MULTIVARIABLE INTEGRATION IN CARTESIAN

Question 1

Find the value of the following multiple integral

$$\int_0^{\pi}\int_0^{\cos\theta} r\sin\theta \, dr\, d\theta .$$

$\boxed{}$, $\boxed{\frac{1}{3}}$

STANDARD EVALUATION AFTER WRITING THE LIMITS EXPLICITLY

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\cos\theta} r\sin\theta \;\; dr\, d\theta$$

$$= \int_{\theta=0}^{\pi}\left[\tfrac{1}{2}r^2\sin\theta\right]_{r=0}^{r=\cos\theta} d\theta = \int_{\theta=0}^{\pi}\tfrac{1}{2}\cos^2\theta\sin\theta - 0 \;\; d\theta$$

$$= \left[-\tfrac{1}{6}\cos^3\theta\right]_0^{\pi} = \tfrac{1}{6}\left[\cos^3\theta\right]_{\pi}^{0} = \tfrac{1}{6}\left[1-(-1)\right]$$

$$= \tfrac{1}{6}\times 2 = \tfrac{1}{3}$$

Question 2

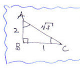
Find the exact value of

$$\int_0^{\frac{\pi}{2}}\int_{\arctan\frac{1}{2}}^{\arctan 2}\int_0^{5} 4x\sin y \;\; dx\, dy\, d\theta .$$

$\boxed{5\pi\sqrt{5}}$

$$\int_0^{\frac{\pi}{2}}\int_{\arctan\frac{1}{2}}^{\arctan 2}\int_0^{5} 4x\sin y \, dx\, dy\, d\theta = \int_0^{\frac{\pi}{2}}\int_{\arctan\frac{1}{2}}^{\arctan 2}\left[2x^2\sin y\right]_0^5 dy\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_{\arctan\frac{1}{2}}^{\arctan 2} 50\sin y \; dy\, d\theta = \int_0^{\frac{\pi}{2}}\left[-50\cos y\right]_{\arctan\frac{1}{2}}^{\arctan 2} d\theta$$



$\tan C = 2$ $\quad \tan A = \frac{1}{2}$

$\cos C = \frac{1}{\sqrt{5}}$ $\quad \cos A = \frac{2}{\sqrt{5}}$

$$= \int_0^{\frac{\pi}{2}} 50\left[\cos y\right]_{\arctan 2}^{\arctan\frac{1}{2}} d\theta = \int_0^{\frac{\pi}{2}} 50\left[\tfrac{2}{\sqrt{5}} - \tfrac{1}{\sqrt{5}}\right] d\theta = \int_0^{\frac{\pi}{2}} \tfrac{50}{\sqrt{5}}\, d\theta$$

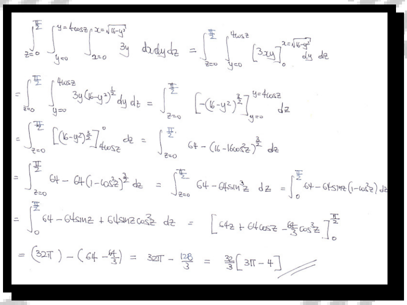
$$= \left[\tfrac{50}{\sqrt{5}}\theta\right]_0^{\frac{\pi}{2}} = \tfrac{50}{\sqrt{5}}\times\left(\tfrac{\pi}{2}-0\right) = \tfrac{25\pi}{\sqrt{5}} = 5\pi\sqrt{5}$$

Question 3

Find the exact value of

$$\int_0^{\frac{\pi}{2}} \int_0^{4\cos z} \int_0^{\sqrt{16-y^2}} 3y \;\; dx\,dy\,dz\,.$$

$$\boxed{\frac{32}{3}[3\pi - 4]}$$



$$\int_{z=0}^{\frac{\pi}{2}} \int_{y=0}^{y=4\cos z} \int_{x=0}^{x=\sqrt{16-y^2}} 3y \;\; dx\,dy\,dz = \int_{z=0}^{\frac{\pi}{2}} \int_{y=0}^{4\cos z} \Big[3xy\Big]_0^{x=\sqrt{16-y^2}} dy\,dz$$

$$= \int_{z=0}^{\frac{\pi}{2}} \int_{y=0}^{4\cos z} 3y(16-y^2)^{\frac{1}{2}} dy\,dz = \int_{z=0}^{\frac{\pi}{2}} \Big[-(16-y^2)^{\frac{3}{2}}\Big]_{y=0}^{y=4\cos z} dz$$

$$= \int_{z=0}^{\frac{\pi}{2}} \Big[(16-y^2)^{\frac{3}{2}}\Big]_{4\cos z}^{0} dz = \int_{z=0}^{\frac{\pi}{2}} 64 - (16-16\cos^2 z)^{\frac{3}{2}} dz$$

$$= \int_{z=0}^{\frac{\pi}{2}} 64 - 64(1-\cos^2 z)^{\frac{3}{2}} dz = \int_{z=0}^{\frac{\pi}{2}} 64 - 64\sin^3 z \; dz = \int_0^{\frac{\pi}{2}} 64 - 64\sin z(1-\cos^2 z)\,dz$$

$$= \int_0^{\frac{\pi}{2}} 64 - 64\sin z + 64\sin z\cos^2 z \; dz = \Big[64z + 64\cos z - \tfrac{64}{3}\cos^3 z\Big]_0^{\frac{\pi}{2}}$$

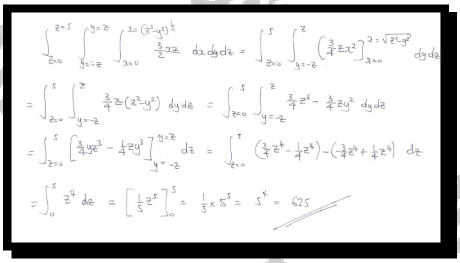
$$= (32\pi) - \left(64 - \tfrac{64}{3}\right) = 32\pi - \frac{128}{3} = \frac{32}{3}[3\pi - 4]$$

Question 4

Find the value of

$$\int_0^5 \int_{-z}^{z} \int_0^{\sqrt{z^2-y^2}} \frac{3}{2}xz \;\; dx\,dy\,dz\,.$$

$$\boxed{625}$$



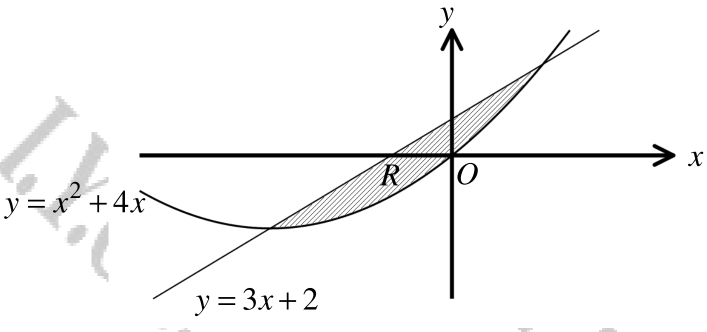
$$\int_{z=0}^{z=5} \int_{y=-z}^{y=z} \int_{x=0}^{x=(z^2-y^2)^{\frac{1}{2}}} \frac{3}{2}xz \;\; dx\,dy\,dz = \int_{z=0}^{5} \int_{y=-z}^{z} \Big[\tfrac{3}{4}zx^2\Big]_{x=0}^{x=\sqrt{z^2-y^2}} dy\,dz$$

$$= \int_{z=0}^{5} \int_{y=-z}^{z} \tfrac{3}{4}z(z^2-y^2)\;dy\,dz = \int_{z=0}^{5} \int_{y=-z}^{z} \tfrac{3}{4}z^3 - \tfrac{3}{4}zy^2 \;dy\,dz$$

$$= \int_{z=0}^{5} \Big[\tfrac{3}{4}yz^3 - \tfrac{1}{4}zy^3\Big]_{y=-z}^{y=z} dz = \int_{z=0}^{5} \left(\tfrac{3}{4}z^4 - \tfrac{1}{4}z^4\right) - \left(-\tfrac{3}{4}z^4 + \tfrac{1}{4}z^4\right) dz$$

$$= \int_0^5 z^4\,dz = \Big[\tfrac{1}{5}z^5\Big]_0^5 = \tfrac{1}{5}\times 5^5 = 5^4 = 625$$

Question 5



The finite region R in the x-y plane, is bounded a curve and a straight line with the following equations.

$$y = x^2 + 4x \quad \text{and} \quad y = 3x + 2 .$$

Use double integration to find

$$\int_R 2 \; dx\,dy .$$

$\boxed{9}$

$$\int_{-2}^{1}\int_{x^2+4x}^{3x+2} 2 \; dy\,dx = \int_{-2}^{1} \left[2y\right]_{y=x^2+4x}^{y=3x+2} dx = \int_{-2}^{1} (6x+4)-(2x^2+8x) \; dx$$

$$= \int_{-2}^{1} -2x^2-2x+4 \; dx = \left[-\tfrac{2}{3}x^3-x^2+4x\right]_{-2}^{1} = \left[\tfrac{2}{3}x^3+x^2-4x\right]_{1}^{-2}$$

$$= \left(-\tfrac{16}{3}+4+8\right)-\left(\tfrac{2}{3}+1-4\right) = -\tfrac{16}{3}+12-\tfrac{2}{3}+3 = 9$$

Question 6

The finite region R in the x-y plane, is defined by the inequalities

$$y \le x^3, \quad y \ge -x \quad \text{and} \quad 2 \le x \le 4.$$

Determine the value of

$$\int_R \frac{xy}{9}\, dx\, dy.$$

$\boxed{450}$

• MUST INTEGRATE y FIRST THEN x

$$\int_{x=2}^{x=4}\int_{y=-x}^{y=x^3} \frac{xy}{9}\, dy\, dx = \int_{x=2}^{4}\left[\frac{1}{18}xy^2\right]_{y=-x}^{y=x^3} dx = \int_2^4 \frac{1}{18}x(x^6) - \frac{1}{18}x(x^2)\, dx$$

$$= \int_2^4 \frac{1}{18}x^7 - \frac{1}{18}x^3\, dx = \frac{1}{18}\left[\frac{1}{8}x^8 - \frac{1}{4}x^4\right]_2^4$$

$$= \frac{1}{18}\left[(8192-64)-(32-4)\right] = \frac{1}{18}\times 8100$$

$$= 450$$

Question 7

Find the value of the following multiple integral

$$\int_0^{\frac{\pi}{4}}\int_{2y}^{\frac{\pi}{2}} \frac{\cos x}{x}\, dx dy.$$

$\boxed{\frac{1}{2}}$

$$\int_{y=0}^{\frac{\pi}{4}}\int_{x=2y}^{x=\frac{\pi}{2}} \frac{\cos x}{x}\, dx\, dy = \ldots\ldots$$

$$= \int_{x=0}^{\frac{\pi}{2}}\int_{y=0}^{y=\frac{1}{2}x} \frac{\cos x}{x}\, dy\, dx$$

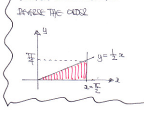
$$= \int_0^{\frac{\pi}{2}} \left[\frac{y\cos x}{x}\right]_{y=0}^{y=\frac{1}{2}x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\cos x\, dx$$

$$= \left[\frac{1}{2}\sin x\right]_0^{\frac{\pi}{2}} = \frac{1}{2}$$

IT IS NOT SENSIBLE TO ATTEMPT AN INTEGRATION W.R.T x — INTEGRAL MIGHT NOT EVEN BE POSSIBLE TO BE INTEGRATED

REVERSE THE ORDER



Question 8

The finite region R in the x-y plane, is bounded a curve and a straight line with the following equations.

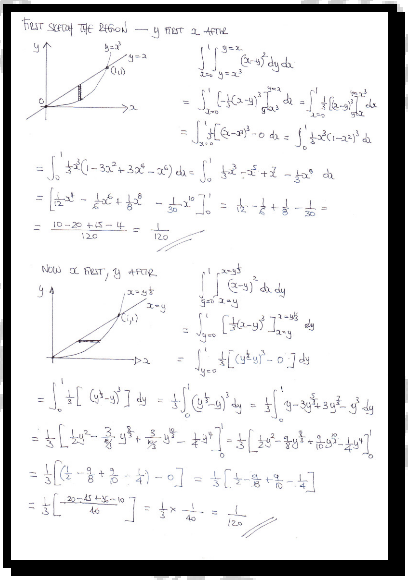
$$y = x^3 \,,\; x \geq 0 \qquad \text{and} \qquad y = x \,,\; x \geq 0 \,.$$

Determine the value for both

$$\iint_R (x-y)^2 \; dy\,dx \qquad \text{and} \qquad \iint_R (x-y)^2 \; dx\,dy \,,$$

showing clearly that they have the same value.

$$\boxed{\frac{1}{120}}$$



Question 9

The finite region R is bounded by the planes with equations

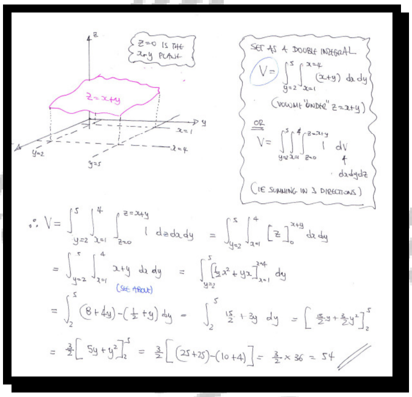
$$z = 0, \quad x = 1, \quad x = 4, \quad y = 2, \quad y = 5,$$

and the surface with equation

$$z = x + y.$$

Determine the volume of R.

