned} &\Rightarrow 2\lambda + 4 = 0 \\ &\Rightarrow 2\lambda = -4 \\ &\Rightarrow \lambda = -2 \\ &\Rightarrow C(6, 3, 10) \end{aligned}$$

• NEXT WRITE AN EXPRESSION FOR THE EQUATION OF  $\ell_2$

$$\begin{aligned} \vec{r}_2 &= (\mathbf{a}_1, p, 2p+2) + \mu(\mathbf{a}_1, b_1, c) \\ \vec{r}_2 &= (p, p, p, 2p+2) \end{aligned}$$

• NOW  $\ell_1 \perp \ell_2$  INTERSECT AT C(6, 3, 10)

$$\begin{aligned} i: \mu a &= 0 \\ j: \mu b + p &= 3 \\ k: \mu c + 2p + 2 &= 10 \end{aligned} \quad \left. \begin{array}{l} \text{looking at } i \\ \text{if } \mu = 0, \mu \neq 0 \\ \text{otherwise, } \mu = 3 \\ \text{and is constant} \\ \text{yields } B \neq 10 \end{array} \right\}$$

$$\therefore a = 0$$

• UNDER ARE ALSO PREPARED, SO THEIR DIRECTION VECTORS MUST DOT TO ZERO

$$(\mathbf{a}_1, b_1, c) \cdot (2, 1, -1) = 0$$

$$2a + b - c = 0$$

$$\underline{b = c}$$

• HENCE THE  $\begin{cases} a \\ b \\ c \end{cases}$  COMPONENT EQUATIONS BECOME

$$\begin{cases} \mu b + p = 3 \\ \mu c + 2p + 2 = 10 \end{cases} \Rightarrow \begin{cases} \mu b + p = 3 \\ \mu b + 2p + 10 = 10 \end{cases}$$

$$\frac{\mu b + 2p + 10 = 10}{\mu b + 2p = -10}$$

$$\frac{\mu b + p = 3}{\mu b + 2p = 6}$$

$$\frac{-10}{-7}$$

$$\mu = 5$$

**Question 148** (\*\*\*\*\*)

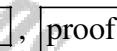
The points  $A$  and  $B$ , have respective position vectors  $\mathbf{a}$  and  $\mathbf{b}$ , relative to a fixed origin  $O$ .

The point  $C$  lies on  $AB$  produced such that  $|AB| : |AC| = 1 : 4$

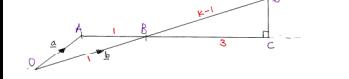
The point  $D$  lies on  $OB$  produced such that  $|OB| : |OD| = 1 : k$ , where  $|OB| : |OD| = 1 : k$  is a scalar constant.

Given that  $AB$  is perpendicular to  $CD$  show that

$$k = \frac{3|\mathbf{a}|^2 - 7\mathbf{a} \cdot \mathbf{b} + 4|\mathbf{b}|^2}{|\mathbf{b}|^2 - \mathbf{a} \cdot \mathbf{b}}.$$

● **DEMO & DIBUJO**



● **DETERMINE SOME VECTORS IN TERMS OF  $\mathbf{a}$  &  $\mathbf{b}$**

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$\overrightarrow{BC} = 3\mathbf{b} - 3\mathbf{a}$$

$$\overrightarrow{BD} = (k-1)\mathbf{b}$$

● **NOW DERIVE AN EXPRESSION FOR  $\overrightarrow{CD}$**

$$\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BD}$$

$$\overrightarrow{CD} = 3\mathbf{a} - 3\mathbf{b} + (k-1)\mathbf{b}$$

● **AS  $CD \perp BC$  THEIR DOT PRODUCT MUST BE ZERO**

$$\Rightarrow \overrightarrow{CD} \cdot \overrightarrow{BC} = 0$$

$$\Rightarrow [3\mathbf{a} - 3\mathbf{b} + (k-1)\mathbf{b}] \cdot (3\mathbf{b} - 3\mathbf{a}) = 0$$

$$\Rightarrow (\mathbf{a} - \mathbf{b}) \cdot [3\mathbf{a} - 3\mathbf{b} + (k-1)\mathbf{b}] = 0$$

$$\Rightarrow (\mathbf{a} - \mathbf{b}) \cdot (3\mathbf{a} - 3\mathbf{b}) + (\mathbf{a} - \mathbf{b}) \cdot (k-1)\mathbf{b} = 0$$

$$\Rightarrow 3\mathbf{a} \cdot \mathbf{a} - 3\mathbf{a} \cdot \mathbf{b} - 3\mathbf{b} \cdot \mathbf{a} + 3\mathbf{b} \cdot \mathbf{b} + k(\mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{b}) - \mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) = 0$$

$$\Rightarrow 3\mathbf{a} \cdot \mathbf{a} - 6\mathbf{a} \cdot \mathbf{b} + 3\mathbf{b} \cdot \mathbf{b} + k(\mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{b}) - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = 0$$

$$\Rightarrow 3|\mathbf{a}|^2 - 7\mathbf{a} \cdot \mathbf{b} + 4|\mathbf{b}|^2 + k(2\mathbf{a} \cdot \mathbf{b} - |\mathbf{b}|^2) = 0$$

$$\Rightarrow k(2\mathbf{a} \cdot \mathbf{b} - |\mathbf{b}|^2) = -3|\mathbf{a}|^2 + 7\mathbf{a} \cdot \mathbf{b} - 4|\mathbf{b}|^2$$

$$\Rightarrow k = \frac{-3|\mathbf{a}|^2 + 7\mathbf{a} \cdot \mathbf{b} + 4|\mathbf{b}|^2}{|\mathbf{b}|^2 - \mathbf{a} \cdot \mathbf{b}}$$

~~ANSWER~~

**Question 149 (\*\*\*\*\*)**

$OAB$  is a triangle and  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OB} = \mathbf{b}$ .

- The point  $C$  lies on  $OB$  so that  $OC : CB = 3 : 1$ .
- The point  $P$  lies on  $AC$  so that  $AP : PC = 2 : 1$ .
- The point  $Q$  lies on  $AB$  so that  $O, P$  and  $Q$  are collinear.

Determine the ratio  $AQ : QB$ .

$$\boxed{\text{Answer}}, \quad AQ : QB = 3 : 2$$

Start by determining scale factor vectors

$$\begin{aligned}\vec{AC} &= \vec{AO} + \vec{OC} = -\mathbf{a} + \frac{3}{4}\mathbf{b} \\ \vec{AP} &= \frac{2}{3}\vec{AC} = \frac{2}{3}(-\mathbf{a} + \frac{3}{4}\mathbf{b}) = -\frac{2}{3}\mathbf{a} + \frac{1}{2}\mathbf{b} \\ \vec{PC} &= \frac{1}{3}\vec{AC} = \frac{1}{3}(-\mathbf{a} + \frac{3}{4}\mathbf{b}) = -\frac{1}{3}\mathbf{a} + \frac{1}{4}\mathbf{b} \\ \vec{AB} &= \vec{OB} + \vec{OA} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}\end{aligned}$$

Let  $\vec{AQ} = \lambda \vec{AB}, 0 < \lambda < 1$

$$\vec{PQ} = \mu \vec{OP}, 0 < \mu < 1$$

Now by looking at the terminal of  $\vec{AQ}$

$$\begin{aligned}\vec{AQ} &= \vec{AP} + \vec{PQ} \\ &\Rightarrow \lambda \vec{AB} = \vec{AP} + \mu \vec{OP} \\ &\Rightarrow \lambda \vec{AB} = \vec{AP} + \mu (\vec{OA} + \vec{OP}) \\ &\Rightarrow \lambda \vec{AB} = \vec{AP} + \mu \vec{OA} + \mu \vec{AB} \\ &\Rightarrow \lambda \vec{AB} = (\mu + 1) \vec{AB} + \mu \vec{OA} \\ &\Rightarrow \lambda (\mathbf{b} - \mathbf{a}) = (\mu + 1)(-\frac{2}{3}\mathbf{a} + \frac{1}{2}\mathbf{b}) + \mu \mathbf{a} \\ &\Rightarrow \lambda \mathbf{b} - \lambda \mathbf{a} = [(\mu + 1)(-\frac{2}{3}\mathbf{a}) + \mu \mathbf{a}] + [(\mu + 1)\frac{1}{2}\mathbf{b}] \\ &\Rightarrow -\lambda \mathbf{a} + \lambda \mathbf{b} = [\frac{4}{3}\mu \mathbf{a} - \frac{2}{3}\mathbf{a}] + [\frac{1}{2}\mathbf{b} + \mu \mathbf{b}] \\ &\Rightarrow -\lambda \mathbf{a} + \lambda \mathbf{b} = \left(\frac{1}{3}\mu - \frac{2}{3}\right)\mathbf{a} + \left(\frac{1}{2}\mu + \frac{1}{2}\right)\mathbf{b}\end{aligned}$$

Equating vectors  $\vec{a}$  &  $\vec{b}$  in both sides

$$\begin{aligned}-\lambda &= \frac{1}{3}\mu - \frac{2}{3} & \lambda &= \frac{1}{2}\mu + \frac{1}{2} \\ -3\lambda &= \mu - 2 & 2\lambda &= \mu + 1 \\ \mu &= 2 - 3\lambda & \mu &= 2\lambda - 1\end{aligned}$$

$$\begin{aligned}-2\lambda - 1 &= 2 - 3\lambda & -2\lambda - 1 &= 2 - 3(2\lambda - 1) \\ 5\lambda &= 3 & 5\lambda &= 3 \\ \lambda &= \frac{3}{5} & \lambda &= \frac{3}{5}\end{aligned}$$

Thus looking in the diagram if  $\lambda = \frac{3}{5}$ ,  $\vec{AQ} = \frac{3}{5}\vec{AB}$

$$AQ : QB = \frac{3}{5} : 2$$

**Question 150 (\*\*\*\*\*)**

Relative to a fixed origin  $O$ , the straight lines  $l_1$  and  $l_2$  have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} 14 \\ 19 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix},$$

where  $\lambda$  and  $\mu$  are scalar parameters.

