

LINE INTEGRALS

LINE INTEGRALS

IN 2 DIMENSIONAL CARTESIAN COORDINATES

Question 1

Evaluate the integral

$$\int_C (x+2y) \, dx,$$

where C is the path along the curve with equation $y = x^2 + 1$, from $(0,1)$ to $(6,37)$.

[174]

Plotted to follow

$$\begin{aligned} \int_C (x+2y) \, dx &= \int_{x=0}^{x=6} x+2(x^2+1) \, dx \\ &= \int_0^6 x+2x^2+2 \, dx \\ &= \left[\frac{1}{2}x^2 + \frac{2}{3}x^3 + 2x \right]_0^6 \\ &= (18 + 144 + 0) - 0 \\ &= 174 \end{aligned}$$

Question 2

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (x^2 - y^2)\mathbf{i} + (2xy)\mathbf{j}.$$

Evaluate the line integral

$$\int_{(-2,-1)}^{(4,2)} \mathbf{F} \cdot d\mathbf{r},$$

along a path joining directly the points with Cartesian coordinates $(-2, -1)$ and $(4, 2)$.

[30]

$$\begin{aligned} \int_{(-2,-1)}^{(4,2)} \mathbf{F} \cdot d\mathbf{r} &= \int_{(-2,-1)}^{(4,2)} (x^2 - y^2, 2xy) \cdot (dx, dy) \\ &= \int_{(-2,-1)}^{(4,2)} (x^2 - y^2) dx + 2xy dy \\ &\quad \dots = \int_{-2}^{4} (x^2 - \frac{1}{4}x^2) dx + 2x(\frac{1}{2}) dx \\ &= \int_{-2}^{4} \frac{3}{4}x^2 + \frac{1}{2}x^2 dx \\ &= \int_{-2}^{4} \frac{5}{4}x^2 dx \\ &= \left[\frac{5}{12}x^3 \right]_{-2}^{4} \\ &= \frac{80}{3} \approx \left(\frac{16}{3} \right) \\ &= 30 \end{aligned}$$

*Note: DIRECTLY \Rightarrow straight line
 $\text{gradient} = \frac{2x}{4-(-2)} = \frac{2x}{6} = \frac{1}{3}x$
 $\text{eqn of line: } y - 2 = \frac{1}{3}(x - 4)$
 $y - 2 = \frac{1}{3}x - \frac{4}{3}$
 $y = \frac{1}{3}x + \frac{2}{3}$
 $dy = \frac{1}{3}dx$*

Question 3

The path along the straight line with equation $y = x + 2$, from $A(0, 2)$ to $B(3, 5)$, is denoted by C .

- a) Evaluate the integral

$$\int_C (x^3 + y) dx + (x - y^3) dy.$$

- b) Show that the integral is independent of the path chosen from A to B .
- c) Verify the independence of the path by evaluating the integral of part (a) along a different path from A to B .

-117

a) $\int_C (x^3 + y) dx + (x - y^3) dy = \int_{x=0}^{x=3} [x^3 + (x+2)] dx + [x - (x+2)^3] dx$
 $= \int_0^3 [x^3 + 2x^2 + 2x] dx = \int_0^3 (-6x^2 - 10x - 6) dx$
 $= \left[-2x^3 - 5x^2 - 6x \right]_0^3 = (-54 - 45 - 18) - 0 = -117$

b) SINCE THIS IS TWO DIMENSIONAL,
 $\int (x^3 + y) dx + (x - y^3) dy$
 $\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy$
 $\frac{\partial F}{\partial x} = 1, \quad \frac{\partial F}{\partial y} = 1$
 $\text{SO } \text{F} \text{ IS NOT DIFFERENTIABLE}$
 $\text{SO } \text{F} \text{ IS INDEPENDENT OF THE PATH}$

c)

THE
 $\int (x^3 + y) dx + (x - y^3) dy$
 $= \int_0^3 x^3 + 2x dx + \int_2^5 3 - y^3 dy$
 $\text{(using } C_1\text{)}$
 $= \left[\frac{1}{4}x^4 + 2x^2 \right]_0^3 + \left[3y - \frac{1}{4}y^4 \right]_2^5$
 $= \left(\frac{81}{4} + 18 \right) - (0) + \left(5 - \frac{625}{4} \right) - (8)$
 $= -117$

Question 4

The path along the perimeter of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C .

Evaluate the integral

$$\oint_C x^2 \, dx - 2xy \, dy.$$

$$-\frac{1}{3}$$

Diagram showing a right-angled triangle C in the first quadrant with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. The hypotenuse is labeled C_1 , the x-axis segment is labeled C_2 , and the y-axis segment is labeled C_3 .

Given conditions:

- $C_1: y=0, \, dy=0 \quad 0 \leq x \leq 1$
- $C_2: y=1-x, \, dy=-dx \quad 0 \leq x \leq 1$
- $C_3: x=0, \, dx=0 \quad 0 \leq y \leq 1$

$$\begin{aligned} \oint_C x^2 \, dx - 2xy \, dy &= \left\{ \int_{C_1} + \int_{C_2} + \int_{C_3} \right\} (x^2 \, dx - 2xy \, dy) \\ &= \int_0^1 x^2 \, dx + \int_1^0 x^2 \, dx - 2(1-x)(-dx) + \int_0^1 0 \, dx - 0 \, dy \\ &= \int_0^1 x^2 \, dx + \int_0^1 -x^2 - 2x(1-x) \, dx \\ &= \int_0^1 x^2 - x^2 - 2x + 2x^2 \, dx \\ &= \left[\frac{2}{3}x^3 - x^2 \right]_0^1 \\ &= \left(\frac{2}{3} - 1 \right) - 0 \\ &= -\frac{1}{3} \end{aligned}$$

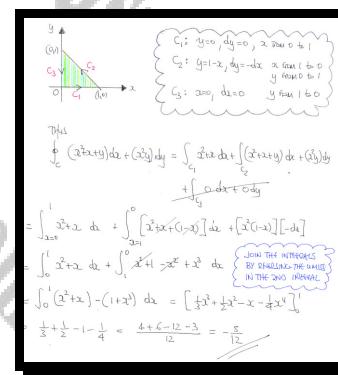
Question 5

The path along the perimeter of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C .

Evaluate the integral

$$\oint_C (x^2 + x + y) dx + (x^2 y) dy.$$

$$-\frac{5}{12}$$



Question 6

The functions F and G are defined as

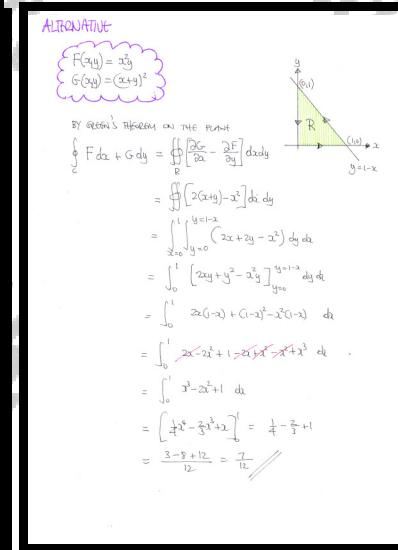
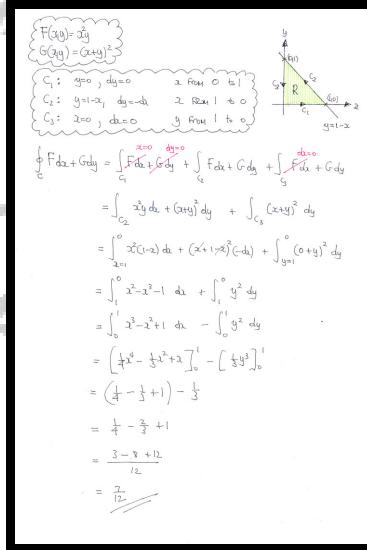
$$F(x, y) = x^2 y \quad \text{and} \quad G(x, y) = (x+y)^2$$

The anticlockwise path along the perimeter of the triangle whose vertices are located at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C .

Evaluate the line integral

$$\int_C F dx + G dy .$$

7
12



Question 7

The anticlockwise path along the perimeter of the square whose vertices are located at the points $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, is denoted by C .

Evaluate the line integral

$$\int_C (x^2 + xy) dx + (x+y)^3 dy.$$

You may not use Green's theorem in this question.

[3]

$$\begin{aligned} \int_C (x^2 + xy) dx + (x+y)^3 dy &= \int_{C_1} x^2 dx + \int_{C_2} (1+y)^3 dy + \int_{C_3} x^2 dx + \int_{C_4} y^3 dy \\ &= \int_{x=0}^1 x^2 dx + \int_{y=0}^1 (1+y)^3 dy - \int_{x=1}^0 x^2 dx + \int_{y=1}^0 y^3 dy \\ &= \int_0^1 x^2 dx + \int_0^1 ((1+y)^3 - y^3) dy \\ &= \int_0^1 -x dx + \int_0^1 1+3y+3y^2 dy \\ &= \left[-\frac{1}{2}x^2 \right]_0^1 + \left[y + \frac{3}{2}y^2 + y^3 \right]_0^1 \\ &= -\frac{1}{2} + (1+\frac{3}{2}+1) = 3 \end{aligned}$$

Question 8

Evaluate the integral

$$\int_{(-1,7)}^{(5,0)} (3y) \, dx + (3x+2y) \, dy,$$

along a path joining the points with Cartesian coordinates $(-1, 7)$ and $(5, 0)$.

[–28]

METHOD A: USE A STRAIGHT LINE

$$I = \int_{(-1,7)}^{(5,0)} 3y \, dx + (3x+2y) \, dy$$

$$dx = \frac{\partial x}{\partial t} dt = \frac{2}{3}t \, dt$$

$$dy = \frac{\partial y}{\partial t} dt = \frac{2}{3}t^2 \, dt$$

$$\frac{\partial f}{\partial y} = 3, \quad \frac{\partial f}{\partial x} = 3 \quad \therefore \text{INDEPENDENT OF THE PATH}$$

Thus

$$\begin{aligned} I &= \int_{(-1,7)}^{(5,0)} 3y \, dx + (3x+2y) \, dy \\ &= \int_{-1}^5 3\left(\frac{2}{3}t^2\right) + (3t + \frac{2}{3}t^2)(\frac{2}{3}t) \, dt \\ &= \int_{-1}^5 -\frac{2}{3}t^2 + \frac{25}{9}t^3 \, dt \\ &= \left[-\frac{2}{9}t^3 + \frac{25}{27}t^4 \right]_{-1}^5 \\ &= \left(-\frac{2}{9}(5)^3 + \frac{25}{27}(5)^4 \right) - \left(-\frac{2}{9}(-1)^3 + \frac{25}{27}(-1)^4 \right) \\ &= \left(-\frac{250}{9} + \frac{1250}{27} \right) - \left(\frac{2}{9} - \frac{25}{27} \right) \\ &= -28 \end{aligned}$$

METHOD B: OR USE 2 STRAIGHT LINES

$$\begin{aligned} C_1: \quad x = -1 \Rightarrow dx = 0 \\ \quad y \text{ runs from } 7 \text{ to } 0 \\ C_2: \quad y = 0 \Rightarrow dy = 0 \\ \quad x \text{ runs from } -1 \text{ to } 5 \\ I = \int_{y=7}^{y=0} -3+2y \, dy + \int_{x=-1}^{x=5} 0 \, dx \end{aligned}$$

METHOD C:

AS THIS IS AN EXACT DIFFERENTIAL

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$3y \, dx + (3x+2y) \, dy$$

$$\therefore f(x,y) = 3xy + y^2$$

(BY INSPECTION)

$$\begin{aligned} I &= \int_{(-1,7)}^{(5,0)} 3y \, dx + (3x+2y) \, dy \\ &= \int_{(-1,7)}^{(5,0)} 1 \, df \\ &= \left[3xy + y^2 \right]_{(-1,7)}^{(5,0)} \\ &= (0+0) - (-21+49) \\ &= -28 \end{aligned}$$

AS SHOWN

Question 9

Evaluate the integral

$$\int_{(1,1)}^{(3,4)} \left(3x^2y^2 \right) dx + \left(2x^3y \right) dy,$$

along a path joining the points with Cartesian coordinates (1,1) and (3,4).

[431]

$I = \int_{(1,1)}^{(3,4)} 3x^2y^2 dx + 2x^3y dy$

 $\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(3x^2y^2) = 6x^2y^2$
 $\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(2x^3y) = 2x^3$
 $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} \Rightarrow$ INDEPENDENT OF PATH

METHOD A

$I = \int_{x=1}^{x=3} 3x^2 dx + \int_{y=1}^{y=4} 2x^3 dy$

 $I = [3x^3]_1^3 + [2x^4]_1^4$
 $I = (27-1) + (482-16)$
 $I = 423$

METHOD B As it is an exact differential

 $I = \int_{(1,1)}^{(3,4)} dF = \left[F(x,y) \right]_{(1,1)}^{(3,4)} = \left[3x^2y^2 + 2x^3y \right]_{(1,1)}^{(3,4)} = (3^2 \cdot 4^2) - (1^2 \cdot 1)$
 $= 72 \times 16 - 1 = 423$

Note: $\frac{\partial F}{\partial x} = 3x^2y^2$, $F = 3x^2y^2 + A(y)$
 $\frac{\partial F}{\partial y} = 2x^3$, $F = 2x^3y + B(x)$ $\therefore A'(y) = B(y) = c$
 $\therefore F(x,y) = 3x^2y + C$

Question 10

$$\mathbf{F}(x, y) = (2xy^2 + \cos x)\mathbf{i} + (2x^2y - \sin y)\mathbf{j}.$$

Show that the vector field \mathbf{F} is conservative, and hence evaluate the integral

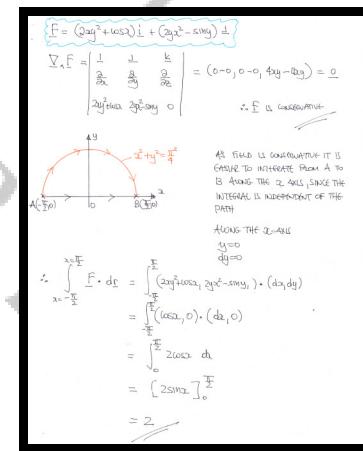
$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the arc of the circle with equation

$$x^2 + y^2 = \frac{\pi^2}{4}, \quad y \geq 0,$$

from $A\left(-\frac{\pi}{2}, 0\right)$ to $B\left(\frac{\pi}{2}, 0\right)$.

[2]



Question 11

In this question α , β and γ are positive constants.

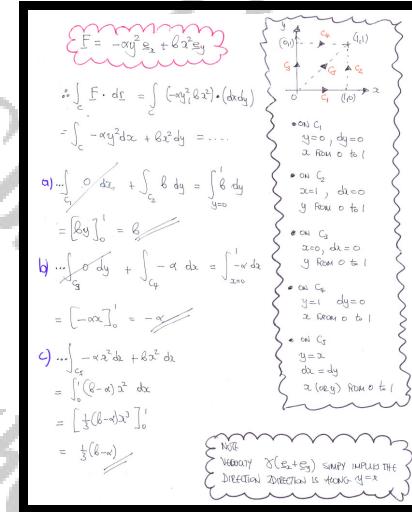
$$\mathbf{F} = -\alpha y^2 \mathbf{e}_x + \beta x^2 \mathbf{e}_y.$$

A particle of mass m is moving on the x - y plane, under the action of \mathbf{F} .

