

# **DIFFERENTIAL EQUATIONS**

## **2<sup>nd</sup> order or higher**

# **2<sup>ND</sup> ORDER WITH CONSTANT COEFFICIENTS**

**Question 1 (\*\*)**

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12(x + e^x).$$

$$[ ] , \quad y = Ae^{-3x} + Be^{-2x} + e^x + 2x - \frac{5}{3}$$

$$\begin{aligned} & \frac{\partial^2}{\partial x^2} + 5\frac{\partial}{\partial x} + 6I_0 = 12(x + e^x) \\ \bullet \text{AUXILIARY EQUATION: } & \lambda^2 + 5\lambda + 6 = 0 \\ & (\lambda + 3)(\lambda + 2) = 0 \\ & \lambda_1 = -3, \quad \lambda_2 = -2 \\ \bullet \text{C.F.: } & y = A e^{-3x} + B e^{-2x} \\ \bullet \text{PARTICULAR INTEGRAL: } & \text{TRY } y = P + Qx + R e^x \\ & \frac{dy}{dx} = P + Re^x \\ & \frac{d^2y}{dx^2} = Re^x \\ & Re^x + 5(P + Re^x) + 6(Q + Rx + Re^x) = 12(x + e^x) \\ & (2R + 5P + 6Q) + (6R + 5Re^x + 6Pe^x) = 12x + 12e^x \\ & R=1, \quad P=2, \quad \left\{ \begin{array}{l} 5P + 6Q = 0 \\ 6R + 5P = 12 \end{array} \right. \\ & Q=-\frac{5}{6} \\ \therefore & y = A e^{-3x} + B e^{-2x} + e^x + 2x - \frac{5}{3} \end{aligned}$$

**Question 2 (\*\*)**

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 13x^2 - x + 22.$$

$$[ ] , \quad y = e^{-3x}(A \cos 2x + B \sin 2x) + x^2 - x + 2$$

$$\begin{aligned} & \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 13x^2 - x + 22 \\ \bullet \text{AUXILIARY EQUATION: } & \lambda^2 + 6\lambda + 13 = 0 \\ & (\lambda + 3)^2 + 4 = 0 \\ & (\lambda + 3)^2 = -4 \\ & \lambda + 3 = \pm 2i \\ & \lambda = -3 \pm 2i \\ \bullet \text{C.F.: } & y = e^{-3x}(A \cos 2x + B \sin 2x) \\ \bullet \text{PARTICULAR INTEGRAL: } & y = P + Qx + R \\ & \left\{ \begin{array}{l} \frac{dy}{dx} = P + Q + R \\ \frac{d^2y}{dx^2} = R \end{array} \right. \\ \text{THUS BY SUBSTITUTING INTO THE O.D.E.} & 2P + 6(Q + R) + 13((P + Qx + R) + R) = 13x^2 - x + 22 \\ & 2P + 6Q + 13R + 13(P + Qx + R) = 13x^2 - x + 22 \\ & 12P + 13Qx + 26R = 13x^2 - x + 22 \\ & \boxed{P=1} \quad \boxed{12P + 13Qx + 26R = 22} \\ & \boxed{Q=-1} \quad \boxed{12 - 13x + 26} \\ & \boxed{R=2} \quad \boxed{(2x+2)} \\ \therefore & y = e^{-3x}(A \cos 2x + B \sin 2x) + x^2 - x + 2 \end{aligned}$$

**Question 3 (\*\*)**

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10\sin x,$$

subject to the boundary conditions  $y = 6$  and  $\frac{dy}{dx} = 5$  at  $x = 0$ .

,  $y = 2e^x + e^{2x} + 3\cos x + \sin x$

<p><math>\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10\sin x \quad y(0)=6, \quad y'(0)=5</math></p> <p><u>Auxiliary equation</u>  <math>(D^2 - 3D + 2) = 0</math>  <math>(D-2)(D-1) = 0</math>  <math>D = 1, 2</math></p> <p><u>COMPLEMENTARY/EQUATION</u>  <math>y = Ae^x + Be^{2x}</math></p> <p><u>PARTICULAR INTEGRAL BY INSPECTION</u>  <math>y = Pe\sin x + Q\cos x</math>  <math>y' = -Pe\sin x + Q\cos x</math>  <math>y'' = -Pe\sin x - Q\cos x</math></p> <p><u>SUBSTITUTE INTO THE D.S.E.</u>  <math>\Rightarrow (-Pe\sin x - Q\cos x) - 3(-Pe\sin x + Q\cos x) + 2(Pe\sin x + Q\cos x) \equiv 10\sin x</math>  <math>\left\{ \begin{array}{l} -Pe\sin x - Q\cos x \\ -3Pe\sin x + 3Q\cos x \\ +2Pe\sin x + 2Q\cos x \end{array} \right\} \equiv 10\sin x</math>  <math>\Rightarrow (P-3Q)\cos x + (3P+Q)\sin x \equiv 10\sin x</math>  <math>\bullet P-3Q=0 \quad P=3Q</math>  <math>\bullet 3P+Q=10 \quad 3(3Q)+Q=10 \quad Q=1 \quad \therefore P=3</math></p>	<p><u>PARTICULAR INTEGRAL</u>  <math>y = 3e\sin x + e\cos x</math></p> <p><u>GENERAL SOLUTION</u>  <math>y = Ae^x + Be^{2x} + 3e\sin x + e\cos x</math></p> <p><u>DIFFERENTIATE w.r.t x &amp; APPLY CONDITIONS</u>  <math>\frac{dy}{dx} = Ae^x + 2Be^{2x} + 3e\cos x + \sin x</math></p> <p><math>\bullet x=0, y=6 \Rightarrow 6 = A + B + 3 + 1 \Rightarrow A + B = 3</math></p> <p><math>\bullet x=0, \frac{dy}{dx}=5 \Rightarrow 5 = A + 2B + 1 \Rightarrow A + 2B = 4</math></p> <p><math>\therefore B = 1 \quad A = 2</math></p> <p><u>Finalizing our answer</u>  <math>y = 2e^x + e^{2x} + 3e\sin x + e\cos x</math></p>
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**Question 4 (\*\*)**

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 6e^x.$$

$$[ ] , y = (A + 2x)e^x + Be^{-2x}$$

START WITH THE AUXILIARY EQUATION

$$\lambda^2 + \lambda - 2 = 0$$
$$(\lambda + 2)(\lambda - 1) = 0$$
$$\lambda = -2, 1$$

COMPLEMENTARY FUNCTION

$$y_c = Ae^{\lambda x} = A e^{-2x}$$

NOW FOR PARTICULAR INTEGRAL WE TRY  $y_p = Pe^x$  AS  $e^x$  IS ALREADY PART OF THE COMPLEMENTARY FUNCTION

$$y_p = Pe^x$$
$$\frac{dy_p}{dx} = Pe^x + Be^{2x} = P(1+2x)e^x$$
$$\frac{d^2y_p}{dx^2} = P(2+2x)e^x + Pe^{2x} = 2Pe^x + Pe^{2x} = Pe^x(2+x)$$

SUBSTITUTE INTO THE D.E.

$$Pe^x(2+x) + P(1+2x)e^x - 2Pe^x = 6e^x$$
$$P[2x^2 + 3x + 1] = 6$$
$$2P = 6$$
$$P = 3$$

GENERAL SOLUTION IS

$$y = Ae^{-2x} + Be^{2x} + 3e^x$$

**Question 5 (\*\*)**

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin 2x,$$

subject to the boundary conditions  $y=1$  and  $\frac{dy}{dx}=-5$  at  $x=0$ .

$$y = 3\cos 2x - \sin 2x - e^{2x} - e^x$$

The working shows the auxiliary equation  $\lambda^2 - 3\lambda + 2 = 0$ , which factors into  $(\lambda-1)(\lambda-2) = 0$ . The roots are  $\lambda_1 = 1$  and  $\lambda_2 = 2$ . The general solution is  $y = Ae^{2x} + Be^x$ . The particular integral is found to be  $y_p = 10\sin 2x - 5\cos 2x$ . Substituting into the original equation, we find  $A = 3$  and  $B = -1$ . Therefore, the solution is  $y = 3\cos 2x - \sin 2x - e^{2x} - e^x$ .

**Question 6 (\*\*)**

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 12(e^{2x} - e^{-2x}).$$

$$y = (A+4x)e^{2x} + Be^{-x} - 3e^{-2x}$$

The working shows the auxiliary equation  $\lambda^2 - \lambda - 2 = 0$ , which factors into  $(\lambda-2)(\lambda+1) = 0$ . The roots are  $\lambda_1 = 2$  and  $\lambda_2 = -1$ . The general solution is  $y = Ae^{2x} + Be^{-x}$ . The particular integral is found to be  $y_p = 12e^{2x} - 3e^{-2x}$ . Substituting into the original equation, we find  $A = 1$  and  $B = -3$ . Therefore, the solution is  $y = (A+4x)e^{2x} + Be^{-x} - 3e^{-2x}$ .

**Question 7 (\*\*)**

$$\frac{d^2y}{dx^2} + y = \sin 2x, \quad \text{with } y=0, \frac{dy}{dx}=0 \text{ at } x=\frac{\pi}{2}.$$

Show that a solution of the above differential equation is

$$y = \frac{2}{3} \cos x(1 - \sin x).$$

**[proof]**

To apply conditions find  $\frac{dy}{dx} = -A\sin x + B\cos x$

Now

- $y = \frac{2}{3} \cos x$ ,  $\frac{dy}{dx} = 0$ ,  $0 = 0 + B = 0$   
∴  $B = 0$
- $y = \frac{2}{3} \cos x$ ,  $\frac{d^2y}{dx^2} = 0$ ,  $0 = 0 - A = -A$   
∴  $A = \frac{2}{3}$

∴  $y = \frac{2}{3} \cos x - \frac{2}{3} \sin x$   
 $y = \frac{2}{3} \cos x - \frac{2}{3} \sin x$   
 $y = \frac{2}{3} \cos x(1 - \sin x)$  ✓ As required

**Question 8** (\*\*\*)

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 6e^{-2x},$$

with  $y = 3$  and  $\frac{dy}{dx} = -2$  at  $x = 0$ .

Show that the solution of the above differential equation is

$$y = 2e^x + (1-2x)e^{-2x}.$$

[ ] , [ ] proof

The image shows two pages of handwritten working for a differential equation problem. The left page starts with the differential equation  $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 6e^{-2x}$ . It notes that the homogeneous part has roots  $\lambda = 1$  and  $\lambda = -2$ , leading to the complementary function  $y = Ae^x + Be^{-2x}$ . It then considers the particular integral  $y_p = Pe^{-2x}$  and finds  $\frac{dy_p}{dx} = Pe^{-2x}$  and  $\frac{d^2y_p}{dx^2} = P(-2)e^{-2x}$ . Substituting into the equation gives  $(P(-2)e^{-2x} - 2Pe^{-2x}) - 2(Pe^{-2x}) = 6e^{-2x}$ , which simplifies to  $-3Pe^{-2x} = 6e^{-2x}$ , so  $P = -2$ . The particular integral is thus  $y_p = -2e^{-2x}$ . The general solution is  $y = Ae^x + Be^{-2x} - 2e^{-2x}$ . The right page shows the application of boundary conditions  $y(0) = 3$  and  $\frac{dy}{dx}(0) = -2$ . Substituting  $x=0$  into the general solution gives  $3 = A + B - 2$ , so  $A + B = 5$ . Differentiating the general solution and substituting  $x=0$  gives  $-2 = A - 2B$ , so  $A = 2B + 2$ . Substituting  $A = 2B + 2$  into  $A + B = 5$  gives  $2B + 2 + B = 5$ , so  $B = 1$  and  $A = 3$ . Therefore, the solution is  $y = 3e^x + e^x - 2e^{-2x}$ .

**Question 9** (\*\*+)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2 y = \frac{1}{4}, \quad k > 0.$$



$$y = A e^{kx} + B x e^{kx} + \frac{1}{4k^2}$$

Determine the complementary function — via auxiliary equation  
 $\lambda^2 - 2k\lambda + k^2 = 0$   
 $(\lambda - k)^2 = 0$   
 $\lambda = k$  (repeated)  
 $\therefore y = A e^{kx} + B x e^{kx}$

To find particular integral try  $y = P$  — constant  
 $\frac{dy}{dx} = \frac{dP}{dx} = 0$   
Sub into the O.D.E.  
 $0 + 0 + k^2 P = \frac{1}{4}$   
 $P = \frac{1}{4k^2}$

∴ General solution  
 $y = A e^{kx} + B x e^{kx} + \frac{1}{4k^2}$

**Question 10 (\*\*+)**

Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x + 3,$$

subject to the conditions  $y = 2$ ,  $\frac{dy}{dx} = -5$  at  $x = 0$ .

$$y = x^2 + x - 4 + 6e^{-x}$$

The image shows handwritten working for solving the differential equation  $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x + 3$  with initial conditions  $y(0) = 2$  and  $y'(0) = -5$ . The working includes:

- Characteristic equation:  $\lambda^2 + \lambda = 2x + 3$
- Particular solution:  $y = Ax^2 + Bx + C$
- TRY:  $y = Bx^2 + Qx$
- SUB:  $\frac{dy}{dx} = 2Bx + Q$
- SUB:  $\frac{d^2y}{dx^2} = 2B$
- SUB into the O.D.E.:  $2B + (2Bx + Q) = 2x + 3$
- Equating coefficients:  $2B = 2$  (gives  $B = 1$ ) and  $2B + Q = 3$  (gives  $Q = 1$ )
- General solution:  $y = A + Bx^2 + Qx + C$
- APPLY CONDITIONS:  $\frac{dy}{dx} = -Bx^2 + 2x + 1$
- $A = 0$ ,  $y = 2 \Rightarrow 2 = A + B$
- $B = 0$ ,  $\frac{dy}{dx} = -3 \Rightarrow -5 = -B + 1$
- $\therefore \boxed{A=0}$ ,  $\boxed{B=1}$ ,  $\boxed{C=2}$ ,  $\boxed{D=-4}$
- $\therefore y = 6e^{-x} + x^2 + 2 - 4$

**Question 11 (\*\*+)**

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 5y = 34\cos 2x,$$

subject to the boundary conditions  $y=18$  and  $\frac{dy}{dx}=0$  at  $x=0$ .

$$y = 2(8e^{-x} + 1)\cos 2x + 8\sin 2x$$

$$\begin{aligned}
 & \frac{\partial^2 y}{\partial x^2} + 2 \frac{\partial y}{\partial x} + 5y = 34\cos 2x \\
 & 2^2 + 2A + 5 = 0 \\
 & (2+1)^2 + 5 = 0 \\
 & (3+1)^2 = -4 \\
 & 3+1 = \pm 2i \\
 & 3 = -1 \pm 2i \\
 & CF: y = e^{-x}(A\cos 2x + B\sin 2x) \\
 & \frac{\partial y}{\partial x} = -e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}(-2A\sin 2x + 2B\cos 2x) \\
 & \frac{\partial^2 y}{\partial x^2} = e^{-x}(A\cos 2x + B\sin 2x) + 2e^{-x}(-2A\sin 2x + 2B\cos 2x) - 4A\sin 2x + 4B\cos 2x \\
 & \left. \begin{aligned}
 & y = e^{-x}(A\cos 2x + B\sin 2x) + 2e^{-x}(-2A\sin 2x + 2B\cos 2x) \\
 & \frac{dy}{dx} = -e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}(-2A\sin 2x + 2B\cos 2x) - 4A\sin 2x + 4B\cos 2x
 \end{aligned} \right\} \\
 & x=0: y=18, A+2=18 \quad \Rightarrow \quad A=16 \\
 & x=0: \frac{dy}{dx}=0, -A+2B=0 \Rightarrow -16+2B=0 \quad \Rightarrow \quad B=8 \\
 & \text{Hence } y = 16e^{-x}\cos 2x + 2e^{-x}\sin 2x + 8\sin 2x \\
 & y = 2(8e^{-x} + 1)\cos 2x + 8\sin 2x
 \end{aligned}$$

**Question 12 (\*\*+)**

The curve  $C$  has a local minimum at the origin and satisfies the differential equation

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 8y = 32x^2.$$

Find an equation for  $C$ .

$$y = e^x (\sin 2x + \cos 2x) + (2x-1)^2$$

Given differential equation:

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 8y = 32x^2$$

Homogeneous equation:

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 8y = 0$$

$$(A+2)^2 + 4B = 0$$

$$(A+2)^2 = -4$$

$$A+2 = \pm 2i$$

$$A = -2 \pm 2i$$

Characteristic equation:  $y = P e^{Ax} + Q x e^{Ax}$

$$y = P e^{-2x} + Q x e^{-2x}$$

$$\frac{dy}{dx} = -2P e^{-2x} + Q e^{-2x} + Q x (-2e^{-2x})$$

$$\frac{d^2y}{dx^2} = 4P e^{-2x} - 4Q e^{-2x} - 4Q x e^{-2x}$$

$$4P e^{-2x} - 4Q e^{-2x} - 4Q x e^{-2x} + 8P e^{-2x} + 8Q x e^{-2x} = 32x^2$$

$$4P e^{-2x} + 8P e^{-2x} - 4Q e^{-2x} + 8Q x e^{-2x} - 4Q x e^{-2x} = 32x^2$$

$$12P e^{-2x} = 32x^2$$

$$P e^{-2x} = \frac{8}{3}x^2$$

$$P = \frac{8}{3}x^2 e^{2x}$$

$$Q e^{-2x} = 0$$

$$Q = 0$$

$$y = e^{-2x} \left( \frac{8}{3}x^2 e^{2x} \right) + C_1 e^{-2x} + C_2 x e^{-2x}$$

$$y = \frac{8}{3}x^2 + C_1 e^{-2x} + C_2 x e^{-2x}$$

Given condition:  $y = e^{-2x} (\sin 2x + \cos 2x) + (2x-1)^2$

$$y = e^{-2x} (\sin 2x + \cos 2x) + 4x^2 - 4x + 1$$

$$y = e^{-2x} (\sin 2x + \cos 2x) + (2x-1)^2$$

Question 13 (\*\*+)

$$\frac{d^2x}{dt^2} + 9x + 12 \sin 3t = 0, \quad t \geq 0,$$

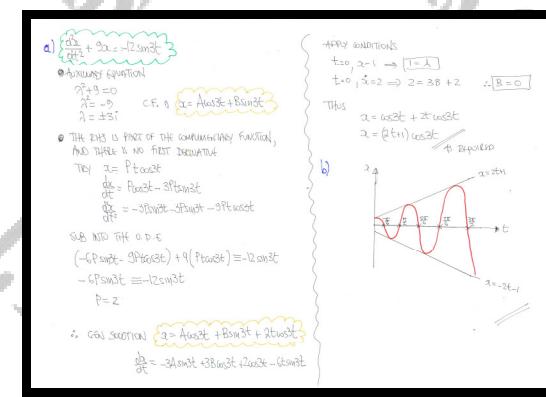
with  $x = 1$ ,  $\frac{dx}{dt} = 2$  at  $t = 0$ .

- a) Show that a solution of the differential equation is

$$x = (2t+1)\cos 3t.$$

- b) Sketch the graph of  $x$ .

proof



Question 14 (\*\*+)

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 4y = 16 + 32e^{2x},$$

with  $y = 8$  and  $\frac{dy}{dx} = 0$  at  $x = 0$ .

Show that the solution of the above differential equation is

$$y = 8 \cosh^2 x.$$

proof

The handwritten proof is organized into several sections:

- Differential Equation:**  $\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 4y = 16 + 32e^{2x}$ . It notes that  $(2-2i)^2 = -4$  and  $(2+2i)^2 = 16 + 32e^{2x}$ .
- Particular Integral:**  $16y = 16 + 32e^{2x}$ . This is solved using the method of undetermined coefficients, resulting in  $y_p = 4e^{2x} + 8e^{-2x}$ .
- Homogeneous Equation:**  $\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 4y = 0$ . The characteristic equation is  $\lambda^2 + 4\lambda + 4 = 0$ , which factors into  $(\lambda + 2)^2 = 0$ , giving the general solution  $y_h = Ae^{2x} + Be^{-2x}$ .
- Applying Initial Conditions:**
  - At  $x=0$ ,  $y=8 \Rightarrow A + B = 8$  (Equation 1)
  - At  $x=0$ ,  $\frac{dy}{dx}=0 \Rightarrow 2A - 2B = 0 \Rightarrow A = B$  (Equation 2)
  - Solving Equations 1 and 2 gives  $A = B = 4$ .
- Final Solution:**  $y = y_h + y_p = 2Ae^{2x} + 2Be^{-2x} + 4e^{2x} + 8e^{-2x}$ . Substituting  $A = B = 4$  into the homogeneous part yields  $y = 8e^{2x} + 8e^{-2x} + 4e^{2x} + 8e^{-2x} = 16e^{2x} + 16e^{-2x} = 16(\cosh 2x)$ .

**Question 15** (\*\*+)

$$\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2 y = 12x e^{kx}, \quad k > 0$$

- a) Find a general solution of the differential equation given that  $y = Px^3 e^{kx}$ , where  $P$  is a constant, is part of the solution.

- b) Given further that  $y = 1$ ,  $\frac{dy}{dx} = 0$  at  $x = 0$  show that

$$y = e^{kx} (2x^3 - kx + 1).$$

$$y = e^{kx} (2x^3 + Ax + B)$$

(a)

$\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2 y = 0$ <b>Auxiliary equation:</b> $\lambda^2 - 2k\lambda + k^2 = 0$ $(\lambda - k)^2 = 0$ $\lambda = k$ (repeated)	<b>Complementary function:</b> $y = A e^{kx} + B x e^{kx}$
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TOP PARTICULAR INTEGRAL TRY  
 $y = P x^3 e^{kx}$   
 $\frac{dy}{dx} = 3P x^2 e^{kx} + P k x^3 e^{kx}$   
 $\frac{d^2y}{dx^2} = 6P x e^{kx} + 3P k x^2 e^{kx} + 3P k^2 x^3 e^{kx} + P k^3 x^4 e^{kx}$

SUB INTO THE O.D.E.  
 $\frac{d^2y}{dx^2} - 2k \frac{dy}{dx} + k^2 y = P k^3 x^4 e^{kx} + P k^2 x^3 e^{kx} + P k x^2 e^{kx}$   
 $-2k \frac{dy}{dx} = -2P k^2 x^3 e^{kx} - 6P k x^2 e^{kx}$   
 $P y = P x^3 e^{kx}$

$\frac{dy}{dx} = \frac{P x^3 e^{kx}}{A x^3 + B x^2 + C x + D}$   
 $\therefore$  GIVE SOLUTION:  $y = A e^{kx} + B x e^{kx} + C x^2 e^{kx} + D x^3 e^{kx}$   
 $y = e^{kx} (A + B x + C x^2 + D x^3)$

(b)

$\frac{dy}{dx} = k e^{kx} (A + B x + C x^2 + D x^3) + e^{kx} (B + 2C x + 3D x^2)$

$x=0 \quad y=1 \Rightarrow 1 = A$   
 $x=0 \quad \frac{dy}{dx}=0 \Rightarrow 0 = B + C \quad \therefore B = -C$   
 $\therefore y = e^{kx} (1 - k x + 2 x^2)$

**Question 16 (\*\*+)**

Show that the solution of the differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 24e^{4x},$$

subject to the boundary conditions  $y = -1$ ,  $\frac{dy}{dx} = -4$  at  $x = 0$ , can be written as

$$y = (12x^2 - 1)e^{4x}.$$

[proof]

The handwritten proof is contained within a black-bordered box. It starts with the differential equation  $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 24e^{4x}$ . A note indicates that the right-hand side is zero at  $x=0$ . The proof then shows that the general solution is  $y = Ae^{4x} + Be^{4x}$ . By applying the boundary condition  $y = -1$  at  $x=0$ , it is found that  $A = -1$ . Then, by differentiating and applying the second boundary condition  $\frac{dy}{dx} = -4$  at  $x=0$ , it is found that  $B = 5$ . Therefore, the solution is  $y = (12x^2 - 1)e^{4x}$ .

**Question 17** (\*\*\*)

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}.$$

a) Find a solution of the differential equation given that  $y=1$ ,  $\frac{dy}{dx}=0$  at  $x=0$ .

b) Sketch the graph of  $y$ .

The sketch must include ...

- the coordinates of any points where the graph meets the coordinate axes.
- the coordinates of any stationary points of the curve.
- clear indications of how the graph looks for large positive or negative values of  $x$ .

$$y = e^{3x} (2x^2 - 3x + 1)$$

(3)  $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}$

AUXILIARY EQUATION  
 $(2^2 - 3)^2 = 0$   
 $(2x - 3)^2 = 0$   
 $2x - 3 = 0$   $\Rightarrow$   $x = \frac{3}{2}$

CORRESPONDING FUNCTION  
 $y = Ae^{3x} + Be^{3x}$

BY PARTIAL FRACTION TDY  
 $y = P + Qx^2e^{3x}$   
 $\frac{dy}{dx} = 2Qx^2e^{3x} + 3Qx^2e^{3x}$   
 $\frac{d^2y}{dx^2} = 2P + 6Qx^2e^{3x} + 6Qx^2e^{3x} + 18Qx^2e^{3x}$

SUB INTO THE D.D.E  
 $\frac{d^2y}{dx^2} = 2P + 12Qx^2e^{3x} + 9Qx^2e^{3x}$   
 $- 6\frac{dy}{dx} = -12Qx^2e^{3x} - 18Qx^2e^{3x}$   
 $+ 9y = 9Qx^2e^{3x}$

ADD TO GET  $2Pe^{3x} = 4e^{3x}$   
 $P = 2$

$\therefore y = Ae^{3x} + Be^{3x} + 2x^2e^{3x}$   
 $y = e^{3x}[A + Be^{3x} + 2x^2]$   
 $\frac{dy}{dx} = 3e^{3x}[A + 3Be^{3x} + 2x^2] + e^{3x}[B + 6x]$

$\bullet$   $x \rightarrow \infty$   $y \rightarrow \infty$   
 $\bullet$   $x \rightarrow -\infty$   $y \rightarrow 0$

$\bullet$   $x=0$   $y=1 \Rightarrow [1=1]$   
 $x=0$   $\frac{dy}{dx}=0 \Rightarrow 0=3A+B \Rightarrow [B=-3]$

$y = e^{3x}[1 - 3x + 2x^2]$

$\bullet$   $y = e^{3x}(2x^2 - 3x + 1)$   
 $y = e^{3x}(2(x - \frac{1}{2})(x - 1))$

AXES INTERCEPTS  $(0,1)$ ,  $(\frac{1}{2},0)$ ,  $(1,0)$

$\bullet$   $\frac{dy}{dx} = 3e^{3x}(2(2x^2 - 3x + 1)) + e^{3x}(4x - 3)$   
 $= e^{3x}(6x^2 - 9x + 3 + 4x - 3)$   
 $= e^{3x}(6x^2 - 5x)$   
 $= 2e^{3x}(3x^2 - 5)$

$\therefore$  T.P.  $x=0$ ,  $y=1$   
 $x=\frac{5}{6}$ ,  $y=\frac{25}{216}$

$\bullet$   $x \rightarrow \infty$   $y \rightarrow +\infty$   
 $x \rightarrow -\infty$   $y \rightarrow 0$

**Question 18      (\*\*\*)+**

The curve with equation  $y = f(x)$  is the solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8\sin 2x.$$

The first two non zero terms in Maclaurin series expansion of  $f(x)$  are  $x + kx^2$ , where  $k$  is a constant.

