

SYSTEMATIC CURVE SKETCHING

Question 1 ()**

The curve C has equation

$$y = \frac{a}{x}, \quad x \neq 0,$$

where a is a positive constant.

- a) Describe geometrically the transformation that maps the graph of $y = \frac{a}{x}$ onto

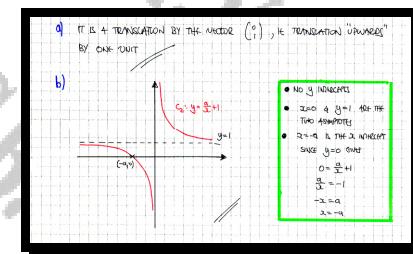
$$\text{the graph of } y = \frac{a}{x} + 1.$$

- b) Sketch the graph of C .

The sketch must include the coordinates of ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of any asymptotes of the curve.

, , translation "upwards" by 1 unit



Question 2 ()**

A curve C has equation

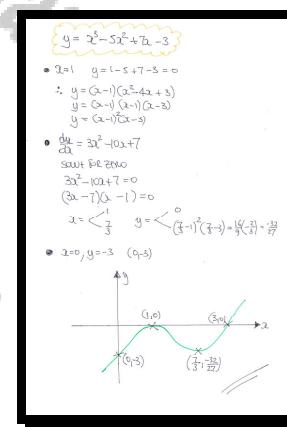
$$y = x^3 - 5x^2 + 7x - 3, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.

graph



Question 3 ()**

A curve C has equation

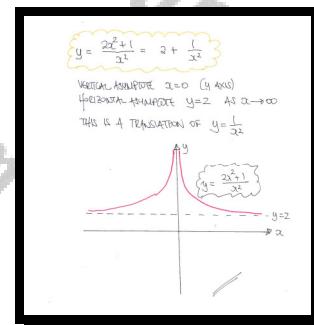
$$y = \frac{2x^2 + 1}{x^2}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

[graph]



Question 4 (*)**

A curve C has equation

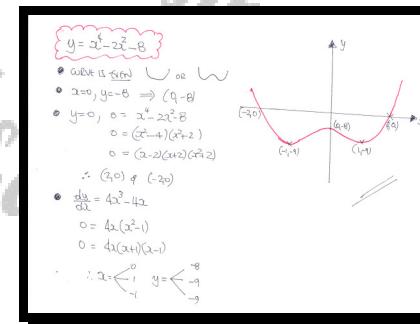
$$y = x^4 - 2x^2 - 8, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.

graph



Question 5 (*)**

A curve C has equation

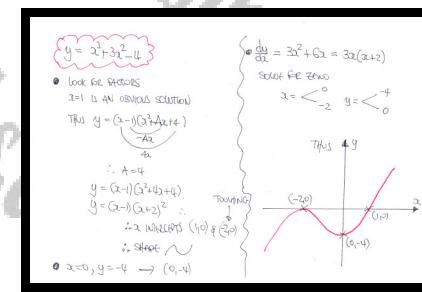
$$y = x^3 + 3x^2 - 4, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.

graph



Question 6 (*)**

A curve C has equation

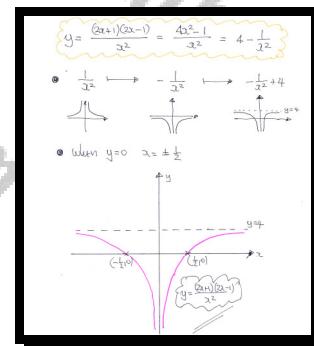
$$y = \frac{(2x+1)(2x-1)}{x^2}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

[graph]



Question 7 (*)**

A curve C has equation

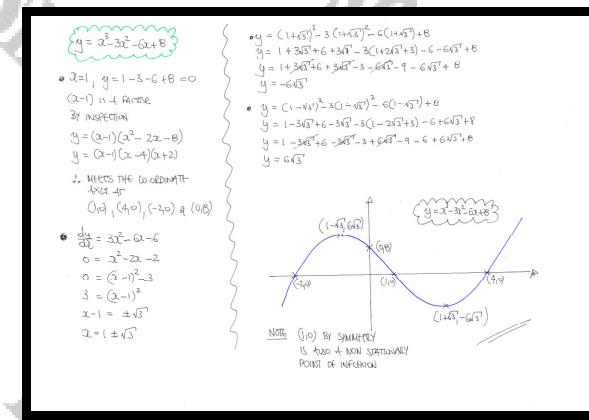
$$y = x^3 - 3x^2 - 6x + 8, \quad x \in \mathbb{R}$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate

- the coordinates of any points where the graph of C meets the coordinate axes.
 - the coordinates of any stationary points.
 - the coordinates of any non stationary turning points.

graph



Question 8 (***)

$$f(x) = \frac{4x-13}{x-3}, \quad x \in \mathbb{R}, \quad x \neq 3.$$

- a) Show that the equation of $f(x)$ can be written as

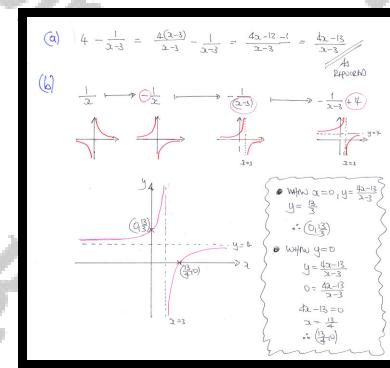
$$f(x) = 4 - \frac{1}{x-3}, \quad x \in \mathbb{R}, \quad x \neq 3.$$

- b) Sketch the graph of $f(x)$.

The sketch must include ...

- ... the coordinates of the points where $f(x)$ meets the coordinate axes.
- ... the equations of any asymptotes of the curve.

graph



Question 9 (*)+**

The curve C has equation

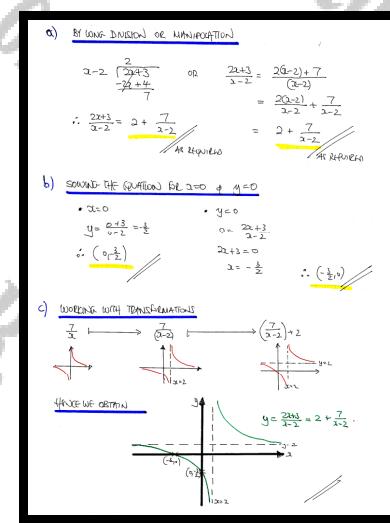
$$y = \frac{2x+3}{x-2}, \quad x \in \mathbb{R}, \quad x \neq 2.$$

- a) Show clearly that

$$\frac{2x+3}{x-2} \equiv 2 + \frac{7}{x-2}.$$

- b) Find the coordinates of the points where C meets the coordinate axes.
 c) Sketch the graph of C showing clearly the equations of any asymptotes.

, $\left(0, -\frac{3}{2}\right), \left(-\frac{3}{2}, 0\right)$



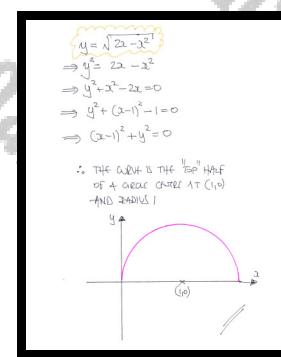
Question 10 (***)

A curve C has equation

$$y = \sqrt{2x - x^2}, \quad x \in \mathbb{R}, \quad 0 \leq x \leq 2.$$

Sketch the graph of C .

graph



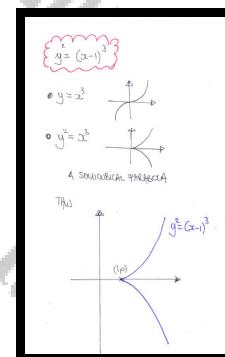
Question 11 (***)

A curve C has equation

$$y^2 = (x-1)^3, \quad x \in \mathbb{R}, \quad x \geq 1.$$

Sketch the graph of C .

graph



Question 12 (***)

$$f(x) = a - \frac{1}{b-x}, \quad x \in \mathbb{R}, \quad x \neq b,$$

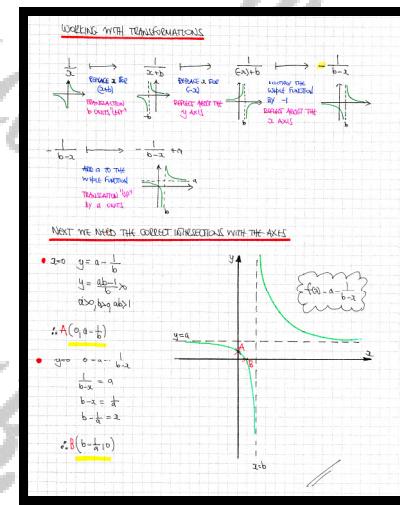
where a and b are positive constants such that $ab > 1$.

Sketch the graph of $f(x)$.

The sketch must include, in terms of a and b , ...

- ... the coordinates of the points where $f(x)$ meets the coordinate axes.
- ... the equations of any asymptotes of the curve.

[] , graph



Question 13 (*)+**

A curve C has equation

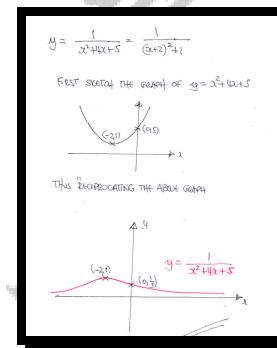
$$y = \frac{1}{x^2 + 4x + 5}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 14 (***)+

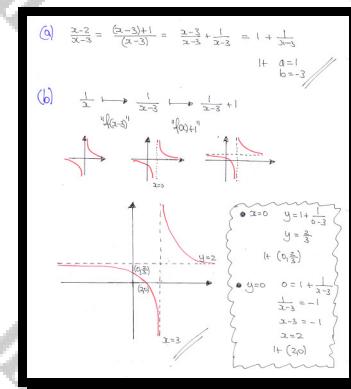
$$f(x) = \frac{x-2}{x-3}, \quad x \in \mathbb{R}, \quad x \neq 3.$$

- a) Express $f(x)$ in the form $f(x) = a + \frac{1}{x+b}$, where a and b are integers.
- b) By considering a series of transformations which map the graph of $\frac{1}{x}$ onto the graph of $f(x)$, sketch the graph of $f(x)$.

The sketch must include ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of the two asymptotes of the curve.

$$\boxed{a = 1}, \quad \boxed{b = -3}$$



Question 15 (***)

A curve C has equation

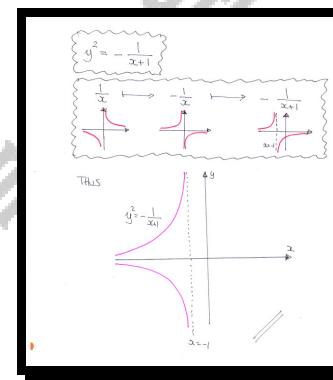
$$y^2 = -\frac{1}{x+1}, \quad x \in \mathbb{R}, \quad x \neq -1.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 16 (***)+

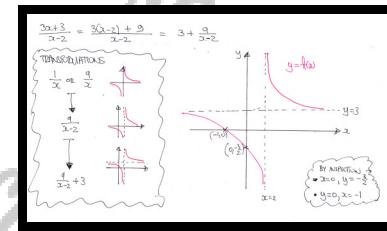
$$f(x) = \frac{3x+3}{x-2}, \quad x \in \mathbb{R}, \quad x \neq 2.$$

Sketch the graph of $f(x)$.

The sketch must include ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of the two asymptotes of the curve.

graph



Question 17 (*)+**

A curve has equation $y = f(x)$ given by

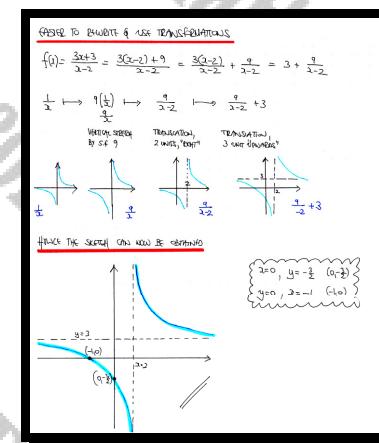
$$f(x) = \frac{3x-1}{x+2}, \quad x \in \mathbb{R}, \quad x \neq -2.$$

Sketch the graph of $f(x)$.

The sketch must include ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of the two asymptotes of the curve.

, graph



Question 18 (***)

A curve C has equation

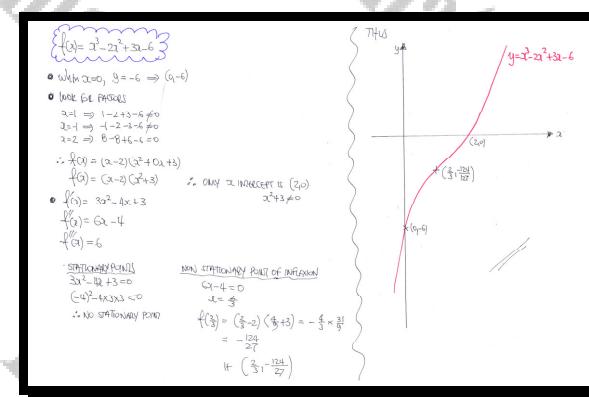
$$y = x^3 - 2x^2 + 3x - 6, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.

graph



Question 19 (***)

A curve C has equation

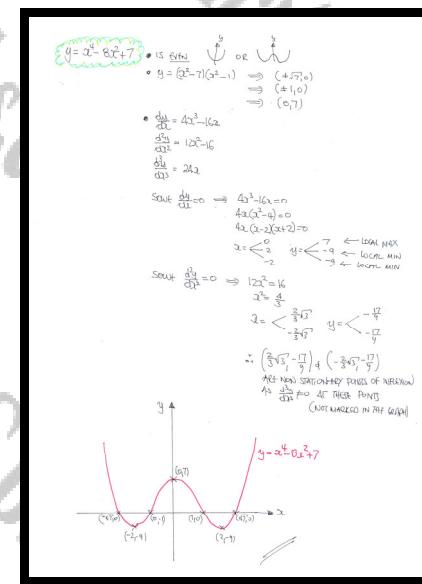
$$y = x^4 - 8x^2 + 7, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.

graph



Question 20 (***)

A curve C has equation

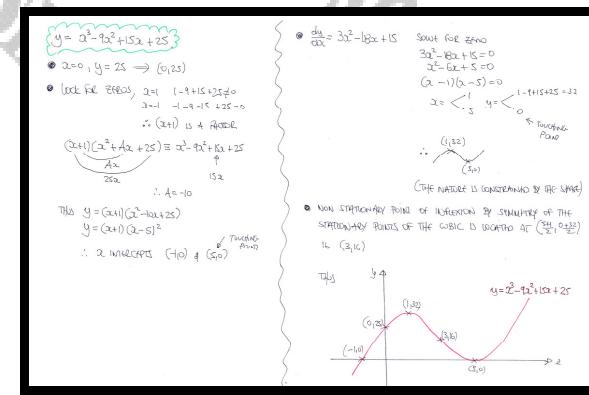
$$y = x^3 - 9x^2 + 15x + 25, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.

graph



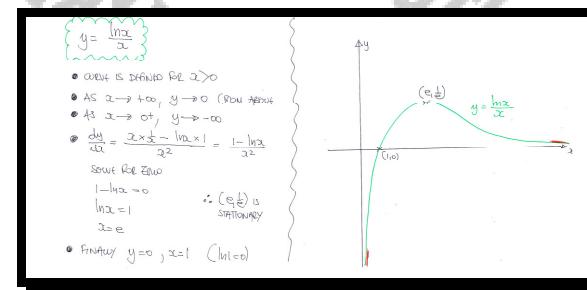
Question 21 (***)+A curve C has equation

$$y = \frac{\ln x}{x}.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.



