

COLLISIONS

Question 1 ()**

A smooth sphere is moving with speed $u \text{ ms}^{-1}$ on a smooth horizontal plane when it strikes at right angles a fixed smooth vertical wall. The sphere is modelled as a particle. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$.

Find the fraction of the kinetic energy is lost by the sphere, as a result of the impact with the wall.

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$$\bullet e = \frac{\text{S.E.P}}{\text{A.P.P.}} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}mu^2} = \frac{\frac{1}{2}m(\frac{1}{3}u)^2}{\frac{1}{2}mu^2} = \frac{\frac{1}{2}m\frac{1}{9}u^2}{\frac{1}{2}mu^2} = \frac{1}{18}$$

$$\therefore V = \frac{1}{3}u$$

$$\bullet \text{KE}_{\text{IMPACT}} = \frac{1}{2}mv^2 = \frac{1}{2}m(\frac{1}{3}u)^2 = \frac{1}{18}mu^2$$

$$\therefore \text{KE}_{\text{LOSS}} = \frac{1}{2}mu^2 - \frac{1}{18}mu^2 = \frac{17}{18}mu^2$$

$$\therefore \text{FRACTION OF KE LOST} = \frac{\frac{17}{18}mu^2}{\frac{1}{2}mu^2} = \frac{17}{9}$$

Question 2 (+)**

A smooth sphere is moving with speed $u \text{ ms}^{-1}$ on a smooth horizontal plane when it strikes at right angles a fixed smooth vertical wall.

One quarter of the kinetic energy is lost by the sphere, as a result of the impact with the wall. The sphere is modelled as a particle.

Find the coefficient of restitution between the sphere and the wall.

$\frac{\sqrt{3}}{2}$

$$\bullet e = \frac{\text{S.E.P}}{\text{A.P.P.}} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}mu^2} = \frac{\frac{1}{2}meu^2}{\frac{1}{2}mu^2} = \frac{e^2}{1}$$

$$\bullet \text{KE}_{\text{IMPACT}} = \frac{1}{2}mv^2 = \frac{1}{2}m(eu)^2 = \frac{1}{2}me^2u^2$$

$$\bullet \text{KE}_{\text{LOSS}} = \frac{1}{2}mu^2 - \frac{1}{2}me^2u^2$$

$$\therefore \frac{\frac{1}{2}mu^2 - \frac{1}{2}me^2u^2}{\frac{1}{2}mu^2} = \frac{1}{4}$$

$$1 - e^2 = \frac{1}{4}$$

$$1 - \frac{1}{4} = e^2$$

$$e^2 = \frac{3}{4}$$

$$e = \frac{\sqrt{3}}{2}$$

Question 3 (*)**

A smooth sphere A of mass 3 kg is moving with speed 4 ms^{-1} on a smooth horizontal plane when it collides directly with a smooth sphere B of mass 5 kg moving with speed 2 ms^{-1} , in the opposite direction as B . The two spheres are modelled as particles and the coefficient of restitution between them is $\frac{5}{6}$.

- a) Calculate the speed of A and the speed of B after the collision.

After the collision between A and B , sphere B collides directly with a smooth vertical wall. The coefficient of restitution between B and the wall is $\frac{1}{5}$.

- b) Find the magnitude of the impulse exerted by the wall onto B .

$$|V_A| = \frac{23}{8} \text{ ms}^{-1}, |V_B| = \frac{17}{8} \text{ ms}^{-1}, |I| = 12.75 \text{ Ns}$$

The handwritten solution is organized into two main sections, (a) and (b), each with multiple steps and calculations:

- Section (a):**
 - Diagram shows two spheres, A and B, on a horizontal surface. Sphere A has mass 3 kg and speed 4 ms⁻¹ to the right. Sphere B has mass 5 kg and speed 2 ms⁻¹ to the left. After collision, they move to the right with speeds x and y respectively. The coefficient of restitution is $\frac{5}{6}$.
 - By conservation of momentum: $(4 \times 3) - (2 \times 5) = 2x + 5y$ and $3x + 5y = 2$.
 - By restitution: $e = \frac{5y}{4x}$ and $\frac{5}{6} = \frac{5y}{4x}$ which simplifies to $y = x - 5$.
 - Solving the system of equations:

$$\begin{aligned} 3x + 5(x-5) &= 2 \\ 3x + 5x - 25 &= 2 \\ 8x &= 27 \\ x &= \frac{27}{8} \end{aligned}$$
 Since $x = \frac{27}{8}$ is greater than 4, sphere A has rebounded. $y = x - 5 = -2.875$.
 - Final speeds: Speed of A is 2.875 ms^{-1} (from rebound). Speed of B is 2.125 ms^{-1} .
- Section (b):**
 - Diagram shows sphere B moving to the right with speed 2.125 ms^{-1} towards a vertical wall. The coefficient of restitution is $\frac{1}{5}$.
 - Speed of B just before impact is "e x approach speed": $e = \frac{1}{5} \times \frac{17}{8} = \frac{17}{40} = 0.425$.
 - Taking "right" as positive:

$$\begin{aligned} I &= \text{mass}_B \times v - \text{mass}_B \times v' \\ I &= -\frac{5}{8} \times 5 - \frac{5}{8} \times 5 \\ I &= \frac{5}{4} \\ |I| &= \frac{5}{4} = 12.75 \text{ Ns} \end{aligned}$$
 - Magnitude of impulse: $|I| = 12.75 \text{ Ns}$.

Question 4 (*)**

Two smooth spheres P and Q of respective masses $3m$ and $4m$ are moving towards each other, both with speed u , when they collide directly. As a result of the collision the direction of the motion of Q is reversed and its speed is halved.

The spheres are modelled as particles moving on a smooth horizontal plane.

- a) Find the coefficient of restitution between the two spheres.

Consequently, sphere Q collides directly with a third sphere R of mass $6m$ which is initially at rest. The collision between Q and R is perfectly elastic.

Sphere R is also modelled as a particle.

- b) Show that after the collision between Q and R , they will be no more collisions between P and Q .

$$e = \frac{3}{4}$$

(a)

By conservation of momentum
 $3mu - 4mu = 3mX + 2mW$
 $-mu = -mX$
 $X = u$ (ie. REVERSE)

$e = \frac{5eP}{4mP} = \frac{5e}{4}$
 $e = \frac{u + \frac{1}{2}u}{2u} = \frac{\frac{3}{2}u}{2u} = \frac{3}{4}$

(b)

By conservation of momentum
 $2mu + 0 = 4mu + 6mW$
 $2Y + 3W = u$

$2Y + 3W = u$
 $2Y + 3Y + \frac{3}{2}u = u$
 $5Y = -\frac{1}{2}u$
 $Y = \frac{1}{10}u$ (ie. IT REVERSES)

By definition
 $e = 1$ (GLOBALLY ELASTIC)
 $W - Y = 1$
 $-\frac{1}{2}u = \frac{1}{2}u$
 $W = Y + \frac{1}{2}u$

Hence after two collisions
 $\overset{u}{\textcirclearrowleft} \textcirclearrowright \overset{\frac{1}{2}u}{\textcirclearrowleft}$ As $\frac{1}{10}u < u$, no more collisions between P & Q

Question 5 (*)**

A smooth sphere P of mass m is moving with speed u on a smooth horizontal plane when it collides directly with a smooth sphere Q of mass $4m$ which is initially at rest. The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

The magnitude of the impulse of P on Q is $\frac{22}{15}mu$.

- a) Find the value of e .

It is now given that $m = 2 \text{ kg}$ and $u = 30 \text{ ms}^{-1}$.

- b) Show that the kinetic energy loss due to the collision is 220 J .

$$e = \frac{5}{6}$$

(a)

IMPULSE ON Q IS $\frac{22}{15}mu$

$$\frac{22}{15}mu = 4mv - 4mu$$

$$v = \frac{11}{15}u$$

BY CONSERVATION OF MOMENTUM

$$mu + 0 = mu' + 4mv$$

$$u = u' + \frac{22}{15}u$$

$$u' = -\frac{7}{15}u$$

Restitution

$$e = \frac{u' - v}{u - v}$$

$$e = \frac{-\frac{7}{15}u - \frac{11}{15}u}{u - \frac{11}{15}u}$$

$$e = \frac{5}{6}$$

(b)

IF $m=2$, & $u=30$

$K.E_{\text{BEFORE}} = \frac{1}{2} \times 2 \times 30^2 + 0 = 900 \text{ J}$

$K.E_{\text{AFTER}} = \frac{1}{2} \times 2 \times (-14)^2 + \frac{1}{2} \times 2 \times 11^2 = 196 + 121 = 317 \text{ J}$

$\therefore \text{LOSS OF } 900 - 317 = 583 \text{ J}$

Question 6 (*)+**

A smooth sphere A of mass $3m$ is moving with speed $5u$ on a smooth horizontal plane. It collides directly with a smooth sphere B of mass m moving with speed $3u$ in the opposite direction as A . The spheres are modelled as particles and the coefficient of restitution between A and B is e .

