

VECTOR PRACTICE

Part A

INTRODUCING

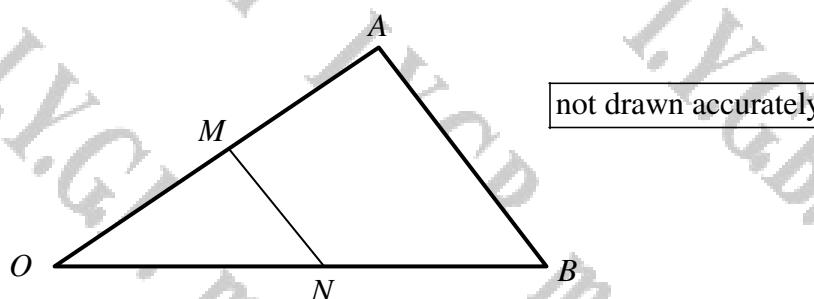
VECTOR ALGEBRA

AND

GEOMETRY

Question 1 ()**

The figure below shows the triangle OAB .

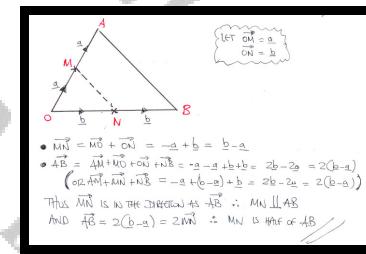


The point M is the midpoint of OA and the point N is the midpoint of OB .

Let $\overrightarrow{OM} = \mathbf{a}$ and $\overrightarrow{ON} = \mathbf{b}$.

By finding simplified expressions for \overrightarrow{MN} and \overrightarrow{AB} , in terms of \mathbf{a} and \mathbf{b} , show that MN is parallel to AB , and half its length.

proof



Question 2 (+)**

$OABC$ is a square.

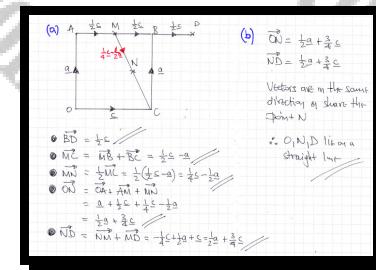
The point M is the midpoint of AB and the point N is the midpoint of MC .

The point D is such so that $\overrightarrow{AD} = \frac{3}{2}\overrightarrow{AB}$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

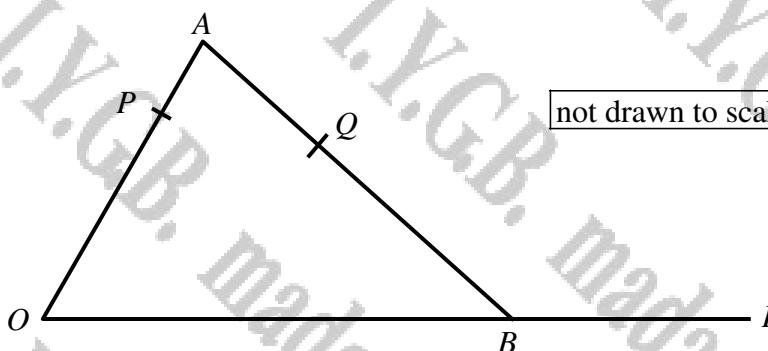
- a) Find simplified expressions, in terms of \mathbf{a} and \mathbf{c} , for each of the vectors \overrightarrow{BD} , \overrightarrow{MC} , \overrightarrow{MN} , \overrightarrow{ON} and \overrightarrow{ND} .
- b) Deduce, showing your reasoning, that O , N and D are collinear.

$$\boxed{\overrightarrow{BD} = \frac{1}{2}\mathbf{c}}, \boxed{\overrightarrow{MC} = \frac{1}{2}\mathbf{c} - \mathbf{a}}, \boxed{\overrightarrow{MN} = \frac{1}{4}\mathbf{c} - \frac{1}{2}\mathbf{a}}, \boxed{\overrightarrow{ON} = \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{c}}, \boxed{\overrightarrow{ND} = \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{c}}$$



Question 3 (*)**

The figure below shows a triangle OAB .

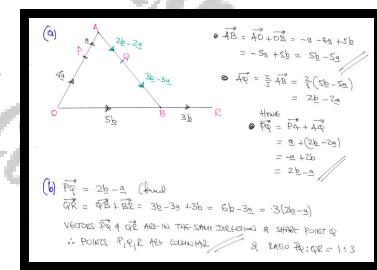


- The point P lies on OA so that $OP:PA = 4:1$.
- The point Q lies on AB so that $AQ:QB = 2:3$.
- The side OB is extended to the point R so that $OB:BR = 5:3$.

Let $\overrightarrow{PA} = \mathbf{a}$ and $\overrightarrow{OB} = 5\mathbf{b}$.

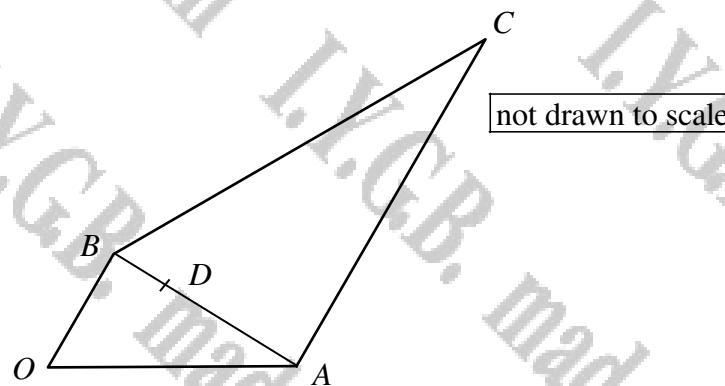
- Find simplified expressions, in terms of \mathbf{a} and \mathbf{b} , for each of the vectors \overrightarrow{AB} , \overrightarrow{AQ} and \overrightarrow{PQ} .
- Deduce, showing your reasoning, that P , Q and R are collinear and state the ratio of $PQ:QR$.

$$\boxed{\overrightarrow{AB} = 5\mathbf{b} - 5\mathbf{a}}, \boxed{\overrightarrow{AQ} = 2\mathbf{b} - 2\mathbf{a}}, \boxed{\overrightarrow{PQ} = 2\mathbf{b} - \mathbf{a}}, \boxed{\overrightarrow{PQ}:QR = 1:3}$$



Question 4 (*)**

The figure below shows a trapezium $OBCA$ where OB is parallel to AC .



The point D lies on BA so that $BD : DA = 1 : 2$.

Let $\overrightarrow{OA} = 4\mathbf{a}$, $\overrightarrow{OB} = 3\mathbf{b}$ and $\overrightarrow{AC} = 6\mathbf{b}$.

- Find simplified expressions, in terms of \mathbf{a} and \mathbf{b} , for each of the vectors \overrightarrow{OC} , \overrightarrow{AB} , \overrightarrow{AD} and \overrightarrow{OD} .
- Deduce, showing your reasoning, that O, D and C are collinear and state the ratio of $OD : DC$.

$$\boxed{\overrightarrow{OC} = 4\mathbf{a} + 6\mathbf{b}}, \boxed{\overrightarrow{AB} = -4\mathbf{a} + 3\mathbf{b}}, \boxed{\overrightarrow{AD} = -\frac{8}{3}\mathbf{a} + 2\mathbf{b}}, \boxed{\overrightarrow{OD} = \frac{4}{3}\mathbf{a} + 2\mathbf{b}}, \boxed{OD : DC = 1 : 2}$$

(a)

- $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = 4\mathbf{a} + 6\mathbf{b}$
- $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -4\mathbf{a} + 3\mathbf{b}$
- $\overrightarrow{AD} = \frac{2}{3}\overrightarrow{AB} = \frac{2}{3}(-4\mathbf{a} + 3\mathbf{b}) = -\frac{8}{3}\mathbf{a} + 2\mathbf{b}$
- $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = 4\mathbf{a} + (-\frac{8}{3}\mathbf{a} + 2\mathbf{b}) = \frac{4}{3}\mathbf{a} + 2\mathbf{b}$

(b)

$$\overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AC} = (\frac{8}{3}\mathbf{a} - 4\mathbf{b}) + 4\mathbf{b} = \frac{8}{3}\mathbf{a} + 4\mathbf{b} = 2(\frac{4}{3}\mathbf{a} + 2\mathbf{b}) = \frac{2}{3}\overrightarrow{OD}$$

AS VECTORS \overrightarrow{DC} & \overrightarrow{OD} ARE IN THE SAME DIRECTION AND SHARE D,
THE POINTS O, D, C ARE COPLANAR
Hence $OD : DC = 1 : 2$

Question 5 (*)**

$OABC$ is a parallelogram and the point M is the midpoint of AB .

The point N lies on the diagonal AC so that $AN : NC = 1 : 2$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

- Find simplified expressions, in terms of \mathbf{a} and \mathbf{c} , for each of the vectors \overrightarrow{AC} , \overrightarrow{AN} , \overrightarrow{ON} and \overrightarrow{NM} .
- Deduce, showing your reasoning, that O , N and M are collinear.

$$\boxed{\overrightarrow{AC} = \mathbf{c} - \mathbf{a}}, \quad \boxed{\overrightarrow{AN} = \frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a}}, \quad \boxed{\overrightarrow{ON} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c}}, \quad \boxed{\overrightarrow{NM} = \frac{1}{3}\mathbf{a} + \frac{1}{6}\mathbf{c}}$$

a) Start by drawing a diagram, and label edges

$\bullet \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\mathbf{a} + \mathbf{c} = \mathbf{c} - \mathbf{a}$
 $\bullet \overrightarrow{AN} = \frac{1}{3}\overrightarrow{AC} = \frac{1}{3}(\mathbf{c} - \mathbf{a}) = \frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a}$
 $\bullet \overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \mathbf{a} + \frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c}$
 $\bullet \overrightarrow{NM} = \overrightarrow{NA} + \overrightarrow{AM} = -\overrightarrow{AN} + \overrightarrow{AM} = -\left(\frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a}\right) + \frac{1}{2}\mathbf{c} = \frac{1}{2}\mathbf{a} + \frac{1}{6}\mathbf{c}$

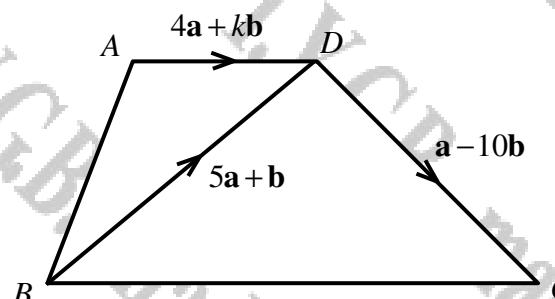
b) Argue as follows

$\overrightarrow{ON} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c} = \frac{1}{3}(2\mathbf{a} + \mathbf{c})$
 $\overrightarrow{NM} = \frac{1}{2}\mathbf{a} + \frac{1}{6}\mathbf{c} = \frac{1}{6}(3\mathbf{a} + \mathbf{c})$

As \overrightarrow{ON} & \overrightarrow{NM} are in the same direction & since the point O , N & M must be collinear

Question 6 (*)+**

The figure below shows a trapezium $OBCA$ where AD is parallel to BC .



The following information is given for this trapezium.

$$\overrightarrow{BD} = 5\mathbf{a} + \mathbf{b}, \quad \overrightarrow{DC} = \mathbf{a} - 10\mathbf{b} \text{ and } \overrightarrow{AD} = 4\mathbf{a} + k\mathbf{b}, \text{ where } k \text{ is an integer.}$$

- Find the value of k .
- Find a simplified expression for \overrightarrow{AB} in terms of \mathbf{a} and \mathbf{b} .

$$k = -6, \quad \boxed{\overrightarrow{AB} = -\mathbf{a} - 7\mathbf{b}}$$

(a)

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BD} + \overrightarrow{DC} \\ &= 5\mathbf{a} + \mathbf{b} + \mathbf{a} - 10\mathbf{b} \\ &= 6\mathbf{a} - 9\mathbf{b} \\ \overrightarrow{AB} &= \overrightarrow{AD} + \overrightarrow{DB} \quad \text{??} \\ &= 4\mathbf{a} + k\mathbf{b} + (-5\mathbf{a} - \mathbf{b}) \\ &\therefore \frac{4}{-5} = \frac{k}{-1} \\ &\therefore k = -6 \quad \text{??} \\ \therefore \overrightarrow{AB} &= -\mathbf{a} - 7\mathbf{b}\end{aligned}$$

(b)

$$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} = 4\mathbf{a} - 6\mathbf{b} - 5\mathbf{a} - \mathbf{b} = -\mathbf{a} - 7\mathbf{b}$$

Question 7 (*)+**

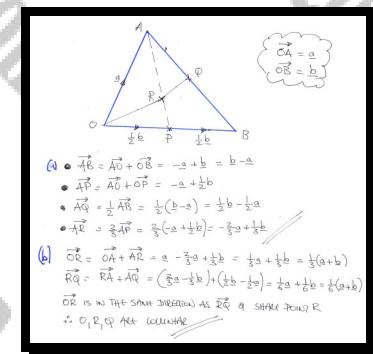
OAB is a triangle with the point P being the midpoint of OB and the point Q being the midpoint of AB .

The point R is such so that $\overrightarrow{AR} = \frac{2}{3}\overrightarrow{AP}$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

- Find simplified expressions, in terms of \mathbf{a} and \mathbf{b} , for each of the vectors \overrightarrow{AB} , \overrightarrow{AP} , \overrightarrow{AQ} and \overrightarrow{AR} .
- By finding simplified expressions, in terms \mathbf{a} and \mathbf{b} , for two more suitable vectors, show that the points O , R and Q are collinear.

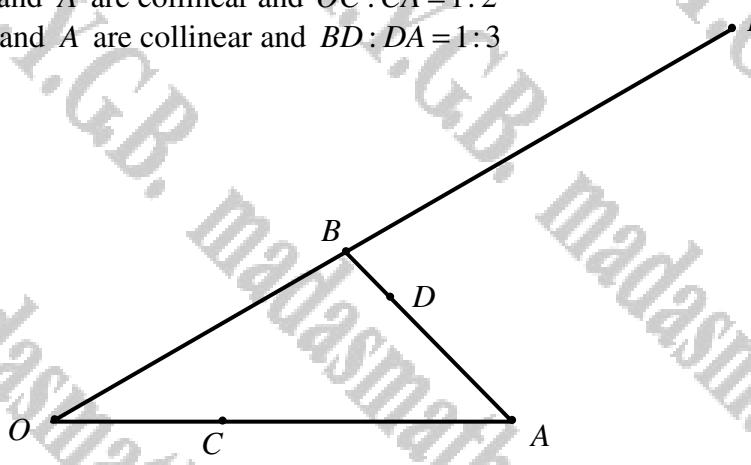
$$\boxed{\overrightarrow{AB} = \mathbf{b} - \mathbf{a}}, \boxed{\overrightarrow{AP} = \frac{1}{2}\mathbf{b} - \mathbf{a}}, \boxed{\overrightarrow{AQ} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}}, \boxed{\overrightarrow{AR} = \frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{a}}$$



Question 8 (***)+

The figure below shows the points O , C , A , D , B and E , which are related as follows.

- O , B and E are collinear and $OB : BE = 1 : 2$
- O , C and A are collinear and $OC : CA = 1 : 2$
- B , D and A are collinear and $BD : DA = 1 : 3$



Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

- Find simplified expressions, in terms of \mathbf{a} and \mathbf{b} , for each of the vectors \overrightarrow{AB} , \overrightarrow{DB} , \overrightarrow{CD} and \overrightarrow{DE} .
- Show that the points C , D and E are collinear, and find the ratio $CD : DE$.
- Show further that BC is parallel to EA , and find the ratio $BC : EA$.

$$\boxed{\overrightarrow{AB} = \mathbf{b} - \mathbf{a}}, \boxed{\overrightarrow{DB} = \frac{1}{4}\mathbf{b} - \frac{1}{4}\mathbf{a}}, \boxed{\overrightarrow{CD} = -\frac{1}{12}\mathbf{a} + \frac{3}{4}\mathbf{b}}, \boxed{\overrightarrow{DE} = -\frac{1}{4}\mathbf{a} + \frac{9}{4}\mathbf{b}}$$

$$\boxed{CD : DE = 1 : 3}, \boxed{BC : EA = 1 : 3}$$

(a)

LET $\overrightarrow{OA} = \mathbf{a}$
 $\overrightarrow{OB} = \mathbf{b}$

$OB : BE = 1 : 2$
 $OC : CA = 1 : 2$
 $BD : DA = 1 : 3$

$\bullet \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$
 $\bullet \overrightarrow{DB} = \frac{1}{4}\mathbf{b} - \frac{1}{4}\mathbf{a} = \frac{1}{4}(\mathbf{b} - \mathbf{a}) = \frac{1}{4}\mathbf{b} - \frac{1}{4}\mathbf{a}$
 $\bullet \overrightarrow{CD} = \overrightarrow{CA} + \overrightarrow{AD} = \overrightarrow{CA} + \frac{3}{4}\overrightarrow{AB}$
 $= \frac{3}{4}\mathbf{a} + \frac{3}{4}(\mathbf{b} - \mathbf{a}) = \frac{3}{4}\mathbf{a} + \frac{3}{4}\mathbf{b} - \frac{3}{4}\mathbf{a} = \frac{3}{4}\mathbf{b}$
 $\bullet \overrightarrow{DE} = \overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DB} + 2\overrightarrow{DB} = 3\overrightarrow{DB} = \frac{3}{4}\mathbf{b} - \frac{3}{4}\mathbf{a} + 2\left(\frac{1}{4}\mathbf{b} - \frac{1}{4}\mathbf{a}\right) = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} + 2\mathbf{b} = -\mathbf{a} + \frac{5}{2}\mathbf{b}$

\therefore VECTORS ARE IN THE SAME DIRECTION AND SHARE THE POINT D
 $\therefore C, D \text{ & } E \text{ ARE COLLINAR}$

$CD : DE = \frac{1}{2} : \frac{5}{2} = \frac{1}{5}$

(b)

$\overrightarrow{CD} = -\frac{1}{12}\mathbf{a} + \frac{3}{4}\mathbf{b} = \frac{1}{12}(-\mathbf{a} + 9\mathbf{b})$
 $\overrightarrow{DE} = -\frac{1}{4}\mathbf{a} + \frac{9}{4}\mathbf{b} = \frac{1}{4}(-\mathbf{a} + 9\mathbf{b})$

\therefore VECTORS ARE IN THE SAME DIRECTION AND SHARE THE POINT E
 $\therefore C, D \text{ & } E \text{ ARE COLLINAR}$

$CD : DE = \frac{1}{12} : \frac{1}{4} = 1 : 3$

(c)

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\mathbf{b} + \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a} - 2\mathbf{b})$
 $\overrightarrow{EA} = \overrightarrow{EB} + \overrightarrow{BA} + \overrightarrow{AC} = -2\mathbf{b} - \mathbf{a} + \mathbf{a} = -2\mathbf{b} = 2(-\mathbf{b}) = 2(\mathbf{a} - 2\mathbf{b})$

\therefore VECTORS ARE IN THE SAME DIRECTION, SO PARALLEL

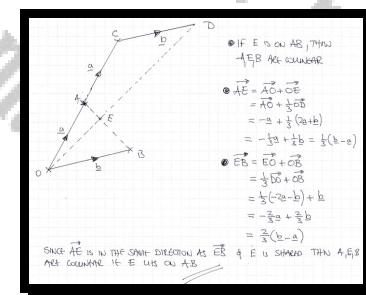
$BC : EA = \frac{1}{2} : 2 = 1 : 4$

Question 9 (**)**

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = 2\mathbf{a}$ and $\overrightarrow{OD} = 2\mathbf{a} + \mathbf{b}$.

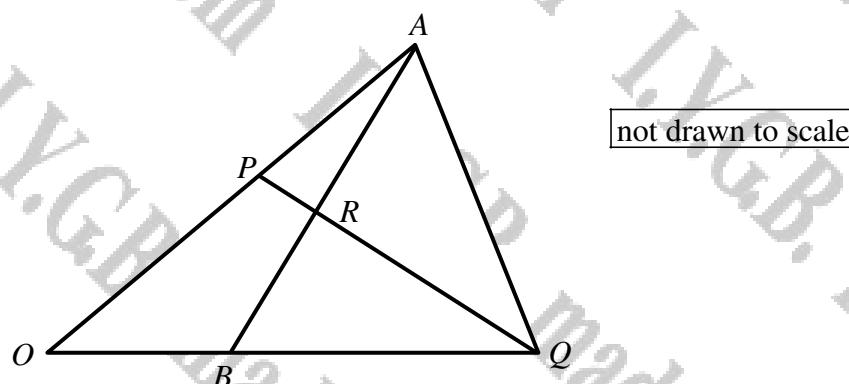
If $\overrightarrow{OE} = \frac{1}{3}\overrightarrow{OD}$ prove that the point E lies on the straight line AB .

proof



Question 10 (**+)**

The figure below shows a triangle OAQ .



