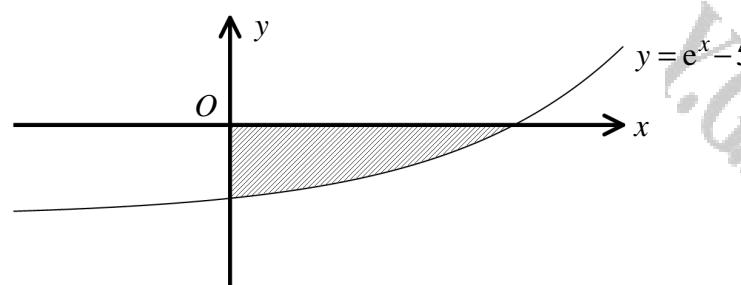


INTEGRATION

FINDING AREAS

Question 1 (**)



The diagram above shows the graph of the curve with equation

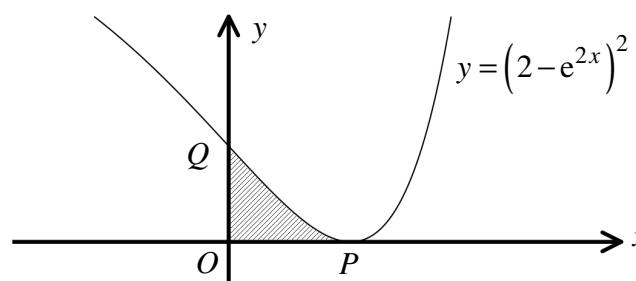
$$y = e^x - 5, \quad x \in \mathbb{R}.$$

Use integration to find the exact area of the finite region bounded by the curve and the coordinate axes.

$$5\ln 5 - 4 \approx 4.05$$

$$\begin{aligned} A &= \int_{-5}^{ln 5} f(x) dx \\ &= \int_{-5}^{ln 5} e^x - 5 dx \\ &= \left[e^x - 5x \right]_{-5}^{ln 5} \\ &= \left(e^{ln 5} - 5 \cdot ln 5 \right) - \left(e^{-(-5)} - 5 \cdot (-5) \right) \\ &= 5 - 5 \ln 5 - 1 = 4 - 5 \ln 5 \end{aligned}$$

Question 2 (**+)



The figure above shows the graph of the curve with equation

$$y = (2 - e^{2x})^2, \quad x \in \mathbb{R}.$$

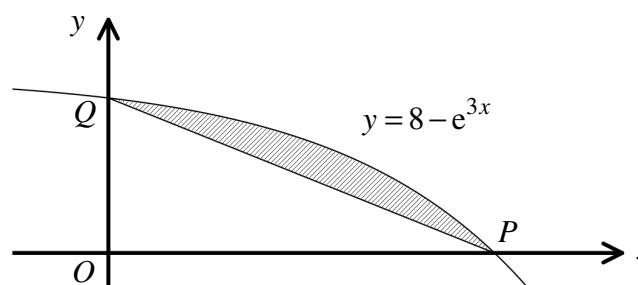
The curve touches the x axis at the point P and crosses the y axis at the point Q .

Find the exact area of the finite region bounded by the curve and the line segments OP and OQ .

$$\boxed{-\frac{5}{4} + \ln 4}$$

REPRODUCED WITH PERMISSION BY
 $\int_0^{\ln 2} (2 - e^{2x})^2 dx$
 $= \int_0^{\ln 2} 4 - 4e^{2x} + e^{4x} dx$
 $= \left[4x - 2e^{2x} + \frac{1}{4}e^{4x} \right]_0^{\ln 2}$
 $= (2\ln 2 - 4 + \frac{1}{4}) - (0 - 2 + \frac{1}{4})$
 $= 2\ln 2 - 3 + 2 - \frac{1}{4}$
 $= -\frac{5}{4} + 2\ln 2$ OR $-\frac{5}{4} + \ln 4$

Question 3 (***)



The diagram above shows the graph of the curve with equation

$$y = 8 - e^{3x}, \quad x \in \mathbb{R}.$$

The curve meets the x -axis at the point P and the y -axis at the point Q .

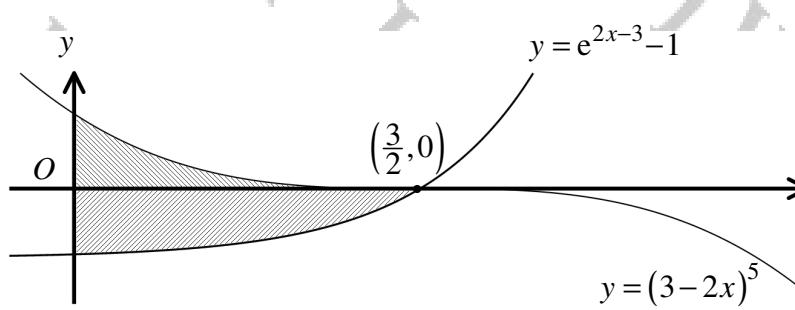
Show that the area of the finite region bounded by the curve and the straight line segment PQ is exactly $\frac{9}{2} \ln 2 - \frac{7}{3}$.

proof

$$\begin{aligned} \text{when } x=0 &: y=7 && \therefore Q(0, 7) \\ \text{when } y=0 &: 0=8-e^{3x} \\ e^{3x} &= 8 \\ 3x &= \ln 8 \\ x &= \frac{\ln 8}{3} \\ x &= \ln 2. \end{aligned} \quad \therefore P(\ln 2, 0)$$

- AREA OF TRIANGLE $\triangle OPQ = \frac{1}{2}|OP||OQ| = \frac{1}{2}\times \ln 2 \times 7 = \frac{7}{2}\ln 2$
- AREA UNDER THE CURVE = $\int_0^{\ln 2} (8 - e^{3x}) dx = \left[8x - \frac{1}{3}e^{3x} \right]_0^{\ln 2} = \left(8\ln 2 - \frac{1}{3}e^{3\ln 2} \right) - \left(0 - \frac{1}{3} \right) = 8\ln 2 - \frac{7}{3}$
- REQUIRED AREA = $(8\ln 2 - \frac{7}{3}) - \frac{7}{2}\ln 2 = \frac{9}{2}\ln 2 - \frac{7}{3}$ \checkmark RECORDED

Question 4 (***)



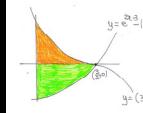
The figure above shows the graphs of the curves with equations

$$y = (3 - 2x)^5 \quad \text{and} \quad y = e^{2x-3} - 1.$$

Both curves meet each other at the point with coordinates $\left(\frac{3}{2}, 0\right)$.

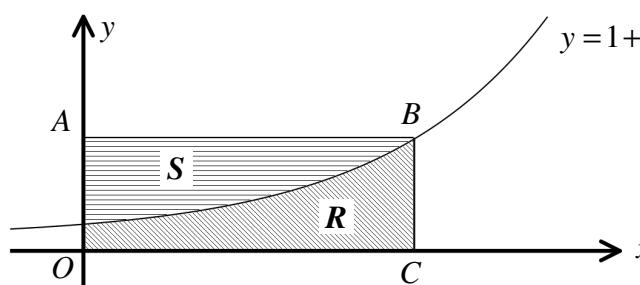
Calculate the area of the shaded region bounded by the two curves and the y axis giving the answer correct to three decimal places.

≈ 61.775



$$\begin{aligned} \text{SHADE AREA} &= \int_{0}^{\frac{3}{2}} (3 - 2x)^5 dx = \left[-\frac{1}{12}(3 - 2x)^6 \right]_0^{\frac{3}{2}} = \frac{1}{12}((3 - 2x)^6) \Big|_0^{\frac{3}{2}} = \frac{1}{12}(27 - 0) = \frac{27}{12} \\ \text{AREA} &= \int_{0}^{\frac{3}{2}} e^{2x-3} dx = \left[\frac{1}{2}e^{2x-3} \right]_0^{\frac{3}{2}} = \left(\frac{1}{2}e^{2 \cdot \frac{3}{2} - 3} \right) - \left(\frac{1}{2}e^{2 \cdot 0 - 3} \right) = -1 - \frac{1}{2}e^{-3} \\ \therefore \text{AREA} &= -1 - \frac{1}{2}e^{-3} + \left(1 + \frac{1}{2}e^3 \right) \approx 61.775 \end{aligned}$$

Question 5 (***)



The figure above shows the graph of the curve with equation

$$y = 1 + e^{2x}, \quad x \in \mathbb{R}.$$

The point C has coordinates $(1, 0)$. The point B lies on the curve so that BC is parallel to the y axis. The point A lies on the y axis so that $OABC$ is a rectangle.

The region R is bounded by the curve, the coordinate axes and the line BC .

The region S is bounded by the curve, the y axis and the line AB .

Show that the area of R is equal to the area of S .

, proof

LOOKING AT THE DIAGRAM

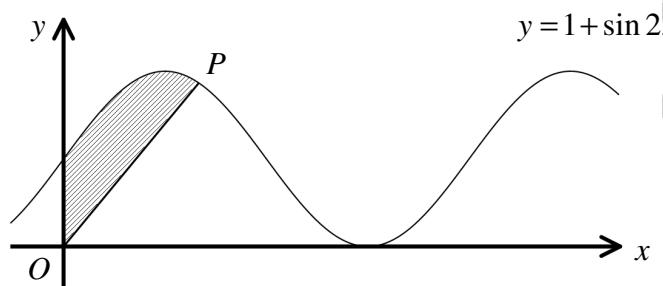
- when $x=1, y=1+e^2$
i.e. $B(1, 1+e^2)$
- AREA OF THE RECTANGLE $OABC$
 $1 \times (1+e^2) = 1+e^2$

NEXT FIND BY INTEGRATION THE AREA OF R

$$\begin{aligned} R &= \int_0^1 (1+e^{2x}) dx = \left[x + \frac{1}{2}e^{2x} \right]_0^1 = (1 + \frac{1}{2}e^2) - (0 + \frac{1}{2}) \\ &= \frac{1}{2} + \frac{1}{2}e^2 = \frac{1}{2}(1+e^2) \end{aligned}$$

i.e. HALF THE AREA OF THE RECTANGLE

$\therefore \text{AREA } R = \text{AREA } S$

Question 6 (***)

The figure above shows the graph of the curve with equation

$$y = 1 + \sin 2x, \quad x \in \mathbb{R}.$$

The point P lies on the curve where $x = \frac{\pi}{3}$.

