

VECTOR OPERATORS

GRADIENT

$$\text{grad } \varphi = \nabla \varphi$$

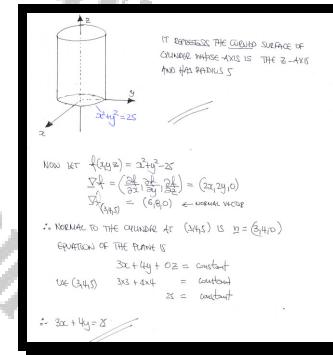
Question 1

A surface S is given by the Cartesian equation

$$x^2 + y^2 = 25.$$

- a) Draw a sketch of S , and describe it geometrically.
- b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(3, 4, 5)$.

$$3x + 4y = 25$$



Question 2

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Show by detailed workings that

$$\nabla r = \frac{\mathbf{r}}{r}.$$

proof

$$\begin{aligned}\nabla r &= \nabla \left[\sqrt{x^2 + y^2 + z^2} \right] = \left[\frac{\partial}{\partial x} \left[\sqrt{x^2 + y^2 + z^2} \right] \right] \hat{i} + \left[\frac{\partial}{\partial y} \left[\sqrt{x^2 + y^2 + z^2} \right] \right] \hat{j} + \left[\frac{\partial}{\partial z} \left[\sqrt{x^2 + y^2 + z^2} \right] \right] \hat{k} \\ &= \left[\frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \times 2x \right] \hat{i} + \left[\frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \times 2y \right] \hat{j} + \left[\frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \times 2z \right] \hat{k} \\ &= \left[\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right] \hat{i} + \left[\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right] \hat{j} + \left[\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right] \hat{k} \\ &= \frac{1}{r} (\hat{x}, \hat{y}, \hat{z}) = \frac{\mathbf{r}}{r}\end{aligned}$$

Question 3

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that \mathbf{a} is a constant vector find

$$\nabla(\mathbf{a} \cdot \mathbf{r}).$$

$$\boxed{\nabla(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a}}$$

$$\begin{aligned}\nabla(\mathbf{a} \cdot \mathbf{r}) &= \nabla[(a_1 a_2 a_3)(x_1 x_2 x_3)] = \nabla(a_1 a_2 a_3) \\ &= \left[\frac{\partial}{\partial x_1}(a_1 a_2), \frac{\partial}{\partial x_2}(a_1 a_2), \frac{\partial}{\partial x_3}(a_1 a_2) \right] = (a_1, a_2, a_3) = \mathbf{a}\end{aligned}$$

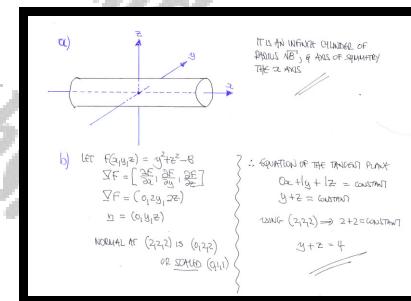
Question 4

A surface S is defined by the Cartesian equation

$$y^2 + z^2 = 8.$$

- a) Draw a sketch of S , and describe it geometrically.
- b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(2, 2, 2)$.

$$\boxed{y + z = 4}$$



Question 5

The scalar function V is defined as

$$V(x, y, z) = (y+z)^2 + y^2(x+y) + xyz + 1.$$

Determine the value of the directional derivative of V at the point $P(1, -1, 1)$, in the direction $-\mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\boxed{\sqrt{3}}$$

- $\nabla V = (y+z)^2 + y^2(x+y) + xyz + 1 = (y+z)^2 + y^2 + xyz + 1$
- $\vec{r}(1, -1, 1)$
- $\vec{u} = (-1, 1, 1)$

Finally: $\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right)$
 $\nabla V = \left[y^2 + 2yz + 2xy + 2y^2 + xyz, 2y + 2z + xy, 2y + z \right]$
 $\nabla V = \left[0, -2 + 2z + 1, 1 \right]$
 $\nabla V \Big|_{(1, -1, 1)} = (0, 2, 1)$

Now: $\vec{u} = (-1, 1, 1)$
 $|\vec{u}| = \sqrt{3}$
 $\hat{u} = \frac{1}{\sqrt{3}}(-1, 1, 1)$

So: DIRECTIONAL DERIVATIVE AT THE REQUIRED POINT AND DIRECTION

$$\nabla V \cdot \hat{u} = (0, 2, 1) \cdot \frac{1}{\sqrt{3}}(-1, 1, 1) = \frac{0+2+1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} \quad \checkmark$$

Question 6

The scalar function φ is defined as

$$\varphi(x, y, z) = e^{x-y} \sin z.$$

Determine the value of the directional derivative of φ at the point $P(1, 1, 0)$, in the direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\boxed{\frac{1}{\sqrt{3}}}$$

- One requires $\nabla \varphi \cdot \hat{u}$, evaluated at $(1, 1, 0)$
- $\nabla \varphi = (e^{x-y} \cos z, -e^{x-y} \sin z, e^{x-y})$
 - $\nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right) = \left[e^{x-y} \cos z, -e^{x-y} \sin z, e^{x-y} \right]$
 - $\vec{u} = (1, 1, 1) \implies \hat{u} = \frac{1}{\sqrt{3}}(1, 1, 1)$
 - $\nabla \varphi \cdot \hat{u} = e^{x-y} \left[\cos z - \sin z \cos z \right] = \frac{1}{\sqrt{3}} e^{x-y} (1, 1, 1)$
 - $\nabla \varphi \cdot \hat{u} = \frac{1}{\sqrt{3}} e^{x-y} \left[\cos z - \sin z \cos z \right] = \frac{1}{\sqrt{3}} e^{x-y} 2 \cos z$
 - $\nabla \varphi \cdot \hat{u} \Big|_{(1, 1, 0)} = \frac{1}{\sqrt{3}} e^{1-1} \cos 0 = \frac{1}{\sqrt{3}} \times 1 = \frac{1}{\sqrt{3}}$

Question 7

The point $P(1,2,3)$ lies on the surface with Cartesian equation

$$2z^2 = 6x^2 + 3y^2.$$

The scalar function u is defined as

$$u(x, y, z) = x^2yz + x^2y.$$

Determine the value of the directional derivative of u at the point P in the direction to the normal at P .

$6\sqrt{3}$

$\text{SURFACE } 2z^2 = 6x^2 + 3y^2$ $\text{let } f(x,y,z) = 2z^2 - 6x^2 - 3y^2$ $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$ $\nabla f = \left[2z \cdot 0, 2z \cdot 0, 2z \cdot 2z \right]$ $\nabla f \Big _{(1,2,3)} = \left[0, 0, 2z \cdot 2z \right]$ $\nabla f \Big _{(1,2,3)} = (16, 0, 2)$	$\text{SURFACE } 2z^2 = 6x^2 + 3y^2$ $\text{let } f(x,y,z) = 2z^2 - 6x^2 - 3y^2$ $\Rightarrow \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$ $\Rightarrow \nabla f = (-12x, -6y, 4z)$ $\Rightarrow \nabla f \Big _{(1,2,3)} = (-12, -12, 8)$ $\text{Take } \hat{n} = (1, 1, 1)$ $\hat{n} = \frac{1}{\sqrt{3}}(1, 1, 1)$
$\therefore \text{Directional derivative at } (1,2,3)$ $\nabla u \cdot \hat{n} = \left(16, 0, 2 \right) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$ $= \frac{16+0+2}{\sqrt{3}}$ $= \frac{18}{\sqrt{3}}$ $= \frac{18\sqrt{3}}{3}$ $= 6\sqrt{3}$	

Question 8

The point $P(1, y_0, z_0)$ lies on both surfaces with Cartesian equations

$$x^2 + y^2 + z^2 = 9 \quad \text{and} \quad z = x^2 + y^2 - 3$$

The two surfaces intersect each other at an angle θ , at the point P

Given further that P lies in the first octant, determine the exact value of $\cos \theta$

$$\cos \theta = \frac{8}{3\sqrt{21}}$$

$\text{P}(\{g, z\})$ $x^2 + y^2 + z^2 = 9$ \Rightarrow $\begin{cases} x^2 + y^2 = 9 - z^2 \\ x^2 + z^2 = 9 \end{cases}$ \Rightarrow $\begin{cases} x^2 + y^2 = 9 - z^2 \\ x^2 + z^2 = 9 + 3z \end{cases}$

* $z^2 + 3 = 9$
 $z^2 = 6 \Rightarrow 0$
 $(z+3)(z-3) = 0$
 $\begin{cases} z = 2 \\ z = -2 \end{cases}$

$\begin{cases} x^2 = 4 \\ y^2 = 4 \end{cases}$ $\begin{cases} x^2 = 4 \\ z^2 = 4 \end{cases}$

$\therefore \text{P}(\{zz\})$

• LET $\begin{cases} x^2 + y^2 + z^2 = 9 \\ x^2 = \left(\frac{2x_1}{3}, \frac{2y_1}{3}, \frac{2z_1}{3}\right)^2 \\ \sum_{i=1}^3 = (2x_1, 2y_1, 2z_1) \\ \underline{\underline{x_1}} = (2x_1, 2y_1, 2z_1) \end{cases}$

$\begin{cases} g(x_1, y_1, z_1) = x^2 + y^2 + z^2 - 9 \\ g(x_1, y_1, z_1) = \left(\frac{2x_1}{3}, \frac{2y_1}{3}, \frac{2z_1}{3}\right)^2 \\ \sum_{i=1}^3 = (2x_1, 2y_1, 2z_1 - 1) \\ \underline{\underline{x_1}} = (2x_1, 2y_1, 2z_1 - 1) \end{cases}$

$\underline{\underline{x_1}} = (2x_1, 2y_1, 2z_1)$ $\underline{\underline{x_1}} = (2x_1, 2y_1, 2z_1 - 1)$

AT P P

$\underline{\underline{x_1}} = (1, 2, 2)$ $\underline{\underline{x_1}} = (2, 1, 1)$

• FOR THE $\underline{\underline{x_1}}$ THAT NOT

$\begin{cases} (1, 2, 2), (2, 1, 1) = [1, 2, 2][2, 1, 1] \cos 67^\circ \\ 2 + 8 - 2 = \sqrt{1+4+4} \cdot \sqrt{4+16+1} \cdot \cos 67^\circ \\ 8 = 3\sqrt{17} \cos 67^\circ \\ \underline{\underline{x_1}} = \frac{8}{3\sqrt{17}} \end{cases}$

Question 9

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that $\varphi(r) = \ln r$, show that

$$\nabla \varphi(r) = \frac{\mathbf{r}}{r^2}.$$

[proof]

$$\varphi(3, 4, 2) = \ln |r| \text{ where } r = (3, 4, 2)$$

$$\varphi(3, 4, 2) = \ln (\sqrt{3^2 + 4^2 + 2^2})$$

$$\frac{d}{dr}\varphi(r) = \frac{1}{r} \ln(\sqrt{3^2 + 4^2 + 2^2})$$

$$\begin{aligned} \nabla \varphi &= \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right) = \left[\frac{1}{r} \cdot \frac{2x}{\sqrt{3^2 + 4^2 + 2^2}}, \frac{1}{r} \cdot \frac{2y}{\sqrt{3^2 + 4^2 + 2^2}}, \frac{1}{r} \cdot \frac{2z}{\sqrt{3^2 + 4^2 + 2^2}} \right] \\ &= \frac{1}{\sqrt{3^2 + 4^2 + 2^2}} [3, 4, 2] = \frac{1}{\sqrt{29}} [3, 4, 2] \end{aligned}$$

Question 10

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Show by detailed workings that

$$\nabla r^3 \equiv 3r\mathbf{r}.$$

[proof]

$$\begin{aligned} \nabla(|\mathbf{r}|^3) &= \nabla((x^2 + y^2 + z^2)^{\frac{3}{2}}) \\ &= \left[\frac{\partial}{\partial x} [(x^2 + y^2 + z^2)^{\frac{3}{2}}], \frac{\partial}{\partial y} [(x^2 + y^2 + z^2)^{\frac{3}{2}}], \frac{\partial}{\partial z} [(x^2 + y^2 + z^2)^{\frac{3}{2}}] \right] \\ &\quad \downarrow \\ &= \frac{3}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}} (2x, 2y, 2z) \\ &\quad \text{AND BY CYCLIC SYMMETRY ...} \\ &= \left[3x(x^2 + y^2 + z^2)^{\frac{1}{2}}, 3y(x^2 + y^2 + z^2)^{\frac{1}{2}}, 3z(x^2 + y^2 + z^2)^{\frac{1}{2}} \right] \\ &= 3(x^2 + y^2 + z^2)^{\frac{1}{2}} [x, y, z] \\ &= 3|\mathbf{r}| \mathbf{r} \end{aligned}$$

Question 11

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that $\psi(r) = \frac{1}{r}$, show that

$$\nabla \psi(r) = -\frac{\mathbf{r}}{r^3}.$$

[proof]

$$\begin{aligned}\Psi(x,y,z) &= \frac{1}{|\mathbf{r}|} & \mathbf{r} &= (x^2+y^2+z^2)^{\frac{1}{2}} \\ \nabla \Psi &= \left(\frac{\partial \Psi}{\partial x}, \frac{\partial \Psi}{\partial y}, \frac{\partial \Psi}{\partial z} \right) = \left[\frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right] \\ &= \frac{-1}{(x^2+y^2+z^2)^{\frac{3}{2}}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{\mathbf{r}}{|\mathbf{r}|^3} = -\frac{\mathbf{r}}{r^3}\end{aligned}$$

Question 12

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Show clearly that

$$\nabla(r^n) = nr^{n-2}\mathbf{r}.$$

[proof]

$$\begin{aligned}\nabla(|\mathbf{r}|^n) &= \nabla((x^2+y^2+z^2)^{\frac{n}{2}}) = \nabla[(x^2+y^2+z^2)^{\frac{n}{2}}] \\ &= \left[\frac{\partial}{\partial x} (x^2+y^2+z^2)^{\frac{n}{2}}, \frac{\partial}{\partial y} (x^2+y^2+z^2)^{\frac{n}{2}}, \frac{\partial}{\partial z} (x^2+y^2+z^2)^{\frac{n}{2}} \right] \\ &= \left[\frac{n}{2}x\cdot 2x (x^2+y^2+z^2)^{\frac{n-2}{2}}, \frac{n}{2}y\cdot 2y (x^2+y^2+z^2)^{\frac{n-2}{2}}, \frac{n}{2}z\cdot 2z (x^2+y^2+z^2)^{\frac{n-2}{2}} \right] \\ &= n(x^2+y^2+z^2)^{\frac{n-2}{2}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = n(x^2+y^2+z^2)^{\frac{n-2}{2}} \mathbf{r} \\ &= n(x^2+y^2+z^2)^{\frac{n-2}{2}} (x^2+y^2+z^2)^{\frac{1}{2}} = n|\mathbf{r}|^{\frac{n-2}{2}}\end{aligned}$$

Question 13

The surface S has Cartesian equation

$$f(x, y, z) = \text{constant},$$

where f is a smooth function.

Given that $\nabla f \neq \mathbf{0}$, show that ∇f is a normal to S .

proof

SUPPOSE THE EQUATION OF THE SURFACE IS $f(x, y, z) = \text{constant}$
LET A CIRCLE C_1 LIE ON THE AREA DESCRIBED SURFACE. A CIRCLE IS A CURVE WHICH POSITION CAN BE GIVEN AS A FUNCTION OF A SINGLE PARAMETER.

say C_1 CAN BE DESCRIBED AS $\mathbf{r} = \mathbf{r}(t)$, t IS A PARAMETER.

ON THE SURFACE C_1

$$f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$$

Differentiating w.r.t t , AND NOTING THAT $f(x, y, z) = \text{constant}$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

$$0 = \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial t} \frac{\partial x}{\partial z} \right) \cdot \left(\frac{\partial z}{\partial t} \frac{\partial x}{\partial t} \frac{\partial x}{\partial z} \right)$$

$$0 = \nabla f \cdot \frac{d\mathbf{r}}{dt}$$

BUT $\frac{d\mathbf{r}}{dt}$ IS A TANGENT TO THE CURVE (CURVE DIFFERENTIATION).

SO ∇f IS PERPENDICULAR TO THIS TANGENT.

THE SAME ARGUMENT CAN BE APPLIED TO ANOTHER CURVE C_2 INTERSECTING C_1 AT SUCH POINT P ON THE SURFACE.

SINCE ∇f WILL BE PERPENDICULAR TO TWO DIFFERENT TANGENTS TO THE SURFACE AT P , THEN ∇f MUST BE A NORMAL TO THE SURFACE AT P .

Question 14

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that $f(r)$ is a differentiable function, show that

$$\nabla f(r) = \frac{\mathbf{r}}{r} f'(r).$$

, proof

MANIPULATE TO EXTRACT

$$\begin{aligned}\nabla f(r) &= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = \left[\frac{\partial f}{\partial r} \frac{\partial r}{\partial x}, \frac{\partial f}{\partial r} \frac{\partial r}{\partial y}, \frac{\partial f}{\partial r} \frac{\partial r}{\partial z} \right] \\ &= \frac{\partial f}{\partial r} \left[\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right]\end{aligned}$$

NOW WE HAVE

$$\begin{aligned}\Rightarrow \mathbf{f} &= (x, y, z) \\ \rightarrow \mathbf{r} &= |\mathbf{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}} \\ \Rightarrow \frac{\partial \mathbf{r}}{\partial x} &= \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(2x) = \frac{x}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{x}{r}\end{aligned}$$

AND SIMILARLY $\frac{\partial \mathbf{r}}{\partial y}$ & $\frac{\partial \mathbf{r}}{\partial z}$ ARE THERE IS CYCLIC SYMMETRY

RETURNING TO THE MAIN LINE WE OBTAIN

$$\begin{aligned}\dots &= \frac{\partial f}{\partial r} \left[\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right] = f'(r) \times \frac{1}{r} (x, y, z) = \frac{1}{r} \mathbf{f}'(r)\end{aligned}$$

To expand

NOTE THE INITIAL MANIPULATION CAN BE THOUGHT OF A STANDARD CHAIN RULE

$$\nabla f(r) = f(r) \nabla \mathbf{r} = f(r) \left[\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right] = \dots \text{AS ABOVE}$$

Question 15

The smooth functions $F(x, y, z)$ and $G(x, y, z)$ are given.

