

# 2<sup>nd</sup> ORDER O.D.E.s SUBSTITUTIONS

**Question 1** (\*\*\*)+

$$2y \frac{d^2y}{dx^2} - 8y \frac{dy}{dx} + 16y^2 = \left(\frac{dy}{dx}\right)^2, \quad y \neq 0,$$

Find the general solution of the above differential equation by using the transformation equation  $t = \sqrt{y}$ .

Give the answer in the form  $y = f(x)$ .

$$y = (A e^{2x} + B x e^{2x})^2$$

$$\begin{aligned} & 2y \frac{d^2y}{dx^2} - 8y \frac{dy}{dx} + 16y^2 = \left(\frac{dy}{dx}\right)^2 \\ \rightarrow & 2t \frac{d^2t}{dt^2} - 8t \frac{dt}{dt} + 16t^2 = \left(\frac{dt}{dx}\right)^2 \\ \rightarrow & 4t \frac{dt}{dx} - 4t \frac{dt}{dx} - 8t \frac{dt}{dx} + 16t^2 = 4t^2 \frac{dt}{dx}^2 \\ \rightarrow & \frac{dt}{dx} - 4t \frac{dt}{dx} + 4t^2 = 0 \\ \bullet & \text{Auxiliary equation} \\ & t^2 - 4t + 4 = 0 \\ & (t-2)^2 = 0 \\ & t = 2 \quad (\text{REMARK}) \\ \therefore & t = A e^{2x} + B x e^{2x} \\ \sqrt{y} &= (A e^{2x} + B x e^{2x})^2 \\ y &= (A e^{2x} + B x e^{2x})^4 \end{aligned}$$

**Question 2** (\*\*\*)

The differential equation

$$x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 3x, \quad x \neq 0,$$

is to be solved subject to the boundary conditions  $y = \frac{3}{2}$ ,  $\frac{dy}{dx} = \frac{1}{2}$  at  $x = 1$ .

- a) Show that the substitution  $v = \frac{dy}{dx}$ , transforms the above differential equation into

$$\frac{dv}{dx} + \frac{2v}{x} = 3.$$

- b) Hence find the solution of the original differential equation, giving the answer in the form  $y = f(x)$ .

$$y = \frac{1}{2} \left( x^2 + \frac{1}{x} + 1 \right)$$

④  $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 3x$   
 $x \frac{dy}{dx} + 2v = 3x$   
 $\frac{dy}{dx} + \frac{2v}{x} = 3$

⑤  $\frac{dv}{dx} + \frac{2v}{x} = 0$   
 $\frac{dv}{dx} = -\frac{2v}{x}$   
 $\int \frac{1}{v} dv = \int -\frac{2}{x} dx$   
 $\ln|v| = -2\ln|x| + C$   
 $|v| = |v| \left( \frac{A}{x^2} \right)$   
 $v = \frac{A}{x^2}$

⑥  $v = \frac{P}{x^2} \dots$   
 $\frac{dv}{dx} = P \frac{-2}{x^3}$   
 $P + \frac{2P}{x^2} (P) \equiv 3$   
 $P + 2P = 3$   
 $P = 1$

∴  $v = \frac{1}{x^2} + C$  (or do it by integrating factor)  
 $\Rightarrow \frac{dy}{dx} = \frac{A}{x^2} + C$   
 $\Rightarrow y = -\frac{A}{x} + \frac{C}{2} + B$

• APPLY CONDITION  $x=1, \frac{dy}{dx}=\frac{1}{2} \Rightarrow \frac{1}{2} = \frac{A}{1} + 1 \Rightarrow A = -\frac{1}{2}$

• APPLY CONDITION  $x=1, y=\frac{3}{2} \Rightarrow \frac{3}{2} = \frac{1}{2} + \frac{C}{2} + B \Rightarrow B = \frac{1}{2}$

∴  $y = \frac{1}{2} \left( 1 + x^2 + \frac{1}{x^2} \right)$

**Question 3    (\*\*\*)+**

The curve  $C$  has equation  $y = f(x)$  and satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 2y(2x^2 - 1) = 3x^3 e^x, \quad x \neq 0$$

is to be solved subject to the boundary conditions  $y = \frac{3}{2}$ ,  $\frac{dy}{dx} = \frac{1}{2}$  at  $x = 1$ .

- a) Show that the substitution  $y = xv$ , where  $v$  is a function of  $x$  transforms the above differential equation into

$$\frac{d^2v}{dx^2} - 4v = 3e^x.$$

It is further given that  $C$  meets the  $x$  axis at  $x = \ln 2$  and has a finite value for  $y$  as  $x$  gets infinitely negatively large.

- b) Express the equation of  $C$  in the form  $y = f(x)$ .

$$y = \frac{1}{2}xe^{2x} - xe^x$$

**a)**

$x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 2y(2x^2 - 1) = 3x^3 e^x$   
 USE THE SUBSTITUTION  
 $x^2 \left[ \frac{d^2(vx)}{dx^2} + 2 \frac{d(vx)}{dx} \right] - 2x(v + x \frac{dv}{dx}) - 2v(2x^2 - 1) = 3x^3 e^x$   
 $x^2 \left[ \frac{d^2v}{dx^2} + 2 \frac{dv}{dx} + v + x \frac{d^2v}{dx^2} + 2x \frac{dv}{dx} - 2v(2x^2 - 1) \right] = 3x^3 e^x$   
 $x^2 \frac{d^2v}{dx^2} + 2x \frac{d^2v}{dx^2} + v + 2x \frac{dv}{dx} - 4xv = 3x^3 e^x$   
 $\frac{d^2v}{dx^2} - 4v = 3e^x$  AS REQUIRED

**b)**

AUXILIARY EQUATION:  $\gamma^2 - 4 = 0$  PARTICULAR SOLUTION: TRY  $V = P e^{2x}$   
 $\gamma = \pm 2$ . SUB INTO THE O.D.E.

$$\begin{aligned} \frac{d^2V}{dx^2} &= P e^{2x} \\ P e^{2x} - 4P e^{2x} &= 3e^x \\ P - 4P &= 3 \\ P &= -\frac{3}{3} \end{aligned}$$

GENERAL SOLUTION:  
 $y = A e^{2x} + B e^{-2x}$   
 $y = A x e^{2x} + B e^{-2x} - x e^{2x}$   
 $y$  IS FINITE AS  $x \rightarrow -\infty$   $\therefore B = 0$   
 $y = A x e^{2x} - x e^{2x}$

GRAPH CROSSES THE  $x$  AXIS AT  $x = \ln 2$ :  
 $0 = A \ln 2 e^{2 \ln 2} - e^{2 \ln 2} \ln 2$   
 $0 = 4A \ln 2 - 2 \ln 2$   
 $0 = 2A - 1$   
 $A = \frac{1}{2}$   
 $\therefore y = \frac{1}{2} x e^{2x} - x e^{2x}$

**Question 4** (\*\*\*)+

Given that if  $x = e^t$  and  $y = f(x)$ , show clearly that ...

a) ...  $x \frac{dy}{dx} = \frac{dy}{dt}$ .

b) ...  $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$ .

The following differential equation is to be solved

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 2 \ln x$$

subject to the boundary conditions  $y = \frac{1}{2}$ ,  $\frac{dy}{dx} = \frac{3}{2}$  at  $x = 1$ .

c) Use the substitution  $x = e^t$  to solve the above differential equation.

$$y = \frac{1}{2} + \frac{1}{2}(2x^2 + 1)\ln x$$

**a)**  $\frac{dy}{dt} = e^t \Rightarrow \frac{dy}{dx} = e^t \cdot \frac{dy}{dt}$

Differentiate with respect to  $t$ :

$$\begin{aligned} \frac{dy}{dt} &= e^t \cdot \frac{dy}{dx} \\ \frac{d^2y}{dt^2} &= e^t \cdot \frac{d}{dt}\left(\frac{dy}{dx}\right) \\ &= e^t \cdot \frac{d}{dt}\left(e^t \cdot \frac{dy}{dt}\right) \\ &= e^{2t} \cdot \frac{d^2y}{dt^2} \end{aligned}$$

Substitute into the original equation:

$$e^{2t} \cdot \frac{d^2y}{dt^2} - 3e^t \cdot \frac{dy}{dt} + 4y = 2 \ln e^t$$

$$e^{2t} \cdot \frac{d^2y}{dt^2} - 3e^t \cdot \frac{dy}{dt} + 4y = 2t$$

**b)** Differentiate (a) with respect to  $x$ :

$$\begin{aligned} \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}\left[\frac{dy}{dt}\right] \\ \frac{d^2y}{dx^2} &= \frac{d}{dt}\left[\frac{dy}{dt}\right] \cdot \frac{dt}{dx} \\ &= \frac{d}{dt}\left[\frac{dy}{dt}\right] \cdot \frac{1}{x} \end{aligned}$$

Substitute into the original equation:

$$\begin{aligned} x^2 \cdot \frac{d^2y}{dx^2} - 3x \cdot \frac{dy}{dx} + 4y &= 2 \ln x \\ x^2 \cdot \frac{d}{dt}\left[\frac{dy}{dt}\right] \cdot \frac{1}{x} - 3x \cdot \frac{dy}{dx} + 4y &= 2 \ln x \\ x \frac{d}{dt}\left[\frac{dy}{dt}\right] - 3x \cdot \frac{dy}{dx} + 4y &= 2 \ln x \end{aligned}$$

**c)** General solution:

$$y = Pe^t + Qt$$

$$\frac{dy}{dt} = Pe^t + Q$$

At  $x = 1$ ,  $y = \frac{1}{2}$ :

$$\begin{aligned} \frac{1}{2} &= P + Q \\ \frac{1}{2} &= Pe^t + Q \\ \frac{1}{2} &= Ae^t + Bt \\ \frac{1}{2} &= Ae^t + Be^t \end{aligned}$$

At  $x = 1$ ,  $\frac{dy}{dx} = \frac{3}{2}$ :

$$\begin{aligned} \frac{3}{2} &= Pe^t + Q \\ \frac{3}{2} &= Ae^t + B \\ \frac{3}{2} &= Ae^t + B \end{aligned}$$

Subtract the first from the second:

$$\begin{aligned} \frac{3}{2} - \frac{1}{2} &= Ae^t + B - Ae^t - B \\ 1 &= B \end{aligned}$$

Substitute into the first:

$$\begin{aligned} \frac{1}{2} &= Ae^t + 1 \\ \frac{1}{2} - 1 &= Ae^t \\ -\frac{1}{2} &= Ae^t \\ -\frac{1}{2} &= Ae^t \end{aligned}$$

Solve for  $A$ :

$$A = -\frac{1}{2e^t}$$

General solution:

$$y = -\frac{1}{2}e^t + Be^t$$

At  $x = 1$ ,  $y = \frac{1}{2}$ :

$$\begin{aligned} \frac{1}{2} &= -\frac{1}{2}e^t + Be^t \\ \frac{1}{2} &= -\frac{1}{2} + B \\ \frac{1}{2} + \frac{1}{2} &= B \\ 1 &= B \end{aligned}$$

Particular solution:

$$y = -\frac{1}{2}e^t + e^t$$

$$y = \frac{1}{2}e^t$$

Final answer:

$$y = \frac{1}{2}e^t$$

**Question 5 (\*\*\*\*)**

The differential equation

$$(x^3 + 1) \frac{d^2y}{dx^2} - 3x^2 \frac{dy}{dx} = 2 - 4x^3,$$

is to be solved subject to the boundary conditions  $y=0$ ,  $\frac{dy}{dx}=4$  at  $x=0$ .

