

SERIES

79 EXAM
QUESTIONS

SUMMATIONS BY FORMULAS

17 BASIC QUESTIONS

Question 1 (**)

Use standard results on summations to find the value of

$$\sum_{r=36}^{48} [(r-1)(3r-2)].$$

, 66638

FIND A SIMPLIFIED EXPRESSION FOR THE SUM OF THE FIRST n TERMS

$$\begin{aligned}\sum_{r=1}^n [(r-1)(3r-2)] &= \sum_{r=1}^n (3r^2 - 5r + 2) \\&= 3 \sum_{r=1}^n r^2 - 5 \sum_{r=1}^n r + 2 \sum_{r=1}^n 1 \\&= 3 \times \frac{1}{6} n(n+1)(2n+1) - 5 \times \frac{1}{2} n(n+1) + 2n \\&= \frac{1}{2} n(n+1)(2n+1) - \frac{5}{2} n(n+1) + 2n \\&= \frac{1}{2} n \left[(2n+1)(n+1) - 5n - 5 + 4 \right] \\&= \frac{1}{2} n [2n^2 + 3n + 1 - 5n - 5 + 4] \\&= \frac{1}{2} n [2n^2 - 2n] \\&= n^2 (n-1)\end{aligned}$$

NOW WE HAVE

$$\begin{aligned}\sum_{r=36}^{48} [(r-1)(3r-2)] &= \sum_{r=1}^{48} [(r-1)(3r-2)] - \sum_{r=1}^{35} [(r-1)(3r-2)] \\&= 48^2 (48-1) - 35^2 (35-1) \\&= 106288 - 41690 \\&= \underline{\underline{66638}}\end{aligned}$$

Question 2 (**)

Use standard results on summations to show that

$$\sum_{r=1}^n r(r+1)(r+5) = \frac{1}{4}n(n+a)(n+b)(n+c),$$

where a , b , and c are positive integers to be found.

, $a = 1, b = 2, c = 7$

EXPANDING AND SIMPLIFYING

$$\sum_{r=1}^n r(r+1)(r+5) = \sum_{r=1}^n (r^3 + 6r^2 + 5r) = \sum_{r=1}^n r^3 + 6\sum_{r=1}^n r^2 + 5\sum_{r=1}^n r$$

USING STANDARD RESULTS

$$\begin{aligned} \sum_{r=1}^n r^3 &= \frac{1}{4}n^2(n+1)^2 \\ \sum_{r=1}^n r^2 &= \frac{1}{6}n(n+1)(2n+1) \\ \sum_{r=1}^n r &= \frac{1}{2}n(n+1) \end{aligned}$$

$$\begin{aligned} \dots &= \frac{1}{4}n^2(n+1)^2 + 6n\left(\frac{1}{2}n(n+1)\right) + 5\left(\frac{1}{2}n(n+1)\right) \\ &= \frac{1}{4}n^2(n+1)^2 + 3n^2(n+1) + \frac{5}{2}n(n+1) \\ &= \frac{1}{4}n(n+1)\left[n(n+1) + 4(2n+1) + 10\right] \\ &= \frac{1}{4}n(n+1)\left[n^2 + n + 8n + 4 + 10\right] \\ &= \frac{1}{4}n(n+1)\left(n^2 + 9n + 14\right) \\ &= \frac{1}{4}n(n+1)(n+2)(n+7) \\ \therefore a &= 1, b = 2, c = 7 \end{aligned}$$

Question 3 ()**

Use standard results on summations to show that

$$\sum_{r=1}^n [r^2(r-1)] = \frac{1}{12}n(n-1)(n+1)(3n+2)+m,$$

where m is an integer to be found.

$$\boxed{\quad}, \boxed{m = -22}$$

EXPAND THE SUMMAND & USE STANDARD RESULTS

$$\begin{aligned} \sum_{r=1}^n r^2(r-1) &= \sum_{r=1}^n (r^3 - r^2) = \left[\sum_{r=1}^n (r^3 - r^2) \right] - (0 + 4 + 8) \\ &= \sum_{r=1}^n (r^3 - r^2) - 22 \\ (\text{using } \sum_{r=1}^n r^3 = \frac{1}{4}n^2(2n+1)) \quad &\quad \text{& } \sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1) \\ \dots &= \frac{1}{4}n^2(2n+1)^2 - \frac{1}{6}n(n+1)(2n+1) - 22 \\ \dots &= \frac{1}{12}n(n+1)[9n^2+30n+40-2(2n+1)] - 22 \\ \dots &= \frac{1}{12}n(n+1)[3n^2+28n+40-2] - 22 \\ \dots &= \frac{1}{12}n(n+1)[3n^2+n-2] - 22 \\ \dots &= \frac{1}{12}n(n+1)(n+1)(3n+2) - 22 \\ \dots &= \underline{\underline{\frac{1}{12}n(n+1)(n+1)(3n+2)}} - 22 \\ \text{ie } m &= -22 \end{aligned}$$

Question 4 ()**

Use standard results on summations to show that

$$\sum_{r=1}^n [r^3(r+1)(r-1)] = \frac{1}{6}n^2(n+1)^2(n-1)(n+2).$$

You may assume without proof that $\sum_{r=1}^n r^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$

, proof

EXPAND THE SUMMAND FIRST

$$\begin{aligned}\sum_{r=1}^n [(r^3(r+1)(r-1))] &= \sum_{r=1}^n [(r^3(r^2-1))] = \sum_{r=1}^n (r^5-r^3) \\ &= \sum_{r=1}^n r^5 - \sum_{r=1}^n r^3\end{aligned}$$

USING STANDARD RESULTS

$$\begin{aligned}\dots &= \frac{1}{12}n^2(n+1)^2(2n^2+2n-1) - \frac{1}{4}n^2(n+1)^2 \\ \dots &= \frac{1}{12}n^2(n+1)^2 [2n^2+2n-3] \\ \dots &= \frac{1}{6}n^2(n+1)^2 (2n^2+2n-3) \\ \dots &= \frac{1}{6}n^2(n+1)^2 (n^2+n-2) \\ \dots &= \frac{1}{6}n^2(n+1)^2 (n-1)(n+2)\end{aligned}$$

AS REQUIRED

Question 5 ()**

$$F(r) \equiv \sum_{n=1}^r [n(n-1)(n+2)].$$

Use standard results on summations express $F(n)$ in fully factorized from.

$$\boxed{\quad}, F(r) = \frac{1}{12} r(r+1)(r-1)(3r+10)$$

START BY EXPANDING THE SUMMAND¹

$$\begin{aligned} F(r) &= \sum_{k=1}^r [(k-1)(k+2)] = \sum_{k=1}^r (k^2 + k - 2) = \sum_{k=1}^r (k^2 + k^2 - 2k) \\ &= \sum_{k=1}^r k^3 + \sum_{k=1}^r k^2 - 2 \sum_{k=1}^r k \end{aligned}$$

USING STANDARD RESULTS ON INFINITE SUMMATIONS

$$\begin{aligned} \Rightarrow F(r) &= \frac{1}{4} r^2 (r+1)^2 + \frac{1}{2} r(r+1)(2r+1) - 2 \times \frac{1}{2} r^2 (r+1) \\ \Rightarrow F(r) &= \frac{1}{4} r^2 (r+1)^2 + \frac{1}{2} r(r+1)(2r+1) - (r^2 r) \\ \Rightarrow F(r) &= \frac{1}{4} r^2 (r+1) [3r^2 + 3r + 2 - 12] \\ \Rightarrow F(r) &= \frac{1}{4} r^2 (r+1) (3r^2 + r - 10) \\ &\quad \text{R} \quad b^2 - 4ac = 1^2 - 4(3)(-10) \\ F(r) &= \frac{1}{4} r^2 (r+1) (3r^2 + 10r - 10) \quad = 161 \quad (\text{Gauss number}) \end{aligned}$$

Question 6 (+)**

Find, in fully simplified factorized form, an expression for the sum of the first n terms of the following series.

$$(5 \times 3) + (11 \times 7) + (17 \times 11) + (23 \times 15) + \dots$$

$$\boxed{\quad}, n^2(8n+7)$$

WRITE THE EXPRESSION IN COMPACT NOTATION

$$\underbrace{(5 \times 3) + (11 \times 7) + (17 \times 11) + \dots}_{n \text{ terms}} = \sum_{k=1}^n (2k-1)(2k+1)$$

USING STANDARD RESULTS

$$\begin{aligned} \sum_{k=1}^n (2k-1)(2k+1) &= \sum_{k=1}^n (4k^2 - 10k + 1) \\ &= 4 \sum_{k=1}^n k^2 - 10 \sum_{k=1}^n k + \sum_{k=1}^n 1 \\ &= 4 \left(\frac{1}{6} n(n+1)(2n+1) \right) - 10 \times \frac{1}{2} n(n+1) + n \\ &= 4 \left(\frac{1}{6} n(n+1)(2n+1) \right) - 5n(n+1) + n \\ &= n \left[4 \left(\frac{1}{6} n(n+1)(2n+1) \right) - 5(n+1) + 1 \right] \\ &= n \left[\frac{1}{3} n^3 + 2n^2 + 4 - 5n - 5 + 1 \right] \\ &= n \left[\frac{1}{3} n^3 + 2n^2 - 4n - 3 \right] \\ &= n^2(8n+7) \end{aligned}$$

Question 7 (*)**

Show by using standard summation results that ...

a) ... $\sum_{r=1}^n (r+1)(r+5) = \frac{1}{6}n(n+7)(2n+7)$.

b) ... $\sum_{r=11}^{40} (r+1)(r+5) = 26495$.

, proof

a) USING STANDARD FORMULEA FOR SUMS

$$\begin{aligned}\sum_{r=1}^n (r+1)(r+5) &= \sum_{r=1}^n (r^2 + 6r + 5) = \sum_{r=1}^n r^2 + 6 \sum_{r=1}^n r + 5 \sum_{r=1}^n 1 \\&= \frac{1}{6}n(n+1)(2n+1) + 6 \times \frac{1}{2}n(n+1) + 5n \\&= \frac{1}{6}n \left[(n+1)(2n+1) + 3n(n+1) + 30 \right] \\&= \frac{1}{6}n \left[2n^2 + 3n + 1 + 3n^2 + 3n + 30 \right] \\&= \frac{1}{6}n \left[5n^2 + 6n + 31 \right] \\&= \frac{1}{6}n (5n+7)(n+7) \quad \text{As required}\end{aligned}$$

b) Using the result of part (a)

$$\begin{aligned}\sum_{r=11}^{40} (r+1)(r+5) &= \sum_{r=1}^{40} (r+1)(r+5) - \sum_{r=1}^{10} (r+1)(r+5) \\&= \frac{1}{6} \times 40(40+7)(2 \times 40+7) - \frac{1}{6} \times 10(10+7)(2 \times 10+7) \\&= 272400 - 7765 \\&= 26495 \quad \text{As required}\end{aligned}$$

Question 8 (***)

Show by using standard summation results that ...

a) ... $\sum_{k=1}^n (k^2 - k - 1) = \frac{1}{3}n(n+2)(n-2)$.

b) ... $\sum_{k=10}^{40} (k^2 - k - 1) = 21049$.

, proof

a) USING THE FORMULAE

$$\begin{aligned}\sum_{k=1}^n k^2 &= \frac{1}{3}n(n+1)(2n+1) \quad \text{&} \quad \sum_{k=1}^n k^3 = \frac{1}{4}n^2(n+1)^2(2n+1) \\ \sum_{k=1}^n (k^2 - k - 1) &= \sum_{k=1}^n k^2 - \sum_{k=1}^n k - \sum_{k=1}^n 1 \\ &= \frac{1}{3}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - n \\ &= \frac{1}{6}n \left[(2n+1)(2n+3) - 3(n+1) - 6 \right] \\ &= \frac{1}{6}n \left[2n^2 + 6n + 1 - 3n - 3 - 6 \right] \\ &= \frac{1}{6}n \left[2n^2 + 3n - 8 \right] \\ &= \frac{1}{3}n (2n^2 + 3n - 8) \\ &= \frac{1}{3}n ((n-2)(n+2))\end{aligned}$$

// AS REQUIRED

b) USING PART (a)

$$\begin{aligned}\sum_{k=10}^{40} (k^2 - k - 1) &= \sum_{k=1}^{40} (k^2 - k - 1) - \sum_{k=1}^9 (k^2 - k - 1) \\ &= \frac{1}{3} \times 40 \times 39 \times 91 - \frac{1}{3} \times 9 \times 7 \times 11 \\ &= 21280 - 201 \\ &= 21079\end{aligned}$$

Question 9 (**+)

Find, in fully factorized form, an expression for the sum

$$\sum_{p=1}^k (p^3 + p^2).$$

L.P. , $\frac{1}{12} k(k+1)(k+2)(3k+1)$

USING STANDARD RESULTS

$$\begin{aligned} \sum_{p=1}^k (p^3 + p^2) &= \frac{1}{6}k(k+1)(2k+1) + \frac{1}{4}k^2(k+1)^2 \\ &= \frac{1}{2}k(k+1) \left[2(2k+1) + 3k(k+1) \right] \\ &= \frac{1}{2}k(k+1) (4k+2 + 3k^2 + 3k) \\ &= \frac{1}{2}k(k+1) (3k^2 + 7k + 2) \\ &= \frac{1}{2}k(k+1)(3k+1)(k+2) // \end{aligned}$$

Question 10 (**+)

Find, in fully factorized form, an expression for the sum

$$\sum_{r=1}^{2n} \left(3r^2 - \frac{1}{2} \right).$$

L.P. , $2n^2(4n+3)$

USING STANDARD SUMMATION FORMULAE

$$\begin{aligned} \sum_{r=1}^{2n} \left(3r^2 - \frac{1}{2} \right) &= 3 \sum_{r=1}^{2n} r^2 - \frac{1}{2} \sum_{r=1}^{2n} 1 \\ &= 3 \times \underbrace{\frac{2n(2n+1)(4n+1)}{6}}_{\sum_{r=1}^{2n} r^2} \left[2(2n)+1 \right] - \frac{1}{2} \times 2n \\ &= n(2n+1)(4n+1) - n \\ &= n \left[(2n+1)(4n+1) - 1 \right] \\ &= n \left[8n^2 + 6n + 1 - 1 \right] \\ &= n(8n^2 + 6n) \\ &= \underline{\underline{n^2(4n+3)}} // \end{aligned}$$

Question 11 (*)**

Use standard results on summations to show that

$$\sum_{r=1}^{n-2} r(r+1)^2 = \frac{1}{12} n(n-1)(n-2)(3n-1).$$

, proof

USING THE STANDARD SUMMATIONS $\sum_{m=1}^n m^n$

$$\begin{aligned} \sum_{r=1}^{n-2} r(r+1)^2 &= \sum_{r=1}^{n-2} (r^2 + 2r + r) \\ &= \sum_{r=1}^{n-2} r^2 + 2\sum_{r=1}^{n-2} r + \sum_{r=1}^{n-2} r \\ &= \frac{1}{6}n^2(n+1)^2 + 2n\left(\frac{1}{2}n(n+1)\right) + \frac{1}{2}n(n+1) \\ &= \frac{1}{6}n^2(n+1)^2 + \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\ &= \frac{1}{12}(n+1)\left[3n(n+1) + 4(n(n+1) + 6) \right] \\ &= \frac{1}{12}n(n+1)\left(3n^2 + 3n + 8n + 4 + 6 \right) \\ &= \frac{1}{12}n(n+1)\left(3n^2 + 11n + 10 \right) \\ &= \frac{1}{12}n(n+1)(n+2)(3n+5) \end{aligned}$$

Finally, let $n \rightarrow n-2$

$$\begin{aligned} \sum_{r=1}^{n-2} r(r+1)^2 &= \frac{1}{12}(n-2)(n-1)(n-2+1)(n-2+2)[3(n-2)+5] \\ &= \frac{1}{12}(n-2)(n-1)(n-2)(3n-1) \\ &= \underline{\underline{\frac{1}{12}n(n-1)(n-2)(3n-1)}} \end{aligned}$$

Question 12 (***)

It is given that

$$\sum_{r=1}^n [(3r+a)(r+2)] \equiv n(n+2)(n+b).$$

Determine the values of each of the constants a and b .

, $a=1$, $b=3$

PROCEED AS FOLLOWS

$$\begin{aligned} &\sum_{r=1}^n (3r+a)(r+2) \equiv n(n+2)(n+b) \\ &\sum_{r=1}^n [3r^2 + (6+a)r + 2a] \equiv n^3 + (3+b)n^2 + 2bn \\ &3\sum_{r=1}^n r^2 + (6+a)\sum_{r=1}^n r + 2a\sum_{r=1}^n 1 \equiv n^3 + (3+b)n^2 + 2bn \\ &\rightarrow 3 \times \frac{n}{6}n^2 + (6+a) \frac{n}{2}n + 2a \equiv n^3 + (3+b)n^2 + 2bn \\ &\rightarrow 3 \times \frac{1}{6}(2n)(3n) + (6+a) \times \frac{1}{2}(2n) + 2an \equiv n^3 + (3+b)n^2 + 2bn \\ &\rightarrow \frac{1}{2}n(2n)(3n) + \frac{1}{2}(6+a)n(2n) + 2an \equiv n^3 + (3+b)n^2 + 2bn \\ &\rightarrow n^3 + \frac{3}{2}n^3 + \frac{1}{2}an^2 + \frac{1}{2}(6+a)n^2 + 2an \equiv n^3 + (3+b)n^2 + 2bn \\ &\text{EXPAND THE RHS FULLY} \\ &\rightarrow n^3 + \frac{3}{2}n^3 + \frac{1}{2}an^2 + \frac{1}{2}(6+a)n^2 + 2an \equiv n^3 + (3+b)n^2 + 2bn \end{aligned}$$

LOOKING AT THE COEFFICIENTS OF n^2 & n

$\bullet \frac{3}{2} + \frac{1}{2}(6+a) = 2+b$ $3 + a + 6 = 4+2b$ $a = 2b - 5$	$\bullet \frac{1}{2} + \frac{1}{2}(6+a) + 2a = 2b$ $1 + 6a + 4a = 4b$ $7 + 10a = 4b$ $6a = 4b$ $b = 3$ $\therefore a = 1$
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Question 13 (*)**

Show clearly that

$$(1 \times 3) + (2 \times 4) + (3 \times 5) + \dots + (n-5)(n-3) = \frac{1}{6}(n+6)(2n+11)(n+5).$$

, proof

WRITE IN SIGMA NOTATION

$$(1 \times 3) + (2 \times 4) + (3 \times 5) + \dots + (n-5)(n-3) = \sum_{r=1}^{n-5} r(r+2)$$

SUM TO n INSTEAD OF n-5

$$\sum_{r=1}^n r(r+2) = \sum_{r=1}^n (r^2 + 2r) = \sum_{r=1}^n r^2 + 2 \sum_{r=1}^n r$$

$$= \frac{1}{6}n(n+1)(2n+1) + 2n \cdot \frac{1}{2}n(n+1)$$

$$= \frac{1}{6}n(n+1)(2n+1) + n(n+1)$$

$$= \frac{1}{6}n(n+1)[(2n+1) + 1]$$

$$= \frac{1}{6}n(n+1)(2n+2)$$

Now using 'standard' notation

$$\begin{aligned} f(r) &= \sum_{k=1}^r k(k+2) = \frac{1}{6}r(r+1)(2r+1) \\ f(r-2) &= \sum_{k=1}^{r-2} k(k+2) = \frac{1}{6}(r-2)(r-1)(2(r-2)+1) \\ &= \frac{1}{6}(r-2)(r-1)(2n-3) \end{aligned}$$

AS required

Question 14 (*)**

Use standard results on summations to show that

$$\sum_{r=1}^n (3r^2 + r - 1) \equiv n^2(n+2).$$

, proof

USING THE UNILINEAR PROPERTY OF THE SIGMA OPERATOR

$$\begin{aligned} \sum_{r=1}^n (3r^2 + r - 1) &= \sum_{r=1}^n 3r^2 + \sum_{r=1}^n r - \sum_{r=1}^n 1 \\ &= 3 \sum_{r=1}^n r^2 + \frac{1}{2}n(n+1) - \frac{n}{2} \\ &= 3 \times \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - n \\ &= \frac{1}{2}n(n+1)(2n+1) + n(n+1) - n \\ &= \frac{1}{2}n[(n+1)(2n+1) + (n+1) - 1] \\ &= \frac{1}{2}n[2n^2 + 3n + 1 + n + 1 - 1] \\ &= \frac{1}{2}n[2n^2 + 4n] \\ &= n[2n^2 + 4n] \\ &= n^2(n+2) \end{aligned}$$

