

PARAMETERIZATION

Question 1

A curve C is defined parametrically

$$(x, y, z) = (3 \cos t, 3 \sin t, 4t), \quad 0 \leq t \leq 5\pi.$$

where t is a parameter.

- a) Sketch the graph of C .
- b) Find the length of C .

25π

4) $(x, y, z) = (3 \cos t, 3 \sin t, 4t)$ $0 \leq t \leq 5\pi$

- IT IS A HELIX
- AXIS OF ROTATION IS THE Z AXIS SPANNING AT $(0,0,0)$ & ENDING AT $(0,0,20\pi)$

b) LENGTH OF C

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(3 \cos t)^2 + (3 \sin t)^2 + 4^2}$$

$$= \sqrt{9 \cos^2 t + 9 \sin^2 t + 16} = \sqrt{9 + 16} = 5$$

$$\therefore s = \int_{t=0}^{5\pi} 1 dt = \int_{t=0}^{5\pi} 5 dt = 25\pi$$

Question 2

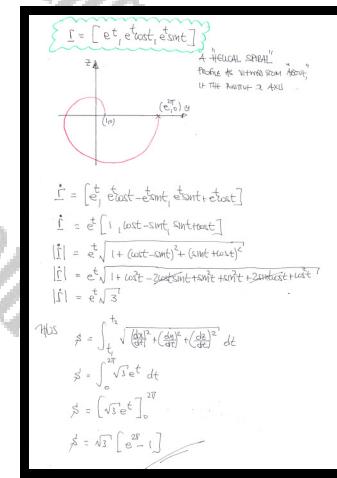
A curve C is defined parametrically

$$(x, y, z) = (e^t, e^t \cos t, e^t \sin t), \quad 0 \leq t \leq 2\pi.$$

where t is a parameter.

Describe the graph of C and find its length.

$$\text{arclength} = \sqrt{3} [e^{2\pi} - 1]$$



Question 3

The position vector of a curve C is given by

$$\mathbf{r}(t) = \cos(\cosh t)\mathbf{i} + \sin(\cosh t)\mathbf{j} + tk\mathbf{k},$$

where t is a scalar parameter with $0 \leq t \leq a$, $a \in \mathbb{R}$.

Determine the length of C .

$$\boxed{\text{arclength} = \sinh a}$$

$$\begin{aligned}\mathbf{r}(t) &= [\cos(\cosh t), \sin(\cosh t), t] \quad 0 \leq t \leq a \\ \dot{\mathbf{r}} &= \int_{t_1}^t \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt \\ \dot{x} &= -\sin(\cosh t) \times \sinh t \Rightarrow \dot{x}^2 = \sin^2(\cosh t) \sinh^2 t \\ \dot{y} &= \cos(\cosh t) \times \sinh t \Rightarrow \dot{y}^2 = \cos^2(\cosh t) \sinh^2 t \\ \dot{z} &= 1 \\ \text{Thus} \quad \dot{s} &= \int_0^a \sqrt{\sin^2(\cosh t) \sinh^2 t + \cos^2(\cosh t) \sinh^2 t + 1} dt \\ \dot{s} &= \int_0^a \sqrt{\sinh^2 t [\sin^2(\cosh t) + \cos^2(\cosh t)] + 1} dt \\ \dot{s} &= \int_0^a \sqrt{\sinh^2 t + 1} dt = \int_0^a \cosh t dt \\ s &= \int \sinh t dt = \sinh t + \text{constant} = \sinh a\end{aligned}$$

Question 4

Evaluate the integral

$$\int_{(1,0)}^{(3,3)} (y+x) dx + (y-x) dy,$$

along the curve with parametric equations

$$x = 2t^2 - 3t + 1 \quad \text{and} \quad y = t^2 - 1.$$

$\boxed{\frac{28}{3}}$

(3,3)

$$\int_{(1,0)}^{(3,3)} (y+x) dx + (y-x) dy = \dots$$

PARAMETRIC

$x = 2t^2 - 3t + 1$ $dx = (4t-3)dt$

$y = t^2 - 1$ $dy = 2t dt$

$t \in [1, 2]$ $\Rightarrow dt = \frac{1}{2} dt$

BY SUBSTITUTION

$$\begin{aligned} & \int_{t=1}^{t=2} (t^2 - 1)(4t-3)dt + (-t^2 + 3t - 1) dt \\ &= \int_1^2 (12t^3 - 2t^2 + 9t - 1) dt \\ &= \left[3t^4 - \frac{2t^3}{3} + 6t^2 - 2t \right]_1^2 \\ &= 48 - \frac{176}{3} + 24 - 4 \\ &= \frac{28}{3} \end{aligned}$$

Question 5

It is given that

$$\mathbf{F}(x, y, z) \equiv \mathbf{j} \wedge \mathbf{r},$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the closed curve given parametrically by

$$\mathbf{R}(t) = (t - t^2)\mathbf{i} + (2t - 2t^2)\mathbf{j} + (t^2 - t^3)\mathbf{k}, \quad 0 \leq t \leq 1.$$

$$\boxed{-\frac{1}{30}}$$

$$\begin{aligned}
 \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = (z\mathbf{i} - y\mathbf{j} - x\mathbf{k}) = (z\mathbf{i} - y\mathbf{j} - x\mathbf{k}) \\
 \text{param.} \\
 C: \quad \frac{d\mathbf{R}}{dt} &= \begin{bmatrix} t - 1 \\ 2t - 2t^2 \\ t^2 - 3t^2 \end{bmatrix} = \begin{bmatrix} 1 - 2t \\ 2 - 4t \\ 2t - 3t^2 \end{bmatrix} \\
 \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (z\mathbf{i} - y\mathbf{j} - x\mathbf{k}) \cdot (dx, dy, dz) \\
 &= \int_C z \, dx - y \, dy - x \, dz \\
 &\stackrel{\circlearrowleft}{=} \int_{t=0}^{t=1} (t^2 - t^3)(1 - 2t) \, dt - (t - t^2)(2t - 3t^2) \, dt \\
 &= \int_0^1 \cancel{t^2} - \cancel{2t^3} - \cancel{t^3} + \cancel{2t^4} - \cancel{2t^2} + \cancel{3t^3} - \cancel{3t^4} \, dt \\
 &= \int_0^1 -t^2 + 2t^3 - t^2 \, dt = \left[-\frac{1}{3}t^3 + \frac{1}{4}t^4 - \frac{1}{3}t^3 \right]_0^1 \\
 &= \left(-\frac{1}{3} + \frac{1}{2} - \frac{1}{3} \right) - (0) = \frac{-6 + 15 - 10}{30} = -\frac{1}{30}
 \end{aligned}$$

Question 6

Evaluate the integral

$$\int_{(1,1,0)}^{(5,3,4)} (3x - 2y) \, dx + (y + z) \, dy + (1 - z^2) \, dz,$$

along the straight line segment joining the points with Cartesian coordinates $(1,1,0)$ and $(5,3,4)$.

$$\boxed{\frac{32}{3}}$$

Firstly $\mathbf{A} = (1,1,0)$ & $\mathbf{b} = (5,3,4)$
 $\mathbf{b} - \mathbf{a} = (5,2,4) - (1,1,0) = (4,1,4)$

PARAMETRIZE THE LINE SEGMENT $\mathbf{r} = (1+4t, 1+t, 4t)$
 $y = 1+t$
 $z = 4t$

$$\begin{aligned} dy &= 1 \, dt \\ dz &= 4 \, dt \\ \Rightarrow \frac{dy}{dt} &= 1 \\ \frac{dz}{dt} &= 4 \end{aligned}$$

Thus

$$\begin{aligned} &\int_{(1,1,0)}^{(5,3,4)} (3x - 2y) \, dx + (y + z) \, dy + (1 - z^2) \, dz \\ &= \int_{t=0}^1 [3(1+4t) - 2(1+t)] \, d(4t) + \int_{t=0}^1 [(1+t) + 4t] \, (2 \, dt) + \int_{t=0}^1 [1 - (4t)^2] \, (4 \, dt) \\ &= \int_0^1 [4(8t+1) + 2(5t+1) + 4(1-16t^2)] \, dt \\ &= \int_0^1 [-40t^2 + 44t + 10] \, dt \\ &= \left[-\frac{40}{3}t^3 + 22t^2 + 10t \right]_0^1 = \left(-\frac{40}{3} + 22 + 10 \right) - (0) = \frac{32}{3} \end{aligned}$$

Question 7

A surface S has equation

$$x^2 + y^2 - z^2 = 1.$$

Find a suitable parameterization of S .

$$\boxed{x = \cos \theta \cosh t, \quad y = \sin \theta \cosh t, \quad z = \sinh t}$$

The note shows the derivation of the parameterization for the surface $x^2 + y^2 - z^2 = 1$. It starts by noting that $x^2 + y^2 = r^2$ in polar coordinates, leading to the parametrization $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$. Then, it shows that the equation becomes $r^2 - z^2 = 1$. Using the identity $\cosh^2 t - \sinh^2 t = 1$, it leads to the parametrization $\begin{cases} r = \cosh t \\ z = \sinh t \end{cases}$. Finally, combining these with the polar coordinates gives the parameterization $\begin{cases} x = \cos \theta \cosh t \\ y = \sin \theta \cosh t \\ z = \sinh t \end{cases}$.

