

# **RESIDUES and APPLICATIONS in SERIES SUMMATION**

**The Residue Theorem can often be used to sum various types of series.**

The following results are valid under some restrictions on  $f(z)$ , which more often than not are satisfied when the series converges.

$$\sum_{r=-\infty}^{\infty} f(r)$$

use  $\oint_{\Gamma_n} f(z) \pi \cot \pi z \, dz$ , where  $\Gamma_n$  is the square with vertices at  $(n + \frac{1}{2})(\pm 1 \pm i)$

$$\sum_{r=-\infty}^{\infty} (-1)^r f(r)$$

use  $\oint_{\Gamma_n} f(z) \pi \operatorname{cosec} \pi z \, dz$ , where  $\Gamma_n$  is the square with vertices at  $(n + \frac{1}{2})(\pm 1 \pm i)$

$$\sum_{r=-\infty}^{\infty} f\left(\frac{2r+1}{2}\right)$$

use  $\oint_{\Gamma_n} f(z) \pi \tan \pi z \, dz$ , where  $\Gamma_n$  is the square with vertices at  $n(\pm 1 \pm i)$

$$\sum_{r=-\infty}^{\infty} (-1)^r f\left(\frac{2r+1}{2}\right)$$

use  $\oint_{\Gamma_n} f(z) \pi \sec \pi z \, dz$ , where  $\Gamma_n$  is the square with vertices at  $n(\pm 1 \pm i)$

## Question 1

$$f(z) = \frac{\pi \cot \pi z}{(a+z)^2}, \quad z \in \mathbb{C}$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=-\infty}^{\infty} \frac{1}{(a+r)^2} = \pi^2 \operatorname{cosec}^2(\pi a), \quad a \notin \mathbb{Z}.$$

proof

**CONSIDER** THE SQUARE OUTLINE WITH VERTICES  $(\operatorname{cosec}(k+1)\pi), k \in \mathbb{Z}$

**THE SIMPLE POLES AT  $Z = i\pi$ ,  $w \in \mathbb{Z}$**

$\lim_{z \rightarrow i\pi} [(z-i\pi)^{-1} f(z)] = \lim_{z \rightarrow i\pi} [(z-i\pi)^{-1} \operatorname{cotan}(z)]$

$= \lim_{z \rightarrow i\pi} \frac{\operatorname{cosec}(z)}{(z-i\pi)^{-1}} = \lim_{z \rightarrow i\pi} \frac{\operatorname{cosec}(z)}{z-i\pi} \cdot \frac{1}{z-i\pi}$

By Definition the derivative of  $\operatorname{cosec}(z)$  at  $z = i\pi$

$= \frac{\operatorname{cosec}(z)'}{(z-i\pi)^2} \times \frac{1}{\operatorname{cosec}(z)} = \frac{1}{(z-i\pi)^2}$

**ANALOGY BY L'HOSPITAL**

$\lim_{z \rightarrow i\pi} \frac{(z-i\pi)^{-1} f(z)}{(z-i\pi)^{-1}} = \lim_{z \rightarrow i\pi} \frac{(z-i\pi)^{-1} \operatorname{cotan}(z)}{(z-i\pi)^{-1}}$

$= \lim_{z \rightarrow i\pi} \left[ \operatorname{cotan}(z) - \frac{1}{(z-i\pi)^2} \operatorname{cotan}'(z) \right] = \operatorname{cotan}(i\pi) - \frac{1}{(i\pi)^2} \operatorname{cotan}'(i\pi)$

$= \frac{1}{(i\pi)^2}$  AS REQUIRED

**BY THE RESIDUE THEOREM**

$\int_C f(z) dz = 2\pi i \times \sum (\text{residues inside } \Gamma)$

$\int_C \operatorname{cotan}(z) dz = 2\pi i \left[ \operatorname{Res}_{z=i\pi} + \sum_{n=1}^{\infty} \frac{1}{(i\pi+n\pi)^2} \right]$

THIS PROOF ATTACHED AT THE END

$$\left| \int_{\Gamma_n} f(z) dz \right| = \left| \int_{\Gamma_n} \frac{T(z)}{(z - w_1)^{1-\alpha}} dz \right| = \left| \int_{\Gamma_n} \frac{T(z)}{z^{1-\alpha}(z - w_1)^{\alpha}} dz \right|$$

$$\leq \int_{\Gamma_n} |T(z)| dz \leq \int_{\Gamma_n} \sqrt{|z|^{2(1-\alpha)} + R^2} dz \leq \frac{1}{\sin(\pi\alpha)} \int_{\Gamma_n} |z|^{1-\alpha} dz$$

$$\leq \int_{\Gamma_n} \frac{|w_1|}{(1+|z|)^{1-\alpha}} dz \leq \frac{|w_1|}{1+|w_1|}$$

$$= \frac{\pi R}{(1+R)^{1-\alpha}}$$

RETURNING TO THE INTEGRAL AS  $n \rightarrow \infty$

SUM UP THE 201 AND PEARLANCE TO GET

$$\sum_{k=1}^{\infty} \frac{1}{k^{1-\alpha}} = \pi^{\alpha} \left[ \frac{\pi^2}{6} \cos \alpha^2 \pi^2 + \frac{w_1}{4} \frac{1}{\sin(\pi\alpha)} \right]$$

**Question 2**

$$f(z) = \frac{\pi \cot \pi z}{(3z+1)(2z+1)}, z \in \mathbb{C}.$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=-\infty}^{\infty} \frac{1}{(3r+1)(2r+1)} = \frac{\pi}{\sqrt{3}}.$$

[ ] , proof

GIVEN:  $\int_{-\infty}^{\infty} \frac{\text{Tanh } z}{(az+b)(cz+d)} dz$  WHERE THE REAL SQUARE CONTOUR WITH VERTICES AT  $(m+\frac{1}{2})(a \pm bi)$ .

THE SINGULARITIES ARE ALREADY NERED ON THE 2 AXES. [COMPLEX + COMPLEX]

THE SINGULARITIES AT  $-b/a$  &  $-d/c$  [DENOMINATOR OF  $f(z)$ ]

COMPLEX PARTIALS

- $\lim_{z \rightarrow -\frac{1}{2}a} \left[ \frac{\text{Tanh } z}{(az+b)(cz+d)} \right] = \lim_{z \rightarrow -\frac{1}{2}a} \left[ \frac{\text{Tanh } z}{(az+b)(cz+1)} \right] = \lim_{z \rightarrow -\frac{1}{2}a} \left[ \frac{\text{Tanh } z}{(az+b)(cz+1)} \right]_{M \in \mathbb{Z}}$
- THIS WILL NOW BE AN INDEFINITE INTEGRAL OF THE TYPE "END ONE PART, DO BY 2 INTEGRATION" RULE
- $= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\text{Tanh } z - \frac{1}{2}(a+bi)}{(az+b)(cz+1)(az+1)} dz = \frac{1}{2\pi i} \frac{\text{constant}}{(a+bi)(a+1)} = \frac{1}{(a+bi)(a+1)}$
- $\bullet \lim_{z \rightarrow -\frac{1}{2}c} \left[ \frac{\text{Tanh } z}{(az+b)(cz+d)} \right] = \lim_{z \rightarrow -\frac{1}{2}c} \left[ \frac{\text{Tanh } z}{(az+1)(cz+1)} \right] = \frac{\text{Tanh } (-\frac{1}{2}c)}{-1} = 0$

BY THE RESIDUE THEOREM WE NOW HAVE

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum(\text{residues inside } \Gamma)$$

$$\int_{\Gamma} \frac{\text{Tanh } z}{(az+b)(cz+d)} dz = 2\pi i \times \left[ \frac{\pi}{iz} + \sum_{n=-\infty}^{\infty} \frac{1}{(an+b)(cn+d)} \right]$$

$$\int_{\Gamma} \frac{\text{Tanh } z}{(az+1)(cz+1)} dz = 2\pi i \times \left[ \frac{\pi}{iz} + \sum_{n=-\infty}^{\infty} \frac{1}{(an+1)(cn+1)} \right]$$

IT CAN BE SHOWN THAT  $\text{COTH } z$  IS BOUNDED ON  $\Gamma$ , i.e.  $|\text{COTH } z| \leq M$  FOR SOME  $M > 0$  - THIS IS SHOWN AS A SEPARATE PROOF AT THE END OF THIS QUESTION

THUS WE HAVE THE FOLLOWING

$$\left| \int_{\Gamma} \frac{\text{Tanh } z}{(az+1)(cz+1)} dz \right| \leq \int_{\Gamma} \left| \frac{\text{Tanh } z}{(az+1)(cz+1)} \right| dz = \left| \int_{\Gamma} \frac{\text{Tanh } z}{(az+1)(cz+1)} dz \right|$$

$$= \int_{\Gamma} \left| \frac{\text{Tanh } z}{(az+1)(cz+1)} \right| dz$$

PROOF THAT  $\text{COTH } z$  IS BOUNDED ON THE SQUARE CONTOUR WITH VERTICES AT  $(m+\frac{1}{2})(a \pm bi)$

$$|\text{COTH } z| = \left| \frac{\text{COT } z}{\text{SINH } z} \right| = \frac{1}{2} \left| \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \right| = \frac{1}{2} \left| \frac{e^{iz} + e^{-iz}}{|e^{iz} - e^{-iz}|} \right| \quad (1)$$

LEMMA:  $|e^{iz}| \leq M$  (BOUNDED)  
LEMMA:  $|e^{-iz}| \geq M-1$  (INCREASING)  
LEMMA:  $|e^{iz} - e^{-iz}| \leq 2M-2$

$$\leq \frac{1}{2} \frac{|e^{iz} + e^{-iz}|}{|e^{iz} - e^{-iz}|} \leq \frac{1}{2} \frac{|e^{iz}| + |e^{-iz}|}{|e^{iz} - e^{-iz}|} \leq \frac{1}{2} \frac{M + M}{2M-2} = \frac{1}{2}$$

DEFINITION:  $\text{tanh } z = \frac{\sinh z}{\cosh z}$

$$= \frac{-1 \times e^{-iz} + 1}{-1 \times e^{-iz} - 1} = \frac{-e^{-iz} + 1}{-e^{-iz} - 1} = \frac{e^{iz} - 1}{e^{iz} + 1}$$

DEFINITION:  $\text{tanh } (-z) = -\text{tanh } z = |\text{tanh } z| < 1$

DEFINITION:  $\text{tanh } z$  IS A DECREASING FUNCTION FOR  $z > 0$ .  
 $\text{tanh } z > \text{tanh } \frac{z}{2} > \text{tanh } \frac{z}{4} \dots$   
 $\text{so } \text{tanh } z > \text{tanh } \frac{z}{2} > \text{tanh } \frac{z}{4} \dots$

NOTE: ON THE "VERTICAL" SIDE OF THE CONTOUR,  $z = \pm(m+\frac{1}{2}) + iy$  WHERE  $-(M+\frac{1}{2}) \leq y \leq M+\frac{1}{2}$  - PROCEED AS BEFORE

$$|\text{COTH } z| \leq \left| \frac{\text{COT } z}{\text{SINH } z} \right| = \left| \frac{\frac{1}{2}(e^{iz} + e^{-iz})}{\frac{1}{2}(e^{iz} - e^{-iz})} \right| = \left| \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \right|$$

$$= \left| \frac{e^{2iz} + 1}{e^{2iz} - 1} \right| = \left| \frac{e^{2iz} + 1}{e^{2iz} - 1} + 1 \right|$$

$$= \left| \frac{e^{2iz} + 1}{e^{2iz} - 1} + 1 \right| = \left| \frac{e^{2iz} + 1 + e^{2iz} - 1}{e^{2iz} - 1} \right| = \left| \frac{2e^{2iz}}{e^{2iz} - 1} \right|$$

$$= \left| \frac{2e^{2iz}}{e^{2iz} - 1} \right| = \left| \frac{2e^{2iz}}{e^{2iz} - 1} \right| = \left| \frac{2e^{2iz}}{e^{2iz} - 1} \right| = 1$$

ON THE HORIZONTAL SIDES OF THE CONTOUR,  $z = 2a(m+\frac{1}{2}) - (b+ci)y$

- $|\text{COT } z| = \left| \frac{1}{e^{iz} - e^{-iz}} \right| = \left| \frac{1}{e^{iz} - e^{-iz}} \right| = \frac{1}{2} \left| \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \right| = \frac{1}{2} \left| \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \right|$
- $|\text{e}^{-iz}| \dots$  BY ANALOGOUS WORKING

$$\therefore |\text{COTH } z| \leq \frac{e^{-iz} + 1}{e^{-iz} - 1} = \frac{2 \text{cosec} [\pi(m+\frac{1}{2})]}{2 \text{cosec} [\pi(m+\frac{1}{2})]}$$

$$= \frac{2 \text{cosec} [\pi(m+\frac{1}{2})]}{2 \text{cosec} [\pi(m+\frac{1}{2})]} = \text{cosec} [\pi(m+\frac{1}{2})] \leq \text{cosec} \frac{\pi}{2} = 2$$

