

SERIES

EXAM QUESTIONS

Question 1 ()**

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3}.$

b) $\sum_{r=1}^{\infty} \frac{1}{2^r + 2^r}.$

, divergent , convergent

$$\begin{aligned} \text{a)} \quad \sum_{n=1}^{\infty} \left[\frac{(n+1)^2}{n^3} \right] &= \sum_{n=1}^{\infty} \left[\frac{n^2 + 2n + 1}{n^3} \right] = \sum_{k=1}^{\infty} \left[\frac{1}{k} + \frac{2}{k^2} + \frac{1}{k^3} \right] \\ &= \sum_{k=1}^{\infty} \left[\frac{1}{k} \right] + 2 \sum_{k=1}^{\infty} \left[\frac{1}{k^2} \right] + \sum_{k=1}^{\infty} \left[\frac{1}{k^3} \right] \\ &\quad \uparrow \text{HARMONIC SERIES DIVERGES} \\ &\therefore \sum_{k=1}^{\infty} \left[\frac{1}{k} \right] \text{ DIVERGES} \end{aligned}$$

$$\text{b)} \quad \sum_{r=1}^{\infty} \left[\frac{1}{2^r + 2^r} \right] < \sum_{r=1}^{\infty} \left[\frac{1}{2^r} \right] = \dots \text{CONVERGES G.P. WITH } r=1 \\ < 1$$

AS THIS IS A SERIES OF POSITIVE TERMS, BECAUSE ABOVE THE PARENTHESIS CONVERGES

$$\therefore \sum_{r=1}^{\infty} \left[\frac{1}{2^r + 2^r} \right] \text{ CONVERGES}$$

Question 2 ()**

Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} k \left(\frac{1}{2} \right)^k.$$

Show a full method, justifying every step in the workings.

convergent

$$\begin{aligned} \sum_{k=1}^{\infty} k \left(\frac{1}{2} \right)^k &= \dots \text{BY THE RATIO TEST} \\ \lim_{k \rightarrow \infty} \left| \frac{U_{k+1}}{U_k} \right| &= \lim_{k \rightarrow \infty} \left| \frac{(k+1) \left(\frac{1}{2} \right)^{k+1}}{k \left(\frac{1}{2} \right)^k} \right| = \lim_{k \rightarrow \infty} \left[\frac{(k+1)}{k} \times \frac{1}{2} \right] = \frac{1}{2} < 1 \\ \therefore \sum_{k=1}^{\infty} k \left(\frac{1}{2} \right)^k &\text{ CONVERGES BY THE RATIO TEST} \end{aligned}$$

Question 3 ()**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{r=3}^{\infty} \frac{\sqrt{r}}{r-2}$.

b) $\sum_{k=1}^{\infty} \frac{1}{(k^4 + 2)\sqrt{k}}$.

[divergent], [convergent]

(a) $\sum_{r=3}^{\infty} \frac{\sqrt{r}}{r-2} > \sum_{r=3}^{\infty} \frac{\sqrt{r}}{r} = \sum_{r=3}^{\infty} \frac{1}{\sqrt{r}}$ WHICH DIVERGES BY THE P-TEST
 $\therefore \sum_{r=3}^{\infty} \frac{\sqrt{r}}{r-2}$ IS DIVERGENT BY COMPARISON

(b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^4+2}\sqrt{k}} < \sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^4}} = \sum_{k=1}^{\infty} \frac{1}{k^{1.5}}$ WHICH CONVERGES BY THE P-TEST
 $\therefore \sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^4+2}\sqrt{k}}$ IS CONVERGENT BY COMPARISON

P-TEST $\sum_{n=1}^{\infty} \frac{1}{n^p} <$ CONVERGES IF $p > 1$
 DIVERGES IF $p \leq 1$

Question 4 ()**

Determine whether the following series converges or diverges.

$$\sum_{r=1}^{\infty} \frac{1}{r^2 + 4r}$$

Show a full method, justifying every step in the workings.

[convergent]

$\sum_{r=1}^{\infty} \left(\frac{1}{r^2 + 4r} \right) < \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$
 $\therefore \sum_{r=1}^{\infty} \frac{1}{r^2 + 4r}$ CONVERGES BY COMPARISON

Question 5 ()**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{r=1}^{\infty} \frac{3r^2}{r^4+2}$.

b) $\sum_{k=1}^{\infty} \left[\frac{\cos^6\left(\frac{\pi}{k}\right)}{6^k} \right]$.

[convergent], [convergent]

(a) $\sum_{r=1}^{\infty} \frac{3r^2}{r^4+2} < \sum_{r=1}^{\infty} \frac{3r^2}{r^4} = 3 \sum_{r=1}^{\infty} \frac{1}{r^2}$ which converges to $3 \times \frac{\pi^2}{6}$
 $\therefore \sum_{r=1}^{\infty} \frac{3r^2}{r^4+2}$ is convergent ✓

(b) $\sum_{k=1}^{\infty} \frac{\cos^6\left(\frac{\pi}{k}\right)}{6^k} < \sum_{k=1}^{\infty} \frac{1}{6^k} = \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \dots = \frac{\frac{1}{6}}{1-\frac{1}{6}} = \frac{1}{5}$
 $\therefore \sum_{k=1}^{\infty} \frac{\cos^6\left(\frac{\pi}{k}\right)}{6^k}$ is bounded

Question 6 ()**

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

Show a full method, justifying every step in the workings.

[convergent]

$\sum_{k=1}^{\infty} \frac{2^k}{k!} = \dots$ BY THE RATIO TEST
 $\lim_{k \rightarrow \infty} \left| \frac{U_{k+1}}{U_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}} \right| = \lim_{k \rightarrow \infty} \left[\frac{2^{k+1}}{2^k} \times \frac{k!}{(k+1)!} \right]$
 $= \lim_{k \rightarrow \infty} \left[2 \times \frac{1}{k+1} \right] = 0 < 1$
 $\therefore \sum_{k=1}^{\infty} \frac{2^k}{k!}$ converges BY THE RATIO TEST

Question 7 ()**

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n.$$

Show a full method, justifying every step in the workings,

convergent

$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n$ BY THE RATIO TEST

$$\lim_{n \rightarrow \infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 \left(\frac{1}{2}\right)^{n+1}}{n^2 \left(\frac{1}{2}\right)^n} \right| = \lim_{n \rightarrow \infty} \left(\left(\frac{n+1}{n}\right)^2 \times \frac{1}{2} \right)$$

$$= \frac{1}{2} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^2 = \frac{1}{2} < 1$$

$\therefore \sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n$ CONVERGES BY THE RATIO TEST

Question 8 ()**

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}.$$

Show a full method, justifying every step in the workings.

divergent

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} > \sum_{n=1}^{\infty} \frac{n^n}{n^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ WHICH DIVERGES}$$

$\therefore \sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$ DIVERGES BY COMPARISON

Question 9 (**)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^2}.$

b) $\sum_{k=1}^{\infty} \frac{\sin k}{k(k+1)}.$

c) $\sum_{r=1}^{\infty} \frac{1}{r} \left(\frac{2}{3}\right)^r.$

[divergent], [convergent], [convergent]

(a) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^2} = \sum_{n=1}^{\infty} 1 + \frac{1}{n^2} = \sum_{n=1}^{\infty} 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} \approx \dots$ Diverges
 ↑
 Diverges to $\frac{\pi^2}{6}$
 ↑
 Converges to $\frac{\pi^2}{6}$

(b) $\sum_{k=1}^{\infty} \frac{\sin k}{k(k+1)} < \sum_{k=1}^{\infty} \frac{| \sin k |}{k(k+1)} < \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k^2+1} < \sum_{k=1}^{\infty} \frac{1}{k^2}$
 ↑
 Converges to $\frac{\pi^2}{6}$

(c) $\sum_{r=1}^{\infty} \left(\frac{2}{3}\right)^r < \sum_{r=1}^{\infty} \left(\frac{3}{2}\right)^r = \frac{\frac{3}{2}}{1-\frac{3}{2}} = 2.$
 ↑
 Converges

Question 10 ()**

Investigate the convergence or divergence of each of the following series justifying every step in the workings.

a) $\sum_{k=1}^{\infty} \left[\frac{\sqrt{k}}{k^2 + 4k + 1} \right].$

b) $\sum_{n=1}^{\infty} \left[\frac{3^n + 2}{2^n + 3} \right].$

, convergent, divergent

q) ATTEMPTING TO SHOW CONVERGENCE BY COMPARISON AS THE USE HAVE $\sum_{k=1}^{\infty} \left[\frac{1}{k^2} \right]$

$$\sum_{k=1}^{\infty} \left[\frac{\sqrt{k}}{k^2 + 4k + 1} \right] < \sum_{k=1}^{\infty} \left[\frac{\sqrt{k^2}}{k^2} \right] = \sum_{k=1}^{\infty} \left[\frac{1}{k} \right]$$

WHICH CONVERGES BY THE "P-TEST"

$$\sum_{n=1}^{\infty} \left[\frac{1}{n^2} \right] \begin{cases} \text{CONVERGES IF } p > 1 \\ \text{DIVERGES IF } p \leq 1 \end{cases}$$

b) BY COMPARISON WE HAVE

$$\sum_{k=1}^{\infty} \left[\frac{3^k + 2}{2^k + 3} \right] < \sum_{k=1}^{\infty} \left[\frac{3^k + 2}{2^k} \right] = \sum_{k=1}^{\infty} \left[\frac{3^k}{2^k} + \frac{2}{2^k} \right]$$

$$= \sum_{k=1}^{\infty} \left[\left(\frac{3}{2} \right)^k \right] + 2 \sum_{k=1}^{\infty} \left[\frac{1}{2^k} \right]$$

↑ DIVERGENT G.P. ↑ CONVERGENT G.P.

$$\therefore \sum_{k=1}^{\infty} \left[\frac{3^k + 2}{2^k + 3} \right] \text{ DIVERGES}$$

ALTERNATIVE

$$\lim_{k \rightarrow \infty} \left[\frac{3^k + 2}{2^k + 2} \right] = \lim_{k \rightarrow \infty} \left[\frac{\left(\frac{3}{2} \right)^k + \frac{2}{3^k}}{1 + \frac{2}{3^k}} \right] \sim \left(\frac{3}{2} \right)^k \text{ as } k \rightarrow \infty$$

(A SIMPLER WAY IS TO EXAMINE THE NECESSARY (BUT NOT SUFFICIENT) CONDITION
IS THAT THE ABOVE LIMIT IS ≥ 1 , WHICH IS NOT THE CASE HERE)

Question 11 ()**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2 + 1}.$$

divergent

<p><u>METHOD A</u> $u_n = \frac{(n+1)^2}{n^2 + 1} > \frac{(n+1)^2}{n^2 + 2n + 1} = \frac{(n+1)^2}{(n+1)^2} = 1$</p> <p>thus $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2 + 1} > \sum_{n=1}^{\infty} 1 = \infty$ WHICH IS UNBOUNDED AS AN ARITHMETIC SERIES</p> <p>$\therefore \sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2 + 1}$ DIVERGES BY THE COMPARISON TEST</p>
<p><u>METHOD B</u> $u_n = \frac{(n+1)^2}{n^2 + 1} = \frac{n^2 + 2n + 1}{n^2 + 1} = \frac{n^2 + 1}{n^2 + 1} + \frac{2n}{n^2 + 1} = 1 + \frac{2n}{n^2 + 1}$</p> <p>$\therefore \sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2 + 1} = \sum_{n=1}^{\infty} \left(1 + \frac{2n}{n^2 + 1}\right) = \sum_{n=1}^{\infty} 1 + \sum_{n=1}^{\infty} \frac{2n}{n^2 + 1}$</p> <p>which diverges as $\sum_{n=1}^{\infty} 1$ is not bounded</p>

Question 12 (+)**

$$u_r = \ln\left(\frac{r}{r+1}\right), r \in \mathbb{N}.$$

Show clearly that $\sum_{r=1}^{\infty} u_r$ is divergent.

proof

$\begin{aligned} \sum_{r=1}^{\infty} \ln\left(\frac{r}{r+1}\right) &= \lim_{k \rightarrow \infty} \left[\sum_{r=1}^k \ln\left(\frac{r}{r+1}\right) \right] = \lim_{k \rightarrow \infty} \left[\ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \dots + \ln\left(\frac{k}{k+1}\right) \right] \\ &= \lim_{k \rightarrow \infty} \left[\ln\left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \dots \cdot \frac{k}{k+1}\right) \right] \\ &= \lim_{k \rightarrow \infty} \left[\ln\left(\frac{1}{k+1}\right) \right] = -\lim_{k \rightarrow \infty} \left[\ln(k+1) \right] \text{ WHICH IS UNBOUNDED} \end{aligned}$ <p>$\therefore \sum_{r=1}^{\infty} u_r$ IS DIVERGENT</p>

Question 13 (+)**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=2}^{\infty} \frac{(n-1)^2}{n^2 - 1}$$

divergent

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{(k-1)^2}{k^2 - 1} &= \sum_{k=2}^{\infty} \frac{k^2 - 2k + 1}{k^2 - 1} > \sum_{k=2}^{\infty} \frac{k^2 - 2k + 1}{k^2 + 1} = \sum_{k=2}^{\infty} \left(\frac{k^2 + 1}{k^2 + 1} - \frac{2k}{k^2 + 1} \right) \\ &= \sum_{k=2}^{\infty} \left(1 - \frac{2k}{k^2 + 1} \right) = \sum_{k=2}^{\infty} 1 - \sum_{k=2}^{\infty} \frac{2k}{k^2 + 1} \end{aligned}$$

↑
which diverges as the first series is an arithmetic progression

Question 14 (*)**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \left[\frac{\sin^2 n}{n(n+1)} \right]$.

b) $\sum_{n=1}^{\infty} \left[\frac{2n}{3n^2 - 4} \right]$.

convergent, divergent

$$\begin{aligned} \text{(a)} \quad u_n &= \frac{\sin^2 n}{n(n+1)} < \frac{1}{n^2 + n} < \frac{1}{n^2}. \\ \text{Now } \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6} \text{ is a converges (standard result).} \\ \therefore \sum_{n=1}^{\infty} \frac{\sin^2 n}{n(n+1)} &\text{ converges by the comparison test.} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad u_n &= \frac{2n}{3n^2 - 4} \approx \frac{2}{3n - \frac{4}{n}} > \frac{2}{3n} = \frac{2}{3} \left(\frac{1}{n} \right) \\ \text{Now } \sum_{n=1}^{\infty} \frac{2}{3n} &= \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges (harmonic series).} \\ \therefore \sum_{n=1}^{\infty} \frac{2n}{3n^2 - 4} &\text{ diverges by the comparison test.} \end{aligned}$$

Question 15 (***)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$.

b) $\sum_{r=1}^{\infty} \frac{\cos r}{2(2^{r-1} + 1)}$.

c) $\sum_{k=1}^{\infty} \frac{k}{k+1}$.

[convergent], [convergent], [divergent]

$\text{(a)} \quad \sum_{k=1}^{\infty} \frac{k}{k^3 + 1} < \sum_{k=1}^{\infty} \frac{k}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \frac{\pi^2}{6}$ <p style="text-align: center;">↓ converges</p>
$\text{(b)} \quad \sum_{r=1}^{\infty} \frac{ \cos r }{2(2^{r-1} + 1)} < \sum_{r=1}^{\infty} \frac{1}{2(2^{r-1} + 1)} < \sum_{r=1}^{\infty} \frac{1}{2^r} = 1$ <p style="text-align: center;">↓ converges (using G-P)</p>
$\text{(c)} \quad \sum_{k=1}^{\infty} \frac{k}{k+1} = \sum_{k=1}^{\infty} \frac{k+1-1}{k+1} = \sum_{k=1}^{\infty} 1 - \frac{1}{k+1} = \sum_{k=1}^{\infty} 1 - \frac{1}{k+1}$ $= \sum_{k=1}^{\infty} 1 = \sum_{k=2}^{\infty} \frac{1}{k} \quad \text{initial terms both removed}$ <p style="text-align: center;">↓ Diverges</p>

Question 16 (***)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n(n+1)^2}.$$

Show a full method, justifying every step in the workings.

convergent

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n(n+1)^2} &\leq \sum_{n=1}^{\infty} \frac{4^{n+1}}{4^n(n+1)^2} = \sum_{n=1}^{\infty} \frac{4 \times 4^n}{4^n(n+1)^2} = \sum_{n=1}^{\infty} \frac{4}{(n+1)^2} \\ &< 4 \sum_{n=1}^{\infty} \frac{1}{n^2} = 4 \times \frac{\pi^2}{6} = \frac{2\pi^2}{3}. \end{aligned}$$

* IT CONVERGES BY COMPARISON

Question 17 (***)

Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{(k \sin k)^2}{2^k}.$$

Show a full method, justifying every step in the workings,

convergent

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(k \sin k)^2}{2^k} &= \sum_{k=1}^{\infty} \frac{k^2 \sin^2 k}{2^k} < \sum_{k=1}^{\infty} \frac{k^2}{2^k} \\ &\stackrel{\text{BY THE RATIO TEST}}{=} \lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\left(\frac{(k+1) \sin(k+1)}{2^{k+1}}\right)^2}{\left(\frac{k \sin k}{2^k}\right)^2} \right| \\ &= \lim_{k \rightarrow \infty} \left(\frac{(k+1)^2}{k^2} \cdot \frac{1}{2^{2k}} \right) = \lim_{k \rightarrow \infty} \left(\left(\frac{k+1}{k}\right)^2 \cdot \frac{1}{2^{2k}} \right) \\ &= \frac{1}{2} < 1. \end{aligned}$$

* SERIES CONVERGES BY COMPARISON TEST FOLLOWED BY THE RATIO TEST

Question 18 (*)**

Evaluate showing clearly your method

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$$

5
2

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1+2^n}{3^n} &= \sum_{n=1}^{\infty} \left(\frac{1}{3^n} + \frac{2^n}{3^n} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n + (2)^n \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n + \sum_{n=1}^{\infty} (2)^n \dots \text{this is an arithGeo sum} \\ &= \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots \right) + \left(\frac{2}{3} + \frac{4}{3^2} + \frac{8}{3^3} + \dots \right) \\ &= \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{1}{2} + 2 = \frac{5}{2} // \end{aligned}$$

Question 19 (*)**

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5^n (n!)^2}{(2n)!}$$

Show a full method, justifying every step in the workings.

divergent

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{5^n (n!)^2}{(2n)!} &= \dots \text{BY THE RATIO TEST} \\ \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{5^{n+1} ((n+1)!)^2}{(2(n+1))!}}{\frac{5^n (n!)^2}{(2n)!}} \right| = \lim_{n \rightarrow \infty} \left[\frac{\frac{5^{n+1} ((n+1)!)^2}{(2(n+1))!}}{\frac{5^n (n!)^2}{(2n)!}} \times \frac{(2n)!}{(2(n+1))!} \right] \\ &= \lim_{n \rightarrow \infty} \left[5 \times \frac{(n+1)^2}{(2n+1)(2n+2)} \times \frac{1}{(2n+1)} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{5n^2 + 10n + 5}{4n^2 + 6n + 2} \right] = \frac{5}{4} > 0 \\ \text{So series DIVERGES} // \end{aligned}$$

Question 20 (***)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{n-2}{2n^2(n+2)}.$

b) $\sum_{k=1}^{\infty} \frac{2}{\sqrt{k^2+k}}.$

[convergent], [divergent]

(a) $\sum_{n=1}^{\infty} \frac{n-2}{2n^2(n+2)} < \sum_{n=1}^{\infty} \frac{n-2}{2n^2 \cdot n} = \sum_{n=1}^{\infty} \frac{n-2}{2n^3} = \sum_{n=1}^{\infty} \frac{1}{2n^2} - \frac{1}{n^3}$

 $= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{n^3}$ AND BOTH SERIES CONVERGE BY THE P-TEST
 $\therefore \sum_{n=1}^{\infty} \frac{n-2}{2n^2(n+2)}$ IS CONVERGENT BY COMPARISON

(b) $\sum_{k=1}^{\infty} \frac{2}{\sqrt{k^2+k}} > \sum_{k=1}^{\infty} \frac{2}{\sqrt{4k^2+4k+4}} = \sum_{k=1}^{\infty} \frac{2}{\sqrt{4(k+1)^2}}$
 $= \sum_{k=1}^{\infty} \frac{2}{2\sqrt{(k+1)^2}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{(k+1)^2}} = \sum_{k=1}^{\infty} \frac{1}{k+1}$
 $= \sum_{k=2}^{\infty} \frac{1}{k}$ WHICH DIVERGES BY THE P-TEST
 $\therefore \sum_{k=1}^{\infty} \frac{2}{\sqrt{k^2+k}}$ DIVERGES
 CP TEST: $\frac{\infty}{\infty} \stackrel{H}{\rightarrow} 0 <$ DIVIDES IF ∞

Question 21 (***)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$.

b) $\sum_{k=1}^{\infty} \frac{k^4}{(k+1)^6}$.

c) $\sum_{r=1}^{\infty} \frac{(r+1)(2r+1)(3r+1)}{r^4}$.

, convergent , , , divergent

q) $\sum_{k=1}^{\infty} \frac{10^k}{k!} = \dots$ BY THE RATIO TEST
 AS ALL THE TERMS ARE POSITIVE WE MAY IGNORE NEGATIVE IN THE TEST

$$\lim_{k \rightarrow \infty} \left[\frac{a_{k+1}}{a_k} \right] = \lim_{k \rightarrow \infty} \left[\frac{10^{k+1}}{(k+1)!} \cdot \frac{k!}{10^k} \right] = \lim_{k \rightarrow \infty} \left[\frac{10}{(k+1)} \cdot \frac{1}{10^k} \right] = 0 < 1$$

SERIES CONVERGES BY THE RATIO TEST

b) $\sum_{k=1}^{\infty} \frac{k^4}{(k+1)^6} < \sum_{k=1}^{\infty} \frac{k^4}{k^6} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

SERIES CONVERGES BY COMPARISON

c) $\sum_{r=1}^{\infty} \frac{(r+1)(2r+1)(3r+1)}{r^4} > \sum_{r=1}^{\infty} \frac{r \times 2r \times 3r}{r^4} = \sum_{r=1}^{\infty} \frac{6r^2}{r^4} = 6 \sum_{r=1}^{\infty} \frac{1}{r^2}$ WHICH DIVERGES

SERIES DIVERGES BY COMPARISON

Question 22 (***)

Determine whether the following series converges or diverges.

$$\sum_{t=1}^{\infty} \frac{(t!)^3}{(3t)!}.$$

Show a full method, justifying every step in the workings.

convergent

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{(t!)^3}{(3t)!} &= \dots \text{ BY THE RATIO TEST} \\ \lim_{t \rightarrow \infty} \left| \frac{U_{t+1}}{U_t} \right| &= \lim_{t \rightarrow \infty} \left| \frac{\frac{(t+1!)^3}{(3(t+1))!}}{\frac{(t!)^3}{(3t)!}} \right| = \lim_{t \rightarrow \infty} \left[\frac{(t+1)!}{(3t+3)!} \times \frac{(t!)^3}{(3t)!} \right] \\ &= \lim_{t \rightarrow \infty} \left[\frac{1}{(3t+3)(3t+2)(3t+1)} \times (t+1)^3 \right] \\ &= \lim_{t \rightarrow \infty} \left[\frac{(t+1)^3}{(3t+3)(3t+2)(3t+1)} \right] = \frac{1}{27} < 1 \end{aligned}$$

∴ SERIES CONVERGES BY THE RATIO TEST

Question 23 (***)

The sum of the first n terms of an arithmetic series with first term a and common difference d , is denoted by S_n .

Simplify fully

$$S_n - 2S_{n+1} + S_{n+2}.$$

$$S_n - 2S_{n+1} + S_{n+2} = d$$

$$\begin{aligned} S_{n+2} - S_{n+1} &= U_{n+2} = a + (n+1)d = a + nd + d \\ S_{n+1} - S_n &= U_{n+1} = a + nd = a + nd \end{aligned}$$

SUBTRACT SIDE BY SIDE

$$S_{n+2} - 2S_{n+1} + S_n = d$$

Question 24 (*)**

It is given that

$$\frac{1}{n} \sum_{r=1}^n x_r = 2 \quad \text{and} \quad \sqrt{\frac{1}{n} \sum_{r=1}^n (x_r)^2 - \frac{1}{n^2} \left(\sum_{r=1}^n x_r \right)^2} = 3.$$

Determine, in terms of n , the value of

$$\sum_{r=1}^n (x_r + 1)^2.$$

$$\boxed{\quad}, \quad \sum_{r=1}^n (x_r + 1)^2 = 18n$$

$$\begin{aligned}
 \frac{\sum_{r=1}^n x_r}{n} &= 2 \\
 \Rightarrow \sum_{r=1}^n x_r &= 2n
 \end{aligned}$$

$$\begin{aligned}
 \sqrt{\frac{\sum_{r=1}^n (x_r)^2}{n} - \left(\frac{\sum_{r=1}^n x_r}{n} \right)^2} &= 3 \\
 \Rightarrow \frac{\sum_{r=1}^n (x_r)^2}{n} - \left(\frac{\sum_{r=1}^n x_r}{n} \right)^2 &= 9 \\
 \Rightarrow \frac{\sum_{r=1}^n (x_r)^2}{n} - \frac{(2n)^2}{n^2} &= 9 \\
 \Rightarrow \frac{\sum_{r=1}^n (x_r)^2}{n} - 4 &= 9 \\
 \Rightarrow \sum_{r=1}^n (x_r)^2 &= 13n
 \end{aligned}$$

• Thus

$$\begin{aligned}
 \sum_{r=1}^n (x_r + 1)^2 &= \sum_{r=1}^n [(x_r)^2 + 2x_r + 1] \\
 &= \sum_{r=1}^n (x_r)^2 + 2 \sum_{r=1}^n x_r + \sum_{r=1}^n 1 \\
 &= 13n + 2(2n) + n \\
 &= 18n
 \end{aligned}$$

Question 25 (*)**

It is given that

$$\sum_{r=1}^{20} [f(r) - 10] = 200 \quad \text{and} \quad \sum_{r=1}^{20} [f(r) - 10]^2 = 2800.$$

Find the value of

$$\sum_{r=1}^{20} [f(r)]^2.$$

$$\boxed{\quad}, \quad \boxed{\sum_{r=1}^{20} [f(r)]^2 = 8800}$$

MANIPULATE THE SUMS AS FOLLOWS

$$\begin{aligned} \sum_{r=1}^{20} (f(r) - 10) &= \left[\sum_{r=1}^{20} f(r) \right] - \sum_{r=1}^{20} 10 \\ 200 &= \left[\sum_{r=1}^{20} f(r) \right] - 10 \times 20 \\ \sum_{r=1}^{20} f(r) &= 400 \end{aligned}$$

NEXT WE TAKE

$$\begin{aligned} \sum_{r=1}^{20} (f(r) - 10)^2 &= \sum_{r=1}^{20} [f(r)]^2 - 20f(r) + 100 \\ 2800 &= \sum_{r=1}^{20} [f(r)]^2 - 20 \sum_{r=1}^{20} f(r) + 100 \sum_{r=1}^{20} 1 \\ 2800 &= \sum_{r=1}^{20} [f(r)]^2 - 20 \times 400 + 100 \times 20 \\ 2800 &= \sum_{r=1}^{20} [f(r)]^2 - 6000 \\ \sum_{r=1}^{20} [f(r)]^2 &= 8800 \end{aligned}$$

Question 26 (*)**

It is given that the following series converges.

$$\sum_{n=1}^{\infty} \frac{(5x)^n}{4n^2}, \quad x \in \mathbb{R}, \quad x > 0.$$

Determine the range of possible values of x .

, $0 < x < \frac{1}{5}$

- This can be done by the ratio test.
- The n th term of the series is given by $u_n = \frac{(5x)^n}{4n^2}$.
- By the ratio test (ignoring moduli as the terms are positive).

$$\frac{u_{n+1}}{u_n} = \frac{(5x)^{n+1}}{4(n+1)^2} \times \frac{4n^2}{(5x)^n} = \frac{5x}{(n+1)^2}$$

$$= \frac{5x}{n^2} \times \frac{n^2}{n^2 + 2n + 1} = \frac{5x}{1 + \frac{2}{n} + \frac{1}{n^2}}$$
- This will converge if

$$\rightarrow \frac{u_{n+1}}{u_n} \rightarrow L, \quad 0 \leq L < 1, \quad \text{as } n \rightarrow \infty$$

$$\rightarrow 5x \rightarrow L, \quad 0 \leq L < 1$$

$$(\text{since } 1 + \frac{2}{n} + \frac{1}{n^2} \rightarrow 1, \text{ as } n \rightarrow \infty)$$
- $0 \leq 5x < 1$

$$0 < x < \frac{1}{5}$$
// ignoring the trivial case

Question 27 (***)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{n+4}{2n^2+6}.$$

divergent

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n+4}{2n^2+6} &\approx \sum_{n=1}^{2} \frac{n+4}{2n^2+6} + \sum_{n=3}^{\infty} \frac{n+4}{2n^2+6} = A + \sum_{n=3}^{\infty} \frac{n+4}{2n^2+6} \\&> A + \sum_{n=3}^{\infty} \frac{n}{2n^2+6} = A + \frac{1}{2} \sum_{n=3}^{\infty} \frac{n}{n^2+3} \\&> A + \frac{1}{2} \sum_{n=3}^{\infty} \frac{n}{n^2+3n} = A + \frac{1}{2} \sum_{n=3}^{\infty} \frac{1}{n+3} \\&= A + \frac{1}{2} \sum_{n=6}^{\infty} \frac{1}{n} \leftarrow \text{which diverges}\end{aligned}$$

$\sum_{n=1}^{\infty} \frac{n+4}{2n^2+6}$ is DIVERGENT

ALTERNATIVE

$$\sum_{n=1}^{\infty} \frac{n+4}{2n^2+6} > \sum_{n=1}^{\infty} \frac{n+4}{2n^2+16n+32} = \sum_{n=1}^{\infty} \frac{n+4}{2(n+4)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n+4}$$
$$= \frac{1}{2} \sum_{n=5}^{\infty} \frac{1}{n} \leftarrow \text{which diverges}$$

Question 28 (*)+**

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\sum_{n=1}^{\infty} \frac{e^n (n!)^2}{(2n)!}$.

b) $\sum_{t=1}^{\infty} \frac{(-1)^{t+1}}{\sqrt{t+1}}$.

[convergent], [convergent]

(2) $\sum_{n=1}^{\infty} \frac{e^n (n!)^2}{(2n)!} = \dots$ BY RATIO TEST
 $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{e^{n+1} ((n+1)!)^2}{(2n+2)!}}{\frac{e^n (n!)^2}{(2n)!}} \right| = \lim_{n \rightarrow \infty} \left[\frac{n+1}{2n+2} \cdot \frac{(2n)!}{(2n+2)!} \cdot \sqrt{n+1} \right]^2$
 $= \lim_{n \rightarrow \infty} \left[\frac{1}{2} \times \frac{1}{(2n+2)(2n+1)} \times (n+1)^2 \right] = \lim_{n \rightarrow \infty} \left[\frac{n(n+1)^2}{2(2n+1)(2n+2)} \right]$
 $= \lim_{n \rightarrow \infty} \left[\frac{n(n+1)}{2(2n+1)} \right] = \frac{1}{4} < 1$
 \therefore SERIES CONVERGES BY THE RATIO TEST

(3) $\sum_{t=1}^{\infty} \frac{(-1)^{t+1}}{\sqrt{t+1}} = \dots$ TEST ALTERNATING IN SIGN
 $\lim_{t \rightarrow \infty} \left(\frac{1}{\sqrt{t+1}} \right) = 0$
 \therefore SERIES CONVERGES BY ALTERNATING-SIGN TEST

Question 29 (***)

By using an algebraic method, find the value of

$$99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2$$

, 5000

Method A

RECOGNISE THE TRINOMIAL

$$\begin{aligned} & 99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2 \\ &= [99^2 + 95^2 + 91^2 + \dots + 3^2] - [97^2 + 93^2 + 89^2 + \dots + 1^2] \\ &= \sum_{r=1}^{25} (4r-1)^2 - \sum_{r=1}^{25} (4r-3)^2 \quad (\text{TERM IN SIGN: } \text{+---+---}) \\ &= \sum_{r=1}^{25} [(4r-1)^2 - (4r-3)^2] \quad (\text{COMMON DIFFERENCE}) \\ &= \sum_{r=1}^{25} (4r-1 + 4r-3)(4r-1 - 4r+3) \quad (\text{DIFFERENCE OF SQUARES}) \\ &= \sum_{r=1}^{25} (8r-4) \times 2 \\ &= \sum_{r=1}^{25} (8r-8) \\ &= 8 \sum_{r=1}^{25} r - 8 \sum_{r=1}^{25} 1 \quad (\text{COMMON DIFFERENCE}) \\ &= 8 \times \frac{25}{2} \times 25 \times 26 - 8 \times 25 \\ &= 5000 - 200 \\ &= 5000 \end{aligned}$$

Method B

RECOGNISE THE TRINOMIAL AS FOLLOWING

$$\begin{aligned} & 99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2 \\ &= (99^2 - 97^2) + (95^2 - 93^2) + (91^2 - 89^2) + \dots + (3^2 - 1^2) \\ &= (99-97)(99+97) + (95-93)(95+93) + (91-89)(91+89) + \dots + (3-1)(3+1) \\ &= 2(96) + 2(98) + 2(102) + \dots + 2(14) \\ &= 2[4 + 12 + 20 + \dots + 100 + 108 + 116] \\ &= 2 \times 4 \left[1 + 3 + 5 + \dots + 45 + 47 + 49 \right] \\ &\rightarrow 8 \sim \text{Arithmetic Progression with } a=1, d=2, n=25 \quad U_n = a + (n-1)d \\ & \qquad \qquad \qquad 49 = 1 + 24 \times 2, \quad 20 = 2a, \quad n=25 \\ &= 8 \times \frac{25}{2} [1+49] \\ &= 8 \times \frac{25 \times 50}{2} \\ &= 5000 \end{aligned}$$

ANSWER

Question 30 (*)+**

Evaluate, showing clearly your method

$$\sum_{n=1}^{\infty} \frac{3^n - 2}{4^{n+1}}.$$

, $\boxed{\frac{7}{12}}$

SPLIT THE SUMMATION AS FOLLOWS

$$\begin{aligned} \sum_{n=1}^{\infty} \left[\frac{3^n - 2}{4^{n+1}} \right] &= \sum_{n=1}^{\infty} \left[\frac{3^n}{4^{n+1}} - \frac{2}{4^{n+1}} \right] = \sum_{n=1}^{\infty} \left[\frac{3^n}{4 \cdot 4^n} \right] - \sum_{n=1}^{\infty} \left[\frac{2}{4 \cdot 4^n} \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{3^n}{4 \cdot 4^n} \right] - \sum_{n=2}^{\infty} \left[\frac{2}{4 \cdot 4^n} \right] \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{3}{4} \right)^n - \frac{1}{4} \sum_{n=2}^{\infty} \left(\frac{1}{4} \right)^n \end{aligned}$$

THIS IS A GP
 $a = \frac{3}{4}$
 $r = \frac{3}{4}$
 $S_{\infty} = \frac{3}{1 - \frac{3}{4}} = 3$

THIS IS A GP
 $a = \frac{1}{4}$
 $r = \frac{1}{4}$
 $S_{\infty} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$

PUTTING ALL THE PIECES TOGETHER

$$\sum_{n=1}^{\infty} \left[\frac{3^n - 2}{4^{n+1}} \right] = \frac{1}{4} \times 3 - \frac{1}{4} \times \frac{1}{3} = \frac{3}{4} - \frac{1}{12} = \frac{7}{12}$$

Question 31 (*)+**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{k=1}^{\infty} \frac{k+2}{4k^2+5}.$$

divergent

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{k+2}{4k^2+5} &> \sum_{k=1}^{\infty} \frac{k+2}{4k^2+4k+16} = \sum_{k=1}^{\infty} \frac{k+2}{4(k^2+k+4)} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{k+2}{k^2+k+4} \\ &= \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k+2}. = \frac{1}{4} \sum_{k=3}^{\infty} \frac{1}{k} \quad \text{without indices} \\ &\therefore \sum_{k=1}^{\infty} \frac{k+2}{4k^2+5} \text{ DIVERGES} \end{aligned}$$

Question 32 (*)+**

Show clearly that

$$\sum_{r=1}^n \left[2r(2r^2 - 3r - 1) + n + 1 \right] = (n+1)^2(n-1)^2.$$

[proof]

$$\begin{aligned}
 \sum_{r=1}^n \left[2r(2r^2 - 3r - 1) + n + 1 \right] &= \sum_{r=1}^n (4r^3 - 6r^2 - 2r + n + 1) \\
 &= 4 \sum_{r=1}^n r^3 - 6 \sum_{r=1}^n r^2 - 2 \sum_{r=1}^n r + \sum_{r=1}^n (n+1) \\
 &= 4 \sum_{r=1}^n r^3 - 6 \sum_{r=1}^n r^2 - 2 \sum_{r=1}^n r + (n+1) \sum_{r=1}^n 1 \\
 \text{NOW USING STANDARD RESULTS} \\
 &= 4 \times \frac{1}{4}n^2(n+1)^2 - 6 \times \frac{1}{6}n(n+1)(2n+1) - 2n \sum_{r=1}^n r(n+1) + (n+1) \times (n+1) \\
 &= n^2(n+1)^2 - n(n+1)(2n+1) - n(n+1) + (n+1)^2 \\
 &= n(n+1)[n(n+1) - (2n+1) - 1] + (n+1)^2 \\
 &= n(n+1)[n^2+n-2n-2] + (n+1)^2 \\
 &= n(n+1)(n^2-n-2) + (n+1)^2 \\
 &= n(n+1)(n+1)(n-2) + (n+1)^2 \\
 &= (n+1)^2[n(n-2)+1] \\
 &= (n+1)^2(n-1)^2 \\
 &\quad \blacksquare \text{ As required}
 \end{aligned}$$

Question 33 (***)+

Consider the infinite series

$$1 - \frac{x^2}{2^2} + \frac{x^4}{4^2 \times 2^2} - \frac{x^6}{6^2 \times 4^2 \times 2^2} + \frac{x^8}{8^2 \times 6^2 \times 4^2 \times 2^2} - \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} (n!)^2} x^{2n}$$

$$\begin{aligned} & 1 - \frac{x^2}{2^2} + \frac{x^4}{4^2 \times 2^2} - \frac{x^6}{6^2 \times 4^2 \times 2^2} + \frac{x^8}{8^2 \times 6^2 \times 4^2 \times 2^2} - \dots \\ & = 1 - \frac{x^2}{2^2(1)^2} + \frac{x^4}{2^2(2\cdot 1)^2} - \frac{x^6}{2^2(3\cdot 2\cdot 1)^2} + \frac{x^8}{2^2(4\cdot 3\cdot 2\cdot 1)^2} - \dots \\ & = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} (1\cdot 1)^2} x^{2n} \end{aligned}$$

Question 34 (***)+

Show clearly by an algebraic method that

$$40^2 - 39^2 + 38^2 - 37^2 + \dots + 2^2 - 1^2 = 820.$$

proof

$$\begin{aligned} & 40^2 - 39^2 + 38^2 - 37^2 + \dots + 2^2 - 1^2 \\ & = (40+39)(40-39) + (36+37)(36-37) + \dots + (2+1)(2-1) \\ & = 79 + 75 + 71 + \dots + 7 + 3 \\ & = 3 + 7 + \dots + 37 + 35 + 33 \\ & = \frac{25}{2} [2(3+1) + 4(4+2)] \leftarrow \frac{25}{2}[2(1+1)+2] \\ & = 10 \times (6+16) \\ & = 820 \end{aligned}$$

Algebraic Proof

Question 35 (****)

By justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt[3]{n^2}}.$$

convergent

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt[3]{n^2}} &= \sum_{n=1}^{\infty} \frac{[\sqrt{n+1} - \sqrt{n}][\sqrt{n+1} + \sqrt{n}]}{\sqrt[3]{n^2} [\sqrt{n+1} + \sqrt{n}]} \\
 &= \sum_{n=1}^{\infty} \frac{(\sqrt{n+1})^2 - n}{\sqrt[3]{n^2} [\sqrt{n+1} + \sqrt{n}]} \\
 &= \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{(n+1)^2 + n^2}} \\
 &< \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2 + n^2}} \\
 &= \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n^2}} \\
 &= \frac{1}{\sqrt[3]{2}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}
 \end{aligned}$$

which converges since

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

CONVERGES IF $p > 1$
DIVERGES IF $p \leq 1$

Question 36 (****)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n!}{n^{n+1}}.$$

Show a full method, justifying every step in the workings.