Use the substitution  $u = \frac{dy}{dx} - 2x$ , where  $u$  is a function of  $x$ , to show that the solution of the above differential equation is

$$y = x^4 + x^2 + 4x.$$

 proof

USING THE SUBSTITUTION GIVEN

$$\begin{aligned} \Rightarrow u &= \frac{dy}{dx} - 2x \\ \Rightarrow \frac{du}{dx} &= 1 + 2x \\ \Rightarrow \frac{du}{dx} &= \frac{du}{dx} + 2x \end{aligned}$$

SUBSTITUTE INTO THE O.D.E.

$$\begin{aligned} \Rightarrow (x^3 + 1) \frac{d^2u}{dx^2} - 3x^2 \frac{du}{dx} &= 2 - 4x^3 \\ \Rightarrow (x^3 + 1) \frac{du}{dx} + 2(2x+1) - 3x^2(u+2x) &= 2 - 4x^3 \\ \Rightarrow (x^3 + 1) \frac{du}{dx} + 2x^3 + 2 - 3x^2(u+2x) &= 2 - 4x^3 \\ \Rightarrow (x^3 + 1) \frac{du}{dx} + 2x^3 + 2 - 3x^2u - 6x^3 &= 2 - 4x^3 \\ \Rightarrow (x^3 + 1) \frac{du}{dx} + 2x^3 + 2 - 3x^2u - 4x^3 &= 2 - 4x^3 \\ \Rightarrow (x^3 + 1) \frac{du}{dx} - 3x^2u &= 2x^3 \\ \Rightarrow (x^3 + 1) \frac{du}{dx} &= 3x^2u \end{aligned}$$

SORT INTO FRACTIONAL FORM

$$\begin{aligned} \Rightarrow \frac{1}{u} du &= \frac{3x^2}{x^3 + 1} dx \\ \Rightarrow \int \frac{1}{u} du &= \int \frac{3x^2}{x^3 + 1} dx \\ \Rightarrow \ln|u| &= \ln|x^3 + 1| + \ln A \\ \Rightarrow |\ln|u|| &= \ln|A(x^3 + 1)| \\ \Rightarrow |u| &= A(x^3 + 1) \end{aligned}$$

DIVIDING THE TRANSFORMATION

$$\begin{aligned} \Rightarrow \frac{du}{dx} - 2x &= A(x^3 + 1) \\ \Rightarrow \frac{du}{dx} &= A(x^3 + 1) + 2x \\ \text{INTEGRATING w.r.t. } x \\ \Rightarrow u &= A\left(\frac{1}{4}x^4 + x\right) + x^2 + B \end{aligned}$$

USING THE CONDITION GIVEN

$$\begin{aligned} x=0, y=0 &\Rightarrow 0=B \\ 2x, \frac{du}{dx} &= 4 \Rightarrow 4=A \\ \therefore u &= \frac{1}{4}(x^4 + 4x^2) + x^2 \\ u &= x^4 + 4x^2 + \underline{\underline{x^2}} \end{aligned}$$

**Question 6** (\*\*\*\*\*)

$$x \frac{d^2y}{dx^2} + (6x+2) \frac{dy}{dx} + 9xy = 27x - 6y.$$

Use the substitution  $u = xy$ , where  $u$  is a function of  $x$ , to find a general solution of the above differential equation.

$$\boxed{\quad}, \quad y = \frac{A}{x} e^{-3x} + B e^{-3x} + 3 - \frac{2}{x}$$

USING THE SUBSTITUTION GIVEN  $u(x) = x \cdot y(x)$

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx}(xy) = \frac{1}{x} \left( x \cdot y'(x) + 1 \cdot y \right) \\ \frac{du}{dx} &= x \cdot \frac{dy}{dx} + y \\ \frac{dy}{dx} &= \frac{du}{dx} - y \\ \frac{d}{dx} \left[ x \frac{du}{dx} \right] &= \frac{d}{dx} \left[ \frac{du}{dx} - y \right] \\ x \frac{d^2u}{dx^2} + u \frac{d^2u}{dx^2} &= \frac{d^2u}{dx^2} - \frac{dy}{dx} \\ x \frac{d^2u}{dx^2} &= \frac{d^2u}{dx^2} - 2 \frac{du}{dx} \end{aligned}$$

DIFFERENTIATE THE ABOVE AGAIN WITH RESPECT TO  $x$

$$\begin{aligned} \rightarrow 2 \frac{d}{dx} \left[ \frac{du}{dx} \right] + x \frac{d^2u}{dx^2} &= 2 \frac{du}{dx} - 2y \\ \rightarrow 2 \frac{d^2u}{dx^2} + x \frac{d^2u}{dx^2} + 2 \frac{du}{dx} + 2y &= 2x - 6y \\ \rightarrow \frac{d^2u}{dx^2} + \frac{x}{2} \frac{d^2u}{dx^2} + 9u &= 2x - 6y \\ \rightarrow \frac{d^2u}{dx^2} + 6 \left( \frac{1}{2}x \right) \frac{d^2u}{dx^2} + 9u &= 2x - 6y \\ \rightarrow \frac{d^2u}{dx^2} + 6 \frac{du}{dx} - 6y + 9u &= 2x - 6y \\ \rightarrow \boxed{\frac{d^2u}{dx^2} + 6 \frac{du}{dx} + 9u = 2x - 6y} \end{aligned}$$

THE AUXILIARY EQUATION FOR THE LHS IS

$$\begin{aligned} A^2 + Ax + 9 &= 0 \\ (A+3)^2 &= 0 \\ A &= -3 \end{aligned}$$

COMPLEMENTARY FACTOR

$$u = Ae^{-3x} + Bx^{-3}$$

PARTICULAR SOLUTION BY INSPECTION

$$\begin{aligned} u &= Px + Q \\ u' &= P \\ u'' &= 0 \\ \therefore 0 + 6P + 9(Px+Q) &= 2x \\ (6P+9Q) + 9Px &= 2x \\ P=3 & \quad Q \quad 6P+9Q=0 \\ 18+9Q=0 & \quad 18+9Q=0 \\ Q=-2 & \quad Q=-2 \end{aligned}$$

Thus we have

$$u(x) = (A + Bx)e^{-3x} + 3x - 2$$

USING THE TRANSMISSION

$$\begin{aligned} 2y - (A + Bx)e^{-3x} + 3x - 2 &= 2y \\ y &= \boxed{\left( \frac{A}{x} + B \right) e^{-3x} + 3 - \frac{2}{x}} \end{aligned}$$

**Question 7 (\*\*\*\*)**

By using the substitution  $z = \frac{dy}{dx}$ , or otherwise, solve the differential equation

$$(x^2 + 1) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 6x^2 + 2,$$

subject to the conditions  $x = 0, y = 2, \frac{dy}{dx} = 1$

$$y = x^2 + 2 + \arctan x$$

The handwritten solution shows the following steps:

- Substitution:  $z = \frac{dy}{dx}, \frac{dz}{dx} = \frac{d^2y}{dx^2}$
- Equation transformation:  $(x^2+1)\frac{dz}{dx} + 2z = 6x^2 + 2$
- Integration factor:  $\frac{dz}{dx} + \frac{2z}{x^2+1} = \frac{6x^2+2}{x^2+1}$
- Integrating factor:  $I.F. = e^{\int \frac{2}{x^2+1} dx} = e^{\ln(x^2+1)} = x^2+1$
- Differential equation:  $\frac{d}{dx}(z(x^2+1)) = \frac{6x^2+2}{x^2+1}(x^2+1)$
- Integration:  $z(x^2+1) = \int 6x^2+2 dx$
- Solution:  $\boxed{z(x^2+1) = 2x^3+2x+C}$
- Condition:  $z(1) = 1 \Rightarrow 1 = 2+C \Rightarrow C = -1$
- Equation:  $\therefore z(x^2+1) = 2x^3+2x-1$
- Condition:  $x=0, y=2 \Rightarrow 2 = 2+C \Rightarrow C=0$
- Equation:  $\therefore z = 2x^3+2x$
- Equation:  $y = \int \frac{2x^3+2x}{x^2+1} dx$
- Equation:  $y = \int \frac{2(x^2+1)+1}{x^2+1} dx$
- Equation:  $y = \int 2x + \frac{1}{x^2+1} dx$
- Equation:  $y = x^2 + \arctan x + D$
- Final answer:  $\boxed{y = x^2 + 2 + \arctan x}$

**Question 8** (\*\*\*\*\*)

$$\frac{d^2y}{dx^2} - (1-6e^x) \frac{dy}{dx} + 10ye^{2x} = 5e^{2x} \sin(2e^x).$$

- a) By using the substitution  $x = \ln t$  or otherwise, show that the above differential equation can be transformed to

$$\frac{d^2y}{dt^2} + 6 \frac{dy}{dt} + 10y = 5 \sin 2t.$$

- b) Hence find a general solution for the original differential equation.