Find the work done by \mathbf{F} on the particle in moving it from the Cartesian origin O to the point $(1,1)$, in each of the following cases.

- a) Directly from O to $(1,0)$, then directly from $(1,0)$ to $(1,1)$.
- b) Directly from O to $(0,1)$, then directly from $(0,1)$ to $(1,1)$.
- c) Moving the particle with velocity $\mathbf{v} = \gamma(\mathbf{e}_x + \mathbf{e}_y)$.

$$W_1 = \beta, \quad W_2 = -\alpha, \quad W_3 = \frac{1}{3}(\beta - \alpha)$$



Question 12

Evaluate the line integral

$$\oint_C \left[y(x+1)e^x dx + x(e^x+1) dy \right],$$

where C is a circle of radius 1, centre at the origin O , traced anticlockwise.

$\boxed{\pi}$

The handwritten solution shows the setup of the integral, the application of Green's Theorem, and the simplification of the resulting double integral to find the area of the unit circle.

Setup:

$$\int_C y(2x+1)e^x dx + x(e^x+1) dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Using Green's Theorem:

$$\begin{aligned} & \int_C y(2x+1)e^x dx + x(e^x+1) dy \\ &= \iint_D (2xe^x + e^x + 1) dx dy \\ &= \iint_D 2xe^x dx dy + \iint_D e^x dx dy \\ &= \iint_D 1 dx dy \end{aligned}$$

Final result:

$$= \text{Area of the circle } x^2 + y^2 = 1 = \pi$$

Question 13

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (\sin x^3 - xy)\mathbf{i} + (x + y^3 \sin y)\mathbf{j}.$$

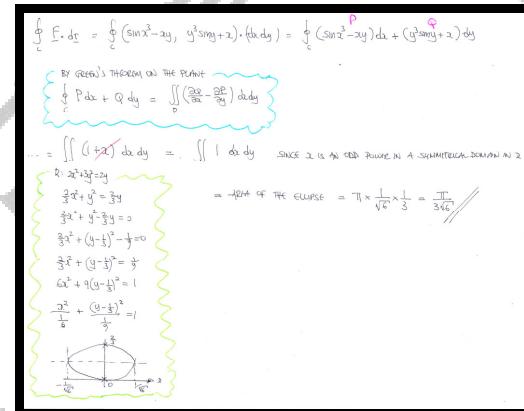
Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the ellipse with Cartesian equation

$$2x^2 + 3y^2 = 2y.$$

$$\boxed{\frac{\pi}{3\sqrt{6}}}$$



Question 14

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = [x \cos x] \mathbf{i} + [15xy + \ln(1+y^3)] \mathbf{j}.$$

Evaluate the line integral

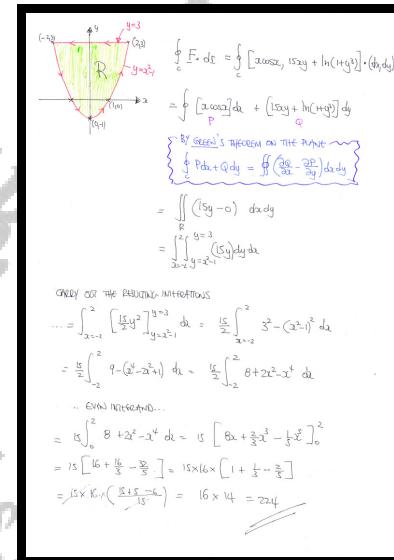
$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the curve

$$\{(x, y) : y = 3, -2 \leq x \leq 2\} \cup \{(x, y) : y = x^2 - 1, -2 \leq x \leq 2\},$$

traced in an anticlockwise direction.

[224]



LINE INTEGRALS

2 DIMENSIONAL PARAMETERIZATIONS

Question 1

The path along the semicircle with equation

$$x^2 + y^2 = 1, \quad x \geq 0$$

from $A(0,1)$ to $B(0,-1)$, is denoted by C .

Evaluate the integral

$$\int_C (x^3 + y^3) \, dx.$$

$\boxed{\frac{3\pi}{8}}$

$$\begin{aligned}
 \int_C (x^3 + y^3) \, dx &= \int_{x=0}^{x=1} x^3 + \sqrt{1-x^2}^3 \, dx + \int_{x=1}^{x=0} x^3 - \sqrt{1-x^2}^3 \, dx \\
 &= \int_0^1 x^3 + (1-x^2)^{\frac{3}{2}} \, dx + \int_0^{-1} x^3 + (1-x^2)^{\frac{3}{2}} \, dx \\
 &= \int_0^1 2(1-x^2)^{\frac{3}{2}} \, dx \\
 \dots \text{BY SUBSTITUTION...} \\
 &\quad \text{Let } u = 1-x^2 \\
 &\quad \text{Then } du = -2x \, dx \\
 &\quad \text{At } x=0, u=1 \\
 &\quad \text{At } x=1, u=0 \\
 &= \int_0^{\frac{\pi}{2}} 2(1-u)^{\frac{3}{2}} \, du \\
 &= \int_0^{\frac{\pi}{2}} 2u^{\frac{3}{2}} \, du \\
 \dots \text{BY THE DEFINITION OF BETA/FINMAIR FUNCTIONS} \\
 &= \int_0^{\frac{\pi}{2}} \frac{2u^{\frac{3}{2}-1}}{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})} u^{\frac{5}{2}-1} \, du \\
 &= B\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \\
 &= \frac{\frac{3}{2}\times\frac{1}{2}\times\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{2!} \\
 &= \frac{\frac{3}{2}\times\frac{1}{2}\times\sqrt{3}\pi}{2} \\
 &= \frac{3\pi}{8}
 \end{aligned}$$

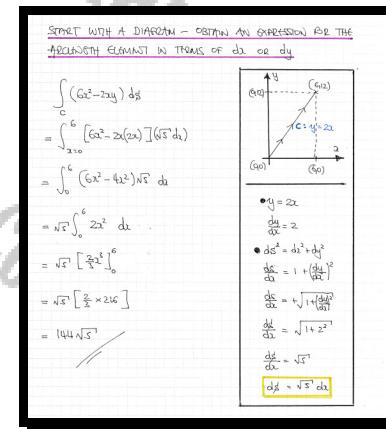
Question 2

Evaluate the integral

$$\int_{(0,0)}^{(6,12)} (6x^2 - 2xy) \, ds,$$

where s is the arclength along the straight line segment from $(0,0)$ to $(6,12)$.

, $144\sqrt{5}$



Question 3

Evaluate the integral

$$\int_{(1,-1)}^{(3,3)} (y+x) \, dx + (y-x) \, dy,$$

along the curve with parametric equations

$$x = 2t^2 - 3t + 1 \quad \text{and} \quad y = t^2 - 1.$$

 [10]

SUBSTITUTING THE INITIAL INTO PARAMETRIC

$$\begin{aligned}
 & \int_{(1,-1)}^{(3,3)} (y+x) \, dx + (y-x) \, dy \\
 & \text{Let } x = 2t^2 - 3t + 1 \quad y = t^2 - 1 \\
 & 0 \leq t \leq 2 \quad (\text{By inspection}) \\
 & \text{Hence we have} \\
 & \int_0^2 (y+x) \, dx + (y-x) \, dy \\
 & = \int_0^2 (t^2 - 1 + 2t^2 - 3t + 1)(4t - 3) \, dt + (t^2 - 1 - 2t^2 + 3t - 1)(2t + 4) \, dt \\
 & = \int_0^2 [(3t^2 - 3t)(4t - 3) + 2t(-t^2 + 3t - 2)] \, dt \\
 & = \int_0^2 [12t^3 - 9t^2 - 12t^2 + 9t - 2t^3 + 6t^2 - 4t] \, dt \\
 & = \int_0^2 [10t^3 - 15t^2 + 5t] \, dt \\
 & = \left[\frac{5}{2}t^4 - 5t^3 + \frac{5}{2}t^2 \right]_0^2 \\
 & = \left(50 - 80 + 10 \right) = 0 \\
 & = 0
 \end{aligned}$$

Question 4

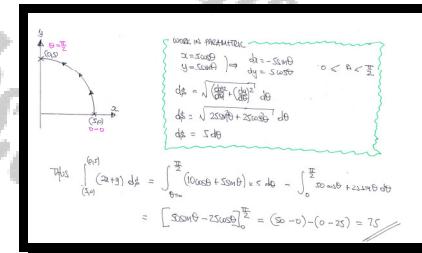
Evaluate the line integral

$$\int_{(5,0)}^{(0,5)} (2x+y) \, ds,$$

where s is the arclength along the quarter circle with equation

$$x^2 + y^2 = 25.$$

[75]



Question 5

Evaluate the line integral

$$\oint_C y^5 \, dx,$$

where C is a circle of radius 2, centre at the origin O , traced anticlockwise.

You may not use Green's theorem in this question.

-40π

The diagram shows a circle of radius 2 centered at the origin O in a Cartesian coordinate system. The circle is traced anticlockwise. A point (x, y) is shown on the circle in the first quadrant. The angle θ is measured from the positive x -axis to the radius vector. The text indicates that $x = 2\cos\theta$ and $y = 2\sin\theta$.

Method 1 (Parametric Equations):

$$\begin{aligned} \oint_C y^5 \, dx &= \int_0^{2\pi} y^5 \, dx \\ &= \int_0^{2\pi} (2\sin\theta)^5 \, d(2\cos\theta) \\ &= \int_{-2}^2 (4-2x)^5 \, dx + \int_{2\pi}^0 (4-x^2)^5 \, dx \\ &\text{(by substitution)} \\ &= \int_2^{-2} -2(4-2x)^{\frac{5}{2}} \, dx = \dots \text{(using } u=4-2x \text{ and } du=-2dx\text{)} \dots = \int_2^{-2} -4(4-x)^{\frac{5}{2}} \, dx \\ &\text{(by substitution)} \\ &= \int_0^{\frac{\pi}{2}} -\frac{1}{4}(4-4\sin^2\theta)^{\frac{3}{2}} (2\cos\theta \, d\theta) = -4 \int_0^{\frac{\pi}{2}} 2x^2 \cos^2\theta \, d\theta \\ &= -128 \int_0^{\frac{\pi}{2}} (\cos^2\theta)^{\frac{2k+1}{2}} (\sin\theta)^{2k-1} \, d\theta = -128 B\left(\frac{3}{2}, \frac{1}{2}\right) \\ &= -128 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = -128 \times \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} \\ &= -128 \times \frac{\frac{1}{2}\times\sqrt{\pi}\times\sqrt{\pi}}{1} = -40\pi \end{aligned}$$

Method 2 (Polar Coordinates):

$$\begin{aligned} \text{Given } x^2+y^2=4 &\quad \oint_C y^5 \, dx = \int_0^{2\pi} (2\sin\theta)^5 (2\cos\theta) \, d\theta = -4 \int_0^{2\pi} \sin^5\theta \, d\theta \\ x=2\cos\theta &\\ y=2\sin\theta &\\ dx=-2\sin\theta \, d\theta &\\ 0 \leq \theta \leq 2\pi &\\ &= -4 \times 4 \int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = -128 B\left(\frac{3}{2}, \frac{1}{2}\right) \\ &= -128 \times \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \dots \text{as above} \\ &= -40\pi \end{aligned}$$

Question 6

Evaluate the line integral

$$\oint_C [y^3 dx + (xy) dy] ,$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

You may not use Green's theorem in this question.

, $\boxed{\frac{3\pi}{4}}$

PROCEED BY PARAMETRIZING THE CIRCULAR PATH

$$\begin{aligned} & \oint_C [y^3 dx + (xy) dy] \\ &= \int_0^{2\pi} \sin^3 \theta (-\cos \theta) d\theta + (\text{coefficient})(\cos \theta) d\theta \\ &= \int_0^{2\pi} (-\sin^3 \theta + (\text{coefficient})) d\theta = \int_0^{2\pi} -\sin^3 \theta d\theta \\ &\quad \text{NO INTEGRATION OVER THESE LIMITS} \\ &+ \int_0^{2\pi} (-\frac{1}{2} - \frac{1}{2}\cos^2 \theta) d\theta = \int_0^{2\pi} (\frac{1}{2}\cos^2 \theta + \frac{1}{2} - \frac{1}{2}\cos^2 \theta) d\theta \\ &\quad \text{NO INTEGRATION OVER THESE LIMITS} \\ &= \int_0^{2\pi} (-\frac{1}{2} - \frac{1}{2}\cos^2 \theta) d\theta = \int_0^{2\pi} [-\frac{1}{2} - \frac{1}{2}(\frac{1}{2} + \frac{1}{2}\cos 2\theta)] d\theta \\ &= \int_0^{2\pi} (-\frac{1}{2} - \frac{1}{4} - \frac{1}{4}\cos 2\theta) d\theta = \int_0^{2\pi} -\frac{3}{8} d\theta \\ &\quad \text{NO INTEGRATION OVER THESE LIMITS} \\ &\therefore -\frac{3}{8} \times 2\pi = -\frac{3\pi}{4} \end{aligned}$$

ALTERNATIVE EVALUATION FROM $\int_0^{2\pi} -\sin^3 \theta d\theta$

$$\begin{aligned} \int_0^{2\pi} -\sin^3 \theta d\theta &= -\int_0^{2\pi} 4\sin^3 \theta d\theta \\ &= -2 \int_0^{2\pi} 2(\sin \theta)^2 2\sin \theta d\theta \\ &= -2 \cdot B(\frac{3}{2}, \frac{1}{2}) \end{aligned}$$

FD 100 GAMMA FUNCTIONS

$$\begin{aligned} &= -2 \times \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{1}{2})}{\Gamma(2)} \\ &= -2 \times \frac{[\frac{1}{2} \times \frac{1}{2} \times \Gamma(\frac{1}{2})] \Gamma(\frac{1}{2})}{2!} \\ &= -2 \times \frac{\frac{1}{2} \times \frac{1}{2} \times \sqrt{\pi}}{2} \\ &= -\frac{3\pi}{4} \end{aligned}$$

// 4 zeros

Question 7

Evaluate the line integral

$$\oint_C [y \, dx + x(2+y) \, dy] ,$$

where C is a circle of radius 1, centre at the origin O , traced anticlockwise.

You may not use Green's theorem in this question.

