

Created by T. Madas

TRIGONOMETRIC EQUATIONS

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Question 1

Solve each of the following trigonometric equations.

- a) $\sec \theta = 4, \quad 0^\circ \leq \theta < 360^\circ$
- b) $3\cot 2x - 1 = 4, \quad 0^\circ \leq x < 180^\circ$
- c) $2\operatorname{cosec} 2y = 10, \quad 0^\circ \leq y < 2\pi$
- d) $8\tan \varphi = \cot^2 \varphi, \quad 0^\circ \leq \varphi < 2\pi$

$$\boxed{\theta \approx 75.5^\circ, 284.5^\circ}, \quad \boxed{x \approx 15.5^\circ, 105.5^\circ}, \quad \boxed{y \approx 0.10^\circ, 1.47^\circ, 3.24^\circ, 4.61^\circ},$$

$$\boxed{\varphi \approx 0.46^\circ, 3.61^\circ}$$

<p>5. (5) $\sin \theta = \frac{4}{5}$</p> $\Rightarrow \cos^2 \theta = \frac{9}{25}$ $\Rightarrow \cos \theta (\frac{1}{5}) = \pm \sqrt{0.36} = \pm 0.6$ $\Rightarrow \theta = 36.9^\circ \pm 360^\circ n \quad n \in \mathbb{Z}, \dots$ $\theta_1 = 36.9^\circ$ $\theta_2 = 283.1^\circ$ <hr/> <p>(4) $3\cot 2x - 1 = 4$</p> $\Rightarrow 3\cot 2x = 5$ $\Rightarrow \cot 2x = \frac{5}{3}$ $\Rightarrow \tan 2x = \frac{3}{5}$ $\Rightarrow \tan(\frac{1}{2}\theta) = 0.6$ $\Rightarrow 2x = 36.9^\circ + 180^\circ n \quad n \in \mathbb{Z}, \dots$ $\Rightarrow x = 18.45^\circ + 90^\circ n$ $\therefore x_1 = 18.45^\circ$ $x_2 = 108.45^\circ$ <hr/> <p>(6) $2\operatorname{cosec} 2y = 10$</p> $\Rightarrow \operatorname{cosec} 2y = 5$ $\Rightarrow \sin 2y = \frac{1}{5}$ $\Rightarrow \sin(\frac{1}{2}\theta) = 0.201$ $\Rightarrow 2y = 11.5^\circ + 360^\circ n \quad n \in \mathbb{Z}, \dots$ $\Rightarrow y = 5.75^\circ + 180^\circ n$ $\therefore y_1 = 5.75^\circ$ $y_2 = 185.75^\circ$ $y_3 = 0.10^\circ, 1.47^\circ, 3.24^\circ, 4.61^\circ$	<p>(5) $2\operatorname{cosec} 2y = 10$</p> $\Rightarrow \operatorname{cosec} 2y = 5$ $\Rightarrow \sin 2y = \frac{1}{5}$ $\Rightarrow \sin(\frac{1}{2}\theta) = 0.201$ $\Rightarrow 2y = 11.5^\circ + 360^\circ n \quad n \in \mathbb{Z}, \dots$ $\Rightarrow y = 5.75^\circ + 180^\circ n$ $\therefore y_1 = 5.75^\circ$ $y_2 = 185.75^\circ$ $y_3 = 0.10^\circ, 1.47^\circ, 3.24^\circ, 4.61^\circ$
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Question 2

Solve each of the following trigonometric equations.

a) $2\sec \theta = 3, \quad 0^\circ \leq \theta < 360^\circ$

b) $\cot 3x = \frac{1}{4}, \quad -90^\circ \leq x < 90^\circ$

c) $5 - \operatorname{cosec} 2y = -1, \quad 0^\circ \leq y < 2\pi$

d) $27 \sin^2 \varphi + 8 \operatorname{cosec} \varphi = 0, \quad 0^\circ \leq \varphi < 2\pi$

$\boxed{\theta \approx 48.2^\circ, 311.8^\circ}, \boxed{x \approx -34.7^\circ, 25.3^\circ, 85.3^\circ}, \boxed{y \approx 0.0837^\circ, 1.49^\circ, 3.23^\circ, 4.63^\circ},$

$\boxed{\varphi \approx 3.87^\circ, 5.55^\circ}$

<p>(a) $2\sec \theta = 3$ $\Rightarrow \sec \theta = \frac{3}{2}$ $\Rightarrow \cos \theta = \frac{2}{3}$ $\operatorname{arccos}(\frac{2}{3}) = 48.2^\circ$ $(\theta = 48.2^\circ \pm 360^\circ) \quad u = 0.837^\circ$ $\theta_1 = 48.2^\circ$ $\theta_2 = 311.8^\circ$</p>	<p>(b) $\cot 3x = \frac{1}{4}$ $\Rightarrow \tan 3x = 4$ $\operatorname{arctan} 4 \approx 75.9^\circ$ $(3x = 75.9^\circ \pm 180^\circ n) \quad 4 = \alpha/3 \dots$ $(x = 25.3^\circ \pm 60^\circ n)$ $x_1 = 25.3^\circ$ $x_2 = 85.3^\circ$ $x_3 = -34.7^\circ$</p>
<p>(c) $5 - \operatorname{cosec} 2y = -1$ $\rightarrow 6 = \operatorname{cosec} 2y$ $\rightarrow \sin 2y = \frac{1}{6}$ $\operatorname{arcsin}(\frac{1}{6}) \approx 0.1674^\circ$ $(2y = 0.1674^\circ \pm 360^\circ n) \quad u = 0.0837^\circ$ $(y = 0.0837^\circ \pm 180^\circ n)$ $y_1 = 0.0837^\circ$ $y_2 = 3.23^\circ$ $y_3 = 1.49^\circ$ $y_4 = 4.63^\circ$</p>	<p>(d) $27 \sin^2 \varphi + 8 \operatorname{cosec} \varphi = 0$ $\Rightarrow 27 \sin^2 \varphi + \frac{8}{\sin \varphi} = 0$ $\Rightarrow 27 \sin^2 \varphi + 8 = 0$ $\Rightarrow \sin^2 \varphi = -\frac{8}{27}$ $\Rightarrow \sin \varphi = \pm \sqrt{-\frac{8}{27}}$ $\operatorname{arcsin}(\pm \frac{2\sqrt{2}}{3}) = -0.785^\circ$ $(\varphi = -0.785^\circ \pm 2\pi n) \quad u = 5.55^\circ$ $\varphi_1 = -0.785^\circ$ $\varphi_2 = 5.55^\circ$</p>

Question 3

Solve each of the following trigonometric equations.

a) $3\sec 2\theta = 7$, $0 \leq \theta < 180^\circ$

b) $2\cot(x - 30^\circ) = 3$, $0 \leq x < 360^\circ$

c) $5 - 2\operatorname{cosec} 3y = 9$, $0 \leq y < \pi$

d) $27\cos\varphi = \sec^2\varphi$, $0 \leq \varphi < 2\pi$

$$\boxed{\theta \approx 32.3^\circ, 147.7^\circ}, \boxed{x \approx 63.7^\circ, 243.7^\circ}, \boxed{y = \frac{7\pi}{18}, \frac{11\pi}{18}}, \boxed{\varphi \approx 1.23^\circ, 5.05^\circ}$$

<p>(a) $3\sec 2\theta = 7$ $\Rightarrow \sec 2\theta = \frac{7}{3}$ $\Rightarrow \cos 2\theta = \frac{3}{7}$ $\Rightarrow \arccos(\frac{3}{7}) = 64.2^\circ$ $(2\theta) = 64.2^\circ \pm 360^\circ$ $(2\theta) = 294.36^\circ \pm 360^\circ$ $(\theta) = 147.18^\circ \pm 180^\circ$ $\therefore \theta = 32.3^\circ, 147.7^\circ$</p>	<p>(c) $5 - 2\operatorname{cosec} 3y = 9$ $\Rightarrow -2\operatorname{cosec} 3y = 4$ $\Rightarrow \operatorname{cosec} 3y = -2$ $\Rightarrow \sin 3y = -\frac{1}{2}$ $(3y) = \frac{7\pi}{6} \pm 2m\pi$ $(3y) = \frac{11\pi}{6} \pm 2m\pi$ $(y) = \frac{7\pi}{18} \pm \frac{2m\pi}{3}$ $y = \frac{7\pi}{18}, \frac{11\pi}{18}$</p>
<p>(b) $2\cot(x - 30) = 3$ $\Rightarrow \cot(x - 30) = \frac{3}{2}$ $\Rightarrow \tan(x - 30) = \frac{2}{3}$ $\Rightarrow \arctan(\frac{2}{3}) = 33.7^\circ$ $x - 30 = 33.7^\circ \pm 180^\circ$ $x = 63.7^\circ \pm 180^\circ$ $x = 63.7^\circ, 243.7^\circ$</p>	<p>(d) $27\cos\varphi = \sec^2\varphi$ $\Rightarrow 27\cos\varphi = \frac{1}{\cos^2\varphi}$ $\Rightarrow 27\cos^3\varphi = 1$ $\Rightarrow \cos^3\varphi = \frac{1}{27}$ $\Rightarrow \cos\varphi = \frac{1}{3}$ $\operatorname{arccos}(\frac{1}{3}) = 1.23$ $\phi = 1.23^\circ \pm 2m\pi$ $\phi = 5.05^\circ \pm 2m\pi$ $\phi = 1.23^\circ, 5.05^\circ$</p>

Question 4

Solve each of the following trigonometric equations.

a) $2\sec\theta - 1 = 9$, $0 \leq \theta < 360^\circ$

b) $2 + 3\cot(x - 20^\circ) = 8$, $0 \leq x < 360^\circ$

c) $14 - 3\operatorname{cosec} 2y = 5$, $0 \leq y < \pi$

d) $4\sin^3\varphi + \frac{1}{8}\operatorname{cosec}^2\varphi = 0$, $0 \leq \varphi < 2\pi$

$$\boxed{\theta \approx 78.5^\circ, 281.5^\circ}, \boxed{x \approx 46.6^\circ, 226.6^\circ}, \boxed{y \approx 0.170^\circ, 1.40^\circ}, \boxed{\varphi = \frac{7\pi}{6}, \frac{11\pi}{6}}$$

<p>(a) $2\sec\theta - 1 = 9$ $\Rightarrow 2\sec\theta = 10$ $\Rightarrow \sec\theta = 5$ $\Rightarrow \cos\theta = \frac{1}{5}$ $\bullet \arccos\left(\frac{1}{5}\right) = 78.5^\circ$ $\therefore \theta = 78.5^\circ \pm 360^\circ n = 0, 132^\circ$ $\theta_1 = 78.5^\circ$ $\theta_2 = 281.5^\circ$</p>	<p>(b) $2 + 3\cot(x - 20^\circ) = 8$ $\Rightarrow 3\cot(x - 20^\circ) = 6$ $\Rightarrow \cot(x - 20^\circ) = 2$ $\Rightarrow \tan(x - 20^\circ) = \frac{1}{2}$ $\bullet \arctan\left(\frac{1}{2}\right) = 26.6^\circ$ $\therefore x - 20^\circ = 26.6^\circ + 180^\circ n$ $x = 46.6^\circ \pm 180^\circ n$ $\therefore x = 46.6^\circ, 226.6^\circ$</p>
<p>(c) $14 - 3\operatorname{cosec} 2y = 5$ $\Rightarrow 9 = 3\operatorname{cosec} 2y$ $\Rightarrow 3 = \operatorname{cosec} 2y$ $\Rightarrow \sin 2y = \frac{1}{3}$ $\arcsin\left(\frac{1}{3}\right) = 0.346^\circ$ $\therefore 2y = 0.346^\circ \pm 2n\pi$ $2y = 2\arcsin\left(\frac{1}{3}\right) \pm 2n\pi$ $y = 0.170^\circ \pm n\pi$ $y = 1.40^\circ \pm n\pi$ $\therefore y_1 = 0.170^\circ$ $y_2 = 1.40^\circ$</p>	<p>(d) $4\sin^3\varphi + \frac{1}{8}\operatorname{cosec}^2\varphi = 0$ $\Rightarrow 4\sin^3\varphi + \frac{1}{8\sin^2\varphi} = 0$ $\Rightarrow 32\sin^5\varphi + 1 = 0$ $\Rightarrow 32\sin^5\varphi = -1$ $\Rightarrow \sin^5\varphi = -\frac{1}{32}$ $\Rightarrow \sin\varphi = -\frac{1}{2}$ $\bullet \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ $\Rightarrow \left(\frac{\pi}{6} < \varphi < 2\pi\right)$ $\varphi = \frac{7\pi}{6} \pm 2n\pi$ $\varphi_1 = \frac{7\pi}{6}$ $\varphi_2 = \frac{11\pi}{6}$</p>

Question 5

Solve each of the following trigonometric equations.

a) $2\sec\theta - 1 = 2\sec\theta \sin^2\theta, \quad 0 \leq \theta < 180^\circ, \quad \theta \neq 90^\circ$

b) $\cos x \cot x + \sin x + 2 \cot x = 0, \quad 0 < x < 360^\circ, \quad x \neq 180^\circ$

c) $(\operatorname{cosec} y - \sin y) \sec^2 y = 2, \quad 0 \leq y < \pi, \quad y \neq \frac{\pi}{2}$

d) $\operatorname{cosec} \varphi - \sin \varphi + 2 \cos^2 \varphi \cot \varphi = 0, \quad 0 < \varphi < 2\pi, \quad \varphi \neq \pi$

$$\boxed{\theta = 60^\circ}, \boxed{x = 120^\circ, 240^\circ}, \boxed{y = \frac{\pi}{6}, \frac{5\pi}{6}}, \boxed{\varphi = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}}$$

(a) $2\sec\theta - 1 = 2\sec\theta \sin^2\theta$

$$\Rightarrow \frac{2}{\cos\theta} - 1 = \frac{2}{\cos\theta} \sin^2\theta$$

cancel $\frac{2}{\cos\theta}$ no solutions

$$\Rightarrow \frac{2}{\cos\theta} - 1 = \frac{2\sin^2\theta}{\cos\theta}$$

$$\Rightarrow 2 - \cos\theta = 2\sin^2\theta$$

$$\Rightarrow 2 - \cos\theta = 2(1 - \cos^2\theta)$$

$$\Rightarrow 2\cos^2\theta - \cos\theta = 0$$

$$\Rightarrow \cos\theta(\cos\theta - 1) = 0$$

$$\cos\theta < \frac{1}{2}$$

(b) $\cos x \cot x + \sin x + 2 \cot x = 0$

$$\Rightarrow \frac{\cos x \cos x}{\sin x} + \sin x + 2 \cot x = 0$$

$$\Rightarrow \frac{\cos^2 x}{\sin x} + \sin x + \frac{2 \cos x}{\sin x} = 0$$

$$\Rightarrow \frac{\cos^2 x + \sin^2 x + 2 \cos x}{\sin x} = 0$$

$$\Rightarrow 1 + 2\cos x = 0$$

$$\Rightarrow \cos x = -\frac{1}{2}$$

$\arcsin(-\frac{1}{2}) \approx 120^\circ$

$x = 120^\circ \pm 360^\circ$

$x = 240^\circ \pm 360^\circ$

$x_1 = 120^\circ$

$x_2 = 240^\circ$

(c) $(\operatorname{cosec} y - \sin y) \sec^2 y = 2$

$$\Rightarrow \left(\frac{1}{\sin y} - \sin y\right) \frac{1}{\cos^2 y} = 2$$

$$\Rightarrow \left(\frac{1 - \sin^2 y}{\sin y}\right) \frac{1}{\cos^2 y} = 2$$

$$\Rightarrow \frac{\cos^2 y}{\sin y} = 2$$

$$\Rightarrow \frac{1}{\sin y} = 2$$

$$\Rightarrow \sin y = \frac{1}{2}$$

$\arcsin(\frac{1}{2}) = \frac{\pi}{6}$

$y \approx \frac{\pi}{6} \pm 2\pi n^\circ$

$y \approx \frac{5\pi}{6} \pm 2\pi n^\circ$

$y_1 = \frac{\pi}{6}$

$y_2 = \frac{5\pi}{6}$

(d) $\operatorname{cosec} \varphi - \sin \varphi + 2 \cos^2 \varphi \cot \varphi = 0$

$$\Rightarrow \frac{1}{\sin \varphi} - \sin \varphi + \frac{2 \cos^2 \varphi \cos \varphi}{\sin \varphi} = 0$$

$$\Rightarrow \frac{1 - \sin^2 \varphi}{\sin \varphi} + \frac{2 \cos^3 \varphi}{\sin \varphi} = 0$$

$$\Rightarrow \frac{\cos^2 \varphi}{\sin \varphi} + \frac{2 \cos^3 \varphi}{\sin \varphi} = 0$$

$$\Rightarrow \cos^2 \varphi + 2 \cos^3 \varphi = 0$$

$$\Rightarrow \cos^2 \varphi (1 + 2\cos\varphi) = 0$$

$\cos\varphi = 0$

$\operatorname{arccos}(0) = \frac{\pi}{2}$

$\operatorname{arccos}(-1) = \frac{3\pi}{2}$

$\frac{\pi}{2} \pm 2\pi n^\circ$

$\frac{3\pi}{2} \pm 2\pi n^\circ$

$\frac{\pi}{2} \pm \frac{3\pi}{2} \pm 2\pi n^\circ$

$n = 0, 1, 2, 3, \dots$

$\therefore \varphi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$

Question 6

Solve each of the following trigonometric equations.

a) $\sec \theta + \cos \theta = \frac{5}{2}$, $0^\circ \leq \theta < 360^\circ$, $\theta \neq 90^\circ, 270^\circ$

b) $\sec x - \cos x = 8(\operatorname{cosec} x - \sin x)$, $0^\circ \leq x < 360^\circ$, $x \neq 90^\circ$

c) $2\cot y - 3\operatorname{cosec} y = 2\sec y \operatorname{cosec} y$, $0 < y < 2\pi$, $y \neq \frac{k\pi}{2}$, $k \in \mathbb{Z}$

d) $(1 + \sec \varphi)(1 - \cos \varphi) = \tan \varphi$, $0 \leq \varphi < 2\pi$, $\varphi \neq \frac{\pi}{2}, \frac{3\pi}{2}$

e) $\frac{\operatorname{cosec}^2 \psi \tan^2 \psi}{\cos \psi} + 8 = 0$, $0^\circ \leq \psi < 360^\circ$, $\psi \neq 90^\circ, 270^\circ$

$\boxed{\theta = 60^\circ, 300^\circ}$, $\boxed{x = 63.4^\circ, 243.6^\circ}$, $\boxed{y = \frac{2\pi}{3}, \frac{4\pi}{3}}$, $\boxed{\varphi = 0, \pi}$, $\boxed{\psi = 120^\circ, 240^\circ}$

Q1 $\sec \theta + \cos \theta = \frac{5}{2}$, $0^\circ \leq \theta < 360^\circ$

$$\begin{aligned} \Rightarrow \frac{1}{\cos \theta} + \cos \theta = \frac{5}{2} \\ \Rightarrow \frac{1 + \cos^2 \theta}{\cos \theta} = \frac{5}{2} \\ \Rightarrow 2 + 2\cos^2 \theta = 5\cos \theta \\ \Rightarrow 2\cos^2 \theta - 5\cos \theta + 2 = 0 \\ \Rightarrow (2\cos \theta - 1)(\cos \theta - 2) = 0 \\ \Rightarrow \cos \theta = \frac{1}{2} \quad \text{or} \quad \cos \theta = 2 \end{aligned}$$