The point  $A$  lies on  $l_1$  and the point  $B$  lies on  $l_2$ , so that the distance  $AB$  is least.

Find the coordinates of  $A$  and the coordinates of  $B$ .

, A(13,1,-1) , B(18,11,9)

• Without using the cross product

$$\begin{cases} (a_1 b_1 c_1) \cdot (2, 0, -1) = 0 \\ (a_1 b_1 c_1) \cdot (-2, 4, -3) = 0 \end{cases} \Rightarrow \begin{cases} 2a_1 - c_1 = 0 \\ -2a_1 + 4b_1 - 3c_1 = 0 \end{cases}$$

LET ONE OF THE VARIABLES TAKE A NON ZERO VALUE. SAY  $c_1 = 2$ .  
 THEN  $2a_1 = c_1 \Rightarrow a_1 = 1$  ;  $-2a_1 + 4b_1 - 6 = 0 \Rightarrow b_1 = 2$  ;  $\begin{cases} a_1 = 1 \\ b_1 = 2 \\ c_1 = 2 \end{cases} \Leftrightarrow (1, 2, 2)$

BY THE CROSS PRODUCT

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 2 & 0 & -1 \\ -2 & 4 & -3 \end{vmatrix} = (0+4, 2+6, 8-0) = (4, 8, 8) \text{ WHICH SCALES TO } (1, 2, 2) \text{ AS ABOVE}$$

$\vec{l}_1 = (7, 1, 2) + \lambda (2, 0, -1) = (2\lambda + 7, 1, 2 - \lambda)$   
 $\vec{l}_2 = (14, 19, 3) + \mu (-2, 4, -3) = (14 - 2\mu, 19 + 4\mu, 3 - 3\mu)$   
 $\vec{AB} = b - a = (14 - 2\mu - 7, 19 + 4\mu - 1, 3 - 3\mu - 2 + \lambda)$   
 $\vec{AB} = (7 - 2\mu - 2\lambda, 19 + 4\mu, 2 - 3\mu + \lambda)$

NOW  $(7 - 2\mu - 2\lambda, 19 + 4\mu, 2 - 3\mu + \lambda) = k(1, 2, 2)$

$\begin{cases} 7 - 2\mu - 2\lambda = k \\ 19 + 4\mu = 2k \\ 2 - 3\mu + \lambda = 2k \end{cases} \Rightarrow \boxed{\begin{cases} k = 2\mu + 3 \\ \lambda = 2k - 2\mu - 7 \end{cases}}$

$\begin{cases} 7 - 2\mu - 2\lambda = 2\mu + 3 \\ 2 - 3\mu + \lambda = 4\mu + 6 \end{cases} \Rightarrow \boxed{\begin{cases} 8\mu + 2\lambda = -2 \\ 2\lambda = 17 \end{cases}}$

$\begin{cases} 2\lambda = 17 \\ 2\lambda = 17 \end{cases} \Rightarrow \boxed{\lambda = 2\frac{1}{2}}$

$\begin{cases} k = 2\mu + 3 \\ \lambda = 2k - 2\mu - 7 \end{cases} \Rightarrow \boxed{\begin{cases} k = 11 \\ \lambda = 11 \end{cases}}$

$\therefore A(13, 1, -1)$   
 $\therefore B(18, 11, 9)$

**Question 151 (\*\*\*\*\*)**

The point  $P$  lies on the straight line  $L_1$ , which is parallel to the vector  $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$  and passes through the point with coordinates  $(10, 3, 7)$ , relative to an origin at  $(0, 0, 0)$ .

The point  $Q$  lies on another straight line  $L_2$ , which is in the direction of the vector  $4\mathbf{i} - \mathbf{j} + \mathbf{k}$  and passes through the point with coordinates  $(9, 1, 0)$ .

The straight line  $L_3$  is perpendicular to both  $L_1$  and  $L_2$ , and meets  $L_1$  and  $L_2$  at the points  $P$  and  $Q$ , respectively.

Find the coordinates of  $P$  and  $Q$ .

,  $P(4, 0, 1)$ ,  $Q(5, 2, -1)$

- WRITE DOWN THE EQUATIONS OF THE TWO LINES  $L_1$  &  $L_2$ 

$$L_1: \quad \vec{r}_1 = (10, 3, 7) + \lambda(2, 1, 2) = (2\lambda + 10, 2\lambda + 3, 2\lambda + 7)$$

$$L_2: \quad \vec{r}_2 = (9, 1, 0) + \mu(4, -1, 1) = (4\mu + 9, 1 - \mu, \mu)$$
- DRAW A DIAGRAM — NOTE THAT THE LINES ARE TO BE SKewed
- LET  $\vec{p} = (2\lambda + 10, 2\lambda + 3, 2\lambda + 7)$  for  $\lambda = a$   
 $\vec{q} = (4\mu + 9, 1 - \mu, \mu)$  for  $\mu = b$ 

$$\vec{PQ} = \vec{q} - \vec{p} = (4\mu + 9, 1 - \mu, \mu) - (2\lambda + 10, 2\lambda + 3, 2\lambda + 7)$$

$$\Rightarrow \vec{PQ} = [4b - 2a - 1, -b + 2a + 2, b - 2a - 7]$$
- NOW  $\vec{PQ}$  IS PERPENDICULAR TO BOTH LINES
$$\begin{cases} (4b - 2a - 1, -b + 2a + 2, b - 2a - 7) \cdot (2, 1, 2) = 0 \\ (4b - 2a - 1, -b + 2a + 2, b - 2a - 7) \cdot (4, -1, 1) = 0 \end{cases} \Rightarrow$$

$$\left. \begin{array}{l} 8b - 4a - 2 = 0 \\ -b + 4a + 4 = 0 \\ 4b - 3a + 4 = 0 \\ b + a + 2 = 0 \\ b - 2a - 7 = 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} 9b - 9a - 18 = 0 \\ 16b - 9a - 9 = 0 \end{array} \right\} \Rightarrow$$

$$\begin{aligned} &\Rightarrow \frac{b - a = 2}{2b - a = 1} \\ &\Rightarrow -b = 1 \\ &\Rightarrow b = -1 \\ &\text{Thus } a = -3 \\ &\therefore \text{Hence } P(-1, 1, 1) \quad Q(5, 2, -1) \end{aligned}$$

**Question 152 (\*\*\*\*\*)**

The straight line  $l_1$ , where  $\lambda$  is a scalar parameter, has vector equation

$$\mathbf{r} = 10\mathbf{i} + 8\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

The points  $A(4,1,3)$  and  $B(6,5,-3)$  lie on the straight line  $l_2$ .

- a) Given that  $l_1$  and  $l_2$  lie on the same plane, show that  $l_1$  is perpendicular to  $l_2$ .

The points  $C$  and  $D$  lie on  $l_1$  so that the resulting quadrilateral  $ACBD$  is a kite, whose line of symmetry is  $l_2$ .

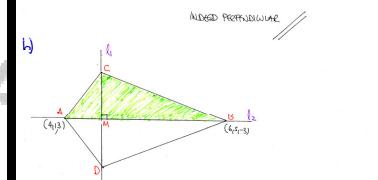
- b) Given further that the area of the kite is  $8\sqrt{42}$  square units, determine the possible coordinates of the points  $C$  and  $D$ .

$$[\quad], \mathbf{r} = 4\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}), |CD| = 8\sqrt{3},$$

$$[C(9,7,4) \text{ & } D(1,-1,-4) \text{ in any order}]$$

a)  $\vec{AB} = \mathbf{b} - \mathbf{a} = (6,5,-3) - (4,1,3) = (2,4,-6)$   
 DIRECTION OF  $l_2$  IS  $(2,4,-6)$   
 $(1,1,1) \cdot (2,4,-6) = 1+2-3=0$

INDED PERPENDICULAR //



START BY A GOOD DIAGRAM — AREA OF THE KITE IS  $8\sqrt{42}$   
 $2[\frac{1}{2}|AB||AC|] = 8\sqrt{42}$   
 $|AB||AC| = 8\sqrt{42}$   
 $|2,4,-6||AC| = 8\sqrt{42}$   
 $2\sqrt{1+16}||AC| = 8\sqrt{42}$   
 $\sqrt{17}||AC| = 4\sqrt{42}$   
 $\sqrt{17}||AC| = 4\sqrt{42}\sqrt{3}$   
 $||AC| = 4\sqrt{42}\sqrt{3}$