You may assume without proof the value of $\lim_{n \rightarrow \infty} \left[\left(\frac{n}{n+1} \right)^n \right]$.

convergent

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n!}{n^{n+1}} &< \sum_{n=1}^{\infty} \frac{n!}{n^n} \quad \text{NOW BY THE RATIO TEST...} \\ \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} \right| = \lim_{n \rightarrow \infty} \left[\frac{(n+1)!}{n!} \times \frac{n^n}{(n+1)^{n+1}} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{(n+1) \cdot n^n}{(n+1)^{n+1}} \right] = \lim_{n \rightarrow \infty} \left[\frac{n^n}{(n+1)^n} \right] = \lim_{n \rightarrow \infty} \left[\left(\frac{n}{n+1} \right)^n \right] \\ &= \frac{1}{e} < 1 \end{aligned}$$

∴ SERIES CONVERGES

NOTES
 $\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = e$
 $\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{-n} = \frac{1}{e}$

Question 37 (*****)

The sum of the first n terms of a series with general term u_n is given by the expression

$$S_n = n^2(n+1)(n+2).$$

a) Find the first term of the series.

b) Show clearly that ...

i. ... $u_n = n(n+1)(4n-1)$

ii. ... $\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2).$

u_1 = 6

(a) $S_n = n^2(n+1)(n+2)$
 $S_1 = U_1 = 1^2(1+1)(1+2) = 1 \times 2 \times 3 = 6$

(b) (i) $U_n = S_n - S_{n-1} = n^2(n+1)(n+2) - (n-1)^2n(n+1)$
 $= n(n+1)[n(n+2) - (n-1)^2]$
 $= n(n+1)(n^2+2n - n^2 + 2n - 1)$
 $= n(n+1)(4n-1)$ as required

(ii) $\sum_{r=n+1}^{2n} u_r = S_{2n} - S_n = [2n^2(2n+1)(2n+2) - n^2(n+1)(n+2)]$
 $= 4n^2(n+1)2(n+1) - n^2(n+1)(n+2)$
 $= 8n^2(n+1)(2n+1) - n^2(n+1)(n+2)$
 $= n^2(n+1)[8(2n+1) - (n+2)]$
 $= n^2(n+1)(15n+6)$
 $= 3n^2(n+1)(5n+2)$ as required

Question 38 (****)

Determine whether the following series converges or diverges.

$$\sum_{t=1}^{\infty} \sqrt[4]{2^t + 5^t}.$$

Show a full method, justifying every step in the workings.

divergent

$$\sum_{t=1}^{\infty} \sqrt[4]{2^t + 5^t}$$

CONSIDER A TERM IN GENERAL TERM
 $\sqrt[4]{2^t + 5^t} > \sqrt[4]{2^t + 2^t} = \sqrt[4]{2 \cdot 2^t} = \sqrt{2} \cdot \sqrt[4]{2^t} = 2 \sqrt[4]{2}$

$\Rightarrow t \rightarrow \infty$ THE GENERAL TERM DOES NOT TEND TO ZERO
 \therefore SERIES CANNOT CONVERGE

Question 39 (****)

$$\sum_{r=1}^n (ar^2 + br + c) \equiv n^3 + 5n^2 + 6n,$$

where a , b and c are integer constants.

Determine the value of a , b and c .

$a = 3$, $b = 7$, $c = 2$

$$\begin{aligned} \sum_{r=1}^n ar^2 + br + c &= a \sum_{r=1}^n r^2 + b \sum_{r=1}^n r + c \sum_{r=1}^n 1 \\ &= \frac{a}{3}n(n+1)(2n+1) + \frac{b}{2}n(n+1) + cn \\ &= \frac{1}{6}n [2an^3 + 3an^2 + 2an + 3bn^2 + bn + 6n] \\ &= \frac{1}{6}n [2an^3 + 3n(2an+1) + (a+3b+6)n] \\ &= \frac{1}{6}an^3 + \frac{1}{2}(a+b)n^2 + \frac{1}{6}(a+3b+6)n \end{aligned}$$

Now $[n(n+1)(n+2)] = n^3 + 3n^2 + 6n$] (cancel)

$$\begin{aligned} \frac{1}{6}a &= 1 & \frac{1}{2}(a+b) &= 5 & \frac{1}{6}(a+3b+6c) &= 6 \\ a &= 6 & a+b &= 10 & a+3b+6c &= 36 \\ \therefore a &= 3 & b &= 7 & 3+21+6c &= 36 \\ & & & & & c = 2 \end{aligned}$$

Question 40 (***)**

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\sum_{n=3}^{\infty} \left[\frac{\sqrt{n}}{n-2} \right].$

b) $\sum_{n=1}^{\infty} \left[\frac{\sqrt{n}}{n+2} \right].$

[divergent], [divergent]

$$\begin{aligned}
 \text{(a)} \quad & \sum_{n=3}^{\infty} \frac{\sqrt{n}}{n-2} > \sum_{n=3}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}} \text{ which DIVERGES BY THE P-TEST} \\
 \text{(b)} \quad & \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+2} = \sum_{n=1}^4 \frac{\sqrt{n}}{n+2} + \sum_{n=5}^{\infty} \frac{\sqrt{n}}{n+2} = A + \sum_{n=5}^{\infty} \frac{\sqrt{n}}{n+2} \\
 & > A + \sum_{n=5}^{\infty} \frac{\sqrt{n}}{n+4+2} = A + \sum_{n=5}^{\infty} \frac{\sqrt{n}}{(n+2)^2} \\
 & > A + \sum_{n=5}^{\infty} \frac{\sqrt{n}}{(\sqrt{n}+\sqrt{4})^2} = A + \sum_{n=5}^{\infty} \frac{1}{4n} \\
 & = A + \frac{1}{2} \sum_{n=5}^{\infty} \frac{1}{n} \text{ which DIVERGES BY THE P-TEST}
 \end{aligned}$$

DIVERGES IF P < 1
DIVERGES IF P > 1

Question 41 (*****)

Show clearly that

$$1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 = -33200.$$

[proof]

$$\begin{aligned}
 1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 &= \sum_{n=1}^{20} (2n-1)^3 - \sum_{n=1}^{20} (2n)^3 = \sum_{n=1}^{20} (2n-1)^3 - (2n)^3 \\
 &= \sum_{n=1}^{20} 8n^2 - 12n^3 + 6n^2 - 1 = 8\sum_{n=1}^{20} n^2 - 12\sum_{n=1}^{20} n^3 \\
 &= -12 \sum_{n=1}^{20} n^3 + 6 \sum_{n=1}^{20} n^2 = -12 \sum_{n=1}^{20} n^3 + 6 \times \frac{20(21)(41)}{6} \\
 &= -12 \times \frac{20(21)(41)}{6} + 6 \times \frac{20(21)(41)}{6} \\
 &= -34440 + 1260 - 20 = -33200
 \end{aligned}$$

At
RHS=0.000

Left-hand side:

$$\begin{aligned}
 1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 &= (1^3 + 2^3 + 3^3 + \dots + 40^3) - 2(2^3 + 4^3 + 6^3 + \dots + 40^3) \\
 &= \sum_{n=1}^{40} n^3 - 2 \times 2^3 \left(1^3 + 2^3 + 3^3 + \dots + 20^3 \right) \\
 &= \sum_{n=1}^{40} n^3 - 16 \sum_{n=1}^{20} n^3 \\
 &= \frac{1}{4} 40^2 41^2 - 16 \times \frac{1}{4} 20^2 21^2 \\
 &= 62400 - 70560 \\
 &= -33200
 \end{aligned}$$

Question 42 (*****)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}.$$

[divergent]

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}} &= \frac{1}{\frac{1}{2}} + \sum_{n=2}^{\infty} \frac{1}{1+\sqrt{n}} = \frac{1}{\frac{1}{2}} + \sum_{n=2}^{\infty} \frac{1-\sqrt{n}}{(1+\sqrt{n})(1-\sqrt{n})} \\
 &= \frac{1}{\frac{1}{2}} + \sum_{n=2}^{\infty} \frac{1-\sqrt{n}}{1-n} = \frac{1}{\frac{1}{2}} + \sum_{n=2}^{\infty} \frac{\sqrt{n}-1}{n-1} \\
 &> \frac{1}{\frac{1}{2}} + \sum_{n=2}^{\infty} \frac{\sqrt{n}-1}{n} = \frac{1}{\frac{1}{2}} + \sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{n} \right) \\
 &= \frac{1}{\frac{1}{2}} + \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}
 \end{aligned}$$

Beweis Intervall

$$\therefore \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}} \text{ Diverges}$$

Question 43 (**)**

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}.$

b) $\sum_{r=1}^{\infty} \frac{1}{\sqrt[3]{r^2 + 4}}.$

c) $\sum_{k=1}^{\infty} \frac{k+10}{k^2+10}.$

d) $\sum_{m=1}^{\infty} \frac{\sqrt{m+2}}{4m^2+1}$

[divergent], [divergent], [divergent], [convergent]

$$\begin{aligned}
 \text{(a)} & \sum_{n=1}^{\infty} \frac{3}{\ln(n+1)} = 3 \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)} > 3 \sum_{n=1}^{\infty} \frac{1}{n \ln n} = 3 \sum_{n=2}^{\infty} \frac{1}{n \ln n} \\
 & \text{Since } \sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ DIVERGES BY COMPARISON WITH THE HARMONIC SERIES} \\
 & \text{WHICH IS UNKNOWN TO DIVERGE} \\
 \text{(b)} & \sum_{r=1}^{\infty} \frac{1}{\sqrt[3]{r^2+4}} > \sum_{r=1}^{\infty} \frac{1}{\sqrt[3]{r^2+4r+4}} = \sum_{r=1}^{\infty} \frac{1}{\sqrt[3]{(r+2)^2}} = \sum_{r=1}^{\infty} \frac{1}{(r+2)^{\frac{2}{3}}} \\
 & = \sum_{r=2}^{\infty} \frac{1}{r^{\frac{2}{3}}} > \sum_{r=2}^{\infty} \frac{1}{r^{\frac{1}{2}}} \text{ WHICH DIVERGES} \\
 & \sum_{r=2}^{\infty} \frac{1}{\sqrt[3]{r^2+4}} \text{ DIVERGES BY COMPARISON} \\
 \text{(c)} & \sum_{k=1}^{\infty} \frac{k+10}{k^2+10} > \sum_{k=1}^{\infty} \frac{k}{k^2+10} > \sum_{k=1}^{\infty} \frac{k}{k^2+8k+16} = \sum_{k=1}^{\infty} \frac{k}{(k+4)^2} \\
 & = \sum_{k=1}^{\infty} \frac{k-4}{k^2} = \sum_{k=5}^{\infty} \frac{1}{k} - 4 \sum_{k=1}^4 \frac{1}{k^2} \\
 & \text{Simplifies} \\
 & \text{Cancels} \\
 & \sum_{k=1}^{\infty} \frac{k+10}{k^2+10} \text{ DIVERGES BY COMPARISON} \\
 \text{(d)} & \sum_{k=1}^{\infty} \frac{\sqrt{m+2}}{4m^2+1} < \sum_{k=1}^{\infty} \frac{\sqrt{m+2}}{4m^2} < \sum_{k=1}^{\infty} \frac{\sqrt{4km+16}}{4m^2} < \sum_{k=1}^{\infty} \frac{4\sqrt{m+4}}{4m^2} \\
 & = \sum_{k=1}^{\infty} \frac{4\sqrt{m+4}}{m^2} < \sum_{k=1}^{\infty} \frac{4\sqrt{2m}}{m^2} = \sum_{k=1}^{\infty} \frac{4\sqrt{2m}}{m^2} = 4\sqrt{2} \sum_{k=1}^{\infty} \frac{1}{k^2} \\
 & < \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ WHICH CONVERGES BY THE P-TEST} \\
 & \text{DIVERGES BY P-TEST} \\
 & \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ CONVERGES BY COMPARISON}
 \end{aligned}$$

Question 44 (****)

The variance $\text{Var}(n)$ of the first n natural numbers is given by

$$\text{Var}(n) = \frac{1}{n} \sum_{r=1}^n r^2 - \left[\frac{1}{n} \sum_{r=1}^n r \right]^2.$$

Determine a simplified expression $\text{Var}(n)$ and hence evaluate $\text{Var}(61)$.

$$\boxed{\text{Var}(n) = \frac{1}{12}(n^2 - 1)}, \quad \boxed{\text{Var}(61) = 310}$$

$$\begin{aligned}\text{Variance} &= \frac{\sum_{r=1}^n r^2}{n} - \left(\frac{\sum_{r=1}^n r}{n} \right)^2 \\ &= \frac{\frac{1}{2}n(n+1)(2n+1)}{n} - \left(\frac{\frac{1}{2}n(n+1)}{n} \right)^2 \\ &= \frac{1}{2}(n+1)(2n+1) - \frac{1}{4}(n+1)^2 \\ &= \frac{1}{12}(n+1)[2(2n+1) - 3(n+1)] \\ &= \frac{1}{12}(n+1)(4n+2 - 3n-3) \\ &= \frac{1}{12}(n+1)(n-1)\end{aligned}$$

If $n=61$

$$\text{Var}(61) = \frac{1}{12} \times 62 \times 60 = 62 \times 5 = 310 //$$

Question 45 (****)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$.

b) $\sum_{r=1}^{\infty} \frac{r!}{10^r}$.

c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$.

d) $\sum_{m=2}^{\infty} \frac{1}{m \ln m}$

[divergent], [divergent], [convergent], [divergent]

(a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} > \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} > \sum_{n=2}^{\infty} \frac{1}{n}$ which diverges
 $\therefore \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ is divergent by comparison

(b) $\sum_{r=1}^{\infty} \frac{r!}{10^r}$ BY THE RATIO TEST $\lim_{r \rightarrow \infty} \left| \frac{U_{r+1}}{U_r} \right| = \lim_{r \rightarrow \infty} \left| \frac{(r+1)!}{r!} \times \frac{10^r}{10^{r+1}} \right| = \lim_{r \rightarrow \infty} \left[\frac{(r+1)!}{r!} \times \frac{10^r}{10^{r+1}} \right] = \infty > 1$
 $\therefore \sum_{r=1}^{\infty} \frac{r!}{10^r}$ DIVERGES BY THE RATIO TEST

(c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

- TERMS ACCUMULATE IN SIGN.
- $\lim_{k \rightarrow \infty} \left[\frac{1}{\sqrt{k}} \right] = 0$

 \therefore SERIES DIVERGES BY THE ALTERNATING-SERIES TEST

(d) $\sum_{m=2}^{\infty} \frac{1}{m \ln m}$

- $\lim_{m \rightarrow \infty} \left[\frac{1}{m \ln m} \right] = 0$
- $\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{n \rightarrow \infty} \int_2^n \frac{1}{x \ln x} dx = \lim_{n \rightarrow \infty} [\ln(\ln x)]_2^n = \text{WHICH IS UNDEFINED}$

 $\therefore \sum_{m=2}^{\infty} \frac{1}{m \ln m}$ DIVERGES BY THE INTEGRAL-TEST

Question 46 (*****)

Consider the infinite series

$$1 - \frac{x^2}{2} + \frac{x^4}{4 \times 2} - \frac{x^6}{6 \times 4 \times 2} + \frac{x^8}{8 \times 6 \times 4 \times 2} - \dots$$

- a) Write the above series in Sigma notation, in its simplest form.

Next consider another infinite series

$$x + \frac{x^3}{3} + \frac{x^5}{5 \times 3} + \frac{x^6}{7 \times 5 \times 3} + \frac{x^9}{9 \times 7 \times 5 \times 3} + \dots$$

- b) Also, write this series in Sigma notation, in its simplest form.

[You are not required to investigate the convergence or the sum of these series.]

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{2^r r!} x^{2r}, \quad \sum_{r=0}^{\infty} \frac{2^r r!}{(2r+1)!} x^{2r}$$

$$\begin{aligned}
 \text{a) } l &= \frac{x^2}{2} + \frac{x^4}{4 \times 2} - \frac{x^6}{6 \times 4 \times 2} + \frac{x^8}{8 \times 6 \times 4 \times 2} - \dots = (-1)^0 x^2 - \sum_{r=1}^{\infty} \frac{(-1)^r}{(2r)!} x^{2r} \\
 &\approx l - \frac{x^2}{2} + \frac{x^4}{2^2 (2!)^2} - \frac{x^6}{2^3 (3!)^2} + \frac{x^8}{2^4 (4!)^2} - \dots = \sum_{r=0}^{\infty} \frac{(-1)^r}{2^r (r!)^2} x^{2r} \\
 \text{b) } l &= x + \frac{x^3}{3} + \frac{x^5}{5 \times 3} + \frac{x^6}{7 \times 5 \times 3} + \dots \\
 &= x + \frac{x^3}{3!} + \frac{4x^5}{5!} + \frac{6x^7}{7!} + \dots + \frac{8x^9}{9!} + \dots \\
 &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{(2x)(2x^2)(2x^4)}{7!} + \frac{8x^9}{9!} + \dots \\
 &= x + \left(\frac{2x(1)}{3!}\right)^2 + \left(\frac{2x(2)}{5!}\right)^2 + \left(\frac{2x(3)}{7!}\right)^2 + \left(\frac{2x(4)}{9!}\right)^2 + \dots \\
 &= \sum_{r=0}^{\infty} \frac{2^r r!}{(2r+1)!} x^{2r+1}
 \end{aligned}$$

Question 47 (****)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^n}$$

Show a full method, justifying every step in the workings.

divergent

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^n} &= \sum_{n=1}^{\infty} \frac{\binom{2n}{n} \cdot \frac{(2n)!}{(2n-1)!}}{2^n} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n} \cdot \frac{(2n)!}{2^n n!}}{2^n} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^n} \cdot \frac{(2n)!}{2^n (n!)^2} \\ \text{BY THE RATIO TEST} \\ \lim_{n \rightarrow \infty} \left| \frac{U_{n+1}}{U_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\frac{\binom{2(n+1)}{n+1}}{2^{n+1}} \cdot \frac{(2n+2)!}{(2n+1)!^2}}{\frac{\binom{2n}{n}}{2^n} \cdot \frac{(2n)!}{(2n-1)!^2}} \right| = \lim_{n \rightarrow \infty} \left[\frac{\frac{2^{n+1}}{2^{n+1}} \cdot \frac{(2n+2)!}{(2n+1)!^2} \cdot \left[\frac{(n+1)(2n+1)}{(2n)!} \right]^2}{\frac{(2n)!}{2^{n+1}} \cdot \frac{(2n+2)!}{(2n+1)!^2}} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} \cdot \frac{(2n+2) \cdot (2n+1)}{(2n+1)^2} \cdot \left[\frac{(n+1)(2n+1)}{(2n)!} \right]^2 \right] = \lim_{n \rightarrow \infty} \left[\frac{(n+1)(2n+1)}{(2n+1)^2} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{2n+1}{2n+1} \right] = 2 > 1 \\ \therefore \text{SERIES DIVERGES} \end{aligned}$$

Question 48 (****)

$$\sum_{r=1}^n \left[\binom{n}{r} x^r (1+x+x^2)^{n-r} \right].$$

Simplify fully the above sum, into a summation free expression

$$(x+1)^{2n} - (x^2 + x - 1)^n$$

$$\begin{aligned} \text{LET } S &= \sum_{r=1}^n \binom{n}{r} x^r (1+x+x^2)^{n-r} \\ \text{Now ADD TO BOTH SIDES } \underbrace{\left(\frac{x}{2} \right)^2 C(1+2x+x^2)^{n-1}}_{P(x)} &= C(n2x+x^2)^n \\ \text{THIS} \\ S + C(1+2x+x^2)^n &= C(1+2x+x^2)^n + \sum_{r=0}^{\infty} \binom{r}{r} x^r C(1+2x+x^2)^{n-r} \\ S + C(1+2x+x^2)^n &= \left(\sum_{r=0}^{\infty} \binom{r}{r} x^r (1+2x+x^2)^{n-r} \right) \xrightarrow{\text{BINOMIAL}} \\ S + C(1+2x+x^2)^n &= [x + C(1+2x+x^2)]^n \\ S + C(1+2x+x^2)^n &= (2x+2x+1)^n \\ S + C(1+2x+x^2)^n &= (2x+1)^{2n} \\ S &= (2x+1)^{2n} - C(1+2x+x^2)^n \\ L.E. \sum_{r=1}^n \binom{n}{r} x^r C(1+2x+x^2)^{n-r} &= (2x+1)^{2n} - (2^2+2x+1)^n // \end{aligned}$$

Question 49 (****+)

Consider the infinite series

$$x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{1 \times 3} + \frac{x^{\frac{5}{2}}}{(1 \times 2)(3 \times 5)} - \frac{x^{\frac{7}{2}}}{(1 \times 2 \times 3)(3 \times 5 \times 7)} + \frac{x^{\frac{9}{2}}}{(1 \times 2 \times 3 \times 4)(3 \times 5 \times 7 \times 9)} - \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{r=0}^{\infty} \left[\frac{(-2)^r}{(2r+1)!} x^{\frac{1}{2}+r} \right]$$

$$\begin{aligned} x^{\frac{1}{2}} &= \frac{x^{\frac{1}{2}}}{1 \times 3} + \frac{x^{\frac{3}{2}}}{(1 \times 2) \times (3 \times 5)} - \frac{x^{\frac{5}{2}}}{(1 \times 2 \times 3) \times (3 \times 5 \times 7)} + \frac{x^{\frac{7}{2}}}{(1 \times 2 \times 3 \times 4) \times (3 \times 5 \times 7 \times 9)} - \dots \\ \text{LOOKING AT THE FIFTH TERM (Exercise 2.9)} \\ \frac{1}{(1 \times 2 \times 3 \times 4) \times (3 \times 5 \times 7 \times 9)} &= \frac{\cancel{1} \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5} \times \cancel{6} \times \cancel{7} \times \cancel{8} \times \cancel{9}}{(1 \times 2 \times 3 \times 4) \times (3 \times 5 \times 7 \times 9) \times 8!} \\ &= \frac{2^8}{(1 \times 2 \times 3 \times 4) \times 8!} = \frac{2^8}{8!} \\ \text{Thus, } \sum_{r=1}^{\infty} \frac{(-2)^r}{(2r+1)!} x^{\frac{1}{2}+r} &= \sum_{r=1}^{\infty} \frac{(-2)^r}{(2r+1)!} \frac{2^{8+r}}{8!} \\ \text{or } \sum_{r=0}^{\infty} \frac{(-2)^r}{(2r+1)!} 2^{\frac{17+r}{2}} &= \end{aligned}$$

Question 50 (**+)**

Consider the infinite series

$$x - \frac{2}{3}(2x^2) + \frac{2 \times 2}{3 \times 5}(3x^3) - \frac{2 \times 2 \times 2}{3 \times 5 \times 7}(4x^4) + \frac{2 \times 2 \times 2 \times 2}{3 \times 5 \times 7 \times 9}(5x^5) - \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{n=0}^{\infty} \left[\frac{2^{2n} \times (-1)^n \times (n+1)! \times x^{n+1}}{(2n+1)!} \right]$$

WORKING AT THE BEGINNING (choose the 4th)

$$\begin{aligned} \frac{2 \times 2 \times 2 \times 2}{3 \times 5 \times 7 \times 9} \times (5x^5) &= \frac{2^4 \times 5^5}{3^4 \times 5^4 \times 7^4 \times 9^4} \times 5^4 \times x^5 \\ &= \frac{2^4 (5 \times 3 \times 7 \times 9)}{9!} \times 5^4 \times (5x^5) \\ &= \frac{2^4 \times 45}{9!} \times 5x^5 \quad (\text{h.s. } 5) \\ \therefore \sum_{k=1}^{20} \frac{2^{2k-2} \times (k-1)! \times k^k \times (-1)^{k-1}}{(2k-1)!} &= \sum_{k=1}^{20} \frac{2^{2k-2} \times [k-1]! \times k^k \times (-1)^{k-1}}{(2k-1)!} \\ \sum_{k=0}^{19} \frac{2^{2k} \times k! \times (k+1)^{k+1} \times (-1)^k}{(2k+1)!} &= \sum_{k=0}^{19} \frac{2^{2k} \times k! \times (k+1)^{k+1} \times (-1)^k}{(2k+1)!} \end{aligned}$$

Question 51 (***)+

Use a suitable method to sum the following series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)}.$$

[14]

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)} &= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n(n+2)} = \dots \text{PARTIAL FRACTION, BY INSPECTION} \\
 &= \sum_{k=1}^{\infty} (-1)^{k+1} \left[-\frac{\frac{1}{k}}{1} + \frac{\frac{1}{k}}{k+2} \right] = \frac{1}{2} \sum_{k=1}^{\infty} \left[\frac{(-1)^{k+1}}{k} - \frac{(-1)^{k+1}}{k+2} \right] \\
 &= \frac{1}{2} \left[1 - \cancel{\frac{1}{2}} + \cancel{-\frac{1}{3}} + \cancel{\frac{1}{4}} + \cancel{-\frac{1}{5}} + \cancel{\frac{1}{6}} + \cancel{-\frac{1}{7}} + \cancel{\frac{1}{8}} + \dots \right] \\
 &= \frac{1}{2} \left[1 - \frac{1}{2} \right] = \frac{1}{4}
 \end{aligned}$$

Question 52 (***)+

Consider the infinite series

$$1 + \frac{2}{1 \times 1} + \frac{6}{(1 \times 2)(1 \times 3)} + \frac{10}{(1 \times 2 \times 3)(1 \times 3 \times 5)} + \frac{15}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)} + \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{r=0}^{\infty} \frac{(r+1)(r+2)2^{r-1}}{(2r)!}$$

$$\begin{aligned}
 1 + \frac{2}{1 \times 1} + \frac{6}{(1 \times 2)(1 \times 3)} + \frac{10}{(1 \times 2 \times 3)(1 \times 3 \times 5)} + \frac{15}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)} + \dots
 \end{aligned}$$

DENOMINATORS ARE TRIANGLE NUMBERS
 $1, 2, 6, 10, 15, 21, \dots$ if $\frac{1}{2}r(r+1)$

LOOKING AT THE FIFTH TERM

$$\begin{aligned}
 \frac{15}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5)} &= \frac{15 \times \cancel{(1 \times 2 \times 3 \times 4)}}{(1 \times 2 \times 3 \times 4) \times \cancel{(1 \times 3 \times 5 \times 7)}} \\
 &= \frac{15 \times 2^3 \times (1 \times 3 \times 5)}{(1 \times 2 \times 3 \times 4) \times 5!} = \frac{15 \times 2^3}{5!} \\
 \therefore \sum_{r=0}^{\infty} \frac{\frac{1}{2}r(r+1) \times 2^{r-1}}{\frac{1}{2}r(r+1)2^r} &= \sum_{r=0}^{\infty} \frac{r(r+1)2^{r-2}}{(2r)!} \\
 \text{or } \sum_{r=0}^{\infty} \frac{\frac{1}{2}r(r+1)2^r}{(2r)!} &= \sum_{r=0}^{\infty} \frac{(r+1)r2^{r-1}}{(2r)!}
 \end{aligned}$$

Question 53 (***)+

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{n(2n+3)^{\frac{1}{2}}}{n^3 + 4}.$$

convergent

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{n(2n+3)^{\frac{1}{2}}}{n^3 + 4} &< \sum_{n=1}^{\infty} \frac{n(2n+3)^{\frac{1}{2}}}{n^3} = \sum_{n=1}^{\infty} \frac{(2n+3)^{\frac{1}{2}}}{n^2} \\ &< \sum_{n=1}^{\infty} \frac{(4n+12)^{\frac{1}{2}}}{n^2} = 2 \sum_{n=1}^{\infty} \frac{(2n+1)^{\frac{1}{2}}}{n^2} \\ &< 2 \sum_{n=1}^{\infty} \frac{(n+n)^{\frac{1}{2}}}{n^2} = 2 \sum_{n=1}^{\infty} \frac{(2n)^{\frac{1}{2}}}{n^2} \\ &= 2\sqrt{2} \sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}}}{n^2} = 2\sqrt{2} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}.\end{aligned}$$

which converges by the p-test

$$\sum \frac{1}{n^p} = \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

Question 54 (***/**+)

A sequence is generated by the function

$$u_r(\theta) \equiv r \sin(\theta + r\pi), r \in \mathbb{N}.$$

Find an expression or the value, whichever is appropriate, for each of the series

a) $\sum_{r=1}^{40} u_r(\theta).$

b) $\sum_{r=1}^{40} \left[u_r\left(\frac{\pi}{6}\right) \right]^2.$

20 sin θ, [5535]

$$\begin{aligned} \sum_{r=1}^{40} r \sin(\theta + r\pi) &= \sin(\theta + \pi) + 2\sin(\theta + 2\pi) + 3\sin(\theta + 3\pi) + \dots \\ &= -\sin\theta + 2\sin\theta - 3\sin\theta + 4\sin\theta - \dots \\ &= (-\sin\theta + 2\sin\theta) + (-3\sin\theta + 4\sin\theta) + \dots \\ &= \sin\theta + \sin\theta + \dots + \sin\theta \\ &\quad \text{20 terms (in pairs)} \\ &= 20\sin\theta \end{aligned}$$

$$\begin{aligned} \sum_{r=1}^{40} r^2 \sin^2(\theta + r\pi) &= [\sin(\theta + \pi)]^2 + [2\sin(\theta + 2\pi)]^2 + [3\sin(\theta + 3\pi)]^2 + \dots \\ &= \sin^2\theta + 4\sin^2\theta + 9\sin^2\theta + \dots + 1600\sin^2\theta \\ &= \sin^2\theta (1 + 4 + 9 + \dots + 1600) \\ &= \sin^2\theta \times \frac{40}{6} \times 41 \times 81 \\ &= \frac{40}{6} \times \frac{40}{6} \times 41 \times 81 \\ &= \frac{40}{6} \times 4 \times 81 \\ &= \frac{40}{6} \times 4 \times 81^2 \\ &= 5 \times 41 \times 27 \\ &= 135 \times 41 \\ &= 5535 \end{aligned}$$



Question 55 (****+)

Consider the infinite series

$$1 + \frac{1}{1 \times 5} + \frac{1}{(1 \times 2)(5 \times 8)} + \frac{1}{(1 \times 2 \times 3)(5 \times 8 \times 11)} + \frac{1}{(1 \times 2 \times 3 \times 4)(5 \times 8 \times 11 \times 14)} + \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{r=1}^{\infty} \frac{\Gamma\left(\frac{5}{3}\right)}{3^{r-1} \times (r-1)! \times \Gamma\left(\frac{3r+2}{3}\right)} = \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{5}{3}\right)}{3^r \times r! \times \Gamma\left(\frac{3r+5}{3}\right)}$$

1 + $\frac{1}{1 \times 5}$ + $\frac{1}{(1 \times 2)(5 \times 8)}$ + $\frac{1}{(1 \times 2 \times 3)(5 \times 8 \times 11)}$ + $\frac{1}{(1 \times 2 \times 3 \times 4)(5 \times 8 \times 11 \times 14)}$ + ...

LOOK AT THE SERIES TO SEE THE PATTERN

$$\begin{aligned} \frac{1}{(1 \times 2 \times 3)(5 \times 8 \times 11)} &= \frac{1}{4! \times 3^3 \times (\frac{5}{3} \times \frac{7}{3} \times \frac{11}{3})} = \frac{\Gamma(\frac{5}{3})}{4! \times 3^3 \times (\frac{5}{3}) \times (\frac{7}{3}) \times (\frac{11}{3})} \\ &= \frac{\Gamma(\frac{5}{3})}{4! \times 3^3 \times \cancel{\Gamma(\frac{5}{3})}} \xrightarrow{\text{variable}} \end{aligned}$$

Thus $\sum_{r=0}^{\infty} \frac{\Gamma(\frac{5}{3})}{(r!) \times 3^r \times \Gamma(\frac{3r+5}{3})} = \sum_{r=0}^{\infty} \frac{\Gamma(\frac{5}{3})}{(r!) \times 3^r \times (\frac{3r+5}{3})}$

Question 56 (***/**+)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}.$

b) $\sum_{k=1}^{\infty} k^2 e^{-k^2}.$

c) $\sum_{r=1}^{\infty} \frac{(-1)^r}{\ln(r+1)}.$

d) $\sum_{m=2}^{\infty} \frac{1}{m(\ln m)^2}$

[divergent], [convergent], [convergent], [convergent]

(a) $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}} + \sum_{n=5}^{\infty} \frac{1}{2+\sqrt{n}} = A + \sum_{n=5}^{\infty} \frac{\sqrt{n}-2}{(\sqrt{n}+2)(\sqrt{n}-2)}$

 $= A + \sum_{n=5}^{\infty} \frac{\sqrt{n}-2}{n-4} > A + \sum_{n=5}^{\infty} \frac{\sqrt{n}-2}{n-5} = A + \sum_{n=5}^{\infty} \frac{1}{n-5} = A + \sum_{n=5}^{\infty} \frac{1}{n-5} - 2 \sum_{n=5}^{\infty} \frac{1}{n}$

Rearranging

$\therefore \sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$ is DIVERGENT BY THE COMPARISON TEST

(b) $\sum_{k=1}^{\infty} k^2 e^{-k^2} =$ BY THE RATIO TEST $\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^2 e^{-(k+1)^2}}{k^2 e^{-k^2}} \right|$
 $= \lim_{k \rightarrow \infty} \left[\frac{(k+1)^2}{k^2} e^{-2k-1+2k} \right] = \lim_{k \rightarrow \infty} \left[(1+\frac{1}{k})^2 e^{-1/k} \right] = 1 \times 0 = 0 < 1$

\therefore SERIES CONVERGES BY THE RATIO TEST

(c) $\sum_{r=1}^{\infty} \frac{(-1)^r}{\ln(r+1)}$

- TERMS ALTERNATE IN SIGN
- $\lim_{r \rightarrow \infty} \frac{1}{\ln(r+1)} = 0$

\therefore SERIES CONVERGES BY THE ALTERNATING SERIES TEST

(d) $\sum_{m=2}^{\infty} \frac{1}{m(\ln m)^2}$

- $\lim_{m \rightarrow \infty} \left[\frac{1}{m(\ln m)^2} \right] = 0$
- $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{a \rightarrow \infty} \int_2^a \frac{1}{x(\ln x)^2} dx$
 $= \lim_{a \rightarrow \infty} \left[-\frac{1}{\ln x} \right]_2^a = \lim_{a \rightarrow \infty} \left[\frac{1}{\ln 2} - \frac{1}{\ln a} \right] = \frac{1}{\ln 2}.$

\therefore SERIES CONVERGES BY THE INTEGRAL TEST

Question 57 (***)+

The following convergent series S is given below

$$S = \sin \theta - \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta \dots$$

By considering the sum to infinity of a suitable geometric series involving the complex exponential function, show that

$$S = \frac{\sin \theta}{10 + 6\cos \theta}$$

proof

$$\begin{aligned} S &= \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta + \dots \\ C &= \cos \theta - \frac{1}{3} \cos 2\theta + \frac{1}{9} \cos 3\theta - \frac{1}{27} \cos 4\theta \dots \\ S &= \sin \theta - \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta \dots \\ \text{Hence} \\ C + iS &= (\cos \theta + i\sin \theta) - \frac{1}{3}(\cos 2\theta + i\sin 2\theta) + \frac{1}{9}(\cos 3\theta + i\sin 3\theta) - \frac{1}{27}(\cos 4\theta + i\sin 4\theta) + \dots \\ C + iS &= e^{i\theta} - \frac{1}{3}e^{i2\theta} + \frac{1}{9}e^{i3\theta} - \frac{1}{27}e^{i4\theta} \dots \end{aligned}$$

GEOMETRIC PROGRESSION WITH FIRST TERM $e^{i\theta}$, COMMON RATIO $(-\frac{1}{3}e^{i\theta})$

$$\begin{aligned} \text{Sum to infinity} &= \frac{e^{i\theta}}{1 - (-\frac{1}{3}e^{i\theta})} = \frac{e^{i\theta}}{1 + \frac{1}{3}e^{i\theta}} = \frac{3e^{i\theta}}{3 + e^{i\theta}} = \frac{3e^{i\theta}(1 + e^{-i\theta})}{(3 + e^{i\theta})(1 + e^{-i\theta})} = \frac{3e^{i\theta} + 3}{9 + 3e^{i\theta} + 3e^{i\theta}} \\ &= \frac{3(\cos \theta + i\sin \theta) + 3}{10 + 6\cos \theta + 3i\sin \theta} = \frac{(\cos \theta + 3) + i(\sin \theta)}{10 + 6\cos \theta} = \frac{(\cos \theta + 3) + i(\sin \theta)}{10 + 6\cos \theta} \\ \text{The required answer is the imaginary part of the expression, i.e. } \sum_{n=0}^{\infty} (-\frac{1}{3})^n \sin n\theta &= \frac{\sin \theta}{10 + 6\cos \theta} \end{aligned}$$

Question 58 (***)+

A sequence of positive integers is generated by

$$u_n = 3^n - 1, \quad n = 1, 2, 3, 4, \dots$$

- a) Write down the first seven terms of this sequence.
- b) Verify that

$$u_{n+1} = 3u_n + 2.$$

- c) Show clearly that ...

i. $\dots \frac{1}{u_{n+1}} < \frac{1}{3} \times \frac{1}{u_n}.$

ii. $\dots \frac{1}{26} + \frac{1}{80} + \frac{1}{242} + \frac{1}{728} + \frac{1}{2186} + \dots < \frac{1}{8} \left[\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots \right]$

- d) Deduce that

$$\sum_{n=1}^{\infty} u_n < \frac{11}{16}.$$

2, 8, 26, 80, 242, 728, 2186, ...

