

PROOF BY INDUCTION

SUMMATION RESULTS

Question 1 ()**

Prove by induction that

$$\sum_{r=1}^n r(r+1) = \frac{1}{3}n(n+1)(n+2), \quad n \geq 1, \quad n \in \mathbb{N}.$$

 , proof

$$\sum_{r=1}^n r(r+1) = \frac{1}{3}n(n+1)(n+2) \quad n \geq 1$$

BASE CASE: IF $n=1$

- LHS = $1 \times 2 = 2$
- RHS = $\frac{1}{3} \times 1 \times 2 \times 3 = 2$

: RESULT HOLDS FOR $n=1$

INDUCTIVE HYPOTHESIS
SUPPOSE THE RESULT HOLDS FOR $n=k, k \in \mathbb{N}$

$$\Rightarrow \sum_{r=1}^k r(r+1) = \frac{1}{3}k(k+1)(k+2)$$

$$\Rightarrow \sum_{r=1}^k r(r+1) + (k+1)(k+2) = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$$

$$\Rightarrow \sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}(k+1)(k+2)[k+2]$$

$$\Rightarrow \sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}(k+1)(k+2+1)(k+2+2)$$

CONCLUSION

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
- SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 2 ()**

Prove by induction that

$$\sum_{r=1}^n r(r+3) = \frac{1}{3}n(n+1)(n+5), \quad n \geq 1, \quad n \in \mathbb{N}.$$

proof

$$\sum_{r=1}^n r(r+3) = \frac{1}{3}n(n+1)(n+5) \quad n \in \mathbb{N}$$

BASE CASE: $n=1$

$$\text{LHS} = \sum_{r=1}^1 r(r+3) = 1 \times 4 = 4$$

$$\text{RHS} = \frac{1}{3} \times 1 \times 2 \times 6 = 4$$

: RESULT HOLDS FOR $n=1$

INDUCTIVE HYPOTHESIS
SUPPOSE THAT THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\sum_{r=1}^k r(r+3) = \frac{1}{3}k(k+1)(k+5)$$

$$\Rightarrow \left[\sum_{r=1}^k r(r+3) \right] + (k+1)(k+4) = \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4)$$

$$\Rightarrow \sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}(k+1)(k+5+1)(k+4)$$

$$\Rightarrow \sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}(k+1)(k+2)(k+6)$$

$$\Rightarrow \sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}(k+1)(k+2+1)(k+6+2)$$

CONCLUSION

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN THE RESULT HOLDS FOR $n=k+1$
- SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 3 (*)**

Prove by induction that

$$\sum_{r=1}^n (r-1)(r+1) = \frac{1}{6}n(n-1)(2n+5), \quad n \geq 1, \quad n \in \mathbb{N}.$$

[proof]

$$\sum_{r=1}^n (r-1)(r+1) = \frac{1}{6}n(n-1)(2n+5)$$

- IF $n=1$ $\Leftrightarrow S=0$ \quad) IS RESULT TRUE FOR $n=1$
 $r=1 \times (1-1)(2 \times 1 + 5) = 0$
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$
 $\sum_{r=1}^k (r-1)(r+1) = \frac{1}{6}k(k-1)(2k+5)$
 $\sum_{r=1}^{k+1} (r-1)(r+1) = (k+1-1)(2k+1) = \frac{1}{6}k(k-1)(2k+5) + (k+1-1)(2k+1)$
 $\sum_{r=1}^{k+1} (r-1)(r+1) = \frac{1}{6}k[(k-1)(2k+5) + 6(k+1)]$
 $\sum_{r=1}^{k+1} (r-1)(r+1) = \frac{1}{6}k[2k^2 + 3k - 5 + 6k + 12]$
 $\sum_{r=1}^{k+1} (r-1)(r+1) = \frac{1}{6}k(2k^2 + 9k + 7)$
 $\sum_{r=1}^{k+1} (r-1)(r+1) = \frac{1}{6}(k+1)(2k+7)$
 $\sum_{r=1}^{k+1} (r-1)(r+1) = \frac{1}{6}(k+1)(k+1-1)(2(2k+7)+5)$
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N} \Rightarrow$ THE RESULT ALSO HOLDS FOR $n=k+1$
 SINCE THE RESULT HOLDS FOR $n=1 \Rightarrow$ IT MUST HOLD $\forall n \in \mathbb{N}$

Question 4 (*)**

Prove by induction that

$$\sum_{r=2}^n r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2), \quad n \geq 2, \quad n \in \mathbb{N}.$$

 , proof

$\sum_{r=2}^n r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2), \quad k \geq 2$

BASE CASE, $n=2$

- LHS = $2^2(2-1) = 4$
- RHS = $\frac{1}{12} \times 2 \times 1 \times 3 \times 8 = 4$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$$\begin{aligned} &\sum_{r=2}^k r^2(r-1) = \frac{1}{12}k(k-1)(k+1)(3k+2) \\ &\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) + (k+1)^2(k+1) = \frac{1}{12}k(k-1)(k+1)(3k+2) + k(k+1)^2 \\ &\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)[(k-1)(3k+2) + 12(k+1)] \\ &\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)(3k^2 - k - 2 + 12k + 12) \\ &\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)(3k^2 + 11k + 10) \\ &\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)(3k+5)(k+2) \\ &\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}(k)(k+1)(k+2+1)[3(k+2)+2] \end{aligned}$$

CONCLUSION

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
- SINCE THE RESULT HOLDS FOR $n=2$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 5 (***)

Prove by induction that

$$1+8+27+64+\dots+n^3 = \frac{1}{4}n^2(n+1)^2, \quad n \geq 1, \quad n \in \mathbb{N}.$$

proof

- IF $n=1$ $\frac{1}{4} \times 1^2 \times (1+1)^2 = 1$ → THE RESULT HOLDS FOR $n=1$
- SUPPOSE THE RESULT HOLDS FOR $n \in \mathbb{N}$

$$\begin{aligned} 1+8+27+\dots+n^3 &= \frac{1}{4}n^2(n+1)^2 \\ 1+8+27+\dots+(n+1)^3 &= \frac{1}{4}n^2(n+1)^2 + (n+1)^3 \\ 1+8+27+\dots+n^3+(n+1)^3 &= \frac{1}{4}n^2(n+1)^2 + (n+1)^3 \\ 1+8+27+\dots+n^3+(n+1)^3 &= \frac{1}{4}(n+1)^2(n+2)^2 \\ 1+8+27+\dots+n^3+(n+1)^3 &= \frac{1}{4}(n+1)^2(n+2)^2 \end{aligned}$$

If the result holds for $n \in \mathbb{N} \Rightarrow$ the result holds for $n+1$
since the result holds for $n \Rightarrow$ the result holds $\forall n \in \mathbb{N}$

Question 6 (***)

Prove by induction that

$$\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1), \quad n \geq 1, \quad n \in \mathbb{N}.$$

proof

$$\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}(2n-1)(2n+1)$$

- IF $n=1$ $LHS = \sum_{r=1}^1 (2r-1)^2 = 1^2 = 1$ $RHS = \frac{1}{3} \times 1 \times 3 = 1$ If result holds for $n=1$
- SUPPOSE THE RESULT HOLDS FOR $n \in \mathbb{N}$

$$\begin{aligned} \sum_{r=1}^{n+1} (2r-1)^2 &= \frac{1}{3}(2n-1)(2n+1) + (2(n+1)-1)^2 \\ \sum_{r=1}^n (2r-1)^2 + (2(n+1)-1)^2 &= \frac{1}{3}(2n-1)(2n+1) + (2(n+1)-1)^2 \\ (2n-1)^2 &= \frac{1}{3}(2n-1)(2n+1) + (2(n+1)-1)^2 \\ (2n-1)^2 &= \frac{1}{3}(2n-1)[(2n+1) + 3(2n+1)] \\ (2n-1)^2 &= \frac{1}{3}(2n-1)(2n^2+6n+3) \\ (2n-1)^2 &= \frac{1}{3}(2n-1)(2n+1)(2n+3) \\ \sum_{r=1}^{n+1} (2r-1)^2 &= \frac{1}{3}(2n-1)(2n+1)(2n+3) \end{aligned}$$

If the result holds for $n \in \mathbb{N} \Rightarrow$ it also holds for $n+1$
since the result holds for $n \Rightarrow$ it must hold for $n+1$

Question 7 (*)**

Prove by induction that

$$\sum_{r=1}^n r(3r-1) = n^2(n+1), \quad n \geq 1, \quad n \in \mathbb{N}.$$

proof

$\sum_{r=1}^n r(3r-1) = n^2(n+1)$

- If $n=1$ LHS = $\sum_{r=1}^1 r(3r-1) = 1 \times 2 = 2$ } \Rightarrow result holds for $n=1$
RHS = $1^2(1+1) = 2$
- SURFACE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$
 $\sum_{r=1}^k r(3r-1) = k^2(k+1)$
 $\sum_{r=1}^{k+1} r(3r-1) = \sum_{r=1}^k r(3r-1) + (k+1)[3(k+1)-1] = k^2(k+1) + (k+1)[3(k+1)-1]$
 $\sum_{r=1}^{k+1} r(3r-1) = k^2(k+1) + (k+1)(3k+2)$
 $\sum_{r=1}^{k+1} r(3r-1) = (k+1)(k+1)(3k+2)$
 $\sum_{r=1}^{k+1} r(3r-1) = (k+1)^2(3k+2)$
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N} \Rightarrow$ result holds for $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1 \Rightarrow$ THE RESULT MUST HOLD FOR $n \in \mathbb{N}$

Question 8 (*)**

Prove by induction that

$$\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

proof

$\sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1}$

- If $n=1$ LHS = $\sum_{r=1}^1 \frac{1}{r(r+1)} = \frac{1}{1 \times 2} = \frac{1}{2}$ } \Rightarrow result holds for $n=1$
RHS = $\frac{1}{1+1} = \frac{1}{2}$
- SURFACE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$
 $\sum_{r=1}^k \frac{1}{r(r+1)} = \frac{k}{k+1}$
 $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} + \dots + \frac{1}{(k+1)(k+2)}$
 $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k(k+1)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$
 $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+1+1}$
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N} \Rightarrow$ it also holds for $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1 \Rightarrow$ IT MUST ALSO HOLD FOR $n \in \mathbb{N}$

Question 9 (*)**

Prove by induction that

$$\sum_{r=1}^n \left(3^{r-1}\right) = \frac{3^n - 1}{2}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

[proof]

• BASE CASE, $n=1$

$$\text{LHS} = \sum_{r=1}^1 \left(3^{r-1}\right) = \frac{3^1 - 1}{2} = 1$$

$$\text{RHS} = \frac{3^1 - 1}{2} = 1$$

RESULT HOLDS FOR $n=1$

• PROVE THAT THE RESULT HOLDS FOR $n+1 \in \mathbb{N}$

$$\begin{aligned} &\rightarrow \sum_{r=1}^{n+1} \left(3^{r-1}\right) = \frac{3^{n+1} - 1}{2} \\ &\rightarrow \left[\sum_{r=1}^n \left(3^{r-1}\right) + 3^{n+1-1} \right] = \frac{3^{n+1} - 1}{2} + 3^n \\ &\rightarrow \sum_{r=1}^n \left(3^{r-1}\right) = \frac{3^{n+1} - 1}{2} + 3^n \\ &\rightarrow \sum_{r=1}^n \left(3^{r-1}\right) = \frac{3^{n+1} - 1}{2} + 2 \times 3^n \\ &\rightarrow \sum_{r=1}^n \left(3^{r-1}\right) = \frac{3 \times 3^n - 1}{2} \\ &\rightarrow \sum_{r=1}^n \left(3^{r-1}\right) = \frac{3^n - 1}{2} \end{aligned}$$

• IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$.
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD, FOR ALL $n \in \mathbb{N}$.

Question 10 (*)+**

Prove by induction that

$$\sum_{r=1}^n \frac{r}{2^r} = 2 - \frac{n+2}{2^n}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

 , proof

START WITH THE BASE CASE, $n=1$:

$$\text{L.H.S.} = \sum_{r=1}^1 \left(\frac{r}{2^r}\right) = \frac{1}{2^1} = \frac{1}{2} \quad \text{R.H.S.} = 2 - \frac{1+2}{2^1} = 2 - \frac{3}{2} = \frac{1}{2}$$

THE RESULT HOLDS FOR $n=1$.

SUPPOSE THAT THE RESULT HELDS FOR $n=k$ ($k \in \mathbb{N}$)

$$\Rightarrow \sum_{r=1}^k \left(\frac{r}{2^r}\right) = 2 - \frac{k+2}{2^k}$$

$$\Rightarrow \left[\sum_{r=1}^{k+1} \left(\frac{r}{2^r}\right) \right] + \frac{k+1}{2^{k+1}} = 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$$

$$\Rightarrow \sum_{r=1}^{k+1} \left(\frac{r}{2^r}\right) = 2 + \left[\frac{k+1}{2^k} - \frac{k+2}{2^{k+1}} \right] = 2 + \left[\frac{(k+1)-2(k+2)}{2^{k+1}} \right]$$

$$\Rightarrow \sum_{r=1}^{k+1} \left(\frac{r}{2^r}\right) = 2 + \frac{-k-3}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$$

$$\Rightarrow \sum_{r=1}^{k+1} \left(\frac{r}{2^r}\right) = 2 - \frac{(k+3)+2}{2^{k+1}}$$

IF THE RESULT HOLDS FOR $n=k$ ($k \in \mathbb{N}$), THEN IT MUST ALSO HOLD FOR $n=k+1$.
SINCE THE RESULT HELDS TRUE FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 11 (***)+

Prove by induction that

$$\sum_{r=1}^n \frac{1}{4r^2-1} = \frac{n}{2n+1}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

 proof

$$\sum_{r=1}^n \left(\frac{1}{4r^2-1} \right) = \frac{n}{2n+1}$$

- TESTING THE BASE CASE, i.e. $n=1$
 $LHS = \sum_{r=1}^1 \frac{1}{4r^2-1} = \frac{1}{4(1)^2-1} = \frac{1}{3}$
 $RHS = \frac{1}{2(1)+1} = \frac{1}{3}$
 ∴ THE RESULT HOLDS FOR $n=1$
- SUPPOSE THAT THE RESULT HOLDS FOR $n=k \in \mathbb{N}$
 $\sum_{r=1}^k \left(\frac{1}{4r^2-1} \right) = \frac{k}{2k+1}$
 $\sum_{r=1}^{k+1} \left(\frac{1}{4r^2-1} \right) = \frac{1}{4(k+1)^2-1} = \frac{k}{2k+1} + \frac{1}{4(k+1)^2-1}$
 $\sum_{r=1}^{k+1} \left(\frac{1}{4r^2-1} \right) = \frac{k}{2k+1} + \frac{1}{[2(k+1)][2(k+1)-1]}$
 $\sum_{r=1}^{k+1} \left(\frac{1}{4r^2-1} \right) = \frac{k}{2k+1} + \frac{1}{(2k+2)(2k)}$
 $\sum_{r=1}^{k+1} \left(\frac{1}{4r^2-1} \right) = \frac{k(2k+2) + 1}{(2k+2)(2k+1)} = \frac{2k^2+3k+1}{(2k+2)(2k+1)} = \frac{(2k+1)(2k+2)}{(2k+2)(2k+1)} = \frac{k+1}{2k+3} = \frac{k+1}{2(k+1)+1}$
 • IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
 SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 12 (***)+

Prove by induction that

$$\sum_{r=1}^n r \times 2^r = 2 + (n-1)2^{n+1}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

 proof

$$\sum_{r=1}^n r \times 2^r = 2 + (n-1)2^{n+1}$$

- BASE CASE $n=1$
 $LHS = 1 \times 2^1 = 2$
 $RHS = 2 + (1-1) \times 2^2 = 2,$ } BUT IT'S $n=1$
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$
 $\sum_{r=1}^k r \times 2^r = 2 + (k-1)2^{k+1}$
 $\Rightarrow \sum_{r=1}^k r \times 2^r = 2 + (k-1)2^{k+1}$
 $\Rightarrow \sum_{r=1}^k r \times 2^r + (k+1) \times 2^{k+1} = 2 + (k-1)2^{k+1} + (k+1) \times 2^{k+1}$
 $\Rightarrow \sum_{r=1}^{k+1} r \times 2^r = 2 + 2^{k+1}((k-1)+(k+1))$
 $\Rightarrow \sum_{r=1}^{k+1} r \times 2^r = 2 + 2^{k+2}k$
 $\Rightarrow \sum_{r=1}^{k+1} r \times 2^r = 2 + 2 \times 2^k k$
 • IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
 SINCE IT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n

Question 13 (*)+**

Prove by induction that

$$\sum_{r=1}^n [(r+1) \times 2^r] = n \times 2^n, \quad n \geq 1, \quad n \in \mathbb{N}.$$

[proof]

• IF $n=1$ $\text{LHS} = (1+1) \times 2^{1-1} = 2 \times 1 = 2$ $\text{RHS} = 1 \times 2^1 = 2$ $\left\{ \begin{array}{l} \text{LHS} = \text{RHS} \\ \text{RHS} \in \mathbb{N} \end{array} \right.$

• SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\sum_{r=1}^k (r+1) 2^{r-1} = k \times 2^k$$

$$\left[\sum_{r=1}^k (r+1) 2^{r-1} \right] + (k+2) 2^k = (k \times 2^k) + (k+2) 2^k$$

$$\sum_{r=1}^{k+1} (r+1) 2^{r-1} = 2^k [k+k+2]$$

$$\sum_{r=1}^{k+1} (r+1) 2^{r-1} = 2^k (2k+2) = 2k 2^k + 2^k$$

$$\sum_{r=1}^{k+1} (r+1) 2^{r-1} = 2^{k+1} (k+1) = (k+1) 2^{k+1}$$

• IF THE RESULT HOLDS FOR $k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$.
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 14 (*)+**

If $n \geq 1$, $n \in \mathbb{N}$, prove by induction that

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1.$$

[proof]

$$(1 \times 1!) + (2 \times 2!) + (3 \times 3!) + \dots + (n \times n!) = (n+1)! - 1$$

$$\sum_{r=1}^n r r! = (n+1)! - 1$$

• IF $n=1$ $\text{LHS} = 1 \times 1! = 1$ $\text{RHS} = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$
 \therefore RESULT HOLDS FOR $n=1$

• SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\sum_{r=1}^k r r! = (k+1)! - 1$$

$$\Rightarrow \sum_{r=1}^k r r! + (k+1) \times (k+1)! = (k+1)! - 1 + (k+1) \times (k+1)!$$

$$\Rightarrow \sum_{r=1}^{k+1} r r! = (k+1)! [1 + (k+1)] - 1$$

$$\Rightarrow \sum_{r=1}^{k+1} r r! = (k+1)! (k+2) - 1$$

$$\Rightarrow \sum_{r=1}^{k+1} r r! = (k+2)! - 1$$

$$\Rightarrow \sum_{r=1}^{k+1} r r! = (k+1+1)! - 1$$

• IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$.
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 15 (*)+**

Prove by induction that

$$\sum_{r=1}^n r \times 2^{-r} = 2 - (n+2)2^{-n}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

[proof]

$\sum_{r=1}^n (r \times 2^{-r}) = 2 - (n+2)2^{-n}$

• IF $n=1$, $1 \times 2^{-1} = \frac{1}{2} = 2 - (1+2)2^{-1} = 2 - 3 \times \frac{1}{2} = \frac{1}{2}$ \Rightarrow RESULT TRUE FOR $n=1$

• SUPPOSE THE RESULT TRUE FOR $n=k \in \mathbb{N}$

$$\sum_{r=1}^k (r \times 2^{-r}) = 2 - (k+2)2^{-k}$$

$$\sum_{r=1}^{k+1} (r \times 2^{-r}) + (k+1)2^{-(k+1)} = 2 - (k+2)2^{-k} + (k+1)2^{-(k+1)}$$

$$\sum_{r=1}^{k+1} (r \times 2^{-r}) = 2 - (k+2)2^{-k} + (k+1)2^{-k} + (k+1)2^{-(k+1)}$$

$$\sum_{r=1}^{k+1} (r \times 2^{-r}) = 2 - (k+2)2^{-k} + \frac{1}{2}(k+1)2^{-k}$$

$$\sum_{r=1}^{k+1} (r \times 2^{-r}) = 2 + \frac{1}{2}2^{-k} \left[(k+1) - 2(k+2) \right]$$

$$\sum_{r=1}^{k+1} (r \times 2^{-r}) = 2 + \frac{1}{2}2^{-k} [-k-3]$$

$$\sum_{r=1}^{k+1} (r \times 2^{-r}) = 2 - 2^{-(k+1)}(k+3)$$

$$\sum_{r=1}^{k+1} (r \times 2^{-r}) = 2 - 2^{-(k+1)}[(k+3)+2]$$

• IF THE RESULT TRUE FOR $n=k \in \mathbb{N}$, THEN IT MUST ALSO TRUE FOR $n=k+1$
SINCE THE RESULT HAS TO BE TRUE FOR $\forall n \in \mathbb{N}$

Question 16 (*****)

Prove by induction that

$$\sum_{r=1}^n \frac{2r^2 - 1}{r^2(r+1)^2} = \frac{n^2}{(n+1)^2}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

 , proof

BASE CASE : $n=1$

- LHS = $\frac{2 \cdot 1^2 - 1}{1^2(1+1)^2} = \frac{1}{4}$
- RHS = $\frac{1^2}{(1+1)^2} = \frac{1}{4}$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$$\begin{aligned} &\textcircled{1} \quad \sum_{r=1}^k \frac{2r^2 - 1}{r^2(r+1)^2} = \frac{k^2}{(k+1)^2} \\ &\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2 - 1}{r^2(r+1)^2} + \frac{2(k+1)^2 - 1}{(k+1)^2(k+2)^2} = \frac{k^2}{(k+1)^2} + \frac{2(k+1)^2 - 1}{(k+1)^2(k+2)^2} \\ &\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2 - 1}{r^2(r+1)^2} = \frac{k^2(k+2)^2 + 2(k+1)^2 - 1}{(k+1)^2(k+2)^2} \\ &\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2 - 1}{r^2(r+1)^2} = \frac{k^2(k+2)^2 + 2k^4 + 4k^2 + 1}{(k+1)^2(k+2)^2} \\ &\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2 - 1}{r^2(r+1)^2} = \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{(k+1)^2(k+2)^2} \\ &\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2 - 1}{r^2(r+1)^2} = \frac{(k+1)^2}{(k+2)^2} \end{aligned}$$

BY INSPECTION

$$(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$$

RETURNING TO THE MAIN LINE

$$\begin{aligned} &\sum_{r=1}^{k+1} \frac{2r^2 - 1}{r^2(r+1)^2} = \frac{(k+1)^4}{(k+1)^2(k+2)^2} \\ &\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2 - 1}{r^2(r+1)^2} = \frac{(k+1)^2}{(k+2)^2} \\ &\Rightarrow \textcircled{2} \quad \sum_{r=1}^{k+1} \frac{2r^2 - 1}{r^2(r+1)^2} = \frac{(k+1)^2}{(k+2)^2} \end{aligned}$$

(CONCLUSION)

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$.
- SINCE THE RESULT HOLDS FOR $n=2$, THEN IT MUST HOLD FOR ALL n .

