

RESIDUES and INTEGRATION

CALCULATIONS OF RESIDUES

Question 1

$$f(z) \equiv \frac{\sin z}{z^2}, z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$.

, $\text{res}(z=0)=1$

• $f(z)$ has a simple pole at $z=0$, which is very easy to find from its Laurent expansion

$$\begin{aligned} f(z) &= \frac{\sin z}{z^2} = \frac{1}{z^2} \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7) \right] \\ &= \frac{1}{z} - \frac{1}{6}z^2 + \frac{1}{120}z^4 + O(z^6) \\ \therefore \text{residue is } 1 \end{aligned}$$

• Alternative is to use the standard method for a simple pole at $z=0$

$$\begin{aligned} \lim_{z \rightarrow 0} \left[\frac{d}{dz} [z^{-2} f(z)] \right] &= \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left[z^{-2} \frac{\sin z}{z^2} \right] \right] \\ &= \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left(\frac{\sin z}{z^2} \right) \right] \\ \text{by L'Hopital rule} \rightarrow &= \lim_{z \rightarrow 0} \left(\frac{\cos z}{2z} \right) \\ &= 1 \quad \text{as required} \end{aligned}$$

Question 2

$$f(z) \equiv e^z z^{-5}, z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$.

, $\text{res}(z=0)=\frac{1}{24}$

• $f(z)$ has a simple pole of order 5 at the origin, which is easy to find directly from its Laurent expansion

$$\begin{aligned} f(z) &= e^z z^{-5} = \frac{e^z}{z^5} = \frac{1}{z^5} \left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots \right] \\ &= \frac{1}{z^5} + \frac{1}{2!z^4} + \frac{1}{3!z^3} + \frac{1}{4!z^2} + \frac{1}{5!z} + \dots \\ \therefore \text{Required residue is } \frac{1}{24} \end{aligned}$$

• Alternative is to use the standard formula for finding a pole of order n at $z=0$

$$\begin{aligned} \text{Res}(z=0) &= \frac{1}{(n-1)!} \lim_{z \rightarrow 0} \left[\frac{d^{n-1}}{dz^{n-1}} (z^{-n} f(z)) \right] \\ \text{Res}(z=0) &= \frac{1}{4!} \lim_{z \rightarrow 0} \left[\frac{d^4}{dz^4} \left(z^{-5} \times e^z \right) \right] \\ \therefore \text{Res}(z=0) &= \frac{1}{24} \lim_{z \rightarrow 0} \left[\frac{d^4}{dz^4} (e^z) \right] \\ \text{Res}(z=0) &= \frac{1}{24} \lim_{z \rightarrow 0} [e^z] \\ \text{Res}(z=0) &= \frac{1}{24} \quad \text{as required} \end{aligned}$$

Question 3

$$f(z) \equiv \frac{z^2 + 2z + 1}{z^2 - 2z + 1}, z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$.

, $\text{res}(z=1) = 4$

FACTORISING THE FUNCTION

$$f(z) = \frac{z^2 + 2z + 1}{z^2 - 2z + 1} = \frac{(z+1)^2}{(z-1)^2}$$

$f(z)$ HAS A DOUBLE POLE AT $z=1$

$$\begin{aligned} \lim_{z \rightarrow 1} \left[\frac{d}{dz} \left[(z-1)^2 f(z) \right] \right] &= \lim_{z \rightarrow 1} \left[\frac{d}{dz} \left[(z-1)^2 \frac{(z+1)^2}{(z-1)^2} \right] \right] \\ &= \lim_{z \rightarrow 1} \left[\frac{d}{dz} (z+1)^2 \right] \\ &= \lim_{z \rightarrow 1} [2(z+1)] \\ &= 4 \end{aligned}$$

Question 4

$$f(z) \equiv \frac{2z+1}{z^2 - z - 2}, z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

, $\text{res}(z=2) = \frac{5}{3}$, $\text{res}(z=-1) = \frac{1}{3}$

START BY FACTORIZING THE DENOMINATOR

$$f(z) = \frac{2z+1}{z^2 - z - 2} = \frac{2z+1}{(z+1)(z-2)}$$

$f(z)$ HAS SIMPLE POLES AT $z=-1$ & AT $z=2$

- $\text{Res}(f, -1) = \lim_{z \rightarrow -1} [(z+1) f(z)] = \lim_{z \rightarrow -1} \left[(z+1) \frac{2z+1}{(z+1)(z-2)} \right]$

$$= \frac{2(-1)+1}{-1+2} = \frac{-1}{1} = -1$$

- $\text{Res}(f, 2) = \lim_{z \rightarrow 2} [(z-2) f(z)] = \lim_{z \rightarrow 2} \left[(z-2) \frac{2z+1}{(z+1)(z-2)} \right]$

$$= \frac{2(2)+1}{2+1} = \frac{5}{3}$$

Question 5

$$f(z) \equiv \frac{z}{2z^2 - 5z + 2}, z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$$\boxed{\quad}, \quad \boxed{\text{res}\left(z = \frac{1}{2}\right) = -\frac{1}{6}}, \quad \boxed{\text{res}(z = 2) = \frac{2}{3}}$$

SIMPLIFY FACTORISING THE DENOMINATOR

$$\begin{aligned} f(z) &= \frac{z}{2z^2 - 5z + 2} = \frac{z}{(2z-1)(z-2)} \\ f(z) \text{ HAS SIMPLE POLES AT } z = \frac{1}{2} \text{ & } z = 2. \\ \bullet \text{Res}\left(\frac{1}{2}; \frac{1}{2}\right) &= \lim_{z \rightarrow \frac{1}{2}} \left[(z - \frac{1}{2})f(z) \right] = \lim_{z \rightarrow \frac{1}{2}} \left[(z - \frac{1}{2}) \cdot \frac{z}{(2z-1)(z-2)} \right] \\ &= \lim_{z \rightarrow \frac{1}{2}} \left[(z - \frac{1}{2}) \times \frac{z}{2(z-1)(z-2)} \right] = \frac{\frac{1}{2} - \frac{1}{2}}{2(-\frac{1}{2})} = \frac{\frac{1}{2}}{-1} \\ &= -\frac{1}{2} \\ \bullet \text{Res}\left(\frac{1}{2}; 2\right) &= \lim_{z \rightarrow 2} \left[(z - 2)f(z) \right] = \lim_{z \rightarrow 2} \left[(z - 2) \cdot \frac{z}{(2z-1)(z-2)} \right] \\ &= \frac{2 - 2}{(2 \cdot 2 - 1)(2 - 2)} = \frac{0}{3} = 0 \end{aligned}$$

Question 6

$$f(z) \equiv \frac{1-e^{iz}}{z^3}, z \in \mathbb{C}.$$

- a) Find the first four terms in the Laurent expansion of $f(z)$ and hence state the residue of the pole of $f(z)$.
- b) Determine the residue of the pole of $f(z)$ by an alternative method

$$\boxed{\text{Method 1}}, \quad \boxed{\text{res}(z=0) = \frac{1}{2}}$$

a) FINDING THE RESIDUE FROM THE LAURENT EXPANSION

$$\begin{aligned} f(z) &= \frac{1-e^{iz}}{z^3} = \frac{1}{z^3} \left[1 - \left[1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \dots \right] \right] \\ &= \frac{1}{z^3} \left[-iz - \frac{1}{2}i^2z^2 - \frac{1}{6}i^3z^3 - \frac{1}{24}i^4z^4 + \dots \right] \\ &= \frac{1}{z^3} \left[-iz + \frac{1}{2}z^2 + \frac{1}{6}iz^3 - \frac{1}{24}z^4 + \dots \right] \\ &= -\frac{1}{2z^2} + \frac{1}{6}z + \frac{1}{6}iz^2 - \frac{1}{24}z^3 + \dots \end{aligned}$$

∴ RESIDUE OF THE DOUBLE POLE AT ZERO IS $\frac{1}{6}$

b) NOT USING THE FORMULA OR THE RESIDUE OF POLE OF ORDER n

$$\text{res}(f; c) = \frac{1}{(n-1)!} \lim_{z \rightarrow c} \left[\frac{d^{n-1}}{dz^{n-1}} [(z-c)^n f(z)] \right]$$

$$\begin{aligned} \text{res}(f; 0) &= \frac{1}{1!} \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left[z^2 \times \frac{1-e^{iz}}{z^3} \right] \right] \quad \text{NOTE THAT IT IS A DOUBLE POLE NOT TRIPLE} \\ &= \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left[\frac{1-e^{iz}}{z} \right] \right] \\ &= \lim_{z \rightarrow 0} \left[\frac{i(-e^{iz}) - (1-e^{iz})}{z^2} \right] = \lim_{z \rightarrow 0} \left[\frac{-ie^{iz} + e^{iz} - 1}{z^2} \right] \\ \text{THIS IS "BECOME ZERO", BY L'HOSPITAL RULE WE OBTAIN} \\ &= \lim_{z \rightarrow 0} \left[\frac{ie^{iz} + ie^{iz} - 0}{2z} \right] = \lim_{z \rightarrow 0} \left[\frac{2ie^{iz}}{2z} \right] \\ &= \lim_{z \rightarrow 0} \left[\frac{ie^{iz}}{z} \right] = \frac{1}{6} \quad \text{AS BEFORE} \end{aligned}$$

Question 7

$$f(z) = \frac{z^2 + 4}{z^3 + 2z^2 + 2z}, z \in \mathbb{C}$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{\text{res}(z=0)=2}, \boxed{\text{res}(z=-1+i)=\frac{1}{2}(-1+3i)}, \boxed{\text{res}(z=-1-i)=-\frac{1}{2}(1+3i)}$$

$\text{res}(z) = \frac{z^2 + 4}{z^3 + 2z^2 + 2z} = \frac{(z^2 + 4)(z - (-1-i))}{z(z^2 + 2z + 2)} = \frac{(z^2 + 4)(z - (-1-i))}{z^2(z + 2 + 2/z)}$

$\therefore \text{SIMPLY POLES AT } z=0, z=-i, z=-1-i$

- $\lim_{z \rightarrow 0} [z^{-1} f(z)] = \lim_{z \rightarrow 0} \left[\frac{z^2 + 4}{z^2(z + 2 + 2/z)} \right] = \frac{4}{2} = 2$
- $\lim_{z \rightarrow -i} \left[\frac{(z+1-i)}{z} \cdot \frac{z^2 + 4}{(z+1-i)(z+2+i)} \right] = \lim_{z \rightarrow -i} \left[\frac{z^2 + 4}{z(z+2+i)} \right]$
 $= \frac{(z+1-i)^2 + 4}{(z+1-i)(z+1+i)} = \frac{-1-2i-1+4}{2i(z+1-i)} = \frac{-4-2i}{-2-2i} = \frac{2+i}{1-i}$
 $= \frac{(2+i)(1+i)}{(1-i)(1+i)} = \frac{-2-2i+i+1}{2} = \frac{-1-3i}{2} = \frac{1}{2}(-1-3i)$
- $\lim_{z \rightarrow -1-i} \left[\frac{(z+1-i)}{z} \cdot \frac{z^2 + 4}{(z+1-i)(z+2+i)} \right] = \lim_{z \rightarrow -1-i} \left[\frac{z^2 + 4}{z(z+2+i)} \right]$
 $= \frac{(-1-i)^2 + 4}{(-1-i)(-1+i)} = \frac{(i+1)^2 + 4}{-i(i-1)} = \frac{(i+1)^2 + 4}{2i(i-1)} = \frac{1+2i-1+4}{-2+2i} = \frac{4+2i}{-2+2i}$
 $= \frac{2+2i}{-1+i} = \frac{(2+i)(1-i)}{(1+i)(1-i)} = \frac{-2-2i+i+1}{2} = \frac{-1-3i}{2} = -\frac{1}{2}(-1-3i)$

Question 8

$$f(z) = \frac{\tan 3z}{z^4}, z \in \mathbb{C}$$

Find the residue of the pole of $f(z)$.

$$\boxed{32}, \boxed{\text{res}(z=0)=9}$$

SIMPLY WITH THE EXPANSION OF TAN

$\tan z = \tan z$	$\tan' z = \sec^2 z = 1 + \tan^2 z = 1 + \tan^2 z$	$\tan'' z = 0$
$\tan z = \sec^2 z$	$\tan' z = 0$	$\tan''' z = 1$
$\tan z = 2\sec^2 z$	$\tan'' z = 0$	$\tan'''' z = 0$
$\tan z = 2\sec^2 z + 2\sec^4 z$	$\tan''' z = 0$	$\tan''''' z = 2$

$\Rightarrow \tan z = 2 + \frac{1}{2!}z^2 + O(z^4)$
 $\Rightarrow \tan' z = 2 + \frac{3}{3!}z^3 + O(z^5)$
 $\Rightarrow \tan'' z = (2z) + \frac{1}{2!}z^2 + O(z^4)$
 $\Rightarrow \tan''' z = 32 + \frac{9z^2}{3!} + O(z^4)$
 $\Rightarrow \frac{1}{z!} \tan''' z = \frac{1}{24} [32 + 9z^2 + O(z^4)]$
 $\Rightarrow f(z) = \frac{2}{23} + \frac{1}{23} O(z^2)$

THE VALUE OF THE THREE TERMS AT THE ORIGIN IS 9

Question 9

$$f(z) \equiv \frac{z^2 - 2z}{(z^2 + 4)(z+1)^2}, z \in \mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{\text{res}(z=2i) = \frac{1}{25}(7+i)}, \boxed{\text{res}(z=-2i) = \frac{1}{25}(7-i)}, \boxed{\text{res}(z=-1) = -\frac{14}{25}}$$

$f(z) = \frac{z^2 - 2z}{(z^2 + 4)(z+1)^2} = \frac{z^2 - 2z}{(z+2i)(z-2i)(z+1)^2}$ (Hence simple poles at $z = \pm 2i$,
Hence double pole at $z = -1$)

• RESIDUE AT $z = 2i$

$$\begin{aligned} & \lim_{z \rightarrow 2i} \left[(z-2i)^2 \frac{z^2 - 2z}{(z+2i)(z-2i)(z+1)^2} \right] = \lim_{z \rightarrow 2i} \left[\frac{z^2 - 2z}{(z+2i)(z+1)^2} \right] = \frac{-4-4i}{4i(1+2i)^2} \\ & = \frac{-1-i}{1+5i} = \frac{(1-i)(1-2i)^2}{1+5i} = \frac{-(1+i)(1-4i-4)}{25i} = \frac{-(1+i)(-3-i)}{25i} \\ & = \frac{(-1+i)(3+4i)}{25i} = \frac{3+4i+3i-4}{25i} = \frac{-1+7i}{25i} = i - \frac{7}{25}i = \frac{7i}{25}. \end{aligned}$$

• RESIDUE AT $z = -2i$

$$\begin{aligned} & \lim_{z \rightarrow -2i} \left[(z+2i)^2 \frac{z^2 - 2z}{(z+2i)(z-2i)(z+1)^2} \right] = \lim_{z \rightarrow -2i} \left[\frac{z^2 - 2z}{(z-2i)(z+1)^2} \right] = \frac{-4+4i}{(1-2i)^2(2i)} \\ & = \frac{1-i}{1+5i} = \frac{(1-i)(1+2i)^2}{1+5i} = \frac{(1-i)(1+4i-4)}{25i} = \frac{(1-i)(-3+4i)}{25i} = \frac{-3+4i+4i+4}{25i} \\ & = \frac{1+7i}{25i} = i - \frac{7}{25}i = \frac{7i}{25}. \end{aligned}$$

• RESIDUE AT $z = -1$

$$\begin{aligned} & \lim_{z \rightarrow -1} \left[\frac{d}{dz} \left[\frac{z^2 - 2z}{(z+2i)(z-2i)} \right] \right] = \lim_{z \rightarrow -1} \left[\frac{d}{dz} \left(\frac{z^2 - 2z}{z^2 + 4} \right) \right] = \lim_{z \rightarrow -1} \left[\frac{(z^2 + 4)(2z-2) - (z^2 - 2z)(2z)}{(z^2 + 4)^2} \right] \\ & = \frac{5(-1)^2 + 2(3)}{5^2} = \frac{-26i}{25} = -\frac{26}{25}i. \end{aligned}$$

Question 10

$$f(z) \equiv \frac{1}{e^z - 1}, z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$, at the origin.

$$\boxed{\text{res}(z=0) = 1}$$

$$\begin{aligned} f(z) &= \frac{1}{e^z - 1} = \frac{1}{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots - 1} = \frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots} \\ &= \frac{1}{z \left[1 + \frac{z}{2!} + O(z^2) \right]} = \frac{1}{z} \left(1 + \frac{z}{2!} + O(z^2) \right)^{-1} \\ &= \frac{1}{z} \left[1 - \frac{1}{2}z + O(z^2) \right] = \frac{1}{z} - \frac{1}{2} + O(z) \\ \therefore \text{Residue } 1 \end{aligned}$$

Question 11

$$f(z) = \frac{z}{(3z^2 - 10iz - 3)^2}, z \in \mathbb{C}$$

Find the residue of each of the two poles of $f(z)$.

$$\boxed{\text{res}(z = 3i) = \frac{5}{256}}, \quad \boxed{\text{res}\left(z = \frac{1}{3}i\right) = -\frac{5}{256}}$$

$\frac{1}{f(z)} = \frac{2}{(3z^2 - 10iz - 3)^2}$

$$\begin{aligned} 3z^2 - 10iz - 3 &= 3\left[z^2 - \frac{10}{3}iz - 1\right] = 3\left[\left(z - \frac{5}{3}i\right)^2 + \frac{25}{9}i^2 - 1\right] \\ &= 3\left[\left(z - \frac{5}{3}i\right)^2 + \frac{25}{9}(-1) - 1\right] = 3\left[\left(z - \frac{5}{3}i\right)^2 - \frac{28}{9}\right] \\ &= 3\left(z - 3i\right)\left(z - \frac{1}{3}i\right) \\ &= (z - 3i)(z - \frac{1}{3}i) \end{aligned}$$

$f(z)$ has double poles at $z = 3i$ & $\frac{1}{3}i$

- $\lim_{z \rightarrow 3i} \left[\frac{1}{z-3i} \left[\frac{(2-3i)^2 - \frac{2}{(3z^2-10iz-3)^2}}{(z-3i)^2} \right] \right] = \lim_{z \rightarrow 3i} \left[\frac{1}{z-3i} \left[\frac{-2}{(z-3i)^2} \right] \right]$
- $= \lim_{z \rightarrow 3i} \left[\frac{(3i-1)^2 - 1}{(3i-1)^4} \right] = \lim_{z \rightarrow 3i} \left[\frac{(3i-1) - 6i}{(3i-1)^3} \right]$
- $= \lim_{z \rightarrow 3i} \left[\frac{-1 - 3i}{(3i-1)^3} \right] = \frac{-1 - 3i}{(3i-1)^3} = \frac{-10i}{(3i)^3} = \frac{-10i}{27i^3} = \frac{5}{27}$
- $\bullet \lim_{z \rightarrow \frac{1}{3}i} \left[\frac{1}{z-\frac{1}{3}i} \left[\frac{(2-\frac{1}{3}i)^2 - \frac{2}{(3z^2-10iz-3)^2}}{(z-\frac{1}{3}i)^2} \right] \right] = \lim_{z \rightarrow \frac{1}{3}i} \left[\frac{1}{z-\frac{1}{3}i} \left[\frac{-2}{(z-\frac{1}{3}i)^2} \right] \right]$
- $= \lim_{z \rightarrow \frac{1}{3}i} \left[\frac{(2-\frac{1}{3}i)^2 - 1}{(2-\frac{1}{3}i)^4} \right] = \frac{1}{2-\frac{1}{3}i} \lim_{z \rightarrow \frac{1}{3}i} \left[\frac{(2-i) - 2i}{(2-\frac{1}{3}i)^3} \right]$
- $= \frac{1}{2} \lim_{z \rightarrow \frac{1}{3}i} \left[\frac{-2-3i}{(2-3i)^3} \right] = -\frac{1}{2} \lim_{z \rightarrow \frac{1}{3}i} \left[\frac{2+3i}{(2-3i)^2} \right] = -\frac{1}{2} \left[\frac{\frac{1}{2} + \frac{3i}{2}}{(\frac{5}{3})^2} \right]$
- $= -\frac{1}{2} + \frac{\frac{3i}{2}}{(\frac{5}{3})^2} = -\frac{1}{2} - \frac{9i}{25} = -\frac{10i}{25} = -\frac{2i}{5}$

Question 12

$$f(z) \equiv \frac{\cot z \coth z}{z^3}, \quad z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$ at $z=0$.

, $\text{res}(z=0) = -\frac{7}{45}$

IT IS BEST TO FIND THIS RESIDUE BY EXPANSION AS $f(z) = \frac{\cot z \coth z}{z^3}$

$$\begin{aligned} f(z) &= \frac{1}{z^3} \times \frac{\cot z}{\sin z} \times \frac{\cosh z}{\sinh z} \\ &= \frac{1}{z^3} \times \left(1 - \frac{z^2}{3} + \frac{z^4}{45} + O(z^6) \right) \times \left(1 + \frac{z^2}{6} + \frac{z^4}{180} + O(z^6) \right) \\ &= \frac{1}{z^3} \times \left(1 - \frac{z^2}{3} + \frac{z^4}{45} + O(z^6) \right) \times \left(1 + \frac{z^2}{6} + \frac{z^4}{180} + O(z^6) \right) \\ &= \frac{1}{z^3} \times \left(1 + \frac{z^2}{18} - \frac{z^4}{135} - \frac{z^6}{405} + O(z^8) \right) \\ &= \frac{1}{z^3} \times \left(1 - \frac{z^2}{18} + O(z^4) \right) \\ &\quad \text{REWRITE IN ORDER TO COMPLETE THE EXPANSION} \\ &= \frac{1}{z^3} \left[\left(1 - \frac{1}{2}z^2 + O(z^4) \right) \left[1 - \frac{1}{18}z^2 + O(z^4) \right]^{-1} \right] \\ &= \frac{1}{z^3} \left[\left(1 - \frac{1}{2}z^2 + O(z^4) \right) \left(1 + \frac{1}{18}z^2 + O(z^4) \right) \right] \\ &= \frac{1}{z^3} \left[1 + \frac{1}{36}z^2 - \frac{1}{36}z^2 + O(z^4) \right] \\ &= \frac{1}{z^3} \left[1 - \frac{7}{36}z^2 + O(z^4) \right] \\ &= \frac{1}{z^3} - \frac{7}{432}z^0 + O(z^2) \end{aligned}$$

$\therefore \text{Residue } 12 = -\frac{7}{45}$

Question 13

$$f(z) \equiv \frac{z^6 + 1}{2z^5 - 5z^4 + 2z^3}, z \in \mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{\text{res}(z = \frac{1}{2}) = -\frac{65}{24}}, \boxed{\text{res}(z = 2) = \frac{65}{24}}, \boxed{\text{res}(z = 0) = \frac{21}{8}}$$

$$f(z) = \frac{z^6 + 1}{2z^5 - 5z^4 + 2z^3} = \frac{z^6 + 1}{2^3(3z^2 - 5z + 2)} = \frac{z^6 + 1}{2^3(3z-1)(z-2)}$$

SHREVE RULES AT $z = \frac{1}{2}$, $z = 2$ & TRIPLE RULE AT $z = 0$

- $\bullet \lim_{z \rightarrow 2} \left[\frac{(z-2)^3}{2^3(3z-1)(z-2)} \right] = \frac{2^6 + 1}{2^3 \times 3} = \frac{65}{24} //$
- $\bullet \lim_{z \rightarrow \frac{1}{2}} \left[\frac{(z-\frac{1}{2})^3}{2^3(3z-1)(z-\frac{1}{2})} \right] = \frac{(\frac{1}{2})^6 + 1}{2^3(\frac{1}{2}-1)} = -\frac{1}{3} + 1 = -\frac{1+64}{24} = -\frac{65}{24} //$
- \bullet FOR THE TRIPLE RULE IT IS EASIER TO GET A LAURENT EXPANSION AROUND $z=0$
 $f(z) = \frac{1+z^6}{2^3(z-2)(z-2)} = \frac{1+z^6}{2^3(-1+2z)\times 2(1-\frac{1}{z}))} = \frac{1}{2^3} \frac{(1+2z)(-1+2z)(1-z)^{-1}}{(-1+2z)(-1+2z)(1-z)^{-1}}$
 $= \dots \frac{1}{2^3} (1+2z)(1+2z+4z^2\dots)(1+\frac{1}{2}z+4z^2\dots)$
 $= \dots \frac{1}{2^3} \times 1 \times (\dots + 2z^2 + 2z^2 + 4z^2\dots) = \dots \frac{8z}{2^3} \dots //$

ANOTHER WAY FOR THE TRIPLE RULE

$$\begin{aligned} &\lim_{z \rightarrow 0} \left[\frac{z^3}{2^3(3z-1)(z-2)} \right] = \frac{1}{2^3} \lim_{z \rightarrow 0} \left[\frac{z^6 + 1}{2^3(3z^2 - 5z + 2)} \right] \\ &= \frac{1}{2^3} \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left(\frac{(2z^2-5z+2)(z^2+1)}{2^3(3z^2-5z+2)} \right) \right] = \frac{1}{2^3} \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left(\frac{2z^4-10z^3+12z^2-4z^2-4z^2+1}{2^3(3z^2-5z+2)^2} \right) \right] \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left(\frac{8z^3-30z^2+12z^2-4z^2-4z^2+1}{2^3(3z^2-5z+2)^2} \right) \right] \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \left[\frac{(24z^2-90z^2+12z^2-4z^2-4z^2+1) - 2(8z^3-25z^2+28z^2-4z^2+1)(2z^2-5z+2)(4z-1)}{(2z^2-5z+2)^4} \right] \\ &= \frac{1}{2} \times \frac{32(1-4)-2 \times 5 \times 2(-5)}{2^4} = -\frac{1}{2} \times \frac{-16+100}{16} = \frac{84}{32} = \frac{21}{8} // \end{aligned}$$

Question 14

$$f(z) \equiv \frac{4}{z^2(1-2i)+6zi-(1+2i)}, z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$$\boxed{\operatorname{res}(z=2-i)=i}, \boxed{\operatorname{res}\left(z=\frac{1}{5}(2-i)\right)=-i}$$

$$f(z) = \frac{4}{z^2(1-2i)+6zi-(1+2i)}$$

BY THE QUADRATIC FORMULA

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1-2i)(1+2i)}}{2(1-2i)} = \frac{-6i \pm \sqrt{-36 + 48i}}{2(1-2i)} = \frac{-6i \pm \sqrt{-12}}{2(1-2i)}$$

$$= \frac{-6i \pm 2i\sqrt{3}}{2(1-2i)} = \frac{(3 \pm 2i)}{1-2i} = \begin{cases} \frac{3+2i}{1-2i} & \text{if } i > 0 \\ \frac{-3+2i}{1-2i} & \text{if } i < 0 \end{cases} = \frac{1}{2}(2-i)$$

$f(z)$ HAS SIMPLE POLES AT $z=2-i$ & $\frac{1}{2}(2-i)$

- $\lim_{z \rightarrow 2-i} \left[(z-2+i) \times \frac{4}{z^2(1-2i)+6zi-(1+2i)} \right] = \frac{0}{0} \Rightarrow \dots$ BY L'HOSPITAL...
- $= \lim_{z \rightarrow 2-i} \left[\frac{4}{2z(1-2i)+6i} \right] = \frac{4}{2(2-i)(1-2i)+6i} = \frac{2}{(2-i)(1-2i)+3i}$
- $= \frac{2}{2-4i-i-2+3i} = \frac{2}{-3i} = -\frac{2}{3}i \Rightarrow \cancel{\text{}}$
- $\lim_{z \rightarrow \frac{1}{2}(2-i)} \left[(z-\frac{1}{2}(2-i)) \times \frac{4}{z^2(1-2i)+6zi-(1+2i)} \right] = \frac{0}{0} \Rightarrow \dots$ BY L'HOSPITAL...
- $= \lim_{z \rightarrow \frac{1}{2}(2-i)} \left[\frac{4}{2z(1-2i)+6i} \right] \approx \frac{4}{2 \times \frac{1}{2}(2-i)(1-2i)+6i} = \frac{10}{(2-i)(1-2i)+15i}$
- $= \frac{10}{2-4i-i-2+15i} = \frac{10}{15i} = \frac{1}{15}i \Rightarrow \cancel{\text{}}$

Question 15

$$f(z) \equiv \frac{ze^{kz}}{z^4 + 1}, z \in \mathbb{C}, k \in \mathbb{R}, k > 0.$$