Show that the area of the finite region bounded by the curve, the y axis and the straight line segment OP is exactly

$$\frac{1}{12}(2\pi + 9 - \pi\sqrt{3}).$$

, proof

Looking at the diagram below,

Area under the curve between $x=0$ & $x=\frac{\pi}{3}$ is given by

$$\int_0^{\frac{\pi}{3}} (1 + \sin 2x) dx = \left[x - \frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{3}} = \left[\frac{\pi}{3} - \frac{1}{2} \left(-\frac{1}{2} \right) \right] - \left[0 - \frac{1}{2} \right] = \frac{\pi}{3} + \frac{1}{4} + \frac{1}{2} = \frac{\pi}{3} + \frac{3}{4}$$

Area of the triangle is given by

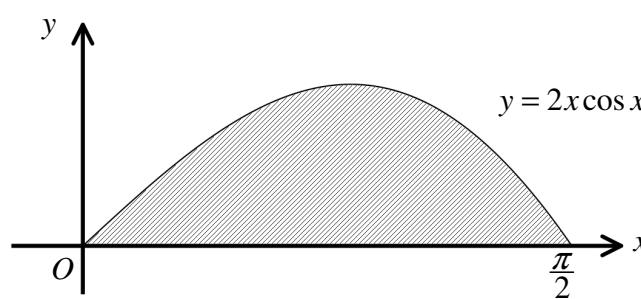
$$\frac{1}{2} \times \frac{\pi}{3} \times \left(1 + \sin \frac{\pi}{3} \right) = \frac{\pi}{6} + \frac{\pi\sqrt{3}}{12}$$

Required area is

$$\left(\frac{\pi}{3} + \frac{3}{4} \right) - \left(\frac{\pi}{6} + \frac{\pi\sqrt{3}}{12} \right) = \frac{\pi}{6} + \frac{3}{4} - \frac{\pi\sqrt{3}}{12} = \frac{1}{12}(2\pi + 9 - \pi\sqrt{3})$$

As required

Question 7 (***)



The figure above shows the graph of the curve with equation

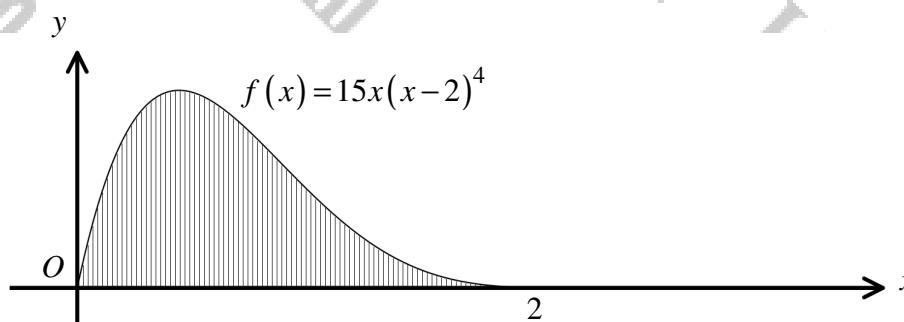
$$y = 2x \cos x, \quad 0 \leq x \leq \frac{\pi}{2}.$$

Use integration by parts to find the exact area of the finite region bounded by the curve and the x axis.

$\boxed{\pi - 2}$

$A_1 = \int_{x_1}^{\frac{\pi}{2}} y(x) dx = \dots \text{ by parts and ignoring limits}$ $A_1 = \int_0^{\frac{\pi}{2}} 2x \cos x dx = 2x \sin x - \int 2 \sin x dx$ $= 2x \sin x + 2 \cos x + C$ <p style="text-align: center;">REINTRODUCE LIMITS</p> $A = \left[2x \sin x + 2 \cos x \right]_0^{\frac{\pi}{2}} = \left[2 \left(\frac{\pi}{2} \right) \sin \frac{\pi}{2} + 2 \cos \frac{\pi}{2} \right] - \left[0 + 2 \cos 0 \right]$ $A = \pi - 2$	$\begin{array}{ c c } \hline x & 2 \\ \hline \sin x & \cos x \\ \hline \end{array}$
---	---

Question 8 (***)



The figure above shows the curve with equation

$$f(x) = 15x(x-2)^4, \quad 0 \leq x \leq 2.$$

- a) Express $f(x+2)$ in the form $Ax^5 + Bx^4$, where A and B are constants.
- b) By using part (a), or otherwise find the value of

$$\int_0^2 f(x) \, dx.$$

- c) Explain geometrically the significance of

$$\int_0^2 f(x) \, dx \equiv \int_{-2}^0 f(x+2) \, dx.$$

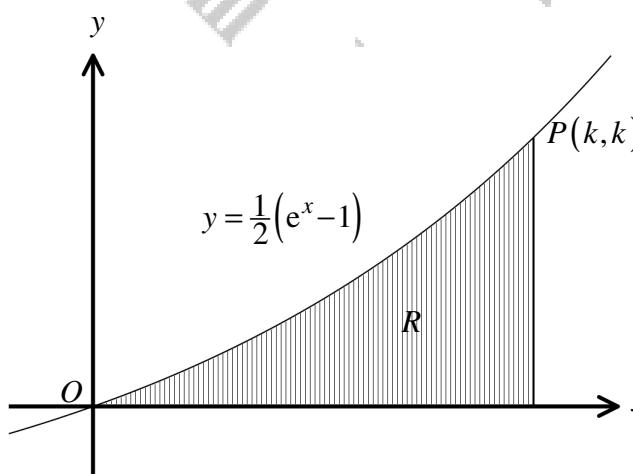
$$f(x+2) = 15x^5 + 30x^4, \quad \int_0^2 f(x) \, dx = 32$$

(a) $f(x) = 15x(x-2)^4$
 $f(x+2) = 15(x+2)(x+2-2)^4$
 $f(x+2) = 15(x+2)x^4$
 $f(x+2) = 15x^5 + 30x^4$

(b) $A = \int_0^2 15x(x-2)^4 \, dx$
 $= \int_{-2}^0 15x^5 + 30x^4 \, dx$
 $= \left[\frac{3}{2}x^4 + 6x^5 \right]_0^{-2}$
 $= 0 - [160 - 192]$
 $= 32$

(c)

Question 9 (***)



The figure above shows the curve C with equation

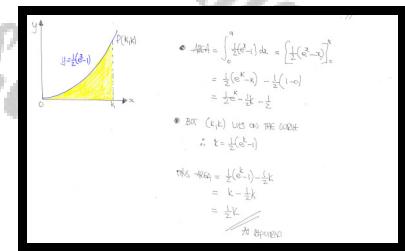
$$y = \frac{1}{2}(e^x - 1), \quad x \in \mathbb{R}.$$

The point $P(k, k)$, where k is a positive constant, lies on C .

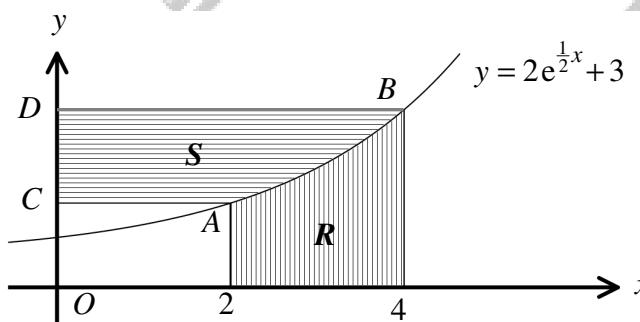
The finite region R , shown shaded in the figure, is bounded by C , the x axis and the line with equation $x = k$.

Show clearly that the area of R is $\frac{1}{2}k$.

proof



Question 10 (***)



The figure above shows the graph of the curve with equation

$$y = 2e^{\frac{1}{2}x} + 3, \quad x \in \mathbb{R}.$$

The points A and B lie on the curve where $x = 2$ and $x = 4$, respectively.

The finite region R is bounded by the curve, the x axis and the lines with equations $x = 2$ and $x = 4$.

- a) Determine the exact area of R .

The points C and D lie on the y axis so that the line segments CA and DB are parallel to the x axis.

The region S is bounded by the curve, the y axis and the line segments CA and DB .

- b) Determine the exact area of S .

area of $R = 4e^2 - 4e + 6$	area of $S = 4e^2$
-----------------------------	--------------------

(a)

area of $R = \int_2^4 2e^{\frac{1}{2}x} + 3 \, dx$
 $= \left[4e^{\frac{1}{2}x} + 3x \right]_2^4$
 $= (4e^2 + 12) - (4e + 6)$
 $= 4e^2 - 4e + 6$

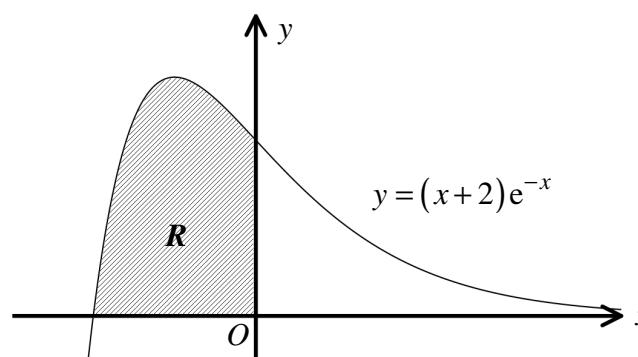
(b)

With $x=2$: $y = 2e+3$
 With $x=4$: $y = 2e^2+3$

area of $S = \frac{2e^2+3}{2} - \frac{2e+3}{2}$

$= \frac{4e^2+6}{4} - \frac{2(2e+3)}{2}$
 $= 4e^2 - 2(2e+3) - (4e^2 - 4e + 6)$
 $= 8e^2 - 12e - 4e^2 + 4e - 6$
 $= 4e^2$

Question 11 (***)



The figure above shows the graph of the curve with equation

$$y = (x+2)e^{-x}, \quad x \in \mathbb{R}.$$

The finite region R , shown shaded in the figure, is bounded by the curve and the coordinate axes.

Use integration by parts to show that the area of R is $e^2 - 3$.

proof

$y = (x+2)e^{-x}$
 $y = 0$
 $x = -2$ BY INSPECTION
 (x_0)

$$\int_{-2}^0 (x+2)e^{-x} dx$$

$$= -(x+2)e^{-x} \Big|_{-2}^0 + \int e^{-x} dx$$

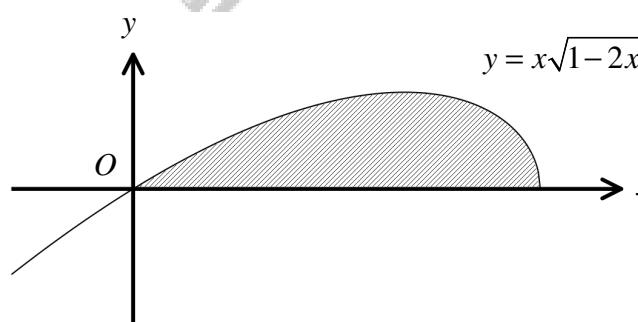
$$= -(x+2)e^{-x} - e^{-x} \Big|_{-2}^0 + C$$

$$\therefore A = \left[-(x+2)e^{-x} - e^{-x} \right]_{-2}^0 = \left[(x+2)e^{-x} + e^{-x} \right]_{-2}^0$$

$$= (0 + e^2) - (2e^{-2}) = e^2 - 3$$

→ see worked

Question 12 (***)



The figure above shows the graph of the curve with equation

$$y = x\sqrt{1-2x}, \quad x \leq \frac{1}{2}.$$

Use integration by substitution to find the area of the finite region bounded by the curve and the x axis.

15

when $y=0$
 $0=x\sqrt{1-2x}$
 $\sqrt{1-2x}=0$ $(x \neq 0)$
 $1-2x=0$
 $2x=1$
 $\therefore A(\frac{1}{2}, 0)$

$A = \int_0^{\frac{1}{2}} x\sqrt{1-2x} dx$

BY SUBSTITUTION

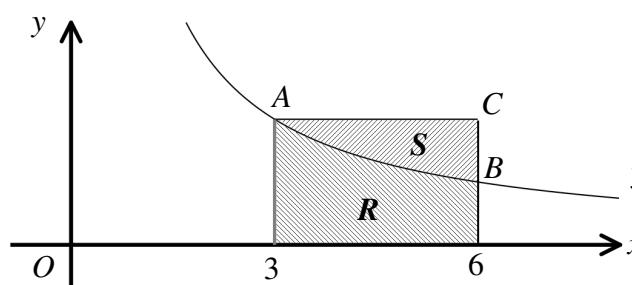
$A = \int_0^{\frac{1}{2}} x u^{\frac{1}{2}} (-\frac{du}{2}) = \int_0^{\frac{1}{2}} \frac{1}{2} u^{\frac{1}{2}} (-(\frac{u}{2})) du$

$A = \frac{1}{4} \left[u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}} \right]_0^{\frac{1}{2}}$

$A = \frac{1}{4} \left[(\frac{1}{8} - \frac{2}{5} \cdot \frac{1}{32}) - 0 \right] = \frac{1}{4} \times \frac{3}{15} = \frac{1}{15}$

$\begin{aligned} u &= 1-2x \\ \frac{du}{dx} &= -2 \\ du &= -\frac{du}{dx} \\ 2 &= 0 \quad a=1 \\ 2 &= \frac{1}{2} \quad a=0 \\ 2 &= -\frac{du}{2} \end{aligned}$

Question 13 (***)



The figure above shows the graph of the curve with equation

$$y = \frac{c}{x},$$

where c is a positive constant.