$\boxed{54}$



Question 10

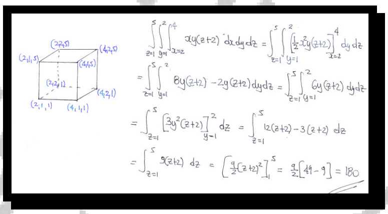
Integrate the function

$$f(x, y, z) \equiv xy(z + 2)$$

over the finite region bounded by the cuboid with coordinates at

$$(2,1,1),\ (4,1,1),\ (4,2,1),\ (2,2,1),\ (2,1,5),\ (4,1,5),\ (4,2,5) \text{ and } (2,2,5).$$

$\boxed{180}$



Question 11

The finite region R in the x-y plane, is bounded by the straight lines with equations

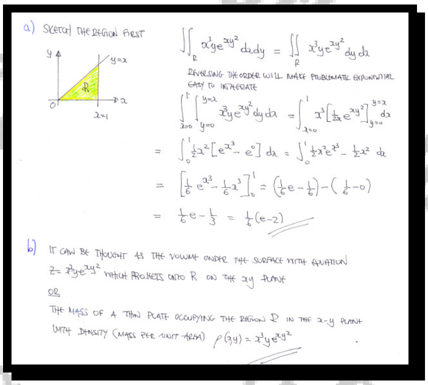
$$y = x, \qquad y = 0 \quad \text{and} \quad x = 1.$$

a) Determine the value of

$$I = \iint_R yx^3 \mathrm{e}^{xy^2} \, dx\, dy.$$

b) Give two distinct possible interpretations for the value of I.

$$\boxed{\frac{1}{6}(\mathrm{e}-2)}$$

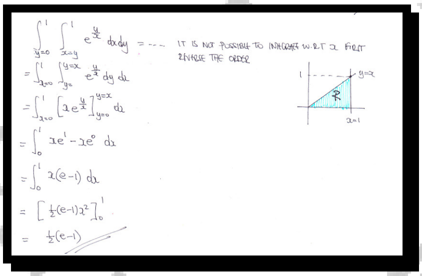


Question 12

Find the exact value of the following multiple integral

$$\int_0^1 \int_y^1 \mathrm{e}^{\frac{y}{x}} \, dx dy.$$

$$\boxed{\frac{1}{2}(\mathrm{e}-1)}$$



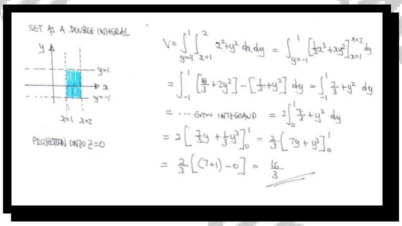
Question 13

The finite region R is defined by the inequalities

$$1 \le x \le 2, \quad -1 \le y \le 1 \quad \text{and} \quad 0 \le z \le x^2 + y^2 .$$

Determine the volume of R.

$$\boxed{\frac{16}{3}}$$

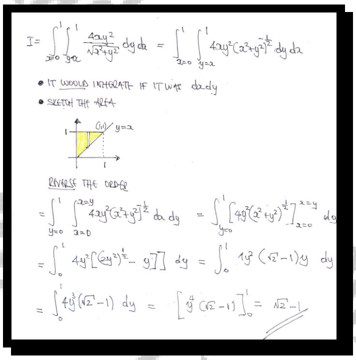


Question 14

Find the value of I in exact surd form.

$$I = \int_{0}^{1}\int_{x}^{1} \frac{4xy^2}{\sqrt{x^2+y^2}} \, dy \, dx .$$

$$\boxed{\sqrt{2}-1}$$



Question 15

The finite region Ω in the x-y plane, is defined by the inequalities

$$x^2 + y^2 \le 1 \qquad \text{and} \qquad x + y \ge 1 .$$

Determine the value of

$$\int_{\Omega} x^3 y \;dx\,dy .$$

$\boxed{}$, $\boxed{\dfrac{1}{30}}$

$$\iint_{\Omega} x^3 y \, dx\,dy = \dots\dots$$

$$= \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} x^3 y \, dy\,dx$$

$$= \int_{x=0}^{1} \left[\tfrac{1}{2}x^3y^2\right]_{y=1-x}^{y=+\sqrt{1-x^2}} dx$$

$$= \int_0^1 \tfrac{1}{2}x^3 \left[(1-x^2)-(1-x)^2\right] dx$$

$$= \int_0^1 \tfrac{1}{2}x^3 \left[1-x^2-(1-2x+x^2)\right] dx$$

$$= \int_0^1 \tfrac{1}{2}x^3 (2x-2x^2) \, dx$$

$$= \int_0^1 x^4 - x^5 \, dx$$

$$= \left[\tfrac{1}{5}x^5 - \tfrac{1}{6}x^6\right]_0^1$$

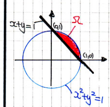
$$= \left(\tfrac{1}{5}-\tfrac{1}{6}\right) - (0)$$

$$= \frac{1}{30}$$

REGION Ω

$x^2+y^2 \le 1$

$x+y \ge 1$



LIMITS

y: FROM $y=1-x$ TO $y=+\sqrt{1-x^2}$

x: FROM $x=0$ TO $x=1$

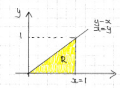
Question 16

Find the value of the following multiple integral

$$\int_0^1 \int_y^1 x\mathrm{e}^{x^3}\ dx\,dy\,.$$

$\boxed{\quad}$, $\boxed{\frac{1}{3}(\mathrm{e}-1)}$

START BY SKETCHING THE REGION OF INTEGRATION IN ORDER TO REVERSE THE ORDER OF INTEGRATION IF NECESSARY



REVERSING THE ORDER

$$\int_{y=0}^{y=1}\int_{x=y}^{x=1} xe^{x^3}\,dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=x} xe^{x^3}\,dy\,dx$$

$$= \int_0^1 \left[xye^{x^3}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 x^2e^{x^3} - 0\ dx$$

$$= \left[\tfrac{1}{3}e^{x^3}\right]_0^1$$

$$= \tfrac{1}{3}e - \tfrac{1}{3}$$

$$= \tfrac{1}{3}(e-1)$$

Question 17

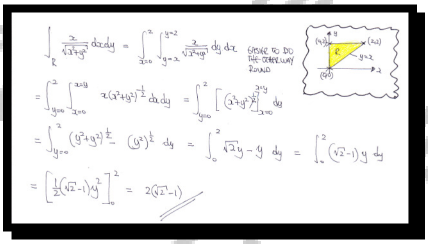
The finite region R in the x-y plane is defined as the rectilinear triangle with vertices at the points with coordinates

$$(0,0), \quad (0,2) \quad \text{and} \quad (2,2).$$

Determine an exact simplified value for

$$\int_R \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy .$$

$$\boxed{2\left(\sqrt{2}-1\right)}$$



$$\int_R \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy = \int_{x=0}^{2}\int_{y=x}^{y=2} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$$ EASIER TO DO THE OTHER WAY ROUND

$$= \int_{y=0}^{2}\int_{x=0}^{x=y} x\left(x^2+y^2\right)^{-\frac{1}{2}}\,dx\,dy = \int_{y=0}^{2}\left[\left(x^2+y^2\right)^{\frac{1}{2}}\right]_{x=0}^{x=y}\,dy$$

$$= \int_{y=0}^{2} \left(y^2+y^2\right)^{\frac{1}{2}} - \left(y^2\right)^{\frac{1}{2}}\,dy = \int_0^2 \sqrt{2}\,y - y\,dy = \int_0^2 \left(\sqrt{2}-1\right)y\,dy$$

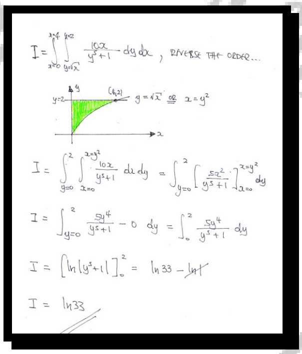
$$= \left[\tfrac{1}{2}\left(\sqrt{2}-1\right)y^2\right]_0^2 = 2\left(\sqrt{2}-1\right)$$

Question 18

Find the value of I in exact logarithmic form.

$$I = \int_0^4 \int_{\sqrt{x}}^{2} \frac{10x}{y^5+1}\,dy\,dx .$$

$$\boxed{\ln 33}$$



$$I = \int_{x=0}^{x=4}\int_{y=\sqrt{x}}^{y=2} \frac{10x}{y^5+1}\,dy\,dx$$, REVERSE THE ORDER...

$$I = \int_{y=0}^{2}\int_{x=0}^{x=y^2} \frac{10x}{y^5+1}\,dx\,dy = \int_{y=0}^{2}\left[\frac{5x^2}{y^5+1}\right]_{x=0}^{x=y^2}\,dy$$

$$I = \int_{y=0}^{2} \frac{5y^4}{y^5+1} - 0\,dy = \int_0^2 \frac{5y^4}{y^5+1}\,dy$$

$$I = \left[\ln\left|y^5+1\right|\right]_0^2 = \ln 33 - \cancel{\ln 1}$$

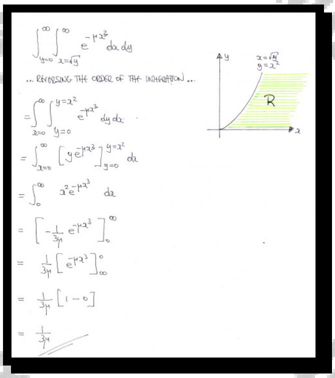
$$I = \ln 33$$

Question 19

Given that μ is a positive constant, find the value of the following integral.

$$\int_0^{\infty} \int_{\sqrt{y}}^{\infty} e^{-\mu x^3} \, dx \, dy .$$

$$\boxed{\frac{1}{3\mu}}$$



$$\int_{y=0}^{\infty} \int_{x=\sqrt{y}}^{\infty} e^{-\mu x^3} \, dx \, dy$$

... REVERSING THE ORDER OF THE INTEGRATION ...

$$= \int_{x=0}^{\infty} \int_{y=0}^{y=x^2} e^{-\mu x^3} \, dy \, dx$$

$$= \int_{x=0}^{\infty} \Big[y e^{-\mu x^3} \Big]_{y=0}^{y=x^2} dx$$

$$= \int_0^{\infty} x^2 e^{-\mu x^3} \, dx$$

$$= \left[-\frac{1}{3\mu} e^{-\mu x^3} \right]_0^{\infty}$$

$$= \frac{1}{3\mu} \left[e^{-\mu x^3} \right]_{\infty}^{0}$$

$$= \frac{1}{3\mu} [1 - 0]$$

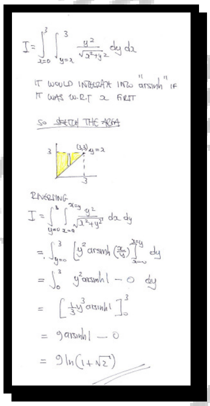
$$= \frac{1}{3\mu}$$

Question 20

Find the value of I in exact logarithmic form.

$$I = \int_0^3 \int_x^3 \frac{y^2}{\sqrt{x^2 + y^2}} \, dy \, dx .$$

$$\boxed{9 \ln\left(1 + \sqrt{2}\right)}$$



$$I = \int_{x=0}^{3} \int_{y=x}^{3} \frac{y^2}{\sqrt{x^2+y^2}} \, dy \, dx$$

IT WOULD INTEGRATE INTO "arsinh" IF IT WAS W.R.T x FIRST

SO SKETCH THE AREA

REVERSING

$$I = \int_{y=0}^{3} \int_{x=0}^{x=y} \frac{y^2}{\sqrt{x^2+y^2}} \, dx \, dy$$

$$= \int_{y=0}^{3} \left[y^2 \operatorname{arsinh}\left(\frac{x}{y}\right) \right]_{x=0}^{x=y} dy$$

$$= \int_0^3 y^2 \operatorname{arsinh} 1 - 0 \, dy$$

$$= \left[\frac{1}{3} y^3 \operatorname{arsinh} 1 \right]_0^3$$

$$= 9 \operatorname{arsinh} 1 - 0$$

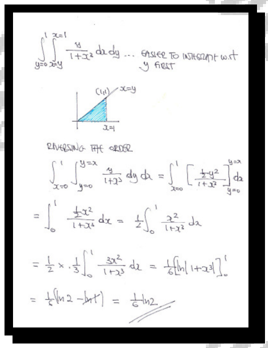
$$= 9 \ln(1 + \sqrt{2})$$

Question 21

Find the value of I in exact logarithmic form.

