

INTEGRATION MIX

$$1. \int 6(4x+3)^{\frac{1}{2}} dx = (4x+3)^{\frac{3}{2}} + C$$

$$\begin{aligned} \int 6(4x+3)^{\frac{1}{2}} dx &= \text{BY INSPECTION} = \frac{6}{3}(4x+3)^{\frac{3}{2}} + C \\ &= \underline{(4x+3)^{\frac{3}{2}}} + C \end{aligned}$$

$$2. \int \frac{12}{\sqrt{3x+1}} dx = 8(3x+1)^{\frac{1}{2}} + C$$

$$\begin{aligned} \int \frac{12}{\sqrt{3x+1}} dx &= \int 12(3x+1)^{-\frac{1}{2}} dx = \text{BY INSPECTION} \\ &\quad (\text{UNQUOTE ANSWER}) \\ &= \frac{12}{2}(3x+1)^{\frac{1}{2}} + C = \underline{8(3x+1)^{\frac{1}{2}}} + C \end{aligned}$$

$$3. \int \frac{12x}{\sqrt{4x-1}} dx = \frac{1}{2}(4x-1)^{\frac{3}{2}} + \frac{3}{2}(4x-1)^{\frac{1}{2}} + C$$

$$\begin{aligned} \int \frac{12x}{\sqrt{4x-1}} dx &= \dots \text{BY SUBSTITUTION} \dots \\ &= \int \frac{12x}{4x-1} \left(\frac{du}{4}\right) = \int \frac{12x}{4x-1} du = \int \frac{3(4x)}{4(4x-1)} du \\ &= \int \frac{3(4x)}{4(4x-1)} du = \frac{3}{4} \int \frac{4x-1}{4x} du \\ &= \frac{3}{4} \int \frac{4x}{4x} + \frac{1}{4x} du = \frac{3}{4} \int u^{\frac{1}{2}} + u^{-\frac{1}{2}} du \\ &= \frac{3}{4} \left[\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} \right] + C = \frac{1}{2}u^{\frac{3}{2}} + \frac{3}{2}u^{\frac{1}{2}} + C \\ &= \frac{1}{2}(4x-1)^{\frac{3}{2}} + \frac{3}{2}(4x-1)^{\frac{1}{2}} + C \end{aligned}$$

ALTERNATIVE SUBSTITUTION

$$\begin{aligned} \int \frac{12x}{\sqrt{4x-1}} dx &= \dots \int \frac{12x}{4} \left(\frac{du}{4}\right) \\ &= \int \frac{12x}{2x} du = \int \frac{3(4x)}{2} du \\ &= \frac{3}{2} \int (4x) du = \frac{3}{2} \left[\frac{2}{3}u^{\frac{3}{2}} + u^{\frac{1}{2}} \right] + C \\ &= \frac{1}{2}u^{\frac{3}{2}} + \frac{3}{2}u^{\frac{1}{2}} + C \\ &= \frac{1}{2}(4x-1)^{\frac{3}{2}} + \frac{3}{2}(4x-1)^{\frac{1}{2}} + C \end{aligned}$$

$u = 4x-1$
 $\frac{du}{dx} = 4$
 $du = \frac{du}{dx} dx$
 $4x = u+1$

$u = (4x-1)^{\frac{1}{2}}$
 $u^2 = 4x-1$
 $2u \frac{du}{dx} = 4$
 $2u du = 4dx$
 $du = \frac{2dx}{2u}$
 $4x = u^2 + 1$

$$4. \int \frac{12}{4x+1} dx = 3\ln|4x+1| + C$$

$$\begin{aligned} \int \frac{12}{4x+1} dx &= \dots \text{RECOGNIZING A SPONDED LOG DIFFERENTIATION} \\ &\quad (\text{UNQUOTE ANSWER}) \\ &= \frac{12}{4} \ln|4x+1| + C = \underline{3\ln|4x+1| + C} \end{aligned}$$

5. $\int \frac{12x}{4x+1} dx = 3x - \frac{3}{4} \ln|4x+1| + C$

$$\begin{aligned} \int \frac{12x}{4x+1} dx &= \dots \text{SUBSTITUTION} \dots = \int \frac{12}{u} \left(\frac{du}{\frac{1}{4}} \right) \\ &= \int \frac{12}{u} du = \int \frac{3(u-1)}{4u} du \\ &= \int \frac{3(u-1)}{4u} du = \frac{3}{4} \int \frac{u-1}{u} du \\ &= \frac{3}{4} \int \frac{u}{u} - \frac{1}{u} du = \frac{3}{4} \left[u - \ln|u| \right] + C \\ &= \frac{3}{4}u - \frac{3}{4}\ln|u| + C = \frac{3}{4}(4x+1) - \frac{3}{4}\ln|4x+1| + C \end{aligned}$$

ALTERNATIVE BY MANIPULATION

$$\begin{aligned} \int \frac{12x}{4x+1} dx &= \int \frac{3(4x+1)-3}{4x+1} dx \\ &= \int \frac{3(4x+1)}{4x+1} dx - \int \frac{3}{4x+1} dx \\ &= \int 3 - \frac{3}{4x+1} dx \\ &= 3x - \frac{3}{4}\ln|4x+1| + C \end{aligned}$$

6. $\int (\sin x + \cos x)^2 dx = x - \frac{1}{2} \cos 2x + C$

$$\begin{aligned} \int (\sin x + \cos x)^2 dx &= \int \sin^2 x + 2\sin x \cos x + \cos^2 x dx \\ &= \int 1 + \sin 2x dx \\ &= x - \frac{1}{2} \sin 2x + C \end{aligned}$$

7. $\int (2\sin x + \operatorname{cosec} x)^2 dx = 6x - \sin 2x - \cot x + C$

$$\begin{aligned} \int (2\sin x + \operatorname{cosec} x)^2 dx &= \int 4\sin^2 x + 4\sin x \operatorname{cosec} x + \operatorname{cosec}^2 x dx \\ &= \int 4\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) + 4\left(\frac{1}{\sin x}\right) + \operatorname{cosec}^2 x dx \\ &= \int 2 - 2\cos 2x + 4 + \operatorname{cosec}^2 x dx \\ &= \int 6 - 2\cos 2x + \operatorname{cosec}^2 x dx \\ &= 6x - \sin 2x - \cot x + C \end{aligned}$$

8. $\int \frac{5}{(3x-1)(2x+1)} dx = \ln \left| \frac{3x-1}{2x+1} \right| + C$

$$\int \frac{5}{(3x-1)(2x+1)} dx = \dots \text{ BY PARTIAL FRACTION } \dots$$

$$\begin{aligned} \frac{5}{(3x-1)(2x+1)} &\equiv \frac{A}{3x-1} + \frac{B}{2x+1} \\ S &\equiv A(2x+1) + B(3x-1) \\ \bullet \text{ IF } 2x+1 \rightarrow S = \frac{5}{2}A \\ &\quad \Rightarrow \frac{5}{2}A \\ \bullet \text{ IF } 3x-1 \rightarrow S = -\frac{5}{2}B \\ &\quad \Rightarrow \frac{5}{2}B \end{aligned}$$

$$\begin{aligned} \dots &= \int \frac{\frac{3}{2}A}{3x-1} - \frac{\frac{5}{2}B}{2x+1} dx = \ln|3x-1| - \ln|2x+1| + C \\ &= \ln \left| \frac{3x-1}{2x+1} \right| + C \end{aligned}$$

9. $\int 4x \sin 2x dx = -2x \cos 2x + \sin 2x + C$

$$\int 4x \sin 2x dx = \dots \text{ INTEGRATION BY PARTS } \dots$$

| | |
|--------------|-----------|
| dx | 4 |
| $\int 4x dx$ | $\sin 2x$ |

$$\begin{aligned} &= -2x \cos 2x - \int -2 \cos 2x dx \\ &= -2x \cos 2x + \int 2 \cos 2x dx \\ &= -2x \cos 2x + \sin 2x + C \end{aligned}$$

10. $\int \frac{7}{4x} dx = \frac{7}{4} \ln|x| + C$

$$\int \frac{7}{4x} dx = \int \frac{7}{4} \times \frac{1}{x} dx = \dots \text{ BY INSERATION } \dots = \frac{7}{4} \ln|x| + C$$

11. $\int \left(x + \frac{2}{x} \right)^2 dx = \frac{1}{3}x^3 + 4x - \frac{4}{x} + C$

$$\begin{aligned} \int \left(x + \frac{2}{x} \right)^2 dx &= \int x^2 + 2x \cdot \frac{2}{x} + \frac{4}{x^2} dx \\ &= \int x^2 + 4 + 4x^{-2} dx \\ &= \frac{1}{3}x^3 + 4x - 4x^{-1} + C \\ &= \frac{1}{3}x^3 + 4x - \frac{4}{x} + C \end{aligned}$$

$$12. \int \frac{10}{(1-4x)^{\frac{7}{2}}} dx = \frac{1}{(1-4x)^{\frac{5}{2}}} + C$$

$$\begin{aligned} \int \frac{10}{(1-4x)^{\frac{7}{2}}} dx &= \int 10(1-4x)^{-\frac{7}{2}} dx = \text{BY RECOGNITION} \\ &= \frac{10}{5}(-4x)^{-\frac{5}{2}} + C = \frac{C(-4)}{(-4x)^{\frac{5}{2}}} + C \\ &= \frac{1}{(-4x)^{\frac{5}{2}}} + C \end{aligned}$$

$$13. \int (1+2\cos x) \sin x dx = \begin{bmatrix} -\cos x - \frac{1}{2}\cos 2x + C \\ -\cos x + \sin^2 x + C \\ -\cos x - \cos^2 x + C \\ -\frac{1}{4}(1+2\cos x)^2 + C \end{bmatrix}$$

$$\begin{aligned} \int (1+2\cos x) \sin x dx &= \int \sin x + 2\cos x \sin x dx \\ &= \int \sin x + \sin 2x dx \\ &= -\cos x - \frac{1}{2}\cos 2x + C \end{aligned}$$

OR ... = $-\cos x + \sin^2 x + C$
 OR ... = $-\cos x - \cos^2 x + C$

ALTERNATIVE: BY RHINE RULE RECOGNITION, OR SUBSTITUTION

$$\begin{aligned} \int (1+2\cos x) \sin x dx &= \dots \\ &= \int u \sin x \left(-\frac{du}{2\sin x} \right) \\ &= \int -\frac{1}{2}u \sin x du \\ &= -\frac{1}{2}u^2 + C \\ &= -\frac{1}{2}(1+2\cos x)^2 + C \end{aligned}$$

SUBSTITUTION
 $u = 1+2\cos x$
 $\frac{du}{dx} = -2\sin x$
 $du = -2\sin x dx$

$$14. \int 2\cos^2 x - \cos\left(\frac{1}{2}x\right) dx = x + \frac{1}{2}\sin 2x - 2\sin\left(\frac{1}{2}x\right) + C$$

$$\begin{aligned} \int 2\cos^2 x - \cos\frac{1}{2}x dx &= \int 2\left(\frac{1+\cos 2x}{2}\right) - \cos\frac{1}{2}x dx \\ &= \int 1 + \cos 2x - \cos\frac{1}{2}x dx \\ &= x + \frac{1}{2}\sin 2x - 2\sin\frac{1}{2}x + C \end{aligned}$$

$$15. \int \frac{14x}{x^2 - 4} dx = 7 \ln|x^2 - 4| + C$$

$\int \frac{14x}{x^2 - 4} dx = 7 \int \frac{2x}{x^2 - 4} dx = 7 \ln|x^2 - 4| + C$

$\int \frac{2x}{x^2 - 4} dx = \ln|x^2 - 4| + C$

ALTERNATIVE BY SUBSTITUTION:

$$\begin{aligned} \int \frac{u}{2x-4} du &= \dots \text{SUBSTITUTION} \dots = \int \frac{u}{u} \left(\frac{du}{2x-4} \right) du \\ &= \int \frac{u}{2x-4} du = 7 \ln|u| + C \quad u = x^2 - 4 \\ &= 7 \ln|x^2 - 4| + C \end{aligned}$$

ALTERNATIVE BY PARTIAL FRACTIONS:

Firstly we have: $\frac{14x}{x^2 - 4} = \frac{14x}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$

$$\begin{bmatrix} 14x \equiv A(x+2) + B(x-2) \end{bmatrix}$$

$$\begin{aligned} \text{If } x=2, \quad 28=4A \rightarrow A=7 \\ \text{If } x=-2, \quad -28=-4B \rightarrow B=7 \end{aligned}$$

Hence the integral becomes:

$$\begin{aligned} \int \frac{14x}{x^2 - 4} dx &= \int \frac{7}{x-2} + \frac{7}{x+2} dx = 7 \ln|x-2| + 7 \ln|x+2| + C \\ &= 7 \left[\ln|x-2| + \ln|x+2| \right] + C \\ &= 7 \ln|x^2 - 4| + C \end{aligned}$$

$$16. \int \frac{9}{2x^2 + x - 1} dx = 3 \ln \left| \frac{2x-1}{x+1} \right| + C$$

$\int \frac{9}{2x^2 + x - 1} dx = \int \frac{9}{(2x-1)(x+1)} dx \dots \text{BY PARTIAL FRACTIONS}$

$\frac{9}{(2x-1)(x+1)} \equiv \frac{A}{2x-1} + \frac{B}{x+1}$

$$\begin{bmatrix} 9 \equiv A(x+1) + B(2x-1) \end{bmatrix}$$

| | |
|-------------|----------------------|
| • If $x=-1$ | • If $x=\frac{1}{2}$ |
| $9 = -3B$ | $9 = \frac{1}{2}A$ |
| $B = -3$ | $A = 6$ |

$$\dots = \int \frac{6}{2x-1} - \frac{3}{x+1} dx = 3 \ln|2x-1| - 3 \ln|x+1| + C$$

$$= 3 \ln \left| \frac{2x-1}{x+1} \right| + C$$

$$17. \int (8x+1)e^{-2x} dx = -\frac{1}{2}(8x+1)e^{-2x} - 2e^{-2x} + C$$

$\int (8x+1)e^{-2x} dx = \dots \text{INTEGRATION BY PARTS}$

$$\dots = -\frac{1}{2}(8x+1)e^{-2x} - \int 4e^{-2x} dx$$

$$\begin{bmatrix} 8x+1 & B \\ e^{-2x} & e^A \end{bmatrix}$$

$$\begin{aligned} &= -\frac{1}{2}(8x+1)e^{-2x} + \int 4e^{-2x} dx \\ &= -\frac{1}{2}(8x+1)e^{-2x} - 2e^{-2x} + C \end{aligned}$$

18. $\int (e^{2x} + e^{-x})^2 dx = \frac{1}{4}e^{4x} + 2e^x - \frac{1}{2}e^{-2x} + C$

$$\begin{aligned}\int (e^{2x} + e^{-x})^2 dx &= \int (e^{2x})^2 + 2(e^{2x})(e^{-x}) + (e^{-x})^2 dx \\ &= \int e^{4x} + 2e^x + e^{-2x} dx \\ &= \frac{1}{4}e^{4x} + 2e^x - \frac{1}{2}e^{-2x} + C\end{aligned}$$

19. $\int \frac{6}{(3x+2)^3} dx = -\frac{1}{(3x+2)^2} + C$

$$\begin{aligned}\int \frac{6}{(3x+2)^3} dx &= \int 6(3x+2)^{-3} dx \quad \text{BY INSPECTION} \\ &= -\frac{6}{2} (3x+2)^{-2} + C = -\frac{1}{(3x+2)^2} + C\end{aligned}$$

20. $\int \frac{6x}{(3x+2)^3} dx = \frac{2}{3} \left[\frac{1}{(3x+2)^2} - \frac{1}{3x+2} \right] + C$

| | |
|--|--|
| $\begin{aligned}\int \frac{6x}{(3x+2)^3} dx &= \dots \quad \text{BY SUBSTITUTION} \\ &= \int \frac{6x}{u^3} \left(\frac{du}{3} \right) \\ &= \int \frac{2u^{-4}}{3u^2} du \\ &= \int \frac{2u^{-4}}{3u^2} du \\ &= \int \frac{2u^{-2}}{3u^3} du \\ &= \frac{2}{3} u^{-1} + \frac{2}{3} u^{-2} + C \\ &= \frac{2}{3} \left[\frac{1}{u^2} - \frac{1}{u} \right] + C \\ &= \frac{2}{3} \left[\frac{1}{(3x+2)^2} - \frac{1}{3x+2} \right] + C\end{aligned}$ | <p>ALTERNATIVE BY PARTIAL FRACTIONS</p> $\frac{6x}{(3x+2)^3} = \frac{A}{(3x+2)} + \frac{B}{(3x+2)^2} + \frac{C}{(3x+2)^3}$ $6x = A(3x+2)^2 + B(3x+2) + C$ <ul style="list-style-type: none"> • If $x = -\frac{2}{3}$ • If $x = 0$ • If $x = -1$ -4 = A 0 = A + B + C -6 = A - B + C 0 = A + 4B + C 0 = A + B + C 0 = A + B + C 4 = 2B + 4C 4 = B + C 2 = -B + C 2 = B + 4C 2 = B + C 2 = -B + C <p>∴ $A = -4$, $B = 0$, $C = 2$</p> $\begin{aligned}\int \frac{6x}{(3x+2)^3} dx &= \int \frac{-4}{(3x+2)} - \frac{4}{(3x+2)^2} dx \\ &= \int 2(3x+2)^{-3} - 4(3x+2)^{-2} dx \\ &= -\frac{2}{3}(3x+2)^{-2} - \frac{4}{3}(3x+2)^{-3} + C \\ &= -\frac{2}{3} \left[\frac{1}{(3x+2)^2} - \frac{1}{(3x+2)^3} \right] + C\end{aligned}$ |
|--|--|

$$21. \int (\sin x - 2\cos x) \sin x \, dx = \begin{bmatrix} \frac{1}{2}x - \frac{1}{4}\sin 2x + \frac{1}{2}\cos 2x + C \\ \frac{1}{2}x - \frac{1}{4}\sin 2x + \cos^2 x + C \\ \frac{1}{2}x - \frac{1}{4}\sin 2x - \sin^2 x + C \end{bmatrix}$$

$$\begin{aligned} \int (\sin x - 2\cos x) \sin x \, dx &= \int \sin^2 x - 2\cos x \sin x \, dx \\ &= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) - \sin 2x \, dx \\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + \frac{1}{2}\cos 2x + C \end{aligned}$$

This can also be integrated directly to either
+ $\sin^2 x$
or
- $\sin^2 x$

$$22. \int \frac{(2+\cos x)\sin 2x}{2\cos x} \, dx = \begin{bmatrix} -\frac{1}{2}(2+\cos x)^2 + C \\ -2\cos x + \frac{1}{2}\sin^2 x + C \\ -2\cos x - \frac{1}{2}\cos^2 x + C \\ -2\cos x - \frac{1}{4}\cos 2x + C \end{bmatrix}$$

$$\begin{aligned} \int \frac{(2+\cos x)\sin 2x}{2\cos x} \, dx &= \int \frac{(2+\cos x)(2\sin x)}{2\cos x} \, dx \\ &= \int (2+\cos x)\sin x \, dx = -\frac{1}{2}(2+\cos x)^2 + C \quad (\text{BY REVERSE CHAIN RULE}) \\ &\quad \downarrow \text{OR CONTINUING WITH TRIGONOMETRIC IDENTITIES} \\ &= \int 2\sin x + \cos x \sin x \, dx = \begin{cases} -2\cos x + \frac{1}{2}\sin^2 x + C \\ -2\cos x - \frac{1}{2}\cos^2 x + C \end{cases} \quad (\text{BY INTEGRATION}) \\ &\quad \downarrow \text{OR ANOTHER APPROACH} \\ &= \int 2\sin x + \frac{1}{2}(2\sin x \cos x) \, dx \\ &= \int 2\sin x + \frac{1}{2}\sin 2x \, dx \\ &\approx -2\cos x - \frac{1}{4}\cos 2x + C \end{aligned}$$

$$23. \int \frac{8x-3}{4x} \, dx = 2x - \frac{3}{4}\ln|x| + C$$

$$\begin{aligned} \int \frac{8x-3}{4x} \, dx &= \int \frac{8x}{4x} - \frac{3}{4x} \, dx = \int 2 - \frac{3}{4x} \, dx \\ &= 2x - \frac{3}{4}\ln|x| + C \end{aligned}$$

24. $\int \frac{6x^2}{x^3+8} dx = 2\ln|x^3+8| + C$

$$\begin{aligned}\int \frac{6x^2}{x^3+8} dx &= 2 \int \frac{3x^2}{x^3+8} dx \quad \leftarrow \int \frac{f(u)}{g(u)} du = \ln|g(u)| + C \\ &= 2\ln|x^3+8| + C \\ \text{ALTERNATIVE BY SUBSTITUTION} \\ \int \frac{6x^2}{x^3+8} dx &= \dots \text{SUBSTITUTION} \dots = \int \frac{2}{u} \left(\frac{du}{3x^2} \right) \\ &= \int \frac{2}{u} du = 2\ln|u| + C \\ &= 2\ln|x^3+8| + C\end{aligned}$$

25. $\int (1+2\cos x)^3 \sin x dx = -\frac{1}{8}(1+2\cos x)^4 + C$

$$\begin{aligned}\int (1+2\cos x)^3 \sin x dx &= \frac{-1}{8}(1+2\cos x)^4 + C \quad (\text{BY REVERSE CHAIN RULE}) \\ \text{ALTERNATIVE BY SUBSTITUTION} \\ \int (1+2\cos x)^3 \sin x dx &(\text{BY SUBSTITUTION}) \dots \\ &= \int (1+2\cos x)^3 \left(\frac{du}{-2\sin x} \right) = \int -\frac{1}{2}u^3 du \\ &= -\frac{1}{8}u^4 + C = -\frac{1}{8}(1+2\cos x)^4 + C \\ [\text{THE SUBSTITUTION } u &= 2\cos x \text{ OR } u = \cos x \text{ ALSO "WORKS"]}\end{aligned}$$

26. $\int \cos \sqrt{x} dx = 2\sqrt{x} \sin \sqrt{x} + 2\cos \sqrt{x} + C$

$$\begin{aligned}\int \cos \sqrt{x} dx &\dots \text{BY SUBSTITUTION} \dots \\ &= \int \cos u (2u du) = \int 2u \cos u du \\ \text{INTEGRATION BY PARTS, NEXT} \\ &= 2u \sin u - \int 2u \sin u du \\ &= 2u \sin u + 2u \cos u + C \\ &= 2\sqrt{x} \sin \sqrt{x} + 2\cos \sqrt{x} + C\end{aligned}$$

27. $\int \frac{3}{2x-1} dx = \frac{3}{2} \ln|2x-1| + C$

$$\int \frac{3}{2x-1} dx = \dots \text{BY INJECTION} \dots = \frac{3}{2} \ln|2x-1| + C$$

28. $\int \frac{10}{(2x+1)^6} dx = -\frac{1}{(2x+1)^5} + C$

$$\begin{aligned}\int \frac{10}{(2x+1)^6} dx &= \int 10(2x+1)^{-6} dx \dots \text{BY INSPECTION} \dots &= \frac{10}{2}(2x+1)^{-5} + C \\ &= -\frac{1}{(2x+1)^5} + C &= -\frac{1}{(2x+1)^5} + C\end{aligned}$$

29. $\int \frac{3x-1}{(2x+1)(x-2)} dx = \frac{1}{2} \ln|2x+1| + \ln|x-2| + C$

$$\begin{aligned}\int \frac{3x-1}{(2x+1)(x-2)} dx &= \dots \text{BY PARTIAL FRACTIONS} \\ \frac{3x-1}{(2x+1)(x-2)} &\equiv \frac{A}{2x+1} + \frac{B}{x-2} \\ 3x-1 &\equiv A(x-2) + B(2x+1) \\ \bullet 14x+2 &\quad \begin{matrix} S=56 \\ B=1 \end{matrix} \quad \bullet 14x-2 &\quad \begin{matrix} 2x=\frac{1}{2} \\ A=1 \end{matrix} \\ &= \int \frac{1}{2x+1} + \frac{1}{x-2} dx = \frac{1}{2} \ln|2x+1| + \ln|x-2| + C\end{aligned}$$

30. $\int (2+\sin x)^2 dx = \frac{9}{2}x - 4\cos x - \frac{1}{4}\sin 2x + C$

$$\begin{aligned}\int (2+\sin x)^2 dx &= \int 4 + 4\sin x + \sin^2 x dx \\ &= \int 4 + 4\sin x + \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx \\ &= \int \frac{9}{2} + 4\sin x - \frac{1}{2}\cos 2x dx \\ &= \frac{9}{2}x - 4\cos x - \frac{1}{4}\sin 2x + C\end{aligned}$$

31. $\int 5(2x-3)^{\frac{1}{4}} dx = 2(2x-3)^{\frac{5}{4}} + C$

$$\begin{aligned}\int 5(2x-3)^{\frac{1}{4}} dx &= \dots \text{BY INSPECTION} \dots + \frac{5}{2}(2x-3)^{\frac{5}{4}} + C \\ &= 2(2x-3)^{\frac{5}{4}} + C\end{aligned}$$

32. $\int \frac{e^{4x} + 3}{e^{3x}} dx = e^x - e^{-3x} + C$

$$\begin{aligned}\int \frac{e^{4x} + 3}{e^{3x}} dx &= \int \frac{e^{4x}}{e^{3x}} + \frac{3}{e^{3x}} dx = \int e^x + 3e^{-3x} dx \\ &= e^x - e^{-3x} + C\end{aligned}$$

33. $\int x e^{5x} dx = \frac{1}{5} x e^{5x} - \frac{1}{25} e^{5x} + C$

$$\begin{aligned}\int x e^{5x} dx &= \dots \text{INTEGRATION BY PARTS} \\ &= \frac{1}{5} x e^{5x} - \int \frac{1}{5} e^{5x} dx \\ &= \frac{1}{5} x e^{5x} - \frac{1}{25} e^{5x} + C\end{aligned}$$

34. $\int \frac{x^3}{x^4 + 2} dx = \frac{1}{4} \ln(x^4 + 2) + C$

$$\begin{aligned}\int \frac{x^3}{x^4 + 2} dx &= \frac{1}{4} \int \frac{du^3}{u^4 + 2} du = \dots \text{OF THE FORM} \\ &\quad \int \frac{f(u)}{g(u)} du = \ln|f(u)| + C \\ &= \frac{1}{4} \ln(u^4 + 2) + C \\ \text{ALTERNATIVE BY SUBSTITUTION: } u &= x^4 + 2 \\ \frac{du}{dx} &= 4x^3 \Rightarrow du = 4x^3 dx \\ \int \frac{x^3}{x^4 + 2} dx &= \int \frac{\frac{1}{4}u^2}{u^4 + 2} \left(\frac{du}{4x^3} \right) = \int \frac{1}{4u} du \\ &= \int \frac{1}{4} \times \frac{1}{u} du = \frac{1}{4} \ln|u| + C \\ &= \frac{1}{4} \ln(x^4 + 2) + C\end{aligned}$$

35. $\int \frac{2}{(x-2)(x-4)} dx = \ln \left| \frac{x-4}{x-2} \right| + C$

$$\begin{aligned}\int \frac{2}{(x-2)(x-4)} dx &= \dots \text{BY PARTIAL FRACTIONS} \dots \\ &= \int \frac{1}{x-4} - \frac{1}{x-2} dx \\ &= \ln|x-4| - \ln|x-2| + C \\ &= \ln \left| \frac{x-4}{x-2} \right| + C\end{aligned}$$

| | | |
|------------------------|-------------------|---------------------------------|
| $\frac{2}{(x-2)(x-4)}$ | \equiv | $\frac{A}{x-2} + \frac{B}{x-4}$ |
| $2 \equiv$ | $A(x-4) + B(x-2)$ | |
| • If $x=4$, $2=2B$ | | $B=1$ |
| • If $x=2$, $2=-2A$ | | $A=-1$ |

36. $\int \frac{3}{4x+1} dx = \frac{3}{4} \ln|4x+1| + C$

$$\int \frac{3}{4x+1} dx = \dots \text{ BY INSPECTION } \dots = \frac{3}{4} \ln|4x+1| + C$$

37. $\int \left(1 + \frac{1}{x}\right)^2 dx = x + 2\ln|x| - \frac{1}{x} + C$

$$\begin{aligned} \int \left(1 + \frac{1}{x}\right)^2 dx &= \int 1 + 2x\ln\frac{1}{x} + \left(\frac{1}{x}\right)^2 dx = \int 1 + \frac{2}{x} + \frac{1}{x^2} dx \\ &= \int 1 + \frac{2}{x} + x^{-2} dx = x + 2\ln|x| - \frac{1}{x} + C \\ &= x + 2\ln|x| - \frac{1}{x} + C \end{aligned}$$

38. $\int \frac{x}{(x^2-1)^3} dx = -\frac{1}{4(x^2-1)^2} + C$

$$\begin{aligned} \int \frac{x}{(x^2-1)^3} dx &= \dots \text{ BY REVERSE CHAIN RULE } \dots = \int 2(x^2-1)^{-3} dx \\ \frac{d}{dx}(x^2-1) &= 2x \end{aligned}$$

$$= -\frac{1}{4}(x^2-1)^{-2} + C = -\frac{1}{4(x^2-1)^2} + C$$

ALTERNATIVE - BY SUBSTITUTION

$$\begin{aligned} \int \frac{x}{(x^2-1)^3} dx &= \int \frac{x}{u^3} \left(\frac{du}{2x} \right) = \int \frac{1}{2u^3} du \quad u = x^2-1 \\ \frac{du}{dx} &= 2x \quad \frac{du}{2x} = \frac{du}{2x} \\ &= \frac{1}{2}u^{-2} + C = -\frac{1}{4}u^{-2} \\ &= -\frac{1}{4(x^2-1)^2} + C \end{aligned}$$

39. $\int \cos x - \sin x dx = \sin x + \cos x + C$

$$\int \cos x - \sin x dx = \sin x + \cos x + C$$

40. $\int \sin x - \cos x \, dx = -\cos x - \sin x + C$

$$\int \sin x - \cos x \, dx = -\cos x - \sin x + C$$

41. $\int \sin(4x+3) \, dx = -\frac{1}{4}\cos(4x+3) + C$

$$\int \sin(4x+3) \, dx = \dots \text{BY INSPECTION} \dots = -\frac{1}{4}\cos(4x+3) + C$$

42. $\int \frac{x}{\sqrt{x+1}} \, dx = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C$

$$\begin{aligned} & \int \frac{x}{\sqrt{x+1}} \, dx = \dots \text{BY SUBSTITUTION} \dots \\ &= \int \frac{x}{u^{\frac{1}{2}}} \, du = \int \frac{u-1}{u^{\frac{1}{2}}} \, du = \int \frac{u}{u^{\frac{1}{2}}} - \frac{1}{u^{\frac{1}{2}}} \, du \\ &= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} \, du = \frac{1}{2}u^{\frac{3}{2}} - \frac{1}{\frac{1}{2}}u^{\frac{1}{2}} + C \\ &= \frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C \end{aligned}$$

$$\begin{aligned} u &= x+1 \\ \frac{du}{dx} &= 1 \\ du &= dx \\ \frac{dx}{du} &= 1 \\ x &= u-1 \end{aligned}$$

$$\begin{aligned} & \int \frac{x}{\sqrt{x+1}} \, dx = \int \frac{x}{u^{\frac{1}{2}}} (2u \, du) = \int 2x \, du \\ &= \int 2(u^{\frac{1}{2}-1}) \, du = \int 2u^{\frac{1}{2}-1} \, du \\ &= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C \end{aligned}$$

$$\begin{aligned} u &= \sqrt{2x+1} \\ u^2 &= 2x+1 \\ 2u \frac{du}{dx} &= 1 \\ du &= \frac{dx}{2u} \\ x &= u^2-1 \end{aligned}$$

43. $\int \cos(5-2x) \, dx = -\frac{1}{2}\sin(5-2x) + C$

$$\int \cos(5-2x) \, dx = -\frac{1}{2}\sin(5-2x) + C$$

44. $\int 3\sin 2x \, dx = -\frac{3}{2}\cos 2x + C$

$$\int 3\sin 2x \, dx = -\frac{3}{2}\cos 2x + C$$

45. $\int (1 + \sec^2 x) \sin x \, dx = -\cos x + \sec x + C$

$$\begin{aligned} \int \sin(1 + \sec^2 x) \, dx &= \int \sin x + \sin \sec^2 x \, dx \\ &= \int \sin x + \frac{\sin x}{\sec^2 x} \, dx \\ &= \int \sin x + \tan x \sec x \, dx \\ &\quad \boxed{\frac{d(\sec x)}{dx} = \sec x \tan x} \\ &= -\cos x + \sec x + C \end{aligned}$$

46. $\int (1 - 2\cos x)^2 \, dx = 3x - 4\sin x + \sin 2x + C$

$$\begin{aligned} \int (1 - 2\cos x)^2 \, dx &= \int 1 - 4\cos x + 4\cos^2 x \, dx \\ &= \int 1 - 4\cos x + 4\left(\frac{1+\cos 2x}{2}\right) \, dx \\ &= \int 1 - 4\cos x + 2 + 2\cos 2x \, dx \\ &= \int 3 - 4\cos x + 2\cos 2x \, dx \\ &= 3x - 4\sin x + \sin 2x + C \end{aligned}$$

47. $\int (1 + \cot^2 x) \sec^2 x \, dx = \tan x - \cot x + C$

$$\begin{aligned} \int \sec(1 + \cot^2 x) \, dx &= \text{use identity } 1 + \cot^2 x = \csc^2 x, \text{ multiply out} \\ &= \int \sec x + \sec x \cot^2 x \, dx \\ &= \int \sec x + \frac{1}{\csc^2 x} \cdot \csc^2 x \, dx \\ &= \int \sec x + \csc^2 x \, dx \\ &= \tan x - \cot x + C \\ &\quad \boxed{\frac{d(\tan x)}{dx} = \sec^2 x} \\ &\quad \boxed{\frac{d(\cot x)}{dx} = -\csc^2 x} \end{aligned}$$

48. $\int 2x\cos 3x \, dx = \frac{2}{3}x\sin 3x + \frac{2}{9}\cos 3x + C$

$$\begin{aligned} \int 2x\cos 3x \, dx &= \dots \text{ INTEGRATION BY PARTS} \\ &= (2x)(\frac{1}{3}\sin 3x) - \int \frac{2}{3}\sin 3x \, dx \\ &= \frac{2}{3}x\sin 3x - \left[-\frac{2}{9}\cos 3x \right] + C \\ &= \frac{2}{3}x\sin 3x + \frac{2}{9}\cos 3x + C \end{aligned}$$

49. $\int \frac{3}{(2+x)(1-x)} \, dx = \ln \left| \frac{x+2}{x-1} \right| + C$

$$\begin{aligned} \int \frac{3}{(2+x)(1-x)} \, dx &= \dots \text{ BY PARTIAL FRACTION} \dots \\ \frac{3}{(2+x)(1-x)} &= \frac{A}{2+x} + \frac{B}{1-x} \\ 3 &\equiv A(-x+1) + B(2+x) \\ \bullet \text{ If } x=1 \Rightarrow 3=3B &\quad \bullet \text{ If } x=-2 \Rightarrow 3=3A \\ \Rightarrow B=1 &\quad \Rightarrow A=1 \\ \dots &= \int \frac{1}{2+x} + \frac{1}{1-x} \, dx = \ln|2+x| - \ln|1-x| + C \\ &= \ln \left| \frac{2+x}{1-x} \right| + C \\ &= \ln \left| \frac{x+2}{x-1} \right| + C \end{aligned}$$

50. $\int 10(3x+1)^4 \, dx = \frac{2}{3}(3x+1)^5 + C$

$$\begin{aligned} \int (3x+1)^5 \, dx &= \dots \text{ BY INSPECTION} \dots = \frac{10}{15}(3x+1)^5 + C \\ &= \frac{2}{3}(3x+1)^5 + C \end{aligned}$$

51. $\int 6(2x+1)^{\frac{1}{2}} \, dx = 2(2x+1)^{\frac{3}{2}} + C$

$$\begin{aligned} \int \frac{1}{\cot x \tan x} \, dx &= \int \frac{1}{\cot x \cdot \frac{\sin x}{\cos x}} \, dx = \int \frac{\cos^2 x}{\sin x} \, dx \\ &= \int \frac{1}{\sin x} \, dx = (\operatorname{cosec} x) + C \quad \text{INTEGRATE} \\ &= -\operatorname{cot} x + C \quad \frac{d}{dx}(\operatorname{cot} x) = -\operatorname{cosec}^2 x \end{aligned}$$

52. $\int \frac{1}{\cos^2 x \tan^2 x} dx = -\cot x + C$

$$\begin{aligned}\int \frac{1}{\cos^2 x \tan^2 x} dx &= \int \frac{1}{\cos^2 x \cdot \frac{\sin^2 x}{\cos^2 x}} dx = \int \frac{1}{\sin^2 x} dx \\&= -\cot x + C \\&\text{ANSWER: } \int \csc^2(\theta) d\theta = -\cot(\theta) + C \quad \boxed{\frac{d}{d\theta}(\cot(\theta)) = -\csc^2(\theta)}\end{aligned}$$

53. $\int \cos x \sin x dx = \begin{bmatrix} -\frac{1}{4} \cos 2x + C \\ -\frac{1}{2} \cos^2 x + C \\ \frac{1}{2} \sin^2 x + C \end{bmatrix}$

$$\begin{aligned}\int \cos x \sin x dx &= \int \frac{1}{2} (\cos x \sin x) dx = \int \frac{1}{2} \sin 2x dx \\&= \frac{1}{4} \cos 2x + C \\&\int \cos x \sin x dx = \frac{1}{2} \sin x + C \quad \text{since } \frac{d}{dx}(\sin x) = \cos x \\&\int \cos x \sin x dx = -\frac{1}{2} \cos x + C \quad \text{since } \frac{d}{dx}(\cos x) = -\sin x\end{aligned}$$

54. $\int \frac{2}{\cos^2 x} dx = 2 \tan x + C$

$$\int \frac{2}{\cos^2 x} dx = \int 2 \sec^2 x dx \quad \text{BY INSPECTION} \dots = 2 \tan x + C$$

55. $\int 2 + 2 \tan^2 x dx = 2 \tan x + C$

$$\begin{aligned}\int 2 + 2 \tan^2 x dx &= \int 2(1 + \tan^2 x) dx = \int 2 \sec^2 x dx \\&\Rightarrow \dots \text{BY INSPECTION} \dots = 2 \tan x + C\end{aligned}$$

56. $\int \frac{1+\cos x}{\sin^2 x} dx = -\cot x - \operatorname{cosec} x + C$

$$\begin{aligned}\int \frac{1+\cos x}{\sin^2 x} dx &= \int \frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x} dx = \int \operatorname{cosec}^2 x + \frac{\cos x}{\sin^2 x} dx \\ &= \int \operatorname{cosec}^2 x + \cot x \operatorname{cosec} x dx = -\operatorname{cot} x - \operatorname{cosec} x + C\end{aligned}$$

$\frac{d}{dx}(\operatorname{cot} x) = -\operatorname{cosec}^2 x$ $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \operatorname{cot} x$

57. $\int \frac{(1+\cos x)^2}{\sin^2 x} dx = -x - 2\cot x - 2\operatorname{cosec} x + C$

$$\begin{aligned}\int \frac{(1+\cos x)^2}{\sin^2 x} dx &= \int \frac{1+2\cos x+\cos^2 x}{\sin^2 x} dx \\ &= \int \frac{1}{\sin^2 x} + \frac{2\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} dx = \int \operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + \operatorname{cosec}^2 x dx \\ &= \int \operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + (\operatorname{cosec}^2 x - 1) dx = \int 2\operatorname{cosec}^2 x + \operatorname{cosec} x dx \\ &\quad \left[\frac{d}{dx}(\operatorname{cot} x) = -\operatorname{cosec}^2 x \quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \operatorname{cot} x \right] \\ &= -2\operatorname{cot} x - 2\operatorname{cosec} x - x + C\end{aligned}$$

58. $\int x \sin 3x dx = -\frac{1}{3}x \cos 3x - \frac{1}{9} \sin 3x + C$

$$\begin{aligned}\int x \sin 3x dx &= \dots \text{ INTEGRATION BY PARTS } \dots \\ &= x \left(\frac{1}{3} \cos 3x \right) - \int \frac{1}{3} \cos 3x dx \\ &= -\frac{1}{3} x \cos 3x + \int x \cos 3x dx \\ &= \underline{-\frac{1}{3} x \cos 3x} - \frac{1}{3} \sin 3x + C\end{aligned}$$

59. $\int \frac{2x}{(2x+1)^3} dx = \frac{1}{4(2x+1)^2} - \frac{1}{2(2x+1)} + C$

$$\begin{aligned}\int \frac{2x}{(2x+1)^3} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{2x}{u^3} \left(\frac{du}{2} \right) = \int \frac{2x}{2u^3} du \\ &= \int \frac{x}{2u^3} du = \int \frac{u}{2u^3} du = \frac{1}{2u^2} du \\ &= \int \frac{1}{2u^2} - \frac{1}{2} u^{-3} du = -\frac{1}{2} u^{-1} + \frac{1}{4} u^{-2} = \frac{1}{4}(2xu)^{-2} - \frac{1}{2}(2xu)^{-1} + C \\ &= \frac{1}{4(2xu)^2} - \frac{1}{2(2xu)} + C\end{aligned}$$

60. $\int (4-5x)^{-1} dx = -\frac{1}{5} \ln|4-5x| + C$

$$\int (4-5x)^{-1} dx = \int \frac{1}{4-5x} dx = -\frac{1}{5} \ln|4-5x| + C$$

61. $\int \frac{1}{4x} dx = \frac{1}{4} \ln|x| + C$

$$\int \frac{1}{4x} dx = \int \frac{1}{4} \cdot \frac{1}{x} dx = \frac{1}{4} \ln|x| + C$$

62. $\int \frac{1}{(x+1)(x+2)} dx = \ln \left| \frac{x+1}{x+2} \right| + C$

$$\int \frac{1}{(x+1)(x+2)} dx = \dots \text{ BY PARTIAL FRACTIONS ...}$$

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x+1)$$

$$\begin{cases} 1 \Rightarrow A(x+2) \\ \text{if } x=1 \Rightarrow 1 = A + 2B \end{cases} \quad \begin{cases} 1 \Rightarrow B(x+1) \\ \text{if } x=-1 \Rightarrow 1 = -A + B \end{cases}$$

$$\dots = \int \frac{1}{x+1} - \frac{1}{x+2} dx = \ln|x+1| - \ln|x+2| + C = \ln \left| \frac{x+1}{x+2} \right| + C$$

63. $\int \frac{x+1}{x} dx = x + \ln|x| + C$

$$\int \frac{x+1}{x} dx = \int \frac{x}{x} + \frac{1}{x} dx = \int 1 + \frac{1}{x} dx = x + \ln|x| + C$$

64. $\int \frac{x}{x+1} dx = x - \ln|x+1| + C$

$$\begin{aligned}\int \frac{x}{x+1} dx &= \dots \text{ BY MANIPULATION } \dots = \int \frac{(x+1)-1}{x+1} dx \\ &= \int \frac{x+1-1}{x+1} dx = \int 1 - \frac{1}{x+1} dx = x - \ln|x+1| + C\end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}\int \frac{x}{x+1} dx &= \int \frac{x}{u} (du) = \int \frac{u-1}{u} du \\ &= \int \frac{u}{u} - \frac{1}{u} du = \int 1 - \frac{1}{u} du \\ &= u - \ln|u| + C \\ &= (x+1) - \ln|x+1| + C\end{aligned}$$

$\frac{dx}{dt} = 1$
 $dx = dt$
 $t = u-1$

65. $\int \frac{4x}{\sqrt{1-2x^2}} dx = -\sqrt{1-2x^2} + C$

$$\begin{aligned}\int \frac{4x}{\sqrt{1-2x^2}} dx &= \int 4x(1-2x^2)^{-\frac{1}{2}} dx \dots \text{ BY REVERSE CHAIN RULE } \dots \\ &= \frac{4x}{-2} (1-2x^2)^{\frac{1}{2}} + C = -(1-2x^2)^{\frac{1}{2}} + C\end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}\int \frac{1}{\sqrt{1-2x^2}} dx &= \int \frac{1}{u^{\frac{1}{2}}} \left(\frac{du}{-4x} \right) \quad u = 1-2x^2 \\ &= \int -u^{-\frac{1}{2}} du = -\frac{1}{\frac{1}{2}} u^{\frac{1}{2}} + C \quad \frac{du}{dx} = -4x \\ &= -2(1-2x^2)^{\frac{1}{2}} + C\end{aligned}$$

THE SUBSTITUTION $u = \sqrt{1-2x^2}$ ALSO WORKS

66. $\int \frac{x+1}{9x^2-1} dx = \frac{2}{9} \ln|3x-1| - \frac{1}{9} \ln|3x+1| + C$

$$\begin{aligned}\int \frac{x+1}{9x^2-1} dx &= \int \frac{x+1}{(3x-1)(3x+1)} dx \dots \text{ BY PARTIAL FRACTIONS } \dots \\ \frac{x+1}{(3x-1)(3x+1)} &\equiv A(3x+1) + B(3x-1) \\ \bullet \text{ IF } 3x+\frac{1}{2} = 2A &\quad \bullet \text{ IF } 3x-\frac{1}{2} = 2B \\ A = \frac{1}{6} &\quad B = -\frac{1}{6}\end{aligned}$$

$$\begin{aligned}&= \int \frac{\frac{1}{6}}{3x-1} - \frac{\frac{1}{6}}{3x+1} dx = \frac{\frac{1}{3}}{3} \ln|3x-1| - \left(-\frac{1}{3} \right) \ln|3x+1| + C \\ &= \frac{1}{6} \ln|3x-1| + \frac{1}{3} \ln|3x+1| + C\end{aligned}$$

67. $\int x \sin 4x \, dx = \frac{1}{16} \sin 4x - \frac{1}{4} x \cos 4x + C$

$$\begin{aligned}\int x \sin 4x \, dx &= \dots \text{ INTEGRATION BY PARTS } \\ &= x \left(-\frac{1}{4} \cos 4x \right) - \int -\frac{1}{4} \cos 4x \, dx \\ &= -\frac{1}{4} x \cos 4x + \int \frac{1}{4} \cos 4x \, dx \\ &= -\frac{1}{4} x \cos 4x + \frac{1}{16} \sin 4x + C\end{aligned}$$

68. $\int \ln x \, dx = x \ln x - x + C$

$$\begin{aligned}\int \ln x \, dx &= \int 1 \times \ln x \, dx = \dots \text{ INTEGRATION BY PARTS } \\ &= x \ln x - \int x \left(\frac{1}{x} \right) dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + C\end{aligned}$$

(THE SUBSTITUTION $u = \ln x \Rightarrow x = e^u$, FOLLOWED BY INTEGRATION BY PARTS ALSO WORKS HERE.)

69. $\int \frac{4}{(2x-7)^2} \, dx = -\frac{2}{2x-7} + C$

$$\begin{aligned}\int \frac{4}{(2x-7)^2} \, dx &= \int 4(2x-7)^{-2} \, dx = \dots \text{ BY INSPECTION } \\ &= \frac{4}{-2} (2x-7)^{-1} + C = -2(2x-7)^{-1} + C = -\frac{2}{2x-7} + C\end{aligned}$$

70. $\int 4 \cos^2 x \, dx = 2x + \sin 2x + C$

$$\begin{aligned}\int 4 \cos^2 x \, dx &= \int 4 \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) \, dx = \int 2 + 2 \cos 2x \, dx \\ &= 2x + \sin 2x + C\end{aligned}$$

$$71. \int (1 + \tan^2 x) \sec^2 x \, dx = \tan x + \frac{1}{3} \tan^3 x + C$$

$$\begin{aligned}\int \sec^2(1+\tan x) \, dx &= \int \sec^2 x + \sec x \tan x \, dx = \dots \text{BY RECOGNITION...} \\ &= \tan x + \frac{1}{3} \tan^3 x + C \\ &\quad \uparrow \\ &\quad \frac{d}{dx}(\tan x) = 3x \tan^2 x \times \sec^2 x\end{aligned}$$

$$72. \int (1 + \tan x) \sec^2 x \, dx = \left[\begin{array}{l} \frac{1}{2}(1 + \tan x)^2 + C \\ \tan x + \frac{1}{2} \tan^2 x + C \\ \tan x + \frac{1}{2} \sec^2 x + C \end{array} \right]$$

$$\begin{aligned}\int \sec^2(1+\tan x) \, dx &= \dots \text{BY RECOGNITION/REARRANGEMENT/DEFINITION} \\ &= \frac{1}{2}(1+\tan x)^2 + C \\ \text{ALTERNATIVE} \\ \int \sec^2(1+\tan x) \, dx &= \int \sec^2 x + \sec x \tan x \, dx = \dots \text{BY RECOGNITION...} \\ &= \tan x + \frac{1}{2} \tan^2 x + C \\ &\quad \uparrow \\ &\quad \frac{d}{dx}(\tan x) = 2 \tan x \times \sec^2 x \\ \text{ANOTHER WAY/ALTERNATIVE} \\ \int \sec^2(1+\tan x) \, dx &= \int \sec^2 x + \sec x \tan x \, dx \\ &= \int \sec^2 x + \sec x (\sec x \tan x) \, dx \\ &= \tan x + \frac{1}{2} \sec^2 x + C \\ &\quad \uparrow \\ &\quad \frac{d}{dx}(\sec^2 x) = 2 \sec x \times \sec x \tan x\end{aligned}$$

$$73. \int \operatorname{cosec}^2(3x+1) \, dx = -\frac{1}{3} \cot(3x+1) + C$$

$$\begin{aligned}\int \operatorname{cosec}^2(3x+1) \, dx &= -\frac{1}{3} \operatorname{at}(3x+1) + C \\ \text{STANDARD DIFFERENTIATION} \quad \frac{d}{dx}(\operatorname{at}) &= -\operatorname{cosec}^2 x\end{aligned}$$

$$74. \int 12 \sec^2(2x+3) \, dx = 6 \tan(2x+3) + C$$

$$\begin{aligned}\int 12 \sec^2(2x+3) \, dx &= 6 \operatorname{at}(2x+3) + C \\ \text{STANDARD DIFFERENTIATION RESULT} \quad \frac{d}{dx}(\operatorname{at}) &= \sec^2 x\end{aligned}$$

$$75. \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = -2\cos \sqrt{x} + C$$

$$\begin{aligned} \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx &= \dots \text{BY REVERSE CHAIN RULE} \dots \int x^{\frac{1}{2}} \sin(x^{\frac{1}{2}}) dx \\ &= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \cos(x^{\frac{1}{2}}) + C = -2\cos(\sqrt{x}) + C \\ &= -2\cos(\sqrt{x}) + C \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned} \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx &= \int \frac{\sin u}{u} (2u^{\frac{1}{2}} du) & u = x^{\frac{1}{2}} \\ &= \int \frac{\sin u}{u} (2u du) = \int 2\sin u du & \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \\ &= -2\cos u + C = -2\cos(\sqrt{x}) + C & du = \frac{1}{2}x^{-\frac{1}{2}} dx \end{aligned}$$

THE SUBSTITUTION $u = \sqrt{x} \Rightarrow u^2 = x$, ALSO WORKS WELL.

$$76. \int 6e^{2x+2} dx = 3e^{2x+2} + C$$

$$\begin{aligned} \int 6e^{2x+2} dx &= \dots \text{BY INSPECTION} \dots = \frac{6}{2} e^{2x+2} + C \\ &= 3e^{2x+2} + C \end{aligned}$$

$$77. \int (1 - \cot^2 x) \sec^2 x dx = \tan x + \cot x + C$$

$$\begin{aligned} \int \sec^2(1 - \cot^2 x) dx &= \int \sec^2 - \csc^2 dx = \int \sec^2 - \frac{1}{\csc^2} (\csc^2) dx \\ &= \int \sec^2 - \frac{1}{\csc^2} dx = \int \sec^2 - \csc^2 dx \\ &= \tan x + \cot x + C \\ \frac{d}{dx}(\tan x) &= \sec^2 x \quad \& \frac{d}{dx}(\cot x) = -\csc^2 x \end{aligned}$$

$$78. \int \frac{\sin x - \cos x}{\sin x + \cos x} dx = -\ln |\sin x + \cos x| + C$$

$$\begin{aligned} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx &= -\int \frac{\cos x - \sin x}{\sin x + \cos x} dx = \dots \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C \\ &= -\ln |\sin x + \cos x| + C \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx &= \int \frac{\sin x - \cos x}{\sin x + \cos x} \frac{du}{\cos x - \sin x} & u = \sin x + \cos x \\ &= \int \frac{-(\cos x - \sin x)}{u} \frac{du}{\cos x - \sin x} = \int -\frac{1}{u} du & \frac{du}{dx} = \cos x - \sin x \\ &= -\ln |u| + C = -\ln |\sin x + \cos x| + C & du = \frac{du}{dx} dx \end{aligned}$$

79. $\int \sec x \tan x \sqrt{1+\sec x} dx = \frac{2}{3}(1+\sec x)^{\frac{3}{2}} + C$

$$\begin{aligned}\int \sec x \tan x \sqrt{1+\sec x} dx &= \int (\sec x + 1)^{\frac{1}{2}} (\sec x) dx \\ &\stackrel{\text{BY PARTIAL FRACTION DECOMPOSITION}}{=} \frac{1}{2\sqrt{2}} (\sec x + 1)^{\frac{3}{2}} + C \\ &= \frac{2}{3}(1+\sec x)^{\frac{3}{2}} + C\end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}\int \sec x \tan x \sqrt{1+\sec x}^2 dx &= \dots \\ &= \int (\sec x) u^{\frac{1}{2}} \frac{du}{\sec x} = \int u^{\frac{1}{2}} du \\ &= \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{3}(1+\sec x)^{\frac{3}{2}} + C\end{aligned}$$

(THE SUBSTITUTIONS $u = \sec x$ OR $u = (1+\sec x)^{\frac{1}{2}}$ AND WORK)

80. $\int \tan 2x \sec 2x dx = \frac{1}{2} \sec 2x + C$

$$\int \tan 2x \sec 2x dx = \dots \stackrel{\text{SIMPLIFIED DIFFERENTIATE}}{=} \frac{1}{2} \sec 2x + C$$

81. $\int x^2 \ln x dx = \frac{1}{3} x^3 \ln|x| - \frac{1}{9} x^3 + C$

$$\begin{aligned}\int x^2 \ln x dx &= \dots \stackrel{\text{INTEGRATION BY PARTS}}{=} \begin{array}{|c|c|} \hline \ln x & \frac{1}{x^2} \\ \hline \end{array} \\ &= (\frac{1}{3}x^3)(\ln x) - \int (\frac{1}{3})(3x^2) dx \\ &= \frac{1}{3}x^3 \ln|x| - \int x^2 dx \\ &= \frac{1}{3}x^3 \ln|x| - \frac{1}{3}x^3 + C\end{aligned}$$

82. $\int \frac{6}{x^2 - 2x - 8} dx = \ln \left| \frac{x-4}{x+2} \right| + C$

$$\begin{aligned}\int \frac{6}{x^2 - 2x - 8} dx &= \int \frac{6}{(x+2)(x-4)} dx = \dots \stackrel{\text{BY PARTIAL FRACTION}}{=} \\ &\begin{array}{|c|} \hline \frac{6}{(x+2)(x-4)} = \frac{A}{x+2} + \frac{B}{x-4} \\ \hline \end{array} \\ &\begin{array}{l} 6 = A(x-4) + B(x+2) \\ \bullet \text{IF } x=4 \Rightarrow 6=6B \quad \bullet \text{IF } x=-2 \Rightarrow 6=-6A \\ \Rightarrow B=1 \quad \Rightarrow A=-1 \end{array} \\ &= \int \frac{-1}{x-4} - \frac{1}{x+2} dx = \ln|x-4| - \ln|x+2| + C = \ln \left| \frac{x-4}{x+2} \right| + C\end{aligned}$$

83. $\int 3\cot^2 x \, dx = -3x - 3\cot x + C$

$$\begin{aligned} \int 3\cot^2 x \, dx &= \int 3(\cot^2 x - 1) \, dx = \int 3\cot^2 x \, dx - 3 \, dx \\ &\quad \boxed{\text{1+}\cot^2 x = \csc^2 x} \\ &= -3\cot x - 3x + C \quad \leftarrow \text{since } \frac{d}{dx}(\cot x) = -\csc^2 x \end{aligned}$$

84. $\int \cos 2x \sin x \, dx =$

| |
|---|
| $\begin{aligned} &-\frac{2}{3}\cos^3 x + \cos x + C \\ &-\frac{1}{6}\cos 3x + \frac{1}{2}\cos x + C \\ &\frac{4}{3}\cos^3 x - \cos 2x \cos x + C \\ &\frac{1}{3}\cos 2x \cos x + \frac{2}{3}\sin 2x \sin x + C \\ &\frac{1}{3}\sin x \sin 2x + \frac{1}{3}\cos x + C \\ &\frac{2}{3}\sin^2 x \cos x + \frac{1}{3}\cos x + C \\ &\frac{1}{3}(1 + 2\sin^2 x)\cos x + C \\ &-\frac{1}{4}\cos 3x + \frac{5}{4}\cos x + C \end{aligned}$ |
|---|

$\int \cos 2x \sin x \, dx = \int (2\sin 2x - 1)\sin x \, dx$

$$= \int 2\sin 2x \sin x \, dx - \int \sin x \, dx = \dots = -\frac{2}{3}\cos^3 x + \cos x + C$$

BY SUBSTITUTION $u = \cos 2x$
OR DIRECTLY DIFFERENTIATE SINCE $\frac{d}{dx}(\cos 2x) = 2\cos 2x(-\sin x)$

STANDARD ALTERNATIVE FOR THIS TYPE OF INTEGRAL

- $\sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x$) subtract
- $\sin(2x+x) - \sin(2x-x) = \sin 2x \cos x - \cos 2x \sin x$

$$\cos 2x \sin x = \frac{1}{2}(\sin 2x - \sin(-2x))$$

$$\int \cos 2x \sin x \, dx = \int \frac{1}{2}\sin 2x - \frac{1}{2}\sin(-2x) \, dx = -\frac{1}{2}\cos 2x + \frac{1}{2}\cos(-2x) + C$$

BY PARTS ONCE & RECOGNITION

$$\begin{aligned} \int \cos 2x \sin x \, dx &= \dots \\ &= -\cos 2x \cos x - \int (\sin 2x)(-2\cos x) \, dx \\ &= -\cos 2x \cos x - \int 2\sin 2x \cos x \, dx \\ &= -\cos 2x \cos x - \int 2(\sin x \cos x) \cos x \, dx \\ &= -\cos 2x \cos x - \int \sin x \cos^2 x \, dx \\ &= -\cos 2x \cos x + \frac{1}{2}\cos^2 x + C \quad \uparrow \\ &\quad \text{Since } \frac{d}{dx}(\frac{1}{2}\cos^2 x) = \frac{1}{2}\sin x \cos(-x) \end{aligned}$$

BY PARTS TWICE

$$\begin{aligned} \int \cos 2x \sin x \, dx &= \dots \\ &= -\cos 2x \cos x - \int (-2\sin 2x)(-\cos x) \, dx \\ &= -\cos 2x \cos x - \int 2\sin 2x \cos x \, dx \\ &= -\cos 2x \cos x - \left[2\sin 2x \sin x - \int (-2\sin 2x) \sin x \, dx \right] \\ &= -\cos 2x \cos x - 2\sin 2x \sin x + 4 \int \cos 2x \sin x \, dx \\ &\text{IF } \int \cos 2x \sin x \, dx = -\cos 2x \cos x - 2\sin 2x \sin x + 4 \int \cos 2x \sin x \, dx \\ &\cos 2x \cos x + 2\sin 2x \sin x = 3 \int \cos 2x \sin x \, dx \\ &\therefore \int \cos 2x \sin x \, dx = \frac{1}{3}[\cos 2x \cos x + 2\sin 2x \sin x] + C \\ &= \frac{1}{3}[\cos(2(x-x)) + \frac{1}{2}\sin 2x \sin x] + \frac{1}{3}\sin 2x \sin x + C \\ &= \frac{1}{3}\cos 2x + \frac{1}{3}\sin 2x \sin x + C \\ &= \frac{1}{3}\cos 2x + \frac{2}{3}\sin x \cos x + C \\ &= \frac{1}{3}\cos 2x (1 + 2\sin^2 x) + C \end{aligned}$$

USING THE IDENTITY $\sin 2x = 2\sin x - \cos 2x$

$$\begin{aligned} \int \cos 2x \sin x \, dx &= \int (-2\sin 2x) \sin x \, dx = \int \sin x - 3\sin x \, dx \\ &\Rightarrow \sin x - 3(\frac{1}{2}\sin 2x - \frac{1}{2}\cos 2x) \, dx = \int \frac{1}{2}\sin 2x - \frac{1}{2}\cos 2x \, dx \\ &= -\frac{1}{2}\cos 2x + \frac{1}{2}\sin 2x + C \end{aligned}$$

85. $\int \frac{\sqrt{x^2+4}}{x} dx = \sqrt{x^2+4} + \ln \left| \frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2} \right| + C$

$$\begin{aligned} & \int \frac{\sqrt{x^2+4}}{x} dx = \dots \text{SUBSTITUTION} \dots \\ & \quad \boxed{u = (x^2+4)^{\frac{1}{2}}} \\ & \quad \boxed{du = \frac{1}{2}x^{-1}dx} \\ & \quad \boxed{2u du = 2x} \\ & \quad \boxed{u du = x dx} \\ & \quad \boxed{dx = \frac{2}{u} du} \\ & \quad \boxed{u^2 du = 4} \\ & = \int 1 + \frac{4}{u^2-4} du = \int 1 + \frac{4}{(u-2)(u+2)} du \\ & \quad \text{BY PARTIAL FRACTIONS} \\ & \quad \frac{4}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2} \\ & \quad 4 = A(u+2) + B(u-2) \\ & \quad \text{IF } u=2, \frac{4}{u-2} \rightarrow \frac{4}{0} \quad \text{IF } u=-2, \frac{4}{u-2} \rightarrow \frac{-16}{-4} \\ & \quad A=1 \quad B=-1 \\ & = \int 1 + \frac{1}{u-2} - \frac{1}{u+2} du = u + \ln|u-2| - \ln|u+2| + C \\ & = u + \ln \left| \frac{u-2}{u+2} \right| + C = \underline{\underline{\sqrt{x^2+4}-2}} + \ln \left| \frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2} \right| + C \end{aligned}$$

86. $\int 7(2x-3)^{\frac{5}{2}} dx = (2x-3)^{\frac{7}{2}} + C$

$$\begin{aligned} & \int 7(2x-3)^{\frac{5}{2}} dx = \dots \text{BY INSPECTION} \dots = \frac{7}{7} (2x-3)^{\frac{7}{2}} + C \\ & = (2x-3)^{\frac{7}{2}} + C \end{aligned}$$

87. $\int \frac{x^2}{4-x^3} dx = -\frac{1}{3} \ln|4-x^3| + C$

$$\begin{aligned} & \int \frac{x^2}{4-x^3} dx = -\frac{1}{3} \int \frac{-3x^2}{4-x^3} dx = -\frac{1}{3} \ln|4-x^3| + C \\ & \quad \text{OF THE FORM } \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C \\ & \text{ALTERNATIVE BY SUBSTITUTION} \\ & \int \frac{x^2}{4-x^3} dx = \int \frac{dx}{u} \left(-\frac{du}{3x^2} \right) = \int -\frac{1}{3u} du \\ & = -\frac{1}{3} \times \frac{1}{3} \ln|u| + C = -\frac{1}{9} \ln|u| + C \\ & = -\frac{1}{9} \ln|4-x^3| + C \end{aligned}$$

88. $\int x \sin\left(\frac{1}{2}x\right) dx = -2x \cos\left(\frac{1}{2}x\right) + 4 \sin\left(\frac{1}{2}x\right) + C$

$$\begin{aligned}\int x \sin\left(\frac{1}{2}x\right) dx &= \dots \text{ INTEGRATION BY PARTS} \\ &= -2x \cos\left(\frac{1}{2}x\right) - \int -2 \cos\left(\frac{1}{2}x\right) dx \\ &= -2x \cos\left(\frac{1}{2}x\right) + \int 2 \cos\left(\frac{1}{2}x\right) dx \\ &= -2x \cos\left(\frac{1}{2}x\right) + 4 \sin\left(\frac{1}{2}x\right) + C\end{aligned}$$

89. $\int \frac{4}{4x^2 + 4x + 1} dx = -\frac{2}{2x+1} + C$

$$\begin{aligned}\int \frac{4}{4x^2 + 4x + 1} dx &= \dots \text{ THIS IS A PERFECT SQUARE} \dots = \int \frac{4}{(2x+1)^2} dx \\ &= 4 \cdot \frac{1}{(2x+1)^2} dx = \frac{4}{(2x+1)^2} + C \\ &= -\frac{2}{2x+1} + C\end{aligned}$$

90. $\int \frac{3}{\sqrt{4x+1}} dx = \frac{3}{2} \sqrt{4x+1} + C$

$$\begin{aligned}\int \frac{3}{\sqrt{4x+1}} dx &= \int 3(4x+1)^{-\frac{1}{2}} dx = \dots \text{ BY INSPECTION} \dots \\ &= \frac{3}{2}(4x+1)^{\frac{1}{2}} + C = \frac{3}{2} \sqrt{4x+1} + C\end{aligned}$$

91. $\int \frac{x}{\sqrt{x-1}} dx = \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$

$$\begin{aligned}\int \frac{x}{\sqrt{x-1}} dx &\dots \text{ BY SUBSTITUTION} \dots \\ &= \int \frac{u+1}{u^{\frac{1}{2}}} du = \int \frac{u}{u^{\frac{1}{2}}} + \frac{1}{u^{\frac{1}{2}}} du \\ &= \int u^{\frac{1}{2}} + u^{-\frac{1}{2}} du = \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C \\ &= \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C\end{aligned}$$

ALTERNATIVE SUBSTITUTION

$$\begin{aligned}\int \frac{x}{\sqrt{x-1}} dx &= \int \frac{x}{\sqrt{u}} (2u du) \\ &= \int 2x du = \int 2(u^2+1) du \\ &= \int 2u^2 + 2 du = \frac{2}{3}u^3 + 2u + C \\ &= \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C\end{aligned}$$

$u = (x-1)^{\frac{1}{2}}$
 $u^2 = x-1$
 $2u \frac{du}{dx} = 1$
 $du = \frac{1}{2u} dx$
 $x = u^2 + 1$

92. $\int \frac{1}{3(x-2)^{\frac{1}{2}}} dx = \frac{2}{3}(x-2)^{\frac{1}{2}} + C$

$$\int \frac{1}{3(x-2)^{\frac{1}{2}}} dx = \dots \text{ BY INSPECTION } \dots = \int \frac{1}{2}(x-2)^{-\frac{1}{2}} dx \\ = \frac{1}{2}(x-2)^{\frac{1}{2}} + C = \frac{1}{2}(x-2)^{\frac{1}{2}} + C$$

93. $\int \frac{6x+3}{2x} dx = 3x + \frac{3}{2} \ln|x| + C$

$$\int \frac{6x+3}{2x} dx = \int \frac{6x}{2x} + \frac{3}{2x} dx = \int 3 + \frac{3}{2} \times \frac{1}{x} dx \\ = 3x + \frac{3}{2} \ln|x| + C$$

94. $\int \frac{4x+1}{2x-5} dx = 2x + \frac{11}{2} \ln|2x-5| + C$

$$\int \frac{4x+1}{2x-5} dx = \dots \text{ BY MULTIPLICATION TO SPLIT THE FRACTION } \\ = \int \frac{2(2x-5)+11}{2x-5} dx = \int \frac{2(2x-5)}{2x-5} + \frac{11}{2x-5} dx \\ = \int 2 + \frac{11}{2x-5} dx = 2x + \frac{11}{2} \ln|2x-5| + C$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned} \int \frac{4x+1}{2x-5} dx &= \int \frac{4x+1}{u} \frac{du}{2} && u = 2x-5 \\ &= \int \frac{2(4x+1)+11}{2u} du &=& \int \frac{2u+11}{2u} du \\ &= \int \frac{2u}{2u} + \frac{11}{2u} du &=& \int 1 + \frac{11}{2u} du \\ &= u + \frac{11}{2} \ln|u| + C &=& (2x-5) + \frac{11}{2} \ln|2x-5| + C \\ & &=& 2x + \frac{11}{2} \ln|2x-5| + C \end{aligned}$$

95. $\int \frac{4x}{x^2-1} dx = 2 \ln|x^2-1| + C$

$$\int \frac{4x}{x^2-1} dx = 2 \int \frac{2x}{x^2-1} dx = \dots = 2 \ln|x^2-1| + C$$

OF THE FORM $\int \frac{f(u)}{g(u)} du = \ln|f(u)| + C$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned} \int \frac{4x}{x^2-1} dx &= \int \frac{2x}{u} \left(\frac{du}{2x} \right) = \int \frac{du}{u} du \\ &= \ln|u| + C = 2 \ln|x^2-1| + C \end{aligned}$$

PARTIAL FRACTION ALSO WORK HERE AFTER FACTORIZATION OF THE DENOMINATOR.

$$96. \int \frac{x^2}{2x-1} dx = \left[\frac{1}{16}(2x-1)^2 + \frac{1}{4}(2x-1) + \frac{1}{8} \ln|2x-1| + C \right]$$

$$\begin{aligned} \int \frac{x^2}{2x-1} dx &= \dots \text{ BY SUBSTITUTION } \\ &= \int \frac{\frac{u^2}{4}}{u-1} \left(\frac{du}{2} \right) = \int \frac{u^2}{8(u-1)} du = \int \frac{u^2}{8u} du \\ &= \int \frac{u^2+2u+1-2u-1}{8u} du = \int \frac{u^2 + \frac{1}{u} + \frac{1}{u^2}}{8u} du \\ &= \int \left(\frac{1}{16}u^2 + \frac{1}{8u} + \frac{1}{8u^2} \right) du = \frac{1}{16}u^3 + \frac{1}{8}\ln|u| + C \\ &= \frac{1}{16}(2x-1)^3 + \frac{1}{8}\ln|2x-1| + C \end{aligned}$$

$u = 2x-1$
 $\frac{du}{dx} = 2$
 $du = \frac{du}{dx} dx$
 $2x-1 = u+1$
 $4x = u^2+2u+1$

$$97. \int \frac{1+\cos^4 x}{\cos^2 x} dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + \tan x + C$$

$$\begin{aligned} \int \frac{1+\cos^4 x}{\cos^2 x} dx &= \int \frac{1}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} dx = \int \sec^2 x + \tan^2 x dx \\ &= \int \sec^2 x + (\frac{1}{2} + \frac{1}{2}\tan 2x) dx \\ &= \int \frac{1}{2} + \frac{1}{2}\sec^2 x + \tan x dx \\ &= \frac{1}{2}x + \frac{1}{2}\ln|\sec x| + \tan x + C \end{aligned}$$

$$98. \int \frac{17-5x}{(2x+3)(2-x)^2} dx = \ln \left| \frac{2x+3}{2-x} \right| + \frac{1}{2-x} + C$$

$$\int \frac{17-5x}{(2x+3)(2-x)^2} dx = \dots \text{ BY PARTIAL FRACTIONS }$$

| |
|---|
| $\frac{17-5x}{(2x+3)(2-x)^2} \equiv \frac{A}{2x+3} + \frac{B}{2-x} + \frac{C}{(2-x)^2}$ $17-5x \equiv A(2-x)^2 + B(2x+3) + C(2-x)(2x+3)$ |
| $\bullet \text{ IF } x=2:$ $7 = 7B$ $B = 1$ |
| $\bullet \text{ IF } x=-\frac{3}{2}:$ $\frac{41}{2} = \frac{A}{2}$ $A = \frac{41}{2}$ |
| $\bullet \text{ IF } x=0:$ $17 = 4A + 3B + 6C$ $17 = 4A + 3 + 6C$ $6 = 6C$ $C = 1$ |

$$\begin{aligned} &= \int \frac{\frac{41}{2}}{2x+3} + \frac{1}{2-x} dx = \ln|2x+3| + (2-x)^{-1} - \ln|2-x| \\ &= \ln \left| \frac{2x+3}{2-x} \right| + \frac{1}{2-x} + C \end{aligned}$$

99. $\int x \sin(2x-1) dx = -\frac{1}{2}x \cos(2x-1) + \frac{1}{4} \sin(2x-1) + C$

$$\begin{aligned}\int 2 \sin(2x-1) dx &= \dots \text{ INTEGRATION BY PARTS } \dots \\ &= -\frac{1}{2}x \cos(2x-1) - \int -\frac{1}{2} \cos(2x-1) dx \\ &= -\frac{1}{2}x \cos(2x-1) + \int \frac{1}{2} \cos(2x-1) dx \\ &= -\frac{1}{2}x \cos(2x-1) + \frac{1}{4} \sin(2x-1) + C\end{aligned}$$

100. $\int 4(3x-2)^3 dx = \frac{1}{3}(3x-2)^4 + C$

$$\begin{aligned}\int 4(3x-2)^3 dx &= \dots \text{ BY INSPECTION } \dots = \frac{4}{12}(3x-2)^4 + C \\ &= \frac{1}{3}(3x-2)^4 + C\end{aligned}$$

101. $\int \sqrt{x} \sqrt{x} dx = \frac{4}{7}x^{\frac{7}{4}} + C$

$$\begin{aligned}\int \sqrt{x} \sqrt{x} dx &= \int (x \cdot x^{\frac{1}{2}})^{\frac{1}{2}} dx = \int (x^{\frac{3}{2}})^{\frac{1}{2}} dx \\ &= \int x^{\frac{3}{4}} dx = \frac{4}{7}x^{\frac{7}{4}} + C\end{aligned}$$

102. $\int \frac{1}{x^2 \sqrt[3]{x^2}} dx = -\frac{3}{5}x^{-\frac{5}{3}} + C$

$$\begin{aligned}\int \frac{1}{x^2 \sqrt[3]{x^2}} dx &= \int \frac{1}{x^2 \cdot x^{\frac{2}{3}}} dx = \int \frac{1}{x^{\frac{8}{3}}} dx \\ &= \int x^{-\frac{8}{3}} dx = -\frac{3}{5}x^{-\frac{5}{3}} + C\end{aligned}$$

103. $\int \frac{3}{\sqrt{2-4x}} dx = -\frac{3}{2}(2-4x)^{\frac{1}{2}} + C$

$$\begin{aligned} \int \frac{3}{\sqrt{2-4x}} dx &= \int 3(2-4x)^{\frac{1}{2}} dx = \dots \text{ BY INSPECTION} \\ &= -\frac{3}{2}(2-4x)^{\frac{1}{2}} + C \end{aligned}$$

104. $\int \frac{1}{1+\cos 2x} dx = \left[\frac{1}{2} \tan x + C \right]_{\frac{1}{2} \operatorname{cosec} 2x - \frac{1}{2} \cot 2x + C}$

$$\begin{aligned} \int \frac{1}{1+\cos 2x} dx &= \int \frac{1}{1+(2\cos^2 x - 1)} dx = \int \frac{1}{2\cos^2 x} dx \\ &= \int \frac{1}{2} \sec^2 x dx = \frac{1}{2} \tan x + C \end{aligned}$$

ALTERNATIVE:

$$\begin{aligned} \int \frac{1}{1+\cos 2x} dx &= \int \frac{(1-\cos 2x)}{(1+\cos 2x)(1-\cos 2x)} dx = \int \frac{1-\cos 2x}{1-\cos^2 2x} dx \\ &= \int \frac{1-\cos 2x}{\sin^2 2x} dx = \int \frac{1}{\sin^2 2x} - \frac{\cos 2x}{\sin^2 2x} dx = \int \operatorname{cosec}^2 2x - \frac{\cot 2x}{\sin^2 2x} dx \\ &= \int \operatorname{cosec}^2 2x - \cot 2x \operatorname{cosec} 2x dx = -\frac{1}{2} \operatorname{cot} 2x + \frac{1}{2} \operatorname{cosec} 2x + C \\ \frac{d}{dx}(\operatorname{cot} 2x) &= -\operatorname{cosec}^2 2x \quad \frac{d}{dx}(\operatorname{cosec} 2x) = -\operatorname{cosec} 2x \operatorname{cot} 2x \end{aligned}$$

105. $\int \frac{2}{2x-x^2} dx = \ln \left| \frac{x}{2-x} \right| + C$

$$\begin{aligned} \int \frac{2}{2x-x^2} dx &= \int \frac{2}{x(2-x)} dx = \dots \text{ BY PARTIAL FRACTIONS} \\ \frac{2}{x(2-x)} &\equiv \frac{A}{x} + \frac{B}{2-x} \\ 2 &\equiv A(2-x) + Bx \\ \begin{array}{ll} \bullet \text{ IF } x=2 & \bullet \text{ IF } x=0 \\ 2 \equiv 2B & 2 \equiv 2A \\ B \equiv 1 & A \equiv 1 \end{array} \\ \int \frac{1}{x} + \frac{1}{2-x} dx &= \ln|x| - \ln|2-x| + C = \ln \left| \frac{x}{2-x} \right| + C \end{aligned}$$

106. $\int \frac{\ln x}{x^2} dx = -\frac{1}{x} \frac{\ln|x|}{x} + C$

$$\begin{aligned}\int \frac{\ln x}{x^2} dx &= \int x^{-2} \ln x dx = \dots \text{ INTEGRATION BY PARTS} \\ &= -\frac{1}{x} \ln|x| - \int -x^{-1}(\frac{1}{x}) dx \\ &= -\frac{1}{x} \ln|x| + \int x^{-2} dx \\ &= -\frac{1}{x} \ln|x| - x^{-1} + C \\ &= -\frac{1}{x} \ln|x| - \frac{1}{x} + C\end{aligned}$$

107. $\int \frac{3x^2}{x^3+1} dx = \ln|x^3+1| + C$

$$\begin{aligned}\int \frac{3x^2}{x^3+1} dx &= \dots \text{ OF THE FORM } \int \frac{f'(u)}{f(u)} du = \ln|f(u)| + C \\ &= \ln|x^3+1| + C\end{aligned}$$

THE SUBSTITUTION $u = x^3+1$ ALSO WORKS WELL.

108. $\int \frac{2x+1}{2x-1} dx = x + \ln|2x-1| + C$

$$\begin{aligned}\int \frac{2x+1}{2x-1} dx &= \text{BY MANIPULATION & SPLITTING} \\ &= \int \frac{(2x-1)+2}{(2x-1)} dx = \int \frac{2x-1}{2x-1} dx + \int \frac{2}{2x-1} dx \\ &= \int 1 + \frac{2}{2x-1} dx = x + \ln|2x-1| + C\end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}\int \frac{2x+1}{2x-1} dx &= \int \frac{2x+1}{u} \cdot \frac{du}{2} \\ &= \int \frac{2x+1}{2u} du = \int \frac{u+2}{2u} du \\ &= \int \frac{1}{2} + \frac{2}{2u} du = \int \frac{1}{2} + \frac{1}{u} du \\ &= \frac{1}{2}x + \ln|u| + C = \frac{1}{2}(2x-1) + \ln|2x-1| + C \\ &= x + \ln|2x-1| + C\end{aligned}$$

(2x-1)

109. $\int \frac{14x+1}{(1-x)(2x+1)} dx = -5\ln|1-x| - 2\ln|2x+1| + C$

$$\begin{aligned} \int \frac{14x+1}{(1-x)(2x+1)} dx &= \dots \text{ BY PARTIAL FRACTION } \\ \frac{14x+1}{(1-x)(2x+1)} &\equiv \frac{A}{1-x} + \frac{B}{2x+1} \\ 14x+1 &\equiv A(2x+1) + B(1-x) \\ \bullet \text{ If } x=1 \Rightarrow 15=3A &\quad \bullet \text{ If } x=-\frac{1}{2} \Rightarrow -6=\frac{3}{2}B \\ \Rightarrow A=5 &\quad \Rightarrow B=-4 \\ \dots &= \int \frac{5}{1-x} - \frac{4}{2x+1} dx = -5\ln|1-x| - \frac{4}{2} \ln|2x+1| + C \\ &= -5\ln|1-x| - 2\ln|2x+1| + C \end{aligned}$$

110. $\int \frac{6x}{\sqrt{2x+3}} dx = (2x+3)^{\frac{3}{2}} - 9(2x+3)^{\frac{1}{2}} + C$

$$\begin{aligned} \int \frac{6x}{\sqrt{2x+3}} dx &= \dots \text{ BY SUBSTITUTION } \\ &= \int \frac{6x}{\sqrt{u^2+3}} \left(\frac{du}{dx} \right) = \int \frac{3(u-3)}{2u^{\frac{1}{2}}} du \\ &= \int \frac{3(u-3)}{2u^{\frac{1}{2}}} du = \int \frac{3u-9}{2u^{\frac{1}{2}}} du \\ &= \int \frac{3u}{2u^{\frac{1}{2}}} - \frac{9}{2u^{\frac{1}{2}}} du = \int \frac{3}{2}u^{\frac{1}{2}} - \frac{9}{2}u^{-\frac{1}{2}} du \\ &= 10u^{\frac{1}{2}} - \frac{9}{2}u^{\frac{1}{2}} + C = (2x+3)^{\frac{3}{2}} - 9(2x+3)^{\frac{1}{2}} + C \end{aligned}$$

ALTERNATIVE SUBSTITUTION

$$\begin{aligned} \int \frac{6x}{\sqrt{2x+3}} dx &= \int \frac{6x}{\sqrt{u^2+3}} (u \, du) \\ &= \int 6x \, du = \int 3(u) \, du = \int 3(u^2-3) \, du \\ &= \int 3u^2 - 9 \, du = u^3 - 9u + C \\ &= (2x+3)^{\frac{3}{2}} - 9(2x+3)^{\frac{1}{2}} + C \end{aligned}$$

111. $\int \frac{\sin^4 x + \cos^2 x}{\sin^2 x} dx = -\frac{1}{2}x - \frac{1}{4}\sin 2x - \cot x + C$

$$\begin{aligned} \int \frac{\sin^4 x + \cos^2 x}{\sin^2 x} dx &= \dots \text{ SPLIT THE FRACTION & USE IDENTITIES} \\ \dots &= \int \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} dx = \int \sin^2 x + \cot^2 x dx \\ &= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x \right) + (\cot^2 x - 1) dx \\ &= \int -\frac{1}{2} + \frac{1}{2}\cos 2x + \cot^2 x dx \quad \text{NOT} \\ &= -\frac{1}{2}x - \frac{1}{4}\sin 2x - \cot x + C \quad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \end{aligned}$$

112. $\int \frac{2}{(x-4)\sqrt{x}} dx = \ln \left| \frac{\sqrt{x}-2}{\sqrt{x}+2} \right| + C$

$$\begin{aligned} \int \frac{2}{(x-4)\sqrt{x}} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int \frac{2}{(u^2-4)u} (2u du) = \int \frac{4}{u^2-4} du \\ &= \int \frac{4}{(u-2)(u+2)} du = \dots \text{ BY PARTIAL FRACTIONS} \\ &\quad \boxed{\frac{4}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}} \\ &\quad A \equiv 4(u+2) + B(u-2) \\ &\quad \bullet \text{ IF } u=2 \Rightarrow 4+4A \quad \bullet \text{ IF } u=-2 \Rightarrow A=-8 \\ &\quad \Rightarrow \boxed{A=1} \quad \Rightarrow \boxed{B=-8} \\ &\dots = \int \frac{1}{u-2} - \frac{1}{u+2} du = \ln|u-2| + \ln|u+2| + C \\ &= \ln \left| \frac{u-2}{u+2} \right| + C = \ln \left| \frac{\sqrt{x}-2}{\sqrt{x}+2} \right| + C \end{aligned}$$

113. $\int \frac{1}{2+\sqrt{x-1}} dx = 2\sqrt{x-1} - 4\ln|2+\sqrt{x-1}| + C$

$$\begin{aligned} \int \frac{1}{2+\sqrt{x-1}} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int \frac{1}{u} (2(u-2) du) = \int \frac{2u-4}{u} du \\ &= \int \frac{2u}{u} - \frac{4}{u} du = \int 2 - \frac{4}{u} du \\ &= 2u - 4\ln|u| + C \\ &= 2[2 + \sqrt{x-1}] - 4\ln|2 + \sqrt{x-1}| + C \\ &= 2\sqrt{x-1} - 4\ln|2 + \sqrt{x-1}| + C \\ &\text{AUTOMATIC SUBSTITUTION} \\ \int \frac{1}{2+\sqrt{x-1}} dx &= \int \frac{1}{2+u} (2u du) \\ &= \int \frac{2u}{u+2} du \quad \leftarrow \text{BY AUTOMATIC SUBSTITUTION} \quad u=u, \text{ in order to save THE PARENTHESIS} \\ &\quad \boxed{u=\sqrt{x-1}} \quad \boxed{u^2=x-1} \\ &= \int \frac{2(u+2)-4}{u+2} du = \int \frac{2(u+2)}{u+2} - \frac{4}{u+2} du \\ &= \int 2 - \frac{4}{u+2} du = 2u - 4\ln|u+2| + C \\ &= 2\sqrt{x-1} - 4\ln|\sqrt{x-1}+2| + C \end{aligned}$$

114. $\int \frac{4}{\sqrt{6x-1}} dx = \frac{4}{3}\sqrt{6x-1} + C$

$$\begin{aligned} \int \frac{1}{\sqrt{6x-1}} dx &= \int 4(6x-1)^{-\frac{1}{2}} dx = \dots \text{ BY SUBSTITUTION} \\ &= \frac{4}{3}(6x-1)^{\frac{1}{2}} + C = \frac{4}{3}\sqrt{6x-1} + C \end{aligned}$$

115. $\int \frac{3e^{2x}}{e^{2x}-1} dx = \frac{3}{2} \ln|e^{2x}-1| + C$

$$\begin{aligned}\int \frac{3e^{2x}}{e^{2x}-1} dx &= 3 \int \frac{e^{2x}}{e^{2x}-1} dx = \frac{3}{2} \int \frac{2e^{2x}}{e^{2x}-1} dx \\ &\stackrel{\text{CAN BE WRITTEN IN THE FORM}}{=} \frac{3}{2} \ln|e^{2x}-1| + C \\ \int \frac{f(x)}{g(x)} dx &= \ln|g(x)| + C \\ \text{ALTERNATIVE BY SUBSTITUTION:} \\ \int \frac{3e^{2x}}{e^{2x}-1} dx &= \int \frac{3e^{2x}}{u} \left(\frac{du}{2e^{2x}} \right) = \int \frac{3}{2} \cdot \frac{1}{u} du \\ &= \frac{3}{2} \ln|u| + C = \frac{3}{2} \ln|e^{2x}-1| + C\end{aligned}$$

116. $\int x \sec^2 x dx = \begin{cases} x \tan x - \ln|\sec x| + C \\ x \tan x + \ln|\cos x| + C \end{cases}$

$$\begin{aligned}\int x \sec^2 x dx &= \dots \text{INTEGRATION BY PARTS} \\ &= x \ln|\sec x| - \int \ln|\sec x| dx \\ &= x \ln|\sec x| - \ln|\sec x| + C \quad \text{STANDARD FORM} \\ &= x \ln|\sec x| - \int \frac{1}{\sec x} \sec x dx \\ &= x \ln|\sec x| + \int \frac{\sec x}{\sec x} dx \\ &= x \ln|\sec x| + \ln|\cos x| + C \\ &= x \ln|\sec x| + \ln|\cos x| + C \quad \text{OR THE FORM "SECANT" DIFFERENTIABLE TO "TAN"}\end{aligned}$$

117. $\int \operatorname{cosec} 2x \cot 2x dx = -\frac{1}{2} \operatorname{cosec} 2x + C$

$$\begin{aligned}\int \operatorname{cosec} 2x \cot 2x dx &= \dots \text{STANDARD DIFFERENTIAL} \\ &= \frac{d}{dx} (\operatorname{cosec} 2x) = -\operatorname{cosec} 2x \cot 2x \\ &= -\frac{1}{2} \operatorname{cosec} 2x + C\end{aligned}$$

118. $\int \tan^2 x \sec^2 x dx = \frac{1}{3} \tan^3 x + C$

$$\begin{aligned}\int \tan^2 x \sec^2 x dx &= \dots \text{BY REVERSE CHAIN RULE SINCE } \frac{d}{dx}(\tan x) = \sec^2 x \\ &= \frac{1}{3} \tan^3 x + C \\ \text{THE SUBSTITUTION: } u &= \tan x \text{ ALSO WORKS WELL}\end{aligned}$$

119. $\int \sin 2x \cosec x \, dx = 2 \sin x + C$

$$\begin{aligned} \int \sin 2x \cosec x \, dx &= \dots \text{SIMPLIFY & INTEGRATE} \\ &= \int (\cancel{\sin 2x}) \left(\frac{1}{\cancel{\sin x}} \right) \, dx \\ &= \int 2 \cosec x \, dx \\ &= 2 \ln |\cosec x| + C \end{aligned}$$

120. $\int (2\cos x - 3\sin x)^2 \, dx = \frac{13}{2}x - \frac{5}{4}\sin 2x + 3\cos 2x + C$

$$\begin{aligned} \int (2\cos x - 3\sin x)^2 \, dx &= \dots \text{EXPAND & USE IDENTITIES} \\ &= \int (4\cos^2 x - 12\cos x \sin x + 9\sin^2 x) \, dx \\ &= \int (4(\frac{1+\cos 2x}{2}) - 6(\sin 2x)) + 9(\frac{1-\cos 2x}{2}) \, dx \\ &= \int 2 + 2\cos 2x - 6\sin 2x + \frac{9}{2} - \frac{9}{2}\cos 2x \, dx \\ &= \int \frac{13}{2} - \frac{5}{2}\cos 2x - 6\sin 2x \, dx = \frac{13}{2}x - \frac{5}{2}\sin 2x + 3\cos 2x + C \end{aligned}$$

121. $\int \frac{\sin x + \tan x}{\cos x} \, dx = \begin{cases} \sec x + \ln|\sec x| + C \\ \sec x - \ln|\cos x| + C \end{cases}$

$$\begin{aligned} \int \frac{\sin x + \tan x}{\cos x} \, dx &= \dots \text{SPLIT & USE TRIGONOMETRIC RESULTS} \\ &= \int \frac{\sin x}{\cos x} \, dx + \int \frac{\tan x}{\cos x} \, dx = \int \tan x + \sec x \, dx \\ &= \ln|\sec x| + \sec x + C \\ &\quad \uparrow \text{SIMPLIFIED RESULT} \quad \uparrow \text{d}(\sec x) = \sec x \tan x \\ &[\sec x - \ln|\cos x| + \sec x + C] \\ &\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{-\sin x}{\cos x} \, dx \\ &= -\ln|\cos x| + C \\ &\quad \uparrow \text{AS IT IS OF THE FORM "BOTTOM" DIFFERENTIALLY TO "TOP"} \end{aligned}$$

122. $\int \frac{1+\sin x}{\cos^2 x} \, dx = \tan x + \sec x + C$

$$\begin{aligned} \int \frac{1+\sin x}{\cos^2 x} \, dx &= \dots \text{SPLIT THE FRACTION & USE TG RESULTS} \\ &= \int \frac{1}{\cos^2 x} \, dx + \int \frac{\sin x}{\cos^2 x} \, dx = \int \sec^2 x \, dx + \int \frac{\sin x}{\cos^2 x} \, dx \\ &= \int \sec^2 x + \tan x \sec x \, dx = \tan x + \sec x + C \\ &\quad \uparrow \text{d}(\tan x) = \sec^2 x \quad \uparrow \frac{d}{dx}(\sec x) = \sec x \tan x \end{aligned}$$

123. $\int e^{\sin x} \cos x \, dx = e^{\sin x} + C$

$$\begin{aligned}\int e^{\sin x} \cos x \, dx &= \dots \text{ BY REVERSE CHAIN RULE SINCE } \frac{d}{dx}(e^{\sin x}) = e^{\sin x} \cos x \\ &= e^{\sin x} + C\end{aligned}$$

THE SUBSTITUTION $u = \sin x$ ALSO WORKS WELL.

124. $\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$

$$\begin{aligned}\int x^2 \sin x \, dx &= \dots \text{ INTEGRATION BY PARTS THREE TIMES} \\ &= -x^2 \cos x - \int 2x \cos x \, dx \\ &= -x^2 \cos x + \int 2x \cos x \, dx \\ &\quad \text{REPEATING THE PROCESS} \\ &= -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx \\ &= -x^2 \cos x + 2x \sin x - 2 \cos x \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C\end{aligned}$$

125. $\int (2x+1)^3 \, dx = \frac{1}{8}(2x+1)^4 + C$

$$\int (2x+1)^3 \, dx = \dots \text{ BY RECOGNITION} \dots = \frac{1}{8}(2x+1)^4 + C$$

126. $\int \frac{\tan^4 x}{\cos^2 x} \, dx = \frac{1}{5} \tan^5 x + C$

$$\begin{aligned}\int \frac{\tan^4 x}{\cos^2 x} \, dx &= \int \tan^3 x \sec x \, dx = \dots \text{ BY REVERSE CHAIN RULE AS} \\ &\quad \frac{d}{dx}(\tan x) = \sec^2 x \\ &= \frac{1}{2} \tan^2 x + C\end{aligned}$$

THE SUBSTITUTION $u = \tan x$ ALSO WORKS WELL.

127. $\int \frac{4x^2 - 6x + 5}{(2-x)(2x-1)^2} dx = -\frac{1}{2x-1} - \ln|2-x| + C$

$$\int \frac{4x^2 - 6x + 5}{(2-x)(2x-1)^2} dx = \dots \text{ BY PARTIAL FRACTIONS }$$

$$4x^2 - 6x + 5 \equiv \frac{A}{2-x} + \frac{B}{(2x-1)} + \frac{C}{(2x-1)^2}$$

$$\begin{aligned} 4x^2 - 6x + 5 &\equiv A(2x-1)^2 + B(2x-1) + C(2-x)(2x-1) \\ \bullet \text{ If } x=2 & \bullet \text{ If } x=1 \\ 4(4) - 6(2) + 5 &= 4(2)(3) - 2B \\ 9 &= 24 - 2B \\ 9 &= 24 - 2B \\ B &= 7.5 \end{aligned}$$

$$\begin{aligned} \bullet \text{ If } x=0 & \bullet \text{ If } 2=0 \\ 4(0) - 6(0) + 5 &= 4 + 2B - C \\ 5 &= 4 + 2B - C \\ 5 &= 4 + 2(7.5) - C \\ C &= 0 \end{aligned}$$

$$\begin{aligned} &= \int \frac{1}{2-x} + \frac{2}{(2x-1)^2} dx = \int \frac{1}{2-x} + 2(2x-1)^{-2} dx \\ &= -\ln|2-x| - (2x-1)^{-1} + C = -\frac{1}{2x-1} - \ln|2-x| + C \end{aligned}$$

128. $\int \frac{3x-1}{2x+3} dx = \frac{3}{2}x - \frac{11}{4}\ln|2x+3| + C$

$$\int \frac{3x-1}{2x+3} dx = \text{MANIPULATING & SPLITTING THE FRACTION}$$

$$\begin{aligned} &= \int \frac{\frac{3}{2}(2x+3) - \frac{11}{2}}{2x+3} dx = \int \frac{\frac{3}{2}(2x+3)}{2x+3} - \frac{\frac{11}{2}}{2x+3} dx \\ &= \int \frac{3}{2} - \frac{11/2}{2x+3} dx = \frac{3}{2}x - \frac{11}{4}\ln|2x+3| + C \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned} \int \frac{3x-1}{2x+3} dx &= \int \frac{3x-1}{4} \left(\frac{du}{dx}\right) dx \\ &= \int \frac{3x-1}{2u} du = \int \frac{\frac{3}{2}u - \frac{1}{2}}{2u} du \\ &= \int \frac{\frac{3}{2}u}{2u} - \frac{\frac{1}{2}}{2u} du = \int \frac{3}{4} - \frac{1}{4} \times \frac{1}{u} du \\ &= \frac{3}{2}u - \frac{1}{4}\ln|u| + C = \frac{3}{2}(2x+3) - \frac{1}{4}\ln|2x+3| + C \\ &= \frac{3}{2}x - \frac{1}{4}\ln|2x+3| + C \end{aligned}$$

$$129. \int \frac{8x^2}{1-2x} dx = \left[\begin{array}{l} -2x^2 - 2x - \ln|1-2x| + C \\ 2(1-2x) - \frac{1}{2}(1-2x)^2 - \ln|1-2x| + C \end{array} \right]$$

$$\begin{aligned} \int \frac{8x^2}{1-2x} dx &= \dots \text{ BY MANIPULATION & SPLITTING THE FRACTION} \\ &= \int \frac{2(1-2x)^2 - 4(1-2x) + 2}{1-2x} dx \\ &= \int \frac{2(1-2x)^2}{1-2x} - \frac{4(1-2x)}{1-2x} + \frac{2}{1-2x} dx \\ &= \int 2(1-2x) - 4 + \frac{2}{1-2x} dx \\ &= \int -4x - 2 + \frac{2}{1-2x} dx \\ &= -2x^2 - 2x - \ln|1-2x| + C \end{aligned}$$

ALTERNATIVE MANIPULATION

$$8x^2 = -4x(1-2x) - 2(1-2x) + 2$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned} \int \frac{8x^2}{1-2x} dx &= \int \frac{8x^2}{u} \left(-\frac{du}{2} \right) & u = 1-2x \\ &= \int -\frac{4x^2}{u} du = \int -\frac{1-2u+u^2}{u} du & \frac{du}{dx} = -2 \\ &= \int -\frac{1}{u} + \frac{2u}{u} - \frac{u^2}{u} du & du = -\frac{du}{2} \\ &= \int -\frac{1}{u} + 2 - u du & 2u = 1-u \\ &= -\ln|u| + 2u - \frac{1}{2}u^2 + C & u^2 = -2u+u^2 \\ &= -\ln|1-2x| + 2(-2x) - \frac{1}{2}(1-2x)^2 + C & \boxed{u^2 = -2u+u^2} \\ &\dots = -\ln|1-2x| \cancel{(2-4x-2)} + 2x - 2x^2 + C & \text{CAS BORKED} \\ &= -\ln|1-2x| - 2x - 2x^2 + C \end{aligned}$$

$$130. \int \frac{10}{(3x+1)^{\frac{3}{2}}} dx = -\frac{20}{3\sqrt{3x+1}} + C$$

$$\begin{aligned} \int \frac{10}{(3x+1)^{\frac{3}{2}}} dx &= \int 10(3x+1)^{-\frac{3}{2}} dx = \dots \text{ BY RECOGNITION} \\ &= \frac{10}{2} (3x+1)^{-\frac{1}{2}} + C = -\frac{20}{3} (3x+1)^{-\frac{1}{2}} + C \\ &= -\frac{20}{3\sqrt{3x+1}} + C \end{aligned}$$

$$131. \int 5^x dx = \frac{5^x}{\ln 5} + C$$

$$\begin{aligned} \int 5^x dx &= \dots \text{ BY RECOGNITION SINCE } \frac{d}{dx}(5^x) = 5^x \ln 5 \\ &= \frac{1}{\ln 5} 5^x + C = \frac{5^x}{\ln 5} + C \end{aligned}$$

132. $\int \sqrt{\sin x \cos^2 x} dx = \frac{2}{3}(\sin x)^{\frac{3}{2}} + C$

$$\begin{aligned}\int \sqrt{\sin x \cos^2 x} dx &= \int (\sin x)^{\frac{1}{2}} \cos x dx \\ &= \dots \text{BY REVERSE CHAIN RULE} \\ &= \frac{2}{3}(\sin x)^{\frac{3}{2}} + C\end{aligned}$$

THE SUBSTITUTION $u = \sin x$, OR $u = \sqrt{\sin x}$ ALSO WORKS

133. $\int (2x+1) \sin(x^2+x+1) dx = -\cos(x^2+x+1) + C$

$$\begin{aligned}\int (2x+1) \sin(x^2+x+1) dx &= \dots \text{BY REVERSE CHAIN RULE SINCE} \\ &\quad \frac{d}{dx}(x^2+x+1) = 2x+1 \\ &= -\cos(x^2+x+1) + C\end{aligned}$$

(THE SUBSTITUTION $u = x^2+x+1$ ALSO WORKS)

134. $\int (2x+1)(x^2+x+1) dx = \left[\frac{1}{2}(x^2+x+1)^2 + C \right]$
 $\qquad\qquad\qquad \left[\frac{1}{2}x^4 + x^3 + \frac{3}{2}x^2 + x + C \right]$

$$\begin{aligned}\int (2x+1)(x^2+x+1) dx &= \text{BY REVERSE CHAIN RULE SINCE} \\ &\quad \frac{d}{dx}(x^2+x+1) = 2x+1 \\ &= \frac{1}{2}(x^2+x+1)^2 + C\end{aligned}$$

(THE SUBSTITUTION $u = x^2+x+1$ ALSO WORKS)

ALTERNATIVE BY EXPANSION ... $\frac{1}{2}x^4 + x^3 + \frac{3}{2}x^2 + x + C$

135. $\int (x+1) \cos(x^2+2x+1) dx = \frac{1}{2} \sin(x^2+2x+1) + C$

$$\begin{aligned}\int (x+1) \cos(x^2+2x+1) dx &= \dots \text{BY REVERSE CHAIN RULE SINCE} \\ &\quad \frac{d}{dx}(x^2+2x+1) = 2x+2 = 2(x+1) \\ &= \frac{1}{2} \sin(x^2+2x+1) + C\end{aligned}$$

(THE SUBSTITUTION $u = x^2+2x+1$, ALSO WORKS)

136. $\int \frac{1}{2+\sqrt{x}} dx = 2\sqrt{x} - 4\ln(2+\sqrt{x}) + C$

$$\begin{aligned} \int \frac{1}{2+\sqrt{x}} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int \frac{1}{u} (2-u) du = \int \frac{2u-1}{u} du \\ &= \int 2 - \frac{1}{u} du = \int 2 - \frac{1}{u} du \\ &= 2u - 4\ln|u| + C = 2[2+\sqrt{x}] - 4\ln(2+\sqrt{x}) + C \\ &= 2\sqrt{x} - 4\ln(2+\sqrt{x}) + C \end{aligned}$$

(THE SUBSTITUTION $u = \sqrt{x}$ ALSO WORKS)

137. $\int \frac{1}{e^x + e^{-x} + 2} dx = -\frac{1}{e^x + 1} + C$

$$\begin{aligned} \int \frac{1}{e^x + e^{-x} + 2} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int \frac{1}{u + \frac{1}{u} + 2} \left(\frac{du}{u}\right) = \int \frac{1}{u^2 + 2u + 1} du \\ &= \int \frac{1}{(u+1)^2} du = \int \frac{1}{(u+1)^2} du \\ &= \int (u+1)^{-2} du = -(u+1)^{-1} + C = -\frac{1}{u+1} + C \end{aligned}$$

138. $\int x^2 \tan(x^3 + 1) dx = \frac{1}{3} \ln|\sec(x^3 + 1)| + C$

$$\begin{aligned} \int x^2 \tan(x^3 + 1) dx &= \dots \text{ BY REVERSE CHAIN RULE SINCE } \frac{d}{dx}(x^3 + 1) = 3x^2 \\ &= \frac{1}{3} \ln|\sec(x^3 + 1)| + C \\ &\quad \text{SINCE } \int \tan u du = -\ln|\sec u| + C \\ &\quad (\text{THE SUBSTITUTION } u = x^3 + 1 \text{ ALSO WORKS}) \end{aligned}$$

$$\int x^2 \ln(x^2+1) dx = \dots$$

BY SUBSTITUTION FIRST

$$= \int x^2 \ln(a) \left(\frac{du}{dx} \right) = \int \frac{1}{2} x^2 \ln u du$$

$$= \int \frac{1}{2} (u-1) \ln u du$$

PROCEED BY INTEGRATION BY PARTS

$$= -\frac{1}{4} (u-1)^2 \ln(u) - \int \frac{1}{4} (u-1)^2 du$$

$$= \frac{1}{4} (u-1)^2 \ln|u| - \int \frac{u^2 - 2u + 1}{4u} du$$

$$= \frac{1}{4} (u-1)^2 \ln|u| - \int \frac{u^2}{4u} - \frac{1}{2} u + \frac{1}{4} du$$

$$= \frac{1}{4} (u-1)^2 \ln|u| - \frac{1}{12} u^3 + \frac{1}{4} u^2 + C$$

$$= \frac{1}{4} [(u-1)^2 - 1] \ln|u| - \frac{1}{12} u^3 + \frac{1}{4} u^2 + C$$

$$= \frac{1}{4} (u^2 - 2u) \ln|u| - \frac{1}{12} (u^4 - 4u^3) + C$$

$$= -\frac{1}{4} (u-2) \ln|u| - \frac{1}{12} u(u-4) + C$$

$$= \frac{1}{4} (\bar{x}^4 + 1) \ln(\bar{x}^2) - \frac{1}{4} (\bar{x}^4 - 2\bar{x}^2 + 3) + C$$

$$= \frac{1}{4} (\bar{x}^4 - 1) \ln(\bar{x}^2) - \frac{1}{4} (\bar{x}^4 - 2\bar{x}^2) + C$$

$$= \frac{1}{4} (\bar{x}^4 - 1) \ln(\bar{x}^2) - \frac{1}{2} \bar{x}^2 (\bar{x}^2 - 2) + C$$

140. $\int \sin^2 x \sec^2 x \, dx = -x + \tan x + C$

$$\begin{aligned} \int \sin^2 x \sec^2 x \, dx &= \int \sin^2 x \cdot \frac{1}{\cos^2 x} \, dx = \int \frac{\sin^2 x}{\cos^2 x} \, dx \\ &= \int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx \\ &= \boxed{\tan x - x + C} \\ &\text{such that } \frac{d}{dx}(\tan x) = \sec^2 x. \end{aligned}$$

141. $\int 3\sec^2 x \sin x \, dx = 3\sec x + C$

$$\int 3\sec^2 x \sin x \, dx = \int 3 \left(\frac{1}{\cos x}\right) \sin x \, dx = \int \frac{3 \sin x}{\cos x} \cdot \frac{1}{\cos x} \, dx$$

$$= \int 3 \tan x \sec x \, dx = \underline{3 \sec x + C}$$

SINCE $\frac{d}{dx}(\sec x) = \sec x \tan x$

142. $\int \frac{1}{x(1+\ln x)^3} dx = -\frac{1}{2(1+\ln x)^2} + C$

$$\begin{aligned}\int \frac{1}{x(1+\ln x)^3} dx &= \dots \text{ BY REVERSE CHAIN RULE SINCE} \\ &\quad \frac{d}{dx}(\ln x) = \frac{1}{x} \\ &= \int \frac{1}{2} (1+\ln x)^{-2} dx \\ &= \frac{1}{2} (1+\ln x)^{-1} + C \\ &= \frac{1}{2(1+\ln x)} + C\end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}\int \frac{1}{x(1+\ln x)^3} dx &= \int x du \quad u = 1+\ln x \\ &= \int \frac{1}{u^3} du = \int u^{-3} du = -\frac{1}{2}u^{-2} + C \\ &= -\frac{1}{2u^2} + C = -\frac{1}{2(1+\ln x)^2} + C.\end{aligned}$$

$u = 1+\ln x$
 $\frac{du}{dx} = \frac{1}{x}$
 $dx = x du$

143. $\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln|x| - \frac{1}{16}x^4 + C$

$$\begin{aligned}\int x^3 \ln x \, dx &= \text{INTEGRATION BY PARTS} \\ &= \frac{1}{4}x^4 \ln|x| - \int \frac{1}{4}x^3 \, dx \\ &= \frac{1}{4}x^4 \ln|x| - \frac{1}{4}x^4 + C\end{aligned}$$

| | |
|------------------|---------------|
| $\ln x$ | $\frac{1}{4}$ |
| $\frac{1}{4}x^3$ | x^4 |

144. $\int x \ln(2x^3) \, dx = \frac{1}{2}x^2 \ln|2x^3| - \frac{3}{4}x^2 + C$

$$\begin{aligned}\int x \ln(2x^3) \, dx &= \dots \text{ INTEGRATION BY PARTS} \\ &= \frac{1}{2}x^2 \ln(2x^3) - \int \frac{1}{2}x^2 \left(\frac{6}{x}\right) \, dx \\ &= \frac{1}{2}x^2 \ln(2x^3) - \int 3x \, dx \\ &= \frac{1}{2}x^2 \ln(2x^3) - \frac{3}{4}x^2 + C\end{aligned}$$

| | |
|------------------|---------------|
| $\ln(2x^3)$ | $\frac{6}{x}$ |
| $\frac{1}{2}x^2$ | x |

145. $\int 4 - \cos^4 x \sin x \, dx = 4x + \frac{1}{5} \cos^5 x + C$

$$\begin{aligned}\int 4 - \cos^4 x \sin x \, dx &= \dots \text{ BY REVERSE CHAIN RULE ...} \\ &= 4x + \frac{1}{5} \cos^5 x + C\end{aligned}$$

CHE SUBSTITUTION: $u = \cos x$ AND $u' = -\sin x$

146. $\int \frac{\cos x}{\sin^3 x} dx = \left[-\frac{1}{2} \cot^2 x + C \right] - \left[-\frac{1}{2} \operatorname{cosec}^2 x + C \right]$

$$\begin{aligned} \int \frac{\cos x}{\sin^2 x} dx &= \int \cos x (\sin x)^{-2} dx = \dots \text{ BY REVERSE CHAIN RULE} \\ &= -\frac{1}{2} (\sin x)^{-2} + C = -\frac{1}{2} \operatorname{cosec}^2 x + C \end{aligned}$$

VARIATIONS BY REVERSE CHAIN RULE

$$\begin{aligned} \int \frac{\cos x}{\sin x} dx &= \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} dx = \int \cot x \operatorname{cosec} x dx \\ &= \left\langle \begin{array}{l} -\frac{1}{2} \cot^2 x + C \leftarrow \frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x \\ -\frac{1}{2} \operatorname{cosec}^2 x + C \leftarrow \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \operatorname{cosec} x \end{array} \right. \end{aligned}$$

THE SUBSTITUTIONS : $u = \sin x$, $u = \cot x$, $u = \operatorname{cosec} x$ ALL WORK WELL

147. $\int \frac{4 \sec^2 x}{\tan x} dx = \left[\begin{array}{l} 4 \ln |\tan x| + C \\ 4 \ln |\sec x| + 4 \ln |\sin x| + C \end{array} \right]$

$$\begin{aligned} \int \frac{4 \sec^2 x}{\tan x} dx &= \dots \text{ BY REVERSE CHAIN RULE} \\ &= 4 \ln |\tan x| + C \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned} \int \frac{4 \sec^2 x}{\tan x} dx &= \int \frac{4 \sec^2 x}{u} \left(\frac{du}{\sec^2 x} \right) dx = \int 4 du \\ &= \int \frac{4}{u} du = 4 \ln u + C \quad \boxed{u = \tan x} \quad \boxed{\frac{du}{dx} = \sec^2 x} \quad \boxed{du = \frac{du}{\sec^2 x}} \\ &= 4 \ln |\tan x| + C \end{aligned}$$

ALTERNATIVE BY TRIGONOMETRIC MANIPULATIONS

$$\begin{aligned} \int \frac{4 \sec^2 x}{\tan x} dx &= \int \frac{4(\tan^2 x + 1)}{\tan x} dx = \int \frac{4 \tan^2 x}{\tan x} + \frac{4}{\tan x} dx \\ &= \int 4 \tan x + 4 \operatorname{cot} x dx = 4 \ln |\sec x| + 4 \ln |\tan x| + C \\ &\dots = [4 \ln |\sec x \sin x| + C = 4 \ln |\tan x| + C] \end{aligned}$$

148. $\int \sec^2 x \tan x \sqrt{1 + \tan x} dx = \frac{2}{5}(1 + \tan x)^{\frac{5}{2}} - \frac{2}{3}(1 + \tan x)^{\frac{3}{2}} + C$

$$\begin{aligned} \int \sec^2 x \tan x \sqrt{1 + \tan x} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int \sec^2 x \tan x \cdot \frac{2u}{\sqrt{u^2-1}} du = \int 2u \tan x du \\ &= \int 2u^2(u^2-1) du = \int 2u^4 - 2u^2 du \\ &= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C \\ &= \frac{2}{5}(1 + \tan x)^{\frac{5}{2}} - \frac{2}{3}(1 + \tan x)^{\frac{3}{2}} + C \end{aligned}$$

THE SUBSTITUTION : $u = 1 + \tan x$ ALSO WORKS

$u = \sqrt{1 + \tan x}$
 $u^2 = 1 + \tan x$
 $2u \frac{du}{dx} = \sec^2 x$
 $du = \frac{u}{\sec^2 x} du$
 $\tan x = u^2 - 1$

149. $\int \frac{\sqrt{1+2\tan x}}{\cos^2 x} dx = \frac{1}{3}(1+2\tan x)^{\frac{3}{2}} + C$

$$\begin{aligned} \int \frac{\sqrt{1+2\tan x}}{\cos^2 x} dx &= \dots \text{ BY REVERSE CHAIN RULE} \\ &= \int (1+2\tan x)^{\frac{1}{2}} \sec^2 x dx = \frac{1}{3}(1+2\tan x)^{\frac{3}{2}} + C \\ \text{ALTERNATIVE BY SUBSTITUTION} \\ \int \frac{u}{\cos^2 u} \sec^2 u du &= \int u^2 du \\ &= \frac{1}{3}u^3 + C = \frac{1}{3}(1+2\tan x)^{\frac{3}{2}} + C \\ \text{THE SUBSTITUTION } u = 1+2\tan x \text{ ALSO WORKS} \end{aligned}$$

$u = \sqrt{1+2\tan x}$
 $u^2 = 1+2\tan x$
 $2u du = 2\tan x \sec^2 x$
 $du = \frac{1}{2}\tan x \sec^2 x dx$

150. $\int \tan^2 x dx = -x + \tan x + C$

$$\int \tan^2 x dx = \int \sec^2 x - 1 dx = \tan x - x + C$$

151. $\int \frac{(1+\sin x)^2}{\cos^2 x} dx = -x + 2\tan x + 2\sec x + C$

$$\begin{aligned} \int \frac{(1+\sin x)^2}{\cos^2 x} dx &= \int \frac{1+2\sin x+\sin^2 x}{\cos^2 x} dx \\ &= \int \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} dx \\ &= \int \sec^2 x + 2\tan x \sec x + \tan^2 x dx \\ &= \int \sec^2 x + 2\tan x \sec x + (\sec^2 x - 1) dx \\ &= \int 2\sec^2 x + 2\tan x \sec x - 1 dx \\ &= 2\tan x + 2\sec x - x + C \end{aligned}$$

152. $\int \frac{\cos^2 x}{1+\sin x} dx = x + \cos x + C$

$$\begin{aligned} \int \frac{\cos^2 x}{1+\sin x} dx &= \int \frac{1-\sin x}{1+\sin x} dx = \int \frac{(1-\sin x)(1+\sin x)}{(1+\sin x)^2} dx \\ &= \int 1-\sin x dx = x + \cos x + C \\ \text{ALTERNATE} \\ \int \frac{\cos^2 x}{1+\sin x} dx &= \int \frac{\cos^2(1-\sin x)}{(1+\sin x)(1-\sin x)} dx \\ &= \int \frac{\cos^2(1-\sin x)}{1-\sin^2 x} dx = \int \frac{\sec^2(1-\sin x)}{\cos^2 x} dx \\ &= \int 1-\sin x dx = x + \cos x + C \end{aligned}$$

153. $\int \frac{1}{1+\cos x} dx = \begin{bmatrix} \cosec x - \cot x + C \\ \tan\left(\frac{1}{2}x\right) + C \end{bmatrix}$

$$\begin{aligned} \int \frac{1}{1+\cos x} dx &= \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)} dx = \int \frac{1-\cos x}{1-\cos^2 x} dx \\ &= \int \frac{1-\cos x}{\sin^2 x} dx = \int \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} dx = \int \cosec^2 x - \frac{\cos x}{\sin^2 x} dx \\ &= \int \cosec x - \cot x \cosec x dx = -\cot x + \cosec x + C \end{aligned}$$

ALTERNATIVE: USING THE DOUBLE ANGLE IDENTITIES

$$\begin{aligned} \int \frac{1}{1+\cos x} dx &= \int \frac{1}{1+2\cos^2\frac{x}{2}-1} dx \quad (\cos^2 x = 2\cos^2\frac{x}{2}-1) \\ &= \int \frac{1}{2\cos^2\frac{x}{2}} dx = \int \frac{1}{2} \sec^2\frac{x}{2} dx \\ &= \tan\frac{x}{2} + C \end{aligned}$$

NOTE: $\cosec x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1-\cos x}{\sin x} = \frac{1-(1-2\sin^2\frac{x}{2})}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \tan\frac{x}{2}$

154. $\int \frac{\cos x}{\sqrt{\sin x}} dx = 2\sqrt{\sin x} + C$

$$\int \frac{\cos x}{\sqrt{\sin x}} dx = \int \cos x (\sin x)^{-\frac{1}{2}} dx \quad \text{... BY REVERSE CHAIN RULE...}$$

$$= 2(\sin x)^{\frac{1}{2}} + C$$

THE SUBSTITUTION $u = \sin x$ OR $u = \sqrt{\sin x}$ BOTH WORK WELL

155. $\int \frac{10x^4}{2x^{\frac{5}{2}}+1} dx = 2x^{\frac{5}{2}} - \ln\left(2x^{\frac{5}{2}}+1\right) + C$

$$\begin{aligned} \int \frac{10x^4}{2x^{\frac{5}{2}}+1} dx &= \dots \text{BY SUBSTITUTION} \\ &= \int \frac{10x^4}{4} \left(\frac{du}{2x^{\frac{5}{2}}} \right) = \int \frac{2x^{\frac{5}{2}}}{u} du \\ &= \int \frac{u-1}{u} du = \int 1 - \frac{1}{u} du \\ &= u - \ln|u| = (2x^{\frac{5}{2}}+1) - \ln(2x^{\frac{5}{2}}+1) + C \\ &= 2x^{\frac{5}{2}} - \ln(2x^{\frac{5}{2}}+1) + C \end{aligned}$$

u = 2x²+1
 $\frac{du}{dx} = 2x^{\frac{5}{2}}$
 $du = \frac{du}{dx} dx = \frac{du}{2x^{\frac{5}{2}}} \cdot 2x^{\frac{5}{2}} dx$
 $2x^{\frac{5}{2}} = u-1$

156. $\int \sin \sqrt{x} \, dx = 2\sin \sqrt{x} - 2\sqrt{x} \cos \sqrt{x} + C$

$$\begin{aligned} \int \sin \sqrt{x} \, dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \sin u (2du) = \int 2u \sin u \, du \\ \text{INTEGRATION BY PARTS NOW} \\ &= -2u \cos u - \int -2u \sin u \, du \\ &= -2u \cos u + \int 2u \sin u \, du \\ &= -2u \cos u + 2u \sin u + C \\ &\Rightarrow 2u \sin u - 2u \cos u + C \end{aligned}$$

157. $\int \frac{x^2}{1-2x} \, dx = \left[-\frac{1}{16}(1-2x)^2 + \frac{1}{4}(1-2x) - \frac{1}{8} \ln|1-2x| + C \right]$
 $\quad \quad \quad -\frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{8} \ln|1-2x| + C$

$$\begin{aligned} \int \frac{x^2}{1-2x} \, dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{x^2}{u} \left(\frac{du}{-2} \right) = \int \frac{-x^2}{2u} \, du \\ &= \int -\frac{\ln^2 u}{2u} \, du = -\frac{1}{2} \int \frac{4u^2}{u} \, du \\ &= -\frac{1}{2} \int \frac{4u^2 - 2u + 1}{u} \, du = -\frac{1}{2} \int 4 - 2 + \frac{1}{u} \, du \\ &= -\frac{1}{2} \left[\frac{1}{2}u^2 - 2u + \ln|u| \right] + C = -\frac{1}{2}(1-2x)^2 + \frac{1}{4}(1-2x) - \frac{1}{2}\ln|1-2x| + C \end{aligned}$$

ALTERNATING BY MANIPULATION & SUB

$$\begin{aligned} \int \frac{x^2}{1-2x} \, dx &= \int \frac{-\frac{1}{2}x(1-2x) - \frac{1}{2}(1-2x) + \frac{1}{2}}{1-2x} \, dx \\ &= \int -\frac{1}{2}x - \frac{1}{4} + \frac{1}{1-2x} \, dx \\ &= -\frac{1}{2}x^2 - \frac{1}{4}x - \frac{1}{2}\ln|1-2x| + C \end{aligned}$$

158. $\int \frac{12}{(1-2x)^5} \, dx = \frac{3}{2(1-2x)^4} + C$

$$\begin{aligned} \int \frac{12}{(1-2x)^5} \, dx &= \int 12(1-2x)^{-5} \, dx = \dots \text{ BY RECOGNITION } \dots \\ &= \frac{12}{8}(1-2x)^{-4} + C = \frac{3}{2}(1-2x)^{-4} + C = \frac{3}{2(1-2x)^4} + C \end{aligned}$$

159. $\int \frac{x^4 + 2x}{x^5 + 5x^2 + 8} dx = \frac{1}{5} \ln|x^5 + 5x^2 + 8| + C$

$$\int \frac{x^4 + 2x}{x^5 + 5x^2 + 8} dx = \frac{1}{5} \int \frac{5x^4 + 10x}{5x^5 + 5x^2 + 8} dx = \dots$$

... or we can $\int f(x) dx = \ln|f(x)| + C$...

