INTEGRAL THEOREMS

Green's Theorem

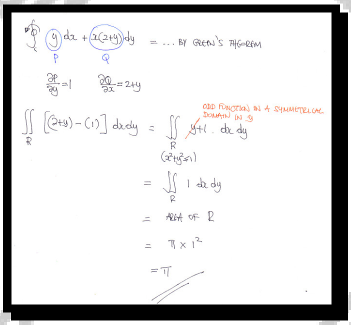
Question 1

Use Green's Theorem on the plane to evaluate the line integral

$$\oint_C \left[y\,dx + x(2+y)\,dy \right],$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

$\boxed{\pi}$



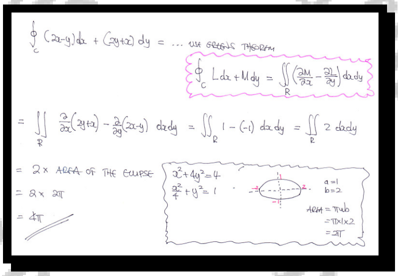
Question 2

Use Green's Theorem on the plane to evaluate the line integral

$$\oint_C (2x-y)\,dx + (2y+x)\,dy,$$

where C is the path around the ellipse with equation $x^2 + 4y^2 = 4$, taken in an anticlockwise direction.

$\boxed{4\pi}$



Question 3

Use Green's Theorem on the plane to evaluate the line integral

$$\oint_C y(x+1)e^x \, dx \; + \; x\left(e^x+1\right) \, dy ,$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

$\boxed{\pi}$

$$\int_C y(x+1)e^x \, dx + x(e^x+1) \, dy \qquad C: x^2+y^2=1$$

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\int_C \underbrace{y(x+1)e^x}_{P} \, dx + \underbrace{x(e^x+1)}_{Q} \, dy$$

$$\frac{\partial Q}{\partial x} = 1(e^x+1) + x(e^x)$$

$$\frac{\partial Q}{\partial x} = xe^x + e^x + 1$$

$$\frac{\partial P}{\partial y} = (x+1)e^x$$

$$= \iint_C (xe^x + e^x + 1) - (xe^x + e^x) \, dx\,dy$$

$$= \iint_C \cancel{xe^x} + \cancel{e^x} + 1 - \cancel{xe^x} - \cancel{e^x} \, dx\,dy$$

$$= \iint_C 1 \, dx\,dy$$

$= \text{AREA OF THE CIRCLE } x^2+y^2=1$

$= \pi$

Question 4

The functions F and G are defined as

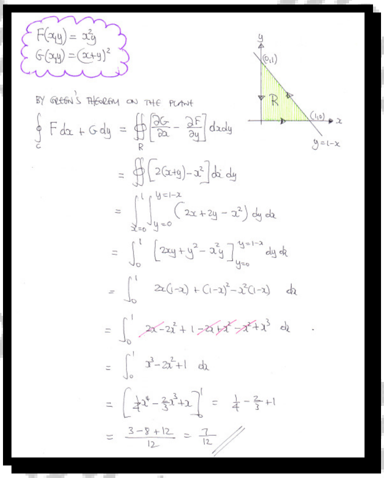
$$F(x,y) = x^2 y \quad \text{and} \quad G(x,y) = (x+y)^2$$

The anticlockwise path along the perimeter of the triangle whose vertices are located at $(0,0)$, $(1,0)$ and $(0,1)$, is denoted by C.

Use Green's Theorem on the plane to evaluate the line integral

$$\int_C (F\,dx + G\,dy).$$

$\boxed{\frac{7}{12}}$



$F(x,y) = x^2y$
$G(x,y) = (x+y)^2$

BY GREEN'S THEOREM ON THE PLANE

$$\oint_C F\,dx + G\,dy = \iint_R \left[\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right] dx\,dy$$

$$= \iint \left[2(x+y) - x^2\right] dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{y=1-x} \left(2x + 2y - x^2\right) dy\,dx$$

$$= \int_0^1 \left[2xy + y^2 - x^2y\right]_{y=0}^{y=1-x} dy\,dx$$

$$= \int_0^1 2x(1-x) + (1-x)^2 - x^2(1-x)\; dx$$

$$= \int_0^1 \cancel{2x} - 2x^2 + 1 \cancel{-2x} \cancel{+x^2} \cancel{-x^2} + x^3\; dx$$

$$= \int_0^1 x^3 - 2x^2 + 1\; dx$$

$$= \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + x\right]_0^1 = \frac{1}{4} - \frac{2}{3} + 1$$

$$= \frac{3 - 8 + 12}{12} = \frac{7}{12}$$

Question 5

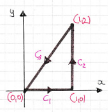
The contour C is the boundary of a triangle with vertices at the points with Cartesian coordinates $(0,0)$, $(1,0)$ and $(1,2)$, traced in an anticlockwise direction.

Verify Green's Theorem on the plane for the line integral

$$\oint_C (3x+4y)\,dx + (5x-2y)\,dy\,.$$

$\boxed{\text{both sides yield 1}}$

STARTING WITH THE LINE INTEGRAL



$$\boxed{\begin{array}{l} C_1:\ y=0,\ dy=0,\ x \text{ RUNS FROM } 0 \text{ TO } 1 \\ C_2:\ x=1,\ dx=0,\ y \text{ RUNS FROM } 0 \text{ TO } 2 \\ C_3:\ y=2x,\ dy=2dx,\ x \text{ RUNS FROM } 1 \text{ TO } 0 \end{array}}$$

HENCE WE NOW HAVE

$$\oint_C (3x+4y)\,dx + (5x-2y)\,dy$$

$$= \underbrace{\int_{x=0}^{x=1} 3x\,dx}_{C_1} + \underbrace{\int_{y=0}^{y=2} 5-2y\,dy}_{C_2} + \underbrace{\int_{x=1}^{x=0} [3x+4(2x)]\,dx + [5x-2(2x)](2dx)}_{C_3}$$

$$= \int_0^1 3x\,dx + \int_0^2 5-2y\,dy + \int_1^0 x\,dx$$

$$= \int_0^1 3x\,dx - \int_0^1 13x\,dx + \int_0^2 5-2y\,dy$$

$$= \int_0^2 5-2y\,dy - \int_0^1 10x\,dx$$

$$= \left[5y-y^2\right]_0^2 - \left[5x^2\right]_0^1$$

$$= \left[(10-4)-0\right] - \left[5-0\right]$$

$$= 6-5$$

$$= 1$$

NEXT GREEN'S THEOREM STATES

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

$$\Rightarrow \iint_{\substack{\text{AREA OF}\\\text{TRIANGLE}}} \left[\frac{\partial}{\partial x}(5x-2y) - \frac{\partial}{\partial y}(3x+4y)\right] dx\,dy$$

$$= \iint_{\substack{\text{AREA OF}\\\text{TRIANGLE}}} (5-4)\,dy\,dx$$

$$= \iint_{\substack{\text{AREA OF}\\\text{TRIANGLE}}} 1\,dx\,dy$$

$$= \text{AREA OF THE TRIANGLE}$$

$$= \tfrac{1}{2}\times 1\times 2$$

$$= 1$$

AND THE THEOREM IS VERIFIED

Question 6

The functions $P(x,y)$ and $Q(x,y)$ have continuous first order partial derivatives.

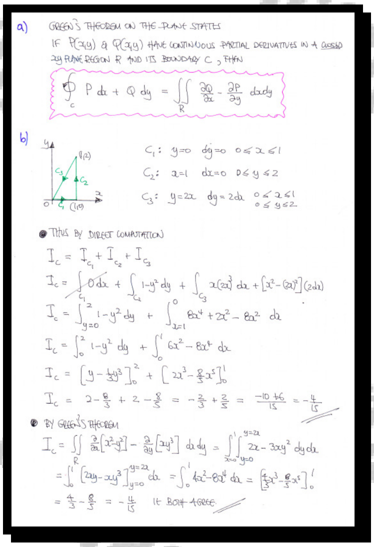
a) State formally Green's theorem in the plane, with reference to P and Q.

The contour C is the boundary of a triangle with vertices at the points with Cartesian coordinates $(0,0)$, $(1,0)$ and $(1,2)$.

b) Verify Green's Theorem on the plane for the line integral

$$\int_C (xy^3)\,dx + (x^2 - y^2)\,dy .$$

$\boxed{\text{both sides yield } -\frac{4}{15}}$



a) GREEN'S THEOREM ON THE PLANE STATES

IF $P(x,y)$ & $Q(x,y)$ HAVE CONTINUOUS PARTIAL DERIVATIVES IN A CLOSED xy PLANE REGION R AND ITS BOUNDARY C, THEN

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy}$$

b)

C_1: $y=0$ $dy=0$ $0 \le x \le 1$

C_2: $x=1$ $dx=0$ $0 \le y \le 2$

C_3: $y=2x$ $dy=2dx$ $0 \le x \le 1$, $0 \le y \le 2$

● THUS BY DIRECT COMPUTATION

$I_C = I_{C_1} + I_{C_2} + I_{C_3}$

$I_C = \cancel{\int_{C_1} 0\,dx} + \int_{C_2} 1-y^2\,dy + \int_{C_3} x(2x)^3\,dx + [x^2-(2x)^2](2dx)$

$I_C = \int_{y=0}^{2} 1-y^2\,dy + \int_{x=1}^{0} 8x^4 + 2x^2 - 8x^2\,dx$

$I_C = \int_0^2 1-y^2\,dy + \int_0^1 6x^2 - 8x^4\,dx$

$I_C = \left[y - \frac{1}{3}y^3\right]_0^2 + \left[2x^3 - \frac{8}{5}x^5\right]_0^1$

$I_C = 2 - \frac{8}{3} + 2 - \frac{8}{5} = -\frac{2}{3} + \frac{2}{5} = \frac{-10+6}{15} = -\frac{4}{15}$

● BY GREEN'S THEOREM

$I_C = \iint_R \frac{\partial}{\partial x}[x^2-y^2] - \frac{\partial}{\partial y}[xy^3]\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{y=2x} 2x - 3xy^2\,dy\,dx$

$= \int_0^1 \left[2xy - xy^3\right]_{y=0}^{y=2x} dx = \int_0^1 4x^2 - 8x^4\,dx = \left[\frac{4}{3}x^3 - \frac{8}{5}x^5\right]_0^1$

$= \frac{4}{3} - \frac{8}{5} = -\frac{4}{15}$ // BOTH AGREE

Question 7

The functions $P(x,y)$ and $Q(x,y)$ have continuous first order partial derivatives.

a) State formally Green's theorem in the plane, with reference to the functions, P and Q.

b) Evaluate the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\left(x^2-7y^2\right)\,dy\,dx\,.$$

c) By considering a line integral over a suitable contour C, use Green's theorem in the plane to independently verify the answer to part **(b)**.

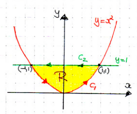
$\boxed{-\frac{56}{15}}$

a) IF $P(x,y)$ & $Q(x,y)$ HAVE CONTINUOUS FIRST ORDER PARTIAL DERIVATIVES IN A REGION R IN THE x-y PLANE AND IN THE CLOSED BOUNDARY WHICH CONTAINS R, THEN

$$\oint_C P\,dx+Q\,dy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

WHERE C IS TRACED ANTICLOCKWISE

b) START WITH A SKETCH SHOWING THE REGION OF INTEGRATION



$$\iint_R x^2-7y^2\,dy\,dx=\int_{x=-1}^{1}\int_{y=x^2}^{y=1}x^2-7y^2\,dy\,dx$$

$$=\int_{x=-1}^{1}\left[x^2y-\frac{7}{3}y^3\right]_{y=x^2}^{y=1}dx$$

$$=\int_{-1}^{1}x^2-\frac{7}{3}-\left(x^4-\frac{7}{3}x^6\right)dx$$

$$=\int_{-1}^{1}\frac{7}{3}x^6-x^4+x^2-\frac{7}{3}\,dx=2\int_{0}^{1}\frac{14}{3}x^6-2x^4+2x^2-\frac{14}{3}\,dx$$

$$=\left[\frac{2}{3}x^7-\frac{2}{5}x^5+\frac{2}{3}x^3-\frac{14}{3}\right]_0^1=\frac{2}{3}-\frac{2}{5}+\frac{2}{3}-\frac{14}{3}=-\frac{2}{5}-\frac{10}{3}=-\frac{56}{15}$$

c) NOW WE NEED TO CHANGE THE INTEGRAND IN A "CURL FORM"

LET $-\frac{\partial P}{\partial y}=x^2-7y^2$

$\frac{\partial P}{\partial y}=7y^2-x^2$

$P(x,y)=\frac{7}{3}y^3-x^2y+F(x)$

$\therefore$ PICK $G(x)$ SUCH THAT $\frac{d}{dx}\left[G(x)\right]=F(x)$

FORMING A LINE INTEGRAL USING GREEN'S THEOREM (TAKE $Q(x,y)=0$)

$$\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\oint_C P\,dx+Q\,dy$$

$$\iint_R x^2-7y^2\,dx\,dy=\oint_C\frac{7}{3}y^3-x^2y+F(x)\,dx$$

- ON C_1
  $y=x^2$
  $dy=2x\,dx$
  $-1\le x\le 1$
- ON C_2
  $y=1$
  $dy=0$
  BUT x RUNS FROM 1 TO -1

SPLIT INTO TWO PARTS C_1 & C_2

$$\dots=\int_{C_1}\frac{7}{3}y^3-x^2y+F(x)\,dx+\int_{C_2}\frac{7}{3}y^3-x^2y+F(x)\,dx$$

$$\dots=\int_{x=-1}^{1}\frac{7}{3}x^6-x^4+F(x)\,dx+\int_{x=1}^{x=-1}\frac{7}{3}-x^2+F(x)\,dx$$

$$=\left[\frac{1}{3}x^7-\frac{1}{5}x^5+G(x)\right]_{-1}^{1}+\left[\frac{7}{3}x-\frac{1}{3}x^3+G(x)\right]_{1}^{-1}$$

$$=\left[\frac{1}{3}-\frac{1}{5}+\cancel{G(1)}\right]-\left[-\frac{1}{3}+\frac{1}{5}+\cancel{G(-1)}\right]+\left[-\frac{7}{3}+\frac{1}{3}+\cancel{G(-1)}\right]-\left[\frac{7}{3}-\frac{1}{3}+\cancel{G(1)}\right]$$

$$=\frac{2}{15}+\frac{2}{15}-2-2=-\frac{56}{15}$$

AS BEFORE

Question 8

The closed curve C bounds the finite region R in the x-y plane defined as

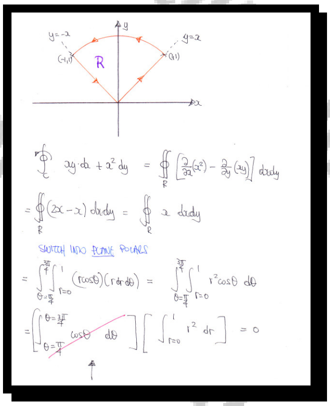
$$R(x,y) = \{x + y \geq 0 \cap x - y \leq 0 \cap x^2 + y^2 \leq 1\}.$$

Evaluate the line integral

$$\oint_C \left(xy\,dx + x^2\,dy\right),$$

where C is traced anticlockwise.

$\boxed{0}$



Question 9

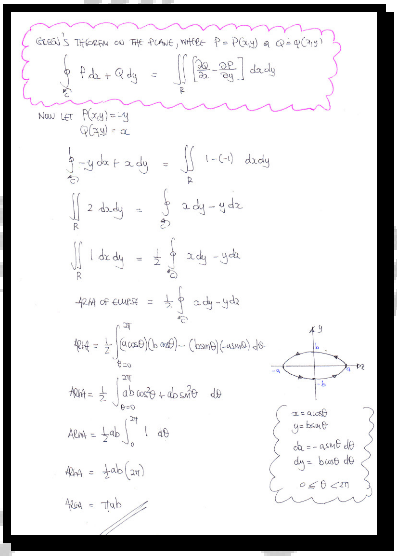
An ellipse has Cartesian equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where a and b positive constants.

Use Green's theorem in the plane, to show that the area of the ellipse is πab.

$\boxed{\text{proof}}$



GREEN'S THEOREM ON THE PLANE, WHERE $P = P(x,y)$ & $Q = Q(x,y)$

$$\oint_C P\,dx + Q\,dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy$$

NOW LET $P(x,y) = -y$
$Q(x,y) = x$

$$\oint_C -y\,dx + x\,dy = \iint_R 1-(-1)\ dx\,dy$$

$$\iint_R 2\,dx\,dy = \oint_C x\,dy - y\,dx$$

$$\iint_R 1\,dx\,dy = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$\text{AREA OF ELLIPSE} = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$\text{AREA} = \frac{1}{2}\int_{\theta=0}^{2\pi} (a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\,d\theta$$

$$\text{AREA} = \frac{1}{2}\int_{\theta=0}^{2\pi} ab\cos^2\theta + ab\sin^2\theta\ \ d\theta$$

$$\text{AREA} = \frac{1}{2}ab\int_0^{2\pi} 1\ d\theta$$

$$\text{AREA} = \frac{1}{2}ab(2\pi)$$

$$\text{AREA} = \pi ab$$

$x = a\cos\theta$
$y = b\sin\theta$
$dx = -a\sin\theta\,d\theta$
$dy = b\cos\theta\,d\theta$
$0 \le \theta < 2\pi$

Question 10

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F} = \left(\sin x^3 - xy\right)\mathbf{i} + \left(x + y^3 \sin y\right)\mathbf{j}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr},$$

where C is the ellipse with cartesian equation

$$2x^2 + 3y^2 = 2y.$$

$\boxed{}$, $\boxed{\dfrac{\pi}{3\sqrt{6}}}$

PROCEED AS FOLLOWS

$$\oint_C \underline{F} \cdot d\underline{r} = \oint_C \left(\sin x^3 - xy,\ y^3 \sin y + x\right) \cdot (dx, dy)$$
$$= \oint_C \left(\sin x^3 - xy\right) dx + \left(y^3 \sin y + x\right) dy$$

NOW GREEN'S THEOREM ON THE PLANE ASSERTS THAT

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

APPLYING IT HERE YIELDS

$$\dots = \iint_R \left[\frac{\partial}{\partial x}\left[y^3 \sin y + x\right] - \frac{\partial}{\partial y}\left[\sin x^3 - xy\right]\right] dx\,dy$$
$$= \iint_R 1 - x \quad dx\,dy$$

NOW LOOKING AT THE REGION R, WHICH IS THE ELLIPSE ANALYSED OPPOSITE WE HAVE

$$= \iint_R 1 \quad dx\,dy$$

AS x IS AN ODD POWER IN A SYMMETRICAL DOMAIN IN x

$$= 1 \times \text{AREA OF THE ELLIPSE}$$
$$= 1 \times \pi \times \frac{1}{3} \times \frac{1}{\sqrt{6}}$$
$$= \frac{\pi}{3\sqrt{6}}$$

R: $2x^2 + 3y^2 = 2y$

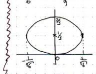
$2x^2 + 3y^2 - 2y = 0$

$\frac{2}{3}x^2 + y^2 - \frac{2}{3}y = 0$

$\frac{2}{3}x^2 + \left(y - \frac{1}{3}\right)^2 = \frac{1}{9}$

$6x^2 + 9\left(y - \frac{1}{3}\right)^2 = 1$

$\frac{x^2}{\frac{1}{6}} + \frac{\left(y - \frac{1}{3}\right)^2}{\frac{1}{9}} = 1$



Question 11

It is given that the vector function $\mathbf{F}$ satisfies

$$\mathbf{F} = \left[x\cos x\right]\mathbf{i} + \left[15xy + \ln\left(1+y^3\right)\right]\mathbf{j}.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr},$$

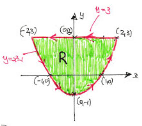
where C is the curve

$$\left\{(x,y): y=3,\ -2 \le x \le 2\right\} \cup \left\{(x,y): y = x^2-1,\ -2 \le x \le 2\right\},$$

traced in an anticlockwise direction.

