

VECTOR PRACTICE

Part B

THE CROSS PRODUCT

Question 1

Find in each of the following cases $\mathbf{a} \wedge \mathbf{b}$, where the vectors \mathbf{a} and \mathbf{b} are

- a) $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$
- b) $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}$
- c) $\mathbf{a} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$
- d) $\mathbf{a} = 7\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$
- e) $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$$\boxed{\mathbf{i} + 3\mathbf{j} - 17\mathbf{k}}, \boxed{-\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}}, \boxed{5\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}}, \boxed{-10\mathbf{i} - 18\mathbf{j} + 22\mathbf{k}}, \boxed{-11\mathbf{i} + 18\mathbf{j} + 17\mathbf{k}}$$

$$\begin{aligned}
 \text{(a)} \quad & (\mathbf{2}\mathbf{i} + 5\mathbf{j} + \mathbf{k}) \wedge (\mathbf{3}\mathbf{i} - \mathbf{j}, \mathbf{0}) = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 3 & -1 & 0 \end{vmatrix} = \begin{bmatrix} 0 - (-1), 3 - 0, -2 - 15 \\ 0, 3 - 0, -2 - 15 \end{bmatrix} = \langle 0, 3, -17 \rangle \\
 \text{(b)} \quad & (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \wedge (\mathbf{3}\mathbf{i} - \mathbf{j}, \mathbf{-1}) = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 3 & -1 & -1 \end{vmatrix} = \begin{bmatrix} 2 - (-1), 3 - (-1), -1 - (-1) \\ 0, 3 - (-1), -1 - (-1) \end{bmatrix} = \langle -1, 4, -7 \rangle \\
 \text{(c)} \quad & (\mathbf{3}\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \wedge (\mathbf{1}\mathbf{i} + \mathbf{3}\mathbf{j}, \mathbf{1}) = \begin{vmatrix} 1 & 3 & 1 \\ 3 & -1 & -2 \\ 1 & 3 & 1 \end{vmatrix} = \begin{bmatrix} -1 - (-4), -2 - 3, 9 - (-2) \\ 0, -2 - 3, 9 - (-2) \end{bmatrix} = \langle 5, -5, 10 \rangle \\
 \text{(d)} \quad & (\mathbf{7}\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \wedge (-\mathbf{i} + \mathbf{3}\mathbf{j}, \mathbf{2}) = \begin{vmatrix} 1 & 3 & 1 \\ 7 & 1 & 4 \\ -1 & 3 & 2 \end{vmatrix} = \begin{bmatrix} 2 - 12, -4 - 14, 21 - (-12) \\ 0, -2 - 12, 21 - (-12) \end{bmatrix} = \langle -10, -18, 22 \rangle \\
 \text{(e)} \quad & (\mathbf{2}\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}) \wedge (\mathbf{3}\mathbf{i} - \mathbf{j}, \mathbf{-3}) = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ -3 & 1 & -3 \end{vmatrix} = \begin{bmatrix} -15 - (-9), 12 - (-24), 2 - (-15) \\ 0, -2 - 12, 2 - (-15) \end{bmatrix} = \langle 15, 18, 17 \rangle
 \end{aligned}$$

Question 2

Find a unit vector perpendicular to both

$$\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

$$\boxed{\frac{1}{\sqrt{330}}(-4\mathbf{i} + 5\mathbf{j} + 17\mathbf{k})}$$

$$\begin{aligned}
 \mathbf{a} \wedge \mathbf{b} &= \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 3 & -1 & -1 \end{vmatrix} = \begin{bmatrix} -5 - (-1), 3 - (-2), -2 - 15 \\ 0, 3 - 0, -2 - 15 \end{bmatrix} = \langle 0, 3, -17 \rangle \\
 \| -4\mathbf{i} + 5\mathbf{j} + 17\mathbf{k} \| &= \sqrt{16 + 25 + 289} = \sqrt{330} \\
 \therefore \text{Required vector is } \frac{1}{\sqrt{330}}(-4\mathbf{i} + 5\mathbf{j} + 17\mathbf{k})
 \end{aligned}$$

Question 3

The vectors \mathbf{a} and \mathbf{b} , are not parallel.

Simplify fully

$$(2\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} - 2\mathbf{b}).$$

$$5\mathbf{b} \wedge \mathbf{a} = -5\mathbf{a} \wedge \mathbf{b}$$

USING THE FACT THAT THE "CROSS PRODUCT" IS DISTRIBUTIVE OVER ADDITION & SUBTRACTION, WE OBTAIN

$$(2\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} - 2\mathbf{b}) = 2\mathbf{a} \wedge \mathbf{a} - 4\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} - 2\mathbf{b} \wedge \mathbf{b}$$

NEXT WE USE THE PROPERTIES

- $\mathbf{u} \wedge \mathbf{u} = \mathbf{0}$ FOR ALL \mathbf{u}
- $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$ FOR ALL $\mathbf{u} \neq \mathbf{v}$

$$\dots = \mathbf{0} + 4\mathbf{b} \wedge \mathbf{a} + \mathbf{b} \wedge \mathbf{a} - \mathbf{0}$$

$$= 5\mathbf{b} \wedge \mathbf{a}$$

[OR indeed $-5\mathbf{a} \wedge \mathbf{b}$]

Question 4

The vectors \mathbf{a} , \mathbf{b} and \mathbf{c} are not parallel.

Simplify fully

$$\mathbf{a} \cdot [\mathbf{b} \wedge (\mathbf{c} + \mathbf{a})].$$

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$$

APPLY THE CROSS-PRODUCT PROPERTY

$$\Rightarrow \mathbf{a} \cdot [\mathbf{b} \wedge (\mathbf{c} + \mathbf{a})] = \mathbf{a} \cdot [\mathbf{b} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{a}] \\ = \mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c} + \mathbf{a} \cdot \mathbf{b} \wedge \mathbf{a}$$

Now $\mathbf{b} \wedge \mathbf{a}$ is perpendicular to \mathbf{a} , so $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{a}) = 0$

$$\therefore \mathbf{a} \cdot [\mathbf{b} \wedge (\mathbf{c} + \mathbf{a})] = \mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c}$$

Question 5

The following vectors are given

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

$$\mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{c} = \mathbf{j} + 3\mathbf{k}$$

- a) Show that the three vectors are coplanar.
- b) Express \mathbf{a} in terms of \mathbf{b} and \mathbf{c} .

$$\boxed{\mathbf{a} = 2\mathbf{b} - \mathbf{c}}$$

| | |
|--|---|
| (a) $\begin{aligned}\mathbf{a} &= (2, 3, -1) \\ \mathbf{b} &= (1, 2, 1) \\ \mathbf{c} &= (0, 1, 3)\end{aligned}$ | $\text{if } \mathbf{a}, \mathbf{b}, \mathbf{c} \text{ are coplanar, } \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ $\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix} = -1 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -3 + 3 = 0$ |
| (b) $\begin{aligned}\mathbf{a} &= 2\mathbf{b} - \mathbf{c} \\ 2(1, 2, 1) + \mathbf{c} &= (2, 3, -1) \\ (2, 2+1, 2+3) &= (2, 3, -1) \\ (2, 3, 3) &= (2, 3, -1)\end{aligned}$ | $\left. \begin{array}{l} \therefore \mathbf{a} = 2\mathbf{b} - \mathbf{c} \\ 2\mathbf{i} + 4\mathbf{j} + \mathbf{k} = (2, 3, -1) \\ 2\mathbf{i} + 4\mathbf{j} + \mathbf{k} = (2, 3, -1) \\ 4\mathbf{j} = 3 \\ \mathbf{k} = -1 \end{array} \right\}$ $\therefore \mathbf{a} = 2\mathbf{b} - \mathbf{c}$ |

Question 6

The following three vectors are given

$$\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\mathbf{c} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$$

where λ is a scalar constant.