$= \frac{1}{5} \ln|x^5 + 5x^2 + 8| + C$

ALTERNATIVE METHOD BY THE SUBSTITUTION, $u = x^5 + 5x^2 + 8$

160. $\int \frac{x}{x-1} dx = x + \ln|x-1| + C$

$$\int \frac{x}{x-1} dx = \dots \text{ BY SUBSTITUTION } \dots$$

$\frac{x}{x-1} (du) = \int \frac{u+1}{u} du = \int 1 + \frac{1}{u} du$

$= u + \ln|u| + C = (x-1) \ln|x-1| + C$

$= x + \ln|x-1| + C$

ALTERNATIVE BY MANIPULATIONS

$$\int \frac{x}{x-1} dx = \int \frac{u+1}{x-1} du = \int \frac{x-1+1}{x-1} du$$

$$= \int 1 + \frac{1}{x-1} du = x + \ln|x-1| + C$$

161. $\int \frac{1}{(1+\sqrt{x})\sqrt{x}} dx = 2 \ln(1+\sqrt{x}) + C$

$$\int \frac{1}{(1+\sqrt{x})\sqrt{x}} dx = \dots \text{ BY SUBSTITUTION } \dots$$

$u = \sqrt{x}$
 $u^2 = x$
 $2u \frac{du}{dx} = 1$
 $2u du = dx$

$$= \int \frac{1}{(1+u)\sqrt{u}} (2u du) = \int \frac{2}{1+u} du$$

$$= 2 \ln|u+1| + C = 2 \ln(\sqrt{x}+1) + C$$

CAN ALSO BE DONE BY "REVERSE CHAIN RULE" BY DIRECT RECOGNITION OF THE FRACTION AS IS DERIVATIVE

162. $\int \frac{2x^3+1}{x^4+2x} dx = \frac{1}{2} \ln|x^4+2x| + C$

$$\int \frac{2x^3+1}{x^4+2x} dx = \frac{1}{2} \int \frac{4x^3+2}{2x^4+4x} dx = \frac{1}{2} \ln|x^4+2x| + C$$

(THIS IS OF THE FORM $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$)

THE SUBSTITUTION $u = x^4+2x$ ALSO WORKS WELL

163. $\int \frac{x+2}{x(x+1)} dx = \ln \left| \frac{x^2}{x+1} \right| + C$

$$\int \frac{dx}{x(x+1)} = \dots \text{ BY PARTIAL FRACTIONS ...}$$

| | |
|---|--|
| $\frac{x+2}{x(x+1)} \equiv \frac{A}{x} + \frac{B}{x+1}$ | $= \int \frac{dx}{x} - \frac{dx}{x+1}$ |
| $x+2 \equiv Ax+Bx+A$ | $= 2\ln x - \ln x+1 + C$ |
| • If $A=0$ | $= \ln x^2 - \ln x+1 + C$ |
| $2=A$ | $= \ln \left \frac{x^2}{x+1} \right + C$ |
| • If $x=-1$ | |
| $x=-B$ | |
| $B=-1$ | |

164. $\int \sin x \ln(\sec x) dx = \begin{bmatrix} -[1 + \ln|\sec x|] \cos x \\ [-1 + \ln|\cos x|] \cos x \end{bmatrix} + C$

$$\int \sin x \ln(\sec x) dx = \dots \text{ INTEGRATION BY PARTS }$$

| | |
|---------------|----------------|
| $\ln(\sec x)$ | $d\ln(\sec x)$ |
| $-\csc x$ | $\sin x$ |

$$= -\csc x \ln(\sec x) - \int \csc x \ln(\sec x) dx$$

$$= \csc x \ln(\csc x) + \int \sin x dx$$

$$= \csc x \ln(\csc x) - \csc x + C = -\csc x [\ln \sec x + 1] + C$$

$$= \csc x [\ln \csc x - 1] + C$$

165. $\int \frac{(1+2\cos x)^2}{3\sin^2 x} dx = -\frac{5}{3} \cot x - \frac{4}{3} \operatorname{cosec} x - \frac{4}{3} x + C$

$$\int \frac{(1+2\cos x)^2}{3\sin^2 x} dx = \int \frac{1+4\cos x+4\cos^2 x}{3\sin^2 x} dx$$

$$= \int \frac{\frac{1}{3}\csc^2 x + \frac{4}{3}\cot x + \frac{4}{3}\cot^2 x}{\sin^2 x} dx$$

$$= \int \frac{\frac{1}{3}\csc^2 x + \frac{4}{3}\cot x + \frac{4}{3}\cot^2 x}{\frac{1}{3}\csc^2 x} dx$$

$$= \int \frac{1}{3} + \frac{4}{3}\cot x + \left(\frac{4}{3}\cot^2 x - \frac{1}{3} \right) dx$$

$$= \int \frac{4}{3}\cot x + \frac{4}{3}\cot^2 x - \frac{1}{3} dx$$

$$= -\frac{4}{3}\ln|\sin x| - \frac{4}{3}\operatorname{cosec} x - \frac{1}{3} x + C$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

166. $\int \frac{1}{x \ln x} dx = \ln|\ln|x|| + C$

$$\begin{aligned}\int \frac{1}{x \ln x} dx &= \int \frac{1}{x} \times \frac{1}{\ln x} dx = \dots \text{ BY RECOGNITION RULE } \\ &= \underline{\ln|\ln x| + C} \\ \text{ALTERNATIVE BY SUBSTITUTION} \\ \int \frac{1}{x \ln x} dx &= \int \frac{1}{x} du = \int \frac{1}{u} du \\ &= \ln|u| + C \\ &= \underline{\ln|\ln x| + C} \quad \boxed{u = \ln x} \\ \frac{du}{dx} &= \frac{1}{x} \\ dx &= x du\end{aligned}$$

167. $\int (2-3x)^{-2} dx = \frac{1}{3(2-3x)} + C$

$$\begin{aligned}\int (2-3x)^{-2} dx &= \dots \text{ BY RECOGNITION } \\ &= \underline{\frac{1}{3}(2-3x)^{-1}} + C = \underline{\frac{1}{3(2-3x)} + C}\end{aligned}$$

168. $\int 2\sec^2 x + \frac{1}{2}\sin 2x dx = 2\tan x - \frac{1}{4}\cos 2x + C$

$$\begin{aligned}\int 2\sec^2 x + \frac{1}{2}\sin 2x dx &= \dots \text{ BY RECOGNITION } \\ &= \underline{2\tan x - \frac{1}{4}\cos 2x + C}\end{aligned}$$

169. $\int \frac{\ln x}{\sqrt{x}} dx = 2\sqrt{x}[\ln|x|-2] + C$

$$\begin{aligned}\int \frac{\ln x}{\sqrt{x}} dx &= \int 2^{\frac{1}{2}} \ln x dx = \dots \text{ INTEGRATION BY PARTS } \\ &= 2x^{\frac{1}{2}} \ln|x| - \int 2x^{\frac{1}{2}} \left(\frac{1}{x}\right) dx \\ &= 2x^{\frac{1}{2}} \ln|x| - \int 2x^{-\frac{1}{2}} dx \\ &= 2x^{\frac{1}{2}} \ln|x| - 4x^{\frac{1}{2}} + C \\ &= 2x^{\frac{1}{2}} [\ln|x|-2] + C \\ &= \underline{2\sqrt{x}[\ln|x|-2] + C}\end{aligned}$$

170. $\int \frac{\cos^4 x}{\sin x} dx = \ln|\tan\left(\frac{1}{2}x\right)| + \cos x + \frac{1}{3}\cos^3 x + C$

$$\begin{aligned} \int \frac{\cos^4 x}{\sin x} dx &= \int \frac{(1-\sin^2 x)^2}{\sin x} dx = \int \frac{1-2\sin^2 x+\sin^4 x}{\sin x} dx \\ &= \int (\csc x - 2\sin x + \sin^2 x) dx \\ &= \int (\csc x - \sin x - \sin x \csc x) dx \\ &\quad \downarrow \text{STANDARD RESULTS} \quad \downarrow \text{TO REDUCE CHAN RULE} \\ &= \ln|\tan\frac{x}{2}| + \cos x + \frac{1}{3}\cos^3 x + C \end{aligned}$$

171. $\int 6\tan^2 x - \sec^2 x dx = 5\tan x - 6x + C$

$$\begin{aligned} \int 6\tan^2 x - \sec^2 x dx &= \int 6(\sec^2 x - 1) - \sec^2 x dx \\ &= \int 5\sec^2 x - 6 dx \\ &= 5\tan x - 6x + C \end{aligned}$$

172. $\int 6\cos^4 x - 2\sin^2 x dx = \frac{1}{2}x + \frac{3}{2}\sin 2x + \frac{1}{8}\sin 4x + C$

$$\begin{aligned} \int 4\cos^2 x - 2\sin^2 x dx &= \int 4(\cos^2 x)^2 - 2\left(\frac{1-\cos 2x}{2}\right) dx \\ &= \int 4\left(\frac{1+\cos 2x}{2}\right)^2 - 1 + \cos 2x dx \\ &= \int 1+2\cos 2x+\cos^2 2x - 1 + \cos 2x dx \\ &= \int 3\cos 2x + \cos^2 2x dx \\ &= \int 3\cos 2x + \left(\frac{1+\cos 4x}{2}\right) dx \\ &= \int \frac{1}{2}+3\cos 2x + \frac{1}{2}\cos 4x dx \\ &= \frac{1}{2}x + \frac{3}{2}\sin 2x + \frac{1}{8}\sin 4x + C \end{aligned}$$

173. $\int \sin 4x \cos 4x \, dx = \begin{bmatrix} -\frac{1}{16} \cos 8x + C \\ \frac{1}{8} \sin^2 4x + C \\ -\frac{1}{8} \cos^2 4x + C \end{bmatrix}$

$$\int \sin 4x \cos 4x \, dx = \int \frac{1}{2} (2 \sin 4x \cos 4x) \, dx = \int \frac{1}{2} \sin 8x \, dx$$

 $= -\frac{1}{16} \cos 8x + C$

ALTERNATIVE BY REVERSE CHAIN RULE

$$\int \sin 4x \cos 4x \, dx = \frac{1}{8} \sin^2 4x + C, \text{ since } \frac{d}{dx} (\sin^2 4x) = 2 \sin 4x (4 \cos 4x)$$

$$\int \sin 4x \cos 4x \, dx = -\frac{1}{8} \cos^2 4x + C, \text{ since } \frac{d}{dx} (\cos^2 4x) = 2 \cos 4x (-4 \sin 4x)$$

174. $\int \frac{1}{\operatorname{cosec} x - \cot x} \, dx = \begin{bmatrix} \ln \left| \frac{\sin x}{\operatorname{cosec} x + \cot x} \right| + C \\ \ln |1 - \cos x| + C \end{bmatrix}$

$$\int \frac{1}{\operatorname{cosec} x - \cot x} \, dx = \int \frac{\operatorname{cosec} x + \cot x}{(\operatorname{cosec} x - \cot x)(\operatorname{cosec} x + \cot x)} \, dx$$

 $= \int \frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec}^2 x - \cot^2 x} \, dx = \int \frac{\operatorname{cosec} x + \cot x}{1 + \cot^2 x} \, dx$
 $\quad \text{Ratio and product rules}$
 $= -\ln |\operatorname{cosec} x + \cot x| + \ln |\operatorname{cosec} x| + C$
 $= \ln \left| \frac{\operatorname{cosec} x}{\operatorname{cosec} x + \cot x} \right| + C$
 $= \ln \left| \frac{\operatorname{cosec} x \operatorname{cosec} x}{\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x} \right| + C = \ln \left| \frac{1 - \cot^2 x}{1 + \cot^2 x} \right| + C$
 $= \ln \left| \frac{(1 - \cot^2 x)(1 + \cot^2 x)}{1 + \cot^2 x} \right| + C = \ln |1 - \cot^2 x| + C$

ALTERNATIVE (VARIATION)

$$\int \frac{1}{\operatorname{cosec} x - \cot x} \, dx = \int \frac{1}{\operatorname{cosec} x - \cot x} \cdot \frac{1}{\operatorname{cosec} x - \cot x} \, dx = \int \frac{\operatorname{cosec} x}{1 - \cot x} \, dx$$

 $\quad \text{of the form } \int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + C$
 $= \ln |1 - \cot x| + C$

175. $\int \frac{x^2}{\sqrt{x-1}} \, dx = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$

$$\int \frac{x^2}{\sqrt{x-1}} \, dx = \dots \text{ BY SUBSTITUTION } \dots$$

$$= \int \frac{2x}{\sqrt{x-1}} (2x \, dx) = \int 2x^2 \, dx$$

 $= \int 2x^2 + 4x^2 + 2 \, dx = \frac{2}{3}x^3 + \frac{4}{3}x^3 + 2x + C$
 $= \frac{2}{3}(x-1)^{\frac{3}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}}$

$$\begin{aligned} u &= \sqrt{x-1} & d\frac{u}{dx} &= 1 \\ du &= \frac{1}{2\sqrt{x-1}} \, dx & dx &= 2\sqrt{x-1} \, du \\ 2\sqrt{x-1} \, du &= dx & 2 &= 2\sqrt{x-1} \\ 2u^2 &= x-1 & 2 &= 2u^2+2 \\ 2u^2+2 &= x & 2 &= 2u^2+2 \end{aligned}$$

[THE SUBSTITUTION $u = \sqrt{x-1}$ ALSO WORKS WELL]

$$176. \int \frac{3e^{2x}}{\sqrt{e^x-1}} dx = 2(e^x-1)^{\frac{3}{2}} + 6(e^x-1)^{\frac{1}{2}} + C$$

$$\begin{aligned} \int \frac{3e^{2x}}{\sqrt{e^x-1}} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{3e^{2x}}{e^x} \left(\frac{2e^x}{e^x} du \right) = \int \frac{6e^{2x}}{e^x} du \\ &= \int 6e^x du = \int 6e^x + 6 du \\ &= 2(e^x-1)^{\frac{3}{2}} + 6(e^x-1)^{\frac{1}{2}} + C \\ &\quad (\text{THE SUBSTITUTION } u=e^x-1 \text{ ALSO WORKS WELL}) \end{aligned}$$

$$177. \int \frac{1}{(2+\sqrt[3]{x})^3 \sqrt{x^2}} dx = 3\ln|2+\sqrt[3]{x}| + C$$

$$\begin{aligned} \int \frac{1}{\sqrt[3]{x^2(2+2\sqrt[3]{x})}} dx &= \dots \text{ BY REVERSE CHAIN RULE / DIFFERENTIATION OF SUBSTITUTION } \dots \\ &= \int \frac{1}{\sqrt[3]{x^2(2+2u)}} (2\sqrt[3]{x} du) = \int \frac{2}{u+2} du \\ &= 2\ln|u+2| + C = 2\ln|2+\sqrt[3]{x}| + C \\ &\quad u=\sqrt[3]{x} \\ &\quad u^2=2 \\ &\quad \frac{2u}{3}du=du \\ &\quad u^2=\sqrt[3]{x}^2 \\ &\quad u^3=\sqrt[3]{x^2} \end{aligned}$$

$$178. \int \frac{4x^7}{x^4+1} dx = x^4 - \ln(x^4+1) + C$$

$$\begin{aligned} \int \frac{4x^7}{x^4+1} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{4u^7}{u^4+1} \times \frac{du}{4u^3} = \int \frac{u^4}{u^4+1} du \\ &= \int \frac{u^4-1}{u^4} du = \int 1 - \frac{1}{u^4} du \\ &= u - \ln|u| + C = (u^4-1) - \ln(u^4+1) + C \\ &= u^4 - \ln(u^4+1) + C \\ &\quad u=x^4+1 \\ &\quad \frac{du}{dx} = 4x^3 \\ &\quad du = 4x^3 dx \\ &\quad u^4 = x^4+1 \\ &\quad u^4-1 = x^4+1-1 = x^4 \\ &\quad \text{ALTERNATIVE BY MANIPULATION / LONG DIVISION} \\ \int \frac{4x^7}{x^4+1} dx &= \int \frac{4x^8(1+u^{-4}) - 4x^8}{x^4+1} dx = \int 4x^8 - \frac{4x^8}{x^4+1} dx \\ &= 4x^8 - \ln(x^4+1) + C \end{aligned}$$

$$179. \int \frac{x}{9x^2+1} dx = \frac{1}{18} \ln(9x^2+1) + C$$

$$\begin{aligned} \int \frac{x}{9x^2+1} dx &= \frac{1}{18} \int \frac{18x}{9x^2+1} dx = \frac{1}{18} \ln(9x^2+1) + C \\ &\quad \text{BY SUBSTITUTION } u=9x^2+1 \text{ ALSO WORKS WELL} \end{aligned}$$

180. $\int \frac{2}{x+\sqrt[3]{x}} dx = 3\ln|x+\sqrt[3]{x}| + C$

$$\begin{aligned} \int \frac{2}{x+\sqrt[3]{x}} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{2u^2}{u^3+u} (3u^2 du) = \int \frac{6u^2}{u^3+u} du = \int \frac{6u^2}{u(u^2+1)} du \\ &= 3 \int \frac{2u}{u^2+1} du \xrightarrow{\substack{f'(u) \\ f(u)}} = 3 \ln(u^2+1) + C \\ &= 3 \ln(x^2+1) + C \end{aligned}$$

181. $\int \frac{3x}{1+\sqrt{x}} dx = \left[\begin{array}{l} 2x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6\ln|1+\sqrt{x}| + C \\ 2(1+\sqrt{x})^3 - 9(1+\sqrt{x})^2 + 18(1+\sqrt{x}) - 6\ln|1+\sqrt{x}| + C \end{array} \right]$

$$\begin{aligned} \int \frac{3x}{1+\sqrt{x}} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{3u^2}{1+u} (2u du) = \int \frac{6u^2}{u+1} du \\ &\quad \text{BY U-DIVISION / MANIPULATION} \\ &= 6 \int \frac{u(u+1)-u(u+1)-1}{u+1} du = 6 \int u^2-u+1-\frac{1}{u+1} du \\ &= 6 \left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln(u+1) \right] + C \\ &= 2x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6\ln(\sqrt{x}+1) + C \\ &\quad \text{ALTERNATIVE SUBSTITUTION} \\ &\int \frac{3x}{1+\sqrt{x}} dx = \int \frac{3(u-1)^2 - [2(u-1)]}{u+1} du \\ &= \int \frac{6(u-1)^3}{u} du = \int \frac{6u^3 - 18u^2 + 18u - 6}{u} du \\ &= \int 6u^2 - 18u + 18 - \frac{6}{u} du \\ &= 2u^3 - 9u^2 + 18u - 6\ln|u| + C \\ &= 2(1+\sqrt{x})^3 - 9(1+\sqrt{x})^2 + 18(1+\sqrt{x}) - 6\ln(1+\sqrt{x}) + C \\ &\quad \text{OR EXPAND RISING CURVES} \\ &= 2 \left[\frac{1}{4}x^2 + 3x + 3x^{\frac{1}{2}} \right] - 9 \left[x + 2x^{\frac{1}{2}} \right] + 18x^{\frac{1}{2}} - 6\ln(1+\sqrt{x}) + C \\ &= 2x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 6\ln(1+\sqrt{x}) + C \\ &- 9x^{\frac{1}{2}} - 18x^{\frac{1}{2}} + 18x^{\frac{1}{2}} \\ &= 2x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6\ln(1+\sqrt{x}) + C \end{aligned}$$

182. $\int \frac{3x^2+2}{4x+1} dx = \left[\begin{array}{l} \frac{3}{8}x^2 - \frac{3}{16}x + \frac{35}{64}\ln|4x+1| + C \\ \frac{3}{128}(4x+1)^2 - \frac{3}{32}(4x+1) + \frac{35}{64}\ln|4x+1| + C \end{array} \right]$

$$\begin{aligned} \int \frac{3x^2+2}{4x+1} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \frac{1}{4} \int \frac{48x^2+32}{u} \left(\frac{du}{4} \right) = \frac{1}{4} \int \frac{3(16x^2)+32}{u} du \\ &= \frac{1}{4} \int \frac{3(6x^2+8x+1)+32}{u} du = \frac{1}{4} \int \frac{3x^2+8x+35}{u} du \\ &= \frac{1}{4} \int 3u^2 - 6u + \frac{35}{u} du = \frac{1}{4} \left[3u^3 - 6u^2 + 35\ln|u| \right] + C \\ &= \frac{3}{16}u^2 - \frac{3}{4}u + \frac{35}{4u} \ln|u| + C = \frac{3}{16}(4x+1)^2 - \frac{3}{4}(4x+1) + \frac{35}{4u} \ln|4x+1| + C \\ &\quad \text{ALTERNATIVE BY MANIPULATION / DIVISION} \\ &\int \frac{3x^2+2}{4x+1} dx = \frac{1}{4} \int \frac{3x^2+2}{x+\frac{1}{4}} dx = \frac{1}{4} \int \frac{3x(x+\frac{1}{4}) - \frac{3}{4}(x+\frac{1}{4}) + \frac{3}{4}+2}{x+\frac{1}{4}} dx \\ &= \frac{1}{4} \int 3x^2 - \frac{3}{4}x + \frac{35}{4} dx = \int \frac{3}{4}x^2 - \frac{3}{16}x + \frac{35}{16} dx \\ &= \frac{3}{8}x^2 - \frac{3}{32}x + \frac{35}{32} \ln|4x+1| + C \end{aligned}$$

183. $\int \frac{3}{x} + \frac{4}{x^2} - \frac{2}{x^3} dx = \frac{1}{x^2} - \frac{4}{x} + 3\ln|x| + C$

$$\int \frac{3}{x} + \frac{4}{x^2} - \frac{2}{x^3} dx = \dots \text{ BY INSPECTION } \dots = \int \frac{3}{x} + 4x^{-2} - 2x^{-3} dx \\ = 3\ln|x| - 4x^{-1} + x^{-2} + C = \frac{1}{x^2} - \frac{4}{x} + 3\ln|x| + C$$

184. $\int 2\sin 2x \cos^2 x dx = -\cos^4 x + C$

$$\int 2\sin 2x \cos^2 x dx = \int 2(2\sin x \cos x) \cos^2 x dx \\ = \int 4\sin x \cos^3 x dx \\ = \dots \text{ BY INVERSE SUBSTITUTION (CANCELLATION)} \\ = -\cos x + C$$

THE SUBSTITUTION $u = \cos x$ ALSO WORKS WELL.

185. $\int \frac{10x^2 - 23x + 11}{(2-3x)(2x-1)^2} dx = -\frac{1}{3}\ln|3x-2| - \frac{1}{2}\ln|2x-1| - \frac{2}{2x-1} + C$

$$\int \frac{10x^2 - 23x + 11}{(2-3x)(2x-1)^2} dx = \dots \text{ BY PARTIAL FRACTIONS}$$

$$\frac{10x^2 - 23x + 11}{(2-3x)(2x-1)^2} = \frac{A}{2-3x} + \frac{B}{(2x-1)} + \frac{C}{(2x-1)^2}$$

$$10x^2 - 23x + 11 = A(2x-1)^2 + B(2x-1) + C(2x-1)^2$$

- IF $x = \frac{1}{2}$ $\frac{A}{2-3(\frac{1}{2})} = \frac{1}{2}$ $A = \frac{1}{2}$
- IF $x = \frac{1}{2}$ $\frac{B}{(2(\frac{1}{2})-1)} = \frac{1}{2}$ $B = \frac{1}{2}$
- IF $x = 0$ $\frac{C}{(2(0)-1)^2} = 11$ $C = 11$

$$5 - \frac{3}{2} + 11 = \frac{1}{2}B \\ 5 - \frac{3}{2} + 12 = \frac{1}{2} \\ 5 + 28 + 2x = 8 \\ 2x = -22 \\ B = -11$$

$$A = \frac{1}{2} \\ B = -11 \\ C = 11$$

$$= \int \frac{\frac{1}{2}}{2-3x} - \frac{1}{2x-1} + \frac{11(2x-1)^2}{(2x-1)^2} dx = -\frac{1}{3}\ln|3x-2| - \frac{1}{2}\ln|2x-1| - \frac{2}{2x-1} + C$$

$$= -\frac{1}{3}\ln|3x-2| - \frac{1}{2}\ln|2x-1| - \frac{2}{2x-1} + C$$

186. $\int \frac{\sec x}{\sec x - \tan x} dx = \tan x + \sec x + C$

$$\int \frac{\sec x}{\sec x - \tan x} dx = \int \frac{\sec x (\sec x + \tan x)}{(\sec x - \tan x)(\sec x + \tan x)} dx \\ = \int \frac{\sec^2 x + \sec x \tan x}{\sec^2 x - \tan^2 x} dx = \int \sec^2 x + \sec x \tan x dx \\ 1 + \tan^2 x = \sec^2 x \\ = \tan x + \sec x + C$$

187. $\int x \cos 6x \, dx = \frac{1}{6}x \sin 6x + \frac{1}{36} \cos 6x + C$

| | |
|---|---|
| $\int x \cos 6x \, dx = \dots$ INTEGRATION BY PARTS $= \frac{1}{6}x \sin 6x - \int \sin 6x \, dx$ $= \frac{1}{6}x \sin 6x + \frac{1}{36} \cos 6x + C$ | $\begin{array}{ c c } \hline a & 1 \\ \hline u & \sin 6x \\ \hline \end{array}$ |
|---|---|

188. $\int \sin x \sin 3x \, dx = \left[\begin{array}{l} \frac{1}{4} \sin 2x - \frac{1}{8} \sin 4x + C \\ \frac{1}{8} \cos x \sin 3x - \frac{3}{8} \sin x \cos 3x + C \end{array} \right]$

| |
|---|
| $\int \sin x \sin 3x \, dx = \dots$ BY TRIG IDENTITIES $= \int \frac{1}{2} \cos 2x - \frac{1}{2} \cos 4x \, dx$ (see below) $= \frac{1}{2} \sin 2x - \frac{1}{8} \sin 4x + C$ $\cos(2x+2) = \cos 2x \cos 2 - \sin 2x \sin 2$) SUBTRACT $\cos(3x-2) = \cos 3x \cos 2 + \sin 3x \sin 2$ $\cos 2x - \cos 2x = -2 \sin 2x \sin 2$ $\sin 3x \sin 2 = \frac{1}{2} \cos 2x - \frac{1}{2} \cos 6x$ |
|---|

ALTERNATIVE BY USING THE TRIPLE ANGLE FOR SINE

$$\begin{aligned} \sin 3x &= \sin(2x+2) = \sin 2x (\cos 2 + \tan 2 \sin 2) \\ &= (2\sin 2 \cos 2) \cos 2 + ((1-\cos^2 2) \sin 2) \\ &= 2\sin 2 \cos^2 2 + \sin 2 - 2\sin^2 2 \\ &= 2\sin 2 (1-\sin^2 2) + \sin 2 - 2\sin^2 2 \\ &= 2\sin 2 - 2\sin^2 2 + \sin 2 - 2\sin^2 2 \\ &= 3\sin 2 - 4\sin^2 2 \\ \\ \int \sin 3x \sin 2x \, dx &= \int \sin 2x (3\sin 2 - 4\sin^2 2) \, dx \\ &= \int 3\sin^2 2x - 4\sin^3 2x \, dx \\ &= \int 3\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) - 4\left(\frac{1}{2} - \frac{1}{2}\cos 4x + \frac{1}{2}\cos 2x\right) \, dx \\ &= \int \frac{3}{2} - \frac{3}{2}\cos 4x - 4\left(\frac{1}{2} - \frac{1}{2}\cos 4x + \frac{1}{2}\cos 2x\right) \, dx \\ &= \int \frac{3}{2} - \frac{3}{2}\cos 4x - 1 + 2\cos 4x - \cos^2 2x \, dx \\ &= \int \frac{3}{2} + \frac{1}{2}\cos 4x - (\cancel{\frac{3}{2}} + \cancel{\cos 4x}) \, dx \end{aligned}$$

| |
|--|
| $\int \sin x \sin 3x \, dx = \dots$ BY PARTS $= -\cos x \sin 3x - \int -\cos x \cos 3x \, dx$ $= -\cos x \sin 3x + \int 3\cos^2 x \cos 3x \, dx$ BY PARTS AGAIN $= -\cos x \sin 3x + \int 3\cos^2 x \cos 3x \, dx - \int -9\sin x \cos^2 x \, dx$ $= -\cos x \sin 3x + 3\cos^2 x \sin 3x + 9 \int \sin^3 x \cos x \, dx$ $\text{COLLECTING THE RESULTS}$ $\Rightarrow \int \sin x \sin 3x \, dx = -\cos x \sin 3x + 3\cos^2 x \sin 3x + 9 \int \sin^3 x \cos x \, dx$ $\Rightarrow \cos x \sin 3x - 3\cos^2 x \sin 3x = 8 \int \sin^3 x \cos x \, dx$ $\Rightarrow \int \sin x \sin 3x \, dx = \frac{1}{8} \cos x \sin 3x - \frac{3}{8} \cos^2 x \sin 3x + C$ $= \frac{1}{8} (\cos x \sin 3x - \cos^2 x \sin 3x) - \frac{3}{8} \cos^2 x \sin 3x + C$ $= \frac{1}{8} \sin(2x-2) - \frac{1}{8} (2\cos^2 x \sin 2x) + C$ $= \frac{1}{8} \sin 2x - \frac{1}{8} (\sin 2x - \sin 4x) + C$ $= \frac{3}{8} \sin 2x - \frac{1}{8} \sin 4x + C$ (as above) |
|--|

189. $\int 4 \cos 3x + \frac{1}{2} \sin 3x \, dx = \frac{4}{3} \sin 3x - \frac{1}{6} \cos 3x + C$

| |
|---|
| $\int 4 \cos 3x + \frac{1}{2} \sin 3x \, dx = \dots$ BY INSPECTION ... $= \frac{4}{3} \sin 3x - \frac{1}{6} \cos 3x + C$ |
|---|

190. $\int \sin^2 6x \, dx = \frac{1}{2}x - \frac{1}{24}\sin 12x + C$

$$\begin{aligned}\int \sin^2 6x \, dx &= \int \frac{1}{2} - \frac{1}{2}\cos 12x \, dx = \frac{1}{2}x - \frac{1}{24}\sin 12x + C \\ \cos 2B &\equiv 1 - 2\sin^2 B \\ \cos 2B &\equiv 1 - 2\sin^2 B \\ 2\sin^2 B &\equiv 1 - \cos 2B \\ \sin^2 B &\equiv \frac{1}{2} - \frac{1}{2}\cos 2B\end{aligned}$$

191. $\int \frac{\cos 2x}{1 - \cos^2 2x} \, dx = -\frac{1}{2}\operatorname{cosec} 2x + C$

$$\begin{aligned}\int \frac{\cos 2x}{1 - \cos^2 2x} \, dx &= \int \frac{\cos 2x}{\sin^2 2x} \, dx = \int \frac{\cos 2x}{\sin^2 2x} \cdot \frac{1}{\sin 2x} \, dx \\ &= \int \cot^2 2x \operatorname{cosec} 2x \, dx = -\frac{1}{2}\operatorname{cosec} 2x + C \\ \frac{d}{dx}(\operatorname{cosec} x) &= -\operatorname{cosec} x \cot x\end{aligned}$$

ALTERNATIVE/VARIATION

$$\begin{aligned}\int \frac{\cos 2x}{1 - \cos^2 2x} \, dx &= \int \frac{\cos 2x}{\sin^2 2x} \, dx = \int (\sin 2x)^{-2} \cos 2x \, dx \\ &\dots \text{BY REVERSE CHAIN RULE OR THE SUBSTITUTION METHOD} \\ &= -\frac{1}{2}(\sin 2x)^{-1} + C = -\frac{1}{2}\operatorname{cosec} 2x + C\end{aligned}$$

192. $\int \frac{1}{\cos^2 x \tan^2 x} \, dx = -\frac{1}{3}\cot^3 x + C$

$$\begin{aligned}\int \frac{1}{\cos^2 x \tan^2 x} \, dx &= \int \frac{1}{\cos x} (\tan x)^2 \, dx = \int (\sec x)^{-2} \tan^2 x \, dx \\ &\dots \text{BY REVERSE CHAIN RULE AS } \frac{d}{dx}(\sec x) = \sec x \\ &= -\frac{1}{3}(\sec x)^{-3} + C = -\frac{1}{3}\sec^3 x + C \\ [\text{THE SUBSTITUTION } u = \tan x \text{ WOULD ALSO WORK}]\end{aligned}$$

193. $\int 3\cos^3 3x \, dx = \sin 3x - \frac{1}{3}\sin^3 3x + C$

$$\begin{aligned}\int 3\cos^3 3x \, dx &= \int 3\cos^2 3x \cos 3x \, dx = \int 3\cos^2 3x (-\sin^2 3x) \, dx \\ &= \int 3\cos 3x - 3\cos 3x \sin^2 3x \, dx \\ &\quad \text{REVERSE CHAIN RULE (INSPECTION)} \\ &= \sin 3x - \frac{1}{3}\sin^3 3x + C\end{aligned}$$

ALTERNATIVE BY SUBSTITUTION $u = \sin 3x$

$$\begin{aligned}\int 3\cos^2 3x \, dx &= \int 3\cos^2 3x \left(\frac{du}{3\cos 3x} \right) \\ &= \int \cos^2 3x \, du \\ &= \int 1 - \sin^2 3x \, du \\ &= \int 1 - u^2 \, du \\ &= u - \frac{1}{3}u^3 + C = \sin 3x - \frac{1}{3}\sin^3 3x + C\end{aligned}$$

194. $\int 3\sec^4 x \, dx = 3\tan x + \tan^3 x + C$

$$\begin{aligned} \int 3\sec^3 u \, du &= \int 3\sec u (\sec^2 u) \, du \\ &= \int 3\sec u (1 + \tan^2 u) \, du \\ &\quad \text{reverse chain rule as } u = \tan x \\ &= 3\tan u + \tan^2 u + C \\ \text{ALTERNATIVE BY THE SUBSTITUTION } u = \tan x \\ \int 3\sec^3 u \, du &= \int 3\sec^3 (\frac{du}{dx}) \, du \\ &= \int 3(1 + \tan^2 u) \, du = \int 3(1 + v^2) \, dv \\ &= \int 3 + 3v^2 \, dv = 3v + v^3 + C \\ &= 3\tan u + \tan^3 u + C \end{aligned}$$

195. $\int 2\sin x \cos 3x \, dx = \left[\begin{array}{l} \frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x + C \\ -2\cos^4 x + 3\cos^2 x + C \\ -2\cos^4 x - 3\sin^2 x + C \\ -2\cos^4 x + \frac{3}{2}\cos 2x + C \\ \frac{1}{4}\cos x \cos 3x + \frac{3}{4}\sin x \sin 3x + C \end{array} \right]$

$$\begin{aligned} \int 2\cos 3x \sin x \, dx &\approx \dots \text{BY TRIGONOMETRIC IDENTITIES (SEE BELOW)} \\ &= \int \sin 3x - \sin 2x \, dx = -\frac{1}{3}\cos 3x + \frac{1}{2}\cos 2x + C \\ \sin 4x &= \sin(3x+2x) = \sin 3x \cos 2x + \cos 3x \sin 2x \\ \sin 2x &= \sin(2(x-\pi)) = -\sin 2x \cos \pi - \cos 2x \sin \pi \\ \sin 4x - 2\sin 2x &= 2\cos 3x \sin 2x \end{aligned}$$

ALTERNATIVE BY USING THE TABLE ANGLE IDENTITY FOR COSINE

$$\begin{aligned} \cos 2x &= \cos(2x+\pi) = \cos 2x \cos \pi - \sin 2x \sin \pi \\ &= (2\cos^2 x - 1)\cos \pi - (2\sin x \cos x)\sin \pi \\ &= 2\cos^2 x - \cos \pi - 2\sin x \cos x \\ &= 2\cos^2 x - \cos 2x - (2\cos x)(\cos x) \\ &= 2\cos^2 x - \cos 2x + 2\cos^2 x \\ &= 4\cos^2 x - 3\cos 2x \end{aligned}$$

$$\begin{aligned} \int 2\cos 3x \sin x \, dx &= \int 2\sin(4x - 3\cos 2x) \, dx \\ &= \int 8\sin x \cos 3x - 6\sin x \cos x \, dx \\ &\dots \text{BY EXPANDING AND GROUPING} \\ &= -2\cos^2 x + 3\cos^3 x + C \\ &\quad \text{OR} \\ &= 2\cos^2 x - 3\sin^2 x + C \\ &\dots = \int 8\cos^2 x \sin 2x - 3\sin^2 x \, dx \\ &= -2\cos^2 x + \frac{3}{2}\cos 2x + C \end{aligned}$$

ALTERNATIVE BY DOUBLE INTEGRATION BY PARTS

$$\begin{aligned} \int 2\cos 3x \sin x \, dx &= \dots \text{Integration by parts} \\ &= 2\cos 3x - \int 6\sin x \cos 3x \, dx \\ &\dots \text{by parts again for this integral} \\ &= 2\cos 3x - \int 6\cos 3x \sin x \, dx \\ &\Rightarrow \int 2\cos 3x \sin x \, dx = -2\cos 3x - \left[6\sin x \cos 3x - \int 18\cos^2 x \sin x \, dx \right] \\ &\Rightarrow \int 2\cos 3x \sin x \, dx = -2\cos 3x - 6\sin x \cos 3x + \int 18\cos^2 x \sin x \, dx \\ &\Rightarrow \int 2\cos 3x \sin x \, dx = -2\cos 3x - 6\sin x \cos 3x + \int 2\cos 3x \sin x \, dx \\ &\Rightarrow 2\cos 3x \sin x + 6\sin x \cos 3x = 18 \int 2\cos^2 x \sin x \, dx \\ &\Rightarrow \int 2\cos 3x \sin x \, dx = \frac{1}{18} \cos 3x \sin x + \frac{3}{2} \sin^2 x \cos 3x + C \\ &= \frac{1}{18}(\cos 3x \sin x + \sin^2 x \cos 3x) + \frac{3}{2} \sin^2 x \cos 3x + C \\ &= \frac{1}{18}(2\cos^2 x + \sin^2 x) + \frac{1}{18}[2\sin^2 x \cos 3x] + C \\ &= \frac{1}{9}\cos 2x + \frac{1}{18}[\cos 2x - \cos 6x] + C \\ &= \frac{1}{9}\cos 2x - \frac{1}{18}\cos 6x + C \quad \text{Cancelling} \end{aligned}$$

196. $\int \frac{2\sin x}{\cos x + \sin x} dx = \left[x - \ln|\cos x + \sin x| + C \right. \\ \left. x + \frac{1}{2} \ln|\sec 2x| - \frac{1}{2} \ln|\sec 2x + \tan 2x| + C \right]$

$$\begin{aligned} \int \frac{2\sin x}{\cos x + \sin x} dx &= \int \frac{(\sec x + \tan x) + (\sin x - \cos x)}{\cos x + \sin x} dx \\ &= \int 1 + \frac{\sin x - \cos x}{\cos x + \sin x} dx = \int 1 - \frac{\sin x + \cos x}{\cos x + \sin x} dx \\ &\text{THIS IS OF THE FORM } \int \frac{f(x)}{f(x)+1} dx = \ln|f(x)| + C \\ &= x - \ln|\cos x + \sin x| + C \\ \text{ANALYSIS BY TRIGONOMETRIC IDENTITIES} \\ \int \frac{2\sin x}{\cos x + \sin x} dx &= \int \frac{2\sin x(\sec x - \tan x)}{(\cos x + \sin x)(\sec x - \tan x)} dx \\ &= \int \frac{2\sin x \sec x - 2\sin x \tan x}{\cos x - \sin x} dx = \int \frac{2\sin x}{\cos x} dx \\ &= \int \frac{\sin x}{\cos x} - 2\left(\frac{1}{\cos x}\right) dx \\ &= \int \frac{\sin x}{\cos x} + \frac{\cos x}{\cos x} - 1 dx \\ &= \int \tan x + 1 - \sec^2 x dx \\ &= x + \frac{1}{2} \ln|\sec 2x| - \frac{1}{2} \ln|\sec 2x + \tan 2x| + C \\ &= x + \frac{1}{2} \ln \left| \frac{\sec 2x}{\sec 2x + \tan 2x} \right| + C \\ &= x + \frac{1}{2} \ln \frac{\sec 2x}{\sec 2x \cos 2x + \tan 2x \cos 2x} + C \\ &= x + \frac{1}{2} \ln \left| \frac{1}{\cos^2 x + \sin^2 x + 2\sin x \cos x} \right| + C \\ &= x + \frac{1}{2} \ln \left| \frac{1}{\cos(2x)} \right| + C = x + \frac{1}{2} \ln|\sec(2x)| + C \\ &= x - \ln|\cos x + \sin x| + C \quad (\text{as } \sec x) \end{aligned}$$

197. $\int \frac{4}{2x-1} + \frac{1}{3-4x} dx = 2\ln|2x-1| - \frac{1}{4}\ln|3-4x| + C$

$$\begin{aligned} \int \frac{4}{2x-1} + \frac{1}{3-4x} dx &= \text{BY INSPECTION...} \\ &= 2\ln|2x-1| - \frac{1}{4}\ln|3-4x| + C \end{aligned}$$

198. $\int x^2 \sin 3x dx = -\frac{1}{3}x^2 \cos 3x + \frac{2}{9}x \sin 3x + \frac{2}{27} \cos 3x + C$

$$\begin{aligned} \int x^2 \sin 3x dx &= \dots \text{ INTEGRATION BY PARTS} \\ &= -\frac{1}{3}x^2 \cos 3x - \int -\frac{2}{3}x \sin 3x dx \\ &= -\frac{1}{3}x^2 \cos 3x + \int \frac{2}{3}x \sin 3x dx \\ &\quad \text{INTEGRATION BY SUBSTITUTION} \\ &= -\frac{1}{3}x^2 \cos 3x + \left[\frac{2}{3}x \sin 3x - \int \frac{2}{3}x \sin 3x dx \right] \\ &= -\frac{1}{3}x^2 \cos 3x + \frac{2}{3}x \sin 3x + \frac{2}{9} \cos 3x + C \end{aligned}$$

199. $\int \operatorname{cosec}^2 2x \, dx = -\frac{1}{2} \cot 2x + C$

$$\begin{aligned}\int \operatorname{cosec}^2 2x \, dx &= \dots \text{BY INTEGRATION BY } \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \\ &= -\frac{1}{2} \cot 2x + C\end{aligned}$$

200. $\int \sin^3 2x \cos 2x \, dx = \frac{1}{8} \sin^4 2x + C$

$$\begin{aligned}\int \sin^3 2x \cos 2x \, dx &= \dots \text{BY REVERSE CHAIN RULE / INTEGRATION} \\ &= \frac{1}{8} \sin^4 2x + C \\ [\text{THE SUBSTITUTION: } u &= \sin 2x. \text{ THIS WORKS BECAUSE:}]\end{aligned}$$

201. $\int \cot^2 3x \, dx = -x - \frac{1}{3} \cot 3x + C$

$$\begin{aligned}\int \cot^2 3x \, dx &= \int (\operatorname{cosec}^2 3x - 1) \, dx = \dots \\ &\quad \text{BY INTEGRATION SINCE } \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \\ &= -\frac{1}{3} \cot 3x - x + C\end{aligned}$$

202. $\int \frac{7}{3x} \, dx = \frac{7}{3} \ln|x| + C$

$$\int \frac{7}{3x} \, dx = \int \frac{7}{3} \times \frac{1}{x} \, dx = \dots \text{BY INTEGRATION} \dots = \frac{7}{3} \ln|x| + C$$

203. $\int \frac{\sin x \cos x}{\sqrt{1-\cos 2x}} dx = \left[\frac{1}{2} \sqrt{1-\cos 2x} + C \right]$

$$\begin{aligned} \int \frac{\sin x \cos x}{\sqrt{1-\cos 2x}} dx &= \frac{1}{2} \int \frac{2\sin x \cos x}{1-\cos 2x} dx = \frac{1}{2} \int \frac{(1-\cos 2x)^{-\frac{1}{2}}}{(1-\cos 2x)} dx \\ &\quad \text{BY RATIONALISING THE DENOMINATOR} \\ &= \frac{1}{2} \int (1-\cos 2x)^{\frac{1}{2}} dx \\ &\quad [\text{THE SUBSTITUTIONS } u = 1-\cos 2x, u = 1-\cos 2x, u = \cos 2x + k \text{ work}] \\ \text{ALTERNATIVE BY TRIGONOMETRIC IDENTITIES} \\ \int \frac{\sin x \cos x}{\sqrt{1-\cos 2x}} dx &= \int \frac{\sin x \cos x}{\sqrt{1-(1-2\sin^2 x)^2}} dx \\ &= \int \frac{\sin x \cos x}{\sqrt{2\sin^2 x - 1}} dx = \int \frac{\sin x \cos x}{\sqrt{2(\sin^2 x)}} dx \quad \text{A. IF } \sin x > 0 \\ &= \int \frac{1}{\sqrt{2}} \cos x dx = \frac{1}{\sqrt{2}} \sin x + C \end{aligned}$$

204. $\int \frac{\sin x \cos x}{1+\cos 2x} dx = \begin{bmatrix} -\frac{1}{4} \ln|1+\cos 2x| + C \\ -\frac{1}{2} \ln|\cos x| + C \\ \frac{1}{2} \ln|\sec x| + C \end{bmatrix}$

$$\begin{aligned} \int \frac{\sin x \cos x}{1+\cos 2x} dx &= \int \frac{\frac{1}{2}(2\sin x \cos x)}{1+\cos 2x} dx = \frac{1}{2} \int \frac{\sin 2x}{1+\cos 2x} dx \\ &= \frac{1}{2} \int \frac{-2\sin 2x}{1+\cos 2x} dx \\ &\quad \text{WHICH IS OF THE FORM } \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C \\ &= -\frac{1}{2} \ln|1+\cos 2x| + C \\ &\quad [\text{THE SUBSTITUTIONS } u = 1+\cos 2x, u = \cos 2x, \text{ A. work}] \\ \text{ALTERNATIVE BY TRIG IDENTITIES} \\ \int \frac{\sin x \cos x}{1+\cos 2x} dx &= \int \frac{\sin x \cos x}{1+(2\cos^2 x - 1)} dx = \int \frac{\sin x \cos x}{2\cos^2 x} dx \\ &= \int \frac{\sin x}{2\cos x} dx = -\frac{1}{2} \int \frac{\sin x}{\cos x} dx = -\frac{1}{2} \ln|\cos x| + C = \frac{1}{2} \ln|\sec x| + C \end{aligned}$$

205. $\int 4(3-2x)^5 dx = -\frac{1}{3}(3-2x)^6 + C$

$$\begin{aligned} \int 4(3-2x)^5 dx &= \dots \text{BY INJECTION} \dots = -\frac{1}{3}(3-2x)^6 + C \\ &= -\frac{1}{3}(3-2x)^6 + C \end{aligned}$$

206. $\int \frac{1+\sin x}{\cos x} dx = \left[\ln|\sec^2 x + \sec x \tan x| + C \right] - \ln|1-\sin x| + C$

$$\begin{aligned} \int \frac{1+\sin x}{\cos x} dx &= \int \frac{1}{\cos x} + \frac{\sin x}{\cos x} dx = \int \sec x + \tan x dx \\ &\quad \text{... "Simplifying terms"} \\ &= \ln|\sec x + \tan x| + C \\ &= (\ln|\sec x + \tan x|) + C \\ &\quad \text{or now further} \\ &= \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C \\ &= \ln\left|\frac{1+\sin x}{1-\sin x}\right| + C = \ln\left|\frac{1+\sin x}{(1-\sin x)(1+\sin x)}\right| + C \\ &= \ln\left|\frac{1}{1-\sin x}\right| + C = -\ln|1-\sin x| + C \end{aligned}$$

ALTERNATIVE: BY TRIGONOMETRIC IDENTIFICATIONS

$$\begin{aligned} \int \frac{1+\sin x}{\cos x} dx &= \int \frac{(1+\sin x)(1-\sin x)}{\cos x(1-\sin x)} dx = \int \frac{1-\sin^2 x}{\cos x(1-\sin x)} dx \\ &= \int \frac{\cos^2 x}{\cos x(1-\sin x)} dx = \int \frac{\cos x}{1-\sin x} dx \\ &= -\int \frac{-\cos x}{1-\sin x} dx = -\ln|1-\sin x| + C \\ &\quad [\text{what is } \frac{d}{dx} \ln|1-\sin x|? \quad \left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}] \end{aligned}$$

207. $\int \sin 2x \sec x dx = -2 \cos x + C$

$$\begin{aligned} \int \sin 2x \sec x dx &= \int (2\sin x \cos x) \times \frac{1}{\cos x} dx = \int 2\sin x dx \\ &= -2\cos x + C \end{aligned}$$

$$208. \int \sin 2x \sin x \, dx = \left[\begin{array}{l} \frac{2}{3} \sin^3 x + C \\ \frac{1}{2} \sin x - \frac{1}{6} \sin 3x + C \\ \frac{1}{2} \cos x \sin 2x - \frac{2}{3} \sin x \cos 2x + C \end{array} \right]$$

$\int \sin 2x \sin x \, dx = \int (\sin 2x) \sin x \, dx = \int 2\sin^2 x \cos x \, dx$

... BY DIFFERENTIATION OF THE
SUBSTITUTION USING ...
 $= \frac{2}{3} \sin^3 x + C$

ALTERNATIVE BY COMPOUND ANGLE IDENTITIES

$\cos 2x \equiv \cos(2x-2) = \cos 2x - \sin 2x \sin 2x \quad \text{SUBTRACT}$
 $(\cos 2x)^2 \equiv \cos^2(2x-2) = \cos^2 2x + \sin^2 2x \quad \text{SUBTRACT}$

$\Rightarrow \cos 2x - \cos 2x \equiv -2 \sin 2x \sin x$
 $\Rightarrow \sin 2x \sin x \equiv -\frac{1}{2} \cos 2x + \frac{1}{2} \cos 2x$
 $\Rightarrow \int \sin 2x \sin x \, dx = \int -\frac{1}{2} \cos 2x + \frac{1}{2} \cos 2x \, dx = \frac{1}{2} \sin 2x - \frac{1}{4} \sin 2x + C$

ALTERNATIVE BY INTEGRATION BY PARTS

$\int \sin 2x \sin x \, dx = -\cos 2x \sin x - \int -2\sin 2x \cos x \, dx$
 $= -\cos 2x \sin x + \int 2\sin 2x \cos x \, dx$
 BY PARTS AGAIN ON THIS SECTION
 $\begin{array}{c} \sin 2x \\ \downarrow \\ \cos 2x \end{array} - 4\sin 2x$

$\int \sin 2x \sin x \, dx = -\cos 2x \sin x + \left[2\sin 2x \cos x - \int -4\sin 2x \cos x \, dx \right]$
 $\int \sin 2x \sin x \, dx = -\cos 2x \sin x + 2\sin 2x \cos x + 4 \int \sin 2x \cos x \, dx$
 $\cos 2x \sin x - 2\sin 2x \cos x = 3 \int \sin 2x \cos x \, dx$
 $\int \sin 2x \sin x \, dx = \frac{1}{3} \cos 2x \sin x - \frac{2}{3} \sin 2x \cos x + C$

$$209. \int x \cos^2 x \, dx = \frac{1}{4}x^2 + \frac{1}{4}x \sin 2x + \frac{1}{8} \cos 2x + C$$

$\int x \cos^2 x \, dx = \int x \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) \, dx = \int \frac{x}{2} \, dx + \int x \cos 2x \, dx$
 BY PARTS
 $\begin{array}{c} \frac{x^2}{2} \\ \downarrow \\ \cos 2x \end{array} \quad \begin{array}{c} \frac{1}{2} \\ \downarrow \\ x \end{array}$

$= \frac{x^2}{2} + \frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \, dx$
 $= \frac{x^2}{2} + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C$

210. $\int \sin(x+1)^{\frac{1}{3}} dx = 3 \left[2 - (x+1)^{\frac{2}{3}} \right] \cos(x+1)^{\frac{1}{3}} + 6(x+1)^{\frac{1}{3}} \sin(x+1)^{\frac{1}{3}} + C$

$\int \sin(3u)^{\frac{1}{3}} du = \dots$ BY SUBSTITUTION FIRST...

$$= \int \sin(u) (3u^2 du) = \int 3u^2 \sin(u) du$$

BY PARTS NEXT

| | |
|--------|-----------|
| $3u^2$ | $\sin(u)$ |
| $6u$ | $\cos(u)$ |

$$= -3u \cos(u) - \int -\cos(u) du$$

$$= -3u \cos(u) + \int \cos(u) du$$

BY PARTS FORMULA FOR THE INTEGRAL

| | |
|-----------|--------|
| $\sin(u)$ | 6 |
| $\cos(u)$ | $3u^2$ |

$$= -3u \cos(u) + [6 \sin(u) - \int 6 \sin(u) du]$$

$$= -3u \cos(u) + 6u \sin(u) + 6 \sin(u) + C$$

$$= (6 - 3u^2) \cos(u) + 6u \sin(u) + C$$

$$= [6 - 3(x+1)^{\frac{2}{3}}] \cos(x+1)^{\frac{1}{3}} + 6(x+1)^{\frac{1}{3}} \sin(x+1)^{\frac{1}{3}} + C$$

$$= 3[2 - (x+1)^{\frac{2}{3}}] \cos(x+1)^{\frac{1}{3}} + 6(x+1)^{\frac{1}{3}} \sin(x+1)^{\frac{1}{3}} + C$$

$u = (x+1)^{\frac{1}{3}}$
 $u' = x+1$
 $du = u^2 - 1$
 $du = 3u^2$
 $du = 3u^2 du$

211. $\int \frac{1 - \ln x}{x^2} dx = \frac{\ln|x|}{x} + C$

$\int \frac{1 - \ln x}{x^2} dx = \int \frac{1}{x^2} - \frac{\ln x}{x^2} dx = \int \frac{1}{x^2} dx - \frac{\ln x}{x^2} dx$ BY PARTS

| | |
|---------|-----------------|
| $\ln x$ | $\frac{1}{x}$ |
| x^2 | $\frac{1}{x^2}$ |

$$= -\frac{1}{x} + \left[x^2 \ln x - \int \frac{1}{x^2} 2x dx \right]$$

$$= -\frac{1}{x} + \frac{|\ln x|}{2} - \int \frac{1}{x^2} dx$$

$$= \frac{\sqrt{x}}{x} + \frac{|\ln x|}{2} + \frac{1}{x^2} + C$$

$$= \underline{\underline{|\ln x| + C}}$$

ALTERNATIVE, STARTING WITH + SUBSTITUTION

| |
|------------------------|
| $u = \ln x$ |
| $du = \frac{1}{x} dx$ |
| $dx = x du$ |
| $x = e^u$ |
| $\frac{1}{x} = e^{-u}$ |

$$\int \frac{1 - \ln x}{x^2} dx = \int \frac{1-u}{x^2} x du$$

$$= \int \frac{1-u}{x^2} du$$

$$= \int (1-u) e^{-u} du$$

BY PARTS

| | |
|----------|----------|
| $1-u$ | -1 |
| e^{-u} | e^{-u} |

$$= -(1-u)e^{-u} - \int e^{-u} du$$

$$= (u-1)e^{-u} + e^{-u} + C$$

$$= ue^{-u} - e^{-u} + C$$

$$= (\ln x) \cdot \frac{1}{x} + C$$

$$= \underline{\underline{\frac{|\ln x|}{x} + C}}$$

212. $\int 4x e^{-\frac{2}{3}x} dx = -3(2x+3)e^{-\frac{2}{3}x} + C$

$\int 4x e^{-\frac{2}{3}x} dx = \dots$ INTEGRATION BY PARTS ...

| | |
|------|---|
| $4x$ | $\frac{4}{\frac{2}{3}e^{\frac{2}{3}x}}$ |
| 4 | $\frac{1}{e^{\frac{2}{3}x}}$ |

$$= -4x e^{-\frac{2}{3}x} - \int -4e^{-\frac{2}{3}x} dx$$

$$= -4x e^{-\frac{2}{3}x} + \int 4e^{-\frac{2}{3}x} dx$$

$$= -4x e^{-\frac{2}{3}x} + 9 e^{-\frac{2}{3}x} + C$$

$$= -3(2x+3)e^{-\frac{2}{3}x} + C$$

213. $\int (e^x + x)^2 dx = \frac{1}{2}e^{2x} + 2xe^x - 2e^x + \frac{1}{3}x^3 + C$

$$\begin{aligned}\int (e^x + x)^2 dx &= \int e^{2x} + 2x^2 + x^2 dx \quad \text{BY PARTS} \\ &= \frac{1}{2}e^{2x} + \left[2x^2 - \int 2x dx \right] + \frac{1}{3}x^3 \\ &= \frac{1}{2}e^{2x} + 2x^2 - 2x^2 + \frac{1}{3}x^3 + C\end{aligned}$$

214. $\int \frac{e^{4x} - e^{-x}}{e^{2x}} dx = \frac{1}{2}e^{2x} + \frac{1}{3}e^{-3x} + C$

$$\begin{aligned}\int \frac{e^{4x} - e^{-x}}{e^{2x}} dx &= \int \frac{e^{2x}}{e^{2x}} - \frac{e^{-x}}{e^{2x}} dx = \int e^{2x} - e^{-3x} dx \\ &= \frac{1}{2}e^{2x} + \frac{1}{3}e^{-3x} + C\end{aligned}$$

215. $\int \frac{e^{\ln x}}{x} dx = x + C$

$$\int \frac{e^{\ln x}}{x} dx = \int \frac{x}{x} dx = \int 1 dx = x + C$$

216. $\int \frac{1}{2(3x+1)^4} dx = -\frac{1}{18(3x+1)^3} + C$

$$\begin{aligned}\int \frac{1}{2(3x+1)^4} dx &= \int \frac{1}{2}(3x+1)^{-4} dx = \dots \text{BY INTEGRATION} \\ &= \frac{1}{-7}(3x+1)^{-3} + C = -\frac{1}{18}(3x+1)^{-3} + C \\ &= -\frac{1}{18(3x+1)^3} + C\end{aligned}$$

217. $\int \frac{\cos(\ln x)}{x} dx = \sin(\ln x) + C$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}\int \frac{\cos(\ln x)}{x} dx &= \int \cos(u) (ze^u du) \\&= \int \cos(u) du \\&= \sin(u) + C \\&= \sin(\ln x) + C\end{aligned}$$

$u = \ln x$
 $\frac{du}{dx} = \frac{1}{x}$
 $dx = x du$

218. $\int \frac{4xe^{2x^2}}{\sqrt{1+2e^{2x^2}}} dx = \sqrt{1+2e^{2x^2}} + C$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}\int \frac{4xe^{2x^2}}{\sqrt{1+2e^{2x^2}}} dx &= \int 4xe^{2x^2} (1+2e^{2x^2})^{-\frac{1}{2}} dx \\&\quad \text{BY THESE OTHER RULE (Quotient)} \\&= (1+2e^{2x^2})^{\frac{1}{2}} + C\end{aligned}$$

$u = \sqrt{1+2e^{2x^2}}$
 $u^2 = 1+2e^{2x^2}$
 $2u \frac{du}{dx} = 4e^{2x^2} dx$
 $du = \frac{2e^{2x^2}}{u} dx$

[THE SUBSTITUTIONS $u = 1+2e^{2x^2}$ OR $u = 2e^{2x^2}$ ALSO WORK WELL]

219. $\int \frac{4}{3(2x+1)} dx = \frac{2}{3} \ln|2x+1| + C$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}\int \frac{4}{3(2x+1)} dx &= \int \frac{4}{3} \left(\frac{1}{2x+1} \right) dx = \frac{4}{3} \times \frac{1}{2} \ln|2x+1| + C \\&= \frac{2}{3} \ln|2x+1| + C\end{aligned}$$

220. $\int \frac{1}{\cos^2 x \tan x} dx = \left[\begin{array}{l} \ln|\tan x| + C \\ -\ln|\cot 2x + \operatorname{cosec} 2x| + C \end{array} \right]$

$$\begin{aligned} \int \frac{1}{\cot x \tan x} dx &= \int \sec x \times \frac{1}{\tan x} dx \dots \text{BY SIMPLE SUBSTITUTION...} \\ &= \ln|\tan x| + C \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned} \int \frac{1}{\cot x \tan x} dx &= \int \frac{1}{\sec^2 x} (\sec x dx) \\ &= \int \frac{1}{\sec x} dx = \ln|\sec x| + C \\ &= \ln|\tan x| + C \end{aligned}$$

ALTERNATIVE BY TRIGONOMETRIC IDENTITIES

$$\begin{aligned} \int \frac{1}{\cot x \tan x} dx &= \int \frac{1}{\cot x \tan x} dx = \int \frac{1}{\sin x \cos x} dx \\ &= \int \frac{\sin x}{\sin^2 x} dx = \int \frac{\sin x}{\sin^2 x} dx = \int 2\sin x \cos x dx \\ &\quad \text{WHICH IS A STANDARD FORM} \\ &= -\ln|\sin x| + \operatorname{cosec} x + C \end{aligned}$$

221. $\int \frac{\sin 2x}{1+\cos x} dx = -2\cos x + 2\ln|1+\cos x| + C$

$$\begin{aligned} \int \frac{\sin 2x}{1+\cos x} dx &= \int \frac{2\sin x \cos x}{1+\cos x} dx = \dots \text{BY SUBSTITUTION} \\ &= \int \frac{2\sin x \cos x}{\frac{1+u}{2}} du = \int \frac{2\sin x}{u} du \\ &= \int \frac{-2(u-1)}{u} du = \int \frac{2u}{u-1} du = \int \frac{2u}{u-1} du \\ &= \int 2 + \frac{2}{u-1} du = -2u + 2\ln|u| + C \\ &= -2\cos x + 2\ln|1+\cos x| + C \\ &= -2\cos x + 2\ln|1+\cos x| + C \end{aligned}$$

222. $\int \sin^3 x dx = \frac{1}{3}\cos^2 x - \cos x + C$

$$\begin{aligned} \int \sin^3 x dx &= \int \sin x \sin^2 x dx = \int \sin x (-\cos^2 x) dx \\ &= \int \sin x - \sin x \cos^2 x dx = \dots \\ &\quad \text{BY REVERSE CHAIN RULE (REDUCTION) IN THE 2nd TERM} \\ &\dots = -\cos x + \frac{1}{3}\cos^3 x + C \end{aligned}$$

ALTERNATIVE BY THE SUBSTITUTION $u = \cos x$

$$\begin{aligned} \int \sin^3 x dx &= \int \sin^3 x \left(\frac{du}{dx} \right) = \int -\sin^2 x du \\ &= \int -(1-\cos^2 x) du = \int (1-u^2) du \\ &= \int u^2 - 1 du = \frac{1}{3}u^3 - u + C \\ &= \frac{1}{3}\cos^3 x - \cos x + C \end{aligned}$$

223. $\int \frac{1}{3} \sin 2x - \frac{1}{2} \cos 3x \, dx = -\frac{1}{6} \cos 2x - \frac{1}{6} \sin 3x + C$

$\int \frac{1}{3} \sin 2x - \frac{1}{2} \cos 3x \, dx = \dots$ STANDARD INTEGRALS
 $= -\frac{1}{6} \cos 2x - \frac{1}{6} \sin 3x + C$

224. $\int \frac{3x}{\sqrt{4-2x^2}} \, dx = -\frac{3}{2}(4-2x^2)^{\frac{1}{2}} + C$

$\int \frac{3x}{\sqrt{4-2x^2}} \, dx = \int 3x(4-2x^2)^{\frac{1}{2}} \, dx = \dots$ BY INVERSE CHAIN RULE REARRANGED
 $= \frac{3}{2}(4-2x^2)^{\frac{1}{2}} + C$

ALTERNATIVE BY SUBSTITUTION
 $\int \frac{3x}{\sqrt{4-2x^2}} \, dx = \int \frac{3x}{\sqrt{-\frac{2}{4}u^2}} \, du = \int -\frac{3}{2}u \, du$
 $= -\frac{3}{2}u^2 + C = -\frac{3}{2}(4-2x^2)^{\frac{1}{2}} + C$

THE SUBSTITUTION $u=4-2x^2$ ALSO WORKS WELL

225. $\int \frac{1}{\sin x \cos^2 x} \, dx = \begin{cases} \ln|\tan(\frac{1}{2}x)| + \sec x + C \\ \sec x - \ln|\cosec x + \cot x| + C \\ \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + \sec x + C \end{cases}$

$\int \frac{1}{\sin x \cos^2 x} \, dx = \int \frac{\cos^2 x \sin x}{\sin x \cos^2 x} \, dx = \int \frac{\cos^2 x}{\sin x} + \frac{\sin x}{\cos^2 x} \, dx$
 $= \int \cosec x + \operatorname{tnx} \sec x = \dots$ STANDARD RESULTS
 $= \ln|\operatorname{tnx} \frac{x}{2}| + \sec x + C$

OR
 $\sec x - \ln|\cosec x + \cot x| + C$

ALTERNATIVE BY SUBSTITUTION & PARTIAL FRACTIONS

$\int \frac{1}{\sin x \cos^2 x} \, dx = \int \frac{1}{\sin x \cos^2 x} (-\frac{du}{\sin x})$
 $= \int \frac{1}{u^2 \cos^2 x} \, du = \int \frac{1}{u^2 (1-u^2)} \, du$
 $= \int \frac{1}{u^2 (1-u^2)} \, du = \int \frac{1}{(u^2+1)(u^2-1)} \, du$

$\frac{1}{(u^2+1)(u^2-1)} \equiv \frac{A}{u^2+1} + \frac{B}{u-1} + \frac{C}{u+1} + \frac{D}{u^2-1}$
 $\boxed{1 \equiv A(u-1)(u+1) + B(u^2+1) + C(u^2-1) + D(u+1)(u-1)}$

IF $k=0$ \bullet $u=u-1$ \bullet $u=u+1$ \bullet $u=u^2-1$ \bullet $u=u^2+1$ \Rightarrow $(u-1)(u+1) \Rightarrow 1=6A+12B+4C+2D$
 $B=-D$ $1=2B$ $1=2C$ $C=\frac{1}{2}$ $A=6-2-2=2$
 $D=0$

$= \int \frac{\frac{2}{u^2-1} - \frac{2}{u^2+1} - \frac{1}{u-1} + \frac{1}{u+1}}{u^2-1} \, du = \frac{1}{2} \int \frac{2}{u^2-1} \, du + \frac{1}{u^2+1} \, du - \frac{1}{u-1} \, du + \frac{1}{u+1} \, du$
 $= \frac{1}{2} \ln \left| \frac{u^2-1}{u^2+1} \right| + \frac{1}{u^2+1} + \frac{1}{2} \ln \left| \frac{u^2-1}{u^2+1} \right| + \sec x + C$
 $= \frac{1}{2} \ln \left| \frac{1-\cos^2 x}{1+\cos^2 x} \right| + \sec x + C = \frac{1}{2} \ln \left| \tan^2 \frac{x}{2} \right| + \sec x + C$
 $= \frac{1}{2} \ln \left| \operatorname{tnx} \frac{x}{2} \right| + \sec x + C = \boxed{\ln \left| \operatorname{tnx} \frac{x}{2} \right| + \sec x + C}$ AS PER Q

226. $\int \frac{3x}{4-2x^2} dx = -\frac{3}{4} \ln|4-2x^2| + C$

$$\begin{aligned}\int \frac{3x}{4-2x^2} dx &= -\frac{3}{4} \int \frac{-4x}{4-2x^2} dx = \dots \int \frac{f(u)}{g(u)} du = \ln|f(u)| + C \\ &= -\frac{3}{4} \ln|4-2x^2| + C\end{aligned}$$

THE SUBSTITUTION $u=4-2x^2$ ALSO WORKS WELL

227. $\int \frac{1-4x}{x(4x-\ln x)^{\frac{3}{2}}} dx = \frac{2}{\sqrt{4x-\ln x}} + C$

$$\begin{aligned}\int \frac{1-4x}{x(4x-\ln x)^{\frac{3}{2}}} dx &= \int \left(\frac{1-4x}{x}\right) (4x-\ln x)^{-\frac{3}{2}} dx = \int -(4-\frac{1}{x})(4x-\ln x)^{-\frac{3}{2}} dx \\ &\quad \text{BY INVERSE CHOOSE ONE / ELIMINATE} \\ &= \frac{1}{x} (4x-\ln x)^{-\frac{1}{2}} + C = \frac{2}{\sqrt{4x-\ln x}} + C\end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}\int \frac{1-4x}{x(4x-\ln x)^{\frac{3}{2}}} dx &= \int \frac{1-4x}{x(4x-\ln x)^{\frac{3}{2}}} \left(\frac{2}{4x-1} dx\right) \\ &= \int -\frac{1}{4x-1} dx = \int -u^{-\frac{1}{2}} du = 2u^{\frac{1}{2}} + C \\ &= 2(4x-\ln x)^{\frac{1}{2}} + C = \frac{2}{\sqrt{4x-\ln x}} + C\end{aligned}$$

THE SUBSTITUTION $u=\sqrt{4x-\ln x}$ ALSO WORKS WELL

228. $\int \frac{\sin x}{\cos^4 x} dx = \frac{1}{3} \sec^3 x + C$

$$\begin{aligned}\int \frac{\sin x}{\cos^2 x} dx &= \int (\cos x)^{-2} \sin x dx = \dots \text{BY INVERSE CHOOSE} \\ &= \frac{1}{2} (\cos x)^{-1} + C = \frac{1}{2 \cos x} + C = \frac{1}{2} \sec^2 x + C\end{aligned}$$

ALTERNATIVE VARIATION OF DIRECT CHOOSE RULE

$$\begin{aligned}\int \frac{\sin x}{\cos^2 x} dx &= \int \frac{\sin x}{\cos x \cos x} dx = \int \frac{\sin x}{\cos x} \frac{1}{\cos x} dx \\ &= \int \sec^2 x (\operatorname{secatan}) dx = \int \sec^2 x dx = \frac{1}{2} \sec^2 x + C\end{aligned}$$

THE SUBSTITUTIONS $u=\cos x$ OR $u=\sec x$ ALSO WORK WELL

229. $\int \frac{\operatorname{cosec}^2 x}{1+\cot x} dx = -\ln|1+\cot x| + C$

$$\begin{aligned}\int \frac{\operatorname{cosec}^2 x}{1+\cot x} dx &= -\int \frac{\operatorname{cosec}^2 x}{1+\cot x} dx = \dots \int \frac{f(u)}{g(u)} du = \ln|f(u)| + C \\ &= -\ln|1+\cot x| + C\end{aligned}$$

THE SUBSTITUTION $u=1+\cot x$ ALSO WORKS WELL

230. $\int x^2 \cos\left(\frac{1}{4}x\right) dx = 4x^2 \sin\left(\frac{1}{4}x\right) + 32 \cos\left(\frac{1}{4}x\right) - 128 \sin\left(\frac{1}{4}x\right) + C$

$$\begin{aligned}\int 2^2 \cos\left(\frac{1}{4}x\right) dx &= \text{INTEGRATION BY PARTS} \\ &= 4x \sin\left(\frac{1}{4}x\right) - \int 4x \sin\left(\frac{1}{4}x\right) dx \\ &\quad \text{INTEGRATION BY PARTS AGAIN} \\ &= 4x \sin\left(\frac{1}{4}x\right) - \left[-32 \cos\left(\frac{1}{4}x\right) - \int -32 \cos\left(\frac{1}{4}x\right) dx \right] \\ &= 4x \sin\left(\frac{1}{4}x\right) + 32 \cos\left(\frac{1}{4}x\right) - \int 32 \cos\left(\frac{1}{4}x\right) dx \\ &= 4x \sin\left(\frac{1}{4}x\right) + 32 \cos\left(\frac{1}{4}x\right) - 128 \sin\left(\frac{1}{4}x\right) + C\end{aligned}$$

231. $\int \frac{3}{(\sqrt{x}-2)(\sqrt{x}+1)} dx = 4 \ln|\sqrt{x}-2| + 2 \ln(\sqrt{x}+1) + C$

$$\begin{aligned}\int \frac{3}{(\sqrt{x}-2)(\sqrt{x}+1)} dx &= \dots \text{BY SUBSTITUTION} \\ &\Rightarrow \int \frac{3}{(u-2)(u+1)} du = \int \frac{6u}{(u-2)(u+1)} du \\ &\quad \text{BY PARTIAL FRACTIONS} \\ &\quad \frac{6u}{(u-2)(u+1)} = \frac{A}{u-2} + \frac{B}{u+1} \\ &\quad u = A(u+1) + B(u-2) \\ &\quad \text{IF } u=1 \Rightarrow 6 = 3A \quad \text{IF } u=2 \Rightarrow 12 = 3A \\ &\quad A=2 \quad \Rightarrow \quad B=4 \\ &= \int \frac{6}{u-2} + \frac{24}{u+1} du = 6 \ln|u-2| + 24 \ln|u+1| + C \\ &= 4 \ln|\sqrt{x}-2| + 24 \ln(\sqrt{x}+1) + C\end{aligned}$$

232. $\int \frac{1}{(\sqrt{x}-2)(\sqrt{x}+2)} dx = \ln|x-4| + C$

$$\int \frac{1}{(\sqrt{x}-2)(\sqrt{x}+2)} dx = \int \frac{1}{x-4} dx = \ln|x-4| + C$$

STANDARD ANTIDERIVATIVE

233. $\int \frac{4-3x}{2x+1} dx = \frac{11}{4} \ln|2x+1| - \frac{3}{2}x + C$

$$\begin{aligned}\int \frac{4-3x}{2x+1} dx &= \text{MANIPULATING & SPLITTING THE FRACTION} \\ &\Rightarrow \int \frac{4-\frac{3}{2}(2x+1)+\frac{3}{2}}{(2x+1)} dx = \int \frac{\frac{11}{2}-\frac{3}{2}(2x+1)}{2x+1} dx = \int \frac{\frac{11}{2}}{2x+1} - \frac{\frac{3}{2}(2x+1)}{2x+1} dx \\ &= \frac{11}{4} \ln|2x+1| - \frac{3}{2}x + C \\ &\quad \text{ALTERNATIVE BY SUBSTITUTION} \\ &\quad \int \frac{4-3x}{2x+1} dx = \int \frac{4-3x}{u} \left(\frac{du}{2} \right) = \int \frac{4-3x}{2u} du \\ &\quad \int \frac{8-6x}{4u} du = \int \frac{11-3x}{4u} du = \int \frac{11}{4u} - \frac{3}{4} du \\ &\quad = \frac{11}{4} \ln|u| - \frac{3}{4}u + C = \frac{11}{4} \ln|2x+1| - \frac{3}{4}(2x+1) + C \\ &\quad = \frac{11}{4} \ln|2x+1| - \frac{3}{2}x - \frac{3}{4} \text{ AS WORK}\end{aligned}$$

234. $\int \left(1 + \frac{1}{x}\right) \sqrt{x} \, dx = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

$$\begin{aligned} \int \sqrt{x} \left(1 + \frac{1}{x}\right) dx &= \int x^{\frac{1}{2}} \left(1 + \frac{1}{x}\right) dx = \int x^{\frac{1}{2}} + x^{-\frac{1}{2}} dx \\ &= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C \end{aligned}$$

235. $\int \frac{\sec x}{\cos x - \sin x} \, dx = \ln|1 - \tan x| + C$

$$\begin{aligned} \int \frac{\sec x}{\cos x - \sin x} dx &= \int \frac{\sec x \sec x}{\cos x - \sin x} dx \\ &= \int \frac{\sec^2 x}{1 - \tan x} dx \\ &\dots \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C \\ &= \ln|1 - \tan x| + C \end{aligned}$$

236. $\int \frac{x^2 - 2}{x^2 - 1} \, dx = x + \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right| + C$

$$\begin{aligned} \int \frac{\frac{d}{dx}(x^2 - 2)}{\frac{d}{dx}(x^2 - 1)} dx &= \int \frac{\frac{d(x^2 - 1)}{dx} - 1}{x^2 - 1} dx = \int 1 - \frac{1}{x^2 - 1} dx \\ [\text{Integrate!}] &= \int 1 - \frac{1}{x^2 - 1} dx = \dots \text{BY PARTIAL FRACTION BY INSPECTION} \\ &= \int \left(1 - \left(\frac{1}{x-1} + \frac{1}{x+1}\right)\right) dx = \int 1 + \frac{1}{x+1} - \frac{1}{x-1} dx \\ &= x + \frac{1}{2} \ln|x+1| - \frac{1}{2} \ln|x-1| + C = x + \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right| + C \end{aligned}$$

237. $\int \frac{1}{x^3 - x^2} \, dx = \frac{1}{x} + \ln \left| \frac{x-1}{x} \right| + C$

$$\begin{aligned} \int \frac{1}{x^3 - x^2} dx &= \int \frac{1}{x^2(x-1)} dx = \dots \text{BY PARTIAL FRACTIONS} \\ \frac{1}{x^2(x-1)} &= \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1} \\ 1 &= Ax(x-1) + B(x-1) + C(x^2) \\ \bullet \text{ If } x=1 &\quad \bullet \text{ If } x=0 &\quad \bullet \text{ If } x=2 \\ 1=0 &\quad A=-B &\quad 1=4+2C \\ A=-B &\quad A=0 &\quad 1=4+2C \\ A=0 &\quad A=0 &\quad C=-1 \end{aligned}$$

$$= \int \frac{1}{x^2} - \frac{1}{x} \, dx = -\frac{1}{x} - \ln|x| + C = \frac{1}{x} + \ln \left| \frac{x-1}{x} \right| + C$$

238. $\int \frac{6x^2}{2x^2-1} dx = 2x^{\frac{3}{2}} + \ln|2x^{\frac{3}{2}} - 1| + C$

$$\begin{aligned} \int \frac{6x^2}{2x^2-1} dx &= \int \frac{3x^2(2x^{\frac{3}{2}}) + 3x^2}{(2x^2-1)} dx = \int 3x^2 + \frac{3x^2}{2x^2-1} dx \\ &= 2x^{\frac{3}{2}} + \ln|2x^{\frac{3}{2}} - 1| + C \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned} \int \frac{6x^2}{2x^2-1} dx &= \int \frac{6x^2}{2x^2} dx = \int \frac{3x^2}{x^2} dx \\ &= \int \frac{3x^2}{x^2} dx = \int 3 dx = 3x + C \\ &\approx (2x^{\frac{3}{2}}) + \ln|2x^{\frac{3}{2}} - 1| + C = 2x^{\frac{3}{2}} + \ln|2x^{\frac{3}{2}} - 1| + C \end{aligned}$$

AS ABOVE

239. $\int \frac{1}{1+\sqrt{x-1}} dx = 2\sqrt{x-1} - 2\ln(1+\sqrt{x-1}) + C$

$$\begin{aligned} \int \frac{1}{1+\sqrt{x-1}} dx &= \dots \text{BY SUBSTITUTION} \dots \\ &= \int \frac{1}{u} [2(u-1) du] = \int \frac{2u-2}{u} du \\ &= \int 2 - \frac{2}{u} du = 2u - 2\ln|u| + C \\ &= 2(\sqrt{x-1}) - 2\ln(\sqrt{x-1}) + C \\ &= 2\sqrt{x-1} - 2\ln(1+\sqrt{x-1}) + C \end{aligned}$$

ALTERNATIVE SUBSTITUTION

$$\begin{aligned} \int \frac{1}{1+\sqrt{x-1}} dx &= \int \frac{1}{1+u} (2u du) = \int \frac{2u}{u+1} du \\ &= \int \frac{2(u+1)-2}{u+1} du = \int \frac{2}{u+1} du \\ &= 2u - 2\ln|u+1| + C = 2\sqrt{x-1} - 2\ln(1+\sqrt{x-1}) + C \end{aligned}$$

240. $\int \frac{x+1}{x-5} dx = x + 6\ln|x-5| + C$

$$\begin{aligned} \int \frac{x+1}{x-5} dx &= \text{BY MANIPULATION & SPLIT} \\ &= \int \left(\frac{(x-5)+6}{x-5} \right) dx = \int 1 + \frac{6}{x-5} dx \\ &= x + 6\ln|x-5| + C \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION $u = x-5$

$$\begin{aligned} \int \frac{x+1}{x-5} dx &= \int \frac{x+1}{u} du = \int \frac{(u+5)+6}{u} du \\ &= \int \frac{u+6}{u} du = \int 1 + \frac{6}{u} du \\ &= u + 6\ln|u| + C \\ &= (x-5) + 6\ln|x-5| + C \end{aligned}$$

241. $\int \frac{(x+2)^2}{3x} dx = \frac{1}{6}x^2 + \frac{4}{3}x + \frac{4}{3}\ln|x| + C$

$$\int \frac{(x+2)^2}{3x} dx = \int \frac{x^2+4x+4}{3x} dx = \int \frac{x^2}{3x} dx + \frac{4}{3} \int \frac{x}{3x} dx \\ \dots \text{STANDARD ANTIDERIVATIVES...} \\ = \frac{1}{6}x^2 + \frac{4}{3}x + \frac{4}{3}\ln|x| + C$$

242. $\int 4e^{-2x} - \frac{1}{3}\sin 3x dx = -2e^{-2x} + \frac{1}{9}\cos 3x + C$

$$\int 4e^{-2x} - \frac{1}{3}\sin 3x dx = \dots \text{STANDARD ANTIDERIVATIVES...} \\ = -2e^{-2x} + \frac{1}{9}\cos 3x + C$$

243. $\int 2x\sec^2 2x dx = \begin{cases} x\tan x + \frac{1}{2}\ln|\cos 2x| + C \\ x\tan x - \frac{1}{2}\ln|\sec 2x| + C \end{cases}$

$$\int 2x\sec^2 2x dx = \dots \text{INTEGRATION BY PARTS...} \\ = x\tan 2x - \int \tan 2x dx \\ \uparrow \\ \text{STANDARD RESULT WHICH CAN BE FOUND:} \\ \int \tan u du = \ln|\sec u| + C = -\ln|\cos u| + C \\ = x\tan 2x + \frac{1}{2}\ln|\cos 2x| + C = x\tan 2x - \frac{1}{2}\ln|\sec 2x| + C$$

244. $\int \cos x \sin^8 x dx = \frac{1}{9}\sin^9 x + C$

$$\int \cos x \sin^8 x dx = \dots \text{BY REVERSE CHAIN RULE (DECOMPOSITION)} \\ = \frac{1}{9}\sin^9 x + C \\ [\text{THE SUBSTITUTION } u=\sin x \text{ ALSO WORKS HERE!}]$$

245. $\int \frac{\sin^5 x}{\cos^7 x} dx = \frac{1}{6} \tan^6 x + C$

$$\begin{aligned}\int \frac{\sin^2 x}{\cos^2 x} dx &= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx = \int \tan x \sec x dx \\ &= \dots \text{ BY REVERSE CHAIN RULE (DEFINITION)} \\ &= \frac{1}{6} \tan^2 x + C\end{aligned}$$

[THE SUBSTITUTION $u = \tan x$, ALSO WORKS WELL]

246. $\int \tan 3x dx = \left[\begin{array}{l} \frac{1}{3} \ln |\sec x| + C \\ -\frac{1}{3} \ln |\cos x| + C \end{array} \right]$

$$\begin{aligned}\int \tan 3x dx &= \int \frac{\sin 3x}{\cos 3x} dx = -\frac{1}{3} \int \frac{3 \sin 3x}{\cos 3x} dx \\ &\quad \uparrow \int f(u) du = \ln |f(u)| + C \\ &= -\frac{1}{3} \ln |\cos 3x| + C = \frac{1}{3} \ln |\sec 3x| + C\end{aligned}$$

[THE SUBSTITUTION $u = \cos 3x$, OR $u = \sec 3x$ ALSO WORKS.]

247. $\int \frac{1}{\sec x - 1} dx = -x - \cot x - \operatorname{cosec} x + C$

$$\begin{aligned}\int \frac{1}{\sec x - 1} dx &= \int \frac{\sec x + 1}{(\sec x - 1)(\sec x + 1)} dx = \int \frac{\sec x + 1}{\sec^2 x} dx \\ &= \int \frac{\sec x}{\tan^2 x} dx \quad [+ \tan x = \sec x] \\ &= \int \frac{\sec x}{\tan^2 x} + \frac{1}{\tan^2 x} dx = \int \sec x dx + \tan x dx \\ &= \int \frac{1}{\tan x} \frac{\sec x}{\sec x} dx + (\sec x - 1) dx \\ &= \int \frac{\sec x}{\sin x} dx + \sec x - 1 dx \\ &= \int \frac{\sec x}{\sin x} dx + \sec x - x + C \\ &= -\operatorname{cosec} x - \cot x - x + C \\ &\quad \boxed{d(\cot x) = -\operatorname{cosec} x \cdot \cot x + d(\cot x) = -\operatorname{cosec}^2 x}\end{aligned}$$

248. $\int \frac{\sin^2 x}{\cos^4 x} dx = \frac{1}{3} \tan^3 x + C$

$$\begin{aligned}\int \frac{\sin^2 x}{\cos^2 x} dx &= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx = \int \tan x \sec x dx \\ &\quad \text{BY REVERSE CHAIN RULE SINCE } \frac{d(\tan x)}{dx} = \sec^2 x \\ &= \frac{1}{3} \tan^3 x + C\end{aligned}$$

[THE SUBSTITUTION $u = \tan x$, ALSO WORKS WELL.]

249. $\int \frac{\sin x \cos x}{1-\cos x} dx = \begin{cases} \cos x + \ln|1-\cos x| + C \\ \cos x - \ln|\cot x + \operatorname{cosec} x| + C \end{cases}$

$\int \frac{\sin x \cos x}{1-\cos x} dx = \dots \text{ BY SUBSTITUTION...}$
 $= \int \frac{\sin x \cos x}{\sin^2 x} du = \int \frac{1-u}{u} du$
 $= \int \frac{1}{u} - 1 du = \ln|u| - u + C$
 $= \ln|1-\cos x| - (1-\cos x) + C$
 $= \operatorname{cosec} x + \ln|1-\cos x| + C$

ALTERNATIVE BY TRIG MANIPULATIONS

$\int \frac{\sin x \cos x}{1-\cos x} dx = \int \frac{\sin x \cos x(1+\cos x)}{(1-\cos x)(1+\cos x)} dx$
 $= \int \frac{\sin x \cos x(1+\cos x)}{\sin^2 x} dx = \int \frac{\cos x(1+\cos x)}{1-\sin^2 x} dx$
 $= \int \frac{\cos x + \cos^2 x}{\sin^2 x} dx = \int \frac{\cos x + 1 - \sin^2 x}{\sin^2 x} dx$
 $= \int \frac{\cos x}{\sin^2 x} + \frac{1}{\sin^2 x} dx = \dots \text{ SIMPLIFIED RESULTS}$
 $= \ln|\sin x| - \ln|\cosec x + \cot x| + \operatorname{cosec} x + C$
 $= \ln|1-\cos x| + C$

250. $\int \frac{(4x-1)^{-1}}{4} dx = -\frac{1}{16} \ln|4x-1| + C$

$\int \frac{(4x-1)^{-1}}{4} dx = \int \frac{1}{4} \left(\frac{1}{4x-1} \right) dx \quad \text{SIMPLIFIED AS FRACTION}$
 $= \frac{1}{16} \ln|4x-1| + C$

251. $\int \frac{e^{2x}-2e^x}{e^x+1} dx = e^x - 3 \ln(e^x+1) + C$

$\int \frac{e^{2x}-2e^x}{e^x+1} dx = \int \frac{\frac{d}{dx}(e^x+1)-3e^x}{e^x+1} dx = \int e^x - \frac{3e^x}{e^x+1} dx$
 $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$
 $= e^x - 3 \ln(e^x+1) + C$

ALTERNATIVE BY SUBSTITUTION

$\int \frac{e^{2x}-2e^x}{e^x+1} dx = \int \frac{e^{2x}-2e^x}{u} \left(\frac{du}{e^x} \right)$
 $= \int \frac{e^x-2}{u} du = \int \frac{(u-1)-2}{u} du$
 $= \int \frac{u-3}{u} du = \int 1 - \frac{3}{u} du$
 $= u - 3 \ln|u| = e^x + 1 - 3 \ln(e^x+1) + C$

252. $\int \frac{2x^2 - 3x + 2}{x-1} dx = x^2 - x + \ln|x-1| + C$

$$\begin{aligned}\int \frac{2x^2 - 3x + 2}{x-1} dx &= \dots \text{ BY LONG DIVISION OR MANIPULATION} \\ &\quad \text{IN ORDER TO SIMPLIFY THE FRACTION} \\ &= \int 2x(x-1) + 1 \cdot \frac{1}{x-1} dx = \int 2x - 1 + \frac{1}{x-1} dx \\ &= x^2 - x + \ln|x-1| + C \\ &\quad \boxed{\text{THE SUBSTITUTION } u=x-1 \text{ ALSO WORKS WELL.]}}\end{aligned}$$

253. $\int 1 - \cot^2 x \ dx = 2x + \cot x + C$

$$\begin{aligned}\int 1 - \cot^2 x \ dx &= \int 1 - (\csc^2 x - 1) dx = \int 2 - \csc^2 x dx \\ &= 2x + \cot x + C \\ &\quad \boxed{\frac{d}{dx}(\cot x) = -\csc^2 x}\end{aligned}$$

254. $\int \frac{x^2 + 1}{x^4 - x^2} dx = \frac{1}{x} + \ln \left| \frac{x-1}{x+1} \right| + C$

$$\begin{aligned}\int \frac{x^2 + 1}{x^4 - x^2} dx &= \int \frac{x^2 + 1}{x^2(x-1)(x+1)} dx = \int \frac{x^2 + 1}{x^2(2x)(x-1)} dx \\ &\quad \text{BY PARTIAL FRACTIONS} \\ \frac{x^2 + 1}{x^2(2x)(x-1)} &\equiv \frac{A}{x^2} + \frac{B}{2x} + \frac{C}{x-1} + \frac{D}{x} \\ x^2 + 1 &\equiv Ax^2(x-1) + Bx(x-1) + C(x^2)(x) + Dx(x)(2x)\end{aligned}$$

| | | | |
|--------------------|--------------------|---------------------|----------------------------|
| \bullet If $x=0$ | \bullet If $x=1$ | \bullet If $x=-1$ | \bullet If $x=2$ |
| $1 = -C$ | $2 = 2B$ | $2 = -2A$ | $5 = 4A + 2B + 3C + 4D$ |
| $C = -1$ | $B = 1$ | $A = -1$ | $5 = -4 + 12 - 3 + 40$ |
| | | | $5 = 54 \Rightarrow D = 0$ |
| | | | $D = 0$ |

$$\begin{aligned}\dots &= \int \frac{1}{x^2} - \frac{1}{2x} - \frac{1}{x-1} dx = \ln|x-1| - \ln|x+1| + \frac{1}{2x} + C \\ &= \frac{1}{2x} + \ln \left| \frac{x-1}{x+1} \right| + C\end{aligned}$$

255. $\int \operatorname{cosec}^4 x \, dx = -\cot x - \frac{1}{3} \cot^3 x + C$

$$\begin{aligned}\int \operatorname{cosec}^4 x \, dx &= \int \operatorname{cosec}^2 x \operatorname{cosec}^2 x \, dx = \int \operatorname{cosec}^2(1+\operatorname{ant}x) \, dx \\ &= \int \operatorname{cosec}^2 u \operatorname{cosec}^2 u \, du \\ &\quad \text{BY INVERSE CHAIN RULE (INTRODUCTORY) } \frac{du}{dx} = -\operatorname{cosec}^2 x \\ &= -u - \frac{1}{3} u^3 + C \\ &= -\cot x - \frac{1}{3} \cot^3 x + C\end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}\int \operatorname{cosec}^4 x \, dx &= \int \operatorname{cosec}^4 \left(\frac{du}{dx}\right) \, dx = \int -\operatorname{cosec}^2 u \, du \\ &= \int -(1+\operatorname{ant}x)^2 \, du = \int -1 - u^2 \, du \\ &= -u - \frac{1}{3} u^3 + C = -\operatorname{cot}x - \frac{1}{3} \operatorname{cot}^3 x + C\end{aligned}$$

$u = \operatorname{cosec} x$
 $\frac{du}{dx} = -\operatorname{cosec}^2 x$
 $dx = -\frac{du}{\operatorname{cosec}^2 x}$

256. $\int x^2 e^{\frac{1}{2}x} \, dx = 2x^2 e^{\frac{1}{2}x} - 8x e^{\frac{1}{2}x} + 16e^{\frac{1}{2}x} + C$

$$\begin{aligned}\int x^2 e^{\frac{1}{2}x} \, dx &= \dots \text{INTEGRATION BY PARTS} \\ &= 2x^2 e^{\frac{1}{2}x} \int x e^{\frac{1}{2}x} \, dx \\ &\quad \text{INTEGRATION BY PARTS AGAIN} \\ &= 2x^2 e^{\frac{1}{2}x} \left[2x e^{\frac{1}{2}x} - \int 2e^{\frac{1}{2}x} \, dx \right] \\ &= 2x^2 e^{\frac{1}{2}x} - 8x e^{\frac{1}{2}x} + \int 8e^{\frac{1}{2}x} \, dx \\ &= 2x^2 e^{\frac{1}{2}x} - 8x e^{\frac{1}{2}x} + 16e^{\frac{1}{2}x} + C = (2x^2 - 8x + 16)e^{\frac{1}{2}x} + C\end{aligned}$$

| | |
|--------------------|--------------------|
| x^2 | $2x$ |
| $e^{\frac{1}{2}x}$ | $e^{\frac{1}{2}x}$ |

| | |
|----------------|-------------------------------|
| $\frac{d}{dx}$ | $\frac{d}{dx}$ |
| $2x$ | $\frac{1}{2}e^{\frac{1}{2}x}$ |

257. $\int (e^x + 2e^{-x})^2 \, dx = \frac{1}{2}e^{2x} + 4x - 2e^{-2x} + C$

$$\begin{aligned}\int (e^x + 2e^{-x})^2 \, dx &= \int (e^x)^2 + 2(e^x)(2e^{-x}) + (2e^{-x})^2 \, dx \\ &= \int e^{2x} + 4 + 4e^{-2x} \, dx \\ &= \frac{1}{2}e^{2x} + 4x - 2e^{-2x} + C\end{aligned}$$

258. $\int e^x \sin(e^x) \, dx = -\cos(e^x) + C$

$$\begin{aligned}\int e^x \sin(e^x) \, dx &= \dots \text{BY INVERSE CHAIN RULE (INTRODUCTORY)} \dots \\ &= -\cos(e^x) + C \\ &\quad [\text{THE SUBSTITUTION } u = e^x \text{ WORKS WITH } \sin(u)]\end{aligned}$$

259. $\int xe^x \, dx = \frac{1}{2}e^x x^2 + C$

$$\int xe^x \, dx = \frac{1}{2}e^x x^2 + C = \underline{\frac{1}{2}e^x x^2 + C} \quad [e^x \text{ is a constant}]$$

260. $\int (2\cos x - 3)^2 \, dx = 11x + \sin 2x - 12\sin x + C$

$$\begin{aligned} \int (2\cos x - 3)^2 \, dx &= \int 4\cos^2 x - 12\cos x + 9 \, dx \\ &= \int 4(\frac{1+\cos 2x}{2}) - 12\cos x + 9 \, dx \\ &= \int 11 + 2\cos 2x - 12\cos x \, dx \\ &= \underline{11x + \sin 2x - 12\sin x + C} \end{aligned}$$

261. $\int \frac{x^2}{x-2} \, dx = \left[\frac{1}{2}x^2 + 2x + 4\ln|x-2| + C \right]$
 $\quad \quad \quad \left[\frac{1}{2}(x-2)^2 + 4(x-2) + 4\ln|x-2| + C \right]$

$$\begin{aligned} \int \frac{x^2}{x-2} \, dx &= \text{BY DIVISION (MANIPULATION) IN ORDER TO SPLIT THE FRACTION} \\ &= \int \frac{x(x-2)+2(x-2)+4}{x-2} \, dx = \int x+2 + \frac{4}{x-2} \, dx \\ &= \underline{\frac{1}{2}x^2 + 2x + 4\ln|x-2| + C} \\ \text{ALTERNATIVE BY SUBSTITUTION} \\ \int \frac{x^2}{x-2} \, dx &= \int \frac{u^2+4u+4}{u-2} \, du = \int u+4 + \frac{4}{u-2} \, du \\ &= \frac{1}{2}u^2 + 4u + 4\ln|u| + C \\ &= \underline{\frac{1}{2}(x-2)^2 + 4(x-2) + 4\ln|x-2| + C} \end{aligned}$$

$u = x-2$
 $\frac{du}{dx} = 1$
 $du = dx$
 $x = u+2$
 $x^2 = (u+2)^2$
 $2^2 = (u+2)(u+4)$

262. $\int \frac{(x+1)e^{\frac{1}{x}}}{x^3} \, dx = -\frac{1}{x}e^{\frac{1}{x}} + C$

$$\begin{aligned} \int \frac{(x+1)e^{\frac{1}{x}}}{x^3} \, dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{x+1}{x^3} e^{\frac{1}{x}} (x^2 \, dx) = \int -\left(\frac{2}{x^2}\right) e^{\frac{1}{x}} \, dx \\ &= \int -\left(1+\frac{1}{x}\right)^{-2} \, dx = \int -(1+u)^{-2} \, du \\ \text{INTEGRATION BY PARTS (D TO F FOLLOW)} \\ &= -(-1+u)^{-1} - \int -e^u \, du \\ &= -(-1+u)^{-1} + \int e^u \, du \\ &= -(-1+u)e^u + e^u + C \\ &= e^u (-1-u+1) + C \\ &= -ue^u + C = \underline{-\frac{1}{x}e^{\frac{1}{x}} + C} \end{aligned}$$

| |
|----------------------------------|
| $u = \frac{1}{x}$ |
| $\frac{du}{dx} = -\frac{1}{x^2}$ |
| $du = -\frac{1}{x^2} \, dx$ |

263. $\int \frac{\sqrt{4x+1}}{x} dx = 2(4x-1)^{\frac{1}{2}} + \ln \left| \frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1} \right| + C$

$$\begin{aligned} & \int \frac{\sqrt{4x+1}}{x} dx = \dots \text{ BY SUBSTITUTION} \\ & = \int \frac{u}{x} \left(\frac{du}{dx} dx \right) = \int \frac{u^2}{2x} du \\ & = \int \frac{2u^2}{4x} du = \int \frac{2u^2}{u^2-1} du \\ & \text{AS THE NUMERATOR IS IMPROVED DIVIDE OR MANIPULATE} \\ & = \int \frac{2(u^2-1)+2}{u^2-1} du = \int 2 + \frac{2}{u^2-1} du \\ & = \int 2 + \frac{2}{(u-1)(u+1)} du = \dots \text{ PARTIAL FRACTION BY INSPECTOR} \dots \\ & = \int 2 + \frac{1}{u-1} - \frac{1}{u+1} du = 2u + \ln|u-1| + \ln|u+1| + C \\ & = 2u + \ln \left| \frac{u-1}{u+1} \right| + C = 2(2x+1)^{\frac{1}{2}} + \ln \left| \frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1} \right| + C \end{aligned}$$

264. $\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx = \frac{x}{\sqrt{1-x^2}} + C$

$$\begin{aligned} & \int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx = \dots \text{ BY A TRIGONOMETRIC SUBSTITUTION} \\ & = \int \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}} (\cos\theta d\theta) = \int \frac{\cos\theta}{(\cos^2\theta)^{\frac{3}{2}}} d\theta \quad u = \sin\theta \\ & = \int \frac{\cos\theta}{\cos^3\theta} d\theta = \int \frac{1}{\cos^2\theta} d\theta = \int \sec^2\theta d\theta \quad \frac{du}{d\theta} = \cos\theta d\theta \\ & = \tan\theta + C = \frac{\sin\theta}{\cos\theta} + C \\ & \text{NOW THE SUBSTITUTION IS IN FACT } \theta = \arcsin x, \text{ SO} \\ & = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} + C = \frac{x}{\sqrt{1-x^2}} + C \\ & \text{[THE SUBSTITUTION } u = \cos\theta / \theta = \arccos x \text{ ALSO WORKS]} \end{aligned}$$

265. $\int \frac{2^x}{2^x+1} dx = \frac{\ln(2^x+1)}{\ln 2} + C$

$$\begin{aligned} & \int \frac{2^x}{2^x+1} dx = \frac{1}{\ln 2} \int \frac{2^x \ln 2}{2^x+1} dx \\ & \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C \\ & = \frac{1}{\ln 2} \ln(2^x+1) + C = \frac{\ln(2^x+1)}{\ln 2} + C \\ & \text{[THE SUBSTITUTION } u = 2^x+1 \text{ ALSO WORKS WELL]} \end{aligned}$$

266. $\int (2x-1)\sqrt{2x-3} dx = \left[\frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C \right] - \left[\frac{1}{3}(2x-1)(2x-3)^{\frac{3}{2}} - \frac{2}{15}(2x-3)^{\frac{5}{2}} + C \right]$

$\int (2x-1)\sqrt{2x-3} dx = \text{BY SUBSTITUTION}$

$$\begin{aligned} &= \int (2x-1)u \left(u du \right) = \int u^2(u^2+3-1) du \\ &= \int u^2(u^2+2) du = \int u^4+2u^2 du \\ &= \frac{1}{5}u^5 + \frac{2}{3}u^3 + C \\ &= \frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C \end{aligned}$$

THE SUBSTITUTION $u = 2x-3$ ALSO WORKS.