- The point P lies on OA so that $OP:OA = 3:5$.
- The point B lies on OQ so that $OB:BQ = 1:2$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

- a) Given that $\overrightarrow{AR} = h\overrightarrow{AB}$, where h is a scalar parameter with $0 < h < 1$, show that

$$\overrightarrow{OR} = (1-h)\mathbf{a} + h\mathbf{b}.$$

- b) Given further that $\overrightarrow{PR} = k\overrightarrow{PQ}$, where k is a scalar parameter with $0 < k < 1$, find a similar expression for \overrightarrow{OR} in terms of k , \mathbf{a} , \mathbf{b} .

- c) Determine ...

i. ... the value of k and the value of h .

ii. ... the ratio of $\overrightarrow{PR}:\overrightarrow{PQ}$.

$$\boxed{\overrightarrow{OR} = \frac{3}{5}(1-k)\mathbf{a} + k\mathbf{b}}, \boxed{k = \frac{1}{6}}, \boxed{h = \frac{1}{2}}, \boxed{PR:PQ = 1:5}$$

a)

b)

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\frac{3}{5}\mathbf{a} + \mathbf{b}$

$\bullet \overrightarrow{PR} = k\overrightarrow{PQ} \Rightarrow \overrightarrow{PR} = k(-\frac{3}{5}\mathbf{a} + \mathbf{b})$

$\bullet \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \frac{3}{5}\mathbf{a} + k(-\frac{3}{5}\mathbf{a} + \mathbf{b})$

$= \frac{3}{5}\mathbf{a} - \frac{3}{5}k\mathbf{a} + k\mathbf{b}$

$= \frac{3}{5}(1-k)\mathbf{a} + k\mathbf{b}$

c)

$\frac{1}{3}k = \frac{2}{5}(1-k) \quad | \cdot 15 \Rightarrow 5 - 5k = 6 - 10k \Rightarrow k = 3k \quad | :3 \Rightarrow k = \frac{1}{3}$

$\Rightarrow 5 - 5k = 5 - 5k \Rightarrow 0 = 0$

$\Rightarrow 2 = 12k \quad | :12 \Rightarrow k = \frac{1}{6}$

$\bullet \overrightarrow{PR} = k\overrightarrow{PQ}$

$\overrightarrow{PR} = \frac{1}{6}\overrightarrow{PQ}$

$\therefore PR:PQ = 1:6$

Question 11 (***)**

OAB is a triangle and $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

- The point C lies on OB so that $OC : CB = 3 : 1$.
- The point P lies on AC so that $AP : PC = 2 : 1$.
- The point Q lies on AB so that O, P and Q are collinear.

Let $\overrightarrow{OQ} = m\overrightarrow{OP}$ and $\overrightarrow{AQ} = n\overrightarrow{AB}$

Find the value of m and the value of n , and hence write down the ratio $AQ : QB$.

$$m = \frac{6}{5}, \quad n = \frac{3}{5}, \quad AQ : QB = 3 : 2$$

Now look at $\triangle APQ$

$$\begin{aligned} \Rightarrow \overrightarrow{AP} + \overrightarrow{PQ} &= \overrightarrow{AQ} \\ \Rightarrow \overrightarrow{AP} + (\overrightarrow{OQ} - \overrightarrow{OP}) &= n\overrightarrow{AB} \\ \Rightarrow \overrightarrow{AP} - \overrightarrow{OP} + m\overrightarrow{OP} &= n\overrightarrow{AB} \\ \Rightarrow \overrightarrow{AP} + (m-1)\overrightarrow{OP} &= n\overrightarrow{AB} \\ \Rightarrow \overrightarrow{AP} + (m-1)(\overrightarrow{OA} + \overrightarrow{OB}) &= n\overrightarrow{AB} \\ \Rightarrow m\overrightarrow{AP} + (m-1)\overrightarrow{OA} + (m-1)\overrightarrow{OB} &= n\overrightarrow{AB} \\ \Rightarrow m(\frac{1}{5}\mathbf{a} + \frac{1}{5}\mathbf{b}) + (m-1)\mathbf{a} &= n\overrightarrow{AB} \\ \Rightarrow (\frac{1}{5}m + \frac{1}{5})\mathbf{a} + (m-1)\mathbf{a} &= n(\mathbf{a}-\mathbf{b}) \\ \Rightarrow (\frac{1}{5}m + m - 1)\mathbf{a} + \frac{1}{5}m\mathbf{b} &= n\mathbf{a} - n\mathbf{b} \\ \Rightarrow (\frac{6}{5}m - 1)\mathbf{a} + \frac{1}{5}m\mathbf{b} &= n\mathbf{a} - n\mathbf{b} \end{aligned}$$

Now look at $\triangle ABC$

$$\begin{aligned} \Rightarrow \overrightarrow{AC} + \overrightarrow{CB} &= \overrightarrow{AB} \\ \Rightarrow \overrightarrow{AO} + \overrightarrow{OC} + \overrightarrow{CB} &= \overrightarrow{AB} \\ \Rightarrow \overrightarrow{AO} + \frac{3}{4}\overrightarrow{OC} + \frac{1}{4}\overrightarrow{CB} &= \overrightarrow{AB} \\ \Rightarrow \overrightarrow{AO} + \frac{3}{4}(\mathbf{a} + \frac{1}{5}\mathbf{b}) + \frac{1}{4}\mathbf{b} &= \overrightarrow{AB} \\ \Rightarrow \overrightarrow{AO} + \frac{3}{4}\mathbf{a} + \frac{1}{20}\mathbf{b} + \frac{1}{4}\mathbf{b} &= \overrightarrow{AB} \\ \Rightarrow \overrightarrow{AO} + \frac{3}{4}\mathbf{a} + \frac{3}{20}\mathbf{b} &= \overrightarrow{AB} \\ \Rightarrow \overrightarrow{AO} = \overrightarrow{AB} - \frac{3}{4}\mathbf{a} - \frac{3}{20}\mathbf{b} &= -\frac{3}{4}\mathbf{a} + \frac{12}{20}\mathbf{b} \\ \Rightarrow \overrightarrow{AO} = -\frac{3}{4}\mathbf{a} + \frac{3}{5}\mathbf{b} &= -\frac{3}{4}\mathbf{a} + \frac{3}{5}\mathbf{b} \\ \Rightarrow \overrightarrow{AO} = \mathbf{b} - \mathbf{a} &= \mathbf{b} - \mathbf{a} \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{5}m - 1 &= -n \\ m - 5 &= -5n \\ m &= 5n \\ m &= 2n \end{aligned}$$

$$\begin{aligned} \frac{1}{5}m - 1 &= -n \\ m - 5 &= -5n \\ 2n - 5 &= -5n \\ 7n &= 5 \\ n &= \frac{5}{7} \\ n &= \frac{5}{7} \end{aligned}$$

AND FINALLY

$$\begin{aligned} \overrightarrow{AQ} &= \frac{3}{5}\overrightarrow{AB} \\ \overrightarrow{AQ} : \overrightarrow{QB} &= 3 : 2 \end{aligned}$$

Question 12

Find the value of λ and μ , given that the vectors \mathbf{a} and \mathbf{b} are not parallel.

- a) $7\lambda\mathbf{a} + 5\lambda\mathbf{b} + 3\mu\mathbf{a} - \mu\mathbf{b} = 5\mathbf{a} + 2\mathbf{b}$
- b) $2\lambda\mathbf{a} + 3\lambda\mathbf{b} + 3\mu\mathbf{a} - 5\mu\mathbf{b} = -5\mathbf{a} + 21\mathbf{b}$
- c) $2\lambda\mathbf{a} + 3\mu\mathbf{b} = 7\mu\mathbf{a} + 11\lambda\mathbf{b} + 57\mathbf{a} + 6\mathbf{b}$
- d) $\lambda\mathbf{a} + 3\lambda\mathbf{b} + \mu\mathbf{b} = 2\mu\mathbf{a} + 5\mathbf{a} + 8\mathbf{b}$

$$\lambda = \frac{1}{2}, \mu = \frac{1}{2}, [\lambda = 2, \mu = -3], [\lambda = -3, \mu = -9], [\lambda = 3, \mu = -1]$$

<p>60) $7\lambda\mathbf{a} + 5\lambda\mathbf{b} + 3\mu\mathbf{a} - \mu\mathbf{b} = 5\mathbf{a} + 2\mathbf{b}$</p> $(7\lambda + 3\mu)\mathbf{a} + (5\lambda - \mu)\mathbf{b} = 5\mathbf{a} + 2\mathbf{b}$ $7\lambda + 3\mu = 5 \quad (1)$ $5\lambda - \mu = 2 \quad (2)$ $7\lambda + 3\mu = 5 \quad (1)$ $5\lambda - \mu = 2 \quad (2)$ $\Rightarrow 8\lambda = 7 \Rightarrow \lambda = \frac{7}{8}$ $\Rightarrow 5\left(\frac{7}{8}\right) - \mu = 2 \Rightarrow \mu = \frac{35}{8} - 2 = \frac{27}{8}$ $\lambda = \frac{7}{8}, \mu = \frac{27}{8}$	<p>61) $2\lambda\mathbf{a} + 3\lambda\mathbf{b} + 3\mu\mathbf{a} - 5\mu\mathbf{b} = -5\mathbf{a} + 21\mathbf{b}$</p> $(2\lambda + 3\mu)\mathbf{a} + (3\lambda - 5\mu)\mathbf{b} = -5\mathbf{a} + 21\mathbf{b}$ $2\lambda + 3\mu = -5 \quad (1)$ $3\lambda - 5\mu = 21 \quad (2)$ $2\lambda + 3\mu = -5 \quad (1)$ $3\lambda - 5\mu = 21 \quad (2)$ $\Rightarrow 5\lambda = 16 \Rightarrow \lambda = \frac{16}{5}$ $\Rightarrow 3\left(\frac{16}{5}\right) - 5\mu = 21 \Rightarrow \mu = \frac{48}{5} - 21 = -\frac{63}{5}$ $\lambda = \frac{16}{5}, \mu = -\frac{63}{5}$
<p>62) $2\lambda\mathbf{a} + 3\lambda\mathbf{b} + 3\mu\mathbf{a} - 5\mu\mathbf{b} = -5\mathbf{a} + 21\mathbf{b}$</p> $(2\lambda + 3\mu)\mathbf{a} + (3\lambda - 5\mu)\mathbf{b} = -5\mathbf{a} + 21\mathbf{b}$ $2\lambda + 3\mu = -5 \quad (1)$ $3\lambda - 5\mu = 21 \quad (2)$ $2\lambda + 3\mu = -5 \quad (1)$ $3\lambda - 5\mu = 21 \quad (2)$ $\Rightarrow 5\lambda = 16 \Rightarrow \lambda = \frac{16}{5}$ $\Rightarrow 3\left(\frac{16}{5}\right) - 5\mu = 21 \Rightarrow \mu = \frac{48}{5} - 21 = -\frac{63}{5}$ $\lambda = \frac{16}{5}, \mu = -\frac{63}{5}$	<p>63) $\lambda\mathbf{a} + 3\lambda\mathbf{b} + \mu\mathbf{b} = 2\mu\mathbf{a} + 5\mathbf{a} + 6\mathbf{b}$</p> $(\lambda + 2\mu)\mathbf{a} + (3\lambda + \mu)\mathbf{b} = 5\mathbf{a} + 6\mathbf{b}$ $\lambda + 2\mu = 5 \quad (1)$ $3\lambda + \mu = 6 \quad (2)$ $\lambda + 2\mu = 5 \quad (1)$ $3\lambda + \mu = 6 \quad (2)$ $\Rightarrow 5\lambda = 11 \Rightarrow \lambda = \frac{11}{5}$ $\Rightarrow 3\left(\frac{11}{5}\right) + \mu = 6 \Rightarrow \mu = \frac{33}{5} - 6 = -\frac{3}{5}$ $\lambda = \frac{11}{5}, \mu = -\frac{3}{5}$

VECTOR COMPONENTS AND 3D-COORDINATES

Question 1

Relative to a fixed origin O , the point A has coordinates $(2,1,-3)$.

The point B is such so that $\overrightarrow{AB} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

Determine the distance of B from O .

$$\boxed{\quad}, \quad |OB| = \sqrt{29}$$

• STARTING WITH A DIAGRAM

• FIND THE COORDINATES OF B

$$\begin{aligned}\rightarrow \overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} \\ \rightarrow \mathbf{b} &= (2, 1, -3) + (3, -1, 5) \\ \rightarrow \mathbf{b} &= (5, 0, 2)\end{aligned}$$

• FINALLY THE DISTANCE OB CAN BE FOUND

$$\begin{aligned}\rightarrow |\overrightarrow{OB}| &= |(5, 0, 2)| \\ \rightarrow |\mathbf{b}| &= \sqrt{5^2 + 0^2 + 2^2} \\ \rightarrow |\mathbf{b}| &= \sqrt{29} \approx 5.39\end{aligned}$$

Question 2

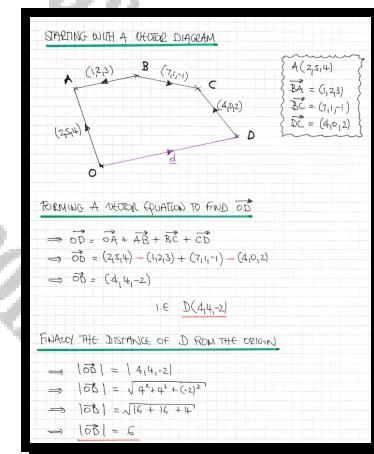
Relative to a fixed origin O , the point A has coordinates $(2, 5, 4)$.

The points B , C and D are such so that

$$\overrightarrow{BA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \overrightarrow{BC} = 7\mathbf{i} + \mathbf{j} - \mathbf{k} \quad \text{and} \quad \overrightarrow{DC} = 4\mathbf{i} + 2\mathbf{k}.$$

Determine the distance of D from the origin.

, $|OD| = 6$



Question 3

Relative to a fixed origin O , the point A has coordinates $(6, -4, 1)$.

The point B is such so that $\overrightarrow{BA} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

If the point M is the midpoint of OB , show that $|\overrightarrow{AM}| = k\sqrt{10}$, where k is a rational constant to be found.

$$\boxed{}, \quad k = \frac{3}{2}$$

PUT THE INFORMATION INTO A DIAGRAM

FIND THE POSITION VECTOR (CO-ORDINATES) OF B

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} \\ \overrightarrow{OB} &= (6, -4, 1) - (1, -1, 3) \\ \overrightarrow{OB} &= (5, -3, -2) \quad \therefore B(5, -3, -2)\end{aligned}$$

NEXT THE CO-ORDINATES OF M

$$\begin{aligned}\overrightarrow{OM} &= \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}(5, -3, -2) = \left(\frac{5}{2}, -\frac{3}{2}, -1\right) \quad \therefore M\left(\frac{5}{2}, -\frac{3}{2}, -1\right)\end{aligned}$$

THEN FIND THE VECTOR \overrightarrow{AM}

$$\begin{aligned}\overrightarrow{AM} &= \overrightarrow{AO} + \overrightarrow{OM} = -(6, -4, 1) + \left(\frac{5}{2}, -\frac{3}{2}, -1\right) \\ \overrightarrow{AM} &= \left(-\frac{7}{2}, \frac{5}{2}, -2\right)\end{aligned}$$

FINALLY THE DISTANCE AM

$$\begin{aligned}|\overrightarrow{AM}| &= \sqrt{\left(-\frac{7}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + (-2)^2} = \sqrt{\frac{49}{4} + \frac{25}{4} + 4} = \sqrt{\frac{90}{4}} = \frac{3}{2}\sqrt{10} \\ &\therefore \boxed{k = \frac{3}{2}}\end{aligned}$$

Question 4

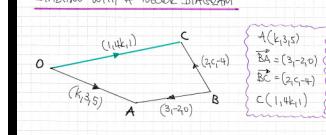
Relative to a fixed origin O , the point A has coordinates $(k, 3, 5)$, where k is a scalar constant.

The points B and C are such so that $\vec{BA} = 3\mathbf{i} - 2\mathbf{j}$ and $\vec{BC} = 2\mathbf{i} + c\mathbf{j} - 4\mathbf{k}$, where c is a scalar constant.

If the coordinates of C are $(1, 4k, 1)$, determine the distance BC .

$$\boxed{\quad}, \quad \boxed{|BC| = \sqrt{29}}$$

STARTING WITH A VECTOR DIAGRAM



$A(k, 3, 5)$
 $\vec{BA} = (3, -2, 0)$
 $\vec{BC} = (2, c, -4)$
 $C(1, 4k, 1)$

FOLLOWING A VECTOR EQUATION

$$\begin{aligned} &\rightarrow \vec{OA} + \vec{AB} + \vec{BC} = \vec{OC} \\ &\rightarrow (k\mathbf{i}, 3\mathbf{j}, 5\mathbf{k}) + (2\mathbf{i} - 2\mathbf{j}, 0) + (2\mathbf{i} + c\mathbf{j} - 4\mathbf{k}) = (1, 4k, 1) \\ &\rightarrow (k+1, c+1, k-4) = (1, 4k, 1) \end{aligned}$$

[1]: $k+1=1 \rightarrow k=0$
[2]: $c+1=4k$
 $c+1=0$
 $c=-1$

FINALLY WE CAN FIND THE DISTANCE BC

$$\begin{aligned} &\rightarrow |\vec{BC}| = |(2, -1, -4)| \\ &\rightarrow |\vec{BC}| = \sqrt{2^2 + (-1)^2 + (-4)^2} \\ &\rightarrow |\vec{BC}| = \sqrt{4+1+16} \\ &\rightarrow |\vec{BC}| = \sqrt{21} \approx 4.59 \end{aligned}$$

Question 5

The points $A(5, -1, 0)$, $B(3, 5, -4)$, $C(12, 2, 8)$ are referred relative to a fixed origin O .

The point D is such so that $\overrightarrow{AD} = 2\overrightarrow{BC}$.

Determine the distance CD .

$$|CD| = \sqrt{458} \approx 21.40$$

Start with A, directed

$A(5, -1, 0)$
 $B(3, 5, -4)$
 $C(12, 2, 8)$
 $AB = 2BC$

FINDING A VECTOR EQUATION

$$\begin{aligned} \Rightarrow \overrightarrow{OD} &= \overrightarrow{OA} + \overrightarrow{AD} \\ &= \overrightarrow{OA} + 2(\overrightarrow{BC}) \\ &\Rightarrow \vec{d} = \vec{a} + 2(\vec{c} - \vec{b}) \\ &\Rightarrow \vec{d} = \vec{a} + 2\vec{c} - 2\vec{b} \\ &\Rightarrow \vec{d} = (5, 1, 0) + 2(12, 2, 8) - 2(3, 5, -4) \\ &\Rightarrow \vec{d} = (23, -7, 24) \end{aligned}$$

FINDING THE DISTANCE CD CAN BE FOUND

$$\begin{aligned} \Rightarrow |\overrightarrow{CD}| &= |\vec{d} - \vec{c}| \\ &= |(23, -7, 24) - (12, 2, 8)| \\ &= |(11, -9, 16)| \\ &= \sqrt{121 + 81 + 256} \\ &= \sqrt{458} \\ &\approx 21.40 \end{aligned}$$

Question 6

The point $A(t, 2, 3)$, where t is a constant, is referred relative to a fixed origin O .

Given that $|\overrightarrow{OA}| = 7$, find the possible values of t .

$$t = \pm 6$$

$$\begin{aligned} \Rightarrow |\overrightarrow{OA}| &= 7 \quad (\text{GIVEN}) \\ \Rightarrow |a| &= 7 \\ \Rightarrow |t, 2, 3| &= 7 \\ \Rightarrow \sqrt{t^2 + 2^2 + 3^2} &= 7 \quad (\text{DEFINITION OF MODULE}) \\ \Rightarrow \sqrt{t^2 + 13} &= 7 \\ \Rightarrow t^2 + 13 &= 49 \\ \Rightarrow t^2 &= 36 \\ \Rightarrow t &= \underline{\underline{\pm 6}} \end{aligned}$$

Question 7

The point $A(3t, 2t, t)$, where t is a constant, is referred relative to a fixed origin O .

Given that $|\overrightarrow{OA}| = 7\sqrt{2}$, find the possible values of t .

$$t = \pm\sqrt{7}$$

$$\begin{aligned} &\Rightarrow |\overrightarrow{OA}| = 7\sqrt{2} \quad (\text{Given}) \\ &\Rightarrow |a| = 7\sqrt{2} \\ &\Rightarrow |3t, 2t, t| = 7\sqrt{2} \\ &\Rightarrow \sqrt{9t^2 + 4t^2 + t^2} = 7\sqrt{2} \quad (\text{DEFINITION OF THE NORM/} \\ &\qquad \text{OF A VECTOR}) \\ &\Rightarrow \sqrt{14t^2} = 7\sqrt{2} \\ &\Rightarrow 14t^2 = 49 \times 2 \\ &\Rightarrow t^2 = 7 \\ &\Rightarrow t = \pm\sqrt{7} \end{aligned}$$

Question 8

The point $A(4, 3, t+2)$, where t is a constant, is referred relative to a fixed origin O .