Question 17 (**)**

Prove by induction that

$$\sum_{r=1}^n [r(r-1)-1] = \frac{1}{3}n(n+2)(n-2), \quad n \geq 1, \quad n \in \mathbb{N}.$$

[proof]

① LHS = $\sum_{k=1}^n [(k(k-1)-1)] = -1$

RHS = $\frac{1}{3}n(n+2)(n-2) = -1$

\therefore Equality holds for $n=1$.

② SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$,

$$\sum_{r=1}^k [r(r-1)-1] = \frac{1}{3}k(k+2)(k-2)$$

$$\sum_{r=1}^{k+1} [r(r-1)-1] + [(k+1)(k+1)-1] = \frac{1}{3}(k+1)(k+3)(k-2) + [(k+1)^2-1]$$

$$\sum_{r=1}^{k+1} [r(r-1)-1] = \left[\frac{1}{3}k(k+2)(k-2) + k^2+k-1 \right]$$

$$= \frac{1}{3}[k^3-4k^2+3k-3]$$

$$= \frac{1}{3}[k^2(3k-4)-k-3]$$

$$= \frac{1}{3}[k(k+3)(3k-2)]$$

$$= \frac{1}{3}(k+3)(k+1)(k-1)$$

$$= \frac{1}{3}(k+1)(k+2)(k-1)$$

$$\sum_{r=1}^{k+1} [r(r-1)-1] = \frac{1}{3}(k+1)(k+2)(k-1)$$

③ IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$.
SINCE IT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 18 (****)

Prove by induction that

$$\sum_{r=1}^n \frac{r \times 2^r}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

 proof

SUPPOSE THAT THE RESULT HOLDS FOR $n=1$

$$L.H.S = \sum_{r=1}^1 \frac{r \times 2^r}{(r+2)!} = \frac{1 \times 2^1}{3!} = \frac{2}{6} = \frac{1}{3}$$

$$R.H.S = 1 - \frac{2^2}{3!} = 1 - \frac{4}{6} = 1 - \frac{2}{3} = \frac{1}{3}$$

LC RESULT HOLDS FOR $n=1$

NEXT SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$,

$$\sum_{r=1}^k \left[\frac{r \times 2^r}{(r+2)!} \right] = 1 - \frac{2^{k+1}}{(k+2)!}$$

$$\sum_{r=1}^{k+1} \left[\frac{r \times 2^r}{(r+2)!} \right] + \frac{(k+1) \times 2^{k+1}}{(k+3)!} = 1 - \frac{2^{k+1}}{(k+2)!} + \frac{(k+1) \times 2^{k+1}}{(k+3)!}$$

$$\sum_{r=1}^{k+1} \left[\frac{r \times 2^r}{(r+2)!} \right] = 1 - \frac{(k+2) \times 2^{k+1}}{(k+2)!} + \frac{(k+1) \times 2^{k+1}}{(k+3)!}$$

$$\sum_{r=1}^{k+1} \left[\frac{r \times 2^r}{(r+2)!} \right] = 1 - \frac{(k+2) \times 2^{k+1}}{(k+3)!} + \frac{(k+1) \times 2^{k+1}}{(k+3)!}$$

$$\sum_{r=1}^{k+1} \left[\frac{r \times 2^r}{(r+2)!} \right] = 1 + \frac{(k+1) \times 2^{k+1} - (k+2) \times 2^{k+1}}{(k+3)!}$$

$$\sum_{r=1}^{k+1} \left[\frac{r \times 2^r}{(r+2)!} \right] = 1 - \frac{-2 \times 2^{k+1}}{(k+3)!} = 1 - \frac{2^{k+2}}{(k+3)!} = 1 - \frac{2^{(k+1)+1}}{(k+3)!+1}$$

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$

SINCE THE RESULT HOLDS FOR $n=1$, THEN THE RESULT HOLDS FOR ALL n

Question 19 (****)

Prove by induction that

$$1^2 + 3^2 + 5^2 + 7^2 + \dots + (2n-1)^2 \equiv \frac{1}{3}n(4n^2-1), \quad n \geq 1, \quad n \in \mathbb{N}.$$

 proof

- IF $n=1$ $\frac{1}{3} \times 1 \times (4 \times 1^2 - 1) = \frac{1}{3} \times (4-1) = 1$ WHICH IS 1^2 .
LC RESULT HOLDS FOR $n=1$
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{1}{3}k(4k^2-1)$$

$$\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + (2k+1)^2 = \frac{1}{3}k(4k^2-1) + (2k+1)^2$$

$$\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(2k+1)[4(k+1)^2 - 1]$$

$$\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(2k+1)(4k^2+8k+3)$$

$$\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(2k+1)(2k^2+4k+3)$$

$$\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(2k+1)(2k+1)(k+1)$$

$$\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(k+1)[4k^2+8k+3]$$

$$\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(k+1)[4(k+2)^2 - 1]$$

$$\Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{1}{3}(k+1)[4(2k+1)^2 - 1]$$

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$ \Rightarrow IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$ \Rightarrow THE RESULT HOLDS FOR ALL $n \in \mathbb{N}$

Question 20 (*****)

Prove by mathematical induction that if n is a positive integer then

$$\sum_{r=1}^n (3r-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1).$$

You may not use other methods of proof in this question.

, proof

ESTABLISH A BASE CASE FOR $n=1$

- LHS = $\sum_{r=1}^1 (3r-2)^2 = (3 \times 1 - 2)^2 = 1$
- RHS = $\frac{1}{2} \times 1 \times (6 \times 1^2 - 3 \times 1 - 1) = \frac{1}{2} \times (6 - 3 - 1) = \frac{1}{2} \times 2 = 1$

I.E THE EQUATION HOLDS FOR $n=1$.

SUPPOSE THAT THE RESULT HOLDS FOR $n_k \in \mathbb{N}$

$$\begin{aligned} \Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 &= \frac{1}{2}k(6k^2 - 3k + 1) \\ \Rightarrow \sum_{r=1}^k (3r-2)^2 + [3(k+1)-2]^2 &= \frac{1}{2}k(6k^2 - 3k + 1) + [3(k+1)-2]^2 \\ \Rightarrow \sum_{r=1}^k (3r-2)^2 &= \frac{1}{2}k(6k^2 - 3k + 1) + (3k+1)^2 \\ \Rightarrow \sum_{r=1}^k (3r-2)^2 &= \frac{1}{2}k(6k^2 - 3k + 1) + 9k^2 + 6k + 1 \\ \Rightarrow \sum_{r=1}^k (3r-2)^2 &= \frac{1}{2}k(6k^2 - 3k + 1) + 9k^2 + 6k + 1 \\ \Rightarrow \sum_{r=1}^k (3r-2)^2 &= \frac{1}{2}k(6k^2 - 3k + 1) + \frac{1}{2}(6k^3 + 12k^2 + 12k + 2) \\ \Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 &= \frac{1}{2}(k+1)\left[\frac{6k^3 + 12k^2 + 12k + 2}{k+1}\right] \\ \text{BY LONG DIVISION OR MANIPULATION} \end{aligned}$$

$$\begin{aligned} \Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 &= \frac{1}{2}(k+1)\left[\frac{6k^2 + 12k + 12}{k+1}\right] \\ \Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 &= \frac{1}{2}(k+1)(6k^2 + 12k + 12) \end{aligned}$$

MANIPULATE FURTHER

$$\begin{aligned} \Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 &= \frac{1}{2}(k+1)(6k^2 + 12k + 12) \\ \Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 &= \frac{1}{2}(k+1)[(6k+1)^2 - 6 + 12k + 2] \\ \Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 &= \frac{1}{2}(k+1)[6(6k+1)^2 - 3k - 4] \\ \Rightarrow \sum_{r=1}^{k+1} (3r-2)^2 &= \frac{1}{2}(k+1)[6(6k+1)^2 - 3(6k+1) - 4] \end{aligned}$$

IF THE RESULT HOLDS FOR $n=k+1$, THEN IT MUST HOLD FOR $n=k+2$.

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n .

Question 21 (***)**

Prove by mathematical induction that if n is a positive integer then

$$\sum_{r=1}^n \frac{3r+2}{r(r+1)(r+2)} = \frac{n(2n+3)}{(n+1)(n+2)}.$$

You may not use other methods of proof in this question.

, proof

ESTABLISH A BASE CASE:

$$\text{LHS} = \sum_{r=1}^1 \frac{3r+2}{r(r+1)(r+2)} = \frac{3(1)+2}{1(1+1)(1+2)} = \frac{5}{6}$$

$$\text{RHS} = \frac{1(2(1)+3)}{(1+1)(1+2)} = \frac{5}{6}$$

} THE RESULT HOLDS FOR $n=1$.

SUPPOSE THAT THE RESULT HOLDS FOR $n=k$ THEN

$$\begin{aligned} \Rightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} &= \frac{k(2k+3)}{(k+1)(k+2)} \\ \Rightarrow \sum_{r=1}^k \frac{3r+2}{r(r+1)(r+2)} + \frac{3(k+1)+2}{(k+1)(k+2)(k+3)} &= \frac{k(2k+3)}{(k+1)(k+2)} + \frac{3(k+1)+2}{(k+1)(k+2)(k+3)} \\ \Rightarrow \sum_{r=1}^k \frac{3r+2}{r(r+1)(r+2)} &= \frac{k(2k+3)}{(k+1)(k+2)(k+3)} + \frac{3k+5}{(k+1)(k+2)(k+3)} \\ \Rightarrow \sum_{r=1}^k \frac{3r+2}{r(r+1)(r+2)} &= \frac{k(2k+3) + (3k+5)}{(k+1)(k+2)(k+3)} \\ \Rightarrow \sum_{r=1}^k \frac{3r+2}{r(r+1)(r+2)} &= \frac{2k^2+9k+7}{(k+1)(k+2)(k+3)} \\ \Rightarrow \sum_{r=1}^k \frac{3r+2}{r(r+1)(r+2)} &= \frac{2k^2+9k+7+k+5}{(k+1)(k+2)(k+3)} \end{aligned}$$

NOW WE "EXPECT" THAT (E.H.) IS A PRIME FOR THE INDUCTION TO WORK

$$\begin{aligned} \Rightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} &= \frac{2k^2+9k+7+(k+1)}{(k+1)(k+2)(k+3)} \\ \Rightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} &= \frac{(k+1)(2k^2+9k+7)}{(k+1)(k+2)(k+3)} \end{aligned}$$

$$\begin{aligned} \Rightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} &= \frac{2k^2+9k+7}{(k+1)(k+2)(k+3)} \\ \Rightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} &= \frac{(k+1)(2k^2+9k+7)}{(k+1)(k+2)(k+3)} \\ \Rightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} &= \frac{(k+1)[2k^2+9k+7]}{(k+1)(k+2)(k+3)} \end{aligned}$$

IF THE RESULT HOLDS FOR $n=k+1$, THEN IT MUST ALSO HOLD FOR $n=k+2$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n

Question 22 (***)**

Prove by induction that

$$\sum_{r=1}^n \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 16 - \left(\frac{1}{2} \right)^{n-1} (n^2 + 5n + 8), \quad n \geq 1, \quad n \in \mathbb{N}.$$

 , proof

<p><u>BASE CASE: $n=1$</u></p> $\sum_{r=1}^1 \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 1 \times 2 \times \left(\frac{1}{2} \right)^0 = 2$ $16 - \left(\frac{1}{2} \right)^0 (1^2 + 5 \times 1 + 8) = 16 - 1 \times 14 = 2$ <p><u>IF THE RESULT HOLDS FOR $n=1$</u></p> <p><u>SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$</u></p> $\sum_{r=1}^k \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 16 - \left(\frac{1}{2} \right)^k (k^2 + 5k + 8)$ $\Rightarrow \sum_{r=1}^{k+1} \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = (k+1)(k+2) \left(\frac{1}{2} \right)^k + \sum_{r=1}^k \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right]$ $\Rightarrow \sum_{r=1}^{k+1} \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 16 + \left(\frac{1}{2} \right)^k ((k+1)(k+2) - (k^2 + 5k + 8))$ $\Rightarrow \sum_{r=1}^{k+1} \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 16 + \left(\frac{1}{2} \right)^k \left[((k+1)k)2 - \left(\frac{1}{2} \right)^k (k^2 + 5k + 8) \right]$ $\Rightarrow \sum_{r=1}^{k+1} \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 16 + \left(\frac{1}{2} \right)^k \left[k^2 + 3k + 2 - 2(k^2 + 5k + 8) \right]$ $\Rightarrow \sum_{r=1}^{k+1} \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 16 + \left(\frac{1}{2} \right)^k \left[k^2 + 3k + 2 - 2k^2 - 10k - 16 \right]$ $\Rightarrow \sum_{r=1}^{k+1} \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 16 + \left(\frac{1}{2} \right)^k \left[-k^2 - 7k - 16 \right]$ $\Rightarrow \sum_{r=1}^{k+1} \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 16 - \left(\frac{1}{2} \right)^{k+1} (k^2 + 7k + 16)$	$\Rightarrow \sum_{r=1}^{k+1} \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 16 - \left(\frac{1}{2} \right)^k \left[\underbrace{(k^2 + 7k + 16)}_{k^2 + 7k + 16} + 8(k+1) + 8 \right]$ $\Rightarrow \sum_{r=1}^{k+1} \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 16 - \left(\frac{1}{2} \right)^k \left[(k+1)^2 + 8(k+1) + 8 \right]$ $\Rightarrow \sum_{r=1}^{k+1} \left[r(r+1) \left(\frac{1}{2} \right)^{r-1} \right] = 16 - \left(\frac{1}{2} \right)^{k+1} (k^2 + 14k + 24)$ <p><u>IF THE RESULT HOLDS FOR $n=k+1$, THEN IT MUST ALSO HOLD FOR $n=k+2$ — SINCE THE RESULT HELD FOR $n=1$</u></p> <p><u>THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$</u></p>
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DIVISIBILITY RESULTS

Question 1 ()**

$$f(n) = 7^n + 5, \quad n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 6, for all $n \in \mathbb{N}$.

[proof]

$f(x) = 7^x + 5$

- $f(0) = 7^0 + 5 = 12$ WHICH IS DIVISIBLE BY 6
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, i.e. $f(k) = 6m$, $m \in \mathbb{N}$
- $f(k+1) - f(k) = [7^{k+1} + 5] - [7^k + 5]$
- $f(k+1) - 6m = 7^{k+1} - 7^k$
- $f(k+1) - 6m = 7^k(7 - 1)$
- $f(k+1) - 6m = 6m + 6 \times 7^k$
- $f(k+1) = 6[m + 7^k]$
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST HOLD FOR $n=k+1$. SINCE THIS RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 2 ()**

$$f(n) = 6^n + 4, \quad n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 5, for all $n \in \mathbb{N}$.

[proof]

$f(x) = 6^x + 4$

- $f(0) = 6^0 + 4 = 10 = 5 \times 2$ IT PROVES IT HOLDS FOR $x=1$
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, i.e. $f(k) = 5m$, WHERE $m \in \mathbb{N}$
- $\Rightarrow f(k+1) - f(k) = [6^{k+1} + 4] - [6^k + 4]$
- $\Rightarrow f(k+1) - 5m = 6^{k+1} - 6^k$
- $\Rightarrow f(k+1) - 5m = 6 \times 6^k - 6^k$
- $\Rightarrow f(k+1) - 5m = 5 \times 6^k$
- $\Rightarrow f(k+1) = 5[m + 6^k]$
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$. SINCE IT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 3 ()**

$$f(n) = 5^n + 3, \quad n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 4, for all $n \in \mathbb{N}$.

proof

$f(1) = 5^1 + 3 = 8$, i.e. divisible by 4

- SUPPOSE THE RESULT HOLDS FOR $n \in \mathbb{N}$, so $f(n) = 4m, m \in \mathbb{Z}$

$$\Rightarrow f((n+1)) - f(n) = [5^{n+1} + 3] - [5^n + 3]$$

$$\Rightarrow f(n+1) - 4m = 5^n \cdot 5 - 5^n$$

$$\Rightarrow f(n+1) = 4m + 4 \cdot 5^n$$

$$\Rightarrow f(n+1) = 4m + 4 \cdot 4m$$

$$\Rightarrow f(n+1) = 4[m + 5^n]$$

• IF THE RESULT HOLDS FOR $n \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n+1$.
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 4 ()**

Prove by induction that for all natural numbers n ,

$$4^{2n} - 1$$

is divisible by 15.

proof

$f(n) = 4^{2n} - 1, \quad n \in \mathbb{N}$

BASE CASE, $n=1$
 $f(1) = 4^{2 \cdot 1} - 1 = 15$, i.e. THE RESULT HOLDS FOR $n=1$

INDUCTIVE HYPOTHESIS
 SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$, i.e. $f(k) = 15m$ WHERE $m \in \mathbb{N}$

$$\Rightarrow f(k+1) - f(k) = [4^{2(k+1)} - 1] - [4^{2k} - 1]$$

$$\Rightarrow f(k+1) - 15m = 4^{2(k+1)} - 4^{2k}$$

$$\Rightarrow f(k+1) - 15m = 4^{2k+2} - 4^{2k}$$

$$\Rightarrow f(k+1) - 15m = 4^2 \times 4^{2k} - 1 \cdot 4^{2k}$$

$$\Rightarrow f(k+1) = 15m + 16 \times 4^{2k} - 4^{2k}$$

$$\Rightarrow f(k+1) = 15m + 15 \times 4^{2k}$$

$$\Rightarrow f(k+1) = 15[m + 4^{2k}]$$

CONCLUSION
 • IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$.
 • SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 5 ()**

Prove by induction that for all natural numbers n ,

$$7^{2n-1} + 1$$

is divisible by 8.

[proof]

Let $f(x) = 7^{2x-1} + 1$

- $f(1) = 7^1 + 1 = 8$ i.e. divisible by 8
- Suppose the result holds for $n \in \mathbb{N}$, i.e. $f(n) = 8m$, $m \in \mathbb{N}$

$$f(n+1) - f(n) = [7^{2n+1} + 1] - [7^{2n-1} + 1]$$

$$f(n+1) - 8m = 7^{2n+1} - 7^{2n-1}$$

$$f(n+1) = 8m + 7^{2n+1} - 7^{2n-1}$$

$$f(n+1) = 8m + 49 \times 7^{2n-1} - 7^{2n-1}$$

$$f(n+1) = 8m + 48 \times 7^{2n-1}$$

$$f(n+1) = 8[m + 6 \times 7^{2n-1}]$$

- If the result holds for $n \in \mathbb{N}$, then it must also hold for $n+1$.
Since the result holds for $n=1$, then it must hold $\forall n \in \mathbb{N}$.

Question 6 ()**

Prove by induction that for all natural numbers n ,

$$3^{2n} + 7 \text{ is divisible by 8.}$$

[proof]

Let $f(x) = 3^{2x} + 7$

- $f(1) = 3^2 + 7 = 9 + 7 = 16$ is divisible by 8
- Suppose that the result holds for $n \in \mathbb{N}$, i.e. $f(n) = 8m$ for $m \in \mathbb{Z}$

$$f(n+1) - f(n) = [3^{2n+2} + 7] - [3^{2n} + 7]$$

$$f(n+1) - 8m = 3^{2n+2} - 3^{2n}$$

$$f(n+1) = 8m + 3^2 \times 3^{2n} - 3^{2n}$$

$$f(n+1) = 8m + 9(3^{2n}) - 3^{2n}$$

$$f(n+1) = 8m + 8(3^{2n})$$

$$f(n+1) = 8[m + 3^{2n}]$$

- If the result holds for $n \in \mathbb{N}$, then it also holds for $n+1$.
Since the result holds for $n=1$, then it must hold $\forall n \in \mathbb{N}$.