- Find, in terms of e and u , the speeds of the two spheres after the collision.
- Show that A cannot possibly reverse direction as a result of the collision.
- Given further that the total kinetic energy of the two spheres after the collision is $\frac{39}{2}mu^2$, show clearly that $e = \frac{1}{4}$

$$V_A = u(3 - 2e), \quad V_B = 3u(1 + 2e)$$

(a)

BY CONSERVATION OF MOMENTUM
 $15mu - 3mu = 3mX + mY$
 $12u = 3X + Y$

BY RESTITUTION
 $e = \frac{X - X_0}{X_0 - Y_0}$
 $e = \frac{Y - X}{3u - X}$
 $-X + Y = 8eu$

SUBTRACT
 $3X + Y = 12u \Rightarrow 4X = 12u - 8eu$
 $-X + Y = 8eu$
 $X = 3u - 2eu$
 $X = u(3 - 2e)$

AND
 $Y = X + 8eu$
 $Y = (3u - 2eu) + 8eu$
 $Y = 3u + 6eu$
 $Y = 3u(1 + 2e)$

(b)

IF $e=0$
 $X = 3u$ (to the right)

IF $e=1$
 $X=u$ (to the right)

WE CANNOT GET A KINETIC ENERGY FOR X IF $e=1$
 \therefore ALWAYS TO THE RIGHT

∴ (GIVEN ENERGY EQUAL)
 $\therefore X > 0$

(c)

$$KE_{\text{TOTAL}} = \frac{39}{2}mu^2$$

$$\Rightarrow \frac{1}{2}(3m)X^2 + \frac{1}{2}(m)Y^2 = \frac{39}{2}mu^2$$

$$\Rightarrow 3X^2 + Y^2 = 39u^2$$

$$\Rightarrow 3[u(3-2e)^2] + [3u(1+2e)^2] = 39u^2$$

$$\Rightarrow 3u^2(3-2e)^2 + 9u^2(1+2e)^2 = 39u^2$$

$$\Rightarrow (3-2e)^2 + 3(1+2e)^2 = 13$$

$$\Rightarrow 9-12e+4e^2 + 12e^2+12e+3 = 13$$

$$\Rightarrow 16e^2 = 1$$

$$\Rightarrow e^2 = \frac{1}{16}$$

$$\Rightarrow e = \frac{1}{4}$$

∴ $e = \frac{1}{4}$ (CORRECT)

Question 7 (*)+**

A smooth sphere P of mass m is moving with speed $4u$ on a smooth horizontal plane. It collides directly with a smooth sphere Q of mass $2m$ moving with speed u in the opposite direction as P . The spheres are modelled as particles and the coefficient of restitution between two spheres is e .

- a) Find, in terms of e and u , the speeds of the two spheres after their collision.

The total kinetic energy of the two spheres after the collision is mu^2 .

- b) Find the value of e and hence show that P is at rest after the collision.

$$V_P = \frac{2}{3}u(1-5e), \quad V_Q = \frac{1}{3}u(5e+2), \quad e = \frac{1}{5}$$

(a)

By conservation of momentum:

$$4mu - 2mu = mv_X + 2mv_Y$$

$$X + 2Y = 2u \quad (1)$$

By restriction:

$$e = \frac{V_Q - V_P}{u}$$

$$e = \frac{-X - Y}{2u}$$

$$-X - Y = 2eu \quad (2)$$

Add equations (1) and (2):

$$3Y = 2u + 2eu$$

$$Y = \frac{1}{3}u(1+5e) \quad (\text{velocity } Q)$$

$$-X - Y = 2eu$$

$$-X = 2u - \frac{1}{3}u(1+5e)$$

$$X = \frac{2}{3}u - \frac{1}{3}u(1+5e) \quad (\text{velocity } P)$$

(b)

Given $V_P = \frac{2}{3}u(1-5e)$

$$kE_{\text{kinetic}} = mu^2$$

$$\rightarrow \frac{1}{2}m(X^2 + \frac{1}{2}(2e)^2)Y^2 = mu^2$$

$$\rightarrow \frac{1}{2}X^2 + Y^2 = u^2$$

$$\rightarrow X^2 + 2Y^2 = 2u^2$$

$$\rightarrow \frac{1}{3}X^2(1-5e)^2 + 2 \times \frac{1}{3}X^2(2+5e)^2 = 2u^2$$

$$\rightarrow \frac{2}{3}(1-5e)^2 + \frac{2}{3}(2+5e)^2 = 2 \quad \checkmark$$

$$\rightarrow 2(1-10e+25e^2) + 4+20e+25e^2 = 9$$

$$\rightarrow 2-20e+50e^2+4+20e+25e^2=9$$

$$\rightarrow 75e^2 = 3$$

$$\rightarrow e^2 = \frac{1}{25}$$

$$\rightarrow e = \pm \frac{1}{5}$$

Final speed in $e = \frac{1}{5}$:

$$X = \frac{2}{3}u(-\frac{1}{5})$$

$$X = \frac{2}{15}u(1-5e)$$

$$X = 0 \quad (\text{velocity } P \text{ is zero})$$

Question 8 (*)+**

A smooth sphere P of mass m is moving with constant speed on a smooth horizontal plane when it collides directly with a smooth sphere Q of mass km , where k is a positive constant, which is initially at rest. After the collision both spheres are moving in the same direction as the original direction of P with speeds u and $4u$.

The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

- Show clearly that $4k = \frac{3}{e} - 1$.
- Hence, find the range of the possible values of k .
- Given further that $k = 2$...
 - calculate the value of e
 - find the kinetic energy loss of the system, due to the collision.

$$k \geq \frac{1}{2}, \quad e = \frac{1}{3}, \quad \text{loss} = 24mu^2$$

(a) 

- BY CONSERVATION OF MOMENTUM $mX + 0 = mu + kmu$
- $X = u + 4ku$
- $X = u(1+4k)$
- $u(1+4k) = \frac{3u}{e}$
- $1+4k = \frac{3}{e}$
- $4k = \frac{3}{e} - 1$ // REASON: $4k > 0$

- BY RESTITUTION $e = \frac{v_f - v_i}{u_f - u_i}$
- $e = \frac{u - 0}{4u - u}$
- $X = \frac{3u}{e}$

(b) $0 < e \leq 1$ CONTACT IS ZERO BECAUSE PARTICLES DID NOT COLLIDE

$$\begin{aligned} &\Rightarrow 1 \leq \frac{1}{e} \leq \infty \\ &\Rightarrow \frac{1}{e} \geq 1 \\ &\Rightarrow \frac{3}{e} \geq 3 \\ &\Rightarrow \frac{3}{e} - 1 \geq 2 \end{aligned}$$

$\left\{ \begin{array}{l} \Rightarrow 4k \geq 2 \\ \Rightarrow k \geq \frac{1}{2} \end{array} \right.$

(c) IF $k=2$ (i) $X = u(1+4k) = 9u$

$$\begin{aligned} 4k &= \frac{3}{e} - 1 \\ 8 &= \frac{3}{e} - 1 \\ 9 &= \frac{3}{e} \\ e &= \frac{1}{3} \end{aligned}$$

$\left\{ \begin{array}{l} \bullet KE_{\text{Initial}} = \frac{1}{2}mX^2 = \frac{1}{2}m(9u)^2 = \frac{81}{2}mu^2 \\ \bullet KE_{\text{Final}} = \frac{1}{2}m\bar{u}^2 + \frac{1}{2}(2mu)(4mu) = \frac{13}{2}mu^2 \\ \therefore \text{A LOSS OF } \frac{18}{2}mu^2 - \frac{13}{2}mu^2 = 25mu^2 \end{array} \right.$

Question 9 (*)+**

A smooth sphere A of mass m is moving with constant speed on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $3m$, which is initially at rest. After the collision A reverses direction and is speed is v .

The two spheres are modelled as particles and the coefficient of restitution between the two spheres is $\frac{3}{4}$.

- a) Determine, in terms of v , the speed of A before the collision and the speed of B after the collision.

Consequently, B collides directly with a third sphere C of mass km , where k is a positive constant. It is further given that C was initially at rest and the coefficient of restitution between B and C is $\frac{6}{7}$.

