

FOURIER SERIES

The Fourier Theorem

If $f(x)$ is a piecewise continuous function on (α, β) , then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

where $a_n = \frac{1}{L} \int_{\alpha}^{\beta} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

$$b_n = \frac{1}{L} \int_{\alpha}^{\beta} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$L = \frac{\beta - \alpha}{2} = \text{half period}$$

Parseval's Identity

$$\frac{1}{L} \int_{\alpha}^{\beta} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

FOURIER SERIES EXPANSIONS

Question 1

$$f(x) = x, \quad x \in \mathbb{R}, \quad -\pi \leq x \leq \pi.$$

$$f(x) = f(x + 2\pi).$$

Determine the Fourier series expansion of $f(x)$.

$$\boxed{\quad}, \quad f(x) = 2 \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} \sin nx}{n} \right]$$

• USING THE STANDARD FOURIER SERIES FORMULA

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \quad L = \text{HALF PERIOD} = \frac{\pi}{2}$$

where $a_n = \frac{1}{L} \int_a^b f(x) dx$

$$a_n = \frac{1}{\frac{\pi}{2}} \int_{-\pi}^{\pi} f(x) \cos \frac{n\pi x}{\frac{\pi}{2}} dx, \quad n = 1, 2, 3, 4, \dots$$

$$b_n = \frac{1}{\frac{\pi}{2}} \int_{-\pi}^{\pi} f(x) \sin \frac{n\pi x}{\frac{\pi}{2}} dx, \quad n = 1, 2, 3, 4, \dots$$

• USING THE ABOVE FORMULAE, WITH $a = -\pi$, $b = \pi$, $f(x) = x$, WE OBTAIN

- $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x dx = 0$ (odd integrand in a symmetric domain)
- $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx dx = 0$ (odd integrand in a symmetric domain)
- $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx$

• PROCEEDED BY INTEGRATION BY PARTS

$$b_n = \frac{2}{\pi} \left\{ \left[\frac{x}{n} \cos nx \right]_0^\pi + \frac{1}{n} \int_0^\pi (\cos nx) dx \right\}$$

$$b_n = \frac{2}{\pi} \left\{ \left[\frac{\cos nx}{n} \right]_0^\pi + \left[\frac{1}{n^2} \sin nx \right]_0^\pi \right\}$$

• FINDING THE FOURIER SERIES

$$\Rightarrow b_n = \frac{2}{\pi} \left[0 - \frac{2 \cos nx}{n} \right]$$

$$\Rightarrow b_n = -\frac{2 \cos nx}{n}$$

$$\Rightarrow b_1 = -\frac{2}{\pi} (-1)^n$$

∴ $f(x) = \sum_{n=1}^{\infty} \left[-\frac{2}{\pi} (-1)^n \sin nx \right]$

$$f(x) = 2 \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \sin nx \right]$$

$$x = 2 \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots \right]$$

Question 2

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
- b) Find the Fourier series of

$$f(x) = 2x, -\pi \leq x \leq \pi.$$

$$2x = \sum_{n=1}^{\infty} \left[\frac{4(-1)^{n+1}}{n} \sin(nx) \right]$$

a) If $f(x)$ is piecewise continuous on $(-L, L)$, $\int_L^0 f(x) dx \neq 0$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

where $a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$ $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ $n = 1, 2, 3, 4, \dots$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, 3, 4, \dots$$

b) $f(x) = 2x$ is odd $\Rightarrow a_n = 0$ for all n

$$b_n = \frac{4}{\pi} \int_{-\pi}^{\pi} (2x) \sin \frac{n\pi x}{\pi} dx = \frac{4}{\pi} \int_{-\pi}^{\pi} 2x \sin nx dx$$

$$= \frac{4}{\pi} \int_{-\pi}^{\pi} 2x \sin nx dx = \dots \text{ by parts}$$

$$= \frac{4}{\pi} \left\{ \left[\frac{1}{n} x \cos nx \right]_{-\pi}^{\pi} + \frac{4}{\pi} \int_{-\pi}^{\pi} x \cos nx dx \right\}$$

$$= \frac{4}{\pi} \left\{ -\frac{\pi}{n} \cos n\pi + \frac{1}{n^2} \left[\sin nx \right]_{-\pi}^{\pi} \right\} = -\frac{4}{n} \cos n\pi$$

$$= -\frac{4}{n} (-1)^n = \frac{4}{n} (-1)^{n+1}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4}{n} (-1)^{n+1} \sin nx$$

It: $2x = 4 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \frac{\sin 5x}{5} - \dots \right]$

Question 3

$$f(t) = \begin{cases} 2t+2 & -1 \leq t \leq 0 \\ 0 & 0 \leq t \leq 1 \end{cases}$$

$$f(t) = f(t+2).$$

Determine the Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{2\cos[(2n-1)\pi t]}{\pi(2n-1)^2} - \frac{\sin(n\pi t)}{n} \right]$$

| | | |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------|
| $\frac{f(t)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)]$ <p style="text-align: center;">$(L = \frac{\pi}{2} \text{ since } \pi/2 \text{ is half period})$</p> | <p>where $a_n = \frac{1}{\pi/2} \int_0^{\pi/2} f(t) dt$</p> <p>$a_0 = \frac{1}{\pi/2} \int_0^{\pi/2} f(t) dt$</p> <p>$b_n = \frac{1}{\pi/2} \int_0^{\pi/2} f(t) \sin(nt) dt$</p> | <p>Since $a_0 = 1$</p> <p>$b_n = 0$</p> <p>Period = π, $L = 1$</p> |
| $f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{2\cos[(2n-1)\pi t]}{\pi(2n-1)^2} - \frac{\sin(n\pi t)}{n} \right]$ | | |

$\bullet a_0 = \frac{1}{\pi/2} \int_0^{\pi/2} f(t) dt = \int_0^{\pi/2} 2t+2 dt + \int_0^{\pi/2} 0 dt = \left[t^2 + 2t \right]_0^{\pi/2} = 0 - [0 - 0] = 1$

$\bullet a_n = \frac{1}{\pi/2} \int_0^{\pi/2} f(t) \cos(nt) dt = \int_0^{\pi/2} (2t+2) \cos(nt) dt + \int_0^{\pi/2} 0 \cos(nt) dt = \dots \text{by parts} \dots$

$= \frac{2}{nt^2} \int_0^{\pi/2} (2t+2) n \sin(nt) dt - \frac{2}{nt} \int_0^{\pi/2} \sin(nt) dt = \frac{2}{nt^2} \left[(2t+2) \sin(nt) \right]_0^{\pi/2} - \frac{2}{nt} \left[-\cos(nt) \right]_0^{\pi/2} < \frac{2}{\pi^2 n^3} \left[1 - (-1)^n \right] < \frac{4}{\pi^2 n^3} \text{ if } n \neq 0$

$\bullet b_n = \frac{1}{\pi/2} \int_0^{\pi/2} f(t) \sin(nt) dt = \int_0^{\pi/2} (2t+2) \sin(nt) dt + \int_0^{\pi/2} 0 \sin(nt) dt = \dots \text{by parts} \dots$

$= -\frac{2}{nt^2} \int_0^{\pi/2} (2t+2) n \cos(nt) dt + \frac{2}{nt} \int_0^{\pi/2} \cos(nt) dt = -\frac{2}{nt^2} \left[(2t+2) \cos(nt) \right]_0^{\pi/2} + \frac{2}{nt} \left[\sin(nt) \right]_0^{\pi/2} = -\frac{2}{nt^2} \left[(\pi/2)(-1)^n - 0 \right] + \frac{2}{nt} \left[0 - 0 \right] = -\frac{2}{nt} \left[(-1)^n - 1 \right]$

