

HEAT EQUATION

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$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t}, \quad \theta = \theta(x, t)$$

One Dimensional

Question 1

A thin rod of length 2 m has temperature $z = 20^\circ\text{C}$ throughout its length.

At time $t = 0$, the temperature z is suddenly dropped to $z = 0^\circ\text{C}$ at both its ends at $x = 0$, and at $x = 2$.

The temperature distribution along the rod $z(x,t)$, satisfies the standard heat equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}, \quad 0 \leq x \leq 2, \quad t \geq 0.$$

Assuming the rod is insulated along its length, determine an expression for $z(x,t)$.

[You must derive the standard solution of the heat equation in variable separate form]

$$\boxed{\text{[]}}, \quad z(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{80}{\pi(2n-1)} \exp\left[-\frac{\pi^2(2n-1)^2 t}{4}\right] \sin\left[\frac{(2n-1)\pi x}{2}\right] \right\}$$

ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM, DIFFERENTIATE AND SUBSTITUTE INTO THE PDE.

$z(t) = X(x)T(t) \Rightarrow \frac{\partial^2 z}{\partial x^2} = X''(x)T(t)$
 $\frac{\partial z}{\partial x} = X'(x)T'(t)$

$\rightarrow \frac{\partial^2 z}{\partial x^2} = -\lambda^2$
 $\Rightarrow X'(x)T(t) = X(x)T'(t)$
 $\Rightarrow X'(x)T(t) = \frac{X''(x)T(t)}{X(x)}$
 $\Rightarrow \frac{X'(x)}{X(x)} = \frac{T'(t)}{T(t)}$

AS THE LHS IS A FRACTION OF x ONLY AND THE RHS IS A FRACTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY A

IF $A=0$ $\bullet \frac{X'(x)}{X(x)} = 0 \quad | \quad \bullet \frac{T'(t)}{T(t)} = 0$
 $X'(x) = 0 \quad | \quad T'(t) = 0$
 $X(x) = C \quad | \quad T(t) = D$

$\therefore z(t) = C(Ax+B) = Bx+C$ 1.E. SIMPLY SHOWN WHICH IS NOT AMBIGUOUS IN THIS PROBLEM

IF $A \neq 0$, $\lambda = \pm p^2$ $\bullet \frac{X'(x)}{X(x)} = p^2 \quad | \quad \bullet \frac{T'(t)}{T(t)} = p^2$
 $X'(x) = p^2 X(x) \quad | \quad T'(t) = p^2 T(t)$
 $X(x) = A \cos(px) + B \sin(px)$

$\therefore z(t) = C_0 e^{i\lambda t} (A \cos(px) + B \sin(px))$
 $\underline{z(t)} = e^{i\lambda t} (P \cos(px) + Q \sin(px))$
 WHICH IS ALSO INAPPROPRIATE SINCE IT IS UNDEFINED AS $t \rightarrow \infty$

IF $A \neq 0$, $\lambda = \pm p^2$

$\bullet \frac{X'(x)}{X(x)} = -p^2 \quad | \quad \bullet \frac{T'(t)}{T(t)} = -p^2$
 $X'(x) = -p^2 X(x) \quad | \quad T'(t) = -p^2 T(t)$
 $X(x) = A \cosh(px) + B \sinh(px)$

$\therefore z(t) = C_0 e^{-i\lambda t} (A \cosh(px) + B \sinh(px))$
 $\underline{z(t)} = e^{-i\lambda t} (P \cosh(px) + Q \sinh(px))$ WHICH IS OK

APPLY INITIAL CONDITION, $z(0,t) = 0$

$\rightarrow 0 = e^{-i\lambda t} \times P \quad | \quad P = 0$
 $\therefore z(t) = Q e^{-i\lambda t} \sinh(px)$

APPLY BOUNDARY CONDITION, $z(0,t) = 20$

$\rightarrow 0 = Q_0 e^{-i\lambda t} \sinh(0) \quad | \quad Q_0 = 0$
 $\rightarrow \sinh(0) = 0 \quad (\text{Q } \neq 0, \text{ OTHERWISE SOLUTION IS TRIVIAL})$
 $\rightarrow 20 = Q_0 e^{-i\lambda t} \sinh(2) \quad | \quad \lambda = 0, 1, 2, 3, \dots$
 $\rightarrow Q_0 = \frac{20}{\sinh(2)} = \frac{20}{3.68} = 5.44$

$\therefore z_v(x,t) = Q_0 e^{-i\lambda t} \sinh(\frac{\pi x}{2})$
 $\underline{z(t)} = \sum_{n=1}^{\infty} [Q_n e^{-i\frac{(2n-1)\pi}{2} t} \sinh(\frac{\pi x}{2})]$

NOTE THAT $\lambda = \pm i\frac{(2n-1)\pi}{2}$ IN [1], PERIOD 2.

APPLY INITIAL CONDITION, $\underline{z(2,t)} = 20$

$\rightarrow 20 = \sum_{n=1}^{\infty} [Q_n e^{-i\frac{(2n-1)\pi}{2} t} \sinh(\frac{\pi x}{2})] \quad | \text{E. A FOURIER SERIES IN } z \text{ IN [2], PERIOD 2.}$

$\Rightarrow Q_1 = \frac{1}{\pi} \int_0^{\pi} 20 \sinh(\frac{\pi x}{2}) dx$
 $\Rightarrow Q_1 = \left[\frac{20}{\pi} \frac{2}{\sinh(\frac{\pi}{2})} \sinh(\frac{\pi x}{2}) \right]_0^{\pi}$
 $\rightarrow Q_1 = \frac{20}{\pi} \frac{2}{\sinh(\frac{\pi}{2})} = \frac{40}{\pi \sinh(\frac{\pi}{2})}$
 $\Rightarrow Q_1 = \frac{40}{\pi} = \frac{40}{3.68} = 10.87$
 $\Rightarrow Q_1 = \begin{cases} 0 & \text{if } k \neq 1 \\ \frac{40}{\pi} & \text{if } k = 1 \end{cases}$

IF $n > 2m+1 \rightarrow 1, 3, 5, \dots$

$\underline{z(t)} = \sum_{n=1}^{\infty} \left[\frac{20}{\sinh(\frac{\pi}{2})} e^{-i\frac{(2n-1)\pi}{2} t} \sinh\left(\frac{\pi x}{2}\right) \right]$

NOTE THAT AS $t \rightarrow \infty$, $\underline{z}(t) \rightarrow 0$. SO WE DO NOT NEED ANY EXPONENTIAL TERM UP FRONT!

Question 2

At time $t < 0$, a long thin rod of length l has temperature distribution $\theta(x)$ given by

$$\theta(x) = xl - x^2.$$

At time $t = 0$ the temperature is suddenly dropped to 0°C at both ends of the rod, and maintained at 0°C for $t \geq 0$.

The temperature distribution along the rod $\theta(x, t)$ satisfies the standard heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq l, \quad t \geq 0,$$

where α is a positive constant.

Assuming the rod is insulated along its length, determine an expression for $\theta(x, t)$ and hence show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

[You must derive the standard solution of the heat equation in variable separate form]

$$\theta(x, t) = \sum_{n=1}^{\infty} \left\{ \frac{8l^2}{(2n-1)^3 \pi^3} \exp\left[-\frac{\alpha^2 \pi^2 (2n-1)^2 t}{l^2} \right] \sin\left[\frac{(2n-1)\pi x}{l} \right] \right\}$$

[solution overleaf]

a)

$$\frac{\partial \Theta}{\partial t} = \frac{1}{a^2} \frac{\partial^2 \Theta}{\partial x^2}$$

Locate solution in variable separate form

$$\Theta(x,t) = X(x)T(t)$$

$$\frac{\partial \Theta}{\partial t} = X(x)T'(t)$$

$$\frac{\partial^2 \Theta}{\partial x^2} = X''(x)T(t)$$

$$\frac{\partial^2 \Theta}{\partial x^2} = X(x)T''(t)$$

Sub into the PDE yields

$$\rightarrow X''(x)T(t) = \frac{1}{a^2} X(x)T''(t)$$

$$\rightarrow \frac{X''(x)}{X(x)} T(t) = \frac{1}{a^2} \frac{X''(x)}{X(x)} T(t)$$

$$\rightarrow \frac{X''(x)}{X(x)} = \frac{1}{a^2} \frac{T''(t)}{T(t)}$$

As the LHS is a function of x only, and the RHS is a function of t only, both sides are at most a constant, say λ , which can be positive, negative or zero.

Hence $\frac{X''(x)}{X(x)} = \lambda \Rightarrow X(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$

$$T(t) = \lambda t + B$$

or $X''(x) = 0 \Rightarrow X(x) = C_1 x + C_2$

$$T(t) = 0 \Rightarrow B = C$$

or $X''(x) = -\lambda^2 X(x) \Rightarrow X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

$$T(t) = -\lambda^2 t + B$$

$\Theta(x,t) = e^{-\lambda^2 t} (A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x))$

$$\Theta(x,t) = e^{-\lambda^2 t} (A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x))$$

b) Now solution Θ is not suitable as it has no initial dependence, i.e. it is a steady state thermal solution.

Solution Θ has to be discounted, as it produces unphysical solutions as $t \rightarrow \infty$

$$\therefore \Theta(x,t) = e^{-\lambda^2 t} (A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x))$$

Boundary and initial conditions

$$\Theta(0,t) = 0 \quad \text{--- (1)}$$

$$\Theta(l,t) = 0 \quad \text{--- (2)}$$

$$\Theta(x,0) = xl - x^2 \quad \text{--- (3)}$$

By (1) $\Rightarrow 0 = e^{-\lambda^2 t} (A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)) \Rightarrow A = 0$

$$\therefore \Theta(x,t) = B e^{-\lambda^2 t} \sin(\sqrt{\lambda}x)$$

By (2) $\Rightarrow 0 = B e^{-\lambda^2 t} \sin(\sqrt{\lambda}l) \Rightarrow \sin(\sqrt{\lambda}l) = 0 \Rightarrow \sqrt{\lambda}l = n\pi \Rightarrow \lambda = \frac{n^2 \pi^2}{l^2}$

$$\therefore \Theta_n(x,t) = B_n e^{-\frac{n^2 \pi^2 t}{l^2}} \sin\left(\frac{n\pi x}{l}\right)$$

or

$$\Theta(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2 \pi^2 t}{l^2}} \sin\left(\frac{n\pi x}{l}\right)$$

By (3) $\Rightarrow xl - x^2 = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2 \pi^2 t}{l^2}} \sin\left(\frac{n\pi x}{l}\right)$

This is a Fourier expansion in x , $0 \leq x \leq l$.

$$\therefore B_n = \frac{1}{l/2} \int_0^{l/2} (xl - x^2) \sin\left(\frac{n\pi x}{l}\right) dx$$

By parts

$$\begin{aligned} & B_n = \frac{2}{\pi} \left[\left[(l-2x)x \sin\left(\frac{n\pi x}{l}\right) \right]_0^{l/2} + \frac{1}{n\pi} \int_0^{l/2} (l-2x) \cos\left(\frac{n\pi x}{l}\right) dx \right] \\ & \Rightarrow B_n = \frac{2}{\pi} \left[\left[(l-2x)x \sin\left(\frac{n\pi x}{l}\right) \right]_0^{l/2} + \frac{2l}{n\pi} \int_0^{l/2} \sin\left(\frac{n\pi x}{l}\right) dx \right] \\ & \Rightarrow B_n = \frac{2}{\pi} \left[\frac{l}{n\pi} \times \frac{2l}{n\pi} \int_0^{l/2} \sin\left(\frac{n\pi x}{l}\right) dx \right] \\ & \Rightarrow B_n = \frac{4l^2}{n^2 \pi^2} \left[-\left[\cos\left(\frac{n\pi x}{l}\right) \right]_0^{l/2} \right] \\ & \Rightarrow B_n = \frac{4l^2}{n^2 \pi^2} \left[\cos\left(\frac{n\pi l}{l}\right) - \cos(0) \right] \\ & \Rightarrow B_n = \frac{4l^2}{n^2 \pi^2} [1 - \cos(n\pi)] \end{aligned}$$

b) Consider $\Theta\left(\frac{1}{2}, 0\right)$ is the initial temperature at the midpoint

$$\rightarrow \frac{l}{2} \left(1 - \frac{l}{2}\right) = \sum_{n=1}^{\infty} \left[\frac{B_n^2 l^2}{n^2 \pi^2} \times \left(1 - \cos\left[\frac{(2n-1)\pi}{2}\right]\right) \right]$$

$$\Rightarrow \frac{l^2}{4} = \sum_{n=1}^{\infty} \left[\frac{B_n^2 l^2}{n^2 \pi^2} \times (-1)^{n+1} \right]$$

$$\Rightarrow \frac{l^2}{4} = \frac{B_1^2}{\pi^2} + \frac{B_3^2}{9\pi^2} + \dots$$

$$\Rightarrow \frac{l^3}{24} = \sum_{n=1}^{\infty} \left[\frac{B_n^2 l^2}{n^2 \pi^2} \right]$$

OR starting from $n=0$

$$\sum_{n=0}^{\infty} \frac{B_n^2 l^2}{n^2 \pi^2} = \frac{\pi^2}{32}$$

Question 3

The temperature distribution $\theta(x,t)$ along a thin bar of length 2 m satisfies the partial differential equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{9} \frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq 2, \quad t \geq 0.$$

Initially the bar has a linear temperature distribution, with temperature 0 °C at one end of the bar where $x=0$ m, and temperature 50 °C at the other end where $x=2$ m.