(a) $u_n = 3^n - 1 \Rightarrow u_0 = 2, u_1 = 8, u_2 = 26, u_3 = 80, u_4 = 242, u_5 = 728, u_6 = 2186, \dots$

(b) $u_{n+1} = 3u_n + 2 = 3(3^n - 1) + 2 = 3^{n+1} - 3 + 2 = 3^{n+1} - 1$

(c) $u_{n+1} = 3u_n + 2$

$$\frac{1}{u_{n+1}} = \frac{1}{3u_n + 2} < \frac{1}{3u_n} = \frac{1}{3} \times \frac{1}{u_n} /$$

$\frac{1}{u_1} < \frac{1}{3} \times \frac{1}{u_0} \Rightarrow \frac{1}{8} < \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} \times \frac{1}{2}$

$$\frac{1}{u_2} < \frac{1}{3} \times \frac{1}{u_1} \Rightarrow \frac{1}{26} < \frac{1}{3} \times \frac{1}{8} < \frac{1}{3} \times \frac{1}{6} = \frac{1}{18} \times \frac{1}{3}$$

$$\frac{1}{u_3} < \frac{1}{3} \times \frac{1}{u_2} \Rightarrow \frac{1}{728} < \frac{1}{3} \times \frac{1}{26} < \frac{1}{3} \times \frac{1}{18} = \frac{1}{54} \times \frac{1}{3}$$

$$\frac{1}{u_4} < \frac{1}{3} \times \frac{1}{u_3} \Rightarrow \frac{1}{2186} < \frac{1}{3} \times \frac{1}{728} < \frac{1}{3} \times \frac{1}{54} = \frac{1}{162} \times \frac{1}{3}$$

$$\frac{1}{u_5} < \frac{1}{3} \times \frac{1}{u_4} \Rightarrow \frac{1}{6558} < \frac{1}{3} \times \frac{1}{2186} < \frac{1}{3} \times \frac{1}{162} = \frac{1}{486} \times \frac{1}{3}$$

$$\dots$$

(d)

$$\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} + \dots < \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \frac{1}{162} + \dots$$

$$\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} + \dots < \frac{1}{6} \left[\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots \right]$$

$$\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} + \dots < \frac{1}{6} \left[\frac{3}{3} \right] = \frac{1}{2}$$

$$\left(\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} + \dots \right) < \frac{1}{2} + \frac{1}{6} + \frac{1}{12}$$

$$\sum_{n=1}^{\infty} \frac{1}{u_n} < \frac{11}{16}$$

Question 59 (***)+

$$u_n = \frac{\sqrt{n}+1}{\sqrt{n^3-n}}, n \in \mathbb{N}, n \geq 5.$$

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of

$$\sum_{n=5}^{\infty} u_n.$$

convergent

$$\begin{aligned}
 u_n &= \frac{\sqrt{n}+1}{\sqrt{n^3-n}} = \frac{(\sqrt{n}+1)(\sqrt{n^3+n})}{(\sqrt{n^3-n})(\sqrt{n^3+n})} = \frac{\sqrt{n^3+n}\sqrt{n^3+n}}{n^3-n^2} = \frac{\sqrt{n^3+3n^2+n}}{n^3-n^2} \\
 &> \frac{n^3+3n^2+n}{n^3} = \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^3} \\
 \therefore \sum_{n=5}^{\infty} u_n &> \sum_{n=5}^{\infty} \frac{1}{n} + 3 \sum_{n=5}^{\infty} \frac{1}{n^2} + \sum_{n=5}^{\infty} \frac{1}{n^3} \\
 &\quad \uparrow \quad \uparrow \quad \uparrow \\
 &\text{DIVERGES} \quad \text{CONVERGES} \quad \text{CONVERGES} \\
 &\text{BY THE P-TEST } \sum_{n=5}^{\infty} \frac{1}{n^p} = \begin{cases} \text{CONVERGES IF } p > 1 \\ \text{DIVERGES IF } p \leq 1 \end{cases} \\
 &\therefore \sum_{n=5}^{\infty} u_n \text{ IS DIVERGENT}
 \end{aligned}$$

Question 60 (***)+

Show clearly that

$$\sum_{n=1}^{\infty} \frac{4n-3}{n!} = e + 3.$$

proof

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{4n-3}{n!} &= \sum_{n=1}^{\infty} \left(\frac{4n}{n!} - \frac{3}{n!} \right) = 4 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} - 3 \sum_{n=1}^{\infty} \frac{1}{n!} \\
 &= 4 \left[1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right] - 3 \left[1 + \frac{1}{2!} + \frac{1}{3!} + \dots \right] \\
 &= 4e - 3(e-1) = e+3
 \end{aligned}$$

Question 61 (****+)

Consider the infinite series

$$x + \frac{x^3}{3^2} + \frac{x^5}{5^2 \times 3^2} + \frac{x^7}{7^2 \times 5^2 \times 3^2} + \frac{x^9}{9^2 \times 7^2 \times 5^2 \times 3^2} + \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{n=0}^{\infty} \left[\frac{2^{2n} (n!)^2}{[(2n+1)!]^2} x^{2n+1} \right]$$

$$\begin{aligned} & x + \frac{x^3}{3^2} + \frac{x^5}{5^2 \times 3^2} + \frac{x^7}{7^2 \times 5^2 \times 3^2} + \frac{x^9}{9^2 \times 7^2 \times 5^2 \times 3^2} + \dots \\ &= x + \left(\frac{x^3}{3^2} + \left(\frac{2^2 x^5}{5^2 \times 3^2} + \left(\frac{2^4 x^7}{7^2 \times 5^2 \times 3^2} + \left(\frac{2^6 x^9}{9^2 \times 7^2 \times 5^2 \times 3^2} \right) x^2 + \dots \right) x^2 + \dots \right) x^2 + \dots \right) \\ &= x + \left(\frac{2^2 (1)^2}{3^2} x^2 + \left(\frac{2^4 (2 \times 1)^2}{5^2 \times 3^2} x^2 + \left(\frac{2^6 (3 \times 2 \times 1)^2}{7^2 \times 5^2 \times 3^2} x^2 + \dots \right) x^2 + \dots \right) x^2 + \dots \right) \\ &= \sum_{n=0}^{\infty} \left(\frac{2^{2n} (n!)^2}{(2n+1)!^2} x^{2n+2} \right) \end{aligned}$$

Question 62 (****+)

Show clearly that

$$\sum_{r=0}^{\infty} \frac{r+4}{(r+2)!} = 3e - 5.$$

proof

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{r+4}{(r+2)!} &= \frac{4}{2!} + \frac{5}{3!} + \frac{6}{4!} + \frac{7}{5!} + \frac{8}{6!} + \dots \\ \text{HENCE WE MAY MANIPULATE IT AS FOLLOWS} \\ \sum_{r=0}^{\infty} \frac{(r+2)+2}{(r+2)!} &= \sum_{r=0}^{\infty} \left[\frac{r+2}{(r+2)!} + \frac{2}{(r+2)!} \right] = \sum_{r=0}^{\infty} \frac{1}{(r+1)!} + 2 \sum_{r=0}^{\infty} \frac{1}{(r+2)!} \\ \text{WE GET A FEW TERMS FROM EACH SUM} \\ &= \left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \right] + 2 \left[\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots \right] \\ &= \left[\underbrace{\frac{1}{1!} + \frac{1}{2!}}_e + \underbrace{\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots}_e \right] + 2 \left[\underbrace{\frac{1}{2!} - \frac{1}{1!}}_e + \underbrace{\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots}_e \right] \\ &= [e(1+e)] + 2[-1-1+e] \\ &= 3e - 5 \end{aligned}$$

Question 63 (***)+

$$I_n = \int_0^{\ln 2} \tanh^n x \ dx, \ n \in \mathbb{N}.$$

By considering a reduction formula for I_n , or otherwise, show clearly that

$$\sum_{r=1}^{\infty} \frac{1}{2r} \left(\frac{3}{5} \right)^{2r} = \ln \left(\frac{5}{4} \right).$$

proof

$$\begin{aligned}
 I_n &= \int_0^{\ln 2} \tanh^n x \ dx = \int_0^{\ln 2} \tanh^{n-2} x \tanh^2 x \ dx = \int_0^{\ln 2} \tanh^{n-2} x (1 - \operatorname{sech}^2 x) dx \\
 &= \int_0^{\ln 2} \tanh^{n-2} x \left[1 - \operatorname{sech}^2 x \right]^{1/2} dx = \int_0^{\ln 2} \tanh^{n-2} x dx - \int_0^{\ln 2} \operatorname{sech}^2 x \tanh^{n-2} x dx \\
 &= I_{n-2} - \frac{1}{n-1} \left[\tanh^{n-1} x \right]_0^{\ln 2} = I_{n-2} - \frac{1}{n-1} (\tanh(\ln 2))^{n-1} \\
 \text{Now } \tanh(\ln 2) &= \frac{2e^2 - 1}{2e^2 + 1} = \frac{4 - 1}{4 + 1} = \frac{3}{5} \\
 \therefore I_n &= I_{n-2} - \frac{1}{n-1} \left(\frac{3}{5} \right)^{n-1} \\
 \frac{1}{n-1} \left(\frac{3}{5} \right)^{n-1} &\equiv I_{n-2} - I_n
 \end{aligned}$$

$$\begin{aligned}
 n=3: \quad \frac{1}{2} \left(\frac{3}{5} \right)^2 &= I_1 - I_2 \\
 n=5: \quad \frac{1}{4} \left(\frac{3}{5} \right)^4 &= I_3 - I_4 \\
 n=7: \quad \frac{1}{6} \left(\frac{3}{5} \right)^6 &= I_5 - I_6 \\
 n=9: \quad \frac{1}{8} \left(\frac{3}{5} \right)^8 &= I_7 - I_8 \\
 &\vdots \\
 \therefore \sum_{n=1}^{\infty} \frac{1}{n-1} \left(\frac{3}{5} \right)^{n-1} &= I_1 = \int_0^{\ln 2} \tanh x dx = \int_0^{\ln 2} \frac{\sinh x}{cosh x} dx \\
 &= \left[\ln(\operatorname{tanh} x) \right]_0^{\ln 2} = \ln(\operatorname{tanh}(\ln 2)) - \ln(\operatorname{tanh} 0) \\
 &= \ln \left[\frac{1}{2} e^{\frac{1}{2} \ln 2 + \frac{1}{2} \ln 2} \right] - \ln 1 = \ln \left[1 + \frac{1}{2} \right] \\
 &= \ln \frac{3}{2}
 \end{aligned}$$

Question 64 (****+)

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\sum_{n=1}^{\infty} \left[\frac{n+2}{n^2+2} \right].$

b) $\sum_{n=1}^{\infty} \left[\frac{2^n + n^2}{3^n} \right].$

[divergent], [convergent]

Q3 $\sum_{n=1}^{\infty} \frac{n+2}{n^2+2} > \sum_{n=1}^{\infty} \frac{n+2}{n^2+n+1} = \sum_{n=1}^{\infty} \frac{n+2}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n+1} > \sum_{n=1}^{\infty} \frac{1}{2n+3}$
 \therefore As $\sum_{n=1}^{\infty} \frac{1}{n+1} > \sum_{n=1}^{\infty} \frac{1}{2n+3}$ is DIVERGENT.
 $\Rightarrow \sum_{n=1}^{\infty} \frac{n+2}{n^2+2}$ is DIVERGENT.

Q4 $\sum_{n=1}^{\infty} \frac{2^n + n^2}{3^n} = \sum_{n=1}^{\infty} \frac{2^n}{3^n} + \sum_{n=1}^{\infty} \frac{n^2}{3^n} = A + \sum_{n=1}^{\infty} \frac{2^n}{3^n}$
 $< A + \sum_{n=1}^{\infty} \frac{2^n}{3^n} = A + 2 \sum_{n=1}^{\infty} \frac{2^n}{3^n} = A + 2 \sum_{n=1}^{\infty} (\frac{2}{3})^n$
 $\text{if } n > 5$
 $\therefore \sum_{n=1}^{\infty} \frac{2^n}{3^n}$ is CONVERGED BY COMPARISON.

ACTUAL WORK
 $\sum_{n=1}^{\infty} \frac{2^n + n^2}{3^n} = \sum_{n=1}^{\infty} (\frac{2}{3})^n + \frac{n^2}{3^n} = \sum_{n=1}^{\infty} (\frac{2}{3})^n + \sum_{n=1}^{\infty} \frac{n^2}{3^n}$
 \therefore CONVERGED BY RATIO TEST.

RATIO TEST
 $\lim_{n \rightarrow \infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{2^{n+1} + (n+1)^2}{3^{n+1}}}{\frac{2^n + n^2}{3^n}} = \frac{\frac{2^{n+1}}{3^{n+1}} + \frac{(n+1)^2}{3^{n+1}}}{\frac{2^n}{3^n} + \frac{n^2}{3^n}} = \left(\frac{2}{3} \right)^2 + \frac{1}{3} < 1$
 $\therefore \sum_{n=1}^{\infty} \frac{2^n + n^2}{3^n}$ CONVERGED BY THE RATIO TEST.

Question 65 (***)+

By considering the McLaurin expansion of $\ln\left(\frac{1+x}{1-x}\right)$ find the value of

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r},$$

giving the final answer as the natural logarithm of an integer.

ln 3

Given $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)$

$$= 2x^3 + \frac{2}{3}x^5 + \frac{2}{5}x^7 + \dots$$

Now $2 \sum_{n=0}^{\infty} \frac{2^{2n+1}}{2n+1} = \ln\left(\frac{1+2}{1-2}\right)$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)4^n} = \sum_{n=0}^{\infty} \frac{(2^n)^3}{2n+1} = 2 \sum_{n=0}^{\infty} \frac{1}{2n+1}$$

$$= 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} = \text{RHS result} \quad \text{LHS result} \dots$$

$$= \ln\left(\frac{2+1}{2-1}\right) = \ln 3$$

Question 66 (***)+

Use partial fractions to sum the following series.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+n^2}.$$

You may assume the series converges.

[], 1

• Since by taking up the summation

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+n^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n^2+2n+1)} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

• Although we have repeated factors the partial fractions can easily be done by inspection

$$= \sum_{n=1}^{\infty} \left[\frac{1}{n^2} - \frac{1}{(n+1)^2} \right]$$

$$= \left(\frac{1}{1^2} - \frac{1}{2^2} \right) + \left(\frac{1}{2^2} - \frac{1}{3^2} \right) + \left(\frac{1}{3^2} - \frac{1}{4^2} \right) + \left(\frac{1}{4^2} - \frac{1}{5^2} \right) \dots$$

$$= 1$$

Question 67 (*)+**

Sum each of the following double series

$$\text{a)} \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[\frac{1}{2^{m+n}} \right].$$

$$\text{b)} \quad \sum_{m=0}^{\infty} \sum_{n=0}^m \left[\frac{1}{2^{m+n}} \right].$$

$$[\quad], [4], \left[\frac{8}{3} \right]$$

$$\begin{aligned}
 & \text{q) } \sum_{n=0}^{\infty} \sum_{k=0}^{2^n} \left[\frac{1}{2^k + 1} \right] = \sum_{k=0}^{\infty} \sum_{n=0}^{2^k} \left[\frac{1}{2^k + 1} \right] = \sum_{k=0}^{\infty} \left[\sum_{n=0}^{2^k} \frac{1}{2^k + 1} \right] \\
 & = \sum_{k=0}^{\infty} \left[\frac{1}{2^k} \cdot \left(1 + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^k+1} \right) \right] \\
 & \stackrel{P.S.}{=} \sum_{k=0}^{\infty} \left(\frac{1}{2^k} \times \frac{2^k}{2^k+1} \right) = \sum_{k=0}^{\infty} \left(\frac{1}{2^k} \times 2 \right) \\
 & = \sum_{k=0}^{\infty} \left(\frac{1}{2^{k-1}} \right) = \left(2 + \frac{1}{2} + \frac{1}{4} + \dots \right) \\
 & \text{P.S. } \frac{2}{1 - \frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4
 \end{aligned}$$

ALTERNATIVES

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left[\frac{1}{2^{n+k}} \right] = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \left[\frac{1}{2^{n+k}} \cdot 2^n \right] = \sum_{k=0}^{\infty} \left[\frac{1}{2^k} \cdot \frac{1}{2^k} \right] = \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \\
 & = \left(\sum_{k=0}^{\infty} \frac{1}{2^k} \right)^2 = \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right]^2 \\
 & = \left[\frac{1}{1 - \frac{1}{2}} \right]^2 = \left(\frac{1}{\frac{1}{2}} \right)^2 = 2^2 = 4
 \end{aligned}$$

$$\begin{aligned}
& \sum_{k=0}^{\infty} \sum_{n=0}^{2^k} \left[\frac{1}{2^{kn}} \right] = \sum_{k=0}^{\infty} \sum_{n=0}^{2^k} \left[\frac{1}{2^{kn}} \times \frac{1}{2^{kn}} \right] = \sum_{k=0}^{\infty} \sum_{n=0}^{2^k} \left[\frac{1}{2^{kn+kn}} \right] \\
& = \sum_{k=1}^{\infty} \sum_{n=1}^{2^k} \left[\frac{1}{2^{kn}} \right] = \sum_{k=0}^{\infty} \left[\frac{1}{2^{kn}} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{kn}} \right) \right] \\
& \quad \text{METHODS} \\
& \frac{d}{dx} \ln \left(\frac{a(x)}{1-x} \right) \\
& = \sum_{k=0}^{\infty} \left[\frac{1}{2^{kn}} \times \frac{1 \cdot (1 - \frac{1}{2^{kn+1}})}{1 - \frac{1}{2^{kn}}} \right] \\
& = \sum_{k=0}^{\infty} \left[\frac{1}{2^{kn}} \times 2 \times \left(1 - \left(\frac{1}{2} \right)^{kn+1} \right) \right] \\
& = \sum_{k=0}^{\infty} \left[\frac{1}{2^{kn}} \left(1 - \frac{1}{2^{kn+1}} \right) \right] \\
& = \sum_{k=0}^{\infty} \left[\frac{1}{2^{kn}} - \frac{1}{2^{kn+1}} \right] \\
& = \sum_{k=0}^{\infty} \left(\frac{1}{2^{kn}} \right) - \sum_{k=0}^{\infty} \left(\frac{1}{2^{kn+1}} \right) \\
& = \left(2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right) - \left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots \right) \\
& = \frac{2}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{4}} \\
& = \frac{2}{\frac{1}{2}} - \frac{1}{\frac{3}{4}} \\
& = \frac{4}{2} - \frac{4}{3} = \frac{2}{3}
\end{aligned}$$

Question 68 (****+)

Consider the infinite series

$$1 - \frac{3x^2}{1 \times 1} + \frac{9x^4}{(1 \times 2)(1 \times 4)} - \frac{27x^6}{(1 \times 2 \times 3)(1 \times 4 \times 7)} + \frac{81x^8}{(1 \times 2 \times 3 \times 4)(1 \times 4 \times 7 \times 10)} + \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1} \Gamma\left(\frac{1}{3}\right)}{(r-1)! \times \Gamma\left(\frac{3r-2}{3}\right)} (x^2)^{r-1} = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{1}{3}\right)}{r! \times \Gamma\left(\frac{3r+1}{3}\right)} x^{2r}$$

Look at the fifth term for a pattern:

$$\begin{aligned} \frac{81x^8}{(1 \times 2 \times 3 \times 4)(1 \times 4 \times 7 \times 10)} &= \frac{(3x^2)^8}{4! \times 3^8 \times (4 \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3})} = \frac{(3x^2)^8 \times \Gamma\left(\frac{1}{3}\right)}{4! \times 3^8 \times \Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(\frac{2}{3}\right) \cdot \Gamma\left(\frac{5}{3}\right)} \\ &= \frac{-3^8 \cdot (2!)^4 \times \Gamma\left(\frac{1}{3}\right)}{4! \times 3^8 \times \Gamma\left(\frac{1}{3}\right)} = \frac{-(2!)^4 \times \Gamma\left(\frac{1}{3}\right)^4}{4! \times \Gamma\left(\frac{1}{3}\right)} \rightarrow \text{Value neg!} \\ \therefore \sum_{r=1}^{\infty} \frac{(3x^2)^r \Gamma\left(\frac{1}{3}\right)}{(r-1)! \Gamma\left(\frac{3r-2}{3}\right)} &= \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{1}{3}\right)}{r! \Gamma\left(\frac{3r+1}{3}\right)} x^{2r} \end{aligned}$$

Question 69 (***)+

$$\sum_{n=1}^{\infty} \left[\frac{n \pm 2}{n^2 \pm 2} \right].$$

Use a comparison test to show that all four series described by the above expression are divergent.

proof

(a) $\sum_{n=1}^{\infty} \frac{n+2}{n^2-2} > \sum_{n=1}^{\infty} \frac{n+2}{n^2} > \sum_{n=1}^{\infty} \frac{3}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ WHICH DIVERGES

(b) $\sum_{n=1}^{\infty} \frac{n-2}{n^2-2} > \sum_{n=1}^{\infty} \frac{n-2}{n^2-4n+4} = \sum_{n=1}^{\infty} \frac{n-2}{(n-2)^2} = \sum_{n=3}^{\infty} \frac{n-2}{n^2} = \sum_{n=3}^{\infty} \frac{1}{n^2}$
 $= \sum_{n=3}^{\infty} \frac{1}{n^2} - 4 \sum_{n=1}^2 \frac{1}{n^2}$
 \uparrow CONVERGES TO $4 \times \frac{\pi^2}{6}$
DIVERGES

(c) $\sum_{n=1}^{\infty} \frac{n+2}{n^2+2} > \sum_{n=1}^{\infty} \frac{n+2}{n^2+4n+4} = \sum_{n=1}^{\infty} \frac{n+2}{(n+2)^2} = \sum_{n=1}^{\infty} \frac{1}{n+2} = \sum_{n=3}^{\infty} \frac{1}{n}$ WHICH DIVERGES

(d) $\sum_{n=1}^{\infty} \frac{n-2}{n^2+2} > \sum_{n=1}^{\infty} \frac{n-2}{n^2+4n+4} = \sum_{n=1}^{\infty} \frac{n-2}{(n+2)^2} = \sum_{n=3}^{\infty} \frac{n-2}{n^2} = \sum_{n=3}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right)$
 $= \sum_{n=3}^{\infty} \frac{1}{n^2} - 4 \sum_{n=1}^2 \frac{1}{n^2}$
 \uparrow CONVERGES TO $4 \times \frac{\pi^2}{6}$
DIVERGES

$\therefore \sum_{n=1}^{\infty} \frac{n-2}{n^2+2}$ IS DIVERGENT

Question 70 (***)+

By showing a detailed method, sum the following series.

$$\frac{\pi^2}{2^2 2!} - \frac{\pi^4}{2^4 4!} + \frac{\pi^6}{2^6 6!} - \frac{\pi^8}{2^8 8!} + \dots + \frac{(-1)^{n+1} \pi^{2n}}{2^{2n} (2n)!} + \dots$$

[1]

$$\frac{1}{2} = \frac{\pi^2}{2^2 2!} - \frac{\pi^4}{4! 2^4} + \frac{\pi^6}{6! 2^6} - \frac{\pi^8}{8! 2^8} + \dots + \frac{\pi^{2n} (-1)^{n+1}}{(2n)! 2^{2n}} + \dots$$

- CONSIDER THE EXPANSION OF $\cos x$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots = 1 - \cos x$$

- LET $x = \frac{\pi}{2}$

$$\frac{(\frac{\pi}{2})^2}{2!} - \frac{(\frac{\pi}{2})^4}{4!} + \frac{(\frac{\pi}{2})^6}{6!} - \frac{(\frac{\pi}{2})^8}{8!} + \dots = 1 - \cos \frac{\pi}{2}$$

$$\therefore \frac{\pi^2}{2^2 2!} - \frac{\pi^4}{2^4 4!} + \frac{\pi^6}{2^6 6!} - \frac{\pi^8}{2^8 8!} + \dots = 1$$

Question 71 (***)+

By showing a detailed method, sum the following series.

$$\frac{2}{1} + \frac{3}{2} + \frac{4}{4} + \frac{5}{8} + \frac{6}{16} + \frac{7}{32} \dots$$

[6]

$$\bullet \text{LET } S = \frac{2}{1} + \frac{3}{2} + \frac{4}{4} + \frac{5}{8} + \frac{6}{16} + \frac{7}{32} + \dots + \frac{n+1}{2^n} + \dots$$

(MULTIPLY THROUGH BY $\frac{1}{2}$)

$$\frac{1}{2}S = \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \dots$$

- 'LINK UP' THE TWO EXPRESSIONS AS FOLLOWS

$$S = \frac{2}{1} + \frac{3}{2} + \frac{4}{4} + \frac{5}{8} + \frac{6}{16} + \frac{7}{32} + \dots$$

$$-\frac{1}{2}S = -\frac{2}{2} - \frac{3}{4} - \frac{4}{8} - \frac{5}{16} - \frac{6}{32} - \dots$$

$$\Rightarrow \frac{1}{2}S = 2 + \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots}_{\text{THIS IS A GEOMETRIC PROGRESSION WITH } a = \frac{1}{2}, r = \frac{1}{2}}$$

$$\therefore S = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$\Rightarrow \frac{1}{2}S = 2 + 1$$

$$\Rightarrow \frac{1}{2}S = 3$$

$$\Rightarrow S = 6$$

Question 72 (*)+**

The positive integer functions f and g are defined as

$$f(n) = \sum_{r=1}^n r^3 \quad \text{and} \quad g(n) = 1 + \sum_{r=1}^n (2r+1).$$

Evaluate

$$\sum_{n=1}^{39} \left[\frac{f(n)}{g(n)} \right].$$

S1, [5135]

$f(n) = \sum_{r=1}^n r^3 = \frac{1}{4}n^2(2n+1)^2$

$g(n) = 1 + \sum_{r=1}^n (2r+1) = 1 + 2\sum_{r=1}^n r + \sum_{r=1}^n 1$

• **DEPENDING ON THE INDIVIDUAL COMPONENTS IS SIMPLIFIED FIRST**

$$\begin{aligned} f(n) &= \sum_{r=1}^n r^3 = \frac{1}{4}n^2(2n+1)^2 \\ g(n) &= 1 + \sum_{r=1}^n (2r+1) = 1 + 2\sum_{r=1}^n r + \sum_{r=1}^n 1 \\ &= 1 + 2 \times \frac{1}{2}n(n+1) + n \\ &= 1 + n(n+1) + n = 1 + n^2 + 2n + n \\ &= n^2 + 2n + 1 = (n+1)^2 \end{aligned}$$

• **HENCE WE HAVE**

$$\begin{aligned} \sum_{n=1}^{39} \frac{f(n)}{g(n)} &= \sum_{n=1}^{39} \frac{\frac{1}{4}n^2(2n+1)^2}{(n+1)^2} = \sum_{n=1}^{39} \frac{1}{4}n^2 \\ &= \frac{1}{4} \times \frac{1}{6}L(2n+1)(2n+3) \Big|_{n=1}^{n=39} \\ &= \frac{1}{24} \times 39 \times 40 \times 7 \\ &= \underline{\underline{5135}} \end{aligned}$$

Question 73 (**+)**

Find the Maclaurin expansion of $\arctan x$, and use it to show that

$$\pi = \sum_{n=0}^{\infty} f(n),$$

for some suitable function f .

$$\boxed{\quad}, \quad \boxed{\pi = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}}$$

START WITH DIFFERENTIATION & INTEGRATION

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

$$\frac{d}{dx}(\arctan x) = \sum_{n=0}^{\infty} [(-1)^n x^{2n}]$$

INTEGRATE WITH RESPECT TO x, ASSUMING INTEGRATION/SUMMATION COINCIDE

$$\arctan x = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx.$$

$$\arctan x = \sum_{n=0}^{\infty} \left[(-1)^n \frac{x^{2n+1}}{2n+1} \right] + C$$

Using $x=0 \Rightarrow 0 = 0 + C$

$$\arctan x = \sum_{n=0}^{\infty} \left[(-1)^n \frac{x^{2n+1}}{2n+1} \right]$$

FINALLY SUBSTITUTE x=1

$$\arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n \times 1^{2n+1}}{2n+1}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$$

if $f(x) = \frac{4(-1)^x}{2x+1}$

Question 74 (*****)

It is given that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} = k^n,$$

where n and k are positive integer constants.

- a) By considering the binomial expansion of $(1+x)^n$, determine the value of k .
- b) By considering the coefficient of x^n in

$$(1+x)^n (1+x)^n \equiv (1+x)^{2n},$$

simplify fully

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n-1}^2 + \binom{n}{n}^2.$$

 , $k = 2$

a) $(1+2x)^n = \binom{n}{0} 1^n 2^0 x^0 + \binom{n}{1} 1^{n-1} 2^1 x^1 + \binom{n}{2} 1^{n-2} 2^2 x^2 + \dots + \binom{n}{n} 1^0 2^n x^n$

$\Rightarrow (1+2x)^n = \binom{n}{0} + \binom{n}{1} 2x + \binom{n}{2} 2^2 x^2 + \binom{n}{3} 2^3 x^3 + \dots + \binom{n}{n} 2^n x^n$

LET $x=1$

$\Rightarrow (1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n}$

$\Rightarrow \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n} = 2^n$

b) $(1+x)(1+\infty)^n \equiv (1+x)^{2n}$

$\Rightarrow \left[\binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{n} x^n \right] \left[\binom{2n}{0} + \binom{2n}{1} x + \binom{2n}{2} x^2 + \dots + \binom{2n}{n} x^n \right]$

$\equiv \binom{2n}{0} + \binom{2n}{1} x + \binom{2n}{2} x^2 + \dots + \binom{2n}{2n} x^{2n}$

LOOKING AT THE COEFFICIENT OF x^n ON BOTH SIDES

$\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \dots + \binom{2n}{n} = \binom{2n}{n}$

BTW FROM THE DEFINITION OF BINOMIAL COEFFICIENTS
 $\binom{n}{r} = \binom{n}{n-r}$ E.g. $\binom{10}{3} = \binom{10}{7}$, $\binom{12}{2} = \binom{12}{10}$, $\binom{12}{3} = \binom{12}{9}$, $\binom{12}{4} = \binom{12}{8}$...

HENCE

$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$

Question 75 (*****)

By considering the binomial expansion of

$$\frac{1}{(1-\cos \theta)^2},$$

sum each of the following series.

- $\sum_{r=1}^{\infty} \left[\frac{r}{2^{r-1}} \right].$

- $\sum_{r=1}^{\infty} \left[\frac{r}{(-2)^{r-1}} \right].$

$$\boxed{\quad}, \quad \sum_{r=1}^{\infty} \left[\frac{r}{2^{r-1}} \right] = 4, \quad \sum_{r=1}^{\infty} \left[\frac{r}{(-2)^{r-1}} \right] = \frac{4}{9}$$

• Start by + SURD SUBSTITUTION, $x = \cos \theta$

$$\begin{aligned} \frac{1}{(1-\cos \theta)^2} &= \frac{1}{(1-x)^2} = (1-x)^{-2} \\ &= 1 + \frac{-2}{1-x}(x) + \frac{-2(-1)}{(1-x)^2}(x)^2 + \frac{(-2)(-1)(-3)}{(1-x)^3}(x)^3 + O(x^4) \\ &= 1 + 2x + 3x^2 + 4x^3 + O(x^4) \quad |x| < 1 \\ &= 1 + 2\cos \theta + 3\cos^2 \theta + 4\cos^3 \theta + \dots \quad |\cos \theta| < 1 \\ &= \sum_{r=1}^{\infty} r(\cos \theta)^{r-1} \end{aligned}$$

• Now $\sum_{r=1}^{\infty} \frac{r}{2^{r-1}} = \dots$ is THE ABOVE EXPANSION WITH $\cos \theta = \frac{1}{2}$

$$\begin{aligned} &= (1, \text{ i.e. } \theta = \frac{\pi}{3}) \\ &= \frac{1}{(1-\frac{1}{2})^2} = \frac{1}{(\frac{1}{2})^2} = \frac{1}{\frac{1}{4}} = 4 \quad // \end{aligned}$$

• AND $\sum_{r=1}^{\infty} \frac{r}{(-2)^{r-1}} = \dots$ is THE ABOVE EXPANSION WITH $\cos \theta = -\frac{1}{2}$

$$\begin{aligned} &= (1, \text{ i.e. } \theta = \frac{2\pi}{3}) \\ &= \frac{1}{(1-(-\frac{1}{2}))^2} = \frac{1}{(\frac{3}{2})^2} = \frac{1}{\frac{9}{4}} = \frac{4}{9} \quad // \end{aligned}$$

Question 76 (*****)

$$f(x) \equiv \frac{1-x}{1+x+x^2+x^3}, -1 < x < 1.$$

Show that $f(x)$ can be written in the form

$$f(x) = g(x) \sum_{r=0}^{\infty} (x^{4r}),$$

where $g(x)$ is a simplified function to be found.

$$\boxed{\quad}, \boxed{g(x) = (1-x)^2}$$

$$\begin{aligned} f(x) &= \frac{1-x}{1+x+x^2+x^3} = \frac{1-x}{(1+x)(1+x^2)} = \frac{1-x}{(1+x)(1+x^2)} \\ &= \frac{(1-x)(1-x)}{(1-x)(1+x)(1+x^2)} = \frac{(1-x)^2}{(1+x)(1+x^2)} \\ &= \frac{(1-x)^2}{1-x^4} \end{aligned}$$

NOW USING STANDARD EXPANSION, OR THE SUM TO INFINITY OF A G.P.

$$\begin{aligned} f(x) &= \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \\ &\dots = (1-x)^{-2} (1 + x^4 + x^8 + x^{12} + \dots) \\ &\dots = (1-x)^2 \sum_{r=0}^{\infty} x^{4r} \end{aligned}$$

LONGER ALTERNATIVE

$$f(x) = \frac{1-x}{(1+x)(1+x^2)(1+x^3)} = \dots = \frac{1-x}{(1+x)(1+x^2)} \dots \text{ NOW PARTIAL FRACTION}$$

$$\begin{aligned} \frac{1-x}{(1+x)(1+x^2)} &\equiv \frac{A}{1+x} + \frac{Bx+C}{1+x^2} \\ 1-x &\equiv A(1+x^2) + (Bx+C)(Bx+C) \\ \text{IF } 2x+1 &\Rightarrow 2=2A \Rightarrow A=1 \\ \text{IF } 2x=0 &\Rightarrow 1=A+C \Rightarrow C=0 \\ \text{IF } x=1 &\Rightarrow 0=2A+2B \Rightarrow B=-1 \end{aligned}$$

THESE WE HAVE

$$\begin{aligned} f(x) &= \frac{1}{1+x} - \frac{x}{1+x^2} \\ f(x) &= (1-x + x^3 - x^7 + \dots) - x(1-x^2 + x^6 - x^{10} + \dots) \\ f(x) &= 1 - x + x^3 - x^5 + x^7 - x^9 + x^{11} - x^{13} + x^{15} - x^{17} + x^{19} - \dots \\ f(x) &= ((1-x)+x^4) + (x^8-2x^4+x^2) + (x^{12}-2x^8+x^{10}) + \dots \\ f(x) &= ((1-2x+x^2)) + (x^8-2x^4+x^2) + (x^{12}-2x^8+x^{10}) + \dots \\ f(x) &= ((1-2x+x^2)) [1 + x^4 + x^8 + x^{12} + \dots] \\ f(x) &= (1-x)^2 \sum_{r=0}^{\infty} x^{4r} \end{aligned}$$

AS SOON AS

Question 77 (*****)

The sum to infinity S of the convergent geometric series is given by

$$S = 1 + x + x^2 + x^3 + x^4 + \dots, \quad |x| < 1,$$

By integrating the above equation between suitable limits, or otherwise, find

$$\sum_{r=1}^{\infty} \left[\frac{1}{r \times 2^r} \right].$$

You may assume that integration and summation commute.

, $\ln 2$

• WRITE THE GEOMETRIC SERIES COMPACTLY

$$\Rightarrow S = 1 + x + x^2 + x^3 + x^4 + \dots \quad |x| < 1$$

$$\Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad |x| < 1$$

• IN ORDER TO PRODUCE THE REQUIRED SERIES WE DIVIDE THE LHS BY x :

$$\Rightarrow \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

$$\Rightarrow \int \left[\sum_{n=1}^{\infty} x^{n-1} \right] dx = \int \frac{1}{1-x} dx$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\int x^{n-1} dx \right] = \left[-\ln(1-x) \right]_0^1$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\frac{x^n}{n} \right]_0^1 = \left[\ln(1-x) \right]_0^1$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\frac{1}{n} \left(\frac{1}{2^n} - 0 \right) \right] = -\ln \frac{1}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\frac{1}{n \cdot 2^n} \right] = -\ln \frac{1}{2}$$

$$\Rightarrow \sum_{r=1}^{\infty} \left[\frac{1}{r \times 2^r} \right] = \ln 2$$

ALTERNATIVE SOLUTION USING STANDARD EXPANSIONS

CONSIDERING THE EXPANSION OF $\ln(1-x)$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

let $x = \frac{1}{2}$

$$\ln \frac{1}{2} = - \left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots \right]$$

$$-\ln 2 = - \left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots \right]$$

$$\sum_{r=1}^{\infty} \left[\frac{1}{r \cdot 2^r} \right] = \ln 2$$

Question 78 (*****)

Show clearly that

$$\sum_{r=1}^{\infty} \frac{r^2}{r!} = 2e.$$

[proof]

$$\begin{aligned}
 \sum_{r=1}^{\infty} \frac{r^2}{r!} &= \left(1 + \sum_{r=1}^{\infty} \frac{r^2}{r!} \right) - \left(1 + \sum_{r=1}^{\infty} \frac{r^2}{r!} \right) + \sum_{r=1}^{\infty} \frac{r^2}{r!} \\
 &= \left(1 + \sum_{r=1}^{\infty} \left(\frac{r^2}{r!} + \frac{1}{r!} \right) \right) - \left(1 + \sum_{r=1}^{\infty} \frac{1}{r!} \right) + \sum_{r=1}^{\infty} \frac{r^2}{r!} \\
 &= \left(1 + \sum_{r=1}^{\infty} \frac{1}{(r-1)!} \right) + \sum_{r=1}^{\infty} \frac{1}{r!} + \sum_{r=1}^{\infty} \frac{r^2}{r!} \\
 &\stackrel{\text{defn}}{=} e + e + e = 3e.
 \end{aligned}$$

Question 79 (***)**

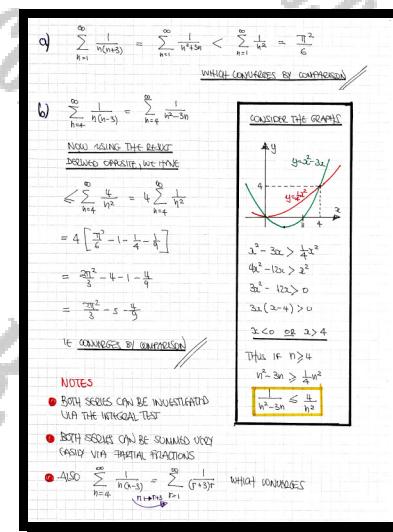
Investigate the convergence or divergence of each of the following two series using standard tests and justifying every step in the workings.

a) $\sum_{n=1}^{\infty} \left[\frac{1}{n(n+3)} \right].$

b) $\sum_{n=4}^{\infty} \left[\frac{1}{n(n-3)} \right].$

You may not conclude simply by summing each the series.

, convergent , convergent



Question 80 (*****)

The finite sum C is given below.

$$C = \sum_{r=0}^n \left[\binom{n}{r} (-1)^r \cos^n \theta \cos r\theta \right].$$

Given that $n \in \mathbb{N}$ determine the 4 possible expressions for C .

Give the answers in exact fully simplified form.

- , $n = 4k, k \in \mathbb{N} : C = \cos n\theta \sin^n \theta$, $n = 4k+1, k \in \mathbb{N} : C = \sin n\theta \sin^n \theta$,
 $n = 4k+2, k \in \mathbb{N} : C = -\cos n\theta \sin^n \theta$, $n = 4k+3, k \in \mathbb{N} : C = -\sin n\theta \sin^n \theta$

$1 - \binom{n}{1} \cos \theta \sin \theta + \binom{n}{2} \cos^2 \theta \sin^2 \theta - \binom{n}{3} \cos^3 \theta \sin^3 \theta + \dots + (-1)^{\frac{n}{2}} \cos^{\frac{n}{2}} \theta \sin^{\frac{n}{2}} \theta$
 $C = 1 - \binom{n}{1} \cos \theta \sin \theta + \binom{n}{2} \cos^2 \theta \sin^2 \theta - \binom{n}{3} \cos^3 \theta \sin^3 \theta + \dots + (-1)^{\frac{n}{2}} \cos^{\frac{n}{2}} \theta \sin^{\frac{n}{2}} \theta$
 $S = -\binom{n}{0} \cos^0 \theta \sin^0 \theta + \binom{n}{1} \cos^1 \theta \sin^1 \theta - \binom{n}{2} \cos^2 \theta \sin^2 \theta + \dots + (-1)^{\frac{n}{2}} \cos^{\frac{n}{2}} \theta \sin^{\frac{n}{2}} \theta$
 $C + iS = 1 - \binom{n}{0} \cos^0 \theta [\sin^0 \theta + i \cos^0 \theta] + \binom{n}{1} \cos^1 \theta [\sin^1 \theta + i \cos^1 \theta] - \binom{n}{2} \cos^2 \theta [\sin^2 \theta + i \cos^2 \theta] + \dots + (-1)^{\frac{n}{2}} \cos^{\frac{n}{2}} \theta [\sin^{\frac{n}{2}} \theta + i \cos^{\frac{n}{2}} \theta]$
 $= 1 - \binom{n}{0} e^{i0\theta} \cos^0 \theta + \binom{n}{1} e^{i1\theta} \cos^1 \theta - \binom{n}{2} e^{i2\theta} \cos^2 \theta + \dots + (-1)^{\frac{n}{2}} e^{i\frac{n}{2}\theta} \cos^{\frac{n}{2}} \theta$
 which is a binomial expansion $(1 - e^{i\theta})^n$
 $= (1 - e^{i\theta} \cos \theta)^n = (1 - \cos(\theta) + i \sin(\theta))^n = (1 - \cos \theta - i \sin \theta)^n$
 $= (\sin \theta - i \cos \theta)^n = \sin^n \theta [\sin \theta - i \cos \theta]^n = (\sin \theta)^n \sin^n \theta [\cos \theta + i \sin \theta]^n$
 $= (-i \sin^n \theta e^{in\theta}) = (-i)^n \sin^n \theta (e^{in\theta}) = (-1)^n \sin^n \theta [\cos(n\theta) + i \sin(n\theta)]$

- If $n = 4k, k \in \mathbb{N}$, $(-1)^n = 1 \Rightarrow C+iS = \cos n\theta \sin^n \theta + i \sin n\theta \sin^n \theta \Rightarrow C = \cos n\theta \sin^n \theta$
- If $n = 4k+1, k \in \mathbb{N}$, $(-1)^n = -1 \Rightarrow C+iS = \sin n\theta \sin^n \theta - i \cos n\theta \sin^n \theta \Rightarrow C = \sin n\theta \sin^n \theta$
- If $n = 4k+2, k \in \mathbb{N}$, $(-1)^n = 1 \Rightarrow C+iS = -\cos n\theta \sin^n \theta - i \sin n\theta \sin^n \theta \Rightarrow C = -\cos n\theta \sin^n \theta$
- If $n = 4k+3, k \in \mathbb{N}$, $(-1)^n = -1 \Rightarrow C+iS = -\sin n\theta \sin^n \theta + i \cos n\theta \sin^n \theta \Rightarrow C = -\sin n\theta \sin^n \theta$

Question 81 (*****)

$$f(x) \equiv \frac{2-3x}{(1-x)(1-2x)}, -\frac{1}{2} < x < \frac{1}{2}.$$

Show that $f(x)$ can be written in the form

$$f(x) = \sum_{r=0}^{\infty} [x^r g(r)],$$

where $g(r)$ is a simplified function to be found.

$$\boxed{\quad}, \boxed{g(r) = 2^r + 1}$$

• SIMPLIFY BY REARRANGING & SPLITTING INTO FRACTIONAL FRACTIONS BY INSPECTION

$$\Rightarrow f(x) = \frac{2-3x}{(1-x)(1-2x)} = (2-3x) \times \frac{1}{(1-x)(1-2x)}$$

$$\Rightarrow f(x) = (2-3x) \left[\frac{1}{1-x} + \frac{1}{1-2x} \right]$$

$$\Rightarrow f(x) = (2-3x) \left[\frac{2}{1-2x} - \frac{1}{1-x} \right]$$

• NEXT WE USE STANDARD EXPANSIONS FOR THE SUM TO INFINITY OF A G.P.

$$\frac{1}{1-x} = 1+t+t^2+t^3+\dots,$$

TO OBTAIN:

$$\Rightarrow f(x) = (2-3x) \left[2 \left(1+2x+4x^2+8x^3+\dots \right) - \left(1+x+x^2+x^3+\dots \right) \right]$$

$$\Rightarrow f(x) = (2-3x) \left[2 + 4x + 8x^2 + 16x^3 + \dots \right]$$

$$\Rightarrow f(x) = (2-3x) (1 + 3x + 7x^2 + 15x^3 + \dots)$$

$$\Rightarrow f(x) = (2-3x) \sum_{r=0}^{\infty} (2^{r+1}) x^r$$

$$\Rightarrow f(x) = 2 \sum_{r=0}^{\infty} (2^{r+1}) x^r - 3 \sum_{r=0}^{\infty} (2^{r+1}) x^{r+1}$$

• ADJUST THE FIRST* EXPANSION AS FOLLOWS

$$\Rightarrow f(x) = 2 + 2 \sum_{r=1}^{\infty} (2^{r+1}) x^r - 3 \sum_{r=0}^{\infty} (2^{r+1}) x^{r+1}$$

• NEXT ADJUST THE FIRST EXPANSION SO IT STARTS FROM R=0 AGAIN

$$f(x) = 2 + 2 \sum_{r=0}^{\infty} (2^{r+1}) x^{r+1} - 3 \sum_{r=0}^{\infty} (2^{r+1}) x^{r+1}$$

$$f(x) = 2 + \sum_{r=0}^{\infty} [2(2^{r+1}) - 3(2^{r+1})] x^{r+1}$$

$$f(x) = 2 + \sum_{r=0}^{\infty} (-4x^{r+1} - 2 - 3x^{r+1} + 3) x^{r+1}$$

$$f(x) = 2 + \sum_{r=0}^{\infty} (2^{r+1} + 1) x^{r+1}$$

• ADJUST THE EXPANSION SO THAT IT STARTS FROM R=1

$$f(x) = 2 + \sum_{r=1}^{\infty} (2^{r+1}) x^r$$

$$f(x) = (2^2 + 1) x^1 + \sum_{r=1}^{\infty} (2^{r+1}) x^r$$

$$f(x) = \sum_{r=1}^{\infty} (2^{r+1}) x^r$$

ANOTHER APPROACH THROUGH NOT AS FORMAL IS AS FOLLOWS

$$f(x) = (2-3x) (1 + 3x + 7x^2 + 15x^3 + \dots) \quad \leftarrow \text{REARRANGE}$$

$$f(x) = 2 + 6x + 14x^2 + 30x^3 + \dots$$

$$\quad - 3x - 9x^2 - 21x^3 - \dots$$

$$f(x) = 2 + 3x + 5x^2 + 9x^3 + \dots$$

WHICH ONE MIGHT DEDUCE IS $\sum_{r=0}^{\infty} (2^{r+1}) x^r$

Question 82 (*****)

Show, by considering standard series, that

$$\int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\pi^2}{12}.$$

You may assume without proof that $\sum_{n=1}^{\infty} \left[\frac{1}{n^2} \right] = \frac{1}{6}\pi^2$

[proof]

$\int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\pi^2}{12}$

• Using the expansion of $\ln(1+x)$

$$\begin{aligned} \int_0^1 \frac{\ln(1+x)}{x} dx &= \int_0^1 \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^n}{n} (-1)^{n+1} dx \\ &= \int_0^1 \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} (-1)^{n+1} dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{n} (-1)^{n+1} dx \\ &= \sum_{n=1}^{\infty} \left[\frac{x^n}{n} (-1)^{n+1} \right]_{x=0}^{x=1} = \sum_{n=1}^{\infty} \left[\frac{1}{n} (-1)^{n+1} - 0 \right] \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \left(-\frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots \right) \end{aligned}$$

• Now $\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}$

$$\begin{aligned} \zeta(2) &\approx 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6} \\ \frac{1}{2^2} \zeta(2) &\approx \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{24} \end{aligned}$$

$\left\{ \begin{aligned} \zeta(2) &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6} \\ -2 \times \frac{1}{2^2} \zeta(2) &= -\frac{2}{2^2} - \frac{2}{3^2} - \frac{2}{4^2} + \dots = -\frac{2\pi^2}{24} \end{aligned} \right\}$

• Hence $\frac{1}{2} \zeta(2) = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots = \frac{\pi^2}{12}$

$$\int_0^1 \frac{\ln(1+x)}{x} dx = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots = \frac{\pi^2}{12}$$

Question 83 (*****)

Show, by a detailed method, that

$$\frac{48}{2 \times 3} + \frac{47}{3 \times 4} + \frac{46}{4 \times 5} \dots + \frac{2}{48 \times 49} + \frac{1}{49 \times 50} = A + B \sum_{r=1}^{50} \frac{1}{r},$$

where A and B are constants to be found.