Question 7 (**)

$$f(n) = 3^{2n} - 1, n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is a multiple of 8, for all $n \in \mathbb{N}$.

proof

$f(n) = 3^{2n} - 1$

- $f(0) = 3^0 - 1 = 0 = 8 \times 0$ IT IS A MULTIPLE OF 8
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, i.e. $f(k) = 8m, m \in \mathbb{N}$

$$\begin{aligned} & \rightarrow f(k+1) - f(k) = [3^{2(k+1)} - 1] - [3^{2k} - 1] \\ & \Rightarrow f(k+1) - 8m = 3^{2k+2} - 3^{2k} \\ & \Rightarrow f(k+1) - 8m = 3^2 \cdot 3^{2k} - 3^{2k} \\ & \Rightarrow f(k+1) - 8m = 9 \cdot 3^{2k} - 3^{2k} \\ & \Rightarrow f(k+1) = 8m + 3^{2k} [9 - 1] \\ & \Rightarrow f(k+1) = 8m + 8 \cdot 3^{2k} = 8[m + 3^{2k}] \text{ IT IS A MULTIPLE OF 8} \end{aligned}$$

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$ \Rightarrow IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=0 \Rightarrow$ IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 8 (**+)

$$f(n) = 4^n + 6n - 1, n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 3, for all $n \in \mathbb{N}$.

proof

$f(n) = 4^n + 6n - 1$

- $f(0) = 4^0 + 6 \cdot 0 - 1 = 9 - 1 = 8 = 3 \times 3$ IT IS A DIVISOR OF 3
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$ i.e. $f(k) = 3m, m \in \mathbb{N}$

$$\begin{aligned} & \rightarrow f(k+1) - f(k) = [4^{k+1} + 6(k+1) - 1] - [4^k + 6k - 1] \\ & \Rightarrow f(k+1) - 3m = 4^{k+1} - 4^k + 6k + 6 - 6k \\ & \Rightarrow f(k+1) - 3m = 4 \cdot 4^k - 4^k + 6 \\ & \Rightarrow f(k+1) = 3m + 3 \cdot 4^k + 6 \\ & \Rightarrow f(k+1) = 3[m + 4^k + 2] \end{aligned}$$

- AS THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
SINCE IT HOLDS FOR $n=0$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 9 (***)

$$f(n) = 5^n + 8n + 3, n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 4, for all $n \in \mathbb{N}$.

proof

$$\boxed{\begin{aligned} f(n) &= 5^n + 8n + 3 \\ \bullet f(0) &= 5^0 + 8(0) + 3 = 16, \text{ i.e. divisible by 4} \\ \bullet \text{SUPPOSE THE RESULT HOLDS FOR } n=k \in \mathbb{N}, \text{i.e. } f(k)=4m \text{ FOR SOME} \\ &\rightarrow f(k+1) - f(k) = [5^{k+1} + 8(k+1) + 3] - [5^k + 8k + 3] \\ &\Rightarrow f(k+1) - 4m = 5^{k+1} + 8k + 11 - 5^k - 8k - 3 \\ &\Rightarrow f(k+1) - 4m = 5^k + 8 + 1 \\ &\Rightarrow f(k+1) = 4m + 5 \cdot 5^k + 8 \\ &\Rightarrow f(k+1) = 4m + 4 \times 5^k + 8 \\ &\Rightarrow f(k+1) = 4(m + 5^k + 2), \text{ i.e. divisible by 4} \\ \bullet \text{IF THE RESULT HOLDS FOR } n=k \in \mathbb{N}, \text{ THEN THE RESULT HOLDS FOR } n=k+1. \\ \text{SINCE THE RESULT HOLDS FOR } n=1, \text{ THEN IT MUST HOLD FOR } n \in \mathbb{N}. \end{aligned}}$$

Question 10 (***)

$$f(n) = 3^{4n} + 2^{4n+2}, n \in \mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 5, for all $n \in \mathbb{N}$.

proof

$$\boxed{\begin{aligned} f(n) &= 3^{4n} + 2^{4n+2} \\ \bullet f(0) &= 3^{4 \cdot 0} + 2^{4 \cdot 0 + 2} = 81 + 4 = 85, \text{ i.e. divisible by 5} \\ \bullet \text{SUPPOSE THAT THE RESULT HOLDS FOR } n=k \in \mathbb{N}, \text{i.e. } f(k)=5m, m \in \mathbb{N} \\ &\rightarrow f(k+1) - f(k) = [3^{4(k+1)} + 2^{4(k+1)+2}] - [3^{4k} + 2^{4k+2}] \\ &\Rightarrow f(k+1) - 5m = 3^{4k+4} + 2^{4k+6} - 3^{4k} - 2^{4k+2} \\ &\Rightarrow f(k+1) - 5m = 3^4 \times 3^{4k} - 3^{4k} + 2^{4k+2} - 2^{4k+2} \\ &\Rightarrow f(k+1) = 5m + 80 \times 3^{4k} + 15 \times 2^{4k+2} \\ &\Rightarrow f(k+1) = 5[m + 16 \times 3^{4k} + 3 \times 2^{4k+2}] \text{ i.e. divisible by 5.} \\ \bullet \text{IF THE RESULT HOLDS FOR } n=k \in \mathbb{N}, \text{ THEN THE RESULT HOLDS FOR } n=k+1. \\ \text{SINCE THE RESULT HOLDS FOR } n=1, \text{ THEN IT MUST HOLD FOR } n \in \mathbb{N}. \end{aligned}}$$

Question 11 (+)**

Prove by induction that for all natural numbers n ,

$$9^n - 5^n$$

is divisible by 4.

[proof]

Let $f(n) = 9^n - 5^n$

- If $n=1$, $9^1 - 5^1 = 9 - 5 = 4$, i.e. THE RESULT HOLDS FOR $n=1$.
- SUPPOSE THE RESULT HOLDS FOR $n=k$ EN, i.e. $f(k) = 4m$ FOR $m \in \mathbb{N}$.
 $f(k+1) - f(k) = (9^{k+1} - 5^{k+1}) - (9^k - 5^k)$
 $f(k+1) - 4m = 9^k \cdot 9 - 5^k + 5^k - 5^{k+1}$
 $f(k+1) - 4m = 9(9^k - 5^k) - 5(5^k)$
 $f(k+1) - 4m = 8(9^k) - 4(5^k)$
 $f(k+1) = 4m + 8(9^k) - 4(5^k)$
 $f(k+1) = 4[m + 2(9^k) - 5^k]$ IS ALSO DIVISIBLE BY 4

• IF THE RESULT FAILS FOR $n=k+1$, THEN THE RESULT FAILS FOR $n=k+1$
 SINCE THE RESULT HOLDS FOR $n=k$, THEN THE RESULT MUST HOLD $\forall n \in \mathbb{N}$

Question 12 (+)**

$$f(n) = (4n+3)5^n - 3, n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 16, for all $n \in \mathbb{N}$.

[proof]

Let $f(n) = (4n+3)5^n - 3$

- $f(1) = 7 \times 5 - 3 = 32$, WHICH IS DIVISIBLE BY 16
- SUPPOSE THE RESULT HOLDS FOR $n=k$ EN, i.e. $f(k) = 16m$, $m \in \mathbb{N}$.
 $f(k+1) - f(k) = [(16k+7)5^{k+1}] - [(4k+3)5^k - 3]$
 $f(k+1) - 16m = (16k+7)5^{k+1} - (4k+3)5^k$
 $f(k+1) = (16m + 5(16k+7)5^k) - (4k+3)5^k$
 $f(k+1) = 16m + (16k+52)5^k$
 $f(k+1) = 16[m + (k+3)5^k]$ IS DIVISIBLE BY 16

• IF THE RESULT FAILS FOR $n=k+1$, THEN IT MUST FAIL FOR $n=k+1$
 SINCE THE RESULT HOLDS FOR $n=k$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 13 (*)**

Prove by induction that the sum of the cubes of any three consecutive positive integers is always divisible by 9.

, **proof**

$f(n) = n^3 + (n+1)^3 + (n+2)^3, \quad n \in \mathbb{N}$

BASE CASE, i.e. $f(1)$

$$f(1) = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36 \text{ IF DIVISIBLE BY } 9$$

INDUCTIVE HYPOTHESIS
SUPPOSE THAT THE RESULT HOLDS FOR $n=k, k \in \mathbb{N}$, i.e. $f(k) = 9A$ WHERE $A \in \mathbb{N}$.

$$\Rightarrow f(k+1) - f(k) = [(k+1)^3 + (k+2)^3 + (k+3)^3] - [k^3 + (k+1)^3 + (k+2)^3]$$

$$\Rightarrow f(k+1) - 9A = (k+1)^3 - k^3$$

$$\Rightarrow f(k+1) - 9A = (k^3 + 3k^2 + 3k + 1) - k^3$$

$$\Rightarrow f(k+1) = 9A + 3k^2 + 3k + 1$$

$$\Rightarrow f(k+1) = 9[4 + k^2 + 3k + 1]$$

CONCLUSION
IF THE RESULT HOLDS FOR $n=k, k \in \mathbb{N}$ THEN IT ALSO HOLDS FOR $n=k+1$. SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 14 (*)**

Prove by induction that for all natural numbers n , such that $n \geq 2$,

$$15^n - 8^{n-2},$$

is divisible by 7.

proof

$f(n) = 15^n - 8^{n-2}$

\bullet $f(2) = 15 - 8^0 = 225 - 1 = 224 = 7 \times 32$
IF DIVISIBLE FOR $n=2$.

SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$,
i.e. $f(k) = 7m$, $m \in \mathbb{N}$

$$\Rightarrow f(k+1) - f(k) = [15^{k+1} - 8^{k-1}] - [15^k - 8^{k-2}]$$

$$\Rightarrow f(k+1) - 7m = 15^{k+1} - 15^k - 8^{k-1} + 8^{k-2}$$

$$\Rightarrow f(k+1) - 7m = 15 \times 15^k - 8 \times 8^{k-2}$$

$$\Rightarrow f(k+1) - 7m = 14 \times 15^k - 7 \times 8^{k-2}$$

$$\Rightarrow f(k+1) = 7[m + 2 \times 15^k - 8^{k-2}]$$

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$.
SINCE IT HOLDS FOR $n=2$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$, $n \geq 2$.

Question 15 (*)**

Prove by induction that for all natural numbers n ,

$$(2n+1)7^n + 11,$$

is divisible by 4.

proof

$f(x) = (2x+1)x^{2^n} + 11$

- $f(1) = (2 \cdot 1 + 1)x^{2^1} + 11 = 3 \cdot 7 + 11 = 21 + 11 = 32$
i.e. divisible by 4
- SURGE THAT THE ZERO HAVES FOR $n \in \mathbb{N}$, i.e. $f(0) = 4m, m \in \mathbb{N}$
 $\Rightarrow f(x) - f(0) = [(2(0)+1)x^{2^n}] - [(2(0)+1)x^0]$
 $\Rightarrow f(x) - 4m = (2x+1)x^{2^n} - (2x+1)x^0$
 $\Rightarrow f(x) - 4m = 7^k [7(2x+1) - (2x+1)]$
 $\Rightarrow f(x) - 4m = 7^k [42x - 42]$
 $\Rightarrow f(x) - 4m = 7^k [42(x - 1)]$
 $\Rightarrow f(x) - 4m = 4m + 7^k (4x+5)$
 $\Rightarrow f(x) = 4[m + 7^k (4x+5)]$ is a multiple of 4
- IF THE RESULT APPLIES FOR $n \in \mathbb{N}$, THEN IT ALSO APPLIES FOR $n+1$
SINCE THE RESULT APPLIES FOR $n=1$, THEN IT MUST APPLIES FOR ALL $n \in \mathbb{N}$

Question 16 (*)**

$$f(n) = 24 \times 2^{4n} + 3^{4n}, n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 5, for all $n \in \mathbb{N}$.

proof

$f(n) = 24 \times 2^{4n} + 3^{4n}$

- $f(1) = 24 \times 2^4 + 3^4 = 24 \times 16 + 81 = 384 + 81 = 465 = 5 \times 93$
i.e. divisible by 5
- SURGE THAT THE ZERO HAVES FOR $n \in \mathbb{N}$, i.e. $f(0) = 5m, m \in \mathbb{N}$
 $\Rightarrow f(x) - f(0) = [24 \times 2^{4n}] - [24 \times 2^0]$
 $\Rightarrow f(x) - 5m = 24 \times 2^{4n} - 24 \times 2^0 + 3^{4n} - 3^0$
 $\Rightarrow f(x) - 5m = 16 \times 24 \times 2^{4n} - 24 \times 2^0 + 81 \times 3^{4n} - 3^0$
 $\Rightarrow f(x) - 5m = 15 \times 24 \times 2^{4n} + 80 \times 3^{4n}$
 $\Rightarrow f(x) = 5m + 15 \times 24 \times 2^{4n} + 80 \times 3^{4n}$
 $\Rightarrow f(x) = 5[m + 3 \times 2^{4n} + 16 \times 3^{4n}]$ i.e. divisible by 5
- IF THE RESULT APPLIES FOR $n \in \mathbb{N}$ \Rightarrow IT ALSO APPLIES FOR $n+1$
SINCE THE RESULT APPLIES FOR $n=1$ \Rightarrow IT MUST APPLIES FOR ALL $n \in \mathbb{N}$

Question 17 (***)

$$f(n) = 4 \times 7^n + 3 \times 5^n + 5, n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 12, for all $n \in \mathbb{N}$.

[proof]

$$\boxed{\begin{aligned} f(0) &= 4 \times 7^0 + 3 \times 5^0 + 5 \\ &= 4 \times 1 + 3 \times 1 + 5 = 12 = 4 \times 3 \quad \text{ie divisible by 12} \\ &\bullet \text{ SUPPOSE THE RESULT HOLDS FOR } n=k \in \mathbb{N}, \text{ ie } f(k)=12m, m \in \mathbb{N} \\ &\Rightarrow f(k+1)-f(k) = [4 \times 7^{k+1} + 3 \times 5^{k+1} + 5] - [4 \times 7^k + 3 \times 5^k + 5] \\ &\Rightarrow f(k+1)-f(k) = 4 \times 7^{k+1} - 4 \times 7^k + 3 \times 5^{k+1} - 3 \times 5^k \\ &\Rightarrow f(k+1)-12m = 28 \times 7^k - 4 \times 7^k + 15 \times 5^k - 3 \times 5^k \\ &\Rightarrow f(k+1) = 12m + 24 \times 7^k + 12 \times 5^k \\ &\Rightarrow f(k+1) = 12[m + 2 \times 7^k + 1 \times 5^k] \quad \text{ie divisible by 12} \\ &\bullet \text{ IF THE RESULT HOLDS FOR } n=k \in \mathbb{N} \Rightarrow \text{ IT ALSO HOLDS FOR } n=k+1 \\ &\text{ SINCE THE RESULT IS TRUE FOR } n=1 \Rightarrow \text{ IT MUST ALSO HOLD FOR } n \end{aligned}}$$

Question 18 (***)

$$f(n) = (2n+1)7^n - 1, n \in \mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 4, for all $n \in \mathbb{N}$.

[proof]

$$\boxed{\begin{aligned} f(0) &= (2 \times 0+1)7^0 - 1 \\ &= 20 - 1 = 20 = 5 \times 4 \quad \text{ie divisible by 4} \\ &\bullet \text{ SUPPOSE THE RESULT HOLDS FOR } n=k \in \mathbb{N}, \\ &\text{ ie } f(k) = 4m, m \in \mathbb{N} \\ &\Rightarrow f(k+1)-f(k) = [(2 \times (k+1))7^{k+1} - 1] - [2 \times k \times 7^k - 1] \\ &\Rightarrow f(k+1)-4m = (2k+2)7^{k+1} - (2k+1)7^k \\ &\Rightarrow f(k+1)-4m = (2k+2)7^k \times 7 - (2k+1)7^k \\ &\Rightarrow f(k+1)-4m = (12k+12)7^k - (14k+7)7^k \\ &\Rightarrow f(k+1)-4m = (12k+12)7^k \\ &\Rightarrow f(k+1) = 4m + 4(3k+3)7^k \\ &\Rightarrow f(k+1) = 4[m + (3k+3)7^k] \\ &\bullet \text{ THE RESULT HOLDS FOR } n=k \in \mathbb{N}, \text{ THEN} \\ &\text{ IT ALSO HOLDS FOR } n=k+1 \\ &\text{ SINCE IT HOLDS FOR } n=1, \text{ THEN IT MUST} \\ &\text{ HOLD FOR } \forall n \in \mathbb{N} \end{aligned}}$$

Question 19 (***)

Prove by induction that for all natural numbers n ,

$$4^n + 6n - 1$$

is divisible by 9.

 [] , proof

$f(n) = 4^n + 6n - 1 \quad n \in \mathbb{N}$

BASE CASE
 $f(1) = 4^1 + 6 \times 1 - 1 = 4 + 6 - 1 = 9$, i.e. result holds for $n=1$

INDUCTIVE HYPOTHESIS
 SUPPOSE THAT THE RESULT \Rightarrow f(n) holds for $n=k$, $k \in \mathbb{N}$ i.e. $f(k) = 9m$
 WHERE $m \in \mathbb{N}$

$$\begin{aligned} \Rightarrow f(k+1) - f(k) &= [4^{k+1} + 6(k+1) - 1] - [4^k + 6k - 1] \\ \Rightarrow f(k+1) - 9m &= 4^{k+1} + 6k + 6 - 4^k - 6k + 1 \\ \Rightarrow f(k+1) - 9m &= 4^{k+1} - 4^k + 6 \\ \Rightarrow f(k+1) - 9m &= 4 \times 4^k - 4^k + 6 \\ \Rightarrow f(k+1) - 9m &= 3 \times 4^k + 6 \\ \Rightarrow f(k+1) &= 9m + 6 + 3[4^k - 6k + 1] \\ \Rightarrow f(k+1) &= 9m + 6 + 3f(k) - 18k + 3 \\ \Rightarrow f(k+1) &= 9m - 18k + 9 + 3(4m) \\ \Rightarrow f(k+1) &= 36m - 18k + 9 \\ \Rightarrow f(k+1) &= 9[4m - 2k + 1] \end{aligned}$$

CONCLUSION
 IF THE RESULT holds for $n=k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
 SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n 

Question 20 (***)+

Prove by induction that for all natural numbers n ,

$$4^{n+1} + 5^{2n-1}$$

is divisible by 21.

, proof

$f(n) = 4^{n+1} + 5^{2n-1}$

- THE BASE CASE , i.e $n=1$
- $f(1) = 4^2 + 5^1 = 16 + 5 = 21 \quad \text{ie DIVISIBLE BY 21}$
- INDUCTIVE HYPOTHESIS
- SUPPOSE THAT $f(n)$ IS DIVISIBLE BY 21 FOR $n=k \in \mathbb{N}$, i.e $f(k)=21m$
- FOR SOME $m \in \mathbb{N}$
- THEN $f(k+1) - f(k) = (4^{k+2} + 5^{2k+1}) - (4^{k+1} + 5^{2k-1})$
- $f(k+1) - 21m = 4 \times 4^{k+1} - 4^{k+1} + 5^k \times 5^2 - 5^{2k-1}$
- $f(k+1) - 21m = 4 \times 4^{k+1} - 4^{k+1} + 25 \times 5^{2k-1} - 5^{2k-1}$
- $f(k+1) - 21m = 3 \times 4^{k+1} + 24 \times 5^{2k-1}$
- BUT $f(k) = 4^{k+1} + 5^{2k-1} \approx 21m$
- $f(k+1) - 21m = [3 \times 4^{k+1} + 3 \times 5^{2k-1}] + 21 \times 5^{2k-1}$
- $f(k+1) - 21m = 3 \times f(k) + 21 \times 5^{2k-1}$
- $f(k+1) = 84m + 21 \times 5^{2k-1}$
- $f(k+1) = 21 \times [4m + 5^{2k-1}]$
- CONCLUSION
- IF $f(k)$ IS DIVISIBLE BY 21 FOR $k \in \mathbb{N}$, SO IS $f(k+1)$. SINCE $f(1)$ IS DIVISIBLE BY 21 FOR ALL $n \in \mathbb{N}$

Question 21 (***)

$$f(n) = 5^{2n} + 3n - 1, \quad n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 9, for all $n \in \mathbb{N}$.

proof

$\text{Let } f(0) = 5^{20} + 3 \cdot 0 - 1$

- $f(0) = 5^2 + 3 \cdot 0 - 1 = 25 + 3 - 1 = 27$, ie divisible by 9
- SUPPOSE THE RESULT HOLDS FOR $n \in \mathbb{N}$, i.e. $f(n) = 9m$, $m \in \mathbb{N}$

$$\Rightarrow f(2n+1) - f(n) = [5^{2n+2} + 3(2n+1) - 1] - [5^{2n} + 3n - 1]$$

$$\Rightarrow f(2n+1) - 9m = 5^{2n+2} + 3(2n+1) - 5^{2n} - 3n + 1$$

$$\Rightarrow f(2n+1) - 9m = 5 \times 5^{2n} + 3(2n+1) - 5^{2n} - 3n + 1$$

$$\Rightarrow f(2n+1) - 9m = 25 \times 5^{2n} - 5^{2n} + 3$$

$$\Rightarrow f(2n+1) - 9m = 24 \times 5^{2n} + 3$$

But $f(2n) = 5^{2n} + 3n - 1$

$$9m = 5^{2n} + 3n - 1$$

$$5^{2n} = 9m - 3n + 1$$

$$\Rightarrow f(2n+1) - 9m = 24 \times [9m - 3n + 1] + 3$$

$$\Rightarrow f(2n+1) = 9m + 9m \times 24 - 72n + 24 + 3$$

$$\Rightarrow f(2n+1) = 9m \times 25 - 72n + 27$$

$$\Rightarrow f(2n+1) = 9[25m - 8n + 3] \quad \text{ie multiple of 9}$$

- IF THE RESULT HOLDS FOR $n \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n+1$ SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n

Question 22 (****)

Prove by induction that 18 is a factor of $4^n + 6n + 8$, for all $n \in \mathbb{N}$.

proof

$f(n) = 4^n + 6n + 8$

- $f(0) = 4^0 + 8 = 18$ ie divisible by 18
- SUPPOSE THAT THE RESULT HOLDS FOR $n \in \mathbb{N}$, i.e. $f(n) = 18k$ WHERE $k \in \mathbb{N}$

$$\Rightarrow f(2n+1) - f(n) = [4^{2n+1} + 6(2n+1) + 8] - [4^n + 6n + 8]$$

$$f(2n+1) - 18k = 4^{2n+1} + 6 + 6(2n+1) - 4^n - 6n - 8$$

$$f(2n+1) - 18k = 4(4^n) - 4^n + 6 + 6(2n+1) - 6n - 8$$

$$f(2n+1) - 18k = 3 \times 4^n + 6 + 6(2n+1) - 6n - 8$$

$$f(2n+1) = 18n + 54n - 18k - 24 + 6$$

$$f(2n+1) = 72n - 18k - 18$$

$$f(2n+1) = 18[4n - k - 1] \quad \text{ie a multiple of 18}$$

But $f(2n) = 4^n + 6n + 8$

$$18k = 4^n + 6n + 8$$

$$18k - 6n - 8 = 4^n$$

$$3 \times 4^n = 34n - 18k - 24$$

- IF THE RESULT HOLDS FOR $n \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n+1$ SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n

Question 23 (****)

Prove by induction that for all natural numbers n ,

$$2^n + 6^n$$

is divisible by 8.