Given that $|\overrightarrow{OA}| = 13$, find the possible values of t .

$$t = 10, -14$$

$$\begin{aligned} &\Rightarrow |\overrightarrow{OA}| = 13 \quad (\text{Given}) \\ &\Rightarrow |a| = 13 \\ &\Rightarrow |4, 3, t+2| = 13 \\ &\Rightarrow \sqrt{16 + 9 + (t+2)^2} = 13 \quad (\text{DEFINITION OF THE NORM/} \\ &\qquad \text{OF A VECTOR}) \\ &\Rightarrow \sqrt{25 + t^2 + 4t + 4} = 13 \\ &\Rightarrow \sqrt{t^2 + 4t + 29} = 13 \\ &\Rightarrow t^2 + 4t + 29 = 169 \\ &\Rightarrow t^2 + 4t - 140 = 0 \\ &\Rightarrow (t+14)(t-10) = 0 \\ &\Rightarrow t = \underline{\underline{-14}} \end{aligned}$$

Question 9

The points $A(t, 3, 2)$ and $B(5, 2, 2t)$, where t is a scalar constant, are referred relative to a fixed origin O .

Given that $|\overrightarrow{AB}| = \sqrt{21}$, find the possible values of t .

$$t = 3, t = \frac{3}{5}$$

A($t, 3, 2$) B($5, 2, 2t$) $|\overrightarrow{AB}| = \sqrt{21}$

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{21} \quad (\text{Given})$$
$$\Rightarrow |\overrightarrow{OB}| = \sqrt{21}$$
$$\Rightarrow |(5, 2, 2t) - (t, 3, 2)| = \sqrt{21}$$
$$\Rightarrow |(5-t, -1, 2t-2)| = \sqrt{21}$$
$$\Rightarrow \sqrt{(5-t)^2 + (-1)^2 + (2t-2)^2} = \sqrt{21} \quad (\text{DEFINITION OF THE MODULUS OF A VECTOR})$$
$$\Rightarrow \sqrt{25-10t+t^2+1+4t^2-8t+4} = \sqrt{21}$$
$$\Rightarrow \sqrt{5t^2-18t+30} = \sqrt{21}$$
$$\Rightarrow 5t^2-18t+30 = 21$$
$$\Rightarrow 5t^2-18t+9 = 0$$
$$\Rightarrow (5t-3)(t-3) = 0$$
$$\Rightarrow t_1 = \frac{3}{5}, t_2 = 3$$

Question 10

The variable points $A(t+1, 6, t)$ and $B(2t+1, t+1, 4)$, where t is a scalar variable, are referred relative to a fixed origin O .

- a) Show that

$$|\vec{AB}| = \sqrt{3t^2 - 18t + 41}.$$

- b) Hence find the shortest distance between A and B , as t varies.

$$|\vec{AB}|_{\min} = \sqrt{14}$$

A($t+1, 6, t$) & B($2t+1, t+1, 4$)

a) $|\vec{AB}| = |\vec{b} - \vec{a}| = |(2t+1, t+1, 4) - (t+1, 6, t)|$
 $= |t+1, t+5, 4-t| = \sqrt{t^2 + (t+5)^2 + (4-t)^2}$
 $= \sqrt{t^2 + t^2 + 10t + 25 + 16 - 8t + t^2}$
 $= \sqrt{3t^2 - 8t + 41}$ AS 2010/10/10

b) BY COMPUTING THE SQUARE OF CALCULUS
 $\Rightarrow |\vec{AB}| = \sqrt{3t^2 - 18t + 41}$
 $\Rightarrow |\vec{AB}| = \sqrt{3(t^2 - 6t + \frac{41}{3})}$
 $\Rightarrow |\vec{AB}| = \sqrt{3(t-3)^2 - 9 + \frac{41}{3}}$
 $\Rightarrow |\vec{AB}| = \sqrt{3(t-3)^2 - 27 + 41}$
 $\Rightarrow |\vec{AB}| = \sqrt{3(t-3)^2 + 14}$
 $\therefore \text{Hence } |\vec{AB}|_{\min} = \sqrt{14}$ (It occurs when t=3)

Question 11

The variable points $A(2t, t, 2)$ and $B(t, 4, 1)$, where t is a scalar variable, are referred relative to a fixed origin O .

- a) Show that

$$|\vec{AB}| = \sqrt{2t^2 - 8t + 17}.$$

- b) Hence find the shortest distance between A and B , as t varies.

$$|\vec{AB}|_{\min} = 3$$

$A(2t, t, 2)$ $B(t, 4, 1)$

a) $|\vec{AB}| = |\vec{b} - \vec{a}| = |(t, 4, 1) - (2t, t, 2)| = |-t, 4-t, -1|$
 $= \sqrt{(t)^2 + (4-t)^2 + (-1)^2} = \sqrt{t^2 + 16 - 8t + t^2 + 1}$
 $= \sqrt{2t^2 - 8t + 17}$ ✓
 As required

b) BY COMPLETING THE SQUARE (OR CALCULUS)
 $\Rightarrow |\vec{AB}| = \sqrt{2t^2 - 8t + 17}$
 $\Rightarrow |\vec{AB}| = \sqrt{2(t^2 - 4t + 8)}$
 $\Rightarrow |\vec{AB}| = \sqrt{2[(t-2)^2 - 4 + 8]}$
 $\Rightarrow |\vec{AB}| = \sqrt{2(t-2)^2 + 4}$
 $\Rightarrow |\vec{AB}| = \sqrt{2(t-2)^2 + 4}$
 HENCE $|\vec{AB}|_{\min} = 3$ ✓ ← $\sqrt{4}$
 which occurs when $t=2$

Question 12

The variable points $A(1, 8, t-1)$ and $B(2t-1, 4, 3t-1)$, where t is a scalar variable, are referred relative to a fixed origin O .

Find the shortest distance between A and B , as t varies.

$$|\overrightarrow{AB}|_{\min} = \sqrt{18}$$

$A(1, 8, t-1) \quad B(2t-1, 4, 3t-1)$

- START BY DETERMINING AN EXPRESSION, IN TERMS OF t , FOR $|\overrightarrow{AB}|$

$$\Rightarrow |\overrightarrow{AB}| = |\overrightarrow{B-A}| = |(2t-1, 4, 3t-1) - (1, 8, t-1)|$$

$$\Rightarrow |\overrightarrow{AB}| = |2t-2, -4, 3t| = \sqrt{(2t-2)^2 + (-4)^2 + (3t)^2}$$

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{4t^2 - 8t + 4 + 16 + 9t^2} = \sqrt{13t^2 - 8t + 20}$$
- TO MINIMIZE THIS DISTANCE PROCEED BY ONE OF TWO METHODS
 - BY CONTRACTING THE SQUARE

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{8(t^2 - t + \frac{1}{2})}$$

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{8((t-\frac{1}{2})^2 - \frac{1}{4} + \frac{1}{2})}$$

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{8(t-\frac{1}{2})^2 + \frac{15}{4}}$$

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{8(t-\frac{1}{2})^2 + 18}$$

$$\therefore |\overrightarrow{AB}|_{\min} = \sqrt{18} = 3\sqrt{2}$$

(It occurs when $t = \frac{1}{2}$)
 - BY CALCULUS
 - Let $f(t) = |\overrightarrow{AB}|^2 = 8t^2 - 8t + 20$
 - $f'(t) = 16t - 8$
 - SOLVE FOR ZERO
 $16t - 8 = 0$
 $16t = 8$
 $t = \frac{1}{2}$
 - $f(\frac{1}{2}) = 8(\frac{1}{2})^2 - 8(\frac{1}{2}) + 20$
 $= 2 - 4 + 20$
 $= 18$
 - $\therefore f(t)_{\min} = |\overrightarrow{AB}|_{\min}^2 = 18$
 - $\therefore |\overrightarrow{AB}|_{\min} = \sqrt{18}$

Question 13

The points $A(4, 2, 3)$, $B(3, 3, -1)$ and $C(6, 0, -1)$ are referred with respect to a fixed origin O .

If A , B , C and the point D form the parallelogram $ABCD$, use vector algebra to find the coordinates of D .

$$D(7, -1, 3)$$

LOOKING AT THE DIAGRAM

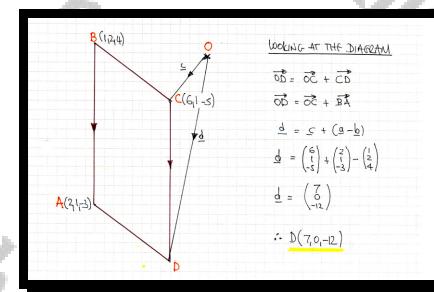
$\vec{OD} = \vec{OA} + \vec{AD}$
 $\vec{OD} = \vec{OA} + \vec{BC}$
 $\vec{d} = \vec{a} + (\vec{c} - \vec{b})$
 $\vec{d} = \left(\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}\right) + \left(\begin{pmatrix} 6 \\ 0 \\ -1 \end{pmatrix}\right) - \left(\begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}\right)$
 $\vec{d} = \left(\begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix}\right)$
 $\therefore D(7, -1, 3)$

Question 14

The points $A(2,1,-3)$, $B(1,2,4)$ and $C(6,1,-5)$ are referred with respect to a fixed origin O .

If A , B , C and the point D form the parallelogram $ABCD$, use vector algebra to find the coordinates of D .

$$D(7,0,-12)$$

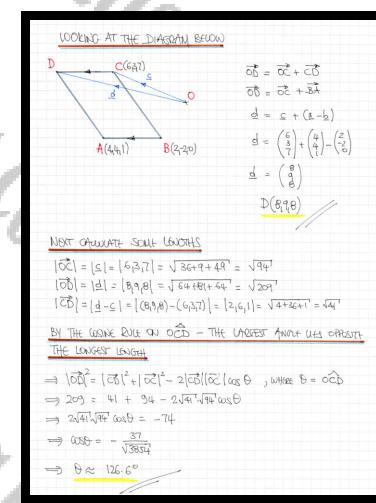


Question 15

The points $A(4,4,1)$, $B(2,-2,0)$ and $C(6,3,7)$ are referred with respect to a fixed origin O .

If A , B , C and the point D form the parallelogram $ABCD$, use vector algebra to find the coordinates of D and hence calculate the angle OCD .

$$D(8,9,8), \angle OCD \approx 126.6^\circ$$



Question 16

The points $A(0, -5, -4)$, $B(2, -1, 2)$ and $C(5, 5, 11)$ are referred with respect to a fixed origin O .

Show that A , B and C are collinear and find the ratio $AB : BC$.

[2:3]

$\bullet A(0, -5, -4) \bullet B(2, -1, 2) \bullet C(5, 5, 11)$

\bullet CALCULATE THE VECTORS \vec{AB} & \vec{BC}

$$\vec{AB} = b - a = (2, -1, 2) - (0, -5, -4) = (2, 4, 6) = \underline{\textcolor{red}{2}}(1, 2, 3)$$

$$\vec{BC} = c - b = (5, 5, 11) - (2, -1, 2) = (3, 6, 9) = \underline{\textcolor{blue}{3}}(1, 2, 3)$$

\bullet AS BOTH \vec{AB} & \vec{BC} ARE IN THE DIRECTION $(1, 2, 3)$ & SINCE THE POINT B \Rightarrow A, B, C ARE COLLINEAR

$$\Rightarrow$$
 REQUIRED RATIO $|\vec{AB}| : |\vec{BC}|$

$$\underline{\textcolor{red}{2}} : \underline{\textcolor{blue}{3}}$$

Question 17

The points $A(9, 10, -5)$, $B(3, 1, 7)$ and $C(-5, -11, 23)$ are referred with respect to a fixed origin O .

Show that A , B and C are collinear and find the ratio $AB : BC$.

[3:4]

$\bullet A(9, 10, -5) \bullet B(3, 1, 7) \bullet C(-5, -11, 23)$

\bullet CALCULATE THE VECTORS \vec{AB} & \vec{BC}

$$\vec{AB} = b - a = (3, 1, 7) - (9, 10, -5) = (-6, -9, 12) = \underline{\textcolor{red}{3}}(-2, -3, 4)$$

$$\vec{BC} = c - b = (-5, -11, 23) - (3, 1, 7) = (-8, -12, 16) = \underline{\textcolor{blue}{4}}(-2, -3, 4)$$

\bullet AS BOTH \vec{AB} & \vec{BC} ARE IN THE SAME DIRECTION $(-2, -3, 4)$ & SINCE THE POINT B \Rightarrow A, B, C ARE COLLINEAR

$$\Rightarrow$$
 REQUIRED RATIO $|\vec{AB}| : |\vec{BC}|$

$$\underline{\textcolor{red}{3}} : \underline{\textcolor{blue}{4}}$$

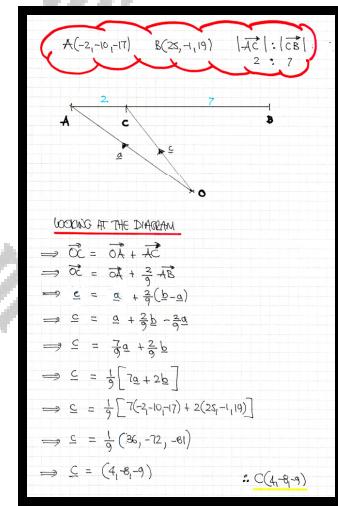
Question 18

The points $A(-2, -10, -17)$ and $B(25, -1, 19)$ are referred with respect to a fixed origin O .

The point C is such so that ACB forms a straight line.

Given further that $\frac{|\overrightarrow{AC}|}{|\overrightarrow{CB}|} = \frac{2}{7}$, determine the coordinates of C .

$$C(4, -8, -9)$$



Question 19

The points $A(-3, -14, -5)$ and $B(1, -4, -1)$ are referred relative to a fixed origin O .

The point C is such so that ABC forms a straight line.

Given further that $\frac{|\overrightarrow{AB}|}{|\overrightarrow{BC}|} = \frac{2}{5}$, determine the coordinates of C .

$$C(11, 21, 9)$$

PUT THE INFORMATION IN A DIAGRAM.

$$\begin{aligned}\Rightarrow \vec{OC} &= \vec{OB} + \vec{BC} \\ \Rightarrow \vec{OC} &= \vec{OB} + \frac{2}{5}\vec{AB} \\ \Rightarrow \underline{C} &= b + \frac{2}{5}(b-a) \\ \Rightarrow \underline{C} &= b + \frac{2}{5}b - \frac{2}{5}a \\ \Rightarrow \underline{C} &= \frac{7}{5}b - \frac{2}{5}a \\ \Rightarrow \underline{C} &= \frac{1}{2}(7b - 2a) \\ \Rightarrow \underline{C} &= \frac{1}{2}[(7(1, -4, -1)) - 2(-3, -14, -5)] \\ \Rightarrow \underline{C} &= \frac{1}{2}(21, 42, 18) \\ \Rightarrow \underline{C} &= (11, 21, 9)\end{aligned}$$

Question 20

The points $A(2, -1, 4)$, $B(0, -5, 10)$, $C(3, 1, 3)$ and $D(6, 7, -8)$ are referred relative to a fixed origin O .

- a) Use vector algebra to show that three of the above four points are collinear.

A triangle is drawn using three of the above four points as its vertices.

- b) Given further that the triangle has the largest possible area, determine, in exact surd form, the length of its shortest side.

$$\boxed{\sqrt{94}}$$

a)

$A(-2, 1, 4)$ $B(0, -5, 10)$ $C(3, 1, 3)$ $D(6, 7, -8)$

- PICK A POINT AT RANDOM AND CALCULATE ALL THREE VECTORS TO THE OTHER 3 POINTS

$\vec{AB} = \vec{b} - \vec{a} = (0, -5, 10) - (-2, 1, 4) = (2, -4, 6) = 2(-1, 2, 3)$
 $\vec{AC} = \vec{c} - \vec{a} = (3, 1, 3) - (-2, 1, 4) = (1, 2, -1) = 1(1, 2, -1)$
 $\vec{AD} = \vec{d} - \vec{a} = (6, 7, -8) - (-2, 1, 4) = (4, 6, -12) = 4(1, 2, -3)$

- HENCE WE HAVE \vec{AB} & \vec{AD} IN "PARALLEL CONFIGURATION"

$\vec{AB} = 2(1, 2, 3) = -2(1, -2, -3)$
 $\vec{AD} = 4(1, 2, -3)$

$\therefore A, B \text{ & } D \text{ ARE COLLINEAR}$

b) • DRAWING A DIAGRAM

- THE LENGTH OF BD IS $6\sqrt{1^2 + 4^2 + 9^2} = 6\sqrt{14^2}$
- ALSO WE HAVE
 - $|\vec{BC}| = |\vec{c} - \vec{b}| = |(3, 1, 3) - (0, -5, 10)| = |(3, 6, -7)| = \sqrt{9 + 36 + 49} = \sqrt{94}$
 - $|\vec{DC}| = |\vec{c} - \vec{d}| = |(3, 1, 3) - (6, 7, -8)| = |-3, -6, 11| = \sqrt{9 + 36 + 121} = \sqrt{166}$

∴ THE SHORTEST SIDE OF THE TRIANGLE WHICH HAS THE LARGEST AREA IS $\sqrt{94}$

Question 21

The points $A(-3,3,a)$, $B(b,b,b-5)$ and $C(c,-2,5)$, where a , b and c are scalar constants, are referred relative to a fixed origin O .

It is further given that A , B and C are collinear and the ratio $|\overrightarrow{AB}| : |\overrightarrow{BC}| = 2 : 3$.

Use vector algebra to find the value of a , the value of b and the value of c .

$$[a,b,c] = [-10, 1, 7]$$

Putting the information in a diagram

$A(-3,3,a)$	$B(b,b,b-5)$	$C(c,-2,5)$
-------------	--------------	-------------

"CALCULATE" THE VECTORS \overrightarrow{AB} & \overrightarrow{BC}

$$\overrightarrow{AB} = b - a = (b, b, b-5) - (-3, 3, a) = (b+3, b-3, b-a-5)$$

$$\overrightarrow{BC} = c - b = (c, -2, 5) - (b, b, b-5) = (c-b, -2-b, 10-b)$$

LOOKING AT $\frac{1}{2}$

$$\frac{b-3}{-2-b} = \frac{2}{3} \Rightarrow 3b-9 = -4-2b \Rightarrow 5b = 5 \Rightarrow b=1$$

LOOKING AT $\frac{1}{3}$

$$\frac{b+3}{c-b} = \frac{2}{3} \Rightarrow 3b+9 = 2c-2b \Rightarrow 3+9 = 2c-2 \Rightarrow 14 = 2c \Rightarrow c=7$$

LOOKING AT $\frac{5}{6}$

$$\frac{b-a-5}{10-b} = \frac{2}{3} \Rightarrow 3b-3a-15 = 20-2b \Rightarrow 3-3a-15 = 20-2 \Rightarrow -3a = 34 \Rightarrow a = -10$$

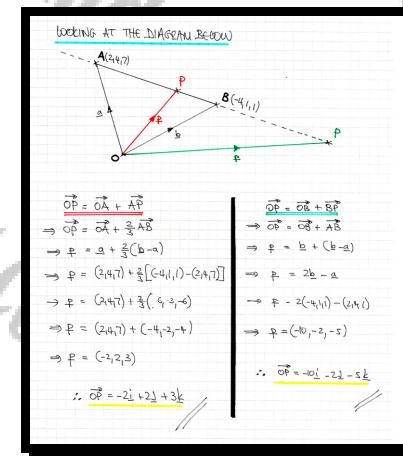
Question 22

With respect to a fixed origin, the points A and B have position vectors $2\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$ and $-4\mathbf{i} + \mathbf{j} + \mathbf{k}$, respectively.

The point P lies on the straight line through A and B .

Find the possible position vectors of P if $|\overrightarrow{AP}| = 2|\overrightarrow{PB}|$.

$$\boxed{\overrightarrow{OP} = \mathbf{p} = -2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}, \boxed{\overrightarrow{OP} = \mathbf{p} = -10\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}}$$



Question 23

With respect to a fixed origin, the points A and B have position vectors $10\mathbf{i} + 9\mathbf{j} - 6\mathbf{k}$ and $6\mathbf{i} - 3\mathbf{j} + 10\mathbf{k}$, respectively.

The position vector of the point C has \mathbf{i} component equal to 2.

The distance of C from both A and B is 12 units.