Q2 $\sec x - \cos x = 8(\operatorname{cosec} x - \sin x)$

$$\begin{aligned} \Rightarrow \frac{1}{\cos x} - \cos x = 8\left(\frac{1}{\sin x} - \sin x\right) \\ \Rightarrow \frac{1 - \cos^2 x}{\cos x} = 8\left(\frac{1 - \sin^2 x}{\sin x}\right) \\ \Rightarrow \frac{\sin^2 x}{\cos x} = \frac{8(1 - \sin^2 x)}{\sin x} \\ \Rightarrow \sin^2 x = 8\cos x \\ \Rightarrow \frac{\sin^2 x}{\cos^2 x} = 8 \end{aligned}$$

Q3 $2\cot y - 3\operatorname{cosec} y = 2\sec y \operatorname{cosec} y$

$$\begin{aligned} \Rightarrow \frac{2\cos y}{\sin y} - \frac{3}{\sin y} = \frac{2}{\cos y} \times \frac{1}{\sin y} \\ \Rightarrow \frac{2\cos y - 3}{\sin y} = \frac{2}{\cos y \sin y} \\ \Rightarrow \frac{(2\cos y - 3)\cos y}{\sin y \cos y} = \frac{2}{\cos y \sin y} \\ \Rightarrow (2\cos^2 y - 3\cos y) = 2 \\ \Rightarrow 2\cos^2 y - 3\cos y - 2 = 0 \\ \Rightarrow (2\cos y + 1)(\cos y - 2) = 0 \\ \Rightarrow \cos y = -\frac{1}{2} \quad \text{or} \quad \cos y = 2 \end{aligned}$$

Q4 $(1 + \sec \varphi)(1 - \cos \varphi) = \tan \varphi$

$$\begin{aligned} \Rightarrow (1 + \frac{1}{\cos \varphi})(1 - \cos \varphi) = \frac{\sin \varphi}{\cos \varphi} \\ \Rightarrow 1 - \cos \varphi + \frac{1}{\cos \varphi} - \frac{1}{\cos^2 \varphi} = \frac{\sin \varphi}{\cos \varphi} \\ \Rightarrow -\cos^2 \varphi + \cos \varphi = \frac{\sin \varphi}{\cos \varphi} \\ \Rightarrow 1 - \cos^2 \varphi = \frac{\sin \varphi}{\cos \varphi} \\ \Rightarrow \sin^2 \varphi = \frac{\sin \varphi}{\cos \varphi} \\ \Rightarrow \sin \varphi (\sin \varphi - 1) = 0 \\ \Rightarrow \sin \varphi = 0 \quad \text{or} \quad \sin \varphi = 1 \end{aligned}$$

Q5 $\frac{\operatorname{cosec}^2 \psi \tan^2 \psi}{\cos \psi} + 8 = 0$

$$\begin{aligned} \Rightarrow \operatorname{cosec}^2 \psi \tan^2 \psi + 8\cos \psi = 0 \\ \Rightarrow \frac{\sin^2 \psi}{\cos^2 \psi} \tan^2 \psi + 8\cos \psi = 0 \\ \Rightarrow \frac{1 - \cos^2 \psi}{\cos^2 \psi} \tan^2 \psi + 8\cos \psi = 0 \\ \Rightarrow \frac{\tan^2 \psi}{\cos^2 \psi} + 8\cos \psi = 0 \\ \Rightarrow 1 + 8\cos^2 \psi = 0 \\ \Rightarrow 8\cos^2 \psi = -1 \\ \Rightarrow \cos^2 \psi = -\frac{1}{8} \\ \Rightarrow \cos \psi = -\frac{1}{2} \end{aligned}$$

Question 7 (hard questions)

Solve each of the following trigonometric equations.

a) $2\sin\theta + 3\sec\theta = 6 + \tan\theta, \quad 0 \leq \theta < 2\pi, \quad \theta \neq \frac{\pi}{2}, \frac{3\pi}{2}$

b) $\sin^2 x \tan x + \cos^2 x \cot x + 2\sin x \cos x = 2, \quad 0 < x < 360^\circ, \quad x \neq 90^\circ, 180^\circ, 270^\circ$

you may use in this part the fact that $[2\sin x \cos x \equiv \sin 2x]$

c) $\sin y(1 + \tan y) + \cos y(1 + \cot y) = 0, \quad 0 < y < 360^\circ, \quad y \neq 90^\circ, 180^\circ, 270^\circ$

d) $\frac{4}{2\sec\varphi - 2\sin\varphi + 1} = \cot\varphi, \quad 0 < \varphi < 2\pi, \quad \varphi \neq \pi$

e) $\frac{\cot\psi}{\operatorname{cosec}\psi - 1} - \frac{\cos\psi}{1 + \sin\psi} = 2, \quad 0 < \psi < 2\pi, \quad \psi \neq \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

f) $\frac{\cot\beta}{\operatorname{cosec}\beta - 1} + \frac{\operatorname{cosec}\beta - 1}{\cot\beta} = 4, \quad 0 \leq x < 360^\circ, \quad x \neq 90^\circ, 180^\circ, 270^\circ$

$$\boxed{\varphi = \frac{\pi}{3}, \frac{5\pi}{3}}, \boxed{x = 45^\circ, 225^\circ}, \boxed{y = 135^\circ, 315^\circ}, \boxed{\psi = \frac{\pi}{6}, \frac{5\pi}{6}}, \boxed{\beta = 60^\circ, 300^\circ}$$

(a) $2\sin\theta + 3\sec\theta = 6 + \tan\theta$

$$\begin{aligned} &\rightarrow 2\sin\theta + \frac{3}{\cos\theta} = 6 + \frac{\sin\theta}{\cos\theta} \\ &\rightarrow 2\sin\theta\cos\theta + 3 = 6\cos\theta + \sin\theta \\ &\rightarrow 2\sin\theta\cos\theta - \sin\theta = 6\cos\theta - 3 \\ &\Rightarrow \sin\theta(2\sin\theta - 1) = 3(2\cos\theta - 1) \\ &\rightarrow \sin\theta(2\sin\theta - 1) - 3(2\cos\theta - 1) = 0 \\ &\Rightarrow (2\sin\theta - 1)(\sin\theta - 3) = 0 \end{aligned}$$

$\left. \begin{array}{l} \sin\theta \leq 3 \\ \cos\theta = \frac{1}{2} \\ \arccos(\frac{1}{2}) \end{array} \right\}$

$\begin{cases} \theta = \frac{\pi}{3} \pm 2m\pi \\ \theta = \frac{4\pi}{3} \pm 2m\pi \end{cases} \quad m \in \{0, 1, 2, \dots\}$

$\theta = \frac{4\pi}{3} \mp \frac{\pi}{3}$

(b) $\sin^2 x \tan x + \cos^2 x \cot x + 2\sin x \cos x = 2$

$$\begin{aligned} &\rightarrow \sin^2 x \tan x + \cos^2 x \cot x + 2\sin x \cos x = 2 \\ &\rightarrow \frac{\sin^2 x}{\cos x} \tan x + \frac{\cos^2 x}{\sin x} \cot x = 2 \\ &\rightarrow \frac{\sin^2 x}{\cos x} \tan x + \frac{\cos^2 x}{\sin x} \cot x = 2\sin x \cos x \\ &\Rightarrow (\sin^2 x + \cos^2 x)^2 = 2\sin x \cos x \\ &\Rightarrow 1^2 = 2\sin x \cos x \\ &\Rightarrow \sin 2x = 1 \end{aligned}$$

$\left. \begin{array}{l} \arcsin(1) = 90^\circ \\ 2x = 90^\circ \pm 360^\circ \\ 2x = 90^\circ \pm 360^\circ \end{array} \right\}$

$\begin{cases} x = 45^\circ \pm 180^\circ \\ x = 225^\circ \end{cases}$

(c) $\sin y(1 + \tan y) + \cos y(1 + \cot y) = 0$

$$\begin{aligned} &\rightarrow \sin y + \sin y \tan y + \cos y + \cos y \cot y = 0 \\ &\rightarrow \sin y + \frac{\sin y}{\cos y} + \cos y + \frac{\cos y}{\sin y} = 0 \quad (\text{cancel } \sin y \cos y) \\ &\Rightarrow \sin^2 y + \cos^2 y + \cos y \sin y + \cos y \sin y = 0 \\ &\Rightarrow \sin^2 y + \cos^2 y + 2\cos y \sin y = 0 \\ &\Rightarrow (\sin y + \cos y)^2 = 0 \\ &\Rightarrow \sin y + \cos y = 0 \\ &\Rightarrow \frac{\sin y}{\cos y} + \frac{\cos y}{\sin y} = 0 \end{aligned}$$

$\left. \begin{array}{l} \tan y = 0 \\ \tan y = -1 \\ \arctan(-1) = 45^\circ \\ y_1 = -45^\circ \pm 180^\circ \\ y_2 = 315^\circ \end{array} \right\}$

(d) $\frac{4}{2\sec\varphi - 2\sin\varphi + 1} = \cot\varphi$

$$\begin{aligned} &\rightarrow \frac{4}{2\sec\varphi - 2\sin\varphi + 1} = \cot\varphi \\ &\rightarrow \frac{4}{\frac{2}{\cos\varphi} - 2\sin\varphi + \frac{1}{\cos\varphi}} = \cot\varphi \\ &\rightarrow \frac{4\cos\varphi}{2 - 2\sin\varphi\cos\varphi + 1} = \cot\varphi \\ &\Rightarrow 4 = \frac{\cos^2\varphi}{1 - \sin\varphi\cos\varphi} = \cot^2\varphi \\ &\Rightarrow 4\cos^2\varphi = 2 - 2\sin\varphi\cos\varphi + \cos^2\varphi \\ &\Rightarrow 4\cos^2\varphi + 2\sin\varphi\cos\varphi = 2 + \cos^2\varphi \\ &\Rightarrow 2\sin\varphi(2 + \cos\varphi) = 2 + \cos^2\varphi \\ &\Rightarrow 2\sin\varphi(2 + \cos\varphi) = (2 + \cos\varphi)\cos\varphi \\ &\Rightarrow (2 + \cos\varphi)(2\sin\varphi - 1) = 0 \\ &\Rightarrow 2\sin\varphi - 1 = 0 \quad \text{or } 2 + \cos\varphi = 0 \\ &\Rightarrow \sin\varphi = \frac{1}{2} \quad \text{or } \cos\varphi = -2 \\ &\Rightarrow \arcsin(\frac{1}{2}) = \frac{\pi}{6} \\ &\left. \begin{array}{l} \varphi = \frac{\pi}{6} \pm 2m\pi \\ \varphi = \frac{5\pi}{6} \pm 2m\pi \end{array} \right\} \\ &\Rightarrow \varphi = \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

(e) $\frac{\cot\psi}{\operatorname{cosec}\psi - 1} - \frac{\cos\psi}{1 + \sin\psi} = 2$

$$\begin{aligned} &\rightarrow \frac{\cot\psi}{\frac{1}{\sin\psi} - 1} - \frac{\cos\psi}{1 + \sin\psi} = 2 \\ &\rightarrow \frac{\cot\psi}{\frac{1 - \sin\psi}{\sin\psi}} - \frac{\cos\psi}{1 + \sin\psi} = 2 \\ &\rightarrow \frac{\cot\psi \sin\psi}{1 - \sin\psi} - \frac{\cos\psi}{1 + \sin\psi} = 2 \\ &\rightarrow \frac{\cos\psi}{\sin\psi} - \frac{\cos\psi}{1 + \sin\psi} = 2 \\ &\Rightarrow \frac{\cos\psi(1 + \sin\psi) - \cos\psi\sin\psi}{\sin\psi(1 + \sin\psi)} = 2 \\ &\Rightarrow \frac{\cos\psi + \cos\psi\sin\psi - \cos\psi\sin\psi}{\sin\psi(1 + \sin\psi)} = 2 \\ &\Rightarrow \frac{\cos\psi}{\sin\psi(1 + \sin\psi)} = 2 \\ &\Rightarrow \frac{\cos\psi}{\sin\psi} = 2(1 + \sin\psi) \\ &\Rightarrow \cot\psi = 2(1 + \sin\psi) \\ &\Rightarrow \cot\psi = 2 \\ &\Rightarrow \operatorname{arccot}(2) = \frac{\pi}{4} \\ &\left. \begin{array}{l} \psi = \frac{\pi}{4} \pm m\pi \\ \psi = \frac{5\pi}{4} \end{array} \right\} \\ &\Rightarrow \psi = \frac{\pi}{4}, \frac{5\pi}{4} \end{aligned}$$

Question 8

Solve each of the following equations.

a) $2 \tan^2 \theta = 11 \sec \theta - 7$, $0^\circ \leq \theta < 360^\circ$

b) $4 \cot^2 x - 9 \operatorname{cosec} x + 6 = 0$, $0^\circ \leq x < 360^\circ$

c) $\sec^2 y + \tan y = 3$, $0^\circ \leq y < 360^\circ$

d) $2 \operatorname{cosec}^2 \varphi + \cot^2 \varphi = 11$, $0^\circ \leq \varphi < 360^\circ$

$\theta = 78.5^\circ, 281.5^\circ$, $x = 30^\circ, 150^\circ$, $y = 45^\circ, 225^\circ$, $y \approx 116.6^\circ, 296.6^\circ$,

$\varphi = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

<p>(a) $2 \tan^2 \theta = 11 \sec \theta - 7$ $\rightarrow 2(\sec^2 \theta - 1) = 11 \sec \theta - 7$ $\rightarrow 2 \sec^2 \theta - 11 \sec \theta + 5 = 0$ $\rightarrow 2 \sec^2 \theta - 11 \sec \theta + 5 = 0$ $\Rightarrow \sec \theta < \frac{\sqrt{13}}{2}$ $\cos \theta = \frac{1}{\sec \theta} < \frac{1}{\sqrt{13}}$</p>	<p>$\operatorname{arccos}\left(\frac{1}{\sqrt{13}}\right) = 78.5^\circ$ $\theta_1 = 78.5^\circ \pm 360n$ $\theta_2 = 281.5^\circ \pm 360n$ $\theta_1 = 78.5^\circ$ $\theta_2 = 281.5^\circ$</p>
<p>(b) $4 \cot^2 x - 9 \operatorname{cosec} x + 6 = 0$ $\rightarrow 4(\operatorname{cosec}^2 x - 1) - 9 \operatorname{cosec} x + 6 = 0$ $\rightarrow 4 \operatorname{cosec}^2 x - 10 \operatorname{cosec} x + 2 = 0$ $\rightarrow (\operatorname{cosec} x - 2)(4 \operatorname{cosec} x - 1) = 0$ $\Rightarrow \operatorname{cosec} x < \frac{1}{2}$ $\sin x > \frac{1}{2}$</p>	<p>$\operatorname{arcsin}\left(\frac{1}{2}\right) = 30^\circ$ $x_1 = 30^\circ + 360n$ $x_2 = 150^\circ + 360n$ $x_1 = 30^\circ$ $x_2 = 150^\circ$</p>
<p>(c) $\sec^2 y + \tan y = 3$ $\Rightarrow (1 + \tan^2 y) + \tan y = 3$ $\Rightarrow \tan^2 y + \tan y - 2 = 0$ $\Rightarrow (\tan y - 1)(\tan y + 2) = 0$ $\tan y = -2$ $\operatorname{arctan}(1) = 45^\circ$ $\operatorname{arctan}(-2) = -63.4^\circ$ $y = 45^\circ \pm 180n$ $y = -63.4^\circ \pm 180n$ $y = 45^\circ, 225^\circ, 116.6^\circ, 296.6^\circ$</p>	<p>(d) $2 \operatorname{cosec}^2 \varphi + \cot^2 \varphi = 11$ $\rightarrow 2(1 + \operatorname{cot}^2 \varphi) + \operatorname{cot}^2 \varphi = 11$ $\rightarrow 3 \operatorname{cot}^2 \varphi + 2 = 11$ $\rightarrow 3 \operatorname{cot}^2 \varphi = 9$ $\rightarrow \operatorname{cot}^2 \varphi = 3$ $\Rightarrow \operatorname{cot} \varphi = \pm \sqrt{3}$ $\Rightarrow \tan \varphi = \pm \frac{1}{\sqrt{3}}$ $\operatorname{arctan}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$ $\operatorname{arctan}\left(-\frac{1}{\sqrt{3}}\right) = -30^\circ$ $\text{Hence } \varphi = 30^\circ \pm 180n$ $\varphi = -30^\circ \pm 180n$ $\therefore \varphi = 30^\circ, 210^\circ, 150^\circ, 330^\circ$</p>

Question 9

Solve each of the following equations.

a) $2\cot^2 \theta - \operatorname{cosec} \theta = \operatorname{cosec}^2 \theta, \quad 0^\circ \leq \theta < 360^\circ$

b) $2\tan^2 x + \sec^2 x = 5\sec x, \quad 0^\circ \leq x < 360^\circ$

c) $3 - \tan^2 y = 3\sec^2 y + 6\sec y, \quad 0^\circ \leq y < 360^\circ$

d) $\tan^2 \varphi = 2\sec \varphi - 1, \quad 0^\circ \leq \varphi < 360^\circ$

$\theta = 30^\circ, 150^\circ, 270^\circ, \quad x = 60^\circ, 300^\circ, \quad y = 120^\circ, 240^\circ, \quad \varphi = 60^\circ, 300^\circ$

<p>(a) $2\cot^2 \theta - \operatorname{cosec} \theta = \operatorname{cosec}^2 \theta$</p> $\begin{aligned} \Rightarrow 2(\operatorname{cosec}^2 \theta - 1) - \operatorname{cosec} \theta &= \operatorname{cosec}^2 \theta \\ \Rightarrow \operatorname{cosec}^2 \theta - \operatorname{cosec} \theta - 2 &= 0 \\ \Rightarrow (\operatorname{cosec} \theta + 1)(\operatorname{cosec} \theta - 2) &= 0 \\ \Rightarrow \operatorname{cosec} \theta &< -1 \quad \text{or} \\ \operatorname{cosec} \theta &> 2 \end{aligned}$ <p>$\sin \theta = \frac{1}{\operatorname{cosec} \theta}$</p> <p>$\theta = 30^\circ, 150^\circ, 270^\circ$</p> <p>$\theta_1 = 30^\circ, \theta_2 = 150^\circ, \theta_3 = 270^\circ$</p>	<p>(b) $2\tan^2 x + \sec^2 x = 5\sec x$</p> $\begin{aligned} \Rightarrow 2(\sec^2 x - 1) + \sec^2 x &= 5\sec x \\ \Rightarrow 2\sec^2 x - 2\sec x + \sec^2 x &= 5\sec x \\ \Rightarrow 3\sec^2 x - 5\sec x - 2 &= 0 \\ \Rightarrow (3\sec x + 1)(\sec x - 2) &= 0 \\ \Rightarrow \sec x &< -\frac{1}{3} \quad \text{or} \\ \sec x &> 2 \end{aligned}$ <p>$\cos x = \frac{1}{\sec x}$</p> <p>$x = 60^\circ, 300^\circ$</p> <p>$x_1 = 60^\circ, x_2 = 300^\circ$</p>	<p>(c) $3 - \tan^2 y = 3\sec^2 y + 6\sec y$</p> $\begin{aligned} \Rightarrow 3 - (\sec^2 y - 1) &= 3\sec^2 y + 6\sec y \\ \Rightarrow 4 - \sec^2 y &= 3\sec^2 y + 6\sec y \\ \Rightarrow 0 &= 4\sec^2 y + 6\sec y - 4 \\ \Rightarrow 0 &= 2\sec^2 y + 3\sec y - 2 \\ \Rightarrow 0 &= (2\sec y - 1)(\sec y + 2) \\ \Rightarrow \sec y &< -\frac{1}{2} \quad \text{or} \\ \sec y &> 2 \end{aligned}$ <p>$\cos y = \frac{1}{\sec y}$</p> <p>$y = 120^\circ, 240^\circ$</p> <p>$y_1 = 120^\circ, y_2 = 240^\circ$</p>	<p>(d) $\tan^2 \varphi = 2\sec \varphi - 1$</p> $\begin{aligned} \Rightarrow \sec^2 \varphi - 2\sec \varphi + 1 &= 0 \\ \Rightarrow (\sec \varphi - 1)^2 &= 0 \\ \Rightarrow \sec \varphi - 1 &= 0 \\ \Rightarrow \sec \varphi &< 1 \quad \text{or} \\ \sec \varphi &> 2 \end{aligned}$ <p>$\cos \varphi = \frac{1}{\sec \varphi}$</p> <p>$\varphi = 60^\circ, 300^\circ$</p> <p>$\varphi_1 = 60^\circ, \varphi_2 = 300^\circ$</p>
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Question 10

Solve each of the following equations.