As required

Question 15 (***)

Use standard results on summations to show that

$$\sum_{n=1}^k (18n^2 + 28n + 5) = k(k+2)(6k+11).$$

 , proof

SPLIT THE SUM INTO INDIVIDUAL TERMS

$$\begin{aligned} \sum_{n=1}^k (18n^2 + 28n + 5) &= \sum_{n=1}^k (18n^2) + \sum_{n=1}^k (28n) + \sum_{n=1}^k 5 \\ &= 18 \sum_{n=1}^k n^2 + 28 \sum_{n=1}^k n + 5 \sum_{n=1}^k 1 \end{aligned}$$

USING STANDARD RESULTS

- $\sum_{n=1}^k n^2 = \frac{1}{6}k(k+1)(2k+1)$
- $\sum_{n=1}^k n = \frac{1}{2}k(k+1)$
- $\sum_{n=1}^k 1 = k$

SUMMING

$$\begin{aligned} &= 18 \times \frac{1}{6}k(k+1)(2k+1) + 28 \times \frac{1}{2}k(k+1) + 5 \times k \\ &= 3k(k+1)(2k+1) + 14k(k+1) + 5k \\ &= k[3(2k+1)(2k+1) + 14(2k+1) + 5] \\ &= k[6k^2+18k+3+14k+14+5] \\ &= k[6k^2+32k+22] \\ &= k(k+2)(6k+11) \end{aligned}$$

Question 16 (***)

Use standard results on summations to find the value of the following sum.

$$\sum_{k=2}^{16} [(k-1)(k+2)].$$

 , 1600

SPLIT BY FINDING AN EXPRESSION FOR THE SUM OF THE FIRST n TERMS
 (NOTE THAT $k=1$ MEANS ZERO)

$$\begin{aligned} \sum_{k=1}^n [(k-1)(k+2)] &= \sum_{k=1}^n [k^2+3k-2] \\ &= \sum_{k=1}^n k^2 + \sum_{k=1}^n 3k - \sum_{k=1}^n 2 \\ &= \frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) - 2 \times n \\ &= \frac{1}{6}n[(4n+1)(2n+1) + 3(n+1) - 12] \\ &= \frac{1}{6}n[2n^2+3n+1 + 6n+3 - 12] \\ &= \frac{1}{6}n(2n^2+9n-8) \\ &= \frac{1}{3}n(2n^2+9n-8) \\ &= \frac{1}{3}n(n-1)(n+8) \end{aligned}$$

Now letting $n=16$

$$\sum_{k=2}^{16} [(k-1)(k+2)] = \frac{1}{3} \times 16 \times 15 \times 20 = 1600$$

Question 17 (*)**

Use standard results on summations to show that

$$\sum_{r=1}^{2n} r^3 - \sum_{r=1}^n (6r-3)^2 \equiv f(n),$$

where $f(n)$ is written as a product of 4 linear factors.

$$\boxed{\quad}, \boxed{f(n) = n(n-1)(2n+1)(2n-3)}$$

(USING STANDARD SUMMATION RESULTS)

$$\begin{aligned}
 \bullet \sum_{r=1}^k r &= \frac{k}{2}(k+1) \\
 \bullet \sum_{r=1}^k r^2 &= \frac{k}{6}k(k+1)(2k+1) \\
 \bullet \sum_{r=1}^k r^3 &= \frac{k^2}{4}(k+1)^2
 \end{aligned}$$

$$\begin{aligned}
 \sum_{r=1}^{2n} r^3 - \sum_{r=1}^n (6r-3)^2 &= \sum_{r=1}^{2n} r^3 - \left[\sum_{r=1}^n (36r^2 - 36r + 9) \right] \\
 &= \sum_{r=1}^{2n} r^3 - 36 \sum_{r=1}^n r^2 + 36 \sum_{r=1}^n r - 9 \sum_{r=1}^n 1 \\
 &= \frac{1}{4} \sum_{r=1}^{2n} (2r+1)^2 - 36 \sum_{r=1}^n (6r(2r+1) + 36(r(r+1) - 9)) \\
 &= r^2(2n+1)^2 - 6r(2n+1)(2n+1) + 18n(n+1) - 9n
 \end{aligned}$$

REDUCE IN PAIRS

$$\begin{aligned}
 &= n(2n+1) \left[n(2n+1) - 6(2n+1) + 9n \right] \\
 &= n(2n+1) \left[n^2 + 4n - 6n - 6 + 9n \right] \\
 &= n(2n+1) \left(n^2 + 3n - 6 \right) + 9n(2n+1) \\
 &= n(2n+1) \left(n^2 + 3n - 6 + 1 \right) \\
 &= n(2n+1) \left[n^2 + 3n - 6 + 1 \right] \\
 &= n(2n+1) \left(2n^2 + 3n - 5 \right) \\
 &= \boxed{n(2n+1)(2n-1)(n+1)}
 \end{aligned}$$

FURTHER SIMPLIFICATION

$$\begin{aligned}
 &= n^2(2n+1)^2 - 6n(2n+1)(2n+1) + 18n(n+1) - 9n \\
 &= n^2(2n+1)^2 - 12n^2(2n+1) - 12n(2n+1) + 18n(n+1) - 9n \\
 &= n^2(4n^2 + 4n + 1) - 12n^2(2n+1) - 12n(2n+1) + 18n(n+1) - 9n \\
 &\quad \text{OR THIS LOOK THE PRODUCT } (2n+1)(2n-1) \text{ IS A FACTOR OF}
 \end{aligned}$$

SUMMATIONS BY FORMULAS

15 STANDARD QUESTIONS

Question 1 (***)+

Find the sum of the first n terms of the series

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 5 + 3 \cdot 4 \cdot 7 + 4 \cdot 5 \cdot 9 + \dots$$

Express the answer as a product of linear factors.

, proof

Start by writing the series in sigma notation

$$(1 \cdot 2 \cdot 3) + (2 \cdot 3 \cdot 5) + (3 \cdot 4 \cdot 7) + (4 \cdot 5 \cdot 9) + \dots = \sum_{n=1}^{\infty} [(n^2)(2n+1)]$$

Expand & simplify

$$\begin{aligned} \sum_{n=1}^{\infty} [(n^2)(2n+1)] &= \sum_{n=1}^{\infty} [2n^3 + 3n^2 + n] \\ &= 2 \sum_{n=1}^{\infty} n^3 + 3 \sum_{n=1}^{\infty} n^2 + \sum_{n=1}^{\infty} n \end{aligned}$$

Final simplified result

$$\begin{aligned} &= 2n^2(n+1)^2 + 3n^2(n+1)(2n+1) + \frac{1}{2}n^2(2n+1) \\ &= \frac{1}{2}n^2(2n+1)^2 + \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n^2(2n+1) \\ &= \frac{1}{2}n(2n+1)[n(2n+1) + 1] \\ &= \frac{1}{2}n(2n+1)[n^2 + 3n + 2] \\ &= \frac{1}{2}n(2n+1)(n+1)(n+2) \\ &= \underline{\underline{\frac{1}{2}n(n+1)^2(n+2)}} \end{aligned}$$

Question 2 (***)+

By using standard results, show that

$$\sum_{r=n+1}^{4n} (2r-1)^2 \equiv n(84n^2 - 1).$$

, proof

PROCEED AS FOLLOWS

$$f(a) = \sum_{r=1}^n (2r-1)^2 = \sum_{r=1}^n [4r^2 - 4r + 1] = 4\sum_{r=1}^n r^2 - 4\sum_{r=1}^n r + \sum_{r=1}^n 1$$

USING STANDARD RESULTS

$$\begin{aligned} f(a) &= 4 \times \frac{1}{6}n(n+1)(2n+1) - 4 \times \frac{1}{2}n(n+1) + n \\ f(a) &= \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n \\ f(0) &= \frac{2}{3}n \left[2(0+1)(0+1) - 6(0+1) + 2 \right] \\ f(0) &= \frac{2}{3}n \left[4n^2 + 4n + 2 - 6n - 6 \right] \\ f(0) &= \frac{2}{3}n (4n^2 - 2) \end{aligned}$$

Now we have

$$\begin{aligned} \sum_{r=n+1}^{4n} (2r-1)^2 &= \sum_{r=1}^{4n} (2r-1)^2 - \sum_{r=1}^n (2r-1)^2 \\ &= f(4n) - f(n) \\ &= \frac{2}{3}(4n) \left[4(4n^2 - 1) \right] - \frac{2}{3}n \left[4(n^2 - 1) \right] \\ &= \frac{2}{3}n (64n^2 - 4) - \frac{2}{3}n (4n^2 - 4) \\ &= \frac{2}{3}n [256n^2 - 4 - 4n^2 + 4] \\ &= \frac{2}{3}n (252n^2 - 3) \\ &= n(84n^2 - 1) \end{aligned}$$

✓ INCORRECT

Question 3 (*)+**

Determine the value of a and the value of b given that

$$\sum_{r=1}^n r(r+a)(r+b) \equiv \frac{1}{12}n(n+1)(n+2)(3n+17).$$

, $a = 1, b = 4$ or the other way round

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$$\begin{aligned} \sum_{r=1}^n r(r+a)(r+b) &= \frac{1}{12}n(n+1)(n+2)(3n+17) \\ \frac{1}{12} \left[r^3 + (ar+b)r^2 + abr \right] &\equiv \frac{1}{12}n(n+1)(n+2)(3n+17) \\ \sum_r r^3 + (ar+b)\sum_r r^2 + ab\sum_r r &\equiv \frac{1}{12}n(n+1)(n+2)(3n+17) \\ \frac{1}{4}n^2(n+1)^2 + \frac{1}{2}(ab)n(n+1)(n+2) + \frac{1}{2}abn(n+1) &\equiv \frac{1}{12}n(n+1)(n+2)(3n+17) \end{aligned}$$

'DIVIDE' A Factor of $n(n+1)$ ALL THE WAY THROUGH

$$\begin{aligned} \frac{1}{4}n(n+1) + \frac{1}{2}(ab)(n+1) + \frac{1}{2}abn &\equiv \frac{1}{12}(n+2)(3n+17) \quad \checkmark \times 12 \\ 3n(n+1) + 2(ab)n(n+1) + 6abn &\equiv (n+2)(3n+17) \\ 3n^2 + 3n + 4(ab)n^2 + 2(ab)n + 6abn &\equiv 3n^2 + 17n + 2n + 34 \\ 3n^2 + [3 + 4(ab)]n + [2(ab) + 6ab] &\equiv 5n^2 + 23n + 34 \end{aligned}$$

FINDING TWO EQUATIONS

$$\begin{array}{lcl} 3 + 4(ab) = 23 & & 2(ab) + 6ab = 34 \\ 4(ab) = 20 & \rightarrow & 10 + 6ab = 34 \\ ab = 5 & & 6ab = 24 \\ & & ab = 4 \end{array}$$

By inspection, substitution, or polynomial root technique

$a = 1 \quad b = 4 \quad (\text{or the other way round})$

Question 4 (*)+**

Find, in fully factorized form, an expression for the following sum.

$$\sum_{r=n}^{2n} (r^3 - 2r).$$

□,	$\sum_{r=n}^{2n} (r^3 - 2r) = \frac{3}{4}n(5n-4)(n+1)^2$
----	--

USING THE STANDARD SUMMATION FORMULAE

$$\begin{aligned} \sum_{r=1}^k r &= \frac{1}{2}k(2k+1) \\ \sum_{r=1}^k r^2 &= \frac{1}{3}k^2(2k+1) \end{aligned}$$

Once we now take

$$\begin{aligned} \sum_{r=1}^{2n} (r^3 - 2r) &= \sum_{r=1}^{2n} r^3 - 2 \sum_{r=1}^{2n} r \\ &= \left[\frac{1}{4}n^2(2n+1)^2 - \frac{1}{2}(2n)(2n+1) - 2(2n) \right] \\ &= n^2(2n+1)^2 - 2n(2n+1) + 4(n-1) \\ &= \frac{1}{4}n \left[4n(2n+1)^2 - 4(n+1)^2 + 4(n-1) \right] \end{aligned}$$

As it will be a mess to expand to fully write it out to factorise the terms, inside the bracket in pairs,

$$\begin{aligned} &= \frac{1}{4}n \left[n \left[4(2n+1)^2 - (n+1)^2 \right] - 4 \left[4n+2 + n-1 \right] \right] \\ &= \frac{1}{4}n \left[n \left[4(4n^2+4n+1) - (n^2+2n+1) \right] - 4(5n+1) \right] \\ &\quad \text{difference of squares} \\ &= \frac{1}{4}n \left[n \left[2(8n+4) - (n-1) \right] - 4(5n+1) \right] \\ &= \frac{1}{4}n \left[n \left[16n+8 - (n-1) \right] - 4(5n+1) \right] \\ &= \frac{1}{4}n \left[n(15n+9) - (20n+4) \right] \\ &= \frac{1}{4}n \left[3n(5n+3) - 4(5n+1) \right] \end{aligned}$$

ALTERNATIVE BY EXPANDING & CANCELING THE TERMS PRECISELY

$$\begin{aligned} &= \frac{1}{4}n \left[4n(2n+1)^2 - n(n-1)^2 - 8(2n+1) + 4(n-1) \right] \\ &= \frac{1}{4}n \left[16n^2+16n+4n^2-4n^2-2n+8 - 16n-8 + 4n-4 \right] \\ &= \frac{1}{4}n \left[20n^2+8n^2-7n-12 \right] \\ &= \frac{1}{4}n \left[28n^2+8n^2-3n-12 \right] \end{aligned}$$

LOCATING FOR FACTORS

$$\begin{array}{ll} n=1 & 5+4-3-4=0 \\ n=1 & -4+4+3-3=0 \\ & \text{Hence } (n+1) \text{ is a factor} \end{array}$$

LONG DIVIDE

$$\begin{array}{rl} \text{H.R.} & 5n^2+8n^2-3n-4 \\ \text{S.R.} & 5n^2+5n \\ & \cancel{5n^2}-\cancel{5n^2} \\ & n^2-3n-4 \\ & \cancel{n^2}-\cancel{n^2} \\ & -4n-4 \\ & \cancel{-4n}-\cancel{-4n} \end{array} \quad \begin{array}{l} \dots = \frac{3}{4}n(n+1)(5n^2+3n-4) \\ = \frac{3}{4}n(n+1)(5n^2+3n-4)(n+1) \\ = \frac{3}{4}n(n+1)^2(5n^2+3n-4) \\ \text{As required} \end{array}$$

Question 5 (***)+

It is thought that for some values of the constants p and q that

$$\sum_{r=1}^n r^2(r+p) \equiv n(n+1)(n+2)(3n+q).$$

Use a detailed method to show that there exist no such values of p and q .

, proof

EXPAND THE LHS & COEFFICIENTS

$$\begin{aligned} &\Rightarrow \sum_{r=1}^n (r^2 + pr^2) = qn(n+1)(n+2)(3n+1) \\ &\Rightarrow \sum_{r=1}^n (r^2 + p^2r^2) = \sum_{r=1}^n r^2 + p \sum_{r=1}^n r^2 = qn(n+1)(n+2)(3n+1) \\ &\Rightarrow \frac{1}{4}n^2(n+1)^2 + \frac{1}{2}pn(n+1)(2n+1) = qn(n+1)(n+2)(3n+1) \\ &\Rightarrow n(n+1) \left[\frac{1}{4}(n+1)^2 + \frac{1}{2}p(2n+1) \right] = q(n+1)(n+2)(3n+1) \\ &\Rightarrow \frac{1}{4}n(n+1) + \frac{1}{2}p(2n+1) = q(n+1)(3n+1) \\ &\Rightarrow \frac{1}{4}n^2 + \frac{1}{4}n + \frac{1}{2}pn + \frac{1}{2}p = (3n^2 + 7n + 2q) \\ &\equiv \frac{1}{4}n^2 + \left(\frac{1}{4} + \frac{1}{2}p\right)n + \frac{1}{2}p \equiv 3qn^2 + 7qn + 2q \end{aligned}$$

NOW SOLVE FOR EACH POWER

$$\begin{aligned} [n^2]: \quad \frac{1}{4} &= 3q \\ q &= \frac{1}{12} \\ [n]: \quad \frac{1}{4} + \frac{1}{2}p &= 7q \\ \frac{1}{4} + \frac{1}{2}p &= \frac{7}{12} \\ 3 + 4p &= 7 \\ 4p &= 4 \\ p &= 1 \end{aligned}$$

BUT NOW $[n]$ YIELDS INCONSISTENCY SINCE $\frac{1}{2}p = 2q$ $\frac{1}{2}p \neq 2$

Question 6 (****)

Use standard results on summations to solve the following equation.

$$\sum_{r=1}^k (r^3 - 1) = 89976.$$

$$\boxed{ } , k = 24$$

STORY BY GETTING A POLYNOMIAL EQUATION USING STANDARD RESULTS

$$\begin{aligned} \sum_{r=1}^k (r^3 - 1) &= \sum_{r=1}^k r^3 - \sum_{r=1}^k 1 = \frac{1}{4}k^2(2k+1)^2 - k \\ &= \frac{1}{4}k(k(2k+1)^2 - 4) = \frac{1}{4}k(k^2 + 3k^2 + k - 4) \end{aligned}$$

NOW $k=1$ IS AN OBVIOUS "ZERO" OF THE CUBIC, SO $(k-1)$ IS A FACTOR

$$\begin{aligned} &= \frac{1}{4}k[(k-1)(k^2 + 3k^2 + k + 4)] \quad \text{OR USE SYNTHETIC ALGEBRAIC DIVISION} \\ &= \frac{1}{4}k(k-1)(k^2 + 3k + 4) \quad \text{R POSSIBLY} \end{aligned}$$

NOW SOLVING BY TRIAL & ERROR AS k IS A POSITIVE INTEGER

$$\begin{aligned} f(1) &= \frac{1}{4}k(1-1)(1^2 + 3 \cdot 1 + 4) \\ f(2) &= \frac{1}{4}k \cdot 2 \times 9 \times (100 + 3 \cdot 4) = 3015 < 89976 \\ f(3) &= \frac{1}{4}k \cdot 3 \times 27 \times (280 + 3 \cdot 4) = 40560 < 89976 \\ f(4) &= \frac{1}{4}k \cdot 4 \times 64 \times (560 + 3 \cdot 4) = 26195 > 89976 \\ f(5) &= \frac{1}{4}k \cdot 5 \times 125 \times (1056 + 3 \cdot 4) = 105600 > 89976 \\ f(6) &= \frac{1}{4}k \cdot 6 \times 216 \times (1656 + 3 \cdot 4) = 89976 \end{aligned}$$

$\therefore k=24 //$

Question 7 (****)

It is given that

$$\sum_{r=1}^n (Ar^3 + Br^2 + Cr) = n(n+1)(n+2)(4n-5).$$

Use a detailed method to find the value of each of the integer constants, A , B and C .