Determine in any order the value of  $k$  and the exact value of  $f\left(\frac{1}{4}\pi\right)$ .

$$[ \quad ], [k=2], \left[ f\left(\frac{1}{4}\pi\right) = \frac{1}{2}(3\pi - 4)e^{\frac{1}{2}\pi} \right]$$

SIMILAR WITH THE D'Alembert's EQUATION

$$x^2 - 2x + 4 = 0$$

$$(x-1)^2 = 0$$

$$x=1$$

$$\therefore C.F. = (A+Bx)e^{2x}$$

PARTICULAR INTEGRAL BY INSPECTION OR D'-OPERATOR

TRY:  $y = P(x)e^{2x} + Q(x)e^{-2x}$

$$y' = 2Pe^{2x} - 2Qe^{-2x}$$

$$y'' = 4Pe^{2x} + 4Qe^{-2x}$$

$$\frac{dy}{dx} = -4Qe^{-2x} - 4Qe^{2x}$$

$$-\frac{d^2y}{dx^2} = 8Qe^{-2x} - 8Qe^{2x}$$

$$+4y = 4Pe^{2x} + 4Qe^{-2x}$$

$$\Rightarrow 8Qe^{-2x} - 8Qe^{2x} = 8Pe^{2x}$$

$$\therefore P=0 \quad Q=1$$

GENERAL SOLUTION IS  $y = (A+Bx)e^{2x} + ce^{2x}$

NOW WE HAVE  $y' = 2e^{2x} + (A+Bx)e^{2x} - 2ce^{2x}$   
 $y'' = 2Be^{2x} + 2Be^{2x} + (A+Bx)e^{2x} - 4ce^{2x}$   
 $= (Bx+14+4B)e^{2x} - 11ce^{2x}$

AND SUBSTITUTING AT  $x=0$   $y_0 = 4+1 \quad y_0' = 2A+8 \quad y_0'' = 48+4A-4$

NOW THE NEXXUOUS STEP IS SOLUTION OF THE O.D.E IS

$$y_0 = y_0 + 2y_0' + \frac{1}{2}y_0'' - \dots$$

$$y_0 = (4+1) + (2A+8)2 + \frac{1}{2}(48+4A-4)2^2$$

$$y_0 = 0 + x + 2x^2$$

COEFFICIENTS OF EXPONENTIAL

$A+1=0$	$2A+B=1$	$2=2A+2B-2$
$A=-1$	$-2+B=1$	$2=-2+C-2$
	$B=3$	$C=2$

FINALLY WE HAVE

$$y = f(x) = (3x-1)e^{2x} + ce^{2x}$$

$$f(\frac{x}{2}) = (\frac{3x}{2}-\frac{1}{2})e^{\frac{x}{2}} + ce^{\frac{x}{2}}$$

$$f(\frac{c}{2}) = \frac{1}{2}(3x-4)e^{\frac{x}{2}}$$

ALGEBRAIC FORM STANDARD EXPANSION

$$y = (4+Bx)e^{2x} + ce^{2x} = (A+Bx)(1+2x+2x^2+\dots) + (1-2x^2+\dots)$$

$$= A+2Ax+2Ax^2+Bx+2Bx^2$$

$$1 - 2x^2$$

$$\Rightarrow (A+1)x^2 + (2Ax+2B)x + (1-2x^2) = \dots$$

$A+1=0$	$2A+B=1$	$2A+2B-2=C$
$A=-1$	$-2+B=1$	$-2+4-2=C$
	$B=3$	$C=2$

etc etc etc

**Question 19** (\*\*\*)+

The function  $y = f(x)$  satisfies the following differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 2e^{-x}(\sin 2x - 2\cos 2x),$$

subject to the boundary conditions  $y = 0$ ,  $\frac{dy}{dx} = 2$  at  $x = 0$ .

Solve the differential equation to show that

$$y = \cosh x \sinh 2x.$$

No credit will be given for verification methods.

[ ] , proof

<p><b>FIND THE COMPLEXIFYING FACTOR FIRST</b></p> $\begin{aligned} & \lambda^2 - 2\lambda + 5 = 0 \\ & (\lambda - 1)^2 + 4 = 0 \\ & (\lambda - 1)^2 = -4 \\ & \lambda - 1 = \pm 2i \\ & \lambda = 1 \pm 2i \end{aligned}$ <p><b>∴ COMPLEXIFYING FACTOR</b></p> $y = e^{\lambda x}(A\cos\lambda x + B\sin\lambda x)$ <p><b>THE PARTICULAR INTEGRAL TRY</b></p> <ul style="list-style-type: none"> <li><math>y = e^{\lambda x}(P\cos\lambda x + Q\sin\lambda x)</math></li> <li><math>\frac{dy}{dx} = e^{\lambda x}(P\lambda\cos\lambda x + Q\lambda\sin\lambda x) + e^{\lambda x}(2P\cos\lambda x - 2Q\sin\lambda x)</math></li> <li><math>= e^{\lambda x}(-P\cos\lambda x - Q\sin\lambda x + 2P\cos\lambda x - 2Q\sin\lambda x)</math></li> <li><math>= e^{\lambda x}[(P+2Q)\sin\lambda x - (2P-Q)\cos\lambda x]</math></li> <li><math>\frac{d^2y}{dx^2} = e^{\lambda x}[(P+2Q)\sin\lambda x - (2P-Q)\cos\lambda x] - e^{\lambda x}[(2P+4Q)\cos\lambda x + (4P-2Q)\sin\lambda x]</math></li> <li><math>= e^{\lambda x}[(P+2Q-4P+2Q)\sin\lambda x + (-2P-Q-2P+4Q)\cos\lambda x]</math></li> <li><math>= e^{\lambda x}[(4Q-3P)\sin\lambda x - (4P+2Q)\cos\lambda x]</math></li> </ul> <p><b>SUB INTO THE ODE</b></p> $\begin{aligned} \frac{d^2y}{dx^2} &= (4Q-3P)e^{\lambda x}\sin\lambda x + (-4P-5Q)e^{\lambda x}\cos\lambda x \\ -2\frac{dy}{dx} &= -(2P+4Q)e^{\lambda x}\sin\lambda x + (-4P+2Q)e^{\lambda x}\cos\lambda x \\ +5y &= (5P+4Q)e^{\lambda x}\sin\lambda x + (4P+2Q)e^{\lambda x}\cos\lambda x \end{aligned}$ $(4P+4Q)e^{\lambda x}\sin\lambda x + (-8P+2Q)e^{\lambda x}\cos\lambda x \equiv 2e^{-x}(\sin 2x - 2\cos 2x)$	<p><b>FOUNDING COEFFICIENTS WE GET</b></p> $\begin{aligned} 4P+4Q &= 2 \\ -8P+2Q &= -4 \end{aligned} \Rightarrow \begin{cases} P = 0 \\ Q = 1 \end{cases} \Rightarrow P = 0, Q = \frac{1}{2}$ <p><b>∴ GENERAL SOLUTION IS</b></p> $y = e^{\lambda x}(A\cos\lambda x + B\sin\lambda x) + \frac{1}{2}e^{\lambda x}\sin 2x$ <p><b>APPLY CONDITIONS - FIRSTLY <math>x=0</math>, <math>y=0</math></b></p> $\begin{aligned} & \Rightarrow 0 = A \\ & \Rightarrow y = B\sin\lambda x + \frac{1}{2}e^{\lambda x}\sin 2x \\ & \Rightarrow y = (B\sin x + \frac{1}{2}e^x)\sin 2x \end{aligned}$ <p><b>DIFFERENTIATE &amp; APPLY THE SECOND CONDITION <math>x=0</math>, <math>\frac{dy}{dx}=2</math></b></p> $\begin{aligned} & \Rightarrow \frac{dy}{dx} = (B\sin x + \frac{1}{2}e^x)\sin 2x + (B\sin x + \frac{1}{2}e^x)(2\cos 2x) \\ & \quad \therefore 2 = (B + \frac{1}{2})x \\ & \quad B = \frac{1}{2} \\ & \therefore y = (\frac{1}{2}\sin x + \frac{1}{2}e^x)\sin 2x \\ & \quad y = \cosh x \sinh 2x \end{aligned}$ <p style="color: yellow;">✓ READING</p>
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# **2<sup>ND</sup> ORDER or HIGHER CAUCHY EULER TYPE**

**Question 1 (\*\*)**

Find the general solution of the following differential equation.

$$4t^2 \frac{d^2x}{dt^2} + 4t \frac{dx}{dt} - x = 0.$$

$$x = At^{\frac{1}{2}} + Bt^{-\frac{1}{2}}$$

TRY A. SOLUTION OF THE FORM  $x = t^n$ , WHERE  $n$  IS A CONSTANT TO BE FOUND

$$\frac{dy}{dt} = nt^{n-1}$$

$$\frac{d^2y}{dt^2} = n(n-1)t^{n-2}$$

• SUB. INTO THE O.D.E

$$\Rightarrow 4t^2 \left[ n(n-1)t^{n-2} \right] + 4t \left[ nt^{n-1} \right] - t^n = 0$$

$$\Rightarrow 4n(n-1)t^{n-2} + 4nt^{n-1} - t^n = 0$$

$$\Rightarrow [4n(n-1) + 4n] t^{n-2} = 0$$

$$\Rightarrow 4n^2 + 4n - 1 = 0$$

$$\Rightarrow (2n+1)(2n-1) = 0$$

$$\therefore n = \pm \frac{1}{2}$$

$$\therefore x = At^{\frac{1}{2}} + Bt^{-\frac{1}{2}}$$

**Question 2 (\*\*+)**

Find the general solution of the following differential equation.

$$4t^2 \frac{d^2y}{dt^2} + 4t \frac{dy}{dt} + y = 0.$$

$$y = P \cos[\ln \sqrt{t}] + P \sin[\ln \sqrt{t}]$$

TRY A. TRIAL SOLUTION OF THE FORM  $y = t^n$

$$\frac{dy}{dt} = nt^{n-1}$$

$$\frac{d^2y}{dt^2} = n(n-1)t^{n-2}$$

• SUB. INTO THE O.D.E

$$\Rightarrow 4t^2 \left[ n(n-1)t^{n-2} \right] + 4t \left[ nt^{n-1} \right] + t^n = 0$$

$$\Rightarrow [4n(n-1) + 4n + 1] t^{n-2} = 0$$

$$\Rightarrow [4n^2 - 4n + 1] t^{n-2} = 0$$

$$\Rightarrow 4n^2 + 1 = 0$$

$$\Rightarrow n = \pm \frac{1}{2}i$$

$$\therefore y = A t^{\frac{1}{2}i} + B t^{-\frac{1}{2}i}$$

$$y = A e^{\frac{1}{2}i \ln t} + B e^{-\frac{1}{2}i \ln t}$$

$$y = A \cos(\ln t) + B \sin(\ln t)$$

$$y = B \cos(\ln t) + A \sin(\ln t)$$

$$y = (A+B) \cos(\ln t) + (A-B) \sin(\ln t)$$

$$y = P \cos(\ln \sqrt{t}) + Q \sin(\ln \sqrt{t})$$

**Question 3** (\*\*\*)

$$x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = 9x^8.$$

Determine the solution of the above differential equation subject to the boundary conditions

$$y = \frac{3}{2}, \frac{dy}{dx} = 2 \text{ at } x = 1.$$

$$\boxed{\quad}, \quad y = \frac{1}{4}x^4(x^4 + 1) + \frac{1}{x}$$

$x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = 9x^8 \quad \text{with } x=1, y=\frac{3}{2}, \frac{dy}{dx}=2$

• ASSUME A SOLUTION OF THE FORM  $y = x^2$

$$\begin{aligned} y &= x^2 \\ y' &= 2x \\ y'' &= 2(2-0)x^{2-2} \end{aligned}$$

• SUBSTITUTE INTO THE L.H.S. OF THE O.D.E. (IGNORE R.H.S.)

$$\begin{aligned} &\Rightarrow x^2 [2(2-0)x^{2-2}] - 2x[2x^{2-1}] - 4[x^2] = 0 \\ &\Rightarrow 2(x-1)x^2 - 2x^3 - 4x^2 = 0 \\ &\Rightarrow [2(x-1) - 2x - 4] x^2 = 0 \\ &\Rightarrow 2x - 3x - 4 = 0 \\ &\Rightarrow (x-4)(x+1) = 0 \\ &\Rightarrow x = -1, 4 \end{aligned}$$

• COMPLEMENTARY FUNCTION  $y = Ax^{-1} + Bx^4$

• PARTICULAR INTEGRAL BY INSPECTION

$$\begin{aligned} y &= Px^8 \\ y' &= 8Px^7 \\ y'' &= 56Px^6 \end{aligned} \Rightarrow \begin{aligned} &x^2[36Px^6] - 2x[8Px^7] - 4Px^8 = 9x^8 \\ &\Rightarrow 36Px^8 - 16Px^9 - 4Px^8 = 9x^8 \\ &\Rightarrow 36P = 9 \\ &\Rightarrow P = \frac{1}{4} \end{aligned}$$

• GENERAL SOLUTION IS

$$y = \frac{A}{x} + Bx^4 + \frac{1}{4}x^8$$

• APPLYING CONDITIONS  $x=1, y=\frac{3}{2}, \frac{dy}{dx}=2$

$$\begin{aligned} y &= \frac{A}{x} + Bx^4 + \frac{1}{4}x^8 \\ \frac{dy}{dx} &= -\frac{A}{x^2} + 4Bx^3 + 2x^7 \end{aligned} \Rightarrow \begin{aligned} \frac{3}{2} &= A + B + \frac{1}{4} \\ 2 &= -A + 4B + 2 \\ 2 &= -A + 4B + 2 \end{aligned} \quad \text{ADDING}$$

$$\begin{aligned} &\Rightarrow \frac{7}{2} = 5B + \frac{9}{4} \\ &\Rightarrow 14 = 20B + 9 \\ &\Rightarrow 5 = 20B \\ &\Rightarrow B = \frac{1}{4} \end{aligned}$$

$\therefore A = 4B \Rightarrow A = 1$

• FINALLY WE HAVE A SOLUTION

$$\begin{aligned} y &= \frac{1}{x} + \frac{1}{4}x^4 + \frac{1}{4}x^8 \\ y &= \frac{1}{x} + \frac{1}{4}x^4(1+x^4) \end{aligned}$$

**Question 4    (\*\*\*)**

Find the general solution of the following differential equation

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - n(n+1)y = 0.$$

$$y = Ax^n + \frac{B}{x^{n+1}}$$

This is a standard "Euler type" equation.  
Let  $y = x^\lambda$   
 $\frac{dy}{dx} = \lambda x^{\lambda-1}$   
 $\frac{d^2y}{dx^2} = \lambda(\lambda-1)x^{\lambda-2}$

SUB. INTO THE O.D.E

$$\begin{aligned} &\Rightarrow x^2 \lambda(\lambda-1)x^{\lambda-2} + 2x[\lambda x^{\lambda-1}] - n(n+1)x^\lambda = 0 \\ &\Rightarrow 2(\lambda-1)x^\lambda + 2\lambda x^\lambda - n(n+1)x^\lambda = 0 \\ &\Rightarrow [2(\lambda-1) + 2\lambda - n(n+1)]x^\lambda = 0 \\ &\Rightarrow 2^2\lambda - n^2 - n = 0 \\ &\Rightarrow 2^2\lambda - n^2 - n = 0 \\ &\Rightarrow (\lambda + \frac{1}{2})^2 - \frac{n^2 + n}{4} = 0 \\ &\Rightarrow (\lambda + \frac{1}{2})^2 = (n + \frac{1}{2})^2 \\ &\Rightarrow \lambda + \frac{1}{2} = \pm (n + \frac{1}{2}) \\ &\Rightarrow \lambda + \frac{1}{2} = \begin{cases} n + \frac{1}{2} \\ -n - \frac{1}{2} \end{cases} \\ &\Rightarrow \lambda = \begin{cases} n \\ -n-1 \end{cases} \end{aligned}$$

∴ General solution  $y = A x^n + \frac{B}{x^{n+1}}$

**Question 5** (\*\*\*)+

Given that if  $x = e^t$  and  $y = f(x)$ , show clearly that ...

a) ...  $x \frac{dy}{dx} = \frac{dy}{dt}$ .

b) ...  $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$ .

The following differential equation is to be solved

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 2 \ln x$$

subject to the boundary conditions  $y = \frac{1}{2}$ ,  $\frac{dy}{dx} = \frac{3}{2}$  at  $x = 1$ .

c) Use the substitution  $x = e^t$  to solve the above differential equation.

$$y = \frac{1}{2} + \frac{1}{2}(2x^2 + 1) \ln x$$

The handwritten solution shows the steps for solving the differential equation  $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 2 \ln x$  using the substitution  $x = e^t$ . It includes the derivation of the derivatives  $\frac{dy}{dt}$  and  $\frac{d^2y}{dt^2}$ , the reduction of the equation to a standard form, the application of auxiliary equations, and the final solution  $y = \frac{1}{2} + \frac{1}{2}(2x^2 + 1) \ln x$ .

**Question 6** (\*\*\*)

$$x^3 \frac{d^2y}{dx^2} - 2x^2 \frac{dy}{dx} - 4xy = 5.$$

Find the solution of the above differential equation subject to the boundary conditions

$$y = 4, \frac{dy}{dx} = 20 \text{ at } x = 0.$$

$$y = 5x^4 - \frac{1}{x}(1 + \ln x)$$

$x^3 \frac{d^2y}{dx^2} - 2x^2 \frac{dy}{dx} - 4xy = 5, \quad x=0, y=4, \frac{dy}{dx}=20$

ASSUME A SOLUTION OF THE FORM  $y = x^a$   
 $\frac{dy}{dx} = ax^{a-1}$   
 $\frac{d^2y}{dx^2} = a(a-1)x^{a-2}$

DIVIDE 2. THROUGH, THEN SUB INTO THE HOMOGENEOUS O.D.E  
 $x^3 \frac{d^2y}{dx^2} - 2x^2 \frac{dy}{dx} = \frac{5}{x}$   
 $\Rightarrow x^2[x^3(a(a-1)x^{a-2}) - 2x[x^2ax^{a-1}]] - 4x^3 = 0$   
 $\Rightarrow [x^5(a^2 - a) - 2x^3]x^2 = 0$   
 $\Rightarrow x^5(a^2 - a) - 2x^3 = 0$   
 $\Rightarrow (x^5 - 2x^3)(a+1) = 0$   
 $\Rightarrow x^3 < -1 \quad \therefore \text{C.F. : } y = Ax^3 + Bx^{-1}$

FIND PARTICULAR INTEGRAL BY  $y = \frac{B}{x}$  (since  $\frac{1}{x}$  is a part of C.F.)  
 $y = -\frac{B}{x^2}$   
 $\frac{dy}{dx} = \frac{2B}{x^3}$   
 $\frac{d^2y}{dx^2} = -\frac{6B}{x^4}$   
 $\therefore -\frac{6B}{x^4}[2x^5 - 2x^3] - 2x\left[\frac{B}{x}\right] - 4\left(\frac{B}{x}\right) = \frac{5}{x}$   
 $\therefore -12Bx^4 + 12Bx^2 - 2Bx^2 - 4B = 5$   
 $\therefore -10Bx^4 + 8Bx^2 - 4B = 5$   
 $\therefore 5Bx^4 - 4Bx^2 - 5 = 0$   
 $\therefore B = 1$   
 $\therefore y = Ax^3 + \frac{1}{x}$   
 $\therefore y = 5x^4 - \frac{1}{x} + \ln x$

**Question 7    (\*\*\*)+**

Find the general solution of the following differential equation.

$$\frac{d^4y}{dx^4} + \frac{2}{x} \frac{d^3y}{dx^3} - \frac{1}{x^2} \frac{d^2y}{dx^2} + \frac{1}{x^3} \frac{dy}{dx} = 0, \quad x > 0.$$

$$, \quad y = A \ln x + Bx^2 + Cx^2 \ln x + D$$

$$\frac{\partial^4 y}{\partial x^4} + \frac{2}{x} \frac{\partial^3 y}{\partial x^3} - \frac{1}{x^2} \frac{\partial^2 y}{\partial x^2} + \frac{1}{x^3} \frac{dy}{dx} = 0$$

$$\Rightarrow x^3 \frac{\partial^4 y}{\partial x^4} + 2x^2 \frac{\partial^3 y}{\partial x^3} - x^2 \frac{\partial^2 y}{\partial x^2} + \frac{dy}{dx} = 0$$

- **Sub (Frobenius) to deduce the order n**

$$\text{Let } z = \frac{dy}{dx}, \frac{dz}{dx}, \frac{d^2z}{dx^2}, \frac{d^3z}{dx^3}, \frac{d^4z}{dx^4}$$

$$\Rightarrow x^3 \frac{\partial^4 z}{\partial x^4} + 2x^2 \frac{\partial^3 z}{\partial x^3} - x^2 \frac{\partial^2 z}{\partial x^2} + z = 0$$

TRY SOLUTION OF THE EQUATION

$$z = x^n, \quad \frac{dz}{dx} = nx^{n-1}, \quad \frac{d^2z}{dx^2} = n(n-1)x^{n-2}, \quad \frac{d^3z}{dx^3} = n(n-1)(n-2)x^{n-3}$$

$$\Rightarrow n(n-1)(n-2)x^n + 2n(n-1)x^{n-1} - nx^n + x = 0$$

$$\Rightarrow n^3 - 3n^2 + 2n + 2x^n - nx^n - n = 0$$

$$\Rightarrow n^3 - n^2 - n + 1 = 0$$

$$\Rightarrow n^3(n-1)(n-2) = 0$$

$$\Rightarrow (n-1)(n-2)(n-1) = 0$$

$$\Rightarrow n = 1 \quad (\text{REJECT})$$

REJECT

$$\Rightarrow z = A x^{-1} + B x^1 + C x^2 \ln x$$

$$\Rightarrow \frac{dz}{dx} = \frac{A}{x} + Bx + Cx^2 \ln x \quad \text{BY PLESS}$$

$$\Rightarrow y = Ax^{-1} + Bx^1 + \int [Bx + Cx^2 \ln x] dx$$

$$\Rightarrow y = Ax^{-1} + Bx^1 + Cx^2 \ln x + Dx^2$$

$$\Rightarrow y = Ax^{-1} + Bx^1 + Cx^2 \ln x + D$$

**Question 8** (\*\*\*)

The curve with equation  $y = f(x)$  satisfies

$$x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 13y = 0, \quad x > 0.$$

By using the substitution  $x = e^t$ , or otherwise, determine an equation for  $y = f(x)$ ,

given further that  $y = 1$  and  $\frac{dy}{dx} = -2$  at  $x = 1$ .

$$y = \frac{\cos(3\ln x)}{x^2}$$

**Using  $x = e^t$**

- $x = e^t$
- $\frac{dx}{dt} = e^t$
- $\frac{d}{dt}(x) = \frac{d}{dt}(e^t) \Rightarrow \frac{d}{dt}(x) = e^t$
- $\Rightarrow \frac{dx}{dt} = e^t \frac{dt}{dx}$
- $\Rightarrow \frac{d^2x}{dt^2} = e^t \frac{d}{dt}\left(\frac{dt}{dx}\right) = e^t \frac{dt}{dx} \cdot \frac{d}{dt}(x) = e^t \cdot e^t = e^{2t}$
- $\Rightarrow \frac{d^2x}{dt^2} = e^{2t} \frac{dx}{dt}$
- $\Rightarrow \frac{d^2x}{dt^2} = \frac{d}{dt}(e^{2t})$
- $\Rightarrow \frac{d^2x}{dt^2} = \frac{d}{dt}(x^2)$
- $\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)^{-1} = \frac{d}{dt}\left(\frac{1}{e^{2t}}\right) = -\frac{2}{e^{2t}}$
- $\Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{e^{2t}} \frac{dx}{dt}$
- $\Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{e^{2t}} \cdot e^t = -\frac{2}{e^t}$
- $\Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{x}$
- $\text{Therefore } 2x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 13y = 0$
- $(2x^2)(-\frac{2}{x}) + 5x(-2) + 13y = 0$
- $\Rightarrow -4x + 5x + 13y = 0$
- $\Rightarrow x + 13y = 0$
- $\Rightarrow 1 + 13y = 0$
- $\Rightarrow 13y = -1$
- $\Rightarrow y = -\frac{1}{13}$

**Using the initial conditions**

- $x = 1 \Rightarrow t = 0$
- $y = 1 \Rightarrow A = 1$
- $\frac{dy}{dx} = -2 \Rightarrow B = -2$
- $B = 0 \Rightarrow C = 0$
- $\therefore y = \frac{A}{x^2}$

**Initial conditions of the form**

- $x = 1 \Rightarrow t = 0$
- $y = 1 \Rightarrow A = 1$
- $\frac{dy}{dx} = -2 \Rightarrow B = -2$
- $B = 0 \Rightarrow C = 0$
- $\therefore y = \frac{A}{x^2}$

**Given  $y = \frac{A}{x^2}$**

- $y = A x^{-2}$
- $\frac{dy}{dx} = A(-2)x^{-3}$
- $\frac{d^2y}{dx^2} = A(2)x^{-4}$
- $\therefore 2x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 13y = 0$
- $2x^2(A(2)x^{-4}) + 5x(A(-2)x^{-3}) + 13(Ax^{-2}) = 0$
- $\Rightarrow 4x^{-2} + 5x^{-2} + 13x^{-2} = 0$
- $\Rightarrow 4 + 5 + 13 = 0$
- $12 = 0$  (Contradiction)
- $\therefore y = \frac{A}{x^2}$

**Given  $\frac{dy}{dx} = -2$**

- $y = Ax^{-2}$
- $\frac{dy}{dx} = -2Ax^{-3}$
- $-2Ax^{-3} = -2$
- $Ax^{-3} = 1$
- $A = x^3$
- $\therefore y = x^3$

**Given  $y = 1$**

- $y = 1 \Rightarrow A = 1$
- $\therefore y = x^3$

**Proceed as above**

- $\therefore y = \frac{A}{x^2}$
- $\therefore A = x^2$
- $\therefore y = x^2$

**Question 9** (\*\*\*)

$$x^2 \frac{d^2y}{dx^2} - 8x \frac{dy}{dx} + 9y = 0, \quad x > 0.$$

Use the fact that  $y = Ax^{\frac{3}{2}}$  satisfies the above differential equation, to find the full solution subject to  $y = 4$  and  $\frac{dy}{dx} = 10$  at  $x = 1$ .