$\therefore f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2\cos[(2n-1)\pi t]}{\pi(2n-1)^2} + \sum_{n=1}^{\infty} \frac{-2}{nt} \sin(nt)$

$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{2\cos[(2n-1)\pi t]}{\pi(2n-1)^2} - \frac{\sin(n\pi t)}{n} \right]$

Question 4

$$f(t) = \begin{cases} 1 + \frac{1}{4}t & -4 \leq t \leq 0 \\ 1 - \frac{1}{4}t & 0 \leq t \leq 4 \end{cases}$$

$$f(t) = f(t+8).$$

Determine the Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\cos\left[\frac{1}{4}(2n-1)\pi t\right]}{(2n-1)^2} \right]$$

QUESTION IF $f(x)$ IS PIECEWISE CONTINUOUS IN $(-L, L)$, THEN

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}] \quad \text{WITH}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad n = 0, 1, 2, 3, \dots$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, 3, \dots$$

HERE

$$f(t) = \begin{cases} 1 + \frac{1}{4}t & -4 \leq t < 0 \\ 1 - \frac{1}{4}t & 0 \leq t \leq 4 \end{cases}$$

HERE $\int_{-4}^4 f(t) dt = \int_{-4}^0 (1 + \frac{1}{4}t) dt + \int_0^4 (1 - \frac{1}{4}t) dt = 0$

• THIS $a_0 = \frac{1}{4} \int_{-4}^4 f(t) dt = \frac{1}{4} \times \text{area of triangle} = \frac{1}{4} \times \frac{1}{2} \times 8 \times 1 = 1$

• $a_n = \frac{1}{4} \int_{-4}^4 f(t) \cos \frac{n\pi t}{4} dt = \frac{1}{4} \int_{-4}^0 (1 + \frac{1}{4}t) \cos \frac{n\pi t}{4} dt + \int_0^4 (1 - \frac{1}{4}t) \cos \frac{n\pi t}{4} dt = \frac{1}{2} \int_{-4}^0 (1 + \frac{1}{4}t) \cos \frac{n\pi t}{4} dt + \frac{1}{2} \int_0^4 (1 - \frac{1}{4}t) \cos \frac{n\pi t}{4} dt$

$$= -\frac{1}{2n\pi} \frac{4}{n} \left[\cos \frac{n\pi t}{4} \right]_0^4 = -\frac{2}{n\pi} \left[\cos 4n\pi - 1 \right] = \frac{2}{n\pi} \left[1 - \cos 4n\pi \right]$$

$$= \frac{2}{n\pi} \cdot 0 = 0 \quad \text{IF } n = 0 \text{ OR } n = 4m \quad \text{RECALL } n \text{ AS } 2n-1$$

$$\frac{4}{n\pi} \left[\frac{1}{4} \sin \frac{n\pi t}{4} \right]_0^4 = \frac{1}{n\pi} \left[\cos \frac{n\pi t}{4} \right]_0^4 = \frac{1}{n\pi} \left[\cos 4n\pi - 1 \right] = \frac{1}{n\pi} \left[1 - \cos 4n\pi \right] = \frac{2}{n\pi}$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{2}{n\pi} \frac{\cos(4n\pi)}{(2n-1)^2} \right] = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(4n\pi)}{(2n-1)^2}$$

$$= \frac{1}{2} + \frac{4}{\pi^2} \left[\frac{\cos(0)}{1^2} + \frac{\cos(16\pi)}{3^2} + \frac{\cos(64\pi)}{5^2} + \frac{\cos(128\pi)}{7^2} + \dots \right]$$

Question 5

The “Top Hat” function is defined as

$$f(x) = \begin{cases} 1 & |x| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |x| \leq \pi \end{cases}$$

for $x \in \mathbb{R}$, $f(x) = f(x+2\pi)$.

Determine the Fourier series expansion of $f(x)$.

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} \cos[(2n-1)x]}{2n-1} \right]$$

The page shows the derivation of the Fourier series for the Top Hat function $f(x) = \begin{cases} 1 & |x| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |x| \leq \pi \end{cases}$. The function is plotted over one period from $-\pi$ to π , showing a rectangular pulse of height 1 between $|x| = \frac{\pi}{2}$ and π .

The derivation uses the formulas for Fourier coefficients:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

For a_0 :

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 dx = \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} 1 dx = \frac{2}{\pi} \times \frac{\pi}{2} = 1$$

For b_n :

$$b_n = 0 \quad \text{SINCE } f(x) \text{ is even} \Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \times \sin(nx) dx = 0$$

For a_n :

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \times \cos(nx) dx = \dots \text{ SINCE } n \text{ is odd} \Rightarrow \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \cos(nx) dx = \frac{2}{\pi n} \left[\sin(nx) \right]_0^{\frac{\pi}{2}} = \frac{2}{\pi n} [\sin(n\frac{\pi}{2}) - 0]$$

$$= \frac{2}{\pi n} \sin(n\frac{\pi}{2}) = \begin{cases} \frac{2}{\pi n} & \text{if } n = 1, 3, 5, \dots \\ 0 & \text{if } n = 2, 4, 6, \dots \end{cases}$$

Thus $f(x) = \frac{1}{2} + \left[\frac{2}{\pi} \cos_2 x - \frac{2}{3\pi} \cos_3 x + \frac{2}{5\pi} \cos_5 x - \frac{2}{7\pi} \cos_7 x + \frac{2}{9\pi} \cos_9 x + \dots \right]$

$$\boxed{f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)}}$$

NOTE IF $2n-1 = 1, 3, 5, \dots$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)} = \frac{\pi}{4}$$

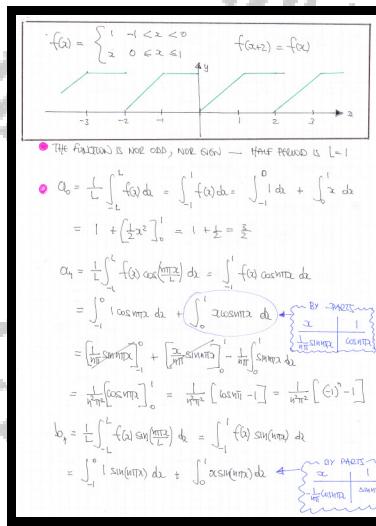
Question 6

$$f(x) = \begin{cases} 1 & -1 \leq x \leq 0 \\ x & 0 \leq x \leq 1 \end{cases}$$

$$f(x+2) = f(x).$$

Determine the Fourier series expansion of $f(x)$.

$$f(x) = \frac{3}{4} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\cos[(2n-1)\pi x]}{(2n-1)^2} \right] - \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin(n\pi x)}{n} \right]$$



$$\begin{aligned} &= \left[-\frac{1}{n\pi} \cos(n\pi x) \right]_0^{-1} + \left[-\frac{x}{n\pi} \cos(n\pi x) \right]_0^1 + \frac{1}{n\pi} \int_0^1 \cos(n\pi x) dx \\ &= \left[\frac{1}{n\pi} \cos(n\pi x) \right]_0^{-1} + \left[\frac{x}{n\pi} \cos(n\pi x) \right]_0^1 + \frac{1}{n\pi^2} \int_0^1 \sin(n\pi x) dx \\ &= \frac{1}{n\pi} (\cos(-n\pi) - \cos(n\pi)) + 0 - \frac{1}{n\pi} \sin(n\pi) \\ &= \frac{1}{n\pi} \cos(n\pi) - \frac{1}{n\pi} = \frac{1}{n\pi} \cos(n\pi) = -\frac{1}{n\pi} \end{aligned}$$