- a) $2\cot^2 \theta + 6 = 9\operatorname{cosec} \theta$, $0^\circ \leq \theta < 360^\circ$
- b) $5\tan^2 x + 16\sec x + 8 = 0$, $0^\circ \leq x < 360^\circ$
- c) $\operatorname{cosec} y + 5\operatorname{cosec}^2 y = 6\cot^2 y$, $0^\circ \leq y < 360^\circ$
- d) $2\tan^2 \varphi = 15\sec \varphi - 9$, $0^\circ \leq \varphi < 360^\circ$

$$\theta \approx 14.5^\circ, 165.5^\circ, x \approx 109.5^\circ, 250.5^\circ, y \approx 19.5^\circ, 160.5^\circ, y = 210^\circ, 330^\circ,$$

$$\varphi \approx 81.8^\circ, 278.2^\circ$$

(a) $2\cot^2 \theta + 6 = 9\operatorname{cosec} \theta$

$$\begin{aligned} &\rightarrow 2\cot^2 \theta - 9\operatorname{cosec} \theta + 6 = 0 \\ &\rightarrow 2\cot^2 \theta - 2 + 6 = 9\operatorname{cosec} \theta \\ &\rightarrow 2\cot^2 \theta - 9\operatorname{cosec} \theta + 4 = 0 \\ &\rightarrow (\operatorname{cosec} \theta - 4)(2\cot^2 \theta - 1) = 0 \\ &\rightarrow \operatorname{cosec} \theta = 4 \quad \text{or} \\ &\rightarrow \sin \theta = \frac{1}{4} \\ &\rightarrow \theta = \arcsin\left(\frac{1}{4}\right) = 14.47^\circ \\ &\rightarrow \theta = 14.47^\circ \pm 360^\circ \quad y = 0, 13.3^\circ, \\ &\rightarrow \theta = 185.52^\circ \pm 360^\circ \quad y = 360^\circ, 473.3^\circ, \\ &\rightarrow \theta = 14.5^\circ, 165.5^\circ \end{aligned}$$

(b) $5\tan^2 x + 16\sec x + 8 = 0$

$$\begin{aligned} &\rightarrow 5\sec^2 x + 16\sec x + 8 = 0 \\ &\rightarrow 5\sec^2 x - 5 + 16\sec x + 8 = 0 \\ &\rightarrow 5\sec^2 x + 16\sec x + 3 = 0 \\ &\rightarrow (5\sec x + 1)(\sec x + 3) = 0 \\ &\rightarrow \sec x = -\frac{1}{5} \quad \text{or} \\ &\rightarrow \sec x = -3 \\ &\rightarrow \sec x = -\frac{1}{5} \\ &\rightarrow \sec x = -3 \\ &\rightarrow \operatorname{arcsec}\left(-\frac{1}{5}\right) = 109.47^\circ \\ &\rightarrow \operatorname{arcsec}\left(-\frac{1}{5}\right) = 250.53^\circ \quad y = 0, 13.3^\circ, \\ &\rightarrow x = 109.47^\circ \pm 360^\circ \quad y = 360^\circ, 473.3^\circ, \\ &\rightarrow x = 250.53^\circ \pm 360^\circ \quad y = 0, 13.3^\circ, \\ &\rightarrow x = 109.5^\circ, 250.5^\circ \end{aligned}$$

(c) $\operatorname{cosec} y + 5\operatorname{cosec}^2 y = 6\cot^2 y$

$$\begin{aligned} &\rightarrow \operatorname{cosec} y + 5\operatorname{cosec}^2 y = 6\operatorname{cosec}^2 y \\ &\rightarrow \operatorname{cosec} y + 5\operatorname{cosec}^2 y = 6\operatorname{cosec}^2 y - 6 \\ &\rightarrow 0 = 5\operatorname{cosec}^2 y - \operatorname{cosec} y - 6 \\ &\rightarrow (5\operatorname{cosec} y + 2)(\operatorname{cosec} y - 3) = 0 \\ &\rightarrow \operatorname{cosec} y = -\frac{2}{5} \quad \text{or} \\ &\rightarrow \operatorname{cosec} y = 3 \\ &\rightarrow \operatorname{arcsec}\left(-\frac{1}{5}\right) = -30^\circ \\ &\rightarrow y = -30^\circ \pm 360^\circ \quad y = 19.5^\circ, 160.5^\circ \\ &\rightarrow y = 330^\circ \pm 360^\circ \quad y = 160.5^\circ, 360^\circ \\ &\rightarrow y = 0, 13.3^\circ, \dots \end{aligned}$$

(d) $2\tan^2 \varphi = 15\sec \varphi - 9$

$$\begin{aligned} &\rightarrow 2\sec^2 \varphi - 2 = 15\sec \varphi - 9 \\ &\rightarrow 2\sec^2 \varphi - 15\sec \varphi + 7 = 0 \\ &\rightarrow (2\sec \varphi - 1)(\sec \varphi - 7) = 0 \\ &\rightarrow \sec \varphi = \frac{1}{2} \quad \text{or} \\ &\rightarrow \sec \varphi = 7 \\ &\rightarrow \operatorname{arcsec}\left(\frac{1}{2}\right) = 91.8^\circ \\ &\rightarrow \operatorname{arcsec}\left(\frac{1}{2}\right) = 278.2^\circ \\ &\rightarrow \varphi = 91.8^\circ, 278.2^\circ \end{aligned}$$

(a) $2\cot^2 \theta + 6 = 9\operatorname{cosec} \theta$

$$\begin{aligned} &\rightarrow 2\cot^2 \theta - 9\operatorname{cosec} \theta + 6 = 0 \\ &\rightarrow 2\cot^2 \theta - 2 + 6 = 9\operatorname{cosec} \theta \\ &\rightarrow 2\cot^2 \theta - 9\operatorname{cosec} \theta + 4 = 0 \\ &\rightarrow (\operatorname{cosec} \theta - 4)(2\cot^2 \theta - 1) = 0 \\ &\rightarrow \operatorname{cosec} \theta = 4 \quad \text{or} \\ &\rightarrow \sin \theta = \frac{1}{4} \\ &\rightarrow \theta = \arcsin\left(\frac{1}{4}\right) = 14.47^\circ \\ &\rightarrow \theta = 14.47^\circ \pm 360^\circ \quad y = 0, 13.3^\circ, \\ &\rightarrow \theta = 185.52^\circ \pm 360^\circ \quad y = 360^\circ, 473.3^\circ, \\ &\rightarrow \theta = 14.5^\circ, 165.5^\circ \end{aligned}$$

(b) $5\tan^2 x + 16\sec x + 8 = 0$

$$\begin{aligned} &\rightarrow 5\sec^2 x + 16\sec x + 8 = 0 \\ &\rightarrow 5\sec^2 x - 5 + 16\sec x + 8 = 0 \\ &\rightarrow 5\sec^2 x + 16\sec x + 3 = 0 \\ &\rightarrow (5\sec x + 1)(\sec x + 3) = 0 \\ &\rightarrow \sec x = -\frac{1}{5} \quad \text{or} \\ &\rightarrow \sec x = -3 \\ &\rightarrow \sec x = -\frac{1}{5} \\ &\rightarrow \sec x = -3 \\ &\rightarrow \operatorname{arcsec}\left(-\frac{1}{5}\right) = 109.47^\circ \\ &\rightarrow \operatorname{arcsec}\left(-\frac{1}{5}\right) = 250.53^\circ \quad y = 0, 13.3^\circ, \\ &\rightarrow x = 109.47^\circ \pm 360^\circ \quad y = 360^\circ, 473.3^\circ, \\ &\rightarrow x = 250.53^\circ \pm 360^\circ \quad y = 0, 13.3^\circ, \\ &\rightarrow x = 109.5^\circ, 250.5^\circ \end{aligned}$$

(c) $\operatorname{cosec} y + 5\operatorname{cosec}^2 y = 6\cot^2 y$

$$\begin{aligned} &\rightarrow \operatorname{cosec} y + 5\operatorname{cosec}^2 y = 6\operatorname{cosec}^2 y \\ &\rightarrow \operatorname{cosec} y + 5\operatorname{cosec}^2 y = 6\operatorname{cosec}^2 y - 6 \\ &\rightarrow 0 = 5\operatorname{cosec}^2 y - \operatorname{cosec} y - 6 \\ &\rightarrow (5\operatorname{cosec} y + 2)(\operatorname{cosec} y - 3) = 0 \\ &\rightarrow \operatorname{cosec} y = -\frac{2}{5} \quad \text{or} \\ &\rightarrow \operatorname{cosec} y = 3 \\ &\rightarrow \operatorname{arcsec}\left(-\frac{1}{5}\right) = -30^\circ \\ &\rightarrow y = -30^\circ \pm 360^\circ \quad y = 19.5^\circ, 160.5^\circ \\ &\rightarrow y = 330^\circ \pm 360^\circ \quad y = 160.5^\circ, 360^\circ \\ &\rightarrow y = 0, 13.3^\circ, \dots \end{aligned}$$

(d) $2\tan^2 \varphi = 15\sec \varphi - 9$

$$\begin{aligned} &\rightarrow 2\sec^2 \varphi - 2 = 15\sec \varphi - 9 \\ &\rightarrow 2\sec^2 \varphi - 15\sec \varphi + 7 = 0 \\ &\rightarrow (2\sec \varphi - 1)(\sec \varphi - 7) = 0 \\ &\rightarrow \sec \varphi = \frac{1}{2} \quad \text{or} \\ &\rightarrow \sec \varphi = 7 \\ &\rightarrow \operatorname{arcsec}\left(\frac{1}{2}\right) = 91.8^\circ \\ &\rightarrow \operatorname{arcsec}\left(\frac{1}{2}\right) = 278.2^\circ \\ &\rightarrow \varphi = 91.8^\circ, 278.2^\circ \end{aligned}$$

Question 11

Solve each of the following equations.

a) $4 \cot^2 \theta = 1 + \operatorname{cosec} \theta, \quad 0^\circ \leq \theta < 360^\circ$

b) $4 \tan^2 x = 19 \sec x + 1, \quad 0^\circ \leq x < 360^\circ$

c) $4 - \cot^2 y = 8 \operatorname{cosec} y + 3 \operatorname{cosec}^2 y, \quad 0^\circ \leq y < 360^\circ$

d) $\sec^2 \varphi = 2 \tan \varphi, \quad 0^\circ \leq \varphi < 360^\circ$

$\theta \approx 53.1^\circ, 126.9^\circ \quad \theta = 270^\circ, \quad [x \approx 78.5^\circ, 281.5^\circ], \quad [y \approx 203.6^\circ, 336.4^\circ],$

$\varphi = 45^\circ, 225^\circ$

(a) $4 \cot^2 \theta = 1 + \operatorname{cosec} \theta$

$$\begin{aligned} \Rightarrow 4(\operatorname{cosec}^2 \theta - 1) &= 1 + \operatorname{cosec} \theta \\ \Rightarrow 4\operatorname{cosec}^2 \theta - 4 &= 1 + \operatorname{cosec} \theta \\ \Rightarrow 4\operatorname{cosec}^2 \theta - \operatorname{cosec} \theta - 5 &= 0 \\ \Rightarrow (4\operatorname{cosec} \theta - 5)(\operatorname{cosec} \theta + 1) &= 0 \\ \Rightarrow \operatorname{cosec} \theta &= \frac{5}{4} \quad \text{or} \quad \operatorname{cosec} \theta = -1 \\ \Rightarrow \sin \theta &= \frac{-4}{5} \quad \text{or} \quad \sin \theta = 1 \end{aligned}$$

From $\operatorname{cosec}(\theta) = 5/4$:
 $\theta = 53.1^\circ, 126.9^\circ$
 $\theta = 270^\circ$

From $\operatorname{cosec}(\theta) = -1$:
 $\theta = 180^\circ$

From $\sin(\theta) = 1$:
 $\theta = 90^\circ$

From $\sin(\theta) = -4/5$:
 $\theta = 203.6^\circ, 336.4^\circ$

(b) $4 \tan^2 x = 19 \sec x + 1$

$$\begin{aligned} \Rightarrow 4(\sec^2 x - 1) &= 19 \sec x + 1 \\ \Rightarrow 4\sec^2 x - 4 &= 19 \sec x + 1 \\ \Rightarrow 4\sec^2 x - 19 \sec x - 5 &= 0 \\ \Rightarrow (4\sec x + 1)(\sec x - 5) &= 0 \\ \Rightarrow \sec x &= -\frac{1}{4} \quad \text{or} \quad \sec x = 5 \\ \Rightarrow \sec x &= \frac{1}{4} \quad \text{or} \quad \sec x = 5 \\ \bullet \operatorname{arccos}\left(\frac{1}{4}\right) &= 78.5^\circ \\ (x &= 78.5^\circ + 360^\circ) \\ x_1 &= 438.5^\circ \\ x_2 &= 281.5^\circ \end{aligned}$$

(c) $4 - \cot^2 y = 8 \operatorname{cosec} y + 3 \operatorname{cosec}^2 y$

$$\begin{aligned} \Rightarrow 4 - (8 \operatorname{cosec} y + 3 \operatorname{cosec}^2 y) &= \cot^2 y \\ \Rightarrow 4 - 8 \operatorname{cosec} y - 3 \operatorname{cosec}^2 y &= \cot^2 y \\ \Rightarrow 4 - 8 \operatorname{cosec} y - 3 \operatorname{cosec}^2 y + 1 &= \cot^2 y + 3 \operatorname{cosec}^2 y \\ \Rightarrow 5 - 8 \operatorname{cosec} y &= 4 \operatorname{cosec}^2 y - 3 \\ \Rightarrow 0 &= (2 \operatorname{cosec} y - 1)(2 \operatorname{cosec} y + 5) \\ \Rightarrow \operatorname{cosec} y &= \frac{1}{2} \quad \text{or} \quad \operatorname{cosec} y = -\frac{5}{2} \\ \sin y &= \frac{1}{2} \quad \text{or} \quad \sin y = -\frac{2}{5} \\ \bullet \operatorname{arcsin}\left(\frac{1}{2}\right) &= 30^\circ \\ (y &= 30^\circ + 360^\circ) \\ y_1 &= 390^\circ \\ y &= 283.95^\circ + 360^\circ \\ y_2 &= 543.95^\circ \end{aligned}$$

(d) $\sec^2 \varphi = 2 \tan \varphi$

$$\begin{aligned} \Rightarrow 1 + \tan^2 \varphi &= 2 \tan \varphi \\ \Rightarrow \tan^2 \varphi - 2 \tan \varphi + 1 &= 0 \\ \Rightarrow (\tan \varphi - 1)^2 &= 0 \\ \tan \varphi &= 1 \end{aligned}$$

From $\operatorname{arctan}(\varphi) = 45^\circ$:
 $\varphi = 45^\circ$

From $\operatorname{arctan}(\varphi) = 180^\circ + 45^\circ$:
 $\varphi = 225^\circ$

Question 12

Solve each of the following equations.

a) $2\tan^2 \theta + 4\tan \theta + 5 = \sec^2 \theta, \quad 0 \leq \theta < 360^\circ$

b) $2\sec^2 x + 2\tan^2 x = 1 + 4\sec x, \quad 0 \leq x < 360^\circ$

c) $6\cot^2 y + 3\operatorname{cosec}^2 y = 2 + 6\cot y, \quad 0 \leq y < 2\pi$

d) $4\operatorname{cosec}^2 \varphi + \cot^2 \varphi = 1 - 9\operatorname{cosec} \varphi, \quad 0 \leq \varphi < 2\pi$

$$\theta \approx 116.6^\circ, 296.6^\circ, \quad x \approx 48.2^\circ, 311.8^\circ, \quad y \approx 1.25^\circ, 4.39^\circ, \quad \varphi = \frac{7\pi}{6}, \frac{11\pi}{6}$$

<p>(a) $2\tan^2 \theta + 4\tan \theta + 5 = \sec^2 \theta$ $\Rightarrow 2\tan^2 \theta + 4\tan \theta + 5 = 1 + \sec^2 \theta$ $\Rightarrow \tan^2 \theta + 4\tan \theta + 4 = 0$ $\Rightarrow (\tan \theta + 2)^2 = 0$ $\Rightarrow \tan \theta = -2$</p> <p>$2\sec^2 x + 2\tan^2 x = 1 + 4\sec x$ $\Rightarrow 2\sec^2 x + 2\tan^2 x - 1 = 4\sec x$ $\Rightarrow 2\sec^2 x + 2\tan^2 x - 1 = 4\sec x$ $\Rightarrow 2\sec^2 x - 4\sec x - 3 = 0$ $\Rightarrow (2\sec x + 1)(2\sec x - 3) = 0$ $\Rightarrow \sec x = -\frac{1}{2}$ or $\sec x = \frac{3}{2}$</p> <p>$6\cot^2 y + 3\operatorname{cosec}^2 y = 2 + 6\cot y$ $\Rightarrow 6\cot^2 y + 3\operatorname{cosec}^2 y - 2 = 6\cot y$ $\Rightarrow 6\cot^2 y + 3\operatorname{cosec}^2 y - 2 = 6\cot y$ $\Rightarrow 9\operatorname{cosec}^2 y - 6\cot y + 1 = 0$ $\Rightarrow (3\operatorname{cosec} y - 1)^2 = 0$ $\Rightarrow \operatorname{cosec} y = \frac{1}{3}$</p> <p>$4\operatorname{cosec}^2 \varphi + \cot^2 \varphi = 1 - 9\operatorname{cosec} \varphi$ $\Rightarrow 4\operatorname{cosec}^2 \varphi + \cot^2 \varphi - 1 = -9\operatorname{cosec} \varphi$ $\Rightarrow 5\operatorname{cosec}^2 \varphi + \cot^2 \varphi - 2 = 0$ $\Rightarrow (5\operatorname{cosec} \varphi - 1)(\operatorname{cosec} \varphi + 2) = 0$ $\Rightarrow \operatorname{cosec} \varphi = \frac{1}{5}$ or $\operatorname{cosec} \varphi = -2$</p>	<p>$\bullet \operatorname{arctan}(-2) = -63.4^\circ$ $\theta_1 = -63.4^\circ + 180^\circ \quad n=1,2,3,..$ $\theta_1 = 116.6^\circ$ $\theta_2 = 296.6^\circ$</p> <p>$\cos x = \frac{\sqrt{3}}{2}$ $\bullet \operatorname{arccos}\left(\frac{\sqrt{3}}{2}\right) = 48.2^\circ$ $x = 48.2^\circ + 360^\circ \quad n=0,1,2,..$ $x = 311.8^\circ + 360^\circ$ $\Delta = 48.2^\circ, 311.8^\circ$</p> <p>$\tan y = 3$ $\bullet \operatorname{arctan}(3) = 124.9^\circ$ $y = 124.9^\circ + \pi n \quad n=0,1,2,..$ $y = 1.25^\circ, 4.39^\circ$</p> <p>$\sin \varphi = -\frac{1}{5}$ $\bullet \operatorname{arcsin}\left(-\frac{1}{5}\right) = -11.3^\circ$ $\phi = -\frac{11.3^\circ}{5} + 2\pi m \quad n=0,1,2,..$ $\phi = \frac{348.7^\circ}{5} + 2\pi m$ $\frac{\phi}{360^\circ} = \frac{348.7^\circ}{1800^\circ}$ $\frac{\phi}{360^\circ} = \frac{3487}{1800}$</p>
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Question 13