,   $y = 4x^{\frac{3}{2}}(1 + \ln x)$

$4x^2 \frac{d^2y}{dx^2} - 8x \frac{dy}{dx} + 9y = 0 \quad y(1) = 4 \quad y'(1) = 10$

IT IS GIVEN THAT A SOLUTION IS  $y_1 = Ax^{\frac{3}{2}}$

TRY A SOLUTION OF THE FORM  $y_2 = x^{\frac{3}{2}}V(x)$

DIFFERENTIATE TWICE

$$\begin{aligned} \frac{dy_2}{dx} &= \frac{3}{2}x^{\frac{1}{2}}V(x) + x^{\frac{1}{2}}V'(x) \\ \frac{d^2y_2}{dx^2} &= \frac{3}{4}x^{-\frac{1}{2}}V(x) + \frac{3}{2}x^{\frac{1}{2}}V'(x) + \frac{3}{2}x^{\frac{1}{2}}V'(x) + x^{\frac{1}{2}}V''(x) \\ &= \frac{3}{4}x^{-\frac{1}{2}}V(x) + 3x^{\frac{1}{2}}V'(x) + x^{\frac{1}{2}}V''(x) \end{aligned}$$

SUB INTO THE O.D.E

$$\begin{aligned} &\Rightarrow 4x^2 \left[ \frac{3}{4}x^{-\frac{1}{2}}V(x) + 3x^{\frac{1}{2}}V'(x) + x^{\frac{1}{2}}V''(x) \right] - 8x \left[ \frac{3}{2}x^{\frac{1}{2}}V(x) + x^{\frac{1}{2}}V'(x) \right] + 9V(x) = 0 \\ &\Rightarrow 3x^{\frac{3}{2}}V(x) + 12x^{\frac{3}{2}}V'(x) + 3x^{\frac{3}{2}}V''(x) - 12x^{\frac{3}{2}}V(x) - 8x^{\frac{3}{2}}V'(x) + 9V(x) = 0 \\ &\Rightarrow 4x^{\frac{3}{2}}V'(x) + 4x^{\frac{3}{2}}V''(x) = 0 \\ &\Rightarrow 4x^{\frac{1}{2}} \frac{dV}{dx} + \frac{d^2V}{dx^2} = 0 \quad (\text{let } P = \frac{dV}{dx}) \\ &\Rightarrow x \frac{dP}{dx} + P = 0 \quad \text{WHERE } P = \frac{dV}{dx} \\ &\Rightarrow x \frac{dP}{dx} = -P \end{aligned}$$

$$\begin{aligned} &\Rightarrow \frac{1}{P} dP = -\frac{1}{x} dx \\ &\Rightarrow \ln|P| = (\ln B - \ln|x|) \\ &\Rightarrow |P| = B \left( \frac{1}{x} \right) \\ &\Rightarrow P = \frac{B}{x} \\ &\Rightarrow \frac{dV}{dx} = \frac{B}{x} \\ &\Rightarrow V = B \ln|x| + C \end{aligned}$$

FINDING THE GENERAL SOLUTION IS

$$\begin{aligned} y &= Ax^{\frac{3}{2}} + x^{\frac{3}{2}}(B \ln x + C) \quad x > 0 \\ y &= Ax^{\frac{3}{2}} + Bx^{\frac{3}{2}} \ln x \\ y &= x^{\frac{3}{2}}(A + B \ln x) \end{aligned}$$

APPLY CONDITION  $y(1) = 4$

$$4 = 1(A + B \cdot 0) \Rightarrow A = 4$$

DIFFERENTIATE AND APPLY  $y'(1) = 10$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3}{2}x^{\frac{1}{2}}(A + B \ln x) + x^{\frac{1}{2}}\left(\frac{B}{x}\right) \\ 10 &= \frac{3}{2}(4) + B \\ B &= 4 \\ \therefore y &= 4x^{\frac{3}{2}}(1 + \ln x) \end{aligned}$$

**Question 10** (\*\*\*\*\*)

Find the general solution of the following differential equation.

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 2x, \quad x > 0.$$

          ,     $y = Ax + B \cos(\ln x) + C \sin(\ln x) + x \ln x$

• LOOKING AT THE LHS OF THE O.D.E WE TRY A SOLUTION OF THE FORM

$$\begin{aligned} y &= x^{\lambda} \\ \frac{dy}{dx} &= \lambda x^{\lambda-1} \\ \frac{d^2y}{dx^2} &= \lambda(\lambda-1)x^{\lambda-2} \\ \frac{d^3y}{dx^3} &= \lambda(\lambda-1)(\lambda-2)x^{\lambda-3} \end{aligned}$$

• SUBSTITUTE INTO THE O.D.E (LHS = 0)

$$\begin{aligned} &\Rightarrow \lambda(\lambda-1)(\lambda-2)x^{\lambda-3} + 2\lambda(\lambda-1)x^{\lambda-2} + \lambda x^{\lambda-1} - x^{\lambda} = 0 \\ &\Rightarrow x^{\lambda} [ \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1) + \lambda - 1 ] = 0 \\ &\Rightarrow (\lambda-1)[ \lambda(\lambda-2) + 2\lambda + 1 ] = 0 \\ &\Rightarrow (\lambda-1)(\lambda+1) = 0 \end{aligned}$$

$$\lambda = \begin{cases} 1 \\ -1 \end{cases} \quad y = Ax^1 + Bx^{-1} + Cx^0$$

• NOW NOTE THAT

$$\begin{aligned} Bx^{-1} + Cx^0 &= B e^{i \ln x} + C e^{i \ln x} = B e^{i \ln x} + C e^{-i \ln x} \\ &= B [\cos(\ln x) + i \sin(\ln x)] + [C \cos(\ln x) - i \sin(\ln x)] \\ &= (B+C) \cos(\ln x) + i(C-B) \sin(\ln x) \\ &= D \cos(\ln x) + E \sin(\ln x) \end{aligned}$$

• FOR PARTICULAR INTEGRAL BY INSPECTION, LET TRY  $y = Px \ln x$ .

$$\begin{aligned} y &= Px \ln x \\ \frac{dy}{dx} &= P + P \ln x \\ \frac{d^2y}{dx^2} &= \frac{P}{x} \\ \frac{d^3y}{dx^3} &= -\frac{P}{x^2} \end{aligned}$$

• SUB INTO THE O.D.E GIVES

$$\begin{aligned} &\Rightarrow -\frac{P}{x^2} + 2Px + Px \ln x - Px \ln x = 2x \\ &\therefore P = 1 \end{aligned}$$

• HENCE THE GENERAL SOLUTION IS

$$y = Ax + B \cos(\ln x) + C \sin(\ln x) + x \ln x$$

**Question 11 (\*\*\*\*)**

Use variation of parameters to determine the specific solution of the following differential equation

$$x^2 \frac{d^2y}{dx^2} - 7x \frac{dy}{dx} + 16y = 16\ln x,$$

given further that  $y = \frac{1}{2}$ ,  $\frac{dy}{dx} = 2$  at  $x = 1$ .

$$y = \frac{1}{2} + (1+x^4)\ln x$$

**ANSWER:**  $y = \frac{1}{2} + (1+x^4)\ln x$

# **2<sup>ND</sup> ORDER ODES WITH MISSING INDEPENDENT VARIABLE**

**Question 1** (\*\*\*\*+)

The curve  $C$ , has gradient  $\frac{2}{9}$  at the point with coordinates  $(\ln 2, \frac{2}{3})$ , and satisfies the differential relationship

$$\frac{d^2y}{dx^2} = (1-2y) \frac{dy}{dx}, \quad y < \frac{1}{2}.$$

Find an equation for  $C$ , giving the answer in the form  $y = f(x)$ .

$$y = \frac{e^x}{1+e^x} = \frac{1}{e^x+e^{-x}} = \frac{1}{2} \operatorname{sech} x$$

**Method 1:**

$\frac{d^2y}{dx^2} = (1-2y) \frac{dy}{dx}$

AT THE INTEGRATION VARIABLE IS MISSING, WE TRY  $P = \frac{dy}{dx}$

THIS DIFFERENTIATING WITH RESPECT TO  $y$

$$\frac{dp}{dy} = \frac{d}{dx} \frac{dy}{dx} = \frac{d^2y}{dx^2} \times \frac{1}{P} \Rightarrow \frac{dp}{dy} = \frac{1}{P} \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{\partial P}{\partial x} = P \frac{\partial^2 y}{\partial x^2}$$

$$\Rightarrow P \frac{\partial^2 y}{\partial x^2} = (1-2y)P$$

BY PARTIAL FRACTION

$$\Rightarrow \frac{dp}{dy} = 1-2y$$

$$\Rightarrow \int dp = \int (1-2y) dy$$

$$\Rightarrow \left| \ln|P| - \ln|y| \right| = x + C$$

$$\Rightarrow P = y - y^2 + A$$

$$\Rightarrow \left| \ln \frac{P}{y} \right| = x + C$$

$$\Rightarrow \frac{dy}{dx} = y - y^2 + A$$

•  $\frac{dy}{dx} = \frac{2}{3}$  AT  $(\ln 2, \frac{2}{3})$

$$\frac{2}{3} = \frac{2}{3} - \frac{4}{9} + A$$

$$\therefore [A=0]$$

$$\Rightarrow \frac{dy}{dx} = y - y^2$$

$$\Rightarrow \frac{1}{y-y^2} dy = 1 dx$$

$$\Rightarrow \int \frac{1}{y(y-1)} dy = \int 1 dx$$

$$\Rightarrow \frac{1}{y-1} - \frac{1}{y} dy = \int 1 dx$$

$$\Rightarrow \left[ \ln \left| \frac{1}{y-1} \right| \right]_2^{\frac{2}{3}} = \left[ x \right]_{\ln 2}^2$$

$$\Rightarrow \ln \left| \frac{1}{y-1} \right| \Big|_{y=2}^{y=\frac{2}{3}} = 2 - \ln 2$$

$$\Rightarrow -\frac{1}{y-1} = e^{2-\ln 2} \dots \text{ WHICH CAN BE REARRANGED TO } y = \frac{e^2}{1+e^2}$$

**ALTERNATIVE METHOD**

$\frac{d^2y}{dx^2} = (1-2y) \frac{dy}{dx}$

INTEGRATE BOTH SIDES WITH RESPECT TO  $x$ , SUBJECT TO  $y = \frac{2}{3}$ ,  $\frac{dy}{dx} = \frac{2}{3}$

$$\Rightarrow \int \frac{d^2y}{dx^2} dx = \int (1-2y) \frac{dy}{dx} dx$$

$$\Rightarrow \frac{dy}{dx} = \left[ y - y^2 \right]_{\frac{2}{3}}^{\frac{2}{3}}$$

$$\Rightarrow \frac{dy}{dx} - \frac{2}{3} = \left( \frac{2}{3} - \frac{4}{9} \right) - \left( \frac{2}{3} - \frac{4}{9} \right)$$

$$\Rightarrow \frac{dy}{dx} = y - y^2$$

(SEPARATE VARIABLES)

$$\Rightarrow \frac{1}{y-y^2} dy = 1 dx$$

$$\Rightarrow \int \frac{1}{y(y-1)} dy = \int 1 dx$$

$$\Rightarrow \frac{1}{y-1} - \frac{1}{y} dy = \int 1 dx$$

$$\Rightarrow \left[ \ln \left| \frac{1}{y-1} \right| \right]_2^{\frac{2}{3}} = \left[ x \right]_{\ln 2}^2$$

$$\Rightarrow \ln \left| \frac{1}{y-1} \right| \Big|_{y=2}^{y=\frac{2}{3}} = 2 - \ln 2$$

$$\Rightarrow -\frac{1}{y-1} = e^{2-\ln 2} \dots \text{ WHICH CAN BE REARRANGED TO } y = \frac{e^2}{1+e^2}$$

**Question 2** (\*\*\*\*)

Use appropriate techniques to solve the following differential equation.

$$\frac{d^2y}{dx^2} = -\frac{144}{y^3}, \quad y(0) = 6, \quad \left.\frac{dy}{dx}\right|_{x=0} = 0.$$

,  $\frac{x^2}{9} + \frac{y^2}{36} = 1$

$\frac{\frac{dy}{dx}}{dx} = -\frac{144}{y^3}$  WE USE THAT

$\Rightarrow \frac{dy}{dx} = -\frac{144}{y^3}$

$\Rightarrow \frac{d}{dx}(\frac{dy}{dx}) = -\frac{144}{y^3}$

$\Rightarrow \frac{dp}{dx} = -\frac{144}{y^3}$

$\Rightarrow \frac{dp}{dy} \cdot \frac{dy}{dx} = -\frac{144}{y^3}$

$\Rightarrow \frac{dp}{dy} \cdot p = -\frac{144}{y^3}$

$\Rightarrow p dp = -\frac{144}{y^3} dy$

INTEGRATING BOTH SIDES WE GET

$\Rightarrow \frac{1}{2}p^2 = -\frac{144}{y^2} + A$

$\Rightarrow p^2 = \frac{-144}{y^2} + B$

$\therefore p^2 = \frac{-144}{y^2} - 4$

$\boxed{2x, t=0 \Rightarrow 0 = 4 + 8 \Rightarrow B = -4}$

$\Rightarrow p^2 = \frac{144 - 4y^2}{y^2}$

$\Rightarrow p = \pm \frac{\sqrt{144 - 4y^2}}{y}$

$\Rightarrow \frac{dy}{dx} = \pm \frac{\sqrt{144 - 4y^2}}{y}$

METHOD OF FINDING

$\Rightarrow \frac{dy}{dx} = \pm \frac{1}{\sqrt{144 - 4y^2}} = \pm y(144 - 4y^2)^{\frac{1}{2}}$

SEPARATE VARIABLES AND INTEGRATE BY INSPECTION

$\Rightarrow \int 1 dx = \int \pm y(144 - 4y^2)^{\frac{1}{2}} dy$

$\Rightarrow x = \pm \frac{1}{4}(144 - 4y^2)^{\frac{1}{2}} + C$

$\boxed{4PQ 2x=0, y=6}$

$\boxed{0=4+8}$

$\boxed{C=0}$

FINALLY WE HAVE

$\Rightarrow x^2 = \left[ \pm \frac{1}{4}(144 - 4y^2)^{\frac{1}{2}} \right]^2$

$\Rightarrow x^2 = \frac{1}{16}(144 - 4y^2)$

$\Rightarrow 16x^2 = 144 - 4y^2$

$\Rightarrow 16x^2 + 4y^2 = 144 \quad \text{OR} \quad \frac{x^2}{9} + \frac{y^2}{36} = 1$

**Question 3** (\*\*\*)+

The curve  $C$ , has a stationary point at  $(0,2)$  and satisfies the differential relationship

$$\frac{d^2y}{dx^2} = \frac{4}{y^3}, \quad y \neq 0.$$

- a) Given further that  $\frac{dy}{dx} \geq 0$  along  $C$ , determine a simplified expression for the Cartesian equation of  $C$ .
- b) Verify by differentiation the answer to part (a).

$$y^2 - x^2 = 4$$

**a)**

$\frac{d^2y}{dx^2} = \frac{4}{y^3}$  STATIONARY POINT AT  $(0,2)$ ,  $\frac{dy}{dx} > 0$

Let  $P = \frac{dy}{dx}$  (Solve the homogeneity initial & reduce)

$$\frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \cdot \frac{dy}{dx} = \frac{d^2y}{dx^2} \cdot P$$

$$\frac{d^2P}{dx^2} = P \frac{dp}{dy}$$

$$\Rightarrow P \frac{dp}{dy} = \frac{4}{y^3}$$

$$\Rightarrow \int P dp = \int \frac{4}{y^3} dy$$

$$\Rightarrow \frac{1}{2}P^2 = -\frac{4}{2y^2} + C$$

$$\Rightarrow P^2 = C - \frac{8}{y^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = C - \frac{8}{y^2}$$

APPLY CONDITION,  $y=2$ ,  $\frac{dy}{dx} > 0$

$$0 = C - \frac{4}{2^2}$$

$$0 = C - \frac{4}{4}$$

$$0 = C - 1$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 1 - \frac{4}{y^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{y^2 - 4}{y^2}$$

$$\Rightarrow \frac{dy}{dx} = \pm \sqrt{\frac{y^2 - 4}{y^2}}$$

$$\Rightarrow \frac{dy}{dx} = \pm \frac{\sqrt{y^2 - 4}}{y}$$

$$\Rightarrow \frac{dy}{\sqrt{y^2 - 4}} = \pm dx$$

**b)**

$y^2 - x^2 = 4$  Differentiate w.r.t.  $x$

$$\Rightarrow 2y \frac{dy}{dx} = 2x$$

$$\Rightarrow y \frac{dy}{dx} = x$$

Differentiate w.r.t.  $x$  again

$$\Rightarrow \frac{d}{dx}\left(y \frac{dy}{dx}\right) + y \frac{d^2y}{dx^2} = 1$$

$$\Rightarrow \frac{dy^2}{dx^2} + y \frac{d^2y}{dx^2} = 1$$

$$\Rightarrow \frac{3y^2}{4} = \frac{3x^2}{4}$$

$$\Rightarrow \frac{3y^2}{4} + y \frac{d^2y}{dx^2} = 1$$

$$\Rightarrow \frac{3x^2}{4} + y \frac{d^2y}{dx^2} = 1$$

$$\Rightarrow x^2 - \frac{4}{3} + y \frac{d^2y}{dx^2} = 1$$

$$\Rightarrow y \frac{d^2y}{dx^2} = \frac{4}{3} - x^2$$

$$\Rightarrow \frac{dy^2}{dx^2} = \frac{4}{3y^2} - x^2$$

As required

**Question 4** (\*\*\*)+

The curve  $C$ , has a stationary point at  $(0,4)$  and satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{2}{y^2}, \quad y \neq 0.$$

- a) Given further that  $\frac{dy}{dx} \geq 0$  along  $C$ , determine a simplified expression for the Cartesian equation of  $C$ , giving the answer in the form  $x = f(y)$ .
- b) Verify by differentiation the answer to part (a).

$$x = 4 \operatorname{arcosh}\left(\frac{1}{2}\sqrt{y}\right) + \sqrt{y^2 - 4y}$$

a)

Let  $P = \frac{dy}{dx}$  (SINCE THE INDEPENDENT VARIABLE IS  $x$ )

$$\frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dy^2} \cdot \frac{dx}{dy} = \frac{d^2y}{dy^2} \times P$$

$$\frac{d^2y}{dy^2} = P \frac{dp}{dy}$$

Then

$$\Rightarrow P \frac{dp}{dy} = \frac{2}{y^2}$$

$$\Rightarrow \int P dp = \int \frac{2}{y^2} dy$$

$$\Rightarrow \frac{1}{2}P^2 = -\frac{2}{y} + C$$

$$\Rightarrow P^2 = C - \frac{4}{y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = C - \frac{4}{y}$$

APPLY CONDITIONS

$$y=4 \Rightarrow \frac{dy}{dx}=0 \Rightarrow 0=C-\frac{4}{4} \Rightarrow C=1$$

$$\Rightarrow \frac{dy}{dx} = \pm \sqrt{C-\frac{4}{y}}$$

$$\Rightarrow \frac{dy}{dx} = \pm \sqrt{\frac{y-4}{y}}$$

$$\Rightarrow \frac{dy}{\sqrt{y-4}} = \pm \frac{dx}{\sqrt{y}}$$

$$\Rightarrow \int \frac{dy}{\sqrt{y-4}} = \pm \int \frac{dx}{\sqrt{y}}$$

$$\Rightarrow \int \frac{dy}{\sqrt{y-4}} = \pm \int \frac{dx}{\sqrt{y}}$$

$$\Rightarrow \int \frac{dy}{\sqrt{y-4}} = \pm \int dx$$

$$\Rightarrow \int \frac{dy}{\sqrt{y-4}} = \pm x + B$$

... SUBSTITUTE ...

b)

$x = 4 \operatorname{arcosh}\left(\frac{1}{2}\sqrt{y}\right) + \sqrt{y^2 - 4y}$

$$\frac{dx}{dy} = 4 \operatorname{arcosh}\left(\frac{1}{2}\sqrt{y}\right)' + (\sqrt{y^2 - 4y})^{\frac{1}{2}}$$

$$\frac{dx}{dy} = \frac{1}{\sqrt{y-4}} \times \frac{1}{2}\sqrt{y} \times y^{\frac{1}{2}} + \frac{1}{2}(y^2 - 4y)^{\frac{1}{2}}(2y - 4)$$

$$\frac{dx}{dy} = \frac{1}{\sqrt{y-4}} + \frac{y-2}{\sqrt{y^2 - 4y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{y-4}} + \frac{y-2}{\sqrt{y^2 - 4y}} = \frac{-2}{\sqrt{y-4}} + \frac{y-2}{\sqrt{y^2 - 4y}} = \frac{y-2}{\sqrt{y-4}}$$

$$\frac{dy}{dx} \times \frac{\sqrt{y-4}}{y} = \frac{y-2}{\sqrt{y^2 - 4y}} \times \frac{\sqrt{y-4}}{y} = \frac{y-2}{\sqrt{y^2 - 4y}} \times \frac{\sqrt{y-4}}{y} = \frac{y-2}{\sqrt{y^2 - 4y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{y}} \left(1 - \frac{4}{y}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{y}} \left(1 - \frac{4}{y}\right)^{\frac{1}{2}} \times \frac{y}{y} \times \frac{dy}{dy} = \frac{y}{\sqrt{y}} \left(1 - \frac{4}{y}\right)^{\frac{1}{2}} \times \frac{dy}{dy}$$

$$\frac{dy}{dx} = \frac{y}{\sqrt{y}} \left(1 - \frac{4}{y}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{2}{y^{\frac{1}{2}}} \quad \text{As required}$$

**Question 5** (\*\*\*)+

The curve  $C$  with Cartesian equation  $f(x, y) = 0$ , satisfies the differential equation

$$(1-y)y'' = (2-y)(y')^2.$$

It is further given that  $y(0) = 0$  and  $y'(0) = 1$

- Determine a simplified expression for the Cartesian equation of  $C$ .
- Verify by differentiation the answer to part (a).

$$x = y e^{-y}$$

a)

(1-y)  $\frac{dy}{dx}$   $\frac{d^2y}{dx^2} = (2-y) \left( \frac{dy}{dx} \right)^2$  ,  $x=0, y=0, \frac{dy}{dx}=1$

SINCE THE INDEPENDENT VARIABLE  $x$  IS MISSING, WE USE THE STANDARD SUBSTITUTION

$\frac{dy}{dx} = p$

DIFF WRT  $y$

$$\Rightarrow \frac{d}{dy} \left( \frac{dy}{dx} \right) = \frac{dp}{dy}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dy}$$

$$\Rightarrow \frac{d^2y}{dx^2} \times \frac{1}{\frac{dp}{dy}} = \frac{dp}{dy}$$

$$\Rightarrow \frac{dp}{dy} = p \frac{dp}{dy}$$

• (1-y)  $p \frac{dp}{dy} = (2-y) p^2$

$$\Rightarrow (1-y) \frac{dp}{dy} = (2-y) p$$

$$\Rightarrow \frac{1}{p} dp = \frac{2-y}{1-y} dy$$

$$\Rightarrow \frac{1}{p} dp = \frac{1-(y-2)}{1-y} dy$$

$$\Rightarrow \int \frac{1}{p} dp = \int \frac{1}{1-y} dy$$

$$\Rightarrow \ln|p| = -\ln|1-y| + y + C$$

$$\Rightarrow p = e^{y-\ln|1-y|+C}$$

$$\Rightarrow p = \frac{Ae^y}{1-y} (A \neq 0)$$

... APPLY CONDITION:  $y=0, p=\frac{dy}{dx}$

1 =  $\frac{A}{1} \approx [A=1]$

THAT  $p = \frac{e^y}{1-y}$

$$\Rightarrow \frac{dy}{dx} = \frac{e^y}{1-y}$$

$$\Rightarrow \int (1-y)e^y dy = \int 1 dx$$

↑ BY PARTS

$\begin{aligned} & \text{Let } u = 1-y, \quad v = e^y \\ & \Rightarrow -u \frac{du}{dy} + v \frac{dv}{dy} = 1 \\ & \Rightarrow -(1-y)e^y - \int e^y dy = x + D \\ & \Rightarrow -e^y(y-1) - e^{y^2} = x + D \\ & \Rightarrow y e^y = x + D \end{aligned}$

APPLY CONDITION  
 $x=0, y=0 \Rightarrow D=0$

b)

$x = y e^{-y}$  OR  $xe^y = y$

DIFF WRT  $y$

$$\frac{dx}{dy} = xe^{-y} + y(-e^{-y})$$

$$\frac{dx}{dy} = e^{-y} - ye^{-y}$$

$$\frac{dx}{dy} = e^{-y}(1-y)$$

$$\frac{dy}{dx} = \frac{e^y}{1-y}$$

(1-y)  $\frac{dy}{dx}$   $\frac{d^2y}{dx^2} = e^y$

DIFF WRT  $x$

$$-\frac{dy}{dx} \frac{d^2y}{dx^2} + (1-y) \frac{d^2y}{dx^2} = \frac{e^y}{1-y} \frac{d^2y}{dx^2}$$

$$-\left(\frac{dy}{dx}\right)^2 + (1-y) \frac{d^2y}{dx^2} = \frac{e^y}{1-y} \frac{d^2y}{dx^2}$$

$$(1-y) \frac{d^2y}{dx^2} = (1-y) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2$$

$$(1-y) \frac{d^2y}{dx^2} = (2-y) \left(\frac{dy}{dx}\right)^2$$

AS REQUIRED

**Question 6 (\*\*\*\*\*)**

The function with equation  $y = f(x)$  satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{y} \left( \frac{dy}{dx} \right)^2 = 2y \ln 3, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 2\ln 3.$$

Solve the above differential equation to show that  $y = 3^{x^2+2x}$ .

