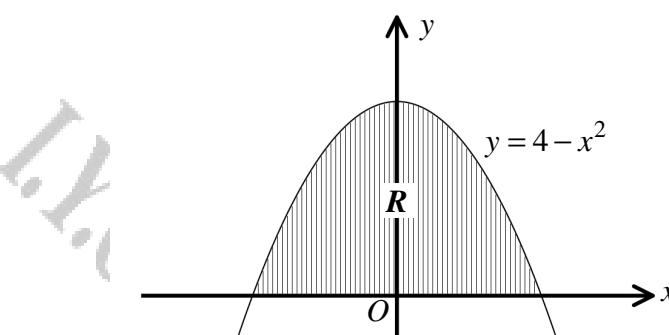


INTEGRATION

VOLUME OF REVOLUTION

Question 1 (***)



The figure above shows the graph of the curve with equation

$$y = 4 - x^2$$

The shaded region R , is bounded by the curve and the x axis.

The region R is rotated through 2π radians about the x axis to form a solid of revolution.

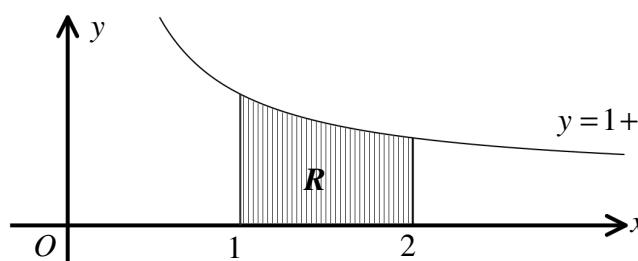
Show that the volume of the solid is $\frac{256\pi}{15}$.

, proof

AS THE QUADRATIC IS SYMMETRICAL CONSIDER THE INTEGRAL BY π
 OR HALF THE AREA (OR THE ENTIRE AREA BY π)

$$\begin{aligned}
 y &= 4 - x^2 = (2-x)(2+x) \\
 \Rightarrow V &= \pi \int_{-2}^{2} [y(x)]^2 dx \\
 \Rightarrow V &= \pi \int_{0}^{2} (4-x^2)^2 dx \\
 \Rightarrow V &= \pi \int_{0}^{2} 16-9x^2+2x^4 dx \\
 \Rightarrow V &= \pi \left[16x - \frac{9}{3}x^3 + \frac{2}{5}x^5 \right]_0^2 \\
 \Rightarrow V &= \pi \left[(32 - \frac{51}{3} + \frac{32}{5}) - 0 \right] \\
 \therefore V &= \frac{256\pi}{15}
 \end{aligned}$$

Question 2 (***)



The figure above shows part of the graph of the curve with equation

$$y = 1 + \frac{2}{x}, \quad x \neq 0.$$

The region R , shown shaded in the figure above, is bounded by the curve, the straight lines with equations $x=1$ and $x=2$, and the x axis.

The region R is rotated through 360° about the x axis to form a solid of revolution.

Show that the volume of the solid is

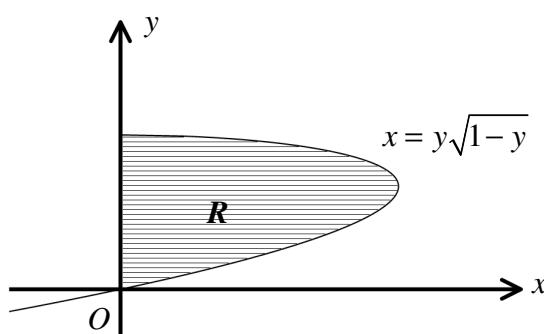
$$\pi(3 + 4\ln 2).$$

, [proof]

A small rectangular window containing a diagram of the shaded region R and the following mathematical derivation:

$$\begin{aligned} V &= \pi \int_{1/x}^{\infty} (y)^2 dx = \pi \int_1^2 (1 + \frac{2}{x})^2 dx \\ &\Leftrightarrow V = \pi \int_1^2 1 + \frac{4}{x} + \frac{4}{x^2} dx \\ &\Leftrightarrow V = \pi \left[x + 4\ln|x| - \frac{4}{x} \right]_1^2 \\ &\Leftrightarrow V = \pi \left[(2 + 4\ln 2 - 2) - (1 + 4\ln 1 - 4) \right] \\ &\Leftrightarrow V = \pi (3 + 4\ln 2) \end{aligned}$$

Question 3 (***)



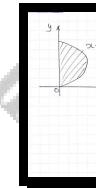
The figure above shows part of the graph of the curve with equation

$$x = y\sqrt{1-y}, \quad y \leq 1.$$

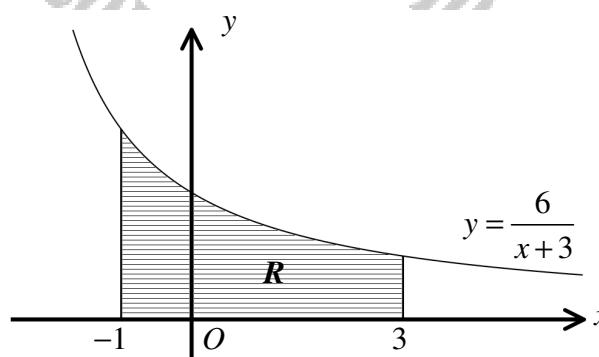
The shaded region R , bounded by the curve and the y axis is rotated through 2π radians about the y axis to form a solid of revolution.

Show that the volume of the solid is $\frac{\pi}{12}$.

proof

	$\begin{aligned} \text{When } x=0, \\ 0 &= y\sqrt{1-y} \\ y &= 1 \end{aligned}$ $\begin{aligned} V &= \pi \int_{y_1}^{y_2} x^2 dy \\ V &= \pi \int_0^1 y^2(1-y) dy \\ V &= \pi \int_0^1 y^2 - y^3 dy \\ V &= \pi \left[\frac{y^3}{3} - \frac{y^4}{4} \right]_0^1 \\ V &= \pi \left[\left(\frac{1}{3} - \frac{1}{4} \right) - 0 \right] \\ V &= \frac{\pi}{12} \end{aligned}$
--	---

Question 4 (***)



The diagram above shows the graph of the curve with equation

$$y = \frac{6}{x+3}, \quad x \neq -3.$$

The region R , shown shaded in the figure above, is bounded by the curve, the coordinate axes and the straight lines with equations $x = -1$ and $x = 3$.

- a) Show that the area of R is exactly $6\ln 3$.

The region R is rotated by 360° about the x axis to form a solid of revolution.

- b) Show that the volume of the solid generated is 12π .

, proof

(a)

$$A = \int_{-1}^3 \frac{6}{x+3} dx = \left[6\ln|x+3| \right]_{-1}^3$$

$$= 6\ln 6 - 6\ln 2 = 6(\ln 6 - \ln 2)$$

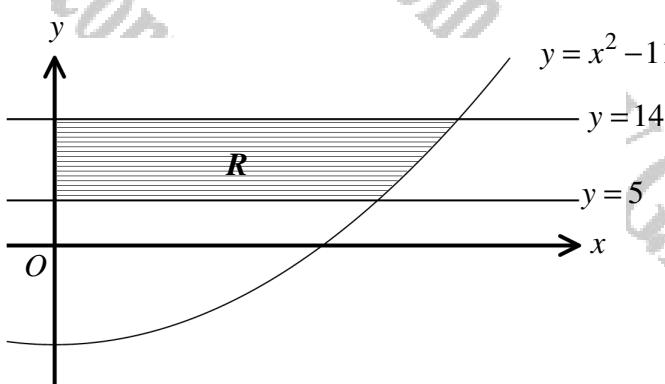
$$= 6\ln(\frac{6}{2}) = 6\ln 3$$

(b)

$$V = \pi \int_{-1}^3 y^2 dx = \pi \int_{-1}^3 \left(\frac{6}{x+3} \right)^2 dx = \pi \int_{-1}^3 \frac{36}{(x+3)^2} dx$$

$$= \pi \int_{-1}^3 36(2x+3)^{-2} dx = \pi \left[-36(2x+3)^{-1} \right]_{-1}^3 = \pi \left[\frac{36}{2x+3} \right]_{-1}^3$$

$$= \pi \left[\frac{36}{2(3)} - \frac{36}{2(-1)} \right] = \pi(18-6) = 12\pi$$

Question 5 (***)

The figure above shows the parabola with equation

$$y = x^2 - 11.$$

The shaded region R , is bounded by the curve, the y axis and the horizontal lines with equations $y = 5$ and $y = 14$.

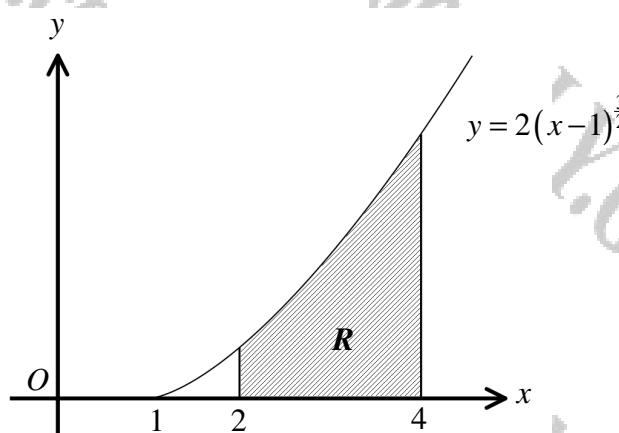
This region R is rotated through 360° about the y axis to form a solid of revolution.

Show that the volume of the solid generated is $\frac{369\pi}{2}$.

[proof]

$$\begin{aligned} V &= \pi \int_{y_1}^{y_2} 2\pi x \, dx \, dy \\ &= \pi \int_{y_1}^{y_2} \pi (y+11)^2 \, dy \\ &= \pi \left[\frac{1}{3} y^3 + 11y^2 \right]_{y_1}^{y_2} \\ &= \pi \left[\left(14^3 + 14^2 \right) - \left(5^3 + 5^2 \right) \right] / 3 \\ &= \frac{369\pi}{2} \end{aligned}$$

Question 6 (***)



The figure above shows part of the curve with equation

$$y = 2(x-1)^{\frac{3}{2}}.$$

The shaded region, labelled as R , bounded by the curve, the x axis and the straight lines with equations $x = 2$ and $x = 4$.

This region is rotated by 2π radians in the x axis, to form a solid of revolution S .

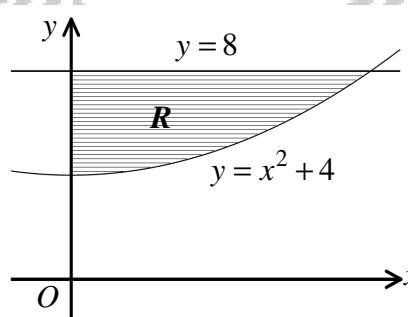
Show that the volume of S is 80π .

[proof]

A diagram illustrating the volume calculation. It shows a cross-section of the solid of revolution, which is a hemisphere. The radius of the hemisphere is labeled as 3. The volume is calculated using the formula for the volume of a hemisphere:

$$V = \pi \int_{x_1}^{x_2} y^2 dx = \pi \int_{2}^{4} 4(x-1)^{\frac{5}{2}} dx = \pi \left[2(x-1)^{\frac{5}{2}} \right]_{2}^{4} = \pi \left[3^{\frac{5}{2}} - 1^{\frac{5}{2}} \right] = 80\pi$$

Question 7 (***)



The figure above shows the graph of the curve C with equation

$$y = x^2 + 4,$$

intersected by the straight line L with equation

$$y = 8.$$

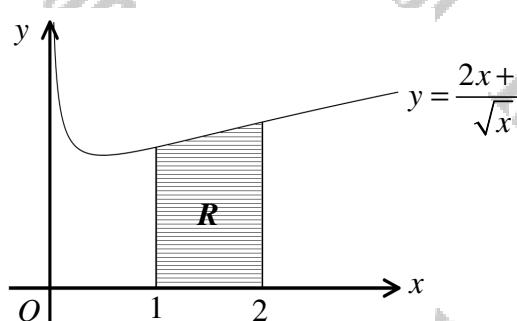
The shaded region R , is bounded by C , the y -axis and L .

Show that when R is rotated through 2π radians about the y -axis it will generate a volume of 8π cubic units.

proof

A diagram showing a small vertical slice of the shaded region R at height y with thickness dx . The volume of this slice is given by $V = \pi y^2 dx$. The region is bounded by the curve $y = x^2 + 4$ and the line $y = 8$.

$$\begin{aligned} V &= \pi \int_{y_1}^{y_2} y^2 dy = \pi \left[\frac{y^3}{3} \right]_{y_1}^{y_2} \\ &= \pi \left[\frac{1}{3}y^3 \right]_4^8 \\ &= \pi [(8^3 - 4^3) / 3] \\ &= 8\pi \end{aligned}$$

Question 8 (***)

The figure above shows the graph of the curve with equation

$$y = \frac{2x+1}{\sqrt{x}}, \quad x > 0.$$

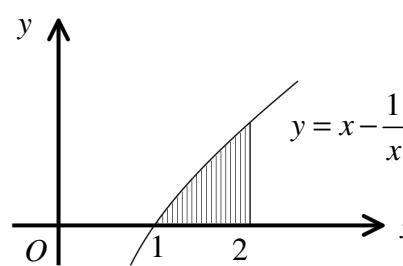
The shaded region R is bounded by the curve, the x axis and the straight lines with equations $x=1$ and $x=2$.

Find the volume that will be generated when R is rotated through 360° in the x axis.

Give the answer in the form $\pi(a+b \ln 2)$, where a and b are integers.

$$\boxed{\pi(10 + \ln 2)}$$

$$\begin{aligned}
 & \text{Diagram: A small rectangle of width } dx \text{ and height } y = \frac{2x+1}{\sqrt{x}} \text{ is shown at position } x. \\
 & \Rightarrow V = \pi \int_{x_1}^{x_2} y^2 dx \\
 & \Rightarrow V = \pi \int_1^2 \left(\frac{2x+1}{\sqrt{x}}\right)^2 dx \\
 & \Rightarrow V = \pi \int_1^2 \frac{4x^2+4x+1}{x} dx \\
 & \Rightarrow V = \pi \int_1^2 (4x+4+\frac{1}{x}) dx \\
 & \Rightarrow V = \pi \left[2x^2 + 4x + \ln x \right]_1^2 \\
 & \Rightarrow V = \pi \left[(8+8+\ln 2) - (2+4+\ln 1) \right] \\
 & \Rightarrow V = \pi [10 + \ln 2]
 \end{aligned}$$

Question 9 (*)**

The figure above shows part of the curve with equation

$$y = x - \frac{1}{x}, \quad x \neq 0.$$

The shaded region bounded by the curve and the straight line with equation $x=2$ is rotated by 360° about the x axis to form a solid of revolution.

Show that this volume is $\frac{5\pi}{6}$.

[proof]

$$\begin{aligned} y^2 &= (x - \frac{1}{x})^2 = x^2 - 2x + \frac{1}{x^2} \\ \therefore y^2 &= x^2 - 2x + \frac{1}{x^2} \\ V &= \pi \int_{1/2}^2 y^2 dx = \pi \int_1^2 x^2 - 2x + \frac{1}{x^2} dx = \pi \left[\frac{1}{3}x^3 - 2x^2 + \frac{1}{x} \right]_1^2 \\ &= \pi \left[\left(\frac{8}{3} - 8 + \frac{1}{2} \right) - \left(\frac{1}{3} - 2 + 1 \right) \right] = \frac{5\pi}{6} // \end{aligned}$$

Question 10 (*)**

The curve C has equation

$$y = 2 + \frac{1}{x}, \quad x > 0.$$

The region bounded by C , the x axis and the lines $x = \frac{1}{2}$, $x = 2$ is rotated through 360° about the x axis.

Show that the volume of the solid formed is

$$\pi\left(\frac{15}{2} + 8\ln 2\right).$$

[proof]

$$\begin{aligned}
 V &= \pi \int_{x_1}^{x_2} y^2 dx = \pi \int_{\frac{1}{2}}^2 \left(4 + \frac{4}{x} + x^{-2}\right) dx \\
 &= \pi \left[4x + 4\ln|x| - x^{-1}\right]_{\frac{1}{2}}^2 \\
 &= \pi \left[4x + 4\ln|x| - \frac{1}{x}\right]_{\frac{1}{2}}^2 \\
 &= \pi \left[\left(8 + 4\ln 2 - \frac{1}{2}\right) - \left(2 + 4\ln \frac{1}{2} - 2\right)\right] \\
 &= \pi \left[\frac{15}{2} + 4\ln 2 - 4\ln \frac{1}{2}\right] \\
 &= \pi \left[\frac{15}{2} + 4\ln 2 + 4\ln 2\right] \\
 &= \pi \left[\frac{15}{2} + 8\ln 2\right]
 \end{aligned}$$

Question 11 (*)**

The curve C has equation

$$y = \sqrt{x} + \frac{4}{\sqrt{x}}, \quad x > 0.$$

The region bounded by C , the x axis and the lines $x=1$, $x=4$ is rotated through 360° about the x axis.

Show that the volume of the solid formed is

$$\frac{\pi}{2}(63 + 64\ln 2).$$

[proof]

$$\begin{aligned} y^2 &= \left(\sqrt{x} + \frac{4}{\sqrt{x}}\right)^2 = (\sqrt{x})^2 + 2\sqrt{x}\left(\frac{4}{\sqrt{x}}\right) + \left(\frac{4}{\sqrt{x}}\right)^2 = x + 8 + \frac{16}{x} \\ \therefore V &= \pi \int_1^4 y^2 dx = \pi \int_1^4 x + 8 + \frac{16}{x} dx = \pi \left[\frac{x^2}{2} + 8x + 16\ln x \right]_1^4 \\ &= \pi \left[(8+32+16\ln 4) - (1+8+16\ln 1) \right] = \pi \left[\frac{63}{2} + 16\ln 4 \right] \\ &= \pi \left[\frac{63}{2} + 32\ln 2 \right] = \frac{\pi}{2} [63 + 64\ln 2]. \end{aligned}$$

Question 12 (*)**

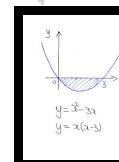
The curve C has equation

$$y = x^2 - 3x.$$

The region bounded by C and the x axis is rotated through 2π radians in the x axis.

Find the exact volume of the solid formed.

$$\frac{81\pi}{10}$$



$$\begin{aligned} V &= \pi \int_0^3 (y(x))^2 dx = \pi \int_0^3 (x^2 - 3x)^2 dx \\ &\Rightarrow V = \pi \int_0^3 (x^4 - 6x^3 + 9x^2) dx \\ &\Rightarrow V = \pi \left[\frac{x^5}{5} - \frac{6x^4}{4} + 3x^3 \right]_0^3 \\ &\Rightarrow V = \pi \left[\left(\frac{243}{5} - \frac{243}{4} + 27 \right) - 0 \right] \\ &\Rightarrow V = \frac{81\pi}{10}. \end{aligned}$$

Question 13 (*)**

The curve C has equation

$$y = x^{\frac{3}{2}} \sqrt{\ln x}, \quad x > 0.$$

The region bounded by C , the x axis and the straight lines with equations $x=1$ and $x=e$ is rotated through 360° about the x axis.

Use integration by parts to show that the volume of the solid formed is

$$\frac{1}{16}\pi(3e^4+1).$$

, proof

$$\begin{aligned} y^2 &= (x^{\frac{3}{2}} \sqrt{\ln x})^2 = x^3 \ln x \\ V &= \pi \int_{\ln 1}^{\ln e} y^2 dx = \pi \int_1^e x^3 \ln x dx \end{aligned}$$

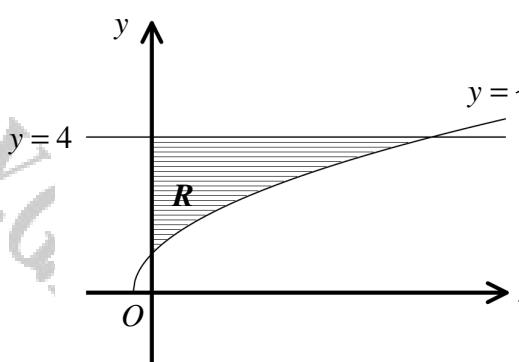
BY PARTS

$\left\{ \begin{array}{l} \ln x \rightarrow \frac{1}{x} \\ \frac{1}{x} dx \rightarrow x^2 \end{array} \right.$

- Integrate π & units
- $\frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 d(\ln x) = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 dx = \frac{1}{2}x^2 \ln x - \frac{1}{12}x^3 + C$
- Hence

$$\begin{aligned} V &= \pi \left[\frac{1}{2}x^2 \ln x - \frac{1}{12}x^3 \right]_1^e = \pi \left[\left(\frac{1}{2}e^2 \ln e - \frac{1}{12}e^3 \right) - \left(\frac{1}{2} \cdot 1^2 - \frac{1}{12} \right) \right] \\ &= \pi \left[\frac{1}{2}e^2 - \frac{1}{6}e^3 + \frac{1}{12} \right] = \pi \left[\frac{3}{2}e^2 + \frac{1}{12} \right] = \frac{1}{16}\pi(3e^4+1) \end{aligned}$$

Question 14 (***)



The curve C has equation

$$y = \sqrt{x+1}, \quad x > -1.$$

The region R is bounded by C , the y axis and the straight line with equation $y=4$ is rotated through 360° about the y axis to form a solid of revolution.