π

$$\begin{aligned} \oint_C y \, dx + x(2+y) \, dy &= \int_0^{2\pi} y \, dx + (2x+y) \, dy \\ \text{PARAMETRIZE: } x &= \cos\theta \quad dx = -\sin\theta \, d\theta \\ y &= \sin\theta \quad dy = \cos\theta \, d\theta \\ 0 \leq \theta \leq 2\pi & \text{ (anticlockwise)} \\ &= \int_{0\pi}^{2\pi} \sin\theta(-\sin\theta) \, d\theta + (2\cos\theta + \cos\theta\sin\theta)(\cos\theta) \, d\theta \\ &= \int_{0\pi}^{2\pi} -\sin^2\theta + 2\cos^2\theta + \cos\theta\sin\theta \, d\theta \\ &= \int_{0\pi}^{2\pi} (\cos^2\theta - \sin^2\theta) + \cos\theta\sin\theta + \cos\theta\sin\theta \, d\theta \\ &= \int_{0\pi}^{2\pi} \cancel{\cos^2\theta} + \cancel{\sin^2\theta} + (1 + \cancel{\cos\theta\sin\theta}) \, d\theta \\ &\quad \text{No contribution since there is zero in } \theta \\ &= \int_{0\pi}^{2\pi} \frac{1}{2} \, d\theta \\ &= \frac{1}{2} \times 2\pi = \pi \end{aligned}$$

Question 8

Evaluate the line integral

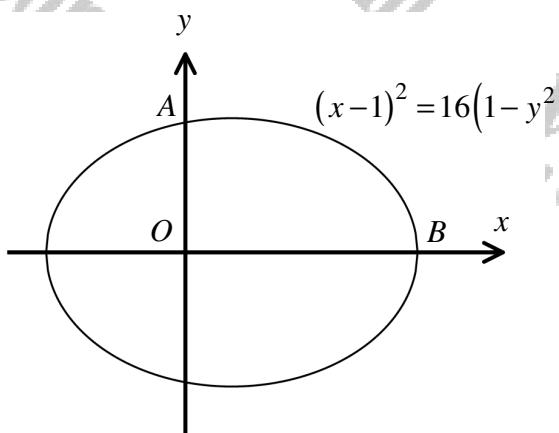
$$\oint_C \frac{x^2y}{x^2+y^2} dx,$$

where C is a circle of radius 1, centre at the origin O , traced anticlockwise.

$$\boxed{\frac{\pi}{4}}$$

PARAMETRIZE THE CIRCLE $x^2+y^2=1$
 $\begin{cases} x = \cos\theta \\ y = \sin\theta \\ 0 \leq \theta \leq 2\pi \\ dx = -\sin\theta d\theta \end{cases}$

$$\begin{aligned}
 & \int_{C=0}^{2\pi} \frac{\frac{d}{dt}(x,y)}{x^2+y^2} (-\sin\theta d\theta) = \int_{0=0}^{2\pi} -\cos\theta \sin\theta d\theta \\
 & = \int_{0=0}^{2\pi} -\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta = \int_{0=0}^{2\pi} -\left(\frac{1}{4} - \frac{1}{4}\cos^2 2\theta\right) d\theta \\
 & = \int_{0=0}^{2\pi} \frac{1}{4}(4\cos^2 2\theta - \frac{1}{4}) d\theta = \int_{0=\frac{\pi}{2}}^{2\pi} \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta \\
 & = \int_{0=0}^{2\pi} \frac{1}{8} + \frac{1}{8}\cos(4\theta) - \frac{1}{4} d\theta = \int_{0=0}^{2\pi} -\frac{1}{8} + \frac{1}{8}\cos(4\theta) d\theta \\
 & = \left[-\frac{1}{8}\theta + \frac{1}{32}\sin(4\theta)\right]_{0=0}^{2\pi} = -\frac{\pi}{4}
 \end{aligned}$$

Question 9

The figure above shows the ellipse with equation

$$(x-1)^2 = 16(1-y^2).$$

The ellipse meets the positive y and x axes at the points A and B , respectively, as shown in the figure.

The elliptic path C is the clockwise section from A to B .

Determine the value of each of the following line integrals.

a) $\int_C \left[(x^2 + xy) \, dx + \left(y^2 + \frac{1}{2}x^2\right) \, dy \right].$

b) $\int_C \left[y^3 \, dx + \frac{1}{16}(x-1)^3 \, dy \right].$

, $\frac{125}{3} - \frac{5}{64}\sqrt{15}$, $\frac{1}{4}\sqrt{15}$

[solution overleaf]

Q4 FINITELY DETERMINING THE COORDINATES OF A & B

$$\begin{aligned} \bullet \quad & g=0 \\ \bullet \quad & 1 = 6(1-y^2) \\ \frac{1}{6} & = 1 - y^2 \\ y^2 & = \frac{5}{6} \\ y & = \pm \sqrt{\frac{5}{6}} \\ A & (0, \pm \sqrt{\frac{5}{6}}) \end{aligned}$$

$$\begin{aligned} \bullet \quad & g=0 \\ \bullet \quad & (x-1)^2 = 6 \\ x-1 & = \pm \sqrt{6} \\ x & = 5 \end{aligned}$$

$$B(5, 0)$$

NEXT WE TEST WHETHER THE INTEGRAL IS INDEPENDENT OF THE PATH.

$$\begin{aligned} \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy &= df \\ \frac{\partial f}{\partial x} dy &= y^2 - \frac{1}{2}x^2 \\ \frac{\partial}{\partial y} (y^2 - \frac{1}{2}x^2) dx &= \frac{\partial}{\partial x} (y^2 - \frac{1}{2}x^2) dy \quad \text{NOT EXACT} \\ \bullet \quad \frac{\partial f}{\partial x} - \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial y} &\rightarrow d(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + f(y) \\ \bullet \quad \frac{\partial f}{\partial y} &= y^2 + \frac{1}{2}x^2 \rightarrow d(x,y) = y^2 + \frac{1}{2}x^2 + g(x) \\ \therefore \quad df(x,y) &= \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}xy + C \end{aligned}$$

HENCE WE OBTAIN

$$\int_4^8 (\frac{1}{2}x^2 + xy) dx + (y^2 + \frac{1}{2}x^2) dy = \int_4^8 df$$

$$= \left[\frac{1}{8}x^3 + \frac{1}{2}xy^2 + \frac{1}{2}\frac{1}{2}x^2y \right]_{(4,0)}^{(8,0)}$$

$$= \frac{1}{8} \times 8^3 - \frac{1}{8} \times (4\sqrt{6})^2$$

$$= \frac{128}{8} - \frac{1}{8} \times \frac{144 \sqrt{6}}{4} \rightarrow \frac{128}{8} - \frac{36\sqrt{6}}{8}$$

B CHECKING FOR PATH INDEPENDENCE FOR Q

$$\frac{\partial f}{\partial y}(x^2) = 2xy \quad \frac{\partial f}{\partial x}(y^2) = \frac{1}{2}(x^2)$$

INTEGRAL IS NOT PATH DEPENDENT ON THE PATH

POLARISATION THE CURVE

$$\Rightarrow (x-1)^2 = 16(1-y^2)$$

$$\Rightarrow (x-1)^2 = 16 - 16y^2$$

$$\Rightarrow (x-1)^2 + 16y^2 = 16$$

$$\Rightarrow \frac{(x-1)^2}{16} + \frac{y^2}{1} = 1$$

$$[\cos^2 \theta + \sin^2 \theta = 1]$$

$$\therefore \cos \theta = \frac{x-1}{4} \quad \text{and} \quad \sin \theta = y$$

$$x = 1 + 4\cos \theta \quad y = \sin \theta$$

$$dx = -4\sin \theta d\theta \quad dy = \cos \theta d\theta$$

REWORKING THE INTEGRAL INTO POLARIC

$$\begin{aligned} A(0, \pm \sqrt{\frac{5}{6}}) &\rightarrow \theta = \arcsin \frac{\sqrt{\frac{5}{6}}}{2} = \arccos \frac{1}{2} \\ B(5, 0) &\rightarrow \theta = 0 \end{aligned}$$

$$\begin{aligned} \int_C (\frac{1}{2}x^2 + \frac{1}{2}xy^2) dy &= \int_0^{\pi/2} \left[\sin^2 \theta (-4\sin \theta) + \frac{1}{16}(1+4\cos \theta - 1)\cos^2 \theta \right] d\theta \\ &= \int_0^{\pi/2} -4\sin^4 \theta + 4\cos^3 \theta d\theta \\ &= \int_0^{\pi/2} 4(\cos^2 \theta - \sin^2 \theta) d\theta \\ &= \int_0^{\pi/2} 4(\cos^2 \theta - \sin^2 \theta) \frac{1 + \cos 2\theta}{2} d\theta \\ &= \int_0^{\pi/2} 4\cos 2\theta d\theta \end{aligned}$$

$$\Rightarrow (x-1)^2 = 16(1-y^2)$$

$$\Rightarrow (x-1)^2 = 16 - 16y^2$$

$$\Rightarrow (x-1)^2 + 16y^2 = 16$$

$$\Rightarrow \frac{(x-1)^2}{16} + \frac{y^2}{1} = 1$$

$$[\cos^2 \theta + \sin^2 \theta = 1]$$

$$\therefore \cos \theta = \frac{x-1}{4} \quad \text{and} \quad \sin \theta = y$$

$$x = 1 + 4\cos \theta \quad y = \sin \theta$$

$$dx = -4\sin \theta d\theta \quad dy = \cos \theta d\theta$$

Question 10

The closed curve C bounds the finite region R in the x - y plane defined as

$$R(x, y) = \{x + y \geq 0 \cap x - y \leq 0 \cap x^2 + y^2 \leq 2\}.$$

Evaluate the line integral

$$\oint_C (xy \, dx + x^2 \, dy),$$

where C is traced anticlockwise.

, 0

Start by sketching the region

PARAMETRIZE EACH SECTION

- $C_1: y = x$
 $dy/dx = 1$
 $a: 0 \rightarrow \sqrt{2}$
- $C_2: x = \sqrt{2}\cos\theta$
 $y = \sqrt{2}\sin\theta$
 $dx = -\sqrt{2}\sin\theta d\theta$
 $dy = \sqrt{2}\cos\theta d\theta$
 $\theta: \text{Row } \frac{\pi}{2} \text{ to } \frac{\pi}{4}$
- $C_3: y = -x$
 $dy/dx = -1$
 $a: \sqrt{2} \rightarrow 0$

$$\begin{aligned} \oint_C (xy \, dx + x^2 \, dy) &= \left[\int_{C_1} + \int_{C_2} + \int_{C_3} \right] [xy \, dx + x^2 \, dy] \\ &= \int_{\sqrt{2}}^0 x^2 \, dx + \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \left[\sqrt{2}\cos\theta(-\sqrt{2}\sin\theta) \, d\theta + \sqrt{2}\cos^2\theta \, d\theta \right] \\ &\quad + \int_{\sqrt{2}}^0 -x^2 \, dx + x^2(-1) \, dx \\ &= \int_0^{\sqrt{2}} 2x^2 \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -2\sqrt{2}\cos\theta\sin\theta \, d\theta + 2\sqrt{2}\cos^2\theta \, d\theta + \int_0^{\sqrt{2}} -2x^2 \, dx \\ &= \int_0^{\sqrt{2}} 2x^2 \, dx + 2\sqrt{2} \left[\sin^2\theta - \cos^2\theta \right] \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_0^{\sqrt{2}} -2x^2 \, dx \\ &= \int_0^{\sqrt{2}} 2x^2 \, dx + \int_{\frac{\pi}{2}}^{\sqrt{2}} 2x^2 \, dx + 2\sqrt{2} \left[\sin^2\theta - \cos^2\theta \right] \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \end{aligned}$$

INTEGRATING BY DISCOURSES

$$\begin{aligned} &= \left[\frac{2}{3}x^3 \right]_0^{\sqrt{2}} + \left[\frac{2}{3}x^3 \right]_{\frac{\pi}{4}}^{\sqrt{2}} + 2\sqrt{2} \left[\sin^2\theta - \cos^2\theta \right] \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \left(\frac{2}{3}(8) \right) + \left(\frac{2}{3}(8) \right) + 2\sqrt{2} \left[(\cos\frac{\pi}{4})^2 - (\sin\frac{\pi}{4})^2 \right] - \left[(\cos\frac{\pi}{2})^2 - (\sin\frac{\pi}{2})^2 \right] \\ &= 2\sqrt{2} \left[\sin\frac{\pi}{2} - \cos\frac{\pi}{2} + \frac{2}{3}\sin\frac{\pi}{2} - \frac{2}{3}\cos\frac{\pi}{2} \right] = 0 \end{aligned}$$

ALTERNATIVE BY GREEN'S THEOREM

$$\begin{aligned} \oint_C P \, dx + Q \, dy &= \oint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \\ \oint_C xy \, dx + x^2 \, dy &= \oint_C \left(\frac{\partial}{\partial x}(\sqrt{2}\cos\theta) - \frac{\partial}{\partial y}(\sqrt{2}\sin\theta) \right) \, dx \\ &= \oint_C 2x \, dx = \oint_C 2 \, dx \end{aligned}$$

Switching into polar coordinates

$$\begin{aligned} &= \int_{0}^{\frac{\pi}{2}} \int_{0}^{\sqrt{2}} (2r\cos\theta) \, r \, dr \, d\theta = \int_{0}^{\frac{\pi}{2}} \int_{0}^{\sqrt{2}} r^2 \cos\theta \, dr \, d\theta \\ &= \int_{0}^{\frac{\pi}{2}} \int_{0}^{\sqrt{2}} \left[\frac{1}{3}r^3 \right]_0^{\sqrt{2}} \cos\theta \, d\theta \, dr = \int_{0}^{\frac{\pi}{2}} \int_{0}^{\sqrt{2}} \frac{8}{3}\cos\theta \, d\theta \, dr \\ &= \frac{2\sqrt{2}}{3} \left[\sin\theta \right]_0^{\frac{\pi}{2}} = \frac{2\sqrt{2}}{3} \left[\sin\frac{\pi}{2} - \sin 0 \right] = 0 \end{aligned}$$

As required

Question 11

Evaluate the line integral

$$\oint_C \left[\arctan\left(\frac{y}{x}\right) dx + \ln(x^2 + y^2) dy \right],$$

where C is the polar rectangle such that $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$, traced anticlockwise.

$[-\pi]$

$$\begin{aligned} & \int_C \left[\arctan\left(\frac{y}{x}\right) dx + \ln(x^2 + y^2) dy \right] \\ &= \int_C \arctan\left(\frac{y}{x}\right) dx + \ln(x^2 + y^2) dy \\ & \text{SPLIT into 4 sections } C_1 \text{ to } C_4 \text{ of length } \Delta\theta \text{ along boundary} \\ &= \int_0^{\pi} \theta(-2\sin\theta d\theta) + (\ln(4)(2\cos\theta d\theta)) \overset{C_2}{\curvearrowleft} \\ & \quad + \\ & \quad \int_0^{\pi} \theta(-\sin\theta d\theta) + \ln(4)\cos\theta d\theta \overset{C_3}{\curvearrowleft} \\ &= \int_0^{\pi} [-2\sin\theta + (3\ln 2)\cos\theta + \theta\cos\theta] d\theta \\ &= \int_0^{\pi} [(3\ln 2)\cos\theta - \theta\sin\theta] d\theta \quad \text{BY PARTS} \\ &= \left[(3\ln 2)\sin\theta \right]_0^{\pi} - \left\{ -\theta\cos\theta \right\}_0^{\pi} + \int_0^{\pi} \theta\sin\theta d\theta \\ &= [0] - \left[\theta\cos\theta \right]_0^{\pi} \\ &= -\pi \end{aligned}$$

$C_1: y=0, dy=0$ $C_2: x=2\cos\theta$ $y=2\sin\theta$ $\theta \text{ from } 0 \text{ to } \pi$ $d\theta = -2\sin\theta d\theta$ $dy = 2\cos\theta d\theta$
$C_3: y=0, dy=0$ $C_4: x=\cos\theta$ $y=\sin\theta$ $\theta \text{ from } \pi \text{ to } 0$ $d\theta = -\sin\theta d\theta$ $dy = \cos\theta d\theta$

Question 12

Evaluate the line integral

$$\oint_C [(2x-y)dx + (2y-x)dy],$$

where C is an ellipse with Cartesian equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

traced anticlockwise.