Show that the sum of the residues of the four poles of $f(z)$, is

$$\sin\left(\frac{k}{\sqrt{2}}\right)\sinh\left(\frac{k}{\sqrt{2}}\right).$$

, proof

IT IS BEST TO WORK WITH EXPONENTIALS IN THIS QUESTION

$$f(z) = \frac{ze^{kz}}{z^4 + 1} \text{ has simple poles at:}$$

$$z^4 + 1 = 0 \Rightarrow z^4 = -1 \Rightarrow z^4 = e^{i\pi(2m+1)}, m=0,1,2,3$$

$$z = e^{i\pi/4}(2m+1)^{1/4} \quad (m=0,1,2,3)$$

$$z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$$

CALCULATE THREE RESIDUES USING A GENERAL METHOD - LET A POLE BE AT $z=z_0$

$$\text{Res}(f; z_0) = \lim_{z \rightarrow z_0} \left[\frac{(z-z_0)f(z)}{z^4 + 1} \right]$$

This will produce "annular cancellation" as $(z-z_0)$ will be a factor of the denominator so we proceed by L'HOSPITAL RULE

$$\text{Res}(f; z_0) = \lim_{z \rightarrow z_0} \left[\frac{\frac{d}{dz}[(z-z_0)e^{kz}]}{4z^3} \right] = \lim_{z \rightarrow z_0} \left[\frac{ze^{kz} + (z-z_0)e^{kz} + (z-z_0)ke^{kz}}{4z^3} \right]$$

$$\text{Res}(f; z_0) = \frac{z_0e^{kz_0}}{4z_0^3} = \boxed{\frac{e^{kz_0}}{4z_0^3}}$$

RECALCULATE THE RESIDUE AT EACH OF THE FOUR POLES

$$z_0 = e^{i\pi/4} \quad \text{comes } \frac{e^{i\pi/4}}{4e^{i\pi/4}} = \boxed{\frac{i(\sqrt{2}+i\sqrt{2})}{4i}}$$

$$z_0 = e^{i3\pi/4} \quad \text{comes } \frac{e^{i3\pi/4}}{4e^{i3\pi/4}} = \boxed{\frac{i(\sqrt{2}-i\sqrt{2})}{4i}}$$

$$z_0 = e^{i5\pi/4} \quad \text{comes } \frac{e^{i5\pi/4}}{4e^{i5\pi/4}} = \boxed{\frac{-i(\sqrt{2}+i\sqrt{2})}{4i}}$$

$$z_0 = e^{i7\pi/4} \quad \text{comes } \frac{e^{i7\pi/4}}{4e^{i7\pi/4}} = \boxed{\frac{-i(\sqrt{2}-i\sqrt{2})}{4i}}$$

ADDING THE 4 RESIDUES - LET $a = \frac{k\pi}{\sqrt{2}} = \frac{k\pi}{4\sqrt{2}}$ FOR SIMPLICITY

$$\text{SUM OF 4 RESIDUES} = \frac{i(\sqrt{2}+i\sqrt{2})}{4i} + \frac{i(\sqrt{2}-i\sqrt{2})}{-4i} + \frac{-i(\sqrt{2}+i\sqrt{2})}{-4i} + \frac{i(\sqrt{2}-i\sqrt{2})}{4i}$$

$$= \frac{1}{4i} \left[e^{ik\pi/4} - e^{-ik\pi/4} - e^{ik\pi/4} + e^{-ik\pi/4} \right]$$

$$= \frac{1}{4i} \left[e^{ik\pi/4} - e^{ik\pi/4} - e^{-ik\pi/4} + e^{-ik\pi/4} \right]$$

$$= \frac{1}{4i} \left[e^{ik\pi/4} (e^{-ik\pi/4} - e^{ik\pi/4}) - e^{-ik\pi/4} (e^{ik\pi/4} - e^{-ik\pi/4}) \right]$$

$$= \frac{1}{4i} \left[(e^{ik\pi/4} - e^{-ik\pi/4})(e^{ik\pi/4} - e^{-ik\pi/4}) \right]$$

$$= \frac{1}{4i} \times 2\sinh(ak\pi) \times 2\sinh(ak\pi)$$

$$= \frac{1}{4i} \times 2^2 \sin(ak\pi) \times 2\sinh(ak\pi)$$

$$= \sin(ak\pi) \sinh(ak\pi)$$

$$= \boxed{\sin\left(\frac{k\pi}{\sqrt{2}}\right) \sinh\left(\frac{k\pi}{\sqrt{2}}\right)}$$

As required

REAL INTEGRALS

UNIT CIRCLE CONTOUR

Question 1

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5} d\theta.$$

V, $\boxed{\quad}$, $-\frac{2\pi}{3}$

SOLN BY USING THE CIRCLE TEST | OE $z = e^{i\theta}, 0 \leq \theta < 2\pi$

- $dz/d\theta = \frac{1}{i}(e^{i\theta})' = e^{i\theta} \left(\frac{i}{2}\right) = \frac{1}{2}(i + \frac{1}{2})$
- $d\theta/dz = \frac{1}{i(e^{i\theta})} = \frac{1}{i(z + \frac{1}{2})} = \frac{1}{iz + \frac{1}{2}}$
- $dz = (z + \frac{1}{2})d\theta$
- $= 2\theta + \frac{1}{2} - 5$
- $= \frac{1}{2}(2\theta^2 - 8\theta + 2)$
- $= \frac{1}{2}(2(\theta - 1)(\theta - 2))$

TELEGRAPHING THE INTEGRAL

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5} d\theta = \int_0^{2\pi} \frac{1}{4\left(\frac{1}{2}(i + \frac{1}{2})\right)} \left(\frac{1}{2}(i + \frac{1}{2}) d\theta\right)$$

$$= \int_0^{2\pi} \frac{-i}{(2\theta - 1)(2\theta - 2)} d\theta$$

$z = e^{i\theta}$
$dz = ie^{i\theta} d\theta$
$d\theta = \frac{dz}{iz}$
$d\theta = -\frac{1}{iz} dz$

NOTE: THE INTEGRAND HAS SINGULAR POINTS AT $z = 2$ & $z = \frac{1}{2}$, OF WHICH ONLY THE ONE AT $z = \frac{1}{2}$ IS INSIDE $|z| = 1$ (THIS IS NOT

$$\lim_{z \rightarrow \frac{1}{2}} [(z - \frac{1}{2}) f(z)] = \lim_{z \rightarrow \frac{1}{2}} [(z - \frac{1}{2}) \frac{-1}{(2(z - 1)(2z - 2))}]$$

$$= \lim_{z \rightarrow \frac{1}{2}} \frac{(z - \frac{1}{2}) \times -1}{2(z - 1)(2z - 2)}$$

$$= \lim_{z \rightarrow \frac{1}{2}} \frac{-1}{2(2z - 2)}$$

$$= \frac{-1}{4}$$

BY THE RESIDUE THEOREM WE HAVE

$$\int_C f(z) dz = 2\pi i \times \sum \text{Residues inside } C$$

$$\int_C \frac{1}{(2z-1)(2z-2)} dz = 2\pi i \times \frac{1}{3} i$$

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5} d\theta = -\frac{2\pi}{3}$$

Question 2

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta.$$

, $\frac{2\pi}{\sqrt{3}}$

$\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta = \int_0^{2\pi} \frac{1}{2+\frac{1}{2}(e^{i\theta}+e^{-i\theta})} d\theta$

THIS IS A STANDARD PARAMETRISATION (ONE THE UNIT CIRCLE IS DRAWN)

IF $z = re^{i\theta}$ ON THIS CIRCLE
 $z = e^{i\theta}$
 $dz = ie^{i\theta} d\theta$
 $dz = i d\theta$
 $d\theta = \frac{dz}{iz}$

TRANSFORM THE INTEGRAL

$$= \int_{\gamma} \frac{1}{2+\frac{1}{2}(z+\frac{1}{z})} \frac{dz}{iz} = \int_{\gamma} \frac{-2i}{4+z+\frac{1}{z}} dz$$

$$= \int_{\gamma} \frac{-2i}{z^2+4z+1} dz = \int_{\gamma} \frac{-2i}{(z+2)^2+3} dz$$

LOOK FOR THE POLES OF $f(z)$

$$f(z) = \frac{-2i}{z^2+4z+1} = \frac{-2i}{(z+1)^2+3} = \frac{-2i}{(z+2-i)(z+2+i)}$$

f(z) HAS SIMPLE POLES AT $z = -2 \pm i\sqrt{3}$, BUT ONLY THE ONE AT $z = -2 + i\sqrt{3}$ IS INSIDE γ — COMPUTE THE RESIDUE

$$\operatorname{Res}(f; z=-2+i\sqrt{3}) = \lim_{z \rightarrow -2+i\sqrt{3}} [(z+2-i)^2] f(z)$$

$$= \lim_{z \rightarrow -2+i\sqrt{3}} \left[(z+2-i)^2 \right] \times \frac{-2i}{(z+2-i)(z+2+i)}$$

$= \frac{-2i}{-2+2i+2i\sqrt{3}} = \frac{-2i}{2+2i\sqrt{3}} = \frac{-i}{1+\sqrt{3}}$

BY THE RESIDUE THEOREM

$$\int_{\gamma} f(z) dz = 2\pi i \times \sum (\text{RESIDUES INSIDE } \gamma)$$

$$\int_{\gamma} \frac{-2i}{z^2+4z+1} dz = 2\pi i \times \left(\frac{-i}{1+\sqrt{3}} \right)$$

$$\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta = \frac{2\pi}{\sqrt{3}}$$

Question 3

By integrating a suitable complex function over an appropriate contour find

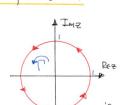
$$\int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta.$$

, $\frac{5\pi}{512}$

• USING THE CONTOUR $|z|=1$ OR $z = e^{i\theta}, 0 \leq \theta < 2\pi$

Thus $z = e^{i\theta}$
 $dz = i e^{i\theta} d\theta$

• HENCE THE INTEGRAL BECOMES

$$\begin{aligned} & \int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta \\ &= \int_0^{2\pi} (e^{i\theta} \cos \theta)^6 \, d\theta = \int_0^{2\pi} \left(\frac{1}{2} \sin 2\theta\right)^6 \, d\theta \\ &= \frac{1}{64} \int_0^{2\pi} \sin^6 2\theta \, d\theta = \frac{1}{64} \int_0^{2\pi} \left[\frac{1}{32}(e^{i2\theta} - e^{-i2\theta})\right]^6 \, d\theta \\ &= \frac{1}{64} \left(\frac{1}{2i}\right)^6 \int_0^{2\pi} \left[(e^{i2\theta})^2 - (e^{-i2\theta})^2\right]^6 \, d\theta \\ &= \frac{1}{64} \times \frac{1}{64} \int_{\Gamma} \left(z^2 - \frac{1}{z^2}\right)^6 \left(\frac{dz}{iz}\right) \quad \leftarrow dz = \frac{dz}{i\theta} \\ &= \frac{1}{64^2} \int_{\Gamma} \frac{1}{z^2} (z^2 - \frac{1}{z^2})^6 \, dz \\ &\quad \bullet \text{ EXPAND BINOMIALLY} \\ &= \frac{i}{256} \int_{\Gamma} \left[\frac{1}{z^2} \left(z^2 - \frac{1}{z^2} \right)^6 + \dots \right] dz \end{aligned}$$


• AS THE INTEGRAND IS IN REAL FORM THE ONLY CONTRIBUTION IS FROM THE $\frac{1}{z^2}$ TERM IF THE RESIDUE IS -20

$$\begin{aligned} & \Rightarrow \int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta = 2\pi i \times \sum \text{residues inside } \Gamma \\ & \Rightarrow \int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta = 2\pi i \times \frac{i}{z^2} \times (-20) \\ & \Rightarrow \int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta = -\frac{20\pi}{2^2} = -\frac{5\pi}{2} \\ & \Rightarrow \int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta = \frac{5\pi}{512} \end{aligned}$$

Question 4

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{1}{5+4\sin\theta} d\theta.$$

, $\frac{2\pi}{3}$

SPLIT THE INTEGRAL INTO A UNIT DISK CENTER AT THE ORIGIN IN $z = 2e^{i\theta}$, THE CIRCLE SHOWN BELOW (RADII AS r)

$$\begin{aligned} & \int_0^{2\pi} \frac{1}{5+4\sin\theta} d\theta \\ &= \int_0^{\pi} \frac{1}{5+4\sin\left(\frac{\theta}{2}\right)} \frac{d\theta}{2} \\ &= \int_0^{\pi} \frac{d\theta}{5+4\sin\left(\frac{\theta}{2}\right)(2-\frac{1}{2})} \\ &= \int_0^{\pi} \frac{d\theta}{4(2+\frac{1}{2}(2-\frac{1}{2}))} = \int_0^{\pi} \frac{d\theta}{5(2+\frac{1}{2}(2-\frac{1}{2}))} \\ &= \int_0^{\pi} \frac{d\theta}{25(2+\frac{1}{2}(2-\frac{1}{2}))} = \int_0^{\pi} \frac{1}{25(2+\frac{1}{2}(2-\frac{1}{2}))} d\theta \end{aligned}$$

FACTORISE THE DENOMINATOR OF $f(z)$ IN THE QUADRATIC FORMULA / COMPUTE THE JACOBIAN

- $b-4ac = (\sin^2\theta - 4\cos\theta) = -25 + 16 = -9$
- SOLVE THE DENOMINATOR FOR θ TO OBTAIN POLES
 $z = \frac{-\sin\theta \pm \sqrt{-9}}{2\cos\theta} = \frac{-\sin\theta \pm 3i}{2\cos\theta} = \begin{cases} -\frac{1}{2}i \\ -2i \end{cases}$
- $f(z) = \frac{1}{25(2+\frac{1}{2}(2-\frac{1}{2}))}$ HAS SOURCE POLES AT $-2i$ & $-\frac{1}{2}i$ BUT ONLY THE ONE AT $-\frac{1}{2}i$ IS IN Γ — NOTE THAT $f(z) = \frac{1}{25(2+\frac{1}{2}(2-\frac{1}{2}))}$

BY THE RESIDUE THEOREM

$$\int_0^{2\pi} \frac{1}{5+4\sin\theta} d\theta = \int_{\Gamma} \frac{1}{25(2+\frac{1}{2}(2-\frac{1}{2}))} dz = 2\pi i \times \sum \text{poles inside } \Gamma$$

COMPUTE THE RESIDUE AT THE POLE

$$\begin{aligned} & \lim_{z \rightarrow -\frac{1}{2}i} [(z + \frac{1}{2}i) f(z)] = \lim_{z \rightarrow -\frac{1}{2}i} \left[(z + \frac{1}{2}i) \frac{1}{25(2+\frac{1}{2}(2-\frac{1}{2}))} \right] \\ &= \frac{(z + \frac{1}{2}i)}{2 \rightarrow -\frac{1}{2}i} \left[\frac{1}{25(2+\frac{1}{2}(2-\frac{1}{2}))} \right] \\ &= \frac{1}{2(-\frac{1}{2}i + 2)} = \frac{1}{3i} \\ & \therefore \int_0^{2\pi} \frac{1}{5+4\sin\theta} d\theta = 2\pi i \times \frac{1}{3i} = \frac{2\pi}{3} \end{aligned}$$

Question 5

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{\sin^2 t}{5 - 4\cos t} dt.$$

$$\boxed{\frac{\pi}{4}}$$

$\int_0^{2\pi} \frac{\sin^2 t}{5 - 4\cos t} dt = \int_0^{2\pi} \frac{-\frac{1}{2}(e^{2it} - 2 + e^{-2it})}{5 - 2(e^{it} + e^{-it})} dt = \int_{\Gamma} \frac{-\frac{1}{2}(z^2 - 2 + \frac{1}{z})}{5 - 2(z + \frac{1}{z})} \frac{dz}{iz} = -\frac{1}{4i} \int_{\Gamma} \frac{z^2 - 2 + \frac{1}{z}}{5 - 2z - \frac{2}{z}} dz$

LET $z = e^{it}$, $0 \leq t < 2\pi$
 $dz = ie^{it}dt = iz dt$
 $dt = \frac{dz}{iz}$
 $\cos t = \frac{1}{2}(e^{it} + e^{-it})$
 $\sin t = \frac{1}{2i}(e^{it} - e^{-it})$
 $\sin^2 t = \frac{1}{4}(e^{2it} - 2 + e^{-2it})$

$\int_{\Gamma} \frac{z^2 - 2 + \frac{1}{z}}{5 - 2z - \frac{2}{z}} dz = -\frac{1}{4i} \int_{\Gamma} \frac{z^4 - 2z^2 + 1}{5z^2 - 2z^2 - 2} dz = -\frac{1}{4i} \int_{\Gamma} \frac{z^4 - 2z^2 + 1}{3z^2 - 2z^2 - 2} dz = -\frac{1}{4i} \int_{\Gamma} \frac{z^4 - 2z^2 + 1}{z^2(2z^2 - 3z + 2)} dz$

NOW IGNORE THIS DOUBLE-POLE AT $z=0$, AND SINGULARITIES AT $z=\frac{1}{2}$ & $z=2$.
 OR VIEW THE ONE AT $z=2$ DOES NOT CONTRIBUTE AS IT IS OUTSIDE Γ

- $\lim_{z \rightarrow \frac{1}{2}} \left[(z-\frac{1}{2}) \frac{z^4 - 2z^2 + 1}{z^2(2z^2 - 3z + 2)} \right] = \lim_{z \rightarrow \frac{1}{2}} \left[\frac{(z-\frac{1}{2})(z^3 + z^2 + z + 1)}{z^2(2z^2 - 3z + 2)} \right] = -\frac{1}{2} = \frac{1}{2}$
- $\lim_{z \rightarrow \infty} \left[\frac{z^4 - 2z^2 + 1}{z^2(2z^2 - 3z + 2)} \right] = \lim_{z \rightarrow \infty} \left[\frac{1}{z^2} \left[z^2 \times \frac{z^2 - 2z + 1}{2z^2 - 3z + 2} \right] \right] = \lim_{z \rightarrow \infty} \left[\frac{(2z^2 - 2z + 1)(z^2 + 1)}{2z^4 - 3z^3 + 2z^2} \right] = \frac{1}{2}$

BY THE RESIDUE THEOREM:

$$\int_{\Gamma} \frac{z^4 - 2z^2 + 1}{z^2(2z^2 - 3z + 2)} dz = 2\pi i \left[-\frac{1}{2} + \frac{1}{2} \right] = \pi i$$

$$\frac{1}{4i} \int_{\Gamma} \frac{z^4 - 2z^2 + 1}{z^2(2z^2 - 3z + 2)} dz = \frac{1}{4i} \times \pi i = \frac{\pi}{4}$$

$$\int_0^{2\pi} \frac{\sin^2 t}{5 - 4\cos t} dt = \frac{\pi}{4}$$

Question 6

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2} d\theta.$$

$$\boxed{\frac{5\pi}{32}}$$

• CONSIDER $z = e^{i\theta}$ i.e. θ lies on the standard unit circle, for $0 < \theta < \pi$

$dz = e^{i\theta} d\theta$

$d\bar{z} = \bar{e}^{-i\theta} d\theta$

$d\theta = \frac{dz}{iz}$

Thus

$$\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2} d\theta = \int_{\Gamma} \frac{1}{\frac{1}{(5-3\sin\theta)^2} \cdot \frac{dz}{iz}} \frac{dz}{iz} = \int_{\Gamma} \frac{-iz}{(5-3\sin\theta)^2} dz$$

Now

$$5-3\sin\theta = 5 - 3\left[\sin^2\theta + \cos^2\theta - 1\right] = 3\left[\cos^2\theta - \frac{4}{3}\sin^2\theta - 1\right]$$

$$= 3\left[\left(\cos\theta - \frac{2}{\sqrt{3}}\sin\theta\right)^2 + \frac{1}{3}\right] = 3\left[\left(\cos\theta - \frac{2}{\sqrt{3}}\right)^2 + \left(\sin\theta\right)^2\right]$$

$$= 3\left[\left(\cos\theta - \frac{2}{\sqrt{3}}\right)\left(\cos\theta + \frac{2}{\sqrt{3}}\right)\right] = 3\left[\cos^2\theta - \frac{4}{3}\sin^2\theta\right]$$

THE INTEGRAND HAS POLES AT $\cos\theta = \pm\frac{2}{\sqrt{3}}$

• CALCULATE RESIDUES

$$\lim_{z \rightarrow \cos\theta} \left[\frac{1}{(5-3\sin z)^2} \cdot \frac{4z^2}{(z-\cos\theta)(z+\cos\theta)} \right] = \frac{4z^2}{3} \lim_{z \rightarrow \cos\theta} \left[\frac{1}{(z-\cos\theta)^2} \right]$$

$$= \frac{4z^2}{3} \lim_{z \rightarrow \cos\theta} \left[\frac{(z-\cos\theta) - z\cos\theta(z-\cos\theta)}{(z-\cos\theta)^2} \right] = \frac{4}{3} \lim_{z \rightarrow \cos\theta} \left[\frac{(-\cos\theta-2z)}{(z-\cos\theta)^2} \right] = \frac{4}{3} \left[\frac{(-\frac{2}{\sqrt{3}})^2}{(\frac{2}{\sqrt{3}})^2} \right]$$

$$= \frac{4}{3} \frac{\frac{4}{3} \times \frac{4}{3}\pi^2}{(\frac{4}{3}\pi^2)^2} = \frac{4}{3} \times \frac{4}{3} \times -\frac{\pi^2}{4\pi^2} = -\frac{\pi^2}{9}$$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} \frac{1}{(5-3\sin z)^2} dz = \text{arg}(i) \times 2\text{residues inside } \Gamma = \text{arg}(i) \times (-\frac{\pi^2}{9}) = \frac{5\pi}{32}$$

$$\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2} d\theta = \frac{5\pi}{32}$$

Question 7

$$I = \int_0^{2\pi} \frac{1}{3 - 2\cos x + \sin x} dx.$$

By integrating a suitable complex function over an appropriate contour find the exact value of I .

$\boxed{\pi}$

DEFINITION TO FIND $I = \int_0^{2\pi} \frac{1}{3 - 2\cos x + \sin x} dx$

- $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ & $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$
- SINGULAR Z ON THE UNIT CIRCLE: $z = e^{it}, \omega \in Q.C.P$

$$I = \int_{\Gamma} \frac{1}{3 - (z + \frac{1}{z}) + \frac{1}{z^2}(z - \frac{1}{z})} \times \frac{dz}{iz}$$

Take CP

$$\int_{\Gamma} \frac{1}{3z^2 - 2z^2 - 1 + \frac{1}{z^2}} dz = \int_{\Gamma} \frac{2}{z^2 - 2z^2 + 6iz - 1 - 2i} dz$$

$$= \int_{\Gamma} \frac{2}{z^2(1-2i) + 6iz - (1+2i)} dz$$

† QUADRATIC FORMULA $Z = \frac{-6i \pm \sqrt{(6i)^2 + 4(1-2i)(1+2i)}}{2(1-2i)}$

$$Z = \frac{-6i \pm \sqrt{-36 + 16i}}{2(1-2i)} = \frac{-6i \pm \sqrt{-16i}}{2(1-2i)}$$

$$Z = \frac{-6i \pm 4i}{2(1-2i)} < \frac{-8i}{2(1-2i)} = \frac{4i}{-4(1-2i)} = \frac{-10i}{2(1-2i)} = \frac{5i}{-1+2i}$$

$$Z = \begin{cases} \frac{(-1-2i)}{(-1+2i)(1-2i)} = \frac{1}{2}(2-i) \\ \frac{5(-2-2i)}{(-1+2i)(1-2i)} = 2-i \end{cases}$$

• ONLY $Z = \frac{1}{2}(2-i)$ IS INSIDE Γ

RECALL: $\lim_{z \rightarrow \frac{1}{2}(2-i)} [(z - \frac{1}{2}(2-i)) f(z)] = \lim_{z \rightarrow \frac{1}{2}(2-i)} \left[\frac{2(z - \frac{1}{2}(2-i))}{z^2(1-2i) + 6iz - (1+2i)} \right]$

$$= \frac{0}{0} \dots \text{BY L'HOSPITAL RULE}$$

$$\lim_{z \rightarrow \frac{1}{2}(2-i)} \left[\frac{2}{z^2(1-2i) + 6iz - (1+2i)} \right] = \frac{2}{2 \cdot \frac{1}{4}(2-1)(1-2i) + 6i \cdot \frac{1}{2}}$$

$$= \frac{10}{2(1-1)(1-2i) + 30i} = \frac{10}{4-8i-2i+4+30i} = \frac{10}{20i} = \frac{1}{2i}$$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum(\text{Residues inside } \Gamma)$$

$$\int_{\Gamma} \frac{2}{z^2(1-2i) + 6iz - (1+2i)} dz = 2\pi i \times \frac{1}{2i}$$

$$\int_0^{2\pi} \frac{1}{3 - 2\cos x + \sin x} dx = \pi$$

Question 8

$$I = \int_0^{2\pi} \frac{\cos 3x}{5 - 4\cos x} dx.$$

By integrating a suitable complex function over an appropriate contour find the exact value of I .

$$\frac{\pi}{12}$$

• INTEGRAL IN THE RE CONVENTION
 $I = \int_0^{\pi} \frac{\cos 3x}{5 - 4\cos x} dx$

- LET $z = e^{i\theta}, \theta \in [0, 2\pi]$
 $dz = ie^{i\theta} d\theta$
 $d\theta = \frac{dz}{iz}$
 $d\theta = \frac{dz}{iz}$
- $\cos 3x = \frac{1}{2}(e^{3ix} + e^{-3ix}) = \frac{1}{2}(z^3 + \frac{1}{z^3})$
 $\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) = \frac{1}{2}(z + \frac{1}{z})$

THUS

$$I = \int_{C(R)} \frac{\frac{1}{2}(z^3 + \frac{1}{z^3})}{5 - 4z(\frac{1}{2}(z + \frac{1}{z}))} dz = \int_{C(R)} \frac{\frac{1}{2}z^2 + \frac{1}{2z^3}}{5 - 2z - \frac{2}{z^2}} dz = \int_{C(R)} \frac{\frac{z^2}{2} + \frac{1}{2z^3}}{(z - \frac{1}{z})(z^2 - 2z - 2)} dz$$

$$I = \int_{C(R)} \frac{z^2 + 1}{2z^2(z^2 - 2z - 2)} dz = \int_{C(R)} \frac{z^2 + 1}{-2z^2(2z^2 - 2z - 2)} dz = \int_{C(R)} \frac{z^2 + 1}{-2z^2(2z^2 - 2z)} dz$$

INTEGRATE WITH SIMPLE POLES AT $\frac{3\pi}{4}, z = \pm i$, & A TINY POLE AT $z = 0$ (INSIDE Γ)

- $\lim_{z \rightarrow i} \left[\frac{z^2 + 1}{-2z^2(2z^2 - 2z)} \right] = \frac{(i)^2 + 1}{-2((i)^2)2(i)(i-1)} = \frac{-1 + 1}{-4i} = 0$
- $\lim_{z \rightarrow 0} \text{Res} \left[\frac{z^2 + 1}{-2z^2(2z^2 - 2z)} \right] = \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left(\frac{z^2 + 1}{-2z^2(2z^2 - 2z)} \right) \right]$
 $= -\frac{1}{4i} \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left(\frac{3z^2 + 1}{-2z^2(2z^2 - 2z)} \right) \right] = -\frac{1}{4i} \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left(\frac{(2z^2 - 2z + 1)(4z^2 - 4z)}{(-2z^2 - 2z)^2} \right) \right]$
 $= -\frac{1}{4i} \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left(\frac{2z^4 - 2z^3 - 2z^2 + 2z^2 - 2z + 1}{(-2z^2 - 2z)^2} \right) \right]$

$= -\frac{1}{4i} \lim_{z \rightarrow 0} \left[\frac{d}{dz} \left(\frac{2z^4 - 2z^3 + 1}{(-2z^2 - 2z)^2} \right) \right] = -\frac{1}{4i} \lim_{z \rightarrow 0} \left[\frac{(8z^3 - 6z^2)(-4z^2 - 4z) - 2(2z^2 + 1)(-8z^3 - 4z^2)}{(-2z^2 - 2z)^4} \right]$

 $= -\frac{1}{4i} \left[\frac{2^2(-4) - 2(2)(2)(0)}{24} \right] = -\frac{1}{4i} \frac{-16 + 16}{48} = \frac{2i}{48}$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{Residue inside } \Gamma)$$

$$\int_{C(R)} \frac{dz}{5 - 4z\cos \theta} = 2\pi i \times \left[-\frac{2i}{4i} + \frac{2i}{48i} \right]$$

$$\int_{C(R)} \frac{dz}{5 - 4z\cos \theta} = \pi \left[\frac{5}{24} - \frac{1}{8} \right] = \frac{35}{12}$$

SEMI CIRCLE CONTOUR

Jordan's Lemma

Suppose that $f(z) \rightarrow 0$ uniformly, as $|z| \rightarrow \infty$, for $0 \leq \arg z \leq \pi$.