At time $t=0$ the temperature is suddenly dropped to 0 °C at both ends of the rod, and maintained at 0 °C for $t \geq 0$.

Assuming the rod is insulated along its length, determine an expression for $\theta(x,t)$ and hence show that

[You must derive the standard solution of the heat equation in variable separate form]

$$\theta(x,t) = \frac{100}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n+1}}{n} \exp\left[-\frac{9n^2\pi^2 t}{4}\right] \sin\left[\frac{n\pi x}{2}\right] \right\}$$

$\frac{\partial \theta}{\partial t} = \frac{1}{9} \frac{\partial^2 \theta}{\partial x^2}$ for $0 < x < 2$, $t > 0$

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM
 $\theta(x,t) = X(x)T(t)$
- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E
 $\frac{\partial \theta}{\partial t} = X(x)T'(t)$ & $\frac{\partial^2 \theta}{\partial x^2} = X''(x)T(x)$
- $\Rightarrow X''(x)T(x) = \frac{1}{9} T'(t)X(x)$
- $\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{9} \frac{T'(t)}{T(x)}$
- AS THE LHS IS A FUNCTION OF x ONLY AND THE RHS IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT
- THIS CONSTANT HAS TO BE NEGATIVE SINCE WE REQUIRE 0 BOUNDARY CONDITIONS IN t , VARIOUS WITH THAT.

e.g. $\lambda > 0$, $T'(t) = \text{constant}$ $\Rightarrow T(t) = \text{constant}$
 $\Rightarrow \lambda > 0$, $\lambda^2 = -\frac{1}{9}$
 $\Rightarrow \lambda = \pm \frac{1}{3}$
 $\Rightarrow T(t) = C e^{\pm \frac{1}{3} t}$

• HAVE LET $\lambda < 0$, SAY $\lambda = -p^2$

$\frac{X''(x)}{X(x)} = -p^2$ $\frac{T'(t)}{T(t)} = -p^2$
 $X''(x) = -p^2 X(x)$ $T'(t) = -p^2 T(t)$
 $X(x) = A \cos px + B \sin px$ $T(t) = C e^{-p^2 t}$

$\theta(x,t) = C e^{-p^2 t} [A \cos px + B \sin px]$

• APPLY CONDITION ①
 $\theta(0,t) = 0 \Rightarrow 0 = A$
 $\theta(x,t) = B e^{-p^2 t} \sin px$

• APPLY CONDITION ②
 $\theta(2,t) = 50 \Rightarrow 0 = B e^{-p^2 t} \sin 2p$
 $\sin 2p = 0 \Rightarrow 2p = n\pi \Rightarrow p = \frac{n\pi}{2}$
 $n = 1, 2, 3, \dots$

$B_n = \frac{100}{n\pi} e^{-\frac{n^2\pi^2 t}{4}}$
 $\theta(x,t) = \sum_{n=1}^{\infty} \left[B_n e^{-\frac{n^2\pi^2 t}{4}} \sin\left(\frac{n\pi x}{2}\right) \right]$

• APPLY CONDITION ③
 $\theta(2,0) = f(x)$
 $f(x)$ IS A LINEAR FUNCTION OF x FROM $x=0$ TO $x=2$

$25x = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right)$
 $16 + \text{FOURIER EXPANSION WITH } L=2$

$\Rightarrow B_1 = \frac{1}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{2}\right) dx$
 $\Rightarrow B_1 = \frac{1}{2} \int_0^2 25x \sin\left(\frac{n\pi x}{2}\right) dx$ (COS EXPANSION)
 $\Rightarrow B_1 = \int_0^2 25x \sin\left(\frac{n\pi x}{2}\right) dx$

INTEGRATION BY PARTS

$\Rightarrow B_1 = \left[-\frac{25x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) \right]_0^2 + \frac{25}{n\pi} \int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx$
 $\Rightarrow B_1 = -\frac{100}{n\pi} \cos\left(\frac{n\pi 2}{2}\right) + \frac{100}{n\pi} \left[\sin\left(\frac{n\pi x}{2}\right) \right]_0^2$
 $B_1 = -\frac{100(-1)^n}{n\pi}$

$\therefore \theta(x,t) = \sum_{n=1}^{\infty} \left[-\frac{100(-1)^n}{n\pi} e^{-\frac{n^2\pi^2 t}{4}} \sin\left(\frac{n\pi x}{2}\right) \right]$

$\theta(x,t) = \frac{100}{\pi} \sum_{n=1}^{\infty} \left[(-1)^{n+1} e^{-\frac{n^2\pi^2 t}{4}} \sin\left(\frac{n\pi x}{2}\right) \right]$

Question 4

The temperature $\Theta(x,t)$ satisfies the one dimensional heat equation

$$\frac{\partial^2 \Theta}{\partial x^2} = 4 \frac{\partial \Theta}{\partial t},$$

where x is a spatial coordinate and t is time, with $t \geq 0$.

For $t < 0$, two thin rods, of lengths 3π and π , have temperatures 0°C and 100°C , respectively. At time $t=0$ the two rods are joined end to end into a single rod of length 4π .

The rods are made of the same material, have perfect thermal contact and are insulated along their length.

Determine an expression for $\Theta(x,t)$, $t \geq 0$.

[You must derive the standard solution of the heat equation in variable separate form]

$$\Theta(x,t) = 25 - \frac{200}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} e^{-\frac{1}{4}n^2 t} \sin\left(\frac{3}{4}n\pi\right) \cos\left(\frac{1}{4}nx\right) \right]$$

$\frac{\partial \Theta}{\partial t} = 4 \frac{\partial^2 \Theta}{\partial x^2}$

• ASSUME A SOLUTION IN DIMINISHING SEPARABLE FORM
 $\Theta(x,t) = X(x)T(t) \Rightarrow \frac{\partial \Theta}{\partial t} = X'(x)T(t)$
 $\frac{\partial^2 \Theta}{\partial x^2} = X''(x)T(t)$

SUBSTITUTE THE P.D.E.
 $X'(x)T(t) = 4X''(x)T(t) \Rightarrow \frac{X''(x)}{X(x)} = \frac{4T'(t)}{T(t)} = k$

AS THE LHS OF THE ABOVE EQUATION IS A FUNCTION OF x ONLY, AND THE RHS IS A FUNCTION OF t ONLY, BOTH SIDES MUST AT LEAST BE A CONSTANT, SAY k .

LET $T(t) = C$
 $X'(x) = Cx + B$ ← STEADY STATE

$X(x) = p^2$
 $X(x) = p^2 X(x)$
 $X(x) = A \cos(px) + B \sin(px)$
 $X(x) = A \cos(px) + B \sin(px)$
 $X(x) = e^{ipx} (\cos(px) + \sin(px))$

$\frac{X''(x)}{X(x)} = -p^2$
 $X''(x) = -p^2 X(x)$
 $X''(x) = -p^2 T(t)$
 $T(t) = Ce^{-pt}$

$X(x) = e^{ipx} (\cos(px) + \sin(px))$
 $X(x) = -p^2 X(x)$
 $X(x) = A \cos(px) + B \sin(px)$
 $X(x) = e^{ipx} (\cos(px) + B \sin(px))$
 $\Theta(x,t) = e^{ipx} (A \cos(px) + B \sin(px))$

• NOW WE ARE REDUCED TO THE PROBLEM → ONLY VALID SOLUTION IS ONE THAT IS TIME INVARIANT & BOUNDARY SET UP AN AUXILIARY FUNCTION G

AS $t \rightarrow \infty$, THE TEMPERATURE WILL DECAY OUT TO AN AVERAGE TEMPERATURE OF $\frac{25(0) + 100\pi}{3\pi + \pi} = \frac{100\pi}{4\pi} = 25$

• BOUNDARY/INITIAL CONDITIONS

GIVEN INSULATED $\begin{cases} \frac{\partial \Theta}{\partial x}(0) = 0 \\ \frac{\partial \Theta}{\partial x}(\pi) = 0 \end{cases}$
 $\Theta(x,0) = \begin{cases} 100 & 0 \leq x \leq 3\pi \\ 25 & 3\pi < x \leq 4\pi \end{cases}$

NEXT WE NEED TO BUILD THE SOLUTION $\Theta(x,t) = 25 + \tilde{\Theta}(x,t)$ WHERE $\tilde{\Theta}(x,t)$ SATISSES $\frac{\partial^2 \tilde{\Theta}}{\partial x^2} = \frac{\partial^2 \Theta}{\partial x^2}$

LET $\tilde{\Theta}(x,t) = 25 + \tilde{G}(x,t)$ WHERE $\tilde{G}(x,t)$ SATISSES $\frac{\partial^2 \tilde{G}}{\partial x^2} = \frac{\partial^2 \Theta}{\partial x^2}$

• TRANSLATING THE CONDITIONS

$\tilde{G}(0,t) = 25 + \tilde{\Theta}(0,t)$
 $\Rightarrow \tilde{G}(0,t) = \tilde{\Theta}(0,t) - 25$
 $\Rightarrow \frac{\partial \tilde{G}}{\partial x}(0,t) = \frac{\partial \tilde{\Theta}}{\partial x}(0,t) \quad \text{②}$
 $\tilde{G}(\pi,t) = 25 + \tilde{\Theta}(\pi,t)$
 $\Rightarrow \tilde{G}(\pi,t) = \tilde{\Theta}(\pi,t) - 25 \quad \text{③}$

STARTING WITH $\tilde{G}(x,t) = e^{-\frac{1}{4}pt} (A \cos(px) + B \sin(px))$

$\frac{\partial \tilde{G}}{\partial x}(0,t) = e^{-\frac{1}{4}pt} (-Ap \sin(0) + Bp \cos(0))$
 $\Rightarrow \tilde{G}(0,t) = e^{-\frac{1}{4}pt} (Bp \cos(0) - Ap \sin(0))$

• BY CONDITION ① $\Rightarrow 0 = e^{-\frac{1}{4}pt} \cos(0) \Rightarrow p = 0$
 $\therefore \tilde{G}(0,t) = 0 = e^{-\frac{1}{4}pt} \cos(0) \Rightarrow \frac{\partial \tilde{G}}{\partial x}(0,t) = -Ap = -\frac{1}{4}p^2 \sin(0)$

• BY CONDITION ② $\Rightarrow 0 = -Ap = -\frac{1}{4}p^2 \sin(0) \Rightarrow p = 0$
 $\Rightarrow n = 0, 1, 2, 3, \dots$ (NOTE IT IS EVEN AND ODD IN \tilde{G})
 $\therefore \tilde{G}(0,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{4}\right)$

• BY CONDITION ③ $\Rightarrow \tilde{G}(\pi,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{4}\right)$

IT IS FURTHER IN PERIODIC 2 π → $x = 4\pi$ IF HALF PERIOD 2π
DO THE A_0 → $A_0 = \frac{1}{4\pi} \int_0^{4\pi} 25 dx = 25$
 $A_1 = \frac{1}{4\pi} \int_0^{4\pi} 25 \cos\left(\frac{\pi x}{4}\right) dx = \frac{1}{2\pi} \int_0^{4\pi} 25 \cos\left(\frac{\pi x}{4}\right) d\left(\frac{\pi x}{4}\right) = 75 \sin\left(\frac{\pi x}{4}\right) \Big|_0^{4\pi} = 0$
 $A_2 = \frac{1}{4\pi} \int_0^{4\pi} 25 \cos\left(\frac{2\pi x}{4}\right) dx = \frac{1}{2\pi} \times \frac{25}{2} \left[\sin\left(\frac{2\pi x}{4}\right) \right] \Big|_0^{4\pi} = -25 \sin\left(\frac{2\pi x}{4}\right) \Big|_0^{4\pi} = -25 \sin(\pi) = 25$
 $\therefore A_2 = \frac{1}{2\pi} \int_0^{4\pi} 25 \cos\left(\frac{2\pi x}{4}\right) dx = \frac{1}{2\pi} \int_0^{4\pi} 25 \cos\left(\frac{\pi x}{2}\right) dx = \frac{1}{2\pi} \times 25 \left[\sin\left(\frac{\pi x}{2}\right) \right] \Big|_0^{4\pi} = 25$
 $\therefore \tilde{G}(0,t) = \sum_{n=0}^{\infty} \frac{200}{n+1} \sin\left(\frac{n\pi x}{4}\right) e^{-\frac{1}{4}(n+1)^2 t}$
 $\tilde{G}(x,t) = 25 - \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{4}\right) e^{-\frac{1}{4}n^2 t}$

Question 5

Solve the heat equation for $u = u(x, t)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}, \quad 0 \leq x \leq 5, \quad t \geq 0,$$

subject to the conditions

$$u(0, t) = 0, \quad u(5, t) = 0 \quad \text{and} \quad u(x, 0) = \sin \pi x - 37 \sin\left(\frac{1}{5}\pi x\right) + 6 \sin\left(\frac{9}{5}\pi x\right).$$

[You must derive the standard solution of the heat equation in variable separate form]

$$u(x, t) = -37 e^{-\frac{1}{25}\pi^2 c^2 t} \sin\left(\frac{1}{5}\pi x\right) + e^{-\pi^2 c^2 t} \sin(\pi x) + 6 e^{-\frac{81}{25}\pi^2 c^2 t} \sin\left(\frac{9}{5}\pi x\right)$$

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$ for $u = u(x, t), \quad 0 \leq x \leq 5, \quad t \geq 0$

SUBJECT TO THE CONDITIONS

- (1) $u(0, t) = 0$
- (2) $u(5, t) = 0$
- (3) $u(x, 0) = \sin(\pi x) - 37 \sin\left(\frac{1}{5}\pi x\right) + 6 \sin\left(\frac{9}{5}\pi x\right)$

ASSUME A SOLUTION IN VARIABLE SEPARATE FORM
 $u(x, t) = X(x)T(t)$

DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= X''(x)T(t) \quad \frac{\partial u}{\partial t} = X(x)T'(t) \\ \Rightarrow X''(x)T(t) &= \frac{1}{c^2} X(x)T'(t) \\ \Rightarrow \frac{X''(x)}{X(x)} T(t) &= \frac{1}{c^2} X(x) T'(t) \\ \Rightarrow \frac{X''(x)}{X(x)} &= \frac{1}{c^2} \frac{T'(t)}{T(t)} \end{aligned}$$

AS THE L.H.S. IS A FUNCTION OF x ONLY, AND THE R.H.S. IS A FUNCTION OF t ONLY, BOTH SIDES MUST BE EQUAL AT ALL x , SAY λ .