, $A = \frac{51}{2}$, $B = -1$

WORKING WITH SIMPLIFIED FRACTION FORM

$$\sum_{k=1}^{49} \frac{49-k}{(k+1)k(2k+1)} = \sum_{k=1}^{49} \left(\frac{50}{k(k+1)} - \frac{5}{k(2k+1)} \right)$$

$$= \left(\frac{50}{1 \cdot 2} - \frac{5}{1 \cdot 3} \right) + \left(\frac{50}{2 \cdot 3} - \frac{5}{2 \cdot 5} \right) + \left(\frac{50}{3 \cdot 4} - \frac{5}{3 \cdot 7} \right) + \dots + \left(\frac{50}{48 \cdot 49} - \frac{5}{48 \cdot 97} \right) + \left(\frac{50}{49 \cdot 50} - \frac{5}{49 \cdot 101} \right)$$

$$= 25 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{49} \right] - \frac{5}{101}$$

$$= 25 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{49} \right] - 1 - \frac{1}{50}$$

$$= 25 - \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{49} + \frac{1}{50} \right]$$

$$= 25 + \frac{1}{2} - \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{49} + \frac{1}{50} \right]$$

$$= \frac{51}{2} - \frac{5}{51}$$

$A = \frac{51}{2}$
 $B = -1$

Question 84 (*****)

$$S = 1 + \frac{2}{4} + \frac{2 \cdot 3}{4 \cdot 8} + \frac{2 \cdot 3 \cdot 4}{4 \cdot 8 \cdot 12} + \frac{2 \cdot 3 \cdot 4 \cdot 5}{4 \cdot 8 \cdot 12 \cdot 16} + \dots$$

By considering a suitable binomial series, or other wise, find the sum to infinity of S .

$$\boxed{\quad}, \quad S_{\infty} = \frac{16}{9}$$

MINIMISE THE SERIES STEP BY STEP.

$$\begin{aligned} \Rightarrow S &= 1 + \frac{2}{4} + \frac{2 \cdot 3 \cdot 4}{4 \cdot 8} + \frac{2 \cdot 3 \cdot 4 \cdot 5}{4 \cdot 8 \cdot 12} + \dots \\ \Rightarrow S &= 1 + \frac{2}{4(1)} + \frac{2(1)}{4^2(2)} + \frac{2(1)(2)}{4^3(2^2)(3)} + \dots \\ \Rightarrow S &= 1 + \frac{2}{4} \left(\frac{1}{4}\right) + \frac{2(1)}{4^2} \left(\frac{1}{4}\right)^2 + \frac{2(1)(2)}{4^3(2^2)} \left(\frac{1}{4}\right)^3 + \dots \end{aligned}$$

Finally we need to "free one" of the sums in order to form a
convergent binomial expansion

$$\begin{aligned} \Rightarrow S &= 1 + \frac{2}{4} \left(\frac{1}{4}\right) + \frac{(1-\cancel{2})}{2!} \left(\frac{1}{4}\right)^2 + \frac{(1-\cancel{2})(1-\cancel{3})}{3!} \left(\frac{1}{4}\right)^3 + \frac{(1-\cancel{2})(1-\cancel{3})(1-\cancel{4})}{4!} \left(\frac{1}{4}\right)^4 + \dots \\ \Rightarrow S &= \left(1 - \frac{1}{4}\right)^{-2} \\ \Rightarrow S &= \left(\frac{3}{4}\right)^{-2} \\ \Rightarrow S &= \left(\frac{3}{4}\right)^{-1} \\ \Rightarrow S &= \frac{4}{3} \end{aligned}$$

Question 85 (*****)

$$3 + 33 + 333 + 3333 + 33333 + \dots$$

Express the sum of the first n terms of the above series in sigma notation.

You are not required to sum the series.

$$S_n = \sum_{r=1}^n \left[\frac{1}{3}(10^r - 1) \right]$$

$$\begin{aligned} S &= 3 + 33 + 333 + 3333 + 33333 + \dots \\ S &= (\frac{3}{9})10^1 + (\frac{3}{9}) \times 99 + (\frac{3}{9}) \times 999 + (\frac{3}{9}) \times 9999 + \dots \\ S &= \frac{3}{9} [0 + 99 + 999 + 9999 + \dots] \\ S &= \frac{1}{3} [(10^1 - 1) + (10^2 - 1) + (10^3 - 1) + (10^4 - 1) + \dots] \\ S &= \frac{1}{3} [(10^1 + 10^2 + 10^3 + 10^4 + \dots) + (-1 - 1 - \dots - 1)] \\ S &= \frac{1}{3} \sum_{r=1}^n (10^r - 1) \end{aligned}$$

↙ n of them

Question 86 (*****)

$$S_n = (2 \times 1!) + (5 \times 2!) + (10 \times 3!) + (17 \times 4!) + \dots + (n^2 + 1)n!$$

Use an appropriate method to show that

$$S_n = n(n+1)!$$

proof

SIMPLY BY WRITING THE SUMS IN SIMPLE NOTATION

$$(2 \times 1!) + (5 \times 2!) + (10 \times 3!) + \dots + [(n^2 + 1)n!] = \sum_{r=1}^n [(r^2 + 1)r!]$$

TRY SOME DIFFERENCES INCLUDING PREVIOUSLY, TRYING TO CANCEL OFF SOMETHING

$$(r+2)! - r! = (r+2)(r+1)r! - r! = r(r+1)r!$$

AS THIS DOES NOT PRODUCE A QUADRATIC TERM IN r WE MAY TRY

$$\begin{aligned} (r+2)! - r! &= (r+2)(r+1)r! - r! \\ (r+2)! - r! &= (r^2 + 3r + 2)r! - r! \\ (r+2)! - r! &= (r^2 + 3r + 1)r! \\ (r+2)! - r! &= (r^2 + 3r)r! + 3r \cdot r! \\ &\quad \uparrow \\ \text{NOTICE } r \cdot r! &\equiv (r+1)r! - r! \end{aligned}$$

$$\begin{aligned} (r+2)! - r! &= (r^2 + 1)r! + 3[(r+1)r! - r!] \\ (r+2)! - r! &= (r^2 + 1)r! + 3(r+1)r! - 3r! \\ (r+2)! &= (r^2 + 1)r! + 3(r+1)r! - 2r! \\ (r+2)! - 3(r+1)r! + 2r! &= (r^2 + 1)r! \end{aligned}$$

HENCE WE GET

$$(r^2 + 1)r! \equiv (r+2)! - 3(r+1)r! + 2r!$$

WRITING THE IDENTITY JUST ESTABLISHED

$$(r^2 + 1)r! \equiv (r+2)! - 3(r+1)r! + 2r!$$

$\begin{aligned} r=1: \quad 2 \times 1! &= 2! = 3 \times 1! + 2 \times 1! \\ r=2: \quad 5 \times 2! &= 4! = 3 \times 2! + 2 \times 2! \\ r=3: \quad 10 \times 3! &= 5! = 3 \times 3! + 2 \times 3! \\ r=4: \quad 17 \times 4! &= 6! = 3 \times 4! + 2 \times 4! \\ &\vdots &\vdots &\vdots &\vdots \\ r=n: \quad [n^2 + 1]n! &= (n+1)! = 3 \times n! + 2 \times n! \\ (r^2 + 1)r! &= (n+1)! = 2 \times (n+1)! + 2 \times n! \end{aligned}$	$\sum_{r=1}^n [(r^2 + 1)r!] = (n+2)! - 2(n+1)! - 3 \times 2! + 2 \times 1! + 2 \times 2!$ $\begin{aligned} &= (n+2)(n+1)! - 2(n+1)! - 6 + 2 + 4 \\ &= (n+2)(n+1)! \\ &= n(n+1)! \end{aligned}$
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Question 87 (*****)

Consider the infinite series

$$1 + \frac{-1}{2 \times 1} x^2 + \frac{-1 \times 1}{4 \times 3 \times 2 \times 1} x^4 + \frac{-1 \times 1 \times 3}{6 \times 5 \times 4 \times 3 \times 2 \times 1} x^6 + \frac{-1 \times 1 \times 3 \times 5}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} x^8 + \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{n=0}^{\infty} \left[\frac{2^{n-1} \Gamma\left(n - \frac{1}{2}\right)}{-\sqrt{\pi} \times (2n)!} x^{2n} \right]$$

$$\begin{aligned}
 & 1 + \frac{-1}{2 \times 1} x^2 + \frac{-1 \times 1}{4 \times 3 \times 2 \times 1} x^4 + \frac{-1 \times 1 \times 3}{6 \times 5 \times 4 \times 3 \times 2 \times 1} x^6 + \dots \\
 \text{looking at } \left[\frac{x^8}{8!} \right] \text{ if we need term } (\text{e.g. } n=4) \text{ if we start from } n=0 \\
 \frac{-1 \times 1 \times 3 \times 5}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} x^8 &= \frac{2^8 \left(-\frac{1}{2} \times \frac{3}{2} \times \frac{5}{2} \right)}{8!} x^8 = \frac{2^8 \times \Gamma(-\frac{1}{2}) \times \frac{3}{2} \times \frac{5}{2} \times \frac{7}{2}}{(7 \frac{1}{2})! 8!} x^8 \\
 \text{Now, } \Gamma(n+1) &= n \Gamma(n) \\
 \text{or } n-\frac{1}{2}, \quad \Gamma(\frac{1}{2}) &= \frac{1}{2} \Gamma(\frac{1}{2}) \\
 \sqrt{\pi} &= \frac{1}{2} \Gamma(\frac{1}{2}) \\
 \Gamma(-\frac{1}{2}) &= -2\sqrt{\pi}
 \end{aligned}$$

$\dots = \frac{2^8 \Gamma(\frac{1}{2})}{-2\sqrt{\pi} 8!} x^8$
 i.e. starting from $n=0$, $\text{e.g. } n=4$
 $\therefore \sum_{k=0}^{\infty} \frac{2^k \Gamma(\frac{2k+1}{2})}{-2\sqrt{\pi} (2k)!} x^{2k}$
 $= \sum_{k=0}^{\infty} \frac{2^{2k} \Gamma(\frac{2k+1}{2})}{-4\sqrt{\pi} (2k)!} x^{2k} \blacksquare$

Question 88 (*****)

$$\frac{3}{1^2+2^2} + \frac{5}{1^2+2^2+3^2} + \frac{7}{1^2+2^2+3^2+4^2} + \frac{9}{1^2+2^2+3^2+4^2+5^2} + \dots,$$

Show, by a detailed method, that the sum of the first 40 terms of the series shown above is $\frac{240}{41}$.

, proof

$$\begin{aligned}
 S_{40} &= \frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \frac{11}{1^2+2^2+3^2+4^2+5^2} + \dots \\
 S_{40} &= \sum_{n=1}^{40} \left[\frac{\frac{1}{2}(2n+1)}{n(n+1)} \right] = \sum_{n=1}^{40} \left[\frac{2n+1}{2n(n+1)(2n+1)} \right] \\
 &= 6 \sum_{n=1}^{40} \frac{1}{n(n+1)} = 6 \sum_{n=1}^{40} \left[\frac{1}{n} - \frac{1}{n+1} \right] \\
 &= 6 \left[\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{40} - \frac{1}{41} \right) \right] \\
 &= 6 \left[1 - \frac{1}{41} \right] = 6 \times \frac{40}{41} = 6 \times \frac{10}{41} = \frac{240}{41}
 \end{aligned}$$

Question 89 (*****)

By showing a detailed method, sum the following series.

$$\frac{1}{2^2 2!} + \frac{1}{2^4 4!} + \frac{1}{2^6 6!} + \frac{1}{2^8 8!} + \dots + \frac{1}{2^{2r} (2r)!} + \dots$$

$$\boxed{\frac{1}{2} \left[e^{\frac{1}{4}} - e^{-\frac{1}{4}} \right]^2 = 2 \sinh^2 \left(\frac{1}{4} \right)}$$

$S = \frac{1}{2^2 2!} + \frac{1}{2^4 4!} + \frac{1}{2^6 6!} + \frac{1}{2^8 8!} + \dots$

Calculate the series expansion of $y = \cosh x$.

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$
$$\cosh \left(\frac{x}{2}\right) = 1 + \frac{\left(\frac{x}{2}\right)^2}{2!} + \frac{\left(\frac{x}{2}\right)^4}{4!} + \frac{\left(\frac{x}{2}\right)^6}{6!} + \frac{\left(\frac{x}{2}\right)^8}{8!} + \dots$$
$$\cosh \left(\frac{x}{2}\right) = 1 + \frac{1}{2^2 2!} + \frac{1}{2^4 4!} + \frac{1}{2^6 6!} + \frac{1}{2^8 8!} + \dots$$
$$\cosh \left(\frac{x}{2}\right) = 1 + S$$
$$S = \cosh \frac{x}{2} - 1$$
$$S = \frac{1}{2} e^{\frac{x}{2}} + \frac{1}{2} e^{-\frac{x}{2}} - 1$$
$$S = \frac{1}{2} (e^{\frac{x}{2}} - e^{-\frac{x}{2}})$$
$$S = \frac{1}{2} (e^x - e^{-x})^2$$

$$S = [1 + 2 \sinh^2 \left(\frac{1}{4} \right)] - 1$$
$$S = 2 \sinh^2 \left(\frac{1}{4} \right)$$

Question 90 (*****)

A function is defined as

$$[x] \equiv \{ \text{the greatest integer less or equal to } x \}.$$

The function f is defined as

$$f(n) = n \left[\frac{3}{5} + \frac{3n}{100} \right], \quad n \in \mathbb{N}.$$

Determine the value of

$$\sum_{n=1}^{82} f(n).$$

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$[x] \equiv \{ \text{GREATEST INTEGER LESS OR EQUAL TO } x \}$

- $f(n) = n \left[\frac{3}{5} + \frac{3n}{100} \right], \quad n \in \mathbb{N}$
- WE NEED TO DETERMINE THE SUM OF THE FIRST 82 TERMS.
WE NEED TO WORK IN BLOCKS
- $\frac{3}{5} + \frac{3n}{100} \leq 1$
 $\frac{3n}{100} \leq \frac{2}{5}$
 $3n \leq 40$
 $n \leq \frac{40}{3} = 13\frac{1}{3}$
 $\therefore \text{THE FIRST } "3\text{ BLOCKS"} \text{ OF } [\dots] \text{ ARE ALL } 13\frac{1}{3}$
 $\therefore \text{THE } "3\text{ BLOCKS"} \text{ OF } [\dots] \text{ FROM } 14\text{ TO } 26\text{ ARE } 14$
 $\therefore \text{THE } "3\text{ BLOCKS"} \text{ OF } [\dots] \text{ FROM } 27\text{ TO } 49\text{ ARE } 15$
 $\therefore \text{THE } "3\text{ BLOCKS"} \text{ OF } [\dots] \text{ FROM } 50\text{ TO } 72\text{ ARE } 16$
 $\therefore \text{THE } "3\text{ BLOCKS"} \text{ OF } [\dots] \text{ FROM } 73\text{ TO } 82\text{ ARE } 17$
- THE "BLOCKS" OF $[\dots]$ ARE ALL 13, 14, 15, 16, 17
- THE "BLOCKS" OF $[\dots]$ FROM 14 TO 26 ARE 14
- THE "BLOCKS" OF $[\dots]$ FROM 27 TO 49 ARE 15
- THE "BLOCKS" OF $[\dots]$ FROM 50 TO 72 ARE 16
- THE "BLOCKS" OF $[\dots]$ FROM 73 TO 82 ARE 17
- EXPLICITLY THE 82nd, 83rd, 84th TERMS OF $[\dots]$ ARE 13, 14, 15
- SUMMING UP THE SERIES

$$\begin{aligned} \sum_{n=1}^{82} f(n) &= \sum_{n=1}^{82} n \left[\frac{3}{5} + \frac{3n}{100} \right] \\ &= \left[\sum_{n=1}^{13} (13n) \right] + \left[\sum_{n=14}^{26} (14n) \right] + \left[\sum_{n=27}^{49} (15n) \right] + \left[\sum_{n=50}^{72} (16n) \right] + \left[\sum_{n=73}^{82} (17n) \right] \\ &\quad 13 = 14 \quad 14 = 14 \times 13 \quad 15 = 15 \times 23 \quad 16 = 16 \times 23 \quad 17 = 17 \times 10 \\ &\quad d = 1 \quad d = 2 \quad d = 1 \quad d = 2 \quad d = 1 \\ &\quad L = 14 \quad L = 26 \quad L = 49 \quad L = 72 \quad L = 82 \\ &\quad n = 13 \quad n = 23 \quad n = 23 \quad n = 23 \quad n = 10 \end{aligned}$$

USING $S_n = \frac{n}{2} [a + l]$

- $\sum_{n=14}^{46} n = \frac{33}{2} [14 + 46] = \frac{33}{2} \times 60 = 33 \times 30 = 990$
- $\sum_{n=47}^{72} 2n = \frac{26}{2} [94 + 158] = \frac{26}{2} \times 252 = 33 \times 26 = \frac{3780}{4158}$
- $\sum_{n=83}^{82} 3n = (80 + 81 + 82) \times 3 = 243 \times 3 = 729$
- $\therefore \sum_{n=1}^{82} n f(n) = 990 + 4158 + 729$
 $= 4158 \quad 3990 \quad 729$
 $= 5877$

Question 91 (*****)

Use partial fractions and a suitable Mclaurin expansion to sum the following series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+3)}.$$

$$\boxed{\frac{2}{3} \ln 2 - \frac{5}{18}}$$

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+3)} &= \sum_{n=1}^{\infty} (-1)^{n+1} \times \frac{1}{n(n+3)} = \dots \text{ PARTIAL FRACTIONS BY INSPECTION } \dots \\
 &= \sum_{n=1}^{\infty} (-1)^{n+1} \times \left[\frac{\frac{1}{3}}{n} - \frac{\frac{1}{3}}{n+2} \right] = \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \frac{(-1)^{n+1}}{n+2} \\
 &= \frac{1}{3} \left[\left(1 - \frac{1}{2} \right) + \left(-\frac{1}{2} + \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(-\frac{1}{4} + \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) + \dots \right] \\
 &= \frac{1}{3} \left[\left(1 - \frac{1}{2} + \frac{1}{3} \right) + 2 \left(-\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right) \right] \\
 &= \frac{5}{18} + \frac{2}{3} \left(-\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots \right)
 \end{aligned}$$

Now $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \dots$
 $\ln(1+x) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$
 $\ln 2 = \left(1 - \frac{1}{2} + \frac{1}{3} \right) + \left(-\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots \right)$
 $\ln 2 - \frac{5}{18} = -\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$
 $= \frac{5}{18} + \frac{2}{3} \left(\ln 2 - \frac{5}{18} \right)$
 $= \frac{2}{3} \ln 2 - \frac{5}{18}$

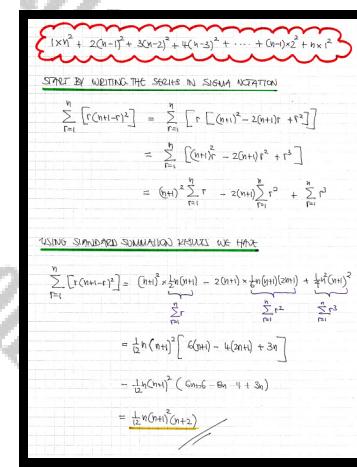
Question 92 (*****)

The function f is defined for $n \in \mathbb{N}$ as

$$f(n) \equiv 1 \times n^2 + 2(n-1)^2 + 3(n-2)^2 + 4(n-3)^2 + \dots + (n-1) \times 2^2 + n \times 1^2.$$

Determine a simplified expression for the sum of $f(n)$, giving the final answer in fully factorized form.

, $f(n) = \frac{1}{12}n(n+2)(n+1)^2$



START BY WRITING THE TERMS IN SUMMATION NOTATION

$$\sum_{r=1}^n [r(n-r)^2] = \sum_{r=1}^n [r((n+r)-2(r+1)r + r^2)]$$

$$= \sum_{r=1}^n [(n+r)r^2 - 2(r+1)r^2 + r^2]$$

$$= (n+r)^2 \sum_{r=1}^n r - 2(n+r) \sum_{r=1}^n r^2 + \sum_{r=1}^n r^2$$

USING STANDARD SUMMATION RESULTS WE HAVE

$$\sum_{r=1}^n [r(n+r)^2] = [n+r]^2 \sum_{r=1}^n r - 2(n+r) \sum_{r=1}^n (r+1)r + \frac{1}{2}r^2(n+r)^2$$

$$= \frac{n^2}{2} \sum_{r=1}^n r^2 - \frac{n^2}{2} \sum_{r=1}^n r^2 + \frac{n^2}{2} \sum_{r=1}^n r^2$$

$$= \frac{1}{2}n(n+1)^2 \left[6(n+1) - 4(n+1) + 3n \right]$$

$$= \frac{1}{2}n(n+1)^2 (3n+6 - 4n + 3n)$$

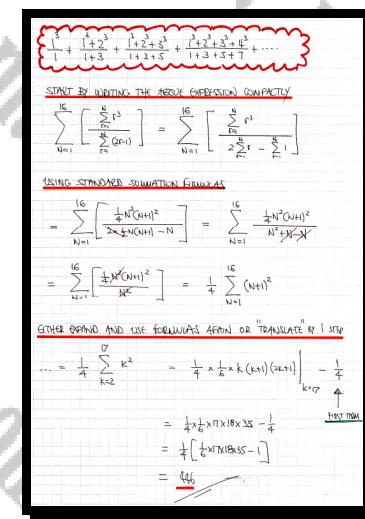
$$= \frac{1}{2}n(n+1)^2 (3n+2)$$

Question 93 (*****)

Find the sum of the first 16 terms of the following series.

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \frac{1^3 + 2^3 + 3^3 + 4^3}{1+3+5+7} + \dots$$

 [446]



The handwritten derivation shows three methods:

- Start by writing the next expression compactly:**
$$\sum_{N=1}^{16} \left[\frac{\frac{1}{2}N^2(2N+1)^2}{\frac{1}{2}N(N+1)} \right] = \sum_{N=1}^{16} \left[\frac{\frac{1}{2}N^2(2N+1)^2}{\frac{1}{2}N^2(N+1)} \right] = \frac{\frac{1}{2}N^2(2N+1)^2}{\frac{1}{2}N^2(N+1)} = \frac{N^2(2N+1)^2}{N^2(N+1)}$$

- Using standard summation formulae as:**
$$= \sum_{N=1}^{16} \left[\frac{\frac{1}{2}N^2(2N+1)^2}{2N^2(N+1)-N} \right] = \sum_{N=1}^{16} \frac{\frac{1}{2}N^2(2N+1)^2}{N^2(2N+1)-N}$$

$$= \sum_{N=1}^{16} \left[\frac{\frac{1}{2}N^2(2N+1)^2}{N^2} \right] = \frac{1}{4} \sum_{N=1}^{16} (N+1)^2$$
- Either expand and use formulae again or "translate" by 1 step:**
$$\dots = \frac{1}{4} \sum_{k=2}^{17} k^2 = \frac{1}{4} \times \frac{1}{6} \times k(k+1)(2k+1) \Big|_{k=17} - \frac{1}{4}$$

$$= \frac{1}{4} \times \frac{1}{6} \times 17 \times 18 \times 35 - \frac{1}{4}$$

$$= \frac{1}{4} \left[\frac{1}{6} \times 17 \times 18 \times 35 - 1 \right]$$

$$= 446$$

Question 94 (*****)

$$S_n = 1 \times 3 + 3 \times 3^2 + 5 \times 3^3 + 7 \times 3^4 + \dots + (2n-1) \times 3^n$$

Find a simplified expression for S_n , giving the answer in the form $A + f(n) \times 3^{n+1}$, where A is an integer and $f(n)$ a linear function of n .

[The standard techniques used for the summation of a geometric series are useful in this question]

$$S_n = 3 + (n-1) \times 3^{n+1}$$

$$\begin{aligned} S_n &= 1 \times 3 + 3 \times 3^2 + 5 \times 3^3 + 7 \times 3^4 + \dots + (2n-1) \times 3^n \\ - 3S_n &= -1 \times 3^2 - 3 \times 3^3 - 5 \times 3^4 - 7 \times 3^5 - \dots - (2n-3) \times 3^n - (2n-1) \times 3^n \\ \Rightarrow -2S_n &= 3 + 2 \left[3^2 + 3^3 + 3^4 + \dots + 3^n \right] - (2n-1) \times 3^{n+1} \\ \Rightarrow -2S_n &= 3 + 2 \left[\underbrace{3^2 + 3^3 + 3^4 + \dots + 3^n}_{\text{G.P. } \begin{cases} a=9 \\ r=3 \\ n=n+1 \end{cases}} \right] - (2n-1) \times 3^{n+1} \\ \Rightarrow -2S_n &= 3 + 2 \left[\frac{9(3^{n-1}-1)}{3-1} \right] - (2n-1) \times 3^{n+1} \\ \Rightarrow -2S_n &= 3 + 9(3^{n-1}-1) - (2n-1) \times 3^{n+1} \\ \Rightarrow -2S_n &= 3 + 3^{n+1} - 9 - (2n-1) \times 3^{n+1} \\ \Rightarrow 2S_n &= 6 + (2n-1) \times 3^{n+1} - 3^{n+1} \\ \Rightarrow 2S_n &\approx 6 + (2n-2) \times 3^{n+1} \\ \Rightarrow S_n &= 3 + (n-1) \times 3^{n+1} \end{aligned}$$

Question 95 (*****)

By showing a detailed method, sum the following series.

$$\sum_{r=1}^{\infty} \left[\frac{2^r}{(r+1)!} \right].$$

$$\boxed{\frac{1}{2}(e^2 - 3)}$$

$$\begin{aligned} S &= \sum_{r=1}^{\infty} \frac{2^r}{(r+1)!} = \frac{2}{2!} + \frac{8}{3!} + \frac{8}{4!} + \frac{16}{5!} + \frac{32}{6!} + \dots \\ \Rightarrow S &= \frac{2^1}{2!} + \frac{2^2}{3!} + \frac{2^3}{4!} + \frac{2^4}{5!} + \frac{2^5}{6!} + \dots \\ \Rightarrow 2S &= \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \dots \\ \Rightarrow 2S + 1 + \frac{2^1}{1!} &= 1 + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \dots \\ \Rightarrow 2S + 3 &= e^2 \\ \Rightarrow S &= \frac{e^2 - 3}{2} \end{aligned}$$

Question 96 (*****)

The r^{th} term of a progression is given by

$$u_r = ak^{r-1},$$

where a and k are constants with $k \neq \pm 1$.

Show clearly that

$$\sum_{r=1}^n (u_r \times u_{r+1}) = \frac{a^2 k (1 - k^{2n})}{1 - k^2}.$$

[proof]

$$\begin{aligned}
 u_r &= ak^{r-1} \Rightarrow \left\{ \begin{array}{l} u_1 = a \\ u_2 = ak \\ u_3 = ak^2 \\ u_4 = ak^3 \\ \dots \\ u_n = ak^{n-1} \end{array} \right\} \\
 \text{Hence} \\
 \sum_{r=1}^n (u_r \times u_{r+1}) &= u_1 u_2 + u_2 u_3 + u_3 u_4 + \dots + u_n u_{n+1} \\
 &= a(ak) + ak(ak^2) + ak^2(ak^3) + \dots + ak^{n-1}(ak^n) \\
 &= a^2 k + a^2 k^3 + a^2 k^5 + \dots + a^2 k^{2n-1} \\
 &= a^2 k \underbrace{[1 + k^2 + k^4 + \dots + k^{2n-2}]}_{\substack{\text{G.P. with } a=1 \\ r=k^2 \\ n \text{ terms}}} \\
 &= a^2 k \times \frac{1(1 - (k^2)^n)}{1 - k^2} \\
 &= \frac{a^2 k (1 - k^{2n})}{1 - k^2} \quad \text{As required}
 \end{aligned}$$

Question 97 (*****)

It is given that the following series converges to a limit L .

$$\sum_{r=1}^{\infty} \left[\frac{2x-1}{x+2} \right]^r.$$

Determine with full justification the range of possible values of L .

, $L > -\frac{1}{2}$

$\sum_{r=1}^{\infty} \left(\frac{2x-1}{x+2} \right)^r = L$

• FIRSTLY THIS IS A GEOMETRIC PROGRESSION, WITH COMMON RATIO $r = \frac{2x-1}{x+2}$

$$S_{\infty} = \frac{a}{1-r} = \frac{2x-1}{1-\frac{2x-1}{x+2}} = \frac{2x-1}{(x+2)-(2x-1)} = \frac{2x-1}{3-x}.$$

• NEXT WE REQUIRE THE RANGE OF VALUES OF x , FOR WHICH THE SUM TO INFINITY EXISTS

$$|r| < 1$$

$$-1 < \frac{2x-1}{x+2} < 1$$

$$\Rightarrow \frac{2x-1}{x+2} < 1$$

$$\Rightarrow \frac{2x-1}{x+2} - 1 < 0$$

$$\Rightarrow \frac{2x-1-2x-2}{x+2} < 0$$

$$\Rightarrow \frac{-3}{x+2} < 0$$

$$\frac{x+3}{x+2} > 0$$

$$\frac{x+3}{x+2} > 0$$

$$\frac{\sqrt{-2}}{-2} < x < \frac{\sqrt{-3}}{-3}$$

$$\therefore -2 < x < 3$$

$\therefore -\frac{1}{2} < x < 3$

• Thus we have

$$L(x) = \frac{2x-1}{3-x}, \quad -\frac{1}{2} < x < 3$$

$$L'(x) = \frac{(3-x)(2x-1) - (-1)(2x-1)}{(3-x)^2} = \frac{2(3-x) + (2x-1)}{(3-x)^2}$$

$$L'(x) = \frac{5}{(3-x)^2} > 0 \quad \text{FOR THE ABOVE DOMAIN}$$

• AS $L(x)$ IS AN INCREASING FUNCTION, THE MINIMUM & MAXIMUM CAN BE EASILY FOUND

$$L(-\frac{1}{2}) = \frac{2(-\frac{1}{2})-1}{3-(-\frac{1}{2})} = \frac{-\frac{5}{2}}{\frac{10}{3}} = -\frac{3}{4}$$

$$L(3) = +\infty$$

$$\therefore L > -\frac{1}{2}$$

Question 98 (*****)

By considering the trigonometric identity for $\tan(A - B)$, with $A = \arctan(n+1)$ and $B = \arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right).$$

You may assume the series converges.

$\square, \boxed{\frac{\pi}{4}}$

CONSIDER THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{\tan[\arctan(n+1)] - \tan[\arctan n]}{1 + \tan[\arctan(n+1)] \tan[\arctan n]}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{(n+1) - n}{1 + (n+1)n}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{1}{n^2+n+1}$$

$$\arctan[\tan(\arctan(n+1) - \arctan n)] = \arctan\left(\frac{1}{n^2+n+1}\right)$$

HENCE THE SUMMATION NOW BECOMES

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right) = \sum_{n=1}^{\infty} [\arctan(n+1) - \arctan n]$$

$$= \sum_{n=1}^{\infty} \arctan(n+1) - \sum_{n=1}^{\infty} \arctan n$$

Initial now gives in a limiting sense

$$\lim_{k \rightarrow \infty} \left[\sum_{n=1}^k \arctan(n+1) - \sum_{n=1}^k \arctan n \right]$$

$$= \lim_{k \rightarrow \infty} \left[\arctan(k+1) - \arctan 1 \right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Question 99 (*****)

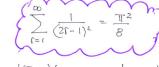
It is given that

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}.$$

By using this fact alone find the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{r^2}.$$

$$\boxed{\frac{\pi^2}{6}}$$



$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

Let $X = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$

$$\frac{1}{2^2}X = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots$$

ADD $\frac{5}{4}X = \left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) + \left(\frac{1}{9^2} + \frac{1}{10^2} + \frac{1}{11^2}\right) + \dots$

$$\frac{5}{4}X = \sum_{r=1}^{\infty} \frac{1}{r^2} - \sum_{r=1}^{\infty} \frac{1}{(2r)^2}$$

$$\frac{5}{4}X = \sum_{r=1}^{\infty} \frac{1}{r^2} - \frac{1}{6^2} \sum_{r=1}^{\infty} \frac{1}{r^2}$$

$$\frac{5}{4}X \times \frac{36}{35} = \frac{5}{4} \times \frac{36}{35}$$

$$\frac{2}{7} \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{4}{5} \times \frac{\pi^2}{8}$$

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{4\pi^2}{35}$$



LET $A = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$

$$\frac{1}{4}A = \frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{8^2} + \frac{1}{11^2} + \dots$$

$$\frac{1}{4}A = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots$$

THUS

$$A - \frac{1}{4}A = 1 + \frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$$

$$\frac{3}{4}A = \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}$$

$$\frac{3}{4} \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{8}$$

$$\frac{6}{7} \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{4 \cdot \pi^2}{35}$$

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$$

Question 100 (***)**

Evaluate the following expression

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{1}{3^{m+n}} \right].$$

Detailed workings must be shown.

, $\frac{9}{4}$

WORK AS FOLLOWS

$$\begin{aligned} \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \left[\frac{1}{3^{m+n}} \right] \right] &= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \left[\frac{1}{3^n \times 3^m} \right] \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{3^n} \sum_{m=0}^{\infty} \left[\frac{1}{3^m} \right] \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{3^n} \left(\underbrace{1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots}_{\text{G.P.}} \right) \right] \\ \text{THIS IS A GEOMETRIC PROGRESSION WITH } a = 1, r = \frac{1}{3} \text{ & } S_\infty = \frac{a}{1-r} \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{3^n} \times \frac{1}{1 - \frac{1}{3}} \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{3}{2} \times \frac{1}{3^n} \right] \\ &= \frac{3}{2} \sum_{n=0}^{\infty} \left[\frac{1}{3^n} \right] \\ &= \frac{3}{2} \left(\underbrace{1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots}_{\text{Sum of G.P. as above with } S_\infty = \frac{1}{1-\frac{1}{3}}} \right) \\ &= \frac{3}{2} \times \frac{3}{2} \\ &= \frac{9}{4} \end{aligned}$$

Question 101 (*****)

$$S = 1 - \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} - \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12 \cdot 16} - \dots$$

Find the sum to infinity of S , by considering the binomial series expansion of $(1+x)^n$ for suitable values of x and n .