THE WE HAVE

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)]$$

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \left[\frac{1}{n\pi^2} [(-1)^n - 1] \cos(n\pi x) - \frac{1}{n\pi} \sin(n\pi x) \right]$$

$$f(x) = \frac{3}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{-2}{(2n-1)^2} \cos((2n-1)\pi x) \right] = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n}$$

$$f(x) = \frac{3}{4} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi x)}{(2n-1)^2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n}$$

Question 7

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

- b) Find the Fourier series of

$$f(x) = x^2, -1 \leq x \leq 1.$$

- c) Hence determine the exact value of

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots$$

$$\boxed{\text{[]}}, \quad x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos(n\pi x) \right], \quad 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots = \frac{\pi^2}{12}$$

a) DIRECTLY FROM THE DEFINITION
IF $f(x)$ IS PIECEWISE CONTINUOUS IN THE INTERVAL $(-L, L)$, THEN

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)]$$

WHERE $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$ $n=1, 2, \dots$

$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$ $n=1, 2, \dots$

b) LET US START BY NOTING THAT $f(x) = x^2, x \in [-1, 1]$ IS EVEN

- As $f(x)$ IS EVEN, ALL $b_n = 0$, AS THE INTEGRAL OF b_n WOULD BE ZERO IN A SYMMETRICAL DOMAIN
- $a_0 = \frac{1}{L} \int_{-L}^L x^2 dx = \dots$ EVEN MOREOVER ... $\int_{-L}^L x^2 dx = \frac{2}{3} L^3$
- $a_1 = \frac{1}{L} \int_{-L}^L x^2 \cos\left(\frac{n\pi x}{L}\right) dx = \dots$ EVEN MOREOVER ... $\int_{-L}^L x^2 \cos\left(\frac{\pi x}{L}\right) dx$

INTEGRATION BY PARTS

$$= \left[\frac{x^2}{2} \sin\left(\frac{\pi x}{L}\right) \right]_0^L - \frac{1}{\pi} \int_0^L 2x \sin\left(\frac{\pi x}{L}\right) dx$$

$$= -\frac{1}{\pi} \int_0^L 2x \sin\left(\frac{\pi x}{L}\right) dx$$

INTEGRATION BY PARTS AGAIN

$$= -\frac{1}{\pi} \left\{ \left[-\frac{2x}{\pi} \cos\left(\frac{\pi x}{L}\right) \right]_0^L + \frac{2}{\pi^2} \int_0^L \cos\left(\frac{\pi x}{L}\right) dx \right\}$$

| | |
|----------------------------------------------------|------|
| $\frac{2x^2}{\pi}$ | u |
| $\frac{1}{\pi} x \sin\left(\frac{\pi x}{L}\right)$ | dv |
| 1 | |

$$\begin{aligned}
 &= -\frac{1}{\pi^2} \left[2x \cos\left(\frac{\pi x}{L}\right) \right]_0^L - \frac{1}{\pi^3} \int_0^L \sin\left(\frac{\pi x}{L}\right) dx \\
 &= \frac{4}{\pi^2 L^2} \left[2 \cos\left(\frac{\pi x}{L}\right) \right]_0^L - \frac{1}{\pi^3 L^3} \left[\sin\left(\frac{\pi x}{L}\right) \right]_0^L \\
 &= \frac{4}{\pi^2 L^2} \left[\cos\left(\frac{\pi L}{L}\right) - 0 \right] = \frac{4 \cos(\pi)}{\pi^2 L^2} = \frac{4(-1)^n}{\pi^2 L^2} \\
 \therefore f(x) &= \frac{2}{3} + \sum_{n=1}^{\infty} \left[\frac{4(-1)^n}{\pi^2 n^2} \cos(n\pi x) \right] \\
 x^2 &= \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos(n\pi x) \right]
 \end{aligned}$$

c) LETTING $x=0$ IN THE ABOVE EXPANSION

$$\begin{aligned}
 \Rightarrow 0^2 &= \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos(0) \right] \\
 \Rightarrow 0 &= \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \right] \\
 \Rightarrow \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} &= -\frac{1}{3} \\
 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} &= -\frac{1}{12} \\
 \Rightarrow -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \dots &= -\frac{1}{12} \\
 \Rightarrow 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots &= \frac{1}{12}
 \end{aligned}$$

Question 8

A function $f(x)$ is defined in an interval $(-\pi, \pi)$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-\pi, \pi)$, giving general expressions for the coefficients of the series.
- b) Find the Fourier series of

$$f(x) = \begin{cases} 0 & -\pi \leq x \leq -\frac{1}{2}\pi \\ 1 & -\frac{1}{2}\pi < x \leq \frac{1}{2}\pi \\ 0 & \frac{1}{2}\pi \leq x \leq \pi \end{cases}$$

- c) Hence determine the exact value of

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

$$f(x) = \frac{1}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} \cos(nx)}{2n-1} \right], \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{\pi}{4}$$

a) If $f(x)$ is piecewise continuous on $(-\pi, \pi)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad n=0, 1, 2, 3, 4, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n=1, 2, 3, 4, \dots$$

b)

- $f(x)$ is even, so b_n will be zero. ($\int_{-\pi}^{\pi} \sin nx dx = 0$)
- $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{-\pi/2} 0 dx = \frac{1}{\pi} x \Big|_{-\pi}^{-\pi/2} = 0$
- $a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{-\pi/2} 0 dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 1 \cdot \cos nx dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos nx dx = \frac{2}{\pi n} \sin nx \Big|_{-\pi/2}^{\pi/2} = \frac{2}{\pi n} [\sin \frac{\pi n}{2} - \sin (-\frac{\pi n}{2})] = \frac{2}{\pi n} \cdot 2 \sin \frac{\pi n}{2} = \frac{4}{\pi n} \sin \frac{\pi n}{2}, \text{ if } n=1, 3, 5, 7, \dots$
- $= \frac{2}{\pi n}, \text{ if } n=0, 2, 4, 6, \dots$

c) Let $x=0$

$$f(0) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow 1 = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow \frac{1}{2} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \frac{\pi}{4}$$

Question 9

A function $f(x)$ is defined in an interval $(\alpha, \alpha+2L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(\alpha, \alpha+2L)$, giving general expressions for the coefficients of the series.

$$f(x) = x, 0 \leq x \leq 4.$$

- b) Find the Fourier series of $f(x)$...

i. ... in the interval $0 \leq x \leq 4$, with period 4.

ii. ... in the interval $0 \leq x \leq 4$, with period 8, by building a suitable “extension” to $f(x)$.