LOOKING AT THE BOUNDARY CONDITIONS (1) & (2), WE REQUIRE A PERIODIC SOLUTION IN x — THIS IMPLIES λ HAS TO BE NEGATIVE (THIS IS FURTHER ASKED BY THE R.H.S. ABOUT WHICH REQUIRES THE COEFFICIENT TO BE NEGATIVE SO WE OBTAIN FINITE SOLUTIONS AS $t \rightarrow \infty$)

HENCE LET $\lambda = -\frac{p^2}{c^2}$

$$\begin{aligned} \frac{X''(x)}{X(x)} &= -\frac{p^2}{c^2} \\ X''(x) &= -p^2 X(x) \\ X(x) &= A \cos px + B \sin px \end{aligned}$$

$u(x, t) = X(x)T(t) = e^{-\frac{p^2}{c^2}t} (A \cos px + B \sin px)$

(CD THE EVEN MODELED AND $A \neq 0$)

APPLY CONDITION (1), $u(0, t) = 0$

$$0 = e^{-\frac{p^2}{c^2}t} A \Rightarrow A = 0$$

$u(x, t) = B e^{-\frac{p^2}{c^2}t} \sin px$

APPLY CONDITION (2), $u(5, t) = 0$

$$0 = B e^{-\frac{p^2}{c^2}t} \sin p5 = B \sin p5 = 0 \Rightarrow p5 = n\pi, \quad n = 1, 2, \dots$$

$$p = \frac{n\pi}{5}, \quad n = 1, 2, \dots$$

$u(x, t) = B_n e^{-\frac{n^2\pi^2}{25}t} \sin \frac{n\pi x}{5}$

OR

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2}{25}t} \sin \frac{n\pi x}{5}$$

FINALLY APPLY THE INITIAL CONDITION (3)

$$u(x, 0) = \sin \pi x - 37 \sin\left(\frac{1}{5}\pi x\right) + 6 \sin\left(\frac{9}{5}\pi x\right) = \sum_{n=1}^{\infty} [B_n \sin\left(\frac{n\pi x}{5}\right)]$$

CONSTANTS

$$\begin{aligned} n=5 &\Rightarrow B_5 \sin \pi x = \sin \pi x \Rightarrow B_5 = 1 \\ n=1 &\Rightarrow B_1 \sin \frac{\pi x}{5} = -37 \sin \frac{\pi x}{5} \Rightarrow B_1 = -37 \\ n=9 &\Rightarrow B_9 \sin \frac{9\pi x}{5} = 6 \sin \frac{9\pi x}{5} \Rightarrow B_9 = 6 \end{aligned}$$

THE REST OF THE "B_n" ARE ZERO

FINALLY WE HAVE A SOLUTION

$$u(x, t) = -37 e^{-\frac{361\pi^2}{25}t} \sin \frac{\pi x}{5} + e^{-\frac{\pi^2}{25}t} \sin \frac{\pi x}{5} + 6 e^{-\frac{81\pi^2}{25}t} \sin \frac{9\pi x}{5}$$

Question 6

A long thin rod of length L has temperature $\theta = 0$ throughout its length.

At time $t = 0$ the temperature is suddenly raised to T_1 at both ends of the rod, at $x = 0$ and at $x = L$.

Both ends of the rod are maintained at temperature T_1 for $t \geq 0$.

The temperature distribution along the rod $\theta(x, t)$ satisfies the standard heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq L, \quad t \geq 0,$$

where α is a positive constant.

Assuming the rod is insulated along its length, determine an expression for $\theta(x, t)$.

[You must derive the standard solution of the heat equation in variable separate form]

$$\boxed{\theta(x, t) = T_1 \left[1 - \frac{4}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{1}{(2n-1)} \exp \left[-\frac{\alpha^2 \pi^2 (2n-1)^2 t}{L^2} \right] \sin \left[\frac{(2n-1)\pi x}{L} \right] \right\} \right]},$$

SOLVING THE HEAT EQUATION BY SEPARATION OF VARIABLES AND IGNORING ANY CONDITIONS AT THIS STAGE

Let $\theta(x, t) = X(x)T(t)$

$$\frac{\partial \theta}{\partial x} = X'(x)T(t)$$

$$\frac{\partial^2 \theta}{\partial x^2} = X''(x)T(t)$$

$$\frac{\partial \theta}{\partial t} = X(x)T'(t)$$

SUBSTITUTE INTO THE D.D.E.

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t} \implies X''(x)T(t) = \frac{1}{\alpha^2} X(x)T'(t)$$

$$\implies \frac{X''(x)}{X(x)} = \frac{1}{\alpha^2} \frac{T'(t)}{T(t)} = \lambda$$

BOTH SIDES OF THE ABOVE EQUATION MUST AT MOST BE A CONSTANT? AS THE LHS IS A FRACTION OF x ONLY AND THE RHS IS A FUNCTION OF t ONLY. THIS CONSTANT MAY BE ZERO, POSITIVE OR NEGATIVE.

- IF $\lambda = 0$,** $X''(x) = 0 \implies X(x) = Ax + B$ $\implies \theta(x, t) = (Ax + B)T(t)$ $\implies T(t) = C$ $\implies \theta(x, t) = Ax + C$ $\implies \theta(x, t) = Ax + B$
- IF $\lambda > 0$, SAY λ^2**

$$\implies \frac{X''(x)}{X(x)} = \lambda^2 \implies \frac{1}{X(x)} = \frac{\lambda^2}{X(x)} \implies X''(x) = \lambda^2 X(x)$$

$$\implies X(x) = A e^{\lambda x} + B e^{-\lambda x}$$

$$\implies \theta(x, t) = A e^{\lambda x} T(t) + B e^{-\lambda x} T(t)$$

THE SOLUTION IS NOT APPLICABLE AS IT PRODUCES UNWANTED TEMPERATURES AT $x = 0$ AS $t \rightarrow \infty$

- IF $\lambda < 0$, SAY $\lambda = -p^2$**

$$\implies \frac{X''(x)}{X(x)} = -p^2 \implies \frac{1}{X(x)} = -p^2 \implies X''(x) = -p^2 X(x)$$

$$\implies X(x) = A \cosh(p^2 x) + B \sinh(p^2 x) \implies \theta(x, t) = C e^{-p^2 t} (A \cosh(p^2 x) + B \sinh(p^2 x))$$

THIS SOLUTION IS "ACCCEPTABLE" TO THE TYPE OF PROBLEM AS $\theta(x, t)$ IS BOUNDED AS $t \rightarrow \infty$

- NOW THE INITIAL & BOUNDARY CONDITIONS NEED TO BE BUILT IN**

LET $\theta(x, 0) = T_1 + \tilde{\theta}(x)$ WHERE $\tilde{\theta}(x)$ SATISFIES $\frac{\partial \tilde{\theta}}{\partial x} = \frac{1}{\alpha^2} \frac{\partial \tilde{\theta}}{\partial t}$

SO THAT $\tilde{\theta}(x) \rightarrow 0$ AS $t \rightarrow \infty$ AND THAT $\tilde{\theta}(0) = T_1$ AS $t \rightarrow \infty$

- AT THE END $x=0$, $\theta(x, 0) = T_1$** $\implies \tilde{\theta}(0) = 0$ AS $t \rightarrow \infty$ **-I**
- AT THE END $x=L$, $\theta(x, 0) = T_1$** $\implies \tilde{\theta}(L) = 0$ AS $t \rightarrow \infty$ **-II**
- INITIAL TEMPERATURE IS ZERO, $\theta(x, 0) = 0$** $\implies \tilde{\theta}(0) = 0$, $\alpha x/L = -I$

APPLYING EACH OF THESE CONDITIONS IN TURN

- IF $\lambda < 0$, SAY $\lambda = -p^2$ (ACCEPTABLE)**
- $\tilde{\theta}(0) = 0 = e^{-p^2 t} (A \cosh(p^2 x) + B \sinh(p^2 x))$**
- $\tilde{\theta}(L) = 0 = e^{-p^2 t} (A \cosh(p^2 L) + B \sinh(p^2 L))$**
- $\tilde{\theta}(0) = 0 = e^{-p^2 t} \sinh(p^2 x)$ FOR ALL $t > 0$** $\therefore p^2 L = m\pi, m \in \mathbb{Z}$ $\therefore p = \frac{m\pi}{L}$
- $\tilde{\theta}(L) = 0 = e^{-p^2 t} \sinh(p^2 L)$** WHICH IS A SOURCE TERM IN (ii) WITH FACTOR $= -1$
- $\therefore B_m = \frac{1}{2} \int_0^L \tilde{\theta}(0) \sinh(p^2 L) \, dx = \frac{1}{2} \int_0^L T_1 \sinh(p^2 L) \, dx = \frac{1}{2} \frac{L}{m\pi} (\cos(m\pi) - 1) = \frac{-2L}{m\pi} (\cos(m\pi) - 1)$
- $\therefore B_{2m+1} = \frac{-4L}{(2m+1)\pi}, m = 1, 3, \dots$
- $\tilde{\theta}(0) = \sum_{m=0}^{\infty} \frac{-4L}{(2m+1)\pi} e^{-\frac{(2m+1)^2 \pi^2 t}{L^2}} \sin \left[\frac{(2m+1)\pi x}{L} \right]$**
- $\theta(x, t) = T_1 - \frac{4L}{\pi} \sum_{m=1}^{\infty} \left[\frac{1}{2m+1} e^{-\frac{(2m+1)^2 \pi^2 t}{L^2}} \sin \left[\frac{(2m+1)\pi x}{L} \right] \right]$**

Question 7

A long thin rod of length L has temperature $\theta = T_1$ throughout its length.

At time $t = 0$ the temperature is suddenly raised to T_2 at one of its ends at $x = 0$, and is maintained at T_2 for $t \geq 0$.

The temperature distribution along the rod $\theta(x, t)$, satisfies the standard heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq L, \quad t \geq 0,$$

where α is a positive constant.

Assuming the rod is insulated along its length, determine an expression for $\theta(x, t)$.

[You must derive the standard solution of the heat equation in variable separate form]

$$\theta(x, t) = T_2 + \frac{4(T_1 - T_2)}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{1}{(2n-1)} \exp \left[-\frac{\alpha^2 \pi^2 (2n-1)^2 t}{4L^2} \right] \sin \left[\frac{(2n-1)\pi x}{2L} \right] \right\}$$

$\frac{\partial \theta}{\partial x} = \frac{1}{\alpha^2} \frac{\partial^2 \theta}{\partial t^2}, \quad \theta = \theta(x, t)$

• LOOK FOR SOLUTIONS IN VARIABLE SEPARATE FORM, $\theta(x, t) = X(x)T(t)$

$\frac{\partial \theta}{\partial t} = X(x)T'(t) \quad \text{and} \quad \frac{\partial^2 \theta}{\partial t^2} = X(x)T''(t)$

SUB INTO THE PDE

$X''(x)T(t) + \frac{1}{\alpha^2} X(x)T''(t)$

$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2} \frac{T''(t)}{T(t)}$

AS THE RHS IS A FUNCTION OF x ONLY AND THE RHS IS A FUNCTION OF t ONLY,

THEN BOTH SIDES ARE AT MOST A CONSTANT, SAY A ,

• IF $A > 0$: $\frac{X''(x)}{X(x)} = p^2 \Rightarrow X(x) = A_1 x + B \quad \left\{ \begin{array}{l} \Rightarrow E(x) = (A_1 x + B)X(x) \\ T'(t) = 0 \Rightarrow T(t) = C \end{array} \right. \Rightarrow E(x) = P(x) + Q$

(THIS PRODUCES AN ESTATE STATE SOLUTION - NOT THAT DESIRED?)