[] , proof

$f(n) = 2^n + 6^n$, $n \in \mathbb{N}$

BASE CASE
 $f(1) = 2^1 + 6^1 = 2 + 6 = 8$, i.e. divisible by 8

INDUCTIVE HYPOTHESIS
 SUPPOSE THAT THE RESULT HOLDS FOR $n = k$; i.e., if $f(k) = 8m$, $m \in \mathbb{N}$.

$$\begin{aligned} \Rightarrow f(k+1) - f(k) &= [2^{k+1} + 6^{k+1}] - [2^k + 6^k] \\ \Rightarrow f(k+1) - 8m &= 2^{k+1} - 2^k + 6^{k+1} - 6^k \\ \Rightarrow f(k+1) - 8m &= 2 \times 2^k - 2^k + 6 \times 6^k - 6^k \\ \Rightarrow f(k+1) - 8m &= 2^k + 5 \times 6^k \\ \Rightarrow f(k+1) - 8m &= [f(k) - 6^k] + 5 \times 6^k \\ \Rightarrow f(k+1) - 8m &= f(k) + 4 \times 6^k \\ \Rightarrow f(k+1) - 8m &= 8m + 4 \times 6 \times 6^{k-1} \\ \Rightarrow f(k+1) &= 16m + 24 \times 6^{k-1} \\ \Rightarrow f(k+1) &= 8[2m + 3 \times 6^{k-1}] \end{aligned}$$

CONCLUSION
 IF THE RESULT HOLDS FOR $n = k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$
 SINCE THE RESULT HOLDS FOR $n = 1$, THEN IT MUST HOLD

Question 24 (***)

Prove by mathematical induction that if n is a positive integer then $3^{2n+3} + 2^{n+3}$ is always divisible by 7.

, proof

Let $f(x) = 3^{2x+3} + 2^{x+3}$, $x \in \mathbb{N}$

Establish a base case

$$f(0) = 3^3 + 2^4 = 27 + 16 = 259 = 37 \times 7$$

ie the result holds for $n=1$

Suppose that the result holds for $n=k$ (i.e. $f(k) = 7A$, $A \in \mathbb{N}$)

$$\begin{aligned} \Rightarrow f(k+1) - f(k) &= [3^{2k+6} + 2^{k+4}] - [3^{2k+3} + 2^{k+3}] \\ \Rightarrow f(k+1) - 7A &= 3^{2k+6} + 2^{k+4} - 3^{2k+3} - 2^{k+3} \\ \Rightarrow f(k+1) - 7A &= 3^3 \cdot 3^{2k+3} + 2^3 \cdot 2^{k+3} - 3^{2k+3} - 2^{k+3} \\ \Rightarrow f(k+1) - 7A &= 9 \times 3^{2k+3} - 3^{2k+3} + 2 \times 2^{k+3} - 2^{k+3} \\ \Rightarrow f(k+1) - 7A &= 8 \times 3^{2k+3} + 2^{k+3} \end{aligned}$$

But

$$\begin{aligned} f(k+1) &= 7A \\ 3^{2k+6} + 2^{k+3} &= 7A \\ 3^3 \cdot 3^{2k+3} + 2^{k+3} &= 7A \end{aligned}$$

$$\begin{aligned} \Rightarrow f(k+1) - 7A &= 8 \times 3^{2k+3} + 7A - 3^{2k+3} \\ \Rightarrow f(k+1) &= 11A + 7 \times 3^{2k+3} \\ \Rightarrow f(k+1) &= 7[2A + 3^{2k+3}] \end{aligned}$$

ie the result holds for $n+1$, then it must also hold for $n+2$

Since the result holds for $n=1$, then it must hold for all $n \in \mathbb{N}$

Question 25 (***)+

Prove by mathematical induction that if n is a positive integer then $5^{n-1} + 11^n$ is always divisible by 6.

, proof

LET $f(n) = 5^{n-1} + 11^n, n \in \mathbb{N}$

ESTABLISH A BASE CASE
 $f(1) = 5^0 + 11^1 = 1 + 11 = 12 \Rightarrow \boxed{6 \mid 12}$

LET THE RESULT HOLD FOR $n=1$
SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, i.e. $f(k) = 6A, A \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [5^{k+1-1} + 11^{k+1}] - [5^{k-1} + 11^k]$
 $\Rightarrow f(k+1) - 6A = 5^k + 11^{k+1} - 5^{k-1} - 11^k$
 $\Rightarrow f(k+1) - 6A = 5^k 5^{-1} + 11 \cdot 11^k - 5^{k-1} - 11^k$
 $\Rightarrow f(k+1) - 6A = 5^k 5^{-1} - 5^{k-1} + 11 \cdot 11^k - 11^k$
 $\Rightarrow f(k+1) - 6A = 4 \times 5^{k-1} + 10 \cdot 11^k$

BUT WE ALSO HAVE $f(k+1) = 6A$
 $5^{k-1} + 11^k = 6A$
 $11^k = 6A - 5^{k-1}$

$\Rightarrow f(k+1) - 6A = 4 \times 5^{k-1} + 10[6A - 5^{k-1}]$
 $\Rightarrow f(k+1) - 6A = 4 \times 5^{k-1} + 60A - 10 \times 5^{k-1}$
 $\Rightarrow f(k+1) = 66A - 6 \times 5^{k-1}$
 $\Rightarrow f(k+1) = 6[11A - 5^{k-1}]$

IF THE RESULT HOLDS FOR $n=k$, THEN IT MUST ALSO HOLD FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 26 (***)+

Prove by the method of induction that

$$3^{3n-2} + 2^{4n-1}, \quad n \in \mathbb{N},$$

is divisible by 11.

[] , proof

$\{f_n\} = 3^{3n-2} + 2^{4n-1}, \quad n \in \mathbb{N}$

BASE CASE, $n=1$
 $f(1) = 3^1 + 2^3 = 3 + 8 = 11$, IE THE RESULT HOLDS FOR $n=1$

INDUCTIVE HYPOTHESIS
SUPPOSE THAT THE RESULT HELD FOR $n=k$, $k \in \mathbb{N}$, I.E. $f(k)=11m, m \in \mathbb{N}$

$$\begin{aligned} \Rightarrow f(2k) - f(k) &= [3^{3(2k)-2} + 2^{4(2k)-1}] - [3^{3k-2} + 2^{4k-1}] \\ \Rightarrow f(2k) - 11m &= 3^{3k+1} + 2^{4k+3} - 3^{3k-2} - 2^{4k-1} \\ \Rightarrow f(2k) - 11m &= 3 \times 3^{3k-2} + 2 \times 2^{4k-1} - 3^{3k-2} - 2^{4k-1} \\ \Rightarrow f(2k) - 11m &= 27k \cdot 3^{3k-2} - 3^{3k-2} + 16 \times 2^{4k-1} - 2^{4k-1} \\ \Rightarrow f(2k) - 11m &= 26 \times 3^{3k-2} + 15 \times 2^{4k-1} \end{aligned}$$

$$\begin{aligned} f(2k) &= 3^{3k-2} + 2^{4k-1} \\ 11m &= 3^{3k-2} + 2^{4k-1} \\ 2^{4k-1} &= 11m - 3^{3k-2} \end{aligned}$$

$$\begin{aligned} \Rightarrow f(2k) - 11m &= 26 \times 3^{3k-2} + 15(11m - 3^{3k-2}) \\ \Rightarrow f(2k+1) - 11m &= 26 \times 3^{3k+1} + 15 \times 3^{3k+1} + 165m \\ \Rightarrow f(2k+1) &= 11 \times 3^{3k+1} + 165m \\ \Rightarrow f(2k+1) &= 11[3^{3k+1} + 15m], \quad \text{IE DIVISIBLE BY 11} \end{aligned}$$

CONCLUSION
IF THE RESULT HOLDS FOR $n=k, k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 27 (***)+

$$f(n) = 8^n - 2^n, n \in \mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 6, for all $n \in \mathbb{N}$.

[proof]

$f(0) = 8^0 - 2^0 = 1$

- $f(0) = 8^0 - 2^0 = 1$ is divisible by 6.
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, i.e. $f(k) = 6m, m \in \mathbb{N}$

$$\begin{aligned} f(k+1) - f(k) &= (8^{k+1} - 2^{k+1}) - (8^k - 2^k) \\ f(k+1) - 6m &= 8^{k+1} - 8^k - 2^{k+1} + 2^k \\ f(k+1) &= 6m + 8x8^k - 8^k - 2x2^k + 2^k \\ f(k+1) &= 6m + 7x8^k - 2^k \end{aligned}$$

Now

$$\begin{aligned} f(k+1) &= 8^k - 2^k \\ 6m &= 8^k - 2^k \\ 6m - 8^k &= -2^k \\ -2^k &= 6m - 8^k \end{aligned}$$

$f(k+1) = 6m + 7x8^k + 6m - 8^k$

$$\begin{aligned} f(k+1) &= 12m + 6x8^k \\ f(k+1) &= 6[2m + 8^k] \end{aligned}$$

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$. SINCE THE RESULT SPINS FOR $n=1$, THEN THE RESULT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 28 (***)+

$$f(n) = 7^n - 2^n, n \in \mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 5, for all $n \in \mathbb{N}$.

[proof]

$f(0) = 7^0 - 2^0$

- $f(0) = 7^0 - 2^0 = 5$ is result holds for $n=0$.
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, i.e. $f(k) = 5m$ for some $m \in \mathbb{N}$

$$\begin{aligned} f(k+1) - f(k) &= (7^{k+1} - 2^{k+1}) - (7^k - 2^k) \\ f(k+1) - 5m &= 7^{k+1} - 7^k - 2^{k+1} + 2^k \\ f(k+1) - 5m &= 7x7^k - 7^k - 2x2^k + 2^k \\ f(k+1) - 5m &= 6x7^k - 2^k \\ f(k+1) &= 5m + 5x7^k + 2^k \end{aligned}$$

$$\begin{aligned} f(k+1) &= 5m + 5x7^k + f(k) \\ f(k+1) &= 5m + 5x7^k + 5m \\ f(k+1) &= 5[2m + 7^k] \end{aligned}$$

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$. SINCE THE RESULT HOLDS FOR $n=0$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 29 (***)+

$$f(n) = n^3 + 5n, \quad n \in \mathbb{N}.$$

- a) Show that $n^2 + n + 2$ is always even for all $n \in \mathbb{N}$.
- b) Hence, prove by induction that $f(n)$ is divisible by 6, for all $n \in \mathbb{N}$.

proof

a) $n^2 + n + 2 = n(k+1) + 2$. If k is even $n(k+1)$ is even
 $n(k+1)+2$ is also even

If k is odd
 $n(k+1)$ is even
 $n(k+1)+2$ is even

$\therefore n^2 + n + 2$ is even for all $n \in \mathbb{N}$

b) $f(n) = n^3 + 5n$

$f(n) = n^3 + 5n$, is divisible by 6

- SUPPOSE THAT THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, i.e. $f(k) = 6m, m \in \mathbb{N}$
- $\Rightarrow f(k+1) - f(k) = [(k+1)^3 + 5(k+1)] - [k^3 + 5k]$
- $\Rightarrow f(k+1) - 6m = (k^3 + 3k^2 + 3k + 2) - (k^3 + 5k)$
- $\Rightarrow f(k+1) = 6m + 3k^2 - 2k + 2$
- $\Rightarrow f(k+1) = 6m + 3(2k + 1)$
- $\Rightarrow f(k+1) = 6m + 6p$ < FROM PART (a) $k+1=2m+1$
- $\Rightarrow f(k+1) = 6(m+p)$ IS DIVISIBLE BY 6
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$
 SINCE THE RESULT HOLDS FOR $n=k$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 30 (***)+

A sequence of positive numbers is given by

$$a_n = 12^{n+1} + 2 \times 5^n, \quad n \in \mathbb{N}.$$

Prove by induction that every term of the sequence is a multiple of 7

proof

$a_1 = 12^{1+1} + 2 \times 5^1 = 144 + 10 = 154 = 7 \times 22$ IS A MULTIPLE OF 7

- $a_1 = 12^2 + 2 \times 5^1 = 144 + 10 = 154 = 7 \times 22$ IS A MULTIPLE OF 7
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, i.e. $a_k = 7m, m \in \mathbb{N}$
- $\Rightarrow a_{k+1} - 7m = (12^{k+1} + 2 \times 5^k) - (7m)$
- $\Rightarrow a_{k+1} - 7m = 12^{k+1} + 2 \times 5^k - 7m$
- $\Rightarrow a_{k+1} - 7m = (12^k \cdot 12 + 2 \times 5^k) - 7m$
- $\Rightarrow a_{k+1} - 7m = 11 \times 12^k + 2 \times 5^k - 7m$
- $\Rightarrow a_{k+1} - 7m = 11 \times 12^k + 4[2 \times 5^k]$
- $\Rightarrow a_{k+1} - 7m = 11 \times 12^k + 4[a_k - 7m]$
- $\Rightarrow a_{k+1} = 7m + 11 \times 12^k + 4a_k$
- $\Rightarrow a_{k+1} = 7m + 11 \times 12^k + 4(7m)$
- $\Rightarrow a_{k+1} = 7[12^k + 5m]$ IS A MULTIPLE OF 7
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$ \Rightarrow IT ALSO HOLDS FOR $n=k+1$
 SINCE THE RESULT HOLDS FOR $n=k$, \Rightarrow IT MUST HOLD $\forall n \in \mathbb{N}$

Question 31 (***)+

$$f(r) = 4 + 6^r, r \in \mathbb{N}.$$

Prove by induction that $f(r)$ is divisible by 10

proof

$f(r) = 4 + 6^r$

- $f(0) = 4 + 6^0 = 10$, it is divisible by 10
- SUPPOSE THE RESULT HOLDS FOR $r=k \in \mathbb{N}$, i.e. $f(k) = 10k$, $k \in \mathbb{N}$

$$\begin{aligned} f(k+1) - f(k) &= (4 + 6^{k+1}) - (4 + 6^k) \\ f(k+1) - 10k &= 6^{k+1} - 6^k \\ f(k+1) - 10k &= 5 \times 6^k && \text{BY } f(k) = 4 + 6^k \\ f(k+1) - 10k &= 5[6^k - 2] && 10k = 4 + 6^k \\ f(k+1) &= 10k + 5(6^k - 2) \\ f(k+1) &= 10[k + 6^k - 2], \text{ it is divisible by 10} \end{aligned}$$

• IF THE RESULT HOLDS FOR $r=k \Rightarrow$ IT ALSO HOLDS FOR $r=k+1$
SINCE IT HOLDS FOR $r=1 \Rightarrow$ IT MUST HOLD $\forall r \in \mathbb{N}$

Question 32 (***)+

Prove by induction that for all natural numbers n , the following expression

$$7^n + 4^n + 1$$

is divisible by 6.

proof

LET $f(n) = 7^n + 4^n + 1$

- $f(0) = 7^0 + 4^0 + 1 = 12 = 2 \times 6$ IT RESULT HOLDS FOR $n=0$
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, i.e. $f(k) = 6m$, $m \in \mathbb{N}$

$$\begin{aligned} \rightarrow f(k+1) - f(k) &= [7^{k+1} + 4^{k+1} + 1] - [7^k + 4^k + 1] \\ \rightarrow f(k+1) - 6m &= 7^{k+1} - 7^k + 4^{k+1} - 4^k \\ \rightarrow f(k+1) - 6m &= 7 \times 7^k - 7^k + 4 \times 4^k - 4^k \\ \rightarrow f(k+1) - 6m &= 6 \times 7^k + 3 \times 4^k \\ \rightarrow f(k+1) - 6m &= 6 \times 7^k + 3 \times 2^{2k} \\ \rightarrow f(k+1) - 6m &= 6 \times 7^k + 3 \times 2^{2k-2} \times 2^2 \\ \rightarrow f(k+1) - 6m &= 6m + 6 \times 7^k + 6 \times 2^{2k-2} \\ \rightarrow f(k+1) &= 6[m + 7^k + 2^{2k-2}] \end{aligned}$$

• IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 33 (*)+**

A sequence of positive numbers is given by

$$u_n = 7^n + 3n + 8, \quad n \in \mathbb{N}.$$

Prove by induction that every term of the sequence is a multiple of 9

proof

$u_k = 7^k + 3k + 8$

- $u_k = 7^k + 3k + 8 = 9 \times 2$ ie. multiple of 9
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, i.e. $u_k = 9m$, $m \in \mathbb{N}$

$$\Rightarrow u_{kn} - u_k = [7^{kn} + 3(kn) + 8] - [7^k + 3k + 8]$$

$$\Rightarrow u_{kn} - 9m = 7^{kn} - 7^k + 3(kn) - 3k + 8 - 8$$

$$\Rightarrow u_{kn} - 9m = 7^{kn} - 7^k + 3$$

$$\Rightarrow u_{kn} - 9m = (7^k)^n - 7^k + 3$$

$$\Rightarrow u_{kn} - 9m = 6x^n + 3$$

$$\Rightarrow u_{kn} - 9m = 6x^n + 3$$

BUT $u_k = 7^k + 3k + 8$

$$7^k = u_k - 3k - 8$$

$$6x^n = u_k - 3k - 48$$

$$\Rightarrow u_{kn} - 9m = 6u_k - 18k - 48 + 3$$

$$\Rightarrow u_{kn} - 9m = 6(9m) - 18k - 45$$

$$\Rightarrow u_{kn} = 54m - 18k - 45$$

$$\Rightarrow u_{kn} = 9[5m - 2k - 5]$$
 ie. multiple of 9

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$ \rightarrow IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=k$ \Rightarrow IT MUST HOLD FOR $n=k+1$

Question 34 (*)+**

$$f(n) = 5^{n+1} - 4n - 5, \quad n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 16

proof

$f(n) = 5^{n+1} - 4n - 5$

- $f(1) = 5^2 - 4(1) - 5 = 25 - 4 - 5 = 16$ ie. divisible by 16
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, i.e. $f(k) = 16m$, $m \in \mathbb{N}$

$$\Rightarrow f(k+1) = [5^{k+2} - 4(k+1) - 5] - [5^{k+1} - 4k - 5]$$

$$\Rightarrow f(k+1) - 16m = 5^{k+2} - 4k - 4 - 5 - 5^{k+1} + 4k + 5$$

$$\Rightarrow f(k+1) - 16m = 5^{k+2} - 5^{k+1} - 4$$

$$\Rightarrow f(k+1) - 16m = 5 \times 5^{k+1} - 5^{k+1} - 4$$

$$\Rightarrow f(k+1) - 16m = 4 \times 5^{k+1} - 4$$

$f(k+1) = 5^{k+2} - 4k - 5$

$$16m = 5^{k+1} - 4k - 5$$

$$64m = 4 \times 5^{k+1} - 16k - 20$$

$$\Rightarrow f(k+1) - 16m = (4 \times 5^{k+1} - 16k - 20) - 4$$

$$\Rightarrow f(k+1) = 80m + 16k + 16$$

$$\Rightarrow f(k+1) = 16[5m + k + 1]$$
 ie. divisible by 16

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=k$, THEN THE RESULT HOLDS FOR ALL n

Question 35 (*)+**

A sequence of positive numbers is given by

$$u_n = 2^{3n+2} + 5^{n+1}, \quad n \in \mathbb{N}.$$

Prove by induction that every term of the sequence is a multiple of 3.