Show that the volume of the solid is $\frac{828\pi}{5}$.

proof

ROTATING THE CURVE AROUND THE y AXIS AT $(0,1)$

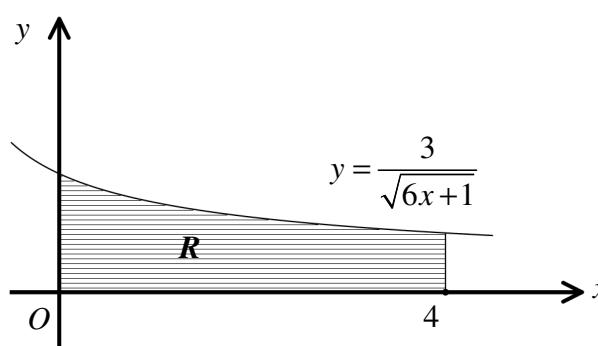
$$\begin{aligned} y &= \sqrt{x+1} \\ y^2 &= x+1 \\ x &= y^2-1 \\ x^2 &= (y^2-1)^2 \\ x^2 &= y^4-2y^2+1 \end{aligned}$$

SETTING UP + SOLVING INTEGRAL, ROOT THE y AXIS

$$\begin{aligned} \text{Volume} &= \int_1^4 \pi x^2 dy = \pi \int_1^4 (y^4-2y^2+1) dy \\ &= \pi \left[\frac{1}{5}y^5 - \frac{2}{3}y^3 + y \right]_1^4 = \pi \left(\frac{1024}{5} - \frac{128}{3} + 4 \right) - \pi \left(\frac{1}{5} - \frac{2}{3} + 1 \right) \\ &= \frac{256}{5}\pi - \frac{8}{3}\pi \\ &= \frac{828}{5}\pi \end{aligned}$$

As required

Question 15 (***)



The graph below shows the curve with equation

$$y = \frac{3}{\sqrt{6x+1}}, \quad x \neq -\frac{1}{6}.$$

The region R , shown in the figure shaded, is bounded by the curve, the coordinate axes and the straight line with equation $x = 4$.

- a) Show that the area of R is 4 square units.

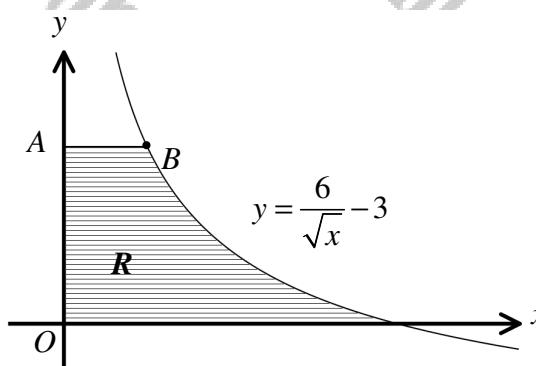
The shaded region R is rotated by 2π radians about the x axis to form a solid of revolution.

- b) Show that the volume of the solid generated is $3\pi \ln 5$.

[proof]

	$(a) \text{ Area} = \int_0^4 \frac{3}{\sqrt{6x+1}} dx = \int_0^4 3(x+1)^{-\frac{1}{2}} dx$ $= \left[3(x+1)^{\frac{1}{2}} \right]_0^4 = 3(5^{\frac{1}{2}} - 1^{\frac{1}{2}}) = 3(\sqrt{5} - 1) = 4$
	$(b) V = \pi \int_0^4 y^2 dx = \pi \int_0^4 \frac{9}{6x+1} dx = \pi \left[\frac{9}{6} \ln 6x+1 \right]_0^4$ $= \frac{3}{2}\pi \left[\ln 25 - \ln 1 \right] = \frac{3}{2}\pi (\ln 25) = 3\pi \ln 5$

Question 16 (***)



The figure above shows part of the graph of the curve with equation

$$y = \frac{6}{\sqrt{x}} - 3, \quad x > 0.$$

The point B lies on the curve where $x = 1$.

The shaded region R is bounded by the curve, the coordinate axes and a straight line segment AB , where AB is parallel to the x axis. The region R is rotated through 2π radians in the y axis to form a solid of revolution.

Show that the volume of this solid is 14π .

, proof

AS THE REVOLUTION IS ABOUT THE y AXIS, REARRANGE THE EQUATION FOR x

$$\Rightarrow x = \frac{36}{y^2} + 3$$

$$\Rightarrow y+3 = \frac{6}{\sqrt{x}}$$

$$\Rightarrow (y+3)^2 = \frac{36}{x}$$

$$\Rightarrow x = \frac{36}{(y+3)^2}$$

$$\Rightarrow x^2 = \frac{1296}{(y+3)^4}$$

BY INSPECTION THE y COORDINATE OF B IS 3

THIS WE HAVE

$$V = \pi \int_{-3}^{3} [x(y)]^2 dy = \pi \int_{-3}^{3} \frac{1296}{(y+3)^4} dy$$

$$= \pi \int_{-3}^{3} 1296(y+3)^{-4} dy = \pi \left[\frac{1296}{-3} (y+3)^{-3} \right]_{-3}^3$$

$$= \pi \left[-\frac{432}{(y+3)^3} \right]_{-3}^3 = 432\pi \left[\frac{1}{(y+3)^3} \right]_3^{-3}$$

$$= 432\pi \left[\frac{1}{27} - \frac{1}{216} \right] = 432\pi \times \frac{7}{216} = \frac{14\pi}{3}$$

As required

Question 17 (*)**

The curve C has equation

$$y = x + \frac{1}{x^2}, \quad x > 0.$$

The region bounded by C , the x axis and the lines $x=1$, $x=2$ is rotated through 360° about the x axis.

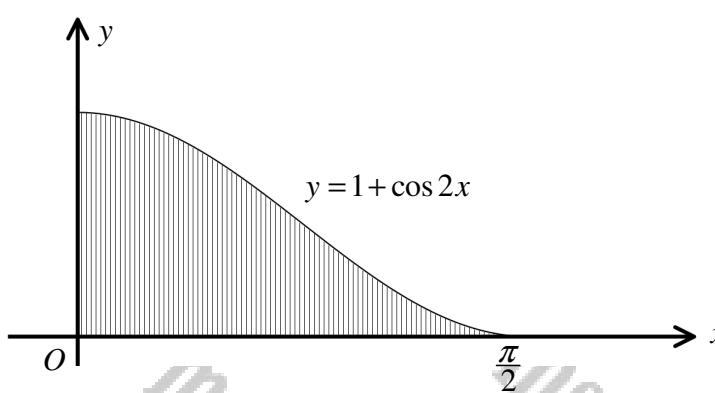
Show that the volume of the solid formed is

$$\pi\left(\frac{21}{8} + 2\ln 2\right).$$

[proof]

$$\begin{aligned} g &= (x + \frac{1}{x^2})^2 = x^2 + 2x(\frac{1}{x^2}) + (\frac{1}{x^2})^2 = x^2 + \frac{2}{x} + \frac{1}{x^4} \\ V &= \pi \int_{x_1}^{x_2} g^2 \, dx = \pi \int_1^2 x^2 + \frac{2}{x} + \frac{1}{x^4} \, dx = \pi \left[\frac{1}{3}x^3 + 2\ln x - \frac{1}{x^3} \right]_1^2 \\ &= \pi \left[\frac{8}{3} + 2\ln 2 - \frac{1}{8} \right]^2 = \pi \left[\left(\frac{8}{3} + 2\ln 2 - \frac{1}{8} \right) - \left(\frac{8}{3} - \frac{1}{8} \right) \right]^2 \\ &= \pi \left[\frac{21}{8} - 2\ln 2 \right] \end{aligned}$$

Question 18 (***)



The figure above shows the graph of the curve with equation

$$y = 1 + \cos 2x, 0 \leq x \leq \frac{\pi}{2}.$$

- a) Show clearly that

$$(1 + \cos 2x)^2 \equiv \frac{3}{2} + 2 \cos 2x + \frac{1}{2} \cos 4x.$$

The shaded region bounded by the curve and the coordinate axes is rotated by 2π radians about the x axis to form a solid of revolution.

- b) Show that the volume of the solid is

$$\frac{3}{4}\pi^2.$$

[2A], [proof]

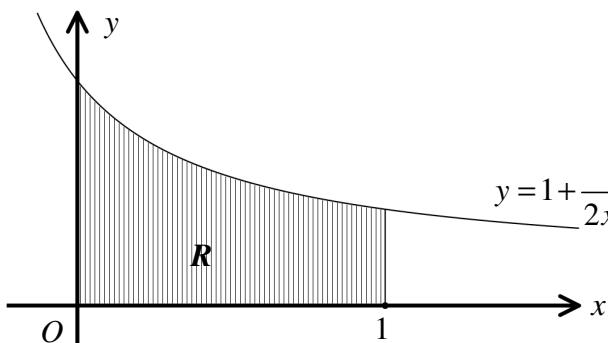
(a)

$$\begin{aligned}
 (1 + \cos 2x)^2 &= 1 + 2\cos 2x + (\cos 2x)^2 \\
 &= (1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2} \cos 4x) \\
 &= \frac{3}{2} + 2\cos 2x + \frac{1}{2} \cos 4x
 \end{aligned}$$

Method: Use double angle formulae.

(b)

$$\begin{aligned}
 V &= \pi \int_{0}^{\pi/2} (y)^2 dx = \pi \int_{0}^{\pi/2} (1 + \cos 2x)^2 dx \\
 V &= \pi \int_{0}^{\pi/2} \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2} \cos 4x \right) dx \\
 V &= \pi \left[\frac{3}{2}x + 2\sin 2x + \frac{1}{8} \sin 4x \right]_0^{\pi/2} \\
 V &= \pi \left[\left(\frac{3}{2} \cdot \frac{\pi}{2} + 0 + 0 \right) - (0) \right] \\
 V &= \frac{3}{4}\pi^2
 \end{aligned}$$

Question 19 (***)

The figure above shows the graph of the curve with equation

$$y = 1 + \frac{6}{2x+1}, \quad x \neq -\frac{1}{2}.$$

- a) Show that

$$\left(1 + \frac{6}{2x+1}\right)^2 \equiv 1 + \frac{A}{2x+1} + \frac{B}{(2x+1)^2},$$

where A and B are constants to be found.

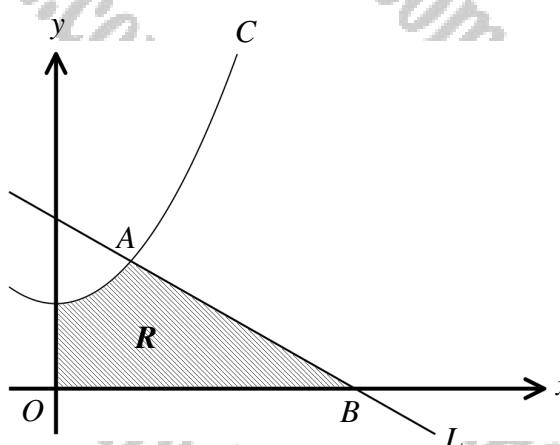
The shaded region, labelled as R , bounded by the curve, the coordinate axes and the line $x=1$ is rotated by 2π radians in the x axis to form a solid of revolution.

- b) Show further that the volume generated is

$$\pi(13 + 6\ln 3).$$

$$\boxed{}, \boxed{A=12}, \boxed{B=36}$$

$$\begin{aligned}
 \text{(a)} \quad & \left(1 + \frac{6}{2x+1}\right)^2 = 1^2 + 2x \cdot \frac{6}{2x+1} + \left(\frac{6}{2x+1}\right)^2 = 1 + \frac{12}{2x+1} + \frac{36}{(2x+1)^2} \\
 \text{(b)} \quad & V = \pi \int_{-0.25}^{1.0} y^2 dx = \pi \int_{-0.25}^{1.0} \left(1 + \frac{12}{2x+1} + \frac{36}{(2x+1)^2}\right)^2 dx = \\
 & = \pi \int_{-0.25}^{1.0} \left[25 + 6\ln|2x+1| - 12(2x+1)^{-1}\right]^2 dx = \pi \int_{-0.25}^{1.0} \left[25 + 6\ln|2x+1| - \frac{144}{(2x+1)^2}\right] dx = \\
 & = \pi \left[(0 + 6\ln 3 - 6) - (0 + 0 + 13) \right] = \pi [13 + 6\ln 3]
 \end{aligned}$$

Question 20 (***)

The figure above shows the graph of the curve C with equation

$$y = x^2 + 2,$$

intersected by the straight line L with equation

$$x + y = 4.$$

The point A is the intersection of C and L . The point B is the point where L meets the x axis.

The region R , shown shaded in the figure above, is bounded by C , L and the coordinate axes. This region is rotated by 360° in the x axis, forming a solid of revolution S .

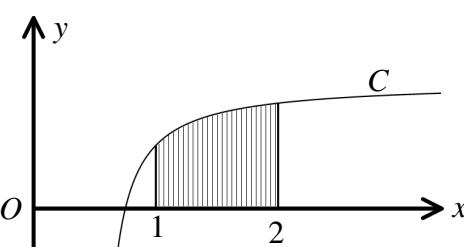
Find an exact value for the volume of S .

$$\frac{218}{15}\pi$$

\bullet $V_1 = \pi \int_{x_1}^{x_2} (y(x))^2 dx$
 $V_1 = \pi \int_{0}^{2} (x^2 + 2)^2 dx = \pi \int_{0}^{2} (x^4 + 4x^2 + 4) dx$
 $V_1 = \pi \left[\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x \right]_0^2$
 $V_1 = \pi \left[\left(\frac{1}{5}(32) + \frac{4}{3}(8) + 8 \right) - 0 \right]$
 $V_1 = \frac{88\pi}{15}$

\bullet $V_2 = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi (3)^2 \times 2 = \pi r^3$
 \uparrow Radius 3, height 2
 $\therefore V = \frac{218}{15}\pi$

Question 21 (***)



The figure above shows part of the graph of the curve C with equation

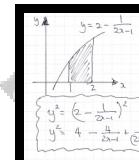
$$y = 2 - \frac{1}{2x-1}, \quad x \neq \frac{1}{2}.$$

The shaded region bounded by C and the straight lines with equations $x=1$ and $x=2$, is rotated by 360° about the x axis, forming a solid of revolution.

Show that the volume of the solid is

$$\pi\left(\frac{13}{3} - 2\ln 3\right).$$

proof



$$y = 2 - \frac{1}{2x-1}$$

$$y^2 = \left(2 - \frac{1}{2x-1}\right)^2$$

$$y^2 = 4 - \frac{4}{2x-1} + \frac{1}{(2x-1)^2}$$

$$V = \int_{1/2}^2 y^2 dx = \int_1^2 4 - \frac{4}{2x-1} + \frac{1}{(2x-1)^2} dx$$

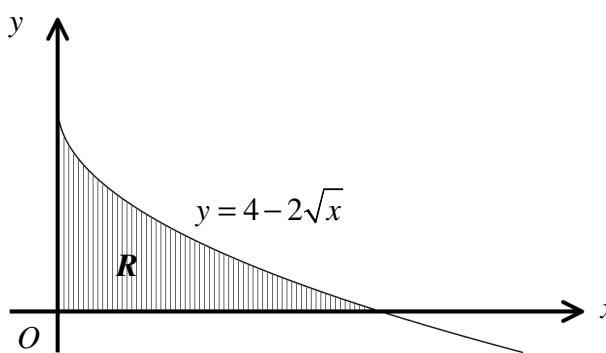
$$= \pi \left[4x - 2\ln|2x-1| - \frac{1}{2} (2x-1)^{-1} \right]_1^2$$

$$= \pi \left[(8 - 2\ln 3 - \frac{1}{2}) - (4 - 2\ln 1 - \frac{1}{2}) \right]$$

$$= \pi \left[8 - 2\ln 3 - \frac{1}{2} - 4 + \frac{1}{2} \right]$$

$$= \pi \left[\frac{13}{3} - 2\ln 3 \right]$$

Question 22 (***)



The figure above shows the graph of the equation

$$y = 4 - 2\sqrt{x}, x \geq 0.$$

The shaded region R , bounded by the curve and the coordinate axes, is rotated through 4 right angles about the y axis to form a solid of revolution.

Show that the volume generated is $\frac{64\pi}{5}$.

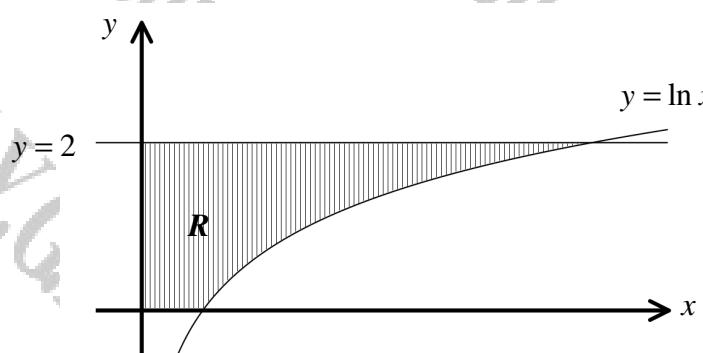
proof

Handwritten working for the volume of revolution proof:

Given: $y = 4 - 2\sqrt{x}$
 $2\sqrt{x} = 4 - y$
 $\sqrt{x} = \frac{4-y}{2}$
 $x = \frac{(4-y)^2}{4}$
 $x^2 = \frac{(4-y)^2}{16}$

Volume formula:
 $V = \pi \int_{y_1}^{y_2} x^2 dy = \pi \int_{0}^{4} \left(\frac{(4-y)^2}{16}\right) dy$
 $= \frac{1}{16}\pi \int_{0}^{4} (4-y)^2 dy = \frac{1}{48}\pi \left[(4-y)^3\right]_0^4 = \frac{1}{48}\pi [64 - 0] = \frac{64\pi}{3}$

Question 23 (***)



The figure below shows the graph of the curve C with equation

$$y = \ln x, \quad x > 0,$$

intersected by the horizontal straight line L with equation

$$y = 2.$$

The shaded region R , bounded by C , L and the coordinate axes, is rotated through 2π radians in the y axis to form a solid of revolution.

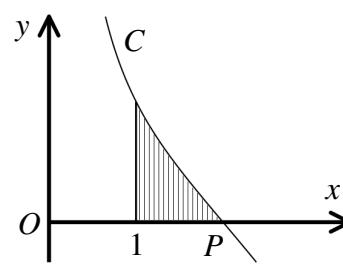
Show that the volume of the solid is

$$\frac{1}{2}\pi(e^4 - 1).$$

proof

$$\begin{aligned} V &= \pi \int_{e}^{2e} x^2 dy = \pi \int_{0}^2 (e^y)^2 dy \\ &\Rightarrow V = \pi \int_{0}^2 e^{2y} dy = \pi \left[\frac{1}{2}e^{2y} \right]_0^2 \\ &\Rightarrow V = \pi \left[e^4 - e^0 \right] = \frac{1}{2}\pi(e^4 - 1) // \end{aligned}$$

Question 24 (***)



The figure above shows part of the curve C with equation

$$y = \frac{2}{x} - \frac{x^2}{4}, \quad x > 0.$$

The curve crosses the x axis at the point P .

The shaded region bounded by the curve, the straight line with equation $x = 1$ and the x axis is rotated by 360° about the x axis to form a solid of revolution.

Show that the volume of the solid is $\frac{71\pi}{80}$.

proof

Question 25 (***)

The curve C lies entirely above the x axis and has equation

$$y = 1 + \frac{1}{2\sqrt{x}}, \quad x \geq 0.$$

- a) Show that

$$y^2 = 1 + \frac{1}{\sqrt{x}} + \frac{1}{4x}.$$

The region R is bounded by the curve, the x axis and the straight lines with equations $x=1$ and $x=4$.

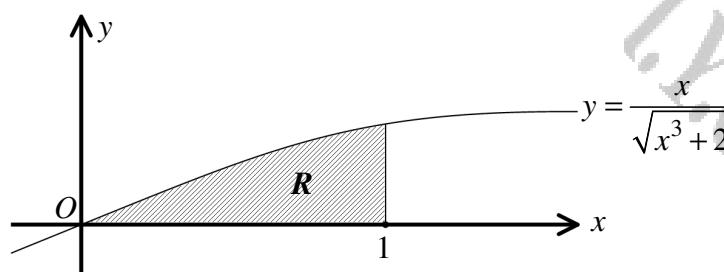
- b) Show that when R is rotated by 360° about the x axis, the solid generated has a volume

$$\pi(5 + \ln \sqrt{2}).$$

proof

$$\begin{aligned}
 \text{(a)} \quad \int_1^4 y^2 dx &= \left(1 + \frac{1}{2\sqrt{x}}\right)^2 = 1^2 + 2 \cdot 1 \cdot \frac{1}{2\sqrt{x}} + \left(\frac{1}{2\sqrt{x}}\right)^2 \\
 &= 1 + \frac{2}{2\sqrt{x}} + \frac{1}{4x} = 1 + \frac{1}{\sqrt{x}} + \frac{1}{4x} \\
 \text{(b)} \quad V &= \pi \int_1^4 y^2 dx = \pi \int_1^4 \left(1 + \frac{1}{\sqrt{x}} + \frac{1}{4x}\right) dx \\
 &= \pi \left[x + 2\sqrt{x} + \frac{1}{4} \ln|x| \right]_1^4 = \pi \left[(4 + 2 + \frac{1}{4} \ln 4) - (1 + 2 + \frac{1}{4} \ln 1) \right] \\
 &= \pi \left(5 + \frac{1}{4} \ln 4 \right) = \pi \left(5 + \frac{1}{2} \ln 2 \right) = \pi (5 + \ln \sqrt{2})
 \end{aligned}$$

Question 26 (***)



The figure above shows part of the curve with equation

$$y = \frac{x}{\sqrt{x^3 + 2}}, \quad x^3 > -2.$$

The shaded region R , bounded by the curve, the x axis and the straight line with equation $x = 1$, is rotated by 360° about the x axis to form a solid of revolution.