- If the three vectors given above are coplanar, find the value of λ .
- Express \mathbf{a} in terms of \mathbf{b} and \mathbf{c} .

$$\boxed{\lambda=1}, \boxed{\mathbf{a}=3\mathbf{c}-\mathbf{b}}$$

a) IF THE VECTORS ARE COPLANAR, THE CROSS PRODUCT OF ANY TWO WILL BE PERPENDICULAR TO THE THIRD

$$\Rightarrow (\mathbf{a}, \mathbf{b}) \cdot \mathbf{c} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 2 \\ 1 & 3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \left| \begin{matrix} 3 & 2 \\ 3 & 1 \end{matrix} \right| - 2 \left| \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \right| + 2 \left| \begin{matrix} 1 & 0 \\ 2 & 3 \end{matrix} \right| = 0$$

$$\Rightarrow (3-6) - 2(1-4) + 2(3-6) = 0$$

$$\Rightarrow -3 + 6 - 12 = 0$$

$$\Rightarrow 3 = 2\lambda$$

$$\therefore \lambda = 1$$

b) SETTING UP AN EQUATION

$$\mathbf{a} = p\mathbf{b} + q\mathbf{c}$$

$$\left(\begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \right) = p \left(\begin{matrix} 2 \\ 3 \\ 1 \end{matrix} \right) + q \left(\begin{matrix} 1 \\ 3 \\ 1 \end{matrix} \right)$$

EQUALISE SAY i, j, k (THE j SHOULD BALANCE)

$$\begin{cases} 2p+q=1 \\ p+q=2 \end{cases} \rightarrow p=-1 \quad \& \quad q=3$$

$$\therefore \mathbf{a} = 3\mathbf{c} - \mathbf{b}$$

Question 7

The vectors \mathbf{a} , \mathbf{b} and \mathbf{c} are such so that

$$\mathbf{c} \wedge \mathbf{a} = \mathbf{i} \quad \text{and} \quad \mathbf{b} \wedge \mathbf{c} = 2\mathbf{k}.$$

Express $(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} + \mathbf{b} + 2\mathbf{c})$ in terms of \mathbf{i} and \mathbf{k} .

$$-2\mathbf{i} + 4\mathbf{k}$$

$$\begin{aligned}(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} + \mathbf{b} + 2\mathbf{c}) &= (\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} + \mathbf{b}) + (\mathbf{a} + \mathbf{b}) \wedge 2\mathbf{c} \\&= 2\mathbf{a} \wedge \mathbf{c} + 2\mathbf{b} \wedge \mathbf{c} \\&= -2\mathbf{i} + 2(2\mathbf{k}) \\&= -2\mathbf{i} + 4\mathbf{k}\end{aligned}$$

CROSS PRODUCT GEOMETRIC APPLICATIONS

Question 1

Find the area of the triangle with vertices at $A(1, -1, 2)$, $B(-1, 2, 1)$ and $C(2, -3, 3)$.

$$\boxed{\frac{1}{2}\sqrt{3}}$$

$$\begin{aligned}
 \vec{AB} &= B-A = (-1-1, 2+1, 1-2) = (-2, 3, -1) \\
 \vec{AC} &= C-A = (2-1, -3+1, 3-2) = (1, -2, 1) \\
 \text{Area} &= \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{vmatrix} \right| = \frac{1}{2} \left| \begin{vmatrix} 1 & 0 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{vmatrix} \right| \\
 &= \frac{1}{2} \sqrt{r^2 + r^2} = \frac{1}{2}\sqrt{3}
 \end{aligned}$$

Question 2

Find the area of the triangle with vertices at $A(2, 1, 1)$, $B(-1, 0, 4)$ and $C(3, -1, -1)$.

$$\boxed{\frac{1}{2}\sqrt{122}}$$

$$\begin{aligned}
 \vec{B} &= (2, 1, 1) \\
 &\in (-1, 0, 4) \\
 &\in (3, -1, -1) \\
 \vec{AB} &= B-A = (-1-2, 0-1, 4-1) = (-3, -1, 3) \\
 \vec{AC} &= C-A = (3-2, -1-1, -1-1) = (1, -2, -2) \\
 \text{Area} &= \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -1 & 3 \\ 1 & -2 & -2 \end{vmatrix} \right| \\
 &= \frac{1}{2} |(0, -3, -7)| \\
 &= \frac{1}{2}\sqrt{0+9+49} \\
 &= \frac{1}{2}\sqrt{58}
 \end{aligned}$$

Question 3

A triangle has vertices at $A(-2, -2, 0)$, $B(6, 8, 6)$ and $C(-6, 8, 12)$.

Find the area of the triangle ABC .

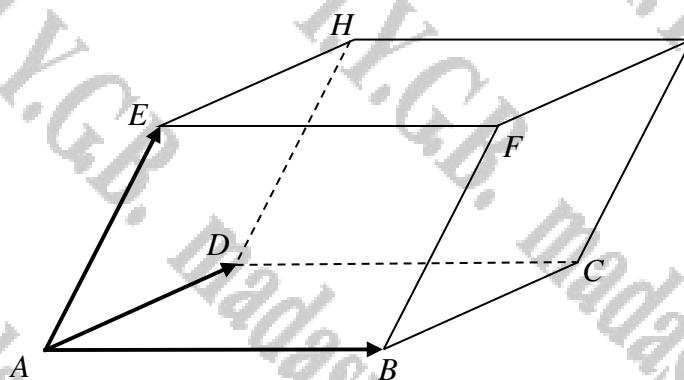
90

$\vec{a} = (-2, -2, 0)$
 $\vec{b} = (6, 8, 6)$
 $\vec{c} = (-6, 8, 12)$

$$\begin{aligned}\vec{AB} &= \vec{b} - \vec{a} = (6, 8, 6) - (-2, -2, 0) = (8, 10, 6) \\ \vec{AC} &= \vec{c} - \vec{a} = (-6, 8, 12) - (-2, -2, 0) = (-4, 10, 12) \\ \text{Area} &= \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} \left| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ 8 & 10 & 6 \\ -4 & 10 & 12 \end{array} \right| \\ &= \frac{1}{2} |(8, -40, -120)| \\ &= \frac{1}{2} \sqrt{3600 + 16000 + 14400} \\ &= 90\end{aligned}$$

Question 4

A parallelepiped has vertices at the points $A(2,1,t)$, $B(3,3,2)$, $D(4,0,5)$ and $E(1,-2,7)$, where t is a scalar constant.



- Calculate $\overrightarrow{AB} \wedge \overrightarrow{AD}$, in terms of t .
- Find the value of $\overrightarrow{AB} \wedge \overrightarrow{AD} \cdot \overrightarrow{AE}$

The volume of the parallelepiped is 22 cubic units.

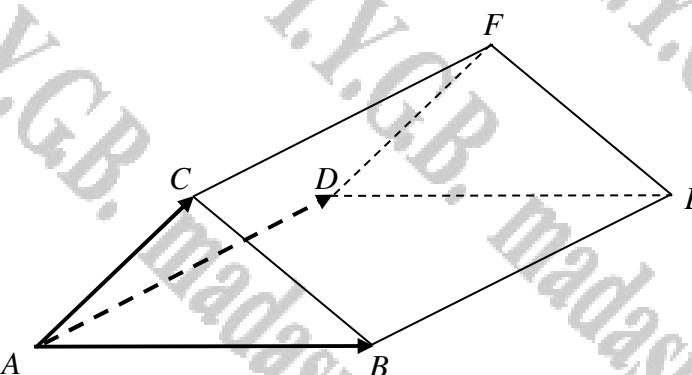
- Determine the possible values of t .

$$(12-3t)\mathbf{i} + (-t-1)\mathbf{j} - 5\mathbf{k}, \quad |11t-44|, \quad t = 2, 6$$

| |
|--|
| $\vec{AB} = \mathbf{i} - \mathbf{j} = (1, 0, 0) - (2, 1, t) = (-1, -1, t)$ $\vec{AD} = \mathbf{i} - \mathbf{k} = (1, 0, 0) - (2, 0, 5) = (-1, 0, -5)$ $\vec{AE} = \begin{bmatrix} 1 & -2 & 7 \\ 2 & 1 & t \end{bmatrix} = (-1, -3, 7-t) = (-1, -3, 7-t)$ |
| $\vec{AB} \wedge \vec{AD} = (-1, -1, t) \times (-1, 0, -5) = (-1, 5, -1)$ $\therefore \vec{AB} \cdot \vec{AD} = (-1, -1, t) \cdot (-1, 5, -1) = 1 + 5 - t = 6 - t$ $\vec{AB} \wedge \vec{AD} \cdot \vec{AE} = (-1, -1, t) \cdot (-1, 5, -1) \cdot (-1, -3, 7-t) = 6t - 12 + 5 + 3t - 35 + 5t = 14t - 42$ |
| $\text{Q) } V = 14t - 42 \quad 14t - 42 = 22 \quad 14t - 42 = 22 \Rightarrow 14t - 42 = 22 \quad 14t - 42 = 22 \Rightarrow 14t - 42 = -22 \Rightarrow t = 2, 6$ |

Question 5

A triangular prism has vertices at the points $A(3,3,3)$, $B(1,3,t)$, $C(5,1,5)$ and $F(8,0,10)$, where t is a scalar constant.