Show that

$$\nabla \left[\frac{F}{G} \right] = \frac{G(\nabla F) - F(\nabla G)}{G^2}.$$

proof

Let $F = F(x, y, z)$
 $G = G(x, y, z)$

- CONSIDER THE \hat{i} component of the gradient
 $\nabla_i \left[\frac{F}{G} \right] = \frac{\partial}{\partial x} \left[\frac{F}{G} \right] = \frac{G \frac{\partial F}{\partial x} - F \frac{\partial G}{\partial x}}{G^2}$
- AND SIMILARLY THE \hat{j} & \hat{k}
- $\therefore \nabla \left[\frac{F}{G} \right] = \left[\frac{G \frac{\partial F}{\partial x} - F \frac{\partial G}{\partial x}}{G^2}, \frac{G \frac{\partial F}{\partial y} - F \frac{\partial G}{\partial y}}{G^2}, \frac{G \frac{\partial F}{\partial z} - F \frac{\partial G}{\partial z}}{G^2} \right]$

$$\begin{aligned} &= \frac{1}{G^2} \left[G \frac{\partial F}{\partial x} - F \frac{\partial G}{\partial x}, G \frac{\partial F}{\partial y} - F \frac{\partial G}{\partial y}, G \frac{\partial F}{\partial z} - F \frac{\partial G}{\partial z} \right] \\ &= \frac{1}{G^2} \left[G \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right] - F \left[\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z} \right] \right] \\ &= \frac{1}{G^2} \left[G \nabla F - F \nabla G \right] \end{aligned}$$

as required

Question 16

$$\Psi(x, y, z) = (x^2 + y^2 + z^2) e^{-\sqrt{x^2 + y^2 + z^2}}$$

Show that

$$\nabla [\Psi(x, y, z)] = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \left(2 - \sqrt{x^2 + y^2 + z^2} \right) e^{-\sqrt{x^2 + y^2 + z^2}}.$$

proof

$\Psi(x, y, z) = (x^2 + y^2 + z^2) e^{-\sqrt{x^2 + y^2 + z^2}}$

Let $r = \sqrt{x^2 + y^2 + z^2}$
 $\vec{r} = \langle x, y, z \rangle$
 $\vec{r} = |\vec{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}} \Rightarrow \Psi(r) = r^2 e^{-r}$

YOU KNOW THE STANDARD FORM OF GRADIENT
 $\nabla [f(r)] = f'(r) \frac{\vec{r}}{r}$

$$\begin{aligned} \nabla [\Psi(r)] &= \Psi'(r) \frac{\vec{r}}{r} = \left[2re^{-r} - r^2 e^{-r} \right] \frac{\vec{r}}{r} \\ &= \left[2e^{-r} - r^2 e^{-r} \right] \vec{r} = (2 - r^2 e^{-r}) \vec{r} \\ \therefore \nabla \Psi(\vec{r}) &= \left[2 - \sqrt{x^2 + y^2 + z^2} \right] e^{-\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle \end{aligned}$$

Question 17

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Show that

$$\nabla \left(9r^2 + \frac{4}{r} - 12\sqrt{r} \right) = 2 \left(3 - 2r^{-\frac{3}{2}} \right) \left(3 + r^{-\frac{3}{2}} \right) \mathbf{r}.$$

proof

By the chain rule

$$\begin{aligned} \nabla \left[9r^2 + \frac{4}{r} - 12\sqrt{r} \right] &= \frac{\partial}{\partial r} \left[9r^2 + \frac{4}{r} - 12\sqrt{r} \right] \\ &= \frac{\partial}{\partial r} \left(9r^2 + \frac{4}{r} - 12\sqrt{r} \right) \left(\frac{\partial}{\partial r} \left[9r^2 + \frac{4}{r} - 12\sqrt{r} \right] \right) \\ &= \frac{\partial}{\partial r} \left(9r^2 + \frac{4}{r} - 12\sqrt{r} \right) \cdot \frac{1}{|\mathbf{r}|} \frac{\partial}{\partial r} \mathbf{r} \end{aligned}$$

Thus

$$\begin{aligned} \nabla \left[9r^2 + \frac{4}{r} - 12\sqrt{r} \right] &= \nabla \left[3|\mathbf{r}|^2 - 2|\mathbf{r}|^{-\frac{3}{2}} \right] \\ &= 2 \left[3|\mathbf{r}| - 2|\mathbf{r}|^{-\frac{5}{2}} \right] \left[3 + |\mathbf{r}|^{-\frac{3}{2}} \right] \frac{\mathbf{r}}{|\mathbf{r}|} \\ &= 2 \left[3 - 2|\mathbf{r}|^{-\frac{3}{2}} \right] \left[3 + |\mathbf{r}|^{-\frac{3}{2}} \right] \mathbf{r} \end{aligned}$$

As required

Question 18

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

The smooth function $f(r)$ satisfies

$$\nabla [f(r)] = 10r^3 \mathbf{r}.$$

Determine a simplified expression for $f(r)$.

$$f(r) = 2r^5 + C$$

Observe

$$\nabla \left[\frac{1}{r} \mathbf{r} \right] \text{ where } \mathbf{r} = |\mathbf{r}| \text{ with } \frac{1}{r} = \frac{1}{|\mathbf{r}|} \text{ and } \frac{\mathbf{r}}{r} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\mathbf{r}}{(\mathbf{r}^2 + \mathbf{u}^2 + \mathbf{v}^2)^{\frac{1}{2}}}.$$

By the chain rule

$$\begin{aligned} \nabla \left[\frac{1}{r} \mathbf{r} \right] &= f'(r) \nabla \mathbf{r} = f'(r) \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \\ &= f'(r) \left[\frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2+y^2+z^2}} \right), \frac{\partial}{\partial y} \left(\frac{y}{\sqrt{x^2+y^2+z^2}} \right), \frac{\partial}{\partial z} \left(\frac{z}{\sqrt{x^2+y^2+z^2}} \right) \right] \\ &= f'(r) \left[\frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{1}{\sqrt{x^2+y^2+z^2}} \right] = \frac{1}{r^2} f'(r) \mathbf{r} \end{aligned}$$

Now

$$10r^3 \mathbf{r} = 10r^3 \left(\frac{\mathbf{r}}{r} \right) = \dots \nabla \left(\frac{1}{r} \mathbf{r} \right) = \frac{1}{r^2} f'(r) \mathbf{r}$$

$$\therefore f'(r) = 2r^5 + C$$

DIVERGENCE

$$\operatorname{div} \mathbf{F} \equiv \nabla \cdot \mathbf{F}$$

Question 1

A Cartesian position vector is denoted by \mathbf{r} .

Determine the value of

$$\nabla \cdot \mathbf{r}.$$

$$\boxed{\nabla \cdot \mathbf{r} = 3}$$

$$\nabla \cdot \mathbf{r} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(x, y, z \right) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

Question 2

A Cartesian position vector is denoted by \mathbf{r} .

Given that \mathbf{a} is a constant vector, find

$$\nabla \cdot (\mathbf{a} \wedge \mathbf{r}).$$

$$\boxed{\nabla \cdot (\mathbf{a} \wedge \mathbf{r}) = 0}$$

$$\begin{aligned}\nabla \cdot (\mathbf{a} \wedge \mathbf{r}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} \\ &= \frac{\partial}{\partial x}(a_2 - a_3y) + \frac{\partial}{\partial y}(a_3 - a_1z) + \frac{\partial}{\partial z}(a_1 - a_2x) \\ &= 0 + 0 + 0 \\ &= 0\end{aligned}$$

Question 3

A Cartesian position vector is denoted by \mathbf{r} .

Given that \mathbf{a} is a constant vector, show that

$$\nabla \cdot [(\mathbf{a} \cdot \mathbf{r}) \mathbf{r}] = 4\mathbf{a} \cdot \mathbf{r}.$$

proof

$$\begin{aligned}
 \nabla \cdot [(\mathbf{a} \cdot \mathbf{r}) \mathbf{r}] &= (\mathbf{a} \cdot \mathbf{r}) \nabla \cdot \mathbf{r} + \nabla (\mathbf{a} \cdot \mathbf{r}) \cdot \mathbf{r} \\
 \nabla \cdot [\mathbf{a} \mathbf{r}] &= \mathbf{a} \nabla \cdot \mathbf{r} + \nabla \cdot [\mathbf{a} \mathbf{r}] \\
 &= [a_1 a_2 a_3] [x \frac{\partial}{\partial x} y \frac{\partial}{\partial y} z \frac{\partial}{\partial z}] + \nabla \cdot (\mathbf{a} \cdot \mathbf{r}) \cdot (a_1 a_2) \\
 &\quad \text{a red bracket highlights } \mathbf{a} \cdot \mathbf{r} \\
 &= (a_1 a_2 + a_2 a_3 + a_3 a_1) [x \frac{\partial}{\partial x} y \frac{\partial}{\partial y} z \frac{\partial}{\partial z}] + (a_1 a_2, a_1 a_3, a_2 a_3) \\
 &= 3(a_1 a_2 + a_2 a_3 + a_3 a_1) + (a_1 a_2, a_1 a_3, a_2 a_3) \\
 &= 4(a_1 a_2 + a_2 a_3 + a_3 a_1) \\
 &= 4(a_1 a_2, a_1 a_3, a_2 a_3) \\
 &= 4 \Delta \cdot \mathbf{r} \\
 &\quad \text{As required}
 \end{aligned}$$

Question 4

A Cartesian position vector is denoted by \mathbf{r} .

Given that \mathbf{a} is a constant vector, show that

$$\nabla \cdot [\mathbf{r} \wedge (\mathbf{r} \wedge \mathbf{a})] = 2\mathbf{a} \cdot \mathbf{r}.$$

proof

$$\begin{aligned}
 \nabla \cdot [\mathbf{r} \wedge (\mathbf{r} \wedge \mathbf{a})] &= (\mathbf{r} \cdot \mathbf{a}) \nabla \cdot \mathbf{r} - \mathbf{r} \cdot \nabla \cdot (\mathbf{r} \wedge \mathbf{a}) \\
 \nabla \cdot (\mathbf{r} \wedge \mathbf{b}) &= \mathbf{b} \cdot (\nabla \cdot \mathbf{r}) - \mathbf{r} \cdot (\nabla \cdot \mathbf{b}) \\
 \nabla \cdot \mathbf{r} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (0, 0, 0) \\
 &= -\mathbf{r} \cdot \nabla \cdot (\mathbf{r} \wedge \mathbf{a}) = \dots \\
 &= -\mathbf{r} \cdot (-2\mathbf{a}) \\
 &= 2\mathbf{r} \cdot \mathbf{a} \\
 &= 2\mathbf{r} \cdot \mathbf{a} \\
 &\quad \text{As required}
 \end{aligned}$$

$$\begin{aligned}
 \bullet \quad \mathbf{r} \cdot \mathbf{a} &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (a_3 - a_2, a_2 - a_3, a_3 - a_1) \\
 &= (a_3 - a_2, a_2 - a_3, a_3 - a_1) \\
 \bullet \quad \nabla \cdot (\mathbf{r} \wedge \mathbf{a}) &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-a_1 - a_2, a_2 - a_3, a_3 - a_1) \\
 &= (-a_1 - a_2, a_2 - a_3, a_3 - a_1) \\
 &= (-2a_1, 2a_2, -2a_3) \\
 &= -2\mathbf{a}
 \end{aligned}$$

Question 5

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show clearly that

$$\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^3} \right) = 0.$$

proof

$\frac{1}{|\mathbf{r}|} = (x^2+y^2+z^2)^{\frac{1}{2}}$

$$\begin{aligned} \nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^3} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \\ &= \frac{\partial}{\partial x} \left[\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right] + \frac{\partial}{\partial y} \left[\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right] + \frac{\partial}{\partial z} \left[\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right] \\ &\quad \downarrow \\ &= \frac{(x^2+y^2+z^2)^{\frac{3}{2}}(-2 + \frac{3}{2}x^2/(x^2+y^2+z^2))}{(x^2+y^2+z^2)^{\frac{5}{2}}} \\ &= \frac{(x^2+y^2+z^2)^{\frac{1}{2}}[(x^2+y^2+z^2)-3x^2]}{(x^2+y^2+z^2)^{\frac{5}{2}}} = \frac{-2x^2y^2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} \end{aligned}$$

As the expression is SIMMETRIC

$$\nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{-2x^2y^2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{-x^2z^2y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{x^2y^2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0$$

ALTERNATE

$$\begin{aligned} \nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^3} &= \nabla \cdot \left[\mathbf{r} \cdot \frac{1}{|\mathbf{r}|^3} \right] = (\nabla \cdot \mathbf{r}) \frac{1}{|\mathbf{r}|^3} + \mathbf{r} \cdot \nabla \left(\frac{1}{|\mathbf{r}|^3} \right) \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (x, y, z) \cdot \frac{1}{|\mathbf{r}|^3} + \mathbf{r} \cdot \left[-\frac{3}{|\mathbf{r}|^4} \nabla |\mathbf{r}| \right] \\ &= 3x \frac{1}{|\mathbf{r}|^3} - \frac{3}{|\mathbf{r}|^4} \mathbf{r} \cdot \nabla |\mathbf{r}| \\ &= \frac{3}{|\mathbf{r}|^3} - \frac{3}{|\mathbf{r}|^3} = 0 \end{aligned}$$

NOTE: $\nabla |\mathbf{r}| = \frac{\mathbf{r}}{|\mathbf{r}|}$

Question 6

A Cartesian position vector is denoted by \mathbf{r} .

Given that \mathbf{m} is a constant vector, show that

$$\nabla \cdot \left(\frac{\mathbf{m} \wedge \mathbf{r}}{|\mathbf{r}|^3} \right) = 0.$$

, proof

FOOTER: Define the components of some vectors

- $\mathbf{m} = (m_1, m_2, m_3)$ constant vector
- $\mathbf{r} = (x, y, z)$ variable position vector
- $\mathbf{m} \wedge \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ m_1 & m_2 & m_3 \\ x & y & z \end{vmatrix} = (m_2 - m_3, m_3 - m_1, m_1 - m_2)$

Putting all the results together

$$\nabla \cdot \left(\frac{\mathbf{m} \wedge \mathbf{r}}{r^3} \right) = \nabla \cdot \left[\frac{m_2 - m_3}{(x^2 + y^2 + z^2)^{3/2}}, \frac{m_3 - m_1}{(x^2 + y^2 + z^2)^{3/2}}, \frac{m_1 - m_2}{(x^2 + y^2 + z^2)^{3/2}} \right]$$

Apply the divergence operator

$$\begin{aligned} \dots &= \frac{\partial}{\partial x} \left[\frac{m_2 - m_3}{(x^2 + y^2 + z^2)^{3/2}} \right] + \frac{\partial}{\partial y} \left[\frac{m_3 - m_1}{(x^2 + y^2 + z^2)^{3/2}} \right] + \frac{\partial}{\partial z} \left[\frac{m_1 - m_2}{(x^2 + y^2 + z^2)^{3/2}} \right] \\ &= (m_2 - m_3) \left(\frac{3}{2} \right) (2(x^2 + z^2))^{-\frac{1}{2}} \\ &\quad (m_3 - m_1) \left(\frac{3}{2} \right) (2(y^2 + z^2))^{-\frac{1}{2}} \\ &\quad (m_1 - m_2) \left(\frac{3}{2} \right) (2(x^2 + y^2))^{-\frac{1}{2}} \\ &= 3(x^2 + y^2 + z^2)^{-\frac{1}{2}} \left[2(m_3 - m_2) + 2(m_1 - m_3) + 2(m_2 - m_1) \right] \\ &= 3(x^2 + y^2 + z^2)^{-\frac{1}{2}} \left[m_{12} - m_{13} + m_{23} - 2m_1 + m_{12} - m_{13} \right] \\ &= 0 \end{aligned}$$

AS REQUIRED

Question 7

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show clearly that

$$\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|} \right) = \frac{2}{|\mathbf{r}|}.$$

[proof]

- THE UNIT VECTOR OF A POSITION VECTOR $\mathbf{r} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ IS
 $\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(\sqrt{x^2+y^2+z^2})}\mathbf{e}$
- TAKING THE DIVERGENCE OF $\hat{\mathbf{r}}$ FROM FIRST PRINCIPLES

$$\begin{aligned} \nabla \cdot \hat{\mathbf{r}} &= \nabla \cdot \left[\frac{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(\sqrt{x^2+y^2+z^2})} \right] \\ &= \left[\frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2+y^2+z^2}} \right) \right] + \left[\frac{\partial}{\partial y} \left(\frac{y}{\sqrt{x^2+y^2+z^2}} \right) \right] + \left[\frac{\partial}{\partial z} \left(\frac{z}{\sqrt{x^2+y^2+z^2}} \right) \right] \\ &= \frac{\left(x^2+y^2+z^2 \right)^{\frac{1}{2}} - 2x^2\left(x^2+y^2+z^2 \right)^{-\frac{3}{2}}}{\left(x^2+y^2+z^2 \right)^{\frac{1}{2}}} + \frac{\left(x^2+y^2+z^2 \right)^{\frac{1}{2}} - 2y^2\left(x^2+y^2+z^2 \right)^{-\frac{3}{2}}}{\left(x^2+y^2+z^2 \right)^{\frac{1}{2}}} \\ &\quad + \frac{\left(x^2+y^2+z^2 \right)^{\frac{1}{2}} - 2z^2\left(x^2+y^2+z^2 \right)^{-\frac{3}{2}}}{\left(x^2+y^2+z^2 \right)^{\frac{1}{2}}} \\ &= \frac{\left(2x^2+2y^2+2z^2 \right)^{\frac{1}{2}} \left(x^2+y^2+z^2 \right)^{-\frac{3}{2}}}{x^2+y^2+z^2} + \frac{\left(2x^2+2y^2+2z^2 \right)^{\frac{1}{2}} \left(x^2+y^2+z^2 \right)^{-\frac{3}{2}}}{x^2+y^2+z^2} \\ &\quad + \frac{\left(2x^2+2y^2+2z^2 \right)^{\frac{1}{2}} \left(x^2+y^2+z^2 \right)^{-\frac{3}{2}}}{x^2+y^2+z^2} \\ &= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{2}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{2}{|\mathbf{r}|} \end{aligned}$$

Question 8

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show, with a detailed method, that

$$\nabla \cdot (\mathbf{r}|\mathbf{r}|^3) = 6|\mathbf{r}|^3.$$

proof

$$\begin{aligned}
 \nabla \cdot (\mathbf{r}^3) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left[(x^2+y^2+z^2)^{\frac{3}{2}} (x, y, z) \right] \\
 &= \frac{\partial}{\partial x} \left[x(x^2+y^2+z^2)^{\frac{3}{2}} \right] + \frac{\partial}{\partial y} \left[y(x^2+y^2+z^2)^{\frac{3}{2}} \right] + \frac{\partial}{\partial z} \left[z(x^2+y^2+z^2)^{\frac{3}{2}} \right] \\
 &\quad \text{By using product rule} \\
 &= (x^2+y^2+z^2)^{\frac{1}{2}} + 3x^2(x^2+y^2+z^2)^{\frac{1}{2}} \\
 &= (x^2+y^2+z^2)^{\frac{1}{2}} [1 + 3(x^2+y^2+z^2)] \\
 &= (x^2+y^2+z^2)^{\frac{1}{2}} (3x^2+y^2+z^2) \\
 &= r(3x^2+y^2+z^2) \\
 &\quad \text{By using similarity we deduce} \\
 \nabla \cdot (\mathbf{r}^3) &= r(x^2+y^2+z^2) + r(x^2+y^2+z^2) + r(x^2+y^2+z^2) \\
 &= r[3x^2+y^2+z^2] \\
 &= 6r(x^2+y^2+z^2) \\
 &= 6r(r^2) \\
 &= 6r^3
 \end{aligned}$$