, $S_{\infty} = \sqrt{\frac{2}{3}}$

SOFT BY COUNTING PROCEDURES IN THE COORDINATES

$$\begin{aligned} \Rightarrow S &= 1 - \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} - \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12 \cdot 16} - \dots \\ \Rightarrow S^2 &= 1 - \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} - \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12 \cdot 16} - \dots \\ \Rightarrow S &= \left(- \frac{2(1)}{4 \cdot 1} + \frac{2(3)(2)}{4^2(8 \cdot 2)} - \frac{2(3)(2)(5)}{4^3(8 \cdot 2 \cdot 4)} + \frac{2(3)(2)(5)(7)}{4^4(8 \cdot 2 \cdot 4 \cdot 8)} - \dots \right) \\ &\quad \text{CREATE WHAT WORKS USE + BINOMIAL EXPANSION} \\ \Rightarrow S &= \left(- \frac{2}{4} \left(\frac{1}{2} \right)^1 + \frac{2}{4^2} \left(\frac{1}{2} \right)^2 - \frac{2}{4^3} \left(\frac{1}{2} \right)^3 + \frac{2}{4^4} \left(\frac{1}{2} \right)^4 - \dots \right) \\ &\quad \text{FINALLY DEAL WITH THE ALIAS SIGNS} \\ \Rightarrow S &= 1 + \frac{-2}{4} \left(\frac{1}{2} \right)^1 + \frac{2}{4^2} \left(\frac{1}{2} \right)^2 + \frac{-16+12}{4^3} \left(\frac{1}{2} \right)^3 + \frac{-12(-2)+12(-2)}{4^4} \left(\frac{1}{2} \right)^4 - \dots \\ \Rightarrow S &= \left(1 + \frac{1}{2} \right)^{-\frac{1}{2}} \quad (\text{BY CONSIDERING THE EXPANSION } (1+x)^{\frac{1}{2}} \text{ WITH } x = \frac{1}{2}) \\ \Rightarrow S &= \left(\sqrt{\frac{1}{2}} \right)^{-1} \\ \therefore 1 - \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} - \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12 \cdot 16} &= \sqrt{\frac{2}{3}} \end{aligned}$$

Question 102 (***)**

Show clearly that

$$\ln\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] = \sum_{n=1}^{\infty} \frac{\sin^{2n-1} x}{2n-1}.$$

[proof]

$$\begin{aligned}
 \ln\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] &= \ln\left[\frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}}\right] = \ln\left[\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right] = \ln\left[\frac{1 + \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1 - \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right] = \ln\left[\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right] \\
 &= \ln\left[\frac{(\cos\frac{x}{2} + \sin\frac{x}{2})(\cos\frac{x}{2} + \sin\frac{x}{2})}{(\cos\frac{x}{2} - \sin\frac{x}{2})(\cos\frac{x}{2} + \sin\frac{x}{2})}\right] = \ln\left[\frac{\cos^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2} + \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\right] \\
 &= \ln\left[\frac{1 + \sin x}{\cos x}\right] = \ln(1 + \sin x) - \ln(\cos x) = \ln(1 + \sin x) - \frac{1}{2}\ln(\cos 2x) \\
 &= \ln(1 + \sin x) - \frac{1}{2}\ln(1 - \sin^2 x)
 \end{aligned}$$

Now, $\ln(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \dots$

$\ln(1-y) = -y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \dots$

$$\begin{aligned}
 &= \ln(1+y) - \frac{1}{2}\ln(1-y^2) \quad \text{write } y = \sin x \\
 &= y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5 - \frac{1}{6}y^6 + \frac{1}{7}y^7 - \frac{1}{8}y^8 + \dots \\
 &\quad - \frac{1}{2}[y^2 - y^4 + y^6 - y^8 + \dots] \\
 &= y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5 - \frac{1}{6}y^6 + \frac{1}{7}y^7 - \frac{1}{8}y^8 + \dots \\
 &= y + \frac{1}{3}y^3 + \frac{1}{5}y^5 + \frac{1}{7}y^7 + \dots \\
 &= \sin x + \frac{1}{3}\sin^3 x + \frac{1}{5}\sin^5 x + \frac{1}{7}\sin^7 x + \dots
 \end{aligned}$$

as required

Question 103 (***)**

The function f is defined as

$$f(n) = \frac{e^{-\lambda} \lambda^n}{n!},$$

where $n = 0, 1, 2, 3, 4, \dots$ and λ is a positive constant.

By showing a detailed method, prove that ...

a) ... $\sum_{n=0}^{\infty} [n f(n)] = \lambda$.

b) ... $\sum_{n=0}^{\infty} [n^2 f(n)] = \lambda^2 + \lambda$.

proof

$f(n) = \frac{e^{-\lambda} \lambda^n}{n!}$

a) $\sum_{n=0}^{\infty} n f(n) = \sum_{n=0}^{\infty} n \left(\frac{e^{-\lambda} \lambda^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{n e^{-\lambda} \lambda^n}{n!}$

BUT THE FIRST TERM IS ZERO, SO WE MAY START THE SUMMATION FROM 1:

$$= \sum_{n=1}^{\infty} \frac{n \lambda^n e^{-\lambda}}{n!} = e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!}$$

$$= e^{-\lambda} \times \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}$$

RE-ARRANGE THE SUMMATION FROM ZERO:

$$= 2e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = 2e^{-\lambda} \times e^{\lambda} = 2$$

b) $\sum_{n=0}^{\infty} n^2 f(n) = \sum_{n=0}^{\infty} \frac{n^2 \lambda^n e^{-\lambda}}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{n^2 \lambda^n}{n!} = \text{FIRST THREE TERMS}$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{n^2 \lambda^n}{n!} = \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{n^2 \lambda^{n-1}}{n!} = \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{n^2 \lambda^{n-1}}{(n-1)!}$$

RE-ARRANGE THE SUMMATION BACK TO ZERO:

$$= \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{(n+1)^2 \lambda^n}{n!} = \lambda e^{-\lambda} \sum_{n=0}^{\infty} \left[\frac{n^2 \lambda^n}{n!} + \frac{2n \lambda^n}{n!} \right]$$

$$= \lambda e^{-\lambda} \left[\frac{2 \lambda^0}{0!} + e^{\lambda} \right]$$

NOTE THAT IN THE SOLUTION A ZERO IS SKIPPED (SEE STEP 10).

$$= \lambda e^{-\lambda} \left[\frac{2 \lambda^0}{0!} + e^{\lambda} \right] = \lambda e^{-\lambda} \left[\frac{2 \lambda^0}{0!} + e^{\lambda} \right]$$

$$= \lambda e^{-\lambda} \left[2 + e^{\lambda} \right] = \lambda e^{-\lambda} \left[2 + e^{\lambda} \right]$$

RE-ARRANGE SUMMATION BACK TO ZERO:

$$= 2e^{-\lambda} \left[2 + e^{\lambda} \right] = 2e^{-\lambda} \left[2 + e^{\lambda} \right]$$

Question 104 (*****)

Find in exact simplified form an exact expression for the sum of the first n terms of the following series

$$1 + 11 + 111 + 1111 + 11111 + \dots$$

$$, \quad S_n = \frac{1}{81} \left[10^{n+1} - 10 - 9n \right]$$

Let $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{N^2} + \dots + \frac{1}{\infty^2}$

 $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \left(\frac{1}{1^2} \times 99 \right) + \left(\frac{1}{2^2} \times 99 \right) + \left(\frac{1}{3^2} \times 99 \right) + \dots + \left(\frac{1}{N^2} \times 99 \right)$
 $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} [9 + 99 + 999 + \dots + \underbrace{999 \dots 999}_{N \text{ digits}}]$
 $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} \left[(10^0 - 1) + (10^1 - 1) + (10^2 - 1) + \dots + (10^{N-1} - 1) \right]$
 $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} \left[\underbrace{(10^0 + 10^1 + 10^2 + \dots + 10^N)}_{N \text{ terms}} - (1 + 1 + \dots + 1) \right]$

GEOMETRIC PROGRESSION

$$a = 10$$

$$r = 10$$

$$\sum_{n=0}^{N-1} ar^n = a \frac{(r^N - 1)}{r - 1}$$

 $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} \left[\frac{10(N \cdot 10^N - 1)}{10 - 1} - (N \times 10) \right]$
 $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} \left[\frac{10}{9} (10^N - 1) - N \right]$
 $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} \left[10 (10^N - 1) - 9N \right]$
 $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} \left[10^{N+1} - 10 - 9N \right]$
 $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} \left[10^{N+1} - 9N - 10 \right]$

ALTERNATIVE BY LOOKING AT DIFFERENT PATTERNS

- Let $S_n = 1 + 11 + 111 + 1111 + \dots + \underbrace{11\dots11}_{n \text{ digits}}$

$$\Rightarrow S_n = 10^0 \leftarrow 1$$

$$10^0 + 10^1 \leftarrow 11$$

$$10^0 + 10^1 + 10^2 \leftarrow 111$$

$$10^0 + 10^1 + 10^2 + 10^3 \leftarrow 1111$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\frac{10^0}{10^0} \downarrow \frac{10^1}{10^1} \downarrow \frac{10^2}{10^2} \downarrow \frac{10^3}{10^3} \downarrow \dots \downarrow \frac{10^n}{10^n} \downarrow \leftarrow \underbrace{11\dots11}_{n \text{ digits}}$$

$$10^0 \text{ G.R. } 10^1 \text{ G.R. } 10^2 \text{ G.R. } 10^3 \text{ G.R. } \dots 10^n \text{ G.R.}$$

$$10^0 = 1 \quad 10^1 = 10 \quad 10^2 = 100 \quad 10^3 = 1000 \quad \dots \quad 10^n = 1000\dots0$$

$$10^0 + 10^1 + 10^2 + 10^3 + \dots + 10^n = 1 + 10 + 100 + 1000 + \dots + 1000\dots0$$

$$= 1 + 10(1 + 10 + 100 + \dots + 10^{n-1})$$

$$= 1 + 10S_{n-1}$$

- NOW LOOKING AT ANY OF THE GR'S. VARIOUSLY DOWN

$$S_n = \frac{10(S_{n-1} + 1)}{10 - 1} = \frac{(10^n - 1)}{10 - 1} = \frac{1}{9}(10^n - 1)$$

$$\Rightarrow S_n = \sum \text{All the G.R.s}$$

$$\Rightarrow S_4 = \frac{1}{9} [10^0 + 10^1 + 10^2 + 10^3] + \frac{1}{9} [10^0 + 10^1 + 10^2] + \dots + \frac{1}{9} [10^0 + 10^1] + \frac{1}{9} [10^0]$$

$$\Rightarrow S_4 = \frac{1}{9} [(10^0 + 10^1 + 10^2 + \dots + 10^3) + 10^3 + 10^2 + \dots + 10^0] - (1 + 1 + 1 + \dots + 1)$$

$$\Rightarrow S_4 = \frac{1}{9} [(10^0 + 10^1 + 10^2 + \dots + 10^3) - n \times 1] \quad \text{↑ terms}$$

$$\Rightarrow S_4 = \frac{1}{9} \times \frac{10(10^3 - 1)}{10 - 1} - n \quad (\text{CANCELLING THE SAME G.R. IN EACH})$$

$$\Rightarrow S_4 = \frac{1}{9} \times \frac{10^{4+1} - 10}{10 - 1} - n$$

$$\Rightarrow S_4 = \frac{1}{9} \times [10^{4+1} - 10 - n] \quad \cancel{\text{ABOVE}}$$

Question 105 (*****)

The product operator \prod , is defined as

$$\prod_{i=1}^k [u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \dots \times u_{k-1} \times u_k.$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right].$$

$$\boxed{S = K}, \quad \boxed{8\sqrt{\frac{5}{4}} - 1}$$

Start by writing a few terms explicitly & look for a pattern

$$\begin{aligned} \sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right] &= \frac{1}{1!} \left(\frac{8-7}{40} \right) + \frac{2}{2!} \left(\frac{15-7}{40} \right)^2 + \frac{3}{3!} \left(\frac{22-7}{40} \right)^3 + \frac{4}{4!} \left(\frac{29-7}{40} \right)^4 + \dots \\ &= \frac{1}{40} + \frac{1}{40} \times \frac{9}{80} + \frac{1}{40} \times \frac{9}{80} \times \frac{17}{160} + \frac{1}{40} \times \frac{9}{80} \times \frac{17}{160} \times \frac{25}{240} + \dots \\ &= \frac{1}{40} + \frac{1 \times 9}{40 \times 80} + \frac{1 \times 9 \times 17}{40 \times 80 \times 160} + \frac{1 \times 9 \times 17 \times 25}{40 \times 80 \times 160 \times 240} \\ &\approx \frac{1}{40} + \frac{1 \times 9}{40 \times (160)} + \frac{1 \times 9 \times 17}{40 \times (160 \times 240)} + \frac{1 \times 9 \times 17 \times 25}{40 \times (160 \times 240 \times 320)} \end{aligned}$$

This resembles a binomial expansion due to the factorials at the denominators. The next item is to create "numbers" of the form $n(n+1)(n+2)\dots(n+3)\dots$

By inspection this will come as $-\frac{1}{8}, -\frac{9}{8}, -\frac{17}{8}, -\frac{25}{8}$

Now try and adjust the signs

$$= \frac{1}{(-1)(-6)1!} + \frac{1 \times 9}{(-1)^2(-5)2!} + \frac{1 \times 9 \times 17}{(-1)^3(-4)3!} + \frac{1 \times 9 \times 17 \times 25}{(-1)^4(-3)4!}$$

$\sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right] = \underbrace{\frac{-1}{1!} \left(-\frac{1}{8} \right) + \frac{(-1)(-5)}{2!} \left(-\frac{1}{8} \right)^2 + \frac{(-1)(-5)(-9)}{3!} \left(-\frac{1}{8} \right)^3 + \frac{(-1)(-5)(-9)(-17)}{4!} \left(-\frac{1}{8} \right)^4 + \dots}_{\text{THIS IS A BINOMIAL EXPRESSION WITH THE } (-1)^k \text{ MISSING AT THE FRONT}}$

$$\begin{aligned} &= \left(-\frac{1}{8} \right)^1 - 1 \\ &= \left(\frac{1}{8} \right)^{-1} - 1 \\ &= \frac{8}{\sqrt{4}} - 1 \end{aligned}$$

Question 106 (*****)

Find the value of

$$\sum_{r=0}^{\infty} \left[\frac{\sin^4(\pi \times 2^{r-2})}{4^r} \right].$$

Hint: Express $\sin^4 \theta$ in terms of $\sin^2 \theta$ and $\sin^2 2\theta$ only.

, $\frac{1}{2}$

• STARTING BY MANIPULATING THE SINE TO THE FOURTH POWER

$$\begin{aligned}\sin^4 \theta &= (\sin \theta)^2 = \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta \right)^2 = \frac{1}{4} - \frac{1}{2} \cos 2\theta + \frac{1}{4} \cos^2 2\theta \\ &= \frac{1}{4} - \frac{1}{2}(1 - 2 \sin^2 \theta) + \frac{1}{4}(1 - \sin^2 2\theta) \\ &= \frac{1}{4} - \frac{1}{2} + 2 \sin^2 \theta + \frac{1}{4} - \frac{1}{4} \sin^2 2\theta \\ &= \sin^2 \theta + \frac{1}{2} \sin^2 2\theta\end{aligned}$$

• NOW WE PROVE BY CONSIDERING THE SUM OF THE FIRST n TERMS

$$\begin{aligned}\sum_{r=0}^n \frac{\sin^4(\pi \times 2^{r-2})}{4^r} &= \sum_{r=0}^n \left[\frac{1}{4^r} \int_0^{\pi} \sin^2(\pi x 2^{r-2}) - \frac{1}{4} \sin^2(\pi x 2^{r-2}) \right] \\ &= \sum_{r=0}^n \left[\frac{1}{2^r} \sin^2(\pi x 2^{r-2}) - \frac{1}{4^r} \sin^2(\pi x 2^{r-1}) \right] \\ &= \frac{\sin^2 \frac{\pi}{4}}{2^0} - \frac{1}{4^0} \sin^2 \frac{\pi}{2} \quad \leftarrow r=0 \\ &\quad \frac{1}{2^1} \sin^2 \frac{\pi}{2} - \frac{1}{4^1} \sin^2 \frac{\pi}{4} \quad \leftarrow r=1 \\ &\quad \frac{1}{2^2} \sin^2 \frac{\pi}{4} - \frac{1}{4^2} \sin^2 \frac{\pi}{8} \quad \leftarrow r=2 \\ &\quad \frac{1}{2^3} \sin^2 \frac{\pi}{8} - \frac{1}{4^3} \sin^2 \frac{\pi}{16} \quad \leftarrow r=3 \\ &\quad \dots \\ &\quad \frac{1}{2^n} \sin^2 \frac{\pi}{2^{n-2}} - \frac{1}{4^n} \sin^2 \frac{\pi}{2^n} \quad \leftarrow r=n \\ &= \sin^2 \frac{\pi}{4} - \frac{1}{4^n} \sin^2(\pi x 2^{-1})\end{aligned}$$

• Hence we have

$$\sum_{r=0}^{\infty} \frac{\sin^4(\pi x 2^{r-2})}{4^r} = \sin^2 \frac{\pi}{4} - \frac{1}{4^\infty} \sin^2(\pi x 2^{-1}) = \frac{1}{2}$$

Question 107 (*****)

Find the sum to infinity of the following series.

$$\frac{1}{1} - \frac{1}{1+4} + \frac{1}{1+4+9} - \frac{1}{1+4+9+16} + \frac{1}{1+4+9+16+25} + \dots$$

You may find the series expansion of $\arctan x$ useful in this question.

, $6(\pi - 3)$

WRITE THE SERIES IN COMPACT³ NOTATION

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(1+4+9+\dots+n^2)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\frac{1}{4}(n^2+2n+1)}$$

IGNORE THE $\frac{1}{4}$ TERM & SPLIT THE REST INTO PARTIAL FRACTIONS BY INSPECT

$$\frac{1}{n(n+1)(2n+1)} = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{2n+1} = \frac{1}{n} + \frac{1}{n+1} - \frac{1}{2n+1}$$

HENCE WE HAVE

$$\dots = \sum_{n=1}^{\infty} \left[(-1)^{n+1} \left(\frac{1}{n} + \frac{1}{n+1} - \frac{1}{2n+1} \right) \right]$$

$$= 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} - 24 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

NEXT CONSIDER EACH TERM OF THE SUMMATION SEPARATELY

- $6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 6 \left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right] = 6G_2$ (from fact)
- $6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} = 6 \left[\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \dots \right]$

$$= -6 \left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right]$$

$$= -6 \left[G_1 - \left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right) \right]$$

$$= 6 - 6 \left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right]$$

$$= 6 - 6G_2$$

NEXT CONSIDER THE SERIES EXPANSION OF ARCTAN

$$\Rightarrow \frac{1}{1-x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

$$\Rightarrow \int \frac{1}{1-x^2} dx = x - \frac{1}{2}x^3 + \frac{1}{4}x^5 - \frac{1}{6}x^7 + \dots + C$$

$$\Rightarrow \arctan x = C + x - \frac{1}{2}x^3 + \frac{1}{4}x^5 - \frac{1}{6}x^7 + \dots$$

$$(C=0 \rightarrow C=0)$$

$$\Rightarrow \arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} x^{2n+1}$$

$$\Rightarrow \arctan 1 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

$$\Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

$$\Rightarrow \pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

$$\Rightarrow G_1 = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

$$\Rightarrow G_2 = 2 \sum_{n=1}^{\infty} \left[1 + \frac{(-1)^{n+1}}{2n+1} \right]$$

$$\Rightarrow G_2 = 24 + 24 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

$$\Rightarrow 24 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} = 4\pi - 24$$

FINALLY COLLECTING ALL THE PARTS

$$\frac{1}{1-\frac{1}{4}x^2} + \frac{1}{1-\frac{1}{4}x^2} + \frac{1}{1-\frac{1}{4}x^2} = \frac{1}{1-\frac{1}{4}x^2} + \frac{1}{1-\frac{1}{4}x^2} + \frac{1}{1-\frac{1}{4}x^2}$$

$$= 6 \frac{\pi}{4} \frac{(-1)^{n+1}}{2n+1} + 6 \frac{\pi}{4} \frac{(-1)^{n+1}}{2n+1} - 24 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

$$= 6G_2 + (\pi - 4\pi) + (6\pi - 24)$$

$$= G_1 - 18 = 6(\pi - 3)$$

Question 108 (*****)

$$f(x) \equiv \frac{1-7x}{(1+x)(1-3x)}, -\frac{1}{3} < x < \frac{1}{3}.$$

Show that $f(x)$ can be written in the form

$$f(x) = 1 - \sum_{r=1}^{\infty} [x^r g(r)],$$

where $g(r)$ is a simplified function to be found.

, $g(r) = 3^r + 2 \times (-1)^{r+1}$

● BY PARTIAL FRACTION OR DIRECT EXPANSION (USING STANDARD EXPANSION) OR THE SUM TO INFINITY OF A GEOMETRIC SERIES.

$$\begin{aligned} (1+x)^{-1} &= 1-x+x^2-x^3+x^4-\dots \\ (1-3x)^{-1} &= 1+2x+2x^2+2x^3+2x^4+\dots \end{aligned}$$

● Thus we have

$$\begin{aligned} f(x) &= (1-7x)(1+x)^{-1}(1-3x)^{-1} \\ &\Rightarrow f(x) = (1-7x)(1-x+x^2-x^3+\dots)(1+2x+2x^2+2x^3+2x^4+\dots) \\ &\Rightarrow f(x) = (1-7x) \left[1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + 729x^6 + \dots \right. \\ &\quad - 2x - 6x^2 - 18x^3 - 54x^4 - 162x^5 - 486x^6 - 1458x^7 - 4374x^8 \\ &\quad + 3x^3 + 9x^4 + 27x^5 + 81x^6 \\ &\quad - 3x^4 - 30x^5 - 93x^6 - 279x^7 \\ &\quad + 3x^5 + 9x^6 \\ &\quad - 3x^6 - 30x^7 \\ &\Rightarrow f(x) = (1-7x)(1+2x+7x^2+20x^3+61x^4+182x^5+\dots) \\ &\Rightarrow f(x) = \frac{1+2x+7x^2+20x^3+61x^4+182x^5+\dots}{1-5x-7x^2-29x^3-79x^4-243x^5} \end{aligned}$$

● BY INSPECTION

\uparrow
 \uparrow
 \uparrow
 \uparrow
 \uparrow
 \uparrow
 \uparrow

● HENCE WE MAY WRITE

$$\begin{aligned} f(x) &= 1 - \sum_{r=1}^{\infty} \left[3^r + 2(-1)^{r+1} \right] x^r \\ f(x) &= 1 - \sum_{r=1}^{\infty} \left[3^r + 2(-1)^{r+1} \right] x^r \end{aligned}$$

Question 109 (*****)

It is given that

$$\zeta(2) = \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}.$$

By using this fact alone find the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

$$\boxed{\frac{\pi^2}{8}}$$

$\zeta(2) \equiv \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$

④ $\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$

$\frac{1}{2^2} \zeta(2) = \frac{1}{2^2} + \frac{1}{2 \cdot 3^2} + \frac{1}{2^2 \cdot 3^2} + \frac{1}{2^4 \cdot 5^2} + \dots$

$\frac{1}{4} \zeta(2) = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots$

⑤ SUBTRACTING

$\zeta(2) - \frac{1}{4} \zeta(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$

$\frac{3}{4} \zeta(2) = \sum_{r=1}^{8} \left(\frac{1}{(2r-1)^2} \right)$

$\frac{3}{4} \times \frac{\pi^2}{6} = \sum_{r=1}^{8} \left(\frac{1}{(2r-1)^2} \right)$

$\sum_{r=1}^{8} \left(\frac{1}{(2r-1)^2} \right) = \frac{\pi^2}{8}$ //

Question 110 (*****)

$$S = \frac{3}{8} + \frac{3 \times 9}{8 \times 16} + \frac{3 \times 9 \times 15}{8 \times 16 \times 24} + \frac{3 \times 9 \times 15 \times 21}{8 \times 16 \times 24 \times 32} + \frac{3 \times 9 \times 15 \times 21 \times 27}{8 \times 16 \times 24 \times 32 \times 40} \dots$$

By considering a suitable binomial expansion, show that $S = 1$.

SOL, proof

$$\begin{aligned} \textcircled{1} \quad S &= \frac{3}{8} + \frac{3 \times 9}{8 \times 16} + \frac{3 \times 9 \times 15}{8 \times 16 \times 24} + \frac{3 \times 9 \times 15 \times 21}{8 \times 16 \times 24 \times 32} + \dots \\ \Rightarrow S &= \frac{3}{8} + \frac{3^2(1 \times 3)}{8^2(1 \times 2)} + \frac{3^3(1 \times 3 \times 5)}{8^3(1 \times 2 \times 3)} + \frac{3^4(1 \times 3 \times 5 \times 7)}{8^4(1 \times 2 \times 3 \times 4)} + \dots \\ \Rightarrow S &= \frac{\frac{1}{2} \times 3}{1!} \left(\frac{1 \times 3}{2}\right)^2 + \frac{\left(\frac{1}{2} \times 3 \times \frac{5}{2}\right) 3^2}{4!} + \frac{\left(\frac{1}{2} \times 3 \times \frac{5}{2} \times \frac{7}{2}\right) 3^3}{4^3} + \dots \\ \Rightarrow S &= \frac{\frac{1}{2}}{1!} \left(\frac{3}{2}\right)^1 + \frac{\frac{1}{2} \times \frac{3}{2}}{2!} \left(\frac{3}{2}\right)^2 + \frac{\frac{1}{2} \times \frac{3}{2} \times \frac{5}{2}}{3!} \left(\frac{3}{2}\right)^3 + \dots \\ \textcircled{2} \quad \text{THIS IS ALMOST A BINOMIAL, EXCEPT FOR THE SIGNS! (BUT) OUGHT TO DECAY BY 1 (GIVE THAT)} \\ \text{BY INSPECTING THE NUMBERS, NOTicing EACH TERM IS STILL DIVISIBLE BY 4, PROve!} \\ \Rightarrow S = \frac{\frac{1}{2}}{1!} \left(-\frac{3}{2}\right)^1 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{2!} \left(-\frac{3}{2}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{3!} \left(-\frac{3}{2}\right)^3 + \dots \\ \textcircled{3} \quad \text{ADDING 1 TO BOTH SIDES TO GET A SIMPLE BINOMIAL} \\ \Rightarrow 1 + S = 1 + \frac{\frac{1}{2}}{1!} \left(-\frac{3}{2}\right)^1 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} \left(-\frac{3}{2}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} \left(-\frac{3}{2}\right)^3 + \dots \\ \Rightarrow 1 + S = \left(1 - \frac{3}{2}\right)^{-\frac{1}{2}} \\ \Rightarrow S = \left(\frac{1}{2}\right)^{\frac{1}{2}} - 1 \\ \Rightarrow S = 2 - 1 = 1 \quad \checkmark \end{aligned}$$

Question 111 (*****)

Sum the following series of infinite terms.

$$\frac{1}{3} + \frac{3}{9} + \frac{7}{27} + \frac{15}{81} + \frac{31}{243} + \frac{63}{729} + \dots$$

3, 2

$$\begin{aligned} \textcircled{1} \quad &\frac{1}{3} + \frac{3}{9} + \frac{7}{27} + \frac{15}{81} + \frac{31}{243} + \frac{63}{729} + \dots \\ \textcircled{2} \quad &\text{EVIDENCE: THE DENOMINATOR IS "POWERS" OF 3} \\ &\text{THE NUMERATORS DO NOT FORM A LINEAR NOR A QUADRATIC — OR ANY YET} \\ &\text{SPOTTED WHAT IT IS ACTUALLY DIFFERENTIAL.} \\ \textcircled{3} \quad &\begin{array}{ccccccc} 3 & 7 & 15 & 31 & 63 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 8 & 16 & 32 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 4 & 8 & 16 & \dots \end{array} \\ \textcircled{4} \quad &\text{WE CAN SEE THAT THE NUMERATOR FORM } 2^n - 1 \\ \textcircled{5} \quad &\text{THIS WE REQUIRE} \\ &\sum_{n=1}^{\infty} \frac{2^n - 1}{3^n} = \sum_{n=1}^{\infty} \left[\left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n \right] = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \\ &\quad \begin{array}{c} \uparrow \\ a = \frac{2}{3} \end{array} \quad \begin{array}{c} \uparrow \\ a = \frac{1}{3} \end{array} \quad \begin{array}{c} \uparrow \\ r = \frac{2}{3} \end{array} \quad \begin{array}{c} \uparrow \\ r = \frac{1}{3} \end{array} \\ \textcircled{6} \quad &\text{USING THE STANDARD FORMULA FOR THE SUM TO INFINITY OF A G.P.} \\ &\text{i.e. } \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \\ \dots &= \frac{\frac{2}{3}}{1 - \frac{2}{3}} - \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} - \frac{\frac{1}{3}}{\frac{2}{3}} = 2 - \frac{1}{2} = \frac{3}{2}. \end{aligned}$$

Question 112 (*****)

Sum the following series of infinite terms.

$$\frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \frac{13}{128} + \dots$$

[] , 2

$\frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \frac{13}{128} + \dots$

THE NUMERATOR IS THE FIBONACCI SERIES, THE DENOMINATOR IS + G.P.
WITH COMMON RATIO 2, OR $\frac{1}{2}$ FOR THE WHOLE FRACTIONAL TERM

LET THE REQUIRED SUM BE S

$$\begin{aligned} S &= \frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \frac{13}{128} + \dots \\ -\frac{1}{2}S &= \quad -\frac{1}{4} - \frac{1}{8} - \frac{2}{16} - \frac{3}{32} - \frac{5}{64} - \frac{8}{128} - \dots \\ -\frac{1}{2}S &= \quad \frac{1}{8} + \frac{1}{16} + \frac{2}{32} + \frac{3}{64} + \frac{5}{128} + \dots \end{aligned}$$

ADDITION: $\frac{1}{2}S = \frac{1}{2}$

$$\therefore S = 2$$

Question 113 (*****)

By considering the simplification of

$$\arctan(2n+1) - \arctan(2n-1),$$

determine the exact value of

$$\sum_{n=1}^{\infty} \left[\arctan\left(\frac{1}{2n^2}\right) \right].$$

$$\boxed{}, \frac{\pi}{4}$$

$\arctan(2n+1) - \arctan(2n-1) = \psi$

- TAKE TANGENTS on both sides

$$\tan[\arctan(2n+1) - \arctan(2n-1)] = \tan\psi$$

$$\frac{(2n+1) - (2n-1)}{1 + (2n+1)(2n-1)} = \tan\psi$$

$$\tan\psi = \frac{2}{1+4n^2-1} = \frac{1}{2n^2}$$

$$\psi = \arctan\left(\frac{1}{2n^2}\right)$$

- Hence $\arctan\left(\frac{1}{2n^2}\right) = \arctan(2n+1) - \arctan(2n-1)$

$n=1$:	$\arctan\left(\frac{1}{2}\right) = \arctan 3 - \arctan 1$
$n=2$:	$\arctan\left(\frac{1}{8}\right) = \arctan 5 - \arctan 3$
$n=3$:	$\arctan\left(\frac{1}{18}\right) = \arctan 7 - \arctan 5$
\vdots	\vdots
$n=k$:	$\arctan\left(\frac{1}{2k^2}\right) = \arctan(2k+1) - \arctan(2k-1)$

- SUMMING

$$\sum_{n=1}^k \arctan\left(\frac{1}{2n^2}\right) = \arctan(2k+1) - \arctan 1$$

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{2n^2}\right) = \lim_{k \rightarrow \infty} [\arctan(2k+1) - \arctan 1]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Question 114 (*****)

Determine the exact value of the following sum.

$$\sum_{n=2}^{20} \left[\frac{n^3 - n^2 + 1}{n^2 - n} \right].$$

 $\frac{4199}{20}$

• START MANIPULATING BY DIVISION EXCLUDING BY PARTIAL FRACTIONS

$$\begin{aligned}\frac{n^3 - n^2 + 1}{n^2 - n} &= \frac{n(n^2 - n) + 1}{n^2 - n} = n + \frac{1}{n(n-1)} = n + \frac{1}{n(n-1)} \\ &= n + \frac{-1}{n} + \frac{1}{n-1} = n + \frac{1}{n-1} - \frac{1}{n}\end{aligned}$$

• THIS WE HAVE:

$$\frac{n^3 - n^2 + 1}{n^2 - n} = n + \frac{1}{n-1} - \frac{1}{n}$$

If $n=2$	$\frac{2^3 - 2^2 + 1}{2^2 - 2} = 2 + \frac{1}{2-1} - \frac{1}{2}$
If $n=3$	$\frac{3^3 - 3^2 + 1}{3^2 - 3} = 3 + \frac{1}{3-1} - \frac{1}{3}$
If $n=4$	$\frac{4^3 - 4^2 + 1}{4^2 - 4} = 4 + \frac{1}{4-1} - \frac{1}{4}$
If $n=5$	$\frac{5^3 - 5^2 + 1}{5^2 - 5} = 5 + \frac{1}{5-1} - \frac{1}{5}$
⋮	⋮
If $n=20$	$\frac{20^3 - 20^2 + 1}{20^2 - 20} = 20 + \frac{1}{20-1} - \frac{1}{20}$

• ADDING

$$\sum_{n=2}^{20} \left[\frac{n^3 - n^2 + 1}{n^2 - n} \right] = \left[\sum_{n=2}^{20} \left(n + \frac{1}{n-1} - \frac{1}{n} \right) \right] + 1 - \frac{1}{20}$$

$$\begin{aligned}&= \frac{13}{2}(2+20) + 1 - \frac{1}{20} \\ &\approx (13 \times 11) + 1 - \frac{1}{20} \\ &= 143 + 1 - \frac{1}{20} \\ &= \frac{286}{20} - \frac{1}{20} \\ &= \frac{4199}{20}\end{aligned}$$

$13 \times 11 = 143 + 1 = 144$

$\frac{286}{20} = \frac{143}{10}$

Question 115 (*****)

$$\sum_{r=1}^{\infty} \left[\frac{1}{r^2} \right] = L.$$

It is given that the above infinite series converges to a limit L .

Find, in terms of L where appropriate, the limit of each of the following infinite series.

a) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots$

b) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$

c) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots$

d) $\frac{1}{1^2} + \frac{1}{2^2} - \frac{8}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - \frac{8}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} - \frac{8}{9^2} + \dots$

, $\frac{1}{4}L$, $\frac{3}{4}L$, $\frac{1}{2}L$, 0

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots = L$

a) MULTIPLY THE GIVEN SERIES BY $\frac{1}{2^2} = \frac{1}{4}$

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \dots = \frac{1}{4}L \quad //$$

b) SUBTRACT THE ABOVE SERIES FROM PART (a). ROW THE SERIES GIVEN

$$\begin{aligned} 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots &= L \\ - \frac{1}{2^2} & \\ \hline 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots &= -\frac{1}{4}L \end{aligned}$$

c) SUBTRACT (a) FROM (b) OR SUBTRACT 2×(a) FROM THE SERIES GIVEN

$$\begin{aligned} 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots &= L \\ - \frac{2}{2^2} & \\ \hline 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots &= -2 \times \frac{1}{4}L \end{aligned}$$

d) FIRSTLY MULTIPLY THE SERIES GIVEN BY $\frac{1}{3^2} = \frac{1}{9}$

$$\frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \dots = \frac{1}{9}L$$

THEN BY SUBTRACTING 9 TIMES THE ABOVE SERIES FROM THE SERIES GIVEN WE OBTAIN

$$\begin{aligned} 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \dots &= L \\ - \frac{9}{3^2} & \\ \hline 1 + \frac{1}{2^2} - \frac{8}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - \frac{8}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} - \frac{8}{9^2} + \dots &= 0 \end{aligned}$$

Question 116 (*****)

Consider the infinite series

$$\frac{2}{2^2} \left(\frac{1}{2} \right) x^2 + \frac{3}{2^2 \times 4^2} \left(\frac{1}{2} + \frac{1}{4} \right) x^4 + \frac{4}{2^2 \times 4^2 \times 6^2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} \right) x^6 + \frac{5}{2^2 \times 4^2 \times 6^2 \times 8^2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} \right) x^8 + \dots$$

Write the above series in Sigma notation, in its simplest form.

[You are not required to investigate its convergence or to sum it.]

$$\sum_{n=1}^{\infty} \left[\frac{n+1}{(n!)^2} \left(\frac{x}{2} \right)^{2n} \sum_{m=1}^n \frac{1}{2m} \right] \text{ or } \sum_{n=1}^{\infty} \sum_{m=1}^n \left[\frac{n+1}{2m(n!)^2} \left(\frac{x}{2} \right)^{2n} \right]$$

$$\begin{aligned} & \frac{2}{2^2} \left(\frac{1}{2} \right) x^2 + \frac{3}{2^2 \times 4^2} \left(\frac{1}{2} + \frac{1}{4} \right) x^4 + \frac{4}{2^2 \times 4^2 \times 6^2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} \right) x^6 + \frac{5}{2^2 \times 4^2 \times 6^2 \times 8^2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} \right) x^8 + \dots \\ & = \frac{2x^2}{2^2} \left(\frac{1}{2} \right) + \frac{3x^4}{2^2 \times 4^2} \left(\frac{1}{2} + \frac{1}{4} \right) + \frac{4x^6}{2^2 \times 4^2 \times 6^2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} \right) + \frac{5x^8}{2^2 \times 4^2 \times 6^2 \times 8^2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} \right) + \dots \\ & = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{2^n n!^2} \sum_{m=1}^n \frac{1}{2m} = \sum_{n=1}^{\infty} \sum_{m=1}^n \frac{(n+1)x^{2n}}{2^n (n!)^2 2m} \\ & = \sum_{n=1}^{\infty} \sum_{m=1}^n \frac{(n+1)x^{2n}}{2^n (n!)^2 2m} \end{aligned}$$

Question 117 (*****)

Determine the sum to infinity of the following series

$$\frac{10}{1!} + \frac{7}{2!} + \frac{4}{3!} + \frac{1}{4!} - \frac{2}{5!} - \frac{5}{6!} + \dots$$

10e-13

$$\begin{aligned} & \frac{10}{1!} + \frac{7}{2!} + \frac{4}{3!} + \frac{1}{4!} - \frac{2}{5!} - \frac{5}{6!} - \dots = \sum_{n=1}^{\infty} \frac{(3-n)}{n!} \\ & = \sum_{n=1}^{\infty} \frac{13}{n!} - 3 \sum_{n=1}^{\infty} \frac{1}{n!} = 13 \sum_{n=1}^{\infty} \frac{1}{n!} - 3 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \\ & = 13 \sum_{n=1}^{\infty} \frac{1}{n!} - 3 \sum_{n=0}^{\infty} \frac{1}{n!} \\ & \text{But } \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots = e \\ & = 13 \left[-1 + \sum_{n=0}^{\infty} \frac{1}{n!} \right] - 3e = 13 \left[-1 + \sum_{n=0}^{\infty} \frac{1}{n!} \right] - 3e \\ & = 13 \left[-1 + e \right] - 3e = 13e - 13 - 3e = 10e - 13 // \end{aligned}$$

Question 118 (*****)

Consider the binomial infinite series expansion

$$(1+ax)^n,$$

where $a \in \mathbb{R}$, $n \in \mathbb{Q}$, $n \notin \mathbb{N}$.

Show that the series converges if $|ax| < 1$.

proof

$\sum_{n \in \mathbb{Q}, n \notin \mathbb{N}} (1+ax)^n$

• GENERAL TERM
 $(1+ax)^n = \sum_{r=0}^{\infty} \frac{n(n-1)(n-2)\dots(n-r)}{r!} (ax)^r$

• TEST FOR CONVERGENCE BY D'ALMENETE'S RATIO TEST

$$\lim_{r \rightarrow \infty} \left| \frac{U_{n+r}}{U_n} \right| = L < 1 \quad \text{For convergence}$$

• Hence

$$\left| \frac{n(n-1)(n-2)\dots(n-(r-1))}{(r+1)!} (ax)^{n+r} \right| = \left| \frac{n(n-1)(n-2)\dots(n-r)}{(r+1)!} (ax)^{n+r} \right|$$

$$= \left| \frac{(n!)^2}{(r+1)!(n-r)!} (ax)^{n+r} \right| = \left| \frac{(n-r)}{(r+1)} (ax) \right| = \left| \frac{n-r}{r+1} \right| |ax|$$

• $\lim_{r \rightarrow \infty} \left| \frac{U_{n+r}}{U_n} \right| = \lim_{r \rightarrow \infty} \left| \frac{n-r}{r+1} \right| |ax| = |ax|$

\uparrow
TEST IS < 1

\therefore For convergence $|ax| < 1$

Question 119 (***)**

The n^{th} term of a series is given recursively by

$$u_{n+1} = \frac{n}{2n+1} u_n, \quad n \in \mathbb{N}, \quad u_1 = 2.$$

- a) Show, by direct manipulation, that

$$u_n = \frac{2^n \times [(n-1)!]^2}{(2n-1)!}.$$

[you may not use proof by induction in this part]

- b) Determine whether $\sum_n u_n$ converges or diverges.

converges

a) $U_{n+1} = \frac{n}{2n+1} U_n \quad U_1 = 2$

$$\begin{aligned} U_{n+1} &= \frac{n}{2n+1} \times \frac{n-1}{2n-1} \times \dots \times \frac{2}{5} \times \frac{1}{3} U_1 \\ U_{n+1} &= \frac{n(n-1)(n-2) \dots \times 2 \times 1}{(2n)(2n-1)(2n-2) \dots \times 7 \times 5 \times 3} U_1 \\ U_{n+1} &= \frac{n!}{(2n+1)(2n-1)(2n-3) \dots \times 7 \times 5 \times 3} \times 2 \\ U_{n+1} &= \frac{n! \times (2n)(2n-1)(2n-2) \dots \times 4 \times 3 \times 2}{(2n+1)(2n-1)(2n-2) \dots \times 7 \times 5 \times 3} \\ U_{n+1} &= \frac{n! \times 2^{2n}}{(2n+1)(2n-1)(2n-3) \dots \times 3 \times 2 \times 1} \times 2 \\ U_{n+1} &= \frac{n! \times 2^{2n} \times (n!)!}{(2n+1)(2n-1)} = \frac{n! \times 2^{2n} \times (n+1) \times n!}{2(2n+1)(2n-1)} \\ U_{n+1} &= \frac{n! \times n! \times 2^{2n} \times 2(2n+1)}{(2n+1)! \times 2(2n+1)} \\ U_{n+1} &= \frac{2^n (n!)^2}{(2n+1)!} \\ \therefore U_n &= \frac{2^n (n!)^2}{(2n-1)!} \end{aligned}$$

b) BY THE RATIO TEST, DIRECTLY FROM THE RECURRANCE RELATION

$$\frac{U_{n+1}}{U_n} = \frac{n}{2n+1}$$

As $n \rightarrow \infty$, $\frac{U_{n+1}}{U_n} \rightarrow \frac{1}{2} < 1$

∴ SERIES CONVERGES

Question 120 (*****)

Determine, in terms of k and n , a simplified expression

$$\sum_{r=2}^n \left[\frac{r(1-k)-1}{r(r-1)k^r} \right].$$

, $\frac{1}{n} \left(\frac{1}{k} \right)^n - \frac{1}{k}$

• **SPLIT BY PARTIAL FRACTIONS**

$$\frac{\Gamma(r-k)-1}{\Gamma(r-1)} \equiv \frac{A}{r} + \frac{B}{r-1}$$

$$\frac{\Gamma(r-k)-1}{\Gamma(r-1)} \equiv A(r-1) + Br$$

If $r=0 \Rightarrow -1 = -A \Rightarrow A=1$
 If $r=1 \Rightarrow -k = B \Rightarrow B=-k$

• **SUMMING** we now have

$$\left(\frac{1}{k} \right)^r \frac{\Gamma(r-k)-1}{\Gamma(r-1)} = \left(\frac{1}{k} \right)^r + \left(\frac{1}{k} \right)^r \left(\frac{k}{r-1} \right)$$

- $r=2$ $\left(\frac{1}{k} \right)^2 \frac{\Gamma(2-k)-1}{\Gamma(1)} = \left(\frac{1}{k} \right)^2 \times \frac{1}{2} - \left(\frac{1}{k} \right)^2 \times \frac{k}{1}$
- $r=3$ $\left(\frac{1}{k} \right)^3 \frac{\Gamma(3-k)-1}{\Gamma(2)} = \left(\frac{1}{k} \right)^3 \times \frac{1}{3} - \left(\frac{1}{k} \right)^3 \times \frac{k}{2}$
- $r=4$ $\left(\frac{1}{k} \right)^4 \frac{\Gamma(4-k)-1}{\Gamma(3)} = \left(\frac{1}{k} \right)^4 \times \frac{1}{4} - \left(\frac{1}{k} \right)^4 \times \frac{k}{3}$
- $r=5$ $\left(\frac{1}{k} \right)^5 \frac{\Gamma(5-k)-1}{\Gamma(4)} = \left(\frac{1}{k} \right)^5 \times \frac{1}{5} - \left(\frac{1}{k} \right)^5 \times \frac{k}{4}$
- ⋮
- $r=n$ $\left(\frac{1}{k} \right)^n \frac{\Gamma(n-k)-1}{\Gamma(n-1)} = \left(\frac{1}{k} \right)^n \times \frac{1}{n} - \left(\frac{1}{k} \right)^n \times \frac{k}{n-1}$

• **ADDING**

$$\sum_{r=2}^n \left[\frac{\Gamma(r-k)-1}{\Gamma(r-1)} \right] = \left(\frac{1}{k} \right)^n \times \frac{1}{n} - \frac{1}{k}$$

Question 121 (*****)

Use an appropriate method to sum the following series

$$\sum_{r=1}^{\infty} \frac{r \times 2^r}{(r+2)!}.$$

You may assume the series converges.