Show that one of the two possible position vectors of C is $2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ and determine the other.

$$\mathbf{c} = 2\mathbf{i} + \frac{61}{25}\mathbf{j} + \frac{2}{25}\mathbf{k}$$

$A(10, 9, -6)$ $B(6, -3, 10)$ $C(2, y, z)$

START BY FINDING \vec{AC} & \vec{BC}

$$\vec{AC} = \underline{c} - \underline{a} = (2, y, z) - (10, 9, -6) = (-8, y-9, z+6)$$

$$\vec{BC} = \underline{c} - \underline{b} = (2, y, z) - (6, -3, 10) = (-4, y+3, z-10)$$

NEXT SET UP EQUALS EXPRESSIONS FOR EACH OF THE MODULI

$$\Rightarrow |-\underline{8}, y-9, z+6| = 12 \quad \Rightarrow |-4, y+3, z-10| = 12$$

$$\Rightarrow \sqrt{64 + (y-9)^2 + (z+6)^2} = 12 \quad \Rightarrow \sqrt{16 + (y+3)^2 + (z-10)^2} = 12$$

$$\Rightarrow 64 + (y-9)^2 + (z+6)^2 = 144 \quad \Rightarrow 16 + (y+3)^2 + (z-10)^2 = 144$$

$$\Rightarrow (y-9)^2 + (z+6)^2 = 80 \quad \Rightarrow (y+3)^2 + (z-10)^2 = 128$$

$$\Rightarrow y^2 - 18y + 81 + z^2 + 12z + 36 = 144 \quad \Rightarrow y^2 + 6y + 9 + z^2 - 20z + 100 = 128$$

$$\Rightarrow y^2 + z^2 - 18y + 12z = -37 \quad \Rightarrow y^2 + z^2 + 6y - 20z = 19$$

SOLVING SIMULTANEOUSLY BY SUBTRACTING THE EQUATIONS

$$\Rightarrow \left\{ \begin{array}{l} y^2 + z^2 - 18y + 12z = -37 \\ y^2 + z^2 + 6y - 20z = 19 \end{array} \right. \Rightarrow \begin{array}{l} 2y^2 - 32y + 38 = 56 \\ 2y = 42 - 7 \end{array} \Rightarrow \begin{array}{l} 2y = 42 + 7 \\ 2y = 49 \end{array} \Rightarrow \underline{y = 42 + 7}$$

TAKE ONE OF THE EQUATIONS SUCH AS

$$\Rightarrow y^2 + z^2 + 6y - 20z = 19 \quad \downarrow \times 9$$

$$\Rightarrow 9y^2 + 9z^2 + 54y - 180z = 171$$

$$\Rightarrow (3y)^2 + 9z^2 + 18(3y) - 180z = 171$$

$$\Rightarrow (3z+7)^2 + 9z^2 + 18(3z+7) - 180z = 171$$

$$\Rightarrow 16z^2 + 56z + 49 + 9z^2 + 72z + 126 - 180z - 171 = 0$$

$$\Rightarrow 25z^2 - 52z + 4 = 0$$

$$\Rightarrow (z-2)(25z+2) = 0$$

$$\Rightarrow z = \begin{cases} 2 \\ -\frac{2}{25} \end{cases}$$

FINALLY, RAISING $3y = 42 + 7$

- IF $z = 2$, $3y = 42 + 7$
- IF $z = -\frac{2}{25}$, $3y = \frac{42}{25} + 7$

$$\begin{array}{l} 3y = 42 + 7 \\ 3y = \frac{42}{25} + 7 \\ 3y = \frac{183}{25} \\ y = \frac{61}{25} \end{array}$$

$\therefore (2, \underline{y}, z) \quad \& \quad (2, \underline{\frac{61}{25}}, \underline{z})$

ANGLES AND VECTORS

Question 1

Find the angle between each pair of vectors.

- a) $4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $8\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$
- b) $3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $-\mathbf{i} - \mathbf{j} + 2\mathbf{k}$
- c) $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$
- d) $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$
- e) $2\mathbf{i} - 7\mathbf{k}$ and $3\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}$

16.0° , 168.6° , 63.6° , 42.7° , 103.2°

$$\begin{aligned}
 & \text{(a)} (4, 1, -3) \cdot (8, 2, -3) = |(4, 1, -3)| |(8, 2, -3)| \cos \theta \\
 & \Rightarrow 32 + 2 + 9 = \sqrt{16+1+9} \sqrt{64+4+9} \cos \theta \\
 & \Rightarrow 43 = \sqrt{26} \sqrt{77} \cos \theta \\
 & \Rightarrow \cos \theta = \frac{43}{\sqrt{26} \sqrt{77}} \\
 & \Rightarrow \theta \approx 16.0^\circ
 \end{aligned}$$

$$\begin{aligned}
 & \text{(b)} (3, 3, -1) \cdot (-1, -1, 2) = |(3, 3, -1)| |(-1, -1, 2)| \cos \theta \\
 & \Rightarrow -3 - 3 - 2 = \sqrt{9+9+1} \sqrt{1+1+4} \cos \theta \\
 & \Rightarrow -8 = \sqrt{19} \sqrt{6} \cos \theta \\
 & \Rightarrow \cos \theta = \frac{-8}{\sqrt{19} \sqrt{6}} \\
 & \Rightarrow \theta \approx 168.6^\circ
 \end{aligned}$$

$$\begin{aligned}
 & \text{(c)} (2, 2, 1) \cdot (1, -2, 2) = |(2, 2, 1)| |(1, -2, 2)| \cos \theta \\
 & \Rightarrow 2 + 4 - 2 = \sqrt{4+4+1} \sqrt{1+4+4} \cos \theta \\
 & \Rightarrow 4 = \sqrt{13} \sqrt{9} \cos \theta \\
 & \Rightarrow \cos \theta = \frac{4}{\sqrt{13} \sqrt{9}} \\
 & \Rightarrow \theta \approx 63.6^\circ
 \end{aligned}$$

$$\begin{aligned}
 & \text{(d)} (6, 2, 3) \cdot (3, -4, 2) = |(6, 2, 3)| |(3, -4, 2)| \cos \theta \\
 & \Rightarrow 18 + 12 - 6 = \sqrt{36+4+9} \sqrt{9+16+4} \cos \theta \\
 & \Rightarrow 36 = \sqrt{49} \sqrt{29} \cos \theta \\
 & \Rightarrow \cos \theta = \frac{36}{\sqrt{49} \sqrt{29}} \\
 & \Rightarrow \theta \approx 42.7^\circ
 \end{aligned}$$

$$\begin{aligned}
 & \text{(e)} (2, 0, -1) \cdot (3, 8, 2) = |(2, 0, -1)| |(3, 8, 2)| \cos \theta \\
 & \Rightarrow 6 + 0 - 2 = \sqrt{4+0+1} \sqrt{9+64+4} \cos \theta \\
 & \Rightarrow -15 = \sqrt{5} \sqrt{73} \cos \theta \\
 & \Rightarrow \cos \theta = \frac{-15}{\sqrt{5} \sqrt{73}}
 \end{aligned}$$

Question 2

Find the angle between each pair of vectors.

- a) $3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$
- b) $4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$
- c) $\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$
- d) $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$
- e) $8\mathbf{i} - 5\mathbf{k}$ and $4\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

41.9° , 92.9° , 144.4° , 58.4° , 73.7°

$\text{(a)} (3, 2, 3) \cdot (2, 1, 4) = (3, 2, 3) (2, 1, 4) \cos \theta$ $\rightarrow 6 - 2 + 12 = \sqrt{3^2 + 2^2 + 3^2} \sqrt{2^2 + 1^2 + 4^2} \cos \theta$ $\rightarrow 16 = \sqrt{22} \sqrt{21} \cos \theta$ $\rightarrow \cos \theta = \frac{16}{\sqrt{22} \sqrt{21}}$ $\rightarrow \theta \approx 41.9^\circ$	$\text{(d)} (2\mathbf{i}) \cdot (\mathbf{i} - 5\mathbf{j}) = \mathbf{i} \mathbf{i} - 5\mathbf{j} \cos \theta$ $\rightarrow 2 - 0 = \sqrt{1} \sqrt{1 + 25} \cos \theta$ $\rightarrow 2 = \sqrt{26} \cos \theta$ $\rightarrow \cos \theta = \frac{2}{\sqrt{26}}$ $\rightarrow \theta \approx 37.4^\circ$
$\text{(b)} (4, 3, 1) \cdot (3, 4, 1) = (4, 3, 1) (3, 4, 1) \cos \theta$ $\rightarrow 12 - 12 + 3 = \sqrt{16 + 9 + 1} \sqrt{9 + 16 + 1} \cos \theta$ $\rightarrow 3 = \sqrt{36} \sqrt{36} \cos \theta$ $\rightarrow 3 = 36 \cos \theta$ $\rightarrow \cos \theta = \frac{1}{12}$ $\rightarrow \theta \approx 85.4^\circ$	$\text{(e)} (2\mathbf{i}) \cdot (\mathbf{i} - 5\mathbf{j}) = \mathbf{i} \mathbf{i} - 5\mathbf{j} \cos \theta$ $\rightarrow 2 - 0 = \sqrt{1} \sqrt{1 + 25} \cos \theta$ $\rightarrow 2 = \sqrt{26} \cos \theta$ $\rightarrow \cos \theta = \frac{2}{\sqrt{26}}$ $\rightarrow \theta \approx 73.7^\circ$
$\text{(c)} (1, 5, 1) \cdot (1, 2, -3) = (1, 5, 1) (1, 2, -3) \cos \theta$ $\rightarrow 1 - 10 - 9 = \sqrt{1 + 25 + 1} \sqrt{1 + 4 + 9} \cos \theta$ $\rightarrow -18 = \sqrt{26} \sqrt{14} \cos \theta$ $\rightarrow \cos \theta = -\frac{18}{\sqrt{26} \sqrt{14}}$ $\rightarrow \theta \approx 144.4^\circ$	$\text{(f)} (8\mathbf{i} - 5\mathbf{k}) \cdot (4\mathbf{i} - 7\mathbf{j}) = (8\mathbf{i} - 5\mathbf{k}) (4\mathbf{i} - 7\mathbf{j}) \cos \theta$ $\rightarrow 32 - 0 - 10 = \sqrt{64 + 25} \sqrt{16 + 49} \cos \theta$ $\rightarrow 22 = \sqrt{89} \sqrt{65} \cos \theta$ $\rightarrow \cos \theta = \frac{22}{\sqrt{89} \sqrt{65}}$ $\rightarrow \theta \approx 58.4^\circ$

Question 3

Find the angle between each pair of vectors.

a) $2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ and $4\mathbf{i} - \mathbf{j} - \mathbf{k}$

b) $4\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$ and $\mathbf{i} - \mathbf{j} - 5\mathbf{k}$

c) $2\mathbf{i} - 6\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + 5\mathbf{j} - \mathbf{k}$

d) $3\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$

e) $3\mathbf{i} - \mathbf{j} - 5\mathbf{k}$ and $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

 93.6° , 31.0° , 150.6° , 20.5° , 123.5°

(a) $(2, 4, 6) \cdot (4, -1, -1) = |2\sqrt{14}| |4\sqrt{2}| |\cos\theta$
 $\Rightarrow 2(-4) + 4(-6) = \sqrt{4+16+36} \sqrt{16+1+1} \cos\theta$
 $\Rightarrow -28 = \sqrt{26} \sqrt{18} \cos\theta$
 $\Rightarrow \cos\theta = -\frac{2}{\sqrt{468}}$
 $\Rightarrow \theta = 135^\circ$

(b) $(4, 2, -7) \cdot (1, -1, -5) = |4\sqrt{17}| |1\sqrt{3}| |\cos\theta$
 $\Rightarrow 4(-1) + 2(-5) = \sqrt{16+4+49} \sqrt{1+1+25} \cos\theta$
 $\Rightarrow -12 = \sqrt{61} \sqrt{31} \cos\theta$
 $\Rightarrow \cos\theta = -\frac{12}{\sqrt{1861}}$
 $\Rightarrow \theta = 31.0^\circ$

(c) $(3, -1, -5) \cdot (1, 1, 2) = |3\sqrt{3}| |1\sqrt{3}| |\cos\theta$
 $\Rightarrow (3)(1) - (-1)(2) = \sqrt{9+1+25} \sqrt{1+1+4} \cos\theta$
 $\Rightarrow 3 - 1 = \sqrt{35} \sqrt{3} \cos\theta$
 $\Rightarrow -2 = \sqrt{35} \sqrt{3} \cos\theta$
 $\Rightarrow \cos\theta = -\frac{2}{\sqrt{105}}$
 $\Rightarrow \theta = 123.5^\circ$

(d) $(2, -1, -1) \cdot (1, 1, 2) = |2\sqrt{2}| |1\sqrt{3}| |\cos\theta$
 $\Rightarrow (2)(1) - (-1)(1) = \sqrt{4+1+1} \sqrt{1+1+4} \cos\theta$
 $\Rightarrow 2 - 1 = \sqrt{6} \sqrt{3} \cos\theta$
 $\Rightarrow \cos\theta = -\frac{1}{\sqrt{18}}$
 $\Rightarrow \theta = 150.6^\circ$

(e) $(2, -6, 1) \cdot (1, 5, -1) = |2\sqrt{36}| |1\sqrt{26}| |\cos\theta$
 $\Rightarrow (2)(1) - 6(5) + (1)(-1) = \sqrt{4+36+1} \sqrt{1+25+1} \cos\theta$
 $\Rightarrow 2 - 30 - 1 = \sqrt{42} \sqrt{27} \cos\theta$
 $\Rightarrow -29 = \sqrt{42} \sqrt{27} \cos\theta$
 $\Rightarrow \cos\theta = -\frac{29}{\sqrt{1134}}$
 $\Rightarrow \theta \approx 150.6^\circ$

Question 4

Find the angle between each pair of vectors.

- a) $\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} - \mathbf{j} + \mathbf{k}$
- b) $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$
- c) $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
- d) $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
- e) $3\mathbf{i} - 4\mathbf{j} - 12\mathbf{k}$ and $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

84.8° , 109.1° , 25.2° , 80.4° , 75.1°

(a) $(1, 3, 1) \cdot (3, -1, 1) = |(1, 3, 1)| |(3, -1, 1)| \cos \theta$
 $\Rightarrow 3 - 3 + 1 = \sqrt{1+9+1} \sqrt{9+1+1} \cos \theta$
 $\Rightarrow 1 = 4\sqrt{4} \cos \theta$
 $\Rightarrow \theta \approx 11.06^\circ$
 $\Rightarrow \cos \theta = \frac{1}{4\sqrt{4}}$
 $\Rightarrow \theta \approx 89.8^\circ$

(b) $(1, -2, 1) \cdot (2, 1, -3) = |(1, -2, 1)| |(2, 1, -3)| \cos \theta$
 $\Rightarrow 2 - 2 - 3 = \sqrt{1+4+1} \sqrt{4+1+9} \cos \theta$
 $\Rightarrow -3 = \sqrt{6} \sqrt{14} \cos \theta$
 $\Rightarrow \cos \theta = -\frac{3}{\sqrt{6} \sqrt{14}}$
 $\Rightarrow \theta \approx 101.1^\circ$

(c) $(3, -4, -12) \cdot (2, 2, -1) = |(3, -4, -12)| |(2, 2, -1)| \cos \theta$
 $\Rightarrow 6 - 8 + 12 = \sqrt{9+16+144} \sqrt{4+4+1} \cos \theta$
 $\Rightarrow 10 = 13\sqrt{5} \cos \theta$
 $\Rightarrow \cos \theta = \frac{10}{13\sqrt{5}}$
 $\Rightarrow \theta = 75.1^\circ$

(d) $(1, -1, -2) \cdot (2, 1, -1) = |(1, -1, -2)| |(2, 1, -1)| \cos \theta$
 $\Rightarrow 1 - 2 - 2 = \sqrt{1+1+4} \sqrt{4+1+1} \cos \theta$
 $\Rightarrow 1 = 6 \cos \theta$
 $\Rightarrow \cos \theta = \frac{1}{6}$
 $\Rightarrow \theta \approx 85.4^\circ$

Question 5

Find the angle $\angle CAB$ for each set of the coordinates given.

a) $A(3,3,4)$, $B(2,7,2)$, $C(5,6,0)$

b) $A(6,1,4)$, $B(1,-1,5)$, $C(4,4,-3)$

c) $A(2,1,-3)$, $B(-1,-1,4)$, $C(0,4,-7)$

d) $A(2,2,1)$, $B(4,0,-3)$, $C(1,-3,2)$

43.2° , 94.0° , 131.3° , 81.0°

<p>(a)</p> $\begin{aligned} \overrightarrow{AB} &= b - a = (1, 4, -2) \\ &= (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= (1, 4, -2) \\ &\ \overrightarrow{AB}\ = \sqrt{(1^2 + 4^2 + (-2)^2)} = \sqrt{1+16+4} = \sqrt{21} \\ &\cos\theta = \frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{\ \overrightarrow{AC}\ \ \overrightarrow{AB}\ } \\ &= \frac{(2, 3, -1) \cdot (1, 4, -2)}{\sqrt{21} \sqrt{21}} \\ &= \frac{2+12-8}{21} = \frac{6}{21} = \frac{2}{7} \\ &\theta = \cos^{-1}\left(\frac{2}{7}\right) \approx 65.3^\circ \end{aligned}$	<p>(b)</p> $\begin{aligned} \overrightarrow{AC} &= c - a = (1, 3, -1) \\ &= (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= (1, 3, -1) \\ &\ \overrightarrow{AC}\ = \sqrt{(1^2 + 3^2 + (-1)^2)} = \sqrt{1+9+1} = \sqrt{11} \\ &\overrightarrow{AB} = b - a = (5, 0, -3) \\ &= (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= (5, 0, -3) \\ &\ \overrightarrow{AB}\ = \sqrt{(5^2 + 0^2 + (-3)^2)} = \sqrt{25+9} = \sqrt{34} \\ &\cos\theta = \frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{\ \overrightarrow{AC}\ \ \overrightarrow{AB}\ } \\ &= \frac{(1, 3, -1) \cdot (5, 0, -3)}{\sqrt{11} \sqrt{34}} \\ &= \frac{5-9}{\sqrt{374}} = \frac{-4}{\sqrt{374}} \\ &\theta = \cos^{-1}\left(\frac{-4}{\sqrt{374}}\right) \approx 104.0^\circ \end{aligned}$
---	---

Question 6

Find the angle $\angle CAB$ for each set of the coordinates given.

a) $A(2, -2, 4)$, $B(1, 3, 2)$, $C(5, -4, 1)$

b) $A(2, 0, 0)$, $B(0, 0, 1)$, $C(3, 1, 3)$

c) $A(1, 5, 3)$, $B(2, 8, 6)$, $C(-1, -10, -5)$

d) $A(5, 0, -2)$, $B(0, -1, 4)$, $C(9, 4, 0)$

105.8° , 82.3° , 162.1° , 104.7°

(a) $A(2, -2, 4)$, $B(1, 3, 2)$, $C(5, -4, 1)$

$$\begin{aligned} \vec{AC} &= \vec{c} - \vec{a} = (5, -4, 1) - (2, -2, 4) = (3, -2, -3) \\ \vec{AB} &= \vec{b} - \vec{a} = (1, 3, 2) - (2, -2, 4) = (-1, 5, -2) \\ \text{Dotting product} \\ (3, -2, -3) \cdot (-1, 5, -2) &= [3, -2, -3] \cdot [-1, 5, -2] = 14 \\ -3 - 10 + 6 &= \sqrt{1^2 + 5^2 + (-3)^2} \sqrt{(-1)^2 + 5^2 + (-2)^2} \cos\theta \\ -7 &= \sqrt{35} \sqrt{27} \cos\theta \\ \cos\theta &= -\frac{7}{\sqrt{35} \sqrt{27}} \\ \theta &\approx 105.8^\circ \end{aligned}$$

(b) $A(2, 0, 0)$, $B(0, 0, 1)$, $C(3, 1, 3)$

$$\begin{aligned} \vec{AC} &= \vec{c} - \vec{a} = (3, 1, 3) - (2, 0, 0) = (1, 1, 3) \\ \vec{AB} &= \vec{b} - \vec{a} = (0, 0, 1) - (2, 0, 0) = (-2, 0, 1) \\ \text{Dotting product} \\ (1, 1, 3) \cdot (-2, 0, 1) &= [1, 1, 3] \cdot [-2, 0, 1] = 14 \\ -2 + 0 + 3 &= \sqrt{1^2 + 1^2 + 3^2} \sqrt{(-2)^2 + 0^2 + 1^2} \cos\theta \\ 1 &= \sqrt{11} \sqrt{5} \cos\theta \\ \cos\theta &= \frac{1}{\sqrt{55}} \\ \theta &\approx 82.3^\circ \end{aligned}$$

(c) $A(1, 5, 3)$, $B(2, 8, 6)$, $C(-1, -10, -5)$

$$\begin{aligned} \vec{AC} &= \vec{c} - \vec{a} = (-1, -10, -5) - (1, 5, 3) = (-2, -15, -8) \\ \vec{AB} &= \vec{b} - \vec{a} = (2, 8, 6) - (1, 5, 3) = (1, 3, 3) \\ \text{Dotting product} \\ (-2, -15, -8) \cdot (1, 3, 3) &= -2 - 45 - 24 = (-1, 5, 3) \cdot (1, 3, 3) = 14 \\ -49 - 24 &= \sqrt{(-2)^2 + (-15)^2 + (-8)^2} \sqrt{(1)^2 + (3)^2 + (3)^2} \cos\theta \\ -71 &= \sqrt{255} \sqrt{27} \cos\theta \\ \cos\theta &= -\frac{71}{\sqrt{255} \sqrt{27}} \quad \therefore \theta = 162.1^\circ \end{aligned}$$

(d) $A(5, 0, -2)$, $B(0, -1, 4)$, $C(9, 4, 0)$

$$\begin{aligned} \vec{AB} &= \vec{b} - \vec{a} = (0, -1, 4) - (5, 0, -2) = (-5, -1, 6) \\ \vec{AC} &= \vec{c} - \vec{a} = (9, 4, 0) - (5, 0, -2) = (4, 4, 2) \\ \text{Dotting product} \\ (-5, -1, 6) \cdot (4, 4, 2) &= (-5, -1, 6) \cdot [4, 4, 2] = 14 \\ -20 - 4 + 12 &= \sqrt{(-5)^2 + (-1)^2 + 6^2} \sqrt{(4)^2 + (4)^2 + (2)^2} \cos\theta \\ -12 &= \sqrt{42} \sqrt{36} \cos\theta \\ \cos\theta &= -\frac{12}{\sqrt{42} \sqrt{36}} \quad \therefore \theta = 104.7^\circ \end{aligned}$$

Question 7

The vectors \mathbf{a} and \mathbf{b} are perpendicular, and λ is a scalar constant.