- b) By modelling C as a smooth particle, find the value of k , given that B is brought to rest after the collision.

$$V_A = \frac{16}{5}v, V_B = \frac{7}{5}v, k = \frac{7}{2}$$

(a)

- By Conservation of Momentum: $mV_A + 0 = -mV_A + 3mV_B$
- $V_A = -V_A + 3V_B$
- $2V_A = 3V_B$
- $V_B = \frac{2}{3}V_A$
- By Restitution: $e = \frac{V_B - V_A}{V_A + V_B}$
- $\frac{3}{4} = \frac{V_B - V_A}{V_A + V_B}$
- $\frac{3}{4} = \frac{\frac{2}{3}V_A - V_A}{V_A + \frac{2}{3}V_A}$
- $\frac{3}{4} = \frac{-\frac{1}{3}V_A}{\frac{5}{3}V_A}$
- $\frac{3}{4} = -\frac{1}{5}$
- $V_A = -\frac{15}{4}v$
- $V_B = \frac{10}{4}v = \frac{5}{2}v$
- SPEED OF B AFTER: $V_B = \frac{5}{2}v$

(b)

- By Conservation of Momentum: $\frac{2}{3}mV_B + 0 = 0 + kmV_C$
- $\frac{2}{3}mV_B = kmV_C$
- $V_C = \frac{2}{3k}V_B$
- By Restitution: $e = \frac{V_C - 0}{V_B + V_C}$
- $\frac{6}{7} = \frac{V_C}{V_B + V_C}$
- $\frac{6}{7} = \frac{\frac{2}{3k}V_B}{V_B + \frac{2}{3k}V_B}$
- $\frac{6}{7} = \frac{\frac{2}{3k}V_B}{\frac{5}{3}V_B}$
- $\frac{6}{7} = \frac{2}{5k}$
- $k = \frac{7}{2}$

Question 10 (*)+**

A smooth sphere A of mass $5m$ is moving with speed $4u$ on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $3m$ moving with speed u , in the same direction as A. The two spheres are modelled as particles and the coefficient of restitution between them is e .

After the collision the speed of B is $\frac{21}{2}eu$.

a) Show clearly that $e = \frac{1}{3}$.

b) Find in terms of m and u the kinetic energy lost, as a result of the collision.

$$\text{kinetic energy lost} = \frac{15}{2}mu^2$$

(a)

- BY CONSERVATION OF MOMENTUM
$$25mu + 3mu = 5uX + \frac{21}{2}eu$$

$$28mu = 5uX + \frac{21}{2}eu$$

(b) BY RESTITUTION

$$e = \frac{21}{2}eu - u}{28mu}$$

$$e = \frac{\frac{15}{2}eu}{28mu}$$

$$e = \frac{\frac{15}{2}eu}{14mu}$$

$$e = \frac{15}{28}$$

(b)

- $K.E_{\text{before}} = \frac{1}{2}(5m)(4u)^2 + \frac{1}{2}(3m)u^2 = 40mu^2 + \frac{9}{2}mu^2 = \frac{83}{2}mu^2$
- $K.E_{\text{after}}$
- (A): $X = \frac{15}{2}eu = \frac{15}{2} \times \frac{1}{3}u = \frac{5}{2}u$
- (B): $\frac{21}{2}eu = \frac{21}{2} \times \frac{1}{3}u = \frac{7}{2}u$
- $K.E_{\text{after}} = \frac{1}{2}(5m)\left(\frac{5}{2}u\right)^2 + \frac{1}{2}(3m)\left(\frac{7}{2}u\right)^2 = \frac{125}{4}mu^2 + \frac{147}{4}mu^2 = 34.25mu^2$
- $\therefore \Delta K.E_{\text{lost}} = \frac{83}{2}mu^2 - 34.25mu^2 = \frac{15}{2}mu^2$

Question 11 (*)+**

A smooth sphere A of mass 2 kg is moving with speed 12 ms^{-1} on a smooth horizontal plane when it collides directly with a smooth sphere B of mass m kg, which is initially at rest. After the collision A reverses direction and is speed is v .

The two spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

- a) Given that the speed of B after the collision is $2v$, show clearly that

$$m = \frac{v+12}{v}.$$

After the collision, $\frac{11}{16}$ of the initial kinetic energy is conserved.

- b) Calculate ...

- i. ... the value of v .
- ii. ... the value of e .

$$v = 3, e = 0.75$$

<p>4)</p> <p>By conservation of momentum</p> $\Rightarrow 2 \times 12 + 0 = -v + 2mv$ $\Rightarrow 24 = -v + 2mv$ $\Rightarrow 12 = mv - v$ $\Rightarrow 12 = mv$ $\Rightarrow m = \frac{12}{v}$	<p>(b) KE before = $\frac{1}{2} \times 2 \times 12^2 = 144$ KE after = $\frac{1}{2} \times 2 \times v^2 + \frac{1}{2} \times m(2v)^2$ $= v^2 + 2mv^2$ $= v^2 + 2(\frac{12}{v})v^2$ $= v^2 + 24v^2$</p> <p>NOW $\frac{v^2 + 24v^2}{144} = \frac{11}{16}$</p> $\Rightarrow v^2 + 24v^2 = 99$ $\Rightarrow v^2 + 24v^2 + 24v = 99$ $\Rightarrow 25v^2 + 24v = 99$ $\Rightarrow v^2 + 24v - 33 = 0$ $\Rightarrow (v-3)(v+11) = 0$ $\Rightarrow v = 3$ <p>5) $e = \frac{v_B}{v_A} = \frac{2v}{12} = \frac{2v}{12} = \frac{1}{6} = 0.75$</p>
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Question 12 (***)

A particle A , of mass 0.2 kg, is travelling in a straight line on a smooth horizontal surface, when it collides with a particle B , of mass 1.5 kg, which is moving on the same surface and in the same direction as A .

The respective speeds of A and B just before the collision are 15 ms^{-1} and 4 ms^{-1} .

The coefficient of restitution between the two particles, e , is such so that the two particles move in the same direction after the collision.

Show that $e < \frac{6}{11}$.

[] , [proof]

Diagram illustrating the collision between particles A and B. Particle A (mass 0.2 kg) moves with an initial velocity of 15 ms^{-1} and a final velocity of $X \text{ ms}^{-1}$. Particle B (mass 1.5 kg) moves with an initial velocity of 4 ms^{-1} and a final velocity of $Y \text{ ms}^{-1}$. The direction of motion is indicated by arrows above the particles.

By Conservation of Momentum:

$$(0.2 \times 15) + (1.5 \times 4) = 0.2X + 1.5Y$$

$$(0.2 \times 15) + (1.5 \times 4) = 0.2X + 1.5Y$$

$$2X + 1.5Y = 90$$

By Considering Restitution:

$$\frac{Y - X}{15 - 4} = e$$

$$Y - X = 11e$$

$$Y = X + 11e$$

It is given that both particles continue in the original direction of motion, of which "B" has to be "A" way from B onwards.

Hence we need an expression for X , then set it positive

$$\Rightarrow 2X + 1.5(X + 11e) = 90$$

$$\Rightarrow 2X + 1.5X + 16.5e = 90$$

$$\Rightarrow 3.5X = 90 - 16.5e$$

$$\Rightarrow X = \frac{15}{7}(6 - 11e)$$

But $X > 0$,

$$6 - 11e > 0$$

$$-11e > -6$$

$$e < \frac{6}{11}$$

Question 13 (**)**

Three small smooth spheres A , B and C , are resting on a straight line, and in that order, on a horizontal surface.

The respective masses of A , B and C , are m , $3m$ and $7m$.

A is projected towards B with speed u and a direct collision takes place.

The coefficient of restitution between A and B is 0.5.

The coefficient of restitution between B and C is e .