proof

$u_1 = 2^{3n+2} + 5^{n+1}$

- If $n=1$, $u_1 = 2^5 + 5^2 = 32 + 25 = 57 = 3 \times 19$ is divisible by 3
- Suppose the result holds for all $n \in \mathbb{N}$, i.e. $u_n = 3m$, $m \in \mathbb{N}$

$$u_{n+1} - u_n = [2^{3(n+1)+2} + 5^{(n+1)+1}] - [2^{3n+2} + 5^{n+1}]$$

$$u_{n+1} - 3m = 2^{3n+4} - 2^{3n+2} + 5^{n+2} - 5^{n+1}$$

$$u_{n+1} - 3m = 2^2 \times 2^{3n+2} - 2^{3n+2} + 5 \times 5^{n+1} - 5^{n+1}$$

$$u_{n+1} - 3m = 7 \times 2^{3n+2} + 4 \times 5^{n+1}$$

$$u_{n+1} - 3m = 7 \times 2^{3n+2} + 3 \times 5^{n+1} + 5^{n+1}$$

(But $u_n = 2^{3n+2} + 5^{n+1}$)

$$u_{n+1} - 3m = 7 \times 2^{3n+2} + 3 \times 5^{n+1} + u_n - 2^{3n+2}$$

$$u_{n+1} - 3m = 6 \times 2^{3n+2} + 3 \times 5^{n+1} + 3m$$

$$u_{n+1} = 6m + 6 \times 2^{3n+2} + 3 \times 5^{n+1}$$

$$u_{n+1} = 3[2m + 2 \times 2^{3n+2} + 5^{n+1}]$$

- If the result holds for $n \in \mathbb{N}$ \Rightarrow it also holds for $n+1$
Since the result holds for $n=1$ \Rightarrow it holds $\forall n \in \mathbb{N}$

Question 36 (*)+**

$$f(n) = 3^{2n+4} - 2^{2n}, \quad n \in \mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 5, for all $n \in \mathbb{N}$.

proof

$f(0) = 3^{2 \times 0+4} - 2^{2 \times 0} = 729$ which is divisible by 5

- Suppose that the result holds for all $n \in \mathbb{N}$, i.e. $f(n) = 5m$ for $m \in \mathbb{N}$

$$f(0) = \frac{f(k)}{5} = \left[3^{2(0+1)+4} - 2^{2(0+1)} \right] = \left[3^{2k+4} - 2^{2k+2} \right]$$

$$f(0) - 5m = 3^{2k+4} - 2^{2k+2} - 5m$$

$$f(0) = 5m + 8(3^{2k+4}) - 3^{2k+4} + 2^{2k+2} - 2^2(2^{2k})$$

$$f(0) = 5m + 8(3^{2k+4}) - 3^{2k+4} + 2^{2k+2} - 2^2(2^{2k})$$

$\frac{f(0)}{5}$

$$f(0) = 5m + 8(3^{2k+4}) - 3^{2k+4} + 2^{2k+2} - 2^2(2^{2k})$$

$$f(0) = 5m + 8(3^{2k+4}) - 3^{2k+4} + 2^{2k+2} - 2^2(2^{2k})$$

$$f(0) = 5m + 8(3^{2k+4}) - 3^{2k+4} + 2^{2k+2} - 2^2(2^{2k})$$

$$f(0) = 5[4m + 3^{2k+4}] \quad \text{is divisible by 5}$$

$\left\{ \begin{array}{l} f(0) = 2^{2k+2} \cdot 3^{2k+4} \\ 5m = 3^{2k+4} - 2^{2k+2} \\ 5m - 3^{2k+4} = -2^{2k} \\ 5m - 3(3^{2k+4}) = -2(3^{2k+4}) \end{array} \right.$

- If the result holds for all $n \in \mathbb{N}$, then it must also hold for $n=k+1$. Since the result holds for $n=1$, then the result holds $\forall n \in \mathbb{N}$

RECURRENCE RELATIONS

Question 1 ()**

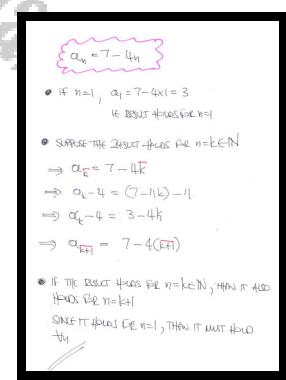
A sequence of integers is defined recursively by the relation

$$a_{n+1} = a_n - 4, \quad a_1 = 3, \quad n = 1, 2, 3, \dots$$

Prove by induction that its n^{th} term is given by

$$a_n = 7 - 4n, \quad n = 1, 2, 3, \dots$$

[proof]



Question 2 ()**

A sequence of integers t_1, t_2, t_3, \dots is given by the recurrence relation

$$t_{n+1} = 3t_n + 2, \quad t_1 = 1, \quad n \in \mathbb{N}.$$

Prove by induction that its n^{th} term of the sequence is given by

$$t_n = 2 \times 3^{n-1} - 1, \quad n \in \mathbb{N}.$$

, proof

↔

BASE CASE
 $t_1 = 1$
 $t_1 = 2 \times 3^{1-1} - 1 = 2 \times 1 - 1 = 1$

$\left\{ \begin{array}{l} \text{result holds for } n=1 \\ \text{result holds for } n=k \\ \text{result holds for } n=k+1 \end{array} \right.$

INDUCTIVE HYPOTHESIS
 SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$.

$$\begin{aligned} &\Rightarrow t_k = 2 \times 3^{k-1} - 1 \\ &\Rightarrow 3t_k = 3[2 \times 3^{k-1} - 1] \\ &\Rightarrow 3t_k = 2 \times 3^k - 3 \\ &\Rightarrow 3t_k + 2 = 2 \times 3^k - 3 + 2 \\ &\Rightarrow t_{k+1} = 2 \times 3^{(k+1)-1} - 1 \end{aligned}$$

CONCLUSION
 IF THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$.
 SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n .

Question 3 ()**

A sequence of integers is defined inductively by the relation

$$a_{n+1} = 3a_n + 4, \quad a_1 = 3, \quad n = 1, 2, 3, \dots$$

Prove by induction that its n^{th} term is given by

$$a_n = 5 \times 3^{n-1} - 2, \quad n = 1, 2, 3, \dots$$

proof

$a_{n+1} = 3a_n + 4$, $a_1 = 3$ IS THE STATEMENT TO PROVE.

- IF $n=1$, $a_1 = 3$. $a_1 = 5 \times 3^0 - 2 = 3$. \checkmark BOTH TRUE OR THE FIRST TERM.
- SUPPOSE THE RESULT HOLDS FOR $n \in \mathbb{N}$.
 $\Rightarrow a_n = 5 \times 3^{n-1} - 2$
 $\Rightarrow 3a_n = 3[5 \times 3^{n-1} - 2]$
 $\rightarrow 3a_n = 15 \times 3^{n-1} - 6$
 $\Rightarrow 3a_n + 4 = 15 \times 3^{n-1} - 2$
 $\Rightarrow a_{n+1} = 5 \times 3^n - 2$
- IF THE RESULT HOLDS FOR $n \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n+1$.
 SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$ \checkmark

Question 4 ()**

The terms of a sequence can be generated by the recurrence relation

$$b_{n+1} = 4b_n + 2, \quad b_1 = 2, \quad n = 1, 2, 3, \dots$$

Prove by induction that the n^{th} term of the sequence is given by

$$b_n = \frac{2}{3}(4^n - 1), \quad n = 1, 2, 3, \dots$$

proof

$b_{n+1} = 4b_n + 2$, $b_1 = 2$, $b_n = \frac{2}{3}(4^n - 1)$

- IF $n=1$, $b_1 = 2$.
 $b_1 = \frac{2}{3}(4^1 - 1) = \frac{2}{3} \times 3 + 2$ \checkmark i.e. RECUR. TERMS FOR $n=1$.
- SUPPOSE THE RESULT HOLDS FOR $n \in \mathbb{N}$.
 $b_n = \frac{2}{3}(4^n - 1)$
 $4b_n = 4 \times \frac{2}{3}(4^n - 1) = \frac{8}{3}(4^{n+1} - 4)$
 $4b_n + 2 = \frac{8}{3}(4^{n+1} - 4) + 2 = \frac{8}{3}(4^{n+1}) - \frac{8}{3} \times 4 + 2 = \frac{8}{3}(4^{n+1}) - \frac{2}{3}$
 $b_{n+1} = \frac{2}{3}(4^{n+1}) - \frac{2}{3}$
 $b_{n+1} = \frac{2}{3}[4^{n+1} - 1]$
- IF THE RESULT HOLDS FOR $n \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n+1$.
 SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$ \checkmark

Question 5 ()**

A sequence is defined by the recurrence relation

$$u_{n+1} = 7u_n - 3, \quad u_1 = 7, \quad n = 1, 2, 3, \dots$$

Prove by induction that its n^{th} term is given by

$$u_n = \frac{1}{2}(13 \times 7^{n-1} + 1), \quad n = 1, 2, 3, \dots$$

proof

$u_{n+1} = 7u_n - 3, \quad u_1 = 7 \quad \text{and} \quad u_n = \frac{1}{2}(13x^{n-1} + 1)$

• $u_1 = 7$
 $u_1 = \frac{1}{2}(13x^0 + 1) = \frac{1}{2}(13 \times 1 + 1) = 7 \quad \left\{ \begin{array}{l} \text{is } u_1 \text{ true?} \\ \text{the first term} \end{array} \right.$

• SUPPOSE THAT THE n^{th} TERM FORMULA WORKS CORRECTLY FOR k^{th} TERM, $k \in \mathbb{N}$.

$u_k = \frac{1}{2}(13x^{k-1} + 1)$
 $7u_k = 7 \times \frac{1}{2}(13x^{k-1} + 1) = \frac{1}{2}(13x^k + 7)$
 $7u_k - 3 = \frac{1}{2}(13x^k + 7) - 3$
 $7u_k - 3 = \frac{1}{2}(13x^k + 4)$
 $u_{k+1} = \frac{1}{2}(13x^k + \frac{4}{2})$
 $u_{k+1} = \frac{1}{2}(13x^k + 1)$
 $u_{k+1} = \frac{1}{2}(13x^{(k+1)-1} + 1)$

• IF THE PRODUCT HAD AN INDEX $m < k+1$, THEN IT ALSO HAD TO BE $m=k+1$. SINCE THE RESULT HOLDS FOR $m=1$, THEN THE PROOF MUST HOLD. \checkmark

Question 6 ()**

A sequence of integers $a_1, a_2, a_3, a_4, \dots$ is given by

$$a_{n+1} = 3a_n + 2, \quad a_1 = 2, \quad n = 1, 2, 3, \dots$$

Prove by induction that its n^{th} term is given by

$$a_n = 3 \times 3^{n-1} - 1, \quad n = 1, 2, 3, \dots$$

proof

The proof is structured as follows:

- Step 1:** Shows the initial condition $a_1 = 2$ and the general formula $a_n = 3 \times 3^{n-1} - 1$. It notes that if $n=1$, then $a_1 = 3^0 + 2 = 3 \times 3^0 - 1 = 2$.
- Step 2:** Assumes the statement is true for $n=k$ (inductive hypothesis). It shows that if $a_k = 3 \times 3^{k-1} - 1$, then $a_{k+1} = 3a_k + 2 = 3(3 \times 3^{k-1} - 1) + 2 = 3^k + 2 = 3 \times 3^k - 1$.
- Step 3:** Concludes that since it holds for $n=1$, it must hold for all $n \in \mathbb{N}$.

Question 7 (*)**

A certain sequence can be generated by the recurrence relation

$$u_{n+1} = \frac{1}{3}(2u_n - 1), \quad u_1 = 1, \quad n = 1, 2, 3, \dots$$

Prove by induction that the n^{th} term of the sequence is given by

$$u_n = 3\left(\frac{2}{3}\right)^n - 1, \quad n = 1, 2, 3, \dots$$

proof

$u_k = 3\left(\frac{2}{3}\right)^k - 1$

- $u_1 = 3\left(\frac{2}{3}\right)^1 - 1 = 2 - 1 = 1$
IF THE FORMULA WORKS FOR $n=k$,
- SUPER THE RECURRENCE FOR $n=k+1$
 $\Rightarrow u_k = 3\left(\frac{2}{3}\right)^k - 1$
 $\Rightarrow 2u_k = 6\left(\frac{2}{3}\right)^k - 2$
 $\Rightarrow 2u_{k+1} = 6\left(\frac{2}{3}\right)^{k+1} - 3$
 $\Rightarrow 3(2u_{k+1}) = 2\left(\frac{2}{3}\right)^{k+1} - 1$
 $\Rightarrow u_{k+1} = 3 \times \frac{2}{3} \times \left(\frac{2}{3}\right)^k - 1$
 $\Rightarrow u_{k+1} = 3 \times \left(\frac{2}{3}\right)^{k+1} - 1$
- IF THE FORMULA WORKS FOR $n=k+1$, THEN IT ALSO WORKS FOR $n=k+2$.
SINCE THE RECURRANCE IS FOR $n \geq 1$, THEN IT MUST HOLD FOR ALL n .

Question 8 (***)+

A sequence is defined recursively by

$$u_{n+1} = \frac{3}{4-u_n}, \quad u_1 = \frac{3}{4}, \quad n=1,2,3,\dots$$

Prove by induction that

$$u_n = \frac{3^{n+1}-3}{3^{n+1}-1}, \quad n=1,2,3,\dots$$

, proof

$u_{n+1} = \frac{3}{4-u_n}$

$u_1 = \frac{3}{4}$

$n=1,2,3,\dots$

\iff

$u_n = \frac{3^n - 3}{3^n - 1}$

$n=1,2,3,\dots$

BASE CASE

$$U_1 = \frac{3}{4}$$

$$U_1 = \frac{3^{1+1}-3}{3^{1+1}-1} = \frac{9-3}{9-1} = \frac{6}{8} = \frac{3}{4} \quad \begin{cases} \text{if } n=1 \\ \text{REDUCT HOMS} \end{cases}$$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $m, k, \in \mathbb{N}$

$$\Rightarrow U_k = \frac{3^{k+1}-3}{3^{k+1}-1}$$

$$\Rightarrow -U_k = -\frac{3^{k+1}-3}{3^{k+1}-1}$$

$$\Rightarrow 4-U_k = 4 - \frac{3^{k+1}-3}{3^{k+1}-1}$$

$$\Rightarrow 4-U_k = \frac{4(3^{k+1}) - (8^{k+1}-5)}{3^{k+1}-1}$$

$$\Rightarrow 4-U_k = \frac{-3 \cdot 3^{k+1} + 1}{3^{k+1}-1}$$

$$\Rightarrow \frac{1}{4-U_k} = \frac{3^{k+1}-1}{3 \cdot 3^{k+1}-1}$$

$$\Rightarrow \frac{3}{4-U_k} = 3 \left[\frac{3^{k+1}-1}{3 \cdot 3^{k+1}-1} \right]$$

$$\Rightarrow U_{k+1} = \frac{3^{k+2}-3}{3^{k+2}-1}$$

$$\Rightarrow U_{k+1} = \frac{3^{k+1}-3}{3^{k+1}-1}$$

CONCLUSION

IF THE RESULT HOLDS FOR $m, k, \in \mathbb{N}$ THEN IT ALSO HOLDS FOR $m+k+1$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 9 (***)+

A sequence is defined recursively by

$$u_{n+1} = u_n + 3k - 2, \quad u_1 = 3, \quad n = 1, 2, 3, \dots$$

Prove by induction that

$$u_n = \frac{1}{2}(3n-1)(n-2) + 4, \quad n = 1, 2, 3, \dots$$

, proof

HOW TO SHOW THAT THE n TH TERM OF THIS RECURSIVE RELATION

IS GIVEN BY

$$u_n = \frac{1}{2}(3n-1)(n-2) + 4$$

PROVE IF $n=1$

$$u_1 = \frac{1}{2} \times 2 \times (1-1) + 4 = -1 + 4 = 3$$

∴ THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\begin{aligned} &\rightarrow u_k = \frac{1}{2}(3k-1)(k-2) + 4 \\ &\rightarrow u_k + 3k - 2 = \frac{1}{2}(3k-1)(k-2) + 4 + 3k - 2 \\ &\rightarrow u_{k+1} = \frac{1}{2}(3k-1)(k-2) + 3k + 2 \\ &\rightarrow u_{k+1} = \frac{1}{2}[3k^2 - 3k - k^2 + 2k + 4] \\ &\rightarrow u_{k+1} = \frac{1}{2}[3k^2 - 2k + 4] \\ &\rightarrow u_{k+1} = \frac{1}{2}[3(k^2 - k + \frac{4}{3})] \\ &\rightarrow u_{k+1} = \frac{1}{2}[3(k^2 - k + 1) + 1] \\ &\rightarrow u_{k+1} = \frac{1}{2}(3(k+1)^2 - 3(k+1) + 1) \\ &\rightarrow u_{k+1} = \frac{1}{2}(3(k+1)^2 - 2) + 4 \end{aligned}$$

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n+1$.

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 10 (***)

A sequence is defined recursively by

$$u_{n+1} = \frac{u_n}{u_n + 1}, \quad u_1 = 2, \quad n \in \mathbb{N}.$$

By writing the above recurrence relation in the form

$$u_{n+1} = A + \frac{B}{u_n + 1},$$

where A and B are integers, use proof by induction to show that

$$u_n = \frac{2}{2n-1}, \quad n \in \mathbb{N}.$$

, proof

$u_{n+1} = \frac{u_n}{u_n + 1}, \quad n \in \mathbb{N}$ \Leftrightarrow $u_n = \frac{2}{2n-1}, \quad n \in \mathbb{N}$

Start by re-writing the recurrence relation

$$u_{n+1} = \frac{u_n}{u_n + 1} = \frac{(u_n) - 1}{(u_n + 1)} = 1 - \frac{1}{u_n + 1}$$

Base case, i.e. $n=1$

$$\begin{cases} u_1 = 2 \\ u_1 = \frac{2}{2(1)-1} = 2 \end{cases}$$

1. E. THE RESULT HOLDS FOR $n=1$

Inductive hypothesis

SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$$\begin{aligned} &\Rightarrow u_k = \frac{2}{2k-1} \\ &\Rightarrow u_k + 1 = \frac{2}{2k-1} + 1 = \frac{2+(2k-1)}{2k-1} = \frac{2k+1}{2k-1} \\ &\Rightarrow \frac{1}{u_k+1} = \frac{2k-1}{2k+1} \\ &\Rightarrow -\frac{1}{u_k+1} = -\frac{2k-1}{2k+1} \\ &\Rightarrow 1 - \frac{1}{u_k+1} = 1 - \frac{2k-1}{2k+1} = \frac{(2k+1)-(2k-1)}{2k+1} = \frac{2}{2k+1} \\ &\Rightarrow u_{k+1} = \frac{2}{2(2k+1)-1} \end{aligned}$$

Conclusion

IF THE RESULT HOLDS FOR $n=k$, THEN IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 11 (*)+**

A sequence is generated by the recurrence relation

$$u_{n+2} = 5u_{n+1} - 6u_n, \quad u_1 = 5, \quad u_2 = 13, \quad n = 1, 2, 3, \dots$$

Prove by induction that n^{th} term of this sequence is given by

$$u_n = 2^n + 3^n, \quad n = 1, 2, 3, \dots$$

[proof]

The proof is contained within a black-bordered box. It starts with the recurrence relation $u_{n+2} = 5u_{n+1} - 6u_n$ and initial conditions $u_1 = 2^1 + 3^1 = 5$ and $u_2 = 2^2 + 3^2 = 13$. A green arrow points from these to the general formula $u_n = 2^n + 3^n$.

Step 1: Verify for $n=1$ and $n=2$:

$$\begin{cases} u_1 = 2^1 + 3^1 = 2 + 3 = 5 \\ u_2 = 2^2 + 3^2 = 4 + 9 = 13 \end{cases} \quad \Rightarrow \text{THE RESULT HOLDS FOR } n=1 \text{ & } n=2.$$

Step 2: Suppose the result holds for two consecutive integers $n=k$ & $n=k+1$, then:

$$\begin{cases} u_k = 2^k + 3^k \\ u_{k+1} = 2^{k+1} + 3^{k+1} \end{cases} \quad \Rightarrow \begin{cases} 5u_{k+1} - 6u_k = -5 \times 2^k + 5 \times 2^{k+1} \\ 5u_{k+1} = 5 \times 2^{k+1} + 5 \times 3^{k+1} \end{cases} \quad \Rightarrow \text{ADD & SIMPLIFY}$$

$$\begin{aligned} 5u_{k+1} - 6u_k &= [5 \times 2^{k+1} - 6 \times 2^k] + [5 \times 3^{k+1} - 6 \times 3^k] \\ u_{k+2} &= [5 \times 2^{k+1} - 6 \times 2^k] + [5 \times 3^{k+1} - 6 \times 3^k] \\ u_{k+2} &= [10 \times 2^k - 6 \times 2^k] + [15 \times 3^k - 6 \times 3^k] \\ u_{k+2} &= 4 \times 2^k + 9 \times 3^k \\ u_{k+2} &= 2^{k+2} + 3^{k+2} \\ u_{k+2} &= 2^{k+2} + 3^{k+2} \end{aligned}$$

Step 3: If the result holds for $n=k$ & $n=k+1$, then it must also hold for $n=k+2$. Since the result holds for $n=1$ & $n=2$, then it must hold $\forall n \in \mathbb{N}$.