Solve each of the following equations.

a) $10\sec^2 \theta = 11\tan \theta + 16, \quad 0^\circ \leq \theta < 360^\circ$

b) $\cot^2 x = 7 - 2\operatorname{cosec} x, \quad 0^\circ \leq x < 360^\circ$

c) $\sec y = 13 - \frac{\tan^2 y + 16}{\sec y}, \quad 0^\circ \leq y < 360^\circ$

d) $(\operatorname{cosec} \varphi + 1)^2 + 2(\cot \varphi - 1)^2 = 9 - 4\cot \varphi, \quad 0^\circ \leq \varphi < 360^\circ$

$\theta \approx 56.3^\circ, 158.2^\circ, 236.3^\circ, 338.2^\circ, \quad x = 30^\circ, 150^\circ, \quad x \approx 194.5^\circ, 344.5^\circ,$

$y \approx 48.2^\circ, 78.5^\circ, 281.5^\circ, 311.8^\circ, \quad \varphi \approx 48.6^\circ, 131.4^\circ, \quad \varphi = 210^\circ, 330^\circ$

<p>(a) $10\sec^2 \theta = 11\tan \theta + 16$</p> $\Rightarrow 10(1 + \tan^2 \theta) = 11\tan \theta + 16$ $\Rightarrow 10 + 10\tan^2 \theta = 11\tan \theta + 16$ $\Rightarrow 10\tan^2 \theta - 11\tan \theta - 6 = 0$ $\Rightarrow (5\tan \theta + 2)(2\tan \theta - 3) = 0$ $\Rightarrow \tan \theta = -\frac{2}{5} \quad \text{or} \quad \tan \theta = \frac{3}{2}$	<p>$\arctan\left(\frac{3}{2}\right) \approx 56.3^\circ$ $\arctan\left(-\frac{2}{5}\right) \approx -21.8^\circ$ $\therefore \theta = 56.3^\circ \pm 180^\circ$ $\theta = -21.8^\circ \pm 180^\circ$ $\therefore \theta = 56.3^\circ, 158.2^\circ, 236.3^\circ, 338.2^\circ$</p>
<p>(b) $\cot^2 x = 7 - 2\operatorname{cosec} x$</p> $\Rightarrow (\operatorname{cosec} x - 1)^2 = 7 - 2\operatorname{cosec} x$ $\Rightarrow (\operatorname{cosec} x - 1)^2 = 7 - 2\operatorname{cosec} x$ $\Rightarrow (\operatorname{cosec} x - 2)(\operatorname{cosec} x + 1) = 0$ $\Rightarrow \operatorname{cosec} x = 2 \quad \text{or} \quad \operatorname{cosec} x = -1$ $\Rightarrow \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1$ $\Rightarrow x = 30^\circ, 150^\circ, 194.5^\circ, 344.5^\circ$	<p>$\operatorname{cosec}\left(\frac{\pi}{6}\right) = 2$ $\operatorname{cosec}\left(\frac{5\pi}{6}\right) = -1$ $\therefore x = 30^\circ, 150^\circ, 194.5^\circ, 344.5^\circ$</p>
<p>(c) $\sec y = 13 - \frac{\tan^2 y + 16}{\sec y}$</p> $\Rightarrow \sec y = 13\sec y - \tan^2 y - 16$ $\Rightarrow \sec y = 13\sec y - (\sec y - 4)^2$ $\Rightarrow \sec y = 13\sec y - \sec^2 y + 8$ $\Rightarrow 2\sec^2 y - 12\sec y + 15 = 0$ $\Rightarrow (2\sec y - 3)(\sec y - 5) = 0$ $\Rightarrow \sec y = \frac{3}{2} \quad \text{or} \quad \sec y = 5$	<p>$\arccos\left(\frac{3}{2}\right) \text{ is undefined}$ $\arccos\left(\frac{5}{13}\right) \approx 66.4^\circ$ $\therefore y = 66.4^\circ \pm 360^\circ$ $y = 311.6^\circ \pm 360^\circ$ $y = 78.5^\circ \pm 360^\circ$ $y = 281.5^\circ \pm 360^\circ$ $\therefore y = 48.2^\circ, 311.6^\circ, 78.5^\circ, 281.5^\circ$</p>

<p>(d) $(\operatorname{cosec} \varphi + 1)^2 + 2(\cot \varphi - 1)^2 = 9 - 4\cot \varphi$</p> $\Rightarrow (\operatorname{cosec}^2 \varphi + 2\operatorname{cosec} \varphi + 1) + 2(\cot^2 \varphi - 2\cot \varphi + 1) = 9 - 4\cot \varphi$ $\Rightarrow \operatorname{cosec}^2 \varphi + 2\operatorname{cosec} \varphi + 1 + 2\cot^2 \varphi - 4\cot \varphi + 2 = 9 - 4\cot \varphi$ $\Rightarrow \operatorname{cosec}^2 \varphi + 2\operatorname{cosec} \varphi + 2\cot^2 \varphi - 6 = 0$ $\Rightarrow \operatorname{cosec}^2 \varphi + 2(\cot^2 \varphi - 1) + 2\operatorname{cosec} \varphi - 6 = 0$ $\Rightarrow 3\operatorname{cosec}^2 \varphi + 2\cot^2 \varphi - 6 = 0$ $\Rightarrow (3\operatorname{cosec}^2 \varphi - 4)(\operatorname{cosec}^2 \varphi + 2) = 0$ $\Rightarrow \operatorname{cosec}^2 \varphi = \frac{4}{3}$ $\Rightarrow \sin^2 \varphi = \frac{3}{4}$ $\Rightarrow \sin \varphi = \pm \frac{\sqrt{3}}{2}$	<p>$\arcsin\left(\frac{\sqrt{3}}{2}\right) = 60^\circ$ $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -60^\circ$ $\therefore \varphi = 60^\circ, 120^\circ, 210^\circ, 330^\circ$</p>
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Question 14

Solve each of the following equations.

a) $3\tan^2 \theta = 8\sec \theta, \quad 0 \leq \theta < 2\pi$

b) $\operatorname{cosec}^2 x = 2\cot x + 9, \quad 0 \leq x < 2\pi$

c) $\operatorname{cosec}^2 y + 7(1 + \operatorname{cosec} y) + \cot^2 y = 0, \quad 0 \leq y < 2\pi$

d) $6\tan \varphi = \frac{2 - 3\sec^2 \varphi}{\tan \varphi - 1}, \quad 0 \leq \varphi < 2\pi$

$\boxed{\theta \approx 1.23^\circ, 5.05^\circ}, \quad \boxed{x \approx 0.245^\circ, 2.68^\circ, 3.39^\circ, 5.82^\circ}, \quad \boxed{y \approx 3.67^\circ, 3.87^\circ, 5.55^\circ, 5.76^\circ},$

$\boxed{\varphi \approx 0.322^\circ, 3.46^\circ}$

(a) $3\tan^2 \theta = 8\sec \theta$ $\arccos\left(\frac{1}{3}\right) = 1.23^\circ$
 $\Rightarrow 3(\sec \theta - 1) = 8\sec \theta$
 $\Rightarrow 3\sec \theta - 3 = 8\sec \theta$
 $\Rightarrow 3\sec \theta - 8\sec \theta = 3$
 $\Rightarrow (3\sec \theta + 1)(8\sec \theta - 3) = 0$
 $\Rightarrow \sec \theta > -\frac{1}{3}$
 $\Rightarrow \sec \theta > -\frac{1}{3}$
 $\Rightarrow \sec \theta > -\frac{1}{3}$
 $\arccos\left(-\frac{1}{3}\right) = 5.05^\circ$
 $\arccos\left(\frac{1}{3}\right) = 1.23^\circ$
 $\theta_1 = 1.23^\circ$
 $\theta_2 = 5.05^\circ$

(b) $\operatorname{cosec}^2 x = 2\cot x + 9$
 $\Rightarrow 1 + \operatorname{cosec}^2 x = 2\cot x + 9$
 $\Rightarrow \operatorname{cosec}^2 x - 2\cot x - 8 = 0$
 $\Rightarrow (\cot x + 2)(\cot x - 4) = 0$
 $\Rightarrow \cot x > -2$
 $\Rightarrow \cot x > -2$
 $\operatorname{arccot} x = -0.465^\circ$
 $\operatorname{arccot} x = 0.245^\circ$
 $x_1 = -0.465^\circ \pm 180^\circ$
 $x_2 = 0.245^\circ \pm 180^\circ$
 $\therefore x = 0.245^\circ, 2.39^\circ, 2.68^\circ, 5.82^\circ$

(c) $\operatorname{cosec}^2 y + 7(1 + \operatorname{cosec} y) + \cot^2 y = 0$
 $\Rightarrow \operatorname{cosec}^2 y + 7 + 7\operatorname{cosec} y + (\operatorname{cosec} y - 1) = 0$
 $\Rightarrow \operatorname{cosec}^2 y + 7 + 7\operatorname{cosec} y + \operatorname{cosec} y - 1 = 0$
 $\Rightarrow 8\operatorname{cosec}^2 y + 8\operatorname{cosec} y - 6 = 0$
 $\Rightarrow (2\operatorname{cosec} y + 3)(4\operatorname{cosec} y - 2) = 0$
 $\operatorname{cosec} y < -\frac{3}{2}$
 $\operatorname{cosec} y = -\frac{1}{2}$
 $\operatorname{arccosec}\left(-\frac{1}{2}\right) = 3.14^\circ$
 $\operatorname{arccosec}\left(\frac{1}{2}\right) = -0.523^\circ$

(d) $6\tan \varphi = \frac{2 - 3\sec^2 \varphi}{\tan \varphi - 1}$ $\arctan\left(\frac{1}{3}\right) = 0.322^\circ$
 $\Rightarrow 6\tan^2 \varphi - 6\sec^2 \varphi = 2 - 3\sec^2 \varphi$
 $\Rightarrow 6\tan^2 \varphi - 6\sec^2 \varphi = 2 - 3(1 + \tan^2 \varphi)$
 $\Rightarrow 6\tan^2 \varphi - 6\sec^2 \varphi = 2 - 3 - 3\tan^2 \varphi$
 $\Rightarrow 9\tan^2 \varphi - 6\tan^2 \varphi + 1 = 0$
 $\Rightarrow (3\tan^2 \varphi - 1)(3\tan^2 \varphi + 1) = 0$
 $\tan^2 \varphi = \frac{1}{3}$
 $\therefore \tan \varphi = \pm \frac{1}{\sqrt{3}}$
 $\therefore \varphi_1 = 0.322^\circ$
 $\varphi_2 = 3.46^\circ$

Question 15

Solve each of the following equations.

a) $5\tan^2\theta - 12\sec\theta + 9 = 0, \quad 0 \leq \theta < 360^\circ$

b) $4\cot^2 x - 11\operatorname{cosec} x + 1 = 0, \quad 0 \leq x < 360^\circ$

c) $\frac{5 + \tan^2 y}{\sec y} = 9 - \sec y, \quad 0 \leq y < 2\pi$

d) $\frac{\sec^2 \varphi - 2}{\tan \varphi} = \frac{\tan \varphi - 1}{2}, \quad 0 \leq \varphi < 2\pi$

$\theta = 60^\circ, 300^\circ, \quad x \approx 19.5^\circ, 160.5^\circ, \quad y \approx 1.32^\circ, 4.97^\circ$

$\varphi \approx 0.785^\circ, 2.03^\circ, 3.93^\circ, 5.18^\circ$

<p>(a) $5\tan^2\theta - 12\sec\theta + 9 = 0$ $\Rightarrow 5(\sec^2\theta - 1) - 12\sec\theta + 9 = 0$ $\Rightarrow 5\sec^2\theta - 5 - 12\sec\theta + 9 = 0$ $\Rightarrow 5\sec^2\theta - 12\sec\theta + 4 = 0$ $\Rightarrow 5(\sec\theta - 2)(\sec\theta - 1) = 0$ $\Rightarrow \sec\theta = 2 \quad \text{or} \quad \sec\theta = 1$ $\therefore \sec(\frac{1}{2}\theta) = 2$ $\Rightarrow \frac{1}{2}\theta = 60^\circ \quad \text{or} \quad \frac{1}{2}\theta = 300^\circ$ $\Rightarrow \theta = 120^\circ \quad \text{or} \quad \theta = 600^\circ$ $\therefore \theta = 60^\circ, 300^\circ$ </p>	<p>(b) $4\cot^2 x - 11\operatorname{cosec} x + 1 = 0$ $\Rightarrow 4(\operatorname{cosec}^2 x - 1) - 11\operatorname{cosec} x + 1 = 0$ $\Rightarrow 4\operatorname{cosec}^2 x - 4 - 11\operatorname{cosec} x + 1 = 0$ $\Rightarrow 4\operatorname{cosec}^2 x - 11\operatorname{cosec} x - 3 = 0$ $\Rightarrow (4\operatorname{cosec} x + 1)(\operatorname{cosec} x - 3) = 0$ $\Rightarrow \operatorname{cosec} x = -\frac{1}{4}$ $\Rightarrow \sin x = -\frac{1}{4}$ $\therefore \operatorname{arcsin}(\frac{1}{4}) = 15.47^\circ$ $\Rightarrow x = 15.47^\circ + 360^\circ$ $\therefore x = 375.47^\circ$ </p>	<p>(c) $\frac{5 + \tan^2 y}{\sec y} = 9 - \sec y$ $\Rightarrow 5 + \tan^2 y = 9\sec y - \sec^2 y$ $\Rightarrow 5 + (\sec^2 y - 1) = 9\sec y - \sec^2 y$ $\Rightarrow 4 + \sec^2 y = 9\sec y - \sec^2 y$ $\Rightarrow 2\sec^2 y - 9\sec y + 4 = 0$ $\Rightarrow (2\sec y - 1)(\sec y - 4) = 0$ $\Rightarrow \sec y = \frac{1}{2} \quad \text{or} \quad \sec y = 4$ $\operatorname{arccos}(\frac{1}{2}) = 60^\circ$ $\therefore y = 60^\circ$ </p>	<p>(d) $\frac{\sec^2 \varphi - 2}{\tan \varphi} = \frac{\tan \varphi - 1}{2}$ $\Rightarrow \sec^2 \varphi - 2 = \frac{1}{2} \tan^2 \varphi - \frac{1}{2}$ $\Rightarrow 2\sec^2 \varphi - 4 = \tan^2 \varphi - 1$ $\Rightarrow 2(1 + \tan^2 \varphi) - 4 = \tan^2 \varphi - 1$ $\Rightarrow \tan^2 \varphi + 2 = 3$ $\Rightarrow (\tan \varphi - 1)(\tan \varphi + 2) = 0$ $\therefore \tan \varphi = 1 \quad \text{or} \quad \tan \varphi = -2$ $\operatorname{arctan}(1) = \frac{\pi}{4} \approx 0.785$ $\operatorname{arctan}(-2) \approx -1.57$ $\therefore \varphi = 0.785^\circ \pm 180^\circ$ $\therefore \varphi = -1.57^\circ \pm 180^\circ$ $\therefore \varphi = 0.785^\circ, 2.03^\circ, 3.93^\circ, 5.18^\circ$ </p>
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Question 16

Solve each of the following equations.

a) $\sec \theta = \frac{1 - \tan^2 \theta}{4 \sec \theta - 9}$, $0 \leq \theta < 2\pi$

b) $\frac{\sec^2 x + 8}{4 - \tan x} = 3 \tan x$, $0 \leq x < 2\pi$

c) $\frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} - 2 = \cot y$, $0 \leq y < 2\pi$

d) $\frac{2 \cot^2 \varphi + 5}{\operatorname{cosec} \varphi} + 2 \operatorname{cosec} \varphi = 13$, $0 \leq \varphi < 2\pi$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$, $x \approx 0.983^\circ, 4.12^\circ$, $y \approx 2.03^\circ, 5.18^\circ$, $\varphi \approx 0.340^\circ, 2.80^\circ$

<p>(a) $\sec \theta = \frac{1 - \tan^2 \theta}{4 \sec \theta - 9}$</p> $\Rightarrow 4 \sec \theta - 9 \sec \theta = 1 - \tan^2 \theta$ $\Rightarrow 4 \sec \theta - 9 \sec \theta = 1 - (\sec^2 \theta)$ $\Rightarrow \sec \theta - 9 \sec \theta = 1 + \sec^2 \theta$ $\Rightarrow \sec \theta - 9 \sec \theta = 2 + \sec^2 \theta$ $\Rightarrow (\sec \theta + 1)(\sec \theta - 2) = 0$ $\Rightarrow \sec \theta + 1 = 0$ (reject) $\Rightarrow \sec \theta - 2 = 0$ $\Rightarrow \sec \theta = 2$ (reject) $\Rightarrow \sec \theta = \sqrt{2}$ or $\sec \theta = -\sqrt{2}$ <p>$\sec \theta = \frac{1}{2}$, $\operatorname{cosec} \theta = \frac{1}{\sqrt{2}}$</p> <p>$\theta = \frac{\pi}{3} \pm 2m^\circ$ or $\theta = 0, 12, 3, \dots$</p> <p>$\theta = \frac{\pi}{6} \pm 2m^\circ$</p> <p>$\theta = \frac{\pi}{3} \pm 2m^\circ$</p>	<p>(b) $\frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} - 2 = \cot y$</p> $\Rightarrow \frac{-2 \operatorname{cosec}^2 y}{2 \cot y} = 2 + \cot y$ $\Rightarrow -2 \operatorname{cosec}^2 y = 4 \cot y + 2 \cot^2 y$ $\Rightarrow 1 - 2 \operatorname{cosec}^2 y = 4 \cot y + 2 \cot^2 y$ $\Rightarrow 1 - 2 \operatorname{cosec}^2 y = 4 \cot y + 2 \cot^2 y$ $\Rightarrow 0 = 4 \cot^2 y + 4 \cot y + 1$ $\Rightarrow 0 = (2 \cot y + 1)^2$ $\Rightarrow 2 \cot y + 1 = 0$ $\Rightarrow 2 \cot y = -1$ $\Rightarrow \cot y = -\frac{1}{2}$ $\Rightarrow y = 107^\circ \pm 2m^\circ$ or $y = 1, 3, 5, \dots$ <p>$y = 107^\circ \pm 2m^\circ$</p> <p>$y = 1, 3, 5, \dots$</p>	<p>(c) $\frac{2 \cot^2 \varphi + 5}{\operatorname{cosec} \varphi} + 2 \operatorname{cosec} \varphi = 13$</p> $\Rightarrow \frac{2 \cot^2 \varphi + 5 + 2 \operatorname{cosec}^2 \varphi}{\operatorname{cosec} \varphi} = 13$ $\Rightarrow 2 \cot^2 \varphi + 5 + 2 \operatorname{cosec}^2 \varphi = 13 \operatorname{cosec} \varphi$ $\Rightarrow 2 \cot^2 \varphi + 3 + 2 \operatorname{cosec}^2 \varphi - 13 \operatorname{cosec} \varphi = 0$ $\Rightarrow 4 \cot^2 \varphi - 13 \operatorname{cosec} \varphi + 3 = 0$ $\Rightarrow (4 \cot^2 \varphi - 1)(\operatorname{cosec} \varphi - 3) = 0$ $\Rightarrow \operatorname{cosec} \varphi = \frac{3}{\sqrt{2}}$ or $\operatorname{cosec} \varphi = \frac{1}{\sqrt{2}}$ <p>$\operatorname{cosec} \varphi = \frac{3}{\sqrt{2}}$</p> <p>$\varphi = 0.340^\circ \pm 2m^\circ$</p> <p>$\varphi = 0.340^\circ$</p>
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Question 17