AS  $\text{cosec } z$  IS A DECREASING FUNCTION FOR  $z > 0$ ,  
 $\text{cosec } z > \text{cosec } \frac{z}{2} > \text{cosec } \frac{z}{4} \dots$   
 $\text{so } \text{cosec } z > \text{cosec } \frac{z}{2} > \text{cosec } \frac{z}{4} \dots$

NOTE: ON THE "VERTICAL" SIDE OF THE CONTOUR,  $z = \pm(m+\frac{1}{2}) + iy$  WHERE  $-(M+\frac{1}{2}) \leq y \leq M+\frac{1}{2}$  - PROCEED AS BEFORE

$$|\text{COT } z| \leq \left| \frac{\text{COT } z}{\text{SINH } z} \right| = \left| \frac{\frac{1}{2}(e^{iz} + e^{-iz})}{\frac{1}{2}(e^{iz} - e^{-iz})} \right| = \left| \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \right|$$

$$= \left| \frac{e^{2iz} + 1}{e^{2iz} - 1} \right| = \left| \frac{e^{2iz} + 1}{e^{2iz} - 1} + 1 \right|$$

$$= \left| \frac{e^{2iz} + 1}{e^{2iz} - 1} + 1 \right| = \left| \frac{e^{2iz} + 1 + e^{2iz} - 1}{e^{2iz} - 1} \right| = \left| \frac{2e^{2iz}}{e^{2iz} - 1} \right|$$

$$= \left| \frac{2e^{2iz}}{e^{2iz} - 1} \right| = \left| \frac{2e^{2iz}}{e^{2iz} - 1} \right| = \left| \frac{2e^{2iz}}{e^{2iz} - 1} \right| = 1$$

DEFINITION:  $\text{tanh } z$  IS BOUNDED ON THE CONTOUR SINCE

"HORIZONTALLY" BY  $\text{cosec } z$   
"VERTICALLY" BY 1

(E)  $M$  (USED EARLIER) IS THE SUPERIOR OF THESE TWO

$M = \sup(|\text{COTH } z|)$

**Question 3**

$$f(z) = \frac{\pi \cot \pi z}{4z^2 - 1}, z \in \mathbb{C}.$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=1}^{\infty} \frac{1}{4r^2 - 1} = \frac{1}{2}.$$

**[proof]**

Consider  $f(z) = \frac{\pi \cot \pi z}{4z^2 - 1}$ . Integrating over the standard square contour  $\Gamma$  with vertices at  $(n+1/2, 0), (n+1/2, 1), (n, 1), (n, 0)$ .

(a) HAS SINGULARITIES AT  $z = \pm 1/2$ . AND SINGULARITIES AT EVERY REAL NUMBER SINCE COT(z) IS CO-SHIFTED.

BY L'HOSPITAL'S RULE:

$$\lim_{z \rightarrow \pm 1/2} \frac{\pi \cot \pi z}{4z^2 - 1} = \frac{\pi \cot \pm \pi/2}{\mp 4(1/4)} = \frac{\mp \pi/2}{\mp 1} = \pm \frac{\pi}{2}$$

BY THE RESIDUE THEOREM:

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum \text{residues inside } \Gamma$$

$$\int_{\Gamma} f(z) dz = 2\pi i \times [0 + \sum_{n=0}^{\infty} \frac{\pi}{4n^2 - 1}]$$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \int_{\Gamma} f(z) dz$$

NOW:

- $\text{Res}_{z=1/2} f(z) = \lim_{z \rightarrow 1/2} \frac{(z-1/2) \text{cot}(z\pi/2)}{2(z-1/2)^2} = \lim_{z \rightarrow 1/2} \frac{\frac{1}{2}\pi \text{cosec}^2(z\pi/2)}{2(z-1/2)} = \frac{\pi}{4}$
- $\text{Res}_{z=-1/2} f(z) = \lim_{z \rightarrow -1/2} \frac{(z+1/2) \text{cot}(z\pi/2)}{2(z+1/2)^2} = \lim_{z \rightarrow -1/2} \frac{\frac{1}{2}\pi \text{cosec}^2(z\pi/2)}{2(z+1/2)} = -\frac{\pi}{4}$
- $\int_{\Gamma} f(z) dz = \int_{\Gamma} \frac{\pi \cot \pi z}{4z^2 - 1} dz = \int_{\Gamma} \frac{\pi \cot \pi z}{4z^2 - 1} dz - \int_{\Gamma} \frac{\pi \cot \pi z}{4z^2 - 1} dz$
- $\int_{\Gamma} f(z) dz = \int_{\Gamma} \frac{\pi \cot \pi z}{4z^2 - 1} dz = \int_{\Gamma} \frac{\pi \cot \pi z}{4z^2 - 1} dz + \int_{\Gamma} \frac{\pi \cot \pi z}{4z^2 - 1} dz = 0$

HENCE ACCORDING TO THE INTEGRAL AND THE RESIDUES:

$$2\pi i \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = 0$$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = 0$$

L.E.  $\dots - \frac{1}{4} + \frac{1}{12} - \frac{1}{28} + \frac{1}{48} - \dots = 0$

2.  $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = 0$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

OR  $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$  # required

PROOF THAT  $\text{cot } \pi z$  IS BOUNDED ON THE SQUARE CONTOUR WITH VECTORS  $(n+1/2)(z \pm 1/2)$   $n = 1, 2, 3, \dots$

- $|\text{cot } \pi z| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \left| \frac{\cos \pi z}{\sqrt{1 - \cos^2 \pi z}} \right| = \left| \frac{\cos \pi z}{\sqrt{1 - \cos^2 \pi z}} \right| \leq \frac{|\cos \pi z| + |\cos \pi z|}{|\cos \pi z| - |\cos \pi z|} = \frac{2|\cos \pi z|}{|\cos \pi z| - |\cos \pi z|} = \frac{2}{|\cos \pi z|} > 2 > 1$  # MINIMISE  $|\cos \pi z|$  #  $|\cos \pi z| > |z| - 1$
- DO THE HORIZONTAL SIDES OF THE CONTOUR:  $z = x + i(0 \pm 1)$ ,  $-1 \leq x \leq 1$ 

$$|\text{cot } \pi z| = |\text{cot}(\pi x \pm \pi i)| = |\text{cot} \pi x \pm \text{cot} \pi i| = |\text{cot} \pi x| = e^{\pi x i} = e^{\pi x i}$$

$$|\text{cot } \pi z| = |\text{cot}(\pi x \pm \pi i)| = |\text{cot} \pi x \pm \text{cot} \pi i| = |\text{cot} \pi x| = e^{\pi x i}$$
- $\therefore |\text{cot } \pi z| = \frac{e^{\pi x i} + e^{-\pi x i}}{e^{\pi x i} - e^{-\pi x i}} = \frac{2\cos(\pi x \pm \pi/2)}{2\sin(\pi x \pm \pi/2)} = \frac{2\cos(\pi x \pm \pi/2)}{2\sin(\pi x \pm \pi/2)} = \frac{|\text{cot}(\pi x \pm \pi/2)|}{|\text{cot}(\pi x \pm \pi/2)|} \approx \text{const}$  (since  $x \approx 0$ )
- NOW ON THE VERTICAL SIDES OF THE CONTOUR:  $z = (n+1/2) + iy$ ,  $-1 \leq y \leq 1$ 

$$|\text{cot } \pi z| = \left| \frac{\cos((n+1/2)\pi) + iy}{\sin((n+1/2)\pi) - iy} \right| = \left| \frac{e^{(n+1/2)\pi} + iy}{e^{(n+1/2)\pi} - iy} \right| = \left| \frac{e^{(n+1/2)\pi} - 1}{e^{(n+1/2)\pi} - 1} \right| = \frac{e^{(n+1/2)\pi} + 1}{e^{(n+1/2)\pi} - 1}$$

$$= \left| \frac{e^{(n+1/2)\pi} - iy + 1}{e^{(n+1/2)\pi} - iy - 1} \right| = \left| \frac{-iy + e^{(n+1/2)\pi} + 1}{-iy + e^{(n+1/2)\pi} - 1} \right| = \left| \frac{e^{(n+1/2)\pi}}{e^{(n+1/2)\pi} - 1} \right| = |\text{tan}(y)| \leq 1$$
- $\therefore \text{cot } \pi z$  IS BOUNDED ON THE CONTOUR (AND IS  $\leq \text{const}$ )

## Question 4

$$f(z) = \frac{\pi \cot \pi z}{z^2}, \quad z \in \mathbb{C}$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$$

proof

**PROOF THAT  $\cot \frac{\pi}{n}$  IS BOUNDED ON THE SQUARE CONCERN WITH VERTICES  $(\pm \frac{1}{n}, \pm \frac{1}{n})$ ,  $n=1,2,3,4,\dots$**

- $|\cot \frac{\pi}{n}| = \left| \frac{\cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} \right| = \left| \frac{\frac{1}{2}(\cos(\frac{\pi}{n}) + i\sin(\frac{\pi}{n}))}{\frac{i}{2}(\sin(\frac{\pi}{n}) - \cos(\frac{\pi}{n}))} \right| = \frac{|\frac{1}{2}\cos(\frac{\pi}{n}) + \frac{i}{2}\sin(\frac{\pi}{n})|}{|\frac{i}{2}\sin(\frac{\pi}{n}) - \frac{1}{2}\cos(\frac{\pi}{n})|} \leq \frac{|\frac{1}{2}\cos(\frac{\pi}{n})| + |\frac{i}{2}\sin(\frac{\pi}{n})|}{|\frac{1}{2}\sin(\frac{\pi}{n})| + |\frac{1}{2}\cos(\frac{\pi}{n})|} = \dots$  NUMERATOR  
 $= \frac{1}{2}|\cos(\frac{\pi}{n})| \leq |\cos(\frac{\pi}{n})|$   
 $= \frac{1}{2}|\sin(\frac{\pi}{n})| \leq |\sin(\frac{\pi}{n})|$   
 $\frac{1}{2}(|\cos(\frac{\pi}{n})| + |\sin(\frac{\pi}{n})|) \leq \frac{1}{2}(1+1) = 1$
- DO THE HYPOTENUSE SIDES OF THE CONCERN:  $Z = x \pm i y = (\cos(\frac{\pi}{n}) \pm i \sin(\frac{\pi}{n}))$ ,  $|x| < |y| < |\cos(\frac{\pi}{n})|$   
 $|\text{hyp}| = \sqrt{x^2 + y^2} = \sqrt{(\cos(\frac{\pi}{n}) \pm i \sin(\frac{\pi}{n}))^2} = \sqrt{[\cos^2(\frac{\pi}{n}) \mp 2\cos(\frac{\pi}{n})\sin(\frac{\pi}{n}) + \sin^2(\frac{\pi}{n})]} = e^{\pm i\pi/2}$   
 $e^{\pm i\pi/2} = \sqrt{[\cos^2(\frac{\pi}{n}) \pm 2\cos(\frac{\pi}{n})\sin(\frac{\pi}{n}) + \sin^2(\frac{\pi}{n})]} = \sqrt{[\cos^2(\frac{\pi}{n}) \pm 2\cos(\frac{\pi}{n})\sin(\frac{\pi}{n}) + \sin^2(\frac{\pi}{n})]} = e^{\pm i(\pi/2)}$
- $|\cot \frac{\pi}{n}| = \dots = \frac{\frac{e^{\pm i\pi/2}}{\cos^2(\frac{\pi}{n})} + \frac{i}{\sin^2(\frac{\pi}{n})}}{\frac{e^{\pm i\pi/2}}{\cos^2(\frac{\pi}{n})} - \frac{i}{\sin^2(\frac{\pi}{n})}} = \frac{2\cos(\frac{\pi}{n}) + i\sin(\frac{\pi}{n})}{2\sin(\frac{\pi}{n}) - i\cos(\frac{\pi}{n})} = \frac{2\cos(\frac{\pi}{n}) + i\sin(\frac{\pi}{n})}{(2\sin(\frac{\pi}{n}) - i\cos(\frac{\pi}{n}))^2} = |\cot(\frac{\pi}{n})| < \sqrt{\frac{4}{4\sin^2(\frac{\pi}{n}) + 1}}$  THE HYPOTENUSE  
SIDES OF THE CONCERN  
ARE  $\sqrt{4\sin^2(\frac{\pi}{n}) + 1}$
- NOW DO THE VERTICAL SIDES OF THE CONCERN:  $Z = \pm(\frac{1}{n} + i\frac{1}{n})$ ,  $-1 < \text{Re}(Z) < 1$   
 $|\text{left}| = \left| \frac{\cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} \right| = \left| \frac{\frac{1}{2}(\cos(\frac{\pi}{n}) + i\sin(\frac{\pi}{n}))}{\frac{i}{2}(\sin(\frac{\pi}{n}) - \cos(\frac{\pi}{n}))} \right| = \frac{|\frac{1}{2}\cos(\frac{\pi}{n}) + \frac{i}{2}\sin(\frac{\pi}{n})|}{|\frac{i}{2}\sin(\frac{\pi}{n}) - \frac{1}{2}\cos(\frac{\pi}{n})|} = \frac{\frac{1}{2}|\cos(\frac{\pi}{n})| + \frac{1}{2}|\sin(\frac{\pi}{n})|}{\frac{1}{2}|\sin(\frac{\pi}{n})| + \frac{1}{2}|\cos(\frac{\pi}{n})|} = \frac{\frac{1}{2}|\cos(\frac{\pi}{n})| + \frac{1}{2}}{\frac{1}{2}|\sin(\frac{\pi}{n})| + \frac{1}{2}} = \frac{|\cos(\frac{\pi}{n})| + 1}{|\sin(\frac{\pi}{n})| + 1} = \frac{|\cos(\frac{\pi}{n})| + 1}{|\tan(\frac{\pi}{n})| + 1} = |\cot(\frac{\pi}{n})| < 1$
- $|\text{right}| = \left| \frac{\cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} \right| = \left| \frac{\frac{1}{2}(\cos(\frac{\pi}{n}) + i\sin(\frac{\pi}{n}))}{\frac{i}{2}(\sin(\frac{\pi}{n}) - \cos(\frac{\pi}{n}))} \right| = \frac{|\frac{1}{2}\cos(\frac{\pi}{n}) + \frac{i}{2}\sin(\frac{\pi}{n})|}{|\frac{i}{2}\sin(\frac{\pi}{n}) - \frac{1}{2}\cos(\frac{\pi}{n})|} = \frac{\frac{1}{2}|\cos(\frac{\pi}{n})| + \frac{1}{2}}{\frac{1}{2}|\sin(\frac{\pi}{n})| + \frac{1}{2}} = \frac{|\cos(\frac{\pi}{n})| + 1}{|\sin(\frac{\pi}{n})| + 1} = \frac{|\cos(\frac{\pi}{n})| + 1}{|\tan(\frac{\pi}{n})| + 1} = |\cot(\frac{\pi}{n})| < 1$
- $|\text{left}| \leq \text{left}$  IS BOUNDED ON THE CONCERN  
true