Illustrate the solution in each case with a sketch.

$$x = 2 - \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} \sin \left(\frac{1}{2} n \pi x \right) \right], \quad x = \frac{8}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \sin \left(\frac{1}{4} n \pi x \right) \right]$$

a) If $f(x)$ is piecewise continuous on $(\alpha, \alpha+2L)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

where

$$a_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, 3, \dots$$

$$b_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

b) $f(x) = x, 0 \leq x \leq 4$, $L = 2$

- $a_0 = \frac{1}{2} \int_0^4 x dx = \frac{1}{4} x^2 \Big|_0^4 = 4$
- $a_n = \frac{1}{2} \int_0^4 x \cos \frac{n\pi x}{2} dx = \dots \text{ by parts}$

$$\begin{aligned} &= \left[\frac{x}{2} \sin \frac{n\pi x}{2} \right]_0^4 - \frac{1}{2} \int_0^4 \sin \frac{n\pi x}{2} dx \\ &= \frac{2}{n\pi} \left[\cos \frac{n\pi x}{2} \right]_0^4 = 0 \end{aligned}$$
- $b_n = \frac{1}{2} \int_0^4 x \sin \frac{n\pi x}{2} dx = \dots \text{ by parts}$

$$\begin{aligned} &= \left[-\frac{1}{2} x \cos \frac{n\pi x}{2} \right]_0^4 + \frac{1}{2} \int_0^4 \cos \frac{n\pi x}{2} dx \\ &= \left[\frac{1}{n\pi} x \sin \frac{n\pi x}{2} \right]_0^4 + \left[\frac{2}{n\pi} \sin \frac{n\pi x}{2} \right]_0^4 \\ &= 0 - \frac{1}{n\pi} 4 \sin 2n\pi = -\frac{4}{n\pi} \end{aligned}$$

Thus $x = 2 + \sum_{n=1}^{\infty} \left[\frac{4}{n\pi} \sin \frac{n\pi x}{2} \right]$

ii. $L = 2 = \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} \sin \frac{n\pi x}{2} \right]$

or $L = 2 = \frac{8}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{4} \right]$

Now build an odd extension

$f(x) = x, 0 \leq x \leq 4, L = 4$

Now $f(x)$ is odd so $a_n = 0 \quad \forall n$ (odd harmonic)

$b_n = \frac{1}{4} \int_{-4}^4 x \sin \frac{n\pi x}{4} dx = \int_0^4 \frac{1}{2} x \sin \frac{n\pi x}{2} dx$

By parts

$$\begin{aligned} &= \left[-\frac{1}{2} x \cos \frac{n\pi x}{2} \right]_0^4 + \left[\frac{1}{2} \sin \frac{n\pi x}{2} \right]_0^4 \\ &= \left[\frac{2}{n\pi} x \sin \frac{n\pi x}{2} \right]_0^4 + \left[\frac{2}{n\pi} \cos \frac{n\pi x}{2} \right]_0^4 \\ &= 0 - \frac{2}{n\pi} 4 \sin 2n\pi = -\frac{8}{n\pi} \end{aligned}$$

Thus $x = 2 + \sum_{n=1}^{\infty} \left[\frac{8}{n\pi} \sin \frac{n\pi x}{4} \right]$

$b_n = -\frac{8}{n\pi} \cos n\pi$

$b_1 = -\frac{8}{\pi} (-1)^1$

$b_2 = \frac{8}{2\pi} (-1)^2$

Thus $x = \sum_{n=1}^{\infty} \left[\frac{8}{n\pi} (-1)^n \sin \frac{n\pi x}{4} \right]$

$x = \frac{8}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{4} \right]$

Question 10

A function $f(x)$ is defined in an interval $(\alpha, \alpha+2L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(\alpha, \alpha+2L)$, giving general expressions for the coefficients of the series.

$$f(x) = x^2, 0 \leq x \leq 1.$$

- b) Find the Fourier series of $f(x)$...

i. ... in the interval $0 \leq x \leq 1$, with period 1.

ii. ... in the interval $0 \leq x \leq 1$, with period 2, by building a suitable "extension" to $f(x)$.

Illustrate the solution in each case with a sketch.

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{\cos(2n\pi x)}{n^2 \pi^2} - \frac{\sin(2n\pi x)}{n\pi} \right], \quad x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos(n\pi x) \right]$$

9) If $f(x)$ is piecewise continuous on $(\alpha, \alpha+2L)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

where

$$a_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \cos \frac{n\pi x}{L} dx, \quad n=1,2,3,\dots$$

$$b_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \sin \frac{n\pi x}{L} dx, \quad n=1,2,3,\dots$$

10) $f(x) = x^2, 0 \leq x \leq 1, L = \frac{1}{2}$

- $a_0 = \frac{1}{\frac{1}{2}} \int_0^1 x^2 dx = \left[\frac{2}{3}x^3 \right]_0^1 = \frac{2}{3}$
- $a_1 = \frac{1}{\frac{1}{2}} \int_0^1 x^2 \cos(2\pi x) dx = \int_0^1 2x^2 \cos(2\pi x) dx$
 - ... BY PARTS ...
 - $\begin{aligned} &= \left[\frac{2x^2}{2\pi} \sin(2\pi x) \right]_0^1 + \frac{2}{2\pi} \int_0^1 x^2 \sin(2\pi x) dx \\ &= -\frac{1}{\pi} \cos(2\pi x) + \frac{2}{2\pi} \int_0^1 x^2 \sin(2\pi x) dx \end{aligned}$
 - ... BY PARTS AGAIN ...
 - $\begin{aligned} &= -\frac{1}{\pi} \cos(2\pi x) + \frac{2}{2\pi} \left[\frac{1}{2\pi} x^2 \sin(2\pi x) \right]_0^1 - \frac{1}{2\pi} \int_0^1 x^2 \sin(2\pi x) dx \\ &= -\frac{1}{\pi} \cos(2\pi x) + \frac{1}{2\pi} \left[-\frac{x^2}{2\pi} \cos(2\pi x) \right]_0^1 \end{aligned}$
 - $\therefore a_1 = -\frac{1}{\pi}$
- ... BY PARTS AGAIN ...

... = $-\frac{2}{\pi} \left\{ -\frac{1}{2\pi} (\cos(2\pi x)) + \frac{1}{4\pi^2} \left[\sin(2\pi x) \right]_0^1 \right\}$

$$\boxed{b_1 = \frac{1}{4\pi^2}}$$

- $b_1 = \frac{1}{\frac{1}{2}} \int_0^1 x^2 \sin(2\pi x) dx = \int_0^1 2x^2 \sin(2\pi x) dx$
 - ... BY PARTS ...
 - $\begin{aligned} &= \left[\frac{2x^2}{2\pi} \sin(2\pi x) \right]_0^1 + \frac{2}{2\pi} \int_0^1 x^2 \cos(2\pi x) dx \\ &= -\frac{1}{\pi} \cos(2\pi x) + \frac{2}{2\pi} \int_0^1 x^2 \cos(2\pi x) dx \end{aligned}$
 - ... BY PARTS AGAIN ...
 - $\begin{aligned} &= -\frac{1}{\pi} \cos(2\pi x) + \frac{2}{2\pi} \left[\frac{1}{2\pi} x^2 \sin(2\pi x) \right]_0^1 - \frac{1}{2\pi} \int_0^1 x^2 \sin(2\pi x) dx \\ &= -\frac{1}{\pi} \cos(2\pi x) + \frac{1}{2\pi} \left[-\frac{x^2}{2\pi} \cos(2\pi x) \right]_0^1 \end{aligned}$
 - $\therefore b_1 = -\frac{1}{\pi}$
- ... BY PARTS AGAIN ...

$\therefore x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2 \pi^2} \cos(2n\pi x) - \frac{1}{n\pi} \sin(2n\pi x) \right]$

Now built an "even" extension

$f(x) = x^2, -1 \leq x \leq 1, L = 1$

- $a_0 = \frac{1}{1} \int_{-1}^1 x^2 dx = \int_0^1 2x^2 dx = \left[\frac{2}{3}x^3 \right]_0^1 = \frac{2}{3}$
- $b_0 = 0, \forall n \neq 0$ (odd functions)
- $a_n = \frac{1}{1} \int_{-1}^1 x^2 \cos(n\pi x) dx = \int_0^1 2x^2 \cos(n\pi x) dx$
 - ... BY PARTS ...
 - $\begin{aligned} &= \left[\frac{2x^2}{n\pi} \sin(n\pi x) \right]_0^1 + \frac{2}{n\pi} \int_0^1 x^2 \sin(n\pi x) dx \\ &= -\frac{2}{n\pi} \cos(n\pi x) + \frac{2}{n\pi} \int_0^1 x^2 \sin(n\pi x) dx \end{aligned}$
 - ... BY PARTS AGAIN ...
 - $\begin{aligned} &= -\frac{2}{n\pi} \cos(n\pi x) + \frac{2}{n\pi} \left[\frac{1}{2\pi} x^2 \sin(n\pi x) \right]_0^1 - \frac{2}{n\pi} \int_0^1 x^2 \sin(n\pi x) dx \\ &= -\frac{2}{n\pi} \cos(n\pi x) + \frac{1}{n\pi} \left[-\frac{x^2}{2\pi} \cos(n\pi x) \right]_0^1 \end{aligned}$
 - $\therefore a_n = \frac{4}{n\pi} (-1)^n$
- ... BY PARTS AGAIN ...