The face ABC is parallel to the face DEF and the lines AD , BE and CF are parallel to each other.

- Calculate $\overrightarrow{AB} \wedge \overrightarrow{AC}$, in terms of t .
- Find the value of $\overrightarrow{AB} \wedge \overrightarrow{AC} \cdot \overrightarrow{AD}$, in terms of t .

The value of t is taken to be 6.

- Determine the volume of the prism for this value of t .
- Explain the geometrical significance if $t = -1$.

$$(2t-6)\mathbf{i} + (2t-2)\mathbf{j} + 4\mathbf{k}, [4t+4], V=14 \text{ cubic units},$$

A, B, C, D are coplanar, so no volume

(1) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (1,3,t) - (3,3,3) = (-2,0,t-3)$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = (5,1,5) - (3,3,3) = (2,-2,2)$

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & t-3 \\ 2 & -2 & 2 \end{vmatrix} = (2t-6, 2t-6+4, 4) = (2t-6, 2t-2, 4)$

(2) $\overrightarrow{AD} = \overrightarrow{CE} = \frac{1}{2}\mathbf{c} - \mathbf{a} = (6,0,10) - (3,3,3) = (3,-1,7)$

$\therefore \overrightarrow{AD} \wedge \overrightarrow{AC} \wedge \overrightarrow{AB} = (2t-6, 2t-2, 4) \cdot (3, -1, 7)$

$= 6t-18 - 2t+2 + 20$

$= 4t+4$

~~✓~~

(3) Volume of prism = $\frac{1}{2} \times \text{parallelipiped} = \frac{1}{2} |\overrightarrow{AB} \wedge \overrightarrow{AC} \wedge \overrightarrow{AD}|$

$= \frac{1}{2} |4t+4| = \frac{1}{2} \times 28 = 14 \text{ units}^3$

~~✓~~

(4) If $t = -1$, prism has no volume, i.e. A, B, C, D are coplanar

Question 6

A tetrahedron has vertices at the points $A(-3,6,4)$, $B(0,11,0)$, $C(4,1,28)$ and $D(7,k,24)$, where k is a scalar constant.

- Calculate the area of the triangle ABC .
- Find the volume of the tetrahedron $ABCD$, in terms of k .

The volume of the tetrahedron is 150 cubic units.

- Determine the possible values of k .

$$\boxed{\text{area} = 75}, \quad \boxed{\text{volume} = \frac{50}{3}|k-6|}, \quad \boxed{|k-6| = 30}$$

$$\begin{aligned}
 \text{(a)} \quad & \vec{AB} \cdot \vec{AC} = [b-a] \cdot [c-a] = [(0,5,0) - (-3,6,4)] \cdot [(4,1,28) - (-3,6,4)] = (3,-1,-4) \cdot (7,-5,24) \\
 &= \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 3 & -1 & -4 \\ 7 & -5 & 24 \end{vmatrix} = (0,0,-100) = 100 \\
 \therefore \text{Area} &= \frac{1}{2} |\vec{AB} \cdot \vec{AC}| = \frac{1}{2} |100| = 50 \quad \boxed{\text{area} = 50}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \vec{AD} = d-a = (7,k,24) - (-3,6,4) = (10,k-6,20) \\
 & \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 3 & -1 & -4 \\ 7 & -5 & 24 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 0 \\ 10 & k-6 & 20 \end{vmatrix} = \frac{1}{3} \begin{vmatrix} 0 & 0 & 0 \\ 10 & k-6 & 20 \\ 100 & 10(k-6) & 200 \end{vmatrix} = 75
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \text{Volume} = \frac{1}{6} |\vec{AD} \cdot (\vec{AB} \times \vec{AC})| \\
 &= \frac{1}{6} |(10,k-6,20) \cdot (10(k-6), 100, 100)| \\
 &= \frac{1}{6} |1000 + 100k - 1200| = \frac{1}{6} |100k - 200| = \frac{50}{3} |k-4| \\
 &= 150 \quad \boxed{|k-6| = 30}
 \end{aligned}$$

Question 7

With respect to a fixed origin O the points A , B and C , have respective coordinates $(6,10,10)$, $(11,14,13)$ and $(k,8,6)$, where k is a constant.

- Given that all the three points lie on a plane which contains the origin, find the value of k .
- Given instead that OA , OB , OC are edges of a parallelepiped of volume 150 cubic units determine the possible values of k .

$$[k = 10], [k = -5, 25]$$

(a)

$$\begin{aligned} \vec{OA}, \vec{OB} &= \begin{bmatrix} 6 & 10 & 10 \\ 11 & 14 & 13 \end{bmatrix} = (6, 10, 10) \\ &\quad \text{Scale to } (1, 1, 1) \\ \therefore \text{Plane: } & 6x + 10y + 10z = 0 \quad \leftarrow \text{Gives Required Plane} \end{aligned}$$

(b)

$$\begin{aligned} \because SK - 10OB + 13OC = 0 \\ SK = 50 \\ k = 10 \end{aligned}$$

$$\begin{aligned} \Rightarrow |\vec{OA} \cdot \vec{OB} \cdot \vec{OC}| &= 150 \\ \Rightarrow |(6, 10, 10) \cdot (11, 14, 13)| &= 150 \\ \Rightarrow |-46 + 250 - 120| &= 150 \\ \Rightarrow |104| &= 150 \\ \Rightarrow |104 - 150| &= 15 \\ \Rightarrow |104| - 150 &= 15 \\ \Rightarrow 104 &= 165 \\ \Rightarrow k &= 16.5 \end{aligned}$$

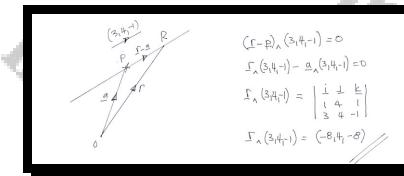
LINES

Question 1

Find an equation of the straight line that passes through the point $P(1,4,1)$ and is parallel to the vector $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

Give the answer in the form $\mathbf{r} \wedge \mathbf{a} = \mathbf{b}$ where \mathbf{a} and \mathbf{b} are constant vectors.

$$\boxed{\mathbf{r} \wedge (3\mathbf{i} + 4\mathbf{j} - \mathbf{k}) = -8\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}}$$

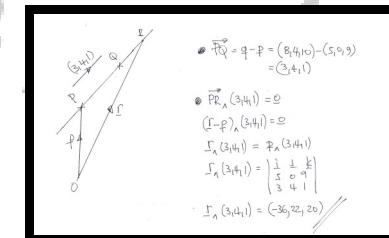


Question 2

Find an equation of the straight line that passes through the points $P(5,0,9)$ and $Q(8,4,10)$.

Give the answer in the form $\mathbf{r} \wedge \mathbf{a} = \mathbf{b}$ where \mathbf{a} and \mathbf{b} are constant vectors.

$$\boxed{\mathbf{r} \wedge (3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = -36\mathbf{i} + 22\mathbf{j} + 20\mathbf{k}}$$



Question 3

A straight line has equation

$$\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}),$$

where λ is a scalar constant.