You may not use Green's theorem in this question.

[0]

$$\begin{aligned}
 \oint_C \vec{F} \cdot d\vec{s} &= \oint_C (2x-y, 2y-x) \cdot (dx, dy) \\
 &= \oint_C (2x-y)dx + (2y-x)dy \\
 \text{PARAMETRIZE THE ELLIPSE} \\
 \frac{x^2}{9} + \frac{y^2}{4} = 1 &\implies \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \\
 &\implies \begin{cases} x = 3\cos\theta \\ y = 2\sin\theta \end{cases}, 0 \leq \theta \leq 2\pi \\
 \dots &= \int_0^{2\pi} ((6\cos\theta - 2\sin\theta) \frac{dx}{d\theta} + (4\sin\theta - 3\cos\theta) \frac{dy}{d\theta}) d\theta \\
 &= \int_0^{2\pi} [6\cos\theta(-2\sin\theta) - 3\sin\theta] + (4\sin\theta - 3\cos\theta)(2\cos\theta) d\theta \\
 &= \int_0^{2\pi} -12\cos\theta\sin\theta + 8\sin^2\theta + 6\cos^2\theta - 6\cos\theta d\theta \\
 &= \int_0^{2\pi} -12\cos\theta\sin\theta - 6(\cos^2\theta - \sin^2\theta) d\theta \\
 &= \int_0^{2\pi} -12\cos\theta\sin\theta - 6\cos2\theta d\theta \\
 &= \int_0^{2\pi} -5\sin2\theta - 6\cos2\theta d\theta = 0 \quad \text{NO PARENTHESIS FOR THESE LIMITS} \\
 \text{OR BY GREEN'S THEOREM} \\
 \oint_C (P dx + Q dy) &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dy dx \\
 &= \iint_D 0 dy dx = 0
 \end{aligned}$$

Question 13

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (x - 3y)\mathbf{i} + (y - 2x)\mathbf{j}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the ellipse with cartesian equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

You may not use Green's theorem in this question.

[6]

$\mathbf{F} = (x - 3y, y - 2x)$ $C: \frac{x^2}{9} + \frac{y^2}{4} = 1$

PARAMETRIZE THE ELLIPSE $\frac{x^2}{9} + \left(\frac{y}{2}\right)^2 = 1$
 $\Rightarrow 3x^2 + y^2 = 1$
 $\Rightarrow 9x^2 + y^2 = 9$
 $\Rightarrow 9x^2 + 4y^2 = 36$

$x = 3\cos t, \quad 0 \leq t < 2\pi$
 $y = 2\sin t$

$dx = -3\sin t dt$
 $dy = 2\cos t dt$

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (x - 3y, y - 2x) \cdot (dx, dy) \\ &= \int_0^{2\pi} (3\cos t - 3\sin t)(-3\sin t) + (2\sin t - 2\cos t)(2\cos t) dt \\ &= \int_{t=0}^{\pi} -9\cos^2 t + 9\sin^2 t + 4\sin^2 t - 12\cos^2 t dt \\ &= \int_{t=0}^{\pi} -5\cos^2 t + 5\sin^2 t - 12\cos^2 t dt \\ &= \int_{t=0}^{\pi} -5\cos^2 t + 5(1 - \cos^2 t) - 12\cos^2 t dt \\ &= \int_{t=0}^{\pi} -5\cancel{\cos^2 t} + 5\cancel{(1 - \cos^2 t)} - 12\cancel{\cos^2 t} dt \\ &\quad \text{NO PARENTHESIS SINCE THESE UNITS} \\ &= \int_0^{\pi} 3 dt \\ &= 2\pi \times 3 \\ &= 6\pi \end{aligned}$$

Question 14

$$\mathbf{F}(x, y) = \left(-\frac{y}{x^2 + y^2} \right) \mathbf{i} + \left(\frac{x}{x^2 + y^2} \right) \mathbf{j}.$$

By considering the line integral of \mathbf{F} over two different suitably parameterized closed paths, show that

$$\int_0^{2\pi} \frac{1}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta = \frac{2\pi}{ab},$$

where a and b are real constants.

You may assume without proof that the line integral of \mathbf{F} yields the same value over any simple closed curve which contains the origin.

[] , proof

SETTING UP THE LINE INTEGRAL: CIRCLE A COUNTER-CLOCKWISE C WINDS ONCE

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \left(-\frac{y}{x^2 + y^2} \right) dx + \left(\frac{x}{x^2 + y^2} \right) dy = \oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

PARAMETRIZING CIRCLE A (ANTICLOCKWISE DIRECTION)

- $x = a \cos \theta$
- $y = a \sin \theta$
- $dx = -a \sin \theta d\theta$
- $dy = a \cos \theta d\theta$
- θ runs from 0 to 2π

$$= \int_0^{2\pi} \frac{-a \sin \theta}{a^2 \cos^2 \theta + a^2 \sin^2 \theta} (-a \sin \theta d\theta) + \frac{a \cos \theta}{a^2 \cos^2 \theta + a^2 \sin^2 \theta} (a \cos \theta d\theta)$$

$$= \int_0^{2\pi} \frac{-a^2 \sin^2 \theta}{a^2 (\cos^2 \theta + \sin^2 \theta)} (-a \sin \theta d\theta) + \int_0^{2\pi} \frac{a^2 \cos^2 \theta}{a^2 (\cos^2 \theta + \sin^2 \theta)} (a \cos \theta d\theta)$$

$$= \int_0^{2\pi} 1 d\theta = 2\pi$$

NOTHIN' DOIN' HERE

- $x = a \cos \theta$
- $y = a \sin \theta$
- $dx = -a \sin \theta d\theta$
- $dy = a \cos \theta d\theta$
- θ runs from 0 to 2π

THUS WE HAVE

$$\begin{aligned} &\Rightarrow \oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = 2\pi \\ &\Rightarrow \int_0^{2\pi} \frac{-a \sin \theta}{a^2 \cos^2 \theta + a^2 \sin^2 \theta} (-a \sin \theta d\theta) + \frac{a \cos \theta}{a^2 \cos^2 \theta + a^2 \sin^2 \theta} (a \cos \theta d\theta) = 2\pi \\ &\Rightarrow \int_0^{2\pi} \frac{a^2 \sin^2 \theta}{a^2 (\cos^2 \theta + \sin^2 \theta)} d\theta + \int_0^{2\pi} \frac{a^2 \cos^2 \theta}{a^2 (\cos^2 \theta + \sin^2 \theta)} d\theta = 2\pi \end{aligned}$$

+ A LITTLE AND INTO THIS SPOT:

$$\mathbf{F} = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

$$\nabla_y \mathbf{F} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{2x}{(x^2+y^2)^2} \\ 0 & 0 & \frac{2y}{(x^2+y^2)^2} \\ 0 & 0 & 0 \end{bmatrix}$$

LOOKING AT THE k COMPONENT

$$\frac{\partial}{\partial x} \left[\frac{-y}{x^2 + y^2} \right] + \frac{\partial}{\partial y} \left[\frac{x}{x^2 + y^2} \right] = \frac{(x^2+y^2)(-2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1-y^2)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2+x^2+y^2}{(x^2+y^2)^2} = 0$$

YET THE INTEGRATION OVER A CLOSED PATH DID NOT HIT ZERO!
WORK FURTHER:

$$\frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x}$$

$\frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x} = \frac{\partial}{\partial y} \left[\frac{-y}{x^2 + y^2} \right] + \frac{\partial}{\partial x} \left[\frac{x}{x^2 + y^2} \right] = \frac{2xy}{(x^2+y^2)^2} + \frac{-2x^2}{(x^2+y^2)^2} = 0$

... IS SIMPLY VERIFIED, SO WORK ON...

... COMPLEX NUMBERS, REVISIT ETC. GOIN TO HAND...

$f(z) = \frac{1}{z^2}$; AND THE INTEGRATION OVER A UNIT CIRCLE AT THE ORIGIN

$$\frac{1}{z} dz = 2\pi i \quad (\text{STANDARD})$$

$$\oint_C \frac{1}{z^2} dz = 2\pi i$$

$$\oint_C \frac{1}{z^2} (dz + idy) = 2\pi i$$

$$\oint_C \frac{1}{z^2} \left(\frac{dx}{dx+iy} + \frac{dy}{dx+iy} \right) (dx + idy) = 2\pi i$$

$$\oint_C \frac{1}{z^2} \left[2dx + idy + idy + ydy \right] = 2\pi i$$

$$\oint_C \left[\frac{2x}{x^2+y^2} dx + \frac{y}{x^2+y^2} dy \right] + \left[\frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \right] = 2\pi i$$

THIS ANSWERS PRECISELY, FOR ANY CLOSED PATH THAT CONTAINS THE ORIGIN

LINE INTEGRALS

IN 3 DIMENSIONS

Question 1

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = \left(x^2 y \right) \mathbf{i} + \left(4xy^2 \right) \mathbf{j} + \left(-6xz \right) \mathbf{k}.$$

Evaluate the line integral

$$\int_{(0,0,0)}^{(10,4,8)} \mathbf{F} \cdot d\mathbf{r}, \quad \text{where } d\mathbf{r} = (dx, dy, dz)^T,$$

along a path given by the parametric equations

$$x = 5t, \quad y = t^2, \quad z = t^3.$$

$$\boxed{-\frac{800}{7}}$$

$$\begin{aligned}
 \mathbf{F}(x,y,z) &= (x^2y + xy^2) \mathbf{i} + (4xy^2) \mathbf{j} + (-6xz) \mathbf{k} \\
 x = 5t &\Rightarrow dx = 5dt \\
 y = t^2 &\Rightarrow dy = 2t dt \\
 z = t^3 &\Rightarrow dz = 3t^2 dt
 \end{aligned}$$

$$\begin{aligned}
 \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \right) \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \\
 &= \int_0^2 \left(25t^3 + 4t^5 + 3t^4 \right) dt = \int_0^2 (5t^4 + 4t^5 + 3t^4) dt \\
 &= \left[5t^5 + 4t^6 + 3t^5 \right]_0^2 = 80t^5 = 80t^5 \\
 &= 800 - \frac{800}{7} = -\frac{800}{7}
 \end{aligned}$$

Question 2

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (x^2 y) \mathbf{i} + (xy^2) \mathbf{j} + (yz) \mathbf{k}.$$

Evaluate the line integral

$$\int_{(0,0,0)}^{(1,2,3)} \mathbf{F} \cdot d\mathbf{r}, \quad \text{where } d\mathbf{r} = (dx, dy, dz)^T,$$

along a path of three straight line segments joining $(0,0,0)$ to $(1,0,0)$, $(1,0,0)$ to $(1,2,0)$ and $(1,2,0)$ to $(1,2,3)$.

$\boxed{\frac{35}{3}}$

LOOKING AT THE PATH IN SEPARATE SEGMENTS

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (x^2 y, xy^2, yz) \cdot (dx, dy, dz) = \int_0^1 x^2 y \, dx + xy^2 \, dy + yz \, dz$$

- From $(0,0,0)$ to $(1,0,0)$ $x=0 \quad dy=0 \quad dz=0$ \Rightarrow RUNS FROM 0 TO 1
- From $(1,0,0)$ to $(1,2,0)$ $x=1 \quad dx=0 \quad dz=0$ \Rightarrow RUNS FROM 0 TO 2
- From $(1,2,0)$ to $(1,2,3)$ $x=1 \quad dz=0 \quad dy=0$ \Rightarrow RUNS FROM 0 TO 3

REASONING IS THE NOTATION

$$\dots = \int_0^{x+1} 0 \, dx + \int_{y=0}^{y=2} x^2 y \, dy + \int_{z=0}^{z=3} xz \, dz$$

$$= \left[\frac{1}{3} x^3 \right]_0^{x+1} + \left[z^2 \right]_0^3 = \frac{8}{3} + 9 = \frac{35}{3}$$

Question 3

It is given that

$$\mathbf{F}(x, y, z) \equiv \mathbf{j} \wedge \mathbf{r},$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the closed curve given parametrically by

$$\mathbf{R}(t) = (t - t^2)\mathbf{i} + (2t - 2t^2)\mathbf{j} + (t^2 - t^3)\mathbf{k}, \quad 0 \leq t \leq 1.$$

$$\boxed{-\frac{1}{30}}$$

$$\begin{aligned}
 \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = (z\mathbf{i} - x\mathbf{j} - y\mathbf{k}) = (z\mathbf{i} - x\mathbf{j}) \\
 \text{param.} \\
 C: \quad \frac{d\mathbf{R}}{dt} &= \left[t - \frac{d}{dt} t^2, 2t - 2t^2, t^2 - t^3 \right] \\
 \frac{dx}{dt} &= \left[1 - 2t, 2 - 4t, 2t - 3t^2 \right] \\
 \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (z\mathbf{i} - x\mathbf{j}) \cdot (dx, dy, dz) \\
 &= \int_C z \, dx - x \, dy \\
 &\stackrel{\circlearrowleft}{=} \int_{t=0}^{t=1} (t^2 - t^3)(1 - 2t) \, dt - (t - t^2)(2t - 3t^2) \, dt \\
 &= \int_0^1 \cancel{t^2} - \cancel{2t^3} - \cancel{t^3} + \cancel{2t^4} - \cancel{2t^2} + \cancel{3t^3} - \cancel{3t^4} \, dt \\
 &= \int_0^1 -t^2 + 2t^3 - t^2 \, dt = \left[-\frac{1}{3}t^3 + \frac{1}{4}t^4 - \frac{1}{3}t^3 \right]_0^1 \\
 &= \left(-\frac{1}{3} + \frac{1}{2} - \frac{1}{3} \right) - (0) = \frac{-6 + 15 - 10}{30} = -\frac{1}{30}
 \end{aligned}$$

Question 4

The simple closed curve C has Cartesian equation

$$x^2 + y^2 = 4, \quad z = 3.$$

Given that $\mathbf{F} = x^2 z \mathbf{i} + y^2 x \mathbf{j} + z^2 y \mathbf{k}$, evaluate the integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

You may not use Green's theorem in this question.