Question 9

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Show that

$$\nabla \cdot (r^n \mathbf{r}) = (n+3)r^n.$$

, proof

• START "EXPANDING" BY THE VECTOR CALCULUS IDENTITY

$$\nabla \cdot (\frac{1}{r} \Delta) = r^2 \nabla \cdot \nabla + \nabla r^2 \cdot \Delta$$

$$\Rightarrow \nabla \cdot (r^n \mathbf{r}) = r^n \nabla \cdot \nabla + r^{n-2} \nabla r^n \cdot \Delta$$

$$= r^n \left[\frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \right] (x,y,z) + \left[\frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial z^2} \right] (x,y,z)$$

$$= r^n \left[1 + 1 + \square \right] + \left[1 + 1 + \square \right]$$

$$= r^n [1 + 1 + \square] + [1 + 1 + \square]$$

• Now $\mathbf{r} = \langle 1, 1, 1 \rangle = \langle x, y, z \rangle = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{\frac{1}{2}}$

$$r^n = (x^2 + y^2 + z^2)^{\frac{n}{2}}$$

$$\frac{\partial}{\partial x} (r^n) = \frac{\partial}{\partial x} ((x^2 + y^2 + z^2)^{\frac{n}{2}}) = n x (x^2 + y^2 + z^2)^{\frac{n-2}{2}}$$

AND SIMILARLY THE REST AS THESE EXPRESSIONS ARE SYMMETRICAL

• TRYING OUT FURTHER, WE GET

$$\nabla \cdot (r^n \mathbf{r}) = 3r^n + r^{n-2}(x^2 + y^2 + z^2)^{\frac{n-2}{2}} + n y (x^2 + y^2 + z^2)^{\frac{n-2}{2}} + n z (x^2 + y^2 + z^2)^{\frac{n-2}{2}}$$

$$= 3r^n + n (x^2 + y^2 + z^2)^{\frac{n-2}{2}} [x^2 + y^2 + z^2]^{\frac{1}{2}}$$

$$= 3r^n + n (x^2 + y^2 + z^2)^{\frac{n-2}{2}}$$

$$= 3r^n + n [(x^2 + y^2 + z^2)^{\frac{n-2}{2}}]^2$$

$$= 3r^n + n r^n$$

$$= (n+3) r^n$$

AS REQUIRED

Question 10

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show, with a detailed method, that

$$\nabla \cdot \left[|\mathbf{r}| \nabla \left(\frac{1}{|\mathbf{r}|^3} \right) \right] = \frac{3}{|\mathbf{r}|^4}.$$

proof

$\nabla \cdot \left[\mathbf{r} \nabla \left(\frac{1}{|\mathbf{r}|^3} \right) \right] = \mathbf{r} \nabla \cdot \nabla \left(\frac{1}{|\mathbf{r}|^3} \right) + \nabla \left(\frac{1}{|\mathbf{r}|^3} \right) \cdot \nabla \mathbf{r}$

$\nabla \left(\frac{1}{|\mathbf{r}|^3} \right) = \frac{1}{4} \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \frac{1}{|\mathbf{r}|^3}$

$\nabla \cdot \left[\mathbf{A} \cdot \nabla \frac{1}{|\mathbf{r}|^3} \right] = \mathbf{A} \cdot \nabla^2 \left(\frac{1}{|\mathbf{r}|^3} \right) + \nabla \left(\frac{1}{|\mathbf{r}|^3} \right) \cdot \nabla \mathbf{A}$

$\nabla^2 \left(\frac{1}{|\mathbf{r}|^3} \right) = \frac{3}{|\mathbf{r}|^4}$

NOW

$$\begin{aligned} \nabla \left(\frac{1}{|\mathbf{r}|^3} \right) &= \nabla \left[\left(x^2 + y^2 + z^2 \right)^{-\frac{3}{2}} \right] \\ &= \left[\frac{\partial}{\partial x} \left(x^2 + y^2 + z^2 \right)^{-\frac{3}{2}} \right] \hat{\mathbf{x}} + \left[\frac{\partial}{\partial y} \left(x^2 + y^2 + z^2 \right)^{-\frac{3}{2}} \right] \hat{\mathbf{y}} + \left[\frac{\partial}{\partial z} \left(x^2 + y^2 + z^2 \right)^{-\frac{3}{2}} \right] \hat{\mathbf{z}} \\ &= \left[-\frac{3x}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \right] \hat{\mathbf{x}} + \left[-\frac{3y}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \right] \hat{\mathbf{y}} + \left[-\frac{3z}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \right] \hat{\mathbf{z}} \end{aligned}$$

INTERCHANGE WITH INDEX 3. AREA

$\frac{\partial}{\partial x} \left[\frac{-3x}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \right] = \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}}(-3)(x^2 + y^2 + z^2)^{\frac{1}{2}}(2x)}{(x^2 + y^2 + z^2)^3}$

$$\begin{aligned} &= \frac{-3(x^2 + y^2 + z^2)^{\frac{1}{2}}(x^2 + y^2 + z^2)^{\frac{1}{2}}(2x)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \\ &= -\frac{3(x^2 + y^2 + z^2)^{\frac{1}{2}}(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \\ &= \frac{12x^2 - 3y^2 - 3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \end{aligned}$$

Thus $\nabla^2 \left(\frac{1}{|\mathbf{r}|^3} \right) = \nabla \cdot \left(\nabla \left(\frac{1}{|\mathbf{r}|^3} \right) \right) = \frac{\partial^2}{\partial x^2} \left(\frac{1}{|\mathbf{r}|^3} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{1}{|\mathbf{r}|^3} \right) + \frac{\partial^2}{\partial z^2} \left(\frac{1}{|\mathbf{r}|^3} \right)$

$$\begin{aligned} &= \frac{12x^2 - 3y^2 - 3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \frac{-3y^2 - 3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} + \frac{-3z^2 - 3y^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \\ &= \frac{6x^2 + 6y^2 + 6z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = \frac{6(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = \frac{6}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \end{aligned}$$

COLLECTING ALL THE RESULTS

$$\begin{aligned} \nabla \cdot \left[\mathbf{r} \nabla \left(\frac{1}{|\mathbf{r}|^3} \right) \right] &= \mathbf{r} \nabla^2 \left(\frac{1}{|\mathbf{r}|^3} \right) + \nabla \left(\frac{1}{|\mathbf{r}|^3} \right) \cdot \nabla \mathbf{r} \\ &= \mathbf{r} \left[\frac{6}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] + \frac{-3}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} (2x)(2x) \cdot \nabla (x^2 + y^2 + z^2)^{\frac{1}{2}} \\ &= \frac{6r}{r^4} - \frac{3}{r^4} (2x)(2x) \cdot \frac{(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \\ &= \frac{6}{r^4} - \frac{3}{r^4} \frac{x^2 + y^2 + z^2}{r} \\ &= \frac{6}{r^4} - \frac{3}{r^4} \\ &= \frac{3}{r^4} \end{aligned}$$

CURL

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$$

Question 1

A Cartesian vector is denoted by

$$\mathbf{A} = A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k},$$

where $A_i = f(x, y, z)$, $i = 1, 2, 3$

Given that A_i are differentiable functions, show that

$$\nabla \cdot \nabla \wedge \mathbf{A} = 0.$$

proof

$$\begin{aligned}\nabla \cdot \nabla \wedge \mathbf{A} &= \nabla \cdot \left| \begin{array}{c|ccc} i & 1 & 2 & 3 \\ \hline A_1 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_2 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ A_3 & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{array} \right| \quad \text{(circled 02)} \\ &= \left(\frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \frac{\partial}{\partial z} \right) - \left(\frac{\partial}{\partial y} \frac{\partial}{\partial x} \right) \frac{\partial}{\partial z} \right) - \left(\frac{\partial}{\partial y} \left(\frac{\partial}{\partial z} \frac{\partial}{\partial x} \right) - \left(\frac{\partial}{\partial z} \frac{\partial}{\partial y} \right) \frac{\partial}{\partial x} \right) \\ &= \cancel{\frac{\partial^3}{\partial x \partial y \partial z}} + \cancel{\frac{\partial^3}{\partial y \partial z \partial x}} + \cancel{\frac{\partial^3}{\partial z \partial x \partial y}} + \cancel{\frac{\partial^3}{\partial y \partial x \partial z}} = 0\end{aligned}$$

Question 2

A function φ is denoted by

$$\varphi = \varphi(x, y, z).$$

Given that φ is differentiable show that

$$\nabla \wedge \nabla \varphi = \mathbf{0}.$$

proof

$$\begin{aligned}\nabla \wedge \nabla \varphi &= \nabla \wedge \left[\frac{\partial \varphi}{\partial x} \mathbf{i} + \frac{\partial \varphi}{\partial y} \mathbf{j} + \frac{\partial \varphi}{\partial z} \mathbf{k} \right] = \left| \begin{array}{c|ccc} \mathbf{i} & 1 & 2 & 3 \\ \hline \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{array} \right| \\ &= \left[\frac{\partial^2 \varphi}{\partial x^2} \frac{\partial}{\partial y} \frac{\partial}{\partial z} - \frac{\partial^2 \varphi}{\partial y^2} \frac{\partial}{\partial x} \frac{\partial}{\partial z} \right] \mathbf{i} + \left[\frac{\partial^2 \varphi}{\partial y^2} \frac{\partial}{\partial z} \frac{\partial}{\partial x} - \frac{\partial^2 \varphi}{\partial z^2} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right] \mathbf{j} + \left[\frac{\partial^2 \varphi}{\partial z^2} \frac{\partial}{\partial x} \frac{\partial}{\partial y} - \frac{\partial^2 \varphi}{\partial x^2} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right] \mathbf{k} \\ &= [0, 0, 0] = \mathbf{0}\end{aligned}$$

Question 3

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Determine the value of

$$\nabla \wedge \mathbf{r}.$$

$$\boxed{\nabla \wedge \mathbf{r} = \mathbf{0}}$$

$$\nabla \wedge \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}, \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}, \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) = (0, 0, 0)$$

Question 4

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that \mathbf{a} is a constant vector, find

$$\nabla \wedge (\mathbf{a} \wedge \mathbf{r}).$$

$$\boxed{\nabla \wedge (\mathbf{a} \wedge \mathbf{r}) = 2\mathbf{a}}$$

$$\begin{aligned} \nabla \wedge (\mathbf{a} \wedge \mathbf{r}) &= \nabla \wedge \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \nabla \wedge [a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix} \\ &= [a_1 z - a_2 y, a_2 x - a_3 z, a_3 y - a_1 x] = (2a_1, 2a_2, 2a_3) \\ &= 2(a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \quad \checkmark \end{aligned}$$

Question 5

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Determine

$$\nabla \wedge (\mathbf{r} + y^2 \mathbf{k}).$$

$$\boxed{\nabla \wedge (\mathbf{r} + y^2 \mathbf{k}) = 2y\mathbf{i}}$$

$\nabla \wedge (\mathbf{r} + y^2 \mathbf{k})$ $\mathbf{r} = (x, y, z)$

Left Column:

$$\begin{aligned} & \nabla \wedge (\mathbf{r} + y^2 \mathbf{k}) \\ &= \nabla \wedge [(x, y, z) + (0, 0, y^2)] \\ &= \nabla \wedge (x, y, z + y^2) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z + y^2 \end{vmatrix} \\ &= [2y - 0, 0 - 0, 0 - 0] \\ &= 2y\mathbf{i} \end{aligned}$$

Right Column:

$$\begin{aligned} & \nabla \wedge (\mathbf{r} + y^2 \mathbf{k}) \\ &= \nabla \cdot \mathbf{r} + \nabla \cdot (\mathbf{k}) \\ &= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot \begin{pmatrix} 0 \\ 0 \\ y^2 \end{pmatrix} \\ &= (0, 0, 0) + (0, 0, 0) \\ &= 2y\mathbf{i} \end{aligned}$$

Question 6

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that \mathbf{a} is a constant vector, show that

$$\nabla \wedge (r^2 \mathbf{a}) \equiv 2\mathbf{r} \wedge \mathbf{a}.$$

proof

$\nabla \wedge (r^2 \mathbf{a}) =$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 + y^2 + z^2)a_1 & (x^2 + y^2 + z^2)a_2 & (x^2 + y^2 + z^2)a_3 \end{vmatrix}$$

$$= [2y^2 - 2z^2, 2z^2 - 2x^2, 2x^2 - 2y^2]$$

$$= 2 [a_3 y - a_2 z, a_2 z - a_1 x, a_1 x - a_3 y]$$

$$= 2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = 2 (x a_2 a_3 - a_1 a_2 a_3)$$

$$= 2 \mathbf{r} \wedge \mathbf{a}$$

$\therefore \text{QED}$

Question 7

A vector function \mathbf{A} is defined as

$$\mathbf{A} = A_1(x, y, z)\mathbf{i} + A_2(x, y, z)\mathbf{j} + A_3(x, y, z)\mathbf{k}.$$

Given that the standard Cartesian position vector is denoted by \mathbf{r} , show that

$$\nabla \cdot (\mathbf{A} \wedge \mathbf{r}) \equiv \mathbf{r} \cdot \nabla \wedge \mathbf{A}.$$

proof

$$\begin{aligned}
 & \nabla \cdot (\mathbf{A} \wedge \mathbf{r}) \quad \text{where } \mathbf{A} = (A_1, A_2, A_3) \quad \text{where } A_i = A_i(x, y, z) \\
 & \quad \Sigma = (x, y, z) \\
 & \nabla \cdot (\mathbf{A} \wedge \mathbf{r}) = \nabla \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix} = \nabla \cdot [A_{2z} - A_{3y}, A_{3x} - A_{2z}, A_{1y} - A_{1x}] \\
 & = \frac{\partial}{\partial x} [A_{2z} - A_{3y}] + \frac{\partial}{\partial y} [A_{3x} - A_{2z}] + \frac{\partial}{\partial z} [A_{1y} - A_{1x}] \\
 & = \frac{\partial}{\partial x} \left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right] + \frac{\partial}{\partial y} \left[\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right] + \frac{\partial}{\partial z} \left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right] \\
 & = (A_1)_{xy} - \left(\frac{\partial^2 A_3}{\partial y \partial z} - \frac{\partial^2 A_2}{\partial z \partial x} \right) + \left(\frac{\partial^2 A_1}{\partial z \partial x} - \frac{\partial^2 A_3}{\partial x \partial y} \right) \\
 & = (A_1)_{xy} - \left(\frac{\partial^2 A_3}{\partial y \partial z} - \frac{\partial^2 A_2}{\partial z \partial x} \right) - \left(\frac{\partial^2 A_1}{\partial z \partial x} - \frac{\partial^2 A_3}{\partial x \partial y} \right) \\
 & = \Sigma \cdot \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_{1x} & A_{1y} & A_{1z} \\ A_{2x} & A_{2y} & A_{2z} \\ A_{3x} & A_{3y} & A_{3z} \end{vmatrix} = \Sigma \cdot \nabla \cdot \mathbf{A} \quad \text{As required}
 \end{aligned}$$

Question 8

The smooth vector functions, \mathbf{A} and \mathbf{B} , are both irrotational.

Show that $\mathbf{A} \wedge \mathbf{B}$ is solenoidal.

proof

$$\begin{aligned}
 & \text{IRROTATIONAL} \Rightarrow \nabla \times \mathbf{A} = \mathbf{0}, \quad \nabla \times \mathbf{B} = \mathbf{0} \\
 & \text{SOLENOIDAL} \Rightarrow \text{DIVERGENCE IS ZERO} \\
 & \text{Now } \nabla \cdot (\mathbf{A} \wedge \mathbf{B}) = \dots \text{ IDENTITY} \\
 & = \frac{\mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})}{\cancel{\mathbf{A} \cdot \mathbf{B}}} \\
 & = \mathbf{0} \\
 & \therefore \text{SOLENOIDAL INDEX}
 \end{aligned}$$

Question 9

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Find the value of

$$\nabla \wedge \left(\frac{\mathbf{r}}{|\mathbf{r}|^2} \right).$$

0

$$\begin{aligned}
 \nabla \wedge \left(\frac{\mathbf{r}}{r^2} \right) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix} \\
 &= \left[\frac{\partial}{\partial y} \left(\frac{z}{x^2+y^2+z^2} \right) - \frac{\partial}{\partial z} \left(\frac{y}{x^2+y^2+z^2} \right) \right] \mathbf{i} \\
 &\quad + \left[\frac{\partial}{\partial z} \left(\frac{x}{x^2+y^2+z^2} \right) - \frac{\partial}{\partial x} \left(\frac{z}{x^2+y^2+z^2} \right) \right] \mathbf{j} \\
 &\quad - \left[\frac{\partial}{\partial x} \left(\frac{y}{x^2+y^2+z^2} \right) - \frac{\partial}{\partial y} \left(\frac{x}{x^2+y^2+z^2} \right) \right] \mathbf{k} \\
 &= \left[\frac{\partial}{\partial y} \left[z(x^2+y^2+z^2)^{-1} \right] - \frac{\partial}{\partial z} \left[y(x^2+y^2+z^2)^{-1} \right] \right] \mathbf{i} \\
 &\quad + \left[\frac{\partial}{\partial z} \left[x(x^2+y^2+z^2)^{-1} \right] - \frac{\partial}{\partial x} \left[z(x^2+y^2+z^2)^{-1} \right] \right] \mathbf{j} \\
 &\quad - \left[\frac{\partial}{\partial x} \left[y(x^2+y^2+z^2)^{-1} \right] - \frac{\partial}{\partial y} \left[x(x^2+y^2+z^2)^{-1} \right] \right] \mathbf{k} \\
 &= \left[-2yz(x^2+y^2+z^2)^{-2} - \left[-2yz(x^2+y^2+z^2)^{-2} \right] \right] \mathbf{i} \\
 &\quad + \left[-2xz(x^2+y^2+z^2)^{-2} - \left[-2xz(x^2+y^2+z^2)^{-2} \right] \right] \mathbf{j} \\
 &\quad + \left[-2xy(x^2+y^2+z^2)^{-2} - \left[-2xy(x^2+y^2+z^2)^{-2} \right] \right] \mathbf{k} \\
 &= \mathbf{0}
 \end{aligned}$$

Question 10

If $\varphi = \varphi(x, y, z)$ is a smooth function, prove that

$$\nabla \wedge (\varphi \nabla \varphi) = \mathbf{0}.$$

proof

• $\nabla \wedge (\varphi \nabla \varphi) = \nabla \wedge (\varphi \Delta) \text{ since } \Delta = \nabla^2$

$$= \varphi (\nabla \wedge \Delta) + (\nabla \varphi \wedge \Delta)$$

$$= \varphi [\nabla \wedge \nabla^2] + \nabla \varphi \wedge \nabla^2$$

$$= 0$$

or

• $\nabla \wedge [\varphi \nabla \varphi] = \begin{vmatrix} 1 & 1 & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi \frac{\partial}{\partial x} & \varphi \frac{\partial}{\partial y} & \varphi \frac{\partial}{\partial z} \end{vmatrix}$

$$= 1 \left[\frac{\partial}{\partial y} \left(\varphi \frac{\partial}{\partial z} \right) - \frac{\partial}{\partial z} \left(\varphi \frac{\partial}{\partial y} \right) \right] + 1 \left[\frac{\partial}{\partial z} \left(\varphi \frac{\partial}{\partial x} \right) - \frac{\partial}{\partial x} \left(\varphi \frac{\partial}{\partial z} \right) \right]$$

$$+ k \left[\frac{\partial}{\partial x} \left(\varphi \frac{\partial}{\partial y} \right) - \frac{\partial}{\partial y} \left(\varphi \frac{\partial}{\partial x} \right) \right]$$

$$= 1 \left[\frac{\partial \varphi}{\partial y} \frac{\partial}{\partial z} + \varphi \frac{\partial^2}{\partial y \partial z} - \frac{\partial \varphi}{\partial z} \frac{\partial}{\partial y} - \varphi \frac{\partial^2}{\partial z \partial y} \right]$$

$$+ 1 \left[\frac{\partial \varphi}{\partial z} \frac{\partial}{\partial x} + \varphi \frac{\partial^2}{\partial z \partial x} - \frac{\partial \varphi}{\partial x} \frac{\partial}{\partial z} - \varphi \frac{\partial^2}{\partial x \partial z} \right]$$

$$+ k \left[\frac{\partial \varphi}{\partial x} \frac{\partial}{\partial y} + \varphi \frac{\partial^2}{\partial x \partial y} - \frac{\partial \varphi}{\partial y} \frac{\partial}{\partial x} - \varphi \frac{\partial^2}{\partial y \partial x} \right]$$

$$= 0 \hat{i} + 0 \hat{j} + 0 \hat{k}$$

Question 11

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Given that \mathbf{a} is a constant vector, find the value of

$$\nabla \wedge \nabla \wedge (\mathbf{r} \wedge \mathbf{a}).$$

□

$\nabla \wedge \nabla \wedge (\mathbf{r} \wedge \mathbf{a}) = 0$ At $\mathbf{r} \cdot \mathbf{a} = 0$, VECTOR OF AT MOST THREE COMPONENTS, SO 2. PARTIAL DIFFERENTIALS WILL BRING IT TO ZERO

or

$$\nabla \wedge \mathbf{a} = \begin{vmatrix} 1 & 1 & k \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = [a_{32}-a_2, a_1-a_3, a_2-a_1]$$

$$\nabla \wedge (\mathbf{r} \wedge \mathbf{a}) = \begin{vmatrix} 1 & 1 & k \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= [-a_1-a_1-a_2, -a_2-a_3, -a_3-a_1]$$

$$= [-2a_1-2a_2-2a_3]$$

$$\nabla \wedge [\nabla \wedge (\mathbf{r} \wedge \mathbf{a})] = \begin{vmatrix} 1 & 1 & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2a_1 & -2a_2 & -2a_3 \end{vmatrix} = 0$$

Question 12

The irrotational vector field \mathbf{F} is given by

$$\mathbf{F} = (8x + 8y + az)\mathbf{i} + (bx + 8y - 4z)\mathbf{j} + (-4x + cy + 2z)\mathbf{k},$$

where a , b and c are scalar constants.