If there is a second collision between A and B , find the range of possible values of e .

$$\boxed{\quad}, \frac{19}{21} < e \leq 1$$

<p>LOOKING AT THE COLLISION BETWEEN A & B</p> <p>By CONSERVATION OF MOMENTUM By CONSIDERING RESTITUTION</p> $\begin{aligned} \Rightarrow mu + 0 &= mv + 3mw \\ \Rightarrow u &= v + 3w \\ \Rightarrow u &= X + 3Y \end{aligned}$ $\begin{aligned} \Rightarrow & \frac{v}{u} = \frac{X}{u} - 3 \\ \Rightarrow & v = \frac{X}{u}u - 3u \\ \Rightarrow & v = \frac{1}{2}u \end{aligned}$ <p><u>ADDITION EQU:</u></p> $\begin{aligned} 4Y &= \frac{3}{2}u \\ Y &= \frac{3}{8}u \end{aligned}$ <p><u>AND ALSO:</u></p> $\begin{aligned} X &= u - 3Y \\ X &= u - 3\left(\frac{3}{8}u\right) \\ X &= \frac{5}{8}u \\ X &= \frac{5}{16}u \end{aligned}$ <p><u>ie A HAS RESONANCE (which) WITH SPEED $\frac{5}{16}u$</u></p>	<p>NEXT THE COLLISION BETWEEN B & C</p> <p>By CONSERVATION OF MOMENTUM By CONSIDERING RESTITUTION</p> $\begin{aligned} \Rightarrow 3v\left(\frac{5}{16}u\right) + 0 &= 3vw + 7w \\ \Rightarrow \frac{15}{16}vu &= 3w + 7w \\ \Rightarrow \frac{15}{16}u &= 10w \\ \Rightarrow w &= \frac{3}{16}u \end{aligned}$ $\begin{aligned} \Rightarrow -v + w &= \frac{3}{8}u \\ \Rightarrow -v + 7w &= \frac{3}{8}u \\ \Rightarrow -v + 7w - 7w &= -\frac{3}{8}u \\ \Rightarrow -v &= -\frac{3}{8}u \end{aligned}$ <p><u>ADDING THE EQUATIONS ABOVE (WE ONLY NEED V)</u></p> $\begin{aligned} \Rightarrow 10v &= \frac{3}{8}u - \frac{3}{8}u \\ \Rightarrow 10v &= \frac{3}{8}u(3 - 7e) \\ \Rightarrow v &= \frac{3}{80}u(3 - 7e) \quad \leftarrow \text{TO THE "RIGHT"} \\ \Rightarrow v &= \frac{3}{80}u(7e - 3) \quad \leftarrow \text{TO THE "LEFT"} \end{aligned}$ <p>For A Collision Between B & A</p> $\begin{aligned} \Rightarrow \frac{3}{8}u(7e - 3) &> \frac{3}{8}u \\ 7e - 3 &> \frac{10}{3} \\ 7e &> \frac{19}{3} \\ \Rightarrow e &> \frac{19}{21} \end{aligned}$ <p>$\therefore \frac{19}{21} < e \leq 1$</p>
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Question 14 (**)**

A smooth sphere P of mass $2m$ is moving with speed u on a smooth horizontal plane when it collides directly with a smooth sphere Q of mass $3m$ which is initially at rest. The spheres are modelled as particles and the coefficient of restitution between the two spheres is $\frac{1}{4}$.

- a) Find, in terms of u , the speeds of P and Q after the collision.

Consequently, sphere Q strikes at right angles a fixed smooth vertical wall, and rebounds at right angles. The coefficient of restitution between Q and the wall is e .

After a second collision between the spheres, P is brought to rest.

- b) Show clearly that $e = \frac{1}{6}$.

$$V_P = \frac{1}{4}u, \quad V_Q = \frac{1}{2}u$$

The handwritten solution is organized into two parts, (a) and (b), each with a diagram and accompanying equations.

Part (a):

- Diagram:** Shows two spheres, P and Q, on a horizontal surface. Sphere P has initial velocity u to the left, and sphere Q is at rest. After the first collision, sphere P moves to the right with velocity X , and sphere Q moves to the right with velocity Y . A vertical arrow labeled "POSITIVE" points to the right.
- Equations:**
 - Conservation of momentum: $2mu = 2mX + 3mY$
 - By substitution: $\frac{1}{4} = \frac{Y-X}{2u}$
 - Rearranging: $-X+Y = \frac{1}{4}u$
 - Another equation from the diagram: $2X+3Y = 2u$
 - Solving the system of equations:

$$\begin{aligned} 2X+3Y &= 2u \\ -X+Y &= \frac{1}{4}u \\ \hline 3Y &= \frac{9}{4}u \\ Y &= \frac{3}{4}u \end{aligned}$$
 - And:

$$\begin{aligned} 2X &= 2u - 3Y \\ 2X &= 2u - \frac{9}{4}u \\ \frac{1}{2}u &= 2X \\ X &= \frac{1}{4}u \end{aligned}$$
- Part (b):**
- Diagram:** Shows sphere Q moving to the right with velocity Y towards a vertical wall. After the collision, sphere Q moves to the left with velocity W . A vertical arrow labeled "POSITIVE" points to the right.
- Equations:**
 - By conservation of momentum: $\frac{1}{2}u = -3mW$
 - By substitution: $\frac{1}{4} = \frac{W-Y}{\frac{1}{2}u}$
 - Rearranging: $W = \frac{1}{6}u + \frac{1}{2}u$
 - Another equation from the diagram: $W = \frac{1}{6}u + \frac{1}{2}u$
 - Solving for e :

$$\begin{aligned} W &= \frac{1}{6}u + \frac{1}{2}u \\ 1-3e &= \frac{1}{6}u + \frac{1}{2}u \\ 1-3e &= 3+e \\ 8-24e &= 3+6e \\ 5 &= 30e \\ e &= \frac{1}{6} \end{aligned}$$

Question 15 (**)**

Two smooth spheres A and B , of respective masses $4m$ and m , are moving in a straight line and in the same direction towards a smooth vertical wall. The speeds of A and B are u and $5u$ respectively. Sphere B hits the wall at right angles and rebounds so that it subsequently collides with A . Immediately after this collision the speed of A is U and the speed of B is V , with the direction of motion of each sphere reversed.

The spheres are modelled as particles and the coefficient of restitution between B and the wall is 0.8.

- a) Show clearly that $V = 4U$.

The total kinetic energy of the two spheres immediately after their collision is $\frac{1}{4}$ of the total kinetic energy of the two spheres immediately before their collision.

- b) Calculate the coefficient of restitution between the two spheres.

$$e = \frac{1}{2}$$

(a)

FIND SPEED OF B AFTER REBOUNDING ON THE WALL IS $\frac{1}{2} \times 5u = 2.5u$

BY CONSERVATION OF MOMENTUM
 $4mu - 4mu = -4mU + mV$
 $0 = V - 4U$
 $V = 4U$
 AS REQUIRED

(b)

K.E. BEFORE = $\frac{1}{2}(4m)u^2 + \frac{1}{2}(m)(5u)^2 = 2mu^2 + 25mu^2 = 10mu^2$
 K.E. AFTER = $\frac{1}{2}(4m)U^2 + \frac{1}{2}mV^2 = 2mU^2 + \frac{1}{2}m(4U)^2 = 10mU^2$

NOW $\text{KE AFTER} = \frac{1}{4} \times \text{KE BEFORE}$
 $10mU^2 = \frac{1}{4} \times 10mu^2$
 $U^2 = \frac{1}{4}u^2$
 $U = \frac{1}{2}u$

SINCE $V = 4U$, $V = 2u$

THUS $e = \frac{5U}{4U} = \frac{U+V}{u+4u} = \frac{\frac{1}{2}u+2u}{5u} = \frac{\frac{5}{2}u}{5u} = \frac{1}{2}$

$\therefore e = \frac{1}{2}$

Question 16 (**)**

Two smooth spheres A and B , of respective masses $2m$ and m , are moving with constant speeds on a smooth horizontal plane, when they collide directly.

The respective speeds of A and B after the collision are $2v$ and $11v$. Before the collision the spheres were moving in opposite directions and after the collision both spheres are moving in the original direction of motion of A .

The two spheres are modelled as particles and the coefficient of restitution between them is e .

- a) Find, in terms of e and v , the speeds of the two spheres before their collision.

The total kinetic energy lost as a result of the collision is $21mv^2$.