, $A = 16$, $B = -3$, $C = -19$

$$\sum_{r=1}^n (Ar^3 + Br^2 + Cr) = n(n+1)(n+2)(4n-5)$$

Plotted as follows

$$f(n) = n(n+1)(n+2)(4n-5)$$

$$f(n-1) = (n-1)n(n)(4(n-1)-5) = n(n-1)(n+1)(4n-9)$$

SUBTRACTING we obtain the n^{th} TERM

$$\begin{aligned} \rightarrow f(n) - f(n-1) &= n(n+1)(n+2)(4n-5) - n(n-1)(n+1)(4n-9) \\ \rightarrow U_n &= n(n+1) [(n+2)(4n-5) - (n-1)(4n-9)] \\ \rightarrow U_4 &= n(n+1) [4n^2 + 3n - 10 - 4n^2 + 13n - 9] \\ \rightarrow U_4 &= n(n+1) (16n - 19) \\ \rightarrow U_4 &= n (16n^2 - 19n + 16n - 19) \\ \rightarrow U_4 &= 16n^3 - 3n^2 - 19n \end{aligned}$$

$\therefore A = 16, B = -3, C = -19$

ALTERNATIVE BY EXPANDING COEFFICIENTS

$$\begin{aligned} \rightarrow \sum_{r=1}^n (Ar^3 + Br^2 + Cr) &= n(n+1)(n+2)(4n-5) \\ \rightarrow A \sum_{r=1}^n r^3 + B \sum_{r=1}^n r^2 + C \sum_{r=1}^n r &\equiv n(n+1)(n+2)(4n-5) \\ \rightarrow \frac{1}{4}A(n+1)^2(n+2)^2 + \frac{1}{2}B(n+1)(2n+1) + \frac{1}{2}Cn(n+1) &\equiv n(n+1)(n+2)(4n-5) \\ \rightarrow n(n+1) \left[\frac{1}{4}A(n+1)^2 + \frac{1}{2}B(2n+1) + \frac{1}{2}C \right] &\equiv n(n+1)(n+2)(4n-5) \end{aligned}$$

Question 8 (***)**

Show by a detailed method that

$$\sum_{r=0}^n \left[2r(2r^2 - 3r - 1) + n + 1 \right] = (n^2 - 1)^2.$$

 , proof

EXPAND THE SUMMATION SO USE ONLY STANDARD RESULTS

$$\begin{aligned}
 & \sum_{r=0}^n \left[2r(2r^2 - 3r - 1) + n + 1 \right] \\
 &= \sum_{r=0}^n \left[4r^3 - 6r^2 - 2r + (n+1) \right] \\
 &= 4 \sum_{r=0}^n r^3 - 6 \sum_{r=0}^n r^2 - 2 \sum_{r=0}^n r + \sum_{r=0}^n (n+1) \\
 &\quad \text{THE SUMMATION HAS NO} \\
 &\quad \text{DEPENDENCE ON } r, \text{ SO IT} \\
 &\quad \text{MAY BE PREPARED!}
 \end{aligned}$$

NOTE THE FACT THAT IN THE FIRST 3 SUMMATIONS IS ZERO,
SO WE MAY START THESE SUMMATIONS FROM 1+

$$\begin{aligned}
 &= 4 \sum_{r=1}^n r^3 - 6 \sum_{r=1}^n r^2 - 2 \sum_{r=1}^n r + (n+1) \sum_{r=0}^n 1 \\
 &\quad \text{USING STANDARD BLOCKS} \\
 &= 4 \times \frac{1}{4} n^2 (n+1)^2 - 6 \times \frac{1}{6} n(n+1)(2n+1) - 2 \times \frac{1}{2} n(n+1) + (n+1)n(n+1) \\
 &= n^2 (n+1)^2 - n(n+1)(2n+1) - n(n+1) + (n+1)^2 n(n+1) \\
 &= n(n+1) \left[n(n+1) - (2n+1) \right] + (n+1)^2 n(n+1) \\
 &= n(n+1) \left[n^2 - n - 2 \right] + (n+1)^2 n(n+1) \\
 &= n(n+1) (n+1)(n-2) + (n+1)^2 n(n+1) \\
 &= (n+1)^2 \left[n(n-2) + 1 \right] \\
 &= (n+1)^2 (n^2 - 2n + 1) \\
 &= (n+1)^2 (n-1)^2 = [(n+1)(n-1)]^2 \\
 &= (n^2 - 1)^2
 \end{aligned}$$

RECALL THAT TO
DO THIS, THREE
ARE THE THREE

AS REQUIRED

Question 9 (*****)

The sum, S_n , of the first n terms of a series whose general term is denoted by u_n is given by the following expression.

$$S_n = n^2(n+1)(n+2).$$

a) Find the first term of the series.

b) Show clearly that ...

i. ... $u_n = n(n+1)(4n-1)$

ii. ... $\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2).$

, $u_1 = 6$

a) TRIVIALLY WE KNOW
 $u_1 = S_1 = 1^2(1+1)(1+2) = 1 \times 2 \times 3 = 6$

b) USING $S_n - S_{n-1} = u_n$

$$\begin{aligned} \Rightarrow u_1 &= 1^2(1+1)(1+2) - (1-1)^2[1(n-1)+1][2(n-1)+2] \\ \Rightarrow u_2 &= 2^2(2+1)(2+2) - 1^2(1+1)[2(2-1)+2] \\ \Rightarrow u_3 &= 3^2(3+1)[3(3-1)+2] \\ \Rightarrow u_4 &= 4^2(4+1)[4(4-1)+2] \\ \Rightarrow u_5 &= 5^2(5+1)[5(5-1)+2] \end{aligned}$$

At 240 words

$$\begin{aligned} \sum_{r=n+1}^{2n} u_r &= S_{2n} - S_n \\ &= (2n)^2(2n+1)(2n+2) - n^2(n+1)(n+2) \\ &= 4n^2(2n+1) \times 2(n+1) - n^2(n+1)(n+2) \\ &= n^2(n+1) [8(2n+1) - (n+2)] \\ &= n^2(n+1) (16n + 8 - n - 2) \\ &= n^2(n+1) (15n + 6) \\ &= 3n^2(n+1)(5n+2) \end{aligned}$$

At 240 words

Question 10 (****)

Use standard summation results to prove that

$$\sum_{r=1}^n (n-r)^2 = \frac{1}{2}n(n-1)(2n-1).$$

 proof

FIRSTLY LET US NOTE THAT $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$

NEXT PROCEED AS FOLLOWS

$$\begin{aligned}\sum_{r=1}^n (n-r)^2 &= \sum_{r=1}^n (N^2 - 2Nr + r^2) \\&\rightarrow \sum_{r=1}^n (N-r)^2 = N^2 \sum_{r=1}^n 1 - 2N \sum_{r=1}^n r + \sum_{r=1}^n r^2 \\&\rightarrow \sum_{r=1}^n (N-r)^2 = N^2 \times N - 2N \times \frac{1}{2}N(N+1) + \frac{1}{6}N(N+1)(2N+1) \\&\Rightarrow \sum_{r=1}^n (N-r)^2 = N^2 - N^2 + \frac{1}{6}N(N+1)(2N+1) \\&\rightarrow \sum_{r=1}^n (N-r)^2 = \frac{1}{6}N [(N+1)(2N+1) - 6N] \\&\rightarrow \sum_{r=1}^n (N-r)^2 = \frac{1}{6}N [2N^2 + 3N + 1 - 6N] \\&\rightarrow \sum_{r=1}^n (N-r)^2 = \frac{1}{6}N [2N^2 - 3N + 1] \\&\rightarrow \sum_{r=1}^n (N-r)^2 = \frac{1}{6}N (2n-1)(2n-1)\end{aligned}$$

Question 11 (**)**

Use standard results on summations to solve the following equation

$$\sum_{r=3}^9 \left[\left(\frac{r}{k} \right)^3 + (r-1)(r+1) \right] = 304.5.$$

$$\boxed{k}, \quad k = 4$$

WORK IN SECONDS

$$\sum_{r=3}^9 \left[\left(\frac{r}{k} \right)^3 + (r-1)(r+1) \right] = \sum_{r=3}^9 \left[\frac{r^3}{k^3} + r^2 - 1 \right] = \sum_{r=3}^9 \frac{r^3}{k^3} + \sum_{r=3}^9 (r^2 - 1)$$

SIMPLIFY AND SIMPLIFY EACH TERM USING $\frac{3}{k} r^2 = \frac{1}{4} k^2 (r+1)^2$

$$\begin{aligned} \sum_{r=3}^9 \frac{r^3}{k^3} &= \frac{1}{k^3} \sum_{r=3}^9 r^3 \\ &= \frac{1}{k^3} \left[\sum_{r=1}^9 r^3 - \sum_{r=1}^2 r^3 \right] \\ &= \frac{1}{k^3} \left[\frac{1}{4} k^2 (k+1)^2 - \frac{1}{4} k^2 (3+2)^2 \right] \\ &= \frac{1}{k^3} \times 204 \end{aligned}$$

SIMILARLY USING $\frac{3}{k^2} r^2 = \frac{1}{4} k^2 (r+1)^2$

$$\begin{aligned} \sum_{r=3}^9 (r^2 - 1) &= \sum_{r=3}^9 r^2 - \sum_{r=3}^9 1 \\ &= \frac{1}{k^2} r^2 - \sum_{r=3}^9 1 \\ &= \frac{1}{k^2} \times 10 \times 19 - (3^2 + 2^2) - 7 \\ &= 285 - 5 - 7 \\ &= 273 \end{aligned}$$

FINALLY WE HAVE

$$\frac{204}{k^3} + 273 = 304.5 \Rightarrow \frac{204}{k^3} = \frac{304.5 - 273}{2} \Rightarrow 204^2 = 4032 \Rightarrow k^3 = 4 \Rightarrow \boxed{k = 4}$$

Question 12 (****)

$$\sum_{r=1}^n (ar^2 + br + c) = n^3 + 5n^2 + 6n,$$

where a , b and c are integer constants.

Determine the value of a , b and c .

$$[a = 3], [b = 7], [c = 2]$$

$$\begin{aligned}
 \sum_{r=1}^n ar^2 + br + c &= a\sum_{r=1}^n r^2 + b\sum_{r=1}^n r + c\sum_{r=1}^n 1 \\
 &= \frac{a}{3}(n(n+1)(2n+1)) + \frac{b}{2}n(n+1) + cn \\
 &= \frac{1}{6}n [2an^2 + 3an + a + 3bn^2 + 3bn + 6c] \\
 &= \frac{1}{6}n [2an^2 + 3n(a+b) + (a+3b+6c)] \\
 &= \frac{1}{6}an^3 + \frac{1}{2}(a+b)n^2 + \frac{1}{6}(a+3b+6c)n
 \end{aligned}$$

Now $[m(n+2)(m+3)] = n^3 + 5n^2 + 6n$] compare

$\frac{1}{6}a = 1$	$\frac{1}{2}(a+b) = 5$	$\frac{1}{6}(a+3b+6c) = 6$
$\therefore a = 3$	$a+b = 10$	$a+3b+6c = 36$
	$b = 7$	$3+21+6c = 36$
		$c = 2$

Question 13 (*)**

The variance $\text{Var}(n)$ of the first n natural numbers is given by

$$\text{Var}(n) = \frac{1}{n} \sum_{r=1}^n r^2 - \left[\frac{1}{n} \sum_{r=1}^n r \right]^2.$$

Determine a simplified expression for $\text{Var}(n)$ and hence evaluate $\text{Var}(61)$.

$$\boxed{\text{Var}(n) = \frac{1}{12}(n^2 - 1)}, \quad \boxed{\text{Var}(61) = 310}$$

Variance $= \frac{\sum_{r=1}^n r^2}{n} - \left(\frac{\sum_{r=1}^n r}{n} \right)^2$

$$= \frac{\frac{1}{2}n(n+1)(2n+1)}{n} - \left(\frac{\frac{1}{2}n(n+1)}{n} \right)^2$$

$$= \frac{1}{2}(n+1)(2n+1) - \frac{1}{4}(n+1)^2$$

$$= \frac{1}{12}(n+1)[2(2n+1) - 3(n+1)]$$

$$= \frac{1}{12}(n+1)(4n+2 - 3n - 3)$$

$$= \frac{1}{12}(n+1)(n-1)$$

If $n=61$

$$\text{Var}(61) = \frac{1}{12} \times 62 \times 60 = 62 \times 5 = 310 //$$

Question 14 (****)

$$f(n) = \sum_{r=1}^n [r^3 - r], \quad n \in \mathbb{N}.$$

a) Use standard summation results to find a fully factorized expression for $f(n)$.

b) Hence solve the equation

$$\sum_{r=5}^{10} [r^3 - r + 6k] - \sum_{r=1}^{12} [r^2 + k^2] = 70$$

$$\boxed{\quad}, \boxed{f(n) = \frac{1}{4}n(n-1)(n+1)(n+2)}, \boxed{k = -12, k = 15}$$

a) $\sum_{r=1}^n (r^3 - r) = \sum_{r=1}^n r^3 - \sum_{r=1}^n r = \frac{1}{4}n^2(2n+1)^2 - \frac{1}{2}n(n+1)$

$$= \frac{1}{4}n(n+1)[2n(n+1) - 2] = \frac{1}{4}n(n+1)(2n^2 + 2n - 2)$$

$$= \frac{1}{4}n(n+1)(n-1)(n+2)$$

b) CALCULATE IN SECTIONS

$$\Rightarrow \sum_{r=5}^9 [r^3 - r + 6k] - \sum_{r=1}^{12} [r^2 + k^2] = 70$$

$$\Rightarrow \sum_{r=5}^{10} [r^3 - r] + 6k \sum_{r=5}^9 1 - \sum_{r=1}^{12} r^2 - \sum_{r=1}^{12} k^2 = 70$$

$$\Rightarrow \left[\sum_{r=1}^{10} (r^3 - r) - \sum_{r=1}^4 (r^3 - r) \right] + 6k \left(\underbrace{(1+1+\dots+1)}_9 \right) - \sum_{r=1}^{12} r^2 - \sum_{r=1}^{12} k^2 = 70$$

$$\Rightarrow \frac{1}{4} \times 9 \times 10 \times 11 \times 12 - \frac{1}{4} \times 3 \times 4 \times 5 \times 6 + 6k \times 6 - \frac{1}{6} \times 12 \times 13 \times 25 - k^2 \times 12 = 70$$

$$\Rightarrow 270 - 90 + 36k - 650 - 12k^2 = 70$$

$$\Rightarrow 0 = 12k^2 - 36k - 260$$

$$\Rightarrow k^2 - 3k - 20 = 0$$

$$\Rightarrow (k-5)(k+4) = 0$$

$$\Rightarrow k = \begin{cases} 5 \\ -4 \end{cases}$$

Question 15 (****)

The function $F(n)$ is defined as

$$F(n) = \sum_{r=1}^n [r(r-1)(n-2)(r+1)] \quad n \in \mathbb{N}.$$

Show with detailed workings that

$$F(2n) - F(n) = \frac{1}{2}n(n^2 - 1)(3n^2 - 4).$$

, proof

SUM THE SERIES AND SIMPLIFY RESULTS

$$\begin{aligned} F(n) &= \sum_{r=1}^n [r(r-1)(n-2)(r+1)] = (n-2) \sum_{r=1}^n [r(r-1)(r+1)] \\ &= (n-2) \sum_{r=1}^n (r^3 - r) = (n-2) \left[\sum_{r=1}^n r^3 - \sum_{r=1}^n r \right] \\ (\text{Simplifying}) \quad \sum_{r=1}^n r^3 &= \frac{1}{4}n^2(n+1)^2 \quad \text{and} \quad \sum_{r=1}^n r = \frac{1}{2}n(n+1) \\ \Rightarrow F(n) &= (n-2) \left[\frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1) \right] \\ \Rightarrow F(n) &= (n-2) \times \frac{1}{4}n(n+1) \left[n(n+1) - 2 \right] \\ \Rightarrow F(n) &= \frac{1}{4}(n-2)n(n+1)(n^2+4n-2) \\ \Rightarrow F(n) &= \frac{1}{4}n(n-1)(n+2)(n^2+4n-2) \\ \Rightarrow F(n) &= \frac{1}{4}n(n^2-1)(n^2+4n-4) \\ \\ \text{Simplifying we have} \\ F(2n) - F(n) &= \frac{1}{4}(2n) \left[(2n)^2-1 \right] \left[(2n)^2-4 \right] - \frac{1}{4}n(n^2-1)(n^2-4) \\ &= \frac{1}{4} \left[2n(4n^2-1)(4n^2-4) - n(n^2-1)(n^2-4) \right] \\ &= \frac{1}{4} \left[8n(4n^2-1)(n^2-1) - n(n^2-1)(n^2-4) \right] \\ &= \frac{1}{4}n(n^2-1) \left[8(4n^2-1) - (n^2-4) \right] \\ &= \frac{1}{4}n(n^2-1) \left[32n^2-8 - n^2+4 \right] \\ &= \frac{1}{4}n(n^2-1) \left[31n^2-4 \right] \quad \cancel{\text{cancel}} \end{aligned}$$

SUMMATIONS BY FORMULAS

5 HARD QUESTIONS

Question 1 (****+)

It is given that

$$\sum_{r=1}^{20} (r-10) = 200 \quad \text{and} \quad \sum_{r=1}^{20} (r-10)^2 = 2800.$$

Find the value of

$$\sum_{r=1}^{20} r^2.$$

$$\boxed{}, \quad \boxed{\sum_{r=1}^{20} r^2 = 8800}$$

USING THE LINEARITY OF THE SIGMA OPERATOR

$$\sum_{r=k}^n [2f(r) + r g(r)] \equiv 2 \sum_{r=k}^n f(r) + \sum_{r=k}^n g(r)$$

MANIPULATING THE FIRST FACT

$$\begin{aligned} \sum_{r=1}^{20} (r-10) &= 200 \\ \sum_{r=1}^{20} r - 10 \sum_{r=1}^{20} 1 &= 200 \\ \sum_{r=1}^{20} r - 10 \times 20 &= 200 \\ \sum_{r=1}^{20} r &= 400 \end{aligned}$$

FINALLY USING THE SECOND FACT

$$\begin{aligned} \sum_{r=1}^{20} (r-10)^2 &= 2800 \\ \sum_{r=1}^{20} (r^2 - 20r + 100) &= 2800 \\ \sum_{r=1}^{20} r^2 - 20 \sum_{r=1}^{20} r + 100 \sum_{r=1}^{20} 1 &= 2800 \\ \sum_{r=1}^{20} r^2 - 20 \times 400 + 100 \times 20 &= 2800 \\ \sum_{r=1}^{20} r^2 - 8000 + 2000 &= 2800 \\ \sum_{r=1}^{20} r^2 &= 8800 \end{aligned}$$

Question 2 (****+)

$$\sum_{r=1}^n (r+a)(r+b) \equiv \frac{1}{3}n(n-1)(n+4),$$

where a and b are integer constants.

Use a clear algebraic method to determine the value of a and the value of b .

 , 2 and -1 (in any order)

USING STANDARD RESULTS & THE PROPERTY OF THE SIGMA OPERATOR

$$\begin{aligned}
 & \sum_{r=1}^n (r+a)(r+b) \equiv \frac{1}{3}n(n-1)(n+4) \\
 \Rightarrow & \sum_{r=1}^n [r^2 + (a+b)r + ab] \equiv \frac{1}{3}n(n-1)(n+4) \\
 \Rightarrow & \sum_{r=1}^n r^2 + (a+b)\sum_{r=1}^n r + ab\sum_{r=1}^n 1 \equiv \frac{1}{3}n(n-1)(n+4) \\
 \Rightarrow & \frac{1}{6}n(n+1)(2n+1) + (a+b)\frac{1}{2}n(n+1) + abn \equiv \frac{1}{3}n(n-1)(n+4) \\
 \Rightarrow & n(n)(2n+1) + 3(a+b)n(n+1) + 6abn \equiv 2n(n-1)(n+4) \\
 \text{DIVIDING BY } n(n-1)(n+4) \text{ AND EXPANDING BOTH SIDES} \\
 \Rightarrow & (n)(2n+1) + 3(a+b)(n+1) + 6ab \equiv 2(n-1)(n+4) \\
 \Rightarrow & 2n^2 + 3n + 1 + 3(a+b)n + 3(a+b) + 6ab \equiv 2n^2 + 8n - 8 \\
 \Rightarrow & 3(a+b)n + 6ab \equiv 3n - 9 \\
 \Rightarrow & 3(a+b)n + 3(a+b) + 6ab \equiv 3n - 9 \\
 \therefore & \frac{3(a+b) = 3}{a+b = 1} \quad \begin{array}{l} 3(a+b) + 6ab = 3 \\ a+b + 2ab = -3 \\ 1 + 2ab = -3 \\ 2ab = -4 \\ \boxed{ab = -2} \end{array} \\
 \text{BY INSPECTION OR SOLVING WE OBTAIN } & a, b = -1, \text{ IN ANY ORDER, AS EQUATIONS ARE SYMMETRIC}
 \end{aligned}$$

Question 3 (*****)

By using an algebraic method, find the value of

$$99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2$$

, 5000

Method A

RECOGNISE THE TRINOMIAL

$$\begin{aligned} & 99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2 \\ &= [99^2 + 95^2 + 93^2 + \dots + 3^2] - [97^2 + 95^2 + 93^2 + \dots + 1^2] \\ &= \sum_{r=1}^{25} (4r-1)^2 - \sum_{r=1}^{25} (4r-3)^2 \quad (\text{TERM IN SIGN: } \text{+---+---}) \\ &= \sum_{r=1}^{25} [(4r-1)^2 - (4r-3)^2] \quad (\text{COMMON DIFFERENCE}) \\ &= \sum_{r=1}^{25} (4r-1 + 4r-3)(4r-1 - 4r+3) \quad (\text{DIFFERENCE OF SQUARES}) \\ &= \sum_{r=1}^{25} (8r-4) \times 2 \\ &= \sum_{r=1}^{25} (8r-8) \\ &= 8 \sum_{r=1}^{25} r - 8 \sum_{r=1}^{25} 1 \quad (\text{COMMON DIFFERENCE}) \\ &= 8 \times \frac{25}{2} \times 25 \times 26 - 8 \times 25 \\ &= 5000 - 200 \\ &= 5000 \end{aligned}$$

