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TRIGONOMETRY EXAM QUESTIONS

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BASIC

QUESTIONS

Question 1 ()**

Given that $\cos x = \sqrt{2} - 1$, show clearly that

$$\cos 2x = 5 - 4\sqrt{2}.$$

proof

$$\begin{aligned}\cos 2x &= 2\cos^2 x - 1 \\ \cos 2x &= 2(\sqrt{2} - 1)^2 - 1 \\ \cos 2x &= 2(2 - 2\sqrt{2} + 1) - 1 \\ \cos 2x &= 4 - 4\sqrt{2} + 2 - 1 \\ \cos 2x &= 5 - 4\sqrt{2}\end{aligned}$$

Question 2 ()**

Show clearly that

$$\frac{\cos(x-y)}{\sin y \cos y} \equiv \frac{\cos x}{\sin y} + \frac{\sin x}{\cos y}.$$

proof

$$\begin{aligned}\text{LHS} &= \frac{\cos(x-y)}{\sin y \cos y} \\ &= \frac{\cos x \cos y + \sin x \sin y}{\sin y \cos y} \\ &= \frac{\cos x}{\sin y} + \frac{\sin x \sin y}{\cos y} \\ &= \frac{\cos x}{\sin y} + \frac{\sin x}{\cos y} \\ &\rightarrow \text{RHS}\end{aligned}$$

Question 3 (+)**

Prove the validity of the trigonometric identity

$$\tan 2\theta \sec \theta \equiv 2 \sin \theta \sec 2\theta.$$

proof

$$\begin{aligned}\text{LHS} &= \frac{\tan 2\theta \sec \theta}{\sin 2\theta \cos 2\theta} \\ &= \frac{\frac{\sin 2\theta}{\cos 2\theta} \times \frac{1}{\cos \theta}}{\sin 2\theta \cos 2\theta} \\ &= \frac{\sin 2\theta}{\cos 2\theta \cos \theta} \\ &= \frac{2 \sin \theta \cos \theta}{\cos 2\theta} \times \frac{\cos 2\theta}{\cos \theta} \\ &= 2 \sin \theta \sec 2\theta \\ &\equiv \text{RHS}\end{aligned}$$

Question 4 (+)**

If $\sin x = \frac{3}{5}$, show clearly that

$$\sec 2x = \frac{25}{7}.$$

$$\sec 2x = \frac{1}{\cos 2x} = \frac{1}{1 - 2\sin^2 x} = \frac{1}{1 - 2\left(\frac{3}{5}\right)^2} = \frac{1}{1 - \frac{18}{25}} = \frac{25}{7} \quad \boxed{\text{QED}}$$

Question 5 (+)**

Solve the following trigonometric equation

$$\cos x \cot x + \sin x + 2 \cot x = 0, \quad 0^\circ \leq x < 360^\circ, \quad x \neq 0^\circ, 180^\circ.$$

$$x = 120^\circ, 240^\circ$$

$$\begin{aligned} \cos x \cot x + \sin x + 2 \cot x &= 0 \\ \Rightarrow (\cos x \times \frac{\cos x}{\sin x}) + \sin x + 2\left(\frac{\cos x}{\sin x}\right) &= 0 \\ \Rightarrow \frac{\cos^2 x}{\sin x} + \sin x + \frac{2 \cos x}{\sin x} &= 0 \\ \Rightarrow \cos x + \sin^2 x + 2 \cos x &= 0 \\ \Rightarrow 1 + 2 \cos x &= 0 \\ \Rightarrow 2 \cos x &= -1 \\ \Rightarrow \cos x &= -\frac{1}{2} \end{aligned} \quad \left. \begin{array}{l} \arccos\left(-\frac{1}{2}\right) = 120^\circ \\ x = 120^\circ + 360^\circ k \\ x = 240^\circ + 360^\circ k, \quad k = 0, 1, 2, \dots \\ x_1 = 120^\circ \\ x_2 = 240^\circ \end{array} \right\}$$

Question 6 (+)**

Simplify fully the following trigonometric expression

$$\frac{\sqrt{2} \cos x^\circ - 2 \sin(45 - x)^\circ}{2 \sin(60 + x)^\circ - \sqrt{3} \cos x^\circ}.$$

$$\boxed{\sqrt{2}}$$

$$\begin{aligned}\frac{\sqrt{2} \cos x - 2 \sin(45 - x)}{2 \sin(60 + x) - \sqrt{3} \cos x} &= \frac{\sqrt{2} \cos x - 2 [\sin 45 \cos x - \cos 45 \sin x]}{2 [\sin 60 \cos x + \cos 60 \sin x] - \sqrt{3} \cos x} \\ &= \frac{\sqrt{2} \cos x - 2 \left[\frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \sin x \right]}{2 \left[\frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x \right] - \sqrt{3} \cos x} \\ &= \frac{\sqrt{2} \cos x - \sqrt{2} \cos x + \sqrt{2} \sin x}{\sqrt{3} \cos x + \sin x - \sqrt{3} \cos x} \\ &= \frac{\sqrt{2} \sin x}{\sin x} = \boxed{\sqrt{2}}\end{aligned}$$

Question 7 (+)**

Solve the following trigonometric equation

$$\pi - 3 \arccos(x+1) = 0.$$

$$\boxed{\quad}, \quad x = -\frac{1}{2}$$

$$\begin{aligned}\pi - 3 \arccos(x+1) &= 0 \\ 3 \arccos(x+1) &= \pi \\ \arccos(x+1) &= \frac{\pi}{3} \\ \cos[\arccos(x+1)] &= \cos\left(\frac{\pi}{3}\right) \\ x+1 &= \frac{1}{2} \\ x &= -\frac{1}{2}\end{aligned}$$

Question 8 (+)**

The angle x is acute so that $\tan x = \frac{1}{2}$.

- a) Find the exact value of $\operatorname{cosec} x$.

It is further given that $\tan(x+y) = 2$, where y is another angle.

- b) Determine the value of $\tan y$.

$$\boxed{\text{[}}, \boxed{\operatorname{cosec} x = \sqrt{5}}, \boxed{\tan y = \frac{3}{4}}$$

<p>(a) $\tan x = \frac{1}{2}$</p>  $\therefore \operatorname{cosec} x = \sqrt{5}$	<p>(b) $\tan(x+y) = 2$</p> $\begin{aligned} \frac{\tan x + \tan y}{1 - \tan x \tan y} &= 2 \\ \frac{\frac{1}{2} + \tan y}{1 - \frac{1}{2} \tan y} &= 2 \\ \frac{1}{2} + \tan y &= 2(1 - \frac{1}{2} \tan y) \\ \frac{1}{2} + \tan y &= 2 - \tan y \\ 2\tan y &= \frac{3}{2} \\ \tan y &= \frac{3}{4} \end{aligned}$
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Question 9 (*)**

$$\operatorname{cosec} \theta + 8 \cos \theta = 0, \quad 0^\circ \leq \theta < 360^\circ.$$

Find the solutions of the above trigonometric equation, giving the answers in degrees correct to one decimal place.

$$\boxed{\text{[}}, \boxed{\theta = 97.2^\circ, 172.8^\circ, 277.2^\circ, 352.8^\circ}$$

$\begin{aligned} \operatorname{cosec} \theta + 8 \cos \theta &= 0 \\ \frac{1}{\sin \theta} + 8 \cos \theta &= 0 \\ 1 + 8 \sin \theta \cos \theta &= 0 \\ 1 + 4(\sin 2\theta) &= 0 \\ 1 + 4 \sin 2\theta &= 0 \\ 4 \sin 2\theta &= -1 \\ \sin 2\theta &= -\frac{1}{4} \end{aligned}$	$\left. \begin{aligned} \operatorname{cosec}(-\theta) &= -\operatorname{cosec} \theta \\ \theta &= 14.49^\circ \pm 360^\circ \\ \theta &= 194.49^\circ \pm 360^\circ \\ \theta &= -7.24^\circ \pm 180^\circ \\ \theta &= 97.24^\circ \pm 180^\circ \end{aligned} \right\} \quad \begin{aligned} \therefore \theta &= 97.2^\circ, 172.8^\circ, 277.2^\circ, 352.8^\circ \end{aligned}$
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Question 10 (***)

$$5\sin 3x \cos x + 5\cos 3x \sin x = 4, \quad 0 \leq x < \pi.$$

Use a compound angle trigonometric identity to find the solutions of the above trigonometric equation, giving the answers in radians correct to two decimal places.

$$\boxed{\text{[]}}, \quad x = 0.23^\circ, 0.55^\circ, 1.80^\circ, 2.12^\circ$$

$$\begin{aligned}
 & 5\sin 3x \cos x + 5\cos 3x \sin x = 4 \\
 \Rightarrow & 5\sin(3x+x) + 5\cos(3x+x) = \frac{4}{5} \\
 \Rightarrow & 5\sin(4x) = \frac{4}{5} \\
 \Rightarrow & \sin 4x = \frac{4}{25} \\
 & \arcsin\left(\frac{4}{25}\right) = 0.427\dots \\
 & 4x = 0.427 \pm 2\pi n, \quad n \in \mathbb{Z}, \quad 3x = 0.232 \pm \frac{2\pi n}{3}, \quad x = 0.077 \pm \frac{\pi n}{9} \\
 & x = 0.23^\circ, 0.55^\circ, 1.80^\circ, 2.12^\circ
 \end{aligned}$$

Question 11 (***)

Prove the validity of the trigonometric identity

$$\frac{1+\tan^2 x}{1-\tan^2 x} \equiv \sec 2x.$$

[proof]

$$\begin{aligned}
 \text{LHS: } & \frac{1+\tan^2 x}{1-\tan^2 x} = \frac{1 + \frac{\sin^2 x}{\cos^2 x}}{1 - \frac{\sin^2 x}{\cos^2 x}} = \dots \text{ MULTIPLY FRACTION BY } \cos^2 x \\
 & = \frac{\cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x} = \frac{1}{\cos 2x} = \sec 2x = \text{RHS}
 \end{aligned}$$

Question 12 (*)**

Solve the following trigonometric equation

$$3\arccot(x - \sqrt{3}) - \pi = 0.$$

$$x = \frac{4}{3}\sqrt{3}$$

$$\begin{aligned} 3\arccot(x - \sqrt{3}) - \pi &= 0 \\ \Rightarrow 3\arccot(x - \sqrt{3}) &= \pi \\ \Rightarrow \arccot(x - \sqrt{3}) &= \frac{\pi}{3} \\ \Rightarrow \cot(\arccot(x - \sqrt{3})) &= \cot\frac{\pi}{3} \\ \Rightarrow x - \sqrt{3} &= \frac{1}{\tan\frac{\pi}{3}} \end{aligned} \quad \left\{ \begin{array}{l} x - \sqrt{3} > \frac{1}{\tan\frac{\pi}{3}} \\ x - \sqrt{3} < \frac{\sqrt{3}}{3} \\ x = \sqrt{3} + \frac{\sqrt{3}}{3} \\ x = \frac{4}{3}\sqrt{3} \end{array} \right.$$

Question 13 (*)**

$$f(x) \equiv \sqrt{3} \sin x + \cos x, \quad 0 \leq x < 2\pi.$$

- a) Express $f(x)$ in the form $R \cos(x - \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.
- b) State the maximum value of $f(x)$ and find the value of x for which this maximum value occurs.
- c) Solve the equation

$$f(x) = \sqrt{3}.$$

$$\boxed{2}, \boxed{f(x) \equiv 2 \cos\left(x - \frac{\pi}{3}\right)}, \boxed{f(x)_{\max} = 2}, \boxed{x = \frac{\pi}{3}}, \boxed{x = \frac{\pi}{6}, \frac{\pi}{2}}$$

$$\begin{aligned} \text{(a)} \quad f(x) &= \sqrt{3} \sin x + \cos x \equiv R \cos(x - \alpha) \\ &\equiv R \cos x \cos \alpha + R \sin x \sin \alpha \\ &\equiv (\cos \alpha) \sin x + (\sin \alpha) \cos x \\ &\therefore \begin{cases} \cos \alpha = \frac{1}{2} \\ \sin \alpha = \frac{\sqrt{3}}{2} \end{cases} \Rightarrow R = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{1+3} = 2 \\ &\cos \alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3} \Rightarrow x = \pi/3 \\ \therefore f(x) &= 2 \cos\left(x - \frac{\pi}{3}\right) \end{aligned}$$

(b) $f(x)_{\max} = 2$ // IT OCCURS WHEN $\cos\left(x - \frac{\pi}{3}\right) = 1$
 $x - \frac{\pi}{3} = 0 \Rightarrow x = \frac{\pi}{3}$ ($0 \leq x < \pi$)

(c) $f(x) = \sqrt{3}$
 $2 \cos\left(x - \frac{\pi}{3}\right) = \sqrt{3}$
 $\cos\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
 $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$
 $\therefore \begin{cases} x - \frac{\pi}{3} = \frac{\pi}{6} + 2\pi m \\ x - \frac{\pi}{3} = \frac{11\pi}{6} + 2\pi m \end{cases} \quad m \in \mathbb{Z}, 1, 2, \dots$
 $\begin{cases} x = \frac{\pi}{6} + 2\pi m \\ x = \frac{11\pi}{6} + 2\pi m \end{cases}$
 $\therefore x = \frac{\pi}{6} + 2\pi m$

Question 14 (*)**

It is given that

$$\frac{1 + \cot^2 x}{\cot x \cosec x} = \sec x.$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\frac{4(1 + \cot^2 x)}{\cot x \cosec x} = \tan^2 x + 5, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

, $x = \frac{\pi}{3}, \frac{5\pi}{3}$

(a) LHS = $\frac{1 + \cot^2 x}{\cot x \cosec x} = \frac{\cot^2 x}{\cot x \cosec x} = \frac{\cot x}{\cosec x} = \frac{\frac{1}{\sin x}}{\frac{1}{\cos x}} = \sec x = \text{RHS}$

(b) $\frac{4(1 + \cot^2 x)}{\cot x \cosec x} = \tan^2 x + 5$

$$\begin{aligned} \Rightarrow 4 \sec x &= \tan^2 x + 5 \\ \Rightarrow 4 \sec x &= (\sec^2 x - 1) + 5 \\ \Rightarrow 0 &= \sec^2 x - 4 \sec x + 4 \\ \Rightarrow 0 &= (\sec x - 2)^2 \\ \Rightarrow \sec x &= 2 \end{aligned}$$

$\left. \begin{aligned} \Rightarrow \cos x &= \frac{1}{2} \\ \cos^{-1}(\frac{1}{2}) &= \frac{\pi}{3} \\ x &= \frac{\pi}{3} + 2m\pi \\ x &= \frac{7\pi}{3} \\ x_1 &= \frac{\pi}{3} \\ x_2 &= \frac{7\pi}{3} \end{aligned} \right\} n = 0, 1, 2, 3, \dots$

Question 15 (*)**

Two vectors \mathbf{a} and \mathbf{b} are given below.

$$\mathbf{a} = (\sin \theta) \mathbf{i} + (2\cos 2\theta) \mathbf{j} + (\sin \theta) \mathbf{k} \quad \text{and} \quad \mathbf{a} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Find the values of θ , $0 \leq \theta < 2\pi$, for which \mathbf{a} is perpendicular to \mathbf{b} .

$$\boxed{\quad}, \quad \boxed{\theta = \frac{1}{6}\pi, \frac{5}{6}\pi, \frac{3}{2}\pi}$$

SOLN BY FORMING A DOT PRODUCT

$$\begin{aligned} \mathbf{a} &= (\sin \theta, 2\cos 2\theta, \sin \theta) \\ \mathbf{b} &= (3, -1, -1) \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathbf{a} \cdot \mathbf{b} &= 0 \\ \Rightarrow (\sin \theta, 2\cos 2\theta, \sin \theta) \cdot (3, -1, -1) &= 0 \\ \Rightarrow 3\sin \theta - 2\cos 2\theta - \sin \theta &= 0 \\ \Rightarrow 2\sin \theta - 2\cos 2\theta &= 0 \\ \Rightarrow \sin \theta - \cos 2\theta &= 0 \end{aligned}$$

BY TRIGONOMETRIC IDENTITIES

$$\begin{aligned} \Rightarrow \sin \theta - (1 - 2\sin^2 \theta) &= 0 \\ \Rightarrow \sin \theta - 1 + 2\sin^2 \theta &= 0 \\ \Rightarrow 2\sin^2 \theta + \sin \theta - 1 &= 0 \\ \Rightarrow (2\sin \theta - 1)(\sin \theta + 1) &= 0 \\ \Rightarrow \sin \theta &= \begin{cases} \frac{1}{2} \\ -1 \end{cases} \\ \Rightarrow \theta &= \begin{cases} \frac{\pi}{6} \pm 2\pi h \\ \frac{3\pi}{2} \pm 2\pi h \\ -\frac{\pi}{6} \pm 2\pi h \end{cases} \quad h = 0, 1, 2, 3, \dots \\ \therefore \theta &= \boxed{\frac{\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}} \end{aligned}$$

Question 16 (***)

It is given that

$$\frac{1-\cos 2\theta}{\sin 2\theta} \equiv \tan \theta, \quad \theta \neq 90k^\circ, k \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence show that

$$\tan 15^\circ = 2 - \sqrt{3}.$$

, proof

(a) $\text{LHS} = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{-(1-\cos 2\theta)}{2\sin \theta \cos \theta} = \frac{2\sin^2 \theta}{2\sin \theta \cos \theta} = \frac{\sin \theta}{\cos \theta}$
 $= \tan \theta = \text{RHS}$

(b) Let $\theta = 15^\circ$
 $\tan \theta \equiv \frac{1-\cos 2\theta}{\sin 2\theta}$
 $\tan 15^\circ = \frac{1-\cos 30^\circ}{\sin 30^\circ} = \frac{1 - \frac{\sqrt{3}}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2}} = \frac{2 - \sqrt{3}}{1} = 2 - \sqrt{3}$

Question 17 (***)

Prove the validity of the trigonometric identity

$$\frac{1}{\sec x - \tan x} - \frac{1}{\sec x + \tan x} \equiv 2 \tan x.$$

proof

$$\begin{aligned} \text{LHS} &= \frac{1}{\sec x - \tan x} - \frac{1}{\sec x + \tan x} \\ &= \frac{(\sec x + \tan x) - (\sec x - \tan x)}{(\sec x - \tan x)(\sec x + \tan x)} \\ &= \frac{\sec x + \tan x - \sec x + \tan x}{\sec x - \tan x} \\ &= \frac{2\tan x}{(1 + \tan^2 x) - \tan^2 x} \\ &= \frac{2\tan x}{1} \\ &= 2\tan x = \text{RHS} \end{aligned}$$

Question 18 (*)**

Solve the trigonometric equation

$$\cos \theta + \sec \theta = \frac{5}{2}, \quad 0^\circ \leq \theta < 360^\circ.$$

$$\theta = 60^\circ, 300^\circ$$

$$\begin{aligned}
 \cos \theta + \sec \theta &= \frac{5}{2} \\
 \Rightarrow \cos \theta + \frac{1}{\cos \theta} &= \frac{5}{2} \\
 \Rightarrow 2\cos^2 \theta + \frac{2}{\cos \theta} &= 5 \\
 \Rightarrow 2\cos^2 \theta + 2 &= 5\cos \theta \\
 \Rightarrow 2\cos^2 \theta - 5\cos \theta + 2 &= 0 \\
 \Rightarrow (2\cos \theta - 1)(\cos \theta - 2) &= 0 \\
 \Rightarrow \cos \theta &= \frac{1}{2} \quad \text{or} \quad \cos \theta = 2
 \end{aligned}
 \quad \left| \begin{array}{l} \cos \theta = \frac{1}{2} \\ \cos(\frac{\pi}{3}) = 0.5 \\ \theta = 60^\circ \pm 360^\circ n \\ \theta = 300^\circ + 360^\circ n \quad n=0,1,2,3, \dots \\ \theta_1 = 60^\circ \\ \theta_2 = 300^\circ // \end{array} \right.$$

Question 19 (*)**

Prove the validity of the trigonometric identity

$$\sqrt{2+2\cos 2\theta} \equiv 2\cos \theta.$$

proof

$$\begin{aligned}
 4\sqrt{3} &= \sqrt{2+2\cos 2\theta}^2 = \sqrt{2+2(2\cos^2 \theta - 1)}^2 = \sqrt{2+4\cos^2 \theta - 2}^2 \\
 &= \sqrt{4\cos^2 \theta} = 2\cos \theta = 24\sqrt{3}
 \end{aligned}$$

Question 20 (*)**

$$y \equiv 2\sqrt{2} \cos x + 2\sqrt{2} \sin x, \quad x \in \mathbb{R}.$$

- a) Express y in the form $R \sin(x+\alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

- b) Solve the equation

$$y = 2 \text{ for } 0 < x < 2\pi.$$

- c) Write down the maximum value of y .

- d) Find the smallest positive value of x for which this maximum value occurs.

$$\boxed{3\pi}, \quad \boxed{y \equiv 4 \sin\left(x + \frac{\pi}{4}\right)}, \quad \boxed{x = \frac{7\pi}{12}, \frac{23\pi}{12}}, \quad \boxed{y_{\max} = 4}, \quad \boxed{x = \frac{\pi}{4}}$$

(a) $2\sqrt{2} \cos x + 2\sqrt{2} \sin x \equiv R \sin(x+\alpha)$
 $\equiv R \sin x \cos \alpha + R \cos x \sin \alpha$
 $\equiv (\cos \alpha) \cos x + (\sin \alpha) \sin x$

$\cos \alpha = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$
 $\sin \alpha = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$

$\therefore R = \sqrt{(\cos \alpha)^2 + (\sin \alpha)^2} = \sqrt{16} = 4$
 $\Rightarrow x + \frac{\pi}{4}$

$\therefore 2\sqrt{2} \cos x + 2\sqrt{2} \sin x = 4 \sin\left(x + \frac{\pi}{4}\right)$

(b) $2\sqrt{2} \cos x + 2\sqrt{2} \sin x = 2$
 $\Rightarrow 4 \sin\left(x + \frac{\pi}{4}\right) = 2$
 $\Rightarrow \sin\left(x + \frac{\pi}{4}\right) = \frac{1}{2}$
 $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$x + \frac{\pi}{4} = \frac{\pi}{6} + 2k\pi$
 $x = -\frac{\pi}{4} + 2k\pi$

$x = -\frac{\pi}{4} + 2\pi$

$\therefore x = \frac{7\pi}{12}, \frac{23\pi}{12}$

(c) Maximum value is 4
This max could be when
 $\sin\left(x + \frac{\pi}{4}\right) = 1$
 $x + \frac{\pi}{4} = \frac{\pi}{2}$
 $x = \frac{\pi}{4}$

Question 21 (*)**

Solve the following trigonometric equation

$$4\sin x \cos x = 1, \quad 0 \leq x < \pi,$$

giving the answers in terms of π .

$$\boxed{\quad}, \quad x = \frac{\pi}{12}, \frac{5\pi}{12}$$

$$\begin{aligned} 4\sin x \cos x &= 1 \\ \Rightarrow 2(\sin x \cos x) &= 1 \\ \Rightarrow 2\sin 2x &= 1 \\ \Rightarrow \sin 2x &= \frac{1}{2} \\ \Rightarrow \sin(2x) &= \frac{1}{2} \\ \arcsin\left(\frac{1}{2}\right) &= \frac{\pi}{6} \end{aligned} \quad \begin{cases} 2x = \pi/6 + 2k\pi \\ 2x = 7\pi/6 + 2k\pi \\ x = \pi/12 + k\pi \\ x = 7\pi/12 + k\pi \end{cases} \quad \begin{aligned} x &= 9, 13, 21, \\ x_1 &= \pi/12 \\ x_2 &= 7\pi/12 \end{aligned}$$

Question 22 (*)**

Solve the following trigonometric equation

$$\frac{\cosec^2 \theta \tan^2 \theta}{\cos \theta} + 8 = 0, \quad 0^\circ \leq \theta < 360^\circ.$$

$$\boxed{\theta = 120^\circ, 240^\circ}$$

$$\begin{aligned} \frac{\cosec^2 \theta \tan^2 \theta}{\cos \theta} + 8 &= 0 \\ \Rightarrow \cosec \theta \tan^2 \theta + 8 \cos \theta &= 0 \\ \Rightarrow \frac{1}{\sin \theta} \times \frac{\sin^2 \theta}{\cos^2 \theta} + 8 \cos \theta &= 0 \\ \Rightarrow \frac{1}{\cos^2 \theta} + 8 \cos \theta &= 0 \\ \Rightarrow 1 + 8 \cos^2 \theta &= 0 \\ \Rightarrow 8 \cos^2 \theta &= -1 \\ \Rightarrow \cos^2 \theta &= -\frac{1}{8} \\ \Rightarrow \cos \theta &= -\frac{1}{2} \end{aligned} \quad \begin{aligned} \cos \theta &= -\frac{1}{2} \\ \arccos\left(-\frac{1}{2}\right) &= 120^\circ \\ (\theta = 120^\circ + 360^\circ) &= 480^\circ \\ (\theta = 240^\circ + 360^\circ) &= 600^\circ \\ \theta_1 &= 120^\circ \\ \theta_2 &= 240^\circ \end{aligned}$$

Question 23 (*)**

Solve the following trigonometric equation

$$8 \tan x = \cot^2 x, \quad 0 \leq x < 2\pi.$$

$$x = 0.464^\circ, 3.61^\circ$$

$$\begin{aligned} 8 \tan x &= \cot^2 x \\ \Rightarrow 8 \tan x &= \frac{1}{\tan^2 x} \\ \Rightarrow 8 \tan^3 x &= 1 \\ \Rightarrow \tan^3 x &= \frac{1}{8} \\ \Rightarrow \tan x &= \frac{1}{2} \end{aligned}$$

Hence $\arctan(\frac{1}{2}) = 0.464^\circ$

- $x = 0.464^\circ + n\pi$ $n = 0, 1, 2, 3, \dots$
- $x_1 = 0.464^\circ$
- $x_2 = 3.61^\circ$

Question 24 (*)**

Prove the validity of the following trigonometric identity

$$\frac{1}{\cos \theta - \sin \theta} - \frac{1}{\cos \theta + \sin \theta} \equiv 2 \sin \theta \sec 2\theta.$$

proof

$$\begin{aligned} LHS &= \frac{1}{\cos \theta - \sin \theta} - \frac{1}{\cos \theta + \sin \theta} = \frac{(\cos \theta + \sin \theta) - (\cos \theta - \sin \theta)}{(\cos \theta - \sin \theta)(\cos \theta + \sin \theta)} \\ &= \frac{2 \sin \theta}{\cos^2 \theta - \sin^2 \theta} = \frac{2 \sin \theta}{\cos 2\theta} = 2 \sin \theta \sec 2\theta = RHS \end{aligned}$$

Question 25 (*)**

It is given that

$$\frac{1+\tan^2 x}{1-\tan^2 x} \equiv \sec 2x.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence solve the equation

$$\frac{1+\tan^2 x}{1-\tan^2 x} + 2 = 0, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

$$\begin{aligned}
 \text{(a)} \quad & \text{LHS} = \frac{1 + \frac{\sin^2 x}{\cos^2 x}}{1 - \frac{\sin^2 x}{\cos^2 x}} = \text{MULTIPLY TOP/BOTTOM BY } \cos^2 x \\
 & = \frac{\cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x} = \frac{1}{\cos 2x} = \sec 2x = \text{RHS}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \frac{1 + \tan^2 x}{1 - \tan^2 x} + 2 = 0 \\
 & \Rightarrow \sec^2 x + 2 = 0 \\
 & \Rightarrow \sec^2 x = -2, \quad \text{NOT POSSIBLE}
 \end{aligned}$$

Question 26 (***)

$$f(x) \equiv \sin x + \sqrt{3} \cos x, \quad 0 \leq x < 2\pi.$$

a) Express $f(x)$ in the form $R \cos(x - \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

b) State the minimum and maximum value of ...

i. ... $f(x)$.

ii. ... $[f(x)]^2$.

iii. ... $\frac{1}{5+f(x)}$.

<input type="text"/>	$f(x) \equiv 2 \cos\left(x - \frac{\pi}{6}\right)$	$[-2, 2]$	$[0, 4]$	$\left[\frac{1}{7}, \frac{1}{3}\right]$
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(a) $f(x) = \sqrt{3} \cos x + \sin x \equiv R \cos(x - \alpha)$
 $\equiv R \cos(\omega x + \phi) \sin(\omega x + \phi) \equiv (R \cos \phi) \cos x + (R \sin \phi) \sin x$

$R \cos \phi = \sqrt{3}$
 $R \sin \phi = 1$

$R = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$
 $\tan \phi = \frac{1}{\sqrt{3}} \Rightarrow \phi = \frac{\pi}{6}$

$\therefore f(x) = 2 \cos\left(x - \frac{\pi}{6}\right)$

(b)

	MIN	MAX	
$f(x)$	-2	2	$\leftarrow 2 \cos(0 - \frac{\pi}{6})$
$[f(x)]^2$	0	4	$\leftarrow 4 \cos^2(0 - \frac{\pi}{6})$
$\frac{1}{5+f(x)}$	$\frac{1}{7}$	$\frac{1}{3}$	$\leftarrow \frac{1}{5+2 \cos(0 - \frac{\pi}{6})}$

Question 27 (***)

$$\cos^2 x + \sin^2 x \equiv 1.$$

a) Starting with the above identity prove that

$$1 + \tan^2 x \equiv \sec^2 x.$$

b) Hence, or otherwise, solve the following trigonometric equation

$$2 \tan^2 x + \sec^2 x = 5 \sec x, \quad 0^\circ \leq x < 360^\circ.$$

, $x = 60^\circ, 300^\circ$

$$\begin{aligned}
 \text{(a)} \quad & \cos^2 \theta + \sin^2 \theta = 1 \\
 & \frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \\
 & 1 + \tan^2 \theta = \sec^2 \theta \\
 \Rightarrow & \sec^2 \theta - 1 = \tan^2 \theta \\
 \Rightarrow & (\sec \theta - 1)(\sec \theta + 1) = \tan^2 \theta \\
 \Rightarrow & (\sec \theta - 1)^2 = \tan^2 \theta \\
 \Rightarrow & \sec^2 \theta - 2 \sec \theta + 1 = \tan^2 \theta \\
 \Rightarrow & \sec^2 \theta - 2 \sec \theta + 1 = \sec^2 \theta - 1 \\
 \Rightarrow & -2 \sec \theta + 2 = 0 \\
 \Rightarrow & \sec \theta = 1 \\
 \Rightarrow & \theta = 0^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & 2 \tan^2 x + \sec^2 x = 5 \sec x \\
 & 2(\sec^2 x - 1) + \sec^2 x = 5 \sec x \\
 & 2\sec^2 x - 2 + \sec^2 x = 5 \sec x \\
 & 3\sec^2 x - 2 = 5 \sec x \\
 & 3\sec^2 x - 5 \sec x - 2 = 0 \\
 & (3\sec x + 1)(\sec x - 2) = 0
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow & \sec x = -\frac{1}{3} \quad (\text{not possible}) \\
 \Rightarrow & \sec x = 2 \\
 \Rightarrow & \cos x = \frac{1}{2} \\
 \Rightarrow & x = 60^\circ \quad (\text{not possible}) \\
 & x = 300^\circ \\
 & x = 60^\circ, 300^\circ
 \end{aligned}$$

Question 28 (***)

Prove the validity of the following trigonometric identity

$$\frac{1 + \cot^2 \theta}{2 \cot \theta} \equiv \operatorname{cosec} 2\theta.$$

, proof

$$\begin{aligned}
 \text{LHS} &= \frac{1 + \cot^2 \theta}{2 \cot \theta} = \frac{\cot^2 \theta}{2 \cot \theta} = \frac{1}{2} \cot \theta \operatorname{cosec} \theta = \frac{1}{2} \times \frac{\sin \theta}{\cos \theta} \times \frac{1}{\sin \theta} \\
 &= \frac{1}{2 \sin \theta \cos \theta} = \frac{1}{\sin 2\theta} = \operatorname{cosec} 2\theta = \text{RHS}
 \end{aligned}$$

$$\begin{aligned}
 \text{OR} \quad \text{LHS} &= \frac{1 + \cot^2 \theta}{2 \cot \theta} = \frac{1 + \frac{\cos^2 \theta}{\sin^2 \theta}}{2 \frac{\cos \theta}{\sin \theta}} = \frac{1 + \frac{\cos^2 \theta}{\sin^2 \theta}}{\frac{2 \cos \theta}{\sin \theta}} = \frac{\sin^2 \theta + \cos^2 \theta}{2 \sin \theta \cos \theta} = \frac{1}{2 \sin \theta \cos \theta} = \operatorname{cosec} 2\theta = \text{RHS}
 \end{aligned}$$

Question 29 (*)**

Prove the validity of the following trigonometric identity

$$\cot x - \tan x \equiv 2 \cot 2x.$$

, proof

$$\begin{aligned} \text{LHS} &= \cot x - \tan x = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{\cos 2x}{\sin x \cos x} = \\ &= \frac{2 \cos^2 x}{2 \sin x \cos x} = \frac{2 \cos x}{\sin x} = 2 \cot 2x = 2x \\ \text{RHS} &= 2 \cot 2x = \frac{2}{\tan 2x} = \frac{2}{\frac{\tan x + \tan x}{1 - \tan x}} = \frac{2(1 - \tan x)}{2 \tan x} = \frac{1 - \tan^2 x}{\tan x} \\ &= \frac{1}{\tan x} - \frac{\tan x}{\tan x} = \cot x - \tan x = \text{LHS} \end{aligned}$$

Question 30 (*)**

It is given that $\arcsin x = \arccos y$.

Show, by a clear method, that

$$x^2 + y^2 = 1.$$

, proof

$$\begin{aligned} &\bullet \arcsin x = \arccos y = \theta \\ &\left\{ \begin{array}{l} \sin \theta = x \\ \cos \theta = y \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \sin^2 \theta = x^2 \\ \cos^2 \theta = y^2 \end{array} \right\} \Rightarrow \sin^2 \theta + \cos^2 \theta = x^2 + y^2 \\ &\therefore x^2 + y^2 = 1 \quad \text{as required} \end{aligned}$$

Question 31 (***)

$$6\sec^2 2x + 5 \tan 2x = 12, \quad 0 \leq \theta < \pi.$$

Find the solutions of the above trigonometric equation, giving the answers in radians correct to two decimal places.

$$\boxed{?}, \quad x = 0.29^\circ, 1.08^\circ, 1.86^\circ, 2.65^\circ$$

$$\begin{aligned}
 & \text{Left side:} \\
 & 6\sec^2 2x + 5 \tan 2x = 12 \\
 & \Rightarrow 6(1 + \tan^2 2x) + 5 \tan 2x = 12 \\
 & \Rightarrow 6 + 6\tan^2 2x + 5 \tan 2x - 6 = 0 \\
 & \Rightarrow 6\tan^2 2x + 5 \tan 2x = 0 \\
 & \Rightarrow \tan 2x = \frac{-5 \pm \sqrt{25 - 4 \times 6 \times 6}}{2 \times 6} \\
 & \Rightarrow \tan 2x = -\frac{5}{2} \quad \text{or} \quad \tan 2x = \frac{1}{2} \\
 & \Rightarrow \tan 2x = -2.5 \quad \text{or} \quad \tan 2x = 0.5 \\
 & \Rightarrow \arctan(-2.5) = -0.983^\circ \\
 & \Rightarrow \arctan(0.5) = 0.588^\circ \\
 \\
 & \text{Right side:} \\
 & 2x = -0.983 + k\pi \\
 & \Rightarrow x = -0.491 + \frac{k\pi}{2} \\
 & \Rightarrow 2x = 0.588 + k\pi \\
 & \Rightarrow x = 0.294 + \frac{k\pi}{2} \\
 \\
 & \therefore x_1 = 1.08^\circ \\
 & x_2 = 0.58^\circ \\
 & x_3 = -0.29^\circ \\
 & x_4 = 1.86^\circ
 \end{aligned}$$

Question 32 (***)

Solve the trigonometric equation

$$4 - 4 \cos 2\theta = \operatorname{cosec} \theta, \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of π .

$$\boxed{?}, \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\begin{aligned}
 & \text{Left side:} \\
 & 4 - 4 \cos 2\theta = \operatorname{cosec} \theta \\
 & \Rightarrow 4 - 4(1 - 2\sin^2 \theta) = \frac{1}{\sin \theta} \\
 & \Rightarrow 4 - 4 + 8\sin^2 \theta = \frac{1}{\sin \theta} \\
 & \Rightarrow 8\sin^2 \theta = 1 \\
 & \Rightarrow \sin^2 \theta = \frac{1}{8} \\
 & \Rightarrow \sin \theta = \pm \frac{1}{\sqrt{8}}
 \end{aligned}
 \quad
 \begin{aligned}
 & \operatorname{cosec}(\frac{\pi}{2}) = \infty \\
 & (\theta = \frac{\pi}{2} \pm 2k\pi) \quad k=0,1,2,3, \dots \\
 & \theta = \frac{\pi}{2}, \frac{3\pi}{2}
 \end{aligned}$$

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STANDARD QUESTIONS

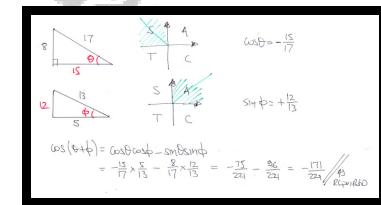
Question 1 (***)+

$$\sin \theta = \frac{8}{17} \quad \text{and} \quad \cos \varphi = \frac{5}{13}.$$

If θ is obtuse and φ is acute, show that

$$\cos(\theta + \varphi) = -\frac{171}{221}.$$

, proof

**Question 2** (***)+

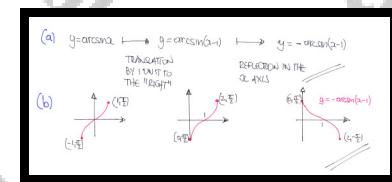
A curve C is defined by the equation

$$y = -\arcsin(x-1), \quad 0 \leq x \leq 2.$$

- Describe the 2 geometric transformations that map the graph of $\arcsin x$ onto the graph of C .
- Sketch the graph of C .

The sketch must include the coordinates of any points where the graph of C meets the coordinate axes and the coordinates of the endpoints of C .

, translation by 1 unit to the right, followed by reflection in the x axis



Question 3 (*)**

It is given that

$$\frac{\sec \theta}{\sec \theta - \cos \theta} \equiv \operatorname{cosec}^2 \theta, \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence solve the equation

$$\frac{\sec \theta}{\sec \theta - \cos \theta} = 4(\operatorname{cosec} \theta - 1), \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of π .

$$\boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}}$$

$(a) \frac{\sec \theta}{\sec \theta - \cos \theta}$ $= \frac{\frac{1}{\cos \theta}}{\frac{1 - \cos^2 \theta}{\cos \theta}}$ $= \frac{\cos \theta}{1 - \cos^2 \theta}$ $= \frac{\cos \theta}{\sin^2 \theta}$ $= \frac{\cos \theta}{\frac{\sin \theta}{\cos \theta} \cdot \frac{\sin \theta}{\cos \theta}}$ $= \frac{\cos \theta}{\tan \theta \cdot \cot \theta}$ $= \frac{1}{\tan \theta \cot \theta}$ $= \operatorname{cosec}^2 \theta$	$(b) \frac{\sec \theta}{\sec \theta - \cos \theta} = 4(\operatorname{cosec} \theta - 1)$ $\Rightarrow \operatorname{cosec}^2 \theta = 4 \operatorname{cosec} \theta - 4$ $\Rightarrow \operatorname{cosec}^2 \theta - 4 \operatorname{cosec} \theta + 4 = 0$ $\Rightarrow (\operatorname{cosec} \theta - 2)^2 = 0$ $\Rightarrow \operatorname{cosec} \theta = 2$ $\Rightarrow \sin \theta = \frac{1}{2}$ $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$ $\theta = \frac{\pi}{6} + 2n\pi \quad n = 0, 1, 2, 3, \dots$ $\theta_1 = \frac{\pi}{6}$ $\theta_2 = \frac{5\pi}{6}$
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Question 4 (*)+**

Solve each of the following trigonometric equations.

i. $2\sec\theta - 1 = 2\sec\theta \sin^2\theta$, $0^\circ \leq \theta < 180^\circ$, $\theta \neq 90^\circ$

ii. $4\cot^2 x - 9\operatorname{cosec} x + 6 = 0$, $0^\circ \leq x < 360^\circ$, $x \neq 0^\circ, 180^\circ$

, $\theta = 60^\circ$, $x = 30^\circ, 150^\circ$

$\begin{aligned} \text{(i)} \quad 2\sec\theta - 1 &= 2\sec\theta \sin^2\theta \\ \Rightarrow \frac{2}{\cos\theta} - 1 &= \frac{2}{\cos\theta} \sin^2\theta \\ \Rightarrow \frac{2}{\cos\theta} - 1 &= \frac{2\sin^2\theta}{\cos\theta} \\ \Rightarrow 2 - \cos\theta &= 2\sin^2\theta \\ \Rightarrow 2 - \cos\theta &= 2(1 - \cos^2\theta) \\ \Rightarrow 2\cos^2\theta - \cos\theta - 2 &= 0 \\ \Rightarrow 2\cos^2\theta - \cos\theta &= 0 \\ \Rightarrow \cos(\cos\theta - 1) &= 0 \\ \Rightarrow \cos\theta = 1 & \text{ or } \cos\theta = -\frac{1}{2} \\ \Rightarrow \theta = 0^\circ & \text{ or } \theta = 120^\circ \end{aligned}$	$\begin{aligned} \text{(ii)} \quad 4\cot^2 x - 9\operatorname{cosec} x + 6 &= 0 \\ \Rightarrow 4(\operatorname{cosec}^2 x - 1) - 9\operatorname{cosec} x + 6 &= 0 \\ \Rightarrow 4\operatorname{cosec}^2 x - 4 - 9\operatorname{cosec} x + 6 &= 0 \\ \Rightarrow 4\operatorname{cosec}^2 x - 9\operatorname{cosec} x + 2 &= 0 \\ \Rightarrow (\operatorname{cosec} x - 1)(4\operatorname{cosec} x - 2) &= 0 \\ \Rightarrow \operatorname{cosec} x - 1 &= 0 \quad \text{or} \quad 4\operatorname{cosec} x - 2 = 0 \\ \Rightarrow \operatorname{cosec} x = 1 & \text{ or } \operatorname{cosec} x = \frac{1}{2} \\ \Rightarrow \sin x = 1 & \text{ or } \sin x = \frac{1}{2} \\ \Rightarrow x = 90^\circ & \text{ or } x = 30^\circ \quad \text{or } x = 150^\circ \end{aligned}$
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Question 5 (*)+**

Prove the validity of each of the following trigonometric identities.

a) $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \equiv \cos 2\theta$.

b) $\frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} \equiv 2 \operatorname{cosec} \theta$.

proof

$\begin{aligned} \text{(a)} \quad \text{RHS} &= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}{\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta}} = \frac{1}{\operatorname{cosec}^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} = \cos^2 \theta - \sin^2 \theta = \cos 2\theta \\ &= \cos 2\theta = \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} = \cos^2 \theta - \sin^2 \theta = \cos 2\theta = \text{LHS} \end{aligned}$ <p>(OR SUBSTITUTE SIN & COS & TRY OUT THE DOUBLE-FRACTION)</p>	$\begin{aligned} \text{(b)} \quad \text{LHS} &= \frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} = \frac{\sin^2 \theta + (1 + \cos \theta)^2}{(1 + \cos \theta) \sin \theta} \\ &= \frac{\sin^2 \theta + 1 + 2\cos \theta + \cos^2 \theta}{(1 + \cos \theta) \sin \theta} = \frac{2 + 2\cos \theta}{(1 + \cos \theta) \sin \theta} \\ &= \frac{2(1 + \cos \theta)}{\sin \theta (1 + \cos \theta)} = \frac{2}{\sin \theta} = 2 \operatorname{cosec} \theta = \text{LHS} \end{aligned}$
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Question 6 (*)+**

It is given that

$$\frac{1+\cos 2\theta}{\sin 2\theta} \equiv \cot \theta, \quad \theta \neq \frac{k\pi}{2}, k \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence solve the equation

$$\operatorname{cosec} 4x + \cot 4x = 1, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

$$\boxed{\quad}, \quad x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

(a) LHS = $\frac{1+\cos 2\theta}{\sin 2\theta} = \frac{1+(2\cos^2 \theta - 1)}{\sin 2\theta} = \frac{2\cos^2 \theta}{\sin 2\theta} = \frac{2\cos^2 \theta}{2\sin \theta \cos \theta} = \frac{\cos \theta}{\sin \theta} = \cot \theta = \text{RHS}$	
(b) $\operatorname{cosec} 4x + \cot 4x = 1$ $\Rightarrow \frac{1}{\sin 4x} + \frac{\cos 4x}{\sin 4x} = 1$ $\Rightarrow \frac{1 + \cos 4x}{\sin 4x} = 1$ From part (a), $\theta \mapsto 2x$ $\Rightarrow \cot 2x = 1$	$\Rightarrow \tan(2x) = 1$ $\arctan(1) = \frac{\pi}{4}$ $2x = \frac{\pi}{4} + n\pi \quad n \in \mathbb{Z}$ $x = \frac{\pi}{8} + \frac{n\pi}{2}$ $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

Question 7 (***)+

$$y \equiv \sqrt{2} \cos \theta - \sqrt{6} \sin \theta, \quad 0 < \theta < 360^\circ.$$

a) Express y in the form $R \cos(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90^\circ$.

b) Solve the equation $y = 2$.

c) Write down the minimum value of ...

i. ... y^2 .

ii. ... $\frac{1}{y^2}$.

 , $y \equiv \sqrt{8} \cos(\theta + 60^\circ)$, $\theta = 255^\circ, 345^\circ$, $\min = 0$, $\min = \frac{1}{8}$

(a) $\sqrt{2} \cos \theta - \sqrt{6} \sin \theta \equiv R \cos(\theta + \alpha)$
 $\equiv R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$
 $\equiv (\cos \alpha) \cos \theta - (\sin \alpha) \sin \theta$

$R \cos \alpha = \sqrt{2}$ $\Rightarrow R = \sqrt{4 + 36} = \sqrt{40} = 2\sqrt{10}$
 $R \sin \alpha = \sqrt{6}$ $\Rightarrow \tan \alpha = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \Rightarrow \alpha = 60^\circ$
 $\therefore y = 2\sqrt{10} \cos(\theta + 60^\circ)$

(b) $\sqrt{2} \cos \theta - \sqrt{6} \sin \theta = 2$
 $\Rightarrow \sqrt{8} \cos(\theta + 60^\circ) = 2$
 $\Rightarrow \cos(\theta + 60^\circ) = \frac{1}{2}$
 $\Rightarrow \theta + 60^\circ = 60^\circ \quad h = 0, 1, 2, 3, \dots$
 $\theta = -15^\circ, 30^\circ, 210^\circ, 225^\circ \quad \therefore \theta_1 = 345^\circ, \theta_2 = 225^\circ$

(c) $y^2 = [\sqrt{8} \cos(\theta + 60^\circ)]^2 = 8 \cos^2(\theta + 60^\circ)$
 $\therefore y_{\min}^2 = 0$

(d) THE MINIMUM VALUE OF $\frac{1}{y^2}$ OCCURS WHEN y^2 IS MAXIMUM IN θ
 $\therefore \left(\frac{1}{y^2}\right)_{\min} = \frac{1}{8}$

Question 8 (***)+

$$\sin A = \frac{12}{13} \quad \text{and} \quad \cos B = \frac{4}{5}.$$

If A is obtuse and B is acute, show clearly that

$$\sin(A+B) = \frac{33}{65}.$$

, **proof**

$$\begin{aligned} \sin A &= \frac{12}{13} & \text{Diagram: } \begin{array}{c} \text{Vertical leg } 12, \text{ Hypotenuse } 13 \\ \text{Angle } A \end{array} & \therefore \cos A = \frac{5}{13} \\ \cos B &= \frac{4}{5} & \text{Diagram: } \begin{array}{c} \text{Horizontal leg } 4, \text{ Hypotenuse } 5 \\ \text{Angle } B \end{array} & \therefore \sin B = \frac{3}{5} \\ \therefore \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ &= \frac{12}{13} \times \frac{4}{5} + \left(\frac{4}{5}\right) \left(\frac{12}{13}\right) \\ &= \frac{48}{65} + \frac{48}{65} = \frac{33}{65} \end{aligned}$$

Question 9 (***)+

By considering the compound angle identity for $\tan(A+B)$, with suitable values for A and B , show that

$$\cot 75^\circ = 2 - \sqrt{3}.$$

, **proof**

$$\begin{aligned} \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ \bullet \cot 75^\circ &= \frac{1}{\tan 75^\circ} = \frac{1}{\tan(45^\circ+30^\circ)} = \frac{1}{\frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30}} = \frac{1}{\frac{1 + \sqrt{3}/3}{1 - \sqrt{3}/3}} \\ &= \frac{1 - \sqrt{3}/3}{1 + \sqrt{3}/3} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} = \frac{(3-\sqrt{3})(3+\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})} = \frac{9-6\sqrt{3}+3}{9-3} \\ &= \frac{(12-6\sqrt{3})}{6} = 2 - \sqrt{3} \end{aligned}$$

Question 10 (*)+**

It is given that

$$\tan \theta + \cot \theta \equiv 2 \operatorname{cosec} 2\theta, \quad \theta \neq \frac{k\pi}{2}, k \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence find, in terms of π , the solutions of the equation

$$\tan \theta + \cot \theta = 4, \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of π .

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

$ \begin{aligned} \text{(a)} \quad & \tan \theta + \cot \theta \\ &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\ &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \\ &= \frac{1}{\sin \theta \cos \theta} \\ &= \frac{2}{2 \sin \theta \cos \theta} \\ &= \frac{2}{\sin 2\theta} \end{aligned} $	$ \begin{aligned} \text{(b)} \quad & \tan \theta + \cot \theta = 4 \\ &\Rightarrow 2 \operatorname{cosec} 2\theta = 4 \\ &\Rightarrow \operatorname{cosec} 2\theta = 2 \\ &\Rightarrow \sin 2\theta = \frac{1}{2} \\ &\Rightarrow \operatorname{cosec} 2\theta = \frac{1}{2} \\ &\Rightarrow 2\theta = \frac{\pi}{6} \pm 2n\pi \\ &\Rightarrow 2\theta = \frac{\pi}{6} \pm 2n\pi \\ &\Rightarrow \theta = \frac{\pi}{12} \pm n\pi \\ &\Rightarrow \theta = \frac{\pi}{12} \pm \frac{n\pi}{2} \\ &\Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12} \end{aligned} $
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Question 11 (*)+**

Solve the following trigonometric equation

$$\sin 2\theta = \tan \theta, \quad 0 \leq \theta \leq 180^\circ.$$

$$\theta = 0^\circ, 45^\circ, 135^\circ, 180^\circ$$

$ \begin{aligned} \sin 2\theta &= \tan \theta \\ \Rightarrow 2 \sin \theta \cos \theta &= \frac{\sin \theta}{\cos \theta} \\ \Rightarrow 2 \sin \theta \cos^2 \theta &= \sin \theta \\ \Rightarrow 2 \sin \theta (\cos^2 \theta - 1) &= 0 \\ \Rightarrow \sin \theta (2\cos^2 \theta - 1) &= 0 \\ \Rightarrow \sin \theta \cos 2\theta &= 0 \end{aligned} $	$ \left\{ \begin{array}{l} \sin \theta = 0 \\ \cos 2\theta = 0 \end{array} \right. \quad \left\{ \begin{array}{l} \sin \theta = 0 \\ \cos \theta = 0 \end{array} \right. \quad \left\{ \begin{array}{l} \cos 2\theta = 0 \\ \cos \theta = 0 \end{array} \right. $	$ \begin{aligned} \sin \theta &= 0 \\ \cos \theta &= 0 \\ \theta &= 0^\circ \in [0^\circ, 180^\circ] \\ \theta &= 90^\circ \in [0^\circ, 180^\circ] \end{aligned} $
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Question 12 (*)**

It is given that

$$\cos(x+30^\circ) + \cos(x-30^\circ) \equiv \sqrt{3} \cos x.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence show that

$$\cos 75^\circ + \cos 15^\circ = \frac{1}{2}\sqrt{6}.$$

proof

(a) LHS = $\cos(x+30^\circ) + \cos(x-30^\circ)$
 $= \cos x \cos 30^\circ - \sin x \sin 30^\circ + \cos x \cos 30^\circ + \sin x \sin 30^\circ$
 $= 2\cos x \cos 30^\circ = 2(\cos x \times \frac{\sqrt{3}}{2}) = \sqrt{3} \cos x = RHS$ //

(b) Let $x=45^\circ$ in
 $\cos(x+30^\circ) + \cos(x-30^\circ) = \sqrt{3} \cos x$
 $\cos(45^\circ+30^\circ) + \cos(45^\circ-30^\circ) = \sqrt{3} \cos 45^\circ$
 $\cos 75^\circ + \cos 15^\circ = \sqrt{3} \times \frac{\sqrt{2}}{2}$
 $\cos 75^\circ + \cos 15^\circ = \frac{1}{2}\sqrt{6}$ // as required

Question 13 (*)**

Prove the validity of each of the following trigonometric identities.

a) $\left(\frac{1+\sin\theta}{\cos\theta}\right)^2 + \left(\frac{1-\sin\theta}{\cos\theta}\right)^2 \equiv 4\tan^2\theta + 2.$

b) $\sin^2\left(\theta + \frac{\pi}{4}\right) + \sin^2\left(\theta - \frac{\pi}{4}\right) \equiv 1.$

proof

(a) LHS = $\left(\frac{1+\sin\theta}{\cos\theta}\right)^2 + \left(\frac{1-\sin\theta}{\cos\theta}\right)^2 = \frac{(1+2\sin\theta+\sin^2\theta)}{\cos^2\theta} + \frac{(1-2\sin\theta+\sin^2\theta)}{\cos^2\theta}$
 $= \frac{2+2\sin\theta}{\cos^2\theta} = \frac{2}{\cos^2\theta} + \frac{2\sin\theta}{\cos^2\theta} = 2\sec^2\theta + 2\tan\theta\sec\theta$
 $= 2(\tan^2\theta + 1) + 2\tan\theta\sec\theta = 4\tan^2\theta + 2 = RHS$ //

(b) LHS = $\sin^2\left(\theta + \frac{\pi}{4}\right) + \sin^2\left(\theta - \frac{\pi}{4}\right) = [\sin\left(\theta + \frac{\pi}{4}\right)]^2 + [\sin\left(\theta - \frac{\pi}{4}\right)]^2$
 $= [\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}]^2 + [\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}]^2$
 $= \left(\frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta\right)^2 + \left(\frac{\sqrt{2}}{2}\sin\theta - \frac{\sqrt{2}}{2}\cos\theta\right)^2$
 $= \frac{1}{2}\sin^2\theta + \frac{1}{2}\cos^2\theta + \frac{1}{2}(\sin^2\theta - \cos^2\theta) + \frac{1}{2}(\cos^2\theta - \sin^2\theta) + \cos^2\theta - \sin^2\theta$
 $= \frac{1}{2}[2\sin^2\theta + 2\cos^2\theta] + \cos^2\theta - \sin^2\theta = 2\sin^2\theta + 2\cos^2\theta = 2$
ALTERNATIVE USE IDENTITY $\sin A = \cos(\frac{\pi}{2} - A)$

Question 14 (*)**

It is given that

$$\frac{1}{\cosec \theta - 1} + \frac{1}{\cosec \theta + 1} \equiv 2 \tan \theta \sec \theta.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence solve the equation

$$\frac{1}{\cosec \theta - 1} + \frac{1}{\cosec \theta + 1} = \frac{1}{2} \cot \theta \sec \theta, \quad 0^\circ \leq \theta < 360^\circ.$$

$$\theta = 26.6^\circ, 153.4^\circ, 206.6^\circ, 333.4^\circ$$

(a)

$$\begin{aligned} \text{LHS} &= \frac{1}{\cosec \theta - 1} + \frac{1}{\cosec \theta + 1} = \frac{(\cosec \theta + 1) + (\cosec \theta - 1)}{(\cosec \theta - 1)(\cosec \theta + 1)} \\ &= \frac{\cosec \theta + 1 + \cosec \theta - 1}{\cosec \theta - 1} = \frac{2 \cosec \theta}{(\cosec \theta - 1) \cosec \theta} = \frac{2 \cosec \theta}{\cosec^2 \theta - 1} = \frac{2 \cosec \theta}{\sin^2 \theta} = \frac{2 \cosec \theta}{\frac{1}{\tan^2 \theta}} = 2 \tan^2 \theta = 2 \tan \theta \sec \theta \\ &= \text{RHS} \end{aligned}$$

(b)

$$\begin{aligned} \frac{1}{\cosec \theta - 1} + \frac{1}{\cosec \theta + 1} &= \frac{1}{2} \cot \theta \sec \theta \\ \Rightarrow 2 \tan \theta \sec \theta &= \frac{1}{2} \cot \theta \sec \theta \\ \Rightarrow 4 \tan \theta \sec \theta &= \cot \theta \sec \theta \\ \Rightarrow 4 \tan \theta \sec \theta - \cot \theta \sec \theta &= 0 \\ \Rightarrow \tan \theta (\tan \theta - \cot \theta) &= 0 \\ \Rightarrow \frac{1}{\cos \theta} (\frac{\sin \theta}{\cos \theta} - \frac{\cos \theta}{\sin \theta}) &= 0 \\ \Rightarrow 4 \tan^2 \theta - 1 &= 0 \\ \Rightarrow 4 \tan^2 \theta &= 1 \\ \Rightarrow \tan^2 \theta &= \frac{1}{4} \\ \Rightarrow \tan \theta &= \pm \frac{1}{2} \\ \Rightarrow \tan \theta &= \frac{1}{2} \quad \text{or} \quad \tan \theta = -\frac{1}{2} \end{aligned}$$

$\bullet \arctan(\frac{1}{2}) = 26.6^\circ \quad \bullet \arctan(-\frac{1}{2}) = -26.6^\circ$

$\theta = 26.6^\circ, 153.4^\circ, 206.6^\circ, 333.4^\circ$

$\theta = 0, 90^\circ, 180^\circ, 270^\circ$

Question 15 (***)

$$\sin x = \frac{12}{13} \text{ and } \cos y = \frac{15}{17}.$$

If x is obtuse and y is acute, show clearly that

$$\sin(x-y) = \frac{220}{221}.$$

, proof

$$\begin{aligned}\sin(x-y) &= \sin x \cos y - \cos x \sin y \\ &= \frac{12}{13} \times \frac{15}{17} - \left(-\frac{5}{13}\right) \times \frac{8}{17} \\ &= \frac{180}{221} + \frac{40}{221} \\ &= \frac{220}{221}\end{aligned}$$

Question 16 (***)

Solve the following trigonometric equation

$$\frac{2+\cos 2x}{3+\sin^2 2x} = \frac{2}{5}, \text{ for } 0^\circ \leq x < 360^\circ,$$

$x = 60^\circ, 120^\circ, 240^\circ, 300^\circ$

$$\begin{aligned}\frac{2+\cos 2x}{3+\sin^2 2x} &= \frac{2}{5} \\ \Rightarrow 10 + 5\cos 2x &= 6 + 2\sin^2 2x \\ \Rightarrow 4 + 5\cos 2x &= 2(1-\cos^2 2x) \\ \Rightarrow 4 + 5\cos 2x &= 2 - 2\cos^2 2x \\ \Rightarrow 2\cos^2 2x + 5\cos 2x + 2 &= 0 \\ \Rightarrow (2\cos 2x + 1)(\cos 2x + 2) &= 0 \\ \Rightarrow \cos 2x &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\arccos\left(-\frac{1}{2}\right) &= 120^\circ \\ 2x &= 120^\circ \pm 360^\circ \\ 2x &= 240^\circ \pm 360^\circ \quad n=0,1,2,3,\dots \\ x &= 60^\circ \pm 180^\circ \\ x &= 120^\circ \pm 180^\circ \\ x_1 &= 60^\circ \\ x_2 &= 240^\circ \\ x_3 &= 120^\circ \\ x_4 &= 300^\circ\end{aligned}$$

Question 17 (***)

It is given that

$$\frac{2 \tan x}{1 + \tan^2 x} \equiv \sin 2x.$$

- a) Prove the validity of the above trigonometric identity.
- b) Use part (a) to show that

$$\tan 15^\circ = 2 - \sqrt{3}.$$

□, proof

$$\begin{aligned}
 \text{(a)} \quad & \tan 15^\circ = \frac{2 \tan x}{1 + \tan^2 x} = \frac{2 \tan x}{\sec^2 x} = \frac{2 \sin x}{\cos^2 x} = \frac{2 \sin x \cos^2 x}{\cos^2 x} = \frac{2 \sin x \cos^2 x}{\cos^2 x} \\
 & = 2 \sin x \cos x = \sin 2x = 2 \sin 15^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \tan 15^\circ = \frac{2 \tan x}{1 + \tan^2 x} = \sin 2x \\
 & \Rightarrow \frac{2 \tan 15^\circ}{1 + \tan^2 15^\circ} = \sin 30^\circ \\
 & \Rightarrow \frac{2 \tan 15^\circ}{1 + \tan^2 15^\circ} = \frac{1}{2} \\
 & \Rightarrow \frac{2t}{1+t^2} = \frac{1}{2} \quad (t = \tan 15^\circ) \\
 & \Rightarrow 4t = 1 + t^2 \\
 & \Rightarrow 0 = t^2 - 4t + 1 \\
 & \Rightarrow (t-2)^2 - 4 + 1 = 0 \\
 & \Rightarrow (t-2)^2 = 3 \\
 & \Rightarrow t-2 = \pm \sqrt{3} \\
 & \Rightarrow t = 2 \pm \sqrt{3}
 \end{aligned}$$

$\therefore \tan 15^\circ < 2 + \sqrt{3} > 1$
 $\therefore \tan 15^\circ < 2 - \sqrt{3} < 1$
 (but $\tan 45^\circ = 1$)
 $\therefore \tan 15^\circ = 2 - \sqrt{3}$

Question 18 (***)

Solve the trigonometric equation

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4, \quad 0 \leq x < 360^\circ.$$

$$x = 15^\circ, 75^\circ, 195^\circ, 255^\circ$$

$$\begin{aligned}
 & \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 4 \\
 & \Rightarrow \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = 4 \\
 & \Rightarrow \frac{1}{\sin x \cos x} = 4 \\
 & \Rightarrow \frac{2}{\sin 2x} = 4 \\
 & \Rightarrow \frac{2}{\sin 2x} = 4 \\
 & \Rightarrow 2 = 4 \sin 2x \\
 & \Rightarrow \sin 2x = \frac{1}{2}
 \end{aligned}$$

$\sin 2x = \frac{1}{2}$
 $\arcsin(\frac{1}{2}) = 30^\circ$
 $2x = 30^\circ \Rightarrow 360^\circ$
 $2x = 150^\circ \Rightarrow 360^\circ$
 $4x = 15^\circ \pm 180^\circ$
 $2x = 75^\circ \pm 180^\circ$
 $2x = 15^\circ$
 $2x = 15^\circ$
 $2x = 15^\circ$
 $2x = 15^\circ$
 $2x = 255^\circ$

Question 19 (***)

Solve the following trigonometric equation

$$\tan(\theta + 45^\circ) = 1 - 2 \tan \theta, \quad 0^\circ \leq \theta < 360^\circ.$$

$$x = 0, 180^\circ, 63.4^\circ, 243.4^\circ$$

$$\begin{aligned}
 \tan(\theta + 45^\circ) &= 1 - 2 \tan \theta \\
 \frac{\tan \theta + \tan 45^\circ}{1 - \tan \theta \tan 45^\circ} &= 1 - 2 \tan \theta \\
 \frac{\tan \theta + 1}{1 - \tan \theta} &= 1 - 2 \tan \theta \\
 \frac{1 + \tan \theta}{1 - \tan \theta} &= 1 - 2 \tan \theta \\
 1 + \tan \theta &= (1 - 2 \tan \theta)(1 - \tan \theta) \\
 1 + \tan \theta &= 1 - \tan \theta - 2 \tan^2 \theta + 2 \tan \theta \\
 0 &= 2 \tan^2 \theta - 4 \tan \theta \\
 0 &= 2 \tan(\theta - 2) \\
 \theta &= 2\pi(\theta - 2)
 \end{aligned}$$

If $T = \frac{\pi}{2}$, $\tan \theta = \frac{\pi}{2}$
 $\arctan 0 = 0$, $\arctan(\frac{\pi}{2}) = 0.57$
 $\theta = 0 + 180^\circ$, $\theta = 63.4^\circ + 180^\circ$
 $n = 1, 3, \dots$
 $\therefore \theta_1 = 0$,
 $\theta_2 = 180^\circ$,
 $\theta_3 = 0.57^\circ$,
 $\theta_4 = 243.4^\circ$

Question 20 (***)

It is given that

$$\tan x \sec x + \operatorname{cosec} x \equiv \operatorname{cosec} x \sec^2 x, \quad x \neq \frac{n\pi}{2}, n \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence show that the equation

$$\tan x \sec x + \operatorname{cosec} x = \frac{1}{2} \sec^2 x$$

has no real solutions.

proof

$$\begin{aligned}
 \text{(Q)} \quad \text{LHS} &= \tan x \sec x + \operatorname{cosec} x = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} + \frac{1}{\sin x} = \frac{\sin x}{\cos^2 x} + \frac{1}{\sin x} \\
 &= \frac{\sin^2 x + \cos^2 x}{\sin x \cos^2 x} = \frac{1}{\sin x \cos^2 x} = \frac{1}{\sin x} \cdot \frac{1}{\cos^2 x} = \operatorname{cosec} x \sec^2 x \\
 &= \text{RHS}
 \end{aligned}$$

(4) $\tan x \sec x + \operatorname{cosec} x = \frac{1}{2} \sec^2 x \quad \left\{ \begin{array}{l} \frac{1}{\cos^2 x} < 0 \text{ or } \sin x = 0 \\ \therefore \text{no real solutions} \end{array} \right.$

Question 21 (***)

Solve each of the following trigonometric equations.

i. $\sin \varphi + \frac{1}{4} \sec \varphi = 0, \quad 0 \leq \varphi < \pi.$

ii. $\cos 2y - 7 \cos y + 4 = 0, \quad 0 \leq y < 360^\circ.$

$$\boxed{\varphi = \frac{7\pi}{12}, \frac{11\pi}{12}}, \boxed{y = 60^\circ, 300^\circ}$$

<p>(a) $\sin \varphi + \frac{1}{4} \sec \varphi = 0$ $\Rightarrow 4 \sin \varphi + \sec \varphi = 0$ $\Rightarrow 4 \sin \varphi + \frac{1}{\cos \varphi} = 0$ $\Rightarrow 4 \sin \varphi \cos \varphi + 1 = 0$ $\Rightarrow 2 \sin 2\varphi + 1 = 0$ $\Rightarrow \sin 2\varphi = -\frac{1}{2}$ $\Rightarrow \arcsin(\frac{1}{2}) = -\frac{\pi}{6}$ $\Rightarrow 2\varphi = -\frac{\pi}{6} \pm 2\pi k, \quad k \in \mathbb{Z}$ $\Rightarrow \varphi = -\frac{\pi}{12} \pm \pi k$ $\Rightarrow \varphi = \frac{7\pi}{12}, \frac{11\pi}{12}$ </p>	<p>(b) $\cos 2y - 7 \cos y + 4 = 0$ $\Rightarrow (2\cos^2 y - 1) - 7\cos y + 4 = 0$ $\Rightarrow 2\cos^2 y - 7\cos y + 3 = 0$ $\Rightarrow (2\cos y - 1)(\cos y - 3) = 0$ $\Rightarrow \cos y = \frac{1}{2}$ $\Rightarrow y = 60^\circ$ $y = 300^\circ$ </p>
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Question 22 (***)

The acute angles α and β satisfy the relationships

$$7 \cot^2 \alpha + 6 \cot \alpha = 1 \quad \text{and} \quad 6 \tan \beta = 8 + \sec^2 \beta.$$

- a) Determine the value of $\tan \alpha$ and the value of $\tan \beta$.

- b) Show clearly that

$$\tan(\alpha + \beta) = -\frac{1}{2},$$

$$\boxed{\tan \alpha = 7}, \boxed{\tan \beta = 3}$$

<p>(a) $7 \cot^2 \alpha + 6 \cot \alpha = 1$ $7 \cot^2 \alpha + 6 \cot \alpha - 1 = 0$ $(\cot \alpha - 1)(\cot \alpha + 1) = 0$ $\cot \alpha = -1 \quad (\cot \alpha = 1 \text{ is not acute})$ $\tan \alpha = -1$ </p>	<p>$6 \tan^2 \beta = 8 + \sec^2 \beta$ $6 \tan^2 \beta = 8 + (1 + \tan^2 \beta)$ $5 \tan^2 \beta = 9$ $\tan^2 \beta = 9/5$ $\tan \beta = 3$</p>
<p>(b) $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{7 + 3}{1 - 7 \times 3} = \frac{10}{-20} = -\frac{1}{2}$</p>	

Question 23 (*)+**

Simplify, showing all steps in the calculation, the following expression

$$\tan(\arctan 3 - \arctan 2),$$

giving the final answer as an exact fraction.

$\frac{1}{7}$

$$\begin{aligned} \tan(\arctan 3 - \arctan 2) &= \frac{\tan(\arctan 3) - \tan(\arctan 2)}{1 + \tan(\arctan 3)\tan(\arctan 2)} = \frac{3 - 2}{1 + 3 \times 2} \\ &= \frac{1}{7} \end{aligned}$$

Question 24 (*)+**

Show clearly that if $x > 0$

$$\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}.$$

[proof]

Method A

Let $\theta = \arctan x \Rightarrow x = \tan \theta$

$\phi = \arctan \frac{1}{x} \Rightarrow \frac{1}{x} = \tan \phi$

$$\begin{aligned} \Rightarrow \psi &= \theta + \phi \\ \Rightarrow \tan \psi &= \tan(\theta + \phi) \\ \Rightarrow \tan \psi &= \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} \\ \Rightarrow \tan \psi &= \frac{x + \frac{1}{x}}{1 - x \cdot \frac{1}{x}} \\ \Rightarrow \tan \psi &= \frac{x + \frac{1}{x}}{0} \\ \Rightarrow \tan \psi &= \infty \\ \Rightarrow \psi &= \dots, -\frac{\pi}{2}, \frac{\pi}{2}, \dots \end{aligned}$$

BUT θ, ϕ ARE ACUTE ANGLES

$\therefore 0 < \theta + \phi < \pi$

$\therefore \psi = \frac{\pi}{2}$

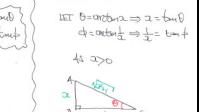
$\therefore \arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$

Method B

LET $\theta = \arctan x \Rightarrow x = \tan \theta$

$\phi = \arctan \frac{1}{x} \Rightarrow \frac{1}{x} = \tan \phi$

$\therefore \theta > 0 > \phi$



BUT $\tan(\hat{A}C) = \frac{1}{x}$

$\therefore \hat{B}AC = \phi$

$\therefore \phi + \theta = \frac{\pi}{2}$

$\therefore \arctan \frac{1}{x} + \arctan \frac{1}{x} = \frac{\pi}{2}$

Question 25 (*)+**

Prove the validity of each of the following trigonometric identities.

a) $\frac{\sec^2 \theta}{1 - \tan^2 \theta} \equiv \sec 2\theta$.

b) $\frac{\cos 4\theta + \cos 2\theta}{\sin 4\theta - \sin 2\theta} \equiv \cot \theta$.

proof

(a) LHS = $\frac{\sec^2 \theta}{1 - \tan^2 \theta} = \frac{\frac{1}{\cos^2 \theta}}{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} = \frac{1}{\cos^2 \theta - \sin^2 \theta} = \frac{1}{\cos 2\theta} = \sec 2\theta$ ✓

(b) LHS = $\frac{\cos 4\theta + \cos 2\theta}{\sin 4\theta - \sin 2\theta} = \frac{2 \cos \frac{4\theta+2\theta}{2} \cos \frac{4\theta-2\theta}{2}}{2 \sin \frac{4\theta+2\theta}{2} \cos \frac{4\theta-2\theta}{2}} = \frac{\cos 3\theta}{\sin \theta} = \cot \theta$ ✓

Question 26 (***)+

It is given that

$$\frac{\sec^2 \theta}{1 - \tan^2 \theta} \equiv \sec 2\theta.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence, or otherwise, solve the equation

$$\frac{\sec^2 2x}{1 - \tan^2 2x} - 2 = 0, \quad 0 \leq x < \frac{\pi}{2},$$

giving the answers in terms of π .

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

$(a) LHS = \frac{\sec^2 \theta}{1 - \tan^2 \theta}$ $= \frac{1}{\cos^2 \theta}$ $= \frac{1}{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}$ $= \frac{1}{\cos^2 \theta - \sin^2 \theta}$ $= \frac{1}{\cos 2\theta}$ $= \sec 2\theta$ $= RHS$	$(b) \frac{\sec^2 2x}{1 - \tan^2 2x} - 2 = 0$ $\Rightarrow \sec 2x - 2 = 0$ $\Rightarrow \sec 2x = 2$ $\Rightarrow \cos 4x = \frac{1}{2}$ $\cos(4x) = \frac{1}{2}$ $4x = \frac{\pi}{3} + 2k\pi \quad k \in \mathbb{Z}$ $4x = \frac{\pi}{3} \quad k = 0$ $x = \frac{\pi}{12} \quad k = 0$ $x = \frac{\pi}{12}, \frac{5\pi}{12}$
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Question 27 (***)

It is given that θ is a reflex angle such that

$$\cos \theta = \frac{2}{3}.$$

Find the exact value of $\sin 2\theta$.

$$\boxed{\sin 2\theta = -\frac{4\sqrt{5}}{9}}$$

$$\begin{aligned} \cos \theta &= \frac{2}{3}, \quad \theta \text{ reflex} \\ \therefore \sin \theta &= -\frac{\sqrt{5}}{3} \\ \therefore \sin 2\theta &= 2 \sin \theta \cos \theta = 2 \left(-\frac{\sqrt{5}}{3}\right) \times \frac{2}{3} = -\frac{4\sqrt{5}}{9} \end{aligned}$$

Question 28 (***)

It is given that

$$\frac{\cosec \theta}{\cosec \theta - \sin \theta} \equiv \sec^2 \theta, \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence solve the equation

$$\frac{\cosec \theta}{\cosec \theta - \sin \theta} + 4(\sec \theta + 1) = 0, \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of π .

$$\boxed{\theta = \frac{2\pi}{3}, \frac{4\pi}{3}}$$

$$\begin{aligned} \text{(a)} \quad LHS &= \frac{\cosec \theta}{\cosec \theta - \sin \theta} = \frac{\frac{1}{\sin \theta}}{\frac{1}{\sin \theta} - \frac{\sin \theta}{\sin \theta}} = \frac{\frac{1}{\sin \theta}}{\frac{1 - \sin^2 \theta}{\sin \theta}} = \frac{\frac{1}{\sin \theta}}{\frac{\cos^2 \theta}{\sin \theta}} = \frac{1}{\cos^2 \theta} = \sec^2 \theta \\ &= \frac{1}{\cos^2 \theta} = \sec^2 \theta \\ \text{(b)} \quad \frac{\cosec \theta}{\cosec \theta - \sin \theta} + 4(\sec \theta + 1) &= 0 \quad \left\{ \begin{array}{l} \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \\ \theta = 2k\pi \pm 2m\pi \quad k = 0, 1, 2, \dots \\ \theta = \frac{2k\pi}{3} + 2m\pi \\ \theta_1 = \frac{2\pi}{3} \\ \theta_2 = \frac{4\pi}{3} \end{array} \right. \\ \Rightarrow \sec \theta + 4 \sec \theta + 4 &= 0 \\ \Rightarrow (\sec \theta + 2)^2 &= 0 \\ \Rightarrow \sec \theta &= -2 \\ \Rightarrow \cos \theta &= -\frac{1}{2} \end{aligned}$$

Question 29 (*)+**

Solve the following trigonometric equation

$$\sin 2\theta = \cot \theta, \quad 0^\circ \leq \theta \leq 180^\circ.$$

$$\boxed{\quad}, \boxed{\theta = 45^\circ, 90^\circ, 135^\circ}$$

Working for Question 29:

$$\begin{aligned} \sin 2\theta &= \cot \theta \\ \Rightarrow 2\sin \theta \cos \theta &= \frac{\cos^2 \theta}{\sin \theta} \\ \Rightarrow 2\sin^2 \theta \cos^2 \theta &= \cos^2 \theta \\ \Rightarrow 2\sin^2 \theta &= 1 \\ \Rightarrow \sin^2 \theta &= \frac{1}{2} \\ \Rightarrow \sin \theta &= \pm \frac{1}{\sqrt{2}} \end{aligned}$$

For $\sin \theta = \frac{1}{\sqrt{2}}$:

$$\begin{aligned} \theta &= 45^\circ + 360^\circ n, \quad n = 0, 1, 2, 3, \dots \\ \theta &= 225^\circ + 360^\circ n, \quad n = 0, 1, 2, 3, \dots \\ \theta &= 45^\circ + 360^\circ n, \quad n = 0, 1, 2, 3, \dots \\ \theta &= 225^\circ + 360^\circ n, \quad n = 0, 1, 2, 3, \dots \end{aligned}$$

For $\sin \theta = -\frac{1}{\sqrt{2}}$:

$$\begin{aligned} \theta &= -45^\circ + 360^\circ n, \quad n = 0, 1, 2, 3, \dots \\ \theta &= 315^\circ + 360^\circ n, \quad n = 0, 1, 2, 3, \dots \end{aligned}$$

Hence $\theta = 45^\circ, 90^\circ, 135^\circ$

Question 30 (*)+**

It is given that

$$(\cosec \theta - \sin \theta) \sec^2 \theta \equiv \cosec \theta, \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}.$$

- Prove the validity of the above trigonometric identity.
- Hence solve the equation

$$(\cosec \theta - \sin \theta) \sec^2 \theta = \sqrt{2}, \quad 0^\circ \leq \theta < 2\pi,$$

giving the answers in terms of π .

$$\boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}}$$

Working for Question 30(a):

$$\begin{aligned} \text{LHS} &= (\cosec \theta - \sin \theta) \sec^2 \theta = \left(\frac{1}{\sin \theta} - \sin \theta \right) \sec^2 \theta = \left(\frac{1 - \sin^2 \theta}{\sin \theta} \right) \sec^2 \theta \\ &= \frac{\cos^2 \theta}{\sin \theta} \times \sec^2 \theta = \frac{\cos^2 \theta}{\sin \theta} \times \frac{1}{\cos^2 \theta} = \cosec \theta = R.H.S \end{aligned}$$

(b)

$$\begin{aligned} (\cosec \theta - \sin \theta) \sec^2 \theta &= \sqrt{2} \\ \Rightarrow \cosec \theta &= \sqrt{2} \\ \Rightarrow \sin \theta &= \frac{1}{\sqrt{2}} \\ \Rightarrow \arcsin \left(\frac{1}{\sqrt{2}} \right) &= \frac{\pi}{4} \end{aligned}$$

From the diagram, $\theta = \frac{\pi}{4} \pm 2n\pi$ and $\theta = \frac{3\pi}{4} \pm 2n\pi$, where $n = 0, 1, 2, 3, \dots$. Hence $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{3\pi}{4}$.

Question 31 (***)+

The constants a and b are such so that

$$\tan a = \frac{1}{3} \quad \text{and} \quad \tan b = \frac{1}{7}.$$

Determine the exact value of $\cot(a - b)$, showing all the steps in the workings.

$$\cot(a - b) = \frac{11}{2}$$

$$\begin{aligned}\cot(a - b) &= \frac{1}{\tan(a - b)} = \frac{1}{\frac{\tan a - \tan b}{1 + \tan a \tan b}} = \frac{1 + \tan a \tan b}{\tan a - \tan b} \\ &= \frac{1 + \frac{1}{3} \times \frac{1}{7}}{\frac{1}{3} - \frac{1}{7}} = \frac{\frac{22}{21}}{\frac{2}{21}} = \frac{11}{2}\end{aligned}$$

Question 32 (***)+

$$f(y) = 6 + 3\cos y + 4\sin y, \quad 0 < y < 2\pi.$$

- a) Express $3\cos y + 4\sin y$ in the form $a\cos(y - b)$, $a > 0$, $0 < b < \frac{\pi}{2}$.

It is further given that for $0 < y < 2\pi$

$$A \leq 2f(2y) \leq B.$$

- b) Determine the value of each of the constants A and B .

$$\boxed{\text{C.P.}}, \quad \boxed{3\cos y + 4\sin y \equiv 5\cos(y - 0.927^\circ)}, \quad \boxed{A = 2}, \quad \boxed{B = 22}$$

$$\begin{aligned}\text{(a)} \quad 3\cos y + 4\sin y &\equiv a\cos(y - b) \\ &\equiv a\cos y \cos b + a\sin y \sin b \\ a\cos b &= 3 \\ a\sin b &= 4 \quad \therefore a = \sqrt{3^2 + 4^2} = 5 \\ \therefore b &= \frac{1}{5} \Rightarrow b = 0.927^\circ \\ \therefore 3\cos y + 4\sin y &\equiv 5\cos(y - 0.927^\circ)\end{aligned}$$

(b)

<ul style="list-style-type: none"> • $-5 \leq 3\cos y + 4\sin y \leq 5$ ($5\cos(y - 0.927^\circ)$) • $1 \leq 6 + 3\cos y + 4\sin y \leq 11$ ($6 + 5\cos(y - 0.927^\circ)$) • $2 \leq 12 + 3\cos y + 4\sin y \leq 22$ ($12 + 5\cos(y - 0.927^\circ)$) 	<ul style="list-style-type: none"> • $2 \leq 12 + 5\cos(y - 0.927^\circ) + 5\sin(y - 0.927^\circ) \leq 22$ ($24 + 5\cos(y - 0.927^\circ)$)
	$\therefore A = 2, \quad B = 22$

Question 33 (*)+**

Use the compound angle identity for $\tan(A + B)$, with suitable values for A and B , to show that

$$\frac{1 + \tan 15^\circ}{1 - \tan 15^\circ} = \tan 60^\circ.$$

proof

$$\begin{aligned}\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ \text{Let } A &= 45^\circ, B = 15^\circ \\ \tan(45^\circ + 15^\circ) &= \frac{\tan 45^\circ + \tan 15^\circ}{1 - \tan 45^\circ \tan 15^\circ} \\ \tan 60^\circ &= \frac{1 + \tan 15^\circ}{1 - \tan 15^\circ} \quad (\tan 45^\circ = 1) \\ \tan 60^\circ &= \frac{1 + \tan 15^\circ}{1 - \tan 15^\circ} //\end{aligned}$$

Question 34 (*)+**

$$f(x) = \cosec x - \sin x, \quad 0 < x < 180^\circ.$$

Show that $f(x) \geq 0$ for the entire domain of the function.

proof

$$\begin{aligned}f(x) &= \cosec x - \sin x \\ f(x) &= \frac{1}{\sin x} - \sin x \\ f(x) &= \frac{\sin x - \sin^2 x}{\sin x} \\ f(x) &= \frac{\sin x(1 - \sin x)}{\sin x} \\ \text{Now } &\begin{array}{l} y = \sin x \\ 0 < x < 180^\circ \\ \sin x > 0 \quad \text{if } 0 < x < 90^\circ \\ \cos x > 0 \\ \therefore f(x) > 0 \end{array}\end{aligned}$$

Question 35 (*)**

Given that $\cos x^\circ = \sin(x - 45)^\circ$, show that

$$\tan x^\circ = 1 + \sqrt{2}.$$

, proof

$$\begin{aligned} \Rightarrow \cos x &= \sin(x - 45) \\ \Rightarrow \cos x &= \sin x \cos 45 - \cos x \sin 45 \\ \Rightarrow \cos x &= \frac{\sqrt{2}}{2} \sin x - \frac{\sqrt{2}}{2} \cos x \\ \Rightarrow 2\cos x &= \sqrt{2} \sin x - \sqrt{2} \cos x \\ \Rightarrow \frac{2\cos x}{\cos x} &= \frac{\sqrt{2} \sin x}{\cos x} - \frac{\sqrt{2} \cos x}{\cos x} \\ \Rightarrow 2 &= \sqrt{2} \tan x - \sqrt{2} \\ \Rightarrow 2 + \sqrt{2} &= \sqrt{2} \tan x \\ \Rightarrow \tan x &= \frac{2 + \sqrt{2}}{\sqrt{2}} \\ \Rightarrow \tan x &= \frac{2}{\sqrt{2}} + 1 \end{aligned}$$

Question 36 (*)**

$$f(x) \equiv \sin x + 2 \cos x.$$

- a) Express $f(x)$ in the form $R \cos(x - \alpha)$, where $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

- b) Hence solve the equation

$$1 + 2 \cot x = \operatorname{cosec} x, \quad 0 < x < 2\pi.$$

$$f(x) = \sqrt{5} \cos(x - 0.464^\circ), \quad x = 1.57^\circ \cup x = 5.64^\circ$$

(a) $\sin x + 2 \cos x \equiv R \cos(x - \alpha)$
 $\equiv R \cos x \cos \alpha + R \sin x \sin \alpha$
 $\equiv (\cos \alpha) \cos x + (\sin \alpha) \sin x$

$\therefore \cos \alpha = 1$ Square and ADD $R = \sqrt{2^2 + 1^2} \rightarrow R = \sqrt{5}$
 $\sin \alpha = 0$ DIVIDE EQUATIONS $\tan \alpha = \frac{1}{2} \rightarrow \alpha = 0.464^\circ$
 $\therefore f(x) = \sqrt{5} \cos(x - 0.464^\circ)$

(b) $1 + 2 \cot x = \operatorname{cosec} x$
 $\Rightarrow 1 + \frac{2 \cos x}{\sin x} = \frac{1}{\sin x}$
 $\Rightarrow \sin x + 2 \cos x = 1$
 $\Rightarrow \sqrt{5} \cos(x - 0.464^\circ) = 1$
 $\Rightarrow \cos(x - 0.464^\circ) = \frac{1}{\sqrt{5}}$

$\operatorname{cosec}(x) = 1/\operatorname{sin} x$
 $\therefore \operatorname{sin} x = \pm \sqrt{5}/\sqrt{5} = \pm 1$
 $\therefore x = 0^\circ \text{ or } 180^\circ$
 $\therefore x = 0^\circ \text{ or } 180^\circ$

Question 37 (***)

$$\frac{\sin 2x}{1 - \cos 2x} = \tan x, \quad 0 < x < 2\pi.$$

Find the solutions of the above trigonometric equation, giving the answers in radians in terms of π .

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\begin{aligned} \frac{\sin 2x}{1 - \cos 2x} &= \tan x \\ \Rightarrow \frac{2\sin x \cos x}{1 - (1 - 2\sin^2 x)} &= \tan x \\ \Rightarrow \frac{2\sin x \cos x}{2\sin^2 x} &= \tan x \\ \Rightarrow \frac{\cos x}{\sin x} &= \tan x \\ \Rightarrow \cot x &= \tan x \\ \Rightarrow \frac{1}{\tan x} &= \tan x \end{aligned} \quad \left. \begin{aligned} \Rightarrow \tan^2 x &= 1 \\ \Rightarrow \tan x &= \pm 1 \\ \left(x = \frac{\pi}{4} \pm \frac{n\pi}{2} \right) &= n\pi/2, n \in \mathbb{Z} \\ \therefore x_1 &= \frac{\pi}{4} \\ x_2 &= \frac{3\pi}{4} \\ x_3 &= \frac{5\pi}{4} \\ x_4 &= \frac{7\pi}{4} \end{aligned} \right\}$$

Question 38 (***)

Given that

$$\tan\left(x + \frac{\pi}{4}\right) = 4 + \tan x,$$

find as an exact surd the exact value of $\tan x$.

$$\tan x = -2 \pm \sqrt{7}$$

$$\begin{aligned} \tan\left(x + \frac{\pi}{4}\right) &= 4 + \tan x \\ \Rightarrow \frac{\tan x + \tan \frac{\pi}{4}}{1 - \tan x \tan \frac{\pi}{4}} &= 4 + \tan x \\ \Rightarrow \frac{\tan x + 1}{1 - \tan x} &= 4 + \tan x \\ \Rightarrow \frac{1 + \tan x}{1 - \tan x} &= 4 + \tan x \\ \Rightarrow 1 + \tan x &= (4 + \tan x)(1 - \tan x) \end{aligned} \quad \left. \begin{aligned} \Rightarrow T + 1 &= 4 - 4T + T^2 - T^2 \\ \Rightarrow T^2 + 4T - 3 &= 0 \\ \Rightarrow (T+2)^2 - 7 &= 0 \\ \Rightarrow (T+2)^2 &= 7 \\ \Rightarrow T+2 &= \pm \sqrt{7} \\ \Rightarrow T &= -2 \pm \sqrt{7} \\ \therefore \tan x &= -2 \pm \sqrt{7} \end{aligned} \right\}$$

Question 39 (***)

$$\frac{1+\tan^2 \theta}{1-\tan^2 \theta} = 2, \quad 0^\circ \leq \theta < 360^\circ.$$

Find the solutions of the above trigonometric equation, giving the answers in degrees.

$$\theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$$

$$\begin{aligned} \frac{1+\tan^2 \theta}{1-\tan^2 \theta} &= 2 \\ \Rightarrow \frac{1 + \frac{\sin^2 \theta}{\cos^2 \theta}}{1 - \frac{\sin^2 \theta}{\cos^2 \theta}} &= 2 \\ \Rightarrow \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta} &= 2 \\ \Rightarrow \frac{1}{\cos 2\theta} &= 2 \\ \Rightarrow \cos 2\theta &= \frac{1}{2} \\ \cos(\frac{\pi}{3}) &= 60^\circ \\ 2\theta &= 60 + 360n \quad n=0,1,2,3,\dots \\ \theta &= 30 + 180n \\ \theta &= 150 + 180n \\ \theta_1 &= 30^\circ \\ \theta_2 &= 150^\circ \\ \theta_3 &= 210^\circ \\ \theta_4 &= 330^\circ \end{aligned}$$

Question 40 (***)

$$3\sec 2\psi - 2\cot 2\psi = 0, \quad 0^\circ \leq \psi < 360^\circ.$$

Find the solutions of the above trigonometric equation, giving the answers in degrees.

$$\psi = 15^\circ, 75^\circ, 195^\circ, 255^\circ$$

$$\begin{aligned} 3\sec 2\psi - 2\cot 2\psi &= 0 \\ \Rightarrow 3\sec 2\psi &= 2\cot 2\psi \\ \Rightarrow \frac{3}{\cos 2\psi} &= \frac{2\cos 2\psi}{\sin 2\psi} \\ \Rightarrow 3\sin 2\psi &= 2\cos^2 2\psi \\ \Rightarrow 3\sin 2\psi &= 2(1 - \sin^2 2\psi) \\ \Rightarrow 3\sin 2\psi &= 2 - 2\sin^2 2\psi \\ \Rightarrow 2\sin^2 2\psi + 3\sin 2\psi - 2 &= 0 \\ \Rightarrow (2\sin 2\psi - 1)(\sin 2\psi + 2) &= 0 \end{aligned}$$

$$\begin{aligned} \sin 2\psi &= \cancel{-\frac{1}{2}} \\ \arcsin(\frac{1}{2}) &= 30^\circ \\ 2\psi &= 30 + 360n \quad n=0,1,2,3,\dots \\ 2\psi &= 150 + 360n \\ \psi &= 15^\circ + 180n \\ \psi &= 75^\circ + 180n \\ \psi &= 15^\circ, 75^\circ, 195^\circ, 255^\circ \end{aligned}$$

Question 41 (***)

$$y = e^{-x} \sin(\sqrt{3}x), \quad x \in \mathbb{R}.$$

Find the exact values of the constants R and α so that

$$\frac{dy}{dx} = R e^{-x} \cos(\sqrt{3}x + \alpha),$$

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

$$R = 2, \quad \alpha = \frac{\pi}{6}$$

$$\begin{aligned}
 y &= e^{-x} \sin(\sqrt{3}x) \\
 \frac{dy}{dx} &= -e^{-x} \sin(\sqrt{3}x) + e^{-x} \times \sqrt{3} \times (\cos(\sqrt{3}x)) \\
 &= -e^{-x} \sin(\sqrt{3}x) + \sqrt{3} e^{-x} \cos(\sqrt{3}x) \\
 &= e^{-x} [-\sin(\sqrt{3}x) + \sqrt{3} \cos(\sqrt{3}x)] \\
 &\equiv \sin(\sqrt{3}x) + \sqrt{3} \cos(\sqrt{3}x) \equiv \cos(\sqrt{3}x + \alpha) \\
 &\equiv R \cos(\sqrt{3}x + \alpha) \quad R \cos(\sqrt{3}x) - R \sin(\sqrt{3}x) \\
 &\equiv R \cos(\sqrt{3}x) \cos(\alpha) - R \sin(\sqrt{3}x) \sin(\alpha) \\
 R \cos \alpha &= \sqrt{3} \\
 R \sin \alpha &= 1 \\
 \Rightarrow R &= \sqrt{(\sqrt{3})^2 + 1} = 2 \\
 \text{Q. } \tan \alpha &= \frac{1}{\sqrt{3}} \quad \therefore \alpha = \frac{\pi}{6} \\
 \therefore y &= e^{-x} \cos(\sqrt{3}x + \frac{\pi}{6}) \\
 y &= 2 e^{-x} \cos\left[\sqrt{3}x + \frac{\pi}{6}\right] \quad R = 2 \\
 &\quad \alpha = \frac{\pi}{6}
 \end{aligned}$$

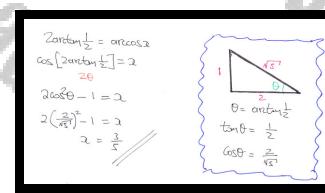
Question 42 (*)+**

Solve the following trigonometric equation

$$2 \arctan\left(\frac{1}{2}\right) = \arccos x,$$

showing clearly all the workings.

$$\boxed{x = \frac{3}{5}}$$



Question 43 (*)+**

The functions f and g are defined as

$$f(x) = 2 \cos x + \sin x, \quad x \in \mathbb{R}.$$

$$g(x) = \frac{5}{x^2 + 5}, \quad x \in \mathbb{R}.$$

- a) Express $f(x)$ in the form $R \sin(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.
- b) Determine the range of $gf(x)$, showing clearly all the relevant workings.

$$\boxed{\quad}, \quad \boxed{f(x) = \sqrt{5} \sin(x + 1.107^\circ)}, \quad \boxed{\frac{1}{2} \leq gf(x) \leq 1}$$

(a) $f(x) = 2 \cos x + \sin x \equiv R \sin(x + \alpha)$
 $\equiv R \sin x \cos \alpha + R \cos x \sin \alpha$
 $\equiv (\cos \alpha) \sin x + (\sin \alpha) \cos x$

Divide both sides by $\sqrt{\cos^2 \alpha + \sin^2 \alpha} = \sqrt{1} = 1$.
 $\therefore f(x) = \sqrt{5} \sin(x + 1.107^\circ)$

(b) $gf(x) = g(\sqrt{5} \sin(x + 1.107^\circ)) = \frac{5}{\sin^2(x + 1.107^\circ) + 5}$

Now
 $0 \leq \sin^2(x + 1.107^\circ) \leq 1$
 $0 \leq \sin^2(x + 1.107^\circ) \leq 1$
 $5 \leq \sin^2(x + 1.107^\circ) + 5 \leq 6$
 $\frac{5}{6} \leq \frac{1}{\sin^2(x + 1.107^\circ) + 5} \leq \frac{1}{5}$
 $\frac{1}{2} \leq \frac{5}{\sin^2(x + 1.107^\circ) + 5} \leq 1 \quad \therefore \frac{1}{2} \leq gf(x) \leq 1$

Question 44 (*)+**

Simplify, showing clearly all the workings, the expression

$$\tan\left[\arctan\frac{1}{3} + \arctan\frac{1}{4}\right],$$

giving the final answer as an exact fraction.

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LET $\theta = \arctan\frac{1}{3}$
 $\tan\theta = \frac{1}{3}$
 $\theta = \arctan\frac{1}{3}$
 $\tan\theta = \frac{1}{3}$

$$\begin{aligned}
 & \tan(\arctan\frac{1}{3} + \arctan\frac{1}{4}) \\
 &= \tan(\theta + \phi) \\
 &= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \\
 &= \frac{\frac{1}{3} + \frac{1}{4}}{1 - \frac{1}{3} \cdot \frac{1}{4}} \\
 &= \frac{\frac{7}{12}}{1 - \frac{1}{12}} = \frac{7}{12-1} = \frac{7}{11} //
 \end{aligned}$$

Question 45 (*)+**

Prove the validity of the following trigonometric identity

$$\frac{\sin 2\varphi}{\sin \varphi} - \frac{\cos 2\varphi}{\cos \varphi} \equiv \sec \varphi.$$

□, proof

$$\begin{aligned}
 LHS &= \frac{\sin 2\varphi}{\sin \varphi} - \frac{\cos 2\varphi}{\cos \varphi} \\
 &= \frac{2\sin\varphi\cos\varphi}{\sin\varphi} - \frac{2\cos^2\varphi - 1}{\cos\varphi} \\
 &= 2\cos\varphi - \frac{2\cos^2\varphi - 1}{\cos\varphi} \\
 &= 2\cos\varphi - 2\cos^2\varphi + \frac{1}{\cos\varphi} \\
 &= \sec\varphi - 2\cos^2\varphi + \frac{1}{\cos\varphi} \\
 &= \sec\varphi - 2\cos^2\varphi + \frac{1}{\cos\varphi} \\
 &= \sec\varphi \\
 &= RHS //
 \end{aligned}$$

Question 46 (***)

It is given that

$$u = \sin \theta + \cos \theta, \quad v = \sin \theta - \cos \theta.$$

Use trigonometric identities to show that

$$u^2 + v^2 = 2.$$

proof

$u = \sin \theta + \cos \theta$
 $v = \sin \theta - \cos \theta$

$\begin{aligned} u+v &= 2\sin \theta \\ u-v &= 2\cos \theta \end{aligned}$ } \Rightarrow
 $(u+v)^2 = 4\sin^2 \theta + 4\cos^2 \theta$ } ADD
 $(u-v)^2 = 4\cos^2 \theta$ } ADD

$(u+v)^2 + (u-v)^2 = 4\sin^2 \theta + 4\cos^2 \theta$
 $u^2 + 2uv + v^2 + u^2 - 2uv + v^2 = 4(\sin^2 \theta + \cos^2 \theta)$
 $2u^2 + 2v^2 = 4$
 $u^2 + v^2 = 2$

~~OR~~ SQUARE & ADD BOTH THE EQUATIONS
 $u^2 = (\sin \theta + \cos \theta)^2 = \sin^2 \theta + 2\sin \theta \cos \theta + \cos^2 \theta$
 $v^2 = (\sin \theta - \cos \theta)^2 = \sin^2 \theta - 2\sin \theta \cos \theta + \cos^2 \theta$
 $u^2 + v^2 = 2\sin^2 \theta + 2\cos^2 \theta$
 $u^2 + v^2 = 2(\sin^2 \theta + \cos^2 \theta)$
 $u^2 + v^2 = 2$

Question 47 (*)+**

Solve the trigonometric equation

$$(\cosec x - \sin x) \sec^2 x = 2, \quad 0 \leq x < \pi, \quad x \neq \frac{\pi}{2},$$

giving the answers in terms of π .

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\begin{aligned} & (\cosec x - \sin x) \sec^2 x = 2 \\ \Rightarrow & \left(\frac{1}{\sin x} - \sin x \right) \sec^2 x = 2 \\ \Rightarrow & \frac{1 - \sin^2 x}{\sin x} \times \sec^2 x = 2 \\ \Rightarrow & \frac{\cos^2 x}{\sin x} \times \sec^2 x = 2 \\ \Rightarrow & \frac{\cos^2 x}{\sin x} \times \frac{1}{\cos^2 x} = 2 \\ \Rightarrow & \cosec x = 2 \end{aligned}$$

$\cosec(\frac{\pi}{6}) = \frac{2}{\sqrt{3}}$
 $x = 70^\circ \approx 20\pi/9$
 $(2, 2) \approx 20\pi/9$
 $\therefore x_1 = \frac{7\pi}{6}$
 $x_2 = \frac{5\pi}{6}$

Question 48 (*)+**

The angle θ is such so that

$$\cot \theta = \frac{1}{3}.$$

Show clearly that

$$\cos \theta = \pm \frac{\sqrt{10}}{10}.$$

proof

$$\begin{aligned} \cot \theta &= \frac{1}{3} \\ \Rightarrow 1 &= 3 \tan \theta \\ \Rightarrow \tan \theta &= \frac{1}{3} \\ \Rightarrow 1 + \tan^2 \theta &= 10 \\ \Rightarrow \sec^2 \theta &= 10 \\ \Rightarrow \sec \theta &= \pm \sqrt{10} \\ \Rightarrow \cos \theta &= \pm \frac{1}{\sqrt{10}} \end{aligned}$$

Question 49 (***)

Use a detailed method to show that

$$\arccos\left(5^{-\frac{1}{2}}\right) + \arccos\left(10^{-\frac{1}{2}}\right) = \frac{3\pi}{4}.$$

Y, proof

METHOD A - USING SINES AND COSINES

$$\begin{aligned} \text{Let } \alpha &= \arccos\frac{1}{\sqrt{5}} + \arccos\frac{1}{\sqrt{10}} \\ \Rightarrow \alpha &= \theta + \phi \\ \Rightarrow \cos\alpha &= \cos(\theta + \phi) \\ \Rightarrow \cos\alpha &= \cos\theta\cos\phi - \sin\theta\sin\phi \\ \Rightarrow \cos\alpha &= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}\frac{3}{\sqrt{10}} \\ \Rightarrow \cos\alpha &= -\frac{1}{\sqrt{50}} - \frac{6}{\sqrt{50}} \\ \Rightarrow \cos\alpha &= -\frac{7}{\sqrt{50}} = -\frac{7}{5\sqrt{2}} > -\frac{1}{\sqrt{2}} \\ \Rightarrow \alpha &= \frac{3\pi}{4} \quad (\text{As } 0 < \theta + \phi < \pi) \\ \therefore \arccos 5^{-\frac{1}{2}} + \arccos 10^{-\frac{1}{2}} &= \frac{3\pi}{4} \end{aligned}$$

METHOD B - USING TRIGONICS

$$\begin{aligned} \Rightarrow \alpha &= \theta + \phi \\ \Rightarrow \tan\alpha &= \tan(\theta + \phi) \\ \Rightarrow \tan\alpha &= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \\ \Rightarrow \tan\alpha &= \frac{2 + 3}{1 - 2 \cdot 3} = \frac{5}{-5} = -1 \\ \Rightarrow \alpha &= \frac{3\pi}{4} \quad (\text{As } 0 < \theta + \phi < \pi) \\ \therefore \arccos 5^{-\frac{1}{2}} + \arccos 10^{-\frac{1}{2}} &= \frac{3\pi}{4} \end{aligned}$$

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Question 50 (***)

A relationship is defined as

$$x = \sin\theta\cos\theta, \quad 0 \leq \theta < 2\pi$$

$$y = 4\cos^2\theta, \quad 0 \leq \theta < 2\pi.$$

Use trigonometric identities to show that

$$16x^2 = y(4 - y).$$

proof

$$\begin{aligned} \begin{array}{l} x = \sin\theta\cos\theta \\ y = 4\cos^2\theta \end{array} &\Rightarrow x^2 = \sin^2\theta\cos^2\theta \\ &\Rightarrow x^2 = (1 - \cos^2\theta)\cos^2\theta \\ &\Rightarrow (1 - x^2)^2 = (1 - \cos^2\theta)(1 - \cos^2\theta) \\ &\Rightarrow 1 - 2x^2 + x^4 = 1 - 2\cos^2\theta + \cos^4\theta \\ &\Rightarrow 2x^2 - x^4 = 2\cos^2\theta - \cos^4\theta \\ &\Rightarrow 2x^2 - x^4 = 2\cos^2\theta(1 - \cos^2\theta) \\ &\Rightarrow 2x^2 - x^4 = 2\cos^2\theta\sin^2\theta \\ &\Rightarrow 2x^2 - x^4 = y(4 - y) \end{aligned}$$

Question 51 (***)

Show clearly that

$$2 \arccos\left(\frac{4}{5}\right) = \arccos\left(\frac{7}{25}\right).$$

[proof]

$2 \arccos\frac{4}{5} = \arccos\frac{7}{25}$ <p style="text-align: center;">LET $\theta = \arccos\frac{4}{5}$ $\cos\theta = \frac{4}{5}$</p> <p style="text-align: center;">Hence $2 \arccos\frac{4}{5} = \arccos\frac{20}{25}$ $\cos[2 \arccos\frac{4}{5}] = \infty$</p>	$\begin{cases} \cos^2\theta - 1 = 0 \\ 2\left(\frac{4}{5}\right)^2 - 1 = 0 \\ 2 \cdot \frac{16}{25} - 1 = 0 \\ 2 \cdot \frac{7}{25} = 0 \end{cases}$ <p style="text-align: center;">\therefore PROVED</p>
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Question 52 (***)

It is given that

$$\sin 3x = 3 \sin x - 4 \sin^3 x.$$

- Prove the validity of the above trigonometric identity, by writing $\sin 3x$ as $\sin(2x + x)$.
- Given that $\sin x = \frac{\sqrt{6}}{6}$, find the exact value of $\sin 3x$.

$$\sin 3x = \frac{7\sqrt{6}}{18}$$

<p>(a) $\sin 3x = \sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$</p> $\begin{aligned} &= (2 \sin x \cos x) \cos x + (1 - 2 \sin^2 x) \sin x \\ &= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x \\ &= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x \\ &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x \\ &= 3 \sin x - 4 \sin^3 x \\ &= \text{RHS} \end{aligned}$	<p>(b) $\sin 3x = 3 \sin x - 4 \sin^3 x$</p> $\begin{aligned} \sin 3x &= 3 \cdot \frac{\sqrt{6}}{6} - 4 \left(\frac{\sqrt{6}}{6}\right)^3 \\ &= \frac{3\sqrt{6}}{6} - 4 \cdot \frac{(\sqrt{6})^3}{6^3} \\ &= \frac{3\sqrt{6}}{6} - \frac{4\sqrt{6}}{6^3} \\ &= \frac{7\sqrt{6}}{18} \end{aligned}$
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Question 53 (*)+**

It is given that

$$4\operatorname{cosec}^2 2\theta - \sec^2 \theta \equiv \operatorname{cosec}^2 \theta, \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence show that if

$$4(\operatorname{cosec}^2 2\theta - 2) = \sec^2 \theta - 2\operatorname{cosec} \theta,$$

then either $\sin \theta = \frac{1}{2}$ or $\sin \theta = -\frac{1}{4}$.

□, proof

(a)

$$\begin{aligned} LHS &= 4\operatorname{cosec}^2 2\theta - \sec^2 \theta = \frac{4}{\sin^2 2\theta} - \frac{1}{\cos^2 \theta} = \frac{4}{(2\sin \theta \cos \theta)^2} - \frac{1}{\cos^2 \theta} \\ &= \frac{4}{(2\sin \theta \cos \theta)^2} - \frac{1}{\cos^2 \theta} = \frac{4}{4\sin^2 \theta \cos^2 \theta} - \frac{1}{\cos^2 \theta} = \frac{1}{\sin^2 \theta \cos^2 \theta} - \frac{1}{\cos^2 \theta} \\ &= \frac{1 - \sin^2 \theta}{\sin^2 \theta \cos^2 \theta} = \frac{\cos^2 \theta}{\sin^2 \theta \cos^2 \theta} = \frac{1}{\sin^2 \theta} = \operatorname{cosec}^2 \theta = RHS \end{aligned}$$

(b)

$$\begin{aligned} &\Rightarrow 4(\operatorname{cosec}^2 2\theta - 2) = \sec^2 \theta - 2\operatorname{cosec} \theta \\ &\Rightarrow 4\operatorname{cosec}^2 2\theta - 8 = \sec^2 \theta - 2\operatorname{cosec} \theta \\ &\Rightarrow (4\operatorname{cosec}^2 2\theta - \sec^2 \theta) + 2\operatorname{cosec} \theta - 8 \\ &\Rightarrow \operatorname{cosec}^2 2\theta + 2\operatorname{cosec} \theta - 8 \\ &\Rightarrow (\operatorname{cosec}^2 \theta - 2)(\operatorname{cosec}^2 \theta + 4) = 0 \end{aligned}$$

↑ $\operatorname{cosec} \theta < 0$
 $\sin \theta = \frac{1}{2}$ $\frac{1}{2}$
 $\sin \theta = -\frac{1}{4}$ $\frac{1}{4}$

Question 54 (*)+**

Show clearly that

$$\arctan \frac{2}{3} + \arctan \frac{5}{12} = \arctan \frac{3}{2}.$$

proof

$$\begin{aligned} &\rightarrow \arctan \frac{2}{3} + \arctan \frac{5}{12} = \alpha \\ &\rightarrow \theta + \phi = \alpha \\ &\rightarrow \tan(\theta + \phi) = \tan \alpha \\ &\rightarrow \tan \alpha = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} \\ &\rightarrow \tan \alpha = \frac{\frac{2}{3} + \frac{5}{12}}{1 - \frac{2}{3} \times \frac{5}{12}} \\ &\rightarrow \tan \alpha = \frac{24 + 15}{36 - 10} \\ &\rightarrow \tan \alpha = \frac{39}{26} \\ &\rightarrow \tan \alpha = \frac{3}{2} \\ &\rightarrow \alpha = \arctan \frac{3}{2}. \quad \text{As required} \end{aligned}$$

ALTERNATIVE

$$\begin{aligned} (3+2i)(2+5i) &= 36 + 15i + 24i - 10 = 26 + 39i \\ \arg(3+2i)(2+5i) &= \arg(26+39i) \\ \arg(3+2i) + \arg(2+5i) &= \arg(26+39i) \\ \arctan \frac{2}{3} + \arctan \frac{5}{12} &= \arctan \frac{3}{2} \\ \therefore \arctan \frac{2}{3} + \arctan \frac{5}{12} &= \arctan \frac{3}{2} \quad \text{As required} \end{aligned}$$

Question 55 (***)

Show clearly that

$$\sin(2 \arctan x) = \frac{2x}{x^2 + 1}.$$

[proof]

$$\begin{aligned} \sin(2\arctan x) &= 2\sin(\arctan x)\cos(\arctan x) \\ &= 2 \left(\frac{x}{\sqrt{1+x^2}} \right) \left(\frac{1}{\sqrt{1+x^2}} \right) \\ &= \frac{2x}{x^2+1} // \end{aligned}$$

Diagram: A right-angled triangle with hypotenuse $\sqrt{1+x^2}$, angle $\theta = \arctan x$, adjacent side x , and opposite side 1 . Labels: $\sin \theta = \frac{1}{\sqrt{1+x^2}}$, $\cos \theta = \frac{x}{\sqrt{1+x^2}}$.

Question 56 (***)

Prove the validity of the following trigonometric identity

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

[proof]

Let $y = \arcsin x$
 $\sin y = x$

Hence
 $\arcsin x + \arccos x$
 $= y + \arccos(\sin y)$
 $= y + \arccos(\cos(\frac{\pi}{2}-y))$
 $= y + (\frac{\pi}{2}-y)$
 $= \frac{\pi}{2} //$

OR Let $f(x) = \arcsin x + \arccos x$

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \\ &\Rightarrow f(x) = 0 \\ &\text{Hence } f(x) = \text{constant} \\ &f(x) = \text{constant} = C \\ &\arcsin x + \arccos x = C \\ &0 + \frac{\pi}{2} = C \\ &C = \frac{\pi}{2} \\ &\therefore \arcsin x + \arccos x = \frac{\pi}{2} \end{aligned}$$

OR

$\theta = \arcsin x$, $\phi = \arccos x$
 $\sin \theta = x$, $\cos \phi = x$

$\rightarrow \arcsin x + \arccos x = \psi$
 $\Rightarrow \theta + \phi = \psi$
 $\Rightarrow \sin(\theta + \phi) = \sin \psi$
 $\Rightarrow \sin \theta \cos \phi + \cos \theta \sin \phi = \sin \psi$
 $\Rightarrow \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \sqrt{1-x^2} = \sin \psi$
 $\Rightarrow x^2 + (1-x^2) = \sin \psi$
 $\Rightarrow \sin \psi = 1$
 $\psi = \frac{\pi}{2} \pm 2k\pi$
 $\therefore \psi = \frac{\pi}{2}$
 $\therefore \arcsin x + \arccos x = \frac{\pi}{2}$

Diagram: Two right-angled triangles. The first has hypotenuse $\sqrt{1-x^2}$ and angle θ at the bottom-left. The second has hypotenuse $\sqrt{1-x^2}$ and angle ϕ at the top-right. Both have adjacent side x and opposite side 1 . Labels: $\sin \theta = x$, $\cos \phi = x$.