$$[ \quad ], \quad y = e^{-3e^x} [ A \cos(e^x) + B \sin(e^x) ] + \frac{1}{6} \sin(2e^x) - \frac{1}{3} \cos(2e^x)$$

a) START BY OBTAIN "REPLACEMENTS" FOR  $\frac{dy}{dx}$  &  $\frac{d^2y}{dx^2}$

- $x = \ln t$   
DIFFERENTIATE w.r.t  $x$   
 $\Rightarrow \frac{dy}{dx} = \frac{1}{t} \frac{dy}{dt}$   
 $\Rightarrow \frac{d^2y}{dx^2} = t \frac{d^2y}{dt^2}$
- ALSO NOTE THAT  
 $x = \ln t$   
 $\frac{dx}{dt} = \frac{1}{t}$   
 $\frac{d}{dt} = t \frac{d}{dx}$

SUBSTITUTING INTO THE O.D.E. AND SIMPLIFY, NOTING FURTHER THAT  $t = e^x$

$$\begin{aligned} &\Rightarrow \frac{d^2y}{dx^2} - (1-6e^x) \frac{dy}{dx} + 10ye^{2x} = 5e^{2x} \sin(2e^x) \\ &\Rightarrow \left[ t \frac{d^2y}{dt^2} + t^2 \frac{d^2y}{dt^2} \right] - (1-6t)(t \frac{dy}{dt}) + 10yt^2 = 5t^2 \sin(2t) \\ &\Rightarrow t^2 \frac{d^2y}{dt^2} + t^2 \frac{d^2y}{dt^2} - (1-6t)t \frac{dy}{dt} + 10t^2y = 5t^2 \sin(2t) \\ &\Rightarrow t^2 \frac{d^2y}{dt^2} + 6t^2 \frac{dy}{dt} + 10t^2y = 5t^2 \sin(2t) \\ &\Rightarrow \frac{d^2y}{dt^2} + 6 \frac{dy}{dt} + 10y = 5 \sin 2t \end{aligned}$$

As required

b) SOLVING THE TRANSFORMED EQUATION

- AUXILIARY EQUATION  
 $\Rightarrow t^2 + 6t + 10 = 0$   
 $\Rightarrow (t+3)^2 - 9+10 = 0$   
 $\Rightarrow (t+3)^2 = -1$   
 $\Rightarrow t+3 = \pm i$   
 $\Rightarrow t = -3 \pm i$
- COMPLEMENTARY FUNCTION  
 $y = e^{3t} (A \cos t + B \sin t)$
- PARTICULAR INTEGRAL  
 $y = P \cos 2t + Q \sin 2t$   
 $\dot{y} = -2P \sin 2t + 2Q \cos 2t$   
 $\ddot{y} = -4P \cos 2t - 4Q \sin 2t$   
 $\ddot{y} = 12P \sin 2t - 12Q \cos 2t$   
 $+10y = 10t \cos 2t + 10t \sin 2t$   
SUB INTO THE O.D.E.  
 $\ddot{y} + 6\dot{y} + 10y = 5 \sin 2t$   
 $(4P+12Q)\cos 2t + (4Q-12P)\sin 2t = 5 \sin 2t$   
 $4P+12Q = 0 \Rightarrow P = -3Q$   
 $4Q-12P = 5 \Rightarrow 4Q+36Q = 5 \Rightarrow 40Q = 5 \Rightarrow Q = \frac{1}{8}$   
 $\Rightarrow P = -\frac{3}{8}$   
ADDING AND CANCELLING

HENCE THE GENERAL SOLUTION CAN BE FOUND

$$\begin{aligned} &\Rightarrow y = e^{3t} (A \cos t + B \sin t) - \frac{3}{8} \cos 2t + \frac{1}{8} \sin 2t \\ &\Rightarrow y = e^{-3e^x} [ A \cos(e^x) + B \sin(e^x) ] - \frac{3}{8} \cos(2e^x) + \frac{1}{8} \sin(2e^x) \end{aligned}$$

**Question 9** (\*\*\*\*)

Use the substitution  $w = \frac{dy}{dx}$  to solve the following differential equation

$$(1-x^2) \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = (1-x)^2, |x| < 1$$

subject to the boundary conditions  $y = -4.5$  and  $\frac{dy}{dx} = -1$  at  $x = 0$ .

Give the answer in the form  $y = \alpha(x-3)^2 + \beta \ln(x+1)$ , where  $\alpha$  and  $\beta$  are constants to be found.

 ,  $y = -\frac{1}{4}(x-3)^2 - 4 \ln(x+1)$

<p><b>USING THE SUBSTITUTION (Part 1)</b></p> $(1-x^2) \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = (1-x)^2$ $(1-x^2) \frac{dw}{dx} + 2w = (1-x)^2$ $\frac{dw}{dx} + \frac{2}{1-x^2} w = \frac{(1-x)^2}{1-x^2}$ $\frac{dw}{dx} + \frac{2}{(1-x)(1+x)} w = \frac{(1-x)^2}{(1-x)(1+x)}$ $\frac{dw}{dx} + \left(\frac{1}{1+x} + \frac{1}{1-x}\right) w = \frac{1-x^2}{1-x^2}$ <p style="text-align: right;"><i>Partial fractions by inspection</i></p> <p><b>FIND THE INTEGRATING FACTOR</b></p> $e^{\int \frac{1}{1+x} + \frac{1}{1-x} dx} = e^{\ln(1+x) - \ln(1-x)} = e^{\ln(\frac{1+x}{1-x})} = \frac{1+x}{1-x}$ <p>Hence we now have</p> $\frac{d}{dx} \left[ w \left( \frac{1+x}{1-x} \right) \right] = \frac{1-x}{1+x} \left( \frac{1+x}{1-x} \right)$ $\frac{d}{dx} \left[ w \left( \frac{1+x}{1-x} \right) \right] = 1$ $w \left( \frac{1+x}{1-x} \right) = \int 1 dx$ $w \left( \frac{1+x}{1-x} \right) = x + A$ <p>Apply condition <math>x=0, w \frac{dy}{dx} = -1</math></p> $-1 = A$ $\therefore w \left( \frac{1+x}{1-x} \right) = x - 1$	$\Rightarrow w = \frac{(1-x)(x+1)}{x+2}$ $\Rightarrow \frac{dw}{dx} = -\frac{(x-1)^2}{x+2}$ $\Rightarrow \frac{dw}{dx} = -\frac{x^2-2x+1}{x+1}$ $\Rightarrow \frac{dw}{dx} = -\frac{x(2x+3)(x+1)+4}{x+1}$ $\Rightarrow \frac{dw}{dx} = -\left[ x^2 - 3x + \frac{4}{x+1} \right]$ $\Rightarrow y = -\left[ \frac{1}{4}x^2 - \frac{3}{2}x + 4 \ln(x+1) \right] + C$ <p><b>INITIAL CONDITION</b> <math>x=0, y=-4.5</math> To obtain <math>C = -4.5</math>.</p> $\Rightarrow y = -\frac{1}{4}x^2 + 3x - 4.5 - 4 \ln(x+1)$ $\Rightarrow y = -\frac{1}{4}(x^2 - 12x + 18) - 4 \ln(x+1)$ $\Rightarrow y = -\frac{1}{4}(x^2 - 12x + 18) - 4 \ln(x+1) \quad //$ <p style="text-align: center;"><i>16. <math>x = -\frac{1}{2}</math> 8 = 4</i> //</p>
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**Question 10** (\*\*\*\*)

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} \tan x - y \sec^4 x = 0.$$

The above differential equation is to be solved by a substitution.

- a) If  $t = \tan x$  show that ...

i. ...  $\frac{dy}{dx} = \frac{dy}{dt} \sec^2 x$

ii. ...  $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} \sec^4 x + 2 \frac{dy}{dt} \sec^2 x \tan x$

- b) Use the results obtained in part (a) to find a general solution of the differential equation in the form  $y = f(x)$ .

 ,  $y = A e^{\tan x} + B e^{-\tan x}$

**i)** DIFFERENTIATING WITH RESPECT TO  $y$

$$t = \tan x \Rightarrow \frac{dt}{dx} = \frac{dt}{dy} \sec^2 x$$

$$\rightarrow \frac{dt}{dy} = \sec^2 x \cdot \frac{dy}{dx}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 x} \frac{dt}{dy}$$

$$\rightarrow \frac{dy}{dx} = \sec^2 x \frac{dt}{dy} \quad \text{as required}$$

**ii)** NON DIFFERENTIATING THE ABOVE EXPRESSION WITH RESPECT TO  $x$

$$\rightarrow \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \sec^2 x \frac{dt}{dy} \right)$$

$$\rightarrow \frac{d^2y}{dx^2} = 2 \sec^2 x \tan x \frac{dt}{dy} + \sec^2 x \frac{d}{dx} \left( \frac{dt}{dy} \right)$$

$$\rightarrow \frac{d^2y}{dx^2} = 2 \sec^2 x \tan x \frac{dt}{dy} + \sec^2 x \frac{d^2t}{dy^2} \frac{dt}{dx}$$

BUT IF  $t = \tan x$

$$\frac{dt}{dx} = \sec^2 x$$

$$\rightarrow \frac{d^2t}{dx^2} = 2 \sec^2 x \tan x \frac{dt}{dy} + \sec^2 x \frac{d^2t}{dy^2} \sec^2 x$$

$$\rightarrow \frac{d^2y}{dx^2} = 2 \frac{dy}{dt} \sec^2 x + 2 \frac{dy}{dt} \sec^2 x \tan x \quad \text{as required}$$

**c)** TRANSFORMING THE GIVEN O.D.E.

$$\rightarrow \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} \tan x - y \sec^4 x = 0$$

$$\rightarrow \left( \frac{d^2y}{dt^2} + 2 \frac{dy}{dt} \sec^2 x \right) \tan x - y \sec^4 x = 0$$

$\rightarrow \frac{d^2y}{dt^2} \sec^2 x - y \sec^4 x = 0$

$$\rightarrow \frac{dy}{dt^2} - y = 0$$

AUXILIARY EQUATION

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

GENERAL SOLUTION IS

$$y = A e^{\lambda t} + B e^{-\lambda t} \quad \text{or } y = Pe^{i\lambda t} + Q\sinht$$

$$y = A e^{\pm t} + B e^{-\pm t} \quad \text{or } y = P\sin(\lambda t) + Q\cos(\lambda t)$$