$\boxed{}$, $\boxed{224}$

START BY SKETCHING THE PATH C



$$\oint_C \underline{F} \cdot d\underline{r}$$

$$= \oint_C \left[x\cos x,\ 15xy + \ln(1+y^3)\right] \cdot \left[dx\ dy\right]$$

$$= \oint_C \left[\underbrace{(x\cos x)}_{P}\,dx + \underbrace{\left[15xy + \ln(1+y^3)\right]}_{Q}\,dy\right]$$

AS THIS INTEGRATION LOOKS LIKE AN IMPOSSIBILITY, INVESTIGATE WHETHER GREEN'S THEOREM ON THE PLANE CAN BE USED

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

$$= \iint_R (15y - 0)\,dx\,dy = \int_{x=-2}^{x=2}\int_{y=x^2-1}^{y=3} 15y\ dy\,dx$$

$$= \int_{-2}^{2} \left[\frac{15}{2}y^2\right]_{y=x^2-1}^{y=3} dx = \frac{15}{2}\int_{-2}^{2} 3^2 - (x^2-1)^2\ dx$$

$$= \frac{15}{2}\int_{-2}^{2} 9 - (x^2-1)^2\ dx$$

THE INTEGRAND IS EVEN IN x

$$= 15\int_0^2 9 - (x^2-1)^2\ dx$$

$$= 15\int_0^2 9 - (x^4 - 2x^2 + 1)\ dx$$

$$= 15\int_0^2 8 + 2x^2 - x^4\ dx$$

$$= \int_0^2 120 + 30x^2 - 15x^4\ dx$$

$$= \left[120x + 10x^3 - 3x^5\right]_0^2$$

$$= (240 + 80 - 96) - (0)$$

$$= 224$$

Gauss' Theorem

also known as the Divergence Theorem

Question 1

$$\mathbf{A}(x,y,z) \equiv (2x+y-z)\mathbf{i} + \left(xy^2z\right)\mathbf{j} + (xy-2yz)\mathbf{k}\,.$$

Evaluate the integral

$$\oiint_S \mathbf{A}\cdot\mathbf{dS}\,,$$

where S is the **closed** surface enclosing the finite region V, defined by

$$-1 \le x \le 2\,, \quad -2 \le y \le 2\,, \quad 1 \le z \le 3\,.$$

$\boxed{48}$

$$\underline{A} = (2x+y-z)\underline{i} + (xy^2z)\underline{j} + (xy-2zy)\underline{k}$$

$\int_S \underline{A}\cdot d\underline{S} = \ldots$ SINCE THE SURFACE IS CLOSED (CUBOID) USE THE DIVERGENCE THEOREM BELOW.

$$\iiint_V \nabla\cdot\underline{A}\, dV = \oiint_S \underline{A}\cdot d\underline{S}$$

$$\ldots = \int_V \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\left(2x+y-z,\, xy^2z,\, xy-2zy\right)\, dV$$

$$= \int_{z=1}^{3}\int_{y=-2}^{2}\int_{x=-1}^{2} 2 + \underbrace{\cancel{2xyz} - \cancel{2y}}_{\text{ODD IN } y}\; dx\,dy\,dz$$

$$= \int_{z=1}^{3}\int_{y=-2}^{2}\int_{x=-1}^{2} 2\; dx\,dy\,dz$$

$= 2 \times$ VOLUME OF THE CUBOID

$$= 2\times(\underset{x}{3}\times\underset{y}{4}\times\underset{z}{2})$$

$$= 48$$

Question 2

The surface S is the sphere with Cartesian equation

$$x^2 + y^2 + z^2 = 1$$

Use the Divergence Theorem to evaluate

$$\oiint_S \left(x^2 + y + z\right) dS .$$

$\boxed{}$, $\boxed{\frac{4}{3}\pi}$

ALTHOUGH THIS IS NOT A FLUX INTEGRAL, IT CAN BE MANIPULATED AS FOLLOWS, SINCE THE SURFACE IS CLOSED AND THE DIVERGENCE THEOREM CAN BE USED

$$\oiint_S x^2+y+z \, dS = \oiint_S (x,1,1)\cdot(x,y,z)\, dS$$

NOW WE HAVE SINCE THE SURFACE IS A SPHERE

$$S: \; x^2+y^2+z^2=1$$
$$f(x,y,z) = x^2+y^2+z^2-1$$
$$\nabla f = (2x,2y,2z)$$
SCALED
$$\underline{n} = (x,y,z)$$
$$|\underline{n}| = \sqrt{x^2+y^2+z^2} = 1$$
$$\therefore \; \boxed{\hat{\underline{n}} = (x,y,z)}$$

RETURNING TO THE INTEGRAL, WE NOW HAVE

$$\ldots = \oiint_S (x,1,1)\cdot\hat{\underline{n}}\, dS = \oiint_S \underline{F}\cdot\hat{\underline{n}}\, dS$$

BY THE DIVERGENCE THEOREM

$$= \iiint_V \underline{\nabla}\cdot\underline{F}\, dV = \iiint_V \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(x,1,1)\, dV$$

$$= \iiint_V 1+0+0 \, dV = \iiint_V 1 \, dV$$

= VOLUME OF THE SPHERE OF RADIUS 1

$$= \frac{4}{3}\pi \times 1^3$$

$$= \frac{4}{3}\pi$$

Question 3

$$\mathbf{F}(x,y,z) \equiv z^2\mathbf{i} + \left(y^2 - x^2\right)\mathbf{j} + \left(x^2 + z^2\right)\mathbf{k}.$$

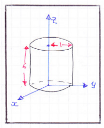
Evaluate the integral

$$\oiint_S \mathbf{F} \cdot \mathbf{dS},$$

where S is the surface of a cylinder of radius 1, whose axis is the z axis, between $z = 0$ and $z = 6$.

$\boxed{36\pi}$

$$\boxed{\underline{F} = \left(z^2, y^2 - x^2, x^2 + z^2\right)}$$



$$\text{FLUX} = \int_S \underline{F} \cdot d\underline{S} = \int_V \underline{\nabla} \cdot \underline{F} \, dV$$

(BY THE DIVERGENCE THEOREM)

$$= \int_V \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(z^2, y^2 - x^2, x^2 + z^2\right) dV$$

$$= \int_V (0 + 2y + 2z) \, dv$$

(USING CYLINDRICAL POLARS)

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{6} (2r\sin\theta + 2z)(r \, dr \, d\theta \, dz)$$

$$= \int_{z=0}^{6} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(\cancel{2r^2\sin\theta} + 2rz\right) dr \, d\theta \, dz$$

FROM THE θ INTEGRATION

$$= \int_{z=0}^{6} \int_{\theta=0}^{2\pi} \left[r^2 z\right]_{r=0}^{1} d\theta \, dz$$

$$= \int_{z=0}^{6} \int_{\theta=0}^{2\pi} z \, d\theta \, dz$$

$$= 2\pi \int_{z=0}^{6} z \, dz$$

$$= 2\pi \left[\tfrac{1}{2} z^2\right]_0^6$$

$$= 36\pi$$

Question 4

$$\mathbf{F}(x, y, z) \equiv xy\mathbf{i} + y\mathbf{j} + 4\mathbf{k}\,.$$

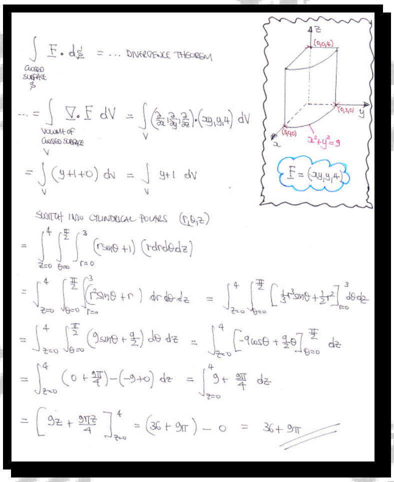
Evaluate the integral

$$\oiint_S \mathbf{F} \cdot \mathbf{dS}\,,$$

where S is the **closed** surface enclosing the finite region V, defined by

$$x^2 + y^2 \le 9\,, \quad x \ge 0\,, \quad y \ge 0\,, \quad 0 \le z \le 4\,.$$

$\boxed{9\pi + 36}$



$\int_{\substack{\text{CLOSED}\\ \text{SURFACE}\\ S}} \underline{F} \cdot d\underline{S} = \dots$ DIVERGENCE THEOREM

$\dots = \int_{\substack{\text{VOLUME OF}\\ \text{CLOSED SURFACE}\\ V}} \underline{\nabla} \cdot \underline{F}\, dV = \int_V \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (xy, y, 4)\, dV$

$= \int_V (y + 1 + 0)\, dV = \int_V y + 1\, dV$

SWITCH INTO CYLINDRICAL POLARS (r, θ, z)

$= \int_{z=0}^{4} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{3} (r\sin\theta + 1)(r\,dr\,d\theta\,dz)$

$= \int_{z=0}^{4} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{3} (r^2\sin\theta + r)\, dr\, d\theta\, dz = \int_{z=0}^{4} \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{1}{3}r^3\sin\theta + \frac{1}{2}r^2\right]_{r=0}^{3} d\theta\, dz$

$= \int_{z=0}^{4} \int_{\theta=0}^{\frac{\pi}{2}} \left(9\sin\theta + \frac{9}{2}\right) d\theta\, dz = \int_{z=0}^{4} \left[-9\cos\theta + \frac{9}{2}\theta\right]_{\theta=0}^{\frac{\pi}{2}} dz$

$= \int_{z=0}^{4} \left(0 + \frac{9\pi}{4}\right) - (-9 + 0)\, dz = \int_{z=0}^{4} 9 + \frac{9\pi}{4}\, dz$

$= \left[9z + \frac{9\pi z}{4}\right]_{z=0}^{4} = (36 + 9\pi) - 0 = 36 + 9\pi$

Question 5

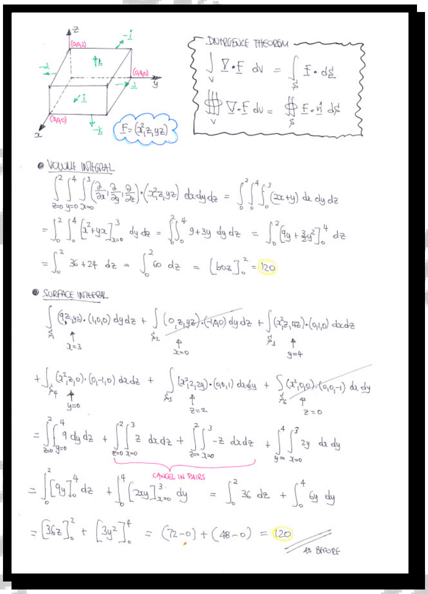
The vector field $\mathbf{F}$ exists inside and around the finite region V, defined by the inequalities

$$0 \le x \le 3, \quad 0 \le y \le 4 \quad \text{and} \quad 0 \le z \le 2 .$$

Use V to verify the Divergence Theorem of Gauss, given further that

$$\mathbf{F}(x,y,z) \equiv x^2\mathbf{i} + z\mathbf{j} + yz\mathbf{k} .$$

$\boxed{\text{both sides yield } 120}$



Question 6

$$\mathbf{F}(x,y,z) \equiv \left(x+y^2\right)\mathbf{i} + (2y+xz)\mathbf{j} + (3z+xyz)\mathbf{k}.$$

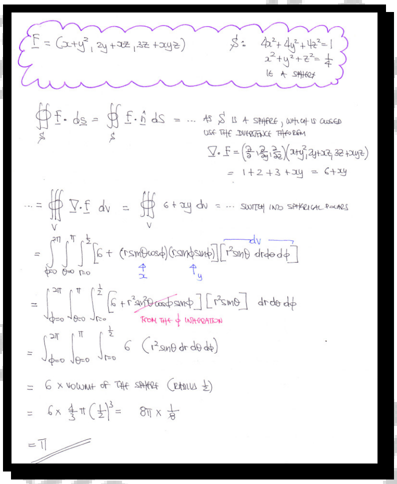
Evaluate the integral

$$\oiint_S \mathbf{F} \cdot \mathbf{dS},$$

where S is the surface with Cartesian equation

$$4x^2+4y^2+4z^2=1.$$

$\boxed{\pi}$



$\underline{F} = (x+y^2, 2y+xz, 3z+xyz)$ $\quad S: \; 4x^2+4y^2+4z^2=1$, $\; x^2+y^2+z^2=\frac{1}{4}$ IS A SPHERE

$\oiint_S \underline{F} \cdot d\underline{S} = \oiint_S \underline{F}\cdot\hat{\underline{n}}\,dS = \dots$ AS S IS A SPHERE, WHICH IS CLOSED USE THE DIVERGENCE THEOREM

$\underline{\nabla}\cdot\underline{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(x+y^2, 2y+xz, 3z+xyz) = 1+2+3+xy = 6+xy$

$\dots = \iiint_V \underline{\nabla}\cdot\underline{F}\,dV = \iiint_V 6+xy\,dV = \dots$ SWITCH INTO SPHERICAL POLARS

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\frac{1}{2}} \Big[6 + \underbrace{(r\sin\theta\cos\phi)}_{x}\underbrace{(r\sin\theta\sin\phi)}_{y}\Big]\Big[\overbrace{r^2\sin\theta\,dr\,d\theta\,d\phi}^{dV}\Big]$

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\frac{1}{2}} \Big[6 + \cancel{r^2\sin^2\theta\cos\phi\sin\phi}\Big]\Big[r^2\sin\theta\Big]\,dr\,d\theta\,d\phi$

FROM THE ϕ INTEGRATION

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\frac{1}{2}} 6\,(r^2\sin\theta\,dr\,d\theta\,d\phi)$

$= 6 \times$ VOLUME OF THE SPHERE (RADIUS $\frac{1}{2}$)

$= 6\times\frac{4}{3}\pi\left(\frac{1}{2}\right)^3 = 8\pi\times\frac{1}{8}$

$= \pi$

Question 7

A smooth vector field $\mathbf{A}$, exists in and on the boundary of a smooth closed surface S, and $\hat{\mathbf{n}}$ is an outward unit vector to S.

a) Show that

$$\int_S \nabla \wedge \mathbf{A} \cdot \hat{\mathbf{n}} \, dS = 0$$

You may find the Divergence Theorem useful in this part.

b) Prove the validity of the result of part **(a)** if

- $\mathbf{A} = xy\,\mathbf{i} + y^2\,\mathbf{j} + zx^2\,\mathbf{k}$
- $S : x^2 + y^2 + z^2 = 1,\ z \geq 0$.

proof

a) BY THE DIVERGENCE THEOREM

$$\iiint_V \nabla \cdot \underline{F} \, dV = \oint_S \underline{F} \cdot d\underline{S}$$ WHERE S IS A CLOSED SURFACE ENCLOSING A VOLUME V

THUS LET $\underline{F} = \nabla \wedge \underline{A}$ FOR SOME VECTOR FIELD $\underline{A}$

SO $$\iiint_V \nabla \cdot (\nabla \wedge \underline{A}) \, dV = \oint_S \nabla \wedge \underline{A} \cdot \hat{\underline{n}} \, dS$$

BUT $\nabla \cdot (\nabla \wedge \underline{A}) = 0$, IDENTITY

$$\therefore \oint_S \nabla \wedge \underline{A} \cdot \hat{\underline{n}} \, dS = 0$$

b) • $\underline{A} = (xy, y^2, zx^2)$

$$\nabla \wedge \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 & zx^2 \end{vmatrix} = (0-0, 0-2xz, 0-x) = (0, -2xz, -x)$$

THE SURFACE IS A HEMISPHERE WITH BASE

LET THE CURVED SURFACE HAVE EQUATION

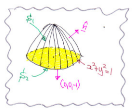
$\phi(x,y,z) = x^2+y^2+z^2-1$

$\nabla\phi = (2x, 2y, 2z)$

$\underline{n} = (x, y, z)$

$|\underline{n}| = \sqrt{x^2+y^2+z^2} = \sqrt{1} = 1$

$\hat{\underline{n}} = (x, y, z)$



• ON THE CURVED SURFACE S_1, $\int_{S_1} \nabla \wedge \underline{A} \cdot \hat{\underline{n}} \, dS$

$$= \int_{S_1} (0, -2xz, -x) \cdot (x, y, z) \, dS = \int_{S_1} (0 - 2xyz - xz) \, dS$$

PROJECT ONTO THE CIRCLE $x^2+y^2 \leq 1$, REGION R

$$= \int_R (-2xyz - xz) \frac{dx\,dy}{\hat{\underline{n}} \cdot \underline{k}} = \int_R (-2xyz - xz) \frac{dx\,dy}{(x,y,z)\cdot(0,0,1)}$$

$$= \int_R -\frac{2xyz + xz}{z} \, dx\,dy = \int_R (-(2xy + x)) \, dx\,dy = 0$$

ODD IN x (AND y) ODD IN x

SINCE THE INTEGRATION IS OVER A SYMMETRICAL DOMAIN IN x, IE THE CIRCLE $x^2+y^2=1$

• ON THE BASE OF THE HEMISPHERE S_2, $\int_{S_2} \nabla \wedge \underline{A} \cdot \hat{\underline{n}} \, dS$

$$= \int_{S_2} (0, -2xz, -x) \cdot (0,0,-1) \, dS = \int_{S_2} x \, dx\,dy = 0$$

ODD POWER OF x IN A SYMMETRICAL DOMAIN

$$\therefore \oint_S \nabla \wedge \underline{A} \cdot d\underline{S} = 0$$

Question 8

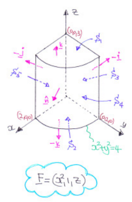
A vector field, $\mathbf{F}$, exists inside and around the finite region V, defined by

$$x^2 + y^2 = 4, \quad x \geq 0, \quad y \geq 0, \quad 0 \leq z \leq 3.$$

Use V to verify the Divergence Theorem of Gauss, given further that

$$\mathbf{F}(x, y, z) \equiv x^2\mathbf{i} + \mathbf{j} + z\mathbf{k}.$$

$\boxed{\text{both sides yield } 3\pi + 16}$



$$\int_V \nabla \cdot \underline{F} \, dV = \int \underline{F} \cdot d\underline{S}$$

$$\iiint_V \nabla \cdot \underline{F} \, dV = \oiint \underline{F} \cdot \hat{\underline{n}} \, dS$$

normal on the curved surface $\hat{n} = \frac{1}{2}(x, y, 0)$

SINCE $f(x,y,z) = x^2 + y^2 - 4$

$\underline{n} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, 0)$

SAY $\underline{n} = (x, y, 0)$

$\hat{\underline{n}} = \frac{(x, y, 0)}{\sqrt{x^2+y^2}} = \frac{1}{2}(x, y, 0)$

- FIRSTLY THE VOLUME INTEGRAL

$$\int_V \nabla \cdot \underline{F} \, dV = \int_V \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x^2, 1, z) \, dV = \int_V 2x + 1 \, dV$$

SWITCH INTO CYLINDRICAL POLARS (r, θ, z)

$$\int_{z=0}^{3} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2} (2r\cos\theta + 1)(r \, dr \, d\theta \, dz) = \int_{z=0}^{3} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2} (2r^2\cos\theta + r) \, dr \, d\theta \, dz$$

$$= \int_{z=0}^{3} \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{2}{3}r^3\cos\theta + \frac{1}{2}r^2\right]_{r=0}^{r=2} = \int_{z=0}^{3} \int_{\theta=0}^{\frac{\pi}{2}} \frac{16}{3}\cos\theta + 2 \, d\theta \, dz$$

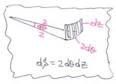
$$= \int_{z=0}^{3} \left[\frac{16}{3}\sin\theta + 2\theta\right]_{\theta=0}^{\frac{\pi}{2}} dz = \int_{z=0}^{3} \left(\frac{16}{3} + \pi\right) - 0 \, dz$$

$$= \int_{z=0}^{3} \frac{16}{3} + \pi \, dz = \left[\frac{16}{3}z + \pi z\right]_0^3 = (16 + 3\pi) - 0 = 3\pi + 16$$

- NEXT THE SURFACE INTEGRAL

$$\int_{S_1} (x^2, 1, 3) \cdot (0,0,1) \, dS + \cancel{\int_{S_2} (x^2, 1, 0) \cdot (0,0,-1) \, dS} + \cancel{\int_{S_3} (0, 1, z) \cdot (-1, 0, 0)} \, dS$$

$$+ \int_{S_4} (x^2, 1, z) \cdot \frac{1}{2}(x, y, 0) \, dS + \int_{S_5} (x^2, 1, z) \cdot (0, -1, 0) \, dS$$



$$= \int_{S_1} 3 \, dS + \int_{S_4} \frac{1}{2}x^3 + \frac{1}{2}y \, dS + \int_{S_5} -1 \, dS$$

$$= (3 \times \text{AREA OF } S_1) + \int_{S_4} \frac{1}{2}x^3 + \frac{1}{2}y \, dS - 1 \times \text{AREA OF } S_5$$