Question 57 (***)

Show clearly that

$$\arctan \frac{1}{3} + \arctan \frac{4}{3} = \arctan 3.$$

proof

$$\begin{aligned}
 \alpha' &= \theta + \phi \\
 \Rightarrow \alpha' &= \arctan \frac{1}{3} + \arctan \frac{4}{3} \\
 \Rightarrow \tan \alpha' &= \frac{1}{3} \left[\arctan \frac{1}{3} + \arctan \frac{4}{3} \right] \\
 \Rightarrow \tan \alpha' &= \frac{\tan(\arctan \frac{1}{3}) + \tan(\arctan \frac{4}{3})}{1 - \tan(\arctan \frac{1}{3}) \cdot \tan(\arctan \frac{4}{3})} \\
 \Rightarrow \tan \alpha' &= \frac{\frac{1}{3} + \frac{4}{3}}{1 - \frac{1}{3} \cdot \frac{4}{3}} \\
 \Rightarrow \tan \alpha' &= \frac{5}{3} \\
 \Rightarrow \tan \alpha &= \frac{5}{3} \\
 \Rightarrow \tan \alpha &= \frac{15}{9 - 4} \\
 \Rightarrow \tan \alpha &= 3 \\
 \Rightarrow \alpha &= \arctan 3
 \end{aligned}$$

Question 58 (***)

Solve the trigonometric equation

$$\arcsin x = \arccos 2x.$$

$$x = \frac{1}{\sqrt{5}}$$

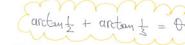
$$\begin{aligned}
 \arcsin x &= \arccos 2x \\
 \Rightarrow \cos(\arcsin x) &= \cos(\arccos 2x) \\
 \Rightarrow \cos(\arcsin x) &= 2x \\
 \text{Now let } S &= \arcsin x \Rightarrow (S = \sin^{-1} x) \\
 \sin \theta &= \sqrt{1-x^2} \\
 \cos(\arcsin x) &= \sqrt{1-x^2} \\
 \Rightarrow \sqrt{1-x^2} &= 2x \\
 \Rightarrow 1-x^2 &= 4x^2 \\
 \Rightarrow 5x^2 &= 1 \\
 \Rightarrow x^2 &= \frac{1}{5} \\
 \Rightarrow x &= \pm \frac{1}{\sqrt{5}}
 \end{aligned}$$

Question 59 (***)+

Using a detailed method, show that

$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{1}{4}\pi.$$

proof



$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \theta$

- Let $A = \arctan \frac{1}{2} \Rightarrow \tan A = \frac{1}{2}$
- $B = \arctan \frac{1}{3} \Rightarrow \tan B = \frac{1}{3}$
- So these equations can be written as $A + B = \theta$

$$\begin{aligned} \Rightarrow \tan(A+B) &= \tan \theta \\ \Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} &= \tan \theta \\ \Rightarrow \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} &= \tan \theta \\ \Rightarrow \tan \theta &= \frac{5}{4} \\ \Rightarrow \tan \theta &= 1 \\ \Rightarrow \theta &= \frac{\pi}{4} \quad (0 < \arctan \frac{1}{2} + \arctan \frac{1}{3} < \pi) \\ \therefore \arctan \frac{1}{2} + \arctan \frac{1}{3} &= \frac{\pi}{4} \quad \text{As required} \end{aligned}$$
Question 60 (***)+

Given that x is measured in radians, use small angle approximations to simplify the following expression.

$$\frac{\cos 7x - 1}{x \sin x}$$

 , $-\frac{49}{2}$

USING THE APPROXIMATIONS FOR SMALL θ IN RADIANS

- $\sin \theta \approx \theta$
- $\cos \theta \approx 1 - \frac{1}{2}\theta^2$

$$\Rightarrow \frac{\cos 7x - 1}{x \sin x} \approx \frac{(1 - \frac{1}{2}(7x)^2) - 1}{x(7x)}$$

$$\approx \frac{-\frac{49}{2}x^2}{7x}$$

$$\approx -\frac{7x}{2}$$

Question 61 (***)

Given that x is measured in radians, use small angle approximations to simplify the following expression.

$$\frac{\cos^2(3x) - 1}{2x \sin\left(\frac{3}{4}x\right)}$$

, -6

SIMPLIFY THE APPROXIMATIONS FOR SMALL θ IN EACH LINE

- $\sin\theta \approx \theta$
- $\cos\theta \approx 1 - \frac{1}{2}\theta^2$

$$\begin{aligned} \frac{\cos^2(3x) - 1}{2x \sin\left(\frac{3}{4}x\right)} &\approx \frac{\left[1 - \frac{1}{2}(3x)^2\right]^2 - 1}{2x \left(\frac{3}{4}x\right)} \\ &\approx \frac{(1 - \frac{9x^2}{2})^2 - 1}{\frac{3}{2}x^2} \\ &\approx \frac{(1 - 9x^2 + \frac{81x^4}{4}) - 1}{\frac{3}{2}x^2} \\ &\approx \frac{-\frac{9x^2}{2}}{\frac{3}{2}x^2} \\ &\approx -6 \end{aligned}$$

ALTERNATIVE

$$\begin{aligned} \frac{\cos^2(3x) - 1}{2x \sin\left(\frac{3}{4}x\right)} &= -\frac{1 - \cos^2(3x)}{2x \sin\left(\frac{3}{4}x\right)} = -\frac{\sin^2(3x)}{2x \sin\left(\frac{3}{4}x\right)} \\ &\approx -\frac{\left(\frac{3}{4}x\right)^2}{2x \left(\frac{3}{4}x\right)} \\ &\approx -\frac{\frac{9x^2}{16}}{\frac{3}{2}x^2} \\ &\approx -6 \end{aligned}$$

Question 62 (*)+**

Prove that

$$2 \arcsin\left(\frac{2}{3}\right) = \arccos\left(\frac{1}{9}\right).$$

V, , proof

METHOD A — 2arcsin $\frac{2}{3}$ = arccos $\frac{1}{9}$

LET $\theta = \arcsin\frac{2}{3}$, SO WE CAN GET RATIOS OFF A TRIANGLE

$\sin\theta = \frac{2}{3}$

THIS $2\theta = 19^\circ$; FOR SIN 4° TO BE FOUND

$\therefore \cos 2\theta = \cos 19^\circ$
 $\Rightarrow 1 - 2\sin^2\theta = \cos 19^\circ$
 $\Rightarrow 1 - 2\left(\frac{2}{3}\right)^2 = \cos 19^\circ$
 $\Rightarrow 1 - \frac{8}{9} = \cos 19^\circ$
 $\Rightarrow \cos 19^\circ = \frac{1}{9}$
 $\Rightarrow \theta = \arccos\frac{1}{9}$

$2\theta = \psi$
 $2\arcsin\frac{2}{3} = \arccos\frac{1}{9}$

METHOD B — $2 \arcsin\frac{2}{3} = \arccos\frac{1}{9}$ (using 1)

$\sin\theta = \frac{2}{3}$ ($\theta = \arcsin\frac{2}{3}$)

$\sin^2\theta = \frac{4}{9}$

$-\sin^2\theta = -\frac{4}{9}$

$-2\sin^2\theta = -\frac{8}{9}$

$1 - 2\sin^2\theta = 1 - \frac{8}{9}$

$\cos^2\theta = \frac{1}{9}$

$2\theta = \arccos\frac{1}{9}$

$2 \arcsin\frac{2}{3} = \arccos\frac{1}{9}$

METHOD C — GEOMETRICAL

$\arcsin\frac{2}{3} = \theta$

$\sin\theta = \frac{2}{3}$

Now By THE PYTHAGOREAN THEOREM

$13 = 3^2 + 2^2 - 2 \times 3 \times 2 \cos 2\theta$
 $13 = 9 + 4 - 12 \cos 2\theta$
 $12 \cos 2\theta = 2$
 $\cos 2\theta = \frac{1}{6}$
 $2\theta = \arccos\frac{1}{6}$

$2 \arcsin\frac{2}{3} = \arccos\frac{1}{9}$

Question 63 (*)**

Solve the following trigonometric equation

$$2 \arctan\left(\frac{3}{x}\right) = \arctan\left(\frac{6x}{25}\right).$$

$$\boxed{x}, \quad x = \pm 6$$

Let $\theta = \arctan\left(\frac{3}{x}\right)$ & $\phi = \arctan\left(\frac{6x}{25}\right)$

$$\begin{aligned} &\Rightarrow 2\arctan\left(\frac{3}{x}\right) = \arctan\left(\frac{6x}{25}\right) \\ &\Rightarrow 2\theta = \phi \\ &\Rightarrow \tan 2\theta = \tan \phi \\ &\Rightarrow \frac{2\tan \theta}{1 - \tan^2 \theta} = \tan \phi \end{aligned}$$

But if $\theta = \arctan\left(\frac{3}{x}\right) \Rightarrow \tan \theta = \frac{3}{x}$

$$\begin{aligned} &\theta = \arctan\left(\frac{3}{x}\right) \Rightarrow \tan \theta = \frac{3}{x} \\ &\phi = \arctan\left(\frac{6x}{25}\right) \Rightarrow \tan \phi = \frac{6x}{25} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \frac{2\left(\frac{3}{x}\right)}{1 - \left(\frac{3}{x}\right)^2} = \frac{6x}{25} \\ &\Rightarrow \frac{\frac{6}{x}}{1 - \frac{9}{x^2}} = \frac{6x}{25} \quad \left. \begin{array}{l} \text{Multiplying top & bottom by } x^2 \\ \text{Double fraction by } x^2 \end{array} \right. \\ &\Rightarrow \frac{6x}{x^2 - 9} = \frac{6x}{25} \quad \left. \begin{array}{l} \text{As } x \neq 0, \text{ we may divide} \\ \text{both sides by } 6x \end{array} \right. \\ &\Rightarrow \frac{6}{x^2 - 9} = \frac{6}{25} \\ &\Rightarrow 2(x^2 - 9) = 25 \\ &\Rightarrow x^2 - 9 = 25 \\ &\Rightarrow x^2 = 34 \\ &\Rightarrow x = \pm \sqrt{34} \end{aligned}$$

Question 64 (*)+**

Show, by detailed workings, that

$$\arctan 2 + \arctan 3 = \frac{3\pi}{4}.$$

proof

Given $\arctan 2 + \arctan 3 = \varphi$

$$\Rightarrow \tan[\arctan 2 + \arctan 3] = \tan \varphi$$

$$\Rightarrow \frac{\tan(\arctan 2) + \tan(\arctan 3)}{1 - \tan(\arctan 2)\tan(\arctan 3)} = \tan \varphi$$

$$\Rightarrow \frac{2+3}{1-2 \cdot 3} = \tan \varphi$$

$$\Rightarrow \tan \varphi = -1$$

$$\Rightarrow \varphi = \arctan(-1) \pm \pi$$

$$\Rightarrow \varphi = \arctan(-1) + \pi \quad \left(\begin{array}{l} 0 < \arctan 2 < \frac{\pi}{2} \\ 0 < \arctan 3 < \frac{\pi}{2} \\ \text{and } 0 < \varphi < \pi \end{array} \right)$$

$$\Rightarrow \varphi = -\frac{\pi}{4} + \pi$$

$$\Rightarrow \arctan 2 + \arctan 3 = \frac{3\pi}{4}$$

ALTERNATIVE BY COMPLEX NUMBERS

LET $z = 1+2i \Rightarrow \arg z = \arctan 2$
 $w = 1+3i \Rightarrow \arg w = \arctan 3$

$$\Rightarrow \arg z + \arg w = \arg((z)(w))$$

$$\Rightarrow \arctan 2 + \arctan 3 = \arg((1+2i)(1+3i))$$

$$\Rightarrow \arctan 2 + \arctan 3 = \arg(1+5i+6)$$

$$\Rightarrow \arctan 2 + \arctan 3 = \arg(-5+5i)$$

$$\Rightarrow \arctan 2 + \arctan 3 = \arctan\left(\frac{5}{-5}\right) + \pi \quad \left(\begin{array}{l} \text{AS THE NUMBER IS IN THE} \\ \text{2ND QUADRANT} \end{array} \right)$$

$$\Rightarrow \arctan 2 + \arctan 3 = \arctan(-1) + \pi$$

$$\Rightarrow \arctan 2 + \arctan 3 = -\frac{\pi}{4} + \pi$$

$$\Rightarrow \arctan 2 + \arctan 3 = \frac{3\pi}{4}$$

Question 65 (*)+**

Find the general solution of the following trigonometric equation

$$2 \arctan(\sin x) = \arctan(\sec x).$$

$$x = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

$2 \arctan(\sin x) = \arctan(\sec x)$
 taking tangents on both sides
 i.e. $\tan 2\theta = \tan \alpha$

$$\Rightarrow \frac{2 \sin x}{1 - \sin^2 x} = \frac{1}{\cos x}$$

$$\Rightarrow \frac{2 \sin x}{\cos^2 x} = \frac{1}{\cos x}$$

$$\Rightarrow \sin x = \cos x$$

$$\Rightarrow \tan x = 1$$

$$\therefore x = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}, 1, 3, 5, \dots$$

Question 66 (****)

$$f(x) = (x^2 + 1)(x - 1), \quad x \in \mathbb{R}.$$

- a) Simplify $f(x)$.
- b) Prove the validity of the trigonometric identity

$$\frac{\cosec \theta - \sin \theta}{\sec \theta - \cos \theta} \equiv \cot^3 \theta.$$

- c) Hence, or otherwise, solve the equation

$$\frac{\cosec \theta - \sin \theta}{\sec \theta - \cos \theta} + \cot \theta = \cosec^2 \theta, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

$$x^3 - x^2 + x - 1, \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\begin{aligned}
 \text{(a)} \quad & (x^2+1)(x-1) = x^3 - x^2 + x - 1 \\
 \text{(b)} \quad & LHS = \frac{\cosec \theta - \sin \theta}{\sec \theta - \cos \theta} = \frac{\frac{1}{\sin \theta} - \sin \theta}{\frac{1}{\cos \theta} - \cos \theta} = \frac{\frac{1 - \sin^2 \theta}{\sin \theta}}{\frac{1 - \cos^2 \theta}{\cos \theta}} = \frac{\frac{\cos^2 \theta}{\sin \theta}}{\frac{\sin^2 \theta}{\cos \theta}} = \frac{\cos^2 \theta}{\sin^2 \theta} = \cot^2 \theta \\
 & \cot^2 \theta = \cot^3 \theta = \cot \theta \\
 \text{(c)} \quad & \frac{\cosec \theta - \sin \theta}{\sec \theta - \cos \theta} + \cot \theta = \cosec^2 \theta \\
 & \cot^2 \theta + \cot \theta = \cosec^2 \theta \\
 & \cot^2 \theta + \cot \theta = 1 + \cot^2 \theta \\
 & \cot^2 \theta - \cot^2 \theta + \cot \theta - 1 = 0 \\
 & (\cot^2 \theta + 1)(\cot \theta - 1) = 0 \\
 & \cot \theta = -1 \quad \text{from part (a)} \\
 & \cot \theta = 1
 \end{aligned}$$

RHS = 1
 tanθ = 1
 arctan 1 = π/4
 θ = π/4 ± nπ, n = 0, 1, 2, 3, ...
 θ₁ = π/4
 θ₂ = 5π/4

Question 67 (**)**

Solve the following trigonometric equation

$$2 \cot \theta - 3 \operatorname{cosec} \theta = 2 \sec \theta \operatorname{cosec} \theta, \quad 0 < \theta < 2\pi, \quad \theta \neq \frac{k\pi}{2}, k \in \mathbb{Z},$$

giving the answers in terms of π .

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\begin{aligned} \Rightarrow 2 \cot \theta - 3 \operatorname{cosec} \theta &= 2 \sec \theta \operatorname{cosec} \theta \\ \Rightarrow 2 \cot \theta - \frac{3}{\sin \theta} &= \frac{2}{\sin \theta \cos \theta} \\ \Rightarrow 2 \cot^2 \theta - 3 &= \frac{2}{\cos^2 \theta} \\ \Rightarrow 2 \cot^2 \theta - 3 \operatorname{cosec}^2 \theta &= 2 \\ \Rightarrow 2 \cot^2 \theta - 3 \operatorname{cosec}^2 \theta - 2 &= 0 \\ \Rightarrow (\cot \theta + 1)(\cot \theta - 2) &= 0 \end{aligned} \quad \left. \begin{array}{l} \cot \theta = -1 \\ \cot \theta = 2 \end{array} \right\} \begin{array}{l} \theta = \frac{\pi}{4} \text{ or } \theta = \frac{5\pi}{4} \\ \theta = \frac{3\pi}{2} \text{ or } \theta = \frac{7\pi}{4} \end{array} \quad \begin{array}{l} \theta = \frac{\pi}{3} \text{ or } \theta = \frac{4\pi}{3} \\ \theta = \frac{2\pi}{3} \text{ or } \theta = \frac{5\pi}{3} \end{array} \quad \begin{array}{l} \theta = 0 \\ \theta = \pi \end{array} \quad \begin{array}{l} \theta = \frac{\pi}{2} \\ \theta = \frac{3\pi}{2} \end{array}$$

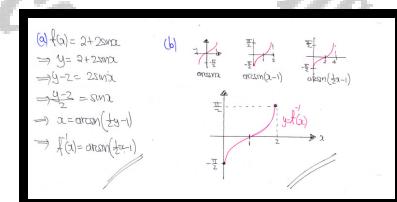
Question 68 (**)**

$$f(x) = 2 + 2 \sin x, \quad -\pi \leq x \leq \pi.$$

- a) Find an expression for $f^{-1}(x)$.
- b) Sketch the graph of $f^{-1}(x)$.

The sketch must include the coordinates of any points where the graph of $f^{-1}(x)$ meet the coordinate axes as well as the coordinates of its endpoints.

$$f^{-1}(x) = \arcsin\left(\frac{1}{2}x - 1\right)$$



Question 69 (**)**

Prove the validity of each of the following trigonometric identities.

a) $2 \cot 2\theta + \tan \theta \equiv \cot \theta$.

b) $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} \equiv 2$.

proof

$\begin{aligned} LHS &= 2\cot 2\theta + \tan \theta \\ &= \frac{2}{\tan 2\theta} + \tan \theta \\ &= \frac{2}{\frac{2\tan \theta}{1-\tan^2 \theta}} + \tan \theta \\ &= \frac{2(1-\tan^2 \theta)}{2\tan \theta} + \tan \theta \\ &= \frac{1-\tan^2 \theta}{\tan \theta} + \tan \theta \\ &= \frac{1}{\tan \theta} - \frac{\tan^2 \theta}{\tan \theta} + \tan \theta \\ &= \cot \theta - \tan \theta + \tan \theta \\ &= \cot \theta \end{aligned}$	$\begin{aligned} RHS &= 2\cot 2\theta + \tan \theta \\ &= \frac{2\cot 2\theta}{\sin 2\theta} + \frac{\tan \theta}{\cos \theta} \\ &= \frac{2\cot 2\theta}{2\sin \theta \cos \theta} + \frac{\tan \theta}{\cos \theta} \\ &= \frac{\cot 2\theta - \sin^2 \theta}{\sin \theta \cos \theta} + \frac{\tan \theta}{\cos \theta} \\ &= \frac{\cot^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta} + \frac{\tan \theta}{\cos \theta} \\ &= \frac{\cot^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta} + \frac{\sin \theta}{\cos^2 \theta} \\ &= \frac{\cot^2 \theta - \sin^2 \theta + \sin \theta}{\sin \theta \cos \theta} \\ &= \frac{\cot^2 \theta + \sin \theta}{\sin \theta \cos \theta} \\ &= \frac{\cot^2 \theta + \sin \theta}{\sin \theta \cos \theta} \\ &= \frac{\cot^2 \theta + \sin \theta}{\sin \theta \cos \theta} \\ &= \cot \theta = RHS \end{aligned}$
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Question 70 (**)**

Show that the following trigonometric equation

$$\tan 2\theta - 3\cot \theta = 0, \quad 0 < \theta < 2\pi,,$$

has six solutions in the interval $0 < \theta < 2\pi$, giving the answers in terms of π .

Answer, $\theta = \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi, \frac{4}{3}\pi, \frac{3}{2}\pi, \frac{5}{3}\pi$

<p>MANIPULATE AS FOLLOW:</p> $\begin{aligned} \tan 2\theta - 3\cot \theta &= 0 \\ \frac{\tan 2\theta}{\cot \theta} - 3 &= \frac{3}{\cot \theta} \end{aligned}$ <p>NOTE THAT THE DENOMINATOR WILL DIVIDE THE DENOMINATORS DO NOT DIVIDE IT, IT IS ONLY A DENOMINATOR EXCLUDING THIS SECTION</p> $\begin{aligned} \rightarrow 2\cot^2 \theta &= 3 + 3\cot^2 \theta \\ \rightarrow 3 &= 4\cot^2 \theta \\ \rightarrow \cot^2 \theta &= \frac{3}{4} \end{aligned}$ <p>COLLECTING ALL THE POSSIBILITIES $\tan^2 \theta = \cot^2 \theta + 1 \Rightarrow \tan^2 \theta = \frac{7}{4}$</p> $\left\{ \begin{array}{l} \theta = \frac{\pi}{3} \pm \frac{\pi}{4} \\ \theta = \frac{\pi}{2} \pm \frac{\pi}{4} \\ \theta = \frac{\pi}{6} \pm \frac{\pi}{4} \\ \theta = \frac{5\pi}{6} \pm \frac{\pi}{4} \end{array} \right. \quad n = 0, 1, 2, 3, 4$ $\therefore \theta = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}$
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Question 71 (****)

$$f(\theta^\circ) \equiv (\sqrt{3} + 1)\cos 2\theta^\circ + (\sqrt{3} - 1)\sin 2\theta^\circ.$$

- a) Express $f(\theta)$ in the form $R \sin(2\theta + \alpha)$, $R > 0$, $0 \leq \theta^\circ < 90^\circ$.
- b) Solve the equation

$$f(\theta^\circ) = 2, \quad 0 \leq \theta^\circ < 360^\circ.$$

, $f(\theta^\circ) \equiv \sqrt{8} \sin(2\theta + 75^\circ)$, $\theta = 30^\circ, 165^\circ, 210^\circ, 345^\circ$

(a) $(\sqrt{3}+1)\cos 2\theta + (\sqrt{3}-1)\sin 2\theta \equiv R \sin(2\theta + \alpha)$
 $\equiv 2\sqrt{2}\cos(\alpha - 2\theta) \sin \alpha$
 $\equiv (2\cos \alpha) \sin 2\theta + (\sin \alpha) \cos 2\theta$

$\therefore R \cos \alpha = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ } $\Rightarrow R^2 = (\sqrt{3}+1)^2 + (\sqrt{3}-1)^2 = 3+2\sqrt{3}+1 = 8+2\sqrt{3}+1 = 8$
 $R = \sqrt{8}$
 $\Rightarrow \tan \alpha = \frac{\sqrt{3}-1}{\sqrt{3}+1} \quad \alpha = 75^\circ$

$\therefore f(\theta) = \sqrt{8} \sin(2\theta + 75^\circ)$

(b) $\begin{cases} f(\theta) = 2 \\ \sqrt{8} \sin(2\theta + 75^\circ) = 2 \\ \sin(2\theta + 75^\circ) = \frac{2}{\sqrt{8}} \\ \sin(2\theta + 75^\circ) = \frac{1}{4}\sqrt{2} \end{cases}$

$2\theta + 75^\circ = 30^\circ \pm 360^\circ$
 $2\theta + 75^\circ = 15^\circ \pm 180^\circ$
 $2\theta = -15^\circ \pm 105^\circ$
 $2\theta = 30^\circ$
 $\theta = 15^\circ$

$2\theta + 75^\circ = 135^\circ \pm 360^\circ$
 $2\theta = 30^\circ$
 $\theta = 15^\circ$

$k=0,1,2,\dots$

Question 72 (**)**

It is given that

$$\frac{\sec x}{1+\sec x} - \frac{\sec x}{1-\sec x} \equiv 2\operatorname{cosec}^2 x.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence, or otherwise, solve the equation

$$\frac{\sec x}{1+\sec x} - \frac{\sec x}{1-\sec x} = 16\sin x, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\begin{aligned}
 \text{(a)} \quad & \frac{\sec x}{1+\sec x} - \frac{\sec x}{1-\sec x} = \frac{\sec(1-\sec x) - \sec(1+\sec x)}{(1+\sec x)(1-\sec x)} \\
 &= \frac{\sec x - \sec^2 x - \sec x - \sec^2 x}{1-\sec^2 x} = \frac{-2\sec^2 x}{1-\sec^2 x} = \frac{-2\sec^2 x}{1-(1+\tan^2 x)} \\
 &= \frac{-2\sec^2 x}{-\tan^2 x} = \frac{2\sec^2 x}{\tan^2 x} = \frac{2}{\operatorname{sin}^2 x} = 2\operatorname{cosec}^2 x = \text{R.H.S.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \frac{\sec x}{1+\sec x} - \frac{\sec x}{1-\sec x} = 16\sin x \quad \Rightarrow \quad \operatorname{sin} x = \frac{1}{2} \\
 & \Rightarrow 2\operatorname{cosec}^2 x = 16\sin x \quad \operatorname{arc sin}(\frac{1}{2}) = \frac{\pi}{6} \\
 & \Rightarrow \operatorname{cosec}^2 x = 8\sin x \\
 & \Rightarrow \frac{1}{\operatorname{sin}^2 x} = 8\sin x \\
 & \Rightarrow 8 = \frac{1}{\operatorname{sin}^2 x} \\
 & \Rightarrow \operatorname{sin} x = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}
 \end{aligned}$$

$x_1 = \frac{\pi}{6} + 2n\pi, n \in \mathbb{Z}, \dots$
 $x_2 = \frac{5\pi}{6} + 2n\pi, n \in \mathbb{Z}, \dots$
 $x_1 = \frac{\pi}{6}$
 $x_2 = \frac{5\pi}{6}$

Question 73 (**)**

Solve the trigonometric equation

$$\operatorname{cosec} \theta - \sin \theta + 2 \cos^2 \theta \cot \theta = 0, \quad 0 < \theta < 2\pi, \quad \theta \neq \pi,$$

giving the answers in terms of π .

$$\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$$

$$\begin{aligned} \operatorname{cosec} \theta - \sin \theta + 2 \cos^2 \theta \cot \theta &= 0 \\ \Rightarrow \frac{1}{\sin \theta} - \sin \theta + 2 \cos^2 \theta \cot \theta &= 0 \\ \Rightarrow \frac{1 - \sin^2 \theta}{\sin \theta} + 2 \cos^2 \theta \cot \theta &= 0 \\ \Rightarrow \frac{\cos^2 \theta}{\sin \theta} + 2 \cos^2 \theta \cot \theta &= 0 \\ \Rightarrow \cos^2 \theta + 2 \cos^2 \theta \cot \theta \sin \theta &= 0 \\ \Rightarrow \cos^2 \theta + 2 \cos^2 \theta \cdot \frac{\cos \theta}{\sin \theta} \sin \theta &= 0 \\ \Rightarrow \cos^2 \theta + 2 \cos^2 \theta \cos \theta &= 0 \\ \Rightarrow \cos^2 \theta (1 + 2 \cos \theta) &= 0 \end{aligned}$$

$\cos \theta < -\frac{1}{2}$

- $\theta \text{ in } [0^\circ, 180^\circ] \Rightarrow \theta = 120^\circ$
- $\theta \in [360^\circ, 540^\circ] \Rightarrow \theta = 300^\circ, 420^\circ$
- $\theta \text{ in } [180^\circ, 360^\circ] \Rightarrow \theta = 240^\circ$
- $\theta = \frac{2\pi}{3} + 2m\pi \Rightarrow \theta = \frac{4\pi}{3}, \frac{8\pi}{3}, \dots$
- $\theta = \frac{4\pi}{3} + 2m\pi \Rightarrow \theta = \frac{10\pi}{3}, \dots$

Question 74 (**)**

$$\sin \theta = \frac{5}{13} \quad \text{and} \quad \sin \varphi = -\frac{7}{25}.$$

If θ is obtuse and φ is such so that $180^\circ < \varphi < 270^\circ$, show that

$$\sin(\theta + \varphi) = -\frac{36}{325}.$$

proof

$\cos \theta = -\frac{12}{13}$

$\cos \varphi = -\frac{24}{25}$

$$\begin{aligned} \sin(\theta + \varphi) &= \sin \theta \cos \varphi + \cos \theta \sin \varphi \\ \sin(\theta + \varphi) &= \frac{5}{13} \left(-\frac{24}{25}\right) + \left(-\frac{12}{13}\right) \left(-\frac{7}{25}\right) \\ \sin(\theta + \varphi) &= -\frac{120}{325} + \frac{84}{325} \\ \sin(\theta + \varphi) &= -\frac{36}{325} \end{aligned}$$

$\Rightarrow \text{DQWOCG}$

Question 75 (**)**

It is given that

$$\sin 2\theta \equiv \frac{2 \tan \theta}{1 + \tan^2 \theta}.$$

- a) Prove the validity of the above trigonometric identity.
 b) By using $\theta = 15^\circ$ in the above identity, show that

$$\tan 15^\circ = 2 - \sqrt{3}.$$

proof

$ \begin{aligned} \text{(a)} \quad \text{RHS} &= \frac{2 \tan \theta}{1 + \tan^2 \theta} \\ &= \frac{2 \tan \theta}{\sec^2 \theta} \\ &= 2 \sin \theta \cos \theta \\ &= 2 \sin 2\theta \\ &= \sin 2\theta \\ &= 2\sqrt{3} \end{aligned} $	$ \begin{aligned} \text{(b)} \quad \text{Let } \theta = 15^\circ \\ \therefore \sin 30^\circ &= \frac{2\sqrt{3}}{1 + \tan 15^\circ} \\ \Rightarrow \frac{1}{2} &= \frac{2\sqrt{3}}{1 + \tan^2 15^\circ} \\ \Rightarrow 1 + \tan^2 15^\circ &= 4\sqrt{3} \\ \Rightarrow \tan^2 15^\circ + 1 &= 0 \\ \Rightarrow (\tan 15^\circ - 1)(\tan 15^\circ + 1) &= 0 \\ \Rightarrow (\tan 15^\circ - 1) &= 0 \\ \Rightarrow \tan 15^\circ &= 1 \\ \Rightarrow \tan 15^\circ &= \frac{\sqrt{3}}{2 - \sqrt{3}} \\ \Rightarrow \tan 15^\circ &= \frac{2\sqrt{3}}{2\sqrt{3} - 3} \\ \Rightarrow \tan 15^\circ &= 2 - \sqrt{3} \end{aligned} $
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Question 76 (**)**

Prove the validity of each of the following trigonometric identities.

$$\text{a) } \frac{2 \sin x \cos x - \cos x}{1 - \sin x + \sin^2 x - \cos^2 x} \equiv \cot x.$$

$$\text{b) } \frac{\sec 2x - 1}{\sec 2x + 1} \equiv \tan^2 x.$$

proof

$ \begin{aligned} \text{(a)} \quad \text{LHS} &= \frac{2 \sin x \cos x - \cos x}{1 - \sin x + \sin^2 x - \cos^2 x} \\ &= \frac{\cos x(2 \sin x - 1)}{-\sin x + \sin^2 x + \cos^2 x} \\ &= \frac{\cos x(2 \sin x - 1)}{\sin x(\cos x - 1)} = \text{RHS} \end{aligned} $	$ \begin{aligned} \text{(b)} \quad \text{LHS} &= \frac{\sec 2x - 1}{\sec 2x + 1} \\ &= \frac{\frac{1}{\cos 2x} - 1}{\frac{1}{\cos 2x} + 1} = \frac{1 - \cos 2x}{1 + \cos 2x} \\ &\approx \frac{1 - (1 - 2 \sin^2 x)}{1 + (2 \cos^2 x - 1)} = \frac{2 \sin^2 x}{2 \cos^2 x} = \tan^2 x = \text{RHS} \end{aligned} $
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Question 77 (**)**

It is given that

$$\cos 3x \equiv 4\cos^3 x - 3\cos x.$$

- a) Prove the validity of the above trigonometric identity by writing $\cos 3x$ as $\cos(2x + x)$.
- b) Hence, or otherwise, solve the trigonometric equation

$$8\cos^3 x - 6\cos x + 1 = 0, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

$$\boxed{x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}}$$

$ \begin{aligned} \text{(a)} \quad & 4\cos^3 x = \cos(2x) \\ & = \cos(2x) \\ & = \cos 2x \cos x - \sin 2x \sin x \\ & = (\cos^2 x - \sin^2 x)(\cos x) - 2\sin x \cos x \sin x \\ & = 2\cos^2 x \cos x - 2\sin^2 x \cos x \\ & = 2\cos^2 x \cos x - 2(1-\cos^2 x) \cos x \\ & = 2\cos^2 x \cos x - 2\cos x + 2\cos^3 x \\ & = 4\cos^3 x - 3\cos x \\ & = \cancel{RHS} \end{aligned} $	$ \begin{aligned} \text{(b)} \quad & 8\cos^3 x - 6\cos x + 1 = 0 \\ & \Rightarrow 8\cos^3 x - 6\cos x = -1 \\ & \Rightarrow 4\cos^3 x - 3\cos x = -\frac{1}{2} \\ & \Rightarrow \cos 3x = -\frac{1}{2} \\ & \arccos(-\frac{1}{2}) = \frac{2\pi}{3} \\ & 3x = \frac{2\pi}{3} + 2\pi n, \quad n \in \mathbb{Z}, \quad 3x = \dots \\ & 3x = \frac{2\pi}{3} + \frac{2\pi}{3} \\ & \left\{ \begin{array}{l} x = \frac{2\pi}{9} + \frac{2\pi}{9} \\ x = \frac{4\pi}{9} \end{array} \right. \\ & x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9} \end{aligned} $
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Question 78 (**)**

It is given that

$$\cos \theta - \theta \sin \theta = 0.9994.$$

Given that the above equation has a solution that is numerically small, show by using a quadratic approximation that $\theta = \pm 0.02^\circ$.

proof

$$\begin{aligned}
 \cos \theta - \theta \sin \theta &= 0.9994 \\
 \left(1 - \frac{\theta^2}{2}\right) - \theta \left(\frac{\theta}{1}\right) &\approx 0.9994 \\
 1 - \frac{\theta^2}{2} - \theta^2 &\approx 0.9994 \\
 -\frac{3\theta^2}{2} &\approx -0.0006 \\
 \theta^2 &\approx 0.0004 \\
 \theta &\approx \pm 0.02^\circ
 \end{aligned}$$

Best Solutions A&E Q&C

Question 79 (****)

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

- a) Use the above trigonometric identity with suitable values for A and B , to show

$$\sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

- b) Hence by using the trigonometric expansion of $\cos(75^\circ + \alpha)$ with a suitable value for α , show that

$$\cos 165^\circ = -\sin 75^\circ.$$

proof

(6) $\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$
 $= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$

(H) $\cos 165^\circ = \cos(90^\circ + 75^\circ) = \cos 90^\circ \cos 75^\circ - \sin 90^\circ \sin 75^\circ$
 $= 0 \times \cos 75^\circ - 1 \times \frac{\sqrt{6} + \sqrt{2}}{4} = -\frac{\sqrt{6} + \sqrt{2}}{4} = -\sin 75^\circ$

Question 80 (****)

Use small angle approximations to show that if x is measured in radians then

$$\frac{1 + \cos x}{1 + \sin\left(\frac{1}{2}x\right)} \approx A + Bx,$$

where A and B are constants to be found.

 , A = 2, B = -1

USING THE STANDARD APPROXIMATIONS FOR SMALL ANGLES

$$\begin{aligned} \frac{1 + \cos x}{1 + \sin\left(\frac{1}{2}x\right)} &\approx \frac{1 + (1 - \frac{1}{2}x^2)}{1 + \frac{1}{2}x} = \frac{2 - \frac{1}{2}x^2}{1 + \frac{1}{2}x} \\ &\approx \frac{4 - x^2}{2 + x} = \frac{(2+x)(2-x)}{2+x} \\ &\approx \frac{2 - x}{2} \end{aligned}$$

AS REQUIRED ie. $A=2, B=-1$

Question 81 (**)**

Solve each of the following trigonometric equations.

i. $6\tan x = \frac{2 - 3\sec^2 x}{\tan x - 1}$, $0 \leq x < 2\pi$, $x \neq \frac{\pi}{4}, \frac{5\pi}{4}$.

ii. $\cos(3\theta - 60^\circ) = \cos(3\theta + 30^\circ)$, $0 \leq \theta \leq 180^\circ$.

, $x \approx 0.322^\circ$, $x \approx 3.46^\circ$, $\theta = 5^\circ, 65^\circ, 125^\circ$

QUESTION

$$\frac{6\tan x}{\tan x - 1} = \frac{2 - 3\sec^2 x}{\tan x - 1} \quad 0 \leq x < 2\pi$$

$$\Rightarrow 6\tan x(\tan x - 1) = 2 - 3\sec^2 x$$

$$\Rightarrow 6\tan^2 x - 6\tan x = 2 - 3\sec^2 x$$

$$\Rightarrow 6\tan^2 x - 6\tan x = 2 - 3(1 + \tan^2 x)$$

$$\Rightarrow 6\tan^2 x - 6\tan x = -1 - 3\tan^2 x$$

$$\Rightarrow 9\tan^2 x - 6\tan x + 1 = 0$$

$$\Rightarrow (\tan x - 1)^2 = 0$$

$$\Rightarrow \tan x = \frac{1}{3}$$

$$\alpha = \arctan\left(\frac{1}{3}\right) \pm n\pi \quad n=0,1,2,3,\dots$$

$$\alpha = 0.322^\circ \pm n\pi$$

$$\alpha = \begin{cases} 0.322^\circ \\ 3.46^\circ \end{cases}$$

METHOD A - USING THE COMPOUND ANGLE IDENTITIES

$$\cos(3\theta - 60^\circ) = \cos(3\theta + 30^\circ)$$

$$\cos 3\theta \cos 60^\circ + \sin 3\theta \sin 60^\circ = \cos 3\theta \cos 30^\circ - \sin 3\theta \sin 30^\circ$$

$$\frac{1}{2}\cos 3\theta + \frac{\sqrt{3}}{2}\sin 3\theta = \frac{\sqrt{3}}{2}\cos 3\theta - \frac{1}{2}\sin 3\theta$$

$$\cos 3\theta + \sqrt{3}\sin 3\theta = \sqrt{3}\cos 3\theta - \sin 3\theta$$

$$\cos 3\theta + \frac{\sqrt{3}\sin 3\theta}{\cos 3\theta} = \frac{\sqrt{3}\cos 3\theta - \sin 3\theta}{\cos 3\theta}$$

$$1 + \sqrt{3}\tan 3\theta = \sqrt{3} - \tan 3\theta$$

QUESTION

$$(\sqrt{3}+1)\tan 3\theta = \sqrt{3}-1$$

$$\tan 3\theta = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$3\theta = \arctan\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) \pm 180n \quad n=0,1,2,3,\dots$$

$$3\theta = 15^\circ \pm 180n$$

$$\theta = 5^\circ \pm 60n$$

$$\theta = \begin{cases} 5^\circ \\ 65^\circ \\ 125^\circ \end{cases}$$

ALTERNATIVE METHOD

$$\cos(3\theta - 60^\circ) = \cos(3\theta + 30^\circ)$$

$$(3\theta - 60^\circ) = (3\theta + 30^\circ) \pm 360n \quad n=0,1,2,3,\dots$$

$$(3\theta - 60^\circ) = 360 - (3\theta + 30^\circ) \pm 360n \quad n=0,1,2,3,\dots$$

$$(3\theta - 60^\circ) \pm 360n$$

$$\theta = 65^\circ \pm 60n$$

$$\theta = \begin{cases} 65^\circ \\ 5^\circ \\ 125^\circ \end{cases}$$

Question 82 (****)

$$\cos \theta = -\frac{3}{5} \quad \text{and} \quad \tan \varphi = \frac{24}{7}.$$

If θ is reflex, and φ is also reflex, show that

$$\sin(\theta - \varphi) = -\frac{44}{125}.$$

proof

$$\begin{aligned} \cos \theta &= -\frac{3}{5} & \sin \theta &= -\frac{4}{5} \\ \tan \theta &= \frac{4}{3} & \cos \varphi &= -\frac{7}{25} \\ \sin(\beta - \varphi) &\approx \sin \theta \cos \varphi - \cos \theta \sin \varphi \\ &= \left(-\frac{4}{5}\right)\left(-\frac{7}{25}\right) - \left(-\frac{3}{5}\right)\left(\frac{24}{25}\right) \\ &= \frac{28}{125} - \frac{72}{125} \\ &= -\frac{44}{125} \end{aligned}$$

Question 83 (****)

Solve the trigonometric equation

$$2(1 - \cos 2\theta) = \tan \theta, \quad 0^\circ \leq \theta \leq 180^\circ.$$

$\theta = 0^\circ, 15^\circ, 75^\circ, 180^\circ$

$$\begin{aligned} 2(1 - \cos 2\theta) &= \tan \theta \\ \rightarrow 2(1 - (1 - 2\sin^2 \theta)) &= \tan \theta \\ \rightarrow 2(2\sin^2 \theta) &= \tan \theta \\ \rightarrow 4\sin^2 \theta &= \tan \theta \\ \rightarrow 4\sin^2 \theta - \tan \theta &= 0 \\ \rightarrow \sin \theta (4\sin \theta - 1) &= 0 \\ \Rightarrow \sin \theta &= 0 \end{aligned}$$

$\bullet \sin \theta = 0 \quad \bullet \sin 2\theta = 0$
 $0^\circ = 0^\circ + 360^\circ \quad 180^\circ = 180^\circ + 360^\circ$
 $0^\circ = 180^\circ + 360^\circ \quad 360^\circ = 180^\circ + 360^\circ$
 $\therefore \theta = 0, 15, 75, 180^\circ$

Question 84 (**)**

A curve has equation

$$y = \pi - \arccos(x+1), -2 \leq x \leq 0.$$

- a) Describe geometrically the 3 transformations that map the graph of

$$y = \arccos x, -1 \leq x \leq 1,$$

onto the graph of

$$y = \pi - \arccos(x+1), -2 \leq x \leq 0.$$

- b) Sketch the graph of

$$y = \pi - \arccos(x+1), -2 \leq x \leq 0.$$

The sketch must include the coordinates of any points where the graph meets the coordinate axes.

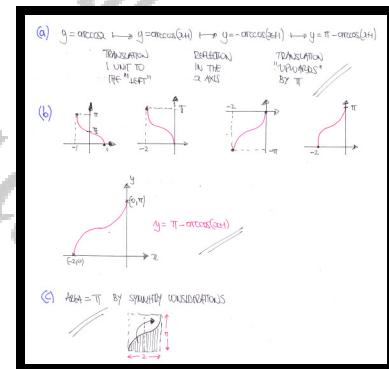
- c) Use symmetry arguments to find the area of the finite region bounded by

$$y = \pi - \arccos(x+1), -2 \leq x \leq 0,$$

and the coordinate axes.

, , translation by 1 unit to the right, followed by reflection in the x axis ,

area = π



Question 85 (**)**

It is given that

$$(\cos x + \sec x)^2 \equiv \cos^2 x + \tan^2 x + 3.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence solve the trigonometric equation

$$\cos^2 x + \tan^2 x = \frac{13}{4}, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

, $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

a) Starting from the left-hand side

$$\begin{aligned} \text{LHS} &= (\cos x + \sec x)^2 = \cos^2 x + 2\cos x \sec x + \sec^2 x \\ &= \cos^2 x + 2\cos x (\frac{1}{\cos x}) + (1 + \tan^2 x) \\ &= \cos^2 x + 2 + 1 + \tan^2 x \\ &= \cos^2 x + \tan^2 x + 3 \\ &= \text{R.H.S.} \end{aligned}$$

b) Using the acute answer

$$\begin{aligned} \Rightarrow \cos^2 x + \tan^2 x &= \frac{13}{4} \\ \Rightarrow \cos^2 x + \tan^2 x + 3 &= \frac{13}{4} + 3 \\ \Rightarrow (\cos x + \sec x)^2 &= \frac{25}{4} \\ \Rightarrow \cos x + \sec x &= \pm \frac{5}{2} \\ \Rightarrow \cos x + \frac{1}{\cos x} &= \pm \frac{5}{2} \\ \Rightarrow \cos^2 x + 1 &= \pm \frac{5}{2} \cos x \\ \Rightarrow 2\cos^2 x + 1 &= \pm 5\cos x \\ \Rightarrow 2\cos^2 x + \sec x + 1 &= 0 \end{aligned}$$

Solve the quadratic

$$\begin{aligned} \Rightarrow (2\cos x + 1)(\cos x + 2) &= 0 \\ \Rightarrow (2\cos x - 1)(\cos x - 2) &= 0 \\ \Rightarrow \cos x = \frac{1}{2} \quad \text{or} \quad \cos x &= 2 \quad (\cancel{\text{as } \cos x \neq 2}) \\ \Rightarrow x &= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

Answers

$\cos x(\frac{\pi}{3}) = \frac{1}{2}$	$\cos x(\frac{2\pi}{3}) = -\frac{1}{2}$
$(x = \frac{\pi}{3} \pm 2\pi)$	$(x = \frac{2\pi}{3} \pm 2\pi)$

Notes in the margin

$x = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$

Question 86 (****)

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$$

a) By using the above identities show that

$$\cos(A+B) + \cos(A-B) \equiv 2 \cos A \cos B.$$

b) Hence show that

$$\cos P + \cos Q \equiv 2 \cos\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

c) Deduce that

$$\frac{\cos 4x + \cos 2x}{2 \cos 3x} \equiv \cos x.$$

proof

(a) $\cos(A+B) = \cos A \cos B - \sin A \sin B$
 $\cos(A-B) = \cos A \cos B + \sin A \sin B$

ADD $\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$

(b) LET $P = A+B$ $\therefore P+Q=2A$ $A = \frac{P+Q}{2}$ $P-Q=2B$
 $Q = A-B$ $\therefore P+Q=2A$ $A = \frac{P+Q}{2}$ $B = \frac{P-Q}{2}$

GIVE INC (a) $\Rightarrow \cos P + \cos Q = 2 \cos\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right)$

(c) LHS = $\frac{\cos 4x + \cos 2x}{2 \cos 3x} = \frac{2 \cos\left(\frac{4x+2x}{2}\right) \cos\left(\frac{4x-2x}{2}\right)}{2 \cos 3x} =$
 $= \frac{2 \cos 3x \cos x}{2 \cos 3x} = \cos x = RHS$

Question 87 (**)**

Solve each of the following trigonometric equations.

i. $\sec \theta = \frac{1 - \tan^2 \theta}{4 \sec \theta - 9}, \quad 0 \leq \theta < 2\pi.$

ii. $2 \cos\left(x + \frac{\pi}{2}\right) + \sin\left(x + \frac{\pi}{3}\right) = 0, \quad 0 \leq x < 2\pi.$

$$\boxed{\theta = \frac{\pi}{3}, \frac{5\pi}{3}}, \quad \boxed{x = \frac{\pi}{6}, \frac{7\pi}{6}}$$

$$\begin{aligned}
 \text{(i) } \sec \theta &= \frac{1 - \tan^2 \theta}{4 \sec \theta - 9} \\
 \Rightarrow 4 \sec \theta - 9 \sec \theta \cdot (1 - \tan^2 \theta) &= 1 - \tan^2 \theta \\
 \Rightarrow 4 \sec \theta - 9 \sec \theta = 1 - \tan^2 \theta \\
 \Rightarrow 4 \sec \theta - 9 \sec \theta = 2 - 2 \sec \theta \\
 \Rightarrow 5 \sec \theta - 9 \sec \theta - 2 = 0 \\
 \Rightarrow (5 \sec \theta + 1)(\sec \theta - 2) = 0 \\
 \Rightarrow \sec \theta &\leq -\frac{2}{5} \\
 \Rightarrow \sec \theta &\leq -\frac{2}{5} \\
 \text{arccos}(\theta) &= \frac{\pi}{3} \\
 (\theta = \frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}), \quad (\theta = \frac{5\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}) \\
 \therefore \theta_1 &= \frac{\pi}{3} \\
 \theta_2 &= \frac{5\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } 2 \cos\left(x + \frac{\pi}{2}\right) + \sin\left(x + \frac{\pi}{3}\right) &= 0 \\
 \Rightarrow 2 \cos\left(x + \frac{\pi}{2}\right) - 2 \cos\left(x + \frac{\pi}{3}\right) + \sin\left(x + \frac{\pi}{3}\right) &= 0 \\
 \Rightarrow -2 \cos x + \frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x &= 0 \\
 \Rightarrow \frac{\sqrt{3}}{2} \sin x - \frac{3}{2} \cos x &= 0 \\
 \Rightarrow \sqrt{3} \sin x &= 3 \cos x \\
 \Rightarrow \tan x &= \frac{\sqrt{3}}{3} \\
 \Rightarrow \arctan\left(\frac{\sqrt{3}}{3}\right) &= \frac{\pi}{6} \\
 \therefore x &= \frac{\pi}{6} + m\pi, \quad m \in \mathbb{Z}, \\
 x_1 &= \frac{\pi}{6} \\
 x_2 &= \frac{7\pi}{6}
 \end{aligned}$$

Question 88 (**)**

Prove the validity of each of the following trigonometric identities.

a) $\frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} \equiv \cot x.$

b) $\tan\left(\theta + \frac{\pi}{4}\right) + \tan\left(\theta - \frac{\pi}{4}\right) \equiv 2 \tan 2\theta.$

proof

$$\begin{aligned}
 \text{(a) LHS} &= \frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} = \frac{2 \cos^2 x - 1 - \cos x + 1}{2 \sin x \cos x - \sin x} = \frac{2 \cos^2 x - \cos x}{\sin(2x) - \sin x} \\
 &= \frac{\cos x(2 \cos x - 1)}{\sin(2x) - \sin x} = \cot x = \text{RHS}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) LHS} &= \tan\left(\theta + \frac{\pi}{4}\right) + \tan\left(\theta - \frac{\pi}{4}\right) = \frac{-\tan\theta + \tan\frac{\pi}{4} + \tan\theta - \tan\frac{\pi}{4}}{1 - \tan\theta \tan\frac{\pi}{4}} \\
 &= \frac{\tan\theta + 1 - \tan\theta - 1}{1 + \tan\theta} = \frac{(\tan\theta + 1)^2 + (\tan\theta - 1)(1 - \tan\theta)}{(1 - \tan\theta)(1 + \tan\theta)} \\
 &= \frac{\tan^2\theta + 2\tan\theta + 1 + 1 - \tan^2\theta - \tan\theta}{1 - \tan^2\theta} = \frac{4\tan\theta}{1 - \tan^2\theta} \\
 &= 2 \left[\frac{2\tan\theta}{1 - \tan^2\theta} \right] = 2 \tan 2\theta = \text{RHS}
 \end{aligned}$$

Question 89 (**)**

It is given that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

- a) Use the above trigonometric identity to express $\tan 2\theta$ in terms of $\tan \theta$.
- b) Hence determine the exact value of $\tan 22.5^\circ$, showing clearly all the relevant workings.

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}, \quad \tan 22.5^\circ = -1 + \sqrt{2}$$

<p>(a) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$</p> <p>Let $A = B = \theta$</p> $\Rightarrow \tan(2\theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta}$ $\Rightarrow \tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$	<p>(b) Let $\theta = 22.5^\circ$</p> $\Rightarrow \tan 45 = \frac{2 \tan 22.5^\circ}{1 - \tan^2 22.5^\circ}$ $\Rightarrow 1 = \frac{2 \tan 22.5^\circ}{1 - \tan^2 22.5^\circ}$ $\Rightarrow 1 - \tan^2 22.5^\circ = 2 \tan 22.5^\circ$ $\Rightarrow 1 - T^2 = 2T$ $\Rightarrow 0 = T^2 + 2T - 1$ $\Rightarrow 0 = (T+1)^2 - 2$ $\Rightarrow 2 = (T+1)^2$ $\Rightarrow \pm \sqrt{2} = T+1$ $\Rightarrow T = -1 \pm \sqrt{2}$ $\Rightarrow \tan 22.5^\circ = \frac{-1 + \sqrt{2}}{-1 - \sqrt{2}} \quad (\text{since } \tan 22.5^\circ > 0)$ $\therefore \tan 22.5^\circ = -1 + \sqrt{2}$
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Question 90 (**)**

Prove the validity of the trigonometric identity

$$\frac{1 - \cos 2x + \sin 2x}{1 + \cos 2x + \sin 2x} \equiv \tan x.$$

proof

$$\begin{aligned} LHS &= \frac{1 - \cos 2x + \sin 2x}{1 + \cos 2x + \sin 2x} = \frac{1 - (1 - 2\sin^2 x) + 2\sin x \cos x}{1 + (2\cos^2 x - 1) + 2\sin x \cos x} = \frac{2\sin^2 x + 2\sin x \cos x}{2\cos^2 x + 2\sin x \cos x} \\ &= \frac{2\sin x(\sin x + \cos x)}{2\cos x(\cos x + \sin x)} = \frac{2\sin x(\cos x + \sin x)}{2\cos x(\cos x + \sin x)} = \tan x = RHS \end{aligned}$$

Question 91 (**)**

In this question it is given that the exact value of $\tan 20^\circ = t$.

- Express $\tan 25^\circ$ in terms of t .
- By using the result of part (a) show that

$$\tan 25^\circ \tan 65^\circ = 1.$$

- Show further that if

$$2\cos(\theta^\circ + 20^\circ) = 5\sin(\theta^\circ - 20^\circ),$$

then

$$\tan \theta = \frac{2+5t}{5+2t}.$$

$$\square, \quad \boxed{\tan 25^\circ = \frac{1-t}{1+t}}$$

(a) $\tan 25^\circ = \tan(45^\circ - 20^\circ) = \frac{\tan 45^\circ - \tan 20^\circ}{1 + \tan 45^\circ \tan 20^\circ} = \frac{1-t}{1+t} = \frac{1-t}{1+t}$

• similarly
 $\tan 65^\circ = \tan(45^\circ + 20^\circ) = \frac{\tan 45^\circ + \tan 20^\circ}{1 - \tan 45^\circ \tan 20^\circ} = \frac{1+t}{1-t} = \frac{1+t}{1-t}$

$\therefore \tan 25^\circ \tan 65^\circ = \frac{1-t}{1+t} \times \frac{1+t}{1-t} = 1$

(a) $2\cos(\theta^\circ + 20^\circ) = 5\sin(\theta^\circ - 20^\circ)$

$\Rightarrow 2\cos(\theta^\circ + 20^\circ) - 5\sin(\theta^\circ - 20^\circ) = 5\sin(\theta^\circ - 20^\circ) - 5\sin(\theta^\circ - 20^\circ)$

$\Rightarrow \frac{2\cos(\theta^\circ + 20^\circ)}{\cos(\theta^\circ + 20^\circ)} - \frac{5\sin(\theta^\circ - 20^\circ)}{\cos(\theta^\circ + 20^\circ)} = \frac{5\sin(\theta^\circ - 20^\circ)}{\cos(\theta^\circ + 20^\circ)} - \frac{5\sin(\theta^\circ - 20^\circ)}{\cos(\theta^\circ + 20^\circ)}$

$\Rightarrow 2 - 2\tan(\theta^\circ + 20^\circ) = 5\tan(\theta^\circ - 20^\circ)$

$\Rightarrow 2 - 2t\tan(\theta^\circ + 20^\circ) = 5t\tan(\theta^\circ - 20^\circ)$

$\Rightarrow 2 - 2t\tan(\theta^\circ + 20^\circ) = 5t\tan(\theta^\circ - 20^\circ)$

$\Rightarrow 2 + 5t\tan(\theta^\circ - 20^\circ) = 5t\tan(\theta^\circ - 20^\circ) + 2t\tan(\theta^\circ - 20^\circ)$

$\Rightarrow (2+5t)\tan(\theta^\circ - 20^\circ) = (5+2t)\tan(\theta^\circ - 20^\circ)$

$\Rightarrow \tan(\theta^\circ - 20^\circ) = \frac{2+5t}{5+2t}$

Question 92 (****)

$$f(x) = -2 + 2 \tan\left(\frac{1}{2}x\right), -\pi \leq x \leq \pi.$$

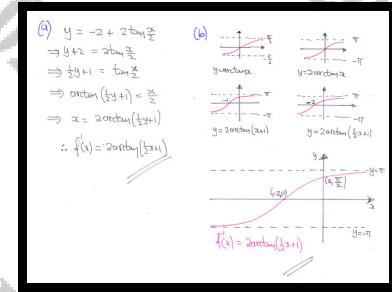
a) Find an expression for $f^{-1}(x)$.

b) Sketch the graph of $f^{-1}(x)$.

The sketch must include ...

- ...the equations of the asymptotes of $f^{-1}(x)$
- ...the coordinates of any points where the graph of $f^{-1}(x)$ meets the coordinate axes.

$$f^{-1}(x) = 2 \arctan\left(\frac{1}{2}x + 1\right)$$



Question 93 (**)**

The function f is defined as

$$f(x) = \frac{1}{1+\tan x}, \quad 0 \leq x < \frac{\pi}{2}.$$

- a) Use differentiation to show that f is a one to one function.
- b) Find a simplified expression for the inverse of f .
- c) Determine the range of f .

$$\boxed{\text{[Answer]}}, \quad \boxed{f^{-1}(x) = \arctan\left(\frac{1-x}{x}\right)}, \quad \boxed{0 < f(x) \leq 1}$$

(a) $f(x) = \frac{1}{1+\tan x} = (1+\tan x)^{-1}$ $f'(x) = -(1+\tan x)^{-2} \times \sec^2 x$ $f'(x) = -\frac{\sec^2 x}{(1+\tan x)^2}$ SINCE $f'(x) < 0$ FOR THE ENTIRE DOMAIN, THE FUNCTION IS DECREASING, SO THE FUNCTION IS ONE TO ONE.	(b) $y = \frac{1}{1+\tan x}$ $1+\tan x = \frac{1}{y}$ $\tan x = \frac{1}{y} - 1$ $\tan x = \frac{1-y}{y}$ $y = \arctan\left(\frac{1-y}{y}\right)$ $\therefore f^{-1}(x) = \arctan\left(\frac{1-x}{x}\right)$	(c) FOR DOMAIN $\tan x > 0$ $1+\tan x > 1$ $0 < \frac{1}{1+\tan x} \leq 1$ $\therefore \text{RANGE } 0 < f(x) \leq 1$
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Question 94 (***)

$$\frac{\cos\left(\frac{1}{2}x\right)}{1 + \sin x} = 0.925.$$

Given that the above equation has a solution that is numerically small, find this solution by using a quadratic approximation.

No credit will be given for solving a trigonometric equation.

, $x \approx 0.08$

SMALL ANGLES APPROXIMATIONS FOR "SMALL ANGLES"

$$\sin \theta \approx \theta$$

$$\cos \theta \approx 1 - \frac{\theta^2}{2}$$

$$\cos(\frac{1}{2}\theta) \approx 1 - \frac{(\frac{1}{2}\theta)^2}{2}$$

$$\approx 1 - \frac{\theta^2}{8}$$

HENCE WE HAVE

$$\Rightarrow \frac{\cos\frac{1}{2}x}{1 + \sin x} = 0.925$$

$$\Rightarrow \frac{1 - \frac{x^2}{8}}{1 + x} = 0.925$$

$$\Rightarrow \frac{8 - x^2}{8 + 8x} = 0.925$$

$$\Rightarrow 7.11 + 7.6x - 8 = x^2$$

$$\Rightarrow x^2 + 7.6x - 0.86 = 0$$

QUADRATIC FORMULA OR COMPLETING THE SQUARE

$$\Rightarrow (x+3.7)^2 - 3.7^2 - 0.86 = 0$$

$$\Rightarrow (x+3.7)^2 = 14.39$$

$$\Rightarrow x+3.7 = \sqrt{14.39} \dots$$

$$\Rightarrow x = \sqrt{14.39} - 3.7 \dots$$

$\therefore x \approx 0.08$

Question 95 (****)

It is given that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x.$$

- a) Prove the validity of the above trigonometric identity, by writing $\sin 3x$ as $\sin(2x+x)$.
- b) Hence, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx.$$

$\frac{2}{3}$

(a) $\begin{aligned} \sin 3x &= \sin(2x+x) = 2\sin x \cos x + \cos^2 x \sin x \\ &= (2\sin x \cos x) \cos x + (1-2\sin^2 x) \sin x \\ &= 2\sin x \cos^2 x + \sin x - 2\sin^3 x \\ &= 2\sin x (1-\sin^2 x) + \sin x - 2\sin^3 x \\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\ &= 3\sin x - 4\sin^3 x \\ &= 2\sin x \end{aligned}$

(b) $\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^3 x \, dx &= \dots \quad \sin x = 3\sin x - 4\sin^3 x \\ 4\sin^3 x &= 3\sin x - \sin^3 x \\ \sin^3 x &= \frac{3}{4}\sin x - \frac{1}{4}\sin^3 x \\ &\geq \dots \int_0^{\frac{\pi}{2}} \frac{3}{4}\sin x - \frac{1}{4}\sin^3 x \, dx \\ &= \left[\frac{3}{4}x \cos x + \frac{1}{4}x \cos^2 x \right]_0^{\frac{\pi}{2}} = \dots \\ &= \left[\frac{3}{4}(0) \cos 0 + \frac{1}{4}(0) \cos^2 0 \right] - \left[\frac{3}{4}(0) \cos 0 + \frac{1}{4}(0) \cos^2 0 \right] \\ &= -\left(-\frac{3}{4} \right) = \frac{3}{4} \end{aligned}$

Question 96 (****)

Solve the trigonometric equation

$$7\sin^2 x + \sin x \cos x = 6, \quad 0^\circ \leq x \leq 360^\circ,$$

giving the answers to the nearest degree.

, $x \approx 63^\circ, 108^\circ, 243^\circ, 288^\circ$

$$\begin{aligned} 7\sin^2 x + 5\sin x \cos x &= 6 \\ 7\sin^2 x + 5\sin x \cos x &= \frac{6}{\sin^2 x} \\ 7\tan^2 x + \tan x &= 6 \quad (\text{divide by } \sin^2 x) \\ 7\tan^2 x + \tan x &= 6(1 + \tan^2 x) \\ 7\tan^2 x + \tan x &= 6 + 6\tan^2 x \\ \tan x + \tan x - 6 &= 0 \\ (\tan x - 2)(\tan x + 3) &= 0 \end{aligned}$$

$\tan x = -3$
 $\arctan x = -3.4$
 $\arctan x(x) = -1.57$

$x = 63^\circ \pm 180^\circ$
 $x = -114^\circ \pm 180^\circ$

$x = 63^\circ, 243^\circ, 108^\circ, 288^\circ$

AUTOMATIK

$$\begin{aligned} \tan x + \sin x \cos x &= 6 \\ 7\sin^2 x + 2\sin x \cos x &= 12 \\ 14\left(\frac{1}{2} + \sin^2 x\right) + \sin x \cos x &= 12 \\ 7 - 7\sin^2 x + \sin x \cos x &= 12 \\ \sin x \cos x - 7\sin^2 x &= 5 \end{aligned}$$

WEIT: $\sin x = 7\sin^2 x$
 $\Rightarrow \sin^2 x = 2\sin x$
 $\Rightarrow \sin x = 2\sin x(1-\sin x)$
 $\therefore \text{AND SOUT-AT APPARATE}$

Question 97 (****)

$$f(x) = \sqrt{3} \cos x - \sin x, \quad x \in \mathbb{R}.$$

- a) Express $f(x)$ in the form $R \cos(x + \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.
- b) State the maximum value of $f(x)$ and find the smallest positive value of x for which this maximum occurs.

The depth of the water, D metres, in a harbour is modelled by the equation

$$D = 13 + \sqrt{3} \cos\left(\frac{\pi t}{6}\right) - \sin\left(\frac{\pi t}{6}\right), \quad 0 \leq t < 24.$$

where t is the time in hours measured since midnight.

- c) State the maximum depth of the water in the harbour and a time when this maximum depth occurs.
- d) Find the times when the depth of the water in the harbour is 12 metres.

$\boxed{}$	$\sqrt{3} \cos x - \sin x \equiv 2 \cos\left(x + \frac{\pi}{6}\right)$	$\max = 2$	$x = \frac{11\pi}{6}$	$D_{\max} = 15$
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11:00/23:00	03:00/07:00/15:00/19:00
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(a) $f(x) = \sqrt{3} \cos x - \sin x$
 $\equiv \sqrt{3} \cos x - \sqrt{1 - \cos^2 x} \sin x$
 $\equiv (\sqrt{3} \cos x)^2 + (\sqrt{1 - \cos^2 x} \sin x)^2 = \sqrt{4} = 2$
 $\therefore f(x) = \sqrt{3} \cos x - \sin x = 2 \cos\left(x + \frac{\pi}{6}\right)$

(b) $f(x)_{\max} = 2$ if $\cos(x + \frac{\pi}{6}) = 1$
 $x + \frac{\pi}{6} = 2k\pi \Rightarrow x = \frac{2k\pi}{3}$
 $x = \frac{11\pi}{6}$

(c) $D = 13 + \sqrt{3} \cos\left(\frac{\pi t}{6}\right) - \sin\left(\frac{\pi t}{6}\right)$
 $D = 13 + 2 \cos\left(\frac{\pi t}{6} + \frac{\pi}{6}\right)$
 $\therefore D_{\max} = 13 + 2 = 15$

(d) $12 = 13 + 2 \cos\left(\frac{\pi t}{6} + \frac{\pi}{6}\right)$
 $\frac{1}{2} = \cos\left(\frac{\pi t}{6} + \frac{\pi}{6}\right)$
 $\frac{\pi t}{6} + \frac{\pi}{6} = 2k\pi \pm 2m\pi \Rightarrow \begin{cases} \frac{\pi t}{6} + \frac{\pi}{6} = 2k\pi \\ \frac{\pi t}{6} + \frac{\pi}{6} = 2m\pi \end{cases} \Rightarrow \begin{cases} t = -1 \pm 12n \\ t = (11 \pm 12n) \end{cases}$
 $t = 11, 1, 23, 3, 19, 15, 7, 13, 17, 19, 15, 11, 23$

Question 98 (****)

$$\sin P = \frac{8}{17} \text{ and } \tan Q = \frac{4}{3}.$$

If P is obtuse and Q is reflex, show that

$$\cos(P-Q) = \frac{13}{85}.$$

proof

$$\begin{aligned} \cos P &= -\frac{15}{17} \\ \cos Q &= -\frac{3}{5} \\ \cos(P-Q) &= \cos P \cos Q + \sin P \sin Q \\ &= -\frac{15}{17} \times \left(-\frac{3}{5}\right) + \frac{8}{17} \left(-\frac{4}{5}\right) = \frac{45}{85} - \frac{32}{85} = \frac{13}{85} \end{aligned}$$

Question 99 (****)

Prove the validity of each of the following trigonometric identities.

$$\text{a) } 2 - 2 \tan x - \frac{2 \tan x}{\tan 2x} \equiv (1 - \tan x)^2.$$

$$\text{b) } \frac{1 - \cos x}{1 + \cos x} \equiv \cot^2\left(\frac{x}{2}\right).$$

proof

$$\begin{aligned} \text{(a)} \quad L.H.S. &= 2 - 2 \tan x - \frac{2 \tan x}{\tan 2x} = 2 - 2 \tan x - \frac{2 \tan x}{\frac{2 \tan x}{1 - \tan^2 x}} \\ &= 2 - 2 \tan x - \frac{2 \tan x}{2 \tan x} = 2 - 2 \tan x - (1 - \tan^2 x) \\ &= 1 - 2 \tan x + \tan^2 x = (1 - \tan x)^2 = R.H.S. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad L.H.S. &= \frac{1 + \cos x}{1 - \cos x} = \frac{1 + (2 \cos^2 \frac{x}{2} - 1)}{1 - (2 \sin^2 \frac{x}{2})} = \frac{2 \cos^2 \frac{x}{2}}{2 \sin^2 \frac{x}{2}} = \cot^2 \frac{x}{2} = R.H.S. \end{aligned}$$

$\cos A = 2 \cos^2 \frac{A}{2} - 1$
 $\cos A = 1 - 2 \sin^2 \frac{A}{2}$
 $\cot A = \frac{\cos A}{\sin A}$

Question 100 (****)

$$\cos\left(x + \frac{\pi}{6}\right) = 4 \cos\left(x - \frac{\pi}{6}\right).$$

Show by using an appropriate compound angle identity that

$$\tan x = -\frac{3}{5}\sqrt{3}.$$

proof

$$\begin{aligned} \cos\left(x + \frac{\pi}{6}\right) &= 4 \cos\left(x - \frac{\pi}{6}\right) \\ \Rightarrow \cos(x\cos\frac{\pi}{6} - \sin x\sin\frac{\pi}{6}) &= 4(\cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6}) \\ \Rightarrow \cos x\cos\frac{\pi}{6} - \frac{1}{2}\sin x &= 2\sqrt{3}\cos x + 2\sin x \\ \Rightarrow \sqrt{3}\cos x - \sin x &= 4\sqrt{3}\cos x + 4\sin x \\ \Rightarrow -3\sqrt{3}\cos x &= 5\sin x \quad (\text{Divide by } \cos x) \\ \Rightarrow -3\sqrt{3} &= 5\tan x \\ \Rightarrow -\frac{3\sqrt{3}}{5} &= \tan x \quad (\text{Divide by } \cos^2 x) \end{aligned}$$

Question 101 (****)

Show clearly that ...

a) ... $\cot A \equiv \frac{\sin(A+B) - \sin(A-B)}{\cos(A-B) - \cos(A+B)}$.

b) ... $\cot 75^\circ = 2 - \sqrt{3}$,
(use part (a) with suitable values of A and B)

proof

$$\begin{aligned} \text{(a)} \quad \cot A &= \frac{\sin(A+B) - \sin(A-B)}{\cos(A+B) - \cos(A-B)} = \frac{\sin A \cos B + \cos A \sin B - \sin A \cos B + \cos A \sin B}{\cos^2 A - \cos^2 B} = \cot A = \cot A // \\ \text{(b)} \quad \cot 75^\circ &= \frac{\sin(75^\circ + 15^\circ) - \sin(75^\circ - 15^\circ)}{\cos(75^\circ + 15^\circ) - \cos(75^\circ - 15^\circ)} \\ \text{LET } A = 75^\circ & \quad \cot 75^\circ = \frac{\sin 90^\circ - \sin 60^\circ}{\cos 90^\circ - \cos 60^\circ} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3} // \end{aligned}$$

Question 102 (*)**

The curves C_1 and C_2 have respective equations

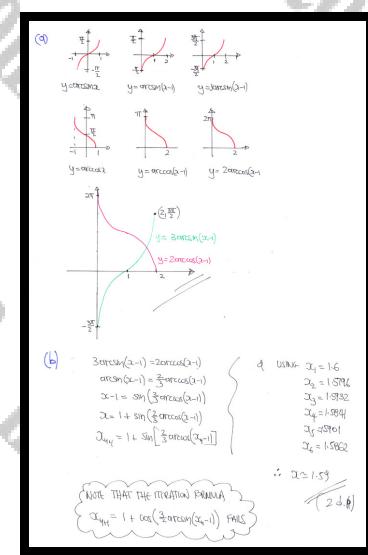
$$y_1 = 3 \arcsin(x-1) \text{ and } y_1 = 2 \arccos(x-1).$$

- a) Sketch in the same diagram the graph of C_1 and the graph of C_2 .

The sketch must include the coordinates of any points where the graphs of C_1 and C_2 meet the coordinate axes as well as the coordinates of the endpoints of the curves.

- b) Use a suitable iteration formula of the form $x_{n+1} = f(x_n)$ with $x_1 = 1.6$ to find the x coordinate of the point of intersection between the graph of C_1 and the graph of C_2 .

$$x \approx 1.59$$



Question 103 (**)**

It is given that

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

- a) Prove the validity of the above trigonometric identity by using the compound angle identities for $\sin(A+B)$ and $\sin(A-B)$.
- b) Hence, or otherwise, solve the trigonometric equation

$$\sin 7x + \sin x = 0, \quad 0 \leq x < \pi,$$

giving the answers in terms of π .

$\boxed{\quad}$	$x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$
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a) Starting from the compound angle identities

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B \end{aligned}$$

$$\Rightarrow \sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

Now let in the L.H.S. of the above expression

$$\begin{aligned} A+B &= P & P &= A+B \\ A-B &= Q & Q &= A-B \end{aligned}$$

Adding the above

$$\begin{aligned} \Rightarrow 2A &= P+Q & 2B &= P-Q \\ \Rightarrow A &= \frac{P+Q}{2} & B &= \frac{P-Q}{2} \end{aligned}$$

Hence we obtain

$$\begin{aligned} \sin(A+B) + \sin(A-B) &= 2 \sin A \cos B \\ \sin P + \sin Q &= 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right) \end{aligned}$$

b) Using part (a) with $P=7x$, $Q=x$

$$\begin{aligned} \Rightarrow \sin 7x + \sin x &= 0 \\ \Rightarrow 2 \sin\left(\frac{7x+x}{2}\right) \cos\left(\frac{7x-x}{2}\right) &= 0 \\ \Rightarrow 2 \sin(4x) \cos(3x) &= 0 \end{aligned}$$

Either $\sin 4x = 0$ or $\cos 3x = 0$

$\sin 3x = 0$

$$\begin{aligned} \arcsin 0 &= 0 \\ 3x &= 0 \pm 2n\pi \\ 3x &= \frac{\pi}{2} \pm 2n\pi \\ x &= \frac{\pi}{6} \pm \frac{2n\pi}{3} \\ x &= \frac{\pi}{6}, \frac{4\pi}{3}, \dots \end{aligned}$$

$\cos 3x = 0$

$$\begin{aligned} \arccos 0 &= \frac{\pi}{2} \\ 3x &= \frac{\pi}{2} \pm 2n\pi \\ 3x &= \frac{\pi}{2} \pm \frac{2n\pi}{3} \\ x &= \frac{\pi}{6} \pm \frac{2n\pi}{3} \\ x &= \frac{\pi}{6}, \frac{4\pi}{3}, \dots \end{aligned}$$

Alternative for part (b) without using part (a)

$$\begin{aligned} \Rightarrow \sin^2 3x + \sin 3x &= 0 \\ \Rightarrow \sin 3x &= -\sin 3x \\ \Rightarrow \sin 3x &= \sin(-3x) \\ \Rightarrow (3x &= -3x \pm 2n\pi) \quad n=0,1,2,\dots \\ \Rightarrow 3x &= \pi - (3x) \pm 2n\pi \\ \Rightarrow (3x &= \pi \pm 2n\pi) \\ \Rightarrow (x &= \frac{\pi}{3} \pm \frac{2n\pi}{3}) \\ \Rightarrow (x &= \frac{\pi}{6} \pm \frac{2n\pi}{3}) \end{aligned}$$

which yields the same solution as before

Question 104 (**)**

Solve each of the following trigonometric equations.

i. $\frac{5 + \tan^2 y}{\sec y} = 9 - \sec y, \quad 0 \leq y < 2\pi, \quad y \neq \frac{\pi}{2}, \frac{3\pi}{2}$.

ii. $\sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right) = \sin\left(\theta + \frac{\pi}{6}\right), \quad 0 \leq \theta < 2\pi$.

$y = 1.32^\circ, 4.97^\circ, \quad \theta = \frac{\pi}{12}, \frac{13\pi}{12}$

$$\begin{aligned}
 \text{(a)} \quad & \frac{5 + \tan^2 y}{\sec y} = 9 - \sec y \\
 \Rightarrow & 5 + \sec y = 9 \sec y - \sec^2 y \\
 \Rightarrow & 5 + (\sec y - 1) = 8 \sec y - \sec^2 y \\
 \Rightarrow & 5 + \sec y - 1 = 8 \sec y - \sec^2 y \\
 \Rightarrow & 2 \sec y - 8 \sec y + 1 = 0 \\
 \Rightarrow & (2 \sec y - 1)(4 \sec y - 1) = 0 \\
 \Rightarrow & 2 \sec y - 1 = 0 \quad \text{or} \quad 4 \sec y - 1 = 0 \\
 \Rightarrow & \sec y = \frac{1}{2} \quad \text{or} \quad \sec y = \frac{1}{4} \\
 \Rightarrow & \sec y = \pm \frac{1}{2} \\
 \Rightarrow & y = 1.32^\circ, 4.97^\circ, 180^\circ, 204^\circ, 234^\circ, 270^\circ, 306^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right) = \sin\left(\theta + \frac{\pi}{6}\right) \\
 \Rightarrow & \sqrt{2} \cos\theta \cos\frac{\pi}{4} - \sqrt{2} \sin\theta \sin\frac{\pi}{4} = \sin\theta \cos\frac{\pi}{6} + \cos\theta \sin\frac{\pi}{6} \\
 \Rightarrow & 2 \cos\theta - 2\sin\theta = \sqrt{3}\sin\theta + \cos\theta \\
 \Rightarrow & 2\cos\theta - \sqrt{3}\sin\theta = \cos\theta + \sqrt{3}\sin\theta \\
 \Rightarrow & 1 = (2 + \sqrt{3})\sin\theta \\
 \Rightarrow & \sin\theta = \frac{1}{2 + \sqrt{3}} \\
 \Rightarrow & \sin\theta = \frac{1}{2 + \sqrt{3}} \cdot \frac{2 - \sqrt{3}}{2 - \sqrt{3}} \\
 \Rightarrow & \sin\theta = \frac{2 - \sqrt{3}}{2} \\
 \Rightarrow & \theta = \frac{\pi}{12} \pm n\pi, \quad n = 0, 1, 2, 3, \dots
 \end{aligned}$$

Question 105 (**)**

Prove the validity of each of the following trigonometric identities.

a) $(\cos x + \sin x)(\operatorname{cosec} x - \sec x) \equiv 2 \cot 2x$.

b) $\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x} \equiv \sec x$.

proof

$$\begin{aligned}
 \text{(a)} \quad & \text{LHS} = (\cos x + \sin x)(\operatorname{cosec} x - \sec x) = \cos x \operatorname{cosec} x - \cos x \sec x - \sin x \operatorname{cosec} x + \sin x \sec x \\
 & = \cos x \cdot \frac{1}{\sin x} - \cancel{\sin x} \operatorname{cosec} x - \frac{1}{\cos x} \sec x + \sin x \cdot \frac{1}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{\cos 2x}{\sin x \cos x} \\
 & = \frac{\cos 2x}{\sin 2x} = \frac{2 \cos 2x}{2 \sin 2x} = 2 \cot 2x = \text{RHS}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \text{LHS} = \frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x} = \frac{\cos x + (1 - \sin x)}{(1 - \sin x) \cos x} = \frac{\cos x + 1 - \sin x}{(1 - \sin x) \cos x} = \frac{\cancel{\cos x} + \cancel{1} - \sin x}{(1 - \sin x) \cos x} = \frac{2 - 2 \sin x}{(1 - \sin x) \cos x} \\
 & = \frac{2(1 - \sin x)}{(1 - \sin x) \cos x} = \frac{2}{\cos x} = 2 \sec x = \text{RHS}
 \end{aligned}$$

Question 106 (****)

It is given that

$$\cos(A+B) + \cos(A-B) \equiv 2\cos A \cos B.$$

- Prove the validity of the above trigonometric identity.
- Hence, or otherwise, solve the trigonometric equation

$$2\cos\left(x+\frac{\pi}{6}\right) = \sec\left(x+\frac{\pi}{2}\right), \quad 0 \leq x \leq \pi,$$

giving the answers in terms of π .

$$\boxed{\quad}, \quad x = \frac{\pi}{2}, \frac{5\pi}{6}$$

(a) $\cos(A+B) + \cos(A-B) = \cos A \cos B - \sin A \sin B + \cos A \cos B + \sin A \sin B$
 $\therefore \cos(A+B) + \cos(A-B) \equiv 2\cos A \cos B$

(b) $2\cos\left(x+\frac{\pi}{6}\right) = \sec\left(x+\frac{\pi}{2}\right)$ $\cos\cos\left(\frac{x}{2}\right) = 1$
 $\Rightarrow 2\cos\left(x+\frac{\pi}{6}\right) = \frac{1}{\cos(x+\frac{\pi}{2})}$ $x+2k\pi = \frac{\pi}{2} \pm 2m\pi$
 $\Rightarrow 2\cos\left(x+\frac{\pi}{6}\right)\cos(x+\frac{\pi}{2}) = 1$ $x+2k\pi = \frac{\pi}{2} \pm 2m\pi$
 $\Rightarrow \cos(x+\frac{\pi}{6})\cos(x+\frac{\pi}{2}) = 1$ $x_1 = -\frac{\pi}{2} \pm 2m\pi$
 $\Rightarrow \cos(x+\frac{\pi}{6}) + \cos(x-\frac{\pi}{6}) = 1$ $x_2 = \frac{\pi}{2} \pm m\pi$
 $\Rightarrow \cos(x+\frac{\pi}{6}) + \cos(-\frac{\pi}{6}) = 1$ $x_1 = \frac{\pi}{2} \pm m\pi$
 $\Rightarrow \cos(x+\frac{\pi}{6}) + \frac{1}{2} = 1$ $x_2 = \frac{\pi}{2} \pm m\pi$
 $\Rightarrow \cos(x+\frac{\pi}{6}) = \frac{1}{2}$

Question 107 (****)

It is given that

$$\cos 2x + \tan x \sin 2x \equiv 1, \quad x \neq 90^\circ (n+1), n \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
 b) Use the above result to solve the trigonometric equation

$$\tan x \sin 2x + 13 \cos x = 8, \quad 0 \leq x < 360^\circ.$$

$$x = 60^\circ, 300^\circ$$

$ \begin{aligned} \text{(a)} \quad & \text{LHS} = \cos 2x + \tan x \sin 2x \\ &= \cos 2x - \sqrt{3} \tan(2x) \frac{\sin 2x}{\cos 2x} \\ &= \cos 2x - \sqrt{3} \tan 2x \\ &= \cos 2x + \sqrt{3} \\ &= 1 \\ &= \text{RHS} \end{aligned} $	$ \begin{aligned} & \Rightarrow 1 - (\cos 2x - \sqrt{3}) + \sqrt{3} \cos 2x = 8 \\ & \Rightarrow 2 - 2 \cos^2 x + \sqrt{3} \cos 2x = 8 \\ & \Rightarrow 0 = 2 \cos^2 x - \sqrt{3} \cos 2x + 6 \\ & \Rightarrow 0 = (2 \cos x - 1)(\cos x - 6) \\ & \Rightarrow \cos x = \frac{1}{2} \\ & \Rightarrow \cos x = \frac{1}{2} \\ & \Rightarrow \cos x = \frac{1}{2} \\ & \Rightarrow x = 60^\circ \pm 360^\circ \\ & \Rightarrow x = 30^\circ \pm 360^\circ \\ & \Rightarrow x = 60^\circ, 300^\circ \end{aligned} $
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Question 108 (**)**

$$3\cos^2 x - \cos x = 1.99375.$$

It is given that the above trigonometric equation has a solution that is numerically small.

Use small angle approximations to find this solution.

No credit will be given for standard solution methods.

, $x \approx \pm 0.05$

Method A

$$\begin{aligned} 3\cos^2 x - \cos x &= 1.99375 \\ \text{USING THE DOUBLE ANGLE FORMULA FOR } \cos 2x &= 2\cos^2 x - 1 \\ \Rightarrow 3\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) - \cos 2x &= 1.99375 \\ \Rightarrow \frac{3}{2} + \frac{3}{2}\cos 2x - \cos 2x &= 1.99375 \\ \Rightarrow \frac{3}{2} + \cos 2x &= 1.99375 \\ \text{USING A GEOMETRIC APPROXIMATION FOR } \cos 2x &\\ \cos 2x &\approx 1 - \frac{x^2}{2} \\ \cos 2x &\approx 1 - \frac{(2x)^2}{2} \approx 1 - \frac{4x^2}{2} \approx 1 - 2x^2 \\ \text{HENCE WE OBTAIN} \\ \Rightarrow 3 + 3\left(1 - 2x^2\right) - 2\left(1 - \frac{2x^2}{2}\right) &= 1.99375 \text{ AS} \\ \Rightarrow 3 + 3 - 6x^2 - 2 + 2x^2 &= 1.99375 \times 2 \\ \Rightarrow 4 - 2x^2 - 1.99375 &= 5x^2 \\ \Rightarrow 8x^2 &= 0.00625 \\ \Rightarrow x^2 &= 0.00078125 \\ \Rightarrow x &= \pm 0.03125 \end{aligned}$$

$\boxed{\pm 0.05}$ / Both sign OK as $\cos 2x$ is even

Method B

$$\begin{aligned} 3\cos^2 x - \cos x &= 1.99375 \\ 3\cos^2 x - \cos x - 1.99375 &= 0 \\ \text{BY THE QUADRATIC FORMULA} \\ \cos 2x &= \frac{-1 \pm \sqrt{1 - 4(3)(-1.99375)}}{6} \\ \cos 2x &= \frac{1 \pm \sqrt{24.975}}{6} \\ \text{NOW SOLVE FOR } x \text{ (USING APPROXIMATION FOR } \cos x) \\ 1 - \frac{x^2}{2} &= \frac{1 \pm \sqrt{24.975}}{6} \\ -\frac{x^2}{2} &= -1 + \frac{1 \pm \sqrt{24.975}}{6} \\ x^2 &= 2\left[\frac{1}{2} - \frac{1 \pm \sqrt{24.975}}{6}\right] \\ x^2 &< 0.0025018778... \\ x &= \sqrt{\frac{1 \pm \sqrt{24.975}}{6}} \\ &= \sqrt{\frac{1 \pm 4.99875}{6}} \\ &= \sqrt{0.416 \text{ or } 8.3333456...} \end{aligned}$$

$\boxed{\pm 0.05}$ / $x \approx 0.05$ to 2d.s.f?

Question 109 (****)

It is given that

$$\frac{2\cot\theta}{1+\cot^2\theta} \equiv \sin 2\theta.$$

- a) Prove the validity of the above trigonometric identity.
 b) Use the above result to solve the trigonometric equation

$$4\cot^2\theta + 1 = 2\sin 2\theta(1 + \cot^2\theta), \quad 0^\circ \leq \theta < 360^\circ.$$

$$\boxed{\quad}, \quad x \approx 63.4^\circ, 243.4^\circ$$

(a) $\frac{2\cot\theta}{1+\cot^2\theta} = \frac{2\cot\theta}{\cot^2\theta+1} = \frac{2\cot\theta}{\cot^2\theta+2} = \frac{2(\cot\theta)}{2(\cot^2\theta+1)} = \frac{\sin 2\theta}{\sin^2\theta+1} = \frac{\sin 2\theta}{\cos^2\theta} = \tan 2\theta$

(b) $2\cot^2\theta + 1 = 2\sin 2\theta(1 + \cot^2\theta)$
 Hence $4\cot^2\theta + 1 = 2\sin 2\theta(1 + \cot^2\theta)$
 $\Rightarrow 4\cot^2\theta + 1 = 2(2\cot\theta)$
 $\Rightarrow 4\cot^2\theta - 4\cot\theta + 1 = 0$
 $\Rightarrow (2\cot\theta - 1)^2 = 0$
 $\Rightarrow \cot\theta = \frac{1}{2}$
 $\Rightarrow \tan\theta = 2$
 $\arctan 2 = 63.4^\circ$
 $\theta = 63.4^\circ + 180^\circ \quad \therefore \theta_1 = 63.4^\circ$
 $\theta = 243.4^\circ \quad \theta_2 = 243.4^\circ$

Question 110 (****)

If $\sin(\theta + \alpha) = 2\sin\theta$, show clearly that

$$\tan\theta = \frac{\sin\alpha}{2-\cos\alpha}.$$

proof

$$\begin{aligned} \sin(\theta + \alpha) &= 2\sin\theta \\ \sin\theta\cos\alpha + \cos\theta\sin\alpha &= 2\sin\theta \\ \cos\theta\sin\alpha &= 2\sin\theta - \sin\theta\cos\alpha \\ \frac{\cos\theta\sin\alpha}{\cos\theta} &= \frac{\sin\theta(2-\cos\alpha)}{\cos\theta} \\ \tan\theta &= \frac{\sin\theta}{2-\cos\alpha} \end{aligned} \quad \therefore \tan\theta = \frac{\sin\alpha}{2-\cos\alpha} \quad \text{as required}$$

Question 111 (**)**

Given that the exact value of $\tan 20^\circ = t$, show that

$$\tan 10^\circ = \frac{-1 + \sqrt{t^2 + 1}}{t}.$$

SYG, proof

PROCEED AS FOLLOWS

$$\begin{aligned}\tan 20^\circ &= \tan(2 \times 10^\circ) = \\ t &= \frac{2 \tan 10^\circ}{1 - \tan^2 10^\circ} \\ t &= \frac{2x}{1 - x^2} \\ \text{where } x &= \tan 10^\circ\end{aligned}$$

DETERMINING x

$$\begin{aligned}\Rightarrow t(1 - x^2) &= 2x \\ \Rightarrow t - tx^2 &= 2x \\ \Rightarrow 0 &= tx^2 + 2x - t \\ \Rightarrow x^2 + \frac{2}{t}x - 1 &= 0\end{aligned}$$

BY THE QUADRATIC FORMULA OR BY COMPLETING THE SQUARE

$$\begin{aligned}\Rightarrow (x + \frac{1}{t})^2 - (\frac{1}{t})^2 - 1 &= 0 \\ \Rightarrow (x + \frac{1}{t})^2 &= 1 + \frac{1}{t^2} \\ \Rightarrow (x + \frac{1}{t})^2 &= \frac{t^2 + 1}{t^2} \\ \Rightarrow x + \frac{1}{t} &= \pm \sqrt{\frac{t^2 + 1}{t^2}} \\ \Rightarrow x &= -\frac{1}{t} \pm \frac{\sqrt{t^2 + 1}}{t} \\ \Rightarrow x &= \frac{-1 \pm \sqrt{t^2 + 1}}{t}\end{aligned}$$

Now $-1 - \sqrt{t^2 + 1} < 0$
and $x = \tan 10^\circ > 0$

$$\therefore x = \frac{-1 + \sqrt{t^2 + 1}}{t}$$

$$\therefore \tan 10^\circ = \frac{-1 + \sqrt{t^2 + 1}}{t}$$

As required

Question 112 (**)**

Use trigonometric algebra to solve the equation

$$\sin \left[\arcsin \frac{1}{4} + \arccos x \right] = 1.$$

x = $\frac{1}{4}$

SOLVING THE EQUATION AS FOLLOWS

$$\begin{aligned}\Rightarrow \sin \left[\arcsin \frac{1}{4} + \arccos x \right] &= 1 \\ \Rightarrow \arcsin \left[\sin \left(\arcsin \frac{1}{4} + \arccos x \right) \right] &= \arcsin 1 \pm 2n\pi \quad (n=0,1,2,\dots) \\ \Rightarrow \arcsin \frac{1}{4} + \arccos x &= \frac{\pi}{2} \pm 2n\pi \\ \Rightarrow \arccos x &= \frac{\pi}{2} - \arcsin \frac{1}{4} \pm 2n\pi\end{aligned}$$

BUT ARCCOS CAN ONLY RETURN VALUES BETWEEN 0 AND PI

$$\begin{aligned}\Rightarrow \arccos x &= \frac{\pi}{2} - \arcsin \frac{1}{4} \\ \Rightarrow x &= \cos \left(\frac{\pi}{2} - \arcsin \frac{1}{4} \right) \\ \text{BUT } \cos \left(\frac{\pi}{2} - \theta \right) &= \sin \theta \\ \Rightarrow x &= \sin \left(\arcsin \frac{1}{4} \right) \\ \Rightarrow x &= \frac{1}{4}\end{aligned}$$

Question 113 (**)**

It is given that

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

- a) Prove the validity of the above trigonometric identity, by using the compound angle identities for $\sin(A+B)$ and $\sin(A-B)$.
- b) Hence, or otherwise, solve the trigonometric equation

$$\sin 4\theta + \sin 2\theta = \cos \theta, \quad 0 \leq \theta < \pi,$$

giving the answers in terms of π .

$$\boxed{\text{[]}}, \quad \theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{\pi}{2}, \frac{13\pi}{18}, \frac{17\pi}{18}$$

(a) $\sin(A+B) = \sin A \cos B + \cos A \sin B$
 $\sin(A-B) = \sin A \cos B - \cos A \sin B$
 $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

Let $\begin{cases} A+B = P \\ A-B = Q \end{cases}$. Add $2A = P+Q \Rightarrow A = \frac{P+Q}{2}$
 $A-B = Q$. Subtract $2B = P-Q \Rightarrow B = \frac{P-Q}{2}$
Hence (a) becomes $\sin P + \sin Q = 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right)$

(b) $\sin 4\theta + \sin 2\theta = \cos \theta$
 $\Rightarrow 2 \sin\left(\frac{2\theta+3\theta}{2}\right) \cos\left(\frac{2\theta-3\theta}{2}\right) = \cos \theta$
 $\Rightarrow 2 \sin 3\theta \cos \theta = \cos \theta$
 $\Rightarrow 2 \sin 3\theta \cos \theta - \cos \theta = 0$
 $\Rightarrow \cos \theta [2 \sin 3\theta - 1] = 0$
 $\cos \theta = 0 \quad \text{or} \quad \sin 3\theta = \frac{1}{2}$
 $\cos \theta = 0 \quad \text{or} \quad \sin \theta = \frac{1}{2}$

$\theta = \frac{\pi}{2} + 2k\pi \quad k=0, 1, 2, \dots$
 $\theta = \frac{\pi}{6} + 2k\pi \quad k=0, 1, 2, \dots$
 $\theta = \frac{5\pi}{6} + 2k\pi \quad k=0, 1, 2, \dots$
 $\theta = \frac{\pi}{3} + 2k\pi \quad k=0, 1, 2, \dots$
 $\theta = \frac{7\pi}{6} + 2k\pi \quad k=0, 1, 2, \dots$

Answers: $\theta = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{18}$

Question 114 (****)

$$\sin \theta = \frac{24}{25} \quad \text{and} \quad \cos \varphi = \frac{15}{17}.$$

If θ is obtuse and φ is reflex, show clearly that

$$\sec(\theta + \varphi) = \frac{425}{87}.$$

proof

Question 115 (****)

Solve each of the following trigonometric equations.

i. $\frac{\sec^2 x - 2}{\tan x} = \frac{\tan x - 1}{2}, \quad 0 \leq x < 2\pi, \quad x \neq \frac{\pi}{2}, \frac{3\pi}{2}$.

ii. $2 \cos 2\theta = 4 \cos \theta - 3, \quad 0 \leq \theta < 360^\circ$.

 , $x = 0.785^\circ, 2.03^\circ, 3.93^\circ, 5.18^\circ$, $\theta = 60^\circ, 300^\circ$

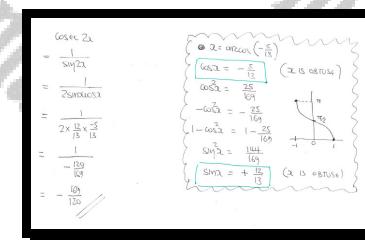
<p>(i) $\frac{\sec^2 x - 2}{\tan x} = \frac{\tan x - 1}{2}$</p> $\Rightarrow 2\sec^2 x - 4 = \tan^2 x - \tan x$ $\Rightarrow 2(1 + \tan^2 x) - 4 = \tan^2 x - \tan x$ $\Rightarrow 2 + 2\tan^2 x - 4 = \tan^2 x - \tan x$ $\Rightarrow \tan^2 x + \tan x - 2 = 0$ $\Rightarrow (\tan x - 1)(\tan x + 2) = 0$ $\tan x < 2$ <p>(• $\tan x = 1 \Rightarrow x = 45^\circ$ • $\tan x = -2 \Rightarrow x = 102^\circ$)</p> $\begin{cases} x_1 = 45^\circ \\ x_2 = 102^\circ \end{cases}$ $x_1 = 0.785^\circ$ $x_2 = 3.93^\circ$ $x_3 = 2.03^\circ$ $x_4 = 5.18^\circ$	<p>(ii) $2\cos 2\theta + 4\cos \theta - 3$</p> $\Rightarrow 2(2\cos^2 \theta - 1) + 4\cos \theta - 3$ $\Rightarrow 4\cos^2 \theta - 2 + 4\cos \theta - 3$ $\Rightarrow 4\cos^2 \theta - 4\cos \theta + 1 = 0$ $\Rightarrow (2\cos \theta - 1)^2 = 0$ $\Rightarrow \cos \theta = \frac{1}{2}$ $\arccos(\frac{1}{2}) = 60^\circ$ $\begin{cases} \theta = 60^\circ \\ \theta = 300^\circ \end{cases}$ $\theta_1 = 60^\circ$ $\theta_2 = 300^\circ$
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Question 116 (****)

$$x = \arccos\left(-\frac{5}{13}\right).$$

Determine the exact value of $\operatorname{cosec} 2x$.

$$\boxed{-\frac{169}{120}}$$



Question 117 (****)

$$f(x) = \sec x, \quad x \in \mathbb{R}, \quad 0 \leq x \leq 4\pi.$$

- a) Sketch the graph of $f(x)$, showing clearly the coordinates of any stationary points and equations of asymptotes.

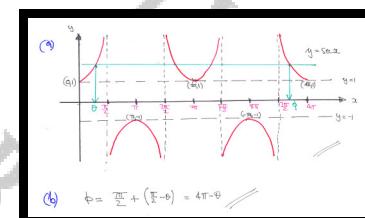
It is now given that

$$\sec \theta = \sec \varphi,$$

where $0 < \theta < \frac{\pi}{2}$ and $\frac{7\pi}{2} < \varphi < 4\pi$.

- b) Express φ in terms of θ .

$$\boxed{\quad}, \quad \boxed{\varphi = 4\pi - \theta}$$



Question 118 (****)

If $\cos x = \frac{1}{3}$, show by detailed workings that

$$\cos x \cos 2x \cos 4x = -\frac{119}{2187}$$

proof

$$\begin{aligned}
 & \text{Cos}2\alpha = \frac{\cos^2\alpha - \sin^2\alpha}{2\sin\alpha\cos\alpha} = \frac{2\cos^2\alpha - 1}{2\sin\alpha\cos\alpha} \\
 & = \frac{2\cos^2\alpha}{2\sin\alpha\cos\alpha} - \frac{1}{2\sin\alpha\cos\alpha} = \frac{\cos 2\alpha}{\sin 2\alpha} - \frac{1}{2\sin 2\alpha} \\
 & = \frac{\cos 2\alpha}{\sin 2\alpha} - \frac{1}{2} \cdot \frac{1}{\cos 2\alpha} = \frac{\cos 2\alpha}{\sin 2\alpha} - \frac{1}{2\cos 2\alpha} \\
 & \quad \text{But } \cos \alpha = \frac{1}{3} \\
 & = 2 \times \frac{1}{3} \times \left[2 \left(\frac{1}{3} \right)^2 - 1 \right] - \frac{1}{2} \left[2 \left(\frac{1}{3} \right)^2 - 1 \right] \\
 & = \frac{2}{3} \left(-\frac{8}{9} \right) - \frac{1}{2} \left(-\frac{7}{9} \right) = -\frac{166}{2157} + \frac{7}{2157} = -\frac{63 + 567}{2157} \\
 & = -\frac{114}{2157}
 \end{aligned}$$

Question 119 (****)

$$4\sin \theta + \cos \theta = 2, \quad 0^\circ \leq \theta < 360^\circ.$$

- a) Show that the above trigonometric equation can be written as

$$16\sin^2 \theta = 4 - 4\cos \theta + \cos^2 \theta.$$

- b) Show further that

$$\cos \theta = \frac{2 \pm 4\sqrt{13}}{17}.$$

- c) Hence, or otherwise, find the **two** values of θ that satisfy the equation

$$4\sin \theta + \cos \theta = 2, \quad 0^\circ \leq \theta < 360^\circ.$$

$$\theta \approx 15.0^\circ, 136.9^\circ$$

$\begin{aligned} \text{(a)} \quad 4\sin \theta + \cos \theta &= 2 \\ \Rightarrow 4\sin \theta &= 2 - \cos \theta \\ \Rightarrow (\cos \theta)^2 &= (2 - \cos \theta)^2 \\ \Rightarrow \cos \theta(2 - \cos \theta) &= 4 - 4\cos \theta + \cos^2 \theta \\ \text{(b)} \quad 16(-\cos \theta) &= 4 - 4\cos \theta + \cos^2 \theta \\ \Rightarrow 16(-\cos \theta) &= 4 - 4\cos \theta + \cos^2 \theta \\ \Rightarrow 0 &= 17\cos^2 \theta - 4\cos \theta - 12 \end{aligned}$ <p style="text-align: center;">BY QUADRATIC FORMULA</p> $\begin{aligned} \cos \theta &= \frac{(-4 \pm \sqrt{(-4)^2 - 4 \cdot 17 \cdot (-12)})}{2 \cdot 17} \\ \cos \theta &= \frac{4 \pm \sqrt{1732}}{34} \\ \cos \theta &= \frac{4 \pm 4\sqrt{13}}{34} \\ \cos \theta &= \frac{2 \pm 2\sqrt{13}}{17} \end{aligned}$	$\begin{aligned} \text{(c)} \quad \cos \theta &= \sqrt{\frac{2+4\sqrt{13}}{17}} \\ &\quad \text{or} \\ &= \sqrt{\frac{2-4\sqrt{13}}{17}} \end{aligned}$ <ul style="list-style-type: none"> $\arccos\left(\frac{2+4\sqrt{13}}{17}\right) = 15.0^\circ$ $\arccos\left(\frac{2-4\sqrt{13}}{17}\right) = 136.9^\circ$ $\begin{aligned} \theta &= 15.0^\circ + 360^\circ k, \quad k \in \mathbb{Z}, \dots \\ \theta &= 136.9^\circ + 360^\circ k, \quad k \in \mathbb{Z}, \dots \\ \theta &= 223.9^\circ + 360^\circ k \end{aligned}$ <p style="text-align: right;">$\cancel{\text{DO NOT SATISFY ORIGINALLY}}$</p>
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Question 120 (****)

It is given that

$$\frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} \equiv \tan x.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence, or otherwise, solve for $0^\circ \leq x < 360^\circ$

$$\frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} = 3 \cot x.$$

$$x = 60^\circ, 120^\circ, 180^\circ, 300^\circ$$

(a) $\begin{aligned} 4 \cot x &= \frac{\sin(2x+30^\circ)}{\cos(2x+30^\circ)+1} = \frac{2\sin x \cos x + \sin x}{(\cos x)^2 + \cos x + 1} = \frac{\sin x(2\cos x + 1)}{2\cos^2 x + \cos x} \\ &= \frac{\sin x(2\cos x + 1)}{\cos x(2\cos x + 1)} = \tan x = 2\sqrt{3} \end{aligned}$	
(b) $\begin{aligned} \frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} &= 3 \cot x \\ \Rightarrow \tan x &= 3 \cot x \\ \Rightarrow \tan x &= \frac{3}{\tan x} \\ \Rightarrow \tan^2 x &= 3 \\ \Rightarrow \tan x &= \pm \sqrt{3} \end{aligned}$	$\left. \begin{array}{l} \bullet \arctan(\sqrt{3}) = 60^\circ \\ \bullet \arctan(-\sqrt{3}) = -60^\circ \\ \bullet x = 60^\circ \pm 180^\circ \\ \bullet x = 60^\circ \pm 180^\circ \quad x = 60^\circ, 120^\circ, 300^\circ \end{array} \right\}$

Question 121 (**)**

It is given that

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

- a) Prove the validity of the above trigonometric identity, by using the compound angle identities for $\sin(A+B)$ and $\sin(A-B)$.
- b) Hence, or otherwise, solve the trigonometric equation

$$\sin \theta - \sin 3\theta + \sin 5\theta = 0, \quad 0^\circ \leq \theta \leq 180^\circ.$$

, $\theta = 0^\circ, 30^\circ, 60^\circ, 120^\circ, 150^\circ, 180^\circ$

(a) $\sin(A+B) = \sin A \cos B + \cos A \sin B$ } by addition
 $\sin(A-B) = \sin A \cos B - \cos A \sin B$ }

$$\begin{aligned} &\sin(A+B) + \sin(A-B) = 2 \sin A \cos B \quad (\cancel{\text{A}}) \\ &\text{Let } A+B = P \quad \text{ADD } A+B = 2A \quad \text{SUBTRACT } A-B = 2B \\ &A+B = P \quad A = \frac{P+Q}{2} \quad 2B = P-Q \\ &\text{Hence (a) becomes} \quad B = \frac{P-Q}{2} \\ &\sin P + \sin Q = 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right) \quad \text{AS Required} \end{aligned}$$

(b) $\sin \theta - \sin 3\theta + \sin 5\theta = 0$
 $\rightarrow \sin \theta + \sin 5\theta - \sin 3\theta = 0$
 $\rightarrow 2 \sin\left(\frac{6\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) - \sin 3\theta = 0$
 $\rightarrow 2 \sin 3\theta \cos(2\theta) - \sin 3\theta = 0 \quad \boxed{\cos 2\theta \cos(-\lambda) = \cos \lambda}$
 $\rightarrow \sin 3\theta [2\cos 2\theta - 1] = 0$
 $\bullet \sin 3\theta = 0 \quad \bullet \cos 2\theta = \frac{1}{2}$
 $\sin 3\theta(0) = 0 \quad \cos 2\theta(0) = 0$
 $3\theta = 0 \pm 360^\circ \quad n=0,1,2,3 \quad 2\theta = 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ, 360^\circ$
 $3\theta = 180^\circ, 360^\circ, 540^\circ, 720^\circ \quad n=1,2,3 \quad \theta = 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ, 360^\circ$
 $\theta = 0 \pm 120^\circ \quad \theta = 0^\circ, 120^\circ, 240^\circ, 360^\circ$
 $\theta = 60^\circ \pm 120^\circ \quad \rightarrow \theta = 0^\circ, 120^\circ, 240^\circ, 360^\circ, 180^\circ, 30^\circ, 150^\circ, 210^\circ, 330^\circ$
 $\rightarrow \theta = 0^\circ, 30^\circ, 60^\circ, 120^\circ, 150^\circ, 180^\circ, 210^\circ, 240^\circ, 300^\circ, 330^\circ, 150^\circ, 30^\circ, 120^\circ, 60^\circ, 30^\circ, 0^\circ$

Question 122 (***)

It is given that

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x$$

- a) Prove the validity of the above trigonometric identity

b) Hence, or otherwise, solve for $0 \leq x < 180^\circ$

$$\frac{\cos 6x}{\sin 3x} + \frac{\sin 6x}{\cos 3x} = 2$$

$$x = 10^\circ, 50^\circ, 130^\circ, 170^\circ$$

$$\begin{aligned}
 \textcircled{4} \quad & \frac{\cos 2x + \sin 2x}{\sin x - \cos x} = \frac{\cos 2x \cos x + \sin 2x \sin x}{\sin x \cos x - \cos x \sin x} = \frac{\cos(2x-3)}{\sin(x-2)} \\
 & = \frac{\cos x}{\sin x} = \cot x = RHS \\
 \text{Aufführung: } & \frac{\cos 2x + \sin 2x}{\sin x - \cos x} = \frac{1 - 2 \sin^2 x + 2 \sin x \cos x}{\sin x - \cos x} = \frac{1 - 2 \sin^2 x}{\sin x - \cos x} + \frac{2 \sin x \cos x}{\sin x - \cos x} \\
 & = \frac{1}{\sin x - \cos x} - 2 \sin x = \cot x - 2 \sin x + 2 \sin x = \cot x = RHS
 \end{aligned}$$

Question 123 (***)

Solve the following trigonometric equation

$$\sin 4y = \sin 2y, \quad 0^\circ \leq y < 180^\circ$$

$$y = 0^\circ, 30^\circ, 90^\circ, 150^\circ$$

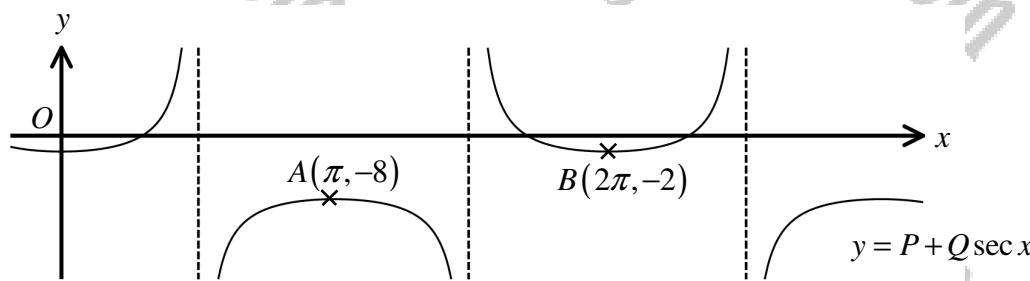
$$\begin{aligned}
 & \text{Simplif. } \sin y = \sin 2y \\
 \Rightarrow & \sin(2x+2y) = \sin 2y \\
 \Rightarrow & 2\sin 2y \cos 2y = \sin 2y \\
 \Rightarrow & 2\sin 2y \cos 2y - \sin 2y = 0 \\
 \Rightarrow & \sin 2y [2\cos 2y - 1] = 0
 \end{aligned}
 \quad \left\{
 \begin{array}{l}
 \sin 2y = 0 \\
 2\cos 2y - 1 = 0
 \end{array}
 \right.
 \quad \left\{
 \begin{array}{l}
 \cos 2y = \frac{1}{2} \\
 2y = \pi k + \frac{\pi}{3}
 \end{array}
 \right.
 \quad \left\{
 \begin{array}{l}
 y = \frac{\pi}{6} + \frac{\pi k}{3} \\
 y = \frac{2\pi}{3} + \frac{3\pi k}{3}
 \end{array}
 \right.$$

Koeffizienten
 $\begin{cases} 2y = 0 \\ 2y = 180^\circ \end{cases}$
 $\begin{cases} y = 0 \\ y = 90^\circ \end{cases}$
 $\begin{cases} y = 0 \\ y = 180^\circ \end{cases}$

y = 0, 90, 30|150°

AKTIVITÄT:
 $\sin y = \sin 2y$
 $\begin{cases} 4y = 2y + 360^\circ \\ 4y = (2y+2) \cdot 360^\circ \end{cases} \quad k=0,1,2,\dots$
 $\begin{cases} 2y = 0 \\ 2y = 180^\circ \end{cases}$
 $\begin{cases} y = 0 \\ y = 90^\circ \end{cases}$

Question 124 (****)



The figure above shows part of the curve with equation

$$y = P + Q \sec x,$$

where P and Q are non zero constants.

The curve has turning points at $A(\pi, -8)$ and $B(2\pi, -2)$.

Determine the value of P and the value of Q .

, $P = -5$, $Q = 3$

$$\begin{aligned} y &= P + Q \sec x \\ (\pi, -8) &\Rightarrow -8 = P + Q \sec \pi \quad | \quad -8 = P - Q \\ (2\pi, -2) &\Rightarrow -2 = P + Q \sec 2\pi \quad | \quad -2 = P + Q \\ 4Q &\Rightarrow 2P = -10 \qquad \text{SUBTRACT} \quad -6 = -2Q \\ P &= -5 \qquad \qquad \qquad Q = 3 \end{aligned}$$

ANALYSIS

- Secular exit between -1 & 1 , i.e. $A 2\pi^o$ of 2
- This creates a gap of Q . (From $x = \pi$ to $x = 2\pi$) so it must have been stretched by factor of 3 in the x direction
- But this means it should have a gap between π & 2π $\rightarrow 6$
- But it has a gap between -8 & -2 , so it must have been translated by 5 units down

$$\therefore y = -5 + 3 \sec x$$

Question 125 (****)

$$\cot A = -\frac{3}{4} \quad \text{and} \quad \cos B = \frac{5}{13}.$$

If A is reflex and B is also reflex, show that

$$\tan(A+B) = \frac{56}{33}.$$

, proof

USE THE INFORMATION GIVEN AS FOLLOWS:

$\cot A = -\frac{3}{4}$

$\tan A = -\frac{3}{4}$

$\cos B = \frac{5}{13}$

But "B is reflex" A positive angle

so $270^\circ < B < 360^\circ$ A11D

so tangent in this range will be negative

$\therefore \tan B = -\frac{12}{5}$

USING THE COMPOUND TANGENT IDENTITY

$$\begin{aligned}\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{-\frac{3}{4} + -\frac{12}{5}}{1 - (-\frac{3}{4})(-\frac{12}{5})} = \frac{-\frac{45}{20} - \frac{48}{20}}{1 - \frac{48}{15}} \\ &= \frac{-93}{15 - 48} = \frac{-93}{-33} = \frac{31}{11} \quad \text{A11D}\end{aligned}$$

Question 126 (**)**

Prove the validity of each of the following trigonometric identities.

a) $\frac{\cot x}{\cosec x - 1} - \frac{\cosec x - 1}{\cot x} \equiv 2 \tan x .$

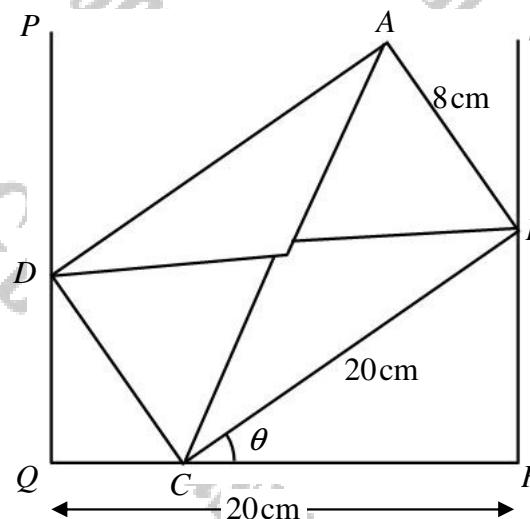
b) $\frac{\sin 3\theta}{\cos \theta} + \frac{\cos 3\theta}{\sin \theta} \equiv 2 \cot 2\theta .$

[proof]

$$\begin{aligned} \text{(a)} \quad & \text{LHS} = \frac{\cot x}{\cosec x - 1} - \frac{\cosec x - 1}{\cot x} = \frac{\cot^2 x - (\cosec x - 1)^2}{\cot x (\cosec x - 1)} = \frac{\cot^2 x - (\cosec^2 x - 2\cosec x + 1)}{\cot x (\cosec x - 1)} \\ & = \frac{\cot^2 x - \cosec^2 x + 2\cosec x - 1}{\cot x (\cosec x - 1)} = \frac{(\cosec x - 1) - \cosec^2 x + 2\cosec x - 1}{\cot x (\cosec x - 1)} = \frac{3\cosec x - 2}{\cot x (\cosec x - 1)} \\ & = \frac{2(\cosec x - 1)}{\cot x (\cosec x - 1)} = \frac{2}{\cot x} = 2\tan x = \text{RHS} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & \text{LHS} = \frac{\sin 3\theta}{\cos \theta} + \frac{\cos 3\theta}{\sin \theta} = \frac{\sin 3\theta \sin \theta + \cos 3\theta \cos \theta}{\cos \theta \sin \theta} \leftarrow \text{Use formula for } \cos(A-B) \right. \\ & = \frac{\cos(3\theta - \theta)}{\cos \theta \sin \theta} = \frac{\cos 2\theta}{\cos \theta \sin \theta} = \frac{2\cos^2 \theta}{2\cos \theta \sin \theta} = \frac{2\cos \theta}{\sin \theta} = 2\tan \theta = \text{RHS} \end{aligned}$$

Question 127 (****)



The figure above shows the cross section of a letter inside a filling slot.

The letter $ABCD$ is modelled as a rectangle with $|AB|=8\text{cm}$ and $|BC|=20\text{cm}$.

The width of the filling slot QR is also 20 cm and the angle BCR is θ .

- a) Show clearly that

$$5\cos\theta + 2\sin\theta = 5.$$

- b) Express $5\cos\theta + 2\sin\theta$ in the form $R\cos(\theta - \alpha)$, $R > 0$, $0 < \alpha < 90^\circ$.

- c) Hence, determine the value of θ .