Show that the solid has a volume of

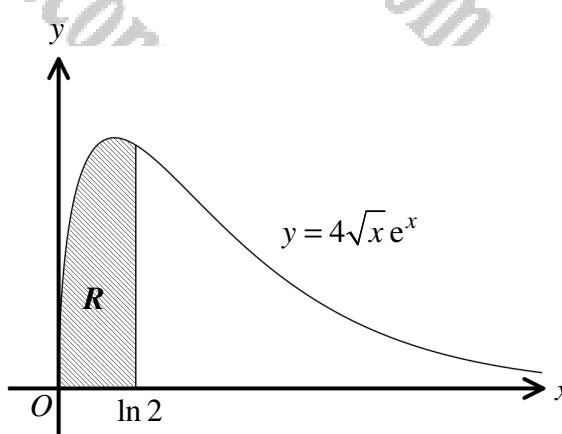
$$\frac{\pi}{3} \ln\left(\frac{3}{2}\right).$$

, proof

$$\begin{aligned} y &= \frac{x}{\sqrt{x^3 + 2}} \\ y^2 &= \left(\frac{x}{\sqrt{x^3 + 2}}\right)^2 = \frac{x^2}{x^3 + 2} \end{aligned}$$

$$\left\{ \begin{aligned} V &= \pi \int_0^1 y^2 \, dx = \pi \int_0^1 \frac{x^2}{x^3 + 2} \, dx \\ &= \pi \times \frac{1}{3} \int_0^1 \frac{3x^2}{x^3 + 2} \, dx \quad (\text{using substitution}) \\ &= \frac{\pi}{3} \left[\ln|x^3 + 2| \right]_0^1 \\ &= \frac{\pi}{3} \left[\ln 3 - \ln 2 \right] \\ &= \frac{\pi}{3} \ln \frac{3}{2} \end{aligned} \right.$$

Question 27 (***)



The figure above shows the graph of the curve C with equation

$$y = 4\sqrt{x} e^x, \quad x \geq 0.$$

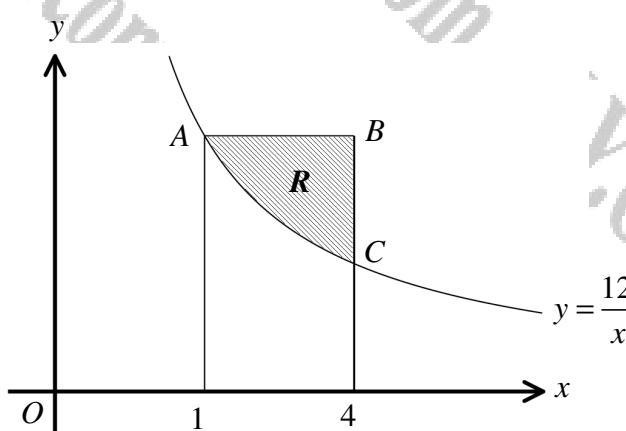
The shaded region R bounded by the curve, the x axis and the vertical straight line with equation $x = \ln 2$, is rotated by 2π radians in the x axis, forming a solid of revolution S .

Find an exact value for the volume of S , giving the answer in the form $\pi(a + b \ln 2)$ where a and b are integers.

, $\boxed{\pi(-12 + 32 \ln 2)}$

<small>USING THE STANDARD FORMULA FOR VOLUME OF REVOLUTION AND THE QL AREA</small>
$V = \pi \int_{x_1}^{x_2} [y(x)]^2 dx$ $V = \pi \int_0^{\ln 2} (4\sqrt{x} e^x)^2 dx = \pi \int_0^{\ln 2} 16x e^{2x} dx$
<small>PROCEEDED BY INTEGRATION BY PARTS</small>
$V = \pi \int_0^{\ln 2} (8x)(2e^{2x}) dx$ $V = \pi \left[(8xe^{2x}) - \int_0^{\ln 2} 8e^{2x} dx \right]$ $V = \pi \left[8xe^{2x} - 4e^{2x} \right]_0^{\ln 2}$ $V = \pi \left[(8\ln 2 e^{\ln 2}) - 4e^{2\ln 2} \right] - (0 - 4)$ $V = \pi [16\ln 2 e - 16e^4] - (-4)$ $V = \pi [16\ln 2 e - 16e^4]$ Q.E.D. Q.E.D.
$\frac{8x}{e^{2x}} \Big _0^{\ln 2}$

Question 28 (***)



The figure above shows part of the graph of the curve with equation

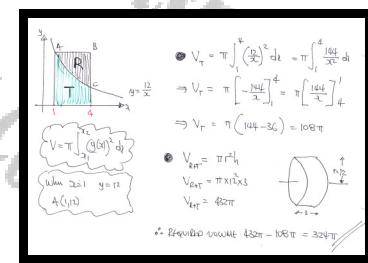
$$y = \frac{12}{x}, \quad x \neq 0.$$

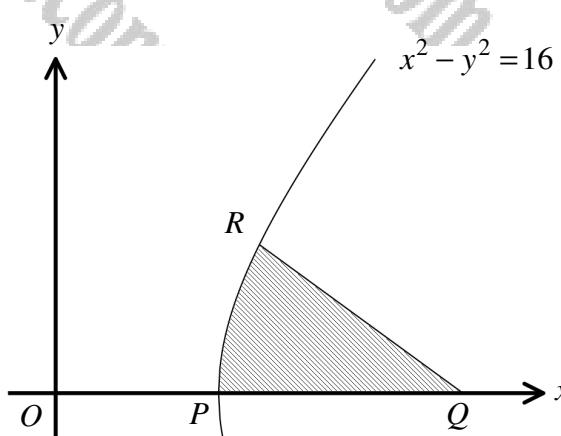
The points A and C lie on the curve where $x=1$ and $x=4$, respectively. The point B is such so that AB is parallel to the x axis and BC is parallel to the y axis.

The region R , shown shaded in the figure above, is bounded by the curve and the straight line segments AB and BC . This region is rotated by 2π radians in the x axis, forming a solid of revolution S .

Find the exact value for the volume of S .

324 π



Question 29 (***)

The figure above shows part of the graph of the hyperbola C with equation

$$x^2 - y^2 = 16.$$

The hyperbola crosses the x axis at $P(4,0)$, the point $R(5,3)$ lies on C and the point $Q(11,0)$ lies on the x axis.

The shaded region bounded by the curve, the x axis and the straight line segment RQ is rotated by 2π radians in the x axis, forming a solid of revolution S .

Find an exact value for the volume of S .

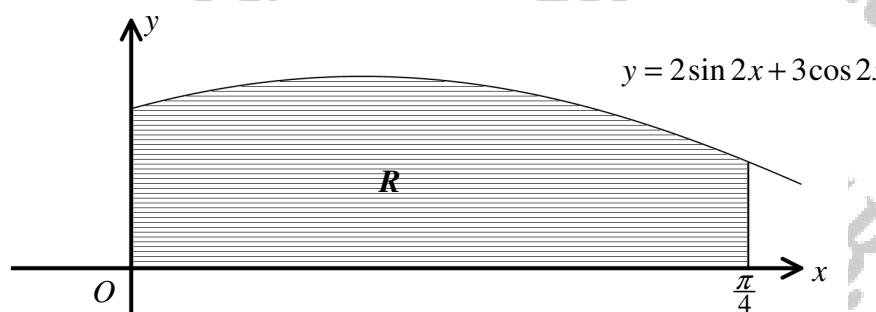
$$\boxed{\frac{67}{3}\pi}$$

Volume of cone = $\frac{1}{3}\pi r^2 h$
 $= \frac{1}{3}\pi \times 3^2 \times 6$
 $= 18\pi$

$V = \int_{x_1}^{x_2} (y(x))^2 dx = \pi \int_{x_1}^{x_2} x^2 - 16 dx = \pi \left[\frac{1}{3}x^3 - 16x \right]_1^{11}$

$\therefore V = \pi \left[\left(\frac{133}{3} - 160 \right) - \left(\frac{11}{3} - 16 \right) \right] = \pi \left[\frac{13}{3} \right]$

$\therefore V_{\text{total}} = \frac{3}{2}\pi + 18\pi = \frac{67}{3}\pi$

Question 30 (***)

The figure above shows part of the curve C , with equation

$$y = 2\sin 2x + 3\cos 2x.$$

- a) Show that

$$y^2 = A + B\cos 4x + C\sin 4x,$$

where A , B and C are constants.

The shaded region R is bounded by the curve, the line $x = \frac{\pi}{4}$ and the coordinate axes.

- b) Find the area of R .

The region R is rotated by 2π radians in the x axis forming a solid of revolution S .

- c) Show that the volume of S is

$$\frac{\pi}{8}(13\pi + 24).$$

, $A = \frac{13}{2}$, $B = \frac{5}{2}$, $C = 6$, area = 2.5

(a) $y^2 = (2\sin 2x + 3\cos 2x)^2 = 4\sin^2 2x + 12\sin 2x\cos 2x + 9\cos^2 2x$
 $= 4(1 - \cos 4x) + 6(2\sin 2x\cos 2x) + 9(1 + \cos 4x)$
 $= 2 - 8\cos 4x + 6\sin 4x + \frac{9}{2} + \frac{9}{2}\cos 4x$
 $= \frac{21}{2} + \frac{5}{2}\cos 4x + 6\sin 4x$

(b) $\text{Area} = \int_0^{\frac{\pi}{4}} 2\sin 2x + 3\cos 2x \, dx = \left[-\cos 2x + \frac{3}{2}\sin 2x \right]_0^{\frac{\pi}{4}}$
 $= \left[\left(0 + \frac{3}{2} \right) - (-1 + 0) \right] = \frac{5}{2} + 1 = \frac{7}{2}$

(c) $V = \pi \int_0^{\frac{\pi}{4}} (y^2)^2 \, dx = \pi \int_0^{\frac{\pi}{4}} \frac{21}{4} + \frac{25}{4}\cos 4x + 6\sin 4x \, dx$
 $= \pi \left[\frac{21}{8}x + \frac{25}{16}\sin 4x - \frac{6}{4}\cos 4x \right]_0^{\frac{\pi}{4}}$
 $= \pi \left(\frac{13\pi}{8} + 0 + \frac{3}{2} \right) - (0 + 0 - \frac{3}{2})$
 $= \pi \left[\frac{13\pi}{8} + 3 \right] = \frac{\pi}{8}(13\pi + 24)$ \square As Required

Question 31 (*)+**

The point P lies on the curve with equation

$$y = x^2, \quad x \geq 0.$$

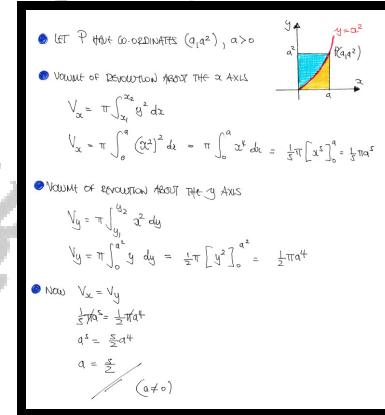
The straight line L_1 is parallel to the x axis and passes through P . The finite region R_1 is bounded by the curve, L_1 and the y axis.

The straight line L_2 is parallel to the y axis and passes through P . The finite region R_2 is bounded by the curve, L_2 and the x axis.

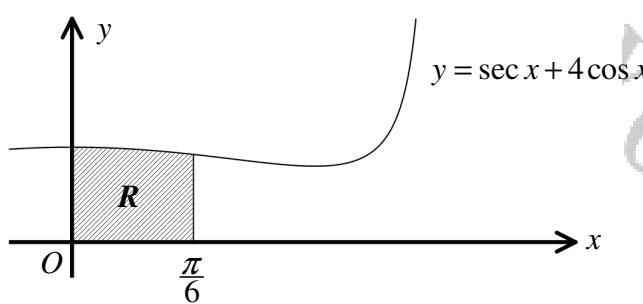
When R_1 is fully revolved about the y axis the volume of the solid formed is equal to the volume of the solid formed when R_2 is fully revolved about the x axis.

Determine the x coordinate of P .

$$\boxed{}, \quad \boxed{x = \frac{5}{2}}$$



Question 32 (*****)



The figure above shows part of the curve with equation

$$y = \sec x + 4 \cos x.$$

The shaded region, labelled R , bounded by the curve, the coordinate axes and the straight line with equation $x = \frac{\pi}{6}$ is rotated by 2π radians in the x axis to form a solid of revolution.

Show that the solid has a volume of

$$\frac{\pi}{3}(8\pi + 7\sqrt{3}).$$

, proof

$$\begin{aligned}
 & \text{Diagram: A shaded region } R \text{ bounded by } y = \sec x + 4 \cos x, x=0, x=\frac{\pi}{6}, \text{ and } y=0. \\
 & \text{Derivation:} \\
 & V = \int_{0}^{\frac{\pi}{6}} (\sec x)^2 dx \\
 & V = \pi \int_{0}^{\frac{\pi}{6}} (\sec x + 4 \cos x)^2 dx \\
 & V = \pi \int_{0}^{\frac{\pi}{6}} (\sec^2 x + 8 \sec x \cos x + 16 \cos^2 x) dx \\
 & V = \pi \int_{0}^{\frac{\pi}{6}} (\sec^2 x + 8 + 16 \cos^2 x) dx \\
 & V = \pi \int_{0}^{\frac{\pi}{6}} (\sec^2 x + 8 + 8 + 8 \sec^2 x) dx \\
 & V = \pi \int_{0}^{\frac{\pi}{6}} (2\sec^2 x + 16) dx \\
 & V = \pi \left[2 \tan x + 16x \right]_0^{\frac{\pi}{6}} \\
 & V = \pi \left[2 \left(\frac{\sqrt{3}}{3} + \frac{\pi}{3} \right) + 16 \cdot \frac{\pi}{6} \right] \\
 & V = \pi \left[\frac{2\sqrt{3}}{3} + \frac{2\pi}{3} + \frac{8\pi}{3} \right] \\
 & V = \pi \left[\frac{2\sqrt{3}}{3} + \frac{10\pi}{3} \right]
 \end{aligned}$$

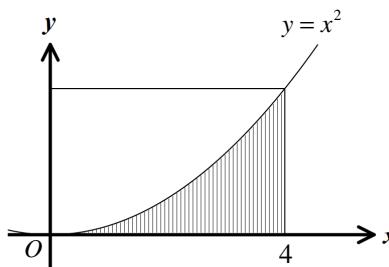
Question 33 (*)**

Figure 1

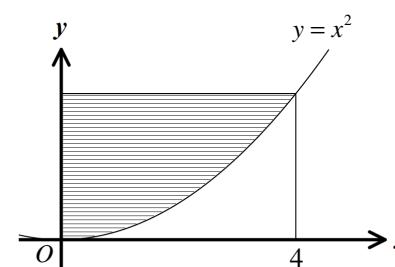


Figure 2

The figures above show part of the parabola with equation

$$y = x^2.$$

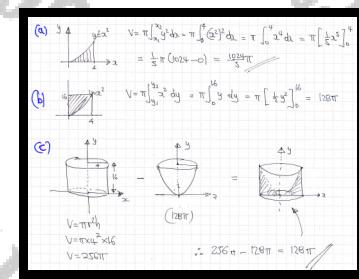
The shaded region, shown in Figure 1, is bounded by the curve, the x axis and the line $x = 4$. This region is revolved by 2π radians about the x axis, to form a solid of revolution.

- a) Show that the solid has a volume of $\frac{1024\pi}{5}$.

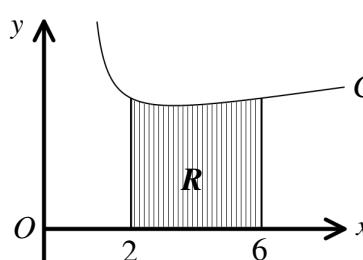
The shaded region, shown in Figure 2, is bounded by the curve, the y axis and a horizontal line originating from a point on the parabola where $x = 4$. This region is revolved by 2π radians about the y axis, to form a solid of revolution.

- b) Show that the solid has a volume of 128π .
- c) Hence find the value of the volume generated when the region shown in figure 1 is revolved by 2π radians about the y axis.

128 π



Question 34 (****)



The figure above shows part of the curve C with equation

$$y = \frac{x+1}{\sqrt{x-1}}, \quad x \geq 1.$$

The shaded region R is bounded by the curve, the x axis and the straight lines with equations $x=2$ and $x=6$. The region R is rotated by 360° about the x axis to form a solid of revolution.

- a) Show that the volume of the solid is
- $$\pi(28 + 4\ln 5).$$

[continues overleaf]

[continued from overleaf]

The solid of part (a) is used to model the wooden leg of a sofa.

The shape of the leg is geometrically similar to the solid of part (a).



- b) Given the height of the leg is 6 cm, determine the volume of the wooden leg to the nearest cubic centimetre.

$$\boxed{\quad}, \approx 365 \text{ cm}^3$$

a) METHOD OF EVALUATION IN CARTESIAN COORDINATES ABOUT THE x-AXIS

$$V = \int_{-1}^{x_1} (\pi y^2) dx = \pi \int_{-1}^x \left(\frac{2x+1}{x+1}\right)^2 dx = \pi \int_{-1}^x \frac{(2x+1)^2}{(x+1)^2} dx$$

BY SUBSTITUTION OR MINIMISATION

- $u = x+1 \Rightarrow x = u-1$
- $du = 1$
- $dx = du$
- $x = -1 \rightarrow u = 0$
- $x = 2 \rightarrow u = 3$
- $3.6 \rightarrow u = 5$

$$\Rightarrow V = \pi \int_{-1}^x \frac{(2x+1)^2}{(x+1)^2} dx$$

$$\Rightarrow V = \pi \int_0^5 \frac{(2u-1)^2}{u^2} du$$

$$\Rightarrow V = \pi \int_0^5 \frac{(4u^2-4u+1)}{u^2} du$$

$$\Rightarrow V = \pi \int_0^5 \left(\frac{4u^2}{u^2} - \frac{4u}{u^2} + \frac{1}{u^2} \right) du$$

$$\Rightarrow V = \pi \int_0^5 \left(4 - \frac{4}{u} + \frac{1}{u^2} \right) du$$

$$\Rightarrow V = \pi \left[4u - 4\ln(u) - \frac{1}{u} \right]_0^5$$

$$\Rightarrow V = \pi \left[20 - 4\ln(5) - 1 \right]$$

$$\Rightarrow V = \pi [20 - 4\ln 5]$$

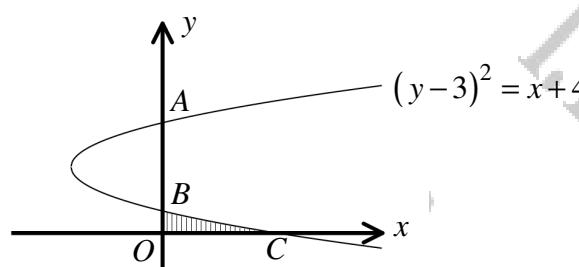
b) LOOKING AT THE SIMILAR SHAPES

$V = V \times (\text{scale factor})^3$

$$V = \pi (8\pi + 4\ln 5) \times (5)^3$$

$$V \approx 365$$

Question 35 (*****)



The figure above shows part of the curve with equation

$$(y-3)^2 = x+4.$$

The curve crosses the coordinate axes at the points A , B and C .

- a) Show that

$$x^2 = y^4 - 12y^3 + 46y^2 - 60y + 25.$$

- b) The shaded region bounded by the curve and the coordinate axes is rotated by 360° about the y axis to form a solid of revolution.

Show that the volume of the solid is $\frac{113\pi}{15}$.

proof

$\text{(a)} (y-3)^2 = x+4$ $(y-3)^2 - 4 = x$ $y^2 - 6y + 9 - 4 = x$ $x = y^2 - 6y + 5$ $x^2 = (y^2 - 6y + 5)^2$ $x^2 = (y^2 - 6y + 5)(y^2 - 6y + 5)$ $x^2 = y^4 - 6y^3 + 15y^2 - 6y^3 + 36y^2 - 30y + 25$ $x^2 = y^4 - 12y^3 + 46y^2 - 60y + 25$	$\text{(b) when } x=0 \quad (y-3)^2 = 4$ $y-3 = \pm 2$ $y = 1 \text{ or } 5$ $\therefore V = \int_{-1}^5 x^2 dy = \int_{-1}^5 (y^4 - 12y^3 + 46y^2 - 60y + 25) dy$ $V = \pi \left[\frac{1}{5}y^5 - 3y^4 + \frac{46}{3}y^3 - 30y^2 + 25y \right]_0^5$ $V = \pi \left[\left(\frac{1}{5}(5)^5 - 3(5)^4 + \frac{46}{3}(5)^3 - 30(5)^2 + 25(5) \right) - 0 \right]$ $V = \pi \left[\left(\frac{1}{5}(3125) - 3(625) + \frac{46}{3}(125) - 30(25) + 125 \right) \right]$ $V = \pi \left[\frac{113}{5} \right]$ $V = \frac{113\pi}{5}$
--	---

Question 36 (***)

The curve C has equation

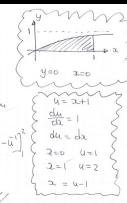
$$y = \frac{x}{x+1}, \quad x \geq 0.$$

The region bounded by the curve, the x axis and the straight line with equation $x=1$ is rotated through 2π radians about the x axis to form a solid of revolution.