NOW WE NEED THE INTERSECTION OF  $l_1$  &  $l_2$   
 $l_1: \mathbf{r} = (10,8,5) + \lambda(1,1,1) = (10+1\lambda, 8+\lambda, 5+\lambda)$   
 $l_2: \mathbf{r}_2 = (4,1,3) + \mu(1,2,-3) = (4+\mu, 1+2\mu, 3-3\mu)$

EQUATE  $1 \& 2$   
 $\begin{cases} 1+2\mu = 10+\lambda \\ 2+4\mu = 8+\lambda \end{cases}$  SUBTRACT "BOTH"  $\Rightarrow -2 = \mu - 3$   
 $\mu = 1 \Rightarrow 2 = -\lambda$

$\therefore \mathbf{U}(5,3,0)$

NOW THE VALUE OF  $\lambda$  AT  $\mathbf{U}$  IS  $-5$   
 THE DIRECTION VECTOR OF  $l_1$  IS  $(1,1,1)$ , WHERE MAGNITUDE IS  $\sqrt{3}$ ; WHICH MEANS THAT EVERY STEP IN  $l_1$  ADVANCES BY  $\sqrt{3}$   
 BUT  $|\mathbf{AC}| = 4\sqrt{42}$  & SIMILARLY  $|\mathbf{AD}| = 4\sqrt{42}$   
 Thus the required points will be produced from  $\lambda = -5 \pm 4$ .

IF  $\lambda = -1 \Rightarrow (9,4,4)$   
 IF  $\lambda = -9 \Rightarrow (1,-1,-4)$  (IN ANY ORDER)

**Question 153 (\*\*\*\*\*)**

The straight line  $L_1$  passes through the points  $A$  and  $B$ , whose respective position vectors relative to a fixed origin  $O$  are

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}.$$

The point  $C$  has position vector  $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ .

The straight line  $L_2$  passes through  $C$  and is parallel to  $L_1$ .

The points  $P$  and  $Q$  both lie on  $L_2$  so that  $|CP| = |CQ| = 2|AB|$ .

Find the area of the quadrilateral with vertices at  $A$ ,  $B$ ,  $P$  and  $Q$ .

 ,  $15\sqrt{2}$

• START WITH A GOOD DIAGRAM — USE  $M$  &  $N$  BE THE MIDPOINTS OF  $CP$  &  $CQ$

• AS ALL THE TRIANGLES IN THE DIAGRAM HAVE EQUAL BASES & THEY ALL HAVE THE SAME HEIGHT (A COMMON VALUE), THESE AREAS ARE EQUAL.

• IT SUFFICES TO FIND THE AREA OF  $T_1$  — LET  $ABC = \theta$

$$\begin{aligned} \vec{AB} &= \vec{b} - \vec{a} = (2, 1, 6) - (1, 2, 5) \\ \vec{BC} &= \vec{c} - \vec{b} = (3, 0, 1) - (2, 1, 6) \\ \vec{CA} &= \vec{a} - \vec{c} = (1, 2, 5) - (3, 0, 1) \end{aligned}$$

• BY THE DOT PRODUCT

$$\vec{AB} \cdot \vec{CA} = |\vec{AB}| |\vec{CA}| \cos \theta$$

$$(1, 1, 5) \cdot (1, 2, 5) = (1, 1, 1) \cdot (-1, 1, 5) \cos \theta$$

$$-1 \cdot 1 + 1 \cdot 2 + 5 \cdot 5 = \sqrt{1+1+25} \cdot \sqrt{1+1+25} \cos \theta$$

$$3 = \sqrt{27} \cos \theta$$

$$3 = 3\sqrt{3} \cos \theta$$

$$\cos \theta = \frac{1}{\sqrt{3}}$$

•  $\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{1}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$

• AREA of  $ABPQ = 5(\text{area of } T_1) = 5 \left[ \frac{1}{2} |\vec{AB}| |\vec{CA}| \sin \theta \right]$

$$= 5 \times \frac{1}{2} \sqrt{27} \times \sqrt{27} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{5}{2} \times 9 \times \frac{\sqrt{2}}{\sqrt{3}} = 15\sqrt{2}$$

• ALTERNATIVE APPROACH (FINDING THE DISTANCE BETWEEN THE LINES)

• START AGAIN WITH A GOOD DIAGRAM — FIND SOME AUXILIARIES

$$\begin{aligned} \vec{AB} &= \vec{b} - \vec{a} = (2, 1, 6) - (1, 2, 5) \\ &= (1, -1, 1) \\ |\vec{AB}| &= \sqrt{1+1+1} = \sqrt{3} \\ \therefore |AB| &= 2|\vec{AB}| = 2\sqrt{3} \end{aligned}$$

• EQUATION OF  $L_1$

$$\vec{r} = (1, 2, 5) + \lambda(-1, 1, 1) = (1+\lambda, 2-\lambda, 2+\lambda)$$

• LET POINT  $W$  LIE ON  $L_1$  SO THAT  $CW \perp L_1$

• SUPPOSE  $W = (x, y, z)$

$$\begin{aligned} \vec{CW} &= \vec{w} - \vec{c} = (x, y, z) - (3, 0, 1) = (x-3, y, z-1) \\ \text{BUT } (x, y, z) &= (1+\lambda, 2-\lambda, 2+\lambda) \\ \therefore \vec{CW} &= (\lambda-2, 2-\lambda, \lambda+1) \\ \vec{CW} &= (\lambda-2, 2-\lambda, \lambda+1) \end{aligned}$$

• BUT  $\vec{CW}$  IS PERPENDICULAR TO  $L_1$

$$(\lambda-2, 2-\lambda, \lambda+1) \cdot (-1, 1, 1) = 0$$

$$\lambda-2+\lambda-2+\lambda+1 = 0$$

$$3\lambda-3 = 0$$

$$\lambda = 1$$

•  $\therefore W = (1, 1, 2)$  i.e.  $W = Q$ .

• HENCE THE HEIGHT OF THE TRAPEZIUM IS  $|\vec{CW}| = |(1, 1, 2)|$

$$\begin{aligned} &= \sqrt{1+1+4} \\ &= \sqrt{2+4} \\ &= \sqrt{6} \end{aligned}$$

• FINALLY THE AREA IS

$$\begin{aligned} \frac{|AB| + |PQ|}{2} \times |\vec{CW}| &= \frac{\sqrt{3} + 4\sqrt{3}}{2} \times \sqrt{6} \\ &= \frac{5\sqrt{3}}{2} \times \sqrt{6} \\ &= 5\sqrt{3} \times \sqrt{42} \\ &= 15\sqrt{2} \\ &\approx 34.64 \end{aligned}$$

**Question 154 (\*\*\*\*\*)**

Relative to a fixed origin  $O$ , the straight line  $l$  passes through the points  $A(a, -3, 6)$ ,  $B(2, b, 2)$  and  $C(3, 3, 0)$ , where  $a$  and  $b$  are constants.

- a) Find the value of  $a$  and the value of  $b$ , and hence find a vector equation of  $l$ .

The points  $P$  and  $Q$  lie on the  $l$  so that  $|OP|=|OQ|$  and  $\angle POQ = 90^\circ$ .

- b) Find the coordinates of  $P$  and the coordinates of  $Q$ .

$$\boxed{\quad}, \boxed{a=0, b=1}, \boxed{\mathbf{r} = -3\mathbf{j} + 6\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})}, \boxed{(1, -1, 4) \text{ & } (3, 3, 0)}$$

**(a)**

$A(a, -3, 6)$ ,  $B(2, b, 2)$ ,  $C(3, 3, 0)$

 $\vec{AB} = \mathbf{t} \times \vec{BC}$ , for some scalar  $t$ 
 $\vec{B}(b, 2, 2) = t \cdot (-1, 1, -2)$ 
 $\begin{bmatrix} b-2 \\ 2-1 \\ 2-0 \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ 
 $\begin{bmatrix} b-2 = -t \\ 2-1 = t \\ 2-0 = -2t \end{bmatrix} \Rightarrow \begin{bmatrix} b = 2-t \\ t = 1 \\ -2t = 0 \end{bmatrix}$ 

Hence  $\begin{cases} 2-t = 1 \\ b = 2-(2-t) \end{cases} \Rightarrow \begin{cases} t = 1 \\ b = 2-1 \end{cases}$

 $\therefore b = 1$ 

**(b)**

$\vec{OP} \perp \vec{OQ}$

$\vec{OP} = \mathbf{r} = (1, -1, 4)$ ,  $\vec{OQ} = (3, 3, 0)$

For some values  $\lambda$  and  $\mu$

 $\vec{OP} = \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ ,  $\vec{OQ} = \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ 

$\vec{OP} \cdot \vec{OQ} = 0$

 $(1, -1, 4) \cdot (3, 3, 0) = 0$ 
 $3 + 4 - 12 = 0 \Rightarrow 7 = 0$ 

$\vec{OP} \cdot \vec{OQ} = 0$

 $\vec{OP}^2 = |\vec{OP}|^2 = 1^2 + (-1)^2 + 4^2 = 18$ 
 $\vec{OQ}^2 = |\vec{OQ}|^2 = 3^2 + 3^2 + 0^2 = 18$ 
 $|\vec{OP}| = \sqrt{18} = 3\sqrt{2}$ ,  $|\vec{OQ}| = \sqrt{18} = 3\sqrt{2}$ 