Question 22 (***)A curve C has equation

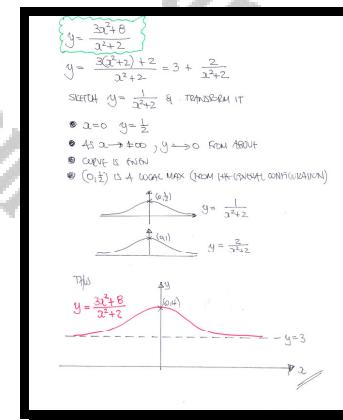
$$y = \frac{3x^2 + 8}{x^2 + 2}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 23 (*)+**

A curve C has equation

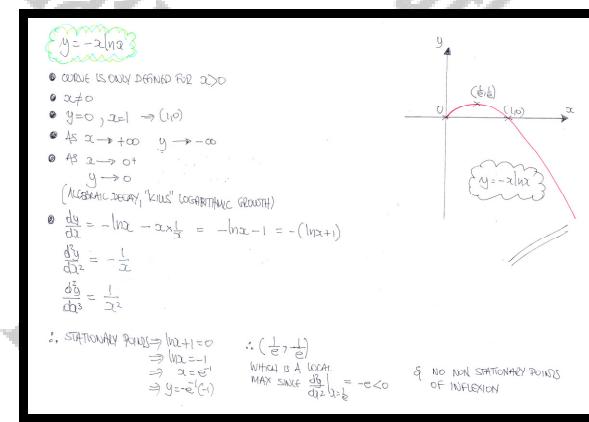
$$y = -x \ln x, \quad x \in \mathbb{R}.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



Question 24 (***)

A curve C has equation

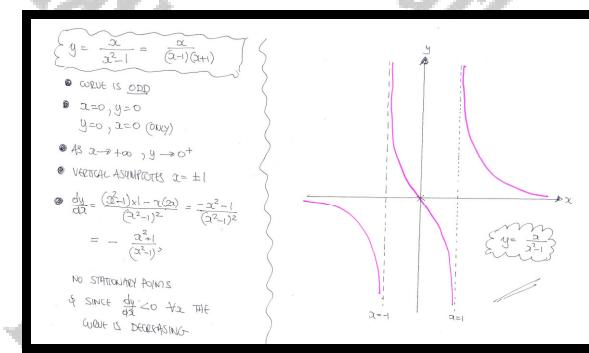
$$y = \frac{x}{x^2 - 1}, \quad x \in \mathbb{R}.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 25 (*)+**

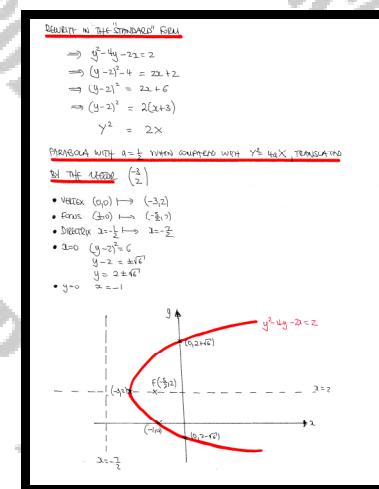
Sketch the parabola with equation

$$y^2 - 4y - 2x = 2.$$

The sketch must include the ...

- a) ... coordinates of points of intersection with the coordinate axes.
- b) ... coordinates of the vertex of the parabola.
- c) ... coordinates of the focus of the parabola.
- d) ... equation of the directrix of the parabola.

, graph



Question 26 (****)

A curve C has equation

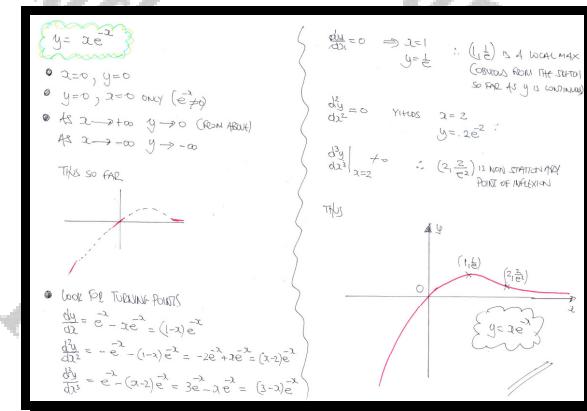
$$y = xe^{-x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

[graph]



Question 27 (****)

A curve C has equation

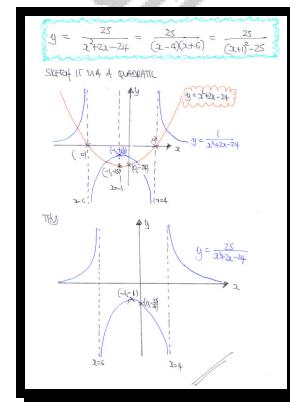
$$y = \frac{25}{x^2 + 2x - 24}, \quad x \in \mathbb{R}.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 28 (****)A curve C has equation

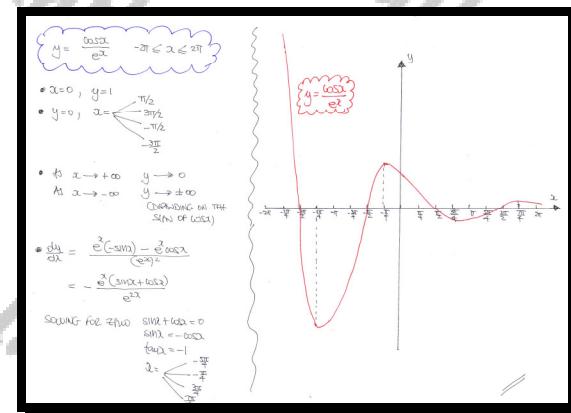
$$y = \frac{\cos x}{e^x}, -2\pi \leq x \leq 2\pi.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the x coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 29 (***)

A curve C has equation

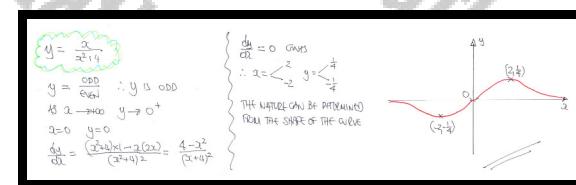
$$y = \frac{x}{x^2 + 4}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 30 (***)

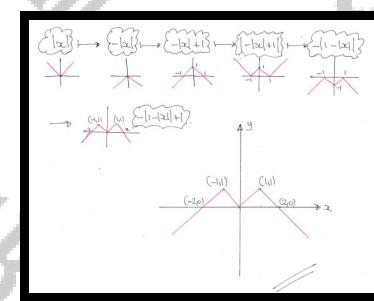
Sketch the graph of

$$y = 1 - |1 - |x||, \quad x \in \mathbb{R}.$$

The sketch must include the coordinates ...

- ... of any points where the graph meets the coordinate axes
- ... of any cusps of the graph.

graph



Question 31 (****)

A curve C has equation

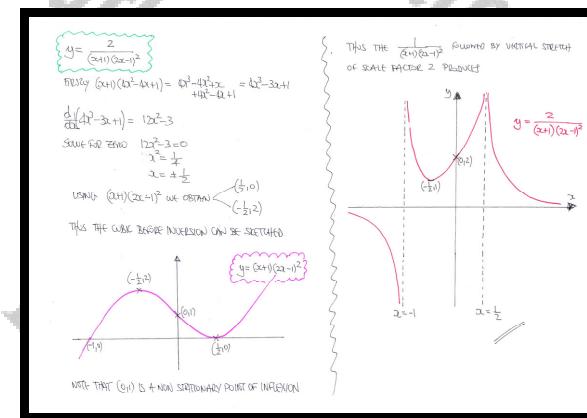
$$y = \frac{2}{(x+1)(2x-1)^2}, \quad x \in \mathbb{R}, \quad x \neq -1, \quad x \neq \frac{1}{2}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 32 (*****)A curve C has equation

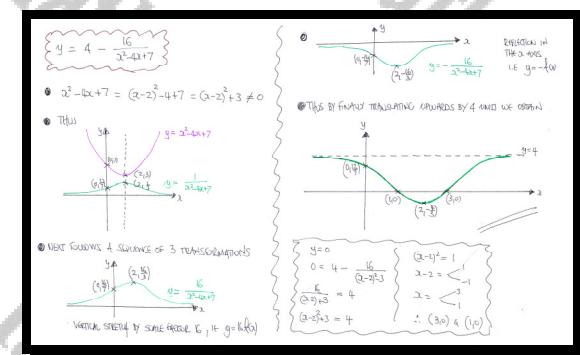
$$y = 4 - \frac{16}{x^2 - 4x + 7}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 33 (****)

A curve C has equation

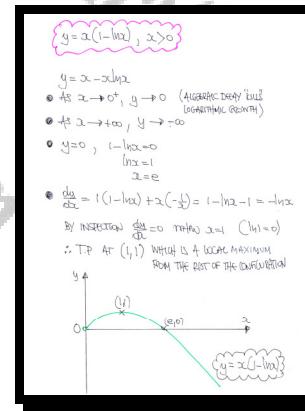
$$y = x(1 - \ln x), \quad x \in \mathbb{R}, \quad x > 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 34 (****)

A curve C has equation

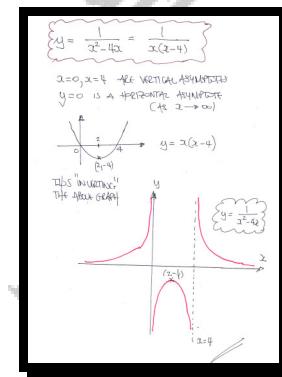
$$y = \frac{1}{x^2 - 4x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 35 (****)

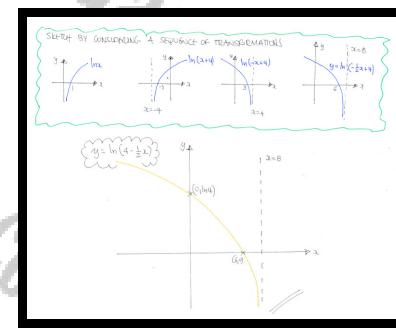
The function f is defined as

$$f : x \mapsto \ln\left(4 - \frac{1}{2}x\right), \quad x \in \mathbb{R}, \quad x < 8.$$

Sketch the graph of f .

Indicate clearly any intersections with the axes and the equation of its asymptote.

graph



Question 36 (****)

A curve C has equation

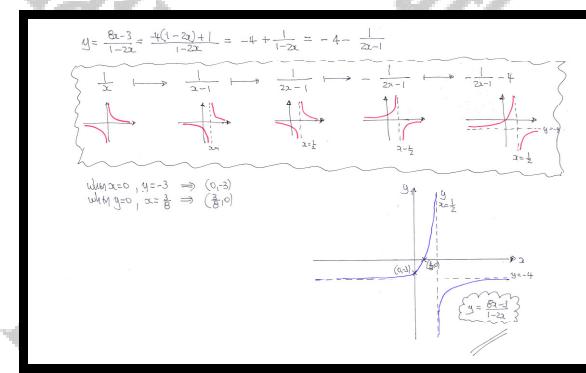
$$y = \frac{8x-3}{1-2x}, \quad x \in \mathbb{R}, \quad x \neq \frac{1}{2}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 37 (****)A curve C has equation

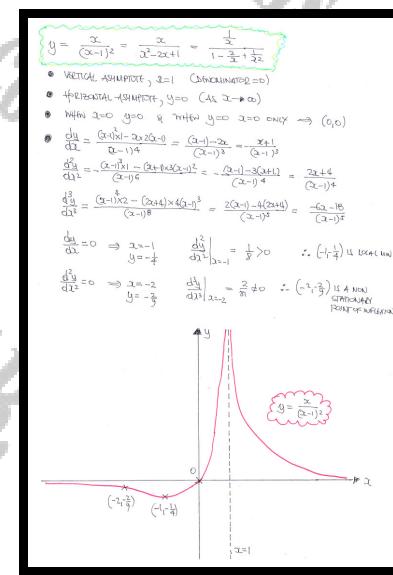
$$y = \frac{x}{(x-1)^2}, x \in \mathbb{R}, x \neq 1.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



Question 38 (***)A curve C has equation

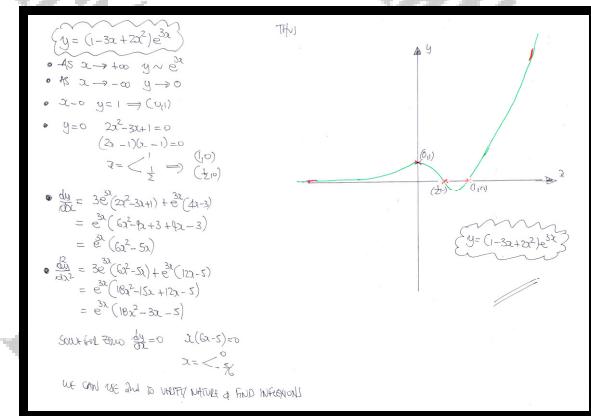
$$y = (1 - 3x + 2x^2)e^{3x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes,
- the coordinates of any stationary points,
- the equations of any asymptotes.

graph



Question 39 (****)

A curve C has equation

$$y = \frac{1}{x^3 - 9x^2 + 24x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

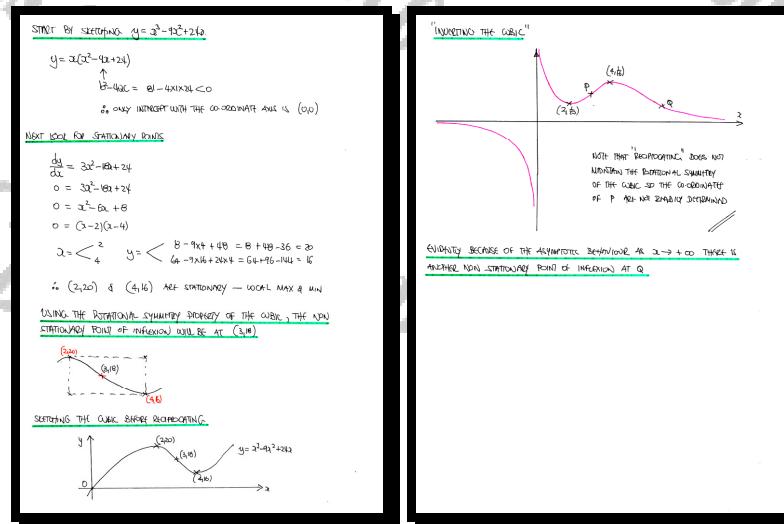
Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

You must further label any non stationary turning points, without explicitly giving their coordinates.