If $\alpha > 0$, then $\int_{\gamma_R} f(z) e^{i\alpha z} dz \rightarrow 0$ as $R \rightarrow \infty$, where $\gamma_R(\theta) = R e^{i\theta}$, for $0 \leq \theta \leq \pi$.

Proof

Given $\epsilon > 0$ we may always pick R_0 , so that if $R > R_0$, $|f(z)| < \epsilon$, $\forall z \in \gamma_R$.

Thus

$$\begin{aligned} \left| \int_{\gamma_R} e^{i\alpha z} f(z) dz \right| &= \left| \int_0^\pi e^{i\alpha R(\cos\theta + i\sin\theta)} f(R e^{i\theta}) i e^{i\theta} d\theta \right| = \\ \left| \int_0^\pi e^{i\alpha R \cos\theta} e^{-\alpha R \sin\theta} f(R e^{i\theta}) i e^{i\theta} d\theta \right| &\leq \int_0^\pi \left| e^{i\alpha R \cos\theta} e^{-\alpha R \sin\theta} f(R e^{i\theta}) i e^{i\theta} \right| d\theta = \\ \int_0^\pi \left| e^{i\alpha R \cos\theta} \right| \left| e^{-\alpha R \sin\theta} \right| \left| f(R e^{i\theta}) \right| \left| i e^{i\theta} \right| d\theta &= \int_0^\pi e^{-\alpha R \sin\theta} \left| f(R e^{i\theta}) \right| d\theta \leq \\ \epsilon R \int_0^\pi e^{-\alpha R \sin\theta} d\theta &= 2\epsilon R \int_0^{\frac{\pi}{2}} e^{-\alpha R \sin\theta} d\theta \quad \left[\text{since } \sin\theta \text{ is even about } \frac{\pi}{2} \right] \end{aligned}$$

Now by **Jordan's Inequality**

$$\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1, \text{ if } 0 < \theta \leq \frac{\pi}{2}$$

$$\sin\theta \geq \frac{2\theta}{\pi} \Rightarrow e^{-\sin\theta} \leq e^{\frac{2\theta}{\pi}}, \text{ if } 0 < \theta \leq \frac{\pi}{2}$$

Hence

$$\begin{aligned} 2\epsilon R \int_0^{\frac{\pi}{2}} e^{-\alpha R \sin\theta} d\theta &\leq 2\epsilon R \int_0^{\frac{\pi}{2}} e^{-\frac{2\alpha R \theta}{\pi}} d\theta = 2\epsilon R \left[-\frac{\pi}{2\alpha R} e^{-\frac{2\alpha R \theta}{\pi}} \right]_0^{\frac{\pi}{2}} = \\ \frac{\epsilon\pi}{\alpha} \left[e^{-\frac{2\alpha R \theta}{\pi}} \right]_0^{\frac{\pi}{2}} &= \frac{\epsilon\pi}{\alpha} \left[1 - e^{-\alpha R} \right] \rightarrow 0 \text{ since as } R \rightarrow \infty, \epsilon \rightarrow 0 \quad \square \end{aligned}$$

Question 1

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx.$$

\boxed{V} , $\boxed{\sqrt{-1}}$, $\boxed{\frac{\pi}{e}}$

• CONSIDER $\oint_{\Gamma} f(z) dz$, WHERE $f(z) = \frac{e^{iz}}{z^2 + 1}$ AND Γ IS THE "STANDARD" SEMICIRCULAR CONTOUR, SHOWN BELOW

• $f(z)$ HAS SIMPLE POLES AT $\pm i$, OF WHICH ONLY THE ONE AT i IS INSIDE Γ .

• CALCULATE THE RESIDUE OF THIS POLE

$$\begin{aligned} \lim_{z \rightarrow i} [(z-i)f(z)] &= \lim_{z \rightarrow i} \left[(z-i) \frac{e^{iz}}{z^2+1} \right] \\ &= \lim_{z \rightarrow i} \left[(z-i) \frac{e^{iz}}{(z-i)(z+i)} \right] \\ &= \frac{e^{i(i)}}{2i} \\ &= \frac{e^{-1}}{2i} \end{aligned}$$

• BY THE RESIDUE THEOREM

$$\begin{aligned} \rightarrow \oint_{\Gamma} f(z) dz &= 2\pi i \times \sum (\text{residues inside } \Gamma) \\ \rightarrow \oint_{\Gamma} \frac{e^{iz}}{z^2+1} dz &= 2\pi i \times \frac{e^{-1}}{2i} \end{aligned}$$

$$\begin{aligned} &\rightarrow \int_{-R}^R \frac{e^{iz}}{1+z^2} dz + \int_R^{\infty} \frac{e^{iz}}{1+z^2} dz = \frac{\pi}{e} \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad z = x+i0 \text{ ALONG THE STRAIGHT LINE} \qquad \qquad \text{ALONG THE ARC} \\ &\bullet \text{ Now } g(z) = \frac{e^{iz}}{1+z^2} \text{ SATISFIES CAUCHY'S CRITERION, SO } \int_{-\infty}^{\infty} g(x) dx \text{ VANISHES.} \\ &\rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx = \frac{\pi}{e} \\ &\rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{1+x^2} dx = \frac{\pi}{e} \\ &\rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{e} \end{aligned}$$

Question 2

By integrating a suitable complex function over an appropriate contour find

$$\int_0^\infty \frac{1}{1+x^2} dx.$$

$$\boxed{\frac{\pi}{2}}$$

• contour: $\int_{\gamma} \frac{1}{1+z^2} dz$, where γ is the contour consisting of the arc $[0, R]$ and the circular arc $Y, |z|=R$, $0 < \theta < \pi$.
 • $dz = iRd\theta$
 • $f(z) = \frac{1}{1+z^2}$, has simple poles at $z=1$, or where $z^2=1$, i.e. inside γ
 • residue at $z=1$: $\lim_{z \rightarrow 1} [f(z)(z-1)] = \lim_{z \rightarrow 1} \left[\frac{1}{(z+1)(z-1)} (z-1) \right] = \frac{1}{2}$

• BY THE RESIDUE THEOREM

$$\int_{\gamma} \frac{1}{1+z^2} dz = 2\pi i \times \sum \text{residues inside } \gamma$$

$$\int_{-R}^R \frac{1}{1+z^2} dz + \int_Y \frac{1}{1+z^2} dz = 2\pi i \times \frac{1}{2!} = 2\pi i$$

• NEED TO SHOW THAT THE INTEGRAL OVER THE ARC $[0, R]$ GOES TO ZERO AS $R \rightarrow \infty$

$$\left| \int_R^\infty \frac{1}{1+z^2} dz \right| = \left| \int_0^\infty \frac{(z+R)^{-1} dz}{1+(z+R)^2} \right| \leq \int_0^\infty \frac{|z+R|^{-1} dz}{1+(z+R)^2} = \int_0^\infty \frac{\frac{R}{z+R}}{1+\frac{R^2}{z+R}} dz = \int_0^\infty \frac{R}{R^2+1} dz = \frac{R}{R^2+1} \int_0^\infty 1 dz$$

$$= \frac{\pi R}{R^2+1} \xrightarrow[R \rightarrow \infty]{} 0$$

• Hence as $R \rightarrow \infty$

$$\int_{-R}^R \frac{1}{1+z^2} dz = \pi i$$

$$2 \int_0^R \frac{1}{1+z^2} dz = \pi i$$

$$\int_0^\infty \frac{1}{1+z^2} dz = \frac{\pi i}{2}$$

$$|w+z| > |w|-|z|$$

$$\frac{1}{|w+z|} < \frac{1}{|w|-|z|}$$

$$\frac{1}{|w+z|} \leq \frac{1}{|w|-|z|}$$

Question 3

By integrating a suitable complex function over an appropriate contour find

$$\int_0^\infty \frac{1}{(x^2+4)^2} dx.$$

, $\frac{\pi}{32}$

CONTOUR $\int_P \frac{1}{(z^2+4)^2} dz$ (CROSS THE SEMICIRCLE COUNTER CLOCKWISE)

$\gamma_R(z) = Re^{i\theta}$
Circ. π
 $z = 2e^{i\theta}$
 $dz = ie^{i\theta} d\theta$

THE INTEGRAND HAS POLES AT $z = \pm 2i$; ONLY THE ONE AT $+2i$ IS INSIDE P — CALCULATE THE RESIDUE OF THIS POLE

$$\begin{aligned} \lim_{z \rightarrow 2i} \left[\frac{d}{dz} (z-2i)^2 f(z) \right] &= \lim_{z \rightarrow 2i} \left[\frac{d}{dz} \left(\frac{1}{(z-2i)^2 (z+2i)^2} \right) \right] \\ &= \lim_{z \rightarrow 2i} \left[\frac{d}{dz} \left(\frac{1}{(z-2i)^2} \right) \right] = \lim_{z \rightarrow 2i} \left[-\frac{2}{(z-2i)^3} \right] \\ &= -\frac{2}{(4i)^3} = -\frac{2}{64i} = \frac{1}{32i} \end{aligned}$$

BY RESIDUE THEOREM

$$\begin{aligned} \Rightarrow \int_P \frac{1}{(z^2+4)^2} dz &= 2\pi i \times \sum \text{Residues inside } P \\ \Rightarrow \int_{-2i}^{2i} \frac{1}{(z^2+4)^2} dz + \int_{-R}^R \frac{1}{(z^2+4)^2} dz &= 2\pi i \times \frac{1}{32i} \\ \Rightarrow \int_0^\pi \frac{1}{(Re^{i\theta})^2 + 4} (Re^{i\theta} d\theta) + \int_R^2 \frac{1}{(z^2+4)^2} dz &= \frac{\pi}{16} \\ \Rightarrow \int_0^\pi \frac{1}{(R^2 e^{2i\theta} + 4)^2} d\theta + \int_R^2 \frac{1}{(z^2+4)^2} dz &= \frac{\pi}{16} \end{aligned}$$

NEXT CONSIDER THE CONTRIBUTION ALONG X AS $R \rightarrow \infty$

$$\begin{aligned} \left| \int_{-R}^R \frac{1}{(z^2+4)^2} dz \right| &= \left| \int_0^\pi \frac{1}{R^2 e^{2i\theta} + 4} d\theta \right| \\ &\leq \int_0^\pi \left| \frac{1}{R^2 e^{2i\theta} + 4} \right| d\theta = \int_0^\pi \frac{|Re^{i\theta}|}{|R^2 e^{2i\theta} + 4|} d\theta \\ &= \int_0^\pi \frac{1}{|Re^{i\theta}|^2 + |4e^{2i\theta}|} d\theta = \int_0^\pi \frac{1}{|Re^{i\theta}|^2 + 16} d\theta \\ &= \int_0^\pi \frac{1 \times R \times 1}{|Re^{i\theta}|^2 + 16} d\theta = \int_0^\pi \frac{R}{|Re^{i\theta}|^2 + 16} d\theta \\ &= \frac{R}{|Re^{i\theta}|^2 + 16} \int_0^\pi 1 d\theta = \frac{\pi R}{|Re^{i\theta}|^2 + 16} = O\left(\frac{1}{R}\right) \end{aligned}$$

where $|Re^{i\theta}| \geq |Re^{i\theta}| - 1$

Hence as $R \rightarrow \infty$

$$\begin{aligned} \int_{-\infty}^\infty \frac{1}{(z^2+4)^2} dz &= \frac{\pi}{4} \quad \text{CROSS CHECK} \\ 2 \int_0^\infty \frac{1}{(z^2+4)^2} dz &= \frac{\pi}{8} \\ \int_0^\infty \frac{1}{(x^2+4)^2} dx &= \frac{\pi}{32} \end{aligned}$$

Question 4

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{1+x^4} dx.$$

$$\frac{\pi\sqrt{2}}{4}$$

• CONSIDER $\int_{\gamma} \frac{1}{1+z^2} dz$ over the contour Γ shown below

$$y(\theta) = Re^{i\theta}, 0 \leq \theta \leq \pi$$

• LOOK FOR POLES
 $1+z^2=0$
 $z^2=-1 \Rightarrow z=i(\pm 1)$
 $z^2=1 \Rightarrow z=\pm i$
 $z_1 = i(\pm 1)$
 OF WHICH THE POLES ARE INSIDE Γ

• COMPUTE PERIODS AS $\lim_{R \rightarrow \infty} \int_{-R}^R \frac{2(z)(z-z_1)}{z^2-1} dz$

BY L'HOSPITAL'S RULE

$$\begin{aligned} &= \lim_{R \rightarrow \infty} \frac{\int_{-R}^R \frac{2(z)(z-i)}{z^2-1} dz}{\int_{-R}^R \frac{2(z)(z+i)}{z^2-1} dz} \\ &= \lim_{R \rightarrow \infty} \frac{\left[\frac{2z^2-2i}{3} \right]_{-R}^R - \lim_{R \rightarrow \infty} \left[\frac{2z^2+2i}{3} \right]_{-R}^R}{\left[\frac{2z^2+2i}{3} \right]_{-R}^R - \lim_{R \rightarrow \infty} \left[\frac{2z^2-2i}{3} \right]_{-R}^R} \\ &= \lim_{R \rightarrow \infty} \frac{\frac{8}{3} - \lim_{R \rightarrow \infty} \left(\frac{2R^2+2i}{3} - \frac{2R^2-2i}{3} \right)}{\frac{8}{3} + \lim_{R \rightarrow \infty} \left(\frac{2R^2+2i}{3} - \frac{2R^2-2i}{3} \right)} \\ &= \frac{\frac{8}{3}}{\frac{8}{3}} = 1 \end{aligned}$$

AT $z = e^{i\frac{\pi}{2}}$

$$\lim_{z \rightarrow e^{i\frac{\pi}{2}}} \left[\frac{1}{1+z^2}(z-e^{i\frac{\pi}{2}}) \right] = 0$$

\Rightarrow A GROWTH RATE OF $2\pi i$

BY L'HOSPITAL'S RULE

$$\begin{aligned} &= \lim_{z \rightarrow e^{i\frac{\pi}{2}}} \frac{\frac{1}{1+z^2}}{\frac{4z}{(1+z^2)^2}} = \frac{\frac{1}{4e^{i\pi/2}}}{\frac{4e^{i\pi/2}}{(1+e^{i\pi/2})^2}} = \frac{1}{4e^{i\pi/2}} \\ &= \frac{1}{4} [\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}] = \frac{1}{4} [0 - i(-1)] = \frac{1}{4} i(-1) \end{aligned}$$

• THIS IS BY THE RESIDUE THEOREM

$$\begin{aligned} &\Rightarrow \int_{\gamma} \frac{1}{1+z^2} dz = 2\pi i \times (\text{RESIDUES INSIDE } \Gamma) \\ &\Rightarrow \int_{\gamma} \left[\frac{1}{z-i} dz + \left[\frac{1}{z+i} dz \right] \right] = 2\pi i \times \left[\frac{i}{2} (1-(-i)) \right] \\ &= \text{PARAMETERIZE } \Gamma: z = e^{i\theta}, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \\ &\Rightarrow \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{e^{i\theta}+i} \frac{ie^{i\theta}}{1+e^{i2\theta}} d\theta = \frac{ie^{i\theta}}{4} \text{ (from } \int \frac{1}{1+e^{i2\theta}} d\theta = \frac{ie^{i\theta}}{4} \text{)} \\ &\Rightarrow \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{e^{i\theta}+i} \frac{ie^{i\theta}}{1+e^{i2\theta}} d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{e^{i\theta}+i} \frac{ie^{i\theta}}{1+e^{i2\theta}} db = -2i \times \frac{\sqrt{2}\pi^2}{4} \\ &\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{i\theta}+i} \frac{ie^{i\theta}}{1+e^{i2\theta}} d\theta + \int_{0}^{\pi} \frac{ie^{i\theta} \cdot ie^{i\theta}}{1+e^{i2\theta}} d\theta = \frac{\sqrt{2}\pi^2}{2} \end{aligned}$$

NOW CONSIDER THE $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$ OF THE INTEGRAL TEST.

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{-1} \frac{1}{1+x^2} dx + \int_{-1}^{1} \frac{1}{1+x^2} dx + \int_{1}^{\infty} \frac{1}{1+x^2} dx \\ &= \int_{-1}^{1} \frac{1}{1+x^2} dx < \int_{-1}^{1} \frac{1}{1+2|x|} dx \quad \text{[because } |x| \geq 1 \text{]} \\ &\quad \uparrow \\ & \boxed{|x+1| \geq |x|-1} \\ & \leq \int_{-1}^{1} \frac{1}{1-|x|} dx \\ &= \int_{-1}^{1} \frac{2}{2-4x} dx = \int_{-1}^{1} \frac{2}{1-4x} dx \\ &= -\frac{1}{2} \cdot \frac{1}{1-4x} \Big|_{-1}^{1} \rightarrow 0 \quad \text{as } x \rightarrow \infty \\ &\text{But AS } x \rightarrow -\infty \\ & \int_{-\infty}^{-1} \frac{1}{1+x^2} dx = \frac{\sqrt{x^2-1}}{2} \Big|_{-\infty}^{-1} = \infty \\ & \int_{-1}^{\infty} \frac{1}{1+x^2} dx = \frac{\sqrt{x^2-1}}{2} \Big|_{-1}^{\infty} = \infty \end{aligned}$$

Question 5

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)(x^2+1)^2} dx.$$

$$\boxed{\frac{\pi}{9}}$$

• CONSIDER $f(z) = \frac{1}{(z^2+4)(z^2+1)^2}$ close the contour Γ shown below

• $f(z)$ has simple poles at $z=2i$ & double poles at $z=i$, or where $z-i = 0$

• All within Γ

• COUNTER clockwise

• DOUBLE POLE AT i $\Rightarrow \lim_{z \rightarrow i} \left[\frac{d}{dz} \left(f(z)(z-i)^2 \right) \right] = \lim_{z \rightarrow i} \left[\frac{d}{dz} \left(\frac{(z-i)^2}{(z-2i)^2(z+i)^2} \right) \right] = \lim_{z \rightarrow i} \left[\frac{d}{dz} \left(\frac{1}{(z-2i)^2(z+i)^2} \right) \right]$

$$= \lim_{z \rightarrow i} \left[\frac{d}{dz} \left(\frac{1}{(z-2i)^2(z+i)^2} \right) \right] = \lim_{z \rightarrow i} \left[-2(z-2i)^{-3}(z+i)^2 - 2(z-2i)^2(z+i)^{-3} \right]$$

$$= \lim_{z \rightarrow i} \left[-2(z-2i)^{-3}(z+i)^2 \left[z(z+i) + (z^2+4) \right] \right] = \lim_{z \rightarrow i} \left[\frac{-2(z-2i)^{-3}(z+i)^2}{(z-2i)^2(z+i)^2} \right] = \frac{-2(-2-1)i}{9+(-2i)} = \frac{1}{3i}$$

SIMPLE POLE AT $2i$ $\Rightarrow \lim_{z \rightarrow 2i} \left[f(z)(z-2i) \right] = \lim_{z \rightarrow 2i} \left[\frac{(z-2i)}{z-2i} \right] = \frac{1}{4i+(-4i)} = \frac{1}{-8i}$

• BY THE RESIDUE THEOREM

$$\int_{\Gamma} |f(z)| dz = 2\pi i \times 5 (\text{Residues about } \Gamma)$$

$$\int_{\Gamma} \frac{1}{(z^2+4)(z^2+1)^2} dz + \int_{\gamma_R} \frac{1}{(z^2+4)(z^2+1)^2} dz = 2\pi i \times \left[\frac{1}{-8i} + \frac{1}{3i} \right]$$

NOW AS $R \rightarrow \infty$ THE CONTRIBUTION OF THE ARC γ_R IS ZERO

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)(x^2+1)^2} dx = \frac{\pi i}{3i}$$

$$\begin{aligned} |\text{arc}| &\geq R \sin \theta & \int_{\gamma_R} |f(z)| dz &\leq \int_{\gamma_R} \left| \frac{1}{(z^2+4)(z^2+1)^2} \right| dz \\ &\geq \int_{\gamma_R} \frac{1}{R^2 \sin^2 \theta} dz && \leq \frac{1}{R^2 \sin^2 \theta} \int_{\gamma_R} dz \\ &= \frac{1}{R^2 \sin^2 \theta} \cdot 2\pi R \sin \theta && \leq \frac{2\pi}{R^2 \sin^2 \theta} \cdot \frac{1}{\sin^2 \theta} \\ &= \frac{2\pi}{R^2} && \rightarrow 0 \quad \text{as } R \rightarrow \infty \end{aligned}$$

Question 6

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 4x + 5)^2} dx.$$

$$\boxed{\frac{\pi}{2}}$$

CONSIDER $\int_{\Gamma} \frac{1}{(z^2 + 4z + 5)^2} dz$ ONCE THE STANDARD SEMICIRCLE INWARD Γ , SHOWN BELOW

\bullet $z^2 + 4z + 5 = (z+2)^2 + 1 = (z+2-i)(z+2+i)$

\bullet THE INTEGRAND HAS SOURCE POLES OF $z = -2 \pm i$ OF WHICH ONLY THE ONE AT $z = -2 + i$ IS INSIDE Γ .
RESIDUE CALCULATION:

$$\begin{aligned} & \lim_{z \rightarrow -2+i} \left[\frac{d}{dz} \left[\frac{1}{(z+2-i)^2} \right] \right] \\ &= \lim_{z \rightarrow -2+i} \left[\frac{d}{dz} \left[\frac{(z+2-i)^2}{(z+2-i)^3} \right] \right] \\ &= \lim_{z \rightarrow -2+i} \left[\frac{d}{dz} \left[\frac{1}{(z+2-i)} \right] \right] \\ &= -\frac{2}{(2i)^2} = -\frac{2}{-4i} = \frac{1}{4i} \end{aligned}$$

\bullet BY THE RESIDUE THEOREM

$$\begin{aligned} \int_{\Gamma} f(z) dz &= 2\pi i \times \text{Sum of residues in } \Gamma \\ \int_{\Gamma} \frac{1}{(z^2 + 4z + 5)^2} dz &= 2\pi i \times \frac{1}{4i} \\ \int_{-R}^R \frac{1}{(z^2 + 4z + 5)^2} dz + \int_0^{\pi} \frac{1}{(Re^{i\theta} + 4Re^{i\theta} + 5)^2} iRe^{i\theta} d\theta &= \frac{\pi}{2} \\ \int_0^{\pi} \frac{1}{(Re^{i\theta} + 4Re^{i\theta} + 5)^2} iRe^{i\theta} d\theta &\leq \frac{\pi}{2} \quad \text{as } R \rightarrow \infty \\ \int_0^{\pi} \frac{1}{(Re^{i\theta} + 4Re^{i\theta} + 5)^2} iRe^{i\theta} d\theta &\leq \frac{2\pi}{2} \quad \text{as } R \rightarrow \infty \end{aligned}$$

\bullet ALSO CONSIDER THE CONTRIBUTION OF THE ARC γ , AS $R \rightarrow \infty$

$$\left| \int_0^{\pi} \frac{iRe^{i\theta}}{(Re^{i\theta} + 4Re^{i\theta} + 5)^2} d\theta \right| \leq \int_0^{\pi} \frac{|iRe^{i\theta}|}{|(Re^{i\theta} + 4Re^{i\theta} + 5)|^2} d\theta$$

Now $|Re^{i\theta}| \geq |z| - |w|$

$$\begin{aligned} & \frac{1}{|(Re^{i\theta})^2|} \leq \frac{1}{(|z| - |w|)^2} \\ & \leq \int_0^{\pi} \frac{R^2}{((R^2)^2 - (4R^2 + 5)^2)} d\theta \end{aligned}$$

$$\begin{aligned} &= \int_0^{\pi} \frac{R}{(R^2 - 4R^2 - 5)^2} d\theta = \frac{R}{(R^2 - 4R^2 - 5)^2} \int_0^{\pi} 1 d\theta \\ &= \frac{\pi R}{(R^2 - 4R^2 - 5)^2} = O\left(\frac{1}{R^3}\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty \\ \text{Thus as } R \rightarrow \infty \\ \int_{-\infty}^{\infty} \frac{1}{(x^2 + 4x + 5)^2} dx &= \frac{\pi}{2}. \end{aligned}$$

Question 7

Given that $k > 0$ find the exact value of

$$\int_{-\infty}^{\infty} \frac{x \cos kx}{x^2 + 2x + 5} dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + 2x + 5} dx.$$

, $\frac{1}{2}\pi e^{-2k}(2\sin k - \cos k)$, $\frac{1}{2}\pi e^{-2k}(\sin k + 2\cos k)$

CONSIDER $f(z) = \frac{e^{izk}}{z^2 + 2z + 5}$ OVER A STANDARD SEMICIRCLE
COUNTER-clockwise, SHOWN BELOW

$$z^2 + 2z + 5 = (z+1)^2 + 4 = (z+1)^2 - (2i)^2 = (z+1-2i)(z+1+2i)$$

$f(z)$ HAS SIMPLE POLES AT $-1 \pm 2i$, BUT ONLY THE ONE AT $-1 + 2i$ IS INSIDE Γ , SO WE NEED ITS RESIDUE

$$\lim_{z \rightarrow -1+2i} \left[(z+1-2i) \frac{e^{izk}}{(z+1+2i)(z+1-2i)} \right] = \frac{(e^{i(-1+2i)k})}{(2i)(-1+2i)} = \frac{e^{-k}(-2k)}{2i} (-1+2i) (cos k - isink)$$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times (\text{sum of residues inside } \Gamma)$$

$$\int_{-R}^R \frac{e^{izk}}{z^2 + 2z + 5} dz + \int_{\Gamma_R}^{\Gamma} \frac{e^{izk}}{z^2 + 2z + 5} dz \rightarrow 2\pi i \times \frac{e^{-k}}{2i} (-1+2i)(cos k - isink)$$

$$\int_{-R}^R \frac{e^{izk}}{z^2 + 2z + 5} dz + \int_{\Gamma_0}^{\Gamma} \frac{e^{izk}}{z^2 + 2z + 5} dz = \frac{\pi}{2} e^{-k} (-1+2i)(cos k - isink)$$

NOW AS $R \rightarrow \infty$, THE CONTRIBUTION OF THE INTEGRAL ALONG Γ_R IS ZERO BY JORDAN'S LEMMA WHICH STATES

If $|g(z)| \rightarrow 0$ as $|z| \rightarrow \infty$, $0 < \arg z = \pi$,
THEN $\int_{\Gamma_R} g(z) e^{izk} dz \rightarrow 0$ as $R \rightarrow \infty$,
SO LONG AS $R \rightarrow \infty$ & $T_R(k) = Re^{ik}$

[Hence $\lim_{R \rightarrow \infty} \int_{\Gamma_R}^{\Gamma} \frac{e^{izk}}{z^2 + 2z + 5} dz = 0$ as $|z| \rightarrow \infty$]

FINALLY WE HAVE AS $R \rightarrow \infty$

$$\int_{-\infty}^{\infty} \frac{e^{izk}}{z^2 + 2z + 5} dz = \frac{\pi}{2} e^{-k} (-1+2i)(cos k - isink)$$

$$\int_{-\infty}^{\infty} 2i e^{izk} dz = \frac{\pi}{2} e^{-k} [(-cos k + 2sink) + i(cos k + sink)]$$

SUMMING EQU & IMAGINARY

$$\int_{-\infty}^{\infty} \frac{2e^{izk}}{z^2 + 2z + 5} dz = \frac{\pi}{2} e^{-k} (2sink - cos k)$$

$$\int_{-\infty}^{\infty} \frac{2e^{izk}}{z^2 + 2z + 5} dz = \frac{\pi}{2} e^{-k} (sink + 2cos k)$$

Question 8

By integrating a suitable complex function over an appropriate contour find

a) ... $\int_0^\infty \frac{\cos ax}{x^2+b^2} dx, a > 0.$

b) ... $\int_0^\infty \frac{\cos ax}{x^2+b^2} dx, a < 0.$

$\frac{\pi e^{-ab}}{2b}, a > 0$	$\frac{\pi e^{ab}}{2b}, a < 0$
---------------------------------	--------------------------------

• CONSIDER $f(z) = \frac{e^{iaz}}{z^2+b^2}$, THE CONTOUR Γ WHICH CONSISTS

- $f(z)$ HAS 2 SIMPLE POLES AT $z = \pm bi$ WHICH ARE INSIDE Γ .
- $\lim_{z \rightarrow \infty} \left| \frac{f(z)}{z^2} \right| = 0$
- $\lim_{z \rightarrow \infty} \left| \frac{zf(z)}{z^2} \right| = 0$

• BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times (\text{RESIDUES INSIDE } \Gamma)$$

$$= \int_{-R}^R \frac{e^{iaz}}{z^2+b^2} dz + \int_R^R \frac{e^{iaz}}{z^2+b^2} dz = 2\pi i \times \frac{\pi e^{ab}}{b} = \frac{\pi e^{ab}}{b}$$