$$I = \int_0^1 \int_y^1 \frac{y}{1+x^3} \, dx \, dy .$$

$$\boxed{\frac{1}{6}\ln 2}$$

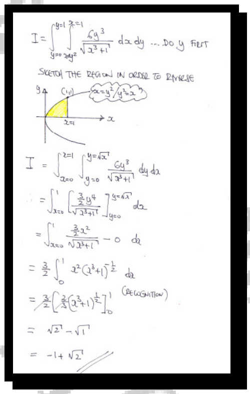


Question 22

Find the value of I in exact surd form.

$$I = \int_0^1 \int_{y^2}^1 \frac{6y^3}{\sqrt{x^3+1}} \, dx \, dy .$$

$$\boxed{\sqrt{2}-1}$$

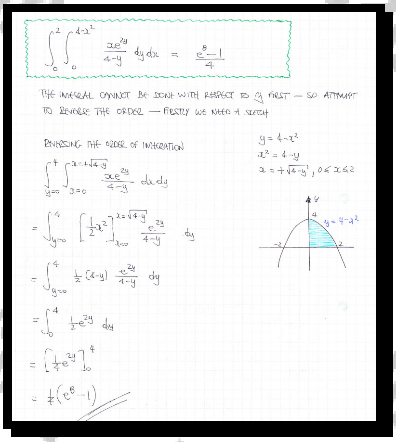


Question 23

Find the value of the following multiple integral

$$\int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y} \, dy \, dx .$$

$$\boxed{\frac{1}{4}\left(e^8 - 1\right)}$$

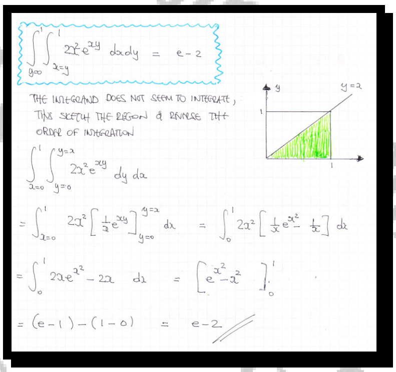


Question 24

Find the value of the following multiple integral

$$\int_0^1 \int_y^1 2x^2 e^{xy} \, dx \, dy .$$

$$\boxed{e - 2}$$

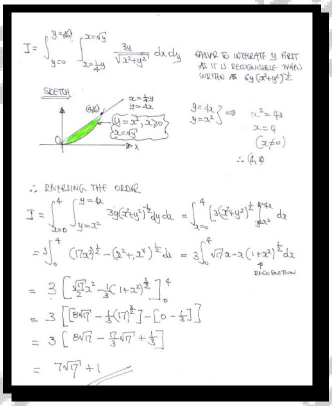


Question 25

Find the value of I in exact surd form.

$$I = \int_0^{16} \int_{\frac{1}{4}y}^{\sqrt{y}} \frac{3y}{\sqrt{x^2+y^2}} \, dx \, dy .$$

$\boxed{1+7\sqrt{17}}$



$I = \int_{y=0}^{y=16} \int_{x=\frac{1}{4}y}^{x=\sqrt{y}} \frac{3y}{\sqrt{x^2+y^2}} \, dx \, dy$ EASIER TO INTEGRATE y FIRST AS IT IS RECOGNISABLE WHEN WRITTEN AS $6y(x^2+y^2)^{-\frac{1}{2}}$

SKETCH

$\left.\begin{array}{l} y=4x \\ y=x^2 \end{array}\right\} \Rightarrow x^2=4x$, $x=4$ $(x \neq 0)$ $\therefore (4,16)$

$\therefore$ REVERSING THE ORDER

$I = \int_{x=0}^{4} \int_{y=x^2}^{y=4x} 3y(x^2+y^2)^{-\frac{1}{2}} \, dy \, dx = \int_{x=0}^{4} \left[3(x^2+y^2)^{\frac{1}{2}}\right]_{y=x^2}^{y=4x} dx$

$= 3\int_0^4 (17x^2)^{\frac{1}{2}} - (x^2+x^4)^{\frac{1}{2}} \, dx = 3\int_0^4 \sqrt{17}x - x(1+x^2)^{\frac{1}{2}} \, dx$

RECOGNITION

$= 3\left[\frac{\sqrt{17}}{2}x^2 - \frac{1}{3}(1+x^2)^{\frac{3}{2}}\right]_0^4$

$= 3\left[\left[8\sqrt{17} - \frac{1}{3}(17)^{\frac{3}{2}}\right] - \left[0 - \frac{1}{3}\right]\right]$

$= 3\left[8\sqrt{17} - \frac{17}{3}\sqrt{17} + \frac{1}{3}\right]$

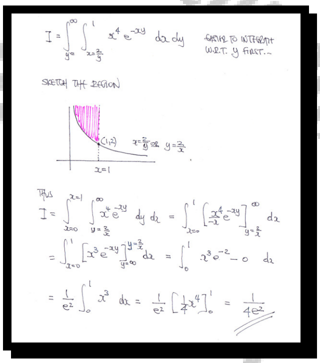
$= 7\sqrt{17} + 1$

Question 26

Find the value of I in exact simplified form.

$$I = \int_2^{\infty} \int_{\frac{2}{y}}^{1} x^4 \mathrm{e}^{-xy} \, dx \, dy .$$

$\boxed{\frac{1}{4\mathrm{e}^2}}$



$I = \int_{y=}^{\infty} \int_{x=\frac{2}{y}}^{1} x^4 e^{-xy} \, dx \, dy$ EASIER TO INTEGRATE W.R.T. y FIRST...

SKETCH THE REGION

THUS

$I = \int_{x=0}^{x=1} \int_{y=\frac{2}{x}}^{\infty} x^4 e^{-xy} \, dy \, dx = \int_{x=0}^{1} \left[\frac{x^4}{-x} e^{-xy}\right]_{y=\frac{2}{x}}^{\infty} dx$

$= \int_{x=0}^{1} \left[x^3 e^{-xy}\right]_{y=\infty}^{y=\frac{2}{x}} dx = \int_0^1 x^3 e^{-2} - 0 \, dx$

$= \frac{1}{e^2}\int_0^1 x^3 \, dx = \frac{1}{e^2}\left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4e^2}$

Question 27

A rectangular block B is bounded in the first octant by the planes with equations

$$x = 3, \quad y = 4, \text{ and } z = 2.$$

The density ρ at any point P within B, is equal to the square of the distance of P from the origin O.

Show that the mass of B is 232.

proof

$$M = \int_V \rho(x,y,z)\,dv = \int_{z=0}^{2}\int_{y=0}^{4}\int_{x=0}^{3} \left[(x^2+y^2+z^2)^{\frac{1}{2}}\right]^2 dx\,dy\,dz = \int_{z=0}^{2}\int_{y=0}^{4}\int_{x=0}^{3} x^2+y^2+z^2\,dx\,dy\,dz$$

$$= \int_{z=0}^{2}\int_{y=0}^{4} \left[\tfrac{1}{3}x^3 + y^2x + z^2x\right]_{x=0}^{3} dy\,dz = \int_{z=0}^{2}\int_{y=0}^{4} 9+3y^2+3z^2\,dy\,dz = \int_{z=0}^{2} \left[9y+y^3+3z^2y\right]_{y=0}^{4} dz$$

$$= \int_{z=0}^{2} 36+64+12z^2\,dz = \int_0^2 100+12z^2\,dz = \left[100z+4z^3\right]_0^2$$

$$= 200 + 32 = 232$$

AS REQUIRED

Question 28

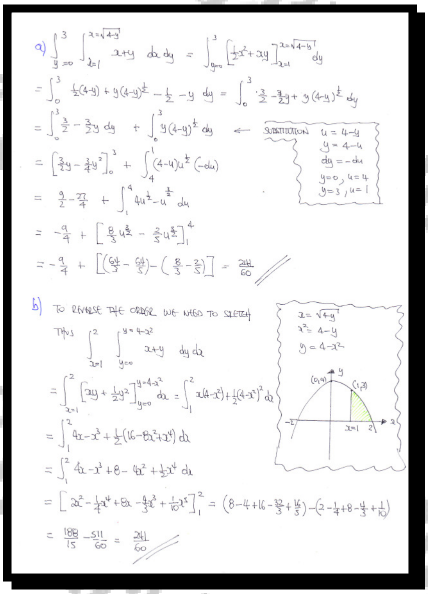
$$I = \int_0^3 \int_1^{\sqrt{4-y}} (x+y) \ dxdy$$

a) Show clearly that $I = \dfrac{241}{60}$.

You may not reverse the order of integration in this part.

b) Carry out the integration in reverse order and hence verify the answer of I, obtained in part (**a**).

$\boxed{\text{proof}}$



a) $\int_{y=0}^{3}\int_{x=1}^{x=\sqrt{4-y}} x+y \ dx\,dy = \int_{y=0}^{3}\left[\frac{1}{2}x^2+xy\right]_{x=1}^{x=\sqrt{4-y}} dy$

$= \int_0^3 \frac{1}{2}(4-y)+y(4-y)^{\frac{1}{2}}-\frac{1}{2}-y \ dy = \int_0^3 \frac{3}{2}-\frac{3}{2}y+y(4-y)^{\frac{1}{2}} \ dy$

$= \int_0^3 \frac{3}{2}-\frac{3}{2}y \ dy + \int_0^3 y(4-y)^{\frac{1}{2}} \ dy \leftarrow$ SUBSTITUTION $u=4-y$, $y=4-u$, $dy=-du$, $y=0,\ u=4$, $y=3,\ u=1$

$= \left[\frac{3}{2}y-\frac{3}{4}y^2\right]_0^3 + \int_4^1 (4-u)u^{\frac{1}{2}}(-du)$

$= \frac{9}{2}-\frac{27}{4} + \int_1^4 4u^{\frac{1}{2}}-u^{\frac{3}{2}} \ du$

$= -\frac{9}{4} + \left[\frac{8}{3}u^{\frac{3}{2}}-\frac{2}{5}u^{\frac{5}{2}}\right]_1^4$

$= -\frac{9}{4} + \left[\left(\frac{64}{3}-\frac{64}{5}\right)-\left(\frac{8}{3}-\frac{2}{5}\right)\right] = \frac{241}{60}$

b) TO REVERSE THE ORDER WE NEED TO SKETCH

$x=\sqrt{4-y}$, $x^2=4-y$, $y=4-x^2$

THUS $\int_{x=1}^{2}\int_{y=0}^{y=4-x^2} x+y \ dy\,dx$

$= \int_{x=1}^{2}\left[xy+\frac{1}{2}y^2\right]_{y=0}^{y=4-x^2} dx = \int_1^2 x(4-x^2)+\frac{1}{2}(4-x^2)^2 \ dx$

$= \int_1^2 4x-x^3+\frac{1}{2}(16-8x^2+x^4) \ dx$

$= \int_1^2 4x-x^3+8-4x^2+\frac{1}{2}x^4 \ dx$

$= \left[2x^2-\frac{1}{4}x^4+8x-\frac{4}{3}x^3+\frac{1}{10}x^5\right]_1^2 = \left(8-4+16-\frac{32}{3}+\frac{16}{5}\right)-\left(2-\frac{1}{4}+8-\frac{4}{3}+\frac{1}{10}\right)$

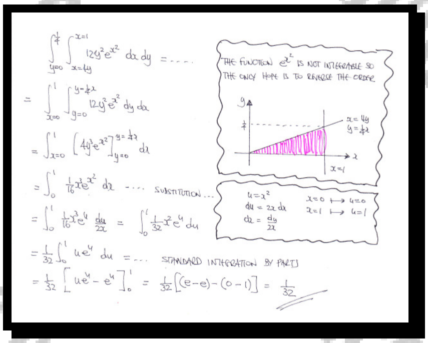
$= \frac{188}{15}-\frac{511}{60} = \frac{241}{60}$

Question 29

Find the value of the following multiple integral

$$\int_{0}^{\frac{1}{4}} \int_{4y}^{1} 12y^2 e^{x^2} \; dxdy .$$

$$\boxed{\frac{1}{32}}$$

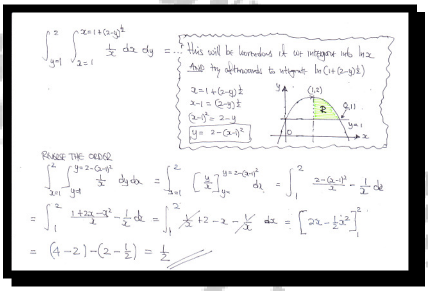


Question 30

Find the value of the following multiple integral

$$\int_{1}^{2} \int_{1}^{1+\sqrt{2-y}} \frac{1}{x} \; dxdy .$$

$$\boxed{\frac{1}{2}}$$



Question 31

The finite region R is in the first octant and is bounded by the plane with equation

$$2x+3y+z=6.$$

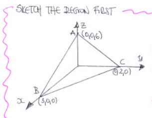
Find the value of the integral

$$\iiint_V x \ dV,$$

where V is the volume of R.