- b) Find the value of e .

$$U_A = v \left(5 + \frac{3}{e} \right), \quad U_B = v \left(\frac{6}{e} - 5 \right), \quad e = \frac{3}{4}$$

(a) By (conservation) of momentum
 $2mX - mY = 2mv + 11mv$
 $2X - Y = 15v \quad |$

By RESTITUTION
 $e = \frac{v_f - v_i}{v_f + v_i}$
 $e = \frac{9v}{X+Y}$
 $X+Y = \frac{9v}{e}$

\downarrow
 $2X - Y = 15v \quad | \text{ Add}$
 $X + Y = \frac{9v}{e}$
 $3X = 15v + \frac{9v}{e}$
 $X = 5v + \frac{3v}{e}$
 $X = v(5 + \frac{3}{e}) \quad \checkmark \text{ SPEED OF A}$

Also by elimination
 $2X - Y = 15v \quad | \text{ SUBTRACT}$
 $2X + 2Y = \frac{18v}{e} \quad |$
 $3Y = \frac{18v}{e} - 15v$
 $Y = \frac{9v}{e} - 5v$
 $Y = v(\frac{9}{e} - 5) \quad \checkmark \text{ SPEED OF B}$

(b) KE_{BEFORE} = $\frac{1}{2}(2m)X^2 + \frac{1}{2}(m)Y^2 = m[v(5 + \frac{3}{e})]^2 + \frac{1}{2}m[v(\frac{9}{e} - 5)]^2$
= $mV^2(5 + \frac{3}{e})^2 + \frac{1}{2}mV^2(\frac{9}{e} - 5)^2$
= $mV^2(25 + \frac{30}{e} + \frac{9}{e^2}) + \frac{1}{2}mV^2(\frac{81}{e^2} - \frac{90}{e} + 25)$
= $mV^2\left[\frac{13}{e^2} + \frac{22}{e}\right]$
KE_{AFTER} = $\frac{1}{2}(2m)(2v)^2 + \frac{1}{2}m(11v)^2 = 4mv^2 + \frac{121}{2}mv^2 = \frac{135}{2}mv^2$

KINETIC ENERGY LOSS = $21mv^2$
 $\Rightarrow mv^2\left[\frac{13}{e^2} + \frac{22}{e}\right] - \frac{135}{2}mv^2 = 21mv^2$
 $\Rightarrow \frac{13}{e^2} + \frac{22}{e} - \frac{135}{2} = 21$
 $\Rightarrow \frac{21}{e^2} = 48 \quad |$

Question 17 (**)**

A smooth sphere A of mass m is moving with speed $4u$ on a smooth horizontal plane. It collides directly with a smooth sphere B of mass $6m$ moving with speed u in the same direction as A . As a result of the impact the direction of motion of A is reversed. The spheres are modelled as particles.

The coefficient of restitution between the two spheres is e .

- Show that after the collision the speed of B is $\frac{1}{7}u(10+3e)$, and find a similar expression for the speed of A .
- Deduce that $e > \frac{5}{9}$.

After the collision, B strikes at right angles a fixed smooth vertical wall, and rebounds also at right angles to the wall. The coefficient of restitution between B and the wall is $\frac{1}{2}$.

- Given that A and B collide again show further that $e < \frac{10}{11}$.

$$\boxed{\frac{2}{7}u(9e-5)}$$

(a)

By CONSERVATION OF MOMENTUM

$$4mu + 6mu = -mu + 6mu$$

$$10mu = -mu + 6mu$$

$$10u = -u + 6u$$

By RESTRICTION
 $e = \frac{\text{SEP}}{\text{APP}}$
 $e = \frac{X+Y}{3u}$

$X+Y = 3eu$

ADD

$$TY = 10u + 3eu$$

$$TY = u(10+3e)$$

$$Y = \frac{1}{7}u(10+3e)$$
 // To be solved

USING $-X+Y = 10u$ \Rightarrow $TX = 18eu - 10u$
 $6X+6Y = 18eu$
 $6X = 2u(9e-5)$
 $X = \frac{2}{7}u(9e-5)$ ✓

(b) $X > 0$ (i.e. it goes to the left)
 $\frac{2}{7}u(9e-5) > 0$
 $9e-5 > 0$
 $e > \frac{5}{9}$ ✓

(c)

Thus, the configuration is

For another collision $\frac{1}{2}Y > X$
 $\frac{1}{2} \times \frac{2}{7}u(10+3e) > \frac{2}{7}u(9e-5)$
 $10+3e > 4(9e-5)$
 $10+3e > 36e-20$
 $-33e > -30$
 $e < \frac{10}{11}$ ✓

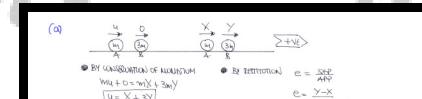
Question 18 (**)**

A smooth sphere A of mass m is moving with speed u on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $3m$ which is at rest.

The two spheres are modelled as particles and the coefficient of restitution between them is e .

- Find, in terms of e and u , the speeds of the two spheres after their collision.
- Given that the direction of A is unchanged after the collision find the range of the possible values of e .
- Given instead that $e = \frac{1}{2}$, show that $\frac{9}{16}$ of the kinetic energy is lost as a result of the collision.

$$V_A = \frac{1}{4}u(1-3e), \quad V_B = \frac{1}{4}u(1+e), \quad 0 \leq e < \frac{1}{3}$$

(a) 

- BY CONSERVATION OF MOMENTUM $mV_A + 0 = mV_A' + 3mV_B'$
 $V_A' = X - 3Y$
- BY ENERGY $e = \frac{V_B - V_A'}{V_A}$
 $e = \frac{Y - X}{X}$
 $e = \frac{Y - X}{X} = \frac{Y}{X} - 1$
 $e = \frac{Y}{X} - 1 = \frac{X - 3Y}{X} = 1 - 3\frac{Y}{X}$
 $e = 1 - 3\frac{Y}{X}$

And $X = u - 3Y$
 $X = u - \frac{3}{2}e(u) = u - \frac{3}{2}u = \frac{3}{2}eu = \frac{3}{2}eu$
 $X = \frac{3}{2}eu$

(b) $X > 0 \Rightarrow \frac{3}{2}eu > 0$
 $1 - 3\frac{Y}{X} > 0$
 $-3\frac{Y}{X} > -1$
 $\frac{Y}{X} < \frac{1}{3}$
 $\therefore 0 \leq e < \frac{1}{3}$

(c) If $e = \frac{1}{2}$
 $X = -\frac{1}{2}u \quad Y = \frac{3}{2}u$

$K.E_{\text{initial}} = \frac{1}{2}mu^2$
 $K.E_{\text{final}} = \frac{1}{2}m(-\frac{1}{2}u)^2 + \frac{1}{2}(3m)(\frac{3}{2}u)^2 = \frac{1}{2}m(-\frac{1}{4}u)^2 + \frac{1}{2}(3m)(\frac{9}{4}u)^2$
 $= \frac{1}{16}mu^2 + \frac{27}{8}mu^2 = \frac{27}{16}mu^2$
 $\therefore \frac{\frac{27}{16}mu^2}{\frac{1}{2}mu^2} \text{ is the amount that remains} \quad \text{ie } \frac{27}{16} \text{ remains}$
 $\therefore 1 - \frac{27}{16} = \frac{9}{16} \text{ is lost}$

Question 19 (**)**

A smooth sphere P of mass $4m$ is moving with speed $2u$ on a smooth horizontal plane. It collides directly with a smooth sphere Q of mass m moving with speed $5u$ in the opposite direction as P . The spheres are modelled as particles and the coefficient of restitution between P and Q is e .

- a) Show that the speed of Q after the collision is $\frac{1}{5}u(3+28e)$.

As a result of the impact the direction of motion of P is reversed after the collision.

- b) Find the range of the possible values of e .

The magnitude of the impulse of Q on P is $10mu$.

- c) Determine the value of e .

$$\boxed{\frac{3}{7} < e \leq 1}, \quad \boxed{e = \frac{11}{14}}$$

(a) Diagrams show sphere P (mass $4m$, speed $2u$) and sphere Q (mass m , speed $5u$) before impact. After impact, P moves with speed $-X$ and Q moves with speed Y . A note indicates P moves to the right after impact.

- By Conservation of Momentum: $4mu - mu = 4mu + muY$
- By RESTITUTION: $e = \frac{Y-X}{2u}$

$$3u = 4X + Y$$

$$3u = 5u - 7eu$$

$$3u - 7eu = 5u$$

$$X = \frac{1}{5}u(3-7e)$$

$$Y = \frac{1}{5}u(3+28e)$$

(b) $X < 0$

$$\frac{1}{5}u(3-7e) < 0$$

$$3-7e < 0$$

$$-7e < -3$$

$$e > \frac{3}{7}$$

$$\therefore \frac{3}{7} < e \leq 1$$

(c) IMPULSE ON P = $-10mu$
 IMPULSE ON Q = $+10mu$
 $10mu = mu(Y - (-5u))$
 $10mu = muY + 5mu$
 $5mu = muY$
 $5u = 3+28e$
 $25 = 3+28e$
 $22 = 28e$
 $\therefore e = \frac{11}{14}$

Question 20 (**)**

A smooth sphere A of mass m is moving with speed $2u$ on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $4m$ which is at rest. As a result of the collision the direction of motion of A is reversed. The two spheres are modelled as particles and the coefficient of restitution between them is e .

- a) Find, in terms of e and u , the speeds of the two spheres after their collision.

After the collision, B strikes at right angles a fixed smooth vertical wall, and rebounds at right angles. The coefficient of restitution between B and the wall is $\frac{5}{6}$.