• AS $R \rightarrow \infty$, THE CONTRIBUTION OF THE ARC γ IS ZERO BY JORDAN'S LEMMA

$$\int_{\gamma} \frac{e^{iaz}}{z^2+b^2} dz = \frac{\pi e^{ab}}{b}$$

$$\int_{-R}^R \frac{\cos az}{z^2+b^2} dz + \int_R^R \frac{\cos az}{z^2+b^2} dz = \frac{\pi e^{ab}}{b}$$

$$2 \int_0^\infty \frac{\cos az}{z^2+b^2} dz = \frac{\pi e^{ab}}{b}$$

$$\int_0^\infty \frac{\cos az}{z^2+b^2} dz = \frac{\pi e^{ab}}{2b} \quad a > 0$$

JORDAN'S LEMMA
IF $|f(z)| \rightarrow 0$ AS $|z| \rightarrow \infty$, AND $\arg z = \pi$ AND $M > 0$
THEN $\int_{\gamma} f(z) e^{iaz} dz \rightarrow 0$ AS $R \rightarrow \infty$ WHERE γ IS $R e^{i\pi} - R e^{i\theta}$ ON THE

• WE NOW CONSIDER $f(z) = \frac{e^{iaz}}{z^2+b^2}$ FOR $a < 0$. HERE THE CONTOUR Γ CONSISTS

- $f(z)$ HAS 4 POLES AT $z = \pm bi$ WHICH ARE INSIDE Γ . AND THIS REQUIRES
- $\lim_{z \rightarrow \infty} \left[\frac{f(z)}{z^2} \right] = 0$
- $\lim_{z \rightarrow \infty} \left[\frac{zf(z)}{z^2} \right] = 0$

NOTE THAT Γ IS NOW TRADED COUNTERCLOCKWISE

• BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = -2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

↑ NOTE THE NEGATIVE AS Γ IS TRADED COUNTERCLOCKWISE

$$\int_{-R}^R \frac{e^{iaz}}{z^2+b^2} dz + \int_R^R \frac{e^{iaz}}{z^2+b^2} dz = -2\pi i \times \frac{\pi e^{ab}}{-2b} = -\frac{\pi e^{ab}}{b}$$

• AS $R \rightarrow \infty$, THE CONTRIBUTION OF THE ARC γ IS ZERO BY JORDAN'S LEMMA

$$\int_{\gamma} \frac{e^{iaz}}{z^2+b^2} dz = \frac{\pi e^{ab}}{b}$$

$$\int_{-R}^R \frac{\cos az}{z^2+b^2} dz + \int_R^R \frac{\cos az}{z^2+b^2} dz = \frac{\pi e^{ab}}{b}$$

$$2 \int_0^\infty \frac{\cos az}{z^2+b^2} dz = \frac{\pi e^{ab}}{b}$$

$$\int_0^\infty \frac{\cos az}{z^2+b^2} dz = \frac{\pi e^{ab}}{2b} \quad a < 0$$

JORDAN'S LEMMA IN THIS CASE
IF $|f(z)| \rightarrow 0$ AS $|z| \rightarrow \infty$ REVERSEWISE, I.E. $z \rightarrow \infty$
THEN $\int_{\gamma} f(z) e^{iaz} dz \rightarrow 0$ AS $R \rightarrow \infty$ WHERE γ IS $R e^{i\pi} - R e^{i\theta} \rightarrow 0$

Question 9

By integrating a suitable complex function over an appropriate contour find

$$\int_0^\infty \frac{x \sin ax}{x^2 + b^2} dx, \quad a > 0.$$

$$\frac{1}{2} \pi e^{-ab}$$

Consider $f(z) = \frac{ze^{iaz}}{z^2 + b^2}, \quad a > 0$, outer
contour Γ .
f(z) has simple poles at $\pm bi$, or in the
 $z = bi$ is inside Γ , with residue
 $\lim_{z \rightarrow bi} \left[\frac{ze^{iaz}}{(z - bi)(z + bi)} \right] =$
 $= \lim_{z \rightarrow bi} \left[\frac{ze^{iaz}}{2bi} \right] = \frac{ib e^{-ab}}{2bi} = \frac{1}{2} e^{-ab}$

By the residue theorem
 $\int_{\Gamma} f(z) dz = 2\pi i \times (\text{residues inside } \Gamma)$
 $\int_{-R}^R \frac{ze^{iaz}}{z^2 + b^2} dz + \int_{\gamma_R} \frac{ze^{iaz}}{z^2 + b^2} dz = 2\pi i \times \frac{1}{2} e^{-ab}$

As $R \rightarrow \infty$, the contribution of the horizontal line γ is zero, by Jordan's lemma
 $\lim_{R \rightarrow \infty} \int_{-R}^R \frac{ze^{iaz}}{z^2 + b^2} dz = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{ze^{iaz}}{z^2 + b^2} dz + i \int_{-\infty}^{\infty} \frac{ze^{izb}}{z^2 + b^2} dz = i\pi e^{-ab}$
 $i\int_{-\infty}^{\infty} \frac{ze^{izb}}{z^2 + b^2} dz = i\pi e^{-ab}$
 $\int_0^{\infty} \frac{ze^{izb}}{z^2 + b^2} dz = \frac{\pi}{2} e^{-ab}$

Jordan's lemma
If $f(z) \rightarrow 0$ as $|z| \rightarrow \infty$ along $2\pi i T$, & in
then $\int_{\gamma_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$, where $\gamma_R(b) = Re^{i\theta}, 0 \leq \theta \leq \pi$

Question 10

By integrating a suitable complex function over an appropriate contour find

$$\int_0^\infty \frac{(1-x^2)\cos \alpha x}{(1+x^2)^2} dx, \quad \alpha > 0.$$

$$\boxed{\frac{1}{2}\pi\alpha e^{-\alpha}}$$

• Consider $f(z) = \frac{(1-z^2)e^{izx}}{(1+z^2)^2}$, $\alpha > 0$. Out the contour Γ :

• $f(z)$ has double poles at $\pm i$, of which only the one at i is inside Γ . Calculate its residue:

$$\begin{aligned} \lim_{z \rightarrow i} \frac{d}{dz} [(z-i)^2 \frac{(1-z^2)e^{izx}}{(1+z^2)^2}] &= \lim_{z \rightarrow i} \frac{d}{dz} \left[\frac{(1-z^2)e^{izx}}{(z+i)^2} \right] \\ &= \lim_{z \rightarrow i} \left[\frac{(2i)(z-2i)e^{izx}}{(z+i)^3} + i(z-2i)e^{izx} - (z-2i)^2 e^{izx} \cdot 2(z+i) \right] \\ &= \lim_{z \rightarrow i} \left[\frac{(2i)^2 e^{izx} [(z-2i)-2i]}{(z+i)^3} - 2(z-2i)^2 e^{izx} \right] \\ &= \frac{2i e^{ix} [2i(-2i)] - 2 \cdot 2i^2 e^{ix}}{(2i)^3} = \frac{-8i e^{ix}}{-8i} = \frac{i e^{ix}}{2i}. \end{aligned}$$

• By the residue theorem

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum(\text{residues inside } \Gamma)$$

$$\int_{-R}^R \frac{(1-x^2)e^{ixx}}{(1+x^2)^2} dx + \int_R^{Re(i)} \frac{(1-z^2)e^{izx}}{(1+z^2)^2} dz \approx 2\pi i \times \frac{e^{ix}}{2i}$$

• As $R \rightarrow \infty$, the contribution of the integral over γ is zero, as it satisfies Jordan's lemma.

• Thus we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{(1-x^2)e^{ixx}}{(1+x^2)^2} dx &= \pi x e^{-ix} \\ \underbrace{\int_{-\infty}^0 \frac{(1-x^2)\cos ixz}{(1+x^2)^2} dz}_0 &+ i \int_{\infty}^{\infty} \frac{(1-x^2)\sin ixz}{(1+x^2)^2} dz = \pi x e^{-ix} \\ 2 \int_0^{\infty} \frac{(1-x^2)\cos ixz}{(1+x^2)^2} dz &= \pi x e^{-ix} \\ \int_0^{\infty} \frac{(1-x^2)\cos ixz}{(1+x^2)^2} dz &= \frac{\pi}{2} x e^{-ix} \end{aligned}$$

Question 11

By integrating a suitable complex function over an appropriate contour find

$$\int_0^\infty \frac{\cos ax}{(x^2 + b^2)^2} dx, \quad a > 0.$$

$$\boxed{\frac{\pi e^{-ab}(ab+1)}{4b^3}}$$

• CONSIDER $\int_{\Gamma} f(z) dz$ where $f(z) = \frac{e^{iz}}{(z^2 + b^2)^2}$.
USE THE COMPLEX PLANE CONCEPT.
• f(z) HAS POLES AT z = bi, OF WHICH THE ONE AT z = -bi IS INSIDE Γ .

• CALCULATING THE RESIDUE OF THIS POLE

$$\begin{aligned} \lim_{z \rightarrow bi} \frac{d}{dz} \int_{\Gamma} f(z) dz &= \lim_{z \rightarrow bi} \frac{d}{dz} \left[\int_0^{\text{arg } z} \frac{e^{iz}}{(z^2 + b^2)^2} dz \right] = \lim_{z \rightarrow bi} \left[\frac{d}{dz} \left(\int_0^{\text{arg } z} \frac{e^{iz}}{(z^2 + b^2)^2} dz \right) \right] \\ &= \lim_{z \rightarrow bi} \frac{\partial}{\partial z} \left(\frac{e^{iz}}{(z^2 + b^2)^2} \right) = \lim_{z \rightarrow bi} \left[\frac{(ie^{iz})(z^2 + b^2)^2 - 2z(e^{iz})(2z)(z^2 + b^2)}{(z^2 + b^2)^4} \right] \\ &= \lim_{z \rightarrow bi} \frac{e^{iz}[(z^2 + ab)^2 - 2z^2]}{(z^2 + b^2)^3} = \frac{e^{-b^2}[-2ab^2 - 2]}{(2b^2)^3} = \frac{-e^{-b^2}(2ab + 2)}{8b^5} \\ &= \frac{(ab+1)e^{-ab}}{4b^3} \end{aligned}$$

• BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{residues inside } \Gamma)$$

$$\int_{-R}^R \frac{e^{ix}}{(x^2 + b^2)^2} dx + \int_{\Gamma_R}^{\infty} \frac{e^{iz}}{(z^2 + b^2)^2} dz = 2\pi i \times \frac{(ab+1)e^{-ab}}{4b^3}$$

AS $R \rightarrow \infty$ THE CONTRIBUTION OF THE ARC γ' IS ZERO BY JORDAN'S LEMMA

$$\begin{aligned} \int_{-\infty}^0 \frac{e^{iz}}{(z^2 + b^2)^2} dz &= \frac{-ie^{-ab}}{2b^3} (ab+1) \\ \int_{-\infty}^0 \frac{\cos ax}{(z^2 + b^2)^2} dz + \int_0^{\infty} \frac{\cos ax}{(z^2 + b^2)^2} dz &= \frac{i e^{-ab}}{2b^3} (ab+1) \\ 2 \int_0^{\infty} \frac{\cos ax}{(z^2 + b^2)^2} dz &= \frac{-ie^{-ab}}{2b^3} (ab+1) \\ \int_0^{\infty} \frac{\cos ax}{(z^2 + b^2)^2} dz &= \frac{i e^{-ab}}{4b^3} (ab+1) \end{aligned}$$

LEADS TO

IF $f(z) \rightarrow 0$ AS $|z| \rightarrow \infty$, $0 \leq \arg z \leq \pi$ & $\gamma(0) = \gamma(\infty)$, $0 \leq \theta \leq \pi$
THEN $\int_X^{\infty} e^{iz} f(z) dz \rightarrow 0$ AS $R \rightarrow \infty$ $\int_R^{\infty} f(z) dz$

Question 12

By integrating a suitable complex function over an appropriate contour find

$$\int_0^\infty \frac{\cos x}{1+x^6} dx.$$

$$\boxed{\frac{\pi}{6e} \left[1 + \sqrt{e} \left[\cos\left(\frac{\sqrt{3}}{2}\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}\right) \right] \right]}$$

• CONSIDER $f(z) = \frac{e^{iz}}{z^6+1}$ ON THE CONTOUR Γ' SHOWN OPPOSITE

• $f(z)$ HAS 6 SIMPLE POLES: $z^6 = -1 \Rightarrow e^{i(2k+2\pi)} = e^{i(2k+2\pi)}$
 $z = e^{i(2k+2\pi)/6}$

OF THESE POLES: $z = e^{i\pi/6}, e^{i5\pi/6} = i^{\pm 1/6}$ ARE INSIDE Γ'

• TO CALCULATE RESIDUES, USE $\lim_{z \rightarrow z_0} [f(z)(z-z_0)]$

CONSIDER A GENERAL CASE:

$\lim_{z \rightarrow z_0} \left[\frac{e^{iz}}{z^6+1} (z-z_0) \right] = \frac{0}{0}$ SINCE $(z-z_0)$ IS A FACTOR OF $1+z^6$

USING L'HOSPITAL RULE

$= \lim_{z \rightarrow z_0} \left[\frac{i e^{iz} (z-z_0) + z^5 e^{iz}}{6z^5} \right] = \frac{i z_0}{6z_0^5}$ (SINCE $z=z_0$ OVER ZERO)

HENCE

POLE AT $z=i^{\pm 1/6}$ HAS RESIDUE: $\frac{i(e^{iz})}{6z^5} = \frac{i(\cos(\pm \pi/6) + i\sin(\pm \pi/6))}{6e^{\pm i\pi/6}} = \frac{i\sqrt{3}-i}{6(1+\sqrt{3})} = \frac{-i}{6(1+\sqrt{3})}$

POLE AT $z=i^{\mp 1/6}$ HAS RESIDUE: $\frac{i(e^{iz})}{6z^5} = \frac{i(\cos(\mp \pi/6) + i\sin(\mp \pi/6))}{6e^{\mp i\pi/6}} = \frac{i\sqrt{3}+i}{6(1+\sqrt{3})} = \frac{i}{6(1+\sqrt{3})}$

POLE AT $z=0$ HAS RESIDUE: $\frac{i(e^{iz})}{6z^5} = \frac{i}{6i} = \frac{1}{6i}$

• NOW BY THE RESIDUE THEOREM: $\int_0^\infty f(x) dx = 2\pi i \times \sum \text{(residues inside } \Gamma')$

• SINCE

$$\int_0^\infty \frac{e^{ix}}{1+x^6} dx + \int_{\Gamma'} \frac{e^{iz}}{1+z^6} dz = 2\pi i \times \left[\frac{\frac{-i}{6(1+\sqrt{3})}}{3(1+\sqrt{3})} + \frac{\frac{i}{6(1+\sqrt{3})}}{3(1+\sqrt{3})} + \frac{1}{6i} \right]$$

AS $z \rightarrow \infty$, THE CONTRIBUTION OF THIS ARC γ' IS ZERO BY JORDAN'S LEMMA

$$\int_0^\infty \frac{e^{ix}}{1+x^6} dx = 2\pi i \times \frac{-i}{3(1+\sqrt{3})} \left[\frac{i\sqrt{3}}{1-i\sqrt{3}} + \frac{i\sqrt{3}}{1+i\sqrt{3}} \right] + \frac{1}{6i}$$

$$\int_0^\infty \frac{e^{iz}}{1+z^6} dz = \frac{2\pi i}{3} \left[\frac{d(-i\sqrt{3})e^{i\sqrt{3}}}{4} + \frac{d(i\sqrt{3})e^{-i\sqrt{3}}}{4} \right] + \frac{i}{3e}$$

$$2 \int_0^\infty \frac{e^{iz}}{1+z^6} dz = \frac{\pi i}{6} \left[i\left(e^{i\sqrt{3}} - e^{-i\sqrt{3}}\right) - i\left(e^{-i\sqrt{3}} - e^{i\sqrt{3}}\right) \right] + \frac{i}{3e}$$

$$2 \int_0^\infty \frac{e^{iz}}{1+z^6} dz = -\frac{\pi i}{6} \left[2\sinh(1\sqrt{3}) + 2i\sinh(1\sqrt{3}) \right] + \frac{i}{3e}$$

$$2 \int_0^\infty \frac{e^{iz}}{1+x^6} dz = -\frac{\pi i}{3} \left[i\cos(1\sqrt{3}) + i^2 \sin(1\sqrt{3}) \right] + \frac{i}{3e}$$

$$2 \int_0^\infty \frac{e^{iz}}{1+x^6} dz = -\frac{\pi i}{3} \left[(\cos(1\sqrt{3}) + i\sin(1\sqrt{3})) \right] + \frac{i}{3e}$$

$$\int_0^\infty \frac{e^{iz}}{1+x^6} dz = \frac{\pi}{6e} \left[1 + \sqrt{e} \left[\cos\left(\frac{\sqrt{3}}{2}\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}\right) \right] \right]$$

JORDAN'S LEMMA

$\int_0^\infty f(x) dx \rightarrow 0$ AS $|z| \rightarrow \infty$ $\Rightarrow \arg(z) \leq \pi$

THUS $\int_0^\infty \frac{e^{iz}}{1+x^6} dz \rightarrow 0 \rightarrow 0 \rightarrow 2 \rightarrow \infty$ $\Rightarrow f(x) = \frac{1}{6e}$

Question 13

By integrating a suitable complex function over an appropriate contour find an exact simplified value for

$$\int_{-\infty}^{\infty} \frac{1}{ax^2 + bx + c} dx,$$

where a , b and c are real constants such that $a > 0$ and $b^2 - 4ac < 0$.

V, $\boxed{\quad}$, $\frac{2\pi}{\sqrt{4ac - b^2}}$

CONSIDER THE INTEGRAL OF $f(z) = \frac{1}{az^2+bz+c}$ AROUND THE CONTOUR Γ

STATION BELOW

USE THE QUADRATIC FORMULA ON THE DENOMINATOR TO FIND THE POLES

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a} \quad \text{WITH } \Delta = b^2 - 4ac < 0$$

$$z_1 = \frac{-b + \sqrt{-\Delta}}{2a}, \quad z_2 = \frac{-b - \sqrt{-\Delta}}{2a}$$

ONE OF THE POLES IS ON THE "TOP" HALF OF THE PLANE AND THE OTHER AT THE BOTTOM HALF - POLE R SURPASSING LENGTH, SO THE POLE AT $z = z_2$ AT THE TOP HALF IS INSIDE Γ

$$\operatorname{Res}[f, z_2] = \lim_{z \rightarrow z_2} [(z - z_2)f(z)] = \lim_{z \rightarrow z_2} \left[\frac{z - z_2}{az^2 + bz + c} \right]$$

BY L'HOSPITAL RULE AS THIS WILL BE A ZERO OVER ZERO LIMIT

$$= \lim_{z \rightarrow z_2} \left[\frac{1}{2az + b} \right] = \frac{1}{2az_2 + b} = \frac{1}{2a\left(\frac{-b + \sqrt{-\Delta}}{2a}\right) + b}$$

$$= \frac{1}{2a\sqrt{-\Delta}} = \frac{1}{\sqrt{4ac - b^2}}$$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} \frac{1}{az^2 + bz + c} dz = 2\pi i \times \sum \text{ (sum of residues of } f \text{ inside } \Gamma)$$

$$\left\{ \int_{-R}^R + \int_{\Gamma_R} \right\} \frac{1}{az^2 + bz + c} dz = 2\pi i \times \frac{1}{\sqrt{4ac - b^2}} = \frac{2\pi i}{\sqrt{4ac - b^2}}$$

PARAMETRIZE AROUND Γ AS $z = Re^{i\theta}$ & ROW 0 TO Γ $dz = iRe^{i\theta}d\theta$

$$\left| \int_{\Gamma} f(z) dz \right| = \left| \int_0^\pi \frac{1}{aR^2e^{2i\theta} + bRe^{i\theta} + c} (iRe^{i\theta} d\theta) \right| = \left| \int_0^\pi \frac{iRe^{i\theta}}{aR^2e^{2i\theta} + bRe^{i\theta} + c} d\theta \right|$$

$$\leq \int_0^\pi \left| \frac{iRe^{i\theta}}{aR^2e^{2i\theta} + bRe^{i\theta} + c} \right| d\theta = \int_0^\pi \frac{|iR| e^{|\theta|}}{|aR^2e^{2i\theta} + bRe^{i\theta} + c|} d\theta$$

$$= \int_0^\pi \frac{R}{|aR^2e^{2i\theta} + bRe^{i\theta} + c|} d\theta \dots \text{ STANDARD INEQUALITY...}$$

$$\leq \int_0^\pi \frac{R}{|aR^2e^{2i\theta} - bRe^{i\theta}|} d\theta \quad \text{IF } |aR^2e^{2i\theta} - bRe^{i\theta}| > 0$$

$$= \int_0^\pi \frac{R}{|aR^2e^{2i\theta}| - |bRe^{i\theta}|} d\theta$$

$$= \int_0^\pi \frac{R}{aR^2 - bR} d\theta = \frac{R}{aR^2 - bR} \int_0^\pi 1 d\theta$$

$$= \frac{\pi R}{aR^2 - bR} = O\left(\frac{1}{R}\right) \rightarrow 0 \text{ AS } R \rightarrow \infty$$

FURTHER AS $R \rightarrow \infty$ WE HAVE

$$\int_{-R}^R \frac{1}{az^2 + bz + c} dz = \frac{2\pi i}{\sqrt{4ac - b^2}}$$

Question 14

$$I = \int_0^\infty \frac{\ln(x^2+1)}{x^2+1} dx.$$

By integrating $\frac{\ln(z+i)}{z^2+1}$ over a semicircular contour find the exact value of I .

$$I = \pi \ln 2$$

Consider $\int_{\Gamma} \frac{\ln(z+i)}{z^2+1} dz$ over the standard semicircular contour Γ , shown below.

(*) Use a branch point at $z = -i$, but the branch cut does not affect the integration.

- If z is a simple pole inside Γ , ($z = i$) very difficult.
 $\lim_{z \rightarrow i} [(z-i)^{-1} \ln(z+i)] = \frac{\ln(2i)}{2i} = \frac{1}{2i} [\ln 2 + \frac{\pi i}{2}]$
- By the residue theorem
 $\int_{\Gamma} f(z) dz = 2\pi i \times \text{sum of residues inside } \Gamma$
 $\int_{\Gamma} \frac{\ln(z+i)}{z^2+1} dz = 2\pi i \left[\frac{1}{2i} (\ln 2 + \frac{\pi i}{2}) \right]$
- FIRST consider the contribution of $f(z)$ onto the arc $\gamma_R(i)$ as $R \rightarrow \infty$

This as $R \rightarrow \infty$

$$\int_{-\infty}^{\infty} f(z) dz = -\pi i \ln 2 + i \frac{\pi^2}{2}$$

We need to parametrise on the arc

$$\int_0^\infty \frac{\ln(z+i)}{z^2+1} dz + \int_{-\infty}^0 \frac{\ln(-z+i)}{z^2+1} dz = -\pi i \ln 2 + i \frac{\pi^2}{2}$$

$$\int_0^\infty \frac{\ln(z+i)}{z^2+1} dz + \frac{\ln(z+i)}{z^2+1} dz = -\pi i \ln 2 + i \frac{\pi^2}{2}$$

$$\int_0^\infty \frac{\ln(z+i)(z+i)}{z^2+1} dz = \pi i \ln 2 + i \frac{\pi^2}{2}$$

$$\int_0^\infty \frac{\ln((z+i)^2)}{z^2+1} dz = -\pi i \ln 2 + i \frac{\pi^2}{2}$$

$$\int_0^\infty \frac{\ln(-z^2-2z)}{z^2+1} dz = \pi i \ln 2 + i \frac{\pi^2}{2}$$

$$\int_0^\infty \frac{\ln(-z^2)}{z^2+1} dz = \pi i \ln 2 + i \frac{\pi^2}{2}$$

$$\int_0^\infty \frac{\ln[(z+i)^2]}{z^2+1} dz = \pi i \ln 2 + i \frac{\pi^2}{2}$$

$$\int_0^\infty \frac{\ln[(z+i)^2 + i\pi]}{z^2+1} dz = \pi i \ln 2 + i \frac{\pi^2}{2}$$

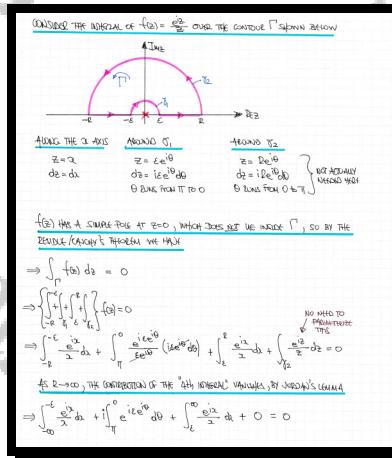
SEMI CIRCLE CONTOUR WITH HOLE

Question 1

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

V, [M.R], proof



NOTICE AS $\epsilon \rightarrow 0$, $e^{i\epsilon\theta} \rightarrow 1$, SO WE HAVE

$$\rightarrow \int_{-\pi}^0 \frac{e^{i\theta}}{\theta} d\theta + 1 \int_{-\pi}^0 \frac{1}{\theta} d\theta + \int_0^\infty \frac{\sin x}{x} dx = 0$$

$$\rightarrow \int_{-\pi}^0 \frac{e^{i\theta}}{\theta} d\theta + i(-\pi) = 0$$

$$\rightarrow \int_{-\pi}^0 \frac{\cos \theta + i \sin \theta}{\theta} d\theta = \pi i$$

$$\rightarrow \int_{-\pi}^0 \frac{\cos \theta}{\theta} d\theta + i \int_{-\pi}^0 \frac{\sin \theta}{\theta} d\theta = \pi i$$

$$\rightarrow \int_{-\pi}^0 \frac{\sin \theta}{\theta} d\theta = \pi$$

AND SINCE $\frac{\sin \theta}{\theta}$ IS ODD

$$\Rightarrow 2 \int_0^\infty \frac{\sin x}{x} dx = \pi$$

$$\rightarrow \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Question 2

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^\infty \frac{\sin x}{x(x^2 + a^2)} dx = \frac{\pi}{2a^2} (1 - e^{-a}).$$

V, , proof

CONSIDER $f(z) = \frac{e^{iz}}{z(z^2 + a^2)}$. INTEGRATE ALONG THE CONTOUR SHOWN BELOW.

$f(z)$ HAS SIMPLE POLES AT $z = ai$ & $z = -ai$, OF WHICH ONLY THE ONE AT $z = ai$ IS INSIDE Γ — COMPUTE THE RESIDUE.

$\text{Res}[f(z); z = ai] = \lim_{z \rightarrow ai} [(z - ai)f(z)] = \lim_{z \rightarrow ai} \left[\frac{e^{iz}}{z^2 - a^2} \right] = \frac{e^{iai}}{2iai} = \frac{e^{-a}}{2ia}$

APPLY THE RESIDUE THEOREM:

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum \{\text{residues of } f \text{ inside } \Gamma\}$$

$$\int_{\Gamma} \left(\frac{e^{iz}}{z^2 - a^2} \right) dz = 2\pi i \times \frac{e^{-a}}{2ia}$$

GO TO $2 \rightarrow \infty$ & LOOKING AT THE INTEGRAL OF $f(z)$ ONCE γ_1 :

$$\int_{\gamma_1} \frac{e^{iz}}{z^2 - a^2} dz \rightarrow 0 \text{ AS } 2 \rightarrow \infty \text{ BY JORDAN'S LEMMA}$$

JORDAN'S LEMMA:
IF $|f(z)| \rightarrow 0$ UNIFORMLY AS $|z| \rightarrow \infty$, FOR $0 < \arg z \leq \pi$,
THEN $\int_{\gamma_R} f(z) dz \rightarrow 0 \rightarrow 2 \rightarrow \infty$, WHERE γ_R IS DEFINED AS $\gamma_R = \gamma_1 \cup \gamma_2$.