ALTERNATIVE USING INTEGRATION BY PARTS

$$\begin{aligned} \int (2x-1)\sqrt{2x-3} dx &= \frac{1}{2}(2x-1)(2x-3)^{\frac{1}{2}} - \int \frac{1}{2}(2x-3)^{\frac{1}{2}} dx \\ &= \frac{1}{2}(2x-1)(2x-3)^{\frac{1}{2}} - \frac{1}{2}(2x-3)^{\frac{3}{2}} + C \end{aligned}$$

ALTERNATIVE BY MANIPULATION

$$\begin{aligned} \int (2x-1)\sqrt{2x-3} dx &= \int [(2x-3)+2](2x-3)^{\frac{1}{2}} dx \\ &= \int (2x-3)^{\frac{1}{2}} + 2(2x-3)^{\frac{1}{2}} dx = \frac{1}{2}(2x-3)^{\frac{3}{2}} + \frac{3}{2}(2x-3)^{\frac{1}{2}} + C \end{aligned}$$

AS ABOVE

267. $\int \frac{9x^5}{\sqrt{x^3+1}} dx = \left[\begin{array}{l} 2(x^3+1)^{\frac{3}{2}} - 6(x^3+1)^{\frac{1}{2}} + C \\ 2x^3(x^3+1)^{\frac{1}{2}} - 4(x^3+1)^{\frac{3}{2}} + C \\ 2(x^3-2)(x^3+1)^{\frac{1}{2}} + C \end{array} \right]$

$\int \frac{q_2^5}{\sqrt{z^2+1}} dz = \text{BY SUBSTITUTION}$

$$\begin{aligned} &= \int \frac{q_2^5}{\sqrt{z^2+1}} \left(\frac{2z^2 dz}{z^2+1} \right) = \int 6z^2 dz \\ &= \int 6u^2 - 6 du = -2u^3 - 6u + C \\ &= 2(z^3+1)^{\frac{3}{2}} - 6(z^3+1)^{\frac{1}{2}} + C \\ &= 2(z^3+1)^{\frac{1}{2}} [2(z^3+1)^{\frac{1}{2}} - 3] + C \\ &= 2(z^3-2)(z^3+1)^{\frac{1}{2}} + C \end{aligned}$$

ALTERNATIVE BY INTEGRATION BY PARTS

$$\begin{aligned} \int \frac{q_2^5}{\sqrt{z^2+1}} dz &= \int (q_2^3)(z)(z(z^2+1)^{-\frac{1}{2}}) dz \\ &= 6z(z^2+1)^{\frac{1}{2}} - \int 18z^2(z^2+1)^{-\frac{1}{2}} dz \\ &= 6z(z^2+1)^{\frac{1}{2}} - 4(z^2+1)^{\frac{1}{2}} + C \\ &= 2(z^3+1)^{\frac{1}{2}} [3z^2 - 2(z^2+1)] + C \\ &= 2(z^3+1)^{\frac{1}{2}} (z^2-2) + C \end{aligned}$$

ALTERNATIVE BY MANIPULATION & RECOGNITION

$$\begin{aligned} \int \frac{q_2^5}{\sqrt{z^2+1}} dz &= \int \frac{q_2^5(z^2+1)-z^2}{(z^2+1)^{\frac{3}{2}}} dz = \int q_2^5(z^2+1)^{-\frac{1}{2}} - q_2^5(z^2+1)^{\frac{1}{2}} dz \\ &= 2(z^3+1)^{\frac{3}{2}} - 6(z^3+1)^{\frac{1}{2}} + C \end{aligned}$$

AS THE SUBSTITUTION METHOD

268. $\int (3\sin x - \cos x)^2 dx = \begin{bmatrix} 5x - \sin 2x + \frac{3}{2}\cos 2x + C \\ 5x - \sin 2x - 3\sin^2 x + C \\ 5x - \sin 2x + 3\cos^2 x + C \end{bmatrix}$

$$\begin{aligned} \int (3\sin x - \cos x)^2 dx &= \int 9\sin^2 x - 6\sin x \cos x + \cos^2 x dx \\ &= \int 9\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) - 3\sin 2x + \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx \\ &= \int 5 - 4\cos 2x - 3\sin 2x dx = \frac{5x - 2\sin 2x + 3\cos 2x + C}{2} \\ \text{VARIATION IF } \int -6\sin x \cos x dx &= \int -3\sin^2 x \\ &\dots = \frac{5x - 2\sin 2x - 3\sin^2 x + C}{2} \\ &\dots = \frac{5x - 2\sin 2x + 3\cos^2 x + C}{2} \end{aligned}$$

269. $\int \frac{4x}{x^2 - 10} dx = 2\ln|x^2 - 10| + C$

$$\begin{aligned} \int \frac{4x}{x^2 - 10} dx &= 2 \int \frac{2x}{x^2 - 10} dx = \frac{2\ln|x^2 - 10| + C}{2} \\ \int \frac{f(x)}{f'(x)} dx &= \ln|f(x)| + C \\ [\text{THE SUBSTITUTION } u = x^2 - 10 \text{ FOR EXP. FRACTIONAL FRACTIONS!}] \\ [\text{WORK IN THIS QUESTION}] \end{aligned}$$

270. $\int \frac{4e^{3x}}{1-e^{3x}} dx = -\frac{4}{3}\ln|1-e^{3x}| + C$

$$\begin{aligned} \int \frac{4e^{3x}}{1-e^{3x}} dx &= -\frac{4}{3} \int \frac{-3e^{3x}}{1-e^{3x}} dx = -\frac{4}{3}\ln|1-e^{3x}| + C \\ \int \frac{f(x)}{f'(x)} dx &= \ln|f(x)| + C \\ [\text{THE SUBSTITUTION } u = 1-e^{3x} \text{ ALSO WORKS WELL}] \end{aligned}$$

271. $\int (1-\cos x)\sin x \cos x dx = \begin{bmatrix} \cos^3 x - \frac{1}{4}\cos 2x + C \\ \cos^3 x - \frac{1}{2}\cos^2 x + C \\ \cos^3 x + \frac{1}{2}\sin^2 x + C \end{bmatrix}$

$$\begin{aligned} \int \sin x \cos x (1-\cos x) dx &= \int \sin x \cos x - \sin x \cos^2 x dx \\ &\quad \text{or } \int \frac{1}{2}\sin 2x - \frac{1}{2}\sin 2x \cos 2x dx \\ \text{BY RECOGNITION (CONVERSE CHAIN RULE)} \\ &= \frac{1}{2}\sin x + \cos x + C = \frac{1}{2}\cos x + \cos^2 x + C = \boxed{\frac{1}{2}\cos x + \cos^2 x + C} \end{aligned}$$

272. $\int (\tan x - 1)^2 dx = \begin{cases} \tan x - 2\ln|\sec x| + C \\ \tan x + 2\ln|\cos x| + C \end{cases}$

$$\begin{aligned} \int (\tan x - 1)^2 dx &= \int \frac{\tan^2 x}{\sec^2 x} - 2\tan x + 1 dx \\ &= \int \frac{\sec^2 x}{\sec^2 x} - 2\tan x dx \quad \dots \text{BOTH ARE STANDARD RESULTS} \\ &= \tan x - 2\ln|\sec x| + C = \tan x + 2\ln|\cos x| + C \end{aligned}$$

273. $\int \frac{1+e^x}{1-e^x} dx = x - 2\ln|1-e^{-x}| + C$

$$\begin{aligned} \int \frac{1+e^x}{1-e^x} dx &= \dots \text{BY SUBSTITUTION} \\ \int \frac{2-u}{u} \cdot \frac{du}{-e^x} &= \int \frac{2-u}{u} \cdot \frac{du}{-e^x} \\ \int \frac{2-u}{u(u-1)} du &= \dots \text{PARTIAL FRACTIONS BY} \\ &\quad \text{INVERSION (GIVE UP)} \\ \int \frac{1}{u-1} - \frac{2}{u} du &= \ln|u-1| - 2\ln|u| + C \\ &= \ln|x+e^x-1| - 2\ln|1-e^x| + C = \underline{x - 2\ln|1-e^{-x}| + C} \\ \text{ALTERNATIVE BY MANIPULATION} \\ \int \frac{1+e^x}{1-e^x} dx &= \int \frac{(1-e^x) + 2e^x}{1-e^x} dx = \int 1 + \frac{2e^x}{1-e^x} dx \\ &= \int 1 - 2\left(\frac{-e^x}{1-e^x}\right) dx \\ &\quad \uparrow \int \frac{f(x)}{f'(x)} dx = \ln|f(x)| + C \\ &= \underline{x - 2\ln|1-e^{-x}| + C} \end{aligned}$$

274. $\int \sin 2x \cos^4 2x dx = -\frac{1}{10} \cos^5 2x + C$

$$\begin{aligned} \int \sin 2x \cos^4 2x dx &= \dots \text{BY DOUBLE CHAIN RULE (COSINE)} \dots \\ &= \underline{-\frac{1}{10} \cos^5 2x + C} \\ [\text{THE SUBSTITUTION } u = \cos 2x \text{ ALSO WORKS HERE}] \end{aligned}$$

275. $\int \frac{1}{\sin x \cos x} dx = \begin{bmatrix} \ln|\sin x| - \ln|\sec x| + C \\ \ln|\sin x| + \ln|\cos x| + C \\ \ln|\tan x| + C \\ -\ln|\cot 2x + \operatorname{cosec} 2x| + C \end{bmatrix}$

$$\begin{aligned} \int \frac{1}{\sin x \cos x} dx &= \int \frac{2}{2 \sin x \cos x} dx = \int \frac{2}{\sin 2x} dx \\ &= \int 2 \operatorname{cosec} 2x dx = -\ln|\operatorname{cosec} 2x + \operatorname{cot} 2x| + C \\ &= \ln|\tan x| + C \end{aligned}$$

BOTH ARE STANDARD RESULTS

ALTERNATIVE MANIPULATION

$$\begin{aligned} \int \frac{1}{\sin x \cos x} dx &= \int \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} dx = \int \operatorname{cosec} x + \operatorname{cot} x dx \\ &= \ln|\sin x| + \ln|\operatorname{sec} x| + C = \ln|\sin x| - \ln|\operatorname{cosec} x| + C \\ &= \ln|\tan x| + C \end{aligned}$$

276. $\int \frac{1}{\sin^2 x \cos x} dx = -\operatorname{cot} x + \ln|\sec x + \tan x| + C$

$$\begin{aligned} \int \frac{1}{\sin^2 x \cos x} dx &= \int \frac{\cos x + \sin x}{\sin^2 x \cos x} dx = \int \frac{\operatorname{cosec} x + \frac{1}{\cos x}}{\sin x} dx \\ &= \int \operatorname{cosec} x + \operatorname{sec} x dx \dots \text{STANDARD RESULTS} \\ &= -\operatorname{cot} x + \ln|\sec x + \tan x| + C \end{aligned}$$

277. $\int \frac{1}{1-\sin x} dx = \tan x + \sec x + C$

$$\begin{aligned} \int \frac{1}{1-\sin x} dx &= \int \frac{1+\sin x}{(1-\sin x)(1+\sin x)} dx = \int \frac{1+\sin x}{1-\sin^2 x} dx \\ &= \int \frac{1+\sin x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} dx \\ &= \int \operatorname{sec}^2 x + \frac{\sin x}{\cos^2 x} dx = \int \operatorname{sec}^2 x + \operatorname{sec} x \operatorname{tang} x dx \\ &= \operatorname{tang} x + \sec x + C \end{aligned}$$

STANDARD ANTI-DIFFERENTIATES

278. $\int \frac{1}{e^x + 1} dx = x - \ln(e^x + 1) + C$

$$\begin{aligned} \int \frac{1}{e^x + 1} dx &= \dots \text{ BY SUBSTITUTION} \dots \\ &= \int \frac{1}{u} \left(\frac{du}{u-1} \right) = \int \frac{1}{u(u-1)} du \\ &\text{PARTIAL FRACTIONS BY INSPECTION (CROSS-OF)} \\ &= \int \frac{1}{u-1} - \frac{1}{u} du = \ln|u-1| - \ln|u| + C \\ &= \ln(e^x) - \ln(e^x + 1) + C = x - \ln(e^x + 1) + C \end{aligned}$$

279. $\int (1+\sin x)\sin 2x \ dx = \begin{bmatrix} \frac{2}{3}\sin^3 x - \cos^2 x + C \\ \frac{2}{3}\sin^3 x + \sin^2 x + C \\ \frac{2}{3}\sin^3 x - \frac{1}{2}\cos 2x + C \end{bmatrix}$

$$\begin{aligned} \int \sin 2x(1+\sin x) dx &= \int 2\sin x \cos x(1+\sin x) dx \\ &= \int 2\sin x \cos x dx + 2\sin^2 x \cos x dx \\ &\text{BY REVERSE CHAIN RULE (INSPECTION)} \\ &= \sin^2 x + \frac{2}{3}\sin^3 x + C = -\cos^2 x + \frac{2}{3}\sin^3 x + C \\ &\text{VALIDATION} \\ &\dots \int 2\sin x \cos x dx = \int \sin 2x + 2\sin^2 x \cos x dx \\ &= -\frac{1}{2}\cos 2x + \frac{2}{3}\sin^3 x + C \end{aligned}$$

280. $\int x \sec x \tan x \ dx = x \sec x - \ln|\sec x + \tan x| + C$

$$\begin{aligned} \int x \sec x \tan x dx &= \dots \\ &\text{INTEGRATION BY PARTS} \\ &= x \sec x - \int \sec x dx \quad \leftarrow \text{STANDARD DERIV} \\ &= x \sec x - \ln|\sec x + \tan x| + C \end{aligned}$$

281. $\int \sin 3x \cos 2x \, dx = \left[-\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + C \right]$

$\int \sin 3x \cos 2x \, dx = \dots$ BY TRIG IDENTITIES

$$\begin{aligned} \sin(3x+2x) &= \sin 5x = \sin 3\cos 2x + \cos 3\sin 2x \\ \sin(3x-2x) &= \sin x = \sin 3\cos 2x - \cos 3\sin 2x \\ \text{ADDING} & \quad \sin 5x + \sin x = 2 \sin 3 \cos 2x \end{aligned}$$

$$= \int \frac{1}{2} \sin 5x + \frac{1}{2} \sin x \, dx = -\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + C$$

ALTERNATIVE BY DOUBLE INTEGRATION BY PARTS

$$\begin{aligned} \int \sin 3x \cos 2x \, dx &= \frac{\sin 3x}{3} \Big| \frac{-\cos 2x}{2} \\ &= \frac{1}{3} \sin 3x \cos 2x - \int \frac{3}{2} \cos 3x \sin 2x \, dx \\ &\quad \text{BY PARTS - AGAIN} \\ &= \frac{3}{2} \cos 3x \Big| \frac{-\frac{1}{2} \sin 2x}{\sin 2x} \\ &= -\frac{3}{4} \cos 3x \end{aligned}$$

$$\begin{aligned} \int \sin 3x \cos 2x \, dx &= \frac{1}{2} \sin 3x \cos 2x - \left[-\frac{3}{4} \cos 3x \cos 2x + \frac{3}{8} \sin 3x \sin 2x \right] \\ \Rightarrow \int \sin 3x \cos 2x \, dx &= \frac{1}{2} \sin 3x \cos 2x + \frac{3}{8} \cos 3x \cos 2x - \frac{3}{8} \int \sin 3x \cos 2x \, dx \\ \Rightarrow \frac{13}{8} \int \sin 3x \cos 2x \, dx &= \frac{1}{2} \sin 3x \cos 2x + \frac{3}{8} \cos 3x \cos 2x \\ \Rightarrow \int \sin 3x \cos 2x \, dx &= \frac{8}{13} \sin 3x \cos 2x + \frac{3}{13} \cos 3x \cos 2x + C \end{aligned}$$

282. $\int \frac{1-2x}{x(2x-\ln x)^2} \, dx = \frac{1}{2x-\ln x} + C$

$$\int \frac{1-2x}{x(2x-\ln x)^2} \, dx = \dots$$
 BY SUBSTITUTION ...
$$\begin{aligned} &= \int \frac{\frac{1-2x}{2x-\ln x}}{(2x-\ln x)^2} \, dx = \int \frac{1}{u^2} \, du \\ &= \frac{1}{u} + C = \frac{1}{2x-\ln x} + C \end{aligned}$$

| |
|---|
| $u = 2x - \ln x$ |
| $\frac{du}{dx} = 2 - \frac{1}{x}$ |
| $\frac{du}{dx} = \frac{2x-1}{x}$ |
| $\frac{du}{dx} = \frac{1-2x}{2x-\ln x}$ |
| $du = \frac{2x-\ln x}{1-2x} \, dx$ |

283. $\int x^2 e^{-\frac{1}{4}x} \, dx = -4e^{-\frac{1}{4}x} [x^2 + 8x + 32] + C$

$$\int x^2 e^{-\frac{1}{4}x} \, dx = \dots$$
 INTEGRATION BY PARTS

| | |
|----------------------|---------------------|
| x^2 | $-\frac{1}{4}x$ |
| $-4e^{\frac{1}{4}x}$ | $e^{-\frac{1}{4}x}$ |

$$\begin{aligned} &= -x^2 e^{-\frac{1}{4}x} - \int -2x e^{-\frac{1}{4}x} \, dx \\ &= -x^2 e^{-\frac{1}{4}x} + \int 8x e^{-\frac{1}{4}x} \, dx \dots \text{INTEGRATION BY PARTS AGAIN} \\ &= -x^2 e^{-\frac{1}{4}x} + \left[-32x e^{-\frac{1}{4}x} - \int -32e^{-\frac{1}{4}x} \, dx \right] \\ &= -x^2 e^{-\frac{1}{4}x} - 32x e^{-\frac{1}{4}x} + \int 32e^{-\frac{1}{4}x} \, dx \\ &= -x^2 e^{-\frac{1}{4}x} - 32x e^{-\frac{1}{4}x} - 128e^{-\frac{1}{4}x} + C \\ &= -4e^{-\frac{1}{4}x} [x^2 + 8x + 32] + C \end{aligned}$$

| | |
|------------------------|---------------------|
| $8x$ | 8 |
| $-32e^{-\frac{1}{4}x}$ | $e^{-\frac{1}{4}x}$ |

284. $\int \frac{x+4}{x-4} dx = x + 8 \ln|x-4| + C$

$$\begin{aligned}\int \frac{2x+8}{x-4} dx &= \int \frac{(2x-4)+8}{(x-4)} dx = \int 1 + \frac{8}{x-4} dx \\ &= 2x + 8 \ln|x-4| + C\end{aligned}$$

[THE SUBSTITUTION $u=x-4$ ALSO WORKS WELL]

285. $\int 3x^2(4-2x^3)^{\frac{3}{2}} dx = -\frac{1}{5}(4-2x^3)^{\frac{5}{2}} + C$

$$\begin{aligned}\int 3x^2(4-2x^3)^{\frac{3}{2}} dx &= \text{BY REVERSE CHAIN RULE (CORRECTION) ...} \\ &= -\frac{1}{5}(4-2x^3)^{\frac{5}{2}} + C\end{aligned}$$

[THE SUBSTITUTION $u=4-2x^3$ OR $u=(4-2x^3)^{\frac{1}{2}}$ ALSO WORKS WELL]

286. $\int x\sqrt{x+1} dx = \left[\begin{array}{l} \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C \\ \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + C \end{array} \right]$

$$\begin{aligned}\int 2\sqrt{x+1}^3 dx &= \dots \text{BY SUBSTITUTION} \dots \\ &\approx \int 2u(2u du) = \int 2u^3 du \\ &= \int 2u^3(u^2-1) du = \int 2u^8 - 2u^2 du \\ &= \frac{2}{9}u^9 - \frac{2}{3}u^3 + C = \frac{2}{9}(2u^3)^3 - \frac{2}{3}(2u^3)^{\frac{3}{2}} + C\end{aligned}$$

ALTERNATIVE BY INTEGRATION BY PARTS

$$\begin{aligned}\int 2\sqrt{x+1}^3 dx &= \frac{2}{3}x(2x+1)^{\frac{3}{2}} - \int \frac{2}{3}(2x+1)^{\frac{1}{2}} dx \\ &= \frac{2}{3}x(2x+1)^{\frac{3}{2}} - \frac{4}{15}(2x+1)^{\frac{5}{2}} + C\end{aligned}$$

287. $\int \frac{x+1}{\sqrt[3]{x^2+2x+3}} dx = \frac{3}{4}(x^2+2x+3)^{\frac{2}{3}} + C$

$$\begin{aligned}\int \frac{x+1}{\sqrt[3]{x^2+2x+3}} dx &= \int (2x+1)(x^2+2x+3)^{\frac{1}{3}} dx \\ &= \int (2x+1)(x^2+2x+3)^{\frac{1}{3}} dx = \dots \text{BY REVERSE CHAIN RULE (INSPECTION)} \\ &= \frac{1}{2}\int (2x+2x+3)^{\frac{4}{3}} + C = \frac{3}{4}(x^2+2x+3)^{\frac{2}{3}} + C\end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}\int \frac{x+1}{\sqrt[3]{x^2+2x+3}} dx &= \int \frac{2x+1}{u} \left(\frac{du}{2x+1} \right) \\ &= \int \frac{1}{2}u du = \frac{3}{4}u^{\frac{4}{3}} + C \\ &= \frac{3}{4}(x^2+2x+3)^{\frac{4}{3}} + C\end{aligned}$$

[THE SUBSTITUTION $u=x^2+2x+3$ ALSO WORKS WELL]

288. $\int \sin 2x \cos x \, dx = \left[-\frac{2}{3} \cos^3 x + C \right. \\ \left. -\frac{1}{6} \cos 3x - \frac{1}{2} \cos x + C \right] \\ \left. -\frac{1}{3}(\sin 2x \sin x + 2 \cos 2x \cos x) + C \right]$

$\int \sin 2x \cos x \, dx = \int (2 \sin x \cos x) \cos x \, dx = \int 2 \sin x \cos^2 x \, dx$
 BY REVERSE CHAIN RULE (INTEGRATION) ... $= -\frac{2}{3} \cos^3 x + C$
 [THE SUBSTITUTION $u = \cos x$ ALSO WORKS WELL]

ALTERNATIVE BY TRIG IDENTITIES

$$\begin{aligned} \sin(2x+2) &= \sin 3x = \sin 2x \cos x + \cos 2x \sin x \\ \sin(2x) &= \frac{\sin 3x}{\cos x} = \frac{\sin 2x \cos x - \cos 2x \sin x}{\cos x} \\ \text{ADDING} &\quad \sin 3x + \sin x = 2 \sin 2x \cos x \\ \text{OR} &\quad \sin 2x \cos x = \frac{1}{2} \sin 3x + \frac{1}{2} \sin x \\ \int \sin 2x \cos x \, dx &= \int \frac{1}{2} \sin 3x + \frac{1}{2} \sin x \, dx \\ &= -\frac{1}{6} \cos 3x - \frac{1}{2} \cos x + C \end{aligned}$$

ALTERNATIVE (AND) INTEGRATION BY PARTS TWICE

| | |
|-----------------------------|--|
| $\int \sin 2x \cos x \, dx$ | $= \sin x \sin x - \int 2 \cos x \sin x \, dx$ |
| | \downarrow |
| | $\begin{array}{ c c } \hline \sin 2x & \cos x \\ \hline \sin x & \cos x \\ \hline \end{array}$ |

| | |
|------------------------------|--|
| $\int 2 \cos x \sin x \, dx$ | $= 2 \cos x \cos x - \int 4 \sin x \, dx$ |
| | \downarrow |
| | $\begin{array}{ c c } \hline 2 \cos x & 4 \sin x \\ \hline -\cos x & \sin x \\ \hline \end{array}$ |

$$\begin{aligned} \Rightarrow \int \sin 2x \cos x \, dx &= \sin x \sin x - \left[-2 \cos x \cos x - \int 4 \sin x \cos x \, dx \right] \\ \Rightarrow \int \sin 2x \cos x \, dx &= \sin x \sin x + 2 \cos x \cos x + \int \sin 2x \cos x \, dx \\ \Rightarrow -3 \int \sin 2x \cos x \, dx &= \sin x \sin x + 2 \cos x \cos x \\ \Rightarrow \int \sin 2x \cos x \, dx &= -\frac{1}{3} [\sin x \sin x + 2 \cos x \cos x] \end{aligned}$$

289. $\int \sin 2x \cos 2x \, dx = \left[\frac{1}{4} \sin^2 2x + C \right. \\ \left. -\frac{1}{4} \cos^2 2x + C \right] \\ \left. -\frac{1}{8} \cos 4x + C \right]$

$\int \sin 2x \cos 2x \, dx = \dots$ BY REVERSE CHAIN RULE (INTEGRATION)
 OR THE SUBSTITUTION $u = \sin 2x$
 OR THE SUBSTITUTION $u = \cos 2x$

$$\dots = \frac{1}{4} \cos^2 2x + C \quad \text{OR} \quad -\frac{1}{4} \cos^2 2x + C$$

ALTERNATIVE BY DOUBLE ANGLES

$$\begin{aligned} \int \sin 2x \cos 2x \, dx &= \int \frac{1}{2} (2 \sin 2x \cos 2x) \, dx = \int \sin 4x \, dx \\ &= -\frac{1}{4} \cos 4x + C \end{aligned}$$

290. $\int \frac{3^x}{3^x + 1} \, dx = \frac{\ln(3^x + 1)}{\ln 3} + C$

$$\begin{aligned} \int \frac{3^x}{3^x + 1} \, dx &= \frac{1}{\ln 3} \int \frac{3^x \ln 3}{3^x + 1} \, dx = \frac{1}{\ln 3} \ln(3^x + 1) + C \\ \text{OR THE FORM} & \quad \int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C \\ &= \frac{\ln(3^x + 1)}{\ln 3} + C \end{aligned}$$

[THE SUBSTITUTION $u = 3^x + 1$ ALSO WORKS WELL]

291. $\int \frac{\tan x}{\tan x - \sec x} dx = \begin{cases} x - \tan x - \frac{1}{2} \tan^2 x + C \\ x - \tan x - \frac{1}{2} \sec^2 x + C \end{cases}$

$$\begin{aligned} \int \frac{\tan x}{\tan x - \sec x} dx &= \int \frac{\tan x (\sec x + \sec x)}{(\tan x - \sec x)(\sec x + \sec x)} dx \\ &= \int \frac{\tan^2 x + \tan x \sec x}{\tan x - \sec x} dx = \int \frac{(\sec x - 1) + \tan x \sec x}{\tan x - \sec x} dx \\ &= \int \frac{\sec x - 1 + \tan x \sec x}{1 - \sec x - \tan x \sec x} dx = \int 1 - \sec x - \tan x \sec x dx \\ &\Rightarrow x - \tan x - \frac{1}{2} \tan^2 x + C \quad \text{Reverse chain rule or substitutions} \\ &x - \tan x - \frac{1}{2} \sec^2 x + C \quad \leftarrow \frac{d}{dx}(\sec x) = 2\sec(x)\tan x \end{aligned}$$

292. $\int \frac{(\ln x)^2}{x} dx = \frac{1}{3}(\ln x)^3 + C$

$$\begin{aligned} \int \frac{(\ln x)^2}{x} dx &= \int \frac{1}{x} (\ln x)^2 dx = \dots \text{BY STANDARD CHAIN RULE (METHOD 1)} \\ &= f(\ln x) + C \\ &[\text{THE SUBSTITUTION } u = \ln x \text{ ALSO WORKS WITH}] \\ &\text{ALTERNATIVE: INTEGRATION BY PARTS} \\ &\Rightarrow \int \frac{(\ln x)^2}{x} dx = (\ln x)^3 - \int \frac{2(\ln x)^2}{x} dx \\ &\Rightarrow \int \frac{(\ln x)^2}{x} dx = (\ln x)^3 - \frac{2(\ln x)^2}{x} \\ &\Rightarrow \int \frac{(\ln x)^2}{x} dx = \frac{1}{3}(\ln x)^3 + C \end{aligned}$$

293. $\int \frac{8(x^2+1)}{(x-3)(x+1)^2} dx = 5\ln|x-3| + 3\ln|x+1| + \frac{2}{x+1} + C$

$$\begin{aligned} \int \frac{8(x^2+1)}{(x-3)(x+1)^2} dx &= \dots \text{BY PARTIAL FRACTIONS} \\ \frac{8(x^2+1)}{(x-3)(x+1)^2} &\equiv \frac{A}{x-3} + \frac{B}{(x+1)^2} + \frac{C}{x+1} \\ 8(x^2+1) &\equiv A(x^2+1) + B(x+1)^2 + C(x+1)(x-3) \\ \bullet (x=3) \rightarrow -1 &\bullet \text{IF } x=1 \\ \frac{16+48}{3-3} = 48 &\quad B=16A \\ B=48 &\quad \frac{2+3}{2+1} = 1 \\ B=48 &\quad B=16A \\ B=4 &\quad B=4 \\ B=4 &\quad B=4 \\ C=3 &\quad C=3 \end{aligned}$$

$$\dots = \int \frac{\frac{2}{x-3}}{x-3} - \frac{\frac{4}{(x+1)^2}}{(x+1)^2} + \frac{\frac{3}{x+1}}{x+1} dx$$

$$= 5\ln|x-3| + 3\ln|x+1| + \frac{2}{x+1} + C$$

294. $\int \sqrt[3]{x} \sqrt{\frac{1}{x}} dx = \frac{6}{7} x^{\frac{7}{6}} + C$

$$\begin{aligned}\int \sqrt[3]{x} \sqrt{\frac{1}{x}} dx &= \int [x x^{-\frac{1}{2}}]^{\frac{1}{3}} dx = \int (x^{\frac{1}{2}})^{\frac{1}{3}} dx \\ &= \int x^{\frac{1}{6}} dx = \frac{6}{7} x^{\frac{7}{6}} + C\end{aligned}$$

295. $\int (x+1)(x^2+2x-1)^4 dx = \frac{1}{10}(x^2+2x-1)^5 + C$

$$\begin{aligned}\int (x+1)(x^2+2x-1)^4 dx &= \frac{1}{2} \int (2x+2)(x^2+2x-1)^4 dx \\ &\text{BY REVERSE CHAIN RULE (INTEGRATION)} \\ &= \frac{1}{10} (x^2+2x-1)^5 + C\end{aligned}$$

AUTOMATIC BY SUBSTITUTION

$$\begin{aligned}\int (x+1)(x^2+2x-1)^4 dx &= \int \frac{du}{dx} u^4 \left(\frac{du}{dx} \right) dx \\ &= \int \frac{1}{2} u^4 du = \frac{1}{10} u^5 + C \\ &= \frac{1}{10} (x^2+2x-1)^5 + C\end{aligned}$$

$$\begin{aligned}u &= x^2+2x-1 \\ \frac{du}{dx} &= 2x+2 \\ du &= (2x+2)dx\end{aligned}$$

296. $\int \sin x \cos^4 x dx = -\frac{1}{5} \cos^5 x + C$

$$\begin{aligned}\int \sin x \cos^4 x dx &= \dots \text{BY REVERSE CHAIN RULE (INTEGRATION)} \\ &= -\frac{1}{5} \cos^5 x + C\end{aligned}$$

[THE SUBSTITUTION $u = \cos x$ ALSO WORKS WELL]

297. $\int \frac{2x+6}{x^2+6x+1} dx = \ln|x^2+6x+1| + C$

$$\begin{aligned}\int \frac{2x+6}{x^2+6x+1} dx &= \dots \text{OR USE LOGIC } \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C \\ &= \ln|x^2+6x+1| + C\end{aligned}$$

298. $\int \frac{1}{x(x-4)} dx = \frac{1}{4} \ln \left| \frac{x-4}{x} \right| + C$

$$\int \frac{1}{z(z-4)} dz = \dots \text{ PARTIAL FRACTIONS BY INVERSE (value op)}$$

$$= \int \frac{\frac{1}{4}}{z-4} - \frac{\frac{1}{4}}{z} dz = \frac{1}{4} \ln |z-4| - \frac{1}{4} \ln |z| + C$$

$$= \frac{1}{4} \ln \left| \frac{|z-4|}{|z|} \right| + C$$

299. $\int \frac{\tan x}{\sqrt{1+\cos 2x}} dx = \pm \frac{1}{\sqrt{2}} \sec x + C$

$$\int \frac{\tan x}{\sqrt{1+\cos 2x}} dx = \int \frac{\tan x}{\sqrt{1+2\cos^2 x - 1}} dx = \int \frac{\tan x}{\sqrt{2\cos^2 x}} dx$$

NOW IF $\cos x > 0$

$$= \int \frac{\tan x}{\sqrt{2} |\cos x|} dx = \int \frac{1}{\sqrt{2}} \tan x \sec x dx = \frac{1}{\sqrt{2}} \sec x + C$$

AND IF $\cos x < 0$

$$= \int \frac{\tan x}{-\sqrt{2} |\cos x|} dx = \int \frac{1}{\sqrt{2}} \tan x \sec x dx = -\frac{1}{\sqrt{2}} \sec x + C$$

COLLECTING RESULTS

$$\dots = \pm \frac{1}{\sqrt{2}} \sec x + C$$

300. $\int \cos^2 x \sin^2 x dx = \frac{1}{8}x - \frac{1}{32} \sin 4x + C$

$$\int \cos^2 x \sin^2 x dx = \int \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx$$

$$= \int \frac{1}{4} - \frac{1}{2} \cos^2 2x dx = \int \frac{1}{4} - \frac{1}{4} \left(\frac{1}{2} + \frac{1}{2} \cos 4x \right) dx$$

$$= \int \frac{1}{4} - \frac{1}{8} - \frac{1}{8} \cos 4x dx = \int \frac{1}{8} - \frac{1}{8} \cos 4x dx$$

$$= \frac{1}{8}x - \frac{1}{32} \sin 4x + C$$

ALTERNATIVE METHOD

$$\int \cos^2 x \sin^2 x dx = \int \frac{1}{4} (4 \cos^2 x \sin^2 x) dx = \int \frac{1}{4} (2 \sin 2x)^2 dx$$

$$= \int \frac{1}{4} (\sin 2x)^2 dx = \int \frac{1}{4} \sin^2 2x dx$$

$$= \int \frac{1}{4} \left(\frac{1}{2} - \frac{1}{2} \cos 4x \right) dx = \int \frac{1}{8} - \frac{1}{8} \cos 4x dx$$

$$= \frac{1}{8}x - \frac{1}{32} \sin 4x + C$$

301. $\int \frac{3x^3 + 5x}{x^2 + 1} dx = \frac{3}{2}x^2 + \ln(x^2 + 1) + C$

$$\begin{aligned} & \int \frac{3x^3 + 5x}{x^2 + 1} dx = \dots \text{BY ALGEBRAIC DIVISION OR SUBSTITUTION} \dots \\ &= \int \frac{3x(x^2+1) + 2x}{x^2+1} dx = \int 3x + \frac{2x}{x^2+1} dx \\ & \quad \text{or the easier } \int \frac{3x}{x^2+1} dx = \ln(1+x^2) + C \\ &= \frac{3}{2}x^2 + \ln(x^2+1) + C \end{aligned}$$

ANOTHER WAY BY INVERSE SUBSTITUTION

$$\begin{aligned} & \int \frac{3x^3 + 5x}{x^2 + 1} dx = \int \frac{3x^3 + 5x}{u} \left(\frac{du}{dx} \right) \\ &= \int \frac{3x^3 + 5x}{2xu} du = \int \frac{3x^2 + 5}{2u} du \\ &= \int \frac{3u^2 + 5}{2u} du = \int \frac{3}{2}u + \frac{5}{2u} du \\ &= \frac{3}{2}u^2 + \ln|u| + C = \frac{3}{2}(u^2) + \ln(u) + C \end{aligned}$$

$u = x^2 + 1$
 $\frac{du}{dx} = 2x$
 $du = \frac{du}{2x} dx$
 $x^2 = u - 1$
 $3x^2 = 3u - 3$
 $3x^2 + 5 = 3u + 2$

302. $\int \sin^2 2x dx = \frac{1}{2}x - \frac{1}{8}\sin 4x + C$

$$\begin{aligned} \int \sin^2 2x dx &= \int \frac{1}{2} - \frac{1}{2}\cos 4x dx = \frac{1}{2}x - \frac{1}{8}\sin 4x + C \\ & \quad \boxed{\begin{array}{l} [\cos 2x] = 1 - \frac{2\sin^2 x}{2} \\ [\cos 4x] = 1 - 2\sin^2 2x \end{array}} \end{aligned}$$

303. $\int \frac{\cos 2x}{\cos^2 x} dx = 2x - \tan x + C$

$$\begin{aligned} \int \frac{\cos 2x}{\cos^2 x} dx &= \int \frac{2\cos^2 x - 1}{\cos^2 x} dx = \int 2 - \frac{1}{\cos^2 x} dx \\ &= \int 2 - \sec^2 x dx = \underline{2x - \tan x + C} \end{aligned}$$

304. $\int \sec^3 x \tan x dx = \frac{1}{3}\sec^3 x + C$

$$\begin{aligned} \int \sec^2 x \tan x dx &= \dots \text{BY REVERSE CHAIN RULE (INTEGRATION), SINCE} \\ & \quad \frac{d}{dx}(\sec x) = \sec x \tan x \quad (\sec^2 x) = \sec x \tan x \\ &= \underline{\sec^3 x + C} \\ & \quad \boxed{\text{[THE SUBSTITUTION } u = \sec x \text{ ALSO WORKS WELL]}}$$

305. $\int x \left[(\ln x)^2 - 1 \right] dx = \frac{1}{4} x^2 \left[2(\ln|x|)^2 - 2\ln|x| + 1 \right] + C$

$$\begin{aligned} & \int x \left[(\ln|x|)^2 - 1 \right] dx = \dots \text{INTEGRATION BY PARTS} \dots \\ & \quad \boxed{\begin{array}{|c|c|} \hline (\ln|x|)^2 - 1 & \frac{1}{2}x^2 \\ \hline \frac{1}{2}x^2 & x \\ \hline \end{array}} \\ & = \frac{1}{2}x^2 \left[(\ln|x|)^2 - 1 \right] - \left(\cancel{\frac{1}{2}x^2} \ln|x| dx \right) \rightarrow \text{INTEGRATION BY PARTS AGAIN} \\ & = \frac{1}{2}x^2 \left[(\ln|x|)^2 - 1 \right] - \frac{1}{2}x^2 \ln|x| + \int \frac{1}{2}x^2 dx \\ & = \frac{1}{2}x^2 \left[(\ln|x|)^2 - 1 \right] - \frac{1}{2}x^2 \ln|x| + \frac{1}{6}x^3 + C \\ & = \frac{1}{2}x^2 \left[(\ln|x|)^2 - 2\ln|x| + 1 \right] + C \end{aligned}$$

306. $\int \frac{x^2}{x^3 + 5} dx = \frac{1}{3} \ln|x^3 + 5| + C$

$$\begin{aligned} & \int \frac{x^2}{x^3 + 5} dx = \dots \text{or in the form } \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C \\ & = \frac{1}{3} \int \frac{3x^2}{x^3 + 5} dx = \frac{1}{3} \ln|x^3 + 5| + C \\ & \quad [\text{THE SUBSTITUTION } u = x^3 + 5 \text{ ALSO WORKS WELL}] \end{aligned}$$

307. $\int \frac{2x+1}{3x-1} dx = \frac{2}{3}x + \frac{5}{9} \ln|3x-1| + C$

$$\begin{aligned} & \int \frac{2x+1}{3x-1} dx = \text{MANIPULATION & SURT} \\ & = \int \frac{\frac{2}{3}(3x-1) + \frac{5}{3}}{(3x-1)} dx = \int \frac{\frac{2}{3}}{3x-1} + \frac{\frac{5}{3}}{3x-1} dx \\ & = \frac{2}{3}x + \frac{5}{3} \ln|3x-1| + C \\ & \quad [\text{THE SUBSTITUTION } u = 3x-1 \text{ IS A BETTER ALTERNATIVE}] \end{aligned}$$

308. $\int x(3+x^2)^4 dx = \frac{1}{10}(3+x^2)^5 + C$

$$\begin{aligned} & \int x(3+x^2)^4 dx = \dots \text{BY INVERSE CHAIN RULE (INVARIATION)} \\ & = \frac{1}{10}(3+x^2)^5 + C \\ & \quad [\text{THE SUBSTITUTION } u = 3+x^2 \text{ ALSO WORKS WELL}] \end{aligned}$$

309. $\int \frac{9}{x^2\sqrt{9-x^2}} dx = -\frac{\sqrt{9-x^2}}{x} + C$

$$\begin{aligned}
 \int \frac{9}{x^2\sqrt{9-x^2}} dx &= \dots \text{ BY * TRIGONOMETRIC SUBSTITUTION} \\
 &= \int \frac{9}{(3\sin\theta)^2\sqrt{9-(3\sin\theta)^2}} (3\cos\theta d\theta) \\
 &= \int \frac{27\cos\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}} d\theta \\
 &= \int \frac{3\cos\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}} d\theta = \int \frac{3\cos\theta}{\sin^2\theta\cos\theta} d\theta \\
 &= \int \frac{3\cos^2\theta}{\sin^2\theta} d\theta = \int \frac{1+2\cos^2\theta}{\sin^2\theta} d\theta = -\cot\theta + C \\
 &= -\frac{\cot\theta}{\sin\theta} + C = -\frac{\sqrt{1-\sin^2\theta}}{\sin\theta} + C \\
 &= -\frac{\sqrt{1-\frac{x^2}{9}}}{\frac{x}{3}} + C = -\frac{\sqrt{\frac{9-x^2}{9}}}{\frac{x}{3}} + C \\
 &= -\frac{\sqrt{9-x^2}}{x} + C = -\frac{\sqrt{9-x^2}}{x} + C //
 \end{aligned}$$

310. $\int \frac{\cos x}{\sqrt{\sin^3 x}} dx = \frac{2}{\sqrt{\sin x}} + C$

$$\begin{aligned}
 \int \frac{\cos x}{\sqrt{\sin^3 x}} dx &= \int (\cos x)(\sin x)^{-\frac{1}{2}} dx \\
 &= \text{BY INVERSE CHAIN RULE (INTEGRATION)} \\
 &= -2(\sin x)^{\frac{1}{2}} + C = -\frac{2}{\sqrt{\sin x}} + C \\
 &[=\text{THE SUBSTITUTION } u=\sin x \text{ OR } u=\sqrt{\sin x} \text{ TWO WAYS}]
 \end{aligned}$$

311. $\int \frac{x-3}{\sqrt{x+1}-2} dx = 2x + \frac{2}{3}(x+1)^{\frac{3}{2}} + C$

$$\begin{aligned}
 \int \frac{x-3}{\sqrt{x+1}-2} dx &= \dots \text{ BY MANIPULATION SINCE } (2x+1)-4=2-3 \\
 &= \int \frac{(2x-3)(\sqrt{x+1}+2)}{(2x+1)-2)(\sqrt{x+1}+2)} dx = \int \frac{(2x-3)(\sqrt{x+1}+2)}{(2x+1)-4} dx \\
 &= \int \frac{(2x-3)(4(x+1)+2)}{(2x+3)(\sqrt{x+1})} dx = \int \frac{C(x+1)^{\frac{1}{2}}+2}{(2x+3)} dx \\
 &= 2x + \frac{2}{3}(x+1)^{\frac{3}{2}} + C \\
 &=\text{ALTERNATIVE BY SUBSTITUTION} \\
 &\int \frac{x-3}{\sqrt{x+1}-2} dx = \int \frac{u^2-4}{u-2} (2u du) \\
 &= \int \frac{2u(u-2)(u+2)}{u-2} du = \int 2u^2+4u du \\
 &= 2u^3+3u^2+C = \frac{2}{3}(x+1)^{\frac{3}{2}}+2(x+1)+C \\
 &= \frac{2}{3}(x+1)^{\frac{3}{2}}+2x+C \\
 &[=\text{THE SUBSTITUTION } u=\sqrt{x+1}-2 \text{ TWO WAYS}]
 \end{aligned}$$

312. $\int \frac{x-4}{x^2-4} dx = \left[\frac{1}{2} \ln|x^2-4| + \ln\left|\frac{x+2}{x-2}\right| + C \right]$

$$\begin{aligned} & \int \frac{x-4}{x^2-4} dx = \int \frac{2x}{x^2-4} dx - \frac{4}{x^2-4} dx \quad \text{BY DIRECTLY INTO PARTIAL FRACTIONS} \\ & = \int \frac{2x}{x^2-4} dx - \int \frac{4}{(x-2)(x+2)} dx \quad \text{CROSSING FRACTION BY INVERSION} \\ & = \frac{1}{2} \ln|x^2-4| - \int \frac{4}{x^2-4} dx = \frac{1}{2} \ln|x^2-4| - \ln|x-2| + \ln|x+2| + C \\ & \text{OR DIRECTLY INTO PARTIAL FRACTIONS BY INVERSION} \\ & \int \frac{x-4}{x^2-4} dx = \int \left(\frac{A}{x-2} + \frac{B}{x+2} \right) dx = \int \frac{\frac{A}{x-2} + \frac{B}{x+2}}{dx} dx \\ & = \frac{1}{2} \ln|2x-4| - \frac{1}{2} \ln|2x+4| + C \end{aligned}$$

313. $\int \frac{2 \sin x}{\cos x + \sin x} dx = \left[\begin{array}{l} x - \ln|\cos x + \sin x| + C \\ x + \frac{1}{2} \ln|\sec 2x| - \frac{1}{2} \ln|\sec 2x + \tan 2x| + C \end{array} \right]$

$$\begin{aligned} & \int \frac{2 \sin x}{\cos x + \sin x} dx = \dots \text{BY MANIPULATION} \\ & = \int \frac{\cos x + \sin x + \sin x - \cos x}{\cos x + \sin x} dx \\ & = \int \frac{\cos x + \sin x}{\cos x + \sin x} dx + \int \frac{\sin x - \cos x}{\cos x + \sin x} dx \rightarrow \int \frac{\cos x}{\cos x + \sin x} dx = \ln|\cos x + \sin x| + C \\ & = x - \ln|\cos x + \sin x| + C \\ & \text{ALTERNATIVE BY TRIGONOMETRIC MANIPULATIONS} \\ & \int \frac{2 \sin x}{\cos x + \sin x} dx = \int \frac{2 \sin x (\cos x - \sin x)}{(\cos x + \sin x)(\cos x - \sin x)} dx \quad \text{cancel} \\ & = \int \frac{2 \sin x \cos x - 2 \sin^2 x}{\cos^2 x - \sin^2 x} dx = \int \frac{\sin 2x + (1 - 2 \sin^2 x) - 1}{\cos 2x} dx \\ & = \int \frac{\sin 2x + \cos 2x - 1}{\cos 2x} dx = \int \frac{\tan 2x + 1 - \sec 2x}{\cos 2x} dx \\ & \text{NOW THESE ARE STANDARD INTEGRALS, WITH OF COURSE ONE EASILY BE PROVEN (NOT HERE)} \\ & = \frac{1}{2} \ln|\sec 2x| + x - \frac{1}{2} \ln|\sec 2x + \tan 2x| + C \\ & = x + \frac{1}{2} \ln \left| \frac{\sec 2x}{\sec 2x + \tan 2x} \right| + C = x - \frac{1}{2} \ln \left| \frac{\sec 2x + \tan 2x}{\sec 2x} \right| + C \\ & = x - \frac{1}{2} \ln \left| 1 + \tan 2x \cos 2x \right| + C \\ & = x - \frac{1}{2} \ln \left| \frac{\cos 2x + \sin 2x}{\cos 2x} \right| + C = x - \frac{1}{2} \ln|1 + \sin 2x| + C \\ & = x - \frac{1}{2} \ln \left| \cos 2x + \sin 2x + 2 \sin x \cos x \right| + C \\ & = x - \frac{1}{2} \ln \left| (\cos x + \sin x)^2 \right| + C \\ & = x - \frac{1}{2} \ln(\cos x + \sin x)^2 + C \quad \text{(AS ABOVE)} \end{aligned}$$

314. $\int (1-x^{-2})^2 dx = 2 + \frac{2}{x} - \frac{1}{3x^3} + C$

$$\begin{aligned} \int (1-x^{-2})^2 dx &= \int (1+x^{-2}+\bar{x}^{-4}) dx = x + 2x^{-1} - \frac{1}{3}x^{-3} + C \\ &= 2 + \frac{2}{x} - \frac{1}{3x^3} + C \end{aligned}$$

315. $\int \frac{\sqrt{\tan x}}{\cos^2 x} dx = \frac{2}{3} \sqrt{\tan^3 x} + C$

$$\begin{aligned}\int \frac{\sqrt{\tan x}}{\cos^2 x} dx &= \int (\tan x)^{\frac{1}{2}} \sec^2 x dx \quad \text{BY DOUBLE ANGLE RULE (UNSTATED)} \\ &= \frac{2}{3} (\tan x)^{\frac{3}{2}} + C = \frac{2}{3} \sqrt{\tan^3 x} + C\end{aligned}$$

[THE SUBSTITUTION $u = \tan x$, OR $u = \sqrt{\tan x}$ ALSO WORK WELL]

316. $\int (3\sin x + \cos x)^2 dx = \begin{bmatrix} 5x - 2\sin 2x - \frac{3}{2}\cos 2x + C \\ 5x - 2\sin 2x + 3\sin^2 x + C \\ 5x - 2\sin 2x - 3\cos^2 x + C \end{bmatrix}$

$$\begin{aligned}\int (3\sin x + \cos x)^2 dx &= \int 9\sin^2 x + 6\sin x \cos x + \cos^2 x dx \\ &= \int \left(\frac{9}{2} - \frac{3\cos 2x}{2} + 3\sin 2x + \frac{1}{2} + \frac{1}{2}\cos 2x\right) dx \\ &= \int 5 - 4\cos 2x + 3\sin 2x dx = 5x - 2\sin 2x - 3\cos 2x + C\end{aligned}$$

RE: IF CONVENIENT TO
3 $\sin^2 x$, OR $-3\cos^2 x$
= 5x - 2\sin 2x + 3\sin^2 x + C
= 5x - 2\sin 2x - 3\cos^2 x + C

317. $\int \frac{\sec^2 x}{(1+\tan x)^3} dx = -\frac{1}{2(1+\tan x)^2} + C$

$$\begin{aligned}\int \frac{\sec^2 x}{(1+\tan x)^3} dx &= \int (1+\tan x)^{-3} \sec^2 x dx \dots \text{BY DOUBLE ANGLE RULE (UNSTATED)} \\ &= -\frac{1}{2}(1+\tan x)^{-2} + C \\ &= -\frac{1}{2(1+\tan x)^2} + C\end{aligned}$$

[THE SUBSTITUTION $u = 1+\tan x$, OR $u = \tan x$ ALSO WORK WELL]

318. $\int \frac{1}{\cos^2 x \sin^2 x} dx = \begin{bmatrix} -2 \cot 2x + C \\ -\cot x + \tan x + C \end{bmatrix}$

$$\begin{aligned} \int \frac{1}{\cos^2 x \sin^2 x} dx &= \int \frac{1}{(1+\cot^2 x)(1-\cot^2 x)} dx \\ &= \int \frac{1}{1-\cot^4 x} dx = \int \frac{4}{1-\cot^2 x} dx = \int \frac{4}{\sin^2 2x} dx \\ &= \int 4 \csc^2 2x dx = -2 \cot 2x + C \end{aligned}$$

ALTERNATIVE CALCULATION:

$$\begin{aligned} \int \frac{1}{\cos^2 x \sin^2 x} dx &= \int \frac{4}{\sin^2 2x} dx = \int \frac{4}{\sin^2 2x} \csc^2 2x dx \\ &= \int \frac{4}{\sin^2 2x} \csc^2 2x dx = \int 4 \cot^2 2x dx = -2 \cot 2x + C \end{aligned}$$

ANOTHER ALTERNATIVE:

$$\begin{aligned} \int \frac{1}{\cos^2 x \sin^2 x} dx &= \int \frac{\cot x + \operatorname{cosec} x}{\cos x \sin x} dx = \dots \text{ SPLITTING THE FRACTION} \dots \\ &= \int \frac{1}{\sin x} + \frac{1}{\cos x} dx = \int \operatorname{cosec} x + \operatorname{sec} x dx \\ &= -\operatorname{ln} |\operatorname{cosec} x| + C \end{aligned}$$

319. $\int \cot 2x dx = \frac{1}{2} \ln |\sin 2x| + C$

$$\begin{aligned} \int \cot 2x dx &= \int \frac{\cos 2x}{\sin 2x} dx = \frac{1}{2} \int \frac{2 \cos 2x}{\sin 2x} dx \\ &\quad \text{OR THE REIN } \int \frac{f'(x)}{f(x)} dx \\ &= \frac{1}{2} \ln |\sin 2x| + C \end{aligned}$$

[The substitution $u = \sin 2x$ also works here]

320. $\int 2^x dx = \frac{2^x}{\ln 2} + C$

$$\begin{aligned} \int 2^x dx &\approx \frac{1}{\ln 2} \int 2^x \ln 2 dx = \frac{2^x}{\ln 2} + C \\ &\quad \boxed{\frac{d}{dx}(2^x) = 2^x \ln 2} \end{aligned}$$

321. $\int \frac{\sin x + \sin x \cos x}{1 - \cos x} dx = \left[\begin{array}{l} \cos x + 2 \ln |1 - \cos x| + C \\ \cos x + 2 \ln |\sin x| - 2 \ln |\cot x + \operatorname{cosec} x| + C \end{array} \right]$

$$\begin{aligned}
 & \int \frac{\sin x + \sin x \cos x}{1 - \cos x} dx = \int \frac{\sin x(1 + \cos x)}{1 - \cos x} dx \\
 &= \int \frac{\sin x(\cos^2 x)}{(1 - \cos x)(1 + \cos x)} dx = \int \frac{\sin x(1 + 2\cos x + \cos^2 x)}{1 - \cos^2 x} dx \\
 &= \int \frac{\sin x(2 + 2\cos x)}{\sin^2 x} dx = \int 2\csc x + 2\cot x dx = \int 2\csc x dx + \int 2\cot x dx \\
 &= \int 2\csc x + 2\cot x + \csc x - \sin x dx = \int 2\csc x + 2\cot x - \sin x dx \\
 &= -2 \ln |\csc x + \cot x| + 2 \ln |\sin x| + \tan x + C \\
 &= \cos x + 2 \ln \left| \frac{\sin x}{\csc x + \cot x} \right| + C = \cos x + 2 \ln \left| \frac{\sin x}{1 + \cos x} \right| + C \\
 &\Rightarrow \text{INCORRECT FOR ALGEBRAIC SIGN IN } \cos x \rightarrow \\
 &= \cos x + 2 \ln \left| \frac{1 - \cos x}{1 + \cos x} \right| + C = \cos x + 2 \ln \left| \frac{(1 - \cos x)^2}{1 + \cos x} \right| + C \\
 &= \cos x + 2 \ln |1 - \cos x| + C
 \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}
 & \int \frac{\sin x + \sin x \cos x}{1 - \cos x} dx = \int \frac{\sin x(1 + \cos x)}{u} \left(\frac{du}{\sin x} \right) \quad u = 1 - \cos x \\
 &= \int \frac{1 + \cos x}{u} du = \int \frac{2 - u}{u} du = \int \frac{2}{u} - 1 du \\
 &= 2 \ln |u| - u + C = 2 \ln |1 - \cos x| - (1 - \cos x) + C \\
 &= \cos x + 2 \ln |1 - \cos x| + C
 \end{aligned}$$

(INTEGRALS OF THE TYPE $\int \frac{\sin x}{1 - \cos x} + \frac{\sin x \cos x}{1 - \cos x} dx$ ALSO WORK, BUT ARE LENGTHY)

322. $\int \frac{1}{\cos^4 x} dx = \tan x + \frac{1}{3} \tan^3 x + C$

$$\begin{aligned}
 & \int \frac{1}{\cos^4 x} dx = \int \sec^2 x dx = \int \sec^2 \sec^2 x dx \\
 &= \int (1 + \tan^2 x) \sec^2 x dx = \int \sec^2 x + \tan^2 x \sec^2 x dx \\
 &\quad \text{BY REVERSE CHAIN RULE (INTRODUCTION)} \\
 &= \sec x + \frac{1}{3} \tan^3 x + C
 \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned}
 & \int \frac{1}{\cos^4 x} dx = \int \frac{1}{\cos^2 x} (\sec^2 x dx) \quad u = \tan x \\
 &= \int \frac{1}{\cos^2 x} du = \int \sec^2 du = \int 1 + \tan^2 du \\
 &= \int 1 + u^2 du = u + \frac{1}{3} u^3 + C \quad \frac{du}{dx} = \frac{1}{\cos^2 x} \\
 &= \tan x + \frac{1}{3} \tan^3 x + C
 \end{aligned}$$

323. $\int \frac{\sin x \cos x}{\sqrt{1+\cos 2x}} dx = \begin{cases} -\frac{1}{2}\sqrt{1+\cos 2x} + C & \\ \left[\begin{array}{l} -\frac{\cos x}{\sqrt{2}} + C \quad \text{if } \cos x > 0 \\ \frac{\cos x}{\sqrt{2}} + C \quad \text{if } \cos x < 0 \end{array} \right] & \end{cases}$

$$\begin{aligned} \int \frac{\sin x \cos x}{\sqrt{1+\cos 2x}} dx &= \int \frac{\frac{1}{2}(\sin 2x)}{\sqrt{1+2\cos^2 x - 1}} dx \\ &= \int \frac{\frac{1}{2}\sin 2x(1+\cos 2x)^{-\frac{1}{2}}}{\sqrt{1+2\cos^2 x - 1}} dx \quad \text{BY REVERSE CHAIN RULE (INSPECTION)} \\ &\approx -\frac{1}{2}(1+\cos 2x)^{\frac{1}{2}} + C = -\frac{1}{2}\sqrt{1+2\cos 2x} + C \\ \text{ALTERNATIVE} \\ \int \frac{\sin x \cos x}{\sqrt{1+\cos 2x}} dx &= \int \frac{\sin x \cos x}{\sqrt{1+2\cos^2 x - 1}} dx = \int \frac{\sin x \cos x}{\sqrt{2\cos^2 x}} dx \\ &= \frac{1}{\sqrt{2}} \int \frac{\sin x \cos x}{|\cos x|} dx = \begin{cases} \frac{1}{\sqrt{2}} \int \sin x dx \quad \text{if } \cos x > 0 \\ -\frac{1}{\sqrt{2}} \int \sin x dx \quad \text{if } \cos x < 0 \end{cases} \\ &= \begin{cases} \frac{1}{\sqrt{2}}\cos x \quad \text{if } \cos x > 0 \\ -\frac{1}{\sqrt{2}}\cos x \quad \text{if } \cos x < 0 \end{cases} \\ [\text{THE SUBSTITUTION } u = \cos 2x, u = (1+\cos 2x)^{\frac{1}{2}}, \text{ AND WORK}] \end{aligned}$$

324. $\int \frac{\ln x^2}{x} dx = (\ln|x|)^2 + C$

$$\begin{aligned} \int \frac{\ln x^2}{x} dx &= \int \frac{2\ln x}{x} dx = \int \frac{2}{x} (\ln x) dx \\ &= (\ln x)^2 + C \\ [\text{THE SUBSTITUTION } u = \ln x \text{ WORKS WELL BEFORE WRITING}] \quad [\ln x^2 AS 2\ln x, OR SIMPLY INTEGRATION BY PARTS] \end{aligned}$$

325. $\int \sin x \cos^2\left(\frac{1}{2}x\right) dx = -\cos^4\left(\frac{1}{2}x\right) + C$

$$\begin{aligned} \int \sin x \cos^2\left(\frac{1}{2}x\right) dx &= \int [2\sin(\frac{1}{2}x)\cos(\frac{1}{2}x)] \cos^2(\frac{1}{2}x) dx \\ &= \int 2\sin(\frac{1}{2}x)\cos^3(\frac{1}{2}x) dx \quad \text{BY REVERSE CHAIN RULE (INSPECTION)} \\ &= ... -\cos^4(\frac{1}{2}x) + C \quad \text{OR USING THE SUBSTITUTION } u=\cos(\frac{1}{2}x) \end{aligned}$$

326. $\int \frac{e^{3x}+1}{e^x+1} dx = \frac{1}{2}e^{2x} - e^x + x + C$

$$\begin{aligned} \int \frac{e^{3x}+1}{e^x+1} dx &= \dots \quad A+B^2 = 4(8)(A-B+C) \\ &= \int \frac{(e^x+1)(e^{2x}-e^x+1)}{e^x+1} dx = \int e^{2x}-e^x+1 dx \\ &= \frac{1}{2}e^{2x}-e^x+x+C \end{aligned}$$

ALTERNATIVE (VERIFICATION) WITH SUBSTITUTION

$$\begin{aligned} \int \frac{e^{3x}}{e^x+1} dx &= \int \frac{u^3+1}{u+1} du \quad u = e^x \\ &\text{NOT BY LONG DIVISION, THE SUM OF CUBES IDENTITY} \\ &\text{FROM THESE, JUST MANIPULATIONS...} \\ &= \int \frac{u^3(u+1)-(u+1)+(u+1)}{u+1} du = \int (u^3-u+1) du \\ &= \int (u-1+\frac{1}{u}) du = \frac{1}{2}u^2-u+\ln|u|+C \\ &= \frac{1}{2}e^{2x}-e^x+\ln|e^x|+C = \underline{\frac{1}{2}e^{2x}-e^x+x+C} \end{aligned}$$

327. $\int \frac{\sqrt{x}}{x-1} dx = 2\sqrt{x} + \ln \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| + C$

$$\begin{aligned} \int \frac{\sqrt{x}}{x-1} dx &= \dots \text{ BY SUBSTITUTION} \quad u = \sqrt{x} \\ &= \int \frac{u}{u^2-1} (2u du) = \int \frac{2u^2}{u^2-1} du \quad 2u du = dx \\ &\text{IMPROPER FRACTION, NEED TO BE DIVIDED OUT} \\ &= \int \frac{2(u^2-1)+2}{u^2-1} du = \int 2 + \frac{2}{u^2-1} du = \int 2 + \frac{2}{(u-1)(u+1)} du \\ &\text{PARTIAL FRACTIONS BY INSPECTION (GONE OB)} \\ &= \int 2 + \frac{1}{u-1} - \frac{1}{u+1} du = 2u + b(\ln|u-1| - \ln|u+1|) + C \\ &= 2u + h(\frac{1}{u+1}) + C = \underline{2\sqrt{x} + \ln \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| + C} \end{aligned}$$

328. $\int 2^x 3^x dx = \frac{6^x}{\ln 6} + C$

$$\begin{aligned} \int 2^x 3^x dx &= \int (2 \cdot 3)^x dx = \int 6^x dx \\ &= \int \frac{1}{\ln 6} (6^x) dx = \underline{\frac{6^x}{\ln 6} + C} \end{aligned}$$

329. $\int 3^{2x+1} dx = \left[\frac{3^{2x+1}}{2\ln 3} + C \right]$

$$\begin{aligned}\int 3^{2x+1} dx &= \frac{1}{2\ln 3} \int 3^{2x+1} \ln 3 \times 2 dx \\ &\quad \text{BY REVERSE CHAIN RULE (EXPLANATION), SO:} \\ \frac{d}{dx}(a^x) &= a^x \ln a \\ &= \frac{3^{2x+1}}{2\ln 3} + C\end{aligned}$$

VARIATION

$$\begin{aligned}\int 3^{2x+1} dx &= \int 3^x \cdot 3^1 dx = 3 \int (3^x)^2 dx = 3 \int 9^x dx \\ &= \frac{3}{\ln 9} \int 9^x dx = \frac{3x9^x}{\ln 9} + C\end{aligned}$$

330. $\int \frac{2x^{\frac{3}{2}}+1}{x^2+2x} dx = \ln(x^2+2\sqrt{x}) + C$

$$\begin{aligned}\int \frac{2x^{\frac{3}{2}}+1}{x^2+2x} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int \frac{2u^{\frac{3}{2}}+1}{u^4+2u^2} (2u du) = \int \frac{du^3+2u}{u^4+2u} du \\ &\quad \text{BY INSPECTION AS THIS IS OF THE FORM } \int \frac{f(u)}{u^2} du = \ln|f(u)| + C \\ &= \ln|u^3+2u| + C = \ln(x^3+2x^2) + C\end{aligned}$$

331. $\int \frac{1}{x(1+\sqrt{x})} dx = 2\ln\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right) + C$

$$\begin{aligned}\int \frac{1}{x(1+\sqrt{x})} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int \frac{1}{u^2(1+u)} (2u du) = \int \frac{2}{u(1+u)} du \\ &\quad \text{PARTIAL FRACTIONS BY INSPECTION (COME ON)} \\ &= \int \frac{2}{u} - \frac{2}{1+u} du = 2\ln|u| - 2\ln|1+u| + C = 2\ln\left|\frac{u}{1+u}\right| + C \\ &= 2\ln\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right) + C \\ &\quad [\text{THE SUBSTITUTION } u=1/\sqrt{x} \text{ ALSO WORKS.}]\\ &\quad \boxed{\begin{array}{l} u = \sqrt{x} \\ u^2 = x \\ 2u du = dx \end{array}}$$

332. $\int \frac{\sqrt{x}}{\sqrt{x}-1} dx = x + 2\sqrt{x} + 2\ln|\sqrt{x}-1| + C$

$$\begin{aligned} \int \frac{\sqrt{x}}{\sqrt{x}-1} dx &= \dots \text{ BY SUBSTITUTION } \\ &= \int \frac{u}{u-1} (2u du) = \int \frac{2u^2}{u-1} du \\ &\text{BY A SECOND SUBSTITUTION OR MANIPULATION (DIVISION)} \\ &= \int \frac{2u(u-1)+2(u-1)+2}{u-1} du = \int 2u+2 + \frac{2}{u-1} du \\ &= u^2 + 2u + 2\ln|u-1| + C = \underline{\underline{u^2 + 2\sqrt{u} + 2\ln|\sqrt{u}-1| + C}} \\ &\text{[THE SUBSTITUTION } u=\sqrt{x} \text{ ALSO WORKS]} \end{aligned}$$

333. $\int \frac{1}{x(1+x^2)} dx = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

$$\begin{aligned} \int \frac{1}{x(1+x^2)} dx &= \dots \text{ BY PARTIAL FRACTION(S) } \\ \frac{1}{x(1+x^2)} &\equiv \frac{A}{x} + \frac{Bx+C}{1+x^2} \\ 1 &\equiv A(x^2+1) + Bx^2 + Cx \\ 1 &\equiv A+Ax^2+Bx^2+Cx \\ 1 &\equiv (A+B)x^2+Cx+A \\ \therefore A &= 1, C = 0, B = -1 \\ &= \int \frac{1}{x} - \frac{x}{1+x^2} dx = \ln|x| - \frac{1}{2}\ln(1+x^2) + C \\ &= \ln|x| - \ln(C(1+x^2))^\frac{1}{2} + C = \underline{\underline{\ln\left|\frac{x}{\sqrt{1+x^2}}\right| + C}} \\ \text{ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION} \\ \int \frac{1}{x(1+x^2)} dx &= \int \frac{1}{x \tan(\theta)(1+\tan^2\theta)} (\sec^2\theta d\theta) \\ &= \int \frac{\sec^2\theta}{x \sin(\theta)} d\theta = \int \frac{1}{x \sin(\theta)} d\theta \\ &= \int \frac{\cos(\theta)}{x \sin(\theta)} d\theta = \ln|\csc(\theta)| + C \\ &= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C \end{aligned}$$

334. $\int \sin^4 x \sin 2x dx = \frac{1}{3} \sin^6 x + C$

$$\begin{aligned} \int \sin^4 x \sin 2x dx &= \int \sin^4 x (2\sin x \cos x) dx = \int 2\sin^5 x \cos x dx \\ &\text{BY REVERSE CHAIN RULE} \\ &= \frac{1}{3} \sin^6 x + C \\ &\text{[THE SUBSTITUTION } u = \sin x \text{ ALSO WORKS]} \end{aligned}$$

335. $\int (\tan x + \cot x)^2 dx = \tan x - \cot x + C$

$$\begin{aligned}\int (\tan x + \cot x)^2 dx &= \int \tan^2 x + 2\tan x \cot x + \cot^2 x dx \\ &= \int (\sec^2 x - 1) + 2\cancel{\sec x \tan x} + (\csc^2 x - 1) dx \\ &= \int \sec^2 x + \csc^2 x dx = \tan x - \cot x + C\end{aligned}$$

336. $\int \frac{x^2 + 2x - 2}{x^2 - 2x + 2} dx = x + \ln(x^2 - 2x + 2) + C$

$$\begin{aligned}\int \frac{x^2 + 2x - 2}{x^2 - 2x + 2} dx &= \dots \text{ BY INSPECTION OR RATIONALISATION} \\ &= \int \frac{(x^2 - 2x + 2) + 4x - 4}{x^2 - 2x + 2} dx = \int 1 + \frac{4x - 4}{x^2 - 2x + 2} dx \\ &= \int 1 + 2 \left(\frac{2x - 2}{x^2 - 2x + 2} \right) dx = x + 2 \ln|x^2 - 2x + 2| + C \\ &\quad \uparrow \text{IF THE FORM } \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C\end{aligned}$$

337. $\int x \sin x \cos x dx = -\frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + C$

$$\begin{aligned}\int x \sin x \cos x dx &= \int x \times \frac{1}{2} (2 \sin x \cos x) dx = \int \frac{1}{2} x \sin 2x dx \\ &\quad \text{INTEGRATION BY PARTS} \\ &= -\frac{1}{2} x \cos 2x - \int -\frac{1}{2} \cos 2x dx \\ &= -\frac{1}{2} x \cos 2x + \int \frac{1}{2} \cos 2x dx \\ &= -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C\end{aligned}$$

338. $\int \sec x \tan^3 x dx = -\frac{1}{3} \sec^3 x - \sec x + C$

$$\begin{aligned}\int \sec x \tan^2 x dx &= \int \sec x \tan x (\sec^2 x) dx = \int \sec x \tan x (\sec^2 x - 1) dx \\ &= \int \sec x \tan x - \sec x \tan^2 x dx = \text{BY ZEROING COMMON FACTOR (INSPECTION)} \\ &\quad \text{AS } \frac{d}{dx}(\sec x) = \sec x \tan x \\ &= -\frac{1}{3} \sec^3 x - \sec x + C \\ &\quad [\text{THE SUBSTITUTION } u = \sec x \text{ WORKS WITH TOO}]\end{aligned}$$

339. $\int \frac{1}{\operatorname{cosec} x - \cot x} dx = \left[\ln |\sin x| - \ln |\operatorname{cosec} x + \cot x| + C \right] \quad \ln |1 - \cos x| + C$

$$\begin{aligned} \int \frac{1}{\operatorname{cosec} x - \cot x} dx &= \int \frac{\operatorname{cosec} x + \cot x}{(\operatorname{cosec} x + \cot x)(\operatorname{cosec} x - \cot x)} dx \\ &= \int \frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec}^2 x - \cot^2 x} dx = \int \frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x} dx \\ &= -\ln |\operatorname{cosec} x + \cot x| + \ln |\sin x| + C \\ &\dots = \ln \left| \frac{\sin x}{\operatorname{cosec} x + \cot x} \right| + C = \ln \left| \frac{\sin x}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos x}{\sin x}} \right| + C \\ &= \ln \left| \frac{\sin^2 x}{\sin^2 x + \cos^2 x} \right| + C = \ln \left| \frac{1 + \cos^2 x}{1 + \cos^2 x} \right| + C \\ &= \ln \left| \frac{(1 - \cos x)(1 + \cos x)}{1 + \cos^2 x} \right| + C = \ln |1 - \cos x| + C \end{aligned}$$

ALTERNATIVE

$$\begin{aligned} \int \frac{1}{\operatorname{cosec} x - \cot x} dx &= \int \frac{1}{\operatorname{cosec} x \operatorname{cosec} x - \cot x \cot x} dx \\ &= \int \frac{\sin^2 x}{1 - \cos^2 x} dx \quad \text{which is of the form } \int \frac{f(u)}{f(u)^2} du = \ln|f(u)| + C \\ &= \ln|1 - \cos x| + C \end{aligned}$$

340. $\int \frac{2x^{\frac{1}{4}} + 1}{4x^{\frac{5}{4}} + 4x} dx = \ln \left(x^{\frac{1}{2}} + x^{\frac{1}{4}} \right) + C$

$$\begin{aligned} \int \frac{2x^{\frac{1}{4}} + 1}{4x^{\frac{5}{4}} + 4x} dx &\dots \text{BY SUBSTITUTION} \dots \quad u = x^{\frac{1}{4}} \\ &= \int \frac{2u + 1}{4u^5 + 4u^4} du = \int \frac{2u + 1}{4u^4(u+1)} du \quad u^4 du = du \\ &\quad \text{WHICH IS OF THE FORM } \int \frac{f(u)}{f(u)^2} du = \ln|f(u)| + C \\ &= \ln|u^2 + u| + C = \ln(x^{\frac{1}{2}} + x^{\frac{1}{4}}) + C \end{aligned}$$

341. $\int \frac{\ln(x-1)}{\sqrt{x}} dx = 2\sqrt{x} \ln|x-1| - 4\sqrt{x} + 2\ln \left| \frac{\sqrt{x}+1}{\sqrt{x}-1} \right| + C$

$$\begin{aligned} \int \frac{\ln(x-1)}{\sqrt{x}} dx &= \int x^{-\frac{1}{2}} \ln(x-1) dx \dots \text{BY PARTS} \quad \begin{array}{|l} u = \ln(x-1) \\ du = \frac{1}{x-1} dx \end{array} \quad \begin{array}{|l} v = x^{-\frac{1}{2}} \\ dv = -\frac{1}{2}x^{-\frac{3}{2}} dx \end{array} \\ &= 2x^{\frac{1}{2}} \ln|x-1| - \int \frac{2x^{\frac{1}{2}}}{x-1} dx \quad \text{BY SUBSTITUTION} \quad \begin{array}{|l} u = x-1 \\ du = dx \end{array} \\ &= 2x^{\frac{1}{2}} \ln|x-1| - \int \frac{2u}{u-1} (2du) \\ &= 2x^{\frac{1}{2}} \ln|x-1| - \int \frac{4u^2}{u^2-1} du \quad \text{MORE} \\ &= 2x^{\frac{1}{2}} \ln|x-1| - \int \frac{4(u^2-1)+4}{u^2-1} du \\ &= 2x^{\frac{1}{2}} \ln|x-1| - \int 4 + \frac{4}{u^2-1} du \\ &= 2x^{\frac{1}{2}} \ln|x-1| - \int 4 + \frac{4}{(u-1)(u+1)} du \\ &\quad \text{PARTIAL FRACTIONS BY INSPECTION} \\ &= 2x^{\frac{1}{2}} \ln|x-1| - \int 1 + \frac{2}{u-1} - \frac{2}{u+1} du \\ &= 2x^{\frac{1}{2}} \ln|x-1| - 4 + 2\ln|u-1| + 2\ln|u+1| + C \\ &= 2x^{\frac{1}{2}} \ln|x-1| - 4 + 2\ln \left| \frac{u+1}{u-1} \right| + C \\ &= 2\sqrt{x} \ln|x-1| - 4\sqrt{x} + 2\ln \left| \frac{\sqrt{x}+1}{\sqrt{x}-1} \right| + C \end{aligned}$$

SUBSTITUTION $u = \sqrt{x}$ FIRST, FOLLOWED BY INTEGRATION BY PARTS ALSO WORKS

342. $\int e^{x+e^x} dx = e^x + C$

$$\int e^{x+e^x} dx = \int e^x \times e^x dx = \dots \text{ BY INDEX CHAIN RULE (ADDITION)}$$

$$= e^x + C$$

[The substitution $u = e^x$ has been made]

343. $\int \frac{1}{x(1+\sqrt{x})^2} dx = 2\ln\left[\frac{\sqrt{x}}{1+\sqrt{x}}\right] + \frac{2}{1+\sqrt{x}} + C$

$$\int \frac{1}{x(1+\sqrt{x})^2} dx = \dots \text{ BY SUBSTITUTION ...}$$

$$= \int \frac{1}{u^2(1+u)^2} (2u du) = \int \frac{2}{u(1+u)^2} du$$

$u = \sqrt{x}$
 $u^2 = x$
 $2u du = dx$

BY PARTIAL FRACTIONS

$$\frac{2}{u(1+u)^2} = \frac{A}{u} + \frac{B}{(1+u)^2} + \frac{C}{1+u}$$

$$2 = A(u+1)^2 + Cu + Bu^2$$

| | | |
|--------------|------------|-------------------|
| • If $u+1=1$ | • If $u=0$ | • If $u=1$ |
| $2 = -B$ | $2 = A$ | $2 = 4A + 2C + B$ |
| $B = -2$ | | $2 = B + 2C - 2$ |
| | | $-4 = 2C$ |
| | | $C = -2$ |

$$\dots = \int \frac{2}{u} - \frac{2}{(1+u)^2} - \frac{2}{1+u} du = 2\ln|u| - 2\ln|1+u| + \frac{2}{1+u} + C$$

$$= 2\ln\left(\frac{u}{1+u}\right) + \frac{2}{1+u} + C = 2\ln\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right) + \frac{2}{1+\sqrt{x}} + C$$

344. $\int x^2 e^{-x} dx = -e^{-x}(x^2 + 2x + 2) + C$

$$\int x^2 e^{-x} dx = \dots \text{ INTEGRATION BY PARTS}$$

$$= -x^2 e^{-x} - \int -2x e^{-x} dx = -x^2 e^{-x} + \int 2x e^{-x} dx$$

$\frac{x^2}{-e^x} \quad \frac{2}{-e^x}$

INTEGRATION BY PARTS AGAIN

$$= -x^2 e^{-x} + [2x e^{-x} - \int 2e^{-x} dx]$$

$$= -x^2 e^{-x} + 2x e^{-x} + \int 2e^{-x} dx$$

$$= -x^2 e^{-x} + 2x e^{-x} - 2e^{-x} + C = -e^{-x}(x^2 + 2x + 2) + C$$

$\frac{2x}{-e^x} \quad \frac{2}{-e^x}$

345. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2e^{\sqrt{x}} + C$

$$\begin{aligned} \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= \int x^{-\frac{1}{2}} e^{x^{\frac{1}{2}}} dx = \dots \text{ BY REVERSE CHAIN RULE (INSPECTION)} \\ &= 2e^{\sqrt{x}} + C = 2e^{\sqrt{x}} + C \end{aligned}$$

[THE SUBSTITUTION $u=\sqrt{x}$ ALSO WORKS WELL.]

346. $\int \sqrt{x} e^{\sqrt{x}} dx = 2e^{\sqrt{x}}(x+2\sqrt{x}+2) + C$

$$\begin{aligned} \int \sqrt{x} e^{\sqrt{x}} dx &= \dots \text{ BY SUBSTITUTION ...} \\ &= \int u e^u (2u du) = \int 2u^2 e^u du \quad \begin{array}{|c|c|} \hline u & = \sqrt{x} \\ u^2 & = x \\ 2u du & = dx \\ \hline \end{array} \\ &= 2u^3 e^u - \int 4u^2 e^u du \quad \begin{array}{|c|c|} \hline 2u^3 & 4u \\ e^u & e^u \\ \hline \end{array} \\ &= 2u^3 e^u - [4u^2 e^u - \int 4e^u du] \quad \text{INTEGRATION BY PARTS - AGAIN} \\ &= 2u^3 e^u - 4u^2 e^u + \int 4e^u du \\ &= 2u^3 e^u - 4u^2 e^u + 4e^u + C \\ &= 2e^{\sqrt{x}}(u^3 - 2u^2 + 2) + C = 2e^{\sqrt{x}}(x - 2\sqrt{x} + 2) + C \end{aligned}$$

347. $\int \frac{4x^2 - x + 1}{(x-1)(2x-1)} dx = 2x + 4\ln|x-1| - \frac{3}{2}\ln|2x-1| + C$

$$\begin{aligned} \int \frac{4x^2 - x + 1}{(x-1)(2x-1)} dx &= \int \frac{4x^2 - x + 1}{2x^2 - 3x + 1} dx \leftarrow \text{IMPROVE} \\ &= \int \frac{2(2x^2 - 3x + 1) + 5x - 1}{2x^2 - 3x + 1} dx = \int 2 + \frac{5x - 1}{2x^2 - 3x + 1} dx \\ &= \int 2 + \frac{\frac{5x-1}{(2x-1)(2x-1)}}{(2x-1)(2x-1)} dx = \dots \text{PARTIAL FRACTIONS, BY INSPECTION} \\ &= \int 2 + \frac{\frac{4}{x-1} - \frac{3}{2x-1}}{(2x-1)(2x-1)} dx = 2x + 4\ln|x-1| - \frac{3}{2}\ln|2x-1| + C \end{aligned}$$

348. $\int e^x \cos x \, dx = \frac{1}{2}e^{-x}(\sin x + \cos x) + C$

$\int e^x \cos x \, dx \dots$ BY PARTS TWICE

| | |
|-------|----------|
| e^x | $\cos x$ |
| SIMP | COS |

| | |
|-------|-------|
| e^x | e^x |
| -INT | SIN |

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \int e^x \sin x \, dx \\ \int e^x \cos x \, dx &= e^x \sin x + \int e^x \cos x \, dx \\ 2 \int e^x \cos x \, dx &= e^x \sin x + e^x \cos x \\ \int e^x \cos x \, dx &= \frac{1}{2}e^x(\sin x + \cos x) + C\end{aligned}$$

ALTERNATIVE BY DIFFERENTIATION

$$\begin{aligned}\Rightarrow \frac{d}{dx}[e^x(A \cos x + B \sin x)] &\equiv e^x \cos x \\ \Rightarrow e^x(A \cos x + B \sin x) + e^x(-A \sin x + B \cos x) &\equiv e^x \cos x \\ \Rightarrow (A+B) \cos x + (B-A) \sin x &\equiv \cos x \\ A+B &= 1 \\ B-A &= 0 \quad \therefore A=B=\frac{1}{2} \\ \therefore \int e^x \cos x \, dx &\approx e^x(\frac{1}{2} \cos x + \frac{1}{2} \sin x) + C\end{aligned}$$

[THE ABOVE INTEGRAL CAN ALSO BE DONE BY COMPLEX NUMBERS – NOT SHOWN HERE]

349. $\int \frac{5^{2x}}{5^{2x}+3} \, dx = \frac{\ln(5^{2x}+3)}{\ln 25} + C$

$$\begin{aligned}\int \frac{5^{2x}}{5^{2x}+3} \, dx &= \frac{1}{2 \ln 5} \int \frac{5^{2x} \cdot 2 \times 5}{5^{2x}+3} \, dx \\ &\text{OF THE FORM } \int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C \\ &\text{SINCE } \frac{d}{dx}[5^{2x}] = 2 \ln 5 \times 5^{2x} + \ln 5 \\ &= \frac{\ln(5^{2x}+3)}{2 \ln 5} + C = \frac{\ln(5^{2x}+3)}{\ln 25} + C\end{aligned}$$

[THE SUBSTITUTION $u = 5^{2x}+3$ ALSO WORKS WELL.]

350. $\int \frac{3x^5}{x^3-1} \, dx = x^3 + \ln|x^3-1| + C$

$$\begin{aligned}\int \frac{3x^5}{x^3-1} \, dx &= \dots \text{BY DIVISION OR MANIPULATION} \\ &= \int \frac{3x^2(x^3-1) + 3x^2}{x^3-1} \, dx = \int 3x^2 + \frac{3x^2}{x^3-1} \, dx \\ &\text{OF THE FORM } \int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C \\ &= \frac{3x^2}{2} + \ln|x^3-1| + C\end{aligned}$$

[THE SUBSTITUTION $u = x^3-1$ ALSO WORKS.]

351. $\int \frac{x-2}{x^2-4x-2} dx = \frac{1}{2} \ln|x^2-4x-2| + C$

$$\begin{aligned}\int \frac{x-2}{x^2-4x-2} dx &= \frac{1}{2} \int \frac{2(x-2)}{x^2-4x-2} dx \\ &= \frac{1}{2} \int \frac{2x-4}{x^2-4x-2} dx \\ \text{THIS IS THE ROW } \int f(x) dx = h(x) + C \\ &= \frac{1}{2} \ln|x^2-4x-2| + C\end{aligned}$$

[THE SUBSTITUTION $u=x^2-4x-2$ ALSO WORKS]

352. $\int \frac{1}{(x-1)\sqrt{x^2-1}} dx = -\sqrt{\frac{x+1}{x-1}} + C$

$$\begin{aligned}\int \frac{1}{(x-1)\sqrt{x^2-1}} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int \frac{1}{\frac{1}{u}\sqrt{1+\frac{2}{u}}} \left(-\frac{1}{u^2} du\right) \\ &= -\int \frac{1}{u \left(\frac{1+2u}{u^2}\right)^{\frac{1}{2}}} du = -\int \frac{1}{(1+2u)^{\frac{1}{2}}} du \\ &= -\int (1+2u)^{-\frac{1}{2}} du = -(1+2u)^{\frac{1}{2}} + C \\ &= -(1+2\left(\frac{1}{x-1}\right))^{\frac{1}{2}} + C = -\sqrt{1+\frac{2}{x-1}} + C = -\sqrt{\frac{x+1}{x-1}} + C \\ &= -\sqrt{\frac{x+1}{x-1}} + C\end{aligned}$$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION

$$\begin{aligned}\int \frac{1}{(x-1)\sqrt{x^2-1}} dx &= \dots \\ &= \int \frac{1}{\sec\theta \tan\theta} (\sec\theta \tan\theta d\theta) \\ &= \int \sec\theta d\theta = \int \frac{\sec(\theta)(\sec\theta + \tan\theta)}{\sec\theta(\sec\theta + \tan\theta)} d\theta \\ &= \int \frac{\sec\theta + \tan\theta}{\sec\theta - 1} d\theta = \int \frac{\sec\theta + \sec\theta \tan\theta}{\sec\theta - 1} d\theta = \int \frac{\sec\theta(1 + \tan\theta)}{\sec\theta - 1} d\theta \\ &= \int \frac{1 + \tan\theta}{\sec\theta} d\theta = \int \frac{1}{\sec\theta} + \frac{\tan\theta}{\sec\theta} d\theta = \int \cos\theta + \cos\theta \tan\theta d\theta \\ &= -\sin\theta - \cos\theta \tan\theta + C = -\left(\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}\right) = -\frac{1+\cos\theta}{\sin\theta} + C \\ &= -\frac{1+\cos\theta}{\sqrt{1-\cos^2\theta}} + C = -\frac{1+\cos\theta}{\sqrt{(1-\cos\theta)(1+\cos\theta)}} + C = -\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} + C \\ &= -\sqrt{\frac{\sec\theta + \cos\theta \tan\theta}{\sec\theta - 1}} + C = -\sqrt{\frac{\sec\theta + 1}{\sec\theta - 1}} + C = -\sqrt{\frac{x+1}{x-1}} + C\end{aligned}$$

353. $\int \frac{3x^3-x^2+10x-3}{x^2+3} dx = \frac{3}{2}x^2 - x + \frac{1}{2} \ln(x^2+3) + C$

$$\begin{aligned}\int \frac{3x^3-x^2+10x-3}{x^2+3} dx &= \dots \text{ BY DIVISION OR MANIPULATION} \\ &= \int \frac{3x(x^2+3)-(x^2+3)+x}{x^2+3} dx = \int 3x-1 + \frac{x}{x^2+3} dx \\ &= \int 3x-1 + \frac{1}{2}(2x^2+3) dx \quad \text{OF THE FORM } \int \frac{f(x)}{g(x)} dx = \ln|g(x)| + C \\ &= \frac{3}{2}x^2 - 2 - \frac{1}{2} \ln(2x^2+3) + C\end{aligned}$$

354. $\int \frac{4}{(4+x^2)^{\frac{3}{2}}} dx = \frac{x}{\sqrt{4+x^2}} + C$

$$\begin{aligned} \int \frac{dx}{(4+x^2)^{\frac{3}{2}}} &= \dots \text{ BY TRIGONOMETRIC SUBSTITUTION} \\ &= \int \frac{4}{(4+4\tan^2\theta)^{\frac{3}{2}}} (2\sec^2\theta d\theta) \\ &= \int \frac{8\sec^2\theta}{(4(1+\tan^2\theta))^{\frac{3}{2}}} d\theta = \int \frac{\sec^2\theta}{(4\sec^2\theta)^{\frac{3}{2}}} d\theta \\ &= \int \frac{\sec^2\theta}{8\sec^3\theta} d\theta = \int \frac{1}{8\sec\theta} d\theta = \int \cos\theta d\theta \\ &= \sin\theta + C = \frac{x}{\sqrt{4+x^2}} + C \end{aligned}$$

355. $\int \frac{(x+1)^2}{x^2+1} dx = x + \ln(x^2+1) + C$

$$\begin{aligned} \int \frac{(x+1)^2}{x^2+1} dx &= \int \frac{x^2+2x+1}{x^2+1} dx = \int \frac{(x^2+1)+2x}{x^2+1} dx \\ &= \int 1 + \frac{2x}{x^2+1} dx = x + \ln(x^2+1) + C \\ &\quad \uparrow \text{OF THE PULL! } \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C \end{aligned}$$

356. $\int \frac{\cos x + \tan x}{1 + \tan^2 x} dx = \sin x - \frac{1}{3} \sin^3 x + \frac{1}{2} \sin^2 x + C$

$$\begin{aligned} \int \frac{\cos x + \tan x}{1 + \tan^2 x} dx &= \int \frac{\cos x + \tan x}{\sec^2 x} dx \\ &= \int \frac{\cos x}{\sec^2 x} + \frac{\tan x}{\sec^2 x} dx = \int \cos x + \tan x \cos x dx \\ &= \int \cos x \cos x dx + \frac{\tan x}{\sec^2 x} \cos^2 x dx = \int \cos x (\cos x - \sin x) + \sin x \cos x dx \\ &= \int \cos x - \cos x \sin x + \sin x \cos x dx \\ &\quad \text{BY REVERSE CHAIN RULE (RECONCILIATION)} \\ &= \sin x - \frac{1}{2}\sin^2 x + \frac{1}{2}\sin^2 x + C \\ &\quad \text{ALTERNATIVE BY SUBSTITUTION} \\ \int \frac{\cos x + \tan x}{1 + \tan^2 x} dx &= \int \frac{\cos x + \tan x}{\sec^2 x} dx \\ &= \int \frac{\cos x + \tan x}{\sec u} \left(\frac{du}{dx}\right) = \int \frac{\cos x + \tan x}{\sec u} du \\ &= \int \cos x + \tan x \cos u du \\ &= \int 1 - \sin^2 u + \frac{\sin u}{\cos u} \cos u du = \int 1 - \sin^2 u + \sin u du \\ &= \int 1 - u^2 + u du = u - \frac{1}{3}u^3 + \frac{1}{2}u^2 + C \\ &= \sin x - \frac{1}{3}\sin^3 x + \frac{1}{2}\sin^2 x + C \end{aligned}$$

357. $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx = \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + C$

$$\begin{aligned} \int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx &= \int x^{\frac{1}{2}}(x^{\frac{1}{2}}+1)^{\frac{3}{2}} dx \\ &\quad \text{BY REVERSE CHAIN RULE (RECOGNITION)} \\ &= \frac{4}{3}(x^{\frac{1}{2}}+1)^{\frac{3}{2}} + C \end{aligned}$$

358. $\int \frac{4x^3\sqrt{x^4+1}}{1+\sqrt{x^4+1}} dx = x^4 - 2\sqrt{x^4+1} + 2\ln(1+\sqrt{x^4+1}) + C$

$$\begin{aligned} \int \frac{4x^3\sqrt{x^4+1}}{1+\sqrt{x^4+1}} dx &= \dots \text{BY SUBSTITUTION} \quad u = \sqrt{2x^4+1} \\ &= \int \frac{2x^3 u}{1+u} \left(\frac{u}{2x^2} du \right) = \int \frac{2u^2}{u+1} du \quad u^2 = x^4 + 1 \\ &\quad \text{MANIPULATE OR LONG-DIVIDE, SO} \\ &\quad \text{ANOTHER SUBSTITUTION IS NOT NEEDED} \\ &= \int \frac{2u(u+1)-2(u+1)+2}{u+1} du = \int 2u - 2 + \frac{2}{u+1} du \\ &= u^2 - 2u + 2\ln|u+1| + C = (2x^4+1) - 2\sqrt{2x^4+1} + 2\ln(1+\sqrt{2x^4+1}) \\ &= 2x^4 - 2\sqrt{2x^4+1} + 2\ln(1+\sqrt{2x^4+1}) + C \\ &\quad \boxed{[\text{THE SUBSTITUTION } u = 1+\sqrt{2x^4+1}]} \end{aligned}$$

359. $\int (\ln x)^2 dx = 2x + x(\ln|x|)^2 - 2x\ln|x| + C$

$$\begin{aligned} \int (\ln x)^2 dx &= \dots \text{INTEGRATION BY PARTS} \quad \begin{array}{c|c} (\ln x)^2 & 2\ln x \\ \hline x & 1 \end{array} \\ &= x(\ln|x|)^2 - \int 2\ln x dx \quad \text{BY PARTS} \\ &= x(\ln|x|)^2 - \left[2x\ln|x| - \int 2 dx \right] \\ &= x(\ln|x|)^2 - 2x\ln|x| + \int 2 dx \\ &= x(\ln|x|)^2 - 2x\ln|x| + 2x + C \end{aligned}$$

360. $\int \sqrt{18\cos x \sin 2x} dx = \begin{cases} \frac{4}{\sqrt{\sin^3 x}} + C & \text{if } \cos x > 0 \\ -\frac{4}{\sqrt{\sin^3 x}} + C & \text{if } \cos x < 0 \end{cases}$

$$\begin{aligned} \int \sqrt{18\cos x \sin 2x} dx &= \int \sqrt{18\cos x (\sin x)^2} dx = \int \sqrt{18\cos x} dx \\ &= \int |\cos x|^{1/2} dx = \dots \quad \text{BY LEAVE OUT THE ABSOLUTE VALUE} \\ &= \begin{cases} 4(\cos x)^{1/2} & \text{if } \cos x > 0 \\ -4(\sin x)^{1/2} & \text{if } \cos x < 0 \end{cases} \rightarrow \begin{cases} \frac{4}{\sqrt{\sin^3 x}} + C & \text{if } \cos x > 0 \\ -\frac{4}{\sqrt{\sin^3 x}} + C & \text{if } \cos x < 0 \end{cases} \end{aligned}$$

361. $\int \frac{1}{\sqrt{x^5+x^2}} dx = \begin{cases} \frac{1}{2} \ln \left| \frac{\sqrt{x^3+1}-1}{\sqrt{x^3+1}+1} \right| + C & \text{if } x > 0 \\ -\frac{1}{2} \ln \left| \frac{\sqrt{x^3+1}-1}{\sqrt{x^3+1}+1} \right| + C & \text{if } x < 0 \end{cases}$

$$\begin{aligned} \int \frac{1}{\sqrt{2x^3+2x^2}} dx &= \int \frac{1}{|x|\sqrt{2x+1}} dx = \dots \quad \text{SUBSTITUTION} \\ \int \frac{1}{|x|} \frac{2u}{3x^2} du &= \frac{2}{3} \int \frac{1}{u^2+1} du \\ \text{IF } x \geq 0 \\ \dots &= \frac{2}{3} \int \frac{1}{u^2+1} du = \frac{2}{3} \int \frac{1}{u^2+1} du \\ &= \frac{2}{3} \int \frac{1}{(u+1)(u+1)} du \quad \dots \text{PARTIAL FRACTION BY INSPECTION} \\ &= \frac{2}{3} \int \frac{1}{u+1} - \frac{1}{u+1} du = \frac{2}{3} \int \frac{1}{u+1} - \frac{1}{u+1} du \\ &= \frac{1}{3} \ln \left| \frac{u+1}{u+1} \right| + C = \frac{1}{3} \ln \left| \frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1} \right| + C, \text{ IF } x \geq 0 \\ \text{IF } x \leq 0 \\ &= -\frac{2}{3} \int \frac{1}{u^2+1} du = \dots = -\frac{1}{3} \ln \left| \frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1} \right| + C, \text{ IF } x \leq 0 \end{aligned}$$

362. $\int \sqrt{\sin^2 x + (\cos x - 1)^2} dx = \begin{cases} 4\cos\left(\frac{1}{2}x\right) + C & \text{if } \sin\left(\frac{1}{2}x\right) < 0 \\ -4\cos\left(\frac{1}{2}x\right) + C & \text{if } \sin\left(\frac{1}{2}x\right) > 0 \end{cases}$

$$\begin{aligned} \int \sqrt{4\sin^2 x + (\cos x - 1)^2} dx &= \int \sqrt{\sin^2 x + 2\cos x - 2\cos x + 1} dx \\ &= \int \sqrt{2 - 2\cos x} dx = \int \sqrt{2 - 2(1 - 2\sin^2 \frac{x}{2})} dx \\ &= \int \sqrt{4\sin^2 \frac{x}{2}} = \int 2|\sin \frac{x}{2}| dx \\ \text{IF } \sin \frac{x}{2} > 0 \dots &= \int 2\sin \frac{x}{2} dx = -4\cos \frac{x}{2} + C \\ \text{IF } \sin \frac{x}{2} < 0 \dots &= \int -2\sin \frac{x}{2} dx = 4\cos \frac{x}{2} + C \end{aligned}$$

363. $\int x(\sin x + \cos x) dx = \begin{bmatrix} x(\sin x - \cos x) + \sin x + \cos x + C \\ (1+x)\sin x + (1-x)\cos x + C \end{bmatrix}$

| | |
|---|--|
| $\int x(\sin x + \cos x) dx = \dots$ INTEGRATION BY PARTS $= x(\cos x - \sin x) - \int -\cos x + \sin x dx$ $= x(\cos x - \sin x) + \int \cos x - \sin x dx$ $= x(\cos x - \sin x) + \sin x + \cos x + C$ $= (1+x)\sin x + (1-x)\cos x + C$ | $\begin{array}{ c c } \hline x & 1 \\ \hline \text{INT. BY PARTS} & \text{SWAP} \\ \hline \end{array}$ |
|---|--|

364. $\int \left(\frac{1}{x^2} + \frac{1}{x^3} \right) e^x dx = -\frac{1}{x} e^x + C$

| | |
|--|--|
| $\int \left(\frac{1}{x^2} + \frac{1}{x^3} \right) e^x dx = \dots$ BY SUBSTITUTION $= \int (u^2 + u^{-3}) e^u (-\frac{1}{u^2} du)$ $= \int -(1+u)e^u du$ $= -(1+u)e^u - \int -e^u du$ $= -(1+u)e^u + \int e^u du$ $= -xe^x - xe^x + x + C$ $= -\frac{1}{x}e^x + C$ | $\begin{array}{ c c } \hline u = \frac{1}{x} & 1 \\ \hline du = -\frac{1}{x^2} dx & dx = -x^2 du \\ \hline \text{INT. BY PARTS} & \text{SWAP} \\ \hline \end{array}$ |
|--|--|

365. $\int \frac{\sec^4 x}{\sqrt{\tan x}} dx = 2(\tan x)^{\frac{1}{2}} + \frac{2}{5}(\tan x)^{\frac{5}{2}} + C$

| |
|--|
| $\int \frac{\sec^4 x}{\sqrt{\tan x}} dx = \int \sec^2 \tan x \tan x^{\frac{1}{2}} dx$ $= \int \sec((1+\tan x)(\tan x))^{\frac{1}{2}} dx = \int \sec^2((\tan x)^{\frac{1}{2}})^2 + \sec^2(\tan x)^{\frac{1}{2}} dx$ $= \int (\sec x)^2 + \frac{1}{2}(\tan x)^2 dx$ $= 2(\tan x)^{\frac{1}{2}} + \frac{2}{5}(\tan x)^{\frac{5}{2}} + C$ <p style="color: green;">THE SUBSTITUTION $u = \tan x$ OR $u = \sqrt{\tan x}$ ALSO WORK WELL</p> |
|--|

366. $\int \frac{\cos x}{(\cos x + \sin x)^3} dx = -\frac{1}{2(1+\tan x)^2} + C$

$$\begin{aligned} & \int \frac{\cos x}{(\cos x + \sin x)^3} dx = \int \frac{\cos x}{\cos x(1 + \frac{\sin x}{\cos x})^3} dx \\ &= \int \sec^2(1 + \tan x)^{-3} dx = \dots \text{ BY REVERSE CHAIN RULE (INTEGRATION)} \\ &= -\frac{1}{2}(1 + \tan x)^{-2} + C = -\frac{1}{2(1+\tan x)^2} + C \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$u = \tan x$
 $du = \sec^2 x dx$
 $dx = \frac{du}{\sec^2 x}$
 $dx = \cos^2 x du$

$$\begin{aligned} & \int \frac{\cos x}{(\cos x + \sin x)^3} dx = \dots \\ &= \int \frac{\cos x}{(\cos x + \sin x)^3} (\cos^2 x du) \\ &= \int \frac{\cos x}{(\cos x + \sin x)^3} du \\ &= \int \left(\frac{\cos x}{\cos x + \sin x} \right)^3 du \\ &= \int \left(\frac{\frac{\cos x}{\cos x}}{1 + \frac{\sin x}{\cos x}} \right)^3 du = \int \left(\frac{1}{1 + \tan x} \right)^3 du \\ &= \int \frac{1}{(1+u)^3} du = \int (1+u)^{-3} du = -\frac{1}{2}(1+u)^{-2} + C \\ &= -\frac{1}{2(1+\tan x)^2} + C \end{aligned}$$

367. $\int \frac{20x}{4-x^2} dx = -10\ln|4-x^2| + C$

$$\begin{aligned} \int \frac{20x}{4-x^2} dx &= -10 \int \frac{-2x}{4-x^2} dx = -10 \ln|4-x^2| + C \\ \text{OR THE FORM } \int \frac{f(x)}{g(x)} dx &= \ln|f(x)| + C \end{aligned}$$

THE SUBSTITUTION $u=4-x^2$ ALSO WORKS WELL, AS WELL AS PARTIAL FRACTION

368. $\int \frac{\sqrt{x}}{1+\sqrt{x}} dx = x - 2\sqrt{x} + 2\ln(1+\sqrt{x}) + C$

$$\begin{aligned} & \int \frac{\sqrt{x}}{1+\sqrt{x}} dx = \dots \text{ BY SUBSTITUTION} \\ &= \int \frac{u-1}{u} (2(u-1) du) = \int 2(u-1)^2 du \\ &= \int 2u^2 - 4u + 2 du = \int 2u-4 + \frac{2}{u} du \\ &= u^2 - 4u + 2\ln|u| + C \\ &= (1+\sqrt{x})^2 - 4(1+\sqrt{x}) + 2\ln(1+\sqrt{x}) + C \\ &= 1 + 2\sqrt{x} + x - 4 - 4\sqrt{x} + 2\ln(1+\sqrt{x}) + C \\ &= x - 2\sqrt{x} + 2\ln(1+\sqrt{x}) + C \end{aligned}$$

$u = 1+\sqrt{x}$
 $\sqrt{x} = u-1$
 $x = (u-1)^2$
 $dx = 2(u-1) du$

369. $\int x(\sec^2 x - \operatorname{cosec}^2 x) dx = x(\tan x + \cot x) - \ln|\tan x| + C$

$$\begin{aligned} & \int x(\sec^2 x - \operatorname{cosec}^2 x) dx = \dots \text{ INTEGRATION BY PARTS} \\ &= x(\tan x + \cot x) - \int \tan x + \cot x dx \quad \begin{array}{|c|c|} \hline x & 1 \\ \hline \tan x + \cot x & \sec^2 x - \operatorname{cosec}^2 x \\ \hline \end{array} \\ & \quad \downarrow \text{STANDARD RESULT} \\ &= x(\tan x + \cot x) - [\ln|\sec x| + \ln|\csc x|] + C \\ &= x(\tan x + \cot x) - \ln\left|\frac{\sec x}{\csc x}\right| + C \\ &= x(\tan x + \cot x) - \ln|\tan x| + C \end{aligned}$$

370. $\int \frac{4x^2 + 4x}{\sqrt{2x+1}} dx = \frac{1}{5}(2x+1)^{\frac{5}{2}} - (2x+1)^{\frac{1}{2}} + C$

$$\begin{aligned} & \int \frac{4x^2 + 4x}{\sqrt{2x+1}} dx = \dots \text{ BY MANIPULATION} \\ &= \int \frac{(4x^2 + 4x) - 1}{(2x+1)^{\frac{1}{2}}} dx = \int \frac{(2x+1)^{\frac{3}{2}} - 1}{(2x+1)^{\frac{1}{2}}} dx \\ &= \int (2x+1)^{\frac{3}{2}} - (2x+1)^{\frac{1}{2}} dx = \underline{\int (2x+1)^{\frac{3}{2}} dx} - \underline{\int (2x+1)^{\frac{1}{2}} dx} + C \\ & \text{THE SUBSTITUTION } u = 2x+1 \text{ OR } u = \sqrt{2x+1} \text{ ALSO WORK WELL} \end{aligned}$$

371. $\int x \tan^2 x dx = -\frac{1}{2}x^2 + x \tan x + \ln|\cos x| + C$

$$\begin{aligned} & \int x \tan^2 x dx = \int x(\sec^2 x - 1) dx = \int x \sec^2 x - x dx \quad \begin{array}{|c|c|} \hline x & 1 \\ \hline \tan x & \sec^2 x \\ \hline \end{array} \\ & \text{INTEGRATION BY PARTS FOR THE FIRST TERM} \\ &= x \tan x - \int \tan x dx - \frac{1}{2}x^2 \\ &= -\frac{1}{2}x^2 + x \tan x - \ln|\sec x| + C \\ &= -\frac{1}{2}x^2 + x \tan x + \ln|\cos x| + C \end{aligned}$$

372. $\int x \cos^2 3x dx = \frac{1}{4}x^2 + \frac{1}{12}x \sin 6x + \frac{1}{72} \cos 6x + C$

$$\begin{aligned} & \int x \cos^2 3x dx = \int x\left(\frac{1}{2} + \frac{1}{2} \cos 6x\right) dx \\ &= \int \frac{1}{2}x + \frac{1}{2}x \cos 6x dx = \frac{1}{2}x^2 + \int \frac{1}{2}x \cos 6x dx \\ & \text{INTEGRATION BY PARTS} \\ &= \frac{1}{2}x^2 + \frac{1}{12}x \sin 6x - \int \frac{1}{2} \sin 6x dx \\ &= \frac{1}{2}x^2 + \frac{1}{12}x \sin 6x + \frac{1}{72} \cos 6x + C \quad \begin{array}{|c|c|} \hline \frac{1}{2}x & \frac{1}{2} \\ \hline \sin 6x & \cos 6x \\ \hline \end{array} \end{aligned}$$

373. $\int \cos x (6\sin x - 2\sin 3x)^{\frac{2}{3}} dx = \frac{4}{3} \sin^3 x + C$

$$\begin{aligned} & \int \cos x (6\sin x - 2\sin 3x)^{\frac{2}{3}} dx \dots \text{BY TRIG IDENTITIES} \\ & 6\sin x - 2\sin(3x) \\ &= 6\sin x \cos 3x + \cos 3x \sin x \\ &= (2\sin x \cos 3x) \cos x + (1 - 2\cos^2 3x) \sin x \\ &= 2\sin x \cos 3x + \sin x - 2\sin^2 3x \\ &= 2\sin x (1 - \sin^2 x) + \sin x - 2\sin^2 3x \\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^2 3x \\ &= 3\sin x - 4\sin^3 x \end{aligned}$$

$$\begin{aligned} &= \int \cos x [6\sin x - 2(3\sin x - 4\sin^3 x)]^{\frac{2}{3}} dx \\ &= \int \cos x [8\sin^3 x]^{\frac{2}{3}} dx = \int \cos x [4\sin^2 x] dx \\ &= \int 4\cos x \sin^2 x dx = \frac{4}{3} \sin^3 x + C \end{aligned}$$

374. $\int \sqrt{x^2 - x^4} dx = \begin{cases} -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C & x > 0 \\ \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C & x < 0 \end{cases}$

$$\begin{aligned} \int \sqrt{x^2 - x^4} dx &= \int \sqrt{x^2(1-x^2)} dx = \int \sqrt{x^2} \sqrt{(1-x^2)} dx \\ &= \int |x| (1-x^2)^{\frac{1}{2}} dx = \begin{cases} \frac{1}{2}(1-x^2)^{\frac{3}{2}} + C & \text{IF } x > 0 \\ \frac{1}{2}(1-x^2)^{\frac{3}{2}} + C & \text{IF } x < 0 \end{cases} \end{aligned}$$

THE SUBSTITUTION $u = 1-x^2$ OR $u = \sqrt{1-x^2}$ MAY ALSO BE USED

375. $\int \frac{x+3}{\sqrt[3]{x^2+6x}} dx = \frac{3}{4}(x^2+6x)^{\frac{2}{3}} + C$

$$\begin{aligned} \int \frac{x+3}{\sqrt[3]{x^2+6x}} dx &= \int (x+3)(x^2+6x)^{-\frac{1}{3}} dx = \frac{1}{2} \int (2x+6)(x^2+6x)^{-\frac{1}{3}} dx \\ &= \dots \text{BY REVERSE CHAIN RULE (CONVERSION)} \\ &= \frac{1}{2} \cdot \frac{3}{2} (x^2+6x)^{\frac{2}{3}} + C = \frac{3}{4} (x^2+6x)^{\frac{2}{3}} + C \end{aligned}$$

THE SUBSTITUTION $u = (x^2+6x)^{\frac{1}{3}}$ ALSO WORKS

$$376. \int \frac{1-x}{1-\sqrt{x}} dx = \left[-\frac{2}{3}(1-\sqrt{x})^3 + 3(1-\sqrt{x})^2 - 4(1-\sqrt{x}) + C \right] \quad \frac{2}{3}x^{\frac{3}{2}} + x + C$$

$$\int \frac{1-x}{1-\sqrt{1-x^2}} dx = \dots$$

BY MANIPULATION (DIFFERENCE OF SQUARES)

$$= \int \frac{(1+\sqrt{1-x^2})(1-\sqrt{1-x^2})}{1-(\sqrt{1-x^2})^2} dx = \int 1+x^{\frac{1}{2}} dx = x + 2x^{\frac{3}{2}} + C$$

ALTERNATIVE BY SUBSTITUTION (LINEAR)

$$\int \frac{1-x}{1-\sqrt{1-x^2}} dx = \int \frac{1-(C(u))^2}{1-u^2} [-2Cu] du$$

$$= \int \frac{\cancel{1-C^2(u^2)}(u^2+u^4)}{u} [-2(Cu)] du$$

$$= \int \frac{2u(1-u^2)}{u} [-2(Cu)(1-u^2)] du$$

$$= -2(2(u-1)(u^2-1)) du = \int -2u^3 + 6u - 4 du$$

$$= -\frac{2}{3}u^3 + 3u^2 + 4u + C - \frac{2}{3}\left[1-(\sqrt{u})^2\right]^2 + 3(1-\sqrt{u})^2 - 4(\sqrt{u}) + C$$

$$= -\frac{2}{3}\left[-3\sqrt{u}^3 + 3\sqrt{u}^2 - 2\sqrt{u}\right] + 3\left[-2\sqrt{u}^2 + 3\right] - 4 + 4\sqrt{u} + C$$

$$= \frac{2}{3}\sqrt{u}^3 - 2\sqrt{u}^2 + 2\sqrt{u} - 6\sqrt{u}^2 + 3\sqrt{u} + 4\sqrt{u} + C$$

$$= \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 2x^{\frac{1}{2}} - 6x^{\frac{3}{2}} + 3x^{\frac{1}{2}} + 4x^{\frac{1}{2}} + C$$

(CAL.CORE)

$$377. \int \frac{1}{\operatorname{cosec} 2x - \cot 2x} dx = \ln|\sin x| + C$$

$$\begin{aligned}
 \int \frac{1}{\cos^2(2x) - \sin^2(2x)} dx &= \int \frac{1}{\cos(2x)\cos(2x) - \sin(2x)\sin(2x)} dx \\
 &= \int \frac{1}{1 - \cos(4x)} dx = \int \frac{2\sin(2x)\cos(2x)}{1 - (\cos(4x))^2} dx = \int \frac{2\sin(2x)\cos(2x)}{2\sin^2(2x)} dx \\
 &= \int \frac{\cos(2x)}{\sin(2x)} dx = \dots \text{ of the type } \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C \\
 &= \ln|\sin(2x)| + C
 \end{aligned}$$

$$378. \int \sec^3 x \tan^5 x \, dx = \frac{1}{7} \sec^7 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^3 x + C$$

$$\begin{aligned}
 & \int \sec^2 \tan^{-1} dx = \dots \text{BY MANIPULATION} \dots \\
 &= \int \sec^2 (\sec^{-1} \tan x) dx = \int \sec^2 (\sec^{-1})^{-1} \sec x dx \\
 &= \int \sec^2 (\sec^2 - 2\sec^2 + 1)^{-1} \sec x dx = \int (\sec^2 - 2\sec^2 + \sec^2)^{-1} \sec x dx \\
 &= \int (\sec^2 - 2\sec^2 + \sec^2) \sec x dx \\
 &\quad \text{BY CANCELLATION RULE AS } \frac{d}{dx}(\sec x) = \sec x \tan x, \\
 &= \cancel{\frac{1}{2}\sec^2} - \frac{2}{3}\sec^2 + \cancel{\frac{1}{2}\sec^2} + C \\
 \\[10pt]
 & \text{ALTERNATIVE BY SUBSTITUTION} \\
 & \int \sec^2 \tan^{-1} dx = \int \sec^2 \tan^{-1} \frac{du}{\sec \tan u} \\
 &= \int \sec^2 \tan^{-1} du = \int \sec^2 (\sec^{-1})^2 du \\
 &= \int (t^2(t^2-1)^2) dt = \int t^6 - 2t^4 + t^2 dt \\
 &= \frac{1}{7}t^7 - \frac{2}{5}t^5 + \frac{1}{3}t^3 + C \\
 &= \frac{1}{7}\sec^7 - \frac{2}{5}\sec^5 + \frac{1}{3}\sec^3 + C
 \end{aligned}$$

a = sec
 da = sec tan x
 dx = $\frac{da}{\sec \tan x}$

THE SUBSTITUTION a = sec x ALSO WORKS WELL

379. $\int \frac{\cosh x}{(\cosh x + \sinh x)^3} dx = \left[-\frac{1}{2(1+\tanh x)^2} + C \right]_{-\frac{1}{4}e^{-2x} - \frac{1}{8}e^{-4x} + C}$

$$\begin{aligned} \int \frac{\cosh x}{(\cosh x + \sinh x)^3} dx &= \dots \text{BY MANIPULATION} = \int \frac{\cosh x}{\cosh^3 x (1 + \tanh x)^3} dx \\ &= \int \frac{1}{\cosh^2 x (1 + \tanh x)^3} dx = \int \operatorname{sech}^2 x (1 + \tanh x)^{-3} dx \\ &\quad \text{BY REVERSE CHAIN RULE (INSPECTION)} \\ &= -\frac{1}{2} (1 + \tanh x)^{-2} + C = -\frac{1}{2(1 + \tanh x)^2} + C \\ &\quad \text{ALTERNATIVE BY EXPANSION} \\ &\int \frac{\cosh x}{(\cosh x + \sinh x)^3} dx = \int \frac{\frac{1}{2}e^x + \frac{1}{4}e^{3x}}{(\frac{1}{2}e^x + \frac{1}{4}e^{3x})^3} dx = \int \frac{\frac{1}{2}e^x + \frac{1}{4}e^{3x}}{(e^x)^3} dx \\ &= \int \frac{\frac{1}{2}e^x + \frac{1}{4}e^{3x}}{e^{3x}} dx = \int \frac{1}{2}e^{-2x} + \frac{1}{4}e^{-6x} dx = \frac{1}{2}e^{-2x} - \frac{1}{8}e^{-6x} + C \end{aligned}$$

380. $\int \cos x \cos^2 3x dx = \frac{1}{2} \sin x + \frac{1}{28} \sin 7x + \frac{1}{20} \sin 5x + C$

$$\begin{aligned} \int \cos x \cos^2 3x dx &= \dots \text{USING DOUBLE ANGLE} = \int \cos x (\frac{1}{2}(1 + \cos 6x)) dx \\ &= \int \frac{1}{2} \cos x + \frac{1}{2} \cos x \cos 6x dx = \dots \text{PRODUCT TO SUM} \\ &\quad \text{OR } \cos(x+6x) = \cos^2 6x = \cos^2 6x - \sin^2 6x \\ &\quad \cos(x+6x) = \cos 6x = \cos 6x \cos x + \sin 6x \sin x \\ &\quad \rightarrow \cos x \cos 6x = 2 \cos x \cos 6x \quad \text{CANCELS} \\ &\quad \rightarrow \frac{1}{2} \cos x \cos 6x = \frac{1}{2} \cos x + \frac{1}{2} \cos 6x \\ &= \dots \int \frac{1}{2} \cos x + \frac{1}{2} \cos 6x dx \\ &= -\frac{1}{2} \sin x + \frac{1}{12} \sin 12x + \frac{1}{12} \sin 6x + C \end{aligned}$$

381. $\int \frac{1}{\sqrt{x^{\frac{3}{2}} + 4x}} dx = 4\sqrt{4 + \sqrt{x}} + C$

$$\begin{aligned} \int \frac{1}{\sqrt{x^{\frac{3}{2}} + 4x}} dx &= \text{BY MANIPULATION} \\ &= \int \frac{1}{\sqrt{x(x^{\frac{1}{2}} + 4)}} dx = \int \frac{1}{x^{\frac{1}{2}}(x^{\frac{1}{2}} + 4)^{\frac{1}{2}}} dx \\ &\quad \text{BY REVERSE CHAIN RULE (INSPECTION)} \\ &\quad \frac{d}{dx}[(x^{\frac{1}{2}} + 4)^{\frac{1}{2}}] = \frac{1}{2}(x^{\frac{1}{2}} + 4)^{-\frac{1}{2}} \cdot \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{4}x^{-\frac{1}{2}}(2x^{\frac{1}{2}} + 4)^{-\frac{1}{2}} \\ &= \frac{1}{4}(x^{\frac{1}{2}} + 4)^{-\frac{1}{2}} + C = \frac{1}{4}\sqrt{x^{\frac{1}{2}} + 4} + C \\ &\quad \text{ALTERNATIVE BY SUBSTITUTION} \\ &\quad \int \frac{1}{\sqrt{x^{\frac{3}{2}} + 4x}} dx = \int \frac{1}{\sqrt{u^{\frac{3}{2}} + 4u^{\frac{1}{2}}}} \cdot (2u du) \quad u = x^{\frac{1}{2}} \\ &= \int \frac{2u}{\sqrt{u^{\frac{3}{2}} + 4u^{\frac{1}{2}}}} du = \int \frac{2u}{u + 4u^{\frac{1}{2}}} du \quad u^{\frac{1}{2}} du = du \\ &\quad \frac{\sqrt{u^3}}{u^{\frac{1}{2}}} = |u| = |x^{\frac{1}{2}}| = x^{\frac{1}{2}} \\ &= \int 2(u^{\frac{1}{2}})^{\frac{1}{2}} du = 4(u^{\frac{1}{2}})^{\frac{1}{2}} + C = 4(x^{\frac{1}{2}})^{\frac{1}{2}} + C \end{aligned}$$

382. $\int \frac{2}{(\cos x + 2\sin x)^2} dx = -\frac{1}{1+2\tan x} + C$

$$\begin{aligned} & \int \frac{2}{(\cos x + 2\sin x)^2} dx \quad \dots \text{MANIPULATE} \quad = \int \frac{2}{\cos^2 x(1 + \frac{2\sin x}{\cos x})^2} dx \\ & = \int \frac{2}{\cos^2 x(1+2\tan x)^2} dx = \int 2(1+2\tan x)^2 \sec^2 x dx \\ & \text{BY DIVIDE OUT} \text{ OR} \text{ CANCELLATION} \text{ OR} \text{ SUBSTITUTION} \\ & = -(1+2\tan x)^{-1} + C = -\frac{1}{1+2\tan x} + C \end{aligned}$$

383. $\int \ln\left(\frac{1}{6}x+2\right) dx = (x+12)\ln|x+12| - (1+\ln 6)x + C$

$$\begin{aligned} & \int \ln\left(\frac{1}{6}x+2\right) dx = \dots \text{BY PARTS, FOLLOWING BY MANIPULATION/SUBSTITUTION} \\ & = 2\ln|\frac{1}{6}x+2| - \int \frac{x}{\frac{1}{6}x+2} dx \\ & = 2\ln|\frac{1}{6}(x+12)| - \int \frac{(x+12)-12}{x+12} dx \\ & = 2\ln|x+12| + 12\ln|x+12| - \int 1 - \frac{12}{x+12} dx \\ & = -2\ln 6 + 2\ln|x+12| - [x - 12\ln|x+12|] + C \\ & = -2\ln 6 - x + 2\ln|x+12| + 12\ln|x+12| + C \\ & = \underline{\underline{(x+12)\ln|x+12|}} - (1+\ln 6)x + C \end{aligned}$$

$$\begin{aligned} & \text{ALTERNATIVE BY SUBSTITUTION FIRST, FOLLOWING BY PARTS} \\ & \int \ln\left(\frac{1}{6}x+2\right) dx = \int (\ln u)(\frac{1}{6} du) \quad \frac{du}{dx} = \frac{1}{6} \quad \frac{d}{dx} = \frac{1}{6} du \\ & = \int 6\ln u du \quad \frac{du}{\frac{1}{6}} = 6 du \\ & \quad \downarrow \quad \boxed{\begin{matrix} \ln u & \frac{1}{6} \\ u & 6 \end{matrix}} \dots = 6u\ln|u| - \int 6 du \\ & = 6u\ln|u| - 6u + C = 6\left(\frac{1}{6}x+2\right)\ln\left|\frac{1}{6}x+2\right| - 6\left(\frac{1}{6}x+2\right) + C \\ & = (x+12)\ln\left|\frac{1}{6}(x+12)\right| - (x+12) + C \\ & = (x+12)\left[\ln\frac{1}{6} + \ln|x+12|\right] - (x+12) + C \\ & = (x+12)\left(\frac{1}{6} + (x+12)\ln|x+12|\right) - (x+12) + C \\ & = -2\ln 6 - 12\ln 6 + (x+12)\ln|x+12| - x - 12 + C \\ & = \underline{\underline{(x+12)\ln|x+12|}} - (1+\ln 6)x + C \end{aligned}$$