$\boxed{\frac{9}{2}}$

SKETCH THE REGION FIRST



PLANE: $2x+3y+z=6$

$z = 6-2x-3y$

LINE BC: GRADIENT $= -\frac{2}{3}$

$y = -\frac{2}{3}x+2$

$$\iiint_V x \ dx\,dy\,dz$$

$$= \int_{x=0}^{3}\int_{y=0}^{-\frac{2}{3}x+2}\int_{z=0}^{z=6-2x-3y} x \ dz\,dy\,dx$$

$$= \int_{x=0}^{3}\int_{y=0}^{y=-\frac{2}{3}x+2} \Big[xz\Big]_{z=0}^{z=6-2x-3y} dy\,dx$$

$$= \int_{x=0}^{3}\int_{y=0}^{y=-\frac{2}{3}x+2} x(6-2x-3y)-0 \ dy\,dx$$

$$= \int_{x=0}^{3}\int_{y=0}^{y=-\frac{2}{3}x+2} 6x-2x^2-3xy \ dy\,dx$$

$$= \int_{x=0}^{3} \Big[6xy-2x^2y-\tfrac{3}{2}xy^2\Big]_{y=0}^{y=-\frac{2}{3}x+2} dx$$

$$= \int_0^3 6x\left(-\tfrac{2}{3}x+2\right)-2x^2\left(-\tfrac{2}{3}x+2\right)-\tfrac{3}{2}x\left(-\tfrac{2}{3}x+2\right)^2-0 \ dx$$

$$= \int_0^3 -4x^2+12x+\tfrac{4}{3}x^3-4x^2-\tfrac{3}{2}x\left[\tfrac{4}{9}x^2-\tfrac{8}{3}x+4\right] dx$$

$$= \int_0^3 \cancel{-4x^2}+12x+\tfrac{4}{3}x^3-4x^2-\tfrac{2}{3}x^3+\cancel{4x^2}-6x \ dx$$

$$= \int_0^3 6x-4x^2+\tfrac{2}{3}x^3 \ dx$$

$$= \Big[3x^2-\tfrac{4}{3}x^3+\tfrac{1}{6}x^4\Big]_0^3$$

$$= 27-36+\tfrac{27}{2}$$

$$= \tfrac{9}{2}$$

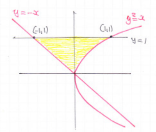
Question 32

$$I = \int_{0}^{1} \int_{-y}^{y^2} 84x^2 \ \ dx\,dy \, .$$

a) Evaluate the above double integral.

b) Verify the answer of part **(a)** by reversing the order of integration.

$\boxed{11}$



a)

$$\int_{y=0}^{1}\int_{x=-y}^{x=y^2} 84x^2 \, dx\,dy$$

$$= \int_{y=0}^{1} \left[28x^3 \right]_{x=-y}^{x=y^2} dy$$

$$= \int_{0}^{1} 28y^6 + 28y^3 \ dy$$

$$= \left[4y^7 + 7y^4 \right]_{0}^{1} = 11$$

b) REVERSING THE ORDER NEEDS TO BE DONE AS TWO SEPARATE INTEGRALS

$$\int_{y=0}^{1}\int_{x=-y}^{x=y^2} 84x^2 \, dx\,dy = \int_{x=-1}^{0}\int_{y=-x}^{y=1} 84x^2 \, dy\,dx + \int_{x=0}^{1}\int_{y=+\sqrt{x}}^{y=1} 84x^2 \, dy\,dx$$

$$= \int_{x=-1}^{0} \left[84x^2 y \right]_{y=-x}^{y=1} dx + \int_{x=0}^{1} \left[84x^2 y \right]_{y=+\sqrt{x}}^{y=1}$$

$$= \int_{-1}^{0} 84x^2 + 84x^3 \ dx + \int_{0}^{1} 84x^2 - 84x^{\frac{5}{2}}$$

$$= \left[28x^3 + 21x^4 \right]_{-1}^{0} + \left[28x^3 - 24x^{\frac{7}{2}} \right]_{0}^{1}$$

$$= \left[0 - (-28+21) \right] + \left[28 - 24 \right]$$

$$= 11$$

Question 33

A surface has equation

$$z \equiv y^3 - xy \, .$$

A cuboid of infinite length is generated. This cuboid intersects the x-y plane at the points with coordinates

$$(1,0,0)\,,\ (3,0,0)\,,\ (3,2,0) \text{ and } (1,2,0)\,.$$

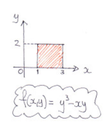
Show clearly that the volume of the finite region inside the cuboid for which

$$z \geq 0 \text{ and } z \leq y^3 - xy\,,$$

is equal to the volume of the finite region inside the cuboid for which

$$z \leq 0 \text{ and } z \geq y^3 - xy\,.$$

$\boxed{0}$



$$\int_{y=0}^{2}\int_{x=1}^{3} y^3 - xy \, dx\, dy = \int_{y=0}^{2} \left[xy^3 - \frac{1}{2}yx^2 \right]_{x=1}^{3}$$

$$= \int_{y=0}^{2} \left(3y^3 - \frac{9}{2}y\right) - \left(y^3 - \frac{1}{2}y\right) dy = \int_0^2 2y^3 - 4y \, dy$$

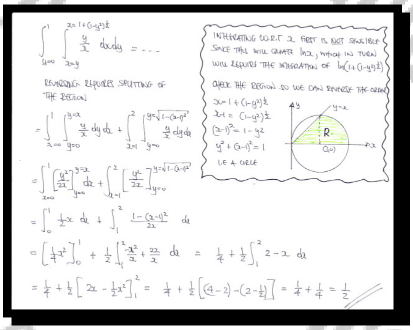
$$= \left[\frac{1}{2}y^4 - 2y^2 \right]_0^2 = (8-8) - (0) = 0$$

Question 34

Find the value of the following multiple integral

$$\int_0^1 \int_y^{1+\sqrt{1-y^2}} \frac{y}{x} \, dx \, dy \, .$$

$$\boxed{\frac{1}{2}}$$



$$\int_{y=0}^{1} \int_{x=y}^{x=1+(1-y^2)^{\frac{1}{2}}} \frac{y}{x} \, dx \, dy = \ldots$$

INTEGRATING W.R.T x FIRST IS NOT SENSIBLE SINCE THIS WILL CREATE $\ln x$, WHICH IN TURN WILL REQUIRE THE INTEGRATION OF $\ln(1+(1-y^2)^{\frac{1}{2}})$

CHECK THE REGION SO WE CAN REVERSE THE ORDER

$x = 1 + (1-y^2)^{\frac{1}{2}}$

$x - 1 = (1-y^2)^{\frac{1}{2}}$

$(x-1)^2 = 1 - y^2$

$y^2 + (x-1)^2 = 1$

I.E A CIRCLE

REVERSING REQUIRES SPLITTING OF THE REGION

$$= \int_{x=0}^{1} \int_{y=0}^{y=x} \frac{y}{x} \, dy \, dx + \int_{x=1}^{2} \int_{y=0}^{y=\sqrt{1-(x-1)^2}} \frac{y}{x} \, dy \, dx$$

$$= \int_{x=0}^{1} \left[\frac{y^2}{2x} \right]_{y=0}^{y=x} dx + \int_{x=1}^{2} \left[\frac{y^2}{2x} \right]_{y=0}^{y=\sqrt{1-(x-1)^2}}$$

$$= \int_0^1 \frac{1}{2}x \, dx + \int_1^2 \frac{1-(x-1)^2}{2x} \, dx$$

$$= \left[\frac{1}{4}x^2 \right]_0^1 + \frac{1}{2} \int_1^2 \frac{-x^2}{x} + \frac{2x}{x} \, dx = \frac{1}{4} + \frac{1}{2} \int_1^2 2 - x \, dx$$

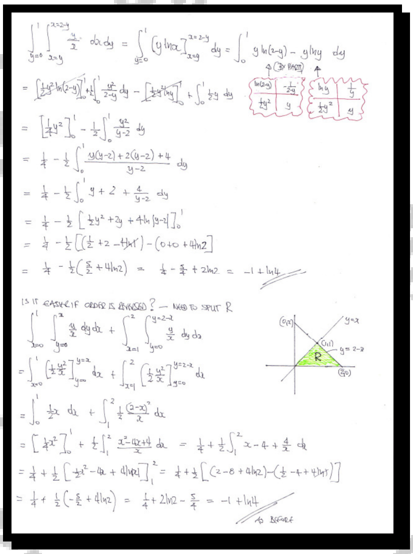
$$= \frac{1}{4} + \frac{1}{2} \left[2x - \frac{1}{2}x^2 \right]_1^2 = \frac{1}{4} + \frac{1}{2} \left[(4-2) - (2-\frac{1}{2}) \right] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Question 35

Find the value of the following multiple integral

$$\int_0^1 \int_y^{2-y} \frac{y}{x} \, dx \, dy \, .$$

$$\boxed{-1 + \ln 4}$$



$$\int_{y=0}^{1} \int_{x=y}^{x=2-y} \frac{y}{x} \, dx \, dy = \int_{y=0}^{1} \left[y \ln x \right]_{x=y}^{x=2-y} dy = \int_0^1 y \ln(2-y) - y \ln y \, dy$$

(BY PARTS)

$\ln(2-y)$	$-\frac{1}{2-y}$
$\frac{1}{2}y^2$	y

$\ln y$	$\frac{1}{y}$
$\frac{1}{2}y^2$	y

$$= \left[\cancel{\tfrac{1}{2}y^2 \ln(2-y)} \right]_0^1 + \frac{1}{2} \int_0^1 \frac{y^2}{2-y} \, dy - \left[\cancel{\tfrac{1}{2}y^2 \ln y} \right]_0^1 + \int_0^1 \frac{1}{2}y \, dy$$

$$= \left[\frac{1}{4}y^2 \right]_0^1 - \frac{1}{2} \int_0^1 \frac{y^2}{y-2} \, dy$$

$$= \frac{1}{4} - \frac{1}{2} \int_0^1 \frac{y(y-2) + 2(y-2) + 4}{y-2} \, dy$$

$$= \frac{1}{4} - \frac{1}{2} \int_0^1 y + 2 + \frac{4}{y-2} \, dy$$

$$= \frac{1}{4} - \frac{1}{2} \left[\frac{1}{2}y^2 + 2y + 4 \ln|y-2| \right]_0^1$$

$$= \frac{1}{4} - \frac{1}{2} \left[(\frac{1}{2} + 2 - \cancel{4\ln 1}) - (0 + 0 + 4\ln 2) \right]$$

$$= \frac{1}{4} - \frac{1}{2} \left(\frac{5}{2} + 4\ln 2 \right) = \frac{1}{4} - \frac{5}{4} + 2\ln 2 = -1 + \ln 4$$

IS IT EASIER IF ORDER IS REVERSED? — NEED TO SPLIT R

$$\int_{x=0}^{1} \int_{y=0}^{x} \frac{y}{x} \, dy \, dx + \int_{x=1}^{2} \int_{y=0}^{y=2-x} \frac{y}{x} \, dy \, dx$$

$$= \int_{x=0}^{1} \left[\frac{1}{2} \frac{y^2}{x} \right]_{y=0}^{y=x} dx + \int_{x=1}^{2} \left[\frac{1}{2} \frac{y^2}{x} \right]_{y=0}^{y=2-x} dx$$

$$= \int_0^1 \frac{1}{2}x \, dx + \int_1^2 \frac{1}{2} \frac{(2-x)^2}{x} \, dx$$

$$= \left[\frac{1}{4}x^2 \right]_0^1 + \frac{1}{2} \int_1^2 \frac{x^2 - 4x + 4}{x} \, dx = \frac{1}{4} + \frac{1}{2} \int_1^2 x - 4 + \frac{4}{x} \, dx$$

$$= \frac{1}{4} + \frac{1}{2} \left[\frac{1}{2}x^2 - 4x + 4\ln|x| \right]_1^2 = \frac{1}{4} + \frac{1}{2} \left[(2 - 8 + 4\ln 2) - (\frac{1}{2} - 4 + \cancel{4\ln 1}) \right]$$

$$= \frac{1}{4} + \frac{1}{2} \left(-\frac{5}{2} + 4\ln 2 \right) = \frac{1}{4} + 2\ln 2 - \frac{5}{4} = -1 + \ln 4$$

AS BEFORE

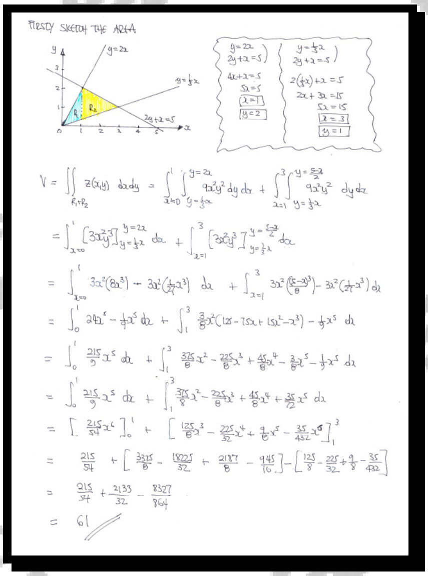
Question 36

The cross section of a right prism is the finite region in the x-y plane, defined by the following inequalities.