NEXT CONSIDER $f(z)$ INTEGRATED OVER γ_2 — PUNCTUATING FIRST

$$\begin{aligned} \int_{\gamma_2} f(z) dz &= \int_0^\pi \frac{e^{i(\alpha e^{i\theta})}}{\alpha e^{i\theta}(z^2 - a^2)} (ie^{i\theta} d\theta) \\ &= -i \int_0^\pi \frac{e^{i\alpha e^{i\theta}}}{\alpha^2 e^{2i\theta} + a^2} d\theta \end{aligned}$$

NOW AS $\epsilon \rightarrow 0$ THIS INTEGRAL TENDS TO

$$-i \int_0^\pi \frac{1}{a^2} d\theta = -\frac{\pi i}{a^2}$$

FINALLY AS $R \rightarrow \infty$ & $\epsilon \rightarrow 0$ WE HAVE — NOT $z = 2$ & $z = -2$:

$$\begin{aligned} &\Rightarrow \int_{\gamma_2} \frac{e^{iz}}{z^2 - a^2} dz + \int_0^\pi \frac{e^{i\alpha e^{i\theta}}}{\alpha^2 e^{2i\theta} + a^2} d\theta - \frac{\pi i}{a^2} = \frac{2\pi i}{a^2}; \\ &\Rightarrow \int_0^\pi \frac{\alpha e^{i\alpha} \cdot e^{i\theta}}{\alpha^2 e^{2i\theta} + a^2} d\theta = \left(\frac{2\pi e^a}{a^2} + \frac{\pi i}{a^2} \right); \\ &\Rightarrow \int_0^\pi \frac{\cos \alpha}{\alpha^2 e^{2i\theta} + a^2} d\theta + i \int_0^\pi \frac{-\sin \alpha}{\alpha^2 e^{2i\theta} + a^2} d\theta = \frac{\pi}{a^2} (1 - e^{-a}); \\ &\Rightarrow \int_0^\pi \frac{\cos \alpha}{\alpha^2 e^{2i\theta} + a^2} d\theta = \frac{\pi}{a^2} (1 - e^{-a}) \\ &\Rightarrow 2 \int_0^\pi \frac{\cos \alpha}{\alpha^2 e^{2i\theta} + a^2} d\theta = \frac{\pi}{a^2} (1 - e^{-a}) \end{aligned}$$

$\therefore \int_0^\infty \frac{\sin x}{x(x^2 + a^2)} dx = \frac{\pi}{2a^2} (1 - e^{-a})$

Question 3

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^\infty \frac{1-\cos x}{x^2} dx = \frac{\pi}{2}.$$

M1, proof

CONSIDER A CONCIAL INTEGRAL FOR $f(z) = \frac{1-e^{iz}}{z^2}$ OVER A SEMICIRCULAR CONTOUR Γ , INWARD AT THE ORIGIN.

- ALONG THE z_1 -ARC: $z = Re^{i\theta}$, $dz = iRe^{i\theta}d\theta$
- ALONG Γ_1 : $z = Re^{i\theta}$, $dz = iRe^{i\theta}d\theta$, θ FROM 0 TO π
- ALONG Γ_2 : $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$, θ FROM 0 TO π

THERE ARE NO POLES INSIDE Γ_1 , BUT ON THE LIMIT AS $R \rightarrow 0$, THE POLE AT $z=0$ WILL FALL OUTSIDE THE BOUNDARY OF Γ . THIS IS KNOWN AS AN EXCEPTIONAL CASE WHERE CONTRIBUTION IS NOT ITS RESIDUE. THE REAL APPROACH IS TO SHOW THAT THE CONTRIBUTION OF $f(z)$ OVER THE BOUNDARY VANISHES AS $R \rightarrow \infty$, BUT FOR THIS PARTICULAR $f(z)$ THIS IS HARD TO SEE! — THE ALTERNATIVE APPROACH IS TO CALCULATE THE ACTUAL CONTRIBUTION OF Γ_2 & USE CAUCHY'S THEOREM (OR DENSITY = ZERO) TO GET THIS VALUE.

BY THE RESIDUE THEOREM (OR CAUCHY'S THEOREM)

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{residues inside } \Gamma) = 0$$

$$\left\{ \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{z_2} \right\} \left[\frac{1-e^{iz}}{z^2} \right] dz = 0$$

START WITH THE CONTRIBUTION OF Γ_2

$$\int_{\Gamma_2} \frac{1-e^{iz}}{z^2} dz = \int_{\Gamma_2} \frac{1}{z^2} dz - \int \frac{e^{iz}}{z^2} dz$$

ESTIMATE EACH OF THESE TWO CONTRIBUTIONS SEPARATELY

- $\left| \int_{\Gamma_2} \frac{1}{z^2} dz \right| = \left| \int_0^\pi \frac{1}{R^2 e^{2i\theta}} (Re^{i\theta}) d\theta \right| = \left| \int_0^\pi \frac{1}{R e^{i\theta}} d\theta \right| < \int_0^\pi \left| \frac{1}{R e^{i\theta}} \right| d\theta$

$$= \int_0^\pi \frac{1}{|R| e^{i\theta}} d\theta = \int_0^\pi \frac{1}{R} d\theta = \frac{1}{R} \rightarrow 0 \text{ as } R \rightarrow \infty$$
- $\int_{\Gamma_2} \frac{e^{iz}}{z^2} dz \rightarrow 0$ AS $R \rightarrow \infty$, BY JORDAN'S LEMMA

NEXT THE CONTRIBUTION OF Γ_1 — NO ESTIMATION BUT A CALCULATION

$$\int_{\Gamma_1} \frac{1-e^{iz}}{z^2} dz = \int_{-\pi}^0 \frac{1-e^{i(\theta+0)}}{(Re^{i\theta})^2} ((Re^{i\theta}) d\theta) + \int_{\pi}^0 \frac{1-e^{i(\theta+0)}}{(Re^{i\theta})^2} ((Re^{i\theta}) d\theta)$$

$$= i \left[\int_{-\pi}^0 \frac{1-e^{i(\theta+0)}}{e^{i\theta}} d\theta + \int_{\pi}^0 \frac{1-e^{i(\theta+0)}}{e^{i\theta}} d\theta \right]$$

$$= -i \left[\int_{-\pi}^0 \left(\frac{1}{e^{i\theta}} + \frac{e^{i\theta}}{e^{i\theta}} + \frac{1}{2}(e^{i\theta})^2 + \frac{1}{2}(e^{i\theta})^3 + \dots \right) d\theta \right]$$

$$= i \left[\int_{-\pi}^0 \frac{-ie^{i\theta} + (e^{i\theta})^2}{e^{i\theta}} d\theta \right] = -i \int_{-\pi}^0 (1 + O(\epsilon)) d\theta$$

$$= -i\pi \text{ AS } \epsilon \rightarrow 0$$

THIS AS $R \rightarrow \infty$ & $\epsilon \rightarrow 0$

$$\left\{ \int_{-z_2}^0 + \int_{z_2}^\infty + \int_{z_2}^{\infty} \right\} f(z) dz = 0$$

$$\left\{ \int_{-z_2}^0 + \int_{z_2}^{\infty} \right\} f(z) dz = \pi$$

$$\int_{-z_2}^{\infty} f(z) dz = \pi$$

$$\int_{z_2}^{\infty} \frac{1-e^{iz}}{z^2} dz = \pi$$

$$2 \int_{z_2}^{\infty} \frac{1-e^{iz}}{z^2} dz = \pi$$

$$\int_{z_2}^{\infty} \frac{1-e^{iz}}{z^2} dz = \frac{\pi}{2}$$

LOOKING AT THE REAL PARTS & REMOVING THE IMAGINARY PARTS

Question 4

$$\int_0^{\infty} \frac{\ln x}{1+x^4} dx$$

- a) Find the value of the above improper integral, by integrating

$$f(z) = \frac{\log z}{1+z^4}, \quad z \in \mathbb{C}.$$

over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

- b) State the value of

$$\int_0^{\infty} \frac{1}{1+x^4} dx$$

$$\left[-\frac{\pi^2 \sqrt{2}}{16}, \frac{\pi \sqrt{2}}{4} \right]$$

• NEXT THE CONTRIBUTION OF THE ARC $T_2(u) - C_2$ AS $\varepsilon \rightarrow 0$

$$\int_{\varepsilon}^R \left| \frac{f(z)}{z^{k+1}} dz \right| = \left| \int_{\varepsilon}^R \frac{\ln z - \frac{i\pi}{2}}{z^{k+1}} dz \right| = \left| \int_{\varepsilon}^R \frac{(1+i)(z^{-k}-1)}{z^{k+1}} dz \right| \leq \int_{\varepsilon}^R \frac{|1+i| |z^{-k}-1|}{|z^{k+1}|} dz$$

BY THE SAME ARGUMENTS AS THE FIRST (shown before)

$$\leq \int_{\varepsilon}^R \frac{|\ln z + i\pi/2|}{|z^{k+1}|} dz = \int_{\varepsilon}^R \frac{|\ln z|}{|z^{k+1}|} dz = \frac{1}{k+1} \int_{\varepsilon}^R |\ln z| dz = \frac{1}{k+1} \left[z^{1-k} + \frac{1}{k+1} z^{1-k} \right]_R^\varepsilon \rightarrow 0 \text{ AS } \varepsilon \rightarrow 0$$

Take the Riemann sum

$$\int_0^R f(z) dz + \int_0^R f(z) dz = -\frac{\pi i}{k+1} + \frac{\pi i}{4} + \int_0^R \frac{\ln x - i\pi}{(1+ix)^k} dx + \int_0^R \frac{\ln x}{(1+ix)^k} dx = -\frac{\pi i \sqrt{2}}{6} + i \frac{\pi \sqrt{2}}{4}$$

$$2 \int_0^R \frac{\ln x}{(1+ix)^k} dx + i \int_0^R \frac{1}{(1+ix)^k} dx = -\frac{\pi i \sqrt{2}}{6} + i \frac{\pi \sqrt{2}}{4}$$

$$\therefore \int_0^R \frac{4ix}{(1+ix)^k} dx = -\frac{4\pi \sqrt{2}}{6} \quad \text{as } \int_0^R \frac{1}{(1+ix)^k} dx = \frac{4\pi \sqrt{2}}{4}$$

PROBLEMS ON AXES

- \bullet $\int_{\infty}^0 \frac{dx}{x^2+1}$
- $= \ln(x+1)$
- $= \ln(\infty)$
- $0 \leq x < \infty$

PROBLEMS ON AXES

- \bullet $\int_0^{\infty} \frac{dx}{(1+x^2)^{3/2}}$
- $= \ln(x+\sqrt{1+x^2})$
- $= \ln(\infty+\sqrt{\infty})$
- $0 \leq x < \infty$

Question 5

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx.$$

Find the value of the above improper integral, by integrating

$$f(z) = \frac{(\log z)^2}{1+z^2}, z \in \mathbb{C},$$

over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

M.M.B.E.

$$\frac{\pi^3}{8}$$

CONTOUR: $f(z) = \frac{(\log z)^2}{z^2+1}$ over a semicircular contour, where the origin is a branch point. If the branch cut is taken arbitrarily in the 3rd or 4th quadrant.

NOTE THAT IF THE BRANCH CUT IS TAKEN ALONG THE $\Im z$ -AXIS, THE INTEGRATION FAILS AS THE REQUIRED INTEGRAL CANCELS OUT.

(b) HAS SIMPLE POLES AT $z = \pm i$, OF WHICH ONLY THE ONE AT $z = i$ IS INSIDE Γ . CALCULATE ITS RESIDUE

$$\lim_{z \rightarrow i} [(z-i)f(z)] = \lim_{z \rightarrow i} \left[\frac{(\log z)^2}{(z+i)(z-i)} \right] = \frac{(\log i)^2}{2i}$$

$$= (\log i + i\arg i)^2 = \frac{(-\pi)^2}{2i} = -\frac{\pi^2}{8i}$$

BY THE RESIDUE THEOREM, WE OBTAIN

$$\int_{\Gamma} f(z) dz = \sum \text{(Residues inside } \Gamma\text{)} \times 2\pi i$$

$$\Rightarrow \int_{-R}^R f(z) dz + \int_R^{-R} f(z) dz = -\frac{\pi^2}{8i} \times 2\pi i = -\frac{\pi^3}{4}$$

APPENDIX: SHADING/UNITING PROCESS FOR THE CONTRIBUTION OF Γ_R .

$$\text{AS } c \rightarrow 0$$

$$\left| \int_{\Gamma_R} f(z) dz \right| \leq \int_{\Gamma_R} |f(z)| dz = \int_{\Gamma_R} \frac{(\log z)^2}{z^2+1} dz$$

ON Γ_R :
 $z = ce^{i\theta}$
 $dz = ie^{i\theta} dz$
 $+ve \text{ dir.}$

$$= \int_{\Gamma_R} \frac{(\log z)^2}{z^2+1} |dz| = \int_{\Gamma_R} \frac{[(\log z)^2 + (i\theta)^2]}{(ce^{i\theta})^2 + 1} |ie^{i\theta} dz|$$

$$= \int_{\Gamma_R} \frac{[(\log z)^2 + (\log z)^2 + \theta^2]}{(c^2e^{2i\theta}) + 1} |ie^{i\theta} dz| = - \int_{\Gamma_R} \frac{[(\log z)^2 + \theta^2]}{(c^2e^{2i\theta}) + 1} |e^{i\theta} dz|$$

USING THE SAME INFORMATION FROM PREVIOUS UNITING PROCESS OF Γ_L :

$$\leq - \int_{\Gamma_R} \frac{2[(\log z)^2 + (\log z)^2]}{(c^2e^{2i\theta}) + 1} |dz| = - \int_{\Gamma_R} \frac{2[(\log z)^2 + \theta^2]}{(c^2e^{2i\theta}) + 1} |dz|$$

$$= -\frac{2}{c^2} \int_0^\pi [\log c]^2 + \theta^2 |e^{i\theta}| d\theta$$

$$= -\frac{2}{c^2} \left[\theta[\log c]^2 + \pi^2[\log c] + \frac{\pi^3}{3} \right] \rightarrow 0 \text{ AS } c \rightarrow 0$$

NOTE THAT $c \rightarrow 0$ FORCE THAT $\log c \rightarrow -\infty$
 $\theta[\log c]^2 \rightarrow +\infty$

NOW CONSIDER THE CONTRIBUTION OF Γ_L AS $z \rightarrow \infty$

$$\left| \int_{\Gamma_L} f(z) dz \right| \leq \int_{\Gamma_L} |f(z)| dz = \int_{\Gamma_L} \frac{(\log z)^2}{z^2+1} dz$$

ON Γ_L :
 $z = re^{i\theta}$
 $dz = ie^{i\theta} dz$
 $-ve \text{ dir.}$

$$= \int_{\Gamma_L} \frac{(\log z)^2}{z^2+1} |dz| = \int_0^\pi \frac{[(\log(re^{i\theta}))^2]}{(re^{i\theta})^2 + 1} |ie^{i\theta} dz|$$

$$= \int_0^\pi \frac{[(\log r)^2 e^{2i\theta}]^2 + [(\theta)^2 e^{2i\theta}]^2}{r^2 e^{2i\theta} + 1} |ie^{i\theta} dz| = \int_0^\pi \frac{r^2[(\log r)^2 + \theta^2]}{r^2 e^{2i\theta} + 1} |e^{i\theta} dz|$$

NO APPLY THE FOLLOWING INEQUALITIES

ON NUMERATOR:
 $|z \pm w| \leq |z| + |w|$
ON DENOMINATOR:
 $|z \pm w| > |z| - |w|$

$$\leq \int_0^\pi \frac{2[(\log r)^2 + (\log r)^2]}{(r^2 e^{2i\theta} - 1)^2} |e^{i\theta} dz| = \int_0^\pi \frac{2[(\log r)^2 + \theta^2]}{(r^2 e^{2i\theta} - 1)^2} |e^{i\theta} dz|$$

$$= \int_0^\pi \frac{2[(\log r)^2 + 2\theta(\log r) + \theta^2]}{(r^2 e^{2i\theta} - 1)^2} |e^{i\theta} dz|$$

$$= \frac{2}{r^2-1} \int_0^\pi (\log r)^2 + 2\theta(\log r) + \theta^2 |e^{i\theta} dz|$$

$$= \frac{2}{r^2-1} \left[(\theta(\log r)^2 + \theta^2(\log r) + \frac{\theta^3}{3}) \right]$$

$$= \frac{2}{r^2-1} \left[\pi^2(\log r)^2 + \pi^2(\log r) + \frac{\pi^3}{3} \right] = 0 \left[\frac{(\log r)^3}{3} \right] \rightarrow 0$$

$$\text{AS } z \rightarrow \infty$$

SUMMING UP THE RESULTS AS $R \rightarrow \infty$ & $c \rightarrow 0$.

ON THE PATHOME: $z = z = ze^{i\theta}$
 $\log z = (\log |z|)e^{i\theta}$
 $\log z = (\log |z|) + i\theta$
 $\log z = (\log |z|) + i\theta$
 $\log z = \log |z|$
 $\log z = \log |z|$

$$\int_0^\infty f(z) dz + \int_{\Gamma_L} f(z) dz = -\frac{\pi^3}{4}$$

$$= \int_0^\infty \frac{(\log z)^2}{z^2+1} dz + \int_0^\infty \frac{(\log z)^2}{z^2+1} dz = -\frac{\pi^3}{4}$$

$$= \int_0^\infty \frac{(\log z)^2 + (\log z)^2}{z^2+1} dz = -\frac{\pi^3}{4}$$

$$= \int_0^\infty \frac{2(\log z)^2 - \theta^2 + (\log z)^2}{z^2+1} dz = -\frac{\pi^3}{4}$$

$$= \int_0^\infty \frac{2(\log z)^2 + 2i\theta(\log z) + \theta^2}{z^2+1} dz = -\frac{\pi^3}{4}$$

$$= 2 \int_0^\infty \frac{(\log z)^2}{z^2+1} dz - \frac{\pi^3}{4} \int_0^\infty \frac{1}{z^2+1} dz + 2i \int_0^\infty \frac{\log z}{z^2+1} dz = \frac{\pi i}{2}$$

SHADING DIRECTION (ANTICLOCKWISE)

$$= 2 \int_0^\infty \frac{(\log z)^2}{z^2+1} dz - \pi^2 \left(\frac{\pi}{2} \right) + 2i \int_0^\infty \frac{\log z}{z^2+1} dz = -\frac{\pi^3}{4} + \pi i$$

ON NEGATIVE: $z = -z = -ze^{i\theta}$
 $\log z = (\log |z|)e^{i\theta}$
 $\log z = (\log |z|) + i\theta$
 $\log z = (\log |z|) + i\theta$
 $\log z = \log |z|$
 $\log z = \log |z|$

$$= 2 \int_0^\infty \frac{(\log z)^2}{z^2+1} dz - \frac{\pi^3}{4} = -\frac{\pi^3}{4}$$

$$= \int_0^\infty \frac{(\log z)^2}{z^2+1} dz - \frac{\pi^3}{4} = -\frac{\pi^3}{8}$$

$$= \int_0^\infty \frac{(\log z)^2}{z^2+1} dz = \frac{\pi^3}{8}$$

Question 6

$$\int_0^\infty \frac{(\ln x)^2}{1+x^4} dx.$$

- a) Find the value of the above improper integral, by integrating

$$f(z) = \frac{(\log z)^2}{1+z^4}, z \in \mathbb{C},$$

over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

$$\left[\text{You may assume without proof that } \int_0^\infty \frac{1}{1+x^4} dx = \frac{\pi\sqrt{2}}{4} \right]$$

- b) State the value of

$$\int_0^\infty \frac{\ln x}{1+x^4} dx.$$

$$\frac{3\pi^3\sqrt{2}}{64}, -\frac{\pi^2\sqrt{2}}{16}$$

CONSIDER $\int_{\Gamma} (\ln z)^2 dz$, WHERE Γ IS A SEMICIRCLE CONTOUR IN THE FIRST QUADRANT BUT THE BRANCH CUT TAKEN IN AN ARBITRARY DIRECTION PARALLEL ON THE 3RD OR 4TH QUADRANT. NOTE THAT THE COMBINED "ENTIRE" CONTOUR WITH THE BRANCH CUT ALONG THE POSITIVE x AXIS FINISHES WITH THE REQUIRED BRANCH CANCELING OUT

(a) HAS SIMPLE POLES INSIDE Γ AT $z = i\sqrt{2}, -i\sqrt{2}$

$$\lim_{z \rightarrow i\sqrt{2}} \frac{(z-i\sqrt{2})(\ln z)^2}{z-i\sqrt{2}} = \frac{0}{0}, \text{ BY L'HOSPITAL} = \lim_{z \rightarrow i\sqrt{2}} \frac{[(\ln z)^2 + 2z(\ln z)]\ln 2}{1} = \frac{[(\ln(i\sqrt{2}))^2 + 2i\sqrt{2}(-i\sqrt{2})]\ln 2}{1} = (i\ln(i\sqrt{2}))^2 + 2i\sqrt{2}\ln 2$$

$$= \frac{-2i^2}{4i\sqrt{2}} = -\frac{i^2}{2\sqrt{2}} = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$$

$$\lim_{z \rightarrow -i\sqrt{2}} \frac{(z+i\sqrt{2})(\ln z)^2}{z+i\sqrt{2}} = \frac{0}{0}, \text{ BY L'HOSPITAL} = \lim_{z \rightarrow -i\sqrt{2}} \frac{[(\ln z)^2 + 2z(\ln z)]\ln 2}{1} = \frac{[(\ln(-i\sqrt{2}))^2 + 2(-i\sqrt{2})(i\sqrt{2})]\ln 2}{1} = (\ln(-i\sqrt{2}))^2 + 2(-i\sqrt{2})\ln 2$$

$$= \frac{-2(-i)^2}{4(-i\sqrt{2})} = -\frac{i^2}{2\sqrt{2}} = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum \text{residues inside } \Gamma$$

$$\left\{ \frac{1}{z-i\sqrt{2}} + \frac{1}{z+i\sqrt{2}} \right\}_{z=i\sqrt{2}} dz = 2\pi i \times \left[-\frac{i\sqrt{2}}{6} + i\frac{5i\sqrt{2}}{6} \right] = -\frac{5\sqrt{2}i}{6} - \frac{5\sqrt{2}i}{6}$$

• NOW CONSIDER THE CONTRIBUTION OF $\gamma_R(z)$ AS $R \rightarrow \infty$

$$\int_{\gamma_R} (\ln z)^2 dz = \int_{\gamma_R} \frac{1}{z-i\sqrt{2}} dz = \int_{\gamma_R} \frac{[(\ln z)^2 + 2z(\ln z)]}{z-i\sqrt{2}} dz = \int_{\gamma_R} \frac{[(\ln R)^2 + 2iR(\ln R)]}{R-i\sqrt{2}} dz$$

$$< \int_{\gamma_R} \frac{|(\ln R)^2 + 2iR(\ln R)|}{|R-i\sqrt{2}|} dz \leq \int_{\gamma_R} \frac{\sqrt{[(\ln R)^2 + 4R^2(\ln R)^2]}}{R-i\sqrt{2}} dz$$

$$= \int_{\gamma_R} \frac{(\ln R)^2 + R^2(\ln R)^2}{R^2 - 2iR\ln R + 2R^2} dz$$

$$= \frac{R}{R^2} \cdot \frac{[(\ln R)^2 + R^2(\ln R)^2]}{1 - 2i\ln R + 2} dz$$

$$= \frac{R}{R^2} \cdot \frac{[(\ln R)^2 + R^2(\ln R)^2]}{1 - 2i\ln R + 2} dz$$

$$= O\left(\frac{(\ln R)^2}{R^2}\right) \rightarrow 0 \text{ AS } R \rightarrow \infty$$

• NEXT CONSIDER THE CONTRIBUTION OF $\gamma_L(z)$ AS $z \rightarrow -\infty$

$$\int_{\gamma_L} (\ln z)^2 dz = \int_{\gamma_L} \frac{1}{z+i\sqrt{2}} dz = \int_{\gamma_L} \frac{[(\ln z)^2 + 2z(\ln z)]}{z+i\sqrt{2}} dz = \int_{\gamma_L} \frac{[(\ln(-z))^2 + 2(-z)(\ln(-z))]}{-z-i\sqrt{2}} dz$$

$$\leq \int_{\gamma_L} \frac{|(\ln(-z))^2 + 2(-z)(\ln(-z))|}{|-z-i\sqrt{2}|} dz \leq \int_{\gamma_L} \frac{\sqrt{[(\ln(-z))^2 + 4z^2(\ln(-z))^2]}}{-z-i\sqrt{2}} dz$$

$$= \int_{\gamma_L} \frac{(\ln(-z))^2 + z^2(\ln(-z))^2}{z^2 + 2i\sqrt{2}z + 2} dz$$

$$= \frac{z}{z^2} \cdot \frac{[(\ln(-z))^2 + z^2(\ln(-z))^2]}{1 + 2i\sqrt{2}/z + 2} dz$$

$$= \frac{z}{z^2} \cdot \left[(\ln(-z))^2 + z^2(\ln(-z))^2 + 2iz\ln(-z) + 2z^2 \right] \Big|_0^{-\infty} = \frac{z}{z^2} \cdot \left[(\ln(-z))^2 + z^2(\ln(-z))^2 \right] \Big|_0^{-\infty} \rightarrow 0 \text{ AS } z \rightarrow 0$$

(SINCE $z \rightarrow 0$ MEANS THAT $\ln(-z) \rightarrow -\infty$)

• THIS AS $R \rightarrow \infty, z \rightarrow 0$

$$\int_{-\infty}^0 \ln(z) dz + \int_0^\infty \ln(z) dz = -\frac{5\sqrt{2}i}{32} - \frac{\sqrt{2}i}{32}$$

$$\int_0^\infty \frac{(\ln z)^2 + z^2(\ln z)^2}{z^2 + 2i\sqrt{2}z + 2} dz = \frac{5\sqrt{2}i}{32} - \frac{\sqrt{2}i}{32}$$

$$2 \int_0^\infty \frac{(\ln z)^2 dz + 2\int_0^\infty \frac{(\ln z)^2 dz - \pi^2}{z^2 + 2i\sqrt{2}z + 2}}{z^2 + 2i\sqrt{2}z + 2} dz = \frac{5\sqrt{2}i}{32} - \frac{\sqrt{2}i}{32}$$

$$2 \int_0^\infty \frac{(\ln z)^2 dz + 2\pi i \int_0^\infty \frac{\ln z}{z^2 + 2i\sqrt{2}z + 2} dz - \frac{\pi^2}{4}}{z^2 + 2i\sqrt{2}z + 2} dz = -\frac{5\sqrt{2}i}{32} - \frac{\sqrt{2}i}{32}$$

$$2 \int_0^\infty \frac{(\ln z)^2 dz + 2\pi i \int_0^\infty \frac{\ln z}{z^2 + 2i\sqrt{2}z + 2} dz - \frac{\pi^2}{4}}{z^2 + 2i\sqrt{2}z + 2} dz = \frac{25\sqrt{2}i}{32} - \frac{\sqrt{2}i}{32}$$

EQUATING REAL & IMAGINARY

$$\int_0^\infty (\ln z)^2 dz = \frac{25\sqrt{2}i}{64} \quad \text{&} \quad \int_0^\infty \frac{\ln z}{z^2 + 2i\sqrt{2}z + 2} dz = -\frac{\sqrt{2}i}{16}$$

KEYHOLE CONTOUR

(Branch Cuts)

Question 1

$$f(z) = \frac{\log z}{1+z^2}, z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\int_0^\infty \frac{1}{1+x^2} dx = \frac{\pi}{2}.$$

proof

Now $f(z) = \frac{\log z}{1+z^2}$ has a **branch point** at $z=0$ because of the $\log z$, so we must make a **branch cut** from $z=0$ to $-\infty$, and it is sensible to do this on the positive x axis (easier to parametrise and easy way to find the value of the real integral).