Question 8

The position vector of a curve C is given by

$$\mathbf{r}(t) = \left(\frac{2}{1+t^2} - 1 \right) \mathbf{i} + \left(\frac{2t}{1+t^2} \right) \mathbf{j},$$

where t is a scalar parameter with $t \in \mathbb{R}$.

Find an expression for the position vector of C , giving the answer in the form

$$\mathbf{r}(s) = f(s) \mathbf{i} + g(s) \mathbf{j},$$

where s is the arc length of a general point on C , measured from the point $(1, 0)$.

$$\boxed{\mathbf{r}(s) = (\cos s) \mathbf{i} + (\sin s) \mathbf{j}}$$

$\mathbf{r}(t) = \left[\frac{2}{1+t^2} - 1 \right] \mathbf{i} + \left[\frac{2t}{1+t^2} \right] \mathbf{j}$

• $x = \frac{2}{1+t^2} - 1 \Rightarrow \dot{x} = \frac{(4t^2)x - 2(2t)}{(1+t^2)^2} = -\frac{4t}{(1+t^2)^2}$

• $y = \frac{2t}{1+t^2} \Rightarrow \dot{y} = \frac{(1+t^2)x^2 - 2x(2t)}{(1+t^2)^2} = \frac{2+2t^2 - 4t^2}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2}$

$\therefore \int_0^t \Rightarrow t=0$

$$\begin{aligned} \int_0^t \sqrt{\dot{x}^2 + \dot{y}^2} dt &= \int_0^t \sqrt{\frac{16t^2}{(1+t^2)^4} + \frac{(2-2t^2)^2}{(1+t^2)^4}} dt \\ &= \int_0^t \sqrt{\frac{16t^2 + 4 - 8t^2 + 4t^4}{(1+t^2)^4}} dt = \int_0^t \sqrt{\frac{4t^4 + 8t^2 + 4}{(1+t^2)^4}} dt \\ &\quad - \int_0^t \sqrt{\frac{4(2t^2+1)}{(1+t^2)^4}} dt = \int_0^t \sqrt{\frac{4(2t^2+1)}{(1+t^2)^4}} dt = \int_0^t \frac{2(2t^2+1)}{(1+t^2)^2} dt \\ &= \int_0^t \frac{2}{1+t^2} dt = \left[2 \arctan t \right]_0^t = 2 \arctan t - \arctan 0 \end{aligned}$$

Thus $\frac{s}{2} = 2 \arctan t$
 $\frac{s}{2} = \arctan t$
 $\tan \frac{s}{2} = t$

• $x = \frac{2}{1+t^2} - 1 = \frac{2 - \frac{s^2}{4}}{1+\frac{s^2}{4}} = \frac{1-\frac{s^2}{4}}{1+\frac{s^2}{4}} = \cos \frac{s}{2}$

• $y = \frac{2t}{1+t^2} = \sin \frac{s}{2}$

These are the little t identities

$\therefore \mathbf{r}(s) = \cos \frac{s}{2} \mathbf{i} + \sin \frac{s}{2} \mathbf{j}$

Question 9

Evaluate the line integral

$$\oint_C y^5 \, dx,$$

where C is a circle of radius 2, centre at the origin O , traced anticlockwise.

You may not use Green's theorem in this question.

-40π

The diagram shows a circle of radius 2 centered at the origin O in a Cartesian coordinate system. The circle is traced anticlockwise. The angle θ is measured from the positive x-axis. The circle passes through points $(2\cos\theta, 2\sin\theta)$ where $0 \leq \theta < 2\pi$.

Method 1 (Parametric Equations):

$$\begin{aligned} \oint_C y^5 \, dx &= \int_0^{2\pi} y^5 \, d\theta \\ &= \int_0^{2\pi} (4-2\cos\theta)^5 \, d\theta \\ &= \int_{-2}^2 (4-x)^5 \, dx + \int_{-2}^2 (4+x)^5 \, dx \\ &= \int_{-2}^2 (4-x)^5 \, dx + \int_2^{-2} (4-x)^5 \, dx \\ &\text{(by symmetry)} \\ &= \int_2^{-2} -2(4-x)^{\frac{5}{2}} \, dx = \dots \text{(using } u=4-x) \dots = \int_2^{-2} -4(4-x)^{\frac{3}{2}} \, dx \\ &\text{(by substitution)} \\ &\quad u = 4-x \\ &\quad du = -dx \\ &\quad dx = -du \\ &\quad x = 2\cos\theta \Rightarrow u = 4-2\cos\theta \\ &\quad \theta = 0 \Rightarrow u = 2 \\ &\quad \theta = 2\pi \Rightarrow u = 4 \\ &= \int_0^{\pi} -4(4-4\sin\theta)^{\frac{3}{2}} (2\cos\theta \, d\theta) = -4 \int_0^{\pi} 2x^2 \sin\theta \, d\theta \\ &= -128 \int_0^{\pi} (\cos\theta)^{2\frac{3}{2}-1} (\sin\theta)^{2\frac{1}{2}-1} \, d\theta = -128 B\left(\frac{3}{2}, \frac{1}{2}\right) \\ &= -128 \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = -128 \times \frac{\frac{1}{2}\pi \times \frac{1}{2}\pi}{\pi} = -128 \times \frac{\frac{1}{2}\pi \times \frac{1}{2}\pi}{\pi} \\ &= -128 \times \frac{\frac{1}{2}\pi \times \sqrt{\pi} \times \sqrt{\pi}}{\pi} = -40\pi \end{aligned}$$

Method 2 (Polar Coordinates):

$$\begin{aligned} \text{Given } x^2 + y^2 = 4 \\ x = 2\cos\theta \\ y = 2\sin\theta \\ dx = -2\sin\theta \, d\theta \\ 0 \leq \theta < 2\pi \end{aligned}$$

$$\begin{aligned} \oint_C y^5 \, dx &= \int_0^{2\pi} (2\sin\theta)^5 (-2\sin\theta) \, d\theta = -16 \int_0^{2\pi} \sin^6 \theta \, d\theta \\ &= -16 \int_0^{2\pi} \frac{1}{2}(1+4\cos^2\theta)(1+4\cos^2\theta)^2 \, d\theta \\ &= -128 B\left(\frac{3}{2}, \frac{1}{2}\right) = \dots \text{as above} \\ &= -40\pi \end{aligned}$$

Question 10

A curve C is defined by $\mathbf{r} = \mathbf{r}(t)$, $0 \leq t \leq 2\pi$ as

$$\mathbf{r}(t) = (x, y, z) = [2(t - \sin t), \sqrt{3} \cos t, 1 + \cos t].$$

Evaluate the integral

$$\int_C z \, ds,$$

where s is the arclength along C .