Solve each of the following trigonometric equations.

a) $\sec \theta = \frac{1 - \tan^2 \theta}{4 \sec \theta - 9}$, $0 \leq \theta < 2\pi$

b) $\frac{\sec^2 x + 8}{4 - \tan x} = 3 \tan x$, $0 \leq x < 2\pi$

c) $\frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} - 2 = \cot y$, $0 \leq y < 2\pi$

d) $\frac{2 \cot^2 \varphi + 5}{\operatorname{cosec} \varphi} + 2 \operatorname{cosec} \varphi = 13$, $0 \leq \varphi < 2\pi$

$\boxed{\theta = \frac{\pi}{3}, \frac{5\pi}{3}}$, $\boxed{x = 0.983^\circ, 4.12^\circ}$, $\boxed{y = 2.03^\circ, 5.18^\circ}$, $\boxed{\varphi = 0.340^\circ, 2.80^\circ}$

<p>(a) $\sec \theta = \frac{1 - \tan^2 \theta}{4 \sec \theta - 9}$</p> $\Rightarrow 4 \sec \theta - 9 \sec \theta = 1 - \tan^2 \theta$ $\Rightarrow 4 \sec \theta - 9 \sec \theta = 1 - (\sec^2 \theta)$ $\Rightarrow \sec \theta - 9 \sec \theta = 1 + \sec^2 \theta$ $\Rightarrow \sec^2 \theta - 9 \sec \theta = 1 + 0$ $\Rightarrow (\sec \theta + 1)(\sec \theta - 9) = 0$ $\Rightarrow \sec \theta + 1 = 0$ or $\sec \theta - 9 = 0$ $\Rightarrow \sec \theta = -1$ or $\sec \theta = 9$ $\Rightarrow \theta = \frac{\pi}{2} \text{ or } \theta = \arccos \left(\frac{1}{9} \right)$ $\theta = \frac{\pi}{2} \text{ or } \theta = 2\pi - \arccos \left(\frac{1}{9} \right)$ $\theta = \frac{\pi}{2} \pm 2m^\circ \quad n=0,1,2,3,\dots$ $\theta = \frac{\pi}{2} \pm 2m^\circ \quad n=0,1,2,3,\dots$	<p>(b) $\frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} - 2 = \cot y$</p> $\Rightarrow 1 - 2 \operatorname{cosec}^2 y = 2 + \cot y$ $\Rightarrow 1 - 2 \operatorname{cosec}^2 y = 4 \cot y + 2 \operatorname{cosec}^2 y$ $\Rightarrow 1 - 2 \operatorname{cosec}^2 y = 4 \operatorname{cot}^2 y + 2 \operatorname{cosec}^2 y$ $\Rightarrow 0 = 4 \operatorname{cot}^2 y + 4 \operatorname{cosec}^2 y + 1$ $\Rightarrow 0 = (2 \operatorname{cot} y)^2 + 4 \operatorname{cosec}^2 y + 1$ $\operatorname{cot} y = \frac{1}{2}$ $\operatorname{cosec} y = 2$ $\arctan(\cdot) = -1.107$ $y = -1.107 \pm 2m^\circ \quad n=0,1,2,3,\dots$ $y = 2.03^\circ$ $y_2 = 5.18^\circ$
<p>(c) $\frac{\sec^2 \varphi + 5}{\operatorname{cosec} \varphi} + 2 \operatorname{cosec} \varphi = 13$</p> $\Rightarrow \sec^2 \varphi + 5 = 13 \operatorname{cosec} \varphi$ $\Rightarrow (1 + \operatorname{cosec} \varphi) \cdot 8 = 13 \operatorname{cosec} \varphi$ $\Rightarrow 8 \operatorname{cosec} \varphi - 12 \operatorname{cosec} \varphi + 8 = 0$ $\Rightarrow (2 \operatorname{cosec} \varphi - 3)^2 = 0$ $\operatorname{cosec} \varphi = \frac{3}{2}$ $\arctan(\cdot) = 0.983^\circ$ $\varphi = 0.983^\circ \pm 2m^\circ \quad n=0,1,2,3,\dots$ $\varphi_1 = 0.983^\circ$ $\varphi_2 = 4.12^\circ$	<p>(d) $\frac{2 \cot^2 \varphi + 5}{\operatorname{cosec} \varphi} + 2 \operatorname{cosec} \varphi = 13$</p> $\Rightarrow 2 \operatorname{cot}^2 \varphi + 5 + 2 \operatorname{cosec}^2 \varphi = 13 \operatorname{cosec} \varphi$ $\Rightarrow 2 (\operatorname{cot}^2 \varphi + 1) + 2 \operatorname{cosec}^2 \varphi - 13 \operatorname{cosec} \varphi = 0$ $\Rightarrow 2 \operatorname{cot}^2 \varphi + 3 + 2 \operatorname{cosec}^2 \varphi - 13 \operatorname{cosec} \varphi = 0$ $\Rightarrow 4 \operatorname{cot}^2 \varphi - 8 \operatorname{cosec} \varphi + 3 = 0$ $\Rightarrow (4 \operatorname{cot}^2 \varphi - 1) \cdot (\operatorname{cosec} \varphi - 3) = 0$ $\operatorname{cot} \varphi = \frac{1}{2}$ or $\operatorname{cosec} \varphi = \frac{1}{3}$ $\arctan(\cdot) = 0.340^\circ$ $\varphi = 0.340^\circ \pm 2m^\circ \quad n=0,1,2,3,\dots$ $\varphi_1 = 0.340^\circ$ $\varphi_2 = 2.80^\circ$

Question 17

Solve each of the following trigonometric equations.

a) $\cos(\theta + 30^\circ) = \sin \theta$, $0 \leq \theta < 360^\circ$

b) $3\cos(x + 30^\circ) = \sin(x - 60^\circ)$, $0 \leq x < 360^\circ$

c) $\sin(y - 30^\circ) = \sin(y + 45^\circ)$, $0 \leq y < 360^\circ$

d) $\sin(\varphi + 30^\circ) = \cos(\varphi - 45^\circ)$, $0 \leq \varphi < 360^\circ$

e) $\cos(\alpha - 60^\circ) = \cos(\alpha - 45^\circ)$, $0 \leq \alpha < 360^\circ$

$$\boxed{\theta = 30^\circ, 210^\circ}, \boxed{x = 60^\circ, 240^\circ}, \boxed{y = 82.5^\circ, 262.5^\circ}, \boxed{\varphi = 52.5^\circ, 232.5^\circ}, \boxed{\alpha = 52.5^\circ, 232.5^\circ}$$

<p>(a) $\cos(\theta + 30^\circ) = \sin \theta$</p> $\Rightarrow \cos \theta \cos 30^\circ - \sin \theta \sin 30^\circ = \sin \theta$ $\Rightarrow \frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta = \sin \theta$ $\Rightarrow \sqrt{3} \cos \theta - \sin \theta = 2 \sin \theta$ $\Rightarrow \sqrt{3} \cos \theta = 3 \sin \theta$ $\Rightarrow \sqrt{3} = 3 \tan \theta$ $\Rightarrow \tan \theta = \frac{\sqrt{3}}{3}$ $\Rightarrow \arctan(\frac{\sqrt{3}}{3}) = 30^\circ$ $\Rightarrow \theta = 30^\circ \text{ or } 180^\circ + 30^\circ = 210^\circ$ $\therefore \theta = 30^\circ, 210^\circ$	<p>(b) $\sin(\varphi + 30^\circ) = \cos(\varphi - 45^\circ)$</p> $\Rightarrow \sin \varphi \cos 30^\circ + \cos \varphi \sin 30^\circ = \cos \varphi \cos 45^\circ + \sin \varphi \sin 45^\circ$ $\Rightarrow \frac{\sqrt{3}}{2} \sin \varphi + \frac{1}{2} \cos \varphi = \frac{\sqrt{2}}{2} \cos \varphi + \frac{\sqrt{2}}{2} \sin \varphi$ $\Rightarrow \frac{\sqrt{3}}{2} \sin \varphi + \cos \varphi = \frac{\sqrt{2}}{2} \cos \varphi + \frac{\sqrt{2}}{2} \sin \varphi$ $\Rightarrow \frac{\sqrt{3}}{2} \sin \varphi - \frac{\sqrt{2}}{2} \sin \varphi = \frac{\sqrt{2}}{2} \cos \varphi - \frac{\sqrt{3}}{2} \cos \varphi$ $\Rightarrow \frac{\sqrt{3}-\sqrt{2}}{2} \sin \varphi = \frac{\sqrt{2}-\sqrt{3}}{2} \cos \varphi$ $\Rightarrow (\sqrt{3}-\sqrt{2}) \sin \varphi = (\sqrt{2}-\sqrt{3}) \cos \varphi$ $\Rightarrow (\sqrt{3}-\sqrt{2})^2 \sin^2 \varphi = (\sqrt{2}-\sqrt{3})^2 \cos^2 \varphi$ $\Rightarrow 1 - 2\sqrt{6} \sin \varphi + 2\sin^2 \varphi = 1 - 2\sqrt{6} \cos \varphi + 2\cos^2 \varphi$ $\Rightarrow 2\sin^2 \varphi - 2\cos^2 \varphi = 2\sqrt{6} \cos \varphi - 2\sqrt{6} \sin \varphi$ $\Rightarrow 2(\sin^2 \varphi - \cos^2 \varphi) = 2\sqrt{6}(\cos \varphi - \sin \varphi)$ $\Rightarrow \sin^2 \varphi - \cos^2 \varphi = \sqrt{6}(\cos \varphi - \sin \varphi)$ $\Rightarrow \tan^2 \varphi = \sqrt{6} \tan \varphi$ $\Rightarrow \tan \varphi = \sqrt{6}$ $\Rightarrow \arctan(\sqrt{6}) = 52.5^\circ$ $\Rightarrow \varphi = 52.5^\circ \text{ or } 180^\circ + 52.5^\circ = 232.5^\circ$ $\therefore \varphi = 52.5^\circ, 232.5^\circ$
<p>(c) $\sin(y - 30^\circ) = \sin(y + 45^\circ)$</p> $\Rightarrow \sin y \cos 30^\circ - \cos y \sin 30^\circ = \sin y \cos 45^\circ + \cos y \sin 45^\circ$ $\Rightarrow \frac{\sqrt{3}}{2} \sin y - \frac{1}{2} \cos y = \frac{\sqrt{2}}{2} \sin y + \frac{\sqrt{2}}{2} \cos y$ $\Rightarrow \frac{\sqrt{3}}{2} \sin y - \frac{\sqrt{2}}{2} \sin y = \frac{\sqrt{2}}{2} \cos y + \frac{\sqrt{3}}{2} \cos y$ $\Rightarrow \frac{\sqrt{3}-\sqrt{2}}{2} \sin y = \frac{\sqrt{2}+\sqrt{3}}{2} \cos y$ $\Rightarrow (\sqrt{3}-\sqrt{2}) \sin y = (\sqrt{2}+\sqrt{3}) \cos y$ $\Rightarrow (\sqrt{3}-\sqrt{2})^2 \sin^2 y = (\sqrt{2}+\sqrt{3})^2 \cos^2 y$ $\Rightarrow 1 - 2\sqrt{6} \sin y + 2\sin^2 y = 1 + 2\sqrt{6} \cos y + 2\cos^2 y$ $\Rightarrow 2\sin^2 y - 2\cos^2 y = 2\sqrt{6} \cos y + 2\sqrt{6} \sin y$ $\Rightarrow 2(\sin^2 y - \cos^2 y) = 2\sqrt{6}(\cos y + \sin y)$ $\Rightarrow \sin^2 y - \cos^2 y = \sqrt{6}(\cos y + \sin y)$ $\Rightarrow \tan^2 y = \sqrt{6} \tan y$ $\Rightarrow \tan y = \sqrt{6}$ $\Rightarrow \arctan(\sqrt{6}) = 82.5^\circ$ $\Rightarrow y = 82.5^\circ \text{ or } 180^\circ + 82.5^\circ = 262.5^\circ$ $\therefore y = 82.5^\circ, 262.5^\circ$	<p>(d) $\sin(\varphi + 30^\circ) = \cos(\varphi - 45^\circ)$</p> $\Rightarrow \sin \varphi \cos 30^\circ + \cos \varphi \sin 30^\circ = \cos \varphi \cos 45^\circ + \sin \varphi \sin 45^\circ$ $\Rightarrow \frac{\sqrt{3}}{2} \sin \varphi + \frac{1}{2} \cos \varphi = \frac{\sqrt{2}}{2} \cos \varphi + \frac{\sqrt{2}}{2} \sin \varphi$ $\Rightarrow \frac{\sqrt{3}}{2} \sin \varphi + \cos \varphi = \frac{\sqrt{2}}{2} \cos \varphi + \frac{\sqrt{2}}{2} \sin \varphi$ $\Rightarrow \frac{\sqrt{3}}{2} \sin \varphi - \frac{\sqrt{2}}{2} \sin \varphi = \frac{\sqrt{2}}{2} \cos \varphi - \frac{\sqrt{3}}{2} \cos \varphi$ $\Rightarrow \frac{\sqrt{3}-\sqrt{2}}{2} \sin \varphi = \frac{\sqrt{2}-\sqrt{3}}{2} \cos \varphi$ $\Rightarrow (\sqrt{3}-\sqrt{2}) \sin \varphi = (\sqrt{2}-\sqrt{3}) \cos \varphi$ $\Rightarrow (\sqrt{3}-\sqrt{2})^2 \sin^2 \varphi = (\sqrt{2}-\sqrt{3})^2 \cos^2 \varphi$ $\Rightarrow 1 - 2\sqrt{6} \sin \varphi + 2\sin^2 \varphi = 1 - 2\sqrt{6} \cos \varphi + 2\cos^2 \varphi$ $\Rightarrow 2\sin^2 \varphi - 2\cos^2 \varphi = 2\sqrt{6} \cos \varphi - 2\sqrt{6} \sin \varphi$ $\Rightarrow 2(\sin^2 \varphi - \cos^2 \varphi) = 2\sqrt{6}(\cos \varphi - \sin \varphi)$ $\Rightarrow \sin^2 \varphi - \cos^2 \varphi = \sqrt{6}(\cos \varphi - \sin \varphi)$ $\Rightarrow \tan^2 \varphi = \sqrt{6} \tan \varphi$ $\Rightarrow \tan \varphi = \sqrt{6}$ $\Rightarrow \arctan(\sqrt{6}) = 52.5^\circ$ $\Rightarrow \varphi = 52.5^\circ \text{ or } 180^\circ + 52.5^\circ = 232.5^\circ$ $\therefore \varphi = 52.5^\circ, 232.5^\circ$
<p>(e) $\cos(\alpha - 60^\circ) = \cos(\alpha - 45^\circ)$</p> $\Rightarrow \cos \alpha \cos 60^\circ + \sin \alpha \sin 60^\circ = \cos \alpha \cos 45^\circ + \sin \alpha \sin 45^\circ$ $\Rightarrow \frac{1}{2} \cos \alpha + \frac{\sqrt{3}}{2} \sin \alpha = \frac{\sqrt{2}}{2} \cos \alpha + \frac{\sqrt{2}}{2} \sin \alpha$ $\Rightarrow \frac{1}{2} \cos \alpha - \frac{\sqrt{2}}{2} \cos \alpha = \frac{\sqrt{2}}{2} \sin \alpha - \frac{\sqrt{3}}{2} \sin \alpha$ $\Rightarrow \frac{1-\sqrt{2}}{2} \cos \alpha = \frac{\sqrt{2}-\sqrt{3}}{2} \sin \alpha$ $\Rightarrow (\sqrt{2}-\sqrt{3}) \cos \alpha = (\sqrt{3}-\sqrt{2}) \sin \alpha$ $\Rightarrow (\sqrt{2}-\sqrt{3})^2 \cos^2 \alpha = (\sqrt{3}-\sqrt{2})^2 \sin^2 \alpha$ $\Rightarrow 1 - 2\sqrt{6} \cos \alpha + 2\cos^2 \alpha = 1 - 2\sqrt{6} \sin \alpha + 2\sin^2 \alpha$ $\Rightarrow 2\cos^2 \alpha - 2\sin^2 \alpha = 2\sqrt{6} \sin \alpha - 2\sqrt{6} \cos \alpha$ $\Rightarrow 2(\cos^2 \alpha - \sin^2 \alpha) = 2\sqrt{6}(\sin \alpha - \cos \alpha)$ $\Rightarrow \cos^2 \alpha - \sin^2 \alpha = \sqrt{6}(\sin \alpha - \cos \alpha)$ $\Rightarrow \tan^2 \alpha = \sqrt{6} \tan \alpha$ $\Rightarrow \tan \alpha = \sqrt{6}$ $\Rightarrow \arctan(\sqrt{6}) = 52.5^\circ$ $\Rightarrow \alpha = 52.5^\circ \text{ or } 180^\circ + 52.5^\circ = 232.5^\circ$ $\therefore \alpha = 52.5^\circ, 232.5^\circ$	

Question 18

Solve each of the following trigonometric equations.

a) $\sin(\theta - 45^\circ) = \sin \theta$, $0 \leq \theta < 360^\circ$

b) $\cos(x - 30^\circ) = \sin(x + 30^\circ)$, $0 \leq x < 360^\circ$

c) $\cos(y - 30^\circ) = \sin(y + 45^\circ)$, $0 \leq y < 360^\circ$

d) $\sin(\varphi - 30^\circ) = \cos(\varphi - 45^\circ)$, $0 \leq \varphi < 360^\circ$

e) $\cos(\alpha - 60^\circ) = \cos(\alpha + 60^\circ)$, $0 \leq \alpha < 360^\circ$

$\theta = 112.5^\circ, 292.5^\circ$, $x = 45^\circ, 225^\circ$, $y = 37.5^\circ, 217.5^\circ$, $\varphi = 82.5^\circ, 262.5^\circ$,