Consider $\int_{\Gamma} f(z) dz$ as Γ is the complex contour below:

Note that the arc length is not zero due to the branch cut

The integration the 2 square roots at $z=0$ is done \square

CALCULATE REAL PART

$$\lim_{R \rightarrow \infty} \left| \int_{\Gamma} f(z) dz \right| = \lim_{R \rightarrow \infty} \left| \int_{\Gamma} \frac{\log |z| + i\arg(z)}{1+z^2} dz \right| = \lim_{R \rightarrow \infty} \left| \int_{\Gamma} \frac{\log |z|}{1+z^2} dz + \int_{\Gamma} \frac{i\arg(z)}{1+z^2} dz \right| = \lim_{R \rightarrow \infty} \left| \int_{\Gamma} \frac{\log |z|}{1+z^2} dz \right| + \lim_{R \rightarrow \infty} \left| \int_{\Gamma} \frac{i\arg(z)}{1+z^2} dz \right|$$

BY THE DIVERGENCE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum \text{(residues inside } \Gamma)$$

$$\int_{\Gamma} \left(\frac{\log |z|}{1+z^2} + i \frac{\arg(z)}{1+z^2} \right) dz = 2\pi i \times \left[\frac{1}{4} - \frac{1}{4} \right] = -\pi^2 i$$

NOW THE CONTRIBUTION OF $\gamma_2(z) = R e^{i\theta}$ AS $R \rightarrow \infty$

LET $z = Re^{i\theta}$, $0 < \theta < 2\pi$

$$\int_{\gamma_2} f(z) dz = \int_0^{2\pi} \frac{\log(R e^{i\theta})}{1+(Re^{i\theta})^2} (Re^{i\theta}) d\theta$$

$$= \int_0^{2\pi} \frac{\log(R e^{i\theta}) + i\theta}{1+R^2 e^{2i\theta}} (Re^{i\theta}) d\theta$$

$$\leq \int_0^{2\pi} \frac{|\log(R e^{i\theta})| + |i\theta|}{1+R^2 e^{2i\theta}} d\theta$$

Now ON NUMERICAL TECHNIQUE INTEGRATE $|w+2| > |w|-|z|$ AND ON THE INTEGRAL $|w| > |z|$

$$\Rightarrow \frac{1}{|w|} < \frac{1}{|z|}$$

Also $\lim_{R \rightarrow \infty} \left| \int_{\gamma_1} f(z) dz \right| = \lim_{R \rightarrow \infty} \left| \int_{\gamma_1} \frac{\log(R e^{i\theta}) + i\theta}{1+(Re^{i\theta})^2} (Re^{i\theta}) d\theta \right| = \lim_{R \rightarrow \infty} \left| \int_{\gamma_1} \frac{\log(R e^{i\theta})}{1+(Re^{i\theta})^2} (Re^{i\theta}) d\theta + \int_{\gamma_1} \frac{i\theta}{1+(Re^{i\theta})^2} (Re^{i\theta}) d\theta \right|$

$$= \lim_{R \rightarrow \infty} \left| \int_{\gamma_1} \frac{\log(R e^{i\theta})}{1+(Re^{i\theta})^2} (Re^{i\theta}) d\theta \right| = \lim_{R \rightarrow \infty} \left| \int_{\gamma_1} \frac{\log(R e^{i\theta})}{1+(Re^{i\theta})^2} (Re^{i\theta}) d\theta + \int_{\gamma_1} \frac{1}{1+(Re^{i\theta})^2} (Re^{i\theta}) d\theta \right|$$

$$= \lim_{R \rightarrow \infty} \left| \int_{\gamma_1} \frac{\log(R e^{i\theta})}{1+(Re^{i\theta})^2} (Re^{i\theta}) d\theta \right| = \lim_{R \rightarrow \infty} \left| \int_{\gamma_1} \frac{\log(R e^{i\theta})}{1+(Re^{i\theta})^2} (Re^{i\theta}) d\theta + \int_{\gamma_1} \frac{1}{1+(Re^{i\theta})^2} (Re^{i\theta}) d\theta \right|$$

BY THE DIVERGENCE THEOREM

$$\int_{\gamma_1} f(z) dz = 2\pi i \times \sum \text{(residues inside } \gamma_1)$$

$$\int_{\gamma_1} \left(\frac{\log(R e^{i\theta})}{1+(Re^{i\theta})^2} + i \frac{1}{1+(Re^{i\theta})^2} \right) dz = 2\pi i \times \left[\frac{1}{4} - \frac{1}{4} \right] = -\pi^2 i$$

NOW THE CONTRIBUTION OF $\gamma_1(z) = \epsilon e^{i\theta}$ AS $\epsilon \rightarrow 0$

LET $z = \epsilon e^{i\theta}$, $0 < \theta < 2\pi$

$$\int_{\gamma_1} f(z) dz = \int_0^{2\pi} \frac{\log(\epsilon e^{i\theta})}{1+(\epsilon e^{i\theta})^2} (\epsilon e^{i\theta}) d\theta$$

$$= \int_0^{2\pi} \frac{\log(\epsilon e^{i\theta}) + i\theta}{1+\epsilon^2 e^{2i\theta}} (\epsilon e^{i\theta}) d\theta$$

$$= \int_0^{2\pi} \frac{\log(\epsilon e^{i\theta}) + i\theta}{1+\epsilon^2 e^{2i\theta}} (\epsilon e^{i\theta}) d\theta \leq \int_0^{2\pi} \frac{|\log(\epsilon e^{i\theta})| + |\theta|}{1+\epsilon^2 e^{2i\theta}} (\epsilon e^{i\theta}) d\theta$$

SUM INTEGRATING, WHICH IS ON γ_1 (SEE EXPLANATION IN PAST SUGGESTED)

$$\leq \int_0^{2\pi} \left| \frac{|\log(\epsilon e^{i\theta})| + |\theta|}{1+\epsilon^2 e^{2i\theta}} \right| d\theta = \int_0^{2\pi} \frac{|\log(\epsilon e^{i\theta})| + |\theta|}{1+\epsilon^2 e^{2i\theta}} d\theta$$

$$= \frac{\epsilon}{\epsilon^2-1} \cdot \epsilon \log(\epsilon) + \frac{\epsilon}{\epsilon^2-1} \int_0^{2\pi} \frac{1}{1+\epsilon^2 e^{2i\theta}} d\theta$$

Question 2

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \pi \operatorname{cosec}(p\pi), \quad 0 < p < 1.$$

V, proof

$f(z) = \frac{z^p}{z+1}$ contains branch-cuts along $0 < \arg z < \pi$, so it has a branch point at $z=0$, so we must make a branch cut (around the positive real axis). It is especially free to do so on the positive real axis, as there is a pole at $z=-1$ and it is easy to parametrize along the \mathbb{R} axis.

Consider $\int_{\Gamma} f(z) dz$ over the "keyhole contour" shown below:

Note that $0 < \arg z < 2\pi$ because of the position of the branch cut.

$f(z)$ has simple poles at $z=1$ and $z=-1$.
 $\operatorname{Res}[f, -1] = \ln(2\pi i) e^{-i\pi}$
 $= \frac{1}{2\pi i} \frac{z^p}{z+1} \Big|_{z=-1} = \frac{1}{2\pi i} \frac{(-1)^p}{-1+1} = 0$
 $\operatorname{Res}[f, 1] = \ln(2\pi i) e^{i\pi}$
 $= \frac{1}{2\pi i} \frac{z^p}{z+1} \Big|_{z=1} = \frac{1}{2\pi i} \frac{1}{1+1} = \frac{1}{4\pi i}$

By the residue theorem,
 $\int_{\Gamma} f(z) dz = 2\pi i \times \sum \text{(residues of } f \text{ inside } \Gamma)$
 $= 2\pi i \left(\frac{1}{4\pi i} + 0 \right) = 2\pi i \times e^{ip\pi}$

Now parametrize z around Ω :

$z = re^{i\theta}$
 $dz = ie^{i\theta} d\theta$
 $d\theta \text{ from } 0 \text{ to } 2\pi$

$\left| \int_{\Gamma} f(z) dz \right| = \left| \int_0^{2\pi} \frac{(re^{i\theta})^p}{re^{i\theta}+1} (ie^{i\theta} d\theta) \right| = \left| \int_0^{2\pi} \frac{1}{2} \frac{r^p}{2} \frac{e^{ip\theta}}{e^{i\theta}+1} d\theta \right|$

$\leq \int_0^{2\pi} \left| \frac{1}{2} r^p e^{ip\theta} \frac{e^{ip\theta}}{e^{i\theta}+1} \right| d\theta = \int_0^{2\pi} \frac{1}{2} \frac{r^p}{2} \frac{|e^{ip\theta}|}{|e^{i\theta}+1|} d\theta$

$= \int_0^{2\pi} \frac{r^p}{2} d\theta \rightarrow |e^{i\theta}+1| \geq |e^{i\theta}| - 1 \geq 1 - 1 = 0$

$\leq \int_0^{2\pi} \frac{r^p}{2} d\theta \leq \frac{r^p}{2} \cdot 2\pi$

$= \int_0^{2\pi} \frac{r^p}{2} d\theta = \frac{r^p}{2} \cdot 2\pi = \frac{r^p}{2} \cdot \int_0^{2\pi} 1 d\theta$

$= \frac{r^p}{2} \cdot \frac{2\pi}{2-1} = O(r^{p+1}) \rightarrow 0 \text{ as } r \rightarrow \infty$

Now parametrize z around Ω :

$z = re^{i\theta}$
 $dz = ie^{i\theta} d\theta$
 $d\theta \text{ from } 0 \text{ to } 2\pi$

$\left| \int_{\Gamma} f(z) dz \right| = \left| \int_0^{2\pi} \frac{(re^{i\theta})^p}{re^{i\theta}+1} (ie^{i\theta} d\theta) \right| = \left| \int_0^{2\pi} \frac{1}{2} \frac{r^p}{2} \frac{e^{ip\theta}}{e^{i\theta}+1} d\theta \right|$

$\leq \int_0^{2\pi} \left| \frac{1}{2} r^p \frac{e^{ip\theta}}{e^{i\theta}+1} \right| d\theta = \int_0^{2\pi} \frac{1}{2} \frac{r^p}{2} \frac{|e^{ip\theta}|}{|e^{i\theta}+1|} d\theta$

$\leq \int_0^{2\pi} \frac{1}{2} \frac{r^p}{2} d\theta = \int_0^{2\pi} \frac{1}{2} \frac{r^p}{2} d\theta = \int_0^{2\pi} \frac{r^p}{2} d\theta$

$= \int_{-1}^0 \frac{r^p}{2} d\theta = \frac{r^p}{2} \cdot 1 = \frac{r^p}{2} \rightarrow 0 \text{ as } r \rightarrow 0$

Finally we have to $\lim_{r \rightarrow 0} \text{ and } r \rightarrow \infty$

$\int_0^\infty f(z) dz + \int_r^\infty f(z) dz = 2\pi i \times e^{ip\pi}$

$\bullet z = re^{i\theta}$
 $\bullet dz = ie^{i\theta} d\theta$
 $\bullet \text{branch cut}$
 $\bullet \text{except } z=0$
 $\bullet \text{but } z=0 \text{ is not on curve}$
 $\bullet z^p = (re^{i\theta})^p = r^p e^{ip\theta}$

Thus we obtain

$\int_0^\infty \frac{2}{2} \frac{r^p}{2} e^{ip\theta} d\theta + \int_0^\infty \frac{2}{2} \frac{r^p}{2} d\theta = 2\pi i \times e^{ip\pi}$

$- e^{ip\pi} \int_0^\infty \frac{2}{2} \frac{r^p}{2} d\theta + \int_0^\infty \frac{2}{2} \frac{r^p}{2} d\theta = 2\pi i \times e^{ip\pi}$

$[1 - e^{ip\pi}] \int_0^\infty \frac{2}{2} \frac{r^p}{2} d\theta = 2\pi i \times e^{ip\pi}$

Final step is

$\int_0^\infty \frac{2}{2} \frac{r^p}{2} d\theta = \frac{2\pi i \times e^{ip\pi}}{1 - e^{ip\pi}} = \frac{2\pi i \times e^{ip\pi}}{e^{ip\pi} - e^{ip\pi}}$

$= \frac{2\pi i}{-[-e^{ip\pi} - e^{ip\pi}]} = -\frac{2\pi i}{-2e^{ip\pi}} = \frac{\pi i}{e^{ip\pi}}$

$= \frac{\pi i}{-2 \sin(p\pi)} = \frac{-\pi i}{2 \sin(p\pi)}$

$= \frac{-\pi i}{2 \sin(p\pi) \cos(p\pi) - 2 \cos(p\pi) \sin(p\pi)} = \frac{-\pi i}{-2 \sin(p\pi)} = \frac{\pi i}{\sin(p\pi)}$

or target

Question 3

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^\infty \frac{x^{p-1}}{1+x^2} dx = \frac{\pi}{2} \operatorname{cosec}\left(\frac{p\pi}{2}\right), \quad 0 < p < 2.$$

proof

Question 4

$$f(z) = \frac{\log z}{(z+1)(z+2)}, z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\int_0^\infty \frac{1}{(x+1)(x+2)} dx = \ln 2.$$

proof

SINCE $f(z) = \frac{\log z}{(z+1)(z+2)}$ IS NOT DEFINED AT ZERO

WE MUST TAKE A **KEY-HOLE** CONTOUR AROUND ZERO TO INTEGRATE AS ANY SMOOTH CONTOUR IN THIS CASE AROUND THE POSITIVE AXIS WHICH IS EASY TO PARAMETRIZE, AND IT IS THE ONLY WAY TO GIVE THE REAL PARTIAL.

• CONSIDER $\oint_\Gamma f(z) dz$ OVER THE CONTOUR SHOWN BELOW

NOTE THAT THE KEY-HOLE IS MADE OUT OF THE POSITIVE PART OF THE REAL AXIS.

- $f(z)$ HAS TWO SIMPLE POLES AT $z=-1$ & $z=-2$. WHICH ARE ALSO INSIDE Γ WITH RESIDUES

$$\text{Res}_{z=-1} [f(z)] = \lim_{z \rightarrow -1} [z + 1] f(z) = \frac{\log(-1)}{-1}$$

$$\text{Res}_{z=-2} [f(z)] = \lim_{z \rightarrow -2} [z + 2] f(z) = \frac{\log(-2)}{-2}$$

$$= -[\log|z| + i\arg(z)]^z_{-\infty} = -\log 2 - i\pi$$

• BY THE RESIDUE THEOREM

$$\oint_\Gamma f(z) dz = -2\pi i \sum \text{Res}(f, z_i)$$

$$= \oint_\Gamma \left(\int_0^{\arg z} \frac{1}{z+2} dz \right) dz = 2\pi i \times \sqrt{2} \times \pi = O(\log 2) + O(\frac{1}{\epsilon})$$

Now consider the contribution along γ : $\int_0^R \frac{1}{z+2} dz$

LET $z = e^{i\theta} \quad 0 < \theta < 2\pi$

$$dz = ie^{i\theta} d\theta$$

$$\left| \int_0^R \frac{1}{z+2} dz \right| = \left| \int_0^{2\pi} \frac{\log(e^{i\theta})}{e^{i\theta}+2} (ie^{i\theta}) d\theta \right|$$

$$= \left| \int_0^{2\pi} \frac{(i\theta + i\pi) \log(e^{i\theta})}{e^{i\theta}+2} d\theta \right|$$

$$\leq \left| \int_0^{2\pi} \frac{|i\theta + i\pi| |e^{i\theta}|}{|e^{i\theta}+2|} d\theta \right|$$

$$= \int_0^{2\pi} \frac{|i\theta + i\pi| |e^{i\theta}|}{|e^{i\theta}+2|} d\theta \leq \int_0^{2\pi} \frac{|i\theta + i\pi| |e^{i\theta}|}{|e^{i\theta}|+2} d\theta$$

• NOTE THE CONTRACTION OF $|e^{i\theta}|$ AS $\theta \rightarrow 0$

LET $z = e^{i\theta} \quad 0 < \theta < 2\pi$

$$dz = ie^{i\theta} d\theta$$

$$\left| \int_0^R \frac{1}{z+2} dz \right| = \left| \int_0^{2\pi} \frac{\log(e^{i\theta})}{e^{i\theta}+2} (ie^{i\theta}) d\theta \right|$$

$$= \left| \int_0^{2\pi} \frac{(i\theta + i\pi) \log(e^{i\theta})}{e^{i\theta}+2} d\theta \right|$$

$$\leq \int_0^{2\pi} \frac{|i\theta + i\pi| |e^{i\theta}|}{|e^{i\theta}+2|} d\theta$$

$$= \int_0^{2\pi} \frac{|i\theta + i\pi| |e^{i\theta}|}{|e^{i\theta}|+2} d\theta$$

BY THE SAME INEQUALITY USED ON γ_1 (CALCULATED ABOVE)

$$\leq \int_0^{2\pi} \frac{|i\theta + i\pi| |e^{i\theta}|}{|e^{i\theta}|+2} d\theta$$

$$= \int_0^{2\pi} \frac{\epsilon |i\theta + i\pi|}{\epsilon + 2} d\theta$$

$$\therefore \int_0^R \frac{1}{z+2} dz = \ln 2.$$

$$\begin{aligned} &= \frac{\epsilon \ln 2}{\epsilon^2 - 2\epsilon - 2} \int_0^{2\pi} 1 d\theta + \frac{\epsilon}{\epsilon^2 - 2\epsilon - 2} \int_0^{2\pi} \theta d\theta \\ &= \frac{2\pi \epsilon \ln 2}{\epsilon^2 - 2\epsilon - 2} + \frac{2\pi^2 \epsilon}{\epsilon^2 - 2\epsilon - 2} \left[\frac{1}{2}\theta^2 \right]_0^{2\pi} \\ &\rightarrow 0 \text{ as } \epsilon \rightarrow 0 \end{aligned}$$

• SINCE DENOMINATOR $\rightarrow 0$

• $\epsilon \rightarrow 0$ SINCE $\epsilon \rightarrow 0$

• FINITE THUS $\ln \epsilon \rightarrow -\infty$

• $\epsilon \rightarrow 0$

Question 5

$$f(z) = \frac{\log z}{(z+a)(z+b)}, z \in \mathbb{C},$$

where $a \in \mathbb{R}^+$, $b \in \mathbb{R}^+$ with $b > a$.

By integrating $f(z)$ over a suitable contour Γ , show that

$$\int_0^\infty \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln\left(\frac{b}{a}\right).$$

[, proof]

Consider $f(z) = \frac{\log z}{(z+a)(z+b)}$ over a suitable contour Γ .

AS $f(z)$ CONTAINS $\log z$ WHICH HAS A BRANCH POINT AT $z=0$, WE MUST TAKE A BRANCH CUT FROM $z=0$ TO INFINITY, AT ANY CONVENIENT DIRECTION.

IN THIS CASE WE MIGHT TAKE THE BRANCH CUT ALONG THE POSITIVE \mathbb{R} AXIS SINCE:

- IT IS CONVENIENT FOR INTEGRATIONS
- THE UNITARITY FROM 0 TO ∞
- AVOID THE SINGULARITIES AT $z=a$ & $z=b$

FOR THE KEYHOLE PATH, SEE THE PICTURE BELOW

THE BRANCH CUT RUNS FROM 0 TO ∞ INSTEAD OF THE CIRCLE Γ BECAUSE OF THE BRANCH CUT.

$y(z) = Re(z) + i0$ $\rightarrow z = Re$
 $y'(z) = 1 + i0 \rightarrow dz = Re^{i0} dz$
 $y(z) = Re^{i0} \rightarrow z = Re$ (From $z=0$ to $z=R$)

FOR THIS SOURCE SINGULARITY AT $z=-a$ & $z=-b$, WHICH BOTH ARE INSIDE Γ

CALCULATE THE RESIDUES AT EACH POLE

- $\lim_{z \rightarrow -a} [z(-a)] \frac{\log z}{(z+a)(z+b)} = \frac{\log(-a)}{-a+b} = \frac{\ln(-a)}{b-a}$
- $\lim_{z \rightarrow -b} [z(-b)] \frac{\log z}{(z+a)(z+b)} = \frac{\log(-b)}{-b+a} = \frac{\ln(-b)}{a-b}$

RESIDUATE TO THE INTEGRAL OVER Γ

$$\begin{aligned} &\leq \int_{\Gamma} \left| \frac{\log z}{(z+a)(z+b)} \right| dz = \int_{\Gamma} \frac{2\pi R |z|}{R^2 - (a+b)z} dz \\ &= \frac{2\pi R}{R^2 - (a+b)R} \int_0^R \left(1 + \frac{2}{R-z} \right) dz + \frac{2\pi R}{R^2 - (a+b)R} \int_R^\infty \frac{1}{z} dz \\ &= \frac{2\pi R \ln R}{R^2 - (a+b)R} + \frac{2\pi R^2}{R^2 - (a+b)R} \\ &= O\left(\frac{R^2}{R}\right) + O\left(\frac{1}{R}\right) \rightarrow 0 \text{ AS } R \rightarrow \infty \end{aligned}$$

NEXT IN ANALOGOUS FASHION WE NEED TO SHOW THAT \int_{Γ} DOES NOT CONTRIBUTE AS $\varepsilon \rightarrow 0$ / $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta$, $R \rightarrow \infty$ AS $\theta \rightarrow 0$

$$\begin{aligned} \left| \int_{\Gamma} f(z) dz \right| &= \left| \int_{\Gamma} \frac{\ln(e^{i\theta})}{(e^{i\theta}+a)(e^{i\theta}+b)} (ie^{i\theta} d\theta) \right| \\ &= \left| - \int_0^{\pi} \frac{(iae + ib)(ie^{i\theta})}{(e^{i\theta}+a)(e^{i\theta}+b)} d\theta \right| \\ &\leq \int_0^{\pi} \frac{|iae + ib||ie^{i\theta}|}{|e^{i\theta}+a||e^{i\theta}+b|} d\theta \\ \text{SINCE INEQUALITY AS IN NOTE 7.1, EXCISE} \\ &\leq \int_0^{\pi} \frac{C(a,b)|ie^{i\theta}|}{|e^{i\theta}+a||e^{i\theta}+b|} d\theta \\ &\leq \int_0^{\pi} \frac{C(a,b)\varepsilon}{\varepsilon^2 - (a+b)\varepsilon} d\theta \\ &= \frac{C(a,b)\varepsilon}{\varepsilon^2 - (a+b)\varepsilon} \int_0^{\pi} 1 d\theta + \frac{C(a,b)\varepsilon}{\varepsilon^2 - (a+b)\varepsilon} \int_0^{\pi} \theta d\theta \\ &= \frac{2\pi C(a,b)\varepsilon}{\varepsilon^2 - (a+b)\varepsilon} + \frac{C(a,b)\varepsilon^2}{\varepsilon^2 - (a+b)\varepsilon} \end{aligned}$$

BY THE RESIDUE THEOREM

$$\begin{aligned} \int_{\Gamma} f(z) dz &= 2\pi i \sum (\text{residue of } f(z) \text{ inside } \Gamma) \\ \int_{\Gamma} f(z) dz &= 2\pi i \left[\frac{\ln(-a)}{b-a} - \frac{\ln(-b)}{b-a} \right] \\ &= \frac{2\pi i}{b-a} \ln\left(\frac{b}{a}\right) \quad (b > a) \end{aligned}$$

NEXT TO SHOW THAT THE CONTRIBUTION ALONG Σ_1 IS ZERO, AS $R \rightarrow \infty$

$$\begin{aligned} &\left| \int_{\Sigma_1} f(z) dz \right| = \left| \int_{\Sigma_1} \frac{\ln(z)}{(z+a)(z+b)} dz \right| \\ &= \left| \int_{\Sigma_1} \frac{(z-a)(z-b)}{(z+a)(z+b)} dz \right| \\ &\leq \int_{\Sigma_1} \left| \frac{(z-a)(z-b)}{(z+a)(z+b)} \right| dz \\ &= \int_{\Sigma_1} \frac{1}{|z-a||z-b|} dz \\ &= \int_{\Sigma_1} \frac{1}{R^2 - (a+b)R + ab} dz \\ &= \int_{\Sigma_1} \frac{1}{R^2 - (a+b)R + ab} R d\theta \end{aligned}$$

NOW USING THE FOLLOWING INEQUALITY

- $|z+w| \leq |z| + |w|$ \rightarrow TRIANGLE INEQUALITY ON THE NUMERATOR
- $|z+w| \geq |z| - |w|$
 $|z+w| \geq |z| - |w| - |w| \quad \text{ON THE DENOMINATOR}$

$$\frac{1}{|z-a| + |z-b|} \leq \frac{1}{|z|-|w|-|w|}$$

$= O(\varepsilon \ln \varepsilon) + O(\varepsilon) \rightarrow 0 \text{ AS } \varepsilon \rightarrow 0$

[DENOMINATORS TEND TO $-ab$ AND $\varepsilon \rightarrow 0$ AS $\ln \varepsilon \rightarrow -\infty$]

THIS IS TO $R \rightarrow \infty$ & $\varepsilon \rightarrow 0$. THE INTEGRAL REDUCES TO

$$\begin{aligned} &\int_0^\infty \frac{\ln z}{(2\pi i)(z)(a+b)} dz + \int_\infty^0 \frac{\ln z}{(2\pi i)(z)(a+b)} dz = -\frac{2\pi i}{b-a} \ln\left(\frac{b}{a}\right) \\ \text{THUS TWO DO NOT CANCEL - WE SHOULD TAKE IT INTO ACCOUNT HERE WHEN } \varepsilon \rightarrow 0 \\ &\text{TOP LINE SEGMENT: } z = e^{i\theta} = 2 \\ &dz = ie^{i\theta} d\theta \\ &\ln z = \ln 2 + i\theta \\ &\text{BOTM LINE SEGMENT: } z = e^{-i\theta} = 2 \\ &dz = -ie^{-i\theta} d\theta \\ &\ln z = \ln 2 - i\theta \\ &\rightarrow \int_0^\infty \frac{\ln z}{(2\pi i)(z)(a+b)} dz + \int_0^\infty \frac{\ln z}{(2\pi i)(z)(a+b)} dz = -\frac{2\pi i}{b-a} \ln\left(\frac{b}{a}\right) \\ &\rightarrow \int_0^\infty \frac{\ln z}{(2\pi i)(z)(a+b)} dz + \int_0^\infty \frac{-\ln z}{(2\pi i)(z)(a+b)} dz = -\frac{2\pi i}{b-a} \ln\left(\frac{b}{a}\right) \\ &\rightarrow \int_0^\infty \frac{\ln z - \ln z}{(2\pi i)(z)(a+b)} dz = -\frac{2\pi i}{b-a} \ln\left(\frac{b}{a}\right) \\ &\rightarrow -2\pi i \int_0^\infty \frac{1}{(2\pi i)(z)(a+b)} dz = -\frac{2\pi i}{b-a} \ln\left(\frac{b}{a}\right) \\ &\rightarrow \int_0^\infty \frac{1}{(2\pi i)(z)(a+b)} dz = \frac{1}{b-a} \ln\left(\frac{b}{a}\right) \end{aligned}$$

Question 6

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^\infty \frac{\sqrt{x}}{1+x^3} dx = \frac{\pi}{3}.$$

proof

$f(z) = \frac{z^{1/3}(z+1)}{z^2+1}$ HAS A SINGULARITY AT $z=0$.
BECUSE OF THE PUNCTURE HOLE, THIS WE MUST MAKE A **ROUNDED** SEMI-CIRCLE AT INFINITY IN ANY ARBITRARY DIRECTION SO THE STANDARD KEYLINE CONTOUR CAN BE USED.

• CONSIDER $\int_{\Gamma} f(z) dz$ OVER THE CONTOUR SHOWN BELOW:

$\int_{\Gamma} f(z) dz$ HAS 3 SINGULARITIES INSIDE Γ : $z^2+1=0$ $\Rightarrow z=\pm i$ & $z=0$.
BECUSE OF THE PUNCTURE HOLE, THIS WE MUST MAKE A **ROUNDED** SEMI-CIRCLE AT INFINITY IN ANY ARBITRARY DIRECTION SO THE STANDARD KEYLINE CONTOUR CAN BE USED.

CALCULATE RESULTS

$$\lim_{R \rightarrow \infty} \left[\int_{\Gamma} f(z) dz \right] = \lim_{R \rightarrow \infty} \left[\frac{1}{2} \int_{-\pi}^{\pi} f(Re^{i\theta}) iRe^{i\theta} d\theta \right]$$

$= \lim_{R \rightarrow \infty} \left[\frac{1}{2} \int_{-\pi}^{\pi} \frac{(Re^{i\theta})(Re^{i\theta}+1)}{R^2+1} iRe^{i\theta} d\theta \right] = \frac{1}{2} \int_{-\pi}^{\pi} \frac{1}{3} d\theta = \frac{\pi}{3}$

$\square \lim_{R \rightarrow \infty} \left[\frac{1}{2} \int_{-\pi}^{\pi} \frac{z^{1/3}(z-i)^{1/3}}{z^2+1} iRe^{i\theta} d\theta \right] = 0$ SINCE $R^{1/3} \ll$ A FAIRLY OF R .