**Now**  $|\vec{OP}| = |\vec{OQ}|$

 $\Rightarrow \sqrt{7p^2 - 2pq + q^2} = \sqrt{9q^2 - 36q + 18}$ 
 $\Rightarrow 7p^2 - 2pq + q^2 = 9q^2 - 36q + 18$ 
 $\Rightarrow p^2 - 4pq = q^2 - 4q$ 
 $\Rightarrow p^2 - q^2 = 4p - 4q$ 
 $\Rightarrow (p-q)(p+q) = 4(p-q)$ 
 $\Rightarrow p+q = 4$ 

Hence  $\begin{cases} pq - 2p - 2q + 5 = 0 \\ p+q = 4 \end{cases}$

 $\Rightarrow (4-p) - 2p - 2(4-p) + 5 = 0$ 
 $\Rightarrow 4p - p^2 - 8 + 2p + 5 = 0$ 
 $\Rightarrow p^2 - 4p - 3 = 0$ 
 $\Rightarrow (p-1)(p-3) = 0$ 
 $\Rightarrow p = 1, 3$ 

$\therefore P(1, -1, 4) \text{ & } Q(3, 3, 0)$   
in either order

**Question 155** (\*\*\*\*\*)

The straight line  $l$  has vector equation

$$\mathbf{r} = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}),$$

where  $\lambda$  is a scalar parameter.

The point  $A$  has coordinates  $(3, 3, -3)$ , relative to a fixed origin  $O$ .

The points  $P$  and  $Q$  lie on the  $l$  so that  $|AP| = |AQ|$ .

Given further that  $\angle PAQ = 90^\circ$ , find the coordinates of  $P$  and the coordinates of  $Q$ .

 ,  $(4, 2, 1)$  &  $(6, 6, -3)$

$\mathbf{l} = (3,3,-3) + \lambda(1,2,-2)$   
 $\mathbf{l} = (3+2\lambda, 3+2\lambda, -3-2\lambda)$

- Let the coordinates of  $P$  &  $Q$   
 $\text{By Eqn } \lambda = p = q$   
 $P(4,2,1)$   
 $Q(6,6,-3)$
- $\vec{AP} = p - 3 = (4+2p, 2p, 3-2p) - (3,3,-3) = (p, 2p, 3-6-2p)$   
 $\vec{AQ} = q - 3 = (6+2q, 2q, -3-2q) - (3,3,-3) = (3+2q, 2q, -3-2q)$
- $\vec{AP} \perp \vec{AQ}$  so their dot product must be zero  
 $(p, 2p, 3-6-2p) \cdot (q, 2q, -3-2q) = 0$   
 $pq + (2p-3)(2q-3) + (3-6-2p)(-3-2q) = 0$   
 $pq + 4pq - 6p - 9q + 9 + 36 - 12p + 4pq = 0$   
 $9pq - 18p - 18q + 45 = 0$   
 $\boxed{9p - 2p - 2q + 5 = 0}$
- Note we need the lengths to be equal  
 $|\vec{AP}| = \sqrt{p^2 + (2p)^2 + (3-6-2p)^2} = \sqrt{p^2 + 4p^2 - 12p + 9 + 36 - 24p + 4p^2} = \sqrt{3p^2 - 36p + 45}$   
 $= 3\sqrt{p^2 - 12p + 15}$   
 $|\vec{AQ}| = 3\sqrt{q^2 + 4q^2 + 9} = 3\sqrt{5q^2 + 9}$

• Finally another equation from  $|\vec{AP}| = |\vec{AQ}|$

$$\sqrt{p^2 + 4q^2 + 5} = \sqrt{q^2 + 4q + 5}$$

$$p^2 + 4q^2 + 5 = q^2 + 4q + 5$$

$$p^2 - q^2 = 4p - 4q$$

$$(p-q)(p+q) = 4(p-q)$$

$$\boxed{p+q = 4}$$

(as  $p \neq q$ , as the points are distinct)

• Solving simultaneously

|         |                        |
|---------|------------------------|
| $p+q=4$ | $pq = 2p - 2q + 5 = 0$ |
|         | $pq - 2(p+q) + 5 = 0$  |
|         | $pq - 2x4 + 5 = 0$     |
|         | $\boxed{pq = 3}$       |

By inspection

$$p=1 \quad q=3 \quad \text{or} \quad q=1 \quad p=3$$

or by solving

∴ the two points are

$$(4, 2, 1) \rightarrow P(4, 2, 1)$$

$$(6, 6, -3) \rightarrow Q(6, 6, -3)$$

on the sketchy board

**Question 156 (\*\*\*\*\*)**

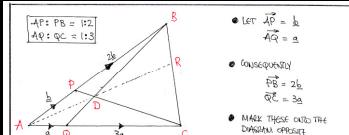
In the acute triangle  $ABC$  the following information is given.

- The point  $P$  lies on  $AB$  so that  $AP:PB = 1:2$ .
- The point  $Q$  lies on  $AC$  so that  $AQ:QC = 1:3$ .
- The point  $D$  is the intersection of  $CP$  and  $BQ$ .

The straight line through  $A$  and  $D$  is extended so that it meets  $BC$  at the point  $R$ .

Determine the ratio  $BR:RC$ .

SOLN,  $BR:RC = 2:5$



• LET  $\vec{AP} = \lambda \vec{a}$   
 $\vec{AQ} = \mu \vec{a}$

• CONSEQUENTLY  
 $\vec{PB} = 2\vec{a}$ ,  
 $\vec{QC} = 3\vec{a}$

• MARK THESE ONTO THE DRAWING ODDSIDE

• FIRST WE CALCULATE SOME LINEAR PATHS SUCH AS

$$\begin{aligned}\vec{BC} &= \vec{BA} + \vec{AC} = -2\vec{a} + 4\vec{a} = 4\vec{a} - 3\vec{b} \\ \vec{BQ} &= \vec{BA} + \vec{AQ} = -2\vec{a} + \mu \vec{a} = (2-\mu)\vec{a} \\ \vec{CP} &= \vec{CA} + \vec{AP} = -4\vec{a} + \lambda \vec{a}.\end{aligned}$$

• NEXT WE LOOK AT THE VECTOR  $\vec{AD}$

$$\begin{cases} \vec{AD} = \vec{AB} + \lambda \vec{BC} & (\text{FOR SOME SCALE } \lambda, 0 < \lambda < 1) \\ \vec{AD} = \vec{AB} + \mu \vec{CP} & (\text{FOR SOME SCALE } \mu, 0 < \mu < 1) \end{cases}$$

$$\Rightarrow \vec{AB} + \lambda \vec{BC} = \vec{AB} + \mu \vec{CP}$$

$$\Rightarrow 3\vec{b} + \lambda(4\vec{a} - 3\vec{b}) = 4\vec{a} + \mu(-4\vec{a} + \lambda\vec{b})$$

$$\Rightarrow (3-4\lambda)\vec{b} + 2\vec{a} = (4-\mu)\vec{a} + \mu\vec{b}$$

$$\begin{cases} \lambda = 4-4\mu \\ \mu = 3-3\lambda \end{cases} \Rightarrow \begin{cases} \mu = 3-3(4-4\mu) \\ \mu = 11\mu \end{cases} \Rightarrow \boxed{\mu = \frac{3}{11}}$$

$$\Rightarrow 2\vec{a} + 4\vec{a} - 4 \times \frac{3}{11}\vec{b} = \frac{44-36}{11}\vec{a} = \frac{8}{11}\vec{a}$$

$$\Rightarrow \boxed{\lambda = \frac{8}{11}}$$

• NEXT WE CONSIDER THE VECTOR  $\vec{AR}$  FROM TWO DIFFERENT PATHS

• FIRSTLY  $\vec{AR} = \vec{AB} + 2\vec{BQ}$   
 $\vec{AB} = 2\vec{a}$ ,  $\vec{BQ} = \frac{2}{3}(2-\mu)\vec{a}$   
 $\vec{AB} = \frac{2}{11}\vec{a} + \frac{4}{3}\vec{b}$

•  $\vec{AR} = \vec{AB} + \vec{BQ}$   
 $= \vec{a} + k\vec{b}$  ( $k$  a scalar,  $0 < k < 1$ )

•  $\vec{AR} = w\vec{AD}$  ( $w$  a scalar,  $w > 1$ )

• FINALLY EQUATING EXPRESSIONS FOR  $\vec{AR}$

$$\begin{aligned}\rightarrow 2\vec{a} + k\vec{b} &= w\vec{a} \\ \rightarrow 2\vec{a} + k(4\vec{a} - 3\vec{b}) &= w\left(\frac{8}{11}\vec{a} + \frac{4}{3}\vec{b}\right) \\ \rightarrow 4\vec{a} + (3-3k)\vec{b} &= \frac{8}{11}w\vec{a} + \frac{4}{3}w\vec{b}\end{aligned}$$

$$\begin{cases} \frac{8}{11}w = 4k \\ 3-3k = \frac{4}{3}w \end{cases} \rightarrow \text{divide } \frac{8}{11}w = \frac{4}{3}k \rightarrow 4 \times 3 \\ \frac{2}{3} = \frac{k}{w} \rightarrow k = \frac{2}{3}w$$

( $w$  IS NOT NEEDED)

• REQUIRED RATIO  $BR:RC = 2:5$

**Question 157 (\*\*\*\*\*)**

Relative to a fixed origin  $O$ , the position vectors of two points  $A$  and  $B$  are denoted by  $\mathbf{a}$  and  $\mathbf{b}$ . The point  $P$  is the foot of the perpendicular from  $O$  to the straight line through  $A$  and  $B$ .