Convert the above equation into Cartesian form.

$$\frac{x-4}{1} = \frac{y-2}{8} = \frac{z-5}{-3}$$

$$\begin{aligned} \mathbf{r} &= (4, 2, 5) + \lambda(1, 8, -3) \\ (x, y, z) &= (4 + \lambda, 2 + 8\lambda, 5 - 3\lambda) \\ \begin{cases} x = 4 + \lambda \\ y = 2 + 8\lambda \\ z = 5 - 3\lambda \end{cases} &\Rightarrow \begin{cases} \lambda = x - 4 \\ \lambda = \frac{y-2}{8} \\ \lambda = \frac{5-z}{-3} \end{cases} \quad \therefore \frac{x-4}{1} = \frac{y-2}{8} = \frac{z-5}{-3} \\ &\text{or} \\ &x - 4 = \frac{y-2}{8} = \frac{z-5}{-3} \end{aligned}$$

Question 4

A straight line has equation

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

where λ is a scalar parameter.

Convert the above equation into Cartesian form.

$$\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z}{2}$$

$$\begin{aligned} \mathbf{r} &= (2, -3, 0) + \lambda(1, -2, 2) \\ (x, y, z) &= (2 + \lambda, -3 - 2\lambda, 2\lambda) \\ \begin{cases} x = 2 + \lambda \\ y = -3 - 2\lambda \\ z = 2\lambda \end{cases} &\Rightarrow \begin{cases} \lambda = x - 2 \\ \lambda = \frac{-y-3}{2} \\ \lambda = \frac{z}{2} \end{cases} \quad \therefore \begin{cases} x - 2 = \frac{-y-3}{2} \\ x - 2 = \frac{z}{2} \end{cases} \\ &x - 2 = \frac{-y-3}{2} = \frac{z}{2} \end{aligned}$$

Question 5

Convert the equation of the straight line

$$\frac{x-3}{2} = \frac{y+2}{3} = \frac{5-z}{7}$$

into a vector equation of the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where \mathbf{a} and \mathbf{b} are constant vectors and λ is a scalar parameter.

$$\boxed{\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 7\mathbf{k})}$$

$$\begin{aligned} \frac{x-3}{2} &= \frac{y+2}{3} = \frac{5-z}{7} = \lambda \\ \frac{x-3}{2} = \lambda & \quad \left. \begin{array}{l} \frac{y+2}{3} = \lambda \\ \frac{5-z}{7} = \lambda \end{array} \right\} \quad \begin{array}{l} x = 2\lambda + 3 \\ y = 3\lambda - 2 \\ z = 5 - 7\lambda \end{array} \quad \therefore \Sigma = (3, 2, 5) + \lambda(2, 3, -7) \\ \frac{y+2}{3} = \lambda & \\ \frac{5-z}{7} = \lambda & \end{aligned}$$

Question 6

A straight line has equation

$$[\mathbf{r} - (5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})] \wedge (7\mathbf{i} + 5\mathbf{k}) = \mathbf{0}$$

Convert the above equation into the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where \mathbf{a} and \mathbf{b} are constant vectors and λ is a scalar parameter.

$$\boxed{\frac{x-5}{7} = \frac{z+3}{5}, \quad y=2}$$



- REWRITE LINE IN PARAMETRIC
 $\Sigma = (5, 2, 2) + \lambda(3, 0, 3)$
- WRITE IN CARTESIAN BY INSPECTION

$$\begin{aligned} 2 &= 5 + \lambda \quad \Rightarrow \quad \lambda = \frac{2-5}{3} \\ y &= 2 \quad \Rightarrow \quad \lambda = \frac{2-2}{3} \\ z &= 2 + 3\lambda \quad \Rightarrow \quad \lambda = \frac{z-2}{3} \end{aligned}$$

$$\therefore \frac{x-5}{3} = \frac{z-2}{3} \text{ AND } y=2$$

OR $5x-25 = 3z+6$ and $y=2$
 $5x = 3z+31 \quad \text{and} \quad y=2$

Question 7

A straight line has equation

$$\mathbf{r} \wedge (2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} - 5\mathbf{j} - 8\mathbf{k})$$

Convert the above equation into the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where \mathbf{a} and \mathbf{b} are constant vectors and λ is a scalar parameter.

$$\boxed{\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) \text{ or } \mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})}$$

$$\begin{aligned}
 \Gamma_A(2, -4, 3) &= (2, -5, -8) \\
 \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 3 \\ 3 & -5 & -8 \end{bmatrix} &= (2, -5, -8) \\
 (3)(4)(2), 2(-2) - 3(1) &= (2, -5, -8) \\
 \text{SWITCHING MATRIX NOTATION} \\
 \begin{bmatrix} 0 & 3 & 4 & 2 \\ -3 & 0 & 2 & -5 \\ 4 & -2 & 0 & -8 \end{bmatrix} R_1(3) \begin{bmatrix} 1 & \frac{1}{2} & 0 & 2 \\ -3 & 0 & 2 & -5 \\ 0 & 3 & 4 & -2 \end{bmatrix} R_2(4) \begin{bmatrix} 1 & \frac{1}{2} & 0 & 2 \\ 0 & \frac{1}{2} & 2 & -2 \\ 0 & 3 & 4 & 2 \end{bmatrix} \\
 R_2(8) \begin{bmatrix} 1 & \frac{1}{2} & 0 & 2 \\ 0 & \frac{1}{2} & \frac{1}{4} & 2 \\ 0 & 3 & 4 & 2 \end{bmatrix} R_3(8) \begin{bmatrix} 1 & \frac{1}{2} & 0 & 2 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 R_3(\frac{-1}{3}) \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{2}{3} \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 \text{Solving, we get} \\
 \begin{cases} x = \frac{5}{3}z \\ y = \frac{2}{3}z \\ z = 2z \end{cases} \Rightarrow \begin{cases} x = \frac{5}{3}z \\ y = \frac{2}{3}z \\ z = 0 + \frac{2}{3}z \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{3}z \\ \frac{2}{3}z \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \\ 1 \end{pmatrix} \\
 \Rightarrow \Gamma = \left(\frac{5}{3}, \frac{2}{3}, 0 \right) + \lambda \left(\frac{2}{3}, 0, 1 \right) \\
 \Gamma = \left(\frac{5}{3}, \frac{2}{3}, 0 \right) + (4+2) \left(\frac{2}{3}, 0, 1 \right) \\
 \Gamma = \left(\frac{5}{3}, \frac{2}{3}, 0 \right) + 1 \left(\frac{2}{3}, 0, 1 \right) + \left(\frac{2}{3}, 0, 2 \right) \\
 \Gamma = (3, 2, 2) + \lambda \left(\frac{2}{3}, 0, 1 \right) \\
 \Gamma = (3, 2, 2) + \lambda (2, 0, 1)
 \end{aligned}$$

Question 8

If the point $A(p, q, 1)$ lies on the straight line with vector equation

$$\mathbf{r} \wedge (2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = (8\mathbf{i} - 7\mathbf{j} - 3\mathbf{k}),$$

find the value of each of the scalar constants p and q .

$$\boxed{p = q = 3}$$

$$\begin{aligned}
 \Gamma_A(2, 1, 3) &= (8, -7, -3) \\
 \Rightarrow (p, q, 1) \wedge (2, 1, 3) &= (8, -7, -3) \\
 \Rightarrow \begin{vmatrix} 1 & p & q & 1 \\ 2 & 1 & 3 & 1 \end{vmatrix} &= (8, -7, -3) \\
 \Rightarrow (2q - 1, 2 - 3q, 1 - 2p) &= (8, -7, -3)
 \end{aligned}$$

This
 $2q - 1 = 8 \quad \frac{q}{2} = 4.5 \quad q = 9$
 $2 - 3q = -7 \quad -3q = -9 \quad q = 3$
 $1 - 2p = -3 \quad -2p = -4 \quad p = 2$
 EQUATIONS SOLVED
 FOR $p = q = 3$