BY L'HOSPITAL

$$\lim_{R \rightarrow \infty} \left[\frac{\frac{1}{3}z^{2/3}(z-i)^{1/3} + z^{1/3} \times 1}{2z^2+1} \right] = \frac{\left(\frac{1}{3}i\right)^{1/3}}{3\left(i\right)^{1/3}}$$

$$= \frac{i\sqrt[3]{i}}{3i\sqrt[3]{i}} = \frac{1}{3}e^{i\pi/4} = \frac{1}{3}i$$

$\square \lim_{R \rightarrow \infty} \left[\frac{1}{2} \int_{-\pi}^{\pi} \frac{z^{1/3}(z+i)^{1/3}}{z^2+1} iRe^{i\theta} d\theta \right] = 0$ BY L'HOSPITAL

$$= \lim_{R \rightarrow \infty} \left[\frac{\frac{1}{3}z^{2/3}(z+i)^{1/3} + z^{1/3} \times 1}{2z^2+1} \right] = \frac{\left(\frac{1}{3}i\right)^{1/3}}{3\left(-i\right)^{1/3}}$$

$$= \frac{-i\sqrt[3]{i}}{3i\sqrt[3]{i}} = \frac{1}{3}e^{-i\pi/4} = \frac{1}{3}i$$

• BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum \text{RESIDUES INSIDE } \Gamma$$

$$\left\{ \begin{array}{l} \int_{\Gamma} f(z) dz = 2\pi i \times \left[\frac{1}{3}i + \frac{1}{3}i \right] \\ \end{array} \right. = 2\pi i \times \left[\frac{1}{3}i - \frac{1}{3}i \right] = 0$$

• CONSIDER THE CONTRIBUTION OF $\int_{\Gamma_R} f(z) dz$

LET $z = Re^{i\theta}$, $0 < \theta < 2\pi$

$$\int_{\Gamma_R} f(z) dz = \int_0^{2\pi} \frac{(Re^{i\theta})(Re^{i\theta}+1)}{R^2+1} iRe^{i\theta} d\theta$$

$$= \int_0^{2\pi} \frac{R^2 e^{i2\theta} (1+e^{i2\theta})}{R^2+1} iRe^{i\theta} d\theta \leq \int_0^{2\pi} \frac{R^2 e^{i2\theta} |1+e^{i2\theta}|}{|R^2+1|} d\theta$$

$$= \int_0^{2\pi} \frac{R^2 e^{i2\theta} R^2}{R^2+1} d\theta \leq \int_0^{2\pi} \frac{R^2 R^2}{R^2-1} d\theta = \frac{R^2}{R-1} \rightarrow 0 \text{ AS } R \rightarrow \infty$$

SINCE $R \rightarrow \infty \rightarrow -1$ NARROWING $\rightarrow -1$ NARROWING $\rightarrow 0$

• THEN AS $R \rightarrow \infty$, $\epsilon \rightarrow 0$

$$\int_{\Gamma} f(z) dz + \int_{\Gamma_R} f(z) dz \approx 2\pi i \times (-i)$$

$\square z = Re^{i\theta}$ ANOTHER SUBSTITUTION
 $dz = ie^{i\theta} d\theta$
 $d\theta = \frac{1}{R} dz$ $\Rightarrow R d\theta = dz$

$$\int_{\Gamma} f(z) dz + \int_{\Gamma_R} f(z) dz = \int_0^{2\pi} \frac{z^{1/3} (z-i)^{1/3}}{z^2+1} dz + \int_0^{2\pi} \frac{z^{1/3} (z+i)^{1/3}}{z^2+1} dz = \frac{2\pi i}{3}$$

Question 7

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx.$$

An attempt is made to find the value of the above improper integral, by integrating

$$f(z) = \frac{(\ln z)^2}{1+z^2}, z \in \mathbb{C},$$

over the standard “keyhole” contour with a branch cut taken on the positive x axis.

- a) Show that such attempt fails.
- b) Calculate the value of the two integrals that can be found during this attempt.

$$\int_0^\infty \frac{1}{1+x^2} dx = \frac{\pi}{2}, \quad \int_0^\infty \frac{\ln x}{1+x^2} dx = 0$$

Now $f(z) = (\ln z)^2$ has a **branch point** at $z=0$. Because of the **branch**, so we must take a **branch cut** from $z=0$ to ∞ . It is sensible to go anti-clockwise along the positive x axis. Thus consider $\int_0^\infty f(z) dz$, where Γ is the standard keyhole contour shown below.

NOTE THAT THE ARGUMENTS ALONG THIS ROAD $= 0$ OR $= 2\pi$ BECAUSE OF THE BRANCH CUT.

$f(z)$ HAS SINGULARITIES AT $\pm i$, BOTH ARE INSIDE Γ . CALCULATE THE RESIDUES

- $\lim_{z \rightarrow i} ((z-i) \frac{(\ln z)^2}{(z-i)^2}) = \lim_{z \rightarrow i} \frac{(\ln z)^2}{z+i} = \frac{(\ln i)^2}{2i}$
- $\lim_{z \rightarrow -i} ((z+i) \frac{(\ln z)^2}{(z+i)^2}) = \lim_{z \rightarrow -i} \frac{(\ln z)^2}{z-i} = -\frac{(\ln (-i))^2}{2i} = -\frac{(\ln 1)^2}{2i} = 0$
- $\lim_{z \rightarrow 0} ((z+it) \frac{(\ln z)^2}{(z+it)^2}) = \lim_{z \rightarrow 0} \frac{(\ln z)^2}{z-it} = \frac{(\ln (-t))^2}{-it} = \frac{(\ln t)^2}{-it} = \frac{(\ln t)^2}{it}$
- ANALYSE BY THE RESIDUE THEOREM

$$\int_P f(z) dz = \text{area} \times \sum \text{Residues inside } \Gamma$$

$$\left\{ \int_{C_R} + \int_{C_r} + \int_{-R}^R \right\} f(z) dz = 2\pi i \times \left[\frac{(\ln t)^2}{it} \right] = 2\pi i$$

CONSIDER THE CONTRIBUTION OF $\int_{C_R} f(z) dz = R e^{i\theta} f(z)$, AS $R \rightarrow \infty$

$$\text{LET } z = Re^{i\theta}, \quad 0 < \theta < 2\pi$$

$$dz = ie^{i\theta} d\theta$$

$$\left| \int_{C_R} f(z) dz \right| = \left| \int_0^{2\pi} \frac{(\ln(Re^{i\theta}))^2}{(Re^{i\theta})^2 + 1} (ie^{i\theta} d\theta) \right|$$

NOTE ON THE NUMERATOR WE USE THE STANDARD DEFINITION OF $\ln(z)$ ON THE DOME DOMAIN: $|x \pm iy| > |y|$ AND $|y| < |w|$

$$\leq \int_0^{2\pi} \frac{|\ln R + iy|^2 |1/(R^2+1)|}{|R^2 e^{i2\theta} + 1|} d\theta \leq \int_0^{2\pi} \frac{R |\ln R + iy|^2}{R^2 - 1} d\theta = \frac{R}{R^2-1} \int_0^{2\pi} (\ln R^2 + 4\pi^2 R^2 + 4\pi^2 R y^2) d\theta$$

$$= \frac{R}{R^2-1} \left[4\pi(\ln R^2 + 4\pi^2 R^2) + \frac{4\pi^2 R^3}{3} y^3 \right] = 0 \left(\frac{\ln R^2}{R^2-1} \right) \rightarrow 0 \text{ AS } R \rightarrow \infty$$

NEXT THE CONTRIBUTION OF $\int_{C_r} f(z) dz = \varepsilon e^{i\theta} f(z)$, AS $\varepsilon \rightarrow 0$

$$\left| \int_{C_r} f(z) dz \right| = \left| \int_0^{2\pi} \frac{(\ln(\varepsilon e^{i\theta}))^2 |1/\varepsilon e^{i\theta}|}{|\varepsilon^2 e^{i2\theta} + 1|} d\theta \right| = \left| - \int_0^{2\pi} \frac{[(\ln \varepsilon + i\theta)^2] (\ln(\varepsilon e^{i\theta}))}{\varepsilon^2 e^{i2\theta} + 1} d\theta \right|$$

$$\leq \int_0^{2\pi} \frac{|\ln \varepsilon + i\theta|^2 |(\ln(\varepsilon e^{i\theta}))|}{|\varepsilon^2 e^{i2\theta} + 1|} d\theta = \dots \text{ SAME INTEGRATION AS THE CASE BEFORE (SEE GREEN BOX ABOVE)}$$

$$= \int_0^{2\pi} \frac{\varepsilon \left[\ln \varepsilon + \theta^2 \right]^2}{\varepsilon^2 - 1} d\theta = \frac{\varepsilon}{\varepsilon^2 - 1} \int_0^{2\pi} (\ln \varepsilon)^2 + 2\ln \varepsilon \cdot \theta^2 + \theta^4 d\theta = \frac{\varepsilon}{\varepsilon^2 - 1} \left[\theta (\ln \varepsilon)^2 + \theta^2 \ln \varepsilon + \frac{1}{3} \theta^3 \right]_0^{2\pi} = \frac{\varepsilon}{\varepsilon^2 - 1} \left[2\pi((\ln \varepsilon)^2 + 4\pi^2 \ln \varepsilon + \frac{8}{3}\pi^3) \right] = \frac{2\pi(\ln \varepsilon)^2}{\varepsilon^2 - 1} + \frac{4\pi^2 \ln \varepsilon}{\varepsilon^2 - 1} + \frac{8\pi^3 \varepsilon}{3(\varepsilon^2 - 1)} \rightarrow 0 \text{ AS } \varepsilon \rightarrow 0$$

SINCE $\frac{1}{\varepsilon^2 - 1} \rightarrow -1$
AS $\varepsilon \rightarrow 0$ PARALLEL TERM $\ln \varepsilon \rightarrow -\infty$ OR $(\ln \varepsilon)^2 \rightarrow \infty$

TOP SEGMENT: $z = 2e^{i\theta}, \quad \theta \in [0, \pi], \quad dz = 2ie^{i\theta} d\theta$
BOTTOM SEGMENT: $z = 2e^{-i\theta}, \quad \theta \in [\pi, 2\pi], \quad dz = -2ie^{-i\theta} d\theta$

$$\log z = \log(2e^{i\theta}) = \log(2e^{-i\theta}) = \ln 2 + i\theta = \ln 2 - i\theta$$

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx + \int_0^\infty \frac{(\ln x + 2\pi i)^2}{1+x^2} dx = 2\pi^3$$

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx - \int_0^\infty \frac{(\ln x + 2\pi i)^2}{1+x^2} dx = 4\pi^3$$

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx - \int_0^\infty \frac{(\ln x + 2\pi i)^2}{1+x^2} dx = 4\pi^3$$

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx = \frac{4\pi^3}{2} = 2\pi^3$$

WE FAIL TO FIND $\int_{1/2}^\infty (\ln x)^2 dx$ USING THIS CONTOUR.

Question 8

Use a substitution followed by integration of a suitable complex function over an appropriate contour, to show that

$$\int_0^{\frac{1}{2}\pi} (\tan x)^\alpha \, dx = \frac{1}{2}\pi \sec\left(\frac{1}{2}\pi\alpha\right), \quad -1 < \alpha < 1$$

proof

$\int_{\gamma} f(z) dz = \dots$ BY SUBSTITUTION METHOD
 $(-1 < x < 1)$

$\int_{\gamma} \frac{e^{iz}}{1+z^2} dz = \int_0^\infty \frac{e^{iy}}{1+y^2} dy$

$y = \text{frame}$
 $z = \text{antiderivative}$
 $dz = \frac{1}{1+y^2} dy$
 $x = \frac{\pi}{2}$
 $dx = 0$
 $y = 0$
 $g = c$

⑥ WAIT AGAIN AS $\int_0^\infty \frac{y''}{1+y^2} dy$ ISN'T CONVERGENT

⑦ NEXT CONSIDER $\int_{\Gamma} \frac{z''}{1+z^2} dz$ WHERE THE COMPLEX CONTOUR SHOWN BELOW

$A(\omega)$

$\Im(z) = R e^{i\theta}$
 $\Re(z) = 0$ from 0 to π

$\Im(z) = r e^{i\theta}$
 $\Re(z) = 0$ from 0 to π

NOTE THAT BECAUSE OF THE BRANCH CUT, INTEGRATION IS ONLY FROM 0 TO 2π

⑧ $\int_{\Gamma} f(z) dz = \int_{R/2}^{\infty} \frac{z''}{1+z^2} dz$ HAS SIMPLE POLES AT $\pm i$

CALCULATE RESIDUES

$\lim_{z \rightarrow i} \left[\frac{z''}{(z-i)(z+i)} \right] = \frac{-i''}{-2i} = \frac{i''}{2i}$

$\lim_{z \rightarrow -i} \left[\frac{(z''')}{(z-i)(z+i)} \right] = \frac{(-i''')}{-2i} = \frac{-i'''}{2i}$

⑥ PROVE BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = \left\{ \int_0^{\infty} \left(\frac{e^{iz}}{z-i} + \int_R^{\infty} \frac{e^{iz}}{z-i} dz \right) - \int_{-R}^0 \frac{e^{iz}}{z-i} dz \right\} - f(0) = 2\pi i \left[\frac{e^{i\infty}}{z-i} - \frac{e^{-i\infty}}{z-i} \right]$$

$$\int_{\Gamma} f(z) dz = \pi i [1 - e^{-i\infty}]$$

⑦ CONSTRUCTION OF γ_1 AS $R \rightarrow \infty$

$$\int_{\gamma_1} f(z) dz = \int_0^{\infty} \frac{e^{iz}}{z-i} dz = \int_0^{\infty} \frac{1}{1+\frac{i}{z}} dz = \int_0^{\infty} \frac{1}{1+\frac{i}{z}} e^{iz} dz = \int_0^{\infty} \frac{1}{1+\frac{i}{z}} e^{iz} dz$$

$$\leq \int_0^{\infty} \frac{1}{z} \left| \frac{e^{iz}}{1+\frac{i}{z}} \right|^{\infty} dz = \int_0^{\infty} \frac{1}{z} \frac{e^{iz+1}}{(1+\frac{i}{z})^2} dz = \int_0^{\infty} \frac{1}{z} \frac{e^{iz+1}}{(1+\frac{1}{z})^2} dz = \int_0^{\infty} \frac{1}{z} \frac{e^{iz+1}}{z^2+2z+1} dz$$

$$\leq \int_0^{\infty} \frac{e^{iz+1}}{z^2+2z+1} dz$$

$|z-ni| \geq |z| - |ni|$

$$\frac{1}{|z-ni|} \leq \frac{1}{|z|-|ni|}$$

$$= \int_0^{\infty} \frac{e^{iz+1}}{|z-ni|^2} dz$$

$$= \int_0^{\infty} \frac{e^{iz+1}}{z^2+1} dz$$

$$= \int_0^{\infty} \frac{e^{iz+1}}{z^2+1} dz$$

$$= \int_0^{\infty} \frac{e^{iz+1}}{1-\frac{1}{z^2}} dz$$

$$= \frac{e^{i\infty-1}}{1-\frac{1}{R^2}} + 2\pi i \rightarrow 0 \quad \text{AS } R \rightarrow \infty \quad \text{SO LONG AS } -\alpha^2 < 0$$

$$\alpha < 1$$

⑧ CONSTRUCTION OF γ_2 AS $\varepsilon \rightarrow 0$

$$\int_{\gamma_2} f(z) dz = \int_0^{\infty} \frac{e^{iz}}{z-i} dz = \int_0^{\infty} \frac{1}{1+\frac{i}{z}} dz = \int_0^{\infty} \frac{1}{1+\frac{i}{z}} e^{iz} dz = \int_0^{\infty} \frac{1}{1+\frac{i}{z}} e^{iz} dz$$

$$\leq \int_{-2\pi}^0 \frac{e^{iz}}{1+E^{\pi/2}} dz = \int_{-2\pi}^0 \frac{e^{iz}}{1+E^{\pi/2}} \left| \frac{e^{iz}}{z-i} \right| dz$$

$\leq \int_{-\infty}^0 \frac{e^{x+1}}{\pi} \left| \frac{e^{ix}}{1-e^{2ix}} - 1 \right| dx = \int_{-\infty}^0 \frac{e^{x+1}}{\pi} \left(\frac{e^{ix}}{1-e^{2ix}} - 1 \right) dx$

same inequality as before

 $= \frac{e^{x+1}}{e^2-1} \int_{-\infty}^0 dx = \frac{e^{x+1}}{e^2-1} (-\pi) = \frac{-\pi e^{x+1}}{1-e^2} \rightarrow 0$

$\text{As } x \rightarrow 0$
 $\frac{-\pi e^{x+1}}{1-e^2} \xrightarrow{x+1 > 0} -\pi > -1$

② FINITE PARAMETRIC LINES (IN THE UNIT $x \rightarrow -\infty$, $R \rightarrow \infty$)

JUST ABOVE THE \Re AXIS JUST BELOW THE \Re AXIS

$$\begin{cases} z = e^{x+i\pi} \\ dz = dx + i\pi dt \\ \text{i.e. } x = z \\ dx = dz \end{cases}$$

$$\begin{cases} z = e^{x-i\pi} \\ dz = dx - i\pi dt \\ \text{i.e. } x = z \\ dx = dz \end{cases}$$

NOW $e^{x-i\pi} = 1$
 EXCEPT IN $(e^{2\pi i t})^\infty$ where t
 OR IS ∞ ANOTHER

③ IN THE UNIT $t \rightarrow -\infty$, $R \rightarrow \infty$

 $\Rightarrow \int_0^\infty \frac{z^x}{1+z^2} dz + \int_{-\infty}^0 \frac{(ze^{2\pi i t})^x}{1+z^2} (ze^{2\pi i t} dz) = \pi \left[i^x - (-1)^x \right]$
 $\Rightarrow \int_0^\infty \frac{z^x}{1+z^2} dz - \int_0^\infty \frac{z^x e^{2\pi i t}}{1+z^2} dz = \pi \left[\left(e^{i\frac{\pi}{2}} \right)^x - \left(e^{-i\frac{\pi}{2}} \right)^x \right]$
 $\Rightarrow \int_0^\infty \frac{z^x}{1+z^2} dz - e^{-2\pi i x} \int_0^\infty \frac{z^x}{1+z^2} dz = \pi \left[\left(e^{i\frac{\pi}{2}} \right)^x - \left(e^{-i\frac{\pi}{2}} \right)^x \right]$
 $\Rightarrow \left[1 - e^{-2\pi i x} \right] \int_0^\infty \frac{z^x}{1+z^2} dz = \pi \left[e^{i\frac{\pi}{2}x} - e^{-i\frac{\pi}{2}x} \right]$

$$\begin{aligned}
& \Rightarrow \left[1 - e^{-2\pi i} \right] \int_0^{\infty} \frac{2^x}{1+2^x} dx = \frac{\pi i e^{i\pi x} \left[e^{-\frac{\pi i x}{2}} - e^{\frac{\pi i x}{2}} \right]}{1 - e^{2\pi i x}} \\
& \Rightarrow \int_0^{\infty} \frac{2^x}{1+2^x} dx = \frac{\pi i e^{i\pi x} \left[-2 \sin\left(\frac{\pi x}{2}\right) \right]}{1 - e^{2\pi i x}} \\
& \Rightarrow \int_0^{\infty} \frac{2^x}{1+2^x} dx = \frac{-2\pi i \sin\left(\frac{\pi x}{2}\right)}{e^{2\pi i x} - e^{-2\pi i x}} \\
& \Rightarrow \int_0^{\infty} \frac{2^x}{1+2^x} dx = \frac{-2\pi i \left(\sin\frac{\pi x}{2} \right)}{2 \cos\left(\pi x\right)} \\
& \Rightarrow \int_0^{\infty} \frac{2^x}{1+2^x} dx = \frac{\pi i \sqrt{2} \sin\frac{\pi x}{2}}{2 \sin\frac{\pi x}{2} \cos\frac{\pi x}{2}} \\
& \Rightarrow \int_0^{\infty} \frac{2^x}{1+2^x} dx = \frac{\pi i \sqrt{2} \sin\frac{\pi x}{2}}{2 \sin\frac{\pi x}{2} \cos\frac{\pi x}{2}} \\
& \Rightarrow \int_0^{\infty} \frac{2^x}{1+2^x} dx = \frac{\pi i}{2 \sin\frac{\pi x}{2}}
\end{aligned}$$

∵ If $-1 < x' < 1$ $\int_0^{\frac{\pi}{2}} \left(\sum_{n=0}^{\infty} (-1)^n x'^n \right) dx = \frac{\pi}{2} \sin\frac{\pi x'}{2}$
 A Biquation

SPECIAL CONTOURS

Question 1

Consider the contour Γ located in the first quadrant, defined as the boundary of a quarter circular sector of radius R , with centre at the origin O .

By integrating a suitable complex function over Γ show that

$$\int_0^\infty \frac{1}{1+x^4} dx = \frac{\pi\sqrt{2}}{4}.$$

, $\frac{\pi\sqrt{2}}{4}$

LET US NOTE THAT THE STANDARD CONTOUR FOR THIS IS A "TOP HALF SEMICIRCLE", HOWEVER WE SHALL USE A QUADRANT CIRCLE HERE AS INSTRUCTED — SEPARATE ATTENTION IS LEFT, BESIDES TO COMPUTE.

CONSIDER $\int_{\Gamma} \frac{1}{1+z^4} dz$ OVER THE CONTOUR SHOWN BELOW

ROLES:

- $z^4 = 100$
- $z^4 = -1$
- $z^4 = e^{i(2k\pi)}$
- $z^4 = e^{i\pi}(e^{i2k\pi})$
- $z = e^{i\pi/4}e^{i2k\pi}$

ONLY POLE INSIDE Γ IS AT $z = e^{i\pi/4}$

CALCULATE THE RESIDUE AT THIS POLE

$$\lim_{z \rightarrow e^{i\pi/4}} [(z - e^{i\pi/4}) \cdot \frac{1}{1+z^4}] = \frac{-ie^{i\pi/4}}{4e^{i\pi/4}}$$

BY TAYLOR, SO $\ln z^4$ HAS ORDER 2 POWER

$$= \lim_{z \rightarrow e^{i\pi/4}} \left[\frac{z - e^{i\pi/4}}{1+z^4} \right] = \lim_{z \rightarrow e^{i\pi/4}} \left[\frac{1}{4z^3} \right] = \frac{1}{4e^{3i\pi/4}} = \frac{1}{4}e^{-i3\pi/4}$$

$$= \frac{1}{4}(\cos 3\pi/4 - i\sin 3\pi/4) = \frac{1}{4} \left[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \right] = -\frac{\sqrt{2}}{8} - i\frac{\sqrt{2}}{8}$$

NOW BY THE RESIDUE THEOREM

$$\int_{\Gamma} \frac{1}{1+z^4} dz = 2\pi i \times \sum \text{(Residues inside } \Gamma)$$

$$\int_{\Gamma} \frac{1}{1+z^4} dz = 2\pi i \times \left(-\frac{\sqrt{2}}{8} - i\frac{\sqrt{2}}{8} \right) = \frac{\pi\sqrt{2}}{4} - \frac{i\pi\sqrt{2}}{4}$$

PROBLEMS FROM SECTION

ALONG Γ	ALONG γ	ALONG THE REC. δ
$z = x$ $dz = dx$ 2. ROW 0 to 1	$z = iy$ $dz = idy$ 3. ROW 2 to 0	$z = e^{i\theta}$ $dz = ie^{i\theta}d\theta$ 0. ROW 0 to 1/2

$\int_{\Gamma} \frac{1}{1+z^4} dz + \int_{\gamma} \frac{1}{1+z^4} dz + \int_{\delta} \frac{1}{1+z^4} dz = \int_0^{\pi/2} \frac{1}{1+e^{4i\theta}} d\theta = \frac{\pi\sqrt{2}}{4} - \frac{i\pi\sqrt{2}}{4};$

$$\int_0^{\pi/2} \frac{1}{1+e^{4i\theta}} d\theta = \int_0^{\pi/2} \frac{1}{1+e^{4i\theta}} dy + \int_0^{\pi/2} \frac{1}{1+e^{4i\theta}} d\theta = \frac{\pi\sqrt{2}}{4} - \frac{i\pi\sqrt{2}}{4};$$

AS $R \rightarrow \infty$, THE INTEGRAL OVER THE REC. VANISHES (SQUEEZE LIM).

$$\int_0^{\pi/2} \frac{1}{1+e^{4i\theta}} dx = \int_0^{\pi/2} \frac{1}{1+e^{4i\theta}} dy = \frac{\pi\sqrt{2}}{4} - i\left(\frac{\pi\sqrt{2}}{4}\right)$$

OTHER REAL OR IMAGINARY PARTS

$$\int_0^{\pi/2} \frac{1}{1+e^{4i\theta}} dx = \frac{\pi\sqrt{2}}{4}$$

FINALLY THE DETERMINING CLAIM THAT THE INTEGRAL OVER γ VANISHES

$$\left| \int_0^{\pi/2} \frac{1}{1+e^{4i\theta}} d\theta \right| \leq \int_0^{\pi/2} \left| \frac{1}{1+e^{4i\theta}} \right| d\theta = \int_0^{\pi/2} \frac{1}{1+e^{4\sin \theta}} d\theta$$

$$= \int_0^{\pi/2} \frac{1+2\sqrt{1-\cos^2 \theta}}{1+e^{4\sin \theta}} d\theta = \int_0^{\pi/2} \frac{2}{1+e^{4\sin \theta}} d\theta$$

USING THE "SQUEEZE INEQUALITY" (NOT THAT $z \neq 0$ IN QUADRANT)

$$\frac{1}{|z+w|} \leq \frac{1}{|z|-|w|}$$

$$\frac{1}{|1+e^{4\sin \theta}|} \leq \frac{1}{|e^{4\sin \theta}|-1} = \frac{1}{|e^{4\sin \theta}| - 1} = \frac{1}{e^{4\sin \theta} - 1} = \frac{1}{e^{4\sin \theta}}$$

HENCE WE FINALLY HAVE

$$\dots = \int_0^{\pi/2} \frac{2}{1+e^{4\sin \theta}} d\theta \leq \int_0^{\pi/2} \frac{2}{e^{4\sin \theta}-1} d\theta = \frac{2}{e^{4\sin \theta}} \int_0^{\pi/2} 1 d\theta$$

$$= \frac{2}{e^{4\sin \theta}} \rightarrow 0 \quad \text{as } \theta \rightarrow \infty$$

AND THE CLAIM IS JUSTIFIED

Question 2

By integrating a suitable complex function over a contour defined as the outline of a circular sector subtending an angle of $\frac{1}{3}\pi$ at the origin, find an exact value for

$$\int_0^\infty \frac{1}{1+x^6} dx.$$

No credit will be given for integration over alternative contours.

V, , $\boxed{\frac{\pi}{3}}$

CONSIDER $f(z) = \frac{1}{z^6+1}$ OVER THE CONTOUR Γ , SHOWN BELOW

$f(z)$ HAS SIMPLE POLES WHEN $z^6+1=0$, i.e. $z^6=-1$, $z=e^{i\pi/6}, e^{i\pi/2}, e^{5i\pi/6}$

ONLY $z=e^{i\pi/6}$ IS INSIDE Γ

$$\operatorname{Res}(f, e^{i\pi/6}) = \lim_{z \rightarrow e^{i\pi/6}} [(z - e^{i\pi/6}) \times \frac{1}{z^6+1}] = \lim_{z \rightarrow e^{i\pi/6}} \left[\frac{z - e^{i\pi/6}}{z^6-1} \right]$$

BY L'HOSPITAL RULE

$$= \lim_{z \rightarrow e^{i\pi/6}} \left[\frac{1}{6z^5} \right] = \frac{1}{6e^{5i\pi/6}} = \frac{1}{6}e^{-5i\pi/6}$$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{residues of } f \text{ inside } \Gamma)$$

$$\left\{ \int_A + \int_B + \int_{\Gamma_R} f(z) dz \right\} = 2\pi i \times \frac{1}{6}e^{-5i\pi/6}$$

NEST LOOKING AT THE CONTRIBUTION OF $f(z)$ AROUND z_0

$$\left| \int_{\Gamma_R} f(z) dz \right| = \left| \int_0^R \frac{1}{(z^6-1)^{1/6}} \times (1+o(1)) dz \right|$$

AROUND z_0
 $z = 2e^{i\theta}$
 $dz = 2ie^{i\theta} d\theta$
 $\theta \text{ FROM } 0 \text{ TO } \pi/6$

NEXT USING THE INEQUALITY $\left| \int_{\Gamma_R} f(z) dz \right| \leq \int_{\Gamma_R} |f(z)| dz$

$$\leq \int_0^R \left| \frac{1}{(z^6-1)^{1/6}} \right| dz = \int_0^R \frac{1}{|z^6-1|^{1/6}} dz = \sqrt[6]{\int_0^R |z^6-1|^2 dz}$$

$$= \int_0^R \frac{1}{(2e^{i\theta})^6-1} dz = \int_0^R \frac{1}{(2e^{i\theta})^6-1} d\theta$$

NEXT USING THE INEQUALITY $|z^6-1| \geq |z^6| - |1| > 0$

$$= \int_0^R \frac{1}{(2e^{i\theta})^6-1} d\theta = \frac{1}{2^6} \int_0^R \frac{1}{(e^{i\theta})^6-1} d\theta = \frac{1}{2^6(6i)} = O(\frac{1}{\theta^5}) \rightarrow 0 \text{ AS } \theta \rightarrow \infty$$

THIS AS $R \rightarrow \infty$ WE HAVE

$$\int_0^\infty \frac{1}{z^6+1} dz + \int_{\Gamma_R} \frac{1}{z^6+1} dz = \frac{1}{3} \pi i \times e^{-5i\pi/6}$$

$z = 2e^{i\theta}, \theta \in [0, \pi/6]$
 $dz = 2ie^{i\theta} d\theta$
 $2\pi i \text{ FROM } 0 \text{ TO } \pi/6$
 $2\pi i \text{ FROM } \pi/6 \text{ TO } 0$

$$\int_0^\infty \frac{1}{z^6+1} dz + \int_{\Gamma_R} \frac{1}{z^6+1} dz = \frac{1}{3} \pi i \times e^{-5i\pi/6}$$

$$\int_0^\infty \frac{1}{z^6+1} dz = \int_0^\infty \frac{1}{(2e^{i\theta})^6-1} dz = \frac{1}{2^6} \pi i \times e^{-5i\pi/6}$$

$$(1 - e^{-5i\pi/6}) \int_0^\infty \frac{1}{z^6+1} dz = \frac{1}{2} \pi i \times e^{-5i\pi/6}$$

$$\int_0^\infty \frac{1}{z^6+1} dz = \frac{\pi i}{3} \times \frac{e^{-5i\pi/6}}{1 - e^{-5i\pi/6}}$$

TIME TO FINISH

$$\int_0^\infty \frac{1}{z^6+1} dz = \frac{\pi i}{3} \times \frac{e^{-5i\pi/6}}{e^{5i\pi/6} - e^{i\pi/6}}$$

$$\int_0^\infty \frac{1}{z^6+1} dz = \frac{\pi i}{3} \times \frac{e^{-5i\pi/6}}{e^{i\pi/6} - e^{5i\pi/6}}$$

$$\int_0^\infty \frac{1}{z^6+1} dz = \frac{\pi i}{3} \times \frac{-e^{-5i\pi/6}}{e^{5i\pi/6} - e^{i\pi/6}}$$

$$\int_0^\infty \frac{1}{z^6+1} dz = \frac{\pi i}{3} \times \frac{-e^{-5i\pi/6}}{2\sin(\pi/3)}$$

$$\int_0^\infty \frac{1}{z^6+1} dz = \frac{\pi i}{3} \times \frac{1}{2\sqrt{3}}$$

$$\int_0^\infty \frac{1}{z^6+1} dz = \frac{\pi i}{6\sqrt{3}}$$

Question 3

By integrating a suitable complex function over an appropriate contour find

$$\int_0^\infty \frac{1}{1+x^3} dx.$$

$$\boxed{\frac{2\pi\sqrt{3}}{9}}$$

The usual semicircular contour does not work well because $\frac{1}{1+z^3}$ is neither odd nor even.