$\frac{32}{3}$

$x = 2(t - \sin t)$
 $y = \sqrt{3} \cos t \quad 0 \leq t \leq 2\pi$
 $z = 1 + \cos t$

• Firstly $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{[2(1 - \cos t)]^2 + [-\sqrt{3}]^2 + [-\sin t]^2} dt$
 $= \sqrt{4(1 - \cos t)^2 + 3\sin^2 t + \sin^2 t} dt = \sqrt{4 - 8\cos t + 4\cos^2 t + 3\sin^2 t + \sin^2 t} dt$
 $= \sqrt{8 - 8\cos t} dt = \sqrt{8(1 - 2\sin^2 \frac{t}{2})} dt$
 $= \sqrt{16\sin^2 \frac{t}{2}} dt = 4\sin \frac{t}{2} dt$

• $\int_C z \, ds = \int_{0}^{2\pi} (1 + \cos t) (4\sin \frac{t}{2}) dt$
 $= \int_{0}^{2\pi} [1 + (2\cos \frac{t}{2} - 1)] [4\sin \frac{t}{2}] dt$
 $= \int_{0}^{2\pi} 8\cos^2 \frac{t}{2} \sin \frac{t}{2} dt$
 $= \left[\frac{8}{3} \times (-2\cos^3 \frac{t}{2}) \right]_0^{2\pi}$
 $= \left[\frac{16}{3} \cos^3 \frac{t}{2} \right]_0^{2\pi}$
 $= \frac{16}{3} - \frac{16}{3} (-1) = \frac{32}{3}$

Question 11

$$V(x, y, z) = 60xyz^2.$$

Evaluate the following integral along C , from $(3,1,1)$ to $(4,3,2)$,

$$\int_C V \, d\mathbf{r}, \quad d\mathbf{r} = (dx, dy, dz)^T,$$

where C is the curve with parametric equations

$$x = t+2, \quad y = 2t-1, \quad z = t.$$

$$1139\mathbf{i} + 2278\mathbf{j} + 1139\mathbf{k}$$

$$V(xyz^2) = 60xyz^2 \quad \text{at} \quad \begin{aligned} x &= t+2 & dx = dt \\ y &= 2t-1 & dy = 2dt \\ z &= t & dz = dt \end{aligned}$$

Thus

$$\begin{aligned} \int_C V \, d\mathbf{r} &= \int_{(3,1,1)}^{(4,3,2)} 60xyz^2 \, (dx, dy, dz)^T = \int_{t=1}^{t=2} 60(x(t)y(t)z(t))^2 \, (dt, 2dt, dt)^T \\ &= \int_{t=1}^{t=2} 60(2t^2+3t^2-2t^2)(1, 2, 1)^T \, dt \\ &= 60C_1(2, 1)^T \int_{t=1}^{t=2} 2t^4+3t^2-2t^2 \, dt \\ &= 60C_1(2, 1)^T \left[\frac{2}{5}t^5 + \frac{3}{3}t^3 - \frac{2}{3}t^3 \right]_{t=1}^{t=2} \\ &= 60C_1(2, 1)^T \left[\left(\frac{64}{5} + 16 - \frac{16}{3} \right) - \left(\frac{2}{5} + \frac{3}{3} - \frac{2}{3} \right) \right] \\ &= 1139C_1(2, 1)^T \\ &\text{ie } 1139\mathbf{i} + 2278\mathbf{j} + 1139\mathbf{k}. \quad \checkmark \end{aligned}$$

Question 12

$$\varphi(x, y, z) \equiv 3x + 2y + z.$$

Evaluate the following integral along C , from $(1, 0, 0)$ to $(2, 2, 1)$,

$$\int_C \varphi \, d\mathbf{r}, \quad d\mathbf{r} = (dx, dy, dz)^T,$$

where C is the curve with parametric equations

$$x = t + 1, \quad y = 2t, \quad z = t^2.$$

$$\boxed{\frac{41}{6}\mathbf{i} + \frac{41}{3}\mathbf{j} + \frac{49}{6}\mathbf{k}}$$

$$\begin{aligned}
 \Phi(3x+2y+z) &= 3x+2y+z \\
 &\quad \begin{array}{l} x=t+1 \Rightarrow dx=dt \\ y=2t \Rightarrow dy=2dt \\ z=t^2 \Rightarrow dz=2t\,dt \end{array} \\
 \text{Thus } \int_C \Phi \, d\mathbf{r} &= \int_{(1,0,0)}^{(2,2,1)} (3x+2y+z) \, (dx, dy, dz) = \int_{t=0}^{t=1} [3(t+1) + 2(2t) + t^2] \, (dt, 2dt, 2t\,dt) \\
 &= \int_{t=0}^{t=1} (5t+3+4t+t^2) \, (1, 2, 2t) \, dt \\
 &= \int_{t=0}^{t=1} (5t+7t+3) \, (1, 2, 2t) \, dt \\
 &= \int_0^1 [5t^2 + 7t^2 + 3t] \, dt = \int_0^1 [12t^2 + 3t] \, dt \\
 &= \left[\frac{4}{3}t^3 + \frac{3}{2}t^2 + 3t \right]_0^1 = \left[\frac{4}{3}t^3 + \frac{7}{2}t^2 + 3t \right]_0^1 \\
 &= \left(\frac{4}{3} + \frac{7}{2} + 3, \frac{4}{3} + \frac{7}{2} + 3, \frac{4}{3} + \frac{7}{2} + 3 \right) \\
 &= \left(\frac{41}{6}, \frac{41}{3}, \frac{49}{6} \right) \\
 &\quad \times \in \boxed{\frac{41}{6}\mathbf{i} + \frac{41}{3}\mathbf{j} + \frac{49}{6}\mathbf{k}}
 \end{aligned}$$

Question 13

$$F(x, y, z) = xyz.$$

Evaluate the following integral along C , from $(1, 0, 0)$ to $(0, 1, 4)$,

$$\int_C F \, d\mathbf{r}, \quad d\mathbf{r} = (dx, dy, dz)^T,$$

where C is the curve with parametric equations

$$x = \cos t, \quad y = \sin t, \quad z = \frac{8t}{\pi}.$$

$$\boxed{\frac{16-12\pi}{9\pi} \mathbf{i} + \frac{16}{9\pi} \mathbf{j} + \frac{8}{\pi} \mathbf{k}}$$

The handwritten solution shows the parametrization of the curve C as $x = \cos t$, $y = \sin t$, $z = \frac{8t}{\pi}$. It then sets up the line integral $\int_C xyz \, d\mathbf{r} = \int_{(1,0,0)}^{(0,1,4)} xyz \, d\mathbf{r}$. The differential $d\mathbf{r} = (\cos t, \sin t, \frac{8}{\pi}) dt$ is used. The integral becomes $\int_0^{\pi/2} (\cos t)(\sin t) \left(\frac{8}{\pi}\right) \cos t \sin t \, dt$. This is simplified to $\frac{8}{\pi} \int_0^{\pi/2} \cos^2 t \sin^2 t \, dt$. The integral is evaluated using substitution $u = \cos t$ and $du = -\sin t \, dt$, resulting in $\frac{8}{\pi} \int_1^0 u^2 (1-u^2) \, du = \frac{8}{\pi} \int_0^1 (u^2 - u^4) \, du$. This is then evaluated using integration by parts or direct substitution to get $\frac{16-12\pi}{9\pi}$.

Question 14

A curve C has equation

$$x^2 + xy + y^2 = 1, \quad 0 \leq x \leq 3.$$

Find a suitable parameterization of C in the form

$$x = A \cos \theta + B \sin \theta \quad \text{and} \quad y = A \cos \theta - B \sin \theta,$$

where A and B are suitable constants.

$$\boxed{x = \frac{1}{\sqrt{3}} \cos \theta + \sin \theta}, \quad \boxed{y = \frac{1}{\sqrt{3}} \cos \theta - \sin \theta}$$

• $x^2 + xy + y^2 = 1$
 $\Rightarrow x^2 + 2xy + y^2 - xy = 1$
 $\Rightarrow (x+y)^2 - xy = 1$

Let $x = \cos \theta + \sin \theta$
 $y = \cos \theta - \sin \theta$

• Then we have
 $(\cos \theta + \sin \theta + \cos \theta - \sin \theta)^2 - (\cos \theta + \sin \theta)(\cos \theta - \sin \theta)$
 $= (2\cos \theta)^2 - (\cos \theta - \sin \theta)$
 $= 4\cos^2 \theta - \cos^2 \theta + \sin^2 \theta$
 $= 3\cos^2 \theta + \sin^2 \theta$

↑
• This is equal to $\frac{1}{3}$ if $\theta = \frac{1}{3}\tan^{-1}(3)$
 $y = \frac{1}{\sqrt{3}}(\cos \theta - \sin \theta)$

Check
$$\left(\frac{1}{\sqrt{3}}(\cos \theta + \sin \theta) + \frac{1}{\sqrt{3}}(\cos \theta - \sin \theta)\right)^2 - \left(\frac{1}{\sqrt{3}}(\cos \theta + \sin \theta)\right)\left(\frac{1}{\sqrt{3}}(\cos \theta - \sin \theta)\right)$$

 $= \left(\frac{2}{\sqrt{3}}\cos \theta\right)^2 - \left(\frac{1}{\sqrt{3}}\cos \theta + \sin \theta\right)$
 $= \frac{4}{3}\cos^2 \theta - \frac{1}{3}\cos^2 \theta + \sin^2 \theta$
 $= \cos^2 \theta + \sin^2 \theta$
 $= 1$

Question 15

A surface S is given parametrically by

$$x = at \cosh \theta, \quad x = bt \sinh \theta, \quad z = t^2,$$

where t and θ are real parameters, and a and b are non zero constants.

