

LEGENDRE'S EQUATION

including Legendre's functions and Legendre's polynomials

Summary on Legendre Functions/Polynomials

Legendre's Differential Equation

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n-1)y = 0, \quad n \in \mathbb{R}.$$

General Solution of Legendre's Differential Equation

$$y = A \left[1 - \frac{(n+1)n}{2!} x^2 + \frac{(n+3)(n+1)n(n-2)}{4!} x^4 - \frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!} x^6 + \dots \right] \\ + \\ B \left[x - \frac{(n+2)(n-1)}{3!} x^3 + \frac{(n+4)(n+2)(n-1)(n-3)}{3!} x^5 - \frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!} x^7 + \dots \right]$$

- If n is an even integer, the first solution terminates after a finite number of terms, while the second one produces an infinite series.
- If n is an odd integer, the second solution terminates after a finite number of terms, while the first solution produces an infinite series.
- The finite solutions are the Legendre Polynomials, also known as solutions of the first kind, denoted by $P_n(x)$.
- The infinite series solutions are known as solutions of the second kind, denoted by $Q_n(x)$.

The second solution $Q_n(x)$ can be written in terms of $P_n(x)$ by

$$Q_n(x) = P_n(x) \int \frac{1}{(1-x^2)(P_n(x))^2} dx$$

The infinite series form for the Legendre's polynomial $P_n(x)$ is given by

$$P_n(x) = \sum_{k=0}^N \left[\frac{(2n-2k)!}{2^n k! (n-k)! (n-2k)!} (-1)^k x^{n-2k} \right],$$

where N is the floor function

$$N = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd} \end{cases}$$

The generating function for the Legendre's polynomial $P_n(x)$ is given by

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)]$$

Question 1

Find the two independent solutions of Legendre's equation

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0, \quad n \in \mathbb{R}.$$

$$y = A \left[1 - \frac{(n+1)n}{2!} x^2 + \frac{(n+3)(n+1)n(n-2)}{4!} x^4 - \frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!} x^6 + \dots \right] \\ + \\ B \left[x - \frac{(n+2)(n-1)}{3!} x^3 + \frac{(n+4)(n+2)(n-1)(n-3)}{3!} x^5 - \frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!} x^7 + \dots \right]$$

$(1-x^2) \frac{\partial^2 y}{\partial x^2} - 2x \frac{\partial y}{\partial x} + n(n+1)y = 0$

$\frac{\partial y}{\partial x} = \sum_{k=0}^{\infty} a_k k x^{k-1}$
 $\frac{\partial^2 y}{\partial x^2} = \sum_{k=0}^{\infty} a_k k(k-1)x^{k-2}$
 $\frac{\partial y}{\partial x^2} = \sum_{k=0}^{\infty} a_k k x^{k-2}$

ASSUME A SOLUTION OF THE FORM
 $y = \sum_{k=0}^{\infty} a_k x^{k+2}$
 $\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k k x^{k+1}$
 $\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k k(k-1) x^{k-2}$

SUB INTO THE ODE
 $\Rightarrow \sum_{k=0}^{\infty} a_k k(k+1) x^{k-2} - 2 \sum_{k=0}^{\infty} a_k k x^{k-2} + n(n+1) \sum_{k=0}^{\infty} a_k x^{k-2} = 0$

THE SUMMEST POWER IN THESE SUMMATIONS IS x^0 & THE HIGHEST IS x^2
 PULL OUT x^0 & x^2
 $\Rightarrow 2a_2^2 + (n+2)a_2 + \sum_{k=4}^{\infty} a_k k(k-1)x^{k-2} - 2a_2 - 2 \sum_{k=2}^{\infty} a_k k x^{k-2} + n(n+2)a_2^2 + n(n+1)a_2 + n(n-1)a_2 = 0$
 $\Rightarrow [2a_2^2 + (n+2)a_2 + n(n+1)a_2] + \sum_{k=4}^{\infty} a_k k(k-1)x^{k-2} - \sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} - 2 \sum_{k=2}^{\infty} a_k k x^{k-2} + n(n+1)a_2^2 = 0$
 (NOTE THERE IS NO INDICIAL EQUATION)

ADJUST THE SUMMATIONS SO THEY ALL START FROM $k=0$:

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^{k-2} - \sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^{k-2} - 2 \sum_{k=0}^{\infty} a_{k+2} (k+2) x^{k-2} + n(n+1) \sum_{k=0}^{\infty} a_{k+2} x^{k-2} = 0$$

FORMING A RECURRANCE EQUATION BY GROUPING POWERS OF x^{k+2} :

$$a_{k+2} (k+2)(k+1) - a_{k+2} (k+2) - 2a_{k+2} (k+1) + n(n+1) a_{k+2} = 0$$

$$a_{k+2} = \frac{(k+2)(k+1) - 2(k+2) - n(n+1)}{(k+1)(k+2)} a_{k+2}$$

$$a_{k+2} = \frac{k(k+1) + 2k - n(n+1)}{(k+1)(k+2)} a_k$$

$$a_{k+2} = \frac{k^2 + k - n(n+1)}{(k+1)(k+2)} a_k$$

$$a_{k+2} = \frac{k(k+1) - (k+1)^2}{(k+1)(k+2)} a_k = - \frac{n(n+1) - k(k+1)}{(k+1)(k+2)} a_k$$

$$a_{k+2} = - \frac{(k+1)(n-k)}{(k+1)(k+2)} a_k$$

where $n(n+1) - k(k+1) = k^2 + k - n^2 - k = k^2 - k^2 = 0$

GENERATING THE FIRST FEW TERMS:

$$k=0 \quad a_0 = - \frac{(n+1)n}{2} a_0$$

$$k=1 \quad a_1 = - \frac{(n+2)(n-1)}{2 \times 3} a_1$$

$$k=2 \quad a_2 = - \frac{(n+3)(n-2)}{3 \times 4} a_2 = \frac{(n+3)(n+1) \times (n-2)}{12 \times 3 \times 2} a_2$$

$$k=3 \quad a_3 = - \frac{(n+4)(n-3)}{4 \times 5} a_3 = - \frac{(n+4)(n+3)(n-1)(n-2)}{72 \times 4 \times 3 \times 2} a_3$$

$$k=4 \quad a_4 = - \frac{(n+5)(n-4)}{5 \times 6} a_4 = - \frac{(n+5)(n+4)(n+3)(n-2)(n-3)}{120 \times 5 \times 4 \times 3 \times 2} a_4$$

$$k=5 \quad a_5 = - \frac{(n+6)(n-5)}{6 \times 7} a_5 = - \frac{(n+6)(n+5)(n+4)(n-1)(n-2)(n-3)}{240 \times 6 \times 5 \times 4 \times 3 \times 2} a_5$$

$$k=6 \quad a_6 = - \frac{(n+7)(n-6)}{7 \times 8} a_6 = - \frac{(n+7)(n+6)(n+5)(n+4)(n-1)(n-2)(n-3)}{1680 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2} a_6$$

WHICH THE FIFTH FERSON COEFFICIENT IS GIVEN IN TERMS OF THE PARAMETER n :

$$y = a_0 \left[1 - \frac{(n+1)n}{2!} x^2 + \frac{(n+3)(n+1)n(n-2)}{4!} x^4 - \frac{(n+5)(n+3)(n+1)n(n-2)(n-4)}{6!} x^6 + \dots \right] \\ + \\ a_1 \left[x - \frac{(n+2)(n-1)}{3!} x^3 + \frac{(n+4)(n+2)(n-1)(n-3)}{5!} x^5 - \frac{(n+6)(n+4)(n+2)(n-1)(n-3)(n-5)}{7!} x^7 + \dots \right]$$

Question 2

Legendre's equation is given below

$$(1-t^2) \frac{d^2w}{dt^2} - 2t \frac{dw}{dt} + n(n+1)w = 0, \quad n \in \mathbb{N}.$$

- a) By assuming a series solution of the form

$$w(t) = \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0,$$

show by a detailed method that

$$a_{r+2} = -\frac{(n-r)(n+r+1)}{(r+2)(r+1)} a_r.$$

- b) By rewriting the recurrence relation of part (a) backwards, and taking the value of a_n as

$$a_n = \prod_{m=1}^n \frac{(2n-2m+1)}{n!},$$

show further that the Legendre's polynomials $P_n(t)$ can be written as

$$P_n(t) = \sum_{k=0}^N \left[\frac{(2n-2k)!}{2^n k! (n-k)! (n-2k)!} (-1)^k t^{n-2k} \right],$$

where N is the floor function

$$N = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd} \end{cases}$$

proof

[solution overleaf]

a)

$$(1-t^2) \frac{d^2y}{dt^2} - 2t \frac{dy}{dt} + n(n+1)y = 0$$

ON INSPECTION THIS IS ANHAROMATIC AT $t=0$, SO WE CAN EXPAND AS POWERS OF $t^{1/2}$

$$\text{LET } y = \sum_{n=0}^{\infty} a_n t^{n/2} \quad \text{then} \quad \frac{dy}{dt} = \sum_{n=0}^{\infty} a_n \frac{1}{2} t^{n/2-1} \quad \text{and} \quad \frac{d^2y}{dt^2} = \sum_{n=0}^{\infty} a_n \frac{1}{4} t^{n/2-2}$$