SWITCH INTO POLARS

$$= 3 \times \frac{1}{4} \times (\pi \times 2^2) + \int_{S_4} \frac{1}{2}(2\cos\theta)^3 + \frac{1}{2}(2\sin\theta) \, (2 \, d\theta \, dz) - (2 \times 3)$$

$$= 3\pi + \int_{z=0}^{3} \int_{\theta=0}^{\frac{\pi}{2}} 8\cos^3\theta + 2\sin\theta \, d\theta \, dz - 6$$

$$= 3\pi - 6 + \int_{\theta=0}^{\frac{\pi}{2}} \int_{z=0}^{3} (8\cos^3\theta + 2\sin\theta) \, dz \, d\theta$$

$$= 3\pi - 6 + \int_{\theta=0}^{\frac{\pi}{2}} \left[(8\cos^3\theta + 2\sin\theta) z\right]_{z=0}^{3} d\theta$$

$$= 3\pi - 6 + \int_{\theta=0}^{\frac{\pi}{2}} 24\cos^3\theta + 6\sin\theta \, d\theta$$

$$= 3\pi - 6 + \int_0^{\frac{\pi}{2}} 24\cos\theta(1 - \sin^2\theta) + 6\sin\theta \, d\theta$$

$$= 3\pi - 6 + \int_0^{\frac{\pi}{2}} 24\cos\theta - 24\cos\theta\sin^2\theta + 6\sin\theta \, d\theta$$

$$= 3\pi - 6 + \left[24\sin\theta - 8\sin^3\theta - 6\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= 3\pi - 6 + \left[(24 - 8 - 0) - (0 - 0 - 6)\right]$$

$$= 3\pi - 6 + 22$$

$$= 3\pi + 16$$

AS BEFORE

Question 9

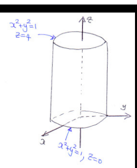
$$\mathbf{F}(x,y,z) \equiv (x+yz)\mathbf{i} + \left(y^3z + x\right)\mathbf{j} + (z+xyz)\mathbf{k}$$

Use the Divergence Theorem of Gauss to find the flux through the **open** surface with Cartesian equation

$$x^2+y^2=1, \quad 0 \le z \le 4.$$

$\boxed{10\pi}$

$\underline{F} = (x+yz,\ y^3z+x,\ z+xyz)$



$\underline{\nabla}\cdot\underline{F} = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\left(x+yz,\ y^3z+x,\ z+xyz\right)$

$\underline{\nabla}\cdot\underline{F} = 1+3y^2z+1+xy$

$\underline{\nabla}\cdot\underline{F} = 2+xy+3zy^2$

$\iiint_V \left[2+\cancel{xy}+3zy^2\right]\,dV$

AS THE x & y INTEGRATION IS IN A SYMMETRICAL DOMAIN, ANY ODD POWERS OF x OR y WILL HAVE NO CONTRIBUTION

- SWITCH INTO CYLINDRICAL POLARS

$= \int_{z=0}^{4}\int_{\theta=0}^{2\pi}\int_{r=0}^{r=1}\left[2+3z(r\sin\theta)^2\right]\left[r\,dr\,d\theta\,dz\right] = \int_{z=0}^{4}\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left[2r+3zr^3\sin^2\theta\right]dr\,d\theta\,dz$

$= \int_{z=0}^{4}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} 2r+3zr^3\left(\tfrac{1}{2}-\cancel{\tfrac{1}{2}\cos 2\theta}\right)dr\,d\theta\,dz$

NO CONTRIBUTION FROM THE θ INTEGRATION IF $0 \le \theta \le 2\pi$

$= \int_{z=0}^{4}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} 2r+\tfrac{3}{2}zr^3\ dr\,d\theta\,dz$

- CARRY OUT THE θ INTEGRATION FIRST

$= 2\pi\int_{z=0}^{4}\int_{r=0}^{1} 2r+\tfrac{3}{2}zr^3\ dr\,dz = 2\pi\int_{z=0}^{4}\left[r^2+\tfrac{3}{8}r^4z\right]_{r=0}^{r=1}dz$

$= 2\pi\int_{z=0}^{4}\left(1+\tfrac{3}{8}z\right)dz = 2\pi\left[z+\tfrac{3}{16}z^2\right]_0^4 = 2\pi\left[(4+3)-0\right]$

$= 14\pi$

NOW FIND THE FLUX THROUGH THE "CIRCULAR" DISCS SO WE CAN CALCULATE THE FLUX THROUGH THE CURVED SURFACE BY THE DIVERGENCE THEOREM

- TOP CAP; CIRCLE $x^2+y^2=1,\ z=4$

$\int_S \underline{F}\cdot\underline{\hat{n}}\,dS = \int_S (x+yz,\ y^3z+x,\ z+xyz)\cdot(0,0,1)\,dS$ $\quad \underline{\hat{n}}=\underline{k}$

$= \int_S z+xyz\ dS$

BUT $z=4$

$= \int_S 4+4xy\ dS$

PROJECT ONTO THE xy PLANE, $dS = dx\,dy$ HERE

$= \int 4+\cancel{4xy}\ dx\,dy$

SYMMETRICAL DOMAIN IN x & y, SO ODD POWERS OF x OR y HAVE NO CONTRIBUTION

$= \int 4\ dx\,dy = 4\times$ AREA OF THE CIRCLE $= 4(\pi\times 1^2)$

$= 4\pi$

- BOTTOM CAP, CIRCLE $x^2+y^2=1,\ z=0$

$\int \underline{F}\cdot\underline{\hat{n}}\,dS = \int (x+yz,\ y^3z+x,\ z+xyz)\cdot(0,0,-1)\,dS$ $\quad \underline{\hat{n}}=-\underline{k}$

$= 0$ $\quad \boxed{\text{SINCE } z=0}$

HENCE BY DIVERGENCE THEOREM

$\Rightarrow$ FLUX THROUGH "TOP" + FLUX THROUGH "BOTTOM" + FLUX THROUGH CURVED SURFACE $= 14\pi$

$\Rightarrow 4\pi + 0 +$ REQUIRED FLUX $= 14\pi$

$\Rightarrow$ REQUIRED FLUX $= 10\pi$

Question 10

A vector field, $\mathbf{F}$, exists inside and around the sphere S, with Cartesian equation

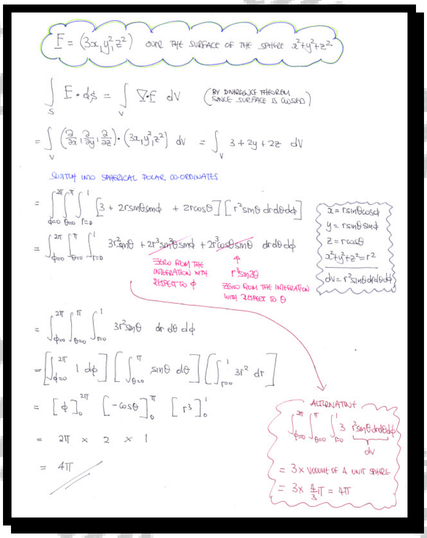
$$x^2 + y^2 + z^2 = 1.$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS},$$

where $\mathbf{F}(x, y, z) = 3x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$.

$\boxed{4\pi}$



Question 11

a) State Gauss' Divergence Theorem for closed surfaces, fully defining all the quantities involved.

b) Verify Gauss' Divergence Theorem for closed surfaces for the vector field

$$\mathbf{F} = xz\mathbf{i} + 2y^2\mathbf{j} + \left(xyz + z^2 + 6\right)\mathbf{k}$$

for the **finite** region defined as

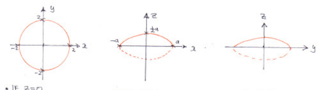
$$x^2 + y^2 + 4z^2 = 4, \; z \geq 0.$$

$\boxed{\text{both sides yield } 3\pi}$

a) $\iiint_V \nabla \cdot \underline{F} \, dV = \oiint \underline{F} \cdot d\underline{S}$

WHERE $\underline{F} = (F_1, F_2, F_3)$, S IS A CLOSED SURFACE ENCLOSING A VOLUME V, $d\underline{S} = \hat{\underline{n}} \, dS$ WHERE $\hat{\underline{n}}$ IS AN "OUTWARD" UNIT NORMAL TO THE SURFACE S

b) FIRSTLY SKETCH THE SURFACE



• IF $z=0$: $x^2+y^2=4$

• IF $y=0$: $x^2+4z^2=4$

• IF $x=0$: $y^2+4z^2=4$

THE VOLUME INTEGRAL FIRST

$\nabla \cdot \underline{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(xz, 2y^2, xyz + z^2 + 6\right)$

$= z + 4y + xy + 2z = xy + 4y + 3z$

THUS

$\iiint_V \nabla \cdot \underline{F} \, dx\,dy\,dz = \ldots$ SWITCH INTO CYLINDRICAL POLAR CO-ORDINATES

$x^2+y^2+4z^2=4$, $r^2+4z^2=4$, $4z^2=4-r^2$, $z^2=\frac{1}{4}(4-r^2)$, $z=\frac{1}{2}(4-r^2)^{\frac{1}{2}}$, $z \geq 0$

$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{z=\frac{1}{2}(4-r^2)^{\frac{1}{2}}} \left[(r\cos\theta)(r\sin\theta) + 4r\sin\theta + 3z\right](r\,dr\,d\theta\,dz)$

$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{z=\frac{1}{2}(4-r^2)^{\frac{1}{2}}} \cancel{r^3\cos\theta\sin\theta} + \cancel{4r^2\sin\theta} + 3zr \, dz\,dr\,d\theta$

$\int_0^{2\pi} \cos\theta\sin\theta \, d\theta = 0$ & $\int_0^{2\pi} \sin\theta \, d\theta = 0$

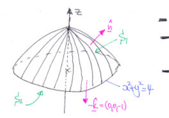
$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left[\frac{3}{2}z^2 r\right]_{z=0}^{z=\frac{1}{2}(4-r^2)^{\frac{1}{2}}} dr\,d\theta$

$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \frac{3}{2}r \times \frac{1}{4}(4-r^2) \, dr\,d\theta = \frac{3}{8}\int_{\theta=0}^{2\pi} \int_{r=0}^{2} r(4-r^2) \, dr\,d\theta$

$= \frac{3}{8}\int_{\theta=0}^{2\pi} \left[-\frac{1}{4}(4-r^2)^2\right]_0^2 d\theta = -\frac{3}{32}\int_{\theta=0}^{2\pi} (0-16) \, d\theta = \frac{3}{2}\int_{\theta=0}^{2\pi} 1 \, d\theta$

$= \frac{3}{2} \times 2\pi = 3\pi$

NEXT THE SURFACE INTEGRAL CONSISTING OF TWO SURFACES



— NOTE THAT THE SURFACE IS CLOSED

— S_1 IS THE ELLIPSOID WITH EQUATION $x^2+y^2+4z^2=4$, $z \geq 0$

— S_2 IS THE "BASE" OF S_1, IE THE CIRCLE $x^2+y^2=4$

• LOOKING AT S_1

$\nabla(x^2+y^2+4z^2-4) = (2x, 2y, 8z)$

$\underline{n} = (x, y, 4z)$

$\hat{\underline{n}} = \frac{(x, y, 4z)}{\sqrt{x^2+y^2+16z^2}}$

$\underline{F} \cdot \hat{\underline{n}} = (xz, 2y^2, xyz + z^2 + 6) \cdot \frac{(x, y, 4z)}{\sqrt{x^2+y^2+16z^2}}$

$= \frac{x^2z + 2y^3 + 4xyz^2 + 4z^3 + 24z}{\sqrt{x^2+y^2+16z^2}}$

THUS

$\int_{S_1} \underline{F} \cdot d\underline{S} = \int_{S_1} \underline{F} \cdot \hat{\underline{n}} \, dS = \int_{S_1} \frac{x^2z + 2y^3 + 4xyz^2 + 4z^3 + 24z}{\sqrt{x^2+y^2+16z^2}} \, dS$

PROJECT ONTO THE x-y PLANE ONTO THE CIRCLE WITH EQUATION $x^2+y^2=4$, THEN USE POLARS

$dS = \frac{dx\,dy}{\hat{n} \cdot \underline{k}} = \frac{dx\,dy}{\frac{(x,y,4z)}{\sqrt{x^2+y^2+16z^2}} \cdot (0,0,1)} = \frac{dx\,dy}{\frac{4z}{\sqrt{x^2+y^2+16z^2}}} = \frac{\sqrt{x^2+y^2+16z^2}}{4z} \, dx\,dy$

THUS PROJECTING ONTO THE REGION R (CIRCLE R: $x^2+y^2=4$)

$= \int_R \frac{x^2z + 2y^3 + 4xyz^2 + 4z^3 + 24z}{\sqrt{x^2+y^2+16z^2}} \frac{\sqrt{x^2+y^2+16z^2}}{4z} \, dx\,dy$

$= \frac{1}{4}\int_R x^2 + \frac{2y^3}{z} + 4xyz + 4z^2 + 24 \, dx\,dy$

$= \frac{1}{4}\int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left[r^2\cos^2\theta + \cancel{\frac{2r^3\sin^3\theta}{\frac{1}{2}\sqrt{4-r^2}}} + \cancel{4r^2\sin\theta\cos\theta \times \frac{1}{2}\sqrt{4-r^2}} + 4 \times \frac{1}{4}(4-r^2) + 24\right] r\,dr\,d\theta$

NOTE FROM EARLIER $z = \frac{1}{2}(4-r^2)^{\frac{1}{2}}$

$\int_0^{2\pi} \sin^3\theta \, d\theta = 0$ $\quad$ $\int_0^{2\pi} \sin\theta\cos\theta \, d\theta$

$= \frac{1}{4}\int_{\theta=0}^{2\pi} \int_{r=0}^{2} r^3\cos^2\theta + 4r - r^3 + 24r \, dr\,d\theta$

$= \frac{1}{4}\int_{\theta=0}^{2\pi} \left[\frac{1}{4}r^4\cos^2\theta + 2r^2 - \frac{1}{4}r^4 + 12r^2\right]_0^2 d\theta$

$= \frac{1}{4}\int_{\theta=0}^{2\pi} 4\cos^2\theta + 8 - 4 + 48 \, d\theta$

$= \frac{1}{4}\int_{\theta=0}^{2\pi} 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) + 52 \, d\theta$

$= \frac{1}{4}\int_{\theta=0}^{2\pi} 54 + 2\cos 2\theta \, d\theta$

$= \frac{1}{4}\left[54\theta + \cancel{\sin 2\theta}\right]_0^{2\pi}$

$= \frac{1}{4} \times 2\pi \times 54$

$= 27\pi$

FINALLY THE SURFACE INTEGRAL ALONG S_2

$\hat{\underline{n}} = (0,0,-1) = -\underline{k}$ (ON S_2)

$\iint_{S_2} \underline{F} \cdot d\underline{S} = \iint_{S_2} \underline{F} \cdot \hat{\underline{n}} \, dS = \iint_R (xz, 2y^2, xyz + z^2 + 6) \cdot (0,0,-1) \, dx\,dy$

$= \iint_R -xyz - z^2 - 6 \, dx\,dy$

$= \iint_R -6 \, dx\,dy$ (SINCE $z=0$ ON S_2)

$= -6 \times$ AREA OF R

$= -6 \times \pi \times 2^2$

$= -24\pi$

HENCE

$\oiint \underline{F} \cdot d\underline{S} = \iint_{S_1} \underline{F} \cdot d\underline{S} + \iint_{S_2} \underline{F} \cdot d\underline{S}$

$= 27\pi - 24\pi$

$= 3\pi$

$= \iiint_V \nabla \cdot \underline{F} \, dV$

Question 12

The region V is defined as

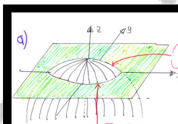
$$x^2 + y^2 + (z+4)^2 \le 25, \; z \ge 0.$$

a) Use cylindrical polar coordinates (r, θ, z) to find the volume of this region.

b) Use Gauss' Divergence Theorem for closed surfaces, with an appropriate vector field, to verify the answer obtained in part **(a)**

$$\boxed{\frac{14}{3}\pi}$$

a)



$x^2+y^2+(z+4)^2=25$ SWITCH INTO C.P.C

$r^2+(z+4)^2=25$

$(z+4)^2=25-r^2$

$z+4=+\sqrt{25-r^2}$

$z=-4+\sqrt{25-r^2}$

$x^2+y^2=9$

$(r=3)$

$$V=\int_V 1\,dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{z=-4+\sqrt{25-r^2}} \underbrace{r\cdot dz\,dr\,d\theta}_{\substack{dV\\ \text{IN CYLINDRICAL POLARS}}} = \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\Big[rz\Big]_{z=0}^{z=-4+\sqrt{25-r^2}}\,dr\,d\theta$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{3} r\left(-4+\sqrt{25-r^2}\right)dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} -4r + r(25-r^2)^{\frac{1}{2}}\,dr\,d\theta$$

$$=\int_{\theta=0}^{2\pi}\left[-2r^2-\tfrac{1}{3}(25-r^2)^{\frac{3}{2}}\right]_{r=0}^{3}d\theta = \int_{\theta=0}^{2\pi}\left[2r^2+\tfrac{1}{3}(25-r^2)^{\frac{3}{2}}\right]_{3}^{0}d\theta$$

$$=\int_{\theta=0}^{2\pi}\left[0+\tfrac{1}{3}(25)^{\frac{3}{2}}\right]-\left[18+\tfrac{1}{3}(16)^{\frac{3}{2}}\right]d\theta$$

$$=\int_{\theta=0}^{2\pi}\left(\frac{125}{3}-18-\frac{64}{3}\right)d\theta = \int_{\theta=0}^{2\pi}\frac{7}{3}\,d\theta$$

$$=\frac{7}{3}\times 2\pi = \frac{14\pi}{3}$$

b) DIVERGENCE THEOREM

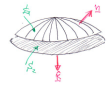
$$\iiint_V \nabla\cdot\underline{F}\,dV = \oiint_S \underline{F}\cdot d\underline{S}$$

PICK A SIMPLE $\underline{F}$ WITH DIVERGENCE 1 (CONST), SAY $\underline{F}=(x,0,0)$

CLOSE THE SPHERICAL CAP WITH A PLANE ON THE xy PLANE

THUS S_1: $x^2+y^2+(z+4)^2=25$, $z\geqslant 4$

S_2: $x^2+y^2=9$



FIND NORMAL TO S_1

LET $f(x,z,y)=x^2+y^2+(z+4)^2-25$

$\nabla f=(2x,2y,2(z+4))$

$\underline{n}=(x,y,z+4)$

$|\underline{n}|=\sqrt{x^2+y^2+(z+4)^2}$

$|\underline{n}|=5$

$\hat{\underline{n}}=\frac{1}{5}(x,y,z+4)$

$$\int_{S_1}\underline{F}\cdot\hat{\underline{n}}\,dS = \int_{S_1}(x,0,0)\cdot\tfrac{1}{5}(x,y,z+4)\,dS = \int_{S_1}\tfrac{1}{5}x^2\,dS$$

PROJECT ONTO THE xy PLANE, ONTO THE CIRCULAR REGION R, $x^2+y^2\leq 9$

$$\dots=\int_R \tfrac{1}{5}x^2\,\frac{dx\,dy}{\hat{\underline{n}}\cdot\underline{k}} = \int_R\tfrac{1}{5}x^2\,\frac{dx\,dy}{\frac{1}{5}(x,y,z+4)\cdot(0,0,1)}$$

$$=\int_R\tfrac{1}{5}x^2\,\frac{dx\,dy}{\frac{1}{5}(z+4)} = \int_R\frac{x^2}{z+4}\,dx\,dy = \int_R\frac{x^2}{+\sqrt{25-x^2-y^2}}\,dx\,dy$$

SWITCH INTO PLANE POLAR CO-ORDINATES

$$\dots=\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\frac{r^2\cos^2\theta}{(25-r^2)^{\frac{1}{2}}}(r\,dr\,d\theta) = \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\frac{r^3\left(\frac{1}{2}+\cancel{\frac{1}{2}\cos 2\theta}\right)}{(25-r^2)^{\frac{1}{2}}}\,dr\,d\theta$$

NO CONTRIBUTION

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\tfrac{1}{2}r^3(25-r^2)^{-\frac{1}{2}}\,dr\,d\theta = 2\pi\int_{r=0}^{3}\tfrac{1}{2}r^3(25-r^2)^{-\frac{1}{2}}\,dr$$

$$=\pi\int_0^3 r^3(25-r^2)^{-\frac{1}{2}}\,dr$$

SUBSTITUTION

$u=(25-r^2)^{\frac{1}{2}}$

$u^2=25-r^2$

$2u\frac{du}{dr}=-2r$

$dr=-\frac{u}{r}\,du$

$r=0$, $u=5$

$r=3$, $u=4$

$$=\pi\int_5^4 r^3u^{-1}\left(-\frac{u}{r}\,du\right)$$

$$=\pi\int_4^5 r^2\,du$$

$$=\pi\int_4^5(25-u^2)\,du = \pi\left[25u-\frac{1}{3}u^3\right]_4^5$$

$$=\pi\left[\left(125-\frac{125}{3}\right)-\left(100-\frac{64}{3}\right)\right]=\frac{14}{3}\pi$$

Now

$$\int_{S_2}\underline{F}\cdot\hat{\underline{n}}\,dS = \int(x,0,0)\cdot(0,0,-1)\,dS = 0$$

HENCE BY THE DIVERGENCE THEOREM

$$\iiint 1\,dV = \frac{14}{3}\pi + 0$$

$$V=\frac{14}{3}\pi$$

AS BEFORE

Question 13

a) State Gauss' Divergence Theorem for closed surfaces, fully defining all the quantities involved.