- b) Given that A and B collide again show that

$$\frac{1}{4} < e < \frac{11}{19}.$$

$$V_A = \frac{2}{5}u(4e-1), \quad V_B = \frac{2}{5}u(1+e)$$

(a) By conservation of momentum:

$$2mu + 0 = -mv_A + 4mv_B$$

$$-X + 4Y = 2u$$

$$-X + 4Y = 2u \quad \text{--- (1)}$$

$$X + Y = 2eu \quad \text{--- (2)}$$

$$5Y = 2u(1+e)$$

$$Y = \frac{2}{5}u(1+e)$$

$$\text{Also } -X + 4Y = 2u \quad ? \quad \text{SURFACE UNDAMAGED}$$

$$4X + 4Y = 8u \quad ? \quad \text{SURFACE UNDAMAGED}$$

$$5X = 8eu - 2u$$

$$X = \frac{2}{5}u(4e-1) \quad \text{Speed of A}$$

By restitution:

$$e = \frac{v_B - v_A}{u}$$

$$e = \frac{X + Y}{2u}$$

$$X + Y = 2eu$$

$$\frac{-X + 4Y}{2u} = \frac{X + Y}{2u}$$

$$-X + 4Y = X + Y$$

$$3Y = 2u$$

$$Y = \frac{2}{3}u$$

(b) After colliding with the wall, the speeds of the two spheres:

$$X \leftarrow \frac{2}{5}u(4e-1) \quad \text{Speed of A}$$

$$5Y \leftarrow \frac{2}{3}u(1+e) \quad \text{Speed of B}$$

$$\text{For another collision: } \frac{2}{5}u(4e-1) > \frac{2}{3}u(1+e)$$

$$5 + 5e > 4e - 6$$

$$-9e > -11$$

$$e < \frac{11}{9}$$

NOTE THAT A REVERSED DIRECTION, SO $X > 0 \Rightarrow \frac{2}{5}u(4e-1) > 0 \Rightarrow 4e-1 > 0 \Rightarrow e > \frac{1}{4}$

$$\therefore \frac{1}{4} < e < \frac{11}{9}$$

Question 21 (**)**

A smooth sphere A of mass $3m$ is moving with speed $3u$ on a smooth horizontal plane. It collides directly with a smooth sphere B of mass $4m$ moving with speed u in the same direction as A . The spheres are modelled as particles and the coefficient of restitution between A and B is $\frac{1}{2}$.

- a) Find, in terms of u , the speeds of the two spheres after their collision.

After the collision between A and B , B collides directly with a third sphere C of mass $2m$ which was at rest on the same smooth horizontal plane as A and B .

The sphere C is also modelled as a particle and the coefficient of restitution between B and C is e .

- b) Given that there are no more collisions between the three spheres find the range of possible values of e .

$$V_A = \frac{9}{7}u, V_B = \frac{16}{7}u, 0 \leq e \leq \frac{5}{16}$$

(a)

Diagram showing spheres A, B, and C on a horizontal surface. Sphere A has initial velocity $3u$ to the right, sphere B has initial velocity u to the right, and sphere C is at rest. After collision, sphere A has velocity V_A , sphere B has velocity V_B , and sphere C has velocity V_C .

By conservation of momentum:

$$3mu + 4mu = 3mV_A + 4mV_B$$

$$3u = 3V_A + 4V_B \quad (1)$$

By restitution:

$$e = \frac{V_B - V_A}{3u - 4V_B}$$

$$\frac{1}{2} = \frac{V_B - V_A}{3u - 4V_B}$$

$$-V_B + V_A = u \quad (2)$$

Solving (1) and (2) simultaneously:

$$3u = 3V_A + 4V_B$$

$$-V_B + V_A = u$$

$$4V_B = 2u$$

$$V_B = \frac{u}{2}$$

Speed of A:

$$V_A = \frac{9}{7}u$$

Speed of B:

$$V_B = \frac{u}{2}$$

(b)

Diagram showing spheres B and C on a horizontal surface. Sphere B has velocity V_B to the right, sphere C is at rest. After collision, sphere B has velocity V_B' and sphere C has velocity V_C .

By conservation of momentum:

$$4mu = 4mV_B + 2mV_C$$

$$2u = 2V_B + V_C \quad (3)$$

By restitution:

$$e = \frac{V_C - V_B'}{V_B - 2u}$$

$$e = \frac{V_C - V_B'}{u}$$

$$-V_B' + V_C = eu \quad (4)$$

Solving (3) and (4) simultaneously:

$$2u = 2V_B + V_C$$

$$-V_B' + V_C = eu$$

$$2V_B + V_C = 2u$$

$$-V_B' + V_C = eu$$

$$2V_B - V_B' = 2u - eu$$

$$V_B(2-e) = 2u - eu$$

$$V_B = \frac{2u}{2-e}$$

$$V_C = \frac{eu}{2-e}$$

$$V_B > V_C$$

$$\frac{2u}{2-e} > \frac{eu}{2-e}$$

$$2 > e$$

$$2 - e > \frac{eu}{2-e}$$

$$2 - e > \frac{e}{2}$$

$$2 - e > \frac{5}{16}$$

$$e < \frac{11}{16}$$

$$0 \leq e \leq \frac{5}{16}$$

No more collisions $\Rightarrow V_C > V_B$

Question 22 (**)**

Three smooth spheres P , Q and R have respective masses $2m$, m and km , where k is a positive constant. The spheres lie at rest in a straight line, in that order, on a smooth horizontal plane. The spheres are modelled as particles. The coefficient of restitution between any pair of spheres is $\frac{1}{2}$.

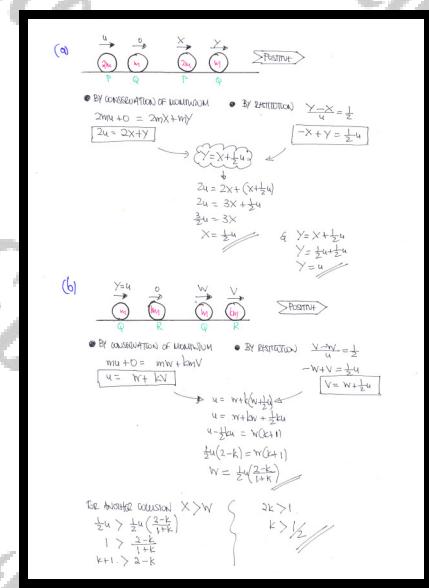
P is then projected towards Q with speed u so that the spheres collide directly.

- a) Find, in terms of u , the speed of P and the speed of Q after the collision.

Consequently there is a collision between Q and R .

- b) Given that there is a third collision between P and Q , show that $k > \frac{1}{2}$.

$$V_P = \frac{1}{2}u, \quad V_Q = u$$



Question 23 (**)**

A smooth sphere P of mass m is moving with speed u on a smooth horizontal plane. It collides directly with a smooth sphere Q of mass $4m$ which is initially at rest. The spheres are modelled as particles and the coefficient of restitution between P and Q is e , where $e > \frac{1}{4}$.

- a) Show that the **speed** of P after the collision is $\frac{1}{5}u(4e-1)$ and find a similar expression for the speed of Q .

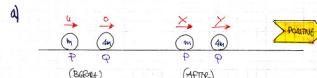
Three smooth spheres A , B and C lie in a straight line in that order on the same smooth horizontal plane. The masses of A and C are $4m$ each, while the mass of B is m . The three spheres are modelled as particles and the coefficient of restitution between any of these spheres is 0.75.

The spheres are initially at rest when B is projected towards C with speed u .