 , $5\cos\theta + 2\sin\theta = \sqrt{29}\cos(\theta - 21.8^\circ)$, $\theta \approx 43.6^\circ$

(a)

$$\begin{aligned} 2+3 &= 5 \\ (2\sin\theta)+(3\cos\theta) &= 5 \\ 2\sin\theta + 3\cos\theta &= 5 \end{aligned}$$

As required

(b)

$$\begin{aligned} 2\sin\theta + 3\cos\theta &\equiv R\cos(\theta - \alpha) \\ &\equiv R(\cos\alpha)\cos\theta + R(\sin\alpha)\sin\theta \\ R\cos\alpha &= 2 \\ R &= \sqrt{5^2 + 3^2} = \sqrt{34} \\ \tan\alpha &= \frac{3}{2} \Rightarrow \alpha \approx 21.8^\circ \\ \therefore 2\sin\theta + 3\cos\theta &= R\cos(\theta - 21.8^\circ) \end{aligned}$$

(c)

$$\begin{aligned} 2\sin\theta + 3\cos\theta &= 5 \\ \frac{\sqrt{34}}{5}\cos(\theta - 21.8^\circ) &= 5 \\ \cos(\theta - 21.8^\circ) &= \frac{5}{\sqrt{34}} \\ \cos(\frac{\theta - 21.8}{\sqrt{34}}) &= 21.8^\circ \end{aligned}$$

$\theta - 21.8^\circ = 21.8^\circ \pm 360^\circ$	$\theta = 21.8^\circ + 360^\circ = 381.8^\circ$
$\theta - 21.8^\circ = 338.2^\circ \pm 360^\circ$	$\theta = 338.2^\circ + 360^\circ = 698.2^\circ$
$\theta = 43.6^\circ \pm 360^\circ$	$\theta = 360^\circ \pm 43.6^\circ$
$\theta = 43.6^\circ$	$\therefore \theta = 43.6^\circ$

Question 128 (**)**

Solve the following trigonometric equation

$$\operatorname{cosec}^4 \theta - \cot^4 \theta = \frac{2}{3} + \sqrt{3} \cot \theta, \quad 0 \leq \theta < 2\pi.$$

, $\theta \approx 1.05^\circ, 1.29^\circ, 4.19^\circ, 4.43^\circ$

USING THE IDENTITY $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta \iff \operatorname{cosec}^2 \theta - \cot^2 \theta = 1$

$$\begin{aligned} \rightarrow (\operatorname{cosec}^2 \theta - \cot^2 \theta) = \frac{2}{3} + \sqrt{3} \cot \theta & \quad \text{DIFERENCE OF} \\ \rightarrow (\operatorname{cosec}^2 \theta - \cot^2 \theta) (\operatorname{cosec}^2 \theta + \cot^2 \theta) = \frac{2}{3} + \sqrt{3} \cot \theta & \quad \text{SUM} \\ \rightarrow 1 = \frac{2}{3} + \sqrt{3} \cot \theta & \\ \rightarrow \operatorname{cosec}^2 \theta + \cot^2 \theta = \frac{2}{3} + \sqrt{3} \cot \theta & \\ \rightarrow 1 + \cot^2 \theta + \cot^2 \theta = \frac{2}{3} + \sqrt{3} \cot \theta & \\ \rightarrow 1 + 2\cot^2 \theta = \frac{2}{3} + \sqrt{3} \cot \theta & \\ \rightarrow 2\cot^2 \theta + 1 = \frac{2}{3} + \sqrt{3} \cot \theta & \\ \rightarrow 6\cot^2 \theta + 3 = 2 + 3\sqrt{3} \cot \theta & \\ \rightarrow 6\cot^2 \theta - 3\sqrt{3} \cot \theta + 1 = 0 & \end{aligned}$$

QUADRATIC FORMULA METHOD

$$\cot \theta = \frac{-3\sqrt{3} \pm \sqrt{27 - 48 + 1}}{2 \times 6} = \frac{3\sqrt{3} \pm 6}{12} = \begin{cases} \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \end{cases}$$

$$\therefore \tan \theta = \begin{cases} \frac{\sqrt{3}}{2} = 45^\circ \\ \frac{3\sqrt{3}}{2} = 120^\circ \end{cases}$$

WORKING 2: SETTING & SOLVING W/ 2ADDITIONS

- $\tan \theta = \sqrt{3}$
- $\theta = \frac{\pi}{3} \pm n\pi$ $n = 0, 1, 2, \dots$
- $\theta = 285.7^\circ \pm n180^\circ$ $n = 0, 1, 2, \dots$

$\theta_1 = 1.05^\circ \left(\frac{\pi}{3}\right)$
 $\theta_2 = 4.19^\circ \left(\frac{\pi}{3}\right)$
 $\theta_3 = 1.29^\circ$
 $\theta_4 = 4.43^\circ$

Question 129 (****)

$$\frac{\cos \theta \cos 2\theta}{\cos \theta + \sin \theta} = \frac{1}{2}, \quad 0 \leq x < 2\pi.$$

Given that $\cos \theta + \sin \theta \neq 0$, find the solutions of the above trigonometric equation, giving the answers in radians in terms of π .

_____ , $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

MULTIPLY ACROSS & TRY

$$\begin{aligned} \Rightarrow 2\cos \theta \cos 2\theta &= \cos \theta + \sin \theta \\ \Rightarrow 2\cos \theta (\cos^2 \theta - \sin^2 \theta) &= \cos \theta + \sin \theta \quad \text{difference of squares} \\ \Rightarrow 2\cos^2 \theta (\cos \theta - \sin \theta)(\cos \theta + \sin \theta) &= \cos \theta + \sin \theta \\ \Rightarrow 2\cos^2 \theta (\cos \theta - \sin \theta) &\neq 0, \quad \text{WE MAY DIVIDE IT THROUGH} \\ \Rightarrow 2\cos^2 \theta (\cos^2 \theta - \sin^2 \theta) &= 1 \\ \Rightarrow 2\cos^2 \theta - 2\cos^2 \sin^2 \theta &= 1 \\ \Rightarrow 2(\frac{1}{2}(1+\cos 2\theta)) - \sin 2\theta &= 1 \quad \text{cancel } 2 \text{ from } 2\cos^2 \theta \\ \Rightarrow \sqrt{1+\cos 2\theta} - \sin 2\theta &= 1 \\ \Rightarrow \cos 2\theta = \sin 2\theta & \quad \text{Divide by } \cos 2\theta \\ \Rightarrow 1 = \tan 2\theta & \\ \bullet 2\theta = \frac{\pi}{4} \pm n\pi & \\ \bullet \theta = \frac{\pi}{8} \pm \frac{n\pi}{2} & \\ \therefore \theta = \underline{\underline{\frac{\pi}{8} + \frac{n\pi}{2}}}, \underline{\underline{\frac{3\pi}{8}}} & \end{aligned}$$

Question 130 (****)

Solve in degrees the following trigonometric equation

$$\tan 2x + \tan x = 0, \quad 0 \leq x < 360^\circ.$$

$x = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$

$$\begin{aligned} \tan 2x + \tan x &= 0 \\ \tan 2x &= -\tan x \\ \Rightarrow \frac{2T}{1-T^2} + T &= 0 \quad (T = \tan x) \\ \Rightarrow 2T + T(1-T^2) &= 0 \\ \Rightarrow 2T + T - T^3 &= 0 \\ \Rightarrow 3T - T^3 &= 0 \\ \Rightarrow T(3 - T^2) &= 0 \\ \Rightarrow T(3 - T^2) &= 0 \\ \Rightarrow \{ \tan x = 0 \} \text{ or } \tan x = 3 & \quad \text{or } \tan x = -3 \\ \{ x = 0^\circ \pm 180n^\circ \} \text{ or } x &= 56.57^\circ, 143.43^\circ, 230.57^\circ, 317.43^\circ \\ x = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ & \quad n = 0, 1, 2, 3, \dots \\ x = 0^\circ & \\ x = 60^\circ & \\ x = 120^\circ & \\ x = 180^\circ & \\ x = 240^\circ & \\ x = 300^\circ & \end{aligned}$$

Question 131 (**)**

It is given that

$$2 \cot 2x + \tan x \equiv \cot x.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence solve the following trigonometric equation

$$\cot x - \tan x = \frac{1}{2} \tan 2x, \quad 0^\circ \leq x < 180^\circ.$$

$$x \approx 31.7^\circ, 58.3^\circ, 121.7^\circ, 148.3^\circ$$

(a) LHS = $2\cot 2x + \tan x = \frac{2\cos 2x}{\sin 2x} + \frac{\sin x}{\cos x} = \frac{2(\cos^2 x - \sin^2 x)}{2\sin x \cos x} + \frac{\sin x}{\cos x}$

OR $\frac{\cos^2 x - \sin^2 x}{\sin x \cos x} + \frac{\sin x}{\cos x} = \cot^2 x - \tan^2 x + \tan x = 2\cot x$

$\cot x - \tan x = \frac{1}{2} \tan 2x$

$\cot x - \tan x = \cot x = 2\cot x$

(b) $\cot x - \tan x = \frac{1}{2} \tan 2x$

$\Rightarrow 2\cot 2x = \frac{1}{2} \tan 2x$ (Cancelling)

$\Rightarrow \frac{2}{\tan 2x} = \frac{\tan 2x}{2}$

$\Rightarrow \tan^2 2x = 4$

$\Rightarrow \tan 2x = \pm 2$

$\Rightarrow \arctan(2) = 63.43^\circ$
 $\Rightarrow \arctan(-2) = -63.43^\circ$

$\bullet 2x = 63.43^\circ + 180^\circ$
 $\bullet 2x = -63.43^\circ + 180^\circ$

$\bullet x = 31.7^\circ \pm 90^\circ$
 $\bullet x = -31.7^\circ \pm 90^\circ$

$x = 31.7^\circ, 121.7^\circ, 58.3^\circ, 148.3^\circ$

Question 132 (**)**

If $\cot \theta = 2$, use the tangent double angle identity to show

$$\tan \theta \cot 2\theta \tan 4\theta = -\frac{9}{7}.$$

You must show detailed workings in this question

[] , proof

$$\begin{aligned} \tan \theta \cot 2\theta \tan 4\theta &= \tan \theta \frac{1}{\cot 2\theta} \frac{2\tan 2\theta}{1-\tan^2 2\theta} \quad [\cot 2\theta = \frac{2\tan \theta}{1-\tan^2 \theta}] \\ &= \frac{2\tan \theta}{1-\tan^2 2\theta} = \frac{2\tan \theta}{1-\left(\frac{2\tan \theta}{1-\tan^2 \theta}\right)^2} = \dots \text{ BUT } \cot \theta = 2 \\ &= \dots \frac{2 \times \frac{1}{2}}{1-\left(\frac{2 \times \frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right)^2} = \frac{1}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} = -\frac{9}{7} \end{aligned}$$

Question 133

(*****)

It is given that

$$(\sec \theta - \cos \theta)(\operatorname{cosec} \theta - \sin \theta) \equiv \sin \theta \cos \theta.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence solve the trigonometric equation

$$(\sec \theta - \cos \theta)(\operatorname{cosec} \theta - \sin \theta) = -\frac{1}{4}, \quad 0^\circ \leq \theta < 360^\circ.$$

$$\theta = 105^\circ, 165^\circ, 285^\circ, 345^\circ$$

$$\begin{aligned} \text{(Q)} \quad & \frac{1}{4} = (\sec \theta - \cos \theta)(\operatorname{cosec} \theta - \sin \theta) = \left(\frac{1}{\cos \theta} - \cos \theta\right)\left(\frac{1}{\sin \theta} - \sin \theta\right) \\ & = \frac{1 - \cos^2 \theta}{\cos \theta} \times \frac{1 - \sin^2 \theta}{\sin \theta} = \frac{\sin^2 \theta}{\cos \theta} \times \frac{\cos^2 \theta}{\sin \theta} = \sin \theta \cos \theta = \frac{1}{4} \\ \text{(L)} \quad & (\sec \theta - \cos \theta)(\operatorname{cosec} \theta - \sin \theta) = -\frac{1}{4} \\ & \Rightarrow \sin \theta \cos \theta = -\frac{1}{4} \\ & \Rightarrow 2 \sin \theta \cos \theta = -\frac{1}{2} \\ & \Rightarrow \sin 2\theta = -\frac{1}{2} \\ & \Rightarrow \sin \theta = -\frac{1}{2} \\ & \Rightarrow \theta = 105^\circ, 345^\circ, 165^\circ, 285^\circ \end{aligned}$$

Question 134 (*****)The three angles of a triangle are denoted by α , β and γ .

Show clearly that ...

$$\text{a) } \dots \sin(\alpha + \beta) = \sin \gamma.$$

$$\text{b) } \dots \sin\left(\frac{\alpha + \beta}{2}\right) = \cos\left(\frac{\gamma}{2}\right).$$

proof

$$\begin{aligned} \text{(Q) (G)} \quad & \alpha + \beta + \gamma = \pi \\ & \alpha + \beta = \pi - \gamma \\ & \sin(\alpha + \beta) = \sin(\pi - \gamma) \\ & \sin(\alpha + \beta) = \sin(\pi) \cos \gamma - \cos(\pi) \sin \gamma \\ & \sin(\alpha + \beta) = \sin \gamma \\ \text{(Q) (T)} \quad & \alpha + \beta + \gamma = \pi \\ & \gamma + \beta = \pi - \alpha \\ & \frac{\gamma + \beta}{2} = \frac{\pi - \alpha}{2} \\ & \sin\left(\frac{\gamma + \beta}{2}\right) = \sin\left(\frac{\pi - \alpha}{2}\right) \\ & \sin\left(\frac{\gamma + \beta}{2}\right) = \sin\frac{\pi}{2} \cos\frac{\alpha}{2} - \cos\frac{\pi}{2} \sin\frac{\alpha}{2} \\ & \sin\left(\frac{\gamma + \beta}{2}\right) = \cos\frac{\alpha}{2} \end{aligned}$$

Question 135 (****)

$$f(x) = \sqrt{3} \sin x + \cos x, \quad x \in \mathbb{R}.$$

- a) Express $f(x)$ in the form $R \cos(x - \alpha)$, $R > 0$, $0 < \alpha < 90^\circ$.
- b) State the maximum value of $f(x)$ and find the smallest positive value of x for which this maximum occurs.

The temperature of the water T °C in a tropical fish tank is modelled by the equation

$$T = 32 + \sqrt{3} \sin(15t)^\circ + \cos(15t)^\circ, \quad 0 \leq t < 24,$$

where t is the time in hours measured since midnight.

- c) State the maximum temperature of the water in the tank and the time when this maximum temperature occurs.
- d) Show that the temperature of the water in the tank reaches 30.5 °C at 13:14 and at 18:46.

[You may not verify the answers in this part]

	$\sqrt{3} \sin x + \cos x \equiv 2 \cos(x - 60^\circ)$	$\max = 2$	$x = 60^\circ$	$T_{\max} = 34$	
					04:00

a) USING THE COMPOUND ANGLE IDENTITY FOR $\cos(A-B)$

$$\begin{aligned} \sqrt{3} \sin x + \cos x &\equiv 2 \cos(2-x) \\ &\equiv 2 \cos(\cos x + 2 \sin x) \\ &\equiv [2 \cos x] \cos 2 + [\cos x] \sin 2 \end{aligned}$$

$\therefore R \cos x = \sqrt{3}$ $\therefore R = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

DIVIDING SIDE BY SIDE: $\frac{\sqrt{3} \sin x}{\sqrt{3}} = \frac{2 \cos(2-x)}{2}$

$$\therefore f(x) = 2 \cos(2-x)$$

ALTERNATIVE BY MANIPULATION

$$\begin{aligned} \sqrt{3} \sin x + \cos x &= 2 \left[\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x \right] = 2 \left[\frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x \right] \\ &= 2 \left[\cos(2-x) + \sin(2-x) \right] = 2 \cos(2-x) \end{aligned}$$

b) MAX VALUE OF $f(x) = 2 \cos(2-x)$ IS 2
 $(2x - 2 \leq 90^\circ \leq 2)$

TO GET THIS MAX VALUE OF 2, $\cos(2-x) = 1$

$$\begin{aligned} \cos(2-x) &= 1 \\ 2-x &= 0 \quad (\text{REVERSE CANT}) \\ x &= 2 \end{aligned}$$

c) INCORPORATING THE SIMILARITY/ANALOGY TO PART (a) & (b)

$$\begin{aligned} T(t) &= 32 + 2 \cos(15t - 60) \\ \therefore T_{\max} &= 32 + 2 \\ T_{\max} &= 34 \end{aligned}$$

Q. 4 MARKS PART (b)

$\frac{1}{15} = \frac{1}{60} \Rightarrow t = 4$

It 04:00

d) SOLVING THE EQUATION $T = 30.5$

$$\begin{aligned} \Rightarrow 30.5 &= 32 + \sqrt{3} \sin(15t) + \cos(15t) \\ \Rightarrow -1.5 &= 2 \cos(15t - 60) \\ \Rightarrow \cos(15t - 60) &= -0.75 \end{aligned}$$

OPTION C $\cos(2-x) = 138.5^\circ$

$$\begin{aligned} \Rightarrow 15t - 60 &= 138.5^\circ \pm 360^\circ \\ \Rightarrow 15t - 60 &= 226.4^\circ \pm 360^\circ \quad n = 0, 1, 2, 3, \dots \end{aligned}$$

$\Rightarrow 15t = 166.5^\circ \pm 360^\circ$

$$\begin{aligned} \Rightarrow 15t &= 206.4^\circ \pm 360^\circ \\ \Rightarrow t &= 13.76 \pm 24 \end{aligned}$$

$\therefore t_1 = 13.76 \quad \therefore 13:14$

$\therefore t_2 = 18.76 \quad \therefore 18:46$

$\leftarrow 0.24 \times 60 = 14.4$

$\leftarrow 0.76 \times 60 = 45.6$

Question 136 (***)

It is given that

$$\tan \theta (1 + \sec 2\theta) \equiv \tan 2\theta$$

- a)** Prove the validity of the above trigonometric identity

b) Hence, or otherwise, solve for $0 \leq \theta < 180^\circ$

$$\tan \theta(1 + \sec 2\theta) = 4 \tan \theta$$

$$\theta = 0^\circ \quad \theta \approx 35.3^\circ, 144.7^\circ$$

Question 137 (***)

By twice applying the identity

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

solve the trigonometric equation

$$\sin x \cos x \cos 2x = \frac{1}{8}, \quad 0 \leq x < \pi$$

$$x = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}$$

$$\begin{aligned}
 & \Rightarrow \sin(\cos x) \cos(2x) = \frac{1}{8} \\
 & \Rightarrow 2\sin(\cos x) \cos(2x) = \frac{1}{4} \\
 & \Rightarrow \sin(2x) \cos(2x) = \frac{1}{4} \\
 & \Rightarrow 2\sin(2x) \cos(2x) = \frac{1}{2} \\
 & \Rightarrow \sin(4x) = \frac{1}{2}
 \end{aligned}
 \quad \left. \begin{array}{l} \text{arcsin}(\frac{1}{2}) = \frac{\pi}{6} \\ \left(\begin{array}{l} 4x = \frac{\pi}{6} + 2k\pi \\ 4x = \frac{5\pi}{6} + 2k\pi \end{array} \right) \\ \left(\begin{array}{l} x = \frac{\pi}{24} + \frac{k\pi}{4} \\ x = \frac{5\pi}{24} + \frac{k\pi}{4} \end{array} \right) \end{array} \right\} \quad \begin{array}{l} \text{for } k=0, 1, 2, \dots \end{array}$$

Question 138 (**)**

$$f(x) = 2.5 \sin 2x + 6 \cos 2x, \quad 0 < x < 2\pi.$$

- a) Express $f(x)$ in the form $R \sin(2x + \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

- b) Determine the value of the constant A so that

$$5 \sin x \cos x + 12 \cos^2 x \equiv f(x) + A.$$

- c) Hence, or otherwise, find the minimum and the maximum value of

$$5 \sin x \cos x + 12 \cos^2 x.$$

$f(x) = 6.5 \sin(2x + 1.176^\circ)$	$A = 6$	$\max = 12.5$	$\min = -0.5$
-------------------------------------	---------	---------------	---------------

(a) $f(x) = 2.5 \sin 2x + 6 \cos 2x \equiv R \sin(2x + \alpha)$
 $\equiv R \sin 2x \cos \alpha + R \cos 2x \sin \alpha$
 $\equiv (R \cos \alpha) \sin 2x + (R \sin \alpha) \cos 2x$

$\begin{cases} R \cos \alpha = 2.5 \\ R \sin \alpha = 6 \end{cases}$ • $R = \sqrt{2.5^2 + 6^2} = 6.5$
 $\tan \alpha = \frac{6}{2.5} \Rightarrow \alpha = 1.176^\circ$
 $\therefore f(x) = 6.5 \sin(2x + 1.176^\circ)$

(b) $5 \sin x \cos x + 12 \cos^2 x = \frac{5}{2} (2 \sin x \cos x) + 6 (2 \cos^2 x)$
 $= \frac{5}{2} \sin 2x + 6 (\cos 2x + 1)$
 $= \frac{5}{2} \sin 2x + 6 \cos 2x + 6$
 $= f(x) + 6 \quad \therefore A = 6$

Arithmetik

$\begin{aligned} 5 \sin x \cos x + 12 \cos^2 x &\equiv f(x) + A \\ 5 \sin x \cos x + 12 \cos^2 x &\equiv 2.5 \sin 2x + 6 \cos 2x + A \\ 5 \sin x \cos x + 12 \cos^2 x &\equiv 2.5 (2 \sin x \cos x) + (6 \cos 2x - 7) + A \\ 5 \sin x \cos x + 12 \cos^2 x &\equiv 5 \sin 2x + 12 \cos 2x \quad \text{Koeffiz. einsetzen} \\ &\therefore A = 6 \end{aligned}$

(c) $5 \sin x \cos x + 12 \cos^2 x = f(x) + 6$
 $= (2.5 \sin 2x + 6 \cos 2x) + 6$
 $= 6.5 \sin(2x + 1.176^\circ) + 6$

$\therefore \max = 6.5 + 6 = 12.5$
 $\min = -6.5 + 6 = -0.5$

Question 139 (**)**

Find the **two** solutions of the trigonometric equation

$$(1 + \sec y)(1 - \cos y) = \tan y, \quad 0 \leq y < 2\pi,$$

giving the answers in terms of π .

$$y = 0, \pi$$

$$\begin{aligned} (1 + \sec y)(1 - \cos y) &= \tan y \\ 1 - \cos y + \sec y - \sec y \cos y &= \tan y \\ 1 - \cos y + \sec y - 1 &= \tan y \\ \sec y - \cos y &= \tan y \\ \frac{1}{\cos y} - \cos y &= \frac{\sin y}{\cos y} \\ 1 - \cos^2 y &= \sin y \\ 1 - (1 - \sin^2 y) &= \sin y \\ 1 - 1 + \sin^2 y &= \sin y \\ \sin^2 y - \sin y &= 0 \\ \sin y (\sin y - 1) &= 0 \\ \sin y &= 0 \text{ or } 1 \end{aligned}$$

$\therefore y = 0, \pi$ ONLY
 $\frac{\pi}{2}$ NOT A SOLUTION
 $\therefore \sec y \tan y$ NOT defined

Question 140 (**)**

Solve each of the following trigonometric equations.

i. $\frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} - 2 = \cot y, \quad 0 < y < 2\pi, \quad y \neq \pi.$

ii. $\cos 2\theta + 6 \cos \theta + 5 = 0, \quad 0 \leq \theta < 360^\circ.$

$$y \approx 2.03^\circ, 5.18^\circ, \quad \theta = 180^\circ$$

$$\begin{aligned} \text{(i)} \quad \frac{1 - 2 \operatorname{cosec}^2 y}{2 \cot y} - 2 &= \cot y \\ 1 - 2 \operatorname{cosec}^2 y - 4 \cot y &= 2 \cot y \\ 1 - 2(1 - \sin^2 y) - 4 \cot y &= 2 \cot y \\ 1 - 2 - 2 \sin^2 y - 4 \cot y &= 2 \cot y \\ 0 &= 4 \cot^2 y + 4 \cot y + 1 \\ 0 &= (2 \cot y + 1)^2 = 0 \\ \cot y &= -\frac{1}{2} \\ \tan y &= -2 \\ \arctan(-2) &= -1.072 \\ \therefore y &= -107^\circ \pm 180^\circ \quad n = 0, 1, 2, 3, \dots \\ y_1 &= 2.03^\circ \\ y_2 &= 5.18^\circ \end{aligned}$$

$\cos 2\theta + 6 \cos \theta + 5 = 0$
 $\Rightarrow (2\cos \theta - 1)(6\cos \theta + 5) = 0$
 $\Rightarrow 2\cos \theta + 5 = 0$
 $\Rightarrow \cos \theta = -\frac{5}{2}$
 $\cos(\theta) = -1$
 $\arccos(-1) = 180^\circ$
 $\theta = 180^\circ \pm 360^\circ \quad n = 0, 1, 2, 3, \dots$
 $\theta = 180^\circ \pm 360^\circ$
 $\therefore \theta = 180^\circ$

Question 141 (**)**

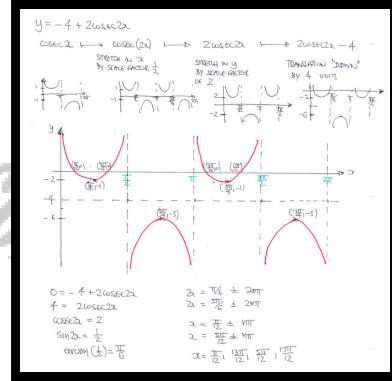
Sketch the graph of

$$y = -4 + 2 \operatorname{cosec} 2x, \quad 0 \leq x \leq 2\pi.$$

The sketch must include

- the equations of any asymptotes to the curve
- the exact coordinates of any stationary points.
- the exact coordinates of any points where the curve crosses the coordinate axes.

graph



Question 142 (*)**

Prove the validity of each of the trigonometric identities.

a) $\operatorname{cosec} \theta - \cot \theta \equiv \tan\left(\frac{\theta}{2}\right)$.

b) $\frac{2 \tan 2x}{\tan 2x - \sin 2x} \equiv \operatorname{cosec}^2 x$.

proof

(a)
$$\begin{aligned} LHS &= \operatorname{cosec} \theta - \cot \theta = \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} = \frac{1 - \cos \theta}{\sin \theta} = \frac{1 - (1 - 2\sin^2 \frac{\theta}{2})}{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}} \\ &= \frac{2\sin^2 \frac{\theta}{2}}{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \tan \frac{\theta}{2} = RHS \end{aligned}$$

$\operatorname{cosec} 2A = 1 - 2\sin^2 \frac{A}{2}$
 $\sin A = 2\sin \frac{A}{2} \cos \frac{A}{2}$
 $\sin 4 = 2\sin^2 \frac{4}{2}$

(b)
$$\begin{aligned} LHS &= \frac{2 \tan 2x}{\tan 2x - \sin 2x} = \frac{2 \left(\frac{\sin 2x}{\cos 2x} \right)}{\frac{\sin 2x}{\cos 2x} - \sin 2x} = \dots \quad \text{Analyze top & bottom By } \cos 2x \\ &= \frac{2 \sin 2x}{\sin 2x - \sin 2x \cos 2x} = \frac{2}{1 - \cos 2x} = \frac{2}{1 - (1 - 2\sin^2 x)} \\ &= \frac{2x}{2\sin^2 x} = \frac{1}{\sin^2 x} = \operatorname{cosec}^2 x = RHS \end{aligned}$$

Question 143 (****)

$$f(x) \equiv 27x^3 - 9x - 2, \quad x \in \mathbb{R}.$$

- a) Show that $(3x+1)$ is a factor of $f(x)$.

It is further given that

$$36\cos 2\theta \cos \theta + 9\sin 2\theta \sin \theta = 4.$$

- b) Find the possible values of $\cos \theta$.

$$\boxed{}, \quad \cos \theta = -\frac{1}{3}, \frac{2}{3}$$

$$\begin{aligned}
 \text{(a)} \quad & f(x) = 27x^3 - 9x - 2 \\
 & f(-\frac{1}{3}) = 27(-\frac{1}{3})^3 - 9(-\frac{1}{3}) - 2 = 27(-\frac{1}{27}) + 3 - 2 = 0 \\
 & \therefore (3x+1) \text{ is a factor of } f(x) // \\
 \text{(b)} \quad & 36\cos 2\theta \cos \theta + 9\sin 2\theta \sin \theta = 4 \\
 & = 2\cos \theta (2\cos^2 \theta + 9\sin \theta \sin \theta) = 4 \\
 & \Rightarrow 72\cos^3 \theta - 36\cos \theta + 18\cos \theta \sin^2 \theta = 4 \\
 & \Rightarrow 72\cos^3 \theta - 36\cos \theta + 18\cos \theta (1 - \cos^2 \theta) = 4 \\
 & \Rightarrow 72\cos^3 \theta - 36\cos \theta - 18\cos \theta = 4 \\
 & \Rightarrow 36\cos^3 \theta - 18\cos \theta = 4 \\
 & \Rightarrow 27\cos^3 \theta - 9\cos \theta - 2 = 0 \\
 & \text{Let } u = \cos \theta \\
 & \Rightarrow 27u^3 - 9u - 2 = 0 \\
 & \Rightarrow (3u+1)(9u^2 - 3u - 2) = 0 \\
 & \Rightarrow (3u+1)(3u+1)(3u-2) = 0 \\
 & \Rightarrow u = -\frac{1}{3} \quad \therefore \cos \theta = -\frac{1}{3} //
 \end{aligned}$$

Question 144 (**)**

It is given that

$$\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \cos 2\theta.$$

- a) Prove the validity of the above trigonometric identity.

- b) Given that $\cos 36^\circ = \frac{1+\sqrt{5}}{4}$, show clearly that

$$\tan^2 18^\circ = \frac{5-2\sqrt{5}}{5}.$$

proof

(a) $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \tan^2 \theta}{\frac{\sin^2 \theta}{\cos^2 \theta}} = \frac{1 - \tan^2 \theta}{\frac{\sin^2 \theta}{\cos^2 \theta}} =$
 $= \cos^2 \theta - \sin^2 \theta = \cos 2\theta = \frac{\sqrt{5}-1}{2} =$
 $= \cos 36^\circ - \sin^2 36^\circ = \cos 36^\circ = 1248$

(b) $\frac{1 - \tan^2 18^\circ}{1 + \tan^2 18^\circ} = \cos 36^\circ$ $\left\{ \begin{array}{l} \Rightarrow 3 - 4T^2 = (S + \sqrt{5})T \\ \Rightarrow T = \frac{3 - \sqrt{5}}{S + \sqrt{5}} \\ \Rightarrow T = \frac{3 - \sqrt{5}}{S + \sqrt{5}} \cdot \frac{S - \sqrt{5}}{S - \sqrt{5}} \\ \Rightarrow T = \frac{-15 - 8\sqrt{5}}{2S} \\ \Rightarrow T = \frac{20 - 8\sqrt{5}}{20} \\ \Rightarrow T = \frac{5 - 2\sqrt{5}}{5} \end{array} \right. \text{ as required}$

Question 145 (****)

$$f(\theta) \equiv 5\cos\theta - 12\sin\theta, \theta \in \mathbb{R}.$$

- a) Express $f(\theta)$ in the form $R\cos(\theta + \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

Give the value of α correct to 3 decimal places.

- b) State the maximum value of $f(\theta)$ and find the smallest positive value of θ for which this maximum occurs.

The pressure P , in suitable units, in a nuclear plant is modelled by the equation

$$P = 20 + 5\cos\left(\frac{4\pi t}{25}\right) - 12\sin\left(\frac{4\pi t}{25}\right), \quad 0 \leq t < 12,$$

where t is the time in hours measured from midnight.

- c) State the maximum pressure in the plant and the value of t when this maximum pressure occurs.

- d) Find the times, to the nearest minute, when $P = 15$.

	$5\cos\theta - 12\sin\theta \equiv 13\cos(\theta + 1.176^\circ)$	$\max = 13$	$\theta = 5.107^\circ$
	$P_{\max} = 33$	$t_{\max} = 10.16$	$01:34/06:15$

a) USING THE STANDARD METHOD

$$\begin{aligned} 5\cos\theta - 12\sin\theta &\equiv R\cos(\theta + \alpha) \\ 5\cos\theta - 12\sin\theta &\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha \\ 5\cos\theta - 12\sin\theta &\equiv (R\cos\alpha)\cos\theta - (R\sin\alpha)\sin\theta \end{aligned}$$

COMPARING & SOLVING

$$\begin{aligned} R\cos\alpha &= 5 \\ R\sin\alpha &= 12 \end{aligned}$$

Squaring & Adding yields

$$R^2 = \sqrt{5^2 + 12^2} = 13$$

Dividing the equations side by side

$$\begin{aligned} \frac{R\cos\alpha}{R\sin\alpha} &= \frac{5}{12} \\ \tan\alpha &= \frac{5}{12} \\ \alpha &\approx 1.176^\circ \end{aligned}$$

$\therefore f(\theta) \approx 13\cos(\theta + 1.176^\circ)$

b) FIND $f(\theta)_{\max}$

$$\begin{aligned} f(\theta)_{\max} &= 13 \\ \text{NEXT, TO GET A MAX OF } 13 & \\ \Rightarrow \cos(\theta + 1.176^\circ) &= 1 \\ \theta + 1.176^\circ &= 0 \\ \theta &= -1.176^\circ + 2\pi \\ \theta &= 5.107^\circ \end{aligned}$$

c) USING PARTS (a) & (b)

$$\begin{aligned} \rightarrow P &= 20 + 5\cos\left(\frac{4\pi t}{25}\right) - 12\sin\left(\frac{4\pi t}{25}\right) \\ \rightarrow P &= 20 + 13\cos\left(\frac{4\pi t}{25} + 1.176^\circ\right) \\ \therefore P_{\max} &= 20 + 13 = 33 \end{aligned}$$

a) FIND FROM PART (b)

$$\begin{aligned} \theta &= 5.107^\circ \\ \frac{4\pi t}{25} &= 5.107^\circ \\ t &\approx 10.16 \end{aligned}$$

d) SOLVING THE EQUATION $P = 15$

$$\begin{aligned} \Rightarrow 15 &= 20 + 13\cos\left(\frac{4\pi t}{25} + 1.176^\circ\right) \\ \Rightarrow -5 &= 13\cos\left(\frac{4\pi t}{25} + 1.176^\circ\right) \\ \Rightarrow \cos\left(\frac{4\pi t}{25} + 1.176^\circ\right) &= -\frac{5}{13} \quad [\arccos\left(-\frac{5}{13}\right) = 14.6558\ldots] \\ \Rightarrow \left(\frac{4\pi t}{25} + 1.176^\circ\right) &= 1.962\ldots \pm 2\pi n \\ \Rightarrow \left(\frac{4\pi t}{25} + 1.176^\circ\right) &= 4.376\ldots \pm 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \\ \Rightarrow \left(\frac{4\pi t}{25}\right) &= 0.91898\ldots \pm 2\pi n \\ \Rightarrow \frac{4\pi t}{25} &= -\pi \quad \pm 2\pi n \\ \Rightarrow t &= 1.5708 \pm \frac{25n}{4\pi} \\ \therefore t &= 6.25 \pm \frac{25n}{4\pi} \end{aligned}$$

$t =$ 

Question 146 (****)

It is given that

$$\frac{\tan 2x - \sin 2x}{\tan 2x} \equiv 2 \sin^2 x, \tan 2x \neq 0.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence find, in terms of π , the solutions of the trigonometric equation

$$\frac{\tan 2x - \sin 2x}{\tan 2x} = 1, \quad 0 \leq x < 2\pi.$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\begin{aligned} \text{(a)} \quad LHS &= \frac{\tan 2x - \sin 2x}{\tan 2x} = \frac{\tan 2x}{\tan 2x} - \frac{\sin 2x}{\tan 2x} = 1 - \frac{\sin 2x}{\cos 2x} \\ &= 1 - \frac{\sin 2x \cos 2x}{\cos 2x} = 1 - \cos 2x = 1 - (1 - 2\sin^2 x) \\ &= 2\sin^2 x = \text{RHS} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{\tan 2x - \sin 2x}{\tan 2x} &= 1 \\ \tan 2x - \sin 2x &= \tan 2x \\ 2\sin 2x &= 1 \\ \sin 2x &= \frac{1}{2} \\ \sin 2x &= \pm \frac{1}{2} \end{aligned}$$

$\bullet \sin(2x) = \frac{1}{2} \quad \bullet \sin(2x) = -\frac{1}{2}$
 $2x = \frac{\pi}{6} + 2k\pi \quad 2x = \frac{7\pi}{6} + 2k\pi$
 $x = \frac{\pi}{12} + k\pi \quad x = \frac{7\pi}{12} + k\pi$
 $(x = \frac{\pi}{12}, \frac{7\pi}{12})$
 $\therefore x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{19\pi}{12}, \frac{31\pi}{12}$

Question 147 (****)

If $\sin x = \frac{12}{13}$ and x is obtuse, show clearly that

$$\cot 2x = \frac{119}{120}.$$

[] , proof

COLLECT ALL THE INFORMATION FIRST

$\sin x = \frac{12}{13}$ \leftarrow AYP
 $\csc x = \frac{13}{12}$ \leftarrow AYP
 cosine rule gives normal value
 $\cos x = -\frac{5}{13}$

USING THE DOUBLE ANGLE FORMULAE

$$\begin{aligned} \cot 2x &= \frac{\cot x}{\sin x} = \frac{1 - 2\sin^2 x}{2\sin x \cos x} = \frac{1 - 2\left(\frac{144}{169}\right)}{2 \cdot \frac{12}{13} \cdot \left(-\frac{5}{13}\right)} = \frac{1 - \frac{288}{169}}{-\frac{120}{13}} = \frac{11}{120} \end{aligned}$$

Question 148 (***)

It is given that

$$\cos x + \sin x \tan 2x \equiv \frac{\cos x}{\cos 2x}$$

- a) Prove the validity of the above trigonometric identity.

b) Hence find, in terms of π , the solutions of the trigonometric equation

$$\cos x + \sin x \tan 2x = 1, \quad 0 \leq x < 2\pi$$

giving the answers in terms of π

$$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

Question 149 (****)

It is given that

$$\frac{\operatorname{cosec} x}{1+\operatorname{cosec} x} - \frac{\operatorname{cosec} x}{1-\operatorname{cosec} x} \equiv 2\sec^2 x, \quad \operatorname{cosec} x \neq \pm 1.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence find the solutions of the trigonometric equation

$$\frac{\operatorname{cosec} x}{1+\operatorname{cosec} x} - \frac{\operatorname{cosec} x}{1-\operatorname{cosec} x} = 5 \tan x, \quad 0 \leq x < 2\pi.$$

$$x \approx 0.464^\circ, 1.11^\circ, 3.61^\circ, 4.25^\circ$$

(a)

$$\begin{aligned} LHS &= \frac{\operatorname{cosec} x}{1+\operatorname{cosec} x} - \frac{\operatorname{cosec} x}{1-\operatorname{cosec} x} = \frac{\operatorname{cosec} x(1-\operatorname{cosec} x) - \operatorname{cosec} x(1+\operatorname{cosec} x)}{(1+\operatorname{cosec} x)(1-\operatorname{cosec} x)} \\ &= \frac{\operatorname{cosec} x - \operatorname{cosec}^2 x - \operatorname{cosec} x - \operatorname{cosec}^2 x}{1-\operatorname{cosec}^2 x} = \frac{-2\operatorname{cosec}^2 x}{1-\operatorname{cosec}^2 x} \\ &= \frac{-2\operatorname{cosec}^2 x}{-\sin^2 x} = \frac{2\operatorname{cosec}^2 x}{\sin^2 x} = \frac{2\operatorname{cosec}^2 x}{\frac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x}} = 2\operatorname{cosec}^2 x \\ &= 2\sec^2 x = R.H.S \end{aligned}$$

(b)

$$\begin{aligned} \frac{\operatorname{cosec} x}{1+\operatorname{cosec} x} - \frac{\operatorname{cosec} x}{1-\operatorname{cosec} x} &= 5 \tan x \\ \Rightarrow 2\operatorname{cosec} x &= 5 \tan x \\ \Rightarrow 2(1/\cos x) &= 5 \tan x \\ \Rightarrow 2/\cos x &= 5 \tan x \\ \Rightarrow 2/\cos x - 5 \tan x + 2 &= 0 \\ \Rightarrow (2/\cos x - 1)(\tan x - 2) &= 0 \end{aligned}$$

$\tan x = \frac{1}{2}$

$$\begin{aligned} \bullet \arctan(\frac{1}{2}) &= 0.464^\circ \\ \bullet \arctan(2) &= 1.107^\circ \\ 2 &= 0.464^\circ + n\pi \\ \therefore 2 &= 1.107^\circ \pm n\pi \quad n=0,1,2,3 \\ x_1 &= 0.464^\circ \\ x_2 &= 3.61^\circ \\ x_3 &= 1.11^\circ \\ x_4 &= 4.25^\circ \end{aligned}$$

Question 150 (****)

$$\cot A = -\frac{3}{4} \text{ and } \cos B = \frac{5}{13}.$$

If A is reflex and B is also reflex, show that

$$\tan(A+B) = \frac{56}{33}.$$

, proof

$$\begin{aligned} \tan B &= \frac{4}{3} \\ \tan A &= \frac{12}{5} \\ \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{-\frac{4}{3} + \frac{12}{5}}{1 - \left(-\frac{4}{3}\right)\left(\frac{12}{5}\right)} = \frac{-\frac{56}{15}}{-\frac{28}{15}} = \frac{56}{33} \end{aligned}$$

Question 151 (****)

Given that

$$\cos\left(x - \frac{\pi}{3}\right) = 2\sin\left(x + \frac{\pi}{3}\right),$$

show clearly that

$$\tan x = -4 - 3\sqrt{3}.$$

proof

$$\begin{aligned} \cos\left(x - \frac{\pi}{3}\right) &= 2\sin\left(x + \frac{\pi}{3}\right) \\ \cos x \cos \frac{\pi}{3} + \sin x \sin \frac{\pi}{3} &= 2\sin x \cos \frac{\pi}{3} + 2\cos x \sin \frac{\pi}{3} \\ \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x &= \sin x + \sqrt{3}\cos x \\ \cos x + \sqrt{3}\sin x &= 2\sin x + 2\sqrt{3}\cos x \\ \frac{\cos x}{\cos x} + \frac{\sqrt{3}\sin x}{\cos x} &= \frac{2\sin x}{\cos x} + \frac{2\sqrt{3}\cos x}{\cos x} \\ 1 + \sqrt{3}\tan x &= 2\tan x + 2\sqrt{3} \\ (2\sqrt{3} - 2)\tan x &= 2\sqrt{3} - 1 \\ \tan x &= \frac{2\sqrt{3} - 1}{2\sqrt{3} - 2} = \frac{(2\sqrt{3} - 1)(\sqrt{3} + 2)}{(2\sqrt{3} - 2)(\sqrt{3} + 2)} = \frac{6 + 3\sqrt{3} - 2}{4(4 - 3)} = -4 - 3\sqrt{3} \end{aligned}$$

Question 152 (**)**

Make x the subject of the equation

$$\arctan(1+x) + \arctan(1-x) = y.$$

$$x = \pm \sqrt{\frac{2}{\tan y}}$$

$$\begin{aligned}
 & \arctan(1+x) + \arctan(1-x) = y \\
 \Rightarrow & \tan[\arctan(1+x) + \arctan(1-x)] = \tan y \\
 \Rightarrow & \frac{(1+x)+(1-x)}{1-(1+x)(1-x)} = \tan y \\
 \Rightarrow & \frac{2}{1-2x} = \tan y \\
 \Rightarrow & \frac{2}{2x} = \tan y \\
 \Rightarrow & \frac{2}{\tan y} = x^2 \\
 \Rightarrow & x = \pm \sqrt{\frac{2}{\tan y}}
 \end{aligned}$$

Question 153 (**)**

It is given that

$$\sin(x+y)\sin(x-y) \equiv \cos^2 y - \cos^2 x.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence, show that

$$\sin\left(\frac{7\pi}{12}\right)\sin\left(\frac{\pi}{12}\right) = \frac{1}{4}.$$

proof

$$\begin{aligned}
 @) \quad & \text{LHS} = \sin(x+y)\sin(x-y) \\
 & = [\sin(x\cos y + y\cos x)] [\sin(x\cos y - y\cos x)] \\
 & \quad (\text{abc}(a-b) \equiv a^2 - b^2) \\
 & = \sin^2 x \cos^2 y - \cos^2 x \sin^2 y \\
 & = (1-\cos^2 x)\cos^2 y - \cos^2 x(1-\sin^2 y) \\
 & = \cos^2 y - \cos^2 x + \cos^2 x \sin^2 y \\
 & = \cos^2 y - \cos^2 x = \text{RHS}
 \end{aligned}$$

$$\begin{aligned}
 @) \quad & \text{Let } x = \frac{\pi}{3}, y = \frac{\pi}{4} \\
 & \rightarrow \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos^2 \frac{\pi}{4} - \cos^2 \frac{\pi}{3} \\
 & \rightarrow \sin\left(\frac{7\pi}{12}\right) \sin\left(\frac{\pi}{12}\right) = \frac{1}{2} - \frac{1}{4} \\
 & \rightarrow \sin\left(\frac{7\pi}{12}\right) \sin\left(\frac{\pi}{12}\right) = \frac{1}{4}
 \end{aligned}$$

Question 154 (**)**

It is given that

$$\frac{1}{\cos x - \sin x} + \frac{1}{\cos x + \sin x} \equiv \frac{2 \sec x}{1 - \tan^2 x}.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence solve the trigonometric equation

$$\frac{1}{\cos x - \sin x} + \frac{1}{\cos x + \sin x} = 2, \quad 0 < x < 2\pi,$$

giving the answers in terms of π .

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

(a) LHS = $\frac{1}{\cos x - \sin x} + \frac{1}{\cos x + \sin x}$
 $= \frac{(\cos x + \sin x) + (\cos x - \sin x)}{(\cos x - \sin x)(\cos x + \sin x)} = \frac{2\cos x}{\cos^2 x - \sin^2 x}$
 $= \frac{2\cos x}{\cos^2 x - \frac{\sin^2 x}{\cos^2 x}} = \frac{2\cos x}{1 - \tan^2 x} = \frac{2 \sec x}{1 - \tan^2 x} = RHS$

(b) $\frac{1}{\cos x - \sin x} + \frac{1}{\cos x + \sin x} = 2$ $\left\{ \begin{array}{l} \sec x < 1 \\ \cos x < -2 \\ \cos x = 0 \\ \cos(x + \frac{\pi}{2}) = \pm \sqrt{1 - \tan^2 x} \\ x = 0 \pm 2\pi n \text{ mod } 2\pi, \dots \\ x = 2\pi n \pm \frac{\pi}{2} \text{ mod } 2\pi, \dots \\ x = \frac{\pi}{2} \pm \frac{2n\pi}{3} \text{ mod } 2\pi, \dots \\ x = \frac{4\pi}{3} \pm \frac{2n\pi}{3} \text{ mod } 2\pi, \dots \\ \therefore x = \frac{4\pi}{3} \mp \frac{2n\pi}{3} \text{ mod } 2\pi, \dots \end{array} \right.$

Question 155 (****)

$$f(x) = \frac{6}{2\cos x + 2\sin x} \text{ for } 0 < x < \pi, x \neq \beta.$$

- a) Express $2\cos x + 2\sin x$ in the form $R\cos(x - \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

The curve with equation $y = f(x)$ has a vertical asymptote at $x = \beta$.

- b) Determine the value of β .
 c) Solve the equation

$$f(3x) - \sqrt{6} = 0,$$

giving the answers in terms of π .

<input type="text"/>	$2\cos x + 2\sin x \equiv 2\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$	$\beta = \frac{3\pi}{4}$	$x = \frac{\pi}{36}, \frac{5\pi}{36}, \frac{25\pi}{36}, \frac{29\pi}{36}$
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(a) By standard transformation ...

$$\begin{aligned} 2\cos x + 2\sin x &= 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right) \\ &= 2\sqrt{2}\left(\cos\left(x - \frac{\pi}{4}\right)\right) \\ &= 2\sqrt{2}\cos\left(x - \frac{\pi}{4}\right) \end{aligned}$$

(b) Asymptote (vertical) $\Rightarrow 2\cos x + 2\sin x = 0$

$$\begin{aligned} &\Rightarrow \cos(x - \frac{\pi}{4}) = 0 \\ &\Rightarrow x - \frac{\pi}{4} = \frac{\pi}{2} \\ &\Rightarrow x = \frac{3\pi}{4} \quad \therefore \beta = \frac{3\pi}{4} \end{aligned}$$

(c) $f(x) = \frac{6}{2\cos x + 2\sin x} = \frac{6}{2\sqrt{2}\cos(x - \frac{\pi}{4})}$

$$\begin{aligned} f(3x) - \sqrt{6} &= 0 \\ \Rightarrow \frac{6}{2\sqrt{2}\cos(3x - \frac{\pi}{4})} - \sqrt{6} &= 0 \\ \Rightarrow \frac{6}{2\sqrt{2}\cos(3x - \frac{\pi}{4})} &= \sqrt{6} \quad \left(3x - \frac{\pi}{4} = \frac{\pi}{2} \pm 2m\pi\right) \\ \Rightarrow \sqrt{2}\cos(3x - \frac{\pi}{4}) &= \sqrt{6} \quad \left(3x - \frac{\pi}{4} = \frac{\pi}{2} \pm 2m\pi\right) \\ \Rightarrow \sqrt{2}\cos(3x - \frac{\pi}{4}) &= \frac{1}{\sqrt{3}} \quad \therefore x_1 = \frac{25\pi}{36} \\ \Rightarrow \cos(3x - \frac{\pi}{4}) &= \frac{\sqrt{3}}{2} \quad x_2 = \frac{29\pi}{36} \\ \bullet \arccos(\frac{\sqrt{3}}{2}) &= \frac{\pi}{6} \quad x_3 = \frac{3\pi}{36} \\ \left(3x - \frac{\pi}{4} = \frac{\pi}{6} \pm 2m\pi\right) \quad & \therefore x_4 = \frac{3\pi}{36} \\ 3x - \frac{\pi}{4} &= \frac{\pi}{6} \pm 2m\pi \\ 3x &= \frac{5\pi}{12} \pm 2m\pi \\ \therefore x &= \frac{5\pi}{36} \pm 2m\pi \end{aligned}$$

Question 156 (****)

It is given that

$$\tan(x+60^\circ) \tan(x-60^\circ) = \frac{\tan^2 x - 3}{1 - 3 \tan^2 x}.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence, or otherwise, solve the trigonometric equation

$$\tan(x+60^\circ) \tan(x-60^\circ) + 11 = 0, \quad 0^\circ \leq x < 360^\circ.$$

$$x \approx 26.6^\circ, 153.4^\circ, 186.6^\circ, 333.4^\circ$$

$$\begin{aligned}
 \text{(a) LHS} &= \tan(x+60^\circ) \tan(x-60^\circ) = \frac{\tan x + \tan 60^\circ}{1 - \tan x \tan 60^\circ} \times \frac{\tan x - \tan 60^\circ}{1 + \tan x \tan 60^\circ} \\
 &= \frac{(\tan x + \sqrt{3})(\tan x - \sqrt{3})}{(1 - \sqrt{3} \tan x)(1 + \sqrt{3} \tan x)} = \frac{-3\tan^2 x - 3}{1 - 3 \tan^2 x} = \text{RHS}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \tan(x+60^\circ) \tan(x-60^\circ) + 11 &\approx 0 \quad \left\{ \begin{array}{l} \bullet \arctan(\frac{x}{\sqrt{3}}) = 26.6 \\ \bullet \arctan(-\frac{x}{\sqrt{3}}) = -26.6 \\ \bullet 2x = 26.6 \pm 180^\circ \quad (\text{using } 180^\circ) \\ \bullet 2x = -26.6 + 180^\circ \\ \bullet x_1 = 26.6^\circ \\ \bullet x_2 = 153.4^\circ \\ \bullet x_3 = 186.6^\circ \\ \bullet x_4 = 333.4^\circ \end{array} \right. \\
 \Rightarrow \frac{\tan^2 x - 3}{1 - 3 \tan^2 x} + 11 &\approx 0 \\
 \Rightarrow \tan^2 x - 3 + 11(1 - 3 \tan^2 x) &\approx 0 \\
 \Rightarrow \tan^2 x - 3 + 11 - 33 \tan^2 x &\approx 0 \\
 \Rightarrow 8 - 32 \tan^2 x &\approx 0 \\
 \Rightarrow \tan^2 x &\approx \frac{1}{4} \\
 \Rightarrow \tan x &\approx \pm \frac{1}{2} \\
 \Rightarrow \tan x &\approx \pm \frac{1}{2}
 \end{aligned}$$

Question 157

(*****)

$$f(x) \equiv \cos x + \sqrt{3} \sin x, \quad x \in \mathbb{R}.$$

- a) Express $f(x)$ in the form $R \cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

- b) Hence solve the equation

$$\cos 2\theta + \sqrt{3} \sin 2\theta = 2 \cos \theta, \quad 0 \leq \theta < 2\pi.$$

$$\boxed{\text{a) } f(x) = 2 \cos\left(x - \frac{\pi}{3}\right)}, \quad \boxed{\text{b) } \theta = \frac{\pi}{3}, \frac{7\pi}{9}, \frac{13\pi}{9}}$$

a) $\cos x + \sqrt{3} \sin x \equiv R \cos(x - \alpha)$
 $\equiv R(\cos x \cos \alpha + \sin x \sin \alpha)$
 $\equiv (\cos \alpha) \cos x + (\sin \alpha) \sin x$

$\cos \alpha = 1 \quad \Rightarrow \quad \text{square of both sides} \quad 2 = \sqrt{1^2 + (\sqrt{3})^2}$
 $2 \sin \alpha = \sqrt{3} \quad \Rightarrow \quad \sin \alpha = \frac{\sqrt{3}}{2}$
 $\tan \alpha = \frac{\sqrt{3}}{1} \quad \Rightarrow \quad \alpha = \frac{\pi}{3}$

$\therefore f(x) = 2 \cos(x - \frac{\pi}{3})$

b) $\cos 2\theta + \sqrt{3} \sin 2\theta = 2 \cos \theta$
 $2 \cos(2\theta - \frac{\pi}{3}) = 2 \cos \theta$
 $\cos(2\theta - \frac{\pi}{3}) = \cos \theta$

$(2\theta - \frac{\pi}{3}) = \textcircled{1} \pm 2\pi n \quad \left. \begin{array}{l} \\ \end{array} \right\} n=0,1,2,3,\dots$
 $(2\theta - \frac{\pi}{3}) = (2\pi \textcircled{2}) \pm 2\pi n$

$(\theta = \frac{\pi}{3} \pm 2\pi n)$
 $(\theta = \frac{\pi}{3} \pm 2\pi n)$

$\theta_1 = \frac{\pi}{3}, \quad \theta_2 = \frac{7\pi}{9}, \quad \theta_3 = \frac{13\pi}{9}$

Question 158 (****)

It is given that

$$\frac{\operatorname{cosec} x - \sin x}{\cot x \cos^2 x} \equiv \sec x.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence, or otherwise, solve the trigonometric equation

$$\tan^2 x - \sec x = \frac{\operatorname{cosec} x - \sin x}{2 \cot x \cos^2 x}, \quad 0^\circ \leq x < 360^\circ.$$

$$x = 60^\circ, 300^\circ$$

(a) LHS = $\frac{\operatorname{cosec} x - \sin x}{\cot x \cos^2 x}$ = $\frac{\frac{1}{\sin x} - \sin x}{\frac{\cos x}{\sin x} \cos^2 x}$ [Multiply top & bottom by $\sin x$]

$$= \frac{1 - \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x = \text{RHS}$$

(b) $2(\sec^2 x - \sec x) = \frac{\operatorname{cosec} x - \sin x}{\cot x \cos^2 x}$ [cancel $\sec^2 x$]

$$\Rightarrow 2\sec x - 2\sec x = \sec x$$

$$\Rightarrow 2(\sec x - 1) - 2\sec x = \sec x$$

$$\Rightarrow -2\sec x - 2 = 0$$

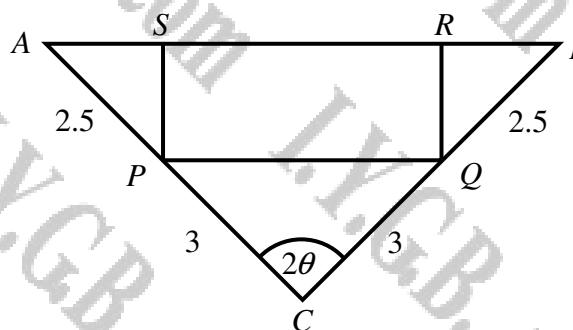
$$\Rightarrow (\sec x + 1)(\sec x - 2) = 0$$

$$\Rightarrow \sec x = -1 \quad \text{or} \quad \sec x = 2$$

$$\Rightarrow \cos x = -\frac{1}{2} \quad \text{or} \quad \cos x = \frac{1}{2}$$

$$\Rightarrow x_1 = 60^\circ \quad x_2 = 300^\circ$$

Question 159 (****)



The figure above shows an isosceles triangle ABC where the angle $ACB = 2\theta$.

A rectangle $PQRS$ is drawn inside ABC , so that S and R lie on AB , P lies on AC and Q lies on BC .

It is further given that $|AP| = |BQ| = 2.5$ and $|PC| = |QC| = 3$.

- a) Show clearly that the perimeter of $PQRS$ is

$$5\cos\theta + 12\sin\theta.$$

- b) Express $5\cos\theta + 12\sin\theta$ in the form $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90^\circ$.

- c) Find the value of θ , given that the perimeter of $PQRS$ is 10.

$$\boxed{\quad}, \quad 5\cos\theta + 12\sin\theta \cong 13\sin(\theta + 22.62)^\circ, \quad \theta \approx 27.7^\circ$$

WORKING AT THE JUNIOR TRIANGLES $\triangle ABC$

$|AP| = 2.5\sin\theta$
 $|BQ| = 2.5\cos\theta$

$\Rightarrow |PQ| = 2(|AP|) = 2 \times 2.5\sin\theta = 5\sin\theta$
 $\Rightarrow |PR| = 2 \times 2.5\cos\theta + 2 \times 2.5\cos\theta = 5\cos\theta$
 $\rightarrow \text{Perimeter} = 5\cos\theta + 5\sin\theta$

AS REQUIRED

USING THE COMPOUND ANGLE IDENTITY FOR $\sin(A+B)$

$5\cos\theta + 5\sin\theta = 5\sin(\theta + \pi)$
 $5\cos\theta + 5\sin\theta = 5\sin\theta\cos\pi + 5\cos\theta\sin\pi$
 $5\cos\theta + 5\sin\theta = (5\cos\theta)\cos\pi + (5\sin\theta)\cos\pi$

$5\cos\theta = 12$
 $5\sin\theta = 5$

Divide the equations side by side

$\therefore \tan\theta = \frac{5}{12}$
 $\theta \approx 22.62^\circ$

$\therefore 5\cos\theta + 5\sin\theta \cong 13\sin(\theta + 22.62)^\circ$

SETTING $P=10$

$\Rightarrow 5\cos\theta + 12\sin\theta = 10$
 $\Rightarrow 12\sin(\theta + 22.62) = 10$
 $\Rightarrow \sin(\theta + 22.62) = \frac{10}{12}$

$\sin(22.62) = 0.3026$

$\Rightarrow (\theta + 22.62) = 18.45^\circ$
 $\Rightarrow \theta + 22.62 = 129.72^\circ \pm 360^\circ$

$\Rightarrow \theta = 27.7^\circ \pm 360^\circ$
 $\Rightarrow \theta = 107.10^\circ \pm 360^\circ$

$\therefore \theta = 27.7^\circ$

Question 160 (****)

$$f(\theta) \equiv \sqrt{3} \sin \theta + \sin \theta, \quad 0 \leq \theta < 2\pi.$$

- a) Express $f(\theta)$ in the form $R \cos(\theta - \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

- b) Find the solutions of the trigonometric equation

$$\sqrt{3} + \sin 2\theta = \sqrt{3} \cos 2\theta + 2 \sin \theta, \quad 0 \leq \theta < 2\pi,$$

giving the answers in radians in terms of π .

$$\boxed{\theta = 0, \frac{2\pi}{3}, \pi}$$

$$\begin{aligned}
 \text{(a)} \quad \sqrt{3} \sin \theta + \cos \theta &\equiv R \cos(\theta - \alpha) \\
 &\equiv R \cos \theta \cos \alpha + R \sin \theta \sin \alpha \\
 &\equiv (R \cos \alpha) \cos \theta + (R \sin \alpha) \sin \theta \\
 R \cos \alpha &= \sqrt{3} \quad \Rightarrow \quad R = \sqrt{(\sqrt{3})^2 + 1^2} = 2 \\
 R \sin \alpha &= 1 \quad \Rightarrow \quad \tan \alpha = \sqrt{3} \quad \alpha = \frac{\pi}{3} \\
 \therefore \sqrt{3} \sin \theta + \cos \theta &= 2 \cos\left(\theta - \frac{\pi}{3}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \sqrt{3} + \sin 2\theta &= \sqrt{3} \cos 2\theta + 2 \sin \theta \\
 \sqrt{3} + 2 \sin \theta \cos \theta &= \sqrt{3}(-2 \sin^2 \theta) + 2 \sin \theta \\
 \sqrt{3} + 2 \sin \theta \cos \theta &= -2\sqrt{3} \sin^2 \theta + 2 \sin \theta \\
 2\sqrt{3} \sin^2 \theta + 2 \sin \theta \cos \theta - 2 \sin \theta &= 0 \\
 2 \sin \theta [\sqrt{3} \sin \theta + \cos \theta - 1] &= 0 \\
 2 \sin \theta [2 \cos\left(\theta - \frac{\pi}{3}\right) - 1] &= 0 \\
 \bullet \sin \theta &= 0 \\
 (\theta = 0 \pm 2\pi n) & \\
 (\theta = \pi \mp 2\pi n) & \\
 \therefore \theta = 0, \pi &
 \end{aligned}$$

$$\begin{aligned}
 \bullet \cos\left(\theta - \frac{\pi}{3}\right) &= \frac{1}{2} \\
 \left(\theta - \frac{\pi}{3} = \frac{\pi}{3} \pm 2\pi n\right) & \\
 \left(\theta = \frac{2\pi}{3} \pm 2\pi n\right) & \\
 \left(\theta = \frac{4\pi}{3} \pm 2\pi n\right) &
 \end{aligned}$$

Question 161 (****)

It is given that

$$\cos P - \cos Q \equiv -2 \sin\left(\frac{P+Q}{2}\right) \sin\left(\frac{P-Q}{2}\right).$$

- a) Prove the validity of the above trigonometric identity by using the compound angle identities for $\cos(A+B)$ and $\cos(A-B)$.

- b) Hence, or otherwise, solve the trigonometric equation

$$\cos 6x + \sin 4x = \cos 2x, \quad 0 \leq x \leq \frac{\pi}{2},$$

giving the answers in terms of π .

$$x = 0, \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{2}$$

(a) $\cos(A+B) = \cos A \cos B - \sin A \sin B$
 $\cos(A-B) = \cos A \cos B + \sin A \sin B$

 $\cos(A+B) - \cos(A-B) = -2 \sin A \sin B \quad \text{Subtract}$

Let $A+B=P$ and $A=P-\frac{P}{2}$
 $A-B=Q$ and $B=\frac{P-A}{2}$

 $\therefore \cos P - \cos Q = -2 \sin\left(\frac{P+A}{2}\right) \sin\left(\frac{P-A}{2}\right)$

(b) $\cos 6x + \sin 4x = \cos 2x$
 $\Rightarrow \cos(6x) - \cos(2x) = -\sin 2x$
 $\Rightarrow -2 \sin\left(\frac{6x+2x}{2}\right) \sin\left(\frac{6x-2x}{2}\right) = -\sin 2x$
 $\Rightarrow -2 \sin 4x \sin 2x = -\sin 2x$
 $\Rightarrow \sin 4x - 2 \sin(2x)\cos(2x) = 0$
 $\Rightarrow \sin 4x(1 - 2\cos^2 2x) = 0$
 $\Rightarrow \sin 4x = 0 \quad \sin 2x = \frac{1}{2}$

$\bullet \sin 4x = 0$
 $(4x = 0 + 2n\pi) \quad n \in \mathbb{Z}$
 $(4x = \pi + 2n\pi) \quad n \in \mathbb{Z}$
 $(x = 0 + \frac{n\pi}{2}) \quad n \in \mathbb{Z}$
 $(x = \frac{\pi}{2} + \frac{n\pi}{2}) \quad n \in \mathbb{Z}$

$\bullet \sin 2x = \frac{1}{2}$
 $(2x = \frac{\pi}{6} + 2n\pi) \quad n \in \mathbb{Z}$
 $(2x = \frac{5\pi}{6} + 2n\pi) \quad n \in \mathbb{Z}$
 $(x = \frac{\pi}{12} + n\pi) \quad n \in \mathbb{Z}$
 $(x = \frac{5\pi}{12} + n\pi) \quad n \in \mathbb{Z}$

$x = 0, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{2}$

Question 162 (**)**

Solve each of the following trigonometric equations.

i. $\frac{\sec^2 x + 8}{4 - \tan x} = 3 \tan x, \quad 0 \leq x < 2\pi, \quad \tan x \neq 4.$

ii. $\cos 2\theta = \sin \theta, \quad 0 \leq \theta < 360^\circ.$

, $x \approx 0.983^\circ, 4.12^\circ$, $\theta = 30^\circ, 150^\circ, 270^\circ$

(i) $\frac{\sec^2 x + 8}{4 - \tan x} = 3 \tan x$
 $4 - \tan x$
 $\rightarrow \sec^2 x + 8 = 12 \tan x - 3 \tan^2 x$
 $\rightarrow (1 + \tan^2 x) + 8 = 12 \tan x - 3 \tan^2 x$
 $\rightarrow 4 \tan^2 x - 12 \tan x + 9 = 0$
 $\rightarrow (2 \tan x - 3)^2 = 0$
 $\tan x = \frac{3}{2}$

$\arctan(\frac{3}{2}) = 0.983^\circ$
 $\theta = 0.983^\circ + n\pi$
 $n = 0, 1, 2, \dots$
 $\therefore \theta = 0.983^\circ$
 $3\theta = 4.12^\circ$

(ii) $\cos 2\theta = \sin \theta$
 $\Rightarrow 1 - 2\sin^2 \theta = \sin \theta$
 $\Rightarrow 1 = 2\sin^2 \theta + \sin \theta - 1$
 $\Rightarrow (2\sin \theta + 1)(\sin \theta - 1) = 0$
 $\Rightarrow \sin \theta = -\frac{1}{2}$
 $\bullet \arcsin(-\frac{1}{2}) = 30^\circ$
 $(\theta = 30^\circ + 360^\circ)$
 $(\theta = 150^\circ + 360^\circ)$
 $\therefore \theta = 30^\circ, 150^\circ, 270^\circ$

$\bullet \arcsin(-\frac{1}{2}) = -30^\circ$
 $(\theta = -30^\circ + 360^\circ)$
 $(\theta = 330^\circ)$
 $\therefore \theta = 330^\circ$

Question 163 (**)**

Solve the following trigonometric equation

$$\tan(\arctan 3x - \arctan 2) + \tan(\arctan 3 - \arctan 2x) = \frac{3}{8}.$$

, $x = \frac{1}{2}$

USING THE IDENTITY FOR $\tan(A-B)$

$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\rightarrow \tan(\arctan 3x - \arctan 2) + \tan(\arctan 3 - \arctan 2x) = \frac{3}{8}$

$\rightarrow \frac{3x - 2}{1 + 3x \cdot 2} + \frac{3 - 2x}{1 + 3 \cdot 2x} = \frac{3}{8}$

$\rightarrow \frac{3x - 2}{1 + 6x} + \frac{3 - 2x}{1 + 6x} = \frac{3}{8}$

$\rightarrow \frac{3x + 3 - 2x - 2}{1 + 6x} = \frac{3}{8}$

$\rightarrow 8x + 8 = 3 + 18x$

$\rightarrow 5 = 10x$

$\Rightarrow x = \frac{1}{2}$

Question 164 (****)

It is given that

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$$

- a) Use the above trigonometric identities show that

$$\cos(A+B) + \cos(A-B) \equiv 2 \cos A \cos B.$$

- b) Hence show further that

$$\cos P + \cos Q \equiv 2 \cos\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

It is further given that

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

- c) Show clearly that

$$\frac{\cos 2x + \cos 2y}{\sin 2x + \sin 2y} \equiv \cot(x+y).$$

- d) Use the above results to show that

$$\cot(52.5^\circ) = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2$$

proof

(a) $\cos(A+B) = \cos A \cos B - \sin A \sin B$
 $\cos(A-B) = \cos A \cos B + \sin A \sin B$
 $\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$ < add the equations

(b) Let $P=A+B$ Add $\Rightarrow \frac{P+Q}{2} = A$
 $Q=A-B$ Subtract $\Rightarrow \frac{P-Q}{2} = B$

Hence $\cos P + \cos Q = 2 \cos\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right)$

(c) RHS = $\frac{\cos 2x + \cos 2y}{\sin 2x + \sin 2y} = \frac{2 \cos(x+y)}{2 \sin(x+y) \cos(x-y)}$ (from (b))
 $= \frac{\cos(x+y)}{\sin(x+y)} = \cot(x+y) = \text{RHS}$

(d) $\frac{\cos 2x + \cos 2y}{\sin 2x + \sin 2y} = \cot(x+y)$ $\left\{ \begin{array}{l} \rightarrow \cot(52.5^\circ) = \frac{1+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \\ \rightarrow \cot(52^\circ) = \frac{(1+\sqrt{2})(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} \end{array} \right.$
 LET $x=30^\circ$ $y=22.5^\circ$ $\left\{ \begin{array}{l} \rightarrow \cot(52.5^\circ) = \frac{\sqrt{3}-\sqrt{2}+1-\sqrt{2}}{1} \\ \rightarrow \cot(52.5^\circ) = \sqrt{3}-\sqrt{2}+1-2 \\ \rightarrow \cot(52.5^\circ) = \sqrt{3}-\sqrt{2}-1 \end{array} \right.$
 $\frac{\cos 2x + \cos 2y}{\sin 2x + \sin 2y} = \cot(x+y)$ $\left\{ \begin{array}{l} \rightarrow \cot(52^\circ) = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \\ \rightarrow \cot(52^\circ) = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \end{array} \right.$
 $\frac{\frac{1}{2} + \frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}} = \cot(52^\circ)$ $\left\{ \begin{array}{l} \rightarrow \cot(52^\circ) = \sqrt{3}+\sqrt{2}-1-2 \\ \rightarrow \cot(52^\circ) = \sqrt{3}-\sqrt{2}-1 \end{array} \right.$

Question 165 (**)**

Solve the trigonometric equation

$$\ln(\cosec \theta) = \ln 4 - \ln(\sec \theta), \quad 0 < \theta < \frac{\pi}{2},$$

giving the answers in terms of π .

$$\boxed{\theta = \frac{\pi}{12}, \frac{5\pi}{12}}$$

$$\begin{aligned} \ln(\cosec \theta) &= \ln 4 - \ln(\sec \theta) \\ \ln(\cosec \theta) + \ln(\sec \theta) &= \ln 4 \\ \ln(\cosec \theta \sec \theta) &= \ln 4 \\ \ln\left(\frac{1}{\sin \theta \cos \theta}\right) &= \ln 4 \\ \ln\left(\frac{2}{2 \sin \theta \cos \theta}\right) &= \ln 4 \\ \ln\left(\frac{2}{\sin 2\theta}\right) &= \ln 4 \\ \frac{2}{\sin 2\theta} &= 4 \end{aligned}$$

$$\begin{aligned} \sin 2\theta &= \frac{1}{2} \\ \arcsin\left(\frac{1}{2}\right) &= \frac{\pi}{6} \\ 2\theta &= \frac{\pi}{6} \pm 2\pi n \quad n=0,1,2,\dots \\ \theta &= \frac{\pi}{12} \pm \pi n \\ \theta_1 &= \frac{\pi}{12} \\ \theta_2 &= \frac{11\pi}{12} \end{aligned}$$

Question 166 (**)**

Solve the trigonometric equation

$$\sin \theta \cos \frac{\pi}{5} = \frac{1}{2} - \cos \theta \sin \frac{\pi}{5}, \quad 2\pi < \theta < 4\pi,$$

giving the answers in terms of π .

$$\boxed{\text{LHS}}, \quad \boxed{\theta = \frac{79\pi}{30}, \frac{119\pi}{30}}$$

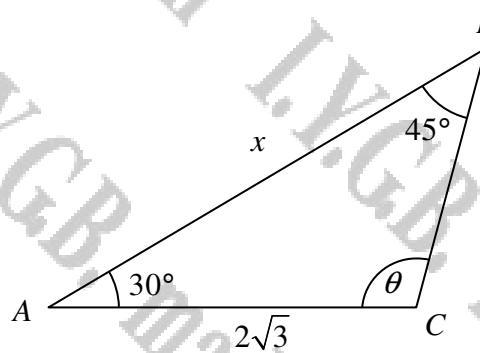
REARRANGE INTO A "COMPONENT" ANGLE IDENTITY

$$\begin{aligned} &\Rightarrow \sin \theta \cos \frac{\pi}{5} = \frac{1}{2} - \cos \theta \sin \frac{\pi}{5} \\ &\rightarrow \sin \theta \cos \frac{\pi}{5} + \cos \theta \sin \frac{\pi}{5} = \frac{1}{2} \\ &\rightarrow \sin\left(\theta + \frac{\pi}{5}\right) = \frac{1}{2} \\ &\qquad\qquad\qquad \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \\ &\qquad\qquad\qquad \theta + \frac{\pi}{5} = \frac{\pi}{6} + 2\pi n \quad n=0,1,2,\dots \\ &\qquad\qquad\qquad \theta = -\frac{5\pi}{6} + 2\pi n \\ &\qquad\qquad\qquad \theta = \frac{11\pi}{6} + 2\pi n \end{aligned}$$

AND FOR $2\pi < \theta < 4\pi$

$$\begin{aligned} \theta_1 &= \frac{11\pi}{6} \\ \theta_2 &= \frac{17\pi}{6} \end{aligned}$$

Question 167 (****) non calculator



The figure above shows a triangle ABC where $|AC| = 2\sqrt{3}$ and $|AB| = x$.

The angles ABC , CAB and BCA are 45° , 30° and θ° , respectively.

- a) By using a suitable compound angle identity show clearly that

$$\sin 105^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

- b) Show without the use of a calculating aid that the exact length of AB , is

$$3 + \sqrt{3}.$$

proof

(a) $\sin(105^\circ) = \sin(60^\circ + 45^\circ) = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \frac{1}{2} \times \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4} // \text{as } 2\sqrt{3} = 2\sqrt{2} \times \sqrt{3}$$

(b) $\frac{x}{\sin B} = \frac{2\sqrt{3}}{\sin 45^\circ} // \text{(By Sine Rule)}$

$$x = \frac{2\sqrt{3} \sin 45^\circ}{\sin 45^\circ} = \frac{2\sqrt{3}}{\sqrt{2}/2} \times \frac{\sqrt{2}/2}{\sqrt{2}/2} = \frac{2\sqrt{3}}{\sqrt{2}} = \frac{2\sqrt{6}}{2} = \sqrt{6}$$

$$= \frac{\sqrt{6} + \sqrt{6}}{\sqrt{2}} = \frac{2\sqrt{2}\sqrt{3}}{\sqrt{2}} = 2 + \frac{\sqrt{6}}{\sqrt{2}} = 2 + \sqrt{3} //$$

Question 168 (****)

$$f(\theta) = 2\cos\theta + 3\sin\theta, \theta \in \mathbb{R}.$$

- a) Express $f(\theta)$ in the form $R\cos(\theta - \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

Give the value of α correct to 3 decimal places.

- b) State the maximum value of $f(\theta)$ and find the smallest positive value of θ for which this maximum occurs.

The temperature T °C in a warehouse is modelled by the equation

$$T = 16 + 2\cos\left(\frac{\pi t}{12}\right) + 3\sin\left(\frac{\pi t}{12}\right), \quad 0 \leq t < 24,$$

where t is the time in hours measured since midnight.

- c) State the maximum temperature in the warehouse and a value of t when this maximum temperature occurs.
- d) Find the times, to the nearest minute using 24 hour clock notation, when the temperature in the warehouse is 17 °C.

$$\boxed{}, \quad \boxed{2\cos\theta + 3\sin\theta \equiv \sqrt{13}\cos(\theta - 0.983^\circ)}, \quad \boxed{\max = \sqrt{13}}, \quad \boxed{\theta = 0.983^\circ},$$

$$\boxed{T_{\max} \approx 19.6}, \quad \boxed{t_{\max} = 3.75}, \quad \boxed{08:41/22:50}$$

<p>a) Approach As PER CUEIL</p> $2\cos\theta + 3\sin\theta \equiv R\cos(\theta - \alpha)$ $2\cos\theta + 3\sin\theta \equiv 2\cos\theta\cos\alpha + 2\sin\theta\sin\alpha$ $2\cos\theta + 3\sin\theta \equiv (\cos\theta)\cos\alpha + (\sin\theta)\sin\alpha$ <p>Simplifying:</p> $\begin{cases} \cos\alpha = 2/3 \\ \sin\alpha = 3/2 \end{cases}$ <p>SQUARE AND ADD:</p> $1 = \sqrt{4/9 + 9/4}$ $1 = \sqrt{13/4}$ $(2 > 3)$ <p>DIVIDE, SUB BY SIN:</p> $\begin{aligned} \cos\alpha &= \frac{2}{\sqrt{13}} \\ \sin\alpha &= \frac{3}{2} \\ \tan\alpha &= \frac{3}{2} \\ \alpha &\approx 0.983^\circ \end{aligned}$ <p>$\therefore f(\theta) \equiv \sqrt{13}\cos(\theta - 0.983^\circ)$</p>	<p>b) $f(\theta) = \sqrt{13}\cos(\theta - 0.983^\circ)$</p> <p>IT OCCURS WHEN: $\cos(\theta - 0.983) = 1$</p> $\theta - 0.983 = 0$ $\theta = 0.983^\circ$ <p>c) LINKS PART (a) & PART (b)</p> $T = 16 + 2\cos\left(\frac{\pi t}{12}\right) + 3\sin\left(\frac{\pi t}{12}\right)$ $T_{\max} = 16 + \sqrt{13} \approx 17.6^\circ\text{C}$ $\begin{aligned} 1 &= 0.983 \\ \frac{\pi t}{12} &= 0.983 \\ t &\approx 3.75 \quad (\text{hours}) \end{aligned}$	<p>d) Finally we take, for $T = 17$</p> $17 = 16 + 2\cos\left(\frac{\pi t}{12}\right) + 3\sin\left(\frac{\pi t}{12}\right)$ $1 = 2\cos\left(\frac{\pi t}{12}\right) + 3\sin\left(\frac{\pi t}{12}\right)$ $\cos\left(\frac{\pi t}{12}\right) = \frac{1}{2}$ <p>COSINE: $\frac{\pi t}{12} = 1.2857^\circ \dots$</p> $\begin{aligned} \frac{\pi t}{12} - 0.983 &= 1.2857^\circ \pm 2\pi^\circ \quad n=0,1,2,3,\dots \\ \frac{\pi t}{12} - 0.983 &= 4.7134^\circ \pm 2\pi^\circ \\ \frac{\pi t}{12} &= 5.7121^\circ \dots \pm 2\pi^\circ \\ \frac{\pi t}{12} &= 7.1111^\circ \dots \pm 2\pi^\circ \\ \frac{\pi t}{12} &= 8.4906^\circ \dots \pm 2\pi^\circ \\ \frac{\pi t}{12} &= 21.813^\circ \dots \pm 2\pi^\circ \end{aligned}$ <p>$\therefore t = \frac{24}{\pi} \approx 7.616$</p> <p>$\therefore t = \frac{24}{\pi} \approx 21.813 \approx 21:50$</p>
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Question 169 (****)

It is given that

$$\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right).$$

- a) Prove the validity of the above trigonometric identity by using the compound angle identities for $\sin(A+B)$ and $\sin(A-B)$.

- b) Hence, or otherwise, solve the equation

$$\sin \theta + \sin 2\theta + \sin 3\theta = 0, \quad 0 \leq \theta < \pi,$$

giving the answers in terms of π .

$$\boxed{\theta = 0, \frac{\pi}{2}, \frac{2\pi}{3}}$$

(a)

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B \\ \sin(A+B) + \sin(A-B) &= 2 \sin A \cos B \quad (\text{By adding}) \end{aligned}$$

Let $P = A+B$ $\therefore P+Q = 2A$ $A = \frac{P+Q}{2}$ $P-Q = B$
 $Q = A-B$ $\therefore A = \frac{P+Q}{2}$ $\frac{P-Q}{2} = B$
 Hence THE PREVIOUS LINE GIVES
 $\sin P + \sin Q = 2 \sin\left(\frac{P+Q}{2}\right) \cos\left(\frac{P-Q}{2}\right)$

(b)

$$\begin{aligned} \sin \theta + \sin 2\theta + \sin 3\theta &= 0 \\ \Rightarrow \sin \theta + \sin(2\theta) + \sin(3\theta) &= 0 \\ \Rightarrow 2 \sin\left(\frac{3\theta+3\theta}{2}\right) \cos\left(\frac{3\theta-3\theta}{2}\right) + \sin 3\theta &= 0 \\ \Rightarrow 2 \sin 3\theta \cos 0 + \sin 3\theta &= 0 \\ \Rightarrow 3 \sin 3\theta (2 \cos 0 + 1) &= 0 \\ \bullet 3 \sin 3\theta = 0 & \bullet 2 \cos 0 = -1 \\ \sin 3\theta = 0 & \cos 0 = \frac{1}{2} \\ 3\theta = 0 \pm 2k\pi & \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3} \\ \frac{3\theta}{3} = 0 \pm 2k\pi & \left(\theta = \frac{\pi}{3} + 2k\pi\right) \\ \theta = 0 \pm \frac{2k\pi}{3} & \left(\theta = \frac{4\pi}{3} + 2k\pi\right) \\ \theta = 0, \frac{2\pi}{3}, -\frac{2\pi}{3} & \left(k = 0, 1, 2, 3, \dots\right) \end{aligned}$$

Question 170 (*)**

Given that

$$64 \cos 2\theta \cos \theta + 32 \sin 2\theta \sin \theta = 27,$$

find the value of $\cos \theta$.

$$\boxed{}, \quad \boxed{\cos \theta = \frac{3}{4}}$$

3004E. DRAW THE DOUBLE ANGLES & THEN

$$\begin{aligned} &\Rightarrow 64(\cos^2 \theta - \sin^2 \theta) + 32(2\sin \theta \cos \theta) \cos \theta = 27 \\ &\Rightarrow 64(\cos^2 \theta - \sin^2 \theta) + 64\sin \theta \cos^2 \theta = 27 \\ &\Rightarrow 128\cos^2 \theta - 64\sin^2 \theta + 64\sin \theta \cos^2 \theta = 27 \\ &\Rightarrow 128\cos^2 \theta - 64\cos^2 \theta + 64(1 - \cos^2 \theta)\cos^2 \theta = 27 \\ &\Rightarrow 128\cos^2 \theta - 64\cos^2 \theta + 64\cos^2 \theta - 64\cos^4 \theta = 27 \\ &\Rightarrow 64\cos^2 \theta = 27 \\ &\Rightarrow \cos^2 \theta = \frac{27}{64} \\ &\Rightarrow \cos \theta = \frac{3}{4} \end{aligned}$$

Question 171 (****)

$$f(x) = 2\sin x + 2\cos x, \quad x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\sin(x+\alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

b) State the minimum and the maximum value of ...

i. ... $y = f\left(x - \frac{\pi}{2}\right)$.

ii. ... $y = 2f(x) + 1$.

iii. ... $y = [f(x)]^2$.

iv. ... $y = \frac{10}{f(x) + 3\sqrt{2}}$.

K, $y = \sqrt{8} \sin\left(\theta + \frac{\pi}{4}\right)$, $[-\sqrt{8}, \sqrt{8}]$, $[-2\sqrt{8}+1, 2\sqrt{8}+1]$, $[0, 8]$,

$[\sqrt{2}, 5\sqrt{2}]$

$\text{(a) } R\sin(2\pi x + 2\pi x) = R\sin(2x + \pi)$ $\equiv R\sin(2x)\cos(\pi) + R\cos(2x)\sin(\pi)$ $\equiv (-2\sin x) + (2\cos x) \cdot 0$ $\therefore R\sin x = 2 \Rightarrow R\sqrt{2^2 + 2^2} = \sqrt{8}$ $\tan x = 1 \Rightarrow x = \frac{\pi}{4}$ $\therefore f(x) = \sqrt{8} \sin\left(x + \frac{\pi}{4}\right)$
(i) MIN = $-\sqrt{8}$ MAX = $\sqrt{8}$ (TRANSLATION RIGHT 2 UNITS) (ii) MIN = $-2\sqrt{8}+1$ MAX = $2\sqrt{8}+1$ (AFFINE IS CO-ORDINATE STRETCH IN Y BY FACTOR OF 2, THEN TRANSLATED UP BY 1 UNIT)
(iii) MIN = 0 MAX = 8 (iv) MIN = $\sqrt{2}$ MAX = $5\sqrt{2}$ $\frac{10}{\sqrt{8} + 3\sqrt{2}} = \frac{10}{\sqrt{8} + 3\sqrt{2}}$

Question 172 (**)**

It is given that θ and φ are such so that

$$\tan \theta = t \quad \text{and} \quad \tan \varphi = t - 1,$$

where t is a constant.

It is further given that

$$\frac{1}{\cos^2 \theta} - \frac{1}{\cos^2 \varphi} = 3.$$

- Show clearly that $t = 2$.
- Determine the exact value of $\tan(\theta + \varphi)$, showing clearly all the steps in the workings.

$$\boxed{}, \quad \boxed{\tan(\theta + \varphi) = -3}$$

a) WORKING AS FOLLOWS

$$\begin{aligned} \Rightarrow \frac{1}{\cos^2 \theta} - \frac{1}{\cos^2 \varphi} &= 3 \\ \Rightarrow \sec^2 \theta - \sec^2 \varphi &= 3 \\ \Rightarrow (1 + \tan^2 \theta) - (1 + \tan^2 \varphi) &= 3 \\ \Rightarrow \tan^2 \theta - \tan^2 \varphi &= 3 \\ \Rightarrow t^2 - (t-1)^2 &= 3 \\ \Rightarrow t^2 - (t^2 - 2t + 1) &= 3 \\ \Rightarrow t^2 - t^2 + 2t - 1 &= 3 \\ \Rightarrow 2t - 1 &= 3 \\ \Rightarrow 2t &= 4 \\ \Rightarrow t &= 2 \end{aligned}$$

b) USING THE TANDED COMPOUND IDENTITY

$$\begin{aligned} \Rightarrow \tan(\theta + \varphi) &= \frac{\tan \theta + \tan \varphi}{1 - \tan \theta \tan \varphi} = \frac{t + (t-1)}{1 - t(t-1)} \\ &= \frac{2t-1}{1-t^2+t} = \frac{2t-1}{1-t^2+2t} = \frac{2(2)-1}{3-2(2)} \\ &= \frac{3}{-1} = -3 \end{aligned}$$

Question 173 (****)

$$\sin 2x + \cos 2x = 1 + \sin x, \quad 0 < x < \frac{\pi}{2}.$$

- a) Show that the above trigonometric equation can be written as

$$\cos x - \sin x = \frac{1}{2}.$$

- b) Express $\cos\left(x + \frac{\pi}{4}\right)$ in the form $R(A \cos x + B \sin x)$, where R , A and B are constants to be found.

- c) Use the results of part (a) and (b) to solve the trigonometric equation

$$\sin 2x + \cos 2x = 1 + \sin x, \quad 0 < x < \frac{\pi}{2}.$$

$$\boxed{R = \frac{\sqrt{2}}{2}}, \quad \boxed{A = 1}, \quad \boxed{B = -1}$$

$(a) \sin 2x + \cos 2x = 1 + \sin x$ $\Rightarrow 2\sin x \cos x + (-2\sin^2 x) = 1 + \sin x$ $\Rightarrow 2\sin x \cos x - 2\sin^2 x = 1 + \sin x$ $\Rightarrow 2\sin x \cos x - 2\sin^2 x = \sin x$ $\text{As } 0 < x < \frac{\pi}{2}, \sin x \neq 0$ $\Rightarrow 2\sin x - 2\sin x = 1$ $\Rightarrow \sin x - \sin x = \frac{1}{2}$ $\cancel{\sin x}$	$(c) \cos x - \sin x = \frac{1}{2}$ $\Rightarrow \frac{\sqrt{2}}{2}(\cos x - \sin x) = \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$ $\Rightarrow \cos x - \sin x = \frac{\sqrt{2}}{4}$ $\cos x - \sin x = 1.205^\circ$ $\cos x = 1.205^\circ + \sin x$ $\cos x = 1.205^\circ + 0.428^\circ$ $x = 4.29^\circ + 2m17^\circ$ $x = 4.29^\circ + 2m17^\circ$ $\therefore x = 0.428^\circ$
$(b) \cos\left(x + \frac{\pi}{4}\right) = \cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4}$ $\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x$ $\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos x - \sin x)$ $\therefore \cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(1)$	$\left(x + \frac{\pi}{4} \right) = 1.205^\circ + 2m17^\circ$ $x + \frac{\pi}{4} = 0.428^\circ + 2m17^\circ$ $x = 0.428^\circ + 2m17^\circ$ $x = 4.29^\circ + 2m17^\circ$ $\therefore x = 0.428^\circ$

Question 174 (****)

$$f(x) = \sin 2x, \quad x \in \mathbb{R}$$

$$g(x) = f\left(x + \frac{\pi}{4}\right) - f\left(x - \frac{\pi}{4}\right), \quad x \in \mathbb{R}.$$

a) Show clearly that

$$g(x) = 2 \cos 2x.$$

b) Express $g'(x)$ in terms of $f(x)$.

$$\boxed{\quad}, \quad \boxed{g'(x) = -4f(x)}$$

a) FROM THE DEFINITION OF $g(x)$ & $f(x)$ AND USING $\sin(A+B)$

$$\begin{aligned} \Rightarrow g(x) &= f\left(x + \frac{\pi}{4}\right) - f\left(x - \frac{\pi}{4}\right) \\ \Rightarrow f(x) &= \sin\left[2\left(x + \frac{\pi}{4}\right)\right] - \sin\left[2\left(x - \frac{\pi}{4}\right)\right] \\ \Rightarrow g(x) &= \sin\left[2x + \frac{\pi}{2}\right] - \sin\left[2x - \frac{\pi}{2}\right] \\ \Rightarrow f(x) &= \sin 2x \cos \frac{\pi}{2} + \cos 2x \sin \frac{\pi}{2} - [\sin 2x \cos \frac{\pi}{2} - \cos 2x \sin \frac{\pi}{2}] \\ \Rightarrow f(x) &= 2 \cos 2x \sin \frac{\pi}{2} \\ \Rightarrow g(x) &= 2 \cos 2x \end{aligned}$$

// AS EXPECTED

b) Differentiation of $g(x)$

$$\begin{aligned} g(x) &= 2 \cos 2x \\ g'(x) &= -4 \sin 2x \\ g'(x) &= -4f(x) \end{aligned}$$

Question 175 (**)**

Solve the trigonometric equation

$$\sec \theta - \cos \theta = 8(\operatorname{cosec} \theta - \sin \theta), \quad 0^\circ \leq \theta < 360^\circ.$$

$$\theta \approx 63.4^\circ, 243.6^\circ$$

$$\begin{aligned}
 \sec \theta - \cos \theta &= 8(\operatorname{cosec} \theta - \sin \theta) \\
 \Rightarrow \frac{1}{\cos \theta} - \cos \theta &= 8\left(\frac{1}{\sin \theta} - \sin \theta\right) \\
 \Rightarrow \frac{1 - \cos^2 \theta}{\cos \theta} &= 8\left(\frac{1 - \sin^2 \theta}{\sin \theta}\right) \\
 \Rightarrow \frac{\sin^2 \theta}{\cos \theta} &= 8 \times \frac{\cos^2 \theta}{\sin \theta} \\
 \Rightarrow \tan^2 \theta &= 8 \cos^2 \theta \\
 \Rightarrow \frac{\tan^2 \theta}{\cos^2 \theta} &= 8
 \end{aligned}
 \quad \begin{aligned}
 \rightarrow \tan^2 \theta &= 8 \\
 \Rightarrow \tan \theta &= 2 \\
 \operatorname{arctan}(2) &= 63.4^\circ \\
 \theta &= 63.4^\circ \pm 180^\circ \quad n=0,1,2,3,4 \\
 \theta_1 &= 63.4^\circ \\
 \theta_2 &= 243.4^\circ //
 \end{aligned}$$

Question 176 (**)**

It is given that

$$\sin 3\theta \equiv 3 \sin \theta - 4 \sin^3 \theta.$$

- Prove the validity of the above trigonometric identity by writing 3θ as $2\theta + \theta$.
- Hence solve the trigonometric equation

$$12 \sin^3 \theta - 9 \sin \theta = 1.5, \quad 0^\circ \leq \theta < 360^\circ.$$

$$\theta = 70^\circ, 110^\circ, 190^\circ, 230^\circ, 310^\circ, 350^\circ$$

$$\begin{aligned}
 \text{(a)} \quad 4\theta &= \sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta \\
 &= (2\sin \theta \cos \theta) \cos \theta + (1 - 2\cos^2 \theta) \sin \theta = 2\sin \theta \cos^2 \theta + \sin \theta - 2\sin \theta \cos^2 \theta \\
 &= 2\sin \theta (-1 + \cos^2 \theta) + \sin \theta + 2\sin \theta \cos^2 \theta = 2\sin \theta - 2\sin \theta \cos^2 \theta + \sin \theta - 2\sin \theta \cos^2 \theta \\
 &= 3\sin \theta - 4\sin \theta \cos^2 \theta = 4\sin \theta
 \end{aligned}
 \quad \begin{aligned}
 \text{(b)} \quad 12 \sin^3 \theta - 9 \sin \theta &= 1.5 \\
 \Rightarrow 48 \sin^3 \theta - 36 \sin \theta &= 6.0 \\
 \Rightarrow 36 \sin \theta - 48 \sin^3 \theta &= -0.5 \\
 \Rightarrow \sin \theta \left(1 - \frac{4}{3} \sin^2 \theta\right) &= -\frac{1}{2} \\
 \operatorname{arcsin}\left(-\frac{1}{2}\right) &= -30^\circ
 \end{aligned}
 \quad \begin{aligned}
 \theta &= -30^\circ \pm 360^\circ \quad n=0,1,2,3,4 \\
 \theta &= 330^\circ \pm 360^\circ \\
 \theta &= -10^\circ \pm 360^\circ \\
 \theta &= 70^\circ \pm 120^\circ \\
 \theta &= 110^\circ, 230^\circ, 350^\circ, 70^\circ, 190^\circ, 310^\circ
 \end{aligned}$$

Question 177 (**)**

Solve the trigonometric equation

$$\frac{\cot \psi}{\operatorname{cosec} \psi - 1} - \frac{\cos \psi}{1 + \sin \psi} = 2, \quad 0 < \psi < 2\pi,$$

giving the answers in terms of π .

$$\boxed{\psi = \frac{\pi}{4}, \frac{5\pi}{4}}$$

Working for Question 177:

$$\begin{aligned} \frac{\cot \psi}{\operatorname{cosec} \psi - 1} - \frac{\cos \psi}{1 + \sin \psi} &= 2 \\ \frac{\cos \psi}{\sin \psi - 1} - \frac{\cos \psi}{1 + \sin \psi} &= 2 \\ \frac{\cos \psi}{\sin \psi - 1} - \frac{\cos \psi}{1 + \sin \psi} &= 2 \\ \text{Multiply the bottom of each by sin } \psi \\ \frac{\cos \psi}{1 - \sin^2 \psi} - \frac{\cos \psi}{1 + \sin \psi} &= 2 \\ \Rightarrow \frac{1}{1 - \sin \psi} - \frac{1}{1 + \sin \psi} &= \frac{2}{\cos \psi} \\ \Rightarrow \frac{(\sin \psi + 1) - (\sin \psi - 1)}{(1 - \sin^2 \psi)(1 + \sin \psi)} &= \frac{2}{\cos \psi} \\ \Rightarrow \frac{2}{(1 - \sin^2 \psi)(1 + \sin \psi)} &= \frac{2}{\cos \psi} \\ \Rightarrow \frac{2}{\cos^2 \psi} &= \frac{2}{\cos \psi} \\ \Rightarrow \cos \psi &= 1 \end{aligned}$$

(Cosine is 1 at $\psi = \frac{\pi}{4}, \frac{5\pi}{4}$)

$\arccos 1 = \frac{\pi}{4}$

$\psi = \frac{\pi}{4} \pm \pi n, n \in \mathbb{Z}$

$\psi = \frac{\pi}{4}, \frac{5\pi}{4}$

Question 178 (**)**

Solve each of the following trigonometric equations.

i. $\frac{2 \cot^2 x + 5}{\operatorname{cosec} x} + 2 \operatorname{cosec} x = 13, \quad 0 \leq x < 2\pi.$

ii. $2 \cos 2\theta = 1 - 2 \sin \theta, \quad 0 \leq \theta < 360^\circ.$

$$\boxed{x \approx 0.340^\circ, 2.80^\circ, \theta = 54^\circ, 126^\circ, 198^\circ, 342^\circ}$$

Working for Question 178:

i. $\frac{2 \cot^2 x + 5}{\operatorname{cosec} x} + 2 \operatorname{cosec} x = 13$

$$\begin{aligned} \Rightarrow 2 \cot^2 x + 5 + 2 \operatorname{cosec}^2 x - 13 \operatorname{cosec} x &= 0 \\ \Rightarrow 2(\operatorname{cosec}^2 x) + 5 + 2 \operatorname{cosec}^2 x - 13 \operatorname{cosec} x &= 0 \\ \Rightarrow 4 \operatorname{cosec}^2 x - 13 \operatorname{cosec} x + 5 &= 0 \\ \Rightarrow (4 \operatorname{cosec} x - 1)(\operatorname{cosec} x - 3) &= 0 \\ \Rightarrow \operatorname{cosec} x = 1 \quad \text{or} \quad \operatorname{cosec} x = 3 \\ \Rightarrow \sin x = \pm 1 \quad \text{or} \quad \sin x = \frac{1}{3} \\ \Rightarrow x_1 = 90^\circ \quad \text{or} \quad x_2 = 2.05^\circ \quad \text{or} \quad x_3 = 2.80^\circ \end{aligned}$$

ii. $2 \cos 2\theta = 1 - 2 \sin \theta, \quad 0 \leq \theta < 360^\circ$

$$\begin{aligned} \Rightarrow 2(1 - 2 \sin^2 \theta) = 1 - 2 \sin \theta \\ \Rightarrow 2 - 4 \sin^2 \theta = 1 - 2 \sin \theta \\ \Rightarrow 1 - 4 \sin^2 \theta + 2 \sin \theta = 0 \\ \Rightarrow \sin \theta = \frac{1 \pm \sqrt{1 - 4}}{2} \\ \Rightarrow \sin \theta = \frac{1 \pm \sqrt{5}}{2} \\ \Rightarrow \sin \theta = \frac{1 \pm \sqrt{5}}{2} \quad \text{or} \quad \sin \theta = \frac{1 \pm \sqrt{5}}{2} \\ \Rightarrow \sin \theta = \frac{1 \pm \sqrt{5}}{2} \quad \text{or} \quad \sin \theta = \frac{1 \pm \sqrt{5}}{2} \\ \Rightarrow \theta = 54^\circ, 126^\circ, 198^\circ, 342^\circ \end{aligned}$$

Question 179 (**)**

Solve the following trigonometric equation

$$\arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{1}{x+1}\right) = \frac{\pi}{4}.$$

$$\boxed{\quad}, \quad x = -1, 2$$

TAN(X + Y) = $\frac{\tan X + \tan Y}{1 - \tan X \tan Y}$

$$\begin{aligned} & \Rightarrow \arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{1}{x+1}\right) = \frac{\pi}{4} \\ & \Rightarrow \tan\left[\arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{1}{x+1}\right)\right] = \tan\frac{\pi}{4} \\ & \Rightarrow \frac{\frac{1}{x} + \frac{1}{x+1}}{1 - \frac{1}{x}(x+1)} = 1 \\ & \text{MULTIPLYING ACROSS} \\ & \Rightarrow \frac{1}{x} + \frac{1}{x+1} = 1 - \frac{1}{2(x+1)} \quad \text{× } 2(x+1) \\ & \Rightarrow (2x+2) + 2 = 2(2x+1) - 1 \\ & \Rightarrow 2x+2 = 2^2+2x-1 \\ & \Rightarrow 0 = x^2-x-2 \\ & \Rightarrow (x-2)(x+1)=0 \\ & \Rightarrow x = -1, 2 \end{aligned}$$

BOTH ARE FINE

- $x = -1$: $\arctan(-1) + \arctan(2) = -\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{4}$
- $x = 2$: $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$

Question 180 (*)**

Solve the following trigonometric equation

$$2 \arctan\left(\frac{3}{x}\right) = \arcsin\left(\frac{6x}{25}\right).$$

$$\boxed{x^2}, \quad x = \pm 4$$

PROOF OF EQUALITY

$$\Rightarrow 2 \arctan\left(\frac{3}{x}\right) = \arcsin\left(\frac{6x}{25}\right)$$

$\theta = \arctan\left(\frac{3}{x}\right)$

$\tan\theta = \frac{3}{x}$

$\frac{3}{x} = \frac{\text{opp}}{\text{adj}}$

$\text{opp} = 3$

$\text{adj} = x$

$\sin\theta = \frac{3}{\sqrt{9+x^2}}$

$\cos\theta = \frac{x}{\sqrt{9+x^2}}$

$\sin\theta = \frac{3}{\sqrt{9+x^2}}$

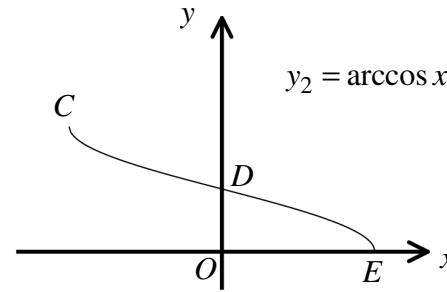
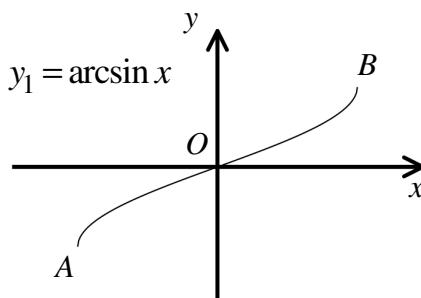
$\cos\theta = \frac{x}{\sqrt{9+x^2}}$

$$\Rightarrow 2\theta = \phi$$
$$\Rightarrow \sin 2\theta = \sin\phi$$
$$\Rightarrow 2\sin\theta\cos\theta = \sin\phi$$

USING THE VALUES FROM ABOVE

$$\Rightarrow 2 \left(\frac{3}{\sqrt{9+x^2}} \right) \left(\frac{x}{\sqrt{9+x^2}} \right) = \frac{6x}{25}$$
$$\Rightarrow \frac{6x}{9+x^2} = \frac{6x}{25}$$
$$\Rightarrow 9+x^2 = 25$$
$$\Rightarrow x^2 = 16$$
$$\Rightarrow x = \boxed{\pm 4}$$

Question 181 (****)



The figures above show the graph of $y_1 = \arcsin x$ and the graph of $y_2 = \arccos x$.

The graph of y_1 has endpoints at A and B .

The graph of y_2 has endpoints at C and E , and D is the point where the graph of y_2 crosses the y axis.

- a) State the coordinates of A , B , C , D and E .

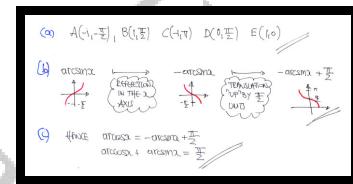
The graph of y_2 can be obtained from the graph of y_1 by a series of two geometric transformations which can be carried out in a specific order.

- b) Describe these two geometric transformations.
c) Deduce using valid arguments that

$$\arcsin x + \arccos x = \text{constant},$$

stating the exact value of this constant.

$A\left(-1, \frac{\pi}{2}\right)$	$B\left(1, \frac{\pi}{2}\right)$	$C(-1, \pi)$	$D\left(0, \frac{\pi}{2}\right)$	$E(1, 0)$	constant $= \frac{\pi}{2}$
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Question 182 (****)

It is given that

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

- a) Use the above trigonometric identity to show that

$$\sin 3x \equiv 3 \sin x - 4 \sin^3 x.$$

- b) Hence find

$$\int \cos x (6 \sin x - 2 \sin 3x)^{\frac{2}{3}} dx.$$

$$\boxed{\quad}, \boxed{\frac{4}{3} \sin^3 x + C}$$

a) Method A: FOIL

$$\begin{aligned}\sin 3x &= \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x \\&= (2\sin x \cos x + \cos^2 x) \sin x \\&= 2\sin x \cos^2 x + \sin x - 2\sin^3 x \\&= 2\sin x (1 - \sin^2 x) + \sin x - 2\sin^3 x \\&= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\&= 3\sin x - 4\sin^3 x.\end{aligned}$$

// as required

b) Using the result of part (a)

$$\begin{aligned}&\int \cos x (6 \sin x - 2 \sin 3x)^{\frac{2}{3}} dx \\&= \int \cos x [6 \sin x - 2(3 \sin x - 4 \sin^3 x)]^{\frac{2}{3}} dx \\&= \int \cos x [6 \sin x - 6 \sin^2 x + 8 \sin^3 x]^{\frac{2}{3}} dx \\&= \int \cos x (6 \sin x)^{\frac{2}{3}} dx \\&= \int \cos x (4 \sin^2 x)^{\frac{1}{3}} dx \\&= \int 4 \cos x \sin^{\frac{2}{3}} x dx \\&\quad \text{BY INTEGRATION BY USING THE SUBSTITUTION: } u = \sin x \\&= \boxed{\frac{4}{3} \sin^3 x + C}\end{aligned}$$

Question 183 (***)

$$f(x) = A \sec 2x + B, \quad 0 \leq x < 2\pi.$$

The graph of $f(x)$, where A and B are non zero constants, passes through the points $\left(\frac{\pi}{2}, -7\right)$ and $(\pi, 1)$.

- Determine the value of A and the value of B .
- Solve the equation

$$f\left(x + \frac{3\pi}{2}\right) = 5.$$

$$\boxed{}, \boxed{A=4}, \boxed{B=-3}, \quad x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

a) Using the points given $(\frac{\pi}{2}, -7)$ & $(\pi, 1)$

$(\frac{\pi}{2}, -7)$	$(\pi, 1)$
$-7 = A \sec \pi + B$	$1 = A \sec \frac{\pi}{2} + B$
$-7 = A + B$	$1 = A + B$

Adding both

$$2B = -6$$

$$B = -3$$

$A = 4$

b) Setting up the equation

$$f(x + \frac{3\pi}{2}) = 5$$

$$4 \sec(2(x + \frac{3\pi}{2})) - 3 = 5$$

$$4 \sec(2x + 3\pi) - 3 = 5$$

$$4 = \sec(2x + 3\pi) = 2$$

$$\cos(2x + 3\pi) = \frac{1}{2}$$

$\cos(2x + 3\pi) = \frac{1}{2}$

$$2x + 3\pi = \frac{\pi}{3} + 2m\pi$$

$$2x = -\frac{8\pi}{3} + 2m\pi$$

$$2x = -\frac{8\pi}{3} + 2m\pi$$

$$x = -\frac{4\pi}{3} + m\pi$$

$$x = \frac{2\pi}{3} + m\pi$$

(Since $0 \leq x < 2\pi$)

$$x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}$$

Question 184 (*)**

It is given that the angles θ , $\frac{\pi}{4}$ and φ are in arithmetic progression.

Show that

$$(\sin \theta - \sin \varphi)^2 + (\cos \theta + \cos \varphi)^2 = k,$$

where k is a constant to be found.

$$\boxed{\quad}, \boxed{k=2}$$

IF IN ARITHMETIC PROGRESSION, IN THE ORDER GIVEN

$$\begin{aligned} \rightarrow \frac{\pi}{4} - \theta &= \frac{\pi}{4} - \varphi \\ \rightarrow \theta + \varphi &= \frac{\pi}{2} \end{aligned}$$

NOW WE HAVE

$$\begin{aligned} &(\sin \theta - \sin \varphi)^2 + (\cos \theta + \cos \varphi)^2 \\ &= \sin^2 \theta - 2 \sin \theta \sin \varphi + \sin^2 \varphi + \cos^2 \theta + 2 \cos \theta \cos \varphi + \cos^2 \varphi \\ &= (\sin^2 \theta + \cos^2 \theta) + (\sin^2 \varphi + \cos^2 \varphi) + 2[\cos \theta \cos \varphi - \sin \theta \sin \varphi] \\ &= 2 + 2[\cos \theta \cos \varphi - \sin \theta \sin \varphi] \\ &= 2 + 2 \cos \left(\theta + \varphi \right) \\ &= 2 + 2 \cos \frac{\pi}{2} \\ &= 2 \end{aligned}$$

//
I.E. $k=2$

Question 185 (****)

It is given that

$$\cos x \cos\left(x + \frac{\pi}{4}\right) - \cos\left(2x - \frac{\pi}{4}\right) = 0.$$

Given further that $x \neq k\pi$, $k \in \mathbb{Z}$, show clearly that $\tan x = 3$

proof

$$\begin{aligned}
 & \bullet \cos x \cos\left(x + \frac{\pi}{4}\right) - \cos\left(2x - \frac{\pi}{4}\right) = 0 \\
 & \Rightarrow \cos x [\cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4}] - [\cos 2x \cos \frac{\pi}{4} + \sin 2x \sin \frac{\pi}{4}] = 0 \\
 & \Rightarrow \cos x [\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x] - [\frac{1}{\sqrt{2}} \cos 2x + \frac{1}{\sqrt{2}} \sin 2x] = 0 \\
 & \Rightarrow \frac{1}{\sqrt{2}} \cos^2 x - \frac{1}{\sqrt{2}} \cos x \sin x - \frac{1}{\sqrt{2}} \cos 2x - \frac{1}{\sqrt{2}} \sin 2x = 0 \\
 & \Rightarrow \cos^2 x - \cos x \sin x - \cos 2x - \sin 2x = 0 \\
 & \Rightarrow \cos^2 x - \cos x \sin x - (2\cos^2 x - 1) - 2\sin x \cos x = 0 \\
 & \Rightarrow 1 - \cos^2 x - 3\sin x \cos x = 0 \\
 & \Rightarrow \sin^2 x - 3\sin x \cos x = 0 \\
 & \Rightarrow \sin x (\sin x - 3\cos x) = 0 \\
 & \Rightarrow \sin x = 0 \text{ or } \sin x = 3\cos x \\
 & \Rightarrow \tan x = 3
 \end{aligned}$$

Question 186 (****)

$$y = \arcsin x, -1 \leq x \leq 1.$$

- a) By expressing $\arccos x$ in terms of y , show that

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

- b) Hence, or otherwise, solve the equation

$$3\arcsin(x-1) = 2\arccos(x-1).$$

$$\boxed{\quad}, \quad x = 1 + \sin\left(\frac{\pi}{5}\right) \approx 1.5878$$

a) Manipulate AS SUGGESTED

$$\begin{aligned} &\Rightarrow y = \arcsin x \\ &\Rightarrow \sin y = x \\ &\Rightarrow x = \sin y \\ &\text{Take "arccos" ON BOTH SIDES} \\ &\Rightarrow \arccos x = \arccos(\sin y) \\ &\Rightarrow \arccos x = \arccos(\cos(\frac{\pi}{2}-y)) \quad \leftarrow \sin A = \cos(\frac{\pi}{2}-A) \\ &\Rightarrow \arccos x = \frac{\pi}{2} - y \\ &\Rightarrow \arccos x = \frac{\pi}{2} - \arccos y \quad \leftarrow \text{As } y = \arcsin x \\ &\Rightarrow \arccos x + \arccos y = \frac{\pi}{2} \quad \text{As required} \end{aligned}$$

b) Using part (a), let $y = 2x-1$

$$\begin{aligned} &\Rightarrow 3\arcsin(2x-1) = 2\arccos(x-1) \\ &\Rightarrow 3\arcsin y = 2\arccos y \\ &\Rightarrow 3\arcsin y = 2[\frac{\pi}{2} - \arccos y] \\ &\Rightarrow 3\arcsin y = \pi - 2\arccos y \\ &\Rightarrow 3\arccos y = \pi - 3\arcsin y \\ &\Rightarrow \arccos y = \frac{\pi}{3} \\ &\Rightarrow y = \cos \frac{\pi}{3} \\ &\Rightarrow 2x-1 = \cos \frac{\pi}{3} \\ &\Rightarrow 2x-1 = \frac{1}{2} \\ &\Rightarrow 2x = 1.5 \\ &\Rightarrow x = 0.75 \end{aligned}$$

Question 187 (**)**

Simplify, showing clearly all the workings, the trigonometric expression

$$\cos^3 \theta \sin \theta - \sin^3 \theta \cos \theta,$$

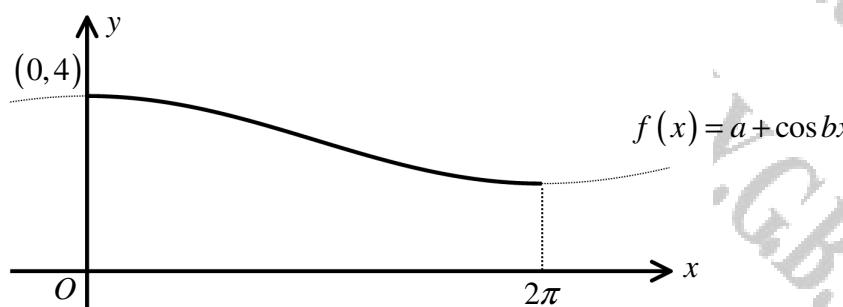
giving the final answer in the form $A \sin k\theta$, where A and k are constants.

$$\boxed{\quad}, \boxed{\frac{1}{4} \sin 4\theta}$$

Process to follow

$$\begin{aligned}& \rightarrow \cos^3 \theta \sin \theta - \sin^3 \theta \cos \theta \\&= \cos \theta \sin \theta [\cos^2 \theta - \sin^2 \theta] \\&= \cos \theta \sin \theta \cos 2\theta \quad [\cos 2\theta \equiv \cos^2 \theta - \sin^2 \theta] \\&= \frac{1}{2} (2 \cos \theta \sin \theta) \cos 2\theta \\&= \frac{1}{2} \sin 2\theta \cos 2\theta \quad [\sin 2\theta \equiv 2 \sin \theta \cos \theta] \\&= \frac{1}{2} \times 2 \sin 2\theta \cos 2\theta \\&= \frac{1}{2} \sin 4\theta \quad \begin{aligned}\sin 4\theta &\equiv \sin(2 \times 2\theta) \\ \sin 4\theta &\equiv 2 \sin 2\theta \cos 2\theta\end{aligned} \\&= \frac{1}{2} \sin 4\theta\end{aligned}$$

Question 188 (****)



The figure above shows the graph of the function

$$f(x) = a + \cos bx, \quad 0 \leq x \leq 2\pi,$$

where a and b are non zero constants.