b) Hence show that for a smooth scalar field $\varphi = \varphi(x, y, z)$,

$$\iiint_V \nabla \varphi \, dV = \oint\!\!\oint_S \varphi \hat{\mathbf{n}} \, dS ,$$

where S is a closed surface enclosing a volume V, and $\hat{\mathbf{n}}$ is an outward unit normal field to S.

c) Evaluate

$$\oint\!\!\oint_S \left(x^2 y + y^2 + z\right) \hat{\mathbf{n}} \, dS ,$$

where S is the paraboloid with equation

$$z = 1 - x^2 - y^2, \ z \geq 0 .$$

$$\boxed{\frac{\pi}{12}(\mathbf{j} + 6\mathbf{k})}$$

a) IF S IS A CLOSED SURFACE ENCLOSING A VOLUME REGION V, AND $\underline{F}$ IS A SMOOTH VECTOR FIELD $\underline{F}$, THEN

$$\iiint_V \nabla \cdot \underline{F} \, dV = \oint\!\!\oint_S \underline{F} \cdot \hat{\underline{n}} \, dS$$

WHERE $\hat{\underline{n}}$ IS AN OUTWARD UNIT NORMAL TO S

b) LET $\underline{F} = \phi(x,y,z)\underline{i} + \phi(x,y,z)\underline{j} + \phi(x,y,z)\underline{k} = (\phi, \phi, \phi)$

WHERE $\phi(x,y,z)$ IS ANY FUNCTION OF x, y & z

BY THE DIVERGENCE THEOREM

$$\iiint_V \nabla \cdot \underline{F} \, dV = \oint\!\!\oint_S \underline{F} \cdot \hat{\underline{n}} \, dS$$

$$\iiint_V \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (\phi, \phi, \phi) \, dV = \oint\!\!\oint_S (\phi, \phi, \phi) \cdot \hat{\underline{n}} \, dS$$

$$\iiint_V \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} \, dV = \oint\!\!\oint_S \phi(1,1,1) \cdot \hat{\underline{n}} \, dS$$

$$\iiint_V \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) \cdot (1,1,1) \, dV \quad \oint\!\!\oint_S \phi(1,1,1) \cdot \hat{\underline{n}} \, dS$$

$$\iiint_V (1,1,1) \cdot \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) dV = \oint\!\!\oint_S (1,1,1) \cdot \hat{\underline{n}} \, \phi \, dS$$

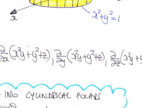
$$(1,1,1) \cdot \iiint_V \nabla \phi \, dV = (1,1,1) \cdot \oint\!\!\oint_S \hat{\underline{n}} \, \phi \, dS$$

AS ϕ IS ARBITRARY

$$\iiint_V \nabla \phi \, dV = \oint\!\!\oint_S \phi \hat{\underline{n}} \, dS$$

AS REQUIRED

c) $\phi(x,y,z) = x^2y + y^2 + z$



$$\oint\!\!\oint_S (x^2y + y^2 + z) \, d\underline{S}$$

$= \ldots$ USING PART (b)

$$= \iiint_V \nabla (x^2y + y^2 + z) \, dV = \iiint_V \left[\frac{\partial}{\partial x}(x^2y + y^2 + z), \frac{\partial}{\partial y}(x^2y + y^2 + z), \frac{\partial}{\partial z}(x^2y + y^2 + z)\right] dV$$

$$= \iiint_V (2xy, \ x^2 + 2y, \ 1) \, dV$$

SWITCH INTO CYLINDRICAL POLARS

$x = r\cos\theta$ $\quad dV = r \, dr \, d\theta \, dz$

$y = r\sin\theta$

$z = z$

$z = 1 - (x^2 + y^2)$

$z = 1 - r^2$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=1-r^2} (2r^2\cos\theta\sin\theta, \ r^2\cos^2\theta + 2r\sin\theta, \ 1)(r \, dr \, d\theta \, dz)$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{z=1-r^2} (\cancel{2r^3\cos\theta\sin\theta}, \ r^3\cos^2\theta + \cancel{2r\sin\theta}, \ r) \, dz \, dr \, d\theta$$

FROM THE θ INTEGRATION $\quad$ FROM THE θ INTEGRATION

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{z=1-r^2} \left(0, \ r^3\left(\tfrac{1}{2} + \cancel{\tfrac{1}{2}\cos 2\theta}\right), \ r\right) dz \, dr \, d\theta$$

FROM THE θ INTEGRATION

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{z=1-r^2} \left(0, \ \tfrac{1}{2}r^3, \ r\right) dz \, dr \, d\theta$$

DO THE θ INTEGRATION FIRST, AS THE INTEGRAND HAS NO θ DEPENDENCE.

$$= 2\pi \int_{r=0}^{1} \int_{z=0}^{z=1-r^2} \left(0, \tfrac{1}{2}r^3, r\right) dz \, dr$$

$$= 2\pi \int_{r=0}^{1} \left[\left(0, \tfrac{1}{2}r^3 z, rz\right)\right]_{z=0}^{z=1-r^2} dr$$

$$= 2\pi \int_{r=0}^{1} \left[0, \tfrac{1}{2}r^3(1-r^2), r(1-r^2)\right] dr$$

$$= 2\pi \int_0^1 \left[0, \tfrac{1}{2}r^3 - \tfrac{1}{2}r^5, r - r^3\right] dr$$

$$= 2\pi \left[\left(0, \tfrac{1}{8}r^4 - \tfrac{1}{12}r^6, \tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right)\right]_0^1$$

$$= 2\pi \left[\left(0, \tfrac{1}{8} - \tfrac{1}{12}, \tfrac{1}{2} - \tfrac{1}{4}\right)\right]$$

$$= 2\pi \left(0, \tfrac{1}{24}, \tfrac{1}{4}\right)$$

$$= \tfrac{\pi}{12}(0, 1, 6)$$

I.E $\dfrac{\pi(\underline{j} + 6\underline{k})}{12}$

Question 14

a) State Gauss' Divergence Theorem for closed surfaces, fully defining all the quantities involved.

The vector field $\mathbf{E}$ s given as

$$\mathbf{E} = \left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}} \left(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\right).$$

b) Show that Gauss' Divergence Theorem for closed surfaces "fails" for $\mathbf{E}$ and the surface with Cartesian equation

$$x^2 + y^2 + z^2 = a^2, \ a > 0.$$

c) Explain carefully why the theorem "fails".

proof

a) $$\iiint_V \nabla \cdot \underline{E} \, dV = \oiint \underline{E} \cdot \hat{\underline{n}} \, dS$$

WHERE S IS A CLOSED SURFACE ENCLOSING A VOLUME REGION V

$\underline{E}$ IS A SMOOTH VECTOR FIELD

$\hat{n}$ IS AN OUTWARD UNIT NORMAL FIELD TO S

b) $$\underline{E} = \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\nabla \cdot \underline{E} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left[\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]$$

CONSIDER

$$\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right] = \frac{(x^2+y^2+z^2)^{\frac{3}{2}} \times 1 - x \times \frac{3}{2} \times (2x)(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{\frac{3}{2}} - 3x^2(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^1 - 3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

& BY CYCLIC SYMMETRY THE OTHER 2

$$\dots = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0$$

HENCE THE DIVERGENCE IS ZERO SO

$$\iiint_V \nabla \cdot \underline{E} \, dV = 0 \quad \text{FOR ALL CLOSED SURFACES}$$

NOW $\Omega: x^2+y^2+z^2 = a^2$

LET $f(x,y,z) = x^2+y^2+z^2-a^2$

$\nabla f = (2x, 2y, 2z)$

LET $\underline{n} = (x,y,z)$

$|\underline{n}| = \sqrt{x^2+y^2+z^2} = a$

$\hat{\underline{n}} = \frac{1}{a}(x,y,z)$

THUS

$$\oiint_\Omega \underline{E} \cdot \hat{\underline{n}} \, dS = \oiint_\Omega \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot \frac{1}{a}(x,y,z) \, dS$$

$$= \oiint_\Omega \frac{(x,y,z)\cdot(x,y,z)}{a^4} \, dS = \oiint_\Omega \frac{x^2+y^2+z^2}{a^4} \, dS = \oiint_\Omega \frac{a^2}{a^4} \, dS$$

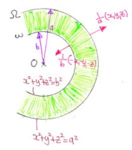
$$= \oiint_\Omega \frac{1}{a^2} \, dS = \frac{1}{a^2} \iint_\Omega 1 \, dS = \frac{1}{a^2}(4\pi a^2) = 4\pi$$

HENCE THE DIVERGENCE THEOREM "FAILS"

c) THE DISCREPANCY OCCURS WHEN $x=y=z=0$ IN $\underline{E}$

TO SHOW THAT IT WORKS EXCLUDE THE ORIGIN

$\nabla \cdot \underline{E} = 0$ SO THE VOLUME INTEGRAL STILL YIELDS ZERO



$$\int_\Omega \underline{E} \cdot \hat{\underline{n}} \, dS = 4\pi$$

(INDEPENDENT OF "a")

$$\int_\omega \underline{E} \cdot \hat{\underline{n}} \, dS = -4\pi$$

(INDEPENDENT OF b)

HENCE THE DIVERGENCE THEOREM "WORKS"

Question 15

The surface S is the sphere with Cartesian equation

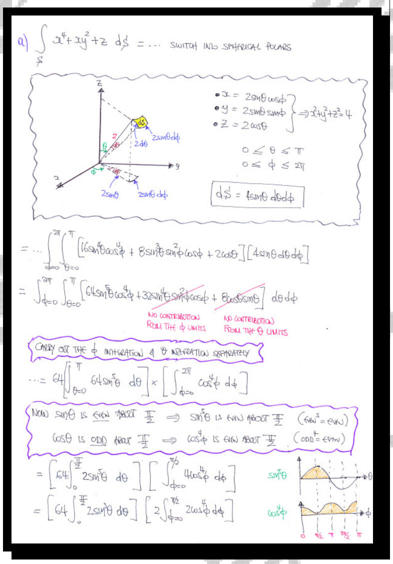
$$x^2 + y^2 + z^2 = 4$$

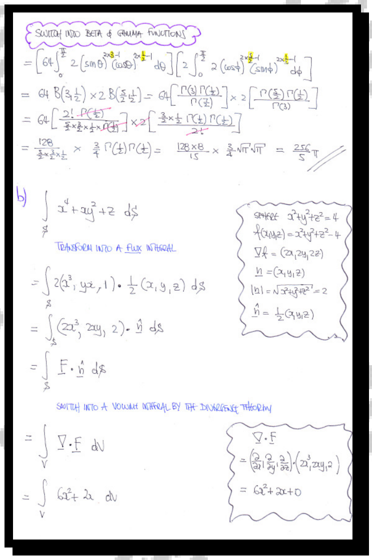
a) By using Spherical Polar coordinates, (r, θ, φ), evaluate by direct integration the following surface integral

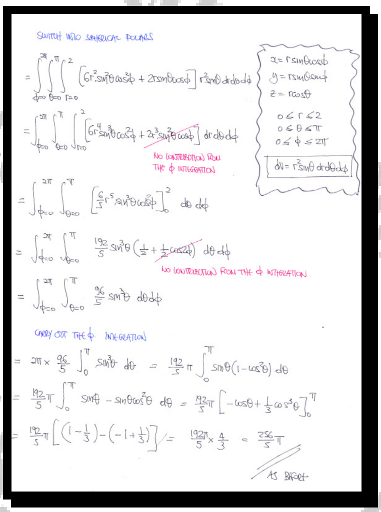
$$I = \oiint_S \left(x^4 + xy^2 + z\right)\, dS .$$

b) Verify the answer of part **(a)** by using the Divergence Theorem.

$$\boxed{\frac{256\pi}{5}}$$







Question 16

The surface Ω is the sphere with Cartesian equation

$$(x-1)^2+(y-1)^2+(z-1)^2=1$$

Use the Divergence Theorem to evaluate

$$\oiint_{\Omega} \left[(x+y)\mathbf{i}+\left(x^2+xy\right)\mathbf{j}+z^2\mathbf{k}\right]\cdot \mathbf{dS},$$

where $\mathbf{dS}$ is a unit surface element on Ω.

$$\boxed{\frac{16}{3}\pi}$$

$$\int_{\Omega} \underline{F}\cdot d\underline{S} = \int_{\Omega} (x+y,\, x^2+xy,\, z^2)\cdot \underline{\hat{n}}\; dS = \dots$$

- TRANSLATE THE ORIGIN AT $(1,1,1)$

$$\left.\begin{matrix} X=x-1 & x=X+1 \\ Y=y-1 & y=Y+1 \\ Z=z-1 & z=Z+1 \end{matrix}\right\} \Rightarrow (x-1)^2+(y-1)^2+(z-1)^2=1$$

BECOMES

$$X^2+Y^2+Z^2=1$$

- AND

$$(x+y,\, x^2+xy,\, z^2)$$
$$= \left[(X+1)+(Y+1),\, (X+1)^2+(X+1)(Y+1),\, (Z+1)^2\right]$$
$$= \left[X+Y+2,\, X^2+2X+1+XY+X+Y+1,\, Z^2+2Z+1\right]$$
$$= \left[X+Y+2,\, X^2+XY+3X+Y+2,\, Z^2+2Z+1\right]$$

- DIVERGENCE $= \frac{\partial}{\partial X}\left[X+Y+2\right]+\frac{\partial}{\partial Y}\left[X^2+XY+3X+Y+2\right]+\frac{\partial}{\partial Z}\left[Z^2+2Z+1\right]$

$$= 1+(X+1)+(2Z+2)$$
$$= X+2Z+4$$

$= \dots$ BY THE DIVERGENCE THEOREM

$$= \int_V X+2Z+4 \; dV$$

- SWITCH INTO SPHERICAL POLARS, BUT FIRST NOTE THAT THE DOMAIN (VOLUME) IS SYMMETRICAL IN X, IN Y AND IN Z $(X^2+Y^2+Z^2=1)$ SO ANY ODD POWERS IN ANY VARIABLE WILL HAVE NO CONTRIBUTION

$$= \int_V \cancel{X}+\cancel{2Z}+4 \; dV$$

$$= \int 4 \; dV \qquad \text{(NO SPHERICAL POLARS ARE ACTUALLY NEEDED)}$$

$= 4 \times$ VOLUME OF UNIT SPHERE

$$= 4 \times \frac{4}{3}\pi$$

$$= \frac{16}{3}\pi$$

Question 17

The vector field $\mathbf{u}$ is given in spherical polar coordinates (r, θ, φ) by

$$\mathbf{u}(r, \theta, \varphi) = \left(r^2 \cos^2 \varphi\right)\hat{\mathbf{r}} + \left(r \cos^2 \varphi\right)\hat{\boldsymbol{\varphi}}.$$

a) Find the flux of $\mathbf{u}$ through a spherical surface of radius R_0.

b) Verify the answer to part **(a)** by calculating an appropriate volume integral.

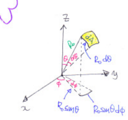
You may assume that in spherical polar coordinates

$$\nabla \cdot \left(A_r, A_\theta, A_\varphi\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \varphi}\left(A_\varphi\right)$$

$$\boxed{2\pi R_0^4}$$

a) $\underline{u} = \left(r^2\cos^2\phi, 0, r\cos^2\phi\right)$ IN S.P.C $\left(\hat{\underline{r}}, \hat{\underline{\theta}}, \hat{\underline{\phi}}\right)$

$\hat{\underline{n}} = \hat{\underline{r}}$



$\boxed{dA = R_0^2 \sin\theta\, d\theta\, d\phi}$

FLUX

$$\iint_S \underline{u} \cdot d\underline{S} = \iint_S \left(r^2\cos^2\phi, 0, r\cos^2\phi\right) \cdot \hat{\underline{n}}\, dS$$

$$= \iint_S \left(R_0^2\cos^2\phi, 0, R_0\cos^2\phi\right) \cdot (1,0,0) \left(R_0^2 \sin\theta\, d\theta\, d\phi\right)$$

$$= R_0^4 \iint_S \cos^2\phi \sin\theta \; d\theta\, d\phi$$

$$= R_0^4 \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta\left(\frac{1}{2} + \cancel{\frac{1}{2}\cos 2\phi}\right) d\theta\, d\phi$$

NO CONTRIBUTION FROM THE ϕ INTEGRATION WITH $[0, 2\pi]$

$$= R_0^4 \left[\int_{\phi=0}^{2\pi} \frac{1}{2}\, d\phi\right]\left[\int_{\theta=0}^{\pi} \sin\theta\, d\theta\right]$$

$$= R_0^4 \times \left[\frac{1}{2} \times 2\pi\right]\left[-\cos\theta\right]_0^{\pi}$$

$$= R_0^4 \times \pi \times \left[\cos\theta\right]_{\pi}^{0}$$

$$= R_0^4 \times \pi \times \left[1 - (-1)\right]$$

$$= 2\pi R_0^4$$

b) IN SPHERICAL POLAR CO-ORDINATES

$$\underline{\nabla} \cdot \underline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(A_\phi\right)$$

$$\underline{\nabla} \cdot \underline{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^4\cos^2\phi\right] + 0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[r\cos^2\phi\right]$$

$$= \frac{1}{r^2}\left(4r^3\cos^2\phi\right) + \frac{1}{r\sin\theta}\left(-2r\cos\phi\sin\phi\right)$$

$$= 4r\cos^2\phi - \frac{2\cos\phi\sin\phi}{\sin\theta}$$

THUS

$$\iiint_V \left[4r\cos^2\phi - \frac{2\cos\phi\sin\phi}{\sin\theta}\right]\left[r^2\sin\theta\, dr\, d\theta\, d\phi\right]$$

$$= \iiint_V \left[4r^3\sin\theta\cos^2\phi - \cancel{2r^2\cos\phi\sin\phi}\right] dr\, d\theta\, d\phi$$

NO CONTRIBUTION FROM THE ϕ INTEGRATION FOR $\phi \in [0, 2\pi]$

$$= \iiint_V 4r^3\sin\theta\left(\frac{1}{2} + \cancel{\frac{1}{2}\cos 2\phi}\right) dr\, d\theta\, d\phi$$

$$= \left[\int_{r=0}^{R_0} 4r^3\, dr\right]\left[\int_{\phi=0}^{2\pi} \frac{1}{2}\, d\phi\right]\left[\int_{\theta=0}^{\pi} \sin\theta\, d\theta\right]$$

$$= \left[r^4\right]_0^{R_0}\left[\frac{1}{2}\phi\right]_0^{2\pi}\left[-\cos\theta\right]_0^{\pi}$$

$$= R_0^4 \times \pi \times \left[\cos\theta\right]_{\pi}^{0}$$

$$= \pi R_0^4\left[1+1\right]$$

$$= 2\pi R_0^4$$

Question 18

a) State Gauss' Divergence Theorem for closed surfaces, fully defining all the quantities involved.

b) Hence show that for a smooth vector field $\mathbf{A} = \mathbf{A}(x, y, z)$, with $\nabla \cdot \mathbf{A} = 0$,

$$\iiint_V \mathbf{A} \, dV = \oiint_S \mathbf{r} \, \mathbf{A} \cdot \hat{\mathbf{n}} \, dS ,$$

where S is a closed surface enclosing a volume V, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $\hat{\mathbf{n}}$ is an outward unit normal field to S.

c) Verify the validity of the result of part **(b)** if $\mathbf{A} = 3\mathbf{i}$ and S is the sphere with equation

$$x^2 + y^2 + z^2 = 1 .$$

$\boxed{\text{both sides yield } 4\pi\mathbf{i}}$

a) $\oiiint_V \nabla \cdot \underline{F} \, dV = \oiint_S \underline{F} \cdot d\underline{S}$, WHERE

S IS A CLOSED SURFACE ENCLOSING A VOLUME V

$\underline{F}$ IS A SMOOTH VECTOR FIELD

$d\underline{S} = \hat{\underline{n}} \, dS$, WITH $\hat{\underline{n}}$ AN OUTWARD UNIT NORMAL TO S

b) STARTING WITH THE DIVERGENCE THEOREM, LET $\underline{F} = (\underline{r} \cdot \underline{c})\underline{A}$

WHERE $\underline{r} = (x, y, z)$

$\underline{c}$ = CONSTANT VECTOR

$\underline{A}$ = VECTOR FIELD, SUCH THAT $\nabla \cdot \underline{A} = 0$

$\Rightarrow \oiiint_V \nabla \cdot [(\underline{r} \cdot \underline{c})\underline{A}] \, dV = \oiint_S (\underline{r} \cdot \underline{c})\underline{A} \cdot d\underline{S}$

NOW $\nabla \cdot (\phi \underline{A}) = \nabla\phi \cdot \underline{A} + \phi \nabla \cdot \underline{A}$