SUBSTITUTE INTO THE O.D.E.

$$(1-t^2) \sum_{n=0}^{\infty} a_n t^{n/2-2} - 2t \sum_{n=0}^{\infty} a_n t^{n/2-1} + n(n+1) \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=2}^{\infty} a_n t^{n/2-2} - 2 \sum_{n=1}^{\infty} a_n t^{n/2-1} + n(n+1) \sum_{n=0}^{\infty} a_n t^n = 0$$

THE LOWER POWER OF t IN THESE SUMMATIONS IS t^0 & THE HIGHEST IS t^2 - PULL OUT OR THE SUBSTITUTION. THIS YIELDS t^0 & t^2

$$\Rightarrow 2a_2 + a_0 + \sum_{n=2}^{\infty} a_n t^{n/2-2} - 2 \sum_{n=1}^{\infty} a_n t^{n/2-1} + n(n+1) \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\Rightarrow [2a_2 + n(n+1)a_0] + [a_0 - 2a_1 + n(n+1)a_1] t + \sum_{n=2}^{\infty} a_n t^{n/2-2} - \sum_{n=2}^{\infty} 2a_n t^{n/2-1} + n(n+1) \sum_{n=2}^{\infty} a_n t^n = 0$$

IGNORE THESE TWO LINES IN THESE EQUATIONS & FOR THE MIDDLE EQUATION ADD:

ALL SUMS FOR $t=0$

$$\sum_{n=0}^{\infty} a_n t^{n/2-2} - \sum_{n=0}^{\infty} a_n t^{n/2-1} + \sum_{n=0}^{\infty} a_n t^n = 0$$

EQUATE POWERS OF t IN THESE SUMMATIONS, SAY $t^{1/2}$ TO OBTAIN A RECURSIVE RELATION

$$\Rightarrow a_{n+2}(n+1)a_0 - a_{n+2}(n+1)a_1 - 2a_1 a_0 + n(n+1)a_{n+2} = 0$$

REFINE DOWN BY t^n

$$\Rightarrow a_{n+2}(n+1)a_0 - a_1 a_0 - 2a_1 a_0 + n(n+1)a_n = 0$$

$$\Rightarrow a_{n+2}(n+1)a_0 = \frac{n(n+1)2a_1 a_0}{(n+1)(n+2)} = \frac{n^2 a_1 a_0}{(n+1)(n+2)}$$

$$\Rightarrow a_{n+2} = \frac{(n+1)(n+2)}{(n+1)(n+2)} a_0$$

$$\Rightarrow a_{n+2} = \frac{(n+1)(n+2)}{(n+1)(n+2)} a_0$$

AS REQUIRED

b) Rewrite the recurrence relation downwards

$$a_r = -\frac{(r+2)(r+1)}{(r+2)(r+1+1)} a_{r+2}$$

NOTE THAT IF $r=0$, $a_{r+2} = a_{r+1} = a_{r+0} = \dots = 0$

Hence

$$r=n-2 \Leftrightarrow a_{n-2} = -\frac{n(n-1)}{2(n-1)} a_n$$

$$r=n-4 \Leftrightarrow a_{n-4} = -\frac{(n-2)(n-3)}{2(n-3)} a_{n-2} = \frac{n(n-1)(n-2)(n-3)}{2 \times (n-1)(n-2)} a_n$$

$$r=n-6 \Leftrightarrow a_{n-6} = -\frac{(n-4)(n-5)}{6(n-5)} a_{n-4} = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2 \times 4 \times 6(n-1)(n-2)(n-3)(n-4)} a_n \text{ ETC}$$

IN THE POLYNOMIAL FORM OF THE SOLUTION OF (LEGENDRE'S EQUATION) SPECIMENS

$$y = \sum_{n=0}^{\infty} a_n t^n = \frac{a_0 t^0}{2(2n)} - \frac{n(n-1)(n-2)(n-3)}{2(2n) \times 2(2n-2) \times 2(2n-4) \times \dots \times 2(2n-(2k))} a_0 t^{2k} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2(2n) \times 2(2n-2) \times 2(2n-4) \times \dots \times 2(2n-(2k))} a_1 t^{2k+2} - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{2(2n) \times 2(2n-2) \times 2(2n-4) \times \dots \times 2(2n-(2k))} a_2 t^{2k+4} + \dots$$

TO NORMALISE THESE POLYNOMIALS WE CHOOSE $a_0 = 1$ SO THAT $P_0(t) = 1$

THE VALUE OF a_n TO PRODUCE THIS IS

$$a_n = \frac{(2n)! (2n+1) \dots (2n+1)}{n!}$$

$$P_n(t) = \frac{(2n)! (2n+1) \dots (2n+1)}{n!} \left[t^n - \frac{n(n-1)}{2(2n)} t^{2n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \times 2(n-1)(2n-2)} t^{2n-4} - \frac{n(n-1)(n-2)(n-3)(n-4)}{2 \times 2(n-1)(2n-2) \times 2(n-3)(2n-4)} t^{2n-6} + \dots \right]$$

THE GENERALISATION OF THIS SERIES STARTING FROM ZERO

$$(t-1)^k = \frac{(2n+1)(2n+3) \dots 3 \times 1}{n!} \left[t^n - \frac{n(n-1)(n-2) \dots (n-2k+1)}{2 \times 2(n-1) \times 2(n-3) \dots (2n-2k+1)} t^{2n-2k} \right]$$

NOW IT IS BEST TO LOOK AT LEAST APPROXIMATIONS WITHIN THE PRODUCT TERM SEPARATELY

- $\bullet (2n+1)(2n+3) \dots 3 \times 1 = \frac{2(2n+1)(2n+3) \dots (2n+2k+1)}{2(2n+2) \dots 2(2n+2k+2)} \times \frac{2(2n+1)(2n+3) \dots (2n+2k+1)}{2(2n+2) \dots 2(2n+2k+2)} = \frac{(2n+1)!}{2^n n(n-1) \dots 2(n+1)} = \frac{(2n)!}{2^{n-1}}$
- $\bullet n(n-1)(n-2) \dots (n-2k+1) = \frac{n(n-1)(n-2) \dots (n-2k+1)}{(n-2k+1)(n-2k+3) \dots (n-2k+1)} = \frac{n!}{(n-2k)!}$
- $\bullet 2n \times 2(n-2) \dots 2(2k) = \frac{2^n (1 \times 3 \times \dots \times 2k)}{n!}$

c) COLLECTING ALL THESE RESULTS INTO THE GENERAL TERM FROM PREVIOUS ONE

$$= t^{-k} (-1)^k \times \frac{\frac{2^{2k}}{2^k k!}}{\frac{n!}{2^k k!}} \times \frac{\frac{(2n)!}{2^k (2n-2k)!}}{\frac{2^k k! \times \frac{2^{2k+1}}{2^{k+1} (k+1)!}}{\frac{2^{2k+1}}{2^{k+1} (k+1)!}}} = \frac{(-1)^k (2n-2k)!}{2^k k! (2n-2k-1)! n!}$$

This

$$P_n(t) = \sum_{k=0}^n (-1)^k \frac{(2n-2k)!}{2^k k! (k+1)! (2n-2k-1)!} t^{2n-2k}$$

IS FRACTION

$$N = \sum_{k=0}^n \frac{2^k}{k+1} \quad \text{ie } N \text{ is large}$$

SINCE THE POLYNOMIAL IS OF DEGREE $n \rightarrow n-2k \geq 0$

$$\begin{aligned} &\Leftrightarrow n-2k \geq 0 \\ &\Leftrightarrow -2k \geq n \\ &\Leftrightarrow k \leq \frac{n}{2} \end{aligned}$$

SO IF n IS EVEN THE SUMMATION THE SUMMATION GOES TO $\frac{n}{2}$

IF n IS ODD, THE SUMMATION GOES UP TO $\frac{n-1}{2}$

Question 3

It can be shown that the Legendre's polynomials $P_n(x)$ can be written as

$$P_n(x) = \sum_{k=0}^N \left[\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} (-1)^k x^{n-2k} \right],$$

where N is the floor function

$$N = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd} \end{cases}$$

Show that the generating function for $P_n(x)$ satisfies

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

proof

① **STARTING FROM A BINOMIAL EXPANSION**

$$(1-2xt+t^2)^{-\frac{1}{2}} = [(1-2xt+t^2)]^{-\frac{1}{2}} = 1 + \frac{1}{2}[-2x(t-t^2)] + \frac{1}{2!}[-2x(t-t^2)]^2 + \frac{1}{3!}[-2x(t-t^2)]^3 + \dots$$

$$= 1 + \frac{1}{2}x(-2t+t^2) + \frac{1}{2!}x^2(-2t+t^2)^2 + \frac{13}{24}x^3(-2t+t^2)^3 + \dots + \frac{1}{24}x^4(-2t+t^2)^4 + \dots$$

② **Now we require the coefficient of t^n in the above binomial expansion**

Firstly: $(2x-t)^n = (2x)^n - n(2x)(t)^1 + \frac{n(n-1)}{2!}(2x)(t)^2 - \frac{n(n-1)(n-2)}{3!}(2x)^3 + \dots$