- Find a Cartesian equation for S .
- Determine an equation of the tangent plane on S at the point with Cartesian coordinates (x_0, y_0, z_0) .

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad 2b^2 x x_0 + 2a^2 y y_0 - a^2 b^2 z = 0$$

$x = at \cosh \theta$
 $y = bt \sinh \theta$
 $z = t^2$

$$\begin{aligned} \frac{\partial}{\partial t} &= t \cosh \theta & \Rightarrow \frac{\partial^2}{\partial t^2} &= t \sinh \theta \\ \frac{\partial}{\partial \theta} &= b t \sinh \theta & \frac{\partial^2}{\partial \theta^2} &= b^2 \sinh \theta \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial \theta^2} &= t^2 \\ z &= \frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial \theta^2} \end{aligned}$$

NOW Let $f(x, y, z) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0$
 $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left(\frac{2x}{a^2}, -\frac{2y}{b^2}, -1 \right)$
Let $\mathbf{N} = \left(\frac{2x}{a^2}, -\frac{2y}{b^2}, -1 \right)$

NOW AT THE POINT (x_0, y_0, z_0) , $\mathbf{N} = \left(\frac{2x_0}{a^2}, -\frac{2y_0}{b^2}, -1 \right)$

EQUATION OF TANGENT PLANE

$$\frac{2x_0}{a^2} x + \left(-\frac{2y_0}{b^2} y \right) - z = \text{constant}$$

Using (x_0, y_0, z_0) to find constant

$$\text{constant} = \frac{2x_0}{a^2} - \frac{2y_0}{b^2} - z_0$$

But this must be zero? (Equation of surface)

\therefore TANGENT PLANE

$$\frac{2x_0}{a^2} x + \frac{2y_0}{b^2} y - z = 0$$

$$\text{OR } 2x_0 a^2 x + 2y_0 b^2 y - a^2 b^2 z = 0$$

Question 16

In standard notation used for tori, r is the radius of the tube and R is the distance of the centre of the tube from the centre of the torus.

The surface of a torus has parametric equations

$$x(\theta, \varphi) = (R + r \cos \theta) \cos \varphi, \quad y(\theta, \varphi) = (R + r \cos \theta) \sin \varphi, \quad z(\theta, \varphi) = r \sin \theta,$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq 2\pi$.

- a) Find a general Cartesian equation for the surface of a torus.

A torus T has Cartesian equation

$$(4 - \sqrt{x^2 + y^2})^2 = 1 - z^2.$$

- b) Use a suitable parameterization of T to find its surface area.

$$\boxed{z^2 + (R - \sqrt{x^2 + y^2})^2 = r^2}, \quad \boxed{\text{area} = (2\pi r)(2\pi R) = 16\pi^2}$$

a) Equations into Cartesian as required

$$\begin{cases} x(\theta, \varphi) = (R + r \cos \theta) \cos \varphi \\ y(\theta, \varphi) = (R + r \cos \theta) \sin \varphi \\ z(\theta, \varphi) = r \sin \theta \end{cases} \quad \begin{matrix} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq 2\pi \end{matrix}$$

$$\begin{aligned} x^2 + y^2 &= (R + r \cos \theta)^2 \cos^2 \varphi + (R + r \cos \theta)^2 \sin^2 \varphi \\ &= (R + r \cos \theta)^2 (\cos^2 \varphi + \sin^2 \varphi) \\ &= (R + r \cos \theta)^2 \\ &\quad + r^2 \cos^2 \theta = R^2 + 2Rr \cos \theta + r^2 \cos^2 \theta = R^2 + r^2(1 + \cos^2 \theta) \\ &\quad - r^2 \cos^2 \theta = R^2 + r^2 \\ &\quad - r^2 \cos^2 \theta = R^2 - r^2 \sin^2 \theta = R^2 - r^2 \\ &\quad r^2 \cos^2 \theta = (R^2 - r^2 \sin^2 \theta)^2 \\ &\quad r^2 \cos^2 \theta = (R^2 - r^2 \sin^2 \theta)^2 + z^2 \\ &\quad r^2 \cos^2 \theta = (R^2 - r^2 \sin^2 \theta)^2 + z^2 \\ &\quad z^2 = (R^2 - r^2 \sin^2 \theta)^2 - r^2 \cos^2 \theta \end{aligned}$$

b) Rearrange the above equation to

$$\begin{aligned} (R^2 - r^2 \sin^2 \theta)^2 - r^2 \cos^2 \theta &= 0 \\ \text{Hence } R^2 - r^2 &= 0 \\ \text{So the parameters become, using part (a)} \\ x(\theta, \varphi) &= (4 + r \cos \theta) \cos \varphi \\ y(\theta, \varphi) &= (4 + r \cos \theta) \sin \varphi \\ z(\theta, \varphi) &= r \sin \theta \quad \text{with } 0 \leq \theta, \varphi \leq 2\pi \end{aligned}$$

$$\begin{aligned} \Gamma(\theta, \varphi) &= [(4 + r \cos \theta) \cos \varphi, (4 + r \cos \theta) \sin \varphi, r \sin \theta] \\ \Gamma(\theta, \varphi) &= [4 \cos \theta + r \cos^2 \theta, 4 \sin \theta + r \cos^2 \theta, r \sin \theta] \end{aligned}$$

$$\frac{\partial \Gamma}{\partial \theta} = [-r \sin \theta, r \sin \theta, 0]$$

$$\frac{\partial \Gamma}{\partial \varphi} = [4 \cos \theta + r \cos 2\theta, 4 \sin \theta + r \cos 2\theta, 0]$$

$$\left| \frac{\partial \Gamma}{\partial \theta} \cdot \frac{\partial \Gamma}{\partial \varphi} \right| = \begin{vmatrix} -r \sin \theta & 4 \cos \theta + r \cos 2\theta & 0 \\ r \sin \theta & 4 \sin \theta + r \cos 2\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

(cancel the signs in the second and the third row)

$$= |0 + 4r \cos \theta + r \cos 2\theta, 4r \sin \theta + r \cos 2\theta - r, 0|$$

$$= |r \sin \theta (4 + r \cos 2\theta), r \cos \theta (4 + r \cos 2\theta), 0|$$

$$= |r \sin \theta (4 + r \cos 2\theta), r \cos \theta (4 + r \cos 2\theta), r \sin \theta (4 + r \cos 2\theta)|$$

$$= |r^2 \sin \theta \cos \theta (4 + r \cos 2\theta), r^2 \sin \theta \cos \theta (4 + r \cos 2\theta), r^2 \sin \theta (4 + r \cos 2\theta)|$$

$$= |r^2 \sin \theta \cos \theta (4 + r \cos 2\theta), r^2 \sin \theta \cos \theta (4 + r \cos 2\theta), r^2 \sin \theta (4 + r \cos 2\theta)|$$

$$= |r^2 \sin \theta \cos \theta (4 + r \cos 2\theta), r^2 \sin \theta \cos \theta (4 + r \cos 2\theta), r^2 \sin \theta (4 + r \cos 2\theta)|$$

$$= |(4 + r \cos 2\theta) \sin \theta \cos \theta, (4 + r \cos 2\theta) \sin \theta \cos \theta, (4 + r \cos 2\theta) \sin \theta|$$

$$= |(4 + r \cos 2\theta) \sqrt{\cos^2 \theta + \sin^2 \theta}, (4 + r \cos 2\theta) \sqrt{\cos^2 \theta + \sin^2 \theta}, (4 + r \cos 2\theta) \sin \theta|$$

$$= |(4 + r \cos 2\theta) \sqrt{1}, (4 + r \cos 2\theta) \sqrt{1}, (4 + r \cos 2\theta) \sin \theta|$$

$$= |(4 + r \cos 2\theta) \sqrt{1}, (4 + r \cos 2\theta) \sqrt{1}, (4 + r \cos 2\theta) \sin \theta|$$

$$= |(4 + r \cos 2\theta) \sqrt{1 + \sin^2 \theta}, (4 + r \cos 2\theta) \sqrt{1 + \sin^2 \theta}, (4 + r \cos 2\theta) \sin \theta|$$