The stationary points $(0, 4)$ and $(2\pi, 2)$ are the endpoints of the graph.

- State the range of $f(x)$ and hence find the value of a and the value of b .
- Find an expression for $f^{-1}(x)$, the inverse function of $f(x)$.
- State the domain and range of $f^{-1}(x)$.
- Find the gradient at the point on $f(x)$ with coordinates $\left(\frac{4\pi}{3}, \frac{5}{2}\right)$.
- State the gradient at the point on $f^{-1}(x)$ with coordinates $\left(\frac{5}{2}, \frac{4\pi}{3}\right)$.

, $[2 \leq f(x) \leq 4]$, $[a = 3, b = \frac{1}{2}]$, $[f^{-1}(x) = 2 \arccos(x-3)]$, $[2 \leq x \leq 4]$,

$[0 \leq f^{-1}(x) \leq 2\pi]$, $[-\frac{\sqrt{3}}{4}]$, $[-\frac{4}{\sqrt{3}}]$

(a) $y = 3 + \cos \frac{1}{2}x$ $y - 3 = \cos \frac{1}{2}x$ $\arccos(y-3) = \frac{1}{2}x$ $x = 2\arccos(y-3)$ $f^{-1}(x) = 2\arccos(x-3)$	(b) $f(0) = 3 + \cos(b \cdot 0)$ $2 = 3 + \cos(b \cdot 0)$ $\arccos(2) = 2\pi/b$ $2\pi/b = \pi$ $2b = 1$ $b = \frac{1}{2}$
(c) $D: 0 \leq x \leq 2\pi$ $R: 0 \leq y \leq 4$ $D: 0 \leq x \leq 2\pi$ $R: 2 \leq y \leq 4$ $D: 2 \leq x \leq 4$ $R: 0 \leq y \leq 2\pi$	(d) $f'(x) = -\frac{1}{2} \sin \left(\frac{1}{2}x\right)$ $f'\left(\frac{4\pi}{3}\right) = -\frac{1}{2} \sin \left(\frac{1}{2} \cdot \frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{4}$
(e) $f'(x) = -\frac{1}{2} \sin \left(\frac{1}{2}x\right)$	(f) $f'(x) = -\frac{1}{2} \sin \left(\frac{1}{2}x\right)$

Question 189 (*)**

Solve the following trigonometric equation.

$$\arctan 2x + \arctan x = \arctan 3, \quad x \in \mathbb{R}.$$

$$\boxed{}, \quad \boxed{x = \frac{1}{2}}$$

$\arctan 2x + \arctan x = \arctan 3$

USING THE COMPOUND ANGLE FORMULA FOR TANGENTS

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\Rightarrow \tan[\arctan 2x + \arctan x] = \tan(\arctan 3)$$
$$\Rightarrow \frac{\tan[\arctan 2x] + \tan[\arctan x]}{1 - \tan[\arctan 2x] \tan[\arctan x]} = 3$$
$$\Rightarrow \frac{2x + x}{1 - 2x^2} = 3$$
$$\Rightarrow 3x = 3(1 - 2x^2)$$
$$\Rightarrow x = 1 - 2x^2$$
$$\Rightarrow 2x^2 + x - 1 = 0$$
$$\Rightarrow (2x-1)(x+1) = 0$$
$$x = \begin{cases} \frac{1}{2} \\ -1 \end{cases}$$

As $\arctan(2x) + \arctan(x) < 0$
 $\arctan 3 > 0$

Question 190 (*)**

The function f is defined below.

$$f(x) \equiv 4\cos x - 3\sin\left(\frac{1}{2}x\right), \quad x \in \mathbb{R}.$$

Show that if θ satisfies the equation

$$4\sin\left(\frac{1}{2}\theta\right) + \sqrt{3} = 0,$$

then $f(\theta) = \frac{1}{4}(a + b\sqrt{3})$, where a and b are integers to be found.

$$\boxed{0^{\circ}}, \boxed{\frac{1}{4}(10 + 3\sqrt{3})}$$

Start by rearranging the equation

$$4\sin\frac{\theta}{2} + \sqrt{3} = 0$$

$$\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{4}$$

We need no solution over an actual range, if we simply manipulate the function

$$\Rightarrow f(\theta) = 4\cos\theta - 3\sin\frac{\theta}{2}$$

$$\Rightarrow f(\theta) = 4(1 - 2\sin^2\frac{\theta}{2}) - 3\sin\frac{\theta}{2}$$

$$\Rightarrow f(\theta) = 4 - 8\sin^2\frac{\theta}{2} - 3\sin\frac{\theta}{2}$$

Evaluating the result at $\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{4}$

$$\begin{aligned} \Rightarrow f(\theta) &= 4 - 8\left(-\frac{\sqrt{3}}{4}\right)^2 - 3\left(-\frac{\sqrt{3}}{4}\right) \\ &= 4 - 8\left(\frac{3}{16}\right) + \frac{3\sqrt{3}}{4} \\ &= 4 - \frac{3}{2} + \frac{3\sqrt{3}}{4} \\ &= \frac{5}{2} + \frac{3\sqrt{3}}{4} \\ &= \frac{10}{4} + \frac{3\sqrt{3}}{4} \\ &= \frac{1}{4}(10 + 3\sqrt{3}) \end{aligned}$$

$\cos 2A \equiv 1 - 2\sin^2 A$
 $\cos 2B \equiv 1 - 2\sin^2 B$
 $\cos A \equiv 1 - 2\sin^2 \frac{A}{2}$

Question 191 (**)**

The obtuse angles A and B , satisfy the following relationships.

$$\cos 2A = \sin B = \frac{1}{3}.$$

Determine the exact value of $\tan(A+B)$.

, $\tan(A+B) = -\sqrt{2}$

• STARTING FROM $\cos 2A = \frac{1}{3}$, NOTING THAT A IS OBTUSE

$$\begin{aligned}\Rightarrow \cos 2A &= 2\cos^2 A - 1 \\ \Rightarrow \frac{1}{3} &= 2\cos^2 A - 1 \\ \Rightarrow \frac{4}{3} &= 2\cos^2 A \\ \Rightarrow \cos^2 A &= \frac{2}{3} \\ \Rightarrow \cos A &= -\sqrt{\frac{2}{3}} \quad (2A \text{ is obtuse})\end{aligned}$$

• HENCE BY A STANDARD RIGHT ANGLED TRIANGLE

$$\therefore \tan A = -\frac{1}{\sqrt{2}}$$

• SIMILARLY $\sin B = \frac{1}{3}$ (B OBTUSE, SO BOTH $\sin B$ & $\tan B$ ARE NEGATIVE)

$$\therefore \tan B = -\frac{1}{\sqrt{2}}$$

• FINALLY BY THE COMPOUND ANGLE IDENTITIES

$$\begin{aligned}\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{1 - (-\frac{1}{\sqrt{2}})(-\frac{1}{\sqrt{2}})} = \frac{-\frac{2}{\sqrt{2}}}{1 - \frac{1}{2}} \\ &= -\frac{\frac{2}{\sqrt{2}}}{\frac{1}{2}} = -\frac{2}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}\end{aligned}$$

Question 192 (**)**

Simplify $(\tan x + \cot x)\sin 2x$ and hence prove that

$$\tan\left(\frac{1}{8}\pi\right) + \tan\left(\frac{5}{12}\pi\right) + \cot\left(\frac{1}{8}\pi\right) + \cot\left(\frac{5}{12}\pi\right) = 4+2\sqrt{2}.$$

, proof

START WITH THE SUSPECTED SIMPLIFICATION

$$\begin{aligned}(\tan x + \cot x)\sin 2x &= (\tan x + \cot x)2\sin x \cos x \\ &= \frac{\sin x}{\cos x} \cdot 2\sin x \cos x + \frac{\cos x}{\sin x} \cdot 2\sin x \cos x \\ &= 2\sin^2 x + 2\cos^2 x \\ &= 2(\sin^2 x + \cos^2 x) \\ &= 2.\end{aligned}$$

Now start the expression as follows, NOTING $\sin \frac{\pi}{4} = \sin \frac{5\pi}{12}$

$$\begin{aligned}(\tan \frac{1}{8}\pi + \cot \frac{1}{8}\pi) \sin \frac{\pi}{4} &= (\tan \frac{1}{8}\pi + \cot \frac{1}{8}\pi) \frac{\sqrt{2}}{2} = 2 \\ (\tan \frac{1}{8}\pi + \cot \frac{1}{8}\pi) \times \frac{1}{\sqrt{2}} &= 2 \\ (\tan \frac{1}{8}\pi + \cot \frac{1}{8}\pi) &= 2\sqrt{2} \\ (\tan \frac{1}{8}\pi + \cot \frac{1}{8}\pi) \times \frac{1}{2} &= 2 \\ \tan \frac{1}{8}\pi + \cot \frac{1}{8}\pi &= 4\end{aligned}$$

$$\therefore \tan \frac{1}{8}\pi + \tan \frac{5\pi}{12} + \cot \frac{1}{8}\pi + \cot \frac{5\pi}{12} = 4+2\sqrt{2}$$

As required

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HARD QUESTIONS

Question 1 (****+)

Find the solutions of the trigonometric equation

$$6 + 13\sin(2\theta + \alpha)^\circ = 5\cos 2\theta^\circ, \quad 0^\circ \leq \theta < 360^\circ$$

where $\tan \alpha^\circ = \frac{5}{12}$, $0^\circ < \alpha < 90^\circ$.

$$\boxed{\quad}, \quad \theta = 105^\circ, 165^\circ, 285^\circ, 345^\circ$$

This equation is about compound angle & NOT about double angles
AS THE ARGUMENT ARE 2θ THROUGHOUT

$$\begin{aligned} \rightarrow 6 + 13\sin(2\theta + \alpha)^\circ &= 5\cos 2\theta^\circ \\ \Rightarrow 6 + 13[\sin 2\theta \cos \alpha + \cos 2\theta \sin \alpha] &= 5\cos 2\theta \\ \Rightarrow 6 + 13(5/12)\cos 2\theta + 13(12/5)\sin 2\theta &= 5\cos 2\theta \end{aligned}$$

Now $\tan \alpha = \frac{5}{12}$ OR ACUTE



BY PYTHAGOREAN

$$\begin{aligned} \sin \alpha &= \frac{5}{13} \\ \cos \alpha &= \frac{12}{13} \end{aligned}$$

RETURNING TO THE MAIN QUES

$$\begin{aligned} \rightarrow 6 + 13(5/12)\cos 2\theta + 13(12/5)\sin 2\theta &= 5\cos 2\theta \\ \rightarrow 6 + 12.5\sin 2\theta + 15\cos 2\theta &= 5\cos 2\theta \\ \rightarrow 12.5\sin 2\theta &= -6 \\ \rightarrow \sin 2\theta &= -\frac{6}{12.5} \\ \arcsin(-\frac{6}{12.5}) &= -36^\circ \end{aligned}$$

$$\begin{aligned} 2\theta &= -36^\circ \pm 360^\circ \\ 2\theta &= 216^\circ \pm 360^\circ \\ \theta &= -18^\circ \pm 180^\circ \\ \theta &= 108^\circ \pm 180^\circ \end{aligned}$$

$\theta = 165^\circ, 345^\circ, 105^\circ, 285^\circ$

Question 2 (****+)

It is given that $\sin 1^\circ \approx 0.8415$ and $\cos 1^\circ \approx 0.5403$.

Show that $\sin(1.01^\circ) = 0.847$, correct to three decimal places.

$$\boxed{\quad}, \quad \text{proof}$$

$$\begin{aligned} \sin(1.01^\circ) &= \sin(1^\circ + 0.01^\circ) = \sin(\cos(0.01) + \cos(1^\circ)\sin(0.01)) \\ &= 0.8415 \times 1 + 0.5403 \times 0.01 = 0.8415 + 0.0054 \\ &= 0.8469 \\ &= 0.847 \quad \checkmark \end{aligned}$$

Question 3 (****+)

It is given that θ satisfies the equation

$$4\tan\theta + \cot\theta = 4.$$

Show clearly that

$$\cos 2\theta = \frac{3}{5}.$$

proof

$$\begin{aligned}
 4\tan\theta + \frac{1}{\tan\theta} &= 4 \\
 \Rightarrow 4\tan^2\theta + 1 &= 4\tan\theta \\
 \Rightarrow 4\tan^2\theta - 4\tan\theta + 1 &= 0 \\
 \Rightarrow (2\tan\theta - 1)^2 &= 0 \\
 \Rightarrow \tan\theta &= \frac{1}{2} \\
 \Rightarrow \tan^2\theta &= \frac{1}{4} \\
 \Rightarrow 1 + \tan^2\theta &= \frac{5}{4}
 \end{aligned}
 \quad
 \begin{aligned}
 \Rightarrow \sec^2\theta &= \frac{5}{4} \\
 \Rightarrow \sec\theta &= \frac{\sqrt{5}}{2} \\
 \Rightarrow \sec\theta &= \frac{2}{\sqrt{5}} \\
 \Rightarrow 2\sec\theta - 1 &= \frac{3}{\sqrt{5}} \\
 \Rightarrow \cos 2\theta &= \frac{3}{5}
 \end{aligned}$$

Question 4 (****+)

Find in radians, correct to two decimal places, the solutions of the trigonometric equation

$$\sec 2x - 3\tan 2x = 2, \quad 0 \leq x < 2\pi.$$

$$x \approx 1.14^\circ, 2.99^\circ, 4.28^\circ, 6.13^\circ$$

$$\begin{aligned}
 \sec 2x - 3\tan 2x &= 2 \\
 \Rightarrow \frac{1}{\cos 2x} - \frac{3\sin 2x}{\cos 2x} &= 2 \\
 \Rightarrow 1 - 3\tan 2x &= 2\cos 2x \\
 \Rightarrow 1 - 2\cos 2x - 3\tan 2x &= 0
 \end{aligned}$$

Now

$$\begin{aligned}
 2\cos 2x + 3\tan 2x &= \text{Rsin}(2x) \\
 2\cos 2x + 3\tan 2x &= \text{Rsin}2x\cos x + \text{Rcos}2x\sin x \\
 &\equiv (\text{Rcos}2x)(\cos x + \text{Rsin}2x)\sin x
 \end{aligned}$$

$$\begin{aligned}
 \text{Rcos}2x &= 3 \\
 \text{Rsin}2x &= 2 \Rightarrow \left\{ \begin{array}{l} 2 = \sqrt{3^2 + 2^2} = \sqrt{13} \\ \tan 2x = \frac{2}{3}, \quad 1/x = 0.588^\circ \end{array} \right.
 \end{aligned}$$

Thus

$$\begin{aligned}
 \Rightarrow \sqrt{13}\sin(2x + 0.588^\circ) &= 1 \\
 \Rightarrow \sin(2x + 0.588^\circ) &= \frac{1}{\sqrt{13}} \\
 \Rightarrow (2x + 0.588^\circ) &= 0.281^\circ \pm 2\pi n^\circ, \quad n = 0, 1, 2, \dots \\
 \Rightarrow (2x + 0.588^\circ) &= 2.85^\circ \pm 2\pi n^\circ \\
 \Rightarrow (2x + 0.588^\circ) &= 0.91^\circ, 3.14^\circ, 6.13^\circ, 8.31^\circ, \dots \\
 \Rightarrow 2x &= 1.14^\circ, 2.99^\circ, 4.28^\circ, 6.13^\circ
 \end{aligned}$$

Question 5 (****+)

Solve in radians the trigonometric equation

$$\sin 8x = \sin 2x, \quad 0 \leq x < \frac{\pi}{2},$$

giving the answers in terms of π .

$$x = 0, \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{3}, \frac{\pi}{2}$$

sin 8x = sin 2x
 $\sin 8x - \sin 2x = 0$
 $2\sin(\frac{8x-2x}{2})\cos(\frac{8x+2x}{2}) = 0$
 $2\sin 3x \cos 5x = 0$
 $\cos 5x = 0 \quad \text{or} \quad \sin 3x = 0$
 $5x = \frac{\pi}{2} + k\pi \quad \text{or} \quad 3x = k\pi$
 $x = \frac{\pi}{10} + \frac{k\pi}{5} \quad \text{or} \quad x = \frac{k\pi}{3}$
 $x = 0, \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{3}, \frac{\pi}{2}$

Question 6 (****+)

Solve in degrees the trigonometric equation

$$\sin 5\theta + \sin 3\theta = 0, \quad 0^\circ \leq \theta < 180^\circ.$$

$$\theta = 0^\circ, 45^\circ, 90^\circ, 135^\circ$$

sin 5θ + sin 3θ = 0
 $\sin 5θ + \sin 3θ = 2\sin \frac{4θ}{2}\cos \frac{2θ}{2} = 0$
 $2\sin \frac{4θ}{2}\cos \frac{2θ}{2} = 0$
 $2\sin 2θ \cos 2θ = 0$
 $\sin 2θ = 0 \quad \text{or} \quad \cos 2θ = 0$
 $2θ = 0 + 360n^\circ \quad \text{or} \quad 2θ = 90 + 360n^\circ$
 $θ = 0 + 180n^\circ \quad \text{or} \quad θ = 45 + 180n^\circ$
 $θ = 0, 45, 90, 135$

Question 7 (**+)**

Solve the following trigonometric equation

$$\frac{\cos 2x}{1 + \cos 2x} = 1 - 2 \tan x, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

$$\boxed{\quad}, \quad x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

Start by reducing the double angles

$$\begin{aligned} \Rightarrow \frac{\cos 2x}{1 + \cos 2x} &= 1 - 2 \tan x \\ \Rightarrow \frac{\cos^2 x - \sin^2 x}{1 + (2 \cos^2 x - 1)} &= 1 - 2 \tan x \\ \Rightarrow \frac{\cos^2 x - \sin^2 x}{2 \cos^2 x} &= 1 - 2 \tan x \\ \Rightarrow \frac{\cos^2 x - \sin^2 x}{\cos^2 x} &= 2 - 4 \tan x \\ \Rightarrow 1 - \tan^2 x &= 2 - 4 \tan x \\ \Rightarrow 0 &= 1 - 4 \tan x + \tan^2 x \end{aligned}$$

The quadratic does not factorise nicely, so proceed by the quadratic formula or by completing the square

$$\begin{aligned} \Rightarrow (4 \tan x - 2)^2 - 4 + 1 &= 0 \\ \Rightarrow (4 \tan x - 2)^2 &= 3 \\ \Rightarrow 4 \tan x - 2 &= \pm \sqrt{3} \\ \Rightarrow 4 \tan x &= \frac{2 + \sqrt{3}}{2 - \sqrt{3}} \Rightarrow \tan x (2 + \sqrt{3}) &= \frac{\sqrt{3}}{12} \end{aligned}$$

Solving separately in radians

- $x = \frac{\pi}{12} \pm n\pi \quad n \in \mathbb{Z}, \dots$
- $x = \frac{7\pi}{12} \pm n\pi \quad n \in \mathbb{Z}, \dots$

$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

Question 8 (****+)

Solve in degrees the trigonometric equation

$$4 \tan(\theta + 60) \tan(\theta - 60) = \sec^2 \theta - 16, \quad 0^\circ \leq \theta < 180^\circ.$$

$$\boxed{\quad}, \quad \theta = 15^\circ, 75^\circ, 105^\circ, 165^\circ$$

USING THE TAN ADDITION FORMULA & THE PYTHAGOREAN IDENTITIES CONCERNING TANGENT, SECANT

$$\begin{aligned} &\rightarrow 4 \tan(\theta + 60) \tan(\theta - 60) = \sec^2 \theta - 16 \\ &\rightarrow 4 \left(\frac{\tan \theta + \tan 60}{1 - \tan \theta \tan 60} \right) \left(\frac{\tan \theta - \tan 60}{1 + \tan \theta \tan 60} \right) = (1 + \tan^2 \theta) - 16 \\ &\rightarrow 4 \left(\frac{\tan \theta + \sqrt{3}}{1 - \sqrt{3} \tan \theta} \right) \left(\frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta} \right) = \tan^2 \theta - 15 \\ &\rightarrow \frac{4(\tan^2 \theta - 3)}{1 - 3\tan^2 \theta} = \tan^2 \theta - 15 \\ &\rightarrow 4T^2 - 12 = T^2 - 3T^2 - 15 + 4ST \\ &\rightarrow 3T^2 - 4ST + 3 = 0 \\ &\rightarrow T^2 - 14T + 1 = 0 \\ &\rightarrow (T-7)^2 - 48 = 0 \\ &\rightarrow (T-7)^2 = 48 \\ &\rightarrow T-7 = \pm \sqrt{48} \end{aligned}$$

$$\begin{aligned} &\rightarrow T = 7 \pm \sqrt{48} \\ &\rightarrow \tan \theta = 7 \pm \sqrt{48} \\ &\rightarrow \tan \theta = \pm \sqrt{7 \pm \sqrt{48}} \\ &\text{INDICATING ALL 4 POSSIBLE ARCTANS} \\ &\left\{ \begin{array}{l} \theta = 15^\circ + 180n \\ \theta = -15^\circ + 180n \\ \theta = 75^\circ + 180n \\ \theta = -75^\circ + 180n \end{array} \right. \quad n=0,1,2,3,\dots \\ &\therefore \theta = 15^\circ, 75^\circ, 105^\circ, 165^\circ \end{aligned}$$

Question 9 (**+)**

Prove the validity of each of the following trigonometric identities.

a) $(\tan \theta + \cot \theta)(\sin \theta + \cot \theta) \equiv \sec \theta + \operatorname{cosec} \theta$.

b) $\tan\left(\theta + \frac{\pi}{4}\right) \equiv \sec 2\theta + \tan 2\theta$.

proof

$$\begin{aligned} \text{(a) LHS} &= (\tan \theta + \cot \theta)(\sin \theta + \cot \theta) \\ &= \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}\right) \left(\sin \theta + \frac{\cos \theta}{\sin \theta}\right) = \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \times (\sin \theta + \cot \theta) \\ &= \frac{\sin \theta + \cot \theta}{\sin \theta \cos \theta} = \frac{\sin \theta}{\sin \theta \cos \theta} + \frac{\cot \theta}{\sin \theta \cos \theta} \\ &= \frac{1}{\cos \theta} + \frac{1}{\sin \theta} = \sec \theta + \operatorname{cosec} \theta = \text{RHS} // \end{aligned}$$

$$\begin{aligned} \text{(b) LHS} &= \tan\left(\theta + \frac{\pi}{4}\right) = \frac{\tan \theta + \tan \frac{\pi}{4}}{1 - \tan \theta \tan \frac{\pi}{4}} = \frac{\tan \theta + 1}{1 - \tan \theta} \\ &= \frac{(\tan \theta + 1)(1 + \tan \theta)}{(1 - \tan \theta)(1 + \tan \theta)} = \frac{1 + 2\tan \theta + \tan^2 \theta}{1 - \tan^2 \theta} \\ &= \frac{\sec^2 \theta}{1 - \tan^2 \theta} + \frac{2\tan \theta}{1 - \tan^2 \theta} = \frac{1}{\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{2\tan \theta}{\cos^2 \theta}} \\ &= \frac{1}{\frac{\cos^2 \theta}{\sin^2 \theta} + \tan^2 \theta} = \frac{1}{\frac{\cos^2 \theta + \sin^2 \theta}{\sin^2 \theta}} = \frac{1}{\frac{1}{\sin^2 \theta}} = \operatorname{cosec}^2 \theta \\ &= \frac{1}{\cos^2 \theta} + \tan^2 \theta = \sec^2 \theta + \tan^2 \theta = \text{RHS} // \end{aligned}$$

MUST NOT DIVIDE BY $\cos^2 \theta$

Question 10 (***)+

$$2 \arctan\left[\frac{1}{x-3}\right] + \arctan\left[\frac{1}{x+2}\right] = \arctan\left[\frac{31}{17}\right].$$

Show that $x=5$ is one of the solutions of the above trigonometric equation and find, in exact surd form, the other two solutions.

$$x = \frac{10 \pm 5\sqrt{190}}{31}$$

$$2 \arctan\left(\frac{1}{x-3}\right) + \arctan\left(\frac{1}{x+2}\right) = \arctan\left(\frac{31}{17}\right)$$

$$\Rightarrow 2 \arctan\left(\frac{1}{x-3}\right) = \arctan\left(\frac{31}{17}\right) - \arctan\left(\frac{1}{x+2}\right)$$

- TAKING TANGENTS ON BOTH SIDES

$$\Rightarrow \frac{2 \left(\frac{1}{x-3}\right)}{1 - \left(\frac{1}{x-3}\right)^2} = \frac{\frac{31}{17} - \frac{1}{x+2}}{1 + \frac{31}{17} \times \frac{1}{x+2}}$$

- TIDYING UP

$$\Rightarrow \frac{2(x-3)}{(x-3)^2 - 1} = \frac{31(2x+2) - 17}{17(2x+2) + 31}$$

$$\Rightarrow \frac{2x-6}{x^2 - 6x + 8} = \frac{31x + 45}{34x + 65}$$

$$\Rightarrow (31x + 45)(x^2 - 6x + 8) = (2x - 6)(34x + 65)$$

$$\Rightarrow 31x^3 - 186x^2 + 248x = 34x^2 + 80x - 102x - 390$$

$$\Rightarrow 31x^3 - 175x^2 - 50x + 750 = 0$$

- BY LONG DIVISION/BY MANUFACTURE

$$\Rightarrow 31x^2(x-5) - 20x(x-5) - 150(x-5) = 0$$

$$\Rightarrow (x-5)(30x^2 - 20x - 150) = 0$$

$\therefore x=5$ OR BY QUADRATIC FORMULA $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\therefore x = \frac{10 \pm 5\sqrt{190}}{31}$$

Question 11 (**+) Non Calculator**

A triangle, ABC has $|AB| = 2\sqrt{3}$ cm, $\angle BAC = 45^\circ$ and $\angle ACB = 60^\circ$.

Determine, in exact simplified surd form, the area of this triangle

, area = $3 + \sqrt{3}$

Given $\hat{A}BC = 75^\circ$

By the Sine Rule we have

$$\frac{|BC|}{\sin 45^\circ} = \frac{|AB|}{\sin 60^\circ}$$

$$|BC| = \frac{|AB| \sin 60^\circ}{\sin 45^\circ}$$

$$|BC| \approx \frac{2\sqrt{3} \times \frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{6}}{\frac{\sqrt{2}}{2}} = \frac{2\sqrt{6}}{\sqrt{2}} = 2\sqrt{\frac{6}{2}} = 2\sqrt{3}$$

Now the area can be found as

$$\begin{aligned} \text{Area} &= \frac{1}{2}|AB||BC|\sin 75^\circ \\ &= \frac{1}{2}(2\sqrt{3})(2\sqrt{3})\sin(45+30)^\circ \\ &= 2\sqrt{3}[\sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ] \\ &= 2\sqrt{3}\left[\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}\right] \\ &= 2\sqrt{3} \times \frac{\sqrt{6} + \sqrt{2}}{4} \\ &= \frac{12 + 2\sqrt{12}}{4} \\ &= \frac{12 + 2(2\sqrt{3})}{4} \\ &= \frac{12 + 4\sqrt{3}}{4} \\ &= 3 + \sqrt{3} \end{aligned}$$

Question 12 (***)+

It is given that

$$\sin^2\left(\theta + \frac{\pi}{4}\right) - \sin^2\left(\theta - \frac{\pi}{4}\right) \equiv \sin 2\theta.$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, show that ...

i. ... $\sin\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta + \frac{\pi}{4}\right) - \sin\left(\theta - \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right) \equiv \cos 2\theta.$

ii. ... $\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{5\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right) = \frac{1}{2}.$

proof

(a)

$$\begin{aligned}
 LHS &= \sin\left(\theta + \frac{\pi}{4}\right) - \sin\left(\theta - \frac{\pi}{4}\right) = \left[\sin\left(\theta + \frac{\pi}{4}\right)\right]^2 - \left[\sin\left(\theta - \frac{\pi}{4}\right)\right]^2 \\
 &= \left(\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right)^2 - \left(\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\right)^2 \\
 &= \left(\frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta\right)^2 - \left(\frac{\sqrt{2}}{2}\sin\theta - \frac{\sqrt{2}}{2}\cos\theta\right)^2 \\
 &= (\cancel{\frac{\sqrt{2}}{2}\sin\theta} + \cancel{\frac{\sqrt{2}}{2}\cos\theta} + \cancel{\frac{\sqrt{2}}{2}\cos\theta}) - (\cancel{\frac{\sqrt{2}}{2}\sin\theta} - \cancel{\frac{\sqrt{2}}{2}\cos\theta} + \cancel{\frac{\sqrt{2}}{2}\cos\theta}) \\
 &= 2\sin\theta\cos\theta = \sin 2\theta = RHS
 \end{aligned}$$

difference of squares

$$\begin{aligned}
 LHS &= \sin^2\left(\theta + \frac{\pi}{4}\right) - \sin^2\left(\theta - \frac{\pi}{4}\right) = \left[\sin\left(\theta + \frac{\pi}{4}\right)\right]\left[\sin\left(\theta + \frac{\pi}{4}\right) - \sin\left(\theta - \frac{\pi}{4}\right)\right] \\
 &= \left[\sin\left(\theta + \frac{\pi}{4}\right)\right]\left[\cancel{\sin\left(\theta + \frac{\pi}{4}\right)}\left(\cancel{\sin\left(\theta + \frac{\pi}{4}\right)} - \cancel{\sin\left(\theta - \frac{\pi}{4}\right)}\right)\right] \\
 &= \cancel{\sin\left(\theta + \frac{\pi}{4}\right)}\left(\cancel{\sin\left(\theta + \frac{\pi}{4}\right)} - \cancel{\sin\left(\theta - \frac{\pi}{4}\right)}\right) = (\cancel{\sqrt{2}\sin\theta})(\cancel{\sqrt{2}\cos\theta}) = 2\sin\theta\cos\theta \\
 &= \sin 2\theta = RHS
 \end{aligned}$$

(b)

$$\begin{aligned}
 \sin 2\theta &= \sin\left(\theta + \frac{\pi}{4}\right) - \sin\left(\theta - \frac{\pi}{4}\right) \\
 \frac{d}{d\theta}(\sin 2\theta) &= \frac{d}{d\theta}[\sin\left(\theta + \frac{\pi}{4}\right) - \sin\left(\theta - \frac{\pi}{4}\right)] \\
 2\cos 2\theta &= 2\sin\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta + \frac{\pi}{4}\right) - 2\sin\left(\theta - \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right) \\
 \text{Divide by 2} \\
 \cos 2\theta &= \sin\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta + \frac{\pi}{4}\right) - \sin\left(\theta - \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right) \\
 \cos 2\theta &= \sin\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cos\frac{\pi}{2} \\
 &= \sin\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cos\frac{\pi}{2} \\
 &\quad \text{cos}(-A) = \cos A \\
 &\quad \sin(-A) = -\sin A
 \end{aligned}$$

Question 13 (***)+

Solve the trigonometric equation

$$(3\sin x + 5\cos x)^2 = 4\cos^2 x, \text{ for } 0 \leq x < 2\pi$$

giving the answers correct to three significant figures

$$x \approx 1.98^c, 2.36^c, 5.12^c, 5.50^c$$

$$3\sin x + \sin 3x = 4\cos^2 x$$

$$3\sin x + \sin 3x < -2\cos x$$

$$3\sin x < -2\cos x$$

$$\text{Divide by } \sin x \text{ to make tangent}$$

$$\tan x = -\frac{2}{3}$$

$$\arctan(-\frac{2}{3})$$

$$= -1.465^\circ \pm 180^\circ$$

$$= -1.465^\circ \pm 107.3^\circ$$

Question 14 (***)+

Solve the following trigonometric equation

$$\tan x + \cot x = 8 \cos 2x, \quad 0 \leq x < \pi$$

where x is measured in radians.

$$\boxed{}, \quad x = \frac{1}{24}\pi, \frac{5}{24}\pi, \frac{13}{24}\pi, \frac{17}{24}\pi$$

MANIPULATE JUST THE LHS OF THE EQUATION

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 8 \cot 2x$$

$$\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = 8 \cot 2x$$

$$\frac{1}{\sin x \cos x} = 8 \cot 2x$$

$$\frac{1}{\sin 2x \cos 2x} = 8 \cot 2x$$

$$\frac{1}{\sin 2x} = 4 \cot 2x$$

$$4 \cot 2x \sin 2x = 1$$

$$2 \csc 2x \cot 2x = \frac{1}{2}$$

$$\sin 2x = \frac{1}{2}$$

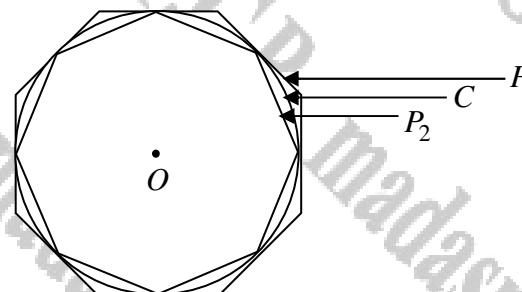
ON THE ORIGIN $\left(\frac{1}{2}, 0\right)$

$\begin{cases} 2x = \frac{\pi}{6} \pm 2k\pi \\ 4x = \frac{\pi}{3} \pm 4k\pi \end{cases} \Rightarrow \begin{cases} x_1 = \frac{\pi}{12} \\ x_2 = \frac{5\pi}{12} \\ x_3 = \frac{11\pi}{12} \\ x_4 = \frac{19\pi}{12} \end{cases}$

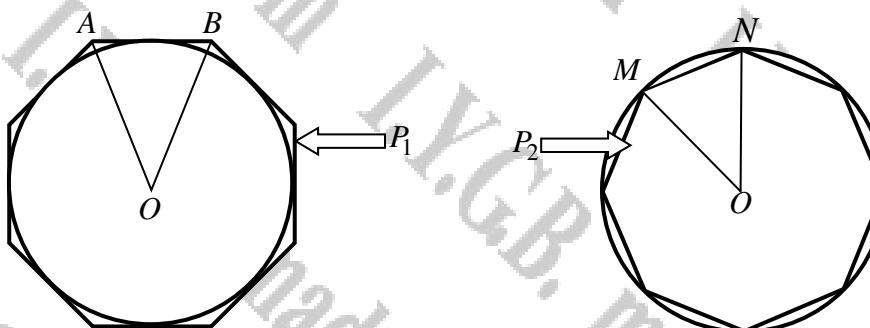
Question 15 (**+)**

The figure below shows a regular octagon P_1 . A circle C is inscribed inside P_1 and another regular octagon P_2 is inscribed inside the circle C .

The three objects have a common centre at O .



The circle C has a radius of 1 unit. The points A and B are consecutive vertices of P_1 , and the points M and N are consecutive vertices of P_2 .



- By considering the triangle OAB , show that the perimeter of the octagon P_1 is $16 \tan \frac{\pi}{8}$.
- Use the triangle OMN in a similar fashion to show that the perimeter of the octagon P_2 is $16 \sin \frac{\pi}{8}$.

[continues overleaf]

[continues from previous page]

- c) Use a standard identity for $\cos 2\theta$ to show that

$$\sin \frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}.$$

- d) Show further that

$$\tan \frac{\pi}{8} = -1 + \sqrt{2}.$$

- e) Deduce from the results obtained so far that

$$3.06 < \pi < 3.31.$$

, proof

a) LOOKING AT THE DIAGRAM

$$360^\circ \div 8 = 45^\circ$$

$$\text{PERIMETER OF OCTAGON} = 8 \times 2 \sin \frac{\pi}{8} = 16 \sin \frac{\pi}{8}$$

b) LOOKING AT THE DIAGRAM AGAIN

$$[MN] = \sin \frac{\pi}{4}$$

$$[ML] = \sin \frac{\pi}{4}$$

$$[NL] = \sin \frac{\pi}{4}$$

$$[MN] = 2 \sin \frac{\pi}{8}$$

$$\text{PERIMETER OF OCTAGON} = 8 \times 2 \sin \frac{\pi}{8} = 16 \sin \frac{\pi}{8}$$

c) USING $\cos 2\theta \equiv 1 - 2\sin^2 \theta$ WITH $\theta = \frac{\pi}{8}$

$$\cos \frac{\pi}{4} = 1 - 2\sin^2 \frac{\pi}{8}$$

$$2\sin^2 \frac{\pi}{8} = 1 - \cos \frac{\pi}{4}$$

$$2\sin^2 \frac{\pi}{8} = 1 - \frac{\sqrt{2}}{2}$$

$$\sin^2 \frac{\pi}{8} = \frac{1 - \sqrt{2}}{2}$$

$$\sin \frac{\pi}{8} = \frac{2 - \sqrt{2}}{2}$$

d) $\sin \frac{\pi}{8} = +\sqrt{\frac{2 - \sqrt{2}}{2}}$ ($\frac{\pi}{8}$ is acute)

$$\sin \frac{\pi}{8} = \frac{1}{2}\sqrt{2 - \sqrt{2}}$$
 // As required

e) USING THE DOUBLE ANGLE IDENTITY FOR $\tan 2\theta$ WITH $\theta = \frac{\pi}{8}$

$$\tan \frac{\pi}{4} = \frac{2 \tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}}$$

$$1 = \frac{2T}{1 - T^2}$$

$$1 - T^2 = 2T$$

$$0 = T^2 + 2T - 1$$

$$(T+1)^2 - 2 = 0$$

$$(T+1)^2 = 2$$

$$T+1 = \pm \sqrt{2}$$

$$T = -1 \pm \sqrt{2}$$

$$\tan \frac{\pi}{8} = -1 \pm \sqrt{2}$$

$$\tan \frac{\pi}{8} = -1 + \sqrt{2}$$
 // ($\frac{\pi}{8}$ is acute & $-1 + \sqrt{2}$ is positive)

Finally we have:

Possible P_8	<	Circumference C / Perimeter P
$16 \sin \frac{\pi}{8}$	<	$2\pi \times 1$
$16 \left(\frac{1}{2} \sqrt{2 - \sqrt{2}} \right)$	<	$16 \sin \frac{\pi}{8}$
3.06	<	2π
3.06	<	π
3.06	<	3.31

Question 16 (****+)

Solve the following trigonometric equation

$$\frac{\cot x}{\cosec x - 1} + \frac{\cosec x - 1}{\cot x} = 4, \quad 0^\circ \leq x < 360^\circ.$$

$$x = 60^\circ, 300^\circ$$

Working for Question 16:

$$\begin{aligned} \frac{\cot x}{\cosec x - 1} + \frac{\cosec x - 1}{\cot x} &= 4 \\ \Rightarrow \cot^2 x + (\cosec x - 1)^2 &= 4(\cosec x - 1)\cot x \\ \Rightarrow \cot^2 x + \cosec^2 x - 2\cosec x + 1 &= 4\cosec x \cot x \\ \Rightarrow 2\cosec^2 x - 2\cosec x + 1 &= 4\cosec x \cot x \\ \Rightarrow 2\cosec x (\cosec x - 1) &= 4\cosec x \cot x \\ \Rightarrow \cosec x - 1 &= 2\cot x \\ \Rightarrow \cosec x &= 2\cot x \\ \Rightarrow \cosec x &= \frac{1}{2} \\ \Rightarrow \cos x &= \pm \sqrt{2} \\ \Rightarrow \cos x &= \pm \frac{1}{2} \\ \Rightarrow x &= 60^\circ \text{ or } 300^\circ \\ x &= 30^\circ + 360^\circ k, k \in \mathbb{Z} \\ x &= 60^\circ, 300^\circ \end{aligned}$$

Question 17 (****+)

$$f(x) = \ln(1 + \sin x), \quad \sin x \neq \pm 1.$$

Show clearly that

$$f(x) - f(-x) = 2 \ln(\sec x + \tan x).$$

[] , proof

Working for Question 17:

$$\begin{aligned} f(x) - f(-x) &= \ln(1 + \sin x) - \ln(1 + \sin(-x)) \\ &= \ln(1 + \sin x) - \ln(1 - \sin x) \\ &= \ln \left(\frac{1 + \sin x}{1 - \sin x} \right) \\ &= \ln \left[\frac{(1 + \sin x)(1 + \sin x)}{(1 - \sin x)(1 + \sin x)} \right] \\ &= \ln \left[\frac{1 + 2\sin x + \sin^2 x}{1 - \sin^2 x} \right] \\ &= \ln \left[\frac{1 + 2\sin x + \sin^2 x}{\cos^2 x} \right] \\ &= \ln \left[\frac{1 + 2\sin x}{\cos x} \right] \\ &= 2 \ln \left(\frac{1 + 2\sin x}{\cos x} \right) \\ &= 2 \ln \left(\frac{1 + 2\sin x}{\cos x} \right) \\ &= 2 \ln(\sec x + \tan x) \end{aligned}$$

Question 18 (***)+

It is given that

$$\cos 3x \equiv 4\cos^3 x - 3\cos x.$$

- a) Prove the validity of the above trigonometric identity.
- b) By differentiating both sides of the above identity with respect to x , show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x.$$

proof

(a) $\text{LHS} = \cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$
 $= (\cos^2 x - 1)\cos x - (2\sin x \cos x)\cos x = 2\cos^3 x - \cos x - 2\cos^2 x \cos x$
 $= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x = 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$
 $= 4\cos^3 x - 3\cos x = 2\cancel{4} \cancel{\cos^3 x} + 4\cos x$

(b) $\cos 3x = \cos(2x + x)$
 $\Rightarrow \frac{d}{dx}(\cos 3x) = \frac{d}{dx}(4\cos^3 x - 3\cos x) \Rightarrow \sin 3x = 4\cos^2 x \cdot (-3) - 4\cos x$
 $\Rightarrow -3\sin 3x = (12\cos^2 x)(-\sin x) - 3\sin x \Rightarrow \sin 3x = 3\sin x - 4\cos^2 x \cdot 3 \Rightarrow -3\sin 3x = -12\cos^2 x \sin x + 3\sin x \Rightarrow \sin 3x = 4\cos^2 x \sin x - \sin x \Rightarrow \sin 3x = 4(1 - \sin^2 x)\sin x - \sin x$

As required

Question 19 (***)+

Prove the validity of each of the following trigonometric identities.

a) $\frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} \equiv \cot x.$

b) $2\cos^4 \theta + \frac{1}{2}\sin^2 2\theta - 1 \equiv \cos 2\theta.$

proof

(a) $\text{LHS} = \frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} = \frac{(\cos 2x)^2 - \cos x + 1}{2\sin x \cos x - \sin x}$
 $= \frac{2\cos^2 x - \cos x}{\sin x(2\cos x - 1)} = \frac{\cos x(2\cos x - 1)}{\sin x(2\cos x - 1)} = \cot x = \text{RHS}$

(b) $\text{LHS} = 2\cos^4 \theta + \frac{1}{2}\sin^2 2\theta - 1 = 2(\cos^2 \theta)^2 + \frac{1}{2}\sin^2 2\theta - 1$
 $\text{Now } \cos 2\theta = 2\cos^2 \theta - 1 \quad \text{and } \sin^2 2\theta = 1 - \cos^2 2\theta$
 $\cos^2 \theta = \frac{1}{2} + \cos 2\theta$
 $= 2\left(\frac{1}{2} + \cos 2\theta\right)^2 + \frac{1}{2}\sin^2 2\theta - 1 = 2\left(\frac{1}{4} + \cos^2 2\theta + \frac{1}{2}\cos 2\theta\right) + \frac{1}{2}\sin^2 2\theta - 1$
 $= \frac{1}{2}(1 + 2\cos 2\theta + (\cos 2\theta)^2) + \frac{1}{2}\sin^2 2\theta - 1$
 $= \frac{1}{2} + 2\cos 2\theta + \frac{1}{2}\cos^2 2\theta + \frac{1}{2}\sin^2 2\theta - 1$
 $= -\frac{1}{2} + \cos 2\theta + \frac{1}{2}(\cos^2 2\theta + \sin^2 2\theta)$
 $= -\frac{1}{2} + \cos 2\theta + \frac{1}{2} = \cos 2\theta = \text{RHS}$

Question 20 (***)+

It is given that

$$\frac{\tan x}{\sec x - 1} - \frac{\sec x - 1}{\tan x} \equiv 2 \cot x, \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence solve the trigonometric equation

$$\frac{\tan 3\theta}{\sec 3\theta - 1} - \frac{\sec 3\theta - 1}{\tan 3\theta} = \frac{2}{\sqrt{3}}, \quad 0 \leq \theta < \pi,$$

giving the answers in terms of π .

$$x = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}$$

$(a) LHS = \frac{\tan x}{\sec x - 1} - \frac{\sec x - 1}{\tan x}$ $= \frac{\tan x - (\sec x - 1)^2}{(\sec x - 1)\tan x}$ $= \frac{\tan x - (\sec^2 x - 2\sec x + 1)}{\tan x (\sec x - 1)}$ $= \frac{\tan x - \sec^2 x + 2\sec x - 1}{\tan x (\sec x - 1)}$ $= \frac{(\sec^2 x) - \sec^2 x + 2\sec x - 1}{\tan x (\sec x - 1)}$ $= \frac{2\sec x - 2}{\tan x (\sec x - 1)}$ $= \frac{2}{\tan x} = 2 \cot x = RHS$	$(b) \frac{\tan 3\theta}{\sec 3\theta - 1} - \frac{\sec 3\theta - 1}{\tan 3\theta} = \frac{2}{\sqrt{3}}$ RHS (a) $\Rightarrow 2 \cot 3\theta = \frac{2}{\sqrt{3}}$ $\Rightarrow \cot 3\theta = \frac{1}{\sqrt{3}}$ $\Rightarrow \tan 3\theta = \sqrt{3}$ $\arctan(\sqrt{3}) = \frac{\pi}{6}$ <ul style="list-style-type: none"> • $3\theta = \frac{\pi}{6} + k\pi$ • $\theta = \frac{\pi}{18} + \frac{k\pi}{3}$ $\therefore \theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{9\pi}{18}$
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Question 21 (***)+

Prove the validity of the following trigonometric identity.

$$\frac{1 + \cos \theta}{1 - \cos \theta} \equiv (\operatorname{cosec} \theta + \cot \theta)^2.$$

proof

$$\begin{aligned}
 LHS &= \frac{1 + \cos \theta}{1 - \cos \theta} = \frac{(1 + \cos \theta)(1 + \cos \theta)}{(1 - \cos \theta)(1 + \cos \theta)} = \frac{1 + 2\cos \theta + \cos^2 \theta}{1 - \cos^2 \theta} = \frac{1 + 2\cos \theta + \cos^2 \theta}{\sin^2 \theta} \\
 &= \frac{1}{\sin^2 \theta} + \frac{2\cos \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{\cos^2 \theta + 2\cos \theta + 1}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} \\
 &= \cos^2 \theta + 2\cos \theta + 1 + \frac{\cos^2 \theta}{\sin^2 \theta} = (\operatorname{cosec} \theta + \cot \theta)^2 = RHS
 \end{aligned}$$

$$QED \text{ (Q.E.D.)} = \frac{(1 + \cos \theta)^2}{1 - \cos^2 \theta} = \frac{(1 + \cos \theta)^2}{\sin^2 \theta} = (\operatorname{cosec} \theta + \cot \theta)^2 = RHS$$

Question 22 (***)+

It is given that

$$\sin 3\theta \equiv 3\sin \theta - 4\sin^3 \theta.$$

- Prove the validity of the above trigonometric identity.
- State the maximum value of

$$f(x) = 6\sin(5x) - 8\sin^3(5x), \quad x \in \mathbb{R},$$

and determine, in degrees, the smallest positive value of x which produces this maximum value.

$$f_{\max}(x) = 2, \quad x = 6^\circ$$

$$\begin{aligned} 4\sin - 8\sin^3 \theta &= \sin(5x+15) = \sin 5\cos 15 + \cos 5\sin 15 \\ &= (2\sin 5\cos 15)\cos 15 + (1 - 2\sin^2 15)\sin 15 = 2\sin 5\cos^2 15 + \sin 5 - 2\sin^3 15 \\ &= 2\sin 5(1 - \sin^2 15) + \sin 5 - 2\sin^3 15 = 2\sin 5 - 2\sin^3 15 + \sin 5 - 2\sin^3 15 \\ &= 3\sin 5 - 4\sin^3 15 = 2\pi \end{aligned}$$

$$\begin{aligned} (b) \quad f(x) &= 6\sin 5x - 8\sin^3 5x \\ f'(x) &= 2[3\sin 5x - 4\sin^2 5x] \\ f'(x) &= 2\sin 15x \\ \therefore f'(x) &= 2 \quad \text{at } x = 0 \text{ when } \sin 15x = 1 \\ 15x &= 90^\circ \\ x &= 6^\circ \end{aligned}$$

Question 23 (**+)**

The obtuse angles θ and φ satisfy the equation

$$\sin 6\theta^\circ + \cos 4\varphi^\circ = -2.$$

Find the possible values of θ and φ .

, $(\theta, \varphi) = (105^\circ, 135^\circ) \cup (165^\circ, 135^\circ)$

AS THE SINE AND COSINE FUNCTIONS "OSCILLATE" BETWEEN -1 & 1, IT IS EVIDENT THAT

$$\sin 6\theta + \cos 4\varphi = -2 \rightarrow \begin{cases} \sin 6\theta = -1 \\ \cos 4\varphi = -1 \end{cases}$$

Therefore we have

- $\sin 6\theta = -1$
 $6\theta = -90^\circ + 360n \quad n=0,1,2,3, \dots$
 $6\theta = 270^\circ + 360n \quad n=0,1,2,3, \dots$
 $\theta = -15^\circ + 60n \quad n=0,1,2,3, \dots$
 $\theta = 45^\circ + 60n \quad n=0,1,2,3, \dots$
 $\theta = 105^\circ \quad n=0$
 $\theta = 165^\circ \quad n=1$
- $\cos 4\varphi = -1$
 $4\varphi = 180^\circ + 360n \quad n=0,1,2,3, \dots$
 $4\varphi = 360^\circ + 360n \quad n=0,1,2,3, \dots$
 $\varphi = 45^\circ + 90n \quad n=0,1,2,3, \dots$
 $\varphi = 135^\circ \quad n=0$

$\therefore (\theta, \varphi) = (105^\circ, 135^\circ) \cup (165^\circ, 135^\circ)$

Question 24 (****+)

It is given that

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

- a) Prove the validity of the above trigonometric identity by using the compound angle formulae for $\sin(A+B)$ and $\cos(A+B)$.

- a) Deduce an exact simplified expression for $\tan\left(\theta - \frac{\pi}{3}\right)$, in terms of $\tan \theta$.
- b) Solve, for $0 \leq \theta < 2\pi$, the trigonometric equation

$$\tan \theta - \sqrt{3} = (1 + \sqrt{3} \tan \theta) \tan(2\pi - \theta),$$

giving the answers in terms of π .

$$\boxed{\tan\left(\theta - \frac{\pi}{3}\right) = \frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta}}, \quad \boxed{\theta = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}}$$

(a) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

Simplify + cancel

$$= \frac{\cos A \cos B + \cos A \cos B}{\cos A \cos B - \sin A \sin B} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Divide by $\cos A \cos B$ (by definition)

(b) $\tan\left(\theta - \frac{\pi}{3}\right) = \frac{\tan \theta - \sqrt{3}}{1 + \tan \theta \tan \frac{\pi}{3}} = \frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta}$

(c) $\tan(\theta - \sqrt{3}) = (1 + \sqrt{3} \tan \theta) \tan(2\pi - \theta)$

$$\Rightarrow \tan(\theta - \sqrt{3}) = (1 + \sqrt{3} \tan \theta) \times \frac{\tan(2\pi - \theta)}{1 + \tan(2\pi - \theta)}$$

$$\Rightarrow \tan(\theta - \sqrt{3}) = (1 + \sqrt{3} \tan \theta) \times (-\tan \theta)$$

$$\Rightarrow \tan(\theta - \sqrt{3}) = -\tan^2 \theta - \sqrt{3} \tan \theta$$

$$\Rightarrow \sqrt{3} \tan^2 \theta + 2\sqrt{3} \tan \theta - 1 = 0$$

$$\Rightarrow \tan \theta + \frac{2}{\sqrt{3}} \tan \theta - 1 = 0$$

$$\Rightarrow \left(\tan \theta + \frac{2}{\sqrt{3}}\right)^2 - \left(\frac{2}{\sqrt{3}}\right)^2 - 1 = 0$$

$$\Rightarrow \left(\tan \theta + \frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$$

$$\Rightarrow -\tan \theta - \frac{2}{\sqrt{3}} = \pm \frac{2}{\sqrt{3}}$$

$$\Rightarrow \tan \theta = -\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{3}}$$

$$\Rightarrow \tan \theta < \frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{3}}$$

$$\bullet \tan \theta = \frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{3}}$$

$$\bullet \tan \theta = \frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{3}}$$

ALTERNATIVE

$$\tan(\theta - \sqrt{3}) = (\tan \theta \tan \sqrt{3}) / \tan(\theta + \sqrt{3})$$

$$\tan(\theta - \sqrt{3}) = \tan(2\pi - \theta)$$

$$\tan(\theta - \frac{\pi}{3}) = \tan(2\pi - \theta)$$

$$(\theta - \frac{\pi}{3}) = (2\pi - \theta) \pm n\pi$$

$$2\theta = \frac{7\pi}{3} \pm n\pi$$

$$\theta = \frac{7\pi}{6} \pm \frac{n\pi}{2}$$

$$\theta_1 = \frac{7\pi}{6} - \frac{n\pi}{2}$$

$$\theta_2 = \frac{7\pi}{6} + \frac{n\pi}{2} - \frac{\pi}{3}$$

$$\theta_3 = \frac{7\pi}{6} + \frac{n\pi}{2}$$

$$\theta_4 = \frac{7\pi}{6} - \frac{\pi}{3}$$

$\Rightarrow \theta = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}$

Question 25 (***)+

Solve the following trigonometric equation

$$\tan 4x - \tan 2x = 0, \quad 0^\circ \leq x < 360^\circ.$$

, $x = 0^\circ, 90^\circ, 180^\circ, 270^\circ$

Method A

$$\begin{aligned} \Rightarrow \tan 4x - \tan 2x &= 0 \\ \Rightarrow \tan 4x &= \tan 2x \\ \Rightarrow 4x &= 2x + 180^\circ \quad n=0,1,2,3,\dots \\ \Rightarrow 2x &= 0^\circ \pm 180^\circ \\ \Rightarrow x &= 0^\circ \pm 90^\circ \\ \underline{x = 0^\circ, 90^\circ, 180^\circ, 270^\circ} \end{aligned}$$

Method B

$$\begin{aligned} \text{using } \tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta} \\ \Rightarrow \tan(2x+2x) - \tan 2x &= 0 \\ \Rightarrow \frac{2\tan 2x}{1-\tan^2 2x} - \tan 2x &= 0 \\ \Rightarrow 2\tan 2x - \tan 2x(1-\tan^2 2x) &= 0 \\ \Rightarrow \tan 2x [2 - (1-\tan^2 2x)] &= 0 \\ \Rightarrow \tan 2x (1 + \tan^2 2x) &= 0 \\ \Rightarrow \tan 2x &= 0 \quad \text{no solutions} \\ \Rightarrow \tan 2x &= 0 \end{aligned}$$
$$\begin{aligned} \Rightarrow 2x &= 0^\circ \pm 180^\circ \quad n=0,1,2,3,\dots \\ \Rightarrow x &= 0^\circ \pm 90^\circ \\ \therefore x &= 0^\circ, 90^\circ, 180^\circ, 270^\circ \quad \text{As required} \end{aligned}$$

Question 26 (***)+

It is given that

$$\cos 3\theta \equiv 4 \cos^3 \theta - 3 \cos \theta.$$

- Prove the validity of the above trigonometric identity.
- Hence solve in degrees the trigonometric equation

$$2 + \cos 6x \sec 2x = 0, \quad 0^\circ \leq x < 180^\circ.$$

$$x = 30^\circ, 60^\circ, 120^\circ, 150^\circ$$

(a) $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$

$$= (\cos^2 \theta - 1)\cos \theta - (2\sin \theta \cos \theta) \sin \theta$$

$$= 2\cos^2 \theta - \cos \theta - 2(1 - \cos^2 \theta) \cos \theta$$

$$= 2\cos^2 \theta - \cos \theta - 2\cos^2 \theta + 2\cos^4 \theta$$

$$= 4\cos^4 \theta - 3\cos^2 \theta$$

✓ is fine

(b) $\cos 6x \sec 2x + 2 = 0$

$$\Rightarrow \cos(3 \times 2x) \sec 2x = -2$$

LHS \neq RHS

$$\Rightarrow \cos 6x \sec 2x = -2$$

$$\Rightarrow (4\cos^4 2x - 3\cos^2 2x) \frac{1}{\cos 2x} = -2$$

$$\Rightarrow 4\cos^2 2x - 3 = -2$$

$$\Rightarrow 4\cos^2 2x = 1$$

$$\Rightarrow \cos^2 2x = \frac{1}{4}$$

$$\Rightarrow \cos 2x = \pm \frac{1}{2}$$

$$\Rightarrow \cos 2x = \pm \frac{1}{2}$$

✓ is fine

x = 0, 1, 2, 3, ...

2x = 60 \pm 360^\circ
 $\Rightarrow 2x = 300 \pm 360^\circ$

or

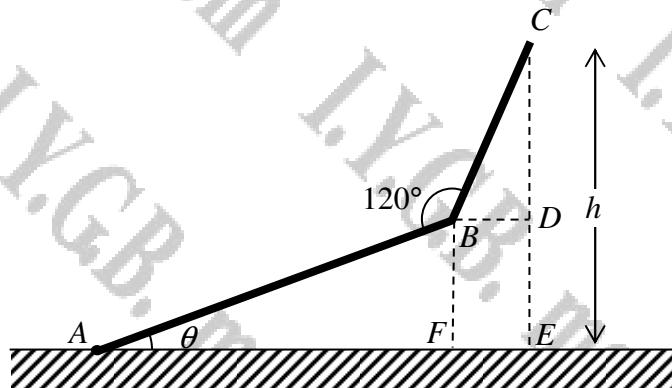
2x = 120 \pm 360^\circ
 $\Rightarrow 2x = 240 \pm 360^\circ$

2x = 30 \pm 180^\circ
 $\Rightarrow x = 60 \pm 180^\circ$

2x = 120 \pm 180^\circ
 $\Rightarrow x = 150 \pm 180^\circ$

2x = 30^\circ, 60^\circ, 120^\circ, 150^\circ

Question 27 (***)+



The figure above shows a rigid rod ABC where $AB = 6$ metres, $BC = 4$ metres and the angle ABC is 120° . The rod is hinged at A so it can be rotated in a vertical plane forming an angle θ° with the horizontal ground.

Let h metres be the height of the point C from the horizontal ground.

a) Show clearly that ...

- $\angle DBC = \theta^\circ + 60^\circ$.
- $h = 8\sin\theta + 2\sqrt{3}\cos\theta$.

b) By expressing h in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90^\circ$, find to the nearest degree, the values of θ when $h = 6$.

, $\theta \approx 20^\circ, 113^\circ$

(a)

By looking around B
 $(10-9+10+6=360^\circ)$
 $\theta + \phi + 360^\circ = 360^\circ$
 $\phi = 60^\circ - \theta$

(b)

$$\begin{aligned} h &= |CD| \cdot |\sin\theta| \\ h &= 4\sqrt{3}\sin\theta + 6\cos\theta \\ h &= 4\sin(60^\circ + \theta) + 6\cos\theta \\ h &= 4\sin(60^\circ)\cos\theta + 4\cos(60^\circ)\sin\theta + 6\cos\theta \\ h &= 2\sqrt{3}\cos\theta + 2\sin\theta + 6\cos\theta \\ h &= 8\sin\theta + 2\sqrt{3}\cos\theta \end{aligned}$$

(b)

$$\begin{aligned} 8\sin\theta + 2\sqrt{3}\cos\theta &= R\cos(\theta - \alpha) \\ &\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha \\ &\equiv (R\cos\theta)\cos\alpha + (R\sin\theta)\sin\alpha \end{aligned}$$

$$\left. \begin{array}{l} R\cos\theta = 8 \\ R\sin\theta = 2\sqrt{3} \end{array} \right\} \Rightarrow R = \sqrt{(8^2)^2 + (2\sqrt{3})^2} = \sqrt{728}$$

$$\tan\theta = \frac{2\sqrt{3}}{8} \Rightarrow \theta \approx 66.4^\circ$$

$$\therefore h = \sqrt{728} \cos(66.4^\circ - 66.7^\circ)$$

$$h = 4\sqrt{3}\cos(66.4^\circ - 66.7^\circ)$$

$$(66.4^\circ - 66.7^\circ) = -0.3^\circ \dots$$

$$(66.4^\circ - 66.7^\circ) = -0.3^\circ \Rightarrow 353.5^\circ \dots$$

$$\left. \begin{array}{l} h = 113.1 \pm 36.9 \\ h = 380.1 \pm 36.9 \end{array} \right\} \therefore \theta = 113^\circ \text{ or } 20^\circ$$

Question 28 (***)+

By considering the expansion of $\tan(2A + A)$, show clearly that

$$\tan 3A \equiv \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

[proof]

$$\begin{aligned} \text{LHS} &= \tan(3A) = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} \\ &= \frac{\frac{2 \tan A + \tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2 \tan A + \tan A}{1 - \tan^2 A} \tan A} = \dots \text{MULTIPLY TOP/BOTTOM BY } (1 - \tan^2 A) \\ &= \frac{2 \tan A + \tan A(1 - \tan^2 A)}{(1 - \tan^2 A) - 2 \tan^2 A} = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} = \text{RHS} \end{aligned}$$

Question 29 (***)+

Show clearly that

$$\frac{\tan^2 x}{\tan^2 x + 2 + \cot^2 x} \equiv \sin^4 x.$$

[proof]

$$\begin{aligned} \text{LHS} &= \frac{\tan^2 x}{\tan^2 x + 2 + \cot^2 x} = \frac{\tan^2 x}{\tan^2 x + 2 \tan x \frac{1}{\tan x} + (\frac{1}{\tan x})^2} \\ &= \frac{\tan^2 x}{(\tan x + \frac{1}{\tan x})^2} = \frac{\tan^2 x}{(\tan^2 x + 1)^2} = \frac{\tan^2 x}{(\sec^2 x)^2} \\ &= \frac{\tan^2 x}{\sec^4 x} = \frac{\tan^2 x}{\frac{1}{\sin^2 x}} = \tan^2 x \sin^2 x \\ &= \frac{\sin^2 x}{\cos^2 x} \cos^4 x = \text{RHS} \quad \square \end{aligned}$$

$$\text{LHS} = \frac{\tan^2 x}{\tan^2 x + 2 + \cot^2 x} = \frac{\sec^2 x - 1}{(\sec^2 x - 1)^2 + \cot^2 x - 1} = \frac{\sec^2 x - 1}{\sec^2 x + \csc^2 x}$$

$$= \frac{\sec^2 x - 1}{\frac{1}{\csc^2 x} + \frac{1}{\sec^2 x}} = \frac{\sec^2 x - \csc^2 x}{\frac{1}{\csc^2 x} + \frac{1}{\sec^2 x}} = \frac{\sec^2 x(1 - \csc^2 x)}{1} = \sin^2 x = \text{RHS}$$

Question 30 (****+)

It is given that

$$\cos 3\theta \equiv 4\cos^3 \theta - 3\cos \theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) Hence, or otherwise, prove that

$$\cos 6\theta \equiv 32\cos^6 \theta - 48\cos^4 \theta + 18\cos^2 \theta - 1.$$

proof

$$\begin{aligned}
 \text{(a)} \quad \cos 3\theta &= \cos(3\theta+0) = \cos 3\theta \cos 0 - \sin 3\theta \sin 0 \\
 &= (2\cos^2 \theta - 1)\cos \theta - (2\sin \theta \cos \theta) \sin 0 \\
 &= 2\cos^3 \theta - \cos \theta - 2\sin \theta \cos^2 \theta \\
 &= 2\cos^3 \theta - \cos \theta - 2\cos \theta + 2\cos^3 \theta \\
 &= 4\cos^3 \theta - 3\cos \theta
 \end{aligned}$$

As required

$$\begin{aligned}
 \text{(b)} \quad \cos 6\theta &= \cos(2 \times 3\theta) = 2\cos^2(3\theta) - 1 \\
 &= 2(\cos^3 \theta - 3\cos \theta)^2 - 1 = 2(4\cos^6 \theta - 24\cos^4 \theta + 36\cos^2 \theta) - 1 \\
 &= 32\cos^6 \theta - 48\cos^4 \theta + 18\cos^2 \theta - 1
 \end{aligned}$$

As required

Question 31 (****+)

Prove the validity of each of the following trigonometric identities.

$$\text{a) } \sin^2(x+y) - \sin^2(x-y) \equiv \sin 2x \sin 2y.$$

$$\text{b) } \frac{\cot 2\theta + \cos 2\theta}{\cot 2\theta} \equiv (\cos \theta + \sin \theta)^2.$$

proof

$$\begin{aligned}
 \text{(i)} \quad LHS &= \sin^2(x+y) - \sin^2(x-y) \\
 &= [\sin(x+y) - \sin(x-y)][\sin(x+y) + \sin(x-y)] \\
 &= [\sin x \cos y + \cos x \sin y - (\sin x \cos y + \cos x \sin y)] \\
 &\quad \times [\sin x \cos y + \cos x \sin y + (\sin x \cos y + \cos x \sin y)] \\
 &= (2\sin x \cos y)(2\cos x \sin y) \\
 &= (2\cos x \sin y)(2\sin x \cos y) \\
 &= 2\sin x \cos y \\
 &= RHS
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad LHS &= \frac{\cot 2\theta + \cos 2\theta}{\cot 2\theta} = \frac{\frac{\cos 2\theta}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta}}{\frac{\cos 2\theta}{\sin 2\theta}} = 1 + \cos 2\theta \tan 2\theta \\
 &= 1 + \cos 2\theta \times \frac{\sin 2\theta}{\cos 2\theta} = 1 + \sin 2\theta = 1 + 2\sin \theta \cos \theta \\
 &= \cos^2 \theta + 2\sin \theta \cos \theta + \sin^2 \theta = (\cos \theta + \sin \theta)^2 = RHS
 \end{aligned}$$

Question 32 (****+)

Show clearly that

$$\tan\left(\theta - \frac{\pi}{4}\right) \equiv \frac{\sin 2\theta - 1}{\cos 2\theta}.$$

proof

$$\begin{aligned}
 \text{LHS} &= \tan\left(\theta - \frac{\pi}{4}\right) = \frac{\tan\theta - \tan\frac{\pi}{4}}{1 + \tan\theta \tan\frac{\pi}{4}} = \frac{\tan\theta - 1}{1 + \tan\theta} = \frac{\frac{\sin\theta}{\cos\theta} - 1}{1 + \frac{\sin\theta}{\cos\theta}} = \frac{\frac{\sin\theta - \cos\theta}{\cos\theta}}{\frac{\cos\theta + \sin\theta}{\cos\theta}} = \frac{(\sin\theta - \cos\theta)(\cos\theta + \sin\theta)}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)} \\
 &= \frac{(\sin^2\theta - \cos^2\theta)}{\cos^2\theta - \sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta - 2\cos^2\theta}{\cos^2\theta - \sin^2\theta} = -\frac{1 - \sin^2\theta}{\cos^2\theta} \\
 &= -\frac{\sin^2\theta - 1}{\cos^2\theta} = -\frac{\cos^2\theta}{\cos^2\theta} = -1
 \end{aligned}$$

Question 33 (****+)

It is given that

$$\cos 3x \equiv 4\cos^3 x - 3\cos x.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence, or otherwise solve the trigonometric equation

$$2 + \cos 6\theta \sec 2\theta = 0, \quad 0^\circ \leq \theta < 360^\circ.$$

 , $\theta = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ, 330^\circ$

a) $\cos 3x = \cos(2x + x)$

$$\begin{aligned}
 &\equiv \cos 2x \cos x - \sin 2x \sin x \\
 &\equiv (2\cos^2 x - 1)\cos x - (2\sin x \cos x) \sin x \\
 &\equiv 2\cos^2 x \cos x - \cos x - 2\sin^2 x \cos x \\
 &\equiv 2\cos^2 x \cos x - \cos x - 2(1 - \cos^2 x) \cos x \\
 &\equiv 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x \\
 &\equiv 4\cos^3 x - 3\cos x
 \end{aligned}$$

As required

b) Proceeding as follows

$$\begin{aligned}
 \cos 3x &\equiv 4\cos^3 x - 3\cos x \\
 \cos 6\theta &\equiv \cos(3x + 3x) = 4\cos^3 2\theta - 3\cos 2\theta
 \end{aligned}$$

TRANSFORMING THE EQUATION

$$\begin{aligned}
 &\Rightarrow 2 + \cos 6\theta \sec 2\theta = 0 \\
 &\Rightarrow 2 + [4\cos^3 2\theta - 3\cos 2\theta] \sec 2\theta = 0 \\
 &\Rightarrow 2 + 4\cos^3 2\theta \sec 2\theta - 3\cos 2\theta \sec 2\theta = 0 \\
 &\Rightarrow 2 + 4\cos^2 2\theta - 3 = 0
 \end{aligned}$$

$$\begin{aligned}
 &\Rightarrow 4\cos^2 2\theta = 1 \\
 &\Rightarrow 4\left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) = 1 \\
 &\Rightarrow 2 + 2\cos 4\theta = 1 \\
 &\Rightarrow 2\cos 4\theta = -1 \\
 &\Rightarrow \cos 4\theta = -\frac{1}{2} \\
 &\Rightarrow \arccos\left(-\frac{1}{2}\right) = 120^\circ \\
 &\Rightarrow \left(\begin{array}{l} 4\theta = 120^\circ \pm 360^\circ \\ 4\theta = 240^\circ \pm 360^\circ \end{array}\right) \quad n=1, 2, 3, \dots \\
 &\Rightarrow \left(\begin{array}{l} \theta = 30^\circ + 90^\circ n \\ \theta = 60^\circ + 90^\circ n \end{array}\right) \\
 &\Rightarrow \theta = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ, 330^\circ
 \end{aligned}$$

Alternative from $4\cos^2 2\theta = 1$

$\cos 2\theta = \pm \frac{1}{2}$

AND SOME REAL THING

Question 34 (***)+

It is given that

$$(\cos x + \sin x)(1 - \sin x \cos x) \equiv \cos^3 x + \sin^3 x.$$

- Prove the validity of the above trigonometric identity.
- Hence find, in terms of π , the solutions of the trigonometric equation

$$\frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} = \frac{3}{4}, \quad 0 \leq x < \pi,$$

giving the answers in terms of π .

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

(a)

$$\begin{aligned}
 4(\cos x + \sin x)(1 - \sin x \cos x) &= (\cos x + \sin x)(\cos^2 x + \sin^2 x - \sin x \cos x) \\
 &= \cos^3 x + \cos x \sin^2 x - \sin x \cos^2 x \\
 &\quad - \cos^2 x \sin x + \sin^2 x \cos x + \sin^2 x \\
 &= \cos^3 x + \sin^3 x \\
 &= R.H.S
 \end{aligned}$$

(b)

$$\begin{aligned}
 \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} &= \frac{3}{4} \\
 \Rightarrow \frac{(\cos x + \sin x)(1 - \sin x \cos x)}{\cos x + \sin x} &= \frac{3}{4} \\
 \Rightarrow 1 - \sin x \cos x &= \frac{3}{4} \\
 \Rightarrow \frac{1}{4} &= \sin x \cos x \\
 \Rightarrow \frac{1}{2} &= \sin 2x
 \end{aligned}$$

$\sin(2x) = \frac{1}{2}$

$2x = \frac{\pi}{6} \pm 2m\pi$
 $x = \frac{\pi}{12} \pm m\pi$
 $x = \frac{\pi}{12} \pm \frac{\pi}{6}$
 $x_1 = \frac{\pi}{12}$
 $x_2 = \frac{7\pi}{12}$

Question 35 (****+)

It is given that

$$\cos(x + 36)^\circ = \sin(x - 54)^\circ.$$

- a) Show clearly without a calculating aid that the above trigonometric equation is equivalent to

$$\tan x^\circ = \tan 54^\circ.$$

- b) Hence solve the trigonometric equation

$$\cos(3y + 36)^\circ = \sin(3y - 54)^\circ, \quad 0 \leq y < 180.$$

$$y = 18, 78, 138$$

(a) $\cos(x + 36) = \sin(x - 54)$

$$\Rightarrow \cos x \cos 36 - \sin x \sin 36 = \sin x \cos 54 - \cos x \sin 54$$

$$\Rightarrow \cos x \cos 36 - \tan x \sin 36 = \tan x \cos 54 - \sin x \cos 54$$

$$\Rightarrow \cos x \cos 36 - \tan x \cos 54 = \tan x \cos 54 - \sin x \cos 54$$

$$\Rightarrow \cos x \cos 36 = \tan x \cos 54$$

$$\Rightarrow \tan x = \tan 54$$

(b) $\tan 3y = \tan 54$
 $3y = 54 + 180n \quad n=0,1,2,3,\dots$
 $y = 18 + 60n$
 $\therefore y = 18^\circ, 78^\circ, 138^\circ //$

Question 36 (**+)**

Solve the trigonometric equation

$$\sin 3x = \cos 2x + \sin x, \quad \text{for } 0 \leq x < \pi,$$

giving the answers in terms of π .

$$x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}$$

ALTERNATIVE
 $\sin 3x = 3\sin x - 4\sin^3 x$
 $\Rightarrow \sin 3x - \sin x = 2\sin^2 x - 4\sin x$
 $\Rightarrow 2\sin x(2\sin x - 1) = 0$
 $\Rightarrow \sin x(2\sin x - 1) = 0$
 $\Rightarrow \sin x = 0 \quad \text{or} \quad \sin x = \frac{1}{2}$
 $(x = \frac{\pi}{2} \pm 2\pi n) \quad (x = \frac{\pi}{6} \pm 2\pi n) \quad (x = \frac{5\pi}{6} \pm 2\pi n)$
 $(x = \frac{\pi}{2} \pm \pi n) \quad (x = \frac{\pi}{6} \pm \pi n) \quad (x = \frac{5\pi}{6} \pm \pi n)$
 $x = 0, \frac{\pi}{2}, \frac{5\pi}{6}, \dots$
 $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}$

Question 37 (**+)**

Solve the trigonometric equation

$$2\sin 2x \tan x + 4\sec 2x + 5 = 0, \quad 0^\circ \leq x < 360^\circ.$$

$$x = 60^\circ, 120^\circ, 240^\circ, 300^\circ$$

ALTERNATIVE BY LETTING $t = \tan x$
 $\sin 2x = \frac{2t}{1+t^2} \quad \cos 2x = \frac{1-t^2}{1+t^2}$
 $\Rightarrow 2\left(\frac{2t}{1+t^2}\right)\left(\frac{t}{1+t^2}\right) + \frac{4}{\cos 2x} + 5 = 0$
 $\Rightarrow 2\left(\frac{2t^2}{1+t^2}\right)t + 4\left(\frac{1+t^2}{1-t^2}\right) + 5 = 0$
 $\Rightarrow \frac{4t^3}{1+t^2} + 4 + 5(1+t^2)^2 = 0$
 $\Rightarrow 4t^3 + 4t^4 + 4 + 5 + 10t^2 = 0$
 $\Rightarrow 4t^3 + 4t^4 + 9 + 10t^2 = 0$
 $\Rightarrow 4t^2(t^2 + 1) + 9 + 10t^2 = 0$
 $\Rightarrow 4t^4 + 4t^2 + 9 + 10t^2 = 0$
 $\Rightarrow 4t^4 + 14t^2 + 9 = 0$
 $\Rightarrow (2t^2 + 3)(2t^2 + 3) = 0$
 $\Rightarrow 2t^2 + 3 = 0 \quad \text{or} \quad 2t^2 + 3 = 0$
 $\Rightarrow t^2 = -\frac{3}{2} \quad \text{or} \quad t^2 = -\frac{3}{2}$
 $\Rightarrow t = \pm \sqrt{-\frac{3}{2}} \quad \text{or} \quad t = \pm \sqrt{-\frac{3}{2}}$
 $\Rightarrow t = \pm i\sqrt{\frac{3}{2}} \quad \text{or} \quad t = \pm i\sqrt{\frac{3}{2}}$
 $\Rightarrow x = 60^\circ \pm 360^\circ n \quad x = -60^\circ \pm 360^\circ n$
 $\Rightarrow x = 120^\circ \pm 360^\circ n \quad x = 240^\circ \pm 360^\circ n$
 $\Rightarrow x = 60^\circ, 120^\circ, 240^\circ, 300^\circ$

Question 38 (***)+

It is given that

$$\tan \theta + \tan \varphi = 3,$$

$$\sin^2 x + 2 \sin x + \sin(\theta + \varphi) = 3 \cos \theta \cos \varphi - 1,$$

for $x \in \mathbb{R}$, $\theta \in \mathbb{R}$, $\varphi \in \mathbb{R}$.

Show that the above relationships imply that

$$\sin x = -1.$$

 , proof

<u>STEPS FROM THE SECOND EQUATION</u> $\rightarrow \tan \theta + \tan \varphi = 3$ $\rightarrow \frac{\sin \theta}{\cos \theta} + \frac{\sin \varphi}{\cos \varphi} = 3$ $\rightarrow \frac{\sin \theta \cos \varphi + \cos \theta \sin \varphi}{\cos \theta \cos \varphi} = 3$ $\rightarrow \sin(\theta + \varphi) = 3 \cos \theta \cos \varphi$ $\rightarrow \sin(0 + \frac{\pi}{2}) = 3 \cos 0 \cos \frac{\pi}{2}$ $\rightarrow \sin(\frac{\pi}{2}) = 3 \cos 0 \cancel{\cos \frac{\pi}{2}}$
<u>NOW THE FIRST EQUATION SIMPLIFIES</u> $\rightarrow \sin^2 x + 2 \sin x + \sin(\theta + \varphi) = 3 \cos \theta \cos \varphi - 1$ $\rightarrow \sin^2 x + 2 \sin x = -1$ $\rightarrow \sin^2 x + 2 \sin x + 1 = 0$ $\rightarrow (\sin x + 1)^2 = 0$ $\rightarrow \sin x + 1 = 0$ $\rightarrow \sin x = -1$ ✓ As required

Question 39 (***)+

A geometric progression has first term $\sin \theta$ and common ratio $\cos \theta$.

- a) Given the value of θ is such so that the progression converges, show that its sum to infinity is $\cot \frac{\theta}{2}$.

A different geometric progression has first term $\cos \theta$ and common ratio $\sin \theta$.

- b) Given the value of θ is such so that this progression also converges, show that its sum to infinity is $\sec \theta + \tan \theta$.

, proof

a) Using the standard formula for the sum to infinity

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad -1 < r < 1$$

$$\sum_{n=0}^{\infty} \sin \theta r^n = \dots$$

SINθ = 20Aθ and SINθ = 2θ

$r = 1 - 2\sin \theta \Rightarrow \text{Q.E.D. } 1 - 2\sin \theta$

$$= \frac{2\sin \theta \cos \frac{\theta}{2}}{1 - (1 - 2\sin \theta)} = \frac{2\sin \theta \cos \frac{\theta}{2}}{2\sin \theta} = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \cot \frac{\theta}{2}$$

As required

b) Using the sum formula with sum & ratio "reversed"

$$\sum_{n=0}^{\infty} ar^n = \frac{ar^0}{1-r^0} = \frac{a(1+r+r^2+\dots)}{(1-r)(1+r)} = \frac{a(1+\sin \theta)}{1-\cos \theta}$$

$$= \frac{\cos \theta(1+\sin \theta)}{\sin \theta} = \frac{1+\sin \theta}{\cos \theta} = \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta}$$

$$= \sec \theta + \tan \theta$$

As required

Question 40 (***)+

Is given that

- $\cos^2 x + \sin^2 x \equiv 1$.
- $\operatorname{cosec} 15^\circ = \sqrt{6} + \sqrt{2}$.

Use these facts **only** to show that

a) $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$.

b) $\cot 15^\circ = 2 + \sqrt{3}$.

proof

(a)

$$\begin{aligned} \cot^2 x + \operatorname{cosec}^2 x &= 1 \\ \frac{\cos^2 x}{\sin^2 x} + \frac{1}{\sin^2 x} &= \frac{1}{\sin^2 x} \\ \cot^2 x + 1 &= \operatorname{cosec}^2 x \end{aligned}$$

$\Rightarrow \cot 15^\circ = \pm \sqrt{1+4\sqrt{3}^2}$

But 15° is acute, so

$\Rightarrow \cot 15^\circ = +\sqrt{1+4\sqrt{3}^2}$

$\Rightarrow \cot 15^\circ = \sqrt{2^2 + 2(2\sqrt{3})^2 + 4(\sqrt{3})^2}$

$(a+b)^2 = a^2 + 2ab + b^2$

$\Rightarrow \cot 15^\circ = \sqrt{(2+\sqrt{3})^2}$

$\Rightarrow \cot 15^\circ = 2+\sqrt{3}$

As required.

Question 41 (***)+

Prove the validity of each of the following trigonometric identities.

a) $\sqrt{1+\sin 2\theta} \equiv \cos \theta + \sin \theta$.

b) $8\cos^4\left(\frac{1}{2}\theta\right) = \cos 2\theta + 4\cos \theta + 3$.

proof

(a)

$$\begin{aligned} LHS &= \sqrt{1+\sin 2\theta} = \sqrt{\cos^2 \theta + \sin^2 \theta + 2\sin \theta \cos \theta} \\ &= \sqrt{(\cos \theta + \sin \theta)^2} = \cos \theta + \sin \theta = RHS \end{aligned}$$

(b)

NOTE: $\cos 2A = 2\cos^2 A - 1$ Hence $\cos^2 A = \frac{1}{2}(1+\cos 2A)$
 $2\cos A = 1 + \cos 2A$ $\cos^2 A = \frac{1}{2}(1+\cos 2A)$

$$\begin{aligned} LHS &= 8\cos^4\left(\frac{1}{2}\theta\right) = 8(\cos^2\theta)^2 = 8\left(\frac{1}{2}(1+\cos 2\theta)\right)^2 \\ &= 8 \times \frac{1}{4}(1+\cos 2\theta)^2 = 2(1+\cos 2\theta)^2 = 2(1+2\cos 2\theta + \cos^2 2\theta) \\ &= 2 + 4\cos 2\theta + 2\cos^2 2\theta = 2 + 4\cos 2\theta + 2\left(\frac{1}{2}(1+\cos 2(2\theta))\right) \\ &= 2 + 4\cos 2\theta + 1 + \cos 4\theta = \cos 2\theta + 4\cos 2\theta + 3 = RHS \end{aligned}$$

Question 42 (**+)**

A relationship between x and y is given by the equations

$$x = \sin 2\theta, \quad 0 < \theta < \pi,$$

$$y = \cot \theta, \quad 0 < \theta < \pi.$$

Use trigonometric identities to show that

$$y(2 - xy) = x.$$

proof

The proof starts with the given equations:

$$x = \sin 2\theta$$

$$y = \cot \theta$$

From $x = \sin 2\theta$, we have:

$$\Rightarrow \sin^2 \theta = \frac{1}{2} \sin 2\theta$$

$$\Rightarrow \sin^2 \theta = \frac{1}{2} \sin \theta \cos \theta$$

$$\Rightarrow \sin^2 \theta = \frac{1}{2} \sin \theta (1 - \sin^2 \theta)$$

Now, $\sin^2 \theta = \sin^2 \theta = \cos^2 \theta - 1$

$$\Rightarrow \sin^2 \theta = \cos^2 \theta$$

$$\Rightarrow \sin^2 \theta = \frac{\cos^2 \theta}{\sin^2 \theta + 1}$$

Thus, $\sin^2 \theta = 4 \times \frac{1}{\sin^2 \theta + 1} \times \left(1 - \frac{\cos^2 \theta}{\sin^2 \theta + 1}\right)$

$$\Rightarrow \sin^2 \theta = \frac{4}{\sin^2 \theta + 1} \times \frac{\sin^2 \theta + 1 - \cos^2 \theta}{\sin^2 \theta + 1}$$

$$\Rightarrow \sin^2 \theta = \frac{4(1 - \cos^2 \theta)}{(\sin^2 \theta + 1)^2}$$

$$\Rightarrow \sin^2 \theta = \frac{4(1 - \cos^2 \theta)}{(\sin^2 \theta + 1)^2} \quad //$$

Given $y = \cot \theta$, we have:

$$\Rightarrow \sin^2 \theta = 2y$$

$$\Rightarrow \sin^2 \theta + 1 = 2y + 1$$

$$\Rightarrow x = 2y - 2y^2$$

$$\Rightarrow x = y(2 - 2y)$$

$$\therefore y(2 - 2y) = x$$

Question 43 (**+)**

Show clearly that

$$\arctan x + \arctan \left(\frac{1-x}{1+x} \right) = \frac{\pi}{4}.$$

proof

Let $\arctan x + \arctan \left(\frac{1-x}{1+x} \right) = \psi$

Hence,

$$\tan \left[\arctan x + \arctan \left(\frac{1-x}{1+x} \right) \right] = \frac{\tan(\arctan x) + \tan \left(\arctan \left(\frac{1-x}{1+x} \right) \right)}{1 - \tan(\arctan x) \tan \left(\arctan \left(\frac{1-x}{1+x} \right) \right)}$$

$$= \frac{x + \frac{1-x}{1+x}}{1 - x \cdot \frac{1-x}{1+x}} = \dots \text{ (apply tan/arctan by C of A)...}$$

$$= \frac{x(1+x) + 1 - x}{(1+x)(1-x)} = \frac{x^2 + x + 1 - x}{1 - x^2 - x^2} = \frac{x^2 + 1}{1 - 2x^2} = 1$$

So, $\theta + \psi = \frac{\pi}{4}$

$$\Rightarrow \tan(\theta + \psi) = \tan \frac{\pi}{4}$$

$$\Rightarrow 1 = \tan \psi$$

$$\Rightarrow \psi = \frac{\pi}{4}$$

Thus $\arctan x + \arctan \left(\frac{1-x}{1+x} \right) = \frac{\pi}{4}$ //

Question 44 (***)+

It is given that

$$\frac{2 \tan x}{\tan x + \sin x} \equiv \sec^2 \left(\frac{x}{2} \right), \quad x \neq n\pi, n \in \mathbb{Z}$$

- a) Prove the validity of the above trigonometric identity
 - b) Hence solve the trigonometric equation

$$\frac{2 \tan x}{\tan x + \sin x} = 4, \quad 0^\circ \leq x < 360^\circ$$

$$x = 120^\circ, 240^\circ$$

$$\begin{aligned}
 \textcircled{a} \quad LHS &= \frac{2 \tan x}{\tan x + \cot x} = \frac{\frac{2 \sin x}{\cos x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} = \frac{2 \sin^2 x}{\sin^2 x + \cos^2 x} = \frac{2 \sin^2 x}{1 + \cos^2 x} = \frac{2}{1 + \left(\frac{2 \cos^2 x - 1}{2}\right)} \\
 &= \frac{2}{\sin^2 x + \cos^2 x} = \frac{2}{1 + \cos 2x} = \frac{2}{1 + \left(2 \cos^2 \frac{x}{2} - 1\right)} \\
 &= \frac{2}{2 \cos^2 \frac{x}{2}} = \frac{1}{\cos^2 \frac{x}{2}} = \sec^2 \left(\frac{x}{2}\right) = RHS
 \end{aligned}$$

Question 45 (***)+

Prove the validity of the following trigonometric identity

$$\frac{\sin x}{1 + \tan x} \equiv \frac{\cos x}{1 + \cot x}$$

proof

$$\begin{aligned} \text{LHS} &= \frac{\sin x}{1 + \tan x} = \frac{\sin x}{1 + \frac{\sin x}{\cos x}} = \frac{\sin x \cdot \cos x}{\cos x + \sin x} = \frac{\cancel{\sin x} \cdot \cos x}{\cos x + \cancel{\sin x}} = \frac{\cos x}{\cos x + \sin x} = \frac{\cos x}{1 + \frac{\sin x}{\cos x}} = \frac{\cos x}{\cos x + \sin x} \\ &= \frac{\cancel{\sin x} \cdot \cos x}{\cos x + \cancel{\sin x}} = \frac{\cos x}{\cos x + 1} = \frac{\cos x}{\cos x + 1} = \text{RHS} \end{aligned}$$

Question 46 (***)+

Let $t = \tan \frac{x}{2}$.

a) Show clearly that ...

i. ... $\sin x = \frac{2t}{1+t^2}$,

ii. ... $\cos x = \frac{1-t^2}{1+t^2}$.

b) Use these results to solve the trigonometric equation

$$5\sin x + 4\cos x = 3, \quad 0^\circ \leq x < 360^\circ.$$

$$x \approx 113.4^\circ, 349.3^\circ$$

$$\begin{aligned} \text{(a)} \quad & \frac{2t}{1+t^2} = \frac{2\sin \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \frac{2\sin \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{2\sin \frac{x}{2}}{1-\cos^2 \frac{x}{2}} = \frac{2\sin \frac{x}{2}}{\sin^2 \frac{x}{2}} = \frac{2}{\cos^2 \frac{x}{2}} \\ & = 2\sin \left(\frac{x}{2}\right) \sec^2 \left(\frac{x}{2}\right) = \sin \left(2 \cdot \frac{x}{2}\right) = \sin x \\ \bullet \quad & \frac{1-t^2}{1+t^2} = \frac{1-\tan^2 \left(\frac{x}{2}\right)}{1+\tan^2 \left(\frac{x}{2}\right)} = \frac{1-\tan^2 \left(\frac{x}{2}\right)}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{1-\tan^2 \left(\frac{x}{2}\right)}{1-\cos^2 \frac{x}{2}} = \frac{\cos^2 \frac{x}{2}}{1-\cos^2 \frac{x}{2}} \\ & = \cos^2 \frac{x}{2} - \tan^2 \frac{x}{2} \cos^2 \frac{x}{2} = \frac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} - \frac{\tan^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \cos^2 \frac{x}{2} \\ & = \cos \left(2 \cdot \frac{x}{2}\right) = \cos x. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & 5\sin x + 4\cos x = 3 \\ & \Rightarrow 5\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) = 3 \\ & \Rightarrow 10t + 4(1-t^2) = 3(1+t^2) \\ & \Rightarrow 10t + 4 - 4t^2 = 3 + 3t^2 \\ & \Rightarrow 10t - 3t^2 = 3 + 3t^2 - 4 \\ & \Rightarrow 10t - 3t^2 = 3t^2 - 1 \\ & \text{Divide by } t^2 \text{ and rearrange:} \\ & \Rightarrow -3t^2 + 10t - 1 = 0 \\ & \Rightarrow t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-10 \pm \sqrt{100 - 4(-3)(-1)}}{-6} = \frac{-10 \pm \sqrt{92}}{-6} = \frac{-10 \pm 2\sqrt{23}}{-6} = \frac{5 \mp \sqrt{23}}{3} \end{aligned}$$

$$\begin{cases} \arctan(1.522) = 56.7^\circ \\ \arctan(-0.833) = -54^\circ \\ \frac{\pi}{2} = 90^\circ \\ \frac{\pi}{2} - 54^\circ = 36.0^\circ \\ n=0,1,2,3,.. \\ t_1 = 5.4 \pm 36.0^\circ \\ t_2 = 113.4^\circ \quad n=0 \\ t_3 = -10.7 \pm 36.0^\circ \\ t_4 = 25.3^\circ \quad n=1 \\ \therefore x = 113.4^\circ, 349.3^\circ \end{cases}$$

Question 47 (**+)**It is given that for $\theta \in \mathbb{R}$, $\varphi \in \mathbb{R}$

$$3 \tan \theta = 4 \tan \varphi.$$

Show that the above relationship implies that

$$\tan(\theta - \varphi) = \frac{\sin 2\theta}{7 + \cos 2\theta}.$$

 , proof

SIMPLIFY BY APPLYING THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$.

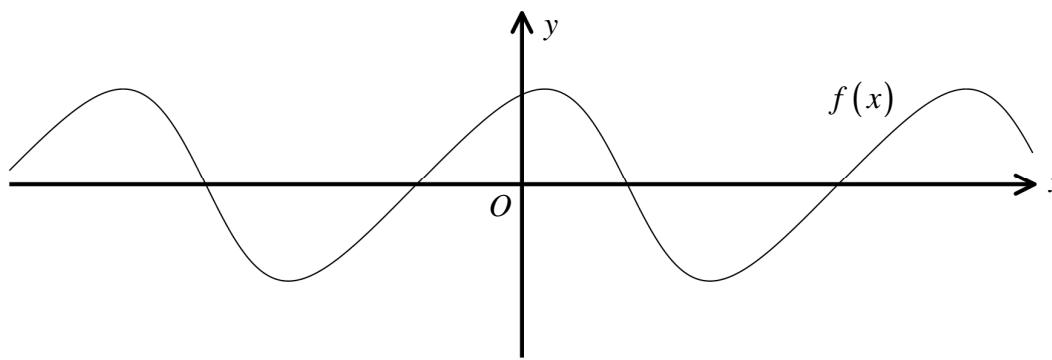
$$\begin{aligned} \Rightarrow \tan(\theta - \varphi) &= \frac{\tan \theta - \tan \varphi}{1 + \tan \theta \tan \varphi} \\ &= \frac{4 \tan \theta - 4 \tan \varphi}{4 + 4 \tan \theta \tan \varphi} \\ \text{BUT } 4 \tan \theta &= 3 \tan \varphi \\ &= \frac{4 \tan \theta - (3 \tan \varphi)}{4 + 4 \tan \theta \tan \varphi} \\ &= \frac{4 \tan \theta - 3 \tan \varphi}{4 + 3 \tan^2 \theta} \\ &= \frac{\tan \theta}{4 + 3 \tan^2 \theta} \\ &= \frac{\tan \theta}{4 + \frac{3 \tan^2 \theta}{\tan^2 \theta}} \\ &= \frac{\tan \theta}{4 + \frac{3 \tan^2 \theta}{\tan^2 \theta}} \\ &= \frac{\tan \theta}{4 + 3} \\ &= \frac{\tan \theta}{7} \end{aligned}$$

MATRIX "TOP 4 BOTTOM" OF THE DOUBLE ANGLE FORMULA BY COSE

$$\begin{aligned} &= \frac{\sin 2\theta}{4 \cos^2 \theta + 2 \sin^2 \theta} \\ &= \frac{2 \sin \theta \cos \theta}{4 \cos^2 \theta + 2 \sin^2 \theta} \\ &= \frac{\cancel{2} \sin \theta \cos \theta}{\cancel{2} \cos^2 \theta + \cancel{2} \sin^2 \theta} \\ &= \frac{\sin 2\theta}{2(4 + 1) \cos^2 \theta + 2(1 - 1) \sin^2 \theta} \\ &= \frac{\sin 2\theta}{4 + 4 \cos^2 \theta + 2 - 2 \sin^2 \theta} \\ &= \frac{\sin 2\theta}{6 + 4 \cos^2 \theta - 2 \sin^2 \theta} \\ &= \frac{\sin 2\theta}{7 + 4 \cos^2 \theta - 2 \sin^2 \theta} \\ &= \frac{\sin 2\theta}{7 + \cos 2\theta} \end{aligned}$$

As required

Question 48 (***)+



The figure above shows part of the graph of the curve with equation

$$f(x) = \frac{\cos x}{3 - \sin x}, x \in \mathbb{R}.$$

Use differentiation to show that

$$-\frac{1}{4}\sqrt{2} \leq f(x) \leq \frac{1}{4}\sqrt{2}.$$

, proof

Differentiate via quotient rule of tidy

$$f'(x) = \frac{(3 - \sin x)(-\cos x) - (\cos x)(-\cos x)}{(3 - \sin x)^2}$$

$$= \frac{-3\sin x + \cos^2 x + \cos^2 x}{(3 - \sin x)^2} = \frac{1 - 3\sin x}{(3 - \sin x)^2}$$

Solving for zero

$$1 - 3\sin x = 0$$

$$\sin x = \frac{1}{3} \quad \leftarrow \text{start key value}$$

Using $\sin^2 x + \cos^2 x = 1$

$$\Rightarrow \cos^2 x = 1 - \sin^2 x$$

$$\Rightarrow \cos x = \pm \sqrt{1 - \left(\frac{1}{3}\right)^2}$$

$$\Rightarrow \cos x = \pm \sqrt{\frac{8}{9}}$$

$$\Rightarrow \cos x = \pm \frac{2\sqrt{2}}{3}$$

Finally using $\sin x = \frac{1}{3}$ with $\cos x = \pm \frac{2\sqrt{2}}{3}$

$$\frac{\cos x}{3 - \sin x} = \frac{\frac{2\sqrt{2}}{3}}{3 - \frac{1}{3}} = \frac{2\sqrt{2}}{9 - 1} = \frac{1}{4}\sqrt{2}$$

$$- \frac{\cos x}{3 - \sin x} = \frac{-\frac{2\sqrt{2}}{3}}{3 - \frac{1}{3}} = \frac{-2\sqrt{2}}{9 - 1} = -\frac{1}{4}\sqrt{2}$$

$$\therefore -\frac{1}{4}\sqrt{2} \leq f(x) \leq \frac{1}{4}\sqrt{2}$$

Question 49 (****+)

It is given that

$$\tan 3x \equiv \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}, \quad x \neq \frac{n\pi}{3}, \quad n = 0, 1, 2, 3, \dots$$

- a) Use the above identity to express $\cot 3x$ in terms of $\cot x$.

- b) Show clearly that

$$\frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} \equiv \cot x, \quad \cos x \neq \frac{1}{2}.$$

- a) Hence, or otherwise, given that $\cos 3x \neq \frac{1}{2}$ solve the trigonometric equation

$$\cos 6x + \sin 6x - \cos 3x - \sin 3x + 1 = 0,$$

for $0 < x < \pi$, giving the answers in terms of π .

, $x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$

a) FIND THE GIVEN IDENTITY

$$\begin{aligned}\tan 3x &= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \\ \frac{1}{\tan 3x} &= \frac{1 - 3\tan^2 x}{3\tan x - \tan^3 x} \\ \cot 3x &= \frac{1 - 3\tan^2 x}{3\tan x - \tan^3 x} \\ \cot 3x &= \frac{1 - \cot^2 x}{\frac{3}{\cot x} - \cot^3 x}\end{aligned}$$

TRYING FOR A SIMPLIFICATION BY CANCELING

$$\cot 3x = \frac{\cot x - 3\cot^2 x}{3\cot x - \cot^3 x}$$

b) USING THE DOUBLE ANGLE IDENTITIES FOR $\cos 2A$ AND $\sin 2A$

$$\begin{aligned}LHS &= \frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} = \frac{(\cos 2x - 1) - \cos x + 1}{2\sin x \cos x - \sin x} \\ &= \frac{2\cos^2 x - \cos x}{2\sin x \cos x - \sin x} = \frac{\cos x(2\cos x - 1)}{\sin x(2\cos x - 1)} = \frac{\cos x}{\sin x} = \cot x = RHS\end{aligned}$$

As required

c) TRYING PART (b) — PART (a) IS NOT ACTUALLY NEEDED!

$$\begin{aligned}\cos 6x + \sin 6x - \cos 3x - \sin 3x + 1 &= 0 \\ \cos 6x - \cos 3x + 1 &= \sin 6x - \sin 3x \\ \cos 6x - \cos 3x + 1 &= -(\sin 6x - \sin 3x) \\ \frac{\cos 6x - \cos 3x + 1}{\sin 6x - \sin 3x} &= -1\end{aligned}$$

THIS IS THE RESULT OF PART (b) WITH $x \mapsto 3x$

$$\Rightarrow \cot 3x = -1$$

$$\Rightarrow \tan 3x = 1$$

$$\arctan(-1) = -\frac{\pi}{4}$$

$$\Rightarrow 3x = -\frac{\pi}{4} + n\pi \quad n \in \mathbb{Z} \setminus \{0\}$$

$$\Rightarrow x = -\frac{\pi}{12} + \frac{n\pi}{3}$$

$\therefore \begin{cases} x_1 = \frac{\pi}{4} \\ x_2 = \frac{7\pi}{12} \\ x_3 = \frac{11\pi}{12} \end{cases}$

All three answers are OK.

Question 50 (***)+

$$f(x) = \frac{\sin 3x}{\cos x} + \frac{\cos 3x}{\sin x}, \quad x \in \mathbb{R}, \quad x \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

- a) Show clearly that

$$f(x) \equiv 2 \cot 2x$$

- b)** Solve the trigonometric equation

$$\frac{1}{4}f(x)+1 = \tan x, \quad 0 \leq x < 2\pi$$

, $x = 0.785^c, 2.94^c, 3.93^c, 6.09^c$

<p>a) AVOID THE THREE-ANGLE FORMULAS !</p> $\begin{aligned} f(x) &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ &= \frac{\cos(A+B)}{\cos A \cos B} = \frac{\cos A \cos B}{\cos A \cos B} = \frac{\cos A \cos B}{\cos A \cos B} = \frac{\cos A \cos B}{\cos A \cos B} = 1 \end{aligned}$ <p><i>AKIBURU</i></p>	<p>b) TRANSFORM SPLITTING THE EQUATION IN $0 < x < 2\pi$</p> $\begin{aligned} \rightarrow \frac{1}{2}f(x) + 1 &= \tan x \\ \rightarrow \frac{1}{2}(10\sin 2x) + 1 &= \tan x \\ \rightarrow \frac{1}{2}\sin 2x + 1 &= \tan x \\ \rightarrow \sin 2x + 2 &= 2\tan x \\ \rightarrow \frac{1}{\tan 2x} + 2 &= 2\tan x \\ \tan 2x &= \frac{2\tan x}{1 - \tan^2 x} \\ \rightarrow \frac{1 - \tan^2 x}{2\tan x} + 2 &= 2\tan x \\ \rightarrow \frac{1 - T^2}{2T} + 2 &= 2T \\ \rightarrow 1 - T^2 + 4T &= 4T^2 \\ \rightarrow 0 &= 5T^2 - 4T - 1 \\ \rightarrow (5T+1)(T-1) &= 0 \end{aligned}$ <p>$\Rightarrow T = -\frac{1}{5}, 1$</p> <p>$\Rightarrow \tan 2x = -\frac{1}{5}$</p> <p>$(0 < \arctan(-\frac{1}{5}) < \frac{\pi}{2})$</p> <p>$\Rightarrow \theta = \arctan(-\frac{1}{5})$</p> <p>$T_1 = 0.785^\circ$</p> <p>$S_1 = 2.427^\circ$</p> <p>$T_2 = 249.4^\circ$</p> <p>$S_2 = 6.08^\circ$</p>
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Question 51 (*)+**

Solve the trigonometric equation

$$\sqrt{3}(\sec x - \tan x) = 1, \quad 0 \leq x \leq 2\pi$$

giving the answers in terms of π .

$$x = \frac{\pi}{6}$$

$$\begin{aligned}
 N\beta (\sec - \tan) &= 1 \\
 \Rightarrow \sec - \tan &= \frac{\sqrt{3}}{3} \\
 \Rightarrow \frac{1}{\cos} - \frac{\sin}{\cos} &= \frac{\sqrt{3}}{3} \\
 \Rightarrow 1 - \sin\beta &= \frac{\sqrt{3}\cos\beta}{3} \\
 \Rightarrow 3 - 3\sin\beta &= \sqrt{3}\cos\beta \\
 \Rightarrow 3 &= 3\sin\beta + \sqrt{3}\cos\beta \\
 \text{Hence } 3\sin\theta + \sqrt{3}\cos\theta &= 3\sin(\theta + \frac{\pi}{6})
 \end{aligned}$$

Question 52 (***)+

Prove the validity of the following trigonometric identity

$$\cot^2 x - \tan^2 x \equiv 4 \cot 2x \cosec 2x$$

proof

$$\begin{aligned}
 LHS &= \cot^2 2x - \tan^2 2x = \frac{\cos^2 2x}{\sin^2 2x} - \frac{\sin^2 2x}{\cos^2 2x} = \frac{\cos^4 2x - \sin^4 2x}{\sin^2 2x \cos^2 2x} \\
 &= \frac{(\cos^2 2x - \sin^2 2x)^2}{\frac{1}{4} \times 4 \sin^2 2x \cos^2 2x} = \frac{(\cos^2 2x - \sin^2 2x)(\cos^2 2x + \sin^2 2x)}{\frac{1}{4} \times (2 \sin^2 2x \cos^2 2x)} \\
 &= \frac{\cos^2 2x}{\frac{1}{4} \sin^2 2x \cos^2 2x} = \frac{4 \cos^2 2x}{\sin^2 2x} = \frac{4 \cos^2 2x}{\sin^2 2x} \times \frac{1}{\sin^2 2x} \\
 &= 4 \cot^2 2x \csc^2 2x = 4H.S
 \end{aligned}$$

Question 53 (***)+

Let $t = \tan \frac{x}{2}$.

a) Show clearly that ...

i. ... $\sin x = \frac{2t}{1+t^2}$,

ii. ... $\cos x = \frac{1-t^2}{1+t^2}$.

b) Use these results to solve the trigonometric equation

$$2\sin x + 3\cos x = 1, \quad 0 \leq x < 2\pi.$$

$$x \approx 1.88^\circ, 5.58^\circ$$

(a) $x = \tan \frac{\alpha}{2}$

$\bullet \sin \alpha = \frac{2t}{1+t^2} = 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right) = \frac{2t}{1+t^2}$

$\bullet \cos \alpha = \frac{1-t^2}{1+t^2} = \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$

(b) $2\sin x + 3\cos x = 1$

$$\begin{aligned} &\rightarrow 2\left(\frac{2t}{1+t^2}\right) + 3\left(\frac{1-t^2}{1+t^2}\right) = 1 \\ &\rightarrow 4t + 3(1-t^2) = 1+t^2 \\ &\rightarrow 4t + 3 - 3t^2 = 1 + t^2 \\ &\Rightarrow 0 = 4t^2 - 4t - 2 \\ &\Rightarrow 0 = 2t^2 - 2t - 1 \\ &\rightarrow \text{quadratic formula} \\ &\rightarrow t = \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

$\left\{ \begin{array}{l} \sin \alpha = \frac{1 \pm \sqrt{5}}{2} = 0.898 \dots \\ \cos \alpha = \frac{1 - \frac{1 \pm \sqrt{5}}{2}}{2} = -0.500 \dots \\ \alpha = 53.9^\circ \pm 180^\circ \\ \alpha = 187.08^\circ \pm 204^\circ \\ \alpha = 0^\circ \dots \\ \frac{\alpha}{2} = 0.8905^\circ \pm 90^\circ \\ \frac{\alpha}{2} = -0.7018^\circ \pm 90^\circ \\ \therefore x = 1.88^\circ, 5.58^\circ \end{array} \right.$

Question 54 (***)+

$$f(x) = \sec x, \quad 0 \leq x < \frac{\pi}{2} \cup \frac{\pi}{2} < x \leq \pi.$$

- a) Sketch in the same diagram the graphs of $f(x)$ and $f^{-1}(x) = \operatorname{arcsec} x$.
- b) State the domain and range of $f^{-1}(x) = \operatorname{arcsec} x$.
- c) Show clearly that $\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$.
- d) Show further that $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{\sqrt{x^4 - x^2}}$.

 , domain: $x \leq -1 \cup x \geq 1$, range: $0 \leq f^{-1}(x) \leq \pi$, $f^{-1}(x) \neq \frac{\pi}{2}$

a) START BY DRAWING SEC X FROM A CO-SINE GRAPH

HENCE DRAWING SEC X AND ITS INVERSE

b) LOOKING AT THE GRAPH ABOVE FOR $y = f(x)$

DOMAIN: $x \in [-1, 0] \cup [0, 1]$

RANGE: $0 \leq f(x) \leq \pi, f(x) \neq \frac{\pi}{2}$

c) DIRECTLY FROM THE INVERSE CURVE

$$\begin{aligned} \Rightarrow y &= \operatorname{arcsec} x \\ \Rightarrow \sec y &= x \\ \Rightarrow \sec y &= \frac{1}{\cos y} \\ \Rightarrow \frac{1}{\cos y} &= \sec y \end{aligned}$$

$\Rightarrow \cos y = \frac{1}{x}$

$$\Rightarrow y = \arccos\left(\frac{1}{x}\right)$$

$\therefore \operatorname{arcsec} x = \operatorname{arcsec}\left(\frac{1}{x}\right)$

d) GRAPH OF STANDARD RESULT & ANSWER OF PART (c)

$$\begin{aligned} \frac{d}{dx}(\operatorname{arcsec} x) &= \frac{d}{dx}(\arccos(\frac{1}{x})) = -\frac{1}{\sqrt{1-(\frac{1}{x})^2}} \times \frac{d}{dx}(\frac{1}{x}) \\ &= -\frac{1}{\sqrt{1-\frac{1}{x^2}}} \times (\frac{1}{x^2}) + \frac{1}{\sqrt{1-\frac{1}{x^2}}} \times \frac{1}{x^2} \\ &= \frac{1}{\sqrt{\frac{2x^2-1}{x^2}}} \times \frac{1}{x^2} = \frac{1}{\sqrt{2x^2-1}} \times \frac{1}{x^2} \end{aligned}$$

REVIEW

OR BY THE INVERSE RULE

$$\begin{aligned} \Rightarrow y &= \operatorname{arcsec} x \\ \Rightarrow \sec y &= x \\ \Rightarrow x &= \sec y \\ \Rightarrow \frac{dx}{dy} &= \sec y \tan y \\ \Rightarrow \left(\frac{dx}{dy}\right)^2 &= \sec^2 y \tan^2 y \\ \Rightarrow \frac{dx}{dy} &= \sec^2 y \tan^2 y \end{aligned}$$

$$\begin{aligned} \Rightarrow \left(\frac{dx}{dy}\right)^2 &= x^2 - 1^2 \\ \Rightarrow \left(\frac{dx}{dy}\right)^2 &= \frac{1}{x^2 - 1} \\ \Rightarrow \frac{dx}{dy} &= \pm \frac{1}{\sqrt{x^2 - 1}} \\ \Rightarrow \frac{dx}{dy} &= \pm \frac{1}{\sqrt{x^2 - 1}} \end{aligned}$$

REVIEW CONCEPT IN THE
SOME DOMAIN (RAM)

Question 55 (***)+

It is given that

$$\sin 3\theta \equiv 3\sin \theta - 4\sin^3 \theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) By differentiating both sides of the above identity with respect to θ , show that

$$\cos 3\theta \equiv 4\cos^3 \theta - 3\cos \theta.$$

- c) Hence show that

$$\tan 3\theta \equiv \frac{3\tan \theta \sec^2 \theta - 4\tan^3 \theta}{4 - 3\sec^2 \theta}.$$

- d) Deduce that

$$\tan 3\theta \equiv \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta}.$$

 , proof

a) Simplifying from the L.H.S.

$$\begin{aligned}
 \text{L.H.S.} &= \sin 3\theta \\
 &= \sin(3\theta + 0) \\
 &= \sin 3\theta \cos 0 + (\cos 3\theta) \sin 0 \\
 &= (2\sin \theta \cos^2 \theta + \cos^3 \theta) + (1 - 2\sin^2 \theta) \sin \theta \\
 &= 2\sin \theta \cos^2 \theta + \sin \theta + 2\sin^2 \theta \\
 &= 2\sin \theta (1 - \sin^2 \theta) + \sin \theta + 2\sin^2 \theta \\
 &= 2\sin \theta - 2\sin^3 \theta + \sin \theta + 2\sin^2 \theta \\
 &= 3\sin \theta - 4\sin^3 \theta \\
 &= \text{R.H.S.}
 \end{aligned}$$

// to remove

b) Differentiating the identity w.r.t. θ .

$$\begin{aligned}
 \frac{d}{d\theta} [\sin 3\theta] &= \frac{d}{d\theta} [3\sin \theta - 4\sin^3 \theta] \\
 3\cos 3\theta &= 3\cos \theta - 12\sin^2 \theta \cos \theta \\
 \cos 3\theta &= \cos \theta - 4\sin^2 \theta (1 - \cos^2 \theta) \\
 \cos 3\theta &= \cos \theta - 4\cos \theta + 4\cos^3 \theta \\
 \cos 3\theta &= 4\cos^3 \theta - 3\cos \theta
 \end{aligned}$$

As required

c) Proved as follows.

$$\begin{aligned}
 \tan 3\theta &= \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\sin \theta - 4\sin^3 \theta}{4\cos \theta - 3\cos \theta} = \frac{\frac{3\sin \theta}{\cos \theta} - \frac{4\sin^3 \theta}{\cos \theta}}{\frac{4\cos \theta}{\cos \theta} - \frac{3\cos \theta}{\cos \theta}} \\
 &= \frac{\frac{3\sin \theta}{\cos \theta} - \frac{4\sin^3 \theta}{\cos \theta}}{4 - 3\cos^2 \theta} = \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta}
 \end{aligned}$$

As required

d) Using the identity $1 + \tan^2 \theta = \sec^2 \theta$ we have

$$\begin{aligned}
 \tan 3\theta &= \frac{3\tan \theta \sec^2 \theta - 4\tan^3 \theta}{4 - 3\sec^2 \theta} = \frac{3\tan \theta (1 + \tan^2 \theta) - 4\tan^3 \theta}{4 - 3(1 + \tan^2 \theta)} \\
 &= \frac{3\tan \theta + 3\tan^3 \theta - 4\tan^3 \theta}{4 - 3 - 3\tan^2 \theta} = \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta}
 \end{aligned}$$

As required

Question 56 (***)+

Prove the validity of the trigonometric identity

$$\frac{1+\sin\theta}{1-\sin\theta} \equiv (\sec\theta + \tan\theta)^2.$$

[proof]

$$\begin{aligned} LHS &= \frac{1+\sin\theta}{1-\sin\theta} = \frac{(1+\sin\theta)(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} = \frac{1+2\sin\theta+\sin^2\theta}{1-\sin^2\theta} \\ &= \frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} \\ &= \sec^2\theta + 2\tan\theta + \tan^2\theta = \sec^2\theta + 2\tan\theta\sec\theta + \tan^2\theta \\ &= (\sec\theta + \tan\theta)^2 = RHS \end{aligned}$$

Question 57 (***)+

Solve the following trigonometric equation.

$$\arctan\left(\frac{x-5}{x-1}\right) + \arctan\left(\frac{x-4}{x-3}\right) = \frac{\pi}{4}, \quad x \in \mathbb{R}.$$

$x = 3 \cup x = 6$

$$\begin{aligned} \text{Let } \theta &= \arctan\left(\frac{3-x}{x-1}\right) \quad \text{and } \phi = \arctan\left(\frac{x-4}{x-3}\right) \\ \Rightarrow \theta + \phi &= \frac{\pi}{4} \\ \Rightarrow \tan(\theta + \phi) &= \tan\frac{\pi}{4} \\ \Rightarrow \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} &= 1 \\ \Rightarrow \frac{\frac{3-x}{x-1} + \frac{x-4}{x-3}}{1 - \frac{3-x}{x-1} \cdot \frac{x-4}{x-3}} &= 1 \\ \Rightarrow \frac{3-x}{x-1} + \frac{x-4}{x-3} &= 1 - \frac{(3-x)(x-4)}{(x-1)(x-3)} \end{aligned}$$

MULTIPLY THROUGH BY $(x-1)(x-3)$

$$\begin{aligned} (x-3)(x-1) + (x-4)(x-1) &= (x-1)(x-3) - (x-2)(x-4) \\ 3x^2 - 10x + 15 + x^2 - 5x + 4 &= x^2 - 4x + 3 - (x^2 - 9x + 12) \\ 2x^2 - 13x + 19 &= -x^2 + 5x - 17 \\ 3x^2 - 18x + 36 &= 0 \\ 3x^2 - 18x + 18 &= 0 \\ (x-3)(3x-6) &= 0 \end{aligned}$$

$\therefore x = \frac{3}{3} \quad \text{OR} \quad 4x = 18$

$$\begin{aligned} \arctan\left(\frac{3-x}{x-1}\right) + \arctan\left(\frac{x-4}{x-3}\right) &= \frac{\pi}{4} \\ \arctan(-1) + \arctan(2) &= -\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

Question 58 (***)+

Given that

$$\sec^2 x - (1 + \sqrt{3}) \tan x + \sqrt{3} = 1,$$

show that either $\tan x = 1$ or $\tan x = \sqrt{3}$.