384. $\int \frac{3x}{1+\sqrt{x-1}} dx = 2(x-1)^{\frac{3}{2}} + 12(x-1)^{\frac{1}{2}} - 3x - 12\ln(1+\sqrt{x-1}) + C$

$$\begin{aligned} \int \frac{3x}{1+\sqrt{x-1}} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int \frac{3x}{u} \cdot 2(u-1) du = \int \frac{6x(u-1)}{u} du \\ &= \int \frac{6(u-1)\left[1+(u-1)^2\right]}{u} du \\ &= \int \frac{6(u-1)+6(u-1)^2}{u} du \\ &= 6 \int \frac{u-1+u^2-2u+1}{u} du = 6 \int \frac{u^2-2u+2}{u} du \\ &= 6 \int u^2 - 2u + 2 - \frac{2}{u} du = 2u^3 - 9u^2 + 24u - 12\ln|u| + C \\ &= 2\left(1+\sqrt{x-1}\right)^3 - 9\left(1+\sqrt{x-1}\right)^2 + 24\left(1+\sqrt{x-1}\right) - 12\ln\left(1+\sqrt{x-1}\right) + C \\ \text{OR TRY FURTHER} \\ &= 2 \left[1 + 3\sqrt{x-1} + 3(x-1) + (x-1)^2 \right] - 9 \left[1 + 2\sqrt{x-1} + (x-1) \right] - 12\ln\left(1+\sqrt{x-1}\right) + C \\ &\quad 24 + 24\sqrt{x-1} \\ &= 2 + 6(x-1)^{\frac{1}{2}} + 6x - 6 + 2(x-1)^2 - 12\ln\left(1+\sqrt{x-1}\right) + C \\ &\quad - 9 - 16(x-1)^{\frac{1}{2}} - 9x + 9 \\ &\quad 24 + 24(x-1)^{\frac{1}{2}} \\ &= 2(x-1)^{\frac{1}{2}} + 12(x-1)^{\frac{1}{2}} - 3x - 12\ln\left(1+\sqrt{x-1}\right) + C \end{aligned}$$

385. $\int \frac{1}{x(x^3-1)} dx = \frac{1}{3}\ln\left|\frac{x^3-1}{x}\right| + C$

$$\begin{aligned} \int \frac{1}{x(x^3-1)} dx &= \text{BY SUBSTITUTION} = \dots \\ &= \int \frac{1}{x(u-1)} \left(\frac{du}{3u^2} \right) = \int \frac{1}{3u^2(u-1)} du \\ &= \int \frac{1}{3u(u-1)} du = \dots \text{ PARTIAL FRACTIONS BY INVERSION (CONTINUATION)} \\ &= \frac{1}{3} \int \frac{1}{u} - \frac{1}{u-1} du = \frac{1}{3} \ln\left|\frac{u-1}{u}\right| + C \approx \frac{1}{3}\ln\left|\frac{x^3-1}{x}\right| + C. \end{aligned}$$

386. $\int x^5(1-x^3)^{\frac{1}{2}} dx = \frac{2}{15}(1-x^3)^{\frac{5}{2}} - \frac{2}{9}(1-x^3)^{\frac{3}{2}} + C$

$$\begin{aligned} \int x^5(1-x^3)^{\frac{1}{2}} dx &= \dots \text{ BY SUBSTITUTION} = \dots \\ &= \int (1-x^3)(1-x^3)^{\frac{1}{2}}(-x^2) + x^2(1-x^3)^{\frac{1}{2}} dx \\ &= \int -x^2(1-x^3)^{\frac{1}{2}} + x^2(1-x^3)^{\frac{1}{2}} dx = \dots \text{ BY REVERSE CHAIN RULE (CONTINUATION)} \\ &= \frac{2}{3}(1-x^3)^{\frac{3}{2}} - \frac{2}{5}(1-x^3)^{\frac{5}{2}} + C \\ \text{ALTERNATIVE BY SUBSTITUTION} \\ &= \int x^2(1-x^3)^{\frac{1}{2}} dx = \int x^2 u \left(\frac{2u}{3u^2} \right) du \\ &= \int -\frac{2}{3}u^3 dx = -\frac{2}{3} \int u^3 (-1-u^2) du \\ &= -\frac{2}{3} \int u^3 - u^5 du = -\frac{2}{3} \left[\frac{1}{4}u^4 - \frac{1}{6}u^6 \right] + C \\ &= -\frac{2}{3}u^3 + \frac{2}{9}u^5 + C \\ &= \frac{2}{3}(1-x^3)^{\frac{3}{2}} - \frac{2}{9}(1-x^3)^{\frac{5}{2}} + C \\ \text{THE SUBSTITUTION } u = 1-x^3 \text{ ALSO WORKS} \end{aligned}$$

387. $\int \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{4}}} dx = 2x^{\frac{1}{2}} + 4x^{\frac{1}{4}} + 4\ln|x^{\frac{1}{4}} - 1| + C$

$$\begin{aligned} \int \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{4}}} dx &= \dots \text{BY SUBSTITUTION} \\ &= \int \frac{1}{u^{\frac{1}{2}} - u} (4u^{\frac{1}{2}} du) = \int \frac{4u^{\frac{1}{2}}}{u(u-1)} du \\ &= \int \frac{4u^{\frac{1}{2}}}{u(u-1)} du \quad \text{NOTICE SUBSTITUTION DIVIDES} \\ &= 4 \int \frac{u(u-1) + (u-1) + 1}{u(u-1)} du = 4 \int (u+1 + \frac{1}{u-1}) du \\ &= 4 \left[\frac{1}{2}u^2 + u + \ln|u-1| \right] + C = 2u^2 + 4u + 4\ln|u-1| + C \\ &= 2x^2 + 4x^{\frac{1}{2}} + 4\ln|x^{\frac{1}{2}} - 1| + C \end{aligned}$$

388. $\int \frac{1}{2x^2 - 2x + 1} dx = \arctan(2x-1) + C$

$$\begin{aligned} \int \frac{1}{2x^2 - 2x + 1} dx &= \int \frac{1}{4x^2 - 4x + 2} dx = \int \frac{1}{(2x-1)^2 + 1} dx \\ &= \dots \text{BY REVERSE CHAIN RULE OR SUBSTITUTION} \quad u=2x-1 \\ &= \arctan(2x-1) + C \end{aligned}$$

389. $\int \frac{1}{\sqrt{-x^2 - 2x}} dx = \arcsin(x+1) + C$

$$\begin{aligned} \int \frac{1}{\sqrt{-x^2 - 2x}} dx &= \int \frac{1}{\sqrt{-(x^2 + 2x)}} dx = \int \frac{1}{\sqrt{1 - (x+1)^2}} dx \\ &= \int \frac{1}{\sqrt{1 - (x+1)^2}} dx \\ &= \dots \text{BY REVERSE CHAIN RULE OR SUBSTITUTION} \quad u=x+1 \\ &= \arcsin(x+1) + C \end{aligned}$$

390. $\int \frac{1}{\sqrt{9x^2 + 12x + 5}} dx = \frac{1}{3} \operatorname{arsinh}(3x+2) + C$

$$\begin{aligned} \int \frac{1}{\sqrt{9x^2 + 12x + 5}} dx &= \int \frac{1}{\sqrt{(3x+2)^2 + 1}} dx = \dots \text{BY REVERSE CHAIN RULE (REVERSE) OR} \\ &\quad \text{SUBSTITUTION} \quad u=3x+2 \\ &= \frac{1}{3} \operatorname{arsinh}(3x+2) + C \end{aligned}$$

391. $\int \frac{1}{\sqrt{16x^2 - 8x}} dx = \frac{1}{4} \operatorname{arcosh}(4x-1) + C$

$$\begin{aligned}\int \frac{1}{\sqrt{16x^2 - 8x}} dx &= \dots \text{COMPLETING THE SQUARE} \dots \\ &= \int \frac{1}{\sqrt{(16x^2 - 8x + 1) - 1^2}} dx = \int \frac{1}{\sqrt{(4x-1)^2 - 1^2}} dx \\ &\stackrel{\text{BY REVERSE CHAIN RULE (INTEGRATION) OR SUBSTITUTION}}{=} u = 4x-1 \\ &= \frac{1}{4} \operatorname{arcosh}(4x-1) + C\end{aligned}$$

392. $\int \sqrt{\frac{x}{1-x}} dx = \arcsin \sqrt{x} - \sqrt{x-x^2} + C$

$$\begin{aligned}\int \sqrt{\frac{x}{1-x}} dx &= \int \frac{\sqrt{x}}{\sqrt{1-x}} dx = \dots \text{SUBSTITUTION} \dots \\ &\stackrel{\text{...}}{=} \int \frac{\sin \theta}{\sqrt{1-\sin^2 \theta}} (2\cos \theta d\theta) \\ &= \int 2\cos^2 \theta d\theta = \int 2\left(\frac{1}{2} - \frac{1}{2}\sin^2 \theta\right) d\theta \\ &= \int 1 - \frac{1}{2}\sin^2 \theta d\theta = \theta - \frac{1}{2}\sin \theta \cos \theta + C \\ &= \theta - \frac{1}{2}\sin \theta \cos \theta + C \\ &= \arcsin \sqrt{x} - \sqrt{x-x^2} + C \\ &= \arcsin \sqrt{x} - \sqrt{x-x^2} + C\end{aligned}$$

393. $\int \frac{4x^2}{4x^2+1} dx = x - \frac{1}{2} \operatorname{arctan} 2x + C$

$$\begin{aligned}\int \frac{4x^2}{4x^2+1} dx &\dots \text{MANIPULATION} \dots = \int \frac{(4x^2+1)-1}{4x^2+1} dx \\ &\stackrel{\text{...}}{=} \int 1 - \frac{1}{4x^2+1} dx = \int 1 - \frac{1}{4x^2+1} dx \\ &\stackrel{\text{...}}{=} \int 1 - \frac{1}{4x^2+1} dx \stackrel{\text{REVERSE CHAIN RULE (INTEGRATION)}}{=} x - \frac{1}{2} \operatorname{arctan} 2x + C\end{aligned}$$

394. $\int e^{\cos x} \sin x \cos x dx = (1 - \cos x)e^{\cos x} + C$

$$\begin{aligned}\int e^{\cos x} \sin x \cos x dx &= \dots \text{INTEGRATION BY PARTS} \dots \\ &= (-\cos x) e^{\cos x} - \int (\sin x) e^{\cos x} dx \\ &= -e^{\cos x} + \int e^{\cos x} \sin x dx \\ &\stackrel{\text{...}}{=} \int e^{\cos x} \sin x dx \stackrel{\text{REVERSE CHAIN RULE (INTEGRATION)}}{=} -e^{\cos x} + e^{\cos x} \sin x \\ &= -e^{\cos x} + e^{\cos x} + e^{\cos x} \sin x + C \\ &= e^{\cos x} (1 - \cos x) + C\end{aligned}$$

395. $\int (9 - 4\sin^2 x)^{\frac{1}{2}} \sin x \, dx = -\frac{9}{4} \arcsin\left(\frac{2}{3}\cos x\right) - \frac{1}{2}(9 - 4\sin^2 x)^{\frac{1}{2}} \cos x + C$

$$\begin{aligned} \int (9 - 4\sin^2 x)^{\frac{1}{2}} \sin x \, dx &= \int 3^{\frac{1}{2}}(-4\sin^2 x)^{\frac{1}{2}} \sin x \, dx \\ &= \int 3(1 - (\frac{2}{3}\cos x))^{\frac{1}{2}} \sin x \, dx \dots \text{BY SUBSTITUTION} \\ &= \int 3[1 - \sin^2 \theta]^{\frac{1}{2}} \sin \theta \, d\theta \quad \begin{array}{l} \sin \theta = \frac{2}{3}\cos x \\ \cos^2 \theta = \frac{1}{3}\sin^2 x \\ d\theta = -\frac{3\cos \theta}{2\sin x} \, dx \end{array} \\ &= \int -\frac{3}{2}\cos \theta \sin \theta \, d\theta \\ &= \int -\frac{3}{2}\cos \theta \cos \theta \, d\theta \\ &= -\frac{3}{2}\theta - \frac{3}{2}\sin \theta \cos \theta + C \\ &= -\frac{3}{2}\arcsin\left(\frac{2}{3}\cos x\right) - \frac{3}{2}\left(\frac{2}{3}\cos x\right)\sqrt{1 - 4\cos^2 x} + C \\ &= -\frac{3}{2}\arcsin\left(\frac{2}{3}\cos x\right) - \frac{1}{2}\arcsin(9 - 4\sin^2 x) + C \end{aligned}$$

396. $\int \frac{2}{x + \sqrt{1-x^2}} \, dx = \arcsin x + \ln(x + \sqrt{1-x^2}) + C$

$$\begin{aligned} \int \frac{2}{x + \sqrt{1-x^2}} \, dx &= \dots \text{SUBSTITUTION} \dots \\ &= \int \frac{2}{\sin \theta + \sqrt{1-\sin^2 \theta}} (\cos \theta \, d\theta) \quad \begin{array}{l} x = \sin \theta \\ \theta = \arcsin x \\ dx = \cos \theta \, d\theta \end{array} \\ &= \int \frac{2\cos \theta}{\sin \theta + \cos \theta} \, d\theta = \int \frac{(2\sin \theta + 2\cos \theta) + (2\cos \theta - 2\sin \theta)}{(2\sin \theta + 2\cos \theta)} \, d\theta \\ &= \int 1 + \frac{2\cos \theta - 2\sin \theta}{\sin \theta + \cos \theta} \, d\theta = \theta + \ln|\sin \theta + \cos \theta| + C \\ &\quad \int \frac{2\cos \theta}{\cos \theta} \, d\theta = \ln|\cos \theta| + C \\ &= \arcsin x + \ln|x + \sqrt{1-x^2}| + C \end{aligned}$$

VARIATION...

$$\begin{aligned} &\dots = \int \frac{2\cos \theta}{\sin \theta + \cos \theta} \, d\theta = \int \frac{2\cos \theta - 2\sin \theta}{\cos \theta - \sin \theta} \, d\theta = \int \frac{2\cos^2 \theta - 2\cos \theta \sin \theta}{\cos^2 \theta - \sin^2 \theta} \, d\theta \\ &= \int \frac{(1 + \tan^2 \theta) - 2\tan \theta}{1 - \tan^2 \theta} \, d\theta = \int \sec^2 \theta + 1 - \tan^2 \theta \, d\theta \\ &= \theta + \frac{1}{2}\ln|\sec \theta + \tan \theta| - \frac{1}{2}\ln|\sec \theta - \tan \theta| + C \\ &= \theta + \frac{1}{2}\ln|1 + \sin 2\theta| + C = \theta + \frac{1}{2}\ln|1 + 2\sin \theta \cos \theta| + C \\ &= \theta + \frac{1}{2}\ln|(2\sin \theta + \cos \theta)^2 + 2\sin^2 \theta \cos^2 \theta| = \theta + \frac{1}{2}\ln[(\cos \theta + \sin \theta)^2] + C \\ &= \theta + \ln|\cos \theta + \sin \theta| + C \dots \\ &= \arcsin x + \ln|x + \sqrt{1-x^2}| + C \end{aligned}$$

ASIDE

397. $\int \frac{1}{\sqrt{x+1} + \sqrt{x-1}} \, dx = \frac{1}{3}(x+1)^{\frac{3}{2}} - \frac{1}{3}(x-1)^{\frac{3}{2}} + C$

$$\begin{aligned} \int \frac{1}{\sqrt{x+1} + \sqrt{x-1}} \, dx &= \dots \text{BY MANIPULATION} \\ &= \int \frac{1((\sqrt{x+1} - \sqrt{x-1})}{(\sqrt{x+1} + \sqrt{x-1})(\sqrt{x+1} - \sqrt{x-1})} \, dx = \int \frac{\sqrt{x+1} - \sqrt{x-1}}{(x+1) - (x-1)} \, dx \\ &= \int \frac{1}{2}(x+1)^{\frac{1}{2}} - \frac{1}{2}(x-1)^{\frac{1}{2}} \, dx = \frac{1}{3}(x+1)^{\frac{3}{2}} - \frac{1}{3}(x-1)^{\frac{3}{2}} + C \end{aligned}$$

398. $\int \cosh 2x \, dx = \frac{1}{2} \sin 2x + C$

$$\int \cosh 2x \, dx = \dots \text{ or } \text{INSPECT} \dots \frac{1}{2} \sin 2x + C$$

399. $\int e^x (\tan x + \sec^2 x) \, dx = e^x \tan x + C$

$$\begin{aligned} \int e^x (\tan x + \sec^2 x) \, dx &= \dots \text{ BY INSPECTION } \dots e^x \tan x + C \\ \frac{d}{dx}(e^x \tan x) &= e^x \tan x + e^x (\sec^2 x) \\ &= e^x (\tan x + \sec^2 x) \end{aligned}$$

400. $\int \frac{\sqrt{x+1}}{x+5} \, dx = 2\sqrt{x+1} - 4 \arctan\left(\frac{\sqrt{x+1}}{2}\right) + C$

$$\begin{aligned} \int \frac{\sqrt{x+1}}{x+5} \, dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{u}{u^2+4} (2u \, du) = \int \frac{2u^2}{u^2+4} \, du \\ &\quad \text{BY MANIPULATION OR SUBSTITUTION } u=2\tan\theta \\ &= \int \frac{2(2\tan^2\theta)}{u^2+4} \, du = \int 2 - \frac{8}{u^2+4} \, du \\ &= 2u - 8 \times \frac{1}{2} \arctan\left(\frac{u}{2}\right) + C \\ &= 2\sqrt{x+1} - 4 \arctan\left(\frac{\sqrt{x+1}}{2}\right) + C \end{aligned}$$

401. $\int \frac{1}{x\sqrt{x^2-1}} dx = \begin{cases} \arctan \sqrt{x^2-1} + C \\ \arcsin \left(\frac{\sqrt{x^2-1}}{x} \right) + C \\ \arccos \left(\frac{1}{x} \right) + C \\ \text{arcsec } x + C \\ 2 \arctan \left(x + \sqrt{x^2-1} \right) + C \end{cases}$

$\int \frac{1}{x\sqrt{x^2-1}} dx = \dots \text{ BY SUBSTITUTION } \dots$

$$\begin{aligned} &= \int \frac{1}{x\sqrt{x^2-1}} \left(\frac{dx}{x} \right) = \int \frac{1}{x^2} du \\ &= \int \frac{1}{1-u^2} du = \arctan u + C \\ &= \arctan \sqrt{x^2-1} + C \end{aligned}$$

NOTE THAT

 $\dots = \arctan \sqrt{x^2-1} + C = \arccos \left(\frac{1}{x} \right) + C = \text{arcsec } x + C = \arctan \left(\frac{\sqrt{x^2-1}}{x} \right) + C \text{ etc}$

ALTERNATIVE APPROACH

$$\begin{aligned} &\int \frac{1}{x\sqrt{x^2-1}} dx = \dots \text{ BY HYPERBOLIC SUBSTITUTION } \dots \\ &= \int \frac{1}{\cosh u \sqrt{\cosh^2 u - 1}} (\sinh u du) = \int \frac{1}{\cosh^2 u} du \\ &= \int \operatorname{sech} u du = \dots \text{ MANY DIFFERENT WAYS THIS CAN BE INTEGRATED - PREFER BY EXPANDING AS A SECOND SUBSTITUTION } \\ &= \int \frac{2}{e^u + e^{-u}} du = \int \frac{2e^u}{e^{2u} - 1} du = \int \frac{2e^u}{(e^u + 1)^2} du \\ &= 2 \operatorname{arctanh}(e^u) + C = 2 \operatorname{arctanh}(\operatorname{arcsec} x) + C \\ &= 2 \operatorname{arctanh}(e^{\ln(x+\sqrt{x^2-1})}) + C = 2 \operatorname{arctanh}(x + \sqrt{x^2-1}) + C \end{aligned}$$

VARIATION

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \dots \text{ BY A TRIGONOMETRIC SUBSTITUTION } \dots$$

$$\begin{aligned} &= \int \frac{1}{\sin \theta \sqrt{1-\sin^2 \theta}} (\cos \theta d\theta) \\ &= \int 1 d\theta = \theta + C = \arctan x + C \\ &= \arctan \frac{1}{x} + C = \arctan \sqrt{x^2-1} + C \end{aligned}$$

ANOTHER TRIGONOMETRIC VARIANT

$$\begin{aligned} &\int \frac{1}{x\sqrt{x^2-1}} dx = \dots \text{ REVERSE SUBSTITUTION } \dots \\ &= \int \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}-1}} \left(-\frac{1}{u^2} du \right) \\ &= \int \frac{1}{\frac{1}{u^2}\sqrt{1-\frac{1}{u^2}}} \left(-\frac{1}{u^2} du \right) = \int -\frac{1}{\sqrt{1-u^2}} du \\ &= -\operatorname{arccosec} u + C = -\operatorname{arccosec} \left(\frac{1}{x} \right) + C \\ &= \operatorname{arcsec} x + C = \arctan \sqrt{x^2-1} + C \end{aligned}$$

402. $\int (1-\cos x) \sqrt{(1+\cos x)^2 + \sin^2 x} dx = \begin{cases} \frac{8}{3} \sin^2 \left(\frac{1}{2}x \right) + C & \text{if } \cos \left(\frac{1}{2}x \right) > 0 \\ -\frac{8}{3} \sin^2 \left(\frac{1}{2}x \right) + C & \text{if } \cos \left(\frac{1}{2}x \right) < 0 \end{cases}$

$$\begin{aligned} &\int (1-\cos x) \sqrt{\sin^2 x + (\cos x + 1)^2} dx = \int (1-\cos x) (\sin x + \cos x + 2\cos x + 1)^{\frac{1}{2}} dx \\ &= \int (1-\cos x) (2+2\cos x)^{\frac{1}{2}} dx = \dots \text{ USING CONV. DOUBLE ANGLE FORMULAE } \dots \\ &= \int [1-(1-2\sin^2 \frac{x}{2})][2\sqrt{2(\cos^2 \frac{x}{2}-1)}]^{\frac{1}{2}} dx \\ &= \int (2\sin^2 \frac{x}{2})(4\cos^2 \frac{x}{2})^{\frac{1}{2}} dx = \int 4 \sin^2 \frac{x}{2} |\cos \frac{x}{2}| dx \\ &= \dots \text{ BY INTEGRATION } \dots \\ &= \begin{cases} \frac{8}{3} \sin^2 \frac{x}{2} + C & \text{if } \cos \frac{x}{2} > 0 \\ -\frac{8}{3} \sin^2 \frac{x}{2} + C & \text{if } \cos \frac{x}{2} < 0 \end{cases} \end{aligned}$$

403. $\int \frac{x+3}{x^2+1} dx = 3\arctan x + \frac{1}{2}\ln(x^2+1) + C$

$$\begin{aligned} \int \frac{x+3}{x^2+1} dx &= \int \frac{x}{x^2+1} dx + \frac{3}{x^2+1} dx = \int \frac{1}{2} \left(\frac{2x}{x^2+1} \right) dx + 3 \left(\frac{1}{x^2+1} \right) dx \\ &\quad \text{BY INSPECTION FOR BOTH PARTS OF THE INTEGRAND} \\ &= \frac{1}{2} \ln(x^2+1) + 3 \arctan x + C \end{aligned}$$

404. $\int \frac{4x^3 - 3x^2 + 2x - 1}{x+1} dx = \frac{4}{3}x^3 - \frac{7}{2}x^2 + 9x - 10\ln|x+1| + C$

$$\begin{aligned} \int \frac{4x^3 - 3x^2 + 2x - 1}{x+1} dx &= \text{BY SUBSTITUTION } u = x+1, \text{ BY ALGEBRAIC DIVISION} \\ &\quad \text{OR FACTORISATION} \\ &= \int \frac{4x^2(x+1) - 7x(x+1) + 9(x+1) - 10}{x+1} dx = \int 4x^2 - 7x + 9 - \frac{10}{x+1} dx \\ &= \frac{4}{3}x^3 - \frac{7}{2}x^2 + 9x - 10\ln|x+1| + C \end{aligned}$$

405. $\int \frac{1}{x^2+x+1} dx = \frac{2}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$

$$\begin{aligned} \int \frac{1}{x^2+x+1} dx &= \text{COMPLETING THE SQUARE IN THE DENOMINATOR} \\ &= \int \frac{4}{4x^2+4x+4} dx = \int \frac{4}{(2x+1)^2+3} dx = \int \frac{4}{(2x+1)^2+(\sqrt{3})^2} dx \\ &\quad \text{BY INSPECTION OR USING THE SUBSTITUTION } u = 2x+1 \\ &= \frac{4}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) \times \frac{1}{2} + C = \frac{2}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C \end{aligned}$$

406. $\int \frac{9}{(9-x^2)^{\frac{3}{2}}} dx = \frac{x}{\sqrt{9-x^2}} + C$

$$\begin{aligned} \int \frac{1}{(9-x^2)^{\frac{3}{2}}} dx &\dots \text{BY A TRIGONOMETRIC SUBSTITUTION} \\ &= \int \frac{1}{(9-9\sin^2\theta)^{\frac{3}{2}}} (3\cos\theta d\theta) \\ &= \int \frac{27\cos\theta}{9(1-\sin^2\theta)^{\frac{3}{2}}} d\theta = \int \frac{27\cos\theta}{9\cos^2\theta} d\theta \\ &= \int \frac{27}{9\cos^2\theta} d\theta = \int \sec^2\theta d\theta \\ &= 9\tan\theta + C = \frac{9}{\sqrt{9-x^2}} + C \end{aligned}$$

$\theta = \arcsin\frac{x}{3}$
 $d\theta = \frac{1}{\sqrt{9-x^2}} dx$
 $\tan\theta = \frac{x}{\sqrt{9-x^2}}$

407. $\int \frac{2}{x^3 - x} dx = \ln \left| \frac{x^2 - 1}{x^2} \right| + C$

$$\begin{aligned}\int \frac{2}{x^3 - x} dx &= \int \frac{2}{x(x+1)(x-1)} dx = \int \frac{2}{x(x+1)(x-1)} dx \\ &\stackrel{\text{PROPER FRACTION BY INVERSION (CAREFUL)}}{=} \int \frac{\frac{2}{x}}{x+1} + \frac{\frac{2}{x-1}}{x-1} dx = \int -\frac{2}{x^2-1} + \frac{1}{x-1} + \frac{1}{x+1} dx \\ &= -2\ln|x| + |\ln|x+1|| + |\ln|x-1|| + C = \ln\left|\frac{x^2-1}{x}\right| + C \\ \int \frac{1}{\sqrt{x^2+36}} dx &= \stackrel{\text{BY INSPECTION / STANDARD RESULT}}{=} \int \frac{1}{\sqrt{x^2+c^2}} dx \\ &= \operatorname{arsinh}\left(\frac{x}{6}\right) + C = \ln\left(2+\sqrt{x^2+36}\right) + C\end{aligned}$$

408. $\int \frac{1}{\sqrt{x^2+36}} dx = \begin{bmatrix} \operatorname{arsinh}\left(\frac{1}{6}x\right) + C \\ \ln\left(x + \sqrt{x^2+36}\right) + C \end{bmatrix}$

409. $\int \frac{x^2}{\sqrt{1-x^2}} dx = \begin{bmatrix} \frac{1}{2} \arcsin x - \frac{1}{2} x \sqrt{1-x^2} + C \\ -\frac{1}{2} \arccos x - \frac{1}{2} x \sqrt{1-x^2} + C \end{bmatrix}$

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{\sin^2 \theta}{\sqrt{1-\sin^2 \theta}} (\cos \theta d\theta) = \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta \\ &= \int \sin^2 \theta d\theta = \dots \text{ BY TRIGONOMETRIC IDENTITIES } \dots \\ &= \int \frac{1}{2} - \frac{1}{2} \cos 2\theta d\theta = -\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C \\ &= \frac{1}{2}\theta - \frac{1}{4}(2\sin \theta \cos \theta) + C = \frac{1}{2}\theta - \frac{1}{2}\sin \theta \cos \theta + C \\ &= \frac{1}{2}\operatorname{arsinx} - \frac{1}{2}\sqrt{1-x^2} + C = -\frac{1}{2}\arccos x - \frac{1}{2}\sqrt{1-x^2} + C \\ &\quad \uparrow \text{ IF THE SUBSTITUTION } \theta = \operatorname{arcsin} x \\ &\quad \text{IS USED INSTEAD}\end{aligned}$$

410. $\int \frac{\sqrt{x}}{x+1} dx = 2\sqrt{x} - 2\arctan(\sqrt{x}) + C$

$$\begin{aligned} & \int \frac{\sqrt{x}}{x+1} dx = \dots \text{ BY SUBSTITUTION } \\ & = \int \frac{u}{u^2+1} (2u du) = \int \frac{2u^2}{u^2+1} du \\ & \quad \text{BY ALGEBRAIC DIVISION OR MANIPULATION} \\ & = \int \frac{2(u^2+1)-2}{u^2+1} du = \int \frac{2(u^2+1)-2}{u^2+1} du \\ & = \int 2 - \frac{2}{u^2+1} du = 2u - 2\arctan(u) + C \\ & = 2\sqrt{x} - 2\arctan(\sqrt{x}) + C \end{aligned}$$

411. $\int \frac{1}{9x^2+64} dx = \frac{1}{24} \arctan\left(\frac{3}{8}x\right) + C$

$$\begin{aligned} \int \frac{1}{9x^2+64} dx &= \int \frac{1}{(3x)^2+8^2} dx = \dots \text{ BY INSPECTION } \\ &= \frac{1}{8} \arctan\left(\frac{3x}{8}\right) \times \frac{1}{3} = \frac{1}{24} \arctan\left(\frac{3x}{8}\right) + C \end{aligned}$$

412. $\int \frac{x}{\sqrt{x^2+1}} dx = \sqrt{x^2+1} + C$

$$\begin{aligned} \int \frac{x}{\sqrt{x^2+1}} dx &= \int x(x+1)^{-\frac{1}{2}} dx \quad \text{BY DIVIDE BY THE ROOT (CANCELLATION)} \\ &= (x+1)^{\frac{1}{2}} + C \\ \text{THE SUBSTITUTIONS } u &= x+1, du = 2x+2dx, x = \sinh\theta \text{ AND } \theta = \text{arsinh}(x+1) \end{aligned}$$

413. $\int \frac{x^2}{\sqrt{x^2+1}} dx = \left[\frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\operatorname{arsinh} x + C \right]$

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2+1}} dx &= \text{BY HYPERBOLIC SUBSTITUTION} \\ &= \int \frac{\sin^2 \theta}{\sqrt{\sin^2 \theta + 1}} (\cos \theta d\theta) = \int \frac{\sin^2 \theta \cos \theta}{\sqrt{\sin^2 \theta + 1}} d\theta \\ &\quad \text{USING IDENTITIES} \\ &= \int \sin^2 \theta d\theta = \int \frac{1}{2} \cos 2\theta - \frac{1}{2} d\theta \\ &= \frac{1}{2} \sin 2\theta - \frac{1}{2}\theta + C \\ &= \frac{1}{2}(2\sin \theta \cos \theta) - \frac{1}{2}\theta + C = \frac{1}{2}\sin(\theta)\sqrt{\sin^2 \theta + 1} - \frac{1}{2}\theta + C \\ &= \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\operatorname{arsinh} x + C \\ &\quad \text{THE TRIGONOMETRIC SUBSTITUTION } u = \operatorname{arsinh} x \text{ ALSO WORKS} \\ &\quad (x = \sinh u) \end{aligned}$$

414. $\int x^3(1-x^2)^{\frac{1}{2}} dx = \left[\begin{array}{l} \frac{1}{5}(1-x^2)^{\frac{5}{2}} - \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C \\ -\frac{1}{3}x^2(1-x^2)^{\frac{3}{2}} - \frac{2}{15}(1-x^2)^{\frac{5}{2}} + C \end{array} \right]$

$$\begin{aligned} \int x^3(1-x^2)^{\frac{1}{2}} dx &= \dots \text{BY SUBSTITUTION} \\ &= \int x^2 u (-\frac{2}{3} du) = \int -u^2 x^2 du \\ &= \int -u^2(1-u^2) du = \int u^4 - u^2 du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}(1-x^2)^{\frac{5}{2}} - \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C \\ &\quad \text{ALTERNATIVE BY MANIPULATION} \\ &= \int x^2(1-x^2)^{\frac{1}{2}} dx = \int -x(-2)(1-x^2)^{\frac{1}{2}} + x(1-x^2)^{\frac{1}{2}} dx \\ &= \int x(1-x^2)^{\frac{1}{2}} - x(1-x^2)^{\frac{1}{2}} dx \quad \text{BY INTERCHANGE} \\ &= -\frac{1}{2}(1-x^2)^{\frac{3}{2}} + \frac{1}{2}(1-x^2)^{\frac{1}{2}} + C \\ &\quad \text{ALTERNATIVE METHOD BY PARTS} \\ &= \int x^2(1-x^2)^{\frac{1}{2}} dx = \int x^2 \left[2x(1-x^2)^{\frac{1}{2}} \right] dx \\ &= -\frac{1}{3}x^2(1-x^2)^{\frac{3}{2}} - \int -2x^2(1-x^2)^{\frac{1}{2}} dx \\ &= -\frac{1}{3}x^2(1-x^2)^{\frac{3}{2}} + \int 2x^2(1-x^2)^{\frac{1}{2}} dx \\ &= -\frac{1}{3}x^2(1-x^2)^{\frac{3}{2}} - \frac{2}{5}(1-x^2)^{\frac{5}{2}} + C \end{aligned}$$

415. $\int \sinh x \cosh x \, dx = \begin{bmatrix} \frac{1}{2} \sinh^2 x + C \\ \frac{1}{2} \cosh^2 x + C \\ \frac{1}{4} \cosh 2x + C \end{bmatrix}$

$\int \sinh x \cosh x \, dx = \dots$ BY REVERSE CHAIN RULE
 $= \frac{1}{2} \sinh^2 x + C \quad \text{OR} \quad \frac{1}{2} \cosh^2 x + C$
 ALTERNATIVE BY SUBSTITUTION
 $\int \sinh x \cosh x \, dx = \int \frac{1}{2} \sinh(2x) \, dx = \frac{1}{4} \cosh 2x + C$

416. $\int \frac{4^x + 2 \times 16^x}{1+16^x} \, dx = \frac{1}{\ln 4} \left[\arctan(4^x) \ln(1+4^{2x}) \right] + C$

$\int \frac{4^x + 2 \times 16^x}{1+16^x} \, dx = \int \frac{4^x + 2 \times 4^{2x}}{1+4^{2x}} \, dx$ $16 = (4^2)^2 = 4^{2x}$
 NOW BY SUBSTITUTION
 $= \frac{1}{\ln 4} \int \frac{1+2u^2}{1+u^2} \, du = \frac{1}{\ln 4} \left[\int \frac{1}{1+u^2} \times \frac{2u^2}{1+u^2} \, du \right]$
 $= \frac{1}{\ln 4} \left[\arctan u + \ln(1+u^2) \right] + C$
 $= \frac{1}{\ln 4} \left[\arctan(4^x) + \ln(1+4^{2x}) \right] + C$

417. $\int \frac{\sin x}{\cos x + \cos^3 x} \, dx = \frac{1}{2} \ln \left(\frac{1+\cos^2 x}{\cos^2 x} \right) + C$

$\int \frac{\sin x}{\cos x + \cos^3 x} \, dx = \dots$ BY SUBSTITUTION
 $= \int \frac{\sin x}{u+u^3} \left(-\frac{du}{\sin x} \right) = \int -\frac{1}{u+u^3} \, du$
 $= -\int \frac{1}{u(u^2+1)} \, du$
 BY PARTIAL FRACTIONS
 $\frac{1}{u(u^2+1)} = \frac{A}{u} + \frac{Bu+C}{u^2+1}$
 $1 = A(u^2+1) + (Bu+C)u$
 $1 = (A+B)u^2 + Cu + A$
 $= -\int \frac{1}{u} \, du - \int \frac{u}{u^2+1} \, du = \int \frac{1}{u^2+1} \, du - \frac{1}{u}$
 $= \frac{1}{2} \ln(u^2+1) - \ln u + C = \frac{1}{2} \left[\ln(u^2+1) - 2 \ln u \right] + C$
 $= \frac{1}{2} \ln(u^2+1) - \ln u^2 + C = \frac{1}{2} \ln \left(\frac{u^2+1}{u^2} \right) + C = \frac{1}{2} \ln \left(\frac{1+\cos^2 x}{\cos^2 x} \right) + C$

418. $\int \sinh \sqrt{x} \, dx = 2\sqrt{x} \cosh \sqrt{x} - 2\sinh \sqrt{x} + C$

$$\begin{aligned} \int \sinh u^2 \, du &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \sinh u \cdot (2u \, du) = \int 2u \sinh u \, du \\ &\text{PROCEEDED BY INTEGRATION BY PARTS} \\ &= 2u \sinh u - \int 2\cosh u \, du \\ &= 2u \sinh u - 2\sinh u + C \\ &= 2\sqrt{x} \cosh \sqrt{x} - 2\sinh \sqrt{x} + C \end{aligned}$$

| |
|-----------------|
| $u = \sqrt{x}$ |
| $u^2 = x$ |
| $2u \, du = dx$ |

| | |
|-----------|-----------|
| $2u$ | 2 |
| $\cosh u$ | $\sinh u$ |

419. $\int \frac{\operatorname{sech} x}{\cosh x - \sinh x} \, dx = \left[\frac{-\ln|1 - \tanh x| + C}{x + \ln(\cosh x) + C} \right]$

$$\begin{aligned} \int \frac{\operatorname{sech} x}{\cosh x - \sinh x} \, dx &= \int \frac{\operatorname{sech} x \operatorname{sech} x}{\cosh x - \sinh x} \, dx \\ &= \int \frac{\operatorname{sech}^2 x}{1 - \tanh x} \, dx = \dots \text{ of the form } \int \frac{f(x)}{g(x)} \, dx = \ln|f(x)| + C \\ &= -\ln|1 - \tanh x| + C \\ &\text{ALTERNATIVE APPROACH:} \\ \int \frac{\operatorname{sech} x}{\cosh x - \sinh x} \, dx &= \int \frac{\operatorname{sech} (\cosh x + \sinh x)}{(\cosh x - \sinh x)(\cosh x + \sinh x)} \, dx \\ &= \int \frac{1 + \operatorname{sech} x}{\cosh x - \sinh x} \, dx = \int 1 + \operatorname{tanh} x \, dx \\ &= \int 1 + \frac{\operatorname{sech} x}{\cosh x} \, dx = x + \ln|\cosh x| + C \\ &\quad \uparrow \\ &\quad \text{again of the form } \int \frac{f(x)}{g(x)} \, dx = \ln|f(x)| + C \end{aligned}$$

420. $\int \frac{1 + \sin x}{1 - \sin x} \, dx = -x + 2 \tan x + 2 \sec x + C$

$$\begin{aligned} \int \frac{1 + \sin x}{1 - \sin x} \, dx &= \int \frac{(1 + \sin x)(1 + \sin x)}{(1 - \sin x)(1 + \sin x)} \, dx = \int \frac{(1 + \sin x)^2}{1 - \sin^2 x} \, dx \\ &= \int \frac{1 + 2\sin x + \sin^2 x}{\cosh^2 x} \, dx = \int \frac{\operatorname{sech}^2 x + \frac{2\sin x}{\cosh x} + \tan^2 x}{\cosh^2 x} \, dx \\ &= \int \operatorname{sech}^2 x + 2\operatorname{tanh} x \operatorname{sech} x + \tan^2 x \, dx = \dots \boxed{1 + \operatorname{tanh}^2 x} \\ &= \int \operatorname{sech}^2 x + 2\operatorname{tanh} x \operatorname{sech} x + \operatorname{sech}^2 x - 1 \, dx \\ &= \int 2\operatorname{sech}^2 x + 2\operatorname{tanh} x \operatorname{sech} x - 1 \, dx. \quad \text{ALL ARE STANDARD INTEGRALS} \\ &= 2\tan x + 2\sec x - x + C \\ &\quad \boxed{\text{THE LITTLE t IDENTITIES! (Gives the twice) ALSO WORK HERE}} \end{aligned}$$

421. $\int \frac{4}{(x^2 - 4)^{\frac{3}{2}}} dx = -\frac{x}{\sqrt{x^2 - 4}} + C$

$$\begin{aligned} \int \frac{4}{(x^2 - 4)^{\frac{3}{2}}} dx &= \dots \text{ BY INSPECTION} \dots \\ &= \int \frac{8x \cos \theta}{(4 \sin^2 \theta - 4)^{\frac{3}{2}}} d\theta = \int \frac{8x \cos \theta}{(4(\sin^2 \theta - 1))^{\frac{3}{2}}} d\theta \\ &= \int \frac{8x \cos \theta}{(4 \sin^2 \theta)^{\frac{3}{2}}} d\theta = \int \frac{8x \cos \theta}{4 \sin^3 \theta d\theta} d\theta \\ &= \int 8x \cos \theta d\theta = -8 \sin \theta + C \\ &= -\frac{8x \sin \theta}{\sin^2 \theta} + C = -\frac{8x}{4 \sin^2 \theta} + C \\ &= -\frac{2x}{\sin^2 \theta} + C = -\frac{2x}{\frac{4}{x^2 - 4}} + C \\ &= -\frac{2(x^2 - 4)}{4} + C = -\frac{x^2 - 4}{2} + C \end{aligned}$$

$x = 2 \sin \theta$
 $d\theta = \frac{dx}{2 \cos \theta}$
 $\sin^2 \theta = \frac{x^2}{4}$
 $\cos^2 \theta = 1 - \frac{x^2}{4}$
 $\sin \theta = \frac{x}{\sqrt{x^2 - 4}}$
 $\sin^2 \theta = \frac{x^2}{x^2 - 4}$

422. $\int 4 \sinh\left(\frac{1}{4}x\right) dx = 16 \cosh\left(\frac{1}{4}x\right) + C$

$$\int 4 \sinh\left(\frac{1}{4}x\right) dx = \dots \text{ BY INSPECTION} \dots = 16 \cosh\left(\frac{1}{4}x\right) + C$$

423. $\int \frac{1-\cos x}{1+\cos x} dx = \begin{bmatrix} -x - 2 \cot x + 2 \operatorname{cosec} x + C \\ -x - 2 \tan\left(\frac{1}{2}x\right) + C \end{bmatrix}$

$$\begin{aligned} \int \frac{1-\cos x}{1+\cos x} dx &= \int \frac{(1-\cos x)(1+\cos x)}{(1+\cos x)(1-\cos x)} dx = \int \frac{1-2\cos x+\cos^2 x}{1-\cos^2 x} dx \\ &= \int \frac{1-2\cos x+\cos^2 x}{\sin^2 x} dx = \int \cos x dx - 2\sin x \cos x + \sin^2 x dx \\ &= \int \cos x dx - 2\sin x \cos x + (\cos x - 1) dx \\ &= \int 2\cos x dx - 2\sin x \cos x - x + C \\ &= 2\sin x + 2\operatorname{cosec} x - x + C \end{aligned}$$

ALTERNATIVE BY USING THE TELEGRAPHIC DOUBLE ANGLE IDENTITIES

$$\begin{aligned} \int \frac{1-\cos x}{1+\cos x} dx &= \int \frac{1 - (1 - 2\sin^2 \frac{x}{2})}{1 + (2\sin^2 \frac{x}{2} - 1)} dx = \int \frac{2\sin^2 \frac{x}{2}}{2\sin^2 \frac{x}{2}} dx \\ &= \int \tan^2 \frac{x}{2} dx = \int \sec^2 \frac{x}{2} - 1 dx \\ &= 2\operatorname{atan} \frac{x}{2} - x + C \end{aligned}$$

ALTERNATIVE BY THE LITTLE t IDENTITIES

$$\begin{aligned} \int \frac{1-\cos x}{1+\cos x} dx &= \int \frac{1 - \frac{1+t^2}{1+t^2}}{1 + \frac{1+t^2}{1+t^2}} \times \left(\frac{2}{1+t^2}\right) dt \quad \text{REMEMBER } \frac{dt}{dx} = \frac{2}{1+t^2} dx \\ &= \int \frac{(1+t^2) - (1-t^2)}{(1+t^2) + (1-t^2)} \times \left(\frac{2}{1+t^2}\right) dt \quad \text{REMEMBER } \frac{dx}{dt} = \frac{2}{1+t^2} dt \\ &= \int \frac{2t^2}{2} \times \frac{2}{1+t^2} dt \\ &= \int \frac{2(1+t^2)}{1+t^2} \times \frac{-2}{1+t^2} dt \\ &= \int 2 - \frac{2}{1+t^2} dt \end{aligned}$$

$$\begin{aligned} &= 2t - 2\operatorname{arctan} \frac{x}{2} + C \\ &= 2\operatorname{atan} \frac{x}{2} - 2\operatorname{arctan}(\operatorname{tan} \frac{x}{2}) + C \\ &= 2\operatorname{atan} \frac{x}{2} - x + C \\ &= 2\operatorname{atan} \frac{x}{2} - x + C \end{aligned}$$

424. $\int \frac{1}{5+4\cos x} dx = \frac{2}{3} \arctan \left[\frac{1}{3} \tan \left(\frac{1}{2}x \right) \right] + C$

$$\begin{aligned} & \int \frac{1}{5+4\cos x} dx = \dots \text{ BY THE SUBSTITUTION} \\ & = \int \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \left(\frac{2}{1+t^2} dt \right) \\ & = \int \frac{2}{5t^2+9} dt \\ & = \int \frac{2}{9(t^2+\frac{5}{9})} dt \\ & = 2 \times \frac{1}{3} \arctan \frac{t}{\sqrt{\frac{5}{9}}} + C \\ & = \frac{2}{3} \arctan \left(\frac{t}{\sqrt{\frac{5}{9}}} \right) + C \\ & = \frac{2}{3} \arctan \left(\frac{t}{\sqrt{\frac{5}{9}}} \right) + C \end{aligned}$$

$t = \tan \frac{x}{2}$
 $\frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{x}{2}$
 $dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$
 $\frac{dt}{dt} = \frac{1}{2} dt$
 $dt = \frac{2}{1+t^2} dt$
 $\sec x = \sqrt{1+t^2}$
 $\cos x = \frac{1}{\sqrt{1+t^2}}$
 $\cos x = \frac{1-t^2}{1+t^2}$

425. $\int \cosh^2 2x dx = \frac{1}{2}x + \frac{1}{2} \sinh 4x + C$

$$\begin{aligned} \int \cosh^2 2x dx &= \int \frac{1}{2} + \frac{1}{2} \cosh 4x dx \\ &= \frac{1}{2}x + \frac{1}{2} \sinh 4x + C \end{aligned}$$

$\cosh^2 2x = \frac{1}{2} + \frac{1}{2} \cosh 4x$
 $\cosh 4x = \frac{1}{2} + \frac{1}{2} \cosh 2x$

426. $\int \frac{1}{\sqrt{3+2x-x^2}} dx = \arcsin \left(\frac{x-1}{2} \right) + C$

$$\begin{aligned} \int \frac{1}{\sqrt{3+2x-x^2}} dx &= \int \frac{1}{\sqrt{4-(x-1)^2}} dx \\ &= \dots \text{ INVERSE SINE RULE} \\ &= \arcsin \left(\frac{x-1}{2} \right) + C \end{aligned}$$

$3+2x-x^2 = -[x^2-2x-3]$
 $= -[(x-1)^2-1]$
 $= 4-(x-1)^2$

427. $\int \frac{1}{1+\sin 2x} dx = \left[\frac{1}{2} \tan 2x - \frac{1}{2} \sec 2x + C \right]$

$$\begin{aligned} \int \frac{1}{1+\sin 2x} dx & \dots \text{BY MANIPULATION, SIMPLIFY} = \int \frac{1(1-\sin 2x)}{(1+\sin 2x)(1-\sin 2x)} dx \\ &= \int \frac{1-\sin 2x}{1-\sin^2 2x} dx = \int \frac{1-\sin 2x}{\cos^2 2x} dx = \int \frac{1}{\cos^2 2x} - \frac{\sin 2x}{\cos^2 2x} dx \\ &= \int \sec^2 2x - \frac{\sin 2x}{\cos^2 2x} dx = \int \sec^2 2x - \sec 2x \tan 2x dx \\ &= \frac{1}{2} \tan 2x - \frac{1}{2} \sec 2x + C \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION, SIMILAR TO THAT IN LITTLE C INTEGRALS

$$\begin{aligned} \int \frac{1}{1+\sin 2x} dx &= \int \frac{1}{1+2\sin x \cos x} dx \\ &= \int \frac{1}{\frac{1}{\cos^2 x} + \frac{2\sin x}{\cos x}} dx = \int \frac{\sec^2 x}{\sec^2 x + 2\tan x} dx \\ &= \int \frac{1 + \tan^2 x}{1 + \tan^2 x + 2\tan x} dx \\ &= \int \frac{1 + \cancel{\tan^2 x}}{1 + \cancel{\tan^2 x} + 2\tan x} \left(\frac{1}{1 + \cancel{\tan^2 x}} dt \right) \\ &= \int \frac{1}{1 + 2\tan x} dt = \int \frac{1}{(1+t)^2} dt \\ &= -\frac{1}{1+t} + C \Rightarrow -\frac{1}{1+\tan x} + C \end{aligned}$$

428. $\int \frac{588}{4x^2 + 49} dx = 42 \arctan\left(\frac{2}{7}x\right) + C$

$$\begin{aligned} \int \frac{\sin}{4x^2 + 49} dx &= \int \frac{147}{x^2 + \frac{49}{4}} dx = \int \frac{147}{x^2 + (\frac{7}{2})^2} dx \\ &\dots \text{BY RECOGNITION (REMEMBER 1/4 * 49 = 1/4 * 7^2)} \\ &= 147 \times \frac{1}{2} \times \arctan\left(\frac{x}{\frac{7}{2}}\right) + C = 42 \arctan\left(\frac{2}{7}x\right) + C \end{aligned}$$

429. $\int \frac{x+1}{x^2 + 9} dx = \frac{1}{2} \ln(x^2 + 9) + \frac{1}{3} \arctan\left(\frac{1}{3}x\right) + C$

$$\begin{aligned} \int \frac{x+1}{x^2 + 9} dx &= \int \frac{\frac{2}{3}x + \frac{1}{3}}{x^2 + 9} dx \\ &\dots \text{LET } u = x^2 + 9 \text{ AND } du = 2x dx \\ &= \frac{1}{3} \ln(x^2 + 9) + \frac{1}{3} \arctan\left(\frac{x}{3}\right) + C \end{aligned}$$

430. $\int \sqrt{e^x - 1} \, dx = 2\sqrt{e^x - 1} - 2\arctan(\sqrt{e^x - 1}) + C$

$$\begin{aligned} \int \sqrt{e^x - 1} \, dx &= \dots \text{ BY SUBSTITUTION } \\ &= \int u \left(\frac{2u}{u^2+1} \, du \right) = \int \frac{2u^2}{u^2+1} \, du \\ &= \int \frac{2(u^2+1)-2}{u^2+1} \, du = \int 2 - \frac{2}{u^2+1} \, du \\ &= 2u - 2\arctan u + C \\ &= 2\sqrt{e^x - 1} - 2\arctan(\sqrt{e^x - 1}) + C \end{aligned}$$

$u = \sqrt{e^x - 1}$
 $u^2 = e^x - 1$
 $2u \, du = e^x \, dx$
 $du = \frac{e^x}{2u} \, dx$
 $dx = \frac{2u}{e^x} \, du$

431. $\int \frac{\sqrt{e^x}}{\sqrt{e^x + e^{-x}}} \, dx = \operatorname{arsinh}(e^x) + C$

$$\begin{aligned} \int \frac{\sqrt{e^x}}{\sqrt{e^x + e^{-x}}} \, dx &= \dots \text{ BY SUBSTITUTION } \\ &= \int \frac{u^{\frac{1}{2}}}{\sqrt{u + \frac{1}{u}}} \left(\frac{1}{u} \, du \right) = \int \frac{u^{\frac{1}{2}}}{(u + \frac{1}{u})^{\frac{1}{2}}} \, du \\ &= \int \frac{1}{u^{\frac{1}{2}}} \times \frac{1}{(u + \frac{1}{u})^{\frac{1}{2}}} \, du = \int \frac{1}{\sqrt{u(u + \frac{1}{u})}} \, du \\ &= \int \frac{1}{\sqrt{u^2 + 1}} \, du = \dots \text{ STANDARD RESULT } \\ &= \operatorname{arsinh} u + C = \operatorname{arsinh}(e^x) + C \end{aligned}$$

$u = e^x$
 $\frac{du}{dx} = e^x$
 $\frac{du}{dx} = u$
 $dx = \frac{1}{u} \, du$

432. $\int \sqrt{1-e^{2x}} \, dx = \sqrt{1-e^{2x}} + \frac{1}{2} \ln \left| \frac{\sqrt{1-e^{2x}}-1}{\sqrt{1-e^{2x}}+1} \right| + C$

$$\begin{aligned} \int \sqrt{1-e^{2x}} \, dx &= \dots \text{ BY SUBSTITUTION } \\ &= \int u \left(\frac{u}{u^2-1} \, du \right) = \int \frac{u^2}{u^2-1} \, du \\ &= \int \frac{(u^2-1)+1}{u^2-1} \, du = \int 1 + \frac{1}{u^2-1} \, du \\ &\quad \dots \text{ BY PARTIAL FRACTION } \\ &= \int 1 + \frac{1}{(u-1)(u+1)} \, du = \int 1 + \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1} \, du = u + \frac{1}{2} \ln|u-1| - \frac{1}{2} \ln|u+1| + C \\ &= u + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C = \frac{1}{2} \sqrt{1-e^{2x}} + \frac{1}{2} \left[\ln \left| \frac{\sqrt{1-e^{2x}}-1}{\sqrt{1-e^{2x}}+1} \right| \right] + C \end{aligned}$$

$u = \sqrt{1-e^{2x}}$
 $u^2 = 1-e^{2x}$
 $2u \, du = -2e^{2x} \, dx$
 $du = -\frac{e^{2x}}{u} \, dx$
 $dx = -\frac{u}{e^{2x}} \, du$

433. $\int \frac{16}{(x^2+4)^{\frac{5}{2}}} dx = \frac{x}{(x^2+4)^{\frac{1}{2}}} - \frac{x^3}{3(x^2+4)^{\frac{3}{2}}} + C$

$$\begin{aligned} \int \frac{16}{(x^2+4)^{\frac{5}{2}}} dx &= \dots \text{ BY TRIGONOMETRIC SUBSTITUTION } \\ &= \int \frac{16}{(4\sin^2\theta)^{\frac{5}{2}}} (2\cos\theta d\theta) = \int \frac{16\cos\theta}{32\sin^5\theta} d\theta \\ &= \int \frac{2\cos\theta}{\sin^5\theta} d\theta = \int \frac{2\cos\theta}{\sin\theta(\sin^4\theta)} d\theta = \int \frac{\cos\theta}{\sin\theta(1-\cos^2\theta)^2} d\theta \\ &= \int \frac{\cos\theta}{\sin\theta - \cos^2\theta} d\theta = \int \frac{\cos\theta}{\sin\theta(1-\cos^2\theta)} d\theta \\ &= \int \cos\theta - \cos^2\theta d\theta = \sin\theta - \frac{1}{3}\cos^3\theta + C \\ &\Rightarrow \frac{2}{(x^2+4)^{\frac{5}{2}}} - \frac{1}{3} \left(\frac{x}{(x^2+4)^{\frac{1}{2}}} \right)^3 + C = \frac{2}{(x^2+4)^{\frac{5}{2}}} - \frac{x^3}{3(x^2+4)^{\frac{3}{2}}} + C \end{aligned}$$

THE HYPERBOLIC SUBSTITUTION $x = 2\sinh\theta$ ALSO WORKS WELL.

434. $\int \sqrt{(1-x^2)^3} dx = \frac{3}{8} \arcsin x + \frac{1}{8} x(5-2x^2)(1-x^2)^{\frac{1}{2}} + C$

$$\begin{aligned} \int \sqrt{(1-x^2)^3} dx &= \dots \text{ BY HYPERBOLIC SUBSTITUTION } \\ &= \int (1-\sin^2\theta)^{\frac{3}{2}} \cos\theta d\theta = \int (\cos^2\theta)^{\frac{3}{2}} \cos\theta d\theta \\ &= \int \cos^4\theta d\theta = \int (\cos^2\theta)^2 d\theta = \int (\frac{1}{2} + \frac{1}{2}\cos 2\theta)^2 d\theta \\ &= \int \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta d\theta \\ &= \int \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4} + \frac{1}{8}\cos 4\theta d\theta = \int \frac{3}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta d\theta \\ &= \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + C = \frac{3}{8}\theta + \frac{1}{4}\sinh 2\theta + \frac{1}{32}\sinh 4\theta + C \\ &= \frac{3}{8}\theta + \frac{1}{4}\sinh 2\theta + \frac{1}{8}\sinh 4\theta + \frac{1}{16}\sinh 8\theta + C \\ &= \frac{3}{8}\arcsin x + \frac{3}{8}x\sqrt{1-x^2} + \frac{1}{8}x(1-x^2)^{\frac{1}{2}}(1-2x^2) + C \\ &= \frac{3}{8}\arcsin x + \frac{1}{8}x\sqrt{1-x^2} \left[4 + (1-x^2)^2 \right] + C = \frac{3}{8}\arcsin x + \frac{1}{8}x(5-2x^2)(1-x^2)^{\frac{1}{2}} + C \end{aligned}$$

435. $\int \cosh x \arctan(\sinh x) dx = \sin x \arctan(\sinh x) - \ln(\cosh x) + C$

$$\begin{aligned} \int \cosh x \arctan(\sinh x) dx &= \dots \text{ INTEGRATION BY PARTS } \\ &= \sinh x \arctan(\sinh x) - \int \frac{\cosh x \arctan(\sinh x)}{1+\sinh^2 x} dx \\ &= \sinh x \arctan(\sinh x) - \int \frac{\cosh x \arctan(\sinh x)}{\cosh^2 x} dx \\ &= \sinh x \arctan(\sinh x) - \int \frac{\sinh x}{\cosh x} dx \\ &= \sinh x \arctan(\sinh x) - [\ln|\cosh x|] + C \end{aligned}$$

$$436. \int \sec x \, dx = \begin{cases} \ln|\sec x + \tan x| + C \\ \ln\left|\tan\left(\frac{1}{4}\pi - \frac{1}{2}x\right)\right| + C \\ \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C \\ \ln\left|\frac{1+\tan\left(\frac{1}{2}x\right)}{1-\tan\left(\frac{1}{2}x\right)}\right| + C \end{cases}$$

$\int \sec x \, dx = \int \frac{\sec x}{\sec x + \tan x} \, dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} \, dx$

This is of the form $\int \frac{f'(u)}{f(u)} \, du = \ln|f(u)| + C$

$= \ln|\sec x + \tan x| + C$

$\int \sec x \, dx = \dots$ ALTERNATIVE BY THE LITTLE t INSIDE (INTEGRATION BY SUBSTITUTION)

$= \int \frac{(1+t^2)}{(1-t^2)} \left(\frac{2}{1+t^2} dt \right) = \int \frac{2}{1-t^2} dt$

$= \int \frac{2}{(1-t)(1+t)} dt$... BY PARTIAL FRACTIONS ...

$= \int \frac{1}{1+t} dt + \frac{1}{1-t} dt$

$= \ln|1+t| - \ln|1-t| + C$

$= \ln\left|\frac{1+t}{1-t}\right| + C = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$

OR TRYING UP FURTHER NOTING THAT $\tan^2\frac{x}{2} = 1$

$= \ln\left|\frac{\tan\frac{x}{2} + 1}{\tan\frac{x}{2} - 1}\right| + C$

$= \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$

ANOTHER ALTERNATIVE

$\int \sec x \, dx = \int \frac{1}{\sec x} \, dx = \int \frac{\sec x}{\sec^2 x} \, dx$

$= \int \frac{\sec x}{1-\tan^2 x} \, dx$

BY SUBSTITUTION

LET $u = \sin x$
 $\frac{du}{dx} = \cos x$
 $du = \cos x \, dx$

$= \int \frac{\sec x}{1-u^2} \left(\frac{du}{\cos x} \right) = \int \frac{1}{(1-u^2)(1+u)} \, du$

PARTIAL FRACTION BY INSPECTION

$= \int \frac{\frac{1}{2u}}{1-u} + \frac{\frac{1}{2u}}{1+u} \, du = \frac{1}{2}\ln|1/u| - \frac{1}{2}\ln|1+u| + C$

$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2}\ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$

$= \frac{1}{2}\ln\left|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right|^2 + C = \ln\left|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right| + C$

$= \ln\left|\frac{\frac{\sqrt{1+u^2}}{1-u}}{\frac{\sqrt{1-u^2}}{1+u}}\right| + C = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$

$$437. \int \frac{1}{\sec x - 1} \, dx = \begin{bmatrix} -x - \cot\left(\frac{1}{2}x\right) + C \\ -x - \cot x - \operatorname{cosec} x + C \end{bmatrix}$$

$\int \frac{1}{\sec x - 1} \, dx = \dots$ BY THE LITTLE t INSIDE ...

NOTICE THAT 4 TERMS OF THE FOUR FRACTION BY 1-1-1-1

$= \int \frac{1}{\frac{1+t^2}{1-t^2} - 1} \left(\frac{2}{1+t^2} dt \right)$

$= \int \frac{1-t^2}{(1+t^2)-(1-t^2)} \times \frac{2}{1+t^2} dt$

$= \int \frac{2(1-t^2)}{2t^2(1+t^2)} dt = \int \frac{1-t^2}{t^2(1+t^2)} dt$... SWAP PARTIAL FRACTIONS IN t², BY INSPECTION

$= \int \frac{1}{t^2} - \frac{2}{t^2+1} dt$

$= -\frac{1}{t} - \text{arctan}(t) + C$

$= -\frac{1}{\tan\frac{x}{2}} - 2\arctan(\tan\frac{x}{2}) + C$

$= -\cot\frac{x}{2} - 2\left(\frac{\pi}{2}\right) + C$

$= -x - \cot\frac{x}{2} + C$

ALTERNATIVE

$\int \frac{1}{\sec x - 1} \, dx = \int \frac{(\sec x + 1)}{(\sec x - 1)(\sec x + 1)} \, dx$

$= \int \frac{\sec x + 1}{\sec^2 x - 1} \, dx = \int \frac{\sec x + 1}{\tan^2 x} \, dx$

$= \int \frac{\sec x}{\tan x} + \frac{1}{\tan^2 x} \, dx$

$\int \frac{1}{\sec x} \times \frac{\sec x}{\sin x} + \cot^2 x \, dx = \int \frac{\sec x}{\sin x} + (\cosec x - 1) \, dx$

$= \int \frac{\sec x}{\sin x} + \frac{1}{\sin x} \, dx = \int \cosec x \sec x + \cosec^2 x - 1 \, dx$

$= -\cosec x - \cot x - x + C$

MULTIPLYING FURTHER

$= -x - (\cosec x + \cot x) + C$

$= -x - \left(\frac{1}{\sin x} + \frac{\cos x}{\sin x} \right) + C$

$= -x - \frac{1+\cos x}{\sin x} + C$

$= -x - \frac{1+(2\cos^2\frac{x}{2}-1)}{2\sin\frac{x}{2}\cos\frac{x}{2}} + C$

$= -x - \frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} + C$

$= -x - \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} + C$

$= -x - \cot\frac{x}{2} + C$

$\cosec^2 x = 2\cosec^2 x - 1$
 $\cosec x = 2\cosec x - 1$
 $\cosec x = 2\cosec x$
 $\cosec x = 2\cosec x$

438. $\int \frac{\sin x}{1-\tan x} dx = \frac{1}{2} \left[-\cos x - \sin x + \frac{1}{\sqrt{2}} \ln \left| \sec \left(x + \frac{1}{4}\pi \right) \right| + \tan \left(x + \frac{1}{4}\pi \right) \right] + C$

$$\begin{aligned} \int \frac{\sin x}{1-\tan x} dx &= \int \frac{\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}} dx = \int \frac{\sin x \cos x}{\cos x - \sin x} dx \\ &\stackrel{\text{multiply top/bottom by } \cos x}{=} -\frac{1}{2} \int \frac{-2\sin x \cos x}{\cos x - \sin x} dx \\ &= -\frac{1}{2} \int \frac{(1-2\sin x \cos x)-1}{\cos x - \sin x} dx \\ &= -\frac{1}{2} \int \frac{\cos x + \sin x - 2\sin x \cos x - 1}{\cos x - \sin x} dx \\ &= -\frac{1}{2} \int \frac{\cos x - \sin x}{\cos x - \sin x} - \frac{1}{\cos x - \sin x} dx \\ &= \frac{1}{2} \int \sin x - \cos x + \frac{1}{\cos x - \sin x} dx \\ &= \frac{1}{2} \int \sin x - \cos x + \frac{1}{\sqrt{2}(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x)} dx \\ &= \frac{1}{2} \int \sin x - \cos x + \frac{1}{\sqrt{2}\cos(x+\frac{\pi}{4})} dx \\ &= \frac{1}{2} \int \sin x - \cos x + \frac{1}{\sqrt{2}\sec(x+\frac{\pi}{4})} dx \\ &= \frac{1}{2} \int \sin x - \cos x + \frac{1}{\sqrt{2}} \sec(x+\frac{\pi}{4}) dx \\ &= \frac{1}{2} \left[-\sin x - \cos x + \frac{1}{\sqrt{2}} \ln \left| \sec(x+\frac{\pi}{4}) \right| + \tan(x+\frac{\pi}{4}) \right] + C \end{aligned}$$

439. $\int \frac{\cos x}{\cos^2 x + 4\sin x - 5} dx = \frac{1}{\sin x - 2} + C$

$$\begin{aligned} \int \frac{\cos x}{\cos x + \sin x - 5} dx &= \int \frac{\cos x}{(1-\sin x) + \sin x - 5} dx \\ &= \int \frac{\cos x}{-\sin x + 4\sin x - 4} dx = \int \frac{\cos x}{\sin x - 4\sin x + 4} dx = -\int \frac{\cos x}{(\sin x - 2)^2} dx \\ &\stackrel{\text{by direct chain rule (differentiation)}}{=} -\int (\sin x - 2)^{-2} \cos x dx \\ &= +(\sin x - 2)^{-1} + C = \frac{1}{\sin x - 2} + C \end{aligned}$$

440. $\int \frac{2\cos x}{\cos x + \sin x} dx = \begin{cases} x + \ln|\cos x + \sin x| + C \\ x + \frac{1}{2} \ln|1 + \sin 2x| + C \end{cases}$

$$\begin{aligned} \int \frac{2\cos x}{\cos x + \sin x} dx &= \int \frac{\cos x + \sin x - \sin x + \cos x}{\cos x + \sin x} dx \\ &= \int \frac{\cos x + \sin x}{\cos x + \sin x} dx + \int \frac{-\sin x + \cos x}{\cos x + \sin x} dx = \int 1 + \frac{-\sin x + \cos x}{\cos x + \sin x} dx \\ &\stackrel{\text{let } u = \int \frac{1}{\cos x + \sin x} dx = \ln|\cos x + \sin x| + C}{=} x + \ln|\cos x + \sin x| + C \\ \text{ALTERNATIVE:} \\ \int \frac{2\cos x}{\cos x + \sin x} dx &= \int \frac{2\cos x (\cos x - \sin x)}{(\cos x + \sin x)(\cos x - \sin x)} dx = \int \frac{2\cos^2 x - 2\cos x \sin x}{\cos x - \sin x} dx \\ &= \int \frac{(1+2\cos 2x) - \sin 2x}{\cos 2x} dx = \int \sec 2x + 1 - \frac{\sin 2x}{\cos 2x} dx \\ &\stackrel{\text{NOTING AS ALL THESE ARE SIMPLY EIGTS}}{=} -\frac{1}{2} \ln|\sec 2x + \tan 2x| + x - \frac{1}{2} \ln|\sec 2x| + C = x + \frac{1}{2} \ln \left| \frac{\sec 2x + \tan 2x}{\sec 2x} \right| + C \\ &= x + \frac{1}{2} \ln \left| 1 + \frac{\tan 2x}{\sec 2x} \right| + C = x + \frac{1}{2} \ln |\sin 2x| + C \\ \text{CONTINUING THIS THE THIRD APPROX:} \\ &= x + \ln \left| 1 + \sin 2x \right|^{\frac{1}{2}} + C = x + \ln \left| \sqrt{\cos^2 x + \sin^2 x + 2\cos x \sin x} \right|^{\frac{1}{2}} + C \\ &= x + \ln \left| \cos x + \sin x \right|^{\frac{1}{2}} + C = x + \ln |\cos x + \sin x| + C \end{aligned}$$

441.
$$\int \frac{18}{(\cos^2 x + 4\sin x - 5)\cos x} dx = \dots$$

$$= \dots - \frac{6}{2 - \sin x} - 8 \ln(2 - \sin x) + 9 \ln|1 - \sin x| - \ln|1 - \sin x| + C$$

WORKED SOLUTION

PROBLEMS BY SUBSTITUTION

$$\begin{aligned} u &= \cos x - 2, \quad du = -\sin x dx, \quad u+2 \\ \frac{du}{dx} &= -\sin x, \quad du = -\sin x dx \end{aligned}$$

$$\begin{aligned} \int \frac{18}{(\cos^2 x + 4\sin x - 5)\cos x} du &= \int \frac{18}{(-\sin^2 x + 4\sin x - 5)\cos x} du \\ &= \int \frac{18}{(-\sin^2 x - 4\sin x + 4)\cos x} du = \int \frac{18}{(-\sin^2 x - 4\sin x + 3)} du \\ &= \int \frac{18}{(-\sin^2 x - 4\sin x + 3)} du = \int \frac{-18}{\sin^2 x + 4\sin x - 3} du \\ &= \int \frac{-18}{u^2 + 4u - 3} du = \int \frac{-18}{u^2 (u^2 + 4u + 3)} du = \int \frac{-18}{u^2 (u+1)(u+3)} du \\ &= \int \frac{-18}{u(u+1)(u+3)} du \end{aligned}$$

BY PARTIAL FRACTION

$$\frac{18}{u(u+1)(u+3)} = \frac{A}{u} + \frac{B}{u+1} + \frac{C}{u+3} + \frac{D}{u+3}$$

$$\boxed{18 = A(u+1)(u+3) + B(u)(u+3) + Cu^2(u+3) + Du^2(u+1)}$$

- If $u=0$, $18 = 3B \Rightarrow B=6$
- If $u=-1$, $18 = -2C \Rightarrow C=-9$
- If $u=-3$, $18 = 6A \Rightarrow A=3$
- If $u=1$, $18 = 8B + 4C + 2D \Rightarrow D=-2$

$$\begin{aligned} &\int \frac{6}{u} du + \frac{18}{u+1} du + \frac{9}{u+3} du = -\frac{6}{u} \ln|u| + 9 \ln|u+1| - \ln|u+3| + C \\ &= -\frac{6}{\sin x - 2} \ln|\sin x - 2| + 9 \ln|\sin x - 1| - \ln|\sin x + 1| + C \\ &= \frac{6}{2 - \sin x} - 8 \ln(2 - \sin x) + 9 \ln|1 - \sin x| - \ln|1 + \sin x| + C \end{aligned}$$

442.
$$\int \frac{\tan x}{1 + \sin^2 x} dx = \frac{1}{4} \ln \left| \frac{1 + \sin^2 x}{\cos^2 x} \right| + C$$

WORKED SOLUTION

PROBLEMS BY SUBSTITUTION

$$\begin{aligned} \int \frac{\tan x}{1 + \sin^2 x} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int \frac{\frac{\sin x}{\cos x}}{1 + \sin^2 x} \cdot \frac{1}{-\sin x} dx \\ &= \int \frac{1}{\cos x} + \frac{\sin x}{2\sin x \cos x} dx \end{aligned}$$

BY PARTIAL FRACTION

$$\begin{aligned} &\int \frac{1}{\cos x} dx + \int \frac{\sin x}{2\sin x \cos x} dx = \int \frac{1}{\cos x} dx + \int \frac{1}{2\sin x} dx \\ &= \int \frac{1}{\cos x} \left(\frac{1}{1 - \sin^2 x} \right) dx = \int \frac{1}{2\sin x} dx = \frac{1}{2} \int \frac{1}{\sin x} dx \\ &= \frac{1}{2} \int \frac{1}{u} du + \frac{1}{2} \int \frac{1}{2u} du = \frac{1}{2} \int \frac{1}{u} du + \frac{1}{4} \int \frac{1}{u} du \\ &= \frac{1}{2} \ln|u| + \frac{1}{4} \ln|u| = \frac{1}{4} \ln \left| \frac{4u}{2 + (1 + \sin^2 x)} \right| + C = \frac{1}{4} \ln \left| \frac{1 + \sin^2 x}{2 + \sin^2 x} \right| + C \\ &= \frac{1}{4} \ln \left| \frac{1 + \sin^2 x}{\cos^2 x} \right| + C \end{aligned}$$

443.
$$\int (x+1)e^{x+1} dx = xe^{x+1} + C$$

WORKED SOLUTION

STANDARD INTEGRATION BY PARTS

$$\begin{aligned} \int (x+1)e^{x+1} dx &= \dots \text{ STANDARD INTEGRATION BY PARTS} \\ &= (x+1)e^{x+1} - \int e^{x+1} dx \\ &= (x+1)e^{x+1} - e^{x+1} + C \\ &= xe^{x+1} + C \end{aligned}$$

$$444. \quad \int \tan x \sec^4 x \, dx = \frac{1}{4} \sec^4 x + C$$

$$\begin{aligned} \int \ln(\sec x) dx &= \int (\ln(\sec x)) \sec x \tan x dx = \dots \text{ BY RECOGNITION SINCE} \\ &\quad \frac{d}{dx}(\sec x) = \sec x \tan x \\ &= \frac{1}{2} \sec^2 x + C \end{aligned}$$

THE SUBSTITUTION $u = \sec x$ ALSO WORKS WELL

$$445. \int \frac{4}{(1-\sin x)\cos x} dx = \ln \left| \frac{1-\sin x}{1+\sin x} \right| + \frac{1}{1-\sin x} + C$$

THE SUBSTITUTION $u = \sin x$. ALSO WORKS WELL.

$\int \frac{4}{\cos^2(1-\sin x)} dx = \dots$ BY SUBSTITUTION ...

$= \int \frac{4}{\cos u \cdot \cos u} \times \frac{du}{-\sin u}$

$= \int \frac{-4}{u \cos^2 u} du = \int \frac{-4}{u(1-\sin^2 u)} du$

$= \int \frac{-4}{u(1-\cos^2 u) + \sin u} du = \int \frac{-4}{u^2(2-u) + \sin u} du = \int \frac{4}{u^2(2-u)} du$

BY PARTIAL FRACTIONS:

$$\frac{4}{u^2(2-u)} \Rightarrow \frac{A}{u} + \frac{B}{u^2} + \frac{C}{2-u}$$

$$4(u^2 - 2u) \Leftrightarrow C(u-2) \equiv 4$$

- * If U < 0 * If U > 2 * If $4 \neq 1$
- $-2u(u-2)$
- $\frac{4u^2}{u^2-2}$
- $\frac{4u^2}{u^2-2} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{2-u}$

$= \int \frac{1}{u} - \frac{2}{u-2} - \frac{1}{u^2} du = \ln|u| + \frac{1}{u} - \ln|u-2| + C$

$= \ln|\frac{u}{u-2}| + \frac{1}{u} + C = \ln|\frac{\sin x}{1-\sin x}| + \frac{1}{\sin x} + C$

$= \ln\left|\frac{1-\sin x}{1+\sin x}\right| + \frac{1}{1-\sin x} + C$

$$446. \int \frac{x+2}{\sqrt{1-4x^2}} dx = \arcsin(2x) - \frac{1}{4}\sqrt{1-4x^2} + C$$

$$\int \frac{x+2}{\sqrt{1-x^2}} dx = \int \frac{\frac{1}{2}}{\sqrt{1-(x^2)^2}} + \frac{\frac{2}{x}}{\sqrt{1-x^2}} dx = \int \frac{1}{2}(1-x^2)^{-\frac{1}{2}} + 2x \cdot \frac{1}{\sqrt{1-x^2}} dx$$

↓ Substitution ↑ Integration by parts
↓ U-substitution ↑ Integration by parts

$$= -\frac{1}{4}(1-x^2)^{\frac{1}{2}} + \arcsin(x) + C$$

$$= \arcsin(2x) - \frac{1}{4}\sqrt{1-4x^2} + C$$

The substitution $2x = \sin(\theta)$ from the beginning also works.

447. $\int \frac{4x+1}{\sqrt{4x^2-9}} dx = \sqrt{4x^2-9} + \frac{1}{2} \operatorname{arcosh}\left(\frac{2x}{3}\right) + C$

$$\begin{aligned} \int \frac{4x+1}{\sqrt{4x^2-9}} dx &= \int \frac{4x}{\sqrt{4x^2-9}} + \frac{1}{\sqrt{4x^2-9}} dx = \int \ln(4x^2-9)^{\frac{1}{2}} + \frac{1}{2\sqrt{4x^2-9}} dx \\ &= \left(\ln(4x^2-9)\right)^{\frac{1}{2}} + \frac{1}{2} \operatorname{arcosh}\left(\frac{2x}{3}\right) + C = \sqrt{4x^2-9} + \frac{1}{2} \operatorname{arcosh}\left(\frac{2x}{3}\right) + C \\ \text{THE SUBSTITUTION } u &= \frac{2x}{3} \operatorname{arcosh}, \text{ FROM THE BEGINNING ALSO } \cos(u) \end{aligned}$$

448. $\int 8x \ln(2x+1) dx = (4x^2 - 1) \ln(2x+1) + 4x - 2x^2 + C$

$$\begin{aligned} \int 8x \ln(2x+1) dx &= \dots \text{INTEGRATION BY PARTS...} \\ &= 4x^2 \ln(2x+1) - \int \frac{8x^2}{2x+1} dx \dots \text{MANIPULATION} \\ &= 4x^2 \ln(2x+1) - \int \frac{4x(2x+1) - 2(2x+1) + 2}{2x+1} dx \\ &= 4x^2 \ln(2x+1) - \int 4x - 2 + \frac{2}{2x+1} dx \\ &= 4x^2 \ln(2x+1) - [2x^2 - 2x + \ln(2x+1)] + C \\ &= 4x^2 \ln(2x+1) - 2x^2 + 2x - \ln(2x+1) + C \\ &= (4x^2 - 1) \ln(2x+1) + 2x - 2x^2 + C \end{aligned}$$

449. $\int \cos 3x \sin 6x dx = \begin{cases} -\frac{2}{9} \cos^2 3x + C \\ -\frac{1}{18} [\cos 9x + 3 \cos 3x] + C \\ -\frac{1}{9} [2 \cos 3x \cos 6x + \sin 3x \sin 6x] + C \end{cases}$

$$\begin{aligned} \int \cos 3x \sin 6x dx &= \int \cos 3x (2 \sin 3x \cos 3x) dx = \int 2 \sin 3x \cos^2 3x dx \\ &= -\frac{2}{3} \cos^3 3x + C \\ \text{ALTERNATIVE BY TRIG IDENTITIES} \\ \sin(3x-3x) &= \sin 3x \cos 3x + \cos 3x \sin 3x \quad \text{ADDITION} \\ \sin(3x-3x) &= \sin 3x \cos 3x - \cos 3x \sin 3x \\ \sin 3x \cos 3x &= 2 \sin 3x \cos 3x \\ \sin 3x \cos 3x &= \frac{1}{2} \sin 6x + \frac{1}{2} \cos 6x \\ \dots &= \int \frac{1}{2} \sin 6x + \frac{1}{2} \cos 6x dx = -\frac{1}{12} \cos 6x - \frac{1}{6} \sin 6x + C \\ \text{ALTERNATIVE BY DOUBLE INTEGRATION BY PARTS} \\ \int \cos 3x \sin 6x dx &= -\frac{1}{6} \cos 6x - \int \frac{1}{6} \sin 6x \cos 3x dx \\ &\quad \downarrow \text{BY PARTS AGAIN!} \\ \int \cos 3x \sin 6x dx &= -\frac{1}{6} \cos 6x - \frac{1}{6} \sin 6x + \frac{1}{6} \int \sin 6x \cos 3x dx \\ 4 \int \cos 3x \sin 6x dx &= -\frac{1}{6} \cos 6x - \frac{1}{6} \sin 6x + \frac{1}{6} \sin 6x + \frac{1}{6} \cos 6x \\ 3 \int \cos 3x \sin 6x dx &= -\frac{1}{6} [\cos 6x + \sin 6x] + C \\ \int \cos 3x \sin 6x dx &= -\frac{1}{18} [\cos 6x + \sin 6x] + C \end{aligned}$$

450. $\int \frac{4}{3+5\cos x} dx = \ln \left| \frac{2+\tan(\frac{1}{2}x)}{2-\tan(\frac{1}{2}x)} \right| + C$

$$\begin{aligned}
 \int \frac{4}{3+5\cos x} dx &= \dots \text{BY THE 'LITTLE t' EXCERPTS} \\
 &= \int \frac{4}{\frac{8-2t^2}{1+t^2}} \sim \frac{2}{1+t^2} dt \\
 &= \int \frac{8}{8-2t^2} dt = \int \frac{4}{4-t^2} dt \\
 &= \int \frac{4}{(2-t)(1+t)} dt \\
 &\quad \text{BY PARTIAL FRACTIONS} \\
 &= \int \frac{1}{2-t} + \frac{1}{2+t} dt \\
 &= -\ln|2-t| + \ln|2+t| + C \\
 &= \ln \left| \frac{2+t}{2-t} \right| + C \\
 &= \ln \left| \frac{2+\tan(\frac{1}{2}x)}{2-\tan(\frac{1}{2}x)} \right| + C
 \end{aligned}$$

451. $\int \arctan x \ dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C$

$$\begin{aligned}
 \int \arctan x \ dx &= \int 1 \times \arctan x \ dx \dots \text{INTEGRATION BY PARTS...} \\
 &= \text{arctan}x \cdot \int \frac{1}{1+x^2} dx - \int \frac{\text{arctan}x}{1+x^2} dx \\
 &= 2\text{arctan}x - \frac{1}{2} \int \frac{2x}{1+x^2} dx \quad \leftarrow \int \frac{f(x)}{g(x)} dx = \ln|f(x)| + C \\
 &= \boxed{\text{arctan}x - \frac{1}{2}\ln(1+x^2) + C}
 \end{aligned}$$

452. $\int \frac{4\cot x}{1+\cos^2 x} dx = \begin{cases} \ln \left| \frac{1-\cos^2 x}{1+\cos^2 x} \right| + C \\ -\ln |\cosec^2 x + \cot^2 x| + C \end{cases}$

$$\begin{aligned}
 \int \frac{4\cot x}{1+\cos^2 x} dx &= \dots \text{BY SUBSTITUTION...} \dots = \int \frac{4\cot x}{1+\cos^2 x} dx \\
 &= \int \frac{\frac{4\cot x}{u}}{u} \frac{du}{-2\sin^2 u} = \int \frac{4\cot x}{2u \sin^2 u} du \\
 &= \int -\frac{2}{u \sin^2 u} du = \int \frac{2}{u(1-u^2)} du \\
 &= \int \frac{-2}{u(u-1)} du = \int \frac{2}{u(u-1)} du \\
 &\quad \text{PARTIAL FRACTIONS BY INSPECTION} \\
 &= \int \frac{1}{u-1} - \frac{1}{u} du = \ln|u-1| - \ln|u| + C = \ln \left| \frac{u-1}{u} \right| + C \\
 &= \ln \left| \frac{1+\cos^2 x}{1+\cos^2 x - 2\cos^2 x} \right| + C = \ln \left| \frac{1+\cos^2 x}{\sin^2 x} \right| + C \\
 &= \boxed{\ln \left| \frac{\cosec^2 x + \cot^2 x}{1+\cos^2 x} \right| + C = -\ln \left| \frac{1+\cos^2 x}{\cosec^2 x + \cot^2 x} \right| + C = -\ln |\cosec^2 x + \cot^2 x| + C}
 \end{aligned}$$

453. $\int \frac{18}{(x^2+9)^2} dx = \frac{1}{3} \arctan\left(\frac{1}{3}x\right) + \frac{x}{x^2+9} + C$

$$\begin{aligned}
 \int \frac{18}{(x^2+1)^2} dx &= \dots \text{USING A TRIGONOMETRIC SUBSTITUTION} \\
 &= \int \frac{18}{(x^2+1)^2} dx = \int \frac{18}{\tan^2 u + 1} \cdot \frac{du}{x^2+1} = \int \frac{18 \sec^2 u}{\tan^2 u + 1} du \\
 &= \int \frac{18 \sec^2 u}{\tan^2 u + 1} du = \int \frac{18}{(\sec u - \tan u)(\sec u + \tan u)} du = \int \frac{18}{\sec u} du \\
 &= \int 18 \sec u du = \frac{1}{2} \int (2 + \sec 2u) du = \frac{1}{2} \int (2 + 2 \cos 2u) du \\
 &= \frac{1}{2} \left(2u + \frac{1}{2} \sin 2u \right) + C = \frac{1}{2} \left(2u + \frac{1}{2} \sin 2u \right) + C \\
 &= \frac{1}{2} u \sin 2u + \frac{1}{4} \sin 2u + C \\
 &= \frac{1}{2} \arctan x \cdot \frac{2}{x^2+1} + \frac{1}{4} \cdot \frac{2}{x^2+1} + C
 \end{aligned}$$

454. $\int e^x \cosh x \, dx = \frac{1}{4}e^{2x} + \frac{1}{2}x + C$

$$\begin{aligned}\int e^x \ln x \, dx &= \int e^x \left(\frac{1}{2}x^2 + \frac{1}{2}e^2 \right) dx = \int \frac{1}{2}e^{2x} + \frac{1}{2}xe^x \, dx \\ &= \frac{1}{2}e^{2x} + \frac{1}{2}xe^x + C.\end{aligned}$$

$$455. \int \frac{(x+1)^{\frac{3}{2}}}{x^2 - x} dx = 2\sqrt{x+1} - \ln \left| \frac{\sqrt{x+1} + 1}{\sqrt{x+1} - 1} \right| + 2\sqrt{2} \operatorname{n} \left| \frac{\sqrt{x+1} - \sqrt{2}}{\sqrt{x+1} + \sqrt{2}} \right|$$

BY SUBSTITUTION, FRACTION BY PARTIAL FRACTIONS

$$\int \frac{(x+1)^{\frac{1}{2}}}{x^2-1} dx = \dots$$

$$> \int \frac{u^{\frac{1}{2}}}{(u-1)(u+1)} (2u du) \quad (u=2x+1)$$

$$= \int \frac{2u^{\frac{1}{2}}}{u(u^2-1)} du \quad \text{INTRODUCE } u^2 \text{ IN DENOMINATOR}$$

$$= \int \frac{2(u^2-1)+4}{u(u^2-1)} du = \int 2 + \frac{4}{u^2-1} du$$

$$\frac{u^2-1}{u^2+u-2} = \frac{u^2+u+u+1-u-1}{u^2+u-2} = \frac{u(u+1)+(u+1)-1}{u(u+1)-(u+1)+2} = \frac{(u+1)(u+1)+(u+1)-1}{(u+1)(u+1)-(u+1)+2} = \frac{(u+1)(u+2)}{(u+1)(u+2)-1} = \frac{(u+1)(u+2)}{(u+1)(u+2)-1} = \frac{(u+1)(u+2)}{(u+1)(u+2)-1} = \frac{(u+1)(u+2)}{(u+1)(u+2)-1}$$

$$\boxed{\frac{u^2-1}{u^2+u-2} = \frac{u^2+u+u+1-u-1}{u^2+u-2} = \frac{u(u+1)+(u+1)-1}{u(u+1)-(u+1)+2} = \frac{(u+1)(u+1)+(u+1)-1}{(u+1)(u+1)-(u+1)+2} = \frac{(u+1)(u+2)}{(u+1)(u+2)-1}}$$

• IF $u=1$ • IF $u=-1$ • IF $u=\sqrt{2}$ • IF $u=-\sqrt{2}$

$$2 = -2k \quad B = 2k \quad B = \frac{8}{\sqrt{2}} = 8\sqrt{2} = -8\sqrt{2} = 40$$

$$A = -1 \quad B = 1 \quad A = \frac{4}{\sqrt{2}} = 2\sqrt{2} \quad C = 2C \quad D = -2C$$

$$= \int 2 - \frac{1}{u-1} + \frac{1}{u+1} + \frac{\frac{2\sqrt{2}}{u-1}C}{u-\sqrt{2}} - \frac{\frac{2\sqrt{2}}{u+1}D}{u+\sqrt{2}} du$$

$$= 2u - \ln|u-1| + \ln|u+1| + 2\sqrt{2} \ln|u-\sqrt{2}| - 2\sqrt{2} \ln|u+\sqrt{2}|$$

$$= 2u - \ln \left| \frac{u+1}{u-1} \right| + 2\sqrt{2} \ln \left| \frac{u-\sqrt{2}}{u+\sqrt{2}} \right| + C$$

$$= 2\cancel{\ln u^2} - \ln \left| \frac{\cancel{u^2}+1}{\cancel{u^2}-1} \right| + 2\sqrt{2} \ln \left| \frac{\cancel{u^2}-4}{\cancel{u^2}+4} \right| + C$$

456. $\int \frac{x^4}{(x^5 - 1)^{\frac{3}{2}}} dx = \frac{2}{5\sqrt{x^5 - 1}} + C$

$$\begin{aligned} \int \frac{u^4}{(u^5 - 1)^{\frac{3}{2}}} du &= \int u^4 (u^5 - 1)^{-\frac{3}{2}} du = \dots \text{BY (METHOD) ...} \\ &= -\frac{2}{5}(u^5 - 1)^{\frac{1}{2}} + C = -\frac{2}{5\sqrt{u^5 - 1}} + C \end{aligned}$$

THE SUBSTITUTION: $u = x^5 - 1$ or $u = \sqrt{x^5 - 1}$ ADD WORDS HERE.

457. $\int \frac{9}{e^x \sqrt{e^{2x} - 9}} dx = \frac{\sqrt{e^{2x} - 9}}{e^x} + C$

$$\begin{aligned} \int \frac{q}{e^x \sqrt{e^{2x} - 9}} dx &= \dots \text{BY SUBSTITUTION} \dots \\ &= \int \frac{q}{dx \sqrt{\frac{1}{e^{-x}} - 9}} \left(\frac{1}{e^x} dx\right) = \int \frac{-9}{\sqrt{1 - 9e^{-2x}}} du = \int -9(u^{-\frac{1}{2}})^2 du \\ &= \int \frac{-81}{u^2} du = \int -81(u^{-2}) du = \int -81(-u^{-1}) du \\ &= (-u^{-1})^2 + C = (-\frac{1}{u})^2 + C \\ &= (\frac{e^{2x} - 9}{e^{2x}})^{\frac{1}{2}} + C = \frac{\sqrt{e^{2x} - 9}}{e^x} + C \end{aligned}$$

ANOTHER APPROACH USING DOUBLE SUBSTITUTIONS

$$\begin{aligned} \int \frac{q}{e^x \sqrt{e^{2x} - 9}} dx &= \dots u = e^x = \int \frac{q}{\sqrt{e^{2x} - 9}} \left(\frac{1}{e^x} dx\right) \\ &= \int \frac{q}{\sqrt{e^{2x} - 9}} du \quad \boxed{u = e^x} \\ &\text{NOW A TRIGONOMETRIC SUBSTITUTION TO FOLLOW} \\ &\text{LET'S TRY} \\ &u = 3\sec\theta \\ &du = 3\sec\theta\tan\theta d\theta \\ &\tan\theta = \sqrt{e^{2x} - 9}/u \end{aligned}$$

ANOTHER APPROACH USING NUMERICAL SUBSTITUTIONS

$$\begin{aligned} \int \frac{q}{e^x \sqrt{e^{2x} - 9}} dx &= \int \frac{q}{\sqrt{3\sec\theta \sqrt{9\sec^2\theta - 1}}} du \\ &= \int \frac{q}{3\sec\theta \sqrt{9\sec^2\theta - 1}} du = \int \frac{q}{3\sec\theta \sqrt{9\sec^2\theta - 1}} d\theta \\ &= \int \frac{1}{\sec\theta} \times \frac{q}{\sqrt{9\sec^2\theta - 1}} d\theta = \int \frac{1}{\cos\theta} \times \frac{q}{\sqrt{9\cos^2\theta - 1}} d\theta \\ &= \int \frac{q}{\cos\theta} d\theta = \frac{q}{\sin\theta} + C = \frac{q\cot\theta}{\cos\theta} + C \\ &= \frac{q\cot(\theta)}{\cos^2\theta} + C = \frac{q(\sec^2\theta - 1)}{\cos^2\theta} + C \\ &= \frac{q}{\cos^2\theta} + C = \frac{q}{\frac{1}{\sin^2\theta}} + C \\ &= q \sin^2\theta + C = q \frac{1 - \cos^2\theta}{\cos^2\theta} + C \\ &= q \frac{\sin^2\theta}{\cos^2\theta} + C = q \frac{u^2}{e^{2x}} + C \end{aligned}$$

$u = \sqrt{e^{2x} - 9}$
 $\therefore \sin\theta = \sqrt{1 - \cos^2\theta}$

458. $\int \frac{1}{\sqrt{1-x-x^2}} dx = \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) + C$

$$\begin{aligned} \int \frac{1}{\sqrt{1-x-x^2}} dx &= \dots \text{COMBINING THE SQUARES} \\ &= \int \frac{1}{\sqrt{(\frac{1}{2}-x)^2 - (\frac{3}{2})^2}} dx \\ &= \arcsin\left(\frac{\frac{1}{2}-x}{\frac{\sqrt{3}}{2}}\right) + C \\ &= \arcsin\left(\frac{2x+1}{\sqrt{5}}\right) + C \end{aligned}$$

$1 - 2x - x^2 = -x^2 - 2x + 1 = -(x^2 + 2x + 1) = -(x+1)^2$

459. $\int \frac{1-3x^3}{\sqrt{1-x^2}} dx = \arcsin x + (2+x^2)\sqrt{1-x^2} + C$

$$\begin{aligned} \int \frac{1-3x^3}{\sqrt{1-x^2}} dx &= \int \frac{1}{\sqrt{1-x^2}} - \frac{3x^2}{\sqrt{1-x^2}} dx \\ &= \arcsin x - 3 \int \frac{x^2}{\sqrt{1-x^2}} dx \\ &= \arcsin x - 3 \int x^2 du \quad (\text{let } u = 1-x^2, du = -2x dx) \\ &= \arcsin x + 3 \int (-u^2) du \\ &= \arcsin x + 3u - u^3 + C \\ &= \arcsin x + 3(1-u^2)^{\frac{1}{2}} - (1-u^2)^{\frac{3}{2}} + C \\ &= \arcsin x + (1-x^2)^{\frac{1}{2}}(3-(1-x^2)^2) + C \\ &= \arcsin x + (2+x^2)\sqrt{1-x^2} + C \end{aligned}$$

ACCOMPLISHED BY TRIGONOMETRIC SUBSTITUTION AND MANIPULATIONS

$$\begin{aligned} \int \frac{1-3x^3}{\sqrt{1-x^2}} dx &= \dots = \int \frac{1-3\sin^3\theta}{\sqrt{1-\sin^2\theta}} (\cos\theta d\theta) \\ &= \int \frac{1-3\sin^3\theta}{\cos\theta} (\cos\theta d\theta) = \int 1-3\sin^3\theta d\theta \\ &= \int 1-3\sin\theta + 3\sin^2\theta \cos\theta d\theta \\ &= \int 1-3\sin\theta + 3\sin\theta \cos^2\theta d\theta \\ &= \theta + 3\sin\theta - \cos\theta + C \\ &= \theta + \cos\theta(3-\sin^2\theta) + C \\ &= \arcsin x + \sqrt{1-x^2} [3-(1-x^2)] + C \\ &= \arcsin x + (1-x^2)^{\frac{1}{2}}(3+x^2) + C \end{aligned}$$

460. $\int \frac{8}{(1+x^2)^3} dx = 3\arctan x + \frac{3x}{1+x^2} + \frac{2x}{(1+x^2)^2} + C$

$$\begin{aligned} \int \frac{8}{(1+x^2)^3} dx &= \dots \text{ (Partial Fractions would need to be done with a more refined technique involving factorising)} \\ &= \int (1+\tan^2\theta)(\sec^2\theta d\theta) = \int \frac{8\sec^2\theta}{\sec^6\theta} d\theta \\ &= \int \frac{8}{\sec^4\theta} d\theta = \int 8\sec^2\theta d\theta - \int 8(\frac{1}{4} + \frac{1}{2}\tan^2\theta)^2 d\theta \\ &= \int 8(\frac{1}{4} + 2\tan^2\theta + \frac{1}{2}\tan^4\theta) d\theta = \int 2 + 4\tan^2\theta + 2\tan^4\theta d\theta \\ &= \int 2 + 4\tan^2\theta + 2(\frac{1}{2}\sec^2\theta) d\theta = \int 3 + 4\tan^2\theta + \sec^2\theta d\theta \\ &= 3\theta + 2\sin 2\theta + \frac{1}{2}\sin 4\theta + C = 3\theta + 4\tan\theta + \frac{1}{2}\tan^2\theta + C \\ &= 3\theta + 4\tan\theta + \sin\theta(\sin\theta - 1) + C \\ &= 3\theta + 4\tan\theta + \sin\theta - \sin\theta\sin\theta + C \\ &= 3\theta + 3\tan\theta + 2\sin\theta\cos\theta + C \\ &= 3\arctan x + 3\left(\frac{2}{\sqrt{1+x^2}}\right) + 2\left(\frac{2}{\sqrt{1+x^2}}\right)^2 + C \\ &= 3\arctan x + \frac{2x}{1+x^2} + \frac{8}{(1+x^2)^2} + C \end{aligned}$$

461. $\int \frac{x^4}{\sqrt{x^{10}-1}} dx = \begin{cases} \frac{1}{5} \operatorname{arsinh} \left[\sqrt{x^{10}-1} \right] + C \\ \frac{1}{5} \ln \left[x^5 + \sqrt{x^{10}-1} \right] + C \end{cases}$

$\int \frac{x^4}{\sqrt{x^{10}-1}} dx = \dots \text{ BY SUBSTITUTION } \dots$

$$\begin{aligned} &= \int \frac{u^{\frac{1}{5}}}{\sqrt{u^2-1}} \left(\frac{5}{u^4} du \right) = \int \frac{5}{u^{\frac{1}{5}}} du = \int \frac{5}{u^{\frac{1}{5}}} \cdot \frac{1}{u^{\frac{4}{5}}} du \\ &= \frac{1}{5} \operatorname{arsinh} u + C = \frac{1}{5} \operatorname{arsinh} \sqrt{x^{10}-1} + C \\ &= \frac{1}{5} \ln \left[\sqrt{x^{10}-1} + \sqrt{x^{10}-1} \right] + C \end{aligned}$$

$u = \sqrt{x^{10}-1}$
 $u^2 = x^{10}-1$
 $2u \frac{du}{dx} = 10x^9$
 $u \frac{du}{dx} = 5x^9$
 $du = \frac{5}{u} x^9 du$
 $u^2 = x^{10}-1$
 $x^2 = \sqrt{u^2-1}$

462. $\int \frac{4}{e^x \sqrt{e^{2x}+4}} dx = -\frac{\sqrt{e^{2x}+4}}{e^x} + C$

$\int \frac{4}{e^x \sqrt{e^{2x}+4}} dx = \dots \text{ BY SUBSTITUTION } \dots$

$$\begin{aligned} &= \int \frac{4}{u \sqrt{u^2+4}} \left(\frac{1}{u} du \right) = \int \frac{-4}{u^2+4} du \\ &= \int \frac{-4}{u^2+4} du = \int \frac{-4}{u^2+4} \frac{du}{u} = \int \frac{-4}{u(u^2+4)} du \\ &= \int -\frac{4}{u(u^2+4)} du = -\frac{4}{u^2+4} + C \\ &= -\left(\frac{4}{u^2+4}\right)^{\frac{1}{2}} + C = -\frac{(4+2u^2)^{\frac{1}{2}}}{u^2+4} + C \\ &= -\frac{(4+2u^2)^{\frac{1}{2}}}{u^2} + C = -\frac{\sqrt{u^2+4}}{u^2} + C \end{aligned}$$

ALTERNATIVE APPROACH (USING A DOUBLE SUBSTITUTION)

$$\begin{aligned} \int \frac{4}{e^x \sqrt{e^{2x}+4}} dx &= \dots u=e^x = \dots \\ \int \frac{4}{u \sqrt{u^2+4}} du &= \int \frac{4}{u^2 \sqrt{u^2+4}} du \end{aligned}$$

NOW A TUTORIALIZED SUBSTITUTION TO FOLLOW

$$\begin{aligned} &= \int \frac{4}{4u^2 \sqrt{u^2+4}} (2u^2 du) \\ &= \int \frac{8u^2 du}{4u^2 \sqrt{u^2+4}} = \int \frac{2u^2}{\sqrt{u^2+4}} du \\ &= \int \frac{2u^2}{\sqrt{u^2+4}} du = \int \frac{2u^2}{\sqrt{u^2+4}} \frac{du}{u} = \int \frac{2u^2}{u \sqrt{u^2+4}} du \\ &= \int \frac{1}{u^2} \times \frac{2u^2}{\sqrt{u^2+4}} du = \int \frac{2u^2}{\sqrt{u^2+4}} du = \int \frac{2u^2}{\sqrt{u^2+4}} \frac{du}{u} = \int \frac{2u^2}{u \sqrt{u^2+4}} du \\ &= \int 2u^2 \cos u du = -2u^2 \sin u + C = -\frac{1}{u^2} + C \\ &= -\frac{\sqrt{u^2+4}}{u} + C = -\frac{\sqrt{e^{2x}+4}}{e^x} + C \end{aligned}$$

$u = e^x$
 $\frac{du}{dx} = e^x$
 $du = e^x dx$
 $dx = \frac{du}{e^x}$

$u = 2\sinh \theta$
 $e^x = 2\sinh \theta$
 $dx = 2\cosh \theta d\theta$
 $du = 2\cosh \theta d\theta$
 $du = \frac{2\cosh \theta}{\sinh^2 \theta} d\theta$
 $du = \frac{2\cosh^2 \theta}{\sinh^2 \theta} d\theta$
 $du = \coth^2 \theta d\theta$

ANOTHER APPROACH (USING HYPERBOLIC SUBSTITUTIONS)

$$\begin{aligned} \int \frac{4}{e^x \sqrt{e^{2x}+4}} dx &= \int 2\cosh \theta \sqrt{4\sinh^2 \theta + 4} \frac{2\cosh \theta}{\sinh^2 \theta} d\theta \\ &= \int 2\cosh \theta \sqrt{4(1+\sinh^2 \theta)} \frac{2\cosh \theta}{\sinh^2 \theta} d\theta \\ &= \int \frac{4\cosh^2 \theta}{\sinh^2 \theta} d\theta = \int \frac{4\cosh^2 \theta}{\sinh^2 \theta} d\theta \\ &= \int \cosh^2 \theta d\theta = -\cosh \theta + C \\ &= -\frac{\cosh \theta}{\sinh \theta} + C = -\frac{2\cosh \theta}{2\sinh \theta} + C \\ &= -\frac{\sqrt{1+\sinh^2 \theta}}{\sinh \theta} + C = -\frac{\sqrt{1+4\sinh^2 \theta}}{2\sinh \theta} + C = -\frac{\sqrt{1+4\cosh^2 \theta}}{2\cosh \theta} + C \end{aligned}$$

$u = \operatorname{arsinh} (\frac{e^x}{2})$
 $e^x = 2\sinh \theta$
 $dx = 2\cosh \theta d\theta$
 $du = 2\cosh \theta d\theta$
 $du = \frac{2\cosh \theta}{\sinh^2 \theta} d\theta$
 $du = \coth^2 \theta d\theta$

463. $\int \frac{2}{1+\sqrt[3]{x}} dx = 3x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 6 \ln \left| 1+x^{\frac{1}{3}} \right| + C$

$\int \frac{2}{1+\sqrt[3]{x}} dx = \dots \text{ BY SUBSTITUTION } \dots$

$$\begin{aligned} &= \int \frac{2}{1+u} (3u^2 du) = \int \frac{6u^2}{1+u} du \\ &\text{NOW BY LONG DIVISION/MANUFACTURE OR ANOTHER SUBSTITUTION} \\ &= \int \frac{6u^2(1+u)-6u-6}{1+u} du = \int 6u-6 + \frac{6u^2}{1+u} du \\ &= 6u^2 - 6u + 6 \ln |1+u| + C = 3u^3 - 6u^2 + 6 \ln |1+u| + C \end{aligned}$$

$u = \sqrt[3]{x}$
 $u^2 = x$
 $x = u^3$
 $dx = 3u^2 du$

NOTE THAT THE SUBSTITUTION $u = (1+x^{\frac{1}{3}})$ ALSO WORKS AND IS EASIER.
 PLEASE ALSO TRY $u = 3x^{\frac{1}{3}} - 6x^{\frac{2}{3}} + 6 \ln |1+x^{\frac{1}{3}}| + C$

464. $\int x \operatorname{arccosh} x \, dx = \frac{1}{2}(2x^2 - 1) \operatorname{arccosh} x - \frac{1}{4}x\sqrt{x^2 - 1} + C$

$\int x \operatorname{arccosh} x \, dx = \dots$ STRONG, EASY INTEGRATION BY PARTS

| | |
|---|-----------------------------|
| $x \operatorname{arccosh} x$ | $\frac{1}{2}\sqrt{x^2 - 1}$ |
| $\frac{d}{dx} x \operatorname{arccosh} x$ | $\frac{1}{2}\sqrt{x^2 - 1}$ |

Now A: INTEGRATION TO SIMPLIFY OR HYPERBOLIC SUBSTITUTION $u = \operatorname{cosh} \theta$

$$= \frac{1}{2} \operatorname{arccosh} x - \frac{1}{2} \int \frac{(2x^2 - 1)}{\sqrt{x^2 - 1}} \, dx = \frac{1}{2} \operatorname{arccosh} x - \frac{1}{2} \int (\sqrt{x^2 - 1} + \frac{1}{\sqrt{x^2 - 1}}) \, dx$$

$$= \frac{1}{2} \operatorname{arccosh} x - \frac{1}{2} \operatorname{arccosh} x - \frac{1}{2} \int (\sqrt{x^2 - 1}) \, dx$$

$$= \frac{1}{2} \operatorname{arccosh} x - \frac{1}{2} \operatorname{arccosh} x - \frac{1}{2} \int \sqrt{x^2 - 1} \sinh \theta \, d\theta$$

$$= \frac{1}{2}(2x^2 - 1) \operatorname{arccosh} x - \frac{1}{2} \int \sinh^2 \theta \, d\theta$$

$$= \frac{1}{2}(2x^2 - 1) \operatorname{arccosh} x - \frac{1}{2} \left[\frac{1}{2} \sinh 2\theta - \frac{1}{2} \theta \right] + C$$

$$= \frac{1}{2}(2x^2 - 1) \operatorname{arccosh} x - \frac{1}{4} \sinh 2\theta + \frac{1}{2} \theta + C$$

$$= \frac{1}{2}(2x^2 - 1) \operatorname{arccosh} x - \frac{1}{4} \operatorname{cosh} \theta \operatorname{sinh} \theta + \frac{1}{2} \operatorname{arccosh} x + C$$

$$= \frac{1}{2}(2x^2 - 1) \operatorname{arccosh} x - \frac{1}{4}x\sqrt{x^2 - 1} + \frac{1}{2} \operatorname{arccosh} x + C$$

$$= \frac{1}{2}(2x^2 - 1) \operatorname{arccosh} x - \frac{1}{4}x\sqrt{x^2 - 1} + C$$

ALTERNATIVE APPROACH SWAPPING (WITH A SUBSTITUTION)

| | |
|--|--|
| $\theta = \operatorname{arccosh} x$ | $d\theta = \operatorname{cosh} \theta \, d\theta$ |
| $\frac{1}{2}\operatorname{arccosh} x \, d\theta$ | $\frac{1}{2}\operatorname{cosh} \theta \, d\theta$ |

$$\int x \operatorname{arccosh} x \, dx = \int (\operatorname{cosh} \theta) \theta (\operatorname{cosh} \theta) \, d\theta = \int \theta \operatorname{cosh}^2 \theta \, d\theta$$

$$= \frac{1}{2} \operatorname{arccosh} x^2 \, d\theta = \dots$$

$$\text{INTEGRATION BY PARTS} \dots$$

$$= \operatorname{arccosh} x^2 - \frac{1}{2} \int \operatorname{cosh} 2\theta \, d\theta = \operatorname{arccosh} x^2 - \frac{1}{2} \sinh 2\theta + C$$

$$= \frac{1}{2} \operatorname{arccosh} x^2 - \frac{1}{2} \sinh 2\theta + C$$

$$= \frac{1}{2} \operatorname{arccosh} x^2 - \frac{1}{2} \operatorname{cosh} \theta \operatorname{sinh} \theta + \frac{1}{2} \theta + C$$

$$= \frac{1}{2} \operatorname{arccosh} x^2 - \frac{1}{4}x\sqrt{x^2 - 1} + \frac{1}{2} \operatorname{arccosh} x + C$$

$$= \frac{1}{2}(2x^2 - 1) \operatorname{arccosh} x - \frac{1}{4}x\sqrt{x^2 - 1} + C$$

465. $\int (x^3 + 5x^2 - 2)e^{2x} \, dx = \frac{1}{8}(4x^3 + 14x^2 - 14x - 1)e^{2x} + C$

$\int (x^3 + 5x^2 - 2)e^{2x} \, dx = \dots$ INTEGRATION BY PARTS ...

| | |
|--------------------------------|----------------------|
| $x^3 + 5x^2 - 2$ | e^{2x} |
| $\frac{d}{dx}(x^3 + 5x^2 - 2)$ | $\frac{d}{dx}e^{2x}$ |

INTEGRATION BY PARTS

$$= \frac{1}{2}(2x^2 - 2)e^{2x} - \frac{1}{2} \int (2x^2 + 10x) e^{2x} \, dx$$

$$= \frac{1}{2}(2x^2 - 2)e^{2x} - \frac{1}{2} \int (2x^2 + 10x) e^{2x} \, dx$$

$$= \frac{1}{2}(2x^2 - 2)e^{2x} - \frac{1}{2} \operatorname{cosh} 2x \operatorname{sinh} 2x + \frac{1}{2} \int (2x^2 + 10x) e^{2x} \, dx$$

$$= \frac{1}{2}(2x^2 - 2)e^{2x} - \frac{1}{2} \operatorname{cosh} 2x \operatorname{sinh} 2x + \frac{1}{2} \operatorname{cosh} 2x \operatorname{sinh} 2x - \frac{1}{2} \int 2e^{2x} \, dx$$

$$= \frac{1}{2}(2x^2 - 2)e^{2x} - \frac{1}{2} \operatorname{cosh} 2x \operatorname{sinh} 2x + \frac{1}{2} \operatorname{cosh} 2x \operatorname{sinh} 2x - \frac{1}{2} e^{2x} + C$$

$$= \frac{1}{2}e^{2x}[(2x^2 - 2) - (2\operatorname{cosh} 2x \operatorname{sinh} 2x) + (2\operatorname{cosh} 2x \operatorname{sinh} 2x) - 1] + C$$

$$= \frac{1}{2}e^{2x}(2x^2 - 3) + C$$

INTEGRATION BY PARTS

| | |
|--|---------------------------------|
| $\operatorname{cosh} 2x$ | $\frac{1}{2}e^{2x}$ |
| $\frac{d}{dx}(\operatorname{cosh} 2x)$ | $\frac{d}{dx}\frac{1}{2}e^{2x}$ |

466. $\int \frac{1}{x\sqrt{x^2-2}} dx = \begin{cases} \frac{1}{\sqrt{2}} \arccos\left[\frac{\sqrt{2}}{x}\right] + C \\ -\frac{1}{\sqrt{2}} \arcsin\left[\frac{\sqrt{2}}{x}\right] + C \\ \frac{1}{\sqrt{2}} \arctan\left[\frac{\sqrt{x^2-2}}{2}\right] + C \end{cases}$

$$\begin{aligned} \int \frac{1}{x\sqrt{x^2-2}} dx &= \dots \text{SINCE } u = x^2 - 2 \\ &= \int \frac{1}{\sqrt{u}} \sqrt{\frac{1}{u-2}} \left(\frac{1}{2u} du\right) = \int \frac{-1}{\sqrt{u(u-2)}} du \\ &= \int \frac{-1}{\sqrt{u-2}} \frac{1}{\sqrt{u}} du = \int \frac{-1}{\sqrt{1-\frac{2}{u}}} du \\ &= \frac{-1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{u-2}{u}}} du = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{2}{u}-1}} du = \frac{1}{\sqrt{2}} \arccos\left(\frac{u}{\sqrt{2}}\right) + C \\ &= \frac{1}{\sqrt{2}} \arccos(\sqrt{2}u) + C = \frac{1}{\sqrt{2}} \arccos\left(\frac{x}{\sqrt{2}}\right) + C \\ &= \frac{1}{\sqrt{2}} \arccos\left(\frac{\sqrt{x^2-2}}{2}\right) + C \\ &= \frac{1}{\sqrt{2}} \arccos\left(\frac{\sqrt{x^2-2}}{2}\right) + C \end{aligned}$$

THE SUBSTITUTION $u = x^2 - 2$ IS ALSO USEFUL.