Show that if  $\mathbf{p}$  denotes the position vector of  $P$ , then

$$\mathbf{p} = \mathbf{a} - \frac{\mathbf{a} \cdot (\mathbf{a} - \mathbf{b})(\mathbf{a} - \mathbf{b})}{|\mathbf{a} - \mathbf{b}|^2}.$$

, proof

• SOLVING BY THE POSITION VECTOR OF THE POINT  $P$

$$\mathbf{p} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

• ALSO  $\overrightarrow{OP} \perp \overrightarrow{AB} \Rightarrow \mathbf{p} \cdot (\mathbf{b} - \mathbf{a}) = 0$

$$(\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})) \cdot (\mathbf{b} - \mathbf{a}) = 0$$

$$\mathbf{a} \cdot (\mathbf{b} - \mathbf{a}) + \lambda(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = 0$$

$$\lambda(\mathbf{b} - \mathbf{a})^2 = -\mathbf{a} \cdot (\mathbf{b} - \mathbf{a})$$

$$\lambda(\mathbf{a} - \mathbf{b})^2 = \mathbf{a} \cdot (\mathbf{a} - \mathbf{b})$$

$$\lambda = \frac{\mathbf{a} \cdot (\mathbf{a} - \mathbf{b})}{|\mathbf{a} - \mathbf{b}|^2}$$

• FINALLY WE OBTAIN

$$\mathbf{p} = \mathbf{a} + \frac{\mathbf{a} \cdot (\mathbf{a} - \mathbf{b})(\mathbf{b} - \mathbf{a})}{|\mathbf{a} - \mathbf{b}|^2}$$

$$\mathbf{p} = \mathbf{a} - \frac{\mathbf{a} \cdot (\mathbf{a} - \mathbf{b})(\mathbf{a} - \mathbf{b})}{|\mathbf{a} - \mathbf{b}|^2}$$

**Question 158** (\*\*\*\*\*)

Relative to a fixed origin  $O$  located at the point with coordinates  $(0,0,0)$ , the points  $A(8,1,4)$  and  $B(4,-1,8)$  are given.

A circle, with centre at the point  $P$  and radius  $r$ , is drawn so that the three sides of the triangle  $OAB$  are tangents to this circle.

Determine the coordinates of  $P$  and the exact value of  $r$ .

$$\boxed{\text{SPJ}}, \boxed{P\left(\frac{9}{2}, 0, \frac{9}{2}\right)}, \boxed{r = \frac{3}{2}\sqrt{2}}$$

• START BY DRAWING THE TRIANGLE

• BY INSPECTION THE TRIANGLE IS ISOSCELES

$$OA = \sqrt{(8-0)^2 + (1-0)^2 + (4-0)^2} = \sqrt{64+1+16} = 9$$

$$OB = \sqrt{(4-0)^2 + (-1-0)^2 + (8-0)^2} = \sqrt{16+1+64} = 9$$

• THE CENTRE OF THE CIRCLE MUST BE ON THE POINT OF INTERSECTION BETWEEN TWO OF ITS ANGLE BISECTORS  
(IF A POINT IS EXTERIOR TO FROM TWO INTERSECTING LINES THEN IT MUST LIE ON THE ANGLE BISECTOR)

• THE MIDPOINT OF AB IS M(6,0,2)

• EQUATION OF LINE THROUGH O & M IS  $L_1 = (0,0,0) + t(6,0,2)$

SCALAR  
 $L_1 = (6t, 0, 2t)$

THIS IS ALSO ANOTHER BISECTOR AS THE TRIANGLE IS ISOSCELES

• NEXT FIND THE LENGTH OF AB

$$AB = \sqrt{(-1-8)^2 + (4-1)^2 + (8-4)^2} = \sqrt{81+9+16} = \sqrt{106}$$

• ENDPT AB RELT R TO 9, SO WE OBTAIN ANOTHER BISECTOR LINE

$$OB = \vec{OA} + \frac{1}{2}(\vec{AB}) = 8\hat{i} + \frac{1}{2}(2\hat{j}-\hat{k}) = \frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} - \frac{1}{2}\hat{k} = \frac{1}{2}(8\hat{i} + \hat{j} - \hat{k})$$

$= \frac{1}{2}[3(4,1,4) - (8,1,4)] = \frac{1}{2}(4,-4,20) = (2,-2,10)$

N IS THE MIDPOINT OF OB  
 $N(1,-1,5)$

$$\vec{AN} = \vec{N} - \vec{A} = (1,-1,5) - (8,1,4) = (7,-2,1)$$

SCALE THE DIRECTION TO (7,-2,1)

GEOMETRY OF THE ANGLE BISECTOR WILL BE

$$L_2 = (1,-1,5) + t(7,-2,1)$$

$$L_2 = (7t+1, 2t-1, 5+t)$$

• SOLVING SIMULTANEOUSLY THE TWO LINES (ANGLE BISECTORS) TO FIND THE CENTRE OF THE CIRCLE

$$\begin{cases} L_1 = (6t, 0, 2t) \\ L_2 = (7t+1, 2t-1, 5+t) \end{cases} \quad \begin{cases} 2t-1=0 \\ 7t+1=6t \end{cases} \quad \begin{cases} t= \frac{1}{2} \\ t=1 \end{cases}$$

$$\begin{cases} 2t=2-t \\ 7t=6t-1 \end{cases} \quad \begin{cases} t=2 \\ t=\frac{1}{2} \end{cases}$$

$$\begin{cases} 3t=1 \\ 7t=1 \end{cases} \quad \begin{cases} t=2 \\ t=\frac{1}{2} \end{cases}$$

∴ CENTRE IS AT  $(\frac{9}{2}, 0, \frac{9}{2})$

• FINALLY TO FIND THE RADIUS OF THE CIRCLE WE CONSIDER THE SUBTLE DISTANCE OF  $(\frac{9}{2}, 0, \frac{9}{2})$  FROM THE LINE OA

- LINE OA:  $(3t,0,2) = (0,0,0) + t(3,0,2)$   
 $(3t,0,2) = (3t, 0, 2t)$

- DISTANCE SQUARED OF THE CENTRAL POINT  $(\frac{9}{2}, 0, \frac{9}{2})$  FROM  $(\frac{9}{2}, 0, \frac{9}{2})$  IS GIVEN BY

$$d^2 = (3t - \frac{9}{2})^2 + 0^2 + (2t - \frac{9}{2})^2$$

$$\frac{d^2}{dt} (d^2) = K(3t - \frac{9}{2}) + 0 + 0(2t - \frac{9}{2})$$

- SOLVE FOR t=0

$$0 = 16(3t - \frac{9}{2}) + 2t + 0(2t - \frac{9}{2})$$

$$0 = 9(3t - \frac{9}{2}) + t + 4(2t - \frac{9}{2})$$

$$0 = 44t - 36 + t + 8t - 18$$

$$54t = 54$$

$$t = \frac{54}{54} = \frac{1}{2}$$

∴  $d^2 = (\frac{9}{2} - \frac{9}{2})^2 + (0 - \frac{9}{2})^2 + (0 - \frac{9}{2})^2$

$$d^2 = (\frac{9}{2} - \frac{9}{2})^2 + \frac{9}{4} + (\frac{9}{2} - \frac{9}{2})^2$$

$$d^2 = (\frac{9}{2} - \frac{9}{2})^2 + \frac{9}{4} + (\frac{9}{2} - \frac{9}{2})^2 = (\frac{9}{2})^2 + \frac{9}{4} + (\frac{9}{2})^2$$

$$d^2 = \frac{25}{4} + \frac{9}{4} + \frac{25}{4} = \frac{59}{4} = \frac{BL}{16} = \frac{9}{2}$$

$$d^2 = \frac{18}{4} = \frac{9}{2}$$

$$d = \frac{3}{2}\sqrt{2}$$

**Question 159 (\*\*\*\*\*)**

The points  $A(-3,1,5)$ ,  $B(1,1,1)$  and  $C(-1,5,-1)$  are three of the vertices of the kite  $ABCD$ , which is circumscribed by a circle.

- a) Given that  $|AB|=|AD|$  and  $|BC|=|DC|$ , find the exact coordinates of  $D$ .

A smaller circle is circumscribed by the kite, and a smaller kite similar to  $ABCD$  is circumscribed by the smaller circle.