Find in each case the possible value(s) of λ .

- a) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \lambda\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + \lambda\mathbf{j} - 5\mathbf{k}$
- b) $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 2\lambda\mathbf{k}$ and $\mathbf{b} = \lambda\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
- c) $\mathbf{a} = 4\lambda\mathbf{i} + (\lambda + 1)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 6\mathbf{j} + 12\mathbf{k}$
- d) $\mathbf{a} = (2\lambda + 2)\mathbf{i} + \mathbf{j} + (\lambda + 1)\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 6\lambda\mathbf{j} + \lambda\mathbf{k}$
- e) $\mathbf{a} = 6\mathbf{i} + (\lambda + 1)\mathbf{j} + (\lambda - 4)\mathbf{k}$ and $\mathbf{b} = \lambda\mathbf{i} + (\lambda - 2)\mathbf{j} + 6\mathbf{k}$

$$\boxed{\lambda_a = 5}, \boxed{\lambda_b = 1}, \boxed{\lambda_c = 9}, \boxed{\lambda_d = 1, -4}, \boxed{\lambda_e = 2, -13}$$

<p>(3) $(2\mathbf{i}, 3\mathbf{j}, \lambda) \cdot (5\mathbf{i}, \lambda\mathbf{j}, -5\mathbf{k}) = 0$ $10 + 3\lambda - 5\lambda = 0$ $10 = 2\lambda$ $\lambda = 5$</p> <p>(4) $(2\mathbf{i}, 2\mathbf{j}, 1, \lambda\mathbf{k}) \cdot (-3\mathbf{i}, \lambda\mathbf{j}, \lambda\mathbf{k}) = 0$ $-4\mathbf{i} \cdot \mathbf{i} + 2\mathbf{j} \cdot \lambda\mathbf{j} + \lambda\mathbf{k} \cdot \lambda\mathbf{k} = 0$ $4\lambda^2 - 4 = 0$ $4(\lambda^2 - 1) = 0$ $\lambda^2 = 1$ $\lambda = \pm 1$</p> <p>(5) $(4\mathbf{i}, -\mathbf{j}, 2\mathbf{k}) \cdot (3\mathbf{i}, \lambda\mathbf{j}, -\mathbf{k}) = 0$ $12\mathbf{i}^2 - 2\lambda\mathbf{j}^2 - 2\mathbf{k}^2 = 0$ $12 - 2\lambda + 2 = 0$ $14 = 2\lambda$ $\lambda = 7$</p> <p>(6) $((4\mathbf{i}, \lambda + 1, 2), (1, -\lambda, 12)) = 0$ $4\mathbf{i} \cdot \mathbf{i} - \lambda\mathbf{j} \cdot \mathbf{j} + 2\mathbf{k} \cdot 12 = 0$ $4\lambda - \lambda^2 - 24 = 0$ $18 = 2\lambda$ $\lambda = 9$</p>	<p>(7) $(2\mathbf{i}, 2\mathbf{j}, 1, \lambda\mathbf{k}) \cdot (5\mathbf{i}, \lambda\mathbf{j}, -5\mathbf{k}) = 0$ $-4\mathbf{i} \cdot \mathbf{i} + 2\mathbf{j} \cdot \lambda\mathbf{j} + \lambda\mathbf{k} \cdot -5\mathbf{k} = 0$ $4\lambda^2 + 3\lambda - 4 = 0$ $(\lambda - 1)(4\lambda + 4) = 0$ $\lambda = 1$ or $\lambda = -4$</p> <p>(8) $(6\mathbf{i}, 2\mathbf{j}, 1, \lambda\mathbf{k}) \cdot (3\mathbf{i}, \lambda\mathbf{j}, -\mathbf{k}) = 0$ $18\mathbf{i}^2 + 2\lambda\mathbf{j}^2 - \lambda\mathbf{k}^2 = 0$ $18 + 2\lambda^2 - \lambda^2 = 0$ $18 + \lambda^2 = 0$ $(\lambda - 2)(\lambda + 18) = 0$ $\lambda = 2$ or $\lambda = -18$</p>
---	---

Question 8

The vectors \mathbf{a} and \mathbf{b} are given by

$$\mathbf{a} = 5\mathbf{i} - 4\mathbf{j} + a\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + b\mathbf{j} - 3\mathbf{k}.$$

- a) If \mathbf{a} and \mathbf{b} are perpendicular find a relationship between a and b .
- b) If instead \mathbf{a} and \mathbf{b} are parallel find the value of a and the value of b .

$$3a + 4b = 10, \quad a = -\frac{8}{5}, \quad b = -\frac{15}{2}$$

(a) If perpendicular $\mathbf{a} \cdot \mathbf{b} = 0$ $\Rightarrow (5, -4, a) \cdot (2, b, -3) = 0$ $\Rightarrow 10 - 4b - 3a = 0$ $\Rightarrow 10 = 4b + 3a$ $\Rightarrow 3a + 4b = 10$
(b) If parallel $\frac{\mathbf{a}}{\ \mathbf{a}\ } = \lambda \frac{\mathbf{b}}{\ \mathbf{b}\ }$ for some λ $\Rightarrow (5, -4, a) = \lambda(2, b, -3)$ $\Rightarrow (5, -4, a) = (2\lambda, b\lambda, -3\lambda)$
(i): $2\lambda = 5 \quad \lambda = \frac{5}{2}$ $2\lambda = \frac{5}{2} \quad \lambda = \frac{5}{2}$ $-4 = b\lambda \quad -4 = \frac{5}{2}b$ $-4 = \frac{5}{2}b \quad b = -\frac{8}{5}$ $-8 = b \quad b = -\frac{8}{5}$ $a = -3\lambda \quad a = -3 \times \frac{5}{2}$ $a = -\frac{15}{2} \quad a = -\frac{15}{2}$

Question 9

Find a vector with **integer** components which is perpendicular to **both** the vectors given below.

(Do not use the cross product)

- a) $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{i} + \mathbf{j} - 3\mathbf{k}$
 - b) $6\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
 - c) $7\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $6\mathbf{i} + \mathbf{j} + 5\mathbf{k}$
 - d) $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$
 - e) $8\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

$$|\mathbf{i}+2\mathbf{j}+\mathbf{k}|, |\mathbf{5i}-8\mathbf{j}-19\mathbf{k}|, |\mathbf{9i}+41\mathbf{j}-19\mathbf{k}|, |-16\mathbf{i}+3\mathbf{j}+11\mathbf{k}|, |\mathbf{i}-22\mathbf{j}-36\mathbf{k}|$$

(a) Let required vector be (x, y, z)
 Then $\begin{cases} (x, y, z) \cdot (2, -3, 4) = 0 \\ (x, y, z) \cdot (1, 1, -1) = 0 \end{cases} \Rightarrow \begin{cases} 2x - 3y + 4z = 0 \\ x + y - z = 0 \end{cases}$

$\boxed{\text{Let } 2x = 1}$ $2x - 3y + 4z = 0 \Rightarrow 2x - 3y = -4 \quad \boxed{1}$
 $x + y - z = 0 \Rightarrow x + y = z \quad \boxed{2}$

(2x) $\begin{cases} 2x - 3y = -4 \\ x + y = z \end{cases} \xrightarrow{\text{Add}} \begin{cases} 3x - 2y = -4 \\ x + y = z \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \\ z = 1 \end{cases}$

\therefore Required vector is $\underline{\underline{1 + 2z + k}}$

(b) Let required vector be (x, y, z)
 Then $\begin{cases} (x, y, z) \cdot (1, -1, 2) = 0 \\ (x, y, z) \cdot (-1, 2, 1) = 0 \end{cases} \Rightarrow \begin{cases} x - y + 2z = 0 \\ -x + 2y + z = 0 \end{cases}$

$\boxed{\text{Let } x = 1}$ $\begin{cases} 1 - y + 2z = 0 \\ -1 + 2y + z = 0 \end{cases} \Rightarrow \begin{cases} -y + 2z = -1 \\ -1 + 2y + z = 0 \end{cases} \Rightarrow \begin{cases} -y + 2z = -1 \\ 2y + z = 1 \end{cases} \Rightarrow$

$x(-1 + 2z) - 3y = 18$ $\begin{array}{l} \text{Add} \quad -5z = 18 \\ \frac{-5z}{5} = \frac{18}{5} \end{array}$

$\therefore \begin{cases} -2y + 15 = 1 \\ -3y = \frac{24}{5} \\ y = -\frac{8}{5} \end{cases}$

\therefore Required vector $\underline{\underline{(1, -\frac{8}{5}, \frac{18}{5})}}$
 Since it by 5
 $(5, -8, 18)$
 $4 \cdot 5 - 18 - 19 \underline{\underline{k}}$

(c) LET THE REQUIRED VECTOR BE $(x_1 y_1 z_1)$

$$\begin{cases} (2, 3, 5)(x_1, -2, 1) = 0 \\ (2, 3, 5)(1, 1, 5) = 0 \end{cases} \Rightarrow \begin{cases} 7x_1 - 2y_1 - z_1 = 0 \\ 6x_1 + y_1 + 5z_1 = 0 \end{cases}$$

$$\begin{matrix} 7x_1 - 2y_1 - z_1 = 0 \\ 6x_1 + y_1 + 5z_1 = 0 \end{matrix} \xrightarrow{\times 2} \begin{matrix} -2y_1 - 2z_1 = -7 \\ 2y_1 + 10z_1 = 12 \end{matrix} \text{ ADD} \quad \boxed{\begin{matrix} 9z_1 = -15 \\ z_1 = -\frac{15}{9} \end{matrix}}$$

Thus $6x_1 + 5z_1 = 0$
 $y_1 = -6 - 5(-\frac{15}{9})$
 $y_1 = -6 + \frac{75}{9}$
 $y_1 = \frac{41}{9}$

\therefore VECTOR IS $(1, \frac{41}{9}, -\frac{15}{9})$
 SCALE IT TO $(9, 41, -15)$
 $\therefore 9i + 41j - 15k$

(d) LET THE REQUIRED VECTOR BE $(x_1 y_1 z_1)$

$$\begin{cases} (2, 3, 5)(1, -2, 1) = 0 \\ (2, 3, 5)(4, 3, 1) = 0 \end{cases} \Rightarrow \begin{cases} 2x_1 - 2y_1 + 2z_1 = 0 \\ 4x_1 + 3y_1 + z_1 = 0 \end{cases}$$

$$\begin{matrix} 2x_1 - 2y_1 + 2z_1 = 0 \\ 4x_1 + 3y_1 + z_1 = 0 \end{matrix} \xrightarrow{\times 2} \begin{matrix} 3x_1 - 6y_1 + 6z_1 = 0 \\ 8x_1 + 6y_1 + 2z_1 = 0 \end{matrix} \text{ ADD}$$
 $11z_1 = 0 \Rightarrow z_1 = 0$
 $x_1 = \frac{16}{11}$
 $2 - 2y_1 + 2 = 0 \Rightarrow y_1 = 2$
 $-\frac{16}{11} + 2y_1 + 0 = 0 \Rightarrow y_1 = \frac{16}{11}$
 $y_1 = \frac{2}{11} = 2y_1$
 $y_1 = \frac{2}{11}$
 $\therefore (-\frac{16}{11}, \frac{2}{11}, 0)$
 SCALE THE VECTOR TO $(-16, 2, 0)$
 $\therefore (-16i + 2j + 0k)$

(c) Let the required vector be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{aligned} (\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}) &= 0 \quad \Rightarrow \quad x + 2y - 2z = 0 \\ (\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}) &= 0 \quad \Rightarrow \quad 5x - 3y + 2z = 0 \quad \text{Let } x = 1 \\ 8 + 4y - 2z &= 0 \quad ? \quad x_2 \\ 6 - 3y + 2z &= 0 \quad x_1 \quad |+| \quad 14 + 4y - 2z = 0 \\ 6 - 3y + 2z &= 0 \quad |+| \quad 22 + 4y = 0 \\ \hline y &= -\underline{\underline{5}} \end{aligned}$$

This $\begin{pmatrix} 8 + 2y - 2z \\ 6 - 3y + 2z \end{pmatrix} = \begin{pmatrix} 8 - 44 + 2 \cdot -5 \\ 6 - 3(-5) + 2(-5) \end{pmatrix} = \begin{pmatrix} -38 \\ 1 \end{pmatrix}$

$\therefore (1, -22, -36)$

I.E. $\overrightarrow{[-22, -36]}$

Question 10

The points A , B and C have coordinates $(1, 2, 1)$, $(5, 1, 4)$ and $(7, 6, 3)$, respectively.

Show that $\angle ABC = 90^\circ$ and hence find the exact area of the triangle ABC .

$$\text{area} = \sqrt{195}$$

$\vec{BA} = \underline{\Delta} \vec{b}_2 = (1, 2, 1) - (5, 1, 4) = (-4, 1, -3)$
 $\vec{BC} = \underline{\Delta} \vec{b}_3 = (7, 6, 3) - (5, 1, 4) = (2, 5, -1)$
 $\vec{BA} \cdot \vec{BC} = (-4, 1, -3) \cdot (2, 5, -1) = -8 + 5 + 3 = 0 \Rightarrow \angle ABC = 90^\circ$

$\bullet |\vec{BA}| = \sqrt{(-4)^2 + 1^2 + (-3)^2} = \sqrt{16 + 1 + 9} = \sqrt{26}$
 $\bullet |\vec{BC}| = \sqrt{2^2 + 5^2 + (-1)^2} = \sqrt{4 + 25 + 1} = \sqrt{30}$

$\text{Area} = \frac{1}{2} |\vec{BA}| |\vec{BC}| = \frac{1}{2} \sqrt{26} \sqrt{30} = \sqrt{13} \sqrt{78} = \sqrt{195}$

THE EQUATION OF A LINE

Question 1

- a) Find a vector equation of the straight line l , that passes through the point $A(7, -1, 2)$ and is in the direction $-2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.
- b) If $B(p, q, 6)$ lies on l find the value of p and the value of q .

$$\boxed{\mathbf{r} = 7\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(-2\mathbf{i} + 3\mathbf{j} + \mathbf{k})}, \boxed{p = -1}, \boxed{q = 11}$$

(a) $\begin{aligned} l &= (\text{fixed point}) + \lambda(\text{direction vector}) \\ l &= (7, -1, 2) + \lambda(-2, 3, 1) \\ (x, y, z) &= (7-2\lambda, 3\lambda-1, \lambda+2) \end{aligned}$

(b) $\begin{aligned} (p, q, 6) &= (7-2\lambda, 3\lambda-1, \lambda+2) \\ 6 &= \lambda+2 \implies \lambda=4 \\ \therefore p &= 7-2\times 4 = -1 \\ \therefore q &= 3\times 4 - 1 = 11 \end{aligned}$

Question 2

- a) Find a vector equation of the straight line l , that passes through the point $A(2, -6, -4)$ and is parallel to the vector $\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$.
- b) If $B(p, 0, q)$ lies on l find the value of p and the value of q .

$$\boxed{\mathbf{r} = 2\mathbf{i} - 6\mathbf{j} - 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})}, \boxed{p = 5}, \boxed{q = 11}$$

(a) $\begin{aligned} l &= (\text{fixed point}) + \lambda(\text{direction vector}) \\ l &= (2, -6, -4) + \lambda(1, 2, 5) \\ (x, y, z) &= (\lambda+2, 2\lambda-6, 5\lambda-4) \end{aligned}$

(b) By inspection \perp : $\begin{aligned} 2+2+5 &= 9 \neq 8 \\ p+2+2 &= 3+2 = 5 \\ q &= 5-4 = 1 \end{aligned}$

Question 3

- a) Find a vector equation of the straight line l , that passes through the point $A(3, -1, 8)$ and is in the direction $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$.
- b) If $B(9, p, q)$ lies on l find the value of p and the value of q .

$$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 8\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}), [p = -10], [q = 23]$$

(a) $\mathbf{r} = (\text{fixed point}) + \lambda(\text{direction vector})$
 $\mathbf{r} = (3, -1, 8) + \lambda(2, -3, 5)$
 $(x, y, z) = (2\lambda + 3, -3\lambda - 1, 5\lambda + 8)$

(b) $(9, p, q) = (2\lambda + 3, -3\lambda - 1, 5\lambda + 8)$
 $(1): 9 = 2\lambda + 3 \Rightarrow \lambda = 3$
 $\therefore 3 = -3\lambda - 1 = -10$
 $\therefore 9 = 5\lambda + 8 = 23$

Question 4

- a) Find a vector equation of the straight line l that passes through the point $A(2, -1, -5)$ and is in the direction $2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$.
- b) If $B(-10, p, q)$ lies on l find the values of p and the value of q .

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} - 5\mathbf{k} + \lambda(2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}), [p = -31], [q = -23]$$

(a) $\mathbf{r} = (\text{fixed point}) + \lambda(\text{direction vector})$
 $\mathbf{r} = (2, -1, -5) + \lambda(2, 5, 3)$
 $(x, y, z) = (2\lambda + 2, 5\lambda - 1, 3\lambda - 5)$

(b) looking at \perp
 $2\lambda + 2 = -10$
 $2\lambda = -12$
 $\lambda = -6$

$\therefore (2): p = 5(-6) - 1 = -31$
 $\therefore (3): q = 3(-6) - 5 = -23$

Question 5

- a) Determine a vector equation of the straight line l that passes through the point $A(4, -1, 2)$ and is in the direction $2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$.

b) If $B(p, 14, q)$ lies on l find the value of p and the value of q .

$$[\mathbf{r} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})], [p = 10], [q = -7]$$

$$\begin{aligned} \text{(a)} \quad \Sigma &= \overline{\left(\text{Primo KNO}_3 \right)^{-1/2} \cdot \left(\text{Keto} \right)} \\ \Gamma &= \left(4, -1, 2 \right) + \lambda \left(2, 5, -3 \right) \\ (3, y, 2) &= \left(24, 4, 2y-1, 2-3y \right) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \left\{ \begin{array}{l} 1+4y = 24 \\ 2+5y = 2y-1 \\ 2-3y = 3 \end{array} \right. \Rightarrow y=3 \end{aligned}$$

$$\begin{aligned} \text{(1):} \quad S_{1-11} &= 11y \Rightarrow y=3 \\ \therefore P &= 2x3+1 = 10 \\ q &= 2-3x3 = -7 \end{aligned}$$

Question 6

- a) Determine a vector equation of the straight line l that passes through the point $A(-7,10,-1)$ and is parallel to the vector $3\mathbf{j} - 4\mathbf{k}$.

b) If $B(p,q,-21)$ lies on l find the value of p and the value of q .

$$\boxed{\mathbf{r} = -7\mathbf{i} + 10\mathbf{j} - \mathbf{k} + \lambda(3\mathbf{j} - 4\mathbf{k})}, \quad \boxed{p = -7}, \quad \boxed{q = 25}$$

$$\textcircled{a} \quad \begin{aligned} S &= (\text{fixed point}) + \lambda(\text{direction vector}) \\ S &= (-7, 10, 1) + \lambda(9, 3, -4) \\ (3, 4, 3) &= (-7, 10, 1) + \lambda(9, 3, -4) \end{aligned} \quad \textcircled{b} \quad \begin{aligned} \text{looking at } \frac{x}{-4} &= -1 - \frac{11}{4} \\ -4x &= -4 - 11 \\ x &= 5 \\ \therefore P &= (-7, 11, 25) \\ q &= 3+4+10 = 25 \end{aligned}$$

Question 7

- a) Determine a vector equation of the straight line l that passes through the point $A(7,1,1)$ and is parallel to the vector $2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$.

- b) If $B(1, p, q)$ lies on l find the value of p and the value of q .

$$\boxed{\mathbf{r} = 7\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - 5\mathbf{j} + \mathbf{k})}, \boxed{p = 16}, \boxed{q = -2}$$

$$\begin{aligned} \text{(a)} \quad & \Gamma = (\text{fixed point}) + \lambda(\text{direction vector}) \\ & \Gamma = (7, 1, 1) + \lambda(2, -5, 1) \\ & \Gamma = (2\lambda + 7, -5\lambda + 1, \lambda + 1) \\ \text{(b)} \quad & C(1, p, q) = (2\lambda + 7, -5\lambda + 1, \lambda + 1) \\ & (1) : 2\lambda + 7 = 1 \\ & \quad \lambda = -3 \\ & \therefore p = 1 - 5(-3) = 16 \\ & \quad q = -3 + 1 = -2 \end{aligned}$$

Question 8

The straight line l passes through the point $A(5, -1, 3)$ and is parallel to the vector $p\mathbf{i} + q\mathbf{j} + 3\mathbf{k}$.