• IF $A < 0$: $\frac{X''(x)}{X(x)} = p^2 \Rightarrow X(x) = p^2 x \quad \left\{ \begin{array}{l} \Rightarrow \frac{1}{\alpha^2} \frac{T''(t)}{T(t)} = p^2 \\ \Rightarrow X'(x) = p^2 X(x) \quad \Rightarrow T'(t) = -\alpha^2 p^2 T(t) \\ \Rightarrow X(x) = A e^{p^2 x} B e^{-\alpha^2 p^2 t} \Rightarrow T(t) = C e^{-\alpha^2 p^2 t} \end{array} \right. \Rightarrow E(x) = C e^{p^2 x} (C e^{-\alpha^2 p^2 t} + Q e^{-\alpha^2 p^2 t})$

(THIS SOLUTION IS NOT APPROPRIATE, SAYS $Q = 0$, AS $t \rightarrow \infty$)

• IF $A = 0$: SAY $p = 0 \Rightarrow \frac{X''(x)}{X(x)} = 0 \Rightarrow X(x) = p^0 x \quad \left\{ \begin{array}{l} \Rightarrow \frac{1}{\alpha^2} \frac{T''(t)}{T(t)} = 0 \\ \Rightarrow X'(x) = 0 \quad \Rightarrow T'(t) = 0 \\ \Rightarrow X(x) = A_0 x + B_0 \end{array} \right. \Rightarrow T(t) = C_0 \Rightarrow E(x) = C_0 x + Q_0$

THIS SOLUTION IS ACCEPTABLE TO THIS TYPE OF PROBLEM

• INITIAL & BOUNDARY CONDITIONS

TEMPERATURE θ_0 AT $x=0$ IS T_0 FOR $t < 0$

TEMPERATURE θ_0 AT $x=L$ IS T_2 FOR $t > 0$

INSULATED AT $x=L$, $\frac{\partial \theta}{\partial x} = 0$ AT $x=L$

WE EXPECT ALL $t \rightarrow \infty$, THE ENTIRE ROD TO REACH T_2

LET $E(x, t) = T_2 + \tilde{E}(x, t)$ WHERE $\tilde{E}(x, t)$ SATISFIES

$\frac{\partial \tilde{E}}{\partial x} = 0 \Rightarrow \frac{\partial \tilde{E}}{\partial x} (x, t) = 0$

(AT $t \rightarrow \infty$, $\tilde{E}(x, t) \rightarrow 0$)

$\tilde{E}(x, t) \rightarrow 0$

$\theta(x, t) = T_2 + \tilde{E}(x, t) \quad (1)$

$\theta(x, 0) = T_0 \Rightarrow \tilde{E}(x, 0) = T_0 - T_2 \quad (2)$

SINCE $\frac{\partial \tilde{E}}{\partial x} = 0$

$\frac{\partial \tilde{E}}{\partial x} (x, t) = 0 \Rightarrow \tilde{E}(x, t) = \int_a^x \tilde{E}'(x') dx' = \int_a^x \tilde{E}_0(x') dx' = \tilde{E}_0(x)$

$\tilde{E}(x, 0) = \tilde{E}_0(x) = T_0 - T_2 \quad (3)$

• FIND $\tilde{E}(x, t) = \tilde{E}_0(x) = \tilde{E}_0(x) \exp(-\alpha^2 t)$

• $\tilde{E}(x, t) = \tilde{E}_0(x) \exp(-\alpha^2 t) \Rightarrow \tilde{E}(x, t) = Q_0 \exp(-\alpha^2 t)$

$Q_0 = \tilde{E}_0(x) \exp(-\alpha^2 t)$

DIFFERENTIATE TO APPLY (2)

$\frac{\partial \tilde{E}}{\partial x} = Q_0 \tilde{E}'_0(x) \exp(-\alpha^2 t)$

• $0 = Q_0 \tilde{E}'_0(x) \exp(-\alpha^2 t) \Rightarrow \tilde{E}'_0(x) = 0$

$\tilde{E}_0(x) = \sum_{n=1}^{\infty} Q_n \left[\sin \left(\frac{n\pi x}{L} \right) \right] \exp \left(-\frac{n^2 \pi^2 \alpha^2 t}{L^2} \right)$

• $T_1 - T_2 = \sum_{n=1}^{\infty} Q_n \sin \left[\frac{n\pi x}{L} \right] \times 1$

WHICH IS A FOURIER SINE EXPANSION IN (2)

WITH $\tilde{E}(x) = T_1 - T_2$

$Q_n = \frac{1}{L} \int_0^L (T_1 - T_2) \sin \left[\frac{n\pi x}{L} \right] dx$

$Q_1 = \frac{1}{L} \int_0^L (T_1 - T_2) \sin \left[\frac{\pi x}{L} \right] dx$

$Q_1 = \frac{2(T_1 - T_2)}{L} \times \int_0^L \sin \left[\frac{\pi x}{L} \right] dx$

$Q_1 = \frac{4(T_1 - T_2)}{\pi L} \left[\int_0^L \sin \left[\frac{\pi x}{L} \right] dx \right]_0^L$

$Q_1 = \frac{4(T_1 - T_2)}{\pi L} \approx 1.33, \dots$

$\therefore \tilde{E}(x, t) = \sum_{n=1}^{\infty} \frac{4(T_1 - T_2)}{\pi n L} \left[\sin \left(\frac{n\pi x}{L} \right) \right] \left[\exp \left(-\frac{n^2 \pi^2 \alpha^2 t}{L^2} \right) \right]$

$\therefore \theta(x, t) = T_2 + \frac{4(T_1 - T_2)}{\pi L} \sum_{n=1}^{\infty} \frac{1}{n^2 \pi^2} \sin \left[\frac{n\pi x}{L} \right] \exp \left[\frac{-n^2 \pi^2 \alpha^2 t}{L^2} \right]$

Question 8

The temperature $u(x,t)$ satisfies the one dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{9} \frac{\partial u}{\partial t}, \quad t \geq 0, \quad 0 \leq x \leq 2$$

where x is a spatial coordinate and t is time.

It is further given that

$$u(0,t) = 0, \quad u(2,t) = 8, \quad u(x,0) = 2x^2$$

Determine an expression for $u(x,t)$.

$$\boxed{\text{ANSWER}} , \quad u(x,t) = 4x - \sum_{k=0}^{\infty} \left[\frac{64}{(2k+1)^3 \pi^3} \exp \left[-\frac{9(2k+1)^2 t}{4} \right] \sin \left[\frac{(2k+1)\pi x}{2} \right] \right]$$

ASSUME A SOLUTION IN CAVARABLE SEPARABLE FORM -- DIFFERENTIATE AND SUBSTITUTE

INTO THE P.D.E

$$u(x,t) = X(x)T(t) \Rightarrow \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = X(x)T'(t)$$

$$\rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{9} X(x)T'(t)$$

$$\rightarrow X''(x)T(t) = \frac{1}{9} X(x)T'(t)$$

$$\rightarrow X''(x)T(t) = \frac{X(x)}{9} X'(x)T(t)$$

$$\rightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{9T(t)}$$

$$\text{AT THE L.H.S. IS A FUNCTION OF } x \text{ AND THE R.H.S. IS A FUNCTION OF } t \text{ ONLY}$$

BOTH SIDES ARE AT MOST A CONSTANT, SAY A

IF $A=0$

- $\frac{X''(x)}{X(x)} = 0$
- $X''(x) = 0$
- $X(x) = Ax+B$

$$\therefore u(x,t) = (Ax+B)C$$

$$\boxed{u(x,t) = Ax+B} \quad \text{--- (i)}$$

IF $A > 0$, SAY $A = p^2$

- $\frac{X''(x)}{X(x)} = p^2$
- $X''(x) = p^2 X(x)$
- $X(x) = A \cos(px) + B \sin(px)$

$$\therefore u(x,t) = e^{pt} (A \cos(px) + B \sin(px)) \quad \text{--- (ii)}$$

IF $A < 0$, SAY $A = -p^2$

- $\frac{X''(x)}{X(x)} = -p^2$
- $X''(x) = -p^2 X(x)$
- $X(x) = A \cosh(px) + B \sinh(px)$

$$\therefore u(x,t) = e^{-pt} (A \cosh(px) + B \sinh(px)) \quad \text{--- (iii)}$$

NOW WE LOOK AT THE BOUNDARY CONDITIONS, IN ORDER TO DETERMINE THE NATURE OF THE SOLUTION

(i) IS IGNORED AS IT PRODUCES A HORIZONTAL DEPENDENT SOLUTION (CUE OTHER SIDE)

(ii) IS DISCARDED AS IT PRODUCES EXPONENTIAL SOLUTIONS AS t INCREASES

(iii) IS THE ONLY VISIBLE SOLUTION FOR THIS PROBLEM

NEXT WE OBSERVE THAT AS $t \rightarrow \infty$, THE TIME-DEPENDENT PARTITION OF THE R.O.D. WILL BE A LINEAR FUNCTION IN x : $u(x,t) \rightarrow u(x)$

SINCE THE BOUNDARY AT $x=0$ AND $x=2$ RESPECTIVELY IS NON-CONSTANT AT 0 AND 2 RESPECTIVELY

BY INSPECTION, AS $t \rightarrow \infty$

$$u(x,t) = u(x) = 4x \quad (\text{BY INSPECTION})$$

WE NEED TO BUILD THIS FEATURE INTO THE SOLUTION

$$u(x,t) = U_0 + U_1(t) \quad \text{WHERE } U_0 \text{ IS THE SOLUTION OF } \frac{\partial^2 U}{\partial x^2} = \frac{1}{9} \frac{\partial U}{\partial t}$$

NEXT WE TRANSFORM THE BOUNDARY & INITIAL CONDITIONS

$$u(0,t) = 0 \Rightarrow 0 = 2x_0 + U(0,t) \Rightarrow U(0,t) = 0$$

$$u(2,t) = 8 \Rightarrow 8 = 4x_2 + U(2,t) \Rightarrow U(2,t) = 8$$

$$u(x,0) = 2x^2 \Rightarrow 2x^2 = 4x_2 + U(x,0) \Rightarrow U(x,0) = 2x^2 - 4x_2$$

APPLY $U(x,t) = 0$ INTO THE GENERAL SOLUTION (iii), FROM PREVIOUS

$$\rightarrow U(x,t) = e^{-pt} (A \cosh(px) + B \sinh(px))$$

$$\rightarrow 0 = e^{-pt} A$$

$$\rightarrow A = 0$$

$$\therefore U(x,t) = B e^{-pt} \sinh(px)$$

APPLY $U(2,t) = 8$ INTO THE SOLUTION ABOVE

$$0 = B e^{-pt} \sinh(2p) \quad (\text{B} \neq 0, \text{ OTHERWISE TRIVIAL SOLUTION})$$

$$5e^{-pt} = \frac{1}{\sinh(2p)} \quad \text{AS } t \rightarrow \infty, \text{ SINH}(2p) \rightarrow \infty$$

$$\therefore U_1(x,t) = B_1 e^{-\frac{pt}{2}} \sin \left(\frac{px}{2} \right)$$

$$\boxed{U(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{pt}{2}} \sin \left(\frac{nx}{2} \right)}$$

NOTE THAT $B_0 = 0$, SINCE ZERO, SO WE MAY OMIT FROM THE SOLUTION

APPLY $U(x,0) = 2x^2$ INTO THE 'LATEST' VERSION OF THE SOLUTION

$$2x^2 = 4x_2 = \sum_{n=1}^{\infty} [B_n \sin \left(\frac{nx}{2} \right)]$$

$$B_1 = \frac{1}{\frac{1}{2} \int_0^2 \sin \left(\frac{nx}{2} \right) dx} \int_0^2 2x^2 \sin \left(\frac{nx}{2} \right) dx$$

[THE PHASE = PHASE OF PHASE, WHERE $n=2$ IS 180°]

INTEGRATE BY PARTS

$2x^2 - 4x_2$	$4x_2$
$\frac{d}{dx} (2x^2) = 4x$	$\frac{d}{dx} (4x_2) = 4$

$$\Rightarrow B_1 = \frac{1}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} \int_0^2 4x^2 \sin \left(\frac{nx}{2} \right) dx$$

$$\Rightarrow B_1 = \frac{2}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} \int_0^2 4x^2 \sin \left(\frac{nx}{2} \right) dx$$

BY PARTS AGAIN

$2x^2 - 4x_2$	$4x_2$
$\frac{d}{dx} (2x^2) = 4x$	$\frac{d}{dx} (4x_2) = 4$

$$\Rightarrow B_1 = \frac{8}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} \int_0^2 4x^2 \sin \left(\frac{nx}{2} \right) dx$$

$$\Rightarrow B_1 = -\frac{16}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} \int_0^2 4x^2 \sin \left(\frac{nx}{2} \right) dx$$

$$\Rightarrow B_1 = \frac{32}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} \left[\cos \left(\frac{nx}{2} \right) \right]_0^2$$

$$\Rightarrow B_1 = \frac{32}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} \left[(-1)^n - 1 \right]$$

$$\Rightarrow B_1 = \frac{32}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} \times \begin{cases} -2 & n \text{ IS ODD} \\ 0 & n \text{ IS EVEN} \end{cases}$$

$$\Rightarrow B_1 = \frac{-64}{(\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx)}$$

THIS IS THE SPECIFIC SOLUTION TO THIS PROBLEM (A) IS FORMED

$$U(x,t) = \sum_{n=1}^{\infty} \left[\frac{64}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} - \frac{(-1)^n - 1}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} \right] \sin \left[\frac{(2n-1)\pi x}{2} \right]$$

$$\text{OR } U(x,t) = \sum_{n=1}^{\infty} \left[\frac{64}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} e^{-\frac{(2n-1)\pi t}{2}} \right] \sin \left[\frac{(2n-1)\pi x}{2} \right]$$

AND HENCE OBTAINING $U(x,t)$ AS

$$\boxed{U(x,t) = 4x - \frac{64}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\frac{1}{2} \int_0^2 4x \sin \left(\frac{nx}{2} \right) dx} \right] \sin \left[\frac{(2n-1)\pi x}{2} \right] e^{-\frac{(2n-1)\pi t}{2}}}$$

Question 9

The temperature $u(x,t)$ in a thin rod of length π satisfies the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}, \quad 0 \leq x \leq \pi, \quad t \geq 0,$$

where c is a positive constant.

The initial temperature distribution of the rod is

$$u(x,0) = \frac{1}{2} \cos(4x), \quad 0 \leq x \leq \pi.$$

For $t \geq 0$, heat is allowed to flow freely along the rod, with the rod including its endpoints insulated.