- b) Determine in exact form the area of the smaller kite.

$$\boxed{D\left(-\frac{33}{7}, \frac{39}{7}, \frac{15}{7}\right)}, \text{ area} = \frac{768(7\sqrt{3}-12)}{7}$$

a) **Start with a diagram**

**EQUATION OF LINE THROUGH A & C**

$$AC = c - a = (-1, 5, -1) - (-3, 1, 5) = (2, 4, -6)$$

SAME DIRECTION IS  $(1, 2, -3)$

$$\Rightarrow \Sigma = (2, 4, -6) + k(1, 2, -3)$$

$$\Rightarrow \Sigma = (k+2, 2k+4, -3k-6)$$

Let the position of  $E$  be  $(x_1, y_1, z_1)$

$$BE = \Sigma - \Sigma_0 = (2, 4, -6) - (-3, 1, 2) = (5, 3, -8)$$

$\vec{EB}$  IS PERPENDICULAR TO  $\vec{AC}$

$$\Rightarrow (5, 3, -8) \cdot (1, 2, -3) = 0$$

$$\Rightarrow 5+6-24=0$$

**Calculation of line through A & C**

$$AC = c - a = (-1, 5, -1) - (-3, 1, 5) = (2, 4, -6)$$

SAME DIRECTION IS  $(1, 2, -3)$

$$\Rightarrow \Sigma = (2, 4, -6) + k(1, 2, -3)$$

$$\Rightarrow \Sigma = (k+2, 2k+4, -3k-6)$$

Let the position of  $E$  be  $(x_1, y_1, z_1)$

$$BE = \Sigma - \Sigma_0 = (2, 4, -6) - (-3, 1, 2) = (5, 3, -8)$$

$\vec{EB}$  IS PERPENDICULAR TO  $\vec{AC}$

$$\Rightarrow (5, 3, -8) \cdot (1, 2, -3) = 0$$

$$\Rightarrow 5+6-24=0$$

**But  $E(x_1, y_1)$  lies on the line through A & C**

$$\Rightarrow (2-3) + 2(2k+1) - (5-3k) = 0$$

$$\Rightarrow -1 + 4k + 2 - 12 + 9k = 0$$

$$\Rightarrow 14k = 16$$

$$\Rightarrow k = \frac{8}{7}$$

$\therefore E\left(\frac{5}{7}, 2\frac{8}{7} + 1, 5 - 3\frac{8}{7}\right)$

$$E\left(\frac{5}{7}, \frac{15}{7}, \frac{15}{7} - 4\right)$$

$$E\left(\frac{5}{7}, \frac{15}{7}, \frac{11}{7}\right)$$

Thus we can now find  $D$  (by inspection)

|   |   |   |
|---|---|---|
| $\begin{matrix} "B" \\   \\ 1 \end{matrix}$   | $\begin{matrix} "E" \\   \\ 1 \end{matrix}$   | $\begin{matrix} "C" \\   \\ 1 \end{matrix}$                   |
| $\begin{matrix} 1 \\ \rightarrow \\ 1+\frac{4}{7} \\   \\ 1+\frac{4}{7} \end{matrix}$ | $\begin{matrix} -\frac{15}{7} \\ \rightarrow \\ -\frac{15}{7} + \frac{11}{7} \\   \\ -\frac{15}{7} + \frac{11}{7} \end{matrix}$ | $\begin{matrix} 0 \\ \rightarrow \\ 0 \\   \\ 0 \end{matrix}$ |
| $\begin{matrix} 1 \\ \rightarrow \\ 1+\frac{4}{7} \\   \\ 1+\frac{4}{7} \end{matrix}$ | $\begin{matrix} \frac{11}{7} \\ \rightarrow \\ \frac{11}{7} + \frac{4}{7} \\   \\ \frac{11}{7} + \frac{4}{7} \end{matrix}$      | $\begin{matrix} 0 \\ \rightarrow \\ 0 \\   \\ 0 \end{matrix}$ |

$\therefore D\left(-\frac{33}{7}, \frac{39}{7}, \frac{15}{7}\right)$

b) **Start with a new diagram**

**IF THE CIRCLE IS INSSCRIBED IN THE KITE, THEN PQQR MUST BE A QUADRILATERAL OF EQUALS OR SIMILARITY**

$$|\overline{AB}| = |\overline{BC}| = |\overline{CD}| = |\overline{DA}| = \sqrt{4+16+16} = \sqrt{32}$$

$$|\overline{BZ}| = |\overline{ZC}| = |\overline{CQ}| = |\overline{QD}| = |\overline{DZ}| = \sqrt{4+16+16} = \sqrt{32}$$

**AND BY SIMILAR TRIANGLES**

$$\frac{|\overline{AP}|}{|\overline{PE}|} = \frac{|\overline{PQ}|}{|\overline{QR}|} \Rightarrow \frac{\sqrt{32}-x}{\sqrt{32}-x} = \frac{x}{\sqrt{32}-x}$$

$$\Rightarrow x = (\sqrt{32}-x)\sqrt{32}-x$$

$$\Rightarrow x^2 = (\sqrt{32}-x)(\sqrt{32}-x)$$

$$\Rightarrow (\sqrt{32}-x)x = \sqrt{32}\sqrt{32}-x^2$$

$$\Rightarrow (4\sqrt{2}+2\sqrt{2})x = 4\sqrt{2}\times 2\sqrt{2}$$

$$\Rightarrow (6\sqrt{2})x = 8\sqrt{2}$$

$$\therefore x = \frac{8\sqrt{2}}{6\sqrt{2}} = \frac{4}{3}$$

**Now the area of the kite is :**

$$2 \times \left( \frac{1}{2} MB \cdot BC \right) = \frac{1}{2} \sqrt{32} \sqrt{32}$$

$$= \sqrt{2} \sqrt{8} \sqrt{4} \sqrt{2}$$

$$= 2 \sqrt{8} \sqrt{4}$$

$$= 16\sqrt{2}$$

**Uniting of HCD**

$$\therefore \text{Area of HCD} = \frac{1}{2} \times 4\sqrt{2} \times 16\sqrt{2} = \frac{1}{2} \times 64\sqrt{4} = 128\sqrt{4} = 256$$

**ALTERNATIVE FOR PART (a)**

Let  $\vec{AE} = q\vec{AC}$  &  $\vec{EC} = r\vec{AC}$

$\Rightarrow$  Then  $\vec{AE} \cdot \vec{EC} = 0$  (because perpendicular)

$$\Rightarrow q\vec{AC} \cdot [r\vec{AC} \cdot \vec{EC}] = 0$$

$$\Rightarrow q\vec{AC} \cdot [\vec{EC} \cdot \vec{EC}] = 0$$

$$\Rightarrow q\vec{AC} \cdot \vec{EC}^2 = 0$$

$$\Rightarrow (q-1)r = 0$$

$$\Rightarrow (q-1)(r-1) = (q-1)(r-1)$$

$$\Rightarrow q(r-1) - q + r - 1 = (q-1)(r-1)$$

$$\Rightarrow qr - q - r + 1 = qr - q - r + 1$$

$$\Rightarrow qr = q + r - 1$$

$$\Rightarrow q = \frac{q + r - 1}{r}$$

$$\Rightarrow q = \frac{(q+1)(r-1) - (q+1) + (r-1) - 1}{(r-1)(r-1)}$$

$$\Rightarrow q = \frac{(q+1)(r-1) - (q+1) + (r-1) - 1}{(r-1)(r-1)}$$

$$\Rightarrow q = \frac{35 - 3 + 3 - 3}{27 - 6 + 3} = \frac{32}{30} = \frac{4}{3}$$

$\therefore \vec{AE} = \frac{4}{3}\vec{AC}$

**NOW THE RADIUS OF THE ORIGINAL CIRCLE WAS  $\sqrt{14}$ .  
THE RADIUS OF Smaller circle was  $8\sqrt{2}-12\sqrt{2}$ .**

- same factor is**  $\frac{8\sqrt{2}-12\sqrt{2}}{\sqrt{14}}$
- Ratio scale factor is**  $\frac{[4(\sqrt{32}-x)^2]^2}{\sqrt{14}^2}$
- Area of original kite was**  $16\sqrt{2}$
- thus the area of the smaller kite will be**

$$\frac{8\sqrt{2}-12\sqrt{2}}{\sqrt{14}} \times 16\sqrt{2} = \frac{8}{\sqrt{14}} \times 16 \times (7-4\sqrt{2})\sqrt{14}$$

$$= \frac{8}{\sqrt{14}} \times 16 \times (7\sqrt{14}-12)$$

$$= \frac{128}{\sqrt{14}}(7\sqrt{14}-12)$$

$$\begin{aligned} \text{let } \vec{AE} = q\vec{AC} \text{ & } \vec{EC} = r\vec{AC} \\ \Rightarrow \text{Then } \vec{AE} \cdot \vec{EC} = 0 \text{ (because perpendicular)} \\ \Rightarrow q\vec{AC} \cdot [r\vec{AC} \cdot \vec{EC}] = 0 \\ \Rightarrow q\vec{AC} \cdot [\vec{EC} \cdot \vec{EC}] = 0 \\ \Rightarrow q\vec{AC} \cdot \vec{EC}^2 = 0 \\ \Rightarrow (q-1)r = 0 \\ \Rightarrow (q-1)(r-1) = (q-1)(r-1) \\ \Rightarrow q(r-1) - q + r - 1 = (q-1)(r-1) \\ \Rightarrow qr - q - r + 1 = qr - q - r + 1 \\ \Rightarrow qr = q + r - 1 \\ \Rightarrow q = \frac{q + r - 1}{r} \\ \Rightarrow q = \frac{(q+1)(r-1) - (q+1) + (r-1) - 1}{(r-1)(r-1)} \\ \Rightarrow q = \frac{(q+1)(r-1) - (q+1) + (r-1) - 1}{(r-1)(r-1)} \\ \Rightarrow q = \frac{35 - 3 + 3 - 3}{27 - 6 + 3} = \frac{32}{30} = \frac{4}{3} \\ \therefore \vec{AE} = \frac{4}{3}\vec{AC} \end{aligned}$$

**Question 160 (\*\*\*\*\*)**

The points  $A(14,1,15)$ ,  $B(8,1,0)$  and  $C(-16,7,-18)$  are three of the vertices of the kite  $ABCD$ . A circle of radius  $r$  is circumscribed by the kite.