## Question 5

$$f(z) = \frac{\pi \cot \pi z}{z^4}, \quad z \in \mathbb{C}$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{\pi^4}{90}.$$

proof

- IT CAN BE SHOWN THAT  $\int_G |f| \leq M$  FOR SOME  $M \in \mathbb{R}$ .  
(IT IS FINITE AT THE VERY END)
- ALSO GEOMETRICALLY  

$$\left| \frac{1}{n} \right| \leq \frac{1}{n+1} \quad \forall z \in G_n$$
- THUS  

$$\left| \int_G \frac{|f(z)|}{n^2} dz \right| \leq \int_G \frac{|f(z)|}{n^2} dz \leq \int_G \frac{M}{(n+1)^2} dz = \frac{M}{(n+1)^2} \cdot n^2 = \frac{Mn}{n+1}$$
LEADER OF THE CONSTANT

$$= \frac{Mn}{n+1} \times \theta(n^{-\frac{1}{2}}) = O(n^{-\frac{1}{2}}) \rightarrow 0 \text{ as } n \rightarrow \infty$$
- RETURNING TO THE INTEGRAL AS  $n \rightarrow +\infty$   
 $\Rightarrow 0 = \lim_{n \rightarrow +\infty} \left[ -\frac{n^2}{48} + \frac{\pi^2 n^2}{48} + \frac{\pi^2}{48} + \frac{1}{n^2} \right] \quad \text{DIVIDE BY } 2\pi i \text{ TO GET}$ 

$$\Rightarrow \sum_{k=0}^{+\infty} \frac{1}{k^2} + \frac{1}{48} = \frac{\pi^2}{48}$$

$$\Rightarrow 2 \sum_{k=0}^{+\infty} \frac{1}{k^2} = \frac{31\pi^2}{48} \quad (\text{SINCE } \frac{1}{k^2} \text{ IS EVEN BOTH SERIES ARE POLYNOMIAL})$$

$$\Rightarrow \sum_{k=0}^{+\infty} \frac{1}{k^2} = \frac{31\pi^2}{96} = \frac{31\pi^2}{72}$$

~~ANSWER~~

$$\text{Q.E.D.} \quad \frac{31\pi^2}{72} = \frac{31\pi^2}{40}$$

## Question 6

$$f(z) = \frac{\pi \cot \pi z}{(z^2 + 1)^2}, \quad z \in \mathbb{C}.$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=1}^{\infty} \frac{1}{(r^2+1)^2} = \frac{1}{4}\pi^2 \operatorname{cosech}^2 \pi + \frac{1}{4}\pi \coth \pi - \frac{1}{2}.$$

[A], proof

$\bullet$  Ques 10  $\int_{-\pi}^{\pi} \left[ -\frac{1}{2}(\cos(2x))^2 - \frac{1}{2}\sin(2x) \right] dx = -\frac{1}{2}\int_0^{2\pi} (\cos^2(u))du - \int_0^{2\pi} \sin(u)du$   
 • Case 1 if  $m = \frac{1}{2}$ ,  $\int_0^{2\pi} \sin(u)du = -\frac{1}{2}\sin(u)|_0^{2\pi} = -\frac{1}{2}\sin(2\pi) = 0$   
 • Case 2 if  $m = \frac{1}{2}$ ,  $\int_0^{2\pi} \sin(u)du = \frac{-\cos(u)}{2}|_0^{2\pi} = -\frac{1}{2}\cos(2\pi) = -\frac{1}{2}$   
 $= \frac{-\pi/2(-\cos(2\pi)) - \pi/2(-\sin(2\pi))}{2\pi} = \frac{\pi/2\cos(2\pi) + \pi/2\sin(2\pi)}{2\pi}$   
 $= -\frac{1}{2}\pi\cos(2\pi) - \frac{1}{2}\pi\sin(2\pi)$   
 $\bullet$  Ques 11  $\int_{-\pi}^{\pi} \left[ \frac{1}{2}(\cos(4x))^2 + \frac{1}{2}\sin(4x) \right] dx = \dots$  ANSWER CONTINUED...  
 $= \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} \left[ -\frac{1}{2}(\cos(2nx))^2 - \frac{1}{2}\sin(2nx) \right] dx = -\frac{1}{2}\int_0^{2\pi} (\cos^2(u))du - \int_0^{2\pi} \sin(u)du$   
 $= -\frac{1}{2}\pi(-\cos(2\pi)) - \pi(-\sin(2\pi)) = -\frac{1}{2}\pi\cos(2\pi) - \pi\sin(2\pi)$   
 $= -\frac{1}{2}\pi\cos(2\pi) - \frac{1}{2}\pi\sin(2\pi)$   
BY THE ROLLE'S THEOREM NOW  
 $\int_{\Gamma} f(x) dx = \pi i \sum \text{(Residues inside } \Gamma)$   
 $\int_{\Gamma} \frac{z^m}{z^2+1} dz = \pi i \sum \left[ -\frac{1}{2}\pi\cos(k\pi) - \frac{1}{2}\pi\sin(k\pi) + \sum_{k=1}^{\infty} \left( \frac{1}{k^2+1} \right) \right]$   
 IT CAN BE SHOWN THAT  $\text{Im}(f(z))$  IS BOUNDED ON  $\Gamma_2$ , i.e.,  $|\text{Im}(f(z))| \leq M$  FOR SOME  $M > 0$  (THIS IS SHOWN AT THE VERY END OF THE QUESTION), AS A SEPARATE PROOF!

WE ALSO HAVE THE FOLLOWING

$$\begin{aligned} |z_1| &\leq \frac{1}{4} + \frac{1}{2} \\ \left|\frac{1}{z_1}\right| &\leq \frac{1}{\frac{1}{4} + \frac{1}{2}} \end{aligned}$$

$$\int_{T_1} \left| \frac{f(z)}{(z-z_1)^n} dz \right| \leq \int_{T_1} \left| \frac{f(z)}{(z-z_1)^n} dz \right| = \int_{T_1} \left| \frac{|f(z)|}{(z-z_1)^n} dz \right|$$

$$\leq \int_{T_1} \frac{\pi M}{\left| \frac{1}{z_1} \right|^n \cdot \left| z-z_1 \right|^{n-1}} |dz|$$

$$\leq \int_{T_1} \frac{\pi M}{\left| \frac{1}{z_1} \right|^n \cdot \left| z-z_1 \right|^{n-1}} |dz|$$

$$= \int_{T_1} \frac{\pi M}{\left( \frac{1}{|z_1|} \right)^n \cdot \left( \frac{|z-z_1|}{|z_1|} \right)^{n-1}} |dz|$$

$$= \frac{\pi M n!}{\left( \frac{1}{|z_1|} \right)^n \cdot \left( \frac{2\pi R}{|z_1|} \right)^{n-1}} = O\left(\frac{1}{R^n}\right) \rightarrow 0 \quad n \rightarrow \infty$$

(length of the contour  $T_1$ )  $\rightarrow$   $B\left(\frac{1}{|z_1|}\right)$

RETURNING TO THE NORMAL AS  $n \rightarrow \infty$

$$\rightarrow 0 = \lim_{n \rightarrow \infty} \left[ -\frac{1}{2\pi i} \operatorname{Res}(z_1) + \frac{1}{2\pi i} \int_{T_n}^{\infty} \frac{f(z)}{(z-z_1)^n} dz \right]$$

$$\rightarrow \sum_{m=1}^{\infty} \frac{1}{(z-z_1)^{m+1}} = \frac{1}{2\pi i} \operatorname{Res}(z_1) + \frac{1}{2\pi i} \int_{T_n}^{\infty} f(z) dz$$

$$\Rightarrow 2 \sum_{n=1}^{\infty} \frac{1}{(z-z_1)^{n+1}} + 1 = \frac{1}{2\pi i} \operatorname{Res}(z_1) + \frac{1}{2\pi i} \int_{T_n}^{\infty} f(z) dz$$

what was in the previous equation

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(z-z_1)^n} = \frac{1}{2\pi i} \operatorname{Res}(z_1) + \frac{1}{2\pi i} \int_{T_n}^{\infty} f(z) dz - \frac{1}{2}$$

PROOF THAT  $\text{COTTE}$  IS BOUNDED ON THE SEMICIRCLE WITH VERTICES AT  $(\pi/2, \pm i\text{COTTE})$ ,  $\Im z = \pm 1/2, \Re z = 0$ .

$$|\text{COTTE}| = \left| \frac{\text{COTTE}}{\text{SINTE}} \right| = \left| \frac{\frac{1}{2}(\text{COTTE} + i\text{COTTE})}{\frac{1}{2}i(\text{COTTE}^2 - 1)} \right| = \left| \frac{\text{COTTE} + i\text{COTTE}}{i(\text{COTTE}^2 - 1)} \right| \quad (|z| = 1)$$

**VIEWING  $1/(1+z^2)$  AS A QUOTIENT**

$$\begin{aligned} |\text{COTTE}| &\geq |\text{COTTE}| / |\text{SINTE}| \\ &= \sqrt{|\text{COTTE}|^2 + |\text{COTTE}|^2} \quad \text{QUOTIENT} \end{aligned}$$

$$\leq \sqrt{\frac{|\text{COTTE}|^2}{|\text{SINTE}|^2} + \frac{|\text{COTTE}|^2}{|\text{SINTE}|^2}} = \sqrt{2|\text{COTTE}|^2} = \sqrt{2}|\text{COTTE}|$$

ON THE HORIZONTAL SLICE OF THE CONTOUR:  $z = 2\pi t(\frac{1}{2} + it) = (\text{COTTE}) + i\text{COTTE}$

- $|\text{COTTE}| = \sqrt{\text{COTTE}^2 + (\text{COTTE})^2} = \sqrt{(\text{COTTE})^2 + (2\pi t)^2} = \sqrt{(\text{COTTE})^2 + 4\pi^2 t^2} = \sqrt{(\text{COTTE})^2 + 4\pi^2}$
- ... ANALOGOUSLY PROVING ...

$$\begin{aligned} |\text{SINTE}| &\leq \frac{\pi/2 - \text{COTTE}}{|\text{COTTE}|} + \frac{\pi/2 + \text{COTTE}}{|\text{COTTE}|} = \frac{2\text{COTTE}}{|\text{COTTE}|} \sqrt{1 + \frac{(\text{COTTE})^2}{(\text{COTTE})^2}} \\ &= \frac{2\text{COTTE}|\text{COTTE}|^{1/2}}{|\text{COTTE}| + \sqrt{(\text{COTTE})^2 + 4\pi^2}} = \text{COTTE}^{1/2} \end{aligned}$$

AS  $\text{COTTE}^{1/2}$  IS A DECREASING FUNCTION

$$\begin{aligned} \text{COTTE}^{1/2} &< 1/2 \\ \text{COTTE} &< 1/4 \end{aligned}$$