$\alpha = 0^\circ, 180^\circ$

(a) $\sin(\theta - 45^\circ) = \sin \theta$

$$\Rightarrow \sin \theta \cos 45^\circ - \cos \theta \sin 45^\circ = \sin \theta$$

$$\Rightarrow \sin \theta \cdot \frac{\sqrt{2}}{2} - \cos \theta \cdot \frac{\sqrt{2}}{2} = \sin \theta$$

$$\Rightarrow \frac{\sqrt{2}}{2} \sin \theta - \frac{\sqrt{2}}{2} \cos \theta = 2\sin \theta$$

$$\Rightarrow \frac{\sqrt{2}\sin \theta}{\cos \theta} - \frac{\sqrt{2}\cos \theta}{\cos \theta} = 2\sin \theta$$

$$\Rightarrow \sqrt{2} \tan \theta - \sqrt{2} = 2\sin \theta$$

$$\Rightarrow \sqrt{2} \tan \theta - 2 = 2\sin \theta$$

$$\Rightarrow (\sqrt{2}-2)\tan \theta = 2$$

$$\Rightarrow \tan \theta = \frac{2}{\sqrt{2}-2}$$

$$\text{arctan} \left(\frac{2}{\sqrt{2}-2} \right) = -75^\circ$$

$$0 - (-75)^\circ \pm (360^\circ) n = 0, 175^\circ, \dots$$

$$\therefore \theta_1 = 175^\circ$$

$$\theta_2 = 212.5^\circ$$

(c) $\cos(y - 30^\circ) = \sin(y + 45^\circ)$

$$\Rightarrow \cos y \cos 30^\circ - \sin y \sin 30^\circ = \sin y \cos 45^\circ + \cos y \sin 45^\circ$$

$$\Rightarrow \frac{\sqrt{3}}{2} \cos y - \frac{1}{2} \sin y = \frac{\sqrt{2}}{2} \sin y + \frac{\sqrt{2}}{2} \cos y$$

$$\Rightarrow \frac{\sqrt{3}}{2} \cos y - \frac{\sqrt{2}}{2} \cos y = \frac{\sqrt{2}}{2} \sin y + \frac{1}{2} \sin y$$

$$\Rightarrow \frac{\sqrt{3}-\sqrt{2}}{2} \cos y = \frac{\sqrt{2}+1}{2} \sin y$$

$$\Rightarrow \sqrt{2} + \sin y = \sqrt{3-\sqrt{2}} \cos y + \sqrt{2}$$

$$\Rightarrow \tan y = \frac{\sqrt{3-\sqrt{2}} \cos y + \sqrt{2}}{\sqrt{2} + \sin y}$$

$$\Rightarrow \tan y = (\sqrt{3-\sqrt{2}})^2 + \sqrt{2}$$

$$\Rightarrow \tan y = \frac{\sqrt{3-\sqrt{2}} \sqrt{3+\sqrt{2}}}{\sqrt{2}-1}$$

$$\text{arctan} \left(\frac{\sqrt{3-\sqrt{2}} \sqrt{3+\sqrt{2}}}{\sqrt{2}-1} \right) = 37.5^\circ$$

$$y_1 = 37.5^\circ \pm 180^\circ n = 0, 175^\circ, \dots$$

$$y_2 = 212.5^\circ$$

(d) $\sin(\varphi - 30^\circ) = \cos(\varphi - 45^\circ)$

$$\Rightarrow \sin \varphi \cos 30^\circ - \cos \varphi \sin 30^\circ = \cos \varphi \cos 45^\circ + \sin \varphi \sin 45^\circ$$

$$\Rightarrow \frac{\sqrt{3}}{2} \sin \varphi - \frac{1}{2} \cos \varphi = \frac{\sqrt{2}}{2} \cos \varphi + \frac{\sqrt{2}}{2} \sin \varphi$$

$$\Rightarrow \frac{\sqrt{3}}{2} \sin \varphi - \frac{\sqrt{2}}{2} \sin \varphi = \frac{\sqrt{2}}{2} \cos \varphi + \frac{1}{2} \cos \varphi$$

$$\Rightarrow \frac{\sqrt{3}-\sqrt{2}}{2} \sin \varphi = \frac{\sqrt{2}+1}{2} \cos \varphi$$

$$\Rightarrow \sqrt{2} + \cos \varphi = \sqrt{3-\sqrt{2}} \sin \varphi + \sqrt{2}$$

$$\Rightarrow \sqrt{2} \cos \varphi - \sqrt{2} = \sqrt{3-\sqrt{2}} \sin \varphi$$

$$\Rightarrow \sqrt{2} \cos \varphi - 1 = \sqrt{2} + \sqrt{2} \sin \varphi$$

$$\Rightarrow \sqrt{2} \cos \varphi - \sqrt{2} \sin \varphi = \sqrt{2} + 1$$

$$\Rightarrow \tan(\varphi - 45^\circ) = \sqrt{2} + 1$$

$$\Rightarrow \tan \varphi \cos 45^\circ - \cos \varphi \sin 45^\circ = \sqrt{2} + 1$$

$$\Rightarrow \tan \varphi \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} = \sqrt{2} + 1$$

$$\text{arctan} \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right) = 82.5^\circ$$

$$\varphi_1 = 82.5^\circ + 180^\circ n = 0, 175^\circ, \dots$$

$$\varphi_2 = 262.5^\circ$$

(b) $\cos(\alpha - 60^\circ) = \cos(\alpha + 60^\circ)$

$$\Rightarrow \cos \alpha \cos 60^\circ + \sin \alpha \sin 60^\circ = \cos \alpha \cos 60^\circ - \sin \alpha \sin 60^\circ$$

$$\Rightarrow 2\sin \alpha \sin 60^\circ = 0$$

$$\Rightarrow \sqrt{3} \sin \alpha = 0$$

$$\Rightarrow \sin \alpha = 0$$

$$\alpha_1 = 0^\circ$$

$$(\alpha_2 = 180^\circ \pm 360^\circ n = 180^\circ, 360^\circ, \dots)$$

$$\alpha_3 = 0^\circ$$

$$\alpha_4 = 180^\circ$$

Question 19

Solve each of the following trigonometric equations.

a) $\sin\left(\theta + \frac{\pi}{4}\right) = \sin\theta, \quad 0 \leq \theta < 2\pi$

b) $\cos\left(x + \frac{\pi}{6}\right) = \cos\left(x + \frac{2\pi}{3}\right), \quad 0 \leq x < 2\pi$

c) $\sin\left(\frac{\pi}{3} - y\right) = \cos\left(y + \frac{5\pi}{6}\right), \quad 0 \leq y < 2\pi \text{ (very hard)}$

d) $2\cos\left(\varphi + \frac{\pi}{2}\right) + \sin\left(\varphi + \frac{\pi}{3}\right) = 0, \quad 0 \leq \varphi < 2\pi$

e) $\sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{6}\right), \quad 0 \leq \alpha < 2\pi$

$$\boxed{\theta = \frac{3\pi}{8}, \frac{11\pi}{8}}, \boxed{x = \frac{7\pi}{12}, \frac{19\pi}{12}}, \boxed{y = \frac{\pi}{2}, \frac{3\pi}{2}}, \boxed{\varphi = \frac{\pi}{6}, \frac{7\pi}{6}}, \boxed{\alpha = \frac{\pi}{12}, \frac{13\pi}{12}}$$

(a) $\sin\left(\theta + \frac{\pi}{4}\right) = \sin\theta$
 $\Rightarrow \sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4} = \sin\theta$
 $\Rightarrow \sin\theta \cdot \frac{\sqrt{2}}{2} + \cos\theta \cdot \frac{\sqrt{2}}{2} = \sin\theta$
 $\Rightarrow \sqrt{2}\sin\theta + \sqrt{2}\cos\theta = 2\sin\theta$
 $\Rightarrow (\sqrt{2}\sin\theta + \sqrt{2})\cos\theta = 2\sin\theta\cos\theta$
 $\Rightarrow \sqrt{2}^2 \sin\theta\cos\theta + \sqrt{2}^2 \cos^2\theta = 2\sin\theta\cos\theta$
 $\Rightarrow 2\sin\theta\cos\theta + \cos^2\theta = 0$
 $\Rightarrow \tan\theta + \frac{\cos\theta}{\sin\theta} = 0$
 $\Rightarrow \tan\theta + \frac{\sqrt{2}}{\sqrt{2}\sin\theta} = 0$
 $\Rightarrow \tan\theta + \frac{1}{\sin\theta} = 0$
 $\Rightarrow \theta = \frac{\pi}{4}k \pm \pi n, \quad n=0,1,2,\dots$
 $\theta_1 = \frac{\pi}{4}, \quad \theta_2 = \frac{5\pi}{4}$

(b) $\cos\left(2 + \frac{\pi}{6}\right) = \cos\left(2 + \frac{2\pi}{3}\right)$
 $\Rightarrow \cos\left(2\cos\frac{\pi}{3} - 2\sin\frac{\pi}{3}\right) = \cos\left(2\cos\frac{\pi}{3} - 2\sin\frac{\pi}{3}\right)$
 $\Rightarrow \cos\left(2 \cdot \frac{1}{2} - 2 \cdot \frac{\sqrt{3}}{2}\right) = \cos\left(2 \cdot \frac{1}{2} - 2 \cdot \frac{\sqrt{3}}{2}\right)$
 $\Rightarrow \sqrt{3}\cos 2 - \sin 2 = -(\cos 2 - \sqrt{3}\sin 2)$
 $\Rightarrow (\sqrt{3}-1)\cos 2 = -1 - \sqrt{3}\sin 2$
 $\Rightarrow \tan 2 = \frac{-1 - \sqrt{3}\sin 2}{\sqrt{3}-1} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$
 $\Rightarrow \arctan\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right) = -\frac{\pi}{12}$
 $\Rightarrow 2 = \frac{-\pi}{12} + k\pi, \quad k \in \mathbb{Z}, \lambda, \dots$
 $\Rightarrow \theta_1 = \frac{\pi}{12}, \quad \theta_2 = \frac{13\pi}{12}$

(c) $\sin\left(\frac{\pi}{3} - y\right) = \cos\left(y + \frac{5\pi}{6}\right)$

$\Rightarrow \sin\frac{\pi}{3}\cos y - \cos\frac{\pi}{3}\sin y = \cos y\cos\frac{5\pi}{6} - \sin y\sin\frac{5\pi}{6}$

$\Rightarrow \frac{\sqrt{3}}{2}\cos y - \frac{1}{2}\sin y = -\frac{\sqrt{3}}{2}\cos y - \frac{1}{2}\sin y$

$\Rightarrow \frac{3\sqrt{3}}{2}\cos y - \sin y = -\frac{\sqrt{3}}{2}\cos y - \sin y$

$\Rightarrow \sqrt{3}\cos y = -\sqrt{3}\cos y$

$\Rightarrow \cos y = 0$

$\Rightarrow y = \frac{\pi}{2} + n\pi, \quad n=0,1,2,\dots$

$y_1 = \frac{\pi}{2}, \quad y_2 = \frac{3\pi}{2}$

(d) $2\cos\left(\varphi + \frac{\pi}{2}\right) + \sin\left(\varphi + \frac{\pi}{3}\right) = 0$

$\Rightarrow 2\cos\left(\varphi + \frac{\pi}{2}\right) - 2\sin\left(\varphi + \frac{\pi}{3}\right) + \sin\left(\varphi + \frac{\pi}{3}\right) + \cos\left(\varphi + \frac{\pi}{2}\right) = 0$

$\Rightarrow -2\sin\varphi + \frac{1}{2}\sin\varphi + \frac{\sqrt{3}}{2}\cos\varphi = 0$

$\Rightarrow \frac{3}{2}\sin\varphi + \frac{\sqrt{3}}{2}\cos\varphi = 0$

$\Rightarrow \sqrt{3}\sin\varphi = -\cos\varphi$

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{\cos\varphi}{\sin\varphi}$

$\Rightarrow \tan\varphi = \frac{\sqrt{3}}{2}$

$\Rightarrow \arctan\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

$\Rightarrow \varphi = \frac{\pi}{6} + n\pi, \quad n=0,1,2,\dots$

$\varphi_1 = \frac{\pi}{6}, \quad \varphi_2 = \frac{7\pi}{6}$

(e) $\sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{6}\right)$

$\Rightarrow \sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right) - \sin\left(\alpha + \frac{\pi}{6}\right) = 0$

$\Rightarrow \cos\alpha - \sin\alpha = \frac{\sqrt{2}}{2}\cos\alpha + \frac{1}{2}\sin\alpha$

$\Rightarrow 2\cos\alpha - 2\sin\alpha = \sqrt{3}\sin\alpha + \cos\alpha$

$\Rightarrow \cos\alpha = (\sqrt{3}+1)\sin\alpha$

$\Rightarrow 1 = (\sqrt{3}+1)\tan\alpha$

$\Rightarrow \tan\alpha = \frac{1}{\sqrt{3}+1}$

$\Rightarrow \arctan\left(\frac{1}{\sqrt{3}+1}\right) = \frac{\pi}{12}$

$\Rightarrow \alpha' = \frac{\pi}{12} + k\pi, \quad k \in \mathbb{Z}, \lambda, \dots$

$\alpha'_1 = \frac{\pi}{12}, \quad \alpha'_2 = \frac{13\pi}{12}$

Question 20

Solve each of the following trigonometric equations.

a) $\sin(\theta - 20^\circ) = \sin(\theta + 60^\circ)$, $0 \leq \theta < 360^\circ$

b) $\cos(x - 35^\circ) = \cos(x - 55^\circ)$, $0 \leq x < 360^\circ$

c) $\sin(y - 48^\circ) = \cos(y + 12^\circ)$, $0 \leq y < 360^\circ$

d) $\sin(\varphi + 72^\circ) = \cos(\varphi - 38^\circ)$, $0 \leq \varphi < 360^\circ$

e) $\cos(\alpha - 36^\circ) = \cos(\alpha - 72^\circ)$, $0 \leq \alpha < 360^\circ$

$\theta = 70^\circ, 250^\circ$, $x = 45^\circ, 225^\circ$, $y = 63^\circ, 243^\circ$, $\varphi = 28^\circ, 208^\circ$, $\alpha = 54^\circ, 234^\circ$

(a) $\sin(\theta - 20^\circ) = \sin(\theta + 60^\circ)$

$$\Rightarrow \sin(\theta - 20^\circ) - \sin(\theta + 60^\circ) = 0 \Rightarrow \sin\theta\cos(-20^\circ) - \cos\theta\sin(-20^\circ) - (\sin\theta\cos(60^\circ) + \cos\theta\sin(60^\circ)) = 0 \Rightarrow \sin\theta(\cos(-20^\circ) - \cos(60^\circ)) = \cos\theta(\sin(-20^\circ) + \sin(60^\circ))$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta}(\cos(-20^\circ) - \cos(60^\circ)) = \tan\theta(\sin(-20^\circ) + \sin(60^\circ))$$

$$\Rightarrow \tan\theta(\cos(-20^\circ) - \cos(60^\circ)) = \sin(-20^\circ) + \sin(60^\circ)$$

$$\Rightarrow \tan\theta = \frac{\sin(-20^\circ) + \sin(60^\circ)}{\cos(-20^\circ) - \cos(60^\circ)} \approx 2.74747\dots$$

arctan(2.74747...) $\approx 70^\circ$

$\bullet \theta = 70^\circ \pm 180m \quad m=0,1,2,3,\dots$

$\therefore \theta_1 = 70^\circ$
 $\theta_2 = 250^\circ$

(b) $\cos(x - 35^\circ) = \cos(x - 55^\circ)$

$$\Rightarrow \cos(x\cos(-35^\circ) - \sin(x)\sin(-35^\circ)) = \cos(x\cos(-55^\circ) + \sin(x)\sin(-55^\circ))$$

$$\Rightarrow \cos(x\cos(-35^\circ) - \sin(x)\cos(55^\circ)) = \sin(x\sin(-35^\circ) + \cos(x)\cos(55^\circ))$$

$$\Rightarrow \cos(x\cos(-35^\circ) - \sin(x)\cos(55^\circ)) = \sin(x\sin(-35^\circ) + \cos(x)\cos(55^\circ))$$

$$\Rightarrow \cos(x\cos(-35^\circ) - \sin(x)\cos(55^\circ)) = \frac{\sin(x)}{\cos(x)}(\sin(-35^\circ) + \cos(-35^\circ))$$

$$\Rightarrow \cos(x\cos(-35^\circ) - \sin(x)\cos(55^\circ)) = \frac{\sin(x)}{\cos(x)}(\sin(-35^\circ) + \cos(-35^\circ))$$

$$\Rightarrow \tan(x) = \frac{\sin(x\cos(-35^\circ) - \sin(x)\cos(55^\circ))}{\cos(x\cos(-35^\circ) - \sin(x)\cos(55^\circ))} = 1$$

arctan(1) $= 45^\circ$

$\bullet x = 45^\circ + 180m \quad m=0,1,2,3,\dots$

$\therefore x_1 = 45^\circ$
 $x_2 = 225^\circ$

(c) $\sin(y - 48^\circ) = \cos(y + 12^\circ)$

$$\Rightarrow \sin(y\cos(-48^\circ) - \cos(y)\sin(-48^\circ)) = \cos(y\cos(12^\circ) + \sin(y)\sin(12^\circ))$$

$$\Rightarrow \sin(y\cos(-48^\circ) - \cos(y)\cos(12^\circ)) = \sin(y\sin(12^\circ) + \cos(y)\sin(12^\circ))$$

$$\Rightarrow \sin(y\cos(-48^\circ) - \cos(y)\cos(12^\circ)) = \cos(y\sin(12^\circ) + \cos(y)\sin(12^\circ))$$

$$\Rightarrow \frac{\sin(y\cos(-48^\circ) - \cos(y)\cos(12^\circ))}{\cos(y\sin(12^\circ) + \cos(y)\sin(12^\circ))} = \frac{\cos(y\sin(12^\circ) + \cos(y)\sin(12^\circ))}{\sin(y\cos(-48^\circ) - \cos(y)\cos(12^\circ))}$$

$$\Rightarrow \tan(y) = \frac{\cos(y\sin(12^\circ) + \cos(y)\sin(12^\circ))}{\sin(y\cos(-48^\circ) - \cos(y)\cos(12^\circ))} = 1$$

arctan(1) $\approx 45^\circ$

$\bullet y = 45^\circ + 180m \quad m=0,1,2,3,\dots$

$\therefore y_1 = 45^\circ$
 $y_2 = 243^\circ$

(d) $\sin(\varphi + 72^\circ) = \cos(\varphi - 38^\circ)$

$$\Rightarrow \sin(\varphi\cos(72^\circ) + \cos(\varphi)\sin(72^\circ)) = \cos(\varphi\cos(-38^\circ) + \sin(\varphi)\sin(-38^\circ))$$

$$\Rightarrow \sin(\varphi\cos(72^\circ) - \sin(\varphi)\cos(38^\circ)) = \cos(\varphi\cos(-38^\circ) - \sin(\varphi)\sin(-38^\circ))$$

$$\Rightarrow \sin(\varphi\cos(72^\circ) - \sin(\varphi)\cos(38^\circ)) = \cos(\varphi\cos(-38^\circ) - \sin(\varphi)\sin(-38^\circ))$$

$$\Rightarrow \frac{\sin(\varphi\cos(72^\circ) - \sin(\varphi)\cos(38^\circ))}{\cos(\varphi\cos(-38^\circ) - \sin(\varphi)\sin(-38^\circ))} = \frac{\cos(\varphi\cos(-38^\circ) - \sin(\varphi)\sin(-38^\circ))}{\sin(\varphi\cos(72^\circ) - \sin(\varphi)\cos(38^\circ))}$$

$$\Rightarrow \tan(\varphi) = \frac{\cos(\varphi\cos(-38^\circ) - \sin(\varphi)\sin(-38^\circ))}{\sin(\varphi\cos(72^\circ) - \sin(\varphi)\cos(38^\circ))} = 1$$

arctan(1) $\approx 45^\circ$

$\bullet \varphi = 45^\circ + 180m \quad m=0,1,2,3,\dots$

$\therefore \varphi_1 = 45^\circ$
 $\varphi_2 = 225^\circ$

(e) $\cos(\alpha - 36^\circ) = \cos(\alpha - 72^\circ)$

$$\Rightarrow \cos(\alpha\cos(-36^\circ) - \sin(\alpha)\sin(-36^\circ)) = \cos(\alpha\cos(-72^\circ) + \sin(\alpha)\sin(-72^\circ))$$

$$\Rightarrow \cos(\alpha\cos(-36^\circ) - \sin(\alpha)\cos(72^\circ)) = \sin(\alpha\sin(-36^\circ) + \cos(\alpha)\sin(72^\circ))$$

$$\Rightarrow \cos(\alpha\cos(-36^\circ) - \sin(\alpha)\cos(72^\circ)) = \cos(\alpha\sin(-36^\circ) + \cos(\alpha)\sin(72^\circ))$$

$$\Rightarrow \frac{\cos(\alpha\cos(-36^\circ) - \sin(\alpha)\cos(72^\circ))}{\cos(\alpha\sin(-36^\circ) + \cos(\alpha)\sin(72^\circ))} = \frac{\cos(\alpha\sin(-36^\circ) + \cos(\alpha)\sin(72^\circ))}{\cos(\alpha\cos(-36^\circ) - \sin(\alpha)\cos(72^\circ))}$$

$$\Rightarrow \tan(\alpha) = \frac{\cos(\alpha\sin(-36^\circ) + \cos(\alpha)\sin(72^\circ))}{\cos(\alpha\cos(-36^\circ) - \sin(\alpha)\cos(72^\circ))} = 1$$

arctan(1) $\approx 45^\circ$

$\bullet \alpha = 45^\circ + 180m \quad m=0,1,2,3,\dots$

$\therefore \alpha_1 = 45^\circ$
 $\alpha_2 = 225^\circ$

Question 21

Solve each of the following trigonometric equations.