③ **THE NEXT TERM IN THE BINOMIAL EXPANSION**

$$\frac{1}{2}x(-2t+t^2)^2 = \frac{1}{2}x(-2t+t^2)^2 - \frac{1}{2}x(-2t+t^2)^2 t + \frac{1}{2}x(-2t+t^2)^2 t^2 + \dots$$

$$= \frac{1}{2}x(-2t+t^2)^2 - \frac{1}{2}x(-2t+t^2)^2 t + \frac{1}{2}x(-2t+t^2)^2 t^2 + \dots$$

④ **THEN THE COEFFICIENT OF t^n**

$$\frac{1}{2}x(-2t+t^2)^n = \frac{1}{2}x(-2t+t^2)^n - \frac{1}{2}x(-2t+t^2)^{n-1} t + \frac{1}{2}x(-2t+t^2)^{n-2} t^2 - \dots$$

$$+ \frac{1}{2}x(-2t+t^2)^{n-3} t^3 - \dots + \frac{1}{2}x(-2t+t^2)^{n-2} t^2 - \dots$$

① **MORE MANIPULATIONS**

$$= \frac{1x3x5\dots(2n-1)}{n!} t^n = \frac{1x3x5\dots(2n-1)}{2(n-1)!} \frac{(2n-2)}{1!} t^{n-2} + \frac{1x3x5\dots(2n-5)}{2^2(n-2)!} \frac{(2n-4)(2n-3)}{2!} t^{n-4} \dots$$

$$= \frac{1x3x5\dots(2n-1)}{n!} t^n = \frac{1x3x5\dots(2n-1)}{2(n-1)!} \frac{n}{2n-1} t^{n-2} + \frac{1x3x5\dots(2n-5)}{2^2(n-2)!} \frac{(2n-4)(2n-3)}{2(n-1)(2n-2)} t^{n-4} \dots$$

$$= \frac{1x3x5\dots(2n-1)}{n!} t^n = \frac{1}{2(n-1)} \int t^n = \frac{1}{2(n-1)} \frac{t^{n+2}}{2(n+2)} + \frac{1x3x5\dots(2n-5)}{2^2(n-2)} \frac{t^{n+4}}{2(n+4)} - \frac{1(n-1)(n-3)(n-5)(n-7)}{2^3(n-3)(2n-3)} \frac{t^{n+6}}{2(n+6)} \dots$$

② **THE GENERAL TERM OF THE ABSOR SERI**, EXTRACTING FROM LHS, IS GIVEN BY

$$(-1)^{n-2k} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{n!} \times \frac{n(n-1)(n-3)\dots(n-2k+1)}{2^k k! (2n-2k-1)(2n-2k)}$$

IT IS BEST TO MANIPULATE TERMS OF THIS EXPRESSION SEPARATELY

- $(2n-1)(2n-3)\dots(2n-2k) = \frac{2n(2n-1)(2n-3)\dots(2n-2k+1)}{2n(2n-1)(2n-3)\dots(2n-2k)} \times \frac{(2n)!}{2^k k! (2n-2k-1)(2n-2k)} = \frac{(2n)!}{2^k k! (2n-2k-1)(2n-2k)}$
- $k(n-1)(n-3)\dots(n-2k+1) = \frac{n(n-1)(n-3)\dots(n-2k+1)}{2^k k! (2n-2k-1)(2n-2k)} = \frac{1}{(n-2k)!}$

• $(2n-1)(2n-3)(2n-5)\dots(2n-2k+1) = \frac{2n(2n-1)(2n-3)(2n-5)(2n-7)\dots(2n-2k+1)}{2n(2n-1)(2n-3)(2n-5)\dots(2n-2k+1)} \times \frac{(2n-2)(2n-4)(2n-6)\dots(2n-2k+2)}{(2n-2)(2n-4)(2n-6)\dots(2n-2k+2)} \times \frac{(2n-1)(2n-3)(2n-5)\dots(2n-2k+1)}{(2n-2)(2n-4)(2n-6)\dots(2n-2k+2)}$

NOTE FROM 2n ALL THE WAY DOWN TO 2, COUNT DOWN IN 2'S, i.e. 10 terms

FROM (2n-2) ALL THE WAY DOWN TO 2, JUST THE 2n-1's

THUS, IS $(n-1)$ TERMS

∴ $n = (n-1)$

∴ $(-1)^{n-2k} \frac{1}{(n-2k)!}$

② **COLLECTING ALL THESE RESULTS INTO THE MAIN EXPRESN, THE TERMS WHERE $k \neq 0$ THEN IS**

$$\dots = (-1)^{n-2k} \frac{1}{2^k k!} \times \frac{n!}{n!} = \frac{(-1)^{n-2k} \frac{1}{2^k k!} \times \frac{2^k k!}{2^k k!} \frac{1}{(n-2k)!}}{2^k k! \frac{1}{(n-2k)!}}$$

SO $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^{n-2k} \frac{1}{(n-2k)!}}{2^k k! (n-2k)!} P_n(x)$

N = ROME FANTASY

Question 4

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

Use this relationship to prove that

$$P_n(-x) = (-1)^n P_n(x).$$

[] , proof

Starting with the generating function for Legendre polynomials, by
dividing all terms by $t^{-\frac{1}{2}}$

$$\rightarrow (1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^{\frac{n}{2}} P_n(x)]$$

$$\rightarrow (1 + 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^{\frac{n}{2}} P_n(-x)]$$

Next we replace t with $-t$

$$\rightarrow (1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [-t]^n P_n(-x)$$

$$\rightarrow \sum_{n=0}^{\infty} [t^n P_n(-x)] = \sum_{n=0}^{\infty} [t^{\frac{n}{2}}]^{-2} P_n(-x)$$

EQUAL COEFFICIENTS OF $t^{\frac{n}{2}}$ IN BOTH SERIES

$$\Rightarrow P_n(x) = (-1)^{\frac{n}{2}} P_n(-x)$$

$$\Rightarrow (-1)^{\frac{n}{2}} P_n(x) = (-1)^{\frac{n}{2}} (-1)^{\frac{n}{2}} P_n(-x)$$

$$\Rightarrow (-1)^{\frac{n}{2}} P_n(x) = (-1)^{\frac{n}{2}} P_n(-x)$$

$$\Rightarrow (-1)^{\frac{n}{2}} P_n(x) = P_n(-x)$$

$\therefore P_n(x) = (-1)^n P_n(-x)$ ✓ AS REQUIRED

Question 5

The generating function $g(x,t)$ for the Legendre's polynomials $P_n(x)$, satisfies

$$g(x,t) = (1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

Use this relationship to prove that

$$\frac{\partial}{\partial x} [g(x,t)] + \frac{\partial}{\partial t} [g(x,t)] = x [g(x,t)]^3.$$

, proof

LET $g(x,t) = (1 - 2xt + t^2)^{-\frac{1}{2}}$

Differentiate g with respect to x

$$\begin{aligned} \frac{\partial g}{\partial x} &= -\frac{1}{2}(1 - 2xt + t^2)^{-\frac{3}{2}} \times (-2t) = t(1 - 2xt + t^2)^{-\frac{3}{2}} \\ &= t[(1 - 2xt + t^2)^{-\frac{1}{2}}]^2 = (x-t)[g(x,t)]^2 \end{aligned}$$

Now differentiate g with respect to t

$$\begin{aligned} \frac{\partial g}{\partial t} &= -\frac{1}{2}(1 - 2xt + t^2)^{-\frac{3}{2}} \times (2x+2t) = (x-t)(1 - 2xt + t^2)^{-\frac{3}{2}} \\ &= (x-t)[(1 - 2xt + t^2)^{-\frac{1}{2}}]^2 = (x-t)[g(x,t)]^3 \end{aligned}$$

Adding and the result follows

$$\begin{aligned} \frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} &= t[g(x,t)]^2 + (x-t)[g(x,t)]^3 \\ \frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} &= x[g(x,t)]^3 \end{aligned}$$

$\therefore \frac{\partial}{\partial x} (104) + \frac{\partial}{\partial t} (104) = \infty [g(x,t)]^3$

$\text{As } 104 > 0$

Question 6

$$f(x) \equiv 10x^3 - 3x^2 + x - 1.$$

Express $f(x)$ as a linear combination of Legendre's polynomials, $P_n(x)$.