The points A and B lie on the curve here $x=3$ and $x=6$ respectively.

The finite region R is bounded by the curve, the x axis and the lines with equations $x=3$ and $x=6$.

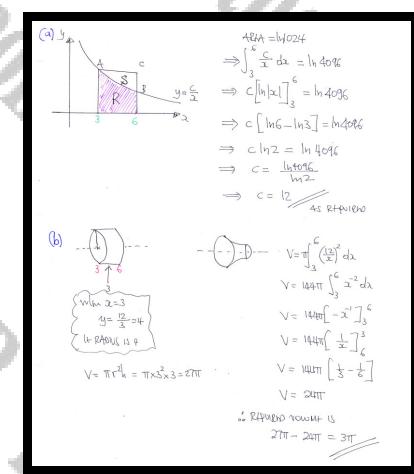
- a) Given that the area of R is $\ln 4096$, show that $c=12$.

The point C is such so that AC is parallel to the x axis and BC is parallel to the y axis. The finite region S is bounded by the curve and the line segments AC and BC .

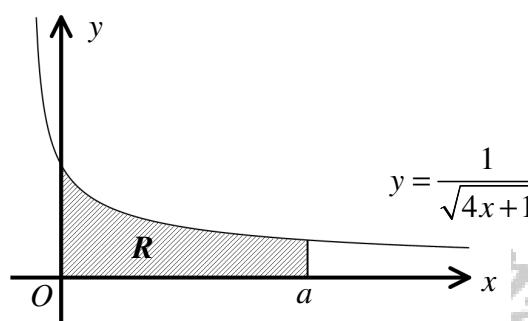
- b) Find the volume of the solid produced when S is complete revolved about the x axis.

[3π]

[solution overleaf]



Question 14 (***)



The figure above shows the graph of the curve with equation

$$y = \frac{1}{\sqrt{4x+1}}, \quad x > -\frac{1}{4}.$$

The finite region R is bounded by the curve, the coordinate axes and the line with equation $x=a$, where a is a positive constant.

When R is revolved by 2π radians in the x axis the resulting solid of revolution has a volume of $\pi \ln 3$, in appropriate cubic units.

Determine the area of R .

area = 4

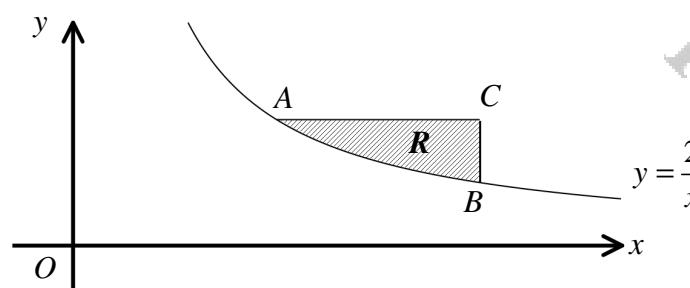
The diagram shows the shaded region R in the first quadrant. The derivation follows:

$$\begin{aligned} V &= \pi \int_{a_1}^{a_2} y^2 dx \\ &= \pi \int_a^a \frac{1}{4x+1} dx \\ &= \pi \left[\frac{1}{4} \ln(4x+1) \right]_a^a \\ &= \frac{1}{4} \ln(4(a+1)) - \frac{1}{4} \ln 4 \\ &= \frac{1}{4} \ln(4(a+1)) \\ &= \ln(4(a+1)) \\ &= \ln 3 \\ &\Rightarrow a+1 = 3 \\ &\Rightarrow a = 20 \end{aligned}$$

Volume calculation:

$$\begin{aligned} \text{Volume} &= \int_0^{\infty} \frac{1}{\sqrt{4x+1}} dx = \int_0^{\infty} (4x+1)^{-\frac{1}{2}} dx = \left[\frac{1}{2} (4x+1)^{\frac{1}{2}} \right]_0^{\infty} \\ &= \frac{1}{2} \left[\sqrt{4a+1} - \sqrt{4a+1} \right] = \frac{1}{2} \times 0 = 4 \end{aligned}$$

Question 15 (***)



The figure above shows the graph of the curve with equation

$$y = \frac{2}{x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

The points A and B lie on the curve. The respective x coordinates of these points are k and $3k$, where k is a positive constant. The point C is such so that AC is parallel to the x axis and BC is parallel to the y axis.

The finite region R is bounded by the curve and the line segments AC and BC .

Show that the area of R is $\ln\left(\frac{1}{9}e^4\right)$.

proof

when $x=k, y=\frac{2}{k}$
 \therefore AREA OF THE RECTANGLE
 $\text{is } \frac{2}{k} \times 2k = 4$

Area under curve ...

$$= \int_k^{3k} \frac{2}{x} dx = \left[2\ln|x| \right]_k^{3k}$$

$$= 2[\ln(3k) - \ln k] = 2\ln\left(\frac{3k}{k}\right)$$

$$= 2\ln 3$$

Difference of A is $4 - 2\ln 3$

$$= 4 - 2\ln 9$$

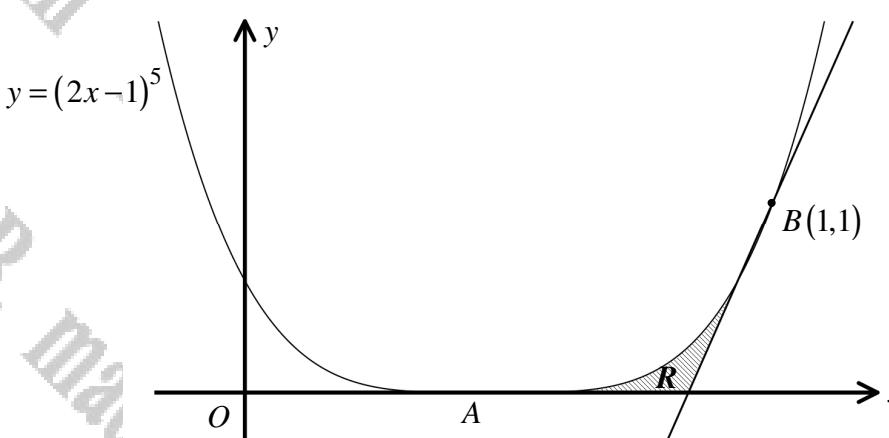
$$= 4\ln e - 2\ln 9$$

$$= \ln e^4 - \ln 81$$

$$= \ln\left(\frac{e^4}{81}\right)$$

\checkmark As required

Question 16 (***)+



The figure above shows the graph of the curve with equation

$$y = (2x - 1)^5, \quad x \in \mathbb{R}.$$

The curve meets the x axis at the point A and the point $B(1, 1)$ lies on the curve.

Show that the area of the finite region R bounded by the curve, the x axis and the tangent to the curve at B is $\frac{1}{30}$.

proof

• $y = (2x - 1)^5$
w.k. $y=0$, $0 = (2x - 1)^5$
 $2x - 1 = 0$
 $x = \frac{1}{2}$
 $A\left(\frac{1}{2}, 0\right)$

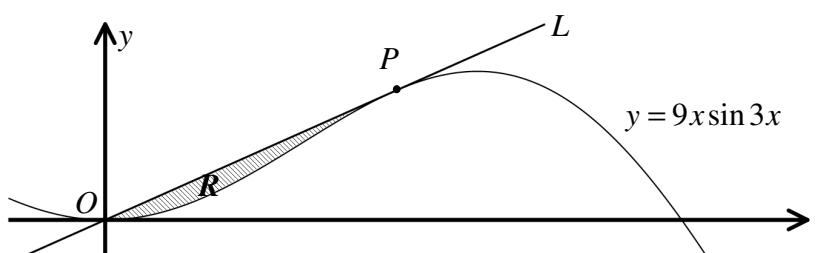
• $y = (2x - 1)^5$
 $\frac{dy}{dx} = 10(2x - 1)^4$
 $\frac{dy}{dx} \Big|_{x=1} = 10$
TANGENT AT $B(1, 1)$ HAS EQUATION
 $y - 1 = 10(x - 1)$
 $y = 10x - 9$
w.k. $y = 0$, $0 = 10x - 9$
 $x = \frac{9}{10}$ $\boxed{\text{tr}\left(\frac{9}{10}, 0\right)}$

$\int_{\frac{1}{2}}^1 (2x - 1)^5 dx$
 $= \left[\frac{1}{12}(2x - 1)^6 \right]_{\frac{1}{2}}^1$
 $= \frac{1}{12}(2 \cdot 1 - 1)^6 - \frac{1}{12}(2 \cdot \frac{1}{2} - 1)^6$
 $= \frac{1}{12} - \frac{1}{12} = 0$

$\int_{\frac{1}{2}}^1 (2x - 1)^5 dx = \int_{\frac{1}{2}}^1 (10x - 9) dx$
 $= \left[\frac{1}{2}(10x - 9)^2 \right]_{\frac{1}{2}}^1$
 $= \frac{1}{2}(10 \cdot 1 - 9)^2 - \frac{1}{2}(10 \cdot \frac{1}{2} - 9)^2$
 $= \frac{1}{2} - \frac{1}{2} = 0$

$\therefore \frac{1}{12} - 0 = \frac{1}{12}$ At Ellipsis

Question 17 (****)



The figure above shows the graph of the curve C with equation

$$y = 9x \sin 3x, \quad x \in \mathbb{R}.$$

The straight line L is the tangent to C at the point P , whose x coordinate is $\frac{\pi}{6}$.

- a) Show that L passes through the origin O .

The finite region R bounded by C and L .

- b) Show further that the area of R is

$$\frac{1}{8}(\pi^2 - 8).$$

[] , proof

a) PROVE IT

$$y = (9x)(\sin 3x)$$

$$\frac{dy}{dx} = 9 \sin 3x + 9x(3 \cos 3x)$$

$$\frac{dy}{dx} = 9(\sin 3x + 3x \cos 3x)$$

When $x = \frac{\pi}{6}$

$$y = (\frac{\pi}{6})(\sin \frac{\pi}{2})$$

$$y = \frac{\pi}{6}$$

$$y(\frac{\pi}{6}) = 9$$

Now we have

$$y - y_0 = m(x - x_0)$$

$$y - \frac{\pi}{6} = 9(x - \frac{\pi}{6})$$

$$y - \frac{\pi}{6} = 9x - \frac{9\pi}{6}$$

$$y = 9x$$

b) WORK AT THE PICTORIAL EQUATION

INTEGRATION BY PARTS

$$\int 9x \sin 3x \, dx = -3 \cos 3x - \int -3 \cos 3x \cdot x \, dx$$

$$= -3 \cos 3x + \int 3x \cos 3x \, dx$$

$$= -3x \cos 3x + 3 \sin 3x + C$$

INSERTING LIMITS OF INTEGRATION

$$\int_0^{\pi/6} 9x \sin 3x \, dx = [\frac{3x \sin 3x - 3 \cos 3x}{3}]_0^{\pi/6}$$

$$= (1 - 3 \cos \frac{\pi}{6}) - (0 - 0)$$

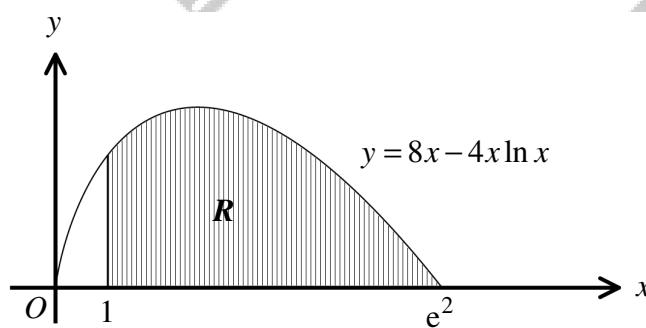
$$= 1$$

∴ REQUIRED AREA = $\frac{\pi^2}{8} - 1$

$$= \frac{1}{8}(\pi^2 - 8)$$

As Required

Question 18 (***)



The figure above shows the graph of the curve with equation

$$y = 8x - 4x \ln x, \quad 0 < x \leq e^2.$$

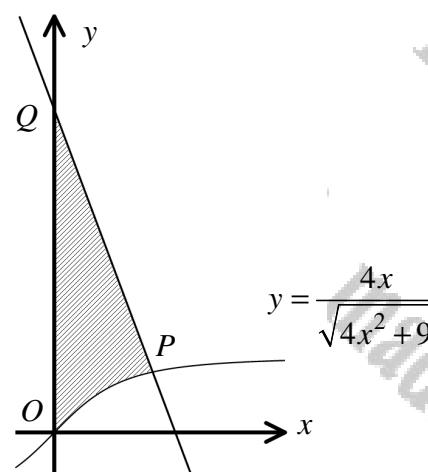
The region R is bounded by the curve, the x axis and the line with equation $x=1$.