$$y \le 2x \ , \quad y \ge \frac{1}{3}x \quad \text{and} \quad 2y + x \le 5 .$$

Show that the volume of the prism, for which $0 \le z \le 9x^2y^2$, is 61 cubic units.

proof



Question 37

A solid is bounded by the surfaces

$$y = x^2, \quad y = x, \quad z = 1 \quad \text{and} \quad z = 2.$$

Find the mass of the solid if its density is given by $\rho(z) = z^3$.

$$\boxed{\frac{5}{8}}$$

$$\text{MASS} = \int_V \rho(x,y,z)\, dV$$

$$= \int_{z=1}^{2}\int_{x=0}^{1}\int_{y=x^2}^{y=x} z^3 \, dy\, dx\, dz$$

$$= \int_{z=1}^{2}\int_{x=0}^{1} \left[z^3 y\right]_{y=x^2}^{y=x} dx\, dz$$

$$= \int_{z=1}^{2}\int_{x=0}^{1} z^3 x - z^3 x^2 \, dx\, dz$$

$$= \int_{z=1}^{2} \left[\tfrac{1}{2}z^3x^2 - \tfrac{1}{3}z^3x^3\right]_{x=0}^{1} dz$$

$$= \int_1^2 \tfrac{1}{2}z^3 - \tfrac{1}{3}z^3 \, dz$$

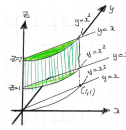
$$= \int_1^2 \tfrac{1}{6}z^3 \, dz$$

$$= \left[\tfrac{1}{24}z^4\right]_1^2$$

$$= \frac{16}{24} - \frac{1}{24}$$

$$= \frac{15}{24}$$

$$= \frac{5}{8}$$



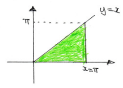
Question 38

Evaluate the integral

$$\int_0^{\pi} \int_y^{\pi} \frac{\sin x}{x} \; dxdy .$$

$\boxed{2}$

$$\int_{y=0}^{\pi} \int_{x=y}^{\pi} \frac{\sin x}{x} \, dx \, dy \; \ldots$$ REVERSE THE ORDER OF INTEGRATION, SO SKETCH THE REGION



$$= \int_{x=0}^{\pi} \int_{y=0}^{y=x} \frac{\sin x}{x} \, dy \, dx$$

$$= \int_{x=0}^{\pi} \left[\frac{\sin x}{x} y \right]_{y=0}^{y=x} dx$$

$$= \int_{x=0}^{\pi} \frac{\sin x}{\cancel{x}} \cdot \cancel{x} - 0 \; dx$$

$$= \int_0^{\pi} \sin x \; dx = \left[-\cos x \right]_0^{\pi} = \left[\cos x \right]_{\pi}^{0}$$

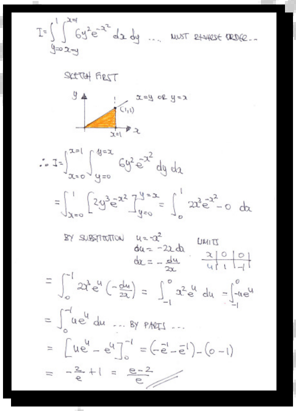
$$= 1 - (-1) = 2$$

Question 39

Find the value of I in exact simplified form.

$$I = \int_0^1 \int_y^1 6y^2 \mathrm{e}^{-x^2} \, dx\, dy .$$

$$\boxed{\frac{\mathrm{e}-2}{\mathrm{e}}}$$

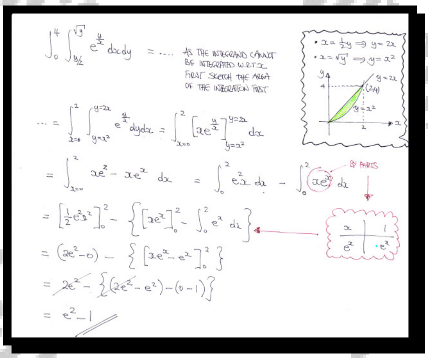


Question 40

Find the value of the following integral.

$$\int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} \mathrm{e}^{\frac{y}{x}} \, dx\, dy .$$

$$\boxed{\mathrm{e}^2 - 1}$$



Question 41

The finite region R is bounded by the curve with Cartesian equation

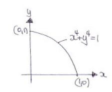
$$x^4 + y^4 = 1, \quad x \geq 0, \quad y \geq 0.$$

Find the value of

$$\iint_R x^3 y^3 \sqrt{1 - x^4 - y^4} \, dx \, dy .$$

$$\boxed{\frac{1}{60}}$$

$$\iint_R x^3 y^3 \sqrt{1-x^4-y^4}\, dx\, dy$$



$$= \int_{y=0}^{1} \int_{x=0}^{x=(1-y^4)^{\frac{1}{4}}} x^3 y^3 (1-x^4-y^4)^{\frac{1}{2}}\, dx\, dy$$

$$= \int_{y=0}^{1} \left[-\tfrac{1}{6} y^3 (1-x^4-y^4)^{\frac{3}{2}} \right]_{x=0}^{x=(1-y^4)^{\frac{1}{4}}} dy$$

$$= \int_{y=0}^{1} \tfrac{1}{6} y^3 \left[(1-x^4-y^4)^{\frac{3}{2}} \right]_{x=(1-y^4)^{\frac{1}{4}}}^{x=0} dy = \int_{y=0}^{1} \tfrac{1}{6} y^3 \left[(1-y^4)^{\frac{3}{2}} - (1-(1-y^4)-y^4)^{\frac{3}{2}} \right] dy$$

$$= \int_0^1 \tfrac{1}{6} y^3 \left[(1-y^4)^{\frac{3}{2}} - (0) \right] dy = \int_0^1 \tfrac{1}{6} y^3 (1-y^4)^{\frac{3}{2}}\, dy$$

$$= \left[-\tfrac{1}{60} (1-y^4)^{\frac{5}{2}} \right]_0^1 = \tfrac{1}{60} \left[(1-y^4)^{\frac{5}{2}} \right]_1^0 = \tfrac{1}{60} \left[1 - 0 \right] = \tfrac{1}{60}$$

Question 42

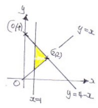
The finite region R in the x-y plane is bounded by the straight lines with equations

$$y = x \ , \quad x = 1 \quad \text{and} \quad y = 4 - x .$$

Determine an exact simplified value for

$$\int_R x(x+y)\mathrm{e}^{x^2} \, dx\, dy .$$

$$\boxed{\mathrm{e}^4 - 4\mathrm{e}}$$



$$\int_{x=1}^{x=2}\int_{y=x}^{y=4-x} x(x+y)e^{x^2}\,dy\,dx$$

$$= \int_{x=1}^{2} \left[xe^{x^2}\left[xy + \tfrac{1}{2}y^2\right]_{y=x}^{y=4-x} \right] dx$$

$$= \int_1^2 xe^{x^2}\left[x(4-x) + \tfrac{1}{2}(4-x)^2 - x^2 - \tfrac{1}{2}x^2\right] dx$$

$$= \int_1^2 xe^{x^2}\left[\cancel{4x} - x^2 + 8 \cancel{-4x} + \cancel{\tfrac{1}{2}x^2} - x^2 \cancel{-\tfrac{1}{2}x^2}\right] dx$$

$$= \int_1^2 xe^{x^2}(8-x^2)\,dx = \int_1^2 8xe^{x^2} - 2x^3e^{x^2}\,dx$$

$$= \int_1^2 8xe^{x^2}\,dx - \int_1^2 2x^3e^{x^2}\,dx \longleftarrow$$

SUBSTITUTION

$u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

x	1	2
u	1	4

$$= \left[4e^{x^2}\right]_1^2 - \int_1^4 2x^3e^u\,\frac{du}{2x}$$

$$= (4e^4 - 4e) - \int_1^4 x^2e^u\,du$$

$$= 4e^4 - 4e - \int_1^4 ue^u\,du \longleftarrow \text{BY PARTS}$$

$$= 4e^4 - 4e - \left[ue^u - e^u\right]_1^4$$

$$= 4e^4 - 4e - \left[(4e^4 - e^4) \cancel{-(e-e)}\right]$$

$$= 4e^4 - 4e - 3e^4$$

$$= e^4 - 4e$$

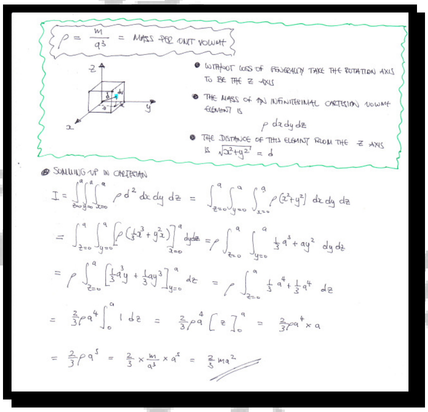
Question 43

A uniform solid cube, of mass m and side length a, is free to rotate about one of its edges, L.

Use multiple integration in Cartesian coordinates, to find the moment of inertia of this cube about L, giving the answer in terms of m and a.

*You may **not** use any standard rules or standard results about moments of inertia in this question apart from the definition of moment of inertia.*

$$\boxed{\frac{2}{3}ma^2}$$



$\rho = \dfrac{m}{a^3}$ = MASS PER UNIT VOLUME

- WITHOUT LOSS OF GENERALITY TAKE THE ROTATION AXIS TO BE THE z AXIS
- THE MASS OF AN INFINITESIMAL CARTESIAN VOLUME ELEMENT IS

$$\rho \, dx \, dy \, dz$$

- THE DISTANCE OF THIS ELEMENT FROM THE z AXIS IS $\sqrt{x^2+y^2} = d$

- SUMMING UP IN CARTESIAN

$$I = \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} \rho d^2 \, dx \, dy \, dz = \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} \rho (x^2+y^2) \, dx \, dy \, dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a}\left[\rho\left(\tfrac{1}{3}x^3 + y^2 x\right)\right]_{x=0}^{a} dy\,dz = \rho\int_{z=0}^{a}\int_{y=0}^{a} \tfrac{1}{3}a^3 + ay^2 \; dy\,dz$$

$$= \rho\int_{z=0}^{a}\left[\tfrac{1}{3}a^3 y + \tfrac{1}{3}ay^3\right]_{y=0}^{a} dz = \rho\int_{z=0}^{a} \tfrac{1}{3}a^4 + \tfrac{1}{3}a^4 \; dz$$

$$= \tfrac{2}{3}\rho a^4 \int_0^a 1 \, dz = \tfrac{2}{3}\rho a^4 \left[z\right]_0^a = \tfrac{2}{3}\rho a^4 \times a$$

$$= \tfrac{2}{3}\rho a^5 = \tfrac{2}{3} \times \frac{m}{a^3} \times a^5 = \tfrac{2}{3}ma^2$$

Question 44

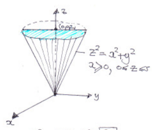
The finite region V is defined as

$$z^2 = x^2 + y^2 \ , \quad x \geq 0 \, , \quad 0 \leq z \leq 2 \, .$$

Use Cartesian integration to determine the value of

$$\int_V 3x \ dV \, .$$

$\boxed{8}$



THE LIMITS ARE

- FROM $x=0$ TO $x=+\sqrt{z^2-y^2}$
- FROM $y=-z$ TO $y=z$
- FROM $z=0$ TO $z=2$

$$\int_{z=0}^{2}\int_{y=-z}^{y=z}\int_{x=0}^{x=\sqrt{z^2-y^2}} 3x \ dx\,dy\,dz = \int_{z=0}^{2}\int_{y=-z}^{y=z}\left[\frac{3}{2}x^2\right]_{x=0}^{x=\sqrt{z^2-y^2}} dy\,dz$$

$$= \int_{z=0}^{2}\int_{y=-z}^{y=z}\frac{3}{2}\left(z^2-y^2\right) dy\,dz = \dots \text{ EVEN INTEGRAND IN } y \dots$$

$$= \int_{z=0}^{2}\int_{y=0}^{z} 3\left(z^2-y^2\right) dy\,dz = \int_{z=0}^{2}\left[3\left(z^2y-\frac{1}{3}y^3\right)\right]_{y=0}^{y=z} dz = \int_0^2\left[3z^2y-y^3\right]_{y=0}^{y=z} dz$$

$$= \int_0^2 3z^3-z^3 \ dz = \int_0^2 2z^3 \ dz = \left[\frac{1}{2}z^4\right]_0^2 = 8$$

Question 45

The finite region R in the x-y plane is defined as

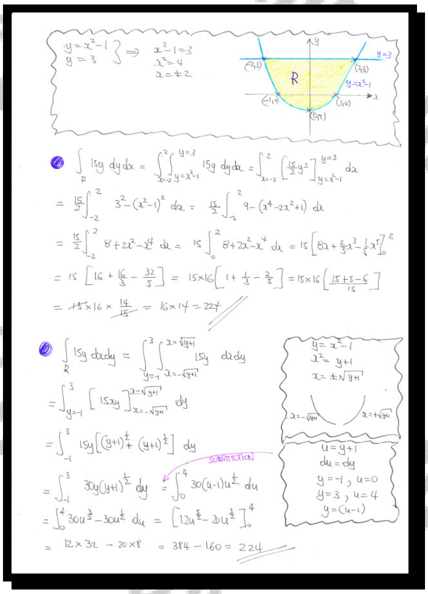
$$R = \left\{ (x, y) : y \le 3 \cap y \ge x^2 - 1 \right\}.$$

a) Evaluate, without reversing the order of integration

$$\int_R 15y \ dy\,dx .$$

b) Verify the answer of part **(a)**, by carrying out the integration in reverse order.