Question 12 (**)**

A sequence is generated by the recurrence relation

$$u_{n+2} = 6u_{n+1} - 8u_n, \quad u_1 = 0, \quad u_2 = 32, \quad n = 1, 2, 3, \dots$$

Prove by induction that n^{th} term of this sequence is given by

$$u_n = 4^{n+1} - 2^{n+3}, \quad n = 1, 2, 3, \dots$$

proof

IS THE STATE AS $U_0 = \frac{4^{0+1} - 2^{0+3}}{4 - 2^3}$

- IF $n=1$: $U_1 = 4^{\frac{1+1}{2}} - 2^{\frac{1+3}{2}} = 16 - 16 = 0$ IT GIVES U_1 CORRECTLY
IF $n=2$: $U_2 = 4^{\frac{2+1}{2}} - 2^{\frac{2+3}{2}} = 64 - 32 = 32$ IT GIVES U_2 CORRECTLY
- SUPPOSE THAT THE k^{th} TERM FORMULA PRODUCES CORRECTLY TWO CONSECUTIVE TERMS OF THE SEQUENCE IS U_k & U_{k+1} FOR $k \in \mathbb{N}$.
THEN

$$\begin{aligned} GU_{k+2} - BU_k &= 6\left[4^{\frac{k+2+1}{2}} - 2^{\frac{k+2+3}{2}}\right] - 6\left[4^{\frac{k+1+1}{2}} - 2^{\frac{k+1+3}{2}}\right] \\ U_{k+2} &= 6\cancel{4^{\frac{k+1+1}{2}}} \cdot \cancel{6 \times 2^{\frac{k+1+3}{2}}} - \cancel{6 \times 4^{\frac{k+1+1}{2}}} + 6 \times 2^{\frac{k+1+3}{2}} \\ U_{k+2} &= \cancel{3 \times 4^{\frac{k+1+1}{2}}} \cdot \cancel{3 \times 2^{\frac{k+1+3}{2}}} - \cancel{3 \times 4^{\frac{k+1+1}{2}}} \times 4^{\frac{k+2+1}{2}} + 2 \times 2^{\frac{k+1+3}{2}} \\ U_{k+2} &= \cancel{\frac{3}{2} \times 4^{\frac{k+1+1}{2}}} \cdot \cancel{3 \times 2^{\frac{k+1+3}{2}}} - \frac{1}{2} \times 4^{\frac{k+1+1}{2}} + 2 \times 2^{\frac{k+1+3}{2}} \\ U_{k+2} &= 4^{\frac{k+2+1}{2}} - 2^{\frac{k+2+3}{2}} \\ U_{k+2} &= 4^{\frac{(k+1)+1}{2}} - 2^{\frac{(k+1)+3}{2}} \end{aligned}$$

COMPARE WITH $U_{k+2} = 4^{\frac{k+2+1}{2}} - 2^{\frac{k+2+3}{2}}$

- IF THE k^{th} TERM FORMULA PRODUCES CORRECTLY ANY TWO CONSECUTIVE TERMS OF THE SEQUENCE, THEN IT PRODUCES CORRECTLY THE NEXT ONE ALSO.
SINCE THE k^{th} TERM FORMULA PRODUCES CORRECTLY THE FIRST TWO TERMS THEN IT PRODUCES CORRECTLY EVERY TERM OF THE SEQUENCE.

Question 13 (***)**

A sequence is generated by the recurrence relation

$$u_{n+2} = u_{n+1} + u_n, \quad u_1 = 0, \quad u_2 = 1, \quad n = 1, 2, 3, \dots$$

Prove by induction that u_{5m} is a multiple of 5, for all $m \in \mathbb{N}$.

, [\[proof\]](#)

BASE CASE FOR u_{5m} : $u_{5m} = u_{5m} + u_{5m}$
 $u_1 = 1, u_2 = 1, u_3 = 2, u_4 = 3, u_5 = 5$

NEED TO PROVE u_{5m} IS A MULTIPLE OF 5

SUPPOSE THAT THE RESULT HOLDS, i.e. u_{5k} IS A MULTIPLE OF 5, i.e. $u_{5k} = 5m$

FOR $n \in \mathbb{N}$

$$\begin{aligned} \Rightarrow u_{5k} &= u_{5k-1} + u_{5k-2} = 5m \\ \Rightarrow u_{5(k+1)} &= (u_{5k+1} + u_{5k}) + u_{5k-1} \\ \Rightarrow u_{5(k+1)} &= (u_{5k+1} + u_{5k+1}) + u_{5k-1} \\ &= 2u_{5k+1} + u_{5k-1} \\ \Rightarrow u_{5(k+1)} &= 2(u_{5k} + u_{5k}) + u_{5k-1} \\ &= 3u_{5k} + 2u_{5k-1} \\ \Rightarrow u_{5(k+1)} &= 3(u_{5k} + u_{5k}) + 2u_{5k-1} \\ &= 5u_{5k+1} + 2u_{5k-1} \\ &= 5u_{5k+1} + 3(5m) \\ &= 5[u_{5k+1} + 3m] \\ &\text{IS A MULTIPLE OF 5} \end{aligned}$$

If the result holds for $n = 5k$, then it holds for $n = 5(k+1)$
 Since it holds for $n = 5$, then it will hold for all multiples of 5

Question 14 (****+)

A sequence of numbers is given by the recurrence relation

$$u_{n+1} = \frac{5u_n - 1}{4u_n + 1}, \quad u_1 = 1, \quad n \in \mathbb{N}, \quad n \geq 1.$$

Prove by induction that the n^{th} term of the sequence is given by

$$u_n = \frac{n+2}{2n+1}.$$

, proof

$U_{k+1} = \frac{5U_k - 1}{4U_k + 1}; \quad U_1 = 1 \implies U_k = \frac{k+2}{2k+1}$

- START THE PROOF BY REARRANGING THE EQUATION

$$U_{k+1} = \frac{\frac{5}{4}(4U_k + 1) - \frac{1}{4}}{4U_k + 1} = \frac{5}{4} - \frac{\frac{1}{4}}{4U_k + 1}$$

$$\therefore U_{k+1} = \frac{5}{4} - \frac{\frac{1}{4}}{4U_k + 1}$$

- BASE CASE

$n=1 \quad U_1 = \frac{1+2}{2(1)+1} = \frac{3}{3} = 1$, TAKE RECURSIVE FORM FOR $n=1$

- INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RECURSIVE FORM FOR $n=k$, $k \in \mathbb{N}$

$$\Rightarrow U_k = \frac{k+2}{2k+1}$$

$$\Rightarrow U_{k+1} = 4\left(\frac{k+2}{2k+1}\right) + 1 = \frac{4k+8}{2k+1} + 1 = \frac{4k+8+2k+1}{2k+1}$$

$$= \frac{6k+9}{2k+1}$$

$$\Rightarrow \frac{1}{4U_{k+1}} = \frac{2k+1}{6k+9}$$

$$\Rightarrow \frac{-\frac{3}{4}}{4U_{k+1}} = -\frac{3}{4} \times \frac{2k+1}{6k+9} = -\frac{3}{4} \times \frac{2k+1}{2(3k+3)} = -\frac{3}{4} \times \frac{1}{3} = -\frac{1}{4}$$

$$\Rightarrow \frac{5}{4} - \frac{\frac{1}{4}}{4U_k + 1} = \frac{5}{4} - \frac{6k+9}{2k+1} = \frac{5(2k+1) - (6k+9)}{2k+1}$$

$$\Rightarrow U_{k+1} = \frac{12k+12}{2k+3} = \frac{k+3}{2k+3}$$

$$\Rightarrow U_{k+1} = \frac{(k+2)+2}{2(2k+1)+1}$$

- CONCLUSION OF THE PROOF

IF THE RECURSIVE FORM FOR $n=k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$. SINCE THE RECURSIVE FORM FOR $n=1$, THEN IT ALSO HOLDS FOR ALL $n \in \mathbb{N}$

Question 15 (***)+

A sequence of numbers is given by the recurrence relation

$$u_{n+1} = \frac{u_n - 5}{3u_n - 7}, \quad u_1 = -1, \quad n \in \mathbb{N}, \quad n \geq 1.$$

Prove by induction that the n^{th} term of the sequence is given by

$$u_n = \frac{2^{n+1} - 5}{2^{n+1} - 3}.$$

[] proof

SOLVE THE RECURRANCE RELN

$$u_{n+1} = \frac{u_n - 5}{3u_n - 7} = \frac{1}{3} \left(\frac{u_n - 5}{u_n - \frac{7}{3}} \right) = \frac{1}{3} \left(\frac{(u_n - \frac{7}{3}) - \frac{2}{3}}{u_n - \frac{7}{3}} \right)$$

$$u_{n+1} = \frac{1}{3} \left[1 - \frac{\frac{2}{3}}{u_n - \frac{7}{3}} \right]$$

$$u_{n+1} = \frac{1}{3} - \frac{\frac{2}{3}}{3u_n - 7}$$

$$\boxed{u_{n+1} = \frac{1}{3} - \frac{2}{3^{n+1}-3}}$$

SOLVE THAT THE RECURRANCE RELN FOR $n=k$ IS

$$\Rightarrow u_k = \frac{2^{k+1} - 5}{2^{k+1} - 3}$$

$$\Rightarrow 9u_{k-21} = 9 \left(\frac{2^{k+1} - 5}{2^{k+1} - 3} \right) - 21$$

$$\Rightarrow 9u_{k-21} = \frac{9 \times 2^{k+1} - 45}{2^{k+1} - 3} - 21$$

$$\Rightarrow 9u_{k-21} = \frac{9 \times 2^{k+1} - 45 - 21 \times 2^{k+1} + 63}{2^{k+1} - 3}$$

$$\Rightarrow 9u_{k-21} = \frac{-12 \times 2^{k+1} + 18}{2^{k+1} - 3}$$

$$\Rightarrow \frac{1}{9u_{k-21}} = \frac{2^{k+1} - 3}{-12 \times 2^{k+1} + 18}$$

$$\Rightarrow -\frac{5}{9u_{k-21}} = \frac{-8 \times 2^{k+1} + 24}{-12 \times 2^{k+1} + 18}$$

$$\Rightarrow \frac{1}{3} - \frac{5}{9u_{k-21}} = \frac{1}{3} + \frac{8 \times 2^{k+1} - 24}{12 \times 2^{k+1} - 36}$$

$$\Rightarrow u_{k+1} = \frac{1}{3} + \frac{4 \times 2^{k+1} - 12}{6 \times 2^{k+1} - 9}$$

$$\Rightarrow u_{k+1} = \frac{6 \times 2^{k+1} - 9 + 12 \times 2^{k+1} - 36}{3(4 \times 2^{k+1} - 9)}$$

$$\Rightarrow u_{k+1} = \frac{18 \times 2^{k+1} - 45}{3 \times 3(4 \times 2^{k+1} - 9)}$$

$$\Rightarrow u_{k+1} = \frac{2 \times 2^{k+1} - 5}{2 \times 2^{k+1} - 3}$$

$$\Rightarrow u_{k+1} = \frac{2^{k+2} - 5}{2^{k+2} - 3} \quad \text{Compare with } u_n = \frac{2^{n+1} - 5}{2^{n+1} - 3}$$

IF THE RECURRANCE RELN FOR $n=k$ IS

TRUE THEN IT ALSO HOLDS FOR $n=k+1$

TRUE IF $n=1$ $u_1 = \frac{2^2 - 5}{2^2 - 3} = \frac{4 - 5}{4 - 3} = \frac{-1}{1} = -1$, IS RECURRANCE RELN

FOR $n=1$, TRUE

THE RECURRANCE RELN FOR ALL $n \in \mathbb{N}$

POWERS OF MATRICES

Question 1 ()**

Prove by induction that

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}, \quad n \geq 1, n \in \mathbb{N}$$

[proof]

• BASE CASE $n=1$
 $LHS = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^1 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$
 $RHS = \begin{pmatrix} 1 & 2^1 - 1 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$
 ✓ RESULT holds for $n=1$

• SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$
 $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix}$
 $\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix}$
 $\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1+0 & 2^k - 1 + 2^k \\ 0+0 & 0+2^{k+1} \end{pmatrix}$
 $\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 2^{k+1} - 1 \\ 0 & 2^{k+1} \end{pmatrix}$
 $\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 2^{k+1} - 1 \\ 0 & 2^{k+1} \end{pmatrix}$

• IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS
 FOR $n=k+1$
 SINCE THE RESULT HOLDS FOR $n=k$, THEN IT MUST HOLD
 FOR ALL $n \in \mathbb{N}$

Question 2 (**)

A transformation where $\mathbb{R}^2 \mapsto \mathbb{R}^2$ is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

- a) Find the elements of the matrices, \mathbf{A}^2 and \mathbf{A}^3 .
- b) Write down a suitable form for \mathbf{A}^n and use the method of proof by induction to prove it.

$$\boxed{\quad}, \quad \mathbf{A}^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad \boxed{\quad}, \quad \mathbf{A}^3 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \quad \boxed{\quad}, \quad \mathbf{A}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

a) CAREFULLY DO THE REQUIRED "MULTIPLICATIONS"

$$\begin{aligned}\mathbf{A}^2 &= \mathbf{A}\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1x1+2x0 & 1x2+2x1 \\ 0x1+1x0 & 0x2+1x1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \\ \mathbf{A}^3 &= \mathbf{A}^2\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1x1+4x0 & 1x2+4x1 \\ 0x1+1x0 & 0x2+1x1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

b) A POSSIBLE FORM OF \mathbf{A}^n MIGHT BE

$$\mathbf{A}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

- IF $n=1$, $\mathbf{A}^1 = \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, so THE RESULT STANDS
- SUPPOSE THAT THE RESULT STANDS FOR $n=k \in \mathbb{N}$

$$\begin{aligned}\Rightarrow \mathbf{A}^k &= \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \\ \Rightarrow \mathbf{A}^k \mathbf{A} &= \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ \Rightarrow \mathbf{A}^{k+1} &= \begin{pmatrix} 1x1+2kx0 & 1x2+2kx1 \\ 0x1+1x0 & 0x2+1x1 \end{pmatrix} \\ \Rightarrow \mathbf{A}^{k+1} &= \begin{pmatrix} 1 & 2(k+1) \\ 0 & 1 \end{pmatrix}\end{aligned}$$

$$\Rightarrow \mathbf{A}^{k+1} = \begin{pmatrix} 1 & 2(k+1) \\ 0 & 1 \end{pmatrix}$$

- IF THE RESULT STANDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO NEEDS FOR $n=k+1$
- SINCE THE RESULT STANDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 3 ()**

Prove by induction that if $n \geq 1$, $n \in \mathbb{N}$, then

$$\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^n = \begin{pmatrix} 1-3n & -n \\ 9n & 3n+1 \end{pmatrix}.$$

[proof]

- IF $\text{M}\left(\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}\right)^1 = \begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}$

$$\begin{pmatrix} 1-3n & -n \\ 9n & 3n+1 \end{pmatrix} = \begin{pmatrix} 1-3(1) & -1 \\ 9(1) & 3(1)+1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}$$

It is true that for $n=1$

- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^k = \begin{pmatrix} 1-3k & -k \\ 9k & 3k+1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^k \begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix} = \begin{pmatrix} 1-3k & -k \\ 9 & 4 \end{pmatrix} \begin{pmatrix} 1-3k & -k \\ 9k & 3k+1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^{k+1} = \begin{pmatrix} -2(-3k)-9k & -2(-3k) \\ 9(-3k)+3k & -9k+4(3k+1) \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^{k+1} = \begin{pmatrix} -2-3k & -k+1 \\ 9-3k & 3k+4 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^{k+1} = \begin{pmatrix} 1-3(k+1) & -(k+1) \\ 9(k+1) & 3(k+1)+1 \end{pmatrix}$$

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$

- SINCE THE RESULT HOLDS FOR $n=1$, THIS IT MUST HOLD $\forall n \in \mathbb{N}$

Question 4 (**)

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$$

Prove by induction that if $n \geq 1$, $n \in \mathbb{N}$, then

$$\mathbf{A}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}.$$

proof

$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$

- IF $n=1$: $\mathbf{A}^1 = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$
 $\mathbf{A}^1 = \begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\begin{aligned}\mathbf{A}^k &= \begin{pmatrix} 3^k & 0 \\ 3(k) & 1 \end{pmatrix} \\ \mathbf{A}^k \mathbf{A}^1 &= \begin{pmatrix} 3^k & 0 \\ 3(k) & 1 \end{pmatrix} \begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix} \\ \mathbf{A}^{k+1} &= \begin{bmatrix} 3 \cdot 3^k & 0 \\ 3 \cdot 3^k + 3(k) & 1 \end{bmatrix} \\ \mathbf{A}^{k+1} &= \begin{bmatrix} 3^{k+1} & 0 \\ 3^{k+1} - 3 & 1 \end{bmatrix} \\ \mathbf{A}^{k+1} &= \begin{bmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{bmatrix}\end{aligned}$$

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$.

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$.

Question 5 ()**

Prove by induction that

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}^n = \begin{pmatrix} 1+4n & 8n \\ -2n & 1-4n \end{pmatrix}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

proof

- IF $n=1$

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}^1 = \begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1+4(1) & 8(1) \\ -2(1) & 1-4(1) \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}$$
THE RESULT HOLDS FOR $n=1$.
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 1+4(k+1) & 8(k+1) \\ -2(k+1) & 1-4(k+1) \end{pmatrix}$$

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 5+4k & 8k \\ -2k-2 & -4k-3 \end{pmatrix} \begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 5+4k+4k-16k & 8k+32k-24k \\ -16k-2+16k & -4k-3+16k \end{pmatrix}$$

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 4k+5 & 8k+8 \\ -2k-2 & -4k-3 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 1+4(k+1) & 8(k+1) \\ -2(k+1) & 1-4(k+1) \end{pmatrix}$$
- IF THE RESULT HOLDS FOR $k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $k+1$. SINCE THE RESULT HOLDS FOR $n=1$ THRU IT MUST HOLD FOR $n=k$.

Question 6 ()**

$$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Prove by induction that

$$\mathbf{M}^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

proof

- IF $n=1$

$$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{M}$$
THE RESULT HOLDS FOR $n=1$.
- SUPPOSE THAT THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\mathbf{M}^{k+1} = \begin{pmatrix} 2+k & k \\ -k & 1-k \end{pmatrix}$$

$$\mathbf{M} \mathbf{M}^k = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2+k & k \\ -k & 1-k \end{pmatrix}$$

$$\mathbf{M}^{k+1} = \begin{pmatrix} 2(2+k)-k & 2k+1-k \\ -2k+1 & -k \end{pmatrix}$$

$$\mathbf{M}^{k+1} = \begin{pmatrix} 4k+2 & k+1 \\ -2k+1 & -k \end{pmatrix}$$

$$\mathbf{M}^{k+1} = \begin{pmatrix} 4k+2 & k+1 \\ -2k+1 & -k \end{pmatrix}$$
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST HOLD FOR $n=k+1$. SINCE THE RESULT HOLDS FOR $n=1$, THEN THE RESULT MUST HOLD FOR $\forall n \in \mathbb{N}$.

Question 7 (***)

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

Prove by induction that

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

proof

• If $n=1$
 $\mathbf{A}^1 = \begin{pmatrix} 2^1 & 3(2^1 - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \mathbf{A}$ i.e. result holds for $n=1$

• SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$.
 $\mathbf{A}^k = \begin{pmatrix} 2^k & 3(2^k - 1) \\ 0 & 1 \end{pmatrix}$
 $\mathbf{A}\mathbf{A}^k = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2^k & 3(2^k - 1) \\ 0 & 1 \end{pmatrix}$
 $\mathbf{A}^{k+1} = \begin{pmatrix} 2 \cdot 2^k + 3 \cdot 0 & 2 \cdot 3(2^k - 1) + 3 \cdot 0 \\ 0 \cdot 2^k + 1 \cdot 0 & 0 \cdot 3(2^k - 1) + 1 \cdot 0 \end{pmatrix}$
 $\mathbf{A}^{k+1} = \begin{pmatrix} 2^{k+1} & 3[2^k - 1] + 3 \\ 0 & 1 \end{pmatrix}$
 $\mathbf{A}^{k+1} = \begin{pmatrix} 2^{k+1} & 3 \cdot 2^k - 3 \\ 0 & 1 \end{pmatrix}$
 $\mathbf{A}^{k+1} = \begin{pmatrix} 2^{k+1} & 3(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$

• IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$.
 SINCE THE PRODUCT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR $\mathbb{N} \geq 1$.

Question 8 (***)

Prove by induction that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}n(n+1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

proof

• IF $n=1$: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ AND $\begin{pmatrix} 1 & 1 & \frac{1}{2} \times 1 \times (1+1) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.
 ∴ RESULT TRUE FOR $n=1$.

• SUPPOSE THE RESULT TRUE FOR $n=k \in \mathbb{N}$.
 $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k & \frac{1}{2}k(k+1) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$
 $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k & \frac{1}{2}k(k+1) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k+1)(k+2) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$
 $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k+1)(k+2) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k+1)(k+2) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$
 $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k+1)(k+2) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$
 • IF THE RESULT HAD BEEN FALSE, THEN IT MUST ALSO HAVE BEEN FALSE SINCE THE RESULT WAS TRUE FOR $n=1$, THEN IT MUST ALSO HAVE BEEN FALSE.