$\therefore x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} (-1)^n \cos(n\pi x)$

Question 11

$$f(x) = \begin{cases} \pi - x & 0 \leq x \leq \pi \\ \pi + x & -\pi < x \leq 0 \end{cases}$$

for $x \in \mathbb{R}$, $f(x) = f(x+2\pi)$.

- a) Determine the Fourier series expansion of $f(x)$.

- b) Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

- c) Show that

$$\sum_{n=0}^{\infty} \left[\frac{\sin \frac{n\pi}{2} - \cos \frac{n\pi}{2}}{(2n+1)^2} \right] = -\frac{\pi^2}{8\sqrt{2}}$$

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\cos[(2n-1)x]}{(2n-1)^2} \right], \quad \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

a) $\hat{f}(0) = \frac{\pi}{2}$

As $f(0)$ is given, there will be no sine present as all $b_n = 0$.

$a_0 = \frac{1}{\pi} \int_0^\pi f(x) dx = \dots = \frac{2}{\pi} \int_0^\pi (\pi - x) dx = \frac{2}{\pi} \left[\pi x - \frac{1}{2}x^2 \right]_0^\pi = \frac{2}{\pi} \times \frac{\pi^2}{2} = \frac{\pi^2}{2}$

$a_n = \frac{1}{\pi} \int_0^\pi f(x) \cos nx dx = \dots = \frac{1}{\pi} \int_0^\pi (\pi - x) \cos nx dx$

$$= \frac{1}{\pi} \int_0^\pi (\pi - x) \cos nx dx = \dots = \text{by parts}$$

$$= \frac{2}{\pi} \left[\frac{1}{n} (\pi - x) \sin nx + \frac{1}{n^2} \int_0^\pi \sin nx dx \right]_0^\pi = \frac{2}{\pi n} \left[(-\cos nx) \right]_0^\pi = \frac{2}{\pi n} \left[(-\cos 0) - (-\cos \pi) \right] = \frac{2}{\pi n} \left[1 - (-1)^n \right]$$

$$= \frac{4}{\pi n} \quad \text{if } n \in \mathbb{O} \\ = 0 \quad \text{if } n \in \mathbb{E}$$

$$a_n = \frac{4}{\pi (2n+1)^2} \quad n \in \mathbb{N}$$

Thus $\hat{f}(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{(2n+1)x}{2} + b_n \sin \frac{(2n+1)x}{2} \right]$

$$\hat{f}(0) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi (2n+1)^2} \cos(2n+1)0 = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n+1)0)}{(2n+1)^2}$$

b) $\hat{f}(0) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n+1)0)}{(2n+1)^2}$

$$\hat{f}(0) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$$

$$\frac{\pi}{2} = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = -\frac{\pi^2}{8}$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{7^2}$$

Let $2n-1 \rightarrow 2n$ $\hat{f}(2n) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n+1)2n)}{(2n+1)^2}$

Thus $\hat{f}(2n) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n+1)(2n+1)2n)}{(2n+1)^2}$

$$\frac{2\pi}{2} = \frac{2\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n+1)(2n+1)2n)}{(2n+1)^2}$$

$$\frac{\pi}{2} = \frac{2\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n^2\pi + 2n\pi^2)}{(2n+1)^2}$$

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{\cos(2n^2\pi + 2n\pi^2)}{(2n+1)^2}$$

Add 50 terms from 2n = 1 to 50 $n = 50 + 1 = 51$

$$2n-1 \rightarrow 2(50+1)-1 = 101$$

$$\cos \frac{2\pi}{2} = \cos(2(50+1)\pi) = \cos(101\pi) = -\cos \pi = -\cos \pi$$

$$\sin \frac{2\pi}{2} = \sin(2(50+1)\pi) = \sin(101\pi) = \sin \pi = \sin \pi$$

$$\therefore -\sin \frac{2\pi}{2} + \cos \frac{2\pi}{2} = -\frac{\pi^2}{8}$$

$$\frac{2\pi}{2} = \frac{-\pi^2}{8} \quad \text{As } 2\pi^2 > 0$$

Question 12

The periodic function f is defined as

$$f(t) = \begin{cases} 0 & -1 \leq t < 0 \\ t^2 & 0 \leq t \leq 1 \end{cases}$$

for $t \in \mathbb{R}$, $f(t) = f(t+2)$.

Determine the Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{6} + \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n \times 2 \cos(n\pi t)}{n^2 \pi^2} + \left[\frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3 \pi^3} [(-1)^n - 1] \right] \sin(n\pi t) \right\}$$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$

where $a_0 = \frac{1}{L} \int_a^{a+2L} f(x) dx$, $a_n = \frac{1}{L} \int_a^{a+2L} f(x) \cos \frac{n\pi x}{L} dx$, $b_n = \frac{1}{L} \int_a^{a+2L} f(x) \sin \frac{n\pi x}{L} dx$, $L = 1$, $n = 1, 2, 3, 4, \dots$

$f(t) = \begin{cases} 0 & -1 \leq t < 0 \\ t^2 & 0 \leq t \leq 1 \end{cases}$ period is 2, half period L is 1

$\bullet a_0 = \frac{1}{1} \int_0^1 f(t) dt = \frac{1}{1} \int_0^1 t^2 dt = \left[\frac{t^3}{3} \right]_0^1 = \frac{1}{3}$

$\bullet a_n = \frac{1}{1} \int_0^1 f(t) \cos \frac{n\pi t}{1} dt = \frac{1}{1} \int_0^1 f(t) \cos nt dt = \int_0^1 t^2 \cos nt dt \dots$ by parts

$= \left[\frac{t^2}{n} \cos nt \right]_0^1 - \frac{2}{n} \int_0^1 t \cos nt dt = -\frac{2}{n} \int_0^1 t \cos nt dt \dots$ by parts

$= -\frac{2}{n^2} \left[t \cos nt \right]_0^1 + \frac{2}{n^2} \int_0^1 \cos nt dt = -\frac{2}{n^2} \left[t \cos nt \right]_0^1 + \frac{2}{n^3} \int_0^1 \cos nt dt$

$\bullet b_n = \frac{1}{1} \int_0^1 f(t) \sin \frac{n\pi t}{1} dt = \frac{1}{1} \int_0^1 f(t) \sin nt dt = \int_0^1 t^2 \sin nt dt$

integration by parts

$+t^2 \quad | \quad nt$
 $- \frac{1}{n} \cos nt \quad | \quad \sin nt$

$= \left[-\frac{1}{n} t^2 \cos nt \right]_0^1 + \frac{2}{n} \int_0^1 t \cos nt dt = -\frac{1}{n} \cos nt + \frac{2}{n^2} \int_0^1 \cos nt dt$

$= \frac{1}{n^2} (-1)^n + \frac{2}{n^3} \int_0^1 \cos nt dt$

by parts again

$\frac{t}{n} \quad | \quad 1$
 $\frac{1}{n^2} \cos nt \quad | \quad \sin nt$

Thus $f(t) = \frac{1}{6} + \sum_{n=1}^{\infty} \left[\frac{-2}{n^2} (-1)^n \cos nt + \left(\frac{1}{n^2} (-1)^n + \frac{2}{n^3} [(-1)^n - 1] \right) \sin nt \right]$

Question 13

$$f(x) = x, \quad x \in \mathbb{R}, \quad 0 \leq x \leq 2\pi.$$

$$f(x) = f(x + 2\pi).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}.$$

$$f(x) = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$$

(a) Given $f(x) = \frac{x}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{nx\pi}{2} + b_n \sin \frac{nx\pi}{2}$. Then $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$, $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$, $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$.