$$= |(4 + r \cos 2\theta) \sqrt{1 + \sin^2 \theta}, (4 + r \cos 2\theta) \sqrt{1 + \sin^2 \theta}, (4 + r \cos 2\theta) \sin \theta|$$

$$= |(4 + r \cos 2\theta) \sqrt{1 + \sin^2 \theta}, (4 + r \cos 2\theta) \sqrt{1 + \sin^2 \theta}, (4 + r \cos 2\theta) \sin \theta|$$

$$= (4 + r \cos 2\theta)$$

Hence the surface element dS in parametric is

$$dS = \left| \frac{\partial \Gamma}{\partial \theta} \cdot \frac{\partial \Gamma}{\partial \varphi} \right| d\theta d\varphi$$

$$dS = (4 + r \cos 2\theta) d\theta d\varphi$$

Finally the area can be found

$$\text{Area} = \int_{0}^{2\pi} \int_{0}^{2\pi} |dS| = \int_{0}^{2\pi} \int_{0}^{2\pi} (4 + r \cos 2\theta) d\theta d\varphi$$

$$= \left[\int_{0}^{2\pi} (4 + r \cos 2\theta) d\theta \right] \left[\int_{0}^{2\pi} d\varphi \right] \quad \text{in spherical polar coords}$$

$$= 2\pi \times 4 \times 2\pi$$

$$= 16\pi^2$$

Please note the "standard" formula is $(2\pi r)(2\pi R)$, which

for this torus is $(4\pi)(2\pi)$, whereas $(2\pi)(2\pi \times 4) = 16\pi^2$

Question 17

A spiral ramp is modelled by the surface S defined by the vector function

$$\mathbf{r}(u, v) = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = (u \cos v)\mathbf{i} + (u \sin v)\mathbf{j} + v\mathbf{k},$$

where $0 \leq u \leq 1$, $0 \leq v \leq 3\pi$.

Determine the value of

$$\int_S \sqrt{x^2 + y^2} \, dS$$

$$\boxed{\pi[\sqrt{8}-1]}$$

$\Sigma(u, v) = [u \cos v, u \sin v, v]$

$0 \leq u \leq 1$
 $0 \leq v \leq 3\pi$

• Firstly we compute the dS element:

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} &= (\cos v, \sin v, 0) \\ \frac{\partial \mathbf{r}}{\partial v} &= (-u \sin v, u \cos v, 1) \\ \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} = [u \sin v, -u \cos v, u \sin^2 v + u \cos^2 v] = [u \sin v, -u \cos v, u]\end{aligned}$$

$\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| = \sqrt{u^2 \sin^2 v + u^2 \cos^2 v + u^2} = \sqrt{1+u^2}$

$\therefore dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv$

$dS = \sqrt{1+u^2} \, du \, dv$

• $\int_S \sqrt{x^2+y^2} \, dS = \int_0^{3\pi} \int_{0 \rightarrow 1} \sqrt{u^2 \cos^2 v + u^2 \sin^2 v} \sqrt{1+u^2} \, du \, dv$

$$\begin{aligned}&= \int_0^{3\pi} \int_{0 \rightarrow 1} u \sqrt{1+u^2} \, du \, dv \\ &= \left[\frac{u}{2} \sqrt{1+u^2} \right]_0^{3\pi} \left[\int_{0 \rightarrow 1} u \sqrt{1+u^2} \, du \right] \\ &= 3\pi \times \left[\frac{1}{2} \left(u^2 \right)^{\frac{1}{2}} \right]_0^{3\pi} \\ &= 3\pi \left[\frac{1}{2} u^2 \right]_0^{3\pi} \\ &= 3\pi \left[\frac{27}{2} - \frac{1}{2} \right] \\ &= 3\pi \left[27 - 1 \right]\end{aligned}$$

Question 18

The surface S is defined by the vector equation

$$\mathbf{F}(u, v) = \left[u \cos v, u \sin v, \frac{1}{u} \right]^T, u \neq 0.$$

Find the area of S lying above the region in the uv plane bounded by the curves

$$v = u^4, \quad v = 2u^4,$$

and the straight lines with equations $u = 3^{\frac{1}{4}}$ and $u = 8^{\frac{1}{4}}$.

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$$\boxed{\mathbf{f}(uv) = \left[u \cos v, u \sin v, \frac{1}{u} \right]^T \quad \begin{matrix} u^4 \leq v \leq u^8 \\ 3^{\frac{1}{4}} \leq u \leq 8^{\frac{1}{4}} \end{matrix}}$$

Given $\frac{\partial \mathbf{F}}{\partial u} = \left[\cos v, \sin v, -\frac{1}{u^2} \right]^T$ $\frac{\partial \mathbf{F}}{\partial v} = \left[u \cos v, u \sin v, 0 \right]^T$ $\Rightarrow \text{find } \frac{\partial \mathbf{F}}{\partial u} \wedge \frac{\partial \mathbf{F}}{\partial v}$ NOT

$$\begin{vmatrix} 1 & 1 & \frac{1}{u^2} \\ \cos v & \sin v & -\frac{1}{u^2} \\ u \cos v & u \sin v & 0 \end{vmatrix} = \begin{vmatrix} 1 & \cos v, \frac{1}{u^2} \sin v, u \end{vmatrix} = \sqrt{\frac{1}{u^2} \cos^2 v + \frac{1}{u^4} \sin^2 v + u^{-2}}$$

$$\begin{vmatrix} \frac{1}{u^2} & \frac{1}{u^2} & \frac{1}{u^2} \\ \cos v & \sin v & 0 \end{vmatrix} = \sqrt{\frac{1}{u^4} + u^{-2}} = \sqrt{\frac{1+u^4}{u^4}} = \frac{1}{u} \sqrt{1+u^4}$$

$$dS = \left\| \frac{\partial \mathbf{F}}{\partial u} \wedge \frac{\partial \mathbf{F}}{\partial v} \right\| du dv$$

$$\boxed{dS = \frac{1}{u} \sqrt{1+u^4} du dv}$$

$$\begin{aligned} \int_D dS &= \int_{3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} \int_{u^4}^{u^8} \frac{1}{u} \sqrt{1+u^4} du dv \\ &= \int_{3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} \int_{u^4}^{u^8} \frac{\sqrt{1+u^4}}{u} \cdot u^{\frac{1}{2}} du dv = \int_{3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} u^{\frac{1}{2}} \sqrt{1+u^4}^{\frac{1}{2}} du \\ &= \int_{3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} u^{\frac{1}{2}} (1+u^4)^{\frac{1}{2}} du = \left[\frac{1}{2} (1+u^4)^{\frac{1}{2}} \right]_{3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} = \frac{1}{2} [27 - 8] \\ &= \frac{15}{2} \end{aligned}$$

Question 19

The surface S is defined by the parametric equations

$$x = t \cosh \theta, \quad y = t \sinh \theta, \quad z = \frac{1}{2}(1-t^2),$$

where t and θ are real parameters such that $0 \leq t \leq 1$ and $0 \leq \theta \leq 1$.

Find, in exact form, the value of

$$\int_S xy \, ds.$$

$$\boxed{\frac{1}{30} \left[\frac{(\cosh 2 + 1)^{\frac{5}{2}} - 1}{\cosh 2} + 1 - 4\sqrt{2} \right] \approx 0.274397\dots}$$

$$\begin{aligned} \text{Given: } & \begin{cases} x(t, \theta) = (t \cosh \theta, t \sinh \theta, \frac{1}{2}(1-t^2)) \\ \frac{\partial x}{\partial t} = (\cosh \theta, \sinh \theta, -t) \\ \frac{\partial x}{\partial \theta} = (\tanh \theta, \text{sech} \theta, 0) \end{cases} \\ \text{T.R.A:} & \begin{bmatrix} 1 & 2 & k \\ \cosh \theta & \sinh \theta & -t \\ \tanh \theta & \text{sech} \theta & 0 \end{bmatrix} = \begin{bmatrix} t \cosh \theta & t \sinh \theta & t \cosh \theta - \tanh \theta \\ t^2 \cosh^2 \theta & t^2 \sinh^2 \theta & -t^2 \\ t^2 \cosh^2 \theta - 1 & t^2 \sinh^2 \theta & 0 \end{bmatrix} \quad \text{since } \cosh^2 \theta - \sinh^2 \theta = 1 \\ \therefore & \left| t^2 \cosh^2 \theta - 1 \right|^{\frac{1}{2}} = \sqrt{t^2 \cosh^2 \theta + t^2 \sinh^2 \theta + t^2} = |t| \sqrt{t^2 \cosh^2 \theta + \sinh^2 \theta + 1} \\ & = |t| \sqrt{t^2 (\cosh^2 \theta + \sinh^2 \theta) + 1} = |t| \sqrt{t^2 \cosh^2 \theta + 1} \quad \text{since } \cosh^2 \theta + \sinh^2 \theta = \cosh 2\theta \\ \therefore & dS = |t| \sqrt{t^2 \cosh^2 \theta + 1} dt \end{aligned}$$