467. $\int \sqrt{\frac{1+x}{1-x}} dx = \begin{cases} \arcsin x - \sqrt{1-x^2} + C \\ -\arccos x - \sqrt{1-x^2} + C \end{cases}$

$$\begin{aligned} \int \frac{1+x}{1-x} dx &= \int \frac{\sqrt{1+x}}{\sqrt{1-x}} dx = \int \frac{\sqrt{1+x}\sqrt{1+x}}{\sqrt{1-x}\sqrt{1+x}} dx = \int \frac{1+x}{\sqrt{1-x^2}} dx \\ &= \int \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} dx + x\int \frac{1}{\sqrt{1-x^2}} dx \\ &= \arcsin x - (1-x^2)^{\frac{1}{2}} + C = -\arccos x - (1-x^2)^{\frac{1}{2}} + C \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION

$$\begin{aligned} \int \frac{1+x}{1-x} dx &= \int \frac{1+2\cos\theta}{1-(2\sin\theta)^2} d\theta = \int \frac{1+2\cos\theta}{1-4\sin^2\theta} d\theta \\ &= \int \frac{1+(2\cos\theta+1)(1-2\sin\theta)}{(2\sin\theta)^2} d\theta \\ &= \int \frac{2\cos\theta+1}{2\sin\theta} d\theta = \int \frac{2\cos\theta}{2\sin\theta} d\theta - \int \frac{1}{2\sin\theta} d\theta \\ &= \int \left(\frac{1}{2} + \frac{1}{2\sin\theta}\right) d\theta = \int -2 - 2\cot^2\theta d\theta \\ &= -2\theta - 2\ln|\sin\theta| + C = -\arccos x - \sqrt{1-x^2} + C \end{aligned}$$

468. $\int \sqrt[3]{3\sin 2x - 2\sin 3x \cos x} dx = -\frac{4}{3} \sqrt[3]{\cos^4 x} + C$

$$\begin{aligned} \int \sqrt[3]{3\sin 2x - 2\sin 3x \cos x} dx &= \dots \text{USING THE THREE ANGLE IDENTITY} \\ &= \int \sqrt[3]{3\sin 2x - 2\sin(3x - 2\pi)} dx \\ &= \int \sqrt[3]{3\sin 2x - 2\sin 3x \cos 2x + 2\cos 3x \sin 2x} dx = \int 2\cos(3x)\sqrt[3]{1-\cos^2 x} dx \\ &= -\frac{2}{3} (\cos 3x)^{\frac{4}{3}} + C = -\frac{2}{3} \sqrt[3]{\cos^4 x} + C \end{aligned}$$

469. $\int \frac{25}{25+x^2} dx = 5 \arctan\left(\frac{1}{5}x\right) + C$

$$\int \frac{25}{25+x^2} dx = \dots \text{ STANDARD ARCTAN INTEGRAL} = 5 \int \frac{1}{5^2+x^2} dx \\ = \frac{25}{5} \arctan \frac{x}{5} + C = \underline{\underline{5 \arctan \frac{x}{5} + C}}$$

470. $\int \frac{4}{x^3+2x} dx = \ln\left[\frac{x^2}{x^2+2}\right] + C$

$$\int \frac{4}{x^3+2x} dx = \int \frac{4}{x(x^2+2)} dx = \dots \text{ BY PARTIAL FRACTIONS}$$

$$\frac{4}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2} \rightarrow 4 = A(x^2+2) + Bx^2 + Cx \\ \rightarrow 4 = (A+B)x^2 + Cx + 2A \rightarrow C=0, A=2, B=-2$$

$$\int \frac{2}{x} - \frac{2x}{x^2+2} dx = 2\ln|x| - \ln|x^2+2| + C = \ln|x|^2 - \ln|x^2+2| + C \\ = \underline{\underline{\ln|x^2+2| + C}}$$

471. $\int \frac{1}{x^2-4} dx = \begin{cases} \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C \\ -\frac{1}{2} \operatorname{artanh} \left[\frac{2+x}{2-x} \right] + C \end{cases}$

$$\int \frac{1}{x^2-4} dx = \int \frac{1}{(x-2)(x+2)} dx = \int \frac{\frac{1}{x-2} - \frac{1}{x+2}}{x-2} dx \\ = \frac{1}{2} \ln|x-2| - \frac{1}{2} \ln|x+2| + C = \frac{1}{2} \ln \left| \frac{x-2}{x+2} \right| + C$$

ALTERNATIVE BY NUMERATING AND STANDARDIZING FOR COTH OR ARCTH

$$\int \frac{1}{2x-4} dx = -\int \frac{1}{4-2x} dx = -\frac{1}{2} \int \frac{-2}{4-2x} dx$$

$$= -\frac{1}{2} \operatorname{artanh} \left(\frac{2x}{2} \right) + C$$

$$= -\frac{1}{2} \ln \left(\frac{1+\frac{2x}{2}}{1-\frac{2x}{2}} \right) + C$$

$$= -\frac{1}{2} \ln \left(\frac{2+x}{2-x} \right) + C$$

$$= \frac{1}{2} \ln \left(\frac{2-x}{2+x} \right) + C$$

472. $\int 4 \coth^2 2x dx = 4x - 2 \coth 2x + C$

$$\int 4 \coth^2 2x dx = \dots \begin{bmatrix} 1+\coth^2 u = \operatorname{cosec}^2 u \\ 1-\coth^2 u = \operatorname{sech}^2 u \end{bmatrix} = \int 4(1+\operatorname{cosec}^2 2x) dx \\ = \int 4 + 4 \operatorname{cosec}^2 2x dx = 4x - 2 \coth 2x + C$$

473. $\int \frac{x^2}{\sqrt{x^2-16}} dx = \begin{cases} 8 \operatorname{arcosh}\left(\frac{1}{4}x\right) + \frac{1}{2}x\sqrt{x^2-16} + C \\ \ln\left(x+\sqrt{x^2-16}\right) + \frac{1}{2}x\sqrt{x^2-16} + C \end{cases}$

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2-16}} dx &= \dots \text{Method of choice} \quad \text{SUGGESTION: PART ...} \\ &= \int \frac{(x^2-16)+16}{\sqrt{x^2-16}} dx = \int (x^2-16)^{\frac{1}{2}} + \frac{16}{\sqrt{x^2-16}} dx = (\operatorname{arcosh}\frac{x}{4}) + \int \frac{16}{\sqrt{x^2-16}} dx - \\ &= (\operatorname{arcosh}\frac{x}{4}) + \int \frac{16(x^2-16)^{-\frac{1}{2}}}{x^2-16} dx \quad \text{Divide top and bottom by } x^2-16 \\ &\rightarrow \operatorname{arcosh}\frac{x}{4} + \int \frac{16}{x^2-16} dx \quad \text{Cancel } x^2-16 \\ &= \operatorname{arcosh}\frac{x}{4} + \int \frac{16}{(x-\frac{4}{\sqrt{x^2-16}})(x+\frac{4}{\sqrt{x^2-16}})} dx \quad \text{Partial fractions} \\ &= \operatorname{arcosh}\frac{x}{4} + \int \frac{A}{x-\frac{4}{\sqrt{x^2-16}}} + \frac{B}{x+\frac{4}{\sqrt{x^2-16}}} dx \\ &= \operatorname{arcosh}\frac{x}{4} + \int \frac{A(x+\frac{4}{\sqrt{x^2-16}}) + B(x-\frac{4}{\sqrt{x^2-16}})}{x^2-16} dx \\ &= \operatorname{arcosh}\frac{x}{4} + A \operatorname{arcosh}\frac{x}{4} + B \operatorname{arcosh}\frac{x}{4} + \frac{1}{2} \times 2 \times \int \frac{1}{x^2-16} dx + C \\ &= B \operatorname{arcosh}\frac{x}{4} + 2A \operatorname{arcosh}\frac{x}{4} + C = B \operatorname{arcosh}\frac{x}{4} + \frac{1}{2}x\sqrt{x^2-16} + C \\ &\quad \boxed{[\operatorname{arcosh}\frac{x}{4} - \ln\left(\frac{x}{4} + \sqrt{\frac{x^2-16}{16}}\right) - \ln\frac{1}{4} + \ln(x+\sqrt{x^2-16})]} \\ &= \boxed{\ln(2+\sqrt{x^2-16}) + \frac{1}{2}x\sqrt{x^2-16} + C} \end{aligned}$$

NOTE THAT SINCE $x=4\operatorname{cosh}\theta$ SEPARATE THEM ALSO WORKS WELL.

474. $\int \frac{\arctan x}{1+x^2} dx = \frac{1}{2}(\arctan x)^2 + C$

$$\int \operatorname{arctan}^2 \frac{dx}{1+x^2} = \frac{1}{2}(\operatorname{arctan} x)^2 + C \quad \text{BY INTEGRATION}$$

THE SUBSTITUTION $u = \operatorname{arctan} x$ WORKS WELL

475. $\int \frac{1}{\sqrt{25-9x^2}} dx = \frac{1}{3} \arcsin\left(\frac{3}{5}x\right) + C$

$$\begin{aligned} \int \frac{1}{\sqrt{25-9x^2}} dx &= \int \frac{1}{3\sqrt{\frac{25}{9}-x^2}} dx = \frac{1}{3} \int \frac{1}{\sqrt{\frac{25}{9}-x^2}} dx \\ &= \dots \text{BY INTEGRATION} \dots = \frac{1}{3} \arcsin\left(\frac{3}{5}x\right) + C \\ &= \frac{1}{3} \arcsin\left(\frac{3}{5}x\right) + C \end{aligned}$$

$$476. \int \frac{\sqrt{x+2}}{\sqrt{x-1}} dx = \begin{cases} 3\ln\left[\sqrt{x+2} + \sqrt{x-1}\right] + \sqrt{x^2+x-2} + C \\ 3\operatorname{arsinh}\sqrt{\frac{x-1}{3}} + \sqrt{x^2+x-2} + C \end{cases}$$

$\int \frac{\sqrt{2x+2}}{\sqrt{2x-1}} dx = \int \frac{(2x+2)^{-1/2}}{(2x-1)^{-1/2}} dx = \int \sqrt{1 + \frac{3}{2x-1}} dx$

Let $\frac{3}{2x-1} = \tan^2\theta \Rightarrow \theta = \operatorname{arctan}\left(\frac{\sqrt{3}}{\sqrt{2x-1}}\right)$
 $d\theta = \frac{3}{(2x-1)^2} dx \Rightarrow dx = \frac{(2x-1)^2}{3} d\theta$
 $dx = -\frac{1}{3} \operatorname{cosec}^2\theta d\theta$

$\int \operatorname{cosec}^2\theta d\theta = \int -\frac{1}{3} \operatorname{cosec}^2\theta d\theta = \int -\operatorname{cosec}^2\theta d\theta$

Now proceed with THIS integral knowing the θ AT THE ROOF

$\int \operatorname{cosec}^2\theta d\theta = \int \operatorname{cosec}\theta \operatorname{cosec}\theta d\theta = -\operatorname{cotan}\theta \operatorname{cosec}\theta d\theta = [\operatorname{cotan}\theta] + \operatorname{cosec}\theta d\theta$

$= \operatorname{cosec}\theta d\theta + [\operatorname{cotan}\theta \operatorname{cosec}\theta d\theta] \quad \text{BY PARTS}$

$= \operatorname{cosec}\theta d\theta + [\operatorname{cotan}\theta \operatorname{cosec}\theta d\theta]$

$\operatorname{cosec}\theta d\theta = \operatorname{cosec}\theta d\theta + (-\operatorname{cotan}\theta) - [\operatorname{cotan}\theta \operatorname{cosec}\theta d\theta]$

1. $\operatorname{cosec}\theta d\theta = [\operatorname{cotan}\theta \operatorname{cosec}\theta d\theta]$

2. $\operatorname{cosec}\theta d\theta = -\operatorname{cotan}\theta \operatorname{cosec}\theta d\theta$

3. $\operatorname{cosec}\theta d\theta = -\operatorname{cotan}\theta \operatorname{cosec}\theta d\theta - \operatorname{cotan}\theta d\theta + C$

-6. $\operatorname{cosec}\theta d\theta = 3\ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + \operatorname{cotan}\theta d\theta + C$

$= 3\ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + 3\ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + C$

$= 3\ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + 3\ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + \sqrt{2x+2} + C$

$= 3\ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + 3\ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + \sqrt{2x+2} + C$

A UNI SUBSTITUTION WHERE $\frac{3}{2x-1} = \tan^2\theta$ THUS THE SIMPLER EQUAT WITH SIMILAR OR SIMILAR MORE INVOLVING METHODS

A SIMPLE APPROACH IS AS FOLLOWS

$\int \sqrt{\frac{2x+2}{2x-1}} dx = \int \frac{\sqrt{2x+2}}{\sqrt{2x-1}} dx \dots \text{START WITH A SIMPLER SUBSTITUTION}$

$\int \frac{\sqrt{2x+2}}{\sqrt{2x-1}} dx = \int \frac{(2x+2)^{-1/2}}{(2x-1)^{-1/2}} dx = \int \sqrt{1 + \frac{3}{2x-1}} dx$

REASON & SIMPLER SUBSTITUTION

$\frac{3}{2x-1} = \tan^2\theta \Rightarrow \theta = \operatorname{arctan}\left(\frac{\sqrt{3}}{\sqrt{2x-1}}\right)$
 $d\theta = \frac{3}{(2x-1)^2} dx \Rightarrow dx = \frac{(2x-1)^2}{3} d\theta$
 $dx = \frac{1}{3} \operatorname{cosec}^2\theta d\theta$

$\int \operatorname{cosec}^2\theta d\theta = \int \frac{1}{3} \operatorname{cosec}^2\theta d\theta = \int \operatorname{cosec}^2\theta d\theta$

TRYING OF FURTHER AND PUSHING THE FINER SUBSTITUTION

$= 3\ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + 3\ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + C$

FINALLY & MORE ELABORATE APPROX

$\int \sqrt{\frac{2x+2}{2x-1}} dx = \int \frac{\sqrt{2x+2}}{\sqrt{2x-1}} dx = \int \sqrt{1 + \frac{3}{2x-1}} dx$

$\int \frac{u}{\sqrt{u^2-1}} du = \int \frac{u}{\sqrt{u^2-1}} du$

$= \frac{1}{2} \operatorname{arsinh}(u^2-1) + C$

$\int \frac{\sqrt{2x+2}}{\sqrt{2x-1}} dx = \int \frac{\sqrt{2x+2}}{\sqrt{2x-1}} dx = \int \sqrt{1 + \frac{3}{2x-1}} dx$

$\frac{3}{2x-1} = \tan^2\theta \Rightarrow \theta = \operatorname{arctan}\left(\frac{\sqrt{3}}{\sqrt{2x-1}}\right)$
 $d\theta = \frac{3}{(2x-1)^2} dx \Rightarrow dx = \frac{(2x-1)^2}{3} d\theta$
 $dx = \frac{1}{3} \operatorname{cosec}^2\theta d\theta$

$\int \operatorname{cosec}^2\theta d\theta = \int \frac{1}{3} \operatorname{cosec}^2\theta d\theta = \int \operatorname{cosec}^2\theta d\theta$

TRYING THE INVERSE OF THE LOGONIC-LOGISTIC TOP TERM BY UNI SUB

$= \frac{3}{2} \ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + \frac{3\sqrt{2x+2}}{2} + C$

$= \frac{3}{2} \ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + \frac{3\sqrt{2x+2}}{2} + C$

$= \frac{3}{2} \ln\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + \frac{3\sqrt{2x+2}}{2} + C$

TRYING TO SOLVE WITH THE OTHER TWO METHODS

$= 3\operatorname{arsinh}\left(\frac{\sqrt{2x+2}}{\sqrt{2x-1}}\right) + \sqrt{2x+2} + C$

$$477. \int \frac{e^x}{\sqrt{e^{2x}-9}} dx = \begin{cases} \ln\left[e^x + \sqrt{e^{2x}-9}\right] + C \\ \operatorname{arcosh}\left[\frac{1}{3}e^x\right] + C \end{cases}$$

$\int \frac{e^x}{\sqrt{e^{2x}-9}} dx = \dots \text{BY INSPECTION}$

$\frac{1}{2} \operatorname{arcosh}(e^x) = \frac{e^x}{\sqrt{e^{2x}-9}}$

$\operatorname{arcosh}(e^x) + C = \frac{2e^x}{\sqrt{e^{2x}-9}}$

$= \ln\left(\frac{1}{2}e^x + \frac{1}{2}\sqrt{e^{2x}-9}\right) + C$

$= \ln\left(\frac{1}{2}e^x + \frac{1}{2}\sqrt{e^{2x}-9}\right) + C$

$= \ln\left(\frac{1}{2}(e^x + \sqrt{e^{2x}-9})\right) + C$

$= \frac{1}{2} \ln(e^x + \sqrt{e^{2x}-9}) + C$

$= \frac{1}{2} \ln(e^x + \sqrt{e^{2x}-9}) + C$

ALTERNATIVE BY SUBSTITUTION

$\int \frac{e^x}{\sqrt{e^{2x}-9}} dx = \int \frac{e^x}{\sqrt{e^{2x}-9}} du = \int \frac{1}{\sqrt{u^2-9}} du$

$= \operatorname{arcosh}\left(\frac{u}{3}\right) + C$

$= \operatorname{arcosh}\left(\frac{1}{3}e^x\right) + C \text{ AS UNI}$

$$478. \int x\sqrt{1-x^2} dx = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$$

$\int x\sqrt{1-x^2} dx = \int x(1-x^2)^{\frac{1}{2}} dx = \frac{1}{2}(1-x^2)^{\frac{3}{2}} + C$

SIMPLY REARRANGE

479. $\int \frac{2^x}{\sqrt{1-4^x}} dx = \frac{\arcsin(2^x)}{\ln 2} + C$

$$\int \frac{2^x}{\sqrt{1-4^x}} dx = \dots$$
 CAN BE RECOGNISED TO BE A SIMILAR SORT BUT IT IS
PROBABLY BETTER TO USE A SUBSTITUTION

$$\begin{aligned} u &= 2^x \\ du &= 2^x \ln 2 \\ du &= \frac{du}{dx} \times \ln 2 \end{aligned}$$

$$\begin{aligned} &= \int \frac{2^x}{\sqrt{1-4^x}} \frac{du}{2^x \ln 2} = \frac{1}{\ln 2} \int \frac{1}{\sqrt{1-u^2}} du \\ &= \text{arcsin}(u) \times \frac{1}{\ln 2} + C = \frac{1}{\ln 2} \times \arcsin(2^x) + C \\ &= \frac{\arcsin(2^x)}{\ln 2} + C \end{aligned}$$

480. $\int \frac{2}{\cos x \sin^3 x} dx = \begin{cases} 2 \ln |\tan x| - \operatorname{cosec}^2 x + C \\ -2 \ln |\cot 2x + \operatorname{cosec} 2x| - \operatorname{cosec}^2 x + C \end{cases}$

$$\int \frac{2}{\cos x \sin^3 x} dx = \dots$$
 BY SUBSTITUTION ... = $\int \frac{2}{\cos x \sin^3 x} dx$

$$\begin{aligned} &= \int \frac{2}{\sin^3 x} \frac{du}{\cos x} dx = \int \frac{2}{u^3 (1-u^2)} du = \int \frac{2}{u^3 (1-u^2)} du \\ &= \int \frac{2}{u^3 (1-u)(1+u)} du \dots$$
 BY PARTIAL FRACTIONS ...

$$\begin{aligned} \frac{2}{u^3 (1-u)(1+u)} &\equiv \frac{A}{u} + \frac{B}{1-u} + \frac{C}{1+u} + \frac{D}{u^3} \\ 2 &\equiv A(u^2(1+u)) + B(u^2(1+u)) + C(u^2(1-u)) + D(u^2(1-u)) \\ &\bullet \text{ IF } u=1 \quad \bullet \text{ IF } u=-1 \quad \bullet \text{ IF } u=0 \\ 2 &\equiv 2A \quad 2 \equiv -2B \quad 2 \equiv 0 \\ &\text{Evaluating} \quad \text{Evaluating} \quad \text{Evaluating} \\ 2 &\equiv A(u^2(1+u)) + B(u^2(1+u)) + D(u^2(1-u)) + C(u^2(1-u)) \\ 2 &\equiv (A-B-E)u^3 + (A+B-D)u^2 + (E-C)u^2 + Du + C \end{aligned}$$

$$\begin{aligned} &= \int \frac{1}{1-u} - \frac{1}{1+u} + \frac{2}{u^3} du = -\ln|1-u| - \ln|1+u| - \frac{1}{u^2} + 2\ln|u| + C \\ &= \ln\left|\frac{1}{1-u}\right| - \frac{1}{u^2} + C = \ln\left|\frac{1}{1+u}\right| - \frac{1}{u^2} + C \\ &= \ln\left|\frac{1}{1-u}\right| - \operatorname{cosec}^2 u + C = 2\ln|u| - \operatorname{cosec}^2 u + C \end{aligned}$$

ACTIONABLE MANIPULATION

$$\begin{aligned} \int \frac{2}{\cos x \sin^3 x} dx &= \int \frac{2 \sin x \cos^2 x}{\sin^3 x} dx = \int \frac{2}{\sin x} + \frac{2 \cos^2 x}{\sin^2 x} dx \\ &= \int \frac{2}{\sin x} + (2 \cos x \operatorname{cosec}^2 x) dx = \int \frac{2}{\sin x} + (2 \cos x \operatorname{cosec}^2 x) dx \\ &= \int \frac{2}{\sin x} + (2 \cos x \operatorname{cosec}^2 x) dx = \int \frac{2}{\sin x} + (2 \cos x \operatorname{cosec}^2 x) dx \\ &= -2 \ln|\sin x| + (\operatorname{cosec}^2 x)^2 + C = -2 \ln|\sin x| + \operatorname{cosec}^2 x - \operatorname{cosec}^2 x + C \\ &= -2 \ln|\sin x| + (\operatorname{cosec}^2 x)^2 + C = -2 \ln\left|\frac{\sin x}{\sin^2 x + \operatorname{cosec}^2 x}\right| + \operatorname{cosec}^2 x + C \\ &= -2 \ln\left|\frac{\sin x}{\operatorname{cosec}^2 x}\right| + \operatorname{cosec}^2 x + C = -2 \ln\left|\frac{\sin x}{\operatorname{cosec}^2 x}\right| + \operatorname{cosec}^2 x + C \\ &= -2 \ln\left|\frac{\sin x}{\operatorname{cosec}^2 x}\right| + \operatorname{cosec}^2 x + C = -2 \ln\left|\frac{\sin x}{\operatorname{cosec}^2 x}\right| + \operatorname{cosec}^2 x + C \end{aligned}$$

481. $\int \frac{1}{1+\sin x + \cos x} dx = \ln|1+\tan\left(\frac{1}{2}x\right)| + C$

$$\begin{aligned} \int \frac{1}{1+\sin x + \cos x} dx &= \dots \text{BY THE "CIRCLE L" METHOD} \\ \begin{array}{l} t = \tan\frac{x}{2} \\ \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} \\ \frac{dt}{dx} = \frac{1}{2}(1+2\sec^2) \\ \frac{dt}{dx} = \frac{1+2t^2}{2} \\ dt = \frac{1+2t^2}{2} dt \end{array} & \begin{array}{l} \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \\ \sin x = 2\frac{\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \times \cos\frac{x}{2} \\ \sin x = 2\tan\frac{x}{2} \times \frac{1}{1+\tan^2\frac{x}{2}} \\ \sin x = \frac{2t}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} \end{array} & \begin{array}{l} \cos x = 2\cos^2\frac{x}{2}-1 \\ \cos x = \frac{2}{1+\tan^2\frac{x}{2}}-1 \\ \cos x = \frac{2}{1+t^2}-1 \\ \cos x = \frac{2(1-t^2)}{1+t^2} \\ \cos x = \frac{2-2t^2}{1+t^2} \end{array} \\ & \begin{aligned} & \times \int \frac{1}{1+\frac{2t}{1+t^2} + \frac{2-2t^2}{1+t^2}} \times \frac{2}{1+t^2} dt = \int \frac{2}{1+2t+2t^2+1-2t^2} dt \\ & = \int \frac{2}{1+2t} dt = \int \frac{1}{1+2t} dt = \ln|1+2t| + C = \underline{\ln|1+2\frac{x}{2}| + C} \end{aligned} \end{aligned}$$

482. $\int \frac{4e^{3x}}{1+e^{2x}} dx = 4e^x + \arctan(e^x) + C$

$$\begin{aligned} \int \frac{4e^{3x}}{1+e^{2x}} dx &= \dots \text{BY SUBSTITUTION} \\ &\left[\begin{array}{l} \frac{du}{dx} = e^{2x} \\ \frac{du}{dx} = \frac{1}{2}e^{2x} \\ du = \frac{1}{2}e^{2x} dx \\ du = \frac{1}{2}e^{2x} dx \end{array} \right] \int \frac{4u^3}{1+u^2} du = \int \frac{4(u^2+1)-4}{u^2+1} du \\ &= \int u - \frac{4}{u^2+1} du = 4u + \arctan u + C \\ &= \underline{4e^x + \arctan(e^x) + C} \end{aligned}$$

483. $\int \frac{e^x}{\sqrt{9-e^{2x}}} dx = \arcsin\left(\frac{1}{3}e^x\right) + C$

$$\begin{aligned} \int \frac{e^x}{\sqrt{9-e^{2x}}} dx &= \dots \text{BY RECOGNITION} \\ &= \arcsin\left(\frac{1}{3}e^x\right) + C \quad \boxed{\begin{array}{l} \frac{d}{du}(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \\ \frac{d}{du}(\arcsin(u^2)) = \frac{1}{\sqrt{1-u^4}} \\ = \frac{1}{\sqrt{1-(\frac{1}{3}e^x)^2}} \\ = \frac{1}{\sqrt{1-\frac{e^{2x}}{9}}} \\ = \frac{1}{\sqrt{\frac{9-e^{2x}}{9}}} \\ = \frac{1}{\sqrt{9-e^{2x}}} \end{array}} \\ \text{THE SUBSTITUTION } u=e^x \text{ ALSO WORKS WELL} \end{aligned}$$

484. $\int \frac{\cos x}{\sqrt{1+\sin^2 x}} dx = \begin{cases} \operatorname{arsinh}(\sin x) + C \\ \operatorname{arcosh}(1+\sin^2 x) + C \end{cases}$

Method 1: By inspection, $\int \frac{\cos x}{\sqrt{1+\sin^2 x}} dx = \operatorname{arsinh}(\sin x) + C$

Method 2: Substitution
 $u = \sqrt{1+\sin^2 x}$
 $u^2 = 1+\sin^2 x$
 $du = 2\sin x \cos x dx$
 $dx = \frac{du}{2\sin x \cos x}$
 $\sin x = u^2 - 1$
 $\sin x = \pm \sqrt{u^2 - 1}$

Method 3: Another substitution
 $u = \sqrt{1+u^2}$
 $du = \frac{1}{\sqrt{1+u^2}} du$
 $du = \frac{u}{\sqrt{1+u^2}} du$

Final Solution: $\int \frac{\cos x}{\sqrt{1+\sin^2 x}} dx = \operatorname{arcosh}(1+\sin^2 x) + C = \operatorname{arcosh}(\sin^2 x + \sqrt{1+\sin^2 x}) + C = \operatorname{arcosh}(\sin 2x) + C$

485. $\int \frac{1}{(e^x+1)(e^x-1)} dx = \frac{1}{2} \tanh\left(\frac{1}{2}x\right) + C$

$\int \frac{1}{(e^x+1)(e^x-1)} dx = \int \frac{1}{e^{2x}+e^x-1} dx = \int \frac{1}{2+e^{-x}+e^{-2x}} dx$

Partial Fractions: $\frac{1}{(e^x+1)(e^x-1)} = \frac{A}{e^x+1} + \frac{B}{e^x-1}$

$A(e^x-1) + B(e^x+1) = 1$

$(A+B)e^x + (B-A) = 1$

$B-A = 1$
 $A+B = 0$

$A = -\frac{1}{2}, B = \frac{1}{2}$

$\int \frac{1}{(e^x+1)(e^x-1)} dx = \int \frac{-\frac{1}{2}}{e^x+1} dx + \int \frac{\frac{1}{2}}{e^x-1} dx = -\frac{1}{2} \operatorname{tanh}^{-1}\left(\frac{e^x}{2}\right) + \frac{1}{2} \operatorname{tanh}^{-1}\left(\frac{e^x}{1}\right) + C = \frac{1}{2} \operatorname{tanh}^{-1}\left(\frac{e^x}{2}\right) + C$

486. $\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}} dx = \frac{x}{\sqrt{1-x^2}} - \arcsin x + C$

$\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}} dx = \int \frac{x^2}{(1-x^2)^{\frac{3}{2}}} dx$

Trigonometric Substitution: $x = \sin \theta, dx = \cos \theta d\theta$

$\int \frac{\sin^2 \theta}{(1-\sin^2 \theta)^{\frac{3}{2}}} \cos \theta d\theta = \int \frac{\sin^2 \theta}{\cos^3 \theta} \cos \theta d\theta = \int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta = \int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta + C = \frac{x}{\sqrt{1-x^2}} - \arcsin x + C$

487. $\int \frac{1}{5\cosh x + 4\sinh x} dx = \frac{2}{3} \arctan(3e^x) + C$

$$\begin{aligned} \int \frac{1}{5e^x + 4e^{-x}} dx &= \dots \text{SPLIT INTO EXPONENTIALS} \\ &= \int \frac{2}{5e^x + 5e^{-x} + e^x - e^{-x}} dx = \int \frac{2}{3e^x + e^{-x}} dx = \dots \\ &\quad \text{BY SUBSTITUTION: } u = e^x, \quad du = e^x dx \\ &\quad \frac{du}{dx} = e^x, \quad e^x dx = du \\ &\quad \frac{du}{e^x} = dx, \quad dx = \frac{du}{e^x} \\ &\quad dx = \frac{du}{u} \end{aligned}$$

$$\begin{aligned} &\dots = \int \frac{2}{3u + \frac{1}{u}} \left(\frac{du}{u}\right) = \int \frac{2}{3u^2 + 1} du \\ &= \frac{2}{3} \int \frac{1}{u^2 + \frac{1}{3}} du = \frac{2}{3} \int \frac{1}{u^2 + (\frac{1}{\sqrt{3}})^2} du \\ &= \frac{2}{3} \times \frac{1}{\frac{1}{\sqrt{3}}} \times \arctan\left(\frac{u}{\frac{1}{\sqrt{3}}}\right) + C = \frac{2}{3} \arctan(u) + C \\ &= \frac{2}{3} \arctan(3e^x) + C \end{aligned}$$

488. $\int \frac{x+3}{\sqrt{4-x^2}} dx = 3\arcsin\left(\frac{1}{2}x\right) - \sqrt{4-x^2} + C$

$$\begin{aligned} \int \frac{3+x}{\sqrt{4-x^2}} dx &\approx \dots \text{SPLIT THE FRACTION} \approx \int \frac{3}{\sqrt{4-x^2}} dx + \frac{x}{\sqrt{4-x^2}} dx \\ &= \int 3(4-x^2)^{-\frac{1}{2}} dx + \frac{x}{\sqrt{4-x^2}} dx \\ &\quad \text{BOTH } \frac{1}{2}(4-x^2)^{\frac{1}{2}} \text{ OR } \arcsin^{-1} \text{ ARE INTEGRABLE} \\ &= -3(4-x^2)^{\frac{1}{2}} + 3\arcsin\left(\frac{x}{2}\right) + C \\ &= 3\arcsin\left(\frac{x}{2}\right) - 3\sqrt{4-x^2} + C. \end{aligned}$$

489. $\int \sqrt{\frac{1-\cos x}{1-\sin x}} dx = \begin{cases} \frac{1}{\sqrt{2}} \left[x + \ln \left[\frac{\sec^2(\frac{1}{2}x)}{\sec^2(\frac{1}{2}x) + 2\tan(\frac{1}{2}x)} \right] \right] + C \\ \frac{1}{\sqrt{2}} \left[x + \ln \left[\frac{1+\tan^2(\frac{1}{2}x)}{\left[1+\tan(\frac{1}{2}x) \right]^2} \right] \right] + C \end{cases}$

$\int \frac{1-\cos x}{1-\sin x} dx = \dots$ BY THE "T" METHOD (SOMETHING)

$t = \tan \frac{x}{2}$
 $\frac{dt}{dx} = \frac{1}{2}(1+\tan^2 \frac{x}{2})$
 $\frac{dt}{dx} = \frac{1}{1+\tan^2 \frac{x}{2}}$
 $dt = \frac{1}{1+\tan^2 \frac{x}{2}} dx$
 $dx = \frac{1}{1+\tan^2 \frac{x}{2}} dt$
 $\sin x = 2t \times \frac{1}{1+\tan^2 \frac{x}{2}}$
 $\sin x = 2t \times \frac{1}{1+t^2}$
 $\sin x = \frac{2t}{1+t^2}$
 $\cos x = \frac{1}{1+t^2}$
 $\cos x = \frac{1}{1+\tan^2 \frac{x}{2}}$
 $\cos x = \sqrt{\frac{1+\tan^2 \frac{x}{2}}{(1+\tan^2 \frac{x}{2})^2}}$
 $\cos x = \frac{1}{1+\tan^2 \frac{x}{2}}$
 $\cos x = \frac{1}{1+t^2}$

OR USE A STANDARD TECHNIQUE

$\int \frac{1-\frac{1-t^2}{1+t^2}}{1-\frac{2t}{1+t^2}} \frac{2}{1+t^2} dt = \int \frac{t^2+1-1+t^2}{(1+t^2)(1-2t)} \frac{2}{1+t^2} dt = \int \frac{2t^2}{(1+t^2)(1-2t)} dt$
 (canceling common factor of the argument of the denominator by 1+t^2)

= $2t \int \frac{1}{(1+t^2)(1-2t)} dt$ BY PARTIAL FRACTION

= $2t \left[\frac{A}{1+t^2} + \frac{B}{1-2t} + \frac{C}{(1-2t)^2} \right] dt$
 $= -2t \left[\frac{A}{1+t^2} + \frac{B}{1-2t} + \frac{1}{2(1-2t)^2} \right] dt + C$
 $= \frac{1}{2} \left[2t \frac{1}{1+t^2} + t \frac{B}{1-2t} - \frac{1}{2} \frac{1}{(1-2t)^2} \right] dt + C$
 $= \frac{1}{2} \left[2t \frac{1}{1+t^2} + t \frac{B}{1-2t} - \frac{1}{2} (1-2t)^{-2} \right] dt + C$
 $= \frac{1}{2} \left[2t \frac{1}{1+t^2} + t \frac{B}{1-2t} + \frac{1}{2} \frac{1}{(1-2t)^2} \right] dt + C$
 $= \frac{1}{2} \left[2t \frac{1}{1+t^2} + t \frac{B}{1-2t} + \frac{1}{2} \frac{1}{(1-2t)^2} \right] dt + C$
 $= \frac{1}{2} \left[2t \frac{1}{1+t^2} + t \frac{B}{1-2t} + \frac{1}{2} \frac{1}{(1-2t)^2} \right] dt + C$
 $= \frac{1}{2} \left[2t + t \left[\frac{B(1-2t)}{(1-2t)^2} \right] + \frac{1}{2} \frac{1}{(1-2t)^2} \right] dt + C$
 $= \frac{1}{2} \left[2t + t \left[\frac{B-2Bt}{(1-2t)^2} \right] + \frac{1}{2} \frac{1}{(1-2t)^2} \right] dt + C$

490. $\int \frac{3\sin x - \cos x + 3}{\cos x + \sin x} dx = x - 2 \ln |\sin x + \cos x| + \frac{3}{\sqrt{2}} \ln |\tan(x + \frac{1}{4}\pi) + \sec(x + \frac{1}{4}\pi)| + C$

$\int \frac{3\sin x - \cos x + 3}{\cos x + \sin x} dx =$ SIMPLIFY AS MUCH AS POSSIBLE

$3\sin x - \cos x \equiv A(\cos x + \sin x) + B(\sin x - \cos x)$
 $3\sin x - \cos x \equiv (4B)x + (A-B)\sin x$

$A+B=1$
 $-A+B=3$ $\Rightarrow A=1$, $B=-2$

$\int (\cos x + \sin x) - 2(\sin x - \cos x) + 3 dx = \int \frac{3\sin x - \cos x}{\cos x + \sin x} dx - 2 \frac{(\sin x - \cos x)}{\cos x + \sin x} dx$

= $\int 1 - 2 \left(\frac{\sin x - \cos x}{\cos x + \sin x} \right) dx + 3 \int \frac{1}{\cos x + \sin x} dx$
 R \rightarrow 2 TRANSFORMATION BY INSPECTION

= $x - 2 \ln |\sin x + \cos x| + \frac{3}{\sqrt{2}} \int \frac{1}{\cos x + \sin x} dx$
 $= x - 2 \ln |\sin x + \cos x| + \frac{3}{\sqrt{2}} \ln |\sec(x + \frac{\pi}{4})|$
 $= x - 2 \ln |\sin x + \cos x| + \frac{3}{\sqrt{2}} \ln |\sec(x + \frac{\pi}{4}) + \tan(x + \frac{\pi}{4})| + C$

THE "T" METHOD (SUBSTITUTION CAN ALSO BE USED HERE)

491. $\int \frac{4x-1}{\sqrt{x^2+16}} dx = 4\sqrt{x^2+16} - \text{arsinh}\left(\frac{1}{4}x\right) + C$

$$\begin{aligned} \int \frac{4x-1}{\sqrt{x^2+16}} dx &= \text{SPLIT AND REARRANGE} = \int \frac{\frac{4x}{4}dx}{\sqrt{x^2+16}} - \frac{1}{\sqrt{x^2+16}} dx \\ &= \int \frac{4x}{(2\sqrt{x^2+16})^2} dx - \frac{1}{\sqrt{x^2+16}} dx = 4\left(\frac{x^2}{4}\right)^{\frac{1}{2}} - \text{arsinh}\left(\frac{x}{4}\right) + C \\ &= 4\sqrt{x^2+16} - \text{arsinh}\left(\frac{x}{4}\right) + C \end{aligned}$$

492. $\int \frac{1}{4x^2+4x+2} dx = \frac{1}{2} \arctan(2x+1) + C$

$$\begin{aligned} \int \frac{1}{4x^2+4x+2} dx &= \frac{1}{4} \int \frac{1}{x^2+x+\frac{1}{4}} dx = \frac{1}{4} \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}} dx \\ &= \frac{1}{4} \int \frac{1}{(\frac{x+1}{2})^2+(\frac{\sqrt{3}}{2})^2} dx = -\frac{1}{2} - \frac{1}{2} \arctan\left(\frac{x+1}{\sqrt{3}}\right) + C \\ &= \frac{1}{2} \arctan(2x+1) + C \end{aligned}$$

493. $\int \frac{1}{1+8\sin^2 x} dx = \frac{1}{3} \arctan(3\tan x) + C$

$$\begin{aligned} \int \frac{1}{1+\tan^2 x} dx &= \dots \text{MANIPULATE AS FOLLOWS} = \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x} dx \\ &= \int \frac{\sec^2 x}{\sec^2 x + \sec^2 x \tan^2 x} dx = \int \frac{\sec^2 x}{(\sec^2 x + \sec^2 x) \tan^2 x} dx \\ &= \int \frac{\sec^2 x}{2\sec^2 x + \tan^2 x} dx = \frac{1}{2} \int \frac{\sec^2 x}{\tan^2 x + \frac{1}{2}} dx \end{aligned}$$

THIS CAN BE RECOGNIZED AS THE INTEGRAL OF $\arctan(3\tan x)$, OR USE THE SUBSTITUTION $u = \tan x$

$$= \frac{1}{2} \int \arctan(3\tan x) dx = \underline{\arctan(3\tan x)} + C$$

NOTE THAT THE 'LITTLE t' SUBSTITUTION $t = \tan x$ MAKES NO DIFFERENCE

494. $\int \frac{1}{\sqrt{\sin x \cos^3 x}} dx = 2\sqrt{\tan x} + C$

$$\begin{aligned} \int \frac{1}{\sqrt{\sin x \cos^3 x}} dx &= \int \frac{1}{\sqrt{\frac{\sin x}{\cos x} \times \cos^2 x}} dx = \int \frac{1}{\sqrt{\tan x \cos^2 x}} dx \\ &= \int \frac{1}{\sqrt{\tan x} \sqrt{\cos^2 x}} dx = \int \frac{1}{\sqrt{\tan x}} |\cos x| dx \\ &\text{BY RECOGNITION OR SUBSTITUTION } u = \sqrt{\tan x} \\ &= 2(\tan x)^{\frac{1}{2}} + C = \underline{2\sqrt{\tan x}} + C \end{aligned}$$

495. $\int \frac{\sin x}{16+9\cos^2 x} dx = -\frac{1}{12} \arctan\left(\frac{3\cos x}{4}\right) + C$

$$\begin{aligned} \int \frac{\sin x}{16+9\cos^2 x} dx &= \text{THIS IS AN OBVIOUS INTEGRAL OF } \arctan(\text{something}) \\ &= \frac{1}{4} \int \frac{\sin x}{4+3\cos^2 x} dx = \frac{1}{3} \int \frac{\sin x}{3+4\cos^2 x} dx \\ &= \frac{1}{3} \times \frac{1}{2} \arctan\left(\frac{\cos x}{2}\right) + C = -\frac{1}{12} \arctan\left(\frac{3\cos x}{4}\right) + C \end{aligned}$$

THE SUBSTITUTION $u = \sin x$ OR $\cos x = \frac{2}{\sqrt{5}} \tan\theta$ BOTH WORK WELL.

496. $\int \sinh^4 x \coth x dx = \frac{1}{4} \sinh^4 x + C$

$$\begin{aligned} \int \sinh^4 x \coth x dx &= \int \sinh^3 x \cosh x dx = \int \sinh^3 x dx \\ &= \frac{1}{4} \sinh^4 x + C \quad (\text{BY INTEGRATION}) \end{aligned}$$

497. $\int \frac{5\cos x - \sin x}{3\cos x + 2\sin x} dx = x + \ln|3\cos x + 2\sin x| + C$

$$\begin{aligned} \int \frac{5\cos x - \sin x}{3\cos x + 2\sin x} dx &= \text{LET THE NUMERATOR IN THE FORM } \\ &\quad A(3\cos x - 2\sin x) + B(2\cos x + 3\sin x) \\ &\quad \text{ie } A(3\cos x - 2\sin x) + B(-3\sin x + 2\cos x) = \text{NUMERATOR} \\ \text{THEN THE NUMERATORS ARE EQUAL IF } & \\ &\quad \begin{aligned} & \int \frac{3\cos x + 2\sin x - 3\cos x - 2\sin x}{3\cos x + 2\sin x} dx \\ &= \int \frac{2\sin x}{3\cos x + 2\sin x} dx \quad \text{OR THE FORM} \\ &= \int \frac{3\cos x + 2\sin x + -3\cos x - 2\sin x}{3\cos x + 2\sin x} dx \\ &= \int 1 + \frac{-2\cos x - 3\sin x}{3\cos x + 2\sin x} dx = \frac{1}{2} + \ln|2\cos x + 3\sin x| + C \end{aligned} \end{aligned}$$

THE QN CAN ALSO BE DONE (OVER WORKY) BY THE SUBSTITUTION $t = \tan \frac{x}{2}$.

498. $\int \frac{3x-4}{x^2+2x+17} dx = \frac{3}{2} \ln(x^2+2x+17) - \frac{7}{2} \arctan\left(\frac{x+1}{2}\right) + C$

$$\begin{aligned} \int \frac{3x-4}{x^2+2x+17} dx &= \text{AS THE DENOMINATOR IS IRREDUCIBLE, COMPLETE THE SQUARE} \\ &= \int \frac{3(x+1)-7}{(x+1)^2+16} dx = \int \frac{3(x+1)}{(x+1)^2+16} dx - \frac{7}{(x+1)^2+16} dx \\ \text{ANOTHER SUBSTITUTION } u=x+1 \text{ (SO } x=u-1 \text{ IS POSSIBLE)} & \\ &= \frac{3}{2} \int \frac{(2u-1)}{u^2+16} du = -7 \int \frac{du}{u^2+16} \quad \text{IF THE FORM } \int \frac{du}{u^2+a^2} \text{ THEN } \frac{1}{a} \arctan\left(\frac{u}{a}\right) \\ &= \frac{3}{2} \ln((2u-1)^2+16) - 7 \arctan\left(\frac{2u-1}{4}\right) \times \frac{1}{4} = \frac{3}{2} \ln(3(u-1)^2+16) - \frac{7}{8} \arctan\left(\frac{2u-1}{4}\right) + C \end{aligned}$$

499. $\int \frac{x+4}{\sqrt{x^2+8x+3}} dx = \sqrt{x^2+8x+3} + C$

$$\begin{aligned} \int \frac{2x+8}{\sqrt{x^2+8x+3}} dx &= \dots \text{NOT-NOT } \frac{d}{dx}(2x+8) = 2+0 = 2(1+0) \\ &= \int (2+4)(x^2+8x+3)^{\frac{1}{2}} dx \\ &\quad \text{BY SUBSTITUTION OR INSPECTION } u = x^2+8x+3 \\ &= \frac{1}{2} \int (2u)(2+4u) du \\ &= (2+4u)^{\frac{1}{2}} + C = \sqrt{2+4u} + C \end{aligned}$$

500. $\int \operatorname{arsinh}(\sqrt{x}) dx = \frac{1}{2}x(2x+1)\operatorname{arsinh}(\sqrt{x}) - \frac{1}{2}\sqrt{x^2+x} + C$

$$\begin{aligned} \int \operatorname{arsinh}(u) du &= \dots \text{BY SUBSTITUTION} \dots \\ &= \int u \operatorname{arsinh}(u) du - \text{INTEGRATION BY PARTS} \\ &= \frac{1}{2}u^2 \operatorname{arsinh}(u) - \int \frac{1}{2}u^2 \operatorname{arsinh}(u) du \\ &= \frac{1}{2}u^2 \operatorname{arsinh}(u) - \frac{1}{2}u \operatorname{arsinh}(u) + C \\ &= \frac{1}{2}u(u+1-2u^2) - \frac{1}{2}u \operatorname{arsinh}(u) + C \\ &\quad \left. \begin{array}{l} u=2x+1 \Rightarrow u^2=(2x+1)^2 \\ u=2x+1 \Rightarrow u-2u^2=2x+1-2(2x+1)^2=2x+1-zu \end{array} \right\} \\ &= \frac{1}{2}(2x+1)(1+2x) - \frac{1}{2}u \operatorname{arsinh}(u) + C \\ &= \frac{1}{2}(2x+1)\operatorname{arsinh}(\sqrt{x}) - \frac{1}{2}\sqrt{x^2+x} + C \\ &\quad \text{ALTERNATIVE SUBSTITUTION IS } u=\sqrt{x} \dots \text{LEADS TO } \int \operatorname{arsinh}(u) du, \text{ EASIER!} \\ &\quad \text{BY INTEGRATION BY PARTS IS AN ALTERNATIVE} \end{aligned}$$

501. $\int \frac{1}{2x^2+7x+3} dx = \frac{1}{5} \ln \left| \frac{2x+1}{x+3} \right| + C$

$$\begin{aligned} \int \frac{1}{2x^2+7x+3} dx &= \int \frac{1}{(2x+1)(x+3)} dx \dots \text{PARTIAL FRACTION BY INSPECTION} \\ &= \int \frac{\frac{1}{2x+1} + \frac{1}{x+3}}{2x+1} dx = \int \frac{\frac{2x}{2x+1} + \frac{1}{x+3}}{2x+1} dx \\ &= \frac{1}{2} \ln|2x+1| - \frac{1}{3} \ln|x+3| + C = \frac{1}{3} \ln \left| \frac{2x+1}{x+3} \right| + C \end{aligned}$$

502. $\int \frac{10 \sinh 2x}{\cosh^2 2x} dx = -5 \operatorname{sech} 2x + C$

$$\begin{aligned} \int \frac{10 \sinh 2x}{\cosh^2 2x} dx &= \int \frac{10 \sinh 2x}{\cosh 2x} \times \frac{1}{\cosh 2x} dx = \int 10 \operatorname{tanh} 2x \operatorname{sech} 2x \\ &\quad \text{now } \frac{d}{dx} [\operatorname{sech} 2x] = -2 \operatorname{sech} 2x \operatorname{tanh} 2x \\ &= -5 \operatorname{sech} 2x + C \end{aligned}$$

503. $\int x \arctan x \, dx = \frac{1}{2}(x^2 + 1) \arctan x - \frac{1}{2}x + C$

$\int x \arctan x \, dx = \dots$ BY SUBSTITUTION ...

$$= \int (\tan \theta) \theta \left(\sec^2 \theta \right) d\theta = \int \theta \tan \theta \sec^2 \theta \, d\theta$$

INTEGRATION BY PARTS

$$= \frac{1}{2}\theta \tan \theta - \frac{1}{2} \int \tan \theta \, d\theta$$

$$= \frac{1}{2}\theta \tan \theta - \frac{1}{2} \int \sec^2 \theta - 1 \, d\theta$$

$$= \frac{1}{2}\theta \tan \theta - \frac{1}{2} \tan \theta - \theta + C$$

$$= \frac{1}{2}\theta (\tan \theta + 1) - \frac{1}{2} \tan \theta + C = \frac{1}{2}(2x^2 + 1) \arctan x - \frac{1}{2}x + C$$

INTEGRATION BY PARTS, AGAIN YIELDS

$$\dots \frac{1}{2}x^2 \arctan x - \int \frac{x^2}{1+x^2} \, dx$$

WHICH WELDS TO THE FIRST INTEGRAL, JUST AS PREDICTED.

504. $\int \frac{\sin(\ln x)}{x^2} \, dx = -\frac{1}{2x} [\cos(\ln x) + \sin(\ln x)] + C$

$\int \frac{\sin(\ln x)}{x^2} \, dx = \dots$ START WITH A SUBSTITUTION

$$u = \ln x, \quad u' = \frac{1}{x}, \quad x = e^u$$

$$= \int \frac{\sin u}{e^{2u}} e^u \, du = \int \frac{\sin u}{e^{2u}} x^2 \, du = \int e^{-2u} \sin u \, du$$

INTEGRATION BY PARTS, TWICE, CONCLUDING WITH AN INTEGRAL

$$\begin{aligned} & \int e^{-2u} (\sin u + 2\cos u) \, du = e^{-2u} \sin u \\ & - e^{-2u} (\sin u + 2\cos u) + 2(-\sin u + 2\cos u) = e^{-2u} \sin u \\ & (4+3)\sin u + (-8-4)\cos u \equiv \sin u \\ & -\frac{1}{2}e^{-2u} \{ \sin u + 2\cos u \} \Rightarrow A = -\frac{1}{2}, \quad B = -\frac{1}{2} \end{aligned}$$

$$\dots -\frac{1}{2}e^{-2u} (\sin u + 2\cos u) + C = -\frac{1}{2}(e^{-2u})^2 (\sin u + 2\cos u) + C$$

$$= -\frac{1}{2}[\cos(\ln x) + \sin(\ln x)] + C$$

505. $\int \frac{x}{(x+1)(x^2+x+1)} \, dx = \frac{1}{2} \ln \left(\frac{x^2+x+1}{x^2+2x+1} \right) + \frac{1}{\sqrt{3}} \arctan \left(\frac{2x+1}{\sqrt{3}} \right) + C$

$\int \frac{z}{\tan(2z^2+1)} \, dz = \dots$ QUADRATIC FACTOR IS irreducible

PARTIAL FRACTION

$$\frac{z}{\sin(2z^2+1)} = \frac{A}{2z+1} + \frac{Bz+C}{2z^2+1}$$

$$\begin{aligned} z &= Az^2+Az+1+Bz^2(Cz+C)+C \\ z &= (A+B)z^2+(Az^2+Cz+C)+C \\ z &= (B-1)z^2+C(Bz+C)+C \end{aligned}$$

IF $2z+1 = 0$, THEN $z = -\frac{1}{2}$

$A = -1, \quad B = 1, \quad C = 1$

$$\begin{aligned} &= \int \frac{-1}{2z+1} + \frac{z+1}{2z^2+1} \, dz = \int \frac{1}{2z+1} + \frac{z+1}{2z^2+1} \, dz \\ &= \int -\frac{1}{2z+1} + \frac{1}{2} \frac{(2z^2+1)}{(2z^2+1)^2} + \frac{1}{2} \frac{2z}{(2z^2+1)^2} \, dz \\ &= -\frac{1}{2} \ln|2z+1| + \frac{1}{2} \ln(2z^2+1) + \int \frac{z}{(2z^2+1)^2} \, dz \\ &= -\frac{1}{2} \ln|2z^2+1| - \ln|2z+1| + \int \frac{z}{(2z^2+1)^2} \, dz \\ &= \frac{1}{2} \left[\ln(2z^2+1) - 2\ln|2z+1| \right] + \frac{1}{2z^2+1} \operatorname{arctan} \left(\frac{2z+1}{\sqrt{3}} \right) + C \\ &= \frac{1}{2} \ln \left(\frac{2z^2+1}{(2z+1)^2} \right) + \frac{1}{2z^2+1} \operatorname{arctan} \left(\frac{2z+1}{\sqrt{3}} \right) + C \end{aligned}$$

506. $\int x(2^x) dx = \frac{2^x}{\ln 2} \left[x - \frac{1}{\ln 2} \right] + C$

$$\begin{aligned}\int x(z^2) dz &= \text{INTEGRATION BY PARTS} \\ &= \frac{x(z^2)}{\ln 2} - \int \frac{1}{\ln 2} z^2 dz \\ &= \frac{x(z^2)}{\ln 2} - \frac{1}{(\ln 2)^2} (z^3) + C \\ &= \frac{z^3}{(\ln 2)^2} \left[x - \frac{1}{\ln 2} \right] + C\end{aligned}$$

507. $\int e^{2x} \sinh x dx = \frac{1}{6} e^{3x} - \frac{1}{2} e^x + C$

$$\begin{aligned}\int e^{2x} \sinh x dx &= \int e^{2x} \left(\frac{e^x}{2} - \frac{e^{-x}}{2} \right) dx = \int \frac{1}{2} e^{3x} - \frac{1}{2} e^x + C \\ &= \frac{1}{6} e^{3x} - \frac{1}{2} e^x + C\end{aligned}$$

508. $\int \frac{x^2}{(x^2+8)^{\frac{3}{2}}} dx = \ln \left[x + \sqrt{x^2+8} \right] - \frac{x}{\sqrt{x^2+8}} + C$

$$\begin{aligned}\int \frac{x^2}{(x^2+8)^{\frac{3}{2}}} dx &= \dots \text{HYPERBOLIC SUBSTITUTION} \dots \\ &= \int \frac{\sinh^2 u}{(\cosh^2 u)^{\frac{3}{2}}} \cdot \frac{2u \cosh u}{\cosh^2 u} du = \int \frac{2u \sinh^2 u}{\cosh^4 u} du \\ &= \int \frac{2u \sinh^2 u}{\cosh^4 u} du = \int \tanh^2 u du = \left(u - \tanh u \right) du \\ &\quad \begin{array}{l} \text{Let } u = \sinh^{-1} x \\ \tanh u = \frac{\sinh u}{\cosh u} = \frac{x}{\sqrt{x^2+8}} \end{array} \\ &= u - \tanh u + C = u - \frac{\sinh u}{\cosh u} + C = u - \frac{\sinh u}{\sqrt{1+\sinh^2 u}} + C \\ &= \arcsinh \left(\frac{x}{\sqrt{8}} \right) - \frac{x}{\sqrt{8}} + C = \ln \left[\frac{x}{\sqrt{8}} + \sqrt{\frac{x^2}{8} + 1} \right] - \frac{3\sqrt{2}x}{8} + C \\ &= \ln \left[\frac{x}{\sqrt{8}} + \sqrt{\frac{x^2}{8} + 1} \right] - \frac{3\sqrt{2}x}{8} + C = \ln \left[\sqrt{8} \left(\frac{x}{\sqrt{8}} + \sqrt{\frac{x^2}{8} + 1} \right) \right] - \frac{3\sqrt{2}x}{8} + C \\ &= \boxed{\ln \left(\frac{x}{\sqrt{8}} + \sqrt{x^2+8} \right) - \frac{3\sqrt{2}x}{8} + C} \quad \boxed{\ln \left[2 + \sqrt{x^2+8} \right] - \frac{3\sqrt{2}x}{8} + C}\end{aligned}$$

509. $\int \ln(1+x^2) dx = x \ln(1+x^2) - 2x + 2 \arctan x + C$

$$\begin{aligned}\int \ln(1+x^2) dx &= \dots \text{INTEGRATION BY PARTS} \dots \\ &= x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx \\ &= x \ln(1+x^2) - 2 \left[\frac{(x^2)-1}{x^2+1} \right] dx = x \ln(1+x^2) - 2 \int 1 - \frac{1}{1+x^2} dx \\ &= x \ln(1+x^2) - 2 \left[x - \arctan x \right] + C \\ &= \boxed{x \ln(1+x^2) - 2x + 2 \arctan x + C}\end{aligned}$$

510. $\int \frac{\cosh 3x}{\sinh^2 3x} dx = -\frac{1}{3} \operatorname{cosech} 3x + C$

$$\begin{aligned}\int \frac{\cosh 3x}{\sinh^2 3x} dx &= \int \frac{\cosh 3x}{\sinh 3x} \times \frac{1}{\sinh 3x} dx = \int \operatorname{cosech} 3x \operatorname{csch} 3x dx \\ \frac{d}{dx} (\operatorname{cosech} 3x) &= -\operatorname{cosech} 3x \operatorname{csch} 3x \\ \dots &= -\frac{1}{3} \operatorname{cosech} 3x + C\end{aligned}$$

511. $\int \frac{4}{5-3\cos x} dx = 2 \arctan \left[2 \tan \left(\frac{1}{2}x \right) \right] + C$

$$\begin{aligned}\int \frac{4}{5-3\cos x} dx &= \dots \text{ By the 'little t' identities} \\ &= \int \frac{4}{5-3\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2t}{1+t^2} dt \\ &= \int \frac{8}{5(1+t^2)-3(1-t^2)} dt \\ &= \int \frac{8}{2+8t^2} dt = \int \frac{8}{1+4t^2} dt \\ &= \int \frac{8}{4(t^2+\frac{1}{4})} dt = \int \frac{1}{t^2+\left(\frac{1}{2}\right)^2} dt \quad \text{(Complete square)} \\ &= \frac{1}{\frac{1}{2}} \operatorname{arctan} \left(\frac{t}{\frac{1}{2}} \right) + C = 2 \operatorname{arctan} 2t + C \\ &= 2 \operatorname{arctan} \left(2 \tan \left(\frac{1}{2}x \right) \right) + C\end{aligned}$$

512. $\int \sinh^2 x dx = \left[\frac{1}{2} \sinh 2x - \frac{1}{2}x + C \right]$
 $= \left[\frac{1}{8} e^{2x} - \frac{1}{2} - \frac{1}{8} e^{-2x} + C \right]$

$$\begin{aligned}\int \sinh^2 x dx &= \dots \text{ By identities} \quad \sinh^2 x = \frac{1}{2} - \frac{1}{2} \cosh 2x \\ &\quad -\sinh^2 x = \frac{1}{2} - \frac{1}{2} \cosh 2x \\ &\quad \int \frac{1}{2} \cosh 2x - \frac{1}{2} x dx = \frac{1}{2} \sinh 2x - \frac{1}{2}x + C \\ \text{ALTERNATIVELY BY SUBSTITUTION:} \\ \int \sinh^2 x dx &= \int \frac{1}{2} [(\partial_z z^2)]^2 dz = \int \frac{1}{2} (z^2 - z + e^{2z}) dz \\ &= \int \frac{1}{2} z^2 dz - \frac{1}{2} z dz + \frac{1}{2} e^{2z} dz = \frac{1}{2} e^{2z} - \frac{1}{2} z + \frac{1}{4} e^{2z} + C\end{aligned}$$

513. $\int \operatorname{sech}^2 4x dx = \frac{1}{4} \tanh 4x + C$

$$\int \operatorname{sech}^2 4x dx = \dots \frac{d}{dz} [\tanh z] \cdot \operatorname{sech}^2 z = \frac{1}{4} \tanh 4x + C$$

514. $\int \frac{1}{\sqrt{64+9x^2}} dx = \frac{1}{3} \operatorname{arsinh}\left(\frac{3}{8}x\right) + C$

$$\begin{aligned} \int \frac{1}{\sqrt{64+9x^2}} dx &= \int \frac{1}{8\sqrt{1+\left(\frac{9}{64}x^2\right)}} dx = \int \frac{1}{8\sqrt{1+\left(\frac{9}{64}x^2\right)}} dx \\ &= \frac{1}{8} \times \operatorname{arsinh}\left(\frac{3}{8}x\right) + C = \frac{1}{8} \operatorname{arsinh}\left(\frac{3}{8}x\right) + C \end{aligned}$$

515. $\int \frac{8}{\sqrt{16x^2-1}} dx = 2 \operatorname{arcosh} 4x + C$

$$\begin{aligned} \int \frac{8}{\sqrt{16x^2-1}} dx &= \int \frac{8}{4\sqrt{4x^2-\frac{1}{4}}} dx = \int \frac{2}{\sqrt{4x^2-\frac{1}{4}}} dx \\ &= 2 \operatorname{arcosh}\left(\frac{2x}{\sqrt{2}}\right) + C = 2 \operatorname{arcosh}(2x) + C \end{aligned}$$

516. $\int \frac{1}{\sqrt{x \cos^2 \sqrt{x}}} dx = 2 \tan \sqrt{x} + C$

$$\begin{aligned} \int \frac{1}{\sqrt{x \cos^2 \sqrt{x}}} dx &= \dots \text{BY RECOGNITION AS} \dots = \int \sqrt{x} \sec^2(x^{\frac{1}{2}}) dx \\ &= 2 \tan x^{\frac{1}{2}} + C = 2 \tan \sqrt{x} + C \end{aligned}$$

THE SUBSTITUTION $u = x^{\frac{1}{2}}$ ALSO WORKS WELL HERE

517. $\int \frac{6}{(1+\cos x)^2} dx = 3 \tan\left(\frac{1}{2}x\right) + \tan^3\left(\frac{1}{2}x\right) + C$

$$\begin{aligned} \int \frac{6}{(1+\cos x)^2} dx &= \int \frac{6}{(1+(2\sin^2 \frac{x}{2}-1))^2} dx = \int \frac{6}{4\sin^4 \frac{x}{2}} dx \\ &= \int \frac{3}{2} \sec^2 \frac{x}{2} dx = \frac{3}{2} \int \sec^2 \frac{x}{2} \sec^2 \frac{x}{2} dx \\ &= \frac{3}{2} \int \sec^2 \left(1+2\sin^2 \frac{x}{2}\right) dx = \frac{3}{2} \int \sec^2 \frac{x}{2} + \sec^2 \frac{x}{2} \tan^2 \frac{x}{2} dx \\ &\quad \text{BY RECOGNITION SINCE } \frac{d}{dx}(\tan x) = \sec^2 x \\ &= \frac{3}{2} \left[2 \tan \frac{x}{2} + \frac{3}{2} \tan^2 \frac{x}{2} \right] + C \\ &= 3 \tan \frac{x}{2} + \tan^3 \frac{x}{2} + C \end{aligned}$$

THE SUBSTITUTION $t = \tan \frac{x}{2}$ ALSO WORKS VERY WELL

518. $\int \sin^3 x \sqrt{(\cos^2 x \sin x)^2 + (\sin^2 x \cos x)^2} dx = \frac{1}{5} \sin^5 x + C$

$$\begin{aligned} \int \sin x \sqrt{\cos^2 x \sin^2 x + \sin^2 x \cos^2 x} dx &= \int \sin x \sqrt{\sin^2 x \cos^2 x + \sin^2 x \cos^2 x} dx \\ &= \int \sin x \sqrt{\sin^2 x \cos^2 x (\cos^2 x + \sin^2 x)} dx = \int \sin x \sqrt{\sin^2 x \cos^2 x} dx \\ &= \int \sin^2 x \cos x dx = \frac{1}{2} \sin^2 x + C \end{aligned}$$

519. $\int \frac{3x^2 - 2x + 5}{(x+3)^2} dx = 3 \ln|x+3| + \frac{20}{x+3} - \frac{19}{(x+3)^2} + C$

$$\begin{aligned} \int \frac{3x^2 - 2x + 5}{(x+3)^2} dx &= \dots \text{ USE SUBSTITUTION...} \\ &\stackrel{u=x+3}{=} \int \frac{3(u-3)^2 - 2(u-3) \cdot 1}{u^2} du = \int \frac{3u^2 - 18u + 27 - 2u + 6}{u^2} du \\ &= \int \frac{3u^2 - 20u + 33}{u^2} du = \int \left(3 + \frac{20}{u^2} + \frac{33}{u^3} \right) du \\ &= 3 \ln|u| + \frac{20}{u} + \frac{33}{u^2} + C = 3 \ln|x+3| + \frac{20}{x+3} - \frac{19}{(x+3)^2} + C \end{aligned}$$

PARTIAL FRACTIONS OF THE FORM $\frac{A}{(ax+b)^2} + \frac{B}{(ax+b)^3}$ ALSO WORK

520. $\int \frac{1}{\sqrt{x^2 + x + 1}} dx = \operatorname{arsinh}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + x + 1}} dx &= \int \frac{1}{\sqrt{(x+\frac{1}{2})^2 + \frac{3}{4}}} dx = \int \frac{1}{\sqrt{(\lambda x + \frac{1}{2})^2 + \frac{3}{4}}} d\lambda \\ &= \dots \text{ USE THE SUBSTITUTION } u = \frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}} \frac{du}{dx} \\ &= \int \frac{1}{\sqrt{u^2 + (\frac{\sqrt{3}}{2})^2}} du \\ &= \operatorname{arsinh}\left(\frac{u}{\frac{\sqrt{3}}{2}}\right) + C = \operatorname{arsinh}\left(\frac{2x+1}{\sqrt{3}}\right) + C \\ &= \operatorname{arsinh}\left(\frac{2x+1}{\sqrt{3}}\right) + C \end{aligned}$$

521. $\int \frac{\sqrt{25-x^2}}{x} dx = \sqrt{25-x^2} + \frac{5}{2} \ln \left| \frac{\sqrt{25-x^2}-5}{\sqrt{25-x^2}+5} \right| + C$

$$\begin{aligned} & \int \frac{\sqrt{25-x^2}}{x} dx = \dots \text{ BY SUBSTITUTION } \dots \\ & = \int \frac{u}{\sqrt{25-u^2}} du = \int \frac{u^2}{u^2+25} du = \int \frac{u^2}{25-u^2} du \\ & = \int \frac{u^2}{u^2-25} du \quad \text{MANIPULATE THE NUMERATOR} \\ & = \int \frac{u^2-25+25}{u^2-25} du = \int 1 + \frac{25}{u^2-25} du \\ & = \int 1 + \frac{25}{(u-5)(u+5)} du = \int 1 + \frac{\frac{5}{u-5} - \frac{5}{u+5}}{u^2-25} du \\ & = u + \frac{5}{2} \ln|u+5| - \frac{5}{2} \ln|u-5| + C = u + \frac{5}{2} \ln \left| \frac{u+5}{u-5} \right| + C \\ & = \sqrt{25-x^2} + \frac{5}{2} \ln \left| \frac{\sqrt{25-x^2}+5}{\sqrt{25-x^2}-5} \right| + C \end{aligned}$$

522. $\int \frac{1}{\sqrt{5-4x-x^2}} dx = \arcsin \left(\frac{x+2}{3} \right) + C$

$$\begin{aligned} \int \frac{1}{\sqrt{5-4x-x^2}} dx &= \int \frac{1}{\sqrt{-1-(x^2+4x-5)}} dx = \int \frac{1}{\sqrt{-1-(x+2)^2+9}} dx \\ &= \int \frac{1}{\sqrt{9-(x+2)^2}} dx \quad \text{BY THE ASTOKEAN } u=x+2, \frac{du}{dx}=1 \\ &= \int \frac{1}{\sqrt{9-u^2}} du \\ &= \arcsin \left(\frac{u}{3} \right) + C \\ &= \arcsin \left(\frac{x+2}{3} \right) + C \end{aligned}$$

523. $\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C$

$$\begin{aligned} & \int e^x \sin x dx = \dots \text{ DOUBLE INTEGRATION BY PARTS } \dots \\ & = e^x \sin x - \int e^x \cos x dx \\ & \quad \text{BY PARTS AGAIN} \\ & \int e^x \cos x dx = e^x \cos x - \int e^x \sin x dx \\ & \int e^x \sin x dx = e^x \sin x - e^x \cos x \\ & 2 \int e^x \sin x dx = e^x (\sin x - \cos x) \\ & \int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C \\ & \text{ANALOGY BY COMPLEX NUMBERS} \\ & \int e^x \sin x dx = \operatorname{Im} \left[\int e^{x+i} dx \right] = \operatorname{Im} \left[\int e^{(x+i)} dx \right] \\ & = \operatorname{Im} \left[\frac{e^{(x+i)}}{x+i} \right] = \operatorname{Im} \left[\frac{e^x}{x+i} e^{ix} \right] \\ & = \frac{1}{2} e^x \operatorname{Im} \left[e^{ix} \right] (\operatorname{cancellation}) = \frac{1}{2} e^x \operatorname{Im} \left[(x-1)(x+1) \right] \\ & = \frac{1}{2} e^x (\sin x - \cos x) \\ & \text{ANALOGY BY DIFFERENTIATION} \\ & \frac{d}{dx} [e^x (\sin x + \cos x)] = e^x (-\sin x + \cos x) + e^x (\sin x + \cos x) \\ & = e^x [(x+1)\cos x + (x-1)\sin x] \\ & \text{NOW } \begin{cases} x+1=0 \\ x-1=1 \end{cases} \Rightarrow \begin{cases} x=-1 \\ x=1 \end{cases} \text{ & } x=-\frac{1}{2} \\ & \therefore \frac{d}{dx} \left[e^x (\frac{1}{2} \sin x - \frac{1}{2} \cos x) \right] = e^x \sin x \\ & \therefore \int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C \end{aligned}$$

524. $\int \frac{2x^2+5x-1}{x^3+x^2-2x} dx = 2\ln|x-1| + \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C$

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x} dx = \int \frac{2x^2-2x-1}{x(x-1)(x+2)} dx = \int \frac{2x^2-2x-1}{x(x-1)(x+2)} dx$$

BY PARTIAL FRACTIONS

$$\frac{2x^2+5x-1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$2x^2+5x-1 \equiv Ax^2 + Bx^2 - Bx + Cx + C - A$$

- If $x=0$ • If $x=1$ • If $x=-2$
 $A=1$ $B=2$ $C=-2$
 $A=1$ $B=1$ $B=0$
 $A=\frac{1}{2}$ $B=\frac{3}{2}$ $C=\frac{1}{2}$

$$= \int \frac{\frac{1}{2}}{x} + \frac{\frac{3}{2}}{x-1} - \frac{\frac{1}{2}}{x+2} dx = \frac{1}{2}\ln|x| + 2\ln|x-1| - \frac{1}{2}\ln|x+2| + C$$

$$= 2\ln|x-1| + \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C$$

525. $\int x^3 e^{x^2} \sin x dx = \frac{1}{2} e^{x^2} (x^2 - 1) + C$

$$\int x^3 e^{x^2} dx = \dots$$

EVALUATE BY $\int x^2 (2e^{x^2}) dx$ **BY PARTS**

$$= \frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} dx$$

$$= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C$$

$$= \frac{1}{2} e^{x^2}(x^2 - 1) + C$$

THE SUBSTITUTION $u=x^2$, FOLLOWED BY INTEGRATION BY PARTS IS ALSO GOOD

526. $\int \frac{\tan \sqrt{x}}{\sqrt{x}} dx = 2\ln|\sec \sqrt{x}| + C$

$$\int \frac{\tan \sqrt{x}}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} \tan(x^{\frac{1}{2}}) dx$$

BY RECOGNITION OR SIMPLIFIED REASONING

$$= 2\ln|\sec x^{\frac{1}{2}}| + C = 2\ln|\sec \sqrt{x}| + C$$

THE SUBSTITUTION $u=\sqrt{x}$ ALSO WORKS WELL

527. $\int \frac{x+2}{(x-2)^2 \sqrt{x}} dx = -\frac{2\sqrt{x}}{x-2} + C$

$\int \frac{x+2}{\sqrt{x}(x-2)^2} dx \dots$

NOTING THAT THE SQUARE ROOT IN THE INTEGRAL
MAY SPLIT INTO A QUOTIENT
AND

$$\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

START BY DIFFERENTIATING A DIVISIBLE QUOTIENT

$$\frac{d}{dx} \left[\frac{ax}{x-2} \right] = \frac{(x-2)x^{\frac{1}{2}} - x^{\frac{1}{2}} \times 1}{(x-2)^2} = \frac{\frac{1}{2}x^{\frac{1}{2}}[(x-2)-2]}{(x-2)^2}$$

$$= \frac{1}{2} \left[\frac{x-2}{\sqrt{x}(x-2)^2} \right]$$

Hence $\int \frac{x-2}{\sqrt{x}(x-2)^2} dx = \frac{-x^{\frac{1}{2}}}{2} + C$

ALTERNATIVE STRATEGY WITH A SUBSTITUTION

$$\int \frac{x+2}{\sqrt{x}(x-2)^2} dx = \int \frac{u^{\frac{1}{2}}}{u^2(u-2)^2} (2u du)$$

$$= \int \frac{2(u^{\frac{1}{2}})}{(u-2)^2} du = \int \frac{2(u^{\frac{1}{2}})}{(u-2)^2(u+2)^2} du$$

BY PARTIAL FRACTION

$$\frac{2(u^{\frac{1}{2}})}{(u-2)^2(u+2)^2} \equiv \frac{A}{u-2} + \frac{B}{(u-2)^2} + \frac{C}{u+2} + \frac{D}{(u+2)^2}$$

$$2(u^{\frac{1}{2}}) \equiv A(u+2)(u-2)^2 + B(u+2)^2 + C(u-2)(u+2)^2 + D(u-2)^2$$

- If $u=2 \Rightarrow B=8B \Rightarrow B=1$
- If $u=-2 \Rightarrow B=BD \Rightarrow B=1$

* If $u=0$ $4 = -2\sqrt{2}A + 2B + 2\sqrt{2}C + 2D$
 $A' = -2\sqrt{2}A + 2 + 2\sqrt{2}C + 2$
 $2\sqrt{2}C = 2\sqrt{2}A$
 $C = A$

- If $u=2\sqrt{2}$ $20 = 16\sqrt{2}A + 18B + 6\sqrt{2}C + 2D$
 $20 = 16\sqrt{2}A + 18 + 6\sqrt{2}A + 2$
 $0 = 24\sqrt{2}A$
 $A = C = 0$

Therefore, we now have

$$\dots = \int \frac{2(u^{\frac{1}{2}}+1)}{(u-2)^2} du = \int \frac{1}{(u-2)^2} + \frac{1}{(u+2)^2} du$$

$$= -\frac{1}{u-2} - \frac{1}{u+2} + C = -\frac{u-2-u+2}{u^2-4} + C$$

$$= \frac{-2u}{u^2-4} + C = \frac{-2\sqrt{x}}{x-2} + C \text{ AS REQUIRED}$$

528. $\int \frac{1}{\sqrt{1+e^x}} dx = \ln \left| \frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1} \right| + C$

$\int \frac{1}{\sqrt{1+e^x}} dx = \dots$ BY SUBSTITUTION

$$= \int \frac{1}{\sqrt{u}} - \frac{2u}{u^2-1} du = \int \frac{2}{(u+1)(u-1)} du$$

PARTIAL FRACTION BY INSPECTION

$$\frac{2}{(u+1)(u-1)} \equiv \frac{A}{u+1} + \frac{B}{u-1}$$

$$2(u^{\frac{1}{2}}) = A(u+2)(u-2)^2 + B(u+2)^2 + C(u-2)(u+2)^2 + D(u-2)^2$$

$$= \int \frac{u+1}{u^2-1} du = \ln|u+1| - \ln|u-1| + C$$

$$= \ln \left| \frac{u+1}{u-1} \right| + C = \ln \left| \frac{\sqrt{1+e^x} + 1}{\sqrt{1+e^x} - 1} \right| + C$$

$$\begin{aligned} u &= \sqrt{1+e^x} \\ u^2 &= 1+e^x \\ 2u du &= e^x du \\ du &= \frac{e^x}{2u} du \\ u^2-1 &= u^2-1 \end{aligned}$$

529. $\int \operatorname{cosec}^3 x \, dx = \left[\frac{1}{2} \ln |\tan(\frac{1}{2}x)| - \frac{1}{2} \operatorname{cosec} x \cot x + C \right]$

$$\begin{aligned}\int \operatorname{cosec}^3 x \, dx &= \int \operatorname{cosec} x \operatorname{cosec}^2 x \, dx = \int (\operatorname{cosec}(1+\operatorname{atn}) \operatorname{cosec}^2(1+\operatorname{atn})) \, dx \\ &= \int \operatorname{cosec} x \, dx + \int \operatorname{cosec} x \operatorname{cosec}^2 x \, dx \\ &= \int \operatorname{cosec} x \, dx + \int (\operatorname{cosec} x)(\operatorname{cosec} x)^2 \, dx \\ &\quad \text{BY PARTS} \\ &\quad \begin{array}{c|c} \operatorname{cosec} x & -\operatorname{cosec} x \\ \hline & \operatorname{cosec} x \operatorname{cosec}^2 x \end{array} \\ \int \operatorname{cosec} x \, dx &= \left[\operatorname{cosec} x \, dx + \left[-\operatorname{atn}(\operatorname{cosec} x) - \int \operatorname{cosec} x \, dx \right] \right] \\ &\quad \text{BY ANOTHER REASON } (\operatorname{cosec} x = \frac{1}{\sin x}, \operatorname{atn}(\operatorname{cosec} x) = \operatorname{atn}(\frac{1}{\sin x})) \\ \int \operatorname{cosec} x \, dx &= \ln |\tan(\frac{x}{2})| - \operatorname{atn}(\operatorname{cosec} x) - \int \operatorname{cosec} x \, dx \\ 2 \int \operatorname{cosec} x \, dx &= \ln |\tan(\frac{x}{2})| - \operatorname{atn}(\operatorname{cosec} x) \\ \int \operatorname{cosec} x \, dx &= \frac{1}{2} \ln |\tan(\frac{x}{2})| - \frac{1}{2} \operatorname{atn}(\operatorname{cosec} x) + C\end{aligned}$$

530. $\int \frac{1}{e^{2x}-3e^x} \, dx = \frac{1}{9} \ln |1-3e^{-x}| - \frac{1}{3} e^{-x} + C$

$$\begin{aligned}\int \frac{1}{e^{2x}-3e^x} \, dx &= \int \frac{1}{e^x(e^x-3)} \, dx \\ &= \int \frac{1}{u(u-3)} \frac{du}{dx} \, dx = \int \frac{1}{u(u-3)} \, du \\ &\quad \text{BY PARTIAL FRACTIONS} \\ \frac{1}{u(u-3)} &= \frac{A}{u} + \frac{B}{u-3} \\ L &\equiv A\operatorname{u}(u-3) + B(u-3) + Cu^2 \\ *u &= 3 \quad *u = 0 \quad *u = 1 \\ 1 &= 9C \quad 1 = -3B \quad 1 = -3A + B \\ C &= \frac{1}{9} \quad B = -\frac{1}{3} \quad (=-3A + \frac{1}{9}) \\ 1 &= -3A + \frac{1}{9} \\ 3A &= -2 \quad A = -\frac{2}{3} \\ u &= \frac{1}{9} \end{aligned}$$

$$\begin{aligned}&= \int -\frac{2}{3u} - \frac{1}{3(u-3)} + \frac{1}{9} \, du = -\frac{2}{3} \ln |u| + \frac{1}{9u} + \frac{1}{3} \ln |u-3| + C \\ &= -\frac{2}{3} \ln \left| \frac{u-3}{u} \right| + \frac{1}{9u} + C = -\frac{2}{3} \ln \left| 1 - \frac{3}{u} \right| + \frac{1}{9u} + C \\ &= -\frac{2}{3} \ln \left| 1 - \frac{3}{e^x} \right| + \frac{1}{9e^x} + C = \frac{1}{9} \ln |1-3e^{-x}| + \frac{1}{3} e^{-x} + C\end{aligned}$$

531. $\int \frac{\cosh x}{1+\sinh x} \, dx = \ln |1+\sinh x| + C$

$$\begin{aligned}\int \frac{\cosh x}{1+\sinh x} \, dx &= \text{OR THE EASY WAY} \quad \int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + C \\ &= \ln |1+\sinh x| + C \\ &\quad \text{OR USE THE SUBSTITUTION } u = 1+\sinh x\end{aligned}$$

532. $\int \frac{1}{6\sinh 2x + 9\cosh 2x} dx = \begin{cases} \frac{\sqrt{5}}{15} \arctan(\sqrt{5} e^{2x}) + C \\ \frac{\sqrt{5}}{15} \arctan\left(\frac{2+3\tanh x}{\sqrt{5}}\right) + C \\ \frac{\sqrt{5}}{30} \arctan\left[\sinh\left(2x + \frac{1}{2}\ln 5\right)\right] + C \end{cases}$

$\int \frac{1}{6\sinh 2x + 9\cosh 2x} dx = \dots$ SWITCHING TO EXPONENTIALS ...

$$= \int \frac{1}{6(e^x - e^{-x}) + (e^x + e^{-x})^2} dx = \int \frac{1}{5e^{2x} + 4e^{-2x}} dx = \frac{5}{2} \int \frac{1}{5e^{2x} + 4} du$$

CONTINUE BY SUBSTITUTION

| | |
|---------------------------------|--|
| $u = e^{2x} \quad u^2 = e^{4x}$ | $du = 2e^{2x} dx \quad 5u^2 + 4 = 5e^{4x} + 4$ |
| $\frac{du}{dx} = 2e^{2x}$ | $5u^2 + 4 = 5e^{4x} + 4$ |
| $\frac{du}{dx} = 2u$ | $du = \frac{2u}{2e^{2x}} dx$ |
| $dx = \frac{du}{2u}$ | |

$$= \frac{5}{2} \int \frac{1}{5u^2 + 4} \frac{du}{2u} = \frac{5}{4} \int \frac{1}{u^2 + \frac{4}{5}} du$$

$$= \frac{5}{4} \int \frac{1}{u^2 + \frac{4}{5}} du = \frac{5}{4} \int \frac{1}{\left(\frac{u^2}{\sqrt{5}} + \frac{2}{\sqrt{5}}\right)^2} du$$

$$= \frac{5}{4} \cdot \frac{1}{\sqrt{5}} \times \arctan\left(\frac{u}{\sqrt{5}}\right) + C = \frac{\sqrt{5}}{8} \arctan\left(\frac{u}{\sqrt{5}}\right) + C$$

ALTERNATIVE BY "LITTLE t" (NOTHETHER APPROVED)

| | |
|---|--|
| $u = t \sinh x \quad \sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2}$ | $du = \frac{2e^x}{2} dt = e^x dt \quad e^x = \frac{1+t}{1-t}$ |
| $\frac{du}{dx} = \sinh x \quad \sinh x = \frac{2e^x}{1+t^2}$ | $\cosh x = \frac{1-t^2}{1+t^2}$ |
| $\frac{du}{dx} = 1-t^2 \quad \sinh x = \frac{2e^x}{1+t^2}$ | $\cosh x = \frac{1-t^2}{1+t^2}$ |
| $dx = \frac{dt}{1-t^2}$ | $\sinh x = \frac{2t}{1+t^2} \quad \cosh x = \frac{1-t^2}{1+t^2}$ |

$$\int \frac{1}{6\sinh 2x + 9\cosh 2x} dx = \int \frac{1}{6\left(\frac{2t}{1+t^2} + 9\left(\frac{1-t^2}{1+t^2}\right)\right)} dt = \int \frac{1}{12t + 9 + 9t^2} dt = \int \frac{1}{9(t + \frac{4}{3})^2 + 1} dt$$

$$= \frac{1}{3} \int \frac{1}{(t + \frac{4}{3})^2 + 1} dt = \frac{1}{3} \int \frac{1}{\left(\frac{t+4}{3}\right)^2 + 1} dt$$

AUGMENT SUBSTITUTION

$$u = t \sinh x \quad du = \frac{2e^x}{2} dt = e^x dt \quad \frac{1}{e^x} du = \frac{1}{2} \arctan\left(\frac{u}{\sqrt{5}}\right) + C$$

$$du = \frac{2e^x}{2} dt = e^x dt \quad \frac{1}{e^x} du = \frac{\sqrt{5}}{10} \arctan\left(\frac{2t+3}{\sqrt{5}}\right) + C$$

$$= \frac{\sqrt{5}}{10} \arctan\left(\frac{2+3\tanh x}{\sqrt{5}}\right) + C$$

UNQUOTE USING "E" TRANSFORMATION"

$$\cosh x + 9\sinh x = \cosh x(1+9) = 2\cosh x(5\sinh x + \cosh x)$$

$$\cosh x = 1 \quad \text{SINCE } t = \sinh x \quad \cosh^2 x = 1 \quad \cosh x = \sqrt{1-t^2}$$

$$2\cosh x = 2 \quad \text{SINCE } t = \sinh x \quad 2\cosh x = 2\sqrt{1-t^2} \quad \cosh x = \sqrt{1-t^2}$$

$$2\cosh x = 2 \quad \text{SINCE } t = \sinh x \quad 2\cosh x = 2\sqrt{1-t^2} \quad \cosh x = \sqrt{1-t^2}$$

$$\int \frac{1}{6\sinh 2x + 9\cosh 2x} dx = \int \frac{1}{6(2t) + 9(2\sqrt{1-t^2})} dt = \frac{1}{3} \int \frac{1}{4t + 3\sqrt{1-t^2}} dt$$

Now $\int \operatorname{sech} x dx = \arctan(\operatorname{sech} x)$

$$= \frac{\sqrt{5}}{15} \arctan\left(\operatorname{sech}(2x + \frac{1}{2}\ln 5)\right) + C = \frac{\sqrt{5}}{30} \arctan\left[\operatorname{sech}\left(2x + \frac{1}{2}\ln 5\right)\right] + C$$

533. $\int \sqrt{1+4\sinh^2 x \cosh^2 x} dx = \frac{1}{2} \sinh 2x + C$

$$\int \frac{1+\tanh^2 x}{\sqrt{1+\tanh^2 x}} dx = \int \sqrt{1+(\operatorname{sech}^2 x)^{-1}} dx$$

$$= \int 1 + \operatorname{sech}^2 x dx = \int \sqrt{\operatorname{cosec}^2 x} dx = \int \operatorname{cosec} x dx$$

$$= \frac{1}{2} \operatorname{Si}(\operatorname{cosec} x) + C$$

534. $\int \frac{\operatorname{arsinh} x}{\sqrt{x^2+1}} dx = \frac{1}{2} (\operatorname{arsinh} x)^2 + C$

$$\int \frac{\operatorname{arsinh} x}{\sqrt{x^2+1}} dx = \dots \text{BY SUBSTITUTION} \dots = \frac{1}{2} (\operatorname{arsinh} x)^2 + C$$

THE SUBSTITUTION $u = \operatorname{arsinh} x$ CAN ALSO BE USED

535. $\int \frac{x}{\sqrt{x^2+1}} dx = \sqrt{x^2+1} + C$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+1}} &= \dots \text{ BY SUBSTITUTION } \dots = x(\sqrt{x^2+1})^{\frac{1}{2}} dx \\ &= (\sqrt{x^2+1})^{\frac{3}{2}} + C \\ &= \sqrt{3x^2+1} + C\end{aligned}$$

THE SUBSTITUTION $u = x^2+1$ OR $u = x^2+1$ ALSO WORKS WELL

536. $\int \sqrt{x^2+4} dx = 2 \ln\left(x + \sqrt{x^2+4}\right) + \frac{1}{2} x \sqrt{x^2+4} + C$

$$\begin{aligned}\int \sqrt{x^2+4} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \sqrt{4u^2+4} du \quad (\text{where } u = \sqrt{x^2+4}) = \int 2u \sqrt{1+\frac{1}{u^2+1}} du \\ &= \int 2u du = \int 4\left(\frac{1}{2} + \frac{1}{2} u^{-1}\right) du \\ &= \int 2 + 2u^{-1} du = 2u + 2\ln|u| + C = 2u + 2\ln|x+2| + C \\ &= 2u + 2\ln\left(1+\sqrt{x^2+4}\right) + C = 2\ln\left(\frac{2}{\sqrt{x^2+4}}\right) + 2\left(1+\frac{x^2}{4}\right)^{\frac{1}{2}} + C \\ &= 2\ln\left[\frac{2}{\sqrt{x^2+4}} + \sqrt{1+\frac{x^2}{4}}\right] + 2\sqrt{1+\frac{x^2}{4}} + C \\ &= 2\ln\left[\frac{2}{\sqrt{x^2+4}} + \sqrt{\frac{4+x^2}{4}}\right] + 2\sqrt{\frac{4+x^2}{4}} + C \\ &= 2\ln\left[\frac{2}{\sqrt{x^2+4}} + \frac{\sqrt{4+x^2}}{2}\right] + \frac{1}{2}\sqrt{4+x^2} + C \\ &= 2\ln\left(\frac{2}{\sqrt{x^2+4}} + \sqrt{1+\frac{x^2}{4}}\right) + \frac{1}{2}\sqrt{4+x^2} + C \\ &= 2\ln\left(x + \sqrt{x^2+4}\right) + \frac{1}{2}x\sqrt{x^2+4} + C\end{aligned}$$

537. $\int \frac{1}{4x^{\frac{1}{2}}(1+x^{\frac{1}{4}})} dx = x^{\frac{1}{4}} - \ln\left(1+x^{\frac{1}{4}}\right) + C$

$$\begin{aligned}\int \frac{1}{4x^{\frac{1}{2}}(1+x^{\frac{1}{4}})} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{1}{4u^{\frac{1}{2}}(1+u)} \left(4u^{\frac{1}{2}} du\right) = \int \frac{4u^{\frac{1}{2}}}{4u^{\frac{1}{2}}(1+u)} du \\ &= \int \frac{1}{u(1+u)} du = \dots \text{ BY PARTIAL FRACTION } \dots = \int \frac{u+1-1}{u(u+1)} du \\ &= \int \left(1 - \frac{1}{u(u+1)}\right) du = u - \ln|u| + C = \underline{u - \ln(1+u^{\frac{1}{4}}) + C}\end{aligned}$$

538. $\int \frac{6x^2}{\sqrt{1-4x^6}} dx = \begin{cases} \arcsin(2x^3) + C \\ \arccos(\sqrt{1-4x^6}) + C \end{cases}$

$$\int \frac{6x^2}{\sqrt{1-4x^6}} dx = \dots \text{ By REVERSE... } \arcsin(2x^3) + C$$

OR BY SUBSTITUTION

$u = \sqrt{1-4x^6}$
 $u^2 = 1-4x^6$
 $2u \frac{du}{dx} = -24x^5$
 $du = \frac{-12x^5}{u} dx$

LOOKING AT THE TRIANGLE BELOW

$\theta = \arccos \sqrt{1-u^2}$
 $\cos \theta = \frac{u}{\sqrt{1-u^2}}$



$2x^2 = \arccos \sqrt{1-u^2} = \arcsin(2x^3)$

539. $\int \frac{x^2}{\sqrt{1-x^2}} dx = \frac{1}{2} \arcsin x + \frac{1}{2} x \sqrt{1-x^2} + C$

$$\int \frac{x^2}{\sqrt{1-x^2}} dx = \dots \text{ BY SUBSTITUTION... }$$

$\theta = \arcsin x$
 $x = \sin \theta$
 $dx = \cos \theta d\theta$

$= \int \frac{\sin^2 \theta}{\sqrt{1-\sin^2 \theta}} (\cos \theta d\theta) = \int \frac{1}{2} - \frac{1}{2} \cos 2\theta d\theta$

$= \frac{1}{2}\theta - \frac{1}{2}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin \theta \cos \theta + C$

$= \frac{1}{2}\theta + \frac{1}{2}\sin \theta \sqrt{1-\sin^2 \theta} + C = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$

THE SUBSTITUTION $\theta = \arcsin x$; $x = \sin \theta$. ALSO WORKS WITH

540. $\int \frac{1}{\sqrt{4-x^2}} dx = \begin{cases} \arcsin\left(\frac{1}{2}x\right) + C \\ -\arccos\left(\frac{1}{2}x\right) + C \end{cases}$

$$\int \frac{1}{\sqrt{4-x^2}} dx = \dots \text{ SIMPLIFIED RESULT... } = \arcsin\left(\frac{x}{2}\right) + C = -\arccos\left(\frac{x}{2}\right) + C$$

541. $\int \left(1 - \frac{1}{x}\right)^2 + \left(\frac{2}{\sqrt{x}}\right)^2 dx = x + \ln x + C$

$$\begin{aligned} \int \left(1 - \frac{2}{x} + \frac{4}{x^2}\right) dx &= \dots \text{ square and tidy} \dots \\ \int \left(1 - \frac{2}{x} + \frac{1}{x} + \frac{3}{x}\right) dx &= \int \left(1 + \frac{1}{x} + \frac{3}{x}\right) dx = \int \sqrt{\left(1 + \frac{1}{x}\right)^2} dx \\ \text{Quadratic without terms square bottom} \\ &= \int \left(1 + \frac{1}{x}\right) dx = x + \ln x + C \end{aligned}$$

542. $\int \sqrt{x^2 - 4} dx = \frac{1}{2} x \sqrt{x^2 - 4} - 2 \ln \left(x + \sqrt{x^2 - 4}\right) + C$

$$\begin{aligned} \int \sqrt{x^2 - 4} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \sqrt{4 \cos^2 \theta - 4} (2 \sin \theta d\theta) = \int 4(\cos^2 \theta - 1) (2 \sin \theta d\theta) \\ &= \int 4 \sin^2 \theta d\theta = \dots \quad \sin \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta \quad \dots = \int 4 \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta - \frac{1}{2} \right) d\theta \\ &= \int 2 \sin^2 \theta - 2 d\theta = \sin 2\theta - 2\theta + C = 2 \sin \theta \cos \theta - 2\theta + C \\ &= 2 \sqrt{\cos^2 \theta - 1} \sin \theta - 2\theta + C = 2\sqrt{\frac{1}{4} - \frac{1}{4} \cos 2\theta} \sin \theta - 2\theta + C \\ &= \frac{1}{2} \sin \theta \sqrt{1 - \cos 2\theta} - 2\theta + C = \frac{1}{2} \sin \theta \sqrt{\frac{1}{2} + \frac{1}{2} \cos 2\theta} - 2\theta + C \\ &= \frac{1}{2} \sin \theta \sqrt{\frac{1}{2} + \frac{1}{2} \cos 2\theta} - 2\theta + C = \frac{1}{2} \sin \theta \sqrt{\frac{1}{2} + \frac{1}{2} \cos 2\theta} - 2\theta + C = \frac{1}{2} \sin \theta \sqrt{\frac{1}{2} + \frac{1}{2} \cos 2\theta} - 2\theta + C = \frac{1}{2} \sin \theta \sqrt{\frac{1}{2} + \frac{1}{2} \cos 2\theta} - 2\theta + C \end{aligned}$$

543. $\int \frac{1}{x + \sqrt{x^2 + 1}} dx = -\frac{1}{2} x^2 + \frac{1}{2} \ln \left(x + \sqrt{x^2 + 1}\right) + \frac{1}{2} x \sqrt{x^2 + 1} + C$

$$\begin{aligned} \int \frac{1}{x + \sqrt{x^2 + 1}} dx &= \dots \text{ SIMPLIFY & CONSIDER MANIPULATION} \dots \\ &= \int \frac{2 - \sqrt{x^2 + 1}}{(x + \sqrt{x^2 + 1})(2 - \sqrt{x^2 + 1})} dx = \int \frac{2 - \sqrt{x^2 + 1}}{2^2 - (x^2 + 1)} dx \\ &= \int \frac{2 - \sqrt{x^2 + 1}}{3} dx = \int -x + \frac{2}{\sqrt{x^2 + 1}} dx \\ &\text{SIMPLIFIED SUBSTITUTION: } u = \sqrt{x^2 + 1}, \quad du = x dx, \quad dx = \frac{du}{x} \\ &= -\frac{1}{2} x^2 + \int (\sinh^{-1} u)^2 (x \cosh u du) = -\frac{1}{2} x^2 \int \cosh u du \\ &= -\frac{1}{2} x^2 \int \frac{1}{2} + \frac{1}{2} \sinh^2 u du = -\frac{1}{2} x^2 + \frac{1}{2} u + \frac{1}{2} \sinh u \cosh u + C \\ &= -\frac{1}{2} x^2 + \frac{1}{2} u + \frac{1}{2} \sinh u \cosh u + C = -\frac{1}{2} x^2 + \frac{1}{2} \theta + \frac{1}{2} \sinh \theta \cosh \theta + C \\ &= -\frac{1}{2} x^2 + \frac{1}{2} \sinh^{-1} x^2 + \frac{1}{2} x \sqrt{x^2 + 1} + C = -\frac{1}{2} x^2 + \frac{1}{2} \sinh^{-1} (x \sqrt{x^2 + 1}) + \frac{1}{2} x \sqrt{x^2 + 1} + C \end{aligned}$$

544. $\int \frac{1}{\sqrt{x^2 + 9}} dx = \begin{cases} \operatorname{arsinh}\left(\frac{1}{3}x\right) + C \\ \ln\left(x + \sqrt{x^2 + 9}\right) + C \end{cases}$

$$\int \frac{1}{\sqrt{z^2+9}} dz = \dots \text{STANDARD RESULT} \dots = \operatorname{arsinh}(z/3) + C$$

$$= \ln\left[\frac{z}{3} + \sqrt{\frac{z^2+9}{9}}\right] + C = \ln\left[\frac{z}{3} + \sqrt{\frac{z^2+9}{z^2+9}}\right] + C$$

$$= \ln\left[\frac{z}{3} + \sqrt{1+\frac{9}{z^2+9}}\right] + C = \ln\left[\frac{z}{3} + \sqrt{1+\frac{9}{z^2+9}}\right] + C$$

$$= \ln\left[\frac{z}{3} + \sqrt{1+\frac{9}{z^2+9}}\right] + C = \ln\left[2z + \sqrt{z^2+9}\right] + C$$

545. $\int \frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}} dx = 4\sqrt{1+\sqrt{x}} + C$

$$\int \frac{1}{\sqrt{z}\sqrt{z+1}} dz = \dots \text{THIS LOOKS LIKE A STANDARD FORMULA} \dots \int \frac{1}{\sqrt{z+1}} dz$$

RECOGNISE AS FAMILIAR

$$\dots = \int z^{\frac{1}{2}}(z+1)^{-\frac{1}{2}} dz = \dots \text{BY RECOGNITION} \dots = \frac{1}{2}(z+1)^{\frac{1}{2}} + C$$

$$= \frac{1}{2}\sqrt{z+1} + C$$

THE SUBSTITUTION $u = \sqrt{z+1}$ **ALSO WORKS WELL**

546. $\int \frac{1}{x(1+x^2)^3} dx = \frac{3+2x^2}{4(1+x^2)^2} \cdot \frac{1}{2} \ln\left(\frac{x^2}{1+x^2}\right) + C$

$\int \frac{1}{x(1+x^2)^3} dx = \dots \text{SINCE BY A SUBSTITUTION} \dots$

 $= \int \frac{1}{x \cdot u^3} \left(\frac{du}{dx}\right) dx = \int \frac{1}{2xu^2} du = \int \frac{1}{2(u-1)^2} du$

BY PARTIAL FRACTION

$$\frac{1}{(u-1)^2} = \frac{A}{u-1} + \frac{B}{u^2} + \frac{C}{u^3} + \frac{D}{u^4}$$

$$1 = A(u^3) + B(u^2) + Cu(u-1) + Du^2(u-1)$$

• IF $u=1$ • IF $u=0$ • IF $u=-1$ • IF $u=\infty$

$$\begin{aligned} 1 &= A + B(u-1) + Cu(u-1) + Du^2(u-1) \\ 1 &= A + (u-1)(B+C) + Cu(u-1) + Du^2(u-1) \\ 1 &= A + (u-1)(B+C) + (C-D)u^2 + C(u-1)u + \\ &\quad 2. Du-1 \quad 4. C-1 \end{aligned}$$

$\therefore \int \frac{1}{(u-1)^2} du = -\frac{1}{u-1} - \frac{1}{u^2} - \frac{1}{u^3} - \frac{1}{u^4} du = \frac{1}{2} \left[\ln|u-1| + \frac{1}{u^2} + \frac{1}{u^3} + \frac{1}{u^4} - \ln|u|\right] + C$

$$= \frac{1}{2} \left[\ln\left|\frac{u-1}{u}\right| + \frac{1+2u}{u^3} \right] + C = \frac{1}{2} \ln\left|\frac{u^2-1}{u^2}\right| + \frac{1+2\left(\frac{1}{u^2}\right)}{u^2} + C$$

$$= \frac{1}{2} \ln\left(\frac{u^2-1}{u^2}\right) + \frac{2u^2+1}{4u^2} + C$$

ALTERNATIVE SUBSTITUTION **WITHOUT** **FRACTION**

$$\int \frac{1}{x(1+x^2)^3} dx = \dots \text{BY SUBSTITUTION} \dots$$

$$- \int \frac{1}{x(1+x^2)^3} \left(-\frac{dx}{x^2}\right) = \int \frac{-1}{x^2(1+x^2)^3} dx$$

$$= \int \frac{-1}{x^2} dx = \int -\frac{1}{x^2} dx = -\frac{1}{x} \int \frac{1}{x^2} dx$$

$$= -\frac{1}{x} \int \frac{u}{u^2-1} du = \frac{1}{2} \int \frac{u}{u^2-1} du = \frac{1}{2} \int \frac{u^2}{u^2-1} du$$

$$= \frac{1}{2} \int \frac{(u(u-1)+(u-1)+1)}{u^2-1} du = \frac{1}{2} \int u+1 + \frac{1}{u-1} du$$

$$= \frac{1}{2} \left[\frac{1}{2}u^2 + u + \ln|u-1| \right] + C$$

$$= \frac{1}{2} \left[\frac{1}{2}\left(\frac{x^2}{1+x^2}\right)^2 + \frac{x^2}{1+x^2} + \ln\left|\frac{x}{1+x^2}\right| \right] + C$$

$$= \frac{1}{2} \left[\frac{x^2}{2(1+x^2)^2} + \frac{1}{1+x^2} + \ln\left|\frac{x}{1+x^2}\right| \right] + C$$

$$= \frac{1+2\left(\frac{x^2}{1+x^2}\right)}{4(1+x^2)^2} + \ln\left(\frac{x^2}{1+x^2}\right) + C = \frac{2x^2+1}{4(1+x^2)^2} + \ln\left(\frac{x^2}{1+x^2}\right) + C$$

ANOTHER APPROACH **USING HYPERBOLIC SUBSTITUTION** CAN BE EXPLORED

$$\int \frac{1}{x(1+x^2)^3} dx = \dots \text{BY SUBSTITUTION} \dots$$

$$\int \frac{1}{\sinh^2(u)\cosh^2(u)} du = \int \frac{\cosh^2(u)-\sinh^2(u)}{\sinh^2(u)\cosh^2(u)} du$$

$$= \int \frac{\cosh^2(u)}{\sinh^2(u)\cosh^2(u)} du = \int \frac{1}{\sinh^2(u)\cosh^2(u)} du$$

$$- \int \frac{1}{\sinh^2(u)\cosh^2(u)} du = \int \frac{\cosh^2(u)-\sinh^2(u)}{\sinh^2(u)\cosh^2(u)} du = \int \frac{\cosh^2(u)-\sinh^2(u)}{\cosh^2(u)} du$$

$$= \int \frac{1}{\sinh^2(u)} du = \frac{\operatorname{sech}^2(u)}{\cosh^2(u)} du$$

$$- \int \frac{1}{\sinh^2(u)} du = -\operatorname{tanh}^2(u) du$$

$$= \int \frac{2}{\cosh^2(u)} du = -\operatorname{tanh}(u) \operatorname{sech}(u) du - (-\operatorname{tanh}(u) \operatorname{sech}(u)) \operatorname{sech}^2(u) du$$

$$= \int \frac{2}{\cosh^2(u)} du = \frac{1}{2} \operatorname{sech}^2(u) + \frac{1}{2} \operatorname{sech}^4(u)$$

$$= \int \frac{2}{\cosh^2(u)} du = \frac{1}{2} \operatorname{sech}^2(u) + \frac{1}{2} \operatorname{sech}^4(u)$$

$$= -\operatorname{arctanh}(\operatorname{cosec}(u)) + \frac{1}{2} \operatorname{sech}^2(u) + \frac{1}{2} \operatorname{sech}^4(u) + C$$

WHICH WILL NORMALLY GIVE THE SAME ANSWER AFTER USING THE INVERSE HYPERBOLIC EQUATION FOR THE ARCTANH

547. $\int \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} dx = \ln(1 + \cos x) - 2 \ln(2 + \cos x) + C$

$$\begin{aligned} & \int \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} dx = \dots \text{BY SUBSTITUTION} \dots \\ & = \int \frac{\sin x \cdot u}{u^2 + 3u + 2} \left(\frac{du}{dx} \right) = - \int \frac{-u}{(u+2)(u+1)} du \\ & \quad \text{BY PARTIAL FRACTIONS (SEE UP)} \\ & = \int \frac{1}{u+1} - \frac{2}{u+2} du = \ln|u+1| - 2\ln|u+2| + C \\ & = \ln(1+\cos x) - 2\ln(2+\cos x) + C \end{aligned}$$

548. $\int \sin(\ln x) dx = \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C$

$$\begin{aligned} & \int \sin(u) du = \dots \text{BY SUBSTITUTION} \dots \\ & = \int \sin(u) \left(e^u du \right) = \int e^u \sin(u) du \\ & \quad \text{NOW BY DOUBLE INTEGRATION, COMPLEX NUMBERS OR} \\ & \quad \text{SUBSTITUTION VIA DIFFERENTIATION} \\ & \frac{d}{du} \left(e^u (\sin u + i \cos u) \right) \equiv e^u \sin u \\ & \stackrel{i}{\frac{d}{du}} (A \sin u + B \cos u) + e^u (A \cos u - B \sin u) \equiv e^u \sin u \\ & (A+iB)e^u \sin u + (A-iB)e^u \cos u \equiv e^u \sin u \\ & A+iB=0 \quad \therefore \quad A=\frac{1}{2} \quad B=-\frac{1}{2} \\ & \Rightarrow \int e^u \sin u du = \frac{1}{2}e^u (\sin u - i \cos u) + C \\ & = \frac{1}{2}e^u (\sin(\ln x) - i \cos(\ln x)) + C \end{aligned}$$

549. $\int -\sinh\left(\frac{1}{2}x\right) dx = -2\cosh\left(\frac{1}{2}x\right) + C$

$$\begin{aligned} \int -\sinh\left(\frac{1}{2}x\right) dx & = \dots \text{STANDARD BASIC} \dots \\ & = -2\cosh\left(\frac{1}{2}x\right) + C \end{aligned}$$

550. $\int (\cos x)[\ln(\sin x)] dx = [1 - \ln|\sin x|] \sin x + C$

$$\begin{aligned} & \int \cos x [\ln(\sin x)] dx = \dots \text{INTEGRATED BY PARTS} \\ & = -\sin x \ln(\sin x) + \int \sin x dx \\ & = -\sin x \ln(\sin x) + \int \cos x dx \\ & = -\sin x \ln(\sin x) + \sin x + C = (1 - \ln|\sin x|) \sin x + C \end{aligned}$$

551. $\int \sqrt{1-x^2} dx = \frac{1}{2} \arcsin x + \frac{1}{2} x \sqrt{1-x^2} + C$

$$\begin{aligned}\int \sqrt{1-x^2} dx &= \dots \text{BY SUBSTITUTION} \dots \\ &= \int \sqrt{1-\sin^2 \theta} (\cos \theta d\theta) = \int \cos^2 \theta d\theta \\ &= \int \frac{1}{2} + \frac{1}{2} \cos 2\theta d\theta = \frac{1}{2}\theta + \frac{1}{4} \sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2} \sin \theta \cos \theta + C \\ &= \frac{1}{2}\theta + \frac{1}{2} \sin \theta \sqrt{1-\sin^2 \theta} + C = \frac{1}{2} \arcsin x + \frac{1}{2} x \sqrt{1-x^2} + C\end{aligned}$$

552. $\int \frac{3}{x^2 - 4x + 13} dx = \arctan\left(\frac{x-2}{3}\right) + C$

$$\begin{aligned}\int \frac{3}{x^2 - 4x + 13} dx &= \int \frac{3}{(x-2)^2 + 9} dx = \text{SPANNED ARCTAN} \text{ IN } (x-2) \\ &= \frac{1}{3} \cdot 3 \times \arctan\left(\frac{x-2}{3}\right) + C = \arctan\left(\frac{x-2}{3}\right) + C\end{aligned}$$

553. $\int \frac{\sin x}{\cos^5 x} dx = \frac{1}{4} \sec^4 x + C$

$$\begin{aligned}\int \frac{\sin x}{\cos x} dx &= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx = \int \frac{1}{\cos x} \sin x dx \\ &= \int (\text{Inversesecant}) \sin x dx \dots \frac{d}{dx}(\text{secant}) = \text{secant} \tan \\ &= \frac{1}{4} \sec^4 x + C\end{aligned}$$

554. $\int 3 \tanh^2 x dx = 3x - 3 \tanh x + C$

$$\begin{aligned}\int 3 \tanh^2 x dx &= \dots \frac{1 + \tanh^2 x = \text{sech}^2 x}{1 - \tanh^2 x = \text{sech}^2 x} \dots = \int 3(1 - \text{sech}^2 x) dx \\ &= \int 3 - 3 \text{sech}^2 x dx = 3x - 3 \tanh x + C \\ &\quad \frac{d}{dx}(\tanh x) = \text{sech}^2 x\end{aligned}$$

555. $\int \frac{1}{\sqrt{x^2 - 4x + 13}} dx = \left[\operatorname{arsinh}\left(\frac{x-2}{3}\right) + C \right] \left[\ln\left[x-2+\sqrt{x^2-4x+13}\right] + C \right]$

$$\begin{aligned} \int \frac{1}{\sqrt{A^2 - 2Ax + B^2}} dx &= \int \frac{1}{\sqrt{(A-x)^2 + C^2}} dx = \text{STANDARD ARCSINH IN } x-2 \\ &= \operatorname{arsinh}\left(\frac{x-2}{3}\right) + C \\ \text{OR IN LOGARITHMS} \\ &= \ln\left[\frac{x-2}{3} + \sqrt{\left(\frac{x-2}{3}\right)^2 + 1}\right] + C = \ln\left[\frac{3x-6}{3} + \frac{1}{3}\sqrt{9x^2-36x+49}\right] + C \\ &= \ln\left[\frac{1}{3}(x-2) + \sqrt{(x-2)^2 + 7^2}\right] + C = \ln\left(\frac{1}{3}(x-2) + \sqrt{(x-2)^2 + 7^2}\right) + C \\ &= \ln\left(x-2 + \sqrt{x^2-4x+13}\right) + C \end{aligned}$$

556. $\int \frac{3}{\sqrt{4-x^4}} dx = \frac{1}{2} \arcsin\left(\frac{1}{2}x^2\right) + C$

$$\begin{aligned} \int \frac{dx}{\sqrt{4-x^4}} &= \dots \text{BY INSPECTION/RECOGNITION OF ARCSINE/ACOS} \\ &= \frac{1}{2} \arcsin\left(\frac{x^2}{2}\right) + C \\ \frac{d}{dx} \left[\arcsin\left(\frac{x^2}{2}\right) \right] &= 2x \cdot \frac{1}{\sqrt{1-\left(\frac{x^2}{2}\right)^2}} = \frac{2x}{\sqrt{4-x^4}} \\ \text{THE SUBSTITUTION } x^2 = 2\sin\theta \text{ & } \sin\theta = \frac{x^2}{2} \text{ ALSO WORKS WELL.} \end{aligned}$$

557. $\int \frac{1}{\sqrt{16x^2-9}} dx = \left[\frac{1}{4} \operatorname{arcosh}\left(\frac{4}{3}x\right) + C \right] \left[\frac{1}{4} \ln\left[4x + \sqrt{16x^2-9}\right] + C \right]$

$$\begin{aligned} \int \frac{1}{\sqrt{16x^2-9}} dx &= -\frac{1}{4} \int \frac{1}{\sqrt{2^2 - \frac{9}{16}x^2}} dx = \frac{1}{4} \int \frac{1}{\sqrt{x^2 - \left(\frac{3}{4}\right)^2}} dx \\ \text{STANDARD ARCCOSH INVERSION} \\ &= \frac{1}{4} \operatorname{arcosh}\left(\frac{4}{3}x\right) + C = \frac{1}{4} \operatorname{arcosh}\left(\frac{4}{3}x\right) + C \\ \text{OR IN LOGARITHMIC FORM} \\ &= \frac{1}{4} \ln\left[\frac{4}{3}x + \sqrt{\left(\frac{4}{3}x\right)^2 - 1}\right] + C = \frac{1}{4} \ln\left[\frac{4}{3}x + \frac{1}{3}\sqrt{16x^2-9}\right] + C \\ &= \frac{1}{4} \ln\left[\frac{1}{3}(4x + \sqrt{16x^2-9})\right] + C \sim \frac{1}{4} \ln\left(\frac{1}{3}\right) + \frac{1}{4} \ln\left(4x + \sqrt{16x^2-9}\right) + C \\ &= \frac{1}{4} \ln\left[4x + \sqrt{16x^2-9}\right] + C \end{aligned}$$

558. $\int \frac{4}{(4x-x^2)^{\frac{3}{2}}} dx = \frac{x-2}{\sqrt{4x-x^2}} + C$

$$\begin{aligned} \int \frac{4}{(4x-x^2)^{\frac{3}{2}}} dx &= \int \frac{4}{[(4x-x^2)+4]-4} dx = \int \frac{4}{(4-(x-2)^2)} dx \\ \text{BY SUBSTITUTION} \quad &x-2 = 2\sin\theta \quad \text{ie } \theta = \arcsin\frac{x-2}{2} \\ \frac{dx}{d\theta} &= 2\cos\theta \quad \text{ie } d\theta = \frac{dx}{2\cos\theta} \\ &= \int \frac{4}{(4-(2\sin\theta)^2)} (2\cos\theta) d\theta = \int \frac{8\cos\theta}{(4(1-\sin^2\theta))^{\frac{3}{2}}} d\theta = \int \frac{8\cos\theta}{8\cos^3\theta} d\theta \\ &= \int \frac{\sec^2\theta}{\cos^2\theta} d\theta = 4\sec\theta + C = \frac{\sec\theta}{\cos\theta} + C = \frac{1}{\sin\theta} + C \\ &= \frac{2-\frac{2}{x}}{\sqrt{1-\frac{(x-2)^2}{4}}} + C = \frac{2(x-2)}{\pm\sqrt{4-(x-2)^2}} + C = \frac{x-2}{\sqrt{4x-x^2}} + C \end{aligned}$$

559. $\int \cosh \sqrt{x} dx = 2\sqrt{x} \sinh \sqrt{x} - 2\cosh \sqrt{x} + C$

$$\begin{aligned} \int \cosh x dx &= \dots \text{STARTING WITH A SUBSTITUTION} \\ -\int \cosh u (du) &= \int \sinh u du \\ \text{INTEGRATION BY PARTS} \quad &\begin{array}{l} u=\sqrt{x} \\ x=u^2 \\ du=\frac{1}{2}u^{-1}dx \end{array} \\ \begin{array}{l} 2u+2 \\ \text{sum} \\ \text{cosh}u \end{array} &= 2u\sinh u - \int \sinh u du \\ &= 2u\sinh u - 2\cosh u + C \\ &= 2\sqrt{x}\sinh \sqrt{x} - 2\cosh \sqrt{x} + C \end{aligned}$$

560. $\int \sinh x \operatorname{sech}^2 x dx = -\operatorname{sech} x + C$

$$\begin{aligned} \int \sinh x \operatorname{sech} x dx &= \int \frac{\sinh x}{\cosh x} \operatorname{sech} x dx = \int \operatorname{sech} x \tanh x dx \\ \text{as } \operatorname{sech} x &= \frac{1}{\cosh x} = -\operatorname{sech} x \operatorname{tanh} x = -\operatorname{sech} x + C \end{aligned}$$

561. $\int \frac{1}{\sqrt{2x-x^2}} dx = \arcsin(x-1) + C$

$$\begin{aligned} \int \frac{1}{\sqrt{2x-x^2}} dx &= \int \frac{1}{\sqrt{1-(x-1)^2}} dx = \int \frac{1}{\sqrt{1-(x-1)^2}} dx \\ &= \int \frac{1}{\sqrt{1-(x-1)^2}} dx \quad \text{STANDARD ARCSINE IN } (0,1) \\ &= \arcsin(x-1) + C \end{aligned}$$

$$562. \quad \int \frac{8}{x^4 - 1} dx = 2 \ln \left| \frac{x-1}{x+1} \right| - 4 \arctan x + C$$

$$563. \quad \int \frac{1}{x(x^2+1)} dx = \left[\frac{1}{2} \ln\left(\frac{x^2}{x^2+1}\right) + C \right]$$