Find the area of the kite and hence or otherwise determine, in exact simplified surd form, the value of  $r$ .

$$\boxed{\text{area} = 270}, \boxed{r = \frac{6}{5}(2\sqrt{26} - \sqrt{29})}$$

$A(14,1,15) \quad B(8,1,0) \quad C(-16,7,-18)$

• FIRST WE CHECK WHICH SIDES OF THE KITE WE HAVE

$$|\vec{AB}| = |\vec{b} - \vec{a}| = |(8,1,0) - (14,1,15)| = |-6,0,-15| = 3|-2,0,-5| = 3\sqrt{4+0+25} = 3\sqrt{29}$$

$$|\vec{BC}| = |\vec{c} - \vec{b}| = |(-16,7,-18) - (8,1,0)| = |-24,6,-18| = 6|-4,1,-3| = 6\sqrt{16+1+9} = 6\sqrt{26}$$

• SO THE KITE HAS THE CONFIGURATION OPPOSITE.

• TO FIND THE AREA, FIND THE EXACT VALUE OF THE SINE OF  $\angle B$ .

DOTTING THE SCALAR VECTORS

$$\Rightarrow \vec{BA} \cdot \vec{BC} = |\vec{BA}| |\vec{BC}| \cos \theta$$

$$\Rightarrow (2\sqrt{29})(-4,1,-3) \cdot \sqrt{26} (\vec{a} \cdot \vec{b})$$

$$\Rightarrow -840 - 15 = \sqrt{26 \cdot 29} \cos \theta$$

$$\Rightarrow \cos \theta = \frac{-23}{\sqrt{774}}$$

$$\Rightarrow \sin \theta = \sqrt{1 - \left(\frac{-23}{\sqrt{774}}\right)^2}$$

$$\Rightarrow \sin \theta = \sqrt{1 - \frac{529}{774}}$$

$$\Rightarrow \sin \theta = \sqrt{\frac{245}{774}}$$

$$\Rightarrow \sin \theta = \frac{15}{\sqrt{774}}$$

• SO THE AREA CAN NOW BE FOUND AS

$$\begin{aligned} & \rightarrow \text{area} = \frac{1}{2} |\vec{AB}| |\vec{BC}| \sin(\angle B) \quad \leftarrow \text{TRIANGLE } BAC \\ & = \frac{1}{2} \times 3\sqrt{29} \times 6\sqrt{26} \times \frac{15}{\sqrt{774}} \\ & = 135 \end{aligned}$$

• SO THE KITE HAS AREA 270 SQUARE UNITS

Now draw & copy diagram of the kite and the circle

LET THE CENTRE OF THE CIRCLE BE LOCATED AT THE POINT P, AND NOTE THAT P IS ALSO THE MIDPOINT OF BD

area  $\triangle SPA + \text{area } BPC = \text{area } ABC$

$$\frac{1}{2} |\vec{AP}| r + \frac{1}{2} |\vec{BC}| r = 135$$

$$\frac{1}{2} \times 3\sqrt{29} r + \frac{1}{2} \times 6\sqrt{26} r = 135$$

$$3\sqrt{29} r + 6\sqrt{26} r = 270$$

$$9\sqrt{29} r + 2\sqrt{62} r = 90$$

$$r [\sqrt{29} + 2\sqrt{62}] = 90$$

$$r = \frac{90}{\sqrt{29} + 2\sqrt{62}}$$

$$r = \frac{90(\sqrt{29} - \sqrt{62})}{4\sqrt{29} - 24} \quad \text{RATIONALISE DENOMINATOR}$$

$$r = \frac{90(4\sqrt{29} - \sqrt{29})}{164 - 24} = \frac{90(3\sqrt{29} - \sqrt{29})}{140} = \frac{90(2\sqrt{26} - \sqrt{29})}{75}$$

$$r = \frac{6}{5}(2\sqrt{26} - \sqrt{29})$$

**Question 161 (\*\*\*\*\*)**

Use a vector method involving the scalar product to prove the validity of the Cosine Rule.

, proof

$$\begin{aligned}
 \Rightarrow s &= a - b \\
 \Rightarrow s \cdot s &= (a - b) \cdot (a - b) \\
 \Rightarrow s \cdot s &= a \cdot a - a \cdot b - b \cdot a + b \cdot b \\
 \Rightarrow s \cdot s &= a^2 - 2ab + b^2 \\
 \Rightarrow s \cdot s &= |a|^2 + |b|^2 - 2|a||b|\cos\theta \\
 \Rightarrow |s|^2 &= |a|^2 + |b|^2 - 2|a||b|\cos\theta \\
 \Rightarrow c^2 &= a^2 + b^2 - 2ab\cos\theta
 \end{aligned}$$

**Question 162 (\*\*\*\*\*)**

Three points in space  $A$ ,  $B$  and  $K$  are such so that  $\overrightarrow{KB} = 2\overrightarrow{AK}$ .

Prove that if  $M$  is a fourth distinct arbitrary point in space, then

$$2|\overrightarrow{MA}|^2 + |\overrightarrow{MB}|^2 - 3|\overrightarrow{MK}|^2 = \text{constant}.$$

, proof

METHOD:

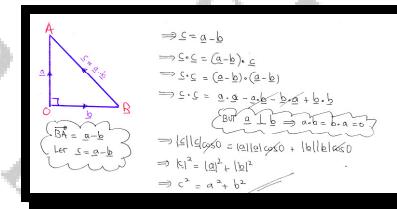
$$\begin{aligned}
 &2|\overrightarrow{MA}|^2 + |\overrightarrow{MB}|^2 - 3|\overrightarrow{MK}|^2 \\
 &= 2|\overrightarrow{MA} + \overrightarrow{AK}|^2 + |\overrightarrow{MB} + \overrightarrow{BK}|^2 - 3|\overrightarrow{MK}|^2 \\
 &= 2(\overrightarrow{MA} + \overrightarrow{AK}) \cdot (\overrightarrow{MA} + \overrightarrow{AK}) + (\overrightarrow{MB} + \overrightarrow{BK}) \cdot (\overrightarrow{MB} + \overrightarrow{BK}) - \\
 &\quad 2|\overrightarrow{MK}|^2 + 2\overrightarrow{MA} \cdot \overrightarrow{AK} + \overrightarrow{AK} \cdot \overrightarrow{AK} + [\overrightarrow{MB} \cdot \overrightarrow{BK} + 2\overrightarrow{MB} \cdot \overrightarrow{BK} + \overrightarrow{BK} \cdot \overrightarrow{BK}] - 3|\overrightarrow{MK}|^2 \\
 &\text{NOTE: } \overrightarrow{MA} \cdot \overrightarrow{AK} = |\overrightarrow{MA}| |\overrightarrow{AK}| \cos 90^\circ = 0 \quad \therefore |\overrightarrow{MA}| = |\overrightarrow{MA}|
 \end{aligned}$$

$$\begin{aligned}
 &= 2[|\overrightarrow{MA}|^2 + 2\overrightarrow{MA} \cdot \overrightarrow{AK} + |\overrightarrow{AK}|^2] + [|\overrightarrow{MB}|^2 - 2\overrightarrow{MB} \cdot \overrightarrow{BK} + |\overrightarrow{BK}|^2] - 3|\overrightarrow{MK}|^2 \\
 &= 2|\overrightarrow{MA}|^2 + 4\overrightarrow{MA} \cdot \overrightarrow{AK} + 2|\overrightarrow{AK}|^2 + |\overrightarrow{MB}|^2 - 2\overrightarrow{MB} \cdot \overrightarrow{BK} + |\overrightarrow{BK}|^2 - 3|\overrightarrow{MK}|^2 \\
 &= 4\overrightarrow{MA} \cdot \overrightarrow{AK} - 2\overrightarrow{MA} \cdot \overrightarrow{BK} + 2|\overrightarrow{AK}|^2 + |\overrightarrow{BK}|^2 \\
 &= 2\overrightarrow{MA} \cdot [2\overrightarrow{AK} - \overrightarrow{BK}] + 2|\overrightarrow{AK}|^2 + |\overrightarrow{BK}|^2 \\
 \underline{\text{WE KNOW THAT: }} \overrightarrow{BK} &= 2\overrightarrow{AK} \\
 &= 2\overrightarrow{MA} \cdot [2\overrightarrow{AK} - 2\overrightarrow{AK}] + 2|\overrightarrow{AK}|^2 + 2|\overrightarrow{AK}|^2 \\
 &= 2|\overrightarrow{AK}|^2 + 2|\overrightarrow{AK}|^2 \\
 &= 2|\overrightarrow{AK}|^2 + 4|\overrightarrow{AK}|^2 \\
 &= 6|\overrightarrow{AK}|^2 \\
 &\text{Hence constant}
 \end{aligned}$$

**Question 163 (\*\*\*\*\*)**

Use a vector method involving the scalar product to prove the validity of Pythagoras' Theorem.