, proof

USING THE SUBSTITUTION  $P = \frac{dy}{dx}$ , AS THE INDEPENDENT VARIABLE IS MISSING

$$\begin{aligned} P &= \frac{dy}{dx} \rightarrow \frac{dp}{dy} = \frac{1}{y} \left( \frac{dy}{dx} \right) = \frac{1}{y} \times \frac{dp}{dx} \\ &\Rightarrow \frac{dp}{dy} = \frac{1}{y} \times \frac{dp}{dx} \\ &\Rightarrow \boxed{\frac{dp}{dx} = P \frac{dp}{dy}} \end{aligned}$$

TRANSFORMING THE O.D.E.

$$\begin{aligned} &\rightarrow \frac{dp}{dx} - \frac{1}{y} \left( \frac{dy}{dx} \right)^2 = 2y \ln 3 \\ &\rightarrow P \frac{dp}{dy} - \frac{1}{y} P^2 = 2y \ln 3 \\ &\rightarrow \frac{dp}{dy} - \frac{P}{y} = \frac{2y \ln 3}{P} \end{aligned}$$

USING ANOTHER SUBSTITUTION  $V = \frac{P}{y}$

$$\begin{aligned} P &= VY \\ \frac{dp}{dy} &= \frac{dV}{dy}y + V \\ \text{TRANSFORMING THE O.D.E. FURTHER} & \end{aligned}$$

$$\begin{aligned} &\Rightarrow \left( \frac{dV}{dy}y + V \right) - V = \frac{2y \ln 3}{P} \\ &\Rightarrow \frac{dV}{dy}y = \frac{2y \ln 3}{V} \\ &\Rightarrow \int V \, dV = (2\ln 3) \int \frac{1}{y} \, dy \\ &\Rightarrow \frac{1}{2}V^2 = (2\ln 3) \ln y + C \end{aligned}$$

TRYING OUT THE UNIT TRANSFORMATION  $y = e^x$

$$\begin{aligned} &\Rightarrow y^2 = (4\ln 3)(\ln y) + C \\ &\Rightarrow (\ln y)^2 = (4\ln 3)(\ln y) + C \\ &\Rightarrow y^2 = (4\ln 3)(x^2 \ln x) + Cx^2 \\ &\Rightarrow x=0, \quad y=1, \quad \frac{dy}{dx} = P = 2\ln 3 \\ &\Rightarrow (2\ln 3) = (4\ln 3)x^2 \ln x + Cx^2 \\ &\Rightarrow C = 4(x_0)^2 \end{aligned}$$

SEPARATE VARIABLES

$$\begin{aligned} &\Rightarrow P^2 = (4\ln 3)^2 \ln y + 4x^2(\ln 3)^2 \\ &\Rightarrow P^2 = 4(\ln 3)^2 [\ln y + b^2] \\ &\Rightarrow P = \frac{dy}{dx} = \sqrt{4(\ln 3)^2 [\ln y + b^2]} \\ &\Rightarrow \frac{dy}{dx} = 2y \sqrt{\ln y + b^2} \\ &\Rightarrow \frac{dy}{dx} = 2y \sqrt{(\ln y)^2 + C(\ln y)^2} \end{aligned}$$

BY SUBSTITUTION OR LOGARITHMIC DIFFERENTIATION

$$\begin{aligned} &\Rightarrow \int \frac{1}{2(\ln y)^2} \, dy = \int 2(\ln y)^2 \, dx \\ &\Rightarrow \int \frac{1}{2} (\ln y)^{-2} \, dy = 2x + A \end{aligned}$$

FINDING OUT THE C.B.M.

$$\begin{aligned} &\Rightarrow 2(\ln y)^{-2} = 2(b^2)^{\frac{1}{2}} + A \\ &\Rightarrow \sqrt{\ln y}^{-1} = 2\sqrt{b^2} + B \\ &\text{TRY CONDITION } x=0, \quad y=1 \\ &\Rightarrow \sqrt{b^2} = 0\sqrt{b^2} + B \\ &\Rightarrow B = b^2 \\ &\Rightarrow \sqrt{\ln y}^{-1} = 2\sqrt{b^2} + b^2 \\ &\Rightarrow \sqrt{\ln y}^{-1} = (2+1)\sqrt{b^2} \\ &\Rightarrow \ln y = (C+1)^2 b^2 \\ &\Rightarrow y = e^{(C+1)^2 b^2} \\ &\Rightarrow y = (e^{b^2})^{(C+1)^2} \\ &\Rightarrow y = 3^{(C+1)^2} \\ &\Rightarrow y = 3^{2+2x+1} \\ &\Rightarrow y = \frac{3^{2+2x+1}}{3} \\ &\Rightarrow y = 3^{2+2x} \end{aligned}$$

// As  $b^2 = 0$

**Question 7 (\*\*\*\*\*)**

The curve with equation  $y = f(x)$  satisfies the differential equation

$$\frac{d^2y}{dx^2} = 6y^2 + 4y, \quad \frac{dy}{dx} \geq 0.$$

If  $y = 3$ ,  $\frac{dy}{dx} = 12$  at  $x = -\frac{1}{2}\ln 3$ , solve the differential equation to show that

$$y = \operatorname{cosech}^2 x.$$

[proof]

$\frac{d^2y}{dx^2} = 6y^2 + 4y$  SUBJECT TO  $x = -\frac{1}{2}\ln 3, y=3, \frac{dy}{dx}=12$

SINCE THE INDEPENDENT VARIABLE IS MISSING WE ASK  $P = \frac{dy}{dx}$

so  $\frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$

Thus  $\frac{dp}{dy} = P \frac{dP}{dy}$

• TAKE O.D.E. TRANSFORMS TO

$$\Rightarrow P \frac{dP}{dy} = 6y^2 + 4y$$

$$\Rightarrow P dP = (6y^2 + 4y) dy$$

$$\Rightarrow \int P dP = \int 6y^2 + 4y dy$$

$$\Rightarrow \frac{1}{2}P^2 = 2y^3 + 4y^2 + C$$

$$\Rightarrow P^2 = 4y^3 + 4y^2 + C$$

• APPLY INITIAL COND.  $y=3, P = \frac{dy}{dx} = 12$

$$144 = 4y^3 + 4y^2 + C$$

$$144 = 108 + 36 + C$$

$$C = 0$$

$$\Rightarrow P^2 = 4y^3 + 4y^2$$

$$\Rightarrow P^2 = 4y^2(y+1)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4y^2(y+1)$$

$$\Rightarrow \frac{dy}{dx} = 2y(y+1)^{\frac{1}{2}} > 0$$

• SEPARATE VARIABLES

$$\Rightarrow \int \frac{1}{(u^2-1)} du = \int 2 dx$$

$$\Rightarrow \int \frac{1}{(u^2-1)} du = \int 2 dx$$

$$\Rightarrow \int \frac{2}{u^2-1} du = \int 2 dx$$

$$\Rightarrow \int \frac{2}{(u-1)(u+1)} du = \int 2 dx$$

BY SUBSTITUTION

$$u = \operatorname{cosech}^{-1} y$$

$$u^2 - 1 = y^2 - 1$$

$$2u du = dy$$

$$y = u^2 - 1$$

• BY PARTIAL FRACTIONS

$$\Rightarrow \int \frac{1}{u-1} - \frac{1}{u+1} du = \int 2 dx$$

$$\Rightarrow \ln|u-1| - \ln|u+1| = 2x + k$$

$$\Rightarrow \ln \left| \frac{u-1}{u+1} \right| = 2x + k$$

$$\Rightarrow \frac{u-1}{u+1} = e^{2x+k}$$

$$\Rightarrow \frac{u-1}{u+1} = Ae^{2x} \quad (A=e^k)$$

$$\Rightarrow u-1 = Ae^{2x} + Ae^{2x}$$

$$\Rightarrow u-Ae^{2x} = Ae^{2x}$$

$$\Rightarrow u(1-Ae^{-2x}) = Ae^{2x}$$

$$\Rightarrow u = \frac{1+Ae^{2x}}{1-Ae^{-2x}}$$

$$\Rightarrow (u+1)^{\frac{1}{2}} = \frac{1+Ae^{2x}}{1-Ae^{-2x}}$$

• APPLY THE INIT. CONDITION

$$\begin{cases} y=3, x = -\frac{1}{2}\ln 3 \quad (\text{i.e. } e^{2x} = \frac{1}{3}) \\ \sqrt{3+1} = \frac{1+\frac{1}{3}A}{1-\frac{1}{3}A} \end{cases}$$

$$2 = \frac{3+A}{3-A}$$

$$6-2A = 3+A$$

$$3 = 3A$$

$$A = 1$$

HENCE

$$\Rightarrow \sqrt{y+1} = \frac{1+e^{2x}}{1-e^{2x}}$$

$$\Rightarrow \sqrt{y+1} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\Rightarrow \sqrt{y+1} = \frac{e^x + e^{-x}}{e^{2x} - e^{-2x}}$$

$$\Rightarrow \sqrt{y+1} = \operatorname{coth} x$$

$$\Rightarrow y+1 = \operatorname{coth}^2 x$$

$$\Rightarrow y = \operatorname{coth}^2 x - 1$$

$$\Rightarrow y = \operatorname{cosech}^2 x$$

which can easily  
be checked against the  
original o.d.e.

**Question 8** (\*\*\*\*\*)

The curve with equation  $y = f(x)$  satisfies the differential equation

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 8y.$$

Given further that the curve has a stationary point at  $(\frac{1}{2}, \frac{1}{4})$ , solve the differential equation to show that

$$y = x^2 + x + \frac{1}{2}.$$

proof

$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 8y$  SUBJECT TO  $y = \frac{1}{4}$  &  $\frac{dy}{dx} = 0$ , AT  $x = \frac{1}{2}$

• Since the independent term is missing (why), let  $p = \frac{dy}{dx}$ .  
 Then  $\frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} \cdot \frac{dt}{dy} = \frac{d^2y}{dt^2} \times \frac{1}{p}$   
 $\therefore \frac{dp}{dy} = \frac{d^2y}{dt^2} \quad \text{and} \quad \frac{dp}{dy} = p \frac{dt}{dy}$

• Now  
 $\Rightarrow p \frac{dp}{dy} + 2p^2 = 8y$   
 $\Rightarrow \frac{dp}{dy} + 2p = \frac{8y}{p}$   
 $\Rightarrow \frac{dp}{dy} + 2p = 8yp^{-1}$  ← STANDARD SEPARABLE TYPE  
 Multiply throughout by  $2p$   
 $\Rightarrow 2p \frac{dp}{dy} + 4p^2 = 16y$   
 $\Rightarrow \frac{d}{dy}(2p^2) + 4y = 16y$   
 $\Rightarrow \frac{d}{dy}(2p^2) + 4y = 16y$

• Now  
 Integrating factor is  
 $e^{\int 4 dy} = e^{4y}$   
 $\Rightarrow \frac{d}{dy}(ze^{4y}) = 16y$   
 $\Rightarrow ze^{4y} = \int 16y dy$  (By parts)

$\Rightarrow 2e^{4y} = \int 16y e^{4y} dy$

$\Rightarrow 2e^{4y} = 4ye^{4y} - \int 4e^{4y} dy$   
 $\Rightarrow 2e^{4y} = 4ye^{4y} - 4e^{4y} + A$   
 $\Rightarrow 2 = (2y-1) + Ae^{-4y}$   
 $\Rightarrow p^2 = 4y-1 + Ae^{-4y}$   
 $\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4y-1 + Ae^{-4y}$

NOW STATIONARY AT  $(\frac{1}{2}, \frac{1}{4}) \Rightarrow 0 = 4 \times \frac{1}{4} - 1 + Ae^{-4 \times \frac{1}{4}}$   
 $0 = 1 - 1 + Ae^{-\frac{1}{4}}$   
 $Ae^{-\frac{1}{4}} = 0$   
 $A = 0$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4y-1$   
 $\Rightarrow \frac{dy}{dx} = \pm (4y-1)^{\frac{1}{2}}$   
 $\Rightarrow \frac{1}{(4y-1)^{\frac{1}{2}}} dy = 1 dx$   
 $\Rightarrow \int \frac{1}{(4y-1)^{\frac{1}{2}}} dy = \int 1 dx$   
 $\Rightarrow \frac{1}{2}(4y-1)^{\frac{1}{2}} = x + C$   
 Now  $(\frac{1}{2}, \frac{1}{4}) \Rightarrow \frac{1}{2} \times 0 = \frac{1}{2} + C$   
 $\Rightarrow 0 = \frac{1}{2} + C$   
 $\Rightarrow C = -\frac{1}{2}$

$\Rightarrow \frac{1}{2}(4y-1)^{\frac{1}{2}} = x + \frac{1}{2}$   
 $\Rightarrow (4y-1)^{\frac{1}{2}} = 2x+1$  (square)  
 $\Rightarrow 4y-1 = 4x^2+4x+1$

$\therefore 4y = 4x^2+4x+2$   
 $\therefore y = x^2+x+\frac{1}{2}$

**Question 9** (\*\*\*\*\*)

The curve with equation  $y = f(x)$  satisfies the differential equation

$$\frac{d^2y}{dx^2} + e^{-y} = 0, \quad \frac{dy}{dx} \geq 0.$$

If  $y = 0$ ,  $\frac{dy}{dx} = -1$  at  $x = \frac{1}{2}\pi$ , solve the differential equation to show that

$$y = \ln(1 - \cos x).$$

[ ] , [proof]

AS THE INDEPENDENT VARIABLE ( $x$ ) IS MISSING, USE THE STANDARD SUBSTITUTION  $p = \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dp} \left( \frac{dp}{dx} \right) = \frac{dy}{dp} \frac{dp}{dx} = \frac{dy}{dp} \times \frac{1}{p}$$

$$\Rightarrow \frac{dy}{dp} = p \frac{dy}{dx}$$

SWITCH INTO THE STANDARD MECHANICS NOTATION FOR ACCELERATION  $\ddot{x} = \frac{d^2x}{dt^2} = V \frac{dv}{dx}$

TRANSFORMING THE O.D.E

$$\Rightarrow \frac{d^2y}{dx^2} + e^{-y} = 0 \quad [x = \bar{x}, y = v, \frac{dy}{dx} = p = v = \dot{x}]$$

$$\Rightarrow p \frac{dp}{dx} = -e^{-y}$$

$$\Rightarrow \int p dp = \int_{\infty}^0 -e^{-y} dy$$

$$\Rightarrow \left[ \frac{1}{2}p^2 \right]_0^{\infty} = \left[ -e^{-y} \right]_0^{\infty}$$

$$\Rightarrow \frac{1}{2}\dot{x}^2 - \frac{1}{2} = -e^{-y} - 1$$

$$\Rightarrow \dot{x}^2 - 1 = -2e^{-y} - 2$$

$$\Rightarrow \dot{x}^2 = \frac{2}{e^y} - 1$$

$$\Rightarrow \left( \frac{dy}{dx} \right)^2 = \frac{2 - e^y}{e^y}$$

$\rightarrow \frac{dy}{dx} = \pm \sqrt{\frac{2 - e^y}{e^y}}$

SWAPPING VARIABLES AGAIN

$$\rightarrow \frac{e^{\frac{dy}{dx}}}{\sqrt{2 - e^y}} dy = 1 dx$$

$$\rightarrow \int_{\frac{\pi}{2}}^x 1 dx = \int_0^3 \frac{e^{\frac{dy}{dx}}}{\sqrt{2 - e^y}} dy$$

USING A TRIGONOMETRIC SUBSTITUTION ON THE INTERVAL IN THE P.D.E

$$\begin{aligned} & \dot{x} = 2\sin\theta \quad \left[ e^{\frac{dy}{dx}} = \sqrt{2 - \sin^2\theta} \quad \theta = \arcsin\left(\frac{e^{\frac{dy}{dx}}}{\sqrt{2}}\right) \right] \\ & \rightarrow e^{\frac{dy}{dx}} dy = 2\sin\theta \cos\theta d\theta \\ & \rightarrow dy = \frac{2\sin\theta \cos\theta}{e^{\frac{dy}{dx}}} d\theta = \frac{2\sin\theta \cos\theta}{2\sin\theta} d\theta = \frac{\cos\theta}{\sin\theta} d\theta = \frac{1}{\tan\theta} d\theta \end{aligned}$$

UNITS TRANSFORM TO

$$\begin{aligned} & y = \int \theta d\theta \\ & y = \theta - \frac{\pi}{2} \end{aligned}$$

RETURNING TO THE O.D.E

$$\begin{aligned} & \Rightarrow \left[ \frac{dy}{dx} \right]_{\frac{\pi}{2}}^x = \int_{\frac{\pi}{2}}^x \frac{\arcsin\left(\frac{e^{\frac{dy}{dx}}}{\sqrt{2}}\right)}{\sqrt{2 - 2\sin^2\theta}} \times \frac{2\cos\theta}{\sin\theta} d\theta \\ & \rightarrow x - \frac{\pi}{2} = \int_{\frac{\pi}{2}}^x \frac{\arcsin\left(\frac{e^{\frac{dy}{dx}}}{\sqrt{2}}\right)}{\sqrt{2 - 2\sin^2\theta}} \times \frac{2\cos\theta}{\sin\theta} d\theta \\ & \rightarrow x - \frac{\pi}{2} = \int_{\frac{\pi}{2}}^x \frac{\arcsin\left(\frac{e^{\frac{dy}{dx}}}{\sqrt{2}}\right)}{2} d\theta \end{aligned}$$

$$\begin{aligned} & x - \frac{\pi}{2} = 2 \left[ \theta - \frac{\pi}{2} \right] \arcsin\left(\frac{e^{\frac{dy}{dx}}}{\sqrt{2}}\right) \\ & \Rightarrow x - \frac{\pi}{2} = 2 \left[ \arcsin\left(\frac{e^{\frac{dy}{dx}}}{\sqrt{2}}\right) - \frac{\pi}{2} \right] \\ & \Rightarrow x = 2\arcsin\left(\frac{e^{\frac{dy}{dx}}}{\sqrt{2}}\right) + \frac{\pi}{2} \\ & \Rightarrow \frac{\pi}{2} = \arcsin\left(\frac{e^{\frac{dy}{dx}}}{\sqrt{2}}\right) \\ & \Rightarrow \sin\frac{\pi}{2} = \frac{e^{\frac{dy}{dx}}}{\sqrt{2}} \\ & \Rightarrow \sin\frac{\pi}{2} = \frac{e^{\frac{dy}{dx}}}{\sqrt{2}} \\ & \Rightarrow 2\sin\frac{\pi}{2} = e^{\frac{dy}{dx}} \\ & \Rightarrow 1 - 2\sin^2\frac{\pi}{2} = 1 - e^{\frac{dy}{dx}} \\ & \Rightarrow \cos\frac{\pi}{2} = 1 - e^{\frac{dy}{dx}} \\ & \Rightarrow \frac{\pi}{2} = 1 - e^{\frac{dy}{dx}} \\ & \Rightarrow y = \ln(1 - \cos\frac{\pi}{2}) \end{aligned}$$

**Question 10** (\*\*\*\*\*)

The curve  $C$ , has gradient 1 at the origin and satisfies the differential relationship

$$\frac{d^2y}{dx^2} \sqrt{1-2y} = \frac{dy}{dx} (3y-2), \quad y < \frac{1}{2}.$$

Find an equation for  $C$ , giving the answer in the form  $y = f(x)$ .

$$y = \frac{\sin x}{1+\sin x} = (\sec x - \tan x) \tan x$$

④  $\frac{d^2y}{dx^2} = \frac{dy}{dx} \left( \frac{3y-2}{\sqrt{1-2y}} \right)$

LET  $P = \frac{dy}{dx}$  DIFF W.R.T.  $y$

$$\frac{dp}{dy} = \frac{dy}{dx} \frac{dx}{dy} = \frac{dy}{dx} \times \frac{1}{\frac{dy}{dx}} = 1$$

$$\frac{dp}{dy} = \frac{3y-2}{\sqrt{1-2y}}$$

$$\frac{d^2y}{dx^2} = P \frac{dp}{dy}$$

Thus

$$x^2 \frac{dp}{dy} = x^2 \frac{3y-2}{\sqrt{1-2y}}$$

$$\Rightarrow \int x^2 dp = \int \frac{3y-2}{\sqrt{1-2y}} dy$$

$$\Rightarrow P = \int \frac{3y-2}{x^2 \sqrt{1-2y}} dy$$

By substitution

$$u = \sqrt{1-2y}$$

$$u^2 = 1-2y$$

$$2u du = -2 dy$$

$$dy = -u du$$

$$\Rightarrow \frac{du}{dx} = \int \frac{3y-2}{u} (-2u du)$$

$$\Rightarrow \frac{du}{dx} = \int 2-3y du$$

$$\Rightarrow \frac{du}{dx} = \int 2-\frac{3}{x^2}(2u) du$$

$$\Rightarrow \frac{du}{dx} = \int 2-\frac{3}{x^2}(u-u^2) du$$

$$\Rightarrow \frac{du}{dx} = \int 2-\frac{3}{x^2}+\frac{3}{x^2}u^2 du$$

$$\Rightarrow \frac{du}{dx} = \int \frac{3}{x^2}u^2 du$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x^2}u^3 + C$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x^2}u(u^2+1) + C$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x^2}\sqrt{1-2y}(1+u^2) + C$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x^2}\sqrt{1-2y}(2-2y) + C$$

$$\Rightarrow \frac{du}{dx} = (1-y)\sqrt{x^2-2y} + C$$

④ ADD constants  
 $y=0, \frac{dy}{dx}=1$   
 $1=1+C$   
 $C=0$

$$\frac{dy}{dx} = (1-y)(1-2y)^{\frac{1}{2}}$$

SEPARATE VARIABLES

$$\int \frac{1}{(1-y)(1-2y)^{\frac{1}{2}}} dy = \int 1 dx$$

OLE & SUBSTITUTION

$$2y = \sin \theta$$

$$2dy = 2\cos \theta d\theta$$

$$dy = \cos \theta d\theta$$

$$\Rightarrow x = \int \frac{\sin \theta \cos \theta d\theta}{(1-\sin^2 \theta)(1-2\sin^2 \theta)^{\frac{1}{2}}}$$

$$\Rightarrow x = \int \frac{2\sin \theta d\theta}{2-\sin^2 \theta}$$

$$\Rightarrow x = \int \frac{2\sin \theta}{\frac{2-\sin^2 \theta}{\sin^2 \theta}} d\theta$$

$$\Rightarrow x = \int \frac{2\sin \theta \sin^2 \theta}{2-\sin^2 \theta} d\theta$$

$$\Rightarrow x = \int \frac{2\sin^3 \theta}{2\sin^2 \theta - 1} d\theta$$

NOW

$$\sin \theta = 2y$$

$$\sqrt{1-4y^2} = \sqrt{1-2y^2}$$

$$\therefore \sec \theta = \frac{1}{\sqrt{1-2y^2}}$$

$$\Rightarrow x = 2 \arctan \left( \frac{1}{\sqrt{1-2y^2}} \right) + K$$

$$\Rightarrow x = -2 \arctan \left( \frac{1}{\sqrt{1-2y^2}} \right) + K$$

APPLY constraint  
 $x=0, y=0 \Rightarrow 0 = -2 \cdot \frac{\pi}{2} + K$   
 $K = \frac{\pi}{2}$

$$x = -2 \arctan \sqrt{1-2y^2} + \frac{\pi}{2}$$

TIDY UP

$$\Rightarrow 2 \arctan \sqrt{1-2y^2} = \frac{\pi}{2} - x$$

$$\Rightarrow \arctan \sqrt{1-2y^2} = \frac{\pi}{4} - \frac{x}{2}$$

$$\Rightarrow \sqrt{1-2y^2} = \tan \left( \frac{\pi}{4} - \frac{x}{2} \right)$$

Tidy further

$$\frac{1-\tan \frac{x}{2}}{1+\tan \frac{x}{2}} = \frac{\cos \frac{x}{2}-\sin \frac{x}{2}}{\cos \frac{x}{2}+\sin \frac{x}{2}} \quad \text{(MULTIPLY TOP & BOTTOM BY } \cos \frac{x}{2} \text{)}$$

$$= \frac{(\cos \frac{x}{2}-\sin \frac{x}{2})(\cos \frac{x}{2}+\sin \frac{x}{2})}{(\cos \frac{x}{2}+\sin \frac{x}{2})(\cos \frac{x}{2}+\sin \frac{x}{2})}$$

$$= \frac{\cos^2 \frac{x}{2}-\sin^2 \frac{x}{2}+2\cos \frac{x}{2}\sin \frac{x}{2}+2\sin \frac{x}{2}\cos \frac{x}{2}}{\cos^2 \frac{x}{2}-\sin^2 \frac{x}{2}}$$

$$= \frac{1-\sin x}{\cos x}$$

$$\Rightarrow \sqrt{1-2y^2} = \frac{(-\sin x)}{\cos x}$$

$$\Rightarrow \sqrt{1-2y^2} = \sec x - \tan x$$

$$\Rightarrow 1-2y^2 = \sec^2 x - 2\tan x \sec x$$

$$\Rightarrow 1-\sec^2 x - \tan^2 x + 2\tan x \sec x = 2y$$

$$\Rightarrow -2\tan^2 x + 2\tan x \sec x = 2y$$

$$\Rightarrow y = \sec x \tan x - \tan^2 x$$

$$\Rightarrow y = \tan x (\sec x - \tan x)$$

ALTERNATIVE ANSWER

$$\Rightarrow \sqrt{1-2y^2} = \frac{1-\sin x}{\cos x}$$

$$\Rightarrow 1-2y^2 = \frac{(1-\sin x)^2}{\cos^2 x}$$

$$\Rightarrow 1-2y^2 = \frac{(1-\sin x)^2}{1-\sin^2 x}$$

$$\Rightarrow 1-2y^2 = \frac{(1-\sin x)^2}{(1-\sin x)(1+\sin x)}$$

$$\Rightarrow 1-2y^2 = \frac{1-\sin x}{1+\sin x}$$

$$\Rightarrow 1 - \frac{1-\sin x}{1+\sin x} = 2y$$

$$\Rightarrow 2y = \frac{1+\sin x - 1+\sin x}{1+\sin x}$$

$$\Rightarrow 2y = \frac{2\sin x}{1+\sin x}$$

$$\Rightarrow y = \frac{\sin x}{1+\sin x}$$

**Question 11** (\*\*\*\*\*)

The curve  $C$ , has gradient  $\frac{1}{8}$  at the point with coordinates  $(1, \frac{1}{2})$  and further satisfies the differential relationship

$$2y^2 \frac{d^2y}{dx^2} + (2y+1)(y-1)^2 \frac{dy}{dx} = 0, \quad y \neq 0.$$

Find an equation for  $C$ , giving the answer in the form  $y = f(x)$ .

$$y = \frac{\sqrt{x}}{1+\sqrt{x}}$$

$2y^2 \frac{d^2y}{dx^2} + (2y+1)(y-1)^2 \frac{dy}{dx} = 0 \quad x=1, y=\frac{1}{2}, \frac{dy}{dx} = \frac{1}{8}$

$$\Rightarrow 2y^2 \frac{d^2y}{dx^2} = -(2y+1)(y-1)^2 \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(2y+1)(y-1)^2}{2y^2} \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2y^2 - 4y^2 + 2y^2 + 1}{2y^2} \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2y^2 - 3y^2 + 1}{2y^2} \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \left(-y + \frac{1}{2} - \frac{1}{2y^2}\right) \frac{dy}{dx}$$

• INTEGRATE BOTH SIDES TOTAL DIRECT TO  $x$   
SUBJECT TO  $y = \frac{1}{2}, \frac{dy}{dx} = \frac{1}{8}$

$$\Rightarrow \int \frac{dy}{\left(-y + \frac{1}{2} - \frac{1}{2y^2}\right)} = \int \frac{dy}{y-1}$$

$$\Rightarrow \left[ \frac{dy}{\left(-y + \frac{1}{2} - \frac{1}{2y^2}\right)} \right]_0^y = \left[ \frac{dy}{y-1} \right]_1^{\frac{1}{2}}$$

$$\Rightarrow \frac{1}{2} \left[ \frac{1}{2} u^{-2} - \frac{1}{2} u^{-1} \right]_0^y = \left[ \ln|u| \right]_1^{\frac{1}{2}}$$

• SEPARATE VARIABLES

$$\Rightarrow \frac{2y}{(y-1)^2} dy = 1 dx$$

• INTEGRATE SUBJECT TO THE CONDITIONS  $x=1, y=\frac{1}{2}$

$$\Rightarrow \int \frac{2y}{(y-1)^2} dy = \int 1 dx$$

• SUBSTITUTION (OR PARTIAL FRACTION)

$$u = 1-y \\ y = 1-u \\ du = -dy \\ y = \frac{1}{2} \rightarrow u = \frac{1}{2} \\ y = -\frac{1}{2} \rightarrow u = -\frac{1}{2}$$

$$\Rightarrow \int \frac{2u}{u^2} du = \int 1 dx$$

$$\Rightarrow \frac{2}{u} = x-1$$

$$\Rightarrow 2u = (x-1)(y^2 - 2y + 1)$$

$$\Rightarrow 2u = (x-1)y^2 - 2(x-1)y + (x-1)$$

$$\Rightarrow 2u = (x-1)y^2 - 2xy + x - 1$$

$$\Rightarrow (x-1)y^2 - 2xy + x = 0$$

FACTORIZE OR COMPLETE THE SQUARE

$$\Rightarrow y^2 - \frac{2x}{x-1}y + \frac{x}{x-1} = 0$$

$$\Rightarrow y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \int_{\frac{1-y}{2}}^{\frac{1-y}{2}} \frac{2-2u}{u^3} (-du) = \int_1^{\frac{1}{2}} 1 du$$

$$\Rightarrow 2 \int_{\frac{1-y}{2}}^{\frac{1-y}{2}} -u^3 + u^2 du = \int_1^{\frac{1}{2}} 1 du$$

$$\Rightarrow 2 \left[ \frac{1}{2} u^2 - u^1 \right]_{\frac{1-y}{2}}^{\frac{1-y}{2}} = \left[ u \right]_1^{\frac{1}{2}}$$

$$\Rightarrow \left[ \frac{1}{4} u^2 - \frac{1}{2} u \right]_{\frac{1-y}{2}}^{\frac{1-y}{2}} = x-1$$

$$\Rightarrow \left[ \frac{1-2u}{u^2} \right]_{\frac{1-y}{2}}^{\frac{1-y}{2}} = x-1$$

$$\Rightarrow \frac{1-2(x-y)}{(1-y)^2} = x-1$$

$$\Rightarrow 2y-1 = (x-1)(y^2 - 2y + 1)$$

$$\Rightarrow 2y-1 = (x-1)y^2 - 2(x-1)y + (x-1)$$

$$\Rightarrow 2y-1 = (x-1)y^2 - 2xy + x - 1$$

$$\Rightarrow (x-1)y^2 - 2xy + x = 0$$

FACTORIZE OR COMPLETE THE SQUARE

$$\Rightarrow y^2 - \frac{2x}{x-1}y + \frac{x}{x-1} = 0$$

$$\Rightarrow \left[ y - \frac{x}{x-1} \right]^2 - \frac{x^2}{(x-1)^2} + \frac{x}{x-1} = 0$$

$$\Rightarrow \left[ y - \frac{x}{x-1} \right]^2 + \frac{-x^2 + 2x(x-1)}{(x-1)^2} = 0$$

$$\Rightarrow \left[ y - \frac{x}{x-1} \right]^2 + \frac{-2x}{(x-1)^2} = 0$$

$$\Rightarrow \left[ y - \frac{x}{x-1} \right]^2 = \frac{2x}{x-1}$$

$$\Rightarrow y - \frac{x}{x-1} = \pm \sqrt{\frac{2x}{x-1}}$$

$$\Rightarrow y = \frac{\frac{x}{x-1} \pm \sqrt{\frac{2x}{x-1}}}{x-1}$$

$$\Rightarrow y = \frac{\sqrt{2x}(x-1) \pm 1}{(x-1)^2}$$

$$\Rightarrow y = \frac{\sqrt{2x} \left[ \sqrt{2x} \pm 1 \right]}{(x-1)^2}$$

$$\Rightarrow y = \sqrt{\frac{2x}{x-1}} \left[ \sqrt{2x} \pm 1 \right]$$

$$\Rightarrow y = \sqrt{\frac{2x}{x-1}} \left[ \sqrt{2x} \pm 1 \right] \text{ NOT } 2x=0 \text{ AT } x=1$$

$$\Rightarrow y = \frac{\sqrt{2x}}{\sqrt{x-1} + 1}$$

ALTERNATIVE REARRANGEMENT

$$\Rightarrow \frac{2y-1}{(y-1)^2} = x-1$$

$$\Rightarrow 2 = \frac{2y-1}{(y-1)^2} + 1$$

$$\Rightarrow 2 = \frac{(2y-1) + (y-1)^2}{(y-1)^2}$$

$$\Rightarrow 2 = \frac{2y-1 + y^2 - 2y + 1}{(y-1)^2}$$

$$\Rightarrow x = \frac{y^2}{(y-1)^2}$$

$$\Rightarrow \frac{y}{y-1} = \sqrt{\frac{y^2}{(y-1)^2}} = \sqrt{\frac{y}{y-1}}$$

EQUATION IS SATISFIED BY  $x=1, y=\frac{1}{2}$   
HENCE

$$\frac{y}{y-1} = -\sqrt{\frac{y}{y-1}}$$

$$y = -y\sqrt{\frac{y}{y-1}} + \sqrt{\frac{y}{y-1}}$$

$$y + y\sqrt{\frac{y}{y-1}} = \sqrt{\frac{y}{y-1}}$$

$$y(1 + \sqrt{\frac{y}{y-1}}) = \sqrt{\frac{y}{y-1}}$$

$$y = \frac{\sqrt{\frac{y}{y-1}}}{1 + \sqrt{\frac{y}{y-1}}} \text{ AS REQUIRED}$$

# **2<sup>ND</sup> ORDER BY SUBSTITUTIONS**

**Question 1** (\*\*\*)

$$2y \frac{d^2y}{dx^2} - 8y \frac{dy}{dx} + 16y^2 = \left( \frac{dy}{dx} \right)^2, \quad y \neq 0,$$

Find the general solution of the above differential equation by using the transformation equation  $t = \sqrt{y}$ .