Determine the exact area of R .

 $e^4 - 5$

$$\begin{aligned} \text{Area} &= \int_1^{e^2} 8x - 4x \ln x \, dx = \int_1^{e^2} 8x \, dx - \int_1^{e^2} 4x \ln x \, dx \\ &\quad \text{INTEGRATION BY PARTS} \\ \int 4x \ln x \, dx &= 2x^2 \ln x - \int 2x \, dx \\ &= 2x^2 \ln x - x^2 + C \quad \begin{array}{|c|c|} \hline \text{Int.} & \frac{1}{2} \\ \hline \text{Diff.} & 4x^2 \\ \hline \end{array} \\ \therefore \text{Area} &= \left[4x^2 \right]_1^{e^2} - \left[2x^2 \ln x - x^2 \right]_1^{e^2} \\ &= (4e^4 - 4) - [(2e^4 \ln e^2 - e^4) - (0 - 1)] \\ &= 4e^4 - 4 - [4e^4 - e^4 + 1] \\ &= e^4 - 5 \end{aligned}$$

Question 19 (****)



The figure above shows the graph of the curve with equation

$$y = \frac{4x}{\sqrt{4x^2 + 9}}, \quad x \in \mathbb{R}.$$

The point P lies on the curve where $x = 2$. The normal to the curve at P meets the y axis at Q .

- a) Show that the y coordinate of Q is $\frac{769}{90}$.

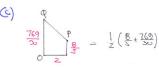
- b) Find the value of

$$\int_0^2 \frac{4x}{\sqrt{4x^2 + 9}} dx.$$

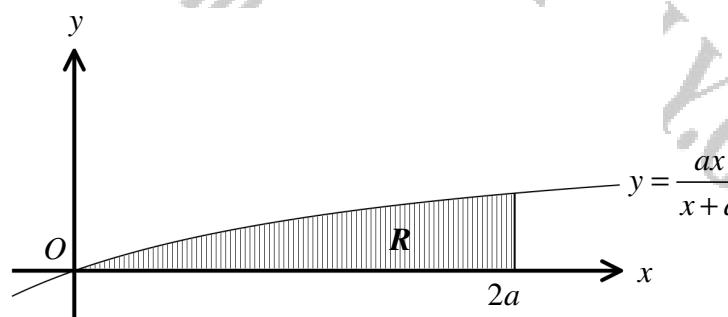
- c) Hence find the area of the finite region bounded by the curve, the y axis and the normal to the curve at P .

<input type="text"/>	, [2],	$\boxed{\frac{733}{90} \approx 8.14}$
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[solution overleaf]

$\text{Q) } y = \frac{4x}{(4x^2+9)^{\frac{1}{2}}}$ $\frac{dy}{dx} = \frac{(4x^2+9)^{\frac{1}{2}} \cdot 4 - 4x \cdot \frac{1}{2} \cdot 2x(4x^2+9)^{-\frac{1}{2}}}{(4x^2+9)^1}$ $\frac{dy}{dx} _{x=2} = \frac{56 + 48 \times \frac{1}{2}}{25} = \frac{56}{25}$ <p>Normal gradient at $x=2$ is $-\frac{25}{56}$</p> <p>When $x=2$, $y = \frac{4x^2}{5} = \frac{16}{5}$ i.e. $(2, \frac{16}{5})$</p> $y - \frac{16}{5} = -\frac{25}{56}(x-2)$ $y = \frac{125}{56} - \frac{25}{56}x = \frac{76}{56}$ <p>when $x=0$, $y = \frac{125}{56} - \frac{25}{56} \times 0 = \frac{76}{56}$</p>	$\text{Q) } \int_0^2 \frac{4x}{\sqrt{4x^2+9}} dx = \dots \text{by inspection or } \text{Q) } \int_0^2 4x(4x^2+9)^{\frac{1}{2}}$ $\text{Q) } \int_0^2 (4x^2+9)^{\frac{1}{2}} dx = [4x^2+9]^{\frac{1}{2}} _0^2 = 5-3=2$  $\text{Q) } \text{Area} = \frac{1}{2} (8 \times 5) \times 2 = \frac{80}{20}$ $\text{Q) } \text{Required Area} = \frac{80}{20} - 2 = \frac{72}{20}$
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Question 20 (****)



The figure above shows the graph of the curve C with equation

$$y = \frac{ax}{x+a}, \quad x \neq -a, \text{ where } a \text{ is a positive constant.}$$

The curve passes through the origin O .

The finite region R is bounded by C , the x axis and the line with equation $x = 2a$.

By using integration by substitution, or otherwise, show that the area of R is

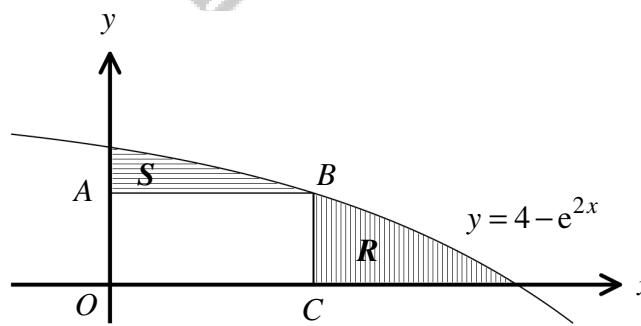
$$a^2(2 - \ln 3).$$

proof

$$\begin{aligned}
 A(R) &= \int_0^{2a} \frac{ax}{x+a} dx = \dots \text{ by substitution} \\
 &= \int_a^{3a} \frac{a(u-a)}{u} du = \int_a^{3a} \frac{au - a^2}{u} du \\
 &= \int_a^{3a} \left(u - \frac{a^2}{u} \right) du = \left[au - a^2 \ln|u| \right]_a^{3a} \\
 &= \left[3a^2 - a^2 \ln(3a) \right] - \left[a^2 - a^2 \ln a \right] \\
 &= 2a^2 + a^2 \ln a - a^2 \ln(3a) \\
 &= a^2(2 + \ln a - \ln(3a)) \\
 &= a^2 \left[2 + \ln \left(\frac{a}{3a} \right) \right] = a^2 \left[2 + \ln \left(\frac{1}{3} \right) \right] = a^2 [2 - \ln 3].
 \end{aligned}$$

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Question 21 (****)



The figure above shows the graph of the curve with equation

$$y = 4 - e^{2x}, \quad x \in \mathbb{R}.$$

The point B lies on the curve where $y = 2$. The point A lies on the y axis and the point C lies on the x axis so that $OABC$ is a rectangle.

The region R is bounded by the curve, the x axis and the line BC .

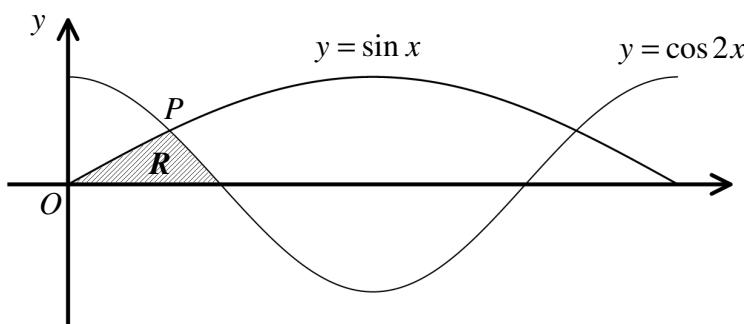
The region S is bounded by the curve, the y axis and the line AB .

Show, by exact calculations, that the area of R is twice as large as the area of S .

proof

<p>AREA OF REGION S</p> $\text{Area of } S = \int_0^{\ln 2} (2 - (4 - e^{2x})) dx = \int_0^{\ln 2} (e^{2x} - 2) dx = \left[\frac{1}{2}e^{2x} - 2x \right]_0^{\ln 2} = \left(\frac{1}{2}e^{2\ln 2} - 2\ln 2 \right) - (0 - 0) = \frac{1}{2} + 4\ln 2$	$\bullet \text{S} + \text{T} + \text{R} = \int_0^{\ln 2} (4 - e^{2x}) dx = \left[4x - \frac{1}{2}e^{2x} \right]_0^{\ln 2} = (4\ln 2 - 2) - (0 - \frac{1}{2}) = -\frac{1}{2} + 4\ln 2$ $\bullet R = \int_{\ln 2}^{\ln 2} 4 - e^{2x} dx = \left[4x - \frac{1}{2}e^{2x} \right]_{\ln 2}^{\ln 2} = (4\ln 2 - 2) - (2\ln 2 - 1) = -1 + 2\ln 2,$ $\bullet \text{P} = (\text{S} + \text{T} + \text{R}) - \text{T} - \text{R} = \left(-\frac{1}{2} + 4\ln 2 \right) - \ln 2 - (-1 + 2\ln 2) = -\frac{1}{2} + \ln 2$ Hence $\text{P} = -\frac{1}{2} + \ln 2$ $\text{P} = -1 + 2\ln 2$ <p>AREA OF R IS TWICE AS LARGE AS S</p>
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Question 22 (*****)



The figure above shows the graphs of the curves with equations

$$y = \cos 2x \text{ and } y = \sin x, \text{ for } 0 \leq x \leq \pi.$$

The point P is the first intersection between the graphs for which $x > 0$.

- a) Show clearly that the x coordinate of P is $\frac{\pi}{6}$.

The finite region R , shown shaded in the figure, is bounded by the two curves and the x axis, and includes the point P on its boundary.

- b) Determine the exact area of R .

, $\boxed{\frac{3}{4}(2-\sqrt{3})}$

a) SOLVING SIMULTANEOUSLY THE EQUATIONS

$$\begin{cases} y = \cos 2x \\ y = \sin x \end{cases} \Rightarrow \begin{cases} \cos 2x = \sin x \\ 1 - 2\sin^2 x = \sin x \\ 0 = 2\sin^2 x + \sin x - 1 \\ 0 = (\sin x + 1)(2\sin x - 1) \end{cases}$$

SIN x < -1

$\sin x = -1$

$\sin x = \frac{1}{2}$

\therefore THE FIRST POSITIVE SOLUTION IS $x = \frac{\pi}{6}$

b) VOLUME OF THE REGION R

$\int_0^{\pi/6} \sin x \, dx + \int_{\pi/6}^{\pi/2} \cos 2x \, dx$

$$= \left[-\cos x \right]_0^{\pi/6} + \left[\frac{1}{2} \sin 2x \right]_{\pi/6}^{\pi/2}$$

$$= \left[-\cos x \right]_0^{\pi/6} + \left[\frac{1}{2} \sin 2x \right]_{\pi/6}^{\pi/2} = -\frac{\sqrt{3}}{2} + 1 + \frac{1}{2} - \frac{\sqrt{3}}{4} = \frac{3}{4}(2-\sqrt{3})$$

Question 23 (****)

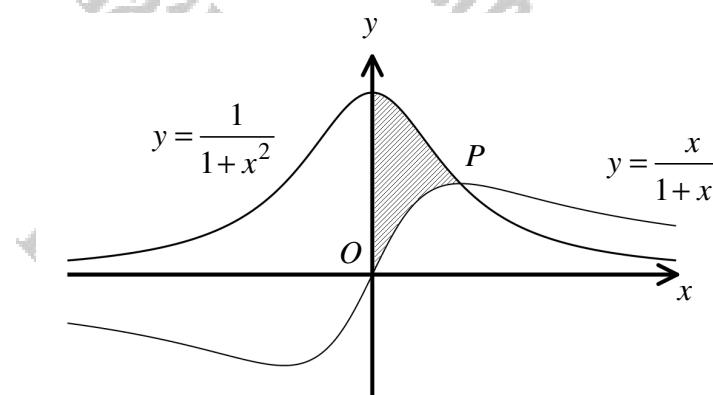
$$y = \arctan x, \quad x \in \mathbb{R}.$$

- a) By rewriting the above equation in the form $x = f(y)$, show clearly that

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

The figure below shows the graphs of the curves with equations

$$y = \frac{1}{1+x^2} \quad \text{and} \quad y = \frac{x}{1+x^2}.$$



The two graphs intersect at the point $P\left(1, \frac{1}{2}\right)$.