Detailed workings must be shown in this question.

, proof

USING $1 + \tan^2 x = \sec^2 x$

$$\begin{aligned} &\Rightarrow \sec^2 x - (1 + \sqrt{3}) \tan x + \sqrt{3} = 1 \\ &\Rightarrow \tan^2 x + 1 - (1 + \sqrt{3}) \tan x + \sqrt{3} = 0 \\ &\Rightarrow \tan^2 x - (1 + \sqrt{3}) \tan x + \sqrt{3} = 0 \end{aligned}$$

BY THE QUADRATIC FORMULA

$$\begin{aligned} &\Rightarrow \tan x = \frac{(1 + \sqrt{3}) \pm \sqrt{(1 + \sqrt{3})^2 - 4(\sqrt{3})}}{2} \\ &\Rightarrow \tan x = \frac{(1 + \sqrt{3}) \pm \sqrt{1 + 2\sqrt{3} + 3 - 4\sqrt{3}}}{2} \\ &\Rightarrow \tan x = \frac{1 + \sqrt{3} \pm \sqrt{-2\sqrt{3} + 2}}{2} \\ &\Rightarrow \tan x = \frac{1 + \sqrt{3} \pm \sqrt{(6\sqrt{3} - 1)}}{2} \\ &\Rightarrow \tan x = \frac{1 + \sqrt{3} \pm \sqrt{(4\sqrt{3} - 1)}}{2} \\ &\Rightarrow \tan x = \frac{1 + \sqrt{3} \pm \sqrt{4\sqrt{3} - 1}}{2} \\ &\Rightarrow \tan x = \frac{2\sqrt{3}}{2} \\ &\Rightarrow \tan x = \sqrt{3} \end{aligned}$$

Question 59 (***)+

It is given that

$$u = \sin 2\theta, \quad v = \cot \theta.$$

Use trigonometric identities to find a simplified expression for u^2 in terms of v .

$$u^2 = \frac{2v}{v^2 + 1}$$

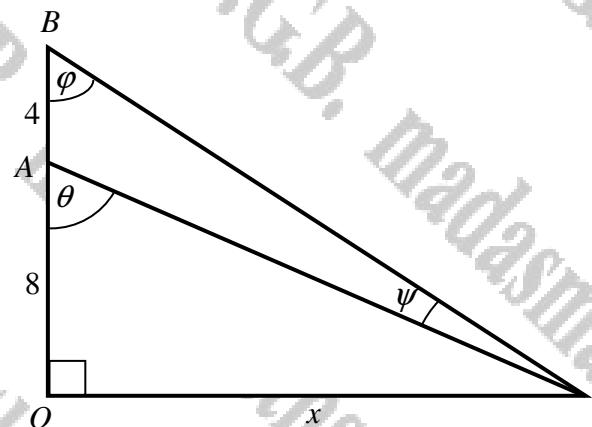
$u = \sin 2\theta$ $u = 2\sin \theta \cos \theta$ $u^2 = 4\sin^2 \theta \cos^2 \theta$ $u^2 = 4\sin^2 \theta (1 - \sin^2 \theta)$	$v = \cot \theta$ $v^2 = \cot^2 \theta$ $v^2 = \cos^2 \theta - 1$ $V^2 + 1 = \cos^2 \theta$ $\cot^2 \theta = \frac{1}{V^2 + 1}$
--	---

$$\begin{aligned} u^2 &= 4 \left(\frac{1}{V^2 + 1} \right) (1 - \frac{1}{V^2 + 1}) \\ u^2 &= 4 \cdot \frac{1}{V^2 + 1} \times \frac{V^2}{V^2 + 1} \\ u^2 &= \frac{4V^2}{(V^2 + 1)^2} \\ u^2 &= \frac{4V}{V^2 + 1} \end{aligned}$$

Question 60 (***)+

The diagram below shows a right angled triangle OBC where $|OC| = x$ and the point A on OB so that $|OA| = 8$, $|AB| = 4$.

The angles OAC , OBC and ACB are denoted by θ , ϕ and ψ respectively.



By considering a relationship between the angles θ , ϕ and ψ , show that

$$\tan \psi = \frac{4x}{96 + x^2} .$$

[] , proof

LOOKING AT THE TRIANGLE

$\hat{B}OC = 180 - \phi$

LOOKING AT $\triangle ABC$

$\phi + (\hat{O} - \phi) + \psi = 180$

$\phi - \phi + \psi = 180$

$\psi = 180 - \phi$

TANGENT-TANGENT BOTH SIDES

$\tan \psi = \tan(180 - \phi)$

$\tan \psi = \frac{\tan \phi - \tan \psi}{1 - \tan \phi \tan \psi}$

$\tan \psi = \frac{\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{3}}{2}}$

$\tan \psi = \frac{\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{2}}{1 - \frac{3}{2}}$

$\tan \psi = \frac{\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$\tan \psi = \frac{\frac{2\sqrt{3}}{6} - \frac{3\sqrt{3}}{6}}{\frac{1}{2}}$

$\tan \psi = \frac{\frac{-\sqrt{3}}{3}}{\frac{1}{2}}$

$\therefore \tan \psi = \frac{4x}{96 + x^2}$ AS REQUIRED

Question 61 (**+)**

Solve the trigonometric equation

$$\sin x \cos x \cos 2x \cos 4x = \frac{\sqrt{2}}{16}, \quad 0 \leq x \leq \frac{\pi}{2},$$

giving the answers in terms of π .

$$\boxed{\quad}, \quad x = \frac{\pi}{32}, \frac{3\pi}{32}, \frac{9\pi}{32}, \frac{11\pi}{32}$$

Using the double angle identity for $\sin 2x$

$$\begin{aligned} &\Rightarrow \sin x \cos x \cos 2x \cos 4x = \frac{\sqrt{2}}{16} \times 2 \\ &\Rightarrow 2\sin x \cos x \cos 2x \cos 4x = \frac{\sqrt{2}}{8} \\ &\Rightarrow \sin 2x \cos 2x \cos 4x = \frac{\sqrt{2}}{8} \\ &\Rightarrow 2\sin 2x \cos 2x \cos 4x = \frac{\sqrt{2}}{4} \\ &\Rightarrow \sin 4x \cos 4x = \frac{\sqrt{2}}{4} \\ &\Rightarrow 2\sin 4x \cos 4x = \frac{\sqrt{2}}{2} \\ &\Rightarrow \sin 8x = \frac{\sqrt{2}}{2} \end{aligned}$$

$\left(\begin{array}{l} 8x = \frac{\pi}{4} \pm 2n\pi \\ 8x = \frac{7\pi}{4} \pm 2n\pi \end{array} \right) \quad n=0,1,2,3,\dots$

$$\left(\begin{array}{l} x = \frac{\pi}{32} \pm \frac{n\pi}{8} \\ x = \frac{7\pi}{32} \pm \frac{n\pi}{8} \end{array} \right)$$

For the range $0 \leq x < \frac{\pi}{2}$

$$x \in \frac{\pi}{32}, \frac{15\pi}{32}, \frac{23\pi}{32}, \frac{31\pi}{32}$$

Question 62 (**+)**

Solve the trigonometric equation

$$\frac{4}{2\sec \phi - 2\sin \phi + 1} = \cot \phi, \quad 0 < \phi < 2\pi, \quad \phi \neq \pi,$$

giving the answers in terms of π .

$$\phi = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\frac{4}{2\sec \phi - 2\sin \phi + 1} = \cot \phi$$

$$\therefore \sec \phi \neq -2$$

$$\sin \phi \neq \frac{1}{2}$$

$$\cos(\frac{\phi}{2}) = \frac{\sqrt{3}}{2}$$

$$\left(\begin{array}{l} \phi = \frac{\pi}{6} \pm 2m\pi \\ \phi = \frac{11\pi}{6} \pm 2m\pi \end{array} \right) \quad m=0,1,2,3,\dots$$

$$\therefore \phi_1 = \frac{\pi}{6}$$

$$\phi_2 = \frac{11\pi}{6}$$

Question 63 (*)+**

Solve the trigonometric equation

$$\tan \theta(1 + \cos 2\theta) = 2\sin^2 2\theta, \quad 0^\circ \leq \theta \leq 90^\circ.$$

$$\boxed{\theta = 0^\circ, 15^\circ, 75^\circ, 90^\circ}$$

Working:

$$\begin{aligned} \tan \theta(1 + \cos 2\theta) &= 2\sin^2 2\theta \\ \tan \theta(1 + 2\cos^2 \theta - 1) &= 2\sin^2 2\theta \\ 2\tan \theta \cos^2 \theta &= 2\sin^2 2\theta \\ 2\tan \theta \cos^2 \theta &= 2\sin^2 2\theta \\ \frac{2\tan \theta \cos^2 \theta}{\cos^2 \theta} &= 2\sin^2 2\theta \\ 2\tan \theta &= 2\sin^2 2\theta \\ 0 &= 2\sin^2 2\theta - \sin 2\theta \\ 0 &= 2\sin 2\theta(\sin 2\theta - 1) \\ \sin 2\theta &= 0 \quad \text{or} \quad \sin 2\theta - 1 = 0 \\ \sin 2\theta &= 0 \\ 2\theta &= 0^\circ \quad \text{or} \quad 2\theta = 180^\circ \\ \theta &= 0^\circ \quad \text{or} \quad \theta = 90^\circ \end{aligned}$$

$\theta = 0^\circ, 15^\circ, 75^\circ, 90^\circ$

Question 64 (*)+**

Solve the trigonometric equation

$$\arcsin x + \arccos \frac{3}{5} = 2\arctan \frac{3}{4}.$$

$$\boxed{\quad}, \quad x = \frac{44}{125}$$

Working:

LET $\alpha = \arccos \frac{3}{5}$, $\beta = \arctan \frac{3}{4}$

TRANSFORM THE EQUATION

$$\begin{aligned} \Rightarrow \arcsin x + \arccos \frac{3}{5} &= 2\arctan \frac{3}{4} \\ \Rightarrow \arcsin x + \theta &= 2\phi \\ \Rightarrow \arcsin x &= 2\phi - \theta \\ \Rightarrow \sin(\arcsin x) &= \sin(2\phi - \theta) \\ \Rightarrow x &= \sin(2\phi - \theta) = \sin 2\phi \cos \theta - \cos 2\phi \sin \theta \end{aligned}$$

USING DOUBLE-ANGLE IDENTITIES $\sin 2\phi = 2\sin \phi \cos \phi$ & $\cos 2\phi = 2\cos^2 \phi - 1$

$$\begin{aligned} \Rightarrow x &= 2\sin \phi \cos \phi - \sin \theta (2\cos^2 \phi - 1) \\ \Rightarrow x &= 2 \left(\frac{3}{5} \right) \left(\frac{4}{5} \right) - \frac{3}{5} \left(2 \cdot \frac{16}{25} - 1 \right) \\ \Rightarrow x &= \frac{24}{25} - \frac{27}{25} \\ \Rightarrow x &= \frac{44}{125} \end{aligned}$$

Question 65 (***)+

It is given that

$$\frac{\cot x}{\cosec x - 1} - \frac{\cos x}{1 + \sin x} \equiv 2 \tan x, \quad x \neq 180^\circ n, \quad n \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence solve the trigonometric equation

$$\frac{\cot 3\theta}{\cosec 3\theta - 1} - \frac{\cos 3\theta}{1 + \sin 3\theta} = 2 \tan \theta, \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of π .

$$\boxed{\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}}$$

(a)

$$\begin{aligned} \text{LHS} &= \frac{\cot x}{\cosec x - 1} - \frac{\cos x}{1 + \sin x} = \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x} - 1} - \frac{\cos x}{1 + \sin x} \quad \text{[Divide top and bottom by } \sin x] \\ &= \frac{\cos x}{1 - \sin x} - \frac{\cos x}{1 + \sin x} = \cos x \left[\frac{1}{1 - \sin x} - \frac{1}{1 + \sin x} \right] \\ &= \cos x \left[\frac{(1 + \sin x) - (1 - \sin x)}{(1 - \sin x)(1 + \sin x)} \right] = \cos x \left[\frac{2\sin x}{1 - \sin^2 x} \right] \\ &= \cos x \frac{2\sin x}{\cos^2 x} = \frac{2\sin x}{\cos x} = 2\tan x = \text{RHS} \end{aligned}$$

(b)

$$\begin{aligned} \frac{\cot 3\theta}{\cosec 3\theta - 1} - \frac{\cos 3\theta}{1 + \sin 3\theta} &= 2\tan 3\theta \\ 2\tan 3\theta &= 2\tan 3\theta \\ \tan 3\theta &= \tan 3\theta \\ 3\theta &= \theta + m\pi \\ 2\theta &= 0 + m\pi \\ \theta &= 0 + \frac{m\pi}{2} \quad \boxed{\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}} \end{aligned}$$

Question 66 (***)+

Prove the validity of the following trigonometric identities.

i. $\frac{\tan 2x}{\tan 2x + \sin 2x} \equiv \frac{1}{2} \sec^2 x$.

ii. $\frac{\tan \varphi}{(1 - \cos \varphi)(1 + \sec \varphi)} \equiv \operatorname{cosec} \varphi$.

proof

$$\begin{aligned}
 \text{(i) LHS} &= \frac{\tan 2x}{\tan 2x + \sin 2x} = \frac{\frac{\sin 2x}{\cos 2x}}{\frac{\sin 2x + \sin 2x}{\cos 2x}} = \text{DIVIDE TOP/BOTTOM OF THE FRACTION BY } \sin 2x \\
 &= \frac{\frac{1}{\cos 2x}}{\frac{1}{\cos 2x} + 1} = \text{CANCEL THE } \sin 2x \text{ IN THE FRACTION} = \frac{1}{1 + \cos 2x} \\
 &= \frac{1}{1 + (2\cos^2 - 1)} = \frac{1}{2\cos^2} = \frac{1}{2} \sec^2 = \text{RHS}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) LHS} &= \frac{\tan \varphi}{(1 - \cos \varphi)(1 + \sec \varphi)} = \frac{\frac{\sin \varphi}{\cos \varphi}}{(1 - \cos \varphi)(1 + \frac{1}{\cos \varphi})} = \frac{\frac{\sin \varphi}{\cos \varphi}}{\frac{\cos \varphi - \cos^2 \varphi}{\cos \varphi}} = \frac{\sin \varphi}{\cos^2 \varphi - \cos \varphi} = \frac{\sin \varphi}{\cos \varphi(\cos \varphi - 1)} \\
 &= \frac{\sin \varphi}{\cos \varphi} = \frac{\sin \varphi}{\sin^2 \varphi} = \frac{1}{\sin \varphi} = \operatorname{cosec} \varphi = \text{RHS}
 \end{aligned}$$

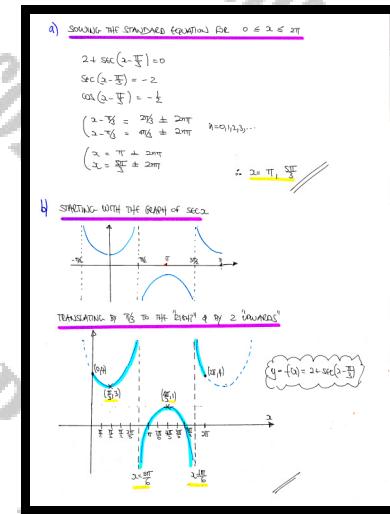
Question 67 (***)+

$$f(x) = 2 + \sec\left(x - \frac{\pi}{3}\right), \quad x \in \mathbb{R}, \quad 0 \leq x \leq 2\pi.$$

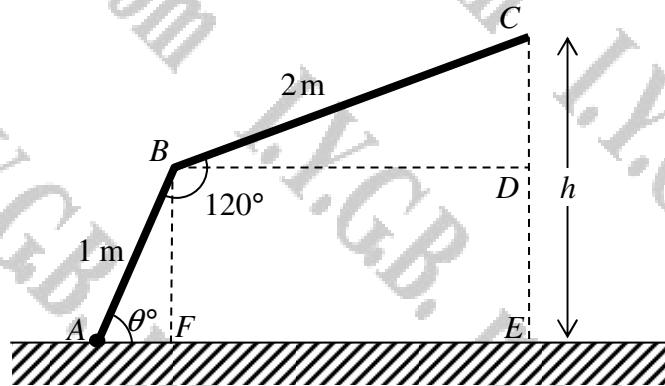
- a) Solve the equation $f(x) = 0$.
- b) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any stationary points, the coordinates of any x or y intercepts and equations of the vertical asymptotes.

$$\boxed{x = \frac{5\pi}{3}}, \quad \boxed{x = \pi}$$



Question 68 (***)+



The figure above shows a rigid rod ABC where AB is 1 metre, BC is 2 metres and the angle ABC is 120° . The rod is hinged at A so it can be rotated in a vertical plane forming an angle θ° with the horizontal ground.

Let h metres be the height of the point C from the horizontal ground.

a) Show that ...

i. $\angle DBC = \theta^\circ - 60^\circ$.

ii. $h = 2 \sin \theta - \sqrt{3} \cos \theta$.

iii. ... when AC is horizontal, $\tan \theta = \frac{\sqrt{3}}{2}$.

b) By expressing h in the form $R \sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90^\circ$, find the maximum value of h and the value of θ when h takes this maximum value.

 , $h_{\max} = \sqrt{7}$, $\theta \approx 130.9^\circ$

(a)

Given: $AB = 1$, $BC = 2$, $\angle ABC = 120^\circ$

$$\begin{aligned} \text{Using } \cos(120^\circ) &= -\frac{1}{2}, \quad \cos(60^\circ) = \frac{1}{2} \\ 1 \cdot \frac{1}{2} + 2 \cdot \left(-\frac{1}{2}\right) &= -\frac{1}{2} \\ 1 - 1 &= -\frac{1}{2} \end{aligned}$$

Using Pythagoras:

$$\begin{aligned} AC^2 &= AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos(120^\circ) \\ AC^2 &= 1 + 4 - 2 \cdot 1 \cdot 2 \cdot \left(-\frac{1}{2}\right) \\ AC^2 &= 1 + 4 + 2 \\ AC^2 &= 7 \end{aligned}$$

Using $\sin^2 \theta + \cos^2 \theta = 1$:

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \sin^2 \theta + \left(-\frac{1}{2}\right)^2 &= 1 \\ \sin^2 \theta + \frac{1}{4} &= 1 \\ \sin^2 \theta &= \frac{3}{4} \\ \sin \theta &= \frac{\sqrt{3}}{2} \end{aligned}$$

Using $\tan \theta = \frac{\sin \theta}{\cos \theta}$:

$$\begin{aligned} \tan \theta &= \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} \\ \tan \theta &= -\sqrt{3} \end{aligned}$$

Using $\theta = 180^\circ - 60^\circ = 120^\circ$:

$$\begin{aligned} \theta &= 180^\circ - 60^\circ \\ \theta &= 120^\circ \end{aligned}$$

(b)

$$\begin{aligned} h &= 2 \sin \theta - \sqrt{3} \cos \theta \\ &\equiv R \sin(\theta - \alpha) \quad \left\{ \begin{array}{l} R = \sqrt{2^2 + (\sqrt{3})^2} = \sqrt{7} \\ \tan \alpha = \frac{\sqrt{3}}{2} \\ \alpha \approx 40.9^\circ \end{array} \right. \\ &\equiv \sqrt{7} \sin(\theta - 40.9^\circ) \end{aligned}$$

$$\begin{aligned} h_{\max} &= \sqrt{7} \sin(90^\circ - 40.9^\circ) \\ &= \sqrt{7} \sin 49.1^\circ \\ &= \sqrt{7} \cdot 0.756 \\ &= 3.5 \end{aligned}$$

Question 69 (***)+

Solve the trigonometric equation

$$2\sin y + 3\sec y = 6 + \tan y, \quad 0 \leq y < 2\pi,$$

giving the answers in terms of π .

$$y = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\begin{aligned} 2\sin y + 3\sec y &= 6 + \tan y \\ 2\sin y + \frac{3}{\cos y} &= 6 + \frac{\sin y}{\cos y} \\ 2\sin y \cos y + 3 &= 6\cos y + \sin y \\ 2\sin y \cos y - 6\cos y &= \sin y - 3 \\ \sin y(2\cos y - 1) - 3(2\cos y - 1) &= 0 \\ (\sin y - 1)(2\cos y - 3) &= 0 \\ \sin y = \frac{1}{2} \text{ or } 2\cos y = 3 \end{aligned}$$

$\arccos(\frac{3}{2}) = \frac{\pi}{3}$
 $y = \frac{\pi}{3} + 2k\pi \quad k=0,1,2,\dots$
 $y = \frac{\pi}{3}$
 $y = \frac{5\pi}{3}$

Question 70 (***)+

$$f(x) = \sec^2 x, x \in \mathbb{R}, x \neq \frac{\pi}{2}(2n+1), n \in \mathbb{N}.$$

Show that if

$$f(x) = \frac{1}{2} f\left(x + \frac{\pi}{4}\right),$$

then either $\sin x = 0$ or $\tan x = 2$.

, proof

WRITE THE EQUATION EXPLICITLY

$$\begin{aligned} \Rightarrow f(x) &= \frac{1}{2} f(x + \frac{\pi}{4}) \\ \Rightarrow 2f(x) &= f(x + \frac{\pi}{4}) \\ \Rightarrow 2\sec^2 x &= \sec^2(x + \frac{\pi}{4}) \\ \Rightarrow \frac{2}{\sin^2 x} &= \frac{1}{\sin^2(x + \frac{\pi}{4})} \\ \Rightarrow \frac{\sin^2 x}{2} &= \sin^2(x + \frac{\pi}{4}) \end{aligned}$$

SOLVE THE COMPLEX NUMBER IDENTITY, $(\cos A + i\sin A)^2$

$$\begin{aligned} \Rightarrow \frac{1}{2}\sin^2 x &= [\cos(x + \frac{\pi}{4}) - i\sin(x + \frac{\pi}{4})]^2 \\ \Rightarrow \frac{1}{2}\sin^2 x &= [\frac{1}{2}\cos x - \frac{i}{2}\sin x]^2 \quad \left\{ \sin(x + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\cos x + i\sin x) \right\} \\ \Rightarrow \frac{1}{2}\sin^2 x &= (\frac{1}{2})^2 (\cos^2 x - \sin^2 x)^2 \\ \Rightarrow \frac{1}{2}\sin^2 x &= \frac{1}{4}(\cos^2 x - \sin^2 x)^2 \end{aligned}$$

FIND TWO DIFFERENT APPROACHES TO FOLLOW

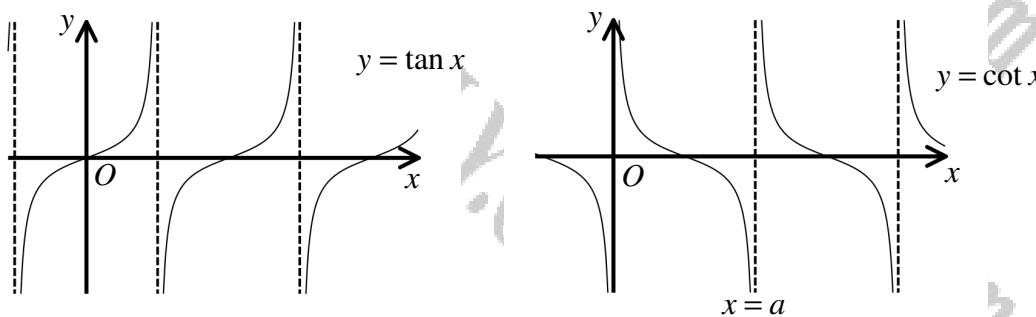
$$\begin{aligned} \cos^2 x - \sin^2 x &= 2\cos x \cos x \\ 0 &= \sin^2 x - 2\sin x \cos x \\ 0 &= \sin x (\sin x - 2\cos x) \\ \frac{0}{\sin x} &= \frac{\sin x (\sin x - 2\cos x)}{\sin x} \\ 0 &= \sin x (\tan x - 2) \end{aligned}$$

$\sin x = 0$ or $\tan x = 2$

$$\begin{aligned} \cos^2 x - \sin^2 x &= -\sin x \\ +\cos^2 x + \cos^2 x &= -\sin x \\ 2\cos^2 x &= -\sin x \\ (\cos x - \cos x)^2 &= -\sin x \\ -\cos x - \cos x &= -\sin x \\ \sin x = 0 & \quad \text{or} \quad \sin x = 2 \end{aligned}$$

$\sin x = 0$ or $\sin x = 2$

Question 71 (****+)



The diagrams above shows part of the graphs of $y = \tan x$ and $y = \cot x$.

- Sketch the graph of $y_1 = -\tan(-x)$ and hence write a simplified expression for y_1 in terms of $\tan x$.
- State the value of a .

The graph of $\cot x$ has vertical asymptotes and the equation of one of them is labelled in the diagram as $x = a$.

- Describe the two geometric transformations.
- Deduce using valid arguments that

$$\cot x = \tan\left(\frac{\pi}{2} - x\right).$$

$a = \pi$, reflection in the x axis/translation to the "right" by $\frac{\pi}{2}$ units

(a) $-\tan(-x)$ is reflection of $\tan x$ in the x axis and in the y axis
(any order)

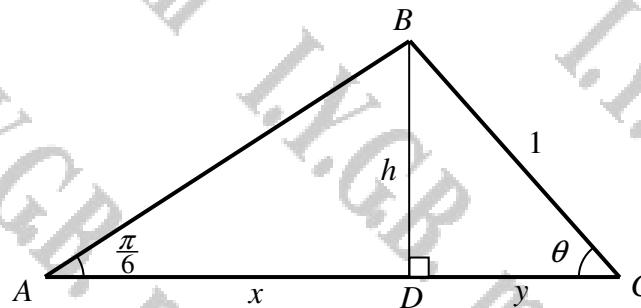
$\therefore -\tan(-x) \equiv \tan x$

(b) $\cot x = \frac{\cos x}{\sin x}$ so first positive value for which $\cot x = 0$

(c) In any order
 $\cot x \rightarrow -\tan(x) \rightarrow -\tan(x - \frac{\pi}{2}) = \cot x$
In the x axis and in the y axis

(d) $\cot x = -\tan(\frac{\pi}{2} - x)$
(parts)
 $\therefore \cot x = \tan(\frac{\pi}{2} - x)$

Question 72 (***)+



The figure above shows a triangle ABC , where $\angle BAC = \frac{\pi}{6}$, $\angle BCA = \theta$ and $|BC| = 1$.

The straight line segment BD , labelled as h , is perpendicular to AC .

Let $AD = x$ and $DC = y$.

- a) By expressing h in terms of θ , and x in terms of h , show that

$$x + y = \sqrt{3} \sin \theta + \cos \theta,$$

and hence deduce that the area of the triangle ABC is given by

$$\sin \theta \sin\left(\theta + \frac{\pi}{6}\right).$$

- b) By using the trigonometric identities for

$$\cos\left[\theta + \left(\theta + \frac{\pi}{6}\right)\right] \quad \text{and} \quad \cos\left[\theta - \left(\theta + \frac{\pi}{6}\right)\right],$$

write a simplified expression for the area of the triangle ABC .

[continues overleaf]

[continued from overleaf]

The value of θ can vary.

- c) By using part (b), deduce that the maximum value of the area of the triangle ABC is

$$\frac{1}{4}(2+\sqrt{3})$$

and this maximum value occurs when $\theta = \frac{5\pi}{12}$.

$$\boxed{\quad}, \sin \theta \sin\left(\theta + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{4} - \frac{1}{2} \cos\left(2\theta + \frac{\pi}{6}\right)$$

a) LOOKING AT $\triangle BDC$

- $h = l \times \sin \theta$
- $y = l \times \cos \theta$

LOOKING AT $\triangle ABD$

$$\begin{aligned} \rightarrow \frac{l}{x} &= \tan \frac{\pi}{6} \\ \rightarrow \frac{\sin \theta}{x} &= \frac{\sqrt{3}}{3} \\ \rightarrow \sqrt{3}x &= 3 \sin \theta \\ \rightarrow x &= \sqrt{3} \sin \theta \end{aligned}$$

$$\therefore x+y = \cos \theta + \sqrt{3} \sin \theta$$

$$\begin{aligned} \therefore \text{Area} &= \frac{1}{2} \text{base} \times \text{height} \\ &= \frac{1}{2} (x+y) h \\ &= \frac{1}{2} (2\sin \theta + \sqrt{3} \sin \theta) \sin \theta \\ &= (\sin \theta + \frac{\sqrt{3}}{2} \sin \theta) \sin \theta \\ &= (\sin \theta \cos \frac{\pi}{6} + \cos \theta \sin \frac{\pi}{6}) \sin \theta \\ &= \sin \left(\theta + \frac{\pi}{6}\right) \sin \theta \end{aligned}$$

LOOKING AT THE COMPOUND ANGLE IDENTITIES FOR $\cos(A+B)$

$$\begin{aligned} \cos\left[\theta + \left(\theta + \frac{\pi}{6}\right)\right] &\equiv \cos \theta \cos\left(\theta + \frac{\pi}{6}\right) - \sin \theta \sin\left(\theta + \frac{\pi}{6}\right) \\ \cos\left[\theta - \left(\theta + \frac{\pi}{6}\right)\right] &\equiv \cos \theta \cos\left(\theta + \frac{\pi}{6}\right) + \sin \theta \sin\left(\theta + \frac{\pi}{6}\right) \end{aligned}$$

SUBTRACTING "NUMBERS"

$$\begin{aligned} \Rightarrow \cos\left(-\frac{\pi}{6}\right) - \cos\left(2\theta + \frac{\pi}{6}\right) &\equiv 2 \sin \theta \sin\left(\theta + \frac{\pi}{6}\right) \\ \Rightarrow \frac{1}{2} \cos\left(\frac{\pi}{6}\right) - \frac{1}{2} \cos\left(2\theta + \frac{\pi}{6}\right) &\equiv \sin \theta \sin\left(\theta + \frac{\pi}{6}\right) \\ \Rightarrow \sin \theta \sin\left(\theta + \frac{\pi}{6}\right) &\equiv \frac{\sqrt{3}}{4} - \frac{1}{2} \cos\left(2\theta + \frac{\pi}{6}\right) \\ \Rightarrow \text{Area} &= \frac{1}{4} \left[\sqrt{3} - 2 \cos\left(2\theta + \frac{\pi}{6}\right) \right] \end{aligned}$$

c) MAXIMUM AREA OCCURS WHEN $\cos\left(2\theta + \frac{\pi}{6}\right) = -1$

$$\begin{aligned} \text{Area}_{\max} &= \frac{1}{4} [\sqrt{3} - 2(-1)] \\ \text{Area}_{\max} &= \frac{1}{4} (\sqrt{3} + 2) \end{aligned}$$

AND FINALLY

$$\begin{aligned} \cos\left(2\theta + \frac{\pi}{6}\right) &= -1 \\ 2\theta + \frac{\pi}{6} &= \pi \\ 2\theta &= \frac{5\pi}{6} \\ \theta &= \frac{5\pi}{12} \end{aligned}$$

Question 73 (**+)**

The height of tide, h meters, in a harbour on a certain day can be modelled by

$$h(t) = 10 + \sqrt{3} \sin(30t)^\circ + \cos(30t)^\circ, \quad 0 \leq t \leq 12,$$

where t is the time in hours since midnight.

- a) Find the time when the high tide and the low tide occur during the morning hours of that day and state the corresponding depth of water in the harbour at these times.

The depth of water in this harbour needs to be at least 8.5 metres for a boat to dock

A boat arrives outside the harbour at high tide and needs five hours to unload.

- b) Show that the boat has to wait until 09:23 to enter the harbour.

[] , high tide of 12 metres at 02:00 , [] , low tide of 8 metres at 08:00

(a)

$$h(t) = 10 + \sqrt{3} \sin(30t)^\circ + \cos(30t)^\circ$$

$$\sqrt{3} \sin(30t)^\circ + \cos(30t)^\circ \equiv R \cos(30t - \alpha)$$

$$= R \cos 30t \cos \alpha + R \sin 30t \sin \alpha$$

$$\equiv (R \cos \alpha) \cos 30t + (R \sin \alpha) \sin 30t$$

R = $\sqrt{1^2 + (\sqrt{3})^2} = 2$
 $\tan \alpha = \sqrt{3} \Rightarrow \alpha = 60^\circ$

$$\therefore h(t) = 10 + 2 \cos(30t - 60)$$

High TIDE is 12 metres $\Rightarrow \cos(30t - 60) = 1$
 $30t - 60 = 0^\circ$
 $30t = 60^\circ$
 $t = 2$

Low Tide is 8 metres $\Rightarrow \cos(30t - 60) = -1$
 $30t - 60 = 180^\circ$
 $30t = 240^\circ$
 $t = 8$

\therefore High Tide 12 metres AT 02:00
 Low Tide 8 metres AT 08:00

(b)

$$h = 8.5$$

$$\Rightarrow 8.5 = 10 + 2 \cos(30t - 60)$$

$$\Rightarrow -1.5 = 2 \cos(30t - 60)$$

$$\Rightarrow \cos(30t - 60) = -0.75$$

$$\Rightarrow \begin{cases} 30t - 60 = 138.59^\circ \pm 360n \\ 30t - 60 = 221.41^\circ \pm 360n \end{cases}$$

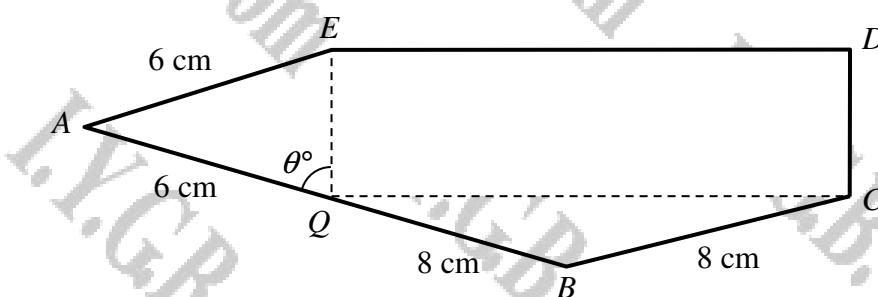
$$\begin{cases} 30t = 198.59^\circ \pm 360n \\ 30t = 281.41^\circ \pm 360n \end{cases}$$

$$\begin{cases} t = 6.62^\circ \pm 12n \\ t = 9.38^\circ \pm 12n \end{cases}$$

$$\begin{cases} t = 6.62^\circ \pm 12n \\ t = 9.38^\circ \pm 12n \end{cases}$$

SHIP ARRIVES AT 02:00
 + 5 hours
 07:00
 IT CAN'T GO IN BEACHING
 When $t = 6.62^\circ$ (06:37)
 IT BOAT ARRIVED
 IT MUST WAIT UNTIL
 $t = 9.38^\circ$
 $10.038 \times 60 = 21.9^\circ$
 i.e. 09:23

Question 74 (****+)



The figure above shows an irregular pentagon $ABCDE$. The lengths of AB , BC and AE are 14 cm, 8 cm and 6 cm respectively.

The point Q lies on AB so that AQ is 6 cm and QB is 8 cm. The point D is then constructed so that $QEDC$ is a rectangle.

Let the angle AQE be θ° and assume that θ° can vary.

- a) Given that P cm and R cm^2 are the perimeter and the area of the pentagon respectively, show that ...

i. ... $P = 28 + 12\cos\theta + 16\sin\theta$.

ii. ... $R = 146\sin 2\theta$.

- b) Hence show that when the pentagon has a maximum area

$$P = 14(2 + \sqrt{2}) \text{ cm}^2.$$

proof

(a)

$\therefore [EQ] = 2(AQ)\sin\theta$
 $= 2 \times 6 \times 6\sin\theta$
 $= 12\sin\theta$
 $\therefore [D] = [EQ] = 12\sin\theta$
 $\bullet [QC] = 2(QB)\sin\theta$
 $= 2 \times 8\sin\theta$
 $= 16\sin\theta$
 $\therefore [EO] = [QC] = 16\sin\theta$

$\therefore \text{PERIMETER} = 6 + 8 + 16 + 12\sin\theta + 16\sin\theta$
 $= 28 + 12\sin\theta + 16\sin\theta$ as required

(b)

$$\begin{aligned} \angle BAE &= \frac{1}{2}AQ([EQ]\sin\theta + \frac{1}{2}[QC]\sin 2\theta + [EQ]\cos\theta) \\ &= \frac{1}{2} \times 6 \times 12\sin\theta\sin\theta + \frac{1}{2} \times 8 \times 8 \sin\theta\cos\theta + 12\sin\theta \times 16\sin\theta \\ &= 36\sin^2\theta + 32\sin\theta\cos\theta + 192\sin^2\theta \\ &= 228\sin^2\theta + 32\sin\theta\cos\theta = 14(2\sin^2\theta\cos\theta) + 32\sin^2\theta \\ &\approx 144\sin^2\theta + 32\sin^2\theta \\ &= 176\sin^2\theta \end{aligned}$$

as required

(c) MAX AREA = 146 (occurs when $\sin 2\theta = 1$)

$$\begin{aligned} 2\theta &= 90^\circ \pm 2n\pi \quad n \in \mathbb{Z}_{\geq 0} \\ \theta &= 45^\circ \pm n\pi \\ \therefore \theta &\leq \frac{\pi}{4} \end{aligned}$$

$\therefore \text{PERIMETER}_{\text{MAX}} = 28 + 12\cos\frac{\pi}{4} + 16\sin\frac{\pi}{4}$
 $= 28 + 6\sqrt{2} + 8\sqrt{2}$
 $= 28 + 14\sqrt{2}$
 $= 14(2 + \sqrt{2})$ as required

Question 75 (****+)

Let $t = \tan\left(\frac{x}{2}\right)$.

a) Show that ...

$$\text{i. } \dots \sin x = \frac{2t}{1+t^2},$$

$$\text{ii. } \dots \cos x = \frac{1-t^2}{1+t^2}.$$

b) Use these results to solve the trigonometric equation

$$5 \sin x - 5 \cos x = 1, \quad 0 \leq x < 2\pi.$$

$$x = 3.79^\circ, 0.927^\circ$$

$$\begin{aligned}
 \text{(a)} \quad \text{RHS} &= \frac{2t}{1+t^2} = \frac{2 \tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \frac{2 \sin \frac{x}{2}}{\cos^2 \frac{x}{2}} = \frac{2 \sin \frac{x}{2}}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} \\
 &= \frac{2 \sin \frac{x}{2}}{2 \cos^2 \frac{x}{2}} = \frac{\sin x}{\cos^2 \frac{x}{2}} = \sin x = \text{LHS} \\
 \text{(b)} \quad \text{LHS} &= \frac{1-t^2}{1+t^2} = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \frac{1-\frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}}{1+\frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}} = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}} = \cos 2\frac{x}{2} = \cos x = \text{RHS}
 \end{aligned}$$

(b) $\sin x - \cos x = 1$

$$\begin{aligned}
 \Rightarrow \left| \frac{\sin x}{\cos^2 \frac{x}{2}} - \frac{\cos x}{\cos^2 \frac{x}{2}} \right| &= 1 \\
 \Rightarrow \left| \frac{\sin x - \cos x}{\cos^2 \frac{x}{2}} \right| &= 1 \Rightarrow t = \pm 1 \\
 \Rightarrow \sin x - \cos x &= 1 \Rightarrow t = \pm 1 \\
 \Rightarrow \sin x - \sin x = 0 &= 0 \\
 \Rightarrow (2t-1)(t+1) &= 0 \\
 \Rightarrow t_1 = \frac{1}{2}, t_2 = -1 &\Rightarrow \begin{cases} \sin \frac{x}{2} = 0.4557 \\ \cos \frac{x}{2} = -1.7410 \\ \text{or} \\ \sin \frac{x}{2} = -0.4557 \\ \cos \frac{x}{2} = -1.7410 \end{cases} \\
 \Rightarrow x_1 = 2(1.57) &= 3.1416 \approx 3.79^\circ \\
 \Rightarrow x_2 = -2(1.57) &= -3.1416 \approx -0.927^\circ \\
 \therefore x_1 &= 3.79^\circ \\
 x_2 &= -0.927^\circ
 \end{cases}
 \end{aligned}$$

Question 76 (***)+

The functions f and g are defined by

$$f(x) \equiv 3\sin x, \quad x \in \mathbb{R}, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$g(x) \equiv 6 - 3x^2, \quad x \in \mathbb{R}.$$

- a) Find an expression for $f^{-1}g(x)$.
- b) Determine the domain of $f^{-1}g(x)$.

$f^{-1}g(x) = \arcsin(2 - x^2)$, $[-\sqrt{3} \leq x \leq -1 \text{ or } 1 \leq x \leq \sqrt{3}]$

a) $f(x) = 3\sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
 $g(x) = 6 - 3x^2, \quad x \in \mathbb{R}$

$\Rightarrow y = 3\sin x$
 $\Rightarrow \frac{y}{3} = \sin x$
 $\Rightarrow x = \arcsin \frac{y}{3}$
 $\therefore f^{-1}(x) = \arcsin \frac{x}{3}$

Now $f^{-1}(g(x)) = f^{-1}(6 - 3x^2)$
 $= \arcsin \left(\frac{6 - 3x^2}{3} \right)$
 $= \arcsin(2 - x^2)$

b) $f(x)$ has domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and range $[-3, 3]$
 $f^{-1}(x)$ has domain $[-3, 3]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$(x \geq 0) \rightarrow g(x) \geq 0$ $x \geq 0$
 $x \geq 0 \rightarrow f(x) \geq 0$

Hence
 $\Rightarrow -3 \leq 6 - 3x^2 \leq 3$
 $\Rightarrow -3 \leq 6 - 3x^2 \leq 3$
 $\Rightarrow -9 \leq -3x^2 \leq -3$
 $\Rightarrow 1 \leq x^2 \leq 3$

$\therefore -\sqrt{3} \leq x \leq \sqrt{3}$
 $x^2 \geq 1 \Rightarrow x \geq 1 \text{ or } x \leq -1$

using $\arcsin(2 - x^2)$
 $-1 \leq 2 - x^2 \leq 1$
 $-3 \leq -x^2 \leq -1$
 $1 \leq x^2 \leq 3 \text{ or}$

$x \in [-\sqrt{3}, -1] \cup [1, \sqrt{3}]$

Question 77 (**+)**

Find the solution of the equation

$$\arctan\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \arctan x.$$

$$\boxed{\text{S.C.}}, \quad x = \frac{\sqrt{3}}{3}$$

$$\begin{aligned}
 &\Rightarrow \arctan\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \arctan x \\
 &\Rightarrow 2\arctan\left(\frac{1-x}{1+x}\right) = \arctan x \\
 &\quad \boxed{\text{Let } \theta = \arctan\left(\frac{1-x}{1+x}\right) \Rightarrow \tan\theta = \frac{1-x}{1+x}} \\
 &\Rightarrow 2\theta = \arctan x \\
 &\Rightarrow \tan 2\theta = \tan(\arctan x) \\
 &\Rightarrow \frac{2\tan\theta}{1 - \tan^2\theta} = x \\
 &\Rightarrow \frac{2\left(\frac{1-x}{1+x}\right)}{1 - \left(\frac{1-x}{1+x}\right)^2} = x \\
 &\Rightarrow \frac{2(1-x)}{1 - \frac{(1-x)^2}{(1+x)^2}} = x \\
 &\quad \text{MULTIPLY TOP AND BOTTOM OF THE FRACTION OF THE LHS BY } (1+x)^2 \\
 &\Rightarrow \frac{2(1-x)(1+x)}{(1+x)^2 - (1-x)^2} = x \\
 &\Rightarrow \frac{2(1-x^2)}{(x^2+2x+1) - (1-2x+x^2)} = x \\
 &\Rightarrow \frac{2(1-x^2)}{4x} = x \\
 &\Rightarrow \frac{1-x^2}{2x} = x \\
 &\Rightarrow 1 - x^2 = 2x^2
 \end{aligned}$$

$$\begin{aligned}
 &\Rightarrow 1 = 3x^2 \\
 &\Rightarrow x^2 = \frac{1}{3} \\
 &\therefore x = \sqrt{\frac{1}{3}} \quad \text{OR} \quad x = -\sqrt{\frac{1}{3}} \\
 &\quad \text{LHS} > 0 \quad \text{RHS} < 0
 \end{aligned}$$

Question 78 (***)+

$$\frac{1}{2}\sin^4 x + \frac{1}{3}\cos^4 x = \frac{1}{5}$$

Show that the above trigonometric equation is equivalent to

$$\tan^2 x = \frac{2}{3}.$$

[] , proof

START MANIPULATING AS follows

$$\begin{aligned}
 &\Rightarrow \frac{1}{2} \sin^2 x + \frac{1}{2} \cos^2 x = \frac{1}{5} \\
 &\Rightarrow \frac{1}{2} (\sin^2 x)^2 + \frac{1}{2} (\cos^2 x)^2 = \frac{1}{5} \\
 &\Rightarrow \frac{1}{2} \left(\frac{1}{4} - \frac{1}{2} \cos 2x \right)^2 + \frac{1}{2} \left(\frac{1}{4} + \frac{1}{2} \cos 2x \right)^2 = \frac{1}{5} \\
 &\Rightarrow \frac{1}{2} \left(\frac{1}{16} - \frac{1}{4} \cos 2x + \frac{1}{4} \cos^2 2x \right) + \frac{1}{2} \left(\frac{1}{16} + \frac{1}{4} \cos 2x + \frac{1}{4} \cos^2 2x \right) = \frac{1}{5} \\
 &\Rightarrow \frac{1}{8} - \frac{1}{4} \cos 2x + \frac{1}{8} (\cos^2 2x + \frac{1}{2}) + \frac{1}{8} \cos 2x + \frac{1}{8} \cos^2 2x = \frac{1}{5} \\
 \text{MULTIPLY THROUGH BY 160, AND TRY AGAIN} \\
 &\Rightarrow (15 - 30 \cos 2x + 15 \cos^2 2x) = 24 \\
 &\Rightarrow 15 - 30 \cos 2x + 10 \cos^2 2x = 0 \\
 &\Rightarrow 25 \cos^2 2x - 10 \cos 2x + 1 = 0 \\
 &\Rightarrow (5 \cos 2x - 1)^2 = 0 \\
 &\Rightarrow \cos 2x = \frac{1}{5} \\
 &\Rightarrow 2 \cos^2 x - 1 = \frac{1}{5} \\
 &\Rightarrow 2 \cos^2 x = \frac{6}{5} \\
 &\Rightarrow \cos^2 x = \frac{3}{5} \\
 &\Rightarrow \sin^2 x = \frac{2}{5} \\
 &\Rightarrow \sin^2 x - 1 = \frac{5}{3} - 1 \\
 &\Rightarrow \frac{1}{\tan^2 x} = \frac{2}{3}
 \end{aligned}$$

Question 79 (****+)

A relationship is defined as

$$x = \sin^2 \theta, \quad 0 \leq \theta < \frac{\pi}{4}.$$

$$y = \tan 2\theta, \quad 0 \leq \theta < \frac{\pi}{4}.$$

Use trigonometric identities to show that

$$y^2 = \frac{4x(1-x)}{(1-2x)^2}.$$

proof

$$\begin{aligned} \text{Given: } x &= \sin^2 \theta \\ y &= \tan 2\theta \\ \Rightarrow y &= \frac{2\sin \theta \cos \theta}{1 - 2\sin^2 \theta} = \frac{2\sin \theta \cos \theta}{\sin^2 \theta + \cos^2 \theta - 2\sin^2 \theta} = \frac{2\sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta} = \frac{2\sin \theta \cos \theta}{\cos 2\theta} \\ \Rightarrow y &= \frac{2\sin \theta}{\cos 2\theta - 1} \\ \Rightarrow y^2 &= \frac{4\sin^2 \theta}{(\cos 2\theta - 1)^2} = \frac{4(\cos^2 \theta - 1)}{(\cos 2\theta - 1)^2} \\ \text{But } \sin^2 \theta &= 1 - \cos^2 \theta \text{ so } \cos^2 \theta = \frac{1 - \sin^2 \theta}{\cos^2 \theta} \\ \Rightarrow y^2 &= \frac{4 \left(\frac{1 - \sin^2 \theta}{\cos^2 \theta} - 1 \right)}{\left(\frac{1 - \sin^2 \theta}{\cos^2 \theta} - 1 \right)^2} = \frac{4 \left(\frac{-\sin^2 \theta}{\cos^2 \theta} \right)}{\left(\frac{-\sin^2 \theta}{\cos^2 \theta} \right)^2} \\ \Rightarrow y^2 &= \frac{4 \sin^2 \theta}{\sin^4 \theta} = \frac{4}{\sin^2 \theta} \end{aligned}$$

Question 80 (****+)

Find the solutions of the trigonometric equation

$$\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0,$$

for which $0 \leq \theta < 180^\circ$.

$$\theta = 22.5^\circ, 45^\circ, 67.5^\circ, 90^\circ, 112.5^\circ, 135^\circ, 157.5^\circ$$

$$\begin{aligned} \cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta &= 0 \\ \cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta &= 0 \\ 2\cos \theta (\cos 3\theta + \cos 5\theta) &= 0 \\ 2\cos \theta [2\cos 4\theta + \cos 2\theta] &= 0 \\ 2\cos \theta \times 2\cos 2\theta \cos 4\theta &= 0 \\ \cos \theta \cos 2\theta \cos 4\theta &= 0 \\ \cos \theta = 0 & \quad \cos 2\theta = 0 \quad \cos 4\theta = 0 \\ \theta = 90^\circ \pm 360n & \quad (2\theta = 90^\circ \pm 360n) \quad (4\theta = 90^\circ \pm 360n) \quad \text{where } n \in \mathbb{Z} \\ \theta = 22.5^\circ \pm 180n & \quad (2\theta = 22.5^\circ \pm 360n) \quad (4\theta = 22.5^\circ \pm 360n) \\ \theta = 45^\circ \pm 180n & \quad (2\theta = 45^\circ \pm 360n) \quad (4\theta = 45^\circ \pm 360n) \\ \theta = 135^\circ \pm 180n & \quad (2\theta = 135^\circ \pm 360n) \quad (4\theta = 135^\circ \pm 360n) \\ \text{Hence: } \theta = 90^\circ, 45^\circ, 22.5^\circ, 112.5^\circ, 67.5^\circ, 157.5^\circ & \end{aligned}$$

Question 81 (***)+

$$\sin 3\theta \equiv 3 \sin \theta - 4 \sin^3 \theta.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence or otherwise solve the trigonometric equation

$$\arcsin x = 3 \arcsin \left(\frac{1}{3} \right).$$

$$\boxed{\quad}, \quad x = \frac{23}{27}$$

(a) $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

$$\begin{aligned} &= 3 \sin \theta \cos^2 \theta + \cos \theta \sin^3 \theta \\ &= 3 \sin \theta (\cos^2 \theta + \sin^2 \theta) + (-2 \sin^3 \theta) \cos \theta \\ &= 3 \sin \theta + \sin \theta - 2 \sin^3 \theta \\ &= 2 \sin \theta (-\sin^2 \theta + \sin \theta - 2 \sin^2 \theta) \\ &= 2 \sin \theta (-3 \sin^2 \theta + 3 \sin \theta) \\ &= 3 \sin \theta - 4 \sin^3 \theta \\ &= 3 \sin \theta \end{aligned}$$

(b) $\arcsin x = 3 \arcsin \frac{1}{3}$

$$\begin{aligned} &\Rightarrow \sin(\arcsin x) = \sin(3 \arcsin \frac{1}{3}) \\ &\Rightarrow x = \sin(3 \arcsin \frac{1}{3}) \\ &\bullet 3 \arcsin \frac{1}{3} = \arcsin \frac{1}{3} \\ &\Rightarrow x = \sin \frac{1}{3} \\ &\Rightarrow x = 3 \sin \frac{1}{3} - 4 \sin^3 \frac{1}{3} \\ &\Rightarrow x = \frac{23}{27} \end{aligned}$$

Question 82 (***)+

Solve the simultaneous equations

$$\arctan x + \arctan y = \arctan 8$$

$$x + y = 2.$$

$$\boxed{\quad}, \quad x = \frac{1}{2}, y = \frac{3}{2}, \text{ in either order}$$

$\arctan x + \arctan y = \arctan 8$

$$\tan(\arctan x + \arctan y) = \tan(\arctan 8)$$

$$\begin{aligned} \frac{x+y}{1-xy} &= 8 & 1-xy &= 2 \\ \frac{2}{1-xy} &= 8 & \downarrow & \downarrow \\ 1-xy &= \frac{1}{8} & \downarrow & \downarrow \\ 1-xy &= \frac{1}{8} & \downarrow & \downarrow \\ 2 &= \frac{1}{8} & \downarrow & \downarrow \\ 16 &= 1 & \downarrow & \downarrow \\ 15 &= 0 & \downarrow & \downarrow \\ 15 &= 0 & \downarrow & \downarrow \\ 0 &= x^2 - 2x + \frac{3}{4} & \downarrow & \downarrow \\ 4x^2 - 8x + 3 &= 0 & \downarrow & \downarrow \\ (2x-3)(2x-1) &= 0 & \downarrow & \downarrow \\ 2x = \frac{3}{2} & \quad 2x = \frac{1}{2} & \downarrow & \downarrow \\ x = \frac{3}{4} & \quad x = \frac{1}{4} & \downarrow & \downarrow \\ \therefore x = \frac{3}{4}, y = \frac{1}{4} & \quad \text{either order} & \downarrow & \downarrow \end{aligned}$$

Question 83 (**+)**

Solve the trigonometric equation

$$4 \tan 2\psi + 3 \cot \psi \sec^2 \psi = 0, \quad 0 \leq \psi < 2\pi,$$

giving the answers in terms of π .

$$\psi = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

$$\begin{aligned}
 4 \tan 2\psi + 3 \cot \psi \sec^2 \psi &= 0 \\
 \Rightarrow 4 \frac{\tan 2\psi}{1 - \tan^2 \psi} + \frac{3}{\tan \psi} \times (1 + \tan^2 \psi) \cot \psi &= 0 \\
 \Rightarrow \frac{8 \tan^2 \psi}{1 - \tan^2 \psi} + \frac{3}{\tan \psi} + 3 \tan \psi \cot \psi &= 0 \\
 \Rightarrow \frac{8T^2}{1-T^2} + \frac{3}{T} + 3T &= 0 \\
 \Rightarrow 8T^2 + 3(1-T^2) + 3T^2(1-T^2) &= 0 \\
 \Rightarrow 8T^2 + 3 - 3T^2 + 3T^2 - 3T^4 &= 0 \\
 \Rightarrow 0 = 3T^4 - 8T^2 + 3 \\
 \Rightarrow (3T^2 + 1)(T^2 - 3) &= 0 \\
 \Rightarrow T^2 = -\frac{3}{3} &\text{ or } T^2 = 1 \\
 \Rightarrow T^2 = 1 &\text{ (since } T^2 \geq 0\text{)}
 \end{aligned}
 \quad \begin{aligned}
 \Rightarrow T = \pm \sqrt{1} &= \pm 1 \\
 \tan \psi = \pm 1 & \\
 \arctan(\sqrt{3}) &= \frac{\pi}{6} \\
 \arctan(-\sqrt{3}) &= -\frac{\pi}{6} \\
 \psi = \frac{\pi}{6} &\pm \frac{n\pi}{2} \\
 \psi = \frac{\pi}{6} &\pm \frac{\pi}{2} \\
 n = 0, 1, 2, 3, \dots &
 \end{aligned}$$

Question 84 (**+)**

Show clearly that

$$2 \arctan\left(\frac{3}{2}\right) + \arctan\left(\frac{12}{5}\right) = \pi.$$

V, proof

$$\begin{aligned}
 \text{Let } \theta = \arctan \frac{3}{2} &\quad \text{And } \phi = \arctan \frac{12}{5} \\
 \tan \theta = \frac{3}{2} &\quad \tan \phi = \frac{12}{5} \\
 \text{Diagram: } \begin{array}{c} \text{Right-angled triangle with hypotenuse } \sqrt{13}, \text{ opposite side } 3, \text{ adjacent side } 2. \\ \sin \theta = \frac{3}{\sqrt{13}}, \cos \theta = \frac{2}{\sqrt{13}} \end{array} & \text{Diagram: } \begin{array}{c} \text{Right-angled triangle with hypotenuse } 13, \text{ opposite side } 12, \text{ adjacent side } 5. \\ \sin \phi = \frac{12}{13}, \cos \phi = \frac{5}{13} \end{array} \\
 \theta = 2\theta + \phi & \\
 \Rightarrow \cos \theta = \cos(2\theta + \phi) & \\
 \Rightarrow \cos \theta = (\cos 2\theta \cos \phi) - (\sin 2\theta \sin \phi) & \\
 \Rightarrow \cos \theta = (2\cos^2 \theta - 1)(\frac{5}{13}) - (2\sin \theta \cos \theta)(\frac{12}{13}) & \\
 \Rightarrow \cos \theta = (2 \cdot \frac{4}{13} - 1)(\frac{5}{13}) - (2 \cdot \frac{3}{\sqrt{13}} \cdot \frac{2}{\sqrt{13}})(\frac{12}{13}) & \\
 \Rightarrow \cos \theta = -\frac{1}{13} \times \frac{13}{13} - \frac{12}{13} \times \frac{12}{13} & \\
 \Rightarrow \cos \theta = -1 & \\
 \Rightarrow \cos \theta = -1 & \\
 \Rightarrow \theta = \dots, -\pi, \pi, 3\pi, \dots & \\
 \text{But } 0 < \theta < \pi & \\
 0 < 2\theta + \phi < \frac{3\pi}{2} & \\
 0 < \phi < \frac{\pi}{2} & \\
 \therefore 2\theta + \phi = \pi & \\
 2 \arctan \frac{3}{2} + \arctan \frac{12}{5} = \pi & \\
 \therefore 2\theta + \phi = \pi &
 \end{aligned}$$

$$\begin{aligned}
 \text{ALTERNATIVE BY COMPLEX NUMBERS} \\
 \text{CONSIDER: } (2+3i)^2(z+1i) &= (4+12i-i)(z+1i) = (-i+12i)(z+1i) \\
 &= -25-60i+40i-144i = -169 \\
 \text{THUS} \\
 \arg[(2+3i)^2(z+1i)] &= \arg(-169) \\
 \arg(2+3i)^2 + \arg(z+1i) &= \pi \\
 2\arg(2+3i) + \arg(z+1i) &= \pi \\
 2\arctan \frac{3}{2} + \arctan \frac{12}{5} &= \pi \\
 \therefore \text{PROVED!} &
 \end{aligned}$$

Question 85 (***)+

Find, in terms of π , the solutions of the trigonometric equation

$$\cos 2x + 3\cos x - 2\cos^2 x - \sqrt[3]{\cos x} = 1, \quad 0 \leq x < 4\pi.$$

$$[] , \quad [] , \quad x = 0, 2\pi$$

• START BY REARRANGING THE ARGUMENT OF $2x$

$$\Rightarrow (\cos 2x + 3\cos x - 2\cos^2 x - \sqrt[3]{\cos x})^{\frac{1}{3}} = 1$$

$$\Rightarrow (\cancel{2\cos^2 x})^{\frac{1}{3}} + 3\cos x - 2\cos^2 x - (\cos x)^{\frac{1}{3}} = 1$$

$$\Rightarrow 3\cos x - (\cos x)^{\frac{1}{3}} - 2 = 0$$

• LET $y = (\cos x)^{\frac{1}{3}}$

$$\Rightarrow 3y^3 - y - 2 = 0$$

• BY INSPECTION $y=1$ IS A ROOT

$$\Rightarrow 3y^2(y-1) + 3y(y-1) + 2(y-1) = 0$$

$$\Rightarrow (3y^2 + 3y + 2)(y-1) = 0$$

$$\uparrow$$

$$b^2 - 4ac = 3^2 - 4 \cdot 3 \cdot 2 < 0$$

• ONLY SOLUTION $y = (\cos x)^{\frac{1}{3}} = 1$

$$\Rightarrow \cos x = 1$$

$$\Rightarrow \begin{cases} x = 0 \pm 2\pi \\ x = 2\pi + 2\pi \end{cases} \quad n=91/93$$

$$\therefore \begin{cases} x_1 = 0 \\ x_2 = 2\pi \end{cases}$$

Question 86 (***)+

Solve the trigonometric equation

$$\cos x + \cos 5x = 0, \quad 0 \leq x < \pi,$$

giving the answers in terms of π .

$$x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$$

$$\begin{aligned} \cos 5x + \cos 5x &= 0 \\ \cos 5x &= -\cos x \\ \cos 5x &= \cos(\pi - x) \\ \Rightarrow 5x &= \pi - x \pm 2n\pi \\ 5x &= \pi - x \pm 2n\pi \\ \Rightarrow (5x &= \pi + 2n\pi) \quad (5x = -\pi + 2n\pi) \\ \therefore x &= \frac{\pi}{5} + \frac{2n\pi}{5} \quad x = \frac{-\pi}{5} + \frac{2n\pi}{5} \end{aligned}$$

AMMULATU USE $\cos A + \cos B = 2\cos \frac{A+B}{2} \cos \frac{A-B}{2}$

Question 87 (***)+

Use the substitution $t = \tan \frac{1}{2}x$ to show that if

$$6 \tan \frac{1}{2}x = 1 + 5 \sin x,$$

then the three possible values of $\tan \frac{1}{2}x$ are

$$1, -\frac{1}{2} \text{ or } -\frac{1}{3}.$$

proof

$6 \tan \frac{1}{2}x = 1 + 5 \sin x$

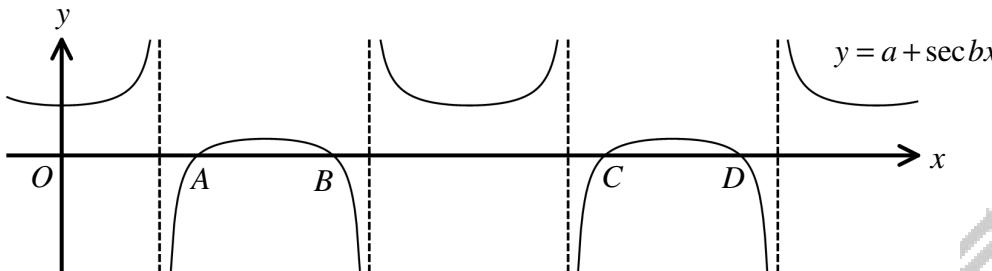
BY INSPECTION $t=1$ IS A SOLUTION, SO $t-1$ IS A FACTOR.

BY LONG DIVISION OR ALGEBRAIC MANIPULATIONS

$$\begin{aligned} & 6t(t-1) + 5t(t+1) + (t-1) \\ & (t-1)(6t+5t+1) = 0 \\ & (t-1)(11t+1) = 0 \\ & t_1 = 1 \quad \text{or} \quad t_2 = -\frac{1}{11} \\ & \therefore \tan \frac{1}{2}x = 1 \quad \text{or} \quad \tan \frac{1}{2}x = -\frac{1}{11} \end{aligned}$$

As required

Question 88 (***)+



The figure above shows part of the graph of

$$y = a + \sec bx,$$

where a and b are positive constants.

The points A , B , C and D are the x intercepts of the graph, with respective coordinates $\left(\frac{\pi}{3}, 0\right)$, $\left(\frac{2\pi}{3}, 0\right)$, $\left(\frac{4\pi}{3}, 0\right)$ and $\left(\frac{5\pi}{3}, 0\right)$.

Determine the value of a and the value of b .

$$a = 2, b = 2$$

$y = a + \sec bx$

- NO REAS AT A & B (OR C & D). THEY ARE 'TO APART'
- THE SECANT FUNCTION HAS A PERIOD OF 2π (LKE COSINE), SO IT HAS BEEN STRETCHED IN x BY SCALE FACTOR OF $\frac{1}{2}$
- $b = 2$

$$\begin{aligned} y &= a + \sec 2x, \text{ using } A(\frac{\pi}{3}) = 0 \\ &\Rightarrow 0 = a + \sec(2 \cdot \frac{\pi}{3}) \\ &\Rightarrow 0 = a + \sec(\frac{4\pi}{3}) \\ &\Rightarrow 0 = a + \frac{1}{\cos(\frac{4\pi}{3})} \\ &\Rightarrow 0 = a + \frac{1}{-\frac{1}{2}} \end{aligned} \quad \left. \begin{aligned} \Rightarrow a &= a - 2, \\ \Rightarrow a &= 2 \end{aligned} \right\}$$

Question 89 (**+)**

The point A lies on the y axis above the origin O and the point B lies on the y axis below the origin O .

The point $C(12,0)$ is at a distance of 20 units from A and at a distance of 13 units from B .

By considering the tangent ratios of $\angle OCA$ and $\angle OCB$, show that the tangent of the angle ACB is exactly $\frac{63}{16}$.

 , proof

• BY PYTHAGORAS
 $|CA| = \sqrt{20^2 - 12^2} = 16$
 $|CB| = \sqrt{13^2 - 12^2} = 5$

• $\tan \theta = \frac{12}{16} = \frac{3}{4}$ from the right angled triangle

$\tan(\angle ACB) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta} = \frac{\frac{3}{4} + \frac{3}{4}}{1 - \frac{3}{4} \times \frac{3}{4}}$

= multiplying top bottom by 16 ... or use calculator
 $= \frac{48 + 16}{32 - 20} = \frac{64}{12} = \frac{16}{3}$ // ≈ 5.3333

Question 90 (**+)**

By using the substitution $t = \tan\left(\frac{1}{2}x\right)$ solve the trigonometric equation

$$3\cos x + 4\sin x = 3 - \tan\left(\frac{1}{2}x\right), \quad 0^\circ \leq x < 360^\circ.$$

$x = 0^\circ, 143.1^\circ$

• $3\cos x + 4\sin x = 3 - \tan\left(\frac{1}{2}x\right)$

$\Rightarrow 3\left(\frac{1-t^2}{1+t^2}\right) + t\left(\frac{2t}{1+t^2}\right) = 3 - t$

$\Rightarrow 3(1-t^2) + 4t = (3-t)(1+t^2)$

$\Rightarrow 3t^2 - 2t^2 + 4t = 3 + 3t^2 - t - t^3$

$\Rightarrow t^3 - 6t^2 + 9t = 0$

$\Rightarrow t(t^2 - 6t + 9) = 0$

$\Rightarrow t(t-3)^2 = 0$

$\Rightarrow t = 0$ or $t = 3$

$\Rightarrow \tan \frac{x}{2} = 0$ or $\tan \frac{x}{2} = 3$

• $\arctan 0 = 0$
 $\arctan 3 = 71.57^\circ$

$\frac{x}{2} = 0^\circ \pm 180^\circ$ $x = 0^\circ, 360^\circ$
 $\frac{x}{2} = 71.57^\circ \pm 180^\circ$
 $x = 0^\circ \pm 360^\circ$
 $x = 143.1^\circ \pm 360^\circ$

$x_1 = 0^\circ$
 $x_2 = 143.1^\circ$

Question 91 (***)+

$$\sin x - \cos x = \sin 2x + \cos 2x - 1, \quad \cos x \neq 0,$$

Show that the above trigonometric equation is equivalent to

$$(\tan x - 1)(\sec x + 2 \tan x) = 0.$$

proof

$$\begin{aligned} \sin x - \cos x &= \sin 2x + \cos 2x - 1 \\ \sin x - \cos x &= 2\sin x \cos x + 1 - 2\cos^2 x - 1 \\ \sin x - \cos x &= 2\sin x \cos x - 2\cos^2 x \\ \sin x - \cos x + 2\cos^2 x - 2\sin x \cos x &= 0 \\ (\sin x - \cos x) + 2\cos x (\sin x - \cos x) &= 0 \\ (\sin x - \cos x)(1 + 2\cos x) &= 0 \\ \left(\frac{\sin x - \cos x}{\cos x} \right) \left(\frac{1 + 2\cos x}{\cos x} \right) &= 0 \\ (\tan x - 1)(\sec x + 2 \tan x) &= 0 \end{aligned}$$

After dividing

Question 92 (***)+

Solve the trigonometric equation

$$\frac{d}{dx} \left(\sqrt{1 - \cos 2x} \right) = 1, \quad 0 \leq x < 2\pi.$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\begin{aligned} \frac{d}{dx} \left[\sqrt{1 - \cos 2x} \right] &= 1 \\ \frac{d}{dx} \left[\sqrt{1 - (1 - 2\sin^2 x)} \right] &= 1 \\ \frac{d}{dx} \left[\sqrt{2\sin^2 x} \right] &= 1 \\ \frac{d}{dx} \left(\sqrt{2}\sin x \right) &= 1 \\ \sqrt{2} \frac{d}{dx} (\sin x) &= 1 \end{aligned}$$

$\Rightarrow \sqrt{2} \cos x = 1$
 $\Rightarrow \cos x = \frac{1}{\sqrt{2}}$
 $\bullet \cos x \left(\frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}}$
 $x = \frac{\pi}{4} \pm 2n\pi$
 $x = \frac{\pi}{4} \pm 2n\pi \quad n \in \mathbb{Z}$
 $x = \frac{\pi}{4}, \frac{7\pi}{4}$

Question 93 (***)+

Given that

$$2\sec^2\left(\frac{x}{2}\right) - \frac{1-\cos x}{\sin x} = 5,$$

find the finite value of $\tan x$.

$$\boxed{\tan x = -\frac{12}{5}}$$

Working for Question 93:

$$\begin{aligned} 2\sec^2\frac{x}{2} - \frac{1-\cos x}{\sin x} &= 5 \\ 2\sec^2\frac{x}{2} - \frac{1-\cos 2\frac{x}{2}}{\sin 2\frac{x}{2}} &= 5 \quad (\text{using } \cos 2\theta = 1 - 2\cos^2\theta) \\ 2\sec^2\frac{x}{2} - \frac{1-(1-2\cos^2\frac{x}{2})}{2\sin\frac{x}{2}\cos\frac{x}{2}} &= 5 \\ 2\sec^2\frac{x}{2} - \frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} &= 5 \\ 2\sec^2\frac{x}{2} - \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} &= 5 \\ 2(1+\tan^2\frac{x}{2}) - \tan\frac{x}{2} &= 5 \\ 2\tan^2\frac{x}{2} - \tan\frac{x}{2} - 3 &= 0 \\ (2\tan\frac{x}{2} + 3)(\tan\frac{x}{2} - 1) &= 0 \end{aligned}$$

Solutions for $\tan\frac{x}{2}$:

$$\begin{aligned} \tan\frac{x}{2} &= -\frac{3}{2} \\ \tan\frac{x}{2} &= 1 \end{aligned}$$

Consequently:

$$\begin{aligned} \tan x &= 2\tan\frac{x}{2} \\ \tan x &= 2\left(-\frac{3}{2}\right) = -3 \quad (\text{if } \tan\frac{x}{2} = -\frac{3}{2}) \\ \tan x &= 2 \quad (\text{if } \tan\frac{x}{2} = 1) \\ \tan x &= \frac{2}{1-\tan^2\frac{x}{2}} = \frac{2}{1-\frac{1}{2}} = 4 \quad (\text{if } \tan\frac{x}{2} = 1) \\ \tan x &= -\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = -\frac{2(-\frac{3}{2})}{1+(-\frac{3}{2})^2} = \frac{3}{5} \quad (\text{if } \tan\frac{x}{2} = -\frac{3}{2}) \\ \tan x &= \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2}{1+\frac{1}{2}} = \frac{4}{3} \quad (\text{if } \tan\frac{x}{2} = 1) \\ \tan x &= -\frac{12}{5} \end{aligned}$$

Question 94 (***)+

$$2\cot 2x + \tan x + 7 = \operatorname{cosec}^2 x, \quad 0 \leq x < \pi.$$

Given that $x \neq \frac{\pi}{2}$, find the solutions of the above trigonometric equation, giving the answers in radians correct to two decimal places.

$$\boxed{x = 0.32^\circ, 2.68^\circ}$$

Working for Question 94:

$$\begin{aligned} 2\cot 2x + \tan x + 7 &= \operatorname{cosec}^2 x \\ \Rightarrow \cot 2x &= \frac{2\tan x}{1-\tan^2 x} \\ \Rightarrow 2\left(\frac{1-\tan^2 x}{2\tan x}\right) + \tan x + 7 &= \operatorname{cosec}^2 x \\ \Rightarrow \frac{1-\tan^2 x}{\tan x} + \tan x + 7 &= \operatorname{cosec}^2 x \quad (\text{canceling } 2) \\ \Rightarrow \frac{\cot^2 x - 1}{\cot x} + \frac{1}{\cot x} + 7 &= \operatorname{cosec}^2 x \\ \Rightarrow \frac{\cot^2 x - 1 + 1}{\cot x} + 7 &= (1 + \cot^2 x) \\ \Rightarrow \cot x + 7 &= 1 + \cot^2 x \\ \Rightarrow \cot x - \cot^2 x - 6 &= 0 \end{aligned}$$

Let $a = \cot x$

$$\Rightarrow (a+2)(a-3) = 0$$

$$\Rightarrow a_1 = -2 \quad a_2 = 3$$

$$\Rightarrow \cot x_1 = -\frac{1}{2} \quad \cot x_2 = 3$$

$$\cot x_1 = -0.5 \quad \cot x_2 = 3$$

$$x_1 = 0.32^\circ \quad x_2 = 2.68^\circ$$

Thus, $x_1 = \frac{\pi}{2}$ is also a solution.
But $x_1 = \frac{\pi}{2}$ is an infinite solution.

Question 95 (*****)

$$\frac{1+\cos x}{1-\cos x} = 3 + \sqrt{8} \operatorname{cosec}\left(\frac{x}{2}\right), \quad 0^\circ \leq x < 720^\circ.$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$x = 30^\circ, 330^\circ, 510^\circ, 570^\circ$$

$$\begin{aligned}
 \frac{1+\cos x}{1-\cos x} &= 3 + \sqrt{8} \operatorname{cosec}\left(\frac{x}{2}\right) \\
 \Rightarrow \frac{1+(2\cos^2\frac{x}{2}-1)}{1-(1-2\sin^2\frac{x}{2})} &= 3 + \sqrt{8} \operatorname{cosec}\frac{x}{2} \\
 \Rightarrow \frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}} &= 3 + \sqrt{8} \operatorname{cosec}\frac{x}{2} \\
 \Rightarrow \cot^2\frac{x}{2} &= 3 + \sqrt{8} \operatorname{cosec}\frac{x}{2} \\
 \Rightarrow \cot^2\frac{x}{2} - 1 &= 3 + \sqrt{8} \operatorname{cosec}\frac{x}{2} \\
 \Rightarrow \cot^2\frac{x}{2} - 1 - 4 &= 3 + \sqrt{8} \operatorname{cosec}\frac{x}{2} \\
 \Rightarrow \cot^2\frac{x}{2} - 5 &= 3 + \sqrt{8} \operatorname{cosec}\frac{x}{2} \\
 \Rightarrow [\cot\frac{x}{2} - \sqrt{2}]\left[\cot\frac{x}{2} + 2\right] &= 0 \\
 \Rightarrow (\cot\frac{x}{2} - \sqrt{2})^2 &= 2 - 4 = 0 \\
 \Rightarrow (\cot\frac{x}{2} - \sqrt{2})^2 &= 0
 \end{aligned}$$

(1)

$$\begin{cases}
 \cot\frac{x}{2} - \sqrt{2} = \pm \sqrt{2} \\
 \cot\frac{x}{2} = \sqrt{2} \pm \sqrt{2} \\
 \sin\frac{x}{2} = \frac{1}{\sqrt{2} \pm \sqrt{2}} \\
 \sin\frac{x}{2} = \frac{\sqrt{2} \mp \sqrt{2}}{2} \\
 \frac{x}{2} = 15^\circ \pm 30^\circ \\
 \frac{x}{2} = 165^\circ \pm 30^\circ \\
 \frac{x}{2} = -75^\circ \pm 30^\circ \\
 \frac{x}{2} = 225^\circ \pm 30^\circ \\
 x = 30^\circ, 210^\circ, 300^\circ, 330^\circ, 510^\circ, 570^\circ
 \end{cases}$$

(2)

$$\begin{cases}
 \cot\frac{x}{2} + 2 = 0 \\
 \cot\frac{x}{2} = -2 \\
 \sin\frac{x}{2} = \frac{1}{\sqrt{2} + 2} \\
 \sin\frac{x}{2} = \frac{-\sqrt{2} + \sqrt{2}}{2} \\
 \frac{x}{2} = 150^\circ \pm 720^\circ \\
 x = 300^\circ, 540^\circ, 510^\circ, 570^\circ
 \end{cases}$$

(3)

$$\begin{cases}
 x = 30^\circ, 210^\circ, 300^\circ, 330^\circ, 510^\circ, 570^\circ
 \end{cases}$$

Question 96 (**+)**

A triangle has vertices at the points with coordinates $A(3,1)$, $B(7,4)$ and $C(10,-4)$.

The acute angle θ is defined as the angle formed between AB and the straight line which is parallel to the y axis and passes through B .

Find the value of $\tan \theta$ and hence show that $\tan(\angle ABC) = \frac{41}{12}$.

$$\boxed{\quad}, \quad \tan \theta = \frac{4}{3}$$

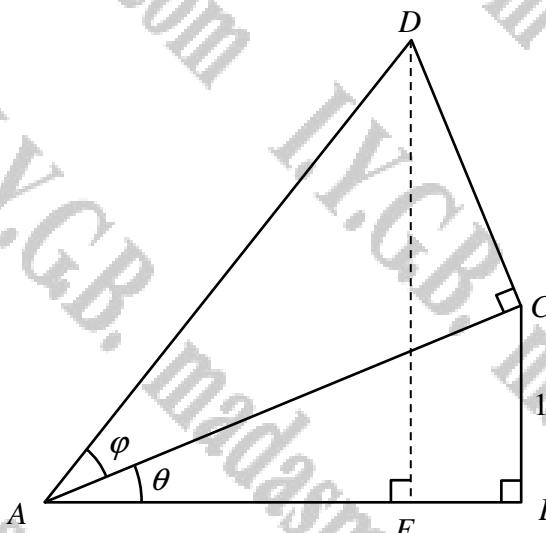
START WITH A DIAGRAM ; $A(3,1)$, $B(7,4)$, $C(10,-4)$

$\tan \theta = \frac{4}{3}$

Hence by using the compound angle identity for $\tan(B+\theta)$

$$\begin{aligned} \Rightarrow \tan(B+\theta) &= \frac{\tan B + \tan \theta}{1 - \tan B \tan \theta} \\ &= \frac{\frac{4}{3} + \frac{3}{5}}{1 - \frac{4}{3} \times \frac{3}{5}} \\ &= \frac{\frac{20}{15} + \frac{9}{15}}{\frac{1}{5}} \\ &= 2 \left(\frac{29}{15} \right) \\ &= 2 \left(\frac{29+9}{24} \right) \\ &= 2 \times \frac{41}{24} \\ &= \frac{41}{12} \end{aligned}$$

Question 97 (***)+



The figure above shows two right angles triangles ABC and ACD . The angles CAB and DAC are denoted by θ and φ , respectively.

The length of BC is 1.

The point E lies on AB so that the angle AED is 90° .

Show clearly that the length of AE is $\cot \theta - \tan \varphi$.

, proof

$\begin{aligned} \frac{ BC }{ AC } &= \sin \theta \\ \frac{1}{ AC } &= \sin \theta \\ AC &= \frac{1}{\sin \theta} \end{aligned}$
$\begin{aligned} \text{LOOKING AT THE RIGHT ANGLED TRIANGLE } ACD \\ \frac{ AC }{ AD } &= \cos \varphi \\ AD &= \frac{ AC }{\cos \varphi} \\ AD &= \frac{1}{\sin \theta \cos \varphi} \end{aligned}$
$\begin{aligned} \text{FINALLY LOOKING AT THE RIGHT ANGLED TRIANGLE } ADE \\ \frac{ AE }{ AD } &= \cos(\theta + \varphi) \\ AE &= AD \cos(\theta + \varphi) \\ AE &= \frac{\cos(\theta + \varphi)}{\sin \theta \cos \varphi} \\ AE &= \frac{\cos \theta \cos \varphi - \sin \theta \sin \varphi}{\sin \theta \cos \varphi} \\ AE &= \frac{\cos \theta \cos \varphi}{\sin \theta \cos \varphi} - \frac{\sin \theta \sin \varphi}{\sin \theta \cos \varphi} \\ AE &= \cot \theta - \tan \varphi \end{aligned}$

Question 98 (***)+

It is given that

$$\sin \varphi = k \sin \theta, \quad k \neq 0, \quad k \neq \pm 1.$$

Show, by a detailed method, that

$$1 + \left(\frac{d\varphi}{d\theta} \right)^2 = (k^2 - 1) \sec^2 \varphi$$

, proof

Simplifying k ≠ 0, k ≠ ±1

Differentiate w.r.t. θ

$$\begin{aligned}
 \Rightarrow \cos \varphi \frac{d\varphi}{d\theta} &= k \cos \theta \\
 \Rightarrow \cos^2 \varphi \left(\frac{d\varphi}{d\theta} \right)^2 &= k^2 \cos^2 \theta \\
 \Rightarrow \cos^2 \varphi \left(\frac{d\varphi}{d\theta} \right)^2 &= k^2 (1 - \sin^2 \theta) \\
 \Rightarrow \cos^2 \varphi \left(\frac{d\varphi}{d\theta} \right)^2 &= k^2 - k^2 \sin^2 \theta \\
 \Rightarrow \cos^2 \varphi \left(\frac{d\varphi}{d\theta} \right)^2 &= k^2 - \sin^2 \varphi \\
 \Rightarrow \left(\frac{d\varphi}{d\theta} \right)^2 &= \frac{k^2 - \sin^2 \varphi}{\cos^2 \varphi} \\
 \Rightarrow \left(\frac{d\varphi}{d\theta} \right)^2 &= \frac{k^2 \cos^2 \varphi}{\cos^2 \varphi} - \frac{\sin^2 \varphi}{\cos^2 \varphi} \\
 \Rightarrow \left(\frac{d\varphi}{d\theta} \right)^2 &= k^2 \sec^2 \varphi - \tan^2 \varphi \\
 \Rightarrow \left(\frac{d\varphi}{d\theta} \right)^2 &= k^2 \sec^2 \varphi - \sec^2 \varphi + 1 \\
 \Rightarrow 1 + \left(\frac{d\varphi}{d\theta} \right)^2 &= (k^2 - 1) \sec^2 \varphi
 \end{aligned}$$

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ENRICHMENT QUESTIONS

Question 1 (*****)

Use trigonometric algebra to find the solution of the following simultaneous equations, in the intervals $0 \leq x < 2\pi$, $0 \leq y < 2\pi$.

$$4\cos y = 3 - 2\sin x \quad \text{and} \quad 4y - 2x = \pi.$$

$$\boxed{\quad}, \quad x = \frac{1}{6}\pi, \quad y = \frac{1}{3}\pi$$

• START BY REARRANGING THE SECOND EQUATION FOR x

$$\begin{aligned} \Rightarrow 4y - 2x &= \pi \\ \Rightarrow 4y &= 2x + \pi \\ \Rightarrow 2y &= x + \frac{\pi}{2} \end{aligned}$$

• SUBSTITUTE INTO THE OTHER

$$\begin{aligned} \Rightarrow 4\cos y &= 3 - 2\sin\left(x + \frac{\pi}{2}\right) \\ \Rightarrow 4\cos y &= 3 - 2\sin\left(\frac{\pi}{2} - 2y\right) \\ \Rightarrow 4\cos y &= 3 + 2\sin(2y) \\ \Rightarrow 4\cos y &= 3 + 2[2\cos^2 y - 1] \\ \Rightarrow 4\cos y &= 3 + 4\cos^2 y - 2 \\ \Rightarrow 0 &= 4\cos^2 y - 4\cos y + 1 \\ \Rightarrow (2\cos y - 1)^2 &= 0 \\ \Rightarrow \cos y &= \frac{1}{2} \end{aligned}$$

• Hence we have

$$\begin{aligned} y &= \dots, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}, \dots \\ x &= \dots, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{4\pi}{6}, \frac{2\pi}{3}, \dots \end{aligned}$$

∴ ONLY SOLUTION IS $(\frac{\pi}{6}, \frac{\pi}{3})$ //

Question 2 (*****)

$$C = \cos\left(\frac{1}{9}\pi\right) \cos\left(\frac{2}{9}\pi\right) \cos\left(\frac{4}{9}\pi\right)$$

Use the **sine double angle identity** for sine to show that $C = \frac{1}{8}$.

[] proof

USING THE DOUBLE ANGLE IDENTITY

$$\sin 2\theta = 2 \sin \theta \cos \theta$$
$$\cos \theta = \frac{\sin(90^\circ - \theta)}{2 \sin \theta \cos \theta}$$

APPLIED TO OUR PRODUCT

$$C = \cos\left(\frac{1}{9}\pi\right) \cos\left(\frac{2}{9}\pi\right) \cos\left(\frac{4}{9}\pi\right) = \frac{\sin\left(\frac{2}{9}\pi\right)}{2 \sin\left(\frac{1}{9}\pi\right)} \times \frac{\sin\left(\frac{4}{9}\pi\right)}{2 \sin\left(\frac{2}{9}\pi\right)} \times \frac{\sin\left(\frac{8}{9}\pi\right)}{2 \sin\left(\frac{4}{9}\pi\right)}$$
$$= \frac{\sin\left(\frac{8}{9}\pi\right)}{8 \sin\left(\frac{1}{9}\pi\right)}$$

NOW $\sin\left(\pi - \theta\right) = \sin\theta$

$$= \frac{\sin\left(\frac{8}{9}\pi\right)}{8 \sin\left(\frac{1}{9}\pi\right)} = \frac{\sin\left(\frac{8}{9}\pi\right)}{8 \sin\left(\frac{1}{9}\pi\right)} = \frac{1}{8}$$

Question 3 (***)**

The acute angles x and y , satisfy the following relationships.

$$2 \tan x = 1 \quad \text{and} \quad \sin(x+y) = \frac{7}{\sqrt{50}}.$$

Determine the possible values of $\tan y$.

$$\boxed{}, \quad \boxed{}, \quad \tan y = \begin{cases} 3 \\ \frac{19}{3} \end{cases}$$

• STARTING WITH x

$$\Rightarrow 2 \tan x = 1$$

$$\Rightarrow \tan x = \frac{1}{2}$$

$$\Rightarrow \text{Acute } x \text{ is } 45^\circ$$

$$\boxed{\sin x = \frac{1}{\sqrt{5}}} \quad \text{or} \quad \boxed{\cos x = \frac{2}{\sqrt{5}}}$$

• NOW USING THE COMPOUND ANGLE IDENTITY FOR $\sin(x+y)$

$$\Rightarrow \sin(x+y) = \frac{7}{\sqrt{50}}$$

$$\Rightarrow \sin x \cos y + \cos x \sin y = \frac{7}{\sqrt{50}}$$

$$\Rightarrow \frac{1}{\sqrt{5}} \cos y + \frac{2}{\sqrt{5}} \sin y = \frac{7}{\sqrt{50}}$$

$$\Rightarrow \sqrt{10} \cos y + 2\sqrt{10} \sin y = 7$$

$$\Rightarrow \sqrt{10} \cos y + 2\sqrt{10} (\pm\sqrt{1-\cos^2 y}) = 7$$

$$\Rightarrow 2\sqrt{10} \sqrt{1-\cos^2 y} = 7 - \sqrt{10} \cos y$$

$$\Rightarrow 4 \times 10 \times (1-\cos^2 y) = (7 - \sqrt{10} \cos y)^2$$

$$\Rightarrow 40 - 40 \cos^2 y = 49 - 14\sqrt{10} \cos y + 10 \cos^2 y$$

$$\Rightarrow 0 = 50 \cos^2 y - 14\sqrt{10} \cos y + 9$$

$$\Rightarrow \cos^2 y - \frac{14\sqrt{10}}{50} \cos y + \frac{9}{50} = 0$$

$$\Rightarrow [\cos y - \frac{7}{\sqrt{50}}\sqrt{10}]^2 = (\frac{7}{\sqrt{50}}\sqrt{10})^2 - \frac{9}{50}$$

• CHECK THE SOLUTIONS AGAINST THE EQUATION BEFORE SOLVING

$$\sqrt{10} \cos y + 2\sqrt{10} \sqrt{1-\cos^2 y} = 7$$

If $\cos y = \frac{7}{\sqrt{50}}\sqrt{10}$

$$\Rightarrow \sqrt{10} \times \frac{7}{\sqrt{50}}\sqrt{10} + 2\sqrt{10} \sqrt{1 - \frac{49}{50} \times 10} = 7$$

$$= \frac{49}{5} + 2\sqrt{10} \sqrt{\frac{1}{50} \times 10}$$

$$= \frac{49}{5} + 2\sqrt{10} \sqrt{\frac{10}{50}}$$

$$= \frac{49}{5} + 2\sqrt{10} \times \frac{\sqrt{10}}{\sqrt{50}}$$

$$= \frac{49}{5} + \frac{20}{5}$$

$$= \frac{69}{5}$$

$$= 13.8$$

If $\cos y = \frac{1}{\sqrt{50}}\sqrt{10}$

$$\Rightarrow \sqrt{10} \times \frac{1}{\sqrt{50}}\sqrt{10} + 2\sqrt{10} \sqrt{1 - \frac{1}{50} \times 10} = 7$$

$$= 1 + 2\sqrt{10} \sqrt{\frac{49}{50} \times 10}$$

$$= 1 + 2\sqrt{10} \times \frac{7}{\sqrt{50}}\sqrt{10}$$

$$= 1 + 14$$

$$= 15$$

• BOTH SOLUTIONS ARE FINE

• FINALLY WE CAN OBTAIN THE TWO POSSIBLE VALUES OF $\tan y$, NOTING FURTHER THAT y IS ACUTE

$$\bullet \sqrt{10}^2 - (\frac{1}{\sqrt{50}}\sqrt{10})^2 = \sqrt{2500 - 10}$$

$$= \sqrt{1490}$$

$$= \sqrt{149 \times 10}$$

$$= \sqrt{149}\sqrt{10}$$

$$\bullet \sqrt{10}^2 - (\frac{7}{\sqrt{50}}\sqrt{10})^2 = \sqrt{100 - 49}$$

$$= \sqrt{51}$$

$$= \sqrt{13 \times 3}$$

$$= \sqrt{13}\sqrt{3}$$

$$\therefore \tan y = \sqrt{\frac{13}{10}} = \frac{\sqrt{13}}{\sqrt{10}} = \frac{\sqrt{130}}{10} = \frac{3}{2}$$

Question 4 (***)**

Two circles, C_1 and C_2 , have respective radii of 4 units and 1 unit and are touching each other externally at the point A .

The coordinates axes are tangents to C_1 , whose centre P lies in the first quadrant.

The x axis is a tangent to C_2 , whose centre Q also lies in the first quadrant.

The straight line l_1 is parallel to the x axis and passes through P .

The straight line l_2 has negative gradient and is a common tangent to C_1 and C_2 , touching C_1 at the point C .

The acute angle formed by PC and l_1 is denoted by ϕ .

$$\text{Show that } \tan \phi = \frac{7}{24}.$$

proof

STARTING WITH A DETAILED DIAGRAM

LOOKING AT THE TRIANGLE NPC : WE OBSERVE THAT IT IS A "3:4:5" TRIANGLE SO $Q(8,0)$
 $\therefore \tan(\angle QPC) = \tan(\theta) = \frac{3}{4}$

LOOKING AT ANGLES:

- LET ϕ BE THE REQUIRED ANGLE, θ BE SUCH THAT $\tan\theta = \frac{3}{4}$
- $\angle NPQ = \angle PQC$ (ALTERNATE ANGLES)
- $\angle BPC = \angle CPO = \theta + \phi$ (ANGLE IN A SEMICIRCLE)
- $\angle POQ = 90^\circ - \theta$ (TRIANGLE NPQ IS A RIGHT ANGLE)
- $\angle NPQ = 90^\circ - \theta = \theta + \phi$

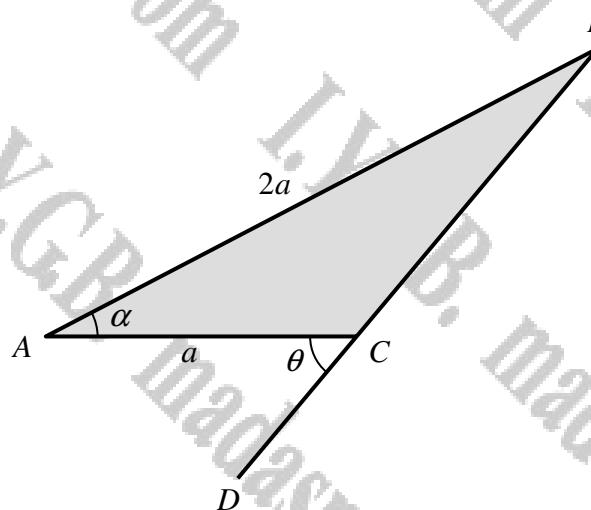
$$\Rightarrow \phi = 90^\circ - 2\theta$$

NOW USING TRIGONOMETRIC IDENTITIES WE HAVE

$$\begin{aligned} \tan \phi &= \tan(90^\circ - 2\theta) \\ &= \cot(2\theta) \quad \text{OR} \quad \frac{\sin(90^\circ - 2\theta)}{\cos(90^\circ - 2\theta)} \\ &= \frac{1}{\tan 2\theta} \\ &= \frac{1}{2\tan\theta} \frac{1}{1 - \tan^2\theta} \\ &= \frac{1 - \tan^2\theta}{2\tan\theta} \\ &= \frac{1 - \left(\frac{3}{4}\right)^2}{2 \times \frac{3}{4}} \\ &= \frac{1 - \frac{9}{16}}{\frac{3}{2}} \times \frac{16}{16} \\ &= \frac{16 - 9}{24} \\ &= \frac{7}{24} \end{aligned}$$

AS REQUIRED

Question 5 (*****)



The figure above shows a triangle ABC , where $|AB|=a$ and $|AC|=2a$.

The angle BAC is α , where $\tan \alpha = \frac{3}{4}$.

The side BC is extended to the point D so that the angle ACD is denoted by θ .

Show clearly that $\theta = \arctan 2$.

 , proof

• START WITH THE DIAGRAM, LET $\angle BAC = \alpha$

• BY THE COSINE RULE ON $\triangle ABC$

$$\Rightarrow |BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos\alpha$$

$$\Rightarrow |BC|^2 = a^2 + (2a)^2 - 2a(2a)\cos\alpha$$

$$\Rightarrow |BC|^2 = 5a^2 - 4a^2\cos\alpha$$

$$\Rightarrow |BC|^2 = 5a^2 - \frac{16a^2}{5}$$

$$\Rightarrow |BC|^2 = \frac{9}{5}a^2$$

$$\Rightarrow |BC| = \frac{3}{\sqrt{5}}a$$


• NEXT BY THE SINE RULE ON $\triangle ABC$

$$\Rightarrow \frac{\sin\alpha}{\frac{3}{\sqrt{5}}a} = \frac{\sin\beta}{a}$$

$$\Rightarrow \frac{1}{\sqrt{5}}\sin\alpha = \frac{3}{5}\sin\beta$$

$$\Rightarrow \frac{3}{5} = \frac{\sqrt{5}}{5}\sin\beta$$

$$\Rightarrow \sin\beta = \frac{\sqrt{5}}{3}$$

• NEXT GET THE EXACT TRIG RATIOS OF β



$$\sin\beta = \frac{3}{5}$$

$$\cos\beta = \frac{4}{5}$$

$$\tan\beta = \frac{3}{4}$$

• FINALLY WE HAVE

$$\Rightarrow \theta = \alpha + \beta$$

$$\Rightarrow \tan\theta = \tan(\alpha+\beta)$$

$$\Rightarrow \tan\theta = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\Rightarrow \tan\theta = \frac{\frac{3}{4} + \frac{3}{4}}{1 - \frac{3}{4} \cdot \frac{3}{4}}$$

$$\Rightarrow \tan\theta = \frac{\frac{3}{2}}{\frac{7}{16}}$$

$$\Rightarrow \tan\theta = \frac{10}{7}$$

$$\Rightarrow \tan\theta = 2$$

$$\Rightarrow \theta = \arctan 2$$

Question 6 (*****)

The acute angles θ , ψ and α satisfy the following equations.

$$4 \tan \theta = \tan \alpha$$

$$(5 + 3 \cos 2\alpha) \tan \psi = 3 \sin 2\alpha.$$

Express $\theta + \psi$, in terms of α .

$$\boxed{\quad}, \boxed{\theta + \psi = \alpha}$$

① STARTING FROM THE SECOND EQUATION

$$\Rightarrow \tan \psi = \frac{3 \sin 2\alpha}{5 + 3 \cos 2\alpha} = \frac{3(2 \sin \alpha \cos \alpha)}{5 + 3(1 - 2 \sin^2 \alpha)}$$

$$= \frac{6 \sin \alpha \cos \alpha}{8 - 6 \sin^2 \alpha} =$$

DIVIDE TOP & BOTTOM BY $\sin^2 \alpha$

$$= \frac{6 \tan \alpha}{8 - 6 \tan^2 \alpha} = \frac{3 \tan \alpha}{4 \sec^2 \alpha - 3 \tan^2 \alpha}$$

$$= \frac{3 \tan \alpha}{4(1 + \tan^2 \alpha) - 3 \tan^2 \alpha} = \frac{3 \tan \alpha}{4 + \tan^2 \alpha}$$

② SUBSTITUTING $T = \tan \alpha$ INTO BOTH EQUATIONS

$$\tan \theta = \frac{1}{4}T$$

$$\tan \psi = \frac{3T}{4 + T^2}$$

③ Hence BY THE COMPOUND ANGLE IDENTITIES

$$\tan(\theta + \psi) = \frac{\tan \theta + \tan \psi}{1 - \tan \theta \tan \psi} = \frac{\frac{1}{4}T + \frac{3T}{4 + T^2}}{1 - \frac{1}{4}T \left(\frac{3T}{4 + T^2} \right)}$$

④ MULTIPLY "TOP & BOTTOM" BY $4(4+T^2)$

$$\dots = \frac{T(4+T^2) + 12T}{4(4+T^2) - 3T^2} = \frac{T^2 + 16T}{T^2 + 16}$$

$$= \frac{T(T+16)}{T^2+16} = T$$

$\therefore \tan(\theta + \psi) = \tan \alpha$

$\therefore \theta + \psi = \alpha$

Question 7 (***)**

The acute angles θ and ϕ satisfy the following equations

$$2\cos\theta = \cos\phi$$

$$2\sin\theta = 3\sin\phi.$$

Show clearly that

$$\theta + \phi = \pi - \arctan \sqrt{15}$$

□, proof

2cos θ = cos ϕ
2sin θ = 3sin ϕ

θ, φ are acute

Start by squaring & adding

$$\begin{aligned} 4\cos^2\theta &= \cos^2\phi \\ 4\sin^2\theta &= 9\sin^2\phi \end{aligned} \quad \Rightarrow \quad 4(\cos^2\theta + \sin^2\theta) = \cos^2\phi + 9\sin^2\phi$$

$$\begin{aligned} A &= \cos^2\theta + 9\sin^2\theta \\ A &= 1 - \sin^2\theta + 9\sin^2\theta \\ 3 &= 8\sin^2\phi \\ \sin^2\phi &= \frac{3}{8} \\ \sin\phi &= \pm\sqrt{\frac{3}{8}} \quad (\phi \text{ is acute}) \end{aligned}$$

Obtain the corresponding value of θ

$$\begin{aligned} \Rightarrow 4\sin\theta &= \pm\sqrt{3} \\ \Rightarrow 4\sin^2\theta &= 9 \times \frac{3}{8} \\ \Rightarrow \sin^2\theta &= \frac{27}{32} \\ \Rightarrow \sin\theta &= \pm\sqrt{\frac{27}{32}} \quad (\theta \text{ is acute}) \end{aligned}$$

Next obtain the exact value of tanθ & tanφ

Using the tan compound identity:

$$\begin{aligned} \tan(\theta + \phi) &= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{\frac{\sqrt{27}}{\sqrt{3}} + \frac{\sqrt{15}}{\sqrt{3}}}{1 - \frac{\sqrt{27}}{\sqrt{3}} \cdot \frac{\sqrt{15}}{\sqrt{3}}} \\ &= \frac{3\sqrt{3}\sqrt{5}}{3} = \frac{\sqrt{27}\sqrt{5}}{3} = \frac{\sqrt{135}}{3} = \frac{3\sqrt{15}}{3} = \sqrt{15} \\ &= \frac{\sqrt{15}\sqrt{15} + \sqrt{15}\sqrt{15}}{-4} = \frac{3\sqrt{15} + \sqrt{15}}{-4} = \frac{4\sqrt{15}}{-4} = -\sqrt{15} \end{aligned}$$

$$\therefore \tan(\theta + \phi) = -\sqrt{15}$$

$$\Rightarrow \theta + \phi = \arctan(-\sqrt{15}) \pm n\pi \quad n \in \{1, 2, 3, \dots\}$$

$$\Rightarrow \theta + \phi = -\arctan\sqrt{15} \pm n\pi$$

But $0 < \theta + \phi < \pi$

$$\begin{aligned} \Rightarrow \theta + \phi &= -\arctan\sqrt{15} + \pi \\ \Rightarrow \theta + \phi &= \pi - \arctan\sqrt{15} \end{aligned}$$

Question 8 (***)**

Determine the range of the following function

$$f(\theta) = \frac{1}{\sin^2 \theta + 3\sin \theta \cos \theta + 5\cos^2 \theta}, \quad \theta \in \mathbb{R}$$

, $\frac{2}{11} \leq f(\theta) \leq 2$

$$\begin{aligned}
 & f(\theta) = \frac{1}{\sin^2 \theta + 3\sin \theta \cos \theta + 5\cos^2 \theta} \quad \theta \in \mathbb{R} \\
 \Rightarrow & f(\theta) = \frac{1}{\sin^2 \theta + \cos^2 \theta + \frac{3}{2}(\sin 2\theta) + 4\cos^2 \theta} \\
 \Rightarrow & f(\theta) = \frac{1}{1 + \frac{3}{2}\sin 2\theta + \frac{1}{4}(1 + \frac{1}{2}\cos 2\theta)} \\
 \Rightarrow & f(\theta) = \frac{1}{1 + \frac{3}{2}\sin 2\theta + 2\cos 2\theta} \\
 \Rightarrow & f(\theta) = \frac{1}{3 + \frac{3}{2}\sin 2\theta + 2\cos 2\theta} \\
 & \xrightarrow{\text{"R TRANSMISSION/HARMONIC FORM}} \\
 & R = \sqrt{(\frac{3}{2})^2 + 2^2} = \sqrt{\frac{25}{4}} = \sqrt{\frac{25}{4}} + \frac{2}{2} \\
 \Rightarrow & f(\theta) = \frac{1}{3 + \frac{3}{2}\cos(2\theta - \alpha)} \\
 \therefore & f(\theta)_{\min} = \frac{1}{3 + \frac{3}{2}} = \frac{2}{6+3} = \frac{2}{9} \\
 & f(\theta)_{\max} = \frac{1}{3 - \frac{3}{2}} = \frac{2}{6-3} = 2. \\
 & \therefore \text{since } \frac{2}{9} \leq f(\theta) \leq 2.
 \end{aligned}$$

Question 9 (*****)

It is given that

$$(a+2)\sin x + (2a-1)\cos x = 2a+1,$$

where a is a non zero constant.

Find the exact value of $\tan\left(\frac{1}{2}x\right)$, giving the answer in terms of a , where appropriate.

$\tan\left(\frac{1}{2}x\right) = \frac{1}{2}$, $\tan\left(\frac{1}{2}x\right) = \frac{1}{a}$

• Using the half tangent identities

$$\sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2} \quad \text{where } t = \tan \frac{x}{2}$$

$$\Rightarrow (a+2)\sin x + (2a-1)\cos x = 2a+1$$

$$\Rightarrow (a+2) \frac{2t}{1+t^2} + (2a-1) \frac{1-t^2}{1+t^2} = 2a+1$$

$$\Rightarrow (a+2) \cdot 2t + (2a-1)(1-t^2) = (2a+1)(1+t^2)$$

$$\Rightarrow 2(a+2)t + (2a-1) - (2a-1)t^2 = (2a+1) + (2a+1)t^2$$

$$\Rightarrow 0 = (2a+1)t^2 + (2a-1)t^2 - 2(a+2)t + (2a+1) - (2a-1)$$

$$\Rightarrow 0 = 4at^2 - 2(a+2)t + 2$$

$$\Rightarrow 2at^2 - (a+2)t + 1 = 0$$

• By factorization

$$\Rightarrow (at-1)(at-1) = 0$$

$$\Rightarrow t = \frac{1}{a}$$

$$\Rightarrow \tan \frac{x}{2} = \frac{1}{a}$$

OR THE QUADRATIC FORMULA

$$t = \frac{-(a+2) \pm \sqrt{(a+2)^2 - 4 \cdot 2a \cdot 1}}{2 \cdot 2a}$$

$$t = \frac{(a+2) \pm \sqrt{a^2 - 4a + 4}}{4a}$$

$$t = \frac{(a+2) \pm \sqrt{a^2 - 4a + 4}}{4a}$$

$$t = \frac{(a+2) \pm \sqrt{(a-2)^2}}{4a}$$

$$t = \frac{a+2 + (a-2)}{4a} = \frac{2a}{4a} = \frac{a}{2}$$

$$t = \frac{a+2 - (a-2)}{4a} = \frac{4}{4a} = \frac{1}{a}$$

$$\Rightarrow t = \frac{1}{a}$$

Question 10 (***)**

Solve the following trigonometric equation.

$$\cos 4x^\circ = \cos 40^\circ + \cos 80^\circ, \quad 0^\circ \leq x \leq 180^\circ.$$

, $x = 5^\circ, 85^\circ, 95^\circ, 175^\circ$

$\cos 4x = \cos 40^\circ + \cos 80^\circ \quad 0^\circ < x < 180^\circ$

- Start by manipulating the right hand side:

$$\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \cos(A-B) &= \cos A \cos B + \sin A \sin B \\ \cos(A+B) + \cos(A-B) &= 2 \cos A \cos B \\ \cos(40^\circ) + \cos(80^\circ) &= 2 \cos 40^\circ \cos 80^\circ \end{aligned}$$

- Hence we have:

$$\begin{aligned} \cos 4x &= 2 \cos 40^\circ \cos 80^\circ \\ \cos 4x &= 2 \times \frac{1}{2} \times \cos 20^\circ \\ \cos 4x &= \cos 20^\circ \end{aligned}$$

$(4x = 20^\circ \pm 360n)$
 $(4x = 340^\circ \pm 360n)$
 $(x = 5^\circ \pm 90n)$
 $(x = 85^\circ \pm 90n)$

$\therefore x = 5^\circ, 85^\circ, 95^\circ, 175^\circ$

Question 11 (***)**

A right circular cone, of radius r and semi-vertical angle θ , lies with one of its generators in contact with a horizontal surface.

The cone is then rolled on the horizontal surface with its vertex at rest, so that the rolling circumference of its base completes a full circle on the surface, while the cone completes N revolutions about its own axis.

Show that $N = \operatorname{cosec} \theta$.

, proof

Let the radius of the cone be r and its height h .

Then looking at the diagram, the circle to be traced has radius $\sqrt{r^2+h^2}$.

This circle has circumference $2\pi\sqrt{r^2+h^2}$.

The circumference of the base is $2\pi r$.

Hence the number of turns is given by

$$\begin{aligned} N &= \frac{2\pi\sqrt{r^2+h^2}}{2\pi r} = \frac{\sqrt{r^2+h^2}}{r} = \frac{h\sqrt{r^2+h^2}}{r^2} \\ &= \frac{\sqrt{(Fr)^2+h^2}}{r^2} = \frac{\sqrt{Fr^2+h^2}}{r^2} = \frac{\sqrt{h^2+\tan^2\theta}}{\tan^2\theta} = \frac{1}{\tan^2\theta} = \frac{\sec^2\theta}{\tan^2\theta} = \frac{\sec^2\theta}{\tan^2\theta} = \frac{\sec^2\theta}{\tan^2\theta} = \operatorname{cosec}^2\theta \end{aligned}$$

As required.