Give the answer in the form  $y = f(x)$ .

$$y = (Ae^{2x} + Bxe^{2x})^2$$

Given differential equation:

$$2y \frac{d^2y}{dx^2} - 8y \frac{dy}{dx} + 16y^2 = \left( \frac{dy}{dx} \right)^2$$

Substitution:  $t = \sqrt{y}$

$$\frac{t^2}{y} \frac{d^2y}{dx^2} - 8t \frac{dy}{dx} + 16t^2y^2 = \left( \frac{dt}{dy} \right)^2$$

$$\frac{t^2}{y} \frac{d^2y}{dx^2} - 8t \frac{dy}{dx} + 16t^2 = \frac{dt}{dy} \frac{dy}{dx}^2$$

$$\frac{t^2}{y} \frac{d^2y}{dx^2} - 8t \frac{dy}{dx} + 16t^2 = \frac{dt}{dy} \frac{dt}{dx}^2$$

$$\frac{t^2}{y} \frac{d^2y}{dx^2} - 8t \frac{dy}{dx} + 16t^2 = \frac{dt}{dy} \frac{dt}{dy}^2$$

$$\frac{d^2t}{dy^2} = 2t \frac{dt}{dy} \frac{dt}{dx}^2 + 2t^2 \frac{dt}{dy}^2$$

$$\frac{d^2t}{dy^2} = 2t \frac{dt}{dy} \frac{dt}{dx}^2 + 2t^2 \frac{dt}{dy}^2$$

• Auxiliary equation:

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2 \text{ (Repeating root)}$$

$$\therefore t = Ae^{2x} + Be^{2x}$$

$$y = t^2 = (Ae^{2x} + Bxe^{2x})^2$$

**Question 2** (\*\*\*)

The differential equation

$$x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 3x, \quad x \neq 0,$$

is to be solved subject to the boundary conditions  $y = \frac{3}{2}$ ,  $\frac{dy}{dx} = \frac{1}{2}$  at  $x = 1$ .

- a) Show that the substitution  $v = \frac{dy}{dx}$ , transforms the above differential equation into

$$\frac{dv}{dx} + \frac{2v}{x} = 3.$$

- b) Hence find the solution of the original differential equation, giving the answer in the form  $y = f(x)$ .

$$y = \frac{1}{2} \left( x^2 + \frac{1}{x} + 1 \right)$$

③  $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 3x$   
 $x \frac{dy}{dx} + 2y = 3x$   
 $\frac{dy}{dx} + \frac{2y}{x} = 3$

④  $\frac{dv}{dx} + \frac{2v}{x} = 0$   
 $\frac{dv}{dx} = -\frac{2v}{x}$   
 $\int \frac{1}{v} dv = \int -\frac{2}{x} dx$   
 $\ln|v| = -2\ln|x| + C$   
 $|v| = |v| \frac{A}{x^2}$   
 $v = \frac{A}{x^2}$

⑤  $V = \frac{dy}{dx} = \frac{A}{x^2}$  (Multiply by integrating factor)  
 $\Rightarrow \frac{dy}{dx} = \frac{A}{x^2} + \alpha$   
 $\Rightarrow y = -\frac{A}{x} + \frac{1}{2}x^2 + B$

• Apply condition  $x=1, \frac{dy}{dx}=\frac{1}{2}$   $\Rightarrow \frac{1}{2} = \frac{A}{1} + 1 \Rightarrow A = -\frac{1}{2}$

• Apply condition  $x=1, y=\frac{3}{2}$   $\Rightarrow \frac{3}{2} = \frac{1}{2} + \frac{1}{2} + B \Rightarrow B = \frac{1}{2}$

$\therefore y = \frac{1}{2} + \frac{1}{2}x^2 + \frac{1}{2x}$   
 $y = \frac{1}{2}(1 + x + x^2)$

**Question 3    (\*\*\*)**

The curve  $C$  has equation  $y = f(x)$  and satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 2y(2x^2 - 1) = 3x^3 e^x, \quad x \neq 0$$

is to be solved subject to the boundary conditions  $y = \frac{3}{2}$ ,  $\frac{dy}{dx} = \frac{1}{2}$  at  $x = 1$ .

- a) Show that the substitution  $y = xv$ , where  $v$  is a function of  $x$  transforms the above differential equation into

$$\frac{d^2v}{dx^2} - 4v = 3e^x.$$

It is further given that  $C$  meets the  $x$  axis at  $x = \ln 2$  and has a finite value for  $y$  as  $x$  gets infinitely negatively large.

- b) Express the equation of  $C$  in the form  $y = f(x)$ .

$$y = \frac{1}{2}x e^{2x} - x e^x$$

**a)**  $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 2y(2x^2 - 1) = 3x^3 e^x$   
 DIVIDE THE EQUATION BY  $x^2$   
 $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} - 2y(2x^2 - 1) = 3x^3$   
 $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} - 2y(2x^2 - 1) = 3x^3$   
 $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} - 2y = 3x^3$   
 $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} - 2y = 3x^3$   
 $\frac{d^2y}{dx^2} - 4v = 3x^3$  AT REQUIRED  
 $\frac{d^2v}{dx^2} - 4v = 3x^3$

**b)** AUXILIARY EQUATION  
 $\lambda^2 - 4 = 0$   
 $\lambda = \pm 2$   
 PARTICULAR SOLUTION  
 $y = P x^2$  SUB INTO THE O.D.E.  
 $P = 1$   
 $P = 4$   
 $P = 4$   
 $y = A x^2 + B x^3 - e^x$   
 $SOLUTION IS FINITE AS  $x \rightarrow -\infty$$   
 $\therefore B = 0$   
 VALUE CHOOSES THE  $x$  AXIS AT  $x = \ln 2$   
 $0 = A \ln 2 e^{2 \ln 2} + B \ln 2 e^{\ln 2}$   
 $0 = A \ln 2 e^2 - 2 \ln 2$   
 $0 = 2A - 1$   
 $A = \frac{1}{2}$   
 $\therefore y = \frac{1}{2}x e^{2x} - x e^x$

**Question 4** (\*\*\*)

The differential equation

$$(x^3 + 1) \frac{d^2y}{dx^2} - 3x^2 \frac{dy}{dx} = 2 - 4x^3,$$

is to be solved subject to the boundary conditions  $y=0$ ,  $\frac{dy}{dx}=4$  at  $x=0$ .

Use the substitution  $u = \frac{dy}{dx} - 2x$ , where  $u$  is a function of  $x$ , to show that the solution of the above differential equation is

$$y = x^4 + x^2 + 4x.$$

[ ] proof

USING THE SUBSTITUTION GIVEN

$$\begin{aligned} \Rightarrow u &= \frac{dy}{dx} - 2x \\ \Rightarrow \frac{du}{dx} &= 0 + 2x \\ \Rightarrow \frac{du}{dx} &= \frac{du}{dt} + 2x \end{aligned}$$

SUBSTITUTE INTO THE O.D.E.

$$\begin{aligned} \Rightarrow (x^3 + 1) \frac{d^2y}{dx^2} - 3x^2 \frac{dy}{dx} &= 2 - 4x^3 \\ \Rightarrow (x^3 + 1) \frac{du}{dx} - 3x^2(u + 2x) &= 2 - 4x^3 \\ \Rightarrow (x^3 + 1) \frac{du}{dx} + 2(x^3 + 1) - 3x^2(u + 2x) &= 2 - 4x^3 \\ \Rightarrow (x^3 + 1) \frac{du}{dx} + 2x^3 + 2 - 3x^2(u + 2x) - 6x^3 &= 2 - 4x^3 \\ \Rightarrow (x^3 + 1) \frac{du}{dx} - 3x^2(u + 2x) &= 4x^3 \\ \Rightarrow (x^3 + 1) \frac{du}{dx} - 3x^2u - 6x^3 &= 4x^3 \end{aligned}$$

SIMPLIFY

$$\begin{aligned} \Rightarrow \frac{1}{u+2} du &= \frac{3x^2}{x^3+1} dx \\ \Rightarrow \int \frac{1}{u+2} du &= \int \frac{3x^2}{x^3+1} dx \\ \Rightarrow \ln|u+2| &= \ln|x^3+1| + \ln A \\ \Rightarrow |u+2| &= |A(x^3+1)| \\ \Rightarrow u+2 &= A(x^3+1) \end{aligned}$$

DIVIDING THE TRANSPOSITION

$$\begin{aligned} \Rightarrow \frac{du}{dx} - 2x &= A(x^3 + 1) \\ \Rightarrow \frac{du}{dx} &= A(x^3 + 1) + 2x \\ \text{INTEGRATING w.r.t. } x \\ \Rightarrow u &= A\left(\frac{x^4}{4} + x\right) + x^2 + B \end{aligned}$$

USING THE CONDITION GIVEN

$$\begin{aligned} x=0, y=0 &\Rightarrow 0=B \\ x=0, \frac{du}{dx}=4 &\Rightarrow 4=A \\ \therefore u &= 4\left(\frac{x^4}{4} + x\right) + x^2 \\ u &= x^4 + 4x^2 \\ u &= x^4 + x^2 + 4x^2 \end{aligned}$$

**Question 5** (\*\*\*\*\*)

$$\frac{d^2y}{dx^2} - (1-6e^x) \frac{dy}{dx} + 10ye^{2x} = 5e^{2x} \sin(2e^x).$$

- a) By using the substitution  $x = \ln t$  or otherwise, show that the above differential equation can be transformed to

$$\frac{d^2y}{dt^2} + 6 \frac{dy}{dt} + 10y = 5 \sin 2t.$$

- b) Hence find a general solution for the original differential equation.

$$[ ] , y = e^{-3e^x} [ A \cos(e^x) + B \sin(e^x) ] + \frac{1}{6} \sin(2e^x) - \frac{1}{3} \cos(2e^x)$$

<p>a) <u>START BY OBTAIN "REPLACEMENTS" FOR <math>\frac{dy}{dx}</math> &amp; <math>\frac{d^2y}{dx^2}</math></u></p> <ul style="list-style-type: none"> <li><math>x = \ln t</math> DIFFERENTIATE W.R.T <math>y</math> <math>\Rightarrow \frac{dy}{dt} = \frac{1}{t} \frac{dy}{dx}</math> <math>\Rightarrow \frac{dy}{dx} = t \frac{dy}{dt}</math></li> <li>ALSO NOTE THAT <math>x = \ln t</math> <math>\frac{dx}{dt} = \frac{1}{t}</math> <math>\frac{dt}{dx} = t</math></li> </ul> <p><u>SUBSTITUTING INTO THE O.D.E. AND SIMPLIFY, NOTING FURTHER THAT <math>t = e^x</math></u></p> $\begin{aligned} &\Rightarrow \frac{d^2y}{dx^2} - (1-6e^x) \frac{dy}{dx} + 10ye^{2x} = 5e^{2x} \sin(2e^x) \\ &\Rightarrow \left[ \frac{d^2y}{dt^2} + t^2 \frac{dy}{dt^2} \right] - (1-6e^x) \left( t \frac{dy}{dt} \right) + 10ye^{2x} = 5e^{2x} \sin(2e^x) \\ &\Rightarrow t^2 \frac{d^2y}{dt^2} + t^2 \frac{dy}{dt^2} - t \frac{dy}{dt} + 6t^2 \frac{dy}{dt} + 10ye^{2x} = 5e^{2x} \sin(2e^x) \\ &\Rightarrow t^2 \frac{d^2y}{dt^2} + 6t^2 \frac{dy}{dt} + 10ye^{2x} = 5e^{2x} \sin(2e^x) \\ &\Rightarrow \frac{d^2y}{dt^2} + 6 \frac{dy}{dt} + 10y = 5 \sin 2t \end{aligned}$ <p style="text-align: right;">As required</p>	<p>b) <u>SOLVING THE TRANSFORMED EQUATION</u></p> <ul style="list-style-type: none"> <li><u>MATERIAL EQUATION</u> <math>\Rightarrow t^2 + 6t + 10 = 0</math> <math>\Rightarrow (t+3)^2 - 4+10 = 0</math> <math>\Rightarrow (t+3)^2 = -1</math> <math>\Rightarrow t+3 = \pm i</math> <math>\Rightarrow t = -3 \pm i</math></li> <li><u>COMPLEMENTARY FUNCTION</u> <math>y = e^{-3t} (A \cos t + B \sin t)</math></li> <li><u>PARTICULAR INTEGRAL</u> <math>y = P \cos 2t + Q \sin 2t</math> <math>\dot{y} = -2P \sin 2t + 2Q \cos 2t</math> <math>\ddot{y} = -4P \cos 2t - 4Q \sin 2t</math> <u>SUB INTO THE O.D.E.</u> <math>\ddot{y} = -4P \cos 2t - 4Q \sin 2t</math> <math>+ 6\dot{y} = 12P \sin 2t - 12Q \cos 2t</math> <math>+ 10y = 10P \cos 2t + 10Q \sin 2t</math> <u>ADDITION AND CANCELLATION</u> <math>(P+2Q)\cos 2t + (Q-2P)\sin 2t \equiv 5 \sin 2t</math> <math>6P+12Q = 0 \Rightarrow P = -2Q</math> <math>6Q-12P = 5 \Rightarrow 6Q+24Q=5</math> <math>\Rightarrow 30Q = 5</math> <math>\Rightarrow Q = \frac{1}{6}</math> <math>\Rightarrow P = -\frac{1}{3}</math></li> </ul> <p><u>HENCE THE GENERAL SOLUTION CAN BE FOUND</u></p> $\begin{aligned} &\Rightarrow y = e^{-3t} (A \cos t + B \sin t) - \frac{1}{3} \cos 2t + \frac{1}{6} \sin 2t \\ &\Rightarrow y = e^{-3e^x} [ A \cos(e^x) + B \sin(e^x) ] - \frac{1}{3} \cos(2e^x) + \frac{1}{6} \sin(2e^x) \end{aligned}$
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**Question 6    (\*\*\*)+**

Solve the differential equation

$$x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 0,$$

subject to the boundary conditions  $y = 2$ ,  $\frac{dy}{dx} = -1$  at  $x = 1$ .

$$y = \frac{2e^{2x}}{e^{2x} + x}$$

The handwritten solution is organized into three columns:

- First Method - Reduce the D.E.**: Shows the original equation  $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 0$  and the condition  $y = 2, \frac{dy}{dx} = -1$  at  $x = 1$ . It then reduces the equation to  $x \frac{dp}{dx} + 2p = 0$ , integrates to get  $p = \frac{C}{x}$ , and then  $\ln p = \ln|x| + \ln C$ , leading to  $p = \frac{C}{x}$ . The general solution is given as  $y = \frac{A}{x} + B$ .
- Second Method - By Inspection**: Starts with the trial solution  $y = x^{\alpha}$ . Substituting into the D.E. gives  $\alpha(\alpha-1)x^{\alpha-2} + 2\alpha x^{\alpha-1} = 0$ , which simplifies to  $(2\alpha-1)x^{\alpha-2} = 0$ . This implies  $\alpha = \frac{1}{2}$ . The second derivative is  $y' = \frac{1}{2}x^{-\frac{1}{2}}$ . Applying the boundary conditions at  $x=1$  leads to  $A=2$  and  $B=-1$ .
- General Solution**: The final general solution is  $y = \frac{A}{x} + Bx^{-1}$ .

**Question 7** (\*\*\*)

$$x \frac{d^2y}{dx^2} + (6x+2) \frac{dy}{dx} + 9xy = 27x - 6y.$$

Use the substitution  $u = xy$ , where  $u$  is a function of  $x$ , to find a general solution of the above differential equation.

$$\boxed{\quad}, \quad y = \frac{A}{x} e^{-3x} + B e^{-3x} + 3 - \frac{2}{x}$$

(USING THE SUBSTITUTION GIVEN  $u(x) = x \cdot y(x)$ )

$$\begin{aligned} \frac{du}{dx}(u) &= \frac{d}{dx}(x \cdot y(u)) \\ \frac{du}{dx} &= 2x \cdot \frac{dy}{dx} + 1 \cdot y \\ \frac{du}{dx} &= 2x \frac{dy}{dx} + y \\ \frac{dy}{dx} &= \frac{du}{dx} - y \end{aligned}$$

Differentiate the above again with respect to  $x$

$$\begin{aligned} \frac{d}{dx}\left[\frac{dy}{dx}\right] &= \frac{d}{dx}\left[\frac{du}{dx} - y\right] \\ 1 \times \frac{d}{dx} + 2 \frac{d^2u}{dx^2} &= \frac{d^2u}{dx^2} - \frac{dy}{dx} \\ 2 \frac{d^2u}{dx^2} &= \frac{d^2u}{dx^2} - 2 \frac{dy}{dx} \end{aligned}$$

Transform the O.D.E.

$$\begin{aligned} \rightarrow 2 \frac{d^2u}{dx^2} + (\cancel{2x} \frac{d}{dx}) + (9 \cancel{u}) &= 27x - 6y \\ \rightarrow \frac{d^2u}{dx^2} + 2 \frac{d}{dx} + 6x \frac{dy}{dx} + 9u &= 27x - 6y \\ \Rightarrow \frac{d^2u}{dx^2} + 6 \frac{du}{dx} + 9u &= 27x - 6y \\ \Rightarrow \frac{d^2u}{dx^2} + 6 \left(\frac{du}{dx} - y\right) + 9u &= 27x - 6y \\ \rightarrow \frac{d^2u}{dx^2} + 6 \frac{du}{dx} - 6y + 9u &= 27x - 6y \\ \rightarrow \frac{d^2u}{dx^2} + 6 \frac{du}{dx} + 9u &= 27x \end{aligned}$$

The auxiliary equation for the L.H.S. is

$$\begin{aligned} A^2 + 6A + 9 &= 0 \\ (A+3)^2 &= 0 \\ A &= -3 \end{aligned}$$

COMPLEMENTARY EQUATION

$$u = Ae^{-3x} + Bxe^{-3x}$$

PARTICULAR SOLUTION BY INSPECTION

$$\begin{aligned} u &= Px + Q \\ u' &= P \\ u'' &= 0 \\ \therefore 0 + 6P + 9(Px+Q) &= 27x \\ (6P+9Q) + 9Px &= 27x \\ P=3 \quad Q & \quad 6P+9Q=0 \\ 18+9Q=0 \quad 18+9Q=0 & \\ Q=-2 & \end{aligned}$$

This we have

$$u(x) = (A + Bx)e^{-3x} + 3x - 2$$

REWRITING THE PARTICULAR SOLUTION

$$\begin{aligned} u &= (A + Bx)e^{-3x} + 3x - 2 \\ y &= \boxed{\left(\frac{A}{x} + B\right)e^{-3x} + 3 - \frac{2}{x}} \end{aligned}$$

**Question 8** (\*\*\*)

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} \tan x - y \sec^4 x = 0.$$

The above differential equation is to be solved by a substitution.

- a) If  $t = \tan x$  show that ...

i. ...  $\frac{dy}{dx} = \frac{dy}{dt} \sec^2 x$

ii. ...  $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} \sec^4 x + 2 \frac{dy}{dt} \sec^2 x \tan x$

- b) Use the results obtained in part (a) to find a general solution of the differential equation in the form  $y = f(x)$ .

,  $y = A e^{\tan x} + B e^{-\tan x}$

a) DIFFERENTIATING WITH RESPECT TO  $y$

$$t = \tan x \Rightarrow \frac{dt}{dx} = \frac{d}{dt}(\tan x)$$

$$\rightarrow \frac{dt}{dx} = \sec^2 x$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 x} \frac{dy}{dt}$$

$$\rightarrow \frac{dy}{dx} = \sec^2 x \frac{dy}{dt}$$

as required

b) NOW DIFFERENTIATING THE ABOVE EXPRESSION WITH RESPECT TO  $x$

$$\rightarrow \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} (\sec^2 x \frac{dy}{dt})$$

$$\rightarrow \frac{d^2y}{dx^2} = 2 \sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d}{dx} \left( \frac{dy}{dt} \right)$$

$$\rightarrow \frac{d^2y}{dx^2} = 2 \sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d^2y}{dt^2} \frac{dt}{dx}$$

BUT IF  $t = \tan x$

$$\frac{dt}{dx} = \sec^2 x$$

$$\rightarrow \frac{d^2y}{dx^2} = 2 \sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d^2y}{dt^2} \sec^2 x$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} \sec^4 x + 2 \frac{dy}{dt} \sec^2 x \tan x$$

as required

c) TRANSFORMING THE GIVEN O.D.E.

$$\rightarrow \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} \tan x - y \sec^4 x = 0$$

$$\rightarrow \left( \frac{d^2y}{dt^2} \sec^4 x + 2 \frac{dy}{dt} \sec^2 x \tan x \right) - 2 \left( \frac{dy}{dt} \sec^2 x \tan x \right) - y \sec^4 x = 0$$

$\Rightarrow \frac{d^2y}{dt^2} - y = 0$

AUXILIARY EQUATION

$$\lambda^2 = 0$$

$$\lambda = \pm 1$$

∴ GENERAL SOLUTION IS

$$y = A e^t + B e^{-t}$$

as  $y = P \sinht + Q \cosh t$

$$y = A e^{\tan x} + B e^{-\tan x}$$

or  $y = P \sin(\tan x) + Q \cosh(\tan x)$

**Question 9    (\*\*\*)+**

Show clearly that the substitution  $z = \sin x$ , transforms the differential equation

$$\frac{d^2y}{dx^2} \cos x + \frac{dy}{dx} \sin x - 2y \cos^3 x = 2\cos^5 x,$$

into the differential equation

$$\frac{d^2y}{dz^2} - 2y = 2(1-z^2)$$

proof

The proof shows the following steps:

- Given:  $\frac{d^2y}{dx^2} \cos x + \frac{dy}{dx} \sin x - 2y \cos^3 x = 2\cos^5 x$
- Substitution:  $z = \sin x$
- Differentiation rules:
  - $\frac{dy}{dx} = \cos x \frac{dy}{dz}$
  - $\frac{d^2y}{dx^2} = \cos^2 x \frac{d^2y}{dz^2} + \cos x \frac{dy}{dz}$
- Product rule application:
  - $\frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\cos x \frac{dy}{dz})$
  - $\frac{dy}{dx} = -\sin x \frac{dy}{dz} + \cos x \frac{d}{dx}(\frac{dy}{dz})$
  - $\frac{dy}{dx} = -\sin x \frac{dy}{dz} + \cos x \frac{d}{dz}(\frac{dy}{dz})$
  - $\frac{dy}{dz} = -\sin x \frac{dy}{dx} + \cos x \frac{d}{dx}(\frac{dy}{dx})$
- Substitution back into the transformed equation:
 
$$\begin{aligned} & \left[ \cos^2 x \frac{d^2y}{dz^2} + \cos x \frac{dy}{dz} \right] \cos x - \sin x \frac{dy}{dz} + \left[ \cos x \frac{dy}{dz} \right] \sin x - 2y \cos^3 x = 2\cos^5 x \\ & \Rightarrow \cos^2 x \frac{d^2y}{dz^2} - \sin x \cos x \frac{dy}{dz} + \sin x \cos x \frac{dy}{dz} - 2y \cos^3 x = 2\cos^5 x \\ & \Rightarrow \cos^2 x \frac{d^2y}{dz^2} - 2y \cos^3 x = 2\cos^5 x \\ & \Rightarrow \frac{d^2y}{dz^2} - 2y = 2(1-z^2) \\ & \Rightarrow \frac{d^2y}{dz^2} - 2y = 2(1-z^2) \end{aligned}$$
- Final note: "As required"

**Question 10    (\*\*\*)+**

By using the substitution  $z = \frac{dy}{dx}$ , or otherwise, solve the differential equation

$$(x^2 + 1) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 6x^2 + 2,$$

subject to the conditions  $x = 0, y = 2, \frac{dy}{dx} = 1$

$$y = x^2 + 2 + \arctan x$$

The handwritten solution shows the following steps:

- Substitution:  $z = \frac{dy}{dx}, \frac{dz}{dx} = \frac{d^2y}{dx^2}$
- Equation after substitution:  $(x^2+1)\frac{dz}{dx} + 2z = 6x^2 + 2$
- Integration factor:  $\frac{dz}{dx} + \frac{2z}{x^2+1} = 6x^2 + 2$
- Integrating factor:  $I.F. = e^{\int \frac{2}{x^2+1} dx} = e^{\ln(x^2+1)} = x^2+1$
- Homogeneous part:  $\frac{d}{dx}(z(x^2+1)) = \frac{d(x^2+2)}{dx}$
- Solution of homogeneous:  $z(x^2+1) = \int x^2+2 dx$
- General solution:  $\boxed{z(x^2+1) = x^3+2x+C}$
- Condition at x=0:  $z=0 \Rightarrow C=0$
- At x=0:  $y=2$
- Applying condition:  $2 = x^3+2x+D$
- Simplifying:  $D=2$
- Final solution:  $y = x^3+2x+2$
- Final boxed answer:  $\boxed{y = x^2 + 2 + \arctan x}$

**Question 11    (\*\*\*\*\*)**

Use the substitution  $z = \sqrt{y}$ , where  $y = f(x)$ , to solve the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - 5 \frac{dy}{dx} + 2y = 0,$$

subject to the boundary conditions  $y = 4$ ,  $\frac{dy}{dx} = 44$  at  $x = 0$ .