- b) Find the exact area of the finite region bounded by the two curves and the y axis, shown shaded in the figure.

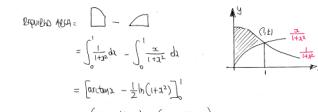
, $\frac{1}{4}(\pi - \ln 4)$

[solution overleaf]

a) USING THE SUGGESTED FORM

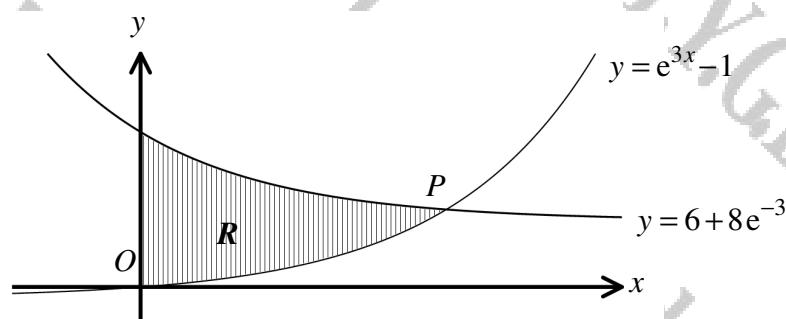
$$\begin{aligned}\Rightarrow y &= \alpha \ln x \\ \Rightarrow \ln x &= \alpha \\ \Rightarrow x &= e^{\alpha x} \\ \Rightarrow \frac{dy}{dx} &= \alpha e^{\alpha x} \\ \Rightarrow \frac{dy}{dx} &= 1 + x y \quad \text{as } x = e^{\alpha x} \\ \Rightarrow \frac{dy}{dx} &= 1 + x^2 \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{1+x^2} \quad \text{if } y \neq 0\end{aligned}$$

b) LOOKING AT THE DIAGRAM

SHADING AREA: 

$$\begin{aligned}\text{Shaded Area: } A &= \int_0^1 \frac{1}{1+x^2} dx - \int_0^1 \frac{x}{1+x^2} dx \\ &= \left[\arctan x - \frac{1}{2} \ln(1+x^2) \right]_0^1 \\ &= \left(\frac{\pi}{4} - \frac{1}{2} \ln 2 \right) - \left(0 - \frac{1}{2} \ln 1 \right) \\ &= \frac{1}{2} \left(\pi - \ln 2 \right)\end{aligned}$$

Question 24 (****)



The figure above shows the graphs of the curves with equations

$$y = e^{3x} - 1 \quad \text{and} \quad y = 6 + 8e^{-3x}.$$

The curves intersect at the point P .

- a) Find the exact coordinates of the point P .

The finite region R is bounded by the two curves and the y axis.

- b) Determine the exact area of R .

$$[(\ln 2, 7)], \text{ area} = 7 \ln 2$$

$\text{(a)} \quad \begin{aligned} y &= e^{3x} - 1 \\ y &= 6 + 8e^{-3x} \end{aligned}$ <p>Solve simultaneously:</p> $\begin{aligned} e^{3x} - 1 &= 6 + 8e^{-3x} \\ e^{3x} - 8e^{-3x} - 7 &= 0 \\ e^{3x} - 8e^{3x} - 7e^{3x} &= 0 \\ (e^{3x})^2 - 8(e^{3x}) - 7 &= 0 \\ (e^{3x})^2 - 7(e^{3x}) - 8 &= 0 \\ (e^{3x} - 8)(e^{3x} + 1) &= 0 \end{aligned}$ $\begin{aligned} e^{3x} - 8 &= 0 \\ e^{3x} &= 8 \\ 3x &= \ln 8 \\ x &= \frac{\ln 8}{3} \end{aligned}$	(b) <p>Region Area:</p> $\begin{aligned} &\int_0^{\ln 2} (6 + 8e^{-3x}) - (e^{3x} - 1) \, dx \\ &= \int_0^{\ln 2} 7 + 8e^{-3x} - e^{3x} \, dx \\ &= \int_0^{\ln 2} 7 + 8e^{-3x} - e^{3x} \, dx \\ &= \left[7x + \frac{8}{3}e^{-3x} - \frac{1}{3}e^{3x} \right]_0^{\ln 2} \\ &= \left[7(\ln 2) + \frac{8}{3}e^{-3\ln 2} - \frac{1}{3}e^{3\ln 2} \right] - \left[7(0) + \frac{8}{3}e^{0} - \frac{1}{3}e^{0} \right] \\ &= 7\ln 2 - \frac{1}{3} - \frac{8}{3} + \frac{1}{3} \\ &= 7\ln 2 \end{aligned}$
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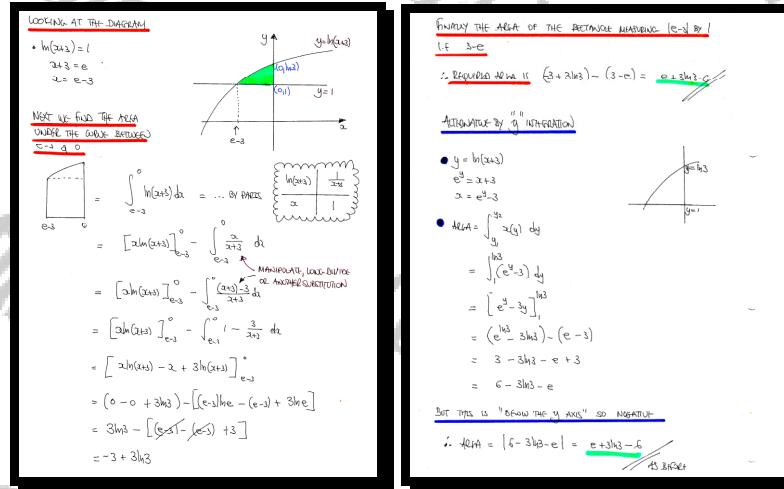
Question 25 (*)+**

The finite region R is bounded by the curve with equation $y = \ln(x+3)$, the y axis and the straight line with equation $y = 1$.

Show, with detailed workings, that the area of R is

$$e - 6 + 3\ln 3.$$

, proof



Question 26 (***)+

The function f is defined in the largest real domain by

$$f(x) \equiv (\ln x)^2, \quad x \in (0, \infty).$$

- a) Sketch the graph of $f(x)$.

The function g is defined as

$$g(x) \equiv \ln x, \quad x \in (0, \infty).$$

- b) Determine in exact simplified form the area of the finite region bounded by the graph of f and the graph of g .

You may assume that $\int \ln x \, dx = x \ln x - x + \text{constant}$.

, 3-e

a) SKETCHING THE GRAPH FROM THE GRAPH OF $y = \ln x$.

b) DETERMINING THE REQUIRED FINITE AREA IN THE Cartesian PLANE.

AREA: $\int_1^e (\ln x - (\ln x)^2) \, dx = \dots$ (WORKING ON PAPER)

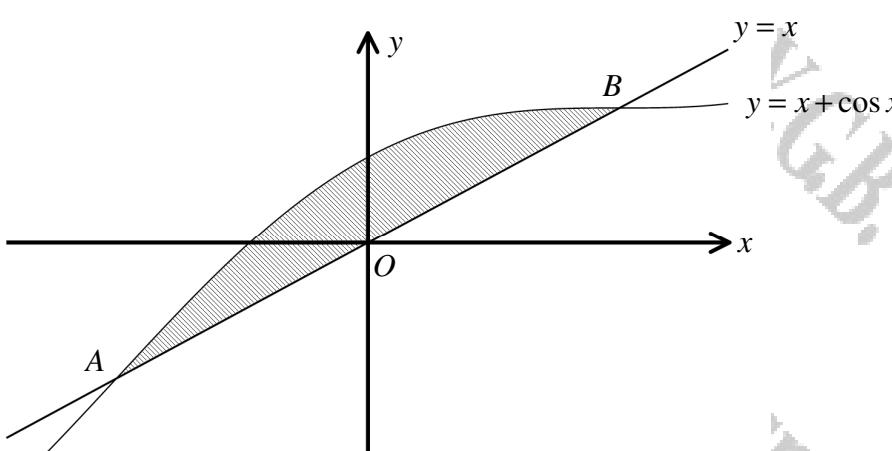
$\ln x - (\ln x)^2$	$\frac{1}{x} - \frac{2}{x} \ln x$
$\frac{1}{2}$	1

$$= \left[2\ln x - 2(\ln x)^2 \right]_1^e - \int_1^e 1 - 2\ln x \, dx$$

USING THE PARTIAL FRACTION RULE: $\int \frac{1}{x} \, dx = \ln x + C$

$$\begin{aligned} &= \left[2\ln x - 2(\ln x)^2 - x + 2(2\ln x - x) \right]_1^e \\ &= \left[2\ln x - 2(\ln x)^2 - x + 2\ln x - 2x \right]_1^e \\ &= \left[2\ln x - 2(\ln x)^2 - 3x \right]_1^e \\ &= (2e - e - 2e) - (0 - 0 - 3) \\ &= 3 - e \end{aligned}$$

Question 27 (*****)



The figure above shows the graph of the curve C with equation

$$y = x + \cos x, \quad x \in \mathbb{R}$$

and the straight line L with equation

$$y = x, \quad x \in \mathbb{R}.$$

Show that the area of the finite region bounded by C and L is 2 square units.