Question 12 (*****)

Solve the following trigonometric equation.

$$2\sqrt{3} \sin\left(x + \frac{7\pi}{12}\right) = 3 \operatorname{cosec}\left(x + \frac{5\pi}{12}\right), \quad 0 \leq x \leq 2\pi.$$

, $x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

$2\sqrt{3} \sin\left(x + \frac{7\pi}{12}\right) = 3 \operatorname{cosec}\left(x + \frac{5\pi}{12}\right), \quad 0 \leq x < 2\pi$

 $\Rightarrow 2\sqrt{3} \sin\left(x + \frac{7\pi}{12}\right) = \frac{3}{\sin\left(x + \frac{5\pi}{12}\right)}$
 $\Rightarrow 2\sin\left(x + \frac{7\pi}{12}\right) \sin\left(x + \frac{5\pi}{12}\right) = \frac{3}{\sqrt{3}}$

Now divide an identity based on the compound angle identities

$$\begin{aligned} \cos(A+B) &\equiv \cos A \cos B - \sin A \sin B \\ \cos(A-B) &\equiv \cos A \cos B + \sin A \sin B \end{aligned} \quad \left. \begin{array}{l} \text{SUBSTITUTE "COPA" } \\ \text{SUBSTITUTE "DOPA"} \end{array} \right\}$$
 $\cos(A+B) - \cos(A-B) \equiv 2 \sin A \sin B$
 $\Rightarrow \cos\left(x + \frac{7\pi}{12}\right) - \cos\left(x - \frac{5\pi}{12}\right) = \cos\left[\left(x + \frac{7\pi}{12}\right) - \left(x - \frac{5\pi}{12}\right)\right] = \frac{3\sqrt{3}}{\sqrt{12}}$
 $\Rightarrow \cos\frac{11\pi}{12} - \cos\left(2x + \pi\right) = \sqrt{3}$
 $\Rightarrow \frac{\sqrt{3}}{2} - \left[\cos 2x \cos \pi - \sin 2x \sin \pi \right] = \sqrt{3}$
 $\Rightarrow \frac{\sqrt{3}}{2} + \cos 2x = \sqrt{3}$
 $\Rightarrow \cos 2x = \frac{\sqrt{3}}{2}$
 $\operatorname{arccos}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

$$\begin{cases} 2x = \frac{\pi}{6} \pm 2n\pi \\ 2x = \frac{4\pi}{3} \pm 2n\pi \end{cases} \quad n=0,1,2,3,\dots$$

$$\begin{cases} x = \frac{\pi}{12} \pm n\pi \\ x = \frac{11\pi}{12} \pm n\pi \end{cases}$$

$$x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

Question 13 (***)**

Solve the following trigonometric equation.

$$\sin(2\theta + 58)^\circ + 2\sin^2(42^\circ) = 1, \quad 0^\circ \leq \theta < 360^\circ.$$

$$\boxed{\quad}, \quad \theta = \{58, 154, 238, 334\}$$

$\sin(2\theta + 58^\circ) + 2\sin^2(42^\circ) = 1 \quad 0^\circ \leq \theta < 360^\circ$

SOLVING THE EQUATION AS FOLLOWS

$$\begin{aligned} \Rightarrow \sin(2\theta + 58^\circ) &= 1 - 2\sin^2 42^\circ \\ \Rightarrow \sin(2\theta + 58^\circ) &= \cos(2 \times 42^\circ) \\ \Rightarrow \sin(2\theta + 58^\circ) &= \cos 84^\circ \\ \Rightarrow \sin(2\theta + 58^\circ) &= \sin(18^\circ - 84^\circ) \\ \Rightarrow \sin(2\theta + 58^\circ) &= \sin(-66^\circ) \end{aligned}$$

EXTRACTING A PRINCIPAL SOLUTION

$$\begin{aligned} \Rightarrow \begin{cases} 2\theta + 58^\circ = 45^\circ \pm 360^\circ \\ 2\theta + 58^\circ = 114^\circ \pm 360^\circ \end{cases} \quad n = 0, 1, 2, 3, \dots \\ \Rightarrow \begin{cases} 2\theta = -52^\circ \pm 360^\circ \\ 2\theta = 116^\circ \pm 360^\circ \end{cases} \\ \Rightarrow \begin{cases} \theta = -26^\circ \pm 180^\circ \\ \theta = 53^\circ \pm 180^\circ \end{cases} \\ \Rightarrow \theta = 154^\circ, 334^\circ, 58^\circ, 238^\circ \\ \Rightarrow \theta = 58^\circ, 154^\circ, 238^\circ, 334^\circ \end{aligned}$$

Question 14 (***)**

Solve the following trigonometric equation

$$\cos\left(\arcsin \frac{1}{4}\right) \sin\left(\arccos x\right) = \frac{1}{4}(4-x), \quad x \in \mathbb{R}.$$

$$\boxed{}, \quad x = \boxed{\frac{1}{4}}$$

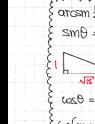
METHOD A

ATTEMPT TO CREATE A SINE COMPOUND IDENTITY

$$\begin{aligned} &\Rightarrow \cos\left(\arcsin \frac{1}{4}\right) \sin\left(\arccos x\right) = \frac{1}{4}(4-x) \\ &\Rightarrow \cos\left(\arcsin \frac{1}{4}\right) \sin\left(\arccos x\right) = 1 - \frac{1}{4}x \\ &\Rightarrow \frac{1}{4}x + \cos\left(\arcsin \frac{1}{4}\right) \sin\left(\arccos x\right) = 1 \\ \text{Let } A &= \arcsin \frac{1}{4}, \quad B = \arccos x \\ &\Rightarrow \sin(A) \cos(B) + \cos(A) \sin(B) = 1 \\ &\Rightarrow \sin\left(\arcsin \frac{1}{4} + \arccos x\right) = 1 \\ &\Rightarrow \arcsin \frac{1}{4} + \arccos x = \frac{\pi}{2} + 2k\pi, \quad k = 0, 1, 2, \dots \\ \text{But } \arccos x \text{ ONLY TAKES VALUES BETWEEN } 0 \text{ & } \pi & \\ &\Rightarrow \arcsin \frac{1}{4} + \arccos x = \frac{\pi}{2} \\ &\Rightarrow \arccos x = \frac{\pi}{2} - \arcsin \frac{1}{4} \\ &\Rightarrow \cos(\arccos x) = \cos\left(\frac{\pi}{2} - \arcsin \frac{1}{4}\right) \\ &\Rightarrow x = \sin\left(\arcsin \frac{1}{4}\right) \quad \text{COS}(\frac{\pi}{2}-B) = \text{SIN} B \\ &\Rightarrow x = \frac{1}{4} \end{aligned}$$

METHOD B

$$\cos\left(\arcsin \frac{1}{4}\right) \sin\left(\arccos x\right) = \frac{1}{4}(4-x)$$

$\arcsin \frac{1}{4} = \theta$  $\sin \theta = \frac{1}{4}$ $\cos \theta = \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4}$ $\cot \theta = \frac{1}{\tan \theta} = \frac{1}{\sqrt{15}}$	$\arccos x = \phi$  $\cos \phi = x$ $\sin \phi = \sqrt{1 - x^2}$ $\tan \phi = \frac{1}{x}$
--	--

$$\begin{aligned} &\Rightarrow \cos(\arcsin \frac{1}{4}) \sin(\arccos x) = \frac{1}{4}(4-x) \\ &\Rightarrow \frac{\sqrt{15}}{4} \sqrt{1-x^2} = \frac{1}{4}(4-x) \\ &\Rightarrow \sqrt{15(1-x^2)} = 4-2x \\ &\Rightarrow 15(1-x^2) = (4-2x)^2 \\ &\Rightarrow 15 - 15x^2 = 16 - 16x + 4x^2 \\ &\Rightarrow 0 = 16x^2 - 16x + 1 \\ &\Rightarrow (4x-1)^2 = 0 \\ &\Rightarrow x = \frac{1}{4} \end{aligned}$$

SOLUTION indeed SATISFIES THE EQUATION BEFORE SPANNING

Question 15 (***)**

It is given that $0 < x < \frac{1}{2}\pi$ and $0 < y < \frac{1}{2}\pi$.

It is further given that

$$\sin(x+y)\sin(x-y) = \frac{5}{36} \quad \text{and} \quad \cos x + \cos y = \frac{5}{6}.$$

Show that $\cos(x-y) = \frac{1+\sqrt{n}}{n}$, where n is a positive integer to be found.

, $n = 6$

Manipulate AS FOLLOWS

$$\begin{aligned} &\Rightarrow \sin(2xy) \sin(2x-y) = \frac{5}{36} \\ &\Rightarrow [\sin(2xy) + \cos(2y)][\sin(2xy) - \cos(2y)] = \frac{5}{36} \\ &\Rightarrow \sin^2(2xy) - \cos^2(2xy) = \frac{5}{36} \\ &\Rightarrow (-\cos 2y)\cos y - \cos(-(-2xy)) = \frac{5}{36} \\ &\Rightarrow (\cos y - \cos 2y)(-\cos 2y + \cos y) = \frac{5}{36} \\ &\Rightarrow (\cos y - \cos 2y)^2 = \frac{5}{36} \\ &\Rightarrow (\cos y - \cos 2y)(\cos y + \cos 2y) = \frac{5}{36} \\ &\text{BUT } \cos 2y + \cos y = \frac{5}{6} \\ &\Rightarrow \frac{1}{2}(\cos y - \cos 2y) = \frac{5}{36} \quad \downarrow \quad \frac{1}{2} \\ &\Rightarrow \cos y - \cos 2y = \frac{1}{6} \end{aligned}$$

SOLVE SIMULTANEOUSLY

$$\begin{aligned} &\cos y + \cos 2y = \frac{5}{6} \quad) \quad \text{ADDING YIELDS} \quad 2\cos y = 1 \\ &\cos y - \cos 2y = \frac{1}{6} \quad) \quad \text{SUBTRACTING} \quad \cos y = \frac{1}{2} \quad \text{A} \quad \cos 2y = \frac{1}{3} \end{aligned}$$

FINALLY SINCE $2y$ IS IN THE 2^{nd} QUADRANT

- $\sin 2y = +\sqrt{1 - \cos^2 2y} = +\sqrt{1 - (\frac{1}{3})^2} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$
- $\cos 2y = +\sqrt{1 - \cos^2 2y} = +\sqrt{1 - (\frac{1}{3})^2} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

THIS IS A COMPOUND ANGLE PROBLEM

$$\begin{aligned} \cos(2y+2) &= \cos 2y \cos y + \sin 2y \sin y = \frac{1}{3} \times \frac{1}{2} + \frac{2\sqrt{2}}{3} \times \frac{1}{2} \sqrt{\frac{2}{3}} \\ &= \frac{1}{6} + \frac{4\sqrt{2}}{6} \\ &= \frac{1}{6}(1+4\sqrt{2}) \end{aligned}$$

Question 16 (***)**

A curve in the x - y plane has equation

$$x^2 + y^2 + 6x \cos \theta - 18y \sin \theta + 45 = 0,$$

where θ is a parameter such that $0 \leq \theta < 2\pi$.

Given that curve represents a circle determine the range of possible values of θ .

$\left\{ \frac{1}{4}\pi < \theta < \frac{3}{4}\pi \right\} \cup \left\{ \frac{5}{4}\pi < \theta < \frac{7}{4}\pi \right\}$

● START BY COMPLETING THE SQUARES

$$\begin{aligned} &\Rightarrow x^2 + y^2 + 6x \cos \theta - 18y \sin \theta + 45 = 0 \\ &\Rightarrow x^2 + 6x \cos \theta + y^2 - 18y \sin \theta + 45 = 0 \\ &\Rightarrow (x + 3 \cos \theta)^2 - 9 \cos^2 \theta + (y - 9 \sin \theta)^2 - 81 \sin^2 \theta + 45 = 0 \\ &\Rightarrow (x + 3 \cos \theta)^2 + (y - 9 \sin \theta)^2 = 9 \cos^2 \theta + 81 \sin^2 \theta - 45 \end{aligned}$$

● NOW IF THIS REPRESENTS A CIRCLE, THEN

$$\begin{aligned} &\Rightarrow 9 \cos^2 \theta + 81 \sin^2 \theta - 45 > 0 \\ &\Rightarrow 9 \cos^2 \theta + 9 \sin^2 \theta - 5 > 0 \\ &\Rightarrow 1 + 8 \sin^2 \theta - 5 > 0 \\ &\Rightarrow 8 \sin^2 \theta > 4 \\ &\Rightarrow \sin^2 \theta > \frac{1}{2} \end{aligned}$$

● Hence $\sin \theta > \frac{1}{\sqrt{2}}$ OR $\sin \theta < -\frac{1}{\sqrt{2}}$

$$\left(\arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \right) \quad \left(\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4} \right)$$

● THIS IN THE REQUIRED RANGE

Question 17 (***)**

Prove the validity of the following trigonometric identity.

$$\frac{1 + \tan \theta \tan 3\theta}{1 + \tan 2\theta \tan 3\theta} \equiv \frac{\cos^2 2\theta}{\cos^2 \theta}.$$

 , proof

Simplifying at the RHS

$$\begin{aligned}
 \text{L.H.S.} &= \frac{1 + \tan \theta \tan 3\theta}{1 + \tan 2\theta \tan 3\theta} = \frac{\frac{1}{\tan 2\theta} + \frac{\tan \theta \tan 3\theta}{\tan 2\theta \tan 3\theta}}{\frac{1}{\tan 2\theta} + \frac{\tan 2\theta \tan 3\theta}{\tan 2\theta \tan 3\theta}} \\
 &= \frac{\cot 2\theta + \tan \theta}{\cot 2\theta + \tan 2\theta} = \frac{\frac{\cos 2\theta}{\sin 2\theta} + \frac{\sin \theta}{\cos \theta}}{\frac{\cos 2\theta}{\sin 2\theta} + \frac{\sin 2\theta}{\cos 2\theta}} \\
 &= \frac{\frac{\cos 2\theta \cos \theta + \sin 2\theta \sin \theta}{\sin 2\theta \cos \theta}}{\frac{\cos 2\theta \sin 2\theta + \sin 2\theta \cos 2\theta}{\cos 2\theta \sin 2\theta}} = \frac{\frac{\cos(2\theta - \theta)}{\sin 2\theta \cos \theta}}{\frac{\cos(2\theta + 2\theta)}{\cos 2\theta \sin 2\theta}} \\
 &= \frac{\frac{\cos \theta}{\sin 2\theta \cos \theta}}{\frac{\cos 4\theta}{\cos 2\theta \sin 2\theta}} = \frac{\cos^2 2\theta \sin 2\theta}{\cos^2 \theta \sin 2\theta} = (\cot \theta)^2 = \text{R.H.S.}
 \end{aligned}$$

Question 18 (***)**

A surveyor views the top of a building, of height h , at an angle of elevation α .

The surveyor walks a distance a , directly towards the building.

From this new position he views the top of the building at an angle of elevation β .

Show that

$$h = \frac{a \sin \alpha \sin \beta}{\sin(\beta - \alpha)}.$$

 , proof

BY SEPARATE TRIANGULARITY ON RIGHT ANGLED TRIANGLES

$$\begin{aligned}
 \tan \alpha &= \frac{h}{x} & \tan \beta &= \frac{h}{x-a} \\
 \cot \alpha &= \frac{x}{h} & \cot \beta &= \frac{x-a}{h}
 \end{aligned}$$

$ \begin{aligned} \text{T.R.S.} \\ \Rightarrow \frac{h}{x-a} &= \cot \alpha \\ \Rightarrow \frac{x-a}{h} &= \cot \alpha \\ \Rightarrow \frac{x}{h} + \frac{a}{h} &= \cot \alpha \\ \Rightarrow \frac{1}{\cot \alpha} + \frac{a}{h} &= \cot \alpha \\ \Rightarrow \cot \alpha + \frac{a}{h} &= \cot \alpha \\ \Rightarrow \frac{a}{h} &= \cot \alpha - \cot \beta \end{aligned} $	$ \begin{aligned} \Rightarrow \frac{a}{h} &= \frac{\cot \alpha - \cot \beta}{\cot \alpha} \\ \Rightarrow \frac{a}{h} &= \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\sin \alpha \cos \beta} \\ \Rightarrow \frac{a}{h} &= \frac{\sin(\alpha - \beta)}{\sin \alpha \cos \beta} \\ \Rightarrow \frac{h}{a} &= \frac{\sin \alpha \cos \beta}{\sin(\alpha - \beta)} \\ \Rightarrow h &= \frac{\sin \alpha \cos \beta}{\sin(\alpha - \beta)} \end{aligned} $ <p style="text-align: right;">As required</p>
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Question 19 (*****)

$$\sin^8 x - \cos^8 x = 1 - \frac{1}{2} \sin^2 2x.$$

Use trigonometric identities to show that the general solution of the above equation is $x = k\pi$, $k \in \mathbb{Z}$.

proof

$$\begin{aligned}
 \sin^8 x - \cos^8 x &= 1 - \frac{1}{2} \sin^2 2x \\
 \Rightarrow (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) &= 1 - \frac{1}{2} \sin^2 2x \\
 \Rightarrow (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)(\sin^2 x + \cos^2 x) &= 1 - \frac{1}{2} \sin^2 2x \\
 \Rightarrow \cos 2x (\sin^2 x + \cos^2 x) &= 1 - \frac{1}{2} \sin^2 2x \\
 \Rightarrow \cos 2x (\sin^2 x + \cos^2 x)^2 &= (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 2x \\
 \Rightarrow \cos 2x (\sin^2 x + \cos^2 x)^2 &= \sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x - 2 \sin^2 2x \\
 \Rightarrow \cos 2x (\sin^2 x + \cos^2 x)^2 &= \cos^4 x + \sin^4 x \\
 \Rightarrow \cos 2x &= 1 \\
 (2x = 0 \pm 2n\pi) &\quad n = 0, 1, 2, 3, \dots \\
 2x = \dots & \\
 x = \dots & \\
 x = k\pi &\quad k \in \mathbb{Z}
 \end{aligned}$$

Question 20 (*****)

It is given that x is a solution of the following equation.

$$\sec x + \tan x = \frac{5}{7}.$$

Without solving the above equation for x , find the value of $\operatorname{cosec} x + \cot x$.

V, , **-6**

$$\begin{aligned}
 \text{Using } 1 + \tan^2 x \equiv \sec^2 x \\
 \Rightarrow \sec x + \tan x = \frac{5}{7} \\
 \Rightarrow (\sec x + \tan x)(\sec x - \tan x) = \frac{5}{7}(\sec x - \tan x) \\
 \Rightarrow \sec x - \tan x = \frac{5}{7}(\sec x - \tan x) \\
 \Rightarrow 1 = \frac{5}{7}(\sec x - \tan x) \\
 \Rightarrow \sec x - \tan x = \frac{7}{5}
 \end{aligned}$$

Take we know that

$$\begin{aligned}
 \sec x + \tan x &= \frac{5}{7} & \text{Hence, } 2 \sec x = \frac{5}{7} + \frac{7}{5} = \frac{25+49}{35} \\
 \sec x - \tan x &= \frac{7}{5} & \sec x = \frac{25}{35} \\
 \sec x &= \frac{25}{35} & \operatorname{cosec} x = \frac{49}{35} \\
 \operatorname{cosec} x &= \frac{49}{35}
 \end{aligned}$$

Similarly

$$\begin{aligned}
 2 \tan x &= \frac{5}{7} - \frac{7}{5} = \frac{25-49}{35} = -\frac{24}{35} \\
 \tan x &= -\frac{12}{35}
 \end{aligned}$$

As we take positive values, by negative tangent we go to the 3rd quadrant — we can still use the standard triangle

$$\begin{aligned}
 \Rightarrow \operatorname{cosec} x + \cot x &= \frac{1}{\operatorname{cosec} x} + \frac{1}{\cot x} \\
 &= \frac{1}{\frac{49}{35}} + \frac{1}{-\frac{12}{35}} \\
 &= -\frac{35}{12} - \frac{35}{12} \\
 &= -\frac{70}{12} \\
 &= -\frac{35}{6}
 \end{aligned}$$

Question 21 (*****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ , and is commonly known as the golden section or golden number.

Solve the equation

$$x \cos\left(\frac{1}{2} \arctan 2\right) = \sqrt{\phi}, \quad x \in \mathbb{R}.$$

Give the answer in the form $\sqrt[n]{m}$, where m and n are positive integers.

V, , $x = \sqrt[4]{5}$

USING A SUBSTITUTION $\theta = \frac{1}{2} \arctan 2$.

$$\begin{aligned} \Rightarrow 2\theta &= \tan 2 \\ \Rightarrow \tan 2\theta &= 2 \\ \Rightarrow \tan^2 2\theta &= 4 \\ \Rightarrow 1 + \tan^2 2\theta &= 5 \\ \Rightarrow \sec^2 2\theta &= 5 \\ \Rightarrow \sec 2\theta &= \sqrt{5} \\ \Rightarrow \cos 2\theta &= \frac{1}{\sqrt{5}} \end{aligned}$$

$$\begin{aligned} \Rightarrow 2\theta &= \tan^{-1} 1 = \frac{1}{\sqrt{3}} \\ \Rightarrow 2\theta &= \frac{\pi}{6} \\ \Rightarrow 2\theta &= 1 + \frac{\pi}{6} \\ \Rightarrow 2\theta &= \frac{5 + \pi}{6} \\ \Rightarrow \cos \theta &= \frac{\sqrt{5 + \pi^2}}{10} \\ \Rightarrow \cos \theta &= +\sqrt{\frac{5 + \pi^2}{10}} \end{aligned}$$

Thus $\cos \theta = \cos\left(\frac{1}{2} \arctan 2\right) = \sqrt{\frac{5 + \pi^2}{10}}$

$$\begin{aligned} \Rightarrow 2\sqrt{\frac{5 + \pi^2}{10}} &= \sqrt{\phi} \\ \Rightarrow \sqrt{\frac{5 + \pi^2}{5}} &= \sqrt{\phi} \\ \Rightarrow \sqrt{\frac{5 + \pi^2}{5}} &= \sqrt{\phi} \end{aligned}$$

$\phi = \frac{1 + \sqrt{5}}{2}$ (quadratic)

$\cos 2\theta = \cos(\pi - \theta) = -\phi = -1$

MINIMISE THE SURE NOW

$$\begin{aligned} \lambda &= \sqrt{\frac{5 + \pi^2}{5}} = \sqrt{\frac{5 + 5\pi^2}{5 + \pi^2}} = \sqrt{\frac{5(1 + \pi^2)}{5 + \pi^2}} = \sqrt{5} \end{aligned}$$

$\lambda = \sqrt[4]{5}$

1. $m=5$
2. $n=4$

Question 22 (*****)

Prove that

$$\arctan\left[\sqrt{\frac{1-x}{1+x}}\right] \equiv \frac{1}{2}\arccos x.$$

V, SP/AV, proof

Work as follows

LET $4x^2 = \theta = \arctan\sqrt{\frac{1-x}{1+x}}$

Manipulate in terms of θ

$$\begin{aligned} \Rightarrow \tan\theta &= \sqrt{\frac{1-x}{1+x}} \\ \Rightarrow \tan\theta &= \frac{1-x}{1+x} \\ \Rightarrow 1+x\tan\theta &= \frac{1-x}{1+x}+1 \\ \Rightarrow \sec^2\theta &= \frac{1-x+1+x}{1+x} \\ \Rightarrow \sec^2\theta &= \frac{2}{1+x} \\ \Rightarrow \cos^2\theta &= \frac{1+x}{2} \end{aligned}$$

Now using: $\cos 2\theta \equiv 2\cos^2\theta - 1$

$$\begin{aligned} \Rightarrow 2\cos^2\theta &= x+1 \\ \Rightarrow 2\cos^2\theta - 1 &= x \\ \Rightarrow \cos 2\theta &= x \\ \Rightarrow 2\theta &= \arccos x \\ \Rightarrow \theta &= \frac{1}{2}\arccos x \\ \Rightarrow \arctan\sqrt{\frac{1-x}{1+x}} &= \frac{1}{2}\arccos x \end{aligned}$$

As required

Question 23 (*****)

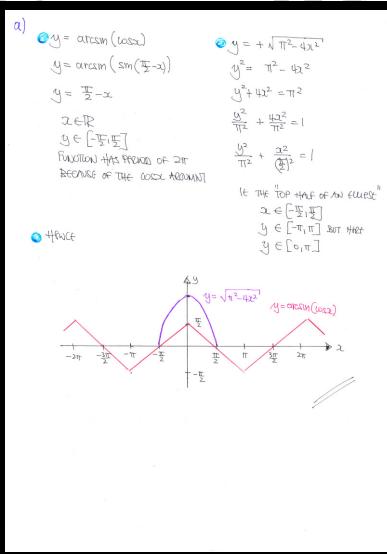
The functions f and g are defined in the largest possible domain by the equations

$$f(x) = \arcsin(\cos x) \quad \text{and} \quad g(x) = \sqrt{\pi^2 - 4x^2}.$$

- a) Sketch the graphs of f and g on the same set of axes.
 b) Use an algebraic method to solve the equation

$$\arcsin(\cos x) = \sqrt{\pi^2 - 4x^2}.$$

$$x = \pm \frac{\pi}{2}$$



b)

$$\begin{aligned} \arcsin(\cos x) &= \sqrt{\pi^2 - 4x^2} \\ \cos x &= \sin(\sqrt{\pi^2 - 4x^2}) \\ \sin(\frac{\pi}{2}-x) &= \sin(\sqrt{\pi^2 - 4x^2}) \\ \Rightarrow \frac{\pi}{2}-x &= \sqrt{\pi^2 - 4x^2} \\ \Rightarrow \frac{\pi^2}{4} - \pi x + x^2 &= \pi^2 - 4x^2 \\ \Rightarrow 5x^2 - \pi x - \frac{\pi^2}{4} &= 0 \\ \Rightarrow 20x^2 - 4\pi x - 3\pi^2 &= 0 \\ \Rightarrow (2x - \pi)(10x + 3\pi) &= 0 \\ x &= \frac{\pi}{2}, -\frac{3\pi}{10} \end{aligned}$$

NOTE THAT BOTH SIDES ARE GIVEN SO POTENTIALLY $\pm \frac{\pi}{2}, 1 \pm \frac{3\pi}{10}$ COULD BE SOLUTIONS BUT DUE TO SQUARING WE MUST CHECK

$$\begin{aligned} \arcsin(\cos(\pm \frac{\pi}{2})) &= \arcsin 0 = 0 & \therefore x = \pm \frac{\pi}{2} \\ \sqrt{\pi^2 - 4(\pm \frac{\pi}{2})^2} &= \sqrt{\pi^2 - \frac{36}{100}\pi^2} = \sqrt{\frac{64}{100}\pi^2} = \frac{8}{5}\pi \end{aligned}$$

NOW WE KNOW FROM THE GRAPH THERE ARE NO MORE SOLUTIONS

$$\begin{aligned} \sqrt{\pi^2 - 4(\pm \frac{3\pi}{10})^2} &= \sqrt{\pi^2 - \frac{36}{100}\pi^2} = \sqrt{\frac{64}{100}\pi^2} = \frac{8}{5}\pi \\ \arcsin(\cos(\pm \frac{3\pi}{10})) &= \arcsin(\cos(\frac{3\pi}{10})) = \arcsin[\sin(\frac{3\pi}{10})] \\ &= \arcsin(\sin \frac{3\pi}{10}) = \frac{3\pi}{10} \\ \frac{3\pi}{10} &\neq \frac{3\pi}{5} \quad \text{SO ONLY SOLUTION} \\ x &= \pm \frac{3\pi}{10} // \end{aligned}$$

Question 24 (***** non calculator

It is given that

$$2\cos \theta + \sin \theta = 1.$$

Determine the possible values of

$$7\cos \theta + 6\sin \theta.$$

 , $7\cos \theta + 6\sin \theta = 2, \quad 7\cos \theta + 6\sin \theta = 6$

METHOD A - BY SQUARING

- Start by squaring the given equation

$$\begin{aligned} &\Rightarrow 2\cos \theta + \sin \theta = 1 \\ &\Rightarrow 2\cos^2 \theta + \sin^2 \theta = 1 \\ &\Rightarrow 4\cos^2 \theta = (1-\sin^2 \theta)^2 \\ &\Rightarrow 4(1-\sin^2 \theta) = 1-2\sin^2 \theta + \sin^4 \theta \\ &\Rightarrow 4-4\sin^2 \theta = 1-2\sin^2 \theta + \sin^4 \theta \\ &\Rightarrow 0 = \sin^4 \theta - 2\sin^2 \theta + 3 \\ &\Rightarrow (\sin^2 \theta - 3)(\sin^2 \theta - 1) = 0 \\ &\Rightarrow \sin^2 \theta = 1 \quad \text{or} \quad \sin^2 \theta = 3 \quad (\text{not possible}) \\ &\Rightarrow \sin \theta = \pm 1 \end{aligned}$$

- Obtain the corresponding solutions for $\cos \theta$

$\cos^2 \theta + \sin^2 \theta = 1$	$\cos^2 \theta + 1 = 1$
$\cos^2 \theta = 0$	$\cos^2 \theta + \frac{1}{2} = 1$
$\cos \theta = 0$	$\cos^2 \theta = \frac{1}{2}$
	$\cos \theta = \pm \frac{1}{\sqrt{2}}$

- Check against the original (due to the squaring)

$\sin \theta = 1, \cos \theta = 0$ is ok	$(2\cos \theta + \sin \theta = 1)$
$\sin \theta = -\frac{1}{2}, \cos \theta = \frac{\sqrt{3}}{2}$ is also ok	$(2\cos \theta - \frac{1}{2} = 1)$
$\sin \theta = -\frac{1}{2}, \cos \theta = -\frac{\sqrt{3}}{2}$ is not ok	$(2\cos \theta - \frac{1}{2} = -\frac{1}{2} \neq 1)$

Thus we obtain

$$7\cos \theta + 6\sin \theta = \boxed{7(\pm 1) + 4(-\frac{1}{2}) = 2 \quad \text{or} \quad 7(\pm 1) + 4(\pm \frac{1}{2}) = 6}$$

METHOD B - BY SOLVING FOR θ

- Write $2\cos \theta + \sin \theta$ in trigonometric form

$$\begin{aligned} 2\cos \theta + \sin \theta &\equiv \sqrt{5}\cos(\theta - \alpha) \\ &\Downarrow \text{BY INSPECTION} \\ &\equiv \sqrt{5}\cos \theta \cos \alpha + \sqrt{5}\sin \theta \sin \alpha \\ &\equiv \sqrt{5}(\cos \theta \cos \alpha + \sin \theta \sin \alpha) \end{aligned}$$

$$\begin{aligned} &\Rightarrow \frac{\sqrt{5}\cos \theta}{\sqrt{5}\cos \alpha} + \frac{\sin \theta}{\cos \alpha} = \frac{1}{\cos \alpha} \\ &\Rightarrow \tan \theta = \frac{1}{\cos \alpha} \\ &\Rightarrow \theta = \arctan \frac{1}{\cos \alpha} \\ &\quad \text{Diagram: } \begin{array}{c} \triangle ABC \\ \angle A = \theta \\ \angle B = \alpha \\ \angle C = 90^\circ \\ AB = 1 \\ AC = \cos \alpha \\ BC = \sin \theta \end{array} \quad \theta = \arctan \frac{BC}{AC} = \arctan \frac{1}{\cos \alpha} \end{aligned}$$

- Returning to the given equation to solve it for θ

$$\begin{aligned} &\Rightarrow 2\cos \theta + \sin \theta = 1 \\ &\Rightarrow \sqrt{5}\cos(\theta - \alpha) = 1 \\ &\Rightarrow \cos(\theta - \alpha) = \frac{1}{\sqrt{5}} \\ &\Rightarrow \theta - \alpha = \arccos \frac{1}{\sqrt{5}} \pm 2\pi n \quad n \in \mathbb{Z}, \alpha \in \dots \\ &\Rightarrow \theta = \alpha + \arccos \frac{1}{\sqrt{5}} \pm 2\pi n \\ &\Rightarrow \theta = \alpha - \arccos \frac{1}{\sqrt{5}} \pm 2\pi n \\ &\Rightarrow \theta = \arccos \frac{1}{\sqrt{5}} + \arccos \frac{1}{\sqrt{5}} \pm 2\pi n \\ &\Rightarrow \theta = \arccos \frac{1}{\sqrt{5}} - \arccos \frac{1}{\sqrt{5}} \pm 2\pi n \end{aligned}$$

- Now substitute each solution into $7\cos \theta + 6\sin \theta$

$$\begin{aligned} 7\cos \theta + 6\sin \theta &= \frac{7}{2} [2\cos \theta + \sin \theta] + \frac{5}{2} \sin \theta \\ &= \frac{7}{2} \times 1 + \frac{5}{2} \sin \theta \\ &= \frac{7}{2} + \frac{5}{2} \sin \theta \end{aligned}$$

- Now let $A = \arccos \frac{1}{\sqrt{5}}$, $B = \arccos \frac{1}{\sqrt{5}}$

Looking at the triangle

$$A + B = \pi$$

Thus we obtain

$$\begin{aligned} \dots \frac{7}{2} + \frac{5}{2} \sin(A+B) &= \frac{7}{2} + \frac{5}{2} \sin \pi \\ &= \frac{7}{2} + \frac{5}{2} \cdot 0 \\ &= 6 \quad \text{Diagram: } \begin{array}{c} \triangle ABC \\ \angle A = \theta \\ \angle B = \alpha \\ \angle C = 90^\circ \\ AB = 1 \\ AC = \cos \theta \\ BC = \sin \theta \end{array} \end{aligned}$$

$$\begin{aligned} \text{OR} \dots \frac{7}{2} + \frac{5}{2} \sin \theta &= \frac{7}{2} + \frac{5}{2} (\sin A \cos B - \cos A \sin B) \\ &= \frac{7}{2} + \frac{5}{2} \left[\frac{1}{\sqrt{5}} \frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}} \frac{1}{\sqrt{5}} \right] \\ &= \frac{7}{2} + \frac{5}{2} \left[\frac{1}{5} - \frac{2}{5} \right] \\ &= \frac{7}{2} + \frac{5}{2} \left(-\frac{1}{5} \right) \\ &= 2. \end{aligned}$$

Question 25 (***)**

The point M is the midpoint of AB , on a triangle ABC .

Given further that

$$\tan[\angle CAM] = \frac{2}{5} \quad \text{and} \quad \tan[\angle CBM] = \frac{2}{3},$$

Use trigonometric identities to find the value of $\tan[\angle CMB]$.

 , $\tan[\angle CMB] = 2$

Start by labelling all the angles in the figure in terms of α, β, θ & γ

Looking at $\triangle ACM$

$$\frac{\sin \alpha}{x} = \frac{\sin(\theta - \alpha)}{y}$$

$$\Rightarrow \frac{x}{y} = \frac{\sin(\theta - \alpha)}{\sin \alpha}$$

$$\Rightarrow \frac{x}{y} = \frac{\sin \theta \cos \alpha - \cos \theta \sin \alpha}{\sin \alpha}$$

$$\Rightarrow \frac{x}{y} = \sin \theta \cot \alpha - \cos \theta$$

$$\Rightarrow \frac{x}{y} = \frac{2}{3} \sin \theta - \cos \theta$$

Looking at $\triangle BCM$

$$\frac{\sin \theta}{x} = \frac{\sin(\gamma - \theta - \beta)}{y}$$

$$\Rightarrow \frac{\sin \theta}{x} = \frac{\sin(\theta + \beta)}{y}$$

$$\Rightarrow \frac{x}{y} = \frac{\sin(\theta + \beta)}{\sin \theta}$$

$$\Rightarrow \frac{x}{y} = \frac{\sin \theta \cos \beta + \cos \theta \sin \beta}{\sin \theta}$$

Simplifying expressions for $\frac{x}{y}$ terms

$$\Rightarrow \frac{x}{y} = \cos \theta + \cot \theta \sin \theta$$

$$\Rightarrow \frac{x}{y} = \cos \theta + \frac{3}{2} \sin \theta$$

$$\Rightarrow \sin \theta = 2 \cos \theta$$

$$\Rightarrow \tan \theta = 2$$

Question 26 (*****)

Show, with detailed workings, that if $\sin 2x = \frac{2}{3}$ then

$$\cos^6 x + \sin^6 x$$

also equals to $\frac{2}{3}$.

V, **□**, **proof**

This response is correct - BUT THE STAFF IS NOW TELETYPEING

$$\begin{aligned} A^2 \pm B^2 &\equiv (A \mp B)(A^2 \pm AB + B^2) \\ \Rightarrow \cos^2 x + \sin^2 x &= (\cos x)^2 + (\sin x)^2 \\ &= (\cos x \cancel{\cos^2 x}) \cancel{(\cos x - \sin x \cos x + \sin^2 x)} \\ &= 1 \times (\cos^2 x - \cos x \sin x + \sin^2 x) \\ &= (\cos x)^2 - (\cos x \sin x) + (\sin x)^2 \end{aligned}$$

NOW REARRANGE THE IDENTITY $(A+B)^2 = A^2 + 2AB + B^2$

$$\begin{aligned} &= [(\cos x)^2 + 2(\cos x)(\sin x) + (\sin x)^2] - 3(\cos x)(\sin x) \\ &= [\cos^2 x + \sin^2 x]^2 - 3(\cos x \sin x)^2 \\ &= 1^2 - \frac{3}{4}(\cos 2x)^2 \\ &= 1 - \frac{3}{4}(\frac{1}{2})^2 \\ &= 1 - \frac{3}{4} \cdot \frac{1}{4} \\ &= 1 - \frac{3}{16} \\ &= \frac{13}{16} \\ &= \frac{2}{3} \quad \text{As required} \end{aligned}$$

Question 27 (***)**

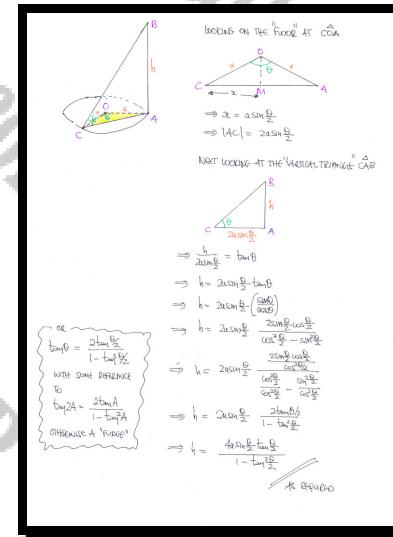
A pole AB , of height h , is standing vertically on level horizontal ground with A on the circumference of a circle of radius a , centred at the point O .

The point C is another point on the circumference of this circle so that $\angle COA = \theta$ and $\angle ACB = \theta$.

Use a detailed method to show that

$$h = \frac{4a \sin\left(\frac{1}{2}\theta\right) \tan\left(\frac{1}{2}\theta\right)}{1 - \tan^2\left(\frac{1}{2}\theta\right)}$$

, proof



Question 28 (*****)

A triangle has angles of 36° , 72° and 72° .

By suitably partitioning this triangle and using similar triangles, show that

$$\sin 18^\circ + \cos 36^\circ = \frac{1}{2}\sqrt{5} \quad \text{and} \quad \sin^2 36^\circ + \cos^2 18^\circ = \frac{5}{4}$$

$$\boxed{}, \boxed{-\frac{9}{16}}$$

SPLIT WITH THE TRIANGLE SUGGESTED: $\triangle ABC$ WHERE $BAC=36^\circ$

- Let $AB=AC=1$
- Let $BC=x$
- BISECT THE ANGLE $\angle A$ WITH BD

THEN TRANSFER ALL THE INFORMATION INTO 3 SEPARATE FIGURES

ALTHOUGH THE TRIANGLES ARE NOT TO SCALE [BOTH] & [C], DRAW THE 3RD FIGURE AND THE INEQUALITY $AB \approx AC$ FROM THE 2ND FIGURE - RESEMBLE FORMS!

BY SIMILAR TRIANGLES

$$\Rightarrow \frac{x}{1-x} = \frac{1}{x}$$

$$\Rightarrow x^2 = 1-x$$

$$\Rightarrow x^2+x-1=0$$

$$\Rightarrow (x+1)(x-1)=0$$

$$\Rightarrow x+1=0 \text{ or } x-1=0$$

$$\Rightarrow x=-1 \text{ or } x=1$$

$$\therefore x=1$$

REDUCE THE ORIGINAL TRIANGLE

- $\sin 18^\circ = \frac{AC}{BC} = \frac{1+\sqrt{5}}{4}$
- $\cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - 2\left(\frac{1+\sqrt{5}}{4}\right)^2 = 1 - 2\left(\frac{1+2\sqrt{5}+5}{16}\right) = 1 - \frac{1+2\sqrt{5}+5}{8} = 1 - \frac{3+2\sqrt{5}}{8} = \frac{4-2\sqrt{5}}{8} = \frac{1-\sqrt{5}}{2}$
- $\sin^2 36^\circ + \cos^2 36^\circ = \frac{1+\sqrt{5}}{4} + \frac{1-\sqrt{5}}{2} = \frac{2\sqrt{5}}{4} = \frac{\sqrt{5}}{2}$

Also we have, using $\cos^2 \theta + \sin^2 \theta = 1$

- $\cos^2 18^\circ = 1 - \sin^2 18^\circ = 1 - \left(\frac{1+\sqrt{5}}{4}\right)^2 = 1 - \frac{1+\sqrt{5}+5}{16} = \frac{16-1-\sqrt{5}-5}{16} = \frac{10-2\sqrt{5}}{16} = \frac{5-\sqrt{5}}{8}$
- $\sin^2 36^\circ = 1 - \cos^2 36^\circ = 1 - \left(\frac{1-\sqrt{5}}{2}\right)^2 = 1 - \frac{1-\sqrt{5}+5}{16} = \frac{16-1+\sqrt{5}-5}{16} = \frac{10+2\sqrt{5}}{16} = \frac{5+\sqrt{5}}{8}$
- $\cos 18^\circ + \sin 36^\circ = \frac{1+\sqrt{5}}{4} + \frac{5+\sqrt{5}}{8} = \frac{5}{8} + \frac{\sqrt{5}}{2}$
- $\cos 18^\circ + \sin 36^\circ = \frac{5}{8} = \frac{1}{2}$ (REQUIRED)

Question 329 (***** non calculator

Solve the trigonometric equation

$$\sin(y-30) = \sin(y-45), \quad 0 \leq y < 360^\circ.$$

$$\boxed{y = 82.5^\circ, 262.5^\circ}$$

$$\begin{aligned} \sin(y-30) &= \sin(y+45) \\ \Rightarrow (y-30) &= (y+45) \pm 360^\circ \quad \text{W.C.}(12, 3, \dots) \\ \Rightarrow y-30 &= 180-(y+45) \pm 360^\circ \\ \Rightarrow \text{INCORRECT!} \quad y-30 &= 135-y \pm 360^\circ \\ \Rightarrow 2y &= 165 \pm 360^\circ \\ \Rightarrow y &= 82.5 \pm 180^\circ \\ \therefore y &= 82.5^\circ, 262.5^\circ \end{aligned}$$

Question 30 (*****)

$$\cot^2 x - \tan^2 x = 8 \cot 2x, \quad 0^\circ < x < 180^\circ.$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$x = 15^\circ, 45^\circ, 75^\circ, 135^\circ$$

$$\begin{aligned}
 \cot^2 x - \tan^2 x &= 8 \cot 2x \\
 \Rightarrow \cot x - \tan x &= \frac{8}{\tan 2x} \\
 \Rightarrow \frac{1}{\tan x} - \frac{1}{\cot x} &= \frac{8(1-\tan^2 x)}{2\tan x} \quad \text{where } T = \tan x \\
 \Rightarrow \frac{1}{T^2} - T^2 &= \frac{8(1-T^2)}{2T} \\
 \Rightarrow 1 - T^4 &= 4T(1-T^2) \\
 \Rightarrow (1-T^2)(1+T^2) &= 4T(1-T^2) \\
 \Rightarrow (1-T^2)(1+T^2) - 4T(1-T^2) &= 0 \\
 \Rightarrow (1-T^2)[(1+T^2)-4T] &= 0 \\
 \Rightarrow (1-T^2)(T^2-4T+1) &= 0 \\
 \Rightarrow (1-T)(1+T)[(T-2)^2-3] &= 0 \\
 \Rightarrow (1-T)(1+T)[(T-2)^2-6\sqrt{3}] &= 0 \\
 \Rightarrow (1-T)(1+T)[T-2-6\sqrt{3}][T-2+6\sqrt{3}] &= 0 \\
 \Rightarrow \tan x &= \begin{cases} 1 \\ -1 \\ \frac{2+\sqrt{3}}{2-\sqrt{3}} \\ \frac{2-\sqrt{3}}{2+\sqrt{3}} \end{cases} \\
 \Rightarrow \begin{cases} x_1 = 45^\circ \pm 180^\circ \\ x_2 = -45^\circ \pm 180^\circ \\ x_3 = 15^\circ \pm 180^\circ \\ x_4 = 75^\circ \pm 180^\circ \end{cases} & \text{where } x_1 = 45^\circ, x_2 = 135^\circ, x_3 = 15^\circ, x_4 = 75^\circ
 \end{aligned}$$

Question 31 (***)**

Solve the following trigonometric equation for $0 \leq x < 360^\circ$

$$\tan x \sec x + \frac{1}{1+\sin x} = \frac{4}{3}.$$

$$x = 30^\circ, 150^\circ, 210^\circ, 330^\circ$$

METHOD THROUGH BY $3(\cos x)$

$$\begin{aligned} & 3\tan x \sec x + \frac{1}{1+\sin x} = \frac{4}{3} \\ \Rightarrow & 3 \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} + 3 = 4(1+\sin x) \\ \Rightarrow & \frac{3\sin x}{\cos^2 x} + 3 = 4(1+\sin x) \\ \Rightarrow & 3\sin x(1+\sin x) + 3\cos^2 x = 4\cos^2 x(1+\sin x) \\ \Rightarrow & 3\sin x + 3\sin^2 x + 3\cos^2 x = 4\cos^2 x(1+\sin x) \\ \Rightarrow & 3\sin x + 3\sin^2 x + 3\cos^2 x = 4(1-\sin^2 x)(1+\sin x) \\ \Rightarrow & 3\sin x + 3 = 4(1-\sin^2 x)(1+\sin x) \\ \Rightarrow & 3(4\cos^2 x + 1) = 4(1-\sin^2 x)(1+\sin x) \\ \Rightarrow & 3\cos^2 x + 1 = 0 \Rightarrow \cos^2 x = 0 \quad \therefore \sec x = \infty \\ \Rightarrow & \text{SO WE MAY DIVIDE IT THROUGH} \\ \Rightarrow & 3 = 4(1-\sin^2 x) \\ \Rightarrow & \frac{3}{4} = 1 - \sin^2 x \\ \Rightarrow & \sin^2 x = \frac{1}{4} \\ \Rightarrow & \sin x = \pm \frac{1}{2} \\ \end{aligned}$$

($x = 30^\circ \pm 360^\circ$) ($x = 150^\circ \pm 360^\circ$) $n=0,1,2,3,\dots$

($x = -30^\circ \pm 360^\circ$) ($x = 210^\circ \pm 360^\circ$) $n=0,1,2,3,$

$x = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

ALTERNATIVE

$$\begin{aligned} & \tan x \sec x + \frac{1}{1+\sin x} = \frac{4}{3} \\ \Rightarrow & \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} + \frac{1}{1+\sin x} = \frac{4}{3} \\ \Rightarrow & \frac{\sin x}{\cos^2 x} + \frac{1}{1+\sin x} = \frac{4}{3} \\ \Rightarrow & \frac{\sin x(1+\sin x) + \cos^2 x}{\cos^2 x(1+\sin x)} = \frac{4}{3} \\ \Rightarrow & \frac{\sin x + \sin^2 x + \cos^2 x}{\cos^2 x(1+\sin x)} = \frac{4}{3} \\ \Rightarrow & \frac{\sin x + 1}{\cos^2 x(1+\sin x)} = \frac{4}{3} \quad (\sin^2 x + \cos^2 x = 1) \end{aligned}$$

$\therefore \frac{1}{\cos^2 x} = \frac{3}{4}$

$\Rightarrow \cos^2 x = \frac{4}{3}$

$\Rightarrow \cos x = \pm \sqrt{\frac{4}{3}}$

$\therefore x = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

Question 32 (***) non calculator**

Given that $\alpha = \arctan \frac{1}{2}$ and $\beta = \arctan \frac{9}{13}$, find the value of $\tan(3\alpha - \beta)$.

$$\tan(3\alpha - \beta) = 1$$

$\bullet \tan 3A = \tan(2A+A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \frac{2\tan A + \tan A}{1 - 2\tan^2 A} = \frac{3\tan A + \tan A}{1 - 3\tan^2 A}$

METHOD TOP OF BOTTOM OF THE FRACTION BY $1-\tan^2 A$...

$$\begin{aligned} & = \frac{3\tan A + \tan A(1-\tan^2 A)}{1 - 3\tan^2 A} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \\ \therefore \tan 3A & = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \\ \text{HENCE } \bullet \tan(3\alpha - \beta) & = \frac{\tan 3\alpha - \tan \beta}{1 + \tan 3\alpha \tan \beta} = \frac{3\tan \alpha - \tan^3 \alpha - \tan \beta}{1 + 3\tan^2 \alpha \tan \beta - \tan \beta} \end{aligned}$$

BUT: $\alpha = \arctan \frac{1}{2} \Rightarrow \tan \alpha = \frac{1}{2}$

$$\begin{aligned} & \beta = \arctan \frac{9}{13} \Rightarrow \tan \beta = \frac{9}{13} \\ \therefore & = \frac{\frac{3}{2} - \frac{1}{8}}{1 + \frac{3}{2} \times \frac{9}{13}} = \frac{\frac{12-1}{8}}{\frac{8+27}{26}} = \frac{\frac{11}{8}}{\frac{35}{26}} = \frac{11}{28} = \frac{11}{28} \times \frac{9}{13} = \frac{99}{364} \\ & = \frac{11 \times 13 - 18}{28 + 99} = \frac{143 - 18}{127} = \frac{125}{127} = 1 \end{aligned}$$

Question 33 (*****)

It is given that for some value of the constant k

$$\cos 4x \equiv 1 + k \sin^2 x \cos^2 x.$$

- a) Determine the value of k .
- b) Hence, or otherwise, show clearly that for $x \in \mathbb{R}$

$$-\frac{3}{4} \leq 4\cos^2 2x - 3\sin^2 x \cos^2 x \leq 4.$$

$$\boxed{\quad}, k = -8$$

(a) $\begin{aligned} \cos 4x &= \cos[2(2x)] \\ &= 1 - 2\sin^2 2x \\ &= 1 - 2(\sin 2x)^2 \\ &= 1 - 2(2\sin x \cos x)^2 \\ &= 1 - 2(4\sin^2 x \cos^2 x) \\ &= 1 - 8\sin^2 x \cos^2 x \end{aligned}$ $\therefore k = -8$

(b) $\begin{aligned} f(x) &= 4\cos^2 2x - 3\sin^2 x \cos^2 x \\ f(x) &= \frac{13}{8} + \frac{19}{8}\sin^2 2x \\ \text{But } -1 \leq \cos 2x \leq 1 \\ f(x)_{\max} &= \frac{13}{8} + \frac{19}{8} \times 1 = 4 \\ f(x)_{\min} &= \frac{13}{8} + \frac{19}{8} \times (-1) = -\frac{3}{4} \\ \therefore -\frac{3}{4} \leq 4\cos^2 2x - 3\sin^2 x \cos^2 x \leq 4 \end{aligned}$

Question 34 (***** non calculator

Solve the following trigonometric equation

$$\cos(\psi - 60^\circ) = \cos(\psi - 45^\circ), \quad 0^\circ \leq \psi < 360^\circ.$$

$$\boxed{\psi = 52.5^\circ, 232.5^\circ}$$

$$\begin{aligned} \cos(\psi - 60^\circ) &= \cos(\psi - 45^\circ) \\ \Rightarrow (\psi - 60) &= \psi - 45 \pm 360^\circ \quad n=0,1,2,3,\dots \\ \Rightarrow \psi - 60 &= +45 - \psi \pm 360^\circ \\ \Rightarrow 2\psi &= 105^\circ \pm 360^\circ \\ \Rightarrow \psi &= 52.5^\circ \pm 180^\circ \\ \therefore \psi &= 52.5^\circ, 232.5^\circ \end{aligned}$$

Question 35 (*****)

$$S = \sin(10^\circ) \sin(30^\circ) \sin(50^\circ) \sin(70^\circ)$$

Use the **sine double angle identity** for sine to show that $S = \frac{1}{16}$.

V, X, proof

SINE WITH THE IDENTITY $\sin\theta = \cos(90^\circ - \theta)$

$$\Rightarrow S_1 = \sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$$
$$\Rightarrow S_1 = \cos 80^\circ \cos 60^\circ \cos 40^\circ \cos 20^\circ$$

USING THE SINE DOUBLE ANGLE

- $\sin 2\theta = 2 \sin \theta \cos \theta$
- $\cos 2\theta = \frac{\sin^2 \theta}{\sin^2 \theta}$

$$\Rightarrow S_1 = \frac{\sin 10^\circ \times \sin 20^\circ}{2 \sin 10^\circ} \times \frac{\sin 50^\circ \times \sin 10^\circ}{2 \sin 50^\circ} \times \frac{\sin 70^\circ \times \sin 20^\circ}{2 \sin 70^\circ}$$
$$\Rightarrow S_1 = \frac{\sin 10^\circ \sin 20^\circ}{16 \sin 10^\circ \sin 20^\circ}$$

But $\sin \theta = \sin(180^\circ - \theta)$

$$\Rightarrow S_1 = \frac{\sin 10^\circ \sin 10^\circ}{16 \sin 10^\circ \sin 10^\circ}$$
$$\Rightarrow S_1 = \frac{1}{16}$$

ANSWER

Question 36 (***)**

Solve the trigonometric equation

$$(\cos 4x + \cos x)^2 + (\sin 4x + \sin x)^2 = 2\sqrt{3} \sin 3x, \quad 0 \leq x < \pi,$$

giving the answers in terms of π .

$$\boxed{\quad}, \boxed{\frac{\pi}{9}, \frac{\pi}{3}, \frac{7\pi}{9}}$$

Start by expanding the LHS

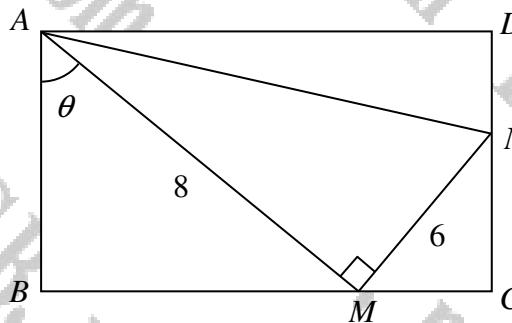
$$\begin{aligned} &\Rightarrow (\cos 4x + \cos x)^2 + (\sin 4x + \sin x)^2 = 2\sqrt{3} \sin 3x \\ &\Rightarrow (\cos^2 4x + 2\cos 4x \cos x + \cos^2 x) + (\sin^2 4x + 2\sin 4x \sin x + \sin^2 x) = 2\sqrt{3} \sin 3x \\ &\Rightarrow 1 + 2[\cos(4x)\cos x + \sin(4x)\sin x] + 1 = 2\sqrt{3} \sin 3x \\ &\Rightarrow 1 + 2\cos(4x-x) + 1 = 2\sqrt{3} \sin 3x \\ &\Rightarrow 2 + 2\cos 3x = 2\sqrt{3} \sin 3x \\ &\Rightarrow 1 + \cos 3x = \sqrt{3} \sin 3x \\ &\Rightarrow \sqrt{3} \sin 3x - \cos 3x = 1 \end{aligned}$$

Now we may proceed by an "E TRANSFORMATION" as follows

$$\begin{aligned} &\Rightarrow \frac{\sqrt{3}}{2} \sin 3x - \frac{1}{2} \cos 3x = \frac{1}{2} \\ &\Rightarrow \cos \frac{\pi}{6} \sin 3x - \sin \frac{\pi}{6} \cos 3x = \frac{1}{2} \\ &\Rightarrow \sin(3x - \frac{\pi}{6}) = \frac{1}{2} \\ &\Rightarrow \arcsin(\frac{1}{2}) = \frac{\pi}{6} \\ &\Rightarrow \begin{cases} 3x - \frac{\pi}{6} = \frac{\pi}{6} \pm 2n\pi \\ 3x - \frac{\pi}{6} = \frac{5\pi}{6} \pm 2n\pi \end{cases} \quad n=0, \pm 1, \dots \end{aligned}$$

$$\begin{aligned} &\Rightarrow \begin{cases} 3x = \frac{\pi}{3} \pm 2n\pi \\ 3x = \pi \pm 2n\pi \end{cases} \\ &\Rightarrow \begin{cases} x = \frac{\pi}{9} \pm \frac{2n\pi}{3} \\ x = \frac{\pi}{3} \pm \frac{2n\pi}{3} \end{cases} \\ &\leftarrow x = 2n^\circ \pm 120^\circ \\ &\downarrow \\ &\therefore x = \frac{\pi}{9}, \frac{\pi}{3}, \frac{7\pi}{9} // \end{aligned}$$

Question 37 (*****)



The figure above shows a rectangle $ABCD$.

The points M and N lie on BC and CD respectively.

The angle AMN is 90° , $|AM|=8$ and $|MN|=6$. The angle BAM is denoted by θ .

- Given that the perimeter of the rectangle $ABCD$ is fixed at 24 units, determine the possible value(s) of θ .
- Given instead that the perimeter of the rectangle $ABCD$ can vary, determine the largest possible area of the triangle ADN .

$$\theta \approx 71.7^\circ, \text{ area}_{\max} = 25$$

a)

By EUCLIDEAN GEOMETRY
 $\angle NMC = \theta$

$$\begin{aligned} & (BC) - (BM) - (NC) = BM\sin\theta + NC\cos\theta \\ & AB = BC \\ & AB + BC = (BS\sin\theta + NC\cos\theta) + (BN\cos\theta) \\ & = BM\sin\theta + NC\cos\theta \\ & \therefore \text{PERIMETER} = 16\sin\theta + 24\cos\theta \\ & 24 = 16\sin\theta + 24\cos\theta \\ & 4\sin\theta + 7\cos\theta = 6 \end{aligned}$$

BY R-TRANSFORMATION

$$\begin{aligned} 4\sin\theta + 7\cos\theta &= R \sin(\theta + \alpha) \\ &\equiv R \sin(\theta + 60.25^\circ) \\ &\equiv (R\cos\alpha + (R\sin\alpha))\cos\theta \\ \therefore R\cos\alpha = 4 &\Rightarrow R = \sqrt{4^2 + 7^2} = \sqrt{65} \\ R\sin\alpha = 7 &\Rightarrow \tan\alpha = \frac{7}{4} \quad \therefore \alpha \approx 60.25^\circ \\ \therefore \sqrt{65}\sin(\theta + 60.25^\circ) &= 6 \\ \sin(\theta + 60.25^\circ) &= 0.794... \\ (\theta + 60.25^\circ) &= 48.91... \quad \therefore \theta = 21.09^\circ \\ (\theta + 60.25^\circ) &= 139.98... \quad \therefore \theta = 79.71^\circ \\ (\theta) &= -12.41 \pm 360^\circ \\ (\theta) &= 71.59^\circ \pm 360^\circ \quad \therefore \theta = 71.7^\circ \quad \text{ONLY POSSIBLE ANGLE} \end{aligned}$$

b)

$$\begin{aligned} \text{Area of } ADN &= \frac{1}{2}|DN||AD| \\ &= \frac{1}{2}[AB - NC][BC]\sin\theta \\ &= \frac{1}{2}(BS\cos\theta)[(BN\cos\theta)]\sin\theta \\ &= \frac{1}{2}[(BS\cos\theta)(BN\cos\theta)\sin\theta] \\ &= \frac{1}{2}[4\sin\theta \cdot 7\cos\theta \sin\theta] \\ &= 7\sin^2\theta + 14\sin\theta\cos\theta \end{aligned}$$

BY R-TRANSFORMATION

$$\begin{aligned} R &= \sqrt{N^2 + 25} = 25 \\ \therefore \text{MAX AREA IS } 25 \end{aligned}$$

Question 38 (***** non calculator

$$\tan 2x^\circ + \tan 2x^\circ \tan 25^\circ = 1 - \tan 25^\circ, \quad 0^\circ \leq x < 360^\circ.$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$x = 10^\circ, 100^\circ, 190^\circ, 280^\circ$$

Question 39 (*****)

$$2 \tan x - \sin 2x = \sin^2 x, \quad 0^\circ \leq x < 360^\circ.$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$x = 0^\circ, 26.6^\circ, 180^\circ, 206.6^\circ$$

Question 40 (***)** non calculator

Solve the following trigonometric equation

$$\sin(\theta - 20^\circ) = \sin(\theta + 60^\circ), \quad 0^\circ \leq \theta < 360^\circ.$$

$$\boxed{\theta = 70^\circ, 250^\circ}$$

$$\begin{aligned} \sin(\theta - 20^\circ) &= \sin(\theta + 60^\circ) \\ \Rightarrow \theta - 20 &= (\theta + 60) \pm 360^\circ \\ \Rightarrow \theta - 20 &= 180 - (\theta + 60) \pm 360^\circ \quad n=0,1,2,3 \\ \Rightarrow \theta - 20 &= 120 - \theta \pm 360^\circ \\ \Rightarrow 2\theta &= 1440 \pm 360^\circ \\ \Rightarrow \theta &= 720 \pm 180^\circ \\ \therefore \theta_1 &= 72^\circ \\ \theta_2 &= 250^\circ \end{aligned}$$

Question 41 (***)**

The three angles in a triangle ABC satisfy the relationship

$$\sin(2A - B) - \sin(B + C) = \cos A \sin(A - B).$$

Show that the triangle ABC is isosceles.

proof

$$\begin{aligned} \sin(2A - B) - \sin(B + C) &= \cos A \sin(A - B) \\ \Rightarrow \sin(2A - B) - \sin(B + 180^\circ - B - A) &= \cos A \sin(A - B) \quad 4+A+B+C=180 \\ \Rightarrow \sin(2A - B) - \sin(180^\circ - A) &= \cos A \sin(A - B) \quad C=180-B-A \\ \Rightarrow \sin(2A - B) - [\sin(180^\circ - A) - \cos A \sin(A - B)] &= \cos A \sin(A - B) \\ \Rightarrow \sin(2A - B) - \sin A + \cos A \sin(A - B) &= \cos A \sin(A - B) \\ \Rightarrow \sin[2A - (A - B)] - \sin A + \cos A \sin(A - B) &= \cos A \sin(A - B) \\ \Rightarrow \sin[A + (A - B)] - \sin A + \cos A \sin(A - B) &= \cos A \sin(A - B) \\ \Rightarrow \sin A \cos(A - B) + \cos A \sin(A - B) - \sin A &= \cos A \sin(A - B) \\ \Rightarrow \sin A [\cos(A - B) - 1] &= 0 \\ \text{Since } \sin A \neq 0 \Rightarrow A = -360^\circ, -180^\circ, 0^\circ, 180^\circ, 360^\circ &\quad \because \sin A \neq 0 \\ \cos(A - B) - 1 &= 0 \\ \cos(A - B) &= 1 \\ A - B &= -360^\circ, 0^\circ, 360^\circ, \dots \\ A - B &= 0 \\ A = B &\quad \therefore \text{isosceles} \end{aligned}$$

Question 42 (***)**

It is given that

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

Use the above trigonometric identity to show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x,$$

and hence find

$$\int \sqrt[3]{3\sin 2x - 2\sin 3x \cos x} dx.$$

$$-\frac{3}{2}\sin^{\frac{4}{3}} x + C$$

$$\begin{aligned}\sin 3x &= \sin(2x+x) = \sin 2x \cos x + (\cos 2x) \sin x \\&= (2\sin x \cos x) \cos x + (1-2\cos^2 x) \sin x \\&= 2\sin x \cos^2 x + \sin x - 2\sin x \\&= 2\sin x (1-\sin^2 x) + \sin x - 2\sin x \\&= 2\sin x - 2\sin^3 x + \sin x - 2\sin x \\&= 3\sin x - 4\sin^3 x.\end{aligned}$$

$$\begin{aligned}&\int (3\sin x - 2\sin^3 x)^{\frac{1}{3}} dx \quad \text{[Substitute } u = 3\sin x - 4\sin^3 x\text{]} \\&= \int (6\sin x \cos x - 2(3\sin x - 4\sin^3 x) \sin x)^{\frac{1}{3}} dx \\&= \int (6\sin x \cos x - 6\sin^2 x + 8\sin^4 x)^{\frac{1}{3}} dx \\&= \int 2\sin x (\cos x)^{\frac{1}{3}} dx \\&= \frac{2}{\frac{3}{2}} (\cos x)^{\frac{4}{3}} + C \\&= -\frac{3}{2}(\cos x)^{\frac{4}{3}} + C = \left(-\frac{3}{2}\cos^{\frac{4}{3}} x + C\right)\end{aligned}$$

Question 43 (*****)

It is given that a , b and c are consecutive terms of an arithmetic progression.

It is further given that

$$a \cos^2 \frac{x}{2} - (2a+c) \sin^2 \frac{x}{2} = a \cos x - b(1+\sin x), \quad x \in \mathbb{R}.$$

Show clearly that

$$\tan x = -1.$$

• IF a, b, c ARE CONSECUTIVE TERMS OF AN ARITHMETIC PROGRESSION

$$\begin{aligned} &\Rightarrow b-a = c-b \\ &\Rightarrow 2b = a+c \\ &\Rightarrow 2b+a = 2a+c \end{aligned}$$

• Hence, we now have

$$\begin{aligned} &\Rightarrow a \cos^2 \frac{x}{2} - (2a+c) \sin^2 \frac{x}{2} = a \cos x - b(1+\sin x) \\ &\Rightarrow a \cos^2 \frac{x}{2} - (2b+a) \sin^2 \frac{x}{2} = a \cos x - b(1+\sin x) \\ &\Rightarrow a \cos^2 \frac{x}{2} - a \sin^2 \frac{x}{2} - 2b \sin^2 \frac{x}{2} = a \cos x - b(1+\sin x) \\ &\Rightarrow a(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}) - 2b \sin^2 \frac{x}{2} = a \cos x - b(1+\sin x) \\ &\quad \text{as } \cos^2 \theta \equiv \cos^2 \theta - \sin^2 \theta \\ &\Rightarrow a \cos x - 2b \sin^2 \frac{x}{2} = a \cos x - b(1+\sin x) \\ &\Rightarrow 2 \sin^2 \frac{x}{2} = 1 + \sin x \\ &\quad \boxed{\cos^2 \theta \equiv 1 - \sin^2 \theta} \\ &\quad \boxed{2 \sin \theta \equiv -1 - \cos \theta} \\ &\Rightarrow 1 - \cos x = 1 + \sin x \\ &\Rightarrow \sin x = -\cos x \\ &\Rightarrow \tan x = -1 \quad \text{As required} \end{aligned}$$

Question 44 (*****)

Prove the validity of each of the following trigonometric identities.

a) $\tan 3x \equiv \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

b) $\frac{2\sec^2 \theta - \cos 2\theta - 1}{2\tan \theta + \sin 2\theta} \equiv \tan \theta$.

proof

(a) LHS = $\tan 3x = \tan(2x+x) = \frac{\tan 2x + \tan x}{1 - (\tan 2x)\tan x} = \frac{2\tan x + \tan x}{1 - 2\tan^2 x} \tan x$

MULTIPLY TOP & BOTTOM OF THE REDUCED FRACTION BY $(1-\tan^2 x)$

$$\begin{aligned} &= \frac{2\tan x + \tan x(1-\tan^2 x)}{1 - \tan^2 x - 2\tan^2 x} = \frac{2\tan x + \tan x - \tan^3 x}{1 - 3\tan^2 x} \\ &= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = \text{RHS} \end{aligned}$$

(b) LHS = $\frac{2\sec^2 \theta - \cos 2\theta - 1}{2\tan \theta + \sin 2\theta} = \frac{2\sec^2 \theta - (2\cos^2 \theta - 1)}{2\tan \theta + 2\sin \theta \cos \theta} = \frac{2\sec^2 \theta - 2\cos^2 \theta + 1}{2\tan \theta + 2\sin \theta \cos \theta}$

MULTIPLY TOP & BOTTOM OF THE REDUCED FRACTION BY $\cos^2 \theta$

$$\begin{aligned} &= \frac{1 - \cos^2 \theta}{\sin^2 \theta + \cos^2 \theta} = \dots \text{Difference of squares} = (1 - \cos^2 \theta) \\ &= \frac{(1 - \cos^2 \theta)(1 + \cos^2 \theta)}{\sin^2 \theta(1 + \cos^2 \theta)} = \frac{1 - \cos^2 \theta}{\sin^2 \theta} = (1 - \cos^2 \theta)(1 + \cos^2 \theta) \\ &= \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta = \text{RHS} \end{aligned}$$

Question 45 (***** non calculator

Solve the trigonometric equation

$$\cos(\psi - 36^\circ) = \cos(\psi - 72^\circ), \quad 0^\circ \leq \psi < 360^\circ$$

$\psi = 54^\circ, 234^\circ$

$$\begin{aligned} \cos(\psi - 36^\circ) &= \cos(\psi - 72^\circ) \\ \Rightarrow \psi - 36^\circ &= \psi - 72^\circ \pm 360^\circ \\ \Rightarrow \psi - 36^\circ &= 72^\circ - \psi \pm 360^\circ \\ &\quad M = 0, 1, 2, \dots \\ \Rightarrow 2\psi &= 108^\circ \pm 360^\circ \\ \psi &= 54^\circ \pm 180^\circ \\ \therefore \psi_1 &= 54^\circ \\ \psi_2 &= 234^\circ \end{aligned}$$

Question 46 (***)**

Solve the trigonometric equation

$$\sin x + \sin 2x + \sin 3x + \sin 4x = 0, \quad 0 \leq x \leq \pi,$$

giving the answers in terms of π .

$$x = 0, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{5}, \pi$$

$$\begin{aligned}
 & \sin x + \sin 2x + \sin 3x + \sin 4x = 0 \\
 \Rightarrow & (\sin x + \sin 4x) + (\sin 2x + \sin 3x) = 0 \\
 \text{using } & \sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \\
 \Rightarrow & 2 \sin \left(\frac{5x}{2} \right) \cos \left(-\frac{3x}{2} \right) + 2 \sin \left(\frac{5x}{2} \right) \cos \left(-\frac{x}{2} \right) = 0 \\
 \Rightarrow & \sin \left(\frac{5x}{2} \right) \cos \left(\frac{3x}{2} \right) + \sin \left(\frac{5x}{2} \right) \cos \left(\frac{x}{2} \right) = 0 \quad (\cos(-\theta) = \cos(\theta)) \\
 \Rightarrow & \sin \left(\frac{5x}{2} \right) [\cos \left(\frac{3x}{2} \right) + \cos \left(\frac{x}{2} \right)] = 0 \\
 \text{using } & \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \\
 \Rightarrow & \sin \left(\frac{5x}{2} \right) \times 2 \cos \left(\frac{2x}{2} \right) \cos \left(\frac{4x}{2} \right) = 0 \\
 \Rightarrow & \cos \left(\frac{5x}{2} \right) \cos \left(\frac{2x}{2} \right) \sin \left(\frac{4x}{2} \right) = 0 \\
 \text{cancel } & \cos \frac{5x}{2} = 0 \quad \text{as all of its solutions fall}\\
 & \text{included in } \cos x = 0 \\
 \left(\alpha \right) & = \frac{\pi}{2} \pm 2\pi k \quad \left(\beta \right) = 0 \pm 2\pi k \\
 \left(\alpha \right) & = \frac{3\pi}{2} \pm 2\pi k \quad \left(\beta \right) = \pi \pm 2\pi k \\
 & (\alpha = 0 \pm \frac{2\pi k}{5}) \\
 & (\alpha = \frac{3\pi}{2} \pm \frac{2\pi k}{5}) \\
 \therefore & \alpha = \frac{\pi}{2}, 0, \frac{4\pi}{5}, \frac{2\pi}{5}, \pi
 \end{aligned}$$

Question 47 (*****)

Use the substitution $t = \tan x$ to show that if

$$10\sin 2x = 3 + 10\tan x,$$

then either $\tan x = \frac{1}{2}$ or $\tan x = \frac{-2 \pm \sqrt{19}}{5}$.

proof

$$\begin{aligned} 10\sin 2x &= 10\tan x + 3 \\ \text{BY } 4\sin^2 t \text{ IDENTITIES IF } t = \tan x, \quad \sin 2x &= \frac{2t}{1+t^2} \\ \rightarrow 10\left(\frac{2t}{1+t^2}\right) &= 10t + 3 \\ \rightarrow \frac{20t}{1+t^2} &= 10t + 3 \\ \rightarrow 20t &= (10t+3)(1+t^2) \\ \rightarrow 20t &= 10t + 10t^3 + 3 + 3t^2 \\ \rightarrow 0 &= 10t^3 + 3t^2 - 10t + 3 \\ \text{Now, } t = \frac{1}{2} &\text{ is a solution} \\ \text{So, } (2t-1) &\text{ is a factor} \end{aligned}$$

• LONG DIVIDE OR MANIPULATE
 $\Rightarrow 5t^2(2t-1) + 4t(2t-1) - 3(2t-1) = 0$
 $\Rightarrow (2t-1)(5t^2 + 4t - 3) = 0$

• BY QUADRATIC FORMULA
 $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
 $t = \frac{-4 \pm \sqrt{4^2 - 4(5)(-3)}}{2(5)}$
 $t = \frac{-4 \pm 2\sqrt{19}}{10}$
 $\therefore \tan x = \frac{1}{2} \text{ or } \frac{-2 \pm \sqrt{19}}{5}$

Question 48 (*****)

Solve the following simultaneous equations.

$$x+y = \frac{\pi}{5}, \quad \cos x + \cos y = 0, \quad 0 \leq x < 2\pi.$$

□, $(x, y) = \left(\frac{3}{5}\pi, -\frac{2}{5}\pi\right) = \left(\frac{8}{5}\pi, -\frac{7}{5}\pi\right)$

$$\begin{aligned} x+y &= \frac{\pi}{5} \\ \cos x + \cos y &= 0 \end{aligned} \quad \Rightarrow \quad y = \frac{\pi}{5} - x$$

- SUBSTITUTE INTO THE OTHER EQUATION
- $\cos x + \cos(\frac{\pi}{5}-x) = 0$
 $\cos x = -\cos(\frac{\pi}{5}-x)$
- MOVE THE MINUS OF THE R.H.S INSIDE USING $\cos(\theta - \theta) = -\cos\theta$
- $\cos x = \cos[\pi - (\frac{\pi}{5}-x)]$
 $\cos x = \cos[\frac{4\pi}{5} + x]$
- SOLVE FOR x
- $x = \pm(\frac{4\pi}{5}) \pm 2n\pi \quad n=0,1,2,\dots$
 $2x = -\frac{8\pi}{5} \pm 2n\pi$
 $x = -\frac{4\pi}{5} \pm n\pi$
- $\therefore x_1 = \frac{4\pi}{5} \quad x_2 = \frac{8\pi}{5}$
- FINISHED USING $y = \frac{\pi}{5} - x$
 $y_1 = -\frac{2\pi}{5} \quad y_2 = -\frac{7\pi}{5}$
- $\therefore \left(\frac{4\pi}{5}, -\frac{2\pi}{5}\right) \text{ & } \left(\frac{8\pi}{5}, -\frac{7\pi}{5}\right)$

Question 49 (*****)

$$f(x) \equiv \frac{1-\sin 2x}{\sin x - \cos x}, \quad x \in \mathbb{R}, \quad \sin x \neq \cos x.$$

a) Show clearly that

$$f(x) \equiv \sin x - \cos x.$$

b) Solve the equation

$$f(x)f(-x) = \cos\left(x - \frac{2\pi}{3}\right), \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

$$x = \frac{2\pi}{9}, \frac{8\pi}{9}, \frac{4\pi}{3}, \frac{14\pi}{9}$$

(a) $f(x) = \frac{1-\sin 2x}{\sin x - \cos x} = \frac{\cos^2 x + \sin^2 x - 2\sin x \cos x}{\sin x - \cos x} = \frac{\sin^2 x - 2\sin x \cos x + \cos^2 x}{\sin x - \cos x}$

 $\therefore f(x) = \frac{(\sin x - \cos x)^2}{\sin x - \cos x}$ ∴ $f(x) = \sin x - \cos x$ (by dividing by $\sin x - \cos x$)

$$\therefore f(x) = \sin x - \cos x$$

$$\therefore f(-x) = \sin(-x) - \cos(-x) = -\sin x - \cos x$$

$$\therefore f(x)f(-x) = (\sin x - \cos x)(-\sin x - \cos x) = -\sin^2 x - \sin x \cos x + \cos x \sin x + \cos^2 x = -\sin^2 x + \cos^2 x = \cos(2x)$$

$$\therefore \cos(2x) = \cos\left(2x - \frac{2\pi}{3}\right)$$

$$\therefore 2x = \left(2x - \frac{2\pi}{3}\right) \pm 2m\pi \quad \text{for } m \in \mathbb{Z}, \dots$$

$$\therefore 2x = \left(\frac{2\pi}{3} - 2x\right) \pm 2m\pi$$

$$\therefore x = \frac{\pi}{3} \pm \frac{m\pi}{2}$$

$$\therefore x = \frac{\pi}{3} \pm \frac{2m\pi}{2} \quad \text{and} \quad x = \frac{\pi}{3} \pm \frac{m\pi}{2}$$

$$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{8\pi}{3}, \frac{4\pi}{3}, \frac{14\pi}{9}$$

Question 50 (***)**

Find the solutions of the trigonometric equation

$$\sin(x+15) + 5\sin x + \sin(x-15) = [\cos(x+15) + 5\cos x + \cos(x-15)] \tan(2x+15),$$

in the range $0^\circ \leq x < 360^\circ$.

$$x = 165^\circ, 345^\circ$$

$$\begin{aligned} \Rightarrow \sin(x+15) + \sin x + \sin(x-15) &= \tan(2x+15) [\cos(x+15) + 5\cos x + \cos(x-15)] \\ \Rightarrow \frac{\sin(x+15)}{\cos(x+15)} + \frac{\sin x}{\cos x} + \frac{\sin(x-15)}{\cos(x-15)} &= \tan(2x+15) \\ \Rightarrow \frac{\sin(x+15)\cos x - \cos(x+15)\sin x}{\cos(x+15)\cos x + \cos(x-15)\sin x} + \frac{\sin x}{\cos x} + \frac{\sin(x-15)\cos x - \cos(x-15)\sin x}{\cos(x-15)\cos x + \cos(x+15)\sin x} &= \tan(2x+15) \\ \Rightarrow \frac{\sin(2x+15)}{\cos(2x+15)} + \frac{\sin x}{\cos x} &= \tan(2x+15) \\ \Rightarrow \frac{\sin(2x+15)}{\cos(2x+15)} &= \tan(2x+15) \\ \Rightarrow \tan x &= \tan(2x+15) \\ \therefore x = (2x+15)^\circ &\pm 180^\circ \quad \text{mod } 180^\circ \\ -x = 15^\circ \pm 180^\circ & \\ x = 15^\circ \pm 180^\circ & \\ \therefore x_1 = 165^\circ & \\ x_2 = 345^\circ & \end{aligned}$$

Question 51 (***)**

Solve the trigonometric equation

$$4\cos(x+30^\circ) = \sec(x-60^\circ), \quad 0^\circ \leq x < 360^\circ.$$

$$x = 45^\circ, 165^\circ, 225^\circ, 345^\circ$$

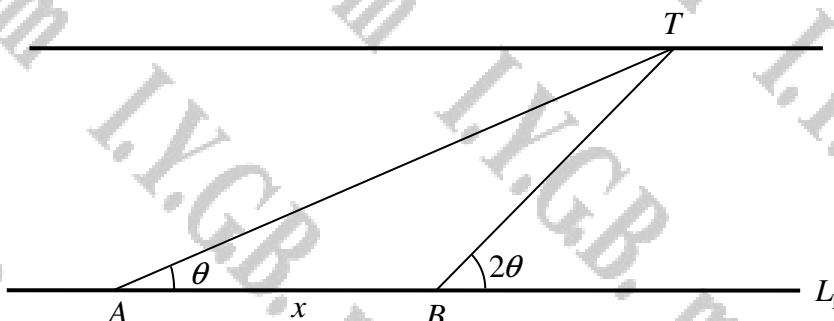
$$\begin{aligned} 4\cos(x+30) &= \sec(x-60) \\ \Rightarrow 4\cos(x+30) &= \frac{1}{\cos(x-60)} \\ \Rightarrow 4[\cos(x+30)\cos(x-60)] &= 1 \\ \Rightarrow 4[\cos(x+30)\cos(x-60) + \cos(x+30)\cos(x-60)] &= 1 \\ \Rightarrow 2[\cos(x+30)\cos(x-60) + \cos(x+30)\cos(x-60)] &= 1 \\ \Rightarrow 2[\cos(x+30)\cos(x-60) + \cos(x+30)\cos(x-60)] &= 1 \\ \Rightarrow 2\cos(x+30)\cos(x-60) &= 1 \\ \Rightarrow \cos(x+30)\cos(x-60) &= \frac{1}{2} \\ \text{cosine } (\frac{1}{2}) &= 60^\circ \\ (2x-30) &= 60^\circ + 360^\circ n \quad n \in \{0, 1, 2, 3, \dots\} \\ 2x-30 &= 300^\circ n + 60^\circ \\ 2x &= 40^\circ + 180^\circ n \\ 2x &= 360^\circ + 180^\circ n \\ x &= 180^\circ + 90^\circ n \end{aligned}$$

$$\begin{aligned} x_1 &= 45^\circ \\ x_2 &= 225^\circ \\ x_3 &= 165^\circ \\ x_4 &= 345^\circ \end{aligned}$$

$$\begin{aligned} \text{ALTERNATIVE:} \\ 4\cos(x+30) &= \sec(x-60) \\ \Rightarrow 4\cos(x+30) &= \frac{1}{\cos(x-60)} \\ \Rightarrow 4(\cos(x+30)\cos(x-60)) &= \frac{1}{\cos(x+30)\cos(x-60)} \\ \Rightarrow 2\sqrt{3}\cos x &- 2\sin x = \frac{1}{2\cos x \sqrt{3}\sin x} \\ \Rightarrow \sqrt{3}\cos^2 x - \sqrt{3}\sin^2 x + 3\sin x \cos x - \sqrt{3}\sin^2 x &= 1 \\ \Rightarrow \sqrt{3}(\cos^2 x - \sin^2 x) + 3\sin x \cos x = 1 \\ \Rightarrow \sqrt{3}(\cos 2x) + 3\sin 2x &= 1 \\ \Rightarrow \sqrt{3}\cos 2x + \sin 2x &= 1 \\ \Rightarrow \frac{\sqrt{3}}{2}\cos 2x + \frac{1}{2}\sin 2x &= \frac{1}{2} \\ \Rightarrow \cos 2x \cos 60^\circ + \sin 2x \sin 60^\circ &= \frac{1}{2} \\ \Rightarrow \cos(2x-30) &= \frac{1}{2} \end{aligned}$$

Question 52 (***)**

In the following question you may not use the sine or the cosine rule.



The figure above shows the plan of a river whose banks are modelled as straight parallel lines L_1 and L_2 .

The points A and B lie on L_1 , so that $|AB| = x$.

A tree is positioned at the point T on L_2 , so that AT and BT subtend angles of θ and 2θ , respectively.

The tree located at T has height h . The angle of elevation of the top of the tree as viewed from A is θ .

Show that

$$h = 2x \sin \theta.$$

, proof

Let the width be y & $|BC| = w$

$$\frac{y}{w} = \tan \theta \quad \frac{y}{x} = \tan \theta$$

$$\frac{y}{w} = \frac{2 \tan \theta}{1 - \tan^2 \theta} \quad \frac{x}{y} = \frac{1}{\tan \theta}$$

$$\frac{w}{y} = \frac{1 - \tan^2 \theta}{2 \tan \theta} \quad \frac{x}{y} = \cot \theta$$

$$\frac{w}{y} + \frac{1 - \tan^2 \theta}{2 \tan \theta} = \cot \theta$$

$$\frac{w}{y} = \cot \theta - \frac{1 - \tan^2 \theta}{2 \tan \theta}$$

$$\frac{w}{y} = \frac{2 \tan \theta - 1 + \tan^2 \theta}{2 \tan \theta}$$

$$\frac{w}{y} = \frac{1 + \tan^2 \theta}{2 \tan \theta}$$

$$\frac{w}{y} = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

$$y = \frac{2w \tan \theta}{1 + \tan^2 \theta}$$

Now angle of elevation, looking at the diagram below

$$\frac{h}{d} = \tan \theta$$

$$d = \frac{h}{\tan \theta}$$

$$d = \frac{2w \tan \theta}{1 + \tan^2 \theta} \times \frac{1}{\tan \theta}$$

$$\frac{h}{d} = \frac{2w \tan^2 \theta}{1 + \tan^2 \theta}$$

$$h = \frac{2w \tan^2 \theta}{1 + \tan^2 \theta} \times d$$

$$h = 2w \sin^2 \theta$$

As required

Question 53 (***)**

By using the trigonometric identity for $\sin 2\theta$, or otherwise, show clearly that

$$\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} = -\frac{1}{8}.$$

proof

$$\begin{aligned}\sin 2\theta &= 2\sin \theta \cos \theta \Rightarrow \cos \theta = \frac{\sin 2\theta}{2\sin \theta} \\ \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} &= \frac{\sin \frac{2\pi}{7}}{2\sin \frac{\pi}{7}} \times \frac{\sin \frac{4\pi}{7}}{2\sin \frac{2\pi}{7}} \times \frac{\sin \frac{8\pi}{7}}{2\sin \frac{4\pi}{7}} = \frac{\sin \frac{8\pi}{7}}{2\sin \frac{\pi}{7}} \\ &= \frac{\sin(\pi + \frac{\pi}{7})}{2\sin \frac{\pi}{7}} = \frac{-\sin \pi \cos \frac{\pi}{7} - \cos \pi \sin \frac{\pi}{7}}{2\sin \frac{\pi}{7}} \\ &= -\frac{\sin \frac{\pi}{7}}{2\sin \frac{\pi}{7}} = -\frac{1}{2} \quad \text{as required}\end{aligned}$$

Question 54 (***)**

Show clearly that

$$\tan \frac{3\pi}{8} - \tan \frac{\pi}{8} - \tan \frac{3\pi}{8} \tan \frac{\pi}{8} = 1.$$

proof

Starting by the L.H.S.

$$\begin{aligned}&\tan \frac{\pi}{8} - \tan \frac{3\pi}{8} - \tan \frac{3\pi}{8} \tan \frac{\pi}{8} \\ &= \tan(\frac{\pi}{8} + \frac{3\pi}{8}) - \tan \frac{3\pi}{8} - \tan \frac{3\pi}{8} [\tan(\frac{\pi}{8} + \frac{3\pi}{8})] \\ &= \tan(\frac{\pi}{8} + \frac{3\pi}{8}) [1 - \tan \frac{3\pi}{8}] - \tan \frac{3\pi}{8} \\ &= \frac{\tan \frac{\pi}{8} + \tan \frac{3\pi}{8}}{1 - \tan \frac{\pi}{8} \tan \frac{3\pi}{8}} [1 - \tan \frac{3\pi}{8}] - \tan \frac{3\pi}{8} \\ &\quad \text{Cancelling terms in brackets} \\ &\quad \text{For } \tan(\frac{\pi}{8} + \frac{3\pi}{8}) = T \text{ & note } \tan^2 \theta \neq 1 \\ &= \frac{1+T}{1-T} [1-T] - T \\ &= 1+T-T \\ &= 1\end{aligned}$$

ALTERNATIVE METHOD

Using the identity $\tan(\frac{\pi}{2} - \theta) = \cot \theta = \frac{1}{\tan \theta}$ **we obtain**

$$\begin{aligned}&\tan \frac{3\pi}{8} - \tan \frac{\pi}{8} - \tan \frac{3\pi}{8} \tan \frac{\pi}{8} \\ &= \tan(\frac{\pi}{2} - \frac{3\pi}{8}) - \tan \frac{\pi}{8} - \tan(\frac{\pi}{2} - \frac{3\pi}{8}) \tan \frac{\pi}{8}\end{aligned}$$

Using $T = \tan \frac{\pi}{8}$ as shown

$$\begin{aligned}&= \frac{1}{\tan \frac{3\pi}{8}} - \tan \frac{\pi}{8} - \frac{1}{\tan \frac{3\pi}{8}} \tan \frac{\pi}{8} \\ &= \frac{1}{T} - T - 1 \\ &= \frac{1-T^2}{T} - 1 \\ &= 2\left(\frac{1-T^2}{2T}\right) - 1 \\ &= 2\left(\frac{1-\tan^2 \frac{\pi}{8}}{2\tan \frac{\pi}{8}}\right) - 1 \quad \text{as } \tan 2\theta = \frac{2\tan \theta}{1-\tan^2 \theta} \\ &= 2 \times \frac{1}{\tan(2 \times \frac{\pi}{8})} - 1 \\ &= 2 \times \frac{1}{\tan \frac{\pi}{4}} - 1 \\ &= 2 \times 1 - 1 \\ &= 1\end{aligned}$$

as required

Question 55 (*****) non calculator

Solve the trigonometric equation

$$\sin(\phi + 30) = \cos(\phi - 45), \quad 0^\circ \leq \phi < 360^\circ.$$

$$\phi = 52.5^\circ, 232.5^\circ$$

$$\begin{aligned} \sin(\phi + 30) &= \cos(\phi - 45) && \text{Sin } A \equiv \cos(90^\circ - A) \\ \Rightarrow \sin[90 - (\phi - 45)] &= \cos(\phi - 45) && \text{using } \sin A \equiv \cos(90^\circ - A) \\ \Rightarrow \cos(90 - \phi) &= \cos(\phi - 45) \\ \Rightarrow (90 - \phi) &= \phi - 45 \pm 360^\circ && \text{Hence } z_1, z_2, \dots \\ (90 - \phi) &= 45 - \phi \pm 360^\circ \\ \Rightarrow (-2\phi) &= 105 \pm 360^\circ && \text{Multiplying by } -1 \\ \Rightarrow 2\phi &= 105 \pm 360^\circ \\ \Rightarrow \phi &= 52.5 \pm 180^\circ \\ \therefore \phi &= 52.5^\circ, 232.5^\circ // \end{aligned}$$

Question 56 (*****)

Eliminate θ from the following pair of equation.

$$\tan \theta + \cot \theta = x^3$$

$$\sec \theta - \cos \theta = y^3$$

Write the answer in the form

$$f(x, y) = 1.$$

$$\boxed{\text{S.P.M.}}, \boxed{x^4y^2 - y^4x^2 = 1}$$

The image shows handwritten mathematical steps for solving the system of trigonometric equations. It starts with the equations:

$$\begin{aligned} \tan \theta + \cot \theta &= x^3 \\ \sec \theta - \cos \theta &= y^3 \end{aligned}$$

It then shows the conversion of cotangent and secant into sine and cosine:

$$\begin{aligned} \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} &= x^3 \\ \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} &= x^3 \\ \frac{1}{\cos \theta \sin \theta} &= x^3 \end{aligned}$$

$$\begin{aligned} \frac{1}{\cos \theta - \sin \theta} &= y^3 \\ \frac{1}{\cos \theta} - \frac{1}{\sin \theta} &= y^3 \\ \frac{\sin \theta - \cos \theta}{\cos \theta \sin \theta} &= y^3 \end{aligned}$$

Next, it multiplies the two expressions side by side:

$$\begin{aligned} \frac{1}{\cos \theta \sin \theta} \times \frac{\sin \theta - \cos \theta}{\cos \theta \sin \theta} &= x^3 y^3 \\ \frac{1}{\cos^2 \theta} &= x^3 y^3 \\ \sec^2 \theta &= \frac{1}{x^3 y^3} \\ \sec \theta &= \frac{1}{xy} \end{aligned}$$

Finally, it substitutes $\sec \theta$ into the second equation:

$$\begin{aligned} \sec \theta - \cos \theta &= y^3 \\ \frac{1}{\cos \theta} - \frac{1}{\cos \theta} &= y^3 \\ \frac{1}{\cos^2 \theta} - 1 &= y^3 \\ \frac{1}{\cos^2 \theta} - \frac{\cos^2 \theta}{\cos^2 \theta} &= y^3 \\ \frac{1 - \cos^2 \theta}{\cos^2 \theta} &= y^3 \\ \frac{\sin^2 \theta}{\cos^2 \theta} &= y^3 \\ \tan^2 \theta &= y^3 \\ \tan \theta &= y^{\frac{3}{2}} \end{aligned}$$

Question 57 (***)**

It is given that

$$\tan^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \equiv \frac{1 - \sin \theta}{1 + \sin \theta}, \quad x \neq \frac{\pi}{2}(4n+3), \quad n \in \mathbb{Z}.$$

- a) Prove the validity of the above trigonometric identity
 b) Hence solve the equation

$$\tan\left(\frac{\pi}{4} - 2x\right) = \sqrt{7 + 4\sqrt{3}}, \quad 0 \leq x < \pi.$$

$$x = \frac{5\pi}{12}, \frac{11\pi}{6}$$

(a) LHS = $\tan^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \left[\frac{\tan\frac{\pi}{4} - \tan\frac{\theta}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{\theta}{2}} \right]^2 = \left[\frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}} \right]^2 = \left[\frac{1 - \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{1 + \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}} \right]^2$

= ~~MULTIPLY NUMERATOR AND DENOMINATOR BY $\cos\frac{\theta}{2}$~~

$$= \left[\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}} \right]^2 = \frac{\cos^2\frac{\theta}{2} - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2}} = \frac{1 - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}{1 + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}$$

$$= \frac{1 - \sin\theta}{1 + \sin\theta} = 2\sqrt{3} \quad \text{✓ REPROVED}$$

(b) $\tan\left(\frac{\pi}{4} - 2x\right) = \sqrt{7 + 4\sqrt{3}}$

$$\Rightarrow \tan\left(\frac{\pi}{4} - 2x\right) = 7 + 4\sqrt{3}$$

using (a) $\tan(0 - 2x) = 7 + 4\sqrt{3}$

$$\Rightarrow \frac{1 - \sin 4x}{1 + \sin 4x} = 7 + 4\sqrt{3}$$

$$\Rightarrow \frac{2 - (1 + \sin 4x)}{1 + \sin 4x} = 7 + 4\sqrt{3}$$

$$\Rightarrow \frac{2}{1 + \sin 4x} - 1 = 7 + 4\sqrt{3}$$

$$\Rightarrow \frac{2}{1 + \sin 4x} = 8 + 4\sqrt{3}$$

$$\Rightarrow \frac{1}{1 + \sin 4x} = 4 + 2\sqrt{3}$$

$$\Rightarrow 1 + \sin 4x = \frac{1}{4 + 2\sqrt{3}}$$

$$\Rightarrow 1 + \sin 4x = \frac{4 - 2\sqrt{3}}{16 - 12}$$

$\therefore \begin{cases} 4x = \frac{\pi}{12} \\ 4x = \frac{11\pi}{12} \end{cases} \quad n = 0, 1, 2, 3, \dots$

$\begin{cases} x = \frac{\pi}{48} + 2\pi n \\ x = \frac{11\pi}{48} + 2\pi n \end{cases}$

$\begin{cases} x = \frac{\pi}{12} + \frac{11\pi}{12} \\ x = \frac{11\pi}{12} + \frac{11\pi}{12} \end{cases}$

$\therefore \begin{cases} x_1 = \frac{\pi}{12} \\ x_2 = \frac{11\pi}{12} \\ x_3 = \frac{\pi}{12} + \frac{11\pi}{12} \\ x_4 = \frac{11\pi}{12} + \frac{11\pi}{12} \end{cases}$

~~DOESN'T SATISFY DOMAIN~~

~~DOESN'T SATISFY DOMAIN~~

Question 58 (***)**

It is given that

$$p = \sin^2 \theta, \quad q = \tan 2\theta.$$

Use trigonometric identities to find a simplified expression for q^2 in terms of p .

$$q^2 = \frac{4p(1-p)}{(1-2p)^2}$$

The image shows handwritten mathematical steps for simplifying the expression $q^2 = \frac{4p(1-p)}{(1-2p)^2}$. It starts with two columns of equations:

Left Column:

- $p = \sin^2 \theta$
- $\Rightarrow \frac{1}{p} = \csc^2 \theta$
- $\Rightarrow \frac{1}{p} - 1 = \csc^2 \theta - 1$
- $\Rightarrow \cot^2 \theta = \frac{1}{p} - 1$
- $\Rightarrow \cot \theta = \frac{1-p}{p}$
- $\Rightarrow \tan^2 \theta = \frac{p}{1-p}$

Right Column:

- $q = \frac{\tan 2\theta}{1-\tan^2 \theta}$
- $\Rightarrow q^2 = \frac{4 \tan^2 \theta}{(1-\tan^2 \theta)^2}$
- $\Rightarrow q^2 = \frac{4 \left(\frac{p}{1-p}\right)}{\left(1-\frac{p}{1-p}\right)^2}$
- $\Rightarrow q^2 = \frac{4p}{\left(\frac{1-p-p}{1-p}\right)^2} = \frac{4p}{\left(\frac{1-2p}{1-p}\right)^2}$
- $\Rightarrow q^2 = \frac{4p}{\frac{(1-2p)^2}{(1-p)^2}}$
- $\Rightarrow q^2 = \frac{4p(1-p)}{(1-2p)^2}$
- Multiply top & bottom of the fraction by $(1-p)$:**
- $\Rightarrow q^2 = \frac{4p(1-p)}{(1-2p)^2}$

Question 659 (*****)

$$\sin 2x - \sqrt{3} \cos 2x = \tan x, \quad 0 \leq x < 2\pi.$$

Find the solutions of the above trigonometric equation, giving the answers terms of π .

$$x = \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{7\pi}{4}$$

$$\begin{aligned}
 & \text{① } \sin 2x - \sqrt{3} \cos 2x = \tan x \\
 & \Rightarrow \sin 2x - \tan x = \sqrt{3} \cos 2x \\
 & \Rightarrow 2\sin 2x - \frac{\sin x}{\cos x} = \sqrt{3} \cos 2x \\
 & \Rightarrow \frac{2\sin 2x - \sin x}{\cos x} = \sqrt{3} \cos 2x \\
 & \Rightarrow \frac{3\sin(2x) - \sin x}{\cos x} = \sqrt{3} \cos 2x \\
 & \Rightarrow \frac{\sin(2x) - \sin x}{\cos x} = \sqrt{3} \cos 2x \\
 & \text{From } \cos 2x = 0 \\
 & \Rightarrow \tan x = \sqrt{3} \\
 & \Rightarrow 2x = \frac{\pi}{2} + 2m\pi \quad \text{or} \quad 2x = \frac{4\pi}{3} + 2m\pi \\
 & \Rightarrow x = \frac{\pi}{4} + m\pi \quad \text{or} \quad x = \frac{2\pi}{3} + m\pi \\
 & \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \\
 & \text{② } \text{Divide by } \sqrt{3} \text{ on LHS} \\
 & \Rightarrow \sin 2x - \sqrt{3} \cos 2x = \tan x \\
 & \Rightarrow \frac{2t}{1+t^2} - \sqrt{3} \frac{(1-t^2)}{1+t^2} = t \\
 & \text{Where } t = \tan x \\
 & \Rightarrow 2t - \sqrt{3}(1-t^2) = t(1+t^2) \\
 & \Rightarrow 2t - \sqrt{3} + \sqrt{3}t^2 = t + t^3 \\
 & \Rightarrow 0 = t^3 - \sqrt{3}t^2 - t + \sqrt{3} \\
 & \Rightarrow 0 = t^2(t-\sqrt{3}) - (t-\sqrt{3}) \\
 & \Rightarrow 0 = (t-\sqrt{3})(t+1)(t-1) \\
 & \Rightarrow (t-\sqrt{3})(t+1)(t+1) = 0 \\
 & \text{So } t = \sqrt{3}, -1 \\
 & \text{From } t = \tan x \\
 & \Rightarrow x = \frac{\pi}{4} \\
 & \Rightarrow x = \frac{3\pi}{4} \\
 & \Rightarrow x = \frac{5\pi}{4} \\
 & \Rightarrow x = \frac{7\pi}{4} \\
 & \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}
 \end{aligned}$$

Question 60 (*****)

$$\sin\left(3x - \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = \sin x, \quad 0 \leq x < \pi.$$

Determine the solutions of the above trigonometric equation, giving the answers in terms of π .

, $x = \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}$

METHOD 1

Start by dividing the identity to combine the sines in the LHS

$$\begin{aligned} \sin(A+B) &\equiv \sin A \cos B + \cos A \sin B \\ \sin(A-B) &\equiv \sin A \cos B - \cos A \sin B \\ \text{ADD} \quad \frac{\sin(A+B)}{\sin(A-B)} &\equiv 2 \sin A \cos B \\ \text{AND} \quad \frac{\sin(A-B)}{\sin(A-B)} &\equiv 2 \sin A \cos B \\ \therefore t = A+B &\rightarrow A = \frac{t-\pi}{2} \\ Q = A-B &\rightarrow B = \frac{P-Q}{2} \\ \text{So } \sin P + \sin Q &\equiv 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2} \end{aligned}$$

Thus we now have

$$\begin{aligned} \rightarrow \sin\left(3x - \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) &= \sin x \\ \rightarrow 2 \sin\left[\frac{(3x-\pi)-x-\pi}{2}\right] \cos\left[\frac{(3x-\pi)+(x-\pi)}{2}\right] &= \sin x \\ \rightarrow 2 \sin\left[\frac{4x-2\pi}{2}\right] \cos\left[\frac{2x}{2}\right] &= \sin x \\ \rightarrow 2 \sin\left(2x - \frac{\pi}{3}\right) \cos x &= \sin x \\ \rightarrow [2 \sin 2x \cos \frac{\pi}{3} - 2 \cos 2x \sin \frac{\pi}{3}] \cos x &= \sin x \\ \rightarrow (\sin 2x - \sqrt{3} \cos 2x) \cos x &= \sin x \\ \Rightarrow \boxed{\sin 2x - \sqrt{3} \cos 2x} &= \sin x \end{aligned}$$

METHOD 2

Now proceed as follows

$$\begin{aligned} \rightarrow \sin 2x - \sqrt{3} \cos 2x &= \sin x \\ \rightarrow 2 \sin 2x \cos x - \frac{\sin 2x}{\cos 2x} &= \sqrt{3} \cos 2x \\ \rightarrow \frac{2 \sin 2x \cos x - \sin 2x}{\cos 2x} &= \sqrt{3} \cos 2x \\ \rightarrow \frac{\sin 2x(2 \cos x - 1)}{\cos 2x} &= \sqrt{3} \cos 2x \\ \rightarrow \tan x \cos 2x - \sqrt{3} \cos 2x &= 0 \\ \rightarrow \tan x \cos 2x - \sqrt{3} \cos 2x &= 0 \\ \rightarrow \cos 2x (\tan x - \sqrt{3}) &= 0 \end{aligned}$$

Hence we now have

$$\begin{aligned} \cos 2x &= 0 \\ \left(2x = \frac{\pi}{2} + 2m\pi \quad m = 0, \pm 1, \dots\right) & \quad \left(\tan x = \sqrt{3} \quad m = 0, 1, 2, \dots\right) \\ \left(2x = \frac{4\pi}{3} + 2m\pi \quad m = 0, \pm 1, \dots\right) & \quad \left(x = \frac{\pi}{3} + m\pi \quad m = 0, 1, 2, \dots\right) \\ \left(2x = \frac{7\pi}{3} + 2m\pi \quad m = 0, \pm 1, \dots\right) & \quad \therefore x = \frac{\pi}{3}, \frac{4\pi}{3} \end{aligned}$$

METHOD 2

This requires knowledge of the "little t" identities starting from the "yellow" boxed expression of Method 1

$$\begin{aligned} \rightarrow \sin 2x - \sqrt{3} \cos 2x &= \sin x \\ \Rightarrow \frac{2t}{1+t^2} - \sqrt{3} \left(\frac{1-t^2}{1+t^2}\right) &= t \\ \Rightarrow 2t - \sqrt{3}(1-t^2) &= t(1+t^2) \\ \Rightarrow 2t - \sqrt{3} + \sqrt{3}t^2 &= t + t^3 \\ \Rightarrow t &= t^3 - \sqrt{3}t^2 - t + \sqrt{3} \\ \Rightarrow 0 &= t^3 - \sqrt{3}t^2 - t + \sqrt{3} \\ \text{Factorise in terms of } t \\ \Rightarrow 0 &= t^2(t - \sqrt{3}) - (t - \sqrt{3}) \\ \Rightarrow 0 &= (t - \sqrt{3})(t-1)(t+1) \\ \Rightarrow (t - \sqrt{3})(t-1) &= 0 \end{aligned}$$

Thus we have

$$\begin{cases} t = \frac{\sqrt{3}}{1} & \Rightarrow \begin{cases} x = \frac{\pi}{3} + m\pi \\ x = \frac{2\pi}{3} + m\pi \\ x = -\frac{\pi}{3} + m\pi \end{cases} \\ t = 1 & \Rightarrow x = \frac{\pi}{4} + m\pi \\ t = -1 & \Rightarrow x = \frac{3\pi}{4} + m\pi \end{cases}$$

$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}$

Question 61 (***)**

The function f is defined as

$$f(x) = \sin\left(x + \frac{7\pi}{12}\right) \sin\left(x + \frac{\pi}{12}\right), \quad 0 \leq x < 2\pi$$

Solve the equation

$$f(x) + f(-x) = f\left(\frac{\pi}{4} - x\right).$$

, $x = \frac{\pi}{24}, \frac{13\pi}{24}, \frac{25\pi}{24}, \frac{37\pi}{24}$

• START BY SIMPLIFYING THE FUNCTION FIRST, BY DEDUCTION TO IDENTITY

$$\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \cos(A-B) &= \cos A \cos B + \sin A \sin B \end{aligned}$$

subtracting "squares" gives

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

• Hence we have

$$\begin{aligned} f(x) &= \sin\left(x + \frac{7\pi}{12}\right) \sin\left(x + \frac{\pi}{12}\right) \\ &= \frac{1}{2} \left[\cos\left(x + \frac{7\pi}{12} - x - \frac{\pi}{12}\right) - \cos\left(x + \frac{7\pi}{12} + x + \frac{\pi}{12}\right) \right] \\ &= \frac{1}{2} \cos\left(\frac{7\pi}{12}\right) - \frac{1}{2} \cos\left(2x + \frac{7\pi}{6}\right) \\ &= -\frac{1}{2} \cos\left(2x + \frac{7\pi}{6}\right) \end{aligned}$$

• NEXT THE EQUATION BECOMES

$$\begin{aligned} &\Rightarrow f(x) + f(-x) = f\left(\frac{\pi}{4} - x\right) \\ &\Rightarrow \frac{1}{2} \cos\left(2x + \frac{7\pi}{6}\right) + \frac{1}{2} \cos\left(-2x - \frac{7\pi}{6}\right) = -\frac{1}{2} \cos\left[\cos\left(\frac{\pi}{4} - x\right) + \frac{7\pi}{6}\right] \\ &\Rightarrow \cos\left(2x + \frac{7\pi}{6}\right) + \cos\left(\frac{\pi}{4} - x - 2x - \frac{7\pi}{6}\right) = \cos\left(\frac{\pi}{4} + \frac{7\pi}{6} - 2x\right) \\ &\Rightarrow \cos\left(2x + \frac{7\pi}{6}\right) + \cos\left(\frac{31\pi}{12} - 2x\right) = \cos\left(\frac{37\pi}{12} - 2x\right) \\ &\Rightarrow \cos\left(2x + \frac{7\pi}{6}\right) + \cos\left(2x - \frac{7\pi}{6}\right) = \cos\left(2x - \frac{7\pi}{6}\right) \\ &\Rightarrow \cos\left(2x + \frac{7\pi}{6}\right) - \sin 2x \cos \frac{7\pi}{6} = \cos\left(2x - \frac{7\pi}{6}\right) \\ &\Rightarrow \cos 2x \cos \frac{7\pi}{6} + \sin 2x \sin \frac{7\pi}{6} = \cos 2x \cos \frac{7\pi}{6} + \sin 2x \sin \frac{7\pi}{6} \\ &\Rightarrow 2\cos 2x \cos \frac{7\pi}{6} = \end{aligned}$$

• DIVIDE THE EQUATION BY $\cos 2x$

$$\begin{aligned} &\Rightarrow 2\cos \frac{7\pi}{6} = \cos \frac{7\pi}{6} + \sin 2x \sin \frac{7\pi}{6} \\ &\Rightarrow 2\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2} + \sin 2x \times \left(-\frac{1}{2}\right) \\ &\Rightarrow -1 = -\frac{\sqrt{3}}{2} - \frac{1}{2} \sin 2x \\ &\Rightarrow -2 = -\sqrt{3} - \sin 2x \\ &\Rightarrow \sin 2x = 2 - \sqrt{3} \\ &\qquad \text{arctan}(2 - \sqrt{3}) = \frac{\pi}{12} \end{aligned}$$

• FINALLY WE OBTAIN

$$\begin{aligned} 2x &= \frac{\pi}{12} \pm n\pi \quad n = 0, 1, 2, 3, \dots \\ x &= \frac{\pi}{24} \pm \frac{n\pi}{2} \\ \therefore x &= \frac{\pi}{24}, \frac{13\pi}{24}, \frac{25\pi}{24}, \frac{37\pi}{24} \end{aligned}$$

Question 62 (*****)

It is given that the three angles of a triangle α , β and γ satisfy the relationship

$$\tan \frac{\alpha}{2} = \left(1 + \tan^2 \frac{\alpha}{2}\right) \sin(\beta - \gamma).$$

Assuming that the triangle is not right angled, show that

$$3 \tan \gamma = \tan \beta.$$

, proof

Start manipulating as follows

$$\begin{aligned} \Rightarrow \tan \frac{\alpha}{2} &= \left[1 + \tan^2 \frac{\alpha}{2}\right] \sin(\beta - \gamma) \\ \Rightarrow \tan \frac{\alpha}{2} &= \sec^2 \frac{\alpha}{2} \sin(\beta - \gamma) \\ \Rightarrow \frac{\tan \frac{\alpha}{2}}{\sec^2 \frac{\alpha}{2}} &= \sin(\beta - \gamma) \\ \Rightarrow \tan \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} &= \sin(\beta - \gamma) \\ \Rightarrow \frac{\sin \frac{\alpha}{2} \times \cos^2 \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} &= \sin(\beta - \gamma) \\ \Rightarrow \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} &= \sin(\beta - \gamma) \\ \Rightarrow 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} &= 2 \sin(\beta - \gamma) \\ \Rightarrow \sin \alpha &= 2 \sin(\beta - \gamma) \end{aligned}$$

Now $\frac{\alpha + \beta + \gamma = \pi}{\alpha = \pi - (\beta + \gamma)}$

$$\begin{aligned} \Rightarrow \sin(\pi - (\beta + \gamma)) &= 2 \sin(\beta - \gamma) \\ \Rightarrow \sin(\pi) - \sin(\beta + \gamma) &= 2 \sin(\beta - \gamma) - 2 \sin(\beta + \gamma) \\ \Rightarrow \sin(\beta \pi) &= 2 \sin(\beta - \gamma) - 2 \sin(\beta + \gamma) \\ \Rightarrow \sin(\beta \pi) + \cancel{\sin(\beta - \gamma)} &= \cancel{2 \sin(\beta - \gamma)} - 2 \sin(\beta + \gamma) \end{aligned}$$

$\Rightarrow \text{cancel } \sin(\beta \pi) = \cancel{\sin(\beta - \gamma)}$

$$\Rightarrow \frac{\cancel{\sin(\beta \pi)}}{\cancel{\sin(\beta - \gamma)}} = \frac{\cancel{\sin(\beta - \gamma)}}{\cancel{\sin(\beta + \gamma)}}$$

cancel + c
cancel - c

$$\Rightarrow 3 \tan \gamma = \tan \beta$$

As required

Question 63 (***** non calculator

Solve the trigonometric equation

$$\sin(y - 48) = \cos(y + 12), \quad 0 \leq y < 360^\circ.$$

$y = 63^\circ, 243^\circ$

$$\begin{aligned} \sin(y - 48) &= \cos(y + 12) \quad \text{sin } A \equiv \cos(90 - A) \\ \Rightarrow \cos[y - (48 - 90)] &= \cos(y + 12) \\ \Rightarrow \cos(138 - y) &= \cos(y + 12) \\ \text{Thus} \quad \Rightarrow (138 - y) &= y + 12 \pm 360^\circ \\ \Rightarrow (138 - y) &= -y - 12 \pm 360^\circ \quad n=0,1,2,3,\dots \\ \Rightarrow (-2y) &= -108 \pm 360^\circ \quad \text{INCONSISTENT} \\ \Rightarrow y &= 63 \pm 180^\circ \\ \therefore y_1 &= 63^\circ \\ y_2 &= 243^\circ // \end{aligned}$$

Question 64 (*****)

The 2nd, 3rd and 4th terms of a geometric series are $\cos\theta$, $\sqrt{2}\sin\theta$ and $\sqrt{3}\tan\theta$, respectively, where $0 < \theta < \frac{\pi}{2}$.

Show clearly that the sum of the first 6 terms of the series is

$$\frac{43}{12}(6 + \sqrt{6}).$$

proof

$U_2 = \cos\theta$ $U_3 = \sqrt{2}\sin\theta$ $U_4 = \sqrt{3}\tan\theta$ $(GP) \Rightarrow U_2 = U_3 \cdot r$ $\Rightarrow \frac{U_3}{U_2} = \frac{U_4}{U_3}$ $\Rightarrow \frac{\sqrt{2}\sin\theta}{\cos\theta} = \frac{\sqrt{3}\tan\theta}{\sqrt{2}\sin\theta}$ $\Rightarrow 2\sin^2\theta = \sqrt{3}\sin\theta \cos\theta$ $\Rightarrow 2\sin\theta = \sqrt{3}\cos\theta$ $\Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$ <div style="border: 1px solid black; padding: 2px; display: inline-block;"> $\boxed{\theta = \frac{\pi}{6}}$ </div> <p style="text-align: center;">ONLY VALUE IN RANGE</p>	$U_2 = \cos\theta = \frac{1}{2}$ $U_3 = \sqrt{2}\sin\theta = \frac{\sqrt{2}}{2}$ $U_4 = \sqrt{3}\tan\theta = \frac{\sqrt{3}}{2}$ $\therefore r = \frac{U_3}{U_2} = \frac{\sqrt{2}/2}{1/2} = \sqrt{2}$ $U_2 = \cos\theta$ $\frac{1}{2} = \alpha \times \sqrt{2}^n$ $\alpha = \frac{1}{2\sqrt{2}^n} = \frac{\sqrt{2}}{2^n}$ $\therefore \frac{1}{2^n} = \frac{\sqrt{2}(\sqrt{2}^{n-1})}{\sqrt{2} - 1}$ $\frac{1}{2^n} = \frac{\sqrt{2}}{2^n} \times \frac{\sqrt{2}(\sqrt{2}^{n-1})}{\sqrt{2} - 1}$ $\frac{1}{2^n} = \frac{2\sqrt{2}}{2^n(\sqrt{2} - 1)}$ $\frac{1}{2^n} = \frac{2\sqrt{2} \times 4(\sqrt{2}^{n-1})}{(2^n)(2^n - 1)}$ $\frac{1}{2^n} = \frac{8\sqrt{2}}{(2^n)(2^n - 1)}$ $\frac{1}{2^n} = \frac{43}{12} \left(1 + \frac{1}{\sqrt{2} - 1}\right)$ $\frac{1}{2^n} = \frac{43}{12} \left(1 + \frac{1}{\sqrt{2} - 1}\right)$ $\frac{1}{2^n} = \frac{43}{12} \left(1 + \frac{1}{\sqrt{2} - 1}\right)$
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Question 65 (*****)

It is given that θ and φ satisfy the relationship

$$\tan \theta = \frac{3 \sin \varphi \cos \varphi}{1 - 3 \sin^2 \varphi}.$$

Show clearly that

$$\tan(\theta - \varphi) = 2 \tan \varphi.$$

, proof

• SIMPLIFY BY EXPRESSION $\tan \theta$ IN TERMS OF $\tan \varphi$ AS REQUIRED.

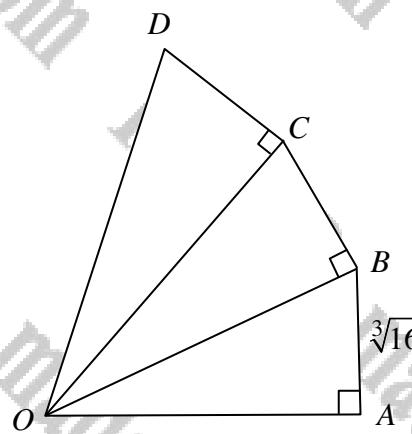
$$\begin{aligned}\tan \theta &= \frac{3 \sin \varphi \cos \varphi}{1 - 3 \sin^2 \varphi} = \frac{\frac{3 \sin \varphi}{\cos \varphi} \cdot \cos^2 \varphi}{\frac{1}{\cos^2 \varphi} - \frac{3 \sin^2 \varphi}{\cos^2 \varphi}} = \frac{3 \tan \varphi}{\frac{1}{\cos^2 \varphi} - 3 \tan^2 \varphi} \\ &= \frac{3 \tan \varphi}{(1 + \tan^2 \varphi) - 3 \tan^2 \varphi} = \boxed{\frac{3 \tan \varphi}{1 - 2 \tan^2 \varphi}}\end{aligned}$$

• FINALLY USING THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$

$$\begin{aligned}\tan(\theta - \varphi) &= \frac{\tan \theta - \tan \varphi}{1 + \tan \theta \tan \varphi} \\ &= \frac{\frac{3 \tan \varphi}{1 - 2 \tan^2 \varphi} - \tan \varphi}{1 + \left(\frac{3 \tan \varphi}{1 - 2 \tan^2 \varphi}\right) \tan \varphi} \\ &= \frac{3 \tan \varphi - \tan \varphi (1 - 2 \tan^2 \varphi)}{(1 - 2 \tan^2 \varphi) + 3 \tan^2 \varphi} \\ &= \frac{3 \tan \varphi - \tan \varphi + 2 \tan^2 \varphi}{1 - 2 \tan^2 \varphi + 3 \tan^2 \varphi} \\ &= \frac{2 \tan \varphi + 2 \tan^2 \varphi}{1 + \tan^2 \varphi} \\ &= \frac{2 \tan \varphi (1 + \tan^2 \varphi)}{1 + \tan^2 \varphi} \\ &= 2 \tan \varphi \quad \text{As Required}\end{aligned}$$

NOTICE THAT BOTH DENOMINATORS ARE CANCELLED OUT

Question 66 (*****)



The figure above shows three right angle triangles, OAB , OBC and OCD .