**Question 11 (\*\*\*\*)**

Show clearly that the substitution  $z = \sin x$ , transforms the differential equation

$$\frac{d^2y}{dx^2} \cos x + \frac{dy}{dx} \sin x - 2y \cos^3 x = 2\cos^5 x,$$

into the differential equation

$$\frac{d^2y}{dz^2} - 2y = 2(1-z^2)$$

proof

$$\begin{aligned}
 & \frac{dy}{dx} = \sin x \\
 & \frac{d^2y}{dx^2} = \cos x \frac{dy}{dx} \\
 & \frac{1}{\cos x} \frac{d^2y}{dx^2} = \frac{dy}{dx} \\
 & \frac{dy}{dx} = \cos x \frac{dy}{dx} \\
 & \frac{dy}{dx} = \tan x
 \end{aligned}$$
  

$$\begin{aligned}
 & z = \sin x \\
 & \frac{dz}{dx} = \cos x \\
 & \frac{d^2z}{dx^2} = -\sin x \\
 & \frac{1}{-\sin x} \frac{d^2z}{dx^2} = \frac{dz}{dx} \\
 & \frac{dz}{dx} = -\csc x
 \end{aligned}$$
  

$$\begin{aligned}
 & \frac{dy}{dx} = \tan x \\
 & \frac{dy}{dx} = \frac{\sin x}{\cos x} \\
 & \frac{dy}{dx} = \frac{-\csc x}{-\csc x} \\
 & \frac{dy}{dx} = 1
 \end{aligned}$$
  

$$\begin{aligned}
 & \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\cos x \frac{dy}{dx}) \\
 & \frac{d^2y}{dx^2} = -\sin x \frac{dy}{dx} + \cos x \frac{d}{dx}(\frac{dy}{dx}) \\
 & \frac{d^2y}{dx^2} = -\sin x \frac{dy}{dx} + \cos x \frac{d}{dx}(\frac{dy}{dx}) \\
 & \frac{d^2y}{dx^2} = -\sin x \frac{dy}{dx} + (\cos x \frac{d}{dx} \frac{dy}{dx}) \\
 & \frac{d^2y}{dx^2} = -\sin x \frac{dy}{dx} + \cos x \frac{d}{dx} \frac{dy}{dx} \\
 & \frac{d^2y}{dx^2} = \cos x \frac{dy}{dx} - \sin x \frac{dy}{dx}
 \end{aligned}$$
  

$$\begin{aligned}
 & \text{Thus} \\
 & \left[ \cos x \frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} \right] \cos x + \left[ \cos x \frac{dy}{dx} \right] \sin x - 2y \cos x = 2\cos^5 x \\
 & \rightarrow \cos x \frac{d^2y}{dx^2} - \sin x \cos x \frac{dy}{dx} + \sin x \cos x \frac{dy}{dx} - 2y \cos x = 2\cos^5 x \\
 & \rightarrow \cos x \frac{d^2y}{dx^2} - 2y \cos x = 2\cos^5 x \\
 & \Rightarrow \frac{d^2y}{dx^2} - 2y = 2\cos^3 x \\
 & \Rightarrow \frac{d^2y}{dx^2} - 2y = 2(1-\sin^2 x) \\
 & \Rightarrow \frac{d^2y}{dx^2} - 2y = 2(1-z^2)
 \end{aligned}$$

↑ 24p

**Question 12** (\*\*\*)+

Use the substitution  $z = \sqrt{y}$ , where  $y = f(x)$ , to solve the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - 5 \frac{dy}{dx} + 2y = 0,$$

subject to the boundary conditions  $y = 4$ ,  $\frac{dy}{dx} = 44$  at  $x = 0$ .

Give the answer in the form  $y = f(x)$ .

$$y = 9e^{6x} - 6e^x + e^{-4x}$$

The image shows handwritten mathematical work for solving the differential equation  $\frac{d^2y}{dx^2} + \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - 5 \frac{dy}{dx} + 2y = 0$  with boundary conditions  $y=4$  and  $\frac{dy}{dx}=44$  at  $x=0$ . The solution involves the substitution  $z = \sqrt{y}$ , leading to a transformed equation and its solution. The final answer is given as  $y = 9e^{6x} - 6e^x + e^{-4x}$ .

**Question 13** (\*\*\*)+

$$2x \frac{d^2y}{dx^2} + \left(1 - 3x^{\frac{1}{2}}\right) \frac{dy}{dx} + y = 0.$$

The above differential equation is to be solved by a substitution.

- a) Given that  $y = f(x)$  and  $t = x^{\frac{1}{2}}$ , show clearly that ...

i. ...  $\frac{dy}{dx} = \frac{1}{2t} \frac{dy}{dt}$ .

ii. ...  $\frac{d^2y}{dx^2} = \frac{1}{4t^2} \frac{d^2y}{dt^2} - \frac{1}{4t^3} \frac{dy}{dt}$ .

- b) Hence show further that the differential equation

$$2x \frac{d^2y}{dx^2} + \left(1 - 3x^{\frac{1}{2}}\right) \frac{dy}{dx} + y = 0,$$

can be transformed to the differential equation

$$\frac{d^2y}{dt^2} - 3 \frac{dy}{dt} + 2y = 0.$$

- c) Find a general solution of the **original** differential equation, giving the answer in the form  $y = f(x)$ .

$$\boxed{y = Ae^{\sqrt{x}} + Be^{2\sqrt{x}}}$$

**a)**  $x = t^2$

$\frac{dy}{dx} = \frac{1}{2t} \frac{dy}{dt}$

$\frac{d^2y}{dx^2} = \frac{1}{2t^2} \frac{d^2y}{dt^2} - \frac{1}{4t^3} \frac{dy}{dt}$

$\frac{dy}{dt} = \frac{1}{2t} \frac{dy}{dx}$

$\frac{d^2y}{dt^2} = \frac{1}{2t^2} \frac{d^2y}{dx^2} + \frac{1}{4t^3} \frac{dy}{dx}$

**b)**  $2x \frac{d^2y}{dx^2} + \left(1 - 3x^{\frac{1}{2}}\right) \frac{dy}{dx} + y = 0$

$\Rightarrow 2t^2 \left[ \frac{1}{2t^2} \frac{d^2y}{dx^2} + \frac{1}{4t^3} \frac{dy}{dx} \right] + \left(1 - 3t^{\frac{1}{2}}\right) \times \frac{1}{2t} \frac{dy}{dx} + y = 0$

$\Rightarrow \frac{1}{2} \frac{d^2y}{dt^2} - \frac{1}{2t} \frac{dy}{dt} + \frac{1}{4t^2} \frac{dy}{dt} - \frac{3}{2} t^{-\frac{1}{2}} \frac{dy}{dt} + y = 0$

$\Rightarrow \frac{1}{2} \frac{d^2y}{dt^2} - \frac{3}{2} \frac{dy}{dt} + y = 0$

$\Rightarrow \frac{d^2y}{dt^2} - 3 \frac{dy}{dt} + 2y = 0$  // required

**c)** Aux equation  
 $x - 3t + 2 = 0$   
 $(t-2)(t-1) = 0$   
 $t=1 \text{ or } t=2$

$\therefore y = Ae^t + Be^{2t}$

But  $t = x^{\frac{1}{2}}$   
 $y = Ae^{\sqrt{x}} + Be^{2\sqrt{x}}$

**Question 14** (\*\*\*)+

Show clearly that the substitution  $z = y^2$ , where  $y = f(x)$ , transforms the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - 5 \frac{dy}{dx} + 2y = 0,$$

into the differential equation

$$\frac{d^2z}{dx^2} - 5 \frac{dz}{dx} + 4z = 0$$

[proof]

$\bullet \quad z = y^2$ Diff w.r.t. x $\frac{dz}{dx} = 2y \frac{dy}{dx}$ $\boxed{\frac{dz}{dx} = \frac{1}{2y} \frac{dy}{dx}}$	$\bullet \quad \frac{dz}{dx} = \frac{1}{2y} \frac{dy}{dx}$ Diff w.r.t. x $\frac{d^2z}{dx^2} = -\frac{1}{2y^2} \frac{dy}{dx} \frac{d^2y}{dx^2} + \frac{1}{2y} \frac{d^2y}{dx^2}$ $\boxed{\frac{d^2z}{dx^2} = \frac{1}{2y} \frac{d^2y}{dx^2} - \frac{1}{2y^2} \frac{dy}{dx} \frac{d^2y}{dx^2}}$
$\bullet \quad \frac{d^2y}{dx^2} + \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - 5 \frac{dy}{dx} + 2y = 0$ $\Rightarrow \frac{1}{2y} \frac{d^2z}{dx^2} - \frac{1}{2y^2} \frac{dy}{dx} \frac{d^2y}{dx^2} + \frac{1}{y} \left( \frac{1}{2y} \frac{dy}{dx} \right)^2 - 5 \left( \frac{1}{2y} \frac{dy}{dx} \right) + 2y = 0$ $\Rightarrow \frac{1}{2y} \frac{d^2z}{dx^2} - \frac{1}{2y^3} \left( \frac{dy}{dx} \right) \frac{d^2y}{dx^2} + \frac{1}{4y^2} \left( \frac{dy}{dx} \right)^2 - \frac{5}{2y} \frac{dy}{dx} + 2y = 0$ $\Rightarrow \frac{1}{2y} \frac{d^2z}{dx^2} - \frac{1}{2y} \left( \frac{dy}{dx} \right)^2 + \frac{1}{4y^2} \left( \frac{dy}{dx} \right)^2 - \frac{5}{2y} \frac{dy}{dx} + 2y = 0$ $\Rightarrow \frac{1}{2y} \frac{d^2z}{dx^2} - \frac{5}{2y} \frac{dy}{dx} + 2y = 0$ $\Rightarrow \frac{d^2z}{dx^2} - 5 \frac{dz}{dx} + 4z = 0$ $\Rightarrow \frac{d^2z}{dx^2} - 5 \frac{dz}{dx} + 4z = 0 \quad \boxed{\text{LHS EQUALS RHS}}$	

**Question 15** (\*\*\*\*+)

Given that if  $x = t^{\frac{1}{2}}$ , where  $y = f(x)$ , show clearly that

a)  $\frac{dy}{dx} = 2t^{\frac{1}{2}} \frac{dy}{dt}$ .

b)  $\frac{d^2y}{dx^2} = 4t \frac{d^2y}{dt^2} + 2 \frac{dy}{dt}$ .

The following differential equation is to be solved

$$x \frac{d^2y}{dx^2} - (8x^2 + 1) \frac{dy}{dx} + 12x^3 y = 12x^5,$$

subject to the boundary conditions  $y = \frac{10}{3}$ ,  $\frac{d^2y}{dx^2} = 10$  at  $x = 0$ .