$\nabla \cdot ((\underline{r} \cdot \underline{c})\underline{A}) = \nabla(\underline{r} \cdot \underline{c}) \cdot \underline{A} + \cancel{(\underline{r} \cdot \underline{c}) \nabla \cdot \underline{A}}$ $\quad \nabla \cdot \underline{A} = 0$

BUT $\underline{r} \cdot \underline{c} = (x, y, z) \cdot (c_1, c_2, c_3) = c_1 x + c_2 y + c_3 z$

$\nabla(\underline{r} \cdot \underline{c}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(c_1 x + c_2 y + c_3 z)$

$\nabla(\underline{r} \cdot \underline{c}) = (c_1, c_2, c_3) = \underline{c}$

$\Rightarrow \oiiint_V \underline{c} \cdot \underline{A} \, dV = \oiint_S (\underline{c} \cdot \underline{r}) \underline{A} \cdot d\underline{S}$

$\Rightarrow \underline{c} \cdot \oiiint_V \underline{A} \, dV = \underline{c} \cdot \oiint_S \underline{r} \underline{A} \cdot d\underline{S}$

$\Rightarrow \oiiint_V \underline{A} \, dV = \oiint_S \underline{r} \underline{A} \cdot d\underline{S}$

SO LONG AS $\nabla \cdot \underline{A} = 0$

c) $\oiiint_V \underline{A} \, dV = \oiint_S \underline{r} \underline{A} \cdot d\underline{S}$

NOW $\underline{A} = 3\underline{i}$ & $S: x^2 + y^2 + z^2 = 1$

- LET $f(x, y, z) = x^2 + y^2 + z^2 - 1$

$\nabla f = (2x, 2y, 2z)$

$\underline{n} = (x, y, z)$

$|\underline{n}| = \sqrt{x^2 + y^2 + z^2} = 1$

$\hat{\underline{n}} = (x, y, z)$

- ALSO IN SPHERICAL POLARS, ON THE SURFACE OF THE UNIT SPHERE

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$0 \leqslant \theta \leqslant \pi$

$0 \leqslant \phi \leqslant 2\pi$

- IF $r = 1$

$x = \sin\theta\cos\phi$

$y = \sin\theta\sin\phi$

$z = \cos\theta$

$\boxed{dS = \sin\theta \, d\theta \, d\phi}$

$0 \leqslant \theta \leqslant \pi$

$0 \leqslant \phi \leqslant 2\pi$

THUS

- L.H.S $= \oiiint_V \underline{A} \, dV = \oiiint_V (3, 0, 0) \, dV = 3\underline{i} \oiiint_V dV = 3\underline{i} \times$ VOLUME OF THE SPHERE

$= 3\underline{i} \times \frac{4}{3}\pi \times 1^3 = 4\pi\underline{i}$

- R.H.S $= \oiint_S \underline{r}\underline{A} \cdot d\underline{S} = \oiint_S (\overset{\underline{r}}{x, y, z})(\overset{\underline{A}}{3, 0, 0}) \cdot (\overset{\hat{\underline{n}}}{x, y, z}) \, dS = \oiint_S (x, y, z)(3x) \, dS$

$= \oiint_S (3x^2, 3xy, 3xz) \, dS = \ldots$ SWITCH INTO SPHERICAL POLARS

$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (3\sin^2\theta\cos^2\phi, 3\sin^2\theta\cos\phi\sin\phi, \sin\theta\cos\theta\cos\phi) \sin\theta \, d\theta \, d\phi$

NO CONTRIBUTIONS FROM THE ϕ INTEGRATION

$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (3\sin^3\theta\cos^2\phi, \cancel{3\sin^3\theta\cos\phi\sin\phi}, \cancel{\sin^2\theta\cos\theta\cos\phi}) \, d\theta \, d\phi$

$= 3\underline{i} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin^3\theta\cos^2\phi \, d\theta \, d\phi$

$= 3\underline{i} \left[\int_0^{\pi} \sin^3\theta \, d\theta\right]\left[\int_0^{2\pi} \cos^2\phi \, d\phi\right]$

$= 3\underline{i} \left[\int_0^{\pi} \sin\theta(1 - \cos^2\theta) \, d\theta\right]\left[\int_0^{2\pi} \frac{1}{2} + \cancel{\frac{1}{2}\cos 2\phi} \, d\phi\right]$

$= 3\underline{i} \left[\int_0^{\pi} \sin\theta - \sin\theta\cos^2\theta \, d\theta\right]\left[\int_0^{2\pi} \frac{1}{2} \, d\phi\right]$

$= 3i \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} \times \left(\frac{1}{2} \times 2\pi\right)$

$= 3\pi i \left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right]$

$= 3\pi i \left[\frac{2}{3} + \frac{2}{3}\right]$

$= 3\pi\underline{i} \times \frac{4}{3}$

$= 4\pi i$

AS REQUIRED

Stokes' Theorem

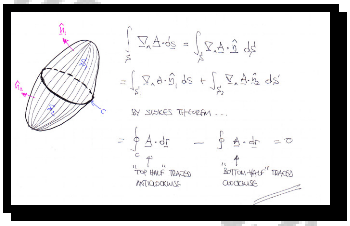
Question 1

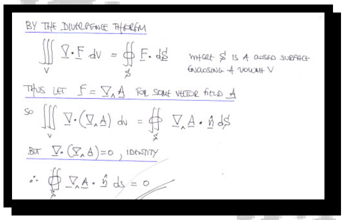
If $\mathbf{F}$ is a smooth vector field, S is a smooth closed surface, and $\hat{\mathbf{n}}$ is an outward unit normal vector to S, show that

$$\int_S \nabla \wedge \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = 0$$

You may find Stokes' Theorem or the Divergence Theorem useful in this question.

$\boxed{\text{proof}}$





Question 2

a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

b) Show that for a smooth scalar field φ and a constant vector $\mathbf{A}$

$$\nabla \wedge (\varphi \mathbf{A}) = \nabla \varphi \wedge \mathbf{A} .$$

The open smooth surface S has boundary c and unit normal field $\hat{\mathbf{n}}$.

c) Use part (a) and (b) to prove

$$\oint_{c} \varphi \, d\mathbf{r} = \int_{S} \hat{\mathbf{n}} \wedge \nabla \varphi \, dS .$$

proof

a) IF S IS AN OPEN SURFACE SURFACE WITH A CLOSED BOUNDARY C AND $\underline{F}$ IS A SMOOTH VECTOR FIELD (CONTINUOUS PARTIAL DERIVATIVES OVER S) THEN THE FOLLOWING RELATIONSHIP HOLDS

$$\iint_{S} \underline{\nabla} \wedge \underline{F} \cdot d\underline{S} = \oint_{C} \underline{F} \cdot d\underline{r}$$

WHERE $d\underline{S} = \hat{\underline{n}}\, dS$, WITH $\hat{\underline{n}}$ IS A UNIT NORMAL TO S, FORMING A RIGHT HAND SET WITH THE DIRECTION OF C

b) LET $\phi = \phi(x,y,z)$

$\underline{A} = (a_1, a_2, a_3)$ = CONSTANT VECTOR

$$\underline{\nabla} \wedge (\phi \underline{A}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi a_1 & \phi a_2 & \phi a_3 \end{vmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \underline{\nabla}\phi \wedge \underline{A}$$

c) BY STOKES THEOREM

$$\iint_{S} \underline{\nabla} \wedge \underline{F} \cdot d\underline{S} = \oint \underline{F} \cdot d\underline{r}$$

LET $\underline{F} = \phi \underline{A}$

$$\Rightarrow \iint_{S} \underline{\nabla} \wedge (\phi \underline{A}) \cdot d\underline{S} = \oint \phi \underline{A} \cdot d\underline{r}$$

PART (b)

$$\Rightarrow \iint_{S} \underline{\nabla}\phi \wedge \underline{A} \cdot \hat{\underline{n}}\, dS = \oint \phi \underline{A} \cdot d\underline{r}$$

NOW

$$\underline{\nabla}\phi \wedge \underline{A} \cdot \hat{\underline{n}} = \begin{vmatrix} n_1 & n_2 & n_3 \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$= +\begin{vmatrix} a_1 & a_2 & a_3 \\ n_1 & n_2 & n_3 \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \underline{A} \cdot \hat{\underline{n}} \wedge \underline{\nabla}\phi$$

THUS

$$\oint_{c} \phi \underline{A} \cdot d\underline{r} = \iint_{S} \underline{\nabla}\phi \wedge \underline{A} \cdot \hat{\underline{n}}\, dS$$

$$\oint_{c} \phi \underline{A} \cdot d\underline{r} = \iint_{S} \underline{A} \cdot \hat{\underline{n}} \wedge \underline{\nabla}\phi \, dS$$

$$\underline{A} \cdot \oint \phi \, d\underline{r} = \underline{A} \cdot \iint_{S} \hat{\underline{n}} \wedge \underline{\nabla}\phi \, dS$$

$$\oint_{c} \phi \, d\underline{r} = \iint_{S} \hat{\underline{n}} \wedge \underline{\nabla}\phi \, dS$$

AS REQUIRED

Question 3

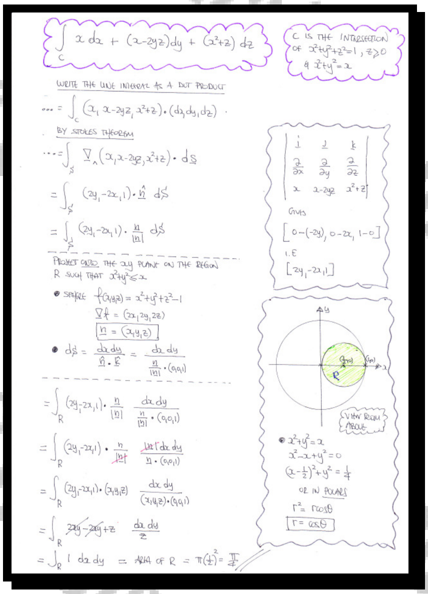
Evaluate the line integral

$$\oint_C \left[x\,dx + \left(x - 2yz\right)dy + \left(x^2 + z\right)dz \right],$$

where C is the intersection of the surfaces with respective Cartesian equations

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0 \qquad \text{and} \qquad x^2 + y^2 = x, \quad z \geq 0.$$

$\boxed{\dfrac{\pi}{4}}$



$\int_C x\,dx + (x-2yz)\,dy + (x^2+z)\,dz$

C IS THE INTERSECTION OF $x^2+y^2+z^2=1$, $z \geq 0$ & $x^2+y^2=x$

WRITE THE LINE INTEGRAL AS A DOT PRODUCT

$\dots = \int_C (x, x-2yz, x^2+z) \cdot (dx, dy, dz)$

BY STOKES THEOREM

$\dots = \int_S \underline{\nabla} \wedge (x, x-2yz, x^2+z) \cdot d\underline{S}$

$= \int_S (2y, -2x, 1) \cdot \underline{\hat{n}}\, dS$

$= \int_S (2y, -2x, 1) \cdot \frac{\underline{n}}{|\underline{n}|}\, dS$

$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x-2yz & x^2+z \end{vmatrix}$

GIVES

$[0-(-2y), 0-2x, 1-0]$

I.E

$[2y, -2x, 1]$

PROJECT ONTO THE xy PLANE ON THE REGION R SUCH THAT $x^2+y^2 \leq x$

- SPHERE $f(x,y,z) = x^2+y^2+z^2-1$

  $\underline{\nabla} f = (2x, 2y, 2z)$

  $\boxed{\underline{n} = (x, y, z)}$

- $dS = \frac{dx\,dy}{\underline{\hat{n}} \cdot \underline{k}} = \frac{dx\,dy}{\frac{\underline{n}}{|\underline{n}|} \cdot (0,0,1)}$

$= \int_R (2y, -2x, 1) \cdot \frac{\underline{n}}{|\underline{n}|} \frac{dx\,dy}{\frac{\underline{n}}{|\underline{n}|} \cdot (0,0,1)}$

$= \int_R (2y, -2x, 1) \cdot \frac{\underline{n}}{\cancel{|\underline{n}|}} \frac{\cancel{|\underline{n}|}\,dx\,dy}{\underline{n} \cdot (0,0,1)}$

$= \int_R (2y, -2x, 1) \cdot (x, y, z) \frac{dx\,dy}{(x,y,z) \cdot (0,0,1)}$

$= \int_R \cancel{2xy} - \cancel{2xy} + z \,\frac{dx\,dy}{z}$

$= \int_R 1\,dx\,dy = \text{AREA OF } R = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$

- $x^2+y^2 = x$

  $x^2 - x + y^2 = 0$

  $\left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$

  OR IN POLARS

  $r^2 = r\cos\theta$

  $\boxed{r = \cos\theta}$

Question 4

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = y^2\,\mathbf{i} + z^2\,\mathbf{j} + x^2\,\mathbf{k}\,.$$

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{dr}\,,$$

where C is the intersection of the surfaces with respective Cartesian equations

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0 \qquad \text{and} \qquad x^2 + y^2 = x, \quad z \geq 0\,.$$

$$\boxed{\frac{\pi}{4}}$$

$$\oint_C \underline{F} \cdot d\underline{r} = -\int_S \underline{\nabla} \wedge \underline{F} \cdot \hat{\underline{n}}\, dS$$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = (-2z, -2x, -2y)$$

$$= -\int_S (-2z, -2x, -2y) \cdot \hat{\underline{n}}\, dS$$

$$= \int_S (2z, 2x, 2y) \cdot (x, y, z)\, dS$$

$$= \int_S (2xz + 2xy + 2yz)\, dS$$

SPHERE HAS EQUATION

$f(x,y,z) = x^2 + y^2 + z^2 - 1$

$\underline{\nabla} f = (2x, 2y, 2z)$

$\underline{n} = (x, y, z)$

$|\underline{n}| = \sqrt{x^2+y^2+z^2} = 1$

$\hat{\underline{n}} = (x, y, z)$

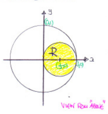
PROJECT ONTO THE xy PLANE, ONTO THE REGION "R" WITH $x^2+y^2 \leq x$

$$= \int_R (2xz + 2xy + 2yz) \underbrace{\frac{dx\,dy}{\hat{\underline{n}} \cdot \underline{k}}}_{dS}$$

$$= \int_R (2xz + 2xy + 2yz) \frac{dx\,dy}{(x,y,z) \cdot (0,0,1)}$$

$$= \int_R (2xz + 2xy + 2yz) \frac{dx\,dy}{z}$$

$$= \int_R \left[2x + \frac{2xy}{z} + 2y\right] dx\,dy$$



$x^2 + y^2 = x$

$x^2 - x + y^2 = 0$

$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$

OR IN POLARS

$r^2 = r\cos\theta$

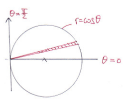
$r = \cos\theta$

NOW R IS A SYMMETRICAL DOMAIN IN y SO ANY ODD POWERS OF y WILL HAVE NO CONTRIBUTION

$$= \int_R \left[2x + \cancel{\frac{2xy}{(1-x^2-y^2)^{\frac{1}{2}}}} + \cancel{2y}\right] dx\,dy$$

$$= \int_R 2x\, dx\,dy \qquad \text{SWITCH INTO PLANE POLARS}$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{r=\cos\theta} 2(r\cos\theta)\, r\,dr\,d\theta$$



$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{r=\cos\theta} 2r^2\cos\theta\, dr\,d\theta$$

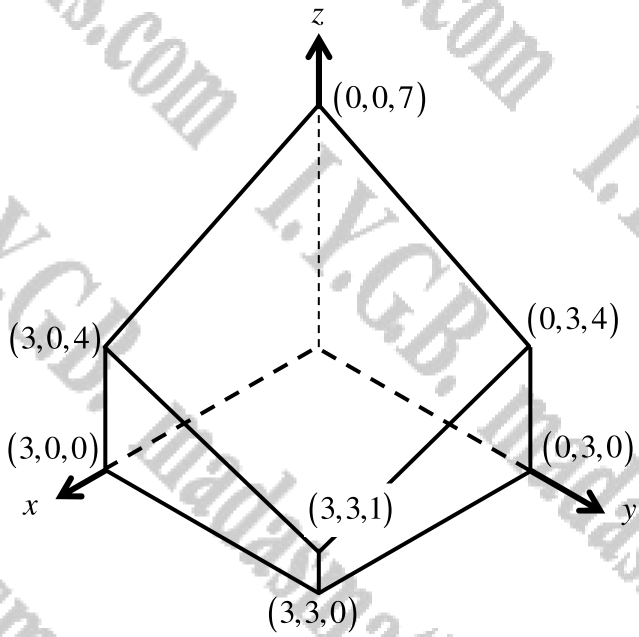
$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{2}{3}r^3\cos\theta\right]_{r=0}^{r=\cos\theta} d\theta = \frac{2}{3}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \overset{\text{EVEN}}{\cos^4\theta}\, d\theta$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} 2\cos^4\theta\, d\theta = \frac{2}{3}\int_0^{\frac{\pi}{2}} 2(\cos\theta)^{2\times\frac{5}{2}-1}(\sin\theta)^{2\times\frac{1}{2}-1}\, d\theta$$

$$= \frac{2}{3}B\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{2}{3}\left[\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)}\right] = \cancel{\frac{2}{3}} \times \frac{\cancel{\frac{3}{2}} \times \frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{2!}$$

$$= \frac{1}{4}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{\pi}\sqrt{\pi} = \frac{\pi}{4}$$

Question 5



The figure above shows the finite region V defined by the intersection of the planes

$$x+y+z=7,\ x=3,\ y=3,\ x=0,\ y=0 \text{ and } z=0.$$

The open surface S encloses V except the plane face with equation $z=0$.

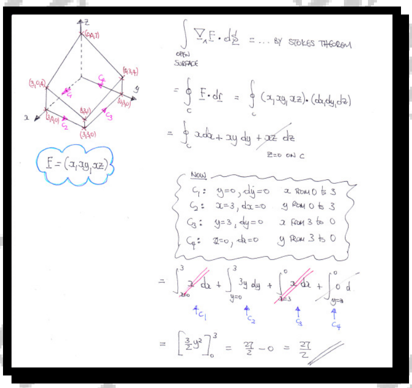
The vector field, $\mathbf{F}(x,y,z) \equiv x\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$, exists on and around S.

Evaluate the surface integral

$$\int_S \nabla \wedge \mathbf{F} \cdot \mathbf{dS},$$

where $\mathbf{dS} = \hat{\mathbf{n}}\,dS$, where $\hat{\mathbf{n}}$ is an outward unit normal vector to S.

$$\boxed{\int_S \nabla \wedge \mathbf{F} \cdot \mathbf{dS} = \frac{27}{2}}$$



Question 6

a) State Stokes' Integral Theorem for open two sided surfaces, fully defining all the quantities involved.

The vector field

$$\mathbf{v} = yz\,\mathbf{k}$$

exists around the open surface S, with closed boundary C.