You may assume

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$,
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$$[f(x) = 4P_3(x) - 2P_2(x) + 7P_1(x) - 2P_0(x)]$$

The image shows handwritten mathematical work for finding the coefficients of a polynomial expansion. It includes:

- Definitions of Legendre polynomials:
 - $P_0(x) = 1$
 - $P_1(x) = x$
 - $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$
 - $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
- Equation setup:

$$\begin{aligned} f(x) &= 10x^3 - 3x^2 + x - 1 \\ &\equiv AP_3(x) + BP_2(x) + CP_1(x) + DP_0(x) \end{aligned}$$
- Substitution of Legendre polynomials:

$$\begin{aligned} 10x^3 - 3x^2 + x - 1 &\equiv A\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) + B\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + Cx + D \\ 10x^3 - 3x^2 + x - 1 &\equiv \frac{5}{2}Ax^3 - \frac{3}{2}Ax \\ &\quad + \frac{3}{2}Bx^2 - \frac{1}{2}B \\ &\quad + Cx + D \end{aligned}$$
- Equating coefficients:

$$\begin{aligned} 10x^3 - 3x^2 + x - 1 &\equiv \frac{5}{2}Ax^3 + \frac{3}{2}Bx^2 + \left(C - \frac{3}{2}A\right)x + \left(D - \frac{1}{2}B\right) \end{aligned}$$
- Solving the system of equations:

$$\begin{aligned} \frac{5}{2}A &= 10 & \frac{3}{2}B &= -3 & C - \frac{3}{2}A &= 1 & D - \frac{1}{2}B &= -1 \\ A &= 4 & B &= -2 & C &= 1 & D &= -1 \\ & & & & C &= 7 & & \\ & & & & & & D &= 2 \end{aligned}$$
- Final result:

$$f(x) = 4P_3(x) - 2P_2(x) + 7P_1(x) - 2P_0(x)$$

Question 7

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

By differentiating the above relationship with respect to t , prove that

$$(2n+1)xP_n(x) - (n+1)P_{n+1}(x) + nP_{n-1}(x) = 0.$$

[] , proof

Starting with the generating function

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)]$$

Differentiate with respect to t .

$$\Rightarrow \frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-\frac{3}{2}}(1-2n+2t) = (1-2xt+t^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-\frac{1}{2}} = (1-2xt+t^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (x-t) \sum_{n=0}^{\infty} [t^n P_n(x)] = (1-2xt+t^2) \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow \sum_{n=0}^{\infty} [t^n P_n(x) - t^{n-1} P_n(x)] = \sum_{n=0}^{\infty} [nt^{n-1} P_n(x) - 2xt^{n-1} P_n(x) + nt^{n-2} P_n(x)]$$

Equating coefficients of t_j say $[t^j]$

$$\Rightarrow 2P_0(x) - P_1(x) = (m+1)P_{m+1}(x) - 2nP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow 0 = (m+1)P_{m+1}(x) - 2nP_n(x) - (n-1)P_{n-1}(x) + P_m(x)$$

$$\Rightarrow 0 = (m+1)P_{m+1}(x) - (2n+2)P_n(x) + nP_{n-1}(x)$$

$$\Rightarrow (m+1)P_{m+1}(x) - (2n+2)P_n(x) + nP_{n-1}(x) = 0$$

AS 2008/09

Question 8

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

By separately differentiating the above relationship once with respect to t and once with respect to x , prove that

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x).$$

[] , [proof]

STRUCTURE WITH THE GENERATING FUNCTION

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)]$$

DIFFERENTIATE FIRST WITH RESPECT TO x & NEXT WITH RESPECT TO t ONLY
(REARRANGED)

$$\left. \begin{aligned} -\frac{1}{2} (1 - 2xt + t^2)^{-\frac{3}{2}} (-2t) &= \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)] \\ -\frac{1}{2} (1 - 2xt + t^2)^{-\frac{3}{2}} (-2) &= \sum_{n=0}^{\infty} [t^n P'_n(x)] \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} (2x-t)(1 - 2xt + t^2)^{-\frac{3}{2}} &= \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)] \\ t(1 - 2xt + t^2)^{-\frac{3}{2}} &= \sum_{n=0}^{\infty} [t^n P'_n(x)] \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} t(2x-t)(1 - 2xt + t^2)^{-\frac{3}{2}} &= t \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)] \\ (2x-t)t(1 - 2xt + t^2)^{-\frac{3}{2}} &= (2x-t) \sum_{n=0}^{\infty} [t^n P'_n(x)] \end{aligned} \right\} \Rightarrow$$

REARRANGING THE RHS OF THE ABOVE EQUATIONS

$$\rightarrow \sum_{n=0}^{\infty} [4t^{2n-1} P'_n(x)] = \sum_{n=0}^{\infty} [t^{2n+1} P'_n(x)] - t^{2n-1} P'_n(x)$$

FINDING "EVEN POWERS OF t " IN THE ABOVE EQUATION, SAY t^{2k}

$$\rightarrow 4t^{2k-1} P'_k(x) = 2t^{2k+1} P'_k(x) - t^{2k-1} P'_k(x)$$

AS REQUIRED

Question 9

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

- a) Use this result to show that

$$P_n(1) = 1.$$

- b) By using the result of part (a) and Legendre's equation, deduce that

$$P'_n(1) = \frac{1}{2}n(n+1).$$

 , proof

a) Starting from the generating function for Legendre's polynomials

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

Letting $x=1$ in the above relationship,

$$\Rightarrow (1 - 2t + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(1)].$$

$$\Rightarrow [(1 - t)^2]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(1)].$$

$$\Rightarrow (1-t)^{-1} = \sum_{n=0}^{\infty} [t^n P_n(1)].$$

$$\Rightarrow 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \dots = P_0(1) + tP_1(1) + t^2P_2(1) + t^3P_3(1) + \dots$$

Hence the result follows by comparison. $P_0(1) = 1$

b) Starting with Legendre's equation, whose solution is $y = P_n(x)$

$$\Rightarrow (1 - 2x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$$

$$\Rightarrow (1 - 2t^2) y'' - 2xy' + n(n+1)y = 0$$

$$\Rightarrow (1 - 2t^2) P_n''(t) - 2tP_n'(t) + n(n+1)P_n(t) = 0$$

Let $t=1$ & note from part (a) $P_0(1) = 1$

$$\Rightarrow -2P_0''(1) + n(n+1) = 0$$

$$\Rightarrow P_0''(1) = \frac{1}{2}n(n+1)$$

AS REQUIRED

Question 10

Use trigonometric identities to show that

$$\sin^2 \theta = \frac{8}{35} P_4(\cos \theta) - \frac{16}{21} P_2(\cos \theta) + \frac{8}{15} P_0(\cos \theta)$$

You may assume

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$,
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

proof

$$\begin{aligned} \sin^4 \theta &= \frac{8}{35} P_4(\cos \theta) - \frac{16}{21} P_2(\cos \theta) + \frac{8}{15} P_0(\cos \theta) \\ \bullet \sin^4 \theta &= (\sin^2 \theta)^2 = (1 - \cos^2 \theta)^2 = 1 - 2\cos^2 \theta + \cos^4 \theta \\ \bullet \text{Now } P_0(x) &= 1 \\ P_1(x) &= \frac{3}{2}x^2 - \frac{1}{2} \\ P_2(x) &= \frac{5}{8}x^4 - \frac{15}{8}x^2 + \frac{3}{8} \\ \Rightarrow 1 - 2x^2 + x^4 &\equiv AP_2(x) + BP_0(x) + CP_1(x) \\ \Rightarrow 1 - 2x^2 + x^4 &\equiv A\left(\frac{5}{8}x^4 - \frac{15}{8}x^2 + \frac{3}{8}\right) + B(1) + C\left(\frac{3}{2}x^2 - \frac{1}{2}\right) \\ \Rightarrow 1 - 2x^2 + x^4 &\equiv \frac{5}{8}Ax^4 - \frac{15}{8}Ax^2 + \frac{3}{8}A + \frac{3}{2}Bx^2 - \frac{1}{2}B \\ \Rightarrow 1 - 2x^2 + x^4 &\equiv \frac{5}{8}Ax^4 + \left(\frac{3}{2}B - \frac{15}{8}A\right)x^2 + \left(C - \frac{1}{2}B + \frac{3}{8}A\right) \end{aligned}$$

$$\begin{array}{lll} \frac{5}{8}A = 1 & \frac{3}{2}B - \frac{15}{8}A = -2 & C - \frac{1}{2}B + \frac{3}{8}A = 1 \\ A = \frac{8}{35} & \frac{3}{2}B - \frac{15}{8} \times \frac{8}{35} = -2 & C - \frac{1}{2} \times \frac{8}{35} + \frac{3}{8} \times \frac{8}{35} = 1 \\ \frac{3}{2}B = \frac{8}{35} & \frac{3}{2}B = -2 & C + \frac{8}{35} + \frac{3}{35} = 1 \\ B = \frac{16}{21} & B = -2 & C = \frac{8}{15} \end{array}$$

TNS

$$\sin^4 \theta = \frac{8}{35} P_4(\cos \theta) - \frac{16}{21} P_2(\cos \theta) + \frac{8}{15} P_0(\cos \theta)$$

Question 11

The generating function $g(x,t)$ for the Legendre's polynomials $P_n(x)$, satisfies

$$g(x,t) = (1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

Verify that $g = g(x,t)$ is a solution of the differential equation

$$t \frac{\partial^2}{\partial t^2} [t g] + \frac{\partial}{\partial x} \left[(1-x^2) \frac{\partial g}{\partial x} \right] = 0.$$

[proof]

\bullet $g(x,t) = \sum_{n=0}^{\infty} [t^n P_n(x)]$

$$\frac{\partial g}{\partial t} = \sum_{n=0}^{\infty} [t^{n+1} P_n(x)]$$

$$\frac{\partial^2 g}{\partial t^2} = \sum_{n=0}^{\infty} [n(n+1) t^{n+2} P_n(x)]$$

$$\frac{\partial^2 g}{\partial t^2} (t) = \sum_{n=0}^{\infty} [n(n+1) t^{n+2} P_n(x)]$$

$$+ \frac{\partial^2}{\partial t^2} (tg) = \sum_{n=0}^{\infty} [n(n+1) t^{n+3} P_n(x)]$$