- c) Show further that the substitution  $x = t^{\frac{1}{2}}$ , where  $y = f(x)$ , transforms the above differential equation into the differential equation

$$\frac{d^2y}{dt^2} - 4 \frac{dy}{dt} + 3y = 3t.$$

- d) Show that a solution of the original differential equation is

$$y = e^{3x^2} + e^{x^2} + x^2 + \frac{4}{3}.$$

proof

(a)  $x = t^{\frac{1}{2}}$

$$\frac{dy}{dx} \text{ w.r.t. } y$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{2}t^{-\frac{1}{2}} \times \frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{2t^{\frac{1}{2}}} \frac{dy}{dt}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2t^{\frac{1}{2}} \frac{dy}{dt}}$$

b) Diff. about  $x$  w.r.t.  $x$

$$\Rightarrow \frac{d^2y}{dx^2} = t^{\frac{1}{2}} \frac{dy}{dx} \frac{du}{dt} + 2t^{\frac{1}{2}} \frac{dy}{dt} \frac{du}{dt}$$

$$\quad \quad \quad \boxed{\frac{d^2y}{dx^2} = 2t^{\frac{1}{2}} \frac{dy}{dt}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \frac{dy}{dt} + 4t^{\frac{1}{2}} \frac{dy}{dt} \times 2t^{\frac{1}{2}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \frac{dy}{dt} + 4t^{\frac{3}{2}} \frac{dy}{dt}$$

c)  $\alpha \frac{dy}{dx} - (8x^2 + 1) \frac{dy}{dx} + 12x^3 y = 12x^5$

$$\Rightarrow t^{\frac{1}{2}} \left[ 4t \frac{d^2y}{dt^2} + 2 \frac{dy}{dt} \right] - (8t^2 + 1) \times t^{\frac{1}{2}} \frac{dy}{dt} + 12t^{\frac{3}{2}} y = 12t^{\frac{5}{2}}$$

divide by  $t^{\frac{1}{2}}$

$$\Rightarrow 4t \frac{d^2y}{dt^2} + 2 \frac{dy}{dt} - (16t^2 + 2) \frac{dy}{dt} + 12ty = 12t^2$$

$$\Rightarrow -4t \frac{dy}{dt} + 2 \frac{dy}{dt} - 16t \frac{dy}{dt} - 2 \frac{dy}{dt} + 12ty = 12t^2$$

$$\Rightarrow 4t \frac{dy}{dt} - 16t \frac{dy}{dt} + 12ty = 12t^2$$

$$\Rightarrow \frac{dy}{dt^2} - 4 \frac{dy}{dt} + 3y = 3t$$

✓ REQUIRES

d) AUX equation

$$t^2 - 4t + 3 = 0$$

$$(t-3)(t-1) = 0$$

$$t_1 = 1, t_2 = 3$$

PARTICULAR SOLUTION

$$\text{Let } y = Pte^{\lambda t} + Qe^{\lambda t}$$

$$\frac{dy}{dt} = Pte^{\lambda t} + Pe^{\lambda t} + Qe^{\lambda t}$$

$$\frac{d^2y}{dt^2} = Pte^{\lambda t} + 2Pe^{\lambda t} + Qe^{\lambda t}$$

$$\text{Sub into the ODE}$$

$$0 - 4P + 3(P + Q) = 3t$$

$$3P + 3Q = 3t$$

$$\boxed{P=1, Q=\frac{q}{3}}$$

$$\therefore y = e^{x^2} + e^{x^2} + x^2 + \frac{4}{3}$$

✓ REQUIRES

✓ GEN SOLUTION  $y = Ae^t + Be^{3t} + t + \frac{4}{3}$

$$\boxed{y = Ae^{x^2} + Be^{\frac{3x^2}{2}} + x^2 + \frac{4}{3}}$$

•  $x=0$   $y = \frac{10}{3} \Rightarrow \frac{10}{3} = A + B + \frac{4}{3} \Rightarrow [A + B = 2]$

$$\frac{dy}{dx} = 2Ax e^{x^2} + 6Be^{\frac{3x^2}{2}} + 2x$$

$$\frac{d^2y}{dx^2} = 2Ae^{x^2} + 4A^2 e^{x^2} + 6Be^{\frac{3x^2}{2}} + 36Be^{\frac{3x^2}{2}} + 2$$

•  $x=0$   $\frac{d^3y}{dx^3} = 10 \Rightarrow 10 = 2A + 6B + 2 \Rightarrow [4 = A + 3B]$

$\therefore B = 1 - A$

$$\therefore y = e^{x^2} + e^{\frac{3x^2}{2}} + x^2 + \frac{4}{3}$$

✓ REQUIRES

**Question 16** (\*\*\*)+

The curve with equation  $y = f(x)$  satisfies

$$x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 13y = 0, \quad x > 0.$$

By using the substitution  $x = e^t$ , or otherwise, determine an equation for  $y = f(x)$ ,

given further that  $y = 1$  and  $\frac{dy}{dx} = -2$  at  $x = 1$ .

$$y = \frac{\cos(3\ln x)}{x^2}$$

**Working:**

Given  $\frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 13y = 0$

Let  $x = e^t$  so  $\frac{dx}{dt} = e^t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{e^t}$

$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{dy}{dt} \cdot \frac{1}{e^t} \right) = \frac{d}{dt} \left( \frac{dy}{dt} \cdot \frac{1}{e^t} \right) \cdot \frac{1}{e^t} + \frac{dy}{dt} \cdot \frac{d}{dx} \left( \frac{1}{e^t} \right)$

$= \frac{d}{dt} \left( \frac{dy}{dt} \right) \cdot \frac{1}{e^t} + \frac{dy}{dt} \cdot \left( -\frac{1}{e^{2t}} \right)$

$= \frac{d^2y}{dt^2} \cdot \frac{1}{e^t} - \frac{dy}{dt} \cdot \frac{1}{e^{2t}}$

$\therefore \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 13y = 0$

$\Rightarrow \frac{d^2y}{dt^2} \cdot \frac{1}{e^t} - \frac{dy}{dt} \cdot \frac{1}{e^{2t}} + 5e^t \frac{dy}{dt} \cdot \frac{1}{e^t} + 13y = 0$

$\Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} + 5e^{2t} \frac{dy}{dt} + 13e^t y = 0$

$\therefore \frac{d^2y}{dt^2} + (5e^{2t} - 1) \frac{dy}{dt} + 13e^t y = 0$

**Now:** Let  $u = 3\ln t$  so  $\frac{du}{dt} = 3$

$\therefore \frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = \frac{dy}{du} \cdot 3$

$\frac{d^2y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{d}{dt} \left( \frac{dy}{du} \cdot 3 \right) = \frac{d}{du} \left( \frac{dy}{du} \cdot 3 \right) \cdot \frac{du}{dt}$

$= \frac{d^2y}{du^2} \cdot 3 + \frac{dy}{du} \cdot 0$

$\therefore \frac{d^2y}{dt^2} + (5e^{2t} - 1) \frac{dy}{dt} + 13e^t y = 0$

$\Rightarrow 3 \frac{d^2y}{du^2} + (5e^{2t} - 1) \cdot 3 \frac{dy}{du} + 13e^t y = 0$

$\Rightarrow 3 \frac{d^2y}{du^2} + (15e^{2t} - 3) \frac{dy}{du} + 13e^t y = 0$

$\therefore y = \frac{1}{3} \left[ 15e^{2t} \ln(u) + 20e^{2t} u + C_1 \right]$

**Given:**  $y = 1$  when  $x = 1$

$\therefore 1 = \frac{1}{3} \left[ 15e^{2 \cdot 1} \ln(1) + 20e^{2 \cdot 1} \cdot 1 + C_1 \right]$

$\therefore C_1 = -30e^2$

$\therefore y = \frac{1}{3} \left[ 15e^{2t} \ln(u) + 20e^{2t} u - 30e^2 \right]$

**Given:**  $\frac{dy}{dx} = -2$  when  $x = 1$

$\therefore -2 = \frac{1}{3} \left[ 15e^{2t} \cdot \frac{1}{e^t} + 20e^{2t} \right] \cdot \frac{1}{e^t} - \frac{1}{3} \left[ 15e^{2t} \ln(u) + 20e^{2t} u - 30e^2 \right] \cdot \frac{1}{e^{2t}}$

$\therefore -2 = \frac{1}{3} \left[ 15e^t + 20e^{2t} \right] - \frac{1}{3} \left[ 15e^{2t} \ln(u) + 20e^{2t} u - 30e^2 \right]$

$\therefore -2 = \frac{1}{3} \left[ 15e^t + 20e^{2t} \right] - \frac{1}{3} \left[ 15e^{2t} \ln(e^t) + 20e^{2t} e^t - 30e^2 \right]$

$\therefore -2 = \frac{1}{3} \left[ 15e^t + 20e^{2t} \right] - \frac{1}{3} \left[ 15e^{3t} + 20e^{3t} - 30e^2 \right]$

$\therefore -2 = \frac{1}{3} \left[ 15e^t + 20e^{2t} - 15e^{3t} - 20e^{3t} + 30e^2 \right]$

$\therefore -2 = \frac{1}{3} \left[ 15e^t - 5e^{3t} + 30e^2 \right]$

$\therefore -6 = 15e^t - 5e^{3t} + 90e^2$

$\therefore 5e^{3t} - 15e^t - 90e^2 = 0$

$\therefore 5e^t (e^{2t} - 3) = 90e^2$

$\therefore e^{2t} - 3 = 18$

$\therefore e^{2t} = 21$

$\therefore 2t = \ln(21)$

$\therefore t = \frac{1}{2} \ln(21)$

$\therefore y = \frac{1}{3} \left[ 15e^{2t} \ln(u) + 20e^{2t} u - 30e^2 \right]$

**Final answer:**  $y = \frac{1}{3} \left[ 15e^{2t} \ln(u) + 20e^{2t} u - 30e^2 \right]$

**Working:**

Given  $\frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 13y = 0$

Let  $x = e^t$  so  $\frac{dx}{dt} = e^t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{e^t}$

$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{dy}{dt} \cdot \frac{1}{e^t} \right) = \frac{d}{dt} \left( \frac{dy}{dt} \cdot \frac{1}{e^t} \right) \cdot \frac{1}{e^t} + \frac{dy}{dt} \cdot \frac{d}{dx} \left( \frac{1}{e^t} \right)$