Question 9 (*)**

Prove by induction that

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2+3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, \quad n \geq 1, n \in \mathbb{N}.$$

proof

• IF $n=1$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = A$$

• SURFACE THE RESULT (THIS IS NOT IN)

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} (1+0)(1+k)+0 & 2+0+\frac{1}{2}(k^2+3k) \\ (0+0)(0+1)+0 & 0+1+k \\ (0+0)(0+0)+1 & 0+0+1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+4k+4) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+4k+4) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+4k+4) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

BUT $\frac{1}{2}(k^2+4k+4) = \frac{1}{2}(k^2+2k+2) + \frac{1}{2}(2k^2+2k+2) = \frac{1}{2}(k^2+2k+2) + \frac{1}{2}(2(k^2+2k+2))$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+2k+2) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

• IF THE RIGHT HAND SIDE IS $n=k+1$ \Rightarrow IT ALSO HOLDS FOR $n=k+2$
SINCE IT HELPS THE $n=k$ \Rightarrow IT HELPS THE $n=k+1$

Question 10 (*****)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Prove by induction that

$$\mathbf{A}^n = n\mathbf{A} - (n-1)\mathbf{I}, \quad n \geq 1, \quad n \in \mathbb{N}.$$

, proof

$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, THEN $\mathbf{A}^n = n\mathbf{A} - (n-1)\mathbf{I}$

- CHECK THE RESULT FOR $n=1$.
 $\mathbf{A}^1 = 1\mathbf{A} - (1-1)\mathbf{I} = \mathbf{A}$ IS THE RESULT TRUE FOR $n=1$.
- SUPPOSE THE RESULT HOLDS FOR $n=k, k \in \mathbb{N}$.
 $\Rightarrow \mathbf{A}^k = k\mathbf{A} - (k-1)\mathbf{I}$
 $\Rightarrow \mathbf{A}^{k+1} = k\mathbf{A}^k - (k-1)\mathbf{I}$
 $\Rightarrow \mathbf{A}^{k+1} = k(k\mathbf{A} - (k-1)\mathbf{I}) - (k-1)\mathbf{I}$
 $\Rightarrow \mathbf{A}^{k+1} = k^2\mathbf{A} - k(k-1)\mathbf{I} - (k-1)\mathbf{I}$
 $\Rightarrow \mathbf{A}^{k+1} = (k+1)\mathbf{A} - [(k+1)-1]\mathbf{I}$
- NOW IN ORDER TO COMPLETE THE MANIPULATION WE NEED TO REPLACE \mathbf{A}^2 WITH SOME LINEAR COMBINATION OF \mathbf{A} & \mathbf{I} .
 $\Rightarrow \mathbf{A}^2 = 2\mathbf{A} + \mathbf{I}$
 $\Rightarrow \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \mathbf{I} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
 $\Rightarrow \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} \lambda+4 & 0 \\ 2\lambda & \lambda+1 \end{pmatrix}$
 $2\lambda=4 \quad \lambda+1=1$
 $\lambda=2 \quad 2+\lambda=1$
 $\underline{\lambda=2}$

$\therefore \mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$

- RETURNING TO THE MANIPULATION OF THE INDUCTION
 $\Rightarrow \mathbf{A}^{k+1} = k[2\mathbf{A} - (k-1)\mathbf{I}] - (k-1)\mathbf{A}$
 $\Rightarrow \mathbf{A}^{k+1} = 2k\mathbf{A} - k\mathbf{I} - k\mathbf{A} + \mathbf{A}$
 $\Rightarrow \mathbf{A}^{k+1} = k\mathbf{A} + \mathbf{A} - k\mathbf{I}$
 $\Rightarrow \mathbf{A}^{k+1} = (k+1)\mathbf{A} - [(k+1)-1]\mathbf{I}$
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN THE RESULT ALSO HOLDS FOR $n=k+1$. SINCE THE RESULT HOLDS FOR $n=1$, THEN THE RESULT MUST HOLD FOR ALL $n \in \mathbb{N}$.

MISCELLANEOUS RESULTS

Question 1 (**+)

De Moivre's theorem states

$$(\cos \theta + i \sin \theta)^n \equiv \cos n\theta + i \sin n\theta, \quad n \in \mathbb{N}.$$

Prove this theorem by induction.

[proof]

- IF $n=1$ $(\cos \theta + i \sin \theta)^1 = (\cos \theta + i \sin \theta) = \cos \theta + i \sin \theta$ IE IT'S TRUE FOR $n=1$
- SUPPOSE THAT IT'S TRUE FOR $n=k \in \mathbb{N}$
 $(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$
 $(\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta) = (\cos k\theta + i \sin k\theta) (\cos \theta + i \sin \theta)$
 $(\cos \theta + i \sin \theta)^{k+1} = (\cos k\theta \cos \theta - \sin k\theta \sin \theta) + i(\sin k\theta \cos \theta + \cos k\theta \sin \theta)$
 $(\cos \theta + i \sin \theta)^{k+1} = [\cos(k\theta + \theta) + i \sin(k\theta + \theta)]$
 $(\cos \theta + i \sin \theta)^{k+1} = \cos(k+1)\theta + i \sin(k+1)\theta$
- IF THE PREDICTION FOR $n=k+1 \in \mathbb{N}$ \Rightarrow IT ALSO HOLDS FOR $n=k+1$
SINCE THE STATEMENT FOR $n=k$ IS TRUE \Rightarrow IT MUST HOLD FOR $n=k+1$

Question 2 (**+)

$$u_n = \frac{3}{7}(8^n - 1), \quad n \in \mathbb{N}.$$

Prove by induction that every term of this sequence is an integer.

[proof]

$U_n = \frac{3}{7}(8^n - 1)$

- $U_1 = \frac{3}{7}(8^1 - 1) = \frac{3}{7} \times 7 = 3$ (IE AN INTEGER)
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, IF U_k IS AN INTEGER, SAY N
 $U_{k+1} - U_k = \frac{3}{7}(8^{k+1} - 8^k)$
 $U_{k+1} - N = \frac{3}{7}[8^k(8 - 1)]$
 $8^k(8 - 1) = \frac{3}{7}[8^k(8^k - 1)]$
 $U_{k+1} - N = \frac{3}{7} \times 7 \times 8^k$
 $U_{k+1} = N + 8^k$ WHICH IS ALSO AN INTEGER
- IF THE PREDICTION FOR $n=k+1 \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=k$, THEN THE RESULT MUST HOLD FOR ALL n

Question 3 (***)

$$\sum_{r=1}^n (2r+1) = (n+1)^2, \quad n \in \mathbb{N}$$

- a) Show that if the above result holds for $n = k$, then it also holds for $n = k + 1$.
- b) Explain why the result is **not** true.

[proof]

a) $\sum_{r=1}^k (2rk+1) = (k+1)^2$

SUPPOSE THE RESULT IS TRUE FOR ALL $n \in \mathbb{N}$

$$\begin{aligned} \sum_{r=1}^{k+1} (2rk+1) &= (\underline{k+1})^2 \\ &= \sum_{r=1}^k (2rk+1) + [2(k+1)+1] = (k+1)^2 + [2(k+1)+1] \\ &= (k+1)^2 + (2k+3) \\ &= k^2 + 4k + 4 \\ &= (k+2)^2 = [\underline{(k+1)+1}]^2 \end{aligned}$$

IF THE RESULT IS TRUE FOR ALL $n \in \mathbb{N}$, THEN IT
MUST ALSO BE TRUE FOR $n = k+1$. \checkmark

b) PROOF IS NOT TRUE BECAUSE THREE CASES ARE MISSING

$\sum_{r=1}^1 (2r+1) = 3 \neq 4$
$\sum_{r=1}^2 (2r+1) = 8 \neq 9$
$\sum_{r=1}^3 (2r+1) = 15 \neq 16$
$\sum_{r=1}^4 (2r+1) = 24 \neq 25$
etc.

Question 4 (*)**

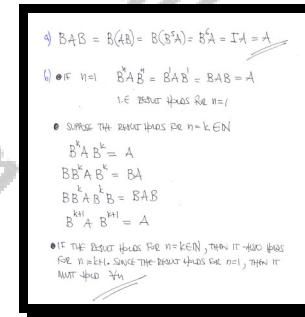
The distinct square matrices \mathbf{A} and \mathbf{B} have the properties

- $\mathbf{AB} = \mathbf{B}^5 \mathbf{A}$
- $\mathbf{B}^6 = \mathbf{I}$

where \mathbf{I} is the identity matrix.

- a) Show that $\mathbf{BAB} = \mathbf{A}$.
- b) Hence prove by induction that $\mathbf{B}^n \mathbf{AB}^n = \mathbf{A}$, for all $n \in \mathbb{N}$.

proof



Question 5 (***)

$$xy + 3y = x.$$

Prove by induction

$$(x+3) \frac{d^n}{dx^n}(y) + (n+1) \frac{d^{n-1}}{dx^{n-1}}(y) = 0.$$

proof

Q4 + 3y - 2 = 0
Diff wrt x
 $y + 2\frac{dy}{dx} + 3\frac{d^2y}{dx^2} - 1 = 0$
 $(x+3)\frac{dy}{dx} + y = 1$

• Suppose that the result holds for $n=k \in \mathbb{N}$
 $(x+3)\frac{dy}{dx} + k\frac{d^k y}{dx^k} = 0$

• Differentiate wrt x once more
 $1\frac{dy}{dx} + (2k)\frac{d^{k+1}y}{dx^{k+1}} + k\frac{d^k y}{dx^k} = 0$
 $(x+3)\frac{d^{k+1}y}{dx^{k+1}} + (k+1)\frac{d^k y}{dx^k} = 0$
 $(k+2)\frac{d^{k+1}y}{dx^{k+1}} + (k+1)\frac{d^k y}{dx^k} = 0$

• If the above holds for $n=k \in \mathbb{N}$, then it also holds for $n=k+1$.
SINCE THE RESULT HOLDS FOR $n=1$,
THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 6 (***)

Bernoulli's inequality asserts that if $a \in \mathbb{R}$, $a > -1$ and $n \in \mathbb{N}$, $n \geq 2$, then

$$(1+a)^n > 1+an.$$

Prove, by induction, the validity of Bernoulli's identity.

,

BERNOULLI INEQUALITY

$$(1+a)^n > 1+an \quad a \in \mathbb{R}, a > -1$$
$$n \in \mathbb{N}, n \geq 2$$

PROOF BY INDUCTION

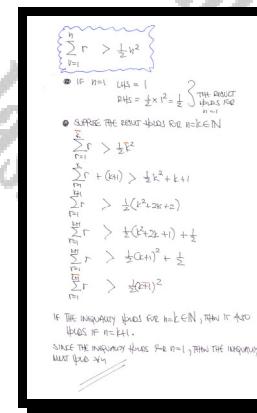
- IF $n=2$, $LHS = (1+a)^2 = a^2 + 2a + 1$
 $RHS = 1+2a$
 $\therefore a^2 + 2a + 1 > 1+2a$, SO THE RESULT IS TRUE FOR $n=2$.
- SUPPOSE THAT THE INEQUALITY HELDS FOR $n=k \in \mathbb{N}$, $k \geq 2$
 $\Rightarrow (1+a)^k > 1+ak$
 $\Rightarrow (1+a)(1+a) > (1+ak)(1+a)$
 $\Rightarrow (1+a)^{k+1} > 1+ak+a+a^2k$
 $\Rightarrow (1+a)^{k+1} > 1+a(k+1)+a^2k > 1+a(k+1)$
 $\Rightarrow (1+a)^{k+1} > 1+a(k+1)$ (Because $a^2k > 0$)
- IF THE INEQUALITY HELDS FOR $n=k \in \mathbb{N}$, $k \geq 2$, THEN IT WILL ALSO HOLD FOR $n=k+1$.
AS THE INEQUALITY HELDS FOR $n=2$, THEN IT MUST HOLD FOR ALL POSITIVE INTEGERS GREATER THAN 2.

Question 7 (***)+

Prove by induction that

$$\sum_{r=1}^n r > \frac{1}{2} n^2, \text{ for } n \geq 1, n \in \mathbb{N}.$$

[proof]

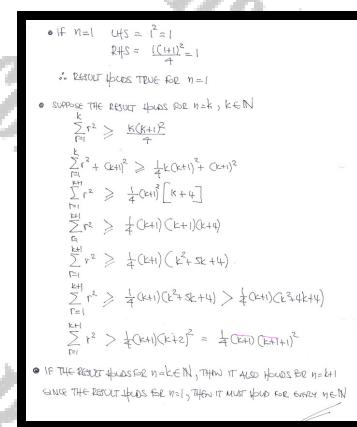


Question 8 (***)+

Prove by induction that

$$\sum_{r=1}^n r^2 \geq \frac{1}{4} n(n+1)^2, \text{ for } n \geq 1, n \in \mathbb{N}.$$

[proof]



Question 9 (**)**

Prove by induction that

$$2^n > 2n, \text{ for } n \geq 3, n \in \mathbb{N}.$$

proof

QUESTION $2^n > 2n$ for $n \in \mathbb{N}, n \geq 3$

- IF $n=3$: $2^3 = 8$, $2 \times 3 = 6$. $8 > 6$ ie result holds for $n=3$
- SUPPOSE THE RESULT HELPS FOR $n=k \in \mathbb{N}, n \geq 4$.

$$\begin{aligned} 2^k &> 2k \Rightarrow 2 \times 2^k > 2 \times 2k \\ 2^{k+1} &> 2k + 2k \\ 2^{k+1} &> 2k + 6 \quad (\text{AT LEAST } k=3) \\ 2^{k+1} &> 2k + 2 \cdot 2 \\ &\text{IF THE RESULT HELPS FOR } n=k \in \mathbb{N}, n \geq 3, \text{ THEN IT ALSO HELPS FOR } n=k+1 \\ &\text{SINCE THE RESULT HELPS FOR } n=3, \text{ THEN IT HELPS FOR } n \in \mathbb{N}, n \geq 3. \end{aligned}$$

Question 10 (**)**

Prove by induction that

$$2^n > n^2, \text{ for } n \geq 5, n \in \mathbb{N}.$$

proof

QUESTION $2^n > n^2$

- IF $n=5$: LHS is $2^5 = 32$? RHS is $5^2 = 25$? ie the result holds for $n=5$
- SUPPOSE THE RESULT HELPS FOR $n=k, k \in \mathbb{N}, k \geq 5$.

$$\begin{aligned} &\rightarrow 2^k > k^2 \\ &\Rightarrow 2 \times 2^k > 2 \times k^2 \\ &\Rightarrow 2^{k+1} > 2k^2 \\ &\Rightarrow 2^{k+1} > k^2 + k^2 \quad \left\{ \text{BE } k \geq 5 \quad [k^2 > 2k] \right. \\ &\Rightarrow 2^{k+1} > k^2 + (2k+1) \\ &\Rightarrow 2^{k+1} > (k+1)^2 \end{aligned}$$

$k^2 - 2k > 1$
 $k^2 - 2k + 1 > 2$
 $(k-1)^2 > 2$
 $k-1 > \sqrt{2}$
 $k-1 > 1.41$
 $k > 1.41$
 $k < 1+1.41$
 $\therefore \text{FOR } k \geq 5 \quad 2^{k+1} > (k+1)^2$
- IF THE RESULT HELPS FOR $n=k, k \in \mathbb{N}, k \geq 5$, THEN THE RESULT ALSO HELPS FOR $n=k+1$.
SINCE THE RESULT HELPS FOR $n=5$, THEN THE RESULT HELPS FOR $n \in \mathbb{N}, n \geq 5$.

Question 11 (**)**

Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$3^n > (n+1)^2.$$

, proof

IF $n \in \mathbb{N}$, $n \geq 3$ THEN $3^n > (n+1)^2$

BASE CASE, $n=3$

L.H.S = $3^3 = 27$
 R.H.S = $(3+1)^2 = 16$ $27 > 16$ SO THE INEQUALITY HOLDS FOR $n=3$.

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE INEQUALITY HOLDS FOR $n=k \in \mathbb{N}$, $k \geq 3$

$$\begin{aligned} \Rightarrow 3^k &> (k+1)^2 \\ \Rightarrow 3 \cdot 3^{k-1} &> 3(k+1)^2 \\ \Rightarrow 3^{k+1} &> 3k^2 + 6k + 3 > k^2 + 4k + 2 \\ \Rightarrow 3^{k+1} &> k^2 + 4k + 2 \quad \text{NOW AS } k \geq 3 \quad 2k+2 > 8 > 4 \\ \Rightarrow 3^{k+1} &> k^2 + 4k + 4 \\ \Rightarrow 3^{k+1} &> (k+2)^2 = [(k+1)+1]^2 \end{aligned}$$

CONCLUSION

IF THE INEQUALITY HOLDS FOR $n=k \in \mathbb{N}$, $k \geq 3$, THEN IT ALSO HOLDS FOR $n=k+1$.

SINCE THE INEQUALITY HOLDS FOR $n=3$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$, $n \geq 3$

Question 12 (**)**

Prove by induction that for all even natural numbers n

$$\frac{d^n}{dx^n}(\sin 3x) = (-1)^{\frac{n}{2}} \times 3^n \times \sin 3x.$$

, proof

CHECK THE BASE CASE, $n=2$

$$\begin{aligned} \frac{d^2}{dx^2}(\sin 3x) &= \frac{d}{dx}(3\cos 3x) = -9\sin 3x \\ (-1)^{\frac{2}{2}} \times 3^2 \times \sin 3x &= (-1) \times 9 \times \sin 3x = -9\sin 3x \end{aligned}$$

IF THE RESULT HOLDS FOR $n=2$,

SUPPOSE THAT THE RESULT HOLDS FOR $n=k = 2m$, $m \in \mathbb{N}$

$$\begin{aligned} \frac{d^k}{dx^k}(\sin 3x) &= (-1)^{\frac{k}{2}} \times 3^{\frac{k}{2}} \times \sin 3x \\ \frac{d^{2m}}{dx^{2m}}(\sin 3x) &= \frac{d}{dx} \left[(-1)^{\frac{k}{2}} \times 3^{\frac{k}{2}} \times \sin 3x \right] = (-1)^{\frac{k}{2}} \times 3^{\frac{k}{2}} \times 3\cos 3x \\ \frac{d^{2m+1}}{dx^{2m+1}}(\sin 3x) &= \frac{d}{dx} [(-1)^{\frac{k}{2}} \times 3^{\frac{k}{2}} \times 3\cos 3x] = (-1)^{\frac{k}{2}} \times 3^{\frac{k}{2}} \times (-18\sin 3x) \\ \frac{d^{2m+2}}{dx^{2m+2}}(\sin 3x) &= (-1)^{\frac{k+1}{2}} \times 3^{\frac{k+1}{2}} \times \sin 3x \\ \frac{d^{2m+2}}{dx^{2m+2}}(\sin 3x) &= (-1)^{\frac{2m+2}{2}} \times 3^{m+1} \times \sin 3x \\ \frac{d^{2m+2}}{dx^{2m+2}}(\sin 3x) &= (-1)^{m+1} \times 3^{m+1} \times \sin 3x \end{aligned}$$

IF THE RESULT HOLDS FOR $n=k=2m$, THEN IT MUST HOLD FOR $n=2m+2 = 2(m+1)$

AS THE RESULT HOLDS FOR $n=2$, THEN IT MUST HOLD FOR ALL EVEN INTEGERS

Question 13 (*)+)**

Prove by induction that for $n \geq 1$, $n \in \mathbb{N}$

$$\prod_{r=1}^n \left(\cos\left(2^{r-1}x\right) \right) = \frac{\sin(2^n x)}{2^n \sin x}.$$

 , proof

WRITE THE \prod EXPANDED EXACTLY

$$\prod_{r=1}^n \left[\cos\left(2^r x\right) \right] = \cos x \cos 2x \cos 4x \dots \cos(2^n x)$$

CHECK THE BASE CASE, i.e. IF $n=1$

$$\text{L.H.S.} = \prod_{r=1}^1 \cos(2^r x) = \cos x$$

$$\text{R.H.S.} = \frac{\sin(2x)}{2 \sin x} = \frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = (\cos x)$$

IF THE RESULT IS TRUE FOR $n=k$

SUPPOSE THAT THE RESULT IS TRUE FOR $n=k$ IN \mathbb{N}

$$\rightarrow \prod_{r=1}^k \left[\cos\left(2^r x\right) \right] = \frac{\sin(2^k x)}{2^k \sin x}$$

$$\rightarrow \prod_{r=1}^{k+1} \left[\cos\left(2^r x\right) \right] \times \cos(2^{k+1} x) = \frac{\sin(2^k x)}{2^k \sin x} \times \cos(2^{k+1} x)$$

$$\rightarrow \prod_{r=1}^{k+1} \left[\cos\left(2^r x\right) \right] = \frac{\sin(2^k x) \cos(2^{k+1} x)}{2^k \sin x} = \frac{2 \sin(2^k x) \cos(2^{k+1} x)}{2 \times 2^k \sin x}$$

$$\rightarrow \prod_{r=1}^{k+1} \left[\cos\left(2^r x\right) \right] = \frac{\sin[2 \times 2^k x]}{2^{k+1} \sin x}$$

$$\Rightarrow \prod_{r=1}^{k+1} \left[\cos\left(2^r x\right) \right] = \frac{\sin(2^{k+1} x)}{2^{k+1} \sin x}$$

IF THE RESULT IS TRUE FOR $n=k$ IN \mathbb{N} , THEN IT MUST ALSO BE TRUE FOR $n=k+1$

SINCE THE RESULT IS TRUE FOR $n=1$, THEN IT MUST ALSO BE TRUE FOR ALL $n \in \mathbb{N}$

Question 14 (****+)

Prove by induction that

$$\cos x + \cos 3x + \cos 5x + \dots + \cos[(2n-1)x] \equiv \frac{\sin(2nx)}{2\sin x}.$$

 , proof

PROVE THE BASE CASE, $n=1$

LHS = $\cos(2x-1) = \cos x$
 RHS = $\frac{\sin(2x)}{2\sin x} = \frac{\sin 2x}{2\sin x} = \frac{2\sin x \cos x}{2\sin x} = \cos x$
 ∴ THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$\sum_{i=1}^k \cos[(2i-1)x] = \frac{\sin(2kx)}{2\sin x}$
 $\cos[(2k+1)x] + \sum_{i=1}^k \cos[(2i-1)x] = \frac{\sin(2kx)}{2\sin x} + \cos[(2k+1)x]$
 $\sum_{i=1}^{k+1} \cos[(2i-1)x] = \frac{\sin(2kx)}{2\sin x} + \cos[(2k+1)x]$
 $\sum_{i=1}^{k+1} \cos[(2i-1)x] = \frac{\sin(2(k+1)x)}{2\sin x}$

PROVE WE NEED TO DEDUCE SOME IDENTITIES

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ } Adding
 $\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$
 $2\sin A \cos B = \sin(A+B) + \sin(A-B)$
 $2\sin A \cos(2kx) = \sin[2x+(2k-1)x] + \sin[2x-(2k-1)x]$
 $2\sin A \cos(2kx) = \sin[(2k+1)x] + \sin[2kx]$

RECURRING TO THE INDUCTION HYPOTHESIS: $\sin(A) = -\sin(-A)$

$\sum_{i=1}^{k+1} \cos[(2i-1)x] = \frac{\sin(2kx)}{2\sin x} + \frac{\sin(2(k+1)x)}{2\sin x} - \frac{\sin(2kx)}{2\sin x}$
 $\sum_{i=1}^{k+1} \cos[(2i-1)x] = \frac{\sin(2(k+1)x)}{2\sin x}$ [COMPARE THE "REVERSE" STEP AT THE START OF THE INDUCTION]

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$
 SINCE THE RESULT HOLDS FOR $n=1$, THEN THE RESULT HOLDS FOR $n \in \mathbb{N}$

REMARK: This proof uses the trigonometric identities $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and $\sin(A-B) = \sin A \cos B - \cos A \sin B$. It also uses the fact that $\sin(A) = -\sin(-A)$. These identities are derived from the sum and difference formulas for sine and cosine. The proof shows that if the result holds for $n=k$, it must also hold for $n=k+1$, thus establishing the result for all $n \in \mathbb{N}$.