, proof



**Question 164 (\*\*\*\*\*)**

Find the modulus of  $6\mathbf{a} - \mathbf{b}$ , given that the equation  $|x\mathbf{a} + \mathbf{b}| = 2\sqrt{3}$  has repeated roots in  $x$ , where  $\mathbf{a}$  and  $\mathbf{b}$  are constant vectors.

,  $4\sqrt{13}$

USING THE DOT PRODUCT AS FOLLOWS

$$\begin{aligned}
 &\Rightarrow |\mathbf{a} + \mathbf{b}| = 2\sqrt{3} \\
 &\Rightarrow |\mathbf{a} + \mathbf{b}|^2 = 12 \\
 &\Rightarrow |\mathbf{a} + \mathbf{b}|(|\mathbf{a} + \mathbf{b}|) = 12 \\
 &\Rightarrow |\mathbf{a} + \mathbf{b}| |\mathbf{a} + \mathbf{b}| \cos 0 = 12 \quad (\cos 0 = 1) \\
 &\Rightarrow (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = 12 \\
 &\Rightarrow \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = 12 \\
 &\Rightarrow |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = 12 \\
 &\text{Now } \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \quad \mathbf{a} \cdot \mathbf{b} = |\mathbf{b}|^2 \\
 &\Rightarrow 4|\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + 16 = 12 \\
 &\Rightarrow 4|\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + 4 = 0 \\
 &\Rightarrow 2|\mathbf{a}|^2 + \mathbf{a} \cdot \mathbf{b} + 2 = 0 \\
 &\text{Now the repeated root we have} \\
 &\Rightarrow (\mathbf{a} \cdot \mathbf{b})^2 - 4 \times 2 \times 2 = 0 \\
 &\Rightarrow (\mathbf{a} \cdot \mathbf{b})^2 = 16 \\
 &\Rightarrow \mathbf{a} \cdot \mathbf{b} = \cancel{-4} \quad (\text{cancel this}) \\
 &\text{Finally we have} \\
 &\Rightarrow |\mathbf{6a} - \mathbf{b}|^2 = |\mathbf{6a} - \mathbf{b}| |\mathbf{6a} - \mathbf{b}| \\
 &= |\mathbf{6a} - \mathbf{b}| (|\mathbf{6a} - \mathbf{b}|) \cos 0 \\
 &= (|\mathbf{6a} - \mathbf{b}|) \cdot (|\mathbf{6a} - \mathbf{b}|)
 \end{aligned}$$

$$\begin{aligned}
 &= 36|\mathbf{a}|^2 - 6\mathbf{a} \cdot \mathbf{b} - 6\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\
 &= 36|\mathbf{a}|^2 - 12\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\
 &= 36|\mathbf{a}|^2 - (2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2) \\
 &= 36 \times 2 \times 2 - 2 \times (-4) + 4 \times 4 \\
 &= 144 + 16 + 16 \\
 &= 208 \\
 &\therefore |\mathbf{6a} - \mathbf{b}| = \sqrt{208} = \sqrt{4 \times 52} = \sqrt{4 \times 4 \times 13} \\
 &= 4\sqrt{13}
 \end{aligned}$$

**Question 165 (\*\*\*\*\*)**

Use a vector method involving the scalar product to prove that an inscribed angle in a circle which corresponds to a diameter is always a right angle.

, proof

Given:  $\vec{OA} = \mathbf{a}, \vec{OB} = \mathbf{b}$   
 $\vec{OA}' = -\mathbf{a}, \vec{OB}' = -\mathbf{b}$

To Prove:  $\vec{OA} \cdot \vec{OB}' = 0$

Proof:

$$\begin{aligned} \vec{OA} \cdot \vec{OB}' &= (\mathbf{a} \cdot -\mathbf{b})(-\mathbf{b} \cdot \mathbf{a}) \\ &= -\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{a} \cdot -\mathbf{b} = \mathbf{a} \cdot \mathbf{b}^2 \\ &= |\mathbf{a}| |\mathbf{b}| \cos 90^\circ = |\mathbf{a}| |\mathbf{b}| \cos 0^\circ \\ &= |\mathbf{a}|^2 \cdot |\mathbf{b}|^2 \\ &= r^2 \cdot r^2 \\ &= 0 \quad \therefore \angle AOB = 90^\circ \end{aligned}$$

**Question 166 (\*\*\*\*\*)**

The vertices of the triangle  $OAB$  have coordinates  $A(6, -18, -6)$ ,  $B(7, -1, 3)$ , where  $O$  is a fixed origin.

The point  $N$  lies on  $OA$  so that  $ON : NA = 1 : 2$ .

The point  $M$  is the midpoint of  $OB$ .

The point  $P$  is the intersection of  $AM$  and  $BN$ .

By using vector methods, or otherwise, determine the coordinates of  $P$ .

,  $P(4, -4, 0)$

SOLVING WITH A DIAGRAM

• BY VECTORIAL

$N(2, -6, -2)$   
 $M\left(\frac{7}{2}, -\frac{1}{2}, \frac{3}{2}\right)$

WORK AS FOLLOWS

$$\begin{aligned} \vec{NP} &= k \vec{NB} \quad , \quad 0 < k < 1 \\ \vec{NP} &= k \left( \mathbf{b} - \mathbf{a} \right) = k \left[ (7, -1, 3) - (6, -18, -6) \right] = k (1, 17, 9) \\ \vec{NP} &= \left( \frac{k}{2}, \frac{17k}{2}, \frac{9k}{2} \right) \end{aligned}$$

NEXT WE WORK AN EXPRESSION FOR  $\vec{MP}$

$$\begin{aligned} \vec{MP} &= \vec{MO} + \vec{OP} + \vec{PO} \\ \vec{MP} &= -\mathbf{m} + \mathbf{n} + (5\mathbf{a} + 18\mathbf{b}) \\ \vec{MP} &= -\left(\frac{7}{2}, -\frac{1}{2}, \frac{3}{2}\right) + (2, -2, 2) + (30, 90, 30) \\ \vec{MP} &= \left( \frac{51}{2}, \frac{87}{2}, \frac{33}{2} \right) \end{aligned}$$

NOT A SIMILAR EXPRESSION FOR  $\vec{PA}$

$$\begin{aligned} \vec{PA} &= \vec{PN} + \vec{NA} \\ \vec{PA} &= -\vec{NP} + 2\vec{NA} \\ \vec{PA} &= (-2\mathbf{a}_1, -2\mathbf{a}_2, -2\mathbf{a}_3) + (2, -4, -4) \\ \vec{PA} &= (-2\mathbf{a}_1 - 2, -2\mathbf{a}_2 - 4, -2\mathbf{a}_3 - 4) \end{aligned}$$

BUT P, M & A ARE COLLINEAR

$$\begin{aligned} \Rightarrow \vec{MP} &= t \vec{MA} \quad \text{FOR SOME SCALAR } t \\ \Rightarrow \left( \frac{51}{2}, \frac{87}{2}, \frac{33}{2} \right) &= t \left( 2, -4, -4 \right) \end{aligned}$$

EQUATE ANY TWO COMPONENTS

$$\begin{aligned} \frac{51}{2} &= -8t + 4 \quad \Rightarrow \quad 51 + 8t = 4 \times \frac{1}{2} \\ \frac{87}{2} &= -8t + 12 \quad \Rightarrow \quad 87 + 8t = 12 \end{aligned}$$

$$\begin{aligned} \Rightarrow 51 + \frac{87}{2} &= \frac{1}{2} - 12 \\ \Rightarrow 87 &= 4 \\ \Rightarrow 21.75 & \end{aligned}$$

USING

$$\begin{aligned} \frac{33}{2} &= -8t + 4 \\ \frac{33}{2} &= -8t + 4 \\ \frac{33}{2} &= \frac{1}{2} + 4 \\ t &= \frac{1}{2} \end{aligned}$$

CHECKING FOR CONSISTENCY THE THIRD COMPONENT (NOT USED ABOVE)

$$\begin{aligned} \frac{51}{2} &= 5t \frac{1}{2} + \frac{33}{2} = 2t \frac{1}{2} - \frac{3}{2} \\ -5t \frac{1}{2} + 4 &= -5 \times \frac{1}{2} + 4 = -\frac{1}{2} - 1 = -\frac{3}{2} \quad (\text{OK since!}) \end{aligned}$$

HENCE WE HAVE

$$\begin{aligned} \vec{OP} &= \vec{ON} + \vec{NP} \\ &= (2, -6, -2) + (15.5, 43.5, 16.5) \\ &= (17.5, 37.5, 14.5) \\ &= (4, -4, 0) \end{aligned}$$

$\therefore P(4, -4, 0)$