Show that

$$u(x,t) = \frac{1}{2} e^{-16c^2 t} \cos 4x.$$

[You must derive the standard solution of the heat equation in variable separate form]

proof

$\frac{\partial u}{\partial t} = \frac{1}{c^2} \frac{\partial^2 u}{\partial x^2}$ for $u(x,t)$ subject to $\frac{\partial u}{\partial x}(0,t) = 0$
 $\frac{\partial u}{\partial x}(\pi,t) = 0$
 $\frac{\partial u}{\partial x}(0,0) = \frac{1}{2} \cos 4x$.

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM
 $u(x,t) = X(x)T(t)$
- DIFFERENTIATE w.r.t. t , SUBSTITUTE INTO THE P.D.E.
 $\frac{\partial u}{\partial t} = X'(x)T(t) \quad \text{&} \quad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$
 $\Rightarrow X''(x)T(t) = \frac{1}{c^2} X(x)T'(t)$
 $\Rightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{c^2} \frac{X(x)T'(t)}{X(x)T(t)}$
 $\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2} \frac{T'(t)}{T(t)}$
- AS THE LHS IS A FUNCTION OF x ONLY AND THE R.H.S. IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT LEAST A CONSTANT λ .
• THIS CONSTANT MAY BE POSITIVE, NEGATIVE OR ZERO.
- LOOKING AT THE R.H.S., λ HAS TO BE NEGATIVE SO WE OBTAIN A BOUNDARY SOLUTION AS $t \rightarrow \infty$, AND LOOKING AT THE R.H.S. λ HAS TO BE NEGATIVE SO WE GET A PERIODIC SOLUTION IN x , REQUIRED BY CONDITION (i) & (ii).
- HENCE WE SET EACH SIDE EQUAL TO A NEGATIVE CONSTANT, SAY $-P^2$
 $\Rightarrow X''(x) = -P^2$
 $\Rightarrow X(x) = -P^2 x + C_1$
 $\Rightarrow X(x) = A \cos Px + B \sin Px$

$\therefore u(x,t) = e^{-P^2 c^2 t} (A \cos Px + B \sin Px)$
(CHEMICAL INSIGHT: C_1 AND C_2 ARE ZERO)

- DIFFERENTIATE THE GENERAL SOLUTION TO APPLY CONDITIONS (i) & (ii)
- BY (i) $\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow 0 = B \sin 0 \Rightarrow B = 0$
- BY (ii) $\frac{\partial u}{\partial x}(\pi,t) = 0 \Rightarrow -A P e^{-P^2 c^2 t} \sin \pi P = 0$
 $\Rightarrow -A P e^{-P^2 c^2 t} \sin \pi P = 0$
 $\Rightarrow \pi P = \pi n \quad \Rightarrow P = n, \quad n = 0, 1, 2, 3, \dots$
- BY (iii) $u(0,0) = \frac{1}{2} \cos 4x \Rightarrow \frac{1}{2} \cos 0 = \sum_{n=0}^{\infty} A_n \cos nx$
 $\therefore A_4 = \frac{1}{2}$ (THE REST ARE ZERO)

$\therefore u(x,t) = \frac{1}{2} e^{-16c^2 t} \cos 4x$

Question 10

The temperature $\theta(x,t)$ in a thin rod of length L satisfies the heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq L, \quad t \geq 0.$$

The initial temperature distribution is

$$\theta(x,0) = \begin{cases} \frac{\theta_0 x}{L} & 0 \leq x < \frac{L}{2} \\ 0 & \frac{L}{2} < x \leq L \end{cases}$$

where θ_0 is a constant.

The endpoints of the rod are maintained at zero temperature for $t \geq 0$.

- a) Assuming the rod is insulated along its length, find an expression for $\theta(x,t)$.
- b) By considering the initial temperature at $x = \frac{L}{2}$, show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

[You must derive the standard solution of the heat equation in variable separate form]

$$\theta(x,t) = \frac{\theta_0}{\pi^2} \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2} \left[2 \sin\left(\frac{n\pi}{2}\right) - n\pi \cos\left(\frac{n\pi}{2}\right) \right] \exp\left[-\frac{n^2\pi^2 t}{L^2}\right] \sin\left[\frac{n\pi x}{L}\right] \right\}$$

[solution overleaf]

a) $\frac{\partial^2 \Theta}{\partial x^2} = \frac{\partial \Theta}{\partial t}, \quad \theta = 0$

ASSUME A SOLUTION IN UNDAMPED HARMONIC FORM
 $\Theta(x,t) = X(x)T(t) \Rightarrow \frac{\partial \Theta}{\partial x} = X'(x)T(t)$
 $\frac{\partial^2 \Theta}{\partial x^2} = X''(x)T(t)$

SUB INTO THE PDE
 $X''(x)T(t) = X(x)T(t)$

Divide by $X(x)T(t)$
 $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$

AS THE LHS OF THIS EQUATION IS A FUNCTION OF x ONLY AND THE RHS IS A FUNCTION OF t ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT A , WHICH MAY BE POSITIVE, NEGATIVE, OR ZERO.

CONSIDER EACH CASE SEPARATELY

Case 1: $X''(x) = 0 \Rightarrow X(x) = Ax + B$
 $T'(t) = 0 \Rightarrow T(t) = C$

ASSUMING CONSTANTS $\Theta(x,t) = Ax + B$ (STATIONARY STATE)

Case 2: $X''(x) = p^2$
 $\frac{X''(x)}{X(x)} = p^2$
 $X(x) = p^2 X(x)$
 $X(x) = A \cos(px) + B \sin(px)$ (COSINE AND SINE)

ASSUMING CONSTANTS $\Theta(x,t) = e^{pt}(A \cos(px) + B \sin(px))$ (H_{II})

Case 3: $X''(x) = -p^2$
 $\frac{X''(x)}{X(x)} = -p^2$
 $X(x) = C e^{-px}$
 $T(t) = C e^{-pt}$

ASSUMING CONSTANTS $\Theta(x,t) = e^{-pt}(A \cos(px) + B \sin(px))$ (H_{III})

EXAMINING THE 3 POSSIBILITIES AND THE NATURE OF THE PROBLEM, SUGGEST THAT U ONLY (H_{II}) THAT COULD MODEL THE PROBLEM, SINCE AS $t \rightarrow \infty$ $\theta \rightarrow 0$ (IN FACT THIS IS PRECISE SO WE DO NOT HAVE TO WRITE THE "EQUALS" Θ).

WE REQUIRE A SOLUTION IN Θ , WHICH PRODUCES 0 TWICE AT THE ENDPOINTS OF THE ROD.

BOUNDARY CONDITIONS

$\Theta(0,t) = 0$ (1)
 $\Theta(L,t) = 0$ (2)

INITIAL CONDITION

$\Theta(x,0) = \begin{cases} 0 & 0 \leq x < L \\ 1 & L \leq x < L \end{cases}$

APPLY (1)
 $\Theta = e^{-pt} \times 1 \Rightarrow 1 = 0$

$\therefore \Theta(x,t) = Be^{pt}$

APPLY (2)
 $0 = Be^{-pL} \sin(pL) \quad pL = \pi k, \quad k = 1, 2, \dots, L \neq 0$
 $p = \frac{\pi k}{L}$

$\therefore \Theta(x,t) = \sum_{k=1}^{\infty} B_k e^{\frac{\pi k}{L} t} \sin\left(\frac{\pi k x}{L}\right)$

APPLY (3)
 $\Theta(0,0) = \sum_{k=1}^{\infty} B_k \sin(0) = 0$
 $\Theta(L,0) = \sum_{k=1}^{\infty} B_k \sin\left(\frac{\pi k L}{L}\right) = \sum_{k=1}^{\infty} B_k \sin(\pi k) = 1$

$\sum_{k=1}^{\infty} B_k \sin(\pi k) = 1$

$B_1 = \frac{1}{\pi} \int_0^L \sin(\pi x) dx = \frac{1}{\pi} \left[-\frac{1}{\pi} \cos(\pi x) \right]_0^L = \frac{1}{\pi} \left[-\frac{1}{\pi} \cos(\pi L) + \frac{1}{\pi} \cos(0) \right] = \frac{1}{\pi} \left[-\frac{1}{\pi} \cos(\pi) + \frac{1}{\pi} \right] = \frac{1}{\pi} \left[\frac{2}{\pi} \right] = \frac{2}{\pi^2}$

$B_2 = \frac{1}{\pi} \int_0^L \sin(2\pi x) dx = \frac{1}{\pi} \left[-\frac{1}{2\pi} \cos(2\pi x) \right]_0^L = \frac{1}{\pi} \left[-\frac{1}{2\pi} \cos(2\pi L) + \frac{1}{2\pi} \cos(0) \right] = \frac{1}{\pi} \left[-\frac{1}{2\pi} \cos(4\pi) + \frac{1}{2\pi} \right] = \frac{1}{\pi} \left[\frac{1}{2\pi} \right] = \frac{1}{2\pi^2}$

$B_3 = \frac{1}{\pi} \int_0^L \sin(3\pi x) dx = \frac{1}{\pi} \left[-\frac{1}{3\pi} \cos(3\pi x) \right]_0^L = \frac{1}{\pi} \left[-\frac{1}{3\pi} \cos(3\pi L) + \frac{1}{3\pi} \cos(0) \right] = \frac{1}{\pi} \left[-\frac{1}{3\pi} \cos(9\pi) + \frac{1}{3\pi} \right] = \frac{1}{\pi} \left[\frac{4}{3\pi} \right] = \frac{4}{3\pi^2}$

$B_4 = \frac{1}{\pi} \int_0^L \sin(4\pi x) dx = \frac{1}{\pi} \left[-\frac{1}{4\pi} \cos(4\pi x) \right]_0^L = \frac{1}{\pi} \left[-\frac{1}{4\pi} \cos(4\pi L) + \frac{1}{4\pi} \cos(0) \right] = \frac{1}{\pi} \left[-\frac{1}{4\pi} \cos(16\pi) + \frac{1}{4\pi} \right] = \frac{1}{\pi} \left[\frac{1}{4\pi} \right] = \frac{1}{4\pi^2}$

b) NOW CONSIDER $\Theta\left(\frac{L}{2}, t\right)$ IN ABOVE EXPRESSION

$= \sum_{k=1}^{\infty} B_k \left[2 \sin\left(\frac{\pi k L}{2}\right) - \sin\left(\frac{\pi k L}{2}\right) \right] \sin\left(\frac{\pi k x}{L}\right)$

$= \sum_{k=1}^{\infty} B_k \left[\frac{1}{2} \sin\left(\frac{\pi k L}{2}\right) - \sin\left(\frac{\pi k L}{2}\right) \right] \sin\left(\frac{\pi k x}{L}\right)$

$= \sum_{k=1}^{\infty} B_k \left[\frac{1}{2} (1 - (-1)^k) \sin\left(\frac{\pi k L}{2}\right) - \sin\left(\frac{\pi k L}{2}\right) \sin\left(\frac{\pi k x}{L}\right) \right]$

$= \sum_{k=1}^{\infty} B_k \frac{2(-1)^{k+1}}{(2k-1)\pi^2} \sin\left(\frac{\pi k L}{2}\right) - \sum_{k=1}^{\infty} B_k \frac{1}{2} \sin\left(\frac{\pi k L}{2}\right) \sin\left(\frac{\pi k x}{L}\right)$

$= \frac{B_1}{\pi^2} \frac{2(-1)^{1+1}}{(2 \cdot 1 - 1)\pi^2} \sin\left(\frac{\pi L}{2}\right) - \sum_{k=2}^{\infty} \frac{B_k}{(2k-1)\pi^2} \sin\left(\frac{\pi k L}{2}\right) \sin\left(\frac{\pi k x}{L}\right)$

$= \frac{B_1}{\pi^2} \frac{2}{\pi^2} \sin\left(\frac{\pi L}{2}\right) - \sum_{k=2}^{\infty} \frac{B_k}{(2k-1)\pi^2} \sin\left(\frac{\pi k L}{2}\right) \sin\left(\frac{\pi k x}{L}\right)$

NOW AT THE MIDDLEPOINT $\Theta\left(\frac{L}{2}, t\right) = \begin{cases} B_1 & \text{IF } x = \frac{L}{2} \\ 0 & \text{IF } x \neq \frac{L}{2} \end{cases}$

AVGAGE IS $\frac{B_1}{2}$

$\sum_{k=1}^{\infty} \frac{B_k}{(2k-1)\pi^2} \sin\left(\frac{\pi k L}{2}\right) = \frac{B_1}{2}$

$\frac{2B_1}{\pi^2} \frac{1}{2} = \frac{B_1}{2}$

$\sum_{k=2}^{\infty} \frac{1}{(2k-1)\pi^2} = \frac{\pi^2}{8}$

Question 11

The temperature $\theta(x,t)$ in a long thin rod of length L satisfies the heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq L, \quad t \geq 0,$$

where α is a positive constant.

The initial temperature distribution of the rod is

$$\theta(x,0) = \sin\left(\frac{\pi x}{L}\right), \quad 0 \leq x \leq L.$$

For $t \geq 0$, heat is allowed to flow freely with the endpoints of the rod insulated.

Show that

$$\theta(x,t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\exp\left(-\frac{4\alpha^2 n^2 \pi^2 t}{L^2}\right) \cos\left(\frac{2n\pi x}{L}\right)}{4n^2 - 1} \right].$$

[You must derive the standard solution of the heat equation in variable separate form]

proof

solution overleaf

Q1 We discuss a time dependent situation, which is bounded as $t \rightarrow \infty$. Hence
 $\Theta(t) = e^{i\omega t} [A \cos(\omega t) + B \sin(\omega t)]$

Boundary & Initial Conditions:

- $\Theta(0) = \sin \frac{\pi x}{L}$
- $\frac{d\Theta}{dt}(0) = 0$
- $\frac{d\Theta}{dx}(L) = 0$

As the LHS is a function of x only, and the right-hand side is a function of t only, then both sides are at most a constant. Say 2.