4π

$\mathbf{F} = x^2 z \mathbf{i} + y^2 x \mathbf{j} + z^2 y \mathbf{k}$

$C: x^2 + y^2 = 4, \quad z = 3$

$z = 3, \quad dz = 0$

$x = 2\cos\theta, \quad dx = -2\sin\theta \, d\theta$

$y = 2\sin\theta, \quad dy = 2\cos\theta \, d\theta$

$0 \leq \theta \leq 2\pi$

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (x^2 z \mathbf{i} + y^2 x \mathbf{j} + z^2 y \mathbf{k}) \cdot (dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}) \, d\theta \\ &= \int_{\theta=0}^{2\pi} (2\cos\theta)^2 \times 3 \times (-2\sin\theta \, d\theta) + (2\sin\theta)^2 (2\cos\theta) (2\cos\theta \, d\theta) \\ &= \int_{\theta=0}^{2\pi} [-24\cos^2\theta \sin\theta + 16\sin^2\theta \cos^2\theta] \, d\theta \\ &\quad \text{ZERO CONTRIBUTION BECAUSE THESE LIMITS} \\ &= \int_{\theta=0}^{2\pi} 16 \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta) \right) (\frac{1}{2} - \frac{1}{2}\cos(2\theta)) \, d\theta = \int_{\theta=0}^{2\pi} 16 \left(\frac{1}{4} - \frac{1}{2}\cos(2\theta) \right) \, d\theta \\ &= \int_{\theta=0}^{2\pi} 4 - 4\cos^2\theta \, d\theta = \int_{\theta=0}^{2\pi} 4 - 4 \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta) \right) \, d\theta \\ &= \int_{\theta=0}^{2\pi} 4 - 2 - 2\cos(2\theta) \, d\theta \\ &\quad \text{ZERO CONTRIBUTION BECAUSE THESE LIMITS} \\ &= \int_0^{2\pi} 2 \, d\theta \\ &= 4\pi \end{aligned}$$

Question 5

$$\mathbf{F} = (xz - y)\mathbf{i} + (xy + z)\mathbf{j} + (x^2 + y^2 + z^2)\mathbf{k}.$$

Determine the work done by \mathbf{F} , when it moves in a complete revolution in a circular path of radius 2 around the z axis, at the level of the plane with equation $z = 6$.

You may not use Green's theorem in this question.

4π

$\mathbf{F} = (xz - y)\mathbf{i} + (xy + z)\mathbf{j} + (x^2 + y^2 + z^2)\mathbf{k}$

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (xz - y, xy + z, x^2 + y^2 + z^2) \cdot (dx, dy, dz)$$

$$= \oint_C (2x - y) dx + (xy + z) dy + (x^2 + y^2) dz$$

\leftarrow Circles about equation $x^2 + y^2 = 4$, $z = 6 \rightarrow dz = 0$

$$= \oint_C (2x - y) dx + (xy + 6) dy$$

\leftarrow Parameterise the circle $x = 2\cos\theta$, $0 \leq \theta < 2\pi$, $y = 2\sin\theta$

$$= \int_{0 \rightarrow 0}^{2\pi} (2x - 2\sin\theta)(-2\sin\theta d\theta) + (4\cos\theta\sin\theta + 6)(2\cos\theta d\theta)$$

$$= \int_{0 \rightarrow 0}^{2\pi} -2\cos\theta\sin^2\theta + 4\sin^2\theta + 8\cos^2\theta\sin\theta + 12\cos\theta d\theta$$

$$= \int_{0 \rightarrow 0}^{2\pi} -12\sin\theta d\theta + \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta \right) + 8\cos\theta\sin\theta + 12\cos\theta d\theta$$

$$= \int_{0 \rightarrow 0}^{2\pi} -12\sin\theta d\theta + 2 - 2\cos 2\theta + 8\cos\theta\sin\theta + 12\cos\theta d\theta$$

$$= \left[12\theta \right]_{0}^{2\pi} = 4\pi$$

Question 6

Evaluate the integral

$$\int_{(1,1,0)}^{(5,3,4)} (3x - 2y) \, dx + (y + z) \, dy + (1 - z^2) \, dz,$$

along the straight line segment joining the points with Cartesian coordinates $(1,1,0)$ and $(5,3,4)$.

, $\frac{32}{3}$

<p>• START BY PARAMETRISING THE LINE SEGMENT USING PARAMETERS</p> $\begin{aligned} \mathbf{a} &= (1,1,0) \\ \mathbf{b} &= (5,3,4) \\ \overrightarrow{AB} &= \mathbf{b} - \mathbf{a} = (5,3,4) - (1,1,0) \\ &= (4,2,4) \\ \mathbf{f} &= (3x - 2y) = C_1(t)x + C_2(t)y \\ \mathbf{f} &= (3t, 2t+2) = (C_1(t), 2C_2(t) + 2) \end{aligned}$	<p>HENCE USE THESE</p> $\begin{cases} x = 4t \\ y = 2t+1 \\ z = 4t \end{cases} \Rightarrow \begin{cases} dx = 4dt \\ dy = 2dt \\ dz = 4dt \end{cases}$ <p style="text-align: center;">$0 \leq t \leq 1$</p> <p style="text-align: center;">$(1,1,0)$ $(5,3,4)$</p>
<p>• DETERMINING THE LINE INTERVAL</p> $\begin{aligned} &\int_{(1,1,0)}^{(5,3,4)} (3x - 2y) \, dx + (y + z) \, dy + (1 - z^2) \, dz \\ &= \int_{t=0}^{t=1} \left[\int_{x=4t}^{x=4t+4} [3(4t) - 2(2t+1)](4dt) + [(2t+1) + 4t](2dt) + [1 - (4t)^2](4dt) \right] dt \\ &= \int_{t=0}^{t=1} \left[(8t+4)(4dt) + (6t+1)(2dt) + (1-16t^2)(4dt) \right] dt \\ &= \int_0^1 (32t^2 + 12t + 12 + 4 - 64t^2) dt \\ &= \int_0^1 (-32t^2 + 12t + 16) dt \\ &= \left[-\frac{32}{3}t^3 + 6t^2 + 16t \right]_0^1 \\ &= \left(-\frac{32}{3} + 12 + 16 \right) - (0) \\ &= \frac{-64 + 60 + 48}{3} = \frac{32}{3} \quad \checkmark \end{aligned}$	

Question 7

$$\mathbf{F}(x, y, z) \equiv yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$$

Show that the vector field \mathbf{F} is conservative, and hence evaluate the integral

$$\int_{(1,1,4)}^{(3,5,10)} \mathbf{F} \cdot d\mathbf{r}$$

1484

$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz \end{vmatrix} = (2xz - 2yz, 2yz - 2xz, 2z^2 - z^2) = (0, 0, 0)$

∴ FIELD IS CONSERVATIVE

If conservative, $\mathbf{F} = -\nabla \phi$, for some scalar ϕ .
By inspection, $\phi = -\frac{1}{3}xyz^3$.

Now, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(1,1,4)}^{(3,5,10)} -\nabla \phi \cdot d\mathbf{r} = -\int_{(1,1,4)}^{(3,5,10)} \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \cdot (dx, dy, dz)$

$$= -\int_{(1,1,4)}^{(3,5,10)} \left(\frac{\partial \phi}{\partial x} \right) dx = -\int_{(1,1,4)}^{(3,5,10)} \left(-2yz^2 \right) dx = -\int_{(1,1,4)}^{(3,5,10)} (-2yz^2) dx$$

$$= [2yz^2]_{(1,1,4)}^{(3,5,10)} = (2 \cdot 5 \cdot 125) - (2 \cdot 1 \cdot 4) = 1250 - 8 = 1242$$

ANALYTICALLY, CROSS THE REGION
DIRECT FROM (1,1,4) TO (3,5,10)
DIRECT USE: $(3,5,10) - (1,1,4)$

$$\int_{(1,1,4)}^{(3,5,10)} \mathbf{F} \cdot d\mathbf{r} = \int_{(1,1,4)}^{(3,5,10)} (yz^2, xz^2, 2xyz) \cdot (dx, dy, dz)$$

$$= \int_{(1,1,4)}^{(3,5,10)} (yz^2) dx + xz^2 dy + 2xyz dz$$

$$= \int_0^2 (2t+22t^2+28t^3) dt + \int_1^5 (2t^2+11t^3+13t^4) dt + \int_4^{10} (2t^3+20t^4+28t^5) dt$$

$$= \int_0^2 (2t+22t^2+28t^3) dt + \int_1^5 (2t^2+11t^3+13t^4) dt + \int_4^{10} (2t^3+20t^4+28t^5) dt$$

$(4t+4)(7t^2+28t+16) = \frac{32t^3+96t^2+64t}{3t^2+12t^2+13t+48}$

$(3t+4)(2t^2+11t+16) = \frac{36t^3+60t^2+24t}{3t^2+10t^2+9t+24}$

$$\dots = \int_0^2 72t^2+22t^3+28t^4+72 dt = \int_0^5 8t^4+75t^3+113t^2+7t dt$$

$$= [(2t^5) + (75t^4) + (113t^3) + (8t^2)] - 0 = 1484$$

ANALYTICALLY IN CARTESIAN

From (1,1,4) to (3,5,10): $\frac{y+1}{2+4} \frac{dy}{dz} = 1 \leq y \leq 5 \quad \leftarrow C_1$

From (3,5,10) to (3,5,4): $\frac{z-4}{2-4} \frac{dz}{dy} = 1 \leq z \leq 5 \quad \leftarrow C_2$

From (3,5,4) to (3,5,10): $\frac{z-3}{3-2} \frac{dz}{dy} = 4 \leq z \leq 10 \quad \leftarrow C_3$

Thus

$$\int_C yz^2 dx + xz^2 dy + 2xyz dz = \int_1^3 16 dz + \int_4^{10} 3x^2 dz + \int_4^{10} 2xyz^2 dz$$

$$= \int_1^3 16 dz + \int_4^5 48 dy + \int_4^{10} 320 dz = \left[16z \right]_1^3 + \left[48y \right]_4^5 + \left[320z \right]_4^{10}$$

$$= (48 - 16) + (240 - 48) + (500 - 240) = 32 + 192 + 260 = 484$$

AS EXPECTED

Question 8

A vector field \mathbf{F} is defined as

$$\mathbf{F}(x, y, z) \equiv [x+yz]\mathbf{i} + [y+xz]\mathbf{j} + [x(y+1)+z^2]\mathbf{k}.$$

The closed path C joins $(0,0,0)$ to $(1,1,1)$, $(1,1,1)$ to $(1,1,0)$, $(1,1,0)$ to $(0,0,0)$, in that order.

By writing

$$\mathbf{F}(x, y, z) = \mathbf{G}(x, y, z) + \mathbf{H}(x, y, z),$$

for some vector functions \mathbf{G} and \mathbf{H} , where $\nabla g(x, y, z) = \mathbf{G}(x, y, z)$ for some smooth scalar function $g(x, y, z)$, evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

$[-\frac{1}{2}]$

$\mathbf{F}(x, y, z) = [x+yz, y+xz, x(y+1)+z^2]$
 $= [xyz, y+xz, x(y+1)+z^2]$
 $= [xyz, y+xz, x^2+y^2+z^2] = [x_1, x_2, x_3]$
 $= \mathbf{G}(x, y, z) + \mathbf{H}(x, y, z)$

Now $\nabla \cdot \mathbf{G} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \mathbf{G} = (x_1, x_2, x_3) = 0$

Thus $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (\mathbf{G} + \mathbf{H}) \cdot d\mathbf{r} = \oint_C \mathbf{G} \cdot d\mathbf{r} + \mathbf{H} \cdot d\mathbf{r}$
 $= \oint_C \nabla \cdot \mathbf{G} \cdot d\mathbf{r} + \mathbf{H} \cdot d\mathbf{r} = \oint_C \mathbf{H} \cdot d\mathbf{r}$ CONSERVATIVITY

SPLIT THE PATH INTO C_1, C_2, C_3
 $\dots = \int_{C_1} \mathbf{H} \cdot d\mathbf{r} + \int_{C_2} \mathbf{H} \cdot d\mathbf{r} + \int_{C_3} \mathbf{H} \cdot d\mathbf{r}$

$C_1: (x_1, y_1, z_1) = (0, 0, 0)$
 $d\mathbf{r} = (dx, dy, dz) = (0, 0, 0)$
 $t \text{ from } 0 \text{ to } 1$

$C_2: (x_1, y_1, z_1) = (1, 1, 1)$
 $d\mathbf{r} = (dx, dy, dz) = (1, 1, 1)$
 $t \text{ from } 1 \text{ to } 0$

$C_3: (x_1, y_1, z_1) = (1, 1, 0)$
 $d\mathbf{r} = (dx, dy, dz) = (0, 0, 1)$
 $t \text{ from } 1 \text{ to } 0$

$$= \left[\frac{1}{2}t^2 \right]_0^1 + \left[t \right]_1^0$$

$$= \left(\frac{1}{2} - 0 \right) + (0 - 1)$$

$$= -\frac{1}{2}$$

Question 9

A vector field \mathbf{F} is defined as

$$\mathbf{F}(x, y, z) \equiv (yz + y^2)\mathbf{i} + (xz + 2xy)\mathbf{j} + (xy + 4z^3)\mathbf{k}.$$

- a) Show that \mathbf{F} is conservative.
- b) Hence evaluate the integral

$$\int_{(0,0,0)}^{(1,1,1)} \mathbf{F} \cdot d\mathbf{r}.$$

[3]

a) IF \mathbf{F} IS CONSERVATIVE $\nabla \cdot \mathbf{F} = 0$

$$\begin{cases} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_x}{\partial x} & \frac{\partial F_y}{\partial y} & \frac{\partial F_z}{\partial z} \\ yz + y^2 & xz + 2xy & xy + 4z^3 \end{cases} = 0$$

$$= \left[\frac{\partial}{\partial x}(yz + y^2) - \frac{\partial}{\partial y}(xz + 2xy), \frac{\partial}{\partial y}(xz + 2xy) - \frac{\partial}{\partial z}(xy + 4z^3), \frac{\partial}{\partial z}(xy + 4z^3) - \frac{\partial}{\partial x}(yz + y^2) \right]$$

$$= [x - x, y - y, 2 + 2x - (x + y)] = [0, 0, 0] = 0$$

INTEGRAL CONSERVATIVE

b) SINCE IT IS CONSERVATIVE ... \mathbf{F} IS $\nabla \phi$, SEE STATEMENT $\phi(1,1,1) - \phi(0,0,0)$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$\Rightarrow \int_{(0,0,0)}^{(1,1,1)} d\phi = \int_{(0,0,0)}^{(1,1,1)} \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right)$$

$$= \int_{(0,0,0)}^{(1,1,1)} \left[xz + 2xy + yz + y^2 + xy + 4z^3 \right] dV$$

$$= \int_{(0,0,0)}^{(1,1,1)} (1z + 2z^2 + z^3) dV$$

$$= 1 + 2 + 1 = 3$$

Question 10

A curve C is defined as

$$(x, y, z) = (\cos 3t, \sin 3t, t), 0 \leq t \leq 2\pi.$$

- a) Sketch the graph of C .

$$\mathbf{F}(x, y, z) \equiv xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}.$$

- b) Determine whether the vector field \mathbf{F} is conservative.
 c) Evaluate the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

$$\boxed{-\frac{\pi}{2}}$$

$(x, y, z) = (\cos t, \sin t, t)$ a) $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$

a)

b) $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = (y_z - z_y)\mathbf{i} - (x_z - z_x)\mathbf{j} + (x_y - y_x)\mathbf{k} = (0, 0, -2)$

IT IS NOT conservative! $\nabla \times \mathbf{F} \neq 0$

c) PARAMETRIZED
 $\begin{cases} x = \cos t, dx = -3\sin 3t \\ y = \sin t, dy = 3\cos 3t \\ z = t, dz = dt \end{cases}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) = \int_0^{2\pi} (xy \, dx + yz \, dy + zx \, dz)$

$= \int_0^{2\pi} [\cos t \sin t(-3\sin 3t) + \tan t(3\cos 3t) + t \cos 3t] \, dt$

$= \int_0^{2\pi} -3\cos^2 t \sin 3t + 3\sin t \tan 3t + t \cos 3t \, dt$

$= \int_0^{2\pi} \frac{3}{2} \sin^2 3t \, dt + \left[\frac{3}{2} \sin t \tan 3t + \frac{1}{3} t \cos 3t \right]_0^{2\pi}$

$= \frac{3}{2} \left[\frac{1}{2} \sin^2 3t \right]_0^{2\pi} + \left[\frac{3}{2} t \cos 3t \right]_0^{2\pi} - \int_0^{2\pi} \frac{1}{2} \sin 3t \cos 3t \, dt + \left[\frac{1}{6} t \cos^2 3t \right]_0^{2\pi}$

$= \frac{3}{4} \left[\tan^2 3t \right]_0^{2\pi} = \frac{3}{4} [0 - 2\pi \times 1] = -\frac{3\pi}{2}$

Question 11

Evaluate the integral

$$\int_{(-1,2,3)}^{(2,0,1)} (3x^2yz + 6x) dx + (x^3z - 8y) dy + (x^3y + 1) dz,$$

along a path joining the points with Cartesian coordinates $(-1, 2, 3)$ and $(2, 0, 1)$.