564. $\int \frac{1}{x\sqrt{1-(\ln x)^2}} dx = \arcsin(\ln x) + C$

PARTIAL FRACTIONS

$$\int \frac{B}{(x-1)(x+1)} dx = \int \frac{\frac{B}{(x-1)(x+1)}}{(B(x-1)+(B(x+1))} dx = \int \frac{\frac{B}{(x-1)(x+1)}}{(Ax+B)+(B(x+1))} dx$$

$$\frac{B}{(Ax+B)+(B(x+1))} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x^2-1}$$

$$B = A(x+1) + B(x-1) + C(x^2-1)$$

- If $x=1$: $B = A(2)+B(-1)+C(0) \Rightarrow B = 2A-B$
- If $x=-1$: $B = A(-2)+B(0)+C(-2) \Rightarrow B = -2A-C$
- If $x=0$: $B = A(1)+B(-1)+C(0) \Rightarrow B = A-B$
- $B = 4A$ $B = -4B$ $B = A-B = D$ $B = 14A + 5B = 3(2D)$
- $A+2$ $B = 0-A = -B$ $B = 2A-2D = -2(A-D)$ $B = 30 - 10 + 3(C-D)$
- $B = 0$ $B = 0$ $B = 0$ $B = 20 - 6C - D$
- $D = 4$ $B = 0$ $B = 0$ $B = 0$

SUM

$$\int \frac{2}{x-1} - \frac{4}{x+1} - \frac{4}{x^2-1} dx = 2\ln|x-1| - 4\ln|x+1| - 4\ln|x+1| + C$$

$$= 2\ln\left|\frac{|x-1|}{|x+1|}\right| - 4\ln|x+1| + C$$

$$\int \frac{1}{3x^2+1} dx = \dots$$

FRACTIONAL INTEGRATION

$$\frac{1}{3x^2+1} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)^2}$$

$$1 = Ax(x^2+1)^2 + (Bx+C)x^2$$

$$1 = (A+B)x^4 + Ax^2 + Cx + A$$

$$A=1, C=0, B=-1$$

$$= \int \frac{1}{x} - \frac{2x}{x^2+1} dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

$$= \frac{1}{2}[\ln x^2 - \ln(x^2+1)] + C = \frac{1}{2}[\ln x^2 - \ln((x^2+1)e^{-\ln x^2})] + C$$

$$= \frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right) + C$$

$$\int \frac{1}{\sqrt{1-(ln x)^2}} dx = \int \frac{1}{x} \lambda \frac{1}{\sqrt{1-(\lambda^2)^2}} d\lambda = \dots \text{ by substitution}$$

$= \operatorname{arcsinh}(\ln x) + C$

THE SUBSTITUTIONS ARE IN THE FORM WHICH WILL

565. $\int \frac{6}{(1+\cos x)^2} dx = 3\tan\left(\frac{1}{2}x\right) + \tan^3\left(\frac{1}{2}x\right) + C$

$$\begin{aligned}\int \frac{6}{(1+2\cos\frac{x}{2}-1)^2} dx &= \int \frac{6}{4\cos^2\frac{x}{2}} dx \\ &= \int \frac{3}{2} \sec^2\frac{x}{2} dx = \int \frac{3}{2} \sec^2(1+2\cot\frac{x}{2}) dx \\ &= \int \frac{3}{2} \sec^2\frac{x}{2} + \frac{3}{2} \sec^2\frac{x}{2} \ln|\sec\frac{x}{2}| dx \\ (\text{BY INTEGRATION}) &= 3\tan\frac{x}{2} + \frac{3}{2}\ln|\sec\frac{x}{2}| + C\end{aligned}$$

566. $\int \frac{x}{1+\sqrt{x}} dx = \frac{2}{3}x^{\frac{3}{2}} - x + 2x^{\frac{1}{2}} - 2\ln(1+\sqrt{x}) + C$

$$\begin{aligned}\int \frac{x}{1+\sqrt{x}} dx &= \dots \text{BY SUBSTITUTION} \dots \\ &= \int \frac{(u-1)^2}{u} (2u du) = \int 2(u-1)^2 du \\ &= 2 \int \frac{u^2-2u+1}{u} du = 2 \int u^2-2u+1 \cdot \frac{1}{u} du \\ &= 2 \left[\frac{u^3}{3} - \frac{2u^2}{2} + 2u - \ln|u| \right] + C \\ &= \frac{2}{3}(1+u^2)^3 - 3C + u^2 + 4C(u^2) - 2\ln(1+u^2) + C \\ (\text{THE SUBSTITUTION } u=1+x^2 \text{ PRODUCES A KNOWN-INTEGRAL.}) \quad u=1+x^2 \\ &= \int \frac{u^2}{1+u} (2u du) = \int \frac{2u^3}{1+u} du - 2 \int \frac{u^2}{u+1} du \\ &= 2 \int \frac{u^2(u+1)-u(u+1)+(u+1)-1}{u+1} du \\ &= 2 \int u^2-u+1 - \frac{1}{u+1} du = 2 \left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1| \right] + C \\ &= \frac{2}{3}u^3 - u^2 + 2u - 2\ln|u+1| + C = \frac{2}{3}x^3 - x^2 + 2x^{\frac{1}{2}} - 2\ln(1+x^2) + C\end{aligned}$$

567. $\int \frac{15}{4\cos x + 3\sin x} dx =$

$$\begin{cases} \ln \left| \frac{2\tan(\frac{1}{2}x) + 1}{\tan(\frac{1}{2}x) - 2} \right| + C \\ \ln \left| \frac{2\sin x + \cos x + 1}{\sin x - \cos x - 1} \right| + C \\ \ln \left| \frac{\sin x - 2\cos x + 2}{1 - \cos x - \sin x} \right| + C \\ \ln \left| \frac{5 + 4\sin x - 3\cos x}{4\cos x + 3\sin x} \right| + C \end{cases}$$

$\int \frac{s}{4\cos x + 3\sin x} dx = \dots$ BY THE TIME IT IS SIMPLY AND DIRECT

$$\begin{aligned} & \text{Let } u = \frac{2\tan(\frac{1}{2}x)}{1 - \cos x} \quad t = \frac{1 - \cos x}{\sin x} \\ & \cos x = \frac{1 - t^2}{1 + t^2} \\ & \sin x = \frac{2t}{1 + t^2} \end{aligned}$$

$$\begin{aligned} & = \int \frac{s}{4(\frac{1-t^2}{1+t^2}) + 3(\frac{2t}{1+t^2})} \times \frac{2}{1+t^2} dt = \int \frac{10}{4(1-t^2) + 3(2t)} dt \\ & = \int \frac{10}{4-4t^2+6t} dt = \int \frac{-5}{2t-2t^2-2} dt = \int \frac{-5}{(2t-1)(t-2)} dt \\ & \text{PARTIAL FRACTIONS BY INSPECTION} \\ & = \int \frac{1}{t-2} + \frac{2}{2t-1} dt = \ln|t-2| + \ln|2t-1| + C \\ & = \ln \left| \frac{2t-1}{t-2} \right| + C = \ln \left| \frac{2\sin x + 1}{\cos x - 2} \right| + C \\ & \text{MANUFACTURE FRACTION WITH SINE} \\ & = \ln \left| \frac{\frac{2\cos x}{\cos x} \cdot \frac{2\sin x + 1}{\cos x}}{\frac{\cos x}{\cos x} \cdot \frac{\cos x - 2}{\cos x}} \right| + C = \ln \left| \frac{2\cos x \cdot 2\sin x + \cos x}{\cos x \cdot \cos x - 2\cos x} \right| + C \\ & = \ln \left| \frac{2\sin x \cos x + \cos^2 x}{\cos^2 x - 2\cos x} \right| + C \\ & = \ln \left| \frac{2\sin x \cos x + \cos^2 x}{\cos^2 x - 2\cos x} \right| + C \\ & \text{OR, INSTEAD OF MULTIPLYING BY } \cos \frac{x}{2}, \text{ MULITPLY BY } \sin \frac{x}{2}. \text{ INSTEAD} \\ & = \dots \ln \left| \frac{2\sin^2 x + \cos^2 x}{\sin^2 x - \cos^2 x} \right| + C = \ln \left| \frac{2\sin^2 x + \cos^2 x}{\sin^2 x - 2\cos^2 x} \right| + C \\ & = \ln \left| \frac{2\sin^2 x + \cos^2 x}{\sin^2 x - \cos^2 x} \right| + C \end{aligned}$$

$= \ln \left| \frac{4\sin^2 x + \sin^2 x}{2\sin^2 x - 3\cos^2 x} \right| + C = \ln \left| \frac{4(1-\frac{1}{4}\cos^2 x) + \cos^2 x}{2(1-\frac{3}{4}\cos^2 x) - 3\cos^2 x} \right| + C$

$= \ln \left| \frac{\sin^2 x - 2\cos^2 x + 2}{t - 2\cos x - \cos^2 x} \right| + C$

ANOTHER ALTERNATIVE

$\cos(2x-\pi) \equiv \cos(2x) + \sin(2x)$

BY INSPECTION

$4\cos x + 3\sin x \equiv 5\cos(2x-\pi)$

$\equiv 5\cos(2x) + 5\sin(2x)$

SO, $\cos x + \sin x = \frac{1}{\sqrt{2}} \cos(2x - \frac{\pi}{4})$

$$\begin{aligned} & \int \frac{s}{4\cos x + 3\sin x} dx = \int \frac{s}{5\cos(2x-\pi)} dx = \int \sec(2x-\pi) dx \\ & = \ln|\sec(2x-\pi) + \tan(2x-\pi)| + C = \ln \left| \frac{1}{\cos(2x-\pi)} + \frac{\sin(2x-\pi)}{\cos(2x-\pi)} \right| + C \\ & = \ln \left| \frac{1 + \sin(2x-\pi)}{\cos(2x-\pi)} \right| + C \\ & = \ln \left| \frac{1 + \sin(2x-\pi) - (\cos x + \sin x)^2}{\cos(2x-\pi) + (\cos x + \sin x)^2} \right| + C \\ & = \ln \left| \frac{1 + \sin(2x-\pi) - (1 + \sin x \cos x + \sin^2 x)}{\cos^2 x + \sin^2 x + \cos x \sin x + \sin x \cos x} \right| + C \\ & = \ln \left| \frac{1 + \sin(2x-\pi) - (1 + \sin x \cos x + \sin^2 x)}{2 + \sin 2x} \right| + C \\ & = \ln \left| \frac{1 + \sin(2x-\pi) - (1 + \sin x \cos x + \sin^2 x)}{4\cos x + 3\sin x} \right| + C \end{aligned}$$

568. $\int e^x (2\cos x - 3\sin x) dx = \frac{1}{2} e^x (5\cos x - \sin x) + C$

$\int e^x (2\cos x - 3\sin x) dx = \dots$ CAN BE SOLVED BY THE INTEGRATION BY PARTS
METHOD OR COMPLEX NUMBERS

OR BY DIFFERENTIATION

$$\begin{aligned} \frac{d}{dx} [e^x (A\cos x + B\sin x)] &= e^x (A\cos x + B\sin x) + e^x (-Ax\sin x + Bx\cos x) \\ &= e^x [(A+B)x\cos x + (B-A)x\sin x] \\ \therefore A+B=2 &\quad B=3 \\ \therefore A=1 & \quad B=3 \\ \therefore \int e^x (2\cos x - 3\sin x) &= \frac{1}{2} e^x (5\cos x - \sin x) + C \end{aligned}$$

569. $\int \log_2 x \, dx = \frac{1}{\ln 2} (x \ln x - x) + C$

$$\begin{aligned}\int \log_2 x \, dx &= \int \frac{\log_2 x}{\ln 2} \, dx = \frac{1}{\ln 2} \int \log_2 x \, dx \\ \text{STANDARD RESULT OF DIFFERENTIATION BY PARTS IN NOTATION } \ln \ln x \\ &= \frac{1}{\ln 2} [x \ln x - x] + C\end{aligned}$$

570. $\int \sqrt{1+\cos 3x} \, dx = \frac{2\sqrt{2}}{3} \sin\left(\frac{3}{2}x\right) + C$

$$\begin{aligned}\int \sqrt{1+\cos 3x} \, dx &= \int \sqrt{1+(2\cos^2 \frac{3}{2}x - 1)} \, dx = \int \sqrt{2\cos^2 \frac{3}{2}x} \, dx \\ \text{USING "TEAR-OFF" APPROX} \\ &= \int \sqrt{2} \cos \frac{3}{2}x \, dx = \frac{\sqrt{2}}{3} \sin \frac{3}{2}x + C\end{aligned}$$

571. $\int \frac{x^3}{1+x^8} \, dx = \frac{1}{4} \arctan(x^4) + C$

$$\begin{aligned}\int \frac{x^3}{1+x^8} \, dx &= \dots \text{BY NUMERICAL SPLIT } \frac{d}{dx}(\arctan(2x)) = \frac{1}{1+4x^2} \times 2 \\ &= \frac{1}{4} \arctan(2x) + C\end{aligned}$$

572. $\int \ln(x+\sqrt{x}) \, dx = x \ln(x+\sqrt{x}) - \ln(1+\sqrt{x}) - x + \sqrt{x} + C$

$$\begin{aligned}\int \ln(x+\sqrt{x}) \, dx &= \dots \text{RELT BY A LOCAL SUBSTITUTION} \\ &= \int \ln(u^2+u) (2u \, du) = \int 2u \ln(u^2+u) \, du \\ \text{NOW DIFFERENTIATION BY PARTS} \\ &= u^2 \ln(u^2+u) - \int \frac{2u^2+2u}{u^2+u} \, du \\ &= u^2 \ln(u^2+u) - \int \frac{2u^2+u}{u^2+u} \, du \\ \text{MANIPULATION OR LONG DIVISION} \\ &= u^2 \ln(u^2+u) - \int \frac{2u^2+u+1-u-1}{u^2+u} \, du \\ &= u^2 \ln(u^2+u) - \int 2u-1 + \frac{1-u}{u^2+u} \, du \\ &= u^2 \ln(u^2+u) - u^2 + u - \ln(u+1) + C \\ &= 2u \ln(u^2+u) - u + \sqrt{u} - \ln(1+\sqrt{u}) + C\end{aligned}$$

573. $\int x \sin x \sin 2x \, dx = \frac{2}{9} (3x \sin^3 x + 3 \cos x - \cos^3 x) + C$

$$\begin{aligned}\int x \sin x \sin 2x \, dx &= \int 2x \sin x (\sin x \cos x) \, dx = \int 2x \sin x \cos x \, dx \\ &\stackrel{\text{INTEGRATION BY PARTS}}{=} \left[2x \sin x - \frac{2}{3} \int \sin^3 x \, dx \right] \\ &= 2x \sin x - \frac{2}{3} \int \sin x (1 - \cos^2 x) \, dx \\ &= 2x \sin x - \frac{2}{3} \int \sin x - \sin x \cos^2 x \, dx \\ &\stackrel{\text{BY SUBSTITUTION}}{=} 2x \sin x - \frac{2}{3} \left[\cos x + \frac{1}{3} \cos^3 x \right] + C \\ &= 2x \sin x + \frac{2}{3} \cos x - \frac{2}{9} \cos^3 x + C \\ &= \frac{2}{9} [3x \sin x + 3 \cos x - \cos^3 x] + C\end{aligned}$$

574. $\int \frac{\cos x}{\sin x + 2 \cos x} \, dx = \frac{2}{5} x + \frac{1}{5} \ln |\sin x + 2 \cos x| + C$

$$\begin{aligned}\int \frac{\cos x}{\sin x + 2 \cos x} \, dx &= \int \frac{(A \cos x + B \sin x)}{\sin x + 2 \cos x} \, dx + \int \frac{B(\cos x - 2 \sin x)}{\sin x + 2 \cos x} \, dx \\ &\stackrel{\text{THE COEFFICIENTES}}{=} A + B \ln |\sin x + 2 \cos x| + C \\ &\quad \text{THIS WILL BE OF THE FORM NEEDED FOR DIFFERENTIATION TO EXP} \\ &\quad [(A-2B)\sin x + (2A+B)\cos x] \equiv \cos x \\ &\quad A-2B=0 \quad | \quad 2A+B=1 \\ &\quad A=2B \quad | \quad 4B+B=1 \\ &\quad B=\frac{1}{5} \quad | \quad A=\frac{2}{5} \\ &\dots = \frac{2}{5} x + \frac{1}{5} \ln |\sin x + 2 \cos x| + C\end{aligned}$$

575. $\int \sqrt{\frac{x-2}{4-x}} \, dx = \arcsin(x-3) - \sqrt{(4-x)(x-2)} + C$

$$\begin{aligned}\int \sqrt{\frac{x-2}{4-x}} \, dx &= \int \sqrt{\frac{(x-2)(x-3)}{(4-x)(x-3)}} \, dx = \int \sqrt{\frac{(x-2)^2}{(4-x)(x-3)^2}} \, dx \\ &= \int \frac{x-2}{\sqrt{(4-x)(x-3)^2}} \, dx = \int \frac{x-2}{\sqrt{-(x^2-7x+12)}} \, dx \\ &= \int \frac{x-2}{\sqrt{1-(x-3)^2}} \, dx = \int \frac{1}{\sqrt{1-(x-3)^2}} \, dx + \frac{x-3}{\sqrt{1-(x-3)^2}} \, dx \\ &\stackrel{\text{STANDARD FORM}}{=} \arcsin(x-3) + \frac{x-3}{\sqrt{1-(x-3)^2}} \, dx \\ &= \int \frac{1}{\sqrt{1-(x-3)^2}} \, dx + (x-3) \left[1 - (x-3)^2 \right]^{-\frac{1}{2}} \, dx \\ &= \arcsin(x-3) - \left[(1 - (x-3)^2)^{\frac{1}{2}} \right] + C \\ &= \arcsin(x-3) - \sqrt{(4-x)(x-2)} + C \quad \text{FROM FORMULA}\end{aligned}$$

576. $\int \frac{\sinh^3 x}{\cosh^2 x} dx = \cosh x + \operatorname{sech} x + C$

$$\begin{aligned}\int \frac{\sinh^3 x}{\cosh^2 x} dx &= \int \frac{\sinh \sinh^2 x}{\cosh^2 x} dx = \int \frac{\sinh(\cosh^2 x - 1)}{\cosh^2 x} dx \\&= \int \frac{\cosh^2 x - 1}{\cosh^2 x} \operatorname{sech} x dx = \operatorname{sech} x - \frac{1}{\cosh x} \operatorname{sech} x dx \\&= \int \operatorname{sech} x - \tanh \operatorname{sech} x dx = \operatorname{cosec} x + \operatorname{sech} x + C\end{aligned}$$

ALTERNATIVE BY SUBSTITUTION:

$$\begin{aligned}\int \frac{\sinh^3 x}{\cosh^2 x} dx &= \int \frac{\sinh^3 u}{u^2} \operatorname{sech} u du = \int \frac{\sinh^3 u}{u^2} du \\&= \int \frac{\cosh^2 u - 1}{u^2} du = \int \frac{u^2 - 1}{u^2} du = \int 1 - \frac{1}{u^2} du \\&\approx u + \frac{1}{u} + C = \operatorname{cosec} x + \frac{1}{\cosh x} + C = \operatorname{cosec} x + \operatorname{sech} x + C\end{aligned}$$

577. $\int \frac{\sec^2 x}{4 + \tan^2 x} dx = \frac{1}{2} \arctan\left(\frac{1}{2} \tan x\right) + C$

$$\begin{aligned}\int \frac{\sec^2 x}{4 + \tan^2 x} dx &= \dots \text{RECOGNISE THAT } \frac{d}{dx}(\operatorname{arctan}(4\tan x)) = \frac{4\sec^2 x}{1+16\tan^2 x} \\&\text{IF } 4\tan x \equiv u \Rightarrow \frac{d}{dx}(\operatorname{arctan}(4\tan x)) = \frac{4\sec^2 x}{1+u^2} \\&= \frac{4\sec^2 x}{1+16\tan^2 x} \\&\therefore \int \frac{\sec^2 x}{4 + \tan^2 x} dx = \frac{1}{4} \operatorname{arctan}(4\tan x) + C\end{aligned}$$

ALTERNATIVE BY SUBSTITUTION:

$$\begin{aligned}\int \frac{\sec^2 x}{4 + \tan^2 x} dx &= \int \frac{\sec^2 x}{4 + u^2} \operatorname{sech} u du = \int \frac{1}{4 + u^2} du \\&\approx \int \frac{1}{u^2+2^2} du = \frac{1}{2} \operatorname{arctan}\left(\frac{u}{2}\right) + C \\&= \frac{1}{2} \operatorname{arctan}\left(\frac{1}{2} \tan x\right) + C\end{aligned}$$

578. $\int \frac{4x+5}{x^2+2x+2} dx = 2\ln(x^2+2x+2) + \arctan(x+1) + C$

$$\begin{aligned}\int \frac{4x+5}{x^2+2x+2} dx &= \dots \text{COMPLETE THE SQUARE AND ARCTAN BY SPLITTING} \\&= \int \frac{(2x+1)^2+4}{(2x+1)^2+1} dx = \int \frac{4x(x+1)+4}{(x+1)^2+1} dx \\&\quad \text{(COMPLEX INTEGRATE(SPLIT INTO TWO))} \\&= 2\ln(x^2+2x+2) + \arctan(x+1) + C \\&= 2\ln(x^2+2x+2) + \arctan(x+1) + C\end{aligned}$$

579. $\int \frac{1}{(x-2)^{\frac{1}{2}}(4-x)^{\frac{1}{2}}} dx = \arcsin(x-3) + C$

$$\begin{aligned} \int \frac{1}{(x-2)(4-x)^{\frac{1}{2}}} dx &= \int \frac{1}{x-2+x+4-x^2} dx = \int \frac{1}{\sqrt{-(x^2-6x+8)}} dx \\ &= \int \sqrt{\frac{1}{-(x-2)(x-4)}} dx = \int \frac{1}{\sqrt{1-(x-3)^2}} dx = \arcsin(x-3) + C \end{aligned}$$

580. $\int \frac{x^3}{x-2} dx = \frac{1}{2}x^3 + x^2 + 4x + 8\ln|x-2| + C$

By long division or manipulation:

$$\begin{aligned} \int \frac{x^3}{x-2} dx &= \int \frac{2(x-2)+2(x-2)+4(x-2)+B}{x-2} dx = \int x^2+4+\frac{B}{x-2} dx \\ &= \int x^2 dx + \int 4 dx + \int \frac{B}{x-2} dx \end{aligned}$$

THE SUBSTITUTION $u=x-2$ ADDS MORE DULL

581. $\int x^5 e^{x^3} dx = \frac{1}{3} e^{x^3} (x^3 - 1) + C$

$$\begin{aligned} \int x^5 e^{x^3} dx &= \int x^2 e^{x^3} dx = \dots \text{ INTEGRATE BY PARTS} \quad \left[\frac{x^3}{e^{x^3}} \right] \left[\frac{dx^3}{dx^2} \right] \\ &= \frac{1}{3} x^2 e^{x^3} - \int x^2 e^{x^3} dx \\ &= \frac{1}{3} x^2 e^{x^3} - \frac{1}{3} e^{x^3} + C \\ &= \frac{1}{3} e^{x^3} (x^3 - 1) + C \end{aligned}$$

THE SECTIONAL $x^2 e^{x^3}$ PART, SWAPS THE WHOLE EXPONENT SO THE INTEGRATION BY PARTS WHICH FOLLOWS IS A FUTILITY

582. $\int \frac{x^2 - 12x - 6}{(x-3)(x^2+2)} dx = 2\ln(x^2+2) - 3\ln|x-3| + C$

$$\begin{aligned} \int \frac{x^2 - 12x - 6}{(x-3)(x^2+2)} dx &= \dots \text{ BY PARTIAL FRACTION} \dots \\ \frac{x^2 - 12x - 6}{(x-3)(x^2+2)} &\approx \frac{A}{x-3} + \frac{Bx+C}{x^2+2} \\ x^2 - 12x - 6 &\equiv A(x^2+2) + (x-3)(Bx+C) \\ x^2 - 12x - 6 &\equiv (A+B)x^2 + (C-3B)x + 2A \end{aligned}$$

Now we have, • IF $A=3$ $\Rightarrow 9-36+6=11A \Rightarrow A=-1$ $\bullet A+B=0 \Rightarrow -3+B=1 \Rightarrow B=4$
 $\bullet C-3B=0 \Rightarrow C-12=0 \Rightarrow C=12$

$$\dots = \int -\frac{2}{x-3} + \frac{4x+12}{x^2+2} dx = \left[2 \left(\frac{2}{x-3} - \frac{1}{x^2+2} \right) dx + 2\ln(x^2+2) - 3\ln|x-3| + C \right]$$

583. $\int \sin x \sin^3\left(\frac{1}{2}x\right) dx = \frac{4}{5} \sin^5\left(\frac{1}{2}x\right) + C$

$$\int \sin x \sin^3\left(\frac{1}{2}x\right) dx = \int (2\sin\frac{x}{2}\cos\frac{x}{2}) \sin^3\frac{x}{2} dx = \int 2\sin^4\frac{x}{2} \cos\frac{x}{2} dx$$

Simplifies to $\sin^3(2\sin\frac{x}{2})$

(Recall)

$$= \frac{1}{2} \times 2 \sin^5\frac{x}{2} + C = \frac{1}{2} \sin^5\frac{x}{2} + C$$

584. $\int \frac{\cot x}{\sqrt{1-\cos 2x}} dx = -\frac{1}{\sqrt{2}} \operatorname{cosec} x + C$

$$\int \frac{\cot x}{\sqrt{1-\cos 2x}} dx = \int \frac{\cot x}{\sqrt{1-(1-2\sin^2 x)}} dx = \int \frac{\cot x}{\sqrt{2\sin^2 x}} dx$$

Recall $\cos 2x = 1 - 2\sin^2 x$

$$= \int \frac{1}{\sqrt{2}} \frac{\cot x}{\sin x} dx = \frac{1}{\sqrt{2}} \int \operatorname{cosec} x \cot x dx = -\frac{1}{\sqrt{2}} \operatorname{cosec} x + C$$

585. $\int \tan^5 x \sec x dx = \frac{1}{5} \sec^5 x - \frac{2}{3} \sec^3 x + \sec x + C$

$$\int \tan^5 x \sec x dx = \int \tan^4 x (\sec x) dx = \int (\sec^2 x)^2 \sec x dx$$

Differential of sec x

$$= \int (\sec^2 x - 2\sec x)(\sec x) dx = \int \sec^3 x (\sec x - 2) dx$$

$$= \frac{1}{3} \sec^3 x - \frac{2}{3} \sec x + \sec x + C$$

586. $\int \frac{2\cosh x \sinh x}{1+\cosh^4 x} dx = \arctan(\cosh^2 x) + C$

$$\int \frac{2\cosh x \sinh x}{1+\cosh^4 x} dx = \dots$$

By Reciprocal of Adj Recip

$$= \arctan(\cosh^2 x) + C$$

Alternative by Substitution

| |
|---|
| $u = \cosh^2 x$ |
| $du = 2\cosh x \sinh x dx$ |
| $du = \frac{2\cosh x \sinh x}{2\cosh^2 x} du$ |

$$\dots = \int \frac{2\cosh x \sinh x}{1+u^2} \times \frac{du}{2\cosh^2 x}$$

$$= \int \frac{1}{1+u^2} du = \arctan u + C$$

$$= \arctan(\cosh^2 x) + C$$

587. $\int \frac{\cos^3 x}{\sin^2 x} dx = \sin x - \operatorname{cosec} x + C$

$$\begin{aligned}\int \frac{\cos^3 x}{\sin^2 x} dx &= \int \frac{\cos x \cos^2 x}{\sin^2 x} dx = \int \frac{\cos x(1-\sin^2 x)}{\sin^2 x} dx \\&= \int \frac{\cos x - \cos x \sin^2 x}{\sin^2 x} dx = \int \frac{\cos x}{\sin^2 x} - \cos x dx \\&= \int (\cot x - \operatorname{cosec} x) dx = -\operatorname{cosec} x + \sin x + C \\&\quad \text{if } (\operatorname{cosec} x) = -\operatorname{cosec} x \quad = \sin x - \operatorname{cosec} x + C\end{aligned}$$

588. $\int 5 \sin\left(\frac{1}{3}x\right) \sqrt{\cos\left(\frac{1}{6}x\right)} dx = -24 \sqrt{\cos^5\left(\frac{1}{6}x\right)} + C$

$$\begin{aligned}\int 5 \sin\left(\frac{1}{3}x\right) \sqrt{\cos\left(\frac{1}{6}x\right)} dx &= \dots \text{NOTE THE DOUBLE ANGLE} \\&= \int 5(2\sin\left(\frac{1}{3}x\right)(\cos\left(\frac{1}{6}x\right))^{\frac{1}{2}})^2 dx = \int 10 \sin^2\left(\frac{1}{3}x\right) \cos\left(\frac{1}{6}x\right)^2 dx \\&\dots \text{BY SUBSTITUTION} \dots \\&= -\left(\cos\left(\frac{1}{6}x\right)\right)^{\frac{5}{2}} 10 \times \frac{1}{3} \times 6 + C = -24 \left(\cos\left(\frac{1}{6}x\right)\right)^{\frac{3}{2}} + C \\&\qquad\qquad\qquad = -24 \sqrt{\cos^2\left(\frac{1}{6}x\right)} + C\end{aligned}$$

589. $\int \arcsin \sqrt{x} dx = \frac{1}{2}(2x-1) \arcsin \sqrt{x} + \frac{1}{2} \sqrt{x-x^2} + C$

$$\begin{aligned}\int \arcsin \sqrt{x} dx &\dots \text{BY SUBSTITUTION} \dots \\&= \int \theta \sin \theta d\theta = \int \theta \sin \theta d\theta \quad \begin{array}{|l} \theta = \arcsin \sqrt{x} \\ d\theta = \frac{1}{\sqrt{1-x^2}} dx \\ dx = \sqrt{1-x^2} d\theta \end{array} \\&\text{INTEGRATION BY PARTS} \\&\quad \begin{array}{|c|c|} \hline \theta & 1 \\ \hline \frac{d\theta}{dx} & \frac{1}{\sqrt{1-x^2}} \\ \hline \end{array} \quad \begin{array}{l} = -\frac{1}{2} \theta \cos \theta + \int \frac{1}{2} \cos \theta d\theta \\ = -\frac{1}{2} \theta (-2\sin \theta) + \frac{1}{2} \sin \theta + C \\ = -\frac{1}{2} \theta + 2\sin \theta + \frac{1}{2} \sin \theta + C \\ = -\frac{1}{2} \theta \sin \theta + 2\cos \theta + \frac{1}{2} \sqrt{1-x^2} + C \\ = \frac{1}{2}(2x-1) \arcsin \sqrt{x} + \frac{1}{2} \sqrt{x-x^2} + C \end{array}\end{aligned}$$

590. $\int \sqrt{x^2 + 2x + 2} dx = \frac{1}{2} \ln \left[x + 1 + \sqrt{x^2 + 2x + 2} \right] + \frac{1}{2}(x+1)\sqrt{x^2 + 2x + 2} + C$

$$\begin{aligned} & \int \sqrt{x^2 + 2x + 2} dx = \dots \text{COMPLETE THE SQUARE AND SUBSTITUTE} \\ & = \int \sqrt{(x+1)^2 + 1} dx = \int \sqrt{u^2 + 1} \cosh u du \\ & = \int u \sinh u du = \int \frac{1}{2} + \frac{1}{2} \tanh^2 u du \\ & = \frac{1}{2}u + \frac{1}{2}\tanh u - C = \frac{1}{2}u + \frac{1}{2}\tanh(u) + C \\ & = \frac{1}{2}(x+1) + \frac{1}{2}\tanh(x+1)(x^2+2x+2)^{\frac{1}{2}} + C \\ & = \frac{1}{2} \ln \left[x+1 + \sqrt{x^2+2x+2} \right] + \frac{1}{2}(\tanh(x+1)(x^2+2x+2)^{\frac{1}{2}}) + C \\ & = \frac{1}{2} \ln(x+1 + \sqrt{x^2+2x+2}) + \frac{1}{2}\tanh(x+1)\sqrt{x^2+2x+2} + C \end{aligned}$$

591. $\int \frac{\cos x}{1+\sin^2 x} dx = \arctan(\sin x) + C$

$$\int \frac{\cos x}{1+\sin^2 x} dx = \dots \text{BY INSPECTING ... } \underline{\arctan(\sin x) + C}$$

THE SUBSTITUTION $u = \sin x$ WORKS WELL.

592. $\int \frac{x^2 - 2x - 2}{1-x^3} dx = \ln \left| \frac{x-1}{x^2+x+1} \right| + C$

$$\begin{aligned} & \int \frac{x^2 - 2x - 2}{1-x^3} dx = \int \frac{x^2 - 2x - 2}{(1-x)(1+x+x^2)} dx \\ & \sim \text{PARTIAL FRACTIONS} \\ & \frac{x^2 - 2x - 2}{(1-x)(1+x+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x+x^2} \\ & x^2 - 2x - 2 = A(1+x+x^2) + (-x)(Bx+C) \\ & x^2 - 2x - 2 = (A-B)x^2 + (A+B-C)x + (C-A) \\ & \begin{array}{l} A=1, \quad 3A=-2 \\ \hline A=-\frac{2}{3}, \quad C=1 \end{array} \\ & x^2 - 2x - 2 = (-1-\frac{2}{3})x^2 + (\frac{2}{3}-1)x + (1-1) \\ & \therefore B=\frac{2}{3}, \quad C=1 \end{aligned}$$

$$\begin{aligned} & = \int \frac{-\frac{1}{3}}{1-x} + \frac{-\frac{2}{3}x+1}{1+x+x^2} dx = \int \frac{\frac{1}{3}}{x-1} - \frac{\frac{2}{3}x+1}{x^2+2x+1} dx \\ & = \ln|x-1| - \ln|x^2+2x+1| + C = \ln \left| \frac{|x-1|}{|x^2+2x+1|} \right| + C \end{aligned}$$

593. $\int \frac{3x^3 - 4x^2 + 5x - 2}{(x-1)(x^2+1)} dx = 3x + \ln \left| \frac{x-1}{x^2+1} \right| + C$

$$\int \frac{3x^3 - 4x^2 + 5x - 2}{(x-1)(x^2+1)} dx = \dots$$
 INVERSE PARTIAL FRACTION WITH IRREDUCIBLE FACTOR

$$\frac{3x^3 - 4x^2 + 5x - 2}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$3x^3 - 4x^2 + 5x - 2 \equiv 3A + (Bx+C)(x^2+1) + (Ax^2+Bx+C)(x-1)$$

- If $x=1$ • If $x=0$ • If $x=2$
- $2A+2 \quad -2 = -3A+C \quad 2A+2$
- $A=1 \quad C=0 \quad -4 = 2C+2$
- $-4 = 2B \quad -4 = 2B$
- $B=-2 \quad B=-2$

$$= \int \frac{3 + \frac{1}{x-1} - \frac{2x}{x^2+1}}{x^2-x-2} dx = x + \ln|x-1| - \ln|x^2+1| + C$$

From boundary
to term

$$= 3x + \ln \left| \frac{x-1}{x^2+1} \right| + C$$

594. $\int \frac{8x^2 - 29x - 10}{(x^2 - x - 2)^2} dx = \frac{x+10}{x^2 - x - 2} + 3 \ln \left| \frac{x-2}{x+1} \right| + C$

$$\int \frac{8x^2 - 29x - 10}{(x^2 - x - 2)^2} dx = \int \frac{8x^2 - 29x - 10}{(x+1)(x-2)^2} dx$$

PARTIAL FRACTIONS WITH TWO IDENTICAL REPEATED FACTORS

$$\frac{8x^2 - 29x - 10}{(x+1)(x-2)^2} \equiv \frac{A}{x+1} + \frac{B}{(x-2)^2} + \frac{C}{x-2}$$

$$8x^2 - 29x - 10 \equiv A(x-2)^2 + B(x+1)^2 + C(x-2)(x+1) + D(x+1)(x-2)$$

- If $x=-2$ • If $x=1$ • If $x=3$
- $32-58-10 = 10 \quad B(-2+1)^2 = 9A \quad 2x-27-10 = A+16C+4D$
- $-56 = -16C-40 \quad 4 = 9A \quad -27 = -16+4C+4D$
- $-56+4C = -36 \quad A = 4 \quad 4C = 16-36$
- $16+4C = -9 \quad C = -2 \quad C+4D = 4$

SECOND EQUATION

$$C+4D = 4 \quad \Rightarrow \quad C+4D = 4 \quad \Rightarrow \quad -16C = 45$$

$$4C+D = -9 \quad \Rightarrow \quad -16-40 = 36 \quad C = -3$$

$$D = 3$$

$$\dots = \int \frac{\frac{3}{x+1} + \frac{1}{(x-2)^2} - \frac{3}{x-2}}{(x+1)(x-2)^2} dx$$

$$= -\frac{3}{x+1} + \frac{4}{x-2} + 3 \ln \left| \frac{x-2}{x+1} \right| + C$$

$$= \frac{4x+10-3x-6}{x^2-x-2} + 3 \ln \left| \frac{x-2}{x+1} \right| + C$$

$$= \frac{x+10}{x^2-x-2} + 3 \ln \left| \frac{x-2}{x+1} \right| + C$$

595. $\int \sqrt{\frac{1-x}{1+x}} dx = \begin{cases} \arcsin x + \sqrt{1-x^2} + C \\ \arccos x - \sqrt{1-x^2} + C \\ -\arcsin x - \sqrt{1-x^2} + C \\ -\arccos x + \sqrt{1-x^2} + C \end{cases}$

$$\begin{aligned} \int \sqrt{\frac{1-x}{1+x}} dx &= \int \sqrt{\frac{(1-x)^2}{(1+x)(1-x)}} dx = \text{REMEMBER MODULUS} = \int \frac{1-x^2}{\sqrt{1-x^2}} dx \\ &\quad \text{DO 2 SEPARATE INTEGRALS AT THE END} \\ &= \int \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} - x \sqrt{1-x^2} dx \\ &= \arcsin x + (-x) \sqrt{1-x^2} + C \\ &= \arcsin x - x \sqrt{1-x^2} + C \quad \leftarrow \text{IF NEGATIVE, } x \mapsto -x \\ \text{ALTERNATIVE BY SUBSTITUTION} \\ 2 = \cos 2\theta \quad 16 = 64 \cos^2 2\theta \\ dx = -2 \sin 2\theta d\theta &\quad \dots = \int \frac{1-\cos 2\theta}{\sqrt{1-\cos 2\theta}} (-2 \sin 2\theta) d\theta \\ &= \int \sqrt{\frac{1-(1-2\sin^2\theta)}{1+2\cos 2\theta}} (-2 \sin 2\theta) d\theta \\ &\quad \text{WHERE MODULUS IS DO THE SEPARATE INTEGRALS} \\ &= \int \frac{\sin^2\theta}{\cos 2\theta} (-4\sin^2\theta) d\theta = \int -4\sin^2\theta d\theta = \int -(3 - \frac{1}{2}\cos 4\theta) d\theta \\ &= \int -2 + 2\cos 2\theta d\theta = -2\theta + \sin 2\theta + C \\ &= -2\theta + \sqrt{1-\cos^2 2\theta} + C = -2(\arccos x) + \sqrt{1-x^2} + C \\ &= -\arccos x + \sqrt{1-x^2} + C \\ &= \arcsin x + \sqrt{1-x^2} + C \quad \text{arcsin x + arccos x = } \frac{\pi}{2} \\ \text{OR IF WE TAKE THE NEGATIVE SQUARE ROOT} \\ &= \tan 2\theta = \frac{\sqrt{1-x^2}}{x} + C \\ &= \arctan x = \sqrt{1-x^2} + C \end{aligned}$$

596. $\int \frac{4 \tan x}{1+\cos^2 x} dx = 2 \ln(1+\sec^2 x) + C$

$$\begin{aligned} \int \frac{4 \tan x}{1+\cos^2 x} dx &\quad \dots \text{BY SUBSTITUTION} \\ &= \int \frac{4 \tan x}{1+\cos^2 x} dx = \int \frac{4 \tan x}{(1+\cos^2 x) \cos^2 x} dx \\ &= \int \frac{4 \tan x}{(1+\cos^2 x) \cos^2 x} dx = \int \frac{4}{\cos^2 x} dx \quad \dots \text{BY PARTIAL FRACTION} \\ &\dots = \int -\frac{4}{u} + \frac{4}{u^2} du \\ &= 2 \ln(u) - 4 \ln(u) + C \\ &= 2 [\ln(u^2) - \ln(u^2)] + C \\ &= 2 \ln\left(\frac{u^2}{u}\right) + C = 2 \ln\left(\frac{1+\cos^2 x}{\cos^2 x}\right) + C \\ &= 2 \ln(\sec^2 x) + C \\ \text{ALTERNATIVE SUBSTITUTION} \\ \int \frac{4 \tan x}{1+\cos^2 x} dx &= \int \frac{4 \tan x}{1+\cos^2 x} dx = \int \frac{4 \sin x}{1+\cos^2 x} dx \\ \boxed{u = 1+\cos^2 x} \\ \frac{du}{dx} = -2\cos x \tan x &\quad \dots = \int \frac{4 \sin x}{u} \times \frac{du}{-2\cos x \tan x} \\ dx = -\frac{du}{2\cos x \tan x} &= \int \frac{-2}{u} du = \int \frac{-2}{u(u-1)} du \\ &= \int \frac{2}{u-1} du \\ \text{BY PARTIAL FRACTION DECOMPOSITION} \\ &= \int \frac{2}{u-1} + \frac{2}{u} du = 2 \ln|u| - 2 \ln|u-1| + C \\ &= 2 [\ln|u| - \ln|u-1|] = 2 \ln\left|\frac{u}{u-1}\right| + C \\ &= 2 \ln\left|\frac{1+\cos^2 x}{(-1+\cos^2 x)}\right| + C = 2 \ln\left|\frac{1+\cos^2 x}{\cos^2 x}\right| + C \\ &= 2 \ln\left(\frac{1+\cos^2 x}{\cos^2 x}\right) + C = 2 \ln(\sec^2 x) + C \end{aligned}$$

597. $\int \frac{1}{x^2\sqrt{x^2-1}} dx = \frac{\sqrt{x^2-1}}{x} + C$

$$\begin{aligned} \int \frac{1}{x^2\sqrt{x^2-1}} dx &= \dots \text{ By substitution } \dots \\ &= \int \frac{1}{x^2\sqrt{x^2-1}} (\sin\theta dx) = \int \frac{\sin\theta}{x^2\sqrt{x^2-1}} d\theta \\ &= \int \sin\theta d\theta = -\cos\theta + C = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{x^2-1}}{x} + C \\ &= \frac{\sqrt{x^2-1}}{x} + C \end{aligned}$$

ACCELERATED SUBSTITUTION FOR THESE TYPE OF INTEGRALS

$$\begin{aligned} \int x^2\sqrt{x^2-1} dx &= \int \frac{1}{\frac{1}{x^2}\sqrt{\frac{x^2-1}{x^2}}} (-\frac{1}{x^2} dx) \\ &= \int \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \frac{1}{x^2} dx = \int \frac{-1}{\sqrt{1-\frac{1}{u^2}}} du \\ &= \int -u(1-u^2)^{-\frac{1}{2}} du = (-u^2)^{\frac{1}{2}} + C = (-\frac{1}{u})^{\frac{1}{2}} + C \\ &= \left(\frac{u^2-1}{u^2}\right)^{\frac{1}{2}} + C = \frac{\sqrt{u^2-1}}{u} + C \end{aligned}$$

598. $\int \frac{1}{(x+1)\sqrt{x+2}} dx = \ln \left| \frac{\sqrt{x+2}-1}{\sqrt{x+2}+1} \right| + C$

$$\begin{aligned} \int \frac{1}{(x+1)\sqrt{x+2}} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{1}{(u+1)\sqrt{u+2}} (du) = \int \frac{2}{u^2+1} du \\ &= \int \frac{2}{(u+1)(u+2)} du \\ &\quad \text{PARTIAL FRACTIONS BY INSPECTION} \\ &= \int \frac{1}{u+1} - \frac{1}{u+2} du = \ln|u+1| - \ln|u+2| + C \\ &= \ln \left| \frac{1}{u+2} \right| + C = \ln \left| \frac{\sqrt{x+2}-1}{\sqrt{x+2}+1} \right| + C \end{aligned}$$

599. $\int \frac{1}{(x+6)(x-2)\sqrt{x+2}} dx = \ln \left| \frac{\sqrt{x+2}-2}{\sqrt{x+2}+2} \right| - 2 \arctan \left(\frac{\sqrt{x+2}}{2} \right) + C$

$$\begin{aligned} \int \frac{16}{(x+6)(x-2)\sqrt{x+2}} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{16}{(u+6)(u-2)\sqrt{u}} (du) = \int \frac{32}{(u^2+4)(u^2-4)} du \\ &\quad \text{PARTIAL FRACTIONS BY INSPECTION} \\ &= \int \frac{\frac{4}{u^2-4} - \frac{4}{u^2+4}}{(u^2-4)(u^2+4)} du = \int \frac{\frac{1}{u-2} - \frac{1}{u+2}}{u^2-4} du \\ &\quad \text{PARTIAL FRACTIONS BY INSPECTION AGAIN} \\ &= \int \frac{1}{u+2} - \frac{1}{u-2} - \frac{4}{u^2-4} du = \ln|u+2| - \ln|u-2| - 2 \arctan \frac{u}{2} + C \\ &= \ln \left| \frac{u-2}{u+2} \right| - 2 \arctan \left(\frac{u}{2} \right) + C = \ln \left| \frac{\sqrt{x+2}-2}{\sqrt{x+2}+2} \right| - 2 \arctan \left(\frac{\sqrt{x+2}}{2} \right) + C \end{aligned}$$

600. $\int \frac{1}{(x^2-1)\sqrt{x}} dx = \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| - \arctan\sqrt{x} + C$

$$601. \int \frac{1}{(x^2+1)\sqrt{x^2-1}} dx = \begin{cases} \frac{\sqrt{2}}{4} \ln \left| \frac{\sqrt{2}x + \sqrt{x^2-1}}{\sqrt{2}x - \sqrt{x^2-1}} \right| + C \\ \frac{\sqrt{2}}{4} \ln \left| \frac{3x^2-1+2x\sqrt{2x^2-2}}{x^2+1} \right| + C \end{cases}$$

$\int \frac{1}{(x^2+1)\sqrt{x^2-1}} dx = \dots$ BY A HYPERBOLIC SUBSTITUTION...
 $\int \frac{1}{(x^2-1)^{1/2}(x^2+1)^{1/2}} dx = \int \frac{1}{x^2-1} dt$
 $\int \frac{1}{1+t^2} dt = \int \frac{2}{3+t^2} dt = \dots$ BY USING t SUBSTITUTION
 $\Rightarrow \int \frac{2}{3+\frac{t^2+1}{t-1}} \cdot \frac{1}{t-1} dt = \int \frac{2}{3+t^2} dt$
 $\Rightarrow \int \frac{2}{t^2-1} dt = \int \frac{1}{t^2-1} dt$
 $\Rightarrow \int \frac{1}{(t^2-1)^{1/2}} dt$ BY INVERSE FUNCTION
 $\Rightarrow \int \frac{\frac{1}{2t}}{\sqrt{t^2-1}} dt = \frac{1}{2\sqrt{t^2-1}} + C$
 $\Rightarrow \frac{1}{2} \ln \left| \frac{\sqrt{t^2-1}}{t} \right| + C = \frac{1}{2} \ln \left| \frac{\sqrt{x^2-1}}{x} \right| + C$
 $\Rightarrow \frac{1}{2} \ln \left| \frac{\sqrt{x^2-1} + \sqrt{x^2-1}}{\sqrt{x^2-1} - \sqrt{x^2-1}} \right| + C = \frac{1}{2} \ln \left| \frac{\sqrt{x^2-1} + \sqrt{x^2-1}}{\sqrt{x^2-1} - \sqrt{x^2-1}} \right| + C$
 $\Rightarrow \frac{1}{2} \ln \left| \frac{\sqrt{x^2-1} + \sqrt{x^2-1}}{\sqrt{x^2-1} - \sqrt{x^2-1}} \right| + C = \frac{1}{2} \ln \left| \frac{\sqrt{x^2-1} + \sqrt{x^2-1}}{\sqrt{x^2-1} - \sqrt{x^2-1}} \right| + C$
 DETERMINE $\frac{(x^2+1)^{1/2}}{(x^2-1)^{1/2}} = \frac{x^2+1}{x^2-1} = \frac{x^2+1+2x\sqrt{x^2-1}+x^2-1}{x^2-1} = \frac{2x^2+2x\sqrt{x^2-1}+x^2-1}{x^2-1} = \frac{3x^2+2x\sqrt{x^2-1}}{x^2-1} + C$
 $\Rightarrow \frac{1}{2} \ln \left| \frac{3x^2+2x\sqrt{x^2-1}}{x^2-1} + C \right|$

ALTERNATIVE WITH ANOTHER SUBSTITUTION

$$\int \frac{1}{(u^2+1)(u^2+2)} du$$

$$= \int \frac{1}{\left(\frac{(u^2+1)^2 - u^2}{(u^2+1)^2}\right) \cdot \frac{2u}{(u^2+1)^2}} du = \frac{2u}{(u^2+1)^2(u^2+2)} du$$

$$= \int \frac{2u}{\left(\frac{(u^2+1)^2 - u^2}{(u^2+1)^2}\right)^{\frac{1}{2}} \left(\frac{(u^2+1)^2 + 2u^2}{(u^2+1)^2}\right)^{\frac{1}{2}}} du = \frac{2u}{(u^2+1)^{\frac{3}{2}}(u^2+2)^{\frac{1}{2}}} du$$

$$= \int \frac{2u}{\frac{u}{u^2+1} \cdot \frac{(u^2+1)^{\frac{3}{2}}}{(u^2+1)^{\frac{1}{2}}} \cdot (u^2+2)^{\frac{1}{2}}} du = \frac{2u}{u(u^2+1)^{\frac{1}{2}}(u^2+2)^{\frac{1}{2}}} du$$

$$= \int \frac{2u}{u^2(u^2+1)^{\frac{1}{2}}(u^2+2)^{\frac{1}{2}}} du = \frac{2u}{u^2(u^2+1)^{\frac{1}{2}}(u^2+2)^{\frac{1}{2}}} du$$

$$= \int \frac{1}{u^2(1+\frac{1}{u^2})(1+\frac{2}{u^2})^{\frac{1}{2}}} du = \frac{1}{u^2} \arcsinh(u) + C$$

$$= \frac{1}{u^2} \arcsinh\left(\sqrt{\frac{u^2+1}{u^2+2}}\right) + C \quad \text{--- PROCEED INTO U FORM}$$

$$= \frac{1}{u^2} \ln\left(\sqrt{\frac{u^2+1}{u^2+2}} + \sqrt{\frac{u^2+1}{u^2+2}+1}\right) = \frac{1}{u^2} \ln\left(\sqrt{\frac{u^2+1}{u^2+2}} + \sqrt{\frac{u^2+3}{u^2+2}}\right)^2$$

$$= \frac{1}{u^2} \ln\left(\sqrt{\frac{u^2+1}{u^2+2}} + \sqrt{\frac{u^2+3}{u^2+2}}\right)^2 = \frac{1}{u^2} \ln\left(\sqrt{\frac{u^2+1}{u^2+2}} + \sqrt{\frac{u^2+3}{u^2+2}}\right)^2$$

$$= \frac{1}{4} \frac{2}{u^3} \ln\left(\frac{\sqrt{u^2+1} + \sqrt{u^2+3}}{\sqrt{u^2+2}}\right) = \frac{1}{4} \frac{2}{u^3} \ln\left[\frac{\sqrt{u^2+1} + 2\sqrt{u^2+2}}{\sqrt{u^2+2}}\right]$$

$$= \frac{1}{4} \frac{2}{u^3} \ln\left[\frac{2\sqrt{u^2+1} + 2\sqrt{u^2+2}}{2\sqrt{u^2+2}}\right] \quad \text{At } B \rightarrow 0^+$$

Created by T. Madas

602. $\int \frac{1}{(x+1)\sqrt{x^2+4x+2}} dx = -\arcsin\left[\frac{x}{(x+1\sqrt{2})}\right] + C$

$u = \frac{x}{x+1}$
 $du = \frac{1}{(x+1)^2} dx$
 $\frac{d^2 du}{dx^2} = \frac{2}{(x+1)^3}$
 $= (x+1)^2 + 2x+1$
 $= x^2+2x+1$
 $= 1 + \frac{2x}{u^2} - 1$
 $= \frac{2}{u^2}$

$\int \frac{1}{(x+1)\sqrt{x^2+4x+2}} dx = \dots \text{ substitute} \dots$

$= \int u \times \frac{1}{\sqrt{\frac{1+2u+u^2}{u^2}}} \times \left(-\frac{1}{(x+1)^2} du\right)$
 $= \int \frac{u}{\sqrt{1+2u+u^2}} \times \frac{1}{u^2} du$
 $= \int -\frac{1}{\sqrt{1+2u+u^2}} du$

Complete the square
 $1+2u+u^2 = (u^2+2u+1) = (u+1)^2$
 $= 2(u+1)^2$

$= \int -\frac{1}{\sqrt{2(u+1)^2}} du = \int -\frac{1}{\sqrt{2(u+1)^2}} du = -\arcsin\left(\frac{u+1}{\sqrt{2}}\right) + C$
 $= -\arcsin\left(\frac{x+1}{\sqrt{2}}\right) + C = -\arcsin\left(\frac{x+1}{\sqrt{2(x+1)}}\right) + C = -\arcsin\left(\frac{\sqrt{2}}{\sqrt{2(x+1)}}\right) + C$

603. $\int \sqrt{36-x^2} dx = 18\arcsin\left(\frac{1}{6}x\right) + \frac{1}{2}x\sqrt{36-x^2} + C$

$\int \sqrt{36-x^2} dx = \dots \text{ By substitution} \dots$
 $\int \sqrt{36-36\sin^2\theta} (6\cos\theta d\theta) = \int 6\cos\theta (\cos^2\theta) (6\cos\theta d\theta)$
 $= 36\cos^3\theta (\cos\theta d\theta) = \int 36\cos^3\theta d\theta = \int 36(1-\sin^2\theta)^{1/2} d\theta = \int 36+18\sin^2\theta d\theta$
 $= 180 + 18\sin^2\theta + C = 180 + 18(1-\cos^2\theta) + C = 180 + (6\sin\theta)(3\cos\theta) + C$

$\cos\theta = \pm\sqrt{1-\frac{x^2}{36}}$
 $\cos\theta = \pm\sqrt{\frac{36-x^2}{36}} = \pm\sqrt{\frac{36-x^2}{36}}$

$= 18\arcsin\left(\frac{x}{6}\right) + x\left(\frac{1}{2}\sqrt{36-x^2}\right) + C > \underline{18\arcsin\left(\frac{x}{6}\right) + \frac{1}{2}x\sqrt{36-x^2} + C}$

604. $\int \frac{x+2}{(x^2+4x+1)^{3/2}} dx = -\sqrt{x^2+4x+1} + C$

$\int \frac{x+2}{(x^2+4x+1)^{3/2}} dx = \dots \text{ By substitution} \dots = \int (x+2)(x^2+4x+1)^{-1/2} dx$
 $= -(x^2+4x+1)^{-1/2} + C = \underline{-\sqrt{x^2+4x+1} + C}$

ALTERNATIVE BY SUBSTITUTION AFTER COMPLETING THE SQUARE ON THE DENOMINATOR

$x^2+4x+1 = (x+2)^2-3$
 $\int \frac{x+2}{((x+2)^2-3)^{3/2}} dx = \int \frac{u}{(u^2-3)^{3/2}} du$
 $= \int u(u^{-2}-3)^{-1/2} du = -\frac{1}{2}(u^{-2}-3)^{-1/2} + C = -\frac{1}{2}\frac{1}{(u^{-2}-3)^{1/2}} + C = -\frac{1}{2}\frac{1}{(x^2+4x+1)^{1/2}} + C$

605. $\int \sqrt{\frac{x}{1-4x}} dx = \frac{1}{8} \arcsin(2\sqrt{x}) - \frac{1}{4} \sqrt{x-4x^2} + C$

$$\begin{aligned} \int \sqrt{\frac{x}{1-4x}} dx &= \dots \text{ BY TRIGONOMETRIC SUBSTITUTION } \dots \\ &= \int \frac{\sqrt{4x}}{\sqrt{1-4x}} (\text{Gordon's do}) = \int \frac{\sqrt{4x}}{\sqrt{4x-4x^2}} (\text{Gordon's do}) dx \\ &= \int \frac{2\sqrt{x}}{2\sqrt{x}(1-\sqrt{x^2})} dx = \int \frac{1}{1-\sqrt{x^2}} dx \\ &\rightarrow \int \frac{1}{1-\frac{1}{4}\sin^2\theta} d\theta = \int \frac{1}{\frac{3}{4}\cos^2\theta} d\theta \\ &= \frac{4}{3} \theta - \frac{4}{3} \arctan\theta + C = \frac{4}{3} \theta - \frac{4}{3} \arctan(\sin\theta) dx \\ &= \frac{4}{3} \arcsin\sqrt{\frac{x}{4}} - \frac{4}{3} \arctan\left(\frac{\sqrt{x}}{\sqrt{4-x}}\right) + C \\ &\approx \frac{4}{3} \arcsin\left(\frac{\sqrt{x}}{2}\right) - \frac{4}{3} \sqrt{4-x} + C \\ &= \frac{4}{3} \arcsin\left(\frac{\sqrt{x}}{2}\right) - \frac{1}{3} \sqrt{4-4x^2} + C \end{aligned}$$

606. $\int \frac{2}{1+\sin x+2\cos x} dx = \ln \left[\frac{\tan\left(\frac{1}{2}x\right)+1}{\tan\left(\frac{1}{2}x\right)-3} \right] + C$

$$\begin{aligned} \int \frac{2}{1+2\sin x+2\cos x} dx &= \dots \text{ BY t-SUBSTITUTION } \dots \\ &= \int \frac{2}{1+\frac{2t}{1+t^2}+2\frac{1-t^2}{1+t^2}} dt = \int \frac{2}{3+2t^2} dt \\ &= \int \frac{4}{(1-t^2)(3+2t^2)} dt = \int \frac{4}{3+2t^2-3t^2} dt \\ &= \int \frac{-4}{t^2-3} dt = \frac{\text{PARTIAL FRACTION}}{\text{BY INTEGRATION}} = \int \frac{-4}{(t-1)(t+2)} dt \\ &= \int \frac{4}{(t-1)(t+2)} dt = \int \frac{1}{t-1} + \frac{1}{t+2} dt \\ &= \ln|t-1| - \ln|t+2| + C = \ln\left|\frac{t-1}{t+2}\right| + C \end{aligned}$$

607. $\int \frac{1}{2(x+1)\sqrt{x^2+x}} dx = \sqrt{\frac{x}{x+1}} + C$

$$\begin{aligned} \int \frac{1}{2(x+1)\sqrt{x^2+x}} dx &= \dots \text{ BY SUBSTITUTION } \dots \\ &= \int \frac{1}{2\left(\frac{u}{u+1}\right)} \sqrt{\frac{u+1}{u}} \left(-\frac{1}{u^2} du\right) \\ &= \int \frac{1}{2} \frac{1}{u} \frac{\sqrt{u+1}}{u} \left(-\frac{1}{u^2}\right) du = \int \frac{-1}{2u\sqrt{u+1}} du \\ &= -\frac{1}{2} \int (u+1)^{-\frac{1}{2}} du = -\frac{1}{2} u(u+1)^{-\frac{1}{2}} + C \\ &= (u+1)^{\frac{1}{2}} + C = \frac{1}{\sqrt{u+1}} + C \\ &= \frac{1}{\sqrt{\frac{u+1}{u}}} + C = \sqrt{\frac{u+1}{u}} + C \end{aligned}$$

- $u = \frac{1}{x}$
- $x = \frac{1}{u}$
- $du = -\frac{1}{u^2} du$
- $x+1 = \frac{u+1}{u}$
- $x^2+1 = \frac{u+1}{u} + 1$
- $x^2+1 = \frac{u+1+u}{u}$
- $x^2+1 = \frac{2u+1}{u}$
- $u+1 = \frac{u+1}{u} + 1$
- $u+1 = \frac{u+1}{u} + 1$

608. $\int \frac{\ln(2x-1)}{2x-1} dx = \frac{1}{2} [\ln(2x-1)]^2 + C$

$$\int \frac{\ln(2x-1)}{2x-1} dx = \dots \text{ BY INTEGRATION } \dots = \frac{1}{2} [\ln(2x-1)]^2 + C$$

TOP SUBSTITUTION $u=2x-1$ AND WORK IT OUT

609. $\int \cos(\ln x) dx = \frac{1}{2} x [\sin(\ln x) + \cos(\ln x)] + C$

$$\int \cos(\ln x) dx \dots \text{ BY SUBSTITUTION } \dots$$

$$\int \cos(u) du = \int e^{iu} \cos(u) du$$

INTRODUCTION BY PARTS, COMPLEX NUMBERS, OR INTEGRATION

$$\frac{d}{du}(e^{iu}(\text{A}u + \text{B}u^2)) = e^{iu} \cos(u)$$

$$ie^{iu}(\text{A}u + \text{B}u^2) + e^{iu}(\text{A} + \text{B}u) = e^{iu} \cos(u)$$

$$(\text{A} - \text{B}u) + (\text{A} + \text{B}u)e^{iu} = e^{iu} \cos(u)$$

$$\begin{aligned} \text{A} + \text{B}u &= 1 \\ \text{A} - \text{B}u &= 0 \\ \text{A} &= \frac{1}{2} \\ \text{B} &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} &= e^{iu} \left(\frac{1}{2} \sin(u) + \frac{1}{2} \cos(u) \right) + C = \frac{1}{2} e^{iu} (\sin(u) + \cos(u)) + C \\ &= \frac{1}{2} x [\sin(\ln x) + \cos(\ln x)] + C \end{aligned}$$

610. $\int \arcsin x dx = x \arcsin x + \sqrt{1-x^2} + C$

$$\int \arcsin x dx = \int x \arcsin x dx = \dots \text{ BY PARTS } \dots$$

OPTIONS | JINX

$$\begin{aligned} &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx \\ &= x \arcsin x - \int x \cdot \frac{1}{2} x^{-\frac{1}{2}} dx \\ &= x \arcsin x + \frac{1}{2} x^{\frac{1}{2}} + C \end{aligned}$$

611. $\int \frac{1}{\sqrt{x^2-16}} dx = \begin{cases} \operatorname{arcosh}\left(\frac{1}{4}x\right) + C \\ \ln\left(x + \sqrt{x^2-16}\right) + C \end{cases}$

$$\begin{aligned} \int \frac{1}{\sqrt{x^2-16}} dx &= \int \frac{1}{\sqrt{x^2-16}} dx = \text{STANDARD RESULT} \\ &= \operatorname{arcosh}\left(\frac{x}{4}\right) + C \\ &= \ln\left(\frac{x}{4} + \sqrt{\frac{x^2}{16}-1}\right) + C = \ln\left(\frac{x}{4} + \sqrt{\frac{x^2-16}{16}}\right) + C \\ &= \ln\left(\frac{x}{4} + \sqrt{\frac{x^2-16}{4}}\right) + C = \ln\left(\frac{x}{4} + \sqrt{\frac{(x+4)(x-4)}{4}}\right) + C \\ &= \ln\left(\frac{x}{4} + \sqrt{\frac{(x+4)(x-4)}{4}}\right) + C = \ln(x\sqrt{x^2-16}) + C \end{aligned}$$

612. $\int \frac{3x^4 + x^3 + 2x^2 + x - 2}{(x+1)x^5} dx = \frac{1-2x+x^2-4x^3}{2x^4} + \ln \left| \frac{x}{x+1} \right| + C$

$$\begin{aligned} \int \frac{3x^4 + x^3 + 2x^2 + x - 2}{(x+1)x^5} dx &= \int \frac{3x^4 + x^3 + 2x^2 + x - 2}{2x^4} \times \frac{1}{x+1} dx \\ \text{Divide by } x^4 \text{ throughout:} \\ &= \int \left(\frac{3}{2}x - \frac{1}{2} + \frac{1}{x^2} + \frac{1}{x^3} - \frac{2}{x^4} \right) \frac{1}{x+1} dx \\ &= \int \left(\frac{3}{2}x - \frac{1}{2} + \frac{1}{x^2} + \frac{1}{x^3} - \frac{2}{x^4} \right) \frac{1}{x+1} dx \\ &= \int \left(\frac{3}{2}x - \frac{1}{2} + \frac{1}{x^2} + \frac{1}{x^3} - \frac{2}{x^4} \right) \frac{1}{x+1} dx \\ &= \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{2x} - \frac{1}{2x^2} + \frac{1}{2x^3} + C \\ &= \frac{1-2x+x^2-4x^3}{2x^4} + \ln \left| \frac{x}{x+1} \right| + C \end{aligned}$$

613. $\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}} dx = e^{\arcsin x} + C$

$$\begin{aligned} \int \frac{e^{\arcsin x}}{\sqrt{1-x^2}} dx &\rightarrow \text{... BY INSPECTION ...} = \frac{e^{\arcsin x}}{2} + C \\ \text{The substitution } u = \arcsin x \text{ AND } du = \frac{dx}{\sqrt{1-x^2}} \end{aligned}$$

614. $\int \sqrt{\frac{3+2\sqrt{x}}{4x}} dx = \frac{1}{3}(3+2\sqrt{x})^{\frac{3}{2}} + C$

$$\begin{aligned} \int \frac{2\sqrt{x+3}}{x} dx &= \int \left(\frac{2\sqrt{x+3}}{x} \right)^{\frac{1}{2}} dx = \int \frac{1}{2x^{\frac{1}{2}}} (2\sqrt{x+3})^{\frac{1}{2}} dx \\ &\rightarrow \text{... IN INSPECTION ...} = \frac{1}{3} (2\sqrt{x+3})^{\frac{3}{2}} + C \\ \text{The substitution } u = \sqrt{x+3} \text{ AND } du = \frac{dx}{2\sqrt{x+3}} \end{aligned}$$

615. $\int \frac{2x+1}{\sqrt{9-x^2}} dx = \arcsin\left(\frac{1}{3}x\right) - 2(9-x^2)^{\frac{1}{2}} + C$

$$\begin{aligned} \int \frac{2x+1}{\sqrt{9-x^2}} dx &= \int \frac{2x}{(9-x^2)^{\frac{1}{2}}} + \frac{1}{(9-x^2)^{\frac{1}{2}}} dx = \int 2x(9-x^2)^{-\frac{1}{2}} + \frac{1}{(9-x^2)^{\frac{1}{2}}} dx \\ &= -2(9-x^2)^{\frac{1}{2}} + \arcsin \frac{x}{3} + C \end{aligned}$$

616. $\int \frac{\sinh x}{\cosh x - 1} dx = \ln(\cosh x - 1) + C$

$$\int \frac{f'(x)}{f(x)} dx = \dots \text{ or the form } \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$= \underline{\ln|\cosh x - 1| + C}$

617. $\int \frac{1}{x\sqrt{1-x^2}} dx = \begin{cases} \frac{1}{2} \ln \left| \frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1} \right| + C \\ \ln \left| \frac{1-\sqrt{1-x^2}}{x} \right| + C \end{cases}$

$$\int \frac{1}{x\sqrt{1-x^2}} dx = \dots \text{ BY SUBSTITUTION ...}$$

$$= \int \frac{1}{\frac{x}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-x^2}} dx} dx = \int \frac{-1}{x^2} dx$$

$$= \int \frac{-1}{1-u^2} du = \int \frac{1}{u^2-1} du = \int \frac{1}{(u-1)(u+1)} du$$

PROBLEMS INVOLVING
FRACTIONS BY INTEGRATION

$$= \int \frac{\frac{1}{u-1} + \frac{1}{u+1}}{u^2-1} du = \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u-1| + C = \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right| + C$$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTIONS

$$\int \frac{1}{x\sqrt{1-x^2}} dx = \int \frac{1}{\sin\theta \sqrt{1-\sin^2\theta}} (\cos\theta d\theta)$$

$$= \int \cos\theta d\theta = -\ln|\sec\theta + \tan\theta| + C$$

$$= -\ln\left|\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right| = -\ln\left|\frac{1+\tan\theta}{\cos\theta}\right| + C = -\ln\left|\frac{(1+\tan\theta)(1-\tan\theta)}{\cos\theta(1-\tan\theta)}\right| + C$$

$$= -\ln\left|\frac{1-\tan^2\theta}{\cos\theta(1-\tan\theta)}\right| = -\ln\left|\frac{\sin^2\theta}{\cos\theta(1-\tan\theta)}\right| = -\ln\left|\frac{\sin\theta}{1-\cos\theta}\right| + C$$

$$= \ln\left|\frac{1-(\cos\theta)^2}{\sin\theta}\right| = \ln\left|\frac{1-(1-\sin^2\theta)^2}{\sin\theta}\right| + C = \ln\left|\frac{1-\sin^4\theta}{\sin\theta}\right| + C = \underline{\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + C}$$

618. $\int \frac{1}{\sqrt{x^2-2x}} dx = \operatorname{arcosh}(x-1) + C$

$$\int \frac{1}{\sqrt{x^2-2x}} dx = \int \frac{1}{\sqrt{(x-1)^2-1}} dx = \int \frac{1}{\sqrt{(x-1)^2-1}} dx$$

... SIMPLIFIED ASIDE: INTEGRAL ...

$$= \operatorname{arcosh}\left(\frac{x-1}{1}\right) + C = \underline{\operatorname{arcosh}(x-1) + C}$$

$$619. \int \frac{3}{(4x+5)\sqrt{1-x^2} - 3(1-x^2)} dx = \begin{cases} \frac{-3x}{4+2x-\sqrt{1-x^2}} + C \\ \frac{2\sqrt{1-x}}{\sqrt{1-x}-3\sqrt{1+x}} + C \end{cases}$$

| | | |
|--|---|--|
| $\int \frac{3}{(2x+1)\sqrt{1-x^2}} dx = \dots \text{BY TRIGONOMETRIC SUBSTITUTION}$ $= \int \frac{3}{(4\cos^2 u - 1)\cos u - 3(\cos u)} (\cos u du)$ $= \int \frac{3\cos u}{(4\cos^2 u + 3\sin^2 u) - 3\cos u} du = \int \frac{3}{4\cos^2 u + 3\sin^2 u} du$ <p>BY THE LITTLE t READING WE NOW HAVE</p> <ul style="list-style-type: none"> • $t = \tan \frac{u}{2}$ $\bullet \, dt = \frac{1}{1+t^2} dt$ $\bullet \, \sin u = \frac{2t}{1+t^2}$ $\bullet \, \cos u = \frac{1-t^2}{1+t^2}$ <p>[THESE ARE FORMULAS STANDARD AND NOT PROVIDED IN EXAM SHEET]</p> $= \int \frac{3}{4\left(\frac{1+t^2}{1-t^2}\right) + 3} \frac{\frac{2}{1+t^2} dt}{1+t^2} = \int \frac{6}{4t^2 + 5t^2 + 2 + 3t^2} dt$ $= \int \frac{6}{2t^2 + 4t + 2} dt = \int \frac{3}{t^2 + 4t + 4} dt = \int \frac{3}{(t+2)^2} dt = -\frac{3}{2(t+2)} + C$ $= -\frac{3}{2t^2 + 4t + 2} + C = -\frac{3}{2\tan^2 \frac{u}{2} + 4} + C = -\frac{3}{4(1-\cos^2 \frac{u}{2}) + 4} + C$ <p>NEED TO REMOVE $\ln \frac{u}{2}$: BACK INTO $\sin u$ ~</p> $\sin u = \frac{2t}{1+t^2}$ $\frac{1+t^2}{1-t^2} = \frac{2t}{1+t^2}$ $1+t^2 = 2t$ $t^2 - 2t + 1 = 0$ $t = \frac{2 \pm \sqrt{1-4t^2}}{2t}$ $t = \frac{2 \pm 2\sqrt{1-t^2}}{2t}$ $t = \frac{1 \pm \sqrt{1-t^2}}{t}$ $\frac{1+t^2}{2t} = \frac{1 \pm \sqrt{1-t^2}}{t}$ $\tan \frac{u}{2} = \frac{1 \pm \sqrt{1-t^2}}{2t}$ | 4 = constant $x = \sin u$ $dx = \cos u du$ | $= -\frac{3\sin u}{4-4\cos^2 u + 2\sin^2 u} + C$ $= -\frac{3\sin u}{4+2\sin^2 u - 4(1-\sin^2 u)}$ <p>4 = constant $\cos u = 1 - \sin^2 u$</p> $= -\frac{3\sin u}{4+2\cos u - \sqrt{1-u^2}} + C$ |
|--|---|--|

ALTERNATIVE SUBSTITUTION

- $\int \frac{1+3x}{1-x} dx$ $\frac{du}{dx} = \frac{(u_1+1)u_2 - u_1(u_2')}{(u_2^2)^2}$ $1 - x^2$
 $u_1 = \frac{1+3x}{1-x}$ $\frac{du}{dx} = \frac{2x + 3}{(1-x)^2}$ $= 1 - \frac{(u_2-1)^2}{(u_2^2-1)^2}$
 $u_2^2 - u_2 = 1 - x$
 $u_2^2 - u_2 = 1 - \frac{1+3x}{1-x}$ $\frac{du}{dx} = \frac{4u}{(u^2-1)^2}$ $= \frac{(u_2-1)(u_2+1)}{(u_2^2-1)^2}$
 $u_2^2 - u_2 = \frac{(1-x)^2 - (1+3x)^2}{(1-x)^2}$ $\frac{du}{dx} = \frac{4u}{(u^2-1)^2} du$ $= \frac{(u_2+1)(u_2-1)}{(u_2^2-1)^2}$
 $\therefore u_2 = \frac{u_2^2 - u_2}{u_2^2 - 1}$ $\frac{du}{dx} = \frac{2u^2 - 2}{(u^2-1)^2}$
 $\frac{du}{dx} = \frac{4u}{(u^2-1)^2}$
 $\therefore u_2 = \frac{u_2^2 - u_2}{u_2^2 - 1}$
- $\int \frac{u^{\frac{3}{2}} + 1}{u^{\frac{1}{2}}} du = 4 \left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} \right) + C = \frac{4u^{\frac{5}{2}} + 8u^{\frac{1}{2}}}{\frac{5}{2}} = \frac{8u^{\frac{5}{2}} + 16u^{\frac{1}{2}}}{5}$

$\int \frac{3}{(2x+1)(1-x^2)} dx = \int \frac{3}{\frac{2x+1}{x+1} \times \frac{2(x+1)}{x-1}} = 3 \times \frac{1}{(2x+1)^2} \times \frac{4x}{(x^2-1)^2} dx$

 $= \int \left[\frac{6}{(2x+1)^2} - \frac{12x}{(x^2-1)^2} \right] dx = \int \frac{6}{2x(4x^2+1)-12x^2} dx$
 $\sim \int \frac{6}{4x^3-6x} dx = \int \frac{6}{9x^2-6x+1} dx = - \int \frac{6}{(3x-1)^2} dx$
 $= -\frac{2}{3x-1} + C = \frac{2}{1-3x} + C = \frac{2}{1-3\sqrt{\frac{1+3x}{1-x}}} + C$
 $= \frac{2}{1-\frac{3(1+3x)}{\sqrt{1-x}}} + C = \frac{2\sqrt{1-x}}{41+18x} + C$

NOTE THAT THE TWO METHODS OBTAINED ARE DIFFERENT BY A CONSTANT
 THE CALCULATIONS ARE NOT SHOWN HERE AS THEY ARE VERY LENGTHY
 (THE CONSTANT DIFFERENCE IS $\frac{1}{2}$)

$$620. \quad \int \frac{1}{x^2\sqrt{4-x^2}} dx = -\frac{\sqrt{4-x^2}}{4x} + C$$

$$\begin{aligned}
 \int \frac{1}{\sqrt{1+4x^2}} dx &= \dots \text{BY SUBSTITUTION} \dots \\
 &= \int \frac{1}{\sqrt{\frac{1}{4}(1+4x^2)}} \left(-\frac{2}{\sqrt{1+4x^2}} dx \right) = \int \frac{-2}{\frac{1}{4}\sqrt{1+4x^2} \cdot 4x \cdot \frac{1}{2} \cdot 8x} dx \\
 &= \int \frac{-8x}{1+4x^2} dx = \int \frac{-8x}{2\sqrt{1+4x^2}} dx \\
 &= \int \frac{-4}{\sqrt{1+4x^2}} dx = \int -\frac{1}{2} \sqrt{1+4x^2}^{-\frac{1}{2}} du = -\frac{1}{2} (1+4x^2)^{-\frac{1}{2}} + C \\
 &= -\frac{1}{2} \left(\frac{1}{\sqrt{1+4x^2}} \right)^{-\frac{1}{2}} + C = -\frac{1}{2} \left(\frac{1}{\sqrt{1+4x^2}} \right)^{\frac{1}{2}} + C = -\frac{\sqrt{1+4x^2}}{4x} + C
 \end{aligned}$$

ALTERNATIVE BY STANDARD TRIGONOMETRIC SUBSTITUTIONS.

$$\begin{aligned}
 \int \frac{1}{\sqrt{1+4x^2}} dx &\rightarrow \dots \rightarrow \int \frac{du}{\sqrt{1+4(u-1)^2}} (2u-2) du \\
 &= \int \frac{2u-2}{\sqrt{4u^2-4u+5}} du = \int \frac{2u-2}{\sqrt{4u(u-1)+5}} du \\
 &= \int \frac{2(u-1)}{\sqrt{4u(u-1)+5}} du = \int \frac{1}{\sqrt{4u(u-1)+5}} du \\
 &= \int \frac{1}{\sqrt{4u^2-4u+5}} du = -\frac{1}{2} \operatorname{arctan} \frac{2u-1}{\sqrt{4u^2-4u+5}} + C \\
 &= -\frac{1}{2} \left(\frac{\sqrt{4x^2-4x+5}}{2x} \right) + C = -\frac{\sqrt{4x^2-4x+5}}{4x} + C
 \end{aligned}$$

$u = \frac{2}{\sqrt{4x^2-4x+5}}$
 $2u = \sqrt{4x^2-4x+5}$
 $du = \frac{2}{\sqrt{4x^2-4x+5}} dx$

$\theta = \operatorname{arctan} \frac{2}{\sqrt{4x^2-4x+5}}$
 $2u = 2\operatorname{arctan} \theta$
 $du = 2\operatorname{arctan} \theta d\theta$

621. $\int \frac{\arcsin x}{\sqrt{1-x^2}} dx = \frac{1}{2}(\arcsin x)^2 + C$

$$\int \frac{\arcsin x}{\sqrt{1-x^2}} dx = \dots \text{ By inspection} \dots = \frac{1}{2}(\arcsin x)^2 + C$$

THE SUBSTITUTION $u = \arcsin x$ ADDS DOUBLE THICK.

622. $\int \frac{3}{x\sqrt{3x^2+2x-1}} dx = \begin{cases} \arccos\left(\frac{1+x}{2x}\right) + C \\ -\arcsin\left(\frac{1+x}{2x}\right) + C \end{cases}$

$$\int \frac{1}{x\sqrt{3x^2+2x-1}} dx = \dots \text{ START WITH A SUBSTITUTION}$$

$$= \int \frac{1}{\frac{1}{2x}\sqrt{\frac{3}{4x^2} + \frac{2}{4x} - 1}} \left(-\frac{1}{2x} dx\right) = \int \frac{-1}{u\sqrt{\frac{3}{4} + \frac{2}{u} - 1}} du$$

$$= \int \frac{-1}{u\sqrt{\frac{3+2u-u^2}{4}}} du = \int \frac{-1}{\sqrt{u(3+2u-u^2)}} du$$

$$= \int \frac{-1}{\sqrt{-u^2+2u+3}} du = \int \frac{-1}{\sqrt{u+2-\frac{u^2-2u}{u}}} du = -\arcan\left(\frac{u+1}{\sqrt{u}}\right) + C$$

$$\text{OR } \arcan\left(\frac{u+1}{\sqrt{u}}\right) + C$$

$$= \arcan\left(\frac{1+x}{2x}\right) + C = \arccan\left(\frac{1+x}{2x}\right) + C \text{ OR } -\arcan\left(\frac{2x+1}{x}\right) + C$$

623. $\int \frac{3}{x\sqrt{x^2+2}} dx = \begin{cases} \frac{\sqrt{2}}{4} \ln \left| \frac{\sqrt{x^2+2}-\sqrt{2}}{\sqrt{x^2+2}+\sqrt{2}} \right| + C \\ \frac{\sqrt{2}}{2} \ln \left| \frac{\sqrt{x^2+2}-\sqrt{2}}{x} \right| + C \end{cases}$

$$\int \frac{1}{x\sqrt{x^2+2}} dx = \text{ BY SUBSTITUTION} \dots$$

$$= \int \frac{1}{xu} \left(\frac{du}{dx}\right) = \int \frac{1}{xu} \frac{du}{dx} = \int \frac{1}{u^2-2} du$$

$$= \int \frac{1}{(u-\sqrt{2})(u+\sqrt{2})} du \text{ PARTIAL FRACTION BY INSPECTION}$$

$$= \int \frac{\frac{A}{u-\sqrt{2}} - \frac{B}{u+\sqrt{2}}}{u^2-2} du = \frac{1}{\sqrt{2}} \left[\ln|u-\sqrt{2}| - \ln|u+\sqrt{2}| \right] + C$$

$$= \frac{\sqrt{2}}{4} \ln \left| \frac{u-\sqrt{2}}{u+\sqrt{2}} \right| + C = \frac{\sqrt{2}}{4} \ln \left| \frac{\sqrt{x^2+2}-\sqrt{2}}{\sqrt{x^2+2}+\sqrt{2}} \right| + C$$

ONCE WE MANUALLY SIMPLIFY

$$= \frac{\sqrt{2}}{4} \ln \left| \frac{(\sqrt{x^2+2}-\sqrt{2})(\sqrt{x^2+2}+\sqrt{2})}{(\sqrt{x^2+2}+\sqrt{2})(\sqrt{x^2+2}-\sqrt{2})} \right| + C = \frac{\sqrt{2}}{4} \ln \left| \frac{(x^2+2)-2}{(\sqrt{x^2+2})^2-2} \right| + C$$

$$= \frac{\sqrt{2}}{4} \ln \left| \frac{x^2}{\frac{(\sqrt{x^2+2})^2-2}{2}} \right|^2 + C = \frac{\sqrt{2}}{2} \ln \left| \frac{\sqrt{x^2+2}-\sqrt{2}}{\sqrt{x^2+2}+\sqrt{2}} \right| + C$$

NOTE THAT THE FORM $\int \frac{1}{xu} du$ CAN BE INTEGRATED BY PARTS AND THEN TURN TO THE SAME ANSWER VIA LOGARITHMIC FORM

624. $\int \frac{1}{1+\sin^2 x} dx = \left[-\frac{1}{\sqrt{2}} \arctan\left(\frac{\cot x}{\sqrt{2}}\right) + C \right] \left[\frac{1}{\sqrt{2}} \arctan\left(\sqrt{2} \tan x\right) + C \right]$

$$\int \frac{1}{1+\sin^2 x} dx = \int \frac{1 \cdot \sec^2 x}{1+\sec^2 x + \tan^2 x} dx = \int \frac{\sec^2 x}{\sec^2 x + 1} dx$$

$$= \int \frac{\cos^2 x}{1+\tan^2 x} dx = \int \frac{\cos^2 x}{2+\cot^2 x} dx = -\frac{1}{\sqrt{2}} \int \frac{-\sqrt{2}\cos^2 x}{(\sqrt{2})^2(1+\tan^2 x)} dt$$

BY REVERSE CHAIN RULE OR SUBSTITUTION

$$\frac{d}{dt} \left[\arctan\left(\frac{\cot x}{\sqrt{2}}\right) \right] = \frac{1}{1+\left(\frac{\cot x}{\sqrt{2}}\right)^2} \times \frac{1}{\sqrt{2}} (-\csc^2 x)$$

$$= \frac{2}{2+\cot^2 x} \times \left(-\frac{\csc^2 x}{\sqrt{2}}\right)$$

$$= -\frac{\sqrt{2}\csc^2 x}{2+\cot^2 x}$$

$$= -\frac{1}{\sqrt{2}} \arctan\left(\frac{\cot x}{\sqrt{2}}\right) + C$$

ALTERNATIVE BY LITTLE t IDENTITIES

$$\int \frac{1}{1+\sin^2 x} dx = \int \frac{1}{1+(\frac{1}{1-\frac{t^2}{1+t^2}})^2} dt = \int \frac{1}{\frac{2}{1-t^2}} \csc^2 x dx$$

$$= \int \frac{2}{3-t^2} dx = \dots \text{SUBSTITUTION} \dots$$

| | | |
|--|---|--|
| $t = \tan x$ $dt = \sec^2 x dx$ $dx = \frac{dt}{\sec^2 x}$ $dx = \frac{dt}{1+t^2}$ $dx = \frac{dt}{1+t^2}$ | $\csc 2x = \csc^2 x - \cot^2 x$ $\csc 2x = (\frac{1}{1-t^2})^2 - (\frac{t}{\sqrt{1-t^2}})^2$ $\csc 2x = \frac{1-t^2}{1-t^2 - t^2}$ $\csc 2x = \frac{1-t^2}{1+t^2}$ | |
|--|---|--|

$$\dots = \int \frac{2}{3 - \frac{1-t^2}{1+t^2} - 1+t^2} dt = \int \frac{2}{3(1+t^2) - (1-t^2)} dt$$

$$= \int \frac{2}{2+4t^2} dt = \int \frac{1}{1+2t^2} dt = \frac{1}{2} \int \frac{1}{t^2+\frac{1}{2}} dt$$

$$= \frac{1}{2} \int \frac{1}{t^2+(\frac{1}{\sqrt{2}})^2} dt = \frac{1}{2} \times \frac{1}{\frac{1}{\sqrt{2}}} \arctan\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \frac{\sqrt{2}}{2} \arctan\left(\sqrt{2} \tan x\right) + C = \underline{\frac{1}{\sqrt{2}} \arctan\left(\sqrt{2} \tan x\right) + C}$$

625. $\int \frac{\sin^2 x}{1+\cos^2 x} dx = \left[-x - \sqrt{2} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C \right] \left[\arctan(\cot x) - \sqrt{2} \arctan\left(\sqrt{2} \cot x\right) + C \right]$

$$\int \frac{\sin^2 x}{1+\cos^2 x} dx = \int \frac{\sin x \cdot \cancel{\sin x}}{\cancel{\cos x} + \cos x \cancel{\cos x}} dx$$

$$= \int \frac{1}{\cancel{\cos x} + \cos x} dx = \int \frac{1}{(1+\cos x) + \cos x} dx$$

$$= \int \frac{1}{1+2\cos x} dx \dots \text{BY SUBSTITUTION} \dots$$

$$= \int \frac{1}{1+2\cos^2 x} (-\frac{du}{dt}) = \int \frac{-1}{(2t+1)(1+t^2)} du$$

BY PARTIAL FRACTIONS, RUDIMENTARY OR
REVERSE CHAIN OR INTEGRATION

$$= \int \frac{-2}{2t+1} - \frac{1}{1+t^2} du$$

$$= \int \frac{1}{1+t^2} - \frac{1}{1+t^2} du$$

$$= \int \frac{1}{1+t^2} du - \frac{1}{1+t^2} du$$

$$= \arctan u - \frac{1}{\sqrt{2}} \arctan(\frac{u}{\sqrt{2}}) + C$$

$$= \arctan(u) - \sqrt{2} \arctan(\frac{u}{\sqrt{2}}) + C$$

$$= \arctan(\cot x) - \sqrt{2} \arctan(\sqrt{2} \cot x) + C$$

ALTERNATIVE APPROACH

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx + \int \frac{\cos^2 x}{1+\cos^2 x} dx = \int \frac{2\sin x \cos x}{1+\cos^2 x} dx$$

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx + \int \frac{(\cos x+1)-1}{1+\cos^2 x} dx = \int \frac{1}{1+\cos^2 x} dx$$

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx + \int 1 - \frac{1}{1+\cos^2 x} dx = \int \frac{1}{1+\cos^2 x} dx$$

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx + \int 1 dx - \int \frac{1}{1+\cos^2 x} dx = \int \frac{1}{1+\cos^2 x} dx$$

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx + x = 2 \int \frac{1}{1+\cos^2 x} dx$$

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx + x = 2 \int \frac{\sec^2 x}{\sec^2 x + \cos^2 x} dx$$

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx + x = 2 \int \frac{\sec^2 x}{\sec^2 x + 1} dx$$

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx + x = 2 \int \frac{\sec^2 x}{(1+\tan^2 x)+1} dx$$

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx + x = 2 \int \frac{\sec^2 x}{2+\tan^2 x} dx$$

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx + x = 2 \int \frac{\sec^2 x}{(1+t^2)^2} dt$$

BY REVERSE CHAIN RULE

(if $\arctan(\frac{u}{\sqrt{2}})$ was

$$\frac{1}{1+t^2} \times \frac{1}{\sqrt{2}} \sec^2 x$$

$$= \frac{\sqrt{2} \sec^2 x}{2+\tan^2 x}$$

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx + x = \frac{2}{\sqrt{2}} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

$$\Rightarrow \int \frac{\sin^2 x}{1+\cos^2 x} dx = -x + \sqrt{2} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

ANOTHER ALTERNATIVE BY THE LITTLE t IDENTITIES

$$\int \frac{\sin^2 x}{1+\cos^2 x} dx = \int \frac{\frac{1}{2} - \frac{1}{2}\cos 2x}{1 + (\frac{1}{2} + \frac{1}{2}\cos 2x)} dx$$

$$= \int \frac{\frac{1}{2} - \frac{1}{2}\cos 2x}{\frac{3}{2} + \frac{1}{2}\cos 2x} dt = \int \frac{1 - \cos 2x}{3 + \cos 2x} dx = \dots \text{SUBSTITUTION}$$

| | | |
|--|--|--|
| $t = \tan x$ $dt = \sec^2 x dx$ $dx = \frac{dt}{\sec^2 x}$ $dx = \frac{dt}{1+t^2}$ $dx = \frac{dt}{1+t^2}$ | $\cos 2x = \cos^2 x - \sin^2 x$ $\cos 2x = (\frac{1}{1+t^2})^2 - (\frac{t}{\sqrt{1+t^2}})^2$ $\cos 2x = \frac{1-t^2}{1+t^2} - \frac{t^2}{1+t^2}$ $\cos 2x = \frac{1-2t^2}{1+t^2}$ | |
|--|--|--|

$$\dots = \int \frac{1 - \frac{1-t^2}{1+t^2}}{3 + \frac{1-2t^2}{1+t^2}} \left(-\frac{dt}{1+t^2} \right) = \int \frac{(1+t^2) - (1-t^2)}{2(1+t^2) + (1-t^2)} \left(-\frac{dt}{1+t^2} \right)$$

(cancel top & bottom of the integrand by $(1+t^2)$)

$$= \int \frac{2t^2}{2t^2+2} \frac{dt}{1+t^2} = \int \frac{t^2}{(1+t^2)(1+t^2)} dt$$

PARTIAL FRACTIONS BY A FOOL WILHELM OR INTEGRATION

$$= \int \frac{2}{1+t^2} - \frac{1}{1+t^2} dt = 2 \int \frac{1}{1+(\frac{t}{\sqrt{2}})^2} dt - \int \frac{1}{1+t^2} dt$$

$$= \frac{2}{\sqrt{2}} \arctan\left(\frac{t}{\sqrt{2}}\right) - \arctan t + C = \sqrt{2} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) - \arctan(\tan x)$$

$$= -x + \sqrt{2} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

626. $\int \frac{2x}{\sqrt{x^4+1}} dx = \begin{cases} \ln[x^2 + \sqrt{x^4+1}] + C \\ \operatorname{arsinh}(x^2) + C \end{cases}$

$$\int \frac{2x}{\sqrt{x^4+1}} dx = \text{BY RECOGNITION, SINCE } \frac{d}{dx}(\operatorname{arsinh}(x^2)) = \frac{1}{\sqrt{x^4+1}} \times 2x$$

$$= \operatorname{arsinh}x^2 + C = \ln(x^2 + \sqrt{x^4+1}) + C$$

OR BY A HYPERBOLIC SUBSTITUTION

$$\int \frac{2x}{\sqrt{x^4+1}} dx = \int \frac{2x}{\sqrt{\sinh^2\theta - 1}} \left(\frac{\cosh\theta}{2x} d\theta \right)$$

$$= \int \frac{\cosh\theta}{\sqrt{\sinh^2\theta - 1}} d\theta = \int 1 d\theta$$

$$= \theta + C = \operatorname{arsinh}x^2 + C = \ln(x^2 + \sqrt{x^4+1}) + C$$

$x^2 = \sinh\theta$
 $(b = \operatorname{arsinh}(x^2))$
 $2x dx = \cosh\theta d\theta$
 $d\theta = \frac{2x dx}{\cosh\theta}$

OR BY A TRIGONOMETRIC SUBSTITUTION

$$\begin{aligned} x^2 &= \tan\theta \\ 2x dx &= \sec^2\theta d\theta \\ d\theta &= \frac{2x dx}{\sec^2\theta} \end{aligned}$$

$$\begin{aligned} \tan\theta &= x^2 \\ \sec\theta &= \sqrt{x^2+1} \\ \sec\theta d\theta &= \frac{1}{\sqrt{x^2+1}} \end{aligned}$$

$$\begin{aligned} \int \frac{2x}{\sqrt{x^4+1}} dx &= \int \frac{2x}{\sqrt{\tan^2\theta + 1}} \left(\frac{\sec\theta}{\sec^2\theta} d\theta \right) \\ &= \int \frac{\sec\theta}{\sqrt{\tan^2\theta + 1}} d\theta = \int \sec\theta d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln|\sqrt{x^2+1} + x^2| + C \end{aligned}$$

627. $\int \frac{1-x^2}{(1+x^2)^2} dx = \frac{x}{1+x^2} + C$

$$\int \frac{1-x^2}{(1+x^2)^2} dx = \dots \text{BY A TRIGONOMETRIC SUBSTITUTION}$$

$$\begin{aligned} &= \int \frac{1-\tan^2\theta}{(1+\tan^2\theta)^2} (\sec^2\theta d\theta) = \int \frac{1-\tan^2\theta}{(\sec^2\theta)^2} \sec^2\theta d\theta \\ &= \int \frac{1-\tan^2\theta}{\sec^2\theta} d\theta = \int \frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta} d\theta \\ &= \int \cos^2\theta - \tan^2\theta \cos^2\theta d\theta = \int \cos^2\theta - \frac{\sin^2\theta}{\cos^2\theta} \cos^2\theta d\theta \\ &= \int \cos^2\theta - \sin^2\theta d\theta = \int \cos 2\theta d\theta \\ &\Rightarrow \frac{1}{2}\sin 2\theta + C = \sin\theta \cos\theta + C \\ &= \frac{2}{1+x^2} \times \frac{1}{\sqrt{1+x^2}} + C = -\frac{x}{1+x^2} + C \end{aligned}$$

$\begin{aligned} x &= \tan\theta \\ dx &= \sec^2\theta d\theta \\ \theta &= \arctan x \\ \sin\theta &= x \\ \sin\theta &= \frac{x}{\sqrt{1+x^2}} \\ \cos\theta &= \frac{1}{\sqrt{1+x^2}} \end{aligned}$

(A HYPERBOLIC SUBSTITUTION DOES NOT WORK HERE - TRY -)

628. $\int \frac{x e^{2x^2}}{\sqrt{1+2e^{2x^2}}} dx = \frac{1}{4} \sqrt{1+2e^{2x^2}} + C$

$$\begin{aligned} \int \frac{xe^{2x^2}}{\sqrt{1+2e^{2x^2}}} dx &= \dots \text{BY INVERSE CHAIN RULE (CORRECT)} \\ &\text{OR} \\ &\text{SUBSTITUTION} \\ &= \int \frac{2xe^{2x^2}}{\sqrt{1+2e^{2x^2}}} \left(\frac{du}{4e^{2x^2}} du \right) \\ &= \int \frac{1}{2} du \\ &= \frac{1}{2}u + C \\ &= \frac{1}{2}\sqrt{1+2e^{2x^2}} + C \\ &\text{(THE SUBSTITUTION } u = 1+2e^{2x^2} \text{ AND WORKS)} \end{aligned}$$

629. $\int \frac{x^5+x-1}{x^3+1} dx = \frac{1}{3}x^3 - \ln|x+1| + C$

$$\begin{aligned} \int \frac{x^5+x-1}{x^3+1} dx &= \dots \text{BY SUBSTITUTION OR MINIMIZATION/DIVISION} \\ &= \int \frac{(2x^2+1)x^3-x^2+x-1}{x^3+1} dx = \int x^2 - \frac{2x^2+x+1}{x^3+1} dx \\ &= \int x^2 - \frac{2x^2+x+1}{(x+1)(x^2-x+1)} dx = \int x^2 - \frac{1}{x+1} dx \\ &= \frac{1}{3}x^3 - \ln|x+1| + C \end{aligned}$$

630. $\int \sin^4 x \cos^4 x dx = \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$

$$\begin{aligned} \int \sin^4 x \cos^4 x dx &= \int \frac{1}{16} (\sin^2 x)^2 (\cos^2 x)^2 dx = \int \frac{1}{16} (2\sin x \cos x)^4 dx \\ &= \int \frac{1}{16} \sin^4 2x dx = \frac{1}{16} \int (\sin^2 2x)^2 dx \\ &= \frac{1}{16} \int \left(\frac{1}{2} - \frac{1}{2}\cos 4x \right)^2 dx = \frac{1}{16} \int \frac{1}{4} - \frac{1}{2}\cos 4x + \frac{1}{4}\cos^2 4x dx \\ &= \frac{1}{16} \int \frac{1}{4} - \frac{1}{2}\cos 4x + \left(\frac{1}{2} + \frac{1}{2}\cos 8x \right) dx \\ &= \frac{1}{16} \int \frac{3}{8}dx - \frac{1}{2}\cos 4x + \frac{1}{8}\cos 8x dx \\ &= \frac{1}{16} \int \frac{3}{8}dx - \frac{1}{2}\cos 4x + \frac{1}{8}\cos 8x dx \\ &= \frac{1}{16} \left[\frac{3}{8}x - \frac{1}{2}\sin 4x + \frac{1}{8}\sin 8x \right] + C \\ &= \frac{1}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C \\ &\text{OR} \\ &\left[\frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C \right] \end{aligned}$$

| | |
|---|---|
| $\cos 2\theta = 2\cos^2 \theta - 1$ | $\cos 2\theta = 1 - 2\sin^2 \theta$ |
| $1 + \cos 2\theta = 2\cos^2 \theta$ | $2\sin^2 \theta = 1 - \cos 2\theta$ |
| $\cos^2 \theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ | $\sin^2 \theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ |
| $\sin^2 \theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ | $\sin 2\theta = \frac{1}{2} - \frac{1}{2}\cos 4\theta$ |

631. $\int \frac{2}{x\sqrt{x^4-1}} dx = \arctan(\sqrt{x^4-1}) + C$

$\int \frac{2}{x\sqrt{x^4-1}} dx = \dots$ BY SUBSTITUTION ...

$$\begin{aligned} &= \int \frac{-x}{2\sqrt{\sec^2\theta - 1}} \left(\frac{\sec\theta \tan\theta}{\sec\theta} d\theta \right) \\ &= \int \frac{\sec\theta \tan\theta}{2\sqrt{\tan^2\theta}} d\theta = \int \frac{\sec\theta \tan\theta}{2\sec\theta} d\theta \\ &= \int \frac{1}{2} d\theta = \theta + C = \arctan(\sqrt{x^4-1}) + C \end{aligned}$$

ALTERNATIVE BY HYPERBOLIC SUBSTITUTION

$$\begin{aligned} \int \frac{2}{x\sqrt{x^4-1}} dx &= \int \frac{-x}{2\sqrt{1+\cosh^2\theta-1}} \left(\frac{\sinh\theta}{\cosh\theta} d\theta \right) \\ &= \int \frac{\sinh\theta}{2\cosh\theta} d\theta = \int \frac{1}{2} d\theta \\ &= \int \frac{1}{\cosh\theta} d\theta = \int \frac{\cosh\theta}{\cosh^2\theta} d\theta \\ &= \int \frac{\cosh\theta}{1+\sinh^2\theta} d\theta \end{aligned}$$

BY REVERSE CHAIN RULE OR HYPERBOLIC SUBSTITUTION
 \downarrow
 $d(\operatorname{arccosh}(\sinh\theta)) = \frac{1}{1+\sinh^2\theta} \times \cosh\theta$

$$\begin{aligned} &= \operatorname{arccosh}(\sinh\theta) + C \\ &= \operatorname{arccosh}(\sqrt{\cosh\theta-1}) + C \\ &= \operatorname{arccosh}(\sqrt{x^4-1}) + C \end{aligned}$$


632. $\int \frac{2x}{\sqrt{x^4-1}} dx = \left[\ln(x^2 + \sqrt{x^4-1}) + C \right]_{\operatorname{arcosh}(x^2) + C}$

$\int \frac{2x}{\sqrt{x^4-1}} dx = \dots$ BY A HYPERBOLIC SUBSTITUTION (OR REVERSE)

$$\begin{aligned} &= \int \frac{2x}{(\cosh^2\theta-1)^{1/2}} \times \frac{\sinh\theta}{\cosh\theta} d\theta = \int \frac{\sinh\theta}{\cosh\theta} d\theta \\ &= \int 1 d\theta = \theta + C = \operatorname{arcosh} x^2 + C \\ &= \ln(x^2 + \sqrt{x^4-1}) + C \end{aligned}$$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION

$$\begin{aligned} \int \frac{2x}{\sqrt{x^4-1}} dx &= \int \frac{2x}{\sqrt{x^2-1}} \frac{\sec\theta \tan\theta}{\sec\theta} d\theta \\ &= \int \frac{\sec\theta \tan\theta}{\sqrt{1-\frac{1}{\tan^2\theta}}} d\theta = \int \sec\theta d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln|x^2 + \sqrt{x^4-1}| + C \end{aligned}$$

$x^2 = \sec\theta$
 $\frac{1}{x^2} = \cos\theta$
 $\theta = \operatorname{arcos}(\frac{1}{x^2})$
 $2x dx = \sec\theta \tan\theta d\theta$
 $dx = \frac{\sec\theta \tan\theta}{2x} d\theta$



633. $\int \frac{\sin^6 x}{\cos^8 x} dx = \frac{1}{7} \tan^7 x + C$

$$\begin{aligned}\int \frac{\sin^6 x}{\cos^8 x} dx &= \int \frac{\sin^2 x}{\cos^2 x} \times \frac{1}{\cos^2 x} dx = \int \tan^2 x \sec^2 x dx \\ &\quad \text{BY COSEC CHAIN RULE AS THE DERIVATIVE OF TANX IS SECX (OR THE SUBSTITUTION SEC = TANX)} \\ &= \frac{1}{7} \tan^7 x + C\end{aligned}$$

634. $\int \cot^2 3x \cosec 3x dx = -\frac{1}{6} [\cot 3x \cosec 3x + \ln |\tan(\frac{3}{2}x)|] + C$

$$\begin{aligned}\int \cot^2 3x \cosec 3x dx &= \int \cot 3x (\cot 3x \cosec 3x) dx \\ &\quad \text{INTEGRATION BY PARTS} \\ &\quad \cot 3x \quad | \quad -\frac{1}{3} \cosec^2 3x \\ &\quad -\frac{1}{3} \cosec 3x \quad | \quad \cot 3x \cosec 3x \\ &= -\int \cot 3x \cosec 3x - \int \cosec^2 3x dx \\ &= -\frac{1}{3} \int \cot 3x \cosec 3x - \int \cosec^2 3x dx \\ &= -\frac{1}{3} \int \cot 3x \cosec 3x - \int \cosec^2 3x + \cosec 3x dx \\ &\rightarrow \int \cot^2 3x \cosec 3x dx = -\frac{1}{3} \int \cot 3x \cosec 3x - \int \cosec^2 3x dx \\ &\rightarrow 2 \int \cot^2 3x \cosec 3x dx = -\frac{2}{3} \int \cot 3x \cosec 3x - \int \cosec^2 3x dx \\ &\rightarrow 2 \int \cot^2 3x \cosec 3x dx = -\frac{2}{3} \int \cot 3x \cosec 3x - \frac{2}{3} \ln |\tan(\frac{3}{2}x)| + C \\ &\rightarrow \int \cot^2 3x \cosec 3x dx = -\frac{1}{3} [\cot 3x \cosec 3x + \ln |\tan(\frac{3}{2}x)|] + C\end{aligned}$$

635. $\int \frac{6 \sinh x}{\sinh x + \cosh x} dx = 3x + \frac{3}{2} e^{-2x} + C$

$$\begin{aligned}\int \frac{6 \sinh x}{\sinh x + \cosh x} dx &= \dots \text{USING EXPONENTIALS} \\ &= \int \frac{6(e^x - e^{-x})}{e^x + e^{-x} + e^x - e^{-x}} dx \\ &= \int \frac{6e^{2x} - 6e^{-2x}}{e^{2x}} dx = \int \frac{6e^{2x}}{e^{2x}} - \frac{6e^{-2x}}{e^{2x}} dx \\ &= \int 6 - 6e^{-4x} dx = \underline{3x + \frac{3}{2} e^{-2x}} + C\end{aligned}$$

$$636. \int (x^2 + x^{-4})^{-2} dx = \frac{1}{6} \left[\arctan(x^3) - \frac{x^3}{1+x^6} \right] + C$$

Method 1: Substitution

$$\int (x^2 + x^{-4})^{-2} dx = \dots \text{TRY FACT} \dots = \int \left(x^2 + \frac{1}{x^4} \right)^{-2} dx = \int \left(\frac{x^4 + 1}{x^4} \right)^{-2} dx$$

$$= \int \left(\frac{x^4 + 1}{x^8} \right)^{-2} dx = \int \frac{x^8}{(x^4 + 1)^2} dx$$

Method 2: Integration by Parts

$$\dots \text{NOT MINIMISE AS FOLLOWS} \dots$$

$$= \int \frac{x^2}{(x^4 + 1)^2} x^3 dx = \frac{1}{6} \int \frac{x^5}{(x^4 + 1)^2} x^3 dx = \frac{1}{6} \left[x^5 (x^4 + 1)^{-1} \right] dx$$

$$\begin{aligned} &= \frac{1}{6} \left[x^5 (x^4 + 1)^{-1} - \int x^5 (-4x^3) (x^4 + 1)^{-2} dx \right] \\ &= \frac{1}{6} \left[-\frac{x^5}{1+x^4} + \left[\frac{3x^2}{1+x^4} dx \right] \right] \\ &\quad \downarrow \text{SIMPLIFY AND ADD } x^2 \\ &= \frac{1}{6} \left[\arctan(x^3) - \frac{x^3}{1+x^4} \right] + C \end{aligned}$$

$$637. \int \frac{12x-1}{(6x^2-x-1)(6x^2-x+5)+10} dx = \arctan(6x^2 - x + 2) + C$$

Substitution:

$$\begin{aligned} \int \frac{12x-1}{(6x^2-x-1)(6x^2-x+5)+10} dx &= \text{TRY SUBSTITUTION} \\ &= \int \frac{12x-1}{u(4u+4)+10} \frac{du}{2x-1} \\ &= \int \frac{1}{u(u+1)+10} du \\ &= \int \frac{1}{(u+5)^2 + 1} du \\ &= \int \frac{1}{(u+5)^2 + 1} du \\ &\quad \text{THIS IS A STANDARD ARCTAN} \\ &= \arctan(u+5) + C \\ &= \arctan(6x^2 - x - 1 + 5) + C \\ &= \arctan(6x^2 - x + 2) + C \end{aligned}$$

638. $\int \sqrt{\frac{x}{x-1}} dx = \frac{1}{6} \left[\arctan(x^3) - \frac{x^3}{1+x^6} \right] + C$

$\int \sqrt{\frac{x}{x-1}} dx = \dots$ BY HYPERBOLIC SUBSTITUTION

$$= \int \sqrt{\frac{\cosh^2 u}{\cosh^2 u - 1}} (2\sinh u \cosh u du)$$

$$= \int \sqrt{\cosh^2 u} (2\sinh u \cosh u du) = \int \cosh u (2\sinh u \cosh u du)$$

$$= \int 2\sinh u du = \int (1 + \tanh^2 u) du$$

$$= u + \frac{1}{2} \operatorname{sech}^2 u + C = u + \operatorname{arctanh}^2 u + C$$

$$= \operatorname{arctanh} u + \sqrt{1 - \tanh^2 u} + C = \ln [\sqrt{2}(\sqrt{u^2 + 1})] + \sqrt{2}u + C$$

ADDITIONAL: BY TRIGONOMETRIC SUBSTITUTION

$$\int \sqrt{\frac{x}{x-1}} dx = \int \frac{\sec u}{\sqrt{2\sin^2 u - 1}} (2\sin u \cos u du)$$

$$= \int \frac{\sec u}{\sqrt{2\sin^2 u - 1}} (2\sin u \cos u du) = \int \frac{\sec u}{\sqrt{2(1-\cos^2 u) - 1}} (2\sin u \cos u du)$$

$$= \int \sec^2 u du \quad \leftarrow \text{THIS & A TRIGONOMETRIC IDENTITY ARE USEFUL FOR THIS}$$

$$\int (2\sin u \cos u) du = 2\cos u du = -2 \int \tan u \sec u du$$

$$= 2\sec u du - 2 \int (\sec^2 u - 1) \sec u du$$

$$= 2\sec u du - 2 \int \sec u - \sec u du$$

$$= \int 2\sec u du = 2\sec u \tan u - 2 \int \sec u du + 2 \int \sec u du$$

$$+ 4 \int \sec u du = 2\sec u \tan u + 2 \ln |\sec u + \tan u| + C$$

$$\int 2\sec u du = \sec u \tan u + \ln |\sec u + \tan u| + C$$

$\therefore \int \sqrt{\frac{x}{x-1}} dx = \dots$ $\int 2\sec u du = \dots$ $\sec u \tan u + \ln |\sec u + \tan u| + C$
 $= \sqrt{2} \sqrt{u^2 + 1} + \ln [\sqrt{2}(\sqrt{u^2 + 1})] + C$
 $= \underline{\underline{\sqrt{2} \sqrt{u^2 + 1} + C}}$ AE MADAS

639. $\int \frac{4x^3 - 12x^2 - 22x - 3}{(4-x)(x+1)} dx = -x^2 - x + 3\ln|x-4| + \frac{1}{2}\ln|2x+1| + C$

$\int \frac{4x^3 - 12x^2 - 22x - 3}{(4-x)(x+1)} dx = \int \frac{4x^3 - 12x^2 - 22x - 3}{-2x^2 + 2x + 4} dx$

BY LONG DIVISION FOLLOWED BY PARTIAL FRACTION

$$\begin{array}{r} -2x^2 + 2x + 4 \\ \overline{4x^3 - 12x^2 - 22x - 3} \\ -4x^3 + 8x^2 + 8x \\ \hline -20x^2 - 22x - 3 \\ -20x^2 - 20x - 8 \\ \hline -2x - 5 \end{array}$$

$$= \int -2x - 1 + \frac{-2x - 1}{-2x^2 + 2x + 4} dx = \int -2x - 1 + \frac{-2x - 1}{2x^2 - 2x - 4} dx$$

$$= \int -2x - 1 + \frac{-2x - 1}{(2x+1)(x-4)} dx = \dots$$

$$\text{BY INTEGRATION} \dots \int -2x - 1 + \frac{3}{2x+1} + \frac{1}{x-4} dx$$

$$= \underline{\underline{-x^2 - x + 3\ln|2x+1| + \frac{1}{2}\ln|x-4| + C}}$$

640. $\int x \ln(x^2 + 1) dx = \frac{1}{2}(x^2 + 1)\ln(x^2 + 1) - \frac{1}{2}x^2 + C$

$\int x \ln(x^2 + 1) dx = \dots$ INTEGRATION BY PARTS

$$= \frac{1}{2}x^2 \ln(x^2 + 1) - \int \frac{2x}{x^2 + 1} dx$$

BY LONG DIVISION OR MANIPULATION

$$= \frac{1}{2}x^2 \ln(x^2 + 1) - \int \frac{2(x^2 + 1) - 2}{x^2 + 1} dx$$

$$= \frac{1}{2}x^2 \ln(x^2 + 1) - \int 2 - \frac{2}{x^2 + 1} dx$$

$$= \frac{1}{2}x^2 \ln(x^2 + 1) - \frac{2}{2}x + \frac{1}{2} \operatorname{arctan} x + C$$

$$= \frac{1}{2}(x^2 + 1)\ln(x^2 + 1) - \frac{1}{2}x^2 + C$$

STARTING WITH THE SUBSTITUTION $(x^2 + 1)$, LEADS TO SIMPLER INTEGRATION
BY PARTS, UNLESS $\int \frac{1}{x^2 + 1} dx$

641. $\int \frac{x^4 + x^3 + 3x - 1}{x(x^2 + 1)^2} dx = \ln \left| \frac{x^2 + 1}{x} \right| + 2 \arctan x + \frac{x}{x^2 + 1} + C$

$\int \frac{x^4+x^3+3x-1}{x(x^2+1)^2} dx = \dots$ SIMPLIFY BY PARTIAL FRACTION

$$\begin{aligned} \frac{x^4+x^3+3x-1}{x(x^2+1)^2} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \\ x^4+x^3+3x-1 &\equiv A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x^2 \\ &\equiv Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx + Dx^3 + Ex^2 \\ 2Bx^4 + 2Cx^2 + Dx^3 + Ex^2 &\equiv Ax^4 + Bx^4 + Dx^3 + Ex^2 + Cx + 2Ax^2 + A \\ D=1, B=2, C=1, D=0, E=2 \end{aligned}$$

IF $x=0 \Rightarrow A=-1$

$$\begin{aligned} &= \int -\frac{1}{x} + \frac{2x+1}{x^2+1} + \frac{2}{(x^2+1)^2} dx \quad \text{BY SUBSTITUTION } x=\tan\theta \\ &= \int -\frac{1}{x} + \frac{2x}{x^2+1} + \frac{1}{(x^2+1)^2} dx \\ &= -\ln|x| + \arctan x + 2 \int \frac{1}{(x^2+1)^2} dx \\ &= \ln|\frac{x^2+1}{x}| + \arctan x + 2 \int \frac{dx}{x^2+1} \\ &= \ln|\frac{x^2+1}{x}| + \arctan x + 2 \left[\frac{1}{2} \arctan x \right] \\ &= \ln|\frac{x^2+1}{x}| + \arctan x + 2 \left[\frac{1}{2} + \frac{1}{2} \arctan x \right] \\ &= \ln|\frac{x^2+1}{x}| + \arctan x + 2 \left[\frac{1}{2} + \frac{1}{2} \arctan x \right] + C \\ &= \ln|\frac{x^2+1}{x}| + \arctan x + \frac{\pi}{4} + \frac{1}{2} \arctan x + C \\ &= \ln|\frac{x^2+1}{x}| + \arctan x + \arctan x + \frac{1}{2} \arctan x + C \\ &= \ln|\frac{x^2+1}{x}| + 2 \arctan x + \frac{3}{4} \arctan x + C \end{aligned}$$

$\theta = \arctan x$
 $x = \tan\theta$
 $A = -\ln|\tan\theta|$
 $B = \frac{2}{1+\tan^2\theta}$
 $C = \frac{1}{(1+\tan^2\theta)^2}$

642. $\int \frac{6}{1+x^3} dx = \ln \left| \frac{x^2+2x+1}{x^2-x+1} \right| + 2\sqrt{3} \arctan \left(\frac{2x-1}{\sqrt{3}} \right) + C$

$\int \frac{6}{1+x^3} dx = \int \frac{6}{(1+x)(-1+x^2)} dx = \int \frac{6}{(x+1)(x^2-x+1)} dx$