$= \frac{d}{dt} \left( \frac{dy}{dt} \right) \cdot \frac{1}{e^t} + \frac{dy}{dt} \cdot \left( -\frac{1}{e^{2t}} \right)$

$= \frac{d^2y}{dt^2} \cdot \frac{1}{e^t} - \frac{dy}{dt} \cdot \frac{1}{e^{2t}}$

$\therefore \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 13y = 0$

$\Rightarrow \frac{d^2y}{dt^2} \cdot \frac{1}{e^t} - \frac{dy}{dt} \cdot \frac{1}{e^{2t}} + 5e^t \frac{dy}{dt} \cdot \frac{1}{e^t} + 13e^t y = 0$

$\Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} + 5e^{2t} \frac{dy}{dt} + 13e^t y = 0$

$\therefore \frac{d^2y}{dt^2} + (5e^{2t} - 1) \frac{dy}{dt} + 13e^t y = 0$

**Now:** Let  $u = 3\ln t$  so  $\frac{du}{dt} = 3$

$\therefore \frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = \frac{dy}{du} \cdot 3$

$\frac{d^2y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{d}{dt} \left( \frac{dy}{du} \cdot 3 \right) = \frac{d}{du} \left( \frac{dy}{du} \cdot 3 \right) \cdot \frac{du}{dt}$

$= \frac{d^2y}{du^2} \cdot 3 + \frac{dy}{du} \cdot 0$

$\therefore \frac{d^2y}{dt^2} + (5e^{2t} - 1) \frac{dy}{dt} + 13e^t y = 0$

$\Rightarrow 3 \frac{d^2y}{du^2} + (5e^{2t} - 1) \cdot 3 \frac{dy}{du} + 13e^t y = 0$

$\Rightarrow 3 \frac{d^2y}{du^2} + (15e^{2t} - 3) \frac{dy}{du} + 13e^t y = 0$

$\therefore y = \frac{1}{3} \left[ 15e^{2t} \ln(u) + 20e^{2t} u + C_1 \right]$

**Given:**  $y = 1$  when  $x = 1$

$\therefore 1 = \frac{1}{3} \left[ 15e^{2 \cdot 1} \ln(1) + 20e^{2 \cdot 1} \cdot 1 + C_1 \right]$

$\therefore C_1 = -30e^2$

$\therefore y = \frac{1}{3} \left[ 15e^{2t} \ln(u) + 20e^{2t} u - 30e^2 \right]$

**Given:**  $\frac{dy}{dx} = -2$  when  $x = 1$

$\therefore -2 = \frac{1}{3} \left[ 15e^{2t} \cdot \frac{1}{e^t} + 20e^{2t} \right] \cdot \frac{1}{e^t} - \frac{1}{3} \left[ 15e^{2t} \ln(u) + 20e^{2t} u - 30e^2 \right] \cdot \frac{1}{e^{2t}}$

$\therefore -2 = \frac{1}{3} \left[ 15e^t + 20e^{2t} \right] - \frac{1}{3} \left[ 15e^{2t} \ln(e^t) + 20e^{2t} e^t - 30e^2 \right]$

$\therefore -2 = \frac{1}{3} \left[ 15e^t + 20e^{2t} - 15e^{3t} - 20e^{3t} + 30e^2 \right]$

$\therefore -2 = \frac{1}{3} \left[ 15e^t - 5e^{3t} + 30e^2 \right]$

$\therefore -6 = 15e^t - 5e^{3t} + 90e^2$

$\therefore 5e^{3t} - 15e^t - 90e^2 = 0$

$\therefore 5e^t (e^{2t} - 3) = 90e^2$

$\therefore e^{2t} - 3 = 18$

$\therefore e^{2t} = 21$

$\therefore 2t = \ln(21)$

$\therefore t = \frac{1}{2} \ln(21)$

$\therefore y = \frac{1}{3} \left[ 15e^{2t} \ln(u) + 20e^{2t} u - 30e^2 \right]$

**Final answer:**  $y = \frac{1}{3} \left[ 15e^{2t} \ln(u) + 20e^{2t} u - 30e^2 \right]$

**Question 17** (\*\*\*\*+)

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} \cot x + 2y \operatorname{cosec}^2 x = 2\cos x - 2\cos^3 x.$$

Use the substitution  $y = z \sin x$ , where  $z$  is a function of  $x$ , to solve the above differential equation subject to the boundary conditions  $y=1$ ,  $\frac{dy}{dx}=0$  at  $x=\frac{\pi}{2}$ .

Give the answer in the form

$$y = a \sin^2 x + b(1 - \sin x) \sin 2x,$$

where  $a$  and  $b$  are constants to be found.

$$a = 1, \quad b = \frac{1}{3}$$

• Now auxiliary equation  $\lambda^2 + 1 = 0$   
 $\lambda = \pm i$

• Particular integral, try  $y = P \sin x$ ,  
 $P = 2P_0 \sin x$   
 $Z = 4P_0 \sin x$

Then  $-4P_0 \sin^2 x + P_0 \sin x \equiv \sin x$   
 $-SP = 1$   
 $P = \frac{1}{3}$

• General solution  $z = A \cos x + B \sin x - \frac{1}{3} \sin x$   
 $\frac{dz}{dx} = A \cos x + B \sin x - \frac{1}{3} \cos x$   
 $y = A \cos x + B \sin x - \frac{1}{3} \cos x \sin x$   
 $y = B \sin x + A \cos x - \frac{1}{3} \sin x \sin x$

Now  $2 - \sqrt{3} \neq 1 \Rightarrow \boxed{1 \text{ B}}$   
 $y = B \sin x + A \cos x - \frac{1}{3} \cos x \sin x$   
 $\frac{dy}{dx} = 2B \cos x + A \sin x - \frac{1}{3} \cos x \sin x - \frac{1}{3} \cos^2 x$   
 $\text{And } 2 + \frac{\pi}{2} \cdot \frac{dy}{dx} = 0 \Rightarrow 0 = 2A - 0 - \frac{2}{3}(-1)$   
 $\Rightarrow 2A = \frac{2}{3}$   
 $\Rightarrow A = \frac{1}{3}$   
 $\therefore y = B \sin x + \left(\frac{1}{3} \cos x - \frac{1}{3} \cos x \sin x\right)$   
 $y = B \sin x + \frac{1}{3} \cos x \left[1 - \sin x\right]$

**Question 18** (\*\*\*)+

The function  $y = f(x)$  satisfies the following relationship.

$$4x \frac{d^2y}{dx^2} + 4x \left( \frac{dy}{dx} \right)^2 + 2 \frac{dy}{dx} - 1 = 0.$$

It is further given that  $x = t^2$  and  $y = \ln v$ .

Show that

$$\frac{d^2v}{dt^2} = v.$$

proof

LET  $x = t^2$  ————— DIFFERENTIATE w.r.t  $t$

$\frac{dx}{dt} = 2t$

$\frac{d}{dt} = \frac{1}{2t} \frac{d}{dx}$

$\frac{d^2}{dt^2} = \left( \frac{1}{2t} \right) \frac{d}{dx} + \frac{1}{2} \left( \frac{d}{dx} \right)^2$

$\frac{d^2}{dt^2} = -\frac{1}{2t^2} \frac{d}{dx} + \frac{1}{2t} \frac{d^2}{dx^2}$

$\frac{d}{dx} = \frac{1}{4t^2} \frac{d}{dt} - \frac{1}{2t} \frac{d^2}{dt^2}$

SUBSTITUTING w.r.t  $x$

$\Rightarrow 4t \left[ \frac{1}{4t^2} \frac{d}{dt} - \frac{1}{2t} \frac{d^2}{dt^2} \right] + 4t^2 \left[ \frac{1}{4t^2} \frac{d}{dt} \right]^2 + 2 \left[ \frac{1}{4t^2} \frac{d}{dt} \right] - 1 = 0$

$\Rightarrow \frac{d}{dt} - \frac{1}{2t} \frac{d^2}{dt^2} + \left( \frac{d}{dt} \right)^2 + \frac{1}{2t} \frac{d}{dt} - 1 = 0$

$\Rightarrow \frac{\frac{dy}{dt}}{dt^2} + \left( \frac{dy}{dt} \right)^2 - 1 = 0$

NEXT w.r.t  $v$

$y = \ln v$

DIFFERENTIATE w.r.t  $t$

$\frac{dy}{dt} = \frac{1}{v} \frac{dv}{dt}$

$\frac{d^2y}{dt^2} = \left( \frac{1}{v} \frac{dv}{dt} \right) \frac{dv}{dt} + \frac{1}{v^2} \frac{d^2v}{dt^2}$

$\frac{d^2y}{dt^2} = \frac{1}{v} \frac{d^2v}{dt^2} - \frac{1}{v^2} \left( \frac{dv}{dt} \right)^2$

FURTHER SUBSTITUTING INTO THE EQUATION

$\Rightarrow \frac{1}{v} \frac{d^2v}{dt^2} - \frac{1}{v^2} \left( \frac{dv}{dt} \right)^2 + \left( \frac{1}{v} \frac{dv}{dt} \right)^2 - 1 = 0$

$\Rightarrow \frac{1}{v} \frac{d^2v}{dt^2} - \frac{1}{v^2} \left( \frac{dv}{dt} \right)^2 + \frac{1}{v^2} \left( \frac{dv}{dt} \right)^2 - 1 = 0$

$\Rightarrow \frac{1}{v} \frac{d^2v}{dt^2} = 1$

$\Rightarrow \frac{d^2v}{dt^2} = v$

**Question 19** (\*\*\*\*+)

$$x \frac{d^2y}{dx^2} - \frac{dy}{dx} - x^3 y + x^5 = 0.$$

Use the substitution  $x = z^{\frac{1}{2}}$ , where  $y = f(x)$ , to find a general solution of the above differential equation.