SO  $\text{MATH}(1/\text{COTTE}) < \infty$

ON THE VERTICAL SIDES OF THE CONTOUR:  $z = (\pi/2) + iy, -(\pi/2) + iy$

$$|\text{COTTE}| = \left| \frac{\text{COTTE}}{\text{SINTE}} \right| = \left| \frac{\frac{1}{2}(\text{COTTE} + i\text{COTTE})}{\frac{1}{2}i(\text{COTTE}^2 - 1)} \right| = \left| \frac{i\text{COTTE} + \text{COTTE}}{i(\text{COTTE}^2 - 1)} \right|$$

$\downarrow$   
 $\approx 1 = 1$

$$\begin{aligned}
 & \left| \frac{e^{2\pi i z}}{e^{2\pi i z} - 1} - 1 \right| = \left| \frac{e^{2\pi i (z + \text{Im}(z) + i\text{Re}(z))} + 1}{e^{2\pi i (z + \text{Im}(z) + i\text{Re}(z))} - 1} \right| \\
 & = \left| \frac{e^{2\pi i (\text{Re}(z) + i\text{Im}(z))} - 1}{e^{2\pi i (\text{Re}(z) + i\text{Im}(z))} + 1} \right| \\
 & = \left| \frac{e^{2\pi i \text{Re}(z)} e^{-2\pi i \text{Im}(z)}}{e^{2\pi i \text{Re}(z)} e^{2\pi i \text{Im}(z)} + 1} \right| \\
 & = \left| \frac{e^{2\pi i \text{Re}(z)}}{e^{2\pi i \text{Re}(z)} e^{2\pi i \text{Im}(z)} + 1} \right| \\
 & = \left| \frac{e^{-2\pi i \text{Re}(z)}}{e^{-2\pi i \text{Re}(z)} e^{2\pi i \text{Im}(z)} + 1} \right| \\
 & = \left| \frac{-e^{-2\pi i \text{Re}(z)}}{e^{-2\pi i \text{Re}(z)} e^{2\pi i \text{Im}(z)} + 1} \right| \\
 & = \left| \frac{e^{2\pi i \text{Re}(z)}}{e^{2\pi i \text{Re}(z)} e^{2\pi i \text{Im}(z)} + 1} \right| \\
 & = \left| \text{tanh}(\pi \text{Re}(z)) \right| \\
 & = \left| -\text{tanh}(\pi \text{Re}(z)) \right| \\
 & = \left| \text{tanh}(\pi \text{Re}(z)) \right| \leq 1
 \end{aligned}$$

∴  $\text{C}_1^{\text{ext}}$  is bounded on the contour.

"Hyperbolic" for  $\text{cosec } \frac{\pi z}{2}$  By 1

"Hyperbolic" for  $\text{cosec } \frac{\pi z}{2}$  By 1

If  $M$  (value radius) is the supremum of these two

$M = \sup \left( \left| \text{cosec} \frac{\pi z}{2} \right| \right)$

**Question 7**

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{(a+z)^2}, z \in \mathbb{C}.$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(a+r)^2} = \pi^2 \operatorname{cosec}(\pi a) \cot(\pi a), a \notin \mathbb{Z}.$$

proof

**Since source alternates in sign we consider the integral of  $f(z)$ . Using the "usual contour" case the standard  $\Gamma$ , for example, is the closed contour with vertices at  $(n+1)(a+i)$ ,  $n \in \mathbb{N}$ .**

**CALCULATE RESIDUES**

$$\begin{aligned} \lim_{z \rightarrow n+1} \left[ \frac{1}{(z-n)^2} \right] dz &= \lim_{z \rightarrow n+1} \left[ \frac{1}{2!} \operatorname{Res}_{z=n+1} \frac{d^2}{dz^2} f(z) \right] \\ &= \lim_{z \rightarrow n+1} \left[ \frac{1}{2!} \operatorname{Res}_{z=n+1} \frac{d^2}{dz^2} \frac{\pi \operatorname{cosec} \pi z}{(a+z)^2} \right] \\ &= -\pi^2 \operatorname{cosec}(\pi a) \cot(\pi a) \end{aligned}$$

NEXT THE SIMPLE POLES AT  $z = -m \pm i$ ,  $m \in \mathbb{N}$

$$\begin{aligned} \lim_{z \rightarrow -m+i} \left[ \frac{1}{(z-m)^2} \right] dz &= \lim_{z \rightarrow -m+i} \left[ \frac{1}{2!} \operatorname{Res}_{z=-m+i} \frac{d^2}{dz^2} \frac{\pi \operatorname{cosec} \pi z}{(a+z)^2} \right] = \frac{\pi}{2}. \\ \text{BY L'HOSPITAL AGAIN THEY ARE} \\ &= \lim_{z \rightarrow -m+i} \left[ \frac{2}{(a+z)^3} \right] dz = \lim_{z \rightarrow -m+i} \left[ \frac{-2\pi^2 \operatorname{cosec} \pi z + 2\pi \operatorname{cosec}^2 \pi z}{(a+z)^4} \right] \\ &= \frac{-2\pi^2 \operatorname{cosec} \pi z}{(a+m)^4} \end{aligned}$$

**BY THE RESIDUE THEOREM**

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum \text{[residues inside } \Gamma]$$

$$\int_{\Gamma} \operatorname{cosec}^2 \pi z dz = 2\pi i \left[ -\pi^2 \operatorname{cosec}(\pi a) + \sum_{n=-\infty}^{\infty} \frac{\pi}{(a+n)^2} \right]$$

**$\bullet$  If  $a = \frac{1}{2}$  then  $f(z)$  has a double pole at  $z = a$ .**

AND SIMPLE POLES AT  $z = -m, m \in \mathbb{N}, -m \neq a$  BECAUSE OF  $\operatorname{cosec} z = \frac{1}{\sin z}$

**• Now  $\operatorname{cosec} z$  is bounded on  $\Gamma_1$ , i.e.  $|\operatorname{cosec} z| \leq M$**

$$\frac{1}{2} \leq n+1 \leq n+\frac{1}{2}$$

**Hence**

$$\begin{aligned} \left| \int_{\Gamma_1} \frac{1}{z} dz \right| &= \left| \int_{\Gamma_1} \frac{\pi \operatorname{cosec} \pi z}{(a+z)^2} dz \right| = \left| \int_{\Gamma_1} \frac{\pi \operatorname{cosec} \pi z}{2^2 (a+2z)^2} dz \right| \\ &\leq \int_{\Gamma_1} \left| \frac{\pi \operatorname{cosec} \pi z}{(a+2z)^2} \right| dz \leq \int_{\Gamma_1} \frac{\pi M}{(a+2z)^2} dz = \frac{\pi M}{(a+2(n+1))^2} \int_{\Gamma_1} dz \\ &\leq \frac{\pi M}{(a+2(n+1))^2} \cdot 2\pi = O\left(\frac{1}{n^2}\right) \xrightarrow{n \rightarrow \infty} 0 \quad \text{As } n \rightarrow \infty \quad \text{Thus } \frac{1}{n+1} \leq \frac{1}{n(n+1)} \end{aligned}$$

**• Now returning to the integral as  $n \rightarrow \infty$**

$$0 = \lim_{n \rightarrow \infty} \int_{\Gamma_n} \frac{\pi \operatorname{cosec} \pi z}{(a+z)^2} dz \quad \text{RECALL THE 2nd OF KRAMER'S}$$

$$\geq \sum_{n=-\infty}^{\infty} \frac{\pi}{(a+n)^2} = \pi^2 \operatorname{cosec}(\pi a) \quad \frac{\pi}{(a+n)^2} = \frac{\pi^2}{(a^2 + 2an + n^2)}$$

**PROOF THAT  $|\operatorname{cosec} z|$  IS BOUNDED ON THE SQUARE CONTOUR WITH VERTICES AT  $(n+1)(a+i)$ ,  $n = 1, 2, 3, \dots$**

**$\bullet$   $|\operatorname{cosec} z| = \left| \frac{1}{\sin z} \right| = \frac{1}{\left| \sin(z-a+ia) \right|} = \frac{|z|}{\left| e^{iz-a+ia} \right|} \leq \frac{2}{\left| (e^{iz-a+ia})^2 - 1 \right|} \quad |z| \geq |z-a+ia| \quad |z-a+ia| \leq |z|$**

Now on the horizontal sections  $z = 2k \pm i(a+1) + ia$ ,  $k \in \mathbb{Z}$

$$\begin{aligned} |\operatorname{cosec} z| &= \left| \frac{1}{\sin z} \right| = \left| \frac{1}{\sin((2k \pm i(a+1)) + ia)} \right| = \left| \frac{1}{\sin((2k \pm 1)a + 2ka \mp 2 + ia)} \right| \\ &= \left| \frac{1}{\sin((2k \pm 1)a + 2ka \mp 2 + ia)} \right| = \left| \frac{1}{\sin((2k \pm 1)a + 2ka \mp 2 + ia)} \right| = \left| \frac{1}{\sin((2k \pm 1)a + 2ka \mp 2 + ia)} \right| \end{aligned}$$

**$\therefore |\operatorname{cosec} z| = \dots = \frac{2}{\left| e^{i(2k \pm 1)a + 2ka \mp 2 + ia} \right|^2} = \frac{2}{\left| e^{i2ka} \left( e^{i(2k \pm 1)a} + 1 \right) \right|^2} = \frac{2}{\left| e^{i2ka} \left( e^{i(2k \pm 1)a} + 1 \right) \right|^2} = \left| \operatorname{cosec}(2ka \pm a) \right| \leq \text{constant}$**

**$\bullet$  ON THE VERTICAL SECTIONS  $z = a(n+1) + ia$ ,  $-n \leq a(n+1) \leq n$**

$$\begin{aligned} |\operatorname{cosec} z| &= \left| \frac{1}{\sin z} \right| = \left| \frac{1}{\frac{1}{2} (e^{iz} - e^{-iz})} \right| = \frac{|z|}{\left| e^{iz} - e^{-iz} \right|} = \frac{2}{\left| e^{i(a(n+1)+ia)} - e^{-i(a(n+1)+ia)} \right|} \\ &= \frac{2}{\left| e^{i(a(n+1)+ia)} - e^{-i(a(n+1)+ia)} \right|} \quad \text{at } |z| = 2 \\ &= \frac{2}{\left| e^{i(a(n+1)+ia)} - e^{-i(a(n+1)+ia)} \right|} = \frac{2}{\left| e^{i(a(n+1)+ia)} - e^{-i(a(n+1)+ia)} \right|} \\ &= \frac{2}{\left| e^{i(a(n+1)+ia)} - e^{-i(a(n+1)+ia)} \right|} = \frac{2}{\left| e^{i(a(n+1)+ia)} - e^{-i(a(n+1)+ia)} \right|} \end{aligned}$$

**$\therefore \operatorname{cosec} z$  IS BOUNDED ON THIS CONTOUR**

## Question 8

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{(2z+1)(3z+1)}, \quad z \in \mathbb{C}.$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r+1)(3r+1)} = \frac{\pi}{3} (2\sqrt{3} - 3).$$

[redacted], proof

As the small subintervals in sign we consider the integral of  $f(x) = \frac{1}{\sqrt{1+x^2}}$ . Instead of the usual  $\Delta x = \frac{b-a}{n}$ , one the rounded square contour  $T_n$  with vertices at  $(n+\frac{1}{2})(k+\frac{1}{2}), n, k \in \mathbb{N}$ .