Here period $T = 2\pi$, $\frac{2\pi}{T} = 1$, $\frac{2}{\pi} = \frac{1}{\pi}$.

- $a_0 = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{1}{\pi} \left[x^2 \right]_0^{\pi} = \frac{1}{\pi} (\pi^2 - 0) = \pi$
- $a_n = \frac{1}{\pi} \int_0^{\pi} x \cos nx dx = \dots \text{parts} = \frac{1}{\pi} \left[\frac{1}{n} x \sin nx \right]_0^{\pi} - \frac{1}{\pi} \int_0^{\pi} \sin nx dx = 0$
- $b_n = \frac{1}{\pi} \int_0^{\pi} x \sin nx dx = \dots \text{parts} = \frac{1}{\pi} \left[-\frac{1}{n} x \cos nx \right]_0^{\pi} + \frac{1}{\pi} \int_0^{\pi} \cos nx dx = -\frac{1}{\pi n} [\pi - 0] = \frac{-1}{n}$

Thus $f(x) = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{2} + 2\pi \frac{\sin x}{2} + \frac{\sin 3x}{3} + \dots$

(b) Let $x = \frac{\pi}{2}$. Then $\frac{\pi}{2} = \pi - 2 \sum_{n=1}^{\infty} \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right]$
 $\frac{\pi}{2} = \pi - 2 \sum_{k=1}^{20} \frac{(-1)^{2k+1}}{2k-1}$
 $\frac{\pi}{2} = 2 \sum_{k=1}^{20} \frac{(-1)^{2k+1}}{2k-1}$

Question 14

A function $f(x)$ is defined in an interval $(-\pi, \pi)$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-\pi, \pi)$, giving general expressions for the coefficients of the series.

- b) Find the Fourier series of

$$f(x) = 3x^2 - \pi^2, -\pi \leq x \leq \pi.$$

- c) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

$$3x^2 - \pi^2 = 12 \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos nx}{n^2} \right], \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

a) If $f(x)$ is piecewise continuous on $(-\pi, \pi)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$\text{where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n = 1, 2, 3, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n = 1, 2, 3, \dots$$

b) $\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} (3x^2 - \pi^2) dx$ (writing is $\int_{-\pi}^{\pi}$)

$$\therefore [b_n = 0] \text{ for all } n \text{ since } \frac{1}{\pi} \int_{-\pi}^{\pi} (3x^2 - \pi^2) \sin nx dx = 0$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} 3x^2 - \pi^2 dx = \frac{2}{\pi} \int_0^{\pi} 3x^2 - \pi^2 dx \\ &= \frac{2}{\pi} \left[x^3 - \pi x^2 \right]_0^{\pi} = \frac{2}{\pi} \left[(\pi^3 - \pi^3) - (0 - 0) \right] = 0 \end{aligned}$$

$$\therefore [a_0 = 0]$$

Finally,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (3x^2 - \pi^2) \cos nx dx, \dots \text{ by parts} \dots$$

$$a_n = \frac{1}{\pi} \left\{ \left[\frac{1}{n} (3x^2 - \pi^2) \sin nx \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 6x \sin nx dx \right\}$$

$$a_n = - \frac{6}{\pi n} \int_{-\pi}^{\pi} x \sin nx dx, \dots \text{ after integrating} \dots$$

$$a_n = \frac{6}{\pi n} \int_0^{\pi} x \sin nx dx, \dots \text{ by parts again} \dots$$

| | | |
|------------------------------|------------------------------|-----------|
| $3x^2 - \pi^2$ | \int_0^{π} | a_n |
| $\frac{d}{dx}(3x^2 - \pi^2)$ | $\frac{d}{dx}(\int_0^{\pi})$ | $\cos nx$ |

| | | |
|-----------------|----------------------------|------------|
| x | \int_0^{π} | a_n |
| $\frac{d}{dx}x$ | $\frac{d}{dx}\int_0^{\pi}$ | $-\sin nx$ |

$$a_n = \frac{12}{\pi n} \left\{ \left[\frac{3x \cos nx}{n} \right]_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx dx \right\}$$

$$a_n = \frac{12}{\pi n} \left[\frac{\pi \cos \pi n}{n} - 0 \right]$$

$$a_n = \frac{12 \pi}{\pi^2 n} (-1)^n$$

$$\therefore f(x) = 3x^2 - \pi^2 = \sum_{n=1}^{\infty} a_n \cos nx$$

$$3x^2 - \pi^2 = \sum_{n=1}^{\infty} \frac{12 \pi}{\pi^2 n} (-1)^n \cos nx$$

$$\text{or } 3x^2 - \pi^2 = \frac{12}{\pi^2} a_0 + \frac{12}{\pi} a_1 \cos x + \frac{12}{\pi} a_2 \cos 2x + \dots$$

c) If $x = 0$

$$-x^2 = \sum_{n=1}^{\infty} \frac{12 \pi}{\pi^2 n} (-1)^n \cos^n x$$

$$-x^2 = \sum_{n=1}^{\infty} \frac{12 \pi}{\pi^2 n} (-1)^n$$

$$\frac{-x^2}{x^2} = \sum_{n=1}^{\infty} \frac{-12 \pi}{\pi^2 n}$$

$$\frac{12 \pi}{\pi^2} = \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\frac{12}{\pi} = \sum_{n=1}^{\infty} \frac{1}{n}$$

Question 15

$$f(x) = |x|, \quad x \in \mathbb{R}, \quad -\pi \leq x \leq \pi.$$

$$f(x) = f(x + 2\pi).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]}{(2n-1)^2}, \quad \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

a)

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$ with $a_n = \frac{1}{L} \int_0^L f(x) dx$

(L = HALF PERIOD)

$a_n = \frac{1}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$

$b_n = \frac{1}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

here $L = \text{HALF PERIOD} = \pi$

$\frac{a_0}{L} = \frac{1}{\pi} \int_0^\pi |x| dx = \frac{1}{\pi} \times 2 \int_0^\pi x dx = \frac{2}{\pi} \times \frac{1}{2} x^2 \Big|_0^\pi = \frac{2}{\pi}$

$a_n = \frac{1}{\pi} \int_0^\pi |x| \cos nx dx = \frac{1}{\pi} \int_0^\pi x \cos nx dx = \frac{1}{\pi} \left[x \cos nx \Big|_0^\pi - \int_0^\pi x \cos nx dx \right] = \frac{1}{\pi} \left[0 - \int_0^\pi x \cos nx dx \right] = \frac{1}{\pi} \int_0^\pi x \cos nx dx = \frac{1}{\pi} \left[x \sin nx \Big|_0^\pi + \int_0^\pi x \sin nx dx \right] = \frac{1}{\pi} \left[0 + \int_0^\pi x \sin nx dx \right] = \frac{1}{\pi} \left[-\frac{x}{n} \cos nx \Big|_0^\pi \right] = -\frac{1}{n\pi} \left[\cos n\pi - \cos 0 \right] = -\frac{1}{n\pi} [(-1)^n - 1] = \frac{1}{n\pi} [1 - (-1)^n]$