The equation of S is

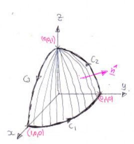
$$z = 1 - x^2 - y^2, \; x \ge 0, \; y \ge 0, \; z \ge 0.$$

b) Use $\mathbf{v}$ and S to verify the validity of Stokes' Theorem.

$$\boxed{\text{both sides yield } \frac{4}{15}}$$

$$\int_S \nabla \wedge \underline{v} \cdot d\underline{S} = \oint_C \underline{v} \cdot d\underline{r}$$

WHERE S = OPEN SURFACE
C = CLOSED BOUNDARY OF S
$\underline{v} = (v_1, v_2, v_3)$
$d\underline{r} = (dx, dy, dz)$
$d\underline{S} = \hat{\underline{n}}\, dS$, WHERE $\hat{\underline{n}}$ IS A UNIT NORMAL TO S, SO THAT $\hat{\underline{n}}$ AND DIRECTION OF INTEGRATION ALONG C, FORM A RIGHT HAND SET



C_1: $x^2+y^2=1$ $\quad z=0$ $\quad dz=0$
C_2: $z=1-y^2$ $\quad x=0$ $\quad dx=0$
C_3: $z=1-x^2$ $\quad y=0$ $\quad dy=0$

$$\oint_C \underline{v} \cdot d\underline{r} = \oint_C yz\hat{\underline{k}} \cdot d\underline{r} = \int_C (0,0,yz)\cdot(dx,dy,dz) = \int_C yz\,dz$$

$$= \Big\{ \cancel{\int_{C_1}} + \int_{C_2} + \cancel{\int_{C_3}} \Big\} yz\,dz = \int_{C_2} y(1-y^2)\,dz$$

$dz=0$ $\quad$ $y=0$

NOW
$z = 1-y^2$
$dz = -2y\,dy$
y FROM 1 to 0

$$= \int_{y=1}^{0} y(1-y^2)(-2y\,dy) = \int_0^1 2y^2 - 2y^4\,dy$$

$$= \left[\frac{2}{3}y^3 - \frac{2}{5}y^5\right]_0^1 = \left(\frac{2}{3} - \frac{2}{5}\right) - (0) = \frac{4}{15}$$

NEXT CARRYING OUT THE CALCULATION OF THE SURFACE INTEGRAL

- $\nabla \wedge (yz\hat{\underline{k}}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & yz \end{vmatrix} = (z,0,0)$

- NORMAL TO SURFACE $z = 1-x^2-y^2$
$x^2+y^2+z-1=0$
LET $f(x,y,z) = x^2+y^2+z-1$

$\underline{n} = \nabla f = (2x, 2y, 1) \leftarrow$ "OUTWARDS" IN 1st OCTANT
$|\underline{n}| = \sqrt{4x^2+4y^2+1} = \sqrt{4(x^2+y^2)+1} = \sqrt{4(1-z)+1} = \sqrt{5-z}$
$\hat{\underline{n}} = \dfrac{(2x,2y,1)}{\sqrt{5-z}}$

- HENCE

$$\int_S \nabla \wedge \underline{v} \cdot d\underline{S} = \int_S (z,0,0)\cdot \hat{\underline{n}}\,dS = \int_S (z,0,0)\cdot \frac{(2x,2y,1)}{\sqrt{5-z}}\,dS$$

$$= \int_S \frac{2xz}{\sqrt{5-z}}\,dS = \dots$$ PROJECT ONTO THE xy PLANE, IE ONTO THE CIRCLE WITH EQUATION $x^2+y^2=1$

$$dS = \frac{dx\,dy}{\hat{\underline{n}}\cdot\hat{\underline{k}}} = \frac{dx\,dy}{\frac{(2x,2y,1)}{\sqrt{5-z}}\cdot(0,0,1)} = \frac{dx\,dy}{\frac{1}{\sqrt{5-z}}} = \sqrt{5-z}\,dx\,dy$$

$$\dots = \int_S \frac{2xz}{\sqrt{5-z}}\,dS = \int_{\substack{\text{CIRCLE } x^2+y^2=1 \\ x>0,\,y>0}} \frac{2xz}{\sqrt{5-z}}\sqrt{5-z}\,dx\,dy = \int_R 2xz\,dx\,dy$$

$$= \int_R 2x(1-x^2-y^2)\,dx\,dy = \dots$$ SWITCH INTO PLANE POLARS

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} 2r\cos\theta\,(1-r^2)\,(r\,dr\,d\theta) = \left[\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta\right]\left[\int_{r=0}^{1} 2r^2 - 2r^4\,dr\right]$$

$$= \Big[\sin\theta\Big]_0^{\frac{\pi}{2}} \left[\frac{2}{3}r^3 - \frac{2}{5}r^5\right]_0^1 = 1 \times \left(\frac{2}{3} - \frac{2}{5}\right)$$

$$= \frac{4}{15}$$

AS BEFORE

Question 7

The vector field

$$\mathbf{F} = z\,\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}$$

exists around the open two sided surface S, with closed boundary C.

S is defined as

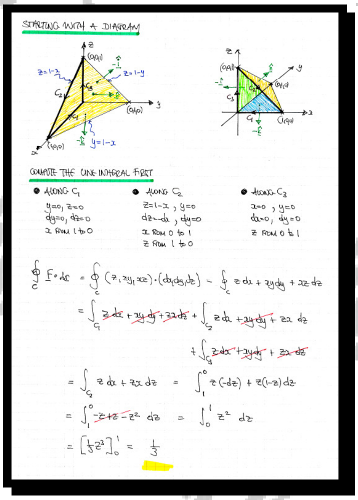
- $x + y + z = 1,\ x \ge 0,\ y \ge 0,\ z \ge 0.$
- $x = 0,\ z \le 1 - y,\ y \ge 0,\ z \ge 0.$
- $z = 0,\ y \le 1 - x,\ x \ge 0,\ y \ge 0.$

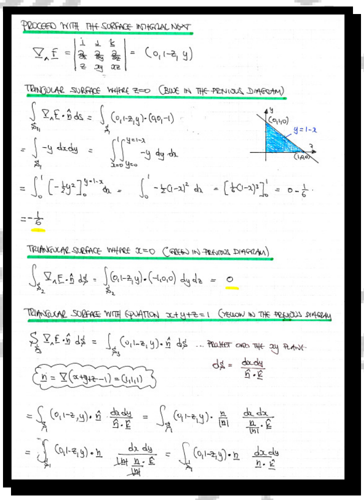
Show that

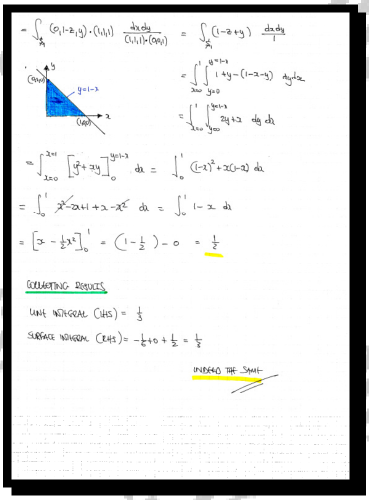
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_S \nabla \wedge \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$

where $\hat{\mathbf{n}}$ is an outward unit normal to S.

$\boxed{}$, $\boxed{\text{both sides yield } \frac{1}{3}}$







Question 8

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = 8z\,\mathbf{i} + 4x\,\mathbf{j} + y\,\mathbf{k}\,.$$

Evaluate the line integral

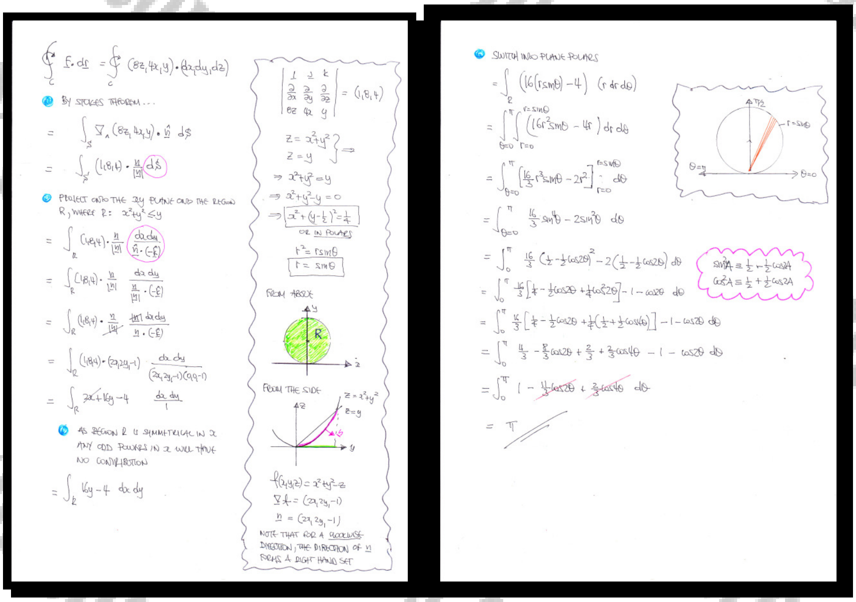
$$\oint_C \mathbf{F} \cdot \mathbf{dr}\,,$$

where C is the intersection of the surfaces with respective Cartesian equations

$$z = y^2 + x^2 \qquad \text{and} \qquad x^2 + y^2 = y\,, \quad z \ge 0\,.$$

You may find Stokes' Theorem useful in this question.

$\boxed{\pi}$



Question 9

The surface S has Cartesian equation

$$(z-1)^2 = x^2 + y^2, \quad 1 \le z \le 3.$$

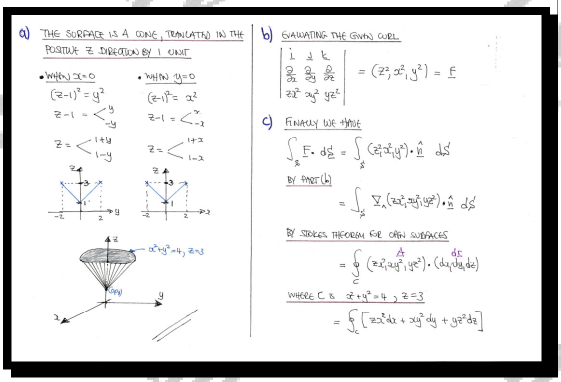
a) Sketch the graph of S.

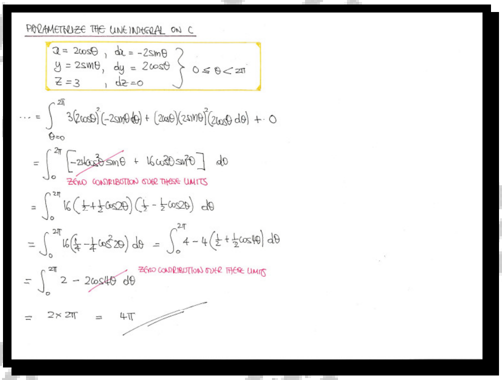
b) Evaluate $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ zx^2 & xy^2 & yz^2 \end{vmatrix}$.

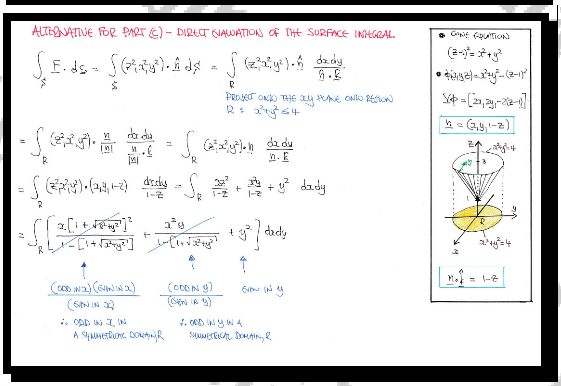
c) Given that $\mathbf{F} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$, evaluate the integral

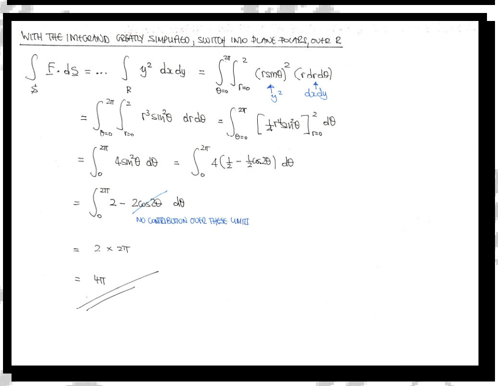
$$\int_S \mathbf{F} \cdot d\mathbf{S}.$$

$\boxed{}$, $\boxed{4\pi}$









Question 10

The vector field $\mathbf{F}$ exists around the open surface S, with closed boundary C.

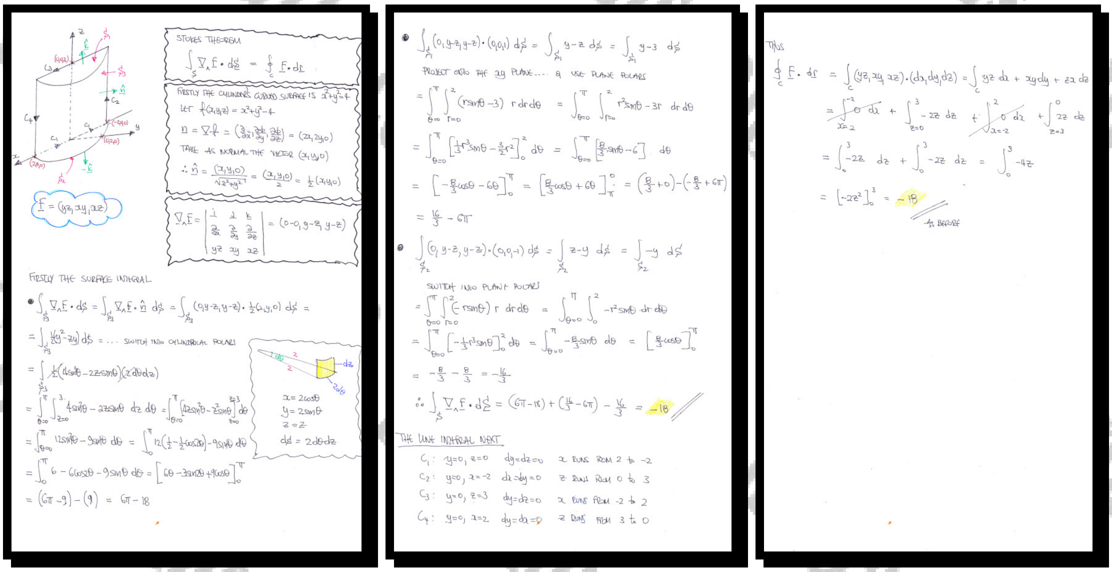
The open surface consists of the following three faces.

- The cylindrical surface $x^2 + y^2 = 4$, $y \geq 0$ and $0 \leq z \leq 3$.
- The plane face $x^2 + y^2 = 4$, $y \geq 0$ and $z = 0$.
- The plane face $x^2 + y^2 = 4$, $y \geq 0$ and $z = 3$.

Use S and C to verify Stokes' Theorem, given further that

$$\mathbf{F}(x, y, z) \equiv yz\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}.$$

$\boxed{\text{both sides yield } -18}$



Question 11

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = 8z\,\mathbf{i} + 4x\,\mathbf{j} + y\,\mathbf{k}\,.$$

Evaluate the line integral

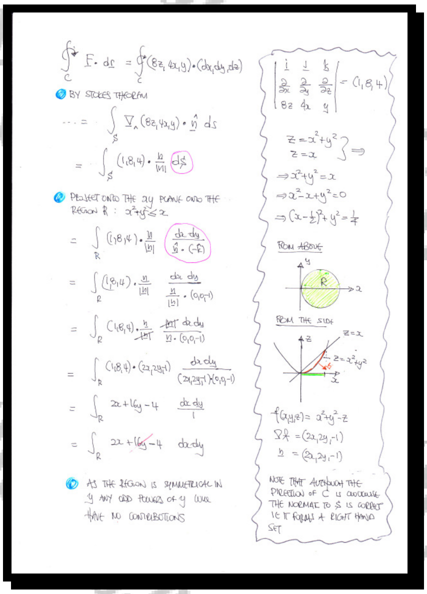
$$\oint_C \mathbf{F} \cdot \mathbf{dr}\,,$$

where C is the intersection of the surfaces with respective Cartesian equations

$$z = x^2 + y^2 \qquad \text{and} \qquad x^2 + y^2 = x\,, \quad z \geq 0\,.$$

You may find Stokes' Theorem useful in this question.

$$\boxed{-\frac{3\pi}{4}}$$



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (8z, 4x, y) \cdot (dx, dy, dz)$$

BY STOKES THEOREM

$$\ldots = \int_S \nabla \wedge (8z, 4x, y) \cdot \hat{\mathbf{n}}\, dS$$

$$= \int_S (1, 8, 4) \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\, dS$$

PROJECT ONTO THE xy PLANE ONTO THE REGION R : $x^2+y^2 \leq x$

$$= \int_R (1,8,4) \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \frac{dx\,dy}{\hat{\mathbf{n}} \cdot (-\hat{\mathbf{k}})}$$

$$= \int_R (1,8,4) \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \frac{dx\,dy}{\frac{\mathbf{n}}{|\mathbf{n}|} \cdot (0,0,-1)}$$

$$= \int_R (1,8,4) \cdot \frac{\mathbf{n}}{\cancel{|\mathbf{n}|}} \frac{\cancel{|\mathbf{n}|}\, dx\,dy}{\mathbf{n} \cdot (0,0,-1)}$$

$$= \int_R (1,8,4) \cdot (2x, 2y, -1) \frac{dx\,dy}{(2x,2y,-1)\cdot(0,0,-1)}$$

$$= \int_R 2x + 16y - 4 \ \frac{dx\,dy}{1}$$

$$= \int_R 2x + \cancel{16y} - 4 \ dx\,dy$$

AS THE REGION IS SYMMETRICAL IN y ANY ODD POWERS OF y WILL HAVE NO CONTRIBUTIONS

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 8z & 4x & y \end{vmatrix} = (1, 8, 4)$$

$$\left.\begin{array}{l} z = x^2+y^2 \\ z = x \end{array}\right\} \Rightarrow$$

$$\Rightarrow x^2+y^2 = x$$

$$\Rightarrow x^2 - x + y^2 = 0$$

$$\Rightarrow (x-\tfrac{1}{2})^2 + y^2 = \tfrac{1}{4}$$

FROM ABOVE

FROM THE SIDE

$z = x$

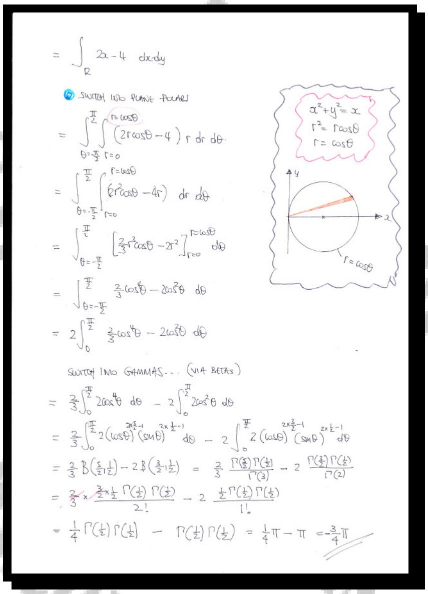
$z = x^2+y^2$

$$f(x,y,z) = x^2+y^2-z$$

$$\nabla f = (2x, 2y, -1)$$

$$\mathbf{n} = (2x, 2y, -1)$$

NOTE THAT ALTHOUGH THE DIRECTION OF C IS ANTICLOCKWISE THE NORMAL TO S IS CORRECT IE IT FORMS A RIGHT HAND SET



$$= \int_R 2x - 4 \ dx\,dy$$

SWITCH INTO PLANE POLARS

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{r=\cos\theta} (2r\cos\theta - 4)\, r\, dr\, d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{r=\cos\theta} (2r^2\cos\theta - 4r)\, dr\, d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\tfrac{2}{3}r^3\cos\theta - 2r^2\right]_{r=0}^{r=\cos\theta} d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \tfrac{2}{3}\cos^4\theta - 2\cos^2\theta \ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \tfrac{2}{3}\cos^4\theta - 2\cos^2\theta \ d\theta$$

SWITCH INTO GAMMAS... (VIA BETAS)

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} 2\cos^4\theta\, d\theta - 2\int_0^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} 2(\cos\theta)^{2\times\frac{5}{2}-1}(\sin\theta)^{2\times\frac{1}{2}-1}\, d\theta - 2\int_0^{\frac{\pi}{2}} 2(\cos\theta)^{2\times\frac{3}{2}-1}(\sin\theta)^{2\times\frac{1}{2}-1}\, d\theta$$

$$= \frac{2}{3}B\left(\tfrac{5}{2},\tfrac{1}{2}\right) - 2B\left(\tfrac{3}{2},\tfrac{1}{2}\right) = \frac{2}{3}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)} - 2\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)}$$

$$= \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{\frac{3}{2}} \times \frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{2!} - 2\frac{\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{1!}$$

$$= \frac{1}{4}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}) - \Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}) = \frac{1}{4}\pi - \pi = -\frac{3}{4}\pi$$

$x^2+y^2 = x$

$r^2 = r\cos\theta$

$r = \cos\theta$

$r = \cos\theta$

Question 12

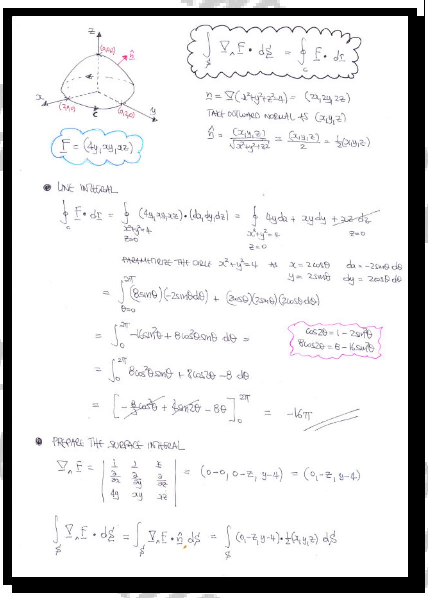
The vector field $\mathbf{F}$ exists around the open surface S, with closed boundary C, whose equation satisfies

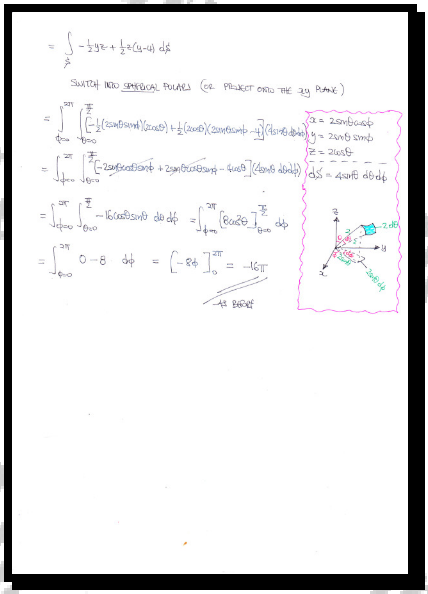
$$x^2 + y^2 + z^2 = 4, \ z \geq 0.$$

Use S and C to verify Stokes' Theorem, given further that

$$\mathbf{F}(x, y, z) \equiv 4y\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}.$$

$\boxed{\text{both sides yield } -16\pi}$





Question 13

The vector field $\mathbf{A}$ exists around the open surface S, with closed boundary C.

$$\mathbf{A} = \left(x^2 y\right)\mathbf{i} + \left(xy + xyz\right)\mathbf{j} + \left(xy + xz^2\right)\mathbf{k}$$

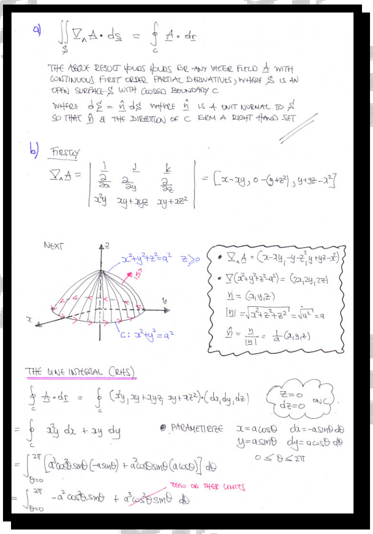
a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

The Cartesian equation of S is

$$x^2 + y^2 + z^2 = a^2, \quad a > 0, \quad z \geq 0.$$

b) Use $\mathbf{A}$ and S to verify the validity of Stokes' Theorem.