[29]

THE INTEGRAL APROPS INDEPENDENT OF THE PATH

Given $\vec{F} = (3x^2yz + 6x, x^3z - 8y, x^3y + 1)$

If independent of path, $\nabla \cdot \vec{F} = 0$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2yz + 6x & x^3z - 8y & x^3y + 1 \end{vmatrix} = \left[x^2 - x_1^2, 3x^2z - 3x_1^2y, 3x^2y \right] = (0, 0, 0)$$

∴ Hence $\int_{(-1,2,3)}^{(2,0,1)} (3x^2yz + 6x) dx + (x^3z - 8y) dy + (x^3y + 1) dz$

$\uparrow \quad \uparrow \quad \uparrow$

$\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz$

$\Rightarrow \begin{cases} f = x_1^2y^2 + 3x_1^2 + A(y_2) \\ f = x_1^3y - 8y_1 + B(x_2) \\ f = x_1^3y + 1 + C(z_2) \end{cases} \Rightarrow f(2,0,1) = 2^3 \cdot 0^2 + 3^2 - 8 \cdot 1 + 1 = 29$

So the answer is $\boxed{29}$

Question 12

A curve C is defined by $\mathbf{r} = \mathbf{r}(t)$, $0 \leq t \leq 2\pi$ as

$$\mathbf{r}(t) = (x, y, z) = [2(t - \sin t), \sqrt{3} \cos t, 1 + \cos t].$$

Evaluate the integral

$$\int_C z \, ds,$$

where s is the arclength along C .

$\frac{32}{3}$

$x = 2(t - \sin t)$
 $y = \sqrt{3} \cos t \quad 0 \leq t \leq 2\pi$
 $z = 1 + \cos t$

• Firstly $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{[2(1 - \cos t)]^2 + [-\sqrt{3}]^2 + [-\sin t]^2} dt$
 $= \sqrt{4(1 - \cos t)^2 + 3\sin^2 t + \sin^2 t} dt = \sqrt{4 - 8\cos t + 4\cos^2 t + 3\sin^2 t + \sin^2 t} dt$
 $= \sqrt{8 - 8\cos t} dt = \sqrt{8(1 - 2\sin^2 \frac{t}{2})} dt$
 $= \sqrt{16\sin^2 \frac{t}{2}} dt = 4\sin \frac{t}{2} dt$

• $\int_C z \, ds = \int_{0}^{2\pi} (1 + \cos t) (4\sin \frac{t}{2}) dt$
 $= \int_{0}^{2\pi} [1 + (2\cos \frac{t}{2} - 1)] [4\sin \frac{t}{2}] dt$
 $= \int_{0}^{2\pi} 8\cos^2 \frac{t}{2} \sin \frac{t}{2} dt$
 $= \left[\frac{8}{3} \times (-2\cos^3 \frac{t}{2}) \right]_0^{2\pi}$
 $= \left[\frac{16}{3} \cos^3 \frac{t}{2} \right]_0^{2\pi}$
 $= \frac{16}{3} - \frac{16}{3} (-1) = \frac{32}{3}$

Question 13

A vector field \mathbf{F} and a scalar field ψ are given.

$$\mathbf{F} = \left(3x^3y\right)\mathbf{i} + \left(15\sqrt{z}\right)\mathbf{j} - \left(\frac{13}{96}xz\right)\mathbf{k} \quad \text{and} \quad \psi(x, y, z) = xe^{\frac{2y}{\sqrt{z}}}.$$

Evaluate the integral

$$\int_{(0,0,0)}^{(2,4,64)} [\mathbf{F} + \nabla \psi] \cdot d\mathbf{r},$$

along the curve with parametric equations

$$x = \sqrt{t}, \quad y = t \quad \text{and} \quad z = t^3.$$

2e - 288

$\mathbf{F} = 3x^3y\mathbf{i} + 15\sqrt{z}\mathbf{j} - \frac{13}{96}xz\mathbf{k}$
 $\psi = 2e^{\frac{2y}{\sqrt{z}}}$

CURVE C:
 $x = t^{\frac{1}{2}} \Rightarrow dx = \frac{1}{2}t^{-\frac{1}{2}}dt$
 $y = t \Rightarrow dy = dt$
 $z = t^3 \Rightarrow dz = 3t^2dt$
 $(0,0,0) \rightarrow (2,4,64)$
 $t=0 \rightarrow t=4$

This

$$\begin{aligned} \int_C (\mathbf{F} + \nabla \psi) \cdot d\mathbf{r} &= \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \nabla \psi \cdot d\mathbf{r} \\ &= \int_{t=0}^4 3x^3y dx + 15z^{\frac{1}{2}}dy - \frac{13}{96}xz dz + \left[\psi \right]_{(0,0,0)}^{(2,4,64)} \\ &= \int_{t=0}^4 (3t^{\frac{3}{2}})^3 t^{\frac{1}{2}} + (15t^{\frac{1}{2}}) \cdot t^3 + (-\frac{13}{96}t^{\frac{3}{2}}t^3 \cdot 3t^2) dt + \left[2e^{\frac{2y}{\sqrt{z}}} \right]_{(0,0,0)}^{(2,4,64)} \\ &= \int_{t=0}^4 \frac{3}{2}t^{12} + 15t^{\frac{7}{2}} - \frac{13}{96}t^8 dt + 2e \\ &= \left[\frac{1}{14}t^{13} + 6t^{\frac{9}{2}} - \frac{13}{96}t^9 \right]_0^4 + 2e \\ &= (32 + 48 \times 2^5 - \frac{1}{6} \times 2^{13}) - (0) + 2e \\ &= 32 + 192 - 2^4 \times 2^8 + 2e \\ &= 224 - 2^8 + 2e \\ &= 224 - 512 + 2e \\ &= 2e - 288 \end{aligned}$$

Question 14

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (3x^2yz + 2z)\mathbf{i} + (x^3z + 2y)\mathbf{j} + (x^3y + 2x)\mathbf{k}.$$

Evaluate the line integral

$$\int_{(-2,2,0)}^{(4,0,1)} \mathbf{F} \cdot d\mathbf{r},$$

along a path joining the points with Cartesian coordinates $(4,0,1)$ and $(-2,2,0)$.

[4]

IT "APPEARS" THAT INTEGRAL IS ~~INDEPENDENT~~ OF THE PATH
IF ~~INDEPENDENT~~ THEN $\nabla \times \mathbf{F} = 0$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xyz+2z & x^3z+2y & x^3y+2x \end{vmatrix} = \left[x^3-x^3, (3x^2y+2)-(3x^2z+2), 3x^2-3x^2 \right] = (0,0,0)$$

\therefore INDEPENDENT OF THE PATH

$$\int_{(-2,2,0)}^{(4,0,1)} \mathbf{F} \cdot d\mathbf{r} = \int_{(-2,2,0)}^{(4,0,1)} (3xyz+2z) \, dx + (x^3z+2y) \, dy + (x^3y+2x) \, dz$$

$$= \int_{(-2,2,0)}^{(4,0,1)} (3xyz+2z) \, dx + (3x^2z+2y) \, dy + (x^3y+2x) \, dz$$

$$d\mathbf{f} = \left(\frac{\partial f}{\partial x} \right) dx + \left(\frac{\partial f}{\partial y} \right) dy + \left(\frac{\partial f}{\partial z} \right) dz$$

$$\left. \begin{aligned} f &= 3xyz+2z \\ &= 3xy^2+2z+1/2(y^3) \\ f &= 3xy^2+2z+C_1(y) \end{aligned} \right\} \Rightarrow f(4,0,1)=3(4)(0)^2+2(0)+C_1(0)$$

$$\left. \begin{aligned} f &= 3xy^2+2z+C_1(y) \\ &= 3(4)(0)^2+2(0)+C_1(0) \\ f &= C_1(0) \end{aligned} \right\} \Rightarrow f(4,0,1)=0$$

$$= \int_{(-2,2,0)}^{(4,0,1)} d\mathbf{f} = \int_{(-2,2,0)}^{(4,0,1)} f(4,0,1) - f(-2,2,0)$$

$$= \int_{(-2,2,0)}^{(4,0,1)} (3(4)(0)^2+2(0)) - (3(-2)(2)^2+2(-2))$$

$$= (0+0+0) - (0+0+4)$$

$$= 4$$

Question 15

It is given that the vector function \mathbf{F} satisfies

$$\mathbf{F} = (1+xy^2)\mathbf{i} + (x+xyz)\mathbf{j} + (y\sin z)\mathbf{k}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the anticlockwise cartesian path

$$x^2 + y^2 = 16, \quad z = 3.$$

You may not use Green's theorem in this question.

16π

$E = (1+xy^2, x+xyz, y\sin z)$

wire: $x^2+y^2=16, z=3$

PARAMETERISE THE CURVE AS

$$\begin{aligned} x &= 4\cos\theta & dx = -4\sin\theta \, d\theta \\ y &= 4\sin\theta & dy = 4\cos\theta \, d\theta \\ z &= 3 & dz = 0 \end{aligned}$$

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} ((1+xy^2, x+xyz, y\sin z)) \cdot (dx, dy, dz) \, d\theta$

$$\begin{aligned} &= \int_{\theta=0}^{2\pi} [1+4\cos^2\theta, 4\cos\theta+4\cos\theta\sin\theta, \sin\theta] \cdot [-4\sin\theta, 4\cos\theta, 0] \, d\theta \\ &= 4 \int_{\theta=0}^{2\pi} -\sin\theta - 4\cos^2\theta + 4\cos^2\theta + 4\cos\theta\sin\theta \, d\theta \\ &= 4 \int_{\theta=0}^{2\pi} 4(\frac{1}{2} + \frac{1}{2}\cos 2\theta) \, d\theta \\ &= 4 \int_{\theta=0}^{2\pi} 2 \, d\theta \\ &= 8 \times 2\pi \\ &= 16\pi \end{aligned}$$

(NO CONCERN OVER THESE LIMITS)

Question 16

Evaluate the line integral

$$\oint_C [x \, dx + (x - 2yz) \, dy + (x^2 + z) \, dz],$$

where C is the intersection of the surfaces with respective Cartesian equations

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0 \quad \text{and} \quad x^2 + y^2 = x, \quad z \geq 0.$$

$$\boxed{\frac{\pi}{4}}$$

$\int_C [x \, dz + (x - 2yz) \, dy + (x^2 + z) \, dz]$

WRITING THE LINE INTEGRAL AS A DOT PRODUCT

$$\dots = \int_C (\mathbf{r}_1 \cdot \mathbf{r}_2 \mathbf{r}_3) \cdot (ds_1 ds_2 ds_3)$$

... BY GREEN'S THEOREM

$$\dots = \iint_R \nabla \cdot (\mathbf{r}_1 \cdot \mathbf{r}_2 \mathbf{r}_3) \cdot d\mathbf{S}$$

$$= \iint_R (\mathbf{r}_3 \cdot \mathbf{r}_2 \mathbf{r}_1) \cdot \frac{\partial}{\partial z} d\mathbf{S}$$

$$= \iint_R (\mathbf{r}_3 \cdot \mathbf{r}_2 \mathbf{r}_1) \cdot \frac{\partial}{\partial z} d\mathbf{S}$$

PLANE R: DRAW THE CIRCLE PLANE ON THE REGION R AND THAT $x^2+y^2 \leq x$

- since $\nabla \cdot (\mathbf{r}_3) = x^2y^2z^2 + z^2 - 1$
- $\nabla \cdot (\mathbf{r}_3) = (2y, 2y, 2z)$
- $d\mathbf{S} = \frac{dA}{\sqrt{1+x^2+y^2}} \mathbf{k}$

$$= \int_R (\mathbf{r}_3 \cdot \mathbf{r}_2 \mathbf{r}_1) \cdot \frac{1}{\sqrt{1+x^2+y^2}} dA$$

$$= \int_R (\mathbf{r}_3 \cdot \mathbf{r}_2 \mathbf{r}_1) \cdot \frac{1}{\sqrt{1+x^2+y^2}} \frac{\partial}{\partial z} dA$$

$$= \int_R (\mathbf{r}_3 \cdot \mathbf{r}_2 \mathbf{r}_1) \cdot \frac{1}{(1+x^2+y^2)^{3/2}} dA$$

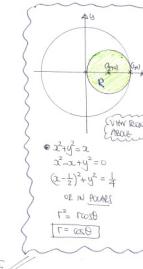
$$= \int_R \mathbf{r}_3 \cdot \mathbf{r}_2 \mathbf{r}_1 \cdot \frac{dx \, dy}{(1+x^2+y^2)^{3/2}}$$

$$= \int_R 1 \, dx \, dy = \text{Area of } R = \pi(\frac{1}{2})^2 = \frac{\pi}{4}$$

C IS THE INTERSECTION OF
THE SURFACE $x^2+y^2+z^2=1, z \geq 0$
 $\& x^2+y^2=x, z \geq 0$

1	2	3
2	$\frac{\partial}{\partial y}$	$\frac{\partial}{\partial z}$
3	$\frac{\partial}{\partial x}$	x^2+y^2

Quadrants:
 $[0, -2y], [0, -2z], [1, -]$
 i.e.
 $[2y, -2z]$



• $x^2+y^2 \leq x$
 $x^2-x^2+y^2=0$
 $(x-\frac{1}{2})^2+y^2=\frac{1}{4}$
 i.e. R
 $r^2 \leq 0.25$
 $r=0.25$

Question 17

It is given that the vector field \mathbf{F} satisfies

$$\mathbf{F} = 8z\mathbf{i} + 4x\mathbf{j} + y\mathbf{k}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the intersection of the surfaces with respective Cartesian equations

$$z = x^2 + y^2 \quad \text{and} \quad z = y.$$

You may **not** use Stokes' Theorem in this question.