V,  $\boxed{\quad}$ ,  $y = A e^{\frac{1}{2}x^2} + B e^{-\frac{1}{2}x^2 + x^2}$

SUBST. WITH THE SUBSTITUTION GIVEN

Given  $x = z^{\frac{1}{2}}$   
 $\frac{dx}{dz} = \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dz}$   
 $\frac{dx}{dz} = \frac{1}{2z^{\frac{1}{2}}}$   
 $\frac{d^2x}{dz^2} = -\frac{1}{4}z^{-\frac{3}{2}}$   
 $\frac{dy}{dx} = \frac{1}{2z^{\frac{1}{2}}}\frac{dy}{dz}$   
 $\frac{d^2y}{dx^2} = -\frac{1}{4}z^{-\frac{5}{2}}\frac{dy}{dz}$

NOW TAKE THE SECOND DERIVATIVES

$$\begin{aligned}\frac{dy}{dz} &= 2z^{\frac{1}{2}}\frac{dy}{dx} \\ \frac{d^2y}{dz^2} &= -\frac{1}{2}z^{-\frac{3}{2}}\frac{d}{dz}\left(\frac{dy}{dx}\right) + 2z^{\frac{1}{2}}\frac{d}{dz}\left(\frac{dy}{dx}\right) \\ \frac{d^2y}{dz^2} &= \frac{d}{dz}\left[\frac{1}{2}z^{\frac{1}{2}}\frac{dy}{dx} + 2z^{\frac{1}{2}}\frac{dy}{dx}\right] \\ \frac{d^2y}{dz^2} &= 2z^{\frac{1}{2}}\left[\frac{1}{2}\frac{d}{dz}\left(z^{\frac{1}{2}}\right)\frac{dy}{dx} + 2z^{\frac{1}{2}}\frac{dy}{dx}\right] \\ \frac{d^2y}{dz^2} &= 2\frac{dy}{dx} + 4z^{\frac{1}{2}}\frac{dy}{dx}\end{aligned}$$

NOW SUBSTITUTE INTO THE O.D.E.

$$\begin{aligned}&\rightarrow 2\frac{d^2y}{dz^2} - \frac{dy}{dz} - x^3 y + x^5 = 0 \\ &\rightarrow 2^{\frac{1}{2}}\left[2\frac{dy}{dx} + 4z^{\frac{1}{2}}\frac{dy}{dx}\right] - 2z^{\frac{1}{2}}\frac{dy}{dx} - z^{\frac{3}{2}} + z^{\frac{5}{2}} = 0 \\ &\rightarrow 2^{\frac{3}{2}}\frac{dy}{dx} + 4z^{\frac{3}{2}}\frac{dy}{dx} - 2z^{\frac{1}{2}}\frac{dy}{dx} - z^{\frac{3}{2}} + z^{\frac{5}{2}} = 0\end{aligned}$$

$\Rightarrow 4z^{\frac{3}{2}}\frac{dy}{dx} - y + z^{\frac{5}{2}} = 0$   
 $\Rightarrow 4\frac{dy}{dz} - y + 2z^{\frac{3}{2}} = 0$

ALGEBRAIC EQUATION FOR  $z^{\frac{3}{2}}\frac{dy}{dz} - y = 2z^{\frac{3}{2}}$

$$\begin{aligned}4z^{\frac{3}{2}} - 1 &= 0 \\ z^{\frac{3}{2}} &= \frac{1}{4} \\ z &= \pm\sqrt{\frac{1}{4}}\end{aligned}$$

PARTIAL INTEGRATION (BY INSPECTION)

$$y = z$$

GENERAL SOLUTION IS

$$y = A e^{\frac{1}{2}z^2} + B e^{-\frac{1}{2}z^2 + z^2}$$

$$y = A e^{\frac{1}{2}z^2} + B e^{z^2 - \frac{1}{2}z^2}$$

**Question 20** (\*\*\*\*\*)

Find the solution of following differential equation

$$\left( \frac{dy}{dx} \right) \left( \frac{d^2y}{dx^2} \right) = \frac{d^3y}{dx^3},$$

subject to the boundary conditions.

$$y\left(-\frac{1}{2}\pi\right) = y'\left(-\frac{1}{2}\pi\right) = 0, \quad y''\left(-\frac{1}{2}\pi\right) = \frac{1}{2}.$$

Given the answer in the form  $y = f(x)$ .

,  $y = 2 \ln \left| \sec \left( \frac{1}{2}x + \frac{1}{4}\pi \right) \right|$

By substitution — let  $p = \frac{dy}{dx}$ , & SEPARATE VARIABLES.

$$\Rightarrow \frac{dp}{dx} \times \frac{d^2y}{dx^2} = \frac{d^3y}{dx^3}$$

$$\Rightarrow p \frac{dp}{dx} = \frac{d^2p}{dx^2}$$

$$\Rightarrow \int p \, dp = \int \frac{d^2p}{dx^2} \, dx$$

$$\Rightarrow \frac{1}{2}p^2 = \frac{d^2p}{dx^2} + A$$

$$\Rightarrow p^2 = 2 \frac{dp}{dx} + A$$

APPLY CONDITION  $x = -\frac{\pi}{2}, \frac{dp}{dx} = 0, \frac{d^2p}{dx^2} = \frac{dp}{dx} = \frac{1}{2}$

$$\Rightarrow 0 = 2x\frac{1}{2} + A \Rightarrow A = -1$$

$$\Rightarrow p^2 = 2\frac{dp}{dx} - 1$$

REARRANGE & SEPARATE VARIABLES AGAIN

$$\Rightarrow p^2 + 1 = 2\frac{dp}{dx}$$

$$\Rightarrow 1 \, dx = \frac{2}{p^2 + 1} \, dp$$

$$\Rightarrow \int \frac{1}{p^2 + 1} \, dp = \int 1 \, dx$$

$$\Rightarrow 2 \arctan p = x + B$$

$$\Rightarrow \arctan p = \frac{1}{2}x + B$$

$$\Rightarrow p = \tan\left(\frac{1}{2}x + B\right)$$

$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

APPLY THE BOUNDARY CONDITION  $x = -\frac{\pi}{2}, \frac{dy}{dx} = 0$

$$\Rightarrow 0 = \tan\left(-\frac{\pi}{2} + \frac{\pi}{4}\right) \Rightarrow \frac{dy}{dx} = 0$$

EASY AS THIS IN FACT COMES FROM PAST PAPERS

$$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$$

Finalise the above BY DIRECT INTEGRATION,

$$\frac{dy}{dx} = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$$

$$y = 2 \ln \left| \sec\left(\frac{1}{2}x + \frac{\pi}{4}\right) \right| + C$$

APPLY THE LAST CONDITION  $x = -\frac{\pi}{2}, y = 0$

$$\Rightarrow 0 = 2 \ln \left| \sec\left(-\frac{\pi}{2} + \frac{\pi}{4}\right) \right| + C \Rightarrow 0 = 2 \ln(1) + C \Rightarrow 0 = 2 \ln 1 + C \Rightarrow C = 0$$

$\therefore y = 2 \ln \left| \sec\left(\frac{1}{2}x + \frac{\pi}{4}\right) \right|$

**Question 21** (\*\*\*\*\*)

Use a suitable substitution to solve the differential equation

$$x^2 \frac{d^2y}{dx^2} - 6y = 2 - 2\ln x - 6(\ln x)^2,$$

subject to the boundary conditions  $y(1) = 1$ ,  $\frac{dy}{dx}(1) = 3$

Give a simplified answer in the form  $y = f(x)$ .

$$\boxed{\text{Answer}}, \quad y = x^3 + (\ln x)^2$$

<p><b>AUXILIARY EQUATION (L.H.S.)</b></p> $t^2 \frac{d^2y}{dt^2} - 6y = 2 - 2\ln t - 6(\ln t)^2 \quad t=1, y=1, \frac{dy}{dt}=3$ <p>LOOKING AT THE R.H.S., WE TRY THE SUBSTITUTION <math>t=\ln x</math></p> <p><math>t=\ln x \Rightarrow x=e^t</math>      DIFF. w.r.t. <math>x</math></p> <p><math>\frac{dt}{dx} = \frac{1}{x}, \frac{dy}{dx} = e^t \frac{dy}{dt}</math></p> <p><math>\frac{d^2y}{dx^2} = \frac{d}{dx} \left( e^t \frac{dy}{dt} \right) = e^t \frac{d^2y}{dt^2} + e^{2t} \frac{dy}{dt} \quad \text{DIFF. w.r.t. } t</math></p> <p><math>\frac{d^2y}{dt^2} = e^{-2t} \frac{d^2y}{dx^2} - e^{-2t} \frac{dy}{dt}</math></p> <p>SUBSTITUTE INTO THE O.D.E. &amp; TRY</p> $\begin{aligned} &\Rightarrow x^2 \frac{d^2y}{dx^2} - 6y = 2 - 2\ln x - 6(\ln x)^2 \\ &\Rightarrow e^{2t} \left[ e^{-2t} \frac{d^2y}{dx^2} - e^{-2t} \frac{dy}{dt} \right] - 6y = 2 - 2t - t^2 \\ &\Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 2 - 2t - t^2 \quad (\text{which can easily be solved}) \end{aligned}$	<p><b>PARTICULAR INTEGRAL (R.H.S.)</b></p> $\begin{aligned} &\Rightarrow t^2 - 2 - 6 = 0 \\ &\Rightarrow (t+2)(t-3) = 0 \\ &\Rightarrow t = -2, 3 \end{aligned}$ <p><b>GENERAL INTEGRAL</b></p> $y = At^2 + Bt^3$ $\begin{aligned} &\Rightarrow 2t^2 - (2t^2 + 4t + 2) - 6(t^2 + 4t + 2) \\ &\equiv 2 - 2t - 6t^2 \\ &\Rightarrow -6t^2 + (-2t^2 - 6t) + (2t - 4t - 2) \\ &\equiv 2 - 2t - 6t^2 \\ &\Rightarrow P=1, \quad -2P-Q=-2, \quad 2P-Q=2 \\ &\Rightarrow -2-6=-2, \quad 2-6=2 \\ &\Rightarrow Q=0, \quad P=1 \end{aligned}$ <p><b>GIVEN SOLUTION:</b> <math>y = Ae^{-2t} + Be^{-3t} + t^2</math></p> <p>REWRITE IN <math>x</math>: <math>y = Ax^3 + Bx^2 + (\ln x)^2</math></p> <p><b>DIFFERENTIATE &amp; APPLY CONDITIONS:</b> <math>x=1, y=1, \frac{dy}{dx}=3</math></p> $\begin{aligned} &\bullet y = \frac{A}{3}x^3 + Bx^2 + (\ln x)^2 \quad \bullet \frac{dy}{dx} = -\frac{2A}{3}x^2 + 2Bx^2 + 2\ln x \\ &1 = A + B \quad 3 = -\frac{2A}{3} + 3B + 2 \\ &2 = 2A + 2B \quad \downarrow \\ &5B = 5 \quad B=1 \quad A=0 \\ &\therefore y = x^3 + (\ln x)^2 \end{aligned}$
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**Question 22** (\*\*\*\*\*)