\bullet $g(x,t) = \sum_{n=0}^{\infty} [t^n P_n(x)]$

$$\frac{\partial g}{\partial x} = \sum_{n=0}^{\infty} [t^n P'_n(x)]$$

$$(1-x^2) \frac{\partial g}{\partial x} = (1-x^2) \sum_{n=0}^{\infty} [t^n P'_n(x)]$$

$$\frac{\partial}{\partial x} \left[(1-x^2) \frac{\partial g}{\partial x} \right] = -2x \sum_{n=0}^{\infty} [t^n P'_n(x)] + (1-x^2) \sum_{n=0}^{\infty} [t^n P''_n(x)]$$

\bullet SUBSTITUTE into the given D.O.E.

$$t \frac{\partial^2}{\partial t^2} (tg) + \frac{\partial}{\partial x} \left[(1-x^2) \frac{\partial g}{\partial x} \right] =$$

$$= \sum_{n=0}^{\infty} [n(n+1) t^{n+3} P_n(x)] - 2x \sum_{n=0}^{\infty} [t^n P'_n(x)] + (1-x^2) \sum_{n=0}^{\infty} [t^n P''_n(x)]$$

$$= \sum_{n=0}^{\infty} [(-2x) P'_n(x) - 2t P'_n(x) + n(n+1) P''_n(x)] t^n$$

$$= 0$$

$P_n(x)$ is a solution of Legendre's equation

Question 12

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

- a) Use this result to show that

$$P_n(-1) = (-1)^n.$$

- b) By using the result of part (a) and Legendre's equation, deduce that

$$P'_n(-1) = \frac{1}{2} n(n+1)(-1)^{n+1}.$$

[] , proof

a) Starting from the generating formula for Legendre's polynomials

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Let $x=-1 \Rightarrow (1+2t+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(-1)$

$$\Rightarrow [(1+t)^2]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(-1)$$

$$\Rightarrow (1+t)^{-1} = \sum_{n=0}^{\infty} t^n P_n(-1)$$

$$\Rightarrow 1-t+t^2-t^3+\dots = \sum_{n=0}^{\infty} t^n P_n(-1)$$

Equating coefficients for t^n on both sides gives $P_n(-1) = (-1)^n$

b) $P_n(x)$ is a solution of Legendre's equation so satisfying with the O.D.E.

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$$

$$(1-x^2) P''_n(x) - 2x P'_n(x) + n(n+1) P_n(x) = 0$$

Let $x=-1$ blocks off the second derivative

$$\Rightarrow 2 P''_n(-1) + n(n+1) P_n(-1) = 0$$

But $P_n(-1) = (-1)^n$

$$\Rightarrow 2 P''_n(-1) + n(n+1) (-1)^n = 0$$

$$\Rightarrow 2 P''_n(-1) = -n(n+1) (-1)^n$$

$$\Rightarrow P''_n(-1) = \frac{1}{2} n(n+1) (-1)^{n+1}$$

Question 13

The Legendre's polynomial $P_n(x)$ is a solution of the differential equation

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0, \quad n \in \mathbb{R}.$$

Show that

$$P'_n(x) = \frac{n(n+1)}{1-x^2} \int_1^x P_n(x) dx.$$

[proof]

$\{(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0\}$

As $P_n(x)$ is a solution of this equation
 $(1-x^2) P''_n(x) - 2x P'_n(x) + n(n+1)P_n(x) = 0$

Integrate the equation with respect to x between a & 1 .
 $\rightarrow \int_a^1 (1-x^2) P''_n(x) dx - 2 \int_a^1 x P'_n(x) dx + n(n+1) \int_a^1 P_n(x) dx = 0$

↓
INTEGRATION BY PARTS

$1-x^2$	$-2x$
$P'_n(x)$	$P_n(x)$

$\Rightarrow [(1-x^2) P'_n(x)]_a^1 + 2 \int_a^1 x P'_n(x) dx - 2 \int_a^1 x P'_n(x) dx + n(n+1) \int_a^1 P_n(x) dx = 0$

$\Rightarrow [(1-x^2) P'_n(x)]_a^1 + n(n+1) \int_a^1 P_n(x) dx = 0$

$\Rightarrow 0 = (1-x^2) P'_n(1) + n(n+1) \int_a^1 P_n(x) dx$

$\Rightarrow n(n+1) \int_a^1 P_n(x) dx = (1-x^2) P'_n(1)$

$\Rightarrow P'_n(1) = \frac{n(n+1)}{1-x^2} \int_a^1 P_n(x) dx$ as required

OR $\int_a^1 P'_n(x) dx = \frac{(1-x^2) P'_n(1)}{n(n+1)}$

Question 14

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

- a) By differentiating the above relationship with respect to t , prove that

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

- b) By separately differentiating the generating function for the Legendre's polynomials once with respect to t and once with respect to x , prove that

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x).$$

- c) Use parts (a) and (b) to show that

$$(2n+1)P_n(x) = P'_{n+1}(x) + P'_{n-1}(x).$$

- d) Use parts (b) and (c) to deduce that

$$(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x).$$

proof

a) Starting from the generating function

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)]$$

Differentiate with respect to t

$$\Rightarrow -\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (2-x)(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (2-x)(1-2xt+t^2)^{-\frac{3}{2}}(1-2x+2t) = \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (2-x)(1-2xt+t^2)^{-\frac{1}{2}} = (1-2xt+t^2) \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (2-x) \sum_{n=0}^{\infty} [t^n P_n(x)] = (1-2xt+t^2) \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow \sum_{n=0}^{\infty} [t^n (2P_n(x) - t^n P'_n(x))] = \sum_{n=0}^{\infty} [nt^{n-1} P_n(x) - 2nt^{n-2} P'_n(x) + nt^{n-1} P'_n(x)]$$

• Equate the coefficients of powers of t , say $[t^n]$

$$\Rightarrow 2P_n(x) - P'_{n-1}(x) = (n+1)P'_{n+1}(x) - 2nP'_{n-1}(x) + (n-1)P'_{n-2}(x)$$

$$\Rightarrow 0 = (n+1)P'_{n+1}(x) - 2(n+1)P'_{n-1}(x) + (n-1)P'_{n-2}(x)$$

$$\Rightarrow (n+1)P'_{n+1}(x) - 2(n+1)P'_{n-1}(x) + nP'_{n-2}(x) = 0$$

As required

b) Starting from the generating function

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)]$$

Differentiate with respect to t , and with respect to x

$$\begin{aligned} -\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) &= \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)] \\ -\frac{1}{2}(1-2xt+t^2)^{-\frac{1}{2}}(2t) &= \sum_{n=0}^{\infty} [t^n P'_n(x)] \end{aligned}$$

$$\begin{aligned} (2-x)(1-2xt+t^2)^{-\frac{3}{2}} &= \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)] \\ t(1-2xt+t^2)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty} [t^n P'_n(x)] \end{aligned}$$

• Multiply the first by t , and the second by $(2-x)$ & equate the terms

$$\begin{aligned} t \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)] &= (2-x) \sum_{n=0}^{\infty} [t^n P'_n(x)] \\ \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)] &= \sum_{n=0}^{\infty} [x t^n P'_n(x) - t^{n+1} P'_n(x)] \end{aligned}$$

• Equate coefficients of powers of t , say $[t^n]$

$$nP_n(x) = xP'_n(x) - P'_{n+1}(x)$$

As required

c) From (a) $(n+1)P'_{n+1}(x) - (n+1)P'_{n-1}(x) + nP'_{n-2}(x) = 0$

Differentiate w.r.t x

$$(2n+1)P'_{n+1}(x) - (2n+1)P'_{n-1}(x) + (2n+1)P'_{n-2}(x) = 0$$

From (b) $2P_n(x) = nP'_n(x) + P'_{n-1}(x)$

$$\Rightarrow (2n+1)P'_{n+1}(x) - (2n+1)P'_{n-1}(x) - (2n+1)(n+1)P'_{n-1}(x) + nP'_{n-2}(x) = 0$$

$$\Rightarrow (n+1)P'_{n+1}(x) - (2n+1)P'_{n-1}(x) - (2n+1)P'_{n-2}(x) = 0$$

$$\Rightarrow (2n+1)P'_{n+1}(x) - (2n+1)P'_{n-1}(x) - P'_{n-2}(x) = 0$$

$$\Rightarrow (2n+1)P'_{n+1}(x) = P'_{n-1}(x) - P'_{n-2}(x)$$

As required

From (b) $nP'_n(x) = xP'_n(x) - P'_{n+1}(x)$

From (b) $(n+1)P'_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

Subtracting the two equations "termwise"

$$(n+1)P'_n(x) = P'_{n+1}(x) - xP'_n(x)$$

As required

Question 15

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

Use this result to show that ...

... if n is even, $P_n(x)$ is an even polynomial in x .