[1]

STARTING FROM THE DEFINITION OF THE EXPONENTIAL SERIES

$$e^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!} ; \text{ where } x \neq 0$$

Divide above by x^2

$$\Rightarrow \frac{e^x}{x^2} = \sum_{n=0}^{\infty} \frac{x^{n-2}}{n!}$$

$$\Rightarrow \frac{e^x}{x^2} = \frac{1}{0!} + \frac{1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$\Rightarrow \frac{e^x}{x^2} = \left(\frac{1}{2!} + \frac{1}{1!} + \frac{1}{0!} \right) + \sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$$

NEXT WE DIFFERENTIATE THE ABOVE EQUATION W.R.T. x

$$\Rightarrow \frac{d}{dx} \left(\frac{e^x}{x^2} \right) = -\frac{2}{x^3} - \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{[x^{n-1}]}{(n+2)!}$$

$$\Rightarrow \frac{e^x(x-2)}{x^3} = -\frac{2}{x^3} - \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{[x^{n-1}]}{(n+2)!}$$

Let $x=2$

$$\Rightarrow 0 = -\frac{1}{4} - \frac{1}{4} + \sum_{n=1}^{\infty} \frac{[x^{n-1}]}{(n+2)!}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{[x^{n-1}]}{(n+2)!} = \frac{1}{2} \times 2$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{[x^{n-1}]}{(n+2)!} = 1$$

Question 122 (*****)

The n^{th} term of a series is given recursively by

$$u_n = \frac{2n}{2n+1} u_{n-1}, \quad n \in \mathbb{N}, \quad u_0 = 1.$$

- a) Show, by direct manipulation, that

$$u_n = \frac{4^n \times (n!)^2}{(2n+1)!}.$$

[you may not use proof by induction in this part]

- b) Determine whether $\sum_n u_n$ converges or diverges.

diverges

<p>a) $u_1 = \frac{2_1}{2+1} u_0 > u_0 = 1$</p> $\begin{aligned} \Rightarrow u_2 &= \frac{2_2}{2+2} \times \frac{2_1}{2+1} u_1 \\ \Rightarrow u_3 &= \frac{2_3}{2+3} \times \frac{2_2}{2+2} \times \frac{2_1}{2+1} u_2 \\ &\dots \\ \Rightarrow u_4 &= \frac{2^{2(2n-1)(2n-3)} \dots 6 \times 4 \times 2}{(2n)(2n-1)(2n-3) \dots 7 \times 5 \times 3} u_0 \\ \Rightarrow u_5 &= \frac{2^{5(2n-1)(2n-3)} \dots 13 \times 11 \times 9}{(2n)(2n-1)(2n-3) \dots 17 \times 15 \times 13} \\ \Rightarrow u_6 &= \frac{2^8 n! \times (2n+1)(2n-3) \dots 17 \times 15 \times 13}{(2n)(2n-1)(2n-3) \dots 21 \times 19 \times 17} \\ \Rightarrow u_7 &= \frac{2^{11} n! \times (2n+1)(2n-3) \dots 23 \times 21}{(2n)(2n-1)(2n-3) \dots 25 \times 23 \times 21} \\ \Rightarrow u_8 &= \frac{2^{14} n! \times (2n+1)(2n-3) \dots 29 \times 27}{(2n)(2n-1)(2n-3) \dots 31 \times 29 \times 27} \\ \Rightarrow u_9 &= \frac{2^{17} n! \times (2n+1)(2n-3) \dots 35 \times 33}{(2n)(2n-1)(2n-3) \dots 37 \times 35 \times 33} \\ \Rightarrow u_{10} &= \frac{2^{20} n! \times (2n+1)(2n-3) \dots 41 \times 39}{(2n)(2n-1)(2n-3) \dots 43 \times 41 \times 39} \end{aligned}$	<p>b) $u_n = \frac{2n}{2n+1} u_{n-1}$</p> $\frac{u_n}{u_{n-1}} = \frac{2n}{2n+1} \rightarrow 1 \text{ as } n \rightarrow \infty$ <p>∴ RATIO TEST FAILS GIVEN THAT ALL TERMS ARE POSITIVE)</p> <p>● BY RATIO'S TEST</p> $\begin{aligned} \lim_{n \rightarrow \infty} \left[n \left(\frac{u_{n+1}}{u_n} - 1 \right) \right] &= \lim_{n \rightarrow \infty} \left[n \left(\frac{2n+1}{2n} - 1 \right) \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{2n+1}{2n} - n \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} \right] = \frac{1}{2} < 1 \\ \therefore \text{SERIES DIVIDES} \end{aligned}$
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Question 123 (*****)

The following convergent series S is given below

$$S = \frac{\sin \theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \dots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S = e^{-\cos \theta} \sin(\sin \theta).$$

, proof

DEFINE SERIES, $C \in \mathbb{C}$, BASED ON COMPLEX NUMBERS.

$$C = \frac{\cos \theta}{1!} - \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} - \frac{\cos 4\theta}{4!} + \dots$$

$$S = \frac{\sin \theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \dots$$

COMBINE TO FORM A COMPLEX EXPONENTIAL SERIES

$$C + iS = \frac{1}{1!}(e^{i\theta} + i\sin \theta) - \frac{1}{2!}(e^{i2\theta} + i\sin 2\theta) + \frac{1}{3!}(e^{i3\theta} + i\sin 3\theta) - \dots$$

$$C + iS = \frac{1}{1!}e^{i\theta} - \frac{1}{2!}e^{i2\theta} + \frac{1}{3!}e^{i3\theta} - \frac{1}{4!}e^{i4\theta} + \dots$$

NOW CONSIDER SOME SIMPLE STANDARD EXPANSIONS

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \dots$$

$$z = \frac{z^2}{2!} + \frac{z^3}{3!} - \frac{z^4}{4!} = 1 - e^{-z}$$

HENCE WE NOW HAVE

$$C + iS = (e^{i\theta}) - \frac{(e^{i\theta})^2}{2!} + \frac{(e^{i\theta})^3}{3!} - \frac{(e^{i\theta})^4}{4!} + \dots$$

$$C + iS = 1 - e^{i\theta}$$

$$C + iS = 1 - e^{(\cos \theta + i\sin \theta)}$$

$$C + iS = 1 - e^{\cos \theta} \times e^{i\sin \theta}$$

$$C + iS = 1 - e^{-\cos \theta} [i\cos(\sin \theta) - i\sin(\sin \theta)]$$

$C + iS = [1 - e^{-\cos \theta} \cos(\sin \theta)] + i[e^{-\cos \theta} \sin(\sin \theta)]$

SELECTING IMAGINARY PART WE OBTAIN

$$\sum_{n=1}^{\infty} \frac{i^n \cos(n\theta)}{n!} = e^{-\cos \theta} \sin(\sin \theta)$$

Question 124 (*****)

$$g(x) \equiv \sum_{r=0}^{\infty} f(x, r) - \frac{1-x}{\sqrt{1-x^2} \sqrt[3]{1-x^3}}, \quad -1 < x < 1.$$

Given that the first term of the series expansion of $g(x)$ is $\frac{1}{5}x^5$, determine in exact simplified form a simplified expression of $f(x, r)$.

, $f(x, r) = \frac{(-x)^r}{r!}$

$\sum_{r=0}^{\infty} f(x, r) - \frac{1-x}{(1-x^2)^{\frac{1}{2}}(1-x^3)^{\frac{1}{3}}} = \frac{1}{5}x^5 + O(x^6)$

- START BY REWRITING THE FRACTION AS FOLLOWS:
 $(1-x)(1-x^2)^{-\frac{1}{2}}(1-x^3)^{-\frac{1}{3}}$
- OBTAIN EACH EXPANSION SEPARATELY

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}(-x)^1 + \frac{-\frac{1}{2}(-\frac{1}{2})}{2!}(-x)^2 + \frac{(-\frac{1}{2})(-\frac{1}{2})(-\frac{1}{2})}{3!}(-x)^3 + O(x^4)$$

$$= 1 + \frac{1}{2}x^1 + \frac{3}{8}x^2 + \frac{5}{16}x^3 + O(x^4)$$

$$(1-x^2)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}(-x^2)^1 + \frac{-\frac{1}{2}(-\frac{1}{2})}{2!}(-x^2)^2 + O(x^3)$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + O(x^5)$$
- COLLECTING ALL THE EXPANSIONS

$$(1-x)\left[1 + \frac{1}{2}x^1 + \frac{3}{8}x^2 + O(x^3)\right]\left[1 + \frac{-\frac{1}{2}}{1}x^1 + \frac{-\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{1}{2})(-\frac{1}{2})}{3!}x^3 + O(x^4)\right]$$

$$= (1-x)\left[1 + \frac{1}{2}x^1 + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{5}{48}x^4 + O(x^5)\right]$$

$$= (1-x)\left(1 + \frac{1}{2}x^1 + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{1}{16}x^4 + O(x^5)\right)$$

$$= \frac{1}{2}x^1 + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{1}{16}x^4 + O(x^5)$$

$$= \frac{-x}{2} - \frac{3}{8}x^2 - \frac{1}{16}x^3 - \frac{3}{16}x^4 + O(x^5)$$

$$= \frac{-x}{2} - \frac{3}{8}x^2 + \frac{1}{16}x^3 - \frac{3}{16}x^4 + O(x^5)$$

• THIS WE NOW HAVE

$$\sum_{r=0}^{\infty} f(x, r) - \left(1 - x + \frac{1}{2}x^2 - \frac{1}{8}x^3 + \frac{1}{24}x^4 - \frac{3}{16}x^5\right) = \frac{1}{5}x^5$$

$$\sum_{r=0}^{\infty} f(x, r) = 1 - x + \frac{1}{2}x^2 - \frac{1}{8}x^3 + \frac{1}{24}x^4 - \frac{1}{160}x^5$$

$$\sum_{r=0}^{\infty} f(x, r) = \frac{x}{5} - \frac{2x}{5} + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{48}x^4 - \frac{1}{8}x^5$$

$$\therefore f(x, r) = \frac{(-x)^r}{r!}$$

Question 125 (*****)

Determine the value of the following infinite convergent sum.

$$\sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right].$$

, $\boxed{\frac{1}{3}}$

Start by partial fractions (by inspection)

$$\frac{4r-1}{r(r-1)} = \frac{1}{r-1} + \frac{3}{r}$$

Hence we now have

$$\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r = \frac{1}{r-1} \left(-\frac{1}{3} \right)^r + \frac{3}{r} \left(-\frac{1}{3} \right)^r$$

- $r=2$ $\frac{7}{2} \left(-\frac{1}{3} \right)^2 = \frac{1}{2} \left(-\frac{1}{3} \right)^2 + \frac{3}{2} \left(-\frac{1}{3} \right)^2 = \frac{1}{2} \left(-\frac{1}{3} \right)^2 + \frac{3}{2} \left(-\frac{1}{3} \right)^2$
- $r=3$ $\frac{11}{6} \left(-\frac{1}{3} \right)^3 = \frac{1}{2} \left(-\frac{1}{3} \right)^3 + \frac{3}{2} \left(-\frac{1}{3} \right)^3 = \frac{1}{2} \left(-\frac{1}{3} \right)^3 + \frac{3}{2} \left(-\frac{1}{3} \right)^3$
- $r=4$ $\frac{15}{24} \left(-\frac{1}{3} \right)^4 = \frac{1}{2} \left(-\frac{1}{3} \right)^4 + \frac{3}{2} \left(-\frac{1}{3} \right)^4 = \frac{1}{2} \left(-\frac{1}{3} \right)^4 + \frac{3}{2} \left(-\frac{1}{3} \right)^4$
- $r=5$ $\frac{19}{120} \left(-\frac{1}{3} \right)^5 = \frac{1}{2} \left(-\frac{1}{3} \right)^5 + \frac{3}{2} \left(-\frac{1}{3} \right)^5 = \frac{1}{2} \left(-\frac{1}{3} \right)^5 + \frac{3}{2} \left(-\frac{1}{3} \right)^5$
- ⋮
- $r=n$ $\frac{4n-1}{n(n-1)} \left(-\frac{1}{3} \right)^n = \frac{1}{n-1} \left(-\frac{1}{3} \right)^n + \frac{3}{n} \left(-\frac{1}{3} \right)^n = \frac{1}{n} \left(-\frac{1}{3} \right)^n + \frac{3}{n} \left(-\frac{1}{3} \right)^n$

Therefore

$$\sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{n} \left(-\frac{1}{3} \right)^n + \frac{3}{n} \right]$$

$$\Rightarrow \sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right] = \frac{1}{3}$$

Question 126 (*****)

Show clearly that

$$\sum_{r=1}^{\infty} \frac{r^2}{2^r} = 6.$$

[proof]

$$\begin{aligned}
 \sum_{r=1}^{\infty} \frac{r^2}{2^r} &= \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \frac{27}{16} + \frac{81}{32} + \frac{243}{64} + \frac{486}{128} + \dots \\
 2S &= 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \frac{243}{32} + \frac{486}{64} + \dots \\
 S &= \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \frac{27}{16} + \frac{81}{32} + \frac{243}{64} + \dots \\
 S' &= 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \frac{243}{32} + \dots \\
 S' &= 1 + \frac{27}{2^7} + \frac{81}{2^8} + \dots \\
 S' &= 1 + \frac{27}{2^7} + \frac{81}{2^8} + \dots \leftarrow \text{standard G.P.} \\
 S' &= 2 + 2 \cdot \frac{27}{2^7} \\
 \frac{S'}{2} &= 2 + 2T \\
 T &= \sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \dots \\
 2T &= 1 + 1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \dots \\
 T &= \frac{1}{2} + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \dots \\
 T &= 1 + \frac{1}{2} + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \dots \\
 T &= 1 + (\text{G.P. with sum to infinity is 1}) \\
 T &= 2 \\
 \therefore S &= 2 + 2T \\
 S &= 2 + 2 \cdot 2 \\
 \frac{S}{2} &= 6
 \end{aligned}$$

ALTERNATIVE USING STANDARD RESULTS FROM STATISTICS

FOR A GEOMETRIC DISTRIBUTION WITH PARAMETER p , $E(X) = \frac{1-p}{p}$, $\text{Var}(X) = \frac{1-p}{p^2}$.

NOW SURFACE THE PROBABILITY OF AN SIGHN IS $\frac{1}{2}$, E.G. TOSING A FAIR COIN AND X IS THE NUMBER OF FLIPS UNTIL YOU GET HEAD

x	: 1	2	3	4	5	...
$P(X=x)$: $\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$...

$$\begin{aligned}
 \text{Thus } E(X) &= \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(2 \times \frac{1}{4}\right) + \left(3 \times \frac{1}{8}\right) + \left(4 \times \frac{1}{16}\right) + \left(5 \times \frac{1}{32}\right) + \dots \\
 &= \sum_{n=1}^{\infty} n \cdot \frac{1}{2^n}
 \end{aligned}$$

$$\begin{aligned}
 \text{But } \text{Var}(X) &= E(X) - (E(X))^2 \\
 \frac{1-\frac{1}{2}}{\frac{1}{2}} &= \sum_{n=1}^{\infty} \frac{n^2}{2^n} - \left(\frac{1}{2}\right)^2 \\
 2 &= \frac{5}{2} \cdot \frac{1}{2^2} - 4 \\
 \sum_{n=1}^{\infty} \frac{n^2}{2^n} &= 6
 \end{aligned}$$

Question 127 (*****)

Show clearly that

$$1 + \frac{1}{24} + \frac{1 \cdot 4}{24 \cdot 48} + \frac{1 \cdot 4 \cdot 7}{24 \cdot 48 \cdot 72} + \frac{1 \cdot 4 \cdot 7 \cdot 10}{24 \cdot 48 \cdot 72 \cdot 96} - \dots = \frac{2}{\sqrt[3]{7}}.$$

[proof]

$$\begin{aligned}
 S &= 1 + \frac{1}{24} + \frac{1 \times 4}{24 \times 48} + \frac{1 \times 4 \times 7}{24 \times 48 \times 72} + \dots \\
 S &= 1 + \frac{1}{24(1)} + \frac{1 \times 4}{24^2(1 \times 2)} + \frac{(1 \times 4 \times 7) \times 10}{24^3(1 \times 2 \times 3 \times 4)} + \dots \\
 S &= 1 + \frac{3(1)}{24(1)} + \frac{3(1)(4)}{24^2(1 \times 2)} + \frac{3(1)(4)(7)}{24^3(1 \times 2 \times 3)} + \dots
 \end{aligned}$$

NOTE FOR BINOMIAL COEFFICIENT SERIES THE POWERS MUST BE DECREASING.

i.e. $(1-x)^n = 1 - nx + \dots$ etc.

THUS ARRANGING THE SIGNS

$$S = 1 + \frac{(1)}{1} \left(-\frac{1}{24}\right)^1 + \frac{(1)(3)}{1 \times 2} \left(-\frac{1}{24}\right)^2 + \frac{(1)(3)(5)}{1 \times 2 \times 3} \left(-\frac{1}{24}\right)^3 + \frac{(1)(3)(5)(7)}{1 \times 2 \times 3 \times 4} \left(-\frac{1}{24}\right)^4 + \dots$$

THIS IS THE BINOMIAL SERIES EXPANSION OF

$$(1 - \frac{1}{24x})^{-\frac{1}{2}} \text{ WITH } x=24 \quad (\text{NOTE THIS SHOWS CONVERGENCE FOR } -8 < x < 8)$$

$$S = \left(1 - \frac{1}{24}\right)^{-\frac{1}{2}} = \left(\frac{23}{24}\right)^{-\frac{1}{2}} = \left(\frac{23}{24}\right)^{\frac{1}{2}} = \frac{\sqrt{23}}{\sqrt{24}} = \frac{\sqrt{23}}{2\sqrt{6}} = \frac{\sqrt{138}}{12}.$$

Question 128 (*****)

$$S_n = \sum_{r=1}^n (r^2 \times 2^r)$$

Use the standard techniques for the summation of a geometric series, to show that

$$S_n = (n^2 - 2n + 3) \times 2^{n+1} - 6.$$

[You may not use proof by induction in this question.]

proof

$$\begin{aligned}
 \sum_{r=1}^n r^2 2^r &= S_n = \{x^3 + 2x^2 + 3x^2 + 4x^3 + 5x^2 + \dots + n^2 x^{n+1}\} \text{ (1)} \\
 -2S_n &= \cancel{-1x^3} - \cancel{2x^2} - \cancel{3x^2} - \cancel{4x^3} - \dots - \cancel{(n-1)x^{n+1}} - n^2 x^{n+2} \\
 -S_n &= \cancel{-1x^3} - \cancel{3x^2} + \cancel{5x^3} + \cancel{7x^4} + \cancel{9x^5} + \dots + \cancel{(2n-1)x^2} - \cancel{n^2 x^{n+1}} \\
 \text{Hence } +2S_n &= \cancel{-1x^3} - \cancel{3x^2} - \cancel{5x^3} - \cancel{7x^4} - \cancel{9x^5} - \dots - \cancel{(2n-3)x^2} - \cancel{(2n-1)x^3} + \cancel{n^2 x^{n+2}} \\
 \text{from (1)} &= 1x^2 + 2x^2 + 2x^2 + 2x^2 + \dots + 2x^2 + (-n^2 - 2n + 1)x^{n+1} + 2n^2 x^{n+1} \\
 S_n &= 2 + 2[1^2 + 2^2 + 3^2 + \dots + n^2] + (n^2 - 2n + 1)x^{n+1} \\
 &\quad \boxed{\text{OF UNIT } \frac{a=1}{r=2} \frac{n=6}{n=3}} \\
 S_1 &= 2 + 2 \left(\frac{n(n+1)}{2-1} \right) + (n^2 - 2n + 1)x^{n+1} \\
 S_2 &= 2 + 2(2^3 - 1) + (n^2 - 2n + 1)x^{n+1} \\
 S_3 &= 2 + 2 \cdot 2^6 \cdot 8 + (n^2 - 2n + 1)x^{n+1} \\
 S_4 &= (n^2 - 2n + 3)x^{n+1} - 6 \\
 &\quad \boxed{\text{AS 24 UNIT}}
 \end{aligned}$$

Question 129 (*****)

By showing a detailed method, sum the following series.

$$\sum_{r=0}^9 [(r+1) \times 11^r \times 10^{9-r}]$$

You may leave the answer in index form.

$$\boxed{\quad}, \quad \boxed{\sum_{r=0}^9 [(r+1) \times 11^r \times 10^{9-r}] = 10^{11}}$$

$$\begin{aligned} & \sum_{r=0}^9 [(r+1) \times 11^r \times 10^{9-r}] = ? \\ \bullet & \text{ WRITE A FEW TERMS OUT AND LOOK FOR A PATTERN} \\ \Rightarrow S &= 1(11^0)(10)^9 + 2(11^1)(10)^8 + 3(11^2)(10)^7 + \dots + 9(11^8)(10)^1 + 10(11^9)(10)^0 \\ \Rightarrow S &= 1(11^0)(10)^9 + 2\left(\frac{11}{10}\right)(10)^8 + 3\left(\frac{11}{10}\right)^2(10)^7 + \dots + 9\left(\frac{11}{10}\right)^8(10)^1 + 10\left(\frac{11}{10}\right)^9 \\ \Rightarrow S &= 10^9 \left[1 + 2\left(\frac{11}{10}\right) + 2\left(\frac{11}{10}\right)^2 + \dots + 9\left(\frac{11}{10}\right)^8 + 10\left(\frac{11}{10}\right)^9 \right] \\ \bullet & \text{ THE SUM IS AN ARITHMETIC GEOMETRIC PROGRESSION - MULTIPLY BY } -\frac{1}{10} \\ \Rightarrow -\frac{1}{10}S &= 10^9 \left[-\left(\frac{11}{10}\right) - 2\left(\frac{11}{10}\right)^2 - \dots - 9\left(\frac{11}{10}\right)^8 - 9\left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10} \right] \\ \bullet & \text{ ADD THE LAST TWO LINES ABOVE} \\ \Rightarrow -\left(1 - \frac{1}{10}\right)S &= 10^9 \left[1 + \left(\frac{11}{10}\right) + \left(\frac{11}{10}\right)^2 + \dots + \left(\frac{11}{10}\right)^8 + \left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10} \right] \\ & \quad \text{THIS IS A G.P.} \\ & \quad \boxed{\frac{a_1}{1-r}} \quad \boxed{S_n = \frac{a(r^n - 1)}{r - 1}} \\ \Rightarrow -\frac{9}{10}S &= 10^9 \times \frac{\left(\frac{11}{10}\right)^9 - 1}{\frac{11}{10} - 1} + 10^9 \times -10\left(\frac{11}{10}\right)^{10} \\ \Rightarrow -\frac{1}{10}S &= 10^9 \times \frac{\left(\frac{11}{10}\right)^9 - 1}{\frac{11}{10}} - 10^9 \times \left(\frac{11}{10}\right)^{10} \\ \Rightarrow -\frac{1}{10}S &= 10^9 \left(\left(\frac{11}{10}\right)^9 - 1 \right) - 10^9 \times \left(\frac{11}{10}\right)^{10} \\ \Rightarrow -\frac{1}{10}S &= 10^9 \times \left(\frac{11}{10}\right)^9 - 10^9 - 10^9 \times \left(\frac{11}{10}\right)^{10} \\ \Rightarrow S &= \boxed{10^{11}} \end{aligned}$$

Question 130 (*****)

Use the ratio test to show that the following series converges

$$\sum_{n=1}^{\infty} \left[\frac{5^n + 1}{n^n + 8} \right].$$

You may assume without proof that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{-n} = \frac{1}{e}$.

, proof

AS ALL THE TERMS ARE POSITIVE WE MAY IGNORE NEGATIVE IN THE RATIO TEST

$$\begin{aligned} \frac{U_{n+1}}{U_n} &= \frac{5^{n+1} + 1}{(n+1)^{n+1} + 8} \times \frac{n^n + 8}{5^n + 1} \\ &< \frac{5^{n+1} + 5}{(n+1)^{n+1} + 8} \times \frac{n^n + 8}{5^n + 1} \\ &= \frac{5(n^n + 1)}{(n+1)^{n+1} + 8} \times \frac{n^n + 8}{5^n + 1} \\ &= \frac{5(n^n + 1)}{(n+1)^{n+1} + 8} \\ &< \frac{5 \times n^n}{(n+1)^{n+1} + 8} \\ &< \frac{A \times n^n}{(n+1)^{n+1}} \quad (\text{for sufficiently large } A) \\ &= A \cdot \frac{n^n}{(n+1)^{n+1}} \\ &= A \cdot \left(\frac{n}{n+1}\right)^n \times \frac{1}{n+1} \\ &= \frac{A}{n+1} \times \left(\frac{n}{n+1}\right)^n = \frac{A}{n+1} \times \left(1 + \frac{1}{n}\right)^{-n} \\ &= \frac{A}{n+1} \times e^{-1} \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

∴ $\sum_{n=1}^{\infty} \left[\frac{5^n + 1}{n^n + 8} \right]$ CONVERGES

Question 131 (*****)

$$f(x) = \frac{1}{\sqrt{1-x}}, -1 < x < 1.$$

a) By manipulating the general term of binomial expansion of $f(x)$ show that

$$f(x) = \sum_{r=0}^{\infty} \binom{2r}{r} \left(\frac{1}{4}x\right)^r.$$

b) Find a similar expression for $\frac{x}{(16-x^2)^{\frac{3}{2}}}$ and show further that

$$\frac{x}{(16-x^2)^{\frac{3}{2}}} = \sum_{r=1}^{\infty} \binom{2r}{r} \left(\frac{1}{16}r\right) \left(\frac{1}{8}x\right)^{2r-1}.$$

c) Determine the exact value of

$$\sum_{r=1}^{\infty} \binom{2r}{r} \left(\frac{5}{32}r\right) \left(\frac{4}{25}\right)^r.$$

$\boxed{\frac{25}{108}}$

a) $(1-x)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}(-x)^1}{1!} + \frac{-\frac{1}{2}(-\frac{1}{2})(-x)^2}{2!} + \frac{-\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-x)^3}{3!} + \dots + \frac{-\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})...(-\frac{1}{2})^{n-1}(-x)^n}{n!} + \dots$
 REWRITE THIS COMPACTLY – PROBABLY IT IS EASIER TO LEAVE THE 1 AT THE BEGINNING OF THE SUMMATION
 $= 1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})...(-\frac{1}{2})^n}{n!} (-x)^n = 1 + \sum_{n=1}^{\infty} \left[\frac{(-\frac{1}{2})^n}{n!} \right] x^n (-1)^{n-1} (-x)^n =$
 $= 1 + \sum_{n=1}^{\infty} \left[\frac{(-\frac{1}{2})^n n! (n+1)(n+3)...(n+2n-1)}{n!} x^n \right] = 1 + \sum_{n=1}^{\infty} \left[\frac{(-\frac{1}{2})^n (n+1)(n+3)...(n+2n-1)}{1} x^n \right] =$
 $= 1 + \sum_{n=1}^{\infty} \left[\frac{(-\frac{1}{2})^n (n+1)(n+3)...(n+2n-1)}{1} \cancel{x(x+2)(x+4)...(x+2n-2)} \left(\frac{1}{2}\right)^n x^n \right]$
 $= 1 + \sum_{n=1}^{\infty} \left[\frac{(-\frac{1}{2})^n (n+1)(n+3)...(n+2n-1)}{2^n 1 \times 3 \times 5 \times ... \times (2n-1)} \left(\frac{1}{2}\right)^n x^n \right] = 1 + \sum_{n=1}^{\infty} \left[\frac{(2n)!}{n! n! 2^n} \left(\frac{1}{2}\right)^n x^n \right]$
 $= 1 + \sum_{n=1}^{\infty} \left[\frac{(2n)!}{n! n! 2^n} \left(\frac{1}{2}x\right)^n \right] = 1 + \sum_{n=1}^{\infty} \left[\left(\frac{2^n}{n!}\right) \left(\frac{1}{2}x\right)^n \right]$
 Note that I added TWO SPACES here.
b) Now $(16-x^2)^{-\frac{3}{2}} = 16^{-\frac{3}{2}} (1 - \frac{1}{16}x^2)^{-\frac{1}{2}} = \frac{1}{4} \sum_{r=0}^{\infty} \left[\binom{r}{2} \left(\frac{1}{16}x^2\right)^r \right] = \frac{1}{4} \sum_{r=0}^{\infty} \left[\binom{r}{2} \left(\frac{1}{16}\right)^r x^{2r} \right]$
 $= \frac{1}{4} \sum_{r=0}^{\infty} \left[\binom{r}{2} \left(\frac{5}{32}\right)^r \right] = \sum_{r=1}^{\infty} \left[\frac{1}{4} \binom{2r}{r} \left(\frac{5}{32}\right)^r \right]$

Differentiate both sides
 $\frac{d}{dx} \left[(16-x^2)^{-\frac{3}{2}} \right] = \frac{d}{dx} \left[\sum_{r=1}^{\infty} \left[\frac{1}{4} \binom{2r}{r} \left(\frac{5}{32}\right)^r x^{2r} \right] \right]$
 $\Rightarrow -\frac{3}{2}(16-x^2)^{-\frac{5}{2}} x = \sum_{r=1}^{\infty} \left[\frac{1}{4} \binom{2r}{r} \times 2r \left(\frac{5}{32}\right)^{r-1} x^{2r-1} \right]$
 $\Rightarrow \frac{-3}{(16-x^2)^{\frac{5}{2}}}{x} = \sum_{r=1}^{\infty} \left[\frac{2r}{4} \binom{2r}{r} \left(\frac{5}{32}\right)^{r-1} x^{2r-1} \right]$
 $\Rightarrow \sum_{r=1}^{\infty} \left[\frac{2r}{4} \binom{2r}{r} \left(\frac{5}{32}\right)^{r-1} x^{2r-1} \right] = \frac{-3}{16-x^2}$
 Now $\frac{2}{4} = \frac{1}{2}$
 $\therefore x = \frac{16}{3}$
 $\therefore \frac{16}{3} = \frac{16}{\left[1 - \left(\frac{1}{16}x^2\right)\right]^{\frac{5}{2}}}$
 $\therefore \frac{16}{3} = \frac{\frac{16}{3}}{64 \left[\left(\frac{16}{3}\right)^2\right]^{\frac{5}{2}}} = \frac{\frac{16}{3}}{64 \times \frac{256}{81}} = \frac{16}{3} \times \frac{21}{128} \times 128$
 $\therefore \frac{16 \times 21}{3 \times 128} = \frac{21}{108}$

Question 132 (*****)

Determine, in terms of n , a simplified expression

$$\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right],$$

and hence, or otherwise, deduce the value of

$$\sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right].$$

□	$\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{1}{6} - \frac{n+5}{(n+5)!}$	$\sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24}$
---	-----------------------------------------------------------------------------------------------	----------------------------------------------------------------------------------

• Start with partial fractions — note that the numerator is a quadratic in r , so we have to try 2 fractions. After:

$$1.6 \quad \frac{r^2 + 9r + 19}{(r+5)!} \equiv \frac{A}{(r+5)!} + \frac{B}{(r+3)!}$$

$$\Rightarrow r^2 + 9r + 19 \equiv A + B(r+5)(r+4)$$

$$\Rightarrow r^2 + 9r + 19 \equiv Br^2 + 9Br + (20B+A)$$

$$\therefore B=1 \text{ & } A=-1$$

• Hence by the method of differences,

$\frac{r^2 + 9r + 19}{(r+5)!} = \frac{1}{(r+5)!} - \frac{1}{(r+3)!}$

$r=1 \quad \frac{1+9+19}{6!} = \frac{1}{4!} - \frac{1}{2!}$
 $r=2 \quad \frac{4+18+19}{7!} = \frac{1}{3!} - \frac{1}{1!}$
 $r=3 \quad \frac{9+27+19}{8!} = \frac{1}{2!} - \frac{1}{0!}$
 $r=4 \quad \frac{16+36+19}{9!} = \frac{1}{1!} - \frac{1}{(-1)!}$
 \vdots
 $r=n-1 \quad \frac{(n+5)(n+4)(n+3)}{(n+4)!} = \frac{1}{(n+2)!} - \frac{1}{(n+1)!}$
 $r=n \quad \frac{n^2 + 9n + 19}{(n+5)!} = \frac{1}{(n+3)!} - \frac{1}{(n+2)!}$

• Therefore

$$\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{1}{4!} + \frac{1}{3!} - \left[\frac{1}{(n+4)!} + \frac{1}{(n+3)!} \right]$$

$$= \frac{5}{24} + \frac{1}{6!} - \left[\frac{n+5}{(n+5)!} + \frac{1}{(n+4)!} \right]$$

• Now decompose as follows

$$\sum_{r=1}^n \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{n+6}{(n+5)!}$$

$\frac{(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)}{(n+6)!} = \frac{5}{24} - \frac{n+6}{(n+5)!}$
 $\sum_{r=1}^n \left[\frac{(r+3)^2 + 7(r+3) + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{n+6}{(n+5)!}$
 $\Rightarrow \sum_{r=2}^n \left[\frac{(r+3)^2 + 7(r+3) + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{n+6}{(n+5)!}$
 $\Rightarrow \sum_{r=2}^n \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{n+6}{(n+5)!}$
 $\Rightarrow \sum_{r=1}^n \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{n+6}{(n+5)!}$
 $\Rightarrow \sum_{r=1}^n \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{1+7+11}{4!} + \frac{5}{24} - \frac{n+6}{(n+5)!}$
 $\Rightarrow \sum_{r=1}^n \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{25}{24} - \frac{n+6}{(n+5)!}$
 $\Rightarrow \sum_{r=1}^n \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{4!} - \frac{n+6}{(n+5)!}$
 $\Rightarrow \sum_{r=1}^{n-1} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24} - \frac{n+6}{(n+5)!}$
 $\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24}$

Question 133 (***)**

The n^{th} term of a series is given recursively by

$$A_n = \frac{a(2n+1)}{2n+4} A_{n-1}, \quad n \in \mathbb{N}, \quad n \geq 1,$$

where a is a positive constant.