, graph



Question 40 (**)**

A curve C has equation

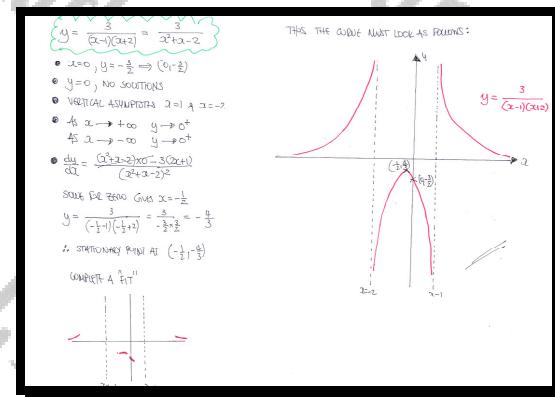
$$y = \frac{3}{(x-1)(x+2)}, \quad x \in \mathbb{R}, \quad x \neq -2, \quad x \neq 1.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 41 (****)A curve C has equation

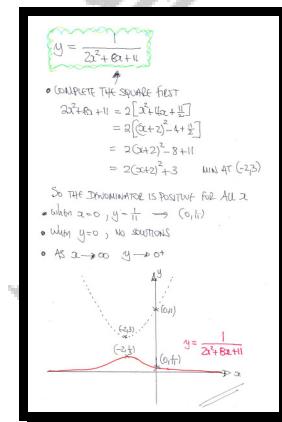
$$y = \frac{1}{2x^2 + 8x + 11}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 42 (****)

Sketch the graph of the curve with equation

$$y(y-2) = 4x+3.$$

The sketch must include the coordinates of any intersections with the axes and the coordinates of the point where the tangent to the curve is parallel to the y axis.

, graph

• START BY COMPLETING THE SQUARE IN y .

$$\begin{aligned} \rightarrow y(y-2) &= 4x+3 \\ \rightarrow y^2 - 2y &= 4x+3 \\ \rightarrow y^2 - 2y + 1 &= 4x+4 \\ \rightarrow (y-1)^2 &= 4(x+1) \end{aligned}$$

• THIS IS A SIMPLE TRANSLATION OF $y^2 = 4x$ BY THE VECTOR $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

ACTIVATOR: WE HAVE

$$\begin{aligned} y-1 &= \pm 2\sqrt{x+1} \\ y &= 1 \pm 2\sqrt{x+1} \end{aligned}$$

- $\sqrt{x^2} \rightarrow \sqrt{2x+1} \rightarrow 2\sqrt{x+1} \rightarrow 2\sqrt{x+1} + 1$
- $\sqrt{x^2} \rightarrow -\sqrt{x^2} \rightarrow -\sqrt{2x+1} \rightarrow -2\sqrt{x+1} \rightarrow -2\sqrt{x+1} + 1$

Question 43 (***)+A curve C has equation

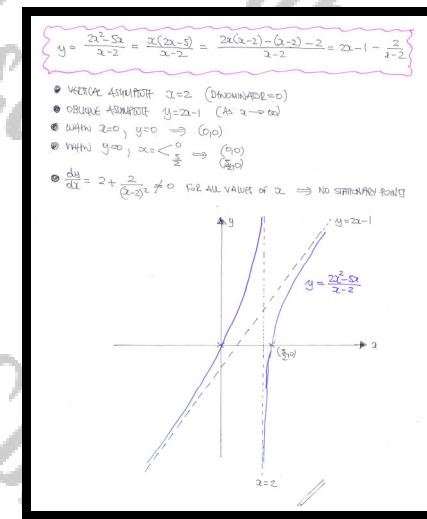
$$y = \frac{2x^2 - 5x}{x-2}, \quad x \in \mathbb{R}, \quad x \neq 2.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 44 (***)+

A curve C has equation

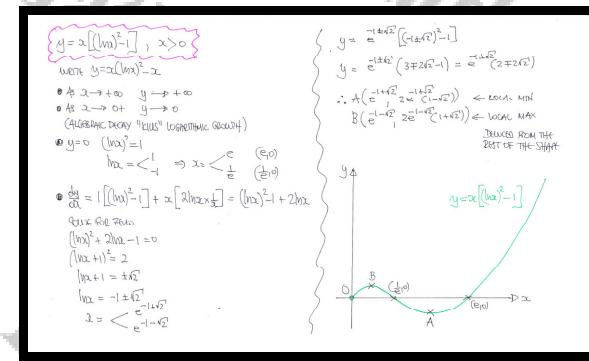
$$y = x \left[(\ln x)^2 - 1 \right], \quad x \in \mathbb{R}, \quad x > 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes,
- the coordinates of any stationary points,
- the equations of any asymptotes,

graph



Question 45 (***)+

A curve C has equation

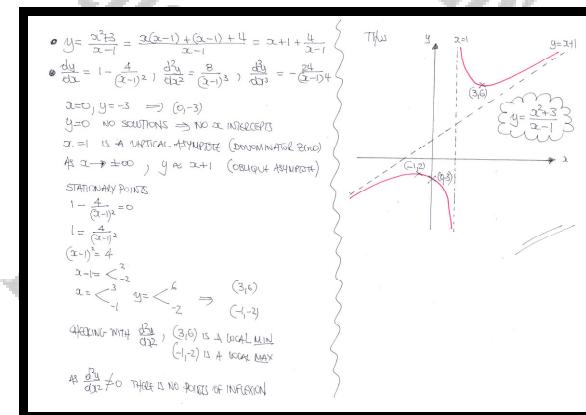
$$y = \frac{x^2 + 3}{x - 1}, \quad x \in \mathbb{R}, \quad x \neq 1.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



\bullet $y = \frac{x^2 + 3}{x - 1} = \frac{x(x-1) + (x-1) + 4}{x-1} = x+1 + \frac{4}{x-1}$
 \bullet $\frac{dy}{dx} = (1 - \frac{4}{(x-1)^2}) \quad \frac{d^2y}{dx^2} = \frac{8}{(x-1)^3} \quad \frac{d^3y}{dx^3} = -\frac{24}{(x-1)^4}$
 $x=0 \Rightarrow y=-3 \rightarrow (0, -3)$
 $y=0 \Rightarrow$ NO SOLUTIONS \Rightarrow NO x-INTERCEPTS
 $x=1$ is a VERTICAL ASYMPTOTE (DISCONTINUITY, 2nd kind)
 $\text{As } x \rightarrow \pm\infty, \quad y \rightarrow x+1$ (Oblique Asymptote)
STATIONARY POINTS
 $1 - \frac{4}{(x-1)^2} = 0$
 $1 = \frac{4}{(x-1)^2}$
 $(x-1)^2 = 4$
 $x-1 = <_1^2$
 $x = <_1^3 \quad y = <_2^4 \Rightarrow (3, 6)$
 $(-1, -2)$
 LOCAL MIN WITH $\frac{d^2y}{dx^2} < 0$, $(-1, -2)$ IS A LOCAL MIN.
 $(3, 6)$ IS A LOCAL MAX.
 $\text{As } \frac{d^3y}{dx^3} \neq 0$, THERE IS NO POINT OF INFLECTION

Question 46 (*)+**

A curve C has equation $y = f(x)$

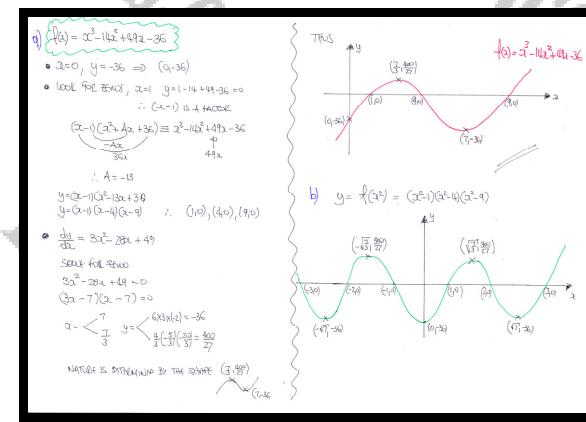
$$f(x) = x^3 - 14x^2 + 49x - 36, \quad x \in \mathbb{R}.$$

- a) Sketch the graph of C .
- b) Use the sketch of part (a) to deduce the graph of $y = f(x^2)$

Each of the sketches must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.

graph



Question 47 (****+)

A curve C has equation

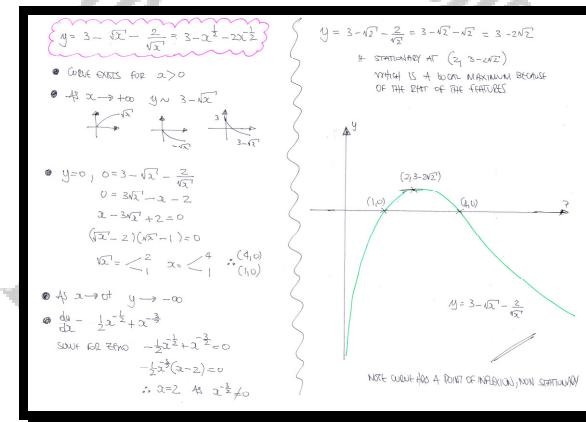
$$y = 3 - \sqrt{x} - \frac{2}{\sqrt{x}}.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

[graph]



Question 48 (****+)

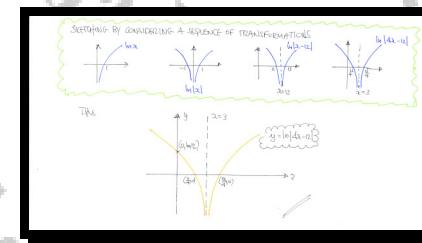
The function f is defined as

$$f : x \mapsto \ln|4x-12|, \quad x \in \mathbb{R}, \quad x \neq 3.$$

Sketch the graph of f .

Indicate clearly any intersections with the axes and the equation of its asymptote.

graph



Question 49 (****+)

A curve C has equation

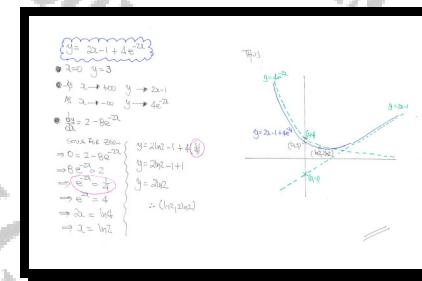
$$y = 2x - 1 + 4e^{-2x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C , indicating clearly the behaviour for large positive and large negative values of x .

The graph must also include the exact coordinates, where appropriate, of ...

- ... any points where the graph of C meets the coordinate axes.
- ... any turning points of C .

graph



Question 50 (***)+

A curve C has equation

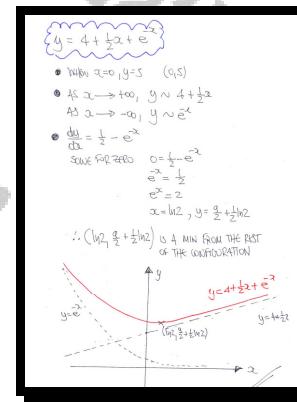
$$y = 4 + \frac{1}{2}x + e^{-x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C , indicating clearly the behaviour for large positive and large negative values of x .

The graph must also include the exact coordinates, where appropriate, ...

- ... of any points where the graph of C meets the coordinate axes.
- ... of any turning points of C .

graph



Question 51 (***)+

A curve C has equation

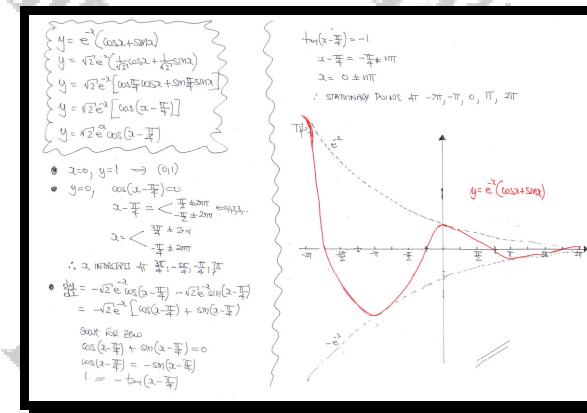
$$y = e^{-x} (\cos x + \sin x), \quad x \in \mathbb{R}.$$

Sketch the graph of C , indicating clearly the behaviour for large positive and large negative values of x .

The graph must also include the exact coordinates, where appropriate, ...

- ... of any points where the graph of C meets the coordinate axes.
- ... of any turning points of C .

graph



Question 52 (***)+

A curve C has equation

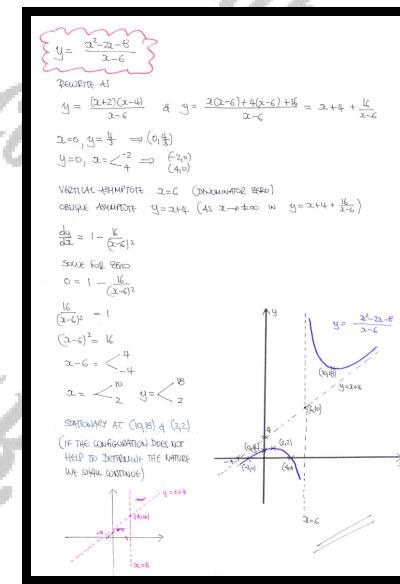
$$y = \frac{x^2 - 2x - 8}{x - 6}, \quad x \in \mathbb{R}, \quad x \neq 6.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

[graph]



Question 53 (****+)A curve C has equation

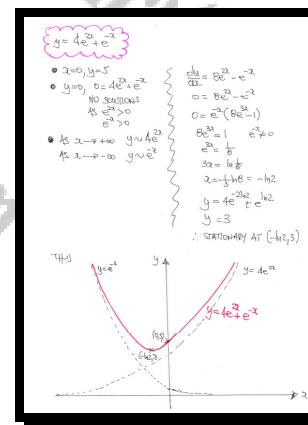
$$y = 4e^{2x} + e^{-x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 54 (***)+A curve C has equation

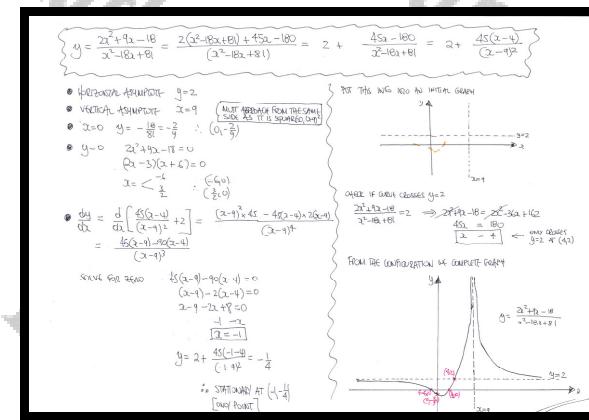
$$y = \frac{2x^2 + 9x - 18}{x^2 - 18x + 81}.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

graph



Question 55 (***)+

A curve C has equation

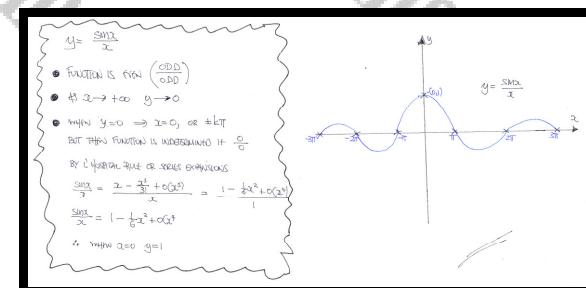
$$y = \frac{\sin x}{x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points, for $-3\pi \leq x \leq 3\pi$.

graph



Question 56 (****+)

A curve C has equation

$$y = 3 \sinh x - 2 \cosh x, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

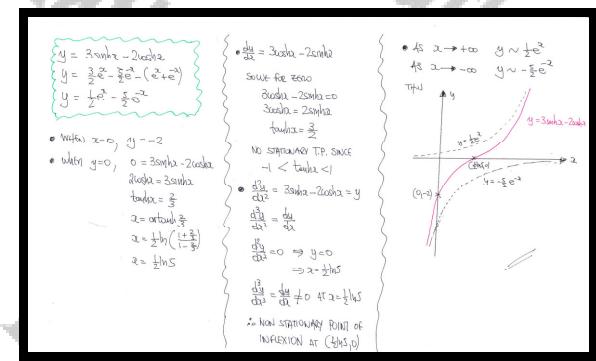
The sketch must include ...