... if n is odd, $P_n(x)$ is an odd polynomial in x .

proof

• USING THE GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

• WE NEED TO INVESTIGATE THE BEHAVIOUR OF $P_n(-x)$

LET $x \mapsto -x$

$$\Rightarrow (1+2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(-x)$$

• NOW REPLACE t BY $-t$

$$\Rightarrow (1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (-t)^n P_n(-x)$$

$$\Rightarrow \sum_{n=0}^{\infty} t^n P_n(x) = \sum_{n=0}^{\infty} t^n (-1)^n P_n(-x)$$

• EXPANDING COEFFICIENTS OF t^n

$$\Rightarrow P_n(x) = P_n(-x)(-1)^n$$

MULTIPLY THIS BY $(-1)^n$ TO THE OTHER SIDE

$$\Rightarrow (-1)^n P_n(x) = (-1)^n (-1)^n P_n(x)$$

$$\Rightarrow (-1)^n P_n(x) = (-1)^{2n} P_n(x)$$

$$\Rightarrow P_n(x) = (-1)^{2n} P_n(x)$$

• CONSIDER SEPARATELY IF $n=2m$ (EVEN) OR IF $n=2m+1$ (ODD)

$$\begin{cases} P_{2m}(-x) = (-1)^{2m} P_m(x) = P_{2m}(x) \\ P_{2m+1}(-x) = (-1)^{2m+1} P_{m+1}(x) = -P_{2m+1}(x) \end{cases}$$

∴ IF n IS EVEN $P_n(x)$ IS AN EVEN POLYNOMIAL
IF n IS ODD $P_n(x)$ IS AN ODD POLYNOMIAL //

Question 16

The generating function for the Legendre's Polynomials $P_n(x)$, satisfies

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

- a) Use this result to show that

$$P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n} n!}.$$

- b) Deduce the value of $P_{2n+1}(0)$.

$$\boxed{P_{2n+1}(0) = 0}$$

a) Starting from the generating function for Legendre Polynomials

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Let $x=0 \Rightarrow (1+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(0)$

Expanding binomially on the L.H.S

$$\Rightarrow (1+t^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}t^2 + \frac{-\frac{1}{2}(\frac{1}{2})}{2!}t^4 + \frac{-\frac{1}{2}(\frac{1}{2})(\frac{3}{2})}{3!}t^6 + \dots$$

$$\Rightarrow (1+t^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}t^2 + \frac{1 \times 3}{2!2!}t^4 - \frac{1 \times 3 \times 5}{2!3!2!}t^6 + \dots - \frac{(-1)^n \times 3 \times 5 \times \dots \times (2n-1)}{2^n n!}t^{2n}$$

Comparing coefficients of t^{2n}

$$\Rightarrow P_{2n}(0) = \frac{(-1)^n (2n-1)(2n-3)\dots 5 \times 3 \times 1}{2^n n!}$$

$$\Rightarrow P_{2n}(0) = \frac{(-1)^n}{2^n n!} \times \frac{2n(2n-2)(2n-4)\dots 6 \times 4 \times 2}{2n(2n-2)(2n-4)\dots 6 \times 4 \times 2}$$

$$\Rightarrow P_{2n}(0) = \frac{(-1)^n}{2^n n!} \times \frac{(2n)!}{2^n n!}$$

$$\Rightarrow P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n} n!}$$

b) Comparing powers of t^{2n+1} in the binomial expansion above we deduce

$$P_{2n+1}(0) = 0$$

Question 17

Legendre's equation is given below

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0, \quad n \in \mathbb{R}.$$

Use the substitution $x = \cos \theta$ to show that

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left[\frac{dy}{d\theta} \sin \theta \right] + n(n+1)y = 0.$$

proof

Q $(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$

LET $x = \cos \theta$ DIFF w.r.t. x $\rightarrow \frac{dx}{d\theta} = -\sin \theta \frac{d\theta}{dx}$

$\frac{dy}{dx} = -\sin \theta \frac{dy}{d\theta}$

NEXT DIFFERENTIATE EXPRESSION WE JUST FOUND w.r.t. x .

$\rightarrow \frac{d^2y}{dx^2} = \cos \theta \sin \theta \frac{d}{d\theta} \frac{dy}{d\theta} - \sin \theta \frac{d}{d\theta} \frac{d^2y}{d\theta^2}$

$\rightarrow \frac{d^2y}{dx^2} = \frac{1}{\sin \theta} \frac{\cos \theta}{\sin \theta} \times \frac{dy}{d\theta} - \frac{1}{\sin \theta} \frac{d^2y}{d\theta^2} \times \frac{d\theta}{dx}$

$\rightarrow \frac{d^2y}{dx^2} = \frac{\cos \theta}{\sin^2 \theta} \left[-\frac{1}{\sin \theta} \frac{dy}{d\theta} \right] - \frac{1}{\sin^2 \theta} \frac{d^2y}{d\theta^2} \times \frac{1}{-\sin \theta}$

$\rightarrow \frac{d^2y}{dx^2} = -\frac{1}{\sin^2 \theta} \left[\cot \theta \frac{dy}{d\theta} + \frac{d^2y}{d\theta^2} \right]$

SUB INTO LEGENDRE'S EQUATION

$\rightarrow (1-\cos^2 \theta) \left[-\frac{1}{\sin^2 \theta} \left[\cot \theta \frac{dy}{d\theta} + \frac{d^2y}{d\theta^2} \right] \right] - 2\cos \theta \left[-\cot \theta \frac{dy}{d\theta} \right] + n(n+1)y = 0$

$\rightarrow \sin^2 \theta \left[\frac{1}{\sin^2 \theta} \left[\frac{d^2y}{d\theta^2} - \cot^2 \theta \frac{dy}{d\theta} \right] \right] + 2\cos \theta \frac{dy}{d\theta} + n(n+1)y = 0$

$\rightarrow \frac{d^2y}{d\theta^2} - \cot^2 \theta \frac{dy}{d\theta} + 2\cos \theta \frac{dy}{d\theta} + n(n+1)y = 0$

$\rightarrow \frac{d^2y}{d\theta^2} + \cot \theta \frac{dy}{d\theta} + n(n+1)y = 0$

Q BY USE OF IDENTIFICATION

$\frac{1}{\sin \theta} \frac{d}{d\theta} \left[\sin \theta \frac{dy}{d\theta} \right] = \frac{1}{\sin \theta} \left[\cos \theta \frac{dy}{d\theta} + \sin \theta \frac{d^2y}{d\theta^2} \right]$

$= \cot \theta \frac{dy}{d\theta} + \frac{d^2y}{d\theta^2}$

$\therefore \frac{d^2y}{d\theta^2} + \cot \theta \frac{dy}{d\theta} = \frac{1}{\sin \theta} \frac{d}{d\theta} \left[\sin \theta \frac{dy}{d\theta} \right]$

∴ THE EQUATION CAN BE WRITTEN AS

$\frac{1}{\sin \theta} \frac{d}{d\theta} \left[\sin \theta \frac{dy}{d\theta} \right] + n(n+1)y = 0$

As required

Question 18

Find the two independent solutions of Legendre's equation

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0.$$

$$y = Ax + B \left[\frac{1}{x} \ln \left(\frac{1+x}{1-x} \right) - 1 \right]$$

This is Legendre's equation with $n(n+1) = 2$, $n+k-2 = 0$, $(n+2)(n+1)$, $k = \begin{cases} -2 \\ 1 \end{cases}$

- THE FINITE SOLUTION IS $A P_1(x) = Ax$
- THE INFINITE-SERIES SOLUTION $Q_0(x)$ SATISFIES

$$Q_0(x) = P_n(x) \int \frac{1}{(1-x^2)^{n+2}} dx$$

$$Q_0(x) = x \int \frac{1}{(1-x^2)^2} dx$$

BY PARTIAL FRACTIONS

$$\frac{1}{(1-x)(1+x)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} + \frac{D}{(1+x)^2}$$

$$1 = Ax(1+x)^2 + B(1-x^2) + Cx^2(1+x) + Dx^2(1-x)$$

If $x=1$ $\Rightarrow 2c=1 \Rightarrow c=\frac{1}{2}$
If $x=0$ $\Rightarrow [B=1]$
If $x=-1 \Rightarrow 2D=1 \Rightarrow D=\frac{1}{2}$
If $x=2 \Rightarrow 1 = -4A - 3 + 6 - 2$ $6A=0$ $A=0$

THIS

$$Q_0(x) = x \int \frac{1}{x^2 + \frac{1}{4}} + \frac{\frac{1}{2}}{1-x^2} dx$$

$$Q_0(x) = x \left[-\frac{1}{x} + \frac{1}{2} \ln|1+x| - \frac{1}{2} \ln|1-x| \right]$$

$$Q_0(x) = -1 + \frac{1}{2}x \ln \left| \frac{1+x}{1-x} \right|$$

HENCE THE GENERAL SOLUTION WILL BE

$$y = A P_1(x) + B Q_0(x)$$

$$y = Ax + B \left[\frac{1}{2}x \ln \left| \frac{1+x}{1-x} \right| - 1 \right]$$

Question 19

The generating function for the Legendre's Polynomials $P_n(x)$, satisfies

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

Using this result, and integrating both sides with respect to t , from 0 to 1, show that

$$\sum_{n=0}^{\infty} \left[\frac{P_n(\cos \theta)}{n+1} \right] = \ln \left[1 + \operatorname{cosec} \left(\frac{1}{2} \theta \right) \right].$$