Show that the volume of the solid is

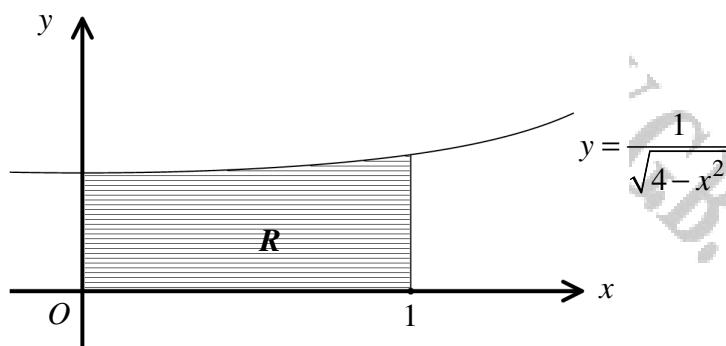
$$\frac{\pi}{2}(3 - 4\ln 2).$$

proof



$$\begin{aligned}
 V &= \pi \int_{-2}^2 (y(x))^2 dx = \pi \int_0^1 \left(\frac{x}{x+1}\right)^2 dx \\
 &= \pi \int_0^1 \frac{x^2}{(x+1)^2} dx = \dots \text{BY SUBSTITUTION} \\
 &= \pi \int_1^2 \frac{(u-1)^2}{u^2} du = \pi \int_1^2 \frac{u^2 - 2u + 1}{u^2} du \\
 &= \pi \int_1^2 \left(1 - \frac{2}{u} + u^{-2}\right) du = \pi \left[u - 2\ln|u| + u^{-1}\right]_1^2 \\
 &= \pi \left[\left(2 - 2\ln 2 - \frac{1}{2}\right) - \left(1 - 2\ln 1 + 1\right)\right] \\
 &= \pi \left(\frac{3}{2} - 2\ln 2\right) = \frac{\pi}{2}(3 - 4\ln 2) // \text{As Required}
 \end{aligned}$$

Question 37 (*****)



The figure above shows part of the curve with equation

$$y = \frac{1}{\sqrt{4-x^2}}, -2 \leq x \leq 2.$$

The shaded region, labelled as R , bounded by the curve, the coordinate axes and the straight line with equation $x=1$ is rotated by 2π radians about the x axis to form a solid of revolution.

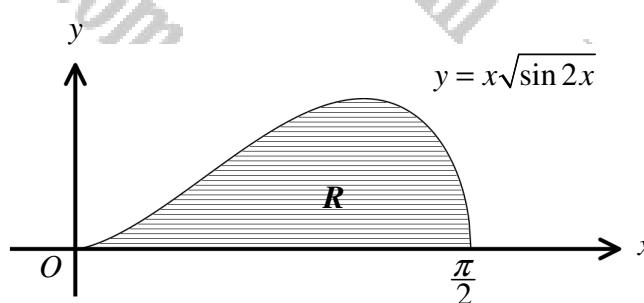
Show that the volume of the solid is

$$\frac{1}{4}\pi \ln 3.$$

proof

$$\begin{aligned}
 V &= \pi \int_{-2}^{2} (y(x))^2 dx = \pi \int_{-2}^{2} \left(\frac{1}{\sqrt{4-x^2}}\right)^2 dx = \pi \int_{-2}^{2} \frac{1}{4-x^2} dx = \text{BY MEANS OF PARTIAL} \\
 &\quad \begin{aligned}
 &\frac{1}{4-2x} \times \frac{1}{(2-x)(2+x)} = \frac{\frac{1}{4-2x}}{2-x} + \frac{\frac{1}{8}}{2+x} \\
 &1 = 4(x-2) + 3(2+2x) \\
 &\frac{1}{4-2x} = \frac{1}{2-x} + \frac{3}{2+x} \Rightarrow 1 = 4x - 8 + 6 + 6x \Rightarrow 1 = 10x - 2 \Rightarrow x = \frac{1}{5} \\
 &\frac{1}{4-2x} = \frac{1}{2-x} + \frac{3}{2+x} \Rightarrow 1 = 4x - 8 + 6 + 6x \Rightarrow 1 = 10x - 2 \Rightarrow x = \frac{1}{5}
 \end{aligned} \\
 &= \pi \left[\frac{1}{2-x} + \frac{3}{2+x} \right]_0^1 = \pi \left[-\frac{1}{2} \ln|2-x| + \frac{3}{2} \ln|2+x| \right]_0^1 \\
 &= \frac{\pi}{2} \left[\ln|\frac{2+2}{2-2}| \right]_0^1 = \frac{\pi}{2} [\ln 3 - \ln 1] = \frac{1}{2}\pi \ln 3
 \end{aligned}$$

Question 38 (*****)



The figure above shows the graph of the curve with equation

$$y = x\sqrt{\sin 2x}, 0 \leq x \leq \frac{\pi}{2}.$$

The shaded region, labelled as R , bounded by the curve and the x axis, is rotated by 360° about the x axis to form a solid of revolution.

Show that the volume of the solid generated is

$$\frac{\pi}{8}(\pi^2 - 4).$$

proof

$$V = \pi \int_{0}^{\frac{\pi}{2}} (y(x))^2 dx = \pi \int_{0}^{\frac{\pi}{2}} (x\sqrt{\sin 2x})^2 dx = \pi \int_{0}^{\frac{\pi}{2}} x^2 \sin 2x dx$$

DOUBLE INTEGRATION BY PARTS... IGNORING LIMITS OF PI

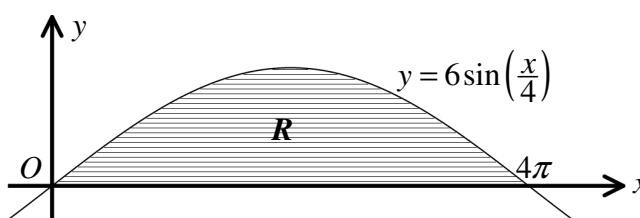
$$\begin{aligned} \int x^2 \sin 2x dx &= -\frac{1}{2}x^2 \cos 2x - \int -x \cdot 2x \cos 2x dx \\ &= -\frac{1}{2}x^2 \cos 2x + \left[2x \cos 2x \right]_0^{\frac{\pi}{2}} \\ &= -\frac{1}{2}x^2 \cos 2x + 2x \cos 2x + \left[\frac{1}{2}x^2 \sin 2x \right]_0^{\frac{\pi}{2}} \\ &= -\frac{1}{2}\pi^2 \cos 2x + 2\pi \cos 2x + \frac{1}{2}\pi^2 \cos 2x + 0 \end{aligned}$$

$$\therefore V = \pi \left[-\frac{1}{2}\pi^2 \cos 2x + 2\pi \cos 2x + \frac{1}{2}\pi^2 \cos 2x \right]_0^{\frac{\pi}{2}}$$

$$= \pi \left[(-\frac{1}{2}\pi^2) \cos \pi + (\frac{1}{2}\pi^2) \cos 0 + \frac{1}{2}\pi^2 \cos \pi \right] = \pi \left[\frac{3}{2}\pi^2 - \frac{1}{2}\pi^2 \right] = \frac{2\pi}{8}(\pi^2 - 4)$$

AS Required

Question 39 (*****)



The figure below shows the graph of the curve with equation

$$y = 6 \sin\left(\frac{x}{4}\right), 0 \leq x \leq 4\pi.$$

The shaded region R , is bounded by the curve and the x axis.

- a) Determine the area of R .

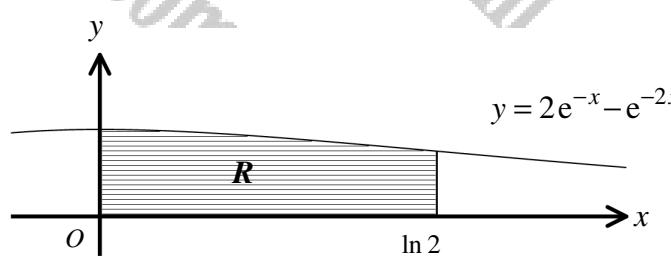
This region R is rotated through 360° about the x axis to form a solid of revolution.

- b) Show that the volume of the solid generated is $72\pi^2$.

, [48 square units]

$\text{(a)} \quad y = 6 \sin\left(\frac{x}{4}\right)$ $A = \int_0^{4\pi} 6 \sin\left(\frac{1}{4}x\right) dx = \left[-6 \cos\left(\frac{1}{4}x\right) \right]_0^{4\pi} = 24 \left[\cos(0) - \cos(4\pi) \right] = 24 [1 - (-1)] = 48$
$\text{(b)} \quad V = \pi \int_0^{4\pi} (6 \sin\left(\frac{1}{4}x\right))^2 dx = \pi \int_0^{4\pi} 36 \sin^2\left(\frac{1}{4}x\right) dx$ $= \pi \int_0^{4\pi} 36 \left[\frac{1}{2} - \frac{1}{2} \cos\left(\frac{1}{2}x\right) \right] dx = \pi \int_0^{4\pi} (18 - 18 \cos\left(\frac{1}{2}x\right)) dx$ $= \pi \left[18x - 36 \sin\left(\frac{1}{2}x\right) \right]_0^{4\pi} = \pi \left[(72\pi - 36 \sin(2\pi)) - (0 - 36 \sin(0)) \right] = 72\pi^2$

Question 40 (*****)



The figure above shows part of the graph of the curve with equation

$$y = 2e^{-x} - e^{-2x}, x \in \mathbb{R}.$$

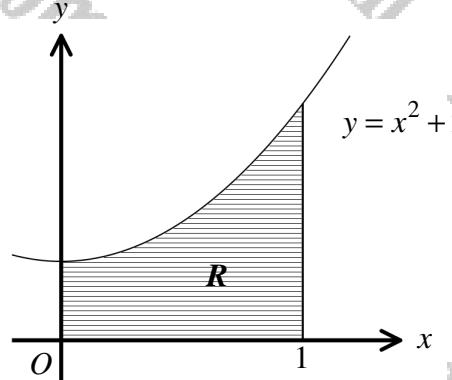
The shaded region R , bounded by the curve, the coordinate axes and the straight line with equation $x = \ln 2$, is rotated through 360° about the x axis to form a solid of revolution.

Show that the volume of the solid generated is exactly $\frac{109}{192}\pi$.

proof

$$\begin{aligned} V &= \pi \int_{\ln 2}^{\infty} y^2 dx = \pi \int_{\ln 2}^{\infty} (2e^{-x} - e^{-2x})^2 dx = \pi \int_{\ln 2}^{\infty} 4e^{-2x} - 4e^{-4x} + e^{-4x} dx \\ &= \pi \left[-2e^{-2x} + \frac{4}{3}e^{-4x} - \frac{1}{4}e^{-4x} \right]_{\ln 2}^{\infty} = \pi \left[2e^{-2\ln 2} - \frac{4}{3}e^{-4\ln 2} + \frac{1}{4}e^{-4\ln 2} \right]_{\ln 2}^0 \\ &= \pi \left[\left(2 - \frac{4}{3} + \frac{1}{4} \right) - \left(\frac{1}{2} - \frac{4}{3} + \frac{1}{4} \right) \right] = \frac{109\pi}{192} \end{aligned}$$

Question 41 (****)



The figure above shows the graph of the curve with equation

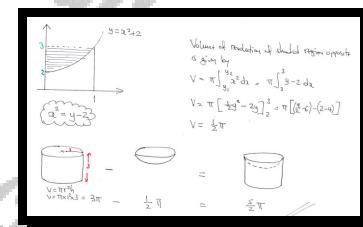
$$y = x^2 + 2.$$

The shaded region R , is bounded by the curve, the coordinate axes and the straight line with equation $x = 1$.

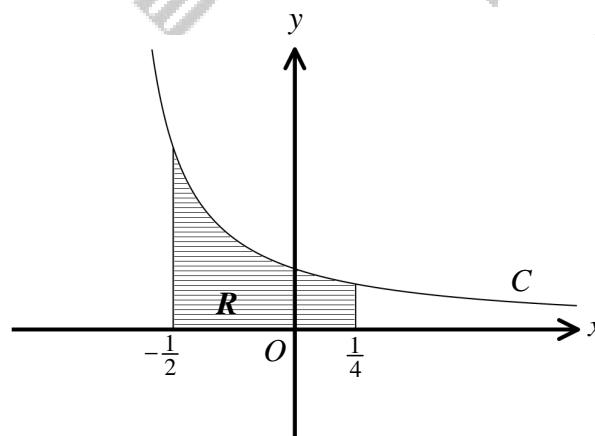
The region R is rotated through 360° about the **y axis** to form a solid of revolution.

Show that the volume of the solid generated is $\frac{5}{2}\pi$ cubic units.

[proof]



Question 42 (****)



The figure above shows part the graph of the curve C , with equation

$$y = \frac{3}{2(4x+3)}, \quad x \neq -\frac{3}{4}.$$

The shaded region R , is bounded by the curve, the x axis and the straight lines with equations $x = -\frac{1}{2}$ and $x = \frac{1}{4}$.

- a) Find the exact area of R .

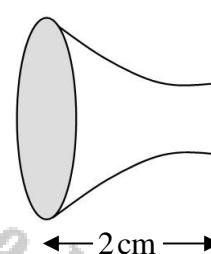
This region R is rotated through 360° about the x axis to form a solid of revolution.

- b) Show that the volume of the solid generated is $\frac{27}{64}\pi$.

[continues overleaf]

[continues from overleaf]

The solid generated in part (b) is used to model a small handle for a drawer.



The solid generated in part (b) and the drawer handle are mathematically similar.

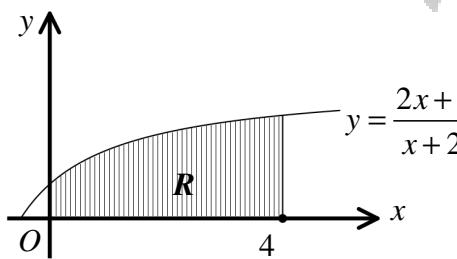
- c) Given that the length of the handle is 2cm , find the exact volume of the handle.

$$\boxed{\text{area} = \frac{3}{4} \ln 2}, \boxed{\text{volume of handle} = 8\pi}$$

$$\begin{aligned}
 \text{(a)} \quad A &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{3}{2}(2x+3)^2 dx = \frac{3}{2} \left[\frac{1}{3} \ln(4x+9) \right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{3}{2} \left[\ln(4x+9) \right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
 &= \frac{3}{2} [\ln 4 - \ln 1] = \frac{3}{2} \ln 4 = \frac{3}{2} (2\ln 2) = \frac{3}{2} \ln 2. \\
 \text{(b)} \quad V &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (4x+3)^2 dx = \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{3}{2}(2x+3) \right)^2 dx = \frac{27\pi}{8} \int_{-\frac{1}{2}}^{\frac{1}{2}} (2x+3)^2 dx \\
 &= \frac{27\pi}{8} \left[\frac{1}{3}(2x+3)^3 \right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{27\pi}{8} \left[\frac{1}{3}(2x+3)^3 \right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{27\pi}{8} \left[1 - \frac{1}{8} \right] \\
 &= \frac{27\pi}{8} \times \frac{7}{8} = \frac{189\pi}{64} \text{ cm}^3 \quad \text{As required}
 \end{aligned}$$

∴ Required volume of handle is $\frac{189\pi}{64}$

Question 43 (*****)



The figure above shows part of the curve with equation

$$y = \frac{2x+1}{x+2}, \quad x \neq -2.$$

- a) Show that

$$\frac{2x+1}{x+2} = A + \frac{B}{x+2},$$

where A and B are constants to be found.

The shaded region, labelled R , bounded by the curve, the coordinate axes and the straight line with equation $x=4$ is rotated by 360° about the x axis to form a solid of revolution.

- b) Show that the volume of revolution is

$$\pi(19 - 12\ln 3).$$

$$A = 2, B = -3$$

Q9

$$y = \frac{2x+1}{x+2} = \frac{2(x+2)-3}{(x+2)} = 2 - \frac{3}{x+2}$$

$A = 2$
 $B = -3$

Q9

$$y^2 = \left(2 - \frac{3}{x+2}\right)^2 = 4 - \frac{12}{x+2} + \frac{9}{(x+2)^2} = 4 - \frac{12}{x+2} + 9(x+2)^{-2}$$

$$V = \pi \int_{-1}^{2x_1} y^2 dx = \pi \int_{-1}^{2x_1} \left[4 - \frac{12}{x+2} + 9(x+2)^{-2}\right] dx = \pi \left[4x_1 - 12 \ln(x_1 + 2) - 9(x_1 + 2)^{-1}\right]_0^{2x_1}$$

$$= \pi \left[4x_1 - 12 \ln(2x_1 + 2) + \frac{9}{2x_1 + 2} - 9\right] = \pi \left[4(4 - 12\ln 6 - \frac{1}{2}) - 9 - 12\ln 2 - \frac{9}{2}\right]$$

$$= \pi \left[16 - 48\ln 6 - \frac{9}{2} + 12\ln 2 + \frac{9}{2}\right] = \pi [9 - 12(\ln 6 - \ln 2)]$$

$$= \pi [9 - 12\ln 3]$$

AS Specimen

Question 44 (***)**

The curve C has equation

$$y = x e^x, \quad x \in \mathbb{R}.$$

The region R is bounded by the curve, the x axis and the vertical straight lines with equations $x=1$ and $x=3$.

- a) Explain why R lies entirely above the x axis.

The region R is rotated by 360° in the x axis to form a solid of revolution.

- b) Show that the volume of this solid is

$$\frac{1}{4}\pi e^2 (13e^4 - 1).$$

proof

(a) $y = x e^x$ $\begin{cases} e^x > 0 \text{ for all } x \\ 1 \leq x \leq 3 \end{cases} \therefore \text{their product is positive}$

(b) $V = \pi \int_{x_1}^{x_2} y^2 dx = \pi \int_1^3 (xe^x)^2 dx$
 $= \pi \int_1^3 x^2 e^{2x} dx = \dots \text{METHOD BY PARTS}$

Let x^2 & e^{2x} be terms
 $= \frac{1}{2}x^2 e^{2x} - \int x^2 e^{2x} dx$ $\begin{cases} x^2 \rightarrow 2x \\ x^2 \rightarrow e^{2x} \end{cases}$

By parts again
 $= \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}x^2 e^{2x} - \int \frac{1}{2}x^2 e^{2x} dx \right]$ $\begin{cases} x^2 \rightarrow 1x \\ x^2 \rightarrow e^{2x} \end{cases}$

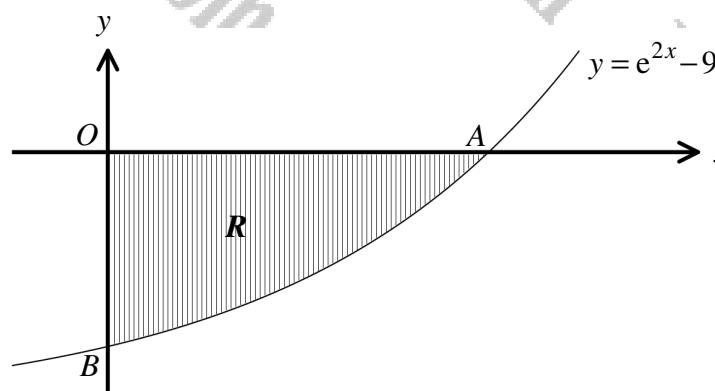
$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x^2 e^{2x} + \int \frac{1}{2}x^2 e^{2x} dx$

$= \frac{1}{2}x^2 e^{2x} + \frac{1}{2}x^2 e^{2x} + C$

$\therefore V = \pi \left[\frac{1}{2}x^2 e^{2x} - \frac{1}{2}x^2 e^{2x} + \frac{1}{2}x^2 e^{2x} \right]^3 = \pi \left[\left(\frac{1}{2}x^2 e^{2x} - \frac{1}{2}x^2 e^{2x} + \frac{1}{2}x^2 e^{2x} \right) - \left(\frac{1}{2}x^2 e^{2x} - \frac{1}{2}x^2 e^{2x} + \frac{1}{2}x^2 e^{2x} \right) \right]$

$\therefore V = \pi \left[\frac{1}{2}x^2 e^{2x} \right] = \frac{1}{2}\pi e^2 (13e^4 - 1)$ as required

Question 45 (*****)



The figure above shows part of the curve with equation

$$y = e^{2x} - 9, \quad x \in \mathbb{R}.$$

The curve crosses the coordinate axes at the points A and B . The shaded region R is bounded by the curve and the coordinate axes.

- a) Determine the exact coordinates of A and B .

The region R is rotated by 2π radians about the x axis to form a solid of revolution.