The function with equation  $y = f(x)$  satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{y} \left( \frac{dy}{dx} \right)^2 = 2y \ln 3, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 2\ln 3.$$

Solve the above differential equation by using the substitution  $p = \frac{dy}{dx}$ , to show that

$$y = 3^{x^2+2x}.$$

[ ] , [ proof]

USING THE SUBSTITUTION  $p = \frac{dy}{dx}$  AS THE INDEPENDENT VARIABLE IS MISSING

$$\begin{aligned} p &= \frac{dy}{dx} \rightarrow \frac{dp}{dy} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2} \times \frac{1}{p} \\ &\Rightarrow \frac{dp}{dy} = \frac{d^2y}{dx^2} \times \frac{1}{p} \\ &\Rightarrow \frac{dp}{dy} = p \frac{d}{dx} \end{aligned}$$

TRANSFORMING THE O.D.E.

$$\begin{aligned} &\rightarrow \frac{dp}{dy} - \frac{1}{p} = 2y \ln 3 \\ &\rightarrow p \frac{dp}{dy} - \frac{1}{y} p^2 = 2y \ln 3 \\ &\rightarrow \frac{dp}{dy} - \frac{p}{y} = \frac{2y \ln 3}{p} \end{aligned}$$

USING ANOTHER SUBSTITUTION  $v = \frac{p}{y}$

$$\begin{aligned} p &= vy \\ \frac{dp}{dy} &= v \frac{dy}{dx} + y \end{aligned}$$

TRANSFORMING THE O.D.E. FURTHER

$$\begin{aligned} &\rightarrow (v \frac{dy}{dx} + y) - vy = \frac{2y \ln 3}{p} \\ &\Rightarrow v \frac{dy}{dy} = \frac{2y \ln 3}{y} \\ &\Rightarrow \int v \, dv = (2\ln 3) \int \frac{1}{y} \, dy \\ &\Rightarrow \frac{1}{2}v^2 = (2\ln 3)(\ln y) + C \end{aligned}$$

$\rightarrow y^2 = (4\ln 3)(\ln y) + C$

APPENDIX: OVER THE LAST TRANSFORMATION IS DIVIDED

$$\begin{aligned} &\rightarrow \left( \frac{dy}{dx} \right)^2 = (4\ln 3)(\ln y) + C \\ &\rightarrow p^2 = (4\ln 3)(y^2 \ln y) + Cy^2 \\ &\text{At } y=0, \quad y=1, \quad \frac{dy}{dx} = p = 2\ln 3 \\ &\rightarrow (2\ln 3)^2 = (4\ln 3)\sqrt{1} \times \ln 1 + C \times 1^2 \\ &\Rightarrow C = 4(\ln 3)^2 \end{aligned}$$

$$\begin{aligned} &\rightarrow 2(\ln 3)^2 = 2(\ln 3)^{\frac{1}{2}} + A \\ &\rightarrow \sqrt{4(\ln 3)^2} = 2\sqrt{(\ln 3)^2} + B \\ &\text{APPENDIX: } 2=0, \quad 4=1 \\ &\rightarrow \sqrt{4(\ln 3)^2} = 0\sqrt{4(\ln 3)^2} + 3 \\ &\rightarrow B = 3 \\ &\text{FINALLY WE GET:} \\ &\rightarrow \sqrt{4(\ln 3)^2} = 2\sqrt{(\ln 3)^2} + 1 \\ &\rightarrow \sqrt{4(\ln 3)^2} = (2+1)\sqrt{(\ln 3)^2} \\ &\rightarrow 2\ln 3 = (2+1)^2 \ln 3 \\ &\rightarrow 2\ln 3 = e^{(2+1)^2 \ln 3} \\ &\rightarrow 2\ln 3 = (e^2)^{2+1} \\ &\rightarrow 2\ln 3 = 2^{2+1} \\ &\rightarrow 2\ln 3 = 3^{2+1} \\ &\rightarrow 2\ln 3 = \frac{3^{2+2+1}}{3} \\ &\rightarrow 2\ln 3 = 3^{2+2+1} \end{aligned}$$

## Question 23 (\*\*\*\*\*)

$$4x \frac{d^2y}{dx^2} + 4x \left( \frac{dy}{dx} \right)^2 + 2 \frac{dy}{dx} = 1.$$

By using the substitution  $t = \sqrt{x}$ , or otherwise, show that the general solution of the above differential equation is

$$y = A - \sqrt{x} + \ln \left[ 1 + B e^{2\sqrt{x}} \right],$$

where  $A$  and  $B$  are arbitrary constants.

,  proof

$4x \frac{d^2y}{dx^2} + 4x \left( \frac{dy}{dx} \right)^2 + 2 \frac{dy}{dx} = 1$

LET  $x = t^2$   $\rightarrow \frac{dy}{dt} = 2t \frac{dy}{dx}$

$\frac{d^2y}{dt^2} = 2t \frac{d}{dt} \left( \frac{dy}{dt} \right) = 2t \frac{d^2y}{dx^2} + 4t^2 \frac{dy}{dx^2}$

$\frac{d^2y}{dt^2} = 2t \frac{d}{dt} \left( \frac{dy}{dt} \right) = 2t \frac{d}{dt} \left( \frac{dy}{dt} \right)^2 + 2t \frac{d}{dt} \left( \frac{dy}{dt} \right)$

$\frac{d^2y}{dt^2} = 2t \left( \frac{d}{dt} \left( \frac{dy}{dt} \right)^2 \right) - \frac{1}{2} \frac{d}{dt} \left( \frac{dy}{dt} \right)^2 + \frac{1}{2} \frac{d}{dt} \left( \frac{dy}{dt} \right)$

$\frac{d^2y}{dt^2} = -\frac{1}{2} \frac{d}{dt} \left( \frac{dy}{dt} \right)^2 + \frac{1}{2} \frac{d}{dt} \left( \frac{dy}{dt} \right)$

$\frac{d^2y}{dt^2} = -\frac{1}{2} \frac{d}{dt} \left( \frac{dy}{dt} \right)^2 + \frac{1}{2} \frac{d}{dt} \left( \frac{dy}{dt} \right)$

$\frac{d^2y}{dt^2} = -\frac{1}{2} \left( \frac{d}{dt} \left( \frac{dy}{dt} \right)^2 \right) + \frac{1}{2} \frac{d}{dt} \left( \frac{dy}{dt} \right)$

$\frac{d^2y}{dt^2} = \frac{1}{2t} \frac{d}{dt} \left( \frac{dy}{dt} \right)^2 - \frac{1}{2} \frac{d}{dt} \left( \frac{dy}{dt} \right)$

SUBSTITUTING INTO THE O.D.E AND SIMPLIFYING

$$\rightarrow 4t \left[ \frac{1}{2t} \frac{d}{dt} \left( \frac{dy}{dt} \right)^2 - \frac{1}{2} \frac{d}{dt} \left( \frac{dy}{dt} \right) \right] + 4t \left[ \frac{dy}{dt} \right]^2 + 2 \left[ \frac{1}{2t} \frac{d}{dt} \left( \frac{dy}{dt} \right) \right] = 1$$

$$\rightarrow \frac{d^2y}{dt^2} - \frac{1}{2} \frac{d}{dt} \left( \frac{dy}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \frac{1}{2} \frac{d}{dt} \left( \frac{dy}{dt} \right) = 1$$

$\Rightarrow \frac{d^2y}{dt^2} + \left( \frac{dy}{dt} \right)^2 = 1$

$\Rightarrow \frac{1}{2} \left( \frac{dy}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = 1$

ANOTHER CRUCIAL SUBSTITUTION IS  $v = \frac{dy}{dt}$

$\Rightarrow \frac{1}{2} \frac{dv^2}{dt^2} + v^2 = 1$

$\Rightarrow \frac{dv}{dt} = 1 - v^2$

$\Rightarrow \frac{1}{1-v^2} dv = dt$

$\Rightarrow \int \frac{1}{1-v^2} dv = \int dt$

$\Rightarrow \int \frac{1}{1-\frac{v^2}{v^2+1}} dv = \int dt$

$\Rightarrow \int \frac{1}{\frac{v^2+1-v^2}{v^2+1}} dv = \int dt$

$\Rightarrow \int \frac{1}{\frac{1}{v^2+1}} dv = \int dt$

$\Rightarrow \int v^2+1 dv = \int dt$

$\Rightarrow \ln|v^2+1| - \ln|v+1| = t + C$

$\Rightarrow \ln|\frac{v^2+1}{v+1}| = t + C$

$\Rightarrow \frac{v^2+1}{v+1} = e^{t+C}$

$\Rightarrow \frac{1-p}{1-p} = Ae^{t+C}$

$\Rightarrow 1+p = Ae^{-t-C}$

$\Rightarrow p(Ae^{-t-C}) = Ae^{-t-C}-1$

$\Rightarrow p = \frac{Ae^{-t-C}-1}{Ae^{-t-C}+1}$

PROVED BY INTEGRATION USING A SUBSTITUTION

$y = \int \frac{1}{4t^2+1} dt$

$y = \int \frac{(v-1)-1}{v} \frac{dv}{2(v-1)}$

$y = \frac{1}{2} \int \frac{v-2}{v(v-1)} dv$

FRACTIONAL FUNCTIONS BY INTEGRATION

$y = \frac{1}{2} \int \frac{\frac{v-2}{v}}{v-1} dv$

$y = \frac{1}{2} \left[ 2 \ln|v| - \ln|v-1| \right] + B$

$y = \ln|Ae^{\frac{2v}{v-1}}| - \frac{1}{2} \ln|v+e^{\frac{2v}{v-1}}| + B$

$y = \ln|Ae^{\frac{2v}{v-1}}| - \frac{1}{2} \ln|A - \frac{1}{2}e^{\frac{2v}{v-1}}| + B$

$y = \ln|Ae^{\frac{2v}{v-1}}| - \frac{1}{2} \ln A - \frac{1}{2} \ln e^{\frac{2v}{v-1}} + B$

$y = \ln|Ae^{\frac{2v}{v-1}}| \left( -\frac{1}{2} + \frac{1}{2}e^{\frac{2v}{v-1}} \right) + B$

$y = \ln|Ae^{\frac{2v}{v-1}}| \left( -\frac{1}{2} + \frac{1}{2}e^{\frac{2v}{v-1}} \right) + B$