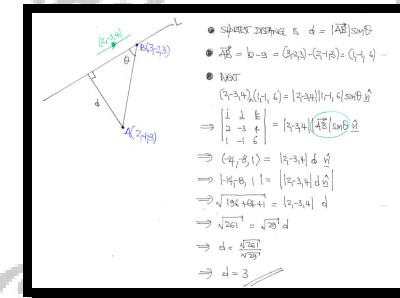
Question 9

The straight line L has equation

$$[\mathbf{r} - (3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})] \wedge (2\mathbf{i} - 3\mathbf{k} + 4\mathbf{k}) = 0.$$

Use a method involving the cross product to show that the shortest distance of the point $(2, -1, -3)$ from L is 3 units.

proof



PLANES

Question 1

Find a Cartesian equation of the plane that passes through the point $A(6, -2, 5)$, and its normal is in the direction $5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

$$5x - 2y + 3z = 49$$

Method A:

$$\vec{A}\vec{n} \cdot \vec{n} = 0$$

$$(5-2, 3) \cdot \vec{n} = 0$$

$$5x - 2y + 3z = 0$$

$$(x_0, y_0, z_0) = (6, -2, 5)$$

$$5x - 2y + 3z = 30 + 4 + 15$$

$$5x - 2y + 3z = 49$$

Method B:

Equation of plane $\vec{n} \cdot \vec{r} = \text{constant}$

$$5x - 2y + 3z = \text{constant}$$
Using $A(6, -2, 5)$

$$5(6) - 2(-2) + 3(5) = \text{constant}$$

$$30 + 4 + 15 = \text{constant}$$

$$\text{constant} = 49$$

$$\therefore 5x - 2y + 3z = 49$$

Question 2

Find a Cartesian equation of the plane that passes through the point $A(5, 1, 2)$, and its normal is in the direction $2\mathbf{i} - 7\mathbf{j} + \mathbf{k}$.

$$2x - 7y + z = 5$$

Method A:

$$\vec{A}\vec{n} \cdot \vec{n} = 0$$

$$(2-7, 1) \cdot \vec{n} = 0$$

$$2x - 7y + z = 0$$

$$(x_0, y_0, z_0) = (5, 1, 2)$$

$$(2(5), 1(1), 2(2)) = (10, 1, 4)$$

$$10 - 7 + 2 = 5$$

$$2x - 7y + z = 5$$

Method B:

Equation of plane $\vec{n} \cdot \vec{r} = \text{constant}$

$$2x - 7y + z = \text{constant}$$
Using $A(5, 1, 2)$

$$2(5) - 7(1) + 2(2) = \text{constant}$$

$$10 - 7 + 2 = \text{constant}$$

$$\text{constant} = 5$$

$$\therefore 2x - 7y + z = 5$$

Question 3

Find a Cartesian equation of the plane that passes through the points

$$A(5, 2, 2), \quad B(-1, 2, 1) \quad \text{and} \quad C(3, -2, -2).$$

$$2x + 11y - 12z = 8$$

Diagram of a 3D coordinate system with points A, B, and C labeled. Vectors \vec{AB} , \vec{AC} , and \vec{BC} are shown.

$\vec{AB} = b - a = (1, 2, 1) - (5, 2, 2) = (-4, 0, -1)$
 $\vec{AC} = c - a = (3, -2, -2) - (5, 2, 2) = (-2, -4, -4)$
 $\vec{n} = \begin{vmatrix} i & j & k \\ -4 & 0 & -1 \\ -2 & -4 & -4 \end{vmatrix} \leftarrow \text{SCALAR MULTIPLE}$
 $\vec{n} = (-2, -11, 8)$

$4\vec{n} = b - a = (1, 2, 1) - (5, 2, 2) = (-4, 0, -1)$
 $4\vec{n} = c - a = (3, -2, -2) - (5, 2, 2) = (-2, -4, -4)$
 $\vec{n} = \begin{vmatrix} i & j & k \\ -4 & 0 & -1 \\ -2 & -4 & -4 \end{vmatrix} \leftarrow \text{SCALAR MULTIPLE}$
 $\vec{n} = (-2, -11, 8)$

$4(-4)x - 2(-11)y + 2(-8)z = \text{constant}$
 $-16x + 44y - 32z = \text{constant}$
 $-16x + 44y - 32z = -8$
 $2x + 11y - 8z = 8$

Question 4

Determine a Cartesian equation of the plane that contains the point $A(9, -1, 0)$ and the straight line with vector equation

$$\mathbf{r} = 5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}),$$

where λ is a scalar parameter.

$$24x + 26y + 9z = 190$$

Diagram of a 3D coordinate system with point A(9, -1, 0) and a line passing through point C(5, 1, 2) with direction vector $\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

$\vec{AC} = c - a = (5, 1, 2) - (9, -1, 0) = (-4, 2, 2)$
 $\vec{AB} = b - a = (1, 2, 1) - (9, -1, 0) = (-8, 3, 1)$
 $\vec{n} = \begin{vmatrix} i & j & k \\ -4 & 2 & 2 \\ -8 & 3 & 1 \end{vmatrix}$
 $\vec{n} = (24, 26, 9)$

$24(x - 9) + 26(y + 1) + 9z = \text{constant}$
 $24x - 216 + 26y + 26 + 9z = \text{constant}$
 $24x + 26y + 9z = 190$
 $\therefore 24x + 26y + 9z = 190$

Question 5

Find a Cartesian equation of the plane that contains the parallel straight lines with vector equations

$$\mathbf{r}_1 = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = 3\mathbf{i} - \mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k}),$$

where λ and μ are scalar parameters.

$$z - x = 3$$

$\bullet \vec{AB} = (3, 2, 4) - (2, 1, 5) = (1, 1, 1)$
 $\bullet \vec{CD} = (2, 0, 4) - (3, 1, 5) = (-1, -1, 1)$
 $\text{Volume} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3$
 $\text{Some volume is } (1, 1, 1)$
 $\text{Hence } z - x = \text{constant}$
 $\text{using } (2, 1, 5) \Rightarrow 2 - 1 = \text{constant}$
 $\Rightarrow \text{constant} = 1$
 $\therefore z - x = 1$

Question 6

Find a Cartesian equation of the plane that contains the intersecting straight lines with vector equations

$$\mathbf{r}_1 = 6\mathbf{i} + 3\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = 6\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k}),$$

where λ and μ are scalar parameters.

$$x - 3y + z = 4$$

$\bullet \text{FIND EQUATION BY CHOOSE DIRECTION OF LINES}$
 $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (-3, 1)$
 $\bullet \text{EQUATION OF PLANE WILL BE}$
 $-x + 3y - z = \text{constant}$
 $\text{use } (6, 2, 4) \text{ to find constant?}$
 $-6 + 6 - 4 = -4$
 $\therefore -x + 3y - z = -4$
 $x - 3y + z = 4$

Question 7

- a) Find a set of parametric equations for the plane that passes through the points $A(2, 4, 1)$, $B(6, 0, -2)$ and $C(0, 1, 7)$.
- b) Eliminate the parameters to obtain a Cartesian equation of the plane.

$$(x, y, z) = (2 - 2\lambda + 4\mu, 4 - 3\lambda - 4\mu, 1 + 6\lambda - 3\mu), \quad [33x + 18y + 20z = 158]$$

(a)

Diagram showing three points A(2, 4, 1), B(6, 0, -2), and C(0, 1, 7) in a 3D coordinate system. Vectors \vec{AB} and \vec{AC} are drawn from the origin.