$\boxed{\text{both yield } 224}$



Question 46

A thin plate occupies the region in the x-y plane with equation

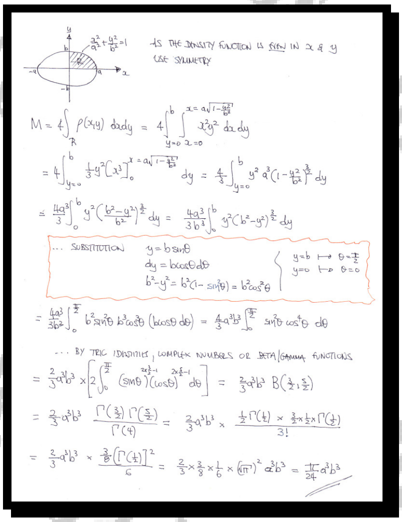
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The mass per unit area of the plate ρ, is given by

$$\rho(x,y)=x^2y^2.$$

Find a simplified expression for the mass of the plate.

$$\boxed{\frac{\pi a^3 b^3}{24}}$$



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

AS THE DENSITY FUNCTION IS EVEN IN x & y USE SYMMETRY

$$M = 4\int_R \rho(x,y)\,dx\,dy = 4\int_{y=0}^{b}\int_{x=0}^{x=a\sqrt{1-\frac{y^2}{b^2}}} x^2y^2\,dx\,dy$$

$$= 4\int_{y=0}^{b} \tfrac{1}{3}y^2\left[x^3\right]_0^{x=a\sqrt{1-\frac{y^2}{b^2}}}\,dy = \frac{4}{3}\int_{y=0}^{b} y^2a^3\left(1-\frac{y^2}{b^2}\right)^{\frac{3}{2}}dy$$

$$= \frac{4a^3}{3}\int_0^b y^2\left(\frac{b^2-y^2}{b^2}\right)^{\frac{3}{2}}dy = \frac{4a^3}{3b^3}\int_0^b y^2(b^2-y^2)^{\frac{3}{2}}dy$$

... SUBSTITUTION $y=b\sin\theta$, $dy=b\cos\theta\,d\theta$, $b^2-y^2=b^2(1-\sin^2\theta)=b^2\cos^2\theta$; $y=b \mapsto \theta=\frac{\pi}{2}$, $y=0 \mapsto \theta=0$

$$= \frac{4a^3}{3b^3}\int_0^{\frac{\pi}{2}} b^2\sin^2\theta\, b^3\cos^3\theta\,(b\cos\theta\,d\theta) = \frac{4}{3}a^3b^3\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^4\theta\,d\theta$$

... BY TRIG IDENTITIES, COMPLEX NUMBERS OR BETA/GAMMA FUNCTIONS

$$= \frac{2}{3}a^3b^3\times\left[2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2\times\frac{3}{2}-1}(\cos\theta)^{2\times\frac{5}{2}-1}d\theta\right] = \frac{2}{3}a^3b^3\,B\left(\tfrac{3}{2},\tfrac{5}{2}\right)$$

$$= \frac{2}{3}a^3b^3\,\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(4)} = \frac{2}{3}a^3b^3\times\frac{\frac{1}{2}\Gamma(\frac{1}{2})\times\frac{3}{2}\times\frac{1}{2}\times\Gamma(\frac{1}{2})}{3!}$$

$$= \frac{2}{3}a^3b^3\times\frac{\frac{3}{8}\left[\Gamma(\frac{1}{2})\right]^2}{6} = \frac{2}{3}\times\frac{3}{8}\times\frac{1}{6}\times(\sqrt{\pi})^2a^3b^3 = \frac{\pi}{24}a^3b^3$$

Question 47

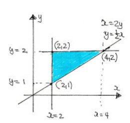
Find the value of I in exact simplified form.

$$I = \int_1^2 \int_2^{2y} \frac{16y}{\left(16 - x^2\right)^{\frac{3}{2}}} \, dx \, dy .$$

$\boxed{}$, $\boxed{\frac{2\pi}{3}}$

- LOOKING AT THE INTEGRAND, IT LOOKS LIKE IT MIGHT BE EASIER TO INTEGRATE WITH RESPECT TO y FIRST
- SKETCH THE REGION IN ORDER TO "REVERSE THE LIMITS"

$$\Rightarrow I = \int_{y=1}^{y=2} \int_{x=2}^{x=2y} \frac{16y}{(16-x^2)^{\frac{3}{2}}} \, dx \, dy$$



$$\Rightarrow I = \int_{x=2}^{x=4} \int_{y=\frac{1}{2}x}^{y=2} \frac{16y}{(16-x^2)^{\frac{3}{2}}} \, dy \, dx$$

$$\Rightarrow I = \int_{x=2}^{x=4} \left[\frac{8y^2}{(16-x^2)^{3/2}} \right]_{y=\frac{1}{2}x}^{y=2} dx$$

$$\Rightarrow I = \int_{x=2}^{x=4} \frac{32 - 8\left(\frac{1}{2}x\right)^2}{(16-x^2)^{3/2}} \, dx = \int_2^4 \frac{32 - 2x^2}{(16-x^2)^{3/2}} \, dx$$

$$\Rightarrow I = \int_2^4 \frac{2(16-x^2)}{(16-x^2)^{3/2}} \, dx = \int_2^4 \frac{2}{\sqrt{16-x^2}} \, dx$$

- THIS IS NOW A STANDARD INTEGRAL (OR USE THE SUBSTITUTION $x = 4\sin\theta$)

$$\Rightarrow I = \left[2\arcsin\left(\frac{x}{4}\right) \right]_2^4 = 2\arcsin 1 - 2\arcsin \frac{1}{2}$$

$$\Rightarrow I = 2 \times \frac{\pi}{2} - 2 \times \frac{\pi}{6} = \pi - \frac{\pi}{3}$$

$$\Rightarrow I = \frac{2\pi}{3}$$

Question 48

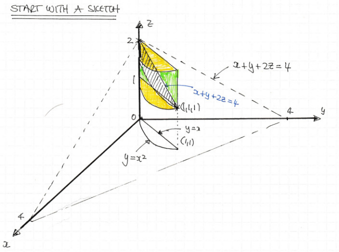
A solid S is bounded by the surfaces

$$y = x^2, \quad y = x, \quad z = 1 \quad \text{and} \quad z = 2.$$

Find the volume of the finite region bounded by S and the plane with equation

$$x + y + 2z = 4.$$

$\boxed{\frac{11}{120}}$


START WITH A SKETCH

THE PLANE $x+y+2z=4$ CUTS THE SOLID INTO TWO CONGRUENT REGIONS

HENCE WE HAVE

$$V = \int_{x=0}^{1}\int_{y=x^2}^{x}\int_{z=1}^{z=\frac{1}{2}(4-x-y)} 1 \, dz\,dy\,dx = \int_{x=0}^{x=1}\int_{y=x^2}^{y=x}\Big[z\Big]_{z=1}^{z=\frac{1}{2}(4-x-y)}$$

$$= \int_{x=0}^{x=1}\int_{y=x^2}^{y=x} \tfrac{1}{2}(4-x-y) - 1 \, dy\,dx = \int_{x=0}^{1}\int_{y=x^2}^{y=x} 1 - \tfrac{1}{2}x - \tfrac{1}{2}y \, dy\,dx$$

$$= \int_{x=0}^{1}\Big[y - \tfrac{1}{2}xy - \tfrac{1}{4}y^2\Big]_{y=x^2}^{y=x} dx = \int_0^1 \left(x - \tfrac{1}{2}x^2 - \tfrac{1}{4}x^2\right) - \left(x^2 - \tfrac{1}{2}x^3 - \tfrac{1}{4}x^4\right) dx$$

$$= \int_0^1 x - \tfrac{7}{4}x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^4 \, dx = \Big[\tfrac{1}{2}x^2 - \tfrac{7}{12}x^3 + \tfrac{1}{8}x^4 + \tfrac{1}{20}x^5\Big]_0^1$$

$$= \tfrac{1}{2} - \tfrac{7}{12} + \tfrac{1}{8} + \tfrac{1}{20} = \frac{60-70+15+6}{120} = \frac{11}{120}$$

Question 49

A thin plate occupies the region in the x-y plane defined by the inequalities

$$0 \le x \le 2 \quad \text{and} \quad 0 \le y \le 2x .$$

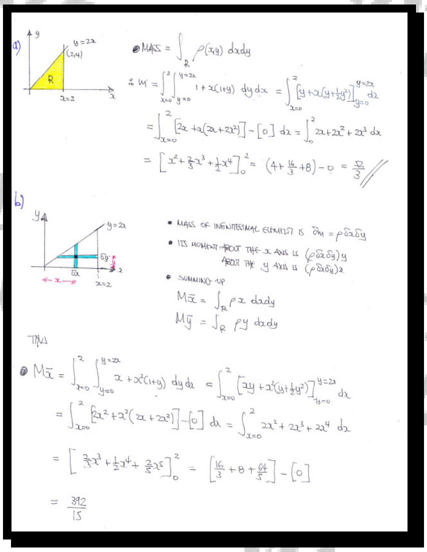
The mass per unit area of the plate ρ, is given by

$$\rho(x, y) = 1 + x(1 + y) .$$

a) Find the mass of the plate.

b) Determine the coordinates of the centre of mass of the plate.

$$\boxed{m = \frac{52}{3}}, \quad \boxed{(\bar{x}, \bar{y}) = \left(\frac{98}{65}, \frac{114}{65}\right)}$$



a)

• MASS $= \int_R \rho(x,y)\, dx\, dy$

$\therefore M = \int_{x=0}^{2} \int_{y=0}^{y=2x} 1 + x(1+y)\, dy\, dx = \int_{x=0}^{2} \left[y + x\left(y + \frac{1}{2}y^2\right) \right]_{y=0}^{y=2x} dx$

$= \int_{x=0}^{2} \left[2x + x(2x + 2x^2)\right] - [0]\, dx = \int_0^2 2x + 2x^2 + 2x^3\, dx$

$= \left[x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4 \right]_0^2 = \left(4 + \frac{16}{3} + 8\right) - 0 = \frac{52}{3}$

b)

• MASS OF INFINITESIMAL ELEMENT IS $\delta m = \rho\, \delta x\, \delta y$

• ITS MOMENT ABOUT THE x AXIS IS $(\rho\, \delta x\, \delta y)y$

ABOUT THE y AXIS IS $(\rho\, \delta x\, \delta y)x$

• SUMMING UP

$M\bar{x} = \int_R \rho x\, dx\, dy$

$M\bar{y} = \int_R \rho y\, dx\, dy$

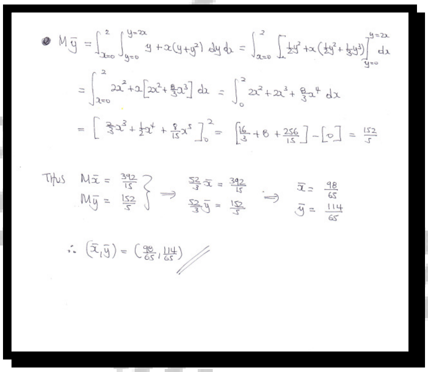
THUS

• $M\bar{x} = \int_{x=0}^{2} \int_{y=0}^{y=2x} x + x^2(1+y)\, dy\, dx = \int_{x=0}^{2} \left[xy + x^2\left(y + \frac{1}{2}y^2\right) \right]_{y=0}^{y=2x} dx$

$= \int_{x=0}^{2} \left[2x^2 + x^2(2x + 2x^2)\right] - [0]\, dx = \int_{x=0}^{2} 2x^2 + 2x^3 + 2x^4\, dx$

$= \left[\frac{2}{3}x^3 + \frac{1}{2}x^4 + \frac{2}{5}x^5 \right]_0^2 = \left[\frac{16}{3} + 8 + \frac{64}{5}\right] - [0]$