- b) Show that after B and C collide, there will be another collision between A and B , and no more collisions between the spheres thereafter.

$$\boxed{\quad}, \quad v_Q = \frac{1}{5}u(1+e)$$

a)



BY CONSERVATION OF MOMENTUM

$$\Rightarrow mu + 0 = mX + 4mY$$

$$\Rightarrow X + 4Y = u$$

BY RESTITUTION

$$e = \frac{v_P - u}{u - v_Q}$$

$$e = \frac{-X - u}{u - Y}$$

$$-X - Y = eu$$

ADDING THE EQUATIONS

$$5Y = (1+e)u$$

$$Y = u(1+e)$$

$$Y = \frac{1}{5}u(4e+1)$$

∴ $v_Q = Y$

$$v_Q = \frac{1}{5}u(4e+1)$$

REASONING

As $e > \frac{1}{4}$, $4e+1 > 2$, so $\frac{1}{5}(4e+1) > \frac{1}{5}2 = \frac{2}{5}$. Since $v_Q > u$, B will collide with C again.

b)



USING MKS (e) WITH $e = 0.75$

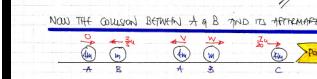
$$X = \frac{1}{5}u(1-0.75) = -\frac{1}{10}u$$

(BECAUSE $X = -V_Q$)

$$Y = \frac{1}{5}u(1+0.75) = \frac{7}{10}u$$

∴ AS B REACHES MAXIMUM DISTANCE BETWEEN A & B

NOW THE COLLISION BETWEEN A & B AND ITS AFTERMATH



BY CONSERVATION OF MOMENTUM

$$0 = \frac{1}{5}mu - \frac{1}{10}uV + mW$$

$$W = \frac{1}{10}u - \frac{1}{5}uV$$

BY RESTITUTION

$$e = \frac{v_A - 0}{0 - v_B}$$

$$0.75 = \frac{v_A - 0}{0 - v_B}$$

$$V + W = \frac{3}{10}u$$

$$4V + 4W = \frac{6}{5}u$$

ADDING THE EQUATIONS

$$5V = \frac{6}{5}u$$

$$V = \frac{6}{25}u$$

(TO THE RIGHT)

AS $\frac{6}{25}u < \frac{7}{10}u$ NO MORE COLLISIONS BETWEEN B & C

Question 24 (**)**

A smooth sphere P of mass m is moving with speed u on a smooth horizontal plane when it collides directly with a smooth sphere Q of mass $3m$, which is initially at rest. The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

- a) Find, in terms of e and u , the speeds of the two spheres after their collision.

It is now given that P reverses direction as a result of the collision.

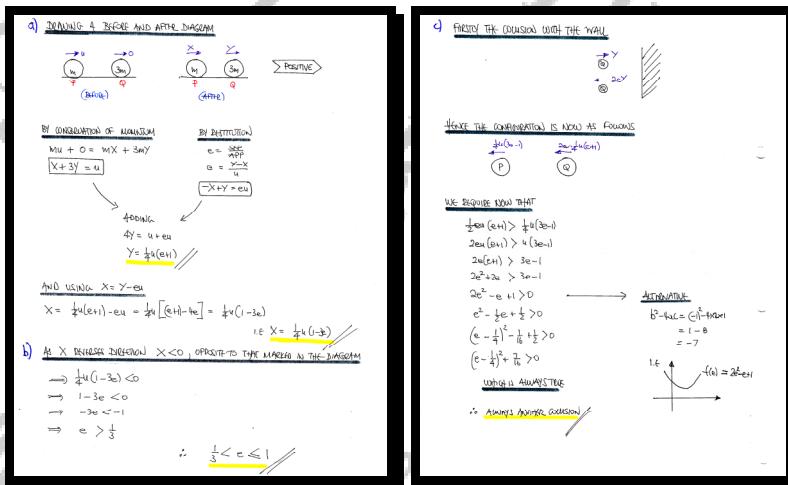
- b) State the range of the possible values of e .

After the collision, Q strikes at right angles a fixed smooth vertical wall, and rebounds at right angles.

The coefficient of restitution between the Q and the wall is $2e$.

- c) Show that there will always be another collision between P and Q .

$$\text{EQUATION}, \quad V_P = \frac{1}{4}u(1-3e), \quad V_Q = \frac{1}{4}u(1+e), \quad \frac{1}{3} < e \leq 1$$



Question 25 (**)**

A smooth sphere A of mass m is moving with speed u on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $3m$ which is initially at rest. The direction of motion of A is reversed as a result of the collision.

The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

- Find, in terms of e and u , the speeds of the two spheres after their collision.
- Find the range of the possible values of the speed of B .

Consequently sphere B strikes at right angles a fixed smooth vertical wall, and rebounds at right angles. The coefficient of restitution between B and the wall is $\frac{1}{4}$.

- Given there is another collision between the spheres show clearly that

$$\frac{1}{3} < e < \frac{5}{11}.$$

$$[] , V_A = \frac{1}{4}u(3e-1) , V_B = \frac{1}{4}u(1+e) , \frac{1}{3}u < V_B < \frac{1}{2}u$$

a) STARTING WITH A BEFORE AND AFTER DIAGRAM

BY CONSERVATION OF MOMENTUM

$$mu + 0 = -mu + 3mv$$

$$u = -X + 3Y$$

BY COEFFICIENT OF RESTITUTION

$$e = \frac{v-u}{u-Y}$$

$$e = \frac{v-u}{u-X+3Y}$$

$$X+Y=eU$$

SOLVE THE EQUATIONS

$$\Rightarrow 5Y = U + eu$$

$$\Rightarrow 4Y = U(1+e)$$

$$\Rightarrow Y = \frac{1}{4}U(1+e)$$

AND FINALLY SOLVE X = 3Y - U

$$X = \frac{3}{4}U(1+e) - U = \frac{1}{4}U[3(1+e) - 4] = \frac{1}{4}U(3e-1)$$

$$X = \frac{1}{4}U(3e-1)$$

← SPEED OF A

b) AS X IS A MEASURE "CORRECTLY" IN THE DIRECTION X>0

$$\Rightarrow \frac{1}{4}U(3e-1) > 0$$

$$\Rightarrow 3e-1 > 0$$

$$\Rightarrow 3e > 1$$

$$\Rightarrow e > \frac{1}{3}$$

THIS WE NOW HAVE

$$\frac{1}{3} < e < 1$$

$$\frac{1}{3} < e < 2$$

$$\frac{1}{3} < e < \infty$$

$$\frac{1}{4} < \frac{1}{4}(1+e) < \frac{1}{2}$$

$$\frac{1}{3} < U < \frac{1}{2}u$$

B REBONDS WITH THE WALL

LATE (CONFUSION) NOTED

ANOTHER CONFUSION NOTED $\frac{1}{4}Y > X$

$$\Rightarrow \frac{1}{4}(U(1+e)) > \frac{1}{4}(3e-1)$$

$$\Rightarrow \frac{1}{4}(1+e) > 3e-1$$

$$\Rightarrow 1+e > 12e-4$$

$$\Rightarrow 5 > 11e$$

$$\Rightarrow e < \frac{5}{11}$$

BY CALCULATING IT WAS FOUND (MAYBE) THAT $\frac{1}{4}(1+e) > \frac{1}{4}(3e-1)$

$$\therefore \frac{1}{3} < e < \frac{5}{11}$$

AS REQUIRED

Question 26 (**)**

A particle, of mass m , lies on a smooth horizontal surface.

Initially the particle is at rest at a point O , which lies midway between a pair of fixed, smooth, parallel vertical walls, which are $2L$ apart. At time $t = 0$ the particle is projected from O with speed u in a direction perpendicular to the walls.

The coefficient of restitution between the particle and each wall is e .

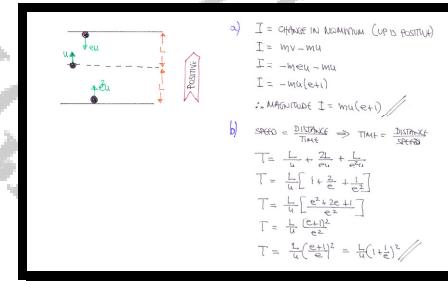
- a) Find, in terms of m , u and e , the magnitude of the impulse on the particle due to the **first** impact with the wall.

The particle returns to O , having bounced off each wall once, at time $t = T$.

- b) Show clearly that

$$T = \frac{L}{u} \left(1 + \frac{1}{e}\right)^2.$$

$$|I| = mu(e+1)$$



Question 27 (**)**

Two particles, A and B , of respective masses m and km , where k is a positive constant, lie on a smooth horizontal surface. Initially the particles are at rest at some point on the surface between a pair of fixed, smooth, parallel vertical walls.

A and B are simultaneously projected, with respective speeds u and $2u$, away from each other in directions perpendicular to the walls. After rebounding from the walls, A and B collide directly with each other.

The coefficient of restitution between **all** collisions in this question is taken to be e .