Determine a smooth scalar function $\varphi(x, y, z)$ such that

$$\nabla \varphi = \mathbf{F}.$$

$$\boxed{\varphi(x, y, z) = (2x + 2y - x)^2 + \text{constant} = 4x^2 + 4y^2 + z^2 - 4xz - 4yz + 8xy + \text{constant}}$$

Given $\mathbf{F} = (8x + 8y + az)\mathbf{i} + (bx + 8y - 4z)\mathbf{j} + (-4x + cy + 2z)\mathbf{k}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 8x + 8y + az & bx + 8y - 4z & -4x + cy + 2z \end{vmatrix} = [-4 - c, 4 + a, b - 8]$$

If irrotational, $\nabla \times \mathbf{F} = 0 \Rightarrow \begin{cases} a = -4 \\ b = 8 \\ c = -4 \end{cases} \Rightarrow \nabla(\nabla \varphi) = 0$

$\nabla \varphi = \frac{\partial \varphi}{\partial x} \mathbf{i} + \frac{\partial \varphi}{\partial y} \mathbf{j} + \frac{\partial \varphi}{\partial z} \mathbf{k}$

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= 8x + 8y - 4z \Rightarrow \varphi(x, y, z) = 4x^2 + 8xy - 4xz + G_1(y, z) \\ \frac{\partial \varphi}{\partial y} &= 8x + 8y - 4z \Rightarrow \varphi(x, y, z) = 8xy + 4y^2 - 4yz + G_2(z, x) \\ \frac{\partial \varphi}{\partial z} &= -4x - 4y + 2z \Rightarrow \varphi(x, y, z) = -4xz - 4yz + 2z^2 + G_3(x, y) \end{aligned}$$

$$\therefore \varphi(x, y, z) = 4x^2 + 4y^2 + z^2 - 4xz - 4yz + 8xy + C$$

Question 13

- a) Define the vector calculus operators grad, div and curl.

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

- b) Determine the vector

$$\nabla \wedge [\mathbf{r} \wedge \mathbf{i} + \nabla(\sin e^{xyz})].$$

$$\boxed{\quad}, \boxed{\nabla \wedge [\mathbf{r} \wedge \mathbf{i} + \nabla(\sin e^{xyz})] = -2\mathbf{i}}$$

a) GRADIENT OF A SMOOTH SCALAR FUNCTION $\phi = \phi(x,y,z)$
 $\nabla \phi = [\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}]$

DIVERGENCE OF A SMOOTH VECTOR FIELD $\mathbf{F} = (F_x, F_y, F_z)$
 $\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

CURL OF A SMOOTH VECTOR FIELD $\mathbf{F} = (F_x, F_y, F_z)$
 $\nabla \wedge \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

b) MANIPULATE AS FOLLOWS
 $\nabla \wedge [\mathbf{r} \wedge \mathbf{i} + \nabla(\sin e^{xyz})] \quad \downarrow \quad \nabla \wedge (\mathbf{A} + \mathbf{B}) = \nabla \wedge \mathbf{A} + \nabla \wedge \mathbf{B}$
 $= \nabla \wedge [\mathbf{r} \wedge \mathbf{i}] + \nabla \wedge \nabla(\sin e^{xyz})$
 $\text{SINCE } \nabla \wedge \nabla \phi = 0$

$$\begin{aligned} &= \nabla \wedge \begin{vmatrix} 1 & y & z \\ x & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \\ &= \nabla \wedge (0, z, -y) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z & -y \end{vmatrix} \\ &= (-1, 0, 0) \\ &= -2\mathbf{i} \end{aligned}$$

Question 14

The smooth functions f and \mathbf{A} are defined as

$$f = f(x, y, z) \quad \text{and} \quad \mathbf{A} = A_1(x, y, z)\mathbf{i} + A_2(x, y, z)\mathbf{j} + A_3(x, y, z)\mathbf{k}.$$

- a) Define the vector calculus operators grad, div and curl with reference to f and \mathbf{A} .

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

- b) Determine the vector

$$\nabla \wedge [\mathbf{r} \wedge \mathbf{i} + (x+y)\mathbf{k}].$$

- c) Show that

$$\nabla \cdot \nabla \left(\frac{1}{r^2} \right) = \frac{2}{r^4}.$$

$$\boxed{}, \boxed{-\mathbf{i} - \mathbf{j}}$$

a) THE GRADIENT OF THE SCALAR FIELD f [FIGURE]

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

THE DIVIGENCE OF THE VECTOR FIELD $\mathbf{A} = [A_1(x,y,z), A_2(x,y,z), A_3(x,y,z)]$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

THE CURL OF THE VECTOR FIELD $\mathbf{A} = [A_1(x,y,z), A_2(x,y,z), A_3(x,y,z)]$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

b) SINCE BY CONG. $\nabla \cdot (\mathbf{a} + \mathbf{b}) = \nabla \cdot \mathbf{a} + \nabla \cdot \mathbf{b}$

$$\nabla \cdot [x\mathbf{i} + (x+y)\mathbf{k}] = \nabla \cdot (x\mathbf{i}) + \nabla \cdot [(x+y)\mathbf{k}]$$

$$= \nabla \cdot \begin{vmatrix} 1 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 2x \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2x \end{vmatrix}$$

$$= \nabla \cdot [0, x, 0] + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2x \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2x \\ 0 & 0 & 0 \end{vmatrix} + (1, 0, 0)$$

$$= (-1, 0, 0) + (1, 0, 0)$$

$$= (0, 0, 0)$$

$$= \boxed{-1 - 1}$$

c) SINCE THE CARTESIAN COMPONENTS

$$\nabla \cdot \nabla \left(\frac{1}{r^2} \right) = \nabla^2 \left(\frac{1}{r^2} \right) = \nabla^2 \left[\frac{1}{x^2 + y^2 + z^2} \right]$$

$$= \frac{\partial^2}{\partial x^2} \left[\frac{1}{x^2 + y^2 + z^2} \right] + \frac{\partial^2}{\partial y^2} \left[\frac{1}{x^2 + y^2 + z^2} \right] + \frac{\partial^2}{\partial z^2} \left[\frac{1}{x^2 + y^2 + z^2} \right]$$

AS THE EXPRESSION HAS CUBIC SIMILARITY CONSIDER ONE OF THE DERIVATIVES

$$\frac{\partial}{\partial x} \left[\frac{(x^2+y^2+z^2)^2}{x^2} \right] = \frac{\partial}{\partial x} \left[-x^2y^2z^2(x^2) \right] = \frac{\partial}{\partial x} \left[-\frac{2xy^2z^2}{(x^2+y^2+z^2)^2} \right]$$

$$= \frac{(3x^2y^2z^2)x^2 - 2x \cdot 2(x^2y^2z^2)(x)}{(x^2+y^2+z^2)^4}$$

$$= \frac{2(x^2y^2z^2)(x^2) - 4x^2y^2z^2}{(x^2+y^2+z^2)^4}$$

$$= \frac{8x^2y^2z^2 - 4x^2y^2z^2}{(x^2+y^2+z^2)^4}$$

$$= \boxed{\frac{4x^2y^2z^2}{(x^2+y^2+z^2)^4}}$$

AND BY SIMILARLY THE OTHER TWO COMPONENTS

$$\nabla \cdot \nabla \left(\frac{1}{r^2} \right) = \frac{6x^2y^2z^2 - 2x^2}{(x^2+y^2+z^2)^3} + \frac{6x^2y^2z^2 - 2y^2}{(x^2+y^2+z^2)^3} + \frac{6x^2y^2z^2 - 2z^2}{(x^2+y^2+z^2)^3}$$

$$= \frac{8x^2y^2z^2 - 2x^2y^2z^2}{(x^2+y^2+z^2)^3} = \frac{6(x^2y^2z^2)(x^2)}{(x^2+y^2+z^2)^3} = \frac{2}{(x^2+y^2+z^2)^3}$$

$$= \boxed{\frac{2}{r^4}}$$

AS REQUIRED

Question 15

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that \mathbf{a} is a constant vector, show that

$$\nabla \wedge [(\mathbf{a} \cdot \mathbf{r}) \mathbf{r}] = \mathbf{a} \wedge \mathbf{r}.$$

proof

$$\begin{aligned}\nabla_{\mathbf{r}} [(\mathbf{a} \cdot \mathbf{r}) \mathbf{r}] &= (\mathbf{a} \cdot \mathbf{r}) \nabla_{\mathbf{r}} \mathbf{r} + \nabla (\mathbf{a} \cdot \mathbf{r}) \wedge \mathbf{r} = \dots \\ \nabla_{\mathbf{r}} (\mathbf{a} \cdot \mathbf{r}) &= \mathbf{a} \nabla_{\mathbf{r}} \cdot \mathbf{r} + \nabla \cdot \mathbf{a} \wedge \mathbf{r} \\ \nabla_{\mathbf{r}} \cdot \mathbf{r} &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} = 3\mathbf{i}_1 + \mathbf{i}_2 \\ \dots &= \nabla (\mathbf{a} \cdot \mathbf{r}) \wedge \mathbf{r} \\ &= \nabla (a_1 a_2 a_3) \wedge (a_1 a_2) \\ &= \nabla (a_1 a_2 + a_2 a_3 + a_3 a_1) \wedge (a_1 a_2) \\ &= \left[\frac{\partial}{\partial x} (a_1 a_2 + a_2 a_3 + a_3 a_1), \frac{\partial}{\partial y} (a_1 a_2 + a_2 a_3 + a_3 a_1), \frac{\partial}{\partial z} (a_1 a_2 + a_2 a_3 + a_3 a_1) \right] \wedge (a_1 a_2) \\ &= (a_1 a_2, a_2 a_3) \wedge (a_1 a_2) \\ &= \mathbf{a} \wedge \mathbf{r}\end{aligned}$$

Question 16

The smooth functions $\mathbf{A} = \mathbf{A}(x, y, z)$, $\mathbf{B} = \mathbf{B}(x, y, z)$ and $\varphi = \varphi(x, y, z)$ are defined as

$$\mathbf{A} = yz\mathbf{i} - x^2y\mathbf{j} + xz^2\mathbf{k}, \quad \mathbf{B} = x^2\mathbf{i} + yz\mathbf{j} - xy\mathbf{k} \quad \text{and} \quad \varphi = xyz.$$

Find, in simplified form, expressions for

a) $(\mathbf{A} \cdot \nabla) \varphi$.

b) $(\mathbf{B} \cdot \nabla) \mathbf{A}$.

c) $(\mathbf{A} \wedge \nabla) \varphi$

$$(\mathbf{A} \cdot \nabla) \varphi = y^2z^2 - x^3yz + x^2yz^2,$$

$$(\mathbf{B} \cdot \nabla) \mathbf{A} = (yz^2 - xy^2)\mathbf{i} + (-2x^3y - x^2yz)\mathbf{j} + (x^2z^2 - 2x^2yz)\mathbf{k},$$

$$(\mathbf{A} \wedge \nabla) \varphi = (-x^3y^2 - x^2z^3)\mathbf{i} + (xyz^3 - xy^2z)\mathbf{j} + (xyz^2 + x^2y^2z)\mathbf{k}$$

a) $(\mathbf{A} \cdot \nabla) \varphi$

$$\begin{aligned} \mathbf{A} &= (yz, -x^2y, xz^2) \quad \mathbf{q} = (x^2, yz, -xy) \quad \varphi = xyz \\ &= \left[yz \frac{\partial}{\partial x} - x^2y \frac{\partial}{\partial y} + xz^2 \frac{\partial}{\partial z} \right] (xyz) \\ &= yz \frac{\partial}{\partial x} (xyz) - x^2y \frac{\partial}{\partial y} (xyz) + xz^2 \frac{\partial}{\partial z} (xyz) \\ &= yz^2 - x^3yz + x^2yz^2 \end{aligned}$$

b) $(\mathbf{B} \cdot \nabla) \mathbf{A}$

$$\begin{aligned} \mathbf{B} &= (x^2, yz, -xy) \quad \mathbf{A} = (yz, -x^2y, xz^2) \\ &= \left[x^2 \frac{\partial}{\partial x} + yz \frac{\partial}{\partial y} - xy \frac{\partial}{\partial z} \right] (yz, -x^2y, xz^2) \\ &= \left[x^2 \frac{\partial}{\partial x} (yz) + yz \frac{\partial}{\partial y} (yz) - xy \frac{\partial}{\partial z} (yz) \right] [yz, -x^2y, xz^2] \\ &\quad + \left[x^2 \frac{\partial}{\partial x} (-x^2y) + yz \frac{\partial}{\partial y} (-x^2y) - xy \frac{\partial}{\partial z} (-x^2y) \right] [-x^2y, xz^2, 0] \\ &\quad + \left[x^2 \frac{\partial}{\partial x} (xz^2) + yz \frac{\partial}{\partial y} (xz^2) - xy \frac{\partial}{\partial z} (xz^2) \right] [xz^2, 0, 0] \\ &= [yz^2, -x^2y^2, -2x^3y - x^2yz, xz^2 - 2x^2yz] \end{aligned}$$

c) $(\mathbf{A} \wedge \nabla) \varphi$

$$\begin{aligned} \mathbf{A} &= \begin{vmatrix} i & j & k \\ yz & -x^2y & xz^2 \\ x^2 & yz & -xy \end{vmatrix} (xyz) \\ &= \left[-x^2y \frac{\partial}{\partial x} - xz^2 \frac{\partial}{\partial y} \right] (xyz^2) - \left[yz^2 \frac{\partial}{\partial x} + xz^2 \frac{\partial}{\partial y} \right] (xyz^2) \\ &= [-x^2y(xz^2) - xz^2(-x^2y), yz^2(xz^2) - yz^2(-x^2y), xz^2(yz^2)] \\ &= [-x^3y^2 - x^2z^3, xyz^3 - xy^2z, xyz^2 + x^2y^2z] \end{aligned}$$

Question 17

- a) Define the vector calculus operators grad, div and curl.
b) Given that $\varphi(x, y, z)$ and $\psi(x, y, z)$ are smooth functions, show that

$$\nabla \wedge [\varphi \nabla \psi] \equiv \nabla \varphi \wedge \nabla \psi .$$

- c) Evaluate**

$$\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left((x+y+z)^3 \mathbf{i} + (4x^3 - yz) \mathbf{j} + (xyz) \mathbf{k} \right) \right] \right] \right].$$

- d) Use the vector function

$$\mathbf{A}(x, y, z) = \left(e^{x+y} \right) \mathbf{i} + \left(x \sin z \right) \mathbf{j} + \left(4\sqrt{x} \right) \mathbf{k}$$

to verify the validity of the identity

$$\nabla \wedge (\nabla \wedge \mathbf{A}) \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\left[\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left((x+y+z)^3 \mathbf{i} + (4x^3 - yz) \mathbf{j} + (xyz) \mathbf{k} \right) \right] \right] \right] = \mathbf{0}$$

4) If $f(x,y,z)$ is a smooth scalar field, then $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$

If $\vec{F} = \left[F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \right]$ is a smooth vector field, then $\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

(i) If $\vec{F} = \nabla f$, then $\nabla \cdot \vec{F} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

5) $\nabla_x [\Phi \Psi] = \nabla_x \left[\Phi \frac{\partial \Psi}{\partial x} + \Psi \frac{\partial \Phi}{\partial x} + \Phi \frac{\partial^2 \Psi}{\partial x^2} \right] = \begin{vmatrix} 1 & \frac{\partial \Psi}{\partial x} & \frac{\partial^2 \Psi}{\partial x^2} \\ \frac{\partial \Phi}{\partial x} & \Phi & \frac{\partial \Phi}{\partial x} \\ \frac{\partial^2 \Phi}{\partial x^2} & \frac{\partial \Phi}{\partial x} & \Phi \end{vmatrix} + \Phi \nabla_x^2 \Psi$

$= \Phi \begin{vmatrix} 1 & \frac{\partial \Psi}{\partial x} & \frac{\partial^2 \Psi}{\partial x^2} \\ \frac{\partial \Phi}{\partial x} & \Phi & \frac{\partial \Phi}{\partial x} \\ \frac{\partial^2 \Phi}{\partial x^2} & \frac{\partial \Phi}{\partial x} & \Phi \end{vmatrix} + \Psi \nabla_x^2 \Phi$

FACTURE OF THE DETERMINANT OF
ROW 1 OF THE MATRIX OF
REVERSE X IN THE SECOND ROW

ACT ON WITH PART
USING THE IDENTITY $\nabla_x (4A) = 4\nabla_x A + \Phi \nabla_x A$

Let $\Phi = -\nabla \Psi$ $\nabla_x (4\nabla \Psi) = 4\nabla_x \nabla \Psi + \Phi \nabla_x \nabla \Psi = 4(\nabla_x \nabla \Psi) - 4(\nabla_x \nabla \Psi) = 0$
Hence $\nabla_x (4\nabla \Psi) = 0$

4) $\nabla_A [\nabla_A [\nabla_A [\nabla_A [(\lambda x_1 y_2)^3, \bar{x}^2 - y_2, 2y_3]^3]] = 0$

PROBLEM 4: "REAL APPLICATIONS" IN OTHER WORDS: 4 PARTIAL DIFFERENTIAL EQUATIONS, POLYNOMIAL FUNCTIONS OF A MAXIMUM DEGREE 3, PLUS VARIOUS