$= \frac{392}{15}$



• $M\bar{y} = \int_{x=0}^{2} \int_{y=0}^{y=2x} y + x(y + y^2)\, dy\, dx = \int_{x=0}^{2} \left[\frac{1}{2}y^2 + x\left(\frac{1}{2}y^2 + \frac{1}{3}y^3\right) \right]_{y=0}^{y=2x} dx$

$= \int_{x=0}^{2} 2x^2 + x\left[2x^2 + \frac{8}{3}x^3\right] dx = \int_0^2 2x^2 + 2x^3 + \frac{8}{3}x^4\, dx$

$= \left[\frac{2}{3}x^3 + \frac{1}{2}x^4 + \frac{8}{15}x^5 \right]_0^2 = \left[\frac{16}{3} + 8 + \frac{256}{15}\right] - [0] = \frac{152}{5}$

Thus $\left. \begin{array}{l} M\bar{x} = \frac{392}{15} \\ M\bar{y} = \frac{152}{5} \end{array} \right\} \Rightarrow \begin{array}{l} \frac{52}{3}\bar{x} = \frac{392}{15} \\ \frac{52}{3}\bar{y} = \frac{152}{5} \end{array} \Rightarrow \begin{array}{l} \bar{x} = \frac{98}{65} \\ \bar{y} = \frac{114}{65} \end{array}$

$\therefore (\bar{x}, \bar{y}) = \left(\frac{98}{65}, \frac{114}{65}\right)$

Question 50

The finite region R in the x-y plane is defined as the rectilinear triangle with vertices at the points with coordinates

$$(0,0),\ (0,1) \text{ and } (1,0).$$

Determine an exact simplified value for

$$\int_R \frac{1}{\sqrt[4]{x^3y^3}}\ dx\,dy.$$

[You may leave the final answer in terms of π and Gamma functions]

$$\boxed{\frac{2}{\sqrt{\pi}}\left[\Gamma\left(\tfrac{1}{4}\right)\right]^2}$$

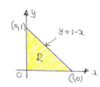
$$\int_{x=0}^{1}\int_{y=0}^{y=1-x} \frac{1}{\sqrt[4]{x^3y^3}}\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{y=1-x} x^{-\frac{3}{4}}y^{-\frac{3}{4}}\,dy\,dx$$

$$= \int_{x=0}^{1}\left[4x^{-\frac{3}{4}}y^{\frac{1}{4}}\right]_0^{1-x}dx = \int_{x=0}^{1} 4x^{-\frac{3}{4}}(1-x)^{\frac{1}{4}}\,dx$$

$$= 4\int_0^1 x^{\frac{1}{4}-1}(1-x)^{\frac{5}{4}-1}\,dx = 4B\left(\tfrac{1}{4},\tfrac{5}{4}\right) = 4\left[\frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{2})}\right]$$

$$= 4\left[\frac{\Gamma(\frac{1}{4})\,\frac{1}{4}\Gamma(\frac{1}{4})}{\frac{1}{2}\Gamma(\frac{1}{2})}\right] = \frac{2\left[\Gamma(\frac{1}{4})\right]^2}{\Gamma(\frac{1}{2})} = \frac{2}{\sqrt{\pi}}\left[\Gamma(\tfrac{1}{4})\right]^2$$



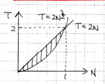
Question 51

$$I = \int_0^1 \int_{2N}^{2N^{\frac{3}{2}}} \frac{N^2 \exp\left[-\frac{1}{2}(T-2)^2\right]}{T^2} \, dT \, dN .$$

Determine, in exact simplified form, the value of I.

$\boxed{}$, $\boxed{I = \frac{1}{24}\left(e^{-2} - 1\right)}$

START WITH A SKETCH OF THE INTEGRATION REGION



NOTE THAT THE "DIRECTION" IN T IS DOWNWARDS SO THIS WILL GENERATE A MINUS, UNLESS WHEN WE REVERSE THE ORDER WE ALSO GO "AGAINST THE FLOW" WITH ONE OF THE INTEGRATIONS

REVERSING THE ORDER WE HAVE

$$\int_0^1 \int_{2N}^{2N^{\frac{3}{2}}} \frac{N^2}{T^2} e^{-\frac{1}{2}(T-2)^2} \, dT \, dN$$

$$= \int_{T=0}^{T=2} \int_{N=\frac{T}{2}}^{N=\left(\frac{T}{2}\right)^{\frac{2}{3}}} -\frac{N^2}{T^2} e^{-\frac{1}{2}(T-2)^2} \, dN \, dT$$

$$= \int_0^2 \left[-\frac{N^3}{3T^2} e^{-\frac{1}{2}(T-2)^2}\right]_{N=\frac{T}{2}}^{N=\left(\frac{T}{2}\right)^{\frac{2}{3}}} dT$$

$$= \int_0^2 -\frac{1}{3T^2}\left(\left(\frac{T}{2}\right)^2 - \left(\frac{T}{2}\right)^3\right) e^{-\frac{1}{2}(T-2)^2} \, dT$$

$$= \int_0^2 -\frac{1}{3T^2}\left(\frac{T^2}{4} - \frac{T^3}{8}\right) e^{-\frac{1}{2}(T-2)^2} \, dT$$

$$= \int_0^2 \left(\frac{T}{24} - \frac{1}{12}\right) e^{-\frac{1}{2}(T-2)^2} \, dT$$

$$= \int_0^2 \frac{1}{24}(T-2) e^{-\frac{1}{2}(T-2)^2} \, dT$$

- $T = 2N$
  $N = \frac{1}{2}T$
- $T = 2N^{\frac{3}{2}}$
  $\frac{1}{2}T = N^{\frac{3}{2}}$
  $\left(\frac{1}{2}T\right)^{\frac{2}{3}} = N$
  $N = \left(\frac{T}{2}\right)^{\frac{2}{3}}$

FINALLY INTEGRATE BY INSPECTION OR SUBSTITUTION

$u = -\frac{1}{2}(T-2)^2$

$\frac{du}{dT} = -(T-2)$

$dT = \frac{du}{-(T-2)}$

$T = 0 \mapsto u = -2$

$T = 2 \mapsto u = 0$

$$\ldots = \int_{-2}^{0} \frac{1}{24}\cancel{(T-2)} e^u \frac{du}{-\cancel{(T-2)}}$$

$$= \int_{-2}^{0} -\frac{1}{24} e^u \, du$$

$$= \int_{0}^{-2} \frac{1}{24} e^u \, du$$

$$= \left[\frac{1}{24} e^u\right]_0^{-2}$$

$$= \frac{1}{24}\left(e^{-2} - 1\right)$$

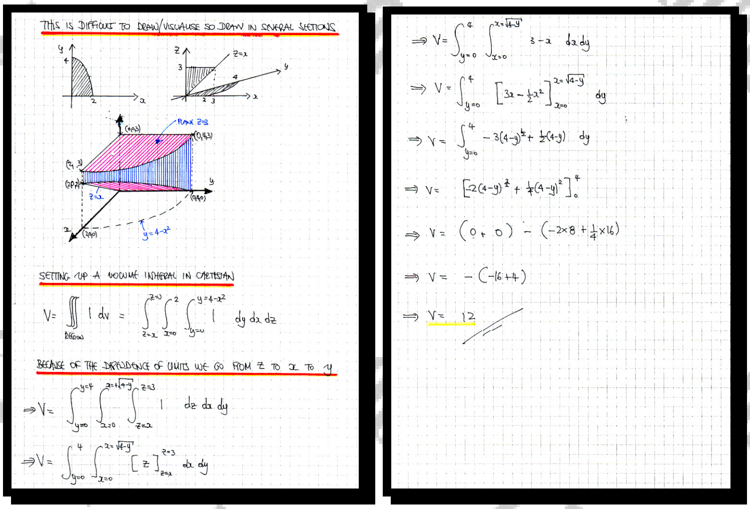
Question 52

The finite region V in the first octant, is bounded by the surfaces with equations

$$y = 4 - x^2 \ , \quad z = x \quad \text{and} \quad z = 3 \, .$$

Find the volume of the solid.

MM, $\boxed{12}$



Question 53

The finite region R is defined as

$$R = \{ (x, y, z) : 0 \le z \le 1 - |x| - |y| \}.$$

Evaluate the triple integral

$$\iiint_R xy + z^2 \; dxdydz .$$

$$\boxed{\frac{1}{15}}$$

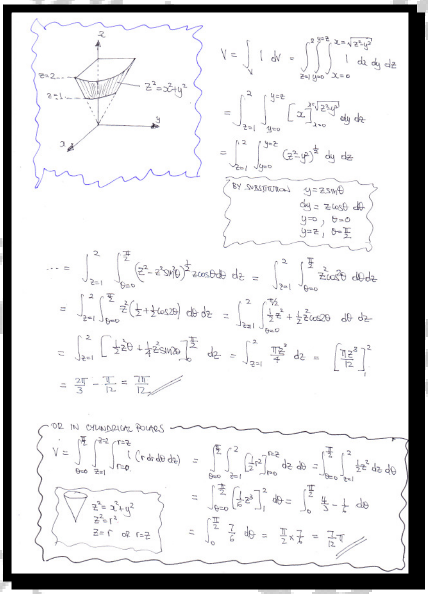
Question 54

A finite region is defined as

$$z^2 = x^2 + y^2, \; x \ge 0, \; y \ge 0, \; 1 \le z \le 2$$

Show by integration in Cartesian that the volume of this region is $\dfrac{7\pi}{12}$.

$\boxed{\text{proof}}$

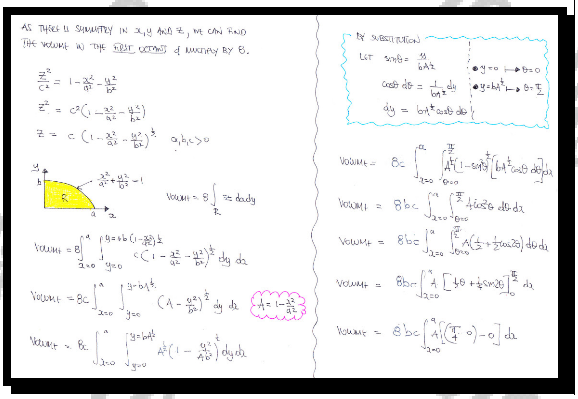


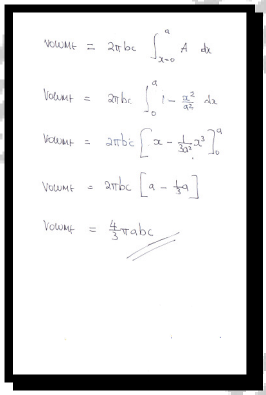
Question 55

Use integration in Cartesian coordinates to find, in exact simplified form, the volume of the ellipsoid with equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

$$\boxed{V=\frac{4}{3}\pi abc}$$





Question 56

The finite region V is defined as

$$V = \left\{ (x, y, z) : x^2 + y^2 + 4z^2 \le 1 \right\}.$$

a) Sketch and name the region V .

b) Evaluate

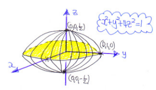
$$\iiint_V (3 - xy + 4y + z) \, dx \, dy \, dz .$$

$\boxed{2\pi}$

a) • $x^2+y^2+4z^2=1$ IS AN ELLIPSOID

$$\frac{x^2}{1^2}+\frac{y^2}{1^2}+\frac{z^2}{(\frac{1}{2})^2}=1$$

IT IS "SQUASHED" IN THE z AXIS BY SCALE FACTOR $\frac{1}{2}$



b) $\iiint_V (3-xy+4y+z)\, dV = \ldots$ LOOKING AT THE SKETCH ...

AS THE DOMAIN (VOLUME REGION) IS SYMMETRICAL IN x, y AND z, ANY ODD POWER OF x, y OR z WILL HAVE NO CONTRIBUTION

$$= \ldots \iiint_V 3 \, dV = 3 \times \text{VOLUME OF THE ELLIPSOID}$$

$$= 3 \times \frac{4}{3}\pi(abc)$$

$$\quad\quad\quad\quad 1 \;\; 1 \;\; \tfrac{1}{2}$$

$$= 4\pi \times (1 \times 1 \times \tfrac{1}{2})$$

$$= 2\pi$$

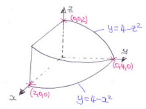
Question 57

The finite region V in the first octant, is bounded by the surfaces with equations

$$y = 4 - x^2 \quad \text{and} \quad y = 4 - z^2 .$$

Find the volume of the solid defined as the interior of V.