... the coordinates of any points where the graph of C meets the coordinate axes.

... the coordinates of any stationary or non stationary turning points.

... the behaviour of the curve for large positive and large negative values of x

graph



Question 57 (****+)

A curve C has equation

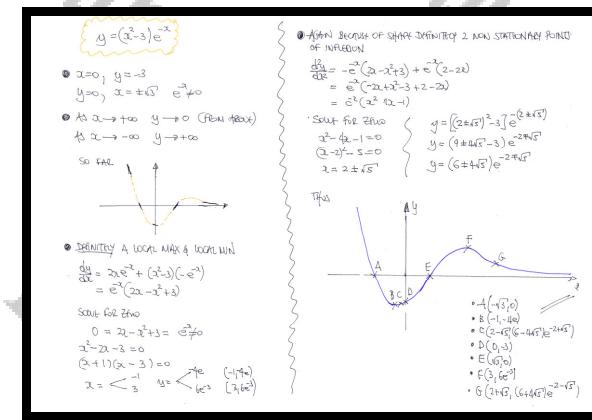
$$y = (x^2 - 3)e^{-x}, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes,
- the coordinates of any stationary points,
- the coordinates of any non stationary turning points,
- the equations of any asymptotes.

[graph]



Question 58 (***)+

A curve C has equation

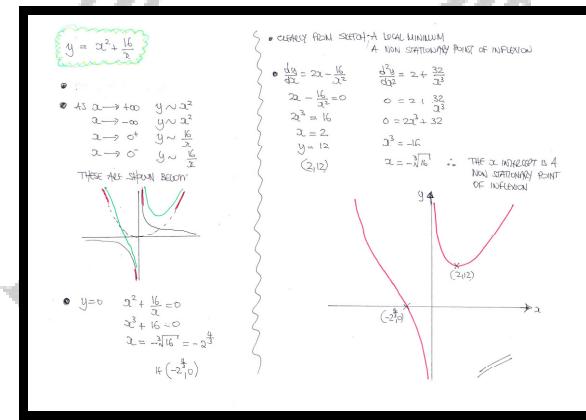
$$y = x^2 + \frac{16}{x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

Sketch the graph of C .

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



Question 59 (***)+

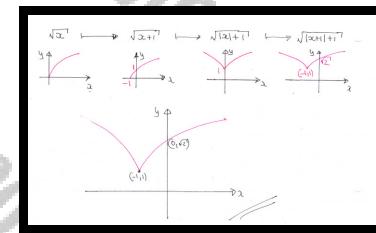
By considering the following sequence of transformations T_1 , T_2 and T_3

$$\sqrt{x} \xrightarrow{T_1} \sqrt{x+1} \xrightarrow{T_2} \sqrt{|x|+1} \xrightarrow{T_3} \sqrt{|x+1|+1}$$

sketch the graph of $y = \sqrt{|x+1|+1}$.

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

graph



Question 60 (***)+

A curve C has equation

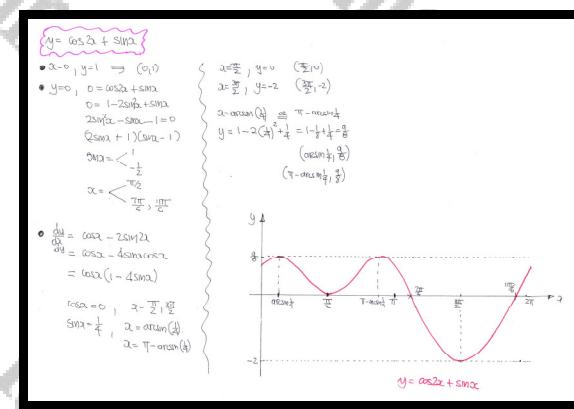
$$y = \cos 2x + \sin x, \quad 0 \leq x \leq 2\pi.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.

graph



Question 61 (***)+

A curve C has equation

$$y = x^4 - 32x + 15, \quad x \in \mathbb{R}.$$

Sketch the graph of C .

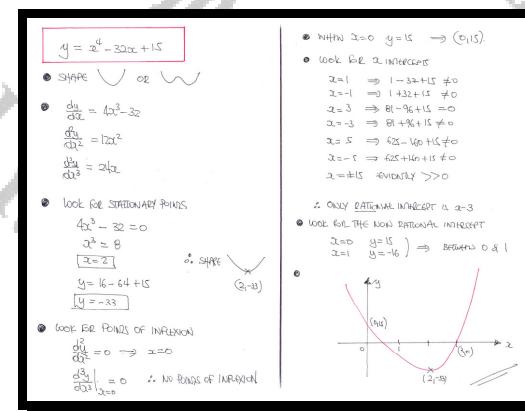
The sketch must include

- the coordinates of any points where the graph of C meets the coordinate axes.

If a coordinate is not rational indicate suitably on the axis an interval of consecutive integers in which the graph meets that particular coordinate axis.

- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

[graph]



Question 62 (***)+

A curve C has equation

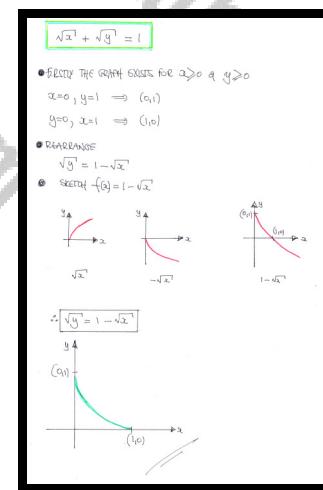
$$\sqrt{y} + \sqrt{x} = 1.$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

graph



Question 63 (*****)

A curve C has equation

$$y^3 - y^2 = x, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}.$$

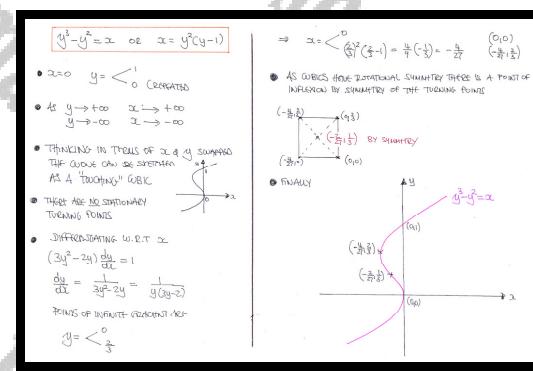
Sketch the graph of C .

The graph must include the coordinates ...

... of any points where the graph of C meets the coordinate axes.

... of the three turning points of C , of which one is a point of inflection.

, graph



Question 64 (*****)

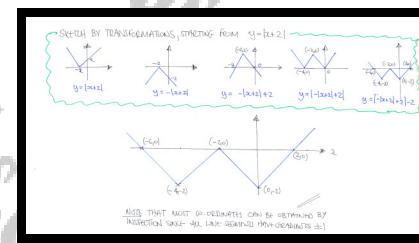
$$f(x) = |2 - |x + 2|| - 2, \quad x \in \mathbb{R}.$$

Sketch the graph of $f(x)$

The sketch must include the coordinates ...

- ... of any points where the graph meets the coordinate axes
- ... of any cusps of the graph.

[graph]



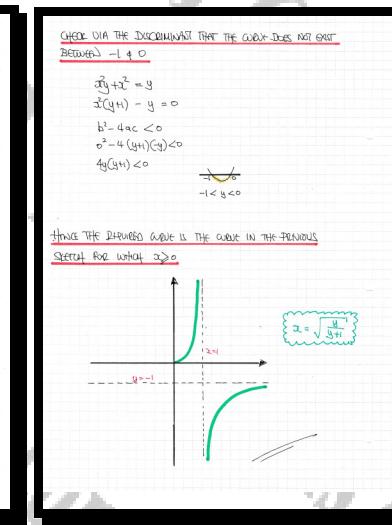
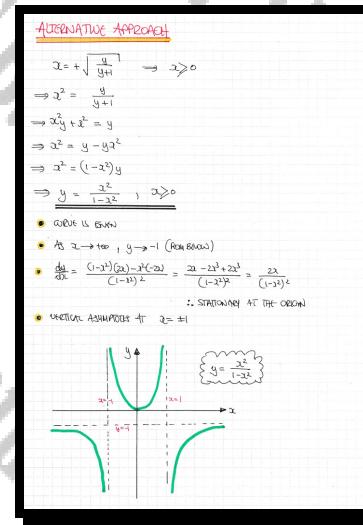
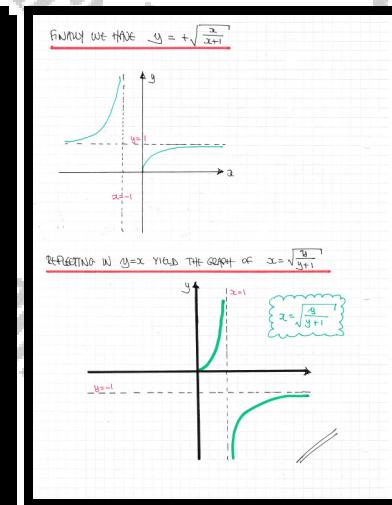
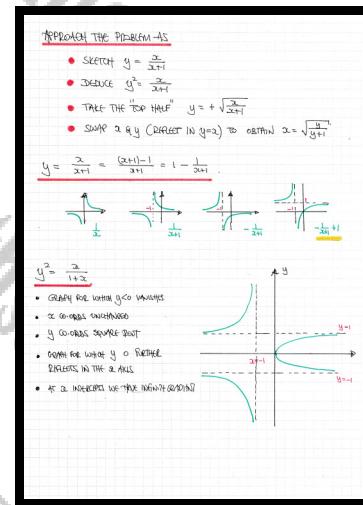
Question 65 (*****)

A curve C is defined in the largest real domain by the equation

$$x = \sqrt{\frac{y}{y+1}}.$$

Sketch a detailed graph of C , fully justifying its key features.

, graph



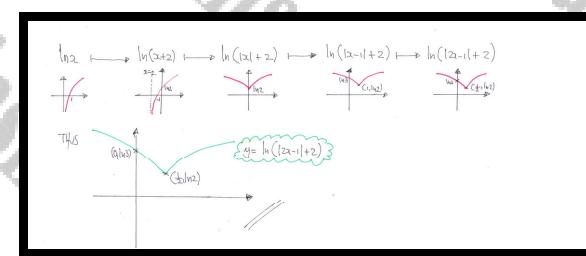
Question 66 (*****)

By considering a sequence of transformations, or otherwise, sketch the graph of

$$y = \ln(|2x - 1| + 2), \quad x \in \mathbb{R}$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

graph



Question 67 (*****)

Sketch the graph of the curve with equation

$$x = y^2 \ln y.$$

The sketch must include ...

- ... the coordinates of any intersections with the axes.
- ... the coordinates of any points where the tangent to the curve is parallel to the coordinate axes.
- ... the coordinates of any points of inflexion.

WE START FROM SKETCHING THE GRAPH OF $y = x^2 \ln x$

- THE CURVE IS ONLY DEFINED FOR $x > 0$
- AS $x \rightarrow 0^+$, $y \rightarrow 0^-$ (AS x^2 TENDS TO ZERO FEWER THAN $\ln x$ TENDS TO $-\infty$)
- AS $x \rightarrow +\infty$, $y \sim x^2$
- WHEN $x=1$, $y=0$
THIS SO FAR ISN'T HINT

LOOKING FOR THE COORDINATES OF THE POINT OF INFLEXION

$$\frac{d^2y}{dx^2} = ((2\ln x + 1) + x(\frac{2}{x})) = 3 + 2\ln x$$

SETTING FOR ZERO

$$3 + 2\ln x = 0$$

$$\ln x = -\frac{3}{2}$$

$$x = e^{-\frac{3}{2}}$$

$$y = (e^{-\frac{3}{2}})^2 (-\frac{3}{2}) = -\frac{3}{2}e^{-3}$$

$$\therefore (e^{-\frac{3}{2}}, -\frac{3}{2}e^{-3})$$

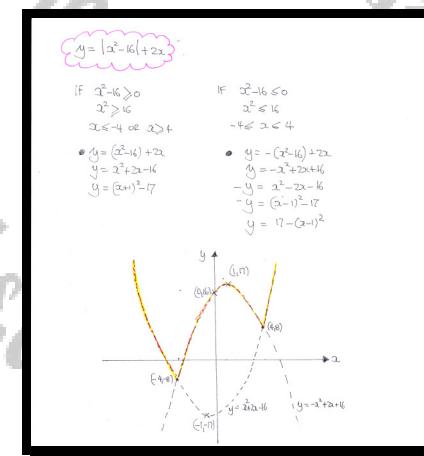
FINALLY THE DESIRED CURVE WILL BE A REFLECTION ABOUT THE LINE $y=x$, i.e. SWAPPING x & y .

Question 68 (*****)

Sketch the graph of

$$y = |x^2 - 16| + 2x, \quad x \in \mathbb{R}.$$

The sketch must include the coordinates of any cusps or any stationary points



Question 69 (*****)

A curve C is defined in the largest possible real number domain and has equation

$$y = \frac{x^3}{1-x^4}.$$

Sketch the graph of C .

The sketch must include

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

, graph

④ FIRSTLY WRITE THE EQUATION IN EXPANDED/FACORIZATED FORM

$$y = \frac{x^3}{1-x^4} = \frac{x^3}{(-x^2)(1+x^2)} = \frac{x^3}{(-x^2)(1+x^2)}$$

• NOTICE THE EQUATION IS ODD, SO IT HAS ROTATIONAL SYMMETRY ABOUT THE ORIGIN. SINCE

$$\frac{\partial y}{\partial x} = \frac{3x^2 - 4x^6}{(1-x^2)^2}$$

SO WE NEED ONLY CONSIDER $x \geq 0$ & REFLATE IT.

• THE ONLY y/x INCLINE IS THE ORIGIN.