$$\begin{aligned} \text{T.R.A:} & \int_S xy \, ds = \int_0^1 \int_{0 \theta=0}^1 (t \cosh \theta)(t \sinh \theta) |t| \sqrt{t^2 \cosh^2 \theta + 1} \, d\theta \, dt \\ & = \int_0^1 \int_{0 \theta=0}^1 t^2 \cosh \theta \sinh \theta (\cosh 2\theta)^{\frac{1}{2}} \, d\theta \, dt \\ & = \int_{-t=0}^1 \int_{0 \theta=0}^1 \frac{1}{2} t^2 \sinh 2\theta (\cosh 2\theta)^{\frac{1}{2}} \, d\theta \, dt \end{aligned}$$

$$\begin{aligned} & = \int_{-t=0}^1 \left[\frac{1}{2} t \left(\cosh 2\theta + 1 \right)^{\frac{3}{2}} \right]_{\theta=0}^1 \, dt \\ & = \int_{-t=0}^1 \frac{1}{2} t (\cosh 2 + 1)^{\frac{3}{2}} - \frac{1}{2} t (\cosh 2)^{\frac{3}{2}} \, dt \\ & = \left[\frac{1}{30 \cosh 2} (\cosh 2 + 1)^{\frac{5}{2}} - \frac{1}{30} (\cosh 2)^{\frac{5}{2}} \right]_0^1 \\ & = \left[\frac{1}{30 \cosh 2} (\cosh 2 + 1)^{\frac{5}{2}} - \frac{1}{30} \times 2^{\frac{5}{2}} \right] - \left[\frac{1}{30 \cosh 2} - \frac{1}{30} \right] \\ & = \frac{1}{30} \left[\frac{1}{\cosh 2} (\cosh 2 + 1)^{\frac{5}{2}} - \frac{1}{\cosh 2} + 1 - 4\sqrt{2} \right] \\ & = \frac{1}{30} \left[\frac{(\cosh 2 + 1)^{\frac{5}{2}} - 1}{\cosh 2} + 1 - 4\sqrt{2} \right] \end{aligned}$$

Question 20

$$\mathbf{F}(x, y, z) \equiv y\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$$

Find the magnitude of the flux through the surface with parametric equations

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u+v)\mathbf{k}, \quad 0 \leq u \leq 1, \quad 1 \leq v \leq 4.$$

All integrations must be carried out in parametric.

, $\boxed{\frac{1}{2}}$

$\boxed{\mathbf{F}(uv) = \begin{pmatrix} u \\ v \\ uv \end{pmatrix} \quad \mathbf{f}(uv) = \begin{pmatrix} u \\ v \\ u+v \end{pmatrix} \quad 0 \leq u \leq 1 \quad 1 \leq v \leq 4}$

FIND AN EXPRESSION FOR THE "ARM FLUX ELEMENT" $d\mathbf{s}$.

- $\frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
- $\frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
- NORMAL = $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}$
- UNIT NORMAL $\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{\begin{pmatrix} -1 & 1 & 1 \end{pmatrix}}{\sqrt{3}}$

COLLECTING THESE RESULTS

$$d\mathbf{s} = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv$$

$$\hat{\mathbf{n}} \cdot d\mathbf{s} = \sqrt{3} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv$$

$$d\hat{\mathbf{s}} = \frac{\hat{\mathbf{n}} \cdot d\mathbf{s}}{|\hat{\mathbf{n}}|} = \left| \frac{\hat{\mathbf{n}} \cdot \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}}{\sqrt{3}} \right| du dv$$

$$d\hat{\mathbf{s}} = \left(\frac{-1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) du dv$$

$$d\hat{\mathbf{s}} = \boxed{(\mathbf{1}, -1, 1)} du dv$$

FINALLY THE FLUX CAN BE CALCULATED

$$\begin{aligned} \text{FLUX} &= \int_S \mathbf{F} \cdot d\mathbf{s} = \int \mathbf{f}(uv) \cdot d\mathbf{s} \\ &= \int_{v=1}^4 \int_{u=0}^1 (G(u, v), (-1, 1, 1)) \cdot d\mathbf{s} dv \\ &= \int_{v=1}^4 \int_{u=0}^1 (-v - u^2 + u + v) \cdot du dv \\ &= \int_{v=1}^4 \int_{u=0}^1 (u - u^2) \cdot du dv \\ &= \int_{v=1}^4 \left[\frac{1}{2}u^2 - \frac{1}{3}u^3 \right]_0^1 dv \\ &= \int_1^4 \left(\frac{1}{2} - \frac{1}{3} \right) dv \\ &= \int_1^4 \frac{1}{6} dv \\ &= \left[\frac{1}{6}v \right]_1^4 \\ &= \frac{2}{3} \\ &= \boxed{\frac{1}{2}} \end{aligned}$$

Question 21

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface represented parametrically by

$$\mathbf{r}(u, v) = \begin{bmatrix} u+v \\ u-v \\ u \end{bmatrix}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 3,$$

and \mathbf{F} is the vector field

$$x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}.$$

All integrations must be carried out in parametric.

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• PREPARE ALL THE NECESSARY ITEMS

- $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$
- $\mathbf{I}(u, v) = (u+v, u-v, u), \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 3$
(THIS IS IN FACT A PLANE THROUGH O)
- $\frac{\partial \mathbf{F}}{\partial u} = (1, 1, 1) \quad \frac{\partial \mathbf{F}}{\partial v} = (1, -1, 0)$
- $\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = (1, 1, -2)$
- $\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = (1, 1, -2)$
- $d\mathbf{s} = \left\| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right\| du dv$

• HENCE WE NOW HAVE IN PARAMETRIC

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S \mathbf{F} \cdot \mathbf{n} d\mathbf{s} \\ &= \int_S \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) d\mathbf{s} \\ &= \int_S \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) du dv \end{aligned}$$

• SUBSTITUTING FULLY INTO THE REQUIRED SURFACE INTEGRAL

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_{u=0}^3 \int_{v=0}^2 \left[(uv)^2 (uv)^2, a^2 \right] \cdot [1, 1, -2] du dv \\ &= \int_{u=0}^3 \int_{v=0}^2 (uv)^2 (uv)^2 + a^2 - 2a^2 du dv \\ &= \int_{u=0}^3 \int_{v=0}^2 u^2 v^2 + a^2 - 2a^2 du dv \\ &= \int_{u=0}^3 \int_{v=0}^2 2u^2 v^2 du dv \\ &= \int_{u=0}^3 \left[2u^2 v^2 \right]_{u=0}^2 du \\ &= \int_{u=0}^3 4u^2 du \\ &= \left[\frac{4}{3}u^3 \right]_0^3 \\ &= \frac{4}{3} \times 27 \\ &= 36 \end{aligned}$$

Question 22

Evaluate the surface integral

$$\int_S z \mathbf{k} \cdot d\mathbf{S},$$

where S is the surface represented parametrically by

$$\mathbf{r}(\theta, \varphi) = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}, \quad 0 \leq \theta \leq \frac{1}{2}\pi, \quad 0 \leq \varphi \leq \frac{1}{2}\pi.$$

All integrations must be carried out in parametric.