Given further that $A_0 = 1$, show that

$$A_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{1}{n+1}.$$



, proof

$A_n = \frac{a(2n+1)}{2n+4} A_{n-1} = \frac{a(2n+1)}{2(2n+2)} A_{n-1}$

• GENERATE A PATTERN FROM THE RECURSIVE RELATION

- $A_1 = \left(\frac{a}{4}\right) \binom{2(1)+1}{2(1)+4} \times \left(\frac{a}{4}\right) \binom{2(1)+1}{2(1)+4} A_{0-1}$
- $A_2 = \left(\frac{a}{4}\right) \binom{2(2)+1}{2(2)+4} \times \left(\frac{a}{4}\right) \binom{2(2)+1}{2(2)+4} \times \left(\frac{a}{4}\right) \binom{2(2)+1}{2(2)+4} A_{0-3}$
- $A_3 = \left(\frac{a}{4}\right) \binom{2(3)+1}{2(3)+4} \times \left(\frac{a}{4}\right) \binom{2(3)+1}{2(3)+4} \times \left(\frac{a}{4}\right) \binom{2(3)+1}{2(3)+4} \times \dots \times \left(\frac{a}{4}\right) \binom{2(3)+1}{2(3)+4} A_{0-5}$
- $A_4 = \left(\frac{a}{4}\right) \binom{2(4)+1}{2(4)+4} \times \left(\frac{a}{4}\right) \binom{2(4)+1}{2(4)+4} \times \left(\frac{a}{4}\right) \binom{2(4)+1}{2(4)+4} \times \dots \times \left(\frac{a}{4}\right) \binom{2(4)+1}{2(4)+4} \times \left(\frac{a}{4}\right) \binom{2(4)+1}{2(4)+4} A_{0-7}$

• Now $A_0 = 1$ so we may compare the expression as follows

$$\Rightarrow A_1 = \left(\frac{a}{4}\right)^1 \frac{(2(1)+1)(2(1)-3) \dots (2(1)-1)}{(2(1)+4)(2(1)+3)(2(1)+2) \dots 2(1)+3}$$

$$\Rightarrow A_2 = \left(\frac{a}{4}\right)^2 \frac{(2(2)+1)(2(2)-3) \dots (2(2)-1)(2(2)-4) \dots 6 \times 5 \times 4 \times 3 \times 2}{(2(2)+4)(2(2)+3)(2(2)+2)(2(2)+1) \dots (2(2)+1)}$$

$$\Rightarrow A_3 = \left(\frac{a}{4}\right)^3 \frac{(2(3)+1)(2(3)-3) \dots (2(3)-1)(2(3)-4) \dots 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(2(3)+4)(2(3)+3)(2(3)+2)(2(3)+1) \dots (2(3)+1)}$$

$$\Rightarrow A_4 = \left(\frac{a}{4}\right)^4 \frac{(2(4)+1)(2(4)-3) \dots (2(4)-1)(2(4)-4) \dots 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(2(4)+4)(2(4)+3)(2(4)+2)(2(4)+1) \dots (2(4)+1)}$$

$$\Rightarrow A_5 = \frac{a^5}{2^5 5!} \times \frac{2(5+1)!}{(5+1)(5+2)!}$$

$$\Rightarrow A_6 = \left(\frac{a}{4}\right)^6 \times \frac{2(6+1)!}{(6+1)(6+2)!}$$

$$\Rightarrow A_7 = \left(\frac{a}{4}\right)^7 \times \frac{2(7+1)!}{(7+1)(7+2)!}$$

$$\Rightarrow A_8 = \left(\frac{a}{4}\right)^8 \times \frac{2(8+1)!}{(8+1)(8+2)!}$$

$$\Rightarrow A_9 = \left(\frac{a}{4}\right)^9 \times \frac{2(9+1)!}{(9+1)(9+2)!}$$

$$\Rightarrow A_{10} = \left(\frac{a}{4}\right)^{10} \times \frac{2(10+1)!}{(10+1)(10+2)!}$$

As required

Question 134 (*****)

By considering the series expansions of $\ln(1-x^2)$ and $\ln\left(\frac{1+x}{1-x}\right)$, or otherwise, find the exact value of the following series.

$$\sum_{r=1}^{\infty} \left[\left(\frac{1}{2r} + \frac{1}{2r+1} \right) \left(\frac{1}{4} \right)^r \right].$$

, $-1 + \frac{1}{2} \ln 12$

• STARTING WITH THE FOLLOWING SERIES EXPANSIONS

$$\ln(1+2x) = 2x - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^5}{5} + \dots \quad |x| < 1$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^5}{5} - \dots \quad |x| < 1$$

• NOW USING THE SUGGESTED SERIES

$$\ln(1-x^2) = -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \frac{1}{4}x^8 - \dots$$

$$\begin{aligned} \ln\left(\frac{1+x}{1-x}\right) &= \ln(1+2x) - \ln(1-x) \\ &= 2x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^5 - \dots \\ &= 2x + \frac{1}{3}x^3 + \frac{1}{3}x^5 + \dots \end{aligned}$$

• WORKING AT THE REQUIRED SERIES

$$\begin{aligned} &\left(\frac{1}{2} + \frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4} + \frac{1}{5}\right)\frac{1}{16} + \left(\frac{1}{6} + \frac{1}{7}\right)\frac{1}{64} + \left(\frac{1}{8} + \frac{1}{9}\right)\frac{1}{256} + \dots \\ &= \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{16} + \frac{1}{3} \times \frac{1}{64} + \frac{1}{8} \times \frac{1}{256} + \dots \\ &= \frac{1}{2} \times \frac{1}{2^2} + \frac{1}{2} \times \frac{1}{2^4} + \frac{1}{3} \times \frac{1}{2^6} + \frac{1}{8} \times \frac{1}{2^8} + \dots \\ &= \frac{1}{2} \times \frac{1}{2^2} + \frac{1}{4} \times \frac{1}{2^4} + \frac{1}{6} \times \frac{1}{2^6} + \frac{1}{16} \times \frac{1}{2^8} + \dots \end{aligned}$$

• NOW MANIPULATE THE SUGGESTED EXPANSION AS REQUIRED

$$-\frac{1}{2x}\ln(1-x^2) = -\frac{2^2}{2} + \frac{2^3}{4} + \frac{2^5}{6} + \frac{2^7}{8} + \dots$$

$$\frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right) = 1 + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{9}x^9 + \dots$$

• ADDING THESE RESULTS WE OBTAIN

$$\frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln(1-x^2) = 1 + \left(\frac{1}{2} + \frac{1}{3}\right)x^3 + \left(\frac{1}{4} + \frac{1}{5}\right)x^5 + \left(\frac{1}{6} + \frac{1}{7}\right)x^7 + \dots$$

• THIS WE HAVE

$$\sum_{r=1}^{\infty} \left(\frac{1}{2r} + \frac{1}{2r+1} \right) \frac{1}{4^r} = \frac{1}{2x} \ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln(1-x^2) - 1, \quad |x| < 1$$

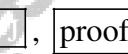
• LET $x = \frac{1}{2}$

$$\begin{aligned} \sum_{r=1}^{\infty} \left(\frac{1}{2r} + \frac{1}{2r+1} \right) \left(\frac{1}{4} \right)^r &= \frac{1}{2x} \ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) - \frac{1}{2}\ln\left(1-\frac{1}{4}\right) - 1 \\ \sum_{r=1}^{\infty} \left(\frac{1}{2r} + \frac{1}{2r+1} \right) \left(\frac{1}{4} \right)^r &= \ln 3 - \frac{1}{2} \ln \frac{3}{2} - 1 \\ &= \frac{1}{2} [\ln 9 - \ln \frac{9}{4}] - 1 \\ &= \frac{1}{2} [\ln 9 + \ln \frac{4}{9}] - 1 \\ &= \frac{1}{2} \ln 12 - 1 \end{aligned}$$

Question 135 (*****)

By considering a suitable binomial expansion, show that

$$\arcsin x = \sum_{r=0}^{\infty} \left[\binom{2r}{r} \frac{2}{2r+1} \left(\frac{x}{2} \right)^{2r+1} \right].$$

Starting from the binomial expansion of $(1-x^2)^{-\frac{1}{2}}$

$$\begin{aligned}\frac{1}{\sqrt{1-x^2}} &= (1-x^2)^{-\frac{1}{2}} = (1-\frac{1}{2}x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}(1-x^2)^{-\frac{1}{2}} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2\sqrt{2}}(1-x^2)^{-\frac{3}{2}} + O(x^4) \\ \frac{1}{\sqrt{1-x^2}} &= (1-\frac{1}{2}x^2)^{-\frac{1}{2}} + \frac{1}{2\sqrt{2}}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2\sqrt{2}}x^4 + O(x^6) \\ \frac{1}{\sqrt{1-x^2}} &= 1 + \frac{1}{2}x^2 + \frac{15}{8}x^4 + \frac{135x^6}{32} + \frac{135x^8}{16} + O(x^8)\end{aligned}$$

Multiplying by x^{-1}

$$\begin{aligned}\frac{1}{\sqrt{1-x^2}} &= 1 + \frac{1}{2}x^2 + \frac{15x^4}{8} + \frac{135x^6}{32} + \frac{135x^8}{16} + O(x^8) \\ \frac{1}{\sqrt{1-x^2}} &= 1 + \frac{21}{12}x^2 + \frac{41}{24}x^4 + \frac{61}{48}x^6 + \frac{61}{48}x^8 + O(x^8) \\ \frac{1}{\sqrt{1-x^2}} &= 1 + \frac{21}{12}x^2 + \frac{41}{24}x^4 + \frac{61}{48}x^6 + \frac{61}{48}x^8 + O(x^8) \\ \frac{1}{\sqrt{1-x^2}} &= 1 + \sum_{n=0}^{\infty} \left[\frac{(2n)!}{(n!)^2} \left(\frac{21}{12} \right)^n \right]\end{aligned}$$

Integrating both sides, within the radius of convergence

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}} dx &= \int \sum_{n=0}^{\infty} \left[\frac{(2n)!}{(n!)^2} \frac{x^{2n+1}}{2^{2n+1}} \right] dx \\ \text{TERM}_0 &= \sum_{n=0}^{\infty} \left[\frac{(2n)!}{(n!)^2} \frac{x^{2n+1}}{2^{2n+1}} \times \frac{1}{2^{2n+1}} \right] + C \quad x=0, \text{ since } C \\ \text{TERM}_1 &= \sum_{n=0}^{\infty} \left[\frac{(2n)!}{(n!)^2} \frac{x^{2n+1}}{2^{2n+1}} \times \frac{1}{2^{2n+1}} \right] \\ \text{TERM}_2 &= \sum_{n=0}^{\infty} \left[\frac{(2n)!}{(n!)^2} \frac{x^{2n+1}}{2^{2n+1}} \times \frac{(\frac{21}{12})^{2n+1}}{2^{2n+1}} \right] \quad \text{As required}\end{aligned}$$

Question 136 (*****)

The product operator \prod , is defined as

$$\prod_{i=1}^k [u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \dots \times u_{k-1} \times u_k.$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right].$$

$$\boxed{8\sqrt{\frac{5}{4}} - 1}$$

Start by writing a few terms explicitly & look for a pattern

$$\begin{aligned} \sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right] &= \frac{1}{1} \left(\frac{8-7}{40} \right) + \frac{2}{2} \left(\frac{15-7}{40} \right) + \frac{3}{3} \left(\frac{22-7}{40} \right) + \frac{4}{4} \left(\frac{29-7}{40} \right) + \dots \\ &= \frac{1}{40} + \frac{1}{40} \times \frac{9}{80} + \frac{1}{40} \times \frac{9}{80} \times \frac{17}{160} + \frac{1}{40} \times \frac{9}{80} \times \frac{17}{160} \times \frac{25}{240} + \dots \\ &= \frac{1}{40} + \frac{1 \times 9}{40 \times 80} + \frac{1 \times 9 \times 17}{40 \times 80 \times 160} + \frac{1 \times 9 \times 17 \times 25}{40 \times 80 \times 160 \times 240} \\ &\dots \end{aligned}$$

This resembles a binomial expansion due to the factorials at the denominator. The next term is to create "numbers" of the form $n(n+1)(n+2)\dots(n+k)$

By inspection this will come as $-\frac{1}{8}, -\frac{3}{8}, -\frac{7}{8}, -\frac{25}{8}$

Now try and adjust the signs

$$= \frac{1}{(8)(8)} (1) + \frac{1 \times 9}{(-8)(8)(2)} + \frac{1 \times 9 \times 17}{(-8)(8)(2)(3)} + \frac{1 \times 9 \times 17 \times 25}{(-8)(8)(2)(3)(4)}$$

$\sum_{k=1}^{\infty} \left[\prod_{r=1}^k \left(\frac{8r-7}{40r} \right) \right] = \underbrace{\frac{-\frac{1}{8}}{1!} (-\frac{1}{8}) + \frac{(\frac{1}{8})(\frac{9}{8})}{2!} (-\frac{1}{8})^2 + \frac{(\frac{1}{8})(\frac{9}{8})(\frac{17}{8})}{3!} (-\frac{1}{8})^3 + \dots}_{\text{THIS IS A BINOMIAL EXPANSION WITH THE } (-1)^k \text{ MISSING AT THE FRONT}}$

$$\begin{aligned} &= \left(-\frac{1}{8} \right)^1 - 1 \\ &= \left(\frac{1}{8} \right)^{-1} - 1 \\ &= \boxed{8\sqrt{\frac{5}{4}} - 1} \end{aligned}$$

Question 137 (*****)

A sequence $u_1, u_2, u_3, u_5, u_6, \dots$ is generated by the recurrence relation

$$n^2 u_{n+1} = (n+1)u_n, \quad n=1, 2, 3, 4, \dots$$

It is further given that

$$\sum_{n=1}^{\infty} u_n = 1.$$

Find in exact form the value of u_1 .

TUTORIAL, $u_1 = \frac{1}{2e}$

START BY RE-WRITING THE RECURSION A, GENERATING A FEW TERMS

$$\begin{aligned} &\rightarrow n^2 u_{n+1} = (n+1)u_n \\ &\rightarrow u_{n+1} = \frac{(n+1)}{n^2} u_n \\ &\bullet u_1 = a \\ &\bullet u_2 = \frac{2}{1^2} a \\ &\bullet u_3 = \frac{3}{2^2} \times \frac{2}{1^2} a \\ &\bullet u_4 = \frac{4}{3^2} \times \frac{3}{2^2} \times \frac{2}{1^2} a \\ &\bullet u_5 = \frac{5}{4^2} \times \frac{4}{3^2} \times \frac{3}{2^2} \times \frac{2}{1^2} a \\ &\vdots \end{aligned}$$

ADDING THESE TERMS (DODGONIAN) WE OBTAIN

$$\begin{aligned} \sum_{n=1}^{20} u_n &= a \left[1 + \frac{2}{1^2} + \frac{3 \cdot 2}{2^2} + \frac{4 \cdot 3 \cdot 2}{3^2} + \frac{5 \cdot 4 \cdot 3 \cdot 2}{4^2} + \dots \right] \\ &\rightarrow 1 = a \left[1 + \frac{2}{(1!)^2} + \frac{3!}{(2!)^2} + \frac{4!}{(3!)^2} + \dots \right] \\ &\rightarrow 1 = a \left[\frac{1!}{(1!)^2} + \frac{2!}{(2!)^2} + \frac{3!}{(3!)^2} + \frac{4!}{(4!)^2} + \dots \right] \\ &\rightarrow 1 = a \left[\frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \frac{4}{4!} + \dots \right] \\ &\rightarrow 1 = a \sum_{n=0}^{\infty} \left(\frac{n!}{n!} \right) \end{aligned}$$

MANIPULATING FURTHER UNDER THE SIGMA NOTATION

$$\rightarrow 1 = a \sum_{n=0}^{\infty} \left(\frac{n}{n!} + \frac{1}{n!} \right)$$

FOR THIS IS ZERO SO IT MAY BE REMOVED

$$\begin{aligned} &\rightarrow 1 = a \left[\sum_{n=1}^{\infty} \frac{n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} \right] \\ &\rightarrow 1 = a \left[\sum_{n=0}^{\infty} \frac{n}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!} \right] \\ &\text{AFTER THE FIRST SUMMATION AS IT GIVES AN INFINITE GEOM } \\ &\rightarrow 1 = a \left[\sum_{n=0}^{\infty} \frac{n}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!} \right] \\ &\rightarrow 1 = 2a \sum_{n=0}^{\infty} \frac{1}{n!} \\ &\rightarrow 1 = 2a e \\ &\rightarrow a = \frac{1}{2e} \\ &\rightarrow u_1 = \frac{1}{2e} \end{aligned}$$

$\boxed{\frac{1}{2e}}$

Question 138 (*****)

Find the sum to infinity of the following convergent series.

$$\frac{1}{4 \times 2!} + \frac{1}{5 \times 3!} + \frac{1}{6 \times 4!} + \frac{1}{7 \times 5!} + \frac{1}{8 \times 6!} + \dots$$

$$\boxed{}, \frac{1}{6}$$

WRITING THE SERIES IN Sigma NOTATION

$$S_{\infty} = \sum_{n=1}^{\infty} \frac{1}{(n+3)(n+1)!}$$

ATTEMPT SUMMATION BY THE METHOD OF DIFFERENCES

TRY $\frac{1}{(n+3)(n+1)!} = \frac{A}{(n+3)!} + \frac{B}{(n+1)!}$

$$1 = A + B(n+3)(n+2)$$

NO A & B CAN SATISFY THE ABOVE

TRY NEXT $\frac{1}{(n+3)(n+1)!} = \frac{A}{(n+3)!} + \frac{B}{(n+2)!}$

$$\Rightarrow \frac{1}{(n+3)(n+1)!} = \frac{A + B(n+3)}{(n+3)!}$$

$$\Rightarrow \frac{n+2}{(n+3)(n+1)!} = \frac{-A + B(n+2)}{(n+3)!}$$

$$\Rightarrow \frac{n+2}{(n+3)!} = \frac{A + B(n+1)}{(n+3)!}$$

$$\Rightarrow n+2 = (A+3B) + Bn$$

$\therefore B=1$ & $A=-1$

HENCE WE NOW HAVE A SUITABLE IDENTITY

$$\frac{1}{(n+3)(n+1)!} = \frac{1}{(n+3)!} - \frac{1}{(n+1)!}$$

- $n=1$: $\frac{1}{4 \times 2!} = \frac{1}{3!} - \frac{1}{1!}$
- $n=2$: $\frac{1}{5 \times 3!} = \frac{1}{4!} - \frac{1}{2!}$
- $n=3$: $\frac{1}{6 \times 4!} = \frac{1}{5!} - \frac{1}{3!}$
- $n=4$: $\frac{1}{7 \times 5!} = \frac{1}{6!} - \frac{1}{4!}$
- ⋮
- $n=N$: $\frac{1}{(N+3)(N+1)!} = \frac{1}{(N+3)!} - \frac{1}{(N+1)!}$

$$\Rightarrow \sum_{n=1}^{N-1} \frac{1}{(n+3)(n+1)!} = \frac{1}{3!} - \frac{1}{(N+1)!}$$

$$\Rightarrow \lim_{N \rightarrow \infty} \left[\sum_{n=1}^{N-1} \frac{1}{(n+3)(n+1)!} \right] = \lim_{N \rightarrow \infty} \left[\frac{1}{3!} - \frac{1}{(N+1)!} \right]$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(n+3)(n+1)!} = \frac{1}{3!} = \frac{1}{6}$$

Question 139 (*****)

- a) Use an appropriate integration method to evaluate the following integral.

$$\int_0^1 x^3 \arctan x \, dx.$$

- b) Obtain an infinite series expansion for $\arctan x$ and use this series expansion to verify the answer obtained for the above integral in part (a).

[you may assume that integration and summation commute]

, $\frac{1}{6}$

a) START BY INTEGRATION BY PARTS

$$\begin{aligned} \int_0^1 x^3 \arctan x \, dx &= [\arctan x]_0^1 - \int_0^1 \frac{x^3}{1+x^2} \, dx \\ &= \frac{\pi}{4}x - 0 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} \, dx \\ &= \frac{\pi}{4} - \frac{1}{2} \int_0^1 \frac{x^2(2x^2+1)-2x^2}{2x^2+1} \, dx \\ &= \frac{\pi}{4} - \frac{1}{2} \int_0^1 (x^2-1 + \frac{2x^2}{2x^2+1}) \, dx \\ &= \frac{\pi}{4} - \frac{1}{2} \left[\frac{2}{3}x^3 - x + \arctan x \right]_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2} \left[\frac{2}{3}(1)^3 - 1 + \arctan 1 \right] \\ &= \frac{\pi}{4} - \frac{1}{2} \left[\frac{2}{3} - \frac{1}{3} + \frac{\pi}{4} \right] \\ &= \frac{\pi}{4} - \frac{1}{6} - \frac{\pi}{8} \\ &= \frac{1}{6} \end{aligned}$$

b) NEED THE EXPANSION OF $\arctan x$.

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1-x^2+x^4-x^6+x^8-\dots$$

INTEGRATE BOTH SIDES GIVES

$$\begin{aligned} \arctan x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots + C \\ \arctan 0 &= 0 \Rightarrow C = 0 \end{aligned}$$

- $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots$
- $\arctan x = \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} x^{2n+1} \right]$

THIS WE KNOW HAVE

$$\begin{aligned} \int_0^1 x^3 \arctan x \, dx &= \int_0^1 x^3 \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} x^{2n+1} \right] \, dx \\ &= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \int_0^1 x^{2n+4} \, dx \right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left[\frac{x^{2n+5}}{2n+5} \right]_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+5)} \left[\frac{1}{2n+5} \right] \end{aligned}$$

NEED TO SUM THIS SERIES BY PARTIAL FRACTION

$$\frac{1}{(2n+1)(2n+5)} = \frac{A}{2n+1} - \frac{B}{2n+5} \quad (\text{BY PARTITION})$$

TRUE WE NOW HAVE

$$\int_0^1 x^3 \arctan x \, dx = \frac{1}{4} \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n+5} \right]$$

INVERSE THE PARTITION

- $n=0$ $\frac{1}{1} - \frac{1}{5}$
- $n=1$ $\frac{1}{3} - \frac{1}{7}$
- $n=2$ $\frac{1}{5} - \frac{1}{9}$
- $n=3$ $\frac{1}{7} - \frac{1}{11}$
- \vdots
- $n=4$ $\frac{1}{9} - \frac{1}{13}$
- $n=5$ $\frac{1}{11} - \frac{1}{15}$

FINALLY WE HAVE THE RESULT

$$\begin{aligned} \int_0^1 x^3 \arctan x \, dx &= \frac{1}{4} \lim_{r \rightarrow \infty} \sum_{n=0}^r \left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n+5} \right] \\ &= \frac{1}{4} \lim_{r \rightarrow \infty} \left[1 - \frac{1}{3} - \frac{(-1)^r}{2r+3} - \frac{(-1)^r}{2r+5} \right] \\ &\quad \uparrow \quad \uparrow \\ &\quad \text{These tend to zero} \\ &= \frac{1}{4} \times \left(1 - \frac{1}{3} \right) \\ &= \frac{1}{4} \times \frac{2}{3} \\ &= \frac{1}{6} \end{aligned}$$

Ans: D

Question 140 (***)**

Find the sum to infinity of the following series.

$$1 + \frac{1}{3 \times 4} + \frac{1}{5 \times 4^2} + \frac{1}{7 \times 4^3} + \frac{1}{9 \times 4^4} + \dots$$

 , $\ln 3$

METHOD A - USING SERIES EXPANSIONS

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$$

$$\ln(1-x) = x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$$

SUBTRACTING THE EXPANSIONS, WE OBTAIN

$$\ln(1+x) - \ln(1-x) = 2x + \frac{3}{2}x^3 + \frac{5}{4}x^5 + O(x^7)$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{x^2}{2} + \frac{x^4}{3} + \frac{x^6}{4} + O(x^8)$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)}$$

NOW, DETERMINE THE RADIUS OF CONVERGENCE, i.e. let $x = \frac{1}{2}$

$$\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = 2 \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2k+1}}{(2k+1)}$$

$$\ln\left(\frac{3}{2}\right) = 2 \sum_{k=0}^{\infty} \left[\frac{1}{(2k+1)2^{2k+1}} \right]$$

$$\ln 3 = \sum_{k=0}^{\infty} \frac{2}{(2k+1)2^{2k+1}}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)2^{2k+1}} = \ln 3$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)2^{2k+1}} = \ln 3$$

$$\therefore 1 + \frac{1}{3 \times 4} + \frac{1}{5 \times 4^2} + \frac{1}{7 \times 4^3} + \frac{1}{9 \times 4^4} + \dots = \ln 3$$

METHOD B - ALTERNATIVE TECHNIQUE

$$\int_0^{\frac{1}{2}} x^k dx = \left[\frac{1}{2k+1} x^{2k+1} \right]_0^{\frac{1}{2}} = \frac{1}{2k+1} \left[\left(\frac{1}{2} \right)^{2k+1} - 0 \right] = \frac{1}{(2k+1)2^{2k+1}}$$

$$= \frac{1}{(2k+1)2^{2k+1}} = \frac{1}{(2k+1)4^k}$$

NOW CONSIDER THE INFINITE SERIES SUM

$$1 + \frac{1}{3 \times 4} + \frac{1}{5 \times 4^2} + \frac{1}{7 \times 4^3} + \dots = \sum_{k=0}^{\infty} \frac{1}{(2k+1)4^k}$$

$$= 2 \times \sum_{k=0}^{\infty} \left[\frac{1}{(2k+1)4^k} \right] = 2 \sum_{k=0}^{\infty} \left[\frac{1}{2(2k+1)2^{2k+1}} \right] = 2 \sum_{k=0}^{\infty} \left[\frac{1}{2} \int_0^{\frac{1}{2}} x^{2k+1} dx \right]$$

INTEGRABLE SITUATION & INTEGRATION

$$\dots = 2 \int_0^{\frac{1}{2}} \left[\sum_{k=0}^{\infty} x^{2k+1} \right] dx = 2 \int_0^{\frac{1}{2}} \left[1 + x^2 + x^4 + x^6 + \dots \right] dx$$

$$= 2 \int_0^{\frac{1}{2}} \frac{1}{1-x^2} dx = \int_0^{\frac{1}{2}} \frac{x}{(1-x)(1+x)} dx$$

$$= \int_0^{\frac{1}{2}} \frac{1}{1+x} + \frac{1}{1-x} dx = \left[\ln(1+x) - \ln(1-x) \right]_0^{\frac{1}{2}}$$

$$= \left(\ln\left(\frac{3}{2}\right) - \ln\left(\frac{1}{2}\right) \right) - \left(\ln(1-0) - \ln(1-0) \right) = \ln\left(\frac{3}{2}\right) = \ln 3$$

Question 141 (***)**

By showing a detailed method, sum the following series.

$$\sum_{n=0}^{\infty} \left[\frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n} \right].$$

, $\frac{3}{2}$

SUM BY TRIGONOMETRIC IDENTITIES

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n} &= \sum_{n=0}^{\infty} \frac{\frac{1}{2} + \frac{1}{2}\cos\left(\frac{1}{3}n\pi\right)}{2^n} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1 + \cos\left(\frac{1}{3}n\pi\right)}{2^n} \\ \text{SPLIT INTO A GEOMETRIC PROGRESSION AND ANOTHER SERIES} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\cos\left(\frac{1}{3}n\pi\right)}{2^n} \\ &= \frac{1}{2} \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right] + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\cos\left(\frac{1}{3}n\pi\right)}{2^n} \end{aligned}$$

Using complex numbers

$$\begin{aligned} S_n &= \frac{1}{1 - \frac{1}{2}} \quad \text{Note that the series converges since } \left| \frac{1}{2} \right| < 1 \\ &= \frac{1}{2} \times \frac{1}{1 - \frac{1}{2}} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\operatorname{Re}[e^{i\frac{1}{3}n\pi}]}{2^n} \\ &= \left[\frac{1}{2} \times \frac{1}{1 - \frac{1}{2}} \right] + \frac{1}{2} \operatorname{Re} \left[\sum_{n=0}^{\infty} \left(\frac{e^{i\frac{1}{3}n\pi}}{2} \right)^n \right] \\ &= 1 + \frac{1}{2} \operatorname{Re} \left[\sum_{n=0}^{\infty} \left(\frac{e^{i\frac{1}{3}n\pi}}{2} \right)^n \right] \end{aligned}$$

Argim taking the sum to infinity of A.G.P.

$$\begin{aligned} &= 1 + \frac{1}{2} \operatorname{Re} \left[1 + \frac{e^{i\frac{1}{3}\pi}}{2} + \frac{e^{i\frac{2}{3}\pi}}{2^2} + \frac{e^{i\frac{3}{3}\pi}}{2^3} + \dots \right] \\ &= 1 + \frac{1}{2} \operatorname{Re} \left[\frac{1}{1 - \frac{e^{i\frac{1}{3}\pi}}{2}} \right] \\ &= 1 + \operatorname{Re} \left[\frac{1}{2 - e^{i\frac{1}{3}\pi}} \right] \end{aligned}$$

MANIPULATE THE EXPRESSION TO EXTRACT THE REAL PART

$$\begin{aligned} &= 1 + \operatorname{Re} \left[\frac{2 - e^{-i\frac{1}{3}\pi}}{(2 - e^{i\frac{1}{3}\pi})(2 - e^{-i\frac{1}{3}\pi})} \right] \\ &= 1 + \operatorname{Re} \left[\frac{2 - e^{-i\frac{1}{3}\pi}}{4 - 2e^{i\frac{2}{3}\pi} - 2e^{-i\frac{2}{3}\pi} + 1} \right] \\ &= 1 + \operatorname{Re} \left[\frac{2 - e^{-i\frac{1}{3}\pi}}{5 - 4\left(\frac{1}{2}e^{i\frac{2}{3}\pi} + \frac{1}{2}e^{-i\frac{2}{3}\pi}\right)} \right] \\ &= 1 + \operatorname{Re} \left[\frac{2 - \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)}{5 - 4\cos\left(\frac{\pi}{3}\right)} \right] \\ &= 1 + \operatorname{Re} \left[\frac{2 - \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}{5 - 4\cos\frac{\pi}{3}} \right] \quad \text{Cosine rule} \\ &= 1 + \operatorname{Re} \left[\frac{\frac{3}{2} + i\frac{\sqrt{3}}{2}}{5} \right] \\ &= 1 + \frac{\frac{3}{2}}{5} \\ &= 1 + \frac{3}{10} \\ &= 1 + \frac{3}{2} \\ &\therefore \sum_{n=0}^{\infty} \left(\frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n} \right) = \frac{3}{2} \end{aligned}$$

Question 142 (*****)

Evaluate the following expression

$$\sum_{k=1}^{\infty} \left[\sum_{r=1}^k r \right]^{-1}.$$

CP , [2]

● REWRITE FOR SIMPLICITY AS FRACTION

$$\sum_{k=1}^{\infty} \left[\sum_{r=1}^k r \right]^{-1} = \sum_{k=1}^{\infty} \left[\frac{1}{\frac{k(k+1)}{2}} \right]$$

$$= \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots$$

● INTRODUCE A FINITE LIMIT FOR THE SUMMATION, SAY n

$$= \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{\frac{k(k+1)}{2}} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{2}{k(k+1)} \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{k(k+1)} \right]$$

● SPLIT INTO TWO FRACTIONS BY INSPECTION

$$= 2 \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \right]$$

$$= 2 \lim_{n \rightarrow \infty} \left[1 - \frac{1}{n+1} \right]$$

$$= 2$$

Question 143 (***)**

The first three terms of a series S are

$$S = 7 + 9x + 8x^2 + \dots$$

The n^{th} term of S is given by

$$A\left(\frac{3}{4}x\right)^n + B\left(\frac{1}{3}x\right)^n,$$

where A and B are non zero constants.

Given that the sum to infinity of S is 19, determine the value of x .

S.P.F., $x = \frac{12}{19}$

S = 7 + 9x + 8x^2 + \dots [A\left(\frac{3}{4}x\right)^n + B\left(\frac{1}{3}x\right)^n]

- If $n=0$ $(A+B)x^0 = 7$
- If $n=1$ $\left(\frac{3}{4}A + \frac{1}{3}B\right)x^1 = 9$

$A+B=7$	$x \times 12$	$9A+7B=63$
$\frac{3}{4}A+\frac{1}{3}B=9$	$\times 12$	$9A+48=108$
		$5B=-45$
		$B=-9$
		$A=16$

Now the series is

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \left[16\left(\frac{3}{4}x\right)^n - 9\left(\frac{1}{3}x\right)^n \right] x^n \\ S &= 16 \sum_{n=0}^{\infty} \left[\left(\frac{3}{4}x\right)^n \right] - 9 \sum_{n=0}^{\infty} \left[\left(\frac{1}{3}x\right)^n \right] \\ S &= 16 \sum_{n=0}^{\infty} \left(\frac{3}{4}x \right)^n - 9 \sum_{n=0}^{\infty} \left(\frac{1}{3}x \right)^n \\ S &= 16 \left[1 + \frac{3}{4}x + \frac{9}{16}x^2 + \frac{27}{64}x^3 + \dots \right] - 9 \left[1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3 + \dots \right] \\ S &= 16x \left(\frac{1}{1-\frac{3}{4}x} \right) - 9 \left(\frac{1}{1-\frac{1}{3}x} \right) \end{aligned}$$

NOW THE SUM TO INFINITY IS 19

$$\begin{aligned} &\rightarrow \frac{16}{1-\frac{3}{4}x} - \frac{9}{1-\frac{1}{3}x} = 19 \\ &\rightarrow \frac{48}{4-3x} - \frac{27}{3-x} = 19 \\ &\rightarrow 64(3-2x) - 27(4-3x) = 19(3-2)(4-3x) \\ &\rightarrow 192 - 64x - 108 + 27x = 19(3x-2)(4-3x) \\ &\rightarrow 84x + 114x = 19(3x^2 - 13x + 8) \\ &\rightarrow 84x + 173x = 57x^2 - 193x + 152 \\ &\rightarrow 0 = 57x^2 - 247x - 172 + 208 - 84 \\ &\Rightarrow 0 = 57x^2 - 59x + 114 \\ &\Rightarrow 0 = 19x^2 - 19x + 48 \\ &\Rightarrow 0 = ((19x-12)(x-4)) \\ &\Rightarrow x = \begin{cases} \frac{12}{19} \\ 4 \end{cases} \end{aligned}$$

BUT IN ORDER TO CONVERGE $| \frac{3}{4}x | < 1 \quad \& \quad | \frac{1}{3}x | < 1 \\ | x | < \frac{4}{3} \quad \& \quad | x | < 3 \\ \therefore x = \frac{12}{19} \quad (\text{as } 4 \text{ is greater than } \frac{4}{3} \text{ or, } 4 > 3) \quad \boxed{x = \frac{12}{19}} \quad \text{(S.P.F.)} \quad \text{if } 4 \text{ is greater than } \frac{4}{3} \text{ or, } 4 > 3 \end{aligned}$

Question 144 (*****)

Find a simplified expression for the following sum

$$\frac{1}{100!} + \frac{1}{99!} + \frac{1}{2!98!} + \frac{1}{3!97!} + \frac{1}{4!96!} + \dots + \frac{1}{2!98!} + \frac{1}{99!} + \frac{1}{100!}$$

SPN

$$\frac{2^{100}}{100!}$$

LET $S = \frac{1}{100!} + \frac{1}{99!} + \frac{1}{2!98!} + \frac{1}{3!97!} + \frac{1}{4!96!} + \dots + \frac{1}{2!98!} + \frac{1}{99!} + \frac{1}{100!}$

$$S = \frac{1}{100!} + \frac{1}{(1!99)} + \frac{1}{2!(98)} + \frac{1}{3!(97)} + \dots + \frac{1}{99!(1)} + \frac{1}{100!(0)}$$

$$S = \sum_{r=0}^{100} \frac{1}{r!(100-r)!}$$

$$S = \frac{1}{100!} \sum_{r=0}^{100} \frac{100!}{r!(100-r)!}$$

$$S = \frac{1}{100!} \sum_{r=0}^{100} \binom{100}{r}$$

$$S = \frac{1}{100!} \sum_{r=0}^{100} \left[\binom{100}{r} \cdot 1^r \cdot 1^{100-r} \right]$$

$$S = \frac{1}{100!} \times (1+1)^{100}$$

$$S = \frac{2^{100}}{100!}$$

NOTE $(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^r b^{n-r}$

Question 145 (*****)

Show by detailed workings that

$$\int_0^\infty \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}$$

proof

1. STARTING BY MULTIPLYING "TOP & BOTTOM" OF THE INTEGRAL BY e^{-x}

$$\int_0^\infty \frac{x}{e^x - 1} dx = \int_0^\infty \frac{xe^{-x}}{1 - e^{-x}} dx$$

2. NOW AS $|e^{-x}| < 1$ WE CAN WRITE: EXPONENTIAL (OR SUM TO INFINITY OF A GEOMETRIC PROGRESSION) AS FOLLOWS

$$\frac{1}{1 - e^{-x}} = 1 + e^{-x} + e^{-2x} + e^{-3x} + \dots$$

3. THIS WE HAVE

$$\dots = \int_0^\infty xe^{-x} [1 + e^{-x} + e^{-2x} + e^{-3x} + \dots] dx$$

$$= \int_0^\infty x [e^{-x} + e^{-2x} + e^{-3x} + \dots] dx$$

$$= \int_0^\infty x \left[\sum_{r=1}^\infty e^{-rx} \right] dx$$

4. INTERCHANGING SUMMATION & INTEGRATION

$$= \sum_{r=1}^\infty \int_0^\infty x e^{-rx} dx$$

5. NOW BY SUBSTITUTION (USING A GAMMA FUNCTION) BY PARTS, DIFFERENTIATION UNDER THE INTEGRAL SIGN OR LAPLACE TRANSFORM

$$= \sum_{r=1}^\infty \int_0^\infty t e^{-rt} dt$$

$$= \sum_{r=1}^\infty \int_0^\infty \frac{1}{r} t^{r-1} dt = \sum_{r=1}^\infty \frac{1}{r!} = \sum_{r=1}^\infty \frac{1}{r^2} = \frac{\pi^2}{6}$$

Question 146 (***)**

Consider the following convergent infinite series.

$$\sum_{r=0}^{\infty} \frac{2^{r+4}}{(r+3)r!}$$

Use appropriate techniques to show that the sum to infinity of the above series is
 $4(e^2 - 1)$

, proof

From the answer, it is evident that the exponential function is involved with some modifications.

Let $\frac{d}{dx} e^x = \frac{d^2}{dx^2} e^x + \frac{d^3}{dx^3} e^x + \frac{d^4}{dx^4} e^x + \dots$
 with $x=2$

Then we have

$$\begin{aligned} \Rightarrow \frac{d}{dx} e^x &= \frac{d^2}{dx^2} e^x + \frac{d^3}{dx^3} e^x + \frac{d^4}{dx^4} e^x + \dots \\ \Rightarrow \frac{d}{dx} \left(\frac{d}{dx} e^x \right) &= \frac{d^3}{dx^3} e^x + \frac{d^4}{dx^4} e^x + \frac{d^5}{dx^5} e^x + \dots \\ \Rightarrow \frac{d}{dx} \left(\frac{d}{dx} e^x \right) &= x! \left[\frac{1}{0!} + \frac{2^2}{1!} + \frac{2^3}{2!} + \frac{2^4}{3!} + \frac{2^5}{4!} + \dots \right] \\ \Rightarrow \frac{d}{dx} \left(\frac{d}{dx} e^x \right) &= x^2 e^x \\ \Rightarrow \frac{d}{dx} \left(\frac{d}{dx} e^x \right) &= \int x^2 e^x dx \end{aligned}$$

↑
Double integration by parts or differentiation under the integral sign

e.g. $\int x^2 e^{kx} dx = \frac{d}{dk} \left[\int e^{kx} dx \right] = \frac{d}{dk} \left[\frac{1}{k} e^{kx} \right]$
 $= \frac{d}{dk} \left[-\frac{1}{k^2} e^{kx} + \frac{2}{k^3} e^{kx} \right]$
 $= \frac{2}{k^3} e^{kx} - \frac{2}{k^2} e^{kx} - \frac{2}{k^3} e^{kx} + \frac{4}{k^4} e^{kx}$
 $\therefore \int x^2 e^{kx} dx = 2e^{kx} - 2e^{kx} - 2e^{kx} + 2e^{kx}$
 $= (x^2 - 2x + 2)e^{kx} + C$

Hence we know that

 $\Rightarrow \frac{d}{dx} e^x = (x^2 - 2x + 2)e^x + C$

To find the constant let $x=0$

 $\Rightarrow \frac{d}{dx} e^x \Big|_{x=0} = 0 \quad \text{Since } \frac{d}{dx} e^x = \frac{d^2}{dx^2} e^x + \frac{d^3}{dx^3} e^x + \dots$
 $\therefore 0 = 2 + C$
 $C = -2$

Therefore we obtain

 $\Rightarrow \frac{d}{dx} e^x = (x^2 - 2x + 2)e^x - 2$
 $\Rightarrow e^x = (x^2 - 2x + 2)e^x - 2x$
 $\Rightarrow \frac{x^2 - 2x + 2}{x^2} e^x - \frac{2x}{x^2} e^x + \frac{2}{x^2} e^x + \dots = (x^2 - 2x + 2)^2 e^{-2x}$
 $\Rightarrow \frac{2}{x^2} + \frac{2x}{x^2} + \frac{2}{x^2} + \frac{2^2}{x^3} + \frac{2^3}{x^4} + \dots = (x^2 - 2x + 2)^2 e^{-2x}$

Now

$$\sum_{n=0}^{\infty} \left[\frac{2^n}{(n!)x^n} \right] = 4e^2 - 4 = 4(e^2 - 1)$$

Question 147 (*****)

A family of infinite geometric series S_k , has first term $\frac{k-1}{k!}$ and common ratio $\frac{1}{k}$, where $k = 3, 4, 5, 6, \dots, 99, 100$.

Find the value of

$$\frac{10^4}{100!} + \sum_{k=3}^{100} [(k-1)(k-2)-1] S_k.$$

[2]

• WRITE THE FIRST FEW TERMS OF THIS GENERAL GEOMETRIC PROGRESSION

$$\Rightarrow S_k = \frac{k-1}{k!} + \frac{k-1}{k!} \cdot \frac{1}{k} + \frac{k-1}{k!} \cdot \frac{1}{k^2} + \frac{k-1}{k!} \cdot \frac{1}{k^3} + \dots$$

$$\Rightarrow S_k = \frac{k-1}{k!} \left[1 + \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \dots \right]$$

SUMMING TO INFINITY USING $\sum_{n=1}^{\infty} \frac{1}{n^k} \approx \frac{1}{k}$

$$\Rightarrow S_k = \frac{k-1}{k!} \times \frac{1}{1-\frac{1}{k}}$$

$$\Rightarrow S_k = \frac{k-1}{k!} \times \frac{k}{k-1}$$

$$\Rightarrow S_k = \frac{1}{k!}$$

$$\Rightarrow S_k = \frac{1}{(k-1)!}$$

• NEXT CONSIDER THE SUMMATION WITH THE GENERAL TERM FOUND

$$\begin{aligned} & \sum_{k=3}^{100} [S_k - \frac{1}{(k-1)!}] \\ &= \sum_{k=3}^{100} \left[\frac{(k-1)(k-2)-1}{(k-1)!} \right] \\ &= \sum_{k=3}^{100} \left[\frac{(k-1)(k-2)}{(k-1)!} - \frac{1}{(k-1)!} \right] \\ &= \sum_{k=3}^{100} \left[\frac{1}{(k-3)!} - \frac{1}{(k-1)!} \right] \end{aligned}$$

• WRITING THE SUM EXPLICITLY IN A "TABLE FORM"

$\frac{1}{0!}$	$-\frac{1}{1!}$	$\frac{1}{2!}$
$\frac{1}{1!}$	$\frac{1}{2!}$	$\frac{1}{3!}$
$\frac{1}{2!}$	$\frac{1}{3!}$	$\frac{1}{4!}$
$\frac{1}{3!}$	$\frac{1}{4!}$	$\frac{1}{5!}$
\vdots	\vdots	\vdots
$\frac{1}{98!}$	$\frac{1}{99!}$	$\frac{1}{99!}$
$\frac{1}{99!}$	$\frac{1}{99!}$	$\frac{1}{100!}$

$$\begin{aligned} & \sum_{k=3}^{100} \left[\frac{1}{(k-3)!} - \frac{1}{(k-1)!} \right] \\ &= \frac{1}{0!} + \frac{1}{1!} - \left(\frac{1}{98!} + \frac{1}{99!} \right) \\ &= 2 - \left(\frac{99+1}{99!} \right) \\ &= 2 - \frac{100}{99!} \end{aligned}$$

• FINALLY ADDING THE TERM AT THE FRONT OF THE SUMMATION

$$\begin{aligned} & \frac{10^4}{100!} + \sum_{k=3}^{100} \left[\frac{(k-1)(k-2)-1}{(k-1)!} \right] S_k \\ &= \frac{10^4}{100!} + 2 - \frac{100}{99!} \\ &= \frac{100^2}{100!} + 2 - \frac{100}{99!} \\ &= \frac{100}{99!} + 2 - \frac{100}{99!} \\ &= 2 \end{aligned}$$

Question 148 (*****)

A discrete random variable X is geometrically distributed with parameter p .