- b) Calculate the volume generated, giving the answer in the form $\pi(p + q \ln 3)$ where p and q are integers.

$$(\ln 3, 0), (0, -8), V = \pi(-52 + 81 \ln 3)$$

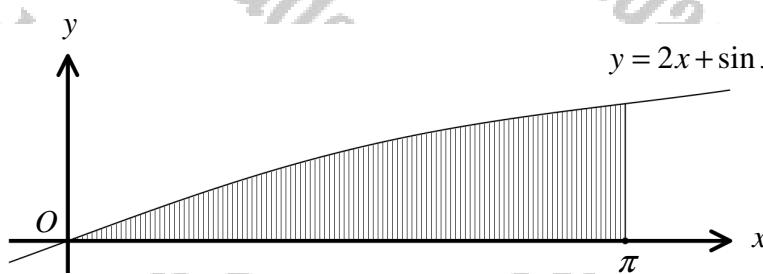
$\text{(a)} \quad \begin{cases} x=0 \\ y=0 \\ y=-8 \end{cases}$ $\begin{aligned} y &= 0 \\ e^{2x} - 9 &= 0 \\ 2x &= \ln 9 \\ x &= \frac{1}{2}\ln 9 \\ A(\ln 3, 0) \end{aligned}$	$\text{(b)} \quad \begin{aligned} V &= \pi \int_{-4}^{-2} y^2 dx = \pi \int_0^{\ln 3} (e^{2x} - 9)^2 dx \\ V &= \pi \int_0^{\ln 3} (e^{4x} - 18e^{2x} + 81) dx \\ V &= \pi \left[\frac{1}{4}e^{4x} - 9e^{2x} + 81x \right]_0^{\ln 3} \\ V &= \pi \left[\left(\frac{1}{4}(e^{4\ln 3}) - 9(e^{2\ln 3}) + 81\ln 3 \right) - (4 - 9) \right] \\ V &= \pi \left[-52 + 81 \ln 3 \right] \end{aligned}$
---	--

Question 46 (***)

Show that

a) $\int_0^\pi 4x \sin x \, dx = 4\pi.$

b) $\int_0^\pi \sin^2 x \, dx = \frac{\pi}{2}.$



The figure above shows part of the curve with equation

$$y = 2x + \sin x.$$

The shaded region bounded by the curve, the x axis and the line $x = \pi$ is rotated by 2π radians about the x axis to form a solid of revolution.

c) Show that the volume of the solid is

$$\frac{1}{6}\pi^2(8\pi^2 + 27).$$

(a) $\int_0^\pi 4x \sin x \, dx = -4x \cos x \Big|_0^\pi - \int -4 \cos x \, dx \quad \left\{ \begin{array}{l} \text{by parts} \\ 4x \rightarrow u \\ -\cos x \rightarrow v \end{array} \right. \\ = -4x \cos x \Big|_0^\pi + 4 \sin x \Big|_0^\pi \\ = [-4\pi \cos \pi + 4 \sin \pi] - [0] = 4\pi. \quad \square$

(b) $\int_0^\pi 2x^2 \, dx = \int_0^\pi \frac{1}{2} \cdot 4x^2 \, dx = \left[\frac{1}{2}x^3 - \frac{1}{6}x^3 \cos 3x \right]_0^\pi = (\frac{1}{2}\pi^3 - 0) - (0) = \frac{1}{2}\pi^3. \quad \square$

(c) $V = \pi \int_{-2}^2 y^2 \, dx = \pi \int_0^\pi (2x + \sin x)^2 \, dx = \pi \int_0^\pi 4x^2 + 4x \sin x + \sin^2 x \, dx$
 $= \pi \int_0^\pi 4x^2 \, dx + \pi \int_0^\pi 4x \sin x \, dx + \pi \int_0^\pi \sin^2 x \, dx$
 $= \pi \left[\frac{4}{3}x^3 \right]_0^\pi + \pi \left(4 \int_0^\pi x \sin x \, dx \right) + \pi \left[\frac{1}{2}\sin^2 x \right]_0^\pi$
 $= \pi \left[\frac{4}{3}\pi^3 \right] + \pi \left[4\pi^2 \right] + \left[\frac{\pi}{2} \right] = \frac{4}{3}\pi^6 + 4\pi^4 + \frac{1}{2}\pi^2$
 $= \frac{4}{3}\pi^6 + \frac{9}{2}\pi^2 \approx \frac{1}{2}\pi^2(8\pi^2 + 27) \quad \square$

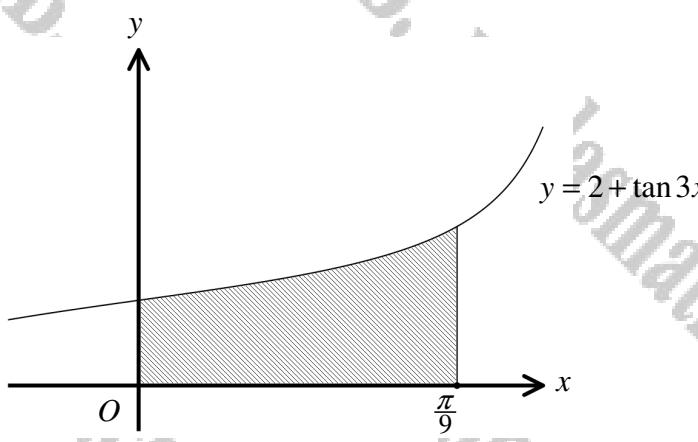
Ap 2014/15

Question 47 (****)

Show that

a) $(2 + \tan 3x)^2 = 3 + 4 \tan 3x + \sec^2 3x$

b) $\int \tan x \, dx = \ln|\sec x| + C$



The figure above shows part of the graph of the curve with equation

$$y = 2 + \tan 3x.$$

The shaded region bounded by the curve the coordinate axes and the line $x = \frac{\pi}{9}$ is rotated by 2π radians about the x axis to form a solid of revolution.

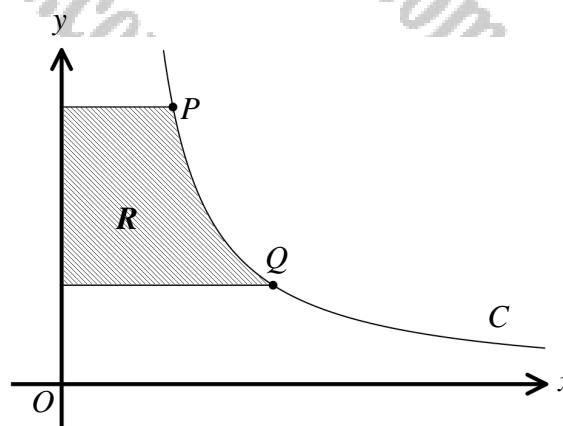
c) Show that the volume of the solid is

$$\frac{\pi}{3}(\pi + 4 \ln 2 + \sqrt{3}).$$

 proof

$$\begin{aligned}
 \text{(a)} \quad & (2 + \tan 3x)^2 = 4 + 4\tan 3x + \tan^2 3x = 4 + 4\tan 3x + (\sec^2 3x - 1) \\
 & = 3 + 4\tan 3x + \sec^2 3x \quad \boxed{\cancel{\text{ }} \cancel{\text{ }}} \\
 \text{(b)} \quad & \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{-\sin x}{\cos x} \, dx = -\ln|\cos x| + C \\
 & = \ln|\sec x|^2 + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C \quad \boxed{\cancel{\text{ }}} \\
 \text{(c)} \quad & V = \pi \int_{0}^{\frac{\pi}{9}} y^2 \, dx = \pi \int_{0}^{\frac{\pi}{9}} (2 + \tan 3x)^2 \, dx = \pi \int_{0}^{\frac{\pi}{9}} (3 + 4\tan 3x + \sec^2 3x) \, dx \\
 & = \pi \left[3x + \frac{4}{3} \ln|\sec 3x| + \frac{1}{3} \tan 3x \right]_0^{\frac{\pi}{9}} \\
 & = \pi \left[\left(\frac{\pi}{3} + \frac{4}{3} \ln 2 + \frac{\sqrt{3}}{3}\right) - (0 + \frac{4}{3} \ln 1 + 0) \right] \\
 & = \frac{\pi}{3} \left(\pi + 4 \ln 2 + \sqrt{3} \right) \quad \boxed{\cancel{\text{ }}} \quad \text{Ans (6 marks)}
 \end{aligned}$$

Question 48 (*****)



The figure above shows the graph of the curve C with equation

$$y = \frac{14}{x-2}, \quad x \neq 2.$$

The points P and Q lie on C where $x = 2.5$ and $x = 3.75$ respectively.

The shaded region R is bounded by the curve and two horizontal lines passing through the points P and Q .

R is rotated by 2π radians about the y axis forming a solid of revolution S .

- a) Find the volume of S , giving the answer in the form $\pi(a + b \ln c)$ where a , b and c are constants.

The solid S is used to model a nuclear station cooling tower.

- b) Given that 1 unit on the axes corresponds to 2 metres on the actual tower, show that the cooling tower has an approximate volume of 4200 m^3 .

$$\boxed{\pi \left(\frac{195}{2} + 56 \ln \left(\frac{7}{2} \right) \right)}$$

(a)

- $\int_{2.5}^{3.75} \frac{14}{x-2} dx$
- $\frac{14}{2} \cdot 2.5 - 28$
- $\frac{14}{2} \cdot 3.75 - 28$
- $\alpha = 2.5$
- $\alpha = \frac{14}{2.5+2}$
- $\alpha = \frac{14}{4.5}$
- $\alpha = 3.111\ldots$
- $\alpha^2 = \frac{100}{20.977777777777776}$
- $\alpha^2 = \left(\frac{\sqrt{20977.77777777777}}{20.97777777777777} \right)^2$
- $\alpha^2 = 4 + \frac{14}{2.5} + \frac{14}{3.75}$

4. Hence $V = \pi \int_{2.5}^{3.75} \alpha(y)^2 dy = \pi \int_{2.5}^{3.75} \frac{14}{2}(y-2)^2 dy = \pi \int_{2.5}^{3.75} 7(y-2)^2 dy$

$$V = \pi \left[\frac{7}{3}y^3 - 14y^2 + 28y \right]_{2.5}^{3.75}$$

$$V = \pi \left[\left(0.125 + 3.75 \times 49 - 77 \right) - \left(0.0625 + 3.125 \times 49 - 77 \right) \right]$$

$$V = \pi \left(\frac{195}{2} + 56 \ln \left(\frac{7}{2} \right) \right)$$

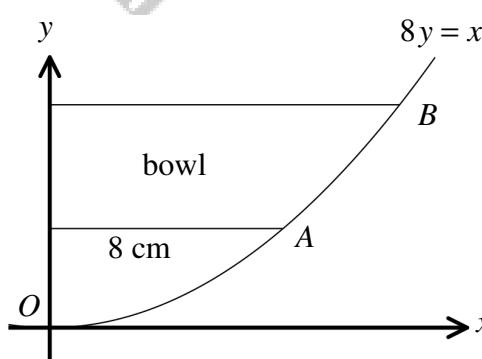
$$V = \pi \left(\frac{195}{2} + 56 \ln \left(\frac{7}{2} \right) \right)$$

(b)

$\frac{1 \text{ unit}}{2 \text{ metres}} = \frac{2 \text{ metres}}{1 \text{ unit}}$

$\therefore V \approx \pi \left(\frac{195}{2} + 56 \ln \left(\frac{7}{2} \right) \right) \times 8 \approx 4200 \text{ m}^3$

Question 49 (*****)



The figure above shows the graph of the curve with equation

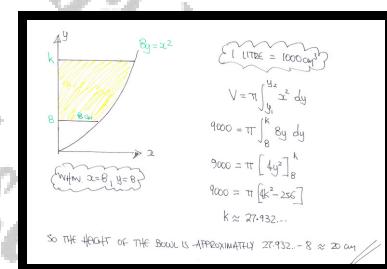
$$8y = x^2, \quad x \geq 0.$$

The points A and B lie on the curve. The curved surface of an open bowl with flat circular base is traced out by the complete revolution of the arc AB about the y axis.

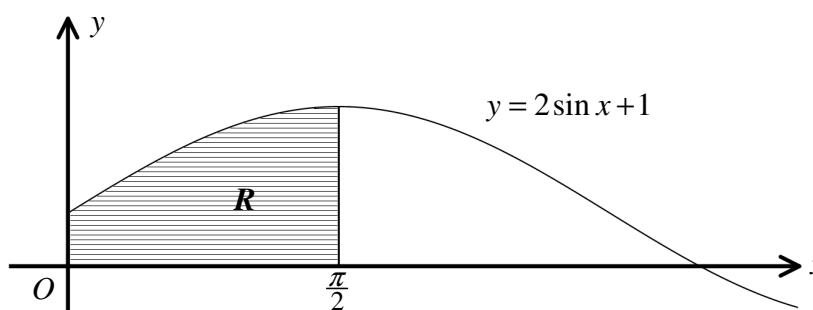
The radius of the flat circular base of the bowl is 8 cm, and its volume is 9 litres.

Find to the nearest cm the height of the bowl.

height ≈ 20 cm



Question 50 (*****)



The figure above shows the graph of the curve C with equation

$$y = 2 \sin x + 1, x \in \mathbb{R}.$$

The shaded region R is bounded by the curve, the line $x = \frac{\pi}{2}$ and the x axis.

- a) Find the exact area of R .

The region R is rotated by 2π radians in the x axis forming a solid of revolution S .

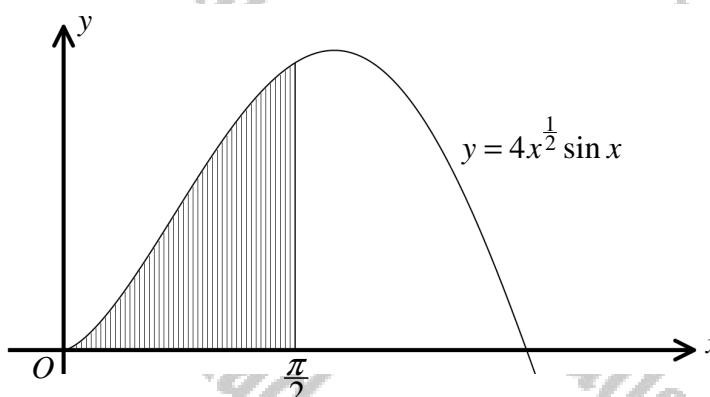
- b) Show that the volume of S is

$$\frac{\pi}{2}(3\pi+8).$$

$$\text{area} = \frac{1}{2}(\pi + 4)$$

$$\begin{aligned}
 \text{(a)} \quad & \text{Area} = \int_0^{\frac{\pi}{2}} 2 \sin x + 1 \, dx = \left[-2 \cos x + x \right]_0^{\frac{\pi}{2}} = (\pi + \frac{\pi}{2}) - (-2 + 0) \\
 & = \frac{\pi}{2} + 2 = \frac{1}{2}(\pi + 4) \\
 \text{(b)} \quad & \text{Volume} = \pi \int_0^{\frac{\pi}{2}} (2 \sin x + 1)^2 \, dx = \pi \int_0^{\frac{\pi}{2}} 4 \sin^2 x + 4 \sin x + 1 \, dx \\
 & = \pi \int_0^{\frac{\pi}{2}} 4 \left(\frac{1 - \cos 2x}{2} \right) + 4 \sin x + 1 \, dx = \pi \int_0^{\frac{\pi}{2}} 2 - 2 \cos 2x + 4 \sin x + 1 \, dx \\
 & = \pi \left[3x - \frac{1}{2} \sin 2x - 4 \cos 2x \right]_0^{\frac{\pi}{2}} = \pi \left[\frac{3\pi}{2} - 0 - 0 \right] - [0 - 4] \\
 & = \pi \left[\frac{3\pi}{2} + 4 \right] = \frac{1}{2}\pi(3\pi + 8)
 \end{aligned}$$

Question 51 (*****)



The figure above shows the graph of the curve with equation

$$y = 4x^{\frac{1}{2}} \sin x .$$

- a) Find the value of $\int_0^{\frac{\pi}{2}} 8x \cos 2x \ dx .$

The shaded region bounded by the curve, the x axis and the straight line with equation $x = \frac{\pi}{2}$ is rotated by 2π radians in the x axis to form a solid of revolution.

- b) Show that the volume of the solid is

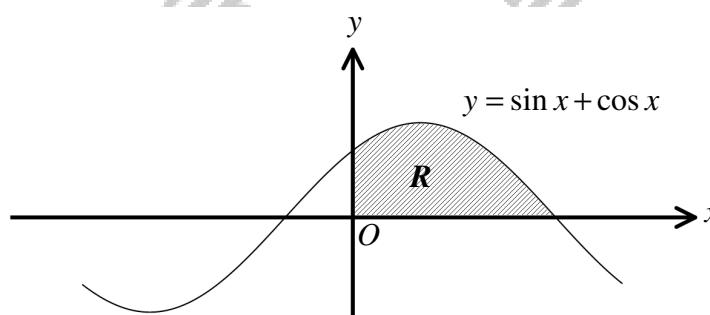
$$\pi(\pi^2 + 4).$$

[4]

(a) $\int_0^{\frac{\pi}{2}} 8x \cos 2x \ dx = 8x \int_0^{\frac{\pi}{2}} \cos 2x \ dx - \int_0^{\frac{\pi}{2}} 8x \cos 2x \ dx = \dots$ untauschen
 $= [4x \sin 2x - \int_0^{\frac{\pi}{2}} 4 \sin 2x \ dx]_0^{\frac{\pi}{2}} = \dots$ untauschen
 $= [4x \sin 2x + 2 \cos 2x]_0^{\frac{\pi}{2}} = (0-2) - (0+2) = -4$

(b) $\Rightarrow V = \pi \int_0^{\frac{\pi}{2}} (y(x))^2 dx$
 $\Rightarrow V = \pi \int_0^{\frac{\pi}{2}} (4x^{\frac{1}{2}} \sin x)^2 dx = \pi \int_0^{\frac{\pi}{2}} 16x \sin^2 x dx$
 $\Rightarrow V = \pi \int_0^{\frac{\pi}{2}} 16x (\frac{1}{2} - \frac{1}{2} \cos 2x) dx$
 $\Rightarrow V = \pi \int_0^{\frac{\pi}{2}} 8x dx - \pi \int_0^{\frac{\pi}{2}} 8x \cos 2x dx$
 $\Rightarrow V = \pi \left[4x^2 \right]_0^{\frac{\pi}{2}} - \pi \int_0^{\frac{\pi}{2}} 8x \cos 2x dx$
 $\Rightarrow V = \pi \left[4x^2 \right]_0^{\frac{\pi}{2}} - (-4)$
 $\Rightarrow V = \pi [4(\frac{\pi}{2})^2 + 4]$
 $\Rightarrow V = \pi(\pi^2 + 4)$

Question 52 (****)



The figure above shows the graph of the curve with equation

$$y = \sin x + \cos x, -\pi \leq x \leq \pi.$$

The finite region R , shown shaded in the figure, is bounded by the curve and the coordinate axes.

When R is revolved by a full turn in the x axis it traces a solid of volume V .

Show clearly that

$$V = \frac{1}{4}\pi(3\pi+2).$$

[] , proof

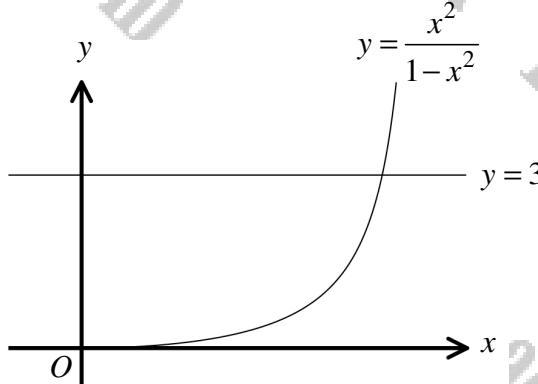
● NEEDED TO FIND THE INTEGRAL LIMITS

$$\begin{aligned} y &= 0 \\ \sin x + \cos x &= 0 \\ \tan x + 1 &= 0 \\ \tan x &= -1 \\ x &= -\frac{\pi}{4} \text{ or } \pi, \quad x = 0, \pi/2, \dots \\ \therefore x &= \frac{3\pi}{4} \end{aligned}$$

By long division

$$\begin{aligned} V &= \pi \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (y^2) dx \\ &= \pi \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin x + \cos x)^2 dx \\ &= \pi \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin^2 x + 2\sin x \cos x + \cos^2 x) dx \\ &= \pi \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (1 + \sin 2x) dx \\ &= \pi \left[x - \frac{1}{2}\cos 2x \right]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \\ &= \pi \left[\left(\frac{3\pi}{4} - 0\right) - \left(-\frac{\pi}{4} - 0\right) \right] \\ &\Rightarrow V = \pi \left(\frac{3\pi}{4} + \frac{\pi}{4} \right) \\ &\Rightarrow V = \frac{1}{4}\pi(3\pi+2) \end{aligned}$$

Question 53 (*****)



The figure above shows part of the graph of the curve with equation $y = \frac{x^2}{1-x^2}$, which passes through the origin O .

The finite area bounded by the curve, the y axis and the straight line with equation $y = 3$, is to be revolved in the y axis by 360° to form a solid of revolution S .

Find an exact value for the volume of S .

, $\pi(3 - \ln 4)$

LOOKING AT THE DIAGRAM

SETTING UP A VOLUME INTEGRAL ABOUT Y

$$\Rightarrow V = \pi \int_{y_1}^{y_2} \pi [x(y)]^2 dy$$

$$\Rightarrow V = \pi \int_0^3 \frac{4y}{1-y} dy$$

$$\Rightarrow V = \pi \int_0^3 \frac{(1+y)-1}{1-y} dy$$

$$\Rightarrow V = \pi \int_0^3 1 - \frac{1}{1-y} dy$$

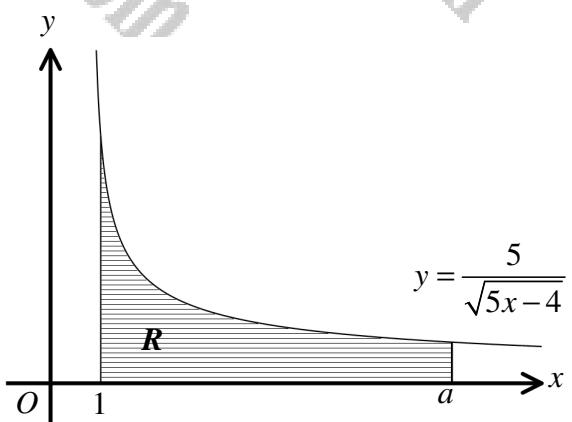
$$\Rightarrow V = \pi \left[y - \ln|1-y| \right]_0^3$$

$$\Rightarrow V = \pi \left[(3 - \ln 4) - (0 - \ln 1) \right]$$

$$\Rightarrow V = \pi (3 - \ln 4)$$

THE SUBSTITUTION: $u = 1-y$

Question 54 (****)



The figure above shows part of the graph of the curve C with equation

$$y = \frac{5}{\sqrt{5x-4}}, \quad x > \frac{4}{5}.$$

The shaded region R is bounded by the curve, the vertical straight lines $x=1$ and $x=a$, and the x axis.

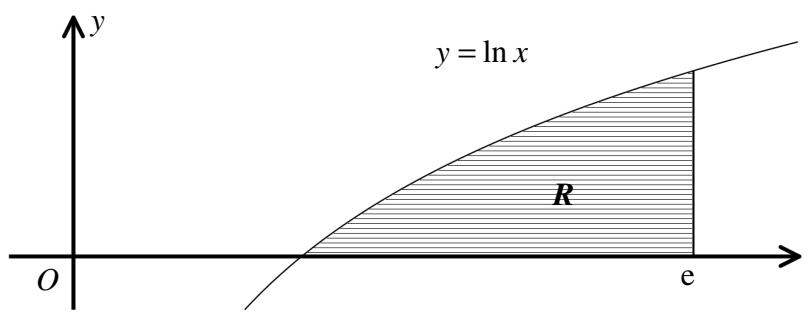
The region R is rotated by 2π radians about the x axis forming a solid of revolution.