$\frac{1}{6}\pi$

$\mathbf{r}(\theta, \varphi) = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}$ $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \varphi \leq \frac{\pi}{2}$

$$\int_S z \mathbf{k} \cdot d\mathbf{S} = \int_S z \mathbf{k} \cdot \hat{n} dS$$

FIND THE UNIT NORMAL TO THE SPHERICAL SURFACE & SWITCH THE INTEGRAND INTO PARAMETRIC

$$\frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix} + \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{bmatrix} -\sin \theta \cos \varphi \\ \sin \theta \cos \varphi \\ 0 \end{bmatrix}$$

$$\therefore \hat{n} = \left| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right| = \begin{vmatrix} 1 & 0 & b \\ \cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta \\ -\sin \theta \cos \varphi & \sin \theta \cos \varphi & 0 \end{vmatrix}$$

$$= \begin{bmatrix} \sin^2 \theta \cos^2 \varphi, \sin^2 \theta \sin^2 \varphi, \cos \theta \sin \theta \cos^2 \varphi + \cos \theta \sin \theta \sin^2 \varphi \\ \sin^2 \theta \cos^2 \varphi, \sin^2 \theta \sin^2 \varphi, \cos \theta \sin \theta \cos^2 \varphi + \cos \theta \sin \theta \sin^2 \varphi \\ \sin^2 \theta \cos^2 \varphi, \sin^2 \theta \sin^2 \varphi, \cos \theta \sin \theta \cos^2 \varphi + \cos \theta \sin \theta \sin^2 \varphi \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2 \theta \cos^2 \varphi, \sin^2 \theta \sin^2 \varphi, \cos \theta \sin \theta \cos^2 \varphi + \cos \theta \sin \theta \sin^2 \varphi \\ \sin^2 \theta \cos^2 \varphi, \sin^2 \theta \sin^2 \varphi, \cos \theta \sin \theta \cos^2 \varphi + \cos \theta \sin \theta \sin^2 \varphi \\ \sin^2 \theta \cos^2 \varphi, \sin^2 \theta \sin^2 \varphi, \cos \theta \sin \theta \cos^2 \varphi + \cos \theta \sin \theta \sin^2 \varphi \end{bmatrix}$$

STARTING WITH A CYLINDER

SPHERE: $x^2 + y^2 + z^2 = a^2$
CYLINDER: $x^2 + y^2 = b^2$ ($a > b$)

AREA OF THE INNER CYLINDRICAL FACE IS GIVEN BY

$$2\pi r H = 2\pi b(2h) = 4\pi b h = 4\pi b (a^2 - b^2)^{\frac{1}{2}}$$

NEXT WE FIND THE AREA OF ONE OF THE SPHERICAL LONGS, SHOWN IN YELLOW - PROJECT THE 'TOP' CAP ($z \geq 0$) ONTO THE xy PLANE

$$\Rightarrow z = (\sqrt{a^2 - x^2 - y^2})^{\frac{1}{2}}$$

$$\frac{\partial z}{\partial x} = -x(\sqrt{a^2 - x^2 - y^2})^{-\frac{1}{2}}, \quad \frac{\partial z}{\partial y} = -y(\sqrt{a^2 - x^2 - y^2})^{-\frac{1}{2}}$$

$$\Rightarrow dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dx dy$$

$$\Rightarrow dS = \sqrt{\frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} + 1} dx dy$$

Question 23

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with parametric equations

$$\mathbf{r}(u, v) = (u \cos v)\mathbf{i} + (u \sin v)\mathbf{j} + u\mathbf{k},$$

such that $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.

All integrations must be carried out in parametric.

$$\boxed{\frac{2}{3}\pi}$$

$\mathbf{F}(u, v) = \begin{bmatrix} u \cos v \\ u \sin v \\ 2u \end{bmatrix} \quad 0 \leq u \leq 1 \\ 0 \leq v \leq 2\pi$

$\mathbf{F}(u, v, z) = (u, u, 2z)$

• **Firstly find the "Jacobian" and the normal.**

- $\frac{\partial \mathbf{r}}{\partial u} = \begin{bmatrix} \cos v \\ \sin v \\ 1 \end{bmatrix}$
- $\frac{\partial \mathbf{r}}{\partial v} = \begin{bmatrix} -u \sin v \\ u \cos v \\ 0 \end{bmatrix}$
- $\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = \begin{bmatrix} u \cos^2 v - u \sin^2 v \\ u \sin v \cos v + u \cos v \sin v \\ u \sin v \end{bmatrix} = \begin{bmatrix} u \\ u \\ u \end{bmatrix} \leftarrow \text{NORMAL}$
- $\left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| = \sqrt{u^2 \cos^2 v + u^2 \sin^2 v + u^2} = u\sqrt{2} \leftarrow \text{"JACOBIAN"}$
- $\hat{n} = \frac{\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}}{\left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right|}$

• **NOW THE FLUX INTEGRAL ON R IS COMPUTED**

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_R \mathbf{F} \cdot \hat{n} \, du \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 \left(u \cos v, u \sin v, 2u \right) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) \, du \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 \left(-u^2 \cos^2 v - u^2 \sin^2 v + 2u^2 \right) \, du \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 -u^2 (1 + \tan^2 v) + 2u^2 \, du \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 u^2 \, du \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \left[\frac{1}{3}u^3 \right]_0^1 \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \frac{1}{3} \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} \times 2\pi$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \boxed{\frac{2\pi}{3}}$$

Question 24

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with parametric equations

$$\mathbf{r}(u, v) = (1 + \sin u \cos v)\mathbf{i} + (\sin u \sin v)\mathbf{j} + (\cos u)\mathbf{k},$$

such that $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$.

All integrations must be carried out in parametric.

[4π]

$\mathbf{F}(u, v) = [1 + \sin u \cos v, \sin u \sin v, \cos u]$ $0 \leq u \leq \pi$
 $0 \leq v \leq 2\pi$
 $\mathbf{F}(u, v) = (x, y, z)$
 $\frac{\partial \mathbf{F}}{\partial u} = [0, \sin u \cos v, \cos u \sin v, -\sin u]$
 $\frac{\partial \mathbf{F}}{\partial v} = [-\sin u \cos v, \sin u \sin v, 0]$
 $\frac{\partial \mathbf{F}}{\partial u} \times \frac{\partial \mathbf{F}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u \sin v & \sin u \sin v & -\sin u \\ -\sin u \cos v & \sin u \sin v & 0 \end{vmatrix}$
 $= [1 + \sin u \cos v, \sin u \sin v, -(\sin u \cos v \sin u \sin v + \cos u \sin v \sin u \sin v)]$
 $= [1 + \sin u \cos v, \sin u \sin v, \cos u \sin u \sin v]$
 $= [\sin u \cos v, \sin u \sin v, \cos u \sin u]$ ← RECALL \mathbf{r}
 $\hat{\mathbf{l}} = \frac{\frac{\partial \mathbf{F}}{\partial u} \times \frac{\partial \mathbf{F}}{\partial v}}{\left| \frac{\partial \mathbf{F}}{\partial u} \times \frac{\partial \mathbf{F}}{\partial v} \right|} = \frac{12}{12} = \hat{\mathbf{l}}$ & $d\mathbf{S} = \left| \frac{\partial \mathbf{F}}{\partial u} \times \frac{\partial \mathbf{F}}{\partial v} \right| du dv$ (cancel)
 NO NEED TO EXPAND $\left| \frac{\partial \mathbf{F}}{\partial u} \times \frac{\partial \mathbf{F}}{\partial v} \right|$ HERE AS IT WILL CANCEL

Hence

$$\begin{aligned}
 \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S \mathbf{F} \cdot \hat{\mathbf{l}} \, d\mathbf{S} = \int_S \mathbf{F}(u, v) \cdot \left(\frac{\frac{\partial \mathbf{F}}{\partial u} \times \frac{\partial \mathbf{F}}{\partial v}}{\left| \frac{\partial \mathbf{F}}{\partial u} \times \frac{\partial \mathbf{F}}{\partial v} \right|} \right) \, du dv \\
 &= \int_0^{\pi} \int_0^{2\pi} [1 + \sin u \cos v, \sin u \sin v, \cos u] \cdot [\sin u \cos v, \sin u \sin v, \cos u] \, du dv \\
 &= \int_0^{\pi} \int_0^{2\pi} \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u \cos^2 v + \sin^2 u \sin^2 v \, du dv \\
 &= \int_0^{\pi} \int_0^{2\pi} \sin^2 u (\cos^2 v + \sin^2 v) + \sin^2 u \cos^2 v \, du dv \\
 &= \int_0^{\pi} \int_0^{2\pi} \sin^2 u + \sin^2 u \cos^2 v \, du dv \\
 &= \int_0^{\pi} \int_0^{2\pi} \sin^2 u (\sin^2 u + \cos^2 u) \, du dv \\
 &= \int_0^{\pi} \int_0^{2\pi} \sin^2 u \, du dv \\
 &= 2\pi \left[-\frac{1}{2}\sin^2 u \right]_0^\pi \\
 &= 2\pi \left[\frac{1}{2}\sin^2 \pi \right] \pi \\
 &= 2\pi [1 - 1] \\
 &= 4\pi
 \end{aligned}$$

Question 25

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with parametric equations

$$\mathbf{r}(u, v) = (u \cos v)\mathbf{i} + (1+u \sin v)\mathbf{j} + (u-1)\mathbf{k},$$

such that $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.