If $B(8, 8, 12)$ lies on l find the value of p and the value of q .

$$\boxed{p = 1}, \boxed{q = 3}$$

$$\begin{aligned} \Gamma &= (\text{fixed point}) + \lambda(\text{direction vector}) \\ \Gamma &= (5, -1, 3) + \lambda(p, q, 3) \\ (8, 8, 12) &= (5p + 5, 5q - 1, 3\lambda + 3) \\ \text{Now looking at } z: 3\lambda + 3 &= 12 \\ \lambda &= 3 \\ \bullet 3p + 5 &= 8 \quad \bullet 5q - 1 = 8 \\ 3p + 5 &= 8 \quad 5q - 1 = 8 \\ p &= 1 \quad q = 3 \end{aligned}$$

Question 9

- a) Determine a vector equation of the straight line l that passes through the point $A(p, 2, 3)$ and is in the direction $6\mathbf{i} - 2\mathbf{j} + q\mathbf{k}$, where p and q are scalar constants.
- b) If l passes through the origin, find the value of p and the value of q .

$$\mathbf{r} = p\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(6\mathbf{i} - 2\mathbf{j} + q\mathbf{k}), [p = -6], [q = -3]$$

$$\begin{aligned}
 \text{(a)} \quad & l = (\text{fixed point}) + \lambda(\text{direction vector}) \\
 & l = (p, 2, 3) + \lambda(6, -2, q) \\
 & l = (p+6\lambda, 2-2\lambda, 3+q\lambda) \\
 \text{(b)} \quad & (0, 0, 0) = (p+6\lambda, 2-2\lambda, 3+q\lambda) \\
 & (1): \quad 2-2\lambda=0 \\
 & \lambda=1 \\
 & (2): \quad p+6\lambda=0 \\
 & p+6=0 \\
 & p=-6 \\
 & (3): \quad 3+q\lambda=0 \\
 & 3+q=0 \\
 & q=-3
 \end{aligned}$$

Question 10

- a) Determine a vector equation of the straight line l that passes through the point $A(5, -1, -3)$ and is parallel to the vector $-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.
- b) If both the points $B(p, 5, q)$ and $C(m, n, 7)$ lie on l , find the distance BC .

$$\mathbf{r} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}), |BC| = 24$$

$$\begin{aligned}
 \text{(a)} \quad & l = (\text{fixed point}) + \lambda(\text{direction vector}) \\
 & l = (5, -1, -3) + \lambda(-1, 2, -2) \\
 & C(5, 2) = (5-\lambda, 2+\lambda, -3-2\lambda) \\
 \text{(b)} \quad & B(5, 5, q) \Rightarrow 2\lambda = 5 \\
 & \lambda = \frac{5}{2} \\
 & \therefore B(5, 5, -9) \\
 & C(m, n, 7) \Rightarrow -3-2\lambda = 7 \\
 & -3-2\left(\frac{5}{2}\right) = 7 \\
 & \therefore C(10, -1, 7)
 \end{aligned}$$

$$\begin{aligned}
 & |BC| = |(5, -1, -3) - (10, -1, 7)| \\
 & = |(5-10, -1-(-1), -3-7)| \\
 & = \sqrt{(-5)^2 + 0^2 + (-10)^2} \\
 & = \sqrt{25 + 100} \\
 & = \sqrt{125} \\
 & = 25
 \end{aligned}$$

Question 11

- a) Determine a vector equation of the straight line l that passes through the points $A(2,1,2)$ and $B(3,-1,5)$.
- b) Given that $P(p,-3,8)$ lies on l find the value of p .

$$\boxed{\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})}, \quad p = 4$$

(a) $\vec{AB} = \mathbf{b} - \mathbf{a} = (3,-1,5) - (2,1,2) = (1,-3,3)$
 $\mathbf{l} = (\text{fixed point}) + \lambda(\text{direction vector})$
 $\mathbf{l} = (2,1,2) + \lambda(1,-3,3)$
 $\mathbf{l} = (2+1\lambda, 1-3\lambda, 2+3\lambda) \quad //$

(b) $(p,-3,8) = (2+1\lambda, 1-3\lambda, 2+3\lambda)$
 $(p,-3,8) = (3+2\lambda, -2\lambda, 3\lambda+2)$
 $\begin{cases} p = 3+2\lambda \\ -3 = -2\lambda \\ 8 = 3\lambda+2 \end{cases} \Rightarrow \begin{cases} \lambda = 2 \\ p = 7 \end{cases}$

Question 12

- a) Find a vector equation of the straight line l that passes through the points $A(1,2,-2)$ and $B(-1,3,1)$.
- b) Given that $P(-5,p,7)$ lies on l find the value of p .

$$\boxed{\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} + 3\mathbf{k})}, \quad p = 5$$

(a) $\vec{AB} = \mathbf{b} - \mathbf{a} = (-1,3,1) - (1,2,-2) = (-2,1,3) \leftarrow \text{use an intercept value}$
 $\mathbf{l} = (\text{fixed point}) + \lambda(\text{direction vector})$
 $\mathbf{l} = (1,2,-2) + \lambda(-2,1,3)$
 $(\mathbf{r}, \mathbf{b}) = (-2\lambda, 1+2\lambda, -2+3\lambda) \quad //$

(b) $(-5,p,7) \in \mathbf{l} \Leftrightarrow \begin{cases} -5 = -2\lambda \\ p = 1+2\lambda \\ 7 = -2+3\lambda \end{cases} \Rightarrow \begin{cases} \lambda = 2.5 \\ p = 5 \\ \lambda = 3 \end{cases} \Rightarrow p = 5$

Question 13

- a) Determine a vector equation of the straight line l that passes through the points $A(-4, 1, -2)$ and $B(5, -1, 2)$.
- b) Given that $P(41, -9, p)$ lies on l find the value of p .

$$\mathbf{r} = \mathbf{i} - 4\mathbf{j} + \mathbf{k} + \lambda(9\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}), [p = 18]$$

(a) $\vec{AB} = \mathbf{b} - \mathbf{a} = (5, -1, 2) - (-4, 1, -2) = (9, -2, 4)$
 $\vec{l} = (9\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) + \lambda(9\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$
 $\vec{l} = (-4, 1, -2) + \lambda(9, -2, 4)$
 $\vec{l} = (5, -1, 2) + \lambda(9, -2, 4)$

(b) $(41, -9, p) = (5, -1, 2) + \lambda(9, -2, 4)$
 $\begin{aligned} 41 - 5 &= 4\lambda & -1 - (-1) &= 4\lambda - 2 \\ 36 &= 4\lambda & 0 &= 4\lambda - 2 \\ 9 &= \lambda & 0 &= 4\lambda - 2 \\ \lambda &= 9 \end{aligned}$

Question 14

- a) Find a vector equation of the straight line l that passes through the points $A(1, 1, -6)$ and $B(3, 2, -9)$.
- b) Given that $P(-3, -1, p)$ lies on l find the value of p .

$$\mathbf{r} = \mathbf{i} + \mathbf{j} - 6\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}), [p = 0]$$

(a) $\vec{AB} = \mathbf{b} - \mathbf{a} = (3, 2, -9) - (1, 1, -6)$
 $= (2, 1, -3) \leftarrow \text{vector from } A \text{ to } B$
 $\vec{l} = (6\mathbf{i} - 6\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$
 $\vec{l} = (1, 1, -6) + \lambda(2, 1, -3)$
 $\vec{l} = (2\lambda + 1, \lambda + 1, -6 - 3\lambda)$

(b) Looking at l
 $2\lambda + 1 = -3$
 $2\lambda = -4$
 $\lambda = -2$
 $\therefore -3 - 1 - 6 = -3(-2) - 6 = 0$

Question 15

- a) Determine a vector equation of the straight line l that passes through the points $A(8, -1, 2)$ and $B(10, 2, 1)$.
- b) Given that $P(20, p, -4)$ lies on l find the value of p .

$$\mathbf{r} = 8\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}), \quad [p = 17]$$

(a) $\vec{AB} = (8, -1, 2) - (10, 2, 1) = (2, 3, -1)$
 $\vec{s} = (\text{fixed point}) + \lambda(\text{direction vector})$
 $\vec{s} = (10, 2, 1) + \lambda(2, 3, -1)$
 $\vec{s} = (20 + 2\lambda, 2 + 3\lambda, 1 - \lambda)$

(b) $(20, 2 + 3\lambda, 1 - \lambda) = (20, p, -4)$
 $\begin{cases} 20 + 2\lambda = 20 \\ 2 + 3\lambda = p \\ 1 - \lambda = -4 \end{cases}$
 $\begin{cases} 2\lambda = 0 \\ 3\lambda = p - 2 \\ \lambda = 5 \end{cases}$
 $\therefore p = 3 \times 5 - 2 = 15 - 2 = 13$

Question 16

- a) Find a vector equation of the straight line l that passes through the points $A(6, -3, 2)$ and $B(5, -1, 3)$.
- b) Given that $P(p, 5, 6)$ lies on l find the value of p .

$$\mathbf{r} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k}), \quad [p = 2]$$

(a) $\vec{AB} = b - a = (5, -1, 3) - (6, -3, 2) = (-1, 2, 1) \leftarrow \text{Direction vector}$
 $\vec{s} = (\text{fixed point}) + \lambda(\text{direction vector})$
 $\vec{s} = (6, -3, 2) + \lambda(-1, 2, 1)$
 $(\text{any}) \vec{c} = (c_1, c_2, c_3)$

(b) looking at $\frac{1}{2}$
 $\frac{21-3}{2} = 5$
 $\frac{2+4}{2} = 3$
 $\therefore p = 6 - 3 = 6 - 4 = 2$

Question 17

- a) Determine a vector equation of the straight line l that passes through the points $A(-2, 4, -4)$ and $B(-17, -1, -14)$.
- b) Given that $P(7, p, q)$ lies on l find the value of p and the value of q .

$$\boxed{\mathbf{r} = -2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})}, \boxed{p = 7}, \boxed{q = 2}$$

a) $\vec{AB} = \mathbf{b} - \mathbf{a} = (-17, -1, -14) - (-2, 4, -4) = (-15, -5, -10)$
 SOLVE THE DIRECTION VECTOR TO $(3, 1, 2)$
 $\mathbf{l} = (\text{fixed point}) + \lambda(\text{direction vector})$
 $\mathbf{l} = (-2, 4, -4) + \lambda(3, 1, 2)$
 ↑
 CONSIDER DE-SCALING BECAUSE IT IS NOT + DIRECTION
 $(3\lambda, \lambda, 2\lambda) = (3\lambda - 2, 4 + \lambda, 2\lambda - 4)$

b) $(7, p, q) \in (3\lambda - 2, 4 + \lambda, 2\lambda - 4)$
 (i): $3\lambda - 2 = 7$ (ii): $4 + \lambda = p$ (iii): $2\lambda - 4 = q$
 $3\lambda = 9$ $\lambda + 4 = p$
 $\lambda = 3$ $\lambda = p - 4$
 $\lambda = 3$ $p = 3 + 4$
 $\lambda = 3$ $p = 7$
 $\lambda = 3$ $q = 2(3) - 4$
 $\lambda = 3$ $q = 2$

Question 18

- a) Find a vector equation of the straight line l that passes through the points $A(8, 6, 2)$ and $B(13, -4, -3)$.
- b) Given that $C(10, p, q)$ lies on l find the value of p and the value of q .

$$\boxed{\mathbf{r} = 8\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})}, \boxed{p = 2}, \boxed{q = 0}$$

a) $\vec{AB} = \mathbf{b} - \mathbf{a} = (13, -4, -3) - (8, 6, 2)$
 $= (5, -10, -5)$
 USE $(-1, 2, 1)$ AS DIRECTION VECTOR
 $\mathbf{l} = (8, 6, 2) + \lambda(-1, 2, 1)$
 $\mathbf{l} = (8, 6, 2) + \lambda(-1, 2, 1)$
 $\mathbf{l} = (8 - \lambda, 6 + 2\lambda, 2 + \lambda)$

b) LOOKING AT $\mathbf{l} : 8 - \lambda = 10$
 $\lambda = -2$
 USE $\lambda = -2$: $p = 2\lambda + 6 = 2$,
 $\lambda = -2$: $q = 2 + \lambda = 0$
 $\lambda = -2$: $\therefore p = 2$, $q = 0$

Question 19

- a) Determine a vector equation of the straight line l that passes through the points $A(6,5,1)$ and $B(4,4,-1)$.
- b) Given the point $C(p,q,q)$ lies on l find the value of p and the value of q .

$$\mathbf{r} = 6\mathbf{i} + 5\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}), [p = 14], [q = 9]$$

(a) $\vec{AB} = \mathbf{b} - \mathbf{a} = (4,4,-1) - (6,5,1) = (-2,-1,-2)$
 Take $(-2,-1,-2)$ as the direction vector
 $\mathbf{s} = (\text{fixed point}) + \lambda(\text{direction vector})$
 $\mathbf{s} = (6,5,1) + \lambda(2,1,2)$
 $\mathbf{s} = (2\lambda + 6, \lambda + 5, 2\lambda + 1)$

(b) $(p,q,q) = (2\lambda + 6, \lambda + 5, 2\lambda + 1)$

$$\begin{aligned} p &= 2\lambda + 6 \\ q &= \lambda + 5 \\ q &= 2\lambda + 1 \end{aligned} \quad \left. \begin{array}{l} \Rightarrow 2\lambda + 5 = 2\lambda + 1 \\ 4 = 2 \end{array} \right\} \quad \left. \begin{array}{l} \therefore p = 2\lambda + 6 \Rightarrow p = 14 \\ d = 4 + 5 \Rightarrow d = 9 \end{array} \right\}$$

Question 20

Given that the points $A(4,6,-2)$, $B(9,1,3)$ and $C(1,p,q)$ lie on a straight line find a vector equation for the straight line and hence find the value of p and the value of q .

$$\mathbf{r} = 4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}), [p = 9], [q = -5]$$

(a) $\vec{AB} = \mathbf{b} - \mathbf{a} = (9,1,3) - (4,6,-2) = (5,-5,5)$ now $(1,-1,1)$ is direction
 $\mathbf{s} = (\text{fixed point}) + \lambda(\text{direction vector})$
 $\mathbf{s} = (4,6,-2) + \lambda(1,-1,1)$
 $(x,y,z) = (4+1\lambda, 6-\lambda, -2+\lambda)$

(b) looking at \mathbf{j} : $6-\lambda = 1$ $\boxed{\lambda = 5}$ $\therefore p = 6+5 = 11$ $\therefore q = 1-5 = -4$ $\therefore q = -5$

Question 21

Show that the straight line with vector equation

$$\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}),$$

where λ is a scalar parameter

and the straight line through the points $A(1, -1, 1)$ and $B(3, 7, 7)$ are parallel.

[proof]

$$\begin{aligned}\vec{AB} &= \mathbf{b} - \mathbf{a} = (3, 7, 7) - (1, -1, 1) = (2, 8, 6) = 2(1, 4, 3) \\ \text{it lies through } A \text{ & } B \text{ has direction } C(1, 4, 3), \text{ which is} \\ \text{THE SAME AS THE direction vector of the other line} \\ \therefore \text{parallel}\end{aligned}$$

Question 22

- a) Find a vector equation of the straight line l that passes through the points $A(5, -2, -7)$ and $B(8, 2, 5)$.
- b) Find the coordinates of the point C which also lies on l with $|AB| = |AC|$,

$$\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} - 7\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}), \quad C(2, -6, -19)$$

$$\begin{aligned}\text{(a)} \quad \vec{AB} &= \mathbf{b} - \mathbf{a} = (8, 2, 5) - (5, -2, -7) = (3, 4, 12) \\ \Gamma &= (5, -2, -7) + \lambda(3, 4, 12) \\ C(2, -6, -19) &= (3, 4, 12) + \lambda(-3, -8, -19) \quad // \\ \text{(b)} \quad &\text{USE FACT THAT IF } |AB| = |AC| \\ &\text{THEN } A \text{ IS THE MIDPOINT OF } BC \\ &\text{& SO } 4 = \frac{1}{2}(2 - 5, -6 + 2, -19 - 7) \\ &\therefore (2, -6, -19) = (2, -4, -19) \quad /\end{aligned}$$

Question 23

- a) Find a vector equation of the straight line l through the points $A(1,4,4)$ and $B(10,1,-2)$.
- b) Find the coordinates of the point C , given that it lies on l so that $|AB|=|AC|$.

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} - 2\mathbf{k}), \quad B(-8, 7, 10)$$

(a) $\vec{AB} = \mathbf{b} - \mathbf{a} = (10, 1, -2) - (1, 4, 4) = (9, -3, -6)$
 Use $(3, -1, -2)$ as direction vector
 $\therefore l = (1, 4, 4) + \lambda(3, -1, -2)$
 $\Gamma = (3\lambda + 1, 4 - \lambda, 4 - 2\lambda)$

(b) $\vec{BC} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
 WORKING THE MIDPOINT OF "BC"
 $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}) = (1, 4, 4)$
 $\therefore \frac{x+8}{2} = 1 \quad \frac{y+7}{2} = 4 \quad \frac{z+10}{2} = 4$
 $x = -6 \quad y = 11 \quad z = 10 \quad \therefore C(-6, 11, 10)$

Question 24

- a) Find a vector equation of the straight line l that passes through the point $A(-1, 4, 6)$ and is parallel to the vector $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
- b) Given that $B(p, q, 1)$ lies on l , find the value of p and the value of q .
- c) Find the coordinates of the point C , given that it lies on l with $|AB|=|AC|$.

$$\mathbf{r} = -\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}), \quad p = -11, \quad q = 14, \quad C(9, -6, 11)$$

(a) $\Gamma = (-1, 4, 6) + \lambda(2, -2, 1)$
 $\Gamma = (2\lambda - 1, 4 - 2\lambda, 2 + \lambda)$

(b) $(p, q, 1) = (2\lambda - 1, 4 - 2\lambda, 2 + \lambda)$
 $\begin{cases} p = 2\lambda - 1 \\ q = 4 - 2\lambda \end{cases} \Rightarrow \begin{cases} \lambda = 1 \\ \lambda = 2 \end{cases}$
 $\therefore \begin{cases} p = -11 \\ q = 14 \end{cases} \Rightarrow \begin{cases} \lambda = 1 \\ \lambda = 2 \end{cases}$

(c) $\Gamma = (-1, 4, 6) + \lambda(2, -2, 1)$
 Using midpoint
 $\lambda = \frac{-1+9}{2}, \frac{4+(-6)}{2}, \frac{6+2}{2}$
 $\left(\frac{2+1}{2}, \frac{4+(-6)}{2}, \frac{6+2}{2}\right) = (-1, 4, 4)$
 $\therefore C(-1, 4, 4)$

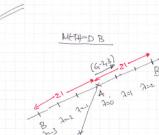
Question 25

- a) Find a vector equation of the straight line l that passes through the point $A(10, 2, -1)$ and is parallel to the vector $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.
- b) Find the two possible sets of coordinates of the point B given that it lies on l and that $|AB| = 21$ units.

$$\boxed{\mathbf{r} = 10\mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})}, \boxed{B(28, -4, 8) \text{ or } B(-8, 8, -10)}$$

(a) $\Sigma = (10, 2, -1) + \lambda(6, -2, 3)$
 $\Sigma = (10+6\lambda, 2-2\lambda, -1+3\lambda)$

(b) Method A
Let $B(2, 1, 2)$
 $\Rightarrow |AB| = 21$
 $\Rightarrow |B-A| = 21$
 $\Rightarrow |(2, 1, 2) - (10, 2, -1)| = 21$
 $\Rightarrow |(2-10, 1-2, 2+1)| = 21$
 $\Rightarrow |(-8, -1, 3)| = 21$
 $\Rightarrow \sqrt{(-8)^2 + (-1)^2 + 3^2} = 21$
 $\Rightarrow \sqrt{64+1+9} = 21$
 $\Rightarrow \sqrt{74} = 21$
 $\Rightarrow 74 = 441$
 $\Rightarrow 21^2 = 441$
 $\Rightarrow 21 = 9$
 $\Rightarrow \lambda = \pm 3$
 $\lambda = 3 \Rightarrow B(28, -4, 8)$
 $\lambda = -3 \Rightarrow B(-8, 8, -10)$



- Length of direction vector is $\sqrt{36+4+9} = 7$
- $21 \div 7 = 3$
- $\lambda = \pm 3$
- ∴ B has coordinates $(28, -4, 8)$

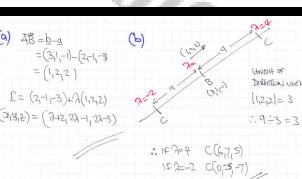
Question 26

- a) Find a vector equation of the straight line l which passes through the points $A(2, -1, -3)$ and $B(3, 1, -1)$.
- b) Find the two possible sets of coordinates of the point C given that it also lies on l and that $|BC| = 9$ units.

$$\boxed{\mathbf{r} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})}, \boxed{C(6, 7, 5) \text{ or } C(0, -5, -7)}$$

(a) $\vec{AB} = b-a$
 $= (3, 1, -1) - (2, -1, -3)$
 $= (1, 2, 2)$

$\Sigma = (2, -1, -3) + \lambda(1, 2, 2)$
 $(1, 2, 2) = (\lambda, 2\lambda, 2\lambda-3)$

(b)

 $\therefore |C-B|=4$
 $|C-B|=2$
 $\therefore C(0, -5, -7)$

Question 27

The point $A(6,1,0)$ lies on the straight line l with equation

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}),$$

where λ is a scalar parameter.