Given further that the direction of motion of A is **not** reversed after colliding with B , show that

$$e < \frac{1-2k}{3k}.$$

 , proof

<ul style="list-style-type: none"> ● STARTING WITH THE COLLISIONS WITH THE WALLS AFTER PREDICTION $v_A = eu$ and $v_B = 2eu$ 	$\Rightarrow -x + y = -3eu \quad \times (-k)$ $\Rightarrow kx - ky = -3keu$
<ul style="list-style-type: none"> ● NEXT THE COLLISION BETWEEN THEM 	\Rightarrow $\begin{aligned} kx - ky &= -3keu \\ x + ky &= eu - 2keu \\ (k+1)x &= eu - 2keu - 3keu \\ x &= \frac{eu(1-2k-3ke)}{(k+1)} \end{aligned}$
<ul style="list-style-type: none"> ● BY CONSERVATION OF MOMENTUM $\Rightarrow m_eu - km_eu = m_x + km_y$ $\Rightarrow eu - 2keu = x + ky$	\Rightarrow $\begin{aligned} \text{BY THE EQUATION OF MOTION} \\ \Rightarrow \frac{y-x}{eu+2eu} &= e \end{aligned}$
<ul style="list-style-type: none"> ● A DOES NOT REVERSE ITS DIRECTION 	$\Rightarrow x > 0$ $\Rightarrow \frac{eu(1-2k-3ke)}{k+1} > 0$ $\Rightarrow 1-2k-3ke > 0 \quad [eu > 0, k > 0]$ $\Rightarrow -3ke > 2k-1$ $\Rightarrow e < \frac{1-2k}{3k}$ AS REQUIRED

Question 28 (**+)**

A smooth sphere A of mass $3m$ is moving with speed $4u$ on a smooth horizontal plane when it collides directly with a smooth sphere B of mass $2m$ which is moving with speed u in the same direction as A . The direction of motion of A is **not** reversed as a result of the collision.

The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

- Show that the speed of B after the collision is $\frac{1}{5}u(14+9e)$.
- Given that the speed of A after the collision is $2u$ show that $e = \frac{2}{3}$.

Consequently sphere B strikes at right angles a fixed smooth vertical wall, and rebounds at right angles. The coefficient of restitution between B and the wall is $\frac{1}{4}$.

The initial collision between A and B takes place at the point P , which is at a distance d from the wall. A second collision between A and B takes place at the point Q .

- Find, in terms of d , the distance of Q from the wall.

$$\boxed{\frac{1}{6}d}$$

(a) Initial velocities: Sphere A (mass $3m$) moves right with speed $4u$; Sphere B (mass $2m$) moves right with speed u . Final velocities: Sphere A moves right with speed $2u$; Sphere B moves right with speed $\frac{1}{5}u(14+9e)$.

By Conservation of Momentum: $12mu + 2mu = 3mX + 2mY$
 $14u = 3X + 2Y$

By Restitution: $e = \frac{v_B - v_A}{u - X}$
 $e = \frac{Y - X}{3u - X}$
 $e = \frac{Y - X}{3u - X} = \frac{1}{4}$
 $4(Y - X) = 3u - X$
 $4Y - 4X = 3u - X$
 $4Y = 3u + 3X$
 $Y = \frac{1}{4}u(14+9e)$ As required

(b) $X = 2u$ So $X = Y - 3eu$
 $2u = \frac{1}{4}u(14+9e) - 3eu$
 $2 = \frac{1}{4}(14+9e) - 3e$
 $10 = 14+9e - 12e$
 $6e = 4$
 $e = \frac{2}{3}$ As required

(c) If $e = \frac{2}{3}$ Then $Y = \frac{1}{4}u(14+9e)$
 $Y = \frac{1}{4}u(14+6)$
 $4Y = 2u$

As B is traveling twice as fast as A after the collision when B reaches the wall, A must be half way to the wall.
 $\frac{1}{2}d$ from the wall

Sphere B rebounds off the wall. Initial speed $\frac{1}{4}(2u) = u$

As A is moving twice as fast as B they will collide $(\frac{1}{2}d + \frac{1}{4}d)$ from the wall.
 $\frac{3}{4}d$ from the wall

Collision will take place $\frac{3}{4}d$ from the wall

Question 29 (**+)**

A small bouncy ball is held at a height of 1.225 m above a smooth horizontal surface. The ball is released from rest and impacts the horizontal surface with speed $V \text{ ms}^{-1}$.

- a) Determine the value of V .

The coefficient of restitution the ball and surface is e .

- b) Show that the time between the second impact and the third impact of the ball and the surface is e^2 .

The bouncy ball takes 7.5 s from the instant it was first released until the instant it comes to rest.

- c) Find the value of e .

$$V = 4.9 \text{ ms}^{-1}, e = \frac{7}{8}$$

(a)

$U = 0 \text{ ms}^{-1}$ $a = 9.8 \text{ ms}^{-2}$ $S = 1.225$ $t = ?$ $V = ?$	$V^2 = U^2 + 2aS$ $V^2 = 0 + 2 \times 9.8 \times 1.225$ $V^2 = 24.01$ $V = 4.9 \text{ ms}^{-1}$
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LOOKING AT THE ENTIRE JOURNEY BETWEEN THE FIRST AND THE SECOND IMPACT

$U = 4.9 \text{ m/s}$
 $a = -9.8 \text{ m/s}^2$
 $S = 0$
 $t = ?$
 $V = ?$

$S = Ut + \frac{1}{2}at^2$
 $0 = 4.9t - 4.9t^2$
 $0 = 4.9t(1 - t)$
 $t = 1 \text{ s}$
 $t < e$

PATRICK REACHES THE GROUND WITH SPEED $4.9e$ (BY CONSIDERATION OF ENERGY) – IT THEN REDUCES BY e (I.e. $4.9e^2$)

LOOKING AT THE JOURNEY BETWEEN THE SECOND AND THIRD IMPACT

$U = 4.9e^2 \text{ m/s}$
 $a = -9.8 \text{ m/s}^2$
 $S = 0$
 $t = ?$
 $V = ?$

$S = Ut + \frac{1}{2}at^2$
 $0 = 4.9e^2t - 4.9t^2$
 $0 = 4.9t(e^2 - t)$
 $t = e^2$
 $t > e^2$

AT REST

(b) FINALLY LOOKING AT THE JOURNEY UP TO THE FIRST IMPACT (SEE PART (a))

$V = 4.9 \text{ m/s}$
 $a = 9.8 \text{ m/s}^2$
 $t = 0.5 \text{ s}$

$S = 0.5 + e + e^2 + e^3 + e^4 + \dots = 7.5$
 $e + e^2 + e^3 + e^4 + \dots = 7$ ← GP WITH $a=e$
 $\frac{S}{1-e} = 7$
 $e = 7 - 7e$
 $8e = 7$
 $e = \frac{7}{8}$

Question 30 (**+)**

A smooth sphere A of mass m is moving with speed u on a smooth horizontal surface when it collides directly with a smooth sphere B of mass $2m$ which is at rest. The spheres are modelled as particles and the coefficient of restitution between the two spheres is e .

The total amount of kinetic energy after the collision is $\frac{11}{64}mu^2$

Determine the value of e .

$$e = \frac{1}{8}$$

The diagram shows two spheres, A and B, on a horizontal surface. Sphere A has mass m and initial velocity u to the right. Sphere B has mass $2m$ and is at rest. After the collision, sphere A has velocity v_A to the right and sphere B has velocity v_B to the right.

By conservation of momentum:

$$mv + 0 = mX + 2mY$$

$$\boxed{X + 2Y = u} \quad \text{--- (1)}$$

By law of conservation of energy:

$$\frac{1}{2}mX^2 + \frac{1}{2}(2m)Y^2 = \frac{11}{64}mu^2$$

$$\Rightarrow \frac{1}{2}X^2 + Y^2 = \frac{11}{32}u^2$$

$$\Rightarrow X^2 + 2Y^2 = \frac{11}{32}u^2$$

But $X = u - 2Y$

$$\Rightarrow (u - 2Y)^2 + 2Y^2 = \frac{11}{32}u^2$$

$$\Rightarrow 4Y^2 - 4uY + u^2 + 2Y^2 = \frac{11}{32}u^2$$

$$\Rightarrow 6Y^2 - 4uY + \frac{11}{32}u^2 = 0$$

$$\Rightarrow 192Y^2 - 128uY + 21u^2 = 0$$

By quadratic formula

$$Y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$Y = \frac{128u \pm \sqrt{(128u)^2 - 4(192)(21u^2)}}{384}$$

$$Y = \frac{128u \pm \sqrt{128u^2}}{384}$$

$$Y = \frac{128u \pm 16u}{384}$$

$$Y = \begin{cases} \frac{144u}{384} \\ \frac{112u}{384} \end{cases} \quad X = \begin{cases} \frac{144u}{384} \\ \frac{112u}{384} \end{cases}$$

But $X < Y$

$$\therefore Y = \frac{112u}{384} \quad X = \frac{112u}{384}$$

So $e = \frac{Y - X}{u}$

$$e = \frac{\frac{112u}{384} - \frac{112u}{384}}{u}$$

$$e = \frac{0}{u}$$

$$e = \frac{1}{8}$$