SIMPLIFICATION OF DENOMINATOR

PARTIAL FRACTION

REDUCE

$$\begin{aligned} \frac{6}{(x+1)(x^2-x+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \\ G &\equiv Ax^2-Ax+A+Bx^2+Cx+B \\ G &\equiv (A+B)x^2-(A-C)x+(A+B) \quad \leftarrow \text{IF } 2=1 \\ G &\equiv (A+B)x^2-(A-C)x+1 \quad \leftarrow \text{IF } A=B \text{ AND } C=0 \\ A=2, B=3, C=0 \Rightarrow & \quad \leftarrow \text{IF } 2=2 \\ G &\equiv 2x^2-3x+1 \quad \leftarrow \text{IF } 2=3 \\ &\Rightarrow 3=2 \\ &\Rightarrow C=0 \end{aligned}$$

ONLY ONE UNKNOWN

$$\begin{aligned} &= \int \frac{2}{x+1} - \frac{2x-4}{x^2-x+1} dx = \int \frac{2}{x+1} - \frac{2x-3}{x^2-x+1} dx \\ &= \int \frac{2}{x+1} dx - \frac{2x-3}{x^2-x+1} dx = \int \frac{2}{x+1} dx - \frac{2x-3}{(x-\frac{1}{2})^2 + \frac{3}{4}} dx \\ &= 2\ln|x+1| - \ln|x^2-x+1| + \int \frac{12}{(x^2-x+1)^2} dx = \int \frac{12}{(x^2-x+1)^2} dx - \int \frac{12}{(x^2-x+1)} dx \\ &= 12 \int \frac{1}{(x^2-x+1)^2} dx + 12 \int \frac{1}{(x^2-x+1)} dx \quad \leftarrow \text{STANDARD ARCTAN}(2x-1) \\ &= 12 \times \frac{1}{2} \times \frac{1}{2} \arctan \left(\frac{2x-1}{\sqrt{3}} \right) + C \\ &= \ln \left| \frac{2x+1}{x^2-x+1} \right| + 2\sqrt{3} \arctan \left(\frac{2x-1}{\sqrt{3}} \right) + C \end{aligned}$$

643. $\int \frac{2\cos x}{\cos x - \sin x} dx = x - \ln|\cos x - \sin x| + C$

$$\begin{aligned}\int \frac{2\cos x}{\cos x - \sin x} dx &= \text{MANIPULATE IN THE FORM } A(\cos x - \sin x) + B \frac{d}{dx}(\cos x - \sin x) \\ &= \int (\cos x - \sin x) \tan x + \sec x dx \\ &= \int \frac{\cos x - \sin x}{\cos x - \sin x} - \frac{\sin x + \cos x}{\cos x - \sin x} dx \quad \text{OF THE FORM } \int \frac{f(x)}{g(x)} dx \\ &= \int 1 - \frac{\sin x + \cos x}{\cos x - \sin x} dx \\ &= x - \ln|\cos x - \sin x| + C\end{aligned}$$

644. $\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$

$$\begin{aligned}\int \sec^3 x dx &= \int \sec x \sec^2 x dx = \text{INTEGRATION BY PARTS} \\ &\quad \begin{array}{c} \sec x \\ \hline \tan x \\ \sec x \end{array} \\ \int \sec^3 x dx &= \sec x \tan x - \int \sec x \tan^2 x dx \\ \int \sec^3 x dx &= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ \int \sec^3 x dx &= \sec x \tan x - \int \sec x - \sec x dx \\ \int \sec^3 x dx &= \sec x \tan x - \int \sec x dx + \int \sec x dx \\ \int \sec^3 x dx &= \sec x \tan x - \left[\sec x + \ln|\sec x + \tan x| \right] + C \\ 2 \int \sec^3 x dx &= \sec x \tan x + \ln|\sec x + \tan x| + C \\ \int \sec^3 x dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C\end{aligned}$$

645. $\int \frac{1}{x+x^4} dx = \frac{1}{3} \ln \left| \frac{x^3}{x^3+1} \right| + C$

$$\begin{aligned}\int \frac{1}{x+x^4} dx &= \int \frac{1}{x(1+x^3)} dx = \int \frac{x^2}{x^3(1+x^3)} dx \\ &\text{NOW USE THE SUBSTITUTION } u = x^3+1 \text{ OR NOTICE THAT THIS IS} \\ &\text{OF THE FORM } \int \frac{1}{t(t+1)} dt \\ &= \int \frac{-\frac{1}{3}x^{-\frac{2}{3}}}{1+x^3} dx = -\frac{1}{3} \ln|1+x^3| + C \\ &= -\frac{1}{3} \ln| \frac{1}{x^3} + 1 | + C = -\frac{1}{3} \ln| \frac{1+x^3}{x^3} | + C \\ &= \frac{1}{3} \ln| \frac{x^3}{1+x^3} | + C \\ &\text{FACTORISING THE DENOMINATOR INTO } x(1+x^2) = x(1+x)(1-x+1) \text{ ALSO} \\ &\text{WORKS, BUT IT IS EASIER LONGER.}\end{aligned}$$

646. $\int \frac{1}{x^2+8x+17} dx = \arctan(x+4) + C$

$$\begin{aligned}\int \frac{1}{x^2+8x+17} dx &= \int \frac{1}{(x+4)^2+17} dx = \int \frac{1}{(x+4)^2+1} dx \\ &\text{STANDARD ARCTAN} \\ &= \arctan(x+4) + C\end{aligned}$$

647. $\int x^2 \cos^3 x \, dx = \dots$

$$\dots = \frac{1}{3} x^2 (3 - \sin^2 x) \sin x + \frac{2}{9} x (6 + \cos^2 x) \cos x - \frac{2}{27} (21 - \sin^2 x) \sin x + C$$

As this is + (more) integration by parts, it is sensible to combine these first:

$$\begin{aligned} \int \cos x \, dx &= \int \sin x' \, dx = \int \cos(1 - \sin x) \, dx = \int (\cos 1 - \cos x) \, dx \\ &= \sin x - \frac{1}{2} \sin^2 x + C \\ \int \sin x \, dx &= \int \cos x' \, dx = \int \sin(1 - \cos x) \, dx = \int (\sin 1 - \sin x) \, dx \\ &= -\cos x + \frac{1}{2} \cos^2 x + C \end{aligned}$$

$\int x \cos^2 x \, dx = \dots$ INTEGRATION BY PARTS

$$\begin{aligned} &= x \left[\frac{1}{2} \sin^2 x \right] - 2 \int \sin x \left(-\frac{1}{2} \sin x \right) \, dx \\ &= \frac{1}{2} x \sin(3 - \sin^2 x) - 2 \int \sin x \cdot \frac{1}{2} \sin x \, dx \\ &= \frac{1}{2} x \sin(3 - \sin^2 x) - \frac{1}{2} \int \sin^2 x \, dx + \frac{1}{2} \int \sin x \, dx \\ &= \frac{1}{2} x \sin(3 - \sin^2 x) - 2 \int \sin x \cos x \, dx - \frac{1}{2} \int \cos x \, dx \\ &\quad + \frac{1}{2} \int \cos x \cdot \frac{1}{2} \sin^2 x \, dx - \int \cos x \cdot \frac{1}{2} \sin^2 x \, dx \\ &= \frac{1}{2} x \sin(3 - \sin^2 x) + 2x \cos x - 2 \sin x - \frac{1}{2} \sin x + \frac{1}{2} \int \cos x \, dx \\ &= \frac{1}{2} x \sin(3 - \sin^2 x) + 2x \cos x - 2 \sin x - \frac{1}{2} \sin x + \frac{1}{2} \int \cos x \, dx + C \\ &= \frac{1}{2} x \sin(3 - \sin^2 x) + \frac{1}{2} \sin x + \frac{1}{2} \cos x - \frac{1}{2} \sin x + \frac{1}{2} \cos x + C \\ &= \frac{1}{2} x \sin(3 - \sin^2 x) + \frac{1}{2} \cos x + \frac{1}{2} \cos x + C \\ &= \frac{1}{2} x \sin(3 - \sin^2 x) + \frac{1}{2} \cos x + \frac{1}{2} \cos x + C \\ &= \frac{1}{2} x \sin(3 - \sin^2 x) + \frac{1}{2} \cos x + \frac{1}{2} \cos x + C \end{aligned}$$

648. $\int \frac{1}{\sqrt{16x^2+1}} \, dx = \begin{cases} \frac{1}{4} \ln \left[4x + \sqrt{16x^2+1} \right] + C \\ \frac{1}{4} \operatorname{arsinh}(4x) + C \end{cases}$

$$\begin{aligned} \int \frac{1}{\sqrt{16x^2+1}} \, dx &= \int \frac{1}{\sqrt{16x^2+1^2}} \, dx = \frac{1}{4} \int \frac{1}{\sqrt{x^2+\left(\frac{1}{4}\right)^2}} \, dx \\ &= \frac{1}{4} \operatorname{arsinh}\left(\frac{x}{\frac{1}{4}}\right) + C = \frac{1}{4} \operatorname{arsinh}(4x) + C \\ &= \frac{1}{4} \ln(4x + \sqrt{16x^2+1}) + C \end{aligned}$$

649. $\int \frac{1}{2 + \cos x} \, dx = \frac{2}{\sqrt{3}} \arctan \left[\frac{1}{\sqrt{3}} \tan \left(\frac{1}{2} x \right) \right] + C$

BY THE little t TRICKS & SUBSTITUTION

$$\begin{aligned} \int \frac{1}{2 + \cos x} \, dx &= \int \frac{1}{2 + \frac{1-t^2}{1+t^2}} \left(\frac{2t}{1+t^2} dt \right) \\ &= \int \frac{2}{2(1+t^2)(1-t^2)} \, dt = \int \frac{2}{t^2+2} \, dt = \int \frac{2}{t^2+2t^2+2} \, dt \\ &= \frac{2}{13} \operatorname{arctan}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{2}{13} \operatorname{arctan}\left[\frac{\tan \frac{1}{2} x}{\sqrt{3}}\right] + C \end{aligned}$$

650. $\int \frac{\sin x}{\sin(x + \frac{1}{4}\pi)} dx = \frac{\sqrt{2}}{2} \left[x - \ln \left| \sin \left(x + \frac{1}{4}\pi \right) \right| \right] + C$

$$\int \frac{\sin x}{\sin(x + \frac{1}{4}\pi)} dx$$
 = FIND A SIMPLE SUBSTITUTION TO FACILITATE STARTING OF THE FRACTION LATER.

$$u = 2x + \frac{\pi}{2}$$

$$du = 2dx$$

$$= \int \frac{\sin(u - \frac{\pi}{4})}{\sin u} du = \int \frac{\sin(\cos \frac{\pi}{4} - \sin \frac{\pi}{4})}{\sin u} du$$

$$= \int (\cos \frac{\pi}{4} - \sin \frac{\pi}{4}) \sin u du = \frac{\sqrt{2}}{2} \int [1 - \sin(u)] \sin u du$$

$$= \frac{\sqrt{2}}{2} \left[u - \ln |\sin u| \right] + C = \frac{\sqrt{2}}{2} \left[x + \frac{\pi}{2} - \ln |\sin(x + \frac{\pi}{4})| \right] + C$$

$$= \frac{\sqrt{2}}{2} \left[x - \ln |\sin(x + \frac{\pi}{4})| \right] + \left(\frac{\sqrt{2}}{2} + C \right) = \frac{\sqrt{2}}{2} \left[x - \ln |\sin(x + \frac{\pi}{4})| \right] + C$$

651. $\int \frac{1}{\tanh x + \tan x} dx = \ln |\sinh x \cos x + \sin x \cosh x| + C$

$$\int \frac{1}{\tanh x + \tan x} dx = \int \frac{1}{\frac{\sinh x + \sin x}{\cosh x + \cos x}} dx$$

$$= \int \frac{\cosh x + \cos x}{\sinh x + \sin x} dx = \int \frac{\cosh x - \sin x}{\sinh x + \cos x} dx$$

NO OBTAIN THE FRACTION

$$\frac{d}{dx} [\sinh x + \cos x] = \cosh x \cos x - \sinh x \sin x + \cosh x \sin x = 2 \cosh x \cos x$$

WHICH MEANS THAT THE INTEGRAND IS OF THE FORM $\frac{f'(u)}{f(u)}$

$$\therefore \int \frac{1}{\tanh x + \tan x} dx = \frac{1}{2} \ln |\sinh x + \cos x| + C$$

652. $\int \frac{(2x+1)^2}{x(x+1)^4} dx = \frac{1}{3(x+1)^3} - \frac{3}{2(x+1)^2} + \frac{1}{x+1} + \ln \left| \frac{x}{x+1} \right| + C$

$$\int \frac{(2x+1)^2}{x(x+1)^4} dx$$
 WE CHOOSE AVOID THE PARTIAL FRACTION, BUT IT IS BETTER TO START WITH A SUBSTITUTION

$$u = 2x+1$$

$$du = 2dx$$

$$= \int \frac{(2u-1+1)^2}{u(u-1)^4} du = \int \frac{(2u-1)^2}{u(u-1)} \times \frac{1}{u^2} du = \int \frac{4u^2-4u+1}{u(u-1)} \frac{1}{u^2} du$$

~LONG DIVISION FRACTION?

$$\begin{array}{r} -1 + 3u - u^2 + u^3 \\ \hline 1 - 4u + 4u^2 \\ -4 + u \\ \hline -3u^2 + u \\ -3u^2 + u \\ \hline 0 \end{array}$$

$$\frac{4u^2-4u+1}{u^2} = \frac{4u^2}{u^2} + \frac{-4u}{u^2} + \frac{1}{u^2}$$

STOP HERE

$$= \int [1 + 3u - u^2 - u^3 + \frac{1}{u^2}] \times \frac{1}{u^2} du = \int \frac{1}{u^2} + \frac{3}{u^3} + \frac{1}{u^4} - \frac{1}{u^5} + \frac{1}{u^6} du$$

$$= \frac{1}{3u^3} - \frac{3}{2u^4} + \frac{1}{u^5} - \ln |u| + \ln(u-1) + C = \frac{1}{3u^3} - \frac{3}{2u^4} + \frac{1}{u^5} + \ln \left| \frac{2x+1}{x} \right| + C$$

653. $\int \frac{1}{x^2(1+x^4)^{\frac{3}{4}}} dx = -\frac{1}{x} (1+x^4)^{\frac{1}{4}} + C$

$$\begin{aligned} \int \frac{1}{x^2(1+x^4)^{\frac{3}{4}}} dx &= \int \frac{1}{x^2(2x^4+1)^{\frac{3}{4}}} dx = \int \frac{1}{x^2 x^{\frac{3}{2}}(2x^4+1)^{\frac{1}{4}}} dx \\ &= \int x^{-\frac{3}{2}}(2x^4+1)^{-\frac{1}{4}} dx \end{aligned}$$

THE SUBSTITUTION $u = 1+2x^4$ WOULD WORK
OR DECIMATION

$$\begin{aligned} &= -(2x^4+1)^{\frac{1}{4}} + C \\ &= -(\frac{x^4+1}{2^4})^{\frac{1}{4}} + C = -\frac{1}{2}(1+x^4)^{\frac{1}{4}} + C \end{aligned}$$

654. $\int \frac{1}{x(x-\sqrt{x^2-1})} dx = x + \sqrt{x^2-1} + \arcsin\left(\frac{1}{x}\right) + C$

$$\begin{aligned} \int \frac{1}{x(x-\sqrt{x^2-1})} dx &= \dots \text{ SIMPL WITH A SIMPLE STANDARD SUBSTITUTION} \\ &= \int \frac{1}{\frac{1}{2}(u+\sqrt{u^2-1})} \left(\frac{1}{u^2} du \right) = \int \frac{-1}{u^2 - \frac{1}{4}u^2 - \frac{1}{4}} \left(\frac{du}{u^2} \right) \\ &= \int \frac{-1}{1-\frac{1}{4}u^2} du = \int \frac{-1}{1-\frac{1}{4}u^2} du \\ &= \int \frac{-1}{1-(\frac{1}{2}\sin\theta)^2} (\cos\theta d\theta) = \int \frac{-\cos\theta}{1-\cos^2\theta} d\theta \\ &= \int \frac{-\cos\theta(1+\cos\theta)}{1-(1+\cos\theta)\cos\theta} d\theta = \int \frac{-\cos\theta-\cos^2\theta}{1-\cos\theta} d\theta \\ &= \int \frac{-\cos\theta-\cos^2\theta}{\sin^2\theta} d\theta = \int -\frac{\cos\theta}{\sin^2\theta} - \cos\theta d\theta \\ &= \int -\frac{\cos\theta}{\sin^2\theta} d\theta = -\frac{1}{2}\frac{d}{d\theta} \left(\frac{1}{\sin\theta} \right) \\ &= \int -\frac{1}{2}\frac{d}{d\theta} \left(\frac{1}{\sin\theta} \right) - \cos\theta d\theta + C = -\frac{1}{2}\frac{d}{d\theta} \left(\frac{1}{\sin\theta} \right) + \cos\theta + C \\ &= \frac{1}{2} \left[\frac{1}{\sin\theta} + \frac{\sqrt{1-\cos^2\theta}}{\sin\theta} + \cos\theta \right] + C = \frac{1}{2} \left[\frac{1}{\sin\theta} + \frac{\sqrt{1-x^2}}{x} + \cos\theta \right] + C \end{aligned}$$

$\frac{1}{u} = \frac{1}{\sin\theta}$
 $u = \sin\theta$
 $du = \cos\theta d\theta$
 $\cos\theta = \frac{1}{u}$
 $d\theta = \frac{du}{\cos\theta} = \frac{du}{\frac{1}{u}} = u du$

655. $\int \frac{\sqrt{x^2-1}}{x} dx = \sqrt{x^2-1} - \arctan\left(\sqrt{x^2-1}\right) + C$

$$\begin{aligned} \int \frac{\sqrt{x^2-1}}{x} dx &= \dots \text{ BY SUBSTITUTION} \\ &= \int \frac{u}{x} \left(\frac{du}{dx} dx \right) = \int \frac{u^2}{x} du = \int \frac{u^2}{u^2+1} du \\ &= \int \frac{(u^2+1)-1}{u^2+1} du = \int 1 - \frac{1}{u^2+1} du \\ &= u - \arctan u + C = \arctan(u) + C \end{aligned}$$

$u = \sqrt{x^2-1}$
 $u^2 = x^2-1$
 $2u du = 2x dx$
 $du = \frac{x}{u} dx$
 $u^2 = \frac{x^2-1}{x^2+1}$

NOTE THAT THE SUBSTITUTION $2x$ IS USED ALSO WORKS FOR IT IS CLEVER

656. $\int \frac{(2 + \tan^2 x) \sec^2 x}{1 + \tan^3 x} dx = \ln |1 + \tan x| + \frac{2}{\sqrt{3}} \arctan\left(\frac{2 \tan x - 1}{\sqrt{3}}\right) + C$

$$\begin{aligned} & \int \frac{(2 + \tan^2 x) \sec^2 x}{1 + \tan^3 x} dx = \dots \text{ proceed with the obvious substitution} \\ & = \int \frac{(2 + u^2) \sec^2 x}{1 + u^3} du = \int \frac{\frac{u^2+2}{u^2+1} du}{1+u^3} = \int \frac{u^2+2}{(u^2+1)(u^2+u+1)} du \\ & = \int \frac{u^2+2}{(u(u+1)(u^2+u+1))} du \quad \text{REARRANGE} \\ & \quad \boxed{\frac{u^2+2}{(u(u+1)(u^2+u+1))} = \frac{A}{u+1} + \frac{Bu+C}{(u^2+1)}} \\ & \quad \text{IF } u=1 \Rightarrow \quad \bullet \quad u=0 \quad \bullet \quad \text{looking at coeff of } u^2 \\ & \quad 3=3A \quad 2=C+A \quad A+B=1 \\ & \quad A=1 \quad C=1 \quad B=0 \\ & = \int \frac{\frac{1}{u+1} + \frac{1}{u^2+1}}{u^2+u+1} du = \int \frac{\frac{1}{u+1} + \frac{4}{4u^2+4u+4}}{u^2+u+1} du \\ & = \int \frac{\frac{1}{u+1} + \frac{4}{(2u+1)^2}}{u^2+u+1} du = \int \frac{\frac{1}{u+1} + \frac{4}{(2u+1)^2} du}{u^2+u+1} \\ & = \ln|u+1| + \frac{4}{\sqrt{3}} \arctan\left(\frac{2u+1}{\sqrt{3}}\right) + C = \ln|1+\tan x| + \frac{4}{\sqrt{3}} \arctan\left(\frac{2\tan x + 1}{\sqrt{3}}\right) + C \end{aligned}$$

657. $\int \frac{1+\sqrt{x+1}}{\sqrt[3]{x+1}+\sqrt{x+1}} dx = x - \frac{6}{5}(x+1)^{\frac{5}{6}} + \frac{3}{2}(x+1)^{\frac{2}{3}} + C$

$$\begin{aligned} & \int \frac{1+\sqrt{x+1}}{\sqrt[3]{x+1}+\sqrt{x+1}} dx = \dots \text{ BY SUBSTITUTION...} \\ & \quad u = (x+1)^{\frac{1}{3}} \quad u^3 = (x+1) \\ & \quad \frac{du}{dx} = \frac{1}{3}(x+1)^{-\frac{2}{3}} \quad du = \frac{1}{3}(x+1)^{-\frac{2}{3}} dx \\ & \quad \int \frac{1+u^3}{u+u^2} \cdot \frac{1}{3}(x+1)^{-\frac{2}{3}} dx = \int \frac{1+u^3}{u(u+1)} du = \int \frac{u^2+u+1}{u(u+1)} du \\ & \quad = \int \frac{u^2+u+1}{u^2+u} du = \int 1 du = u \\ & \quad = u^3 - \frac{5}{3}u^{\frac{9}{2}} + \frac{3}{2}u^{\frac{7}{2}} + C = (x+1)^{\frac{1}{3}} - \frac{5}{3}(x+1)^{\frac{9}{2}} + \frac{3}{2}(x+1)^{\frac{7}{2}} + C \\ & \quad = x - \frac{5}{3}(x+1)^{\frac{9}{2}} + \frac{3}{2}(x+1)^{\frac{7}{2}} + C \end{aligned}$$

658. $\int \operatorname{sech} x dx = \begin{cases} 2 \arctan(e^x) + C \\ \arctan(\sinh x) + C \\ 2 \arctan\left(\tan \frac{1}{2} x\right) + C \end{cases}$

$$\begin{aligned} & \int \operatorname{sech} x dx = \int \frac{1}{\cosh x} dx = \int \frac{1}{\frac{e^x+e^{-x}}{2}} dx = \int \frac{2}{e^x+e^{-x}} dx = \int \frac{2e^x}{e^{2x}+1} dx \\ & = \int \frac{2e^x}{e^{2x}+1} dx = \dots \text{ SUBSTITUTION } u = e^x \text{ OR INSPECTION} \\ & = 2 \operatorname{arctan}(e^x) + C \\ & \text{ALTERNATIVE APPROACH} \\ & \int \operatorname{sech} x dx = \int \frac{\cosh x}{\cosh^2 x} dx = \int \frac{\cosh x}{1+\sinh^2 x} dx \\ & \quad \text{APPROX BY SUBSTITUTION, INSPECTION OR INSPECTION} \\ & = \operatorname{arctan}(\sinh x) + C \\ & \text{ALTERNATIVE BY THE LITTLE KNOTS IN PARAMETRIC} \\ & \quad \boxed{\begin{aligned} & \text{to tanh } \frac{x}{2} dx = \frac{1}{1+t^2} dt \quad \operatorname{cosec} \frac{1-t^2}{1+t^2} \\ & \text{to sinh } \frac{x}{2} dx = \frac{t}{1-t^2} dt \quad \operatorname{cosec} \frac{1+t^2}{1-t^2} \end{aligned}} \\ & \int \operatorname{sech} x dx = \int \frac{1}{\cosh x} dx = \int \frac{\frac{1-t^2}{1+t^2}}{\frac{1+t^2}{1-t^2}} dt = \int \frac{2}{1+t^2} dt \\ & = 2 \operatorname{arctan} t + C = 2 \operatorname{arctan}(\tanh \frac{x}{2}) + C \end{aligned}$$

659. $\int \frac{2x-3}{\sqrt{x^2-9}} dx = \begin{cases} 2\sqrt{x^2-9} - 3\operatorname{arcosh}\left(\frac{1}{3}x\right) + C \\ 2\sqrt{x^2-9} - 3\ln\left(x + \sqrt{x^2-9}\right) + C \end{cases}$

$$\begin{aligned} \int \frac{2x-3}{\sqrt{x^2-9}} dx &= \int \frac{2x}{\sqrt{x^2-9}} dx - \frac{3}{\sqrt{x^2-9}} dx \\ &= \int 2x \cdot \frac{1}{\sqrt{x^2-9}} dx - \frac{3}{\sqrt{x^2-9}} dx \\ &\quad \text{DUPLICATE} \quad \text{STANDARD FORM} \\ u &= x^2-9 \\ &= 2(x^2-9)^{-\frac{1}{2}} - 3\operatorname{arcosh}\left(\frac{1}{3}x\right) + C \\ &= 2\sqrt{x^2-9} - 3\ln\left(\frac{x}{\sqrt{x^2-9}} + \sqrt{1-\frac{9}{x^2}}\right) + C \\ &= 2\sqrt{x^2-9} - 3\ln\left(\frac{x}{\sqrt{x^2-9}} + \sqrt{1-\frac{9}{x^2}}\right) + C \\ &= 2\sqrt{x^2-9} - 3\ln\left(x + \sqrt{x^2-9}\right) + C \end{aligned}$$

660. $\int \frac{1}{1+\tan x} dx = \frac{1}{2}x + \frac{1}{2}\ln|\cos x + \sin x| + C$

$$\begin{aligned} \int \frac{1}{1+\tan x} dx &= \int \frac{1}{1+\frac{\sin x}{\cos x}} dx = \int \frac{\cos x}{\cos x + \sin x} dx \\ &\quad \text{MANUFACTURE TO DIFFERENT } -(1/\cos x)(\cos x) + \frac{1}{2}(1/\cos x)(\cos x) \\ &= \frac{1}{2} \int \frac{2\cos x}{\cos x + \sin x} dx = \frac{1}{2} \int (\cos x + \sin x + \cos x - \sin x) dx \\ &= \frac{1}{2} \int 1 + \frac{-\sin x + \cos x}{\cos x + \sin x} dx = \frac{1}{2} \int [x + \ln|\cos x + \sin x|] + C \\ &\quad \text{DUPLICATE} \\ &= \frac{1}{2}x + \frac{1}{2}\ln|\cos x + \sin x| + C \end{aligned}$$

661. $\int \frac{\sqrt{x}}{1-x^3} dx = \begin{cases} \frac{2}{3}\operatorname{artanh}\left(x^{\frac{3}{2}}\right) + C \\ \frac{1}{3}\ln\left|\frac{1+x^{\frac{3}{2}}}{1-x^{\frac{3}{2}}}\right| + C \end{cases}$

$$\begin{aligned} \int \frac{\sqrt{x}}{1-x^3} dx &= \dots \text{BY INSPECTION} \dots \int \frac{x^{\frac{1}{2}}}{1-x^3} dx = \dots \\ \text{NOW } \frac{d}{dx}(\operatorname{artanh} x) &= \frac{1}{1-x^2} \Rightarrow \frac{d}{dx}(\operatorname{artanh} x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}} \times \frac{1}{1-x^3} \\ \dots &= \frac{3}{2}x^{\frac{1}{2}}\operatorname{artanh} x^{\frac{3}{2}} + C = \frac{3}{2} \times \frac{1}{2} \left[\frac{x^{\frac{3}{2}}}{1-x^3} \right] = \frac{3}{4} \left[\frac{x^{\frac{3}{2}}}{1-x^3} \right] + C \end{aligned}$$

ALTERNATIVE BY SUBSTITUTION AND PARTIAL FRACTION

$$\begin{aligned} \int \frac{\sqrt{x}}{1-x^3} dx &= \int \frac{\sqrt{x}}{1-u^3} \times \frac{2}{3}u^{-\frac{1}{2}} du \\ &= \int \frac{2}{3} \frac{\sqrt{x}}{1-u^3} \cdot \frac{du}{u^{\frac{1}{2}}} = \frac{2}{3} \int \frac{1}{u^{\frac{1}{2}}(1-u^3)} du \\ &\quad \text{SUBSTITUTE TO AVOID USE OF PARTIAL FRACTIONS} \\ &= \frac{2}{3} \int \frac{1}{(1-u^3)u^{\frac{1}{2}}} du = \frac{2}{3} \int \frac{\frac{1}{u^{\frac{1}{2}}}}{1-u^3} du = \frac{2}{3} \int \frac{1}{u^{\frac{1}{2}}} - \frac{1}{1-u^3} du \\ &= \frac{2}{3} [\ln(u^{\frac{1}{2}}) - \ln(1-u^3)] = \frac{2}{3} \ln\left(\frac{u^{\frac{1}{2}}}{1-u^3}\right) + C = \frac{2}{3} \ln\left(\frac{\sqrt{x}}{1-x^3}\right) + C \end{aligned}$$

AS SOON AS

662. $\int \sqrt{2^x - 1} dx = \frac{2}{\ln 2} \left[\sqrt{2^x - 1} - \arctan(\sqrt{2^x - 1}) \right] + C$

$$\begin{aligned} \int \sqrt{2^x - 1} dx &= \text{BY SUBSTITUTION} \\ &= \int u - \frac{2u}{2^x} du = \frac{1}{\ln 2} \int \frac{2^x}{2^x} du \\ &= \frac{1}{\ln 2} \int 2^x du = \frac{2}{\ln 2} \int \frac{2^x}{2^x+1} du = \frac{2}{\ln 2} \int \frac{(2^x+1)-1}{2^x+1} du \\ &= \frac{2}{\ln 2} \int 1 - \frac{1}{2^x+1} du = \frac{2}{\ln 2} \left[u - \arctan(u) \right] + C \\ &= \frac{2}{\ln 2} \left[\sqrt{2^x - 1} - \arctan(\sqrt{2^x - 1}) \right] + C \end{aligned}$$

THE SUBSTITUTION $u = 2^x$ AND WORKS WELL.

663. $\int \frac{4 \sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2} dx = \frac{1}{3} \sec 3x + C$

$$\begin{aligned} \int \frac{4 \sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2} dx &= \text{EXPAND THE DENOMINATOR & SIMPLIFY} \\ &= \frac{4 \sin 3x}{(\cos^2 7x + 2 \cos 7x \cos x + \cos^2 x) + (\sin^2 7x + 2 \sin 7x \sin x + \sin^2 x)} \\ &= \frac{4 \sin 3x}{2 + 2 \cos(7x-x)} \\ &= \frac{4 \sin 3x}{2 + 2 \cos 6x} \\ &= \frac{4 \sin 3x}{2(1 + \cos 6x)} \\ &= 2 \sin 3x \\ \dots \int \frac{4 \sin 3x}{2 \sin 3x} dx &= \int \frac{2 \sin 3x}{\cos 3x} \frac{1}{\cos 3x} dx = \int \tan 3x \sec 3x dx \\ &= \frac{1}{3} \sec 3x + C \end{aligned}$$

664. $\int \frac{4x}{4+x^4} dx = \arctan\left(\frac{1}{2}x^2\right) + C$

$$\begin{aligned} \int \frac{4x}{4+x^4} dx &= \dots \text{BY SUBSTITUTION} = \dots \arctan\left(\frac{1}{2}x^2\right) + C \\ \frac{d}{dx}(\arctan x^2) &= \frac{2x}{1+x^4} \\ \frac{d}{dx}(\arctan(3x^2)) &= \frac{6x}{1+(3x^2)^2} = \frac{6x}{1+9x^4} \end{aligned}$$

THE SUBSTITUTION $x = \sqrt{3}u$ AND WORKS WELL.

665. $\int \frac{12x^2}{(x^2-1)^2} dx = -\frac{6x}{x^2-1} + 3 \ln \left| \frac{x-1}{x+1} \right| + C$

$$\int \frac{12x^2}{(x^2-1)^2} dx = \int \frac{12x^2}{(2-x)(2+x)^2} dx = \dots \text{BY PARTIAL FRACTION}$$

$$\frac{12x^2}{(2-x)(2+x)^2} = \frac{A}{2-x} + \frac{B}{2+x} + \frac{C}{(2+x)^2} + \frac{D}{(2+x)^3}$$

$$12x^2 = A(2+x)^2 + B(2+x)(2+x)^2 + C(2-x)(2+x)^2 + D(2-x)(2+x)^3$$

| | | | |
|-------------------|-------------------|-------------------|---------------------------|
| • If $x=1$ | • If $x=-1$ | • If $x=0$ | • If $x=2$ |
| $12 = 4A$ | $12 = 4B$ | $0 = 4C + 8$ | $48 = 8C + 48 + 16C + 32$ |
| $A = 3$ | $B = 3$ | $C = -2$ | $48 = 24 + 16C + 32$ |
| $\frac{4}{4} = 1$ | $\frac{4}{4} = 1$ | $\frac{8}{8} = 1$ | $48 = 48$ |

$$\int \frac{3}{2-x} dx + \int \frac{3}{2+x} dx + \int \frac{-2}{(2+x)^2} dx + \int \frac{8}{(2+x)^3} dx = -\frac{3}{2-x} + \frac{3}{2+x} + 2\ln|2+x| - 3\ln|2-x| + C$$

$$-\rightarrow 2\left(\frac{1}{2+x} + \frac{1}{2-x}\right) + 2\left(\ln|2+x| - \ln|2-x|\right) + C \rightarrow -2\left(\frac{2x}{2^2-x^2}\right) + 2\ln\left|\frac{2+x}{2-x}\right| + C$$

$$= -\frac{4x}{x^2-4} + 2\ln\left|\frac{2+x}{2-x}\right| + C$$

THE SUBSTITUTIONS ARE EASIER, LESS WORK AND USES LESS UNNECESSARY MANIPULATIONS.

666. $\int \frac{2x+x^2 \cot x}{x^2 + \operatorname{cosec} x} dx = \ln|x+ x^2 \sin x| + C$

$$\int \frac{2x+x^2 \cot x}{x^2 + \operatorname{cosec} x} dx = \int \frac{2x \operatorname{cosec} x + x^2 \cot x}{x^2 + \operatorname{cosec} x} dx = \int \frac{2x \operatorname{cosec} x + x^2 \cot x}{x^2 + \operatorname{cosec} x} dx$$

BY INTEGRATION BY PARTS $\Rightarrow \frac{d}{dx}(2x \operatorname{cosec} x) = 2 \operatorname{cosec} x + 2x \operatorname{cosec} x \cot x$, so $\rightarrow \int \frac{d}{dx}(2x \operatorname{cosec} x) dx = \operatorname{cosec} x + 2x \operatorname{cosec} x \cot x$

$$= \operatorname{cosec} x + 2x \operatorname{cosec} x \cot x + C$$

THE SUBSTITUTION $u = \operatorname{cosec} x$ AND WORKS

667. $\int \frac{1-\tan x}{1+\tan x} dx = \ln|\cos x + \sin x| + C$

$$\int \frac{1-\tan x}{1+\tan x} dx = \int \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}} dx = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

BY INTEGRATION BY PARTS \Rightarrow OR THE FORM $\int \frac{du}{u}$

$$= \ln|\cos x + \sin x| + C$$

668. $\int \frac{\sqrt{\tan x}}{\sin 2x} dx = \sqrt{\tan x} + C$

$\int \frac{\sqrt{\tan x}}{\sin 2x} dx = \dots$ BY SUBSTITUTION...

$$\begin{aligned} &= \int \frac{1}{\sin 2x} \sqrt{\tan x} dx = \int \frac{\sec^2 u}{\sin 2u} du \\ &= \int \frac{u^2}{\tan u} du = \int \frac{u^2}{u} du = u + C \\ &= \sqrt{\tan x} + C \end{aligned}$$

ALTERNATIVE SUBSTITUTION/MANIPULATION

$$\begin{aligned} \int \frac{\sqrt{\tan x}}{\sin 2x} dx &= \int \frac{\sqrt{\tan x}}{2\sin x \cos x} \cdot \frac{1}{\cos x} dx = \int \frac{\sqrt{\tan x}}{2\sin x (\cos x)^2} dx \\ &= \int \frac{1}{2\sin x (\cos x)^2} dx = \int \frac{1}{2\sin x \cos^2 x} dx \\ &= \int \frac{1}{2\sin x \cos^2 x} dx = \int \frac{1}{2\sin x (\cos x)^2} dx \\ &\Rightarrow \text{SUBSTITUTION } u = \tan x \text{ OR REVERSE} \\ &= (\tan x)^{\frac{1}{2}} + C \end{aligned}$$

669. $\int \frac{16}{(x^2+4)^2} dx = \frac{2x}{x^2+4} + \arctan\left(\frac{1}{2}x\right) + C$

$\int \frac{16}{(x^2+4)^2} dx = \dots$ BY SUBSTITUTION ...

$$\begin{aligned} &= \int \frac{16}{(4(\tan^2 \theta))^2} 2\tan \theta d\theta = \int \frac{2\tan \theta}{16\tan^2 \theta} d\theta = \int \frac{2}{\sec^2 \theta} d\theta \\ &= \int 2 \operatorname{d}\! \theta = \int 2(\pm 1 + \tan^2 \theta) d\theta = \int (1 + \tan^2 \theta) d\theta \\ &= \theta + \frac{1}{2}\tan^2 \theta + C = \theta + \sin^2 \theta \cos^2 \theta \\ &= \arctan \frac{x}{2} + \left(\frac{x^2}{x^2+4}\right) + C \\ &= \arctan \frac{x}{2} + \frac{x^2}{x^2+4} + C \end{aligned}$$

$\theta = \arctan \frac{x}{2}$
 $\sin \theta = \frac{x}{\sqrt{x^2+4}}$
 $\cos \theta = \frac{2}{\sqrt{x^2+4}}$

670. $\int \operatorname{arcsec} x dx = \begin{cases} x \operatorname{arcsec} x - \operatorname{arcosh} x + C \\ x \operatorname{arcsec} x - \ln\left(x + \sqrt{x^2-1}\right) + C \end{cases}$

$\int \operatorname{arcsec} x dx =$ BY SUBSTITUTION

$$\begin{aligned} &= \int \theta (\sec \theta \tan \theta) d\theta = \dots \text{ INTEGRATION BY PARTS} \\ &= \theta \sec \theta - \int \sec \theta d\theta = \sec \theta - \ln|\sec \theta + \tan \theta| + C \\ &= \operatorname{arcsec} x - \ln|x + \sqrt{x^2-1}| + C \\ &= \operatorname{arcsec} x - \ln(x + \sqrt{x^2-1}) + C \\ &= \operatorname{arcsec} x - \operatorname{arcosh} x + C \end{aligned}$$

ALTERNATIVE: BY PARTS (OR OTHER WAYS)
THIS ALSO REQUIRES KNOWLEDGE OF INTEGRATION OF

$$\frac{1}{x\sqrt{x^2-1}}$$

$$\begin{aligned} \int \operatorname{arcsec} x dx &= \int x \operatorname{arcsec} x dx = \dots \text{ BY PARTS} \\ &= x \operatorname{arcsec} x - \int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{arcsec} x - \operatorname{cosec} x + C \end{aligned}$$

$$671. \int \frac{x^2}{1-2\sqrt{x}} dx = -\frac{1}{5}x^{\frac{5}{2}} - \frac{1}{8}x^2 - \frac{1}{12}x^{\frac{3}{2}} - \frac{1}{16}x - \frac{1}{32}\ln|1-2\sqrt{x}| + C$$

$$= \frac{1}{32} \left[\frac{1}{5}u^5 - \frac{5}{4}u^4 + \frac{10}{3}u^3 - 5u^2 + 5u - \ln|u| \right] + C, \quad u = 1-2\sqrt{x}$$

$$\begin{aligned} & \int \frac{x^2}{1-2\sqrt{x}} dx = \dots \text{BY SUBSTITUTION} \dots \\ & \int \frac{u^2}{1-2\sqrt{u}} du = \int \frac{(u-1)^2}{u} du = \int \frac{(u-1)^2}{u} du = \int \frac{(u-1)^2}{u} du \\ & = \int u^2 - 2u + 1 du = 10u^2 - 10u + 5 - \frac{1}{3}u^3 du \\ & = \frac{1}{3}u^3 - 5u^2 + 10u^2 - 10u + 5 - \frac{1}{3}u^3 du \\ & = \frac{1}{3}u^3 + \frac{5}{3}u^2 - 5u^2 - 10u + 5 + C \\ & = \frac{1}{3}u^3 - \frac{10}{3}u^2 - 10u + 5 + C \\ & = \frac{1}{3}(u-1)^3 - \frac{10}{3}(u-1)^2 - 10(u-1) + 5 + C \\ & \text{DIV. EVERYTHING BY } (u-1)^2 \\ & = -\frac{1}{3}u - \frac{10}{3}u - \frac{10}{u-1} + 5 + C \end{aligned}$$

$$672. \int \frac{1-x}{(1+x)^2 \sqrt{x}} dx = \frac{2\sqrt{x}}{x+1} + C$$

$$\begin{aligned} & \int \frac{1-x}{\sqrt{x(1+x)}} dx = \dots \text{BY SUBSTITUTION} \dots \\ & \int \frac{1-x}{u(1+u)} du = \int \frac{-2(1-u)}{(1+u)^2} du \\ & \text{A TRIGONOMETRIC SUBSTITUTION TO FOLLOW} \\ & -\int \frac{2(1-u)}{(1+u)^2} du = \int \frac{2u}{(1+u)^2} du \\ & = \int \frac{2u}{u^2+2u+1} du = 2 \int \frac{1}{u+1} - \frac{2u}{u^2+2u+1} du \\ & = 2 \int \frac{1}{u+1} du - \frac{2u}{u^2+2u+1} du = 2 \int \frac{1}{u+1} - \frac{2u}{(u+1)^2} du \\ & = \int 2u du = 2u^2 + C \\ & = 2u^2 \tan^{-1} u + C = 2 \left(\frac{u}{\sqrt{u^2+1}} \right) \left(\frac{1}{\sqrt{u^2+1}} \right) + C \\ & = \frac{2u}{u^2+1} + C = \frac{2\sqrt{x}}{x+1} + C \end{aligned}$$

ALTERNATIVE APPROACH: APPLY THE FIRST SUBSTITUTION

$$\int \frac{1-x}{\sqrt{x(1+x)}} dx = \dots = \int \frac{2(1-u)}{(1+u)^2} du$$

PROCEEDS BY PARTIAL FRACTIONS

$$\frac{2(1-u)}{(1+u)^2} = \frac{A+u}{1+u} + \frac{C+D}{(1+u)^2}$$

$$2(1-u) = (A+u)(1+u) + C+Du$$

- IF $u=1$
- IF $u=0$
- $2=8+D$
- $2=8+D$
- $D=-6$
- $D=-6$
- $D=2A$
- $D=2A$
- $D=0$
- $D=0$
- $D=0$
- $D=0$

$$= 2 \left(\frac{u}{\sqrt{u^2+1}} \right) \left(\frac{1}{\sqrt{u^2+1}} \right) + C = \frac{2u}{u^2+1} + C = \frac{2\sqrt{x}}{x+1} + C$$

$$\begin{aligned} & \int \frac{4}{(1+u)^2} du = -\frac{4}{1+u} \leftarrow \text{THIS IS THE SAME AS THE FIRST ONE, SO USE IT} \\ & = \int \frac{4}{(1+u)^2} du = -2\arctan u \\ & = \int \frac{4}{u^2+2u+1} du = -2\arctan u \\ & = \int 4u du = -2\arctan u \\ & = \int 4 \left(\frac{1}{u^2+2u+1} \right) du = -2\arctan u \\ & = \int 2 \left(\frac{2}{u^2+2u+1} \right) du = -2\arctan u \\ & = 2u + \sin 2u = -2\arctan u + C \\ & = 2u + 2u \tan^{-1} u + C = 2u \tan^{-1} u + C \\ & = 2u \tan^{-1} u + 2u \frac{1}{\sqrt{u^2+1}} + C = \frac{2u}{\sqrt{u^2+1}} + C \end{aligned}$$

673. $\int 2e^x \sqrt{e^{2x}-1} dx = \begin{cases} e^x \sqrt{e^{2x}-1} - \operatorname{arcosh} e^x + C \\ e^x \sqrt{e^{2x}-1} - \ln(e^x + \sqrt{e^{2x}-1}) + C \end{cases}$

$$\int 2e^x \sqrt{e^{2x}-1} dx = \dots \text{ BY SUBSTITUTION}$$

$$= \int 2e^x \operatorname{arcosh}(e^x) dx = \int 2e^x \operatorname{arcosh} u du$$

$$= \int 2 \left(\frac{1}{2} \operatorname{arcosh} u - \frac{1}{2} \right) du = \int \operatorname{arcosh} u - 1 du$$

$$= \frac{1}{2} \operatorname{arcosh} u - u + C = \operatorname{sinh} \operatorname{arcosh} u - u + C$$

$$= \operatorname{cosh} u \sqrt{\operatorname{cosh}^2 u - 1} - u + C \quad \text{NOTE THAT HERE } \operatorname{cosh} u = \sqrt{\operatorname{cosh}^2 u - 1}$$

$$= \operatorname{cosh} u \sqrt{e^{2x}-1} - u + C = e^x / \sqrt{e^{2x}-1} - \ln(e^x + \sqrt{e^{2x}-1}) + C$$

NOTICE THAT THE SUBSTITUTION $u = e^x$ PRODUCES $\int 2\sqrt{u} u du$, WHICH MATCHES \rightarrow EASY ARCSH SUBSTITUTION INTEGRATE

674. $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}} dx = 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} - 6 \ln|1+x^{\frac{1}{6}}| + C$

$$\int \frac{1}{\sqrt{x^2+u^2}} dx = \dots \text{ BY SUBSTITUTION} \dots$$

$$= \int u^{-\frac{1}{2}} \operatorname{cosh}^2(u) du = \int \frac{u^2}{u^2+u^2} du = \int \frac{u^2}{2u^2+u^2} du$$

$$= \int \frac{u^2(u+1)-u(u+1)+(u+1)-1}{u(u+1)} du$$

$$= 6 \int u^2 - u + 1 - \frac{1}{u+1} du = -2u^3 - 3u^2 + Cu - 6 \ln|u+1| + C$$

$$= 2x^{\frac{3}{2}} - 3x^{\frac{2}{3}} + Cx^{\frac{1}{3}} - 6 \ln|1+x^{\frac{1}{6}}| + C$$

675. $\int \frac{1}{\sqrt{1-\sin 2x}} dx = -\sqrt{2} \ln|\tan(\frac{1}{8}\pi - \frac{1}{2}x)| + C$

$$\int \frac{1}{\sqrt{1-\sin 2x}} dx = \int \frac{1}{\sqrt{1-(1-\cos(\frac{\pi}{4}-x))}} dx \quad \text{NOW } \cos(\frac{\pi}{4}-x) = \frac{1}{\sqrt{2}} \cos(\frac{\pi}{4}) + \frac{1}{\sqrt{2}} \sin(x)$$

$$= \int \frac{1}{\sqrt{2\sin^2(\frac{\pi}{4}-x)}} dx = \int \frac{1}{\sqrt{2}\sin(\frac{\pi}{4}-x)} dx$$

$$= \int \frac{1}{\sqrt{2}} \frac{1}{\sin(\frac{\pi}{4}-x)} dx \quad \text{USING ANGULAR SUM IDENTITY}$$

$$= \frac{\sqrt{2}}{2} \int \operatorname{cosec}(\frac{\pi}{4}-x) dx = -\sqrt{2} \ln|\operatorname{tang}(\frac{\pi}{4}-x)| + C$$

676. $\int e^x (3\sec^2 x + 2\sec^2 x \tan x + 2 \tan x) dx = e^x (\sec^2 x + 2 \tan x) + C$

$$\begin{aligned} & \int e^x (3\sec^2 x + 2\sec^2 x \tan x + 2 \tan x) dx \\ & \text{Differentiate } e^x \text{ with respect to } x \rightarrow e^x \quad \text{From differentiation + result } e^x(\sec^2 x + 2 \tan x) \\ & \frac{d}{dx}(e^x) = e^x \quad \text{By INSERCTION} \\ & \frac{d}{dx}(3\sec^2 x + 2\sec^2 x \tan x + 2 \tan x) = 3\sec^2 x + 2\sec^2 x \tan x + 2 \tan x \quad \times 2 \quad \text{From differentiation + product } e^x(3\sec^2 x + 2\sec^2 x \tan x + 2 \tan x) \\ & \therefore = 2e^x \tan x + e^x \sec^2 x + C \\ & = e^x (\sec^2 x + 2 \tan x) + C \end{aligned}$$

677. $\int \sqrt[3]{\tan x} dx = \frac{1}{4} \ln \left| \frac{\tan^{\frac{4}{3}} x - \tan^{\frac{2}{3}} x + 1}{\tan^{\frac{4}{3}} x - 2 \tan^{\frac{2}{3}} x + 1} \right| + \frac{\sqrt{3}}{2} \arctan \left[\frac{2 \tan^{\frac{2}{3}} x - 1}{\sqrt{3}} \right] + C$

$$\begin{aligned} & \int \sqrt[3]{\tan x} dx = \text{SET UP THE INTEGRAL OF THE CUBE ROOT} \\ & = \int \left(\frac{3t^2}{1+t^3} dt \right) = \int \frac{3t^2}{1+t^3} dt \\ & \text{THIS IS } \frac{d}{dt}(\arctan(t^3)) \text{ AFTER INTEGRATION BY PARTS (ANS)} \\ & \text{BY } \int u dv = \int u dv + \int v du \\ & \text{ANOTHER SUBSTITUTION FOLLOWING TO DESIRE THE PRIMARY} \\ & = \int \frac{3t^2}{1+t^3} \frac{du}{dt} dt = \int \frac{3t^2}{1+t^3} du = \int \frac{3t^2}{2(1+u)} du \\ & = \frac{3}{2} \int \frac{u}{1+u^2} du = \frac{3}{2} \int \frac{u}{(1+u)(1-u)} du \\ & = \frac{3}{2} \int \frac{u}{(1+u)(1-u)} du = \frac{3}{2} \int \frac{u}{(1+u)(1-u)} du \\ & \text{PARTIAL FRACTION: NEXT - note } (1+u)(1-u) \text{ IS AN IRREDUCIBLE QUADRATIC} \\ & \frac{u}{(1+u)(1-u)} = \frac{A}{1+u} + \frac{B}{1-u} \quad u^2+1 \\ & u = A(u^2-u) + (1+u)(B+Cu) \\ & \bullet \text{ IF } u=1 \quad \bullet \text{ IF } u=-1 \quad \bullet \text{ IF } u=0 \\ & -1 = 3A \quad 0 = A+C \quad 1 = A+2B+2C \\ & A = \frac{1}{3} \quad 0 = A+C \quad 1 = A+2B+2C \\ & C = \frac{1}{3} \quad C = -A \quad 1 = A+2(-A)+2C \\ & B = \frac{2}{3} \quad B = -\frac{1}{3} \quad 1 = A-\frac{2}{3}+2C \\ & = \frac{3}{2} \int \frac{-\frac{1}{3}}{1+u} du + \frac{\frac{2}{3}u + \frac{1}{3}}{1-u} du = \frac{3}{2} \int \frac{1}{1+u^2} du + \frac{u+1}{u^2-1} du \\ & \text{MANIPULATE EQUALLY TO} \\ & = \int \frac{1}{u^2+1} du + \frac{1}{2} \int \frac{-u+1}{u^2-1} du = -\frac{1}{2} \ln|u+1| + \frac{1}{2} \int \frac{2u+2}{u^2-1} du \\ & = -\frac{1}{2} \ln|u+1| + \frac{1}{2} \int \frac{(2u+2)+3}{u^2-1} du = -\frac{1}{2} \ln|u+1| + \frac{1}{2} \int \frac{2u+2}{u^2-1} du + \frac{3}{2} \int \frac{1}{u^2-1} du \\ & \text{OF THE FORM } \frac{1}{u^2-1} \text{ AND } \frac{3}{2} \int \frac{1}{u^2-1} du = \frac{3}{2} \arctan \left(\frac{u}{\sqrt{2}} \right) + C \end{aligned}$$

$$\begin{aligned} & = -\frac{1}{2} \ln|u+1| + \frac{1}{2} \ln(2-u) + \frac{3}{2} \int \frac{1}{(u+\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} du \\ & \text{NO NEED AS IT IS REDUNDANT} \\ & = \frac{1}{2} \left[\ln(2-u) - 2 \ln|u+1| \right] + \frac{3}{2} \int \frac{1}{(u+\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} du \\ & = \frac{1}{2} \left[\ln|2-u| - \ln|u+1| \right] + \frac{3}{2} \int \frac{1}{(u+\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} du \\ & \text{SIMPLIFIED INTEGRATION} \\ & = -\frac{1}{2} \ln \left| \frac{2-u}{u+1} \right| + \frac{3}{2} \cdot \frac{1}{\sqrt{2}} \arctan \left(\frac{u+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \right) + C \\ & = \frac{1}{2} \ln \left| \frac{2-u}{u+1} \right| + \frac{\sqrt{2}}{2} \arctan \left(\frac{2u+1}{\sqrt{2}} \right) + C \\ & = \frac{1}{2} \ln \left| \frac{2-x}{x+1} \right| + \frac{\sqrt{2}}{2} \arctan \left(\frac{2x+1}{\sqrt{2}} \right) + C \\ & = \frac{1}{2} \ln \left| \frac{4x^2+4x+1}{x^2+2x+1} \right| + \frac{\sqrt{2}}{2} \arctan \left[\frac{2x+1}{\sqrt{2}} \right] + C \end{aligned}$$

678. $\int \frac{x(2-3x)}{x^2+e^{3x}} dx = \ln(1+x^2 e^{-3x}) + C$

$$\begin{aligned} & \int \frac{2(2-3x)}{e^{3x}+x^2} dx = \text{SIMPLIFY THE INTEGRAND} \quad \int \frac{3(2-3x)e^{3x}}{1+x^2e^{3x}} dx \\ & = \int \frac{2e^{3x}-3xe^{3x}}{1+x^2e^{3x}} dx \\ & \text{WHAT IS THE FORM} \\ & \int \frac{f(x)}{g(x)} dx = \ln|f(x)| + C \\ & \frac{d}{dx}(1+x^2e^{3x}) = 0+2x \cdot e^{3x} + x^2 \cdot 3e^{3x} \\ & = \ln(1+x^2e^{3x}) + C \\ & \text{THE SUBSTITUTION } u = 1+x^2e^{3x} \text{ ALSO WORKS} \end{aligned}$$

$$679. \quad \int \frac{(x+6)}{(x+2)^2} dx = \frac{2x^{\frac{3}{2}}}{x+2} + C$$

ALTERNATIVE APPROACH

$$\int \frac{(x+6)^{10}}{(x^2+1)^8} dx = \dots \text{start with a trigonometric substitution} \dots$$

| | |
|---|--|
| $x = 2\tan\theta \quad \frac{dx}{d\theta} = 2\sec^2\theta$ $\sqrt{x^2 + 1} = \sqrt{\tan^2\theta + 1} = \sec\theta$ $\sec\theta = \frac{\sqrt{x^2 + 1}}{\sqrt{x^2}}$ $\cos\theta = \frac{x}{\sqrt{x^2 + 1}}$ | |
|---|--|

$$\begin{aligned}
 &= \int \frac{\cancel{(x^2+1)^8} \cdot \cancel{(x+6)^{10}}}{(2\tan\theta+1)^2} d\theta \\
 &= \sqrt{2} \int \frac{\cancel{(x^2+1)^8} \cdot \cancel{(x+6)^{10}}}{(2\tan\theta+1)^2} d\theta = \sqrt{2} \int \frac{\cancel{(x^2+1)^8} \cdot \cancel{(x+6)^{10}}}{\cancel{\sec^2\theta}} d\theta \\
 &= \sqrt{2} \int \frac{2^{10}\tan^{10}\theta \cdot \sec^{10}\theta}{\sec^2\theta} d\theta = \sqrt{2} \int (2^{10}\tan^8\theta + 2^{10}\tan^6\theta + 2^{10}\tan^4\theta + 2^{10}\tan^2\theta + 1) \sec^8\theta d\theta \\
 &= \sqrt{2} \int (2^{10}\tan^8\theta + 2^{10}\tan^6\theta + 3 \cdot 2^{10}\tan^4\theta + 1) \sec^8\theta d\theta \\
 &= \sqrt{2} \int (2^{10}\tan^8\theta + 2^{10}\tan^6\theta + 3 \cdot 2^{10}\tan^4\theta + 1) \sec^8\theta d\theta \\
 &= \sqrt{2} \int (2^{10}\tan^8\theta + 2^{10}\tan^6\theta + 3 \cdot 2^{10}\tan^4\theta + 1) \sec^8\theta d\theta \\
 &= \sqrt{2} \int (2^{10}\tan^8\theta - 2^{10}\tan^6\theta) + C = \sqrt{2} \int 2^{10}\tan^4\theta(\tan^4\theta - 1) + C \\
 &= \sqrt{2} \int 2^{10}\tan^4\theta(\tan^4\theta - 1) + C = \sqrt{2} \int (\tan^4\theta - 1)\tan^4\theta d\theta + C \\
 &= \sqrt{2} \left[\frac{2\tan^5\theta}{5} - \frac{2\tan^3\theta}{3} \right] + C = \sqrt{2} \left[\frac{2\tan^{\frac{5}{2}}}{5} - \frac{2\tan^{\frac{3}{2}}}{3} \right] + C \\
 &= 2\tan^{\frac{1}{2}} \left[\frac{2\tan^{\frac{3}{2}}}{3} - \frac{2\tan^{\frac{1}{2}}}{5} \right] + C = 2\tan^{\frac{1}{2}} \left[\frac{2}{3} \cdot \frac{4}{25} \right] + C = 2\tan^{\frac{1}{2}} \left[\frac{8}{75} \right] + C \\
 &= 2^{\frac{1}{2}} \left(\frac{2^{\frac{1}{2}}}{3^{\frac{1}{2}}} \right) + C = \frac{2^{\frac{3}{2}}}{3^{\frac{1}{2}}} + C
 \end{aligned}$$

at first

THE SUBSTITUTION $x = 2\tan\theta$ ALSO WORKS

680. $\int \frac{9}{(4x^2 - 24x + 27)^{\frac{3}{2}}} dx = \frac{3-x}{\sqrt{4x^2 - 24x + 27}} + C$

681. $\int \frac{\sqrt{x-x^2}}{x^4} dx = -\frac{2}{3}\left(\frac{1}{x}-1\right)^{\frac{3}{2}} - \frac{2}{5}\left(\frac{1}{x}-1\right)^{\frac{5}{2}} + C$

$$\begin{aligned} & \int \frac{\sqrt{\frac{1-x^2}{x^2}} dx}{x^4} = \dots \text{BY SUBSTITUTION} \\ & = \int \frac{\sqrt{\frac{1}{x^2} - \frac{1}{x^2}}}{x^4} \left(-\frac{1}{x^2} dx\right) = \int \frac{\sqrt{\frac{1-x^2}{x^2}}}{x^4} \left(-\frac{1}{x^2} dx\right) \\ & = \int u^2 \frac{\sqrt{\frac{1-u^2}{u^2}}}{u^4} du = \int -u(u-1)^{\frac{1}{2}} du \quad \text{... ANOTHER SUBSTITUTION OR MANIPULATION} \\ & = \int [1-(u-1)^2]^{(u-1)^{\frac{1}{2}}} du = \int -(u-1)^{\frac{1}{2}} - (u-1)^{\frac{3}{2}} du = -\frac{2}{3}(u-1)^{\frac{3}{2}} + \frac{1}{2}(u-1)^{\frac{1}{2}} + C \\ & = -\frac{2}{3}\left(\frac{1}{x}-1\right)^{\frac{3}{2}} - \frac{1}{2}\left(\frac{1}{x}-1\right)^{\frac{1}{2}} + C \end{aligned}$$

682. $\int \frac{16}{(x^2-2x+5)^2} dx = \arctan\left[\frac{1}{2}(x-1)\right] + \frac{2(x-1)}{x^2-2x+5} + C$

$$\begin{aligned} & \int \frac{16}{(x^2-2x+5)^2} dx = \int \frac{16}{[(x-1)^2+4]^2} dx \quad \text{... BY SUBSTITUTION} \\ & \begin{array}{l} x-1=2\sin\theta \\ dx=2\cos\theta d\theta \\ 0+\arctan\left(\frac{3-1}{2}\right) \\ \tan\theta=\frac{3-1}{2} \end{array} \quad \begin{array}{l} \sin\theta=\frac{3-1}{\sqrt{3^2+1^2}} \\ \cos\theta=\frac{2}{\sqrt{3^2+1^2}} \end{array} \\ & = \int \frac{16}{(4\cos^2\theta+4)^2} (2\cos\theta d\theta) = \int \frac{32\cos^3\theta}{16(4\cos^2\theta+4)^2} d\theta = \int \frac{2\cos^2\theta}{\sin^2\theta} d\theta \\ & = \int \frac{2}{\sin^2\theta} d\theta = \int 2\cot^2\theta d\theta = \int 1+\csc^2\theta d\theta = \theta + \frac{1}{2}\csc 2\theta + C \\ & = \theta + \arctan\frac{3-1}{2} + C = \arctan\left(\frac{3-1}{2}\right) + \frac{2(x-1)}{\sqrt{x^2-2x+5}} + C \\ & = \arctan\left(\frac{x-1}{2}\right) + \frac{2(x-1)}{\sqrt{x^2-2x+5}} + C \end{aligned}$$

683. $\int \frac{\tanh x}{\operatorname{sech}^2 x - \tanh^2 x} dx = \frac{1}{2} \ln|1 - \sinh^2 x| + C$

$$\begin{aligned} & \int \frac{\tanh x}{\operatorname{sech}^2 x - \tanh^2 x} dx = \int \frac{\frac{\sinh x}{\cosh x}}{\frac{1}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x}} dx = \int \frac{\cosh x \sinh x}{1 - \sinh^2 x} dx \\ & = \frac{1}{2} \int \frac{2\sinh x \cosh x}{1 - \sinh^2 x} dx \quad \text{or use } \frac{d}{dx} \ln|\frac{\sinh x}{\cosh x}| = \frac{1}{\cosh^2 x} \\ & = \frac{1}{2} \ln|\cosh x| + C \end{aligned}$$

684. $\int \frac{1}{x\sqrt{x^2+x+4}} dx = -\frac{1}{2} \ln \left| 1 + \frac{8}{x} + \frac{4}{x}\sqrt{x^2+x+4} \right| + C$

$$\begin{aligned} & \int \frac{1}{x\sqrt{x^2+x+4}} dx = \dots \text{ by substitution} \dots \\ & \int \frac{1}{x\sqrt{\frac{1}{4}(4x^2+4x+16)}} dx = \int \frac{-1}{x\sqrt{1+\frac{1}{4}(4x^2+4x+15)}} dx \\ & \int \frac{1}{\sqrt{\frac{1}{4}(4x^2+4x+15)}} dx = \int \frac{1}{\sqrt{\frac{1}{4}(4x^2+4x+15)}} dx \\ & = \frac{1}{2} \int \frac{1}{\sqrt{1+\frac{1}{4}(4x^2+4x+15)}} dx = -\frac{1}{2} \int \frac{1}{\sqrt{(2x+1)^2+15}} dx \\ & = -\frac{1}{2} \operatorname{arcsinh} \left(\frac{2x+1}{\sqrt{15}} \right) + C = -\frac{1}{2} \operatorname{arccosh} \left(\frac{2x+1}{\sqrt{15}} \right) + C \\ & = -\frac{1}{2} \ln \left[\frac{2x+1}{\sqrt{15}} + \sqrt{\frac{(2x+1)^2}{15}+1} \right] + C = -\frac{1}{2} \ln \left[\frac{2x+1}{\sqrt{15}} + \sqrt{4x^2+4x+15} \right] + C \\ & = -\frac{1}{2} \ln \left[\frac{2x+1}{\sqrt{15}} + \sqrt{4(x^2+x)+15} \right] + C = -\frac{1}{2} \ln \left[\frac{2x+1}{\sqrt{15}} + \sqrt{4(x+1)^2+11} \right] + C \\ & = -\frac{1}{2} \ln \left[\frac{2x+1}{\sqrt{15}} + \sqrt{4(x+1)^2+11} \right] + C = -\frac{1}{2} \ln \left[2x+1 + \sqrt{4(x+1)^2+11} \right] + C \\ & = -\frac{1}{2} \ln \left[2x+1 + \sqrt{\frac{4}{3}(x+1)^2+11} \right] + C = -\frac{1}{2} \ln \left[1 + \frac{2}{\sqrt{3}} + \sqrt{\frac{4(x+1)^2+33}{3}} \right] + C \\ & = -\frac{1}{2} \ln \left[1 + \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sqrt{x^2+x+4} \right] + C \end{aligned}$$

685. $\int \frac{x}{(5-4x-x^2)^{\frac{3}{2}}} dx = \frac{5-4x}{9\sqrt{5-4x-x^2}} + C$

$$\begin{aligned} & \int \frac{x}{(5-4x-x^2)^{\frac{3}{2}}} dx = \dots \text{ by trigonometric substitution and the square in the denominator is simplified} \\ & 5-4x-x^2 = -[x^2+4x-5] = -[(2x+3)^2-14] = 9-(2x+3)^2 = 3[1-(\frac{2x+3}{3})^2] \\ & = 9(\cos^2 \theta) \quad \theta = \arccos(\frac{2x+3}{3}) \rightarrow 2x+3 = 3\cos \theta \\ & = 9(\cos^2 \theta) \quad 2 = 2\cos \theta \\ & dx = 3\sin \theta d\theta \quad d\theta = \frac{1}{3\sin \theta} d\theta \\ & \dots = \int \frac{-2+3\sin \theta}{(3\cos^2 \theta)^{\frac{3}{2}}} (3\sin \theta d\theta) = \int \frac{-2+3\sin \theta}{27\cos^3 \theta} d\theta = \int \frac{-2+3\sin \theta}{9\cos^3 \theta} d\theta \\ & = \frac{1}{9} \int \frac{3\sin \theta}{\cos^3 \theta} - \frac{2}{\cos^3 \theta} d\theta = \frac{1}{9} \int \frac{3\sin \theta}{\cos^2 \theta} \frac{1}{\cos \theta} d\theta - 2\operatorname{cosec}^2 \theta d\theta \\ & = \frac{1}{9} \left[3 \operatorname{ln} |\operatorname{cosec} \theta| - 2 \operatorname{cot} \theta \right] d\theta = \frac{1}{9} \left[3 \operatorname{ln} \left| \frac{1}{\cos \theta} \right| - 2 \operatorname{cot} \theta \right] + C \\ & = \frac{1}{9} \left[\frac{3}{\cos \theta} - \frac{2 \operatorname{cot} \theta}{\cos^2 \theta} \right] + C = \frac{1}{9} \left[\frac{3-2 \frac{\cos \theta}{\sin^2 \theta}}{\cos^2 \theta} \right] + C \\ & = \frac{1}{9} \left[\frac{3-2 \frac{2 \operatorname{cosec}^2 \theta}{\sin^2 \theta}}{\cos^2 \theta} \right] + C = \frac{1}{9} \left[\frac{9-2(3\operatorname{cosec}^2 \theta)}{9\cos^2 \theta} \right] + C \\ & = \frac{1}{9} \left[\frac{9-2 \frac{2(3\operatorname{cosec}^2 \theta)}{9}}{9\cos^2 \theta} \right] + C = \frac{1}{9} \left[\frac{5-2 \frac{2(3\operatorname{cosec}^2 \theta)}{9}}{9\cos^2 \theta} \right] + C \end{aligned}$$

686. $\int (1-x^2)^{\frac{5}{2}} dx = \frac{1}{48}x(8x^4 - 26x^2 + 33)(1-x^2)^{\frac{1}{2}} + \frac{5}{16}\arcsin x + C$

$\int (1-x^2)^{\frac{5}{2}} dx = \dots$ STANDARD TRIGONOMETRIC SUBSTITUTION

$$= \int (1-\sin^2\theta)^{\frac{5}{2}} (\cos\theta d\theta) = \int (\cos^2\theta)^{\frac{5}{2}} (\cos\theta d\theta)$$

$$= \int \cos^6\theta d\theta$$

BY COMPLEX NUMBERS TO GET THE IDENTITY FOR $\cos^6\theta$
OR A REDUCTION FORMULA/DO IN X FOR SIMPLICITY

| | |
|----------------|----------------|
| $\cos^2\theta$ | $\cos^4\theta$ |
| $\cos^2\theta$ | $\cos^6\theta$ |

$$\begin{aligned} I_1 &= \int \cos^6\theta d\theta = \int \cos^4\theta \cos^2\theta d\theta \quad \text{BY PARTS} \\ I_1 &= \sin\theta \cos^5\theta + \int \cos^2\theta \sin^2\theta d\theta \\ I_1 &= \sin\theta \cos^5\theta + (x-1) \int (1-\cos^2\theta)^{\frac{1}{2}} d\theta \\ I_1 &= \sin\theta \cos^5\theta + (x-1) \int \cos^3\theta - \cos\theta d\theta \\ I_1 &= 2\sin\theta \cos^5\theta + (x-1) I_2 \\ 2I_1 &= \sin\theta \cos^5\theta + (x-1) I_2 \\ 2I_1 &= \frac{1}{2} \sin\theta \cos^5\theta + \frac{x-1}{2} I_2 \\ I_2 &= \frac{1}{2} \sin\theta \cos^5\theta + \frac{x-1}{2} \int \cos^3\theta - \cos\theta d\theta \\ I_2 &= \frac{1}{2} \sin\theta \cos^5\theta + \frac{x-1}{2} \left[\frac{1}{2} \sin\theta \cos^4\theta + \frac{1}{2} \cos^2\theta \right] \\ I_2 &= \frac{1}{2} \sin\theta \cos^5\theta + \frac{x-1}{2} \sin\theta \cos^4\theta + \frac{x-1}{4} \cos^2\theta \\ I_2 &= \frac{1}{2} \sin\theta \cos^5\theta + \frac{x-1}{2} \sin\theta \cos^4\theta + \frac{x-1}{4} \cos^2\theta + C \\ \therefore \int \cos^6\theta d\theta &= \frac{1}{2} \sin\theta \cos^5\theta + \frac{x-1}{2} \sin\theta \cos^4\theta + \frac{x-1}{4} \cos^2\theta + C \\ &= \frac{1}{2} x(1-x^2)^{\frac{5}{2}} + \frac{x-1}{2} x(1-x^2)^{\frac{3}{2}} + \frac{x-1}{4} \arcsin x + C \\ &= \frac{1}{2} x(1-x^2)^{\frac{3}{2}} [4x(1-x^2)^2 + 2(x-1)^2 + 2x] + \frac{x-1}{4} \arcsin x + C \\ &= \frac{1}{2} x(1-x^2)^{\frac{3}{2}} [16x^4 - 22x^2 + 16 + 2x^2 - 4x + 2] + \frac{x-1}{4} \arcsin x + C \\ &= \frac{1}{2} x(1-x^2)^{\frac{3}{2}} (16x^4 - 20x^2 + 6x + 2) + \frac{x-1}{4} \arcsin x + C \\ &= \frac{1}{2} x(1-x^2)^{\frac{3}{2}} (8x^4 - 10x^2 + 33) + \frac{x-1}{4} \arcsin x + C \end{aligned}$$

687. $\int \frac{x}{x^4+x^2+1} dx = \begin{cases} \frac{1}{\sqrt{3}} \arctan\left(\frac{2x^2+1}{\sqrt{3}}\right) + C \\ \frac{1}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C \end{cases}$

$\int \frac{x}{x^4+x^2+1} dx = \int \frac{4x}{4x^4+4x^2+4} dx = \int \frac{4x}{(2x^2+1)^2+3} dx = \frac{1}{2} \int \frac{2x}{(2x^2+1)^2+1} dx$

BY SUBSTITUTION
 $u = \tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)$
 $du = \frac{4x}{\sqrt{3}(2x^2+1)} dx$
 $2x dx = \frac{\sqrt{3}(2x^2+1)}{4} du$
 $dx = \frac{\sqrt{3}(2x^2+1)}{8x} du$

$$\begin{aligned} &= \frac{1}{2} \int \frac{\sqrt{3}(2x^2+1)}{8x} du = \frac{\sqrt{3}}{16} \int du = \frac{\sqrt{3}}{16} \arctan\left(\frac{2x^2+1}{\sqrt{3}}\right) + C \\ \text{ALTERNATIVE BY FACTORISING THE DENOMINATOR} & \quad \dots \text{BOTH IRREDUCIBLE PARTIAL FRACTIONS} \\ &= \int \frac{\frac{2}{x^2+1}}{x^2+2x+1} dx + \frac{\frac{2}{x^2-1}}{x^2-2x+1} dx \\ &= \int \frac{\frac{2}{x^2+1}}{(2x+1)^2} dx + \frac{\frac{2}{x^2-1}}{(2x-1)^2} dx \quad \text{WICHARE TO STANDARD ARCSIN} \\ &= \frac{2}{2\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{2}{2\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C \\ &= \frac{1}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C \quad (\text{CONSTANTS DIFFER BY } + \text{ CONSTANT}) \end{aligned}$$

688. $\int \frac{8}{(x^2+1)^3} dx = 3\arctan x + \frac{3x^3+5x}{(x^2+1)^2} + C$

$\int \frac{8}{(x^2+1)^3} dx = \dots$ STANDARD TECHNIQUE (CONTINUATION)

$$\begin{aligned} &= \int \frac{8 \sec^2 \theta}{(\tan^2 \theta + 1)^3} (\sec \theta d\theta) = \int \frac{8 \sec^2 \theta}{(\sec^2 \theta)^3} d\theta \\ &= \int \frac{8}{\sec^4 \theta} d\theta = \int 8 \cos^4 \theta d\theta = \int 8 \left(\frac{1}{4} + \frac{1}{2} \cos 2\theta + \frac{1}{4} \right) d\theta \\ &= \int 8 \left(\frac{1}{4} + \frac{1}{2} \cos 2\theta + \frac{1}{4} \right) d\theta = \int 2 + 4 \cos 2\theta + 2 \cos 4\theta d\theta \\ &= \int 2 + 4 \cos 2\theta + 2 \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta \\ &= \int 3 + 4 \cos 2\theta + \cos 4\theta d\theta \\ &= 3\theta + 2 \sin 2\theta + \frac{1}{4} \sin 4\theta + C = 3\theta + 4 \sin \theta \cos \theta + \frac{1}{2} \sin 2\theta \cos 2\theta + C \\ &= 3\theta + 4 \sin \theta \cos \theta + \sin 2\theta (\cos^2 \theta - \sin^2 \theta) \\ &= 3\arctan x + 4 \left(\frac{1}{\sqrt{1+x^2}} \right) \left(\frac{1}{\sqrt{1+x^2}} \right) + \left(\frac{2-x^2}{(1+x^2)^2} \right) \left(\frac{1-x^2}{\sqrt{1+x^2}} \right) \left[\frac{2}{x^2+1} - \frac{2x^2}{x^2+1} \right] + C \\ &= 3\arctan x + \frac{4x}{x^2+1} + \frac{2-x^2}{x^2+1} + C \\ &= 3\arctan x + \frac{4x}{x^2+1} + \frac{2(x^2-1)}{(x^2+1)^2} + C \\ &= 3\arctan x + \frac{4x(x^2+1)+(x^2-1)}{(x^2+1)^2} + C \\ &= 3\arctan x + \frac{5x^3+3x}{(x^2+1)^2} + C \\ &= 3\arctan x + \frac{3x+5x}{(x^2+1)^2} + C \end{aligned}$$

689. $\int \frac{1+\cos x}{\sin^2 x} dx = \frac{1}{4} \ln \left| \frac{1-\cos x}{1+\cos x} \right| + \frac{1}{2 \cos x - 2} + C$

$\int \frac{1+\cos x}{\sin^2 x} dx = \dots$ BY SUBSTITUTION

$$\begin{aligned} &\text{Let } u = 1 + \cos x, \quad \frac{du}{dx} = -\sin x, \quad dx = -\frac{du}{\sin x}, \quad \cos x = u-1 \\ &= \int \frac{u}{\sin^2 x} (-\frac{du}{\sin x}) = \int \frac{-u}{(1-u)^2} du = \int \frac{-u}{(u-1)^2} du \\ &= \int \frac{-u}{(1-(u-1)^2)^2} du = \int \frac{-u}{u^2-2u+2} du \\ &= \int \frac{-u}{u(u-2)} du = \int \frac{-u}{u(u-2)} du = \int \frac{-1}{u-2} du \end{aligned}$$

PROBLEMS BY PARTIAL FRACTION

$$\begin{aligned} \frac{-1}{u(u-2)} &= \frac{A}{u} + \frac{B}{u-2} = \frac{A(u-2) + B(u)}{u(u-2)} \\ &= \frac{(A+B)u - 2A}{u(u-2)} + B = A(u-2) + B(u) \\ &\bullet \text{ If } u=2, \quad 2B+1 = 0 \Rightarrow B = -\frac{1}{2} \\ &\bullet \text{ If } u=0, \quad -2A = 0 \Rightarrow A=0 \\ &\bullet \text{ If } u=1, \quad -1+A+2B+C = 0 \Rightarrow A=1, B=-\frac{1}{2}, C=0 \end{aligned}$$

INTEGRATE BY SPLITTING THE FRACTION

$$\int \frac{1+\cos x}{\sin^2 x} dx = \int \frac{1}{\sin^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx = \int \csc^2 x dx + \int [\csc x \cot x]^2 dx$$

"Cancels!"

690. $\int \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = \frac{\sqrt{x^4 + 1}}{x} + C$

$$\begin{aligned} \int \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx &= \dots \text{ SHADDED TRIGONOMETRIC SUBSTITUTION} \\ &= \int \frac{\tan^4 \theta - 1}{\tan^2 \theta \sqrt{\tan^4 \theta + 1}} (\frac{1}{\sec^2 \theta} \tan^2 \theta d\theta) \\ &= \int \frac{\sec^2 \theta (\tan^2 \theta - 1)}{\sec^2 \theta \tan^2 \theta} d\theta = \int \frac{\sec^2 \theta (\tan^2 \theta - 1)}{\tan^2 \theta} d\theta \\ &= \frac{1}{2} \int \sec^2 \theta (\tan^2 \theta - 1) d\theta \\ &= \frac{1}{2} \int \sec^2 \theta \tan^2 \theta - \sec^2 \theta d\theta \\ &= \frac{1}{2} \left(\frac{1}{2} \tan^3 \theta - \frac{1}{2} \tan \theta \right) \\ &= \frac{1}{2} \int \frac{\tan^3 \theta}{\cos^2 \theta} - \frac{\tan \theta}{\cos^2 \theta} d\theta = \frac{1}{2} \int \frac{\sin^3 \theta - \cos^2 \theta}{\cos^2 \theta} d\theta \\ &= \frac{1}{2} \int \frac{(\sin \theta - \cos \theta)(\sin^2 \theta + \cos^2 \theta)}{\cos^2 \theta} d\theta = \frac{1}{2} \int (-\cos 2\theta) d\theta \quad \text{BY IDENTIFICATION} \\ &= (\sin 2\theta)^{-1} + C = (\sin 2\theta)^{-1} + C = \left[\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{1}{\cos^2 \theta} \right]^{-1} + C \\ &= \left(\frac{\sin^2 \theta}{\cos^2 \theta} + 1 \right)^{-1} + C = \left(\frac{\cos^2 \theta}{\cos^2 \theta + \sin^2 \theta} \right)^{-1} + C = \frac{1}{\cos^2 \theta + \sin^2 \theta} + C = \frac{1}{1} + C = 1 + C \end{aligned}$$

$x = \frac{1}{2} \tan \theta$
 $d\theta = -\frac{1}{2} x^2 dx$

ALTERNATIVE APPROACH BY NON-INTEGRATION TO CREATE NOTATION BY PARTS

$$\begin{aligned} \int \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx &= \int \frac{2x^3 - 2x}{x^2 \sqrt{x^4 + 1}} dx = \int \frac{2x^4 - 2x^2}{x^2 \sqrt{x^4 + 1}} dx - \int \frac{2x^2}{x^2 \sqrt{x^4 + 1}} dx \\ &= \int \frac{2x^2}{\sqrt{x^4 + 1}} dx = \int \frac{2x^2}{\sqrt{x^2(x^2 + 1)}} dx \\ &\quad [\text{NOT-WISE IDEA HERE WITH } \frac{2x^2}{\sqrt{x^2}} \text{ & } \frac{2x^2}{\sqrt{x^2}} \text{ WHICH MEANT} \\ &\quad \text{NOT-SIMPLIFIABLE, NOT-EASY}] \\ &= \int \frac{2x^2}{\sqrt{x^2(1 + \frac{1}{x^2})}} dx = \int \frac{1}{\sqrt{1 + \frac{1}{x^2}}} dx^2 \\ &\quad \text{IN PARENSES} \\ &= \int \frac{1}{\sqrt{\frac{1}{x^2} + 1}} dx^2 = \int \frac{1}{\frac{1}{x^2} + 1} dx^2 \\ &= \int \frac{2x^2}{x^2 + 1} dx = \left[-\frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} + \int 2x^2(x^2 + 1)^{-\frac{1}{2}} dx \right] \\ &= \int \frac{2x^2}{x^2 + 1} dx = \int \frac{2x^2}{x^2 + 1} dx - \int \frac{2x^2}{x^2 + 1} dx \\ &= \frac{\sqrt{x^2 + 1}}{x} + C \quad \text{NOT AGAIN!} \end{aligned}$$

691. $\int \frac{x+9}{(x^2+2x+2)^3} dx = 3\arctan(x+1) + \frac{12x^3 + 36x^2 + 56x + 31}{(x^2+2x+2)^2} + C$

$$\begin{aligned} \int \frac{2x+9}{(x^2+2x+2)^3} dx &= \int \frac{2x+9}{(x^2+2x+2)^3} dx = \int \frac{2}{(x^2+2x+2)^2} dx = \int \frac{2}{(2\arctan(x+1))^2} dx \\ &\quad \text{SPLIT THE INTEGRAL INTO TWO - FIRST SECTION BY REVERSING, SECOND SECTION BY SUBSTITUTION} \\ &= -\frac{1}{2} (2\arctan(x+1))^2 + \int \frac{2}{(2\arctan(x+1))^2} (2\sec^2(x+1)) dx \\ &= -\frac{1}{2} (2\arctan(x+1))^2 + \int \frac{2\sec^2(x+1)}{(2\arctan(x+1))^2} dx \\ &\quad \text{Tidy the second integral?} \\ &\dots \int 2\sec^2(x+1) dx = \int 2 \left(\frac{1}{2} + \frac{1}{2}\tan^2(x+1) \right) dx \\ &= \int 2 \left(\frac{1}{2} + \frac{1}{2}\tan^2(x+1) + \frac{1}{2}\sec^2(x+1) \right) dx \\ &= \left(2 + 4\arctan(x+1) + 2\ln|\sec(x+1)| \right) \\ &= \left(2 + 4\arctan(x+1) + 2 \left(\frac{1}{2} + \frac{1}{2}\tan^2(x+1) \right) \right) \\ &= \int 2 + 4\arctan(x+1) + \ln|\sec(x+1)| dx = 2x + 2\arctan(x+1) + \frac{1}{2}\ln|\sec(x+1)| + C \\ &= 2x + 4\arctan(x+1) + \frac{1}{2}\ln|\sec(x+1)| + C \\ &= 2x + 4\arctan(x+1) + \ln|\sec(x+1)| + C \\ &= 3\arctan(x+1) + 4 \left(\frac{2x+9}{(x^2+2x+2)^2} \right) \left[\left(\frac{x+1}{x^2+2x+2} \right)^2 - \left(\frac{x+1}{x^2+2x+2} \right)^3 \right] + C \\ &= 3\arctan(x+1) + \frac{4(x+1)}{x^2+2x+2} + \frac{2x+1}{x^2+2x+2} - \frac{2x+1}{x^2+2x+2} + C \\ &= 3\arctan(x+1) + \frac{4(x+1)}{x^2+2x+2} + C \\ &= 3\arctan(x+1) + \frac{4(x+1)(x^2+2x+2) - 3(x+1)(x^2+2x+2)}{(x^2+2x+2)^2} + C \\ &= 3\arctan(x+1) + \frac{(3x+1)(x^2+2x+2)}{(x^2+2x+2)^2} + C \\ &\quad \text{COLLECTING TERMS} \\ &\int \frac{2x+9}{(x^2+2x+2)^3} dx = -\frac{1}{2} (2\arctan(x+1))^2 + 3\arctan(x+1) + \frac{\ln|\sec(x+1)|}{(x^2+2x+2)^2} + C \\ &= 3\arctan(x+1) + \frac{12x^3 + 36x^2 + 56x + 31}{(x^2+2x+2)^2} + C \end{aligned}$$

692. $\int \frac{\sec^2 x - 1}{\sec^3 x + \tan^3 x} dx = \frac{1}{3} \ln(1 + \sin^3 x) + C$

$$\begin{aligned}\int \frac{\sec^2 x - 1}{\sec^3 x + \tan^3 x} dx &= \dots \text{ SOURCE IND. SINCE Q. COULD} \\ &= \left[\int \frac{\sec x}{\sec^2 x + \tan^2 x} dx \right] + \text{some} \\ &= \dots \text{ USE TOP Q. ACTION OF THE RATIONAL IN EACH} \\ &= \dots \text{ OF THE FORM } \int \frac{f(u)}{g(u)} du = \int \frac{f'(u)}{1 + g'(u)} du = \frac{1}{g(u)} [f(u) + C].\end{aligned}$$

693. $\int \tan x \sec 2x dx = -\frac{1}{2} \ln|1 - \tan^2 x| + C$

$$\begin{aligned}\int \tan x \sec 2x dx &= \int \frac{\tan x}{\cos 2x} dx = \int \frac{\tan x}{\cos^2 x - \sin^2 x} dx \\ &= \int \frac{\frac{\sin x}{\cos x}}{1 - \tan^2 x} dx = -\frac{1}{2} \int \frac{-2 \tan x \sec^2 x}{1 - \tan^2 x} dx \\ &= \dots \text{ OF THE FORM } \int \frac{f(u)}{g(u)} du = -\frac{1}{2} \ln|1 - \tan^2 x| + C.\end{aligned}$$

694. $\int \frac{5 \cos x}{(2 + \sin x)(3 + 4 \sin x)} dx = \ln \left| \frac{3 + 4 \sin x}{2 + \sin x} \right| + C$

$$\begin{aligned}\int \frac{5 \cos x}{(2 + \sin x)(3 + 4 \sin x)} dx &= \dots \text{ GIVE SPLITTING} \\ &= \int \frac{5 \cos x}{4 \sin^2 x + 11 \sin x + 6} dx \\ &\quad \text{INT. SUBSTITUTION} \\ &= \int \frac{5}{4u^2 + 11u + 6} du \\ &= \int \frac{5}{4u^2 + 8u + 3} du \\ &= \int \frac{5}{4(u+2)^2 - 13} du \\ &= \ln|4(u+2)| - \ln|u+2| + C = \ln \left| \frac{4u+8}{u+2} \right| + C = \ln \left| \frac{3+4u}{2+u} \right| + C \\ &= \ln \left| \frac{3+4\sin x}{2+\sin x} \right| + C\end{aligned}$$

695. $\int \frac{x^2 + 3x + 3}{(x+1)^3} (e^{-x} \sin x) dx = \frac{-e^{-x} [(x+2)\sin x + (x+1)\cos x]}{2(x+1)^2} + C$

$\int \frac{x^2 + 3x + 3}{(x+1)^3} e^{-x} \sin x dx = \dots$ START WITH FRACTIONAL FRACTIONS

$$\begin{aligned} \frac{x^2 + 3x + 3}{(x+1)^3} &\equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} \\ x^2 + 3x + 3 &\equiv A(x+1)^2 + B(x+1) + C \\ x^2 + 3x + 3 &\equiv Ax^2 + 2Ax + A + Bx + B + C \quad A=1, B=1, C=1 \end{aligned}$$

$$= \int \left[\frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{1}{(x+1)^3} \right] e^{-x} \sin x dx = \int \frac{1}{x+1} e^{-x} \sin x dx + \int \frac{1}{(x+1)^2} e^{-x} \sin x dx + \int \frac{1}{(x+1)^3} e^{-x} \sin x dx$$

SEE FIND THE INTEGRAL OF $e^{-x} \sin x$ [DEMO MODE / SEARCH NUMBER OF WORKS]

$$\begin{aligned} \frac{d}{dx} \left[e^{-x} (\cos x - \sin x) \right] &= e^{-x} (-\cos x + \sin x) + e^{-x} (\cos x - \sin x) \\ &= e^{-x} [(\cos x - \sin x) + (\cos x - \sin x)] \\ &\quad + B = -\frac{1}{2} \end{aligned}$$

NEXT INTEGRATION BY PARTS FOR THE FRACTIONAL FRACTION

- $\int \frac{1}{x+1} e^{-x} \sin x dx = -\frac{1}{2(x+1)^2} (\cos x - \sin x) - \int \frac{1}{(x+1)^2} e^{-x} (\cos x - \sin x) dx$ NOTICE THIS PART
- $\int \frac{1}{(x+1)^2} e^{-x} \sin x dx = -\frac{1}{2(x+1)^3} \sin x + \frac{1}{2} \int \frac{1}{(x+1)^3} e^{-x} (\cos x - \sin x) dx$ NOTICE THIS PART AGAIN

$$\begin{aligned} \frac{d}{dx} \left[\frac{1}{(x+1)^3} \right] &= -\frac{3}{(x+1)^4} \\ \frac{d}{dx} \left[e^{-x} \sin x \right] &= -e^{-x} \sin x + e^{-x} \cos x = e^{-x} (\cos x - \sin x) \end{aligned}$$

COLLECTING THE RESULTS SO FAR

$$\begin{aligned} &= \int \frac{1}{x+1} e^{-x} \sin x dx + \int \frac{1}{(x+1)^2} e^{-x} \sin x dx + \int \frac{1}{(x+1)^3} e^{-x} \sin x dx \\ &= -\frac{1}{2(x+1)} e^{-x} (\cos x + \sin x) - \frac{1}{2} \int \frac{1}{(x+1)^2} e^{-x} (\cos x + \sin x) dx \\ &\quad + \int \frac{1}{(x+1)^3} e^{-x} (\cos x - \sin x) dx \\ &\quad - \frac{1}{2(x+1)^2} e^{-x} \sin x \\ &\quad + \frac{1}{2} \int \frac{1}{(x+1)^4} e^{-x} (\cos x - \sin x) dx \\ &= -\frac{1}{2} \left(\frac{e^{-x}}{(x+1)^2} \right) [(\cos x + \sin x) + \cos x - \sin x] \\ &= -\frac{e^{-x}}{2(x+1)^2} [2\cos x] + C \\ &= -\frac{e^{-x}}{2(x+1)^2} [(\cos x + \sin x) + (\cos x - \sin x)] + C \\ &= -\frac{e^{-x}}{2(x+1)^2} [2\cos x] + C \\ &= -\frac{e^{-x}}{2(x+1)^2} [\cos x \sin x + \cos x \cos x] + C \\ &= -\frac{e^{-x}}{2(x+1)^2} [\cos x \sin x + \cos^2 x] + C \end{aligned}$$

696. $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \begin{bmatrix} -3x + \tan x - \cot x + C \\ -3x - 2 \cot 2x + C \end{bmatrix}$

$$\begin{aligned} \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \frac{(1-\sin^2 x)^2 + (1-\cos^2 x)^2}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{1-2\sin^2 x+\sin^4 x}{\sin^2 x} + \frac{1-2\cos^2 x+\cos^4 x}{\sin^2 x} dx \\ &= \int \sin^2 x - 2 + \cos^2 x + \cos^2 x - 2 + \sin^2 x dx \\ &= \int \sin^2 x + \cos^2 x - 4 + 1 dx = \int \sin^2 x + \cos^2 x - 3 dx \\ &= \tan x - \cot x - 3x + C \end{aligned}$$

ADDITION AND SUBTRACTION OF CUBES $A^3 + B^3 = (A+B)(A^2 - AB + B^2)$

$$\begin{aligned} \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx &= \int \frac{(\sin^2 x)^2 + (\cos^2 x)^2}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{(\sin^2 x + \cos^2 x)(\sin^2 x - \sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} - \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x}{\cos^2 x} - 1 + \frac{\cos^2 x}{\sin^2 x} dx \\ &= \int \tan^2 x - 1 + \cot^2 x dx = \int (\sec^2 x - 1) + (\csc^2 x - 1) dx \\ &= \int \sec^2 x + \csc^2 x - 3 dx = \tan x - \cot x - 3x + C \end{aligned}$$

697. $\int \frac{3\sin^2 x \cos^2 x}{(\cos^3 x - \sin^3 x)^2} dx = \frac{1}{1-\tan^3 x} + C$

$$\begin{aligned} \int \frac{3\sin^2 x \cos^2 x}{(\cos^3 x - \sin^3 x)^2} dx &= \int \frac{3\sin^2 x \cos^2 x}{\cos^6 x (1 - \frac{\sin^3 x}{\cos^3 x})^2} dx \\ &= \int \frac{3\sin^2 x \cos^2 x}{\cos^6 x (1 - \tan^3 x)^2} dx = \int \frac{3\sin^2 x \cos^2 x}{(1 - \tan^3 x)^2} dx \\ &\text{By substitution or substitution } u = \tan x \\ &= (1 - \tan^3 x)^{-1} + C = \frac{1}{1 - \tan^3 x} + C \end{aligned}$$

698. $\int \frac{\sin^8 x - \cos^8 x}{1 - \frac{1}{2}\sin^2 2x} dx = \frac{1}{2}\sin 2x + C$

$$\begin{aligned} \int \frac{\sin^8 x - \cos^8 x}{1 - \frac{1}{2}\sin^2 2x} dx &= \int \frac{(\sin^4 x)^2 - (\cos^4 x)^2}{1 - \frac{1}{2}(2\sin x \cos x)^2} dx \\ &= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - \frac{1}{2}(2\sin x \cos x)^2} dx = \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - 2\sin^2 x \cos^2 x} dx \\ &= \int \frac{(\sin^4 x - \cos^4 x)(\sin^2 x + \cos^2 x)^2}{(1)^2 - 2\sin^2 x \cos^2 x} dx = \int \frac{(\sin^4 x - \cos^4 x)(\sin^2 x + \cos^2 x)^2}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x} dx \\ &= \int \frac{(\sin^4 x - \cos^4 x)(\sin^2 x + \cos^2 x)^2}{\cos^4 x + 2\sin^2 x \cos^2 x + \sin^4 x - 2\sin^2 x \cos^2 x} dx \\ &= \int \frac{(\sin^4 x - \cos^4 x)(\sin^2 x + \cos^2 x)^2}{\cos^4 x + \sin^4 x} dx = \int \frac{\sin^4 x - \cos^4 x}{\cos^4 x + \sin^4 x} dx \\ &= \int (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) dx = \int \cos 2x dx \\ &= \frac{1}{2}\sin 2x + C \end{aligned}$$

699. $\int 12x^2 \arctan x dx = 4x^3 \arctan x - 2x^2 + 2\ln(x^2 + 1) + C$

ATTEMPT THIS WITHOUT SUBSTITUTIONS/TEILS, JUST A BIT OF COMMON SENSE

$$\begin{aligned} \int [12x^2 \arctan x] dx &= 12x^2 \arctan x + \text{bit } \times \frac{1}{1+x^2} \\ \text{INTEGRATE WITH RESPECT TO } x \\ \int \frac{d}{dx} [\arctan x] dx &= \int 12x^2 \arctan x dx + \int \frac{dx^2}{1+x^2} dx \\ d[\arctan x] &= \int 12x^2 \arctan x dx + \int \frac{d(x^2+1)+1}{x^2+1} dx \\ d[\arctan x] &= \int 12x^2 \arctan x dx + \int \frac{dx}{x^2+1} dx + \int \frac{1}{x^2+1} dx \\ d[\arctan x] &= \int 12x^2 \arctan x dx + \int \frac{dx}{x^2+1} dx + 2\int \frac{dx}{x^2+1} dx \\ \text{EQUATE TO GET CHEAT} \\ \int 12x^2 \arctan x dx &= 4x^3 \arctan x - 2x^2 - 2\ln(x^2+1) \end{aligned}$$

700. $\int \sqrt{\tan x} \ dx = \dots$

$$\dots = \frac{\sqrt{2}}{4} \ln \left| \frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1} \right| + \frac{\sqrt{2}}{2} \arctan \left[\frac{\sqrt{2 \tan x}}{1 - \tan x} \right] + C$$

$$= \frac{\sqrt{2}}{4} \ln \left| \frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1} \right| + \frac{\sqrt{2}}{2} \arctan \left[\frac{-1 + \tan x}{\sqrt{2 \tan x}} \right] + C$$

SIMPLY SUBSTITUTIONS, AVOID SPLITTING THE INTEGRAL

$$\int \frac{du}{u^2 - 2u + 2} = \int \frac{du}{(u-1)^2 + 1} = \int \frac{1}{(u-1)^2 + 1} du$$

$$u = u-1 \quad u = u-1$$

$$du = du \quad du = du$$

$$u = v+1 \quad u = v+1$$

$$= \int \frac{1}{v^2+1} dv = -\int \frac{1}{v^2+1} dv$$

$$= \frac{1}{\sqrt{v^2+1}} \int \frac{1}{v^2+1} dv + \frac{1}{\sqrt{v^2+1}} \int \frac{1}{v^2+1} dv = -\int \frac{1}{v^2+1} dv + \frac{1}{\sqrt{v^2+1}} \int \frac{1}{v^2+1} dv$$

$$= \frac{1}{\sqrt{v^2+1}} \arctan(v) + \frac{\sqrt{v^2+1}}{v^2+1} dv = \frac{1}{\sqrt{v^2+1}} \arctan(v) + C$$

$$= \frac{1}{\sqrt{(u-1)^2+1}} \arctan(u-1) + \frac{\sqrt{(u-1)^2+1}}{(u-1)^2+1} du = C$$

θ = arctan(v) + arctan(u)

$$\Rightarrow \tan θ = \frac{v+1}{1-vv} = \frac{u-1+1}{1-(u-1)(u-1)}$$

$$\Rightarrow \tan θ = \frac{u}{1-u+1} = \frac{u}{2-u} = \frac{u}{2-u}$$

$$\Rightarrow \sec^2 θ = \frac{u^2}{(2-u)^2} = \frac{u^2}{4-4u+u^2}$$

$$\Rightarrow \sec θ = \frac{u}{\sqrt{4-4u+u^2}}$$

$$\Rightarrow \sec θ = \frac{u}{\sqrt{1-u^2}}$$

$$\Rightarrow u = \sec θ \cdot \frac{1}{\sqrt{1-\tan^2 θ}}$$

$$= \frac{1}{\sqrt{1-\tan^2 θ}} \left[\frac{1-\tan^2 θ}{1+\tan^2 θ} \right] + \frac{1}{\sqrt{1-\tan^2 θ}} \arctan \left(\frac{1-\tan^2 θ}{1+\tan^2 θ} \right) + C$$

$$= \frac{1}{\sqrt{1-\tan^2 θ}} \left[\frac{1-\tan^2 θ}{1+\tan^2 θ} \right] + \frac{1}{\sqrt{1-\tan^2 θ}} \arctan \left(\frac{1-\tan^2 θ}{1+\tan^2 θ} \right) + C$$

$$= \frac{1}{\sqrt{1-\tan^2 θ}} \left[\frac{1-\frac{u^2}{4-4u+u^2}}{1+\frac{u^2}{4-4u+u^2}} \right] + \frac{1}{\sqrt{1-\tan^2 θ}} \arctan \left(\frac{1-\frac{u^2}{4-4u+u^2}}{1+\frac{u^2}{4-4u+u^2}} \right) + C$$

$$= \frac{1}{\sqrt{1-\tan^2 θ}} \left[\frac{4-4u+u^2-u^2}{4-4u+u^2+u^2} \right] + \frac{1}{\sqrt{1-\tan^2 θ}} \arctan \left(\frac{4-4u+u^2-u^2}{4-4u+u^2+u^2} \right) + C$$

QUICKER ALTERNATIVE (AFTER THE INITIAL SUBSTITUTION)

$$\int \sqrt{1 + \tan^2 x} dx = \dots = \int \frac{2x}{1+x^2} dx = \int \frac{2}{\frac{1}{x^2+1}} dx$$

$$= \int \frac{1 + \frac{1}{x^2} - \frac{1}{x^2}}{1+\frac{1}{x^2}} dx = \int \frac{1}{1+\frac{1}{x^2}} dx + \int \frac{\frac{1}{x^2}}{1+\frac{1}{x^2}} dx$$

$$= \int \frac{1 - \frac{1}{x^2}}{(x+\frac{1}{x})(x-\frac{1}{x})} dx + \int \frac{\frac{1}{x^2}}{(x-\frac{1}{x})(x+\frac{1}{x})} dx$$

$$\begin{array}{c} v = x - \frac{1}{x} \\ dv = (1 - \frac{1}{x^2})dx \end{array} \quad \begin{array}{c} w = \frac{1}{x} - \frac{1}{v} \\ dw = (1 + \frac{1}{x^2})dx \end{array}$$

$$= \int \frac{dv}{\sqrt{v^2+2}} + \int \frac{dw}{w^2+2} = \int \frac{1}{\sqrt{w^2+2}} dw + \int \frac{1}{(w^2+2)^{1/2}} dw$$

$$= \int \frac{1}{w\sqrt{w^2+2}} dw + \int \frac{1}{w\sqrt{w^2+2}} dw$$

$$= \frac{1}{w\sqrt{2}} \int \frac{1}{\sqrt{1-\frac{4}{w^2}}} dw = \frac{1}{w\sqrt{2}} \int \frac{1}{\sqrt{1-\frac{4}{w^2}}} dw$$

$$= \frac{\sqrt{2}}{4} \left[\ln|w - \sqrt{w^2-4}| - \ln|w + \sqrt{w^2-4}| \right] + \frac{1}{w\sqrt{2}} \arctan\left(\frac{2w}{\sqrt{w^2-4}}\right) + C$$

$$= \frac{\sqrt{2}}{4} \ln \left| \frac{w - \sqrt{w^2-4}}{w + \sqrt{w^2-4}} \right| + \frac{1}{w\sqrt{2}} \arctan\left(\frac{2w}{\sqrt{w^2-4}}\right) + C$$

$$= \frac{\sqrt{2}}{4} \ln \frac{x - \sqrt{x^2-1}}{x + \sqrt{x^2-1}} + \frac{1}{x\sqrt{2}} \arctan\left(\frac{2x}{\sqrt{x^2-1}}\right) + C$$

$$= \frac{\sqrt{2}}{4} \ln \frac{\frac{1}{x} - \sqrt{\frac{1}{x^2}-1}}{\frac{1}{x} + \sqrt{\frac{1}{x^2}-1}} + \frac{1}{\frac{1}{x}\sqrt{2}} \arctan\left(\frac{\frac{2}{x}}{\sqrt{\frac{1}{x^2}-1}}\right) + C$$

$$= \frac{\sqrt{2}}{4} \ln \frac{\frac{1}{x} - \sqrt{\frac{1-x^2}{x^2}}}{\frac{1}{x} + \sqrt{\frac{1-x^2}{x^2}}} + \frac{1}{\frac{1}{x}\sqrt{2}} \arctan\left(\frac{\frac{2}{x}}{\sqrt{\frac{1-x^2}{x^2}}}\right) + C$$

$$= \frac{\sqrt{2}}{4} \ln \frac{\frac{1}{x} - \sqrt{\frac{1-x^2}{x^2}+1}}{\frac{1}{x} + \sqrt{\frac{1-x^2}{x^2}+1}} + \frac{1}{\frac{1}{x}\sqrt{2}} \arctan\left(\frac{\frac{2}{x}}{\sqrt{\frac{1-x^2}{x^2}+1}}\right) + C.$$

DRAFTED FROM THAT PREVIOUS
NOTATION BY $\approx 75\%$

(b) $\arctan x = \arctan\left(\frac{y}{x}\right) = \arctan y + \arctan\left(\frac{1}{x}\right)$

$$\tan y = \frac{y}{x} \Rightarrow \frac{x+y}{1-yx} = \frac{2+y}{1-y}$$

$$\tan y = \frac{2+y}{1-y}$$

$$y = \frac{2+y}{1-y}$$

701. $\int \frac{\cot^3 x}{1+2\cot^2 x+2\cot^4 x} dx = -\frac{1}{4} \ln(1+\cos^4 x) + C$

$$\int \frac{c_1^2 c_2}{1 + 2c_1^2 x + 2c_2^2 x^2} dx = \dots$$

SWAPPING INTO SUMS OF COLUMNS ...

$$\int \frac{\frac{c_1^2}{x^2}}{1 + 2c_1^2 x + 2c_2^2 x^2} dx = \dots$$

MULTIPLY TOP & BOTTOM BY x^2

$$\int \frac{\frac{c_1^2 x^2}{x^2}}{1 + 2c_1^2 x + 2c_2^2 x^2} dx = \int \frac{\frac{c_1^2 x^2}{(1 - c_1^2 x^2)^2 + 2c_1^2 x(1 - c_1^2 x^2) + 2c_2^2 x^2}}{dx}$$

CANCELS OUT TERMS ONLY AS THERE IS A x^2 TERM IN THE NUMERATOR

$$= \int \frac{c_1^2 x^2}{1 - 2c_1^2 x^2 + c_1^2 + 2c_1^2 x^2 - 2c_1^2 x^2 + 2c_2^2 x^2} dx$$

$$= \int \frac{c_1^2 x^2}{1 + c_2^2 x^2} dx = \frac{1}{\pi} \int \frac{4c_1^2 x^2}{1 + c_2^2 x^2} dx = \frac{4}{\pi} \ln(1 + c_2^2 x^2) + C$$

VARIATION) OF THE ABOVE

$$\int \frac{\frac{c_1^2}{x^2}}{1 + 2c_1^2 x + 2c_2^2 x^2} dx = \int \frac{\frac{c_1^2}{(1 + 2c_1^2 x + c_2^2 x^2) + c_2^2 x^2}}{dx}$$

$$= \int \frac{\frac{c_1^2}{x^2}}{(1 + c_2^2 x^2)^2 + c_2^2 x^2} dx = \int \frac{\frac{c_1^2}{c_2^2 x^2 + c_2^2 x^2}}{c_2^2 x^2 + c_2^2 x^2} dx$$

$$= \int \frac{\frac{c_1^2 x^2}{c_2^2}}{\frac{c_2^2}{x^2} + \frac{c_2^2}{x^2}} dx = \dots$$

MULTIPLY TOP & BOTTOM BY x^2 ...

$$= \int \frac{\frac{c_1^2 x^2}{c_2^2}}{1 + c_2^2 x^2} dx = \frac{1}{\pi} \int \frac{4c_1^2 x^2}{1 + c_2^2 x^2} dx = \frac{4}{\pi} \ln(1 + c_2^2 x^2) + C$$

+ A CONSTANT

702. $\int (2x^2 + 1)e^{x^2} dx = xe^{x^2} + C$

LOOKING AT THE EXPONENTIAL FUNCTION
 $\frac{d}{dx}[e^{x+\frac{1}{x}}] = e^{x+\frac{1}{x}} \left(1 - \frac{1}{x^2}\right)$ → EQUATES TO REVERSE THIS EXPRESSION IN THE INTEGRAND

$$\begin{aligned} \int (2x^2 + 1)e^{x^2} dx &= \int [1 + 2\left(1 - \frac{1}{x^2}\right)] e^{x^2} dx \\ &= \int e^{x^2} dx + 2 \int \left(1 - \frac{1}{x^2}\right) e^{x^2} dx \\ &\quad + \int e^{x^2} dx + \int 2 \left(1 - \frac{1}{x^2}\right) e^{x^2} dx \\ &\quad \text{INTEGRATION BY PARTS} \\ &= \int e^{x^2} dx + 2e^{x^2} - \int e^{x^2} dx \\ &= 2e^{x^2} + C \end{aligned}$$

703. $\int \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx = xe^{x+\frac{1}{x}} + C$

LOOKING AT THE EXPONENTIAL FUNCTION
 $\frac{d}{dx}[e^{x+\frac{1}{x}}] = e^{x+\frac{1}{x}} \left(1 - \frac{1}{x^2}\right)$ → EQUATES TO REVERSE THIS EXPRESSION IN THE INTEGRAND

$$\begin{aligned} \int (2x^2 + 1)e^{x^2} dx &= \int [1 + 2\left(1 - \frac{1}{x^2}\right)] e^{x^2} dx \\ &= \int e^{x^2} dx + 2 \int \left(1 - \frac{1}{x^2}\right) e^{x^2} dx \\ &\quad + \int e^{x^2} dx + \int 2 \left(1 - \frac{1}{x^2}\right) e^{x^2} dx \\ &\quad \text{INTEGRATION BY PARTS} \\ &= \int e^{x^2} dx + 2e^{x^2} - \int e^{x^2} dx \\ &= 2e^{x^2} + C \end{aligned}$$

704. $\int \frac{1}{\cos^6 x + \sin^6 x} dx = \arctan\left(\frac{1}{2} \tan 2x\right) + C$

$$\begin{aligned} \int \frac{1}{\cos^6 x + \sin^6 x} dx &= \int \frac{1}{(\cos^2 x)^3 + (\sin^2 x)^3} dx \\ &= \int \frac{1}{(\cos^2 x + \sin^2 x)(\cos^2 x - \cos^2 x \sin^2 x + \sin^2 x)} dx \\ &= \int \frac{1}{\cos^2 x - \cos^2 x \sin^2 x + \sin^2 x} dx \\ \text{USING THE SUM OF CUBES IDENTITY } A^3 + B^3 &\equiv (A+B)(A^2 - AB + B^2) \\ (A-B)^3 &= A^3 - 3A^2B + 3AB^2 - B^3 \\ (A-B)^2(A+B) &= A^3 - AB^2 + B^3 \\ (A-B)^2(A+B) - A^3 + AB^2 + B^3 &= -AB^2 \\ &= \int \frac{1}{\cos^2 x - \cos^2 x \sin^2 x + \sin^2 x} dx = \int \frac{1}{\cos^2 x + \sin^2 x} dx \\ &= \int \frac{1}{\cos^2 x + \sin^2 x} dx = \int \frac{1}{1} dx \\ &= \int 1 dx \\ \text{AT THIS STAGE WE CAN WRITE } \cos^2 x + \sin^2 x &= 1 \\ \text{AND USE FORMULAE, BUT IT IS FAR EASIER TO MANIPULATE AS POWERS} \\ &= \int \frac{4 \cos^2 x}{4 \cos^2 x + \sin^2 x} dx = \int \frac{4 \cos^2 x}{4 + \sin^2 x} dx \\ \text{ANOTHER SUBSTITUTION, SAY } t = \tan x, \text{ OR INSERTION } \sim \\ \frac{dt}{dx} &= \sec^2 x = \frac{1}{\cos^2 x} \times \sec^2 x \times x = \frac{4 \cos^2 x}{4 + \sin^2 x} \\ &= \arctan\left(\frac{1}{2} \tan x\right) + C \end{aligned}$$

705. $\int \sqrt{e^x - 1} dx = 2\sqrt{e^x - 1} - 2\arctan\left(\sqrt{e^x - 1}\right) + C$

$$\begin{aligned} \int \sqrt{e^x - 1} dx &= \dots \text{ BY SUBSTITUTION ...} \\ &= \int u \left(\frac{du}{e^x} \right) = \int \frac{2u^2}{e^x} du \\ &= \int \frac{2u^2}{u^2+1} du = \int \frac{2(u^2+1)-2}{u^2+1} du \\ &= \int 2 - \frac{2}{u^2+1} du = -2u - 2\arctan u + C \\ &= 2u\sqrt{e^x - 1} - 2\arctan\sqrt{e^x - 1} + C \\ \text{ALTERNATIVE BY HYPERBOLIC SUBSTITUTION } u = \cosh(x) &= \dots \\ \int \sqrt{e^x - 1} dx &= \int \sqrt{\cosh^2 u - 1} (2\sinh u du) \\ &= \int 2\sinh u \cosh u du = \int 2\sinh u \cdot \frac{1}{\cosh u} du = \int 2(\tanh u - 1) du \\ &= \int 2\cosh u - \frac{2}{\cosh u} du = \int 2\cosh u - \frac{2\cosh u}{\cosh^2 u} du \\ &= \int 2\cosh u - \frac{2\cosh u}{1 + \sinh^2 u} du = 2\cosh u - 2\operatorname{atanh}(\sinh u) + C \\ &= 2\sqrt{e^x - 1} - 2\arctanh\sqrt{e^x - 1} + C \quad \text{BY WORKS} \\ \text{BY SUBSTITUTION } e^x = \sec^2 u &= \int \sqrt{\cosh^2 u - 1} (2\sinh u du) \\ &= \int 2\sinh u du = \int 2\cosh u - 2\cosh u du \\ &= 2\cosh u - 2\cosh u + C = 2\sqrt{e^x - 1} - 2\arctanh\sqrt{e^x - 1} + C \quad \text{AL. WORKS} \\ \begin{array}{l} u^2 = \cosh^2 u \\ u^2 - 1 = \cosh^2 u - 1 \\ \frac{u^2 - 1}{u^2} = \frac{\cosh^2 u - 1}{\cosh^2 u} \\ 1 - \frac{1}{u^2} = \frac{1 - \cosh^{-2} u}{\cosh^{-2} u} \end{array} & \begin{array}{l} u = \operatorname{arcsec} e^x \\ e^x = \sec u \\ \cosh u = \sec u \\ \cosh u = \frac{1}{\cos u} \\ \cosh u = \frac{1}{\sqrt{1 - \sin^2 u}} \end{array} \end{aligned}$$

706. $\int \frac{1}{\sqrt{e^x - 1}} dx = 2 \arctan(\sqrt{e^x - 1}) + C$

$$\begin{aligned} \int \frac{1}{\sqrt{e^x - 1}} dx &= \text{BY SUBSTITUTION} \\ &= \int \frac{1}{\sqrt{u}} du = \int \frac{2}{\sqrt{u}} du \\ &= 2 \arctan u + C = 2 \arctan \sqrt{e^x - 1} + C \\ \text{THUS SUBSTITUTION: } u &= \arctan(e^x) \text{ & } u = \arctan(e^{2x}) \\ \text{ALSO WORK, BUT NOT "UNWANTED".} \end{aligned}$$

707. $\int \frac{1}{x^4 + 3x^2 + 2} dx = \ln(x^2 + 2) - \frac{1}{2} \ln(x^2 + 1) + C$

$$\begin{aligned} \int \frac{x^3}{x^4 + 3x^2 + 2} dx &= \int \frac{x^3}{(x^2 + 1)(x^2 + 2)} dx = \dots \text{REDUCIBLE PARTIAL FRACTIONS} \\ \frac{x^3}{(x^2+1)(x^2+2)} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} \\ x^3 &= (Ax+B)(x^2+2) + (Cx+D)(x^2+1) \\ x^3 &= [Ax^3 + Bx^2 + 2Ax + 2B] + [Cx^3 + Dx^2 + Cx + D] \\ Ax^3 + Bx^2 + 2Ax + 2B &= x^3 + Dx^2 + Cx + D \\ Ax^3 + Bx^2 + 2Ax + 2B &= x^3 + Dx^2 + Cx + D \\ A+C=1, B=0 & \Rightarrow A=1, C=0 \\ 2A+C=0 & \Rightarrow B=0 \\ 2B+D=0 & \Rightarrow D=0 \\ 2B+D=0 & \Rightarrow B=D=0 \\ & \Rightarrow A=1, C=0, B=0, D=0 \\ & \Rightarrow \int \frac{-x}{x^2+1} + \frac{2x}{x^2+2} dx = -\frac{1}{2} \ln(x^2+1) + \ln(x^2+2) + C \\ & = \ln(x^2+2) - \frac{1}{2} \ln(x^2+1) \end{aligned}$$

708. $\int x \arccos\left[\frac{1-x^2}{1+x^2}\right] dx = -x + (1+x^2) \arctan x + C$

$$\begin{aligned} \text{THE ARGUMENT OF THE ARCCOS IS VERY STRANGE!} \\ \text{LET } u = \tan \frac{\theta}{2} \text{ OR } 45^\circ - \text{SUBSTITUTE!} \text{ BE CAREFUL!} \\ \bullet \frac{1-x^2}{1+x^2} = \frac{1-\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}} = \frac{1-\sin^2 \theta}{\cos^2 \theta} = \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta} = \cos^2 \theta - \sin^2 \theta = \cos 2\theta \\ \bullet dx = \sec^2 \frac{\theta}{2} \cdot \frac{1}{2} d\theta \\ & \Rightarrow \sec^2 \frac{\theta}{2} = x^2 + 1 \\ \int x \arccos\left(\frac{1-x^2}{1+x^2}\right) dx &= \int \tan \frac{\theta}{2} \arccos(\cos 2\theta) \left(\frac{1}{2} \sec^2 \frac{\theta}{2} d\theta\right) \\ &= \int \frac{1}{2} \theta \left[\ln \left| \frac{1+\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}} \right| \right] d\theta = \dots \text{INTEGRATION BY PARTS} \dots \\ & \begin{array}{c|c} \frac{1}{2} \theta & \frac{1}{2} \\ \hline \ln \left| \frac{1+\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}} \right| & -\frac{1}{2} \tan \frac{\theta}{2} \end{array} \\ &= \frac{1}{2} \theta \ln \frac{1+\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}} - \left(\frac{1}{2} \ln \frac{1+\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}} - \frac{1}{2} \right) \tan \frac{\theta}{2} - \frac{1}{2} d\theta \\ &= \frac{1}{2} \theta \ln \frac{1+\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}} - \frac{1}{2} \ln \frac{1+\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}} + \frac{1}{2} \theta + C \\ &= \frac{1}{2} \theta \left(\tan \frac{\theta}{2} + 1 \right) - \tan \frac{\theta}{2} + C = \frac{1}{2} \theta \sec^2 \frac{\theta}{2} - \frac{1}{2} \tan \frac{\theta}{2} + C \\ &= \frac{1}{2} (\sec \theta) (\tan \frac{\theta}{2}) - x + C = -x + (1+x^2) \arctan x + C \end{aligned}$$

709. $\int 2x^2 \sec^2 x \tan x \, dx = x^2 \sec^2 x - 2x \sec x + 2 \ln |\sec x| + C$

$$\int 2x \sec x \, dx = \dots$$

INTEGRATION BY PARTS

| | |
|-------------------------|-----------------------|
| x^2 | $2x$ |
| $\frac{d}{dx} x^2 = 2x$ | $\frac{d}{dx} 2x = 2$ |

$$= x^2 \sec x - \int 2x \sec^2 x \, dx$$

$$= x^2 \sec x - \int 2x (\sec^2 x - 1) \, dx = x^2 \sec x - \int 2x \sec^2 x \, dx - \int 2x \, dx$$

$$= x^2 \sec x + \int 2x \, dx - \int 2x \sec^2 x \, dx$$

$$= 2x^2 \sec x + 2x - \int 2x \sec^2 x \, dx$$

$$= 2x^2 \sec x + 2x - 2x \tan x + 2 \ln |\sec x| + C$$

$$= 2x^2 \sec x - 2x \tan x + 2 \ln |\sec x| + C$$

(Alternative: Change coordinate)

$$\int 2x \sec x \, dx = \dots$$

BY PARTS AGAIN

| | |
|-------------------------|-----------------------|
| x^2 | $2x$ |
| $\frac{d}{dx} x^2 = 2x$ | $\frac{d}{dx} 2x = 2$ |

$$= x^2 \sec x - \int 2x \sec^2 x \, dx$$

BY PARTS AGAIN, REASON TO ABOVE

$$= x^2 \sec^2 x - [2x \tan x - \int 2x \tan x \, dx]$$

$$= x^2 \sec^2 x - 2x \tan x + \int 2x \tan x \, dx$$

$$= 2x^2 \sec x - 2x \tan x + 2 \ln |\sec x| + C$$

+1 MARK

710. $\int \ln(\sqrt{x} + \sqrt{x+1}) \, dx = \frac{1}{2}(2x+1) \ln\left[\sqrt{x} + \sqrt{x+1}\right] - \frac{1}{2}\sqrt{x^2+x} + C$

$$\int \ln(\sqrt{x} + \sqrt{x+1}) \, dx = \dots$$

This is surely the answer!! ... SUBSTITUTION

$$= \int \theta (\cosh \theta d\theta) = \int \theta \sinh \theta d\theta$$

INTEGRATION BY PARTS OR "INSIDE OUT"

| |
|---|
| $\theta = \operatorname{arctanh} \frac{x}{\sqrt{1-x^2}}$ |
| $d\theta = \frac{1}{\sqrt{1-x^2}} dx$ |
| $\sinh \theta = \frac{x}{\sqrt{1-x^2}}$ |
| $d\theta = \frac{1}{\sqrt{1-x^2}} dx = \frac{x}{\sqrt{1-x^2}} \cosh \theta d\theta$ |
| $x = \sinh \theta$ |
| $x^2 = \sinh^2 \theta$ |
| $x^2 + x = \sinh^2 \theta + \sinh \theta \cosh \theta = \cosh^2 \theta$ |

$$\int \theta (\cosh \theta d\theta) = 1 \times (\sinh \theta) + \theta \times 2 \sinh \theta$$

$$\int \theta (\cosh \theta d\theta) = \int \sinh \theta d\theta + 2 \int \sinh \theta d\theta$$

$$\sinh \theta = \frac{1}{2} \sinh 2\theta + 2 \int \sinh \theta d\theta$$

$$\int \sinh \theta d\theta = \frac{1}{2} \sinh 2\theta - \frac{1}{2} \sinh 2\theta + C$$

$$\rightarrow \int \ln(\sqrt{x} + \sqrt{x+1}) \, dx = \frac{1}{2} \theta (\sinh \theta + \sinh^2 \theta) - \frac{1}{2} \sinh^2 \theta + C$$

$$= \frac{1}{2} \operatorname{arctanh} \frac{x}{\sqrt{1-x^2}} \left[2x + 1 + x^2\right] - \frac{1}{2} \frac{x^2}{\sqrt{1-x^2}} + C$$

$$= \frac{1}{2} \operatorname{arctanh} \frac{x}{\sqrt{1-x^2}} \left[\sqrt{x^2+1}\right] - \frac{1}{2} \frac{x^2}{\sqrt{1-x^2}} + C$$

THE SUBSTITUTION $\theta = \operatorname{arctanh} \frac{x}{\sqrt{1-x^2}}$ ALSO WORKS BUT IT IS EXTREMELY UGLY!!!