π

The handwritten solution shows the following steps:

- Equations of the surfaces: $Z = x^2 + y^2$ and $Z = y$. From these, we find $y = x^2 + y^2$, $x^2 - y^2 = 0$, and $x^2 = y^2 \Rightarrow x = \pm y$.
- Parameterize the circle: $x = \frac{1}{2}\cos\theta$, $y = \frac{1}{2}\sin\theta$, $z = \frac{1}{2} + \frac{1}{2}\sin\theta$. Radius $\frac{1}{2}$.
- Thus, $\mathbf{F}(t) = \left[\frac{3}{2}\cos\theta, \frac{4}{2}\sin\theta, \frac{1}{2} + \frac{1}{2}\sin\theta \right]$, $\mathbf{F}'(t) = \left[-\frac{3}{2}\sin\theta, \frac{4}{2}\cos\theta, \frac{1}{2}\cos\theta \right]$.
- Line integral setup: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (\mathbf{F}(t) \cdot d\mathbf{r}) \cdot dt$.
- Integration: $= \int_0^{2\pi} \left[\left(\frac{3}{2}\cos\theta, \frac{4}{2}\sin\theta, \frac{1}{2} + \frac{1}{2}\sin\theta \right) \cdot \left(-\frac{3}{2}\sin\theta, \frac{4}{2}\cos\theta, \frac{1}{2}\cos\theta \right) \right] dt$
- After simplification, we get: $= \int_0^{2\pi} \left[-\frac{9}{4}\cos\theta\sin\theta - 2\sin^2\theta + \frac{1}{2}\cos^2\theta + \frac{1}{2}\cos\theta\sin\theta \right] dt$
- Integrating term by term: $= \int_0^{2\pi} \left[-\frac{9}{4}\cos\theta\sin\theta - 2\sin^2\theta \right] dt + \int_0^{2\pi} \left[\frac{1}{2}\cos^2\theta + \frac{1}{2}\cos\theta\sin\theta \right] dt$
- Final result: $= \frac{1}{2} \times 2\pi = \pi$.

Question 18

It is given that the vector field \mathbf{F} satisfies

$$\mathbf{F} = y^2 \mathbf{i} + z^2 \mathbf{j} + x^2 \mathbf{k}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the intersection of the surfaces with respective Cartesian equations

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0 \quad \text{and} \quad x^2 + y^2 = x, \quad z \geq 0.$$

You may **not** use Stokes' Theorem in this question.

$$\boxed{\frac{\pi}{4}}$$

[solution overleaf]

1. DRAW THE FIGURE AS NEEDED
REMEMBER "How much" the xy part?
 $x^2 + y^2 = 1$
 $x^2 + y^2 = 0$
 $x^2 + y^2 = \frac{1}{4}$

2. SOLVE SIMULTANEOUSLY TO DETERMINE THE CIRCLE
 $x^2 + y^2 = 1$
 $x^2 + y^2 = 0$
 $x^2 + y^2 = \frac{1}{4}$ SUBTRACT
 $x^2 + y^2 = -\frac{3}{4}$

3. SPECIAL PROFILES OF $x^2 + y^2 = 2$ (REGARDLESS OF WHETHER IT IS PART OF ONE CIRCLE)
PARAMETRIC: $x = t^2$, $y = t$
PARAMETRIC: $x = 1-t^2$, $y = t$
PARAMETRIC: $x = 1-t^2$, $y = t$
 $\therefore x^2 + y^2 = 2$

4. TO SET THE PROFILES OF THE CURVES
 $x(t) = [-t, \sqrt{1-t^2}]$, $y(t) = [0, t]$
 $x'(t) = [-1, \frac{-t}{\sqrt{1-t^2}}]$, $(dy/dx)_{t=0} = 1$
HOWEVER THE CLOCK WISE LOOKS LIKE AN INVERSE TILT
REVERSE THIN VERTICES OUT

$x(t) = [t, \sqrt{1-t^2}]$, $y(t) = [0, t]$

SPLIT THE PATH INTO TWO CLOCKWISE PATHS:

$W_1 = \int_{-\pi/2}^{\pi} t^2 C(-t)^2 C(2t) + t^2 (-\frac{1-t^2}{\sqrt{1-t^2}})^2 C(2t) + (1-t^2) dt$
 $W_2 = \int_{\pi/2}^{3\pi/2} t^2 C(-t)^2 C(2t) + t^2 (\frac{1-t^2}{\sqrt{1-t^2}})^2 C(2t) + (1-t^2) dt$

ADD THE TOTAL WORK: $W = W_1 + W_2$
REMEMBER THE LINES IN THE W_i 'S CANCEL OUT THE
FIRST & THIRD LINES & DOUBLE UP THE '2ND'
WAH! WHICH GIVES:

$W = \int_0^1 \frac{2t^2 \frac{dt^2}{1-t^2}}{(1-t^2)^2} dt = \int_0^1 \frac{4t^2-2t^4}{(1-t^2)^2} dt$

BY SUBSTITUTION LET
 $t = \sin \theta$
 $dt = \cos \theta d\theta$
 $t=0 \rightarrow \theta=0$
 $t=1 \rightarrow \theta=\frac{\pi}{2}$

$W = \int_0^{\frac{\pi}{2}} \frac{4 \sin^2 \theta - 2 \sin^4 \theta}{(\cos^2 \theta)^2} d\theta = \int_0^{\frac{\pi}{2}} 2 \sin^2 \theta d\theta = \text{SMALL NON-BETA FUNCTIONS}$

$B(\alpha, \beta) = \int_0^{\frac{\pi}{2}} 2 \sin^\alpha \theta (\cos^\beta \theta)^2 d\theta$

$= 2 \int_0^{\frac{\pi}{2}} 2 \sin^\alpha \theta (\cos^\beta \theta)^2 d\theta = \int_0^{\frac{\pi}{2}} 2 \sin^{\alpha-1} \theta (\cos^{\beta+1} \theta)^2 d\theta = 2 B(\frac{\alpha}{2}, \frac{\beta}{2}) - B(\frac{\alpha}{2}, \frac{\beta}{2})$

$= 2 \left[\frac{[\Gamma(\frac{\alpha}{2}) \Gamma(\frac{\beta}{2})]}{\Gamma(\alpha+\beta)} \right] - \left[\frac{\Gamma(\frac{\alpha}{2}) \Gamma(\frac{\beta}{2})}{\Gamma(\alpha)} \right] = 2 \frac{[\frac{1}{2} \Gamma(\frac{1}{2})] \Gamma(\frac{1}{2})}{2^{\frac{1}{2}}} - \frac{[\frac{1}{2} \Gamma(\frac{1}{2})] \Gamma(\frac{1}{2})}{1!} = \frac{\frac{1}{2} [\Gamma(\frac{1}{2})]^2 - \frac{1}{2} [\Gamma(\frac{1}{2})]^2}{2^{\frac{1}{2}}} = \frac{1}{4} \Gamma(\frac{1}{2})^2$

$\bullet B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}$
 $\bullet \Gamma(\alpha) = (\alpha-1) \Gamma(\alpha-1)$
 $\bullet \Gamma(\frac{1}{2}) = \sqrt{\pi}$
 $\bullet \Gamma(n) = (n-1)!, n \in \mathbb{N}$

ALTERNATIVE PARAMETRIZATION BY POLARS

1. $x^2 + y^2 = 1$
 $r^2 = 1 \cos \theta$
 $r = \cos \theta$ θ FROM $\frac{\pi}{2}$ TO $-\frac{\pi}{2}$
AS IT IS COUNTERCLOCKWISE

$x = r \cos \theta = (\cos \theta)/\cos \theta = \cos^2 \theta$
 $y = r \sin \theta = (\cos \theta)\sin \theta = -\cos \theta \sin \theta$

2. $x^2 + y^2 = 1 - (x^2 + y^2)$
 $2x^2 = 1 - 1 = 0$
 $x^2 = \sin^2 \theta$
 $\text{FOR } x \geq 0 \quad \therefore z = \sin \theta, \text{ FOR } \theta \text{ BETWEEN } \frac{\pi}{2}, 0$
 $z = -\sin \theta, \text{ FOR } \theta \text{ BETWEEN } 0, -\frac{\pi}{2}$

3. NOW
 $\theta \in \text{FROM } \frac{\pi}{2} \text{ TO } 0$
 $x(t) = [\cos \theta, \cos \theta \sin \theta]$
 $y(t) = [\cos \theta \sin \theta, -\cos \theta]$
 $x'(t) = [-\sin \theta, \cos \theta, \cos \theta]$
 $y'(t) = [\sin \theta, \cos \theta, -\cos \theta]$

1. THIS TIME THE INTEGRATION MUST BE DONE IN 2 SECTIONS

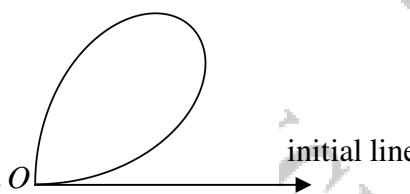
$W = \int_E \mathbf{F} \cdot d\mathbf{r} = \int_C (y^2, x^2, 1) \cdot (dx, dy, dz) = \int_C y^2 dx + x^2 dy + z^2 dz$ BECAUSE OF THE SINESES ONLY THOSE THAT WILL BE DIFFERENT IS THE dz TERM

THUS

$$\begin{aligned} &= \int_{\frac{\pi}{2}}^0 \left[(\cos^2 \theta)(-\sin \theta) + \sin^2 \theta \cos \theta \right] d\theta + \int_0^{\pi} \frac{\cos^4 \theta \cos^2 \theta}{\sin^2 \theta} d\theta + \int_{\pi}^{\frac{3\pi}{2}} \frac{\cos^2 \theta (-\cos \theta)}{\sin^2 \theta} d\theta \\ &= \int_{\frac{\pi}{2}}^0 (-2 \sin \theta \cos \theta + \sin^2 \theta \cos \theta) d\theta + \int_0^{\pi} \cos^6 \theta d\theta + \int_{\pi}^{\frac{3\pi}{2}} \cos^2 \theta (-\cos \theta) d\theta \\ &= \int_{\frac{\pi}{2}}^0 (-2 \sin \theta \cos \theta + \sin^2 \theta \cos \theta) d\theta + \int_0^{\pi} \cos^6 \theta d\theta + \int_{\pi}^{\frac{3\pi}{2}} \cos^2 \theta (-\cos \theta) d\theta \\ &= \int_{\frac{\pi}{2}}^0 \sin^2 \cos^2 \theta + \sin^4 \theta d\theta = 2 \int_{\frac{\pi}{2}}^0 \sin^2 \cos^2 \theta + \sin^4 \theta d\theta \quad \text{CANCEL} \\ &\bullet \text{BY BETTA GAMMA FUNCTIONS OR TRIG IDENTITIES} \\ &= \int_{\frac{\pi}{2}}^0 -2(\sin \theta)^2 \cos^2 \theta + 2(\sin \theta)^4 \cos^2 \theta d\theta = -B(\frac{3}{2}, \frac{3}{2}) + B(\frac{5}{2}, \frac{5}{2}) \\ &= -\frac{\Gamma(\frac{3}{2}) \Gamma(\frac{3}{2})}{\Gamma(3)} + \frac{\Gamma(\frac{5}{2}) \Gamma(\frac{5}{2})}{\Gamma(5)} = \frac{\frac{1}{2} \times \frac{1}{2} \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2})}{2!} + \frac{\frac{3}{2} \times \frac{1}{2} \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2})}{2!} \\ &= -\frac{1}{8} \pi + \frac{3}{8} \pi = \frac{\pi}{4} \end{aligned}$$

LINE INTEGRALS

IN POLAR COORDINATES

Question 1

The figure above shows the closed curve C with polar equation

$$r = \sin 2\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

The vector field \mathbf{F} is given in plane polar coordinates (r, θ) by

$$\mathbf{F}(r, \theta) = (r^2 \cos \theta \sin \theta) \hat{r} + (r \cos \theta) \hat{\theta}.$$

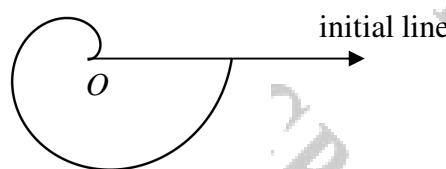
Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

**8
15**

$\mathbf{F}(r, \theta) = [r^2 \cos \theta \sin \theta \hat{r} + r \cos \theta \hat{\theta}]$
 $r(\theta) = \sin 2\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$
 $\mathbf{F} \cdot d\mathbf{r} = \int (r^2 \cos \theta \sin \theta) (r, r d\theta) \cdot (dr, r d\theta)$
 $= \int_0^{\frac{\pi}{2}} r^2 \cos \theta \sin \theta dr + r^2 \cos \theta d\theta$
 $= \int_0^{\frac{\pi}{2}} r^2 \sin 2\theta dr + r^2 \cos \theta d\theta$
 $= \int_0^{\frac{\pi}{2}} (\sin^2 \theta \sin 2\theta + \sin^2 \theta \cos^2 \theta) d\theta$
 $= \int_0^{\frac{\pi}{2}} (\sin^2 \theta \sin 2\theta + 4 \sin^2 \theta \cos^2 \theta) d\theta$
 $= \int_0^{\frac{\pi}{2}} (\sin^2 \theta \sin 2\theta + 4 \sin^2 \theta (1 - \sin^2 \theta)) d\theta$
 $= \int_0^{\frac{\pi}{2}} (\sin^2 \theta \sin 2\theta + 4 \sin^2 \theta - 4 \sin^4 \theta) d\theta$
 $= \left[\frac{1}{8} \sin^2 2\theta + \frac{4}{3} \sin^3 \theta - \frac{4}{3} \sin^5 \theta \right]_0^{\frac{\pi}{2}}$
 $= \left(0 + \frac{4}{3} - \frac{4}{3} \right) - (0)$
 $= \frac{8}{15}$

Question 2



The figure above shows the curve C with polar equation

$$r = \theta, 0 \leq \theta \leq 2\pi.$$

The vector field \mathbf{F} is given in Cartesian coordinates by

$$\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}.$$

Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

$$[2\pi^2]$$

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x_i y_j) \cdot (dx_i dy_j) = \dots$