a) $\sin 2\theta = \tan \theta$, $0 \leq \theta \leq 180^\circ$

b) $2 \sin 2x = \cos x$, $0 \leq x < 180^\circ$

c) $\sin 2y + \sin y = 0$, $0 \leq y < 360^\circ$

d) $4 \sin \varphi \cos \varphi = 1$, $0 \leq \varphi < \pi$

$$\boxed{\theta = 0^\circ, 45^\circ, 135^\circ, 180^\circ}, \boxed{x = 90^\circ, x \approx 14.5^\circ, 165.5^\circ}, \boxed{y = 0^\circ, 120^\circ, 180^\circ, 240^\circ},$$

$$\boxed{\varphi = \frac{\pi}{12}, \frac{5\pi}{12}}$$

<p>(a) $\sin 2\theta = \tan \theta$ $\Rightarrow 2\sin \theta \cos \theta = \frac{\sin \theta}{\cos \theta}$ $\Rightarrow 2\sin \theta \cos^2 \theta = \sin \theta$ $\Rightarrow 2\sin \theta (\cos^2 \theta - 1) = 0$ $\Rightarrow \sin \theta (2\cos^2 \theta - 1) = 0$ $\Rightarrow \sin \theta (2\cos^2 \theta - 1) = 0$</p>	<p>$\bullet \sin 2\theta = 0$ $\Rightarrow 2\sin \theta \cos \theta = 0$ $\Rightarrow \sin \theta (2\cos^2 \theta - 1) = 0$ $\Rightarrow \sin \theta (2\cos^2 \theta - 1) = 0$</p>	<p>$\bullet \cos 2\theta = 0$ $\Rightarrow \cos^2 \theta - 1 = 0$ $\Rightarrow \cos^2 \theta = 1$ $\Rightarrow \cos \theta = \pm 1$</p>
<p>(b) $2 \sin 2x = \cos x$ $\Rightarrow 2(2\sin x \cos x) = \cos x$ $\Rightarrow 4\sin x \cos x - \cos x = 0$ $\Rightarrow \cos x(4\sin x - 1) = 0$</p>	<p>$\bullet \cos x = 0$ $\Rightarrow \cos^2 x = 0$ $\Rightarrow \cos x = 0$</p>	<p>$\bullet \sin 2x = \frac{1}{2}$ $\Rightarrow \sin^2 x = \frac{1}{4}$ $\Rightarrow \sin x = \pm \frac{1}{2}$</p>
<p>(c) $\sin 2y + \sin y = 0$ $\Rightarrow 2\sin y \cos y + \sin y = 0$ $\Rightarrow \sin y(2\cos y + 1) = 0$</p>	<p>$\bullet \sin y = 0$ $\Rightarrow \sin^2 y = 0$ $\Rightarrow \sin y = 0$</p>	<p>$\bullet \cos y = -\frac{1}{2}$ $\Rightarrow \cos^2 y = \frac{1}{4}$ $\Rightarrow \cos y = \pm \frac{1}{2}$</p>
<p>(d) $4 \sin \varphi \cos \varphi = 1$ $\Rightarrow 2(2\sin \varphi \cos \varphi) = 1$ $\Rightarrow 2\sin 2\varphi = 1$ $\Rightarrow \sin 2\varphi = \frac{1}{2}$</p>	<p>$\bullet \sin 2\varphi = 0$ $\Rightarrow 2\sin \varphi \cos \varphi = 0$ $\Rightarrow \sin \varphi \cos \varphi = 0$</p>	<p>$\bullet \cos 2\varphi = \frac{1}{2}$ $\Rightarrow \cos^2 \varphi = \frac{1}{4}$ $\Rightarrow \cos \varphi = \pm \frac{1}{2}$</p>

$\therefore y = 0^\circ, 180^\circ, 45^\circ, 135^\circ$

$\therefore x = 90^\circ, 14.5^\circ, 165.5^\circ$

$\therefore y = 0^\circ, 120^\circ, 180^\circ, 240^\circ$

$\therefore \varphi = 0^\circ, 180^\circ, 14.5^\circ, 135^\circ$

Question 22

Solve each of the following trigonometric equations.

a) $2\sin 2\theta = \cot \theta$, $0 \leq \theta \leq \pi$

b) $3\sin 2x = 2\cos x$, $0 \leq x < 180^\circ$

c) $\sin 4y = \sin 2y$, $0 \leq y < 180^\circ$

d) $\sin \varphi + \frac{1}{4}\sec \varphi = 0$, $0 \leq \varphi < \pi$

$$\boxed{\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}}, \boxed{x = 90^\circ, x \approx 19.5^\circ, 160.5^\circ}, \boxed{y = 0^\circ, 30^\circ, 90^\circ, 150^\circ}, \boxed{\varphi = \frac{7\pi}{12}, \frac{11\pi}{12}}$$

<p>(a) $2\sin 2\theta = \cot \theta$</p> $\Rightarrow 2(2\sin \theta \cos \theta) = \frac{\cos \theta}{\sin \theta}$ $\Rightarrow 4\sin \theta \cos^2 \theta = \cos \theta$ $\Rightarrow 4\sin \theta \cos^2 \theta - \cos \theta = 0$ $\Rightarrow \cos \theta(4\sin \theta - 1) = 0$ <p>• $\cos \theta = 0$</p> $\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$ <p>• $4\sin \theta - 1 = 0$</p> $\sin \theta = \frac{1}{4}$ $\sin \theta = \pm \frac{1}{4}$ $\sin(\theta) = \pm \frac{1}{4}$ $\theta = \arcsin(\pm \frac{1}{4})$ $\theta = \pm \frac{\pi}{6} + 2k\pi$ $\theta = \pm \frac{\pi}{6} + 2m\pi$ $\theta = \pm \frac{\pi}{6} + 3k\pi$ $\theta = \pm \frac{\pi}{6} + 3m\pi$ $\therefore \theta = 0, \pm 30^\circ, \dots$	<p>(b) $3\sin 2x = 2\cos x$</p> $\Rightarrow 3(2\sin x \cos x) = 2\cos x$ $\Rightarrow 6\sin x \cos x = 2\cos x$ $\Rightarrow 6\sin x \cos x - 2\cos x = 0$ $\Rightarrow 2\cos x(3\sin x - 1) = 0$ <p>• $\cos x = 0$</p> $\Rightarrow \arccos(x) = 90^\circ$ $x = 90^\circ, 270^\circ, 450^\circ, 630^\circ, \dots$ <p>• $3\sin x - 1 = 0$</p> $\sin x = \frac{1}{3}$ $\sin x = \pm \frac{1}{3}$ $\sin(x) = \pm \frac{1}{3}$ $x = \arcsin(\pm \frac{1}{3})$ $x = 19.5^\circ, 160.5^\circ, 160.5^\circ, 330^\circ, \dots$ $\therefore x = 19.5^\circ, 330^\circ, 160.5^\circ, \dots$
<p>(c) $\sin 4y = \sin 2y$</p> $\Rightarrow \sin(2\pi y) = \sin 2y$ $\Rightarrow 2\pi y = 2y + 2k\pi$ $\Rightarrow 2\pi y - 2y = 2k\pi$ $\Rightarrow \sin(2\pi y - 2y) = 0$ <p>• $\sin 2\pi y = 0$</p> $\arcsin(0) = 0^\circ$ $2y = 0 + 360^\circ$ $2y = 180^\circ + 360^\circ$ $2y = 30^\circ + 360^\circ$ $2y = 390^\circ$ $y = 195^\circ$ <p>• $2\pi y - 2y = 0$</p> $2\pi y - 2y = 360^\circ$ $(2\pi - 2)y = 360^\circ$ $2y = 360^\circ$ $y = 180^\circ$ $\therefore y = 0^\circ, 30^\circ, 150^\circ$	<p>(d) $\sin \varphi + \frac{1}{4}\sec \varphi = 0$</p> $\Rightarrow 4\sin \varphi + \sec \varphi = 0$ $\Rightarrow 4\sin \varphi + \frac{1}{\cos \varphi} = 0$ $\Rightarrow 4\sin \varphi \cos \varphi + 1 = 0$ $\Rightarrow 2(2\sin \varphi \cos \varphi) + 1 = 0$ $\Rightarrow 2\sin 2\varphi + 1 = 0$ <p>• $\sin 2\varphi = -\frac{1}{2}$</p> $\arcsin(-\frac{1}{2}) = -30^\circ$ $2\varphi = -\frac{\pi}{6} + 2m\pi$ $\varphi = -\frac{\pi}{12} + m\pi$ $\varphi = \frac{11\pi}{12}, \frac{19\pi}{12}, \dots$

Question 23

Solve each of the following trigonometric equations.

a) $\cos \theta - \sin 2\theta = 0$, $0 \leq \theta \leq 360^\circ$

b) $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4$, $0 \leq x < 360^\circ$

c) $2\cos y = 2\tan y \sin y + \sec y$, $0 \leq y < 2\pi$

d) $2\cos \varphi + \operatorname{cosec} \varphi = 0$, $0 \leq \varphi < 2\pi$

$$\boxed{\theta = 30^\circ, 90^\circ, 150^\circ, 270^\circ}, \boxed{x = 15^\circ, 75^\circ, 195^\circ, 255^\circ}, \boxed{y = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}},$$

$$\boxed{\varphi = \frac{3\pi}{4}, \frac{7\pi}{4}}$$

<p>(a) $\cos \theta - \sin 2\theta = 0$</p> $\begin{aligned} \cos \theta - 2\sin \theta \cos \theta &= 0 \\ \cos \theta(1 - 2\sin \theta) &= 0 \\ \cos \theta = 0 \quad \text{or} \quad \sin \theta = \frac{1}{2} \end{aligned}$	<p>$\arccos(\frac{1}{2}) = 30^\circ$</p> $\begin{cases} 1 = 30 + 360n \\ \frac{1}{2} = 15 + 360n \end{cases}$ $\begin{cases} \theta = 30 + 360n \\ \theta = 15 + 360n \end{cases}$ $\theta = 30^\circ, 150^\circ, 270^\circ$
<p>(b) $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4$</p> $\begin{aligned} \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} &= 4 \\ \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} &= 4 \\ \frac{1}{\sin x \cos x} &= 4 \\ \frac{2}{\sin 2x} &= 4 \\ \Rightarrow \sin 2x &= 1 \end{aligned}$	<p>$\tan x = 4$</p> $\begin{cases} 4 = 30 + 360n \\ 4 = 15 + 360n \end{cases}$ $\begin{cases} x = 15^\circ + 360n \\ x = 75^\circ + 360n \end{cases}$ $x = 15^\circ, 75^\circ, 195^\circ, 255^\circ$
<p>(c) $2\cos y = 2\tan y \sin y + \sec y$</p> $\begin{aligned} 2\cos y &= 2\frac{\sin y}{\cos y} \sin y + \frac{1}{\cos y} \\ \Rightarrow 2\cos^2 y &= 2\sin^2 y + 1 \\ \Rightarrow 2\cos^2 y - 2\sin^2 y &= 1 \\ \Rightarrow 2(\cos^2 y - \sin^2 y) &= 1 \\ \Rightarrow 2\cos 2y &= 1 \end{aligned}$	<p>$\cos 2y = \frac{1}{2}$</p> $\begin{cases} \frac{1}{2} = \frac{\pi}{3} + 2\pi n \\ \frac{1}{2} = \frac{4\pi}{3} + 2\pi n \end{cases}$ $\begin{cases} 2y = \frac{\pi}{3} + 2\pi n \\ 2y = \frac{4\pi}{3} + 2\pi n \end{cases}$ $y = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
<p>(d) $2\cos \varphi + \operatorname{cosec} \varphi = 0$</p> $\begin{aligned} 2\cos \varphi + \frac{1}{\sin \varphi} &= 0 \\ \Rightarrow 2\cos \varphi \sin \varphi + 1 &= 0 \\ \Rightarrow \sin 2\varphi + 1 &= 0 \\ \Rightarrow \sin 2\varphi &= -1 \end{aligned}$	<p>$\sin 2\varphi = -\frac{1}{2}$</p> $\begin{cases} -\frac{1}{2} = -\frac{\pi}{6} + 2\pi n \\ -\frac{1}{2} = \frac{7\pi}{6} + 2\pi n \end{cases}$ $\begin{cases} 2\varphi = \frac{\pi}{6} + 2\pi n \\ 2\varphi = \frac{7\pi}{6} + 2\pi n \end{cases}$ $\varphi = \frac{\pi}{12}, \frac{7\pi}{12}$

Question 24

Solve each of the following trigonometric equations.

a) $2\cos 2\theta = 1 + \cos \theta$, $0^\circ \leq \theta < 360^\circ$

b) $\cos 2x + 3\sin x = 2$, $0^\circ \leq x < 360^\circ$

c) $\cos 2y + \sin y = 0$, $0^\circ \leq y < 360^\circ$

d) $2(1 - \cos 2\varphi) = \tan \varphi$, $0^\circ \leq \varphi < 180^\circ$

$\boxed{\theta = 0^\circ, \theta \approx 138.6^\circ, 221.4^\circ}, \boxed{x = 30^\circ, 90^\circ, 150^\circ}, \boxed{y = 90^\circ, 210^\circ, 330^\circ},$

$\boxed{\varphi = 0^\circ, 15^\circ, 75^\circ}$

<p>(a) $2\cos 2\theta = 1 + \cos \theta$ $\Rightarrow 2(2\cos^2 \theta - 1) = 1 + \cos \theta$ $\Rightarrow 4\cos^2 \theta - 2 = 1 + \cos \theta$ $\Rightarrow 4\cos^2 \theta - \cos \theta - 3 = 0$ $\Rightarrow (\cos \theta - 1)(4\cos \theta + 3) = 0$ $\Rightarrow \cos \theta = 1$ or $\cos \theta = -\frac{3}{4}$ $\Rightarrow \theta = 0^\circ \pm 360^\circ$ or $\theta = 214.4^\circ \pm 360^\circ$ $\Rightarrow \theta = 0^\circ, 138.6^\circ, 221.4^\circ$</p>	<p>$\bullet \cos \theta = 1$ $\cos \theta = 0$ $\theta = 0^\circ \pm 360^\circ$ $\theta = 360^\circ, 330^\circ$</p>	<p>$\bullet \cos \theta = -\frac{3}{4}$ $\cos \theta = -0.75$ $\theta = 191.3^\circ, 351.3^\circ$</p>
<p>(b) $\cos 2x + 3\sin x = 2$ $\Rightarrow 1 - 2\sin^2 x + 3\sin x = 2$ $\Rightarrow -2\sin^2 x + 3\sin x - 1 = 0$ $\Rightarrow 2\sin^2 x - 3\sin x + 1 = 0$ $\Rightarrow (2\sin x - 1)(\sin x - 1) = 0$ $\Rightarrow \sin x = \frac{1}{2}$ or $\sin x = 1$ $\Rightarrow \sin x = \frac{1}{2}$ or $\sin x = -\frac{1}{2}$ $\Rightarrow x = 30^\circ, 150^\circ, 210^\circ$</p>	<p>$\bullet \sin x = \frac{1}{2}$ $\sin x = 0.5$ $\sin x = -0.5$ $x = 30^\circ \pm 360^\circ$ $x = 150^\circ \pm 360^\circ$ $x = 90^\circ \pm 360^\circ$</p>	<p>$\bullet \sin x = 1$ $\sin x = 0$ $x = 90^\circ \pm 360^\circ$ $x = 270^\circ \pm 360^\circ$</p>
<p>(c) $\cos 2y + \sin y = 0$ $\Rightarrow 1 - 2\sin^2 y + \sin y = 0$ $\Rightarrow 0 = 2\sin^2 y - \sin y - 1$ $\Rightarrow (2\sin y + 1)(\sin y - 1) = 0$ $\Rightarrow \sin y = -\frac{1}{2}$ or $\sin y = 1$ $\Rightarrow \sin y = -\frac{1}{2}$ or $\sin y = 0$ $\Rightarrow y = 90^\circ \pm 360^\circ$</p>	<p>$\bullet \sin y = -\frac{1}{2}$ $\sin y = -0.5$ $\sin y = 0$ $y = 90^\circ \pm 360^\circ$</p>	<p>$\bullet \sin y = 1$ $\sin y = 0$ $y = 90^\circ$</p>
<p>(d) $2(1 - \cos 2\varphi) = \tan \varphi$ $\Rightarrow 2[1 - (1 - 2\sin^2 \varphi)] = \frac{\sin \varphi}{\cos \varphi}$ $\Rightarrow 2[1 - 1 + 2\sin^2 \varphi] = \frac{\sin \varphi}{\cos \varphi}$ $\Rightarrow 4\sin^2 \varphi = \frac{\sin \varphi}{\cos \varphi}$ $\Rightarrow 4\sin^2 \varphi \cos \varphi = \sin \varphi$ $\Rightarrow 4\sin^2 \varphi \cos \varphi - \sin \varphi = 0$ $\Rightarrow \sin \varphi(4\sin \varphi \cos \varphi - 1) = 0$ $\Rightarrow \sin \varphi(2\sin 2\varphi - 1) = 0$</p>	<p>$\bullet \sin \varphi = 0$ $\sin \varphi = 0$ $\varphi = 0^\circ \pm 360^\circ$ $\varphi = 180^\circ \pm 360^\circ$ $\varphi = 180^\circ, 360^\circ$</p>	<p>$\bullet \sin 2\varphi = \frac{1}{2}$ $\sin 2\varphi = 0.5$ $\sin 2\varphi = -0.5$ $2\varphi = 30^\circ \pm 360^\circ$ $2\varphi = 150^\circ \pm 360^\circ$ $\varphi = 15^\circ, 180^\circ$ $\varphi = 75^\circ, 180^\circ$</p>

Question 25

Solve each of the following trigonometric equations.