Give the answer in the form  $y = f(x)$ .

$$y = 9e^{6x} - 6e^x + e^{-4x}$$

The image shows handwritten mathematical work for solving the differential equation  $\frac{d^2y}{dx^2} + \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - 5 \frac{dy}{dx} + 2y = 0$  using the substitution  $z = \sqrt{y}$ . The working includes the following steps:

- Substitution:  $z = \sqrt{y} \Rightarrow y = z^2$
- Derivatives:  $\frac{dy}{dx} = 2z \frac{dz}{dx}$ ,  $\frac{d^2y}{dx^2} = 2 \frac{dz}{dx} + 2z \frac{d^2z}{dx^2}$
- Equation transformation:  $2z \frac{d^2z}{dx^2} + 2z \frac{d^2z}{dx^2} - 5(2z \frac{dz}{dx}) + 2z^2 = 0$
- Further simplification:  $4z \frac{d^2z}{dx^2} - 10z \frac{dz}{dx} + 2z^2 = 0$
- Simplifying by dividing by  $2z$ :  $2 \frac{d^2z}{dx^2} - 5 \frac{dz}{dx} + z = 0$
- Characteristic equation:  $2\lambda^2 - 5\lambda + 1 = 0$
- Solving for  $\lambda$ :  $(2\lambda - 1)(\lambda - 1) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = \frac{1}{2}$
- General solution for  $z$ :  $z = C_1 e^x + C_2 e^{\frac{x}{2}}$
- Boundary conditions at  $x=0$ :  $z = 2$  and  $\frac{dz}{dx} = 44$
- Solving for constants:  $2 = C_1 + C_2$  and  $44 = C_1 + \frac{1}{2}C_2$
- Simultaneous equations:  $C_1 = 15, C_2 = -11$
- Final solution for  $z$ :  $z = 15e^x - 11e^{\frac{x}{2}}$
- Final solution for  $y$ :  $y = (15e^x - 11e^{\frac{x}{2}})^2$

**Question 12** (\*\*\*\*\*)

$$2x \frac{d^2y}{dx^2} + \left(1 - 3x^{\frac{1}{2}}\right) \frac{dy}{dx} + y = 0.$$

The above differential equation is to be solved by a substitution.

- a) Given that  $y = f(x)$  and  $t = x^{\frac{1}{2}}$ , show clearly that ...

i. ...  $\frac{dy}{dx} = \frac{1}{2t} \frac{dy}{dt}$ .

ii. ...  $\frac{d^2y}{dx^2} = \frac{1}{4t^2} \frac{d^2y}{dt^2} - \frac{1}{4t^3} \frac{dy}{dt}$ .

- b) Hence show further that the differential equation

$$2x \frac{d^2y}{dx^2} + \left(1 - 3x^{\frac{1}{2}}\right) \frac{dy}{dx} + y = 0,$$

can be transformed to the differential equation

$$\frac{d^2y}{dt^2} - 3 \frac{dy}{dt} + 2y = 0.$$

- c) Find a general solution of the **original** differential equation, giving the answer in the form  $y = f(x)$ .

$$y = Ae^{\sqrt{x}} + Be^{2\sqrt{x}}$$

**a)** Substitution steps:

$$\begin{aligned} t &= x^{\frac{1}{2}} \\ \frac{dt}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} \frac{dx}{dt} \\ \frac{dy}{dx} &= \frac{1}{2t} \frac{dy}{dt} \\ \frac{d^2y}{dx^2} &= \frac{1}{2t^2} \frac{d^2y}{dt^2} - \frac{1}{4t^3} \frac{dy}{dt} \end{aligned}$$

**b)** Substitution into the original equation:

$$\begin{aligned} 2x \frac{d^2y}{dx^2} + \left(1 - 3x^{\frac{1}{2}}\right) \frac{dy}{dx} + y &= 0 \\ \Rightarrow 2t^2 \left[ \frac{1}{2t^2} \frac{d^2y}{dt^2} - \frac{1}{4t^3} \frac{dy}{dt} \right] + \left(1 - 3t\right) \times \frac{1}{2t} \frac{dy}{dt} + y &= 0 \\ \Rightarrow \frac{1}{2} \frac{d^2y}{dt^2} - \frac{1}{4t} \frac{dy}{dt} + \frac{1}{2t} \frac{dy}{dt} - \frac{3}{2} \frac{dy}{dt} + y &= 0 \\ \Rightarrow \frac{1}{2} \frac{d^2y}{dt^2} - \frac{3}{2} \frac{dy}{dt} + y &= 0 \\ \Rightarrow \frac{d^2y}{dt^2} - 3 \frac{dy}{dt} + 2y &= 0 \end{aligned}$$

**c) Aux equations:**

$$\begin{aligned} x - 2t + 2 &= 0 \\ (x-2)(t-1) &= 0 \\ t = 1 &\Rightarrow 2 \end{aligned}$$

$$\therefore y = Ae^t + Be^{2t}$$

$$\text{or } L = x^{\frac{1}{2}} = \sqrt{x}$$

$$y = Ae^{\sqrt{x}} + Be^{2\sqrt{x}}$$

**Question 13    (\*\*\*)**

Show clearly that the substitution  $z = y^2$ , where  $y = f(x)$ , transforms the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - 5 \frac{dy}{dx} + 2y = 0,$$

into the differential equation

$$\frac{d^2z}{dx^2} - 5 \frac{dz}{dx} + 4z = 0$$

**proof**

$\bullet \quad \begin{array}{l} z = y^2 \\ \text{Diff w.r.t } x \\ \frac{dz}{dx} = 2y \frac{dy}{dx} \\ \boxed{\frac{dz}{dx} = \frac{1}{2y} \frac{dy}{dx}} \end{array}$	$\bullet \quad \begin{array}{l} \frac{dz}{dx} = \frac{1}{2y} \frac{dy}{dx} \\ \text{Diff w.r.t } x \\ \frac{d^2z}{dx^2} = -\frac{1}{2y} \frac{d}{dx} \frac{dy}{dx} + \frac{1}{2y} \frac{d^2y}{dx^2} \\ \boxed{\frac{d^2z}{dx^2} = \frac{1}{2y} \frac{d^2y}{dx^2} - \frac{1}{2y^2} \frac{d}{dx} \frac{dy}{dx}} \end{array}$
$\bullet \quad \begin{aligned} & \frac{d^2y}{dx^2} + \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - 5 \frac{dy}{dx} + 2y = 0 \\ \Rightarrow & \frac{1}{2y} \frac{d^2z}{dx^2} - \frac{1}{2y^2} \frac{d}{dx} \frac{dy}{dx} + \frac{1}{y} \left( \frac{1}{2y} \frac{dy}{dx} \right)^2 - 5 \left( \frac{1}{2y} \frac{dy}{dx} \right) + 2y = 0 \\ \Rightarrow & \frac{1}{2y} \frac{d^2z}{dx^2} - \frac{1}{2y^3} \left( \frac{1}{2y} \frac{dy}{dx} \right) \frac{d}{dx} \frac{dy}{dx} + \frac{1}{4y^2} \left( \frac{dy}{dx} \right)^2 - \frac{5}{2y} \frac{dy}{dx} + 2y = 0 \\ \Rightarrow & \frac{1}{2y} \frac{d^2z}{dx^2} - \frac{1}{4y^4} \left( \frac{dy}{dx} \right)^2 + \frac{1}{4y^2} \left( \frac{dy}{dx} \right)^2 - \frac{5}{2y} \frac{dy}{dx} + 2y = 0 \\ \Rightarrow & \frac{1}{2y} \frac{d^2z}{dx^2} - \frac{5}{2y} \frac{dy}{dx} + 2y = 0 \\ \Rightarrow & \frac{d^2z}{dx^2} - 5 \frac{dz}{dx} + 4z = 0 \\ \Rightarrow & \frac{d^2z}{dx^2} - 5 \frac{dz}{dx} + 4z = 0 \quad \cancel{\text{✓ P.D.F.}} \end{aligned}$	

**Question 14** (\*\*\*\*\*)

Given that if  $x = t^{\frac{1}{2}}$ , where  $y = f(x)$ , show clearly that

a)  $\frac{dy}{dx} = 2t^{\frac{1}{2}} \frac{dy}{dt}$ .

b)  $\frac{d^2y}{dx^2} = 4t \frac{d^2y}{dt^2} + 2 \frac{dy}{dt}$ .

The following differential equation is to be solved

$$x \frac{d^2y}{dx^2} - (8x^2 + 1) \frac{dy}{dx} + 12x^3 y = 12x^5,$$

subject to the boundary conditions  $y = \frac{10}{3}$ ,  $\frac{d^2y}{dx^2} = 10$  at  $x = 0$ .

- c) Show further that the substitution  $x = t^{\frac{1}{2}}$ , where  $y = f(x)$ , transforms the above differential equation into the differential equation

$$\frac{d^2y}{dt^2} - 4 \frac{dy}{dt} + 3y = 3t.$$

- d) Show that a solution of the original differential equation is

$$y = e^{3x^2} + e^{x^2} + x^2 + \frac{4}{3}.$$

proof

The image contains handwritten working for Question 14. It shows the substitution  $x = t^{\frac{1}{2}}$  and the resulting differential equation in terms of  $t$ . The working includes steps for differentiating  $y$  with respect to  $x$  and  $t$ , and transforming the original equation into the form  $\frac{d^2y}{dt^2} - 4 \frac{dy}{dt} + 3y = 3t$ . It also shows the general solution  $y = A e^{3t^2} + B e^{t^2} + t + \frac{4}{3}$  and the application of boundary conditions to find  $A$  and  $B$ .

**Question 15** (\*\*\*\*\*)

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} \cot x + 2y \operatorname{cosec}^2 x = 2 \cos x - 2 \cos^3 x.$$

Use the substitution  $y = z \sin x$ , where  $z$  is a function of  $x$ , to solve the above differential equation subject to the boundary conditions  $y=1$ ,  $\frac{dy}{dx}=0$  at  $x=\frac{\pi}{2}$ .

Give the answer in the form

$$y = a \sin^2 x + b(1 - \sin x) \sin 2x,$$

where  $a$  and  $b$  are constants to be found.

$$\boxed{a=1}, \boxed{b=\frac{1}{3}}$$

The image shows handwritten mathematical work for solving the differential equation. It starts with the auxiliary equation  $z^2 + 1 = 0$  which has roots  $z = i$  and  $z = -i$ . The general solution for  $z$  is given as  $z = A \cos x + B \sin x$ . The boundary condition  $y=1$  at  $x=\frac{\pi}{2}$  leads to  $A = 1$ . The second derivative condition  $\frac{dy}{dx}=0$  at  $x=\frac{\pi}{2}$  leads to  $B = \frac{1}{3}$ . Therefore, the solution for  $y$  is  $y = \sin^2 x + \frac{1}{3}(1 - \sin x) \sin 2x$ .

**Question 16** (\*\*\*\*\*)

$$x \frac{d^2y}{dx^2} - \frac{dy}{dx} - x^3 y + x^5 = 0.$$

Use the substitution  $x = z^{\frac{1}{2}}$ , where  $y = f(x)$ , to find a general solution of the above differential equation.

V,  ,  $y = A e^{\frac{1}{2}x^2} + B e^{-\frac{1}{2}x^2} + x^2$

SIMPLIFY WITH THE SUBSTITUTIONS GIVEN

$\bullet \quad x = z^{\frac{1}{2}}$	$\bullet \quad x = z^{\frac{1}{2}}$
$\frac{dy}{dz} = \frac{1}{2}z^{-\frac{1}{2}} \frac{dy}{dx}$	$\frac{dy}{dz} = \frac{1}{2}z^{-\frac{1}{2}}$
$\frac{d^2y}{dz^2} = \frac{1}{2} \frac{d}{dz} \left( \frac{dy}{dz} \right) = \frac{1}{2}z^{-\frac{3}{2}} \frac{d^2y}{dx^2}$	$\frac{d^2y}{dz^2} = \frac{1}{2}z^{-\frac{3}{2}} \frac{d^2y}{dx^2}$
$\frac{dy}{dx} = 2z^{\frac{1}{2}} \frac{dy}{dz}$	$\frac{dy}{dx} = 2z^{\frac{1}{2}}$
$\frac{d^2y}{dx^2} = \frac{1}{2}z^{-\frac{1}{2}} \frac{d^2y}{dz^2}$	$\frac{d^2y}{dx^2} = \frac{1}{2}z^{-\frac{1}{2}}$

NOW THE SECOND DERIVATIVES

$$\begin{aligned}\frac{d^2y}{dx^2} &= 2z^{\frac{1}{2}} \frac{d^2y}{dz^2} \\ \frac{d^3y}{dx^3} &= z^{\frac{1}{2}} \frac{d}{dz} \left( \frac{d^2y}{dz^2} \right) + 2z^{\frac{1}{2}} \frac{d^2y}{dz^2} \frac{d}{dz} \\ \frac{d^3y}{dz^3} &= \frac{1}{2}z^{-\frac{1}{2}} \left[ \frac{d}{dz} \left( \frac{d^2y}{dz^2} \right) + 2z^{\frac{1}{2}} \frac{d^2y}{dz^2} \right] \\ \frac{d^3y}{dz^3} &= 2z^{\frac{1}{2}} \left[ \frac{1}{2}z^{-\frac{3}{2}} \frac{d^2y}{dz^2} + 2z^{\frac{1}{2}} \frac{d^2y}{dz^2} \right] \\ \frac{d^3y}{dz^3} &= 2 \frac{dy}{dz} + 12z^{\frac{1}{2}} \frac{d^2y}{dz^2}.\end{aligned}$$

NOW SUBSTITUTE INTO THE O.D.E.

$$\begin{aligned}\Rightarrow 2 \frac{d^2y}{dz^2} - \frac{dy}{dz} + x^5 &= 0 \\ \Rightarrow 2^{\frac{1}{2}} \left[ 2 \frac{dy}{dz} + 12z^{\frac{1}{2}} \frac{d^2y}{dz^2} \right] - 2z^{\frac{1}{2}} \frac{dy}{dz} - z^{\frac{3}{2}} + z^{\frac{15}{2}} &= 0 \\ \Rightarrow 2^{\frac{1}{2}} \frac{dy}{dz} + 2^{\frac{1}{2}} \frac{d^2y}{dz^2} - 2z^{\frac{1}{2}} \frac{dy}{dz} - z^{\frac{3}{2}} + z^{\frac{15}{2}} &= 0\end{aligned}$$

$$\begin{aligned}\Rightarrow 4^{\frac{1}{2}} \frac{dy}{dz} - 3z^{\frac{1}{2}} + z^{\frac{15}{2}} &= 0 \\ \Rightarrow 4 \frac{dy}{dz} - 3z + z^7 &= 0 \\ \text{Auxiliary equation for } 4 \frac{dy}{dz} - 3z - z^7 &= 0 \\ 4k^3 - 1 &= 0 \\ k^3 &= \frac{1}{4} \\ k &= \pm \sqrt[3]{\frac{1}{4}} \\ \text{Particular integral (by inspection)} \\ y &= z\end{aligned}$$

GENERAL SOLUTION

$$y = A e^{\frac{1}{2}z^2} + B e^{-\frac{1}{2}z^2} + z$$

*(Note: The term  $z^7$  is omitted from the general solution as it is a particular integral.)*

**Question 17** (\*\*\*\*)

Use a suitable substitution to solve the differential equation

$$x^2 \frac{d^2y}{dx^2} - 6y = 2 - 2\ln x - 6(\ln x)^2,$$

subject to the boundary conditions  $y(1) = 1$ ,  $\frac{dy}{dx}(1) = 3$

Give a simplified answer in the form  $y = f(x)$ .

,  $y = x^3 + (\ln x)^2$

$\frac{d^2y}{dx^2} - 6y = 2 - 2\ln x - 6(\ln x)^2$

LOOKING AT THE R.H.S., WE TRY THE SUBSTITUTION  $t = \ln x$

$t = \ln x \Rightarrow x = e^t$  DEFINITION OF t

$\frac{dt}{dx} = \frac{1}{x}$  DEFINITION OF dx/dt

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$

$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dt} \cdot \frac{1}{x} \right) \cdot \frac{dt}{dx}$  DEFINITION OF d^2y/dx^2

$\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) \cdot \frac{1}{x} + \frac{dy}{dt} \cdot \frac{d}{dt} \left( \frac{1}{x} \right) \cdot \frac{dt}{dx}$

$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} \cdot \frac{1}{x} + \frac{dy}{dt} \cdot \left( -\frac{1}{x^2} \right) \cdot \frac{1}{x}$

SUBSTITUTE INTO THE O.D.E. & TRY

$\Rightarrow \frac{d^2y}{dx^2} - 6y = 2 - 2\ln x - 6(\ln x)^2$

$\Rightarrow \frac{d^2y}{dt^2} \left[ e^{-2t} \frac{dy}{dt} - \frac{6}{x^2} \right] - 6y = 2 - 2t - t^2$

$\Rightarrow \frac{d^2y}{dt^2} \left[ e^{-2t} \frac{dy}{dt} - \frac{6}{e^{2t}} \right] - 6y = 2 - 2t - t^2$  (which can safely be solved)

AUXILIARY EQUATION (L.H.S.)

$\Rightarrow t^2 - 2 - 6 = 0$

$\Rightarrow (t+2)(t-3) = 0$

$\Rightarrow t = -2, 3$

PARTICULAR INTEGRAL (R.H.S.)

- $y = Pt^2 + Qt + R$
- $\frac{dy}{dt} = 2Pt + Q$
- $\frac{d^2y}{dt^2} = 2P$

$\Rightarrow 2P - (2Pt+Q) - 6(Pt^2+Qt+R) = 2 - 2t - t^2$

$\equiv 2 - 2t - t^2$

$\Rightarrow P=1 \quad -2P-Q=-2 \quad 2P-Q-6R=2$

$\Rightarrow -2-Q=-2 \quad 2-Q-6R=2$

$\Rightarrow Q=0 \quad R=0$

GENERAL SOLUTION:  $y = At^2 + Bt^3 + C$

REWRITE IN 2:  $y = At^2 + Bt^3 + (Ct)^2$

DIFFERENTIALS & APPLY CONDITIONS:  $y(1) = 1 \Rightarrow A + B + C = 1$

- $y = \frac{d}{dt} \left( At^2 + Bt^3 + (Ct)^2 \right)$
- $1 = A + B$
- $2 = 2A + 3B$

$\Rightarrow 5B = 5 \Rightarrow B = 1 \quad A = 0$

$\therefore y = x^3 + (\ln x)^2$

**Question 18** (\*\*\*\*)

Use a suitable trigonometric substitution to solve the following differential equation

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 4.$$

$$y = 3x - \cos(\arcsin x)$$

$(1-x^2) \frac{\partial^2 y}{\partial x^2} - x \frac{\partial y}{\partial x} + y = 0 \quad x=0, y=1, \frac{\partial y}{\partial x}=4$

LET  $u = \arcsin x \Rightarrow x = \sin u$

$\frac{\partial y}{\partial x} = \cos u \frac{\partial y}{\partial u}$ $\frac{\partial y}{\partial u} = \frac{1}{\cos u} \frac{\partial y}{\partial x}$ $\frac{\partial^2 y}{\partial x^2} = \sec^2 u \frac{\partial^2 y}{\partial u^2}$	<b>Differentiate w.r.t. <math>x</math></b> $\frac{\partial y}{\partial x} = \sec u \frac{\partial y}{\partial u} + \tan u \frac{\partial^2 y}{\partial u^2}$ $\frac{\partial y}{\partial u^2} = \sec u \frac{\partial}{\partial x} \left[ \frac{\partial y}{\partial u} + \tan u \frac{\partial^2 y}{\partial u^2} \right]$ $\frac{\partial^2 y}{\partial x^2} = \sec^3 u \frac{\partial^2 y}{\partial u^2}$
--	---

SUB INTO THE O.D.E TO OBTAIN

$$(1 - \sin^2 u) \sec^2 u \left[ \frac{\partial^2 y}{\partial u^2} + \tan^2 u \right] - \sin u \sec u \frac{\partial y}{\partial u} + y = 0$$

$$\cos^2 u \sec^2 u \left[ \frac{\partial^2 y}{\partial u^2} + \tan^2 u \right] - \tan u \frac{\partial y}{\partial u} + y = 0$$

$$\cos^2 u \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial u^2} - \tan u \frac{\partial y}{\partial u} + y = 0$$

$$\frac{\partial^2 y}{\partial u^2} + y = 0$$

THIS IS A "SIMPLE HARMONIC MOTION" O.D.E WITH GENERAL SOLUTION

$$y = A \cos u + B \sin u$$

$$y = A \cos(\arcsin x) + B \sin(\arcsin x)$$

$$y = A \cos(\arcsin x) + Bx$$

**Question 19** (\*\*\*\*\*)

$$4x \frac{d^2y}{dx^2} + 4x \left( \frac{dy}{dx} \right)^2 + 2 \frac{dy}{dx} = 1.$$

By using the substitution  $t = \sqrt{x}$ , or otherwise, show that the general solution of the above differential equation is

$$y = A - \sqrt{x} + \ln \left[ 1 + B e^{2\sqrt{x}} \right],$$

where  $A$  and  $B$  are arbitrary constants.

[ ] , [ ] proof

$$\begin{aligned} & 4x \frac{d^2y}{dx^2} + 4x \left( \frac{dy}{dx} \right)^2 + 2 \frac{dy}{dx} = 1 \\ \text{LET } & t = \sqrt{x} \quad \text{DIFFERENTIATE } \frac{dt}{dx} = \frac{1}{2\sqrt{x}} \quad \frac{dt}{dx} = \frac{1}{2t} \\ & \frac{dy}{dx} = 2t \frac{dy}{dt} \quad \frac{d^2y}{dx^2} = 2 \frac{dy}{dt} \frac{dt}{dx} = 2 \frac{dy}{dt} \cdot \frac{1}{2t} = \frac{dy}{dt} \\ & \text{DIFFERENTIATE } \frac{dy}{dt} \text{ WITH RESPECT TO } t \\ & \frac{d^2y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = -\frac{1}{2t^2} \frac{dy}{dt} + \frac{1}{2} \frac{d^2y}{dt^2} \\ & \frac{d^2y}{dt^2} = -\frac{1}{2t^2} \frac{dy}{dt} + \frac{1}{2} \frac{d^2y}{dt^2} \\ & \frac{d^2y}{dt^2} = -\frac{1}{2t^2} \frac{dy}{dt} + \frac{1}{2} \frac{d^2y}{dt^2} \\ & \frac{d^2y}{dt^2} = -\frac{1}{2t^2} \left( \frac{dy}{dt} \right) + \frac{1}{2} \frac{d^2y}{dt^2} \\ & \frac{d^2y}{dt^2} = \frac{1}{4t^2} \frac{dy}{dt} - \frac{1}{2t^2} \frac{d^2y}{dt^2} \\ \text{SUBSTITUTION INTO THE O.D.E AND SIMPLIFY} \\ \Rightarrow & 4t \left[ \frac{1}{4t^2} \frac{dy}{dt} - \frac{1}{2t^2} \frac{d^2y}{dt^2} \right] + 4t \left( \frac{dy}{dt} \right)^2 + 2 \left( \frac{dy}{dt} \right) = 1 \\ \Rightarrow & \frac{dy}{dt} - \frac{1}{t} \frac{d^2y}{dt^2} + \left( \frac{dy}{dt} \right)^2 + \frac{1}{t} \frac{dy}{dt} = 1 \end{aligned}$$

$$\begin{aligned} & \Rightarrow \frac{\frac{dy}{dt}}{dt} + \left( \frac{dy}{dt} \right)^2 = 1 \\ \rightarrow & \text{ANOTHER CRUCIAL SUBSTITUTION IS } \Rightarrow -\frac{1}{dt} = \frac{dy}{dt} \\ & \Rightarrow \frac{dt}{dt} + P^2 = 1 \\ & \Rightarrow \frac{dt}{dt} = 1 - P^2 \\ & \Rightarrow \frac{1}{1-P^2} dt = 1 dt \\ & \Rightarrow \int \frac{1}{1-P^2} dt = \int 1 dt \\ & \Rightarrow \int \frac{1}{1-\frac{dy}{dt}^2} dt = \int 1 dt \\ & \Rightarrow \int \frac{1}{1-\frac{dy}{dt}^2} dt = \int 1 dt \\ & \Rightarrow \int \frac{1}{1-\frac{dy}{dt}^2} dt = \int 1 dt \\ & \Rightarrow \ln|1-\frac{dy}{dt}| = \ln|1-y| = -dt + C \\ & \Rightarrow \ln|\frac{dy}{dt}| = dt - C \\ & \Rightarrow \frac{dy}{dt} = Ae^{dt-C} \\ & \Rightarrow dy = Ae^{dt-C} dt \\ & \Rightarrow dy = A e^{dt-C} dt \\ & \Rightarrow dy = A e^{dt} dt \\ & \Rightarrow dy = A e^{dt} dt \quad |C = \ln A \\ & \Rightarrow \frac{dy}{dt} = A e^{dt} \end{aligned}$$

$$\begin{aligned} & \text{PROVED BY INTEGRATION USING A SUBSTITUTION} \\ \rightarrow & y = \int \frac{A e^{dt}}{A e^{dt} + 1} dt \\ \rightarrow & y = \int \frac{(A-1)+1}{A+e^{-dt}} dt = \int \frac{A-1}{A+e^{-dt}} dt + \int \frac{1}{A+e^{-dt}} dt \\ \rightarrow & y = \frac{1}{A} \int \frac{V-2}{V} dv \quad V = A e^{dt} + 1 \\ & \frac{dv}{dt} = 2A e^{dt} \quad dt = \frac{dv}{2A e^{dt}} \\ & \frac{dt}{dt} = \frac{dv}{2A e^{dt}} \\ & \frac{dt}{dt} = \frac{dv}{2(A+e^{-dt})} \\ & \text{PARTIAL FRACTIONS BY INSPECTION} \\ \rightarrow & y = \frac{1}{A} \left[ \frac{2}{V} - \frac{1}{V-1} dv \right] + B \\ \rightarrow & y = \frac{1}{A} \left[ 2 \ln|V| - \ln|V-1| \right] + B \\ \rightarrow & y = \ln|A e^{dt} + 1| - \frac{1}{A} \ln|A e^{dt} + 1|^2 + B \\ \rightarrow & y = \ln|A e^{dt} + 1| - \frac{1}{A} \ln|A e^{dt} + 1|^2 + B \\ \rightarrow & y = \ln|A e^{dt} + 1| - \frac{1}{A} \ln|A e^{dt} + 1|^2 + B \\ \rightarrow & y = \ln|A e^{dt} + 1| - \frac{1}{A} \ln|A e^{dt} + 1|^2 + B \end{aligned}$$

# VARIOUS TYPES

**Question 1** (\*\*+)

Find the general solution of the following differential equation.

$$\frac{d^4\psi}{dx^4} + 2\lambda \frac{d^2\psi}{dx^2} + \lambda^4 \psi = 0.$$

$$\boxed{\psi = A \cos \lambda x + B \sin \lambda x}$$

Characteristic equation:  
 $\frac{d^4\psi}{dx^4} + 2\lambda \frac{d^2\psi}{dx^2} + \lambda^4 \psi = 0$   
 $\lambda^4 + 2\lambda^2 + 1^4 = 0$   
 $(\lambda^2 + 1^2)^2 = 0$   
 $\lambda^2 + 1^2 = 0$   
 $\lambda^2 = -1^2$   
 $\lambda = \pm i$

$\therefore \psi = A \cos \lambda x + B \sin \lambda x$

**Question 2** (\*\*\*)

Solve the differential equation

$$\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 1,$$

given that  $y = -\frac{1}{4}$  and  $\frac{dy}{dx} = 1$  at  $x = 0$ , giving the answer in the form  $y = f(x)$ .