Method B

RECOGNISE THE TRINOMIAL AS FOLLOWING

$$\begin{aligned} & 99^2 - 97^2 + 95^2 - 93^2 + \dots + 3^2 - 1^2 \\ &= (99^2 - 97^2) + (95^2 - 93^2) + (91^2 - 89^2) + \dots + (3^2 - 1^2) \\ &= (99-97)(99+97) + (95-93)(95+93) + (91-89)(91+89) + \dots + (3-1)(3+1) \\ &= 2(98) + 2(96) + 2(94) + 2(92) + \dots + 2(4) \\ &= 2[4 + 12 + 20 + \dots + 100 + 108 + 116] \\ &= 2 \times 4 \left[1 + 3 + 5 + \dots + 45 + 47 + 49 \right] \\ &\rightarrow 8 \sim \text{Arithmetic Progression with } a=1, d=2, n=25 \quad U_n = a + (n-1)d \\ & \qquad \qquad \qquad 49 = 1 + 24 \times 2, \quad 20 = 2a, \quad 25 = 25 \\ &= 8 \times \frac{25}{2} [1+49] \\ &= 8 \times \frac{25 \times 50}{2} \\ &= 5000 \end{aligned}$$

ANSWER

Question 4 (***)+

Show clearly that

$$1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 = -33200.$$

S, proof

Method A

$$\begin{aligned} 1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 &= (1^3 + 2^3 + 3^3 + 4^3 + \dots + 39^3) - (2^3 + 4^3 + 6^3 + \dots + 40^3) \\ &= \sum_{i=1}^{20} (2i-1)^3 - \sum_{i=1}^{20} (2i)^3 \\ &= \sum_{i=1}^{20} [(2i-1)^3 - (2i)^3] \\ &= \sum_{i=1}^{20} [8i^3 - 12i^2 + 6i - 1] \\ &= -12 \sum_{i=1}^{20} i^2 + 6 \sum_{i=1}^{20} i - \sum_{i=1}^{20} 1 \end{aligned}$$

USING STANDARD SUMMATION RESULTS

$$\sum_{i=1}^n i^2 = \frac{1}{6} n(n+1)(2n+1) \quad \sum_{i=1}^n i = \frac{1}{2} n(n+1)$$

$$\begin{aligned} \dots &= -12 \times \frac{1}{6} \times 20 \times 21 \times 41 + 6 \times \frac{1}{2} \times 20 \times 21 - 20 \\ &= -2 \times 20 \times 21 \times 41 + 3 \times 20 \times 21 - 20 \\ &= 20 [-2 \times 21 \times 41 + 3 \times 21 - 1] \\ &\approx 20 [21(-84) + 3] \\ &= 20 [-1660] \\ &= -33200 \end{aligned}$$

Method B

$$\begin{aligned} 1^3 - 2^3 + 3^3 - 4^3 + \dots - 40^3 &= (1^3 + 2^3 + 3^3 + \dots + 40^3) - 2(2^3 + 4^3 + \dots + 40^3) \\ &= \sum_{i=1}^{40} i^3 - 2 \sum_{i=1}^{20} (2i)^3 \\ &= \frac{40}{3} i^3 - 16 \sum_{i=1}^{20} i^3 \end{aligned}$$

USING THE STANDARD SUMMATION FORMULA

$$\sum_{i=1}^n i^3 = \frac{1}{4} n^2 (n+1)^2$$

$$\begin{aligned} \dots &= \frac{1}{4} \times 40^2 \times 41^2 - 16 \times \frac{1}{4} \times 20^2 \times 21^2 \\ &= (\frac{1}{4} \times 40^2 \times 41^2 - 4 \times 20^2 \times 21^2) \\ &= 20^2 \times 41^2 - 4 \times 20^2 \times 21^2 \\ &= 20^2 [41^2 - 4 \times 21^2] \end{aligned}$$

QUICK CALCULATIONS

41	21	441	1764	83
41	21	441	1764	83
16	12	144	1764	83
8	6	64	1764	83
1681	1296	144	1764	83

$$\begin{aligned} \dots &= 400 (1681 - 4 \times 1296) \\ &= 400 (1681 - 1764) \\ &= 400 \times (-83) \\ &= -33200 \end{aligned}$$

Question 5 (***)**

The positive integer functions f and g are defined as

$$f(n) = \sum_{r=1}^n r^3 \quad \text{and} \quad g(n) = 1 + \sum_{r=1}^n (2r+1).$$

Evaluate

$$\sum_{n=1}^{39} \left[\frac{f(n)}{g(n)} \right].$$

S1, [5135]

$f(n) = \sum_{r=1}^n r^3$ $g(n) = 1 + \sum_{r=1}^n (2r+1)$

DEFINITION THE "INDIVIDUAL COMPONENTS" IS SIMPLIFIED FORM.

$\bullet f(n) = \sum_{r=1}^n r^3 = \frac{1}{4}n^2(2n+1)^2$

$\bullet g(n) = 1 + \sum_{r=1}^n (2r+1) = 1 + 2\sum_{r=1}^n r + \sum_{r=1}^n 1$

$= (1 + 2 \times \frac{1}{2}n(n+1)) + n$

$= (1 + n(n+1)) + n = (1+n^2+n+n)$

$= n^2 + 2n + 1 = (n+1)^2$

HENCE WE HAVE

$$\sum_{n=1}^{39} \frac{f(n)}{g(n)} = \sum_{n=1}^{39} \frac{\frac{1}{4}n^2(2n+1)^2}{(n+1)^2} = \sum_{n=1}^{39} \frac{1}{4}n^2$$

$$= \frac{1}{4} \times \frac{1}{6}n(n+1)(2n+1) \Big|_{n=39}$$

$$= \frac{1}{24} \times 39 \times 40 \times 79$$

$$= 5135$$

SUMMATIONS BY FORMULAS

8 ENRICHMENT QUESTIONS

Question 1 (*****)

Use standard summation results to prove that

$$\sum_{r=n}^{2n} (n-r)^2 = \sum_{r=1}^n r^2.$$

 , proof

EXPAND AND Tidy

$$\begin{aligned}
 \sum_{r=1}^{2n} (n-r)^2 &= \sum_{r=1}^{2n} (n^2 - 2nr + r^2) \\
 &= n^2 \sum_{r=1}^{2n} 1 - 2n \sum_{r=1}^{2n} r + \sum_{r=1}^{2n} r^2 \\
 &= n^2 [2n - (n+1)] - 2n \left[\frac{1}{2}n(2n+1) - \frac{1}{2}(n-1)(n+1) \right] \\
 &\quad + \frac{1}{6}(n(n+1))(2n+1) - \frac{1}{6}(n-1)(n+1)(2n-2) \\
 &= n^2(n+1) - 2n \left[n(2n+1) - \frac{1}{2}n(n-1) \right] + \frac{1}{6}n(n+1)(2n+1) - \frac{1}{6}n(n-1)(2n-2) \\
 &= n^2(n+1) - 2n^2(2n+1) + n^2(n-1) + \frac{1}{6}n(n+1)(2n+1) - \frac{1}{6}n(n-1)(2n-2)
 \end{aligned}$$

Factorise $\frac{1}{6}n$ at first

$$\begin{aligned}
 &= \frac{1}{6}n \left[6n(n+1) - 12n(2n+1) + 2(n(n-1) + 2(2n+1)(2n+1) - (n-1)(2n-1)) \right] \\
 &= \frac{1}{6}n \left[6n^2 + 6n - 24n^2 - 24n + 4n^2 + 4n + 12n^2 + 12n - 3n + 3n - 1 \right] \\
 &= \frac{1}{6}n \left[2n^2 + 3n + 1 \right] \\
 &= \frac{1}{6}n(n+1)(2n+1) \\
 &= \frac{1}{6}n r^2
 \end{aligned}$$

As required

Question 2 (*****)

Find the sum of the first 16 terms of the following series.

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \frac{1^3 + 2^3 + 3^3 + 4^3}{1+3+5+7} + \dots$$

 [446]

Start by writing the above expression compactly

$$\sum_{N=1}^{16} \left[\frac{\frac{1}{N} \sum_{k=1}^N k^3}{\frac{1}{2}N(N+1)} \right] = \sum_{N=1}^{16} \left[\frac{\frac{1}{N} \sum_{k=1}^N k^3}{\frac{1}{2}N^2(N+1)} \right]$$

Using standard summation formulae

$$= \sum_{N=1}^{16} \left[\frac{\frac{1}{N} N^2(N+1)^2}{\frac{1}{2}N^2(N+1)} \right] = \sum_{N=1}^{16} \frac{1}{2}(N+1)^2$$

$$= \sum_{N=1}^{16} \left[\frac{\pm N^2(N+1)^2}{N^2} \right] = \frac{1}{4} \sum_{N=1}^{16} (N+1)^2$$

either expand and use formulae again or "translate" by 1 step

$$\dots = \frac{1}{4} \sum_{k=2}^{17} k^2 = \frac{1}{4} \times \frac{1}{6} \times k(k+1)(2k+1) \Big|_{k=17} - \frac{1}{4}$$

$$= \frac{1}{4} \times \frac{1}{6} \times 17 \times 18 \times 35 - \frac{1}{4}$$

$$= \frac{1}{4} \left[\frac{1}{6} \times 17 \times 18 \times 35 - 1 \right]$$

$$= \underline{\underline{446}}$$

Question 3 (*****)

The function f is defined for $n \in \mathbb{N}$ as

$$f(n) \equiv 1 \times n^2 + 2(n-1)^2 + 3(n-2)^2 + 4(n-3)^2 + \dots + (n-1) \times 2^2 + n \times 1^2.$$

Determine a simplified expression for the sum of $f(n)$, giving the final answer in fully factorized form.

, $f(n) = \frac{1}{12}n(n+2)(n+1)^2$

1 $\times n^2 + 2(n-1)^2 + 3(n-2)^2 + 4(n-3)^2 + \dots + (n-1) \times 2^2 + n \times 1^2$

START BY WRITING THE SUM IN SIGMA NOTATION

$$\sum_{r=1}^n [(Cr+1)r]^2 = \sum_{r=1}^n [r^2 (Cr+1)^2 - 2(Cr+1)r + r^2]$$

$$= \sum_{r=1}^n [(Cr+1)^2 r^2 - 2(Cr+1)r + r^2]$$

$$= (Cr+1)^2 \sum_{r=1}^n r^2 - 2(Cr+1) \sum_{r=1}^n r^2 + \sum_{r=1}^n r^2$$

USING STANDARD SUMMATION RESULTS, WE HAVE

$$\sum_{r=1}^n [(Cr+1)r]^2 = (Cr+1)^2 \sum_{r=1}^n r^2 - 2(Cr+1) \times \sum_{r=1}^n (Cr+1)(Cr+2) + \frac{1}{3} \sum_{r=1}^n r^3$$

$$= \sum_{r=1}^n Cr^2 + \sum_{r=1}^n Cr^3 + \sum_{r=1}^n Cr^2 + \sum_{r=1}^n Cr^3$$

$$= \frac{1}{12}n(Cr+1)^2 \left[6(Cr+1) - 4(Cr+1) + 3n \right]$$

$$= \frac{1}{12}n(Cr+1)^2 (6n+6 - 8n + 3n)$$

$$= \frac{1}{12}n(Cr+1)^2 (n+2)$$

Question 4 (*****)

Use an algebraic method justifying each step, to find the greatest value of k , $k \in \mathbb{N}$, which satisfies the following inequality.

$$\sum_{r=k+1}^{80} \left[\frac{r-1}{\log_8(16)} \right] > 100\,000.$$

, $k = 48$

Start by manipulating the LHS:

$$\frac{1}{\log_8 16} = \log_k 8^r = r \log_k 8 = r \times \frac{3}{4}$$

$\boxed{\log_b a = \frac{1}{\log_a b}}$

SINCE $16^{\frac{3}{4}} = 8$

Summing from $r=1$ to 80 , to get a general expression:

$$\begin{aligned} \sum_{r=1}^{80} \left[\frac{r-1}{\log_8 16} \right] &= \sum_{r=1}^{80} \left[\frac{3}{4}(r-1) \right] \\ &= \frac{3}{4} \sum_{r=1}^{80} [r^2 - r] \\ &= \frac{3}{4} \left[\frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) \right] \\ &= \frac{3}{4} \left[\frac{1}{6}n(n+1)[(2n+1) - 3] \right] \\ &= \frac{1}{8}n(n+1)(2n-2) \\ &= \frac{1}{8}n(n+1)(n-1) \\ &= \frac{1}{8}n(n^2-1) \end{aligned}$$

Returning to the inequality:

$$\begin{aligned} \Rightarrow \sum_{r=k+1}^{80} \left[\frac{r-1}{\log_8 16} \right] &> 100\,000 \\ \Rightarrow \frac{1}{8} \left[80(8k^2-1) \right] - \frac{1}{8}k(k^2-1) &> 100\,000 \end{aligned}$$

$$\begin{aligned} \Rightarrow 80(8k^2-1) - k(k^2-1) &> 400\,000 \\ \Rightarrow k(3k^2-1) - 80(8k^2-1) &< -400\,000 \\ \Rightarrow k^3 - k - 511920 &< -400\,000 \\ \Rightarrow k(k+1)(k-1) &< 111920 \end{aligned}$$

$k \in \mathbb{N}$ but if $k \in \mathbb{R}$

So $f(x) = k(x+1)(x-1)$ is increasing for $k > 0$

By trial & error noting that $f(x) \approx x^3$

$$\begin{cases} f(40) = 40 \times 41 \times 39 = 63960 < 111920 \\ f(50) = 50 \times 51 \times 49 = 124950 > 111920 \\ f(48) = 48 \times 49 \times 48 = 117600 > 111920 \\ f(46) = 46 \times 47 \times 46 = 110544 < 111920 \end{cases} \therefore k = 48$$

Question 5 (*****)

Use algebra to find the sum of the first 100 terms of the following sequence.

$$7, 12, 19, 28, 39, 52, 67, 84, 103, \dots$$

, $f(n) = \frac{1}{12}n(n+2)(n+1)^2$

• INVESTIGATING THE PATTERN FURTHER BY DIFFERENCING

AS THE SECOND DIFFERENCES ARE CONSTANT, THIS IS A QUADRATIC PATTERN, WHERE QUADRATIC COEFFICIENT IS HALF THE CONSTANT SECOND DIFFERENCE, i.e. $U_n = \frac{1}{2}n^2 + cn + b$

• SUPPOSE THE n^{th} TERM OF THE SEQUENCE/SERIES WAS JUST n^2

n^2	1	4	9	16	25	36
"OUR SERIES"	7	12	19	28	39	52
	+6	+8	+10	+12	+14	+16
						$\leftarrow 2n+4$

• HENCE THE REQUIRED n^{th} TERM IS

$$U_n = n^2 + 2n + 4$$

• THIS WE REQUIRE TO FIND

$$\sum_{n=1}^{100} (n^2 + 2n + 4) \quad \text{WHEN } k=100$$

• USING THE STANDARD SUMMATION FORMULAE IN K AND SUBSTITUTE $K=100$ AT THE END

$$\begin{aligned} \sum_{n=1}^k (n^2 + 2n + 4) &\approx \sum_{n=1}^k n^2 + 2 \sum_{n=1}^k n + 4 \sum_{n=1}^k 1 \\ &= \frac{1}{6}k(k+1)(2k+1) + 2 \times \frac{1}{2}k(k+1) + 4k \\ &= \frac{1}{6}k(k+1)[(2k+1)+6] + 4k \\ &= \frac{1}{6}k(k+1)(2k+7) + 4k \end{aligned}$$

• LET $K=100$ AND WE OBTAIN

$$\sum_{n=1}^{100} (n^2 + 2n + 4) = \frac{1}{6} \times 100 \times 101 \times 207 + 4 \times 100 = 348\,830$$

Question 6 (***)**

Evaluate the following expression

$$\sum_{n=1}^9 \sum_{m=n+1}^{2n} [2m+n].$$

Detailed workings must be shown.

V, , 1185

WORKED SOLUTION

PROCEEDED AS FOLLOWS

$$\sum_{n=1}^9 \left[\sum_{m=n+1}^{2n} (2m+n) \right] = \sum_{n=1}^9 \left[2 \sum_{m=n+1}^{2n} m + \sum_{n=1}^9 n \right]$$

USING STANDARD SUMMATION RESULTS

$$= \sum_{n=1}^9 \left[2 \times \frac{1}{2} (2n)(2n+1) - 2 \times \frac{1}{2} n(n+1) + n(2n-n) \right]$$
$$= \sum_{n=1}^9 \left[4n^2 + 2n - n^2 - n + 2n^2 \right] = \sum_{n=1}^9 (4n^2 + 4n)$$
$$= 4 \sum_{n=1}^9 n^2 + \sum_{n=1}^9 4n$$

FINALLY (AND MORE STANDARD SUMMATION RESULTS)

$$= 4 \times \left(\frac{1}{6} \times 9 \times 10 \times 19 \right) + \frac{1}{2} \times 7 \times 10$$
$$= 60 \times 19 + 45$$
$$= 1140 + 45$$
$$= 1185$$

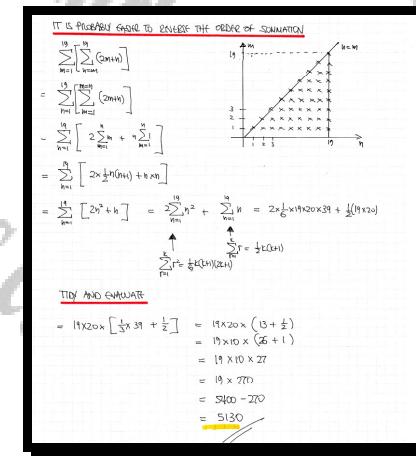
Question 7 (*****)

Evaluate the following expression

$$\sum_{m=1}^{19} \sum_{n=m}^{19} [2m+n].$$

You may find reversing the order of summation useful in this question

V, , 5130



Question 8 (*****)

The function f is defined as

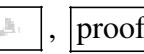
$$f(n, y) \equiv \sum_{x=1}^n \frac{x^2 y^x}{k}, \quad n \in \mathbb{N}, \quad y \in \mathbb{R}$$

$$\text{where } k = \sum_{r=1}^n r^2.$$

Use standard results on series to show that

$$\left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{3n^4 + 6n^3 - n^2 - 4n - 4}{20(2n+1)^2}.$$

$$\text{You may assume without proof } \sum_{r=1}^n r^4 = \frac{1}{30} n(n+1)(6n^3 + 9n^2 + n - 1).$$

 , 

USING STANDARD SUMMATION RESULTS

$$k = \sum_{x=1}^n x^2 = \frac{1}{5} n(n+1)(2n+1) \quad \text{K DOES NOT DEPEND ON } y, \text{ SO USE ONLY FULL FORM OF THE SUMMATION}$$

$$\therefore f(n, y) = \sum_{x=1}^n x^2 y^x = \frac{1}{5} \sum_{x=1}^n (xy)^2.$$

Differentiation with respect to y - K does not depend on y , ignore

$$\frac{df}{dy} = \frac{1}{5} \sum_{x=1}^n [2(x^2 y^{x-1})] = \frac{2}{5} \sum_{x=1}^n (x^2 y^{x-1})$$

$$\frac{d^2 f}{dy^2} = \frac{1}{5} \sum_{x=1}^n [2^2 (x^2 - 1)x y^{x-2}] = \frac{4}{5} \sum_{x=1}^n x^2 y^{x-2} \quad \leftarrow \text{NO NEED TO SIMPLIFY YET}$$

Another differentiation with respect to y - K ignored

$$\frac{df}{dy} = \frac{1}{5} \sum_{x=1}^n (x^2 y^{x-1})$$

$$\frac{d^2 f}{dy^2} = \frac{1}{5} \sum_{x=1}^n [x^2 (x-1)y^{x-2}] = \frac{1}{5} \sum_{x=1}^n (x^2 - x)y^{x-2}$$

$$\frac{d^3 f}{dy^3} = \frac{1}{5} \sum_{x=1}^n x^2 y^{x-3} = \frac{1}{5} \sum_{x=1}^n x^2 y^{x-3} \quad \leftarrow \text{NO NEED TO SIMPLIFY YET}$$