a) $\cos 2\theta = 1 + \sin \theta$, $0^\circ \leq \theta < 360^\circ$

b) $\cos 2x + 3\cos x = 1$, $0 \leq x < 2\pi$

c) $3\cos 2y = 1 - \sin y$, $0^\circ \leq y < 360^\circ$

d) $2\cos \varphi + 1 = \sin\left(\frac{1}{2}\varphi\right)$, $0^\circ \leq \varphi < 360^\circ$

$$\theta = 0^\circ, 180^\circ, 210^\circ, 330^\circ, \boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}, \boxed{y \approx 41.8^\circ, 138.2^\circ \quad y = 210^\circ, 330^\circ},$$

$$\boxed{\varphi = 97.2^\circ, 262.8^\circ}$$

<p>(a) $\cos 2\theta = 1 + \sin \theta$</p> $\begin{aligned} \rightarrow 1 - 2\sin^2 \theta &= 1 + \sin \theta \\ \rightarrow -2\sin^2 \theta &= 1 + \sin \theta \\ \rightarrow 0 &= 2\sin^2 \theta + \sin \theta \\ \rightarrow 0 &= \sin \theta (2\sin \theta + 1) \\ \rightarrow \sin \theta &= 0 \quad \text{or } 2\sin \theta + 1 = 0 \\ \rightarrow \theta &= 0^\circ, 180^\circ, 210^\circ, 330^\circ \end{aligned}$	<p>$\cos 2\theta = 0$</p> $\begin{aligned} \theta &= 0^\circ \pm 360^\circ \\ &= 180^\circ \pm 360^\circ \\ &= -360^\circ \pm 360^\circ \\ &= 210^\circ \pm 360^\circ \\ &= 0^\circ, 180^\circ, 210^\circ, 330^\circ \end{aligned}$	<p>$\cos 2\theta = -\sin \theta$</p> $\begin{aligned} \theta &= 90^\circ \pm 360^\circ \\ &= 270^\circ \pm 360^\circ \\ &= -270^\circ \pm 360^\circ \\ &= 90^\circ, 270^\circ, 210^\circ, 330^\circ \end{aligned}$
<p>(b) $\cos 2x + 3\cos x = 1$</p> $\begin{aligned} \rightarrow 2\cos^2 x - 1 + 3\cos x &= 1 \\ \rightarrow 2\cos^2 x + 3\cos x - 2 &= 0 \\ \rightarrow (2\cos x - 1)(\cos x + 2) &= 0 \\ \rightarrow \cos x &= \frac{1}{2} \quad \text{or } \cos x = -2 \quad (\text{not possible}) \\ \rightarrow x &= 60^\circ, 300^\circ \end{aligned}$	<p>$\cos x = \frac{1}{2}$</p> $\begin{aligned} x &= 60^\circ \pm 360^\circ \\ &= 300^\circ \pm 360^\circ \\ &= 240^\circ, 360^\circ \end{aligned}$	<p>$\cos x = -2$</p> $\begin{aligned} x &= \frac{\pi}{3}, \frac{4\pi}{3} \quad (\text{not possible}) \\ \therefore x &= 60^\circ, 240^\circ, 300^\circ, 360^\circ \end{aligned}$
<p>(c) $3\cos 2y = 1 - \sin y$</p> $\begin{aligned} \rightarrow 3(1 - 2\sin^2 y) &= 1 - \sin y \\ \rightarrow 3 - 6\sin^2 y &= 1 - \sin y \\ \rightarrow 0 &= 6\sin^2 y - \sin y - 2 \\ \rightarrow (3\sin y - 2)(2\sin y + 1) &= 0 \\ \sin y &= \frac{2}{3} \quad \text{or } \sin y = -\frac{1}{2} \\ \rightarrow y &= 41.8^\circ, 210^\circ, 138.2^\circ, 330^\circ \end{aligned}$	<p>$\cos 2y = 41.8^\circ$</p> $\begin{aligned} y &= 41.8^\circ \pm 360^\circ \\ &= -360^\circ \pm 360^\circ \\ &= 210^\circ \pm 360^\circ \\ &= 210^\circ, 330^\circ \end{aligned}$	<p>$\cos 2y = -\sin y$</p> $\begin{aligned} \theta &= 90^\circ \pm 360^\circ \\ &= 270^\circ \pm 360^\circ \\ &= -270^\circ \pm 360^\circ \\ &= 90^\circ, 270^\circ, 210^\circ, 330^\circ \end{aligned}$
<p>(d) $2\cos \varphi + 1 = \sin\left(\frac{1}{2}\varphi\right)$</p> $\begin{aligned} \rightarrow 2\left(1 - 2\sin^2 \frac{\varphi}{2}\right) + 1 &= \sin\left(\frac{\varphi}{2}\right) \\ \rightarrow 2 - 4\sin^2 \frac{\varphi}{2} + 1 &= \sin\left(\frac{\varphi}{2}\right) \\ \rightarrow 0 &= 4\sin^2 \frac{\varphi}{2} + \sin \frac{\varphi}{2} - 3 \\ \rightarrow (4\sin \frac{\varphi}{2} - 3)(\sin \frac{\varphi}{2} + 1) &= 0 \\ \rightarrow \sin \frac{\varphi}{2} &= \frac{3}{4} \quad \text{or } \sin \frac{\varphi}{2} = -1 \\ \rightarrow \varphi &= 97.2^\circ, 262.8^\circ \end{aligned}$	<p>$\cos \varphi = -90^\circ$</p> $\begin{aligned} \frac{\varphi}{2} &= 90^\circ \pm 360^\circ \\ &= 270^\circ \pm 360^\circ \\ &= -270^\circ \pm 360^\circ \\ &= 90^\circ, 270^\circ, 210^\circ, 330^\circ \end{aligned}$	<p>$\sin \frac{\varphi}{2} = 45.6^\circ$</p> $\begin{aligned} \frac{\varphi}{2} &= 45.6^\circ \pm 360^\circ \\ &= 134.4^\circ \pm 360^\circ \\ &= 345.6^\circ \pm 360^\circ \\ &= 228.8^\circ \pm 360^\circ \\ &= 97.2^\circ, 262.8^\circ \end{aligned}$

Question 26

- a) $\cos 2\theta - 7 \sin \theta - 4 = 0, \quad 0 \leq \theta < 360^\circ$
- b) $3 \cos 2x = \sin x + 2, \quad 0 \leq x < 360^\circ$
- c) $3 \cos 2y = 7 \cos y, \quad 0 \leq y < 360^\circ$
- d) $\cos 2\varphi = \sin \varphi, \quad 0 \leq \varphi < 360^\circ$

$$\boxed{\theta = 210^\circ, 330^\circ}, \boxed{x \approx 19.5^\circ, 160.5^\circ \quad x = 210^\circ, 330^\circ}, \boxed{y \approx 109.5^\circ, 250.5^\circ},$$

$$\boxed{\varphi = 30^\circ, 150^\circ, 270^\circ}$$

(a) $\cos 2\theta - 7 \sin \theta - 4 = 0$
 $\Rightarrow (-2\sin^2 \theta) - 7\sin \theta - 4 = 0$
 $\Rightarrow -2\sin^2 \theta - 7\sin \theta - 4 = 0$
 $\Rightarrow 2\sin^2 \theta + 7\sin \theta + 4 = 0$
 $\Rightarrow (2\sin \theta + 1)(\sin \theta + 4) = 0$
 $\Rightarrow \sin \theta = -\frac{1}{2}$
 $\text{or } \sin \theta = -4$
 $\sin \theta = -\frac{1}{2}$
 $\theta = -30^\circ \pm 360^\circ (\text{min} + 360^\circ)$
 $\theta = 330^\circ$
 $\theta = 210^\circ$

(b) $3 \cos 2x = \sin x + 2$
 $\Rightarrow 3(-2\sin^2 x) = \sin x + 2$
 $\Rightarrow -6\sin^2 x = \sin x + 2$
 $\Rightarrow -6\sin^2 x - \sin x - 2 = 0$
 $\Rightarrow 6\sin^2 x + \sin x + 2 = 0$
 $\Rightarrow (3\sin x + 1)(2\sin x + 1) = 0$
 $\Rightarrow \sin x = -\frac{1}{3}$
 $\text{or } \sin x = -\frac{1}{2}$

(c) $3 \cos 2y = 7 \cos y$
 $\Rightarrow 3(2\sin^2 y) - 7\cos y = 0$
 $\Rightarrow 6\sin^2 y - 7\cos y = 0$
 $\Rightarrow (6\sin y - 7\cos y)(\sin y + 1) = 0$
 $\Rightarrow \sin y = \frac{7}{6}$
 $\text{or } \sin y = -1$
 $\sin y = \frac{7}{6}$
 $y = 109.5^\circ$
 $y = 250.5^\circ$

(d) $\cos 2\varphi = \sin \varphi$
 $\Rightarrow 1 - 2\sin^2 \varphi = \sin \varphi$
 $\Rightarrow -2\sin^2 \varphi - \sin \varphi + 1 = 0$
 $\Rightarrow 2\sin^2 \varphi + \sin \varphi - 1 = 0$
 $\Rightarrow (2\sin \varphi - 1)(\sin \varphi + 1) = 0$
 $\Rightarrow \sin \varphi = \frac{1}{2}$
 $\text{or } \sin \varphi = -1$
 $\sin \varphi = \frac{1}{2}$
 $\varphi = 30^\circ$
 $\varphi = 150^\circ$
 $\varphi = 210^\circ$

Question 27

Solve each of the following trigonometric equations.

a) $3\cos 2\theta - 5\sin \theta = 4, \quad 0 \leq \theta < 360^\circ$

b) $3\cos 2x = 1 - \sin x, \quad 0 \leq x < 360^\circ$

c) $\cos 2y - 7\cos y + 4 = 0, \quad 0 \leq y < 360^\circ$

d) $\cos 2\varphi + 6\cos \varphi + 5 = 0, \quad 0 \leq \varphi < 360^\circ$

$\boxed{\theta = 210^\circ, 330^\circ \quad \theta \approx 199.5^\circ, 340.5^\circ}, \quad x \approx 41.8^\circ, 138.2^\circ \quad x = 210^\circ, 330^\circ}$

$\boxed{y = 60^\circ, 300^\circ}, \quad \boxed{\varphi = 180^\circ}$

<p>(a) $3\cos 2\theta - 5\sin \theta = 4$ $3(1 - 2\sin^2 \theta) - 5\sin \theta = 4$ $3 - 6\sin^2 \theta - 5\sin \theta = 4$ $0 = 6\sin^2 \theta + 5\sin \theta + 1$ $(3\sin \theta + 1)(2\sin \theta + 1) = 0$</p>	<p>$\bullet \sin \theta = -\frac{1}{3}$ $\arcsin(-\frac{1}{3}) = -19.47^\circ$</p>	<p>$\bullet \sin \theta = -\frac{1}{2}$ $\arcsin(-\frac{1}{2}) = -30^\circ$</p>
$\left\{ \begin{array}{l} \theta = -19.47 \pm 360^\circ \\ \theta = 199.47 \pm 360^\circ \\ (199.47, -19.47) \end{array} \right.$		
$\therefore \theta = 340.5^\circ, 199.5^\circ, 330^\circ, 210^\circ$		
<p>(b) $3\cos 2x = 1 - \sin x$ $3(1 - 2\sin^2 x) = 1 - \sin x$ $3 - 6\sin^2 x = 1 - \sin x$ $0 = 6\sin^2 x - \sin x - 2$ $0 = (3\sin x - 2)(2\sin x + 1)$</p>	<p>$\bullet \sin x = \frac{2}{3}$ $\arcsin(\frac{2}{3}) = 41.8^\circ$</p>	<p>$\bullet \sin x = -\frac{1}{2}$ $\arcsin(-\frac{1}{2}) = -30^\circ$</p>
$\left\{ \begin{array}{l} x = 41.8^\circ \pm 360^\circ \\ x = 180^\circ \pm 360^\circ \\ (180^\circ, 41.8^\circ) \end{array} \right.$		
$\therefore x = 41.8^\circ, 180^\circ, 330^\circ, 210^\circ$		
<p>(c) $\cos 2y - 7\cos y + 4 = 0$ $(2\cos^2 y - 1) - 7\cos y + 4 = 0$ $2\cos^2 y - 7\cos y + 3 = 0$ $(2\cos y - 1)(\cos y - 3) = 0$ $\cos y = \frac{1}{2}$</p>	<p>$\arccos(\frac{1}{2}) = 60^\circ$ $y = 60^\circ \pm 360^\circ$ $y = 300^\circ \pm 360^\circ$ $y = 60^\circ, 300^\circ$</p>	<p>$\bullet \cos y = -1$ $\arccos(-1) = 180^\circ$ $\phi = 180^\circ \pm 360^\circ$ $\phi = 180^\circ \pm 360^\circ$ $\phi = 180^\circ$</p>
$\left\{ \begin{array}{l} y = 60^\circ, 300^\circ \\ \phi = 180^\circ \end{array} \right.$		

Question 28

Solve each of the following trigonometric equations.

a) $\cos 2\theta = 7 \cos \theta + 3$, $0^\circ \leq \theta < 360^\circ$

b) $2 \cos 2x = 4 \cos x - 3$, $0^\circ \leq x < 360^\circ$

c) $6 \cos 2y + 5 \cos y + 3 = 0$, $0^\circ \leq y < 360^\circ$

d) $5 \cos 2\varphi + 22 \sin \varphi = 9$, $0^\circ \leq \varphi < 360^\circ$

$$\boxed{\theta = 120^\circ, 240^\circ}, \boxed{x = 60^\circ, 300^\circ}, \boxed{y \approx 70.5^\circ, 138.6^\circ, 221.4^\circ, 289.5^\circ},$$

$$\boxed{\varphi \approx 11.5^\circ, 168.5^\circ}$$

<p>(a) $\cos 2\theta = 7 \cos \theta + 3$ $\Rightarrow 2\cos^2 \theta - 1 = 7 \cos \theta + 3$ $\Rightarrow 2\cos^2 \theta - 7 \cos \theta - 4 = 0$ $\Rightarrow (2\cos \theta + 1)(\cos \theta - 4) = 0$ $\Rightarrow \cos \theta = -\frac{1}{2}$</p>	<p>$\cos \theta = -\frac{1}{2}$ $\cos(\frac{\pi}{3}) = -\frac{1}{2}$ $\theta = 120^\circ, 240^\circ$ $n=1,2,3, \dots$ $\theta = 120^\circ, 240^\circ$</p>
<p>(b) $2 \cos 2x = 4 \cos x - 3$ $\Rightarrow 2(2\cos^2 x - 1) = 4 \cos x - 3$ $\Rightarrow 4\cos^2 x - 2 = 4 \cos x - 3$ $\Rightarrow 4\cos^2 x - 4 \cos x + 1 = 0$ $\Rightarrow (2\cos x - 1)^2 = 0$ $\cos x = \frac{1}{2}$</p>	<p>$\cos x = \frac{1}{2}$ $\cos(\frac{\pi}{3}) = \frac{1}{2}$ $x = 60^\circ, 300^\circ$ $n=1,2,3, \dots$ $x = 60^\circ, 300^\circ$</p>
<p>(c) $6 \cos 2y + 5 \cos y + 3 = 0$ $\Rightarrow 6(2\cos^2 y - 1) + 5 \cos y + 3 = 0$ $\Rightarrow 12\cos^2 y - 6 + 5 \cos y + 3 = 0$ $\Rightarrow 12\cos^2 y + 5 \cos y - 3 = 0$ $\Rightarrow (3\cos y - 1)(4\cos y + 3) = 0$ $\cos y = -\frac{1}{3}$</p>	<p>$\cos y = -\frac{1}{3}$ $\cos(\frac{2\pi}{3}) = -\frac{1}{2}$ $y = 120^\circ, 240^\circ$ $(y = 210^\circ, 330^\circ)$ $y_1 = 120^\circ, 240^\circ$ $y_2 = 210^\circ, 330^\circ$</p>
<p>(d) $5 \cos 2\varphi + 22 \sin \varphi = 9$ $\Rightarrow 5(-2\sin^2 \varphi) + 22 \sin \varphi = 9$ $\Rightarrow 5 - 10\sin^2 \varphi + 22 \sin \varphi = 9$ $\Rightarrow 0 = 10\sin^2 \varphi - 22 \sin \varphi + 4$ $\Rightarrow 0 = 5\sin^2 \varphi - 11\sin \varphi + 2$ $\Rightarrow 0 = (\sin \varphi - 1)(5\sin \varphi - 2)$ $\sin \varphi = \frac{1}{5}$</p>	<p>$\sin(\frac{\pi}{6}) = \frac{1}{2}$ $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ $\varphi = 11.5^\circ, 168.5^\circ$ $\varphi = 60^\circ, 300^\circ$ $n=1,2,3, \dots$ $\varphi = 11.5^\circ, 168.5^\circ$</p>

Question 29

Solve each of the following trigonometric equations.

a) $\cos 2\theta + 9 \sin \theta + 4 = 0, \quad 0 \leq \theta < 360^\circ$

b) $3\cos 2x = 9 - 14 \cos x, \quad 0 \leq x < 360^\circ$

c) $2\cos 2y + 7 \cos y = 0, \quad 0 \leq y < 360^\circ$

d) $2\cos 2\varphi = 1 - 2 \sin \varphi, \quad 0 \leq \varphi < 360^\circ$

$\boxed{\theta = 210^\circ, 330^\circ}, \boxed{x \approx 48.2^\circ, 311.8^\circ}, \boxed{y \approx 75.5^\circ, 284.5^\circ}, \boxed{\varphi = 54^\circ, 126^\circ, 198^\circ, 342^\circ}$

<p>(a) $\cos 2\theta + 9 \sin \theta + 4 = 0$ $\rightarrow (1 - 2\sin^2 \theta) + 9\sin \theta + 4 = 0$ $\Rightarrow 0 = 2\sin^2 \theta - 9\sin \theta - 5$ $\Rightarrow (2\sin \theta + 1)(\sin \theta - 5) = 0$ $\Rightarrow \sin \theta = -\frac{1}{2}$</p>	<p>$\bullet \sin \theta = -\frac{1}{2}, \cos(\frac{\pi}{2}) = -30$ $(\theta = -30^\circ \pm 360^\circ)$ $(\theta = 210^\circ \pm 360^\circ)$ $\therefore \theta = 210^\circ, 330^\circ$</p>
<p>(b) $3\cos 2x = 9 - 14 \cos x$ $\rightarrow 3(2\cos^2 x - 1) = 9 - 14 \cos x$ $\Rightarrow 6\cos^2 x - 3 = 9 - 14 \cos x$ $\Rightarrow 6\cos^2 x + 14\cos x - 12 = 0$ $\Rightarrow 3\cos^2 x + 7\cos x - 6 = 0$ $\Rightarrow (3\cos x - 2)(\cos x + 3) = 0$ $\cos x = -3$</p>	<p>$\bullet \cos x = \frac{3}{2}, \cos(\frac{\pi}{2}) = 45^\circ$ $(x = 45^\circ \pm 360^\circ)$ $(x = 315^\circ \pm 360^\circ)$ $x_1 = 45^\circ$ $x_2 = 315^\circ$</p>
<p>(c) $2\cos 2y + 7 \cos y = 0$ $2(2\cos^2 y - 1) + 7 \cos y = 0$ $4\cos^2 y - 2 + 7 \cos y = 0$ $4\cos^2 y + 7 \cos y - 2 = 0$ $(4\cos y - 1)(\cos y + 2) = 0$ $\cos y = -2$</p>	<p>$\bullet \cos(\frac{\pi}{2}) = 75^\circ$ $(y = 75^\circ \pm 360^\circ)$ $(y = 285^\circ \pm 360^\circ)$ $y_1 = 75^\circ$ $y_2 = 285^\circ$</p>
<p>(d) $2\cos 2\varphi = 1 - 2 \sin \varphi$ $2(2\sin^2 \varphi - 1) = 1 - 2 \sin \varphi$ $2 - 4\sin^2 \varphi = 1 - 2 \sin \varphi$ $0 = 4\sin^2 \varphi - 2 \sin \varphi - 1$ $(4\sin \varphi - 1)(\sin \varphi + 1) = 0$ $\sin \varphi = \frac{1 \pm \sqrt{5}}{2}$ $\sin \varphi = \frac{1 \pm \sqrt{5}}{2}$</p>	<p>$\bullet \cos(\frac{1}{4}\pi) = 50^\circ$ $(\varphi = 50^\circ \pm 360^\circ)$ $(\varphi = -40^\circ \pm 360^\circ)$ $(\varphi = 190^\circ \pm 360^\circ)$ $\therefore \varphi = 50^\circ, 126^\circ, 242^\circ, 342^\circ$</p>

Question 30

Solve each of the following trigonometric equations.

a) $\tan \theta(1 + \cos 2\theta) = 2\sin^2 2\theta, \quad 0 \leq \theta \leq 90^\circ$

b) $4\tan 2\varphi + 3\cot \varphi \sec^2 \varphi = 0, \quad 0 \leq \varphi < 2\pi$ (hard)

$$\boxed{\theta = 0^\circ, 15^\circ, 75^\circ, 90^\circ}, \quad \boxed{\varphi = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$$