$\vec{AB} = b - a = (6, 0, -2) - (2, 4, 1) = (4, -4, -3)$
 $\vec{AC} = c - a = (0, 1, 7) - (2, 4, 1) = (-2, -3, 6)$

$\vec{AB} = \vec{a}_1 + \lambda \vec{a}_2 + \mu \vec{a}_3$
 $(3, 3, 3) = (2, 4, 1) + \lambda(-2, -3, 6) + \mu(4, -4, -3)$
 $(2, 1, 7) = (2, 4, 1) + \lambda(-2, -3, 6) + \mu(4, -4, -3)$

(cancel terms)

(b)

$\begin{cases} x = 2 - 2\lambda + 4\mu \\ y = 4 - 3\lambda - 4\mu \\ z = 1 + 6\lambda - 3\mu \end{cases}$ Since $(1 + 6\lambda - 3\mu)(2\lambda - 1) \Rightarrow 2\lambda = 4\mu - 2$
 $\frac{2\lambda = 4\mu - 2}{2 = 1 + 6\mu - 3}$

$\begin{cases} y = 4 - 3\mu - 2(1 + 6\mu - 3) \\ z = 1 + 6\mu + 6(1 + 6\mu - 3) \end{cases} \Rightarrow \begin{cases} y = 4 - 3\mu - 2 - 18\mu + 6 \\ z = -1 - 3\mu + 6 + 36\mu - 18 \end{cases} \Rightarrow$

$\begin{cases} y = 1 - 21\mu - 3x \\ z = 7 + 33\mu - 3x \end{cases} \Rightarrow \begin{cases} y = 1 - 21\mu - 3x \\ 7 + 33\mu - 3x \end{cases} \Rightarrow$

$\begin{cases} 9y - 9 - 63\mu + 27x \\ 102 + 70\mu - 30x \end{cases} \Rightarrow$

$9y + 102 = 79 - 33x$
 $18y + 204 = 158 - 66x$
 $33x + 18y + 202 = 158$

Created by T. Madas

GEOMETRIC PROBLEMS

(WITH PLANES AND LINES)

Created by T. Madas

Question 1

Find the coordinates of the point of intersection of the plane with equation

$$x + 2y + 3z = 4$$

and the straight line with equation

$$\mathbf{r} = -\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k}),$$

where λ is a scalar parameter.

$$(1, 3, -1)$$

$$\begin{aligned} x + 2y + 3z &= 4 \\ (1, 3, -1) + (-1, 2, 5) &= (0, 5, 4) \\ (0, 5, 4) &\Rightarrow (0-1) + 2(3+2) + 3(-1+5) = 4 \\ 2 + 10 + 12 &= 4 \\ 24 &= 4 \\ \text{No solution} \end{aligned}$$

Question 2

Find the coordinates of the point of intersection of the plane with equation

$$3x + 2y - 7z = 2$$

and the straight line with equation

$$\mathbf{r} = 9\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}),$$

where λ is a scalar parameter.

$$(5, -10, -1)$$

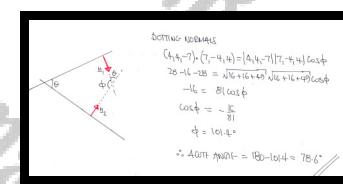
$$\begin{aligned} 3x + 2y - 7z &= 2 \\ \mathbf{r} &= (1, 2, 7) + \lambda(1, 3, 2) \\ (2, 4, 5) &= (1+8, 3+2, 2+7) \\ 3(1+8) + 2(3+2) - 7(2+7) &= 2 \\ 31 + 27 + 6 - 14 = 49 &= 2 \\ -20 &= 2 \\ z &= -4 \\ \text{Substitute into plane equation} \\ 3(1+8) + 2(3+2) - 7(2+7) &= 2 \\ 31 + 27 + 6 - 14 = 49 &= 2 \\ -20 &= 2 \\ z &= -4 \\ \text{From line equation} \\ x &= -4 + 1 \\ x &= -3 \\ y &= 3(-4) + 2 \\ y &= -12 + 2 \\ y &= -10 \\ z &= -4 \end{aligned}$$

Question 3

Find the size of the acute angle formed by the planes with Cartesian equations

$$4x + 4y - 7z = 13 \quad \text{and} \quad 7x - 4y + 4z = 6.$$

78.6°

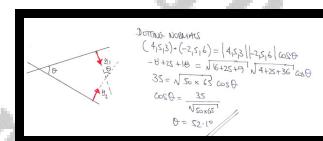


Question 4

Find the size of the acute angle between the planes with Cartesian equations

$$4x + 5y + 3z = 82 \quad \text{and} \quad -2x + 5y + 6z = 124.$$

52.1°



Question 5

Find the size of the acute angle between the plane with equation

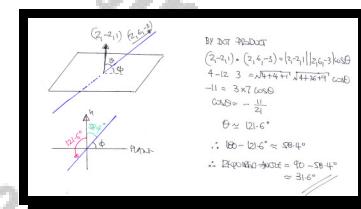
$$2x - 2y + z = 12$$

and the straight line with equation

$$\mathbf{r} = 7\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}),$$

where λ is a scalar parameter.

31.6°



Question 6

Find the size of the acute angle formed between the plane with Cartesian equation

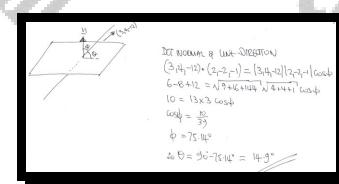
$$2x - 2y - z = 2$$

and the straight line with vector equation

$$\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}),$$

where λ is a scalar parameter.

14.9°



Question 7

Find the size of the acute angle between the plane with equation

$$3x - 2y + z = 5$$

and the straight line with equation

$$\mathbf{r} = -3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}),$$

where λ is a scalar parameter.

52.6°

Dotting directly a normal
 $(2, -3, 1) \cdot (3, 2, 1) = |(2, -3, 1)| |(3, 2, 1)| \cos \theta$
 $6 + 6 + 1 = \sqrt{14} \sqrt{14} \cos \theta$
 $13 = \sqrt{14} \sqrt{14} \cos \theta$
 $\cos \theta = \frac{13}{14\sqrt{2}}$
 $\frac{1}{\sqrt{2}} = \cos 45^\circ$
 $\theta = 45^\circ - 37.4^\circ = 52.6^\circ$

Question 8

Find shortest distance of the origin O from the plane with equation

$$4x + 3y - 5z = 20.$$

2 $\sqrt{2}$

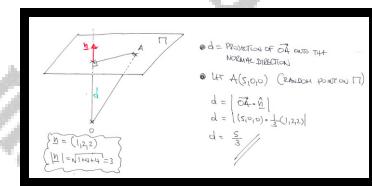
• By inspection $A(5, 0, 0)$ is on the plane
 $d = \text{projection of } \overrightarrow{OA} \text{ onto the normal direction } \mathbf{n}$
 $\mathbf{n} = [4, 3, -5]$
 $d = \left| \frac{\overrightarrow{OA} \cdot \mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{25}{\sqrt{50}} \right| = \frac{25}{5\sqrt{2}} = \frac{5}{\sqrt{2}} = 2\sqrt{2}$

Question 9

Find shortest distance of the origin O from the plane with equation

$$x + 2y + 2z = 5.$$

$\boxed{\frac{5}{3}}$

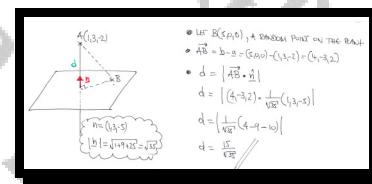


Question 10

Find shortest distance from the point $A(1, 3, -2)$ to the plane with Cartesian equation

$$x + 3y - 5z = 5.$$

$\boxed{\frac{15}{\sqrt{35}}}$

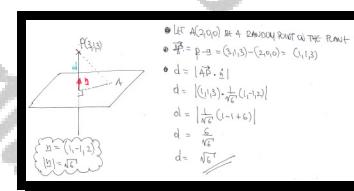


Question 11

Find shortest distance from the point $P(3,1,3)$ to the plane with Cartesian equation

$$x - y + 2z = 2.$$

$\boxed{\sqrt{6}}$

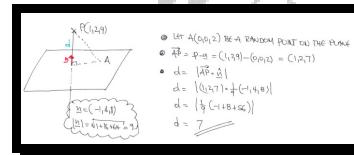


Question 12

Find shortest distance of the point $P(1,2,9)$ from the plane with Cartesian equation

$$-x + 4y + 8z = 16.$$

$\boxed{7}$



Question 13

Find the distance between the parallel planes with Cartesian equations

$$2x + 6y + 3z = 70 \quad \text{and} \quad 2x + 6y + 3z = 14.$$

[8]