, proof

FIND THE x -COORDINATES OF THE POINTS OF INTERSECTION A & B

$$\begin{aligned} y &= x \\ y &= x + \cos x \end{aligned} \Rightarrow x = x + \cos x$$

$$\Rightarrow \cos x = 0$$

$$\Rightarrow x = -\frac{\pi}{2} \quad (\text{BY INSPECTION})$$

HENCE THE REQUIRED AREA IS GIVEN BY THE INTEGRAL

$$\int_{-\frac{\pi}{2}}^0 (x + \cos x) - x \, dx = \int_{-\frac{\pi}{2}}^0 \cos x \, dx$$

$$= 2 \left[\sin x \right]_{-\frac{\pi}{2}}^0$$

$$= [2\sin \frac{\pi}{2} - 2\sin(-\frac{\pi}{2})]$$

$$= 2(1 - (-1)) = 4$$

AN ALTERNATIVE METHOD

AND ESTIMATING THE COORDINATES OF A & B

THE POINTS SEEN ON THE GRAPHS ARE APPROXIMATELY AS FOLLOWS

$$A(-\frac{\pi}{2}, -\frac{\pi}{2}) \quad B(\frac{\pi}{2}, \frac{\pi}{2})$$

$$y = x + \cos x + \frac{\pi}{2} \quad y = x + \frac{\pi}{2}$$

THE AREA UNDER THE CURVE $x + \cos x + \frac{\pi}{2}$ IS

$$\int_{-\frac{\pi}{2}}^0 (x + \cos x + \frac{\pi}{2}) \, dx = \int_0^{\frac{\pi}{2}} 2\sin x + \frac{\pi}{2} \, dx$$

$$[2\sin x + \frac{\pi}{2}]_{-\frac{\pi}{2}}^0 = (2 + \frac{1}{2}\pi^2) - (0) = 2 + \frac{\pi^2}{2}$$

AREA OF THE TRAPEZOID

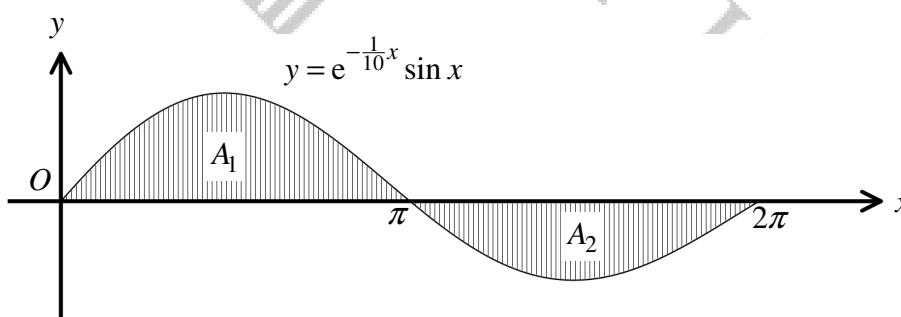
$$\frac{1}{2} \times \pi \times \pi = \frac{1}{2}\pi^2$$

HENCE THE REQUIRED AREA IS

$$2 + \frac{\pi^2}{2} - \frac{1}{2}\pi^2 = 2$$

AS DESIRED

Question 28 (*****)



The figure above shows the curve with equation

$$y = e^{-\frac{1}{10}x} \sin x, \quad 0 \leq x \leq 2\pi.$$

The curve meets the x axis at the origin and at the points $(\pi, 0)$ and $(2\pi, 0)$.

The finite region bounded by the curve for $0 \leq x \leq \pi$ and the x axis is denoted by A_1 , and similarly A_2 denotes the finite region bounded by the curve for $\pi \leq x \leq 2\pi$ and the x axis.

By considering a suitable translation of the curve determine with justification the ratio of the areas of A_1 and A_2 .

	$\frac{A_1}{A_2} = e^{-\frac{\pi}{10}}$	

Starting from A_2 :

$$A_2 = \int_{\pi}^{2\pi} e^{-\frac{1}{10}x} \sin x \, dx$$

TRANSLATE "UP" BY π BY THE SUBSTITUTION

$$\begin{aligned} A_2 &= \int_{\pi}^{2\pi} e^{-\frac{1}{10}(x-\pi)} \sin(x-\pi) \, dx \\ A_2 &= \int_{0}^{\pi} e^{-\frac{1}{10}x} e^{-\frac{1}{10}\pi} \sin(x+\pi) \, dx \end{aligned}$$

$\begin{aligned} dx &= X + \pi \\ \frac{dx}{dx} &= 1 \\ dx &= dX \\ x=0 &\mapsto X=\pi \\ x=\pi &\mapsto X=0 \end{aligned}$

USING THE COMPOUND ANGLE FORMULA

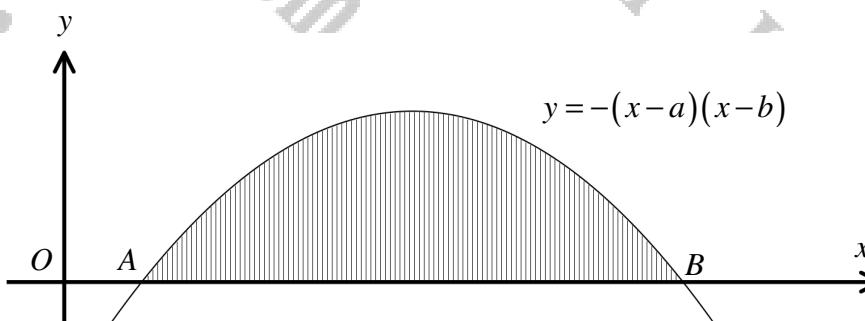
$$\sin(X+\pi) = \sin X \cos \pi + \cos X \sin \pi = -\sin X$$

$$\begin{aligned} A_2 &= \int_{0}^{\pi} e^{-\frac{1}{10}x} e^{-\frac{1}{10}\pi} (-\sin X) \, dx \\ A_2 &= -e^{-\frac{1}{10}\pi} \int_{0}^{\pi} e^{-\frac{1}{10}x} \sin X \, dx \end{aligned}$$

IGNORING THE AREA AS THIS MERELY INDICATES THAT THE AREA IS BELOW THE X -AXIS AND REDEFINING $X \mapsto -x$

$$\frac{A_2}{A_1} = \left| \frac{-e^{-\frac{1}{10}\pi} \int_{0}^{\pi} e^{-\frac{1}{10}x} \sin X \, dx}{\int_{0}^{\pi} e^{-\frac{1}{10}x} \sin X \, dx} \right| = e^{\frac{1}{10}\pi}$$

Question 29 (*****)



The figure above shows the parabola with equation

$$y = -(x-a)(x-b), \quad b > a > 0.$$

The curve meets the x -axis at the points A and B .

- a) Show that the area of the finite region R , bounded by the parabola and the x -axis is

$$\frac{1}{6}(b-a)^3.$$

The midpoint of AB is N . The point M is the maximum point of the parabola.

- b) Show clearly that the area of R is given by

$$k|AB||MN|,$$

where k is a constant to be found.

, $k = \frac{2}{3}$

<p>a) <small>AREA INTEGRATION BY PARTS</small></p> $\begin{aligned} \text{Area } R &= \int_a^b -(x-a)(x-b) \, dx \\ &= \text{Integrating w.r.t. } x \dots \\ &= -\frac{1}{2}(x-a)(x-b)^2 - \int_a^b \frac{1}{2}(x-b)^2 \, dx \\ &= -\frac{1}{2}(x-a)(x-b)^2 + \int_a^b \frac{1}{2}(x-b)^2 \, dx \\ &= -\frac{1}{2}(x-a)(x-b)^2 + \frac{1}{6}(x-b)^3 \\ &= \dots \text{Evaluating limits} \\ &= \left[-\frac{1}{2}(x-a)(x-b)^2 + \frac{1}{6}(x-b)^3 \right]_a^b \\ &= (b-a) - (a+\frac{1}{6}(b-a)^2) \\ &= -\frac{1}{6}(a-b)^3 \\ &= \frac{1}{6}(b-a)^3 \end{aligned}$	<p>b)</p> <ul style="list-style-type: none"> By symmetry $N(\frac{a+b}{2}, 0)$ Using $y = -(x-a)(x-b)$ $\begin{aligned} y_N &= \left[\frac{a+b}{2} \right] \left[\frac{a+b}{2} - a \right] \\ y_M &= -\frac{1}{2} \left[\frac{a+b}{2} \right] \left(\frac{a+b}{2} - b \right) \\ y_M &= \frac{1}{2} (a-b)^2 \end{aligned}$ <p>Now $AB = k MN$</p> $\begin{aligned} AB &= k(b-a) \\ &= k(b-a) \times \frac{1}{2}(b-a)^2 \\ &= \frac{1}{2}k(b-a)^3 \end{aligned}$ <p>Comparing $\frac{1}{6}(b-a)^3$ and $\frac{1}{2}k(b-a)^3$</p> $k = \frac{2}{3}$
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Question 30 (*****)

Determine, in exact simplified form, the area of the finite region bounded by the curves with equations

$$y = 1 + \sqrt{x}, \quad x \in \mathbb{R}, \quad x \geq 0.$$

$$y = 4^{\frac{x}{9}}, \quad x \in \mathbb{R}.$$

$\boxed{27\left(1 - \frac{1}{\ln 4}\right)}$

$y = 1 + \sqrt{x}, \quad x \geq 0$

$y = 4^{\frac{x}{9}}, \quad x \in \mathbb{R}$

- NEED THE INTERSECTIONS OF THE TWO GRAPHS
- BY INSPECTION (Q1) IS ONE SUCH POINT AND FROM THE SKETCH IT IS evident THAT THE TWO GRAPHS INTERSECT ONCE MORE — BY TRYING A FEW INTEG. VALUES, WE SEE THAT THE REQUIRED POINT IS $(9, 4)$.
- HENCE THE REQUIRED AREA IS GIVEN BY

$$\begin{aligned} A &= \int_0^9 (1 + \sqrt{x}) - 4^{\frac{x}{9}} \, dx = \int_0^9 (1 + x^{\frac{1}{2}} - 4^{\frac{x}{9}}) \, dx \\ &= \left[x + \frac{2}{3}x^{\frac{3}{2}} - 4^{\frac{9}{10}} \times \frac{1}{\ln 4} \times \frac{1}{9}x^{\frac{9}{10}} \right]_0^9 \\ &= \left[x + \frac{2}{3}x^{\frac{3}{2}} - \frac{9}{\ln 4}x^{\frac{9}{10}} \right]_0^9 \\ &= \left(9 + \frac{2}{3} \times 9^{\frac{3}{2}} - \frac{9}{\ln 4} \times 9^{\frac{9}{10}} \right) - (0 + 0 - \frac{9}{\ln 4} \times 0) \\ &= 9 + 18 - \frac{36}{\ln 4} + \frac{9}{\ln 4} \\ &= 27 - \frac{27}{\ln 4} \\ &= 27 \left(1 - \frac{1}{\ln 4} \right) // \end{aligned}$$

Question 31 (*****)

The straight line L with equation $y = kx$, where k is a positive constant, meets the curve C with equation $y = xe^{-2x}$, at the point P .

The tangent to C at P meets the x axis at the point Q .

Given that $|OP| = |PQ|$, find in exact simplified form the area of the finite region bounded by C and L .

$$\boxed{\quad}, \boxed{\text{area} = \frac{1}{2}(1 - 5e^{-2})}$$

Straight with a brief sketch of the cubic

$y = xe^{-2x}$ $x=0, y=0$
 $y > 0$ if $x > 0$
 $\text{At } x \rightarrow \infty, y \rightarrow 0$

FIND THE COORDINATES OF P

$$\begin{aligned} y = xe^{-2x} \\ y = kx \end{aligned} \Rightarrow \begin{aligned} xe^{-2x} &= kx \\ e^{-2x} &= k \quad (x \neq 0) \\ -2x &= \ln k \\ x &= -\frac{1}{2}\ln k \end{aligned}$$

$\therefore P(-\frac{1}{2}\ln k, -\frac{1}{2}k\ln k)$

FIND THE GRADIENT OF THE TANGENT AT P

$$\begin{aligned} \frac{dy}{dx} &= e^{-2x} - 2xe^{-2x} = e^{-2x}(1 - 2x) \\ \left. \frac{dy}{dx} \right|_{x=-\frac{1}{2}\ln k} &= k[1 - 2(-\frac{1}{2}\ln k)] = k[1 + \ln k] \end{aligned}$$

AS THE TANGENT & THE LINE $y = kx$ HAVE THE SAME INCLINATION TO THE x AXIS, THEIR GRADIENTS HAVE THE SAME MAGNITUDE AND OPPOSITE SIGNS, THEREFORE

$$k[1 + \ln k] = -k$$

$$1 + \ln k = -1 \quad (\text{Cancelling } k)$$

$\therefore \ln k = -2$
 INVERSE
 $\Rightarrow k = e^{-2}$

FINALLY THE REQUIRED AREA CAN BE FOUND BY INTEGRATION

IF $k = e^{-2}$ $f(x) = e^{-2x}$

$$\begin{aligned} \text{AREA} &= \int_0^1 xe^{-2x} - e^{-2x} dx \\ &= \left[-\frac{1}{2}xe^{-2x} \right]_0^1 + \left[\frac{1}{2}e^{-2x} \right]_0^1 - \left[\frac{1}{2}e^{-2x} \right]_0^1 \\ &= \left[-\frac{1}{2}xe^{-2x} - \frac{1}{2}e^{-2x} - \frac{1}{2}e^{-2x} \right]_0^1 \\ &= \frac{1}{4} \left[2xe^{-2x} + e^{-2x} + 2e^{-2x} \right]_0^1 \\ &= \frac{1}{4} \left[(0 + 1 + 0) - (2e^2 + e^{-2} + 2e^{-2}) \right] \\ &= \frac{1}{4} [1 - 5e^{-2}] \end{aligned}$$