It is given that $\angle AOB = \angle BOC = \angle COD$ and $|OD| = 2|OA|$.

Given further that the length of AB is $3\sqrt{16}$, determine the length of DC .

$$\boxed{\quad}, \boxed{|DC| = 4}$$

Firstly by similar triangles/trigonometry

$$\rightarrow \frac{x}{y} = \frac{1}{2} = \frac{\frac{1}{2}}{z} = \cos\theta$$

$$\rightarrow \frac{x}{y} = \cos\theta \text{ & } \frac{1}{2} = \cos\theta$$

$$\rightarrow \frac{x}{y} \times \frac{1}{2} = \cos^2\theta$$

$$\rightarrow \frac{1}{2} = \cos^2\theta$$

Now we have by similar triangles

$$\rightarrow \frac{z}{x} = \cos^2\theta \text{ & } \frac{z}{2x} = \cos\theta$$

$$\rightarrow \frac{z}{x} = \cos^2\theta \cos\theta$$

$$\rightarrow \frac{z}{x} = \cos^3\theta$$

$$\rightarrow \cos\theta = (\frac{1}{2})^{\frac{1}{3}}$$

$$\rightarrow \cos\theta = 2^{\frac{1}{3}}x$$

Now obtain scale factor

$$\frac{z}{y} = \cos\theta$$

$$y = \frac{z}{\cos\theta}$$

$$y = \frac{z}{2^{\frac{1}{3}}x}$$

$$y = 2^{\frac{1}{3}}x$$

As scale factor of $B : OD$

$$\frac{2x}{y} = \frac{2x}{2^{\frac{1}{3}}x} = 2^{\frac{2}{3}}$$

$$\therefore |DC| = 2^{\frac{2}{3}} |AB|$$

$$= 2^{\frac{2}{3}} \times 2^{\frac{1}{3}} = 2^{\frac{1}{3}}$$

$$= 2^{\frac{1}{3}} \times 2^{\frac{1}{3}} = 4$$

Question 67 (***)**

Solve the trigonometric equation

$$8 \cos 2x \cos 4x \cos 8x + 1 = 0, \quad 0 < x < \frac{\pi}{2}.$$

$$\boxed{x = \frac{\pi}{14}, \frac{\pi}{9}, \frac{3\pi}{14}, \frac{2\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{14}, \frac{4\pi}{9}}$$

$8 \cos 2x \cos 4x \cos 8x + 1 = 0 \quad 0 < x < \frac{\pi}{2}$

• START BY REASONING THE SINE-IDEST AND FORMULA
 $\sin 2A \equiv 2 \sin A \cos A$

$$\cos A = \frac{\sin 2A}{2 \sin A}$$

• THIS WE MAY WRITE THE EQUATION AS FOLLOWS

$$\begin{aligned} &\rightarrow 8 \cos 2x \cos 4x \cos 8x = -1 \\ &\rightarrow 8 \times \frac{\sin 4x}{2 \sin 2x} \times \frac{\sin 8x}{2 \sin 4x} \times \frac{\sin 16x}{2 \sin 8x} = -1 \\ &\Rightarrow \frac{\sin 16x}{\sin 2x} = -1 \\ &\Rightarrow \sin 16x = -\sin 2x \\ &\Rightarrow \sin 16x = \sin(-2x) \\ &\Rightarrow \begin{cases} 16x = -2x + 2\pi n \\ 16x = (\pi + 2x) + 2\pi n \end{cases} \quad n=0,1,2,\dots \\ &\Rightarrow \begin{cases} 14x = 0 + 2\pi n \\ 14x = \pi + 2\pi n \end{cases} \\ &\Rightarrow \begin{cases} x = 0 \pm \frac{2\pi n}{14} \\ x = \frac{\pi}{14} \pm \frac{2\pi n}{14} \end{cases} \quad (n=0,1,2,\dots) \\ &\text{COLLECTING THE SOLUTIONS: } x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{14}, \frac{4\pi}{9} \end{aligned}$$

Question 68 (***)**

Find, in terms of π , the solutions of the equation

$$\sqrt{x} \frac{d}{dx} (\sqrt{x} + 2 \cos \sqrt{x}) = 1, \quad 0 \leq x < 4\pi^2.$$

$$\boxed{x = \frac{49\pi^2}{36}, \frac{121\pi^2}{36}}$$

$$\begin{aligned} x^{\frac{1}{2}} \frac{d}{dx} [x^{\frac{1}{2}} + 2 \cos x^{\frac{1}{2}}] &= 1 \\ x^{\frac{1}{2}} \left[\frac{1}{2}x^{-\frac{1}{2}} + 2 \cdot \frac{1}{2}x^{-\frac{1}{2}}(-\sin x^{\frac{1}{2}}) \right] &= 1 \\ x^{\frac{1}{2}} \left[\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} \sin x^{\frac{1}{2}} \right] &= 1 \\ \frac{1}{2} - \sin \frac{1}{2} &= 1 \\ \sin \frac{1}{2} &= -\frac{1}{2} \\ \sin \theta &= -\frac{1}{2} \\ \sin(\theta - \frac{\pi}{2}) &= -\frac{1}{2} \end{aligned}$$

$\theta = -\frac{\pi}{6} \pm 2k\pi \quad k=0,1,2,\dots$
 $\theta = \frac{7\pi}{6} + 2k\pi \quad k=0,1,2,\dots$
 $\theta = -\frac{5\pi}{6}, \frac{7\pi}{6}, \frac{19\pi}{6}, \dots$
 $\theta = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{19\pi}{6}, \dots$
 $x = \frac{49\pi^2}{36}, \frac{121\pi^2}{36}$

Question 69 (*****)

Find in terms of π the solutions of the trigonometric equation

$$\cot\left(\frac{\pi}{2}\cos x\right)=1, \quad 0 \leq x < 2\pi.$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\begin{aligned} \cot\left(\frac{\pi}{2}\cos x\right) &= 1 \\ \Rightarrow \tan\left(\frac{\pi}{2}\cos x\right) &= 1 \\ \Rightarrow \frac{\pi}{2}\cos x &= \frac{\pi}{4} + n\pi \quad (n \in \mathbb{Z}) \\ \Rightarrow \cos x &= \frac{1}{2} \pm n \\ \Rightarrow \cos x &= \frac{1}{2} \end{aligned} \quad \begin{aligned} \Rightarrow \cos x = -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots \\ \Rightarrow \cos x = \frac{1}{2} \\ \Rightarrow x = \frac{\pi}{3} + 2k\pi \\ \Rightarrow x = \frac{\pi}{3} + 2k\pi \quad k \in \mathbb{Z} \\ \therefore x = \frac{\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

Question 70 (*****)

Solve the trigonometric equation

$$8\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right) = \cos 3x, \quad 0 \leq x < \frac{\pi}{2},$$

giving the answer in terms of π .

$$\boxed{\quad}, \quad x = \frac{\pi}{54}$$

$$\begin{aligned} 8\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right) &= \cos 3x \\ \Rightarrow 4\left(2\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\right) &= \cos 3x \\ \Rightarrow 4\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right) &= \cos 3x \\ \Rightarrow 2\left(2\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\right) &= \cos 3x \\ \Rightarrow 2\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{18}\right) &= \cos 3x \\ \Rightarrow \sin\left(\frac{\pi}{9}\right) &= \cos 3x \\ \Rightarrow \cos\left(\frac{\pi}{3} - \frac{4\pi}{9}\right) &= \cos 3x \\ \Rightarrow \cos 3x &= \cos\left(\frac{3\pi}{10} - \frac{4\pi}{9}\right) \\ \Rightarrow \cos 3x &= \cos\left(\frac{7\pi}{90}\right) \\ \Rightarrow 3x &= \frac{7\pi}{90} \quad (\text{Set } 3x = A) \\ \Rightarrow x &= -\frac{7\pi}{270} \pm 20k\pi \quad (k \in \mathbb{Z}) \\ \Rightarrow x &= -\frac{7\pi}{270} \pm \frac{20\pi}{3} \quad (k \in \mathbb{Z}) \\ \Rightarrow x &= \frac{593\pi}{270} \quad (k = 0) \end{aligned}$$

ONLY SOLUTION IN RANGE IS $\frac{7\pi}{90}$

Question 71 (***)**

Solve the trigonometric equation

$$\sin(2\theta^\circ + 20^\circ) + \cos(3\theta^\circ + 50^\circ) = 0, \quad 0 \leq \theta \leq 360^\circ.$$

$$\theta = 40^\circ, 60^\circ, 112^\circ, 184^\circ, 256^\circ, 328^\circ$$

$$\begin{aligned}
 & \rightarrow \sin(2\theta+20) + \cos(3\theta+50) = 0 \\
 & \rightarrow \cos(3\theta+50) = -\sin(2\theta+20) \\
 & \rightarrow \cos(3\theta+50) = \sin(-2\theta-20) \quad \{ \sin(-A) \equiv -\sin A \} \\
 & \Rightarrow \cos(3\theta+50) = \cos(10^\circ - (-2\theta-20)) \quad \leftarrow \sin A = \cos(90^\circ - A) \\
 & \Rightarrow \cos(3\theta+50) = \cos(10^\circ + 2\theta) \\
 & \quad (3\theta+50) = 10 + 2\theta \pm 360^\circ \quad k=0,1,2,3 \\
 & \quad (3\theta+50) = 360 - (10 + 2\theta) \pm 360^\circ \\
 & \quad (3\theta+50) = 360 \pm 360^\circ \\
 & \quad (3\theta+50) = 60 \pm 360^\circ \\
 & \quad (3\theta+50) = 40 \pm 360^\circ \quad \therefore \theta = 40^\circ, 60^\circ, 112^\circ, 184^\circ, 256^\circ, 328^\circ
 \end{aligned}$$

Question 72 (***)**

A curve has equation

$$y = \ln \left[\tan \left(x + \frac{\pi}{4} \right) \right], \quad -\frac{\pi}{4} < x < \frac{\pi}{4}.$$

Show clearly that

$$\frac{dy}{dx} = 2 \sec 2x.$$

[proof]

$$\begin{aligned}
 y &= \ln(\tan(x + \frac{\pi}{4})) = \ln \left[\frac{\tan x + \tan \frac{\pi}{4}}{1 - \tan x \tan \frac{\pi}{4}} \right] = \ln \left[\frac{\tan x + 1}{1 - \tan x} \right] \\
 \therefore y &= \ln(1 + \tan x) - \ln(1 - \tan x) \\
 \frac{dy}{dx} &= \frac{1}{1 + \tan x} \times \sec^2 x - \frac{1}{1 - \tan x} \times (-\sec^2 x) = \frac{\sec^2 x}{1 + \tan x} + \frac{\sec^2 x}{1 - \tan x} \\
 &= \sec^2 x \left[\frac{1}{1 + \tan x} + \frac{1}{1 - \tan x} \right] = \sec^2 x \left[\frac{1 - \tan x + 1 + \tan x}{(1 + \tan x)(1 - \tan x)} \right] \\
 &= \frac{2 \sec^2 x}{1 - \tan^2 x} = \text{cancel top bottom by } \sec^2 x \\
 &= \frac{2 \sec^2 x \sec^2 x}{\sec^2 x - \sec^2 x \tan^2 x} = \frac{2}{\sec^2 x - \sec^2 x \tan^2 x} = \frac{2}{\sec^2 x - \sec^2 x \frac{\sin^2 x}{\cos^2 x}} \\
 &= \frac{2}{\sec^2 x} = 2 \sec 2x
 \end{aligned}$$

$$\begin{aligned}
 \text{ALTERNATE} \\
 y &= \ln \left[\tan \left(x + \frac{\pi}{4} \right) \right] \\
 \frac{dy}{dx} &= \frac{1}{\tan \left(x + \frac{\pi}{4} \right)} \times \sec^2 \left(x + \frac{\pi}{4} \right) = \frac{1}{\frac{\sin \left(x + \frac{\pi}{4} \right)}{\cos \left(x + \frac{\pi}{4} \right)}} \times \frac{1}{\cos^2 \left(x + \frac{\pi}{4} \right)} \\
 &= \frac{\cos \left(x + \frac{\pi}{4} \right)}{\sin \left(x + \frac{\pi}{4} \right)} \times \frac{1}{\cos^2 \left(x + \frac{\pi}{4} \right)} = \frac{1}{\sin \left(x + \frac{\pi}{4} \right) \cos \left(x + \frac{\pi}{4} \right)} = \frac{2}{2 \sin \left(x + \frac{\pi}{4} \right) \cos \left(x + \frac{\pi}{4} \right)} \\
 &= \frac{2}{\sin 2 \left(x + \frac{\pi}{4} \right)} = \frac{2}{\sin \left(2x + \frac{\pi}{2} \right)} = \frac{2}{2 \sin x \cos x + \cos 2x \sin x} \\
 &= \frac{2}{\cos 2x} = 2 \sec 2x
 \end{aligned}$$

Question 73 (***)**

Prove the validity of each of the following trigonometric identities.

i. $\sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta \equiv \tan \theta + \cot \theta$.

ii. $\sin \theta(1 + \tan \theta) + \cos \theta(1 + \cot \theta) \equiv \sec \theta + \operatorname{cosec} \theta$.

 ,  proof

i) L.H.S. = $\sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta$

$$\begin{aligned} &= \sin^2 \theta \left(\frac{\sin \theta}{\cos \theta}\right) + \cos^2 \theta \left(\frac{\cos \theta}{\sin \theta}\right) + 2 \sin \theta \cos \theta \\ &= \frac{\sin^4 \theta}{\cos \theta} + 2 \sin \theta \cos \theta + \frac{\cos^4 \theta}{\sin \theta} \\ &= \frac{\sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta + \cos^4 \theta}{\cos \theta \sin \theta} \\ &= \frac{(\sin^2 \theta + \cos^2 \theta)^2}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta} \\ &= \frac{\cos^2 \theta + \sin^2 \theta}{\cos \theta \sin \theta} = \frac{\cos^2 \theta}{\cos \theta \sin \theta} + \frac{\sin^2 \theta}{\cos \theta \sin \theta} \\ &= \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} = \cot \theta + \tan \theta = \text{R.H.S.} // \end{aligned}$$

ii) L.H.S. = $\sin \theta(1 + \tan \theta) + \cos \theta(1 + \cot \theta)$

$$\begin{aligned} &= \sin \theta + \sin \theta \tan \theta + \cos \theta + \cos \theta \cot \theta \\ &= \sin \theta + \frac{\sin^2 \theta}{\cos \theta} + \cos \theta + \frac{\cos^2 \theta}{\sin \theta} \\ &= \frac{\sin^2 \theta \cos \theta + \sin \theta \cos^2 \theta + \cos^2 \theta \sin \theta + \sin \theta \cos \theta}{\cos \theta \sin \theta} \\ &= \frac{\sin^2 \theta (\cos \theta + \sin \theta) + \cos^2 \theta (\cos \theta + \sin \theta)}{\cos \theta \sin \theta} \\ &= \frac{(\cos \theta + \sin \theta)(\sin^2 \theta + \cos^2 \theta)}{\cos \theta \sin \theta} = \frac{\cos \theta + \sin \theta}{\cos \theta \sin \theta} \\ &= \frac{\cos \theta}{\cos \theta \sin \theta} + \frac{\sin \theta}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta} = \sec \theta + \operatorname{cosec} \theta = \text{R.H.S.} // \end{aligned}$$

Question 74 (*****)

Given the simultaneous equations

$$3 \tan \theta + 4 \tan \varphi = 8$$

$$\theta + \varphi = \frac{\pi}{2},$$

find the possible values of $\tan \theta$ and $\tan \varphi$.

$$[\tan \theta, \tan \varphi] = \left[2, \frac{1}{2} \right] = \left[\frac{2}{3}, \frac{3}{2} \right]$$

Working:

$$\begin{aligned} 3 \tan \theta + 4 \tan \varphi &= 8 \\ \theta + \varphi &= \frac{\pi}{2} \end{aligned}$$

From the second equation, $\tan(\theta + \varphi) = \tan \frac{\pi}{2}$ which is undefined. This implies $\tan \theta \neq -\tan \varphi$. Substituting $\tan \varphi = \frac{1}{\tan \theta}$ into the first equation:

$$3 \tan \theta + 4 \cdot \frac{1}{\tan \theta} = 8$$
$$3 \tan^2 \theta + 4 = 8 \tan \theta$$
$$3 \tan^2 \theta - 8 \tan \theta + 4 = 0$$
$$(3 \tan \theta - 2)(\tan \theta - 2) = 0$$
$$\tan \theta = 2 \quad \text{or} \quad \tan \theta = \frac{2}{3}$$

Hence either $\tan \theta = 2$, $\tan \varphi = \frac{1}{2}$ or $\tan \theta = \frac{2}{3}$, $\tan \varphi = \frac{3}{2}$

Question 75 (*****)

A relationship between x and y is given by the equations

$$x = \operatorname{cosec} \theta - \sin \theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$y = \sec \theta - \cos \theta, \quad 0 < \theta < \frac{\pi}{2}.$$

Use trigonometric identities to show that

$$y^2 x^2 \left(x^{\frac{2}{3}} + y^{\frac{2}{3}} \right)^3 = 1.$$

proof

$$\begin{aligned} x &= \operatorname{cosec} \theta - \sin \theta \\ y &= \sec \theta - \cos \theta \end{aligned} \Rightarrow \begin{aligned} \frac{x}{\sin \theta} &= \frac{1}{\sin \theta} - \sin \theta = \frac{1 - \sin^2 \theta}{\sin \theta} = \frac{\cos^2 \theta}{\sin \theta} = \frac{\cos \theta}{\sin \theta} = \tan \theta \\ \frac{y}{\cos \theta} &= \frac{1}{\cos \theta} - \cos \theta = \frac{1 - \cos^2 \theta}{\cos \theta} = \frac{\sin^2 \theta}{\cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta \end{aligned} \therefore \tan \theta = \left(\frac{y}{x} \right)^{\frac{1}{2}}$$

$$\begin{aligned} y^2 x^2 &= \left(\frac{y}{x} \right)^2 \left(\frac{x}{\sin \theta} \right)^2 = \frac{\tan^2 \theta}{\sin^2 \theta} = \frac{\tan^2 \theta}{1 - \cos^2 \theta} = \frac{\tan^2 \theta}{\sin^2 \theta} = \text{MUST USE } \sin^2 \theta + \cos^2 \theta = 1 \\ y^2 &= \tan^2 \theta \sin^2 \theta \\ y^2 &= \tan^2 \theta \sin^2 \theta \\ y^2 &= \tan^2 \theta (1 - \cos^2 \theta) \\ y^2 &= \tan^2 \theta \left(1 - \frac{1}{\sec^2 \theta} \right) \\ y^2 &= \tan^2 \theta \left(\frac{1 - \sec^2 \theta}{\sec^2 \theta} \right) \\ y^2 &= \tan^2 \theta \left(\frac{-\tan^2 \theta}{\sec^2 \theta} \right) \\ y^2 &= \frac{\tan^4 \theta}{\sec^2 \theta} \end{aligned}$$

Question 76 (*****)

Solve the trigonometric equation

$$\sec x + \sqrt{3} \operatorname{cosec} x = 4, \quad 0 \leq x < \pi,$$

giving the answers in terms of π .

$$\theta = \frac{2\pi}{9}, \frac{\pi}{3}, \frac{8\pi}{9}$$

$$\begin{aligned}
 \sec x + \sqrt{3} \operatorname{cosec} x &= 4 \\
 \Rightarrow \frac{1}{\cos x} + \frac{\sqrt{3}}{\sin x} &= 4 \\
 \Rightarrow \sin x + \sqrt{3} \cos x &= 4 \cos x \sin x \\
 \Rightarrow \sin x + \sqrt{3} \cos x &= 2 \sin x \cos x \\
 \Rightarrow 2 \left[\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x \right] &= 2 \sin x \cos x \\
 \Rightarrow 2 \left[\sin \frac{\pi}{6} \sin x + \cos \frac{\pi}{6} \cos x \right] &= 2 \sin x \cos x \\
 \Rightarrow 2 \sin \left(x + \frac{\pi}{6} \right) &= 2 \sin x \cos x \\
 \Rightarrow \sin \left(2x + \frac{\pi}{3} \right) &= \sin 2x \\
 \Rightarrow \left(2x + \frac{\pi}{3} \right) &= 2x + 2n\pi \\
 \Rightarrow 2x + \frac{\pi}{3} &= \pi - 2x + 2n\pi \\
 \Rightarrow 4x = \frac{2\pi}{3} + 2n\pi & \\
 \Rightarrow x = \frac{\pi}{6} + \frac{n\pi}{2} & \\
 \Rightarrow x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{8\pi}{9} &
 \end{aligned}$$

$$t = \tan \frac{x}{2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\frac{1-t^2}{1+t^2} + \frac{\sqrt{3}(1+t)}{2t} = 4$$

$$2t(1+t) + \sqrt{3}(1+t)(1-t) = 8t(1-t)$$

$$2t + 2t^2 + \sqrt{3} - \sqrt{3}t^2 = 8t - 8t^2$$

$$0 = \sqrt{3}t^2 - 10t^2 + 8t - \sqrt{3}$$

DIFFICULT TO FACTORISE

$$\frac{\sqrt{3}}{3} \text{ IS A SOLUTION}$$

$$(3t - \sqrt{3}) \text{ IS A ROOT}$$

GCF ETC.

Question 77 (*****)

It is given that

$$\cos\left(\frac{\pi}{12}\right) \equiv \cos\left(\frac{\pi}{4}\right)[\cos(A\pi) + \cos(B\pi)],$$

where A and B are constants such that $0 < A < 1$ and $0 < B < 1$.

Determine the value of A and the value of B .

$$A = \frac{1}{6}, \quad B = \frac{2}{3}$$

$$\begin{aligned}
 \sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6} \\
 &= \cos \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6} \\
 &= \cos \frac{\pi}{4} \left[\cos \frac{\pi}{6} - \sin \frac{\pi}{6} \right] \\
 &= \cos \frac{\pi}{4} \left[\cos \frac{\pi}{6} + \sin \left(-\frac{\pi}{6}\right) \right] \\
 &= \cos \frac{\pi}{4} \left[\cos \frac{\pi}{6} + \cos \left(\frac{\pi}{6} - \pi\right) \right] \\
 &= \cos \frac{\pi}{4} \left[\cos \frac{\pi}{6} + \cos \frac{4\pi}{3} \right]
 \end{aligned}$$

Question 78 (*****)

Find the only finite solution of the trigonometric equation

$$\arcsin\left(\frac{x}{x-1}\right) + 2\arctan\left(\frac{1}{x+1}\right) = \frac{\pi}{2}.$$

$$x = 0$$

$\arcsin\left(\frac{x}{x-1}\right) + 2\arctan\left(\frac{1}{x+1}\right) = \frac{\pi}{2}$

 $\Rightarrow 2\arctan\left(\frac{1}{x+1}\right) = \frac{\pi}{2} - \arcsin\left(\frac{x}{x-1}\right)$
 $\Rightarrow \theta = \frac{\pi}{2} - \phi$
 $\Rightarrow \cos 2\theta = \cos\left(\frac{\pi}{2} - \phi\right)$
 $\Rightarrow 1 - 2\sin^2\theta = \cos\frac{\pi}{2}\cos\phi + \sin\frac{\pi}{2}\sin\phi$
 $\Rightarrow 1 - 2\left(\frac{1}{\sqrt{2(x+1)}}\right)^2 = \sin\phi$
 $\Rightarrow 1 - \frac{2}{x+2+2} = \frac{x}{x-1}$
 $\Rightarrow (x^2+2x+2)(x-1) - 2(x-1) = x(x^2+2x+2)$
 $\Rightarrow x^3+2x^2+2x - 2x^2-2x-2 = x^3+2x^2+2x$
 $\Rightarrow x^3+2x^2+2x - 2x^2-2x-2 = x^3+2x^2+2x$
 $x \neq 0 \quad (\text{finite solutions only})$
 $\Rightarrow 0 = x^2+4x$
 $\Rightarrow x(x+4) = 0$
 $\Rightarrow x = -4$
 $\therefore x = -4 \quad \text{arcsin}\left(\frac{1}{3}\right) + 2\arctan\left(-\frac{1}{3}\right) \neq \frac{\pi}{2}$
 $\therefore x = 0$

Question 79 (*****)

Prove the validity of the following trigonometric identity

$$\sin^4 \theta + \cos^4 \theta \equiv \frac{1}{2}(2 - \sin^2 2\theta).$$

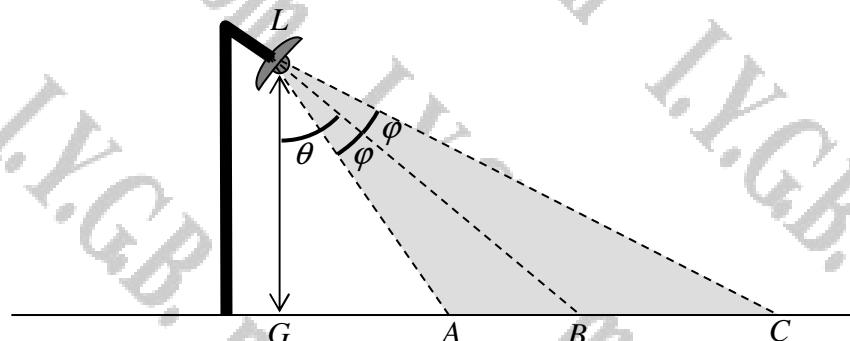
LHS = $\sin^4 \theta + \cos^4 \theta = (\sin^2 \theta)^2 + (\cos^2 \theta)^2$

 $= \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)^2 + \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)^2$
 $= \left(\frac{1}{4} - \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta\right) + \left(\frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta\right)$
 $= \frac{1}{2} + \frac{1}{2}\cos^2 2\theta$
 $= \frac{1}{2} + \frac{1}{2}(1 - \sin^2 2\theta) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\sin^2 2\theta = 1 - \frac{1}{2}\sin^2 2\theta$
 $= \frac{1}{2}(2 - \sin^2 2\theta) = \text{RHS}$

Hence proved

LHS = $\sin^4 \theta + \cos^4 \theta = (\sin^2 \theta + 2\cos^2 \theta \sin^2 \theta + \cos^4 \theta) - 2\sin^2 \theta \cos^2 \theta$
 $= [(\sin^2 \theta)^2 - 2\sin^2 \theta \cos^2 \theta + (\cos^2 \theta)^2] - 2(\sin^2 \theta \cos^2 \theta)$
 $= (\sin^2 \theta + \cos^2 \theta)^2 - 2\left[\frac{1}{4} \times 2\sin^2 \theta \cos^2 \theta\right]^2$
 $= 1^2 - 2 \times \left(\frac{1}{4}\sin^2 2\theta\right)^2 = 1 - 2 \times \frac{1}{16}\sin^4 2\theta = \text{RHS}$

Question 80 (*****)



The figure above shows a spotlight L , beaming down on level ground, where L is mounted at a height of 12 metres and the point G is directly below L .

The bulb emits light in the shape of a cone whose axis of symmetry LB is angled at θ° to the vertical.

The beam is φ° wide all the way round the axis of symmetry of the light cone

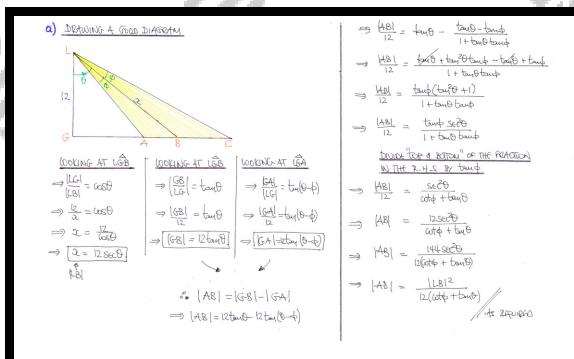
- a) Show that the length of AB is

$$|AB| = \frac{12 \sec^2 \theta}{\cot \varphi + \tan \theta}$$

The lengths of LG and AB are $8\sqrt{3}$ metres and 8 metres, respectively.

- b) Show further that $\tan \varphi = \frac{1}{11}(6 + \sqrt{3})$.

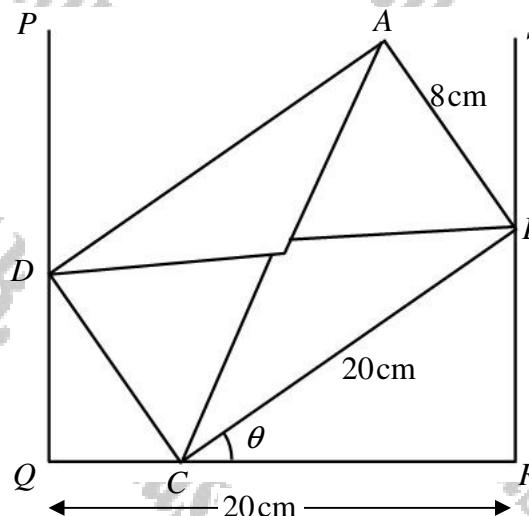
SFR, proof



W $|B| = B\sqrt{3}$ & $|AB| = b$

- Find θ first
- $2 = |AB| = \sqrt{3} \sec \theta$
 $\Rightarrow \sqrt{3} \sec \theta = 2 \sec \theta$
 $\Rightarrow \sec \theta = \frac{2}{\sqrt{3}}$
 $\Rightarrow \cot \theta = \frac{\sqrt{3}}{2}$
 $\Rightarrow \cot \theta = \frac{3\sqrt{3}}{4\sqrt{3}}$
 $\Rightarrow \cot \theta = \frac{3\sqrt{3}}{4}$
 $\Rightarrow \cot \theta = \frac{\sqrt{3}}{2}$
 $\Rightarrow \theta = 30^\circ \text{ (Ans)}$
- $|AB|^2 = \frac{(18)^2}{12(\cot^2 \theta + \csc^2 \theta)}$
 $\Rightarrow B = \frac{(8\sqrt{3})^2}{12(\cot^2 \theta + \csc^2 \theta)}$
 $\Rightarrow \sqrt{3}B = \frac{8\sqrt{3} \times 3}{12(\cot^2 \theta + \csc^2 \theta)}$
 $\Rightarrow \frac{1}{2} = \frac{1}{\cot^2 \theta + \csc^2 \theta}$
 $\Rightarrow 2 = \cot^2 \theta + \csc^2 \theta$
 $\Rightarrow 2 = \cot^2 \theta + \frac{1}{\cot^2 \theta}$
 $\Rightarrow 2 - \frac{1}{\cot^2 \theta} = \cot^2 \theta$
 $\Rightarrow \cot^2 \theta = \frac{1}{2 - \frac{1}{\cot^2 \theta}} = \frac{\sqrt{3}^2}{2(2\sqrt{3}-1)} = \frac{3}{4(2\sqrt{3}-1)} = \frac{3}{8(2\sqrt{3}-1)}$
 $\Rightarrow \cot \theta = \frac{\sqrt{3}}{2\sqrt{3}-1} = \frac{6+3\sqrt{3}}{12-1} = \frac{6+3\sqrt{3}}{11}$

Question 81 (*****)



The figure above shows the cross section of a letter inside a filling slot.

The letter $ABCD$ is modelled as a rectangle with $|AB| = 8\text{cm}$ and $|BC| = 20\text{cm}$.

The width of the filling slot QR is also 20 cm and the angle BCR is θ .

Determine the value of θ .

$$\boxed{\quad}, \quad \theta \approx 43.6^\circ$$

• START WITH A DIAGRAM TO WRITE ANGLES

$$|\angle QCL| + |\angle CLB| = 180^\circ$$

$$|\angle DCB| + |\angle CBA| = 20$$

$$20\sin\theta + 20\cos\theta = S$$

• PROCEED WITH THE "R - TRANSFORMATION" IN SINT

$$20\sin\theta + 20\cos\theta = R\sin(\theta+90^\circ)$$

$$20\sin\theta + 20\cos\theta = R\sin\theta\cos90^\circ + R\cos\theta\sin90^\circ$$

$$\begin{cases} R\cos\theta = 2 \\ R\sin\theta = S \end{cases} \Rightarrow R = \sqrt{2^2 + S^2} = \sqrt{241}$$

$$\tan\theta = \frac{S}{2} \Rightarrow \theta \approx 43.6^\circ$$

• SOLVING THE EQUATION, FOR $0 < \theta < 90$

$$\Rightarrow 20\sin\theta + 20\cos\theta = S$$

$$\Rightarrow 20\sin(\theta+90^\circ) = S$$

$$\Rightarrow \sin(\theta+90^\circ) = \frac{S}{20}$$

$$\Rightarrow \begin{cases} \theta+90^\circ = 30^\circ \\ \theta+90^\circ = 150^\circ \end{cases} \Rightarrow \theta = 0^\circ \pm 30^\circ$$

$$\Rightarrow \begin{cases} \theta = 43.6^\circ \pm 30^\circ \\ \theta = 43.6^\circ \end{cases}$$

∴ ONLY PHYSICAL ANGLE IS 43.6°

Question 82 (***)**

Solve the following trigonometric equation

$$\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta} = 1, \quad 0 \leq \theta < 2\pi.$$

, $\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

① FIRST REARRANGE THE SQUARE ROOTS, BY THE COSEC DOUBLE ANGLE FORMULA

$$\rightarrow \sqrt{1+(2\cos^2\theta-1)} - \sqrt{1-(1-2\sin^2\theta)} = 1$$

$$\rightarrow \sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta} = 1$$

$$\rightarrow \sqrt{2}|\cos\theta| - \sqrt{2}|\sin\theta| = 1$$

② NOW IN EACH QUADRANT WE HAVE

$0 \leq \theta \leq \frac{\pi}{2}$	$\sqrt{2}\cos\theta - \sqrt{2}\sin\theta = 1$
$\frac{\pi}{2} \leq \theta \leq \pi$	$-\sqrt{2}\cos\theta - \sqrt{2}\sin\theta = 1$
$\pi \leq \theta \leq \frac{3\pi}{2}$	$-\sqrt{2}\cos\theta + \sqrt{2}\sin\theta = 1$
$\frac{3\pi}{2} \leq \theta \leq 2\pi$	$\sqrt{2}\cos\theta + \sqrt{2}\sin\theta = 1$

③ SOLVING SEPARATELY IN EACH QUADRANT, STARTING WITH $0 \leq \theta \leq \frac{\pi}{2}$

$$\rightarrow \sqrt{2}\cos\theta - \sqrt{2}\sin\theta = 1$$

$$\Rightarrow \frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta = \frac{1}{2}$$

$$\Rightarrow \cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4} = \frac{1}{2}$$

$$\Rightarrow \cos(\theta + \frac{\pi}{4}) = \frac{1}{2}$$

$$\Rightarrow \left(\theta + \frac{\pi}{4} = \frac{\pi}{3} \text{ or } 2\pi \right) \quad n=0,1,2,3,...$$

$$\Rightarrow \left(\theta = \frac{\pi}{12} \pm 2k\pi \right)$$

$$\Rightarrow \left(\theta = \frac{11\pi}{12} \pm 2k\pi \right)$$

④ FOR $\frac{\pi}{2} \leq \theta \leq \pi$

$$\rightarrow -\sqrt{2}\cos\theta - \sqrt{2}\sin\theta = 1$$

$$\Rightarrow \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta = -\frac{1}{2}$$

$$\Rightarrow \cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4} = -\frac{1}{2}$$

$$\Rightarrow \cos(\theta - \frac{\pi}{4}) = -\frac{1}{2}$$

$$\Rightarrow \left(\theta - \frac{\pi}{4} = \frac{4\pi}{3} \text{ or } 2\pi \right) \quad n=0,1,2,3,...$$

$$\Rightarrow \left(\theta = \frac{19\pi}{12} \pm 2k\pi \right)$$

⑤ FOR $\pi \leq \theta \leq \frac{3\pi}{2}$

$$\rightarrow -\sqrt{2}\cos\theta + \sqrt{2}\sin\theta = 1$$

$$\Rightarrow -\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta = \frac{1}{2}$$

$$\Rightarrow \frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta = \frac{1}{2}$$

$$\Rightarrow \cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4} = \frac{1}{2}$$

$$\Rightarrow \cos(\theta + \frac{\pi}{4}) = \frac{1}{2}$$

$$\Rightarrow \left(\theta + \frac{\pi}{4} = \frac{\pi}{3} \text{ or } 2\pi \right) \quad n=0,1,2,3,...$$

$$\Rightarrow \left(\theta = \frac{11\pi}{12} \pm 2k\pi \right)$$

⑥ FOR $\frac{3\pi}{2} \leq \theta \leq 2\pi$

$$\rightarrow \sqrt{2}\cos\theta + \sqrt{2}\sin\theta = 1$$

$$\Rightarrow \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta = \frac{1}{2}$$

$$\Rightarrow \cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4} = \frac{1}{2}$$

$$\Rightarrow \cos(\theta - \frac{\pi}{4}) = \frac{1}{2}$$

$$\Rightarrow \left(\theta - \frac{\pi}{4} = \frac{\pi}{3} \pm 2k\pi \right) \quad n=0,1,2,3,...$$

$$\Rightarrow \left(\theta = \frac{19\pi}{12} \pm 2k\pi \right)$$

∴ $\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

Question 83 (***)**

Prove the validity of the following trigonometric identity.

$$\frac{\sin 4\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} = 4\cos 3\theta, \quad \theta \neq n\pi.$$

S K, proof

• STARTING ON THE LEFT-HAND-SIDE

$$\begin{aligned}\frac{\sin 4\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} &= \frac{2\sin 2\theta \cos 2\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} \\&= \frac{2(2\sin \theta \cos \theta) \cos 2\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} \\&= \frac{4\sin \theta \cos^2 \theta \cos 2\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} \\&= 4\cos^2 \theta \cos 2\theta - 8\sin^2 \theta \cos \theta \\&= 4\cos^2 \theta \cos 2\theta - 4(2\sin^2 \theta \cos \theta) \sin \theta \\&= 4[\cos^2 \theta \cos 2\theta - \sin^2 \theta \cos \theta] \\&= 4 \cos(2\theta + \theta) \\&= 4\cos 3\theta \\&= \text{R.H.S.} \end{aligned}$$

• ALTERNATIVE CALCULATION STARTING WITH THE LEFT-HAND-SIDE TERM

$$\begin{aligned}\frac{\sin 4\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} &= \frac{2\sin 2\theta \cos 2\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} \\&= \frac{2(2\sin^2 \theta \cos \theta) \cos 2\theta - 8\sin^3 \theta \cos \theta}{\sin \theta} \end{aligned}$$

$$\begin{aligned}&= 4\cos^2 \theta \cos 2\theta - 8\sin^2 \theta \cos \theta \\&= 4\cos^2 \theta (2\cos^2 \theta - 1) - 8(1 - \cos^2 \theta) \cos \theta \\&= 8\cos^4 \theta - 4\cos^2 \theta - 8\cos \theta + 8\cos^2 \theta \\&= 16\cos^4 \theta - 12\cos^2 \theta \\&= 4[4\cos^2 \theta - 3\cos^2 \theta] \\&= 4\cos 2\theta \quad \xrightarrow{\text{R.H.S.}} \\&\cos 3\theta \equiv \cos(2\theta + \theta) \\&\equiv \cos 2\theta \cos \theta - \sin 2\theta \sin \theta \\&\equiv (2\cos^2 \theta - 1)\cos \theta - (2\sin \theta \cos \theta) \sin \theta \\&\equiv 2\cos^3 \theta - \cos \theta - 2\sin^2 \theta \cos \theta \\&\equiv 2\cos^3 \theta - \cos \theta - 2(1 - \cos^2 \theta) \cos \theta \\&\equiv 2\cos^3 \theta - \cos \theta - 2\cos \theta + 2\cos^3 \theta \\&\equiv 4\cos^3 \theta - 3\cos \theta \\&<\text{AS USED ABOVE} \end{aligned}$$

Question 84 (***)**

The three angles of a triangle are denoted by α , β and γ .

a) Show clearly that ...

i. ... $\sin(\alpha + \beta) = \sin \gamma$.

ii. ... $\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\gamma}{2}\right)$.

iii. ... $\sin \alpha + \sin \beta + \sin(\alpha + \beta) = 2 \sin\left(\frac{\alpha+\beta}{2}\right) [\cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\alpha-\beta}{2}\right)]$.

iv. ... $\sin \alpha + \sin \beta + \sin \gamma = 4 \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$.

b) By using (iv) with suitable values for α , β and γ , show that

$$\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

 , proof

a) i) $\alpha + \beta + \gamma = \pi$
 $\rightarrow \alpha + \beta = \pi - \gamma$
 $\rightarrow \sin(\alpha + \beta) = \sin(\pi - \gamma)$
 $\rightarrow \sin(\alpha + \beta) = \sin \gamma$ ✓
 $\Rightarrow \sin(\alpha + \beta) = \sin \gamma$ ✓ As required

ii) $\alpha + \beta + \gamma = \pi$
 $\rightarrow \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \frac{\pi}{2}$
 $\rightarrow \alpha + \beta = \frac{\pi}{2} - \frac{\gamma}{2}$
 $\rightarrow \sin\left(\frac{\alpha+\beta}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{\gamma}{2}\right)$
 $\rightarrow \sin\left(\frac{\alpha+\beta}{2}\right) = \sin\frac{\pi}{2} \cos\frac{\gamma}{2} - \cos\frac{\pi}{2} \sin\frac{\gamma}{2}$
 $\Rightarrow \sin\left(\frac{\alpha+\beta}{2}\right) = \cos\frac{\gamma}{2}$ ✓ As required

iii) $\sin(\alpha) + \sin(\beta) + \sin(\alpha + \beta)$
Using $\sin P + \sin Q \equiv 2 \sin\frac{P+Q}{2} \cos\frac{P-Q}{2}$
 $= 2 \sin\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha-\beta}{2}\right) + \sin\left[\frac{\pi}{2} - \left(\frac{\alpha-\beta}{2}\right)\right]$
Using $\sin 2A \equiv 2 \sin A \cos A$
 $= 2 \sin\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha-\beta}{2}\right) + 2 \sin\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha+\beta}{2}\right)$
 $= 2 \sin\left(\frac{\alpha+\beta}{2}\right) \left[\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}\right]$ ✓ As required

(iii) COMBINING THE CASE 3 RESULTS

$$\begin{aligned} \sin \alpha + \sin \beta + \sin \gamma &= \sin \alpha + \sin \beta + \sin(\alpha + \beta) \\ &= 2 \sin\left(\frac{\alpha}{2}\right) \left[\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right) \right] \\ &\quad \text{Using } \cos P + \cos Q = 2 \cos\frac{P+Q}{2} \cos\frac{P-Q}{2} \\ &= 2 \cos\frac{\alpha}{2} \times 2 \cos\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha-\beta}{2}\right) \\ &= 2 \cos\frac{\alpha}{2} \times 2 \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \quad \leftarrow \text{cancel } \cos \alpha \\ &= 4 \cos^2\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2} \quad \text{As required} \end{aligned}$$

b) Using $\sin \alpha + \sin \beta + \sin \gamma = 4 \cos^2\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$
With $\alpha = \frac{\pi}{2}$
 $\beta = \frac{\pi}{3}$
 $\gamma = \frac{\pi}{4}$
 $\sin\frac{\pi}{2} + \sin\frac{\pi}{3} + \sin\frac{\pi}{4} = 4 \cos^2\frac{\pi}{4} \cos\frac{\pi}{6} \cos\frac{\pi}{12}$
 $1 + \frac{\sqrt{3}}{2} + \frac{1}{2} = 4 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} \times \cos\frac{\pi}{12}$
 $\frac{3 + \sqrt{3}}{2} = \sqrt{2} \cos\frac{\pi}{12}$
 $\cos\frac{\pi}{12} = \frac{3 + \sqrt{3}}{2\sqrt{2}}$
 $\cos\frac{\pi}{12} = \frac{3\sqrt{2} + \sqrt{6}}{4} = \frac{3\sqrt{2} + 3\sqrt{2}}{4}$
 $\cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$ ✓

Question 85 (***)**

Simplify, showing all steps in the calculation, the expression

$$\arctan \frac{4}{3} + \arctan 2 - \arctan 3,$$

giving the answer in terms of π .

$$\boxed{\frac{\pi}{4}}$$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ & $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\therefore \tan(A+B-C) = \frac{\tan A + \tan B - \tan C + \tan A \tan B \tan C}{1 - \tan A \tan B - \tan C + \tan A \tan B \tan C}$$

Multiplying bottom by $1 - \tan A \tan B$

$$\tan(A+B-C) = \frac{\tan A + \tan B - \tan C(1 - \tan A \tan B)}{1 - \tan A \tan B + (\tan A + \tan B)\tan C}$$

$$\tan(A+B-C) = \frac{\tan A + \tan B - \tan C + \tan A \tan B \tan C}{1 - \tan A \tan B + \tan A \tan C + \tan B \tan C}$$

Let $A = \arctan \frac{4}{3} \Rightarrow \tan A = \frac{4}{3}$
 $B = \arctan 2 \Rightarrow \tan B = 2$
 $C = \arctan 3 \Rightarrow \tan C = 3$

Now $\tan(A+B-C) = \frac{\frac{4}{3} + 2 - 3 + \frac{4}{3} \times 2 \times 3}{1 - \frac{4}{3} \times 2 + \frac{4}{3} \times 3 + 2 \times 3} = \frac{25/3}{25/3} = 1$

$$\therefore A+B-C = \arctan 1$$

$$A+B-C = \frac{\pi}{4}$$

$$\therefore \arctan \frac{4}{3} + \arctan 2 - \arctan 3 = \frac{\pi}{4}$$

Alternative:

$$\frac{(3+4i)(1+2i)}{1+3i} = \frac{3+6i+4i-8}{1+3i} = \frac{-5+10i}{1+3i} = \frac{(-5+10i)(1-3i)}{(1+3i)(1-3i)}$$

$$= \frac{-5+15i+10i+30}{10} = \frac{25+25i}{10} = \frac{25}{10} + \frac{25}{10}i$$

Hence

$$\arg\left[\frac{(3+4i)(1+2i)}{1+3i}\right] = \arg\left(\frac{25}{10} + \frac{25}{10}i\right)$$

$$\arg(3+4i) + \arg(1+2i) - \arg(1+3i) = \arg\left(\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}}i\right)$$

$$\arctan \frac{2}{\sqrt{5}} + \arctan 2 - \arctan 3 = \arctan 1$$

$$\arctan \frac{4}{3} + \arctan 2 - \arctan 3 = \frac{\pi}{4}$$

AS Required

Question 86 (***)**

Solve the trigonometric equation

$$\sec x + \operatorname{cosec} x = 2\sqrt{2}, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

□, $x = \frac{\pi}{4}, \frac{11\pi}{12}, \frac{19\pi}{12}$

② $\sec x + \operatorname{cosec} x = 2\sqrt{2} \quad 0 \leq x < 2\pi$

$$\Rightarrow \frac{1}{\cos x} + \frac{1}{\sin x} = 2\sqrt{2}$$

Multiply through by sine cosine.

$$\Rightarrow \sin x + \cos x = 2\sqrt{2} \sin x \cos x$$

$$\Rightarrow (\sin x + \cos x)^2 = (2\sqrt{2} \sin x \cos x)^2$$

$$\Rightarrow \sin^2 x + 2\sin x \cos x + \cos^2 x = 8\sin^2 x \cos^2 x$$

$$\Rightarrow 1 + 2\sin x \cos x = 2(2\sin x \cos x)^2$$

• This is a quadratic in $2\sin x \cos x = \sin 2x$

$$\Rightarrow 1 + \sin 2x = 2\sin^2 2x$$

$$\Rightarrow 2\sin^2 2x - \sin 2x - 1 = 0$$

$$\Rightarrow (2\sin 2x + 1)(\sin 2x - 1) = 0$$

$$\Rightarrow \sin 2x = -\frac{1}{2}$$

(Leave in degrees, because it will be easier to check the solutions.)

$$2x = 90^\circ \pm 360^\circ \quad \text{or} \quad 2x = -90^\circ \pm 360^\circ$$

$$2x = 90^\circ \pm 360^\circ \quad \text{or} \quad 2x = 210^\circ \pm 360^\circ$$

$$(x = 45^\circ \pm 180^\circ) \quad \text{or} \quad (x = -45^\circ \pm 180^\circ)$$

• COLLECTING SOLUTIONS FOR $0 \leq x < 2\pi$ OR $0 \leq x < 360^\circ$

$$x = 45^\circ, 225^\circ, 165^\circ, 305^\circ, 105^\circ, 285^\circ$$

• $x = 45^\circ$ IS OK $\sec 45^\circ + \operatorname{cosec} 45^\circ = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

• $x = 225^\circ$ IS NOT OK $\sec 225^\circ + \operatorname{cosec} 225^\circ < 0$

TO CHECK THE REST

$$\sec 15^\circ = \frac{1}{\cos 15^\circ} = \frac{1}{\cos(45^\circ - 30^\circ)} = \frac{1}{\cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ} = \frac{1}{\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}} = \frac{1}{\frac{\sqrt{6} + \sqrt{2}}{4}} = \sqrt{6} + \sqrt{2}$$

$$\operatorname{cosec} 15^\circ = \frac{1}{\sin 15^\circ} = \frac{1}{\sin(45^\circ - 30^\circ)} = \frac{1}{\sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ} = \frac{1}{\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}} = \frac{1}{\frac{\sqrt{6} - \sqrt{2}}{4}} = \frac{4}{\sqrt{6} - \sqrt{2}} = \frac{4(\sqrt{6} + \sqrt{2})}{4} = \sqrt{6} + \sqrt{2}$$

THIS $\sec 15^\circ = \operatorname{cosec} 15^\circ = \sqrt{6} + \sqrt{2}$
 $\sec 15^\circ = \operatorname{cosec} 15^\circ = \sqrt{6} + \sqrt{2}$

Finally we have 3 valid solutions in the interval
 $x = 45^\circ, 165^\circ, 285^\circ$
 $x = \frac{45}{12}, \frac{165}{12}, \frac{285}{12}$

$\frac{45+15}{12} = \frac{60}{12} = 5$

Question 87 (***)**

Prove the validity of the trigonometric identity

$$\tan^2\left(\frac{3\pi}{4} - 2x\right) \equiv \frac{1 + \sin 4x}{1 - \sin 4x}, \quad x \neq \frac{\pi}{3}(4n+1), \quad n \in \mathbb{Z}$$

proof

$$\begin{aligned} LHS &= \tan^2\left(\frac{3\pi}{4} - 2x\right) = \left[\tan\left(\frac{3\pi}{4} - 2x\right)\right]^2 = \left[\frac{\tan\frac{3\pi}{4} - \tan 2x}{1 + \tan\frac{3\pi}{4}\tan 2x}\right]^2 \\ &= \left[\frac{-1 - \tan 2x}{1 - \tan 2x}\right]^2 = \left[\frac{1 + \tan 2x}{1 - \tan 2x}\right]^2 = \left[\frac{1 + \frac{\sin 2x}{\cos 2x}}{1 - \frac{\sin 2x}{\cos 2x}}\right]^2 = \left[\frac{\cos 2x + \sin 2x}{\cos 2x - \sin 2x}\right]^2 \quad \text{Multiplying top and bottom by } \cos 2x \\ &= \left[\frac{\cos^2 2x + 2\sin 2x \cos 2x + \sin^2 2x}{\cos^2 2x - 2\sin 2x \cos 2x + \sin^2 2x}\right] \\ &= \frac{1 + \sin 4x}{1 - \sin 4x} = RHS \end{aligned}$$

Question 88 (***)**

Use algebra to find, in terms of π , the solution of the trigonometric equation

$$x^2 - 8\pi x + 2 - 2\cos x + 16\pi^2 = 0, \quad x \in \mathbb{R}.$$

$$\boxed{\text{S }}, \quad x = 4\pi$$

$$\begin{aligned} & x^2 - 8\pi x + 2 - 2\cos x + 16\pi^2 = 0 \\ \Rightarrow & x^2 - 8\pi x + 2 - 2(1 - 2\sin^2 \frac{x}{2}) + 16\pi^2 = 0 \\ & \cos 2x = 1 - 2\sin^2 \frac{x}{2} \\ & \sin x = \pm \frac{1 - \cos 2x}{2} \\ \Rightarrow & x^2 - 8\pi x + 4\sin^2 \frac{x}{2} + 16\pi^2 = 0 \\ \Rightarrow & (x - 4\pi)^2 + 4\sin^2 \frac{x}{2} = 0 \\ \Rightarrow & (x - 4\pi)^2 = -4\sin^2 \frac{x}{2} \end{aligned}$$

• BOTH TERMS MUST BE ≥ 0
• THE FIRST TERM CAN ONLY BE ZERO IF $x = 4\pi$
• $x = 4\pi$ MAKES THE SECOND TERM ZERO
HENCE THE ONLY LEGIT SOLUTION IS $x = 4\pi$

Question 89 (*****)

The piecewise continuous function f is given below.

$$f(x) \equiv \begin{cases} \sin x^\circ & 0 \leq x < 360 \\ \sin 2x^\circ & 360 \leq x < 720 \\ \sin 3x^\circ & 720 \leq x < 1080 \end{cases}$$

- a) Sketch the graph of $f(x)$.

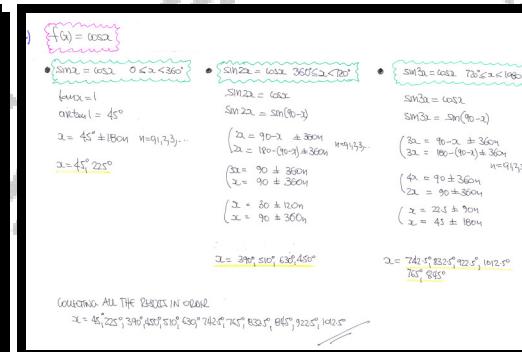
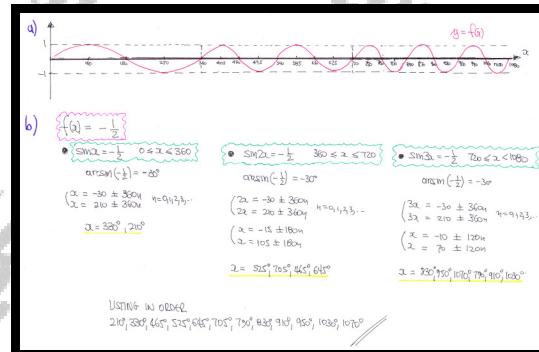
- b)** Solve the equation ...

$$\text{i. } \dots f(x) = -\frac{1}{2}.$$

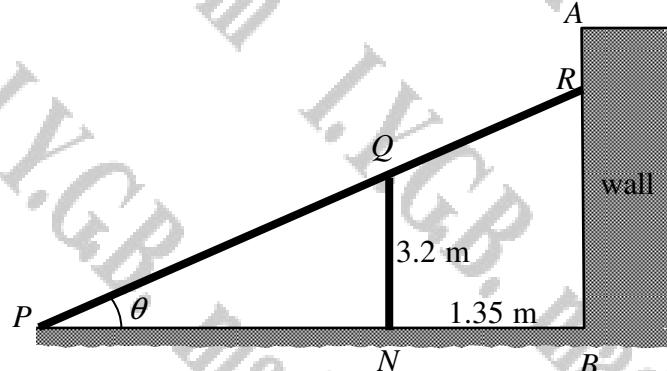
$$\text{ii. } \dots f(x) = \cos x.$$

, $x = 210, 330, 465, 525, 645, 705, 790, 830, 910, 950, 1030, 1070$,

$$x = 45, 225, 390, 450, 510, 630, 735, 742.5, 832.5, 915, 922.5, 1012.5$$



Question 90 (*****)



The figure above shows the wall AB of a certain structure, which is supported by a straight rigid beam PR , where P is on level ground and R is at some point on the wall.

In order to increase the rigidity of the support, the beam is rested on a steady pole NQ , of height 3.2 metres.

The pole is placed at a distance of 1.35 metres from the bottom of the wall B .

The beam PR is forming an acute angle θ with the horizontal ground PNB .

The angle θ is chosen so that the length of the beam PR , is least.

Determine the least value for the length of the beam PR , assuming that R lies on the wall, fully justifying that this is indeed the minimum value.

 , [6.25]

BY SIMPLE TRIGONOMETRY ON RIGHT-ANGLED TRIANGLE

$$\sin \theta = \frac{3.2}{y} \quad \cos \theta = \frac{1.35}{y}$$

$$y_1 = \frac{3.2}{\sin \theta} \quad y_2 = \frac{1.35}{\cos \theta}$$

$$y_1 = 3.2 \csc \theta \quad y_2 = 1.35 \sec \theta$$

• LET $|PR| = y = y_1 + y_2$

$$\Rightarrow y = 3.2 \csc \theta + 1.35 \sec \theta$$

$$\Rightarrow \frac{dy}{d\theta} = -3.2 \csc \theta \cot \theta + (1.35 \sec \theta \tan \theta)$$

$$\Rightarrow \frac{dy}{d\theta} = -\frac{3.2 \csc^2 \theta}{\sin^2 \theta} + \frac{1.35 \sec^2 \theta}{\cos^2 \theta}$$

• SET FOR ZERO

$$\Rightarrow 0 = -\frac{3.2 \csc^2 \theta}{\sin^2 \theta} + \frac{1.35 \sec^2 \theta}{\cos^2 \theta}$$

$$\Rightarrow 0 = -3.2 \csc^2 \theta + 1.35 \sec^2 \theta$$

$$\Rightarrow 0 = -3.2 + 1.35 \tan^2 \theta \quad (\text{using } \theta)$$

• FINALLY $0 = 3.2 \times \frac{4}{3} + 1.35 \times \frac{5}{3} = (0.8 \times 5) + (0.45) \times 5 = 4 + 2.25 = 6.25 \text{ m}$

• AT $\theta \rightarrow 0^\circ \Rightarrow |PR| = 3.2 \csc \theta + 1.35 \sec \theta$
 AT $\theta \rightarrow 90^\circ \Rightarrow |PR| \rightarrow \infty$ (because of cosec)
 AT $\theta \rightarrow 90^\circ \Rightarrow |PR| \rightarrow \infty$ (because of sec)

$\therefore |PR| = 6.25 \text{ m} \rightarrow \text{MINIMUM}$

Question 91 (*****)

The functions f and g are defined by

$$f(x) \equiv \cos x, x \in \mathbb{R}, 0 \leq x \leq \pi$$

$$g(x) \equiv 1 - x^2, x \in \mathbb{R}.$$

- a) Solve the equation

$$fg(x) = \frac{1}{2}.$$

- b) Determine the values of x for which $f^{-1}g(x)$ is **not** defined.

$$\boxed{\text{?}}, \quad \boxed{x = \pm \sqrt{1 - \frac{\pi}{6}}}, \quad \boxed{x < -\sqrt{2} \text{ or } x > \sqrt{2}}$$

a) $f(x) = \cos x, 0 \leq x \leq \pi \quad g(x) = 1 - x^2, x \in \mathbb{R}$

$$\begin{aligned} fg(x) &= f(g(x)) = f(1 - x^2) = \cos(1 - x^2) \\ &\rightarrow (\cos(1 - x^2)) = \sqrt{\frac{1}{2}} \\ &\rightarrow \cos(\sqrt{\frac{1}{2}}) = \frac{\pi}{6} \\ &\rightarrow (1 - x^2) = \frac{\pi}{6} \pm 2\pi k \quad k = 0, 1, 2, \dots \\ &\rightarrow 1 - x^2 = \frac{\pi}{6} \pm 2\pi k \\ &\rightarrow x^2 = 1 - \frac{\pi}{6} \pm 2\pi k \end{aligned}$$

Diagram:

$$\begin{array}{c} \xrightarrow{\text{?}} \boxed{g(x)} \xrightarrow{\text{?}} \boxed{f(g(x))} \xrightarrow{\text{?}} \boxed{f(x)} \xrightarrow{\text{?}} \boxed{\cos x} \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \text{?} \quad y \leq 1 \quad -1 \leq x^2 \leq 1 \quad 0 \leq x^2 \leq 1 \quad -1 \leq x \leq 1 \end{array}$$

$\therefore x^2 = 1 - \frac{\pi}{6} \pm 2\pi k$

Firstly if $f(x) = \cos x, 0 \leq x \leq \pi$
 Then $f'(x) = -\sin x, -1 \leq x \leq 1$
 [We do not really need to work out $f'(g(x))$]

$\begin{array}{c} \xrightarrow{\text{?}} \boxed{g(x)} \xrightarrow{\text{?}} \boxed{f(g(x))} \xrightarrow{\text{?}} \boxed{f(x)} \xrightarrow{\text{?}} \boxed{\cos x} \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \text{?} \quad y \leq 1 \quad -1 \leq x^2 \leq 1 \quad 0 \leq x^2 \leq 1 \quad -1 \leq x \leq 1 \end{array}$

COMPOSITION WILL BE UNDEFINED IF

- 1 $\leq g(x) < 1$
- 1 $\leq -x^2 < 1$
- 2 $\leq x^2 < 1$
- 0 $\leq x^2 < 2$
- $-\sqrt{2} \leq x < \sqrt{2}$

\therefore IT WILL NOT BE DEFINED IF

$$x < -\sqrt{2} \text{ or } x > \sqrt{2}$$

Question 92 (*****)

$$f(x) \equiv \cos^2 x + \sin^2 x, \quad x \in \mathbb{R}.$$

- a) Determine an expression for $f'(x)$ and find the value of $f\left(\frac{1}{2}\pi\right)$.

- b) By using the results of part (a) only, show that

$$\cos^2 x + \sin^2 x \equiv 1.$$

$$f'(x) = 0, \quad f\left(\frac{1}{2}\pi\right) = 1$$

(a) $f(x) = \cos^2 x + \sin^2 x$
 $f'(x) = -2\cos x \sin x + 2\sin x \cos x = 0$
 $\therefore f'(x) = \cos^2 x + \sin^2 x = 1$

(b) Since $f'(x) = 0, \forall x \in \mathbb{R} \Rightarrow f(x) = \text{constant} = c$
Since $f'(x) = 1 \Rightarrow c = 1$
 $\therefore f(x) = 1, \forall x \in \mathbb{R}$
 $\cos^2 x + \sin^2 x = 1 \Rightarrow x \in \mathbb{R}$

Question 93 (*****)

Prove the validity of each of the following trigonometric identities.

a) $\frac{\sqrt{2-2\cos x}}{\sin x} \equiv \sec \frac{x}{2}$.

b) $\sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta \equiv \tan \theta + \cot \theta$.

proof

(a) LHS = $\frac{\sqrt{2-2\cos x}}{\sin x}$
 $= \frac{\sqrt{2(1-\cos x)}}{\sin x}$
 $= \frac{\sqrt{2\cdot 2\sin^2 \frac{x}{2}}}{2\sin \frac{x}{2} \cos \frac{x}{2}}$
 $= \frac{\sqrt{4\sin^2 \frac{x}{2}}}{2\sin \frac{x}{2} \cos \frac{x}{2}} = \frac{2\sin \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} = \frac{1}{\cos \frac{x}{2}}$
 $= \sec \frac{x}{2} = RHS$

(b) LHS = $\sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta = \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} + 2 \sin \theta \cos \theta$
 $= \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{(\sin \theta)^2 + (\cos \theta)^2}{\sin^2 \theta \cos^2 \theta} = \frac{1}{\sin^2 \theta \cos^2 \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta \cos^2 \theta} = \frac{1}{\sin^2 \theta \cos^2 \theta}$
 $= \frac{\sin^2 \theta}{\sin^2 \theta \cos^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta \cos^2 \theta} = \frac{\cos^2 \theta + \sin^2 \theta}{\sin^2 \theta \cos^2 \theta} = \cot \theta + \tan \theta = RHS$

Question 94 (*****)

Solve the trigonometric equation

$$2 \arctan(x-2) + \arcsin\left(\frac{1-x}{1+x}\right) = \frac{\pi}{2}.$$

, $x = 4$

Firstly rewrite the inverse trigonometric functions as angles:

$\theta = \arctan(x-2)$
 $\tan\theta = x-2$



BY PYTHAGOREAN THEOREM
 $\tan^2\theta + 1^2 = \sqrt{1+x^2}$

$\phi = \arcsin\left(\frac{1-x}{1+x}\right)$
 $\sin\phi = \frac{1-x}{1+x}$



BY PYTHAGOREAN THEOREM
 $(x-2)^2 + 1^2 = \sqrt{1+x^2}$

Hence we may rewrite the equation as follows:

$$\Rightarrow 2\theta + \phi = \frac{\pi}{2}$$

$$\Rightarrow 2\theta = \frac{\pi}{2} - \phi$$

Take the cosine of the equation, because of the RHS:

$$\Rightarrow \cos 2\theta = \cos\left(\frac{\pi}{2} - \phi\right)$$

$$\Rightarrow 2\cos^2\theta - 1 = \sin\phi$$

$$\Rightarrow 2\left(\frac{1}{1+x^2+4x+5}\right) - 1 = \frac{1-x}{1+x}$$

$$\Rightarrow \frac{2}{x^2+4x+5} - 1 = \frac{1-x}{1+x}$$

$$\Rightarrow \frac{2-x^2+4x+5}{x^2+4x+5} = \frac{1-x}{1+x}$$

$$\Rightarrow \frac{-x^2+4x+3}{x^2+4x+5} = \frac{1-x}{1+x}$$

$$\Rightarrow \frac{x^2-4x-3}{x^2+4x+5} = \frac{x-1}{1+x}$$

$$\Rightarrow (x+1)(x^2-4x+3) = (x-1)(x^2+4x+5)$$

$$\Rightarrow x^3-4x^2+3x = x^3+4x^2+5x$$

$$\Rightarrow -8x^2-2x = -2x^2+4x+5$$

$$\Rightarrow x^2-5x-5 = x^2-4x+5$$

($\pm i\sqrt{10}$ is definitely NOT a solution of the original equation)

$$\Rightarrow 2x^2-10x+8 = 0$$

$$\Rightarrow x^2-5x+4 = 0$$

$$\Rightarrow (x-1)(x-4) = 0$$

$$\Rightarrow x = 1, 4$$

Checking the solutions against the original:

If $x=4$:

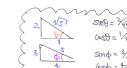
$$\Rightarrow 2\arctan 2 + \arcsin\left(-\frac{1}{5}\right) = \frac{\pi}{2}$$

$$\Rightarrow 2\arctan 2 - \arcsin\frac{1}{5} = \frac{\pi}{2}$$

$$\Rightarrow 2\theta - \phi = \frac{\pi}{2}$$

$$\Rightarrow \cos(2\theta - \phi) = \cos\frac{\pi}{2}$$

$$\Rightarrow \cos(2\theta)\cos\phi + \sin 2\theta \sin\phi = 0$$

$$\Rightarrow (2\cos^2\theta - 1)(\cos\phi + 2\sin\theta \sin\phi) = 0$$


$$\Rightarrow \left[2\left(\frac{1}{5}\right)-1\right]\times\frac{1}{5} + 2\left(\frac{2}{\sqrt{17}}\right)\left(\frac{1}{\sqrt{5}}\right)\frac{1}{5} = \cos\phi$$

$$\Rightarrow -\frac{3}{5}\times\frac{1}{5} + \frac{4}{5}\times\frac{1}{5} = \cos\phi$$

$$\Rightarrow \cos\phi = 0$$

$$\Rightarrow \phi = \frac{\pi}{2}$$

If $x=1$:

$$\Rightarrow 2\arctan(-1) + \arcsin(0) = 2\left(-\frac{\pi}{4}\right) + 0 = -\frac{\pi}{2}$$

The only solution is $x=4$

Question 95 (*****)

By considering the solution of trigonometric equation

$$\sin(x-30)^\circ = \cos(x-45)^\circ,$$

find, in degrees, the exact value of $\arctan\left[\frac{1+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right]$.

, 82.5°

$\boxed{\sin(x-30) = \cos(x-45)}$

• FIRST SOLVE THE EQUATION BY EXPANDING VIA THE COMPOUND ANGLE IDENTITIES FOR $\sin(A-B)$ & $\cos(A-B)$

$$\Rightarrow \sin(x-30) = \cos(x-45) = \cos x \cos 45 + \sin x \sin 45$$

$$\Rightarrow \frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x = \frac{\sqrt{2}}{2} \cos x + \frac{\sqrt{2}}{2} \sin x \times 2$$

$$\Rightarrow \sqrt{3} \sin x - \cos x = \sqrt{2} \cos x + \sqrt{2} \sin x$$

$$\Rightarrow \sqrt{3} \sin x - 1 = \sqrt{2} + \sqrt{2} \sin x \quad \div \cos x$$

$$\Rightarrow (\sqrt{3}-\sqrt{2}) \tan x = \sqrt{2} + 1$$

$$\Rightarrow \tan x = \frac{\sqrt{2}+1}{\sqrt{3}-\sqrt{2}}$$

$$\Rightarrow x = \arctan\left(\frac{\sqrt{2}+1}{\sqrt{3}-\sqrt{2}}\right) \quad (\text{PRINCIPAL VALUE})$$

• NEXT CONSIDER AN ALTERNATIVE SOLUTION FOR THE SAME EQUATION, STARTING WITH A "SIN TO COSINE" (OR INDEED THE OTHER WAY AROUND) MANIPULATION

$$\Rightarrow \sin(x-30) = \cos(x-45)$$

$$\Rightarrow \sin(x-30) = \sin[90-(x-45)] \quad (\sin A \equiv \sin(90-A))$$

$$\Rightarrow \sin(x-30) = \sin(135-x)$$

• HENCE $\Rightarrow (x-30) = 135-x \pm 360n \quad n=0,1,2,3,\dots$

$$\Rightarrow (x-30) = 135-x \pm 360n$$

$$\Rightarrow (-30) = 45+x \pm 360n \quad \text{VALUES NORMING}$$

$$\Rightarrow 2x = 165 \pm 360n$$

\uparrow
PRINCIPAL VALUE FOR $\sin(-90, 90)$

$$\therefore \arctan\left(\frac{1+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right) = 82.5^\circ$$

ALTERNATIVE VARIATION

$$\Rightarrow \sin(x-30) = \cos(x-45) \quad \rightarrow \boxed{\sin A \equiv \cos(90-A)}$$

$$\Rightarrow \cos[90-(x-45)] = \cos(x-45)$$

$$\Rightarrow \cos(90-x) = \cos(x-45)$$

$$\Rightarrow (120-x) = x-45 \pm 360n \quad n=0,1,2,\dots$$

$$\Rightarrow (120-x) = 45-x \pm 360n \quad \text{VALUES NORMING}$$

$$\Rightarrow -2x = -165 \pm 360n$$

$$\Rightarrow x = 82.5 \pm 180n$$

SOME PRINCIPAL VALUE FOR $\cos(-9, 90)$

Question 96 (***)**

Simplify, showing all steps in the calculation, the expression

$$\arctan 8 + \arctan 2 + \arctan \frac{2}{3},$$

giving the answer in terms of π .

, π

• **SIMPLIFY WITH THE COMPOUND ANGLE IDENTITY**

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

• **EXPAND THE IDENTITY**

$$\begin{aligned} \tan(A+B+C) &= \tan[(A+B)+C] = \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C} \\ &= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B} \tan C} \\ &\quad \text{MULTIPLY TOP AND BOTTOM BY } 1 - \tan A \tan B \\ &= \frac{(\tan A + \tan B) + \tan C(1 - \tan A \tan B)}{1 - (\tan A + \tan B) - (\tan A + \tan B)\tan C} \\ \tan(A+B+C) &= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}. \end{aligned}$$

• **NOW LET**

$$\begin{aligned} A &= \arctan 8 \implies \tan A = 8 \\ B &= \arctan 2 \implies \tan B = 2 \\ C &= \arctan \frac{2}{3} \implies \tan C = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \Rightarrow \tan(A+B+C) &= \frac{8 + 2 + \frac{2}{3} - 8 \times 2 \times \frac{2}{3}}{1 - (8 \times 2) - (8 \times \frac{2}{3}) - (2 \times \frac{2}{3})} \\ &= \frac{10 + \frac{2}{3} - \frac{32}{3}}{1 - 16 - \frac{16}{3} - \frac{4}{3}} \\ &= \frac{30 + 2 - 32}{3 - 48 - 14 - 4} = 0 \end{aligned}$$

• **INSERTING**

$$A+B+C = 0 \pm n\pi \quad n=0,1,2,3,\dots$$

$$A+B+C = -2\pi, -\pi, 0, \pi, 2\pi, \dots$$

$$\arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \dots -2\pi, -\pi, 0, \pi, 2\pi, \dots$$

AS A, B, C ARE ALL ACUTE $0 < A+B+C < \frac{3\pi}{2}$

$$\therefore \arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \pi$$

ALTERNATIVE BY COMPLEX NUMBERS

CONSIDER THE FOLLOWING

$$z = (1+8i)(1+2i)(3+2i) = (1+8i)(3+2i+6i-4)$$

$$z = [(1+8i)(-1+6i)](3+2i)$$

$$z = -1+8i-8i-4i$$

$$z = -4i$$

TAKING ARGUMENT IN THE FOLLOWING EXPRESSION

$$\begin{aligned} &\Rightarrow ((1+8i)(1+2i)(3+2i))(-i) = -4i \\ &\Rightarrow \arg((1+8i)(1+2i)(3+2i)) + \arg(-4i) \\ &\Rightarrow \arg(1+8i) + \arg(1+2i) + \arg(3+2i) = \arg(-4i) \\ &\Rightarrow \arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \arg(-4i) \\ &\therefore \arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \pi \end{aligned}$$

Question 97 (***)**

Given that $0 < \theta < \frac{1}{2}\pi$, $0 < \varphi < \frac{1}{2}\pi$, solve the following simultaneous equations.

$$5\cos\theta + 2\tan\varphi = 5 \quad \text{and} \quad 5\sin\theta + \cot\varphi = 5.$$

Give the answers in exact form in terms of inverse trigonometric functions.

$$\boxed{\text{S - M}}, \quad [\theta, \varphi] = \left[\arccos \frac{3}{5}, \arctan 1 \right], \quad [\theta, \varphi] = \left[\arccos \frac{4}{5}, \arctan \frac{1}{2} \right],$$

$$\boxed{[\theta, \varphi] = \left[\arccos \left[\frac{1}{10}(3+\sqrt{41}) \right], \arctan \left[\frac{1}{4}(7+\sqrt{41}) \right] \right]}$$

Solve $\begin{cases} 5\cos\theta + 2\tan\varphi = 5 \\ 5\sin\theta + \cot\varphi = 5 \end{cases} \Rightarrow \begin{cases} \sin\theta = 5 - 2\tan\varphi \\ \sin\theta = 5 - \cot\varphi \end{cases}$

• Squaring & adding gives

$$\begin{aligned} 25\cos^2\theta &= (5 - 2\tan\varphi)^2 \\ 25\cos^2\theta &= (5 - \cot\varphi)^2 \end{aligned}$$

$$\Rightarrow 25(\cos^2\theta + \sin^2\theta) = (25 - 20\tan\varphi + 4\tan^2\varphi) + (25 - 10\cot\varphi + \cot^2\varphi)$$

$$\Rightarrow 25 = 25 - 20\tan\varphi + 4\tan^2\varphi + 25 - 10\cot\varphi + \cot^2\varphi$$

• Let $T = \tan\varphi$

$$\begin{aligned} \Rightarrow 0 &= -20T + 4T^2 + 25 - \frac{10}{T} + \frac{1}{T^2} \\ \Rightarrow 0 &= -20T^3 + 4T^4 + 25T^2 - 10T + 1 \\ \Rightarrow 4T^4 - 20T^3 + 25T^2 - 10T + 1 &= 0 \end{aligned}$$

• By inspection, $T=1$ is a solution — divide it out

$$\begin{aligned} \Rightarrow 4T^3(T-1) - 10T^2(T-1) + 9T(T-1) - 10(T-1) &= 0 \\ \Rightarrow (T-1)(4T^3 - 10T^2 + 9T - 10) &= 0 \end{aligned}$$

• Look for any more solutions. $T = \frac{1}{2}$ or $T = \frac{1}{4}$

$$\begin{aligned} 4\left(\frac{1}{2}\right)^3 - 10\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right) - 1 &= \frac{1}{2} - \frac{5}{2} + \frac{9}{2} - 1 \\ &= 0. \end{aligned}$$

i.e. $(2T-1)$ is a factor

$$\Rightarrow (T-1)[2^2(2T-1) - 10T(2T-1) + (2T-1)] = 0$$

$\rightarrow (T-1)(2T-1)(2T^2-7T+1) = 0$

• Now by the quadratic formula

$$T = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$T = \frac{7 \pm \sqrt{49 - 4(2)(1)}}{4}$$

$$T = \frac{7 \pm \sqrt{41}}{4}$$

$\Rightarrow \tan\varphi = \begin{cases} \frac{1}{2} \\ \frac{7 + \sqrt{41}}{4} \\ \frac{7 - \sqrt{41}}{4} \end{cases}$

• Looking at the first equation

$$\begin{aligned} 5\cos\theta &= 5 - 2\tan\varphi \\ \cos\theta &= 1 - \frac{2}{5}\tan\varphi \end{aligned}$$

- If $\tan\varphi = 1$ • If $\tan\varphi = \frac{1}{2}$
- $\cos\theta = \frac{3}{5}$ $\cos\theta = \frac{4}{5}$

• If $\tan\varphi = \frac{7 + \sqrt{41}}{4}$

$$\begin{aligned} \cos\theta &= 1 - \frac{2}{5}\left(\frac{7 + \sqrt{41}}{4}\right) \\ \cos\theta &= 1 - \frac{7}{10} - \frac{1}{10}\sqrt{41} \\ \cos\theta &= \frac{3}{10} + \frac{1}{10}\sqrt{41} \\ \cos\theta &= \frac{1}{10}(3 + \sqrt{41}) \end{aligned}$$

• Hence we obtain

$$(\theta, \varphi) = \begin{cases} \arccos \frac{3}{5}, \arctan 1 \\ \arccos \frac{4}{5}, \arctan \frac{1}{2} \\ \arccos \left[\frac{1}{10}(3 + \sqrt{41}) \right], \arctan \left[\frac{1}{4}(7 + \sqrt{41}) \right] \end{cases}$$

Question 98 (*****)

Solve the trigonometric equation

$$\sqrt{3} \cos\left(x + \frac{\pi}{5}\right) = \sin\left(x + \frac{\pi}{5}\right), \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

$x = \frac{2\pi}{15}, \frac{17\pi}{15}$

$$\begin{aligned}
 \sqrt{3} \cos(x + \frac{\pi}{5}) &= \sin(x + \frac{\pi}{5}) \\
 \Rightarrow \sqrt{3} \cos(x + \frac{\pi}{5}) - \sin(x + \frac{\pi}{5}) &= 0 \\
 \Rightarrow \sqrt{3} \cos(x + \frac{\pi}{5}) - \frac{1}{2} \sin(x + \frac{\pi}{5}) &= 0 \\
 \Rightarrow (\cos\frac{\sqrt{3}}{2} \cos(x + \frac{\pi}{5})) - \sin\frac{1}{2} \sin(x + \frac{\pi}{5}) &= 0 \\
 \Rightarrow \cos\left(x + \frac{\pi}{5} + \frac{\pi}{6}\right) &= 0 \\
 \Rightarrow \cos\left(x + \frac{11\pi}{30}\right) &= 0 \\
 \Rightarrow \left(x + \frac{11\pi}{30}\right) &= \frac{\pi}{2} \pm 2n\pi \quad n \in \mathbb{Z}, \dots \\
 \Rightarrow \left(x + \frac{11\pi}{30}\right) &= \frac{3\pi}{2} \pm 2n\pi \\
 \Rightarrow \begin{cases} x = \frac{2\pi}{15} \pm 2n\pi \\ x = \frac{17\pi}{15} \pm 2n\pi \end{cases} & \left\{ \begin{array}{l} \frac{1}{2} - \frac{11}{30} = \frac{15}{30} - \frac{11}{30} = \frac{4}{30} \\ \frac{3}{2} - \frac{11}{30} = \frac{45}{30} - \frac{11}{30} = \frac{34}{30} \end{array} \right.
 \end{aligned}$$

Question 99 (***)**

Solve the following trigonometric equation

$$\arcsin 2x + \arccos x = \frac{5\pi}{6}.$$

$$\boxed{\quad}, \quad x = \frac{1}{2}$$

Worked Solution:

Let $\theta = \arcsin 2x$ and $\phi = \arccos x$.

$\sin \theta = 2x$ and $\cos \phi = x$.

From the right-angled triangle, we have:

- $\sin \theta = \frac{1}{\sqrt{1-4x^2}}$
- $\cos \phi = \frac{1}{\sqrt{1-x^2}}$

Hence the above equation becomes:

$$\theta + \phi = \frac{5\pi}{6}$$

$$\sin(\theta + \phi) = \sin \frac{5\pi}{6}$$

$$\sin \theta \cos \phi + \cos \theta \sin \phi = \frac{1}{2}$$

$$(2x)\left(\frac{1}{\sqrt{1-4x^2}}\right) + \left(\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{1}{\sqrt{1-4x^2}}\right) = \frac{1}{2}$$

$$2x^2 + \sqrt{(1-4x^2)(1-4x^2)} = \frac{1}{2}$$

$$2x^2 + \sqrt{1-5x^2+4x^4} = \frac{1}{2}$$

$$4x^2 + 2\sqrt{4x^2-5x^2+1} = 1$$

$$2\sqrt{4x^2-5x^2+1} = 1-4x^2$$

$$4(4x^2-5x^2+1) = (1-4x^2)^2$$

$\Rightarrow 16x^2 - 20x^2 + 4 = 1 - 8x^2 + 1$ $\checkmark x \neq \pm 1$

$$\Rightarrow -2x^2 + 4 = -8x^2 + 1$$

$$\Rightarrow 3 = 12x^2$$

$$\Rightarrow x^2 = \frac{1}{4}$$

$$\Rightarrow x = \pm \frac{1}{2}$$

$\arcsin 1 + \arccos \frac{1}{2} = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

$$\arcsin(-1) + \arccos(-\frac{1}{2}) = -\frac{\pi}{2} + \frac{2\pi}{3} = \frac{\pi}{6}$$

Question 100 (*****) non calculator

Find, in degrees, the solutions of the trigonometric equation

$$2\cos(x+10)^\circ = \frac{\cos(x+22)^\circ}{\sin(x+10)^\circ}, \quad 0^\circ \leq x \leq 360^\circ.$$

, $x = 16^\circ, 92^\circ, 136^\circ, 256^\circ$

Working:

$$\begin{aligned} 2\cos(x+10) &= \frac{\cos(x+22)}{\sin(x+10)} \\ \Rightarrow 2\cos(x+10)\sin(x+10) &= \cos(x+22) \quad | \quad 2\sin x \cos x \equiv \sin 2x \\ \Rightarrow \sin(2x+20) &= \cos(x+22) \\ \Rightarrow \sin(2x+20) &= \cos(90 - (x+22)) \quad | \quad \cos x \equiv \sin(90-x) \\ \Rightarrow \sin(2x+20) &= \sin(48-x) \\ \text{Solving we now obtain} \\ \bullet 2x+20 &= 60-x + 360n \quad | \quad 2x+20 = 180 - (8-2) + 360n \\ 2x &= 16 \pm 120n \quad | \quad n=0,1,2,3, \dots \\ 2x &\approx 16 \pm 360n \\ x &= 16^\circ, 92^\circ, 136^\circ, 256^\circ \end{aligned}$$

Question 101 (*****)

It is given that for $\theta \neq (4k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$,

$$\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \equiv \tan \theta + \sec \theta.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence find a similar expression for $\tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$.

You are now given the equation

$$\tan x - \tan(x - \alpha) = 2 \tan x,$$

where α is a constant.

- c) Express $\tan x$ in terms of trigonometric functions involving α only.
 d) Hence solve the trigonometric equation

$$\tan x - \tan\left(x - \frac{3\pi}{5}\right) = 2 \tan\frac{3\pi}{5}, \quad 0 \leq x < 2\pi,$$

giving the answers in terms of π .

$$\boxed{\quad}, \quad \boxed{\tan x = \tan \alpha \pm \sec \alpha}, \quad \boxed{x = \frac{\pi}{20}, \frac{11\pi}{20}, \frac{21\pi}{20}, \frac{31\pi}{20}}$$

a) LHS = $\tan\left(\frac{\alpha - x}{2}\right) = \frac{\tan\frac{\alpha}{2} + \tan\frac{x}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{x}{2}} = \frac{-\tan\frac{\alpha}{2} + 1}{1 - \tan\frac{\alpha}{2}\tan\frac{x}{2}}$
 $= \frac{\tan\frac{\alpha}{2} + \sec^2\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{x}{2}} = \dots$ (NOTICE THE BOTTOM OF THE FRACTION BY CANCELLING)

 $= \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}} = \frac{(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2})(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2})}{(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2})(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2})}$
 $= \frac{\sin^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}} = \frac{1 + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}$
 $= \frac{1 + \sec^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}} = \frac{1 + \sec^2\frac{\alpha}{2}}{\sec^2\frac{\alpha}{2}} = \sec^2\frac{\alpha}{2} + \tan^2\frac{\alpha}{2} = 1 + \tan^2\frac{\alpha}{2}$

b) FOLLOW ANALOGOUS STEPS

REPLACE α WITH $B - \pi$
 $\tan\left(\frac{B - \pi}{2} - x\right) = \sec(B - \pi) + \tan(B - \pi)$
 $\tan\left(\frac{B}{2} - \frac{\pi}{2} - x\right) = -\sec B + \tan B$

c) $\tan x = \tan(2x - \alpha) = 2\tan x$
 $\Rightarrow \tan x - \frac{\tan x - \tan x}{1 + \tan x \tan x} = 2\tan x$
 $\Rightarrow \tan x + 2\tan^2 x - \tan x + \tan x = 2\tan x + 2\tan^2 x$
 $\Rightarrow \tan x \tan x - 2\tan x \tan x - \tan x = 0$
 $\Rightarrow \tan^2 x - 2\tan x \tan x - 1 = 0$

$$\begin{aligned} &\Rightarrow (\tan x - \tan x)^2 - \tan^2 x = 0 \\ &\Rightarrow (\tan x - \tan x)^2 = 1 + \tan^2 x \\ &\Rightarrow (\tan x - \tan x)^2 = \sec^2 x \\ &\Rightarrow \tan x - \tan x = \pm \sec x \\ &\Rightarrow \tan x = \sqrt{\tan^2 x - \sec^2 x} // \end{aligned}$$

d) $\tan x - \tan\left(x - \frac{3\pi}{5}\right) = 2 \tan\frac{3\pi}{5}$
 $\tan x = \sqrt{\tan^2\frac{3\pi}{5} + \sec^2\frac{3\pi}{5}} = \tan\left(\frac{3\pi}{5} + \frac{\pi}{4}\right) = \tan\frac{17\pi}{20}$
 $\tan x = \sqrt{\tan^2\frac{3\pi}{5} - \sec^2\frac{3\pi}{5}} = \tan\left(\frac{3\pi}{5} - \frac{\pi}{4}\right) = \tan\frac{7\pi}{20}$ (Cancelling)
 THIS SUGGESTS
 $(\tan x = \tan\frac{17\pi}{20} \Rightarrow x = \frac{17\pi}{20} + n\pi), n = 0, 1, 2, 3, \dots$
 $(\tan x = \tan\frac{7\pi}{20} \Rightarrow x = \frac{7\pi}{20} + n\pi), n = 0, 1, 2, 3, \dots$
 $\therefore x = \frac{17\pi}{20}, \frac{17\pi}{20} + \frac{21\pi}{20}, \frac{17\pi}{20} + \frac{31\pi}{20}$

Question 102 (*****)

It is given that θ and φ satisfy the simultaneous equations

$$\frac{\sin 2\theta}{1 + \sin \theta} = 1 - \sin \varphi,$$

$$\theta + \varphi = \pi$$

where $0 < \theta < \pi$, $\theta \neq \frac{\pi}{2}$, $0 < \varphi < \pi$.

a) Determine the value of $\tan \theta$.

b) Show clearly that

$$\tan(3\theta + 5\varphi) = -\frac{4}{3}.$$

$$\boxed{\square}, \quad \boxed{\tan \theta = \frac{1}{2}}$$

(a)

$$\begin{aligned} \frac{\sin 2\theta}{1 + \sin \theta} = 1 - \sin \varphi &\quad \left. \begin{array}{l} \text{?} \\ \theta + \varphi = \pi \end{array} \right\} \Rightarrow \frac{\sin 2\theta}{1 + \sin \theta} = 1 - \sin(\pi - \theta) \\ &\Rightarrow \frac{\sin 2\theta}{1 + \sin \theta} = 1 - [\sin(\pi - \theta) - \cos(\pi - \theta)] \\ &\Rightarrow \frac{\sin 2\theta}{1 + \sin \theta} = 1 - \sin \theta \\ &\Rightarrow \sin 2\theta = (1 - \sin \theta)(1 + \sin \theta) \\ &\Rightarrow 2\sin \theta \cos \theta = 1 - \sin^2 \theta \\ &\Rightarrow 2\sin \theta \cos \theta = \cos^2 \theta \\ &\Rightarrow 2\sin \theta = \cos \theta \quad (\cos \theta \neq 0, \theta \neq \frac{\pi}{2}) \\ &\Rightarrow \tan \theta = \frac{1}{2} \end{aligned}$$

(b)

$$\begin{aligned} \tan(3\theta + 5\varphi) &= \tan(3\theta + 5(\pi - \theta)) = \tan(3\theta + 5\pi - 5\theta) \\ &= \tan(-2\theta + 5\pi) = \frac{1}{-\tan(2\theta)} \quad \text{[tan has period } \pi\text{]} \\ &= -\tan 2\theta \quad \text{[tan is odd]} \\ &= \frac{-2\tan \theta}{1 - \tan^2 \theta} \\ &= \frac{-2 \times \frac{1}{2}}{1 - \frac{1}{4}} = \frac{-1}{\frac{3}{4}} = -\frac{4}{3} \quad \text{[AS REQUIRED]} \end{aligned}$$

Question 103 (***)**

The angle θ satisfies the equation

$$\tan \theta \tan 2\theta = \sum_{r=0}^{\infty} 2 \cos^r 2\theta.$$

Given that $0 < \theta < \frac{\pi}{2}$, find the exact value of $\tan \theta$.

$$\boxed{}, \quad \boxed{\tan \theta = 3^{-\frac{1}{4}}}$$

$$\begin{aligned} \tan \theta \tan 2\theta &= \sum_{r=0}^{\infty} 2 \cos^r 2\theta \\ \Rightarrow \tan \theta \tan 2\theta &= 2 \left[1 + \cos 2\theta + \cos^2 2\theta + \cos^3 2\theta + \cos^4 2\theta + \dots \right] \\ \text{As } 0 < \theta < \frac{\pi}{2}, \quad 0 < \cos 2\theta < 1 \\ \text{THE G.P. ON THE R.H.S. CONVERGES TO } S_{\infty} = \frac{a}{1-r} \\ \Rightarrow \tan \theta \tan 2\theta &= 2 \times \frac{1}{1 - \cos 2\theta} \\ \Rightarrow \tan \theta \left(\frac{2 \tan 2\theta}{1 - \tan^2 2\theta} \right) &= \frac{2}{1 - (1 - 2 \sin^2 \theta)} \\ \Rightarrow \frac{2 \tan 2\theta}{1 - \tan^2 2\theta} &\approx \frac{2}{2 \sin^2 \theta} \\ \Rightarrow \frac{2 \tan 2\theta}{1 - \tan^2 2\theta} &= 2 \sin^2 \theta \\ \Rightarrow \frac{2 \tan 2\theta}{1 - \tan^2 2\theta} &= 1 + \cos^2 \theta \\ \Rightarrow \frac{2 \tan 2\theta}{1 - \tan^2 2\theta} &= 1 + \frac{1}{\tan^2 \theta} \\ \text{LET } T = \tan^2 \theta \\ \Rightarrow \frac{2T}{1-T} &= 1 + \frac{1}{T} \\ \Rightarrow 2T^2 &= T(1-T) + (1-T) \end{aligned}$$

$$\begin{aligned} \Rightarrow 2T^2 &= T - T^2 + 1 - T \\ \Rightarrow 3T^2 &= 1 \\ \Rightarrow T^2 &= \frac{1}{3} \\ \Rightarrow (\tan^2 \theta)^2 &= \frac{1}{3} \\ \Rightarrow \tan^4 \theta &= \frac{1}{3} \\ \Rightarrow \tan^2 \theta &= 3^{-\frac{1}{2}} \\ \Rightarrow \tan \theta &= \sqrt{3^{-\frac{1}{2}}} \quad (\tan \theta > 0 \text{ as } 0 < \theta < \frac{\pi}{2}) \end{aligned}$$

Question 104 (***)**

Use trigonometric algebra to fully simplify

$$\arctan \left[\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} \right], \quad 0 < x < \frac{\pi}{4},$$

giving the final answer in terms of x .

$$\boxed{}, \quad \frac{1}{2}x$$

$$\arctan \left[\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} \right], \quad 0 < x < \frac{\pi}{4}$$

• SIMPLIFY BY CONJUGATING THE DENOMINATOR.

$$\begin{aligned} &= \arctan \left[\frac{(1+\sin x) - \sqrt{1-\sin^2 x}) (1+\sin x) + \sqrt{1-\sin^2 x})}{(1+\sin x) + \sqrt{1-\sin^2 x}) (1+\sin x) - \sqrt{1-\sin^2 x})} \right] \\ &= \arctan \left[\frac{(1+\sin x) - 2\sqrt{1-\sin^2 x}) (1+\sin x) + (1-\sin x)}{(1+\sin x) - \sin x} \right] \\ &= \arctan \left[\frac{2 - 2\sqrt{1-\sin^2 x}}{2\sin x} \right] \\ &= \arctan \left[\frac{2 - 2\sqrt{\cos^2 x}}{2\sin x} \right] \\ &= \arctan \left[\frac{1 - \cos x}{\sin x} \right] \end{aligned}$$

• The student could possibly produce an argument in triangles if we use the double angle formulae

$$\begin{aligned} &= \arctan \left[\frac{(1 - \cos 2x) - 2\cos^2 x}{2\sin^2 x - \cos^2 x} \right] \\ &= \arctan \left[\frac{2\sin^2 \frac{x}{2} - 2\cos^2 \frac{x}{2}}{2\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2}} \right] \\ &= \arctan \left[\frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \right] \\ &= \arctan \left(\tan \frac{x}{2} \right) \\ &= \frac{x}{2} \end{aligned}$$

$\cos 2x = 1 - 2\sin^2 x$
 $\cos 2x = 1 - 2\cos^2 x$
 $\sin 2x = 2\sin x \cos x$
 $\sin^2 x = 2\sin^2 \frac{x}{2} \cos^2 \frac{x}{2}$

Question 105 (***)**

Solve the following trigonometric equation, for $0 \leq x < 2\pi$.

$$2\cos x \sin^2 x - 2\cos^2 x \sin x + \cos^2 x - 4\sin^2 x + 3\cos x \sin x + 2\sin x - 2\cos x = 0.$$

, $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}$

Quadratic - $2\cos^2 x - 2\cos x \sin x + \sin^2 x - 4\sin^2 x + 3\cos x \sin x + 2\sin x - 2\cos x = 0$

- LOOKING AT THE COEFFICIENTS OF THIS EQUATION IN GROUPS OF POWERS
- $2\cos^2 x - 2\cos x \sin x$ (CUBE IN COEFFICIENT)
- $\sin^2 x - 4\sin^2 x + 3\cos x \sin x$ (QUADRATIC IN COEFFICIENT)
- $2\sin x - 2\cos x$ (LINEAR IN COEFFICIENT)

BY INSPECTION IF $\cos x = \sin x$ THEN THE EQUATION IS SATISFIED AS WE FOUND A SOLUTION & HAVE A FACTOR $(\cos x - \sin x)$

REWRITE THE LHS COMPACTLY

$$2\cos^2 x - 2\cos^2 x + \cos^2 x - 4\sin^2 x + 3\cos x \sin x + 2\sin x - 2\cos x = 0$$

FACOTRIZE IN PAIRS BY INSPECTION

$$\begin{aligned} &\Rightarrow 2\cos(\frac{\pi}{2} - c) + c^2 - 4\sin^2 c + 2(c - \frac{\pi}{2}) = 0 \\ &\Rightarrow 2\cos(\frac{\pi}{2} - c) + c^2 + 3c - 4\sin^2 c + 2(c - \frac{\pi}{2}) = 0 \\ &\Rightarrow 2\cos(\frac{\pi}{2} - c) + (c - \frac{\pi}{2})(c + 4\sin^2 c) + 2(c - \frac{\pi}{2}) = 0 \\ &\Rightarrow 2\cos(\frac{\pi}{2} - c) - (c - \frac{\pi}{2})(c + 4\sin^2 c) + 2(c - \frac{\pi}{2}) = 0 \\ &\Rightarrow (\frac{\pi}{2} - c)[2\cos(\frac{\pi}{2} - c) + (c + 4\sin^2 c) + 2] = 0 \\ &\Rightarrow (\frac{\pi}{2} - c)(2\cos(\frac{\pi}{2} - c) + 4\sin^2 c + 2) = 0 \\ \text{FACTOIZE IN PAIRS AGAIN (RATIO } \frac{2\pi - 1}{2} \text{ OR } \frac{-4+2}{2} \text{)} \\ &\Rightarrow (\frac{\pi}{2} - c)[c(2\sin^2 c - 2) + 2] = 0 \end{aligned}$$

$\Rightarrow (\frac{\pi}{2} - c)(2\sin^2 c - 2) = 0$

EITHER $\sin^2 c = \cos^2 c$
OR $\sin^2 c = \frac{1}{2}$
OR $\cos^2 c = 2$

HENCE WE CAN NOW FIND SOLUTIONS

$\sin^2 c = \cos^2 c$	$\sin^2 c = \frac{1}{2}$
$\sin c = \cos c$	$c = \frac{\pi}{4}, \frac{5\pi}{4}$
$\tan c = 1$	

$c = \frac{\pi}{4}, \frac{5\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

Question 106 (*****)

Use trigonometric algebra to solve the equation

$$\arctan x + 2 \operatorname{arccot} x = \frac{2\pi}{3}.$$

You may assume that $\operatorname{arccot} x$ is the inverse function for the part of $\cot x$ for which $0 \leq x \leq \pi$.

$$\boxed{\quad}, x = \sqrt{3}$$

arctan x + 2 arccot x = $\frac{2\pi}{3}$

Using the trigonometric identity $\operatorname{arctan}\left(\frac{a}{b}\right) \equiv \operatorname{arccot}\left(\frac{b}{a}\right)$ without loss of generality, easily verifiable by a right angled triangle:

$$\begin{aligned} \tan \theta &= \frac{a}{b} \Rightarrow \theta = \operatorname{arctan} \frac{a}{b} \\ \operatorname{arccot} \frac{a}{b} &= \frac{1}{\operatorname{cot} \theta} = \frac{a}{b} \\ \operatorname{cot} \theta &= \frac{b}{a} \Rightarrow \theta = \operatorname{arccot} \frac{b}{a} \end{aligned}$$

$$\Rightarrow \operatorname{arctan} x + 2 \operatorname{arccot} \frac{x}{2} = \frac{2\pi}{3}$$

$$\Rightarrow \theta + 2\phi = \frac{2\pi}{3}$$

Taking tangents on both sides of the equation:

$$\Rightarrow \tan[\operatorname{arctan} \frac{x}{2} + 2 \operatorname{arccot} \frac{x}{2}] = \tan \frac{2\pi}{3}$$

$$\Rightarrow \frac{\tan(\operatorname{arctan} \frac{x}{2}) + \tan(2 \operatorname{arccot} \frac{x}{2})}{1 - \tan(\operatorname{arctan} \frac{x}{2}) \tan(2 \operatorname{arccot} \frac{x}{2})} = -\sqrt{3}$$

$$\Rightarrow \frac{\frac{x}{2} + \tan(\operatorname{arccot} \frac{x}{2})}{1 - x \operatorname{arctan}(\operatorname{arccot} \frac{x}{2})} = -\sqrt{3}$$

Applying the tangent double-angle identity $\tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta}$:

$$\Rightarrow \frac{2 + \frac{2 \tan(\operatorname{arccot} \frac{x}{2})}{1 - \tan^2(\operatorname{arccot} \frac{x}{2})}}{1 - x \operatorname{arctan}(\operatorname{arccot} \frac{x}{2})} = -\sqrt{3}$$

Multiplying top & bottom of the double fraction by $1 - \frac{1}{x^2}$:

$$\Rightarrow \frac{x + \frac{\frac{2}{x}}{1 - \frac{1}{x^2}}}{1 - 2\left[1 - \frac{1}{x^2}\right]} = -\sqrt{3}$$

$$\Rightarrow \frac{x + \frac{2x}{x^2 - 1}}{1 - \frac{2}{x^2}} = -\sqrt{3}$$

Multiplying top & bottom of the double fraction by $1 - \frac{1}{x^2}$:

$$\Rightarrow \frac{x\left(1 - \frac{1}{x^2}\right) + \frac{2}{x}}{\left(1 - \frac{1}{x^2}\right) - 2} = -\sqrt{3}$$

$$\Rightarrow \frac{2 - \frac{1}{x^2} + \frac{2}{x}}{1 - \frac{1}{x^2} - 2} = -\sqrt{3}$$

$$\Rightarrow \frac{2 + \frac{2}{x}}{-1 - \frac{1}{x^2}} = -\sqrt{3}$$

Multiplying top & bottom of the double fraction by x^2 :

$$\Rightarrow \frac{2^3 + x}{-x^2 - 1} = -\sqrt{3}$$

$$\Rightarrow -\frac{x(x^2 + 1)}{x^2 + 1} = -\sqrt{3}$$

$$\Rightarrow x = \sqrt{3} \quad (x \neq 0)$$

Question 107 (*****)

The distinct acute angles θ and φ , $\theta > \varphi$ satisfy the equation

$$f(\theta, \varphi) = g(\theta, \varphi) \tan \varphi.$$

where the functions f and g are defined as

$$f(\theta, \varphi) \equiv \sin(\theta - \varphi) \quad \text{and} \quad g(\theta, \varphi) \equiv \cos(\theta - \varphi) - 2 \tan \varphi \sin(\theta - \varphi)$$

Use trigonometric identities to show that

$$\tan \theta = 2 \tan \varphi.$$



WE ARE GIVEN THAT $f(\theta, \phi) = g(\theta, \phi)$ AND

$$\Rightarrow \sin(\theta - \phi) = [\cos(\theta - \phi) - 2\text{tan}\theta \sin(\theta - \phi)] \text{ tan}\phi$$

$$\Rightarrow \frac{\sin(\theta - \phi)}{\cos(\theta - \phi)} = \left[1 - 2\tan\theta \frac{\sin(\theta - \phi)}{\cos(\theta - \phi)} \right] \tan\phi$$

$$\Rightarrow \tan(\theta - \phi) = [1 - 2\tan\theta \tan(\theta - \phi)] \tan\phi$$

$$\Rightarrow \tan(\theta - \phi) = \tan\phi - 2\tan^2\theta \tan(\theta - \phi)$$

$$\Rightarrow \tan(\theta - \phi) + 2\tan^2\theta \tan(\theta - \phi) = \tan\phi$$

$$\rightarrow \tan(\theta - \phi) [1 + 2\tan^2\theta] = \tan\phi$$

$$\rightarrow (\tan\theta - \tan\phi)(1 + 2\tan^2\theta) = \tan\phi$$

$$\Rightarrow (\tan\theta - \tan\phi)(1 + 2\tan^2\theta) = \tan\phi(1 + 2\tan^2\theta)$$

$$\Rightarrow \tan\theta + 2\tan\theta\tan\phi - \tan\phi - 2\tan^3\theta = \tan\phi + \tan^3\theta\tan\phi$$

$$\Rightarrow \tan\theta + \tan^2\theta\tan\phi = 2\tan\phi + 2\tan^3\phi$$

$$\Rightarrow \tan\theta(1 + \tan^2\phi) = 2\tan\phi(1 + \frac{1}{1 + \tan^2\phi})$$

$$\Rightarrow \tan\theta = 2\tan\phi$$

As Required

ALTERNATIVE BY EXPANDING INTO SINES & COSINES

$$\Rightarrow \sin(\theta - \phi) = [\cos(\theta - \phi) - 2\cos(\theta - \phi)\sin(\theta - \phi)] \text{ bunch}$$

$$\Rightarrow \sin(\theta - \phi) = [\cos(\theta - \phi) - \frac{2\cos(\theta - \phi)\sin(\theta - \phi)}{\cos(\theta - \phi)}] \text{ bunch}$$

$$\Rightarrow \sin(\theta - \phi) = \frac{1}{\cos(\theta - \phi)} [\cos(\theta - \phi)\cos(\theta - \phi) - 2\cos(\theta - \phi)\sin(\theta - \phi)] \text{ bunch}$$

$$\Rightarrow \sin(\theta - \phi) = \frac{\text{bunch}}{\cos(\theta - \phi)} [\cos^2(\theta - \phi) - \sin^2(\theta - \phi) - 2\cos(\theta - \phi)\sin(\theta - \phi)]$$

$$\Rightarrow \sin(\theta - \phi) = \frac{\text{bunch}}{\cos(\theta - \phi)} [\cos(\theta + (\theta - \phi)) - \sin(\theta + (\theta - \phi))]$$

$$\Rightarrow \sin(\theta - \phi) = \frac{\text{bunch}}{\cos(\theta - \phi)} [\cos(\theta) - \sin(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)]$$

$$\Rightarrow \cos^2 \sin(\theta - \phi) = \sin^2 \cos(\theta) - \sin^2 \sin(\theta - \phi)$$

$$\Rightarrow \sin(\theta - \phi) \cos^2 \theta + \sin(\theta - \phi) \sin^2 \theta = \sin^2 \cos(\theta)$$

$$\Rightarrow \sin(\theta - \phi) [\cos^2 + \sin^2] = \sin^2 \cos(\theta)$$

$$\Rightarrow \sin(\theta - \phi) \cancel{\cos^2 + \sin^2} = \sin^2 \cos(\theta)$$

$$\Rightarrow \sin \theta \cos \phi - \cos \theta \sin \phi = \sin^2 \cos(\theta)$$

$$\Rightarrow \sin \theta \cos \phi = \sin^2 \cos(\theta) + \cos \theta \sin \phi$$

$$\Rightarrow \frac{\sin \theta}{\cos \theta} = \frac{\sin^2 \cos(\theta) + \cos \theta \sin \phi}{\cos \theta}$$

$$\Rightarrow \tan \theta = 2 \frac{\sin \cos(\theta) + \sin \phi}{\cos \theta}$$

Question 108 (***)**

Solve the following trigonometric equation, for $0 \leq \theta < 2\pi$.

$$3\cos^2 \theta - \sin^2 \theta - \sqrt{3} \cos \theta - \sin \theta = 0.$$

, $\theta = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

$3\cos^2 \theta - \sin^2 \theta - \sqrt{3} \cos \theta - \sin \theta = 0$

• SOLVE BY FACTORISING IN Pairs

$$\Rightarrow ((\sqrt{3}\cos \theta)^2 - (\sin \theta)^2) - [\sqrt{3}\cos \theta + \sin \theta] = 0$$

$$\Rightarrow (\sqrt{3}\cos \theta - \sin \theta)(\sqrt{3}\cos \theta + \sin \theta) - (\sqrt{3}\cos \theta + \sin \theta) = 0$$

$$\Rightarrow (\sqrt{3}\cos \theta + \sin \theta)[(\sqrt{3}\cos \theta - \sin \theta) - 1] = 0$$

• EITHER

$$\begin{aligned} \sqrt{3}\cos \theta + \sin \theta &= 0 \\ \Rightarrow \sqrt{3}\cos \theta &= -\sin \theta \\ \Rightarrow \sqrt{3} &= -\tan \theta \\ \Rightarrow \tan \theta &= -\sqrt{3} \\ \Rightarrow \theta &= -\frac{\pi}{6} \pm 2m\pi \quad n \in \mathbb{Z} \end{aligned}$$

• OR

$$\begin{aligned} \sqrt{3}\cos \theta - \sin \theta &= 1 \\ \Rightarrow \frac{\sqrt{3}}{2}\cos \theta - \frac{1}{2}\sin \theta &= \frac{1}{2} \\ \Rightarrow \cos \theta \left(\frac{\sqrt{3}}{2} \right) - \sin \theta \left(\frac{1}{2} \right) &= \frac{1}{2} \\ \Rightarrow \cos \left(\theta + \frac{\pi}{6} \right) &= \frac{1}{2} \\ \Rightarrow \theta + \frac{\pi}{6} &= \frac{\pi}{3} \pm 2m\pi \quad n \in \mathbb{Z} \\ \Rightarrow \theta &= \frac{\pi}{6} \pm 2m\pi \\ \Rightarrow \theta &= \frac{\pi}{2} \pm 2m\pi \end{aligned}$$

• COLLECTING ALL THE SOLUTIONS FOR $0 \leq \theta < 2\pi$

$$\theta = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$$

Question 109 (***)**

Solve the following trigonometric equation, for $0 < x < 2\pi$.

$$\sin x \sin 2x + \sin 2x \sin 3x + \sin 3x \sin 4x = 0.$$

$$\boxed{\quad}, \quad x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$\sin x \sin 2x + \sin 2x \sin 3x + \sin 3x \sin 4x = 0 \quad 0 < x < 2\pi$

LOOKING AT THE ABOVE WE MAY ATTEMPT TO FACTORISE $\sin 2x$ OUT OF THE GIVEN EXPRESSION, LEAVING BY $\sin 2x$, AFTER FACTORISING $\sin 3x$ OUT OF THE LAST TWO TERMS, & WE'RE LEFT WITH $\sin 2x$ IN TERMS OF $\sin x$.

$$\begin{aligned} &\Rightarrow \sin x \sin 2x + \sin 2x \left[\sin 3x + \sin 4x \right] = 0 \\ &\Rightarrow \sin x \sin 2x + \sin 2x \left[\sin 2x + 2\sin 2x \cos 2x \right] = 0 \\ &\Rightarrow \sin 2x \left[\sin x + \sin 3x (1 + 2\cos 2x) \right] = 0 \end{aligned}$$

NEXT WE REQUIRE THE THREE ANGLE IDENTITY, WHICH IF WE DO NOT REMEMBER WE'D HAVE TO QUOTE.

$$\begin{aligned} \sin 3x &= \sin(2x + x) \\ &= (\sin 2x \cos x) + (\cos 2x \sin x) \\ &= 2\sin 2x \cos x + \sin x - 2\sin^2 x \\ &= 2\sin 2x \cos x + \sin x - 2\sin^2 x \\ &= 2\sin 2x \cos x + \sin x - 2\sin^2 x \\ &= 3\sin 2x - 4\sin^2 x \end{aligned}$$

HENCE WE CAN EASILY FACTORISE $\sin 2x$ OUT

$$\begin{aligned} &\Rightarrow \sin 2x \left[\sin x + (3\sin 2x - 4\sin^2 x)(1 + 2\cos 2x) \right] = 0 \\ &\Rightarrow \sin 2x \sin x \left[1 + (3 - 4\sin^2 x)(1 + 2\cos 2x) \right] = 0 \\ &\Rightarrow \sin 2x \sin x \left[1 + (3 - 4\sin^2 x)(1 + 2(1 - 2\sin^2 x)) \right] = 0 \end{aligned}$$

$\rightarrow \sin^2 2x \left[1 + (3 - 4\sin^2 x)(1 + 2 - 4\sin^2 x) \right] = 0$
 $\rightarrow \sin 2x \sin x \left[1 + (3 - 4\sin^2 x)(3 - 4\sin^2 x) \right] = 0$
 $\rightarrow \sin 2x \sin x \left[1 + (3 - 4\sin^2 x)^2 \right] = 0$

Since $\sin 2x = 0$ or $\sin x = 0$ or $1 + (3 - 4\sin^2 x)^2 \neq 0$ (as it is at least 1)

As ALL THE SOLUTIONS OF $\sin x = 0$ ARE INCLUDED IN THE SOLUTION OF $\sin 2x = 0$, WE FINALLY OBTAIN

$$\begin{aligned} &\rightarrow \sin 2x = 0 \\ &\rightarrow 2x = k\pi, \quad k \in \mathbb{Z} \\ &\rightarrow x = \frac{k\pi}{2} \\ &\therefore x = \frac{\pi}{2}, \pi, \frac{3\pi}{2} \quad \text{for } 0 < x < 2\pi \end{aligned}$$

Question 110 (***)**

Solve the following trigonometric equation, for $0 < x < \frac{1}{2}\pi$.

$$4\cos x \cos 2x \cos 5x + 1 = 0.$$

$$\boxed{\quad}, \quad x = \frac{1}{8}\pi, \frac{1}{5}\pi, \frac{3}{8}\pi, \frac{2}{5}\pi$$

$4\cos x \cos 2x \cos 5x + 1 = 0 \quad 0 \leq x < \frac{\pi}{2}$

 $\Rightarrow 4\cos x \cos 2x \cos 5x = -1$

- $2\pi - \frac{1}{2}\pi$ is not a solution of the equation ($\cos(2\pi) = 1$)
so if we multiply both sides by since we introduce this solution)

 $\Rightarrow 4\sin x \sin 2x \sin 5x = -\sin x$
 $\Rightarrow 2\sin 2x \cos 2x \cos 5x = -\sin x$
 $\Rightarrow \sin^2 2x = -\sin x$

- Now dividing at the LHS

$$\begin{aligned} \sin(4x + 5x) &= \sin(6x)\sin(5x) + \cos(6x)\sin(5x) \\ \sin(4x - 5x) &= \sin(6x)\cos(5x) - \cos(6x)\sin(5x) \\ \sin^2 2x + \sin(x) &= 2\sin^2 2x \cos 5x \end{aligned}$$
 $\therefore \sin^2 2x = \frac{1}{2}\sin^2 x - \frac{1}{2}\sin x$

- Reverting is the solution

$$\begin{aligned} \frac{1}{2}\sin^2 x - \frac{1}{2}\sin x &= -\sin x \\ \frac{1}{2}\sin^2 x &= -\frac{1}{2}\sin x \\ \sin^2 x &= -\sin x \\ \sin x &= \sin(-x) \end{aligned}$$

$$\begin{aligned} \Rightarrow (9x = -x \pm 2n\pi) \quad n=0,1,2,\dots \\ 9x = \pi k \pm 2n\pi \\ \Rightarrow (9x = \pi \pm 2n\pi) \\ 8x = \frac{\pi}{9} \pm \frac{n\pi}{2} \\ \Rightarrow (x = 0 \pm \frac{n\pi}{8}) \\ x = \frac{\pi}{8} \pm \frac{n\pi}{4} \\ \therefore x = 0, \frac{\pi}{8}, \frac{2\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8} \\ \text{but } 0 \text{ is not a solution (introduced)} \\ \therefore x = \frac{\pi}{8}, \frac{2\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8} \end{aligned}$$

Question 111 (***)**

Solve the following trigonometric equation.

$$2\cot 2x = \sec x \sec\left(\frac{2\pi}{15}\right) + 2\tan\left(\frac{2\pi}{15}\right), \quad 0 \leq x < 2\pi.$$

Give the answers in the form $k\pi$, where k is rational.