Given that the area of R is 10 square units, show that the volume of the solid formed is $10\pi \ln 6$ cubic units.

[] proof

Volume = $\pi \int_{x_1}^{x_2} y^2 dx$
 $\Rightarrow V = \pi \int_1^a \left(\frac{5}{\sqrt{5x-4}}\right)^2 dx$
 $\Rightarrow V = \pi \int_1^a \frac{25}{5x-4} dx$
 $\Rightarrow V = \pi \left[5 \ln|5x-4| \right]_1^a$
 $\Rightarrow V = 5\pi \left[\ln 36 - \ln 1 \right]$
 $\Rightarrow V = 5\pi \ln 36$
 $\Rightarrow V = 5\pi \ln 6^2$
 $\Rightarrow V = 10\pi \ln 6$
as required

Question 55 (*****)



The figure above shows the graph of

$$y = \ln x, \quad x > 0.$$

The shaded region R is bounded by the curve, the line $x = e$ and the x axis.

R is rotated by 2π radians about the y axis, forming a solid of revolution S .

Show that the volume of S is

$$\frac{1}{2}\pi(e^2+1).$$

proof

LOOKING AT THE DIAGRAM

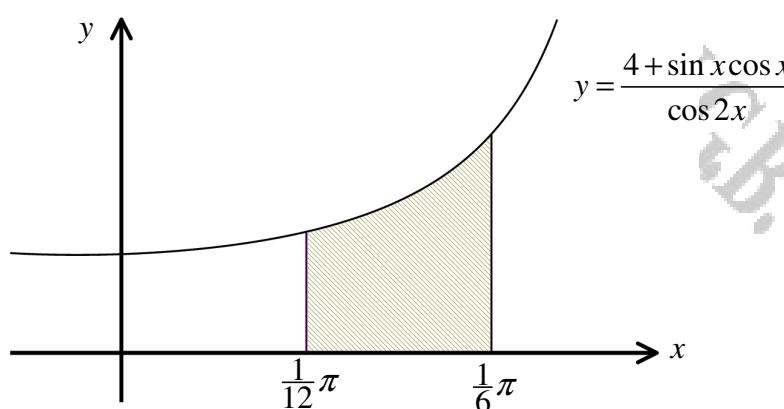
The diagram shows a small vertical strip of width dx at position x with height $y = \ln x$. When this strip is rotated 360 degrees about the y -axis, it forms a cylindrical shell. The radius of this shell is x , and its thickness is dx . The volume of this shell is given by $\pi x^2 dx$. The total volume V is found by integrating from $x=1$ to $x=e$:

$$V = \int_1^e \pi x^2 dx = \pi \int_1^e x^2 dx = \pi \left[\frac{x^3}{3} \right]_1^e = \pi \left[\frac{e^3}{3} - \frac{1}{3} \right] = \frac{1}{3}\pi(e^3 - 1)$$

However, we find that:

$$V = \pi e^2 - \pi \left[e^2 - \frac{1}{3}e^3 + \frac{1}{3} \right] = \pi \left[e^2 - \frac{1}{3}e^3 + \frac{1}{3} \right] = \pi \left[\frac{1}{3}e^3 + \frac{1}{3} \right] = \frac{1}{3}\pi(e^3 + 1) \text{ as required}$$

Question 56 (****)



The figure above shows part of the graph of the curve with equation

$$y = \frac{4 + \sin x \cos x}{\cos 2x}.$$

The finite area bounded by the curve, the x axis and the straight lines with equations $x = \frac{1}{12}\pi$ and $x = \frac{1}{6}\pi$, shown shaded in the figure, is fully revolved about the x axis, forming a solid, S .

Calculate the volume of S , correct to 3 significant figures.

, $V \approx 34.6$

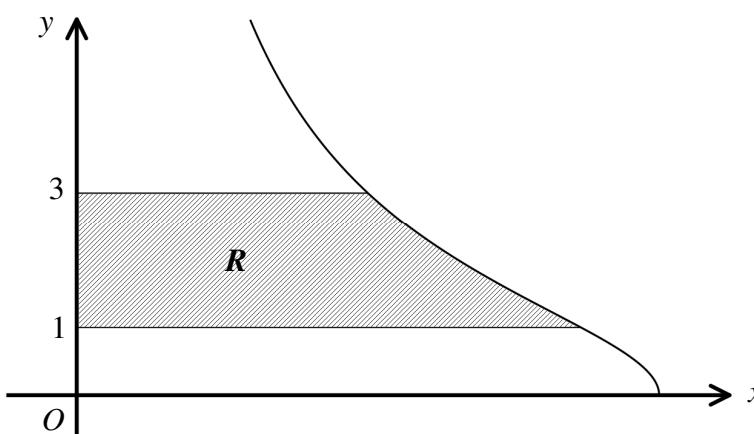
STORY BY MANIPULATING THE FUNCTION

$$\begin{aligned} y &= \frac{4 + \sin x \cos x}{\cos 2x} = \frac{4 + \frac{1}{2} \sin 2x}{\cos 2x} \\ &= \frac{4}{\cos 2x} + \frac{1}{2} \tan 2x = 4 \sec 2x + \frac{1}{2} \tan 2x \end{aligned}$$

HENCE THE VOLUME OF REVOLUTION IS GIVEN BY

$$\begin{aligned} \rightarrow V &= \pi \int_{\frac{1}{12}\pi}^{\frac{1}{6}\pi} (4 \sec^2 2x + \frac{1}{2} \tan^2 2x)^2 dx \\ \rightarrow V &= \pi \int_{\frac{1}{12}\pi}^{\frac{1}{6}\pi} 16 \sec^4 2x + 16 \sec^2 2x \tan^2 2x + \frac{1}{4} \tan^4 2x dx \\ \rightarrow V &= \pi \int_{\frac{1}{12}\pi}^{\frac{1}{6}\pi} 16 \sec^2 2x + 16 \sec^2 2x \tan^2 2x + \frac{1}{4} (\sec^2 2x - 1) dx \\ \rightarrow V &= \pi \int_{\frac{1}{12}\pi}^{\frac{1}{6}\pi} \frac{65}{4} \sec^2 2x + 4 \sec^2 2x \tan^2 2x - \frac{1}{4} dx \\ \rightarrow V &= \pi \left[\frac{65}{8} \tan 2x + 2 \sec 2x - \frac{1}{4} x \right]_{\frac{1}{12}\pi}^{\frac{1}{6}\pi} \\ \rightarrow V &= \pi \left[\left(\frac{65}{8} \sqrt{3} + 4 - \frac{\pi}{24} \right) - \left(\frac{65}{8} \sqrt{5} + \frac{4\pi}{3} - \frac{\pi}{48} \right) \right] \\ \Rightarrow V &= \pi \left[\frac{41}{12} \sqrt{3} - \frac{\pi}{48} \right] \approx 34.6 \end{aligned}$$

Question 57 (*****)



The figure above shows the curve with parametric equations

$$x = 2\cos^2 \theta, \quad y = \sqrt{3} \tan \theta, \quad 0 \leq \theta < \frac{\pi}{2}.$$

The finite region R shown shaded in the figure, bounded by the y axis, and the straight lines with equations $y=1$ and $y=3$.

Use integration in parametric to show that the volume of the solid formed when R is fully revolved about the y axis is $\frac{\pi^2}{\sqrt{3}}$.

, proof

WORKING IN PARAMETRIC

$$V = \pi \int_{y_1}^{y_2} [x(y)]^2 dy = \pi \int_{\theta_1}^{\theta_2} [2\cos^2 \theta]^2 d\theta$$

EXPANDING THE INTEGRAL

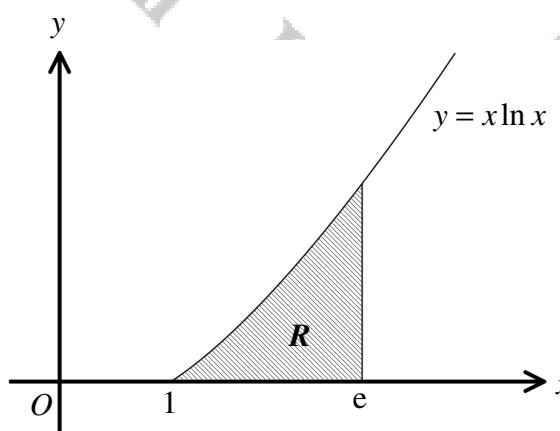
$$\begin{aligned} y_1 &= 1 & y_2 &= 3 \\ \sqrt{3} \tan \theta &= 1 & \sqrt{3} \tan \theta &= 3 \\ \tan \theta &= \frac{1}{\sqrt{3}} & \tan \theta &= \sqrt{3} \\ \theta &= \frac{\pi}{6} & \theta &= \frac{\pi}{3} \end{aligned}$$

NOTICE WE HAVE

$$\begin{aligned} V &= \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (2\cos \theta)^2 (\sqrt{3} \tan \theta) d\theta = 4\sqrt{3}\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^4 \theta \tan \theta d\theta \\ &= 4\sqrt{3}\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^4 \theta d\theta = 4\sqrt{3}\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} + \frac{1}{2} \cos 2\theta d\theta \\ &= 4\sqrt{3}\pi \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= 4\sqrt{3}\pi \left[\left(\frac{\pi}{6} + \frac{1}{4}\sin \frac{\pi}{3} \right) - \left(\frac{\pi}{12} + \frac{1}{4}\sin \frac{\pi}{6} \right) \right] \\ &= 4\sqrt{3}\pi \times \frac{\pi}{12} \\ &= \frac{(\sqrt{3}\pi)^2}{3} = \frac{\sqrt{3}\pi^2}{3} = \frac{3\pi^2}{3\sqrt{3}} \\ &= \frac{\pi^2}{\sqrt{3}} \end{aligned}$$

AS EXPECTED

Question 58 (***)+



The figure above shows the graph of the curve C with equation

$$y = x \ln x, \quad x \geq 1.$$

The shaded region R is bounded by the curve, the x axis and the vertical line $x = e$.

The region R is rotated by 2π radians in the x axis forming a solid of revolution S .

Find an exact value for the volume of S .

, $\frac{\pi}{27}(5e^3 - 2)$

Using the standard result for volume of revolution in 2

$$\Rightarrow V = \pi \int_{x_1}^{x_2} [y(x)]^2 dx = \pi \int_1^e (x \ln x)^2 dx = \pi \int_1^e x^2 (\ln x)^2 dx$$

CONTINUE BY INTEGRATION BY PARTS & IGNORING THE 4 LIMITS

$$\int x^2 (\ln x)^2 dx = \int x^2 (\ln x)^2 - \int \frac{2}{x} x^2 \ln x dx = \left[\frac{(\ln x)^2}{2} \right] \Big|_{\frac{1}{2}}^e - \int \frac{2}{x} x^2 \ln x dx$$

BY PARTS AGAIN ON THIS INTEGRAL

$$\dots = \frac{1}{2} x^2 (\ln x)^2 - \left[\frac{2}{3} x^2 \ln x - \int \frac{2}{x} x^2 \ln x dx \right]$$

$$\dots = \frac{1}{2} x^2 (\ln x)^2 - \frac{2}{3} x^2 \ln x + \int \frac{2}{x} x^2 \ln x dx$$

$$\dots = \frac{1}{2} x^2 (\ln x)^2 - \frac{2}{3} x^2 \ln x - \frac{2}{3} x^2 + C$$

RETURNING TO THE MAIN LIMIT

$$V = \pi \left[\frac{1}{2} x^2 (\ln x)^2 - \frac{2}{3} x^2 \ln x + \frac{2}{3} x^2 \right]_1^e$$

$$V = \pi \left[\left(\frac{1}{2} e^2 - \frac{2}{3} e^2 + \frac{2}{3} e^2 \right) - (0 - 0 + \frac{2}{3}) \right]$$

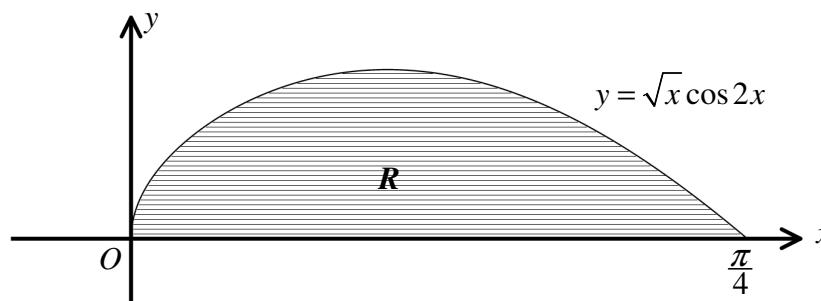
$$V = \pi \left[\frac{5}{2} e^2 - \frac{2}{3} \right]$$

$$V = \frac{\pi}{27} (5e^3 - 2)$$

Question 59 (***)+

$$f(x) = \frac{1}{8}(4x + \sin 4x), \quad x \in \mathbb{R}, \quad 0 \leq x \leq \frac{\pi}{4}.$$

- a) Show that $f'(x) = \cos^2 2x$.



The figure above shows part of the graph of a curve C with equation

$$y = \sqrt{x} \cos 2x, \quad x > 0.$$

The curve meets the x axis at the origin and at the point where $x = \frac{\pi}{4}$.

The shaded region R is bounded by the curve and the x axis. The region R is rotated by 2π radians about the x axis, forming a solid of revolution S .

- b) Show further that the volume of S is

$$\frac{\pi}{64}(\pi^2 - 4).$$

proof

(a)

$$f(x) = \frac{1}{8}(4x + \sin 4x)$$

$$f'(x) = \frac{1}{8}(4 + 4\cos 4x)$$

$$f'(x) = \frac{1}{2} + \frac{1}{2}\cos 4x$$

$$f'(x) = \cos^2 2x$$

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos 4A = 2\cos^2 2A - 1$$

$$1 + \cos 4A = 2\cos^2 2A$$

$$\frac{1}{2} + \frac{1}{2}\cos 4A = \cos^2 2A$$

(b)

$$V = \pi \int_0^{\frac{\pi}{4}} (y(x))^2 dx = \pi \int_0^{\frac{\pi}{4}} (\sqrt{x} \cos 2x)^2 dx = \pi \int_0^{\frac{\pi}{4}} x \cos^2 2x dx$$

Integrating ... by parts ...

$$\int u dv = uv - \int v du$$

$$= \frac{1}{2}x^2(4\cos 4x) - \int 4x \cos 4x dx$$

$$= \frac{1}{2}x^2 + \frac{1}{8}x \sin 4x - \left(\frac{1}{2}x^2 - \frac{1}{8}\cos 4x \right) + C$$

$$= \frac{1}{2}x^2 + \frac{1}{8}x \sin 4x - \frac{1}{2}x^2 + \frac{1}{8}\cos 4x + C$$

$$= \frac{1}{8}x^2 + \frac{1}{8}x \sin 4x + \frac{1}{8}\cos 4x + C$$

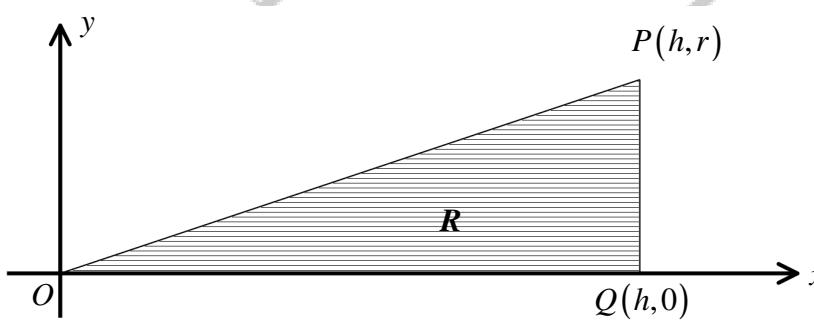
$$\therefore V = \pi \left[\frac{1}{8}x^2 + \frac{1}{8}x \sin 4x + \frac{1}{8}\cos 4x \right]_0^{\frac{\pi}{4}}$$

$$V = \pi \left[\left(\frac{\pi^2}{64} + 0 - \frac{1}{8} \right) - (0 + 0 + \frac{1}{8}) \right]$$

$$V = \pi \left[\frac{\pi^2}{64} - \frac{1}{8} \right]$$

$$V = \frac{\pi}{64}(\pi^2 - 4)$$

Question 60 (***)+



The figure above shows the straight line segment OP , joining the origin to the point $P(h, r)$, where h and r are positive coordinates.

The point $Q(h, 0)$ lies on the x axis.

The shaded region R is bounded by the straight line segments OP , PQ and OQ .

The region R is rotated by 2π radians in the x axis to form a solid cone of height h and radius r .

Show by integration that the volume of the cone V is given by

$$V = \frac{1}{3}\pi r^2 h.$$

proof

• Gradient of OP is $\frac{r}{h}$

• $y = \frac{r}{h}x \leftarrow (\text{Int. op})$

$$\begin{aligned} V &= \pi \int_{-h}^{h} (y dx)^2 dx = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2}x^2 dx \\ &= \frac{mr^2}{h^2} \int_0^h x^2 dx = \frac{mr^2}{h^2} \left[\frac{1}{3}x^3\right]_0^h \\ &= \frac{mr^2}{h^2} \left[\frac{1}{3}h^3 - 0\right] = \frac{1}{3}\pi r^2 h \end{aligned}$$

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Question 61 (*)+**

A finite region R is defined by the inequalities

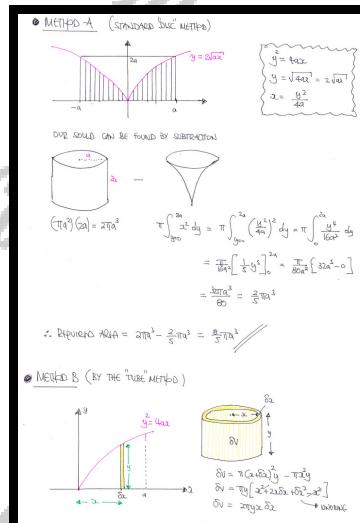
$$y^2 \leq 4ax, \quad 0 \leq x \leq a, \quad y \geq 0,$$

where a is a positive constant.

The region R is rotated by 2π radians in the y axis forming a solid of revolution.

Determine, in terms of π and a , the exact volume of this solid.

, $\frac{8}{5}\pi a^3$



-VOLUME SUMMING UP

$V = \sum 2\pi xy \Delta x$

TAKING LIMITS WE OBTAIN

$$\Rightarrow V = \int_{-a}^{a} 2\pi x (2\sqrt{ax}) dx$$

$$\Rightarrow V = 4\pi a^{\frac{3}{2}} \int_{-a}^{a} x^{\frac{3}{2}} dx$$

$$\Rightarrow V = 4\pi a^{\frac{3}{2}} \left[\frac{2}{5} x^{\frac{5}{2}} \right]_0^a$$

$$\Rightarrow V = 4\pi a^{\frac{3}{2}} \left[\frac{2}{5} a^{\frac{5}{2}} - 0 \right]$$

$$\Rightarrow V = 4\pi a^{\frac{3}{2}} \times \frac{2}{5} a^{\frac{5}{2}}$$

$$\Rightarrow V = \frac{8}{5} \pi a^3$$

ANSWER

Question 62 (*)+**

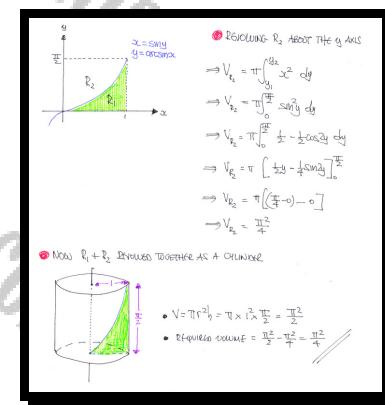
The finite region R is defined by the inequalities

$$y \leq \arcsin x, \quad x \leq 1, \quad y \geq 0.$$

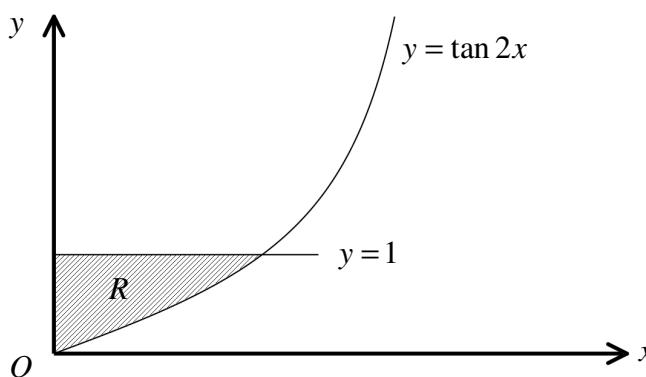
The region R is rotated by 2π radians in the y axis forming a solid of revolution.

Determine the exact volume of this solid.

, $\frac{1}{4}\pi^2$



Question 63 (***)+



The figure above shows the graph of the curve with equation

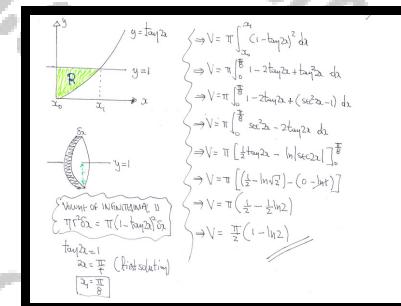
$$y = \tan 2x, \quad 0 \leq x \leq \frac{\pi}{4}.$$

The finite region R is bounded by the curve, the y axis and the horizontal line with equation $y=1$.

The region R is rotated by 2π radians about the straight line with equation $y=1$ forming a solid of revolution.

Determine an exact volume for this solid.

$$\boxed{\text{cu}}, \quad \boxed{\frac{\pi}{2}(1-\ln 2)}$$



Question 64 (***)+

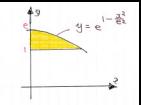
A curve C has equation

$$y = e^{1 - \frac{x^2}{e^2}}, \quad x \in \mathbb{R},$$

The finite region bounded by C , the y axis and straight line with equation $y=1$, is revolved by 2π radians about the y axis, forming a solid of revolution.