$\Pi_1: 2x + 6y + 3z = 70$
 $\Pi_2: 2x + 6y + 3z = 14$

$P_1: A(0,0,0) \text{ on } \Pi_1$
 $P_2: B(4,2,0) \text{ on } \Pi_2$

$\bullet \vec{AB} = \frac{1}{\sqrt{1+36+9}}(4,2,0)$
 $\vec{AB} = \frac{1}{\sqrt{46}}(4,2,0)$

$\bullet \frac{\vec{AB}}{|\vec{AB}|} = \frac{1}{\sqrt{46}}(4,2,0)$
 $\frac{d}{\sqrt{46}} = \frac{1}{\sqrt{46}}(4,2,0)$

$\therefore d = \left| \frac{\vec{AB}}{|\vec{AB}|} \cdot \frac{1}{\sqrt{46}} \right| = \left| (4,2,0) \cdot \frac{1}{\sqrt{46}}(4,2,0) \right| = \frac{1}{\sqrt{46}} |48| = \frac{1}{\sqrt{46}} \cdot 4\sqrt{6} = \frac{4\sqrt{6}}{\sqrt{46}} = \frac{4\sqrt{6}}{\sqrt{23}}$

Question 14

The straight line with vector equation

$$\mathbf{r} = (\lambda + 5)\mathbf{i} + (2 - \lambda)\mathbf{j} + (\lambda + 2)\mathbf{k},$$

where λ is a scalar parameter, is parallel to the plane with Cartesian equation

$$x + 2y + z = 10.$$

Find the distance between the plane and the straight line.

$\frac{\sqrt{6}}{6}$

$\Pi: x + 2y + z = 10$
 $A(5,2,2), B(6,3,1)$

\bullet PICK POINTS ON LINE AND PLANE
 $A(5,2,2), a(6,3,1)$

$\bullet \vec{AB} = b - a = (6,3,1) - (5,2,2) = (1,1,-1)$

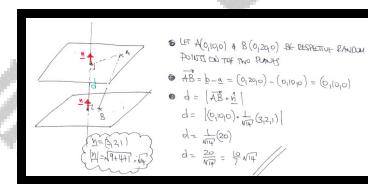
$\bullet d = \left| \vec{AB} \cdot \frac{1}{\sqrt{1+36+9}}(1,1,-1) \right|$
 $d = \left| (6,3,1) \cdot \frac{1}{\sqrt{46}}(1,1,-1) \right|$
 $d = \frac{1}{\sqrt{46}} |6 - 3 - 1| = \frac{2}{\sqrt{46}}$
 $d = \frac{\sqrt{46}}{6}$

Question 15

Find the distance between the parallel planes with Cartesian equations

$$3x + 2y + z = 20 \quad \text{and} \quad 3x + 2y + z = 40.$$

$$\frac{10}{7}\sqrt{14}$$



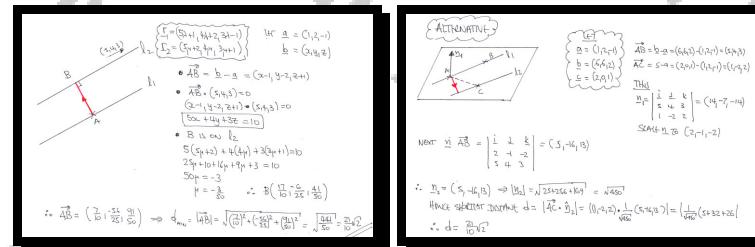
Question 16

Find the distance between the parallel straight lines with vector equations

$$\mathbf{r}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = 2\mathbf{i} + \mathbf{k} + \mu(5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}),$$

where λ and μ are scalar parameters.

$$\frac{21}{10}\sqrt{2}$$

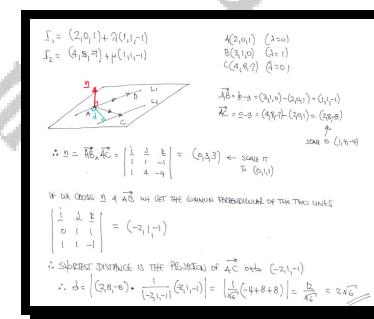


Question 17

Find the distance between the parallel straight lines with vector equations

$$[\mathbf{r} - (2\mathbf{i} + \mathbf{k})] \wedge (\mathbf{i} + \mathbf{j} - \mathbf{k}) = \mathbf{0} \quad \text{and} \quad x - 4 = y - 8 = -z - 7.$$

2 $\sqrt{6}$



$L_1: (2,0,1) + \lambda(1,1,-1)$
 $L_2: (4,0,2) + \mu(1,1,-1)$
 $A(2,0,1) \quad (\lambda=0)$
 $B(3,1,0) \quad (\lambda=1)$
 $C(4,0,2) \quad (\mu=0)$
 $D(5,1,1) \quad (\mu=1)$
 $\vec{AB} = \vec{B}-\vec{A} = (3,1,0)-(2,0,1) = (1,1,-1)$
 $\vec{AC} = \vec{C}-\vec{A} = (4,0,2)-(2,0,1) = (2,0,1)$
 $n = \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} = (2,3,2) \leftarrow \text{SCALE BY } \frac{1}{\sqrt{6}}$
 $\text{IF WE CROSS BY } 4 \sqrt{6} \text{ WE GET THE COMMON PERPENDICULAR OF THE TWO LINES}$
 $\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = (-2,1,-1)$
 $\therefore \text{SHORTEST DISTANCE IS THE PROJECTION OF } \vec{AC} \text{ ONTO } (-2,1,-1)$
 $\therefore d = |(2,0,2) \cdot \frac{1}{\sqrt{6}}(2,1,-1)| = \left| \frac{1}{\sqrt{6}}(-4+0+2) \right| = \frac{2}{\sqrt{6}} = 2\sqrt{\frac{1}{3}}$

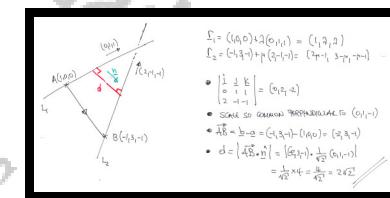
Question 18

Find the shortest distance between the skew straight lines with vector equations

$$\mathbf{r}_1 = \mathbf{i} + \lambda(\mathbf{j} + \mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = -\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - \mathbf{k}),$$

where λ and μ are scalar parameters.

2 $\sqrt{2}$



$L_1: (1,0,0) + \lambda(0,1,1) = (1, \lambda, \lambda)$
 $L_2: (2,1,1) + \mu(2,1,-1) = (2+\mu, 1+\mu, 1-\mu)$
 $A(1,0,0) \quad (\lambda=0)$
 $B(2,1,1) \quad (\mu=1)$
 $\vec{AB} = \vec{B}-\vec{A} = (2,1,1)-(1,0,0) = (1,1,1)$
 $\vec{AC} = \vec{C}-\vec{A} = (2,1,-1)-(1,0,0) = (1,1,-1)$
 $n = \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = (2,2,2) \leftarrow \text{SCALE SO COMMON PERPENDICULAR IS } (1,1,-1)$
 $\therefore \vec{AB} = 1\vec{n} = (1,1,1)$
 $\therefore d = \left| \frac{1}{\sqrt{3}} \vec{AB} \cdot \frac{1}{\sqrt{3}} \vec{AC} \right| = \left| \frac{1}{3} (1,1,1) \cdot (1,1,-1) \right| = \frac{1}{3} \sqrt{3} = \frac{1}{\sqrt{3}}$

Question 19

Find the shortest distance between the skew straight lines with vector equations

$$\mathbf{r}_1 = 7\mathbf{i} + \lambda(7\mathbf{i} - 10\mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \mu(\mathbf{i} + 3\mathbf{j} - \mathbf{k}),$$

where λ and μ are scalar parameters.

$$\boxed{\frac{6}{5}\sqrt{6}}$$

LOOKING AT THE DISSON BEGUN

- FIND A COMMON PERPENDICULAR.

$$\begin{vmatrix} 1 & 1 & 1 \\ 7 & 3 & -10 \\ 1 & 3 & 1 \end{vmatrix} = (20, -3, 2)$$
- SCALE THE PERPENDICULAR ALONG TO
 $\hat{n} = (0, -1, 1)$ SO THAT $|n| = \sqrt{100+1+49} = \sqrt{150}$
- $\vec{AB} = b - a = (3, 3, 1) - (7, 0, 0) = (-4, 3, 1)$

PROJECTING AB onto the direction of n will yield the shortest DISTANCE, d

$$d = |\vec{AB} \cdot \hat{n}| = |(-4, 3, 1) \cdot \frac{1}{\sqrt{150}}(0, -1, 1)| = \frac{1}{\sqrt{150}} |-40 - 3 + 1|$$

$$= \frac{46}{\sqrt{150}} = \frac{46}{5}\sqrt{\frac{6}{15}}$$

Question 20

Find the shortest distance between the skew straight lines with vector equations

$$\mathbf{r}_1 = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(\mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{k}),$$

where λ and μ are scalar parameters.

$$\boxed{\frac{5}{\sqrt{14}}}$$

FIND COMMON PERPENDICULAR.