So we consider $\int_{\Gamma} \frac{1}{1+z^3} dz$ over the contour shown below

- PARAMETRISATION:**
 - $Y_1: z = re^{i\theta}, dz = idr e^{i\theta}, r < 2, \theta \in [0, 2\pi]$
 - $Y_2: z = Re^{i\theta}, dz = iRe^{i\theta}, 0 < \theta < 2\pi$
 - $Y_3: z = t + i0, dz = dt, t \in [-R, R]$
- INTEGRAL HAS 3 SIMPLE POLES:** $z^3 = -1 \Rightarrow z = e^{i(\pi/3 + 2k\pi)}, k \in \mathbb{Z}$
 $z = e^{i\pi/3}(2k+1)$
 $= e^{i\pi/3} \sqrt[3]{2} e^{i\pi/3}$
 Only pole inside Γ
- CALCULATE RESIDUE BY L'HOSPITAL**
 $\lim_{z \rightarrow e^{i\pi/3}} \left[\frac{1}{(z+1)(z-e^{i\pi/3})} \right] = \frac{0}{0}$ as $z \rightarrow e^{i\pi/3}$ is a factor of $1+z^3$
 $\lim_{z \rightarrow e^{i\pi/3}} \left[\frac{z-e^{i\pi/3}}{(z+1)^2} \right] = \lim_{z \rightarrow e^{i\pi/3}} \left[\frac{1}{2z+2} \right] = \frac{1}{3e^{i\pi/3}} = \frac{1}{3e^{i\pi/3}}$
 $= \frac{e^{i\pi/3}}{3e^{i\pi/3}} = \frac{e^{i\pi/3} + i0e^{i\pi/3}}{-3} = -\frac{1}{3}(i\sqrt{3})$

• BY THE RESIDUE THEOREM
 $\rightarrow \int_{\Gamma} \frac{1}{1+z^3} dz = 2\pi i \times \frac{1}{3} (\text{Residue inside } \Gamma)$

$\rightarrow \int_{Y_1} + \int_{Y_2} + \int_{Y_3} \frac{1}{1+z^3} dz = 2\pi i \times \frac{1}{3} (i\sqrt{3})$

$\rightarrow \int_0^\infty \frac{1}{1+t^3} dt + \int_0^\infty \frac{iRe^{i\theta}}{1+R^3e^{i2\theta}} dt + \int_0^\infty \frac{i\sqrt{3}e^{i\pi/3}dt}{1+3e^{i2t}} = -\frac{\pi i}{3}(i\sqrt{3})$

• NOW TAKE LIMITS AS $R \rightarrow \infty$, TO SEE IF ANYTHING VANSHES

LOOKING AT THE INTERVAL CIRCLE Y_2
 NOTE: $|z+1| \geq |z|-1$
 $\frac{|z|^2}{|z+1|^2} \leq \frac{|z|^2}{|z|-1}$ \circlearrowleft

$\left| \int_{Y_2} \frac{1}{1+z^3} dz \right| \leq \int_{Y_2} \frac{|z|^2}{|z|-1} dt = \int_0^\infty \frac{|z|^2}{|z|-1} dt$

$\lesssim \int_0^\infty \frac{R^2}{R^2(3e^{i2t}-1)} dt = \int_0^\infty \frac{1}{3e^{i2t}-1} dt$
 $= \frac{1}{3e^{i\pi/3}} \int_0^\infty \frac{1}{1-3e^{-i2t}} dt = \frac{1}{3e^{i\pi/3}} \times \frac{\pi}{3} \rightarrow 0$
 $\rightarrow R \rightarrow \infty$

THIS SO FAR AS $R \rightarrow \infty$
 $\int_0^\infty \frac{1}{1+t^3} dt + \int_{\infty}^0 \frac{e^{i\pi/3}}{1+t^3} dt = -\frac{\pi i}{3}(i\sqrt{3})$
 $\int_0^\infty \frac{1}{1+t^3} dt - i\frac{\pi}{3} \int_0^\infty \frac{1}{1+t^3} dt = \frac{\pi}{3}(-1+i\sqrt{3})$

EQUATING REAL PARTS & IMAGINARY PARTS
 $\int_0^\infty \frac{1}{1+t^3} dt = (\cos \frac{\pi}{3} + i\sin \frac{\pi}{3}) \int_0^\infty \frac{1}{1+t^3} dt = \frac{\pi}{3}(1+i\sqrt{3})$
 $\int_0^\infty \frac{1}{1+t^3} dt - i\frac{\pi}{3} \int_0^\infty \frac{1}{1+t^3} dt = \frac{\pi}{3}(-1+i\sqrt{3})$
 $\frac{2}{3} \int_0^\infty \frac{1}{1+t^3} dt - i\frac{\pi}{3} \int_0^\infty \frac{1}{1+t^3} dt = \frac{\pi}{3}(i\sqrt{3}-1)$

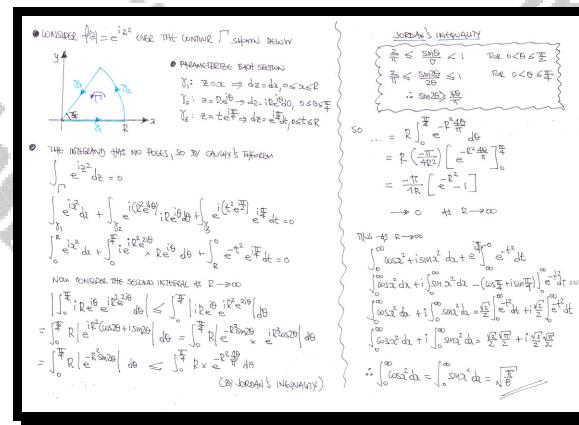
$\frac{2}{3} \int_0^\infty \frac{1}{1+t^3} dt = \frac{\pi}{3} \frac{i\sqrt{3}}{1-i\sqrt{3}}$
 $\frac{2}{3} \int_0^\infty \frac{1}{1+t^3} dt = \frac{\pi}{3} \frac{i\sqrt{3}}{2\sqrt{3}}$
 $\int_0^\infty \frac{1}{1+t^3} dt = \frac{\pi}{3} \frac{i\sqrt{3}}{2\sqrt{3}}$
 $\int_0^\infty \frac{1}{1+t^3} dt = \frac{\pi}{3}\sqrt{3}$

Question 4

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^\infty \cos(x^2) dx = \int_0^\infty \sin(x^2) dx = \sqrt{\frac{\pi}{8}}$$

proof



Question 5

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^\infty \frac{\ln x}{a^2 + x^2} dx = \frac{\pi \ln a}{2a}.$$

proof

$f(z) = \frac{\log z}{z^2 - a^2}$. It's a branch point at $z=0$.

Because of the $\arg z$, choose the standard contour with $\arg z = 0$ along the positive $\text{Re } z$ axis, because the real branch cut, so we will only find $\frac{1}{2\pi i} \int_{\Gamma} f(z) dz$.

The $\arg z$ must be real, $\text{Im } z = 0$, and in any direction, so consider $\int_L f(z) dz$ where L is the contour shown below, & the branch cut in some arbitrary direction.

\bullet BY THE RESIDUE THEOREM

$$\int_L f(z) dz = 2\pi i \times \sum \text{(residues inside } \Gamma)$$

$$\left[\int_{-R}^{-\epsilon} + \int_L + \int_\epsilon^R \right] f(z) dz = 2\pi i \times \frac{\ln a + i\frac{\pi}{2}}{2a}$$

\bullet NOT THE CONTRIBUTION OF $\int_R^\infty \frac{1}{z^2 - a^2} dz$ AS $R \rightarrow \infty$

Let $z = Re^{i\theta}, 0 \leq \theta \leq \pi$

$$\left| \int_R^\infty \frac{1}{z^2 - a^2} dz \right| = \left| \int_0^\pi \frac{\ln(Re^{i\theta}) - \ln(ae^{i\theta})}{R^2 e^{2i\theta} - a^2} d\theta \right|$$

$$= \left| \int_0^\pi \frac{(\ln R + i\theta) - (\ln a + i\theta)}{R^2 e^{2i\theta} - a^2} d\theta \right|$$

$$\leq \int_0^\pi \frac{|\ln R + i\theta| |\ln a + i\theta|}{R^2 e^{2i\theta} - a^2} d\theta$$

Now can manipulate the term like this:

$$|\ln R + i\theta| \leq |\ln R| + |\theta|$$

$$|\ln a + i\theta| \leq |\ln a| + |\theta|$$

$$\leq \int_0^\pi \frac{(|\ln R| + |\theta|)(|\ln a| + |\theta|)}{R^2 e^{2i\theta} - a^2} d\theta = \int_0^\pi \frac{(\ln R + \theta)(\ln a + \theta)}{R^2 e^{2i\theta} - a^2} d\theta$$

CALCULATE THE RESIDE:

$$\lim_{z \rightarrow a} \int_L f(z) dz = \lim_{z \rightarrow a} \int_L \frac{1}{z-a} dz = \lim_{z \rightarrow a} \int_L \frac{1}{z-a} dz$$

$$= \frac{1}{a} \lim_{z \rightarrow a} \int_L \frac{1}{z-a} dz = \frac{1}{a}$$

$$= \frac{\pi i \ln a}{a^2 - a^2} + \frac{R^2 \times \frac{1}{2\pi} \times 2\pi i}{a^2 - a^2} = 0 \left(\frac{\ln a}{a^2} \right) + O\left(\frac{1}{R}\right)$$

$$\rightarrow 0 \text{ AS } R \rightarrow \infty$$

\bullet NOT THE CONTRIBUTION OF $\int_0^\infty \frac{1}{z^2 - a^2} dz = \int_0^\infty \frac{1}{z^2} dz$ AS $\epsilon \rightarrow 0$

$$\left| \int_0^\epsilon \frac{1}{z^2 - a^2} dz \right| = \left| \int_0^\epsilon \frac{\ln(z^2) - \ln(a^2)}{z^2 - a^2} dz \right|$$

$$= \left| \int_0^\epsilon \frac{(\ln z + i2\theta) - (\ln a + i2\theta)}{z^2 - a^2} dz \right|$$

$$\leq \int_0^\epsilon \frac{|\ln z + i2\theta| |\ln a + i2\theta|}{z^2 - a^2} dz$$

Same manipulation done again as the one in Γ (see previous). The two inequalities in the green boxes:

$$\begin{aligned} &\leq \int_0^\epsilon \frac{(|\ln z| + |2\theta|)(|\ln a| + |2\theta|)}{z^2 - a^2} dz = \int_0^\epsilon \frac{(|\ln z| + |2\theta|) \epsilon}{z^2 - a^2} dz \\ &= \frac{\epsilon \ln z}{a^2 - z^2} \int_0^\epsilon \frac{1}{z^2} dz + \frac{\epsilon}{a^2 - z^2} \int_0^\epsilon 2\theta dz \\ &= \frac{\epsilon \ln z}{a^2 - z^2} \rightarrow \frac{\epsilon}{a^2 - z^2} \rightarrow 0 \text{ AS } \epsilon \rightarrow 0 \end{aligned}$$

SINCE THE DENOMINATOR $\rightarrow a^2$

FIRST NUMERATOR $\rightarrow 0$ SINCE $\epsilon \rightarrow 0$ FASTER THAN $\ln z \rightarrow -\infty$

SECOND NUMERATOR $\rightarrow 0$ SINCE IT JUST SIMPLY ϵ

Question 6

By integrating a suitable complex function over an appropriate contour show that

$$\int_{-\infty}^{\infty} \operatorname{sech} x \, dx = \pi.$$

proof

• $\int_{\Gamma} \operatorname{sech} z \, dz = \int_{\Gamma} \frac{1}{2e^z + e^{-z}} \, dz = \int_{\Gamma} \frac{z}{e^{2z} + 1} \, dz$

• NEED TO FIND POLES INSIDE Γ $\Rightarrow 2z = i\pi k + i(2k+1)\pi$
 $\Rightarrow e^{2z} + 1 = 0$
 $\Rightarrow e^{2z} = -1$
 $\Rightarrow z = \frac{i\pi}{2}(2k+1)$
 $\Rightarrow 2z = i\pi(2k+1)$

• NOW CONSIDER $\int_{\Gamma} f(z) \, dz$, $f(z) = \operatorname{sech} z$, WHERE Γ IS THE COMPLEX PLANE. SEE FIGURE.

• THERE ARE 4 SIMPLE POLES INSIDE Γ WITH RESIDUES:
 $\lim_{z \rightarrow \pm i\pi/2} [f(z)(z - \pm i\pi/2)] = \lim_{z \rightarrow \pm i\pi/2} \frac{z}{e^{2z} + 1} = \frac{0}{0}$ (SINGULAR POINT)
 \Rightarrow 4 RESIDUES OF $f(z)$

• BY 1/4THRING RULE
 $= \lim_{z \rightarrow i\pi/2} \left[\frac{2}{e^{2z} + 1} \right] = \frac{2}{e^{2(i\pi/2)} + 1} = \frac{2}{e^{i\pi} + 1} = \frac{1}{e^{i\pi} - 1} = \frac{1}{-e^{\pi} - 1} = -\frac{1}{e^{\pi} + 1}$

• BY THE RESIDUE THEOREM
 $\int_{\Gamma} f(z) \, dz = 2\pi i \times 5$ (BOTTOM INSIDE Γ)
 $\Rightarrow \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} f(z) \, dz = 2\pi i(-5) = -10\pi i$

• CONSIDER THE CONTRIBUTION ALONG Γ_4 AS $R \rightarrow \infty$
 $\Rightarrow z = Re^{i\theta}, 0 < \theta < \pi$

• $\left| \int_{\Gamma_4} f(z) \, dz \right| = \left| \int_{\Gamma_4} \frac{z}{e^{2z} + 1} \, dz \right| \leq \int_{\Gamma_4} \left| \frac{2z}{e^{2z} + 1} \right| \, dz$

$\leq \int_{\Gamma_4} \frac{2}{|e^{2z}| + 1} \, dz \leq \int_{\Gamma_4} \frac{2}{e^{2R} + 1} \, dz \leq \frac{2}{e^{2R}} \int_{\Gamma_4} 1 \, dz = \frac{2}{e^{2R}}$

$\Rightarrow \int_{\Gamma_4} f(z) \, dz = \frac{2}{e^{2R}} \int_{\Gamma_4} 1 \, dz = \frac{2}{e^{2R}} \rightarrow 0 \text{ AS } R \rightarrow \infty$

• CONSIDER THE CONTRIBUTION ALONG Γ_1 AS $R \rightarrow \infty$
 $\Rightarrow z = Re^{i\theta}, \pi < \theta < 2\pi$

• $\int_{\Gamma_1} f(z) \, dz + \int_{\Gamma_2} f(z) \, dz = 2\pi i$

$\Rightarrow \int_{\Gamma_1} \frac{2}{e^{2z} + 1} \, dz + \int_{\Gamma_2} \frac{2}{e^{2z} + 1} \, dz = 2\pi i$

$\Rightarrow \int_{\Gamma_1} \frac{2}{e^{2z} + 1} \, dz + \int_{\Gamma_2} \frac{2}{e^{2z} + 1} \, dz = 2\pi i$

$\Rightarrow \int_{\Gamma_1} \frac{2}{e^{2z} + 1} \, dz + \int_{\Gamma_2} \frac{2}{e^{2z} + 1} \, dz = 2\pi i$

$\Rightarrow 2 \int_{\Gamma_1} \frac{2}{e^{2z} + 1} \, dz = 2\pi i$

$\Rightarrow \int_{\Gamma_1} \operatorname{sech} z \, dz = \pi$

• $\int_{\Gamma} \frac{2}{e^{2z} + 1} \, dz = 2\pi i \times 5$ (BOTTOM INSIDE Γ)
 $\Rightarrow \int_{\Gamma_1} \frac{2}{e^{2z} + 1} \, dz = -\frac{1}{2\pi i} \int_{\Gamma} \frac{2}{e^{2z} + 1} \, dz$
 $\Rightarrow -\frac{2}{e^{2z} + 1} \rightarrow 0 \text{ AS } z \rightarrow \infty$

• SO AS $R \rightarrow \infty$ & $R \rightarrow \infty$
 $\int_{\Gamma_1} f(z) \, dz + \int_{\Gamma_2} f(z) \, dz = 2\pi i$

$\Rightarrow \int_{\Gamma_1} \frac{2}{e^{2z} + 1} \, dz + \int_{\Gamma_2} \frac{2}{e^{2z} + 1} \, dz = 2\pi i$

$\Rightarrow \int_{\Gamma_1} \frac{2}{e^{2z} + 1} \, dz + \int_{\Gamma_2} \frac{2}{e^{2z} + 1} \, dz = 2\pi i$

$\Rightarrow \int_{\Gamma_1} \frac{2}{e^{2z} + 1} \, dz + \int_{\Gamma_2} \frac{2}{e^{2z} + 1} \, dz = 2\pi i$

$\Rightarrow 2 \int_{\Gamma_1} \frac{2}{e^{2z} + 1} \, dz = 2\pi i$

$\Rightarrow \int_{\Gamma_1} \operatorname{sech} z \, dz = \pi$

Question 7

It is required to evaluate the integral

$$\int_0^\infty e^{-x^2} \cos x \, dx.$$

- a) Show that the above integral can be written as

$$\frac{1}{2} e^{-\frac{1}{4}} \int_{-\infty}^\infty e^{-(x+\frac{1}{2}i)^2} dx$$

- b) By integrating the complex function $f(z) = e^{-z^2}$, over a rectangular contour with vertices at $(-R, 0)$, $(R, 0)$, $(R, \frac{1}{2}i)$ and $(-R, \frac{1}{2}i)$, show that

$$\int_0^\infty e^{-x^2} \cos x \, dx = \frac{1}{2} e^{-\frac{1}{4}} \sqrt{\pi}.$$

You may assume without proof that

$$\int_0^\infty e^{-x^2} dx = \frac{1}{2} \sqrt{\pi}.$$

proof

a) $\int_0^\infty e^{-x^2} \cos x \, dx = \frac{1}{2} \int_{-\infty}^\infty e^{-x^2} [(\cos x - i \sin x)] \, dx = \frac{1}{2} \int_{-\infty}^\infty e^{-x^2} e^{-ix} \, dx$

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$$= \frac{1}{2} \int_{-\infty}^\infty e^{-x^2-i x} \, dx = \frac{1}{2} \int_{-\infty}^\infty e^{-(x+\frac{1}{2}i)^2} \, dx = \frac{1}{2} \int_{-\infty}^\infty e^{-(x+\frac{1}{2}i)^2} \, dx$$

$$= \frac{1}{2} \int_{-\infty}^\infty e^{-(x+\frac{1}{2}i)^2} \, dx = \frac{1}{2} e^{-\frac{1}{4}} \int_{-\infty}^\infty e^{-(x+\frac{1}{2}i)^2} \, dx$$

b) Now consider $\int_{-\infty}^\infty e^{-z^2} dz$, where γ is the contour below

As $f(z)$ has no poles inside γ , by Cauchy's theorem

$$\int_{-\infty}^\infty e^{-z^2} dz = 0$$

$$\Rightarrow \int_{-R}^R e^{-z^2} dz + \int_{y=\frac{1}{2}i}^{\frac{1}{2}i} e^{-(z+\frac{1}{2}i)^2} (idz) + \int_R^{\frac{1}{2}i} e^{-(z+\frac{1}{2}i)^2} dz + \int_{y=\frac{1}{2}i}^{-R} e^{-(z+\frac{1}{2}i)^2} (idz) = 0$$

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$$\left\{ \begin{array}{l} dz = dz \\ dz = idz \\ dz = dz \\ dz = idz \end{array} \right\} \left\{ \begin{array}{l} z = z \\ z = z+1 \\ z = z \\ z = z-1 \end{array} \right\} \left\{ \begin{array}{l} dz = dz \\ dz = dz \\ dz = dz \\ dz = idz \end{array} \right\} \left\{ \begin{array}{l} z = z \\ z = z \\ z = z \\ z = z \end{array} \right\}$$

As $R \rightarrow \infty$, the sum of the integrals vanish as they are $O(e^{-R})$

Thus as $R \rightarrow \infty$

$$\int_{-\infty}^\infty e^{-z^2} dz + \int_{-\infty}^\infty e^{-(z+\frac{1}{2}i)^2} dz = 0$$

$$\Rightarrow \int_{-\infty}^\infty e^{-(z+\frac{1}{2}i)^2} dz = - \int_{-\infty}^\infty e^{-z^2} dz$$

$$\Rightarrow \int_{-\infty}^\infty e^{-(z+\frac{1}{2}i)^2} dz = \int_{-\infty}^\infty e^{-z^2} dz$$

$$\Rightarrow \int_{-\infty}^\infty e^{-(z+\frac{1}{2}i)^2} dz = \sqrt{\pi}$$

$$\Rightarrow \frac{1}{2} e^{-\frac{1}{4}} \int_{-\infty}^\infty e^{-(z+\frac{1}{2}i)^2} dz = \frac{1}{2} e^{-\frac{1}{4}} \sqrt{\pi}$$

$$\Rightarrow \int_0^\infty e^{-z^2} \cos z \, dz = \frac{1}{2} e^{-\frac{1}{4}} \sqrt{\pi}$$

Question 8

$$f(z) \equiv \frac{1}{z}, z \in \mathbb{C}, z \neq 0.$$

By considering the integral of $f(z)$ over two different suitably parameterized closed paths, show that

$$\int_0^{2\pi} \frac{1}{9\cos^2 \theta + 4\sin^2 \theta} d\theta = \frac{\pi}{3}.$$

□, proof

PICK A PATH WHICH CONTAINS THE SINGULARITY AT THE ORIGIN, SAY A UNIT CIRCLE IN THE DIRECTION

$$\oint_C \frac{1}{z} dz = 2\pi i \quad \leftarrow \text{STANDARD RESULT}$$

OR PARAMETRIZE AS $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta$, θ FROM 0 TO 2π

$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} (ie^{i\theta}) d\theta = \int_0^{2\pi} i d\theta = 2\pi i$$

NEXT CONSIDER ANOTHER PATH FOLIOWING 0, SAY Γ

$$\oint_{\Gamma} \frac{1}{z} dz = \oint_{\Gamma} \frac{1}{z-i} (dz+idz) = \oint_{\Gamma} \frac{z-i}{(z-i)^2} (dz+idz) = 2\pi i$$

SPLIT INTO PARTS & WORK WITH INDEPENDENTLY

$$\oint_{\Gamma} \frac{1}{z-i} [zdz + idz - iydz - idy] = 2\pi i$$

$$\oint_{\Gamma} \frac{zdz}{z-i} + \oint_{\Gamma} \frac{-idy}{z-i} = 2\pi i$$

NO REAL PATH IN THIS

$$\oint_{\Gamma} \frac{-idy + idz}{z-i} = 2\pi i$$

Now let Γ be a parametrized curve as follows

- $x = 3\cos\theta$
- $y = 2\sin\theta$
- $dx = -3\sin\theta d\theta$
- $dy = 2\cos\theta d\theta$
- $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \rightarrow \int_0^{2\pi} \frac{-2\sin\theta (-3\sin\theta d\theta) + 3\cos\theta (2\cos\theta d\theta)}{9\cos^2\theta + 4\sin^2\theta} d\theta &= 2\pi \\ \rightarrow \int_0^{2\pi} \frac{6\sin^2\theta + 6\cos^2\theta}{9\cos^2\theta + 4\sin^2\theta} d\theta &= 2\pi \\ \rightarrow \int_0^{2\pi} \frac{6}{9\cos^2\theta + 4\sin^2\theta} d\theta &= 2\pi \\ \rightarrow \int_0^{2\pi} \frac{6}{9\cos^2\theta + 4\sin^2\theta} d\theta &= 2\pi \\ \therefore \int_0^{2\pi} \frac{1}{9\cos^2\theta + 4\sin^2\theta} d\theta &= \frac{\pi}{3} \end{aligned}$$

Question 9

The complex number $z = c + a\cos\theta + i b\sin\theta$, $0 \leq \theta < 2\pi$, traces a closed contour C , where a , b and c are positive real numbers with $a > c$.

By considering

$$\oint_C \frac{1}{z} dz,$$

show that

$$\int_0^{2\pi} \frac{a + c\cos\theta}{(c + a\cos\theta)^2 + (b\sin\theta)^2} d\theta = \frac{2\pi}{b}.$$

proof

$\int_0^{2\pi} \frac{a + c\cos\theta}{(c + a\cos\theta)^2 + b^2\sin^2\theta} d\theta = \frac{2\pi}{b}$, $a, b, c \in \mathbb{R}$, $a > c$

Let $z = c + a\cos\theta + i b\sin\theta$, $\theta \in [0, 2\pi]$

$dz = -a\sin\theta d\theta + ib\cos\theta d\theta$
 $dz = (-a\sin\theta + ib\cos\theta) d\theta$

Now considering the integral curve

$$\Rightarrow \oint_C \frac{1}{z} dz = 2\pi i \quad (\text{c is inside } C \text{ as } a > c)$$

$$\Rightarrow \int_0^{2\pi} \frac{1}{[c + a\cos\theta] + i[b\sin\theta]} (a\cos\theta + ib\sin\theta) d\theta = 2\pi i$$

$$\Rightarrow \int_0^{2\pi} \frac{[-a\sin\theta + ib\cos\theta][[(c + a\cos\theta) - i b\sin\theta]]}{[(c + a\cos\theta) + ib\sin\theta][(c + a\cos\theta) - i b\sin\theta]} d\theta = 2\pi i$$

$$\Rightarrow \int_0^{2\pi} \frac{(-a\sin\theta + ib\cos\theta)[(c + a\cos\theta) - i b\sin\theta]}{(c + a\cos\theta)^2 + b^2\sin^2\theta} d\theta = 2\pi i$$

$$\Rightarrow \int_0^{2\pi} \frac{[-a(c + a\cos\theta)\cos\theta + b^2\sin^2\theta] + i[c\cos\theta + a\cos^2\theta + b\sin\theta\cos\theta]}{(c + a\cos\theta)^2 + b^2\sin^2\theta} d\theta = 2\pi i$$

Now separating the integral into real & imaginary

$$\int_0^{2\pi} \frac{(b^2 - a^2)a\cos\theta - a\cos^2\theta}{(c + a\cos\theta)^2 + b^2\sin^2\theta} d\theta + i \int_0^{2\pi} \frac{ab + b\cos\theta}{(c + a\cos\theta)^2 + b^2\sin^2\theta} d\theta = 2\pi i$$

The real integral times zero
looking at the imaginary integral

$$ib \int_0^{2\pi} \frac{a + c\cos\theta}{(c + a\cos\theta)^2 + b^2\sin^2\theta} d\theta = 2\pi i$$

$$\int_0^{2\pi} \frac{a + c\cos\theta}{(c + a\cos\theta)^2 + b^2\sin^2\theta} d\theta = \frac{2\pi}{b}$$