Find an exact simplified value for the volume of this solid.

$$\boxed{S^2}, \quad \boxed{V = \pi e^2 (e-2)}$$



• AS THE REVOLUTION IS IN THE y AXIS, START BY REARRANGING THE FORMULA
 $\Rightarrow y = e^{1 - \frac{x^2}{e^2}}$
 $\Rightarrow \ln y = 1 - \frac{x^2}{e^2}$
 $\Rightarrow \frac{x^2}{e^2} = 1 - \ln y$
 $\Rightarrow x^2 = e^2(1 - \ln y)$

• NEXT THE CURVE OF REVOLUTION ABOUT y IS FURNISHED BY
 $\Rightarrow V = \pi \int_{y_1}^{y_2} (x(y))^2 dy$
 $\Rightarrow V = \pi \int_1^e e^2(1 - \ln y) dy$

• NEXT NOTE THAT $\int \ln x dx = x \ln x - x + C$, either by COMMON KNOWLEDGE, OR INTEGRATION BY PARTS
 $\Rightarrow V = \pi e^2 \int_1^e 1 - \ln y dy$
 $\Rightarrow V = \pi e^2 \left[y - (y \ln y - y) \right]_1^e$
 $\Rightarrow V = \pi e^2 \left[2y - y \ln y \right]_1^e$
 $\Rightarrow V = \pi e^2 \left[(2e - e \ln e) - (2 - \ln 1) \right]$
 $\Rightarrow V = \pi e^2 (2e - e - 2)$
 $\Rightarrow V = \pi e^2 (e-2)$

Question 65 (***)+

A curve has equation

$$y = \ln(4-x), \quad x \in \mathbb{R}, \quad x \neq 4.$$

The finite region bounded by the curve, the x axis and the straight line with equation $x = 2$, is revolved by 2π radians in the y axis.

Find the exact volume of the solid formed.

ANSWER, $V = \frac{1}{2}\pi(24\ln 2 - 13)$

Start by sketching the graph.

Identify the region to be revolved.

REVOLUTED IN THE y AXIS AND THE REGION OUTSIDE IN GREEN ARE SHOWN.

$$\begin{aligned} y &= \ln(4-x) \\ 0 &\leq 4-x \\ x &\leq 4-x \\ x^2 &= 16 - 8x + e^{2x} \end{aligned}$$

$$\bullet V = \pi \int_{-1}^{1} [h(y)]^2 dy$$

$$\Rightarrow V = \pi \int_{-1}^{\ln 2} [16 - 8e^{2y} + e^{4y}] dy$$

$$\Rightarrow V = \pi \left[16y - 8e^{2y} + \frac{1}{2}e^{4y} \right]_0^{\ln 2}$$

$$\Rightarrow V = \pi \left[(16\ln 2 - 8) - (0 - 8 + \frac{1}{2}) \right]$$

$$\Rightarrow V = \pi \left[16\ln 2 - 16 + \frac{15}{2} \right]$$

$$\Rightarrow V = \pi \left[16\ln 2 - \frac{23}{2} \right]$$

Volume of revolution of the "yellow" region, i.e. a cylinder.

$$V = \pi r^2 h$$

$$V = \pi \times 2^2 \times \ln 2$$

$$V = 4\pi \ln 2$$

Required volume is given by:

$$V = \pi \left[16\ln 2 - \frac{23}{2} \right] - 4\pi \ln 2$$

$$V = 16\pi \ln 2 - \frac{45\pi}{2} - 4\pi \ln 2$$

$$V = 12\pi \ln 2 - \frac{43\pi}{2}$$

$$V = \frac{\pi}{2} [24\ln 2 - 13]$$

Question 66 (*****)

The finite region R is bounded by the coordinate axes and the curve with equation

$$y = \arccos x, -1 \leq x \leq 1.$$

The region R is rotated by 2π radians in the x axis forming a solid of revolution.

Determine the exact volume of this solid.

, $\boxed{\pi^2 - 2\pi}$

$$\text{VOLUME} = \pi \int_{x_1}^{x_2} [g(x)]^2 dx$$

$$\Rightarrow V = \pi \int_{-1}^1 (\arccos x)^2 dx$$

BY SUBSTITUTION

$$\Rightarrow V \approx \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta^2 (-\sin \theta) d\theta$$

$$\Rightarrow V = \pi \int_{0}^{\frac{\pi}{2}} \theta^2 \sin \theta d\theta$$

NOW BY PARTS (DIVIDE IT UP!!)

$$\Rightarrow \frac{d}{d\theta} V = \left[\theta^2 \sin \theta \right]_{0}^{\frac{\pi}{2}} + \int_{0}^{\frac{\pi}{2}} 2\theta \sin \theta d\theta$$

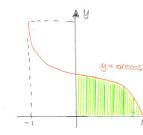
BY PARTS AGAIN

$$\Rightarrow \frac{d}{d\theta} V = \left[2\theta \sin \theta \right]_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} 2\sin \theta d\theta$$

$$\Rightarrow \frac{d}{d\theta} V = (\pi - 0) + \left[2\cos \theta \right]_{0}^{\frac{\pi}{2}}$$

$$\Rightarrow \frac{d}{d\theta} V = \pi + (0 - 2)$$

$$\Rightarrow V = \pi - 2$$

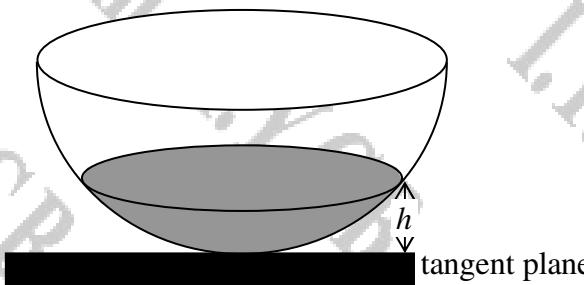


$\theta = \arccos x$
 $\cos \theta = x$
 $dx = -\sin \theta d\theta$
 $x=0 \mapsto \theta=\frac{\pi}{2}$
 $x=1 \mapsto \theta=0$

θ^2	2.0
$-\sin \theta$	$\cos \theta$

2.0	2
$\sin \theta$	$\cos \theta$

Question 67 (*****)



The figure above shows a hemispherical bowl of radius r containing water to a height h . The water in the bowl is in the shape of a minor spherical segment.

It is required to find a formula for the volume of a minor spherical segment as a function of the radius r and the distance of its plane face from the tangent plane, h .

The circle with equation

$$x^2 + y^2 = r^2, \quad x \geq 0$$

is to be used to find a formula for the volume of a minor spherical segment.

Show by integration that the volume V of the minor spherical segment is given by

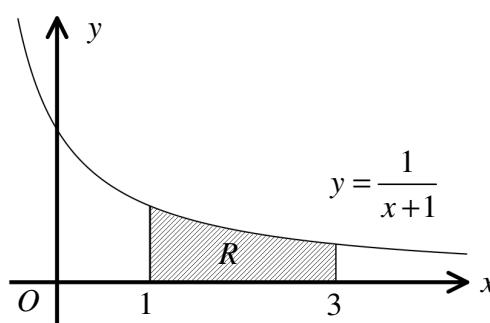
$$V = \frac{1}{3}\pi h^2 (3r - h),$$

where r is the radius of the sphere or hemisphere and h is the distance of its plane face from the tangent plane.

proof

<p>Without loss of generality start with the circle $x^2 + y^2 = r^2, x \geq 0, y \geq 0$</p> <p>Revolving the shaded region in the above diagram about the x-axis</p> $\Rightarrow V = \pi \int_{0}^{2\theta} [y^2] dx$ $\Rightarrow V = \pi \int_{0}^{2\theta} (r^2 - x^2) dx$ $\Rightarrow V = \pi \left[r^2 x - \frac{1}{3} x^3 \right]_{x=0}^{x=r}$	$\Rightarrow V = \pi \left[\left(r^2 - \frac{1}{3} r^3 \right) - \left(r^2 (r-h) - \frac{1}{3} (r-h)^3 \right) \right]$ $\Rightarrow V = \pi \left[\frac{2}{3} r^3 - r^2 (r-h) + \frac{1}{3} (r-h)^3 \right]$ $\Rightarrow V = \pi \left[\frac{2}{3} r^3 - r^3 + r^2 h + \frac{1}{3} (r-3rh+3h^2-r^2) \right]$ $\Rightarrow V = \pi \left[\frac{1}{3} r^3 + r^2 h + \frac{1}{3} r^2 h^2 \right]$ $\Rightarrow V = \frac{1}{3}\pi h^2 (3r - h)$ <p style="text-align: right;">as required</p>
--	--

Question 68 (*****)



The figure above shows the graph of the curve with equation

$$y = \frac{1}{x+1}, \quad x \in \mathbb{R}, \quad x = -1.$$

The finite region R is bounded by the curve, the x axis and the lines with equations $x=1$ and $x=3$.

Determine the exact volume of the solid formed when the region R is revolved by 2π radians about ...

- a) ... the y axis.
- b) ... the straight line with equation $x=3$.

, ,

(a)

$V = \pi r^2 h = \pi \times 3^2 \times \frac{1}{4} = \frac{3}{4}\pi$

Volume of base² solid

$$V = \pi \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{y}-1\right)^2 dy = \pi \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{y^2}-\frac{2}{y}+1\right) dy$$

$$V = \pi \left[y - \frac{1}{y} - 2\ln y \right]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$V = \pi \left[\frac{1}{2} - 2\ln \frac{1}{2} \right] - \left[\frac{1}{4} - 2\ln \frac{1}{4} \right]$$

$$V = \pi \left[\frac{1}{2} + 2\ln 2 - 2\ln \frac{1}{4} \right]$$

$$V = \pi \left[\frac{1}{2} + 4\ln 2 \right]$$

Reversed volume

$$V = \pi r^2 h = \pi \times 1^2 \times \frac{1}{2} = \frac{1}{2}\pi$$

Required volume

$$V = \frac{3}{4}\pi + \pi \left[\frac{1}{2} + 4\ln 2 \right] - \frac{1}{2}\pi$$

$$= 4\pi - \pi \ln 4$$

$$= \pi(4 - \ln 4)$$

(b)

TRANSLATE BY 3 UNITS TO THE LEFT

$$V = \pi \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{y}-4\right)^2 dy = \pi \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{y^2}-\frac{8}{y}+16\right) dy$$

$$= \pi \left[-\frac{1}{y} - 8\ln y + 16y \right]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \pi \left[\left(-\frac{1}{2} - 8\ln \frac{1}{2} + 16\right) - \left(-4 - 8\ln \frac{1}{4} + 16\right) \right]$$

$$= \pi \left[6 + 8\ln 2 - 8\ln \frac{1}{4} \right] = \pi \left[6 + 8\ln 2 \right] = 4\pi(1 + \ln 2)$$

Reversed volume

$$V = 2\pi - \pi \left[\frac{1}{2} - 8\ln 2 \right] = 4\pi + 8\ln 2\pi = 4\pi \left(1 + 2\ln 2 \right) = 4\pi(1 + \ln 4)$$

Alternative (can be done by the shell method)

Shells

$$V = \int_{2\pi}^{3\pi} 2\pi yz \, dz = \int_{2\pi}^{3\pi} 2\pi \frac{1}{z+1} z \, dz$$

$$= 2\pi \int_{2\pi}^{3\pi} \frac{1}{z+1} z \, dz = 2\pi \int_{2\pi}^{3\pi} \frac{z^2}{z+1} \, dz$$

$$= 2\pi \int_{2\pi}^{3\pi} \left(z - \frac{1}{z+1} \right) dz = 2\pi \left[z^2 - \ln(z+1) \right]_{2\pi}^{3\pi}$$

$$= 2\pi \left[(3 - \ln 4) - (4 - \ln 5) \right]$$

$$= 2\pi \left[2 + \ln \frac{5}{4} \right]$$

$$= 2\pi \left[2 + \ln 2 \right]$$

$$= \pi \left[4 + 2\ln 2 \right]$$

$$= \pi(4 - \ln 4)$$

Volume

$$V = \pi r^2 h = \pi \times 1^2 \times 2\pi = 2\pi\pi = 2\pi^2$$

Question 69 (*****)

The finite region R is bounded by the curve with equation

$$y = \sin x, 0 \leq x \leq \pi,$$

and the straight line with equation $y = \frac{1}{3}$.

The region R is rotated by 2π radians in the straight line with equation $y = \frac{1}{3}$ forming a solid of revolution.

Determine the exact volume of this solid.

$$\boxed{\text{SOLN: } V = \frac{\pi}{18} \left[11\pi - 22 \arcsin\left(\frac{1}{3}\right) - 12\sqrt{2} \right]}$$

TO ROTATE THE CURVE "DOWN" BY $\frac{1}{3}$, SO THE REVOLUTION IS AROUND THE x AXIS, AND THE REVOLVED CURVE HAS EQUATION

$$y = \sin x - \frac{1}{3}$$

$$\Rightarrow V = \pi \int_{x_1}^{x_2} y^2 dx = \pi \int_{\arcsin\frac{1}{3}}^{\pi - \arcsin\frac{1}{3}} (\sin x - \frac{1}{3})^2 dx$$

$$\Rightarrow V = \pi \int_{\arcsin\frac{1}{3}}^{\pi - \arcsin\frac{1}{3}} (\sin^2 x - \frac{2}{3}\sin x + \frac{1}{9}) dx$$

USE SUBSTITUTION, TO MAKE ONE OF THE INTEGRALS NICER

$$\Rightarrow V = 2\pi \int_{\arcsin\frac{1}{3}}^{\pi/2} \frac{1}{2} - \frac{1}{3}\cos x - \frac{1}{9}\sin x + \frac{1}{9} dx$$

$$\Rightarrow V = 2\pi \int_{\arcsin\frac{1}{3}}^{\pi/2} \frac{1}{2} - \frac{1}{3}\cos x - \frac{1}{9}\sin x + \frac{1}{9} dx$$

$$\Rightarrow V = 2\pi \int_{\arcsin\frac{1}{3}}^{\pi/2} \frac{1}{18}(9 - 6\cos x - 2\sin x - 1) dx$$

$$\Rightarrow V = 2\pi \left[\frac{1}{18}(9x - 6\sin x - 2\cos x - x) \right]_{\arcsin\frac{1}{3}}^{\pi/2}$$

$$\Rightarrow V = \left[\frac{11\pi}{18} + \frac{2}{3}\cos x - \frac{1}{9}\sin x \right]_{\arcsin\frac{1}{3}}^{\pi/2}$$

NOW

$$\theta = \arcsin\frac{1}{3}$$

$$\sin\theta = \frac{1}{3}$$

$$\therefore \cos(\arcsin\frac{1}{3}) = \cos(\arccos\frac{\sqrt{8}}{3}) = \frac{\sqrt{8}}{3}$$

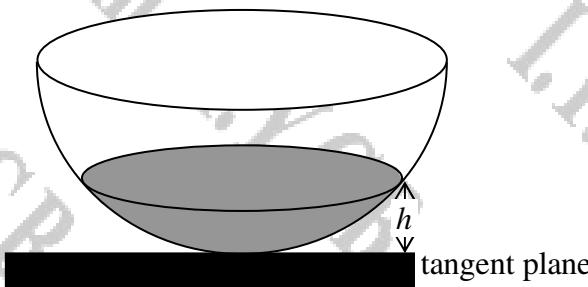
$$\Rightarrow V = 2\pi \left[\left(\frac{11\pi}{18} + \left(-\frac{1}{18}\sin\theta + \frac{2}{3}\sqrt{\frac{8}{3}} - \frac{1}{2} + \frac{1}{3}\sqrt{\frac{8}{3}} \right) \right) \right]$$

$$\Rightarrow V = 2\pi \left[\frac{11\pi}{18} + \frac{1}{18} \sin\theta - \frac{2}{3}\sqrt{\frac{8}{3}} + \frac{1}{6}\sqrt{\frac{8}{3}} \right]$$

$$\Rightarrow V = 2\pi \left[\frac{11\pi}{18} + \left(\frac{1}{18} \sin\theta - \frac{1}{3}\sqrt{2} \right) \right]$$

$$\Rightarrow V = \frac{11\pi}{18} \left[11\pi - 2\sin\theta - 12\sqrt{2} \right]$$

Question 70 (*****)



The figure above shows a hemispherical bowl of radius r containing water to a height h . The water in the bowl is in the shape of a minor spherical segment. It is required to find a formula for the volume of a minor spherical segment as a function of the radius r and the distance of its plane face from the tangent plane, h .

Show by integration that the volume V of the minor spherical segment is given by

$$V = \frac{1}{3}\pi h^2 (3r - h),$$

where r is the radius of the sphere or hemisphere and h is the distance of its plane face from the tangent plane.

, proof

WITHOUT LOSS OF GENERALITY START WITH THE EQUATION $x^2 + y^2 = r^2$, $x > 0$, $y > 0$

REVIEWING THE SHADeD REGION IN THE ABOVE DIAGRAM ABOUT THE x-AXIS

$$\Rightarrow V = \pi \int_{x_1}^{x_2} [y(x)]^2 dx$$

$$\Rightarrow V = \pi \int_{x_1}^{x_2} (r^2 - x^2) dx$$

$$\Rightarrow V = \pi \int_{x_1}^{x_2} [r^2 - x^2] dx$$

$$\Rightarrow V = \pi \left[\left(r^2 x - \frac{1}{3} x^3 \right) \Big|_{x_1}^{x_2} - \left(r^2(x-h) - \frac{1}{3}(x-h)^3 \right) \Big|_{x_1}^{x_2} \right]$$

$$\Rightarrow V = \pi \left[\frac{2}{3}x^3 - r^2(x-h) + \frac{1}{3}(x-h)^3 \right]$$

$$\Rightarrow V = \pi \left[\frac{2}{3}x^3 - r^2x + r^2h + \frac{1}{3}(x^3 - 3x^2h + 3xh^2 - h^3) \right]$$

$$\Rightarrow V = \pi \left[\frac{1}{3}x^3 + r^2h + \frac{1}{3}x^3 - r^2x + x^2h - \frac{1}{3}h^3 \right]$$

$$\Rightarrow V = \pi \left[\frac{2}{3}x^3 - \frac{1}{3}h^3 \right]$$

$$\Rightarrow V = \frac{1}{3}\pi h^2 (3r - h)$$

as required

Question 71 (*****)

A curve has equation

$$y = \frac{8}{x^2 - 4x + 8}, \quad x \in \mathbb{R}.$$

The finite region R is bounded by the curve, the y axis and the tangent to the curve at the stationary point of the curve.

Determine, in simplified exact form, the volume of the solid formed when R is fully revolved about the y axis.

, $V = 4\pi[2 - \pi + 2\ln 2]$

Start with a quick sketch of $y = \frac{8}{x^2 - 4x + 8} = \frac{8}{(x-2)^2 + 4}$.

- At $x = 2 \Rightarrow y = 0$ (removable)
- $y > 0$ for all x
- $x = 2$ is a local minimum
- Highest value of y occurs when $x = 0$
 $\therefore y = 2$

Endeavour the equation for x^2

$$\begin{aligned} \Rightarrow y &= \frac{8}{(x-2)^2 + 4} \\ \Rightarrow (x-2)^2 + 4 &= \frac{8}{y} \\ \Rightarrow (x-2)^2 &= \frac{8}{y} - 4 \\ \Rightarrow x-2 &= \pm \sqrt{\frac{8-y}{y}} \\ \Rightarrow x &= 2 \pm 2\sqrt{\frac{8-y}{y}} \\ \Rightarrow x^2 &= 4 - 8\sqrt{\frac{8-y}{y}} + 4\left(\frac{8-y}{y}\right) \\ \Rightarrow x^2 &= 4 - 8\sqrt{\frac{8-y}{y}} + 4\left(\frac{8-y}{y}\right) \end{aligned}$$

$$\Rightarrow x^2 = 4 - 8\sqrt{\frac{8-y}{y}} + \frac{32-4y}{y} = 4$$

Now the volume can be found

$$\begin{aligned} \Rightarrow V &= \pi \int_{y_1}^{y_2} [x(y)]^2 dy = \pi \int_{y_1}^2 \frac{8}{y} - 8\sqrt{\frac{8-y}{y}} dy \\ \Rightarrow V &= 8\pi \left(\frac{8}{y} - \sqrt{\frac{8-y}{y}} \right) dy \\ \Rightarrow V &= 8\pi \left[\ln y \right]_1^2 - 8\pi \int_1^2 \sqrt{\frac{8-y}{y}} dy \\ \Rightarrow V &= 8\pi [\ln 2 - \ln 1] - 8\pi \int_1^2 \sqrt{\frac{8-y}{y}} dy \end{aligned}$$

$y = 2\sin\theta$
 $dy = 2\cos\theta d\theta$
 $\sin\theta = \frac{y}{\sqrt{y^2+1}}$
 $\theta = \arcsin(\frac{y}{\sqrt{y^2+1}})$
 $y = 1 \mapsto \theta = \frac{\pi}{4}$
 $y = 2 \mapsto \theta = \frac{\pi}{2}$

$$\begin{aligned} \Rightarrow V &= 8\pi \ln 2 - 8\pi \int_{\pi/4}^{\pi/2} \sqrt{\frac{8-2\sin\theta}{2\sin\theta}} (2\cos\theta d\theta) \\ \Rightarrow V &= 8\pi \ln 2 - 8\pi \int_{\pi/4}^{\pi/2} \sqrt{\frac{1-2\sin\theta}{\sin\theta}} (2\cos\theta d\theta) \\ \Rightarrow V &= 8\pi \ln 2 - 8\pi \int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\sin\theta} (2\cos\theta d\theta) \end{aligned}$$

By trigonometric identities we have

$$\begin{aligned} \Rightarrow V &= 8\pi \ln 2 - 8\pi \int_{\pi/4}^{\pi/2} 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta \\ \Rightarrow V &= 8\pi \ln 2 - 8\pi \left[\frac{2}{3} + 2\sin 2\theta \right]_{\pi/4}^{\pi/2} \\ \Rightarrow V &= 8\pi \ln 2 - 8\pi \left[2\theta + \sin 2\theta \right]_{\pi/4}^{\pi/2} \\ \Rightarrow V &= 8\pi \ln 2 - 8\pi \left[(\pi/2 + 0) - (\pi/4 + 1) \right] \\ \Rightarrow V &= 8\pi \ln 2 - 8\pi \left[\frac{\pi}{4} - 1 \right] \\ \Rightarrow V &= 8\pi \ln 2 - 4\pi^2 + 8\pi \\ \Rightarrow V &= 4\pi[2\ln 2 - \pi + 2] \end{aligned}$$

Question 72 (*****)

A spherical cap of depth a is removed from a sphere of radius na , where n is a positive constant, such that $n > \frac{1}{2}$. The volume of the spherical cap is less than half the volume of the sphere.