$\int_{-1}^{n+1/2} \int_{-1}^{k+1/2} \frac{1}{\sqrt{1+x^2}} dx dy$

Calculus practice:

- $\lim_{n \rightarrow \infty} \left[ \int_{-1}^{n+1/2} \int_{-1}^{k+1/2} \frac{1}{\sqrt{1+x^2}} dx dy \right] = \lim_{n \rightarrow \infty} \left[ \text{const} - \frac{\text{arctan}(x)}{2\sqrt{1+x^2}} \Big|_{-1}^{n+1/2} \right]$
- $= \frac{\pi}{2} - \frac{\arctan(n+1/2)}{2\sqrt{n+1/2}} = \frac{-\arctan(n+1/2)}{2\sqrt{n+1/2}}$

$\bullet \lim_{x \rightarrow \infty} \frac{[(x+\frac{1}{2})^{\frac{1}{3}} - x^{\frac{1}{3}}]}{x^{\frac{1}{3}}} = \lim_{x \rightarrow \infty} \frac{[(x+\frac{1}{2})^{\frac{1}{3}} - x^{\frac{1}{3}}]}{x^{\frac{1}{3}}} \cdot \frac{\frac{1}{3}(x^{\frac{2}{3}} + x^{\frac{1}{3}} + \frac{1}{2})}{\frac{1}{3}(x^{\frac{2}{3}} + x^{\frac{1}{3}} + \frac{1}{2})}$   
 $= \frac{-\frac{1}{2}x^{-\frac{2}{3}}}{-\frac{2}{3}x^{-\frac{2}{3}}} = -\frac{1}{2}x^{-\frac{2}{3}} = -\frac{1}{2}(\frac{1}{x^{\frac{2}{3}}}) = -\frac{1}{2}(\frac{1}{x^{\frac{2}{3}}})$

$\bullet \lim_{x \rightarrow \infty} \frac{[(2x+1)^{\frac{1}{2}} - (x+2)^{\frac{1}{2}}]}{x^{\frac{1}{2}}} = \lim_{x \rightarrow \infty} \frac{[(2x+1)^{\frac{1}{2}} - (x+2)^{\frac{1}{2}}]}{x^{\frac{1}{2}}} \cdot \frac{(2x+1)^{\frac{1}{2}} + (x+2)^{\frac{1}{2}}}{(2x+1)^{\frac{1}{2}} + (x+2)^{\frac{1}{2}}}$   
 $= \lim_{x \rightarrow \infty} \frac{x^{\frac{1}{2}}((2x+1)^{\frac{1}{2}} - (x+2)^{\frac{1}{2}})}{(2x+1)^{\frac{1}{2}} + (x+2)^{\frac{1}{2}}}$

THIS IS AN INDETERMINATE FORM OF THE TYPE ZERO OVER INFINITY<sup>1/2</sup> / HORIZONTAL

$= \lim_{x \rightarrow \infty} \frac{x^{\frac{1}{2}}}{((2x+1)^{\frac{1}{2}} + (x+2)^{\frac{1}{2}}) + ((2x+1)^{\frac{1}{2}} - (x+2)^{\frac{1}{2}})}$   
 $= \frac{x^{\frac{1}{2}}}{(2x+1)^{\frac{1}{2}} + (x+2)^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{(2x+1)^{\frac{1}{2}} + (x+2)^{\frac{1}{2}}}$   
 $= \frac{(x^{\frac{1}{2}})^2}{(2x+1)^{\frac{1}{2}} + (x+2)^{\frac{1}{2}}} = \frac{x}{(2x+1)^{\frac{1}{2}} + (x+2)^{\frac{1}{2}}}$

By THE ESTIMATE THEOREM WE GET THIS.

$\int_0^{\infty} f(x) dx = \pi i \sum (\text{residues inside } G)$

$\int_0^{\infty} \frac{\operatorname{Res}(f, z_k)}{(z-z_k)^{n-k+1}} dz = \pi i \left[ -\frac{1}{k-1} + \frac{\frac{2\pi i}{k-1} \operatorname{Res}(f, z_k)}{k-1} + \frac{\frac{2\pi i}{k-1} \operatorname{Res}'(f, z_k)}{(k-1)(k-2)} + \dots + \frac{\frac{2\pi i}{k-1} \operatorname{Res}^{(k-1)}(f, z_k)}{(k-1)!} \right]$

IT WILL BE SHOWN AT THE END OF THE QUESTION THAT  $\operatorname{Residue}$  IS BOUNDED AND  
I.E.  $|\operatorname{Residue}| \leq M$  FOR SOME  $M \in \mathbb{R}$ .

THUS WE HAVE THE FOLLOWING

$|z_k| \geq n + \frac{L}{k} \quad (\text{SEE DIAGRAM})$   
 $|\operatorname{Residue}| \leq \frac{M}{n+1}$



USING STANDARD INTEGRALS

$$\int_{\frac{1}{2}}^{\infty} \frac{\text{TAN}(x)}{\sin(x) \cdot \cos(x)} dx \leq \int_{\frac{1}{2}}^{\infty} \frac{\text{TAN}(x)^2}{\sin^2(x) \cdot \cos(x)} dx = \int_{\frac{1}{2}}^{\infty} \frac{\text{TAN}(x)^2}{(\cos^2(x) - 1) \cdot \cos(x)} dx$$

$\begin{aligned} |\text{TAN}(x)| &> |\cos(x) - 1| \\ |\sin(x)| &\leq |\cos(x) - 1| \\ \frac{1}{|\sin(x)|} &\leq \frac{1}{|\cos(x) - 1|} \end{aligned}$

$$\begin{aligned} &\leq \int_{\frac{1}{2}}^{\infty} \frac{\text{TAN}(x)^2}{(\cos^2(x) - 1)^2} dx \quad \text{as } \frac{1}{|\cos(x) - 1|} \text{ is decreasing.} \\ &\leq \int_{\frac{1}{2}}^{\infty} \frac{\text{TAN}(x)^2}{(\cos^2(x) - 1)^2} dx \\ &= \frac{-\text{TAN}(x)}{(\cos(x) - 1)^2} + \int_{\frac{1}{2}}^{\infty} |\text{TAN}(x)| dx \\ &= \frac{-\text{TAN}(x)}{(\cos(x) - 1)^2} + \int_{\frac{1}{2}}^{\infty} |\cos(x) - 1| dx \\ &= \frac{-\text{TAN}(x)}{(\cos(x) - 1)^2} + \int_{\frac{1}{2}}^{\infty} \frac{|\cos(x) - 1|}{\cos^2(x) - 1} dx \\ &= \frac{-\text{TAN}(x)}{(\cos(x) - 1)^2} + \int_{\frac{1}{2}}^{\infty} \frac{|\cos(x) - 1|}{(\cos(x) - 1)(\cos(x) + 1)} dx \\ &= \frac{1}{\cos(x) + 1} + \int_{\frac{1}{2}}^{\infty} \frac{1}{(\cos(x) - 1)(\cos(x) + 1)} dx \\ &= \frac{1}{\cos(x) + 1} + \int_{\frac{1}{2}}^{\infty} \frac{1}{2\sin(x)} dx \\ &= \frac{1}{\cos(x) + 1} + \frac{1}{2} \left[ \ln|\tan(x) + \sec(x)| \right]_{\frac{1}{2}}^{\infty} \\ &= \frac{1}{\cos(x) + 1} + \frac{1}{2} \left[ \ln|\tan(\infty) + \sec(\infty)| - \ln|\tan(\frac{1}{2}) + \sec(\frac{1}{2})| \right] \\ &= \frac{1}{\cos(x) + 1} + \frac{1}{2} \left[ \ln|\infty + \infty| - \ln|\tan(\frac{1}{2}) + \sec(\frac{1}{2})| \right] \\ &= \frac{1}{\cos(x) + 1} + \frac{1}{2} \left[ \infty - \ln|\tan(\frac{1}{2}) + \sec(\frac{1}{2})| \right] \\ &= \infty \end{aligned}$$

RELATING TO THE INTEGRAL AS  $x \rightarrow \infty$

$$\int_{\frac{1}{2}}^{\infty} \frac{\text{TAN}(x)}{\sin(x) \cdot \cos(x)} dx = \omega \pi^2 \times \left[ \pi - \frac{2\pi}{\sqrt{3}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(\sin(x) \cdot \cos(x))} \right]$$

$$\omega = \frac{1}{\cos(x) + 1} \times \left[ \pi - \frac{2\pi}{\sqrt{3}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(\sin(x) \cdot \cos(x))} \right]$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(\sin(x) \cdot \cos(x))} = \frac{2\pi}{\sqrt{3}}\pi - \pi = \frac{\pi}{3}\pi - \pi = \frac{\pi}{3}[2\pi^2 - 3]$$

$$\therefore \sum_{n=0}^{\infty} \frac{(-1)^n}{(\sin(x) \cdot \cos(x))} = \frac{\pi}{3}[2\pi^2 - 3]$$

**PROOF THAT  $\text{SOSCTE} \approx \text{ROUND}(n)$  ON THE SQUARE CONCER, WITH MERTICES AT  $(\text{CnH})_{\text{CnH}}$ ,  $n = 1, 2, 3, \dots$**

$|\text{SOSCTE}| = \left| \frac{1}{\sin(\pi x)} \right| = \left| \frac{1}{\sin(\pi \cdot \frac{n}{n+1}))} \right| = \left| \frac{\pi n}{\pi(n+1)} \right|$

$$= \frac{|2n|}{|e^{\pi i n} - e^{\pi i (n+1)}} = \frac{2}{|e^{\pi i n} - e^{\pi i (n+1)}} \leq \frac{2}{\left| \frac{1}{(1 - e^{-\pi})^{n+1}} \right|}$$

$\frac{1}{(1 - e^{-\pi})^{n+1}} > \frac{1}{(1 - e^{-\pi})^n}$ 
 $\frac{1}{(1 - e^{-\pi})^n} < \frac{1}{(1 - e^{-\pi})^{n+1}}$

**ON THE HORIZONTAL SECTIONS OF THE CONCER,  $2 \leq 2n/(n+1) \leq (n+1)/n \leq 2$**

- $|e^{2\pi i x}| = \left| e^{2\pi i n/(n+1)} \right| = \left| e^{2\pi i n/(n+1)} \right| = \left| e^{2\pi i n/(n+1)} \right| = \left| e^{2\pi i n/(n+1)} \right|$
- $|e^{2\pi i x}| = \text{HARMONIC MEASURES} = \text{SOSCTE}$

$\therefore |\text{SOSCTE}| \leq \frac{2}{|e^{\pi i n} - e^{\pi i (n+1)}|} = \frac{2}{|2\sin(\pi \cdot \frac{n}{n+1})|}$

$$= \frac{2}{|2\sin(\pi \cdot \frac{n}{n+1})|} = |\operatorname{arcsin}(\frac{n}{n+1})| \leq \operatorname{arcsin} \frac{n}{n+1}$$

As  $\operatorname{arcsin}$  - concave & decreasing function

$\operatorname{arcsin} x > \operatorname{arcsin} y > \operatorname{arcsin} z \dots$

but  $x = n$ ,  $y = n+1$ ,  $z = n+2$

so  $\operatorname{arcsin} n > \operatorname{arcsin} n+1 > \operatorname{arcsin} n+2$

**ON THE VERTICAL SECTIONS OF THE CONCER,  $2 = \operatorname{arctan}(n+1) > -(\pi/2) < \operatorname{arctan}(n)$**

$|\text{SOSCTE}| = \left| \frac{1}{\sin(\pi x)} \right| = \left| \frac{1}{\frac{1}{2}(e^{\pi i x} - e^{-\pi i x})} \right| = \frac{1}{|e^{\pi i x} - e^{-\pi i x}|}$

NOTE THE FOLLOWING PATTERN

$$= \frac{2}{e^{iz}(\cos(z) + iz)} = \frac{2}{e^{iz}\cos(z) - e^{-iz}\sin(z)}$$

$$= \frac{e^{iz}\cos(z) - e^{-iz}\sin(z)}{e^{iz}\cos(z) + e^{-iz}\sin(z)} = \frac{z'}{|z^2 + \cos^2 z|}$$

LOGICALLY IS DIVIDED ON THE CONTOUR

"ANTI-CLOCKWISE" SURROUNDED BY  $|z| = R$  }  
 "CLOCKWISE" SURROUNDED BY  $|z| = 1$  }  
 i.e. M (contour) is the  
 SUPPLEMENT OF THESE TWO

M IS SUBCONTOUR!

## Question 9

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{4z^2 - 1}, \quad z \in \mathbb{C}$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{4r^2 - 1} = \frac{1}{4}(2 - \pi)$$

proof

Since the SIMS algorithm, we consider  $f(x) = \frac{\ln(2x+1)}{2x+1}$

OR THE STANDARD SQUARE CUTTING  $\Gamma_1$ , WITH VERTICES AT  $(\frac{1}{n+1}, f(\frac{1}{n+1})), n \in \mathbb{N}$

$(1/(n+1), 1/(n+1))$

$(1/(n+1), 1/(n+2))$

$(1/(n+2), 1/(n+1))$

$(1/(n+1), 1/(n+2))$

KEE THIS SAME  
RED AREA  $\frac{1}{n+1} \cdot \frac{1}{n+2}$

AT EACH REPE-  
VAL ON THE X,  
BUT BECAUSE OF  
CIRCLE IS CENTER

$\int_{\Gamma_1} \frac{1}{2x+1} dx = \ln(2x+1)$

BY THE RESUME THEOREM

$\int_{\Gamma_1} \frac{1}{2x+1} dx = \ln(2x+1) + \sum_{k=1}^n \text{Rectangles below } \Gamma_k$

$\int_{\Gamma_1} \frac{1}{2x+1} dx = 2\ln 2 + \left[ \frac{1}{2} + \sum_{k=1}^{n-1} \frac{1}{2(n+k+1)} \right]$

$\int_{\Gamma_1} \frac{1}{2x+1} dx = 2\ln 2 + \left[ \frac{1}{2} + \sum_{k=1}^{n-1} \frac{1}{2(n+k+1)} \right]$

CALCULATE PREVIOUS

- $\int_{\Gamma_1} (\frac{1}{2x+1}) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{2(n+k+1)} = \lim_{n \rightarrow \infty} \frac{1}{2} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} - \dots - \frac{1}{2(n+n)}$
- $= \frac{1}{2} \ln(2)$
- $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{2(n+k+1)} = \lim_{n \rightarrow \infty} \frac{1}{2} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} - \dots - \frac{1}{2(n+n)}$
- $= \frac{1}{2} \ln(2) - \frac{1}{4}$