$\boxed{\text{both sides yield } -\frac{1}{4}\pi a^4}$



a) $\iint_S \nabla \wedge \underline{A} \cdot d\underline{S} = \oint_C \underline{A} \cdot d\underline{r}$

THE ABOVE RESULT HOLDS HOLDS FOR ANY VECTOR FIELD $\underline{A}$ WITH CONTINUOUS FIRST ORDER PARTIAL DERIVATIVES, WHERE S IS AN OPEN SURFACE S WITH CLOSED BOUNDARY C

WHERE $d\underline{S} = \hat{n}\, dS$ WHERE $\hat{n}$ IS A UNIT NORMAL TO S SO THAT $\hat{n}$ & THE DIRECTION OF C FORM A RIGHT HAND SET

b) FIRSTLY

$\nabla \wedge \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xy+xyz & xy+xz^2 \end{vmatrix} = \left[x - xy,\ 0 - (y+z^2),\ y + yz - x^2\right]$

NEXT

$x^2+y^2+z^2=a^2 \quad z \geq 0$

S

$C: x^2+y^2=a^2$

- $\nabla \wedge \underline{A} = (x - xy, -y-z^2, y+yz-x^2)$
- $\nabla(x^2+y^2+z^2-a^2) = (2x, 2y, 2z)$

$\underline{n} = (x,y,z)$

$|\underline{n}| = \sqrt{x^2+z^2+z^2} = \sqrt{a^2} = a$

$\hat{n} = \frac{\underline{n}}{|\underline{n}|} = \frac{1}{a}(x,y,z)$

THE LINE INTEGRAL (RHS)

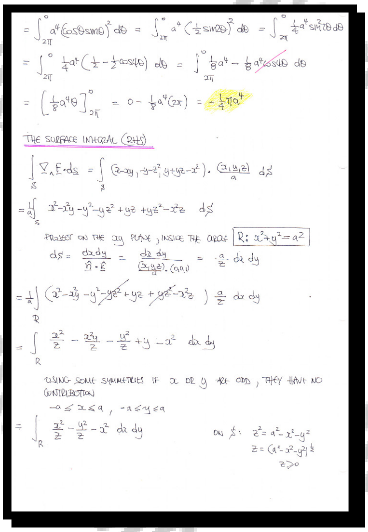
$\oint_C \underline{A} \cdot d\underline{r} = \oint_C (x^2y, xy+xyz, xy+xz^2) \cdot (dx, dy, dz)$ $\quad z=0,\ dz=0$ ON C

$= \oint_C x^2y\, dx + xy\, dy$ $\quad$ • PARAMETRIZE $x = a\cos\theta \quad dx = -a\sin\theta\, d\theta$, $y = a\sin\theta \quad dy = a\cos\theta\, d\theta$, $0 \leq \theta \leq 2\pi$

$= \int_{\theta=0}^{2\pi} \left[a^3\cos^2\theta\sin\theta(-a\sin\theta) + a^2\cos\theta\sin\theta(a\cos\theta)\right] d\theta$

ZERO ON THESE LIMITS

$= \int_{\theta=0}^{2\pi} -a^4\cos^2\theta\sin^2\theta + \cancel{a^3\cos^2\theta\sin\theta}\, d\theta$



$= \int_{2\pi}^{0} a^4(\cos\theta\sin\theta)^2\, d\theta = \int_{2\pi}^{0} a^4\left(\frac{1}{2}\sin 2\theta\right)^2 d\theta = \int_{2\pi}^{0} \frac{1}{4}a^4\sin^2 2\theta\, d\theta$

$= \int_{2\pi}^{0} \frac{1}{4}a^4\left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta = \int_{2\pi}^{0} \frac{1}{8}a^4 - \cancel{\frac{1}{8}a^4\cos 4\theta}\, d\theta$

$= \left[\frac{1}{8}a^4\theta\right]_{2\pi}^{0} = 0 - \frac{1}{8}a^4(2\pi) = -\frac{1}{4}\pi a^4$

THE SURFACE INTEGRAL (RHS)

$\int_S \nabla \wedge \underline{F} \cdot d\underline{S} = \int_S (x - xy, -y-z^2, y+yz-x^2) \cdot \frac{(x,y,z)}{a}\, dS$

$= \frac{1}{a}\int_S x^2 - x^2y - y^2 - yz^2 + yz + yz^2 - x^2z\, dS$

PROJECT ON THE xy PLANE, INSIDE THE AREA $\boxed{R: x^2+y^2=a^2}$

$dS = \frac{dx\,dy}{\hat{n} \cdot \underline{k}} = \frac{dx\,dy}{\frac{(x,y,z)}{a} \cdot (0,0,1)} = \frac{a}{z}\, dx\,dy$

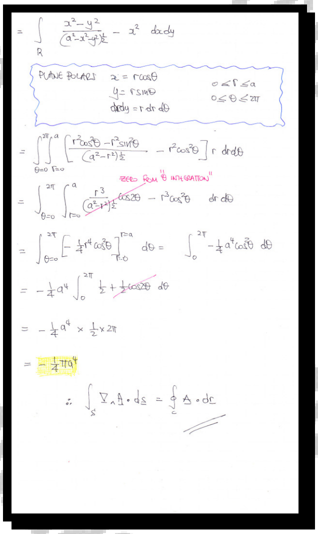
$= \frac{1}{a}\int_R \left(x^2 - x^2y - y^2 - \cancel{yz^2} + yz + \cancel{yz^2} - x^2z\right) \frac{a}{z}\, dx\,dy$

$= \int_R \frac{x^2}{z} - \frac{x^2y}{z} - \frac{y^2}{z} + y - x^2\, dx\,dy$

USING SOME SYMMETRIES IF x OR y ARE ODD, THEY HAVE NO CONTRIBUTION

$-a \leq x \leq a,\ -a \leq y \leq a$

$= \int_R \frac{x^2}{z} - \frac{y^2}{z} - x^2\, dx\,dy$ $\quad$ ON S: $z^2 = a^2 - x^2 - y^2$, $z = (a^2-x^2-y^2)^{\frac{1}{2}}$, $z \geq 0$



$= \int_R \frac{x^2 - y^2}{(a^2-x^2-y^2)^{\frac{1}{2}}} - x^2\, dx\,dy$

PLANE POLARS $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$ $\quad 0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$

$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \left[\frac{r^2\cos^2\theta - r^2\sin^2\theta}{(a^2-r^2)^{\frac{1}{2}}} - r^2\cos^2\theta\right] r\, dr\,d\theta$

ZERO FROM "θ INTEGRATION"

$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \cancel{\frac{r^3}{(a^2-r^2)^{\frac{1}{2}}}\cos 2\theta} - r^3\cos^2\theta\, dr\,d\theta$

$= \int_{\theta=0}^{2\pi}\left[-\frac{1}{4}r^4\cos^2\theta\right]_{r=0}^{r=a} d\theta = \int_0^{2\pi} -\frac{1}{4}a^4\cos^2\theta\, d\theta$

$= -\frac{1}{4}a^4\int_0^{2\pi} \frac{1}{2} + \cancel{\frac{1}{2}\cos 2\theta}\, d\theta$

$= -\frac{1}{4}a^4 \times \frac{1}{2} \times 2\pi$

$= -\frac{1}{4}\pi a^4$

$\therefore \int_S \nabla \wedge \underline{A} \cdot d\underline{S} = \oint_C \underline{A} \cdot d\underline{r}$

Question 14

The smooth vector field $\mathbf{F}$ exists around the open, two sided, surface S, with closed boundary C.

a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

b) Hence show, that if φ is a smooth scalar field defined everywhere, and C is any path between two fixed points, then

$$\int_C \nabla \varphi \cdot d\mathbf{r},$$

is independent of the path of C.

c) Given further that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ evaluate

$$\int_C \left[\frac{\mathbf{r}}{|\mathbf{r}|^3} + x\mathbf{i} \right] \cdot d\mathbf{r},$$

where C is the straight line segment from $(2,1,2)$ to $(6,3,2)$.

$\boxed{}$, $\boxed{\dfrac{340}{21}}$

a) STOKES' THEOREM ASSERTS THAT

$$\iint_S \nabla \wedge \underline{F} \cdot \underline{\hat{n}}\, dS = \oint_C \underline{F} \cdot d\underline{r}$$

WHERE
- S IS AN OPEN TWO SIDED SURFACE WITH A CLOSED BOUNDARY C
- $\underline{F}$ IS A SMOOTH VECTOR FIELD
- $\underline{\hat{n}}$ IS A UNIT NORMAL TO S, SO THAT THE DIRECTION OF C & $\underline{\hat{n}}$ FORM A RIGHT HAND SET
- $d\underline{r} = (dx, dy, dz)$

b) USING STOKES THEOREM WITH $\underline{F} = \nabla\phi$

$$\Rightarrow \iint_S \nabla \wedge \underline{F} \cdot \underline{\hat{n}}\, dS = \oint_C \underline{F} \cdot d\underline{r}$$

$$\Rightarrow \iint_S \underbrace{\left[\nabla \wedge \nabla\phi\right]}_{\text{ZERO AS THIS IS A STANDARD VECTOR CALCULUS IDENTITY}} \cdot \underline{\hat{n}}\, dS = \oint_C \nabla\phi \cdot d\underline{r}$$

$$\Rightarrow 0 = \oint_C \nabla\phi \cdot d\underline{r}$$

$$\Rightarrow \int_{\substack{C_1 \\ (A \text{ to } B)}} \nabla\phi \cdot d\underline{r} + \int_{\substack{C_2 \\ (B \text{ to } A)}} \nabla\phi \cdot d\underline{r} = 0 \qquad C = C_1 + C_2$$

$$\Rightarrow \int_{\substack{C_1 \\ (A \text{ to } B)}} \nabla\phi \cdot d\underline{r} = -\int_{\substack{C_2 \\ (A \text{ to } B)}} \nabla\phi \cdot d\underline{r}$$

I.E. INDEPENDENT OF THE PATH FROM A TO B

c) PROCEED AS FOLLOWS, LOOKING AT PART OF THE INTEGRAND

- $\dfrac{\underline{r}}{|\underline{r}|^3} = \dfrac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \left[x(x^2+y^2+z^2)^{-\frac{3}{2}}, y(x^2+y^2+z^2)^{-\frac{3}{2}}, z(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left[-(x^2+y^2+z^2)^{-\frac{1}{2}}\right] = \nabla\left(-(x^2+y^2+z^2)^{-\frac{1}{2}}\right)$$

- $x\underline{i} = (x,0,0) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{1}{2}x^2\right) = \nabla\left(\frac{1}{2}x^2\right)$

THUS WE NOW HAVE

$$\int_{(2,1,2)}^{(6,3,2)} \left[\frac{\underline{r}}{|\underline{r}^3|} + x\underline{i}\right] \cdot d\underline{r} = \int \nabla \overbrace{\left[\frac{1}{2}x^2 - (x^2+y^2+z^2)^{-\frac{1}{2}}\right]}^{\phi(x,y,z)} \cdot d\underline{r}$$

$$\boxed{\begin{aligned} \nabla\phi \cdot d\underline{r} &= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \cdot (dx, dy, dz) \\ &= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi \end{aligned}}$$

$$\dots = \left[\frac{1}{2}x^2 - \frac{1}{\sqrt{x^2+y^2+z^2}}\right]_{(2,1,2)}^{(6,3,2)} \longleftarrow \int_{(2,1,2)}^{(6,3,2)} 1\, d\phi = \left[\phi\right]_{(2,1,2)}^{(6,3,2)}$$

$$= \left(18 - \frac{1}{7}\right) - \left(2 - \frac{1}{3}\right)$$

$$= 16 - \frac{1}{7} + \frac{1}{3}$$

$$= 16\frac{4}{21} \quad \left(\frac{340}{21}\right)$$

Question 15

The smooth vector field $\mathbf{F}$ exists around the open, two sided, surface S, with closed boundary C.

a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

b) Hence show that

$$\int_S \hat{\mathbf{n}} \wedge \nabla\varphi \, dS = \oint_C \varphi \, d\mathbf{r},$$

where φ is a smooth scalar function and $\hat{\mathbf{n}}$ is a unit normal vector to S.

The Cartesian equation of S is

$$z = x^2 + y^2, \quad z \le 1.$$

c) Use $\varphi(x, y, z) = y$ and S to verify the result of part (**b**).

$\boxed{\text{both sides yield } \pi\mathbf{i}}$

a) IF S IS AN OPEN SURFACE WITH CLOSED BOUNDARY C, AND $\underline{F}$ IS A VECTOR FIELD WITH CONTINUOUS PARTIAL DERIVATIVES OVER S, THEN

$$\iint_S \nabla \wedge \underline{F} \cdot d\underline{S} = \oint_C \underline{F} \cdot d\underline{r}$$

WHERE $d\underline{S} = \hat{\underline{n}}\, dS$, WITH $\hat{\underline{n}}$ THE UNIT NORMAL FIELD TO S, SO THAT $\hat{\underline{n}}$ & THE DIRECTION OF C, FORMS A "RIGHT HAND" SET

b) $$\iint_S \nabla \wedge \underline{F} \cdot \hat{\underline{n}}\, dS = \oint_C \underline{F} \cdot d\underline{r}$$

LET $\underline{F} = \phi\underline{A}$ WHERE $\underline{A}$ IS A CONSTANT VECTOR

$$\nabla \wedge \underline{F} = \nabla \wedge (\phi\underline{A}) = \nabla\phi \wedge \underline{A} + \cancel{\phi \nabla \wedge \underline{A}}$$ AS $\underline{A}$ IS CONSTANT

$$\Rightarrow \iint_S \nabla\phi \wedge \underline{A} \cdot \hat{\underline{n}}\, dS = \oint \phi\underline{A} \cdot d\underline{r}$$

$$\nabla\phi \wedge \underline{A} \cdot \hat{\underline{n}} = -\underline{A} \wedge \nabla\phi \cdot \hat{\underline{n}} = (-1)(-1)\,\hat{\underline{n}} \wedge \nabla\phi \cdot \underline{A}$$
CYCLIC PROPERTY OF SCALAR TRIPLE PRODUCT

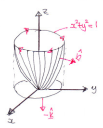
$$\Rightarrow \iint_S \hat{\underline{n}} \wedge \nabla\phi \cdot \underline{A}\, dS = \int \phi\underline{A} \cdot d\underline{r}$$

$$\Rightarrow \iint_S \underline{A} \cdot \hat{\underline{n}} \wedge \nabla\phi\, dS = \int \underline{A}\phi \cdot d\underline{r}$$

$$\Rightarrow \underline{A} \cdot \iint_S \hat{\underline{n}} \wedge \nabla\phi\, dS = \underline{A} \cdot \int \phi\, d\underline{r}$$

$$\Rightarrow \iint_S \hat{\underline{n}} \wedge \nabla\phi\, dS = \int \phi\, d\underline{r}$$ AS REQUIRED

c)



- LET $V(x,y,z) = x^2+y^2-z$
  $\nabla V = (2x, 2y, -1)$
  $\underline{n} = (2x, 2y, -1)$
  $|\underline{n}| = \sqrt{4x^2+4y^2+1}$
  $\hat{\underline{n}} = \dfrac{(2x, 2y, -1)}{\sqrt{4x^2+4y^2+1}}$
- $\phi(x,y,z) = y$
  $\nabla\phi = (0,1,0)$

- NOW

$$\hat{\underline{n}} \wedge \nabla\phi = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{2x}{(4x^2+4y^2+1)^{\frac{1}{2}}} & \frac{2y}{(4x^2+4y^2+1)} & \frac{-1}{(4x^2+4y^2+1)^{\frac{1}{2}}} \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \left[\frac{1}{(4x^2+4y^2+1)^{\frac{1}{2}}}, 0, \frac{2x}{(4x^2+4y^2+1)^{\frac{1}{2}}} \right]$$

$$\int_S \underline{n} \wedge \nabla\phi\, dS = \dots$$ PROJECT ONTO THE x-y PLANE ONTO $x^2+y^2 \le 1$ (REGION R) $\boxed{dS = \frac{dx\,dy}{\hat{\underline{n}} \cdot (-\hat{\underline{k}})}}$

$$= \int_S \left[\frac{1}{|\underline{n}|}, 0, \frac{2x}{|\underline{n}|}\right] dS = \int_R \left[\frac{1}{|\underline{n}|}, 0, \frac{2x}{|\underline{n}|}\right] \frac{dx\,dy}{\frac{\underline{n}}{|\underline{n}|} \cdot (0,0,-1)}$$

$$= \int_R \left[\frac{1}{|\underline{n}|}, 0, \frac{2x}{|\underline{n}|}\right] \frac{|\underline{n}|}{\underline{n} \cdot (0,0,-1)}\, dx\,dy = \int_R (1,0,2x) \frac{dx\,dy}{(2x,2y,-1)\cdot(0,0,-1)}$$

$$= \int_R (1,0,2x)\, dx\,dy$$

SWITCH INTO PLANE POLARS

$$= \int_R (1, 0, 2r\cos\theta)(r\,dr\,d\theta)$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (r, 0, \cancel{2r^2\cos\theta})\, dr\, d\theta$$ ZERO FROM THE θ INTEGRATION

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\underline{i}\, dr\, d\theta$$

$$= \underline{i} \int_{\theta=0}^{2\pi} \left[\tfrac{1}{2}r^2\right]_0^1 d\theta = \underline{i} \int_0^{2\pi} \tfrac{1}{2}\, d\theta = \pi\underline{i}$$

- $$\int_C \phi\, d\underline{r} = \int_C y\,(dx, dy, dz)$$

$$= \int_{t=0}^{2\pi} \sin t\,(\sin t, \cos t, 0)\, dt = \int_0^{2\pi} (-\sin^2 t, \cancel{\sin t\cos t}, 0)\, dt$$ NO CONTRIBUTION OVER THESE LIMITS

PARAMETERIZE CLOCKWISE
$x^2+y^2=1$, $z=1$, $dz=0$
$x = -\cos t$, $dx = \sin t\, dt$
$y = \sin t$, $dy = \cos t\, dt$
CURVE STARTS AT $(-1,0)$ & TRACES CLOCKWISE

$$= \int_0^{2\pi} \left(\tfrac{1}{2} - \cancel{\tfrac{1}{2}\cos 2t}\right)\underline{i}\, dt$$ NO CONTRIBUTION

$$= \int_0^{2\pi} \tfrac{1}{2}\underline{i}\, dt = \pi\underline{i}$$ AS BEFORE

Question 16

The vector field $\mathbf{F}$ exists around the open surface S, with closed boundary C.

a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved.

b) Hence show that

$$\int_S \hat{\mathbf{n}} \wedge \nabla \varphi \, dS = \oint_C \varphi \, d\mathbf{r},$$

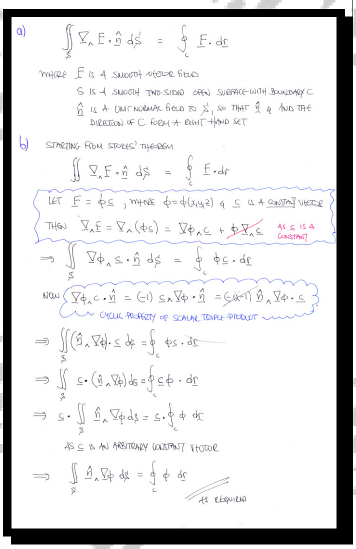
where φ is a smooth scalar function and $\hat{\mathbf{n}}$ is unit normal vector to S.