[proof]

Starting from the generating function for Legendre's Polynomials

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Integrate both sides with respect to t , from 0 to 1, to get

$$\int_0^1 \frac{1}{\sqrt{1-2xt+t^2}} dt = \int_0^1 \left[\sum_{n=0}^{\infty} t^n P_n(x) \right] dt$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{1-2xt+t^2}} dt = \sum_{n=0}^{\infty} P_n(x) \int_0^1 t^n dt$$

Let $x = \omega \sin \theta$

$$\int_0^1 \frac{1}{\sqrt{1-2t\omega \sin \theta + t^2}} dt = \sum_{n=0}^{\infty} P_n(\omega \sin \theta) \int_0^1 t^n dt$$

$$\int_0^1 \frac{1}{\sqrt{(t-\omega \sin \theta)^2 + \omega^2 \cos^2 \theta}} dt = \sum_{n=0}^{\infty} P_n(\omega \sin \theta) \left[\frac{t}{n+1} \right]_0^1$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{(t-\omega \sin \theta)^2 + \omega^2 \cos^2 \theta}} dt = \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} = \int_0^1 \frac{1}{\sqrt{(t-\omega \sin \theta)^2 + \omega^2 \cos^2 \theta}} dt$$

Let $u = t - \omega \sin \theta$
 $du = dt$
 $t = u + \omega \sin \theta$
 $t = 1$
 $u = 1 - \omega \sin \theta$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} = \int_{-\omega \sin \theta}^{1-\omega \sin \theta} \frac{1}{\sqrt{u^2 + \omega^2 \cos^2 \theta}} du$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} &= \left[\operatorname{arsinh} \left(\frac{u}{\sin \theta} \right) \right]_{-\omega \sin \theta}^{1-\omega \sin \theta} \\ \Rightarrow \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} &= \left[\ln \left(\frac{u}{\sin \theta} + \sqrt{\frac{u^2}{\sin^2 \theta} + 1} \right) \right]_{-\omega \sin \theta}^{1-\omega \sin \theta} \\ \Rightarrow \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} &= \left[\ln \left(\frac{u + \sqrt{u^2 + \sin^2 \theta}}{\sin \theta} \right) \right]_{-\omega \sin \theta}^{1-\omega \sin \theta} \\ \Rightarrow \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} &= \ln \left[\frac{(-\omega \sin \theta + \sqrt{(-\omega \sin \theta)^2 + \sin^2 \theta})}{\sin \theta} \right] \\ \Rightarrow \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} &- \ln \left[(-\omega \sin \theta + \sqrt{(-\omega \sin \theta)^2 + \sin^2 \theta}) \right] \\ \Rightarrow \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} &= \ln \left[\frac{(-\omega \sin \theta + \sqrt{(-\omega \sin \theta)^2 + \sin^2 \theta}) + \sin \theta}{1 - \cos \theta} \right] \\ \Rightarrow \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} &= \ln \left[\frac{1 - \cos \theta + \sqrt{1 - 2\cos \theta}}{1 - \cos \theta} \right] \\ \Rightarrow \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} &= \ln \left[\frac{(1 - \cos \theta) + \sqrt{1 - 2\cos \theta}}{1 - \cos \theta} \right] \\ \Rightarrow \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} &= \ln \left[\frac{\sqrt{1 - (1 - 2\cos \theta)^2} + \sqrt{2}}{\sqrt{1 - (1 - 2\cos \theta)^2}} \right] \\ \Rightarrow \sum_{n=0}^{\infty} \frac{P_n(\omega \sin \theta)}{n+1} &= \ln \left[\frac{\sqrt{2\sin^2 \theta + \omega^2}}{\sqrt{2\sin^2 \theta}} \right] \\ \therefore 1 + \frac{1}{2}P_1(\omega \sin \theta) + \frac{1}{3}P_2(\omega \sin \theta) + \dots + \frac{1}{n+1}P_n(\omega \sin \theta) &= \ln \left[\frac{1 + \sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} \right] \end{aligned}$$

Question 20

The generating function g for the Legendre's polynomials $P_n(x)$, satisfies

$$g(x,t) = (1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)].$$

- a) By differentiating g with respect to t , prove that

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

- b) By differentiating g once with respect to t and once with respect to x , prove that

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x).$$

- c) Use parts (a) and (b) to show that

$$(2n+1)P_n(x) = P'_{n+1}(x) + P'_{n-1}(x).$$

- d) Use parts (b) and (c) to deduce that

$$(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x).$$

- e) Use parts (b) and (d) to show that

$$(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)].$$

- f) Use parts (a) and (e) to show that

$$(1-x^2)P'_n(x) = (n+1)[xP_n(x) - P_{n+1}(x)]$$

proof

[solution overleaf]

a) STARTING FROM THE GENERATING FUNCTION

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)]$$

DIFFERENTIATE WITH RESPECT TO t

$$\Rightarrow -\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) = \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (2-x)(1-2xt+t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (2-x)(1-2xt+t^2)^{-\frac{3}{2}}(1-2xt+t^2) = (1-2xt+t^2) \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (2-x)(1-2xt+t^2)^{-\frac{1}{2}} = (1-2xt+t^2) \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow (2-x) \sum_{n=0}^{\infty} [t^n P_n(x)] = (1-2xt+t^2) \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$\Rightarrow \sum_{n=0}^{\infty} [t^n P_n(x) - t^{n-1} P_n(x)] = \sum_{n=0}^{\infty} [nt^{n-1} P_n(x) - nt^{n-2} P_n(x) + nt^{n-1} P_n(x)]$$

• EXPAND

• QUOTE THE COEFFICIENTS OF POWERS OF t , i.e., say $[t^n]$

$$2P_0(x) - P_1(x) = (n+1)P_1(x) - 2nP_0(x) + C(n-1)P_0(x)$$

$$\Rightarrow 0 = (n+1)P_1(x) - 2(n+1)P_0(x) + (n-1)P_0(x)$$

$$\Rightarrow (n+1)P_1(x) - (2n+1)P_0(x) + nP_0(x) = 0$$

AS EXPANDED

b) STARTING FROM THE GENERATING FUNCTION

$$(1-2xt+t^2)^{\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P_n(x)]$$

DIFFERENTIATE WITH RESPECT TO t , AND WITH RESPECT TO x

$$\begin{aligned} -\frac{1}{2}(1-2xt+t^2)^{\frac{3}{2}}(-2x+2t) &= \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)] \\ -\frac{1}{2}(1-2xt+t^2)^{\frac{1}{2}}(2x) &= \sum_{n=0}^{\infty} [t^n P'_n(x)] \end{aligned} \Rightarrow$$

$$(2x)(1-2xt+t^2)^{\frac{1}{2}} = \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)]$$

$$t(1-2xt+t^2)^{\frac{1}{2}} = \sum_{n=0}^{\infty} [t^n P'_n(x)] \Rightarrow$$

MULTIPLY THE FIRST BY t , AND THE SECOND BY $(2x-t)$ & EXPAND THE RHS

$$t \sum_{n=0}^{\infty} [nt^{n-1} P_n(x)] = (2x-t) \sum_{n=0}^{\infty} [t^n P'_n(x)]$$

$$\sum_{n=0}^{\infty} [nt^{n-1} P_n(x)] = \sum_{n=0}^{\infty} [2xt^n P'_n(x) - t^{n+1} P'_n(x)]$$

• EXPAND

• QUOTE COEFFICIENTS OF POWERS OF t , i.e., say $[t^n]$

$$n P'_n(x) = 2x P'_n(x) - P'_{n+1}(x)$$

c) From (a) $(n+1)P_{n+1}(x) - C(n+1)xP_n(x) + nP'_{n+1}(x) = 0$

DIFFERENTIATE w.r.t. x

$$(n+1)P'_{n+1}(x) - (n+1)xP'_n(x) - (2n+1)xP'_n(x) + nP''_{n+1}(x) = 0$$

From (b) $2xP'_n(x) = nP'_n(x) + P'_{n+1}(x)$

$$\Rightarrow (n+1)P'_{n+1}(x) - (2n+1)xP'_n(x) - (2n+1)nP'_n(x) + nP''_{n+1}(x) = 0$$

$$\Rightarrow (n+1)P'_{n+1}(x) - (2n+1)xP'_n(x) - (2n+1)nP'_n(x) + nP''_{n+1}(x) = 0$$

$$\Rightarrow P''_{n+1}(x) - (2n+1)xP'_n(x) - P'_{n+1}(x) = 0$$

$$\Rightarrow (2n+1)xP'_n(x) = P''_{n+1}(x) - P'_{n+1}(x)$$

AS EXPANDED

d) From (b) $nP'_n(x) = 2xP'_n(x) - P'_{n+1}(x)$

From (a) $(n+1)P'_n(x) = P'_{n+1}(x) - P'_{n+2}(x)$

• REPLACE P'_n BY $n-1$ IN THE SECOND EQUATION

$$P'_n(x) = 2xP'_n(x) - P'_{n+1}(x)$$

$$nP'_n(x) = P'_{n+1}(x) - 2xP'_{n+2}(x)$$

• MULTIPLY THE TWO EQUATIONS BY x

$$n x P'_n(x) = x^2 P'_n(x) - x P'_{n+1}(x)$$

$$n x P'_n(x) = P'_{n+1}(x) - 2x P'_{n+2}(x)$$

• SUBTRACT QUOTES!