If the point B also lies on l so that the distance AB is 15 units find the possible coordinates of the point B .

$$B(16, -4, 10) \text{ or } B(-4, 6, -10)$$

Given $\Gamma = (2, -4, -4) + \lambda(2, -1, 2)$
 $\Gamma = (2+2\lambda, -4-\lambda, -4+2\lambda)$

- THE LENGTH OF THE DIRECTION VECTOR
 $\|\Gamma - A\| = \sqrt{(2+2\lambda-6)^2 + (-4-\lambda-1)^2 + (-4+2\lambda)^2} = 3$
- SO $\sqrt{2+2\lambda-6, -4-\lambda-1, -4+2\lambda}$ FROM A
 $\Rightarrow 3 = \sqrt{2+2\lambda-6}$
- $|2+2\lambda-6| = 3$
- $A(6, 1, 0) \Rightarrow \lambda = 2$

ANALYTICAL METHOD

- LET $B(x, y, z)$
 $\Rightarrow \vec{AB} = \vec{b} - \vec{a} = (x, y, z) - (6, 1, 0)$
 $\Rightarrow \vec{AB} = (x-6, y-1, z)$
- $3\vec{AB}$ LIES ON l
 $(x-6, y-1, z) = 2(2, -1, 2)$
 $x-6 = 4z$
 $y-1 = -2z$
 $z = 2z$
- $\Rightarrow AB = (2z-6, -2z-1, 2z)$
 $\Rightarrow \vec{AB} = (2z-6, -2z-1, 2z)$
 $\Rightarrow \vec{AB} = (2z-6, -2z-1, 2z)$
 $\Rightarrow |AB| = \sqrt{(2z-6)^2 + (-2z-1)^2 + (2z)^2} = 15$

$\Rightarrow 15 = \sqrt{(2z-6)^2 + (-2z-1)^2 + (2z)^2}$
 $\Rightarrow 15 = \sqrt{4z^2 - 24z + 36 + 4z^2 + 4z + 1 + 4z^2}$
 $\Rightarrow 15 = \sqrt{12z^2 - 20z + 37}$
 $\Rightarrow 225 = 12z^2 - 20z + 37$
 $\Rightarrow 0 = 12z^2 - 20z - 188$
 $\Rightarrow 0 = 3z^2 - 5z - 46$
 $\Rightarrow 0 = (3z+14)(z-4)$
 $\Rightarrow z = -\frac{14}{3} \text{ or } z = 4$

$\therefore z = 4 \Rightarrow B(16, -4, 10)$
 $\therefore z = -\frac{14}{3} \Rightarrow B(-4, 6, -10)$

Question 28

The point $A(7, 0, -4)$ lies on the straight line l with equation

$$\mathbf{r} = 5\mathbf{i} - \mathbf{j} - 5\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + \mathbf{k}),$$

where λ is a scalar parameter.

If the point B also lies on l so that $|AB| = \sqrt{96}$, find the possible coordinates of B .

$$B(15, 4, 0) \text{ or } B(-1, -4, -8)$$

$\Gamma = (7, 0, -4) + \lambda(2, 1, 1)$
 $\Gamma = (7+2\lambda, 0+\lambda, -4+\lambda)$

- $A(7, 0, -4) \Rightarrow \lambda = 2$
- $\sqrt{96} = \sqrt{64}$
 $\therefore 96 = 64\lambda^2 + 4$
 $\therefore \lambda^2 = 8$
 $\therefore \lambda = \pm\sqrt{8}$
- $B(15, 4, 0)$ or $B(-1, -4, -8)$

Question 29

For each of the pairs of the straight lines shown below,

- ... prove that they intersect.
- ... find the coordinates of their point of intersection.
- ... calculate the acute angle between them.

a) $\mathbf{r}_1 = 3\mathbf{i} + 5\mathbf{j} - 6\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$
 $\mathbf{r}_2 = 2\mathbf{i} + 15\mathbf{j} + 17\mathbf{k} + \mu(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

b) $\mathbf{r}_1 = 14\mathbf{i} + 6\mathbf{j} - 4\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$
 $\mathbf{r}_2 = 10\mathbf{i} + 8\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - 5\mathbf{k})$

c) $\mathbf{r}_1 = 8\mathbf{i} + \mathbf{j} + 7\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$
 $\mathbf{r}_2 = 5\mathbf{i} + 7\mathbf{k} + \mu(\mathbf{i} - 3\mathbf{j} + 8\mathbf{k})$

(9,8,3) & 56.9°, (18,4,-12) & 23.6°, (4,3,-1) & 20.2°

<p>(a)</p> $\begin{aligned}\mathbf{l}_1 &= (3, 5, -6) + \lambda(2, 1, 3) = (2, 13, 15, 31) \\ \mathbf{l}_2 &= (2, 5, 7) + \mu(-1, 2, 1) = (2, 11, 15, 21)\end{aligned}$ <p>• PROVE LINES INTERSECT</p> <p>(1): $21 = 5 + 15 \lambda \Rightarrow \lambda = 1$ $20 = 3 + 2\lambda \Rightarrow \lambda = 1$ $21 = 7 + \mu \Rightarrow \mu = 14$</p> <p>• CHECK LINES ARE NOT PARALLEL</p> <p>• CROSS PRODUCT</p> <p>$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2(2+3) - 1(-2+0) = 12 \neq 0$</p> <p>• DETERMINE ANGLE</p> <p>$\begin{aligned} \mathbf{v}_1 &= (2, 13, 15, 31) - (2, 11, 15, 21) \\ &= (0, 2, 0, 10) \\ \mathbf{v}_1 &= \sqrt{0^2 + 2^2 + 0^2 + 10^2} = \sqrt{104} \\ \mathbf{v}_2 &= (2, 11, 15, 21) - (2, 5, 7) \\ &= (0, 6, 8, 14) \\ \mathbf{v}_2 &= \sqrt{0^2 + 6^2 + 8^2 + 14^2} = \sqrt{280} \end{aligned}$</p> <p>$\cos \theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{ \mathbf{v}_1 \mathbf{v}_2 } = \frac{2(0) + 13(6) + 15(8) + 31(14)}{\sqrt{104} \sqrt{280}} = \frac{362}{\sqrt{104} \sqrt{280}}$</p> <p>$\theta = \cos^{-1} \left(\frac{362}{\sqrt{104} \sqrt{280}} \right) = 56.9^\circ$</p>	<p>(b)</p> $\begin{aligned}\mathbf{l}_1 &= (6, 4, -7) + \lambda(-2, 1, 4) = (-2, 2, 14, 4, -1) \\ \mathbf{l}_2 &= (6, 9, 5) + \mu(3, 1, -2) = (6, 12, 11, 5, -5)\end{aligned}$ <p>• PROVE LINES INTERSECT</p> <p>(1): $12 = 4 + \lambda \Rightarrow \lambda = 8$ $15 = 9 + \mu \Rightarrow \mu = 6$</p> <p>• CHECK LINES ARE NOT PARALLEL</p> <p>• CROSS PRODUCT</p> <p>$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} = -2(1+4) - 3(-2+0) = -12 \neq 0$</p> <p>• DETERMINE ANGLE</p> <p>$\begin{aligned} \mathbf{v}_1 &= (-2, 2, 14, 4, -1) - (-2, 4, -7) \\ &= (0, -2, 17, 8, 3) \\ \mathbf{v}_1 &= \sqrt{0^2 + (-2)^2 + 17^2 + 8^2 + 3^2} = \sqrt{302} \\ \mathbf{v}_2 &= (-2, 4, -7) - (6, 9, 5) \\ &= (8, -6, -12) \\ \mathbf{v}_2 &= \sqrt{8^2 + (-6)^2 + (-12)^2} = \sqrt{224} \end{aligned}$</p> <p>$\cos \theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{ \mathbf{v}_1 \mathbf{v}_2 } = \frac{0 + (-2)(-6) + 17(-12) + 8(-12) + (-1)(8)}{\sqrt{302} \sqrt{224}} = \frac{-216}{\sqrt{302} \sqrt{224}}$</p> <p>$\theta = \cos^{-1} \left(\frac{-216}{\sqrt{302} \sqrt{224}} \right) = 23.6^\circ$</p>	<p>(c)</p> $\begin{aligned}\mathbf{l}_1 &= (6, 1, -7) + \lambda(-2, 1, -1) = (-2, 2, 14, 4, -1) \\ \mathbf{l}_2 &= (5, 7, 1) + \mu(3, 1, 2) = (6, 12, 11, 5, -5)\end{aligned}$ <p>• PROVE LINES INTERSECT</p> <p>(1): $12 = 1 + \lambda \Rightarrow \lambda = 11$ $15 = 7 + \mu \Rightarrow \mu = 8$</p> <p>• CHECK LINES ARE NOT PARALLEL</p> <p>• CROSS PRODUCT</p> <p>$\begin{vmatrix} -2 & 1 & -1 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -2(1+2) - 3(-1+0) = -1 \neq 0$</p> <p>• DETERMINE ANGLE</p> <p>$\begin{aligned} \mathbf{v}_1 &= (-2, 2, 14, 4, -1) - (-2, 1, -7) \\ &= (0, 1, 15, 5, 6) \\ \mathbf{v}_1 &= \sqrt{0^2 + 1^2 + 15^2 + 5^2 + 6^2} = \sqrt{247} \\ \mathbf{v}_2 &= (-2, 1, -7) - (5, 7, 1) \\ &= (3, -6, -14) \\ \mathbf{v}_2 &= \sqrt{3^2 + (-6)^2 + (-14)^2} = \sqrt{213} \end{aligned}$</p> <p>$\cos \theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{ \mathbf{v}_1 \mathbf{v}_2 } = \frac{0 + 1(-6) + 15(-14) + 5(-14) + (-1)(3)}{\sqrt{247} \sqrt{213}} = \frac{-105}{\sqrt{247} \sqrt{213}}$</p> <p>$\theta = \cos^{-1} \left(\frac{-105}{\sqrt{247} \sqrt{213}} \right) = 20.2^\circ$</p>
---	--	---

Question 30

For each of the pairs of the straight lines shown below,

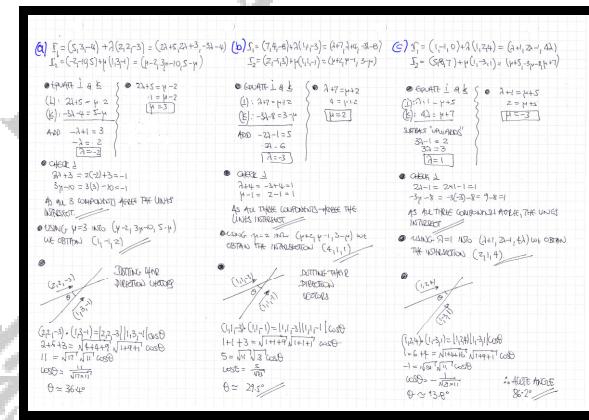
- ... prove that they intersect.
- ... find the coordinates of their point of intersection.
- ... calculate the acute angle between them.

a) $\mathbf{r}_1 = 5\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$
 $\mathbf{r}_2 = -2\mathbf{i} - 10\mathbf{j} + 5\mathbf{k} + \mu(\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

b) $\mathbf{r}_1 = 7\mathbf{i} + 4\mathbf{j} - 8\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - 3\mathbf{k})$
 $\mathbf{r}_2 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$

c) $\mathbf{r}_1 = \mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$
 $\mathbf{r}_2 = 5\mathbf{i} - 8\mathbf{j} + 7\mathbf{k} + \mu(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$

$(1, -1, 2)$ & 36.4° , $(4, 1, 1)$ & 29.5° , $(2, 1, 4)$ & 86.2°



Question 31

For each of the pairs of the straight lines shown below

- i. ... prove that they intersect.
 - ii. ... find the coordinates of their point of intersection
 - iii. ... calculate the acute angle between them.

$$\text{a) } \begin{aligned} \mathbf{r}_1 &= 4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \\ \mathbf{r}_2 &= 2\mathbf{i} + 11\mathbf{j} - \mathbf{k} + \mu(5\mathbf{i} + 2\mathbf{k}) \end{aligned}$$

b) $\mathbf{r}_1 = 9\mathbf{i} + 15\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

$$\mathbf{c}) \quad \mathbf{r}_1 = 8\mathbf{i} + 6\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

$$\boxed{(7,11,1) \text{ & } 57.9^\circ}, \boxed{(-1,5,-5) \text{ & } 90^\circ}, \boxed{(11,0,7) \text{ & } 45^\circ}$$

⑮ $\Sigma_1 = \{(4,5,2), (1,7,1), (-1,4,2), (2,11,3)\}$
 $\Sigma_2 = \{(-2,1,1), (5,2,3), (4,5,1), (2,4,1)\}$

QUESTION 2: 2x2

(1) $1 \times 1 + 2 \times 1 + 3 \times 1 = 6$ $\therefore 2 \times 3$
(2) $2 \times 1 + 1 \times 2 + 1 \times 1 = 5$ $\therefore 3 \times 2$
(3) $1 \times 2 + 2 \times 2 + 1 \times 1 = 7$ $\therefore 3 \times 3$

CHECK: $\begin{cases} 2 - 2 - 3 = 1 \\ 4 - 1 - 2 + 1 = 1 \end{cases}$

AS ALL 3 COMBINATIONS AGREE THE UNITS INDICATED ARE CORRECT.

USING UNIT NO. 3 NO. (2,4,1) WE GET THE
 NORMAL VECTOR AT $(2,4,1)$

QUESTION 3: 2x2

PUTTING THE DIRECTING VECTORS

$\begin{aligned} \vec{v}_1 &= \langle 1, 1, 1 \rangle, \vec{v}_2 = \langle 2, 1, 1 \rangle, \vec{v}_3 = \langle 1, 1, 1 \rangle \\ &\text{S. A. O. D. } \sqrt{1+1+1} = \sqrt{3}, \sqrt{1+1+1} = \sqrt{3}, \sqrt{1+1+1} = \sqrt{3} \end{aligned}$

$\therefore \vec{v}_1 = \frac{1}{\sqrt{3}} \vec{i} + \frac{1}{\sqrt{3}} \vec{j} + \frac{1}{\sqrt{3}} \vec{k}$
 $\vec{v}_2 = \frac{2}{\sqrt{3}} \vec{i} + \frac{1}{\sqrt{3}} \vec{j} + \frac{1}{\sqrt{3}} \vec{k}$
 $\vec{v}_3 = \frac{1}{\sqrt{3}} \vec{i} + \frac{1}{\sqrt{3}} \vec{j} + \frac{1}{\sqrt{3}} \vec{k}$

(b) $\Sigma_1 = \{(1,0,1), \sqrt{2}(2,-1), (2,11,3)\}$
 $\Sigma_2 = \{(-1,7,1), (-3,7,4), (-3,2,1)\}$

QUESTION 3: 2x2

(1) $-1 + 3 = 2 \neq 1$
(2) $2 + 11 + 3 = 16 \neq 1$
(3) $1 + 7 + 1 = 9 \neq 1$

SUBTRACT UNNECESSARY TERMS

$\begin{cases} -1 + 3 = 2 \\ 2 + 11 + 3 = 16 \\ 1 + 7 + 1 = 9 \end{cases}$

CHECK: $\begin{cases} 2 + 16 + 9 = 27 \\ -3 + 7 - 3 = 7 \end{cases}$

ALL 3 COMBINATIONS AGREE SO UNITS INDICATED ARE CORRECT.

USING UNIT NO. 2 NO. (2,11,3) WE GET THE
 NORMAL VECTOR AT $(2,11,3)$

QUESTION 4: 2x2

DIRECTING THE DIRECTING VECTORS

$\begin{aligned} \vec{v}_1 &= \langle 2, 1, 1 \rangle, \vec{v}_2 = \langle -3, 1, 1 \rangle, \vec{v}_3 = \langle 3, 2, 1 \rangle \\ &\text{S. A. O. D. } \sqrt{4+1+1} = \sqrt{6}, \sqrt{9+1+1} = \sqrt{11}, \sqrt{9+4+1} = \sqrt{14} \end{aligned}$

$\therefore \vec{v}_1 = \frac{2}{\sqrt{6}} \vec{i} + \frac{1}{\sqrt{6}} \vec{j} + \frac{1}{\sqrt{6}} \vec{k}$
 $\vec{v}_2 = \frac{-3}{\sqrt{11}} \vec{i} + \frac{1}{\sqrt{11}} \vec{j} + \frac{1}{\sqrt{11}} \vec{k}$
 $\vec{v}_3 = \frac{3}{\sqrt{14}} \vec{i} + \frac{2}{\sqrt{14}} \vec{j} + \frac{1}{\sqrt{14}} \vec{k}$

(c) $\Sigma_1 = \{(2,1,1), \sqrt{2}(2,-1), (2,11,3)\}$
 $\Sigma_2 = \{(-1,7,1), (-3,7,4), (-3,2,1)\}$

QUESTION 2: 2x2

(1) $2 + 11 - 17 = -4$
(2) $-4 - 28 + 24 = -8$
(3) $28 + 24 - 17 = 35$

ANS: $\therefore -4 + 9 = 5$

QUESTION 3: 2x2

$\begin{cases} 2 + 17 - 17 = 2 \\ -4 - 28 + 17 = -13 \\ 28 + 17 - 17 = 28 \end{cases}$

ANS: $\therefore 2 + 28 = 30$

QUESTION 4: 2x2

$\begin{cases} 2 + 18 - 30 = 11 \\ 3 + 25 - 45 = 11 \end{cases}$

ANS: 3 COMBINATIONS AGREE SO UNITS INDICATED ARE CORRECT.

USING UNIT NO. 2 NO. (2,18,30) WE GET THE
 NORMAL VECTOR AT $(2,18,30)$

QUESTION 5: 2x2

DIRECTING THE DIRECTING VECTORS

$\begin{aligned} \vec{v}_1 &= \langle 2, 1, 1 \rangle, \vec{v}_2 = \langle -3, 1, 1 \rangle, \vec{v}_3 = \langle 3, 2, 1 \rangle \\ &\text{S. A. O. D. } \sqrt{4+1+1} = \sqrt{6}, \sqrt{9+1+1} = \sqrt{11}, \sqrt{9+4+1} = \sqrt{14} \end{aligned}$

$\therefore \vec{v}_1 = \frac{2}{\sqrt{6}} \vec{i} + \frac{1}{\sqrt{6}} \vec{j} + \frac{1}{\sqrt{6}} \vec{k}$
 $\vec{v}_2 = \frac{-3}{\sqrt{11}} \vec{i} + \frac{1}{\sqrt{11}} \vec{j} + \frac{1}{\sqrt{11}} \vec{k}$
 $\vec{v}_3 = \frac{3}{\sqrt{14}} \vec{i} + \frac{2}{\sqrt{14}} \vec{j} + \frac{1}{\sqrt{14}} \vec{k}$

$\angle B = 155^\circ$

$\angle A = 100^\circ$ ANGLE = $180^\circ - 135^\circ = 45^\circ$

Question 32

For each of the pairs of the straight lines shown below,

- ... prove that they intersect.
- ... find the coordinates of their point of intersection.
- ... calculate the acute angle between them.

a) $\mathbf{r}_1 = -2\mathbf{i} + 7\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$
 $\mathbf{r}_2 = 3\mathbf{i} + 5\mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

b) $\mathbf{r}_1 = 8\mathbf{i} - 3\mathbf{j} - 7\mathbf{k} + \lambda(-6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$
 $\mathbf{r}_2 = 8\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

c) $\mathbf{r}_1 = 6\mathbf{i} - 2\mathbf{j} + 6\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$
 $\mathbf{r}_2 = 6\mathbf{i} + 10\mathbf{j} - 12\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$

$(2, 3, 5) \text{ & } 63.6^\circ$, $(2, -1, -4) \text{ & } 79.0^\circ$, $(2, 2, 0) \text{ & } 44.5^\circ$

Q1 $\Gamma_1 = (2, 7, 5) + \lambda(2, -2, 1) = (2, 7 - 2\lambda, 5 + \lambda)$
 $\Gamma_2 = (3, 5, 5) + \mu(1, 2, 2) = (3 + \mu, 5 + 2\mu, 5 - 2\mu)$

- Given $\lambda \neq \mu$ implies:
- $7 - 2\lambda = 5 + \mu$
 $\lambda = 3 - \frac{1}{2}\mu$
 $\text{ADD } -2\lambda = \mu$
 $2 - 3 - \frac{1}{2}\mu = \mu$
 $\frac{5}{2} = \frac{3}{2}\mu$
 $\mu = \frac{5}{3}$
- Check:

 - $2\lambda + 2 = 2\mu + 2$
 $\frac{1}{2}(5 - 2) = \frac{1}{2}(5 - 2)$
 - As all three components agree the lines intersect.