$b_n = \frac{1}{\pi} \int_0^\pi |x| \sin nx dx = \frac{1}{\pi} \int_0^\pi x \sin nx dx = \frac{1}{\pi} \left[x \cos nx \Big|_0^\pi - \int_0^\pi x \cos nx dx \right] = \frac{1}{\pi} \left[0 - \int_0^\pi x \cos nx dx \right] = \frac{1}{\pi} \left[-\frac{x}{n} \sin nx \Big|_0^\pi \right] = -\frac{1}{n\pi} \left[\sin n\pi - \sin 0 \right] = -\frac{1}{n\pi} [0 - 0] = 0$

let $n = 2m-1, \quad m \in \mathbb{N}$

$a_{2m-1} = \frac{1}{(2m-1)\pi}$

$b_n = \frac{1}{\pi} \int_0^\pi |x| \sin nx dx = 0 \quad \Rightarrow \text{INTERMID IS 0 IN A SYMMETRICAL DIRECTION}$

thus

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]}{(2n-1)^2}$

I.E. $|x| = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \dots \right]$

b)

let $x = 0$

$0 = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right]$

$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{2}$

$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{16}$

Question 16

$$f(x) = x, \quad x \in \mathbb{R}, \quad -1 \leq x \leq 1.$$

$$f(x) = f(x+2).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n}.$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n \pi x}{n},$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n} = \frac{1}{2}$$

(a) $f(x) = \frac{x}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{1} + b_n \sin \frac{n\pi x}{1}$

$$a_0 = \frac{2}{\pi} \int_0^1 x dx$$

$$a_n = \frac{2}{\pi} \int_0^1 x \cos \frac{n\pi x}{1} dx$$

$$b_n = \frac{2}{\pi} \int_0^1 x \sin \frac{n\pi x}{1} dx$$

• $a_0 = \frac{2}{\pi} \int_0^1 x dx = 0$ (COULD HAVE USED AN INTEGRAL FROM -1 TO 1)

• $a_n = \frac{2}{\pi} \int_0^1 x \cos nx dx = 0$ (COULD HAVE USED AN INTEGRAL FROM -1 TO 1)

• $b_1 = \frac{2}{\pi} \int_0^1 x \sin \pi x dx = \int_0^1 x \sin \pi x dx = \dots$ by parts ... $\int_0^1 x \sin \pi x dx = 2 \left[-\frac{1}{\pi} \sin \pi x \right]_0^1 + \frac{1}{\pi} \int_0^1 \cos \pi x dx$

$$= 2 \left[-\frac{1}{\pi} \cos \pi x \right]_0^1 + \frac{1}{\pi} \int_0^1 \cos \pi x dx = 2 \left(-\frac{1}{\pi} (-1)^{n+1} \right) = \frac{2}{\pi} (-1)^{n+1}$$

Thus $a_1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \pi x = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Given $f(x) = x, \quad x \in \mathbb{C} \setminus \{1\}$

 $T=2, \quad \frac{2}{1}=1, \quad 2\pi \frac{1}{1} = \pi$

Question 17

$$f(x) = x^2, \quad x \in \mathbb{R}, \quad -2 \leq x \leq 2.$$

$$f(x) = f(x+4).$$

Determine the Fourier series expansion of $f(x)$.

$$f(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{1}{2}n\pi x\right)$$

$f(x) = x^2, \quad -2 \leq x \leq 2. \quad f(x) = f(x+4)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi x}{2}\right) + B_n \sin\left(\frac{n\pi x}{2}\right) \right], \quad L = \text{HALF PERIOD}$$

$$a_n = \frac{1}{L} \int_a^{a+2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, 3, 4, \dots$$

$$b_n = \frac{1}{L} \int_a^{a+2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, 4, \dots$$

Now, $f(x)$ is even, so b_n are all zero. As $\sin\frac{n\pi x}{2}$ is odd.

Evaluating, $a_0 = 4$ & $L = 2$.

$$\Rightarrow a_0 = \frac{1}{2} \int_{-2}^2 x^2 dx = \int_0^2 x^2 dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$$

$$\Rightarrow a_1 = \frac{1}{2} \int_{-2}^2 x^2 \cos\left(\frac{n\pi x}{2}\right) dx = \int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right) dx$$

$$\begin{aligned} &\left. \frac{\partial^2}{\partial x^2} \cos\left(\frac{n\pi x}{2}\right) \right|_0^2 \\ &= \frac{2x^2}{\frac{n^2\pi^2}{4}} \Big|_0^2 = \frac{8}{n^2\pi^2} \end{aligned}$$

$$\Rightarrow a_2 = \frac{2}{n^2\pi^2} \int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right) dx = \frac{4}{n^2\pi^2} \int_0^2 x^2 dx = \frac{16}{n^2\pi^2}$$

$$\Rightarrow a_3 = -\frac{4}{n^2\pi^2} \int_0^2 x^2 dx = -\frac{16}{n^2\pi^2}$$

$$\Rightarrow a_n = -\frac{4}{n^2\pi^2} \left\{ -\frac{2}{n\pi} \left[2 \cos\left(\frac{n\pi x}{2}\right) \right]_0^2 + \frac{2}{n\pi} \int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right) dx \right\}$$

$$\Rightarrow a_1 = \frac{8}{n^2\pi^2} (2 \cos(n\pi)) - \frac{8}{n^2\pi^2} \int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right) dx.$$

$$\Rightarrow a_2 = \frac{16(-1)^2}{n^2\pi^2} - \frac{8}{n^2\pi^2} \times \frac{2}{n\pi} \int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right) dx$$

$$\Rightarrow a_3 = \frac{16(-1)^3}{n^2\pi^2}$$

Find the Fourier series of $f(x) = x^2, f(x) = f(x+4), -2 \leq x \leq 2$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{(2n-1)^2}{n^2\pi^2} \cos\left(\frac{(2n-1)\pi x}{2}\right) \right]$$

$$f(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{2}\right) \right]$$

Question 18

A function $f(x)$ is defined in the interval $(-\pi, \pi)$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-\pi, \pi)$, giving general expressions for the coefficients of the series.

- b) Find the Fourier series of

$$f(x) = x, -\pi \leq x \leq \pi.$$

- c) Hence determine the exact value of

$$g(x) = x^2, -\pi \leq x \leq \pi.$$

$$f(x) = 2 \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} \sin nx}{n} \right], \quad g(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos nx}{n^2} \right]$$

a) If $f(x)$ is piecewise continuous on $(-\pi, \pi)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Hence $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$

$n = 1, 2, 3, 4, \dots$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n = 1, 2, 3, 4, \dots$$

b) Now $f(x) = x, -\pi < x < \pi$ is odd

• Hence $a_0 = 0$. As the integrand will be odd in a symmetric interval

• $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx$

... By parts ...