4) $\nabla_A (\nabla_A A) = \nabla (\nabla_A A) - \nabla^2 A$

Left side:

$$\text{LHS} = \nabla_A \begin{bmatrix} 1 & \frac{1}{x_1} & \frac{1}{y_2} \\ \frac{1}{y_2} & \frac{1}{y_2} & \frac{1}{y_3} \\ \frac{1}{y_3} & \frac{1}{y_3} & \frac{1}{x_1} \end{bmatrix} = \nabla_A \begin{bmatrix} 0 & -2xy_2 & 0 - x_1^{-\frac{1}{2}} \\ 0 - 2xy_2 & 0 - x_1^{-\frac{1}{2}} & 2y_3x_1^{-\frac{1}{2}} \\ 0 - x_1^{-\frac{1}{2}} & 2y_3x_1^{-\frac{1}{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{x_1} & \frac{1}{x_1} & \frac{1}{x_1} \\ \frac{1}{x_1} & \frac{1}{x_1} & \frac{1}{x_1} \\ \frac{1}{x_1} & \frac{1}{x_1} & \frac{1}{x_1} \end{bmatrix} = \begin{bmatrix} -2x_1^{-\frac{1}{2}} & 0 & -2x_1^{-\frac{1}{2}} \\ 0 & -2x_1^{-\frac{1}{2}} & 0 \\ -2x_1^{-\frac{1}{2}} & 0 & -2x_1^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} -e^{-\frac{1}{2}} & 0 & -e^{-\frac{1}{2}} \\ 0 & -e^{-\frac{1}{2}} & 0 \\ -e^{-\frac{1}{2}} & 0 & -e^{-\frac{1}{2}} \end{bmatrix}$$

Right side:

$$\text{RHS} = \nabla \left[\begin{bmatrix} \frac{1}{x_1} & \frac{1}{x_1} & \frac{1}{x_1} \\ \frac{1}{x_1} & \frac{1}{x_1} & \frac{1}{x_1} \\ \frac{1}{x_1} & \frac{1}{x_1} & \frac{1}{x_1} \end{bmatrix} - \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} \\ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} \\ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} \end{bmatrix} e^{2x_1} \cdot 2x_1^{-\frac{1}{2}}, 4x_1^{\frac{1}{2}} \right]$$

$$= \nabla \left[\left[e^{2x_1} + 0x_1^{\frac{1}{2}} \right] - \left[\frac{\partial^2}{\partial x_1^2} e^{2x_1} + \frac{\partial^2}{\partial y_2^2} e^{2x_1} + \frac{\partial^2}{\partial y_3^2} e^{2x_1} \right] \cdot \frac{\partial}{\partial x_1} (2x_1^{-\frac{1}{2}}) + \frac{\partial}{\partial y_2} (2x_1^{-\frac{1}{2}}) + \frac{\partial}{\partial y_3} (2x_1^{-\frac{1}{2}}) \cdot \frac{\partial^2}{\partial x_1^2} e^{2x_1} + \frac{\partial^2}{\partial y_2^2} e^{2x_1} + \frac{\partial^2}{\partial y_3^2} e^{2x_1} \right] e^{2x_1} \cdot 2x_1^{-\frac{1}{2}}, 4x_1^{\frac{1}{2}}$$

$$= \left[e^{2x_1} \cdot \frac{\partial}{\partial x_1} \left(e^{2x_1} + 0x_1^{\frac{1}{2}} \right) \right] - \left[\frac{\partial^2}{\partial x_1^2} e^{2x_1} + \frac{\partial^2}{\partial y_2^2} e^{2x_1} + \frac{\partial^2}{\partial y_3^2} e^{2x_1} \right] \left[\frac{\partial}{\partial x_1} (2x_1^{-\frac{1}{2}}) + \frac{\partial}{\partial y_2} (2x_1^{-\frac{1}{2}}) + \frac{\partial}{\partial y_3} (2x_1^{-\frac{1}{2}}) \right] e^{2x_1} \cdot 2x_1^{-\frac{1}{2}}, 4x_1^{\frac{1}{2}}$$

$$= \left[e^{2x_1} \cdot \frac{\partial}{\partial x_1} \left(e^{2x_1} + 0x_1^{\frac{1}{2}} \right) \right] - \left[\frac{\partial^2}{\partial x_1^2} e^{2x_1} + \frac{\partial^2}{\partial y_2^2} e^{2x_1} + \frac{\partial^2}{\partial y_3^2} e^{2x_1} \right] \left[0 + 0 + 0 - 2x_1^{-\frac{1}{2}} - x_1^{-\frac{3}{2}} + 0 + 0 \right]$$

$$= \left[e^{2x_1} \cdot \frac{\partial}{\partial x_1} \left(e^{2x_1} + 0x_1^{\frac{1}{2}} \right) \right] - \left[\frac{\partial^2}{\partial x_1^2} e^{2x_1} + \frac{\partial^2}{\partial y_2^2} e^{2x_1} + \frac{\partial^2}{\partial y_3^2} e^{2x_1} \right] \left[-x_1^{-\frac{1}{2}} + e^{2x_1} + 2x_1^{-\frac{1}{2}} \cdot x_1^{-\frac{1}{2}} \right]$$

Question 18

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that $f(r)$ is a differentiable function, show that

$$\nabla \wedge [\mathbf{r} f(r)] = \mathbf{0}.$$

proof

$$\begin{aligned} \nabla \wedge [\mathbf{r} f(r)] &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \hat{x} \frac{\partial}{\partial r} & \hat{y} \frac{\partial}{\partial r} & \hat{z} \frac{\partial}{\partial r} \end{vmatrix} \\ &= \left[\hat{x} \left[\frac{\partial^2}{\partial r^2} \right] - \hat{y} \left[\frac{\partial^2}{\partial r^2} \right], \hat{y} \left[\frac{\partial^2}{\partial r^2} \right] - \hat{x} \left[\frac{\partial^2}{\partial r^2} \right], \hat{z} \left[\frac{\partial^2}{\partial r^2} \right] - \hat{z} \left[\frac{\partial^2}{\partial r^2} \right] \right] \\ &= \dots \\ \text{EXPRESS IN COMPONENTS} \\ \hat{i} &: [\hat{x} \left(\frac{\partial^2}{\partial r^2} \right) - \hat{y} \left(\frac{\partial^2}{\partial r^2} \right)] - [\hat{y} \left(\frac{\partial^2}{\partial r^2} \right) + \hat{x} \left(\frac{\partial^2}{\partial r^2} \right)] \\ \hat{j} &: [\hat{y} \left(\frac{\partial^2}{\partial r^2} \right) + \hat{x} \left(\frac{\partial^2}{\partial r^2} \right)] - [\hat{y} \left(\frac{\partial^2}{\partial r^2} \right) + \hat{x} \left(\frac{\partial^2}{\partial r^2} \right)] \\ \hat{k} &: [\hat{z} \left(\frac{\partial^2}{\partial r^2} \right) + \hat{z} \left(\frac{\partial^2}{\partial r^2} \right)] - [\hat{z} \left(\frac{\partial^2}{\partial r^2} \right) + \hat{z} \left(\frac{\partial^2}{\partial r^2} \right)] \\ \text{HENCE} \quad \frac{\partial^2}{\partial r^2} &= \frac{\partial^2 \hat{x}}{\partial r^2} = f'(r) \times \frac{\partial}{\partial r} \left[(\hat{x}^2 + \hat{y}^2)^{\frac{1}{2}} \right] = f'(r) \times \frac{1}{2} (\hat{x}^2 + \hat{y}^2)^{-\frac{1}{2}} \cdot 2\hat{x} \\ &= f'(r) \times \frac{\hat{x}}{(\hat{x}^2 + \hat{y}^2)^{\frac{1}{2}}} = f'(r) \frac{\hat{x}}{r} \quad \& \quad \frac{\partial^2}{\partial r^2} = f'(r) \frac{\hat{x}}{r} \\ \text{SIMILARLY} \quad \frac{\partial^2}{\partial r^2} &= f'(r) \frac{\hat{y}}{r} \quad \& \quad \frac{\partial^2}{\partial r^2} = f'(r) \frac{\hat{z}}{r} \\ \hat{i} &: \hat{x} \frac{\partial^2}{\partial r^2} - \hat{y} \frac{\partial^2}{\partial r^2} = \hat{x} \frac{\hat{x}}{r} - \hat{y} \frac{\hat{y}}{r} = 0 \\ \hat{j} &: \hat{y} \frac{\partial^2}{\partial r^2} - \hat{z} \frac{\partial^2}{\partial r^2} = \hat{y} \frac{\hat{y}}{r} - \hat{z} \frac{\hat{z}}{r} = 0 \\ \hat{k} &: \hat{z} \frac{\partial^2}{\partial r^2} - \hat{x} \frac{\partial^2}{\partial r^2} = \hat{z} \frac{\hat{z}}{r} - \hat{x} \frac{\hat{x}}{r} = 0 \\ \text{HENCE} \quad \nabla \wedge [\mathbf{r} f(r)] &= 0 \end{aligned}$$

ALTERNATIVE

$$\begin{aligned} \nabla \wedge [\mathbf{r} f(r)] &= \nabla \wedge \hat{x} f(r) \hat{i} + \hat{y} f(r) \nabla \wedge \hat{i} \\ &= \left(\frac{\partial}{\partial x} \hat{x} \frac{\partial}{\partial r} \right) \hat{i} f(r) + \hat{y} f(r) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix} \\ &= \frac{\partial \hat{x}}{\partial x} \frac{\partial}{\partial r} \hat{i} f(r) + \frac{\partial \hat{x}}{\partial y} \frac{\partial}{\partial r} \hat{j} f(r) + \frac{\partial \hat{x}}{\partial z} \frac{\partial}{\partial r} \hat{k} f(r) \\ &= \frac{\partial \hat{x}}{\partial x} \left[\frac{\partial}{\partial r} \left(\frac{\hat{x}}{r} \right) \right] + \frac{\partial \hat{x}}{\partial y} \left[\frac{\partial}{\partial r} \left(\frac{\hat{y}}{r} \right) \right] + \frac{\partial \hat{x}}{\partial z} \left[\frac{\partial}{\partial r} \left(\frac{\hat{z}}{r} \right) \right] \wedge \hat{i} \\ &= \frac{\partial \hat{x}}{\partial x} \frac{\hat{x}}{r^2} \wedge \hat{i} \\ &= \frac{1}{r} \frac{\partial \hat{x}}{\partial r} \hat{i} \wedge \hat{i} \\ &= 0 \end{aligned}$$

NOTE $\frac{\partial \hat{x}}{\partial x} = \frac{\partial}{\partial x} \left[\frac{\hat{x}}{(\hat{x}^2 + \hat{y}^2)^{\frac{1}{2}}} \right] = \frac{1}{2} (\hat{x}^2 + \hat{y}^2)^{-\frac{1}{2}} \hat{x} \hat{x} = \frac{\hat{x}}{(\hat{x}^2 + \hat{y}^2)^{\frac{1}{2}}} = \frac{\hat{x}}{r}$
AND SIMILARLY $\frac{\partial \hat{y}}{\partial y} = \frac{\partial}{\partial y} \left[\frac{\hat{y}}{(\hat{x}^2 + \hat{y}^2)^{\frac{1}{2}}} \right] = \frac{\hat{y}}{(\hat{x}^2 + \hat{y}^2)^{\frac{1}{2}}} = \frac{\hat{y}}{r}$
 $\frac{\partial \hat{z}}{\partial z} = \frac{\partial}{\partial z} \left[\frac{\hat{z}}{(\hat{x}^2 + \hat{y}^2)^{\frac{1}{2}}} \right] = \frac{\hat{z}}{(\hat{x}^2 + \hat{y}^2)^{\frac{1}{2}}} = \frac{\hat{z}}{r}$

Question 19

Given that $\mathbf{A} = \mathbf{A}(x, y, z)$ is a twice differentiable vector function, show that

$$\nabla \wedge (\nabla \wedge \mathbf{A}) \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

[proof]

$$\begin{aligned}\nabla \wedge (\nabla \wedge \mathbf{A}) &= \nabla \cdot (\nabla \wedge \mathbf{A}) - \nabla^2 \mathbf{A} \\ \nabla \wedge (\nabla \wedge \mathbf{A}) &= \nabla \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \nabla \cdot \left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}, \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right] \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial^2 A_3}{\partial x^2} & \frac{\partial^2 A_2}{\partial x \partial y} & \frac{\partial^2 A_1}{\partial x \partial z} \\ \frac{\partial^2 A_3}{\partial y^2} & \frac{\partial^2 A_2}{\partial y \partial z} & \frac{\partial^2 A_1}{\partial y \partial x} \\ \frac{\partial^2 A_3}{\partial z^2} & \frac{\partial^2 A_2}{\partial z \partial x} & \frac{\partial^2 A_1}{\partial z \partial y} \end{vmatrix} \\ &= \left(\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2} \right) \hat{i} \\ &\quad + \left(\frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2} - \frac{\partial^2 A_1}{\partial x^2} \right) \hat{j} \\ &\quad + \left(\frac{\partial^2 A_3}{\partial z^2} - \frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} \right) \hat{k} \\ &\quad \text{Discusses ADD, SUBTRACTING OR ADDING/REARRANGING TERMS} \\ &= \left(\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2} + \left(\frac{\partial^2 A_3}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} \right) \right) \hat{i} \\ &= \left[-\frac{\partial^2 A_3}{\partial x^2} \left(\frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} \right) - \frac{\partial^2 A_1}{\partial y^2} \left(\frac{\partial^2 A_3}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} \right) + \frac{\partial^2 A_2}{\partial y^2} \left(\frac{\partial^2 A_3}{\partial x^2} + \frac{\partial^2 A_1}{\partial z^2} \right) \right] \hat{j} \\ &= \left[-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2} + \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_3}{\partial y^2} \right] \hat{k} \\ &= \left(\frac{\partial^2}{\partial x^2} \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) A_1 \hat{i} + \left(\frac{\partial^2}{\partial y^2} \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} \right) A_2 \hat{j} + \left(\frac{\partial^2}{\partial z^2} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) A_3 \hat{k} \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{\partial^2}{\partial x^2} \left[\frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} \right] \hat{i} + \frac{\partial^2}{\partial y^2} \left[\frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 A_2}{\partial x^2} \right] \hat{j} + \frac{\partial^2}{\partial z^2} \left[\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_3}{\partial y^2} \right] \hat{k} \right) \right)\end{aligned}$$

$$\begin{aligned}&= \left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) \\ &\quad + \frac{\partial}{\partial x} (\nabla \cdot \mathbf{A}) \hat{i} + \frac{\partial}{\partial y} (\nabla \cdot \mathbf{A}) \hat{j} + \frac{\partial}{\partial z} (\nabla \cdot \mathbf{A}) \hat{k} \\ &= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (A_1 A_2 A_3) + \left(\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right) (\nabla \cdot \mathbf{A}) \\ &= -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})\end{aligned}$$

∴ $\nabla \wedge (\nabla \wedge \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ // At step 10

ALTERNATIVE

$$\begin{aligned}\nabla \wedge (\nabla \wedge \mathbf{A}) &= (\mathbf{A} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{A} \\ \text{Let } \mathbf{A} = \nabla & \quad \text{B} = \nabla \quad \text{C} = \nabla \\ \mathbf{B} = \nabla & \quad \nabla \cdot (\nabla \wedge \mathbf{A}) = (\nabla \cdot \nabla) \nabla - (\nabla \cdot \nabla) \mathbf{A} \\ \mathbf{C} = \nabla & \quad \therefore \nabla \cdot (\nabla \wedge \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}\end{aligned}$$

Question 20

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show that $\mathbf{r} f(|\mathbf{r}|)$ is irrotational, where $f(|\mathbf{r}|)$ is differentiable function of $|\mathbf{r}|$.

proof

$\mathbf{F} = \mathbf{r} f(|\mathbf{r}|)$, i.e. $\mathbf{F} \propto |\mathbf{r}|$ & DIFFERENTIABLE

- $\nabla_A \cdot \mathbf{F} = \nabla_A \cdot (\mathbf{r} f(|\mathbf{r}|)) = \dots$ (using $\nabla_A(\phi A) = \nabla_A \phi A + \phi (\nabla_A \cdot \mathbf{A})$)

$$= \nabla_A \cdot \mathbf{r} + f(|\mathbf{r}|) \nabla_A \cdot \mathbf{r}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\begin{array}{c} x \\ y \\ z \end{array} \right) + f(|\mathbf{r}|) \cdot 0$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (0, 0, 0)$$

Now

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} [x^2 + y^2 + z^2] = x(x^2 + y^2 + z^2)^{\frac{1}{2}} f'$$

$$\frac{\partial f}{\partial y} = \dots = y(x^2 + y^2 + z^2)^{\frac{1}{2}} f'$$

$$\frac{\partial f}{\partial z} = \dots = z(x^2 + y^2 + z^2)^{\frac{1}{2}} f'$$

- $\dots = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \cdot (0, 0, 0)$

$$= \frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} (0, 0, 0) \cdot (0, 0, 0)$$

$$= 0$$

\therefore IRROTATIONAL

Question 21

The vector function \mathbf{E} satisfies

$$\mathbf{E} = \frac{\mathbf{r}}{|\mathbf{r}|^2}, |\mathbf{r}| \neq 0,$$

where $\mathbf{r}(x, y, z) = xi + yj + zk$.