If $\lambda > 0$, say $\lambda = p^2$

- $X'(0) = 0 \Rightarrow C = 0$
- $X(0) = Ax + B \Rightarrow B = 0$
- $\therefore \Theta(0) = (Ax + B)e^{i\omega t} = Ax \Rightarrow A \neq 0$

If $\lambda < 0$, say $\lambda = -p^2$

- $X'(0) = -p^2 X(0)$
- $X(0) = Ax + B \sinh(p^2 t)$
- $T(t) = Ce^{-p^2 t}$ as exponential
- $\therefore \Theta(t) = Ce^{-p^2 t} [Ax + B \sinh(p^2 t)]$

If $\lambda = 0$, say $\lambda = p^2$

- $X'(0) = p^2 X(0)$
- $X(0) = Ax + B \cosh(p^2 t)$
- $T(t) = Ce^{p^2 t}$
- $\therefore \Theta(t) = Ce^{p^2 t} [Ax + B \cosh(p^2 t)]$

Q2 Hence

$$A_0 = \frac{1}{L} \int_0^L \sin \frac{\pi x}{L} dx = \frac{\pi}{L} \left[\frac{x}{\pi} \sin \frac{\pi x}{L} \right]_0^L = \frac{\pi}{L} \left[1 - \cos \frac{\pi x}{L} \right]_0^L$$

$$\therefore A_0 = \frac{\pi}{L} \quad (\text{since } A_0 = \frac{A_0}{L})$$

Q3

$$A_1 = \frac{1}{L} \int_0^L \sin \frac{\pi x}{L} \cos \frac{\pi x}{L} dx = \frac{\pi}{L} \int_0^L \sin \frac{\pi x}{L} \cos \frac{\pi x}{L} dx$$

$$\begin{aligned} \sin \frac{\pi x}{L} &= \frac{e^{i\pi x/L} - e^{-i\pi x/L}}{2i} \\ \sin(\pi x/L) \cos(\pi x/L) &= \frac{e^{i\pi x/L} - e^{-i\pi x/L}}{2i} \cdot \frac{e^{i\pi x/L} + e^{-i\pi x/L}}{2i} \\ &= \frac{1}{4} \int_0^L (e^{i\pi x/L} - e^{-i\pi x/L})(e^{i\pi x/L} + e^{-i\pi x/L}) dx \\ &= \frac{1}{L} \int_0^L \sin(2\pi x/L) dx - \frac{1}{L} \int_0^L \sin(0) dx \end{aligned}$$

Note that the integrals are zero from zero, hence no problem to now zero them from the cosine.

By **Q1**

$$\begin{aligned} \sin \frac{\pi x}{L} &= \sum_{k=0}^{\infty} A_k \cos \frac{k\pi x}{L} \\ \sin \frac{\pi x}{L} &= A_1 + \sum_{k=2}^{\infty} A_k \cos \frac{k\pi x}{L} \\ \sin \frac{\pi x}{L} &= \left(\frac{B_0}{2} \right) + \sum_{k=2}^{\infty} A_k \cos \frac{k\pi x}{L} \end{aligned}$$

Hence expansion in x , $0 \leq x \leq L$.

So for $n > 2$

$$\begin{aligned} &= \frac{1}{L} \times \frac{1}{n!} \int_0^L \left[-\cos \left(\frac{k\pi x}{L} \right) \right] dx - \frac{1}{L} \times \frac{1}{(n-1)!} \int_0^L \left[-\cos \left(\frac{(n-1)\pi x}{L} \right) \right] dx \\ &= \frac{1}{(n-1)!} \left[\cos \left(\frac{k\pi x}{L} \right) \right]_0^L - \frac{1}{(n-1)!} \left[\cos \left(\frac{(n-1)\pi x}{L} \right) \right]_0^L \\ &= \frac{1}{(n-1)!} \left[1 - \cos \left(\frac{k\pi n}{L} \right) \right] - \frac{1}{(n-1)!} \left[1 - \cos \left(\frac{(n-1)\pi n}{L} \right) \right] \\ &\quad \text{cos}(\pi(n-1)) = \cos(\pi(n-1)) = \cos(\pi n - \pi) = \cos(\pi n) - \cos(\pi) = -\cos(\pi) \\ &\quad \cos(\pi(n-1)) = \cos(\pi n - \pi) = \cos(\pi n) - \cos(\pi) = -\cos(\pi) \end{aligned}$$

Q4

$$\begin{aligned} A_1 &= \frac{1}{(n-1)!} \left[1 - \left(\cos(n\pi) \right) - \frac{1}{(n-1)!} \left[1 - \left(\cos(n\pi) \right) \right] \right] \\ A_1 &= \frac{1}{(n-1)!} \left[1 + \cos(n\pi) \right] - \frac{1}{(n-1)!} \left[1 + \cos(n\pi) \right] \\ A_1 &= 1 + \cos(n\pi) \left[\frac{1}{(n-1)!} - \frac{1}{(n-1)!} \right] \\ A_1 &= 1 + \cos(n\pi) \left[\frac{(n-1)-(n-1)}{(n-1)(n-2)} \right] \\ A_1 &= \frac{1 + \cos(n\pi) \times -\frac{2}{n-1}}{n-1} \\ A_1 &= \frac{1 + (-1)^n}{n-1} \times \frac{-2}{n-1} \\ A_1 &= -\frac{2}{n(n-1)} (1 + (-1)^n) \quad \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{4}{(n-1)^2} & \text{if } n \text{ is even} \end{cases} \\ \therefore \text{With it all } A_m &= -\frac{4}{m(m-1)} \quad m > 1 \end{aligned}$$

Q5

$$\Theta(t) = \frac{2}{L} + \sum_{m=1}^{\infty} \left[\frac{-4}{m(m-1)} \omega_m \left(\frac{2\pi x}{L} \right) e^{-\frac{4\pi^2 m^2 t}{L^2}} \right]$$

$$\Theta(t) = \frac{2}{L} - \frac{4}{L} \sum_{m=1}^{\infty} \left[\frac{\omega_m}{4m^2-1} \cos \left(\frac{2\pi mx}{L} \right) \right]$$

Question 12

A long thin rod AB , of length L , has constant temperature $\theta = 0$ throughout its length. Another long thin rod CD , also of length L , has constant temperature $\theta = 100$ throughout its length.

At time $t = 0$ the temperature the ends B and C are brought into full contact, while the ends A and D are maintained at respective temperatures $\theta = 0$ and $\theta = 100$.

Show that, for $t \geq 0$, the temperature $\theta(t)$ of the point where the two rods are joined satisfies

$$50 - \frac{100}{\pi} \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \exp\left(-\frac{\alpha^2 \pi^2 (2n+1)^2 t}{4L^2}\right) \right].$$

You may assume

- $ABCD$ is a straight line
- The rods are insulated along their lengths
- The temperature distribution along the rod $\theta(x, t)$ satisfies the standard heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq L, \quad t \geq 0,$$

where α is a positive constant.

[You must derive the standard solution of the heat equation in variable separate form]

proof

[solution overleaf]

QUESTION

Assume a solution is uniquely determined if $\frac{\partial^2 \theta}{\partial t^2} = X''(t)T(t)$ implies $\frac{\partial \theta}{\partial t} = X(t)T'(t)$.

SUB INTO THE PDE:

$$X''(t)T(t) + \frac{1}{L^2} \frac{\partial^2 \theta}{\partial t^2} = X''(t)T(t)$$

$$\Rightarrow \frac{\partial^2 \theta}{\partial t^2} = X(t)T'(t)$$

NOTE: THE RHS IS A FUNCTION OF t ONLY AND THE LHS IS A FUNCTION OF x ONLY. SO BOTH SIDES HAVE AT MOST 4 CONDITIONS, SAY A , WHICH COULD BE NEGATIVE, POSITIVE OR ZERO.

ANSWER

$$X''(t) = 0 \quad \text{and} \quad T'(t) = 0$$

$$X(t) = Ax + B$$

$$T(t) = C$$

ABSORBING CONSTANTS

(i) $\theta(x,t) = Ax + B$ ← STATIONARY STATE

ANSWER

$$X''(t) = p^2 \quad \text{and} \quad \frac{1}{L^2} \frac{\partial^2 \theta}{\partial t^2} = p^2$$

$$X(t) = p^2 X(t)$$

$$T'(t) = -p^2 T(t)$$

$$X(t) = Acosp + Bsinp$$

$$T(t) = Ce^{-p^2 t}$$

ABSORBING C INTO A & B

(ii) $\theta(x,t) = e^{-p^2 t} (Acosp + Bsinp)$

ANSWER

CONSIDER THE STATIONARY STATE (i) SOLUTION AS IT HAS NO TIME-DEPENDENCE. IGNORE (ii) AS IT PRODUCES UNBOUNDED SOLUTIONS AT $t \rightarrow \infty$.

SKETCH

ANSWER

PLACE BOUNDARY CONDITIONS AT $x=0$ AND $x=L$ AT $t=0$ AND $x=L$ AT $t=T$. AND $x=0$ AT $t=0$.

ASSUME SOLUTION TO

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{L^2} \frac{\partial^2 \theta}{\partial t^2} \quad \text{At} \quad \theta(0,t) = e^{-p^2 t} [Acos p t + Bsin p t]$$

SOLUTIONS

$$\theta(0,t) = 0 \quad t > 0 \quad \text{--- (1)}$$

$$\theta(2L,t) = 100 \quad t > 0 \quad \text{--- (2)}$$

$$\theta(0,t) = \sum_{n=0}^{\infty} 0.5 \sin(n\pi x/L) \quad \text{--- (3)}$$

WE NEED TO SOLVE A CONDITION IN THE SOLUTION AS $t \rightarrow \infty$

AS $t \rightarrow \infty$ AT $x=0$ $\theta=0$ AND AT $x=2L$ $\theta=100$ \Rightarrow θ INVERSELY UNIFORM RELATIONSHIP

ANSWER

$$\theta(2L,t) = \frac{300}{L} + \tilde{\theta}(2L,t) \Leftrightarrow \tilde{\theta}(2L,t) = \frac{300}{L} - \theta(2L,t)$$

WITH $\frac{\partial^2 \tilde{\theta}}{\partial t^2} = \frac{1}{L^2} \frac{\partial^2 \tilde{\theta}}{\partial x^2} \quad \text{and} \quad \tilde{\theta}(2L,t) \rightarrow 0 \quad (\text{As } t \rightarrow \infty)$

CORRECT CONDITIONS

$$\tilde{\theta}(0,t) = \theta(0,t) - \frac{300}{L} \quad \Rightarrow \quad \tilde{\theta}(0,t) = 0 \quad \text{--- (1')}$$

$$\tilde{\theta}(2L,t) = \theta(2L,t) - \frac{300}{L} = 100 - 100 \Rightarrow \tilde{\theta}(2L,t) = 0 \quad \text{--- (2')}$$

$$\tilde{\theta}(2L,t) = \theta(2L,t) - \frac{300}{L} \Rightarrow \tilde{\theta}(2L,t) = \begin{cases} -\frac{300}{L} \cos(n\pi x/L) & 0 \leq x \leq L \\ 0 & L < x \leq 2L \end{cases} \quad \text{--- (3')}$$

TRY CONDITIONS TO $\tilde{\theta}(0,t) = e^{-p^2 t} (Acosp + Bsinp)$

(i) $\tilde{\theta}(0,t) = 0$

$$0 = 4e^{-p^2 t} \Rightarrow A = 0$$

$\therefore \tilde{\theta}(0,t) = Be^{-p^2 t} \sin p t$

(ii) $\tilde{\theta}(2L,t) = 0$

$$0 = Be^{-p^2 t} \sin 2pL = 2pL = n\pi L, n=0,1,2,3,\dots$$

$\therefore p = \frac{n\pi}{L}, n \neq 0$ (PHYSICAL REASONING)

$$\therefore \tilde{\theta}(0,t) = \sum_{n=1}^{\infty} \left[B_n e^{-\frac{n^2 \pi^2 t}{L^2}} \sin \left(\frac{n\pi x}{L} \right) \right] \quad (\text{cos, remains 0})$$

(iii) $\int_0^{2L} \left[\frac{300}{L} - \tilde{\theta}(0,t) \right]^2 dx = 0$

$$\int_0^{2L} \left[\frac{300}{L} - \sum_{n=1}^{\infty} B_n e^{-\frac{n^2 \pi^2 t}{L^2}} \sin \left(\frac{n\pi x}{L} \right) \right]^2 dx = 0$$

THIS IS A FOURIER SERIES WITH PERIOD $2L$

$$\therefore \tilde{\theta}(0,t) = \frac{300}{L} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2L}$$

$\Rightarrow B_1 = \frac{1}{L} \int_0^{2L} \left[\frac{300}{L} - \sum_{n=1}^{\infty} B_n \sin \left(\frac{n\pi x}{2L} \right) \right] dx + \frac{1}{L} \int_L^{2L} \left[\frac{300}{L} - \sum_{n=1}^{\infty} B_n \sin \left(\frac{n\pi x}{2L} \right) \right] dx$

$$\Rightarrow B_1 = \frac{1}{L} \int_0^{2L} \left[-500 \sin \left(\frac{\pi x}{2L} \right) \right] dx + \frac{1}{L} \int_L^{2L} \left[500 \sin \left(\frac{\pi x}{2L} \right) \right] dx$$

$$\Rightarrow B_1 = -\frac{500}{L} \int_0^{2L} \sin \left(\frac{\pi x}{2L} \right) dx + \frac{500}{L} \int_L^{2L} \sin \left(\frac{\pi x}{2L} \right) dx$$