Substitution into the expression (cont)

$$\begin{aligned} & \left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2 \\ &= \frac{1}{5} \sum_{x=1}^n x^4 - \frac{1}{5} \sum_{x=1}^n x^2 + \frac{1}{5} \sum_{x=1}^n x^2 - \left[\frac{1}{5} \sum_{x=1}^n x^2 \right]^2 \\ &= \frac{1}{5} \sum_{x=1}^n x^4 - \frac{1}{5} \left[\sum_{x=1}^n x^2 \right]^2 \end{aligned}$$

SUBSTITUTE $K = \frac{1}{5} n(n+1)(2n+1)$ & SIMPLIFY

$$\begin{aligned} & \cdots = \frac{1}{5} \sum_{x=1}^n x^4 - \frac{1}{5} \left[\frac{1}{5} n(n+1)(2n+1) \right]^2 - \frac{\left[\frac{1}{5} n(n+1)(2n+1) \right]^2}{\left[\frac{1}{5} n(n+1)(2n+1) \right]^2} \\ &= \frac{1}{5} \frac{n(n+1)(2n+1)}{5(2n+1)} - \frac{1}{25} \frac{n^2(n+1)^2}{(2n+1)^2} \\ &= \frac{n^2(n+1)}{5(2n+1)} - \frac{9n^2(n+1)^2}{25(2n+1)^2} \\ &= \frac{45n^4(6n^3 + 9n^2 + n - 1) - 5n^4(9n^2 + 18n + 1)}{25(2n+1)^2} \\ &\quad \text{THE DENOMINATOR IS NOT WHAT WE DESIRE, SO TAKE THE NUMERATOR} \\ & (3n^4(6n^3 + 9n^2 + n - 1) - 45n^4(9n^2 + 18n + 1)) \\ &= \frac{45n^4 + 70n^5 + 6n^6 - 8n^4}{24n^4 + 36n^5 + 4n^6 - 4} - 45n^4 - 90n^5 - 45n^2 \\ &= 45n^4(9n^2 + 9n^3 + 4) - 45n^4 - 90n^5 - 45n^2 \\ &= 3n^4 + 9n^5 - 9n^2 - 4n + 4 \\ &\quad \text{LE THE DESIRED NUMERATOR} \\ & \therefore \left. \frac{d^2 f}{dy^2} \right|_{y=1} + \left. \frac{df}{dy} \right|_{y=1} - \left[\left. \frac{df}{dy} \right|_{y=1} \right]^2 = \frac{3n^4 + 6n^5 - n^2 - 4n - 4}{20(2n+1)^2} \end{aligned}$$

SUMMATIONS

METHOD OF DIFFERENCES

8 BASIC QUESTIONS

Question 1 (**)

$$f(r) = \frac{5}{(5r-1)(5r+4)}, \quad r \in \mathbb{N}$$

a) Express $f(r)$ into partial fractions

b) Hence show that

$$\sum_{r=1}^n f(r) = \frac{5n}{4(5n+4)}.$$

$$f(r) = \frac{1}{5r-1} - \frac{1}{5r+4}$$

$$\begin{aligned}
 \text{(a)} \quad f(r) &= \frac{5}{(5r-1)(5r+4)} = \frac{\frac{5}{5r+4}}{5r-1} + \frac{\frac{5}{5r-1}}{5r+4} \\
 &= \frac{1}{5r-1} - \frac{1}{5r+4} \\
 \text{(b)} \quad \left\{ \frac{5}{(5r-1)(5r+4)} \right\} &\equiv \frac{1}{5r-1} - \frac{1}{5r+4} \\
 \text{if } r=1 \quad \cancel{\frac{5}{(5r-1)(5r+4)}} &= \cancel{\frac{1}{5r-1}} - \cancel{\frac{1}{5r+4}} \\
 \text{if } r=2 \quad \cancel{\frac{5}{(5r-1)(5r+4)}} &= \cancel{\frac{1}{5r-1}} - \cancel{\frac{1}{5r+4}} \\
 \text{if } r=3 \quad \cancel{\frac{5}{(5r-1)(5r+4)}} &= \cancel{\frac{1}{5r-1}} - \cancel{\frac{1}{5r+4}} \\
 \vdots &\vdots \\
 \text{if } r=9 \quad \cancel{\frac{5}{(5r-1)(5r+4)}} &= \cancel{\frac{1}{5r-1}} - \cancel{\frac{1}{5r+4}} \\
 \sum_{r=1}^n \frac{5}{(5r-1)(5r+4)} &= \frac{1}{5} - \frac{1}{5n+4} \\
 \sum_{r=1}^n f(r) &= \frac{5n+4-5}{4(5n+4)} = \frac{5n}{4(5n+4)}
 \end{aligned}$$

Question 2 (**)

a) Show carefully that

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}.$$

b) Hence use the method of differences to find

$$\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2}.$$

$$\boxed{1 - \frac{1}{(n+1)^2}}$$

(a) $\sum_{r=1}^n \frac{1}{r^2(r+1)^2} = \frac{1}{1^2} - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - 1^2}{1^2(n+1)^2} = \frac{n^2 + 2n}{1^2(n+1)^2} = \frac{n(n+2)}{1^2(n+1)^2} = \frac{n(n+2)}{n^2(n+1)^2} = \frac{n+2}{n(n+1)^2}$

(b) $\frac{2r+1}{r^2(r+1)^2} \equiv \frac{1}{r^2} - \frac{1}{(r+1)^2}$

T_{n1}	$\frac{3}{1^2(n+1)^2} = \frac{1}{1^2} - \frac{1}{2^2}$
T_{n2}	$\frac{5}{2^2(3^2)} = \frac{1}{2^2} - \frac{1}{3^2}$
T_{n3}	$\frac{7}{2^2(3^2)} = \frac{1}{3^2} - \frac{1}{4^2}$
\vdots	\vdots
T_{nn}	$\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$

$$\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

Question 3 ()**

a) Show carefully that

$$\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}.$$

b) Hence find

$$\sum_{r=1}^n \frac{r}{(r+1)!}.$$

$$1 - \frac{1}{(n+1)!}$$

$$\begin{aligned}
 \text{(a)} \quad & \frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r!}{(r+1)r!} - \frac{1}{(r+1)!} = \frac{r!}{(r+1)!} - \frac{1}{(r+1)!} = \frac{r!}{(r+1)!} \\
 \text{(b)} \quad & \boxed{\frac{r!}{(r+1)!}} = \frac{1}{r!} - \frac{1}{(r+1)!} \\
 \text{E1: } & \Rightarrow \frac{1}{2!} = \frac{1}{1!} - \cancel{\frac{1}{2!}} \\
 \text{E2: } & \Rightarrow \frac{1}{3!} = \cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}} \\
 \text{E3: } & \Rightarrow \frac{1}{4!} = \cancel{\frac{1}{3!}} - \cancel{\frac{1}{4!}} \\
 & \vdots \\
 \text{En: } & \Rightarrow \frac{1}{(n+1)!} = \cancel{\frac{1}{n!}} - \cancel{\frac{1}{(n+1)!}} \\
 \text{Add: } & \sum_{r=1}^n \frac{r!}{(r+1)!} = 1 - \cancel{\frac{1}{(n+1)!}}
 \end{aligned}$$

Question 4 (***)

$$f(r) = \frac{1}{r(r+2)}, r \in \mathbb{N}$$

a) Express $f(r)$ into partial fractions.

b) Hence show that

$$\sum_{r=1}^{30} f(r) = \frac{1425}{1984}.$$

$$f(r) = \frac{1}{2r} - \frac{1}{2(r+2)}$$

(a) $\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$

(b) $\frac{2}{r(r+2)} \equiv \frac{1}{r} - \frac{1}{r+2}$

$\begin{aligned} & \Rightarrow 2 \sum_{r=1}^{30} \frac{1}{r(r+2)} = \frac{1425}{992} \\ & \text{• } r=1: \frac{2}{1 \cdot 3} = \frac{1}{1} - \frac{1}{3} \\ & \text{• } r=2: \frac{2}{2 \cdot 4} = \frac{1}{2} - \frac{1}{4} \\ & \text{• } r=3: \frac{2}{3 \cdot 5} = \frac{1}{3} - \frac{1}{5} \\ & \text{• } r=4: \frac{2}{4 \cdot 6} = \frac{1}{4} - \frac{1}{6} \\ & \vdots \\ & \text{• } r=29: \frac{2}{29 \cdot 31} = \frac{1}{29} - \frac{1}{31} \\ & \text{• } r=30: \frac{2}{30 \cdot 32} = \frac{1}{30} - \frac{1}{32} \\ & \Rightarrow \sum_{r=1}^{30} \frac{2}{r(r+2)} = 1 + \frac{1}{2} - \frac{1}{31} - \frac{1}{32} \end{aligned}$

Question 5 (***)

$$f(r) = \frac{2}{(r+1)(r+3)}, r \in \mathbb{N}$$

a) Express $f(r)$ into partial fractions

b) Use the method of differences to find

$$\sum_{r=1}^n f(r).$$

c) Hence evaluate

$$\sum_{r=8}^{\infty} f(r).$$

$$f(r) = \frac{1}{r+1} - \frac{1}{r+3}, \quad \sum_{r=1}^n f(r) = \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}, \quad \sum_{r=8}^{\infty} f(r) = \frac{19}{90}$$

(a) $\frac{2}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3}$

$$\frac{2}{(r+1)(r+3)} = A(r+3) + B(r+1)$$

- If $r=-1$, $2 = 2A \Rightarrow A=1$
- If $r=-3$, $2 = -2B \Rightarrow B=-1$

$$\therefore \frac{2}{(r+1)(r+3)} = \frac{1}{r+1} - \frac{1}{r+3}$$

(b) $\frac{2}{(r+1)(r+3)} = \frac{1}{r+1} - \frac{1}{r+3}$

- If $r=1$, $\frac{2}{2 \times 4} = \frac{1}{2} - \frac{1}{4}$
- If $r=2$, $\frac{2}{3 \times 5} = \frac{1}{3} - \frac{1}{5}$
- If $r=3$, $\frac{2}{4 \times 6} = \frac{1}{4} - \frac{1}{6}$
- If $r=4$, $\frac{2}{5 \times 7} = \frac{1}{5} - \frac{1}{7}$
- ⋮
- If $r=n-1$, $\frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}$
- If $r=n$, $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$

$$\sum_{r=1}^n \frac{2}{(r+1)(r+3)} = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{5}{6} - \left[\frac{1}{n+2} + \frac{1}{n+3} \right]$$

$$= \frac{1}{3} + \frac{1}{10}$$

$$= \frac{19}{36}$$

Question 6 (***)

a) Simplify $\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$ into a single fraction.

b) Hence show that

$$\sum_{r=1}^{20} \left[\frac{1}{r(r+1)(r+2)} \right] = \frac{115}{462}.$$

$$\boxed{\frac{2}{r(r+1)(r+2)}}, \quad \boxed{\frac{2}{r(r+1)(r+2)}}$$

a) OBTAIN A COMMON DENOMINATOR AND ADD

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{(r+2) - r}{(r+1)(r+2)} = \frac{2}{(r+1)(r+2)}$$

b) USES THE IDENTITY PROVEN IN PART (a)

$$\frac{2}{r(r+1)(r+2)} = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

- If $r=1$: $\frac{2}{1 \times 2 \times 3} = \frac{1}{1 \times 2} - \frac{1}{2 \times 3}$
- If $r=2$: $\frac{2}{2 \times 3 \times 4} = \frac{1}{2 \times 3} - \frac{1}{3 \times 4}$
- If $r=3$: $\frac{2}{3 \times 4 \times 5} = \frac{1}{3 \times 4} - \frac{1}{4 \times 5}$
- ⋮
- If $r=20$: $\frac{2}{20 \times 21 \times 22} = \frac{1}{20 \times 21} - \frac{1}{21 \times 22}$

$$\Rightarrow \sum_{r=1}^{20} \left[\frac{2}{r(r+1)(r+2)} \right] = \frac{1}{1 \times 2} - \frac{1}{21 \times 22}$$

$$\Rightarrow \sum_{r=1}^{20} \left[\frac{1}{r(r+1)(r+2)} \right] = \frac{1}{2} - \frac{1}{462}$$

$$\Rightarrow \sum_{r=1}^{20} \left[\frac{1}{r(r+1)(r+2)} \right] = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{462} \right) = \frac{115}{462}$$

✓ ANSWER

Question 7 (*)**

$$f(r) \equiv r^2(r+1)^2 - (r-1)^2 r^2, \quad r \in \mathbb{N}.$$

- a) Simplify $f(r)$ as far as possible.
 b) Use the method of differences to show that

$$\sum_{r=1}^{20} r^3 = 44100.$$

$$\boxed{\text{[]}}, \quad \boxed{f(r) = 4r^3}$$

a)
$$\begin{aligned} f(r) &= r^2(r+1)^2 - (r-1)^2 r^2 \\ &= r^2 [(r+1)^2 - (r-1)^2] \\ &= r^2 (r+1+r-1)(r+1-r+1) \\ &= r^2 \times 2r \times 2 \\ &= 4r^3 \end{aligned}$$

b) USING PART (a)

$4r^3$	\equiv	$r^2(r+1)^2 - (r-1)^2 r^2$
IF $r=1$	4×1^3	$= 1^2 \times 2^2 - 0^2 \times 1^2$
IF $r=2$	4×2^3	$= 2^2 \times 3^2 - 1^2 \times 2^2$
IF $r=3$	4×3^3	$= 3^2 \times 4^2 - 2^2 \times 3^2$
IF $r=4$	4×4^3	$= 4^2 \times 5^2 - 3^2 \times 4^2$
⋮	⋮	⋮
IF $r=20$	4×20^3	$= 20^2 \times 21^2 - (19 \times 20)^2$

ADDITIVE

$$\begin{aligned} &\cancel{4 \times 1^3} + \cancel{4 \times 2^3} + \dots + \cancel{4 \times 20^3} \\ \Rightarrow 4 \times &20^3 = 20^2 \times 21^2 \\ \Rightarrow 4 \times &20^3 = 20^2 \times 21^2 \\ \Rightarrow 4 \times &20^3 = \frac{20^2 \times 21^2}{4} \\ \Rightarrow &\sum_{r=1}^{20} r^3 = 44100 \end{aligned}$$

Question 8 (***)

$$f(r) = \frac{1}{r(r+2)}, \quad r \in \mathbb{N}.$$

- a) Express $f(r)$ in partial fractions.
 b) Hence prove, by the method of differences, that

$$\sum_{r=1}^n f(r) = \frac{n(An+B)}{4(n+1)(n+2)},$$

where A and B are constants to be found.

, $A = 3$, $B = 5$

a) BY INVERSION (ROUTE OF METHOD OF SPLITTING)

$$\begin{aligned} f(r) &= \frac{1}{r(r+2)} = \frac{\frac{1}{2}}{r} + \frac{-\frac{1}{2}}{r+2} = \frac{\frac{1}{2}}{r} - \frac{\frac{1}{2}}{r+2} \\ &= \frac{1}{r} - \frac{1}{2(r+2)} \end{aligned}$$

b) SETTING PART (a) AS AN IDENTITY

$\frac{2}{r(r+2)} \equiv \frac{1}{r} - \frac{1}{2(r+2)}$
--

<ul style="list-style-type: none"> • $r=1$: $\frac{2}{1 \cdot 3} \equiv \frac{1}{1} - \frac{1}{2 \cdot 3}$ • $r=2$: $\frac{2}{2 \cdot 4} \equiv \frac{1}{2} - \frac{1}{2 \cdot 4}$ • $r=3$: $\frac{2}{3 \cdot 5} \equiv \frac{1}{3} - \frac{1}{2 \cdot 5}$ • $r=4$: $\frac{2}{4 \cdot 6} \equiv \frac{1}{4} - \frac{1}{2 \cdot 6}$ • $r=5$: $\frac{2}{5 \cdot 7} \equiv \frac{1}{5} - \frac{1}{2 \cdot 7}$ ⋮ • $r=4$: $\frac{2}{(6-1)(6+1)} \equiv \frac{1}{6-1} - \frac{1}{2(6+1)}$ • $r=n$: $\frac{2}{n(n+2)} \equiv \frac{1}{n} - \frac{1}{2(n+2)}$
--

$$\begin{aligned} &\Rightarrow \sum_{r=1}^n \frac{2}{r(r+2)} = \frac{2n^2 + 9n + 6 - 4n - 6}{2(n+1)(n+2)} \\ &\Rightarrow 2 \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{3n^2 + 5n}{2(n+1)(n+2)} \\ &\Rightarrow \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)} \end{aligned}$$

ie $A=3$
 $B=5$

SUMMATIONS

METHOD OF DIFFERENCES
8 STANDARD QUESTIONS

Question 1 (***)+

Use the method of differences to show that

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}.$$

[P] [R], proof

USING PARTIAL FRACTIONS

$$\frac{1}{k(k+1)(k+2)} \equiv \frac{\frac{1}{(k+2)}}{k} - \frac{\frac{1}{(k+1)}}{k+1} + \frac{\frac{1}{k}}{k+2}$$

$$\frac{1}{k(k+1)(k+2)} \equiv \frac{\frac{1}{k}}{k} - \frac{\frac{1}{k+1}}{k+1} + \frac{\frac{1}{k+2}}{k+2}$$

Doubling the basic identity for simplicity

$$\frac{2}{k(k+1)(k+2)} \equiv \frac{\frac{1}{k}}{k} - \frac{\frac{1}{k+1}}{k+1} + \frac{\frac{1}{k+2}}{k+2}$$

- $k=1$ $\frac{2}{1 \times 2 \times 3} \Rightarrow \frac{1}{1} - \frac{1}{2} + \frac{1}{3}$
- $k=2$ $\frac{2}{2 \times 3 \times 4} \Rightarrow \frac{1}{2} - \frac{1}{3} + \frac{1}{4}$
- $k=3$ $\frac{2}{3 \times 4 \times 5} \Rightarrow \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$
- $k=4$ $\frac{2}{4 \times 5 \times 6} \Rightarrow \frac{1}{4} - \frac{1}{5} + \frac{1}{6}$
- $k=n-1$ $\frac{2}{(n-1)n(n+1)} \Rightarrow \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1}$
- $k=n$ $\frac{2}{n(n+1)(n+2)} \Rightarrow \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2}$

$$\rightarrow \sum_{k=1}^n \frac{2}{k(k+1)(k+2)} = \frac{1}{1} - \frac{1}{n+1} + \frac{1}{n+2}$$

$$\rightarrow 2 \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{n^2 + 3n}{2(n+1)(n+2)}$$

$$\rightarrow \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{n(n+3)}{4(n+1)(n+2)} \quad // \text{as required}$$

$$\Rightarrow 2 \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{n^2 + 3n}{2(n+1)(n+2)}$$

$$\Rightarrow 2 \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{n^2 + 3n}{2(n+1)(n+2)}$$

$$\rightarrow \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{n(n+3)}{4(n+1)(n+2)} \quad // \text{as required}$$

Question 2 (***)+

$$u_r = \frac{1}{6}r(r+1)(4r+11), r \in \mathbb{N}.$$

- a) Simplify $u_r - u_{r-1}$ as far as possible.
- b) By using the method of differences, or otherwise, find the sum of the first 100 terms of the following series.

$$(1 \times 5) + (2 \times 7) + (3 \times 9) + (4 \times 11) + \dots$$

, $r(2r+3)$, [691850]

a)
$$\begin{aligned} u_r - u_{r-1} &= \frac{1}{6}r(r+1)(4r+11) - \frac{1}{6}(r-1)r(4(r-1)+11) \\ &= \frac{1}{6}r[(r+1)(4r+11) - (r-1)(4r+7)] \\ &= \frac{1}{6}r[4r^2 + 15r + 11 - 4r^2 - 3r + 7] \\ &= \frac{1}{6}r(12r + 18) \\ &= r(2r+3) \end{aligned}$$

b) Proof by All Following

$$(1 \times 5) + (2 \times 7) + (3 \times 9) + (4 \times 11) + \dots + (100 \times 203)$$

$$\Rightarrow u_r - u_{r-1} \equiv r(2r+3)$$

<ul style="list-style-type: none"> • $r=1$ $u_1 - u_0 = 1 \times 5$ • $r=2$ $u_2 - u_1 = 2 \times 7$ • $r=3$ $u_3 - u_2 = 3 \times 9$ • $r=4$ $u_4 - u_3 = 4 \times 11$ ⋮ • $r=100$ $u_{100} - u_{99} = 100 \times 203$ 	$\begin{aligned} \Rightarrow u_{100} - u_0 &= (1 \times 5) + (2 \times 7) + (3 \times 9) + \dots + (100 \times 203) \\ \Rightarrow \frac{1}{6} \times 100 \times 101 \times 411 - 0 &= \sum_{n=1}^{100} n(2n+3) \\ \Rightarrow \sum_{n=1}^{100} (2n+3) &= 691850 \end{aligned}$
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Question 3 (***)+

$$f(r) = \frac{1}{(r+1)(r-1)}, r \in \mathbb{N}.$$

a) Express $f(r)$ into partial fractions.