- Using $\lambda = 2$ into $(2, 7 - 2\lambda, 5 + \lambda)$ we obtain intersection point $(2, 3, 5)$.
- Using the direction vectors:

 - $(2, -2, 1)$, $(1, 2, 2)$ $\Rightarrow (2, -2, 1) \times (1, 2, 2) = (2, -2, 1) \cdot (-4, 4, 6) = 6$
 - $\Rightarrow 2\lambda + 2 = 2\mu + 2$
 $\Rightarrow -4\lambda + 4 = -4\mu + 4$
 $\Rightarrow 4\lambda - 4 = 4\mu - 4$
 $\Rightarrow \lambda = \mu$
 $\therefore \text{acute angle is } 180^\circ - 180^\circ = 0^\circ$

Q2 $\Gamma_1 = (6, 3, 7) + \lambda(-4, 2, 3) = (6 - 4\lambda, 3 + 2\lambda, 7 + 3\lambda)$
 $\Gamma_2 = (8, 2, 1) + \mu(2, 1, 2) = (8 + 2\mu, 2 + \mu, 1 + 2\mu)$

 - Given $\lambda \neq \mu$ implies:
 - $6 - 4\lambda = 8 + 2\mu$
 $\lambda = -1 - \frac{1}{2}\mu$
 $\text{SUBTRACT } 15 = 6$
 $-9 - 4\lambda = 2\mu + 8$
 $-9 - 4(-1 - \frac{1}{2}\mu) = 2\mu + 8$
 $1 = 4\mu + 8$
 $\mu = -\frac{7}{4}$
 - Check λ :

 - $6 - 4\lambda = 8 + 2\mu + 8$
 $6 - 4(-1 - \frac{1}{2}\mu) = 8 + 2(-\frac{7}{4}) + 8$
 $6 + 4 = 8 - \frac{7}{2} + 8$
 $10 = 15 - \frac{7}{2}$
 $10 = 10$
 - As all three components agree the lines intersect.

 - Using $\lambda = -1$ into $(6 - 4\lambda, 3 + 2\lambda, 7 + 3\lambda)$ we obtain the intersection point $(2, 2, 0)$.
 - Using the direction vectors:

 - $(-4, 2, 3)$, $(2, 1, 2)$ $\Rightarrow (-4, 2, 3) \times (2, 1, 2) = (-4, 2, 3) \cdot (4, 1, 4) = 10$
 - $\Rightarrow -4\lambda + 2 = 2\mu + 2$
 $\Rightarrow -8\lambda + 4 = 4\mu + 4$
 $\Rightarrow -8\lambda = 4\mu$
 $\Rightarrow \lambda = -\frac{1}{2}\mu$
 $\Rightarrow \lambda = -\frac{1}{2}(-\frac{7}{4})$
 $\Rightarrow \lambda = \frac{7}{8}$
 $\therefore \text{acute angle is } 180^\circ - 10^\circ = 170^\circ$

Q3 $\Gamma_1 = (6, 2, 1) + \lambda(2, -2, 3) = (6 + 2\lambda, 2 - 2\lambda, 1 + 3\lambda)$
 $\Gamma_2 = (4, 1, 2) + \mu(1, 2, 1) = (4 + \mu, 1 + 2\mu, 2 + \mu)$

 - Given $\lambda \neq \mu$ implies:
 - $6 + 2\lambda = 4 + \mu$
 $\lambda = -1 + \frac{1}{2}\mu$
 $\text{SUBTRACT } 12 = 0$
 $-6 + 2\lambda = \mu + 4$
 $-6 + 2(-1 + \frac{1}{2}\mu) = \mu + 4$
 $-8 + \mu = \mu + 4$
 $-8 = 4$
 $\mu = -12$
 - Check λ :

 - $6 + 2\lambda = 4 + \mu + 4$
 $6 + 2(-1 + \frac{1}{2}\mu) = 4 + (-12) + 4$
 $6 - 2 = -8 + 4$
 $6 = -6$
 - As all three components agree the lines intersect.

 - Using $\mu = -12$ into $(6 + 2\lambda, 2 - 2\lambda, 1 + 3\lambda)$ we obtain intersection point $(2, 2, 0)$.
 - Using the direction vectors:

 - $(2, -2, 3)$, $(1, 2, 1)$ $\Rightarrow (2, -2, 3) \times (1, 2, 1) = (2, -2, 3) \cdot (-4, 4, 2) = 10$
 - $\Rightarrow -4\lambda + 2 = 2\mu + 2$
 $\Rightarrow -8\lambda + 4 = 4\mu + 4$
 $\Rightarrow -8\lambda = 4\mu$
 $\Rightarrow \lambda = -\frac{1}{2}\mu$
 $\Rightarrow \lambda = -\frac{1}{2}(-12)$
 $\Rightarrow \lambda = 6$
 $\therefore \text{acute angle is } 180^\circ - 10^\circ = 170^\circ$

Question 33

Prove that the straight line through $A(3,8,9)$ and $B(5,12,11)$ intersects with the straight line through $C(-5,6,8)$ and $D(13,0,5)$.

Find the point of intersection and the acute angle between the two straight lines.

$(1, 4, 7)$, 86.3°

Given points:

- $A = (3, 8, 9)$
- $B = (5, 12, 11)$
- $C = (-5, 6, 8)$
- $D = (13, 0, 5)$

Vector calculations:

- $\vec{AB} = \vec{b} - \vec{a} = (5, 12, 11) - (3, 8, 9) = (2, 4, 2)$
- $\vec{CD} = \vec{d} - \vec{c} = (13, 0, 5) - (-5, 6, 8) = (18, -6, -3)$
- \vec{AB} scaled by 9: $9\vec{AB} = (18, 12, 18)$
- $\vec{I}_1 = (3, 8, 9) + 9(2, 4, 2) = (3+18, 21+12, 9+18) = (21, 33, 27)$
- $\vec{I}_2 = (3, 8, 9) + t(6, 2, -1) = (3+6t, 8+2t, 9-t)$ scaled by 3: $3\vec{CD} = (54, -18, -9)$
- $\vec{I}_1 = \vec{I}_2 \Rightarrow 21 = 3 + 6t \Rightarrow t = 3$
- $\vec{I}_1 = (21, 33, 27)$
- $\vec{I}_2 = (3+6(3), 8+2(3), 9-3) = (21, 14, 6)$
- \therefore Lines intersect at $(21, 14, 6)$
- $\angle ABD = 2$ (using $(21, 21, 27)$ and $(13, 0, 5)$) we obtain $(1, 4, 7)$
- \bullet Check $\frac{1}{2}$:
 $\begin{cases} 2(18+2)-2=2 \\ 2(3+6t+18)=2 \\ 2(18+2)-2=2 \end{cases} \quad \begin{cases} 2+6=8 \\ 2+6=8+13 \\ 2+6=8+13 \end{cases}$
 Substituted:
 $\begin{cases} 1=7 \\ 7=14 \\ 1=7 \end{cases}$
 \therefore Lines intersect
- \bullet Check $\frac{1}{2}$:
 $\begin{cases} 2(18+2)-2=2 \\ 2(3+6t+18)=2 \\ 2(18+2)-2=2 \end{cases} \quad \begin{cases} 2+6=8 \\ 2+6=8+13 \\ 2+6=8+13 \end{cases}$
 Substituted:
 $\begin{cases} 1=7 \\ 7=14 \\ 1=7 \end{cases}$
 \therefore Lines intersect
- Diagram: Shows two 3D vectors originating from the same point, labeled \vec{a} and \vec{b} . The angle between them is labeled θ .
- Dot product: $\vec{a} \cdot \vec{b} = 1(1) + 2(2) + 3(3) = 14$
- Magnitude: $\|\vec{a}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$
- Magnitude: $\|\vec{b}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$
- Cosine of angle: $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} = \frac{14}{\sqrt{14} \sqrt{14}} = 1$
- Angle: $\theta = 86.3^\circ$

SHORTEST DISTANCES INVOLVING LINES

Question 1

The straight line l has vector equation

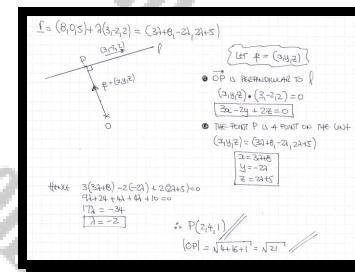
$$\mathbf{r} = 8\mathbf{i} + 5\mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}),$$

where λ is a scalar parameter.

The point P lies on l so that OP is perpendicular to l , where O is the origin.

Find the coordinates of P and the distance OP .

$$P(2, 4, 1), |OP| = \sqrt{21}$$



Question 2

The straight line l has vector equation

$$\mathbf{r} = 2\mathbf{i} - 9\mathbf{j} - 6\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$$

where λ is a scalar parameter.

The point P lies on l so that OP is perpendicular to l , where O is the origin.

Find the coordinates of P and the distance OP .

$$P(4, -1, 0), |OP| = \sqrt{17}$$

Diagram shows a 3D Cartesian coordinate system with axes i, j, k. A line l passes through points $(1, 0, 0)$ and $(0, 1, 0)$. A point P is shown on the line l with coordinates $(4, -1, 0)$.

Equation of line l :

$$\mathbf{r} = (1, 0, 0) + \lambda(0, 1, 0)$$

Let $\mathbf{p} = (4, -1, 0)$

Given $\mathbf{p} \perp \mathbf{r}$

$$\mathbf{p} \cdot \mathbf{r} = 0$$
$$(4, -1, 0) \cdot (1, 0, 0) = 0$$
$$4 + 0 - 0 = 0$$

\mathbf{p} lies on the line l

$$(4, -1, 0) = (1, 0, 0) + \lambda(0, 1, 0)$$
$$3 + \lambda = 4$$
$$\lambda = 1$$
$$4 - 1 = -1$$
$$0 = 0$$

Solving simultaneously:

$$(4, -1, 0) + (0, 1, 0) + 3(0, 1, 0) = 0$$
$$4 + 0 + 3(-1, 0, 0) = 0$$
$$4 + 0 - 3 = 0$$
$$4 - 3 = 0$$
$$\lambda = 1$$
$$\therefore P(4, -1, 0)$$

d. distance $|OP| = \sqrt{(4^2 + (-1)^2 + 0^2)}$

$$= \sqrt{16 + 1}$$
$$= \sqrt{17}$$

Question 3

The straight line l has vector equation

$$\mathbf{r} = 9\mathbf{i} + 11\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 4\mathbf{j} - \mathbf{k})$$

where λ is a scalar parameter.

The point P lies on l so that OP is perpendicular to l , where O is the origin.

Find the coordinates of P and the distance OP .

$$P(3, -1, 2), |OP| = \sqrt{14}$$

Given the vector equation of the line l : $\mathbf{r} = (9\mathbf{i} + 11\mathbf{j} - \mathbf{k}) + \lambda(2\mathbf{i} + 4\mathbf{j} - \mathbf{k})$. Let $P = (3, -1, 2)$. Since $OP \perp l$, we have $(3\mathbf{i} + 4\mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) = 0$. This gives $(3, -1, 2) \cdot (2, 4, -1) = 0$. Solving for λ gives $2(3+9) + 4(-1+1) - (-2-1) = 0 \Rightarrow 24 + 4 - 2 = 0 \Rightarrow 24 = 2 \Rightarrow \lambda = 12$. Therefore, $P(3, -1, 2)$. Then, $|OP| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$.

Question 4

The straight line l has vector equation

$$\mathbf{r} = -16\mathbf{i} + 10\mathbf{j} - 7\mathbf{k} + \lambda(6\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

where λ is a scalar parameter.

The point P lies on l so that OP is perpendicular to l , where O is the origin.

Find the coordinates of P and the distance OP .

$$P(2, 4, -4), |OP| = 6$$

Working:

$$\begin{aligned} l &= (-16\mathbf{i}, 10\mathbf{j}, -7\mathbf{k}) + \lambda(6\mathbf{i}, -2\mathbf{j}, \mathbf{k}) \\ l &= (2(-8), 10 - 2\lambda, -7 + \lambda) \\ \text{Let } P &= (x, y, z) \\ \bullet \quad OP \perp l &\Rightarrow \text{Dot product is zero} \\ (2(-8), 10 - 2\lambda, -7) \cdot (x, y, z) &= 0 \\ 16 - 2x - 2y + 2z &= 0 \end{aligned}$$

\bullet THE POINT P IS A ROOT OF THE UNIT

$$(x, y, z) = (-1, -4, 2)$$

$$\begin{cases} x = -1 - 16 \\ y = -4 - 10 \\ z = 2 - 7 \end{cases}$$

$$\begin{cases} x = -17 \\ y = -14 \\ z = -5 \end{cases}$$

Scaling: SIMPLY DIVIDE

$$\begin{cases} (-17)/(-2) = 8.5 \\ (-14)/(-2) = 7 \\ (-5)/(-2) = 2.5 \end{cases}$$

$$\begin{cases} x = 8.5 \\ y = 7 \\ z = 2.5 \end{cases}$$

$$\therefore P(2, 4, -4)$$

$$\therefore |OP| = \sqrt{2^2 + 4^2 + (-4)^2} = 6$$

Question 5

The straight line l has vector equation

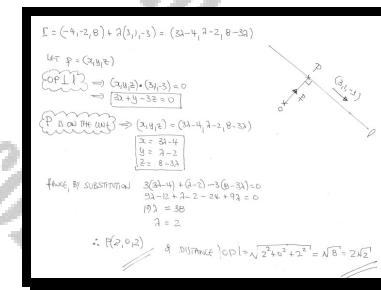
$$\mathbf{r} = -4\mathbf{i} - 2\mathbf{j} + 8\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} - 3\mathbf{k})$$

where λ is a scalar parameter.

The point P lies on l so that OP is perpendicular to l , where O is the origin.

Find the coordinates of P and the distance OP .

$$P(2, 0, 2), |OP| = 2\sqrt{2}$$



$$\Gamma = (-4, -2, 8) + \lambda(3, 1, -3) = (3\lambda - 4, \lambda - 2, 8 - 3\lambda)$$

$$\text{Let } \mathbf{p} = (x, y, z)$$

$$\text{So } \Gamma \text{ is } \Rightarrow (x, y, z) = (3\lambda - 4, \lambda - 2, 8 - 3\lambda)$$

$$\Rightarrow 3x + y - 3z = 0$$

$$\text{P is on the line } \Rightarrow (x, y, z) = (3\lambda - 4, \lambda - 2, 8 - 3\lambda)$$

$$\begin{cases} x = 3\lambda - 4 \\ y = \lambda - 2 \\ z = 8 - 3\lambda \end{cases}$$

$$\text{From, by substitution } 3(3\lambda - 4) - (\lambda - 2) - 3(8 - 3\lambda) = 0$$

$$9\lambda - 12 - \lambda + 2 - 24 + 9\lambda = 0$$

$$19\lambda = 38$$

$$\lambda = 2$$

$$\therefore P(2, 0, 2)$$

$$\therefore \text{DISTANCE } |OP| = \sqrt{2^2 + 0^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

Question 6

The straight line l has vector equation

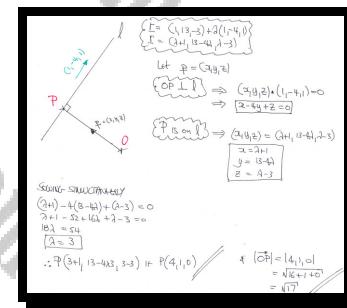
$$\mathbf{r} = \mathbf{i} + 13\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - 4\mathbf{j} + \mathbf{k})$$

where λ is a scalar parameter.

The point P lies on l so that OP is perpendicular to l , where O is the origin.

Find the coordinates of P and the distance OP .

$$P(4,1,0), |OP| = \sqrt{17}$$



$$\begin{aligned}
 & \text{Let } \vec{OP} = (x, y, z) \\
 & \vec{OP} \perp l \Rightarrow (x, y, z) \cdot (1, -4, 1) = 0 \\
 & \Rightarrow x - 4y + z = 0 \\
 & \text{Given } \vec{OP} = (4, 1, 0) \\
 & \Rightarrow (4, 1, 0) \cdot (1, -4, 1) = 0 \\
 & \Rightarrow 4 - 4 + 0 = 0 \\
 & \Rightarrow 0 = 0 \quad \text{True} \\
 & \text{Solving simultaneously:} \\
 & \begin{cases} x - 4y + z = 0 \\ x = 4 \\ 4 - 4y + z = 0 \end{cases} \\
 & \Rightarrow 4 - 4y + z = 0 \\
 & \Rightarrow z = 4y \\
 & \Rightarrow z = 4 \\
 & \Rightarrow \boxed{\lambda = 1} \\
 & \therefore \vec{OP} = (4, 1, 0) \quad \text{if } \vec{OP} = (4, 1, 0) \\
 & \quad \times |\vec{OP}| = |(4, 1, 0)| \\
 & \quad = \sqrt{16 + 1 + 0} \\
 & \quad = \sqrt{17}
 \end{aligned}$$

Question 7

The straight line l has vector equation

$$\mathbf{r} = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$$

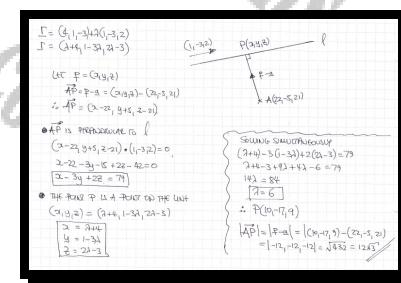
where λ is a scalar parameter.

The point A has coordinates $(22, -5, 21)$.

The point P lies on l so that AP is perpendicular to l .

Find the coordinates of P and the distance AP .

$$P(10, -17, 9), |AP| = 12\sqrt{3}$$



Question 8

The straight line l has vector equation

$$\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + \lambda(-\mathbf{i} + 2\mathbf{k}),$$

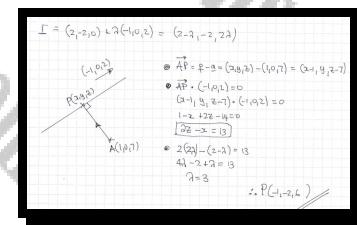
where λ is a scalar parameter.

The point A has coordinates $(1, 0, 7)$.

The point P lies on l so that AP is perpendicular to l .

Find the coordinates of P .

$$P(-1, -2, 6)$$



Question 9

The straight line l has vector equation

$$\mathbf{r} = 17\mathbf{i} + 6\mathbf{j} + 47\mathbf{k} + \lambda(\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}),$$

where λ is a scalar parameter.

The point A has coordinates $(-15, 16, 12)$.

The point P lies on l so that AP is perpendicular to l .

Find the coordinates of P and the distance AP .

$$P(15, -8, 35), |AP| = \sqrt{2005}$$

The diagram shows a 3D Cartesian coordinate system with a line l passing through point $A(-15, 16, 12)$. A point $P(x, y, z)$ is shown on the line l , and a vector \vec{AP} is drawn from A to P . The line l is defined by the vector equation $\mathbf{r} = 17\mathbf{i} + 6\mathbf{j} + 47\mathbf{k} + \lambda(\mathbf{i} + 7\mathbf{j} + 6\mathbf{k})$. The vector \vec{AP} is given as $\vec{AP} = \mathbf{r} - \mathbf{a} = (x, y, z) - (-15, 16, 12) = (x+15, y-16, z-12)$. Since \vec{AP} is perpendicular to the line l , the dot product $\vec{AP} \cdot \mathbf{d}_l = 0$, where $\mathbf{d}_l = (\mathbf{i} + 7\mathbf{j} + 6\mathbf{k})$. This leads to the equation $(x+15)(1) + (y-16)(7) + (z-12)(6) = 0$, which simplifies to $x+15 + 7y-112 + 6z-72 = 0$, or $x+7y+6z = 167$. Solving simultaneously with the line equation, we get $x=15$, $y=-8$, and $z=35$, so $P(15, -8, 35)$. The distance $|AP|$ is calculated as $\sqrt{(15+15)^2 + (-8-16)^2 + (35-12)^2} = \sqrt{900 + 576 + 529} = \sqrt{2005}$.

Question 10

The parallel lines l_1 and l_2 have respective vector equations

$$\mathbf{r}_1 = \mathbf{i} + 6\mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$$

$$\mathbf{r}_2 = 8\mathbf{i} + \mathbf{j} + 25\mathbf{k} + \mu(3\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$$

where λ and μ are scalar parameters.

Find the distance between l_1 and l_2 .

$$15\sqrt{2}$$

$\ell_1: \mathbf{r} = (1, 6, 1) + \lambda(3, 5, -4)$
 $\ell_2: \mathbf{r} = (1, 7, 1) + \mu(3, 5, -4)$
 Let $\mathbf{a} = (3, 5, -4)$ be on ℓ_1
 Let $\mathbf{b} = (3, 5, -4)$ be on ℓ_2
 $\therefore |\mathbf{AB}| = |\mathbf{a} - \mathbf{b}| = |(3, 5, -4) - (1, 7, 1)| = |(2, -2, -3)|$
 $\therefore |\mathbf{AB}| = \sqrt{2^2 + (-2)^2 + (-3)^2} = \sqrt{4 + 4 + 9} = \sqrt{17}$
 SECOND: BY SUBSTITUTION
 $3(3\lambda + 1) = 3(\mu + 7) - 10 + 4\mu = 29$
 $9\lambda + 3 = 3\mu + 21 - 10 + 4\mu = 29$
 $9\lambda + 3 = 7\mu + 11$
 $9\lambda = 7\mu + 8$
 $\lambda = \frac{7}{9}\mu + \frac{8}{9}$
 $\therefore \mathbf{B}(1, 7, 1)$
 Hence $|\mathbf{AB}| = \sqrt{(3, 5, -4) - (1, 7, 1)} = \sqrt{17}$

Question 11

The parallel lines l_1 and l_2 have respective vector equations

$$\mathbf{r}_1 = 8\mathbf{i} + 3\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{k})$$

$$\mathbf{r}_2 = -\mathbf{i} + \mathbf{j} + \mu(2\mathbf{i} - \mathbf{k})$$

where λ and μ are scalar parameters.

Find the distance between l_1 and l_2 .

[3]