Hence according to the integral & residues

$$\sum_{n=1}^{\infty} \left[ \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{4k+1} \right] = 0$$

$$\sum_{n=-\infty}^{-1} \frac{(-1)^n}{4n+1} = -\frac{\pi}{2}$$

$$\dots -\frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{9} - \dots = -\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \dots = -\frac{\pi}{2}$$

$$\therefore -1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{4n+1} = -\frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+1} = \frac{1}{2}(1-\pi)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+1} = \frac{1}{2}(2-\pi) \quad \checkmark$$

PROOF THAT  $\text{WSECT}_{123}$  IS BOUNDED ON THE SQUARE SCTOR WITH VERTICES AT  $(n+\frac{1}{2})(k+i)$ ,  $n=1,2,3, \dots$

①  $|\text{WSECT}_{12}| = \left| \frac{1}{\int_{\gamma}^{\gamma_1} dz} \right| = \frac{1}{\left| \frac{1}{2} \int_{\gamma}^{\gamma_1} (e^{iz} - e^{-iz}) \right|} = \frac{|z_1|}{\left| e^{iz_1} - e^{-iz_1} \right|} \leq \frac{2}{\left| e^{iz_1} - e^{-iz_1} \right|} \quad \begin{array}{l} |\bar{z}_1 - w| > \left| \bar{z}_1 - \bar{w}_1 \right| \\ |\bar{z}_1 - \bar{w}_1| \leq |\bar{z}_1 - w| \end{array}$

Now on the horizontal sections,  $z = x + i(n+\frac{1}{2})$ ,  $|x| \leq 2 < |z|$

$$\left| \frac{e^{iz}}{e^{-iz}} \right| = e^{i\pi n} \left| \frac{e^{ix}}{e^{ix}} \right| = e^{i\pi n} \left| \frac{e^{ix}}{e^{ix} - e^{-ix}} \right| = e^{i\pi n} \left| \frac{e^{ix}}{e^{ix}(1 - e^{-2ix})} \right| = e^{i\pi n} \left| \frac{1}{1 - e^{-2ix}} \right| = e^{i\pi n} \left| \frac{1}{1 - e^{-2ix}} \right| = e^{i\pi n} \left| \frac{1}{1 - e^{-2ix}} \right|$$

$\therefore |\text{WSECT}_{12}| = \dots = \frac{2}{\left| e^{i\pi(n+\frac{1}{2})} - e^{i\pi(n+\frac{1}{2})} \right|} = \frac{2}{\left| 2 \sin \left[ \frac{\pi}{2} (n+\frac{1}{2}) \right] \right|} = \frac{2}{\left| 4 \sin \left[ \frac{\pi}{2} (n+\frac{1}{2}) \right] \right|} = \left| \frac{\text{cosec}((n+\frac{1}{2})\pi)}{\sin((n+\frac{1}{2})\pi)} \right|$

② On the vertical sections,  $z = (n+\frac{1}{2}) + iy$ ,  $\omega_2 \leq y \leq \omega_3 = (n+\frac{1}{2})$

$$\left| \text{WSECT}_1 \right| = \left| \frac{1}{\sin(\pi z)} \right| = \frac{1}{\left| \frac{1}{2i} (e^{iz} - e^{-iz}) \right|} = \frac{|z_1|}{\left| e^{iz_1} - e^{-iz_1} \right|} = \frac{2}{\left| e^{i(n+\frac{1}{2})} - e^{-i(n+\frac{1}{2})} + i(y-\omega_2) \right|}$$

$$= \frac{2}{\left| \frac{e^{i(n+\frac{1}{2})}}{e^{i(n+\frac{1}{2})} - e^{-i(n+\frac{1}{2})}} \right|^2 \cdot \sqrt{1 + (y-\omega_2)^2}} \quad \text{(Red)} = \frac{2}{\left| e^{i(n+\frac{1}{2})} - e^{-i(n+\frac{1}{2})} \right|^2 \cdot \sqrt{1 + (y-\omega_2)^2}} = \frac{2}{\left| e^{i(n+\frac{1}{2})} - e^{-i(n+\frac{1}{2})} \right|^2 \cdot \sqrt{1 + (y-\omega_2)^2}}$$

$$= \frac{2}{\left( 2 \sin \left[ \frac{\pi}{2} (n+\frac{1}{2}) \right] \right)^2 \cdot \sqrt{1 + (y-\omega_2)^2}} = \frac{2}{(2 \sin((n+\frac{1}{2})\pi))^2 \cdot \sqrt{1 + (y-\omega_2)^2}} = \frac{2}{(2 \sin((n+\frac{1}{2})\pi))^2 \cdot \sqrt{1 + (y-\omega_2)^2}} = \frac{2}{(2 \sin((n+\frac{1}{2})\pi))^2 \cdot \sqrt{1 + (y-\omega_2)^2}}$$

$\therefore \text{WSECT}_1 \leq \text{WSECT}_{123}$  (because  $\omega_2 > \omega_3$ )

WSECT<sub>123</sub> IS BOUNDED ON THE WEDGE

WSECT<sub>123</sub> IS BOUNDED ON THE WEDGE

## Question 10

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{z^2}, \quad z \in \mathbb{C}$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{r^2} = -\frac{1}{12} \pi^2$$

proof

CALCULATE DIAMONDS

- $\int_{a=1}^{b=4} (2x-1) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n [(2x_i - 1) \frac{1}{2^n}] \sin \frac{\pi i}{n} = 0$

BY HOSPITAL RULE...

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \frac{2x_i - 1}{\sin \frac{\pi i}{n}} \right] = \frac{2}{\sin \frac{\pi}{n}} \approx \frac{2}{\frac{\pi}{n}} = \frac{2n}{\pi} \\ &= \frac{2(4-1)}{\pi} = \frac{12}{\pi} \end{aligned}$$

TRAPEZOIDAL RULE

- $\int_{a=1}^{b=4} (2x-1) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \frac{h}{2} \left( f(x_i) + f(x_{i+1}) \right) \right] = \lim_{n \rightarrow \infty} \left[ \frac{1}{2} \cdot \frac{1}{n} \left( f(x_1) + f(x_2) + \dots + f(x_n) \right) \right]$
- $= \frac{1}{2} \cdot \frac{1}{n} \left( \frac{1}{n} \left( \frac{1}{2} \sin \frac{\pi}{n} + \dots + \frac{1}{2} \sin \frac{(n-1)\pi}{n} \right) \right) = \text{CROSS OUT THE UNNECESSARY }$
- $\frac{h}{2} \cdot \frac{1}{n} \left( \frac{1}{n} \left( \frac{1}{2} \sin \frac{\pi}{n} + \dots + \frac{1}{2} \sin \frac{(n-1)\pi}{n} \right) \right) = \frac{1}{2} \cdot \frac{1}{n} \cdot \frac{1}{n} \left( \frac{1}{2} \sin \frac{\pi}{n} + \dots + \frac{1}{2} \sin \frac{(n-1)\pi}{n} \right)$
- $= \frac{1}{2} \cdot \frac{1}{n} \left( \frac{1}{2} \left( 1 - \frac{\pi^2}{n^2} + O(n^{-2}) \right) \right)^{-1}$
- $= \frac{1}{2n} \left[ 1 + \frac{\pi^2}{8n^2} + O(n^{-1}) \right]$
- $= \frac{1}{2n} + \frac{\pi^2}{16n^3} + O(n^{-2})$
- $\therefore \text{RESULT AT } n=400 = \frac{12}{\pi}$

BY THE DEFINITION THEOREM

$$\int_{a=1}^{b=4} (2x-1) dx = \pi^2 n^2 / 8 + \text{COSINES INSIDE } I_n$$

$$\int_{a=1}^{b=4} \frac{\text{TRAPEZOIDAL RULE}}{2^n} dx = \pi^2 n^2 / 8 + \sum_{i=1}^{n-1} \frac{\sin \frac{\pi i}{n}}{n^2}$$

- KNOW IT CAN BE SHOWN THAT  $\cos(\pi z)$  IS BOUNDED ON  $\Gamma_1$   
i.e.  $|\cos(\pi z)| \leq M$  FOR SOME  $M \in \mathbb{R}$  (IT IS SHOWN AT THE VERY END)
- ALSO OBVIOUSLY  

$$\left| \frac{1}{z} \right| \geq \frac{1}{|z| + k}$$

- THIS  

$$\left| \int_{\Gamma_1} \frac{\cos(\pi z)}{z^2 - 1} dz \right| \leq \int_{\Gamma_1} \left| \frac{\cos(\pi z)}{z^2 - 1} \right| |dz| \leq \int_{\Gamma_1} \frac{\pi M}{|z|^2 - 1} |dz| = \frac{\pi M}{k^2 - 1} \int_{\Gamma_1} \frac{1}{|z|} |dz| \quad (\text{length of } \Gamma_1)$$

$$= \frac{\pi M \times \operatorname{length}(\Gamma_1)}{(k^2 - 1)} = C(k) \rightarrow 0 \text{ as } n \rightarrow \infty$$
- REGARDING TO THE INTEGRAL AS  $n \rightarrow \infty$   

$$\rightarrow \int_{-\infty}^{\infty} \cos(\pi x) dx = \int_{-\infty}^{\infty} \left( \frac{e^{i\pi x}}{2} + \frac{e^{-i\pi x}}{2} \right) dx \quad [\text{DIVIDE BY } 2\pi \text{ & REARRANGE}]$$

$$\sum_{n=1}^{+\infty} \int_{-R}^R \frac{e^{i\pi x}}{x^2 - 1} dx + \sum_{n=1}^{+\infty} \int_{-R}^R \frac{e^{-i\pi x}}{x^2 - 1} dx = -\frac{\pi^2}{16} \quad (-\left( \dots + \frac{1}{\frac{1}{4} - \frac{1}{n^2}} + \frac{1}{\frac{1}{4} + \frac{1}{n^2}} \right) + \left( -\frac{1}{\frac{1}{4} - \frac{1}{n^2}} - \frac{1}{\frac{1}{4} + \frac{1}{n^2}} \right))$$

$$\Rightarrow 2 \int_{-R}^R \frac{e^{i\pi x}}{x^2 - 1} dx = -\frac{\pi^2}{16}$$

$$\rightarrow \frac{1}{2} \int_{-R}^R \frac{e^{i\pi x}}{x^2 - 1} dx \approx \frac{-\pi^2}{12}$$

$$\therefore \frac{1}{2} \int_{-R}^R \frac{e^{i\pi x}}{x^2 - 1} dx = \frac{-\pi^2}{12}$$

PROOF THAT  $\text{Cosec}z$  IS BOUNDED ON THE SQUARE CONTOUR WITH VERTICES AT  $(n+\frac{1}{2})(k+i)$ ,  $n=1,2,3,\dots$

- ①  $|\text{Cosec}z| = \left| \frac{1}{\sin z} \right| = \frac{1}{\left| \frac{z\pi i}{e^{iz}-e^{-iz}} \right|} = \frac{|z\pi i|}{\left| e^{iz}-e^{-iz} \right|} \leq \frac{2|z\pi i|}{\left| e^{iz}+e^{-iz} \right|} = \frac{2|z\pi i|}{\left| e^{iz}(1+e^{-2iz}) \right|} \leq \frac{2|z\pi i|}{\left| e^{iz} \right| \cdot \left| 1-e^{-2iz} \right|} \quad \boxed{\left| e^{iz}w \right| \geq \left| \left( e^{iz}-1 \right)w \right|}$
- Now on the horizontal sections,  $z = x + i(n+\frac{1}{2})$ ,  $|z\pi i| \leq 2 < e^{\frac{\pi}{2}}$ 
  - $\left| e^{iz} \right| = \left| e^{ix} \left( e^{in\frac{\pi}{2}} + e^{-in\frac{\pi}{2}} \right) \right| = \left| e^{ix} \right| \cdot \left| e^{in\frac{\pi}{2}} + e^{-in\frac{\pi}{2}} \right| \leq e^{\frac{\pi}{2}}$
  - $\left| e^{-iz} \right| = \left| e^{-ix} \left( e^{-in\frac{\pi}{2}} + e^{in\frac{\pi}{2}} \right) \right| = \left| e^{-ix} \right| \cdot \left| e^{-in\frac{\pi}{2}} + e^{in\frac{\pi}{2}} \right| \leq e^{-\frac{\pi}{2}}$
- $\therefore |\text{Cosec}z| = \dots \frac{2}{\left| e^{iz(n+\frac{1}{2})} - e^{-iz(n+\frac{1}{2})} \right|} \approx \frac{2}{\left| 2\sinh \left[ e^{iz(n+\frac{1}{2})} \right] \right|} = \frac{2}{\left| 4\sinh \left[ e^{iz(n+\frac{1}{2})} \right] \right|} = \frac{2}{\left| \text{cosec} \left[ (n+\frac{1}{2})\pi \right] \right|} \quad \begin{array}{l} \text{cosec is decreasing function} \\ \text{cosec} > \text{cosec } \frac{\pi}{2} \end{array}$
- ② On the vertical sections,  $z = (n+\frac{1}{2}) + iy$ ,  $|z\pi i| \leq 2 < e^{\frac{\pi}{2}}$ 

$$\begin{aligned} |\text{cosec}z| &= \left| \frac{1}{\sin z} \right| = \frac{1}{\left| \frac{z\pi i}{e^{iz}-e^{-iz}} \right|} = \frac{|z\pi i|}{\left| e^{iz}-e^{-iz} \right|} = \frac{2}{\left| e^{iz(n+\frac{1}{2})+iy} - e^{-iz(n+\frac{1}{2})+iy} \right|} \\ &= \frac{2}{\left| e^{iz(n+\frac{1}{2})} e^{iy} - e^{-iz(n+\frac{1}{2})} e^{-iy} \right|} \quad \text{④} \quad \frac{2}{\left| e^{iz(n+\frac{1}{2})} e^{iy} - e^{-iz(n+\frac{1}{2})} e^{-iy} \right|} \\ &= \frac{2}{\left| e^{iy} \left( e^{iz(n+\frac{1}{2})} - e^{-iz(n+\frac{1}{2})} \right) \right|} = \frac{2}{\left| e^{iy} \right| \cdot \left| e^{iz(n+\frac{1}{2})} - e^{-iz(n+\frac{1}{2})} \right|} \\ &= \frac{2}{2\left| e^{iy} \right|} = \frac{1}{2\left| e^{iy} \right|} = \text{Security} \approx 1 \end{aligned}$$
- ∴  $\text{Cosec}z$  IS BOUNDED ON THIS CONTOUR  
where  $w = \text{cosec}z$