b) Hence show that

$$\sum_{r=2}^n \frac{1}{r^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}.$$

c) State the value of

$$\sum_{r=2}^{\infty} \frac{1}{r^2-1}$$

$$\boxed{\quad}, f(r) = \frac{1}{2(r-1)} - \frac{1}{2(r+1)}, \boxed{\frac{3}{4}}$$

a) $f(r) = \frac{1}{(r+1)(r-1)} = \frac{\frac{1}{2}}{r-1} - \frac{\frac{1}{2}}{r+1} = \frac{1}{2(r-1)} - \frac{1}{2(r+1)}$

By cross multiplying (canceling)

b) using part (a) "double" for simplicity

$$\frac{2}{r^2-1} \equiv \frac{2}{(r+1)(r-1)} \equiv \frac{1}{r-1} - \frac{1}{r+1}$$

• $r=2$	$\frac{2}{2^2-1} \approx \frac{1}{\cancel{1}} - \frac{1}{\cancel{3}}$
• $r=3$	$\frac{2}{3^2-1} \approx \frac{1}{\cancel{2}} - \frac{1}{\cancel{4}}$
• $r=4$	$\frac{2}{4^2-1} \approx \frac{1}{\cancel{3}} - \frac{1}{\cancel{5}}$
• $r=5$	$\frac{2}{5^2-1} \approx \frac{1}{\cancel{4}} - \frac{1}{\cancel{6}}$
• $r=6$	$\frac{2}{6^2-1} \approx \frac{1}{\cancel{5}} - \frac{1}{\cancel{7}}$
⋮	⋮
• $r=n-1$	$\frac{2}{(n-1)^2-1} \approx \frac{1}{\cancel{n-2}} - \frac{1}{\cancel{n}}$
• $r=n$	$\frac{2}{n^2-1} \approx \frac{1}{\cancel{n-1}} - \frac{1}{\cancel{n+1}}$

$$\begin{aligned} \sum_{r=2}^{\infty} \frac{2}{r^2-1} &= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \\ 2 \sum_{r=2}^{\infty} \frac{1}{r^2-1} &= \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1} \\ \sum_{r=2}^{\infty} \frac{1}{r^2-1} &= \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)} \end{aligned}$$

c) As $n \rightarrow \infty$ the sum tends to $\frac{3}{4}$

Question 4 (***)

$$f(r) = \frac{2}{r(r+1)(r+2)}, r \in \mathbb{N}.$$

a) Express $f(r)$ into partial fractions.

b) Hence show that

$$\sum_{r=1}^n f(r) = \frac{1}{2} - \frac{1}{(n+1)(n+2)}.$$

c) Find the value of the convergent infinite sum

$$\frac{1}{5 \times 6 \times 7} + \frac{1}{6 \times 7 \times 8} + \frac{1}{7 \times 8 \times 9} + \dots$$

, $f(r) = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$, $\boxed{\frac{1}{60}}$

a) BY INSPECTION (CAREFUL)

$$f(r) = \frac{2}{r(r+1)(r+2)} = \frac{\frac{2}{r+2}}{r} + \frac{\frac{2}{r+1}}{r+2} - \frac{\frac{2}{r+1}}{r+2}$$

$$f(r) = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$$

b) USING PFD

$$f(r) = \frac{2}{r(r+1)(r+2)} = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$$

- $r=1$ $f(1) = \frac{2}{1 \times 2 \times 3} = -\frac{2}{3}$
- $r=2$ $f(2) = \frac{2}{2 \times 3 \times 4} = \frac{1}{4}$
- $r=3$ $f(3) = \frac{2}{3 \times 4 \times 5} = -\frac{2}{5}$
- $r=4$ $f(4) = \frac{2}{4 \times 5 \times 6} = \frac{1}{6}$
- $r=5$ $f(5) = \frac{2}{5 \times 6 \times 7} = -\frac{2}{7}$
- ⋮
- $r=n-1$ $f(n-1) = \frac{2}{(n-1)n(n+1)} = \frac{1}{n} - \frac{2}{n+1}$
- $r=n$ $f(n) = \frac{2}{n(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$

$$\Rightarrow \sum_{r=1}^n f(r) = \sum_{r=1}^n \frac{2}{r(r+1)(r+2)} = \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

$$\Rightarrow \sum_{r=1}^n f(r) = \frac{1}{2} + \frac{-(1/n+1)+(1/n+2)}{(n+1)(n+2)}$$

c) $\sum_{r=1}^n f(r) = \frac{1}{2} - \frac{1}{(n+1)(n+2)}$ As $n \rightarrow \infty$ $\frac{1}{(n+1)(n+2)} \rightarrow 0$

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{2} \quad (\text{as } n \rightarrow \infty \text{ } \frac{1}{(n+1)(n+2)} \rightarrow 0)$$

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$$

$$\Rightarrow \frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \frac{1}{4 \times 5 \times 6} + \dots + \sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$$

$$\Rightarrow \frac{1}{6} + \frac{1}{24} + \frac{1}{60} + \frac{1}{120} + \dots + \sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$$

$$\Rightarrow \sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{60}$$

Question 5 (***)+

Use the method of differences to show that

$$\frac{1}{1 \times 2 \times 3} + \frac{4}{2 \times 3 \times 4} + \frac{7}{3 \times 4 \times 5} + \dots + \frac{3n-2}{n(n+1)(n+2)} = \frac{n^2}{(n+1)(n+2)}$$

[, proof]

PROOF BY PARTIAL FRACTIONS (CARRY ON FROM M16B)

$$\begin{aligned}\frac{3n-2}{n(n+1)(n+2)} &= \frac{-\frac{2}{n+2}}{n} + \frac{\frac{1}{n+1}}{n+1} + \frac{-\frac{8}{n+2}}{n+2} \\ &= -\frac{1}{n} + \frac{8}{n+1} - \frac{4}{n+2}\end{aligned}$$

SETTING UP THE METHOD OF DIFFERENCES BASED ON THE ABOVE RESULT

$\frac{3t-2}{t(t+1)(t+2)} \equiv -\frac{1}{t} + \frac{8}{t+1} - \frac{4}{t+2}$
• IF $t=1$
$\frac{3(1)-2}{1(1+1)(1+2)} = -\frac{1}{1} + \frac{8}{1+1} - \frac{4}{1+2}$
• IF $t=2$
$\frac{4}{2(2+1)(2+2)} = -\frac{1}{2} + \frac{8}{2+1} - \frac{4}{2+2}$
• IF $t=3$
$\frac{7}{3(3+1)(3+2)} = -\frac{1}{3} + \frac{8}{3+1} - \frac{4}{3+2}$
• IF $t=4$
$\frac{10}{4(4+1)(4+2)} = -\frac{1}{4} + \frac{8}{4+1} - \frac{4}{4+2}$
⋮
• IF $t=n-1$
$\frac{3(n-1)-2}{(n-1)(n-1+1)(n-1+2)} = -\frac{1}{n-1} + \frac{8}{n-1+1} - \frac{4}{n-1+2}$
• IF $t=n$
$\frac{3n-2}{n(n+1)(n+2)} = -\frac{1}{n} + \frac{8}{n+1} - \frac{4}{n+2}$

$$\sum_{t=1}^n \frac{3t-2}{t(t+1)(t+2)} = (-1 + \frac{8}{2} - \frac{4}{3}) + \left(\frac{1}{2+1} - \frac{4}{3+2} \right) = 1 + \frac{1}{n+1} - \frac{4}{n+2}$$

$$\begin{aligned}&= 1 + \frac{1}{n+1} - \frac{4}{n+2} \\ &= \frac{(n+1)(n+2) + (n+2) - 4(n+1)}{(n+1)(n+2)} \\ &= \frac{n^2 + 3n + 2 + n + 2 - 4n - 4}{(n+1)(n+2)} \\ &= \frac{n^2}{(n+1)(n+2)}\end{aligned}$$

Question 6 (***)+

It is given that

$$\frac{2k+7}{(2k+1)(2k+3)(2k+5)} \equiv \frac{3}{4(2k+1)} - \frac{1}{(2k+3)} + \frac{1}{4(2k+5)}.$$

Use the method of differences to find a simplified expression for

$$\frac{7}{1 \times 3 \times 5} + \frac{9}{3 \times 5 \times 7} + \frac{11}{5 \times 7 \times 9} + \dots + \frac{2n+7}{(2n+1)(2n+3)(2n+5)}.$$

Give your answer in the form $\frac{2}{3} - f(n)$, where $f(n)$ is a single simplified fraction.

, $f(n) = -\frac{n+3}{(2n+3)(2n+5)}$

REWRITE THE PARTIAL FRACTION IDENTITY

$$\frac{2k+7}{(2k+1)(2k+3)(2k+5)} \equiv \frac{\frac{3}{2k+1}}{(2k+3)} + \frac{\frac{1}{2k+5}}{(2k+3)} - \frac{1}{2k+3}$$

- If $k=0$ $2HS = \frac{3}{2} + \frac{1}{5} - \frac{1}{3}$
- If $k=1$ $2HS = \frac{3}{3} + \frac{1}{7} - \frac{1}{5}$
- If $k=2$ $2HS = \frac{3}{5} + \frac{1}{9} - \frac{1}{7}$
- If $k=3$ $2HS = \frac{3}{7} + \frac{1}{11} - \frac{1}{9}$
- If $k=4$ $2HS = \frac{3}{9} + \frac{1}{13} - \frac{1}{11}$
- ⋮
- If $k=n-2$ $2HS < \frac{3}{2n-3} + \frac{1}{2n-1} - \frac{1}{2n-1}$
- If $k=n-1$ $2HS < \frac{3}{2n-1} + \frac{1}{2n+1} - \frac{1}{2n-1}$
- If $k=n$ $2HS = \frac{3}{2n+1} + \frac{1}{2n+5} - \frac{1}{2n+3}$

$$\therefore \sum_{k=0}^n \frac{2k+7}{(2k+1)(2k+3)(2k+5)} = \left(\frac{3}{2} + \frac{1}{5} - \frac{1}{3} \right) + \frac{16}{2n+3} - \frac{1}{2n+3} + \frac{16}{2n+5}$$

$$= \frac{2}{3} + \frac{16}{2n+5} - \frac{16}{2n+3}$$

$$= \frac{2}{3} + \frac{1}{2} \cdot \frac{16(2n+3) - 16(2n+5)}{(2n+3)(2n+5)}$$

$$= \frac{2}{3} - \frac{16}{4(2n+3)(2n+5)} = \frac{2}{3} - \frac{n+3}{(2n+3)(2n+5)}$$

Question 7 (****)

Use the method of differences to find a simplified expression for the first n terms of the following series.

$$\frac{1}{1 \times 3} + \frac{2}{3 \times 5} + \frac{3}{5 \times 7} + \frac{4}{7 \times 9} + \dots$$

Give your answer in the form $\frac{1}{4} - f(n)$, where $f(n)$ is a single simplified fraction.

$$[\quad], \quad f(n) = \frac{(-1)^n}{4(2n+1)}$$

WORK FIRSTLY IN SIMPLE FRACTION

$$\frac{1}{1 \times 3} - \frac{2}{3 \times 5} + \frac{3}{5 \times 7} - \frac{4}{7 \times 9} + \dots \quad \sum_{r=1}^n \frac{(-1)^r r}{(2r-1)(2r+1)}$$

n terms

IGNORING THE (-1)ⁿ⁺¹ TERM IN THE SUMMATION, OBTAIN THE PARTIAL FRACTIONS

$$\frac{r}{(2r-1)(2r+1)} = \frac{\frac{1}{2}}{2r-1} + \frac{\frac{1}{2}}{2r+1} = \frac{1}{2r-1} + \frac{1}{2r+1}$$

$$\frac{dr}{(2r-1)(2r+1)} = \frac{1}{2r-1} + \frac{1}{2r+1}$$

You are asked for different values of r

r=1	$\frac{1}{1 \times 3} = \frac{1}{2} - \frac{1}{4}$	$\frac{1}{3 \times 5} = \frac{1}{2} - \frac{1}{8}$
r=2	$\frac{2}{3 \times 5} = \frac{1}{2} - \frac{1}{12}$	$\frac{2}{5 \times 7} = \frac{1}{2} - \frac{1}{20}$
r=3	$\frac{3}{5 \times 7} = \frac{1}{2} - \frac{1}{30}$	$\frac{3}{7 \times 9} = \frac{1}{2} - \frac{1}{48}$
r=4	$\frac{4}{7 \times 9} = \frac{1}{2} - \frac{1}{72}$	$\frac{4}{9 \times 11} = \frac{1}{2} - \frac{1}{144}$
⋮	⋮	⋮
r=n	$\frac{n}{(2n-1)(2n+1)} = \frac{1}{2} - \frac{1}{4n^2}$	$\frac{n}{(2n-1)(2n+1)} = \frac{1}{2} - \frac{1}{4n^2}$

ADD BOTH SIDES

$$\Rightarrow \sum_{r=1}^n \left[\frac{(-1)^r r}{(2r-1)(2r+1)} \right] = 1 + \frac{(-1)^{n+1}}{4n^2}$$

$$\Rightarrow \sum_{r=1}^n \frac{(-1)^r r}{(2r-1)(2r+1)} = 1 - \frac{(-1)^n}{4n^2}$$

$$\Rightarrow \sum_{r=1}^n \frac{(-1)^r r}{(2r-1)(2r+1)} = \frac{1}{4} - \frac{(-1)^n}{4(2n+1)}$$

Question 8 (*****)

$$f(r) = \frac{1}{\sqrt{r+2} + \sqrt{r}}, \quad r \geq 0.$$

a) Rationalize the denominator of $f(r)$.

b) Find an expression for

$$\sum_{r=1}^n f(r).$$

c) Show clearly that

$$\sum_{r=1}^{48} f(r) = 3 + 2\sqrt{2}$$

$\boxed{\text{_____}}$, $f(r) = \frac{\sqrt{r+2} - \sqrt{r}}{2}$	$\sum_{r=1}^n f(r) = \frac{1}{2}(\sqrt{n+2} + \sqrt{n+1} - \sqrt{2} - 1)$
---	---

a) USING STANDARD SQUIDS

$$\frac{1}{\sqrt{r+2} + \sqrt{r}} = \frac{\sqrt{r+2} - \sqrt{r}}{(\sqrt{r+2} + \sqrt{r})(\sqrt{r+2} - \sqrt{r})} = \frac{\sqrt{r+2} - \sqrt{r}}{r+2 - r} = \frac{1}{2}(\sqrt{r+2} - \sqrt{r})$$

b) USING PART (a)

$$\frac{2}{\sqrt{r+2} + \sqrt{r}} = \sqrt{r+2} - \sqrt{r}$$

Tra 1:	$\frac{2}{\sqrt{2} + \sqrt{1}} = \sqrt{2} - \sqrt{1}$
Tra 2:	$\frac{2}{\sqrt{3} + \sqrt{2}} = \sqrt{3} - \sqrt{2}$
Tra 3:	$\frac{2}{\sqrt{4} + \sqrt{3}} = \sqrt{4} - \sqrt{3}$
Tra 4:	$\frac{2}{\sqrt{5} + \sqrt{4}} = \sqrt{5} - \sqrt{4}$
⋮	⋮
Tra n-1:	$\frac{2}{\sqrt{n+1} + \sqrt{n}} = \sqrt{n+1} - \sqrt{n}$
Tra n:	$\frac{2}{\sqrt{n+2} + \sqrt{n+1}} = \sqrt{n+2} - \sqrt{n+1}$

$$\Rightarrow \sum_{r=1}^{n-1} \frac{2}{\sqrt{r+2} + \sqrt{r}} = \sqrt{n+2} + \sqrt{n+1} - \sqrt{2} - 1 \quad <\text{ADDING BOTH SIDES}$$

$$\Rightarrow \sum_{r=1}^{n-1} \frac{1}{\sqrt{r+2} + \sqrt{r}} = \frac{1}{2}(\sqrt{n+2} + \sqrt{n+1} - \sqrt{2} - 1)$$

$$\Rightarrow \sum_{r=1}^n f(r) = \frac{1}{2}(\sqrt{n+2} + \sqrt{n+1} - \sqrt{2} - 1)$$

c) LET $n = 48$ IN PART (b)

$$\begin{aligned} \sum_{r=1}^{48} f(r) &= \frac{1}{2} [\sqrt{50} + \sqrt{49} - \sqrt{2} - 1] \\ &= \frac{1}{2} [5\sqrt{2} + 7 - \sqrt{2} - 1] \\ &= \frac{1}{2} [4\sqrt{2} + 6] \\ &= 3 + 2\sqrt{2} \quad \text{≈ 8.84} \end{aligned}$$

SUMMATIONS

METHOD OF DIFFERENCES

3 HARD QUESTIONS

Question 1 (****+)

Consider the following infinite convergent series.

$$\frac{3}{1 \times 2} - \frac{5}{2 \times 3} + \frac{7}{3 \times 4} - \frac{9}{4 \times 5} + \frac{11}{5 \times 6} - \dots$$

- a) Use the method of differences, to find the sum of this series.
- b) Verify the answer of part (a) by using a method based on the Maclaurin expansion of $\ln(1+x)$.

V, , **1**

a) Start by rewriting the general term in sigma notation

$$\frac{3}{1 \times 2} - \frac{5}{2 \times 3} + \frac{7}{3 \times 4} - \frac{9}{4 \times 5} + \frac{11}{5 \times 6} - \dots = \sum_{n=1}^{\infty} [(-1)^{n+1} \frac{(2n+1)}{(n)(n+1)}]$$

USING $(-1)^{n+1}$ EXPAND THE TERM INTO PARTIAL FRACTIONS BY CROSS-CANCEL

$$\frac{2n+1}{n(n+1)} = \frac{1}{n} + \frac{1}{n+1}$$

Now, we have

P _{n1}	P _{n2}	P _{n3}
$\frac{-3}{1 \times 2} =$	$\frac{1}{1} + \frac{1}{2}$	
$\frac{-5}{2 \times 3} =$	$\frac{1}{2} + \frac{1}{3}$	
$\frac{7}{3 \times 4} =$	$\frac{1}{3} + \frac{1}{4}$	
$\frac{-9}{4 \times 5} =$	$\frac{1}{4} + \frac{1}{5}$	
\vdots	\vdots	\vdots
$\frac{(2n+1)}{n(n+1)} =$	$\frac{1}{n} + \frac{1}{n+1}$	

$$\sum_{n=1}^{\infty} \left[(-1)^{n+1} \frac{1}{n} + (-1)^{n+1} \frac{1}{n+1} \right] = 1 + \lim_{n \rightarrow \infty} \left(-\frac{1}{n+1} \right)$$

As $n \rightarrow \infty$ THE SUM TO INFINITY IS $\boxed{1}$

b) Working at the expansion of $\ln(1+x)$, valid for $-1 < x \leq 1$

- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$
- LET $x=1$
- $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

USING THE PARTIAL FRACTIONS FROM PART (a)

$$\sum_{n=1}^{\infty} \left[(-1)^{n+1} \frac{1}{n} + (-1)^{n+1} \frac{1}{n+1} \right] = \sum_{n=1}^{\infty} \left[(-1)^{n+1} \left(\frac{1}{n} + \frac{1}{n+1} \right) \right]$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

RE-INDUCTING AND MANIPULATING

$$\begin{aligned} &= \ln 2 + \left[-\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \right] \\ &= \ln 2 + \left[1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \right] \\ &= \ln 2 + (1 - \ln 2) \\ &= \boxed{1} \end{aligned}$$

ALTERNATIVE TO RE-INDUCTING & MANIPULATING

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \\ S &= -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \\ 1 - \frac{S}{2} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \\ 1 - \frac{S}{2} &= \ln 2 \\ S &= 1 - \ln 2 \quad \text{AS REQS} \end{aligned}$$

Question 2 (****+)

Use partial fractions to sum the following series.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4 + 2n^3 + n^2}.$$

You may assume that the series converges.