Question 15 (****+)

Prove by induction that every positive integer power of 5 can be written as the sum of squares of two distinct positive integers.

 , proof

START BY INVESTIGATING SOME BASE CASES

IF $n=1$ $5^1 = 2^2 + 1^2$ i.e. result holds for $n=1$
 IF $n=2$ $5^2 = 3^2 + 4^2$ i.e. result holds for $n=2$

SUPPOSE THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$\Rightarrow x^2 + y^2 = 5^k$, WHERE x, y ARE DISTINCT INTEGERS
 $\Rightarrow 2(x^2 + y^2) = 2 \times 5^k$
 $\Rightarrow 2x^2 + 2y^2 = 5^k \times 5^k$
 $\Rightarrow (5x)^2 + (5y)^2 = 5^{2k+2}$
 [AS x, y ARE DISTINCT POSITIVE INTEGERS, SO
 WOULD $5x \neq 5y$]

IF THE RESULT HOLDS FOR $n=k$, IT WILL ALSO HOLD FOR $n=k+2$
 BUT THE RESULT HOLDS FOR $n=1$, SO IT MUST HOLD FOR ALL ODD INTEGERS
 POWERS OF 5
 AND AS THE RESULT HOLDS FOR $n=2$, IT MUST ALSO HOLD FOR ALL EVEN
 INTEGERS POWERS OF 5

∴ THE RESULT HOLDS FOR ALL $n \in \mathbb{N}$

Question 16 (*****)

Prove by induction that

$$\frac{d^n}{dx^n} (\mathrm{e}^x \cos x) = 2^{\frac{1}{2}n} \mathrm{e}^x \cos\left(x + \frac{n\pi}{4}\right), \quad n \geq 1, n \in \mathbb{N}.$$

S.P.², proof

$\frac{d^n}{dx^n} (\mathrm{e}^x \cos x) = 2^{\frac{1}{2}n} \mathrm{e}^x \cos\left(x + \frac{n\pi}{4}\right), \quad n \geq 1, n \in \mathbb{N}$

BASE CASE, n=1

- $\frac{d}{dx} (\mathrm{e}^x \cos x) = \mathrm{e}^x \cos x + \mathrm{e}^x (-\sin x) = \mathrm{e}^x [\cos x - \sin x]$
- $\text{R.H.S.} = 2^{\frac{1}{2}1} \mathrm{e}^x \cos\left(x + \frac{\pi}{4}\right) = 2^{\frac{1}{2}} \mathrm{e}^x \cos\left(x + \frac{\pi}{4}\right)$
 $= \sqrt{2} \mathrm{e}^x [\cos x \times \frac{1}{\sqrt{2}} - \sin x \times \frac{1}{\sqrt{2}}]$
 $= \sqrt{2} \mathrm{e}^x [\cos x - \sin x]$
 $= \mathrm{e}^x [\cos x - \sin x]$
i.e. RESULT HOLDS FOR n=1

INDUCTION HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n=k, k \in \mathbb{N}$

$$\Rightarrow \frac{d^k}{dx^k} (\mathrm{e}^x \cos x) = 2^{\frac{1}{2}k} \mathrm{e}^x \cos\left(x + \frac{k\pi}{4}\right)$$

$$\Rightarrow \frac{d}{dx} \left(\frac{d^k}{dx^k} (\mathrm{e}^x \cos x) \right) = \frac{d}{dx} \left[2^{\frac{1}{2}k} \mathrm{e}^x \cos\left(x + \frac{k\pi}{4}\right) \right]$$

$$\Rightarrow \frac{d^{k+1}}{dx^{k+1}} (\mathrm{e}^x \cos x) = 2^{\frac{1}{2}(k+1)} \left[\mathrm{e}^x \cos\left(x + \frac{(k+1)\pi}{4}\right) - 2^{\frac{1}{2}k} \sin\left(x + \frac{k\pi}{4}\right) \cdot \frac{1}{\sqrt{2}} \right]$$

$$\Rightarrow \frac{d^{k+1}}{dx^{k+1}} (\mathrm{e}^x \cos x) = 2^{\frac{1}{2}(k+1)} \mathrm{e}^x \cos\left(x + \frac{(k+1)\pi}{4}\right)$$

$$\Rightarrow \frac{d^{k+1}}{dx^{k+1}} (\mathrm{e}^x \cos x) = 2^{\frac{1}{2}(k+1)} \mathrm{e}^x \left[\cos\left(x + \frac{(k+1)\pi}{4}\right) - \sin\left(x + \frac{(k+1)\pi}{4}\right) \cdot \frac{1}{\sqrt{2}} \right]$$

$$\Rightarrow \frac{d^{k+1}}{dx^{k+1}} (\mathrm{e}^x \cos x) = 2^{\frac{1}{2}(k+1)} \mathrm{e}^x \cos\left(x + \frac{(k+1)\pi}{4}\right)$$

CONCLUSION

IF THE RESULT HOLDS FOR $n=k, k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 17 (*****)

It is given that for $n \in \mathbb{N}$

$$u_{n+1} = \frac{7u_n + 12}{u_n + 3}, \quad u_1 = 7.$$

Prove by induction that

$$u_n > 6.$$

, proof

$$u_{n+1} = \frac{7u_n + 12}{u_n + 3}, \quad u_1 = 7$$

• START BY REWRITING THE RECURRANCE RELATION AS FOLLOWS

$$u_{n+1} = \frac{7(u_n + 3) - 9}{u_n + 3} = 7 - \frac{9}{u_n + 3}$$

• SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$$\begin{aligned} &\Rightarrow u_k > 6 \\ &\Rightarrow u_k + 3 > 9 \\ &\Rightarrow \frac{9}{u_k + 3} < 1 \\ &\Rightarrow \frac{9}{u_k + 3} < 1 \\ &\Rightarrow -\frac{9}{u_k + 3} > -1 \\ &\Rightarrow 7 - \frac{9}{u_k + 3} > 6 \\ &\Rightarrow u_{k+1} > 6 \end{aligned}$$

• IF THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $k+1$ AS THE RESULT HOLDS FOR $n=1$ ($u_1 = 7$), THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 18 (*****)

Prove by induction that every positive integer power of 14 can be written as the sum of squares of three distinct positive integers.

, proof

• IF $n=1$ $1^2 + 2^2 + 3^2 = 14^1$, i.e. RESULT HOLDS FOR $n=1$
 $\frac{1^2 + 2^2 + 3^2}{14^2} = \frac{1}{14^2}$, i.e. RESULT NOT FOR $n=2$.

• SUPPOSE THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$$\begin{aligned} &\rightarrow 1^2 + 2^2 + 3^2 = 14^k \quad \text{WHERE } 1, 2, 3 \text{ ARE DISTINCT INTEGERS} \\ &\rightarrow 14^k (1^2 + 2^2 + 3^2) = 14^k \times 14^k \\ &\rightarrow 14^{2k} + 14^{2k} + 14^{2k} = 14^{2k+2} \\ &\Rightarrow (14^k)^2 + (14^k)^2 + (14^k)^2 = 14^{2k+2} \end{aligned}$$

IF $2, 3, 2$ ARE DISTINCT THEN $14^k, 14^k, 14^k$ ARE ALSO DISTINCT

• IF THE RESULT HOLDS FOR $n=k$, THEN IT MUST ALSO HOLD FOR $n=k+2$.
 SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL ODD INTEGERS.
 SINCE THE RESULT HOLDS FOR $n=2$, THEN IT MUST HOLD FOR ALL EVEN INTEGERS.
 THIS THE RESULT HOLDS $\forall n \in \mathbb{N}$

Question 19 (*****)

It is given that for $n \in \mathbb{N}$

$$U_n = \frac{2n}{2n+1} U_{n-1}, \quad U_1 = \frac{2}{3}.$$

Prove by induction that

$$U_n \leq \left(\frac{2n}{2n+1} \right)^n.$$

, proof

BASIC CASE (TO PROVE $U_1 \leq \left(\frac{2}{3} \right)^1$ FOR THE STRICT INEQUALITY)

IF $n=1$: $U_1 = \frac{2}{3}$ $U_1 = \frac{2 \cdot 1}{2 \cdot 1 + 1} = \frac{2}{3}$

IF $n=2$: $U_2 = \frac{2 \cdot 2}{2 \cdot 2 + 1} U_1$ $U_2 = \left(\frac{2 \cdot 2}{2 \cdot 2 + 1} \right)^2$
 $U_2 = \frac{4}{3} \cdot \frac{2}{3}$ $U_2 = \left(\frac{4}{3} \right)^2$
 $U_2 = \frac{16}{9}$ $U_2 = \frac{16}{9}$

∴ THE RESULT HOLDS FOR $n=1$ & $n=2$.

NOW SUPPOSE THAT THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$U_k \leq \left(\frac{2k}{2k+1} \right)^k$

WE HAVE NO SUCCESSOR

$U_{k+1} = \frac{2(k+1)}{2(k+1)+1} U_k = \frac{2k+2}{2k+3} U_k \leq \frac{2k+2}{2k+3} \left(\frac{2k}{2k+1} \right)^k$

REASON: $\frac{2k+2}{2k+3} < 1$

NOW WE HAVE TO PROVE THAT

$\frac{2k+2}{2k+3} < \frac{2k+2}{2k+1}$

DEFN: $f(x) = \frac{2x+2}{2x+3} - \frac{2x+2}{2x+1}, k \in \mathbb{N}$

$f'(x) = \frac{(2x+2)(2x+1) - 2x(2x+3)}{(2x+3)(2x+1)} = \frac{4x^2 + 8x + 2 - 4x^2 - 6x}{(2x+3)(2x+1)} = \frac{2x + 2}{(2x+3)(2x+1)} > 0$

$f(0) > 0 \Rightarrow \frac{2k+2}{2k+3} - \frac{2k+2}{2k+1} > 0$
 $\Rightarrow \frac{2k+2}{2k+3} < \frac{2k+2}{2k+1}$

RETURNS TO THE MAIN UNIT OF THE INDUCTION

$U_{k+1} = \dots = \frac{2k+2}{2k+3} \left(\frac{2k}{2k+1} \right)^k$
 $< \frac{2k+2}{2k+3} \left(\frac{2k+2}{2k+1} \right)^k$
 $= \left(\frac{2k+2}{2k+3} \right)^{k+1}$
 $= \left[\frac{2(k+1)}{2(k+1)+1} \right]^{k+1}$

∴ $U_{k+1} \leq \left[\frac{2(k+1)}{2(k+1)+1} \right]^{k+1}$

IF THE RESULT HOLDS FOR $n=k$, THEN IT MUST HOLD FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN THE RESULT HOLDS FOR $n \in \mathbb{N}$

Question 20 (*****)

Prove by induction that

$$\frac{d^n}{dx^n} \left(e^x \sin(\sqrt{3}x) \right) = 2^n e^x \sin\left(\sqrt{3}x + \frac{n\pi}{3}\right), \quad n \geq 1, n \in \mathbb{N}.$$

ISP, proof

$\frac{d}{dx} \left[e^x \sin(\sqrt{3}x) \right] = 2^1 e^x \sin\left(\sqrt{3}x + \frac{\pi}{3}\right)$

- IF $n=1$ $\frac{d}{dx} \left[e^x \sin(\sqrt{3}x) \right] = e^x \sin(\sqrt{3}x) + \sqrt{3} e^x \cos(\sqrt{3}x)$
 $= e^x \left[\sin(\sqrt{3}x) + \sqrt{3} \cos(\sqrt{3}x) \right]$
 $= 2e^x \left[\frac{1}{2} \sin(\sqrt{3}x) + \frac{\sqrt{3}}{2} \cos(\sqrt{3}x) \right]$
 $= 2e^x \left[\cos\left(\sqrt{3}x - \frac{\pi}{6}\right) \right]$
 $= 2e^x \sin\left(\sqrt{3}x + \frac{\pi}{3}\right)$

If THE RESULT APPLIES FOR $n=1$

- SUPPOSE THE RESULT APPLIES FOR $n=k \in \mathbb{N}$

$$\frac{d^k}{dx^k} \left[e^x \sin(\sqrt{3}x) \right] = 2^k e^x \sin\left(\sqrt{3}x + k\frac{\pi}{3}\right)$$

DIFFERENTIATE AGAIN W.R.T. x

$$\frac{d^{k+1}}{dx^{k+1}} \left[e^x \sin(\sqrt{3}x) \right] = 2^k e^x \sin\left(\sqrt{3}x + \frac{k\pi}{3}\right) + 2^k e^x \times \sqrt{3} \cos\left(\sqrt{3}x + \frac{k\pi}{3}\right)$$
 $= 2^{k+1} e^x \left[\sin\left(\sqrt{3}x + \frac{k\pi}{3}\right) + \sqrt{3} \cos\left(\sqrt{3}x + \frac{k\pi}{3}\right) \right]$
 $= 2^{k+1} e^x \left[\cos\left(\sqrt{3}x - \frac{(k-1)\pi}{3}\right) + \sin\left(\sqrt{3}x + \frac{(k+1)\pi}{3}\right) \right]$
 $= 2^{k+1} e^x \sin\left(\sqrt{3}x + \frac{(k+1)\pi}{3}\right)$
 $= 2^{k+1} e^x \sin\left(\sqrt{3}x + \frac{(k+1)\pi}{3}\right)$

If THE RESULT APPLIES FOR $n=k \in \mathbb{N} \Rightarrow$ THE RESULT ALSO APPLIES FOR $n=k+1$
SINCE THE RESULT APPLIES FOR $n=1 \Rightarrow$ THE RESULT HOLDS $\forall n \in \mathbb{N}$

Question 21 (*****)

The function $f(x)$ is defined by

$$f(x) = 2 - \frac{1}{x}, \quad x \in \mathbb{R}, x \neq 0.$$

a) Prove that

$$f^n(x) = \frac{(n+1)x - n}{nx - (n-1)}, \quad n \geq 1,$$

where $f^n(x)$ denotes the n^{th} composition of $f(x)$ by itself.

b) State an expression for the domain of $f^n(x)$.

$$\boxed{\quad}, \quad x \in \mathbb{R}, x \neq \frac{n-1}{n}$$

$$\begin{aligned} \bullet f^1(x) &= \frac{(1+1)x - 1}{1x - (1-1)} = \frac{2x - 1}{x} = 2 - \frac{1}{x} = f(x) \\ \bullet f^2(x) &= f(f(x)) = f\left(2 - \frac{1}{x}\right) = 2 - \frac{1}{2 - \frac{1}{x}} = 2 - \frac{x}{2x - 1} = \frac{4x - 2 - 1}{2x - 1} \\ &= \frac{3x - 1}{2x - 1}. \end{aligned}$$

$$\begin{aligned} \text{Also } f^3(x) &= \frac{(2+1)x - 2}{2x - (2-1)} = \frac{3x - 2}{2x - 1} \quad \Rightarrow \text{ result holds for } n=1,2. \end{aligned}$$

SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\begin{aligned} \bullet f^k(x) &= \frac{(k+1)x - k}{kx - (k-1)} \\ \bullet f^{k+1}(x) &= f(f^k(x)) = f\left(\frac{(k+1)x - k}{kx - (k-1)}\right) = 2 - \frac{1}{\frac{(k+1)x - k}{kx - (k-1)}} = 2 - \frac{kx - (k-1)}{(k+1)x - k} \\ &= \frac{2k+2x - 2k - kx + (k-1)}{(k+1)x - k} = \frac{(k+2)x - k - 1}{(k+1)x - k} = \frac{(k+1)x - (k+1)}{(k+1)x - (k+1)} \end{aligned}$$

THIS IS THE RESULT FOR $n=k+1 \in \mathbb{N} \Rightarrow$ THE RESULT ALSO HOLDS FOR $n=k+1$

GIVE THE RESULT HOLDS FOR ALL $n \in \mathbb{N} \Rightarrow$ THE DOMAIN MUST HOLD FOR $n \in \mathbb{N}$

(b) RESTRICTION IN DOMAIN OF $f(x)$ IS 1 NEGATIVE

$$\therefore nx - (n-1) \neq 0$$

$$\therefore x \neq \frac{n-1}{n}$$

$$\therefore \text{dom}_1, \text{dom}_{n-1}$$

Question 22 (*****)

Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$n^{n+1} > (n+1)^n,$$

and hence deduce that if $n \in \mathbb{N}$, $n \geq 3$, then

$$\sqrt[n]{n} > \sqrt[n+1]{n+1}$$

 , proof

IF $n \in \mathbb{N}, n \geq 3$, THEN $n^{n+1} > (n+1)^n$

BASE CASE, $n=3$

L.H.S. = $3^4 = 81$
R.H.S. = $4^3 = 64$ $81 > 64$, SO THE RESULT IS TRUE FOR $n=3$

INDUCTIVE HYPOTHESIS
SUPPOSE THAT THE RESULT HOLDS FOR $n=k \geq 3$, $k \in \mathbb{N}$

$\Rightarrow k^{k+1} > (k+1)^k$
 $\Rightarrow k^{k+1} (k+1)^{k+2} > (k+1)^k (k+1)^{k+2}$
 $\Rightarrow k^{k+1} (k+1)^{k+2} > (k+1)^{2k+2}$
 $\Rightarrow (k+1)^{2k+2} > \frac{(k+1)^{2k+2}}{k^{k+1}}$

NOW WE NEED TO SHOW THAT

$\frac{(k+1)^{2k+2}}{k^{k+1}} \geq (k+2)^{k+1} \Rightarrow (k+1)^{2k+2} > k^{k+1} \cdot (k+2)^{k+1}$
 $\Rightarrow [(k+1)^2]^{k+1} > [k(k+2)]^{k+1}$
 $\Rightarrow (k+1)^2 > k(k+2)$
 $\Rightarrow k^2 + 2k + 1 > k^2 + 2k$
WHICH IS TRUE

REFERENCING TO THE MAIN LINE OF THE INDUCTIVE HYPOTHESIS

- IF $k^{k+1} > (k+1)^k$
 $\dots \dots \dots$
 $\therefore (k+1)^{k+2} > \frac{(k+1)^{k+2}}{k^{k+1}} > (k+2)^{k+1}$
 i.e. $(k+1)^{k+2} > [(k+1)+1]^{k+1}$

CONCLUSION
IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, WITH $n \geq 3$, THEN IT MUST ALSO HOLD FOR $n=n+1$.
AS THE RESULT HOLDS FOR $n=3$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$, WITH $n \geq 3$

FINALLY WE HAVE

$n^{n+1} > (n+1)^n \quad n \in \mathbb{N}, n \geq 3$
 $\Rightarrow (n^{\frac{1}{n}})^{n(n+1)} > [(n+1)^{\frac{1}{n+1}}]^{n(n+1)}$
 $\Rightarrow [n^{\frac{1}{n}}]^{n^2+n} > [(n+1)^{\frac{1}{n+1}}]^{n^2+n}$
 $\Rightarrow \sqrt[n]{n^n} > \sqrt[n+1]{n+1}$