All integrations must be carried out in parametric.

$$\boxed{\frac{1}{3}\pi}$$

The handwritten solution shows the following steps:

- Parametric equations: $\mathbf{r}(u, v) = (u \cos v)\mathbf{i} + (1+u \sin v)\mathbf{j} + (u-1)\mathbf{k}$ for $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.
- Surface normal: $\hat{\mathbf{n}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = \frac{1}{\sqrt{2u^2 + 2}} \begin{pmatrix} \sin v & -\cos v & 1 \\ u \cos v & u \sin v & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
- Cross product of partial derivatives: $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = \begin{pmatrix} u \cos v & -u \sin v & 0 \\ -u \sin v & u \cos v & 0 \\ 0 & 0 & u \end{pmatrix} = u \begin{pmatrix} \cos v & -\sin v & 0 \\ -\sin v & -\cos v & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
- Dot product: $\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = u \cos v + 1 + u \sin v - 1 = u \cos v + u \sin v = u(\cos v + \sin v)$.
- Surface integral: $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{2\pi} u(\cos v + \sin v) \frac{1}{\sqrt{2u^2 + 2}} du dv = \int_0^1 \int_0^{2\pi} \frac{u(\cos v + \sin v)}{\sqrt{2u^2 + 2}} du dv$. This integral is evaluated using substitution $u^2 + 1 = t$ and $dt = 2u du$.
- Final result: $= \int_0^1 \int_0^{2\pi} \frac{u(\cos v + \sin v)}{\sqrt{2t - 2}} dt dv = \int_0^1 \int_0^{2\pi} \frac{u(\cos v + \sin v)}{\sqrt{2(t-1)}} dt dv = \int_0^1 \int_0^{2\pi} u \cos v + u \sin v dt dv = \int_0^1 u \left[\frac{1}{2} \sin^2 v - \frac{1}{2} \cos^2 v \right]_0^1 dv = \int_0^1 u \left(-\frac{1}{2} \right) dv = \frac{1}{2} \pi u = \frac{1}{3}\pi$.

Question 26

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with parametric equations

$$\mathbf{r}(\theta, \varphi) = [(4 + \cos \theta)\cos \varphi]\mathbf{i} + [(4 + \cos \theta)\sin \varphi]\mathbf{j} + (\sin \theta)\mathbf{k},$$

such that $0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq 2\pi$.

All integrations must be carried out in parametric.

$$24\pi^2$$

$\mathbf{F}(\theta, \varphi) = [4 + \cos \theta]\cos \varphi, [4 + \cos \theta]\sin \varphi, \sin \theta$ $\mathbf{F}(x, y, z) = (x, y, z)$

$\frac{\partial \mathbf{r}}{\partial \theta} = [-\sin \theta \cos \varphi, -\sin \theta \sin \varphi, \cos \theta]$ $\frac{\partial \mathbf{r}}{\partial \varphi} = [-4\cos \theta \cos \varphi, -4\cos \theta \sin \varphi, 0]$

$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta \cos \varphi & -\sin \theta \sin \varphi & \cos \theta \\ -4\cos \theta \cos \varphi & -4\cos \theta \sin \varphi & 0 \end{vmatrix}$

$= [0 - 4\sin^2 \theta \cos^2 \varphi - 4\sin^2 \theta \sin^2 \varphi, -4\sin \theta \cos \theta \cos^2 \varphi - 4\sin \theta \cos \theta \sin^2 \varphi, -4\cos^2 \theta \cos \varphi]$

$= [-4\sin^2 \theta \cos^2 \varphi, -4\sin^2 \theta \sin^2 \varphi, -4\cos^2 \theta \cos \varphi]$

$= [-4\cos^2 \theta \cos^2 \varphi, -4\cos^2 \theta \sin^2 \varphi, -4\cos^2 \theta \cos \varphi]$

$= (4 + \cos \theta)[-\cos^2 \theta, -\sin^2 \theta, -\sin \theta]$

$= (4 + \cos \theta)[-x^2, -y^2, -z]$ ← normal vector \mathbf{n}

$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{2\hat{\mathbf{i}}}{\sqrt{36}}, \frac{2\hat{\mathbf{j}}}{\sqrt{36}}, \frac{-\hat{\mathbf{k}}}{\sqrt{36}}$

$d\mathbf{S} = \left| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right| d\theta d\varphi$ ← magnitude of $d\mathbf{S}$

Note that evaluation of $\left| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right|$ is NOT NEEDED IN THIS TYPE OF QUESTION, AS IT WILL CANCEL

NOW

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S \mathbf{F} \cdot \hat{\mathbf{n}} d\mathbf{S} = \int \mathbf{F}(\theta, \varphi) \cdot \left(\frac{2\hat{\mathbf{i}}}{\sqrt{36}}, \frac{2\hat{\mathbf{j}}}{\sqrt{36}}, \frac{-\hat{\mathbf{k}}}{\sqrt{36}} \right) d\theta d\varphi \\ &= \int_S [(4 + \cos \theta) \cos \theta, (4 + \cos \theta) \sin \theta, \sin \theta] \cdot [\cos \theta \hat{\mathbf{i}}, \sin \theta \hat{\mathbf{j}}, -\hat{\mathbf{k}}] (4 + \cos \theta) d\theta d\varphi \\ &= \int_0^{2\pi} \int_0^{2\pi} [(4 + \cos \theta) \cos^2 \theta + (4 + \cos \theta) \sin \theta \cos \theta + \sin^2 \theta] (4 + \cos \theta) d\theta d\varphi \\ &= \int_0^{2\pi} \int_0^{2\pi} (4 + \cos \theta) [4\cos^2 \theta + \sin^2 \theta] d\theta d\varphi \\ &= \int_0^{2\pi} \int_0^{2\pi} (4 + \cos \theta) [4\cos^2 \theta + \sin^2 \theta] d\theta d\varphi \\ &= \int_0^{2\pi} \int_0^{2\pi} (-4 + \cos \theta)(4\cos^2 \theta + 1) d\theta d\varphi = - \int_0^{2\pi} \int_0^{2\pi} 16\cos^5 \theta + 4 + 4\cos^2 \theta + \cos \theta d\theta d\varphi \quad \text{NO CONTRACTION IN } \theta \\ &= - \int_{\theta=0}^{2\pi} \int_{\theta=0}^{2\pi} 16\cos^5 \theta + 4 + 4\cos^2 \theta + \cos \theta d\theta d\varphi \quad \text{NO CONTRACTION IN } \theta \\ &= - \int_{\theta=0}^{2\pi} 6 d\theta = -6\theta \Big|_{\theta=0}^{2\pi} \\ &= -6 \times 2\pi = -24\pi^2 \\ \therefore \left| \int_S \mathbf{F} \cdot d\mathbf{S} \right| &= 24\pi^2 \end{aligned}$$

Question 27

It is given that the vector field \mathbf{F} satisfies

$$\mathbf{F} = 2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0,$$

cut off by the cylinder with cartesian equation

$$x^2 + y^2 = x.$$

You **must** find a suitable parameterization for S , and carry out the **integration in parametric**, without using any integral theorems.

$\frac{\pi}{4}$

The notes are organized into three main sections:

- Diagram and Initial Equations:** Shows a 3D plot of the cylinder $x^2 + y^2 = x$ and the sphere $x^2 + y^2 + z^2 = 1$. It also shows a 2D projection of the cylinder onto the xy-plane. Equations include $\mathbf{F} = (2y, 2x, 1)$, $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 = x$, and $(x-1/2)^2 + y^2 = 1/4$.
- Parameterization and Surface Integral:** Details the parameterization of the surface S as $\mathbf{r}(r, \theta) = \left(\frac{1}{2}r(\cos \theta, \sin \theta), \frac{1}{2}r\cos \theta, \sqrt{1 - \frac{1}{4}r^2(\cos^2 \theta + \sin^2 \theta)} \right)^T$ for $0 < r < 1$ and $0 < \theta < 2\pi$. The surface integral is given as $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F}(\mathbf{r}(r, \theta)) \cdot (\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}) dA$.
- Integration Steps:** The notes show the calculation of the cross product and the resulting integral. It includes a substitution $u = \frac{1}{2}r(\cos \theta + \sin \theta)$ and simplifies the integrand to $\frac{1}{2}r^2 \sin \theta \left[1 - \frac{1}{4}r^2(1 + \cos 2\theta) \right]^{-1/2}$. The final result is $\frac{\pi}{4}$.