$$n x P'_n(x) - n x P'_n(x) = (1-2x^2)P'_{n+1}(x)$$

$$\therefore (1-2x^2)P'_{n+1}(x) = n [P'_{n+1}(x) - x P'_{n+2}(x)]$$

AS EXPANDED

d) From (b) $nP'_n(x) = 2xP'_n(x) - P'_{n+1}(x)$

From (a) $(n+1)P'_n(x) = P'_{n+1}(x) - P'_{n+2}(x)$

• SUBTRACT THE TWO EQUATIONS 'QUOTES'

$$(n+1)P'_n(x) = P'_{n+1}(x) - 2xP'_{n+2}(x)$$

• EXPAND

From (a) $(2n+1)xP'_n(x) = (n+1)P'_{n+1}(x) + nP'_{n+2}(x)$

From (b) $(1-2x^2)P'_{n+1}(x) = n [P'_{n+1}(x) - x P'_{n+2}(x)]$

• REWRITE AS

$$(n+1)2xP'_n(x) + n2xP'_{n+1}(x) = (n+1)P'_{n+1}(x) + nP'_{n+2}(x) \Rightarrow$$

$$(1-2x^2)P'_{n+1}(x) = nP'_{n+1}(x) - nxP'_{n+2}(x) \Rightarrow$$

• Tidy so the equations aren't identical R.H.S.

$$(n+1)2xP'_n(x) - (n+1)P'_{n+1}(x) = nP'_{n+1}(x) - nxP'_{n+2}(x) \Rightarrow$$

$$(1-2x^2)P'_{n+1}(x) = nP'_{n+1}(x) - nxP'_{n+2}(x) \Rightarrow$$

$$\therefore (1-2x^2)P'_{n+1}(x) = (n+1)2xP'_n(x) - (n+1)nP'_{n+2}(x) \Rightarrow$$

$$(1-2x^2)P'_{n+1}(x) = (n+1)[2xP'_n(x) - nP'_{n+2}(x)]$$

AS EXPANDED

Question 21

Find one series solution for the Legendre's equation

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0, \quad n \in \mathbb{R},$$

about $x=1$.

$$y = A \sum_{r=0}^{\infty} \left[\frac{(n+r)!}{(n-r)!} \times \frac{1}{(r!)^2} \times \left(\frac{x-1}{2} \right)^{2r} \right]$$

$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$

• USE A SUBSTITUTION $t = x-1 \Rightarrow$ DEPENDS UNCHANGED
 $(1-(t+1)^2) \frac{d^2y}{dt^2} - 2(t+1) \frac{dy}{dt} + n(n+1)y = 0$

$- (t^2+2t) \frac{d^2y}{dt^2} - 2(t+1) \frac{dy}{dt} + n(n+1)y = 0$

$\frac{\frac{d^2y}{dt^2}}{t^2+2t} + \frac{2(t+1)}{t(t+2)} - \frac{n(n+1)}{t(t+1)} y = 0$ MULTIPLY BY -1

\uparrow

SUM RULES AT TWO, SO EXPAND BY
REARRANGE, & CHANGE BACK TO $x-1$
AFFORDS

• ASSUME A SOLUTION OF THE FORM $y = \sum_{r=0}^{\infty} a_r t^{r+c}$, $a_r \neq 0, c \in \mathbb{R}$

$\frac{dy}{dt} = \sum_{r=0}^{\infty} r a_r t^{r+c-1}$

$\frac{d^2y}{dt^2} = \sum_{r=0}^{\infty} r(r-1) a_r t^{r+c-2}$

• SOLVE THE O.D.E. (NOTE WE MULTIPLIED BY -1)
 $\Rightarrow (t^2+2t)(\sum_{r=0}^{\infty} a_r (r+c)(r+c-1)t^{r+c-2}) + (2t+2)\sum_{r=0}^{\infty} a_r (r+c)t^{r+c-1} - n(n+1)\sum_{r=0}^{\infty} a_r t^{r+c}$

$\Rightarrow \sum_{r=0}^{\infty} a_r (r+c)(r+c-1)t^{r+c} + \sum_{r=0}^{\infty} 2a_r (r+c)(r+c-1)t^{r+c-1} + \sum_{r=0}^{\infty} 2a_r (r+c)t^{r+c}$

$+ \sum_{r=0}^{\infty} 2a_r (r+c)t^{r+c-1} - n(n+1) \sum_{r=0}^{\infty} a_r t^{r+c}$

• WORKING FOR THE LOWEST POWER OF t IS t^{-6} & THE HIGHEST IS t^6
 PULL THE LOWEST POWER OF t OUT OF THE SUMMATIONS.

$$\Rightarrow [2a_0(c(c-1)) + 2a_0c]t^{-5} + \sum_{r=0}^{\infty} a_r (r(c-1))t^{r-4}$$

$$+ \frac{2}{2!} [2a_1(c(c-1))t^{r-3}] + \sum_{r=0}^{\infty} a_r (r(c-1))t^{r-2}$$

$$+ \frac{2}{3!} [2a_2(c(c-1))t^{r-1}] + \sum_{r=0}^{\infty} a_r (r(c-1))t^r$$

$$- n(n+1) \sum_{r=0}^{\infty} a_r t^{r+1} = 0$$

• INDICIAL EQUATION $2a_0[c^2 - c + c]t^{-4} = 0$

$c = 0 \quad a_0 \neq 0$

$c = 0 \quad (\text{CARRIES})$

• ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$\sum_{r=0}^{\infty} [a_r (r(c-1)) + 2a_r (r(c-1))t + 2a_r (r(c-1))t^2 + 2a_r (r(c-1))t^3 + 2a_r (r(c-1))t^4]t^r = 0$

THUS

$\Rightarrow a_r [(r+c)(r+c-1) + 2(r+c) - n(n+1)] = -[2(r+c)(r+c+1) + 2(r+c+1)] a_{r+1}$

$\Rightarrow a_{r+1} = -\frac{(r+c)(r+c-1) + 2(r+c) - n(n+1)}{2(r+c)(r+c+1) + 2(r+c+1)} a_r$

$\Rightarrow a_{r+1} = -\frac{(r+c)(r+c-1+2) - n(n+1)}{2(r+c)(r+c+1)} a_r$

So

$$a_{r+1} = -\frac{(r+c)(r+c+1) - n(n+1)}{2(r+c)(r+c+1)^2} a_r$$

• IF $c=0$ THIS RELATION BECOMES

$$a_{r+1} = -\frac{r(r+1) - n(n+1)}{2(n+1)^2} a_r$$

$$a_{r+1} = \frac{(r-r)(r+r+1)}{2(r+r+1)^2} a_r$$

$\begin{cases} \text{NOTE: } \\ = r(r+1) - n(n+1) \\ = r^2 - r^2 - r - n \\ = (r-n)(r+n+1) \\ = -(n-r)(r+n+1) \end{cases}$

• E.g. $a_1 = \frac{n(n+1)}{2(n+1)^2} a_0$

E.g. $a_2 = \frac{(n-1)(n+2)}{2 \times 2^2} a_1 = \frac{n(n-1)(n+2)(n+3)}{2^3 \times (n+1)^2} a_0$

E.g. $a_3 = \frac{(n-2)(n+3)}{2 \times 3^2} a_2 = \frac{(n-2)(n-1)(n)(n+1)(n+2)(n+3)}{2^4 \times (n+2)^2} a_0$

E.g. $a_4 = \frac{(n-3)(n+4)}{2 \times 4^2} a_3 = \frac{(n-3)(n-2)(n-1)n(n+1)(n+2)(n+3)(n+4)}{2^5 \times (n+3)^2} a_0$

$= \frac{(n-4)!}{(n-4)!} = \frac{\Gamma(n+5)}{\Gamma(n-3)}$

SO THE $\frac{1}{k!}$ TERM WILL BE

$$a_k = \frac{(n+k)!}{(n-k)!} \times \frac{a_0}{2^k (k+1)^2}$$

• THUS

$$y = \sum_{r=0}^{\infty} a_r t^{r+c}$$

$$y = \sum_{r=0}^{\infty} \left[\frac{(n+r)!}{(n-r)!} \times \frac{a_0 t^{r+c}}{(r+1)^2} \right]$$

$$y = a_0 \sum_{r=0}^{\infty} \frac{(n+r)!}{(n-r)!} \times \frac{1}{(r+1)^2} \times \left(\frac{t}{2} \right)^{r+c}$$

• INVERTING BACK INTO x , WE OBTAIN ONE SOLUTION

$$y = A \sum_{r=0}^{\infty} \left[\frac{(n+r)!}{(n-r)!} \times \frac{1}{(r+1)^2} \times \left(\frac{x-1}{2} \right)^{r+c} \right]$$