• LOOKING AT THE BEGGS OF THE DENOMINATOR, WE ONLY HAVE TWO VERTICAL ASYMPTOTES AT $x = \pm 1$ ($(x^2 + 1) = 0$)

AS $x \rightarrow +\infty$, $y \rightarrow 0^+$ (IE FROM BELOW)
 $x \rightarrow -\infty$, $y \rightarrow 0^+$ (IE FROM ABOVE) (CONFERRED FROM CONTINUITY)

• LOOK FOR STATIONARITY (TURNING POINTS)

$$\frac{dy}{dx} = \frac{(-x^2)(3x^2) - 2x^5(-4x^3)}{(1-x^2)^2} = \frac{-3x^4 - 2x^5}{(1-x^2)^2} = \frac{-x^4(3+2x)}{(1-x^2)^2}$$

$$\frac{dy}{dx} = \frac{3x^3(3x+2)}{(1-x^2)^2}$$

SETTING TO ZERO YIELDS $x=0$ (REMEMBER $x^2 + 3 \neq 0$)

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{3x^3(3x+2)}{(1-x^2)^2} \right] = \frac{(6x^2+6x)((-x^2)^2 + 8x^2(1-x^2)(3x+2))}{(1-x^2)^4}$$

$$= \frac{(1-x^2)(6x^2+6x) + 8x^2(3x^2+3x)}{(1-x^2)^3}$$

$$= \frac{6x^5+6x^3-6x^9-8x^7-8x^5+24x^5}{(1-x^2)^3} = \frac{2x^5+24x^3}{(1-x^2)^3}$$

$\left. \frac{dy}{dx} \right|_{x=0} = 0$ SO POSSIBLY INFLECTION AT THE ORIGIN

DIFFERENTIATE AGAIN

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{2x^5+24x^3}{(1-x^2)^3} \right] = \frac{(10x^4+72x^2+6)(1-x^2)^3 + 3(1-x^2)^2(2x^4+18x^2+6x)}{(1-x^2)^6}$$

$$\left. \frac{d^2y}{dx^2} \right|_{x=0} = \frac{6x^4+12x^2+6}{1-x^2} = 6 \neq 0 \therefore \text{ORIGIN IS A STATIONARY POINT OF INFLECTION}$$

• THE BEHAVIOR OF THE CURVE AS IT APPROACHES THE VERTICAL ASYMPTOTES CAN BE DETERMINED FROM THE PENCIL COMPUTATION INFORMATION WE HAVE SO FAR

AS $x \rightarrow 1^-$, $y \rightarrow +\infty$
AS $x \rightarrow -1^+$, $y \rightarrow -\infty$

• THIS WE CAN NOW SKETCH

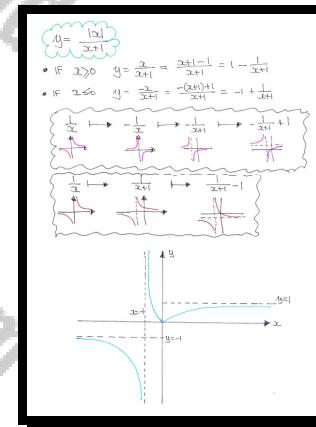
Question 70 (*****)

Sketch the graph of

$$y = \frac{|x|}{x+1}, \quad x \in \mathbb{R}.$$

The sketch must include the equations of any asymptotes of the curve, and the coordinates of any points where the curve meets the coordinate axes.

graph



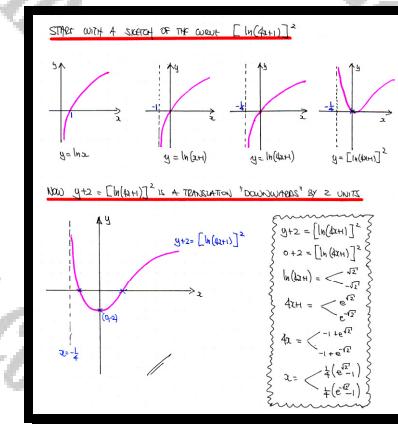
Question 71 (*****)

The curve C has equation

$$y+2 = [\ln(4x+1)]^2, \quad x \in \mathbb{R}, \quad x \geq -\frac{1}{4}.$$

Sketch a detailed graph of C .

graph



Question 72 (*****)

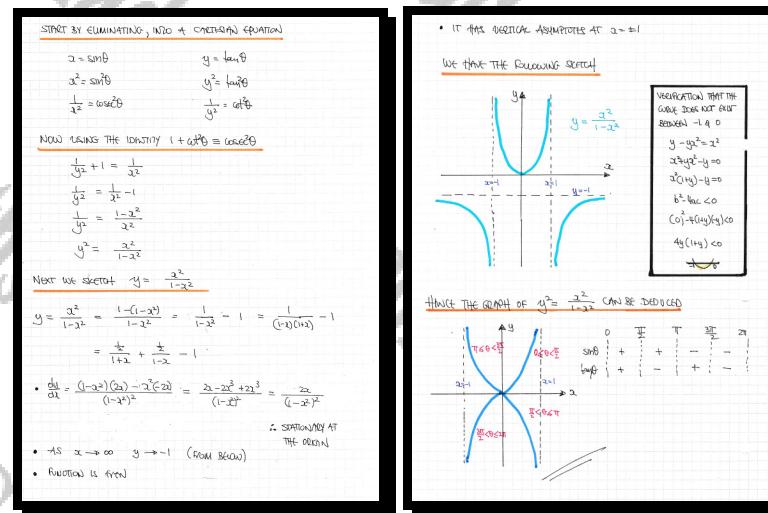
A curve C is defined in the largest real domain by the parametric equations

$$x = \sin \theta, \quad y = \tan \theta.$$

Sketch a detailed graph of C , fully justifying its key features.

The sketch must include the range of values of θ , which produces each section of C .

, graph



Question 73 (*****)

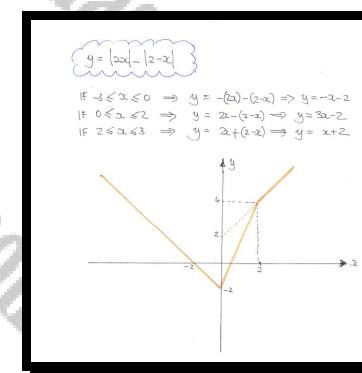
Sketch the graph of

$$y = |2x| - |2-x|, \quad x \in \mathbb{R}.$$

The sketch must include the coordinates of any points where the curve meets the coordinate axes.

[No credit will be given to non analytical sketches based on plotting coordinates]

graph



Question 74 (*****)

A curve C has equation

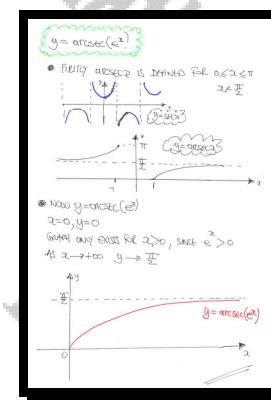
$$y = \operatorname{arcsec}(e^x).$$

Sketch the graph of C , for the largest possible domain.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes,
- the coordinates of any stationary points,
- the equations of any asymptotes.

graph



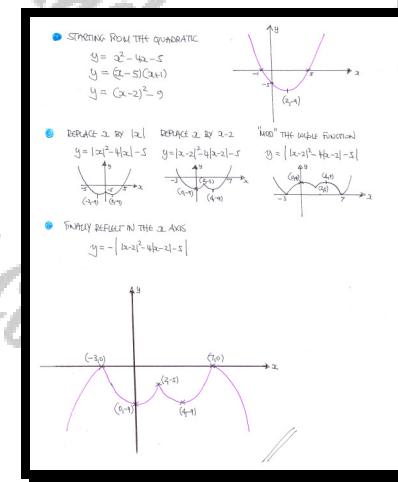
Question 75 (*****)

By considering a sequence of four transformations, or otherwise, sketch the graph of

$$y = -|x-2|^2 - 4|x-2|-5.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

, , ,



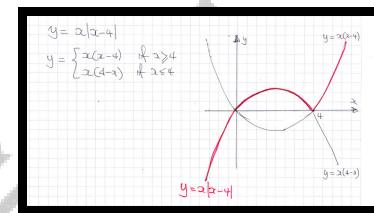
Question 76 (*****)

By considering the graphs of two separate curves, or otherwise, sketch the graph of

$$y = x|x-4|.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

graph

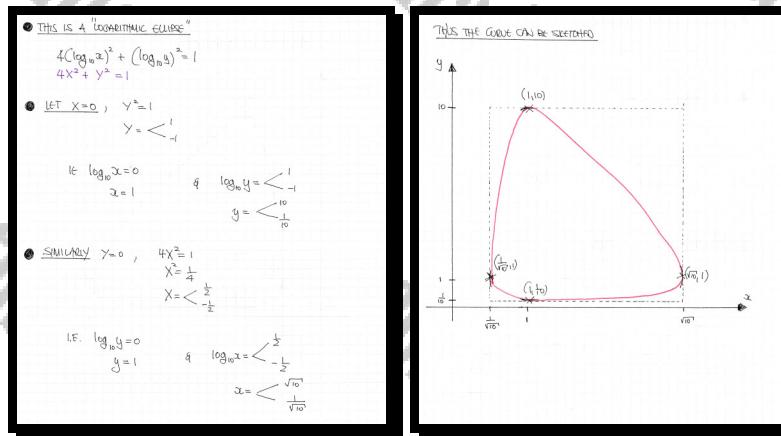


Question 77 (*****)

Sketch the graph of the curve with equation

$$4[\log_{10} x]^2 + [\log_{10} y]^2 = 1, \quad x > 0, \quad y > 0.$$

The sketch must include the coordinates of any points where the tangent to the curve is parallel to the coordinate axes.

 , graph


Question 78 (*****)

Sketch the graph of

$$\left[x + \sqrt{x^2 + 4} \right] \left[y + \sqrt{y^2 + 1} \right] = 2, \quad x \in (-\infty, \infty), \quad y \in (-\infty, \infty)$$

You must show a detailed method in this question

, proof

LOOKING AT THE EQUATION

- y -then is the argument of \ln . i.e. $\ln(y + \sqrt{y^2 + 1})$
- x -then also looks like a similar \ln argument

$$\Rightarrow (x + \sqrt{x^2 + 4})(y + \sqrt{y^2 + 1}) = 2$$

$$\Rightarrow \ln[(x + \sqrt{x^2 + 4})(y + \sqrt{y^2 + 1})] = \ln 2$$

$$\Rightarrow \ln(x + \sqrt{x^2 + 4}) + \ln(y + \sqrt{y^2 + 1}) = \ln 2$$

$$\Rightarrow \ln(x + \sqrt{x^2 + 4}) + \text{arcsinh}(y) = \ln 2.$$

MINIMISE THE LOGARITHM, SO THE RADICAL HAS " y^2 " INSTEAD OF "4"

$$\Rightarrow \ln[2 + 2\sqrt{(x^2 + 1)}] + \text{arcsinh}(y) = \ln 2.$$

$$\Rightarrow \ln[2(1 + \sqrt{(x^2 + 1)})] + \text{arcsinh}(y) = \ln 2$$

$$\Rightarrow \ln[2] + \ln[1 + \sqrt{(x^2 + 1)}] + \text{arcsinh}(y) = \ln 2$$

$$\Rightarrow \text{arcsinh}(y) + \text{arcsinh}(x) = 0$$

$$\Rightarrow \text{arcsinh}(y) = -\text{arcsinh}(x)$$

BUT arcsinh IS AN ODD FUNCTION

$$\Rightarrow \text{arcsinh}(-x) = -\text{arcsinh}(x)$$

BUT THIS IS A ONE TO ONE MAPPING

$$\Rightarrow -x = -y$$

$$\Rightarrow y = x$$

ALTERNATIVE WITHOUT HYPERBOLICS

$$\left[x + \sqrt{x^2 + 4} \right] \left[y + \sqrt{y^2 + 1} \right] = 2.$$

BUT $4 = x + \sqrt{x^2 + 4}$

$$\Rightarrow u(y + \sqrt{y^2 + 1}) = 2.$$

$$\Rightarrow y + \sqrt{y^2 + 1} = \frac{2}{u}$$

$$\Rightarrow \sqrt{y^2 + 1} = \frac{2}{u} - y$$

$$\Rightarrow y^2 + 1 = \frac{4}{u^2} - \frac{4}{u}y + y^2$$

$$\Rightarrow u^2 = 4 - 4uy$$

$$\Rightarrow 4uy = 4 - u^2$$

$$\Rightarrow y = \frac{1}{u} - \frac{u}{4}$$

COMBINING 2 ANSWERS

$$y = \frac{1}{4} - \frac{1}{4}u = -\frac{1}{4}x + \frac{1}{4}\sqrt{2x^2 + 4} - \frac{1}{4}[x + \sqrt{x^2 + 4}] = -\frac{1}{4}x + \frac{1}{4}\sqrt{2x^2 + 4} - \frac{1}{4}x - \frac{1}{4}\sqrt{2x^2 + 4} = -\frac{1}{2}x$$

$\therefore y = -\frac{1}{2}x$ AS DEDUCED AND THE GRAPH FOLLOWS

Question 79 (*****)

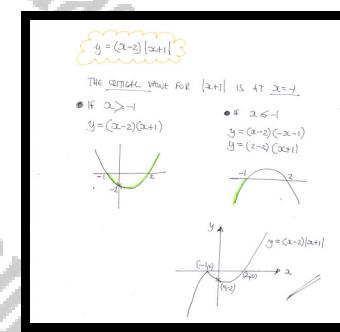
By considering the graphs of two separate curves, or otherwise, sketch the graph of

$$y = (x-2)|x+1|.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[No credit will be given to non analytical sketches based on plotting coordinates]

graph



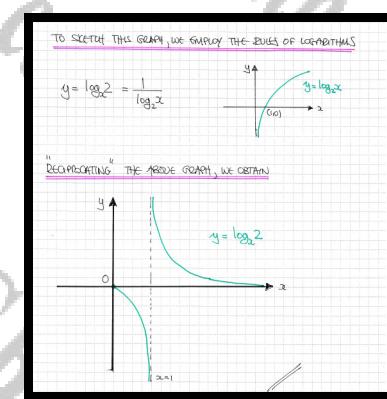
Question 80 (*****)

A curve C is defined in the largest real domain by the equation

$$y = \log_x 2.$$

Sketch a detailed graph of C .

, graph



Question 81 (*****)

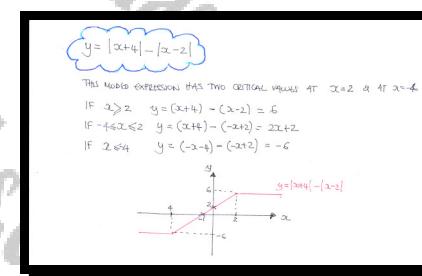
By considering the graphs of three separate lines, or otherwise, sketch the graph of

$$y = |x+4| - |x-2|$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[No credit will be given to non analytical sketches based on plotting coordinates]

graph



Question 82 (*****)

A curve C is defined in the largest real domain by the equation

$$y = \frac{4x^2 - 25}{(2x-1)(x-2)(x+3)}.$$

- a) Sketch the graph of C .