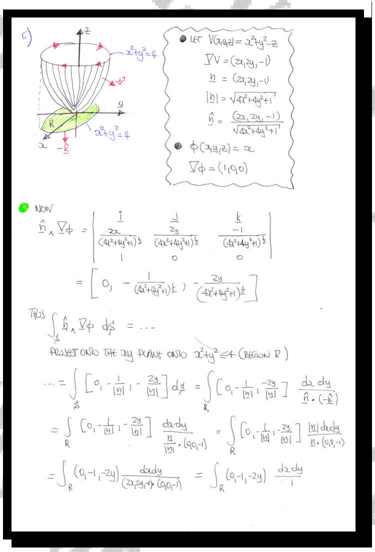
The Cartesian equation of S is

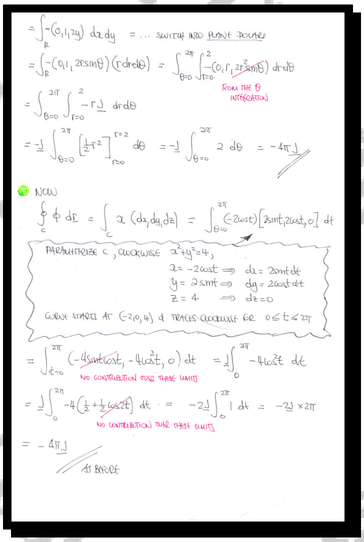
$$z = x^2 + y^2, \quad z \le 4.$$

c) Use $\varphi(x, y, z) = x$ and S to verify the result of part **(b)**.

$\boxed{\text{both sides yield } -4\pi\mathbf{j}}$







Question 17

$\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are vector fields.

a) Prove the validity of the vector identity

$$\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) \equiv \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

b) Given further that $\mathbf{c}$ is a constant vector and $\mathbf{A}$ a smooth vector field, find a simplified expression for

$$\nabla \wedge (\mathbf{c} \wedge \mathbf{A}).$$

An open two sided surface S has boundary C.

c) Use Stokes' Integral Theorem and the result obtained in part **(b)** to show that

$$\int_S (\mathbf{dS} \wedge \nabla) \wedge \mathbf{A} = \oint_C d\mathbf{r} \wedge \mathbf{A},$$

where $\mathbf{dS} = \hat{\mathbf{n}}\, dS$ with $\hat{\mathbf{n}}$ a unit normal vector to S, and $\mathbf{dr} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$.

$$\boxed{\nabla \wedge (\mathbf{c} \wedge \mathbf{A}) = \mathbf{c}(\nabla \cdot \mathbf{A}) - (\mathbf{c} \cdot \nabla)\mathbf{A}}$$

a) $\underline{A} \wedge (\underline{B} \wedge \underline{C}) = (A_1, A_2, A_3) \wedge \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

$= (A_1, A_2, A_3) \wedge [B_2C_3 - C_2B_3, B_3C_1 - B_1C_3, B_1C_2 - C_1B_2]$

$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A_1 & A_2 & A_3 \\ B_2C_3 - C_2B_3 & B_3C_1 - B_1C_3 & B_1C_2 - C_1B_2 \end{vmatrix}$

$= [A_2B_1C_2 - A_2C_1B_2 - A_3B_3C_1 + A_3B_1C_3]\underline{i}$
$[A_3B_2C_3 - A_3C_2B_3 - A_1B_1C_2 + A_1C_1B_2]\underline{j}$
$[A_1B_3C_1 - A_1B_1C_3 - A_2B_2C_3 + A_2C_2B_3]\underline{k}$

REGROUP TERMS

$= \begin{Bmatrix} (A_2B_1C_2 + A_3B_1C_3)\underline{i} \\ (A_3B_2C_3 + A_1C_1B_2)\underline{j} \\ (A_1B_3C_1 + A_2C_2B_3)\underline{k} \end{Bmatrix} - \begin{Bmatrix} (A_2B_2C_1 + A_3B_3C_1)\underline{i} \\ (A_3B_3C_2 + A_1B_1C_2)\underline{j} \\ (A_1B_1C_3 + A_2B_2C_3)\underline{k} \end{Bmatrix}$

$= \begin{bmatrix} (A_2C_2 + A_3C_3)B_1\underline{i} \\ (A_3C_3 + A_1C_1)B_2\underline{j} \\ (A_1C_1 + A_2C_2)B_3\underline{k} \end{bmatrix} - \begin{bmatrix} (A_2B_2 + A_3B_3)C_1\underline{i} \\ (A_3B_3 + A_1B_1)C_2\underline{j} \\ (A_1B_1 + A_2B_2)C_3\underline{k} \end{bmatrix}$

$= \begin{bmatrix} (A_1C_1 + A_2C_2 + A_3C_3)B_1\underline{i} \\ (A_1C_1 + A_2C_2 + A_3C_3)B_2\underline{j} \\ (A_1C_1 + A_2C_2 + A_3C_3)B_3\underline{k} \end{bmatrix} - \begin{bmatrix} (A_1B_1 + A_2B_2 + A_3B_3)C_1\underline{i} \\ (A_1B_1 + A_2B_2 + A_3B_3)C_2\underline{j} \\ (A_1B_1 + A_2B_2 + A_3B_3)C_3\underline{k} \end{bmatrix}$

$= [(\underline{A} \cdot \underline{C})B_1\underline{i} + (\underline{A} \cdot \underline{C})B_2\underline{j} + (\underline{A} \cdot \underline{C})B_3\underline{k}] - [(\underline{A} \cdot \underline{B})C_1\underline{i} + (\underline{A} \cdot \underline{B})C_2\underline{j} + (\underline{A} \cdot \underline{B})C_3\underline{k}]$

$= (\underline{A} \cdot \underline{C})[B_1, B_2, B_3] - [\underline{A} \cdot \underline{B}][C_1, C_2, C_3]$

$= (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$

AS REQUIRED

b) $\underline{A} \wedge (\underline{B} \wedge \underline{C}) = \underline{B}(\underline{A} \cdot \underline{C}) - (\underline{A} \cdot \underline{B})\underline{C}$

LET $\underline{A} = $ "$\underline{\nabla}$"
$\underline{B} = $ "$\underline{c}$"
$\underline{C} = $ "$\underline{A}$"

$\underline{\nabla} \wedge (\underline{c} \wedge \underline{A}) = \underline{c}(\underline{\nabla} \cdot \underline{A}) - (\underline{\nabla} \cdot \underline{c})\underline{A}$

$\underline{\nabla} \wedge (\underline{c} \wedge \underline{A}) = \underline{c}(\underline{\nabla} \cdot \underline{A}) - (\underline{c} \cdot \underline{\nabla})\underline{A}$

c) BY STOKES THEOREM

$\int_S \underline{\nabla} \wedge \underline{F} \cdot d\underline{S} = \oint_C \underline{F} \cdot d\underline{r}$

$\boxed{\text{LET } \underline{F} = \underline{c} \wedge \underline{A}}$

$\int_S \underline{\nabla} \wedge (\underline{c} \wedge \underline{A}) \cdot d\underline{S} = \oint_C \underline{c} \wedge \underline{A} \cdot d\underline{r}$

$\int_S \underline{c}(\underline{\nabla} \cdot \underline{A}) \cdot d\underline{S} - (\underline{c} \cdot \underline{\nabla})\underline{A} \cdot d\underline{S} = \oint_C -d\underline{r} \wedge \underline{A} \cdot \underline{c}$

$\int_S (\underline{c} \cdot d\underline{S})(\underline{\nabla} \cdot \underline{A}) - (\underline{c} \cdot \underline{\nabla})(\underline{A} \cdot d\underline{S}) = -\oint_C \underline{c} \cdot d\underline{r} \wedge \underline{A}$

$\cancel{\underline{c} \cdot} \int_S \underbrace{d\underline{S}(\underline{\nabla} \cdot \underline{A}) - \underline{\nabla}(\underline{A} \cdot d\underline{S})}_{\underline{b}(\underline{a} \cdot \underline{c}) \; - \; (\underline{a} \cdot \underline{b})} = -\cancel{\underline{c} \cdot} \int_C d\underline{r} \wedge \underline{A}$

USING (b) BACKWARDS

$\int_S \underline{A} \wedge (d\underline{S} \wedge \underline{\nabla}) = -\oint_C d\underline{r} \wedge \underline{A}$

$\underline{a} \wedge (\underline{b} \wedge \underline{c})$

$-\int_S (d\underline{S} \wedge \underline{\nabla}) \wedge \underline{A} = -\oint_C d\underline{r} \wedge \underline{A}$

$\int_S (d\underline{S} \wedge \underline{\nabla}) \wedge \underline{A} = \oint d\underline{r} \wedge \underline{A}$

Question 18

An open two sided surface S has boundary C.

It is further given that $\mathbf{a}$ is a constant vector and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

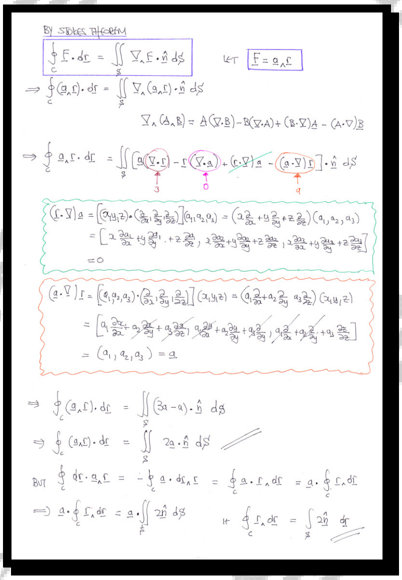
Show that

a) $$\int_S 2\mathbf{a}\cdot\hat{\mathbf{n}}\ dS = \oint_C \mathbf{a}\wedge\mathbf{r}\cdot d\mathbf{r}\ .$$

b) $$\int_S 2\hat{\mathbf{n}}\ dS = \oint_C \mathbf{r}\wedge d\mathbf{r}\ .$$

where $\hat{\mathbf{n}}$ a unit normal vector to S, and $\mathbf{dr} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$.

proof



Question 19

$$\mathbf{A} = 2\mathbf{i} - \mathbf{j} + (4y - 3)\mathbf{k} .$$

The vector field $\mathbf{A}$ exist around the surface S with Cartesian equation

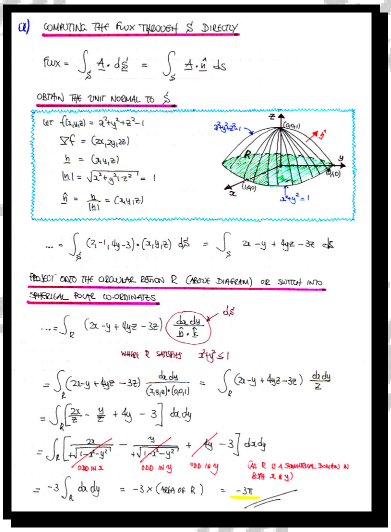
$$x^2 + y^2 + z^2 = 1, \ z \geq 0 .$$

a) Determine the flux of $\mathbf{A}$ through S, where the normal unit field to S is denoted by $\hat{\mathbf{n}}$, such that $\hat{\mathbf{n}} \cdot \mathbf{k} \geq 0$.

b) Obtain the answer of part **(a)** by using the Divergence Theorem.

c) Use Stokes' Theorem to get an expression for the flux of $\mathbf{A}$ through S, as a line integral, and hence verify the answer of part **(a)**.

$\boxed{}$, $\boxed{\text{flux} = -3\pi}$


a) COMPUTING THE FLUX THROUGH S DIRECTLY

$\text{flux} = \int_S \underline{A} \cdot d\underline{S} = \int_S \underline{A} \cdot \hat{n} \, dS$

OBTAIN THE UNIT NORMAL TO S

LET $f(x,y,z) = x^2+y^2+z^2-1$

$\nabla f = (2x, 2y, 2z)$

$\underline{n} = (x,y,z)$

$|\underline{n}| = \sqrt{x^2+y^2+z^2} = 1$

$\hat{n} = \frac{\underline{n}}{|\underline{n}|} = (x,y,z)$

$\dots = \int_S (2,-1,4y-3) \cdot (x,y,z) \, dS = \int_S 2x - y + 4yz - 3z \, dS$

PROJECT ONTO THE CIRCULAR REGION R (ABOVE DIAGRAM) OR SWITCH INTO SPHERICAL POLAR CO-ORDINATES

$\dots = \int_R (2x - y + 4yz - 3z) \frac{dx\,dy}{\hat{n} \cdot \hat{k}}$

WHERE R SATISFIES $x^2+y^2 \leq 1$

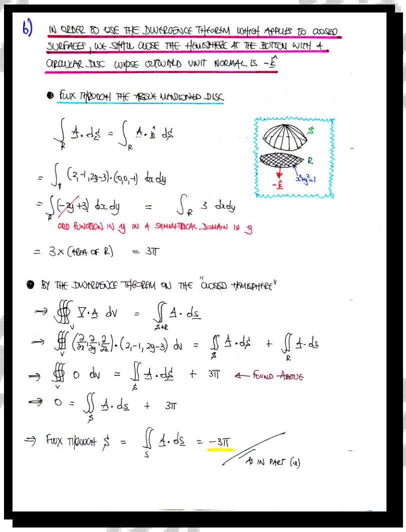
$= \int_R (2x - y + 4yz - 3z) \frac{dx\,dy}{(x,y,z)\cdot(0,0,1)} = \int_R (2x - y + 4yz - 3z) \frac{dx\,dy}{z}$

$= \int_R \left[\frac{2x}{z} - \frac{y}{z} + 4y - 3 \right] dx\,dy$

$= \int_R \left[\frac{2x}{+\sqrt{1-x^2-y^2}} - \frac{y}{+\sqrt{1-x^2-y^2}} + 4y - 3 \right] dx\,dy$

ODD IN x — ODD IN y — ODD IN y — AS R IS A SYMMETRICAL DOMAIN IN BOTH x & y

$= -3 \int_R dx\,dy = -3 \times (\text{AREA OF } R) = -3\pi$


b) IN ORDER TO USE THE DIVERGENCE THEOREM WHICH APPLIES TO CLOSED SURFACES, WE SHALL CLOSE THE HEMISPHERE AT THE BOTTOM WITH A CIRCULAR DISC WHOSE OUTWARD UNIT NORMAL IS $-\underline{k}$

● FLUX THROUGH THE ABOVE MENTIONED DISC

$\int \underline{A} \cdot d\underline{S} = \int_R \underline{A} \cdot \hat{n} \, dS$

$= \int_R (2,-1,4y-3) \cdot (0,0,-1) \, dx\,dy$

$= \int_R (-4y + 3) \, dx\,dy = \int_R 3 \, dx\,dy$

ODD FUNCTION IN y IN A SYMMETRICAL DOMAIN IN y

$= 3 \times (\text{AREA OF } R) = 3\pi$

● BY THE DIVERGENCE THEOREM ON THE "CLOSED HEMISPHERE"

$\Rightarrow \iiint_V \nabla \cdot \underline{A} \, dV = \oint_{S+R} \underline{A} \cdot d\underline{S}$

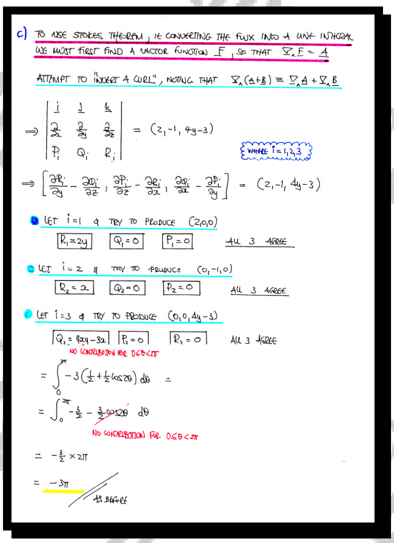
$\Rightarrow \iiint_V \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (2,-1,4y-3) \, dV = \iint_S \underline{A} \cdot d\underline{S} + \iint_R \underline{A} \cdot d\underline{S}$

$\Rightarrow \iiint_V 0 \, dV = \iint_S \underline{A} \cdot d\underline{S} + 3\pi$ ← FOUND ABOVE

$\Rightarrow 0 = \iint_S \underline{A} \cdot d\underline{S} + 3\pi$

$\Rightarrow$ FLUX THROUGH S $= \iint_S \underline{A} \cdot d\underline{S} = -3\pi$

AS IN PART (a)


c) TO USE STOKES THEOREM, IE CONVERTING THE FLUX INTO A LINE INTEGRAL WE MUST FIRST FIND A VECTOR FUNCTION $\underline{F}$, SO THAT $\nabla \wedge \underline{F} = \underline{A}$

ATTEMPT TO "INVERT A CURL", NOTING THAT $\nabla \wedge (\underline{A}+\underline{B}) = \nabla \wedge \underline{A} + \nabla \wedge \underline{B}$

$$\Rightarrow \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P_i & Q_i & R_i \end{vmatrix} = (2,-1,4y-3)$$

WHERE $i = 1,2,3$

$$\Rightarrow \left[\frac{\partial R_i}{\partial y} - \frac{\partial Q_i}{\partial z}, \frac{\partial P_i}{\partial z} - \frac{\partial R_i}{\partial x}, \frac{\partial Q_i}{\partial x} - \frac{\partial P_i}{\partial y} \right] = (2,-1,4y-3)$$

● LET $i=1$ & TRY TO PRODUCE $(2,0,0)$

$\boxed{R_1 = 2y}$ $\boxed{Q_1 = 0}$ $\boxed{P_1 = 0}$ ALL 3 AGREE

● LET $i=2$ & TRY TO PRODUCE $(0,-1,0)$

$\boxed{R_2 = x}$ $\boxed{Q_2 = 0}$ $\boxed{P_2 = 0}$ ALL 3 AGREE

● LET $i=3$ & TRY TO PRODUCE $(0,0,4y-3)$

$\boxed{Q_3 = 4xy - 3x}$ $\boxed{P_3 = 0}$ $\boxed{R_3 = 0}$ ALL 3 AGREE

NO CONTRIBUTION FOR $0 \leq \theta < 2\pi$

$= \int_0^{2\pi} -3\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta =$

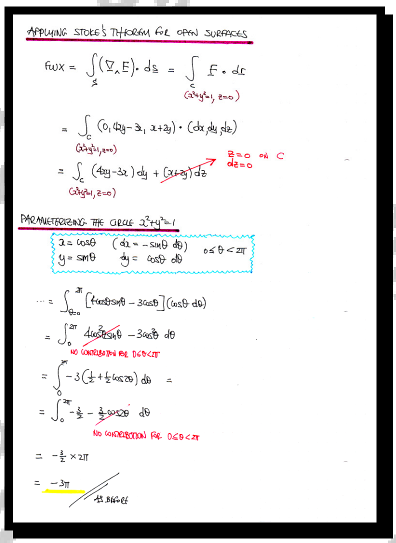
$= \int_0^{2\pi} -\frac{3}{2} - \frac{3}{2}\cos 2\theta \, d\theta$

NO CONTRIBUTION FOR $0 \leq \theta < 2\pi$

$= -\frac{3}{2} \times 2\pi$

$= -3\pi$

AS BEFORE


APPLYING STOKE'S THEOREM FOR OPEN SURFACES

$\text{FLUX} = \int_S (\nabla \wedge \underline{F}) \cdot d\underline{S} = \oint_C \underline{F} \cdot d\underline{r}$

$(x^2+y^2=1, z=0)$

$= \oint_C (0, 4xy - 3x, x + 2y) \cdot (dx, dy, dz)$

$(x^2+y^2=1, z=0)$

$= \oint_C (4xy - 3x) \, dy + \cancel{(x+2y)\,dz}$

$z=0$ ON C, $dz = 0$

$(x^2+y^2=1, z=0)$

PARAMETERIZING THE CIRCLE $x^2+y^2=1$

$x = \cos\theta \quad (dx = -\sin\theta \, d\theta)$

$y = \sin\theta \quad dy = \cos\theta \, d\theta$

$0 \leq \theta < 2\pi$

$\dots = \int_{\theta=0}^{2\pi} \left[4\cos\theta\sin\theta - 3\cos\theta \right] (\cos\theta \, d\theta)$

$= \int_0^{2\pi} 4\cos^2\theta\sin\theta - 3\cos^2\theta \, d\theta$

NO CONTRIBUTION FOR $0 \leq \theta < 2\pi$

$= \int_0^{2\pi} -3\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta =$

$= \int_0^{2\pi} -\frac{3}{2} - \frac{3}{2}\cos 2\theta \, d\theta$

NO CONTRIBUTION FOR $0 \leq \theta < 2\pi$

$= -\frac{3}{2} \times 2\pi$

$= -3\pi$

AS BEFORE