BY PARTS

x	1
$-e^{-2x}$	e^{-2}

Question 32 (***)

Two curves are defined in the largest possible real number domain and have equations

$$y^2 = \frac{4(4-x)}{x} \quad \text{and} \quad x^2 = \frac{4(4-y)}{y}$$

- a) Show that the two curves have one, and only one, common point which is also a point of common tangency.
 - b) Find the exact value of the area enclosed by the common tangent to the curves, and either of the two curves.

$$\boxed{}, \boxed{2(\pi - 3)}$$

a)

$$\begin{aligned} y^2 &= \frac{4(4-x)}{x} \quad \text{or} \quad x^2 = \frac{4(4-y)}{3} \\ \text{SOLVING SIMULTANEOUSLY AT POINTS} \\ \begin{cases} 3y^2 = 4(4-x) \\ 2x^2 = 4(4-y) \end{cases} &\Rightarrow \text{DIVIDE } \frac{y^2}{x^2} = \frac{4-x}{4-y} \end{aligned}$$

b) TRYING UP WT CHARTIN

$$\begin{aligned} 4y - y^2 &= 4x - x^2 \\ 2^2 - 4x &= y^2 - 4y \\ x - 2 + 4 &= y^2 - 4y + 4 \\ (x-2)^2 &= (y-2)^2 \\ x-2 &= \begin{cases} y-2 \\ 2-y \end{cases} \\ y &= \begin{cases} x \\ 4-x \end{cases} \end{aligned}$$

c) THE CUBES MEET ALONG THE STRAIGHT LINES (CUBES) WITH EQUATIONS

$$\begin{aligned} y &= x \quad \text{or} \quad y = 4-x \quad \text{OR INSPECTION AT } (2,2) \\ \text{CUT OF THESE HAS TO BE THE COMMON TANGENT} \\ \text{at } (2,2) &\Rightarrow x^2 = \frac{4(4-x)}{x} \\ &\Rightarrow x^2 = 16 - 4x \\ &\Rightarrow x^2 + 4x - 16 = 0 \quad \text{OR INSPECTION AT } x=2 \\ &\Rightarrow (x+4)(x-4) = 0 \quad \text{OR A SOLUTION} \end{aligned}$$

d) ONLY ONE COMMON POINT WHICH IS A POINT OF COMMON TANGENCY BETWEEN THE TWO CUBES

e) IT MAY BE HELPFUL TO SKETCH THE TWO CUBES AND THEIR COMMON TANGENT ALTHOUGH IT IS NOT NECESSARY

CROSSING WE SEE 2 OF THE POSSIBLE CONGRUENCES EXISTING $y = x$

$$y = \frac{4(4-x)}{x}$$

f) WE FINALLY COMPUTE THE AREA UNDER THE CURVE BETWEEN $x=2$ & $x=4$.

g) IF THE AREA IS GREATER THAN 2 (CRM OF REQUIRE) WE SUBTRACT 2 (GREEN CUBE)

h) IF THE AREA IS LESS THAN 2, WE SUBTRACT IT FROM 2, THIS GIVES US

i) PLS LET MEED TO FIND THE VALUE OF

$$\begin{aligned} \int_2^4 \sqrt{\frac{4(4-x)}{x}} dx &\quad \text{OR} \quad \text{BY SUBSTITUTION} \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{4(4-4x^2)}{x^2}} dx \quad (\text{BASICALY } dx = -4x dx) \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{4(1-\frac{1}{x^2})}{x^2}} dx \quad (\text{BASICALY } dx) \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{2}{\sqrt{x^2}} \sqrt{1-\frac{1}{x^2}} dx \quad (\text{BASICALY } dx) \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{2}{x} \sqrt{1-\frac{1}{x^2}} dx \quad (\text{BASICALY } dx) \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{2}{x} \frac{\sqrt{x^2-1}}{x} dx \quad (\text{BASICALY } dx) \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{2}{x^2} \sqrt{x^2-1} dx \quad (\text{BASICALY } dx) \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{16}{x^2} \cos^2 \theta dx = \int_{\frac{1}{2}}^{\frac{1}{2}} 16 \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta\right) d\theta \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}} 8 + 8 \cos 2\theta d\theta = \left[8\theta + 4 \sin 2\theta \right]_{\frac{1}{2}}^{\frac{1}{2}} \end{aligned}$$

j) $(x-2)(x^2+2x+8) = 0$

ONLY INSPECTION WITH $y=2$ IS $(2,2)$
BUT $y=2$ IS NOT A TANGENT

$$\begin{aligned} y = 4-x &\Rightarrow (4-x)^2 = \frac{4(4-x)}{x} \\ &\Rightarrow x(4-x)^2 = 4(4-x) \\ &\Rightarrow 2(x-2)^2 = 4(4-x) = 0 \\ &\Rightarrow (x-2)[2(4-x)-4] = 0 \\ &\Rightarrow (x-2)(2x^2-4x-8) = 0 \\ &\Rightarrow (x-4)(x+2)^2 = 0 \\ &\Rightarrow (x-4) \quad (x+2)^2 \end{aligned}$$

k) $\frac{1}{2}(4-x) \leftarrow \text{REVERSE}$

l) ONLY ONE COMMON POINT WHICH IS A POINT OF COMMON TANGENCY BETWEEN THE TWO CUBES

m) IT MAY BE HELPFUL TO SKETCH THE TWO CUBES AND THEIR COMMON TANGENT ALTHOUGH IT IS NOT NECESSARY

CROSSING WE SEE 2 OF THE POSSIBLE CONGRUENCES EXISTING $y = x$

$$y = \frac{4(4-x)}{x}$$

n) WE FINALLY COMPUTE THE AREA UNDER THE CURVE BETWEEN $x=2$ & $x=4$

$$(4x+4) - (2x+4) = 2x-4 > 2 \quad (\text{OR } 2x>2x)$$

o) REQUIRED AREA = $(2x-4) - \frac{1}{2} \times 2 \times 2 = 2x-6$

p) NOTE: WE MAY SKETCH AS FOLLOWS

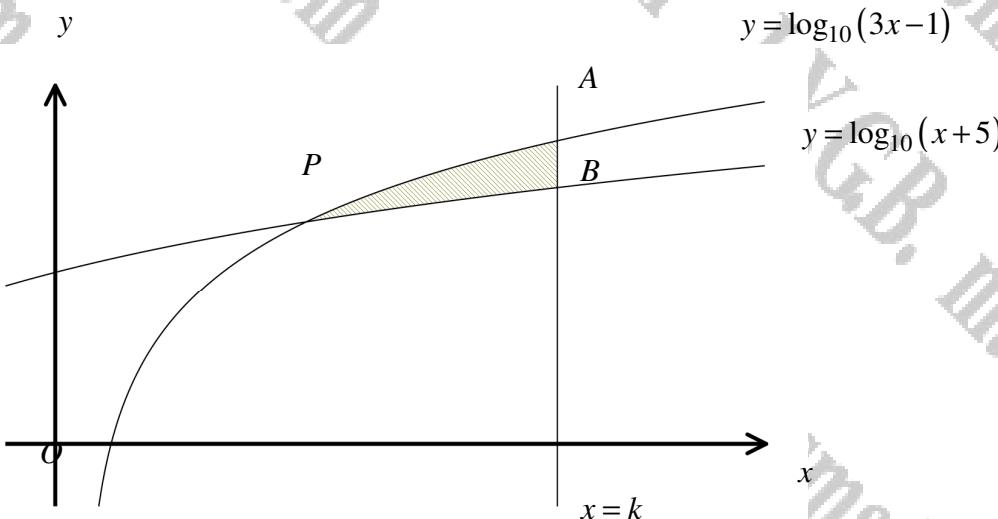
$$\begin{aligned} y &= \frac{4(4-x)}{x} = \frac{4}{x} \left(\frac{4}{x}-1 \right) \\ &= \frac{16}{x^2} - 4 \end{aligned}$$

$$\begin{aligned} y^2 &= \frac{16}{x^2} - 4 \quad \rightarrow \text{PART OF CURVE BELOW } x=4 \text{ AND ABOVE } x=2 \\ &= \frac{16}{x^2} \quad \rightarrow \text{PART OF CURVE AREA } x=2 \text{ AND PARTIAL AREA BELOW } x=2 \\ &= 2 \times 4 \quad \rightarrow \text{2 TIMES OF PARTIAL AREA} \\ &= 8 \end{aligned}$$

$x^2 = \frac{4(4-x)}{y}$ IS A LINEAR FUNCTION IN THE LINE $y=x$ ($x & y$ ARE ALIKE)

HENCE WE OBTAIN

Question 33 (*****)



The figure above shows the graphs of the curves with equations

$$y = \log_{10}(3x-1), \quad \text{and} \quad y = \log_{10}(x+5).$$

The two curves intersect at the point P

The straight line with equation $x = k$, $k > 3$, meets the graph of $y = \log_{10}(3x-1)$ at the point A and the graph of $y = \log_{10}(x+5)$ at the point B , so that $|AB| = \frac{1}{2}$.

Determine the value of the area of the finite region, bounded by the two curves and the straight line $x = k$, shown shaded in the above figure.

 , area = $\frac{8}{3}$

<p>• FIRST FIND THE x COORDINATE OF P BY EQUATION $\Rightarrow \log_{10}(3x-1) = \log_{10}(x+5)$ $\Rightarrow 3x-1 = 10^{x+5}$ $\Rightarrow \frac{3x-1}{10^{x+5}} = 1$ $\Rightarrow \frac{3x-1}{10^x \cdot 10^5} = 1$ $\Rightarrow \frac{3x-1}{10^x} = 10^5$ $\Rightarrow 3x-1 = 10^6$ $\Rightarrow 3x = 10^6 + 1$ $\Rightarrow x = \frac{10^6 + 1}{3}$ (y is not needed)</p> <p>• NEXT WE FIND x_2 $\Rightarrow \log_{10}(3x-1) - \log_{10}(x+5) = \frac{1}{2}$ $\Rightarrow \log_{10}\left(\frac{3x-1}{x+5}\right) = \frac{1}{2}$ $\Rightarrow \frac{3x-1}{x+5} = 10^{\frac{1}{2}}$ $\Rightarrow \frac{3x-1}{x+5} = \sqrt{10}$ $\Rightarrow 3x-1 = \sqrt{10}x + \sqrt{10}$ $\Rightarrow 3x - \sqrt{10}x = \sqrt{10} + 1$ $\Rightarrow x = \frac{\sqrt{10} + 1}{3 - \sqrt{10}}$</p>	$\Rightarrow x = \frac{10^6 + 1}{3}$ (y is not needed) <p>• NEXT WE FIND THE REQUIRED AREA BY INTEGRATION</p> $\Rightarrow \text{AREA} = \int_3^k \log_{10}(3x-1) \, dx - \int_3^k \log_{10}(x+5) \, dx$ $\Rightarrow \text{AREA} = \int_3^k \frac{1}{\ln 10} \frac{1}{3} (3x-1) \, dx - \int_3^k \frac{1}{\ln 10} (x+5) \, dx$ $\Rightarrow \text{AREA} = \frac{1}{3\ln 10} \int_3^k (3x-1) \, dx - \frac{1}{\ln 10} \int_3^k (x+5) \, dx$ $\Rightarrow \text{AREA} = \frac{1}{3\ln 10} \left[3x^2/2 - x \right]_3^k - \frac{1}{\ln 10} \left[x^2/2 + 5x \right]_3^k$ $\Rightarrow \text{AREA} = \frac{1}{3\ln 10} \left[3(k^2/2 - 3^2/2) - (3k - 10) \right] - \frac{1}{\ln 10} \left[(k^2/2 + 5k) - (9/2 + 15) \right]$ $\Rightarrow \text{AREA} = \frac{1}{6\ln 10} (3k^2 - 3k^2 - 18k + 24) - \frac{1}{2\ln 10} (k^2 + 10k - 34)$ $\Rightarrow \text{AREA} = \frac{1}{6\ln 10} [18k^2 - 24k^2 - 24] - \frac{1}{2\ln 10} [6k^2 + 24k^2 - 34]$ $\Rightarrow \text{AREA} = \frac{1}{6\ln 10} [-6k^2 - 24] - \frac{1}{2\ln 10} [6k^2 - 8]$ $\Rightarrow \text{AREA} = \frac{1}{6\ln 10} [6k^2 - 12] - \frac{1}{2\ln 10} [24k^2 - 8]$ $\Rightarrow \text{AREA} = \frac{1}{3\ln 10} [3k^2 - 6k^2 + 4]$ $\Rightarrow \text{AREA} = \frac{8k^2}{3\ln 10}$ $\Rightarrow \text{AREA} = \frac{8}{3}$
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Question 34 (*****)