Show that \mathbf{E} is irrotational, and find a smooth scalar function $\varphi(|\mathbf{r}|)$, with $\varphi(k) = 0$, so that $\mathbf{E} = -\nabla\varphi(|\mathbf{r}|)$.

$$\varphi(|\mathbf{r}|) = \ln\left(\frac{k}{|\mathbf{r}|}\right)$$

$$\begin{aligned} \mathbf{E} &= \frac{\mathbf{r}}{r^2} = \frac{(x, y, z)}{x^2+y^2+z^2} = \left[\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2} \right] \\ \bullet \nabla \times \mathbf{E} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(x^2+y^2+z^2)^{-1} & y(x^2+y^2+z^2)^{-1} & z(x^2+y^2+z^2)^{-1} \end{vmatrix} \\ &= \left[-2yz(x^2+y^2+z^2)^{-2} + 2yz(x^2+y^2+z^2)^{-2} \right] \\ &\quad + \left[-2xz(x^2+y^2+z^2)^{-2} + 2xz(x^2+y^2+z^2)^{-2} \right] \\ &\quad + \left[-2xy(x^2+y^2+z^2)^{-2} + 2xy(x^2+y^2+z^2)^{-2} \right] \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \end{aligned}$$

$$\bullet \mathbf{E} = -\nabla\varphi$$

$$-\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right) = \left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right)$$

$$\bullet \frac{\partial \mathbf{E}}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right), \frac{\partial \mathbf{E}}{\partial y} = -\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right), \frac{\partial \mathbf{E}}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)$$

$$\begin{cases} \varphi = -\frac{1}{2}\ln(x^2+y^2+z^2) + C_1(x, y) \\ \varphi = -\frac{1}{2}\ln(x^2+y^2+z^2) + C_2(y, z) \\ \varphi = -\frac{1}{2}\ln(x^2+y^2+z^2) + C_3(x, z) \end{cases} \Rightarrow F(x, y) = G(y, z) = H(x, z) = \text{constant}$$

$$\therefore \varphi(x, y, z) = \ln\left(\frac{1}{x^2+y^2+z^2}\right) + \text{constant} = \ln\left(\frac{1}{r}\right) + \text{constant}$$

$$\varphi(r) = \ln\left(\frac{1}{r}\right) + \text{constant}$$

$$\because \varphi(0) = 0 \Rightarrow 0 = \ln\left(\frac{1}{r}\right) + \text{constant} \Rightarrow \text{constant} = -\ln k = \ln k$$

$$\therefore \varphi(r) = \ln\left(\frac{1}{r}\right) + \ln k = \ln\left(\frac{1}{r}k\right)$$

Question 22

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that \mathbf{a} is a constant vector, show that

$$\nabla \wedge \left(\mathbf{a} \wedge \frac{\mathbf{r}}{r^3} \right) = \frac{3\mathbf{a} \cdot \mathbf{r}}{r^5} \mathbf{r} - \frac{\mathbf{a}}{r^3}.$$

proof

$\nabla_A \left(\mathbf{a} \wedge \frac{\mathbf{r}}{r^3} \right)$ ACTIVEMATH

NOTE THAT $(\mathbf{b}, \nabla) \mathbf{A} \equiv [\mathbf{b}, \nabla] \mathbf{A} - (\mathbf{b} \cdot \nabla) \mathbf{A} + (\mathbf{A} \cdot \nabla) \mathbf{b} - \mathbf{A} (\nabla \cdot \mathbf{b})$

$\dots = \left(\frac{\partial}{\partial r} \left(\frac{\mathbf{r}}{r^3} \right) \right) \mathbf{a} - \frac{1}{r^2} (\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla) \frac{\mathbf{r}}{r^3} + \mathbf{a} (\nabla \cdot \frac{\mathbf{r}}{r^3})$
SINCE \mathbf{a} IS CONSTANT

• $-\mathbf{a} (\nabla \cdot \frac{\mathbf{r}}{r^3}) = -\left[a_1 \frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) + a_2 \frac{\partial}{\partial y} \left(\frac{y}{r^3} \right) + a_3 \frac{\partial}{\partial z} \left(\frac{z}{r^3} \right) \right] = -a_1 \frac{\partial}{\partial x} \left(\frac{-k}{r^3} \right) - a_2 \frac{\partial}{\partial y} \left(\frac{-k}{r^3} \right) - a_3 \frac{\partial}{\partial z} \left(\frac{-k}{r^3} \right)$
= $\left[a_1 \frac{\partial}{\partial x} \left(\frac{x}{(x^2+y^2+z^2)^{3/2}} \right) - a_1 \frac{\partial}{\partial x} \left(\frac{-k}{(x^2+y^2+z^2)^{3/2}} \right) - a_2 \frac{\partial}{\partial y} \left(\frac{y}{(x^2+y^2+z^2)^{3/2}} \right) - a_2 \frac{\partial}{\partial y} \left(\frac{-k}{(x^2+y^2+z^2)^{3/2}} \right) - a_3 \frac{\partial}{\partial z} \left(\frac{z}{(x^2+y^2+z^2)^{3/2}} \right) - a_3 \frac{\partial}{\partial z} \left(\frac{-k}{(x^2+y^2+z^2)^{3/2}} \right) \right] + \dots$
Now
 $\frac{\partial}{\partial x} \left[\frac{x}{(x^2+y^2+z^2)^{3/2}} \right] = \frac{(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^{3/2}} - 2(x \cdot 2x)(x^2+y^2+z^2)^{-3/2} = \frac{x^2+2yz^2}{(x^2+y^2+z^2)^{5/2}}$
 $\frac{\partial}{\partial y} \left[\frac{y}{(x^2+y^2+z^2)^{3/2}} \right] = -2xy(x^2+y^2+z^2)^{-3/2} = \frac{-2xy}{(x^2+y^2+z^2)^{5/2}}$
 $\frac{\partial}{\partial z} \left[\frac{z}{(x^2+y^2+z^2)^{3/2}} \right] = -2xz(x^2+y^2+z^2)^{-3/2} = \frac{-2xz}{(x^2+y^2+z^2)^{5/2}}$
 $= \left[-a_1 \frac{x^2+2yz^2}{(x^2+y^2+z^2)^{5/2}} - a_1 \frac{-2xy}{(x^2+y^2+z^2)^{5/2}} - a_2 \frac{-2xz}{(x^2+y^2+z^2)^{5/2}} \right] + \left[a_1 \frac{3xy}{(x^2+y^2+z^2)^{5/2}} - a_2 \frac{3yz^2}{(x^2+y^2+z^2)^{5/2}} - a_3 \frac{3xz^2}{(x^2+y^2+z^2)^{5/2}} \right]$

$\nabla_A \left(\mathbf{a} \wedge \frac{\mathbf{r}}{r^3} \right) = \frac{1}{(x^2+y^2+z^2)^{5/2}} \left[a_1 3xyz + a_2 3yz - a_1 (x^2+y^2+z^2) + a_1 3xz + a_2 3xy - a_3 (x^2+y^2+z^2) \right]$

• $\mathbf{a} \left(\nabla \cdot \frac{\mathbf{r}}{r^3} \right) = \mathbf{a} \left(\frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) \right) \left(\frac{x}{(x^2+y^2+z^2)^{3/2}} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r^3} \right) \left(\frac{y}{(x^2+y^2+z^2)^{3/2}} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r^3} \right) \left(\frac{z}{(x^2+y^2+z^2)^{3/2}} \right) \quad \leftarrow \text{WALKS OUT MAKE}$
= $\mathbf{a} \left[\frac{-3x^2-2z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{3y^2-z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{3x^2-y^2}{(x^2+y^2+z^2)^{5/2}} \right] = 0$

• This $\nabla_A \left(\mathbf{a} \wedge \frac{\mathbf{r}}{r^3} \right) = \left[a_1 (3xy) + a_2 (3yz) - a_3 (3xz) + 3a_1^2 - a_1 (x^2+y^2+z^2), a_1 (3xz) + a_2 (3xy) - a_3 (x^2+y^2+z^2) \right]$
 $= \left[\frac{3a_1 x + 3a_2 y + 3a_3 z}{(x^2+y^2+z^2)^{5/2}}, -\frac{a_1}{(x^2+y^2+z^2)^{5/2}} (3a_2 + 3a_3) + \frac{3a_2 x + 3a_3 y}{(x^2+y^2+z^2)^{5/2}}, -\frac{a_2}{(x^2+y^2+z^2)^{5/2}} (3a_1 + 3a_3) + \frac{3a_1 x + 3a_2 y}{(x^2+y^2+z^2)^{5/2}} \right]$
 $= \left[\frac{3a_1 x}{r^5} - \frac{a_1}{r^3}, \frac{3a_2 x}{r^5} - \frac{a_2}{r^3}, \frac{3a_3 x}{r^5} - \frac{a_3}{r^3} \right]$
 $= \left[\frac{3a_1 x}{r^5} \frac{a_1}{r^3}, \frac{3a_2 x}{r^5} \frac{a_2}{r^3}, \frac{3a_3 x}{r^5} \frac{a_3}{r^3} \right]$
 $= \frac{3a_1 x}{r^5} (a_1 a_2 a_3) - \frac{a_1}{r^3} (a_1 a_2 a_3)$
 $= \frac{3a_1 x}{r^5} \frac{a_1}{r^3} = \frac{a_1}{r^3}$
= $\frac{a_1}{r^3} (a_1 a_2 a_3)$

$\nabla_A \left(\mathbf{a} \wedge \frac{\mathbf{r}}{r^3} \right)$ ACTIVEMATH

• $\mathbf{a} = (a_1, a_2, a_3)$ ACTIVEMATH
 $\frac{\partial}{\partial r} = \frac{\partial}{\partial \sqrt{x^2+y^2+z^2}} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \frac{\partial a_2 - a_3}{\partial x} \frac{\partial a_3 - a_1}{\partial y} \frac{\partial a_1 - a_2}{\partial z} = \frac{\partial a_2 - a_3}{(x^2+y^2+z^2)^{1/2}} \frac{\partial a_3 - a_1}{(x^2+y^2+z^2)^{1/2}} \frac{\partial a_1 - a_2}{(x^2+y^2+z^2)^{1/2}}$

• $\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 - a_3 & a_1 - a_2 & a_3 - a_1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \dots \text{WORK COMPUTED BY CALCULATOR}$

• $\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 - a_2 & a_3 - a_1 & a_2 - a_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \frac{(x^2+y^2+z^2)^{1/2} - (a_1-a_2)(x^2+y^2+z^2)^{-1/2} - (a_2-a_3)(x^2+y^2+z^2)^{-1/2} - (a_3-a_1)(x^2+y^2+z^2)^{-1/2}}{(x^2+y^2+z^2)^{3/2}}$
 $= \frac{a_1(x^2+y^2+z^2) - (a_1-a_2)(x^2+y^2+z^2) - (a_2-a_3)(x^2+y^2+z^2) - (a_3-a_1)(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2a_1(x^2+y^2+z^2) - 2a_1^2 - 2a_2^2 - 2a_3^2}{(x^2+y^2+z^2)^{3/2}} = a_1(2x^2 - a_1^2 - 2)$
 $= a_1(2x^2 - a_1^2 - 2) + 2a_2^2 + 2a_3^2$

• $\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 - a_1 & a_3 - a_2 & a_1 - a_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \frac{(x^2+y^2+z^2)^{1/2} - (a_2-a_1)(x^2+y^2+z^2)^{-1/2} - (a_3-a_2)(x^2+y^2+z^2)^{-1/2} - (a_1-a_3)(x^2+y^2+z^2)^{-1/2}}{(x^2+y^2+z^2)^{3/2}}$
 $= \frac{a_2(x^2+y^2+z^2) - (a_2-a_1)(x^2+y^2+z^2) - (a_3-a_2)(x^2+y^2+z^2) - (a_1-a_3)(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2a_2(x^2+y^2+z^2) - 2a_2^2 - 2a_3^2}{(x^2+y^2+z^2)^{3/2}} = a_2(2x^2 - a_2^2 - 2)$

$\nabla_A \left(\mathbf{a} \wedge \frac{\mathbf{r}}{r^3} \right) = \frac{a_1(-x^2+y^2+z^2) + 3a_2 x y + 3a_3 x z}{(x^2+y^2+z^2)^5}$
BY CROSS SUMMING:

• $a_1 \left(\frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) \right) \left(\frac{x}{(x^2+y^2+z^2)^{3/2}} \right) = a_1 \left(\frac{-3x^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \right) + 3a_2 x y + 3a_3 x z = \frac{a_1(-x^2+y^2+z^2) + 3a_2 x y + 3a_3 x z}{(x^2+y^2+z^2)^5}$
 $= a_1 \left(\frac{\partial}{\partial y} \left(\frac{y}{r^3} \right) \right) \left(\frac{y}{(x^2+y^2+z^2)^{3/2}} \right) = a_1 \left(\frac{-3y^2-z^2}{(x^2+y^2+z^2)^{5/2}} \right) + 3a_2 x y + 3a_3 x z = \frac{a_1(-x^2+y^2+z^2) + 3a_2 x y + 3a_3 x z}{(x^2+y^2+z^2)^5}$
 $= a_1 \left(\frac{\partial}{\partial z} \left(\frac{z}{r^3} \right) \right) \left(\frac{z}{(x^2+y^2+z^2)^{3/2}} \right) = a_1 \left(\frac{-3z^2-x^2}{(x^2+y^2+z^2)^{5/2}} \right) + 3a_2 x y + 3a_3 x z = \frac{a_1(-x^2+y^2+z^2) + 3a_2 x y + 3a_3 x z}{(x^2+y^2+z^2)^5}$
 $= 3a_2 \left(\frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) \right) \left(\frac{x}{(x^2+y^2+z^2)^{3/2}} \right) = 3a_2 \left(\frac{-3x^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \right) + 3a_2 x y + 3a_3 x z = \frac{3a_2(-x^2+y^2+z^2) + 3a_2 x y + 3a_3 x z}{(x^2+y^2+z^2)^5}$
 $= 3a_2 \left(\frac{\partial}{\partial y} \left(\frac{y}{r^3} \right) \right) \left(\frac{y}{(x^2+y^2+z^2)^{3/2}} \right) = 3a_2 \left(\frac{-3y^2-z^2}{(x^2+y^2+z^2)^{5/2}} \right) + 3a_2 x y + 3a_3 x z = \frac{3a_2(-x^2+y^2+z^2) + 3a_2 x y + 3a_3 x z}{(x^2+y^2+z^2)^5}$
 $= 3a_2 \left(\frac{\partial}{\partial z} \left(\frac{z}{r^3} \right) \right) \left(\frac{z}{(x^2+y^2+z^2)^{3/2}} \right) = 3a_2 \left(\frac{-3z^2-x^2}{(x^2+y^2+z^2)^{5/2}} \right) + 3a_2 x y + 3a_3 x z = \frac{3a_2(-x^2+y^2+z^2) + 3a_2 x y + 3a_3 x z}{(x^2+y^2+z^2)^5}$
 $= 3a_3 \left(\frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) \right) \left(\frac{x}{(x^2+y^2+z^2)^{3/2}} \right) = 3a_3 \left(\frac{-3x^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \right) + 3a_3 x y + 3a_1 x z = \frac{3a_3(-x^2+y^2+z^2) + 3a_3 x y + 3a_1 x z}{(x^2+y^2+z^2)^5}$
 $= 3a_3 \left(\frac{\partial}{\partial y} \left(\frac{y}{r^3} \right) \right) \left(\frac{y}{(x^2+y^2+z^2)^{3/2}} \right) = 3a_3 \left(\frac{-3y^2-z^2}{(x^2+y^2+z^2)^{5/2}} \right) + 3a_3 x y + 3a_1 x z = \frac{3a_3(-x^2+y^2+z^2) + 3a_3 x y + 3a_1 x z}{(x^2+y^2+z^2)^5}$
 $= 3a_3 \left(\frac{\partial}{\partial z} \left(\frac{z}{r^3} \right) \right) \left(\frac{z}{(x^2+y^2+z^2)^{3/2}} \right) = 3a_3 \left(\frac{-3z^2-x^2}{(x^2+y^2+z^2)^{5/2}} \right) + 3a_3 x y + 3a_1 x z = \frac{3a_3(-x^2+y^2+z^2) + 3a_3 x y + 3a_1 x z}{(x^2+y^2+z^2)^5}$
• $\nabla_A \left(\mathbf{a} \wedge \frac{\mathbf{r}}{r^3} \right) = \frac{3a_1 x y + 3a_2 x z + 3a_3 x y}{r^5} = \frac{a_1}{r^3} (a_1 a_2 a_3)$

Question 23

$$\nabla \wedge (\mathbf{A} \wedge \mathbf{B}) \equiv (\mathbf{B} \cdot \nabla) \mathbf{A} + (\nabla \cdot \mathbf{B}) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} - (\nabla \cdot \mathbf{A}) \mathbf{B}.$$

- a) Given that $\mathbf{A} = \mathbf{A}(x, y, z)$ and $\mathbf{B} = \mathbf{B}(x, y, z)$ are smooth vector functions, use index summation notation to prove the validity of the above vector identity.
- b) Verify the validity of the vector identity if

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \quad \text{and} \quad \mathbf{B} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

both sides yield $2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$

a) *Verify the k^{th} component of $\Delta_A(\mathbf{A} \wedge \mathbf{B})_k = \epsilon_{ijk} A_i B_j$*

NEXT THE k^{th} component of $\nabla_A(\mathbf{A} \wedge \mathbf{B})_k$

$$[\nabla_A(\mathbf{A} \wedge \mathbf{B})]_k = \epsilon_{ijk} \frac{\partial}{\partial x_j} [\epsilon_{lmk} A_l B_m]$$

$$= \epsilon_{ijk} \epsilon_{ilm} \frac{\partial}{\partial x_j} [A_l B_m]$$

Since i, k, l, m are the indices involved, it is ok to swap at the ends, so the commutivity can be applied.

$$= -\epsilon_{ilm} \epsilon_{ijk} \frac{\partial}{\partial x_j} [A_l B_m]$$

$$= \left[\frac{\partial}{\partial x_k} \left(\frac{\partial}{\partial x_j} [A_l B_m] \right) \right]_{l=i}$$

$$= \left[\frac{\partial}{\partial x_k} \left(\frac{\partial A_l}{\partial x_j} B_m + A_l \frac{\partial B_m}{\partial x_j} \right) \right]_{l=i}$$

...USING THE SUBSTITUTION PROPERTY OF ϵ ...

$$= \frac{\partial^2 A_l}{\partial x_k \partial x_j} B_m - \frac{\partial A_l}{\partial x_j} \frac{\partial B_m}{\partial x_k}$$

...PROVE DUE ...

$$= \frac{\partial^2 A_l}{\partial x_k \partial x_j} B_m + \frac{\partial A_l}{\partial x_j} \frac{\partial B_m}{\partial x_k} - \frac{\partial^2 A_l}{\partial x_j \partial x_k} B_m$$

...PROVE TO CENTER MAXIMAUM PROPERTY ...

$$= B_m \frac{\partial^2 A_l}{\partial x_k \partial x_j} + A_l \frac{\partial^2 B_m}{\partial x_k \partial x_j} - A_l \frac{\partial^2 B_m}{\partial x_j \partial x_k}$$

$$= \left[(\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \cdot (\nabla \cdot \mathbf{B}) - \mathbf{B} \cdot (\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla) \mathbf{B} \right]_k$$

• RHS

$$\nabla_A(\mathbf{A} \wedge \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} + (\nabla \cdot \mathbf{B}) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} - (\nabla \cdot \mathbf{A}) \mathbf{B}$$

b) $\mathbf{A} = (1, 2, -3) \quad \mathbf{B} = (x, y, z)$

LHS

$$\Delta_A \mathbf{B} = \begin{vmatrix} 1 & 2 & -3 \\ 1 & 2 & -3 \\ x & y & z \end{vmatrix} = (2x+3y, -3x-2y, y-2z)$$

$$\nabla_A(\mathbf{A} \wedge \mathbf{B}) = \begin{vmatrix} 1 & 1 & -1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y & -3x-2y & y-2z \end{vmatrix} = (14, 24, -3)$$

RHS *EVALUATE EACH OF THE FIVE TERMS SEPARATELY*

$$(\mathbf{B} \cdot \nabla) \mathbf{A} = \left[x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right] (1, 2, -3) = (0, 0, 0)$$

$$(\nabla \cdot \mathbf{B}) \mathbf{A} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (x, y, z) (1, 2, -3) = (3, 5, 1)$$

$$-(\nabla \cdot \mathbf{A}) \mathbf{B} = - \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (1, 2, -3) (x, y, z) = - (0, 1, 2, -3) = (0, 0, 0)$$

$$-(\mathbf{A} \cdot \nabla) \mathbf{B} = - \left[\frac{\partial}{\partial x} (x) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z) \right] (1, 2, -3) = (-1, 2, 3)$$

ABOVE THE 4 TERMS, IGNORING THE ZERO TERM FROM $(0, 0, 0)$

AT THE RHS

LAPLACIAN

$$\nabla \cdot \nabla \varphi = \nabla^2 \varphi$$

Question 1

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Determine the value of

$$\operatorname{div} \left[\operatorname{grad} \left(\frac{1}{r} \right) \right], r \neq 0.$$

$$\boxed{\operatorname{div} \left[\operatorname{grad} \left(\frac{1}{r} \right) \right] = 0}$$

$$\begin{aligned}
 \nabla \cdot \nabla \left(\frac{1}{r} \right) &= \nabla^2 \left(\frac{1}{\sqrt{x^2+y^2+z^2}} \right) = \nabla^2 \left[(x^2+y^2+z^2)^{-\frac{1}{2}} \right] \\
 &= \frac{\partial^2}{\partial x^2} \left[(x^2+y^2+z^2)^{-\frac{1}{2}} \right] + \frac{\partial^2}{\partial y^2} \left[(x^2+y^2+z^2)^{-\frac{1}{2}} \right] + \frac{\partial^2}{\partial z^2} \left[(x^2+y^2+z^2)^{-\frac{1}{2}} \right]
 \end{aligned}$$

Look at one of the components:

$$\begin{aligned}
 \frac{\partial^2}{\partial x^2} \left[(x^2+y^2+z^2)^{-\frac{1}{2}} \right] &= \frac{\partial}{\partial x} \left[-\frac{2}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right] \\
 &= \frac{\partial}{\partial x} \left[-\frac{2}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right] \\
 &= -\frac{(2(x^2+y^2+z^2))^{\frac{1}{2}}(1-2 \times \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}})(2x)}{(x^2+y^2+z^2)^{\frac{5}{2}}} \\
 &= -\frac{(2(x^2+y^2+z^2))^{-\frac{1}{2}}(2x)(2x^2+2y^2+2z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}} \\
 &= -\frac{(2x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}} \\
 &= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}
 \end{aligned}$$

As expression is symmetric,

$$\begin{aligned}
 \Rightarrow \nabla \cdot \nabla \left(\frac{1}{r} \right) &= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{2z^2-y^2-x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} \\
 \Rightarrow \nabla \cdot \nabla \left(\frac{1}{r} \right) &= 0
 \end{aligned}$$

Question 2

A smooth scalar field is denoted by $\varphi = \varphi(x, y, z)$ and a smooth vector field is denoted by $\mathbf{A} = \mathbf{A}(x, y, z)$.