The sketch must include the equations of any asymptotes of C and the coordinates of any point where C meets the coordinate axes. Any turning points, including points of inflection, must be clearly indicated but their coordinates need **not** be found.

- b) Hence sketch on separate set of axes the graph of ...

a) $\dots y^2 = \frac{4x^2 - 25}{(2x-1)(x-2)(x+3)}.$

b) $\dots y = \frac{4x - 25}{(2\sqrt{x}-1)(\sqrt{x}-2)(\sqrt{x}+3)}.$

, graph

a) STANDARD INFORMATION ONCE WE DIVIDE THE EQUATION

$$y = \frac{4x^2 - 25}{(2x+3)(2x-1)(2x-2)} = \frac{(2x+5)(2x-5)}{(2x+3)(2x-1)(2x-2)}$$

- $x=0 \Rightarrow y = \frac{-25}{3(-2)} = -\frac{25}{6} \Rightarrow (0, -\frac{25}{6})$
- $y=0 \Rightarrow 2x = \pm \frac{5}{2} \Rightarrow (\pm \frac{5}{4}, 0)$
- VERTICAL ASYMPTOTES (DENOMINATOR ZERO) $\Rightarrow x = -\frac{3}{2}, 1, -2$
- HORIZONTAL ASYMPTOTE (AS $x \rightarrow \pm\infty$) $\Rightarrow y = 4$
- AS THERE ARE NO SQUARED BRACKETS IN THE DENOMINATOR, THE CURVE MUST REAPPEAR ON OPPOSITE SIDES OF THE VERTICAL ASYMPTOTES

THERE IS ENOUGH INFORMATION TO SKETCH THE CURVE

b) $y = \frac{4x^2 - 25}{(2x-1)(\sqrt{x}-2)(\sqrt{x}+3)}$

- ON THIS, UNHANDED
- y HAS THREE "SQUARE ROOTS"
- PART OF CURVE FOR WHICH $x < 0$ UNDEFINED, i.e. PART OF CURVE $y^2 > 0$
- BRANCHES IN THE x AXIS
- $\frac{dy}{dx} = \infty$ AT $x = \text{INTERCEPT}$

$y = \frac{4x - 25}{(\sqrt{x}-1)(\sqrt{x}-2)(\sqrt{x}+3)}$

- 2. INVERSE SQUARE
- y INVERSE UNDEFINED
- PART OF CURVE FOR WHICH $x < 0$ UNDEFINED

Question 83 (*****)

A curve is defined in the largest real domain by the equation

$$f(x) = -x^2 + 8x - 12.$$

Sketch on separate set of axes detailed graph of ...

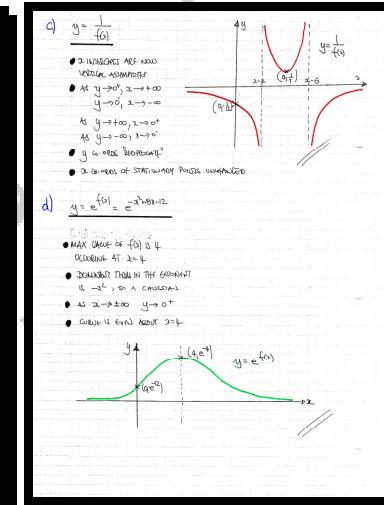
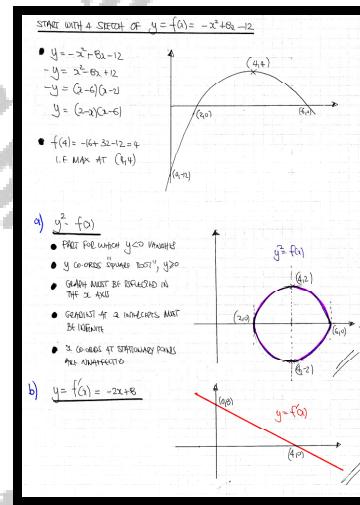
a) ... $y^2 = f(x)$.

b) ... $y = f'(x)$.

c) ... $y = \frac{1}{f(x)}$.

d) ... $y^2 = e^{f(x)}$.

, graph



Question 84 (*****)

On a clearly labelled set of axes, draw a detailed sketch of the graph of

$$y = (\arcsin x)^2 \arccos x, -1 \leq x \leq 1.$$

graph

<p>$y = (\arcsin x)^2 \arccos x, -1 \leq x \leq 1$</p> <p>THE GRAPH CLEARLY EXISTS FOR $-1 \leq x \leq 1$</p> $\frac{dy}{dx} = (\arcsin x)^2 \arccos x = \left(\frac{\pi}{2}\right)^2 \times 0 = 0$ $\frac{dy}{dx} \Big _{x=-1} = [\arcsin(-1)]^2 \arccos(-1) = \left(-\frac{\pi}{2}\right)^2 (\pi) = \frac{\pi^3}{4}$ <p>NEXT look for stationary points — requires y or simplicity</p> $y = (\arcsin x)^2 \arccos x$ $y = (\arcsin x)^2 \left[\frac{\pi}{2} - \arccos x\right]$ $y = \frac{1}{2}(\arcsin x)^2 - (\arccos x)^2$ $\frac{dy}{dx} = \frac{1}{2}(\arcsin x) \cdot \frac{1}{\sqrt{1-x^2}} - 2(\arccos x) \cdot \frac{1}{1-x^2}$ $\frac{dy}{dx} = \frac{\arcsin x}{\sqrt{1-x^2}} \left[\frac{1}{2} - 3\arccos x \right]$ <p>SEARCHING FOR ZERO WE OBTAIN</p> <table style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 50%; vertical-align: top; padding: 10px;"> $\arcsin x = 0$ $x = 0$ </td> <td style="width: 50%; vertical-align: top; padding: 10px;"> $\frac{1}{2} - 3\arccos x = 0$ $\arccos x = \frac{1}{6}$ $x = \cos \frac{1}{6}$ $y \Big _{x=0} = (\arcsin 0)^2 \arccos(0)$ $y \Big _{x=0} = 0$ </td> </tr> </table>	$\arcsin x = 0$ $x = 0$	$\frac{1}{2} - 3\arccos x = 0$ $\arccos x = \frac{1}{6}$ $x = \cos \frac{1}{6}$ $y \Big _{x=0} = (\arcsin 0)^2 \arccos(0)$ $y \Big _{x=0} = 0$	<p>COLLECTING SIGHT OF THESE RESULTS</p> <table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th>x</th> <th>y</th> <th>DESCRIPTION</th> </tr> </thead> <tbody> <tr> <td>-1</td> <td>$-\frac{\pi^3}{4}$</td> <td>ACTUAL MAXIMUM (END POINT)</td> </tr> <tr> <td>0</td> <td>0</td> <td>LOCAL STATIONARY MINIMUM</td> </tr> <tr> <td>$\frac{\pi}{6}$</td> <td>$\frac{\pi^3}{4}$</td> <td>LOCAL STATIONARY MAXIMUM</td> </tr> <tr> <td>1</td> <td>0</td> <td>ACTUAL MINIMUM (END POINT)</td> </tr> </tbody> </table> <p>FINALLY THE CURVE CAN BE SKETCHED</p>	x	y	DESCRIPTION	-1	$-\frac{\pi^3}{4}$	ACTUAL MAXIMUM (END POINT)	0	0	LOCAL STATIONARY MINIMUM	$\frac{\pi}{6}$	$\frac{\pi^3}{4}$	LOCAL STATIONARY MAXIMUM	1	0	ACTUAL MINIMUM (END POINT)
$\arcsin x = 0$ $x = 0$	$\frac{1}{2} - 3\arccos x = 0$ $\arccos x = \frac{1}{6}$ $x = \cos \frac{1}{6}$ $y \Big _{x=0} = (\arcsin 0)^2 \arccos(0)$ $y \Big _{x=0} = 0$																	
x	y	DESCRIPTION																
-1	$-\frac{\pi^3}{4}$	ACTUAL MAXIMUM (END POINT)																
0	0	LOCAL STATIONARY MINIMUM																
$\frac{\pi}{6}$	$\frac{\pi^3}{4}$	LOCAL STATIONARY MAXIMUM																
1	0	ACTUAL MINIMUM (END POINT)																

Question 85 (*****)

By considering the graphs of three separate lines, or otherwise, sketch the graph of

$$y = |x-4| + |x+1|$$

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusp of the curve.

[No credit will be given to non analytical sketches based on plotting coordinates]

graph

$y = x-4 + x+1 = \begin{cases} 4x-3 & (x > 4) \\ 5 & (-1 \leq x \leq 4) \\ -4x+3 & (x < -1) \end{cases}$	$(x-4) + (x+1) = 2x-3$ $-(x-4) + (x+1) = 5$ $-(x-4) - (x+1) = -2x+3$
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Question 86 (*****)

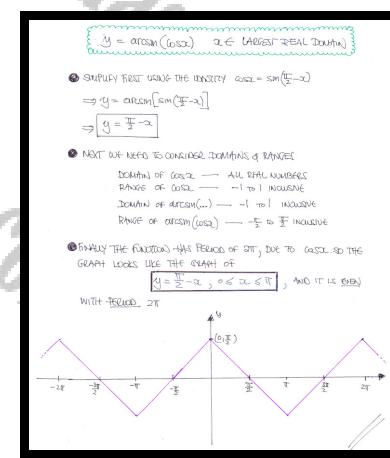
Sketch the graph of

$$f(x) = \arcsin(\cos x),$$

in the largest domain that the function is defined.

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusps of the curve.

, graph



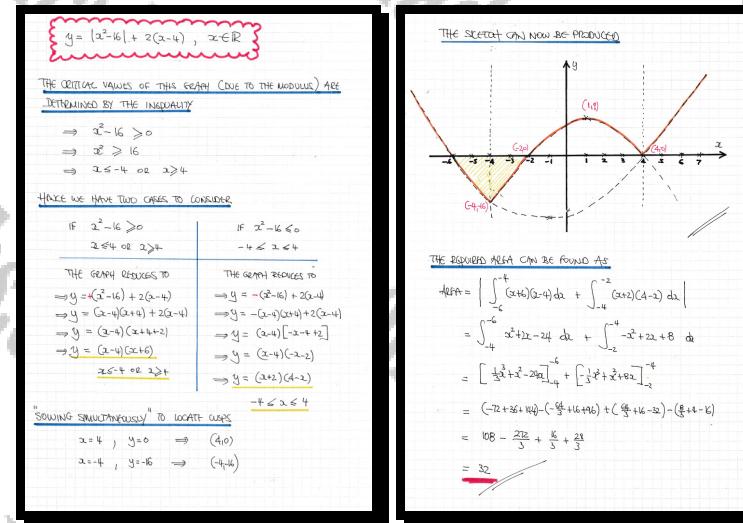
Question 87 (*****)

The curve C has equation

$$y = |x^2 - 16| + 2(x - 4), \quad x \in \mathbb{R}.$$

Sketch a detailed graph of C and hence show that the area of the finite region bounded by C and the x axis, for which $y < 0$, is 32 square units.

, proof



Question 88 (*****)

A curve C is defined in the largest real domain by the equation

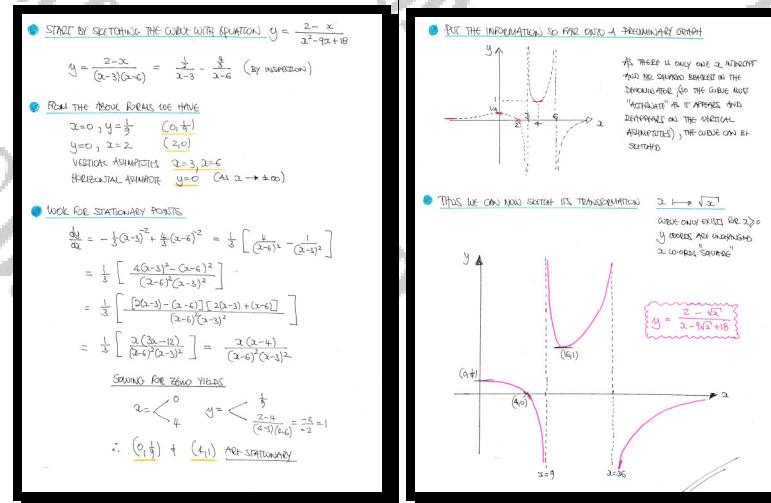
$$y = \frac{2-\sqrt{x}}{x-9\sqrt{x}+18}.$$

Sketch the graph of C .

The sketch must include

- ... the equations of any asymptotes of C .
- ... the coordinates of any point where C meets the coordinate axes.
- ... the coordinates any stationary points of C .

, graph



Question 89 (*****)

A curve C is defined parametrically by

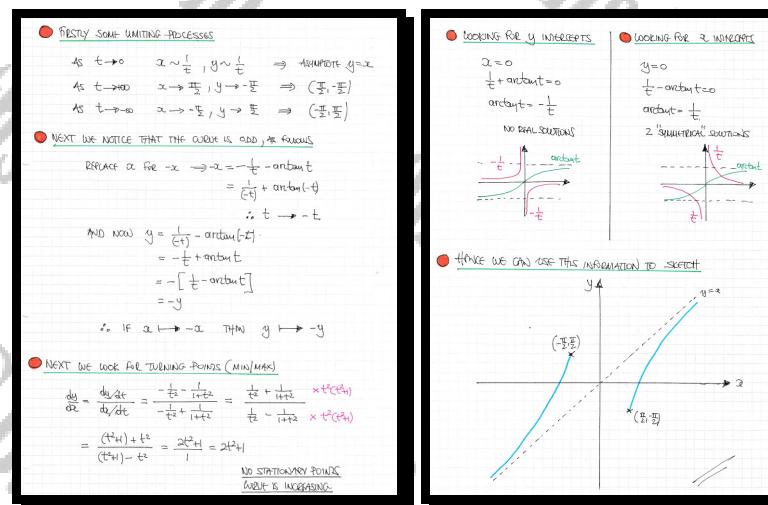
$$x = \frac{1}{t} + \arctan t, \quad y = \frac{1}{t} - \arctan t, \quad t \in \mathbb{R}, t \neq 0.$$

Sketch the graph of C .

Indicate the equations of any asymptotes, stationary points and any endpoints.

You need not mark the coordinates of any intersections with the axes.

, graph



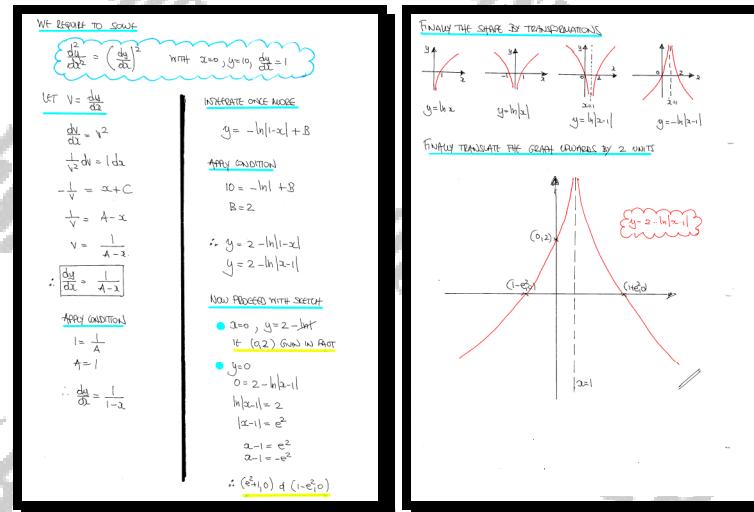
Question 90 (*****)

It is required to sketch the curve with equation $y = f(x)$, defined over the set of real numbers, in the greatest domain.