The remainder of the sphere is moulded to a right circular cone whose base is equal to that of the circular plane face of the spherical cap removed.

Given that the height of the cone is ma , where m is a positive constant, show that

$$m = (n + p)(2n + q),$$

where p and q are integers to be found.

, $m = (n+1)(2n-1)$

SET UP THE VOLUME OF THE SPHERICAL CAP AS A VOLUME OF REVOLUTION OF THE曲面 $x^2+y^2=r^2$, $r>x$, about the x -axis

$$V_{\text{sph}} = \pi \int_{-r}^{r} [y(x)]^2 dx$$

$$V_{\text{cap}} = \pi \int_{-r}^{a} r^2 - x^2 dx$$

$$V_{\text{cap}} = \pi \left[r^2x - \frac{1}{3}x^3 \right]_{-r}^r$$

TIDY THE EXPRESSION

$$V_{\text{cap}} = \pi \left[r^2 \left(x - \frac{1}{3}x^3 \right) \right]_{-r}^r = r^2(r-a) + \frac{1}{3}(r-a)^3$$

$$V_{\text{cap}} = \pi \left[r^2 - \frac{1}{3}a^3 \right] = \pi \left[r^2 + \frac{1}{3}(r^3 - 3ra^2 + 3r^2a - a^3) \right]$$

$$V_{\text{cap}} = \pi \left[\frac{2}{3}r^3 + \frac{1}{3}a^3 \right] = \pi \left[\frac{2}{3}r^3 + \frac{1}{3}(4r^3 - 3ra^2 + 3r^2a - a^3) \right]$$

$$V_{\text{cap}} = \pi \left[\frac{10}{3}r^3 - \frac{1}{3}a^3 \right] = \frac{1}{3}\pi a^3 \left[3r - a \right]$$

NOW REMOVING THE CAP (diagram opposite)

BY PYTHAGORAS

$$|AB|^2 = (na^2 - (ra-a)^2) = a^2 \left[n^2 - (n-1)^2 \right] = a^2 (n^2 - n^2 + 2n - 1) = 2na^2 - 2na + a^2$$

NOW THE VOLUME OF THE CAP

$$V_{\text{cap}} = \frac{1}{3}\pi a^3 (3ra - a) = \frac{1}{3}\pi a^3 (3ra - 1)$$

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi n^3 a^3$$

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h^3 = \frac{1}{3}\pi r^2 (3ra - 1)$$

$$= \frac{1}{3}\pi r^2 [4ra - (3r-1)]$$

$$= \frac{1}{3}\pi r^2 [4ra - 3r + 1]$$

FINDING

$$\frac{1}{3}\pi r^2 h^3 = \frac{1}{3}\pi a^3 (4ra - 3r + 1)$$

$$\Rightarrow \frac{1}{3}\pi [r^2(2r-1)](ma) = \frac{1}{3}\pi r^3 (4a^2 - 3a + 1)$$

$$\Rightarrow \frac{1}{3}\pi r^3 (4a^2 - 3a + 1) = \frac{1}{3}\pi r^3 (4a^2 - 3a + 1)$$

$$\Rightarrow m(2a-1) = 4a^2 - 3a + 1$$

$$\Rightarrow m = \frac{4a^2 - 3a + 1}{2a-1} \quad \text{← BY INSPECTION } m=1 \text{ IS A SOLUTION}$$

$$\Rightarrow m = \frac{4a^2 - 3a + 1}{2a-1}$$

$$\Rightarrow m = \frac{(2a+1)(2a-1)}{2a-1}$$

$$\Rightarrow m = \frac{(2a+1)(2a-1)^2}{2a-1}$$

$$\Rightarrow m = (2a+1)(2a-1)$$

~~$\therefore m = (2a+1)(2a-1)$~~ \checkmark \checkmark

Question 73 (*****)

A curve has equation

$$y^2 = \ln|3x-12|, \quad x \in \mathbb{R}, \quad x \neq 4.$$

The finite region bounded by the curve, the x axis and the straight line with equation $y=1$, is revolved by 2π radians in the x axis.

Find the exact volume of the solid formed.

$$\boxed{\text{Answer}}, \quad V = \frac{2}{3}\pi(e-1)$$

• START WITH A SKETCH OF THE CURVE

• IDENTIFY THE REGION TO BE REVOLVED

$y^2 = \ln|3x-12|$
 $|y| = \sqrt{\ln|3x-12|}$
 $e = \sqrt{|3x-12|}$
 $3x-12 = e^2 \Rightarrow x = \frac{e^2+12}{3}$
 $3x-12 = e \Rightarrow x = \frac{e+12}{3}$
 $\Rightarrow x = \frac{e+12}{3} - \frac{12-e^2}{3}$

• FIND THE DIAMETER OF A CHORUS OF RADIUS 1 AT A POINT $\frac{3}{2}e$

$\Rightarrow V = \pi \int_{\frac{3}{2}e}^e \pi r^2 dx = \pi \int_{\frac{3}{2}e}^e \ln u du$
 $\Rightarrow V = \frac{2}{3}\pi \left[u \ln u - u \right]_{\frac{3}{2}e}^e$
 $\Rightarrow V = \frac{2}{3}\pi \left[(e \ln e - e) - (\frac{3}{2}e \ln \frac{3}{2}e - \frac{3}{2}e) \right]$
 $\Rightarrow V = \frac{2}{3}\pi(e-1)$

• FINALLY CONSIDER THE FOLLOWING REVOLUTION IN THE x AXIS, THUS DOUBLE

$\Rightarrow V = \pi \int_{-\frac{3}{2}e}^{\frac{3}{2}e} [y(x)]^2 dx$
 $\Rightarrow V = \pi \left(\frac{2}{3} \int_{-\frac{3}{2}e}^{\frac{3}{2}e} \ln(2x-3e) dx \right)$

• BY A. SUBSTITUTION

$\Rightarrow V = \pi \int_{-1}^1 \ln u \frac{du}{-3} = \frac{\pi}{3} \int_{-1}^e \ln u du$

• BY PARTS OR QUOTING STANDARD RESULTS

$\Rightarrow V = \frac{\pi}{3} \left[u \ln u - u \right]_1^e$
 $\Rightarrow V = \frac{\pi}{3} \left[(e \ln e - e) - (0 - 1) \right]$
 $\Rightarrow V = \frac{2}{3}\pi(e-1)$

• HENCE THE REQUIRED VOLUME IS GIVEN BY

$\Rightarrow V = \frac{2\pi e}{3} - \left(2 \times \frac{\pi}{3} \right)$
 $\Rightarrow V = \frac{2\pi e}{3} - \frac{2\pi}{3}$
 $\Rightarrow V = \frac{2\pi}{3}(e-1)$

Question 74 (*****)

The finite region R is bounded by the curve with equation $x = \cos y^2$, the y axis and the straight line with equation $y = \frac{1}{2}\sqrt{\pi}$.

Determine, in exact simplified form, the volume of the solid formed by revolving R by a full turn in the x axis.

, $\boxed{\frac{\pi}{2}(2-\sqrt{2})}$

● LEARN HOW TO FIND THE CO-ORDINATES OF POINT P₁ IN THE DIAGRAM

$x = \cos y^2$
 $\cos x = y^2$
 $y = \pm\sqrt{\arccos x}$
 $y = \sqrt{\arccos x}$ IS MARKED IN ORANGE

$y = +\sqrt{\arccos x}$ $y = -\sqrt{\arccos x}$

$\arccos x = \frac{\pi}{4}$
 $x = \cos \frac{\pi}{4}$
 $x = \frac{\sqrt{2}}{2} \quad \therefore P_1(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

● VOLUME OF REVOLUTION OF THE YELLOW REGION (CYLINDER)

$$V = \pi r^2 h = \pi (\frac{1}{2}\sqrt{\pi})^2 (\frac{1}{2}\sqrt{2})$$

$$= \pi \times \frac{1}{4}\pi \times \frac{1}{2}\sqrt{2}$$

$$= \frac{1}{8}\pi^2\sqrt{2}$$

● BY INTEGRATION NOT - FIND THE VOLUME OF REVOLUTION OF THE GREEN & YELLOW REGIONS TOGETHER

$$V = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y^2 dx = \pi \int_{0}^{\frac{\pi}{2}} \arccos x dx$$

$$= \pi \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} (\theta - \sin \theta d\theta) = \pi \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \theta \sin \theta d\theta$$

BY PARTS NEXT

θ	1
$-\cos \theta$	$\sin \theta$

$$\dots = \pi \left\{ [-\theta \cos \theta]_{\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta \right\}$$

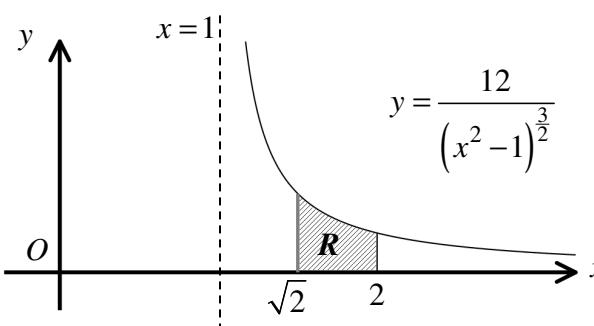
$$= \pi \left[\sin \theta - \theta \cos \theta \right]_{\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi \left[1 - \left(\frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{2} \right) \right]$$

$$= \pi \left[1 - \frac{\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{2} \right] = \pi - \frac{\pi\sqrt{2}}{2} + \frac{\pi^2\sqrt{2}}{8}$$

● HOW THE REQUIRED VOLUME IS

$$V = \left(\pi - \frac{\pi\sqrt{2}}{2} + \frac{\pi^2\sqrt{2}}{8} \right) - \frac{\pi^2\sqrt{2}}{8}$$

$$V = \frac{\pi}{2} [2 - \sqrt{2}]$$

Question 75 (*****)

The figure above shows the curve with equation

$$y = \frac{12}{(x^2 - 1)^{\frac{3}{2}}}, \quad x > 1.$$

The region R , bounded the curve, the x axis and the straight lines with equations $x = \sqrt{2}$ and $x = 2$, is revolved by a full turn about the x axis, forming a solid S .

- a) Show that the volume of S is given by

$$144\pi \int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} \csc^2 \theta \cot^4 \theta \, d\theta.$$

- b) Hence find an exact simplified expression for the volume of S .

	$V = 2\pi \left[14 - 9\sqrt{2} + 27 \ln \left(\frac{1+\sqrt{2}}{\sqrt{3}} \right) \right]$
--	--

a) Starting by the volume formula

$$V = \pi \int_{x_1}^{x_2} y^2 \, dx = \pi \int_{\sqrt{2}}^2 \left(\frac{12}{(x^2 - 1)^{\frac{3}{2}}} \right)^2 \, dx = 144\pi \int_{\sqrt{2}}^2 \frac{1}{x^2 - 1} \, dx$$

• By substitution (trigonometric or otherwise)

$\alpha = \sec \theta$	$\csc^2 \theta = -3\cot \theta \csc \theta$
$d\alpha = \sec \theta \tan \theta \, d\theta$	$-3\cot \theta \csc \theta = -3\csc^2 \theta$
$2 = \sec \theta$	$\theta = \frac{\pi}{3}$
$\sec \theta = 2$	$\csc \theta = \frac{1}{2}$
$\cot \theta = \frac{1}{2}$	$\csc^2 \theta = \frac{1}{4}$
$\theta = \frac{\pi}{3}$	$\tan \theta = \sqrt{3}$

$$\dots = 144\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{(\sec^2 \theta - 1)^{\frac{3}{2}}} (\sec \theta \tan \theta \, d\theta)$$

$$= 144\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sec^2 \theta \tan^2 \theta}{(\tan^2 \theta)^{\frac{3}{2}}} \, d\theta = 144\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sec^2 \theta}{\tan^3 \theta} \, d\theta$$

$$= 144\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sec^2 \theta}{\tan^2 \theta \cot \theta} \, d\theta = 144\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\tan^2 \theta \cot \theta} \, d\theta$$

$$= 144\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \csc^2 \theta \, d\theta$$

b) Regarding what the units ...

$$I = \int \csc^2 \theta \, d\theta = \int (\csc \theta \cot \theta) \csc \theta \, d\theta$$

BY PARTS

$\csc \theta$	$-3\cot \theta \csc \theta$
$-\csc \theta$	$\csc \theta \cot \theta$

$$\Rightarrow I = -\csc \theta \cot \theta - 3 \int \cot \theta \csc^2 \theta \, d\theta$$

$$\Rightarrow I = -\csc \theta \cot \theta - 3 \int \cot \theta \csc \theta \, d\theta$$

$$\Rightarrow I = -\csc \theta \cot \theta - 3 \int \csc \theta \, d\theta$$

$$\Rightarrow I = -\csc \theta \cot \theta - 3 \int \csc \theta \, d\theta$$

$$\Rightarrow I = -\csc \theta \cot \theta - 3I$$

$$\Rightarrow 4I = -\csc \theta \cot \theta$$

$$\Rightarrow I = -\frac{1}{4} \csc \theta \cot \theta - \frac{3}{4} \int \cot \theta \csc^2 \theta \, d\theta$$

Now let $J = \int \cot \theta \csc^2 \theta \, d\theta = \int \cot \theta (\csc \theta \cot \theta) \, d\theta$

BY PARTS AGAIN

$\csc \theta$	$-3\cot \theta \csc \theta$
$-\csc \theta$	$\csc \theta \cot \theta$

$$\Rightarrow J = -\csc \theta \cot \theta - \int \csc \theta \, d\theta$$

$$\Rightarrow J = -\csc \theta \cot \theta - \int \csc \theta (\csc \theta \cot \theta) \, d\theta$$

$$\Rightarrow J = -\csc \theta \cot \theta - \int (\csc^2 \theta + \csc \theta) \, d\theta$$

$$\Rightarrow J = -\csc \theta \cot \theta - J$$

$$\Rightarrow 2J = -\csc \theta \cot \theta - \int \csc \theta \, d\theta$$

• COMBINING THE TWO "DOBLE UNRELATED" RESULTS

$$\Rightarrow I = -\frac{1}{4} \csc \theta \cot \theta - \frac{3}{4} \int \cot \theta \csc^2 \theta \, d\theta - \frac{1}{2} \int \csc \theta \, d\theta$$

$$\Rightarrow I = -\frac{1}{4} \csc \theta \cot \theta + \frac{3}{4} \int \cot \theta \csc^2 \theta \, d\theta + \frac{1}{2} \int \csc \theta \, d\theta$$

$$\Rightarrow I = \frac{3}{4} \csc \theta \cot \theta - \frac{3}{4} \int \cot \theta \csc^2 \theta \, d\theta + \frac{1}{2} \int \csc \theta \, d\theta + C$$

• RETURNING TO LIMITS OF THE INTEGRAL AT THE POINT

$$\Rightarrow V = 144\pi \left[\frac{1}{4} \csc \theta \cot \theta - \frac{3}{4} \int \cot \theta \csc^2 \theta \, d\theta - \frac{1}{2} \int \csc \theta \, d\theta \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$\Rightarrow V = \pi \left[144 \left(\frac{1}{4} \csc \theta \cot \theta - \frac{3}{4} \int \cot \theta \csc^2 \theta \, d\theta - \frac{1}{2} \int \csc \theta \, d\theta \right) \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$\Rightarrow V = \pi \left[\left(\frac{1}{4} \times \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}} - \frac{3}{4} \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{3}} \right) - \left(144 \left(\frac{1}{4} \csc \theta \cot \theta - \frac{3}{4} \int \cot \theta \csc^2 \theta \, d\theta - \frac{1}{2} \int \csc \theta \, d\theta \right) \right) \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$\Rightarrow V = \pi \left[\left(\frac{1}{4} \times \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}} - \frac{3}{4} \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{3}} \right) - \left(144 \left(\frac{1}{4} \times \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}} - \frac{3}{4} \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{3}} \right) \right) \right]$$

$$\Rightarrow V = \pi \left[28 - 18\sqrt{2} + 27 \ln \left(\frac{1+\sqrt{2}}{\sqrt{3}} \right) \right]$$

$$\Rightarrow V = 2\pi \left[14 - 9\sqrt{2} + 27 \ln \left(\frac{1+\sqrt{2}}{\sqrt{3}} \right) \right]$$

Question 76 (***)**

A curve C and a straight line L have respective equations

$$y = x^2 \quad \text{and} \quad y = x.$$

The finite region bounded by C and L is rotated around L by a full turn, forming a solid of revolution S .

Find, in exact form, the volume of S .

$\boxed{\frac{\pi\sqrt{2}}{60}}$

① CONSIDER AN INFINITESIMAL SLICE OF THICKNESS du AT THE POINT (x_1, y_1) ON THE CURVE $y = x^2$.
The radius of the slice is x_1 . The height of the slice is dx_1 . The volume of the slice is $\pi(x_1)^2 dx_1$.

② FIRSTLY $\tan\theta = 1$ (GRADIENT OF $y = x$)
 $\tan\theta = 2x_1$ (CALCULATE AS IT IS THE GRADIENT OF THE TANGENT AT THE POINT $P(x_1, y_1)$)
 $\frac{dx_1}{du} = \frac{1}{2x_1}$

③ NEXT $\frac{dx_1}{du} = \frac{1}{2x_1} = \frac{1}{1+2u^2} = \frac{1}{1+u^2}$
 $\sin\theta = \frac{u}{\sqrt{1+u^2}}$
 $\cos\theta = \frac{1}{\sqrt{1+u^2}}$

④ WORKING AT AN ENLARGED SCALE OF THE REGION TO MAKE LIFE EASIER
 $180 - \theta - (\tan^{-1} u) = 90 - \theta$
 $du = |x_1| = |x_1| = \sqrt{1 + (x_1^2 - u^2)}$
 $du = \sqrt{1 + (x_1^2 - u^2)} \sin\theta$
 $du = \frac{1}{\cos\theta} [\sqrt{1 + (x_1^2 - u^2)} \sin\theta]$
 $du = [\csc\theta + \tan\theta \sin\theta] du$
 $du = [\csc\theta + 2u \sin\theta] du$

⑤ NEXT $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{u}{\sqrt{1+u^2}}} = \frac{1}{u\sqrt{1+u^2}}$
 $\sin\theta = \frac{u}{\sqrt{1+u^2}}$
 $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{u}{\sqrt{1+u^2}}} = \frac{1}{\sqrt{1+u^2}}$

⑥ RETURNING TO THE FORMULATION OF THE VOLUME ELEMENT
 $du = [\csc\theta + 2u \sin\theta] du$
 $du = \frac{1}{\sqrt{1+u^2}} [1 + 2u^2] du$
 $du = \frac{1}{\sqrt{1+u^2}} (1 + 2u^2) du$

⑦ FINALLY NEED THE RADIUS OF THE DISC, i.e. THE HEIGHT OF THE INFINITESIMAL SLICE du :
 $H_1 = \sqrt{R^2 - x_1^2}$
 $H_1 = \sqrt{1 + u^2}$
 $H_1 = \sqrt{(x_1^2 - u^2) + 1}$
 $H_1 = \frac{1}{\sqrt{1+u^2}} (x_1^2 - u^2)$

⑧ SUMMING UP ALL THE DISCS, i.e. THE REGION:
 $V = \sum_{u=0}^{1/\sqrt{2}} \pi(H_1)^2 du$
 $V = \sum_{u=0}^{1/\sqrt{2}} \pi \left[\sqrt{1 + u^2} (x_1^2 - u^2) \right]^2 \frac{1}{\sqrt{1+u^2}} (1 + 2u^2) du$

⑨ TAKING LIMITS:
 $V = \pi \frac{1}{2\sqrt{2}} \int_{0}^{1/\sqrt{2}} (x_1^2 - u^2)(1+2u^2) du$
 $V = \frac{\pi}{2\sqrt{2}} \int_{0}^{1/\sqrt{2}} ((1+2u^2)(x_1^2 - u^2) + 2u^2) du$
 $V = \frac{\pi}{2\sqrt{2}} \int_{0}^{1/\sqrt{2}} \frac{x_1^2 - 3u^2 + 2u^4}{x_1^2 - u^2 + 2u^2} du$
 $V = \frac{\pi}{2\sqrt{2}} \int_{0}^{1/\sqrt{2}} x_1^2 - 3u^2 + 2u^4 du$
 $V = \frac{\pi}{2\sqrt{2}} \left[\frac{1}{3}u^3 - \frac{3}{2}u^4 + \frac{2}{5}u^5 \right]_0^{1/\sqrt{2}}$
 $V = \frac{\pi}{2\sqrt{2}} \left(\frac{1}{3} \cdot \frac{1}{8} - \frac{3}{2} \cdot \frac{1}{16} + \frac{2}{5} \cdot \frac{1}{32} \right)$
 $V = \frac{\pi}{2\sqrt{2}} \cdot \frac{1}{16}$
 $V = \frac{\pi\sqrt{2}}{60}$

Created by T. Madas