- a) Use the standard definitions of vector operators to show that

$$\nabla \cdot (\varphi \mathbf{A}) = \nabla \varphi \cdot \mathbf{A} + \varphi \nabla \cdot \mathbf{A}.$$

- b) Given further that f and g are functions of x , y and z , whose second partial derivatives exist, deduce that

$$\nabla \cdot (f \nabla g - g \nabla f) = f \nabla^2 g - g \nabla^2 f.$$

[M12E], proof

a) Define some quantities first

$$\mathbf{A} = (A_1(x,y,z), A_2(x,y,z), A_3(x,y,z)) \quad \text{and} \quad \varphi = \varphi(x,y,z)$$

Then we have

$$\begin{aligned} \nabla \cdot (\varphi \mathbf{A}) &= \nabla \cdot (\varphi A_1, \varphi A_2, \varphi A_3) \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (\varphi A_1, \varphi A_2, \varphi A_3) \\ &\stackrel{\text{BY THE PRODUCT RULE}}{=} \frac{\partial}{\partial x}(\varphi A_1) + \frac{\partial}{\partial y}(\varphi A_2) + \frac{\partial}{\partial z}(\varphi A_3) \\ &= \frac{\partial \varphi}{\partial x} A_1 + \varphi \frac{\partial A_1}{\partial x} + \frac{\partial \varphi}{\partial y} A_2 + \varphi \frac{\partial A_2}{\partial y} + \frac{\partial \varphi}{\partial z} A_3 + \varphi \frac{\partial A_3}{\partial z} \\ &= \left[\frac{\partial \varphi}{\partial x} A_1 + \frac{\partial \varphi}{\partial y} A_2 + \frac{\partial \varphi}{\partial z} A_3 \right] + \varphi \left[\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right] \\ &= \left(\frac{\partial \varphi}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial}{\partial y} + \frac{\partial \varphi}{\partial z} \frac{\partial}{\partial z} \right) \cdot (A_1, A_2, A_3) + \varphi \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \\ &= \nabla \varphi \cdot \mathbf{A} + \varphi \nabla \cdot \mathbf{A} \quad \text{as required} \end{aligned}$$

b) Proceed using the result of part (a)

$$\begin{aligned} \nabla \cdot [f \nabla g - g \nabla f] &= \nabla \cdot [f \nabla g] - \nabla \cdot [g \nabla f] \\ &= \nabla f \cdot \nabla g + f \nabla \cdot \nabla g - g \nabla \cdot \nabla f - g \nabla \cdot \nabla f \\ &= \cancel{\nabla f \cdot \nabla g} + \cancel{f \nabla \cdot \nabla g} - \cancel{-g \nabla \cdot \nabla f} - g \nabla \cdot \nabla f \\ &= f \nabla^2 g - g \nabla^2 f \quad \text{as required} \end{aligned}$$

Question 3

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$, $r \neq 0$.

Show that

$$\nabla^2(\ln r) = \frac{1}{r^2}.$$

[proof]

$$\begin{aligned}\nabla^2(\ln r) &= \nabla \cdot \nabla(\ln r) = \nabla \cdot \left(\frac{\partial}{\partial x} \ln r, \frac{\partial}{\partial y} \ln r, \frac{\partial}{\partial z} \ln r \right) = \dots \\ \text{Now } \frac{\partial(\ln r)}{\partial x} &= \frac{\partial}{\partial x} \left[\ln(x^2+y^2+z^2)^{-\frac{1}{2}} \right] = \frac{\partial}{\partial x} \left[\frac{1}{2} \ln(x^2+y^2+z^2) \right] \\ &= \frac{1}{2} \times \frac{1}{x^2+y^2+z^2} \times 2x = \frac{x}{x^2+y^2+z^2} = \frac{x}{r^2}. \\ \text{SINCE THE EXPRESSION IS 'BY CYCLIC SYMMETRY'} \\ \frac{\partial}{\partial x}(\ln r) &= \frac{x}{r^2}, \quad \frac{\partial}{\partial y}(\ln r) = \frac{y}{r^2}, \quad \frac{\partial}{\partial z}(\ln r) = \frac{z}{r^2}. \\ \dots &= \nabla \cdot \left[\frac{\partial}{\partial x} \left(\frac{x}{r^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r^2} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r^2} \right) \right] = \left[\frac{\partial}{\partial x} \frac{\partial}{\partial x} \left(\frac{x}{r^2} \right) + \frac{\partial}{\partial y} \frac{\partial}{\partial y} \left(\frac{x}{r^2} \right) + \frac{\partial}{\partial z} \frac{\partial}{\partial z} \left(\frac{x}{r^2} \right) \right] \\ &= \frac{\partial}{\partial x} \left(\frac{x}{r^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r^2} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r^2} \right) = \dots \\ \frac{\partial}{\partial x} \left(\frac{x}{r^2} \right) &= \frac{\partial}{\partial x} \left[\frac{x}{x^2+y^2+z^2} \right] = \frac{(x^2+y^2+z^2)x - x(2x)}{(x^2+y^2+z^2)^2} \\ &= \frac{y^2z^2-x^2}{(x^2+y^2+z^2)^2} = \frac{y^2z^2-x^2}{r^4}. \\ \text{SIMILARLY THE 'OTHER TWO' BY CYCLIC SYMMETRY} \\ &= \frac{y^2z^2-y^2}{r^4} + \frac{x^2z^2-z^2}{r^4} + \frac{x^2y^2-y^2}{r^4} \\ &= \frac{-x^2y^2+x^2z^2}{r^4} = -\frac{x^2}{r^4} = \frac{1}{r^2}. \quad \text{AS REQUIRED.}\end{aligned}$$

Question 4

It is given that $\varphi = \varphi(x, y, z)$ and $\psi = \psi(x, y, z)$ are twice differentiable functions.

Show, with a detailed method, that

$$\nabla^2(\varphi\psi) = \psi\nabla^2\varphi + 2\nabla\varphi \cdot \nabla\psi + \varphi\nabla^2\psi.$$

[proof]

$$\begin{aligned}\nabla^2(\varphi\psi) &= \frac{\partial^2}{\partial x^2}(\varphi\psi) + \frac{\partial^2}{\partial y^2}(\varphi\psi) + \frac{\partial^2}{\partial z^2}(\varphi\psi) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\varphi + \frac{\partial \varphi}{\partial x}\psi\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\varphi + \frac{\partial \varphi}{\partial y}\psi\right) + \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z}\varphi + \frac{\partial \varphi}{\partial z}\psi\right) \\ &= \frac{\partial^2\psi}{\partial x^2}\varphi + \frac{\partial \psi}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial^2\psi}{\partial y^2}\varphi + \frac{\partial \psi}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial^2\psi}{\partial z^2}\varphi + \frac{\partial \psi}{\partial z}\frac{\partial \varphi}{\partial z} \\ &\quad + \frac{\partial^2\varphi}{\partial x^2}\psi + \frac{\partial \varphi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial^2\varphi}{\partial y^2}\psi + \frac{\partial \varphi}{\partial y}\frac{\partial \psi}{\partial y} + \frac{\partial^2\varphi}{\partial z^2}\psi + \frac{\partial \varphi}{\partial z}\frac{\partial \psi}{\partial z} \\ &= \psi\left[\frac{\partial^2\varphi}{\partial x^2}, \frac{\partial^2\varphi}{\partial y^2}, \frac{\partial^2\varphi}{\partial z^2}\right] + 2\left[\frac{\partial \varphi}{\partial x}\frac{\partial \psi}{\partial x}, \frac{\partial \varphi}{\partial y}\frac{\partial \psi}{\partial y}, \frac{\partial \varphi}{\partial z}\frac{\partial \psi}{\partial z}\right] \\ &\quad + \varphi\left[\frac{\partial^2\psi}{\partial x^2}, \frac{\partial^2\psi}{\partial y^2}, \frac{\partial^2\psi}{\partial z^2}\right] \\ &= \psi\nabla^2\varphi + 2\nabla\varphi \cdot \nabla\psi + \varphi\nabla^2\psi.\end{aligned}$$

Question 5

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show clearly that

$$\nabla^2(|\mathbf{r}|^n) = n(n+1)|\mathbf{r}|^{n-2}.$$

proof

$\nabla^2(r^n) = \nabla^2[(x^2+y^2+z^2)^{\frac{n}{2}}]$

• $\nabla^2(r^n) = \frac{\partial^2}{\partial x^2}[(x^2+y^2+z^2)^{\frac{n}{2}}] + \frac{\partial^2}{\partial y^2}[(x^2+y^2+z^2)^{\frac{n}{2}}] + \frac{\partial^2}{\partial z^2}[(x^2+y^2+z^2)^{\frac{n}{2}}]$

• Now $\frac{\partial^2}{\partial x^2}[(x^2+y^2+z^2)^{\frac{n}{2}}] = \frac{\partial}{\partial x}\left[n(x^2+y^2+z^2)^{\frac{n-1}{2}}\right]$
 $= \frac{\partial}{\partial x}\left[n(x^2+y^2+z^2)^{\frac{n-1}{2}}\right]$
 $= n(x^2+y^2+z^2)^{\frac{n-1}{2}} + n(\frac{1}{2})(x^2+y^2+z^2)^{\frac{n-3}{2}(n-1)}$
 $= n(x^2+y^2+z^2)^{\frac{n-1}{2}} + n(n-2)(x^2+y^2+z^2)^{\frac{n-3}{2}(n-1)}$
 $= n(x^2+y^2+z^2)^{\frac{n-1}{2}} + n(n-2)x^2(x^2+y^2+z^2)^{\frac{n-3}{2}(n-1)}$

• ADD BY SIMILARLY
 $\frac{\partial^2}{\partial y^2}[(x^2+y^2+z^2)^{\frac{n}{2}}] = n(x^2+y^2+z^2)^{\frac{n-1}{2}} + n(n-2)y^2(x^2+y^2+z^2)^{\frac{n-3}{2}(n-1)}$
 $\frac{\partial^2}{\partial z^2}[(x^2+y^2+z^2)^{\frac{n}{2}}] = n(x^2+y^2+z^2)^{\frac{n-1}{2}} + n(n-2)z^2(x^2+y^2+z^2)^{\frac{n-3}{2}(n-1)}$

• ADDING THE EXPRESSIONS
 $\nabla^2(r^n) = 3n(x^2+y^2+z^2)^{\frac{n-1}{2}(n-1)} + n(n-2)(x^2+y^2+z^2)^{\frac{n-3}{2}(n-1)}[x^2+y^2+z^2]$
 $= 3n(x^2+y^2+z^2)^{\frac{n-1}{2}(n-1)} + n(n-2)(x^2+y^2+z^2)^{\frac{n-3}{2}(n-1)}$
 $= [3n+n^2-2n](x^2+y^2+z^2)^{\frac{n-1}{2}(n-1)}$
 $= (n^2+n)\int[(x^2+y^2+z^2)^{\frac{n}{2}}]$
 $= n(n+1)r^{n-2}$

✓ 20 MARKS

Question 6

It is given that

$$\varphi(x, y, z) = z + \sinh x \sin y.$$

- a) Verify that φ is a solution of Laplace's equation

$$\nabla^2 \varphi = 0.$$

- b) Hence find a vector field \mathbf{F} , so that

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \wedge \mathbf{F} = \mathbf{0}.$$

- c) Verify that \mathbf{F} found in part (b) is solenoidal and irrotational.

$$\boxed{\mathbf{F} = (\cosh x \sin y) \mathbf{i} + (\sinh x \cos y) \mathbf{j} + \mathbf{k}}$$

a) $\varphi(x, y, z) = z + \sinh x \sin y$

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= \sinh x \sin y \\ \frac{\partial \varphi}{\partial y} &= \sinh x (-\cos y) = -\cosh x \sin y \\ \frac{\partial \varphi}{\partial z} &= 0 \end{aligned} \quad \left. \begin{aligned} \nabla^2 \varphi &= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \\ &= \cosh x \sin y - \cosh x \sin y + 0 \\ &= 0 \end{aligned} \right\}$$

b) $\nabla^2 \varphi = \nabla(\nabla \varphi) = 0$
 $\nabla(\nabla \varphi) = 0 \rightarrow \text{DIV}(\text{Grad } \varphi) = 0 \quad \leftarrow (\text{DEFN})$
 ALSO $\nabla \times (\nabla \varphi) = 0 \leftarrow (\text{IDENTITY})$

$$\therefore \nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right) = [\text{cosh } x \sin y, \text{sinh } x \cos y, 1]$$

$$\therefore \mathbf{F} = [\text{cosh } x \sin y, \text{sinh } x \cos y, 1]$$

c) $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} [\text{cosh } x \sin y] + \frac{\partial}{\partial y} [\text{sinh } x \cos y] + \frac{\partial}{\partial z} [1]$
 $= \text{sinh } x \sin y - \text{sinh } x \cos y + 0$
 $= 0 \quad \therefore \text{SOLENOIDAL}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \text{cosh } x \sin y & \text{sinh } x \cos y & 1 \end{vmatrix} =$$

$$= [\text{cosh } x \sin y - \text{sinh } x \cos y, 0 - 0, \text{sinh } x \cos y - \text{cosh } x \sin y] = (0, 0, 0)$$

$$\therefore \text{IRROTATIONAL}$$

Question 7

A Cartesian position vector is denoted by \mathbf{r} and $r = |\mathbf{r}|$.

Given that $f(r)$ is a twice differentiable function, show that

$$\nabla^2 f = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}, \quad r \neq 0.$$

[proof]

$\Gamma = (x, y, z)$
 $r = |\Gamma| = (\sqrt{x^2+y^2+z^2})^{\frac{1}{2}}$

$\nabla^2 f(\Gamma) = \nabla \cdot \nabla f = \nabla \cdot [\frac{\partial f}{\partial x} \frac{\partial}{\partial x}, \frac{\partial f}{\partial y} \frac{\partial}{\partial y}, \frac{\partial f}{\partial z} \frac{\partial}{\partial z}] = \nabla \cdot [\frac{\partial^2 f}{\partial x^2} \frac{\partial}{\partial x}, \frac{\partial^2 f}{\partial x \partial y} \frac{\partial}{\partial y}, \frac{\partial^2 f}{\partial x \partial z} \frac{\partial}{\partial z} + \frac{\partial^2 f}{\partial y^2} \frac{\partial}{\partial y}, \frac{\partial^2 f}{\partial y \partial z} \frac{\partial}{\partial z} + \frac{\partial^2 f}{\partial z^2} \frac{\partial}{\partial z}]$

$= [\frac{\partial^2 f}{\partial x^2} \frac{\partial}{\partial x}, \frac{\partial^2 f}{\partial x \partial y} \frac{\partial}{\partial y}, \frac{\partial^2 f}{\partial x \partial z} \frac{\partial}{\partial z}] + [\frac{\partial^2 f}{\partial y^2} \frac{\partial}{\partial y}, \frac{\partial^2 f}{\partial y \partial z} \frac{\partial}{\partial z}] + [\frac{\partial^2 f}{\partial z^2} \frac{\partial}{\partial z}]$

$= \frac{\partial^2 f}{\partial x^2} \frac{\partial}{\partial x} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial}{\partial y} + \frac{\partial^2 f}{\partial x \partial z} \frac{\partial}{\partial z} + \frac{\partial^2 f}{\partial y^2} \frac{\partial}{\partial y} + \frac{\partial^2 f}{\partial y \partial z} \frac{\partial}{\partial z} + \frac{\partial^2 f}{\partial z^2} \frac{\partial}{\partial z}$

$= \frac{\partial^2 f}{\partial x^2} \left[\frac{x^2}{r^2} \right] + \frac{\partial^2 f}{\partial y^2} \left[\frac{y^2}{r^2} \right] + \frac{\partial^2 f}{\partial z^2} \left[\frac{z^2}{r^2} \right] + \frac{\partial^2 f}{\partial x \partial y} \left[\frac{2xy}{r^2} \right] + \frac{\partial^2 f}{\partial x \partial z} \left[\frac{2xz}{r^2} \right] + \frac{\partial^2 f}{\partial y \partial z} \left[\frac{2yz}{r^2} \right]$

$= \frac{\partial^2 f}{\partial x^2} \left[\frac{x^2}{r^2} \right] + \frac{\partial^2 f}{\partial y^2} \left[\frac{y^2}{r^2} \right] + \frac{\partial^2 f}{\partial z^2} \left[\frac{z^2}{r^2} \right] + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{2}{r^2} \frac{df}{dr}$

$\therefore \frac{\partial^2 f}{\partial r^2} = \frac{2}{r^2} \frac{df}{dr}$

AND FINALLY THE OTHER TWO BY SIMILARITY

Question 8

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show, with a detailed method, that

$$\nabla^2 \left[\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^2} \right) \right] = \frac{2}{|\mathbf{r}|^4}.$$

proof

• $\nabla^2 \left[\nabla \cdot \left(\frac{\mathbf{r}}{r^2} \right) \right] = \nabla^2 \left[\left(\frac{\partial}{\partial x} \frac{x}{r^2} \right) \cdot \left(-\frac{2xy^2z^2}{x^2+y^2+z^2} \right) \right]$