$$\boxed{y = \frac{1}{2} [2x - e^{-2x}]}$$

Integrate again:  
 $y = \frac{1}{2}x - \frac{1}{4}e^{-2x} + D$

Apply condition:  
 $x=0, y=-\frac{1}{4}$   
 $-\frac{1}{4} = -\frac{1}{4} + D$   
 $D=0$

$\therefore y = \frac{1}{2}x - \frac{1}{4}e^{-2x}$   
 $y = \frac{1}{2}[2x - e^{-2x}]$

Integrating factor:  
 $\frac{dp}{dx} + 2p = 1$   
 $p = \frac{1}{2}x + \frac{1}{2}e^{-2x}$   
 $\frac{dp}{dx} = 1 - 2p$   
 $\frac{dp}{dx} = 1 - 2(\frac{1}{2}x + \frac{1}{2}e^{-2x})$   
 $\int \frac{-2}{1-2p} dp = \int -2 dx$   
 $\ln|1-2p| = -2x + C$   
 $1-2p = e^{-2x+C}$   
 $1-2p = Ae^{-2x}$  (A=e<sup>C</sup>)  
 $1+Ae^{-2x} = 2p$   
 $p = \frac{1}{2} + Be^{-2x}$   
 $\frac{dy}{dx} = \frac{1}{2} + Be^{-2x}$

Apply condition:  
 $x=0, \frac{dy}{dx} = 1$   
 $1 = \frac{1}{2} + B \Rightarrow B = \frac{1}{2}$   
 $\frac{dy}{dx} = \frac{1}{2} + \frac{1}{2}e^{-2x}$

**Question 3** (\*\*\*)

Solve the differential equation

$$\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 = 1,$$

given that  $y=0$  and  $\frac{dy}{dx}=\frac{1}{6}$  at  $x=0$ , giving the answer in the form  $y=f(x)$ .

$$y = \frac{1}{4} \ln \left[ \frac{1+2e^{4x}}{3} \right] - \frac{1}{2}x$$

**QUESTION**

$$\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 = 1$$

LET  $P = \frac{dy}{dx}$

$$\frac{dp}{dx} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{dp}{dx} + 4P^2 = 1$$

$$\Rightarrow \frac{dp}{dx} = 1 - 4P^2$$

$$\Rightarrow \frac{1}{1-4P^2} dP = 1 dx$$

$$\Rightarrow \frac{1}{(1-2P)(1+2P)} dP = 1 dx$$

**BY PARTIAL FRACTIONS**

$$\frac{1}{1-2P} + \frac{1}{1+2P} dP = 1 dx$$

$$\Rightarrow \int \frac{1}{1-2P} + \frac{1}{1+2P} dP = \int 1 dx$$

$$\Rightarrow \frac{1}{2} \ln|1+2P| - \frac{1}{2} \ln|1-2P| = 2x + C$$

$$\Rightarrow \frac{1}{2} \ln \left| \frac{1+2P}{1-2P} \right| = 2x + C$$

$$\Rightarrow \frac{1+2P}{1-2P} = e^{4x+C}$$

$$\Rightarrow \frac{1+2P}{1-2P} = Ae^{4x}$$

**ANSWER**

$$\Rightarrow 1+2P = ((-2P)A)e^{-4x}$$

$$\Rightarrow 1+2P = Ae^{-4x} - 2Pe^{-4x}$$

$$\Rightarrow 2P + 2Pe^{-4x} = Ae^{-4x} - 1$$

$$\Rightarrow 2P(1+e^{-4x}) = Ae^{-4x} - 1$$

$$\Rightarrow P = \frac{1}{2} \left( \frac{Ae^{-4x}-1}{Ae^{-4x}+1} \right)$$

$$\Rightarrow \frac{dp}{dx} = \frac{1}{2} \left( \frac{Ae^{-4x}-1}{Ae^{-4x}+1} \right)$$

**APPLY CONDITION**

$$2x = \frac{d^2y}{dx^2} + \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{2} \left( \frac{A-1}{A+1} \right)$$

$$\frac{1}{3} = \frac{A-1}{A+1}$$

$$A+1 = 3A-3$$

$$4 = 2A$$

$$A = 2$$

$$\Rightarrow \frac{dp}{dx} = \frac{1}{2} \left( \frac{2e^{-8x}-1}{2e^{-8x}+1} \right)$$

$$\Rightarrow y = \frac{1}{2} \int \frac{2e^{-8x}}{2e^{-8x}+1} dx$$

**BY SUBSTITUTION**

$$u = 2e^{-8x} \Rightarrow \frac{du}{dx} = -16e^{-8x}$$

$$du = -16e^{-8x} dx$$

$$dx = \frac{du}{-16e^{-8x}}$$

$$dx = \frac{du}{4e^{-8x}}$$

**ANSWER**

$$y = \frac{1}{2} \int \frac{(u-1)}{u} \times \frac{1}{4(u-1)} du$$

$$y = \frac{1}{8} \int \frac{u-2}{u(u-1)} du$$

**BY PARTIAL FRACTION DECOMPOSITION**

$$\Rightarrow y = \frac{1}{8} \int \frac{2}{u} - \frac{1}{u-1} du$$

$$\Rightarrow y = \frac{1}{8} \left[ 2 \ln u - \ln(u-1) \right] + D$$

$$\Rightarrow y = \frac{1}{8} \ln \left| \frac{u^2}{u-1} \right| + D$$

$$\Rightarrow y = \frac{1}{8} \ln \left| \frac{(2e^{-8x})^2}{2e^{-8x}-1} \right| + D$$

$$\Rightarrow y = \frac{1}{8} \ln \left| \frac{(2e^{-8x}+1)^2}{e^{-8x}} \right| + E$$

$$\Rightarrow y = \frac{1}{4} \ln(2e^{-8x}+1) - \frac{1}{4} \ln(e^{-8x}) + E$$

$$\Rightarrow y = \frac{1}{4} \ln(2e^{-8x}+1) - \frac{1}{4} \ln e^{-8x} + E$$

APPLY CONDITION  $x=0, y=0$

$$\Rightarrow 0 = \frac{1}{4} \ln 3 + E$$

$$E = -\frac{1}{4} \ln 3$$

$$\therefore y = \frac{1}{4} \ln(2e^{-8x}+1) - \frac{1}{4} \ln 3 - \frac{1}{4} \ln e^{-8x}$$

$$y = \frac{1}{4} \ln \left( \frac{2e^{-8x}+1}{3} \right) - \frac{1}{4} x$$

**Question 4** (\*\*\*)

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1.$$

Given that  $y = \frac{dy}{dx} = 0$  at  $x=0$ , show that

$$y = -x + \ln\left[\frac{1}{2}(1+e^{2x})\right].$$

proof

**REDUCE THE O.D.E. INTO A FIRST ORDER AS FRACTION**

$$\rightarrow \frac{1}{dx} \left[ \frac{dy}{dx} \right] + \left[ \frac{dy}{dx} \right]^2 = 1 \quad \text{let } p = \frac{dy}{dx}$$

$$\rightarrow \frac{dp}{dx} + p^2 = 1$$

$$\rightarrow \frac{dp}{dx} = 1 - p^2$$

**SEPARATE VARIABLES & INTEGRATE SUBJECT TO  $x=0$ :  $\frac{dp}{dx} = p = 0$**

$$\rightarrow \frac{1}{1-p^2} dp = 1 dx$$

$$\rightarrow \int_0^p \frac{1}{1-p^2} dp = \int_0^x 1 dx$$

$$\rightarrow \int_0^p \frac{1}{(1+p)(1-p)} dp = \int_0^x 1 dx$$

**PARTIAL FRACTION BY INSPECTOR (Leave up) ON THE LHS**

$$\rightarrow \int_0^p \frac{\frac{1}{2}}{1+p} + \frac{\frac{1}{2}}{1-p} dp = \int_0^x 1 dx$$

$$\rightarrow \int_0^p \frac{\frac{1}{2}}{1+p} + \frac{\frac{1}{2}}{1-p} dp = \int_0^x 2 dx$$

$$\rightarrow \left[ \ln|1+p| + \ln|1-p| \right]_0^p = [2x]_0^x$$

$$\rightarrow \left[ \ln(1+p) + \ln(1-p) \right] - \left[ \ln 1 + \ln 1 \right] = 2x$$

$\rightarrow \ln \left[ \frac{1+p}{1-p} \right] = 2x$

$$\rightarrow \frac{1+p}{1-p} = e^{2x}$$

$$\rightarrow 1+p = e^{2x} - pe^{2x}$$

$$\rightarrow pe^{2x} + p = e^{2x} - 1$$

$$\rightarrow p(e^{2x} + 1) = e^{2x} - 1$$

$$\rightarrow p = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\rightarrow \frac{dy}{dx} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

**SUBSTITUTION**

- $u = e^{2x} + 1$
- $e^{2x} = u-1$
- $du = 2e^{2x} dx$
- $dx = \frac{du}{2e^{2x}}$

$$\therefore \text{PARTIAL FRACTIONS}$$

$$\rightarrow 2y = \int_2^{e^{2x}+1} \frac{u-2}{u(u-1)} du$$

$$\rightarrow 2y = \int_2^{e^{2x}+1} \frac{2}{u-1} - \frac{1}{u} du$$

$$\rightarrow 2y = \left[ 2\ln|u-1| - \ln u \right]_2^{e^{2x}+1}$$

**VARIATION USING HYPERBOLIC FUNCTIONS FROM THIS EXPRESSION onwards**

$$\rightarrow 2y = \left[ 2\ln(e^{2x}+1) - \ln(e^{2x}) \right] - \left[ 2\ln 2 - \ln 1 \right]$$

$$\rightarrow 2y = 2\ln(e^{2x}+1) - 2x - 2\ln 2$$

$$\rightarrow y = \ln(e^{2x}+1) - \ln 2 - x$$

$$\rightarrow y = \ln\left[\frac{e^{2x}+1}{2e^x}\right] - x$$

**WHICH MEANS SINCE**

$$\Rightarrow y = \ln\left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-x}\right] = \ln\left[e^{2x}\left(\frac{1}{2}e^{2x} + \frac{1}{2}\right)\right]$$

$$= \ln e^{2x} + \ln\left[\frac{1}{2}e^{2x} + \frac{1}{2}\right] = -x + \ln\left[\frac{1}{2}(e^{2x}+1)\right]$$

**Question 5    (\*\*\*)+**

The function with equation  $y = f(x)$  satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{2}{2x-1} \left( 1 - \frac{dy}{dx} \right), \quad y(0) = 1, \quad \frac{dy}{dx}(0) = -1.$$

Solve the above differential equation giving the answer in the form  $y = f(x)$ .

$$y = x + 1 + \ln|2x-1|$$

• Start by rearranging the O.D.E as follows

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2}{2x-1} \left( 1 - \frac{dy}{dx} \right)$$

$$\Rightarrow (2x-1) \frac{d^2y}{dx^2} = 2 \left( 1 - \frac{dy}{dx} \right)$$

$$\Rightarrow (2x-1) \frac{d^2y}{dx^2} = 2 - 2 \frac{dy}{dx}$$

$$\Rightarrow (2x-1) \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 2$$

• By inspection the L.H.S is a perfect differential

$$\Rightarrow \frac{d}{dx} \left[ (2x-1) \frac{dy}{dx} \right] = 2$$

$$\Rightarrow (2x-1) \frac{dy}{dx} = 2x + A$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x+A}{2x-1}$$

• Now  $\frac{dy}{dx} = -1$  AT  $x=0$ , gives  $A=-1$

$$\Rightarrow \frac{dy}{dx} = \frac{2x+1}{2x-1}$$

$$\Rightarrow y = \int \frac{2x+1}{2x-1} dx$$

$$\Rightarrow y = \int \frac{(2x-1)+2}{2x-1} dx$$

$$\Rightarrow y = \int 1 + \frac{2}{2x-1} dx$$

$$\Rightarrow y = x + \ln|2x-1| + B$$

NOW IF  $x=0, y=1 \Rightarrow B=1$

$$\therefore y = x + \ln|2x-1| + 1$$

**Question 6** (\*\*\*\*\*)

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2)y = 0.$$

The above differential equation is known as modified Bessel's Equation.

Use the Frobenius method to show that the general solution of this differential equation, for  $n = \frac{1}{2}$ , is

$$y = x^{-\frac{1}{2}} [A \cosh x + B \sinh x].$$

proof

If  $p = -\frac{1}{2}$

$$\begin{aligned} & a_1 \left[ x^{\frac{1}{2}} p + p^2 + \frac{1}{4} \right] = 0 \\ & a_1 \left[ \frac{1}{2} + 1 + \frac{1}{4} \right] = 0 \\ & a_1 x = 0 \end{aligned}$$

∴  $a_1$  IS UNDETERMINED

ASSUME A SOLUTION OF THE FORM  $y = \sum_{n=0}^{\infty} a_n x^{n+p}$ ,  $a_0 \neq 0$

$$\begin{aligned} \frac{dy}{dx} &= \sum_{n=0}^{\infty} a_n (n+p) (n+p-1) x^{n+p-1} \\ \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} a_n (n+p)(n+p-1)(n+p-2) x^{n+p-2} \end{aligned}$$

Substitute into the O.D.E.

$$\begin{aligned} & \sum_{n=0}^{\infty} a_n (n+p)(n+p-1)x^{n+p} + \sum_{n=0}^{\infty} a_n (n+p)x^{n+p} - \sum_{n=0}^{\infty} a_n x^{n+p+2} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{n+p} = 0 \\ & \text{With } p = -\frac{1}{2} \text{ THE LOWEST POWER OF } x \text{ IS } x^{\frac{1}{2}}, \text{ AND THE HIGHEST } x^{n+p}. \\ & \text{pull } x^{\frac{1}{2}} \text{ AND } x^{n+p} \text{ OUT OF THE SUMMATIONS} \end{aligned}$$

[Initial Equation]

$$\begin{aligned} & \left[ a_0 x^{-\frac{1}{2}} (p(p-1)) x^p + \left[ a_0 (pn+p) + a_1 (n+1) - \frac{1}{4} a_0 \right] x^{p+\frac{1}{2}} \right. \\ & \quad \left. + \sum_{n=2}^{\infty} a_n (n+p)(n+p-1)x^{n+p} + \sum_{n=2}^{\infty} a_n (n+p)x^{n+p} - \sum_{n=2}^{\infty} a_n x^{n+p+2} - \frac{1}{4} \sum_{n=2}^{\infty} a_n x^{n+p} \right] = 0 \end{aligned}$$

Initial Equation,  $a_0 \neq 0$

$$p(p-1) + p - \frac{1}{4} = 0$$

$$p^2 - \frac{1}{4} = 0$$

$$p = \pm \frac{1}{2}$$

Two distinct solutions differing by an integer

CHECK THE NEXT VALUE OF THE UNDETERMINED COEFFICIENTS

$$\begin{aligned} & \left[ (p+1)p + (p+1) - \frac{1}{4} \right] a_1 = 0 \\ & \left[ p^2 + p + 1 - \frac{1}{4} \right] a_1 = 0 \\ & \left[ p^2 + 2p + \frac{5}{4} \right] a_1 = 0 \end{aligned}$$

∴  $a_1 = 0$

ADJUST THE SUMMATIONS SO THEY ALL START FROM  $n=0$

$$\begin{aligned} & \sum_{n=0}^{\infty} a_{n+2} (n+p)(n+p-1)x^{n+p} + \sum_{n=0}^{\infty} a_n (n+p)x^{n+p} - \sum_{n=0}^{\infty} a_n x^{n+p+2} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{n+p} = 0 \\ & a_{n+2} \left[ (n+p)(n+p-1)x^{n+p} + (n+p)x^{n+p} - \frac{1}{4} x^{n+p+2} \right] = a_p \\ & a_{n+2} \left[ 4(n+p)(n+p-1)x^{n+p} + 4(n+p)x^{n+p} - 1 \right] = 4a_p \\ & a_{n+2} = \frac{4a_p}{4(n+p)(n+p-1)} \end{aligned}$$

Now the above solution will be simplified from  $p = -\frac{1}{2}$ .  
( $p = \pm \frac{1}{2}$  ALREADY PROVIDED PART OF THE SOLUTION)

ADJUST THE SUMMATIONS SO THEY ALL START FROM  $n=0$

$$\begin{aligned} & a_0 = 0 \quad a_0 = \frac{a_0}{x^{\frac{1}{2}}} \\ & a_1 = 0 \quad a_1 = \frac{a_1}{x^{\frac{3}{2}}} \\ & a_2 = 0 \quad a_2 = \frac{a_2}{x^{\frac{5}{2}}} = \frac{a_2}{16x^{\frac{1}{2}}x^{\frac{3}{2}}} \\ & a_3 = 0 \quad a_3 = \frac{a_3}{x^{\frac{7}{2}}} = \frac{a_3}{142x^{\frac{3}{2}}x^{\frac{5}{2}}} \\ & a_4 = 0 \quad a_4 = \frac{a_4}{x^{\frac{9}{2}}} = \frac{a_4}{142x^{\frac{5}{2}}x^{\frac{7}{2}}} \\ & a_5 = 0 \quad a_5 = \frac{a_5}{x^{\frac{11}{2}}} = \frac{a_5}{142x^{\frac{7}{2}}x^{\frac{9}{2}}} \\ & a_6 = 0 \quad a_6 = \frac{a_6}{x^{\frac{13}{2}}} = \frac{a_6}{142x^{\frac{9}{2}}x^{\frac{11}{2}}} \quad \text{etc.} \\ & \text{etc.} \end{aligned}$$

Therefore

$$\begin{aligned} & a_0 = \frac{1}{2} \left[ a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + \dots \right] \\ & = a_0^2 + 16a_0^2 + 15 \\ & = (2a_0)^2 + 15 \\ & = \left[ 2(x+\frac{1}{2}) \right] \left[ 2(x+\frac{1}{2}) + 15 \right] \\ & = \left[ 2x + 1 \right] \left[ 2x + 17 \right] \\ & \text{AT } p = -\frac{1}{2} \\ & = \left[ 2x - 1 + \frac{1}{2} \right] \left[ 2x - 1 + \frac{1}{2} + 15 \right] \\ & = (2x-1)(2x+15) \\ & = 4(x^2-1)(2x+15) \\ & = 4(2x^3+15x^2-4x-15) \\ & \therefore a_{n+2} = \frac{4a_p}{4(n+p)(n+p-1)} \end{aligned}$$

$$\begin{aligned} & a_0 = \frac{a_0}{x^{\frac{1}{2}}} \\ & a_1 = \frac{a_1}{x^{\frac{3}{2}}} \\ & a_2 = \frac{a_2}{x^{\frac{5}{2}}} = \frac{a_2}{16x^{\frac{1}{2}}x^{\frac{3}{2}}} \\ & a_3 = \frac{a_3}{x^{\frac{7}{2}}} = \frac{a_3}{142x^{\frac{3}{2}}x^{\frac{5}{2}}} \\ & a_4 = \frac{a_4}{x^{\frac{9}{2}}} = \frac{a_4}{142x^{\frac{5}{2}}x^{\frac{7}{2}}} \\ & a_5 = \frac{a_5}{x^{\frac{11}{2}}} = \frac{a_5}{142x^{\frac{7}{2}}x^{\frac{9}{2}}} \\ & a_6 = \frac{a_6}{x^{\frac{13}{2}}} = \frac{a_6}{142x^{\frac{9}{2}}x^{\frac{11}{2}}} \quad \text{etc.} \\ & \text{etc.} \end{aligned}$$

Therefore

$$\begin{aligned} & a_0 = \frac{1}{2} \left[ a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + \dots \right] \\ & = a_0^2 + 16a_0^2 + 15 \\ & = (2a_0)^2 + 15 \\ & = \left[ 2(x+\frac{1}{2}) \right] \left[ 2(x+\frac{1}{2}) + 15 \right] \\ & = \left[ 2x + 1 \right] \left[ 2x + 17 \right] \\ & \text{AT } p = -\frac{1}{2} \\ & = \left[ 2x - 1 + \frac{1}{2} \right] \left[ 2x - 1 + \frac{1}{2} + 15 \right] \\ & = (2x-1)(2x+15) \\ & = 4(x^2-1)(2x+15) \\ & = 4(2x^3+15x^2-4x-15) \\ & \therefore a_{n+2} = \frac{4a_p}{4(n+p)(n+p-1)} \end{aligned}$$

$y = \frac{1}{2} \cosh x + \frac{1}{2} \sinh x$

**Question 7 (\*\*\*\*\*)**

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$4x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + (3 - 4x^2)y = 0.$$

Give the final answer in terms of elementary functions.

$$y = \sqrt{x} (A \cosh x + B \sinh x)$$

ANSWER A: SERIES SOLUTION OF THE EQUATION

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0, \quad n \in \mathbb{N}$$

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

SUB. INTO THE O.D.E

$$\sum_{n=0}^{\infty} 4a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} 4a_n n x^{n-1} + \sum_{n=0}^{\infty} 3a_n x^{n-2} - \sum_{n=0}^{\infty} 4a_n x^{n-2} = 0$$

WEILNG. INTO THE LOWEST POWER OF  $x$  IS  $x^0$  & THE HIGHEST  $x^{n-2}$ .  
PULL  $x^0$ ,  $x^{n-1}$  OUT OF THE SUMMATIONS

$$(4a_1 x^0) - 4a_2 x^0 + 3a_0 x^0 + [(4a_2 x^0) - 4a_3 x^0] x^1 + [3a_1 x^0] x^2 + \dots + [4a_n x^0] x^{n-2} - \sum_{n=2}^{\infty} 4a_n n x^{n-2} + \sum_{n=2}^{\infty} 3a_n x^{n-2} - \sum_{n=2}^{\infty} 4a_n x^{n-2} = 0$$

WEILNG. QUOTED (Lower Power)

$$[4a_1 x^0] - 4a_2 x^0 + 3a_0 x^0 = a_0 \neq 0$$

$$4p^2 - 4p + 3 = 0$$

$$4p^2 - 4p + 3 = 0$$

$$(2p-3)(2p+1) = 0$$

$$p = \begin{cases} \frac{3}{2} \\ -\frac{1}{2} \end{cases}$$

TWO DISTINCT ROOTS NOT DIFFERENT BY AN INTEG.

ANALYZE THE SUMMATIONS SO THAT ALL START FROM  $n=0$

$$\sum_{n=0}^{\infty} 4a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} 4a_n n(n-1)x^{n-1} + \sum_{n=0}^{\infty} 3a_n x^{n-2} - \sum_{n=0}^{\infty} 4a_n x^{n-2} = 0$$

HENCE COMPUTING POWERS

$$\frac{4(4a_1 x^0)(x^0+1)}{4a_0} - 4(x^0+1) + \frac{3a_0 x^0}{x^0} - 4a_0 x^0 = 0$$

$$a_{012} = \frac{4a_0}{4(x^0+1)(x^0+1)+3}$$

TRY & GET  $\rightarrow$   $x^0 = k$

$$4(k(k+1))x^0 - 4(k+2)x^0 + 3 = 4k^2 + 12k + 6 - 4k - 8 + 3 = 4k^2 + 8k + 3 = (2k+3)(2k+1)$$

$$\therefore a_{012} = \frac{4a_0}{2(k+1)(k+1)}$$

$$a_{012} = \frac{4a_0}{2(1+1)(1+1)}$$

KNOW IF  $p = \frac{1}{2}$

$$a_{012} = \frac{4a_0}{(2(1+\frac{1}{2})(2+\frac{1}{2}))}$$

$$a_{012} = \frac{4a_0}{(2+\frac{1}{2})(2+\frac{1}{2})}$$

IF  $p = \frac{1}{2} \rightarrow -4a_0 = 0 \rightarrow a_0 = 0$

IF  $p = -\frac{1}{2} \rightarrow -4a_0 = 0 \rightarrow a_0 = 0$

HENCE

$$y_1 = x^{\frac{1}{2}} [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots]$$

$$y_1 = x^{\frac{1}{2}} [a_0 + \frac{a_1}{2!} x^2 + \frac{a_2}{4!} x^4 + \frac{a_3}{6!} x^6 + \dots]$$

$$y_1 = x^{\frac{1}{2}} \left[ 1 + \frac{a_1}{2!} + \frac{a_2}{4!} + \frac{a_3}{6!} \right]$$

$$y_1 = A \sqrt{x} \cosh x$$

IF  $p = \frac{3}{2}$

$$a_{012} = \frac{4a_0}{(2(1+\frac{3}{2})(2+\frac{3}{2}))}$$

$$a_{012} = \frac{4a_0}{(2+\frac{3}{2})(2+\frac{3}{2})}$$

IF:  $a_0 = \frac{a_0}{2k+2} = 0$

IF:  $a_0 = \frac{a_0}{4k+3} = 0$

IF:  $a_0 = \frac{a_0}{2k+1} = 0$

IF:  $a_0 = \frac{a_0}{7k+5} = 0$

IF:  $a_0 = \frac{a_0}{12k+9} = 0$

IF:  $a_0 = \frac{a_0}{20k+13} = 0$

IF:  $a_0 = \frac{a_0}{30k+25} = 0$

IF:  $a_0 = \frac{a_0}{42k+37} = 0$

IF:  $a_0 = \frac{a_0}{56k+49} = 0$

IF:  $a_0 = \frac{a_0}{72k+61} = 0$

IF:  $a_0 = \frac{a_0}{90k+73} = 0$

IF:  $a_0 = \frac{a_0}{112k+85} = 0$

IF:  $a_0 = \frac{a_0}{132k+97} = 0$

IF:  $a_0 = \frac{a_0}{156k+109} = 0$

IF:  $a_0 = \frac{a_0}{182k+121} = 0$

IF:  $a_0 = \frac{a_0}{210k+133} = 0$

IF:  $a_0 = \frac{a_0}{240k+145} = 0$

IF:  $a_0 = \frac{a_0}{272k+157} = 0$

IF:  $a_0 = \frac{a_0}{306k+169} = 0$

IF:  $a_0 = \frac{a_0}{342k+181} = 0$

IF:  $a_0 = \frac{a_0}{380k+193} = 0$

IF:  $a_0 = \frac{a_0}{420k+205} = 0$

IF:  $a_0 = \frac{a_0}{462k+217} = 0$

IF:  $a_0 = \frac{a_0}{506k+229} = 0$

IF:  $a_0 = \frac{a_0}{552k+241} = 0$

IF:  $a_0 = \frac{a_0}{600k+253} = 0$

IF:  $a_0 = \frac{a_0}{650k+265} = 0$

IF:  $a_0 = \frac{a_0}{702k+277} = 0$

IF:  $a_0 = \frac{a_0}{756k+289} = 0$

IF:  $a_0 = \frac{a_0}{812k+301} = 0$

IF:  $a_0 = \frac{a_0}{870k+313} = 0$

IF:  $a_0 = \frac{a_0}{930k+325} = 0$

IF:  $a_0 = \frac{a_0}{992k+337} = 0$

IF:  $a_0 = \frac{a_0}{1056k+349} = 0$

IF:  $a_0 = \frac{a_0}{1120k+361} = 0$

IF:  $a_0 = \frac{a_0}{1184k+373} = 0$

IF:  $a_0 = \frac{a_0}{1256k+385} = 0$

IF:  $a_0 = \frac{a_0}{1336k+397} = 0$

IF:  $a_0 = \frac{a_0}{1424k+409} = 0$

IF:  $a_0 = \frac{a_0}{1516k+421} = 0$

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IF:  $a_0 = \frac{a_0}{1712k+445} = 0$

IF:  $a_0 = \frac{a_0}{1816k+457} = 0$

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IF:  $a_0 = \frac{a_0}{2152k+493} = 0$

IF:  $a_0 = \frac{a_0}{2272k+505} = 0$

IF:  $a_0 = \frac{a_0}{2396k+517} = 0$

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IF:  $a_0 = \frac{a_0}{3076k+577} = 0$

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IF:  $a_0 = \frac{a_0}{6352k+793} = 0$