Show that ...

a) ... $E(X) = \frac{1}{p}$

b) ... $E(X^2) = \frac{1-p}{p^2}$

proof

a) Let $X \sim \text{Geo}(\frac{1}{p})$, $0 < p < 1$

x	1	2	3	4	5	...
$P(X=x)$	p	$(1-p)p$	$(1-p)^2p$	$(1-p)^3p$	$(1-p)^4p$...

$E(X) = p + 2p(1-p) + 3p(1-p)^2 + 4p(1-p)^3 + \dots$

$E(X^2) = p[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \dots]$

Multiply the above line by $-(1-p)$

$-E(X) = p[-(1-p) - 2(1-p)^2 - 3(1-p)^3 - 4(1-p)^4 - \dots]$

$E(X) = p[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + 5(1-p)^4 + \dots]$

Adding the two lines above we obtain

$\boxed{E(X) = \frac{1}{1-(1-p)} = \frac{1}{p}}$

ALTERNATIVE VARIATION

HAVING FORMED AN EXPRESSION FOR EXPECTATION AS BEFORE

$E(X) = p[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \dots]$

LET $q = 1-p$

$\Rightarrow E(X) = (1-q)[1 + 2q + 3q^2 + 4q^3 + \dots]$

$\Rightarrow E(X) = (1-q) \frac{d}{dq} [q + q^2 + q^3 + q^4 + \dots]$

considering G.P with $a=q$, $r=q$

$\Rightarrow E(X) = (1-q) \frac{d}{dq} \left[\frac{q}{1-q} \right]$

$\Rightarrow E(X) = (1-q) \times \frac{(1-q) \times 1 - q(1-q)}{(1-q)^2}$

$\Rightarrow E(X) = (1-q) \times \frac{1}{(1-q)}$

$\Rightarrow E(X) = \frac{1}{1-q}$

b) NEXT THE VARIANCE

$E(X^2) = [1^2 \times p] + [2^2 \times p(1-p)] + [3^2 \times p(1-p)^2] + [4^2 \times p(1-p)^3] + \dots$

$E(X^2) = p[1 + 4(1-p) + 9(1-p)^2 + 16(1-p)^3 + 25(1-p)^4 + \dots]$

MINIFY THE EXPRESSION BY $-(1-p)$

$-(1-p) E(X^2) = p[-(1-p) - 9(1-p)^2 - 16(1-p)^3 - 25(1-p)^4 - \dots]$

$E(X^2) = p[1 + 4(1-p) + 9(1-p)^2 + 16(1-p)^3 + 25(1-p)^4 + \dots]$

ADDING THE TWO LINES ABOVE

$\boxed{[1 - (1-p)] E(X^2) = p[1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + 9(1-p)^4 + \dots]}$

$E(X^2) = 1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + 9(1-p)^4 + \dots$

MINIFY THE ABOVE LINE AGAIN BY $-(1-p)$

$-(1-p) E(X^2) = -[(1-p) - 3(1-p)^2 - 5(1-p)^3 - 7(1-p)^4 - 9(1-p)^5 - \dots]$

$E(X^2) = 1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + 9(1-p)^4 + 11(1-p)^5 + \dots$

ADDING AGAIN THE TWO LINES ABOVE

$\boxed{[1 - (1-p)] E(X^2) = 1 + 2(1-p) + 2(1-p)^2 + 2(1-p)^3 + 2(1-p)^4 + \dots}$

$E(X^2) = 1 + 2 \left[\frac{1}{1-(1-p)} + (1-p) + (1-p)^2 + (1-p)^3 + \dots \right]$

THIS IS A GEOMETRIC PROGRESSION WITH $a=1$, $r=1-p$

$\therefore E(X^2) = 1 + 2 \times \frac{1}{1-(1-p)} = \frac{1-p}{p}$

• FINALLY THE VARIANCE

$\text{Var}(X) = E(X^2) - [E(X)]^2$

$= \frac{2-p}{p^2} - \left(\frac{1}{p} \right)^2$

$= \frac{2-p}{p^2} - \frac{1}{p^2}$

$= \frac{1-p}{p^2}$

Question 149 (***)**

Find the sum to infinity of the following convergent series

$$1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \frac{5^3}{5!} + \frac{6^3}{6!} + \dots$$

5e

$1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \frac{5^3}{5!} + \dots = ?$

Now expand the numerator - split the fraction - cancel down and readjust the domain variable.

$$\begin{aligned} S &= 1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \dots = \sum_{n=1}^{\infty} \frac{n^3}{n!} \quad \text{[cancel down]} \\ &= \sum_{n=0}^{\infty} \frac{n^3}{(n+1)!} = \sum_{n=0}^{\infty} \frac{(n+1)^3}{(n+1)!} \end{aligned}$$

Now expand the numerator - split the fraction - cancel down and readjust the domain variable.

$$\begin{aligned} \Rightarrow S &= \sum_{n=0}^{\infty} \frac{n^3 + 2n^2 + 2n + 1}{n!} = \left[\sum_{n=0}^{\infty} \frac{n^2}{n!} \right] + \left[2 \sum_{n=0}^{\infty} \frac{n}{n!} \right] + \left[\sum_{n=0}^{\infty} \frac{1}{n!} \right] \\ \Rightarrow S &= \left[\sum_{n=0}^{\infty} \frac{n^2}{n!} \right] + \left[2 \sum_{n=0}^{\infty} \frac{n}{n!} \right] + e \quad \text{[less than 200]} \\ \Rightarrow S &= \left[\sum_{n=0}^{\infty} \frac{n}{(n-1)!} \right] + 2 \left[\sum_{n=0}^{\infty} \frac{1}{(n-1)!} \right] + e \\ \Rightarrow S &= \left[\sum_{n=0}^{\infty} \frac{n+1}{n!} \right] + 2 \left[\sum_{n=0}^{\infty} \frac{1}{n!} \right] + e \quad \text{[cancel]} \\ \Rightarrow S &= \left[\sum_{n=1}^{\infty} \frac{n}{n!} \right] + 2 \left[\sum_{n=0}^{\infty} \frac{1}{n!} \right] + e \quad \text{[cancel]} \\ \Rightarrow S &= \left[\sum_{n=1}^{\infty} \frac{n}{n!} \right] + 2e + e \quad \text{[cancel]} \\ \Rightarrow S &= \left[\sum_{n=1}^{\infty} \frac{n}{n!} \right] + 4e \quad \text{[cancel]} \end{aligned}$$

$$\begin{aligned} \Rightarrow S &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} + 4e \\ \Rightarrow S &= \sum_{n=0}^{\infty} \frac{1}{n!} + 4e \\ \Rightarrow S &= e + 4e \\ \therefore S &= 1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \dots = 5e \end{aligned}$$

Question 150 (***)**

The function f is defined as

$$f(n, x) \equiv \sum_{r=0}^n \binom{n}{r} r x^r (1-x)^{n-r},$$

where $n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$.

Show that $f(n, x) \equiv nx$.

, proof

$$f(x) = \sum_{r=0}^n \left[\binom{n}{r} r x^r (1-x)^{n-r} \right]$$

$$\sum_{r=0}^n \left[\binom{n}{r} r x^r (1-x)^{n-r} \right] = \sum_{r=0}^n \left[\frac{n!}{r!(n-r)!} \times r \times x^r (1-x)^{n-r} \right]$$

- THE FIRST TERM IS ACTUALLY ZERO
- $= \sum_{r=1}^n \left[\frac{n!}{r!(n-r)!} \times r \times x^r (1-x)^{n-r} \right]$
- FACTORIZE OUT OF THE SUMMATION THE "GIVEN"
- $= n! \sum_{r=1}^n \frac{(n-1)!}{(r-1)!(n-r)!} \times r \times x^r (1-x)^{n-r}$
- $= n! \sum_{r=1}^n \frac{(n-1)!}{(r-1)!(n-r)!} x^{r-1} (1-x)^{n-r}$
- ADJUST THE SUMMATION SO IT STARTS FROM $r=0$

IE

$$\begin{aligned} & \sum_{r=1}^n = \sum_{r=1}^{n-1} + \sum_{r=n}^n \\ & \sum_{r=1}^{n-1} = \sum_{r=0}^{n-1} \\ & \sum_{r=n}^n = \sum_{r=0}^n \end{aligned}$$

$$\begin{aligned} & = n! \sum_{r=0}^n \frac{(n-1)!}{(r-1)!(n-r)!} x^{r-1} (1-x)^{n-r} \\ & = n! \sum_{r=0}^n \frac{(n-1)!}{(r-1)!(n-r)!} x^r (1-x)^{n-r} \end{aligned}$$

$$\begin{aligned} & = n! \sum_{k=0}^n \frac{\frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k}}{(k-1)!(n-k)!} \\ & = n! \sum_{k=0}^n \left[\binom{n}{k} x^k (1-x)^{n-k} \right] \\ & \bullet \text{ BUT THIS IS THE DEFINITION OF A BINOMIAL EXPANSION} \\ & = n! x \left[1 + (1-x) \right]^n \\ & = n! x [1]^n \\ & = n! x \end{aligned}$$

~~As Required~~

Question 151 (***)**

The binomial probability distribution $X \sim B(n, p)$ satisfies

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r},$$

where $r = 0, 1, 2, 3, \dots, n$ and $0 < p < 1$.

The expectation of X is defined as

$$E(X) \equiv \sum_{r=0}^n [r P(X = r)]$$

Show that

$$E(X) = np.$$

 , proof

$$\begin{aligned} P(X=r) &= \binom{n}{r} p^r (1-p)^{n-r} \quad \text{from } X \sim B(n, p) \\ E(X) &= \sum_{r=0}^n r P(X=r) = \sum_{r=0}^n r \binom{n}{r} p^r (1-p)^{n-r} \\ &= \sum_{r=1}^n r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} \quad (\text{since } r \geq 0) \\ &= np \sum_{r=1}^n \frac{(n-1)!}{(r-1)!(n-r)!} p^{r-1} (1-p)^{n-r} \\ &\quad \text{ADD THE SUMMATION "ONE DOWN" SO } r \text{ STOPS FROM } 0 \text{ TO } n-1 \\ &= np \sum_{r=0}^{n-1} \frac{n!}{(r+1)!(n-r-1)!} \frac{p^{r+1}}{(1-p)} \frac{(n-1)!}{(r-1)!(n-r)!} (1-p)^{n-r-1} \\ &= np \sum_{r=0}^{n-1} \frac{n!}{(r+1)!(n-r-1)!} p^{r+1} (1-p)^{n-r-1} \\ &= np (p + (1-p))^n \\ &= np \times 1^n \\ &= np \end{aligned}$$

Question 152 (***)**

The function f is defined in terms of the real constants, a , b and c , by

$$f(x) = (a + bx + cx^2)(1-x)^{-3}, \quad x \in \mathbb{R}, \quad |x| < 1.$$

a) Show that

$$f(x) = a + (3a+b)x + \frac{1}{2} \sum_{n=2}^{\infty} \left[[a(n+1)(n+2) + bn(n+1) + cn(n-1)] x^n \right].$$

b) Use the expression of part (a) to deduce the value of

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

 , [6]

a) Starting with the binomial expansion of $(1-x)^{-3}$

$$\begin{aligned} \rightarrow (1-x)^{-3} &= 1 + \frac{3}{1}(-x)^1 + \frac{3 \cdot 2}{1 \cdot 2}(-x)^2 + \frac{3 \cdot 2 \cdot 5}{1 \cdot 2 \cdot 3}(-x)^3 + \frac{3 \cdot 2 \cdot 5 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4}(-x)^4 + \dots \\ \rightarrow (1-x)^{-3} &= 1 + \frac{3}{1}x + \frac{3 \cdot 2}{1 \cdot 2}x^2 + \frac{3 \cdot 2 \cdot 5}{1 \cdot 2 \cdot 3}x^3 + \frac{3 \cdot 2 \cdot 5 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \dots \\ &\quad + \frac{3 \cdot 2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdots (n+2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots n} x^n \end{aligned}$$

• Thus the coefficient of x^2 is

$$\frac{1}{2} \cdot \frac{(2 \cdot 3 \cdot 5 \cdot 8 \cdots (n+2))}{n!} = \frac{1}{2} \cdot \frac{(n+2)!}{n!} = \frac{1}{2} \cdot \frac{(n+1)(n+2)}{2!} x^2$$

$$(1-x)^{-3} = \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{(n+1)(n+2)}{n!} x^n$$

• Thus (looking now at $f(x)$)

$$\begin{aligned} \rightarrow f(x) &= (a + bx + cx^2)(1-x)^{-3} = (a + bx + cx^2) \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{(n+1)(n+2)}{n!} x^n \\ \rightarrow f(x) &= \frac{1}{2} \sum_{n=0}^{\infty} (n+1)(n+2)x^n + \frac{1}{2} \sum_{n=0}^{\infty} b(n+1)(n+2)x^{n+1} + \frac{1}{2} \sum_{n=0}^{\infty} c(n+1)(n+2)x^{n+2} \\ \rightarrow f(x) &= \frac{1}{2} \cdot x \cdot 2 \cdot x^0 + \frac{1}{2} \cdot b \cdot x \cdot 2 \cdot x^1 \\ &\quad + \frac{1}{2} \cdot c \cdot x \cdot 2 \cdot x^2 \\ &\quad + \frac{1}{2} \sum_{n=2}^{\infty} (n+1)(n+2)x^n + \frac{1}{2} \sum_{n=1}^{\infty} (n+1)(n+2)x^{n+1} + \frac{1}{2} \sum_{n=0}^{\infty} (n+1)(n+2)x^{n+2} \end{aligned}$$

• Add up the summations so they all start from $n=2$.

$$\begin{aligned} \rightarrow f(x) &= a + (3a+b)x + \frac{1}{2} \sum_{n=2}^{\infty} [(n+1)(n+2) + bn(n+1) + cn(n-1)] x^n \\ &\quad + \frac{1}{2}b \sum_{n=2}^{\infty} (n+1)x^n \\ &\quad + \frac{1}{2}c \sum_{n=2}^{\infty} (n+1)x^n \end{aligned}$$

b) Now looking at the coefficient of x^4 (multiplies the $\frac{1}{2}$ into the sum)

$$\begin{pmatrix} \frac{1}{2} \cdot 4! + \frac{3}{2} \cdot 3! + 1 \\ \frac{1}{2} \cdot 3! + \frac{3}{2} \cdot 2! \\ \frac{1}{2} \cdot 2! - \frac{1}{2} \cdot 1! \end{pmatrix} x^4$$

Look for this to reduce to x^2 .

$$\begin{cases} a=0 \\ \frac{1}{2}b - \frac{1}{2}c = 0 \\ b=c \end{cases} \Rightarrow \begin{cases} \frac{1}{2}b + \frac{1}{2}c = 1 \\ \frac{1}{2}b + \frac{1}{2}c = 1 \\ b=c=1 \end{cases}$$

$$\rightarrow f(x) = (a + bx + cx^2)(1-x)^{-3} = a + (3a+b)x$$

$$\begin{aligned} &\quad + \frac{1}{2} \sum_{n=2}^{\infty} [(n+1)(n+2) + bn(n+1) + cn(n-1)] x^n \\ &\bullet \text{ Let } a=0, b=1, c=1 \\ \rightarrow f(x) &= (x+x^2)(1-x)^{-3} = x + \sum_{n=2}^{\infty} n^2 x^n \\ \rightarrow f(x) &= \left(\frac{1}{2} + \frac{1}{4}\right)(1-x)^{-3} = \frac{1}{2} + \sum_{n=2}^{\infty} n^2 \left(\frac{1}{2}\right)^n \\ \Rightarrow & \frac{5}{4} \times \left(\frac{1}{2}\right)^{-3} = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{2} \cdot \frac{1}{2}^n \\ \Rightarrow & \frac{5}{4} \times 8 = \frac{1}{2} + \frac{5}{2} \cdot \frac{1}{2}^2 \\ \Rightarrow & 6 = \sum_{n=2}^{\infty} \frac{5}{2} \cdot \frac{1}{2}^n \quad (4 \cdot \frac{1}{2}^3 = \frac{1}{2}) \\ \therefore & \sum_{n=1}^{\infty} \frac{5}{2} \cdot \frac{1}{2}^n = 6 \end{aligned}$$

Question 153 (***)**

The function f is defined by

$$f(x) \equiv \sum_{n=1}^{\infty} [nx^n], \quad x \in \mathbb{R}, \quad |x| < 1.$$

Use the above function to find the sum to infinity of the following series.

$$\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \frac{5}{243} + \dots$$

$$\boxed{}, \boxed{\frac{3}{4}}$$

$\boxed{\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \frac{5}{243} + \dots}$

REASONING IN SIGMA NOTATION OR PARENTHESIS

$$\sum_{n=1}^{\infty} \left[\frac{n}{3^n} \right] = \sum_{n=1}^{\infty} \left[n \times \frac{1}{3^n} \right] = \sum_{n=1}^{\infty} \left[n \times 3^{-n} \right] = \sum_{n=1}^{\infty} \left[n \times \left(\frac{1}{3} \right)^n \right]$$

THE LAST EXPRESSION LOOKS LIKE A 'DIFFERENTIATION'

LET $f(x) = \sum_{n=1}^{\infty} \left[n x^n \right]$ (IF $x = \frac{1}{3}$ USE THE SUM)

$$f(x) = \sum_{n=1}^{\infty} \left[n x^n + x^{n-1} \right] = x \sum_{n=1}^{\infty} \left[n x^{n-1} \right]$$

$$f(x) = x \cdot \frac{d}{dx} \left[\sum_{n=1}^{\infty} x^n \right]$$

$$f(x) = x \cdot \frac{d}{dx} \left[x + x^2 + x^3 + x^4 + \dots \right]$$

↑ CONNECT G.P. WITH $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$

$$f(x) = x \cdot \frac{d}{dx} \left[\frac{x}{1-x} \right]$$

$$f(x) = x \left[\frac{(1-x)(1) - x(-1)}{(1-x)^2} \right] = x \left(\frac{1-x+x^2}{(1-x)^2} \right) = \frac{x}{(1-x)^2}$$

$f'(x) = f(x)$

$$f(\frac{1}{3}) = \frac{\frac{1}{3}}{(1-\frac{1}{3})^2} = \frac{\frac{1}{3}}{\frac{4}{9}} = \frac{3}{4} \quad \boxed{\frac{3}{4}}$$

Question 154 (***)**

Find the value of $x \in \mathbb{R}$ in the following equation

$$\sum_{n=0}^{\infty} \left[\frac{n(n-1)(n-2)(n-3)}{2^{n+k}} \right] = 3.$$

 , $k = 4$

LET US NOTE THAT THE FIRST 4 TERMS OF THE SERIES ARE ZERO, SO

$$\Rightarrow \sum_{n=0}^{\infty} \left[\frac{n(n-1)(n-2)(n-3)}{2^n \cdot 2^k} \right] = 3$$

$$\Rightarrow \frac{1}{2^k} \sum_{n=4}^{\infty} \left[n(n-1)(n-2)(n-3) \left(\frac{1}{2} \right)^n \right] = 3$$

$$\Rightarrow \sum_{n=4}^{\infty} \left[n(n-1)(n-2)(n-3) \left(\frac{1}{2} \right)^n \right] = 3 \times 2^k$$

Let $f(x) = \sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3)x^n]$

$$\Rightarrow f(x) = \sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3)x^n \times x]$$

$$\Rightarrow f(x) = x^4 \sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3)x^{n-4}]$$

$$\Rightarrow f(x) = x^4 \frac{d^4}{dx^4} \left[\sum_{n=4}^{\infty} x^n \right]$$

$$\Rightarrow f(x) = x^4 \frac{d^4}{dx^4} \left[x + x^2 + x^3 + x^4 + \dots \right] \quad |x| < 1$$

$$\Rightarrow f(x) = x^4 \frac{d^4}{dx^4} \left[\frac{x}{1-x} \right] \quad \leftarrow \left\{ \frac{d}{dx} = \frac{1}{1-x} \right\}$$

$$\Rightarrow f(x) = x^4 \frac{d^4}{dx^4} \left[\frac{(1-x)^4 - 1}{(1-x)^3} \right] \quad \leftarrow \text{MANIPULATE THE TERM}$$

$$\Rightarrow f(x) = x^4 \frac{d^4}{dx^4} \left[-1 + \frac{1}{(1-x)^3} \right]$$

$f(x) = x^4 \frac{d^4}{dx^4} \left[-1 + (-1 \cdot x)^{-1} \right]$

$$f(x) = x^4 \left[2 \times 3 \times 4 \times (-1 \cdot x)^{-2} \right]$$

HENCE WE HAVE

$$f(\frac{1}{2}) = \sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3) (\frac{1}{2})^n] = \frac{24 \times (\frac{1}{2})^4}{(-\frac{1}{2})^2}$$

$$\sum_{n=4}^{\infty} [n(n-1)(n-2)(n-3) (\frac{1}{2})^n] = \frac{24 \times 0.0625}{0.25} = \frac{24}{8} = 48$$

HENCE WE HAVE

$$48 = 3 \times 2^k$$

$$16 = 2^k$$

 , $k = 4$

Question 155 (***)**

Evaluate the following expression

$$\sum_{n=0}^{\infty} \sum_{m=0}^n \left[\frac{1}{2^{m+n}} \right].$$

Detailed workings must be shown.

, $\frac{8}{3}$

Work As Follows

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^n \left[\frac{1}{2^{m+n}} \right] &= \sum_{n=0}^{\infty} \left[\sum_{m=0}^n \left(\frac{1}{2^m} \times \frac{1}{2^n} \right) \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \sum_{m=0}^n \left(\frac{1}{2^m} \right) \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} \right) \right] \end{aligned}$$

GP with $a=1$, $r=\frac{1}{2}$
no. terms = $n+1$

$$\begin{aligned} &\therefore \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \times \frac{a(1-r^n)}{1-r} \right] \\ &\therefore \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \times \frac{1(1-(\frac{1}{2})^{n+1})}{1-\frac{1}{2}} \right] \end{aligned}$$

Thus we simplify to

$$\begin{aligned} &= \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \times \frac{1(1-(\frac{1}{2})^{n+1})}{1-\frac{1}{2}} \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \times 2 \times \left(1 - \left(\frac{1}{2} \right)^{n+1} \right) \right] \\ &= 2 \sum_{n=0}^{\infty} \left[\frac{1}{2^n} \left(1 - \left(\frac{1}{2} \right)^{n+1} \right) \right] \\ &= 2 \sum_{n=0}^{\infty} \left[\frac{1}{2^n} - \frac{1}{2^{n+1}} \right] \end{aligned}$$

WITH THE GEOMETRIC PROGRESSIONS EXPLICITLY

$$\begin{aligned} \dots &= 2 \sum_{n=0}^{\infty} \frac{1}{2^n} - 2 \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \\ &= 2 \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right] - 2 \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \right] \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad a=1 \qquad \qquad \qquad a=\frac{1}{2} \qquad \qquad r=\frac{1}{2} \end{aligned}$$

USING $\sum_{n=0}^{\infty} \frac{a}{r^n} = \frac{a}{1-r}$ IN EACH CASE

$$\begin{aligned} &= 2 \times \frac{1}{1-\frac{1}{2}} - 2 \times \frac{\frac{1}{2}}{1-\frac{1}{2}} \\ &= 2 \times \frac{1}{\frac{1}{2}} - 2 \times \frac{\frac{1}{2}}{\frac{1}{2}} \\ &= 2 \times 2 - \frac{2}{2} \\ &= 4 - 1 \\ &= \underline{\underline{\frac{8}{3}}} \end{aligned}$$

Question 156 (*****)

It is given that for $x \in \mathbb{R}$, $-\frac{1}{k} < x < \frac{1}{k}$, $k > 0$,

$$f(x, k) \equiv \frac{k+1}{(1-x)(1+kx)}.$$

Given further that

$$f(x, k) \equiv \sum_{r=0}^{\infty} [a_r x^r],$$

where a_r are functions of k , show that

$$\sum_{r=0}^{\infty} [a_r^2 x^r] = \frac{(1-kx)(1+k)^2}{(1-x)(1+kx)(1-k^2 x)}.$$

You may assume that $\sum_{r=0}^{\infty} [a_r^2 x^r]$ converges.

□, proof

$f(x, k) = \frac{k+1}{(1-x)(1+kx)}$ $-\frac{1}{k} < x < \frac{1}{k}$ $\Rightarrow k > 0$

• EXPAND a EXPAND BINOMIALLY

$$\Rightarrow f(x, k) = (1+k)(1-x)(1+kx)^{-1} = (1+k) \left[1 + x + x^2 + x^3 + x^4 + O(x^5) \right] \left[1 - kx + kx^2 - kx^3 + kx^4 + O(x^5) \right]$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+kx)^{-1} = 1 - kx + kx^2 - kx^3 + kx^4 + \dots$$

• SIMPLIFY THE EXPRESSION

$$\Rightarrow f(x, k) = (1+k) \left[1 - kx + k^2x^2 - k^2x^3 + k^3x^4 - \dots \right]$$

$$= \begin{bmatrix} 1 - kx + k^2x^2 - k^2x^3 + k^3x^4 - \dots \\ x^2 - kx^2 + k^2x^2 - \dots \\ x^3 - kx^3 + k^2x^3 - \dots \\ x^4 - kx^4 + \dots \\ \vdots \end{bmatrix}$$

$$\Rightarrow f(x, k) = (1+k) \left[1 + (-k)x + (1+k^2)x^2 + (1-k+k^2+k^3)x^3 + (1+k+k^2+k^3+k^4)x^4 + \dots \right]$$

• USES THE IDENTITY $a^{m+n} = a^m a^n + a^{m-1} b^2 + a^{m-2} b^4 + \dots + b^n (a-b)$

$$\Rightarrow f(x, k) = \left[\frac{a_0}{1} + \frac{a_1}{1-kx} x + \left(\frac{a_2}{1-k^2 x} x^2 + \frac{a_3}{1-k^3 x} x^3 + \dots + \frac{a_n}{1-k^n x} x^n \right) \right] x^2 + O(x^3)$$

$$\Rightarrow f(x, k) = \sum_{n=0}^{\infty} \left[\left(\frac{a_0}{1} + \frac{a_1}{1-kx} x + \dots \right) x^n \right] x^2$$

$$\Rightarrow f(x, k) = \sum_{n=0}^{\infty} \left[\left(1 + kx + k^2x^2 + \dots \right) x^n \right] x^2$$

• NEXT CONSIDER THE REQUIRED SERIES

$$\Rightarrow g(x, k) = \sum_{n=0}^{\infty} \left[\left(1 + kx + k^2x^2 + \dots \right)^2 x^n \right] = \sum_{n=0}^{\infty} \left[\left(1 + 2kx + k^2x^2 + \dots \right) x^n \right]$$

$$\Rightarrow g(x, k) = \sum_{n=0}^{\infty} [x^n] + 2k \sum_{n=0}^{\infty} [kx^n] x^n + k^2 \sum_{n=0}^{\infty} [k^2 x^n] x^n$$

$$\Rightarrow g(x, k) = \sum_{n=0}^{\infty} (x^n) + 2k \sum_{n=0}^{\infty} (kx^n) + k^2 \sum_{n=0}^{\infty} (k^2 x^n)$$

• NEXT RECALCULATING THE STANDARD EXPANSION WE USED EARLIER

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-x)^n$$

$$\Rightarrow g(x, k) = \frac{1}{1-x} + \frac{-2k}{1+kx} + \frac{-k^2}{1-k^2x}$$

$$\Rightarrow g(x, k) = \frac{(1+k)(1-kx) + 2k(-x)(1-kx) + k^2(1-k)(1+kx)}{(1-x)(1+kx)(1-kx)}$$

$$\Rightarrow g(x, k) = \frac{\cancel{1+k}x - \cancel{1-kx} - \cancel{k^2x^2}}{\cancel{(1-x)}\cancel{(1+kx)}\cancel{(1-kx)}}$$

$$\Rightarrow g(x, k) = \frac{(x^2+2kx)-(x^2+k^2x^2)x}{(1-x)(1+kx)(1-kx)}$$

$$\Rightarrow g(x, k) = \frac{(k^2+2k)x - (k^2+k)x^2}{(1-x)(1+kx)(1-kx)}$$

$$\Rightarrow g(x, k) = \frac{(k^2+2k+1)(-kx)}{(1-x)(1+kx)(1-kx)}$$

$$\Rightarrow g(x, k) = \frac{(k^2+2k+1)(-k)}{(1-x)(1+k)(1-k)}$$

Question 157 (*****)

It is given that

- ♦ $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{1}{4}\pi$
- ♦ $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots = \frac{1}{12}\pi^2$
- ♦ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln 2$

Assuming the following integral converges find its exact value.

$$\int_0^1 (\ln x)(\arctan x) dx .$$

[you may assume that integration and summation commute]

, $\frac{1}{48}[\pi^2 - 12\pi + 24\ln 2]$

IT IS UNLIKELY THAT THIS INTEGRAL HAS A CLOSED FORM IN TERMS OF ELEMENTARY FUNCTIONS IN INDEFINITE FORM—USE SERIES METHOD

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

INTEGRATING WITH RESPECT TO x

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + C$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad x > 0, C = 0$$

YOU DETERMINE TO THE INTEGRAL, & SUMMATION AND SUMMATE!

$$\int_0^1 (\arctan x)(\ln x) dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \times \ln x dx$$

$$= \sum_{n=0}^{\infty} \left[(-1)^n \int_0^1 x^{2n+1} \ln x dx \right]$$

INTEGRATE BY PARTS INSIDE THE SUM

$$\begin{aligned} \int_0^1 x^{2n+1} \ln x dx &= \left[\frac{x^{2n+2}}{2n+2} \right]_0^1 - \int_0^1 \frac{2n+1}{2n+2} x^{2n+1} dx \\ &= \frac{1}{2n+2} \left[x^{2n+2} \ln x \right]_0^1 - \int_0^1 \frac{2n+1}{2n+2} x^{2n+1} dx \\ &\stackrel{\text{Let } u = 2n+1}{=} \frac{1}{2n+2} \left[x^{2n+2} \ln x \right]_0^1 - \int_0^1 \frac{2n+1}{2n+2} x^{2n+1} dx \\ &\stackrel{\text{Let } u = 2n+1}{=} \frac{1}{2n+2} \left[x^{2n+2} \ln x \right]_0^1 - \int_0^1 \frac{2n+1}{2n+2} x^{2n+1} dx \\ &\stackrel{\text{Let } u = 2n+1}{=} \frac{1}{2n+2} \left[x^{2n+2} \ln x \right]_0^1 - \int_0^1 \frac{2n+1}{2n+2} x^{2n+1} dx \end{aligned}$$

SUMMING UP SO FAR

$$\int_0^1 (\arctan x)(\ln x) dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+2}}{(2n+2)(2n+1)} \right] = \frac{1}{2} \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+2}}{(2n+1)(2n+1)} \right]$$

DETERMINE SOME PARTIAL FRACTIONS

$$\frac{1}{(2n+1)(2n+1)} = \frac{A}{2n+1} + \frac{B}{2n+1} + \frac{C}{(2n+1)^2}$$

$$\begin{aligned} 1 &= A(2n+1)^2 + B(2n+1) + C(2n+1)^2 \\ 1 &= 4n^2 + 4n + 1 + A + 2Bn + B + 4Cn^2 + 4Cn + C \\ 1 &= 4n^2 + 4n + 1 + A + 2Bn + B + 4Cn^2 + 4Cn + C \\ 1 &= 4n^2 + 4n + 1 + A + 2Bn + B + 4Cn^2 + 4Cn + C \\ 1 &= 4n^2 + 4n + 1 + A + 2Bn + B + 4Cn^2 + 4Cn + C \\ 1 &= 4n^2 + 4n + 1 + A + 2Bn + B + 4Cn^2 + 4Cn + C \end{aligned}$$

THIS WE KNOW HAVE

$$\begin{aligned} \int_0^1 (\arctan x)(\ln x) dx &= \frac{1}{2} \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n+1)^2} + \frac{-2(-1)^n}{(2n+1)} + \frac{(-1)^n}{2(2n+1)} \right] \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n+1)} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} - \frac{3}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \right] \end{aligned}$$

LOOKING AT THE RESULTS (PNT)

$$\begin{aligned} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots &= \frac{1}{4}\pi \\ 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots &= \frac{1}{12}\pi^2 \\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots &= \ln 2 \end{aligned}$$

FINALLY WE HAVE

$$\begin{aligned} \int_0^1 (\arctan x)(\ln x) dx &= \frac{1}{2} \left(\frac{\pi^2}{12} \right) + \frac{1}{2} \ln 2 - \frac{1}{4}\pi \\ &= \frac{\pi^2}{24} - \frac{1}{4}\pi + \frac{1}{2} \ln 2 \\ &= \underline{\underline{\frac{1}{48}[\pi^2 - 12\pi + 24\ln 2]}} \end{aligned}$$

Question 158 (***)**

Given that p and q are positive, show that the natural logarithm of their arithmetic mean exceeds the arithmetic mean of their natural logarithms by

$$\sum_{r=1}^{\infty} \left[\frac{2}{2r-1} \left(\frac{\sqrt{p} - \sqrt{q}}{\sqrt{p} + \sqrt{q}} \right)^{4r-2} \right].$$

You may find the series expansion of $\operatorname{artanh}(x^2)$ useful in this question.

[] , proof

<p>● SUMMING FROM THE SERIES EXPANSION OF $\operatorname{artanh} x$ IN LOG FORM</p> $\Rightarrow \operatorname{artanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = \frac{1}{2} \left[\ln(1+x) - \ln(1-x) \right]$ $= \frac{1}{2} \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \dots \right]$ $\Rightarrow \operatorname{artanh} x = \frac{1}{2} \left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \dots \right) - \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \dots \right) \right]$ $\Rightarrow \operatorname{artanh} x = \frac{1}{2} \left[2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \dots \right]$ $\Rightarrow \operatorname{artanh} x = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \dots$ $\Rightarrow \operatorname{artanh}(x^2) = x^2 + \frac{1}{3}x^6 + \frac{1}{5}x^10 + \frac{1}{7}x^14 + \dots$ $\therefore \operatorname{artanh}(x^2) = \sum_{r=1}^{\infty} \left[\frac{2}{2r-1} x^{4r-2} \right] = \frac{1}{2} \ln \left(\frac{1+x^2}{1-x^2} \right)$ <p>● NEXT LET $x = \frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}$, IN THE ARGUMENT OF THE LOG ABOVE.</p> $\Rightarrow \frac{1+x^2}{1-x^2} = \frac{1 + \left(\frac{(\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2} \right)^2}{1 - \left(\frac{(\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2} \right)^2}$ <p style="color: red; margin-left: 20px;">REDUCE TOP & BOTTOM OF THE FRACTION BY</p> $(1+x^2) = \frac{(\sqrt{p}+\sqrt{q})^2 + (\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2 - (\sqrt{p}-\sqrt{q})^2}$ $(1-x^2) = \frac{p+2\sqrt{pq}+q + p-2\sqrt{pq}+q}{p+2\sqrt{pq}-q - p+2\sqrt{pq}-q}$ $\frac{1+x^2}{1-x^2} = \frac{2p+4}{4\sqrt{pq}} = \frac{p+q}{2\sqrt{pq}}$

● **POTTING ALL THE RESULTS TOGETHER**

$$\sum_{r=1}^{\infty} \left[\frac{2}{2r-1} x^{4r-2} \right] = \frac{1}{2} \ln \left[\frac{1+x^2}{1-x^2} \right]$$

$$\sum_{r=1}^{\infty} \left[\frac{1}{2r-1} \left(\frac{(\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2} \right)^{4r-2} \right] = \frac{1}{2} \ln \left(\frac{p+q}{2\sqrt{pq}} \right)$$

$$2 \sum_{r=1}^{\infty} \left[\frac{1}{2r-1} \left(\frac{(\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2} \right)^{4r-2} \right] = \ln \left[\frac{p+q}{2\sqrt{pq}} \right]$$

$$\sum_{r=1}^{\infty} \left[\frac{2}{2r-1} \left(\frac{(\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2} \right)^{4r-2} \right] = \ln \left(\frac{p+q}{2} \right) - \ln \sqrt{pq}$$

$$\sum_{r=1}^{\infty} \left[\frac{2}{2r-1} \left(\frac{(\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2} \right)^{4r-2} \right] = \ln \left(\frac{p+q}{2} \right) - \frac{1}{2} \ln(pq)$$

THUS WE FINALLY HAVE THE DESIRED RESULT

$$\ln \left(\frac{p+q}{2} \right) - \frac{\ln(p+q)}{2} = \sum_{r=1}^{\infty} \left[\frac{2}{2r-1} \left(\frac{(\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2} \right)^{4r-2} \right]$$

Question 159 (*****)

Show that

$$1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \frac{x^{12}}{12!} + \frac{x^{15}}{15!} + \dots = \frac{1}{3} \left[e^x + 2e^{\frac{1}{2}x} \cos\left(\frac{1}{2}\sqrt{3}x\right) \right].$$

You may find useful in this question the fact that if $z = e^{i\frac{2\pi}{3}}$ then $1+z+z^2=0$.

, **proof**

• **LET** $z = e^{i\frac{2\pi}{3}}$

• **THEN** $z^2 = e^{i\frac{4\pi}{3}}$, $1 \cdot z = e^{i\frac{2\pi}{3}} \cdot z^2 = z^3 = z^6 = \dots$
 $z^3 = 1$, $1 \cdot z^2 = z^4 = z^7 = z^{10} = z^{13} = \dots$
 $z^6 = z^9 = z^{12} = z^{15} = \dots$
 $z^9 = z^{12} = z^{15} = \dots$
 $z^{12} = z^{15} = \dots$
 $z^{15} = z^{18} = \dots$

• $z^3 + z^6 + 1 = 0$ AS THEY ARE THE CUBE ROOTS OF UNITY

• **NEXT CONSIDER THESE EXPONENTIAL EXPANSIONS**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots$$

$$e^{zx} = 1 + zx + \frac{(zx)^2}{2!} + \frac{(zx)^3}{3!} + \frac{(zx)^4}{4!} + \frac{(zx)^5}{5!} + \frac{(zx)^6}{6!} + \frac{(zx)^7}{7!} + \dots$$

$$e^{zx^2} = 1 + zx^2 + \frac{(zx^2)^2}{2!} + \frac{(zx^2)^3}{3!} + \frac{(zx^2)^4}{4!} + \frac{(zx^2)^5}{5!} + \frac{(zx^2)^6}{6!} + \frac{(zx^2)^7}{7!} + \dots$$

• **WE CAN WRITE THIS AS**

$$\begin{aligned} \frac{e^x}{e^{zx^2}} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots \\ e^{zx^2} &= 1 + zx^2 + \frac{(zx^2)^2}{2!} + \frac{(zx^2)^3}{3!} + \frac{(zx^2)^4}{4!} + \frac{(zx^2)^5}{5!} + \frac{(zx^2)^6}{6!} + \frac{(zx^2)^7}{7!} + \dots \\ \frac{e^x}{e^{zx^2} + e^{2x^2}} &= 1 + (1+2z^2)x^2 + \frac{(1+2z^2)^2}{2!} + \frac{(1+2z^2)^3}{3!} + \frac{(1+2z^2)^4}{4!} + \frac{(1+2z^2)^5}{5!} + \dots \end{aligned}$$

• **TRYING OUT THE EXPANSION**

$$\begin{aligned} e^x + e^{2x^2} + e^{3x^2} &= 3 + 3\left(\frac{x^2}{2!}\right) + 3\left(\frac{x^4}{4!}\right) + 3\left(\frac{x^6}{6!}\right) + \dots \\ z^6 + z^9 + z^{12} + z^{15} + z^{18} &= 3 + 3\left(\frac{x^2}{2!}\right) + 3\left(\frac{x^4}{4!}\right) + \dots \\ z^3 + z^6 + z^9 + z^{12} + z^{15} &= 3 + 3\left(\frac{x^2}{2!}\right) + 3\left(\frac{x^4}{4!}\right) + \dots \\ z^6 + z^9 + z^{12} + z^{15} &= 3 + 3\left(\frac{x^2}{2!}\right) + 3\left(\frac{x^4}{4!}\right) + \dots \\ 3\sum_{n=0}^{\infty} \frac{x^n}{(2n)!} &= 3 + 3e^{2x^2} \left[e^{3x^2} + e^{-3x^2} \right] = 3 + 3e^{-\frac{1}{2}x} \left[2\cosh\left(\frac{1}{2}\sqrt{3}x\right) \right] \\ 3\sum_{n=0}^{\infty} \frac{x^n}{(2n)!} &= 3 + 3e^{2x^2} \left[\cosh\left(\frac{\sqrt{3}x}{2}\right) \right] \\ \sum_{n=0}^{\infty} \frac{x^n}{(2n)!} &= \frac{1}{3} \left[e^x + 2e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}x}{2}\right) \right] \end{aligned}$$