BY PARTS

x	1
$-\frac{500}{L} \cos \left(\frac{\pi x}{2L} \right)$	$\sin \left(\frac{\pi x}{2L} \right)$

$$\Rightarrow B_1 = -\frac{500}{L} \left[\frac{1}{\pi} \left(\cos \left(\frac{\pi x}{2L} \right) \right) \right]_0^{2L} + \frac{500}{L} \left[\frac{1}{\pi} \left(\cos \left(\frac{\pi x}{2L} \right) \right) \right]_L^{2L} = -\frac{1000}{\pi L} \left[\cos \left(\frac{\pi x}{2L} \right) \right]_0^{2L}$$

$$\Rightarrow B_1 = -\frac{1000}{\pi L} \left[\cos(0) - \cos(\pi) \right] = \frac{2000}{\pi L} \cos \left(\frac{\pi x}{2L} \right)$$

$$\Rightarrow B_1 = \frac{200}{\pi} \cos \left(\frac{\pi x}{2L} \right) + \frac{100}{\pi} \int_0^{2L} \cos^2 \left(\frac{\pi x}{2L} \right) dx$$

$$\Rightarrow B_1 = \frac{200}{\pi} \cos \left(\frac{\pi x}{2L} \right) + \frac{100}{\pi} \left[\frac{1}{2} x + \frac{1}{2} \sin \left(\frac{\pi x}{2L} \right) \right]_0^{2L}$$

AT FIRST IT APPEARS LOGGED SINCE IT SEE'S IT MUST NOT BE FG

$$\therefore \tilde{\theta}(0,t) = \frac{300}{L} + \sum_{n=1}^{\infty} \left[\frac{100}{\pi} \left(\cos \left(\frac{n\pi x}{2L} \right) + \frac{200}{\pi} \cos^2 \left(\frac{n\pi x}{2L} \right) \right) \right]$$

NOW AT $x=L$

$$\theta(4,t) = \frac{300}{L} + \sum_{n=1}^{\infty} \frac{100}{\pi} \int_0^L \left[\cos \left(\frac{n\pi x}{2L} \right) + \frac{200}{\pi} \cos^2 \left(\frac{n\pi x}{2L} \right) \right] dx$$

$$\theta(4,t) = 300 + \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^L \left[\cos \left(\frac{n\pi x}{2L} \right) + \frac{200}{\pi} \cos^2 \left(\frac{n\pi x}{2L} \right) \right] dx$$

LOOK FOR A PATTERN IN $\cos(n\pi x/2L)$

n	1	2	3	4	5	6	7	8
$\cos(n\pi x/2L)$	-1	0	1	0	-1	0	1	0

SO SIN TERMS ARE ZERO, $n = 2m+1$ & ADD THE TERM (i)

$\therefore \tilde{\theta}(4,t) = 300 + \frac{100}{\pi} \sum_{i=1}^{\infty} \left[\frac{(-1)^{i+1}}{2i+1} \exp \left[-\frac{(2i+1)^2 \pi^2 t}{4L^2} \right] \right]$

ABOVE THE SUMMATION SO IT STOPS AT $i=4$

$$\theta(4,t) = 300 + \frac{100}{\pi} \sum_{i=1}^4 \left[\frac{(-1)^{i+1}}{2i+1} \exp \left[-\frac{(2i+1)^2 \pi^2 t}{4L^2} \right] \right]$$

$$\theta(4,t) = 300 - \frac{100}{\pi} \sum_{i=1}^4 \left[\frac{(-1)^{i+1}}{2i+1} \exp \left[-\frac{(2i+1)^2 \pi^2 t}{4L^2} \right] \right]$$

OR

$$\theta(4,t) = 300 - \frac{100}{\pi} \sum_{i=0}^3 \left[\frac{(-1)^{i+1}}{2i+1} \exp \left[-\frac{(2i+1)^2 \pi^2 t}{4L^2} \right] \right]$$

Question 13

The one dimensional heat equation is given by

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq L, \quad t \geq 0,$$

where α is a positive constant, known as the thermal diffusivity.

- a) Obtain a general solution of the above equation by trying a solution in variable separable form.

A long thin rod AB , of length $2L$, has its endpoints, at $x=0$ and $x=2L$, maintained at constant temperature $\theta=0$ and its midpoint is maintained at temperature $\theta=100$, until a steady temperature distribution $\Theta(x)$ is reached throughout its length.

- b) Show that

$$\Theta(x) = \begin{cases} \frac{100}{L}x & 0 \leq x < L \\ \frac{100}{L}(2L-x) & L < x \leq 2L \end{cases}$$

- c) Prove that

$$\int_L^{2L} \Theta(x) \sin\left(\frac{n\pi x}{2L}\right) dx = (-1)^{n+1} \int_0^L \Theta(x) \sin\left(\frac{n\pi x}{2L}\right) dx$$

At $t=0$, the heat source which was maintaining the midpoint of the rod at $\theta=100$ is removed, but its endpoints are still maintained at $\theta=0$. The rod is insulated throughout its length and allowed to cool.

- d) Show that for $t \geq 0$, the temperature $\theta(t)$ of the midpoint of the rod satisfies

$$\frac{800}{\pi^2} \sum_{n=0}^{\infty} \left[\frac{1}{(2n+1)^2} \exp\left(-\frac{\alpha^2 \pi^2 (2n+1)^2 t}{4L^2}\right) \right].$$

proof

[solution overleaf]

Q $\frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$

Assume a solution is of the form $X(t)T(t)$

$$\Theta(t) = X(t)T(t)$$

$$\frac{d\Theta}{dt}(t) = X'(t)T(t) + X(t)T'(t)$$

See into the P.D.E.

$$\Rightarrow X'(t)T(t) = \frac{1}{x^2} X(t)T'(t)$$

$$\Rightarrow X(t)T'(t) = \frac{1}{x^2} X(t)T(t)$$

$$\Rightarrow \frac{X'(t)}{X(t)} = \frac{1}{x^2} \frac{T'(t)}{T(t)}$$

Now the LHS of this equation is a function of t only and the RHS is a function of x only. Hence both sides are at most a constant A , which may be negative, positive or zero.

$$A=0 \quad X(t)=0 \Rightarrow X(t)=A_1 t + A_2$$

$$T(t)=0 \Rightarrow T(t)=C$$

$$\therefore \Theta(t) = X(t)T(t) = Cx + Q$$

It is clearly zero at the boundaries

If $2 < 0, A = -P^2$

$$\frac{X''(t)}{X(t)} = -P^2 \Rightarrow X'(t) = -P^2 X(t)$$

$$\Rightarrow X(t) = A_1 \cos Pt + B_1 \sin Pt$$

$$\frac{1}{x^2} \frac{T'(t)}{T(t)} = -P^2 \Rightarrow T'(t) = -P^2 T(t)$$

$$\Rightarrow T(t) = C_2 e^{-P^2 t}$$

$$\therefore \Theta(t) = X(t)T(t) = \frac{C_2}{x^2} (P \cos Pt + Q \sin Pt)$$

If $A > 0, A = P^2$

$$\frac{X''(t)}{X(t)} = P^2 \Rightarrow X'(t) = P^2 X(t)$$

$$\Rightarrow X(t) = A_1 e^{Pt} + B_1 e^{-Pt}$$

$$\frac{1}{x^2} \frac{T'(t)}{T(t)} = P^2 \Rightarrow T'(t) = A_2 e^{Pt}$$

$$\Rightarrow T(t) = C_2 e^{P^2 t}$$

$$\therefore \Theta(t) = X(t)T(t) = e^{\frac{P^2 t}{x^2}} (P \cos Pt + Q \sin Pt)$$

This is not an actual physical solution, as it predicts unbounded solutions as $t \rightarrow \infty$.

b) We expect a linear relationship to $t \rightarrow \infty$

1st section
gradient = $\frac{100}{L}$
 $\therefore \theta = \frac{100}{L} x$

"2nd section"

$$\text{gradient} = \frac{0-100}{2L-L} = -\frac{100}{L}$$

Thus $\theta - \theta_0 = -\frac{100}{L} (x-L)$

$$\theta - 0 = -\frac{100}{L} (x-2L)$$

$$\therefore \theta = \frac{100}{L} (2L-x)$$

$\therefore \Theta(x) = \begin{cases} \frac{100}{L} x & 0 \leq x \leq L \\ \frac{100}{L} (2L-x) & L \leq x \leq 2L \end{cases}$

$\int_0^L \theta(y) \sin \frac{n\pi y}{2L} dy \dots$ by substitution

$$= \int_0^L \theta(y) \sin \left[n\pi \left(\frac{y}{2L} - \frac{100}{L} y \right) \right] dy$$

$$= \int_0^L \theta(y) \left[\sin \frac{n\pi y}{2L} - (-1)^n \cos \frac{n\pi y}{2L} \right] dy$$

$$= \int_0^L \theta(y) \left[\sin \frac{n\pi y}{2L} - (-1)^n \cos \frac{n\pi y}{2L} \right] dy$$

$$= \Theta(y) \int_0^L \left[\sin \frac{n\pi y}{2L} - (-1)^n \cos \frac{n\pi y}{2L} \right] dy$$

$$= (-1)^n \int_0^L \Theta(y) \sin \frac{n\pi y}{2L} dy$$

Also $\Theta(y) = \begin{cases} \frac{100}{L} (2L-y) & 0 \leq y \leq 2L \\ 0 & 2L < y \leq L \end{cases}$

$$\theta(y) = \begin{cases} \frac{100}{L} y & 0 \leq y \leq L \\ 0 & L < y \leq 2L \end{cases}$$

d) Now apply conditions

$$\Theta(0) = 0 \quad \text{(I)}$$

$$\Theta(2L) = 0 \quad \text{(II)}$$

$$\Theta(x_0) = \begin{cases} \frac{100}{L} x_0 & 0 \leq x_0 \leq L \\ \frac{100}{L} (2L-x_0) & L \leq x_0 \leq 2L \end{cases} \quad \text{(III)}$$

Start with

$$\Theta(x_0) = e^{\frac{-P^2 t}{x^2}} (P \cos Pt + Q \sin Pt)$$

By (I) $0 = e^{\frac{-P^2 t}{x^2}} P \Rightarrow P = 0$

$\Theta(t) = Q e^{\frac{-P^2 t}{x^2}}$

By (II) $0 = Q e^{\frac{-P^2 t}{x^2}} \sin(2PL) \Rightarrow 2PL = n\pi t$

$$P = \frac{n\pi}{2L} \quad n = 1, 2, \dots$$

($n=0$ does not fit in this equation)

$$\Theta(x_0) = \sum_{n=1}^{\infty} Q_n e^{\frac{-n^2 \pi^2 t}{4L^2}} \sin \frac{n\pi x_0}{2L}$$

By (III)

$$\Theta(x_0) = \sum_{n=1}^{\infty} Q_n \sin \frac{n\pi x_0}{2L}$$

which is a Fourier in series with three terms

$$\Rightarrow Q_1 = \int_0^{2L} \Theta(x) \sin \frac{n\pi x}{2L} dx$$

$$\Rightarrow Q_1 = \int_0^L \Theta(x) \sin \frac{n\pi x}{2L} dx + \int_L^{2L} \Theta(x) \sin \frac{n\pi x}{2L} dx$$

using part c

$$\Rightarrow Q_1 = \int_0^L \Theta(x) \sin \frac{n\pi x}{2L} dx + \int_0^{L-n\pi} \Theta(x) \sin \frac{n\pi x}{2L} dx$$

$$\Rightarrow Q_1 = \int_0^L \left[1 + (-1)^n \right] \int_0^L \Theta(x) \sin \frac{n\pi x}{2L} dx$$

$$\Rightarrow Q_1 = \frac{100}{L} \left[1 + (-1)^n \right] \int_0^L \sin \frac{n\pi x}{2L} dx$$

BY PARTS

$$\int_0^L \frac{x}{2L} \cos \frac{n\pi x}{2L} dx = \frac{1}{2L} \left[x \cos \frac{n\pi x}{2L} \right]_0^L - \frac{1}{2L} \int_0^L \cos \frac{n\pi x}{2L} dx$$

$$\Rightarrow Q_1 = \frac{100}{L} \left[1 + (-1)^n \right] \left\{ \frac{2L}{n\pi} \left[x \cos \frac{n\pi x}{2L} \right]_0^L - \frac{2L}{n\pi^2} \int_0^L \cos \frac{n\pi x}{2L} dx \right\}$$

$$\Rightarrow Q_1 = \frac{100}{L} \left[1 + (-1)^n \right] \left\{ \frac{2L}{n\pi} \left[x \cos \frac{n\pi x}{2L} \right]_0^L + \frac{2L}{n\pi^2} \left[\sin \frac{n\pi x}{2L} \right]_0^L \right\}$$

$$\Rightarrow Q_1 = \frac{100}{L} \left[1 + (-1)^n \right] \left\{ \frac{2L}{n\pi} \left[x \cos \frac{n\pi x}{2L} \right]_0^L + \frac{4L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right\}$$

$$\Rightarrow Q_1 = \frac{200}{L^2\pi^2} \left[1 + (-1)^n \right] \left[2 \sin \frac{n\pi}{2} - n\pi \cos \frac{n\pi}{2} \right]$$

Now looking at $A = 1 + (-1)^n$

IF n is even $\Rightarrow 0$
IF n is odd $\Rightarrow 2$

$$\Theta(x) = \sum_{n=1}^{\infty} \frac{200}{L^2\pi^2 n^2} \left[2 \sin \frac{(2m+1)\pi}{2} - (2m+1)\pi \cos \frac{(2m+1)\pi}{2} \right] e^{-\frac{n^2 \pi^2 t}{4L^2}} \sin \frac{n\pi x}{2L}$$

Now $\sin \frac{(2m+1)\pi}{2} = \frac{1}{2} \sin(2m+1)\pi = 0$

$$\Theta(x) = \sum_{n=1}^{\infty} \frac{400}{\pi^2 n^2} \left[2 \sin \frac{(2m+1)\pi}{2} \right] e^{-\frac{n^2 \pi^2 t}{4L^2}}$$

$$(E=1)$$

$$\Theta(x) = \sum_{n=1}^{\infty} \frac{800}{\pi^2 n^2} e^{-\frac{n^2 \pi^2 t}{4L^2}}$$

HEAT EQUATION

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t}, \quad \theta = \theta(x, y, t)$$

Two Dimensional

Question 1

The temperature distribution, $\theta(x, y, t)$, on a square plate satisfies the equation

$$\nabla^2 \theta = \frac{1}{\alpha^2} \frac{\partial^2 \theta}{\partial t^2}, \quad 0 \leq x \leq L, \quad 0 \leq y \leq L, \quad t \geq 0.$$

Find a general solution for $\theta(x, y, t)$, which is periodic in x and in y .