Start w/o polar

$$x_i = r \cos \theta$$

$$dx_i = -r \sin \theta d\theta$$

$$y_j = r \sin \theta$$

$$dy_j = r \cos \theta d\theta$$

$$\dots = \int_C (r \cos \theta, r \sin \theta) \cdot (-r \sin \theta d\theta, r \cos \theta d\theta) =$$

$$= \int_C r \cos^2 \theta d\theta - r^2 \sin \theta \cos \theta d\theta + r \sin^2 \theta d\theta + r^2 \sin \theta \cos \theta d\theta$$

$$= \int_C r d\theta$$

Now on the curve $r = \theta, 0 \leq \theta < 2\pi$ or $0 \leq r < 2\pi$

$$= \int_{r=0}^{2\pi} r d\theta$$

or write $\int_{\theta=0}^{2\pi} \theta d\theta$
Since $d\theta = d\theta$

$$= \left[\frac{1}{2}\theta^2 \right]_0^{2\pi} = 2\pi^2$$

DE FROM BASIC PRINCIPLES

FIRST, DRAW SOME BASIC QUANTITIES

$$\mathbf{F} = (x_i y_j) \mathbf{i} - (r \sin \theta) \mathbf{j}$$

$$d\mathbf{r} = (r \cos \theta) \mathbf{i} + (r \sin \theta) \mathbf{j}$$

Now $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x_i y_j) \cdot (dx_i dy_j) = \int_C (x_i^2 + y_j^2) \cdot (dx_i + dy_j)$

$$= \int_C \left[r \cos \theta \hat{i} + r \sin \theta \hat{j} \right] \cdot \left[r \cos \theta \hat{i} + r \sin \theta \hat{j} \right] = \int_C [r^2 \cos^2 \theta + r^2 \sin^2 \theta] d\theta$$

$$= \int_C [r^2 + 0] d\theta = \int_C r d\theta$$

$\therefore \theta = \theta \Rightarrow d\theta = d\theta$

$$= \int_{\theta=0}^{2\pi} \theta d\theta = \left[\frac{1}{2}\theta^2 \right]_0^{2\pi} = 2\pi^2$$

ALTERNATIVE BY POLARS

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x_i y_j) \cdot (dx_i dy_j) = \int_C x_i dx_i + y_j dy_j$$

$x_i = r \cos \theta = \theta \cos \theta$
 $y_j = r \sin \theta = \theta \sin \theta$
 $dx_i = (\cos \theta - \theta \sin \theta) d\theta$
 $dy_j = (\sin \theta + \theta \cos \theta) d\theta$

$$= \int_C [(\theta \cos \theta)(\cos \theta - \theta \sin \theta) + (\theta \sin \theta)(\sin \theta + \theta \cos \theta)] d\theta$$

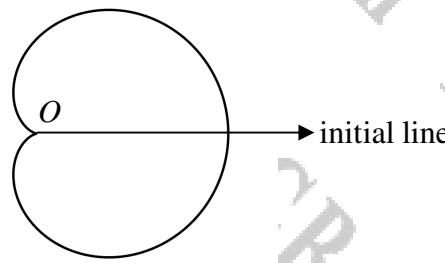
$$= \int_C [\theta \cos^2 \theta - \theta^2 \cos \theta \sin \theta + \theta \sin^2 \theta + \theta^2 \sin \theta \cos \theta] d\theta$$

$$= \int_C \theta (\cos^2 \theta + \sin^2 \theta) d\theta$$

$$= \int_{\theta=0}^{2\pi} \theta d\theta$$

$$= \left[\frac{1}{2}\theta^2 \right]_0^{2\pi} = 2\pi^2$$

Question 3



The figure above shows the closed curve C with polar equation

$$r = 1 + \cos \theta, \quad 0 \leq \theta \leq 2\pi.$$

The vector field \mathbf{F} is given in Cartesian coordinates by

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

[3π]

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C (-y\mathbf{i} + x\mathbf{j}) \cdot (dx, dy) = \dots \\ \text{Start from point } (1, 0) \Rightarrow \theta = 0 &\Rightarrow dx = \cos \theta d\theta = -\sin \theta d\theta \\ y = \sin \theta &\Rightarrow dy = \sin \theta d\theta + \cos \theta d\theta \\ &= \int_0^{2\pi} (-\sin \theta, \cos \theta) \cdot [\cos \theta d\theta - \sin \theta d\theta, \sin \theta d\theta + \cos \theta d\theta] \\ &= \int_0^{2\pi} -\sin^2 \theta d\theta + \sin^2 \theta d\theta + r \cos \theta \sin \theta d\theta + r \sin \theta \cos \theta d\theta + r^2 \cos^2 \theta d\theta \\ &= \int_0^{2\pi} r^2 d\theta \\ \text{But } r = 1 + \cos \theta, \quad 0 \leq \theta < 2\pi &\\ &= \int_0^{2\pi} (1 + \cos \theta)^2 d\theta = \int_{\theta=0}^{2\pi} 1 + 2\cos \theta + \cos^2 \theta d\theta \quad \text{NO CONTRADICTION SEE Q1(a)} \\ &= \int_0^{2\pi} 1 + \frac{1}{2} + \frac{1}{2} \cos 2\theta d\theta \\ \text{NO CONTRADICTION ONE THREE UNITS} &\\ &= \int_0^{2\pi} \frac{3}{2} d\theta \\ &= 3\pi \end{aligned}$$

ALTERNATIVE APPROACH

$$\begin{aligned} \mathbf{F} &= (\cos \theta)\mathbf{i} - (\sin \theta)\mathbf{j} \\ \mathbf{dr}/d\theta &= (\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j} \\ \text{TANGENT VECTOR} &= dr/d\theta = \mathbf{dr}/d\theta \hat{\mathbf{r}} \end{aligned}$$

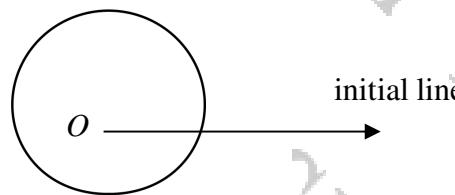
Now,

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (-y\mathbf{i} + x\mathbf{j}) \cdot (dx, dy) = \int_C (-y_1 + x_2) \cdot (dx_1 + dx_2) \\ &= \int_C -\sin \theta [\cos \theta - \sin \theta] + \cos \theta [\sin \theta + \cos \theta] \cdot [dr \hat{\mathbf{r}} + r d\theta \hat{\mathbf{r}}] \\ &= \int_C -r \sin \theta [\cos \theta - \sin \theta] + r \cos \theta [\sin \theta + \cos \theta] \cdot [dr \hat{\mathbf{r}} + r d\theta \hat{\mathbf{r}}] \\ &= \int_C r \hat{\mathbf{r}} \cdot (dr \hat{\mathbf{r}} + r d\theta \hat{\mathbf{r}}) = \int_C r^2 d\theta = \int_{\theta=0}^{2\pi} (1 + \cos \theta)^2 d\theta \\ &= \int_0^{2\pi} (1 + 2\cos \theta + \cos^2 \theta) d\theta = \int_0^{2\pi} 1 + \frac{1}{2} + \frac{1}{2} \cos 2\theta d\theta \\ &= \int_0^{2\pi} \frac{3}{2} d\theta = 3\pi \quad \text{NO CONTRADICTION ONE THREE UNITS} \end{aligned}$$

ALTERNATIVE APPROACH

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (-y\mathbf{i} + x\mathbf{j}) \cdot (dx, dy) = \int_C -y dx + x dy \\ \mathbf{r} = r \cos \theta \mathbf{i} &+ (1 + \cos \theta) \sin \theta \mathbf{j} = \cos \theta + \cos^2 \theta \\ y = r \sin \theta = (1 + \cos \theta) \sin \theta &= \sin \theta + \cos \theta \sin \theta \\ dx = -r \sin \theta d\theta - 2\cos \theta \sin \theta d\theta &= (-\sin \theta - \sin 2\theta) d\theta \\ dy = (\cos \theta + \cos^2 \theta - \sin \theta) d\theta &= (\cos \theta + \cos 2\theta) d\theta \\ &= \int_0^{2\pi} -r \sin \theta (-\sin \theta - \sin 2\theta) d\theta + \cos \theta (\cos \theta + \cos 2\theta) d\theta \\ &= \int_0^{2\pi} r [\sin^2 \theta + \sin 2\theta + \cos^2 \theta + \cos 2\theta] d\theta \\ &= \int_0^{2\pi} (1 + \cos 2\theta)(1 + \sin 2\theta) d\theta \\ &= \int_0^{2\pi} (1 + \cos 2\theta)(1 + \cos(2\theta - \pi)) d\theta \\ &= \int_0^{2\pi} (1 + \cos 2\theta)(1 + \cos 2\theta) d\theta \\ &= \int_0^{2\pi} 1 + 2\cos 2\theta + \cos^2 2\theta d\theta \\ &= \int_0^{2\pi} 1 + \frac{1}{2} + \frac{1}{2} \cos 4\theta d\theta \\ &= \int_0^{2\pi} \frac{3}{2} d\theta = 3\pi \quad \text{APPROX} \end{aligned}$$

Question 4



The figure above shows the closed curve C with polar equation

$$r = 3 + \sin \theta, \quad 0 \leq \theta \leq 2\pi.$$

The vector field \mathbf{F} is given in Cartesian coordinates by

$$\mathbf{F}(x, y) = (x + y)\mathbf{i} + (-x + y)\mathbf{j}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

[19π]

METHOD A:

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C (x+y)(dx) + (-x+y)(dy) \\ &= \oint_C (r \cos \theta + r \sin \theta)(r \cos \theta dr + r \sin \theta d\theta) \\ &= \oint_C (r^2 \cos^2 \theta + r^2 \cos \theta \sin \theta) dr + (r^2 \cos \theta \sin \theta - r^2 \sin^2 \theta) d\theta \\ &= \oint_C r^2 (\cos^2 \theta + \cos \theta \sin \theta) dr + r^2 (\cos \theta \sin \theta - \sin^2 \theta) d\theta \\ &= \oint_C r^2 (1 + \sin 2\theta) dr - r^2 (\cos 2\theta) d\theta \\ &= \int_0^{2\pi} r^2 (1 + \sin 2\theta) d\theta - \int_0^{2\pi} r^2 (\cos 2\theta) d\theta \\ &= \int_0^{2\pi} r^2 d\theta + \int_0^{2\pi} r^2 \sin 2\theta d\theta - \int_0^{2\pi} r^2 \cos 2\theta d\theta \\ &\text{from here we have} \\ &= \int_0^{2\pi} (3 + \sin 2\theta)(r^2) d\theta - (3 \cos 2\theta)(r^2) d\theta \\ &= \int_0^{2\pi} 3r^2 d\theta + \sin 2\theta r^2 d\theta - 3 \cos 2\theta r^2 d\theta \\ &= \int_0^{2\pi} 3r^2 d\theta + \sin 2\theta r^2 d\theta - 3 - \frac{1}{2} \cos 4\theta r^2 d\theta \\ &= \int_0^{2\pi} 3r^2 d\theta + \sin 2\theta r^2 d\theta - 3 - \left(\frac{1}{2} - \frac{1}{2} \cos 4\theta\right) r^2 d\theta \\ &= \int_0^{2\pi} 3r^2 d\theta - \frac{1}{2} r^2 d\theta \end{aligned}$$

(REMEMBER FOR THESE LINES)

METHOD B: *Starting with some auxiliary in the diagram below.*

IN ITS UNIT THE POSITION VECTOR WILL BE $(r \cos \theta \hat{i} + r \sin \theta \hat{j})$

REGARDING TO THE POLAR LINE INTEGRAL

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (x+y)(dx) + (-x+y)(dy)$$

$$= \oint_C [(r \cos \theta + r \sin \theta)] [r \cos \theta dr + r \sin \theta d\theta]$$

$$= \oint_C [(r \cos \theta + r \sin \theta)(r \cos \theta dr + r \sin \theta d\theta)] + [(r \cos \theta + r \sin \theta)] [r \cos \theta dr + r \sin \theta d\theta]$$

$$= \oint_C [r^2 \cos^2 \theta + r^2 \cos \theta \sin \theta] dr + [r^2 \cos \theta \sin \theta - r^2 \sin^2 \theta] d\theta$$

$$= \oint_C [r^2 (\cos^2 \theta + \cos \theta \sin \theta) - r^2 (\cos \theta \sin \theta - \sin^2 \theta)] dr + [r^2 \cos \theta dr - r^2 \sin \theta d\theta]$$

$$= \oint_C [r^2 (1 + \sin 2\theta) - r^2 (\cos 2\theta)] dr + [r^2 \cos \theta dr - r^2 \sin \theta d\theta]$$

$$= \int_0^{2\pi} [r^2 (1 + \sin 2\theta) - r^2 (\cos 2\theta)] d\theta + \int_0^{2\pi} [r^2 \cos \theta dr - r^2 \sin \theta d\theta]$$

$$= \int_0^{2\pi} r^2 (1 + \sin 2\theta) d\theta - \int_0^{2\pi} r^2 (\cos 2\theta) d\theta + \int_0^{2\pi} r^2 \cos \theta dr - \int_0^{2\pi} r^2 \sin \theta d\theta$$

WHICH FROM THIS POINT ENDURES NEEDS WITH ANGLE A

METHOD C: *Start by parametrising directly from the points*

$$\begin{aligned} x &= r \cos \theta & x = (3 + \sin \theta) \cos \theta & \Rightarrow x = 3\cos \theta + \sin \theta \cos \theta \\ y &= r \sin \theta & y = (3 + \sin \theta) \sin \theta & \Rightarrow y = 3\sin \theta + \cos \theta \sin \theta \end{aligned}$$

HOICE WE KNOW HAVE

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C (x+y)(dx) + (-x+y)(dy) = \oint_C (2\cos \theta + 2\sin \theta)(dx) + (2\sin \theta + 2\cos \theta)(dy) \\ &= \oint_C [(2\cos \theta + 2\sin \theta)(-3\sin \theta + \cos \theta) + (2\sin \theta + 2\cos \theta)(3\cos \theta + \sin \theta)] d\theta \\ &= \oint_C [-3\sin^2 \theta - 2\cos^2 \theta - 3\cos^2 \theta + 2\cos \theta \sin \theta - 3\sin^2 \theta + 2\cos \theta \sin \theta + 3\cos^2 \theta + 2\cos \theta \sin \theta] d\theta \\ &= \oint_C [(-\sin^2 \theta + \cos^2 \theta) - (\sin^2 \theta - \cos^2 \theta) - 3(\cos^2 \theta + \sin^2 \theta)] d\theta \\ &= \oint_C [(\cos^2 \theta - \sin^2 \theta) - \sin(2\theta) - 3] d\theta \\ &= \int_0^{2\pi} (3\cos^2 \theta - \sin(2\theta) - 3) d\theta \\ &= \int_0^{2\pi} (3(3\cos^2 \theta - \cos(2\theta) - 2)) d\theta \\ &= \int_0^{2\pi} 3(3\cos^2 \theta - 3\cos^2 \theta + \cos(2\theta) - 2\cos(2\theta) - 2) d\theta \quad (\text{in brackets one term less}) \\ &= \int_0^{2\pi} -9 - (\frac{1}{2} - \frac{1}{2}\cos 4\theta) d\theta = \int_0^{2\pi} -9 + \frac{1}{2}\cos 4\theta d\theta \quad (\text{in brackets one term less}) \\ &= \int_0^{2\pi} -9 - \frac{9}{2} d\theta = -\frac{9}{2}(-2\pi) = 19\pi \end{aligned}$$

At 30/41