## Question 11

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{z^4}, \quad z \in \mathbb{C}$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^4} = \frac{7\pi^4}{720}$$

proof

**SINCE THE TRAINS ARE ALIGNSING IN SENT IT COULD BE**

**THEORETICALLY**

**SEE IF WE CAN USE THE SIMILAR SOURCE CONSTRUCTION WITH VECTORS AT  $(x_1)$ ,  $(x_2)$ ,  $(x_3)$**

**WITH  $\text{RECT}(x_1)$ ,  $\text{RECT}(x_2)$ ,  $\text{RECT}(x_3)$**

**WE GET**

**WE GET**

**(a) WE HAVE POLARISATION**

**GIVEN WORKED VALUE AS THE X-AXIS BECAUSE OF CONCRETE = 2000**

**SOURCE AT ZERO**

**AT ZERO THERE IS A TORSIONAL LOAD, SO IT'S**

**ROTATING IN THE SOURCE**

**ROTATING IN THE SOURCE**

**CALCULATE RESIDUES**

**1.  $\lim_{x \rightarrow x_1^+} \frac{1}{x - x_1} \cdot \text{RECT}(x) = \lim_{x \rightarrow x_1^+} \frac{1}{x - x_1} \cdot \frac{\text{RECT}(x)}{x^2 + 1}$**

**WE GET 2/3**

**2.  $\lim_{x \rightarrow x_2^+} \frac{-\frac{x}{2} \cdot \text{RECT}(x)}{x - x_2} = \frac{0}{0} = \text{BY L'HOSPITAL}$**

**WE GET 2/3**

**3.  $\lim_{x \rightarrow x_3^+} \frac{\frac{x^2}{4} \cdot \text{RECT}(x)}{x - x_3} = \frac{0}{0} = \text{BY L'HOSPITAL}$**

**WE GET 2/3**

**TO CALCULATE THE RESIDUE OF A POLE OF ORDER 3, IT IS PREFERABLY BETTER TO GET IT FROM ITS LAURENT EXPANSION ABOUT  $x = 0$**

**$\frac{1}{z-a} = \frac{1}{z} + \frac{a}{z^2} + \dots = \frac{1}{z} - \frac{a}{z^2} + \frac{a^2}{z^3} - \frac{a^3}{z^4} + \dots$**

**$= \frac{1}{z^3} + \frac{a}{z^2} + \left[ -\frac{a^2}{z} + \frac{a^3}{z^2} - a^4 \right]$**

**$= \frac{1}{z^3} \left[ 1 - \left( \frac{a^2}{z} - \frac{a^3}{z^2} + a^4 \right) \right]$**

**$= \frac{1}{z^3} \left[ 1 + \left( \frac{-a^2}{z} + \frac{-a^3}{z^2} + a^4 \right) + \left( \frac{-a^2}{z} - \frac{-a^3}{z^2} + a^4 \right)^2 + O(z^4) \right]$**

**$= \frac{1}{z^3} \left[ 1 + \frac{-a^2}{z} + \frac{-a^3}{z^2} + \frac{a^4}{z^3} + O(z^4) \right]$**

**$= \dots = \frac{1}{z^3} \left[ \frac{-a^2}{z} + \frac{-a^3}{z^2} + \frac{a^4}{z^3} \right] = \frac{-a^2}{3z^2} - \frac{a^3}{3z^3} + \frac{a^4}{3z^4}$**

**∴ RESIDUE IS  $\frac{-a^2}{3z^2}$**

**BY THE RESIDUE THEOREM**

**$\int_{\Gamma} f(z) dz = 2\pi i \sum \text{(residues inside } \Gamma_n)$**

**$\int_{\Gamma} \frac{1}{z-a} dz = 2\pi i \left[ \frac{\text{RECT}(x_1)}{x_1^2 + 1} + \sum_{n=2}^3 \frac{\text{RECT}(x_n)}{x_n^3 + 1} \right]$**

- It can be shown that  $\int_{\text{rectangle}} f(x) dx \leq M$ , for some  $M$ .  
 (It is shown at the very end)
- Also consider  $\int_1^2 f(x) dx$   
 $|x| \geq \frac{1}{2} \Rightarrow x \in \left[\frac{1}{2}, 2\right]$   
 $\frac{1}{|x|} \leq \frac{1}{\frac{1}{2}}, \frac{1}{2}$
- This  

$$\left| \int_1^2 \frac{\sin \frac{1}{x}}{x^2} dx \right| \leq \int_1^2 \left| \frac{\sin \frac{1}{x}}{x^2} \right| dx \leq \int_1^2 \frac{1}{x^2} dx = \left[ -\frac{1}{x} \right]_1^2 = \frac{1}{2} \rightarrow O(\frac{1}{n}) \rightarrow 0 \text{ As } n \rightarrow \infty$$
- REASONING TO THE INTEGRAL AS  $n \rightarrow \infty$   
 $\Rightarrow 0 = \lim_{n \rightarrow \infty} \left[ \frac{\pi}{360} + \sum_{k=1}^{n-1} \frac{\sin \frac{k\pi}{n}}{\left(\frac{k\pi}{n}\right)^2} + \frac{\sin \frac{n\pi}{n}}{\left(\frac{n\pi}{n}\right)^2} \right] \text{ DUE TO 3rd Q. RIBONCE}$   
 $\Rightarrow \sum_{k=1}^{n-1} \frac{\sin \frac{k\pi}{n}}{\left(\frac{k\pi}{n}\right)^2} + \frac{\sin \frac{n\pi}{n}}{\left(\frac{n\pi}{n}\right)^2} = -\frac{\pi^2}{360}$   
 $\Rightarrow 2 \sum_{k=1}^{n-1} \frac{\sin \frac{k\pi}{n}}{\left(\frac{k\pi}{n}\right)^2} = -\frac{\pi^2}{180} \quad (\text{SINCE } \frac{1}{n} \text{ IS FINE})$   
 $\Rightarrow \frac{2}{n} \sum_{k=1}^{n-1} \frac{\sin \frac{k\pi}{n}}{\left(\frac{k\pi}{n}\right)^2} = -\frac{\pi^2}{180}$   
 $\Rightarrow \sum_{k=1}^{n-1} \frac{\sin \frac{k\pi}{n}}{\left(\frac{k\pi}{n}\right)^2} = \frac{\pi^2}{360}$

## Question 12

$$f(z) = \frac{\pi \tan \pi z}{z^4}, \quad z \in \mathbb{C}$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^4} = \frac{\pi^4}{96}$$

[redacted], proof

At the times  $t = n\pi$  in this interval, we consider the interval of  $\{f(t)\}$ . There are  $n+1$  points in this interval. The second smallest,  $f(n)$ , has vertices at the point  $y = f(n)$ ,  $x \in \mathbb{Q}$ .

**Task:**

- (a) Has simple poles at the "half integer values" because of ~~bottom part~~ ~~bottom part~~ ~~bottom part~~
- (b) Has a triple pole at the origin because of the ~~top~~ ~~top~~ (removable pole or not?)

**Comments:**

- $\lim_{k \rightarrow \infty} \int_{-n+\frac{1}{2}}^{n+\frac{1}{2}} \left[ \frac{1}{z - (n + \frac{1}{2})} f(z) \right] dz = \lim_{k \rightarrow \infty} \int_{-n+\frac{1}{2}}^{n+\frac{1}{2}} \frac{(1/(n + \frac{1}{2})) \cdot k \cdot 1}{z - (n + \frac{1}{2})} dz$
- =  $\lim_{k \rightarrow \infty} \int_{-n+\frac{1}{2}}^{n+\frac{1}{2}} \frac{(1/(n + \frac{1}{2})) \cdot k \cdot 1}{z - (n + \frac{1}{2})} dz = \dots$  THIS INTEGRATION FROM "ZERO TO ZERO" TYPE 30 BY HANSHOT'S RULE
- =  $\lim_{k \rightarrow \infty} \int_{-n+\frac{1}{2}}^{n+\frac{1}{2}} \frac{\text{TOP PART} + \text{TOP PART} + \text{TOP PART}}{4(n + \frac{1}{2})^2 - z^2} dz = \frac{\text{TOP PART}}{4(n + \frac{1}{2})^2} = \frac{1}{(n + \frac{1}{2})^2}$

10) To find the residue of the pole at the origin is required to get it from the Laurent expansion of  $f(z)$

$$\begin{aligned} f(z) &= \frac{1}{z^2 - 4z + 3} = \frac{1}{(z-1)(z-3)} \\ &= \frac{A}{z-1} + \frac{B}{z-3} \\ &= \frac{A(z-3) + B(z-1)}{(z-1)(z-3)} \\ &= \frac{(A+B)z - (3A+B)}{(z-1)(z-3)} \end{aligned}$$

From  $f(z) = \frac{1}{z^2 - 4z + 3}$   
 $\text{where } f(z) = \frac{1}{z^2} + \frac{4z}{z^3} + O(z^3)$

$$\begin{aligned} f(z) &= \frac{1}{z^2} + \frac{4z}{z^3} + O(z^3) \\ &= \frac{1}{z^2} + \frac{4}{z^3} + O(z^3) \\ &\therefore \text{Res}(f, z=1) = \frac{4}{z^3} \end{aligned}$$

Now, by the residue theorem

$$\begin{aligned} \int_{\Gamma} f(z) dz &= 2\pi i \times (\text{residues inside } \Gamma) \\ \int_{\Gamma} \frac{\frac{1}{z^2 - 4z + 3}}{z^n} dz &= 2\pi i \left[ \sum_{k=1}^{n-1} \left( \frac{4}{k+1} \right) + \frac{1}{n} \right] \\ \int_{\Gamma_1} \frac{\frac{1}{z^2 - 4z + 3}}{z^n} dz &= 2\pi i \left[ \frac{16}{2} + \sum_{k=1}^{n-1} \left( \frac{4}{k+1} \right) \right] \end{aligned}$$

IT WILL BE SHOWN AT THE END OF THE QUESTION THAT  $\text{Res}(f, z=1) = 4$ ,  
 $\therefore \text{Res}(f, z=1) = 4$  for some  $n \in \mathbb{N}$ .