Define any constants used.

$$\boxed{\theta(x, y, t) = e^{-p^2 \alpha^2 t} [A \cos qx + B \sin qx] [C \cos ky + D \sin ky], \quad p^2 = q^2 + k^2}$$

$\nabla^2 \theta = \frac{1}{\alpha^2} \frac{\partial^2 \theta}{\partial t^2}$ RR $\theta = \theta(x, y, t)$ $0 \leq x \leq L$
 $0 \leq y \leq L$
 $t \geq 0$

- Assume a solution in separable form
 $\theta(x, y, t) = X(x)Y(y)T(t)$
- Differentiate and substitute into the P.D.E
 $\frac{\partial^2 \theta}{\partial x^2} = X''(x)Y(y)T(t), \quad \frac{\partial^2 \theta}{\partial y^2} = X(x)Y''(y)T(t), \quad \frac{\partial^2 \theta}{\partial t^2} = X(x)Y(y)T''(t)$
 $\rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{1}{\alpha^2} \frac{T''(t)}{T(t)}$
 $\rightarrow X''(x)Y(y)T(t) + X(x)Y''(y)T(t) = \frac{1}{\alpha^2} X(x)Y(y)T''(t)$
 $\rightarrow \frac{X''(x)Y(y)T(t)}{X(x)Y(y)T(t)} + \frac{X(x)Y''(y)T(t)}{X(x)Y(y)T(t)} = \frac{1}{\alpha^2} \frac{X(x)Y(y)T''(t)}{X(x)Y(y)T(t)}$
 $\rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{1}{\alpha^2} \frac{T''(t)}{T(t)}$
- Both sides are at most a constant λ , as the L.H.S. is a function of x & y only and the R.H.S. is a function of t only
- Furthermore looking at the R.H.S. this constant must be negative, otherwise we will produce solutions which are unbounded as $t \rightarrow \infty$. Let $\lambda = -p^2$
- $\frac{X''(x)}{X(x)} = -p^2$
 $T''(t) = -p^2 \alpha^2 T(t)$
 $T(t) = A e^{-p^2 \alpha^2 t}$

Returning to the LHS

$$\begin{aligned} X''(x) + \frac{Y''(y)}{Y(y)} &= -p^2 \\ \frac{X''(x)}{X(x)} &= -p^2 - \frac{Y''(y)}{Y(y)} \end{aligned}$$

- Both sides are at most a constant λ , as the L.H.S. is a function of x only and the R.H.S. is a function of y only
- Again this constant must be negative, as we are looking for periodic solutions in x & y . As indicated $\rightarrow \frac{Y''(y)}{Y(y)} = -q^2 \Rightarrow$ At the boundaries
- Let $p = -q^2$
- $\frac{X''(x)}{X(x)} = -q^2$
 $X''(x) = -q^2 X(x)$
 $\frac{Y''(y)}{Y(y)} = -q^2$
 $Y''(y) = q^2 Y(y)$
- As per this constraint must be negative
- Let $q^2 = p^2 - k^2$
 $p^2 = q^2 + k^2$
- $\frac{Y''(y)}{Y(y)} = -k^2$
 $Y''(y) = -k^2 Y(y)$
 $Y(y) = \text{Cosky} + \text{Sinky}$
- Hence the general solution will be

$$\theta(x, y, t) = X(x)Y(y)T(t) = e^{-p^2 \alpha^2 t} (\text{Cosky}(\cos qx) (\text{Cosky} + \text{Sinky}))$$

where $(p^2 = q^2 + k^2)$

HEAT EQUATION

Miscellaneous Questions

Question 1

The smooth function $u = u(x, t)$ satisfies the diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t},$$

where α is a positive constant.

Show by differentiation that $u(x, t) = A \operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)$, where A is a non zero constant, satisfies the diffusion equation.

You may assume that

- $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2} d\xi$.

- $\frac{d}{dw} \left[\int_0^w f(z) dz \right] = f(w)$.

proof

$\frac{\partial \theta}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t}$

$\theta(x, t) = A \operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)$

• STARTING BY THE DEFINITION OF THE ERROR FUNCTION

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2} d\xi$$

• NEXT WE WRITE THAT FROM THE FUNDAMENTAL THEOREM OF CALCULUS

$$\frac{d}{dw} \left[\int_0^w f(z) dz \right] = f(w)$$

• REWRITE THE ERROR FUNCTION AS $\theta(x, t)$

$$\theta(x, t) = A \operatorname{erf}\left[\frac{2x}{2\alpha\sqrt{t}}\right] = \frac{2A}{\sqrt{\pi t}} \int_0^{\frac{2x}{2\alpha\sqrt{t}}} e^{-z^2} dz$$

• DIFFERENTIATING W.R.T x

$$\begin{aligned} \frac{\partial \theta}{\partial x} &= A \times \frac{2}{2\alpha\sqrt{t}} \times \frac{2}{\sqrt{\pi t}} \times -\left(\frac{2x}{2\alpha\sqrt{t}}\right)^2 \\ \frac{\partial \theta}{\partial x^2} &= \frac{A}{\alpha^2\sqrt{\pi t}} e^{-\frac{2x^2}{4\alpha^2 t}} \\ \frac{\partial \theta}{\partial x^2} &= -\frac{A}{\alpha^2\sqrt{\pi t}} e^{-\frac{2x^2}{4\alpha^2 t}} \times -\frac{2x}{4\alpha^2 t} \\ \frac{\partial \theta}{\partial x^2} &= \frac{Ax}{2\alpha^3\sqrt{\pi t}} e^{-\frac{2x^2}{4\alpha^2 t}} \end{aligned}$$

• DIFFERENTIATING W.R.T t

$$\begin{aligned} \frac{\partial \theta}{\partial t} &= A \times -\frac{1}{2} \times \frac{2}{2\alpha\sqrt{t}} \times \frac{2}{\sqrt{\pi t}} \times \left(\frac{2x}{2\alpha\sqrt{t}}\right)^2 \\ \frac{\partial \theta}{\partial t} &= -\frac{Ax}{2\alpha^3\sqrt{\pi t}} e^{-\frac{2x^2}{4\alpha^2 t}} \end{aligned}$$

• COLLECTING ALL THE RESULTS

$$\frac{\partial \theta}{\partial x^2} = -\frac{Ax}{\alpha^2\sqrt{\pi t}} e^{-\frac{2x^2}{4\alpha^2 t}}$$

$$\frac{1}{\alpha^2} \frac{\partial \theta}{\partial t} = \frac{1}{\alpha^2} \left[-\frac{Ax}{\alpha^2\sqrt{\pi t}} e^{-\frac{2x^2}{4\alpha^2 t}} \right] = -\frac{Ax}{2\alpha^3\sqrt{\pi t}} e^{-\frac{2x^2}{4\alpha^2 t}}$$

∴ THE P.D.E IS SATISFIED

Question 2

The function $u = u(x, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2u = 0,$$

subject to the conditions

$$u(0, t) = 0, \quad u(1, t) = 0 \quad \text{and} \quad u(x, 0) = 1.$$

Use the substitution $u(x, t) = e^{kt} w(x, t)$, with a suitable value for the constant k , to find a simplified expression for $u(x, t)$.

$$u(x, t) = \frac{2e^{2t}}{\pi} \sum_{n=1}^{\infty} \left[\frac{\exp[-(2n-1)^2 \pi^2 t] \sin[(2n-1)\pi x]}{2n-1} \right]$$

$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2u = 0 \quad 0 < x < 1 \quad t > 0$
WHERE $u(x, t)$ IS SUBJECT TO THE CONDITIONS
 $u(x, 0) = 0 \rightarrow 1$
 $u(1, t) = 0 \rightarrow 2$
 $u(0, t) = 1 \rightarrow 3$

• USE THE SUBSTITUTION (CHANGE OF VARIABLES) AS FOLLOWS
 $u(x, t) = e^{kt} w(x, t)$
 $\frac{\partial u}{\partial t} = k e^{kt} w + e^{kt} \frac{\partial w}{\partial t}$
 $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2}$

• SUBSTITUTING THE P.D.E.
 $k e^{kt} w + e^{kt} \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} - 2e^{kt} w = 0$
AND IF $k \neq 0$, THE P.D.E. BECOMES TO
 $\frac{\partial w}{\partial t} = -k^2 w$

• THE CONDITIONS ALSO TURN INTO
1 → $w(x, 0) = 0 \rightarrow e^{k \cdot 0} w(x, 0) = 0 \rightarrow w(x, 0) = 0 \rightarrow 1$
2 → $w(1, t) = 0 \rightarrow e^{kt} w(1, t) = 0 \rightarrow w(1, t) = 0 \rightarrow 2$
3 → $w(0, t) = 1 \rightarrow e^{kt} w(0, t) = 1 \rightarrow w(0, t) = 1 \rightarrow 3$
i.e. ESSENTIALLY UNCHANGED

• NOW ASSUME A SOLUTION IN SIMPLIFIED SEPARABLE FORM
 $w(x, t) = X(x)T(t)$

• DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.
 $\frac{\partial w}{\partial t} = X(x)T'(t)$ $\frac{\partial^2 w}{\partial x^2} = X''(x)T(t)$

$$\begin{aligned} \rightarrow X'(x)T(t) &= X(x)T'(t) \\ \rightarrow X(x)T'(t) &= X''(x)T(t) \\ \rightarrow \frac{X'(x)}{X(x)} &= \frac{T'(t)}{T(t)} \end{aligned}$$

• AS THE LHS IS A FUNCTION OF x ONLY AND THE RHS IS A FUNCTION OF t ONLY, BOTH MUST BE AN AT MOST A CONSTANT, SAY A , WHICH COULD BE POSITIVE, NEGATIVE OR ZERO

• AS WE REQUIRE A NON-ZERO SOLUTION IN x^A FROM THE CONDITIONS
① & ② $\rightarrow A$ HAS TO BE NEGATIVE

LET $A = -k^2$

$$\begin{aligned} \frac{X'(x)}{X(x)} &= -k^2 \\ X(x) &= -k^2 x \\ X(x) &= -k^2 x + C_1 \\ X(x) &= A \cos(kx) + B \sin(kx) \end{aligned}$$

$T'(t) = -k^2 T(t)$
 $T(t) = C_2 e^{-k^2 t}$

$\therefore w(x, t) = e^{-k^2 t} (A \cos(kx) + B \sin(kx))$
(AFTER C_1 AND C_2)

• APPLY CONDITION ①
 $w(0, t) = 0 \Rightarrow A \cdot e^{-k^2 t} = 0 \Rightarrow A = 0$
 $w(x, t) = B_1 x \sin(kx)$

• APPLY CONDITION ②
 $w(1, t) = 0 \Rightarrow B_1 \cdot e^{-k^2 t} \sin(k) = 0 \Rightarrow k = n\pi, n \in \mathbb{Z}, \dots$ (B ≠ 0)
 $w(x, t) = B_2 e^{-k^2 t} \sin(n\pi x)$

• APPLY CONDITION ③
 $w(x, 0) = 1 \Rightarrow \sum_{n=1}^{\infty} [B_n e^{-k^2 t} \sin(n\pi x)] \Big|_{t=0} = 1$
THIS IS A FOURIER SERIES WITH FREQUENCY $n\pi$.
PICK $t = 0$ AS THE TEST EDITION
 $B_1 = \int_0^1 1 \times \sin(n\pi x) dx$
 $B_2 = \int_0^1 \frac{1}{n\pi} \sin(n\pi x) dx$
 $B_n = \frac{1}{n\pi} \int_0^1 [\sin(n\pi x)]^2 dx$

• $B_n = \frac{1}{n\pi} (1 - (-1)^n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$

 $\therefore w(x, t) = \sum_{n=1}^{\infty} \left[\frac{2}{n\pi} e^{-\frac{n^2 \pi^2 t}{2}} \sin((n\pi)x) \right]$
 $\therefore u(x, t) = e^{\frac{k^2 t}{2}} w(x, t)$
 $\therefore u(x, t) = \frac{2e^{\frac{k^2 t}{2}}}{\pi} \sum_{n=1}^{\infty} \left[\exp\left(-\frac{(n\pi)^2 t^2}{2}\right) \sin((n\pi)x) \right]$