The curve has the property that at every point on the curve, the second derivative equals to the first derivative **squared**.

Showing all the relevant details, sketch the graph of $y = f(x)$, given further that the curve passes through the point $(0, 2)$ and the gradient at that point is 1.

, graph



Question 91 (*****)

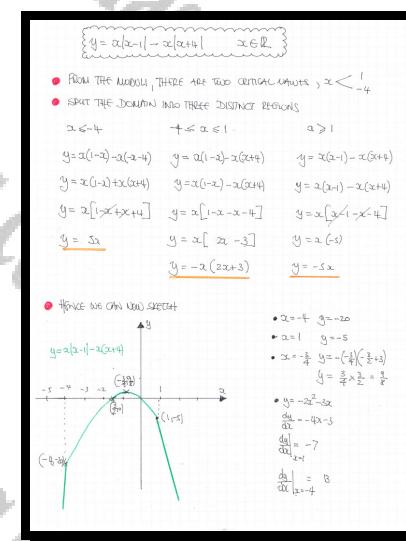
Sketch the graph of

$$y = x|x-1| - x|x+4|, \quad x \in \mathbb{R}.$$

Indicate the coordinates of any intersections with the axes, and the coordinates of any cusps of the curve.

[No credit will be given to non analytical sketches based on plotting coordinates]

, graph

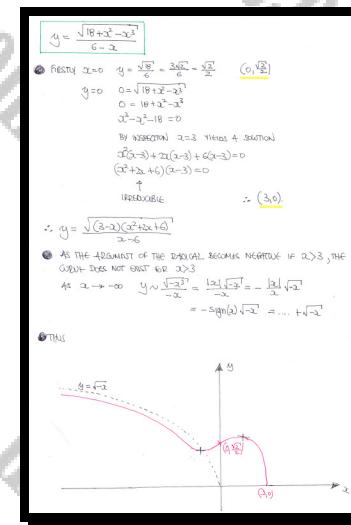


Question 92 (*****)A curve C has equation

$$y = \frac{\sqrt{18+x^2-x^3}}{6-x}.$$

It is given that C has two stationary points whose x coordinates have opposite signs.Sketch the graph of C , for the largest possible domain.

- The sketch must include, in exact form where appropriate the coordinates of any points where the graph of C meets the coordinate axes the equations of any asymptotes.
- You need not find the coordinates of the stationary points of C .

SPEL , graph


Question 93 (*****)

Sketch the curve with equation

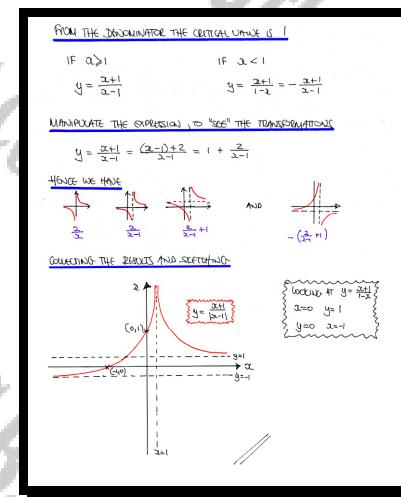
$$y = \frac{x+1}{|x-1|}, \quad x \in \mathbb{R}, \quad x \neq 1.$$

The sketch must include ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of the asymptotes of the curve.

[No credit will be given to non analytical sketches based on plotting coordinates]

, graph



Question 94 (*****)

Sketch in separate sets of axes detailed graphs of the following curves, fully justifying their key features.

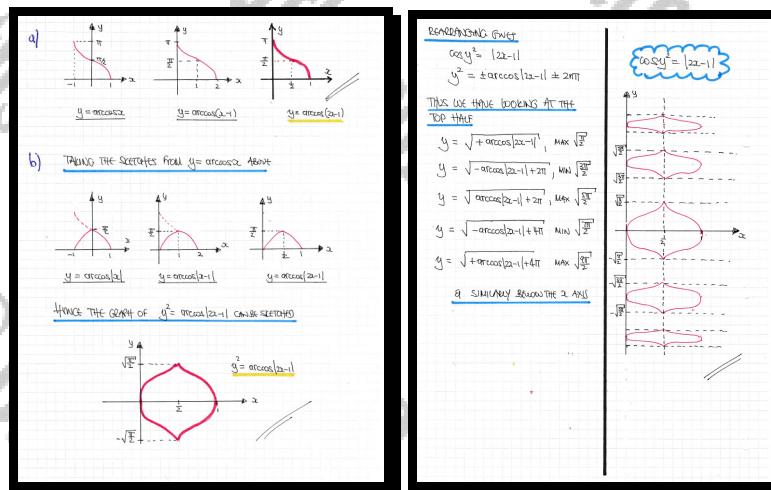
a) $y = \arccos(2x - 1)$.

b) $y^2 = \arccos|2x - 1|$.

c) $\cos y^2 = |2x - 1|$

You may assume that each curve is defined in the largest real domain.

, graph



Question 95 (*****)

Sketch the graph of the curve with equation

$$y = x\sqrt{|\ln|x||}, \quad x \in \mathbb{R}.$$

The sketch must include the coordinates of ...

... any points where the curve meets the coordinate axes.

... any stationary points.

, graph

$y = x\sqrt{|\ln|x||}, \quad x \in \mathbb{R}$

- We notice that the function is odd, since x is odd and $\sqrt{|\ln|x|}|$ is even due to the modulus in the argument of $\ln|x|$ - hence we need only consider $x > 0$, i.e. $y = x\sqrt{|\ln x|}$, and 2 times to zero points than $\ln x$ tends to $-\infty$ twice to zero.
- When $x=0$, $y=0$ as $y = 0 \times |\ln 0|$, and 2 times to zero points than $\ln x$ tends to $-\infty$.
- By inspection when $x=1$, $y=0$.
- The critical value for the modulus that is $x=1$, so we consider two cases separately.
- If $0 < x < 1$
$$y = x\sqrt{-\ln x} = x(-\ln x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = (-\ln x)^{\frac{1}{2}} + \frac{1}{2}x(-\ln x)^{-\frac{1}{2}}(\frac{1}{x})$$

$$= (-\ln x)^{\frac{1}{2}} - \frac{1}{2}(-\ln x)^{-\frac{1}{2}}$$

$$= \frac{1}{2}(-\ln x)^{\frac{1}{2}}[2(-\ln x) - 1]$$

$$= -\frac{2\ln x + 1}{2\sqrt{-\ln x}}$$

If $x > 1$

$$y = x\sqrt{\ln x} = x(\ln x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = (\ln x)^{\frac{1}{2}} + \frac{1}{2}x(\ln x)^{-\frac{1}{2}}(\frac{1}{x})$$

$$= (\ln x)^{\frac{1}{2}} + \frac{1}{2}(\ln x)^{-\frac{1}{2}}$$

$$= \frac{1}{2}(\ln x)^{\frac{1}{2}}[2(\ln x) + 1]$$

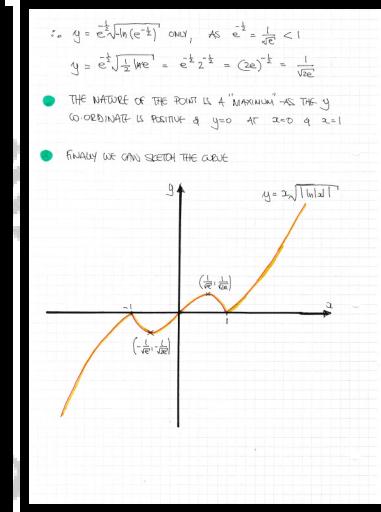
$$= \frac{2\ln x + 1}{2\sqrt{\ln x}}$$

- Solving for zero, either of the above two equations gives

$$\Rightarrow 2\ln x = -1$$

$$\Rightarrow \ln x = -\frac{1}{2}$$

$$\Rightarrow x = e^{-\frac{1}{2}}$$



Question 96 (*****)

The distinct points A and B lie on the curve with equation

$$\ln(x-y) = \ln x + \ln y, \quad x \in (0, \infty), \quad y \in (0, \infty).$$

- Determine possible coordinates for A and B , further verifying that these coordinates indeed satisfy the above given equation.
- Sketch the curve, showing clearly all the relevant details.

$$[\text{ }], [A(4,2)], [B\left(\frac{9}{2}, \frac{3}{2}\right)]$$

a) EXPONENTIATING BOTH SIDES OF THE EQUATION

$$\begin{aligned} \ln(x-y) &= \ln x + \ln y \\ e^{\ln(x-y)} &= e^{\ln x + \ln y} \\ x-y &= e^{\ln x} \cdot e^{\ln y} \\ x-y &= x \cdot y \\ xy - y^2 &= x \\ xy - x &= y^2 \\ x(y-1) &= y^2 \\ x = \frac{y^2}{y-1} & \quad (\text{DIVIDE BY } y \text{ WITH 2 DIVISIONS}) \end{aligned}$$

PICKING SOME SHAPABLE VALUES OF y

$$\begin{aligned} y=2 &\Rightarrow x = \frac{4}{2-1} = 4 \quad \therefore A(4,2) \\ \bullet \ln(2-2) &= \ln(0) = 0 \\ \bullet \ln 2 - \ln 2 &= \ln 1 - \ln 2 = \ln\left(\frac{1}{2}\right) = \ln 2 \\ y=\frac{3}{2} &\Rightarrow x = \frac{\frac{9}{4}}{\frac{1}{2}} = \frac{9}{2} \quad \therefore B\left(\frac{9}{2}, \frac{3}{2}\right) \\ \bullet \ln\left(\frac{3}{2}\right) &= \ln\left(\frac{9}{4} - \frac{9}{4}\right) = \ln 0 \\ \bullet \ln \frac{9}{4} - \ln \frac{9}{4} &= \ln \frac{9}{4} - \ln \frac{9}{4} = \ln\left(\frac{9}{4} \cdot \frac{9}{4}\right) = \ln 9 \end{aligned}$$

INCORPORATE THE ABOVE AS POSSIBLE COORDINATES

b) TO SKETCH THIS CURVE WE NEED TO CONSIDER THAT $x > 0$

$$x - \frac{y^2}{y-1} = y(x-1) + (y-1) + 1 = y + 1 + \frac{1}{y-1}$$

NOW CONSIDER THE GRAPH OF $y = 2x + \frac{1}{x-1}$

- FOR $x > 1$, $y \rightarrow \infty$
- FOR $x < 1$, $y \rightarrow -\infty$
- AS $x \rightarrow 1^+$, $y \rightarrow \infty$
- AS $x \rightarrow -\infty$, $y \rightarrow -\infty$
- $y' = 2 - \frac{1}{(x-1)^2}$
- $0 = 2 - \frac{1}{(x-1)^2}$
- $x = 2 \Rightarrow y = 3$

NOW REFLECTING IN $y=x$ (SWAPPING x & y)

- $x > 0$
- $y > 0$
- $x < y$
- $\frac{y-1}{y-1} > 1 \Rightarrow y > 0$
- $y > 1 - \frac{1}{x}$

SKETCH IS SHOWN IN BOLD IN THE FIRST QUADRANT

Question 97 (*****)

A curve C is defined in the largest real domain by the equation

$$y = -\sqrt{\frac{x(1-x)}{4-x^2}}.$$

Sketch the graph of C .

The sketch must include

- ... the equations of any asymptotes of C .
- ... the coordinates of any point where C meets the coordinate axes.
- ... the coordinates of the stationary points of C , giving the answer in the form $\left[2k + k\sqrt{3}, -\frac{1}{2k}(\sqrt{3k} + \sqrt{k})\right]$, where k is a positive integer.

, graph

Start by considering the sum of $y_1 = \frac{x(1-x)}{4-x^2}$

$$\begin{aligned} y_1 &= \frac{x-x^2}{4-x^2} = \frac{x(1-x)}{(2-x)(2+x)} = \frac{2(1-x)}{(2-x)(2+x)} = \frac{2-x}{2^2-4x} \\ &= \frac{2^2-4x}{2^2-4x} + \frac{-2x+4}{2^2-4x} = 1 - \frac{2x-4}{2^2-4x} = 1 + \frac{4-2x}{2^2-4x} \end{aligned}$$

From the above rows we deduce directly

- $x=0, y=0$ (c.o.)
- $y=0, x < 0$ (c.o.)
- $y=0, x > 2$ (c.o.)
- VERTICAL ASYMPTOTES $x=2, -2$. (EXCLUDING 0 AND 1)
- HORIZONTAL ASYMPTOTE $y=1$ ($4x \rightarrow \pm\infty$)

Next look for stationary points

$$\begin{aligned} \frac{d}{dx}\left[1 + \frac{4-2x}{2^2-4x}\right] &= \frac{(2-x)(1) - (4-2x)(2)}{(2^2-4x)^2} = \frac{4x^2-8x+2x^2}{(2^2-4x)^2} \\ &= \frac{2x^2-6x+4}{(2^2-4x)^2} \end{aligned}$$

Solving for zero terms

$$\begin{aligned} x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \\ x = 1, 2 &\quad (\text{excluded}) \\ x = 4 \pm 2\sqrt{3} & \end{aligned}$$

And

$$\begin{aligned} x^2 &= 16 \pm 16\sqrt{3} + 12 \\ x^2 &= 28 \pm 16\sqrt{3} \end{aligned}$$

Thus $\frac{x^2-2x}{2^2-4x} = \frac{(28 \pm 16\sqrt{3}) - (4 \pm 2\sqrt{3})}{28 \pm 16\sqrt{3} - 4}$

$$\begin{aligned} &< \frac{24 \pm 14\sqrt{3}}{24 \pm 14\sqrt{3}} = \frac{24-14\sqrt{3}}{24-14\sqrt{3}} = \frac{1}{2}(1-\sqrt{3}) \\ &\frac{12-7\sqrt{3}}{12-7\sqrt{3}} = \frac{12-7\sqrt{3}}{12-7\sqrt{3}} = \frac{1}{2}(2-\sqrt{3}) \end{aligned}$$

Hence the stationary points are $(2 \pm \sqrt{3}, \frac{1}{2}(2 \mp \sqrt{3}))$

Thus we have so far

Note that

$$\begin{aligned} \frac{1}{4}(2+\sqrt{3})^2 &= \frac{1}{16}(4+4\sqrt{3}) = \frac{1}{16}(1^2 + 2x(1-x) + (1-x)^2) \\ &= \frac{1}{16}(1+2x+2x^2) \end{aligned}$$

Similarly

$$\frac{1}{4}(2-\sqrt{3})^2 = \frac{1}{16}(4-4\sqrt{3})$$

Thus $\sqrt{\frac{1}{16}(4x^2+1)} = \frac{1}{4\sqrt{2}}(4x+1) = \frac{1}{4}(2+2x)$ and similarly $\sqrt{\frac{1}{16}(4x^2-1)} = \frac{1}{4}(2-2x)$

Finally the required curve is the bottom half of $y = \frac{x(1-x)}{4-x^2}$

Question 98 (*****)

The curve C has equation

$$y = A \ln|x| + Bx^2 + x, \quad x \in \mathbb{R},$$

where A and B are non zero constants.