S , $k = \frac{1}{2}, \frac{3}{2}, \frac{11}{90}, \frac{71}{90}, \frac{131}{90}, \frac{41}{30}$

$$2\cot 2x = \sec x \sec\left(\frac{2\pi}{15}\right) + 2\tan\left(\frac{2\pi}{15}\right), \quad 0 \leq x < 2\pi$$

- START BY SWAPPING EVERYTHING INTO SINES & COSINES
$$\frac{2\cot 2x}{\sin 2x} = \frac{1}{\cos x} \frac{1}{\cos\left(\frac{2\pi}{15}\right)} + \frac{2\sin\frac{2\pi}{15}}{\cos x}$$

- MULTIPLY THROUGH BY $\sin 2x \cos\left(\frac{2\pi}{15}\right)$
$$\Rightarrow 2\cot 2x \cos\left(\frac{2\pi}{15}\right) = \sin 2x + 2\sin\frac{2\pi}{15} \sin 2x \cos\left(\frac{2\pi}{15}\right)$$

- EXPAND & SIMPLIFY
$$\Rightarrow 2\cot 2x \cos\left(\frac{2\pi}{15}\right) = 2\sin x \cos x + \frac{2\sin\frac{2\pi}{15}}{\cos\left(\frac{2\pi}{15}\right)} \sin 2x \cos\left(\frac{2\pi}{15}\right)$$

- EVIDENTLY WEAL 11 + A SOLUTION (DIVIDE IT BY 2 AND ACCOUNT FOR IT AT THE END)
$$\Rightarrow \cos 2x \cos\left(\frac{2\pi}{15}\right) = \sin x + \frac{\sin\frac{2\pi}{15}}{\cos\left(\frac{2\pi}{15}\right)} \sin 2x$$

- $\cos 2x \cos\left(\frac{2\pi}{15}\right) - \sin x \sin\frac{2\pi}{15} = \sin 2x$

- $\Rightarrow \cos(2x + \frac{2\pi}{15}) = \sin x$

- $\Rightarrow \cos(2x + \frac{2\pi}{15}) = \cos(\frac{\pi}{2} - x)$ $\sin A \equiv \cos(\frac{\pi}{2} - A)$

- SETTING UP A GENERAL SOLUTION

$$\Rightarrow \begin{cases} 2x + \frac{2\pi}{15} = \frac{\pi}{2} - x \pm 2m\pi \\ 2x + \frac{2\pi}{15} = \pi - \frac{\pi}{2} \pm 2m\pi \end{cases} \quad m=0, 1, 2, 3, \dots$$

- GRADE TO WORK IN DEGREES & SWITCH TO RADIANS AT THE END

$$\Rightarrow \begin{cases} 2x + 24^\circ = 90^\circ - x \pm 360^\circ \\ 2x + 24^\circ = 180^\circ - 72^\circ \pm 360^\circ \end{cases}$$

$$\Rightarrow \begin{cases} 3x = 45^\circ \pm 360^\circ \\ x = -114^\circ \pm 360^\circ \end{cases}$$

- $\Rightarrow \begin{cases} x = 22^\circ \pm 120^\circ \\ x = 216^\circ \pm 360^\circ \end{cases}$

- SOLUTIONS IN DEGREES: $22^\circ, 142^\circ, 202^\circ, 246^\circ$

- TO THAT WE NEED TO ADD THE SOLUTIONS OF $\cos x = 0$

$$\therefore k = \frac{1}{2}(22^\circ + 142^\circ + 202^\circ + 246^\circ) + \frac{2\pi}{30} //$$

Question 112 (***)**

Use trigonometric algebra to fully simplify

$$2 \arctan\left(\frac{1}{5}\right) + \arccos\left(\frac{7}{5\sqrt{2}}\right) + \arctan\left(\frac{1}{8}\right),$$

giving the final answer in terms of π .

, $\frac{\pi}{4}$

$2\arctan\left(\frac{1}{5}\right) + \arccos\left(\frac{7}{5\sqrt{2}}\right) + 2\arctan\left(\frac{1}{8}\right) = \psi$

$\tan\alpha = \frac{1}{5}$

$\cos\beta = \frac{1}{5\sqrt{2}}$

$\tan\gamma = \frac{1}{8}$

WORKING WITH TRIGONIC AS FOLLOWS:

$$\begin{aligned} &\Rightarrow 2\theta + \alpha + 2\delta = \pi \\ &\Rightarrow 2\theta + 2\delta = \pi - \alpha \\ &\Rightarrow \tan(2\theta + 2\delta) = \tan(\pi - \alpha) \\ &\Rightarrow \frac{\tan 2\theta + \tan 2\delta}{1 - \tan 2\theta \tan 2\delta} = \frac{\tan \pi - \tan \alpha}{1 - \tan \pi \tan \alpha} \\ &\Rightarrow \frac{2\tan\theta + 2\tan\delta}{1 - 2\tan\theta \times 2\tan\delta} = \frac{-\tan\alpha - \tan\pi}{1 - \tan\pi \tan\alpha} \\ &\Rightarrow \frac{2\tan\theta + 2\tan\delta}{1 - 2\tan\theta \times 2\tan\delta} = \frac{\tan\pi - \tan\alpha}{1 - \tan\pi \tan\alpha} \end{aligned}$$

SUBSTITUTING VALUES IN

$$\frac{\frac{2}{5} + \frac{1}{4}}{1 - \frac{2}{5} \times \frac{1}{4}} = \frac{\tan\pi - \frac{1}{5}}{1 + \tan\pi \times \frac{1}{5}}$$

$$\begin{aligned} &\Rightarrow \frac{\frac{10}{25} + \frac{16}{40}}{1 - \frac{10}{25} \times \frac{16}{40}} = \frac{7\tan\pi - 1}{7 + \tan\pi} \\ &\Rightarrow \frac{\frac{36}{120} + \frac{16}{63}}{1 - \frac{36}{120} \times \frac{16}{63}} = \frac{7\tan\pi - 1}{7 + \tan\pi} \\ &\Rightarrow \frac{315 + 192}{720 - 80} = \frac{7\tan\pi - 1}{7 + \tan\pi} \\ &\Rightarrow \frac{3}{4} = \frac{7\tan\pi - 1}{7 + \tan\pi} \\ &\Rightarrow 21 + 3\tan\pi = 28\tan\pi - 4 \\ &\Rightarrow 25\tan\pi = 25 \\ &\Rightarrow \tan\pi = 1 \\ &\Rightarrow \pi = \frac{\pi}{4} // \end{aligned}$$

ALTERNATIVE BY COMPLEX NUMBERS

CONSIDER THE EXPRESSION

$$\begin{aligned} &(5i)^2(7+i)(6+i)^2 \\ &= (25+10i-1)(7+i)(49+12i-1) \\ &= (24+10i)(7+i)(48+12i) \\ &= 2(12+5i)(7+i)(63+16i) \end{aligned}$$

$$\begin{aligned} &= 2(-84+42i+351-5)(63+16i) \\ &= 2(-71+47i)(63+16i) \\ &= 2(497i+1264i^2+246i-752) \\ &= 2(4225+4225i) \\ &= 8450(1+i) \end{aligned}$$

$\arg[(63+16i)(63+16i)] = \arg[8450(1+i)]$

$\arg(1+i)^2 + \arg(1+i) + \arg(63+16i) = \arg 8450 + \arg(1+i)$

$2\arg(1+i) + \arg(1+i) + \arg(63+16i) = \arg 8450 + \arg(1+i)$

$2\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right) + 2\arctan\left(\frac{1}{5}\right) = 0 + \arctan 1$

$2\arctan\frac{1}{5} + \arctan\frac{1}{8} + 2\arctan\frac{1}{5} = \frac{\pi}{4} //$

Question 113 (***)**

It is given that

$$\arctan x + \arctan y + \arctan z = \frac{\pi}{2}$$

Show that x , y and z satisfy the relationship

$$xy + yz + zx = 1.$$

, proof

• WORK IN STAGES

$$\begin{aligned} &\Rightarrow \arctan x + \arctan y = \frac{\pi}{2} \\ &\Rightarrow \theta + \phi = \frac{\pi}{2} \\ &\Rightarrow \tan(\theta+\phi) = \tan\frac{\pi}{2} \\ &\Rightarrow \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \tan\frac{\pi}{2} \\ &\Rightarrow \frac{x+y}{1-xy} = \tan\frac{\pi}{2} \\ &\Rightarrow \frac{x+y}{1-xy} = \infty \\ &\text{THIS} \\ &\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \end{aligned}$$

• NOW USE THE IDENTITY IN THE BOX WITH THE ARCTAN

$$\begin{aligned} &\Rightarrow \arctan x + \arctan y + \arctan z = \frac{\pi}{2} \\ &\Rightarrow \arctan\left(\frac{x+y}{1-xy}\right) + \arctan z = \frac{\pi}{2} \\ &\Rightarrow \arctan\left[\frac{\left(\frac{x+y}{1-xy}\right) + z}{1 - \left(\frac{x+y}{1-xy}\right)z}\right] = \frac{\pi}{2} \\ &\text{TAKING TANGENTS ON BOTH SIDES} \\ &\Rightarrow \frac{\frac{x+y}{1-xy} + z}{1 - z\left(\frac{x+y}{1-xy}\right)} = \infty \end{aligned}$$

• AS THE FRACTION IS INFINITE, THE DENOMINATOR MUST BE ZERO

$$\begin{aligned} &\Rightarrow 1 - z\left(\frac{x+y}{1-xy}\right) = 0 \\ &\Rightarrow 1 - \frac{zx+zy}{1-xy} = 0 \\ &\Rightarrow 1-xy - (zx+zy) = 0 \\ &\Rightarrow 1-xy-zx-zy=0 \\ &\Rightarrow xy+yz+zx=1 \end{aligned}$$

✓ Raghbir

Question 114 (***)**

Use a fully detailed method to show that

$$\arctan[\sqrt{6} + \sqrt{3} - \sqrt{2} - 2] = 37.5^\circ.$$

, proof

$37.5^\circ = \arctan(\sqrt{6} + \sqrt{3} - \sqrt{2} - 2)$

METHOD A — BY CONSTRUCTING A SUITABLE EQUATION

- Start from a general solution for a "man" equation
 $\Rightarrow [2] = 37.5^\circ + 180n, n = 0, 1, 2, \dots$
 $\Rightarrow 2x = 75^\circ + 360n$
 $\Rightarrow (2x - 45^\circ) = (3x - 2) + 360n$
- This could then been a sine or cosine equation, general solution
 $\Rightarrow \cos(3x - 45^\circ) = \cos(3x - 2)$
- Obtain by combining three identities and solve
 $\Rightarrow \cos^2(x) + \sin^2(x) = (\cos x)^2 + 2\cos x \sin x$
 $\Rightarrow \frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x = (\frac{\sqrt{2}}{2}\cos x + \frac{1}{2}\sin x) \times 2$
 $\Rightarrow (\sqrt{2}\cos x + \sqrt{2}\sin x) = 4\sqrt{2}\cos x + \sin x$
 $\Rightarrow \sqrt{2} + \sqrt{2}\tan x = \sqrt{2} + \tan x \div \cos x$
 $\Rightarrow (\sqrt{2} - 1)\tan x = \sqrt{2} - \sqrt{2}$
 $\Rightarrow \tan x = \frac{\sqrt{2} - \sqrt{2}}{\sqrt{2} - 1}$
 $\Rightarrow \tan x = \frac{(\sqrt{2} - \sqrt{2})(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)}$
 $\Rightarrow \tan x = \frac{(\sqrt{2} + \sqrt{2}) - 2 - (\sqrt{2} - 2)}{2 - 1}$
 $\Rightarrow x = \arctan(\sqrt{2} + \sqrt{3} - \sqrt{2} - 2) \approx 180n, n = 0, 1, 2, \dots$
 $\therefore \arctan(\sqrt{6} + \sqrt{3} - \sqrt{2} - 2) = 37.5^\circ$

SUMMARY — BY CONSTRUCTING THE SECOND WAY
THE COMPOUND ANGLE OF THE SECOND ANGLE IS UNKNOWN

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

- $\tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}}$
 $= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = \frac{(\sqrt{3} + 1)^2}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{3 + 2\sqrt{3} + 1}{2} = \frac{4 + 2\sqrt{3}}{2}$
 $= 2 + \sqrt{3}$
- $\tan 75^\circ = \frac{2 \tan 37.5^\circ}{1 - \tan^2 37.5^\circ}$
 $\Rightarrow 2 + \sqrt{3} = \frac{2T}{1 - T^2}$
 $\Rightarrow 1 - T^2 = \frac{2T}{2 + \sqrt{3}}$
 $\Rightarrow T^2 - 1 = -\frac{2T}{2 + \sqrt{3}}$
 $\Rightarrow T^2 + \frac{2T}{2 + \sqrt{3}} - 1 = 0$
 $\Rightarrow \left(T + \frac{1}{2 + \sqrt{3}}\right)^2 - \frac{1}{(2 + \sqrt{3})^2} - 1 = 0$
 $\Rightarrow \left(T + \frac{1}{2 + \sqrt{3}}\right)^2 - \frac{1}{4 - 1} - 1 = 0$
 $\Rightarrow \left(T + \frac{1}{2 + \sqrt{3}}\right)^2 - \frac{2 - \sqrt{3}}{3} - 1 = 0$

$\Rightarrow (T + 2 - \sqrt{3})^2 = \frac{(2 - \sqrt{3})^2}{3} + 1$
 $\Rightarrow (T + 2 - \sqrt{3})^2 = 4 - 4\sqrt{3} + 3 + 1$
 $\Rightarrow (T + 2 - \sqrt{3})^2 = 8 - 4\sqrt{3}$
 $\Rightarrow (T + 2 - \sqrt{3})^2 = 2[4 - 2\sqrt{3}]$
 $\Rightarrow (T + 2 - \sqrt{3})^2 = 2[(T - 2\sqrt{3})^2]$
 $\Rightarrow (T + 2 - \sqrt{3}) = 2\sqrt{4 - 2\sqrt{3} + \sqrt{3}^2}$
 $\Rightarrow T + 2 - \sqrt{3} = \begin{cases} \sqrt{4 - 2\sqrt{3}} & = \sqrt{2} - \sqrt{2} \\ -\sqrt{4 - 2\sqrt{3}} & = -\sqrt{2} + \sqrt{2} \end{cases}$
 $\Rightarrow T = \begin{cases} \sqrt{2} + \sqrt{2} - \sqrt{2} - 2 & > 0 \\ -\sqrt{2} + \sqrt{2} - \sqrt{2} - 2 & < 0 \end{cases}$
 $\Rightarrow \tan(37.5^\circ) = \sqrt{2} + \sqrt{2} - \sqrt{2} - 2$
 $\Rightarrow \arctan(\sqrt{2} + \sqrt{3} - \sqrt{2} - 2) = 37.5^\circ$

VARIATION TO METHOD B

- Use $\tan 45^\circ = \frac{2 \tan 22.5^\circ}{1 - \tan 22.5^\circ}$ to find $\tan 22.5^\circ = \sqrt{2} - 1$
- Use $\tan 30^\circ = \frac{2 \tan 15^\circ}{1 - \tan 15^\circ}$ to find $\tan 15^\circ = 2 - \sqrt{3}$
- $\tan 37.5^\circ = \tan((15^\circ + 22.5^\circ)/2)$ q. use compound

Question 115 (*****)

Use a trigonometric algebra to solve the following equation

$$(\arctan x)^2 + (\operatorname{arccot} x)^2 = \frac{5\pi^2}{8}$$

You may assume that $y = \operatorname{arccot} x$ is the inverse function of $y = \cot x$, $0 \leq x \leq \pi$

, $x = -1$

$$(\arctan x)^2 + (\operatorname{arccot} x)^2 = \frac{\pi^2}{8}$$

$$\Rightarrow \theta^2 + \phi^2 = \frac{\pi^2}{8}$$

LOOKING AT THE DIFFERENT CASES

$$\Rightarrow \theta^2 + (\frac{\pi}{2} - \phi)^2 = \frac{\pi^2}{8}$$

$$\Rightarrow \theta^2 + \frac{\pi^2}{4} - \pi\phi + \phi^2 = \frac{\pi^2}{8}$$

$$\Rightarrow 2\theta^2 - \pi\phi + \frac{\pi^2}{4} = 0$$

$$\Rightarrow 16\theta^2 - 8\pi\phi - 3\pi^2 = 0$$

$$\Rightarrow (4\theta - 3\pi)(4\theta + \pi) = 0$$

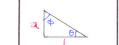
$$\Rightarrow \theta = \begin{cases} \frac{-\pi}{4} \\ \frac{3\pi}{4} \end{cases}$$

$$\Rightarrow \arctan x = \begin{cases} -\frac{\pi}{4} \\ \frac{3\pi}{4} \end{cases}$$

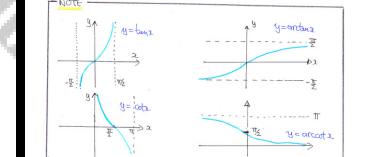
$$\Rightarrow x = \tan(-\frac{\pi}{4})$$

$$\Rightarrow x = -1$$

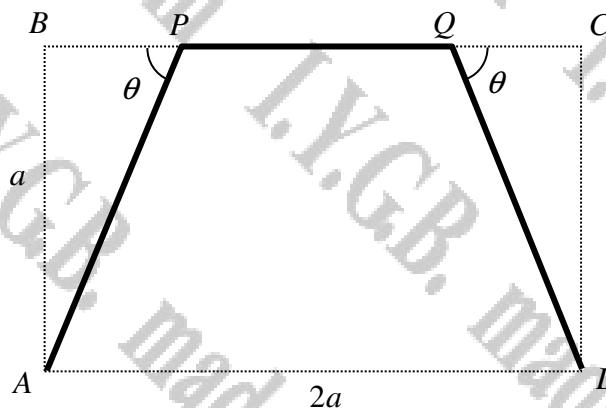
LET $\theta = \arctan x$
 $\tan \theta = x$



LET $\phi = \operatorname{arccot} x$
 $\cot \phi = x$
 $\tan \phi = \frac{1}{x}$
 $\theta + \phi = \frac{\pi}{2}$



Question 116 (*****)



The figure above shows a network $APQD$ inside a rectangle $ABCD$, where $|AB| = a$ and $|BC| = 2a$. The endpoints of the network A and D are fixed. The points P and Q are variable so that they always lie on BC with $|AP| = |QD|$. The angles BPA and CQD are both equal to θ . A particle travels with constant speed v on the sections AP and QD , and with constant speed $\frac{5}{3}v$ on the section PQ .

Let T be the total time for the journey $APQD$.

Given that the positions of the points P and Q can be varied as appropriate, show that the minimum value of T is $\frac{14a}{5v}$, fully justifying that this is the minimum value.

, proof

Now we can express d_3 also in terms of a & θ

$$d_3 = 2a - 2d_1 = 2a - 2a \cos\theta = 2a(1-\cos\theta)$$

SPEED = DISTANCE / TIME \Leftrightarrow TIME = DISTANCE / SPEED

$$t_2 = \frac{d_2}{v} \quad \text{& similarly} \quad t_3 = \frac{d_3}{v} = \frac{2a(1-\cos\theta)}{v}$$

TOTAL TIME $T = 2t_2 + t_3$

$$\Rightarrow T = \frac{2a}{v} \sec\theta + \frac{2a(1-\cos\theta)}{v}$$

$$\Rightarrow T = \frac{2a}{v} [\sec\theta + 1 - \cos\theta]$$

$$\Rightarrow T = \frac{2a}{v} [1 + \sec\theta - \cos\theta]$$

DIFFERENTIATE AND SET TO ZERO

$$\Rightarrow \frac{dT}{d\theta} = \frac{2a}{v^2} [-\sec\theta\sin\theta + \sec^2\theta]$$

$$\Rightarrow 0 = \frac{2a}{v^2} [3\sec\theta - \sec\theta\sin\theta]$$

$$\Rightarrow 0 = \frac{2a}{v^2} \sec\theta [\sec\theta - \sin\theta]$$

$$3\sec\theta - \sin\theta = 0 \quad (\cos\theta \neq 0)$$

$$\Rightarrow \frac{3}{\sin\theta} = \frac{\sin\theta}{\cos\theta} = 0$$

$$\Rightarrow \frac{3}{\sin\theta} = \frac{\cos\theta}{\sin\theta} = 0$$

$$\Rightarrow \cos\theta = \frac{3}{3}$$

CHECK THE NATURE OF THIS STATIONARY VALUE

$$\frac{d^2T}{d\theta^2} = \frac{2a}{v^3} [\sec\theta\sin\theta + \sec^2\theta - \cos\theta\sin\theta]$$

NOW

$\begin{array}{l} 4 \\ \sqrt{3} \\ \hline 5 \end{array}$

$$\cos\theta = \frac{3}{5}, \sec\theta = \frac{5}{3}$$

$$\sin\theta = \frac{4}{5}, \tan\theta = \frac{4}{3}$$

$$\frac{d^2T}{d\theta^2} = \frac{2a}{v^3} \left[5 \times \frac{3}{5} \times \frac{4}{5} + 5 \times \frac{25}{9} - 4 \times \frac{3}{5} \times \frac{4}{5} \right]$$

$$= \frac{2a}{v^3} \left[24 + 625 - 450 \right] = \frac{2a}{v^3} \times \frac{420}{24} > 0$$

\therefore A (LOCAL) MINIMUM //

$$T_{\min} = \frac{2a}{v} \left[1 + \frac{5}{3} - 3 \times \frac{3}{4} + 1 \right]$$

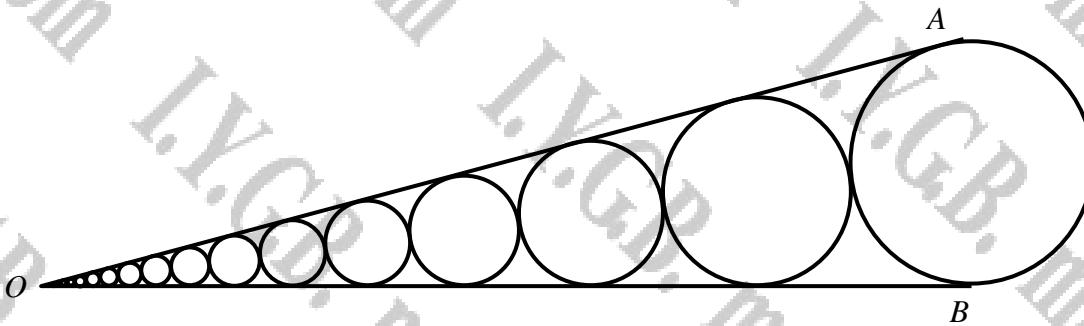
$$= \frac{2a}{v} \left[\frac{9}{4} - \frac{9}{4} + 3 \right]$$

$$= \frac{2a}{v} [7]$$

$$= \frac{14a}{5v}$$

As required

Question 117 (*****)



The figure above shows an infinite sequence of circles of decreasing radius, the radius of the larger circle being $\frac{4}{3}$.

The centres of these circles lie on a straight line. The straight lines OA and OB are tangents to every circle in the sequence, the angle AOB denoted by 2θ .

Given that the total area of these circles is 2π , determine the value of θ .

$$\boxed{\quad}, \theta = \frac{1}{6}\pi$$

Start with a G.P. difference

Looking at the figure above the set

$$\sin\theta = \frac{r_{n+1}}{r_n} \quad \text{and} \quad \sin\theta = \frac{r_n}{r_n + r_{n+1}}$$

Eliminate r_n

$$r_n = \frac{r_{n+1}}{\sin\theta} \implies \sin\theta = \frac{r_n}{r_{n+1} + r_n} = \frac{r_n}{r_n(1 + \frac{1}{r_{n+1}})} = \frac{1}{1 + \frac{1}{r_{n+1}}} \implies r_n + r_n \sin^2\theta + r_{n+1} \sin^2\theta = r_n$$

$$\implies \frac{r_{n+1}}{r_n} + \sin^2\theta + \frac{r_{n+1}}{r_n} \sin^2\theta = 1$$

Let "R" denote $\frac{r_{n+1}}{r_n}$

$$\implies R + \sin^2\theta + R\sin^2\theta = 1 \implies R(1 + \sin^2\theta) = 1 - \sin^2\theta \implies R(1 + \sin^2\theta) = -\sin^2\theta \implies R = \frac{-\sin^2\theta}{1 + \sin^2\theta} \implies \frac{r_{n+1}}{r_n} = \frac{-\sin^2\theta}{1 + \sin^2\theta} \therefore \text{A geometric progression which converges}$$

At the radius form a G.P. with common ratio so will the areas of the circles

$$\Rightarrow AOB = \pi r^2 + \pi r_1^2 + \pi r_2^2 + \pi r_3^2 + \pi r_4^2 + \dots$$

$$\Rightarrow AOB = \pi [a^2 + (ar)^2 + (ar^2)^2 + (ar^3)^2 + (ar^4)^2 + \dots]$$

$$\Rightarrow AOB = \pi \left[a^2 + a^2 r^2 + a^2 r^4 + a^2 r^6 + a^2 r^8 + \dots \right]$$

$$\Rightarrow AOB = \pi a^2 \left[1 + r^2 + r^4 + r^6 + r^8 + \dots \right]$$

$$\Rightarrow 2\pi = \pi \left(\frac{1}{1 - r^2} \right) \quad \text{Let } S_n = \frac{a(1 - r^n)}{1 - r} \text{ (GEOMETRIC PROGRESSION)}$$

$$\Rightarrow 2 = \frac{1}{1 - r^2}$$

$$\Rightarrow q = \frac{1}{1 - r^2}$$

$$\Rightarrow q(1 - r^2) = 2$$

$$\Rightarrow q(1 - \frac{1}{q}) = 2$$

$$\Rightarrow q - 1 = 2$$

$$\Rightarrow q = 3$$

$$\Rightarrow \frac{q(1 + 2qr + qr^2)}{(1 + qr)^2} = 8$$

$$\Rightarrow \frac{3(1 + 2 \cdot 3r + 3r^2)}{(1 + 3r)^2} = 8$$

$$\Rightarrow \frac{9 + 18r + 9r^2}{1 + 6r + 9r^2} = 8$$

$$\Rightarrow 9 + 18r + 9r^2 = 8 + 48r + 72r^2$$

$$\Rightarrow 27r^2 - 30r + 1 = 0$$

$$\Rightarrow (27r - 1)(r - 1) = 0$$

$$\Rightarrow r = \frac{1}{27} \quad \text{or} \quad r = 1$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

Question 118 (*****)

$$\theta = \arctan \left[\frac{\cos 66^\circ - \sin 48^\circ}{\cos 48^\circ - \sin 66^\circ} \right].$$

Show by detailed working that $\theta = -12^\circ$.

proof

METHOD 1

$$\begin{aligned} \arctan \left[\frac{\cos 66^\circ - \sin 18^\circ}{\cos 48^\circ + \sin 66^\circ} \right] &= \arctan \left[\frac{\sin(90^\circ - 66^\circ) - \sin 18^\circ}{\sin(90^\circ - 48^\circ) + \sin 66^\circ} \right] \\ &= \arctan \left[\frac{\sin 24^\circ - \sin 18^\circ}{\sin 42^\circ + \sin 66^\circ} \right] = \dots \quad (\text{cancel } \sin) \end{aligned}$$

Now from the compound angle identities

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B \end{aligned}$$

Adding & Subtracting

$$\begin{aligned} \sin(A+B) + \sin(A-B) &= 2 \sin A \cos B \\ \sin(A+B) - \sin(A-B) &= 2 \cos A \sin B \end{aligned}$$

Letting $P=A+B$ & $Q=A-B$

$$\begin{aligned} \sin P + \sin Q &= 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2} \\ \sin P - \sin Q &= 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2} \end{aligned}$$

Referring to the "unit line"

$$\dots \arctan \left[\frac{2 \cos \frac{24^\circ - 18^\circ}{2} \sin \frac{24^\circ + 18^\circ}{2}}{2 \cos \frac{42^\circ + 66^\circ}{2} \sin \frac{42^\circ - 66^\circ}{2}} \right] = \arctan \left[\frac{\cos 36^\circ \sin(-12^\circ)}{\sin 54^\circ \cos(-12^\circ)} \right]$$

Now $\sin(-A) = -\sin A$ & $\cos(-A) = \cos A$

METHOD 2

$$\begin{aligned} \text{LET } \theta &= \arctan \left[\frac{\cos 66^\circ - \sin 18^\circ}{\cos 48^\circ + \sin 66^\circ} \right], -90^\circ < \theta < 90^\circ \\ \Rightarrow \tan \theta &= \frac{\cos 66^\circ - \sin 18^\circ}{\cos 48^\circ + \sin 66^\circ} \\ \Rightarrow \frac{\sin \theta}{\cos \theta} &= \frac{\cos 66^\circ - \sin 18^\circ}{\cos 48^\circ + \sin 66^\circ} \\ \Rightarrow \sin \theta \cos 66^\circ + \sin 66^\circ \cos 18^\circ &= \cos 66^\circ \cos 48^\circ - \sin 66^\circ \sin 18^\circ \\ \Rightarrow \sin \theta \cos 66^\circ + \cos 66^\circ \sin 18^\circ &= \cos 66^\circ \cos 48^\circ - \sin 66^\circ \sin 18^\circ \\ \Rightarrow \sin(\theta + 18^\circ) &= \cos(66^\circ - 48^\circ) \\ \Rightarrow \cos[90^\circ - (\theta + 18^\circ)] &= \cos(6 + 66^\circ) \\ \uparrow \quad \text{cancel } \cos \quad \text{cancel } 90^\circ \\ \sin A &= \cos(6 + 66^\circ) \end{aligned}$$

$$\begin{aligned} \Rightarrow \cos(42^\circ - \theta) &= \cos(6 + 66^\circ) \\ \text{Cosine is even so } \cos(-A) &\equiv \cos A \\ \Rightarrow \cos(6 - 42^\circ) &= \cos(6 + 66^\circ) \\ \text{SOLVING THE EQUATION IN PERIODIC FORM} \\ \begin{cases} 6 - 42^\circ = 6 + 66^\circ \pm 360^\circ \\ 0 - 42^\circ = -6 - 66^\circ \pm 360^\circ \end{cases}, \quad \text{4 solns...} \end{aligned}$$

THE FIRST EQUATION YIELDS NOTHING, BUT THE SECOND EQUATION CAN PRODUCE 2 VALID SOLUTIONS

$$\begin{aligned} \Rightarrow 2\theta &= -24^\circ \pm 360^\circ \\ \Rightarrow \theta &= -12^\circ \pm 180^\circ \\ \Rightarrow \arctan \left[\frac{\cos 66^\circ - \sin 18^\circ}{\cos 48^\circ + \sin 66^\circ} \right] &= -12^\circ, -108^\circ, 12^\circ, 186^\circ \quad (\text{cancel } 180^\circ) \\ \uparrow \quad \text{ONLY ONE IN RANGE} \\ -10 < \theta < 90^\circ \end{aligned}$$

Question 119 (*****)

$$f(x) = \arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right), \quad x \in \mathbb{R}.$$

Show clearly that ...

a) ... $f'(x) = 0$.

b) ... $\arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right) \equiv k\pi$, stating the value of the constant k .

$$\boxed{k = \frac{1}{2}}$$

(a) Let $f(x) = \arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right)$

$$f(x) = \arctan(3x) + \arcsin[(3x)^{-1}]$$

$$f'(x) = \frac{3}{1+(3x)^2} + \frac{1}{\sqrt{1-(3x)^2}} \times \left(\frac{1}{2}\right)(3x)(9x^2+1)^{-\frac{3}{2}}$$

$$f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\sqrt{1-9x^2}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$$

$$f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\sqrt{9x^2+1}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$$

$$f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\sqrt{9x^2+1}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$$

$$f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\sqrt{9x^2+1}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$$

$$f'(x) = \frac{3}{1+9x^2} - 3\sqrt{9x^2+1} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$$

$$f'(x) = \frac{3}{1+9x^2} - \frac{3}{9x^2+1}$$

$$f'(x) = 0$$

(b) $f(x) = \text{constant}$
 $f(0) = \arctan(0) + \arcsin(1) = \frac{\pi}{2}$
 $\therefore \arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right) = \frac{\pi}{2}$

Question 120 (*****)

It is given that

$$\cot x - 2 \cot 2x \equiv \tan x.$$

- a) Prove the validity of the above trigonometric identity.
 b) Hence, or otherwise, show that

$$\sum_{r=1}^{10} \frac{1}{2^{r-1}} \tan\left(\frac{x}{2^r}\right) = \frac{1}{512} \cot\left(\frac{x}{1024}\right) - 2 \cot x.$$

, proof

a) STARTING WITH THE LEFT HAND SIDE

$$\begin{aligned} L.H.S &= \cot x - 2 \cot 2x = \cot x - \frac{2}{\tan 2x} = \cot x - \frac{2}{1 - \tan^2 x} \\ &= \cot x - \frac{2(1 - \tan^2 x)}{2 \tan x} = \frac{\cot x \tan x - (1 - \tan^2 x)}{\tan x} \\ &= \frac{1 - 1 + \tan^2 x}{\tan x} = \frac{\tan^2 x}{\tan x} = R.H.S. \quad \checkmark \end{aligned}$$

b) NOW USING THE INDUCTION BY RECURSION

$\begin{aligned} \cot x - 2 \cot 2x &= \tan x \\ \cot \frac{x}{2} - 2 \cot x &= \tan \frac{x}{2} \\ \cot \frac{x}{4} - 2 \cot \frac{x}{2} &= \tan \frac{x}{4} \\ \cot \frac{x}{8} - 2 \cot \frac{x}{4} &= \tan \frac{x}{8} \\ \vdots & \vdots \\ \cot \frac{x}{2^n} - 2 \cot \frac{x}{2^{n-1}} &= \tan \frac{x}{2^n} \end{aligned}$	$\begin{array}{c} \cancel{\cot \frac{x}{2} - 2 \cot x = \tan \frac{x}} \\ \cancel{\cot \frac{x}{4} - 2 \cot \frac{x}{2} = \tan \frac{x}{4}} \\ \cancel{\cot \frac{x}{8} - 2 \cot \frac{x}{4} = \tan \frac{x}{8}} \\ \vdots \\ \cancel{\cot \frac{x}{2^n} - 2 \cot \frac{x}{2^{n-1}} = \tan \frac{x}{2^n}} \end{array}$
ADDING THE SET $\frac{1}{2^1} \cot \frac{x}{2^1} - 2 \cot x = \sum_{n=1}^{\infty} \left[\frac{1}{2^n} \cot \left(\frac{x}{2^n} \right) \right]$	

HENCE WE OBTAIN IF $N=10$

$$\begin{aligned} \sum_{n=1}^{10} \left[\frac{1}{2^n} \cot \left(\frac{x}{2^n} \right) \right] &= \frac{1}{2^1} \cot \left(\frac{x}{2^1} \right) - 2 \cot x \\ &= \frac{1}{512} \cot \left(\frac{x}{1024} \right) - 2 \cot x \quad \text{As required} \end{aligned}$$

Question 121 (***)**

Given the trigonometric equation

$$\frac{\sin(x-\alpha)}{\cos(x-\alpha) - 2 \tan \alpha \sin(x-\alpha)} = \tan \alpha,$$

show clearly that

$$\tan x = 2 \tan \alpha.$$

proof

$$\begin{aligned}
 & \frac{\sin(x-\alpha)}{\cos(x-\alpha) - 2 \tan \alpha \sin(x-\alpha)} = \tan \alpha \\
 \Rightarrow & \sin(x-\alpha) = \tan \alpha \cos(x-\alpha) - 2 \tan^2 \alpha \sin(x-\alpha) \\
 \text{DIVIDE THROUGH BY } & \cos(x-\alpha) \\
 \Rightarrow & \tan(x-\alpha) = \tan \alpha - 2 \tan^2 \alpha \tan(x-\alpha) \\
 \Rightarrow & \tan(x-\alpha) + 2 \tan^2 \alpha \tan(x-\alpha) = \tan \alpha \\
 \Rightarrow & \tan(x-\alpha)[1 + 2 \tan^2 \alpha] = \tan \alpha \\
 \Rightarrow & \frac{\tan x - \tan \alpha}{1 + 2 \tan^2 \alpha} [1 + 2 \tan^2 \alpha] = \tan \alpha \\
 \Rightarrow & (\tan x - \tan \alpha)(1 + 2 \tan^2 \alpha) = \tan \alpha [1 + 2 \tan^2 \alpha] \\
 \Rightarrow & \tan x + 2 \tan^2 \alpha \tan x - \tan x - 2 \tan^3 \alpha = \tan \alpha + \tan^3 \alpha \\
 \Rightarrow & \tan^2 \alpha \tan x - 2 \tan^3 \alpha + \tan x - 2 \tan \alpha = 0 \\
 \Rightarrow & \tan x (\tan^2 \alpha - 2 \tan^2 \alpha) + (\tan x - 2 \tan \alpha) = 0 \\
 \Rightarrow & (\tan x - 2 \tan \alpha)(\tan^2 \alpha + 1) = 0 \\
 \Rightarrow & \tan x - 2 \tan \alpha = 0 \\
 \Rightarrow & \tan x = 2 \tan \alpha \\
 \text{As required.}
 \end{aligned}$$

Question 122 (***)**

Solve the trigonometric equation

$$\sin x \cos x + \frac{1}{2} = \cos^2\left(\frac{x}{2} - \frac{\pi}{6}\right), \quad 0 \leq x < \pi,$$

giving the answers in terms of π .

, $x = \frac{\pi}{6}, \frac{5\pi}{18}, \frac{17\pi}{18}$

$$\begin{aligned}
 \sin x \cos x + \frac{1}{2} &= \cos^2\left(\frac{x}{2} - \frac{\pi}{6}\right) \\
 \Rightarrow 2\sin x \cos x + 1 &= 2\cos^2\left(\frac{x}{2} - \frac{\pi}{6}\right) \\
 \Rightarrow \sin 2x + 1 &= 2\cos^2\left(\frac{x}{2} - \frac{\pi}{6}\right) \\
 \Rightarrow \sin 2x &= 2\cos^2\left(\frac{x}{2} - \frac{\pi}{6}\right) - 1 \\
 \Rightarrow \sin 2x &= \cos\left[2\left(\frac{x}{2} - \frac{\pi}{6}\right)\right] \\
 \Rightarrow \sin 2x &= \cos\left(x - \frac{\pi}{3}\right) \\
 \Rightarrow \sin 2x &= \sin\left[\frac{\pi}{2} - \left(x - \frac{\pi}{3}\right)\right] \\
 \Rightarrow \sin 2x &= \sin\left[\frac{5\pi}{6} - x\right]
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow 2x &= \left(\frac{5\pi}{6} - x\right) + 2k\pi, \quad k \in \mathbb{Z} \\
 \Rightarrow 2x &= \frac{5\pi}{6} - x + 2k\pi \\
 \Rightarrow 3x &= \frac{5\pi}{6} + 2k\pi \\
 \Rightarrow x &= \frac{5\pi}{18} + \frac{2k\pi}{3} \\
 \Rightarrow x &= \frac{5\pi}{18} + 2m\pi \\
 \Rightarrow x &= \frac{5\pi}{18} \\
 \Rightarrow x &= \frac{5\pi}{18}, \frac{17\pi}{18}
 \end{aligned}$$

Question 123 (***)**

Prove the validity of the following trigonometric identities.

a) $\cos^4 \theta + \sin^4 \theta \equiv \frac{1}{4}(3 + 4\cos 4\theta)$.

b) $32\sin^2 x \cos^4 x \equiv 2 + \cos 2x - 2\cos 4x - \cos 6x$.

proof

$$\begin{aligned} LHS &= \frac{1}{4}(3 + 2\cos 4\theta) = \frac{1}{4}(3 + 2\cos^2 2\theta - 1) \\ &= \frac{1}{4}(2 + 2\cos^2 2\theta) = \frac{1}{2}(1 + \cos^2 2\theta) \\ &= \frac{1}{2}(1 + (2\cos^2 \theta - 1)^2) = \frac{1}{2}[1 + 4\cos^2 \theta - 4\cos^2 \theta + 1] \\ &= \frac{1}{2}[4\cos^2 \theta - 4\cos^2 \theta + 2] = 2\cos^4 \theta - 2\cos^2 \theta + 1 \\ &= \cos^4 \theta + (\cos^2 \theta - 2\cos^2 \theta + 1) = \cos^4 \theta + (1 - \cos^2 \theta)^2 \\ &= \cos^4 \theta + (\sin^2 \theta)^2 = \cos^4 \theta + \sin^4 \theta = RHS \end{aligned}$$

$$\begin{aligned} RHS &= \cos^4 \theta + \sin^4 \theta = (\cos^2 \theta + 2\sin^2 \theta \cos^2 \theta + \sin^2 \theta)^2 - 2\sin^2 \theta \cos^2 \theta = 1^2 - \frac{1}{2}(2\sin \theta \cos \theta)^2 \\ &= (1 - \frac{1}{2}\sin^2 2\theta)^2 - 2\sin^2 \theta \cos^2 \theta = 1 - \frac{1}{2}(2\sin \theta \cos \theta)^2 \\ &= 1 - \frac{1}{2}(\frac{1}{2} - \frac{1}{2}\cos 4\theta) \\ &= 1 - \frac{1}{4} + \frac{1}{4}\cos 4\theta \\ &= \frac{3}{4} + \frac{1}{4}\cos 4\theta \\ &= \frac{1}{4}(3 + \cos 4\theta) = RHS \end{aligned}$$

$$\begin{aligned} LHS &= 32\sin^2 x \cos^4 x = 8(4\sin^2 x \cos^2 x) \cos^2 x = 8(2\sin x \cos x)^2 \cos^2 x \\ &= 8(\sin 2x)^2 \cos^2 x = 8\sin^2 2x \cos^2 x \\ &\quad \boxed{\begin{array}{l} \cos 2A = 2\cos^2 A - 1 \\ 1 + \cos 2A = 2\cos^2 A \\ \cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A \end{array}} \quad \boxed{\begin{array}{l} \cos 2A = 1 - 2\sin^2 A \\ 2\sin^2 A = 1 - \cos 2A \\ \sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A \end{array}} \\ &= 8 \times (\frac{1}{2} - \frac{1}{2}\cos 4x)(\frac{1}{2} + \frac{1}{2}\cos 2x) \\ &= 2(1 - \cos 4x)(1 + \cos 2x) \\ &= 2 + 2\cos 2x - 2\cos 4x - \boxed{2\cos 2x \cos 4x} \\ &\quad \boxed{\begin{array}{l} \cos(A+B) = \cos A \cos B - \sin A \sin B \\ \cos(A-B) = \cos A \cos B + \sin A \sin B \end{array}} \\ &\quad \boxed{\begin{array}{l} \cos 6x = \cos 4x \cos 2x - \sin 4x \sin 2x \\ \cos 4x = \cos 2x \cos 2x + \sin 2x \sin 2x \end{array}} \\ &\quad \boxed{\begin{array}{l} \cos 6x + \cos 2x = 2\cos 4x \cos 2x \\ = 2 + 2\cos 2x - 2\cos 4x - \cos 6x \end{array}} \\ &= 2 + 2\cos 2x - 2\cos 4x - \cos 6x \\ &= RHS \\ &\quad \boxed{\text{As Required}} \end{aligned}$$

Question 124 (***)**

Show clearly that

$$4 \arccot 2 + \arctan\left(\frac{24}{7}\right) = \pi.$$

, proof

Let $\theta = \arccot 2$

$$\cot \theta = 2 \Rightarrow \tan \theta = \frac{1}{2}$$

$$\sin \theta = \frac{1}{\sqrt{5}}, \cos \theta = \frac{2}{\sqrt{5}}$$

Let $\phi = \arctan \frac{24}{7}$

$$\tan \phi = \frac{24}{7} \Rightarrow \sin \phi = \frac{24}{25}, \cos \phi = \frac{7}{25}$$

Now $\theta + \phi = \arccot 2 + \arctan \frac{24}{7}$

Let $\psi = \theta + \phi$

$$\cot \psi = \cot(\theta + \phi) = \frac{\cot \theta \cot \phi - 1}{\cot \theta + \cot \phi} = \frac{2 \cdot \frac{1}{2} - 1}{2 + \frac{7}{2}} = -\frac{1}{5}$$

$$\tan \psi = \frac{1}{\cot \psi} = 5$$

$$\sin \psi = \frac{5}{\sqrt{26}}, \cos \psi = -\frac{1}{\sqrt{26}}$$

$$\psi = \dots -\pi, \pi, \dots$$

$$3\pi < \theta + \phi < 4\pi \text{ (from } 0 < \theta < \pi, 0 < \phi < \pi)$$

$$\therefore \psi = \pi$$

$$\therefore 4 \arccot 2 + \arctan \frac{24}{7} = \pi$$

ALTERNATIVE BY COMPLEX NUMBERS

$$4 \arccot 2 + \arctan \frac{24}{7} = 4 \arctan \frac{1}{2} + \arctan \frac{24}{7}$$

$$\text{Consider } (2+i)^4(7+24i) - (4+i)^2(7+24i) = (3+i)(5+i)(7+24i)$$

$$= (7+24i)(5+i)(7+24i) = (-7+24i)(5+i)(7+24i)$$

$$= -19 - 16i + 35i - 576 = -625$$

Thus $\arg[(2+i)^4(7+24i)] = \arg(-625)$

$$\arg(2+i)^4 + \arg(7+24i) = \pi$$

$$\arg(2+i) + \arg(7+24i) = \pi$$

$$4\arg(2+i) + \arctan \frac{24}{7} = \pi$$

$$4\arccot 2 + \arctan \frac{24}{7} = \pi$$

Question 125 (***)**

It is given that if $x \neq y$, $x \neq 0$, $y \neq 0$,

$$\tan(x+y) = 2\tan(x-y).$$

Show clearly that

$$\frac{\sin 2x}{\sin 2y} = 3.$$

, proof

$$\begin{aligned}
 & \tan(2xy) = 2\tan(x-y) \\
 \Rightarrow & \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{2(\tan x - \tan y)}{1 + \tan x \tan y} \\
 \Rightarrow & (\tan x + \tan y)(1 + \tan x \tan y) = 2(\tan x - \tan y)(1 - \tan x \tan y) \\
 \Rightarrow & \tan^2 x + \tan x \tan y + \tan y + \tan^2 y = 2\tan x - 2\tan^2 x \tan y + 2\tan^2 y \tan x \\
 \Rightarrow & 0 = \tan x - 3\tan y - 3\tan^2 x \tan y + \tan^2 y \tan x \\
 \Rightarrow & \tan x - 3\tan y - 3\tan^2 x \tan y + \tan^2 y \tan x = 0 \\
 \Rightarrow & \tan x(1 + \tan^2 y) - 3\tan y(1 + \tan^2 x) = 0 \\
 \Rightarrow & \tan x \sec^2 y - 3\tan y \sec^2 x = 0 \\
 \Rightarrow & \frac{\tan x \sec^2 y}{\tan y \sec^2 x} - 3 = 0 \\
 \Rightarrow & \frac{\tan x}{\tan y} \times \frac{\sec^2 y}{\sec^2 x} = 3 \\
 \Rightarrow & \frac{\sin x \cos y}{\sin y \cos x} \times \frac{\cos^2 y}{\cos^2 x} = 3 \\
 \Rightarrow & \frac{\sin x \cos y}{\cos x \sin y} \times \frac{\cos^2 y}{\cos^2 x} = 3 \\
 \Rightarrow & \frac{\sin x \cos y}{\sin y \cos x} = 3
 \end{aligned}$$

$\Rightarrow \frac{2\sin x \cos y}{2\sin y \cos x} = 3$
 $\Rightarrow \frac{\sin 2x}{\sin 2y} = 3$
 ↪ QED

Question 126 (***)**

A triangle ABC is such so that $\angle BAC = \frac{1}{6}\pi$ and $|BC| = 1$.

Show that the maximum value of the area of the triangle ABC is

$$\frac{1}{4}(2 + \sqrt{3}).$$

, proof

STARTING WITH A DIAGRAM

LOOKING AT $\triangle BDC$

- $h = |x|\sin\theta \Rightarrow h = x\sin\theta \Rightarrow \frac{h}{x} = \tan\frac{\theta}{2}$
- $y = |x|\cos\theta \Rightarrow y = x\cos\theta \Rightarrow \frac{y}{x} = \frac{1}{\tan\theta} \Rightarrow \theta = \frac{\pi}{2} - \tan^{-1}\frac{y}{x} \Rightarrow \theta = \frac{\pi}{2} - \tan^{-1}\frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

LOOKING AT $\triangle ABD$

- $b = \sqrt{x^2 + h^2} \Rightarrow b = \sqrt{x^2 + x^2\sin^2\theta} \Rightarrow b = x\sqrt{1 + \sin^2\theta} \Rightarrow b = x\sqrt{1 + \frac{1}{4}\sin^2\theta} \Rightarrow b = x\sqrt{\frac{5}{4}\sin^2\theta} \Rightarrow b = \frac{x\sqrt{5}}{2}\sin\theta$

HENCE THE AREA OF THE TRIANGLE $\triangle ABC$

$$\begin{aligned} \rightarrow \text{AREA} &= \frac{1}{2}|AC||BD| \\ \rightarrow \text{AREA} &= \frac{1}{2}(x^2)y \\ \rightarrow \text{AREA} &= \frac{1}{2}(x^2\sin\theta + x^2\cos\theta)\sin\theta \\ \rightarrow \text{AREA} &= \left(\frac{x^2}{2}\sin^2\theta + \frac{1}{2}x^2\cos^2\theta\right)\sin\theta \\ \rightarrow \text{AREA} &= \left(x^2\sin^2\theta + \frac{1}{2}x^2\cos^2\theta\right)\sin\theta \\ \rightarrow \text{AREA} &= \sin\left(\theta + \frac{\pi}{6}\right)\sin\theta \end{aligned}$$

BY DIFFERENTIATION

$$\begin{aligned} f(\theta) &= \sin\theta \sin\left(\theta + \frac{\pi}{6}\right) \\ f'(\theta) &= \cos\theta \sin\left(\theta + \frac{\pi}{6}\right) + \sin\theta \cos\left(\theta + \frac{\pi}{6}\right) \\ f'(\theta) &= \sin\left(\theta + \left(\theta + \frac{\pi}{6}\right)\right) \\ f'(\theta) &= \sin(2\theta + \frac{\pi}{6}) \end{aligned}$$

SELLING FOR ZERO, ONLY LOOKING AS $\theta > 0$

$$\begin{aligned} 2\theta + \frac{\pi}{6} &= 0, \pi, 2\pi, \dots \\ 2\theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \dots \\ \theta &= \frac{\pi}{12}, \frac{5\pi}{12}, \dots \end{aligned}$$

HENCE AREA_{\max}

$$\begin{aligned} &= \sin\frac{\pi}{12} \sin\left(\frac{\pi}{12} + \frac{\pi}{6}\right) \\ &= \sin\frac{\pi}{12} \sin\frac{5\pi}{12} \quad \boxed{\sin\frac{\pi}{12} \equiv \sin\frac{5\pi}{12}} \\ &= \sin^2\frac{\pi}{12} \\ &= \frac{1}{2} - \frac{1}{2}\cos\left(2\cdot\frac{\pi}{12}\right) \quad \boxed{\cos 2\theta = 1 - 2\sin^2\theta} \\ &= \frac{1}{2} - \frac{1}{2}\cos\frac{\pi}{6} \quad \boxed{\sin^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta} \\ &= \frac{1}{2} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) \\ &= \frac{1}{2} + \frac{\sqrt{3}}{4} \\ &= \frac{1}{4}(2 + \sqrt{3}) \end{aligned}$$

ALTERNATIVE WITHOUT DIFFERENTIATION

$$\begin{aligned} \text{AREA} &= \sin\theta \sin\left(\theta + \frac{\pi}{6}\right) \\ (\text{LOOKING AT THE COMBINING TRIG IDENTITIES FOR } \sin(A+B)) \\ \cos[\theta + (\theta + \frac{\pi}{6})] &= \cos\theta\cos(\theta + \frac{\pi}{6}) - \sin\theta\sin(\theta + \frac{\pi}{6}) \\ \cos[\theta - (\theta + \frac{\pi}{6})] &= \cos\theta\cos(\theta + \frac{\pi}{6}) + \sin\theta\sin(\theta + \frac{\pi}{6}) \end{aligned}$$

DIVIDING BY 2

$$\begin{aligned} \rightarrow \cos(-\frac{\pi}{6}) - \cos(2\theta + \frac{\pi}{6}) &\equiv 2\sin\theta\sin(\theta + \frac{\pi}{6}) \\ \rightarrow \frac{1}{2}\cos(\frac{\pi}{6}) - \frac{1}{2}\cos(2\theta + \frac{\pi}{6}) &\equiv 2\sin\theta\sin(\theta + \frac{\pi}{6}) \\ \rightarrow \sin\theta\sin(\theta + \frac{\pi}{6}) &= \frac{1}{2}(\frac{\sqrt{3}}{2}) - \frac{1}{2}\cos(2\theta + \frac{\pi}{6}) \\ \rightarrow \text{AREA} &= \frac{\sqrt{3}}{4} - \frac{1}{2}\cos(2\theta + \frac{\pi}{6}) \\ \rightarrow \text{AREA}_{\max} &= \frac{\sqrt{3}}{4} - \frac{1}{2}(-1) \quad \{ \rightarrow -1 < \cos(2\theta + \frac{\pi}{6}) < 1 \} \\ \rightarrow \text{AREA}_{\max} &= \frac{\sqrt{3}}{4} + \frac{1}{2} \\ \rightarrow \text{AREA}_{\max} &= \frac{1}{4}(2 + \sqrt{3}) \end{aligned}$$

Question 127 (***)**

It is given that the angles A , B and C are the three angles of a triangle ABC with $B \neq 90^\circ$.

Given further that

$$\sin A - \sin(B-C) = \frac{\cos(B-C)}{\tan B},$$

show that the triangle ABC is right angled.

, proof

$$\begin{aligned}
 & \boxed{\sin A - \sin(B-C) = \frac{\cos(B-C)}{\tan B}} \quad \boxed{A+B+C=180^\circ} \\
 \Rightarrow & \tan B [\sin A - \sin(B-C)] = \cos(B-C) \\
 & \uparrow \\
 & \boxed{\sin P - \sin Q \equiv 2\cos \frac{P+Q}{2} \sin \frac{P-Q}{2}} \\
 \Rightarrow & \tan B \left[2\cos \frac{A+B-C}{2} \sin \frac{A-B+C}{2} \right] = \cos(B-C) \\
 \Rightarrow & \tan B \left[2\cos \frac{A+B+C-2C}{2} \sin \frac{A+B+C-2B}{2} \right] = \cos(B-C) \\
 \Rightarrow & 2\tan B \left[\cos \frac{180-2C}{2} \sin \frac{180-2B}{2} \right] = \cos(B-C) \\
 \Rightarrow & 2\tan B \cos(90-C) \sin(90-B) = \cos(B-C) \\
 \Rightarrow & 2 \frac{\sin B}{\cos B} \times \sin C \times \cos C = \cos(B-C) \\
 & \text{cos}(90-\theta) \equiv \sin \theta \\
 & \text{sin}(90-\theta) \equiv \cos \theta \\
 \Rightarrow & 2\sin B \sin C = \cos(B-C) \\
 \Rightarrow & 2\sin B \sin C = \cos B \cos C - \sin B \sin C \\
 \Rightarrow & 0 = \cos B \cos C - \sin B \sin C \\
 \Rightarrow & \cos(B+C) = 0 \\
 \Rightarrow & B+C = 90^\circ \\
 \therefore & A = 90^\circ
 \end{aligned}$$

THE TRIANGLE IS RIGHT ANGLED AT A

Question 128 (***)**

Solve the following trigonometric equation

$$\arctan\left[x \cos\left(2 \arcsin \frac{1}{x}\right)\right] = \frac{1}{4}\pi.$$

$$\boxed{\quad}, \quad x = -1, \quad x = 2$$

arctan\left[x \cos\left(2 \arcsin \frac{1}{x}\right)\right] = \frac{1}{4}\pi

• TAKING ARCTAN ON BOTH SIDES OF THE EQUATION

$$\begin{aligned} &\Rightarrow \arctan\left(2 \arcsin \frac{1}{x}\right) = \frac{1}{4}\pi \\ &\Rightarrow \cos\left(2 \arcsin \frac{1}{x}\right) = \frac{1}{2} \\ &\Rightarrow 2 \arcsin \frac{1}{x} = \pm \arccos \frac{1}{2} \pm 2n\pi \quad n=0,1,2,\dots \\ &\Rightarrow 2 \arcsin \frac{1}{x} = \pm \left(\frac{\pi}{3} - \arccos \frac{1}{2}\right) \pm 2n\pi \\ &\qquad\qquad\qquad \text{arccos } + \arccos \text{ in } \frac{\pi}{3} \\ &\Rightarrow 2 \arcsin \frac{1}{x} = \begin{cases} \frac{\pi}{6} - \arccos \frac{1}{2} \pm 2n\pi \\ -\frac{\pi}{6} + \arccos \frac{1}{2} \pm 2n\pi \end{cases} \end{aligned}$$

• DEAL WITH EACH POSSIBILITY SEPARATELY

$$\begin{aligned} &\left\{ \begin{array}{l} 2 \arcsin \frac{1}{x} = \frac{\pi}{6} \pm 2n\pi \\ \arcsin \frac{1}{x} = \frac{\pi}{12} \pm n\pi \end{array} \right. \\ &\rightarrow \left\{ \begin{array}{l} \arcsin \frac{1}{x} = \frac{\pi}{12} \pm \frac{3}{2}\pi \\ \arcsin \frac{1}{x} = -\frac{11}{12}\pi \pm 2n\pi \end{array} \right. \end{aligned}$$

• BUT THE ARCSINE FUNCTION IS BOUNDED, i.e. $-\frac{\pi}{2} \leq \arcsin A \leq \frac{\pi}{2}$.
HENCE WE OBTAIN

$\bullet \arcsin \frac{1}{x} = \frac{\pi}{12}$ $\frac{1}{x} = \sin \frac{\pi}{12}$ $\frac{1}{x} = \frac{1}{2}$ $x = 2$	$\bullet \arcsin \frac{1}{x} = -\frac{11}{12}\pi$ $\frac{1}{x} = \sin(-\frac{11}{12}\pi)$ $\frac{1}{x} = -1$ $x = -1$
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Question 129 (***)**

Sketch the graph of

$$y = (\arcsin x)^2 \arccos x, -1 \leq x \leq 1$$

, graph

THE GRAPH CLEARLY EXISTS FOR $-1 \leq x \leq 1$

$$y|_{x=0} = (\arcsin 0)^2 \arccos 0 = \left(\frac{\pi}{2}\right)^2 \times 0 = 0$$

$$y|_{x=-1} = [\arcsin(-1)]^2 \arccos(-1) = \left(-\frac{\pi}{2}\right)^2 \pi = \frac{\pi^3}{4}$$

NEXT LOOK FOR STATIONARY POINTS — REMEMBER y IS SMOOTH

$$y = (\arcsin x)^2 \arccos x$$

$$y = (\arcsin x)^2 [\pi - \arccos x]$$

$$y = \pi(\arcsin x)^2 - (\arcsin x)^3$$

$$\frac{dy}{dx} = \pi(\arcsin x) \times \frac{1}{\sqrt{1-x^2}} - 3(\arcsin x)^2 \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{\arcsin x}{\sqrt{1-x^2}} [\pi - 3\arcsin x]$$

SEARCHING FOR ZEROES WE OBTAIN

$\arcsin x = 0$ $x = 0$	$\pi - \arccos x = 0$ $\arccos x = \frac{\pi}{2}$ $x = \cos \frac{\pi}{2} = 0$
----------------------------	--

$$y|_0 = (\arcsin 0)^2 \arccos(0) = 0$$

$$y|_0 = 0$$

COLLECTING SOME OF THESE POINTS

x	y	Description
-1	$-\frac{\pi^3}{4}$	AESTHETIC MAXIMUM (END POINT)
0	0	LOCAL STATIONARY MINIMUM
$\frac{\pi}{2}$	$\frac{\pi^3}{4}$	LOCAL STATIONARY MAXIMUM
1	0	AESTHETIC MINIMUM (END POINT)

FINISHING THE CURVE CAN BE SKETCHED

Question 130 (***)**

Solve the following trigonometric equation

$$\sin[\arccot(x+1)] = \cos(\arctan x).$$

You may assume that $y = \arccot x$ is the inverse function for $y = \cot x$, $0 \leq x \leq \pi$.

$$\boxed{\quad}, \quad x = -\frac{1}{2}$$

SIN(COT(X+1)) = COS(ARCTAN(X))

• USING THE IDENTITY $\cos A \equiv \sin(\frac{\pi}{2} - A)$
 $\Rightarrow \sin[\arccot(x+1)] = \sin[\frac{\pi}{2} - \arctan x]$

• NOW THERE ARE TWO POSSIBILITIES

$\Rightarrow \arccot(x+1) = \frac{\pi}{2} - \arctan x$ $\Rightarrow \arctan(x+1) + \arctan x = \frac{\pi}{2}$	$\frac{\pi}{2} \rightarrow \arccot(x+1) = \pi - (\frac{\pi}{2} - \arctan x)$ $\Rightarrow \arccot(x+1) = \frac{\pi}{2} + \arctan x$ $\Rightarrow \arctan(x+1) - \arctan x = \frac{\pi}{2}$
--	--

• NOW USING THE IDENTITY $\arccot A \equiv \arctan(\frac{1}{A})$

$\Rightarrow \arctan(\frac{1}{x+1}) + \arctan x = \frac{\pi}{2}$	$\Rightarrow \arctan(\frac{1}{x+1}) - \arctan x = \frac{\pi}{2}$
--	--

• TAKING TANGENTS ON BOTH SIDES IN EACH OF THE TWO EQUATIONS

$\Rightarrow \tan[\arctan(\frac{1}{x+1}) + \arctan x] = \tan \frac{\pi}{2}$ $\Rightarrow \frac{\frac{1}{x+1} + x}{1 - \frac{1}{x+1} \cdot x} = \infty$ $\Rightarrow \frac{1 + x(x+1)}{2x+1 - x^2} = \infty$ $\Rightarrow 1 + x^2 + 2x = \infty$ $\Rightarrow x^2 + 2x + 1 = \infty$ $\text{WHERE } x = \pm \infty$	$\Rightarrow \tan[\arctan(\frac{1}{x+1}) - \arctan x] = \tan \frac{\pi}{2}$ $\Rightarrow \frac{\frac{1}{x+1} - x}{1 + \frac{1}{x+1} \cdot x} = \infty$ $\Rightarrow \frac{1 - x(x+1)}{2x+1 + x^2} = \infty$ $\Rightarrow \frac{1 - x^2 - x}{2x+1} = \infty$ $\Rightarrow 2x+1 = 0$ $\Rightarrow x = -\frac{1}{2}$
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Question 131 (***)**

The function f is defined in the largest possible real domain, contained in the interval $(-2\pi, 2\pi)$, and its equation is

$$f(x) \equiv \ln \left[\tan \left(\frac{1}{8}\pi - \frac{1}{2}x \right) \right].$$

a) Find the domain of f .

b) Show that $f'(x) \equiv \frac{k}{\sqrt{1-\sin 2x}}$, for some constant k .

$$\boxed{\quad}, \boxed{(-2\pi, -\frac{7}{4}\pi) \cup (-\frac{3}{4}\pi, \frac{1}{4}\pi) \cup (\frac{5}{4}\pi, 2\pi)}$$

a) FOR THE FUNCTION TO BE DEFINED, THE ARGUMENT OF THE LOGARITHM
MUST BE NON NEGATIVE – LOOK AT THE GRAPH OR TUTORIAL

$$\tan x > 0 \Rightarrow 0 < x < \frac{\pi}{2} \quad -\pi < x < -\frac{\pi}{2}$$

$$\pi < x < \frac{3\pi}{2} \quad -2\pi < x < -\frac{3\pi}{2}$$

USING TRANSFORMATIONS

$$x \mapsto \frac{1}{8}\pi - \frac{1}{2}x \quad x \mapsto -x$$

$$-\pi < x < -\frac{7}{4}\pi \quad -\frac{7}{4}\pi < x < -3\pi - \frac{\pi}{2}$$

$$-\pi - \frac{\pi}{2} < x < -\frac{5}{4}\pi \quad -\pi - \frac{3\pi}{2} < x < -\pi - \frac{\pi}{2}$$

$$0 - \frac{\pi}{2} < x < -\frac{3}{4}\pi \quad 0 - \pi < x < -\pi - \frac{\pi}{2}$$

$$\pi - \frac{\pi}{2} < x < \frac{3}{4}\pi \quad \pi - \pi - \frac{\pi}{2} < x < \pi - \frac{\pi}{2}$$

$$\tan(\frac{1}{8}\pi - \frac{1}{2}x) \quad \tan(\frac{1}{8}\pi - \frac{1}{2}x) \quad \tan(\frac{1}{8}\pi - \frac{1}{2}x)$$

SIMPLIFYING THESE RANGES

∴ DOMAIN is $\boxed{(-2\pi, -\frac{7}{4}\pi) \cup (-\frac{5}{4}\pi, -\frac{3}{4}\pi) \cup (\frac{1}{4}\pi, \frac{3}{4}\pi)}$

b) CAREFULLY DO THE DIFFERENTIATION

$$\begin{aligned} \frac{d}{dx} \left[\ln \left(\tan \left(\frac{1}{8}\pi - \frac{1}{2}x \right) \right) \right] &= \frac{1}{\tan \left(\frac{1}{8}\pi - \frac{1}{2}x \right)} \times \tan^2 \left(\frac{1}{8}\pi - \frac{1}{2}x \right) \\ &= \frac{\cos \left(\frac{1}{8}\pi - \frac{1}{2}x \right)}{\sin \left(\frac{1}{8}\pi - \frac{1}{2}x \right)} \times \frac{1}{\cos^2 \left(\frac{1}{8}\pi - \frac{1}{2}x \right)} \end{aligned}$$

$$= \frac{1}{\sin \left(\frac{1}{8}\pi - \frac{1}{2}x \right) \cos \left(\frac{1}{8}\pi - \frac{1}{2}x \right)} = \frac{2}{2\sin \left(\frac{1}{8}\pi - \frac{1}{2}x \right) \cos \left(\frac{1}{8}\pi - \frac{1}{2}x \right)}$$

USING $\sin 2A = 2\sin A \cos A$

$$= \frac{2}{\sin \left(\frac{1}{4}\pi - x \right)} = \frac{2}{\sqrt{2} \sin \left(\frac{1}{4}\pi - x \right)}$$

DOING $\sin^2 A + \cos^2 A = 1$

$$= \frac{2}{\sqrt{1 - \cos^2 \left(\frac{1}{4}\pi - x \right)}} = \frac{2}{\sqrt{2} \sqrt{1 - \cos \left(\frac{1}{4}\pi - x \right)}}$$

But $\cos \left(\frac{1}{4}\pi - A \right) = \sin A$

$$= \frac{2\sqrt{2}}{\sqrt{1 - \sin^2 x}} = \frac{2\sqrt{2}}{\sqrt{1 - \sin^2 x}} \quad \text{AS REQUIRED}$$

Question 132 (***)**

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ , and is commonly known as the golden section or golden number.

It can be shown, and you **may assume in this question**, that $\cos\left(\frac{1}{5}\pi\right) = \frac{1}{2}\phi$.

Use trigonometric identities to show that

$$\tan\left(\frac{1}{5}\pi\right)\tan\left(\frac{2}{5}\pi\right)\tan\left(\frac{3}{5}\pi\right)\tan\left(\frac{4}{5}\pi\right) = 5.$$

You may not use complex numbers in this question.

[10], proof

PROCEED AS FOLLOWS

$\tan(\frac{1}{5}\pi)\tan(\frac{2}{5}\pi)\tan(\frac{3}{5}\pi)\tan(\frac{4}{5}\pi)$
 $= [\tan(\frac{\pi}{5})][\tan(\frac{2\pi}{5})][\tan(\frac{3\pi}{5})][-\tan(\frac{4\pi}{5})]$
 $= \tan^2(\frac{\pi}{5})\tan(\frac{2\pi}{5})$
Switch TO SINES AND COSINES
 $= \frac{\sin^2(\frac{\pi}{5})}{\cos^2(\frac{\pi}{5})} \times \frac{\sin(2\pi/5)}{\cos(2\pi/5)}$
 \dots
USE THE DOUBLE ANGLE FORMULA SINCE $\theta = 2\alpha/2$
 $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
 $\sin(2\pi/5) = 2\sin(\pi/5)\cos(\pi/5)$
 $\cos(2\pi/5) = 2\cos^2(\pi/5) - 1$
 \dots
 $= \frac{\sin^2(\frac{\pi}{5})}{\cos^2(\frac{\pi}{5})} \times \frac{[2\sin(\frac{\pi}{5})\cos(\frac{\pi}{5})]^2}{[2\cos^2(\frac{\pi}{5}) - 1]^2} = \frac{4\sin^2(\frac{\pi}{5})\cos^2(\frac{\pi}{5})}{4[\cos^2(\frac{\pi}{5}) - 1]^2}$
USE THE PROPERTY OF $\phi^2 = \phi + 1$ (CHARACTER OF $\phi^2 = \phi + 1$)
 $= \frac{(4 - 4\phi)^2}{(4\phi - 2)^2} = \frac{(2 - \phi)^2}{(\phi - 1)^2} = \frac{4(1 - \phi^2)}{\phi^2 - 2\phi + 1} = \frac{4(1 - \phi^2)}{\phi^2 - 2\phi + 1}$
Divide by $\phi^2 - 2\phi + 1$
 $= \frac{(4 + 1) - 4\phi + 9}{(4 + 1) - 2\phi + 1} = \frac{10 - 4\phi}{2 - \phi} = \frac{5(2 - \phi)}{2 - \phi} = 5$
As required

Question 133 (***)**

It is given that θ , α and β are distinct real numbers which satisfy.

$$\tan(\theta - \alpha) + \tan(\theta - \beta) = x$$

$$\cot(\theta - \alpha) + \cot(\theta - \beta) = y.$$

Find, in exact simplified form, an expression for $\tan(\alpha - \beta)$, in terms of x and y .

$$\boxed{\quad}, \quad \tan(\alpha - \beta) = \pm \frac{\sqrt{x^2 y^2 - 4xy}}{x + y}$$

$\tan(\theta - \alpha) + \tan(\theta - \beta) = x$
 $\cot(\theta - \alpha) + \cot(\theta - \beta) = y$

• START BY SCALE REARRANGING — LET $A = \tan(\theta - \alpha)$
 $B = \tan(\theta - \beta)$

$$\begin{aligned} A + B &= x \\ \frac{1}{A} + \frac{1}{B} &= y \end{aligned} \Rightarrow \begin{cases} A+B=x \\ \frac{1}{A}+\frac{1}{B}=y \end{cases} \text{ or } AB=\frac{x}{y}$$

• NEXT WE MAY PROCEED AS FOLLOWS

$$\begin{aligned} \tan(\alpha - \beta) &= \tan(\alpha - \theta + \theta - \beta) \\ &= \tan[(\theta - \beta) - (\theta - \alpha)] \\ &= \frac{\tan(\theta - \beta) - \tan(\theta - \alpha)}{1 + \tan(\theta - \beta)\tan(\theta - \alpha)} \\ &= \frac{B - A}{1 + BA} \\ &= \frac{\pm\sqrt{(B-A)^2}}{1 + AB} \\ &= \frac{\pm\sqrt{B^2 - 2AB + A^2}}{1 + AB} \\ &= \frac{\pm\sqrt{(A^2 + 2AB + B^2) - 4AB}}{1 + AB} \\ &= \frac{\pm\sqrt{(A^2 + 2AB + B^2) - 4AB}}{1 + AB} \end{aligned}$$

$$\begin{aligned} &= \frac{\pm\sqrt{(A+B)^2 - 4AB}}{1 + AB} \\ &= \frac{\pm\sqrt{x^2 - 4\left(\frac{x}{y}\right)^2}}{1 + \frac{2x}{y}} \\ &= \frac{\pm y\sqrt{x^2 - \frac{4x^2}{y^2}}}{y + x} \\ &= \frac{\pm y\sqrt{\frac{y^2x^2 - 4x^2}{y^2}}}{x + y} \\ &= \frac{\pm\sqrt{xyx^2 - 4xy^2}}{x + y} \end{aligned}$$

Question 134 (***)**

By considering the trigonometric identity for $\tan(A - B)$, with $A = \arctan(n+1)$ and $B = \arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right).$$

You may assume the series converges.

$$\boxed{}, \boxed{\frac{\pi}{4}}$$

CONSIDER THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{\tan[\arctan(n+1)] - \tan[\arctan n]}{1 + \tan[\arctan(n+1)] \tan[\arctan n]}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{(n+1) - n}{1 + (n+1)n}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{1}{n^2+n+1}$$

$$\arctan\left(\frac{1}{n^2+n+1}\right) = \arctan\left(\frac{1}{n^2+n+1}\right)$$

HENCE THE SUMMATION NOW BECOMES

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right) = \sum_{n=1}^{\infty} [\arctan(n+1) - \arctan n]$$

$$= \sum_{n=1}^{\infty} \arctan(n+1) - \sum_{n=1}^{\infty} \arctan n$$

which now gives us a useful sense

$$\lim_{k \rightarrow \infty} \left[\sum_{n=1}^k \arctan(n+1) - \sum_{n=1}^k \arctan n \right]$$

$$= \lim_{k \rightarrow \infty} \left[\arctan(k+1) - \arctan 1 \right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Question 135 (*****)

It is given that

$$(\arcsin x)^3 + (\arccos x)^3 = k\pi^3, |x| \leq 1,$$

for some constant k .

- a) Show that a necessary but not sufficient condition for the above equation to have solutions is that

$$k \geq \frac{1}{32}.$$

- b) Solve the equation given that it only has one solution.

- c) Given instead that that $k = \frac{7}{96}$, find the two solutions of the equation, giving the answers in the form $x = \sin(a\pi)$, where $a \in \mathbb{Q}$.

<input type="text"/>	$x = \frac{\sqrt{2}}{2}$	$x = \sin\left(\frac{\pi}{12}\right), \quad x = \sin\left(\frac{5\pi}{12}\right)$
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a) $(\arcsinx)^3 + (\arccos x)^3 = k\pi^3$

Using the identity $\arcsinx + \arccos x = \frac{\pi}{2}$

$$\rightarrow (\arcsinx)^3 + \left(\frac{\pi}{2} - \arcsinx\right)^3 = k\pi^3$$

$$\rightarrow (\arcsinx)^3 + \frac{\pi^3}{8} - \frac{3\pi^2}{4}(\arcsinx) + \frac{27}{8}(\arcsinx)^3 - (\arcsinx)^5 = k\pi^3$$

$$\rightarrow \frac{35}{8}(\arcsinx)^3 - \frac{3\pi^2}{4}(\arcsinx) + \frac{\pi^3}{8} - k\pi^3 = 0$$

$$\rightarrow \frac{3}{2}(\arcsinx)^3 - \frac{3\pi^2}{4}(\arcsinx) + \frac{\pi^3}{8} - k\pi^3 = 0$$

$$\rightarrow \frac{3}{2}(\arcsinx)^3 - \frac{3\pi^2}{4}(\arcsinx) + \frac{\pi^3}{8}(1 - 8k) = 0$$

$$\rightarrow (\arcsinx)^3 - \frac{\pi^2}{4}(\arcsinx) + \frac{\pi^3}{16}(1 - 8k) = 0$$

For real solutions, $b^2 - 4ac \geq 0$

$$\Rightarrow \frac{\pi^2}{4} - 4 \times \frac{\pi^2}{16}(1 - 8k) \geq 0$$

$$\Rightarrow \frac{1}{4} - \frac{1}{4}(1 - 8k) \geq 0$$

$$\Rightarrow \frac{1}{4} - \frac{1}{4} + \frac{8}{4}k \geq 0$$

$$\Rightarrow 3 - 4 + 32k \geq 0$$

$$\Rightarrow 32k \geq 1$$

$$\Rightarrow k \geq \frac{1}{32}$$

This condition is necessary but not sufficient as $k > \frac{1}{32}$ may produce solutions such as $|\arcsinx| > 1$ without DO NOT EXIST OUTSIDE THE INTERVALS

b) If there is only 1 solution $\Rightarrow k = \frac{1}{32}$

$$\rightarrow (\arcsinx)^2 - \frac{\pi^2}{4}(\arcsinx) + \frac{\pi^3}{16}\left(1 - \frac{8}{8}\right) = 0$$

$$\rightarrow (\arcsinx)^2 - \frac{\pi^2}{4}(\arcsinx) + \frac{\pi^3}{16} \times \frac{3}{2} = 0$$

$$\rightarrow (\arcsinx)^2 - \frac{\pi^2}{4}(\arcsinx) + \frac{3\pi^2}{32} = 0$$

$$\rightarrow (\arcsinx - \frac{\pi}{4})^2 = 0$$

$$\Rightarrow \arcsinx = \frac{\pi}{4}$$

$$\Rightarrow x = \sin\left(\frac{\pi}{12}\right)$$

c) Final if $k > \frac{7}{96}$

$$\rightarrow (\arcsinx)^2 - \frac{\pi^2}{4}(\arcsinx) + \frac{\pi^3}{16}\left(1 - \frac{8}{\frac{7}{96}}\right) = 0$$

$$\rightarrow (\arcsinx)^2 - \frac{\pi^2}{4}(\arcsinx) + \frac{7\pi^2}{192}\left(1 - \frac{8}{7}\right) = 0$$

$$\rightarrow (\arcsinx)^2 - \frac{\pi^2}{4}(\arcsinx) + \frac{56\pi^2}{144} = 0$$

$$\rightarrow \left[\arcsinx - \frac{\pi}{4}\right]^2 - \frac{\pi^2}{16} + \frac{56\pi^2}{144} = 0$$

$$\rightarrow \left[\arcsinx - \frac{\pi}{4}\right]^2 + \frac{52\pi^2}{144} = 0$$

$$\rightarrow \left[\arcsinx - \frac{\pi}{4}\right]^2 = -\frac{52\pi^2}{144} = 0$$

$$\rightarrow \left[\arcsinx - \frac{\pi}{4}\right]^2 = \frac{52\pi^2}{144}$$

$$\rightarrow \arcsinx - \frac{\pi}{4} = \pm\sqrt{\frac{52\pi^2}{144}}$$

$$\Rightarrow \arcsinx = \frac{\pi}{4} \pm \sqrt{\frac{52\pi^2}{144}} = \frac{\pi}{4} \pm \frac{\sqrt{52}\pi}{12}$$

$$\Rightarrow \arcsinx = \frac{\pi}{4} \pm \frac{\sqrt{13}\pi}{6} = \frac{\pi}{4} \pm \frac{\pi\sqrt{13}}{6}$$

$$\Rightarrow x = \sin\left(\frac{\pi}{4} \pm \frac{\pi\sqrt{13}}{6}\right)$$

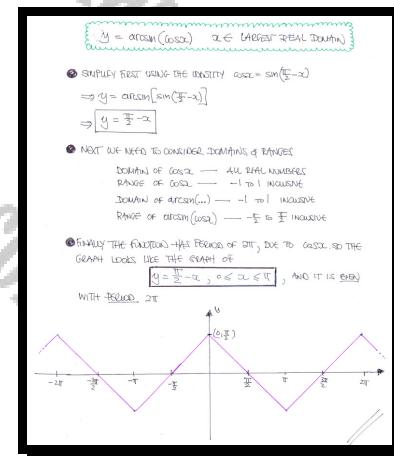
Question 136 (***)**

Sketch the graph of

$$f(x) = \arcsin(\cos x),$$

in the largest domain that the function is defined.

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusps of the curve.

 , graph


Question 137 (*****)

It is given that

$$\arctan 2 + \arctan A + \arctan B = \pi.$$

It is further given that A and B are distinct positive real numbers other than unity.

Determine a pair of possible values for A and B .

$$\boxed{}, \boxed{5} \text{ & } \boxed{\frac{7}{9}}$$

• AS THE PROBLEM IS NOT UNIQUE, LET $A=5$

$$\arctan 2 + \arctan 5 + \arctan B = \pi$$

• USING COMPLEX NUMBERS

$$(1+2i)(1+5i)z = 1+5i+2i-10 = -9+7i$$

• IN POLAR FORM

$$(1+2i)(1+5i)z = \text{NEGATIVE REAL NUMBER (SAY } -1\text{ AT THIS STAGE)}$$

$$\Rightarrow (1+2i)(1+5i)z = -1$$

$$\Rightarrow (-9+7i)z = -1$$

$$\Rightarrow (9-7i)z = 1$$

$$\Rightarrow z = \frac{1}{9-7i}$$

$$\Rightarrow z = \frac{9+7i}{80+49}$$

$$\Rightarrow z = \frac{9+7i}{130}$$

• SO FAR WE HAVE

$$\Rightarrow (1+2i)(1+5i)\left(\frac{9+7i}{130}\right) = -1$$

$$\Rightarrow (1+2i)(1+5i)(9+7i) = -130$$

$$\Rightarrow \arg[(1+2i)(1+5i)(9+7i)] = -180^\circ$$

$$\Rightarrow \arg(1+2i) + \arg(1+5i) + \arg(9+7i) = \arg(-130)$$

$$\Rightarrow \arctan 2 + \arctan 5 + \arctan \frac{7}{9} = \pi$$

Question 138 (*****)

Find the value of

$$\sum_{r=0}^{\infty} \left[\frac{\sin^4(\pi \times 2^{r-2})}{4^r} \right].$$

Hint: Express $\sin^4 \theta$ in terms of $\sin^2 \theta$ and $\sin^2 2\theta$ only.

, $\frac{1}{2}$

Now we think by manipulating the sum to the right:

$$\begin{aligned} \sin^4 \theta &= (\sin^2 \theta)^2 = \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta \right)^2 = \frac{1}{4} - \frac{1}{2} \cos 2\theta + \frac{1}{4} \cos^2 2\theta \\ &= \frac{1}{4} - \frac{1}{2}(1 - 2 \sin^2 \theta) + \frac{1}{4}(1 - \sin^2 2\theta) \\ &= \frac{1}{4} - \frac{1}{2} + \sin^2 \theta + \frac{1}{4} - \frac{1}{4} \sin^2 2\theta \\ &= \sin^2 \theta - \frac{1}{4} \sin^2 2\theta \end{aligned}$$

Now we think by considering the sum of the first n terms:

$$\begin{aligned} \sum_{r=0}^{n-1} \frac{\sin^4(\pi \times 2^{r-2})}{4^r} &= \sum_{r=0}^{n-1} \left[\frac{1}{4^r} \left[\sin^2(\pi \times 2^{r-2}) - \frac{1}{4} \sin^2(\pi \times 2^{r-1}) \right] \right] \\ &= \sum_{r=0}^{n-1} \left[\frac{1}{4^r} \sin^2(\pi \times 2^{r-2}) - \frac{1}{4^{r+1}} \sin^2(\pi \times 2^{r-1}) \right] \\ &= \frac{\sin^2 \frac{\pi}{4}}{4^0} - \frac{1}{4^1} \sin^2 \frac{\pi}{2} \quad \leftarrow r=0 \\ &\quad \frac{1}{4^1} \sin^2 \frac{\pi}{2} - \frac{1}{4^2} \sin^2 \frac{\pi}{4} \quad \leftarrow r=1 \\ &\quad \frac{1}{4^2} \sin^2 \frac{\pi}{4} - \frac{1}{4^3} \sin^2 \frac{\pi}{8} \quad \leftarrow r=2 \\ &\quad \vdots \\ &\quad \frac{1}{4^{n-1}} \sin^2 \frac{\pi}{2^{n-2}} - \frac{1}{4^n} \sin^2 \frac{\pi}{2^{n-1}} \quad \leftarrow r=n-1 \\ &= \sin^2 \frac{\pi}{4} - \frac{1}{4^n} \sin^2(\pi \times 2^{-n}) \end{aligned}$$

Hence we think

$$\sum_{r=0}^{\infty} \frac{\sin^4(\pi \times 2^{r-2})}{4^r} = \sin^2 \frac{\pi}{4} = \frac{(\sqrt{2})^2}{4} = \frac{1}{2}$$

Question 139 (***)**

The triangle ABC is isosceles with $|AB|=|AC|$ and $\angle BAC = 36^\circ$.

The angle bisector of $\angle ABC$ meets AC at the point D .

By using trigonometry in the above construction, or otherwise, show that

$$\cos 36^\circ = \frac{1}{2}(1 + \sqrt{5}).$$

proof

Start with a good diagram, marking all the angles.

If $|AB|=x$, q. $|DC|=y$

Then $|BC|=|AC|=x+y$

$|BD|=|AD|=x$

$|CD|=|BC|=x$

Now looking at the triangle ABD

$$|AB|^2 = 2|DB|\cos 36^\circ$$

$$x^2 = 2x^2\cos 36^\circ$$

$$x^2 = 2x^2\cos 36^\circ$$

Next looking at the triangle BDC

$$|BC|^2 = 2|DC|\cos 72^\circ$$

$$x^2 = 2x(y)\cos 72^\circ$$

Substituting the last two equations and tidy up

$$x^2 = 2x\cos 36^\circ$$

$$y = 2x\cos 72^\circ$$

$$\rightarrow x^2 = 2x\cos 36^\circ - 2x\cos 72^\circ$$

$$\rightarrow x^2 = 2x(\cos 36^\circ - \cos 72^\circ)$$

Using the cosine double angle formula we finally obtain

$$1 = 2\cos^2 36^\circ - 2[\cos 2 \cdot 36^\circ - 1]$$

$$1 = 2\cos^2 36^\circ - 4\cos^2 36^\circ + 2$$

$$\Rightarrow 4\cos^2 36^\circ - 2\cos 2 \cdot 36^\circ - 1 = 0$$

$$\Rightarrow \cos 2 \cdot 36^\circ = \frac{2 \pm \sqrt{(2)^2 - 4 \times 4(-1)}}{2 \times 4}$$

$$\Rightarrow \cos 2 \cdot 36^\circ = \frac{2 \pm \sqrt{36}}{8}$$

$$\Rightarrow \cos 2 \cdot 36^\circ = \frac{2 \pm 2\sqrt{9}}{8}$$

$$\Rightarrow \cos 2 \cdot 36^\circ = \frac{1 \pm \sqrt{9}}{4}$$

$$\Rightarrow \cos 2 \cdot 36^\circ = \frac{1 + \sqrt{9}}{4} \quad \text{ ~~$\frac{1 - \sqrt{9}}{4} < 0$~~

$$\Rightarrow \cos 2 \cdot 36^\circ = \frac{1 + \sqrt{5}}{4}$$

$$\Rightarrow \cos 36^\circ = \frac{1 + \sqrt{5}}{2}$$

$\cos 36^\circ > 0$$$

Question 140 (*****)

$$f(x) = \frac{\cos 2x}{\sqrt{1 + \sin 2x}}, \quad x \in \mathbb{R}, \quad \sin 2x \neq -1.$$

- a) Express $f(x)$ in the form

$$f(x) = \frac{g(x)g(-x)}{|g(x)|},$$

where $g(x)$ is a function to be found.

- b) Sketch the graph of $f(x)$ for $-2\pi \leq x \leq 2\pi$.

- c) Hence solve the trigonometric equation

$$\sqrt{2}f(x) = 1, \quad -2\pi \leq x \leq 2\pi.$$

	$g(x) = \cos x + \sin x$	$x = -\frac{23}{12}\pi, -\frac{11}{12}\pi, \frac{1}{12}\pi, \frac{13}{12}\pi$
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a) MANIPULATE AS FOLLOWS

$$f(x) = \frac{\cos 2x}{\sqrt{1 + \sin 2x}}, \quad x \in \mathbb{R}, \quad \sin 2x \neq -1$$

$$\rightarrow f(x) = \frac{\cos 2x - \sin 2x}{\sqrt{\cos^2 x + \sin^2 x + 2\cos x \sin x}}$$

$$\rightarrow f(x) = \frac{(\cos x + \sin x)(\cos x - \sin x)}{\sqrt{(\cos x + \sin x)^2}}$$

$$\rightarrow f(x) = \frac{(\cos x + \sin x)[\cos(x) + \sin(-x)]}{|\cos x + \sin x|}$$

$$\rightarrow f(x) = \frac{g(x)g(-x)}{|g(x)|} \quad \text{where } g(x) = \cos x + \sin x.$$

b) LOOKING AT THE PLOT OF $f(x)$ ABOVE

- If $g(x) > 0$ $f(x) = \frac{g(x)g(-x)}{g(x)} = g(-x) = \cos x - \sin x$
- If $g(x) < 0$ $f(x) = \frac{g(x)g(-x)}{-g(x)} = -g(-x) = \sin x - \cos x$

PROCEED AS FOLLOWS

$$g(x) = \cos x + \sin x = \sqrt{2} \left[\frac{1}{\sqrt{2}} \cos x + \frac{1}{\sqrt{2}} \sin x \right]$$

$$= \sqrt{2} \left[\sin \frac{\pi}{4} \cos x + \cos \frac{\pi}{4} \sin x \right] = \sqrt{2} \sin \left(x + \frac{\pi}{4} \right)$$

THIS $f(x) = \cos x + \sin x = \sqrt{2} \sin \left(x + \frac{\pi}{4} \right)$

$$\begin{aligned} g(-x) &= \cos x - \sin x = \sqrt{2} \sin \left(-x + \frac{\pi}{4} \right) \\ &= -\sqrt{2} \sin \left(x - \frac{\pi}{4} \right) \\ -g(-x) &= \sin x - \cos x = \sqrt{2} \sin \left(x - \frac{\pi}{4} \right) \end{aligned}$$

START WITH THE GRAPH OF $g(x) = \sqrt{2} \sin \left(x + \frac{\pi}{4} \right)$

c) FINALLY SOLVING THE EQUATION

$$\begin{aligned} \rightarrow \sqrt{2}f(x) &= 1 \\ \rightarrow f(x) &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \Rightarrow \left(\frac{-\sqrt{2} \sin \left(x - \frac{\pi}{4} \right)}{\sqrt{2} \sin \left(x - \frac{\pi}{4} \right)} \right) &= \frac{1}{\sqrt{2}} \quad \leftarrow g(-x) \\ \Rightarrow \left(\frac{\sin \left(x - \frac{\pi}{4} \right)}{\sin \left(x - \frac{\pi}{4} \right)} \right) &= \frac{1}{\sqrt{2}} \quad \leftarrow -g(-x) \end{aligned}$$

$$\Rightarrow \left(\frac{\sin \left(x - \frac{\pi}{4} \right)}{\sin \left(x - \frac{\pi}{4} \right)} \right) = -\frac{1}{\sqrt{2}} \quad \leftarrow g(-x) \quad \leftarrow -g(-x)$$

$$\begin{aligned} \Rightarrow \left(\frac{2 - \frac{\pi}{4}}{2 - \frac{\pi}{4}} \right) &= -\frac{1}{\sqrt{2}} \pm 2\pi n \quad n \in \mathbb{Z}/2\mathbb{Z}, \dots \\ \Rightarrow \left(\frac{x - \frac{\pi}{4}}{x - \frac{\pi}{4}} \right) &= \frac{\pi}{4} \pm 2\pi n \quad \leftarrow g(-x) \\ \Rightarrow \left(\frac{x - \frac{\pi}{4}}{x - \frac{\pi}{4}} \right) &= \frac{3\pi}{4} \pm 2\pi n \quad \leftarrow -g(-x) \end{aligned}$$

$$\begin{aligned} \Rightarrow \left(x - \frac{\pi}{4} \right) &= \frac{\pi}{4} \pm 2\pi n \quad \leftarrow g(-x) \\ x &= \frac{\pi}{4} \pm 2\pi n \quad \leftarrow -g(-x) \\ x &= \frac{\pi}{4} \pm \frac{\pi}{2} \end{aligned}$$

LOOKING AT GRAPHS TO DETERMINE SOLUTIONS IN RANGE

$$\begin{aligned} x &= \frac{\pi}{4} \pm \frac{\pi}{2} \pi - \frac{\pi}{4} - \frac{\pi}{2} \pi \\ x &= \frac{5\pi}{12} \quad \frac{7\pi}{12} \quad \frac{11\pi}{12} \quad \frac{13\pi}{12} \end{aligned}$$

$$\therefore x = \frac{\pi}{4} \pm \frac{\pi}{2}$$

Question 141 (***)**

Prove that for all x such that $-1 \leq x \leq 1$

$$\arccos x + \arccos \left[\frac{1}{2} \left(x + \sqrt{3-3x^2} \right) \right] = \frac{\pi}{3}$$

 , proof

arccos x + arccos $\left(\frac{x+\sqrt{3-3x^2}}{2} \right)$ = $\frac{\pi}{3}$

• LET $\theta = \arccos x$
 $\cos \theta = x$
 $\sin \theta = \sqrt{1-x^2}$

• LET $\phi = \arccos \left(\frac{x+\sqrt{3-3x^2}}{2} \right)$
 $\cos \phi = \frac{x+\sqrt{3-3x^2}}{2}$

• WE NEED TO FIND THE EXACT VALUE OF $\sin \phi$ SO WE USE PYTHAGORAS IN THE "SECOND" TRIANGLE TO FIND y
 $\Rightarrow y = \sqrt{4 - (x + \sqrt{3-3x^2})^2}$
 $\Rightarrow y = \sqrt{4 - (x^2 + 2x\sqrt{3-3x^2} + 3-3x^2)}$
 $\Rightarrow y = \sqrt{4 - x^2 - 2x\sqrt{3-3x^2} - 3 + 3x^2}$
 $\Rightarrow y = \sqrt{1 + 2x^2 - 2x\sqrt{3-3x^2}}$

• ATTEMPTING TO SQUARE ROOT THE ARGUMENT OF THE RADICAL BY INSPECTION
 $\Rightarrow 1 + 2x^2 - 2x\sqrt{3-3x^2} \equiv (Ax + \sqrt{1-x^2})^2$

SIMPLIFY AND EXPAND COEFFICIENTS
 $\Rightarrow 1 + 2x^2 - 2x\sqrt{3-3x^2} \equiv Ax^2 + 2Ax\sqrt{1-x^2} + 1 - x^2$
 $\Rightarrow 1 + 2x^2 - 2x\sqrt{3-3x^2} \equiv (A^2-1)x^2 + 2Ax\sqrt{1-x^2} + 1$

• THIS EASILY WORKS IF $A = \sqrt{3}$
 $\Rightarrow y = \sqrt{3x + \sqrt{1-x^2}}$
 $\Rightarrow \sin \phi = \frac{y}{2}$
 $\Rightarrow \sin \phi = \frac{\sqrt{3x + \sqrt{1-x^2}}}{2}$

• RETURNING TO THE ORIGINAL EXPRESSION AND REWRITE AS FOLLOWS
 $\arccos x + \arccos \left(\frac{x+\sqrt{3-3x^2}}{2} \right) = \frac{\pi}{3}$
 $\rightarrow \theta + \phi = \frac{\pi}{3}$
 $\rightarrow \cos(\theta + \phi) = \cos \frac{\pi}{3}$
 $\rightarrow \cos \theta \cos \phi - \sin \theta \sin \phi = \cos \frac{\pi}{3}$
 $\rightarrow \frac{x + \sqrt{3-3x^2}}{2} - \sqrt{1-x^2} \left[\frac{\sqrt{3x + \sqrt{1-x^2}}}{2} \right] = \cos \frac{\pi}{3}$
 $\rightarrow \frac{x^2 + 2x\sqrt{3-3x^2} - \sqrt{3x + \sqrt{1-x^2}} + (1-x^2)}{2} = \cos \frac{\pi}{3}$
 $\rightarrow \frac{2x^2 + 1 - x^2}{2} = \cos \frac{\pi}{3}$
 $\rightarrow \cos \frac{\pi}{3} = \frac{1}{2}$
 $\Rightarrow \frac{1}{2} = \frac{1}{2}$
 $\therefore \arccos x + \arccos \left(\frac{x+\sqrt{3-3x^2}}{2} \right) = \frac{\pi}{3}$

Question 142 (***)**

The function f is defined as

$$f(x) \equiv \sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x, \quad x \in \left(0, \frac{1}{2}\pi\right).$$

Determine with full justification the range of f .

, $f(x) \in [2+3\sqrt{2}, \infty)$

REWRITE THE FUNCTION IN SINES & COSINES.

$$\begin{aligned} f(x) &= \sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x \\ f(x) &= \sin x + \cos x + \frac{\sin x}{\cos x} + \frac{1}{\cos x} + \frac{1}{\sin x} \\ f(x) &= \sin x + \cos x + \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} + \frac{\sin x + \cos x}{\cos x \sin x} \\ f(x) &= \sin x + \cos x + \frac{2(\sin x + \cos x)}{\cos x \sin x} + \frac{\sin x + \cos x}{\cos x \sin x} \\ f(x) &= \sin x + \cos x + \frac{2}{\sin x \cos x} + \frac{(\sin x + \cos x)^2}{\sin x \cos x} \end{aligned}$$

Now $\sin x + \cos x$ & $\sin x \cos x$ are related to θ as follows

$$\begin{aligned} \text{Let } g(\theta) &= \sin x + \cos x \\ [g(\theta)]^2 &= (\sin x + \cos x)^2 \\ g^2 &= \sin^2 x + 2\sin x \cos x + \cos^2 x \\ g^2 &= 1 + \sin 2x \\ \sin 2x &= g^2 - 1 \end{aligned}$$

Rewrite the function $f(x)$ in terms of $g(\theta)$

$$\begin{aligned} f(x) &= g + \frac{2}{g^2-1} + \frac{2}{g-1} \\ f(x) &= g + \frac{2(g+1)}{g^2-1} \\ f(x) &= g + \frac{2(g+1)}{(g-1)(g+1)} \\ f(x) &= g + \frac{2}{g-1} \end{aligned}$$

Note that $\sin x + \cos x = g$ and $\sin x \cos x = \frac{g^2-1}{2}$

NOW USING CALCULUS

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left[g + \frac{2}{g-1} \right] = \left[-\frac{2}{(g-1)^2} \right] \times \frac{dg}{dx} \\ f'(x) &= \left[-\frac{2}{(g-1)^2} \right] \times \frac{1}{\sin x \cos x} \\ f'(x) &= \left[1 - \frac{2}{(g-1)^2} \right] \times (\sec x - \sin x) \end{aligned}$$

Now solving $f'(x) = 0$

- CASE 1: $1 - \frac{2}{(g-1)^2} = 0$
 $1 = \frac{2}{(g-1)^2}$
 $(g-1)^2 = 2$
 $g-1 = \pm\sqrt{2}$
 $g = 1 \pm \sqrt{2}$
- CASE 2: $\sec x - \sin x = 0$
 $\cos x = \sin x$
 $\tan x = 1$
 $x = \frac{\pi}{4}$ or $\sin x$

Now note that

- $f(0) \rightarrow \infty$ as $x \rightarrow 0$
 (due to $\cot x$ & $\sec x$)
- $f(\pi) \rightarrow -\infty$ as $x \rightarrow \pi$
 (due to $\operatorname{cosec} x$)

$\therefore x = \frac{\pi}{4}$ is a stationary minimum value (true minimum)

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{\sqrt{2}-1} + 1 + \sqrt{2} + \frac{2}{\sqrt{2}} \\ &= 2 + 3\sqrt{2} \end{aligned}$$

$\therefore \text{range } f(x) \in [2+3\sqrt{2}, \infty)$

Question 143 (*****)

It is given that

- the angles A , B and C are the three angles of a triangle ABC .
- the angles A , B and C are in an increasing arithmetic progression, in that order.
- The lengths of the triangle ABC , opposite each of the angles A , B and C are denoted by a , b and c .

Show that

$$\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A = \sqrt{3}.$$

 , proof

START BY A DILEMMA

$A+B+C = 180^\circ$

ANGLES ARE IN ARITHMETIC PROGRESSION IMPLIES

$B-A = C-B$

SOLVE THESE TWO EQUATIONS

$$\begin{aligned} A+B+C &= 180^\circ \\ 2B &= C+A \quad \rightarrow \\ 2A+2B+2C &= 360^\circ \\ 2B &= C+A \quad \rightarrow \\ \Rightarrow 2A+C+A+2C &= 360^\circ \\ \Rightarrow 3A+3C &= 360^\circ \\ \Rightarrow A+C &= 120^\circ \\ \Rightarrow 2B &= 120^\circ \\ \Rightarrow B &= 60^\circ \end{aligned}$$

NEXT BY THE SINE RULE

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \quad (\text{SOME WORKING})$$

∴

$$\begin{aligned} \frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A &= \frac{a}{c} [2\sin C \cos C] + \frac{c}{a} [2\sin A \cos A] \\ &= 2 \sin C [\cos C] + 2 \sin A [\cos A] \\ &= 2k \cos C + 2k \cos A \\ &= 2k[\cos C + \cos A] \quad (\text{NOTICE AT DILEMMA}) \\ &= 2kb = 2 \left(\frac{\sin B}{b} \right) b = 2 \sin B \\ &= 2 \sin 60^\circ = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} \end{aligned}$$

Question 144 (*****)

Find, in exact surd form, the only real solution of the following trigonometric equation

$$\arcsin(2x-1) - \arccos x = \frac{\pi}{6}.$$

The rejection of any additional solutions must be fully justified.

$$\boxed{}, \quad x = \frac{1}{2} - \frac{1}{6}\sqrt{6}$$

$$\begin{aligned}
 & \text{circum}(2x-1) = \text{circosz} = \frac{\pi i}{6} \\
 \Rightarrow & \text{circum}(2x-1) = \frac{\pi i}{6} + \text{circosz} \\
 \Rightarrow & \sin[\text{circum}(2x-1)] = \sin\left[\frac{\pi i}{6} + \text{circosz}\right] \\
 \Rightarrow & 2x-1 = \sin\frac{\pi i}{6} \cos(\text{circosz}) + \cos\frac{\pi i}{6} \sin(\text{circosz}) \\
 \Rightarrow & 2x-1 = \frac{1}{2}i \cdot 2 + \sqrt{3} \sin(\text{circosz}) \\
 \Rightarrow & 4x-2 = 2 + \sqrt{3} \sin(\text{circosz}) \\
 \Rightarrow & 3x-2 = \sqrt{3} \sin(\text{circosz}) \\
 \boxed{\text{Let } \Theta = \text{circosz}} \\
 & \omega \Theta = x \\
 & \omega^2 \Theta = x^2 \\
 & 1 - \omega^2 x^2 = 1 - x^2 \\
 & \sin^2(\text{circosz}) = 1 - x^2 \\
 & \sin^2(\text{circosz}) = 1 - 2x \\
 \Rightarrow & (3x-2)^2 = 3 \sin^2(\text{circosz}) \\
 \Rightarrow & 9x^2 - 12x + 4 = 3(1 - 2x) \\
 \Rightarrow & 9x^2 - 12x + 4 = 3 - 3x^2 \\
 \Rightarrow & 12x^2 - 12x + 1 = 0 \\
 \Rightarrow & 48x^2 - 48x + \frac{1}{3} = 0
 \end{aligned}$$

$$\begin{aligned}
 &\Rightarrow 4x^2 - 4x + 1 - \frac{2}{3} = 0 \\
 &\Rightarrow (2x-1)^2 = \frac{2}{3} \\
 &\Rightarrow 2x-1 = \pm \sqrt{\frac{2}{3}} \\
 &\Rightarrow 2x = 1 \pm \frac{\sqrt{6}}{3} \\
 &\Rightarrow x = \frac{1}{2} \pm \frac{\sqrt{6}}{6} \\
 &\Rightarrow x = \begin{cases} \frac{1}{2} + \frac{\sqrt{6}}{6} \\ \frac{1}{2} - \frac{\sqrt{6}}{6} \end{cases}
 \end{aligned}$$

Now we need to check these solutions (but to square根)

$$\begin{aligned}
 &\Rightarrow \arcsin\left(2\left(\frac{1}{2} + \frac{\sqrt{6}}{6}\right) - 1\right) + \arccos\left(\frac{1}{2} + \frac{\sqrt{6}}{6}\right) = \pi \\
 &\Rightarrow \arcsin\left(\frac{1}{2} + \frac{\sqrt{6}}{6}\right) + \arccos\left(\frac{1}{2} + \frac{\sqrt{6}}{6}\right) = \pi \\
 &\Rightarrow \arccos\left(\frac{1}{2} + \frac{\sqrt{6}}{6}\right) + \arccos\left(\frac{\sqrt{6}}{6}\right) = \pi
 \end{aligned}$$

$\tan \varphi = \frac{6\sqrt{2} - (\sqrt{2} + i\sqrt{12})}{3 + i\sqrt{2} + 4\sqrt{2}i - 6}$ $= \frac{6\sqrt{2} - \sqrt{2}(1 + 3i)}{-3 + i\sqrt{2} + 4\sqrt{2}i}$ $= \frac{\sqrt{2}(5 - 1 - 3i)}{-3 + i\sqrt{2} + 4\sqrt{2}i}$ $= \frac{4\sqrt{2}}{-3 + i\sqrt{2} + 4\sqrt{2}i}$ $= \frac{4\sqrt{2}}{\sqrt{2}(1 - 2i + 4i^2)}$ $= \frac{4}{1 - 2i + 4(-1)}$ $= \frac{4}{-3 - 2i}$ $= \frac{4(-3 + 2i)}{(-3 - 2i)(-3 + 2i)}$ $= \frac{4(-3 + 2i)}{9 - 4i^2}$ $= \frac{4(-3 + 2i)}{13}$ $\therefore \tan \varphi = \frac{4(-3 + 2i)}{13}$
--

Question 145 (***)**

Given that n is an integer such that $n > 3$, use a detailed method to solve the following trigonometric equation.

$$\frac{1}{\sin\left[\frac{2\pi}{n}\right]} + \frac{1}{\sin\left[\frac{3\pi}{n}\right]} = \frac{1}{\sin\left[\frac{\pi}{n}\right]}.$$

$$\boxed{\quad}, \boxed{n=7}$$

$$\begin{aligned}
 & \frac{1}{\sin\frac{2\pi}{n}} + \frac{1}{\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{\pi}{n}} \quad n \in \mathbb{N}, n > 3 \\
 \Rightarrow & \frac{1}{\sin\frac{2\pi}{n}} = \frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}} \\
 \Rightarrow & \frac{1}{\sin\frac{2\pi}{n}} = \frac{\sin\frac{\pi}{n} - \sin\frac{3\pi}{n}}{\sin\frac{\pi}{n} \cdot \sin\frac{3\pi}{n}} \quad \begin{array}{l} \text{• } \sin(A+B) = \sin A \cos B + \cos A \sin B \\ \text{• } \sin(A-B) = \sin A \cos B - \cos A \sin B \end{array} \\
 \Rightarrow & \frac{1}{\sin\frac{2\pi}{n}} = \frac{2\cos\frac{2\pi}{n} \cdot \sin\frac{-2\pi}{n}}{\sin\frac{\pi}{n} \cdot \sin\frac{3\pi}{n}} \\
 \Rightarrow & \sin\frac{2\pi}{n} = 2\cos\frac{2\pi}{n} \sin\frac{-2\pi}{n} \\
 \Rightarrow & \sin\frac{2\pi}{n} = \sin\frac{-2\pi}{n} \\
 \Rightarrow & \sin\frac{2\pi}{n} = 0 \quad \text{--- no solutions for } n > 3 \\
 \Rightarrow & \frac{2\pi}{n} = 0 \pm 2m\pi \quad \text{--- no solutions for } n > 3 \\
 \Rightarrow & \frac{2\pi}{n} = \pi \pm 2m\pi \quad \text{--- no solutions for } n > 3 \\
 \therefore & n=7
 \end{aligned}$$

Question 146 (***)**

Find, in terms of π , the general solution of the equation

$$(x+y)^2 + 4(x+y)\cos(x-y) + 4 = 0.$$

$(x, y) = \left(-1 + \frac{k\pi}{2}, -1 - \frac{k\pi}{2}\right), k = \text{even}$, $(x, y) = \left(1 + \frac{k\pi}{2}, 1 - \frac{k\pi}{2}\right), k = \text{odd}$

ORGANISING THE EQUATION

• $y = \text{even}$

$$\Rightarrow (x+y)^2 + 4(x+y)\cos(x-y) + 4 = 0$$

$$\Rightarrow [(x+y) + 2\cos(x-y)]^2 - [2\sin(x-y)]^2 + 4 = 0$$

$$\Rightarrow [(x+y) + 2\cos(x-y)]^2 - 4\sin^2(x-y) + 4 = 0$$

$$\Rightarrow [(x+y) + 2\cos(x-y)]^2 + [1 - 4\sin^2(x-y)] = 0$$

$$\Rightarrow [(x+y) + 2\cos(x-y)]^2 + 4\sin^2(x-y) = 0$$

$$\Rightarrow [(x+y) + 2\cos(x-y)]^2 + [2\sin(2x-y)]^2 = 0$$

• SINCE THE BRACKETS ARE REAL QUANTITY (AND ONLY BE AFFECTED IF BOTH BRACKETS ARE ZERO)

$$[(x+y) + 2\cos(x-y)] = 0 \quad \text{AND} \quad \sin(2x-y) = 0$$

$$x-y = n\pi, \quad n \in \mathbb{Z}$$

• NOW THERE ARE TWO CASES TO CONSIDER: $n = \text{ODD}$ OR $n = \text{EVEN}$

- IF $n = \text{EVEN}$:
 $x-y = 2\pi, 4\pi, 6\pi, \dots$
 $\cos(x-y) = 1$
 $\therefore x+y + 2 = 0$
 $x+y = -2$
- IF $n = \text{ODD}$:
 $x-y = \pi, 3\pi, 5\pi, \dots$
 $\cos(x-y) = -1$
 $\therefore x+y - 2 = 0$
 $x+y = 2$

• $y = \text{odd}$

$$\Rightarrow (x+y)^2 + 4(x+y)\cos(x-y) + 4 = 0$$

$$\Rightarrow (x+y + 2\cos(x-y))^2 - (2\sin(x-y))^2 + 4 = 0$$

$$\Rightarrow (x+y + 2\cos(x-y))^2 - 4\sin^2(x-y) + 4 = 0$$

$$\Rightarrow (x+y + 2\cos(x-y))^2 + [1 - 4\sin^2(x-y)] = 0$$

$$\Rightarrow (x+y + 2\cos(x-y))^2 + 4\sin^2(x-y) = 0$$

$$\Rightarrow (x+y + 2\cos(x-y))^2 + [2\sin(2x-y)]^2 = 0$$

• ADDING:
 $2x = -2 + n\pi$
 $x = -1 + \frac{n\pi}{2}$

• SUBTRACTING:
 $2y = 2 - n\pi$
 $y = 1 - \frac{n\pi}{2}$

• HENCE THE GENERAL SOLUTION OF THE EQUATION IS GIVEN BY

$x = -1 + \frac{n\pi}{2}$	IF n IS AN EVEN INTEGER
$y = 1 - \frac{n\pi}{2}$	
$x = 1 + \frac{n\pi}{2}$	IF n IS AN ODD INTEGER
$y = 1 - \frac{n\pi}{2}$	

Question 147 (*****)

Find the general solution of the following equation

$$\frac{d}{dx} \left[\int_{\frac{1}{6}\pi}^{\sqrt{2x}} \sin(t^2) + \cos(2t^2) \, dt \right] = -\sqrt{\frac{2}{x}}, \quad x \in \mathbb{R}.$$

$$[\quad], \quad x = \frac{1}{4}\pi(4k-1) \quad k \in \mathbb{Z}$$

PROCEED BY LEIBNIZ INTEGRAL RULE & NOT $\frac{d}{dx} \int f(x) \, dx = 0$

$$\begin{aligned} \frac{d}{dx} \left[\int_{\frac{1}{6}\pi}^{\sqrt{2x}} \sin(t^2) + \cos(2t^2) \, dt \right] &= -\sqrt{\frac{2}{x}} \\ \Rightarrow \quad \sin((\sqrt{2x})^2) \times \frac{d}{dx}(\sqrt{2x}) + \cos((2\sqrt{2x})^2) &= -\sqrt{\frac{2}{x}} \\ \Rightarrow \quad [\sin(2x) + \cos(4x)] \times \frac{d}{dx}(\sqrt{2x}) &= -\sqrt{\frac{2}{x}} \\ \Rightarrow \quad (\sin 2x + \cos 4x) \times \frac{1}{2}\sqrt{2}x^{-\frac{1}{2}} &\cancel{x^{\frac{1}{2}}} = -\cancel{\sqrt{\frac{2}{x}}} \cancel{x^{\frac{1}{2}}} \\ \Rightarrow \quad \sin 2x + \cos 4x &= -2 \end{aligned}$$

NO IDENTITIES NEEDED HERE – JUST NEED A COMMON SOLUTION

- $\sin 2x = -1$

$$\begin{aligned} 2x &= -\frac{\pi}{2} + 2n\pi \quad n=0,1,2,\dots \\ 2x &= -\frac{\pi}{2} [1+4n] \\ 2 &= \frac{1}{2}\pi(1+4n) \quad n = -\frac{1}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, \dots \end{aligned}$$
- $\cos 4x = -1$

$$\begin{aligned} 4x &= \pi + 2n\pi \quad n=0,1,2,\dots \\ 4x &= \pi(1+2n) \\ x &= \frac{1}{4}\pi(1+2n) \quad n = -\frac{1}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, \dots \end{aligned}$$

THE COMMON SOLUTIONS ARE THOSE OF THE SAME

$$\therefore x = \frac{1}{4}\pi(4k-1) \quad k \in \mathbb{Z}$$