[1]

Start by tidying up the summation

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4 + 2n^3 + n^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n^2 + 2n + 1)} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

Although we have repeated factors, the partial fractions can easily be done by inspection

$$= \sum_{n=1}^{\infty} \left[\frac{1}{n^2} - \frac{1}{(n+1)^2} \right]$$
$$= \left(\frac{1}{1^2} - \frac{1}{2^2} \right) + \left(\frac{1}{2^2} - \frac{1}{3^2} \right) + \left(\frac{1}{3^2} - \frac{1}{4^2} \right) + \left(\frac{1}{4^2} - \frac{1}{5^2} \right) \dots$$
$$= 1$$

Question 3 (****+)

It is given that

$$f(r) = \frac{6r^4 + 6r^3 - ar^2 - ar + 1}{r(r+1)}, \quad r \in \mathbb{N},$$

where a is a non zero constant.

It is further given that

$$\sum_{r=1}^n f(r) = \frac{n^2(n+2)(2n+1)}{n+1}.$$

Determine the value of a .

, $a = 2$

MANIPULATE $f(r)$ AS FOLLOWS

$$\begin{aligned} f(r) &= \frac{6r^4 + 6r^3 - ar^2 - ar + 1}{r(r+1)} = \frac{6r^3(r+1) - ar(r+1) + 1}{r(r+1)} \\ &= 6r^3 - a + \frac{1}{r(r+1)} \quad \text{PARTIAL FRACTION IS INSPECTED} \\ &= 6r^3 - a + \frac{1}{r} - \frac{1}{r+1} \end{aligned}$$

NOW PROCEED BY THE METHOD OF DIFFERENCES

$$\begin{aligned} f(1) &\equiv 6 \cancel{r^3} - a + \cancel{\frac{1}{r}} - \cancel{\frac{1}{r+1}} \\ f(2) &\equiv 6 \cancel{r^3} - a + \cancel{\frac{1}{r}} - \cancel{\frac{1}{r+1}} \\ f(3) &\equiv 6 \cancel{r^3} - a + \cancel{\frac{1}{r}} - \cancel{\frac{1}{r+1}} \\ f(4) &\equiv 6 \cancel{r^3} - a + \cancel{\frac{1}{r}} - \cancel{\frac{1}{r+1}} \\ &\vdots \\ f(k) &\equiv 6 \cancel{r^3} - a + \cancel{\frac{1}{r}} - \cancel{\frac{1}{r+1}} \end{aligned}$$

ADD $\sum_{r=1}^k f(r) = 6 \sum_{r=1}^k r^3 - ka + 1 - \frac{1}{k+1} \quad \text{ADD}$

$$\begin{aligned} &= 6 \times \frac{k^2(k+1)^2}{4} - ak + 1 - \frac{1}{k+1} \\ &= \frac{6k^2(k+1)}{4} - ak + \frac{1}{k+1} \\ &= \frac{3k^2(k+1)}{2} - ak(k+1) + \frac{1}{k+1} \end{aligned}$$

COMPARING NUMERATORS

$$\begin{aligned} \frac{n^2(n+2)(2n+1)}{n+1} &\equiv \frac{n(n+1)(2n+1) - an(n+1) + n}{n+1} \\ n^2(2n^2+3n+2) &\equiv n(2n+1)(n^2+2n+1) - an(n+1) + n \\ n(2n^2+3n+2) &\equiv (2n+1)(n^2+2n+1) - a(n+1) + 1 \\ 2n^3+3n^2+2n &\equiv 2n^3+4n^2+2n \\ &\quad -an - a \\ &\quad +1 \\ 2n^3+3n^2+2n &\equiv 2n^3+4n^2+2n - (2-a) \\ \therefore 4-a &= 2 \quad a = 2 \\ a = 2 & \quad a = 2 \\ \therefore a &= 2 \end{aligned}$$

SUMMATIONS

METHOD OF DIFFERENCES

15 ENRICHMENT QUESTIONS

Question 1 (*****)

Determine the exact value of the following sum.

$$\sum_{n=2}^{20} \left[\frac{n^3 - n^2 + 1}{n^2 - n} \right].$$

 , $\frac{4199}{20}$

• START MANIPULATING BY DIVISION EXCLUDING BY PARTIAL FRACTIONS

$$\begin{aligned}\frac{n^3 - n^2 + 1}{n^2 - n} &= \frac{n(n^2 - n) + 1}{n^2 - n} = n + \frac{1}{n(n-1)} = n + \frac{1}{n(n-1)} \\ &= n + \frac{-1}{n} + 1 + \frac{1}{n-1} = n + \frac{1}{n-1} - \frac{1}{n}\end{aligned}$$

• THIS WE HAVE

$\frac{n^3 - n^2 + 1}{n^2 - n} = n + \frac{1}{n-1} - \frac{1}{n}$

If $n=2$: $\frac{2^3 - 2^2 + 1}{2^2 - 2} = 2 + \frac{1}{1} - \frac{1}{2}$

If $n=3$: $\frac{3^3 - 3^2 + 1}{3^2 - 3} = 3 + \frac{1}{2} - \frac{1}{3}$

If $n=4$: $\frac{4^3 - 4^2 + 1}{4^2 - 4} = 4 + \frac{1}{3} - \frac{1}{4}$

If $n=5$: $\frac{5^3 - 5^2 + 1}{5^2 - 5} = 5 + \frac{1}{4} - \frac{1}{5}$

⋮

If $n=20$: $\frac{20^3 - 20^2 + 1}{20^2 - 20} = 20 + \frac{1}{19} - \frac{1}{20}$

• ADDING:

$$\sum_{n=2}^{20} \left[\frac{n^3 - n^2 + 1}{n^2 - n} \right] = \left[\sum_{n=2}^{20} \left(n + \frac{1}{n-1} - \frac{1}{n} \right) \right] + 1 - \frac{1}{20}$$

$$= \frac{19}{2}(2+20) + 1 - \frac{1}{20}$$

$$= (19 \times 11) + 1 - \frac{1}{20}$$

$$= 210 - \frac{1}{20}$$

$$= \frac{4199}{20}$$

$19 \times 11 = 190 + 19 = 209$

$210 - \frac{1}{20} = \frac{210}{20} = \frac{210}{200}$

Question 2 (*****)

$$f(x, n) = \sum_{r=1}^n \left[\frac{1}{(x-1)^r} \right], \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

By observing the simplification of

$$\frac{1}{(x-2)(x-1)^r} - \frac{1}{(x-2)(x-1)^{r+1}}$$

find a simplified expression for $f(x, n)$.

$$\boxed{\text{Simplify}} \quad f(x, n) = \frac{1}{x-2} - \frac{1}{(x-2)(x-1)^n}$$

START WITH THE SIMPLIFICATION

$$\frac{1}{(x-1)^n} - \frac{1}{(x-1)^{n+1}(x-2)} = \frac{(x-1)-1}{(x-1)^n(x-2)} = \frac{x-2}{(x-1)^n(x-2)}$$

Therefore we have

$$\frac{1}{(x-1)^n} \equiv \frac{1}{(x-2)} - \frac{1}{(x-1)^{n+1}(x-2)}$$

- If $n=0$: $\frac{1}{(x-1)^0} = \frac{1}{x-2} - \frac{1}{(x-1)(x-2)}$
- If $n=1$: $\frac{1}{(x-1)^1} = \frac{1}{(x-2)} - \frac{1}{(x-1)^2(x-2)}$
- If $n=2$: $\frac{1}{(x-1)^2} = \frac{1}{(x-2)} - \frac{1}{(x-1)^3(x-2)}$
- If $n=3$: $\frac{1}{(x-1)^3} = \frac{1}{(x-2)} - \frac{1}{(x-1)^4(x-2)}$
- ⋮
- If $n=n-1$: $\frac{1}{(x-1)^{n-1}} = \frac{1}{(x-2)} - \frac{1}{(x-1)^n(x-2)}$

$$\Rightarrow \sum_{r=0}^{n-1} \frac{1}{(x-1)^{n-1}} = \frac{1}{x-2} - \frac{1}{(x-1)^n(x-2)}$$

$$\Rightarrow \sum_{r=1}^n \frac{1}{(x-1)^r} = \frac{1}{x-2} - \frac{1}{(x-1)^n(x-2)}$$

Question 3 (*****)

Determine, in terms of k and n , a simplified expression for

$$\sum_{r=2}^n \left[\frac{r(1-k)-1}{r(r-1)k^r} \right].$$

, $\frac{1}{n} \left(\frac{1}{k} \right)^n - \frac{1}{k}$

• SPLIT BY PARTIAL FRACTIONS

$$\frac{r(1-k)-1}{r(r-1)} = \frac{A}{r} + \frac{B}{r-1}$$

$$r(1-k)-1 = A(r-1) + Br$$

If $r=0 \Rightarrow -1 = -A \Rightarrow A=1$
If $r=1 \Rightarrow -k = B \Rightarrow B=-k$

• HENCE WE KNOW THAT

$$\left(\frac{1}{k} \right)^r \frac{r(1-k)-1}{r(r-1)} = \left(\frac{1}{k} \right)^r \frac{1}{r} - \left(\frac{1}{k} \right)^r \left(\frac{k}{r-1} \right)$$

- $r=2$ $\left(\frac{1}{k} \right)^2 \frac{2(1-k)-1}{2(2-1)} = \left(\frac{1}{k} \right)^2 \times \frac{1}{2} - \left(\frac{1}{k} \right)^2 \times \frac{k}{1}$
- $r=3$ $\left(\frac{1}{k} \right)^3 \frac{3(1-k)-1}{3(3-1)} = \left(\frac{1}{k} \right)^3 \times \frac{1}{3} - \left(\frac{1}{k} \right)^3 \times \frac{k}{2}$
- $r=4$ $\left(\frac{1}{k} \right)^4 \frac{4(1-k)-1}{4(4-1)} = \left(\frac{1}{k} \right)^4 \times \frac{1}{4} - \left(\frac{1}{k} \right)^4 \times \frac{k}{3}$
- $r=5$ $\left(\frac{1}{k} \right)^5 \frac{5(1-k)-1}{5(5-1)} = \left(\frac{1}{k} \right)^5 \times \frac{1}{5} - \left(\frac{1}{k} \right)^5 \times \frac{k}{4}$
- ⋮
- $r=n$ $\left(\frac{1}{k} \right)^n \frac{n(1-k)-1}{n(n-1)} = \left(\frac{1}{k} \right)^n \times \frac{1}{n} - \left(\frac{1}{k} \right)^n \times \frac{k}{n-1}$

• ADDING

$$\sum_{r=2}^n \left[\frac{r(1-k)-1}{r(r-1)k^r} \right] = \left(\frac{1}{k} \right)^2 \times \frac{1}{2} - \frac{1}{k}$$

Question 4 (***)**

Determine the value of the following infinite convergent sum.

$$\sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right].$$

, $\boxed{\frac{1}{3}}$

Start by partial fractions (by inspection)

$$\frac{4r-1}{r(r-1)} = \frac{1}{r-1} + \frac{3}{r}$$

Hence we now have

$$\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r = \frac{1}{r-1} \left(-\frac{1}{3} \right)^r + \frac{3}{r} \left(-\frac{1}{3} \right)^r$$

- r=2: $\frac{7}{2 \cdot 1} \left(-\frac{1}{3} \right)^2 = \frac{1}{2} \left(-\frac{1}{3} \right)^2 + \frac{3}{2} \left(-\frac{1}{3} \right)^2 = \frac{1}{2} \left(-\frac{1}{3} \right)^2 + \frac{1}{2} \left(-\frac{1}{3} \right)^2$
- r=3: $\frac{11}{3 \cdot 2} \left(-\frac{1}{3} \right)^3 = \frac{1}{2} \left(-\frac{1}{3} \right)^3 + \frac{3}{2} \left(-\frac{1}{3} \right)^3 = \frac{1}{2} \left(-\frac{1}{3} \right)^3 + \frac{1}{2} \left(-\frac{1}{3} \right)^3$
- r=4: $\frac{15}{4 \cdot 3} \left(-\frac{1}{3} \right)^4 = \frac{1}{2} \left(-\frac{1}{3} \right)^4 + \frac{3}{2} \left(-\frac{1}{3} \right)^4 = \frac{1}{2} \left(-\frac{1}{3} \right)^4 + \frac{1}{2} \left(-\frac{1}{3} \right)^4$
- r=5: $\frac{19}{5 \cdot 4} \left(-\frac{1}{3} \right)^5 = \frac{1}{2} \left(-\frac{1}{3} \right)^5 + \frac{3}{2} \left(-\frac{1}{3} \right)^5 = \frac{1}{2} \left(-\frac{1}{3} \right)^5 + \frac{1}{2} \left(-\frac{1}{3} \right)^5$
- ⋮
- r=n: $\frac{4r-1}{n(n-1)} \left(-\frac{1}{3} \right)^n = \frac{1}{n-1} \left(-\frac{1}{3} \right)^n + \frac{3}{n} \left(-\frac{1}{3} \right)^n = \frac{1}{n} \left(-\frac{1}{3} \right)^n + \frac{1}{n} \left(-\frac{1}{3} \right)^n$

Therefore

$$\sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{n} \left(-\frac{1}{3} \right)^n + \frac{1}{n} \right]$$

$$\Rightarrow \sum_{r=2}^{\infty} \left[\frac{4r-1}{r(r-1)} \left(-\frac{1}{3} \right)^r \right] = \frac{1}{3}$$

Question 5 (*****)

Determine a simplified expression, in the form $\ln[f(n)]$, for the following sum.

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right].$$

, $\ln \left[\frac{2 \times 3^{N-1}}{N(N+1)} \right]$

• Start by partial fractions in the integrand (by inspection)

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right] = \sum_{r=2}^N \left[\int_2^r \frac{2}{(x-1)(x+1)} dx \right]$$

$$= \sum_{r=2}^N \left[\int_2^r \frac{1}{x-1} - \frac{1}{x+1} dx \right] = \sum_{r=2}^N \left[\ln|x-1| - \ln|x+1| \right]_{x=2}^{x=r}$$

• Writing the terms explicitly, looking for patterns

$$= \sum_{r=2}^N (\ln(r-1) - \ln(r+1)) - (\ln 1 - \ln 3)$$

$$= \sum_{r=2}^N \left[\ln(r-1) - \ln(r+1) + \ln 3 \right]$$

$$= \begin{aligned} & \ln 1 - \ln 3 + \ln 3 && \leftarrow r=2 \\ & \cancel{\ln 2} - \ln 4 + \ln 3 && \leftarrow r=3 \\ & \ln 3 - \cancel{\ln 5} + \ln 3 && \leftarrow r=4 \\ & \ln 4 - \cancel{\ln 6} + \ln 3 && \leftarrow r=5 \\ & \vdots && \end{aligned} \quad \{ (N-1) \text{ terms}$$

$$\begin{aligned} & \ln(2 \cdot 3) - \ln N + \ln 3 && \leftarrow r=N-1 \\ & \ln(N!) - \ln(N!) + \ln 3 && \leftarrow r=N \end{aligned}$$

• Adding

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right] = \ln 2 - \ln N - \ln(N+1) + (N-1)\ln 3$$

$$= \ln 2 + (N-1)\ln 3 - (\ln N + \ln(N+1))$$

$$= \ln \left[\frac{2 \cdot 3^{N-1}}{N(N+1)} \right]$$

Question 6 (*****)

Show, by a detailed method, that

$$\frac{48}{2 \times 3} + \frac{47}{3 \times 4} + \frac{46}{4 \times 5} \dots + \frac{2}{48 \times 49} + \frac{1}{49 \times 50} = A + B \sum_{r=1}^{50} \frac{1}{r},$$

where A and B are constants to be found.

, $A = \frac{51}{2}$, $B = -1$

Method: USE SIMPLIFIED NOTATION

$$\sum_{k=1}^{49} \frac{49-k}{(k+1)k(2k)} = \sum_{k=1}^{49} \left(\frac{50}{k(k+1)} - \frac{5}{k(2k)} \right)$$

REWRITING IN PARTIAL FRACTIONS

$$= \sum_{k=1}^{49} \left(\frac{50}{k(k+1)} - \frac{5}{k(2k)} \right)$$

$$= 25 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{49} \right] - \frac{5}{20}$$

$$= 25 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{49} \right] - 1 - \frac{1}{20}$$

$$= 25 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{49} + \frac{1}{50} \right]$$

$$= 25 + \frac{1}{2} - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{49} + \frac{1}{50} \right]$$

$$= \frac{51}{2} - \frac{51}{50}$$

$A = \frac{51}{2}$
 $B = -1$

Question 7 (*****)

$$\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + \frac{9}{1^2 + 2^2 + 3^2 + 4^2} + \frac{11}{1^2 + 2^2 + 3^2 + 4^2 + 5^2} + \dots,$$

Show, by a detailed method, that the sum of the first 40 terms of this series shown above is $\frac{240}{41}$.

, proof

$$\begin{aligned}
 S_{40} &= \frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \frac{11}{1^2+2^2+3^2+4^2+5^2} + \dots \\
 S_{40} &= \sum_{n=1}^{40} \left[\frac{2n+1}{\sum_{k=1}^n k^2} \right] = \sum_{n=1}^{40} \left[\frac{2n+1}{\frac{1}{6}n(n+1)(2n+1)} \right] \\
 &= 6 \sum_{n=1}^{40} \frac{1}{n(n+1)} = 6 \sum_{n=1}^{40} \left[\frac{1}{n} - \frac{1}{n+1} \right] \\
 &= 6 \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{40} - \frac{1}{41}\right) \right] \\
 &= 6 \left[1 - \frac{1}{41} \right] = 6 \times \frac{40}{41} = 6 \times \frac{60}{41} = \frac{240}{41}
 \end{aligned}$$

Question 8 (*****)

By considering the simplification of

$$\arctan(2n+1) - \arctan(2n-1),$$

determine the exact value of

$$\sum_{n=1}^{\infty} \left[\arctan\left(\frac{1}{2n^2}\right) \right].$$

$\boxed{}, \frac{\pi}{4}$

$\arctan(2n+1) - \arctan(2n-1) = \psi$

- TAKE TANGENTS ON BOTH SIDES

$$\tan[\arctan(2n+1) - \arctan(2n-1)] = \tan\psi$$

$$\frac{(2n+1) - (2n-1)}{1 + (2n+1)(2n-1)} = \tan\psi$$

$$\tan\psi = \frac{2}{1 + 4n^2 - 1} = \frac{1}{2n^2}$$

$$\psi = \arctan\left(\frac{1}{2n^2}\right)$$

- Hence $\arctan\left(\frac{1}{2n^2}\right) = \arctan(2n+1) - \arctan(2n-1)$

$n=1:$	$\arctan\left(\frac{1}{2}\right) = \arctan 3 - \arctan 1$
$n=2:$	$\arctan\left(\frac{1}{8}\right) = \arctan 5 - \arctan 3$
$n=3:$	$\arctan\left(\frac{1}{18}\right) = \arctan 7 - \arctan 5$
\vdots	\vdots
$n=k:$	$\arctan\left(\frac{1}{2k^2}\right) = \arctan(2k+1) - \arctan(2k-1)$

- SUMMING

$$\sum_{n=1}^k \arctan\left(\frac{1}{2n^2}\right) = \arctan(2k+1) - \arctan 1$$

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{2n^2}\right) = \lim_{k \rightarrow \infty} [\arctan(2k+1) - \arctan 1]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Question 9 (*****)

$$S_n = (2 \times 1!) + (5 \times 2!) + (10 \times 3!) + (17 \times 4!) + \dots + (n^2 + 1)n$$

Use an appropriate method to show that

$$S_n = n(n+1)$$

[redacted], proof

TRY BY WRITING THE SERIES IN SUMMATION NOTATION

$$(2x+1)! - (2x+2)! + (2x+3)! - (2x+4)! + \dots + (-1)^{n-1} (2x+n)! = \sum_{r=1}^n (-1)^{r-1} (2x+r)!$$

TRY SOME DIFFERENT INDIVIDUAL PROBLEMS, TRYING TO OBTAIN THE SUMMATION

$$(r+1)! - r! = (r+1)r! - r! = r(r+1)!$$

AS THIS DOES NOT PROVE A QUADRATIC TERM IN r , WE MAY TRY

$$(r+2)! - r! = (r+1)(r+2)r! - r! \\ (r+2)r! - r! = (r^2+3r+2)r! - r! \\ (r+2)r! - r! = (r^2+3r+1)r! \\ (r+2)r! - r! = (r^2+3r+1)r!$$

\uparrow

$$r(r+1)r! \equiv (r+1)r! - r!$$

$$(r+2)r! - r! = (r+1)r! + 3[r(r+1) - r!] \\ (r+2)r! - r! = (r^2+3r+1)r! - 3r! \\ (r+2)r! = (r^2+3r+1)r! - 2r! \\ (r+2)r! - 3(r+1)r! + 2r! = (r+1)r!$$

HENCE WE HAVE

$$(r+1)r! \equiv (r+2) - 3(r+1) + 2r!$$

WRITING THE IDENTITY JUST DEMONSTRATED

$$(r^k)(r)! \equiv (r+2)! - 3(r+1)! + 2r!$$

$r=1$	$2 \times 1! = 2! - 3 \times 2! + 2 \times 1!$
$r=2$	$5 \times 2! = 4! - 3 \times 3! + 2 \times 2!$
$r=3$	$10 \times 3! = 5! - 3 \times 4! + 2 \times 3!$
$r=4$	$17 \times 4! = 6! - 3 \times 5! + 2 \times 4!$
	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$r=n$	$(n+2)^2 \cdot n! = (n+1)! - 3(n+1)! + 2(n+1)!$
$r=n$	$(n+3) \cdot n! = (n+2)! - 3(n+2)! + 2(n+2)!$
$\sum_{r=1}^n [r^k] n! =$	$(n+2)! - 2(n+1)! - 3n! + 2n! + 2n2!$
	$= (n+2)(n+1)! - 2(n+1)! - 6n+4$
	$= (n+2-2)(n+1)!$
	$= n(n+1)!$

Question 10 (*****)

By considering the trigonometric identity for $\tan(A - B)$, with $A = \arctan(n+1)$ and $B = \arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right).$$

You may assume the series converges.