711. $\int \frac{3\sin x + 4\cos x}{2\sin x - \cos x} dx = \frac{2}{5}x + \frac{11}{5}\ln|2\sin x - 4\cos x| + C$

$$712. \int \frac{4}{x\sqrt{x^4+1}} dx = \begin{cases} \ln \left[\frac{\sqrt{x^4+1}-1}{\sqrt{x^4+1}+1} \right] + C \\ 2 \ln \left[\frac{1-\sqrt{x^4+1}}{x^2} \right] + C \end{cases}$$

Integrating by Substitution

$$\int \frac{4}{x\sqrt{x^2+1}} dx = \dots$$

TRANSFORMED SUBSTITUTION

$$= \int \frac{4}{x\sqrt{\frac{u^2+1}{u^2}}} du = \int \frac{4xu}{x\sqrt{u^2+1}} du$$

$$= \int \frac{4u}{\sqrt{u^2+1}} du = \int \frac{4\sec u}{\tan u} du$$

$$= \int \frac{2}{\sin u} \cdot \frac{4\sec u}{\tan u} du = \int \frac{8\sec^2 u}{\sin u} du$$

$$= \begin{cases} 2\ln|\csc u| + C \\ 2\ln|(\sec u - \cot u)| + C \end{cases}$$


REVERSING THE SUBSTITUTIONS

$$\dots = 2\ln|\csc \frac{1}{2}\theta| + C$$

$$= 2\ln\left|\frac{1+\sqrt{1+2x^2}}{x}\right| + C$$

OR

$$\dots = 2\ln|\csc(\theta) - \cot(\theta)| + C$$

$$= 2\ln\left|\frac{1}{x} - \frac{\sqrt{x^2+1}}{x}\right| + C$$

$$= 2\ln\left|\frac{1-x\sqrt{x^2+1}}{x}\right| + C$$

$$= 2\ln\left|\frac{x(1+x\sqrt{x^2+1})}{x}\right| + C$$

T = $\frac{x}{\sqrt{x^2+1}}$

T = $\frac{1+x\sqrt{x^2+1}}{\sqrt{x^2+1}}$

$\tan \frac{\theta}{2} = \frac{1+x\sqrt{x^2+1}}{\sqrt{x^2+1}}$

$\theta = 2\tan^{-1}(\tan \frac{\theta}{2})$

$\theta = 2\tan^{-1}(\frac{1+x\sqrt{x^2+1}}{\sqrt{x^2+1}})$

$\theta = 2\tan^{-1}(\frac{x}{\sqrt{x^2+1}})$

MANIPULATE THE FRACTION IN THE INTEGRAND AS FOLLOWS:

$$\frac{1}{\sin(2x\pi) - \cos x} dx = \frac{2\cos x + \sin x}{2\cos^2 x - \cos x}$$

$$3\sin x + 4\cos x \equiv A(2\sin x - \cos x) + B(2\cos x + \sin x)$$

$$\equiv 2A\sin x - A\cos x + 2B\cos x + B\sin x$$

$$\equiv (2A+B)\sin x + (-A+2B)\cos x$$

$$\begin{cases} 2A+B=3 \\ -A+2B=4 \end{cases} \Rightarrow \begin{cases} 2A+3=7 \\ -2A+8=8 \end{cases} \Rightarrow \begin{cases} 5B=11 \\ B=\frac{11}{5} \end{cases}$$

$$\Rightarrow \boxed{B=\frac{11}{5}}$$

AND SINCE

$$\begin{aligned} -A+2B &= 4 \\ -A+\frac{22}{5} &= 4 \\ -5A+22 &= 20 \\ 2 &= 5A \\ A &= \frac{2}{5} \end{aligned}$$

$$\dots = \int \frac{3(2\sin x - \cos x) + \frac{66}{5}(2\cos x + \sin x)}{2\cos^2 x - \cos x} dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} \frac{25\sin x - 6\cos x}{2\cos x - \cos x} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{20\cos x + 5\sin x}{2\cos^2 x - \cos x} dx$$

$$= \frac{2}{5} \int \left(15x + \frac{6}{5} \right) \frac{2\cos x + \sin x}{\sin^2 x - \cos x} dx$$

$$\rightarrow \text{OF THE TYPE } \int \frac{f(x)}{g(x)} dx$$

$$= \frac{3}{5}x + \frac{1}{5}\ln|2\sin x - \cos x| + C$$

ALTERNATIVE TO HYPERBOLIC SUBSTITUTION

$\int \frac{4}{x^2\sqrt{x^2+1}} dx = \int \frac{4}{x^2\sqrt{x^2+1+x^2-1}} (2x dx)$

$= \int \frac{4}{x^2\sqrt{2x^2+1}} dx = \int \frac{2}{x\sqrt{2x^2+1}} dx$

$= \int \frac{2}{x\sqrt{2(x^2+1)}} dx = \dots$ BY STANDARD SUBST.

OC. MANIPULATIONS AS FOLLOWS

$$\dots = \int \frac{2}{x\sqrt{2x^2+1}} dx = \int \frac{2x^2}{x\sqrt{2x^2+1}} dx = \int \frac{2x^2}{\cancel{x}\sqrt{2(x^2+1)}} dx$$

PARTIAL FRACTIONS AFTER HYPERBOLIC SUBSTITUTION $y = \sqrt{x^2+1}$ OR
INTEGRATION BY PARTS

$\int \frac{2x^2}{\cancel{x}\sqrt{2(x^2+1)}} dx = \int \frac{2}{\cancel{x}\sqrt{2(y^2-1)}} dy = \int \frac{2}{y\sqrt{2(y^2-1)}} dy$

 $= \int \frac{1}{y\sqrt{y^2-1}} dy = \frac{1}{\sqrt{y^2-1}} dy$
 $= \ln|y - \sqrt{y^2-1}| - \ln|y + \sqrt{y^2-1}| = \ln\left|\frac{y-\sqrt{y^2-1}}{y+\sqrt{y^2-1}}\right| + C$
 $= \ln\left|\frac{\sqrt{x^2+1}-1}{x+\sqrt{x^2+1}}\right| + C$
 $= \ln\left|\frac{\sqrt{x^2+1}-1}{\cancel{x}\sqrt{x^2+1}+1}\right| + C$

[EXPRESSIONS ARE EQUIVALENT]

713. $\int \frac{(3x^2 + 5x)\sqrt{x}}{(x+1)^2} dx = \frac{2x^2}{x+1} + C$

$\theta = \arctan(\frac{y}{x})$
 $\tan\theta = \frac{y}{x}$
 $x = \cos\theta$
 $y = \sin\theta$

$$\begin{aligned} & \int \frac{(3x^2 + 5x)\sqrt{x}}{(x+1)^2} dx \dots \text{BY TRIGONOMETRIC SUBSTITUTION} \\ &= \int \frac{(3\cos^2\theta + 5\cos\theta)\sqrt{\cos^2\theta}}{(\cos^2\theta + 1)^2} d\theta \\ &= \int \frac{3\cos^2\theta + 5\cos\theta}{\sin^2\theta} d\theta \\ &= \int \frac{3(1 - \sin^2\theta) + 5\cos\theta}{\sin^2\theta} d\theta \\ &= \int \frac{3 - 3\sin^2\theta + 5\cos\theta}{\sin^2\theta} d\theta \\ &= \int \frac{3(1 - \sin^2\theta) + 10 - 10\sin^2\theta + 10\cos\theta}{\sin^2\theta} d\theta \\ &= \int \frac{6 - 10\sin^2\theta + 10\cos\theta}{\sin^2\theta} d\theta + 10 - 20\cos\theta + 10\sin^2\theta d\theta \\ &= \int \frac{6\cos^2\theta - 10\sin^2\theta + 10\cos\theta}{\sin^2\theta} d\theta + 10 - 20\cos\theta + 10\sin^2\theta d\theta \\ &= \int \frac{6\cos^2\theta(1 + \tan^2\theta)}{\sin^2\theta} d\theta - 10\cos\theta + 10\sin^2\theta d\theta \\ &= \int 6\cos^2\theta \cdot \frac{1}{\sin^2\theta} d\theta - 10\cos\theta + 10\sin^2\theta d\theta \\ &= \int 6\cos^2\theta \cdot \frac{1}{1 - \cos^2\theta} d\theta - 10\cos\theta + 10\sin^2\theta d\theta \\ &= \int \frac{6\cos^2\theta}{1 - \cos^2\theta} d\theta - 10\cos\theta + 10\sin^2\theta d\theta \\ &= \int \frac{6\cos^2\theta}{\sin^2\theta} d\theta - 10\cos\theta + 10\sin^2\theta d\theta \\ &= \int 6 \left[\frac{\cos^2\theta}{\sin^2\theta} + 1 \right] d\theta - 10\cos\theta + 10\sin^2\theta d\theta \\ &= 6 \left[\frac{\cos^2\theta}{\sin^2\theta} + 1 \right] + C - 10\cos\theta + 10\sin^2\theta d\theta \\ &= 6 \left[\frac{1 - \sin^2\theta}{\sin^2\theta} + 1 \right] + C - 10\cos\theta + 10\sin^2\theta d\theta \\ &= \frac{20}{\sin^2\theta} + C - 10\cos\theta + 10\sin^2\theta d\theta \\ &= \frac{20}{\sin^2\theta} + C \end{aligned}$$

714. $\int \frac{4}{(x^2 - 4)^{\frac{3}{2}}} dx = -\frac{x}{\sqrt{x^2 - 4}} + C$

BY A TRIGONOMETRIC SUBSTITUTION

$\theta = \arcsin\frac{y}{x}$
 $x = 2\cos\theta$
 $dx = -2\sin\theta \cos\theta d\theta$
 $\frac{y}{x} = \sin\theta$
 $\cos\theta = \frac{x}{\sqrt{x^2 - 4}}$

$$\begin{aligned} & \int \frac{4}{(x^2 - 4)^{\frac{3}{2}}} dx = \int \frac{4}{(4\cos^2\theta - 4)^{\frac{3}{2}}} (2\sin\theta \cos\theta d\theta) \\ &= \int \frac{4}{8(\sin^2\theta)^{\frac{3}{2}}} (2\sin\theta \cos\theta d\theta) = \int \frac{8\sin\theta \cos\theta}{8(\sin^2\theta)^{\frac{3}{2}}} d\theta \\ &= \int \frac{8\sin\theta \cos\theta}{8\sin^3\theta} d\theta = \int \frac{8\cos\theta}{8\sin^2\theta} d\theta = \int \frac{8\cos\theta}{8(1 - \cos^2\theta)} d\theta \\ &- \int \frac{8\cos\theta}{8\sin^2\theta} \cdot \frac{\sin\theta}{\sin\theta} d\theta = \int \frac{8\cos\theta}{8\sin^2\theta} \cdot \frac{\sin\theta}{8\sin^2\theta} d\theta = \int \frac{8\cos\theta}{64\sin^3\theta} d\theta = \int \frac{1}{8\sin^2\theta} d\theta \\ &= -\left(\frac{1}{8}\right) \frac{1}{\sin\theta} + C = -\frac{1}{8\sin\theta} + C = -\frac{x}{8\sqrt{x^2 - 4}} + C \end{aligned}$$

AUGMENTATIVE FOR $\int \frac{\cos\theta}{\sin^2\theta} d\theta$

$$\begin{aligned} & \int \frac{\cos\theta}{\sin^2\theta} d\theta = \int \frac{\cos\theta}{\sin\theta \cdot \sin\theta} d\theta = \int (\cot\theta \operatorname{cosec}\theta) d\theta \\ &= -\operatorname{cosec}\theta + C \end{aligned}$$

SINCE $\frac{d}{dx}(\operatorname{cosec}\theta) = -\operatorname{cosec}\theta \cot\theta$

ALTERNATIVE BY HYPERBOLIC SUBSTITUTION

$\theta = \operatorname{arccosh}\frac{x}{2}$
 $\cosh\theta = \frac{x}{2}$
 $\sinh\theta = \sqrt{\cosh^2\theta - 1}$
 $\frac{d}{dx}(\operatorname{cosh}\theta) = \sinh\theta$
 $\frac{d}{dx}(\operatorname{cosh}\theta) = \frac{1}{2}$
 $\frac{d}{dx}(2\operatorname{cosh}\theta) = \frac{1}{2}$
 $\frac{d}{dx}(2\operatorname{cosh}\theta) = \frac{1}{2} \operatorname{cosh}\theta$
 $\operatorname{cosh}\theta = \frac{x^2 - 4}{4}$
 $\sinh\theta = \pm \sqrt{\frac{x^2 - 4}{4}}$

$$\begin{aligned} & \int \frac{4}{(x^2 - 4)^{\frac{3}{2}}} dx = \int \frac{4}{(4\operatorname{cosh}^2\theta - 4)^{\frac{3}{2}}} (2\operatorname{cosh}\theta \operatorname{sinh}\theta d\theta) = \int \frac{8\operatorname{cosh}\theta \operatorname{sinh}\theta}{4(\operatorname{cosh}^2\theta - 1)^{\frac{3}{2}}} d\theta \\ &= \int \frac{8\operatorname{cosh}\theta \operatorname{sinh}\theta}{8(\operatorname{sinh}^2\theta)^{\frac{3}{2}}} d\theta = \int \frac{8\operatorname{cosh}\theta \operatorname{sinh}\theta}{8(\operatorname{sinh}^2\theta)^{\frac{3}{2}}} d\theta \\ &= \int \operatorname{cosh}\theta d\theta = -\operatorname{sinh}\theta + C \\ &= -\frac{\operatorname{sinh}\theta}{\operatorname{cosh}\theta} + C = -\frac{\sqrt{x^2 - 4}}{\frac{x}{2}} + C \\ &= -\frac{\sqrt{x^2 - 4}}{x} + C \end{aligned}$$

AS ANSWER

$$715. \int \frac{1}{(1-x)\sqrt{1-2x}} dx = -2 \arctan(\sqrt{1-2x}) + C$$

BY TRIGONOMETRIC SUBSTITUTIONS

$\theta = \arcsin(\sqrt{2x})$

$\sin\theta = \sqrt{2x}$

$2x = \sin^2\theta$

$2dx = 2\sin\theta\cos\theta d\theta$

$dx = \frac{\sin\theta\cos\theta}{2} d\theta$

$\sec\theta = \frac{1}{\sqrt{1-2x}}$

$\int \frac{1}{(1-x)\sqrt{1-2x}} dx = \int \frac{1}{(1-\frac{1}{\sin^2\theta})\sqrt{1-\frac{2\sin^2\theta}{\sin^2\theta}}} (\frac{\sin\theta\cos\theta}{2} d\theta)$

$= \int \frac{\sin\theta\cos\theta}{2\cos^2\theta} d\theta = \int \frac{2\sin^2\theta\cos\theta}{2\cos^2\theta} d\theta$

$= \int \frac{2\sin^2\theta\cos\theta}{2\cos^2\theta - \sin^2\theta} d\theta = \int \frac{2\sin^2\theta\cos\theta}{2\cos^2\theta - \cos^2\theta} d\theta$

WORKING AT THE NUMERATOR, $\frac{d}{d\theta}(\sec\theta) = \sec\theta\tan\theta$

$= \int \frac{2\sin^2\theta\sec\theta\tan\theta}{2\cos^2\theta - (\sec^2\theta - 1)} d\theta = \int \frac{2\sin^2\theta\sec\theta\tan\theta}{\sec^2\theta - 1} d\theta$

(USING SUBSTITUTION OR RESOLUTION)

$= 2\arctan(\sec\theta) + C = 2\arctan\left(\frac{1}{\sqrt{1-2x}}\right) + C$

$= 2\arctan(\sqrt{1-2x}) + C = 2\left[\frac{\pi}{2} - \arctan(\sqrt{1-2x})\right] + C$

$= -2\arctan(\sqrt{1-2x}) + (\pi+C) = -2\arctan(\sqrt{1-2x}) + C$

"AT THE END"

- $\arctan(\frac{1}{x}) = \arccot x \Leftrightarrow \arctan x = \arccot \frac{1}{x}$
- $\arctan x + \arccot x = \frac{\pi}{2}$

$$716. \int \sqrt{\tan x - \sqrt{\cot x}} dx = -\sqrt{2} \ln [\cos x + \sin x + \sqrt{\sin 2x}] + C$$

$\int \sqrt{\tan x - \sqrt{\cot x}} dx = \int \frac{\sqrt{\csc x}}{\sqrt{\cot x}} dx = \int \frac{\sqrt{\csc x}}{\sqrt{\frac{\csc x - \cos x}{\sin x}}} dx = \int \frac{\sqrt{\csc x}}{\sqrt{\frac{\sin x}{\csc x - \cos x}}} dx = \int \frac{\sqrt{\csc x}}{\sqrt{\frac{1}{\csc x - \cos x}}} dx$

MANIPULATE THE DENOMINATOR AS FOLLOWS — $\csc x$ WAS NOT NECESSARILY $\sqrt{\csc x} = \sqrt{\csc x + 1 - 1} = \sqrt{(\csc x)^2 + \csc x + \sin x - 1}$

$= \sqrt{(\csc x + \sin x)^2 - 1}$

NOTE THAT $\frac{d}{dx}(\csc x + \sin x) = \csc x - \sin x$ = "NUMERATOR!"

$u = \csc x + \sin x$
 $du = \csc x - \sin x dx$
 $dx = \frac{du}{\csc x - \sin x}$

$\dots = \sqrt{2} \int \frac{\sin x - \cos x}{\sqrt{(csc x + sin x)^2 - 1}} du = \sqrt{2} \int \frac{\frac{\sin x - \cos x}{\csc x - \sin x} du}{\sqrt{\frac{1}{\csc x - \sin x}}} = \sqrt{2} \int \frac{\sin x - \cos x}{\sqrt{\csc x - \sin x}} du$

$\dots = -\sqrt{2} \int \frac{1}{\sqrt{\csc x - \sin x}} du = -\sqrt{2} \arctan u + C = -\sqrt{2} \ln [1 + \sqrt{\frac{1}{\csc x - \sin x}}] + C$

$= -\sqrt{2} \ln [\csc x + \sin x + \sqrt{(\csc x)^2 + \csc x \sin x - 1}] + C$

$= -\sqrt{2} \ln [\csc x + \sin x + \sqrt{\csc^2 x + \csc x \sin x - 1}] + C$

$= -\sqrt{2} \ln [\csc x + \sin x + \sqrt{\sin 2x}] + C$

717. $\int \sqrt{(1+x)(5-x)} dx = \frac{9}{2} \arcsin\left[\frac{1}{3}(x-2)\right] + \frac{1}{2}(x-2)\sqrt{(1+x)(5-x)} + C$

$$\begin{aligned} \int \sqrt{(1+x)(5-x)} dx &= \int \sqrt{5-2+5x-x^2} dx = \int \sqrt{5+4x-x^2} dx \\ &= \int \sqrt{-(x^2-4x-5)} dx = \int \sqrt{-(x-2)^2-13} dx = \int \sqrt{3-(x-2)^2} dx \\ &\text{By TRIGONOMETRIC SUBSTITUTION} \\ \theta &= \arcsin\left(\frac{x-2}{\sqrt{3}}\right) \\ x-2 &= \sqrt{3}\sin\theta \\ dx &= \sqrt{3}\cos\theta d\theta \end{aligned}$$

$$\begin{aligned} &\int \sqrt{3-\sin^2\theta} (\sqrt{3}\cos\theta) d\theta = \int \sqrt{3(1-\sin^2\theta)} (\sqrt{3}\cos\theta) d\theta \\ &= \int \sqrt{3}\cos^2\theta (\sqrt{3}\cos\theta) d\theta = \int (3\cos^3\theta) d\theta \\ &= \int 3\cos^2\theta d\theta = \int 3\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta = \int \frac{3}{2} + \frac{3}{2}\cos 2\theta d\theta \\ &= \frac{3}{2}\theta + \frac{3}{4}\sin 2\theta + C = \frac{3}{2}\theta + \frac{3}{2}\sin\theta\cos\theta + C \\ &= \frac{3}{2}\arcsin\left(\frac{x-2}{\sqrt{3}}\right) + \frac{3}{2}(x-2)\sqrt{1-(x-2)^2} + C \\ &= \underline{\underline{\frac{3}{2}\arcsin\left(\frac{x-2}{\sqrt{3}}\right) + \frac{1}{2}(x-2)\sqrt{(1+x)(5-x)}}} + C \end{aligned}$$

718. $\int \sqrt[3]{\frac{8}{x} + \frac{8}{x^3}} dx = \dots$

$$3(x^2+1)^{\frac{1}{3}} + \frac{1}{2} \ln \left[\frac{(x^2+1)^{\frac{2}{3}} - 2(x^2+1)^{\frac{1}{3}} + 1}{(x^2+1)^{\frac{2}{3}} + (x^2+1)^{\frac{1}{3}} + 1} \right] - \sqrt{3} \arctan \left[\frac{2(x^2+1)^{\frac{1}{3}} + 1}{\sqrt{3}} \right] + C$$

$$\begin{aligned} \int \sqrt[3]{\frac{8}{x} + \frac{8}{x^3}} dx &= \int 2 \left(\frac{1}{x} + \frac{1}{x^3} \right)^{\frac{1}{3}} dx = \int 2 \left(\frac{x^2+1}{x^3} \right)^{\frac{1}{3}} dx = \int \frac{2}{x} (x^2+1)^{\frac{1}{3}} dx \\ &\text{By SUBSTITUTION TO REMOVE THE CUBE ROOT} \\ \bullet u &= (x^2+1)^{\frac{1}{3}} \quad \bullet x^2 dx = 2u^2 du \\ \bullet u^3 &= x^3+1 \quad \bullet du = \frac{3u^2}{2x^2} dx \\ \bullet x^2 &= u^3-1 \\ &= \int \frac{2}{x} \left(u \right)^{\frac{1}{3}} \left(\frac{3u^2}{2x^2} dx \right) = \int \frac{6u^3}{2x^2} dx = \int \frac{3u^3}{u^3-1} du \\ &= \int \frac{3(u^3-1)+3}{u^3-1} du = \int 3 + \frac{3}{u^3-1} du = \int 3 + \frac{3}{(u-1)(u^2+u+1)} du \\ &\text{HENCE USE} \\ &\frac{3}{(u-1)(u^2+u+1)} \equiv \frac{A}{u-1} + \frac{Bu+C}{u^2+u+1} \\ 3 &\equiv A(u^2+u+1) + (Bu+C)(u-1) \\ 3 &\equiv A(u^2+u+1) + (Bu-C)(u-1) \\ &\bullet \text{IF } u=1 \quad \bullet \text{IF } u=0 \quad \bullet \text{IF } u=2 \\ 3 &\equiv 3A \quad 3=A-C \quad 3=7A+2C \\ A=1 &\quad 3=-C \quad 3=7A+2B-2 \\ C=-2 &\quad 3=7+2B-2 \quad 3=7+2B-2 \\ B=-1 &\quad 3=7+2B-2 \quad 3=7+2B-2 \\ &= 3 + \int \frac{1}{u-1} du = -\frac{u+2}{(u^2+u+1)} du \\ &= \int 3 + \frac{1}{u-1} du = -\frac{1}{(u^2+u+1)} du \\ &= \int 3 du + \int \frac{1}{u-1} du = -\frac{1}{2} \int \frac{1}{(u^2+u+1)} du = -\frac{3}{2} \int \frac{1}{u^2+u+1} du \\ &= 3u + \ln|u-1| - \frac{1}{2} \ln|u^2+u+1| = -\frac{3}{2} \int \frac{4}{4u^2+4u+4} du \\ &= 3u + \frac{1}{2} \ln|u-1| - \frac{1}{2} \ln|u^2+u+1| = -\frac{3}{2} \int \frac{4}{(2u+1)^2+3} du \end{aligned}$$

$$\begin{aligned} &= 3u + \frac{1}{2} [\ln(u^2+u+1) - \ln(u^2+u+1)] - \frac{3}{2} \int \frac{4}{(2u+1)^2+3} du \\ &= 3u + \frac{1}{2} \ln\left(\frac{u^2+u+1}{u^2+u+1}\right) - \frac{3}{2} \int \frac{4}{(2u+1)^2+3} du \\ &\text{By RECOGNITION OR SUBSTITUTION} \\ \bullet u &= 2u+1 \\ \frac{du}{du} &= 2 \\ du &= \frac{1}{2} du \\ &= 3u + \frac{1}{2} \ln\left(\frac{u^2+2u+1}{u^2+2u+1}\right) - \frac{3}{2} \int \frac{4}{(2u+1)^2+3} du \\ &= 3u + \frac{1}{2} \ln\left(\frac{u^2+2u+1}{u^2+2u+1}\right) - 3 \times \frac{1}{\sqrt{3}} \arctan\left(\frac{2u+1}{\sqrt{3}}\right) + C \\ &= 3u + \frac{1}{2} \ln\left(\frac{u^2+2u+1}{u^2+2u+1}\right) - \sqrt{3} \arctan\left(\frac{2u+1}{\sqrt{3}}\right) + C \\ &= 3(x^2+1)^{\frac{1}{3}} + \frac{1}{2} \ln\left[\frac{(x^2+1)^{\frac{2}{3}}-2(x^2+1)^{\frac{1}{3}}+1}{(x^2+1)^{\frac{2}{3}}+(x^2+1)^{\frac{1}{3}}+1}\right] - \sqrt{3} \arctan\left[\frac{2(x^2+1)^{\frac{1}{3}}+1}{\sqrt{3}}\right] + C \end{aligned}$$

719. $\int \frac{1-x}{\sqrt{x}(x+1)^2} dx = \frac{2\sqrt{x}}{x+1} + C$

$$\begin{aligned}
 & \int \frac{1-x}{\sqrt{x}(x+1)^2} dx \dots \text{BY TRIGONOMETRIC SUBSTITUTION} \\
 &= \int \frac{1-\tan^2\theta}{\sqrt{\tan^2\theta}(\sec^2\theta)^2} \cdot 2\sec^2\theta \tan\theta d\theta \\
 &= \int \frac{1-\tan^2\theta}{2\tan\theta} d\theta \\
 &= \int \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{\frac{2\sin\theta}{\cos\theta}} d\theta \\
 &= 2 \int \frac{\cos^2\theta - \sin^2\theta}{\sin\theta} d\theta \\
 &= \int 2\cos 2\theta d\theta \\
 &= \sin 2\theta + C \\
 &= 2\sin\theta \cos\theta + C \\
 &= 2 \frac{\sin\theta}{\cos\theta} \times \cos^2\theta + C \\
 &= 2\tan\theta \times \frac{1}{1+\tan^2\theta} + C \\
 &= 2\tan\theta \times \frac{1}{1+\frac{4x}{1-x}} + C \\
 &= \frac{2\sqrt{x}}{1-x} + C
 \end{aligned}$$

720. $\int 24\sin x \sin 2x \sin 3x dx = \left[3\sin^2 2x + 4\cos^3 2x - 6\cos 2x + C \right. \\ \left. - \frac{3}{2}\cos 4x + 4\cos^3 2x - 6\cos 2x + C \right]$

$$\begin{aligned}
 & \int 24\sin x \sin 2x \sin 3x dx \dots \text{BY TRIGONOMETRIC IDENTITIES} \\
 & \cos(2x+2) = \cos 2x \cos 2 - \sin 2x \sin 2 \\
 & \cos(2x-2) = \cos 2x \cos 2 - \sin 2x \sin 2 \quad \text{SUBTRACT} \\
 & \cos(3x)-\cos 2x = -2\sin 3x \sin 2 \\
 & 2\sin 3x \sin 2 = \cos 2x - \cos 3x \\
 & \dots \int (2\sin 2x)(2\sin 3x \sin 2) dx = \int 2\sin 2x (\cos 2x - \cos 3x) dx \\
 &= \int 12\sin^2 2x \cos 2x - 12\sin^2 2x \cos 3x dx \\
 & \quad \uparrow \text{REMEMBER} \quad \uparrow \text{REMEMBER} \\
 & \quad \sin 2x = 2\sin x \cos x \quad \cos 2x = 2\cos^2 x - 1 \\
 & \quad \sin 3x = 3\sin x \cos^2 x \quad \cos 3x = 4\cos^3 x - 3\cos x \\
 &= \int 6\sin^2 2x - 12\sin^2 2x \cos 3x dx \\
 & \quad \uparrow \text{REMEMBER} \\
 & \quad 3\sin^2 2x \\
 &= \int 6\sin^2 2x - 24\sin^2 2x \cos 2x + 12\sin^2 2x dx \\
 & \quad \uparrow \text{REMEMBER} \quad \uparrow \text{REMEMBER} \\
 &= \left[-\frac{2}{3}\cos 2x + \frac{4\cos^3 2x}{3} - 6\cos 2x + C \right. \\
 & \quad \left. - \frac{3}{2}\cos 4x + 4\cos^3 2x - 6\cos 2x + C \right]
 \end{aligned}$$

721. $\int \frac{\sec^2 x}{\sqrt{\sec x + \tan x}} dx = (\sec x + \tan x)^{\frac{1}{2}} - \frac{1}{3}(\sec x + \tan x)^{-\frac{3}{2}} + C$

WORKING AT THE DOCUMENT "IN THE DOCUMENT"

$\frac{d}{dx}(\sec x + \tan x) = \sec x + \tan x$

$$\begin{aligned} & \int \frac{\sec^2 x}{\sqrt{\sec x + \tan x}} dx \\ &= \int \frac{\sec^2 x - \sec x \tan x - \sec x \tan x}{\sqrt{\sec x + \tan x}} dx \\ &= \int \frac{\sec^2 x - \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec x - \sec x \tan x}{\sqrt{\sec x + \tan x}} dx \\ &= \frac{1}{2} \int \frac{\sec x + \sec x \tan^2 x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec x (\sec x - \tan x)(\sec x + \tan x)}{\sqrt{\sec x + \tan x} (4\sec^2 x - 4\sec x \tan x)} dx \\ &= \frac{1}{2} \int \frac{\sec x + \sec x \tan^2 x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec x (\sec x - \tan x)}{(\sec x + \tan x)^{\frac{3}{2}}} dx \\ &= \frac{1}{2} \int \frac{\sec x + \sec x \tan^2 x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec x}{(\sec x + \tan x)^{\frac{3}{2}}} dx \\ &= \frac{1}{2} \int \frac{\sec x \sec x \tan^2 x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec x (\sec x + \tan x) \tan x}{(\sec x + \tan x)^{\frac{3}{2}}} dx \\ &= \frac{1}{2} \int \frac{\sec x \sec x \tan x}{(\sec x + \tan x)^{\frac{3}{2}}} dx + \frac{1}{2} \int \frac{\sec x (\sec x + \tan x) \tan x}{(\sec x + \tan x)^{\frac{3}{2}}} dx \\ &= \frac{1}{2} \int \frac{\sec x \sec x \tan x}{(\sec x + \tan x)^{\frac{3}{2}}} dx + \frac{1}{2} \int \frac{\sec x (\sec x + \tan x) \tan x}{(\sec x + \tan x)^{\frac{3}{2}}} dx \\ & \text{THESE ARE TWO "EXACT DIFFERENTIATES" OF THE ARGUMENT OF THE RADICAL} \\ &= \frac{1}{2} \int (\sec x + \tan x) (\sec x + \tan x)^{\frac{1}{2}} dx + \frac{1}{2} \int (\sec x + \tan x) (\sec x + \tan x)^{-\frac{1}{2}} dx \\ &= \frac{1}{2} \times \frac{1}{\frac{1}{2}} (\sec x + \tan x)^{\frac{1}{2}} + \frac{1}{2} \times \frac{1}{\frac{1}{2}} (\sec x + \tan x)^{-\frac{1}{2}} + C \\ &= \underline{\underline{(\sec x + \tan x)^{\frac{1}{2}} - \frac{1}{3}(\sec x + \tan x)^{-\frac{3}{2}} + C}}$$

722. $\int \ln(\ln x) + \frac{1}{(\ln x)^2} dx = x \ln(\ln x) - \frac{x}{\ln x} + C$

$\int \ln(\ln x) + \frac{1}{(\ln x)^2} dx = \int \ln(\ln x) dx + \int \frac{1}{(\ln x)^2} dx$

PROCEEDED BY INTEGRATION BY PARTS IN EACH NONREAL REPETITIVE, WITH THE IDEA OF CANCELING THESE OUT

- $\bullet \int x \ln(\ln x) dx = x \ln(\ln x) - \int \frac{1}{\ln x} dx$
- $\bullet \int \frac{1}{\ln x} dx = \frac{2}{(\ln x)^2} + 2 \int \frac{1}{(\ln x)^2} dx$
- $\bullet \int \frac{1}{\ln x} dx = \frac{2}{(\ln x)^2} + \int \frac{1}{(\ln x)^2} dx$
- $\bullet \int \frac{1}{(\ln x)^2} dx = -\frac{2}{\ln x} + \int \frac{1}{\ln x} dx$

NOT NONREAL SO SIMPLY WE NEED TO CANCEL

ONE MORE NOTIFICATION BY PARTS. MIGHT "HAPPEN" $\int \frac{1}{(\ln x)^2} dx = \frac{1}{(\ln x)^2}$

FINALLY WE FINISH BY ADDING.

$$\begin{aligned} \int \ln(\ln x) dx &= x \ln(\ln x) - \int \frac{1}{\ln x} dx \quad \leftarrow \text{BY PARTS} \\ \int \frac{1}{(\ln x)^2} dx &= \frac{2}{(\ln x)^2} + \int \frac{1}{(\ln x)^2} dx \quad \leftarrow \text{BY PARTS} \\ \int \ln(\ln x) + \frac{1}{(\ln x)^2} dx &= x \ln(\ln x) - \frac{2}{\ln x} + C \end{aligned}$$

723. $\int \sqrt{(2x+5)(2x-3)} dx = \dots$

$$\frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4\ln\left[2x+1+\sqrt{(2x+5)(2x-3)}\right] + C$$

SUBSTITUTION:

$$\begin{aligned} & \int \sqrt{(2x+5)(2x-3)} dx = \int \sqrt{4x^2 + 16x - 15} dx = \int \sqrt{(2x+4)^2 - 16} dx \\ & \bullet \theta = \arccos\left(\frac{2x+4}{4}\right) \quad (4x+8)^2 - (4\sin\theta)^2 = 16 \\ & \bullet 2x+4 = 4\cos\theta \quad 16\cos^2\theta - 16\sin^2\theta = 16 \\ & \bullet d\theta = -\frac{1}{2}(2x+4)dx \quad 16\cos\theta = 16\cos\theta \\ & \bullet dx = -2\sin\theta d\theta \end{aligned}$$

$$\begin{aligned} & = \int \sqrt{(4\cos\theta)^2 - 16} \cdot (-2\sin\theta d\theta) \cdot \frac{1}{2} \cdot \frac{1}{\cos\theta} d\theta = \int \frac{8\sin^2\theta}{\cos\theta} d\theta \\ & = \int 8\sin\theta d\theta = \int 8\sin\theta d\theta = -8\cos\theta \\ & = 8\cos\theta - 4\cos\theta = -4\cos\theta \\ & = 2\sin\theta - 4\cos\theta + C = 4\sin\theta\cos\theta - 4\cos\theta + C \\ & = 4\sin\theta\cos\theta - 4\cos\theta + C \\ & = \frac{1}{2}(2x+4)\sqrt{(2x+5)(2x-3)} - 4\cos\theta + C \\ & = \frac{1}{2}(2x+4) \times \sqrt{(2x+5)(2x-3)} - 4\cos\left(\frac{2x+4}{4}\right) + C \\ & = \frac{1}{2}(2x+4)\sqrt{4x^2 + 16x - 15} - 4\ln\left[\frac{2x+4}{2} + \sqrt{\frac{(2x+4)^2 - 16}{16}}\right] + C \\ & = \frac{1}{2}(2x+4)\sqrt{(2x+5)(2x-3)} - 4\ln\left[\frac{2x+4}{2} + \sqrt{2x+5}\sqrt{(2x-3)}\right] + C \\ & = \frac{1}{2}(2x+4)\sqrt{(2x+5)(2x-3)} - 4\ln\left[\frac{1}{2}[2x+1 + \sqrt{(2x+5)(2x-3)}]\right] + C \\ & = \frac{1}{2}(2x+4)\sqrt{(2x+5)(2x-3)} - 4\ln\left[2x+1 + \sqrt{(2x+5)(2x-3)}\right] + C \\ & = \frac{1}{2}(2x+4)\sqrt{(2x+5)(2x-3)} - 4\ln\left[2x+1 + \sqrt{(2x+5)(2x-3)}\right] + C \end{aligned}$$

724. $\int \frac{3x}{x-\sqrt{x^2-1}} dx = x^3 + (x^2-1)^{\frac{3}{2}} + C$

ALTERNATIVE BY HYPERBOLIC SUBSTITUTION:

$$\begin{aligned} & \int \frac{3x}{x-\sqrt{x^2-1}} dx = \int \frac{3x(2x+\sqrt{x^2-1})}{2x-\sqrt{x^2-1}(2x+\sqrt{x^2-1})} dx \\ & = \int \frac{3x^2 + 3x\sqrt{x^2-1}}{2x^2 - (x^2-1)} dx = \int \frac{3x^2 + 3x\sqrt{x^2-1}}{x^2} dx = \int 3x^2 + 3x(x^{-1})^2 dx \\ & = 3x^2 + (x^2-1)^{\frac{3}{2}} + C \end{aligned}$$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION:

$$\begin{aligned} & \int \frac{3x}{x-\sqrt{x^2-1}} dx = \int \frac{3x\sec^2\theta}{\sec\theta - \sqrt{\sec^2\theta - 1}} (\sec\theta\tan\theta) d\theta \\ & = \int \frac{3x\sec^2\theta}{\sec\theta - \tan\theta} (\sec\theta\tan\theta) d\theta = \int \frac{3x\sec^2\theta(\sec\theta + \tan\theta)}{\sec\theta - \tan\theta} (\sec\theta\tan\theta) d\theta \\ & = \int \frac{3x\sec^2\theta(\sec\theta + \tan\theta)}{\sec\theta - \tan\theta} (\sec\theta\tan\theta) d\theta = 3\sec^2\theta\tan^2\theta + 3x\sec^2\theta\tan^2\theta d\theta \\ & = \sec^2\theta + \tan^2\theta + C = \sec^2\theta + (\sec^2\theta - 1)^{\frac{3}{2}} + C \\ & = 2\sec^2\theta + C = 2\sec^2\theta + (+\sqrt{\sec^2\theta - 1})^3 + C \\ & = 2^2 + (x^2-1)^{\frac{3}{2}} + C \end{aligned}$$

$$725. \int \frac{(x^2+1)e^x}{(x+1)^2} dx = \left[\frac{x-1}{x+1} \right] e^x + C$$

726. $\int \frac{(x+3)\sqrt{x}}{(x+1)^2} dx = \frac{2x^{\frac{3}{2}}}{x+1} + C$

$\int \frac{(x+3)^{\frac{1}{2}}}{(x+1)^2} dx = \dots$ By TRIGONOMETRIC SUBSTITUTION

- * $\theta = \arctan x$
- * $\tan \theta = x$
- * $x = \tan \theta$
- * $dx = \sec^2 \theta d\theta$

$$= \int \frac{(\tan \theta + 3)^{\frac{1}{2}} \sec^2 \theta}{(\tan^2 \theta + 1)^2} d\theta = \int \frac{(\tan \theta + 3)^{\frac{1}{2}} \sec^2 \theta}{\sec^4 \theta} d\theta$$

$$= \int \frac{2(\tan \theta + 3)^{\frac{1}{2}}}{\sin^2 \theta} d\theta = \int 2(\tan \theta + 3)^{\frac{1}{2}} \csc^2 \theta d\theta$$

$$= \int 2 \left(\frac{\sin \theta}{\cos \theta} + 3 \right)^{\frac{1}{2}} \csc^2 \theta d\theta = \int \frac{2 \sin \theta}{\cos^2 \theta} + \csc \theta d\theta$$

$$= \int \frac{2(1 - \cos^2 \theta)}{\cos^2 \theta} + 6 \left(\frac{1}{\cos \theta} \right) d\theta = \int \frac{2 - 2\cos^2 \theta}{\cos^2 \theta} + 6 \sec \theta d\theta$$

$$= \int 2 \sec^2 \theta - 4 + 6 \sec \theta d\theta = 2 \tan \theta - 4\theta + 6 \ln |\sec \theta + \tan \theta| + C$$

$$= \int 2 \sec^2 \theta - 4 + 6 \ln |\sec \theta + \tan \theta| - 2 \sec 2\theta + 4\theta$$

$$= \int 2 \sec^2 \theta + 6 \ln |\sec \theta + \tan \theta| - 3 \sec 2\theta d\theta = \int 2 \sec^2 \theta - 2 \sin 2\theta d\theta + C$$

$$= 2 \tan \theta - \sin 2\theta + C = 2 \tan \theta - 2 \sin \theta \cos \theta + C$$

$$= 2 \tan \theta - \frac{2 \sin \theta}{\cos^2 \theta} + C = 2 \frac{\sin \theta}{\cos \theta} - \frac{2 \sin \theta}{\cos^2 \theta} + C$$

$$= 2 \sec \theta \left[1 - \frac{1}{\cos^2 \theta} \right] + C = 2 \sec \theta \left[\frac{1 + \tan^2 \theta}{\cos^2 \theta} \right] + C$$

$$= 2 \sec^3 \theta + C = \frac{2 \sec^3 \theta}{\sec^2 \theta + 1} + C$$

UNIQUE ALTERNATIVES BY SUBSTITUTION & PARTIAL FRACTIONS

$$\begin{aligned} u &= \sqrt{x} & \int \frac{(x+2)^{1/2}}{x^2} dx &= \int \frac{(u+2)^{1/2}}{u^2} du \\ u^2 &= x & = \int \frac{2(u+2)^{1/2}}{u^2(1+u^2)} du &= \int \frac{2u^{1/2}(u+2)^{1/2}}{u^2(u^2+1)} du \\ 2u du &= dx & & \\ \\ &= \int \frac{(u^2+2u+2)^{1/2} - (u^2+1)^{1/2}}{(u^2+1)^{1/2}} du & & \\ &= 2 \int \left(1 + \frac{2u^2}{(u^2+1)^2} \right) du & & \\ & & \downarrow & \\ & & \text{PARTIAL FRACTION, LET } u = u^2 & \\ \frac{2u^2}{(u^2+1)^2} &= \frac{A}{(u^2+1)} + \frac{B}{u^2+1} & A=-2 & \\ 2u^2 &\equiv A + B(u^2+1) & B=1 & \\ 2u^2 &\equiv B(u^2+1) + A(u^2+1) & & \\ \frac{2u^2}{(u^2+1)^2} &= \frac{-2}{(u^2+1)} + \frac{1}{u^2+1} & & \\ \\ &= \int 2 + \frac{2}{u^2+1} - \frac{2}{(u^2+1)^2} du & & \text{ADDITIVE SUBSTITUTION} \\ & & & \begin{array}{l} \text{B=constant} \\ \text{u=tan}\theta \\ \text{du}=\sec^2\theta d\theta \end{array} \\ \\ &= \int 2 + \frac{2}{u^2+1} du - \int \frac{2}{(u^2+1)^2} du & & \\ &= \int 2 + \frac{2}{u^2+1} du - \int \frac{2u^2}{(u^2+1)^3} du & & \\ &- \int 2 + \frac{2}{u^2+1} du & & \int \frac{1}{u^2+1} du = \arctan\theta \\ &= \int 2 + \frac{2}{u^2+1} du - \int \frac{4}{(u^2+1)^2} du & & \\ &= \int 2 + \frac{2}{u^2+1} du - \int 2 + \frac{2}{u^2+1} du & & \int \frac{4}{(u^2+1)^2} du = 2\arctan\theta \end{aligned}$$

PARTIAL FRACTION

$$\int \frac{(x^2+1)^2}{(x+1)^2} dx = \int \frac{\frac{2x^2+2}{(x+1)^2}}{(x+1)^2} dx$$

$$\frac{x^2+1}{(x+1)^2} = A + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$x^2+1 \equiv A(x+1)^2 + B(x+1) + C$$

$$x^2+1 \equiv Ax^2 + (A+2)x + (C+1)$$

$$Ax^2 + (A+2)x + (C+1) = x^2 + 1$$

$$A=1$$

$$C=-2$$

$$B=2$$

$$\int \left[\frac{1}{(x+1)^2} - \frac{2}{x+1} \right] e^x dx = \int \left[e^x - \frac{2e^x}{(x+1)^2} - \frac{2e^x}{x+1} \right] dx$$

INTEGRATE AT THESE TWO

$$\frac{1}{(x+1)^2} \quad \frac{1}{(x+1)}$$

NOTES: BY PARTS, WE ONLY
LEAVE THE SIMPLEST. IT IS

$$= \int e^x dx + 2 \int \frac{e^x}{(x+1)^2} dx - 2 \int \frac{e^x}{x+1} dx$$

$$= \frac{e^x}{x+1} + 2 \left[-\frac{e^x}{(x+1)^2} + \int \frac{e^x}{(x+1)^3} dx \right] - 2 \int \frac{e^x}{x+1} dx$$

$$= \frac{e^x}{x+1} + 2 \left[\int \frac{e^x}{(x+1)^3} dx - \left[\frac{1}{(x+1)^2} + \int \frac{e^x}{(x+1)^2} dx \right] \right]$$

$$= \frac{e^x}{x+1} - \frac{2e^x}{(x+1)^2} + C = e^x \left[1 - \frac{2}{(x+1)^2} \right] + C$$

$$= e^x \left[\frac{2x+1}{x+1} \right] + C = \underline{\underline{\left(\frac{2x+1}{x+1} \right) e^x + C}}$$

$$\begin{aligned}
&= 2u + 2u \arctan u - 2u - 5u \sqrt{u} + C \\
&= 2u + 2u \arctan u - 2u \arctan u - 2u \sqrt{u} + C \\
&= 2u - 2u \sqrt{u} + C = 2u - 2u \frac{\sqrt{u}}{1+u^2} + C = 2u - \frac{2u^{\frac{3}{2}}}{1+u^2} + C \\
&= 2u \left(1 - \frac{1}{1+u^2}\right) + C = 2u \left(\frac{u^2-1}{1+u^2}\right) + C = \frac{2u^3-2u}{1+u^2} + C \\
&= \frac{2u^3}{1+u^2} + C \quad \text{as before}
\end{aligned}$$

SUBSTITUTION WITHOUT FRACTION POSSIBLY EASIER

$$\begin{aligned}
&= 2u + \int \frac{2(u^2-1)}{(u^2+1)^2} (2u^2 du) \\
&= 2u + \int \frac{2(u^2-1)}{u^2+1} 2u^2 du \quad \begin{cases} \theta = \arctan u \\ u = \tan \theta \\ du = \sec^2 \theta d\theta \end{cases} \\
&= 2u + 2 \int \frac{u^2-1}{u^2+1} du = 2u + 2 \int (\tan^2 \theta - 1) \sec^2 \theta d\theta \\
&= 2u + 2 \int \frac{u^2-1}{u^2+1} du = 2u + 2 \int \frac{u^2-1}{u^2+1} du = 2u + 2 \int u^2 du - 2u \int du \\
&= 2u + 2 \int u^2 du = 2u - 2u^2 + C \\
&= 2u - 2u \arctan u + C = 2u - \frac{2u \arctan u}{1+u^2} + C \\
&= 2u - 2u \arctan u + C = 2u - \frac{2u \arctan u}{1+u^2} + C \\
&= 2u \left(1 - \frac{1}{1+u^2}\right) + C = 2u \left(\frac{u^2-1}{1+u^2}\right) + C \\
&= 2u \left(\frac{u^2}{u^2+1}\right) + C = \frac{2u^3}{u^2+1} + C \\
&= \frac{2u^3}{u^2+1} + C \quad \text{as before}
\end{aligned}$$

727. $\int \frac{x^2+2}{(2\cos x + x \sin x)^2} dx = \tan\left[x - \arctan\left(\frac{1}{2}x\right)\right] + C$

$\int \frac{x^2+2}{(2\cos x + x \sin x)^2} dx = \dots$ LOOKING AT THE DENOMINATOR

$2\cos x + x \sin x = 2\cos x - 2\cos x \tan x + 2\cos x \sin x$

$2\cos x = 2 \Rightarrow \tan x = \sqrt{x^2+4}$ & $\tan x = \frac{x}{2} - \frac{1}{2}x$
 $\theta = \arctan\left(\frac{1}{2}x\right)$

$\dots = \int \frac{x^2+2}{4\sqrt{x^2+4} \cos(x - \arctan(\frac{1}{2}x))^2} dx$
 $= \int \frac{x^2+2}{(x^2+4) \cos^2(x - \arctan(\frac{1}{2}x))} dx = \int \frac{(x^2+2) \sec^2(x - \arctan(\frac{1}{2}x))}{x^2+4} dx$

NOW "ALGEBRAIC SUBSTITUTION"

$u = x - \arctan\left(\frac{1}{2}x\right)$
 $\frac{du}{dx} = 1 - \frac{\frac{1}{2}}{1+\frac{1}{4}x^2} = 1 - \frac{2}{4+x^2} = \frac{4+x^2-2}{4+x^2} = \frac{2x^2+2}{4+x^2}$
 $du = \frac{2x}{4+x^2} dx$

$= \int \sec^2(u) \times \frac{2x^2+2}{4+x^2} du = \int \sec^2 u du$
 $= \tan u + C = \tan\left(x - \arctan\left(\frac{1}{2}x\right)\right) + C$

728. $\int \frac{[\ln(x^2+1) - 2\ln x]\sqrt{x^2+1}}{x^4} dx = \frac{2}{9x^3}(x^2+1)^{\frac{3}{2}} \left[1 - 3\ln\left(\frac{x^2+1}{x^2}\right)\right] + C$

$\int \frac{[\ln(x^2+1) - 2\ln x]\sqrt{x^2+1}}{x^4} dx = \dots$

$= \int \ln\left(\frac{x^2+1}{x^2}\right) \frac{\sqrt{x^2+1}}{x^4} dx = \int \ln\left(\frac{x^2+1}{x^2}\right) \frac{1}{x^2} dx$
 $= \int \ln\left(\frac{x^2+1}{x^2}\right) \frac{1}{x^2+1} dx = \int \sqrt{1+\frac{1}{x^2}} \ln\left(\frac{1+x}{x}\right) \frac{1}{x^2} dx$

BY SUBSTITUTION

$u = \sqrt{1+\frac{1}{x^2}}$
 $u^2 = 1 + \frac{1}{x^2}$
 $x^2 = \frac{1}{u^2-1}$
 $2x dx = -\frac{2}{x^3} dx$
 $dx = -\frac{u^3}{2} du$

$\dots = \int u \ln u^2 \frac{1}{x^2} (-\frac{u^3}{2} du)$
 $= \int -u^5 \ln u^2 du$
 $= \int 2u^5 \ln u du$

INTEGRATION BY PARTS OR SIMILAR

$\frac{d}{du}(\ln u) = \frac{1}{u} du$
 $u^2 du = \int 2u^5 du + \int u^2 du$
 $u^2 du = \frac{2}{3}u^3 du + \frac{1}{2}u^2$
 $\frac{1}{u} du = \int u^2 du + \frac{1}{2}u^2$
 $\frac{1}{u^2} du = -\frac{1}{2}u^{-1}$
 $\int 2u^5 du = -\frac{2}{3}u^6 + \frac{1}{2}u^4$

$\therefore \int \frac{[\ln(x^2+1) - 2\ln x]\sqrt{x^2+1}}{x^4} dx = \frac{2}{3}u^4 \left[1 - 3\ln u\right] + C$
 $= \frac{2}{3}\left(1 + \frac{1}{u^2}\right)^4 \left[1 - 3\ln\left(1 + \frac{1}{u^2}\right)\right] + C$
 $= \frac{2}{3}\left(\frac{x^2+1}{x^2}\right)^4 \left[1 - 3\ln\left(1 + \frac{1}{x^2}\right)\right] + C$
 $= \frac{2}{3}(x^2+1)^{\frac{4}{2}} \left[1 - 3\ln\left(1 + \frac{1}{x^2}\right)\right] + C$

729. $\int \frac{\arcsin \sqrt{x} - \arccos \sqrt{x}}{\arcsin \sqrt{x} + \arccos \sqrt{x}} dx = \frac{2}{\pi} \left[\sqrt{x-x^2} + (2x-1) \arcsin \sqrt{x} \right] - x + C$

$\int \frac{\arcsin \sqrt{x} - \arccos \sqrt{x}}{\arcsin \sqrt{x} + \arccos \sqrt{x}} dx \dots$ SIMPLIFY BY SUBSTITUTION
 $y = \sqrt{x}$
 $dy = \frac{1}{2\sqrt{x}} dx$

 $= \int \frac{\arcsin y - \arccos y}{\arcsin y + \arccos y} (2y dy) \quad \text{arcsin A + arccos A = } \frac{\pi}{2}$
 $= \int \frac{\arcsin y - (\frac{\pi}{2} - \arccos y)}{\arcsin y + \arccos y} (2y dy)$
 $= \int \frac{2y(\arccos y - \frac{\pi}{2})}{\arcsin y + \arccos y} dy = \int 4y \sin y - 2y dy$
 $= -y^2 + \frac{4}{\pi} \int y \sin y dy$

NOW INTEGRATION BY PARTS OR SUBSTITUTION FOLLOWED BY INTEGRATION BY PARTS AGAIN

 $y = \arccos y$
 $dy = -\sin y dy$
 $\cos y = \sqrt{1-y^2}$
 $= -y^2 + \frac{4}{\pi} \int (\frac{1}{2} \sin y) dy$
 $= -y^2 + \frac{4}{\pi} \int 4y \sin y dy$

NOW INTEGRATION BY PARTS OR SIMILAR MANIPULATION

 $\Rightarrow \int \frac{4}{\pi} \int 4y \sin y dy = 16y \sin y - 20y \cos y$
 $\Rightarrow \int \frac{4}{\pi} \int 20y \cos y dy = 20y \cos y - 40y \sin y$
 $\Rightarrow \int \frac{4}{\pi} \int 20y \cos y dy = [20y \cos y] - \int 40y \sin y dy$

$\Rightarrow 20y \cos y = \sin 2y - \int 40y \sin 2y dy$
 $\Rightarrow \int 40y \sin 2y dy = \sin 2y - 20y \cos 2y + C$

RETURNING TO THE INTEGRAL - MAIN LINE

 $\int \frac{\arcsin \sqrt{x} - \arccos \sqrt{x}}{\arcsin \sqrt{x} + \arccos \sqrt{x}} dx = -y^2 + \frac{4}{\pi} \int 16y \sin 2y dy$
 $= -y^2 + \frac{4}{\pi} \left[24y \cos 2y - 2y(1-2y^2) \right] + C$
 $= -y^2 + \frac{2}{\pi} \left[24y \cos 2y - 6(1-2y^2) \right] + C$
 $= -y^2 + \frac{2}{\pi} \left[y \sqrt{1-y^2} - (\arccos y)(1-y^2) \right] + C$
 $= -y^2 + \frac{2}{\pi} \left[y \sqrt{1-y^2} + (2y^2-1) \arccos y \right] + C$
 $= -2x + \frac{2}{\pi} \left[\sqrt{x} \sqrt{1-x} + (2x-1) \arccos \sqrt{x} \right] + C$
 $= \frac{2}{\pi} \left[\sqrt{x-x^2} + (2x-1) \arccos \sqrt{x} \right] - x + C$

730. $\int \frac{e^{3x}(6x-5)}{(2x-1)^4} dx = \frac{e^{3x}}{2x-1} + C$

$\int \frac{e^{3x}(6x-5)}{(2x-1)^4} dx \dots$ INTEGRATION BY PARTS
 $\begin{array}{|c|c|} \hline & \frac{3}{2} e^{3x} & \frac{3e^{3x}(2x-5)}{(2x-1)^2} \\ \hline & (2x-1)^2 & \frac{2e^{3x}(2x-1)}{(2x-1)^3} \\ \hline \end{array}$

 $= -\frac{3}{2} e^{3x} \left(\frac{6x-5}{(2x-1)^2} \right) + \frac{3}{2} e^{3x} dx$
 $= -\frac{3}{2} e^{3x} \left[3 - \frac{6x-5}{2x-1} \right] + C$
 $= -\frac{3}{2} e^{3x} \left[\frac{2x^2-10x+15}{2x-1} \right] + C = \frac{3}{2} e^{3x} \left(\frac{-x^2+5x-5}{2x-1} \right) + C = \frac{e^{3x}}{2x-1} + C$

ALTERNATIVE STRATEGY WITH PARTIAL FRACTIONS IN THE NUMERATOR

 $\bullet \frac{6x-5}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2}$
 $\bullet 6x-5 = A(2x-1) + B$
 $\bullet 6x-5 \equiv 2Ax + (B-A)$

RETURNING TO THE INTEGRAL

 $\int \frac{e^{3x}(6x-5)}{(2x-1)^4} dx = \int \frac{3e^{3x}}{2x-1} - \frac{2e^{3x}}{(2x-1)^2} dx$
 $= \int \frac{3e^{3x}}{2x-1} dx - \int \frac{2e^{3x}}{(2x-1)^2} dx$

NOW INTEGRATION BY PARTS ON THE FIRST INTEGRAL ONLY (OF THE SECOND INTEGRAL ONLY)

 $\begin{array}{|c|c|} \hline & \frac{3}{2} e^{3x} & \frac{e^{3x}}{2x-1} - \int \frac{-2e^{3x}}{(2x-1)^2} dx \\ \hline & (2x-1)^2 & \frac{2e^{3x}}{(2x-1)^3} \\ \hline \end{array}$
 $= \left[\frac{e^{3x}}{2x-1} - \int \frac{-2e^{3x}}{(2x-1)^2} dx \right] - \int \frac{2e^{3x}}{(2x-1)^2} dx$
 $= \frac{e^{3x}}{2x-1} + \int \frac{2e^{3x}}{(2x-1)^3} dx - \int \frac{2e^{3x}}{(2x-1)^2} dx$
 $= \frac{e^{3x}}{2x-1} + C$

$$731. \quad \int \frac{1-2x}{x^{\frac{2}{3}}(x+1)^4} dx = \frac{3x^{\frac{1}{3}}}{x+1} + C$$

$$732. \int \frac{1}{x^4+1} dx = \begin{cases} \frac{\sqrt{2}}{8} \ln \left[\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right] + \frac{\sqrt{2}}{4} \arctan \left[\frac{\sqrt{2}x}{1-x^2} \right] + C \\ \frac{\sqrt{2}}{8} \ln \left[\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right] + \frac{\sqrt{2}}{4} \arctan \left[\frac{x^2 - 1}{\sqrt{2}x} \right] + C \end{cases}$$

CONTINUATION BY PARTIAL FRACTION

$$\int \frac{1-2x}{2x^2(2x+1)^2} dx = \int \frac{1-2x}{2x^2} \left(\frac{1}{(2x+1)^2} \right) dx$$

$$\begin{aligned} \frac{1-2x}{(2x+1)^2} &= \frac{A}{2x+1} + \frac{B}{(2x+1)^2} \\ 1-2x &= A(2x+1) + B \\ 1-2x &= Ax + A+B \end{aligned}$$

$$A=-2, B=3$$

$$\int \frac{3x^{\frac{1}{2}}}{(2x+1)^2} - \frac{2x^{\frac{1}{2}}}{2x+1} dx = \int \frac{3x^{\frac{1}{2}}}{(2x+1)^2} dx - \int \frac{2x^{\frac{1}{2}}}{2x+1} dx.$$

Now integration by parts on only one of these two integrals, say on the second one

$$\begin{aligned} &= \int \frac{3x^{\frac{1}{2}}}{(2x+1)^2} dx - \left[\frac{3x^{\frac{1}{2}}}{2x+1} - \int \frac{2x^{\frac{1}{2}}}{(2x+1)^2} dx \right] \\ &= \int \frac{3x^{\frac{1}{2}}}{(2x+1)^2} dx - \frac{3x^{\frac{1}{2}}}{2x+1} + \int \frac{2x^{\frac{1}{2}}}{(2x+1)^2} dx \\ &= -\frac{3x^{\frac{1}{2}}}{2x+1} + \int \frac{3x^{\frac{1}{2}} - 6x^{\frac{1}{2}}}{(2x+1)^2} dx \end{aligned}$$

COLLECTING ALL THE RESULTS

$$\begin{aligned} \int \frac{1-2x}{2x^2(2x+1)^2} dx &= -\frac{3x^{\frac{1}{2}}}{2x+1} + \int \frac{3x^{\frac{1}{2}}(1-2x)}{(2x+1)^2} dx \\ \int \frac{1-2x}{2x^2(2x+1)^2} dx &= 3 \int \frac{3x^{\frac{1}{2}}(1-2x)}{(2x+1)^2} dx - \frac{3x^{\frac{1}{2}}}{2x+1} \\ -2 \int \frac{1-2x}{2x^2(2x+1)^2} dx &= -\frac{6x^{\frac{1}{2}}}{2x+1} + C \\ \int \frac{1-2x}{2x^2(2x+1)^2} dx &= -\frac{3x^{\frac{1}{2}}}{2x+1} + C \end{aligned}$$

THE SUBSTITUTION $x^{\frac{1}{2}}=u$, RELENTLESSLY ESTIMATING LENGTHY FRACTIONAL
MANIPULATIONS & MORE SUBSTITUTIONS AND MORE!

$$\begin{aligned}
&= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx = -\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx \\
&= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1 + \frac{1}{x^2}}{(x - \frac{1}{x})^2 + 2} dx = -\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1 - \frac{1}{x^2}}{(x - \frac{1}{x})^2 + 2} dx \\
&\quad \text{Let } u = x - \frac{1}{x}, \quad du = (1 + \frac{1}{x^2})dx \\
&\quad \text{Let } v = x + \frac{1}{x}, \quad dv = (1 - \frac{1}{x^2})dx \\
&= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{du}{(v+2)} = -\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dv}{(v-2)} \\
&= \frac{1}{2} \arctan \left(\frac{u}{\sqrt{v}} \right) = -\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(v-2)(v+2)} dv \\
&= \frac{1}{4} \arctan \left(\frac{x-1}{\sqrt{x^2+1}} \right) - \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{v-2} - \frac{1}{v+2} dv \\
&= \frac{1}{4} \arctan \left(\frac{x-1}{\sqrt{x^2+1}} \right) - \frac{1}{4} \left[\ln|v-2| - \ln|v+2| \right] + C \\
&= \frac{1}{4} \arctan \left(\frac{x-1}{\sqrt{x^2+1}} \right) - \frac{1}{4} \left[\ln \left| \frac{x-1}{x+1} \right| \right] + C \\
&= \frac{1}{4} \arctan \left(\frac{x-1}{\sqrt{x^2+1}} \right) - \frac{1}{4} \left[\ln \left(\frac{x-1}{x+1} \right) \right] + C \\
&= \frac{1}{4} \arctan \left(\frac{x-1}{\sqrt{x^2+1}} \right) - \frac{1}{4} \left[\ln \left(\frac{x^2-1}{x^2+1} \right) \right] + C \\
&= \frac{1}{4} \arctan \left(\frac{x-1}{\sqrt{x^2+1}} \right) + \frac{1}{4} \left[\ln \left(\frac{x^2+1}{x^2-1} \right) \right] + C \\
&\quad \uparrow \text{ DIFFERENCE FROM PREVIOUS WORKING, AS IT DIFFERS BY } \pm \frac{\pi}{4} \text{ RADIANS}
\end{aligned}$$

733. $\int \frac{x^2}{(x \cos x - \sin x)^2} dx = \begin{cases} \frac{x \tan x + 1}{x - \tan x} + C \\ \frac{x \sin x + \cos x}{x \cos x - \sin x} + C \end{cases}$

Solution:

TRANSFORM THE INTEGRAND
 $x \cos x - \sin x = \sec(x)(\cos x - \tan x)$
 $\cos x - \tan x = \frac{\cos x}{\sec x} = \frac{1}{\sec x}$
 $\sec x = 1 \Rightarrow \frac{1}{\sec x} = 0$
 $\tan x = \frac{1}{\sec x} = \frac{1}{\sqrt{1+x^2}}$
 $x = \frac{1}{\sqrt{1+x^2}}$

$$\begin{aligned} \int \frac{x^2}{(\cos x - \tan x)^2} dx &= \int \frac{x^2}{(\sec x)^2 (\cos x - \tan x)^2} dx \\ &= \int \frac{x^2}{(\sec x)^2 (\sec x - \tan x)^2} dx \\ &= \int \sec^2(x) dx = \int \sec^2(x + \arctan \frac{1}{x}) \left(\frac{x}{1+x^2} \right) dx \end{aligned}$$

Now by a substitution

$$\begin{aligned} u &= x + \arctan \frac{1}{x} \\ du &= \left[1 + \frac{1}{1+x^2} \right] dx = \left[1 - \frac{1}{x^2+1} \right] dx = \left[\frac{2x^2+1}{x^2+1} \right] dx \\ du &= \frac{2x^2+1}{x^2+1} dx \end{aligned}$$

$$\begin{aligned} \int \sec^2(u) du &= \tan u + C = \tan(x + \arctan \frac{1}{x}) + C \\ &= \frac{\tan x + \tan(\arctan \frac{1}{x})}{1 - \tan x \tan(\arctan \frac{1}{x})} + C = \frac{\tan x + \frac{1}{x}}{1 - \frac{1}{x} \tan x} + C \\ &= \frac{2 \tan x + 1}{x - \tan x} + C \\ &= \frac{2 \tan x \sec x + \sec x}{\sec x - \tan x} + C = \frac{2 \sec^2 x + \sec x}{\sec x - \tan x} + C \end{aligned}$$

734. $\int \frac{1}{x^6+1} dx = \begin{cases} \frac{1}{2} \arctan \left[\frac{x^2-1}{x} \right] - \frac{1}{3} \arctan x^3 - \frac{\sqrt{3}}{12} \ln \left[\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1} \right] + C \\ \frac{1}{2} \arctan \left[\frac{x^2-1}{x} \right] - \frac{1}{3} \arctan x^3 + \frac{\sqrt{3}}{6} \operatorname{artanh} \left[\frac{x^2+1}{\sqrt{3}} \right] + C \end{cases}$

Solution:

$$\begin{aligned} \int \frac{1}{x^6+1} dx &\approx \int \frac{1}{(x^2+1)^3} dx = \int \frac{1}{(x^2+1)(x^2-x^2+1)} dx \\ &= \int \frac{(x^2+1)-x^2}{(x^2+1)(x^2-x^2+1)} dx < \int \frac{x^2+1}{(x^2+1)(x^2-x^2+1)} dx = \frac{x^2}{x^2-x^2+1} dx \\ &= \int \frac{1}{x^2-x^2+1} dx = \int \frac{x^2}{x^2+1} dx \quad \text{RECOGNISE THE FORM } (\sec^2 x) \\ &= \int \frac{-2x}{x^2-1+x^2} dx = -\frac{2x}{2x^2} = -\frac{1}{x} \tan x^2 \\ &\quad \text{BY THE FORM } \frac{u'}{u^2} du = -\frac{1}{u} du \\ &= \frac{1}{2} \int \frac{-2x}{x^2-1+x^2} dx = -\frac{1}{2} \int \frac{1}{\sec^2 x} dx = -\frac{1}{2} \operatorname{arctan} x^2 \\ &= \frac{1}{2} \int \frac{-\frac{1}{x^2}}{x^2-1+x^2} dx + \frac{1}{2} \int \frac{\frac{1}{x^2}}{x^2-1+x^2} dx = -\frac{1}{2} \operatorname{arctan} x^2 \\ &= \frac{1}{2} \int \frac{1+\frac{1}{x^2}}{x^2-1+x^2} dx + \frac{1+\frac{1}{x^2}}{2x^2-1+x^2} dx = -\frac{1}{2} \operatorname{arctan} x^2 \\ &= \frac{1}{2} \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}} dx - \frac{1}{2} \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}} dx = -\frac{1}{2} \operatorname{arctan} x^2 \\ &= \frac{1}{2} \int \frac{\left(1+\frac{1}{x^2}\right) dx}{(x^2-1+\frac{1}{x^2})} dx - \frac{1}{2} \int \frac{\left(1+\frac{1}{x^2}\right) dx}{(x^2-1+\frac{1}{x^2})-3} = -\frac{1}{2} \operatorname{arctan} x^2 \\ &\quad \uparrow \quad \uparrow \\ &\quad u = x - \frac{1}{x} \quad v = 2x \cdot \frac{1}{x} \\ &\quad du = \left(1 + \frac{1}{x^2}\right) dx \quad dv = (1 - \frac{1}{x^2}) dx \\ &= \frac{1}{2} \int \frac{\frac{1}{x^2+1} dx}{\frac{1}{x^2+1} + \frac{1}{x^2-1}} du - \frac{1}{2} \int \frac{\frac{1}{x^2+1} dx}{\frac{2x^2-1}{x^2+1} + \frac{1}{x^2-1}} du = -\frac{1}{2} \operatorname{arctan} x^2 \\ &= \frac{1}{2} \operatorname{arctan} u - \frac{1}{2} \int \frac{1}{(v^2-1)(v+1)} dv = -\frac{1}{2} \operatorname{arctan} x^2 \end{aligned}$$

PARTIAL FRACTION BY METHOD:

$$\begin{aligned} &= \frac{1}{2} \operatorname{arctan} u - \frac{1}{3} \operatorname{arctan} v^2 - \frac{1}{2} \int \frac{1}{(v^2-1)(v+1)} dv \\ &= \frac{1}{2} \operatorname{arctan}(x - \frac{1}{x}) - \frac{1}{2} \operatorname{arctan} x^2 - \frac{1}{2} \int \frac{\frac{1}{2x}}{v-1} - \frac{1}{2x} dv \\ &= \frac{1}{2} \operatorname{arctan}(\frac{x^2-1}{x}) - \frac{1}{2} \operatorname{arctan} x^2 + \frac{1}{4x} \int \frac{1}{v-1} - \frac{1}{v+1} dv \\ &= \frac{1}{2} \operatorname{arctan}(\frac{x^2-1}{x}) - \frac{1}{2} \operatorname{arctan} x^2 - \frac{1}{12} \left[\ln|v-1| - \ln|v+1| \right] + C \\ &= \frac{1}{2} \operatorname{arctan}(\frac{x^2-1}{x}) - \frac{1}{2} \operatorname{arctan} x^2 - \frac{1}{12} \left[\ln|\frac{v-1}{v+1}| \right] + C \\ &= \frac{1}{2} \operatorname{arctan}(\frac{x^2-1}{x}) - \frac{1}{2} \operatorname{arctan} x^2 - \frac{1}{12} \left[\frac{2x^2-1}{2x^2+1} \right] + C \\ &= \frac{1}{2} \operatorname{arctan}(\frac{x^2-1}{x}) - \frac{1}{2} \operatorname{arctan} x^2 - \frac{1}{12} \left[\frac{2x^2-1}{2x^2+1} \right] + \frac{1}{6} \operatorname{artanh}(\frac{2x^2+1}{\sqrt{3}}) + C \end{aligned}$$

Note that

$$\begin{aligned} \int \frac{1}{\sqrt{1-v^2}} dv &= \int \frac{1}{\sqrt{1-(\frac{v-1}{v+1})^2}} dv \\ &= -\int \frac{1}{(\frac{v-1}{v+1})^2-1} dv = -\frac{1}{v^2-1} \operatorname{arctanh}(\frac{v-1}{v+1}) \\ \int \frac{1}{\sqrt{1-x^2}} dx &= \frac{1}{x} \operatorname{arcsinh} \frac{1}{x} \\ u &= \frac{v-1}{v+1} \quad v = 2x \cdot \frac{1}{x} \\ du = \left(1 + \frac{1}{x^2}\right) dx \quad dv = (1 - \frac{1}{x^2}) dx \\ &= -\frac{1}{2} \int \frac{1}{(v^2-1)(v+1)} dv + \frac{1}{6} \operatorname{artanh}(\frac{2x^2+1}{\sqrt{3}}) \end{aligned}$$

735. $\int \frac{x^2+1}{x^4-x^2+1} dx = \arctan\left(\frac{x^2-1}{x}\right) + C$

$$\begin{aligned} \int \frac{x^2+1}{x^4-x^2+1} dx &= \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}} dx \\ &= \int \frac{1+\frac{1}{x^2}}{(x-\frac{1}{x})^2+1} dx = \text{BY ANOTHER SUBSTITUTION OR, INTEGRATION} \\ &= \arctan(x-\frac{1}{x}) + C = \underline{\arctan\left(\frac{x^2-1}{x}\right) + C} \end{aligned}$$

736. $\int \frac{x^2 \arctan x}{1+x^2} dx = x \arctan x - \frac{1}{2}(\arctan x)^2 - \frac{1}{2} \ln(x^2+1) + C$

$$\begin{aligned} \int \frac{x^2 \arctan x}{1+x^2} dx &= \dots \text{SPLIT TERM + SUBSTITUTION} \\ &= \int \frac{\arctan x}{1+x^2} (sec^2 dx) = \int \theta \tan \theta d\theta \\ &= \int \theta (\sec^2 - 1) d\theta = \int \sec^2 \theta - \theta d\theta \\ &= \int -\theta d\theta + \int \sec^2 d\theta \leftarrow \text{IN Pairs} \\ &= -\frac{1}{2}\theta^2 + \theta \tan \theta - \int \tan \theta d\theta \\ &= \theta \tan \theta - \frac{1}{2}\theta^2 - \ln|\sec \theta| + C \\ &= \arctan x - \frac{1}{2}(\arctan x)^2 - \ln|\sqrt{x^2+1}| + C \\ &= \underline{\arctan x - \frac{1}{2}(\arctan x)^2 - \frac{1}{2}\ln(2x)} + C \end{aligned}$$

737. $\int \frac{x}{1+x^4} dx = \frac{1}{2} \arctan x^2 + C$

$$\begin{aligned} \int \frac{x}{1+x^2} dx &= \text{BY RECOGNITION! OR LET } u=x^2 \\ \frac{du}{dx} (\arctan x^2) &= \frac{2x}{1+x^2} \\ \therefore &= \underline{\frac{1}{2} \arctan x^2 + C} \end{aligned}$$

738. $\int \frac{1}{1-\cot x} dx = \frac{1}{2}x + \frac{1}{2} \ln|\sin x - \cos x| + C$

$$\begin{aligned} \int \frac{1}{1-\cot x} dx &= \int \frac{\cot x}{1-\cot x} dx = \frac{1}{2} \int \frac{2\cot x}{\sin x - \cos x} dx \\ &= \frac{1}{2} \int \frac{(\sin x - \cos x) + (\sin x + \cos x)}{\sin x - \cos x} dx = \frac{1}{2} \int 1 + \frac{\sin x + \cos x}{\sin x - \cos x} dx \\ &= \frac{1}{2}x + \frac{1}{2} \int \frac{\sin x + \cos x}{\sin x - \cos x} dx = \frac{1}{2}x + \underline{4\ln|\sin x - \cos x|} + C \end{aligned}$$

739. $\int \frac{\sin(x^{-2})}{x^5} dx = \frac{1}{2} \left[\frac{1}{x^2} \cos\left(\frac{1}{x^2}\right) - \sin\left(\frac{1}{x^2}\right) \right] + C$

$$\begin{aligned} \int \frac{\sin(x^{-2})}{x^5} dx &= \dots \text{SUBSTITUTION} \dots \\ &= \int \frac{\sin(u)}{u^5} du = \int -\frac{\cos(u)}{u^4} du \\ &= \int -\frac{1}{u^4} \sin(u) du = \dots \text{INTEGRATION BY PARTS} \\ &= \frac{1}{u^3} \sin(u) - \int \frac{3}{u^2} \cos(u) du = \frac{1}{u^3} \sin(u) - \frac{3}{u^2} \sin(u) \\ &= \frac{1}{u^3} \left[\frac{1}{2} \cos\left(\frac{1}{u}\right) - \sin\left(\frac{1}{u}\right) \right] + C \end{aligned}$$

$u = \frac{1}{x^2}$
 $du = -\frac{2}{x^3} dx$
 $dx = -\frac{x^3}{2} du$
 $\frac{1}{u^3} = \frac{1}{x^6}$
 $\frac{3}{u^2} = \frac{3}{x^4}$

740. $\int 3^{\ln x} dx = \frac{x(3^{\ln x})}{1+\ln 3} + C$

$$\begin{aligned} \int 3^{\ln x} dx &= \dots \text{SUBSTITUTION} \dots \\ &= \int 3^{u^2} du = \int (3e)^u du \quad u = \ln x, \quad \frac{du}{dx} = e^u \\ &\quad \text{using } \frac{d}{du}(e^u) = e^u \ln a \\ &\quad \int e^u du = \frac{1}{\ln a} e^u + C \\ &= \frac{1}{\ln(3e)} (3e)^u + C = \frac{3^u e^u}{\ln(3e)} + C = \frac{3^{\ln x} x}{\ln(3e)+1} + C \\ &= \frac{3^{\ln x}}{1+\ln 3} + C \end{aligned}$$

741. $\int \tan^5 x dx = \frac{1}{4} \sec^4 x - \sec^2 x + \ln|\sec x| + C$

$$\begin{aligned} \int \tan^5 x dx &= \int \tan x \tan^4 x dx = \int \tan x (\sec^2 x - 1)^2 dx \\ &= \int \tan x (\sec^4 x - 2\sec^2 x + 1) dx = \int (\tan x \sec^4 x - 2\tan x \sec^2 x + \tan x) dx \\ &= \int \tan x \sec^4 x dx - 2 \int \tan x \sec^2 x dx + \int \tan x dx \quad \dots \text{BY REARRANGING SINCE} \\ &\quad \frac{d}{dx}(\sec x) = \sec x \tan x \\ &= \frac{1}{4} \sec^4 x - \sec^2 x + \ln|\sec x| + C \end{aligned}$$

742. $\int \tan^3 x \sec^3 x dx = \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$

$$\begin{aligned} \int \tan^3 x \sec^3 x dx &= \int \tan x \tan^2 x \sec^3 x dx = \int \tan x (\sec^2 x - 1) \sec^3 x dx \\ &= (\sec x) \sec^2 x - \sec x \dots \text{now } \frac{d}{dx}(\sec x) = \sec x \tan x \\ &= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C \end{aligned}$$

743. $\int \sec x \operatorname{cosec}^3 x \, dx = \ln|\tan x| - \frac{1}{2} \operatorname{cosec}^2 x + C$

$$\begin{aligned}\int \sec x \operatorname{cosec}^3 x \, dx &= \int \frac{1}{\operatorname{cosec} x \operatorname{cosec}^2 x} \, dx = \int \frac{\operatorname{cosec} x \operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} \, dx \\ &= \int \frac{\operatorname{cosec} x + \operatorname{cosec}^2 x}{\operatorname{cosec} x} \, dx = \int (\operatorname{cosec} x)^{-1} + \operatorname{cosec} x \, dx \\ &= -\frac{1}{2}(\operatorname{cosec} x)^2 + \ln|\operatorname{cosec} x| + C = -\frac{1}{2} \operatorname{cosec}^2 x + \ln|\operatorname{cosec} x| + C \\ &= \ln|\operatorname{tang} x| - \frac{1}{2} \operatorname{cosec}^2 x + C\end{aligned}$$

NOTE THAT $\int \frac{dx}{\operatorname{cosec} x} = \int \frac{z}{\operatorname{cosec} z} dz = \int 2 \operatorname{cosec} z \, dz$

$$\begin{aligned}&= -\ln|\operatorname{cosec} z + \operatorname{cotan} z| + C = -\ln\left|\frac{\operatorname{cosec} z + \operatorname{cotan} z}{\operatorname{cosec} z}\right| + C \\ &= -\ln\left|\frac{2\operatorname{cosec} z + 1}{2\operatorname{cosec}^2 z}\right| + C = -\ln\left|\frac{2\operatorname{cosec} z}{2\operatorname{cosec}^2 z}\right| + C \\ &= -\ln\left|\frac{\operatorname{cosec} z}{\operatorname{cosec}^2 z}\right| + C = -\ln|\operatorname{cosec} z| + C \\ &= \ln|\operatorname{tang} z| + C\end{aligned}$$

REASON

744. $\int \frac{\cos x}{\cos 2x} \, dx = \frac{1}{4} \sqrt{2} \ln \left| \frac{1+\sqrt{2} \sin x}{1-\sqrt{2} \sin x} \right| + C$

$$\begin{aligned}\int \frac{\cos x}{\cos 2x} \, dx &= \int \frac{\cos x}{1-2\sin^2 x} \, dx = \frac{1}{2} \int \frac{\cos x}{\frac{1}{2}-\sin^2 x} \, dx \\ &\quad \text{PARTIAL FRACTION ALIVE SUBSTITUTION } u=\sin x \text{ OR} \\ &\quad \frac{du}{dx}(\operatorname{arbtanh}(u)) = \frac{1}{1-u^2}, \quad \frac{du}{dx}(\operatorname{arbtanh}(u)) = \frac{u}{1-u^2} \\ &= \frac{1}{2} \int \frac{1}{\frac{1}{2}-u^2} \, du = \frac{1}{2} \int \frac{1}{(\frac{1}{2})^2-u^2} \, du = \frac{1}{2} \times \frac{1}{\frac{1}{2}} \operatorname{arbtanh}\left(\frac{u}{\sqrt{\frac{1}{2}}}\right) + C \\ &= \frac{2}{\sqrt{2}} \operatorname{arbtanh}\left(\frac{u}{\sqrt{\frac{1}{2}}}\right) + C = \frac{2}{\sqrt{2}} \operatorname{arbtanh}\left(\frac{2\sin x}{\sqrt{2}}\right) + C \\ &= \frac{2}{\sqrt{2}} \times \ln \left| \frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x} \right| + C = \frac{1}{\sqrt{2}} \ln \left| \frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x} \right| + C\end{aligned}$$

745. $\int \sec^6 x \, dx = \tan x + \frac{2}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C$

$$\begin{aligned}\int \sec^6 x \, dx &= \dots \quad \text{BY MANIPULATION OR SUBSTITUTION } u=\operatorname{tang} x \\ &= \int \sec^2 x \sec^4 x \, dx = \int \sec^2 x (1+\tan^2 x)^2 \, dx \\ &= \int \sec^2 x (1+2\tan^2 x+\tan^4 x) \, dx \\ &= \underline{\tan x} + \underline{2\tan^3 x} + \underline{\tan^5 x} \, dx\end{aligned}$$

746. $\int \frac{\ln x}{\sqrt[3]{x}} dx = \frac{3}{4} x^{\frac{3}{2}} (2 \ln x - 3) + C$

$$\begin{aligned} \int \frac{\ln x}{\sqrt[3]{x}} dx &= \dots \text{ BY SUBSTITUTION} \dots \\ &= \int \frac{du}{\sqrt[3]{e^{3u}}} (2e^{3u} du) = \int \frac{9u e^{3u}}{e^{6u}} du \\ &= \int 9u e^{2u} du \quad \dots \text{ INTEGRATION BY PARTS} \dots \\ &= \frac{9}{2} u e^{2u} - \int \frac{9}{2} e^{2u} du = \frac{9}{2} u e^{2u} - \frac{9}{4} e^{2u} + C \\ &= \frac{9}{4} e^{2u} (2u - 1) + C = \frac{9}{4} e^{\frac{3}{2}} (\frac{3}{2} \ln x - 1) + C \\ &= \frac{3}{4} x^{\frac{3}{2}} (3 \ln x - 3) + C \end{aligned}$$

747. $\int 4x\sqrt{x^4+1} dx = \ln[x^2 + \sqrt{x^4+1}] + x^2\sqrt{x^4+1} + C$

$$\begin{aligned} \int 4x\sqrt{x^4+1} dx &= \dots \text{ BY TRIGONOMETRIC SUBSTITUTION} \dots \\ &= \int \sqrt{\tan^2 \theta + (\sec^2 \theta)} d\theta = \int \sec \theta (2\sec \theta) d\theta \\ &= 2 \int \sec^2 \theta d\theta \\ &\text{NEXT WE REQUIRE THE INTEGRAL OF } \sec^2 \theta \\ &I = \int \sec^2 \theta d\theta = \int \sec \theta \sec \theta d\theta \\ &I = \sec \theta (\tan \theta) + \int \sec^2 \theta d\theta \\ &I = \int \sec^2 \theta d\theta + \int \sec^2 \theta d\theta \\ &I = \ln|\sec \theta + \tan \theta| + \int (\sec^2 \theta) \tan \theta d\theta \quad \leftarrow \text{BY PARTS} \\ &I = \ln|\sec \theta + \tan \theta| + \sec \theta \tan \theta - \int \sec^2 \theta d\theta \\ &I = \ln|\sec \theta + \tan \theta| + \sec \theta \tan \theta - I \\ &2I = \ln|\sec \theta + \tan \theta| + \sec \theta \tan \theta + C \\ &2 \int \sec^2 \theta d\theta = \ln|\sec \theta + \tan \theta| + \sec \theta \tan \theta + C \\ &= \ln[x^2 + \sqrt{x^4+1}] + x^2\sqrt{x^4+1} + C \\ &\therefore \int 4x\sqrt{x^4+1} dx = \ln[x^2 + \sqrt{x^4+1}] + x^2\sqrt{x^4+1} + C \end{aligned}$$

748. $\int \frac{1}{x[1+3\sin^2(\ln x)]} dx = \frac{1}{2} \arctan[2\tan(\ln x)] + C$

$$\begin{aligned}
 & \int \frac{1}{x(1+3\sin^2(\ln x))} dx \dots \text{COSINE SUBSTITUTION} \\
 &= \int \frac{1}{x(1+3\sec^2(u))} (du) = \int \frac{1}{x(3+\sec^2(u))} du \\
 &= \int \frac{\sec u}{1+3\sec^2(u)} du = \int \frac{\sec u}{3+\sec^2(u)} du = \int \frac{\sec u}{1+2\sec^2(u)+3\sec^2(u)} du \\
 &= \int \frac{\sec u}{1+2\tan^2(u)} du = \frac{1}{2} \int \frac{2\sec u}{\sec^2(u)+\frac{1}{2}} du = \frac{1}{2} \int \frac{2\sec u}{1+\tan^2(u)+\frac{1}{2}} du \\
 &\quad \text{(TANINE SUBSTITUTION) } u = \tan u \Rightarrow du = \sec^2(u) du \Rightarrow \sec u du = \frac{1}{2} \tan(u) du + C
 \end{aligned}$$

AS REF

$$\begin{aligned}
 &= \frac{1}{2} \times \frac{1}{2} \arctan\left(\frac{\tan u}{\frac{1}{2}}\right) + C = \frac{1}{2} \arctan(2\tan(u)) + C = \frac{1}{2} \arctan(2\tan(\ln x)) + C \\
 &\quad \text{ALTERNATIVE (NO-WEAVING) OF LINE 14: INITIAL SUBSTITUTION} \\
 &= \int \frac{1}{1+3\sec^2(u)} du = \int \frac{1}{1+3(1-\frac{1}{2}\cos^2(u))} du = \int \frac{1}{\frac{5}{2}-\frac{3}{2}\cos^2(u)} du \\
 &= \int \frac{2}{5-3\cos^2(u)} du = \dots \text{"URGE TO IDENTITIES"} \\
 &= \int \frac{2}{5-3\left(\frac{1-\frac{1}{2}\cos^2(t)}{1+\frac{1}{2}\cos^2(t)}\right)} du = \int \frac{2}{\frac{7}{2}-\frac{3}{2}\cos^2(t)} du \\
 &= \int \frac{2}{\frac{5}{2}+3+3\cos^2(t)} dt = \int \frac{2}{\frac{5}{2}+2} dt \\
 &= \int \frac{2}{\frac{9}{2}+3} dt = \frac{1}{\frac{9}{2}+3} dt \\
 &= \frac{2}{\frac{15}{2}} \int \frac{1}{\frac{9}{2}+3} dt = \frac{2}{\frac{15}{2}} \arctan\left(\frac{t}{\sqrt{\frac{9}{2}}}\right) + C \\
 &= -\frac{1}{\frac{15}{2}} \arctan\left(\frac{t}{\sqrt{\frac{9}{2}}}\right) + C = -\frac{1}{\frac{15}{2}} \arctan\left(\frac{\ln x}{\sqrt{\frac{9}{2}}}\right) + C \\
 &= \frac{1}{\frac{15}{2}} \arctan\left(2\tan(\ln x)\right) + C
 \end{aligned}$$

AS REF

749. $\int \ln[2x+\sqrt{4x^2-1}] dx = \begin{cases} x \operatorname{arcosh} 2x - \frac{1}{2} \sqrt{4x^2-1} + C \\ x \ln[2x+\sqrt{4x^2-1}] - \frac{1}{2} \sqrt{4x^2-1} + C \end{cases}$

$$\begin{aligned}
 & \int \ln(\sec x + \sqrt{\sec^2 x - 1}) dx = \dots \text{THIS IS THE EXPONENTIAL FORM OF THE LEFT} \\
 &= \int \sec x \ln(\sec x + \sqrt{\sec^2 x - 1}) dx = \dots \text{INTEGRATION BY PARTS} \\
 &= \sec x \ln(\sec x + \sqrt{\sec^2 x - 1}) - \int \sec x (\ln(\sec x + \sqrt{\sec^2 x - 1}))' dx \\
 &= \sec x \ln(\sec x + \sqrt{\sec^2 x - 1}) - \int \sec x \frac{1}{\sec x + \sqrt{\sec^2 x - 1}} \cdot (\sec x \tan x + \frac{1}{2} \cdot \frac{2\sec x \tan x}{\sec x + \sqrt{\sec^2 x - 1}}) dx \\
 &= \sec x \ln(\sec x + \sqrt{\sec^2 x - 1}) - \int \frac{\sec x \tan x}{\sec x + \sqrt{\sec^2 x - 1}} dx \\
 &= \sec x \ln(\sec x + \sqrt{\sec^2 x - 1}) - \int \frac{\sec x \tan x}{\sec x + \sqrt{\sec^2 x - 1}} dx
 \end{aligned}$$

750. $\int \ln(x^2 - 1) dx = (x-1)\ln|x^2 - 1| + 2[x - \ln|x+1|] + C$

$\int \ln(\sec^2 x) dx \dots$ BY SUBSTITUTION...

$$\begin{aligned} &= \int \ln(\sec^2 x) (\sec^2 x \tan x) dx = \int \ln(\sec^2 x) \sec^2 x \tan x dx \\ &= \int 2\ln(\sec x) \sec^2 x \tan x dx \dots \text{INTEGRATION BY PARTS} \\ &= 2\sec x \ln(\sec x) - \int 2\sec^2 x \ln(\sec x) dx \\ &= 2\sec x \ln(\sec x) - \int \frac{2\sec^2 x}{\sec x} \ln(\sec x) dx \\ &= 2\sec x \ln(\sec x) - \int \frac{2\sec x}{\sin x} \ln(\sec x) dx \\ &= 2\sec x \ln(\sec x) - \int \frac{2\sec x}{\sin x} dx + \frac{2\sec x}{\sin x} dx \\ &= 2\sec x \ln(\sec x) - \int \frac{2\sec x}{\sin x} dx + 2\sec x dx \\ &= 2\sec x \ln(\sec x) - 2\sec x + 2\ln|\csc x + \cot x| + C \\ &= 2\ln|\sec x| - 2x + 2\ln\left|\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right| + C \\ &= 2\ln|\sec x| - 2x + \ln\left|\frac{1+\cos x}{\sin x}\right| + C \\ &= 2\ln|\sec x| - 2x + 2\ln|\sec x| - \ln|\sec x| + C \\ &= (2-1)\ln|\sec x| + 2(x-\ln|\sec x|) + C \end{aligned}$$

INTEGRATION BY PARTS IS A LOT MORE EFFICIENT HERE - AS SHOWN
USING THE EASIER EQUATION $\ln(\sec^2 x)$

751. $\int \frac{e^x \sqrt{e^x - 1}}{e^x - 2} dx = 2\sqrt{e^x - 1} + \ln\left|\frac{\sqrt{e^x - 1} - 1}{\sqrt{e^x - 1} + 1}\right| + C$

$\int \frac{e^x \sqrt{e^x - 1}}{e^x - 2} dx = \dots$ BY SUBSTITUTION...

$$\begin{aligned} &= \int \frac{e^x u}{e^x - 2} du = \int \frac{2u^2}{u^2 - 1} du \\ &= \int \frac{2(u^2 - 1) + 2}{u^2 - 1} du = \int 2 + \frac{2}{(u^2 - 1)} du \\ &= \int 2 + \frac{2}{u^2 - 1} - \frac{2}{u^2 - 1} du \\ &= 2u + \ln|u^2 - 1| - \ln|u^2 - 1| + C = 2u + \ln\left|\frac{u^2 - 1}{u^2 + 1}\right| + C \\ &= 2\sqrt{e^x - 1} + \ln\left|\frac{\sqrt{e^x - 1} - 1}{\sqrt{e^x - 1} + 1}\right| + C \end{aligned}$$

752. $\int \frac{1}{x^2 \sqrt{x^2 + 1}} dx = -\frac{\sqrt{x^2 + 1}}{x} + C$

$\int \frac{1}{x^2 \sqrt{x^2 + 1}} dx = \dots$ BY TRIGONOMETRIC SUBSTITUTION

$$\begin{aligned} &= \int \frac{1}{\tan^2 \theta + \sec^2 \theta} (\sec^2 \theta) d\theta = \int \frac{\sec^2 \theta}{\tan^2 \theta + \sec^2 \theta} d\theta \\ &= \int \frac{\sec^2 \theta}{\tan^2 \theta + \sec^2 \theta} d\theta = \int \frac{1}{\cos^2 \theta} \frac{\cos^2 \theta}{\sin^2 \theta + \cos^2 \theta} d\theta = \int \frac{1}{\sin^2 \theta} d\theta \\ &= \int \frac{\cos(\sin \theta)^2}{\sin \theta} d\theta = -\cos(\sin \theta)^{-1} + C = -\frac{1}{\sin \theta} + C \\ &= -\frac{\sqrt{x^2 + 1}}{x} + C \end{aligned}$$

753. $\int \frac{\sqrt{x^2+4}}{x^2} dx = \ln\left[x + \sqrt{x^2+4}\right] - \frac{\sqrt{x^2+4}}{x} + C$

$\int \frac{\sqrt{x^2+4}}{x^2} dx = \dots$ BY HYPERBOLIC SUBSTITUTION
 $= \int \frac{\sqrt{z^2+1}}{z^2} dz$ (where $z = 2\sinh\theta$)
 $= \int \frac{(2\cosh\theta)(2\sinh\theta)}{4\sinh^2\theta} d\theta = \int \frac{2\cosh\theta}{1-\sinh^2\theta} d\theta = \int \frac{2\cosh\theta}{\cosh^2\theta} d\theta = \int \frac{2}{\cosh\theta} d\theta = \int 2\cosh\theta d\theta = 2\sinh\theta + C$
 $= 2\sinh^{-1}\frac{x}{2} + C = 2\sinh^{-1}\frac{x}{2} + \frac{2\sqrt{x^2+4}}{x} + C$
 $= \ln\left[\frac{x}{2} + \sqrt{\frac{x^2+4}{4}}\right] + C = \ln\left[\frac{x}{2} + \sqrt{\frac{x^2+4}{4}}\right] + \frac{\sqrt{x^2+4}}{x} + C$
 $= \boxed{\ln\left[\frac{x}{2} + \sqrt{\frac{x^2+4}{4}}\right] + \frac{\sqrt{x^2+4}}{x} + C = \ln\left[\frac{x}{2} + \sqrt{\frac{x^2+4}{4}}\right] + \frac{\sqrt{x^2+4}}{x} + C}$

754. $\int \frac{1+\cos x}{1-\cos x} dx = \begin{cases} -x - 2\cot\left(\frac{1}{2}x\right) + C \\ -x - 2\cot x - \operatorname{cosec} x + C \end{cases}$

$\int \frac{1+\cos x}{1-\cos x} dx = \int \frac{1+(2\cos^2\frac{x}{2}-1)}{1-(1-2\cos^2\frac{x}{2})} dx = \int \frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}} dx = \int \operatorname{cot}^2\frac{x}{2} dx$
 $= \int \operatorname{cosec}^2\frac{x}{2}-1 dx = -2\operatorname{cot}\frac{x}{2} - x + C$
ANOTHER:
 $\int \frac{1+\cos x}{1-\cos x} dx = \int \frac{(1+\cos x)(1+\cos x)}{(1-\cos x)(1-\cos x)} dx = \int \frac{1+2\cos x+\cos^2 x}{1-\cos x} dx$
 $= \int \frac{1}{\sin^2 x} + \frac{2\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} dx = \int \operatorname{cosec}^2 x + 2\operatorname{cosec}(x)\operatorname{cot} x + \operatorname{cot}^2 x dx$
 $= \int \operatorname{cosec}^2 x + 2\operatorname{cosec}(x)\operatorname{cot} x + \operatorname{cot}^2 x - 1 + 2\operatorname{cosec}^2 x dx$
 $= -2\operatorname{cot} x - x - 2\operatorname{cosec}^2 x + C = -2\operatorname{cot} x - x - \frac{2}{\sin^2 x} + C$
 $= \boxed{-2\operatorname{cot} x - x - 2\operatorname{cosec} x + C}$

755. $\int \tan x \ln(\sec x) dx = \frac{1}{2} [\ln(\sec x)]^2 + C$

$\int \tan x \ln(\sec x) dx = \dots$ BY DECOMPOSITION ... = $\frac{1}{2} (\ln(\sec x))^2 + C$

OR SUBSTITUTION

| |
|--|
| $u = \ln(\sec x)$ |
| $\frac{du}{dx} = \frac{1}{\sec x} (\sec x \tan x)$ |
| $du = \tan x dx$ |
| $dx = \frac{du}{\tan x}$ |

$\int \tan x \ln(\sec x) dx$
 $= \int \tan x \cdot u \frac{du}{\tan x} = \int u du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}[\ln(\sec x)]^2 + C}$

756. $\int \frac{\arctan x}{x^2} dx = -\frac{\arctan x}{x} + \ln|x| - \frac{1}{2} \ln(x^2 + 1) + C$

$$\begin{aligned}\int \frac{\arctan x}{x^2} dx &= \dots \text{INTEGRATION BY PARTS OR MANUFACTURE} \\ \int \left(\frac{1}{x^2} \arctan x \right) dx &= -\frac{1}{2x} \arctan x + \frac{1}{2} \int \frac{1}{x^3} dx \\ \int \arctan x dx &= \int -\frac{1}{2x} \arctan x dx + \int \frac{1}{2x^3} dx + \int \frac{-2}{x^2} dx \\ \int \frac{\arctan x}{x^2} dx &= \underline{-\frac{1}{2} \arctan x + \ln|x| - \frac{1}{2} \ln(x^2 + 1) + C} \end{aligned}$$

757. $\int \frac{\sin x \cos x + x \ln x}{x \cos^2 x} dx = (\ln x)(\tan x) + C$

$$\begin{aligned}\int \frac{\sin x \cos x + x \ln x}{x \cos^2 x} dx &= \int \frac{\sin x \cos x}{x \cos^2 x} dx + \int \frac{x \ln x}{x \cos^2 x} dx \\ &= \int \frac{1}{2} \tan x dx + (\ln x) \sec x dx \\ &= \int \tan x dx + (\ln x) \sec x dx \\ \text{INTEGRATE BY PARTS ONE OF THE TWO INTEGRALS - USE THE OTHER} \\ \text{Int. } \frac{1}{2} \tan x &= \int \frac{1}{2} \tan x dx + (\ln x) \sec x - \int \sec x dx \\ \text{Int. } \ln x &= \underline{\ln x \tan x + C} \end{aligned}$$

758. $\int \frac{x^2(x^4+1)}{\sqrt[4]{x^4+2}} dx = \frac{1}{6}(x^8+2x^4)^{\frac{3}{4}} + C$

$$\begin{aligned}\int \frac{x^{2+4/4}}{\sqrt[4]{x^4+2}} dx &\dots \text{MANUFACTURE AS FOLLOWS} \dots \int 2(G(x))(G(x))^{-1/4} dx \\ &= \int (G(x))^4 d(G(x))^{1/4} dx = \int (G^2(x))^2(G(x))^{-1/4} d(G(x)) dx \\ &= \int (G^2(x))[(G(x))^{-1/4}]^2 d(G(x)) = \int (G^2(x))(G(x))^{-1/2} d(G(x)) \\ \text{NOTE THAT } \frac{d}{dx}(x^2+2x) &= 2x^2+2x^2 = 2(G^2+2G) \\ &= \frac{1}{2} \int 2(G^2+2G) G^2(G(x))^{-1/2} d(G(x)) \\ \text{BY SUBSTITUTION (CHOOSE CHAIN RULE) OR THE SUBSTITUTION } u &= \sqrt[4]{x^4+2} \\ &= \frac{1}{2} \times \frac{2}{3} (G^2+2G)^{3/2} + C = \underline{\frac{1}{3} (x^8+2x^4)^{3/4} + C} \end{aligned}$$

759. $\int \sqrt{1+\tan x} dx =$

$$\begin{aligned} & \frac{1}{\sqrt{2\sqrt{2}-2}} \arctan \left[\frac{1-\sqrt{2}+\tan x}{\sqrt{(2\sqrt{2}-2)(1+\tan x)}} \right] - \frac{1}{\sqrt{2\sqrt{2}-2}} \operatorname{artanh} \left[\frac{1+\sqrt{2}+\tan x}{\sqrt{(2\sqrt{2}+2)(1+\tan x)}} \right] + C \\ & \frac{1}{\sqrt{2\sqrt{2}-2}} \arctan \left[\frac{1-\sqrt{2}+\tan x}{\sqrt{(2\sqrt{2}-2)(1+\tan x)}} \right] - \frac{1}{\sqrt{2\sqrt{2}-2}} \ln \left[\frac{1+\sqrt{2}+\tan x - \sqrt{(2\sqrt{2}+2)(1+\tan x)}}{1+\sqrt{2}+\tan x + \sqrt{(2\sqrt{2}+2)(1+\tan x)}} \right] + C \end{aligned}$$

$\int \sqrt{1+\tan x} dx \dots \text{By substitution}$

$$\begin{aligned} &= \int \left(\frac{dt}{t^2-2t+2} \right) dt = \int \frac{dt}{t^2-2t+2} \\ &= \int \frac{2}{t^2-2+\frac{2}{t^2}} dt \\ &= \int \left(\frac{1}{t^2-2} + \frac{1}{t^2+2} \right) dt + \int \frac{1}{t^2+2} dt \\ &\quad dt = \frac{2}{t(t-2)} dt \\ &\quad dt = \frac{2}{t-2} dt \end{aligned}$$

PREPARE THESE INTEGRALS FOR THE "CROSS-TIE" METHOD

$$\begin{aligned} (t-\frac{2}{t})^2 &= t^2-2t+\frac{4}{t^2} & (t+\frac{2}{t})^2 &= t^2+2t+\frac{4}{t^2} \\ (t-\frac{2}{t})^2-2t &= t^2-4t+\frac{4}{t^2} & (t+\frac{2}{t})^2-2t &= t^2+4t+\frac{4}{t^2} \\ t^2+\frac{4}{t^2}-2t &= (t-\frac{2}{t})^2-2t=2 & t^2+\frac{4}{t^2}+2t &= (t+\frac{2}{t})^2-2t=2t^2+2 \\ t^2+\frac{4}{t^2}-2 &= (t-\frac{2}{t})^2+a^2 & t^2+\frac{4}{t^2}+2 &= (t+\frac{2}{t})^2+b^2 \end{aligned}$$

where $a^2=2t^2-2$
where $b^2=2t^2+2$

RETURNING TO THE TWO INTEGRALS NOTING THAT THESE INTEGRALISTS ARE FAMILIAR

$$\begin{aligned} &= \int \frac{1+\frac{2}{t^2}}{t^2-\frac{4}{t^2}-2} dt + \int \frac{1-\frac{2}{t^2}}{t^2-\frac{4}{t^2}-2} dt \\ &= \int \frac{1+\frac{2}{t^2}}{(t-\frac{2}{t})^2-a^2} dt + \int \frac{1-\frac{2}{t^2}}{(t-\frac{2}{t})^2-a^2} dt \\ &= \int \frac{du}{u^2-a^2} + \int \frac{dv}{v^2-b^2} \end{aligned}$$

NOW THE CROSS-TIE TAKES STANDARD SUBSTITUTIONS

$$\begin{aligned} u &= t-\frac{2}{t} & v &= t+\frac{2}{t} \\ du/dt &= \left(1+\frac{2}{t^2}\right)dt & dv/dt &= \left(-\frac{2}{t^2}\right)dt \\ &= \int \frac{du}{u^2-a^2} + \int \frac{dv}{v^2-b^2} \end{aligned}$$

NOW PROCEED ON THE SECOND INTEGRAL BY PARTIAL FRACTIONS OR
INTERFERENCE TO ARCTANH/ARCOTH

$$\begin{aligned} &= \int \frac{du}{u^2-a^2} - \int \frac{dv}{v^2-b^2} \\ &\quad \boxed{\int \frac{1}{u^2-a^2} du = \frac{1}{a} \operatorname{arctanh}(\frac{u}{a}) + C} \\ &= \frac{1}{b} \operatorname{arctanh}(\frac{u}{b}) - \frac{1}{b} \operatorname{arctanh}(\frac{v}{b}) + C \\ &= \frac{1}{b} \operatorname{arctanh}(\frac{t-\frac{2}{t}}{b}) - \frac{1}{b} \operatorname{arctanh}(\frac{t+\frac{2}{t}}{b}) + C \\ &= \frac{1}{\sqrt{2t^2-2}} \operatorname{arctanh}(\frac{t-\frac{2}{t}+b}{\sqrt{2t^2-2}}) - \frac{1}{\sqrt{2t^2+2}} \operatorname{arctanh}(\frac{t+\frac{2}{t}+b}{\sqrt{2t^2+2}}) + C \\ &= \frac{1}{\sqrt{2t^2-2}} \operatorname{arctanh}(\frac{t-\frac{2}{t}+b}{\sqrt{2t^2-2}}) - \frac{1}{\sqrt{2t^2+2}} \operatorname{arctanh}(\frac{t+\frac{2}{t}+b}{\sqrt{2t^2+2}}) + C \end{aligned}$$

ANALOGUE BY PARTIAL FRACTIONS FOR THE "SECOND" INTEGRAL

$$\begin{aligned} &\int \frac{1}{v^2-b^2} dv = \int \frac{1}{(t+\frac{2}{t})(t-\frac{2}{t})} dv = \int \frac{dt}{t^2-2} - \frac{2}{\sqrt{b^2+t^2}} dt \\ &= \frac{1}{2b} \int \frac{1}{t-2} - \frac{1}{t+2} dt = \frac{1}{2b} \ln \left| \frac{t-2}{t+2} \right| + C = \frac{1}{2b} \ln \left| \frac{t+\frac{2}{t}-2}{t+\frac{2}{t}+2} \right| + C \\ &= \frac{1}{2b^2} \ln \left| \frac{t^2-4t+\frac{4}{t^2}}{t^2+4t+\frac{4}{t^2}} \right| = \frac{1}{2b^2} \ln \left| \frac{1+4t+\frac{4}{t^2}-4t^2-4t+\frac{4}{t^2}}{1+4t+\frac{4}{t^2}+4t^2+4t+\frac{4}{t^2}} \right| + C \\ &= \frac{1}{2b^2} \ln \left| \frac{1+4t+\frac{4}{t^2}-4(t^2+1)+\frac{4}{t^2}}{1+4t+\frac{4}{t^2}+4(t^2+1)+\frac{4}{t^2}} \right| + C \\ &= \frac{1}{2b^2} \ln \left| \frac{1+4t+\frac{4}{t^2}-4(t^2+1)+\frac{4}{t^2}}{1+4t+\frac{4}{t^2}+4(t^2+1)+\frac{4}{t^2}} \right| + C \end{aligned}$$

$$\therefore = \frac{1}{\sqrt{2t^2-2}} \operatorname{arctanh} \left[\frac{t-\frac{2}{t}+b}{\sqrt{2t^2-2}(t+\frac{2}{t})} \right] - \frac{1}{\sqrt{2t^2+2}} \operatorname{arctanh} \left[\frac{t+\frac{2}{t}+b}{\sqrt{2t^2+2}(t+\frac{2}{t})} \right] + C$$