THUS WE HAVE

$$\begin{aligned} |z| &\geq n, \quad (\text{COUNTER}) \\ |z| &\leq 1, \quad (\text{ANTI-COUNTER}) \end{aligned}$$

$$\int_{\Omega} \frac{|\text{tr}_{\partial\Omega} u|^2}{2^k} d\sigma \leq \int_{\Omega} \frac{|\text{tr}_{\partial\Omega} u|^2}{2^k} |d\sigma| \leq \int_{\Omega} \frac{\pi M}{2^k} \cdot \sin \theta d\sigma$$

$$= \frac{\pi M}{2^k} \sum_{n=0}^{M-1} |d\sigma| = \frac{\pi M}{2^k} = O(\frac{1}{2^k}) \rightarrow \text{as } n \rightarrow \infty$$

using the formula  
\$\sin x \approx x\$ for small \$x\$

FURTHER REDUCING TO THE INTEGRAL AS \$n \rightarrow \infty\$

$$\Rightarrow \int_{\Omega} \frac{\pi |\text{tr}_{\partial\Omega} u|^2}{2^k} d\sigma = \pi n \left[ \frac{\frac{1}{3}^k - \frac{1}{(2n+1)^k}}{3} \right] = \pi n \left[ \frac{\frac{1}{3}^k - \frac{16}{(2n+1)^k}}{3} \right]$$

$$\Rightarrow 0 = \lim_{n \rightarrow \infty} \left[ \frac{\frac{1}{3}^k - \frac{16}{(2n+1)^k}}{3} \right] \quad \text{CHANGE SUMMATION VARIABLE TO } r$$

$$\Rightarrow \sum_{r=0}^{\infty} \frac{k}{(2r+1)^k} = \frac{\frac{1}{3}^k}{3}$$

$$\Rightarrow 2 \cdot \frac{\frac{1}{3}^k}{3} \cdot \frac{16}{(2r+1)^k} = \frac{\frac{1}{3}^k}{3}$$

$$\Rightarrow 32 \sum_{r=0}^{\infty} \frac{1}{(2r+1)^k} = \frac{\frac{1}{3}^k}{3}$$

$$\Rightarrow \frac{\frac{1}{3}^k}{36} = \frac{\frac{1}{3}^k}{36} \quad \text{AN EQUATION}$$

**PROOF THAT  $\text{Im} z = \text{Im} w$  IS REACHED ON THE SQUARE CONTOUR  $\Gamma_1$ , WITH VERTEXES**

$$4i, -i(\pm 1), -i, i(\pm 1), 4i$$

**DEFINITION**

$$\left| \text{Im} z = \text{Im} w \right| = \left| \frac{\sin z}{\sin w} \right| = \frac{\left| e^{iz} - e^{-iz} \right|}{\left| e^{iw} - e^{-iw} \right|} = \frac{\left| e^{iz} - e^{-iz} \right|}{\left| e^{iw} + e^{-iw} \right|}$$

**DENOMINATOR**

$$\left| e^{iw} \right| = \left| e^{i(w+1)} \cdot e^{-i} \right| = \left| e^{i(w+1)} \right| \cdot \left| e^{-i} \right|$$

**DENOMINATOR**

$$\left| e^{iw} \right| \geqslant \left| e^{-i(w+1)} \right|$$

$$\frac{1}{\left| e^{iw} \right|} \leq \frac{1}{\left| e^{-i(w+1)} \right|}$$

**...  $\Rightarrow$**

$$\frac{\left| e^{iz} - e^{-iz} \right|}{\left| e^{iw} + e^{-iw} \right|} \leq \frac{\left| e^{iz} - e^{-iz} \right|}{\left| e^{iz} + e^{-iz} \right|}$$

**ON THE "HORIZONTAL" SECTIONS OF THE CONTOUR,  $z = 3e^{it}$ ,  $t \in [0, \pi]$**

- $\left| e^{iz} \right|^2 = \left| e^{i(3e^{it})} \right|^2 = \left| e^{i3e^{it}} \right|^2 = \left[ e^{i3e^{it}} \right] \left[ e^{-i3e^{it}} \right] = e^{i6e^{it}}$
- $\left| e^{-iz} \right|^2 = \left| e^{-i(3e^{it})} \right|^2 = \left| e^{-i3e^{it}} \right|^2 = \left( e^{-i3e^{it}} \right) \left( e^{i3e^{it}} \right) = e^{i6e^{it}}$

**...  $\Rightarrow$**

$$\frac{\left| e^{iz} - e^{-iz} \right|}{\left| e^{iz} + e^{-iz} \right|} = \frac{\left| e^{iz} - e^{-iz} \right|}{\left| e^{iz} - e^{-iz} \right|} = \frac{2e^{i6e^{it}}}{2e^{i6e^{it}}} = 1$$

$= \frac{2\sin(z\pi)}{2\sin(w\pi)} = \frac{2\sin(3e^{it}\pi)}{2\sin(w\pi)} = \sin(3e^{it}\pi) \leq \sin(x)$

**AS  $y = \sin x$  IS A DECREASING FUNCTION, THE GREATER VALUE OF  $\text{Im} z$  COMES**

Finally we have

DO THE 'VERTICAL SECTIONS' OF THE CONTROL,  $x = \sin \theta + iy \rightarrow y \leq 0$

$$|\text{tanh} z| = \left| \frac{\sinh z}{\cosh z} \right| = \left| \frac{\left( e^{\theta+i\pi} - e^{-\theta+i\pi} \right)}{e^{\theta+i\pi} + e^{-\theta+i\pi}} \right| = \left| \frac{e^{2\theta i} - 1}{e^{2\theta i} + 1} \right|$$

MATRIX FORM OF EQUATION BY  $e^{2\theta i}$

$$\begin{aligned} &= \left| \frac{e^{2\theta i} - 1}{e^{2\theta i} + 1} \right| = \left| \frac{e^{2\theta i}(\sin \theta + i \cos \theta) - 1}{e^{2\theta i}(\sin \theta + i \cos \theta) + 1} \right| = \left| \frac{e^{2\theta i}\sin \theta - 1}{e^{2\theta i}\sin \theta + 2i \cos \theta + 1} \right| \\ &= \left| \frac{e^{2\theta i}\sin \theta - 1}{e^{2\theta i}\sin \theta + 2i \cos \theta + 1} \right| \quad \begin{array}{l} \text{Let } \frac{e^{2\theta i}}{\sin \theta} = \frac{e^{2\theta i}}{\sin \theta} \\ \text{Then } \frac{e^{2\theta i}}{\sin \theta} = \frac{e^{2\theta i}(2\cos \theta + 1) + 2i \cos \theta}{2\cos \theta + 1} \end{array} \\ &= \left| \frac{\frac{e^{2\theta i}}{\sin \theta} - 1}{\frac{e^{2\theta i}}{\sin \theta} + 1} \right| \\ &= \left| \frac{\frac{e^{2\theta i}}{\sin \theta} - 1}{\frac{e^{2\theta i}}{\sin \theta} + 1} \right| = \left| \frac{1 - e^{-2\theta i}}{1 + e^{-2\theta i}} \right| = \left| \frac{e^{2\theta i} - 1}{e^{2\theta i} + 1} \right| \\ &\Rightarrow |\text{tanh}(z_0)| < 1 \end{aligned}$$

"GRADUATION"  $\cdot$  RIGHT IS BOUNDED BY  $\text{CONE}$       ]  $\Rightarrow M$  (LEAD-FACADE) IS THE SUPERIUM OF THESE TWO  
 "VERTICALS"  $\cdot$  RIGHT IS BOUNDED BY 1

LEMMA 1 IS BOUNDED ON THE CONTROL

Question 13

$$f(z) = \frac{\pi \sec \pi z}{z^3}, z \in \mathbb{C}.$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^3} = \frac{\pi^3}{32}.$$

 , proof

AS THE TERM GOES UP IN TWOS IN THE DENOMINATOR, IF THE SERIES ATTACHES WE CONSIDER THE INTEGRAL OF  $f(z) = \frac{\pi \sec \pi z}{z^3}$ , OVER THE SQUARE CONTOUR  $\Gamma_1$  WITH VERTEXES AT  $(n+1/2)$ ,  $n \in \mathbb{Z}$

(a) THIS SOURCE ROLLS AT "HALF INTEGERS" VERTICES BECAUSE OF THE SECTORS  $\frac{1}{z^3}$   
[OR HAS A DOUBLE ROLL AT THE ORIGIN] BECAUSE OF THE  $\frac{1}{z^3}$

CALCULATE RESIDUES AT THESE POINTS

- $\lim_{z \rightarrow n+1/2} [z-(n+1/2)] f(z) = \lim_{z \rightarrow n+1/2} \frac{(2n+1)\pi}{z^3 \cos(\pi z)} = \lim_{z \rightarrow n+1/2} \frac{(2n+1)\pi}{z^3}$   
THIS WILL BE AN INDEFINITE FORM OF THE RESIDUE TEST TYPE, SO BY L'HOSPITAL
- $\lim_{z \rightarrow n+1/2} \left[ \frac{\pi}{z^3 \cos(\pi z) - \text{residue}} \right] = -\frac{\pi^2}{2\cos(\pi(n+1/2))} = -\frac{\pi^2}{2\sin^2(\pi/2)} = -\frac{\pi^2}{2}$

$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^2}{2}$

$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^2}{16}$

$\Rightarrow \dots -\frac{\pi^2}{3^3} + \frac{\pi^2}{5^3} - \frac{\pi^2}{7^3} + \dots + 1 - \frac{\pi^2}{27} + \frac{\pi^2}{49} - \frac{\pi^2}{81} + \dots = \frac{\pi^2}{16}$

$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^2}{32}$  OR  $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^2}{32}$

PROOF THAT SECTOR IS BOUNDED ON THE SQUARE CONTOUR  $\Gamma_1$  WITH VERTICES AT  $(n+1/2)$ ,  $n \in \mathbb{Z}$ ...

$| \text{SECTOR} | = \left| \frac{1}{z^3} \right| = \left| \frac{1}{(2n+1)^2 z^2} \right| = \frac{2}{(2n+1)^2} \leq \frac{2}{(1/n)^2} = \frac{2}{n^2}$

ON THE "HORIZONTAL" SECTIONS OF THE CONTOUR,  $z = \alpha + iy$ ,  $-y < y < 0$

- $| e^{iz^2} | = | e^{i(\alpha^2+y^2)} | = | e^{i\alpha^2} | | e^{iy^2} | = | e^{i\alpha^2} | = e^{-y^2}$
- $| e^{iz^2} | = | e^{i(\alpha+iy)^2} | = | e^{i\alpha^2} | | e^{iy^2} | = e^{-y^2}$

$\dots \leq \frac{2}{(1/n)^2 - (1/n)^2} = \frac{2}{1/n^2 - 1/n^2} = \frac{2}{2/n^2} = \frac{\pi^2}{4n^2}$

=  $\text{constant}/n^2 \leq \text{constant}$  ( $n \in \mathbb{N}$ , SEE DIAGRAM)

• TO GET THE RESIDUE OF THE TERM DUE AT THE ORIGIN, WE GET IT DIRECTLY FROM ITS UNARY EXPANSION

$f(z) = \frac{\pi \sec \pi z}{z^3} = \frac{\pi}{z^3} \left[ \frac{1}{\cos(\pi z)} \right] = \frac{\pi}{z^3} \left[ 1 - \frac{\pi^2}{2} z^2 + O(z^4) \right]^{-1}$

$= \frac{\pi}{z^3} \left[ 1 + \frac{\pi^2}{2} z^2 + O(z^4) \right] = \dots + \frac{\pi^2}{2z^2} + \dots$  IF PREDICT  $\frac{\pi^2}{2}$

NOW BY THE RESIDUE THEOREM, WE HAVE

$\int_{\Gamma_1} f(z) dz = 2\pi i \times \sum \text{ (residues of } f \text{ inside } \Gamma_1)$

$\int_{\Gamma_1} \frac{\pi \sec \pi z}{z^3} dz = 2\pi i \left[ \frac{\pi^2}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \right]$

IT WILL BE SHOWN AT THE END OF THE QUESTION THAT, SINCE  $\pi$  IS BOUNDED ON  $\Gamma_1$ , i.e.  $|\sec \pi z| \leq M$ , THE SAME AS  $M/2$  - USING THIS FACT WE HAVE

$| \text{SECTOR} | \leq \int_{\Gamma_1} \left| \frac{\pi \sec \pi z}{z^3} \right| dz \leq \int_{\Gamma_1} \frac{M}{|z|^3} dz = \frac{M}{n^2} \int_{\Gamma_1} |z| dz = \frac{M\pi}{n^2} \rightarrow 0$  AS  $n \rightarrow \infty$

FINALLY RETURNING TO THE WORKING AS  $n \rightarrow \infty$

$\Rightarrow \int_{\Gamma_1} \frac{\pi \sec \pi z}{z^3} dz = 2\pi i \left[ \frac{\pi^2}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \right]$

$\Rightarrow 0 = 2\pi i \left[ \frac{\pi^2}{2} - 8 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \right]$

CD) THE "UNARY" SECTION OF THE CONTOUR,  $z = \alpha + iy$ ,  $-y < y < 0$

$| \text{SECTOR} | = \left| \frac{1}{z^3} \right| = \frac{1}{|z|^3} = \frac{2}{e^{3y}}$

$= \frac{2}{(e^{3y})^2 + (e^{3y})^2} = \frac{2}{(e^{3y})^2 + (e^{3y})^2}$

$= \frac{2}{2e^{6y}} = \frac{1}{e^{6y}} = \frac{2}{e^{3y} e^{3y}} = \frac{2}{e^{3y} e^{3y}}$

$= \frac{2}{(e^{3y})^2} = \frac{2}{(e^{3y})^2} = \frac{2}{e^{3y} e^{3y}}$

$= \text{constant} \leq 1$

"HORIZONTALLY", SECTOR IS BOUNDED ON BOTH SIDES BY  $\text{constant}$   $\Rightarrow$   $M$  (AND SIMILARLY)  $\Rightarrow$   $M$  (AND SIMILARLY)  $\Rightarrow$  THE SUMMATION OF THESE TWO VALUES

$\therefore$  SECTOR IS BOUNDED ON  $\Gamma_1$