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = (2, 3, -1) \leftrightarrow (2, 3, -1) = \frac{(2, 3, -1)}{\sqrt{4+9+1} = \sqrt{14}}$$

- $\vec{AB} = b - a = (1, 2, 3) - (2, -1, 1) = (-1, 3, 2)$
- $d = |\vec{AB} \cdot \hat{n}| = |(-1, 3, 2) \cdot \frac{1}{\sqrt{14}}(2, 3, -1)|$
 $= \frac{1}{\sqrt{14}} (-2 + 9 - 2)$
 $= \frac{5}{\sqrt{14}}$

Question 21

Find the intersection of the planes with Cartesian equations

$$2x - 2y - z = 2 \quad \text{and} \quad x - 3y + z = 5,$$

giving the answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$,

where \mathbf{a} and \mathbf{b} are constant vectors and λ is a scalar parameter.

$$\boxed{\mathbf{r} = -\mathbf{i} - 2\mathbf{j} + \lambda(5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})}$$

$$\begin{aligned} & \left(\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 2 & -2 & -1 & 2 \\ 0 & 1 & -\frac{3}{2} & -2 \end{array} \right) \xrightarrow{\text{R}_2 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 1 & -\frac{3}{2} & -2 \\ 2 & -2 & -1 & 2 \end{array} \right) \xrightarrow{\text{R}_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 1 & -\frac{3}{2} & -2 \\ 0 & 0 & \frac{5}{2} & 2 \end{array} \right) \xrightarrow{\text{R}_3 \cdot \frac{2}{5}} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 1 & -\frac{3}{2} & -2 \\ 0 & 0 & 1 & \frac{4}{5} \end{array} \right) \\ & \left(\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 1 & -\frac{3}{2} & -2 \\ 0 & 0 & 1 & \frac{4}{5} \end{array} \right) \xrightarrow{\text{R}_1 - R_3} \left(\begin{array}{ccc|c} 1 & -3 & 1 & \frac{21}{5} \\ 0 & 1 & -\frac{3}{2} & -2 \\ 0 & 0 & 1 & \frac{4}{5} \end{array} \right) \xrightarrow{\text{R}_1 \cdot (-1)} \left(\begin{array}{ccc|c} -1 & 3 & -1 & -\frac{21}{5} \\ 0 & 1 & -\frac{3}{2} & -2 \\ 0 & 0 & 1 & \frac{4}{5} \end{array} \right) \\ & \therefore \Gamma = (-1, 0, 0) + \lambda(-\frac{3}{2}, \frac{1}{2}, \frac{4}{5}) \\ & \Gamma = C(-1, 0, 0) + \lambda(5, 3, 4) // \end{aligned}$$

ALTERNATIVE

DIRECTION OF LINE OF INTERSECTION:

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -2 & -1 \\ 0 & 1 & -3 \end{vmatrix} = (-1, -3, -4)$$

A POINT ON THE LINE, SAY $Z=0$:

$$\begin{cases} 2x - 2y - 2 = 2 \\ 2x - 2y - 5 = 0 \end{cases} \Rightarrow \frac{2x - 2y - 2 = 2}{2x - 2y - 5 = 0} \Rightarrow \frac{3 = 3}{y = -2}$$

$$\therefore \frac{2x - 2(-2) = 2}{x = 3} \Rightarrow \frac{2x = 6}{x = 3}$$

$$\therefore \Gamma = (-1, 0, 0) + \lambda(3, -2, 4)$$

Question 22

Show that the planes with Cartesian equations

$$4x + 5y + 3z = 82 \quad \text{and} \quad -2x + 5y + 6z = 124$$

intersect along the straight line with equation

$$\mathbf{r} = (\lambda - 6)\mathbf{i} + (20 - 2\lambda)\mathbf{j} + (2\lambda + 2)\mathbf{k},$$

where λ is a scalar parameter.

proof

$$\begin{aligned} & \bullet \text{PLANES INTERSECT AT DIRECTION: } \begin{vmatrix} 1 & 1 & 4 \\ 4 & 5 & 3 \\ 2 & -2 & 6 \end{vmatrix} = (15, -30, 30) \\ & \bullet \text{SOKE THE DIRECTION TO: } (1, -2, 2) \\ & \bullet \text{FOR } \begin{cases} 4x + 5y + 3z = 82 \\ -2x + 5y + 6z = 124 \end{cases} \text{ LET } \begin{cases} z=2 \\ y=-2 \end{cases} \Rightarrow \begin{cases} 4x + 5(-2) = 76 \\ -2x + 5(-2) = 112 \end{cases} \\ & \quad \text{SIMPLIFY } \begin{cases} 4x - 10 = 76 \\ -2x - 10 = 112 \end{cases} \Rightarrow \begin{cases} 4x = 86 \\ -2x = 122 \end{cases} \\ & \quad \text{HENCE } \begin{cases} 4x = 86 \\ -2x = 122 \end{cases} \Rightarrow \begin{cases} x = 21.5 \\ x = -61 \end{cases} \Rightarrow \begin{cases} x = 21.5 \\ z = 2 \end{cases} \\ & \bullet \text{INTERSECTION L.I. } \Gamma = (-6, 21.5, 2) + \lambda(1, -2, 2) \\ & \Gamma = (2 - 6, 21.5 - 21, 2 + 2\lambda) \end{aligned}$$

Question 23

The planes Π_1 and Π_2 have Cartesian equations:

$$\Pi_1 : x - 2y + 2z = 0$$

$$\Pi_2 : 3x - 2y - z = 5$$

Show that the two planes intersect along the straight line with Cartesian equation

$$\frac{x-4}{6} = \frac{y-3}{7} = \frac{z-1}{4}$$

proof

| |
|---|
| <ul style="list-style-type: none"> • PLANES INTERSECT ACROSS DIRECTION $\begin{vmatrix} 1 & 2 & 2 \\ 1 & -2 & -2 \\ 3 & -2 & -1 \end{vmatrix} = (6)(A)$ |
| <ul style="list-style-type: none"> • $2-2y+2z=0$ Let $\boxed{z=1}$ $\frac{2-2y+2z=0}{2-2y-1=5}$ $\left\{ \begin{array}{l} 4x=6-2y \\ 4z=2z+5 \end{array} \right.$ |
| <ul style="list-style-type: none"> • $3x-2y-2=5$ $\cancel{3x-2y-1=5}$ $\left\{ \begin{array}{l} 4x=6-2y \\ 4z=2z+5 \end{array} \right.$ |
| <ul style="list-style-type: none"> • $2x+y=5$ $y=5-2x$ $\left\{ \begin{array}{l} 4x=6-2y \\ 4z=2z+5 \end{array} \right.$ |
| <ul style="list-style-type: none"> • $2x+y=5$ $y=5-2x$ $\left\{ \begin{array}{l} 4x=6-2y \\ 4z=2z+5 \end{array} \right.$ |
| <ul style="list-style-type: none"> • $\Sigma = (1, 3) + \lambda(6, 2, 4)$ $(2+4\lambda, 7+2\lambda, 4\lambda+1)$ |
| <ul style="list-style-type: none"> • $\Sigma = (1, 3) + \lambda(6, 2, 4)$ $(2+4\lambda, 7+2\lambda, 4\lambda+1)$ |