$\boxed{8}$



$$V = \int_{x=0}^{2} \int_{y=0}^{y=4-x^2} \int_{z=0}^{z=\sqrt{4-y}} 1 \, dz \, dy \, dx$$

$$V = \int_{x=0}^{2} \int_{y=0}^{y=4-x^2} \Big[z \Big]_{z=0}^{z=\sqrt{4-y}} dy \, dx$$

$$V = \int_{x=0}^{2} \int_{y=0}^{y=4-x^2} (4-y)^{\frac{1}{2}} \, dy \, dx$$

$$V = \int_{x=0}^{2} \left[-\frac{2}{3}(4-y)^{\frac{3}{2}} \right]_{y=0}^{y=4-x^2} dx$$

$$V = \int_{x=0}^{2} -\frac{2}{3}\left[x^3 - 8 \right] dx$$

$$V = \left[\frac{16}{3}x - \frac{1}{6}x^4 \right]_0^2$$

$$V = \frac{32}{3} - \frac{8}{3}$$

$$V = 8$$

Question 58

$$I=\int_{0}^{\pi}\int_{0}^{\sin x} 4y \;\; dy\,dx\,.$$

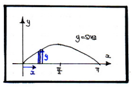
a) Determine the exact value of I.

b) Verify the answer of part **(a)** by reversing the order of integration.

$\boxed{}$, $\boxed{I=\pi}$

a) LOOKING AT THE INTEGRATION REGION

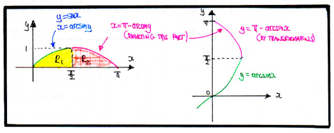
$$\int_{x=0}^{\pi}\int_{y=0}^{y=\sin x} 4y \;\; dy\,dx = \int_{x=0}^{\pi}\Big[2y^2\Big]_{y=0}^{y=\sin x}\,dx$$



$$= \int_0^{\pi} 2\sin^2 x\,dx = \int_0^{\pi} 1-\cos 2x \;\; dx$$

$$= \Big[x-\tfrac{1}{2}\sin 2x\Big]_0^{\pi} = \pi$$

b) REVERSING THE ORDER REQUIRES SPLITTING THE AREA OF INTEGRATION



$$\underbrace{\int_{y=0}^{1}\int_{x=\arcsin y}^{x=\frac{\pi}{2}} 4y \;\; dx\,dy}_{R_1} + \underbrace{\int_{y=0}^{1}\int_{x=\frac{\pi}{2}}^{\pi-\arcsin y} 4y \;\; dx\,dy}_{R_2}$$

$$= \int_0^1 \Big[4yx\Big]_{x=\arcsin y}^{x=\frac{\pi}{2}}\,dy + \int_0^1 \Big[4xy\Big]_{x=\frac{\pi}{2}}^{x=\pi-\arcsin y}\,dy$$

$$= \int_0^1 2\pi y - 4y\arcsin y \;\; dy + \int_0^1 4y(\pi-\arcsin y) - 2\pi y \;\; dy$$

$$= \int_0^1 \cancel{2\pi y} - 4y\arcsin y + 4\pi y - 4y\arcsin y - \cancel{2\pi y} \;\; dy$$

$$= \int_0^1 4\pi y - 8y\arcsin y \;\; dy$$

$$= \Big[2\pi y^2\Big]_0^1 - \int_0^1 8y\arcsin y \;\; dy$$

BY SUBSTITUTION FOLLOWED BY INTEGRATION BY PARTS

$\theta = \arcsin y$
$y = \sin\theta$
$dy = \cos\theta\,d\theta$
$y=0 \mapsto \theta = 0$
$y=1 \mapsto \theta = \frac{\pi}{2}$

$$\dots = 2\pi - \int_0^{\frac{\pi}{2}} (8\sin\theta)\theta\,(\cos\theta\,d\theta)$$

$$= 2\pi - \int_0^{\pi/2} 8\theta\sin\theta\cos\theta\,d\theta$$

$$- \int_0^{\frac{\pi}{2}} 4\theta\sin 2\theta\,d\theta$$

BY PARTS

4θ	4
$-\frac{1}{2}\cos 2\theta$	$\sin 2\theta$

$$= 2\pi - \Big\{\Big[-2\theta\cos 2\theta\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -2\cos 2\theta\,d\theta\Big\}$$

$$= 2\pi - \Big\{\Big[-2\theta\cos 2\theta + \sin 2\theta\Big]_0^{\pi/2}\Big\}$$

$$= 2\pi - \Big\{\Big[-2\times\tfrac{\pi}{2}\times(-1) + 0\Big] - \Big[0\Big]\Big\}$$

$$= 2\pi - \pi$$

$$= \pi$$

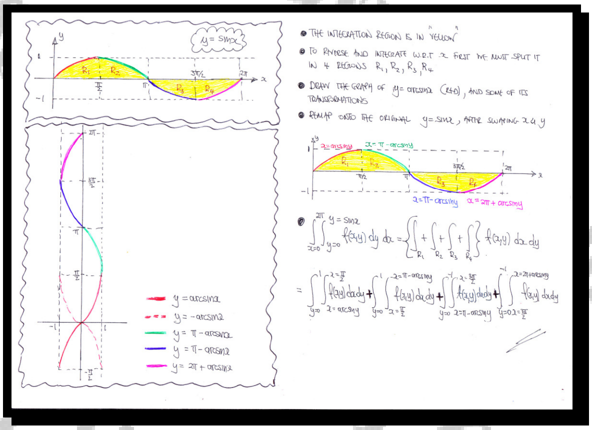
AS BEFORE

Question 59

$$I = \int_{0}^{2\pi} \int_{0}^{\sin x} f(x,y) \; dy \, dx \, .$$

Write an expression for I, by reversing the order of integration.

$$\boxed{I = \left\{ \int_{0}^{1} \int_{\arcsin y}^{\frac{\pi}{2}} + \int_{0}^{1} \int_{\frac{\pi}{2}}^{\pi - \arcsin y} + \int_{0}^{-1} \int_{\pi - \arcsin y}^{\frac{3\pi}{2}} + \int_{0}^{-1} \int_{\frac{3\pi}{2}}^{2\pi - \arcsin y} \right\} f(x,y) \; dx \, dy}$$



Question 60

$$I = \int_0^1 \int_0^{1-x} xy\left(1-x-y\right)^{\frac{1}{2}} \, dy dx \, .$$

Determine the exact value of I by transforming it into an expression involving Beta functions.

$$\boxed{\frac{16}{945}}$$

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x} xy(1-x-y)^{\frac{1}{2}}\,dydx = \int_{x=0}^{1} x\,dx\int_{y=0}^{y=1-x} y(1-x-y)^{\frac{1}{2}}\,dy = \dots$$

LOOK FOR A LINEAR SUBSTITUTION TO REMAP THE LIMITS OF y TO 0 & 1

$u = Ay + B$

$y=0,\ u=0 \Rightarrow 0 = 0 + B \Rightarrow B = 0$

$y=1-x,\ u=1 \Rightarrow 1 = A(1-x) \Rightarrow A = \frac{1}{1-x}$

$\therefore\ u = \frac{1}{1-x}y$ OR $y = u(1-x)$

$dy = (1-x)\,du$

LIMITS BY CONSTRUCTION

$$= \dots \int_{x=0}^{1} x\,dx \int_{u=0}^{1} u(1-x)\left(1-x-(1-x)u\right)^{\frac{1}{2}}(1-x)\,du$$

$$= \int_{x=0}^{1} x\,dx \int_{u=0}^{1} u(1-x)^2\left[(1-x)-(1-x)u\right]^{\frac{1}{2}}\,du$$

$$= \int_{x=0}^{1} x\,dx \int_{u=0}^{1} u(1-x)^2 \times (1-x)^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\,du$$

$$= \int_0^1 x(1-x)^{\frac{5}{2}}\,dx \int_{u=0}^{1} u(1-u)^{\frac{1}{2}}\,du \qquad \text{COMPLETELY SEPARATED}$$

$$= \left[\int_0^1 x(1-x)^{\frac{5}{2}}\,dx\right]\left[\int_0^1 u(1-u)^{\frac{1}{2}}\,du\right]$$

$$= B\left(2,\tfrac{7}{2}\right) \times B\left(2,\tfrac{3}{2}\right)$$

$$= \frac{\Gamma(2)\cancel{\Gamma\left(\frac{7}{2}\right)}}{\Gamma\left(\frac{11}{2}\right)} \times \frac{\Gamma(2)\Gamma\left(\frac{3}{2}\right)}{\cancel{\Gamma\left(\frac{7}{2}\right)}} = \frac{\Gamma(2)\Gamma(2)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{\cancel{\Gamma\left(\frac{3}{2}\right)}}{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\cancel{\Gamma\left(\frac{3}{2}\right)}} = \frac{16}{63\times 15} = \frac{16}{945}$$

Question 61

A finite region R defined by the inequalities

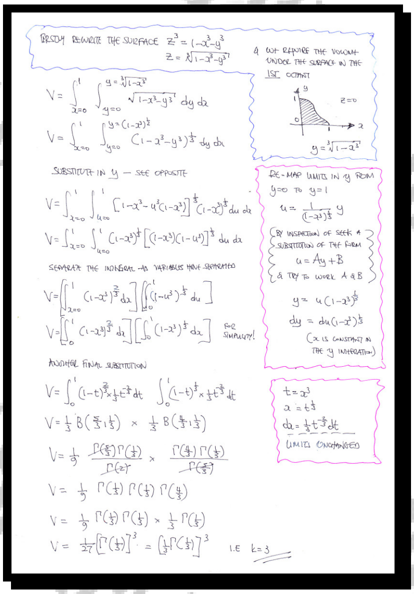
$$x^3 + y^3 + z^3 \le 1, \quad x \ge 0, \quad y \ge 0, \quad z \ge 0.$$

Show that the volume of R is

$$\left[\frac{1}{k}\Gamma\left(\frac{1}{k}\right)\right]^k,$$

where k is a positive integer to be found.

$\boxed{k = 3}$


FIRSTLY REWRITE THE SURFACE $z^3 = 1 - x^3 - y^3$

$z = \sqrt[3]{1-x^3-y^3}$

& WE REQUIRE THE VOLUME UNDER THE SURFACE IN THE 1ST OCTANT

$z=0$

$y = \sqrt[3]{1-x^3}$

$$V = \int_{x=0}^{1} \int_{y=0}^{y=\sqrt[3]{1-x^3}} \sqrt{1-x^3-y^3}\, dy\, dx$$

$$V = \int_{x=0}^{1} \int_{y=0}^{y=(1-x^3)^{\frac{1}{2}}} (1-x^3-y^3)^{\frac{1}{3}}\, dy\, dx$$

SUBSTITUTE IN y — SEE OPPOSITE

RE-MAP LIMITS IN y FROM $y=0$ TO $y=1$

$u = \frac{1}{(1-x^3)^{\frac{1}{3}}} y$

(BY INSPECTION OF SEEK A SUBSTITUTION OF THE FORM $u = Ay + B$ & TRY TO WORK A & B)

$y = u(1-x^3)^{\frac{1}{3}}$

$dy = du(1-x^3)^{\frac{1}{3}}$

(x IS CONSTANT IN THE y INTEGRATION)

$$V = \int_{x=0}^{1} \int_{u=0}^{1} \left[1-x^3-u^3(1-x^3)\right]^{\frac{1}{3}} (1-x^3)^{\frac{1}{3}}\, du\, dx$$

$$V = \int_{x=0}^{1} \int_{u=0}^{1} (1-x^3)^{\frac{1}{3}} \left[(1-x^3)(1-u^3)\right]^{\frac{1}{3}}\, du\, dx$$

SEPARATE THE INTEGRAL AS VARIABLES HAVE SEPARATED

$$V = \left[\int_{x=0}^{1} (1-x^3)^{\frac{2}{3}}\, dx\right]\left[\int_{u=0}^{1} (1-u^3)^{\frac{1}{3}}\, du\right]$$

$$V = \left[\int_{0}^{1} (1-x^3)^{\frac{2}{3}}\, dx\right]\left[\int_{0}^{1} (1-x^3)^{\frac{1}{3}}\, dx\right]$$ FOR SIMPLICITY!

ANOTHER FINAL SUBSTITUTION

$t = x^3$

$x = t^{\frac{1}{3}}$

$dx = \frac{1}{3}t^{-\frac{2}{3}}\, dt$

LIMITS UNCHANGED

$$V = \int_0^1 (1-t)^{\frac{2}{3}} \times \frac{1}{3}t^{-\frac{2}{3}}\, dt \quad \int_0^1 (1-t)^{\frac{1}{3}} \times \frac{1}{3}t^{-\frac{2}{3}}\, dt$$

$$V = \frac{1}{3}B\left(\frac{5}{3}, \frac{1}{3}\right) \times \frac{1}{3}B\left(\frac{4}{3}, \frac{1}{3}\right)$$

$$V = \frac{1}{9}\,\frac{\cancel{\Gamma\left(\frac{5}{3}\right)}\Gamma\left(\frac{1}{3}\right)}{\cancel{\Gamma(2)}} \times \frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\cancel{\Gamma\left(\frac{5}{3}\right)}}$$

$$V = \frac{1}{9}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)$$

$$V = \frac{1}{9}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right) \times \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

$$V = \frac{1}{27}\left[\Gamma\left(\frac{1}{3}\right)\right]^3 = \left[\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\right]^3$$ I.E $k=3$