The curve has stationary points at $x = -1$ and at $x = 2$.

Sketch the graph of C .

The sketch must include ...

- ... the coordinates of all the stationary points.
- ... the equations of the asymptotes of the curve.

You need not find any intercepts with the coordinate axes.

, graph

• CONSIDER THE GRAPH IN TWO SEPARATE SECTIONS

• IF $x > 0$
 $y = A \ln x + Bx^2 + x$
 $\frac{dy}{dx} = \frac{A}{x} + 2Bx + 1$

• SPORADICALLY AT $x=2$
 $0 = \frac{A}{2} + 4B + 1$
 $0 = A + 8B + 2$
 $A + 8B = -2$

• IF $x < 0$
 $y = A \ln(-x) + Bx^2 + x$
 $\frac{dy}{dx} = \frac{A}{-x} + 2Bx + 1$

• STATIONARITY AT $x=-1$
 $0 = \frac{A}{-1} + 2B(-1) + 1$
 $0 = -A - 2B + 1$
 $A + 2B = 1$

• SOLVING SIMULTANEOUSLY THE TWO EQUATIONS, WE OBTAIN ON SUBTRACTION
 $\begin{array}{l} 5B = -3 \\ B = -\frac{3}{5} \end{array}$ $A = 2$

• WE NOW HAVE THE EQUATION OF THE CURVE
 $y = 2 \ln|x| - \frac{3}{5}x^2 + x$

• FURTHER FEATURES OF THE CURVE

- VERTICAL ASYMPTOTE $x=0$ (y -axis), TENDING TO $-\infty$
- AS $|x| \rightarrow \pm\infty$ $y \sim -\frac{3}{5}x^2 + x$
 $\sim -\frac{3}{5}(x^2 - \frac{5}{3}x)$
 $\sim -\frac{3}{5}[x^2 - (\frac{5}{3})^2]$
 $\sim -\frac{3}{5}(x - \frac{5}{3})^2$
- WHEN $x=2$, $y = 2 \ln 2 - \frac{3}{5}(2)^2 + 2 = 2(2 \ln 2)$
- WHEN $x=-1$, $y = 2 \ln 1 - \frac{3}{5}(-1)^2 - 1 = (-1, -\frac{3}{5})$

• A SKETCH CAN NOW BE PRODUCED AS THE NATURE OF THE STATIONARY POINTS CAN BE DETERMINED BY THE ALREADY DETERMINED FEATURES.

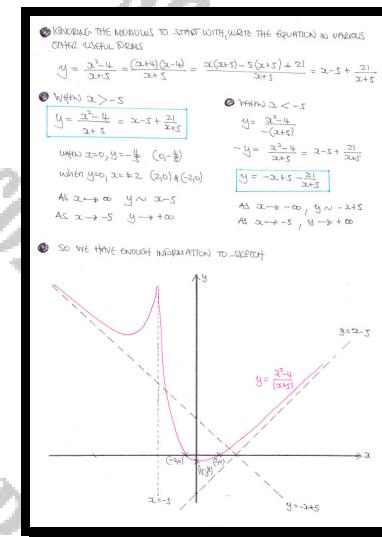
Question 99 (*****)

Sketch the curve with equation

$$y = \frac{x^2 - 4}{|x+5|}, \quad x \in \mathbb{R}, \quad x \neq -5.$$

The sketch must include ...

- ... the coordinates of all the points where the curve meets the coordinate axes.
- ... the equations of the asymptotes of the curve.

 graph


Question 100 (***)**

A general curve C has equation

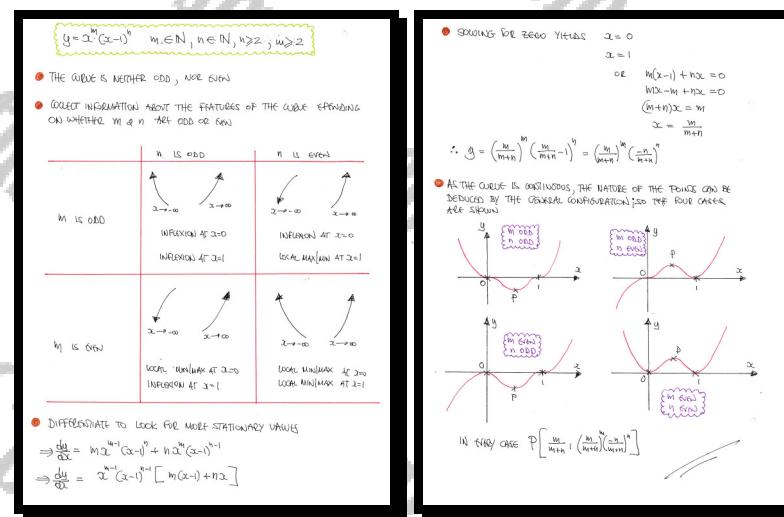
$$y = x^m(x-1)^n,$$

where $x \in \mathbb{R}$, $m \in \mathbb{N}$, $m \geq 2$, $n \in \mathbb{N}$, $n \geq 2$.

Sketch in four separate axes, the 4 separate shapes which C can take, $m \geq 2$.

The sketches must contain the coordinates of any stationary points.

graph



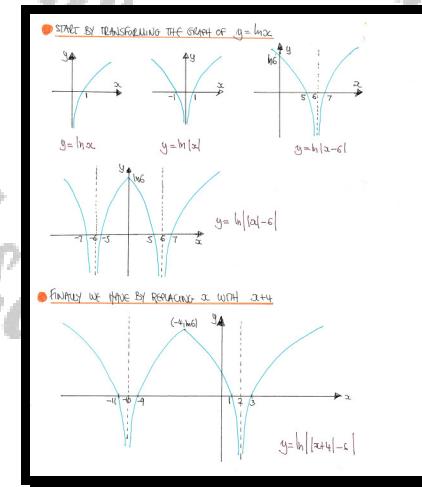
Question 101 (*****)

Sketch, in the largest real domain, the graph of

$$y = \ln|x+4|-6.$$

Indicate the coordinates of any intersections with the axes, the equations of any asymptotes and the coordinates of any cusps of the curve.

, graph



Question 102 (*****)

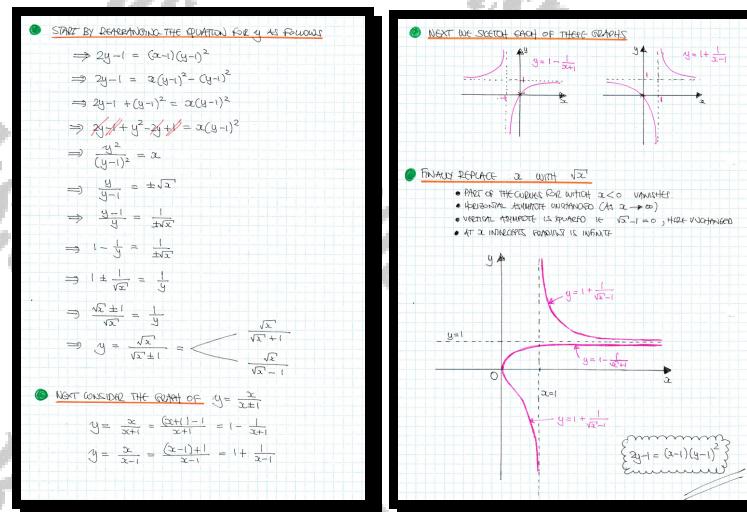
A curve C is defined, in the largest possible real domain, by the Cartesian equation

$$2y-1 = (x-1)(y-1)^2.$$

By expressing the above equation in the form $y = f(x)$, sketch the graph of C .

Indicate the equations of any asymptotes, stationary points and any intersections with the coordinate axes.

, graph



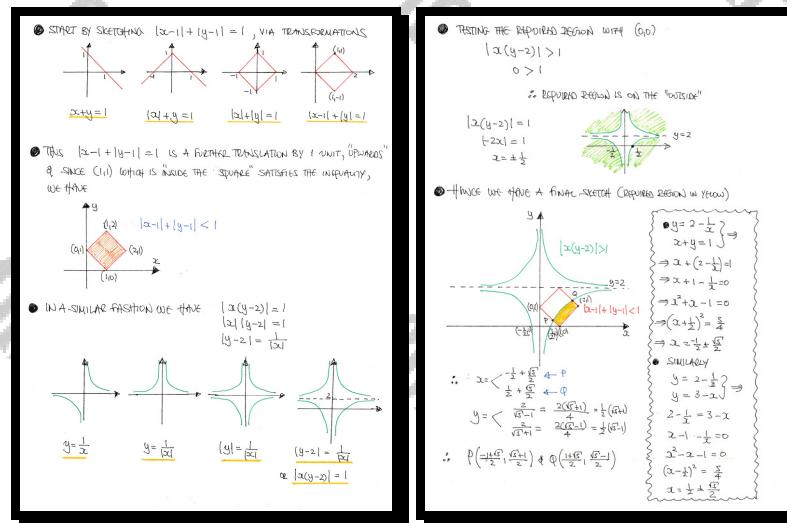
Question 103 (*****)

A finite region in the x - y plane is defined by the inequalities

$$|x-1| + |y-1| < 1 \quad \text{and} \quad |x(y-2)| > 1.$$

Sketch in detail this region, showing clearly any relevant coordinates.

, graph



Question 104 (*****)

The curve C is defined in the greatest real domain by the equation

$$y = \frac{x}{(y-2)(y+1)(y-3)}.$$

a) Show that

$$\frac{dy}{dx} = \frac{1}{2(y-1)(ay^2 + by + c)},$$

where a , b and c are integers to be found.

- b) Determine the exact value of the gradient at the points on C , where $x = 40$.
 c) Sketch the graph of C .

The sketch must include the coordinates of any points where C meets the coordinate axes, the coordinates of the points of infinite gradient. You must also find, with a full algebraic method, the line of symmetry of C .

$$\boxed{\quad}, \boxed{a=2, b=-4, c=-3}, \boxed{\pm \frac{1}{78}}$$

a) MANIPULATE THE EQUATION AS FOLLOWS

$$\Rightarrow y(y-2)(y+1)(y-3) = x$$

$$\Rightarrow x = (y^2-2y)(y^2+1)$$

$$\Rightarrow x = (y^2-2y)^2 - 3(y^2-2y)$$

DIFFERENTIATE WITH RESPECT TO y

$$\Rightarrow \frac{dx}{dy} = 2(y^2-2y)(2y-2) - 3(2y-2)$$

$$\Rightarrow \frac{dx}{dy} = (2y-2)[2(y^2-2y)-3]$$

$$\Rightarrow \frac{dx}{dy} = 2(y-1)(C^2-4y-3)$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{2(y-1)(2y^2-4y-3)}$$

$\Rightarrow (y^2-2y-8)(y^2-2y+5) = 0$

 $\Rightarrow (y+4)(y-2)(y^2-2y+5) = 0$

(REDUCE BY $(y-2)$ AS $y \neq 2$)

 $\therefore y < 4$

USING THE RESULT FROM PART (a)

$$\frac{dy}{dx} \Big|_{y=4} = \frac{1}{2(3)(32-16-3)} = \frac{1}{6 \times 13} = \frac{1}{78}$$

$$\frac{dy}{dx} \Big|_{y=2} = \frac{1}{2(-3)(8+8-3)} = \frac{1}{-6 \times 13} = -\frac{1}{78}$$

(c) COLLECTING ALL THE INFORMATION FOR THE SKETCH

- $x=0 \Rightarrow y=0, -1, 2, 3$
- $y=0 \Rightarrow x=0$
- $\frac{dy}{dx}=0 \Rightarrow$ NO SOLUTIONS
- $\frac{dy}{dx}=\infty \Rightarrow y=1$ OR $\frac{2y^2-4y-3}{y^2-2y+5} = 0$
 $y^2-2y-\frac{3}{5} = 0$
 $(y-1)^2-\frac{8}{5} = 0$
 $y = 1 \pm \sqrt{\frac{8}{5}}$

b) LOOKING AT THE EXPRESSION FROM ABOVE WITH $x=40$

$$\Rightarrow x = (y^2-2y)^2 - 3(y^2-2y)$$

$$\Rightarrow 40 = (y^2-2y)^2 - 3(y^2-2y)$$

$$\Rightarrow 0 = (y^2-2y)^2 - 3(y^2-2y) - 40$$

$$\Rightarrow 0 = [(y^2-2y) - 8][(y^2-2y) + 5]$$

USING $a = (y^2-2y)^2 - 3(y^2-2y)$

- IF $y=1$ $x = (-1)^2 - 2(-1)$
 $x = 1+3$
 $x = 4$ $\text{at } (4, 1)$
- IF $y = 1 \pm \sqrt{\frac{8}{5}}$ $x = 1 \pm \sqrt{16}$
 $y^2-2y = \frac{7}{5} \pm \sqrt{\frac{8}{5}} - 2(1 \pm \sqrt{\frac{8}{5}})$
 $= \frac{3}{5} \pm \sqrt{\frac{8}{5}} \mp \sqrt{\frac{8}{5}}$
 $= \frac{3}{5}$
 $x = (y^2-2y)^2 - 3(y^2-2y)$
 $x = (\frac{3}{5})^2 - 3(\frac{3}{5})$
 $x = \frac{9}{25} - \frac{9}{5}$
 $x = -\frac{36}{25}$
 $\therefore \left(-\frac{3}{5}, 1 \pm \sqrt{\frac{8}{5}}\right) \text{ & } \left(-\frac{3}{5}, 1 \pm \frac{1}{5}\sqrt{16}\right)$

• WRITE THE CURVE AS

$$x = y(y-2)(y+1)(y-3)$$

THE CURVE IS EVEN ABOUT THE LINE $y=1$ SINCE

$$x = (2-y)(3-y)(-y+1)(-y-3)$$

$$x = (-y-1)(-y)(-y+1)(-y-3)$$

$$x = y(y-2)(y+1)(y-3)$$

ALTERNATIVE IN 3 STAGES

$$x = y(y-2)(y+1)(y-3)$$

$$x = (y+1)(y-2)(y+1+1)(y+1-3)$$

$$x = (1+y)(3-y)(4y+2)(y-2)$$

$$x = (-y+1)(3-y)(4y+2)(y-2)$$

$$x = [-(y-1)](-y+1)(4y+2)(y-2)$$

$$x = (-y+1)(-y+1)(4y+2)(y-2)$$

$$x = (y-1)(y-1)(4y+2)(y-2)$$

$$x = y(y-2)(y+1)(y-3)$$

FINALLY A SKETCH CAN BE PRODUCED