The curve C has equation

$$y + 2 = [\ln(4x+1)]^2, \quad x \in \mathbb{R}, \quad x \geq -\frac{1}{4}.$$

Sketch the graph of C and hence determine, in exact simplified form, the area of the finite region bounded by C , for which $x \geq 0$, and the coordinate axes.

 , area = $\frac{1}{2}(\sqrt{2} - 1)e^{\sqrt{2}}$

Start by sketching the curve starting with $[\ln(4x+1)]^2$.

Hence we have the graph of $y+2 = [\ln(4x+1)]^2$.

Now to find the required area by integration in x .

$$\Rightarrow \text{Area} = \int_{-\frac{1}{4}}^0 [\ln(4x+1)]^2 - 2 \, dx$$

$$\Rightarrow \text{Area} = \int_{-\frac{1}{4}}^0 [\ln(4x+1)]^2 \, dx - 2 \, dx$$

... by substitution, before attempting integration by parts...

$$u = 4x+1 \quad \frac{du}{dx} = 4 \quad u = 4x+1 \rightarrow u = e^{\sqrt{2}}$$

$$\Rightarrow \text{Area} = \int_{e^{\sqrt{2}-1}}^1 [\ln(u)]^2 \left(\frac{1}{4} du \right) - \int_{e^{\sqrt{2}-1}}^0 2 \, dx$$

$$\Rightarrow \text{Area} = \frac{1}{2}(e^{\sqrt{2}}) + \frac{1}{4} \int_{e^{\sqrt{2}-1}}^1 [\ln u]^2 du$$

... by parts...

$(\ln u)^2$	$\frac{2}{u} \ln u$
u	1

$$\Rightarrow \text{Area} = \frac{1}{2}(e^{\sqrt{2}}) + \frac{1}{4} \left\{ [\ln u]^2 \Big|_{e^{\sqrt{2}-1}}^1 - \int_{e^{\sqrt{2}-1}}^1 2 \ln u \, du \right\}$$

$$\Rightarrow \text{Area} = \frac{1}{2}(e^{\sqrt{2}}) + \frac{1}{4} \left\{ 0 - e^{\sqrt{2}} - \left[[\ln u] - u \right] \Big|_{e^{\sqrt{2}-1}}^1 \right\}$$

By parts or noting that $\int \ln u \, du = u \ln u - u + C$

$$\Rightarrow \text{Area} = \frac{1}{2}[e^{\sqrt{2}}] - \frac{1}{2}e^{\sqrt{2}} - \frac{1}{2}[(0-1) - (e^{\sqrt{2}} - e^{\sqrt{2}})]$$

$$\Rightarrow \text{Area} = \frac{1}{2}e^{\sqrt{2}} - \frac{1}{2} - \frac{1}{2}\int e^{\sqrt{2}} - e^{\sqrt{2}} \, du$$

$$\Rightarrow \text{Area} = \frac{1}{2}\sqrt{2}e^{\sqrt{2}} - \frac{1}{2}e^{\sqrt{2}}$$

$$\Rightarrow \text{Area} = \frac{1}{2}e^{\sqrt{2}}(\sqrt{2} - 1)$$

Alternative integration parallel to the y -axis

Start by rearranging the equation.

$$y+2 = [\ln(4x+1)]^2$$

$$[\ln(4x+1)]^2 = \begin{cases} < 0 & \rightarrow y < -2 \\ > 0 & \rightarrow y > -2 \end{cases}$$

$$4x+1 = \begin{cases} e^{\sqrt{y+2}} & \rightarrow 4x > 0 \\ e^{-\sqrt{y+2}} & \rightarrow 4x < 0 \end{cases}$$

$$x = \begin{cases} \frac{1}{4}e^{\sqrt{y+2}} - \frac{1}{4} & \rightarrow x > 0 \\ \frac{1}{4}e^{-\sqrt{y+2}} - \frac{1}{4} & \rightarrow x < 0 \end{cases}$$

Thus integrating w.r.t y , from $y=-2$ to $y=0$

$$\Rightarrow \text{Area} = \int_{-2}^0 \frac{1}{4}e^{\sqrt{y+2}} - \frac{1}{4} dy$$

... substitute $u = \sqrt{y+2}$

$u = \sqrt{y+2}$	$u^2 = y+2$
$u^2 - 2 = y$	$u^2 = u^2 - 2$
$dy = 2u \, du$	$dy = 2u \, du$
$y = u^2 - 2$	$y = u^2 - 2$

$$\Rightarrow \text{Area} = \int_{-2}^0 \frac{1}{4}e^u (2u \, du)$$

$$\Rightarrow \text{Area} = 0 - \frac{1}{2} + \frac{1}{4} \int_{-2}^0 2u e^u \, du$$

Integration by parts

$$\Rightarrow \text{Area} = -\frac{1}{2} + \frac{1}{4} \left\{ \left[2u e^u \right]_{-2}^0 - \int_{-2}^0 2e^u \, du \right\}$$

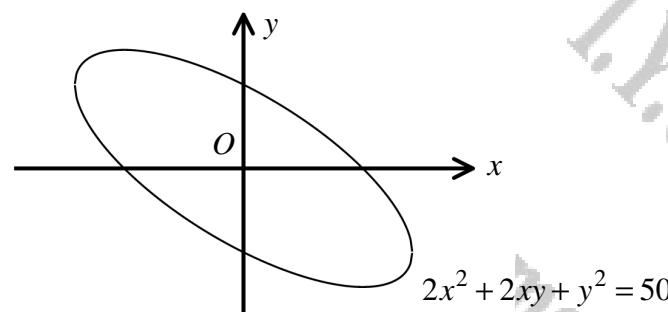
$$\Rightarrow \text{Area} = -\frac{1}{2} + \frac{1}{4} [2(e^0 - e^{-2}) - (0 - 2)]$$

$$\Rightarrow \text{Area} = -\frac{1}{2} + \frac{1}{4} [2(e^0 - e^{-2}) + 2]$$

$$\Rightarrow \text{Area} = -\frac{1}{2} + \frac{1}{2}e^0 - \frac{1}{2}e^{-2} + \frac{1}{2}$$

$$\Rightarrow \text{Area} = \frac{1}{2}e^0(\sqrt{2} - 1)$$

Question 35 (*****)



The figure above shows the curve with equation

$$2x^2 + 2xy + y^2 = 50.$$

Determine the area of the finite region bounded by the x axis and the part of the curve for which $y \geq 0$.

 , 25π

FIRSTLY PRODUCE A SKETCH WITH x & y INTERCEPTS

NEXT FIND THE CO-ORDINATES OF THE POINT P (CIRCLE/TANGENT)

$$2x^2 + 2xy + y^2 = 50$$

$$\frac{\partial}{\partial x}(2x^2 + 2xy + y^2) = 0$$

$$4x + 2y + 2x\frac{dy}{dx} + 2y\frac{dx}{dx} = 0$$

$$2(2y+2x) = -2(2x+y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-4x-2y}{2x+y} \leftarrow \text{INFINITE COORDINATE} \Rightarrow \text{SYMMETRICAL TANGENT}$$

SOLVING SIMULTANEOUSLY WITH THE EQUATION OF THE CIRCLE WE OBTAIN

$$y+2=0 \quad 2x^2 + 2(-2) + (-2)^2 = 50$$

$$y=-2 \quad 2^2 = 50$$

$$x = \pm 5\sqrt{2}$$

$$\therefore P(5\sqrt{2}, -2)$$

NEXT REARRANGE THE EQUATION OF THE CIRCLE IN THE FORM $y = f(x)$

$$\Rightarrow y^2 + 2xy + x^2 = 50$$

$$\Rightarrow (y+x)^2 - x^2 + x^2 = 50$$

$$\Rightarrow (y+x)^2 = 50 - x^2$$

$$\Rightarrow y+x = \pm\sqrt{50-x^2}$$

THIS WE NOW HAVE THE FIGURE OPPOSITE - IN BOTH CASES WE HAVE TO INTEGRATE $\sqrt{50-x^2}$, SO PREPARE THIS FIRST

$$y = -x \pm \sqrt{50-x^2}$$

$$\int_{-5\sqrt{2}}^{5\sqrt{2}} -x \pm \sqrt{50-x^2} dx = \dots \text{... BY SUBSTITUTION}$$

$$x = \sqrt{50}\sin\theta \quad \frac{dx}{d\theta} = \sqrt{50}\cos\theta$$

$$= \int_{B_1}^{B_2} \sqrt{50 - 50\sin^2\theta} (\sqrt{50}\cos\theta d\theta)$$

$$= \int_{B_1}^{B_2} \sqrt{50(1-\sin^2\theta)} \sqrt{50}\cos\theta d\theta$$

$$= \int_{B_1}^{B_2} 50\cos^2\theta d\theta = \int_{B_1}^{B_2} 25 + 25\cos2\theta d\theta$$

$$= \left[25\theta + \frac{25}{2}\sin2\theta \right]_{B_1}^{B_2}$$

HENCE THE REQUIRED AREA CAN BE FOUND

$$\Delta = \int_{-5\sqrt{2}}^{5\sqrt{2}} -x + \sqrt{50-x^2} dx - \int_{-5\sqrt{2}}^{5\sqrt{2}} -x - \sqrt{50-x^2} dx$$

CHANGING THE LIMITS IN THE SUBSTITUTION INTEGRALS

- * $x = 5\sqrt{2}\sin\theta$
- $x=5 \rightarrow \theta = \frac{\pi}{4}$
- $x=-5 \rightarrow \theta = -\frac{\pi}{4}$
- $x=-5\sqrt{2} \rightarrow \theta = -\frac{\pi}{2}$

$$= \left[\frac{1}{2}x^2 \right]_{-5\sqrt{2}}^{5\sqrt{2}} + \left[\frac{1}{2}x^2 \right]_{-5\sqrt{2}}^{-5\sqrt{2}} + \left[25\theta + \frac{25}{2}\sin2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} + \left[25\theta + \frac{25}{2}\sin2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}}$$

$$= \left[\frac{25}{2}\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} + \left[\frac{25}{2}\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} + \left[\left(\frac{25}{4} + \frac{25}{4} \right) - \left(-\frac{25}{2} + 0 \right) \right]$$

$$+ \left[\left(\frac{25}{2} - \frac{25}{2} \right) - \left(\frac{25}{2} + 0 \right) \right]$$

$$= 25\pi$$