IF:  $a_0 = \frac{a_0}{6572k+805} = 0$

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IF:  $a_0 = \frac{a_0}{37760k+1933} = 0$

IF:  $a_0 = \frac{a_0}{38120k+1945} = 0$

IF:  $a_0 = \frac{a_0}{38480k+1957} = 0$

IF:  $a_0 = \frac{a_0}{38840k+1969} = 0$

IF:  $a_0 = \frac{a_0}{39200k+1981} = 0$

IF:  $a_0 = \frac{a_0}{39560k+1993} = 0$

IF:  $a_0 = \frac{a_0}{39920k+2005} = 0$

IF:  $a_0 = \frac{a_0}{40280k+2017} = 0$

IF:  $a_0 = \frac{a_0}{40640k+2029} = 0$

IF:  $a_0 = \frac{a_0}{41000k+2041} = 0$

IF:  $a_0 = \frac{a_0}{41360k+2053} = 0$

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IF:  $a_0 = \frac{a_0}{42440k+2089} = 0$

IF:  $a_0 = \frac{a_0}{42800k+2101} = 0$

IF:  $a_0 = \frac{a_0}{43160k+2113} = 0$

IF:  $a_0 = \frac{a_0}{43520k+2125} = 0$

IF:  $a_0 = \frac{a_0}{43880k+2137} = 0$

IF:  $a_0 = \frac{a_0}{44240k+2149} = 0$

IF:  $a_0 = \frac{a_0}{44600k+2161} = 0$

IF:  $a_0 = \frac{a_0}{44960k+2173} = 0$

IF:  $a_0 = \frac{a_0}{45320k+2185} = 0$

IF:  $a_0 = \frac{a_0}{45680k+2197} = 0$

IF:  $a_0 = \frac{a_0}{46040k+2209} = 0$

IF:  $a_0 = \frac{a_0}{46400k+2221} = 0$

IF:  $a_0 = \frac{a_0}{46760k+2233} = 0$

IF:  $a_0 = \frac{a_0}{47120k+2245} = 0$

IF:  $a_0 = \frac{a_0}{47480k+2257} = 0$

IF:  $a_0 = \frac{a_0}{47840k+2269} = 0$

IF:  $a_0 = \frac{a_0}{48200k+2281} = 0$

IF:  $a_0 = \frac{a_0}{48560k+2293} = 0$

IF:  $a_0 = \frac{a_0}{48920k+2305} = 0$

IF:  $a_0 = \frac{a_0}{49280k+2317} = 0$

IF:  $a_0 = \frac{a_0}{49640k+2329} = 0$

IF:  $a_0 = \frac{a_0}{50000k+2341} = 0$

IF:  $a_0 = \frac{a_0}{50360k+2353} = 0$

IF:  $a_0 = \frac{a_0}{50720k+2365} = 0$

IF:  $a_0 = \frac{a_0}{51080k+2377} = 0$

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IF:  $a_0 = \frac{a_0}{51800k+2401} = 0$

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IF:  $a_0 = \frac{a_0}{53960k+2473} = 0$

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IF:  $a_0 = \frac{a_0}{56120k+2545} = 0$

IF:  $a_0 = \frac{a_0}{56480k+2557} = 0$

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IF:  $a_0 = \frac{a_0}{58280k+2617} = 0$

IF:  $a_0 = \frac{a_0}{58640k+2629} = 0$

IF:  $a_0 = \frac{a_0}{59000k+2641} = 0$

IF:  $a_0 = \frac{a_0}{59360k+2653} = 0$

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IF:  $a_0 = \frac{a_0}{60080k+2677} = 0$

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IF:  $a_0 = \frac{a_0}{60800k+2701} = 0$

IF:  $a_0 = \frac{a_0}{61160k+2713} = 0$

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IF:  $a_0 = \frac{a_0}{62960k+2773} = 0$

IF:  $a_0 = \frac{a_0}{63320k+2785} = 0$

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IF:  $a_0 = \frac{a_0}{64040k+2809} = 0$

IF:  $a_0 = \frac{a_0}{64400k+2821} = 0$

IF:  $a_0 = \frac{a_0}{64760k+2833} = 0$

IF:  $a_0 = \frac{a_0}{65120k+2845} = 0$

IF:  $a_0 = \frac{a_0}{65480k+2857} = 0$

IF:  $a_0 = \frac{a_0}{65840k+2869} = 0$

IF:  $a_0 = \frac{a_0}{66200k+2881} = 0$

IF:  $a_0 = \frac{a_0}{66560k+2893} = 0$

IF:  $a_0 = \frac{a_0}{66920k+2905} = 0$

IF:  $a_0 = \frac{a_0}{67280k+2917} = 0$

IF:  $a_0 = \frac{a_0}{67640k+2929} = 0$

IF:  $a_0 = \frac{a_0}{68000k+2941} = 0$

IF:  $a_0 = \frac{a_0}{68360k+2953} = 0$

IF:  $a_0 = \frac{a_0}{68720k+2965} = 0$

IF:  $a_0 = \frac{a_0}{69080k+2977} = 0$

IF:  $a_0 = \frac{a_0}{69440k+2989} = 0</math$

**Question 8** (\*\*\*\*)

Find the solution of following differential equation

$$\left( \frac{dy}{dx} \right) \left( \frac{d^2y}{dx^2} \right) = \frac{d^3y}{dx^3},$$

subject to the boundary conditions.

$$y\left(-\frac{1}{2}\pi\right) = y'\left(-\frac{1}{2}\pi\right) = 0, \quad y''\left(-\frac{1}{2}\pi\right) = \frac{1}{2}.$$

Given the answer in the form  $y = f(x)$ .

□,  $y = 2 \ln \left| \sec \left( \frac{1}{2}x + \frac{1}{4}\pi \right) \right|$

BY SUBSTITUTION — LET  $p = \frac{dy}{dx}$  & SEPARATE VARIABLES.

$$\begin{aligned} \Rightarrow \frac{dp}{dx} \times \frac{dp}{dx} = \frac{d^3y}{dx^3} \\ \Rightarrow p \frac{dp}{dx} = \frac{d^2y}{dx^2} \\ \Rightarrow \int p \, dp = \int \frac{d^2y}{dx^2} \, dx \\ \Rightarrow \frac{1}{2}p^2 = \frac{dy}{dx} + A \\ \Rightarrow p^2 = 2\frac{dy}{dx} + A \\ \text{APPLY CONDITION } x = -\frac{\pi}{2}, \frac{dy}{dx} = p = 0, \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{1}{2} \\ \Rightarrow 0 = 2x\frac{1}{2} + A \\ \Rightarrow A = -1 \\ \Rightarrow p^2 = 2\frac{dp}{dx} - 1 \end{aligned}$$

REARRANGE & SEPARATE VARIABLES AGAIN

$$\begin{aligned} \Rightarrow p^2 + 1 = 2\frac{dp}{dx} \\ \Rightarrow 1 \, dx = \frac{2}{p^2+1} \, dp \\ \Rightarrow \int \frac{2}{p^2+1} \, dp = \int 1 \, dx \\ \Rightarrow 2 \operatorname{arctan} p = x + B \\ \Rightarrow \operatorname{arctan} p = \frac{1}{2}x + B \\ \Rightarrow p = \tan\left(\frac{1}{2}x + B\right) \end{aligned}$$

APPLY THE BOUNDARY CONDITION  $x = -\frac{\pi}{2}, \frac{dy}{dx} = 0$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} = \tan\left(\frac{1}{2}x + B\right) \\ \Rightarrow 0 = \tan\left(-\frac{\pi}{4} + B\right) \\ \Rightarrow B = \frac{\pi}{4} \text{ EASY AS THIS IS} \\ \text{FACT FROM PREVIOUS PAPER} \\ \Rightarrow \frac{dy}{dx} = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right) \end{aligned}$$

FINALISE WE HAVE BY DIRECT INTEGRATION

$$\begin{aligned} \frac{dy}{dx} &= \ln\left(\frac{1}{2}x + \frac{\pi}{4}\right) \\ y &= 2 \ln \left| \sec\left(\frac{1}{2}x + \frac{\pi}{4}\right) \right| + C \end{aligned}$$

APPLY THE LAST CONDITION,  $x = -\frac{\pi}{2}, y = 0$

$$\begin{aligned} \Rightarrow 0 &= 2 \ln \left| \sec\left(-\frac{\pi}{4} + \frac{\pi}{4}\right) \right| + C \\ \Rightarrow 0 &= 2 \ln(1) + C \\ \Rightarrow 0 &= 2C \\ \Rightarrow C &= 0 \\ \therefore y &= 2 \ln \left| \sec\left(\frac{1}{2}x + \frac{\pi}{4}\right) \right| \end{aligned}$$

**Question 9** (\*\*\*)+

A curve has a stationary point at  $(-\frac{1}{2}, -\frac{1}{2})$ .

The rate of change of the gradient function of the curve is given by

$$x + y + 2,$$

where  $x + y + 2 > 0$ .

Determine the equation of the curve, giving the answer in the form  $y = f(x)$ .

$$\boxed{\quad}, \quad y = e^{x+\frac{1}{2}} - x - 2$$

$$\frac{dy}{dx} = xy+2 \quad \frac{dy}{dx} = 0 \text{ AT } x = -\frac{1}{2}, y = -\frac{1}{2}$$

SIMPLIFY WITH AN OBVIOUS SUBSTITUTION

$$\Rightarrow V = x+y+2 \quad \text{WITH} \quad V=1 \text{ AT } (-\frac{1}{2}, -\frac{1}{2}) \quad \frac{dV}{dx} = 0$$

$$\Rightarrow \frac{dV}{dx} = 1 + \frac{dy}{dx} \quad \text{WITH} \quad \frac{dy}{dx} = 1 \text{ AT } (-\frac{1}{2}, -\frac{1}{2}), \frac{dV}{dx} = 0$$

$$\Rightarrow \frac{dV}{dx} = \frac{dy}{dx}$$

MAKE THE O.D.E. TRANSFORMS FOR  $V = V(x)$

$$\frac{dy}{dx} = V$$

THIS O.D.E. HAS THE INDEPENDENT VARIABLE MISSING, SO WE PROCEED WITH THE STANDARD SUBSTITUTION

$$\frac{dy}{dx} = \frac{dy}{dx}$$

DIFFERENTIATE WITH RESPECT TO  $V$

$$\frac{dv}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \cdot \frac{1}{dx} = \frac{d^2y}{dx^2} \cdot \frac{1}{V} \rightarrow \frac{dy}{dx} = V \frac{dv}{dx}$$

TRANSFORM THE 2ND ORDER O.D.E. TO A FIRST ORDER SEPARABLE

$$\Rightarrow P \frac{dv}{dx} = V$$

$$\Rightarrow P dv = V dx$$

$$\Rightarrow \frac{1}{2} p^2 = \frac{1}{2} V^2 + C$$

$$\Rightarrow p^2 = V^2 + C \quad \leftarrow \text{WHEN } V=1, P=\frac{dy}{dx}=1$$

$$\Rightarrow p = +V \quad \leftarrow V=x+y+2 = \frac{dy}{dx} > 0$$

$$\Rightarrow \frac{dy}{dx} = V$$

$$\Rightarrow \frac{1}{V} dv = 1 dx$$

INTEGRATE SUBJECT TO THE CONDITION  $x = -\frac{1}{2}, V = 1$

$$\Rightarrow [\ln|V|]_1^V = \left[ \frac{x}{2} \right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$\Rightarrow \ln|V| - \ln 1 = \frac{1}{2} - \frac{1}{2}$$

$$\Rightarrow V = e^{\frac{1}{2}}$$

$$\Rightarrow y + x + 2 = e^{\frac{1}{2}x}$$

$$\Rightarrow y = e^{\frac{1}{2}x} - x - 2$$

**Question 10** (\*\*\*)+

Solve the following differential equation

$$y \frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^2 + 2y \frac{dy}{dx} = 0, \quad y(0) = 2, \quad \frac{dy}{dx}(0) = -\frac{1}{2}.$$

Give the answer in the form  $y^2 = f(x)$ .

$y^2 = 3 + e^{-2x}$

$y \frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^2 + 2y \frac{dy}{dx} = 0 \quad x=0, y=2, \frac{dy}{dx}=-\frac{1}{2}$

BECOME WE ATTEMPT A SUBSTITUTION, WE FIND OUT THAT THE THIS IS A D.E. OF THIS FORM

$$\frac{dy}{dx} \left( y \frac{dy}{dx} + \left( \frac{dy}{dx} \right)^2 + 2y \frac{dy}{dx} \right) = 0$$

$$\frac{dy}{dx} \left( y^2 + 2y \right) = 0$$

HENCE WE MAY REWRITE AS

$$\frac{dy}{dx} [y^2 + 2y] = 0$$

$$\Rightarrow y^2 + 2y = C$$

ATY CONDITION  $y=2, \frac{dy}{dx}=-\frac{1}{2}$

$$\Rightarrow 2(-\frac{1}{2}) + 2^2 = C$$

$$\Rightarrow C=3$$

$$\Rightarrow y^2 + 2y = 3$$

PROCEED BY SEPARATION OF VARIABLES

$$\Rightarrow y \frac{dy}{dx} = 3 - y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3-y^2}{y}$$

$$\Rightarrow \frac{y}{3-y^2} dy = 1 dx$$

$$\Rightarrow \int \frac{2y}{3-y^2} dy = \int -2 dx$$

$$\Rightarrow \ln|3-y^2| = -2x + A$$

$$\Rightarrow 3-y^2 = e^{-2x+A}$$

$$\Rightarrow 3-y^2 = e^{-2x} \times e^A$$

$$\Rightarrow 3-y^2 = Be^{-2x}$$

$$\Rightarrow y^2 = 3+Be^{-2x}$$

ATY FWTN CONDITION,  $x=0, y=2$

$$\Rightarrow 4 = 3+B$$

$$\Rightarrow B=1$$

$$\therefore y^2 = 3 + e^{-2x}$$

NOTE THAT THE SUBSTITUTION  $y=\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=3-\frac{y^2}{y}$  WHICH

$$y \frac{dy}{dx} + \frac{y^2}{y} = -2$$

BY INTEGRATING FACTOR & CONDITN  $y=2, p+\frac{dy}{dx}=-\frac{1}{2}$  WE OBTAIN

$$\frac{dy}{dx} = -\frac{1}{2} + \frac{3}{y}$$

$$\frac{dy}{dx} = \frac{3-y^2}{y}$$

WHICH MATCHES WITH THIS SOLUTION

**Question 11    (\*\*\*)+**

By writing  $\frac{dy}{dx} = p$  and seeking a suitable factorization find a general solution for the non linear differential equation

$$\left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx} \left( \frac{x^2 - y^2}{xy} \right) + 1.$$

Give the solution in the form  $F(x, y)G(x, y) = 0$ .

$$(xy + A)(x^2 - y^2 + B) = 0$$

Save  $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1$   
 $y^2 - x^2 = C_1$   
 $\ln y = \ln x + \ln C_2$   
 $y = \frac{C_2}{x}$   
 $y_2 = C_2$

$\therefore (2y + A)(x^2 - y^2 + B) = 0$   
IS THE GENERAL SOLUTION

$\frac{dy}{dx} = \frac{\frac{dy}{dx}x^2 - y^2 + 1}{\frac{dy}{dx}}$   
 $\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dx} \left[ \frac{x^2 - y^2}{xy} \right] + 1}{\frac{dy}{dx}}$   
 $\Rightarrow \frac{dy}{dx} = p \frac{x^2 - y^2}{xy} - p \frac{y^2}{x} + 1$   
 $\Rightarrow p^2 - p \frac{y^2}{x} + p \frac{y^2}{x} - 1 = 0$   
 $\Rightarrow 4p^2 - px^2 + py^2 - xy = 0$   
 $\Rightarrow (4p + x)(px + y) = 0$   
 $\Rightarrow p = \begin{cases} -\frac{x}{4} \\ -\frac{y}{x} \end{cases}$   
 $\Rightarrow \frac{dy}{dx} = \begin{cases} -\frac{x}{4} \\ -\frac{y}{x} \end{cases}$

Then  $\frac{y}{x} \frac{dy}{dx} = x \frac{dx}{x}$   
 $\frac{y}{x} dy = -\frac{1}{x} dx$

**Question 12** (\*\*\*)+

By writing  $\frac{dy}{dx} = p$  and seeking a suitable factorization find a general solution for the non linear differential equation

$$\left(\frac{dy}{dx}\right)^2 + y \frac{dy}{dx} = x^2 + xy.$$

Give the solution in the form  $F(x, y)G(x, y) = 0$ .

$$\boxed{(2y - x^2 + A)(x + y - 1 + Be^{-x}) = 0}$$

Handwritten working for Question 12:

Given differential equation:  $\left(\frac{dy}{dx}\right)^2 + y \frac{dy}{dx} = x^2 + xy$

Let  $p = \frac{dy}{dx}$

$p^2 + yp = x^2 + xy$   
 $p^2 - x^2 + yp - xy = 0$   
 $(p-x)(p+x) + y(p-x) = 0$   
 $(p-x)[1+x+\frac{y}{p}] = 0$

SPLIT INTO 2 LINEAR ODES

- $\frac{dp}{dx} - x = 0$   
 $\frac{dp}{dx} = x$   
 $dp = x dx$   
 $p = \frac{1}{2}x^2 + C_1$   
 $2p = x^2 + C_1$
- $\frac{dy}{dx} + x + y = 0$   
 $\frac{dy}{dx} + y = -x$   
 $\int \frac{dy}{dx} = -x$   
 $\frac{dy}{dx} = -xe^x$   
 $\frac{d}{dx}(ye^x) = -xe^x$   
 $ye^x = \int -xe^x dx$   
 BY PARTS  
 $ye^x = -xe^x + e^x + C_2$   
 $y = -x + 1 + Ce^{-x}$

GENERAL SOLUTION  
 $(2y - x^2 + A)(x + y - 1 + Be^{-x}) = 0$

**Question 13** (\*\*\*\*\*)

A curve  $C$  is described implicitly by the equation

$$xy^2 = e^y.$$

- a) Show, by a detailed method, that

$$(y^2 - 2y) \frac{d^2y}{dx^2} + (y^2 - 2) \left( \frac{dy}{dx} \right)^2 - 4y^3 \frac{dy}{dx} e^{-y} = 0.$$

- b) Use an analytical method, with suitable boundary conditions, to obtain the equation of  $C$  by solving the above differential equation.

V, [ ] , proof

a) DIFFERENTIATE WITH RESPECT TO  $x$

$$\begin{aligned} &\Rightarrow \frac{d}{dx}(xy^2) = \frac{d}{dx}(e^y) \\ &\Rightarrow xy^2 + 2xy\frac{dy}{dx} - e^y \frac{dy}{dx} = 0 \\ &\Rightarrow y^2 + 2y\frac{dy}{dx} = e^y \frac{dy}{dx} \end{aligned}$$

DIFFERENTIATE AGAIN WITH RESPECT TO  $x$  - THREE PRODUCT RULE IS NEEDED

$$\begin{aligned} &\Rightarrow 2y\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} = e^y \left( \frac{dy}{dx} \right)^2 + e^y \frac{d^2y}{dx^2} \\ &\Rightarrow 4y\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} = e^y \left[ \left( \frac{dy}{dx} \right)^2 + \frac{d^2y}{dx^2} \right] \\ &\Rightarrow 4y\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^2 + \frac{d^2y}{dx^2} = e^y \left[ \left( \frac{dy}{dx} \right)^2 + \frac{d^2y}{dx^2} \right] \\ &\Rightarrow 4y\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} \left[ \frac{1}{y^2} \left( \frac{dy}{dx} \right)^2 + \frac{1}{y^2} \frac{d^2y}{dx^2} \right] = e^y \left[ \left( \frac{dy}{dx} \right)^2 + \frac{d^2y}{dx^2} \right] \end{aligned}$$

MULTIPLY THROUGH BY  $y^2$  & SIMPLIFY

$$\begin{aligned} &\Rightarrow 4y^3 \frac{dy}{dx} + 2 \left( \frac{dy}{dx} \right)^2 + 2y \frac{d^2y}{dx^2} = y^2 \left( \frac{dy}{dx} \right)^2 + y^2 \frac{d^2y}{dx^2} \\ &\Rightarrow 0 = (y^2 - 2) \frac{dy}{dx} + (y^2 - 2) \frac{d^2y}{dx^2} - 4y^3 \frac{dy}{dx} \quad \text{As } 2y^2 \neq 0 \\ &\Rightarrow (y^2 - 2) \frac{dy}{dx} + (y^2 - 2) \frac{d^2y}{dx^2} - 4y^3 \frac{dy}{dx} = 0 \end{aligned}$$

b) IGNORE FIRST TWO BOUNDARY CONDITIONS BASED ON THE EQUATION OF PART (a). SAY

$$\begin{aligned} x = c, y = 1 \quad &\Rightarrow \frac{dy}{dx} = -\frac{1}{c} \\ &\Rightarrow \frac{d^2y}{dx^2} = 0 \\ &\Rightarrow \frac{dy}{dx} = -\frac{1}{c} \end{aligned}$$

$$\begin{aligned} &\Rightarrow pe^{y/(p-2y)} = \int dy \\ &\Rightarrow pe^{y/(p-2y)} = y + A \\ &\Rightarrow p = \frac{y+A}{e^{y/(p-2y)}} \Rightarrow e^{-y/(p-2y)} \\ &\Rightarrow \frac{dy}{dx} = \frac{(y+A)e^{-y/(p-2y)}}{y-2y} \end{aligned}$$

APPLY THE CONDITION  $y=1, \frac{dy}{dx}=-\frac{1}{c}$

$$\begin{aligned} -\frac{1}{c} &= \frac{1+A}{1-2} \\ -\frac{1}{c} &= -1+A \\ -1 &= -1+A \\ A &= 0 \\ \therefore \frac{dy}{dx} &= \frac{y^2e^{-y}}{y-2y} \end{aligned}$$

SIMPLIFYING QUADRATIC

$$\begin{aligned} &\Rightarrow \left( \frac{y^2e^{-y}}{y-2y} \right) dy = -\frac{1}{c} dx \\ &\Rightarrow \left( \frac{y^2e^{-y}}{y^2} \right) e^y dy = -\frac{1}{c} dx \\ &\Rightarrow \int \left( \frac{1}{y} - \frac{2}{y^2} \right) e^y dy = -\frac{1}{c} dx \\ &\Rightarrow \int \frac{1}{y} e^y dy - \int \frac{2}{y^2} e^y dy = -\frac{1}{c} dx \end{aligned}$$

AS THE O.D.E HAS NO  $x$ , PREVIOUSLY TRY THE STANDARD SUBSTITUTION

$$\begin{aligned} p &= \frac{dy}{dx} \\ \frac{dp}{dy} &= \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\frac{dy}{dx} \cdot \frac{dx}{dy} = \frac{d}{dy} \times \frac{1}{p} \\ \therefore \frac{dp}{dy} &= \frac{1}{p^2} \end{aligned}$$

TRANSFORM THE O.D.E

$$\begin{aligned} &\Rightarrow (y^2 - 2y) \frac{dy}{dx} + (y^2 - 2) \left( \frac{dy}{dx} \right)^2 - 4y^3 \frac{dy}{dx} = 0 \\ &\Rightarrow (y^2 - 2y) \frac{dy}{dx} + (y^2 - 2) p^2 - 4y^3 p = 0 \\ &\Rightarrow (y^2 - 2y) \frac{dy}{dx} + (y^2 - 2)p - 4y^3 p = 0 \\ &\Rightarrow \frac{dp}{dy} + \frac{y^2 - 2y}{y^2 - 2y} p = \frac{4y^3 p^2}{y^2 - 2y} \end{aligned}$$

NEXT LOOK FOR AN INTEGRATING FACTOR - PARTIAL FRACTIONS BY INSPECTION

$$\begin{aligned} &\int \frac{y^2 - 2y}{y^2 - 2y} dy = \int \frac{y^2 - 2y - 2y^2 + 2y}{y^2 - 2y} dy = \int 1 + \frac{2y - 2}{y^2 - 2y} dy \\ &= \int 1 + \frac{1}{y} + \frac{1}{y-2} dy = e^{\int 1 + \frac{1}{y} + \frac{1}{y-2} dy} = e^{y + \ln y + \ln(y-2)} \\ &= e^y e^{\ln y} e^{\ln(y-2)} = e^y (y^2 - 2y) = y^2 (y-2) \end{aligned}$$

$$\begin{aligned} &\Rightarrow \frac{1}{y^2 (y-2)} \left[ p^2 e^{y/(p-2y)} \right] = \frac{4y^3 p^2}{y^2 (y-2)} \times e^y (y^2 - 2y) \\ &\Rightarrow \frac{1}{y^2 (y-2)} \left[ p^2 e^{y/(p-2y)} \right] = 4y^3 \end{aligned}$$

NOW INTEGRATE BY PARTS - ONLY ONE OF THE INTEGRALS OF THE L.H.S., SAY THE FIRST ONE

$$\begin{aligned} &\Rightarrow \left[ \frac{1}{2} y^2 e^y + \left[ \frac{1}{2} \frac{e^y}{y^2} dy \right] \right] - \left[ 2y^3 e^y \right] = 2y^3 B \quad \boxed{\frac{1}{2} y^2 e^y} \quad \boxed{-\frac{2}{y^3} e^y} \\ &\Rightarrow \frac{y^3}{y^2} e^y = 2y^3 B \quad \Rightarrow B = \frac{1}{2} \end{aligned}$$

ADD THE FINAL CONDITION  $x=c, y=1$

$$\begin{aligned} &\Rightarrow \frac{e^1}{1^2} = c + B \\ &\Rightarrow e = c + \frac{1}{2} \\ &\Rightarrow B = \frac{1}{2} \\ \therefore \frac{e^y}{y^2} &= 2 \\ y^2 e^y &= 2^3 \end{aligned}$$

to expand

**Question 14** (\*\*\*\*\*)

Find a general solution of the following differential equation.

$$y = x \frac{dy}{dx} + e^{\frac{dy}{dx}}.$$

$$\boxed{y = x}, \quad \boxed{(y + Ax + B)(y - x \ln x + Cx) = 0}$$

$y = x \frac{dy}{dx} + e^{\frac{dy}{dx}}$

START BY DIFFERENTIATING THE O.D.E. WITH RESPECT TO  $x$

$$\Rightarrow \frac{dy}{dx} = [x \frac{dy}{dx} + e^{\frac{dy}{dx}}] + e^{\frac{dy}{dx}} \times \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dx} + x \frac{d^2y}{dx^2} + e^{\frac{dy}{dx}} \times \frac{d^2y}{dx^2}$$

$$\Rightarrow 0 = \frac{d^2y}{dx^2} \left[ x + e^{\frac{dy}{dx}} \right]$$

NOW REARRANGING THE ORIGINAL O.D.E

$$\Rightarrow 0 = \frac{d^2y}{dx^2} \left[ x + (y - x \frac{dy}{dx}) \right]$$

THIS WE HAVE TWO SEPARATE O.D.E TO SOLVE

- $\frac{dy}{dx} = 0 \Rightarrow y = Ax + B$
- $x + y - x \frac{dy}{dx} = 0$

$$\Rightarrow x \frac{dy}{dx} - y = x$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 1$$

INTEGRATING FACTOR =  $e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$

$$\Rightarrow \frac{d}{dx} \left( \frac{y}{x} \right) = \frac{1}{x}$$

$$\Rightarrow \frac{y}{x} = \int \frac{1}{x} dx$$

$$\Rightarrow \frac{y}{x} = \ln|x| + C$$

$$\Rightarrow y = x \ln x + Cx$$

COMBINING THE SOLUTIONS WE HAVE

$$y = \begin{cases} Ax + B \\ x \ln x + Cx \end{cases}$$

THIS CAN BE WRITTEN AS

$$\Rightarrow (y - Ax - B)(y - x \ln x - C) = 0$$

$$\Rightarrow (y + Px + Q)(y - x \ln x + R) = 0$$