, , $\frac{\pi}{4}$

CONSIDER THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$

$$\begin{aligned} \Rightarrow \tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \\ \Rightarrow \tan(\arctan(n+1) - \arctan n) &= \frac{\tan(\arctan(n+1)) - \tan(\arctan n)}{1 + \tan(\arctan(n+1)) \tan(\arctan n)} \\ \Rightarrow \tan(\arctan(n+1) - \arctan n) &= \frac{(n+1)\cdot n}{1 + (n+1)n} \\ \Rightarrow \tan(\arctan(n+1) - \arctan n) &= \frac{n}{n^2+n+1} \\ \Rightarrow \arctan[\tan(\arctan(n+1) - \arctan n)] &= \arctan\left(\frac{1}{n^2+n+1}\right) \\ \Rightarrow \arctan(n+1) - \arctan n &= \arctan\left(\frac{1}{n^2+n+1}\right) \end{aligned}$$

DENOTE THE SUMMATION NOW AS

$$\begin{aligned} \sum_{k=1}^{\infty} \arctan\left(\frac{1}{k^2+k+1}\right) &= \sum_{k=1}^{\infty} [\arctan(k+1) - \arctan k] \\ &= \sum_{k=1}^{\infty} \arctan(k+1) - \sum_{k=1}^{\infty} \arctan k \\ &= \lim_{k \rightarrow \infty} \left[\sum_{k=1}^{\infty} \arctan(k+1) - \sum_{k=1}^{\infty} \arctan k \right] \end{aligned}$$

PROCEED AS FOLLOWS - NOTE THE "CANCELLATION"

$$\begin{aligned} \lim_{k \rightarrow \infty} \left[\arctan(k+1) - \arctan k \right] &+ \left[\arctan k - \arctan(k-1) \right] \\ &+ \left[\arctan(k-1) - \arctan(k-2) \right] + \dots \\ &\vdots \\ &+ \left[\arctan 3 - \arctan 2 \right] \\ &\arctan 2 - \arctan 1 \end{aligned}$$

$$\begin{aligned} &= \lim_{k \rightarrow \infty} [\arctan(k+1) - \arctan 1] \\ &= \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{4} \end{aligned}$$

Question 11 (*****)

Determine, in terms of n , a simplified expression

$$\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right],$$

and hence, or otherwise, deduce the value of

$$\sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right].$$

_____	$\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{1}{6} - \frac{n+5}{(n+5)!}$	$\sum_{r=1}^{\infty} \left[\frac{r^2 + 7r + 11}{(r+4)!} \right] = \frac{5}{24}$
-------	---	--

► Start with partial fractions — note that the numerator is a quadratic in r , so we have to try 2 fractions.

i.e. $\frac{r^2 + 9r + 19}{(r+5)!} \equiv \frac{A}{(r+5)!} + \frac{B}{(r+3)!}$

$$\Rightarrow r^2 + 9r + 19 \equiv A + B(r+5)(r+4)$$

$$\Rightarrow r^2 + 9r + 19 \equiv Br^2 + 9Br + 20B + (2AB)r + A$$

$$\therefore B=1 \text{ & } A=-1$$

► Hence by the method of differences,

$\frac{r^2 + 9r + 19}{(r+5)!} = \frac{1}{(r+5)!} - \frac{1}{(r+3)!}$
--

$r=1 \quad \frac{1+9+19}{6!} = \frac{1}{4!} - \frac{1}{6!}$
 $r=2 \quad \frac{4+18+19}{7!} = \frac{1}{3!} - \frac{1}{7!}$
 $r=3 \quad \frac{9+27+19}{8!} = \frac{1}{2!} - \frac{1}{8!}$
 $r=4 \quad \frac{16+36+19}{9!} = \frac{1}{1!} - \frac{1}{9!}$
 \vdots
 $r=n-1 \quad \frac{(n-1)^2 + 9(n-1) + 19}{(n+4)!} = \frac{1}{(n+4)!} - \frac{1}{(n+2)!}$
 $r=n \quad \frac{n^2 + 9n + 19}{(n+5)!} = \frac{1}{(n+5)!} - \frac{1}{(n+3)!}$

► Adding, $\sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] = \frac{1}{4!} + \frac{1}{3!} - \left[\frac{1}{(n+4)!} + \frac{1}{(n+3)!} \right]$

► Now proceed as follows:

$$\begin{aligned} \sum_{r=1}^n \left[\frac{r^2 + 9r + 19}{(r+5)!} \right] &= \frac{6}{5!} - \frac{n+6}{(n+5)!} \\ &\stackrel{\text{CROSS OUT THE } 6\text{'S}}{=} \frac{6}{5!} - \frac{n+6}{(n+5)!} \\ &\Rightarrow \sum_{r=2}^n \left[\frac{(r-1)^2 + 9(r-1) + 19}{(r+4)!} \right] = \frac{6}{5!} - \frac{n+6}{(n+5)!} \\ &\Rightarrow \sum_{r=2}^n \left[\frac{r^2 - 2r + 19}{(r+4)!} \right] = \frac{6}{5!} - \frac{n+6}{(n+5)!} \\ &\Rightarrow \sum_{r=2}^n \left[\frac{r^2 - 2r + 11}{(r+4)!} \right] = \frac{6}{5!} - \frac{n+6}{(n+5)!} \\ &\Rightarrow \sum_{r=1}^n \left[\frac{r^2 - 2r + 11}{(r+4)!} \right] = \frac{1+7+11}{5!} + \frac{6}{5!} - \frac{n+6}{(n+5)!} \\ &\Rightarrow \sum_{r=1}^n \left[\frac{r^2 - 2r + 11}{(r+4)!} \right] = \frac{28}{5!} - \frac{n+6}{(n+5)!} \\ &\Rightarrow \sum_{r=1}^n \left[\frac{r^2 - 2r + 11}{(r+4)!} \right] = \frac{5}{4!} - \frac{n+6}{(n+5)!} \\ &\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 - 2r + 11}{(r+4)!} \right] = \lim_{n \rightarrow \infty} \left[\frac{5}{24} - \frac{n+6}{(n+5)!} \right] \\ &\Rightarrow \sum_{r=1}^{\infty} \left[\frac{r^2 - 2r + 11}{(r+4)!} \right] = \frac{5}{24} \end{aligned}$$

Question 12 (*****)

A sequence is defined as

$$u_{r+1} = u_r + \frac{2r}{r^4 + r^2 + 1}, \quad u_1 = 0, \quad r \in \mathbb{N}.$$

Determine the exact value of u_{61} .

, $u_{61} = \frac{3660}{3661}$

$\therefore u_6 = ?$, $u_{r+1} = u_r + \frac{2r}{r^4 + r^2 + 1}$, $u_1 = 0$

• START BY FACTORISING $r^4 + r^2 + 1$ ALSO TWO QUADRATIC TRAILS.

$$(r^2 + r + 1)(r^2 - r + 1) = r^4 - r^2 + r^2 - r + 1$$

$$\frac{r^2 - r + 1}{r^4 - r^2 + r^2 - r + 1}$$

• BY INSPECTION.

$$\frac{1}{r^2 - r + 1} = \frac{1}{r^2 + r + 1} = \frac{(r^2 + r + 1) - (r^2 - r + 1)}{(r^2 + r + 1)(r^2 - r + 1)} = \frac{2r}{r^4 + r^2 + 1}$$

• REWRITE THE ABOVE EXPRESSION AS FOLLOWS

$u_{r+1} - u_r =$	$\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1}$
1 $u_2 - u_1 =$	$1 - \frac{1}{3}$
2 $u_3 - u_2 =$	$\frac{1}{3} - \frac{1}{7}$
3 $u_4 - u_3 =$	$\frac{1}{7} - \frac{1}{13}$
4 $u_5 - u_4 =$	$\frac{1}{13} - \frac{1}{21}$
5 $u_6 - u_5 =$	\dots
6 $u_7 - u_6 =$	$\frac{1}{21} - \frac{1}{31}$
7 $u_8 - u_7 =$	\dots
8 $u_9 - u_8 =$	$\frac{1}{31} - \frac{1}{37}$
9 $u_{10} - u_9 =$	\dots
10 $u_{11} - u_{10} =$	$\frac{1}{37} - \frac{1}{43}$
11 $u_{12} - u_{11} =$	\dots
12 $u_{13} - u_{12} =$	$\frac{1}{43} - \frac{1}{47}$
13 $u_{14} - u_{13} =$	\dots
14 $u_{15} - u_{14} =$	$\frac{1}{47} - \frac{1}{53}$
15 $u_{16} - u_{15} =$	\dots
16 $u_{17} - u_{16} =$	$\frac{1}{53} - \frac{1}{59}$
17 $u_{18} - u_{17} =$	\dots
18 $u_{19} - u_{18} =$	$\frac{1}{59} - \frac{1}{65}$
19 $u_{20} - u_{19} =$	\dots
20 $u_{21} - u_{20} =$	$\frac{1}{65} - \frac{1}{71}$
21 $u_{22} - u_{21} =$	\dots
22 $u_{23} - u_{22} =$	$\frac{1}{71} - \frac{1}{77}$
23 $u_{24} - u_{23} =$	\dots
24 $u_{25} - u_{24} =$	$\frac{1}{77} - \frac{1}{83}$
25 $u_{26} - u_{25} =$	\dots
26 $u_{27} - u_{26} =$	$\frac{1}{83} - \frac{1}{89}$
27 $u_{28} - u_{27} =$	\dots
28 $u_{29} - u_{28} =$	$\frac{1}{89} - \frac{1}{95}$
29 $u_{30} - u_{29} =$	\dots
30 $u_{31} - u_{30} =$	$\frac{1}{95} - \frac{1}{101}$
31 $u_{32} - u_{31} =$	\dots
32 $u_{33} - u_{32} =$	$\frac{1}{101} - \frac{1}{107}$
33 $u_{34} - u_{33} =$	\dots
34 $u_{35} - u_{34} =$	$\frac{1}{107} - \frac{1}{113}$
35 $u_{36} - u_{35} =$	\dots
36 $u_{37} - u_{36} =$	$\frac{1}{113} - \frac{1}{119}$
37 $u_{38} - u_{37} =$	\dots
38 $u_{39} - u_{38} =$	$\frac{1}{119} - \frac{1}{125}$
39 $u_{40} - u_{39} =$	\dots
40 $u_{41} - u_{40} =$	$\frac{1}{125} - \frac{1}{131}$
41 $u_{42} - u_{41} =$	\dots
42 $u_{43} - u_{42} =$	$\frac{1}{131} - \frac{1}{137}$
43 $u_{44} - u_{43} =$	\dots
44 $u_{45} - u_{44} =$	$\frac{1}{137} - \frac{1}{143}$
45 $u_{46} - u_{45} =$	\dots
46 $u_{47} - u_{46} =$	$\frac{1}{143} - \frac{1}{149}$
47 $u_{48} - u_{47} =$	\dots
48 $u_{49} - u_{48} =$	$\frac{1}{149} - \frac{1}{155}$
49 $u_{50} - u_{49} =$	\dots
50 $u_{51} - u_{50} =$	$\frac{1}{155} - \frac{1}{161}$
51 $u_{52} - u_{51} =$	\dots
52 $u_{53} - u_{52} =$	$\frac{1}{161} - \frac{1}{167}$
53 $u_{54} - u_{53} =$	\dots
54 $u_{55} - u_{54} =$	$\frac{1}{167} - \frac{1}{173}$
55 $u_{56} - u_{55} =$	\dots
56 $u_{57} - u_{56} =$	$\frac{1}{173} - \frac{1}{179}$
57 $u_{58} - u_{57} =$	\dots
58 $u_{59} - u_{58} =$	$\frac{1}{179} - \frac{1}{185}$
59 $u_{60} - u_{59} =$	\dots
60 $u_{61} - u_{60} =$	$\frac{1}{185} - \frac{1}{191}$

ANSWER $u_6 = 0 \Rightarrow 1 - \frac{1}{191} = \frac{1}{191} \therefore u_{61} = \frac{3660}{3661}$

Question 13 (*****)

Find the value of

$$\sum_{r=0}^{\infty} \left[\frac{\sin^4(\pi \times 2^{r-2})}{4^r} \right].$$

Hint: Express $\sin^4 \theta$ in terms of $\sin^2 \theta$ and $\sin^2 2\theta$ only.

, $\frac{1}{2}$

• START BY MANIPULATING THE SUM TO THE FURTHER

$$\sin^4 \theta = (\sin^2 \theta)^2 = \left(\frac{1}{4} - \frac{1}{4} \cos 2\theta \right)^2 = \frac{1}{16} - \frac{1}{8} \cos 2\theta + \frac{1}{16} \cos^2 2\theta$$

$$= \frac{1}{16} - \frac{1}{8}(1 - 2 \sin^2 \theta) + \frac{1}{16}(1 - \sin^2 2\theta)$$

$$= \frac{1}{16} - \frac{1}{8} + \frac{1}{4} \sin^2 \theta + \frac{1}{16} - \frac{1}{16} \sin^2 2\theta$$

$$= \sin^2 \theta - \frac{1}{16} \sin^2 2\theta$$

• NOW WE NOTE BY CONSIDERING THE SUM OF THE FIRST n TERMS

$$\sum_{r=0}^{n-1} \frac{\sin^4(\pi \times 2^{r-2})}{4^r} = \sum_{r=0}^{n-1} \left[\frac{1}{4^r} \left[\sin^2(\pi \times 2^{r-2}) - \frac{1}{16} \sin^2(\pi \times 2^{r-2}) \right] \right]$$

$$= \sum_{r=0}^{n-1} \left[\frac{1}{2^r} \sin^2(\pi \times 2^{r-2}) - \frac{1}{4^r} \sin^2(\pi \times 2^{r-2}) \right]$$

$$= \frac{\sin^2 \frac{\pi}{4}}{2^0} - \frac{1}{4^0} \sin^2 \frac{\pi}{4} \quad \leftarrow r=0$$

$$\frac{1}{2^1} \sin^2 \frac{\pi}{2} - \frac{1}{4^1} \sin^2 \frac{\pi}{2} \quad \leftarrow r=1$$

$$\frac{1}{2^2} \sin^2 \frac{\pi}{4} - \frac{1}{4^2} \sin^2 \frac{\pi}{4} \quad \leftarrow r=2$$

$$\frac{1}{2^3} \sin^2 \frac{\pi}{2} - \frac{1}{4^3} \sin^2 \frac{\pi}{2} \quad \leftarrow r=3$$

$$\vdots$$

$$\frac{1}{2^n} \sin^2 \frac{\pi}{4} - \frac{1}{4^n} \sin^2 \frac{\pi}{4} \quad \leftarrow r=n$$

$$= \sin^2 \frac{\pi}{4} - \frac{1}{4^n} \sin^2(\pi \times 2^{n-2})$$

• Hence we have

$$\sum_{r=0}^{n-1} \frac{\sin^4(\pi \times 2^{r-2})}{4^r} = \sin^2 \frac{\pi}{4} - \frac{1}{4^n} \left(\frac{\pi}{2} \right)^2 = \frac{1}{2}$$

Question 14 (*****)

Find the sum to infinity of the following convergent series.

$$\frac{1}{4 \times 2!} + \frac{1}{5 \times 3!} + \frac{1}{6 \times 4!} + \frac{1}{7 \times 5!} + \frac{1}{8 \times 6!} + \dots$$

, $\frac{1}{6}$

$\frac{1}{4 \times 2!} + \frac{1}{5 \times 3!} + \frac{1}{6 \times 4!} + \frac{1}{7 \times 5!} + \frac{1}{8 \times 6!} + \dots$

WRITING THE SERIES IN SIGMA NOTATION

$$S_{\infty} = \sum_{r=1}^{\infty} \frac{1}{(r+3)(r+2)!}$$

ATTEMPT SUMMATION BY THE METHOD OF DIFFERENCES

TRY $\frac{1}{(r+3)(r+1)!} = \frac{A}{(r+3)!} + \frac{B}{(r+1)!}$

$$1 = A + B(r+3)(r+2)$$

NO A & B CAN SATISFY THE ABOVE

TRY NEXT $\frac{1}{(r+3)(r+1)!} = \frac{A}{(r+3)!} + \frac{B}{(r+2)!}$

$$\Rightarrow \frac{1}{(r+3)(r+2)(r+1)!} = \frac{A + B(r+2)}{(r+3)!}$$

$$\Rightarrow \frac{r+2}{(r+3)(r+2)(r+1)!} = \frac{A + B(r+3)}{(r+4)!}$$

$$\Rightarrow \frac{r+2}{(r+3)!} = \frac{A + B(r+3)}{(r+3)!}$$

$$\Rightarrow r+2 = (A+3B) + Br$$

$\therefore B=1$ & $A=-1$

HENCE WE NOW HAVE A SURPRISE IDENTITY

$$\frac{1}{(r+3)(r+2)!} \equiv \frac{1}{(r+2)!} - \frac{1}{(r+3)!}$$

- $r=1$: $\frac{1}{4 \times 2!} = \frac{1}{3!} - \frac{1}{4!}$
- $r=2$: $\frac{1}{5 \times 3!} = \frac{1}{4!} - \frac{1}{5!}$
- $r=3$: $\frac{1}{6 \times 4!} = \frac{1}{5!} - \frac{1}{6!}$
- $r=4$: $\frac{1}{7 \times 5!} = \frac{1}{6!} - \frac{1}{7!}$
- ⋮
- $r=N$: $\frac{1}{(N+3)(N+2)!} = \frac{1}{(N+1)!} - \frac{1}{(N+3)!}$

$$\Rightarrow \sum_{r=1}^{N-1} \frac{1}{(r+3)(r+2)!} = \frac{1}{3!} - \frac{1}{(N+3)!}$$

$$\Rightarrow \lim_{N \rightarrow \infty} \left[\sum_{r=1}^{N-1} \frac{1}{(r+3)(r+2)!} \right] = \lim_{N \rightarrow \infty} \left[\frac{1}{3!} - \frac{1}{(N+3)!} \right]$$

$$\Rightarrow \sum_{r=1}^{\infty} \frac{1}{(r+3)(r+2)!} = \frac{1}{3!} = \frac{1}{6}$$

Question 15 (*****)

Evaluate the following expression

$$\sum_{k=1}^{\infty} \left[\sum_{r=1}^k r \right]^{-1}.$$

QP , [2]

● REWRITE FOR SIMPLICITY AS FOLLOWS

$$\sum_{k=1}^{\infty} \left[\sum_{r=1}^k r \right]^{-1} = \sum_{k=1}^{\infty} \left[\frac{1}{\frac{k(k+1)}{2}} \right] = \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

● INTRODUCE A FINITE LIMIT FOR THE SUMMATION, SAY n

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{1}{\frac{r(r+1)}{2}} \right] \\ &= \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{2}{r(r+1)} \right] \\ &\approx 2 \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{1}{r(r+1)} \right] \end{aligned}$$

● SPOT INTO TWO FRACTIONS BY INTEGRATION

$$\begin{aligned} &= 2 \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{1}{r} - \frac{1}{r+1} \right] \\ &= 2 \lim_{n \rightarrow \infty} \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \right] \\ &= 2 \lim_{n \rightarrow \infty} \left[1 - \frac{1}{n+1} \right] \\ &\approx 2 \end{aligned}$$