• $\left(\frac{\partial}{\partial x} \frac{x}{r^2} \right) \cdot \left(-\frac{2xy^2z^2}{x^2+y^2+z^2} \right) = \frac{2}{r^4(x^2+y^2+z^2)} \left[\frac{2}{x} \left(\frac{y^2z^2}{x^2+y^2+z^2} \right) - \frac{2}{x^2+y^2+z^2} \left(2xy^2z^2 \right) \right] = \frac{2}{r^4(x^2+y^2+z^2)} \left[\frac{2(y^2z^2)x^2 - 2(x^2y^2z^2)}{(x^2+y^2+z^2)^2} \right] = \frac{2(x^2y^2z^2)(x^2 - y^2 - z^2)}{(x^2+y^2+z^2)^3}$

• By cyclic symmetry,

$$\begin{aligned} \nabla^2 \left(\frac{x}{r^2} \right) &= \frac{\partial^2 \frac{x}{r^2}}{\partial x^2} + \frac{\partial^2 \frac{x}{r^2}}{\partial y^2} + \frac{\partial^2 \frac{x}{r^2}}{\partial z^2} \\ &= \frac{2x^2y^2z^2}{(x^2+y^2+z^2)^3} = \frac{2}{x^2+y^2+z^2} = \frac{2}{r^2}. \end{aligned}$$

• $\nabla^2 \left(\nabla \cdot \frac{\mathbf{r}}{r^2} \right) = \frac{\partial^2}{\partial x^2} \left(x \frac{y^2z^2}{x^2+y^2+z^2} \right) + \frac{\partial^2}{\partial y^2} \left(x \frac{y^2z^2}{x^2+y^2+z^2} \right) + \frac{\partial^2}{\partial z^2} \left(x \frac{y^2z^2}{x^2+y^2+z^2} \right)$

$\frac{\partial^2}{\partial x^2} \left[-2x \left(\frac{y^2z^2}{x^2+y^2+z^2} \right)^2 \right]$

$$\begin{aligned} &= -2(x^2+y^2+z^2)^2 \cdot 2x \left(\frac{y^2z^2}{x^2+y^2+z^2} \right)^3 \\ &= 2 \left(\frac{y^2z^2}{x^2+y^2+z^2} \right)^3 \left[-2x(3y^2z^2) + 4x^3 \right] \\ &= \frac{2}{(x^2+y^2+z^2)} (3x^2y^2z^2 - 2x^4) = \frac{2(x^2y^2z^2 - x^4)}{(x^2+y^2+z^2)^3} \end{aligned}$$

• This by cyclic symmetry holds,

$$\begin{aligned} \nabla^2 \left[\nabla \cdot \frac{\mathbf{r}}{r^2} \right] &= \frac{2(x^2y^2z^2 - x^4)}{(x^2+y^2+z^2)^3} + \frac{2(x^2y^2z^2 - y^4)}{(x^2+y^2+z^2)^3} + \frac{2(x^2y^2z^2 - z^4)}{(x^2+y^2+z^2)^3} \\ &= \frac{2(x^2y^2z^2 - x^4)}{(x^2+y^2+z^2)^3} = \frac{2(x^2y^2z^2 - x^4)}{(x^2+y^2+z^2)^3} = \frac{2}{r^4} \end{aligned}$$

Question 9

The scalar function φ is given below in terms of the non zero constants λ and μ .

$$\varphi(x, y, z) = e^x \sin(\lambda y + \lambda z) \cos(\mu y - \mu z).$$

Given that

$$\nabla \cdot (\nabla \varphi) = 0,$$

show, with a detailed method, that $\lambda^2 + \mu^2 = \frac{1}{2}$.

proof

$\nabla^2 \varphi = 0 \Rightarrow \left(e^x \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \right) = 0$

$\nabla \cdot (\nabla \varphi) = 0$

• $\nabla(\lambda y + \lambda z) = e^x \sin(\lambda y + \lambda z) \cos(\mu y - \mu z)$

• ~~ANSWER~~ $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$
 $\sin(A-B) \equiv \sin A \cos B - \cos A \sin B$

AND $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$
 $\sin A \cos B \equiv \frac{1}{2} \sin(A+B) + \frac{1}{2} \sin(A-B)$

Hence $\nabla(\lambda y + \lambda z) = \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A+B)] + \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A-B)]$

• $\nabla \cdot \nabla \varphi = \left[\frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A+B)] + \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A-B)] \right]_x + \left[\frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A+B)] + \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A-B)] \right]_y + \left[\frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A+B)] + \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A-B)] \right]_z$

• $\nabla \cdot \nabla \varphi = \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A+B)] + \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A-B)] - \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A+B)] + \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A-B)] - \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A+B)] + \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A-B)]$

$$= \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A+B)] \left[1 - 2^2 - 2\lambda^2 - \mu^2 - 2^2 + 2\lambda^2 - \mu^2 \right] + \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A-B)] \left[1 - \lambda^2 + 2\lambda^2 - \mu^2 - \lambda^2 - \mu^2 \right]$$

$$= \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A+B)] \left[1 - 2\lambda^2 - \mu^2 \right] + \frac{1}{2} e^x \sin[(\lambda y + \lambda z) + (A-B)] \left[1 - 2\lambda^2 - \mu^2 \right]$$

$$= \frac{1}{2} e^x (1 - 2\lambda^2 - 2\mu^2) \left[\sin[(\lambda y + \lambda z) + (A+B)] + \sin[(\lambda y + \lambda z) + (A-B)] \right]$$

$$= \frac{e^x}{2} (1 - 2\lambda^2 - 2\mu^2) \left[2 \sin[(\lambda y + \lambda z) + (A+B)] \right] \stackrel{\text{ENVERTING BACK}}{\rightarrow}$$

$$\therefore \Psi(\lambda, \mu) = (-2\lambda^2 - 2\mu^2)$$

∴ A SOLUTION IF $1 - 2\lambda^2 - 2\mu^2 = 0$
 $1 - 2\lambda^2 - 2\mu^2 = 1$

Question 10

It is given that

$$\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) \equiv \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

- a) Use the above identity to find a simplified expression for $\nabla \wedge (\nabla \wedge \mathbf{F})$, where \mathbf{F} is a smooth vector field.

The smooth vector fields \mathbf{E} and \mathbf{H} , satisfy the following relationships.

- $\nabla \cdot \mathbf{E} = 0$
- $\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}$
- $\nabla \cdot \mathbf{H} = 0$
- $\nabla \wedge \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}$

- b) Show that \mathbf{E} and \mathbf{H} , satisfy the wave equation.

$$\nabla^2 \mathbf{U} = \frac{\partial^2 \mathbf{U}}{\partial t^2}.$$

$$\boxed{\nabla \wedge (\nabla \wedge \mathbf{F}) \equiv \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}}$$

a) $\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) \equiv \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$

Let $\mathbf{A} = \nabla$
 $\mathbf{B} = \nabla$
 $\mathbf{C} = \mathbf{F}$

$\nabla \cdot (\nabla \cdot \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla) \mathbf{F}$
 $\nabla \cdot (\nabla \cdot \mathbf{E}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

b) $\nabla \cdot \mathbf{E} = 0 \quad \nabla \cdot \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}$
 $\nabla \cdot \mathbf{H} = 0 \quad \nabla \cdot \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}$

$\bullet \quad \nabla \cdot \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}$
 $\Rightarrow \nabla \cdot (\nabla \cdot \mathbf{E}) = \nabla \cdot (-\frac{\partial \mathbf{H}}{\partial t})$
 $\Rightarrow \nabla \cdot (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} [\nabla \cdot \mathbf{H}] \Rightarrow \nabla \cdot (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \frac{\partial}{\partial t} [\nabla \cdot \mathbf{E}]$
 $\Rightarrow -\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} [\frac{\partial \mathbf{E}}{\partial t}]$
 $\Rightarrow -\nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$
 $\Rightarrow \nabla^2 \mathbf{E} = -\frac{\partial^2 \mathbf{E}}{\partial t^2}$

$\bullet \quad \nabla \cdot \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}$
 $\Rightarrow \nabla \cdot (\nabla \cdot \mathbf{H}) = \nabla \cdot (\frac{\partial \mathbf{E}}{\partial t})$
 $\Rightarrow \nabla \cdot (\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = \frac{\partial}{\partial t} [\nabla \cdot \mathbf{E}] \Rightarrow \nabla \cdot (\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = \frac{\partial}{\partial t} [\frac{\partial \mathbf{E}}{\partial t}]$
 $\Rightarrow -\nabla^2 \mathbf{H} = \frac{\partial}{\partial t} [-\frac{\partial \mathbf{H}}{\partial t}]$
 $\Rightarrow -\nabla^2 \mathbf{H} = -\frac{\partial^2 \mathbf{H}}{\partial t^2}$
 $\Rightarrow \nabla^2 \mathbf{H} = \frac{\partial^2 \mathbf{H}}{\partial t^2}$

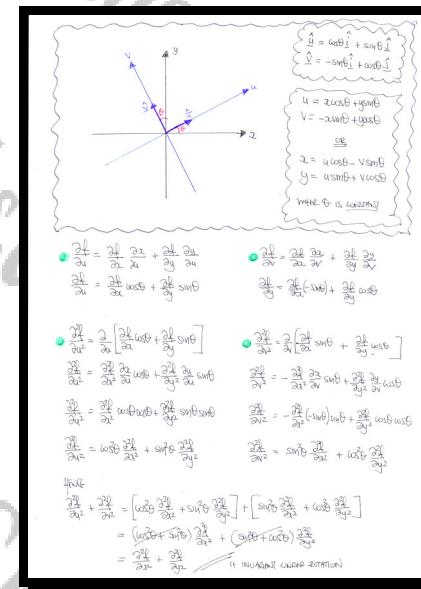
Question 11

The Laplacian operator ∇^2 in the standard two dimensional Cartesian system of coordinates is defined as

$$\nabla^2(\) \equiv \frac{\partial^2}{\partial x^2}(\) + \frac{\partial^2}{\partial y^2}(\).$$

Show that the two dimensional Laplacian operator is invariant under rotation.

proof



Question 12

It is given that if \mathbf{F} is a smooth vector field, then

$$\nabla \wedge (\nabla \wedge \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$

- a) By using index summation notation, or otherwise, prove the validity of the above vector identity.

Maxwell's equations for the electric field \mathbf{E} and magnetic field \mathbf{B} , satisfy

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

where ρ is the electric charge density, \mathbf{J} is the current density, and μ_0 and ϵ_0 are positive constants.

- b) Show that $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$.
- c) Given further that $\rho = 0$ and $\mathbf{J} = \mathbf{0}$, show also that

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

proof

a) consider the i th component of $\nabla \wedge (\nabla \wedge \mathbf{E})$

$$[\nabla_{k\ell m} \nabla_i (\epsilon_{ijk} \nabla_j F_\ell)]$$

$$= \epsilon_{ijk} \epsilon_{ikl} \frac{\partial}{\partial x_l} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_n} F_j$$

SUM K & M IN THE FIRST LAM-UNITS THEOREM

$$= -\epsilon_{kij} \epsilon_{ikl} \frac{\partial^2}{\partial x_l \partial x_m} F_j$$

use + LAGRANGE'S THEOREM

$$= -\left| \begin{array}{c} \delta_{kl} \delta_{ij} \\ \delta_{kl} \delta_{ij} \end{array} \right| \frac{\partial^2}{\partial x_l \partial x_m} F_j$$

$$= -[\delta_{ij} \delta_{kl} - \delta_{il} \delta_{kj}] \frac{\partial^2}{\partial x_l \partial x_m} F_j$$

$$= -\delta_{ij} \delta_{kl} \frac{\partial^2}{\partial x_l \partial x_m} F_j + \delta_{il} \delta_{kj} \frac{\partial^2}{\partial x_l \partial x_m} F_j$$

BY THE LAGRANGE'S PROPERTY

$$= -\frac{\partial^2}{\partial x_l \partial x_m} F_j + \frac{\partial^2}{\partial x_l \partial x_m} F_j$$

$$= -\nabla^2 F_j + \frac{\partial^2}{\partial x_l \partial x_m} F_j$$

$$= -\nabla^2 F_j + \frac{2}{\partial x_l} (\frac{\partial^2}{\partial x_l \partial x_m} F_j)$$

$$= -\nabla^2 F_j + \frac{2}{\partial x_l} (\nabla^2 F_j)$$

$$= [\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}]_j$$

$$\therefore \nabla \wedge (\nabla \wedge \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

AS REQUIRED

b) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \quad (i)$

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (ii)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (iii)$$

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (iv)$$

• SUMMING WITH (i)

$$\nabla \wedge \mathbf{B} = \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad (v)$$

• TRACE THE CURL OF THE ABOVE EQUATION

$$\Rightarrow \nabla \wedge (\nabla \wedge \mathbf{B}) = \nabla \wedge \left(\mu_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

• BY THE IDENTITY OF PART (a)

$$\Rightarrow \nabla \wedge (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0 \frac{\partial}{\partial t} (\nabla \cdot \mathbf{B})$$

$$\Rightarrow -\nabla^2 \mathbf{B} = \mu_0 \frac{\partial}{\partial t} \left[-\frac{\partial \mathbf{E}}{\partial t} \right]$$

$$\Rightarrow -\nabla^2 \mathbf{B} = -\mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \mathbf{B} = \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

AS REQUIRED

• SUMMING WITH (ii)

$$\nabla \wedge \mathbf{B} = \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad (vi)$$

• TRACE THE CURL OF THE ABOVE EQUATION

$$\Rightarrow \nabla \wedge (\nabla \cdot \mathbf{E}) = \nabla \wedge \left(-\frac{\partial \mathbf{B}}{\partial t} \right)$$

• BY THE IDENTITY OF PART (a)

$$\Rightarrow \nabla \wedge (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B})$$

$$\Rightarrow -\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \left[\mu_0 \frac{\partial \mathbf{B}}{\partial t} \right]$$

$$\Rightarrow \nabla^2 \mathbf{E} = \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

AS REQUIRED

Question 13

Show that the Laplacian operator in the standard two dimensional Polar system of coordinates is given by

$$\nabla^2(\) \equiv \frac{\partial^2}{\partial r^2}(\) + \frac{1}{r} \frac{\partial}{\partial r}(\) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}(\).$$

proof

$\nabla^2 \phi =$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$\bullet \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{2r}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \theta^2}$

$\bullet \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{2r}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \theta^2}$

$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{2r}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \theta^2}$

$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{2r}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \theta^2}$

$\frac{\partial^2 \phi}{\partial x^2} = \cos^2 \theta \frac{\partial^2 \phi}{\partial r^2} - \sin^2 \theta \frac{\partial^2 \phi}{\partial r^2}$ or to operator $\left[\frac{\partial}{\partial r} = \cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta} \right]$

$\frac{\partial^2 \phi}{\partial y^2} = \sin^2 \theta \frac{\partial^2 \phi}{\partial r^2} + \cos^2 \theta \frac{\partial^2 \phi}{\partial r^2}$ or to operator $\left[\frac{\partial}{\partial r} = \sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta} \right]$

Now the second derivatives

$\bullet \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial r} \left(\frac{\partial \phi}{\partial r} \right) = \left(\cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta} \right) \left(\cos \theta \frac{\partial \phi}{\partial r} - \sin \theta \frac{\partial \phi}{\partial \theta} \right)$

$= \cos^2 \theta \left(\cos \theta \frac{\partial^2 \phi}{\partial r^2} + \cos \theta \frac{\partial \phi}{\partial r} \left(-\frac{\partial \phi}{\partial r} \right) - \frac{\partial \phi}{\partial r} \left(\cos \theta \frac{\partial \phi}{\partial r} \right) + \sin^2 \theta \frac{\partial^2 \phi}{\partial \theta^2} \right)$

$= \cos^2 \theta \frac{\partial^2 \phi}{\partial r^2} - \cos \theta \sin \theta \frac{\partial}{\partial r} \left(\frac{\partial \phi}{\partial r} \right) - \sin^2 \theta \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial \phi}{\partial r} \right) + \sin^2 \theta \frac{\partial^2 \phi}{\partial \theta^2}$

Product rule Product rule Product rule

$$= \cos^2 \theta \frac{\partial^2 \phi}{\partial r^2} - \cos \theta \sin \theta \frac{\partial^2 \phi}{\partial r^2} + \frac{2r}{r} \cos \theta \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial r} + \frac{2r}{r} \sin^2 \theta \frac{\partial^2 \phi}{\partial \theta^2}$$
 $+ \frac{2r}{r} \cos \theta \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \theta} + \frac{2r}{r} \sin \theta \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial r} - \frac{2r}{r} \cos \theta \frac{\partial^2 \phi}{\partial \theta^2} + \cos^2 \theta \frac{\partial^2 \phi}{\partial r^2}$
 $+ \frac{2r}{r} \cos \theta \frac{\partial^2 \phi}{\partial r^2} - \cos \theta \sin \theta \frac{\partial^2 \phi}{\partial r^2} + \frac{2r}{r} \cos \theta \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial r} + \frac{2r}{r} \sin^2 \theta \frac{\partial^2 \phi}{\partial \theta^2}$
 $+ \frac{2r}{r} \cos \theta \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial \theta} + \frac{2r}{r} \sin \theta \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial r} - \frac{2r}{r} \cos \theta \frac{\partial^2 \phi}{\partial \theta^2} + \cos^2 \theta \frac{\partial^2 \phi}{\partial r^2}$

cancel out

 $= -\frac{2r}{r} \cos \theta \frac{\partial^2 \phi}{\partial r^2} + \sin^2 \theta \left[\frac{\partial^2 \phi}{\partial r^2} + \frac{2r}{r} \frac{\partial \phi}{\partial r} \right] + \frac{2r}{r} \left[\cos \theta \frac{\partial^2 \phi}{\partial r^2} + \sin^2 \theta \frac{\partial^2 \phi}{\partial \theta^2} \right]$
 $+ \frac{2r}{r} \left[-\cos \theta \frac{\partial^2 \phi}{\partial r^2} + \cos^2 \theta \frac{\partial^2 \phi}{\partial \theta^2} \right]$
 $= -\frac{2r}{r} \cos \theta \frac{\partial^2 \phi}{\partial r^2} - \sin^2 \theta \frac{\partial^2 \phi}{\partial r^2} + \frac{2r}{r} \cos^2 \theta \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cos^2 \theta}{r} \frac{\partial^2 \phi}{\partial r^2} + \frac{\sin^2 \theta}{r} \frac{\partial^2 \phi}{\partial \theta^2}$
 $= \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$

$$= \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{2r}{r} \cos \theta \frac{\partial \phi}{\partial r} + \frac{2r}{r} \sin \theta \frac{\partial \phi}{\partial \theta} - \frac{2r}{r} \cos \theta \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2}$$

cancel

$$\frac{\partial^2 \phi}{\partial r^2} = \left(\cos^2 \theta \frac{\partial^2 \phi}{\partial r^2} + \frac{2r}{r} \cos \theta \frac{\partial \phi}{\partial r} + \frac{2r}{r} \sin \theta \frac{\partial \phi}{\partial \theta} - \frac{2r}{r} \cos \theta \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} \right) \frac{1}{\cos^2 \theta + \frac{2r}{r} \frac{\partial \phi}{\partial r} + \frac{2r}{r} \frac{\partial \phi}{\partial \theta} - \frac{2r}{r} \cos \theta \frac{\partial^2 \phi}{\partial \theta^2}}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$$

or to operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \equiv \frac{2}{r^2} + \frac{1}{r} \frac{2}{r} + \frac{1}{r^2} \frac{2}{r^2}$$