$$= \frac{2}{\pi} \left[\left[-x \cos nx \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} x \cos nx dx \right] = \frac{2}{\pi n} \left[x \cos nx \right]_0^{\pi}$$

$$= \frac{2}{\pi n} \left[0 - (-\pi) \cos \pi \right] = \frac{2}{\pi n} \times (-\pi) (-1) = \frac{2}{n} (-1)^{n+1}$$

$$\text{Hence } f(x) = x = \frac{2}{\pi} \sin 2x - \frac{2}{\pi} \sin 4x + \frac{2}{3} \sin 3x - \frac{2}{\pi} \sin 6x + \dots$$

$$f(x) = x = 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \frac{1}{5} \sin 5x \dots \right)$$

$$f(x) = x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$$\frac{d}{dx}(f(x)) = 2x$$

$$\frac{d}{dx}(f(x)) = 2f(x)$$

$$g(x) = 2 \int_0^x f(t) dt$$

$$g(x) = 2 \int_0^x \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots \right) dx$$

$$g(x) = 2x \left(-\cos x + \frac{1}{2} \cos 2x - \frac{1}{3} \cos 3x + \frac{1}{4} \cos 4x - \dots \right) + C$$

$$g(x) = -x^2 \left[\cos x - \frac{1}{4} \cos 2x + \frac{1}{9} \cos 3x - \frac{1}{16} \cos 4x \dots \right] + C$$

To find the constant

• Check at $x=0$

$$0 = -4 \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{16} + \dots \right] + C$$

$$0 = -4 \times \frac{11}{12} + C$$

$$0 = -\frac{11}{3} + C$$

$$C = \frac{11}{3}$$

• Or directly

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{3\pi} \int_0^{\pi} 3x^2 dx$$

$$= \frac{2}{3\pi} \times \frac{\pi^2}{3} = \frac{\pi^2}{3}$$

$$\therefore \frac{a_0}{2} = \frac{\pi^2}{3}$$

• Hence $g(x) = x^2 - \frac{\pi^2}{3} + \left[\cos x - \frac{1}{2} \cos 2x + \frac{1}{3} \cos 3x - \frac{1}{4} \cos 4x \dots \right]$

$$g(x) = x^2 - \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx$$

Question 19

$$f(x) = x^2, \quad x \in \mathbb{R}, \quad 0 \leq x \leq 1.$$

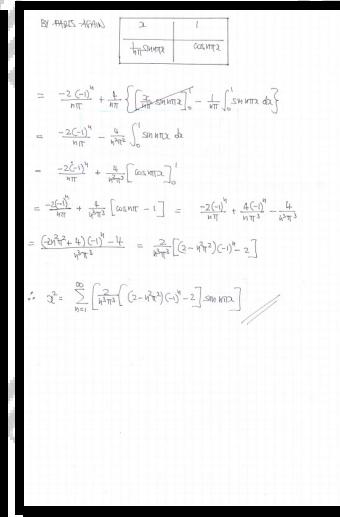
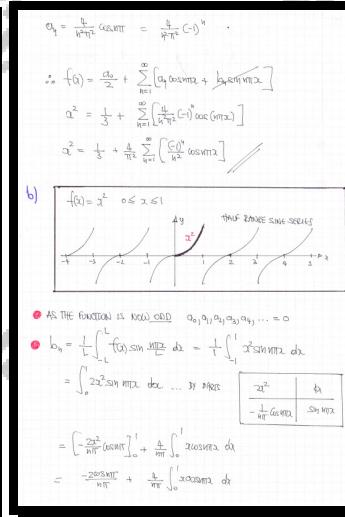
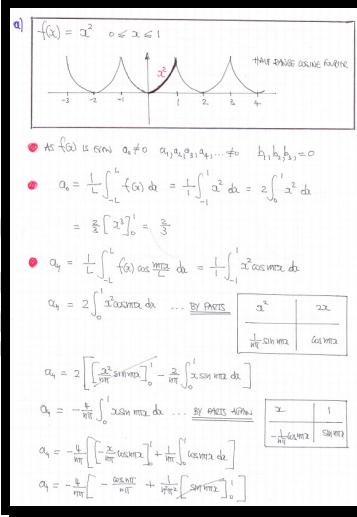
Determine the Fourier series of $f(x)$ as

a) ... as half range cosine expansion.

b) ... as half range sine expansion.

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos(n\pi x) \right],$$

$$f(x) = \frac{2}{\pi^3} \sum_{n=1}^{\infty} \left[\frac{2}{n^3} \left[(2 - n^2\pi^2)(-1)^n - 2 \right] \sin(n\pi x) \right]$$



Question 20

$$f(x) = \begin{cases} \pi - x & 0 \leq x \leq \pi \\ 0 & \pi \leq x \leq 2\pi \end{cases}, \quad x \in \mathbb{R}.$$

$$f(x) = f(x + 2\pi).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

$$f(x) = \frac{\pi}{4} + \sum_{m=1}^{\infty} \left[\frac{2\cos[(2m-1)x]}{\pi(2m-1)^2} + \frac{\sin mx}{m} \right], \quad \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

Worked Solution:

Given $f(x) = \begin{cases} \pi - x & 0 \leq x \leq \pi \\ 0 & \pi \leq x \leq 2\pi \end{cases}$, $L = \pi$ (Half Period)

$\hat{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{(2n-1)\pi x}{L} + b_n \sin \frac{(2n-1)\pi x}{L}]$

$a_n = \frac{1}{\pi} \int_0^{\pi} f(x) \cos \frac{(2n-1)\pi x}{\pi} dx = \begin{cases} 0 & n \text{ is even} \\ \frac{1}{\pi} \int_0^{\pi} (\pi - x) \cos((2n-1)x) dx & n \text{ is odd} \end{cases}$

$b_n = \frac{1}{\pi} \int_0^{\pi} f(x) \sin \frac{(2n-1)\pi x}{\pi} dx$

$a_0 = \frac{1}{\pi} \int_0^{\pi} \pi dx = \frac{1}{\pi} [\pi x]_0^{\pi} = \pi$

$a_1 = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \cos x dx + \frac{1}{\pi} \int_0^{\pi} x \cos x dx$

$= \frac{1}{\pi} \left\{ \left[\pi \sin x \right]_0^{\pi} + \left[x \sin x \right]_0^{\pi} \right\}$

$= \frac{1}{\pi} \left\{ \left[\pi \sin \pi \right] - \left[\pi \sin 0 \right] + \left[\pi x \sin x \right]_0^{\pi} \right\}$

$= \frac{1}{\pi} \left\{ 0 - 0 + \left[\pi x \sin x \right]_0^{\pi} \right\}$

$= \frac{1}{\pi} \left\{ \pi \pi \sin \pi \right\} = -\pi$

$a_2 = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \cos 2x dx + \frac{1}{\pi} \int_0^{\pi} x \cos 2x dx$

$= \frac{1}{\pi} \left\{ \left[\frac{1}{2}(\pi - x) \sin 2x \right]_0^{\pi} + \left[\frac{1}{2}x \sin 2x \right]_0^{\pi} \right\}$

$= \frac{1}{\pi} \left\{ \left[\frac{1}{2}(\pi - \pi) \sin 2\pi \right] - \left[\frac{1}{2}\pi \sin 0 \right] \right\}$

$= \frac{1}{\pi} \left\{ 0 - \frac{1}{2}\pi \right\} = -\frac{1}{2}\pi$

$a_3 = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \cos 3x dx + \frac{1}{\pi} \int_0^{\pi} x \cos 3x dx$

$= \frac{1}{\pi} \left\{ \left[\frac{1}{3}(\pi - x) \sin 3x \right]_0^{\pi} + \left[\frac{1}{3}x \sin 3x \right]_0^{\pi} \right\}$

$= \frac{1}{\pi} \left\{ \left[\frac{1}{3}(\pi - \pi) \sin 3\pi \right] - \left[\frac{1}{3}\pi \sin 0 \right] \right\}$

$= \frac{1}{\pi} \left\{ 0 - 0 \right\} = 0$

$a_4 = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \cos 4x dx + \frac{1}{\pi} \int_0^{\pi} x \cos 4x dx$

$= \frac{1}{\pi} \left\{ \left[\frac{1}{4}(\pi - x) \sin 4x \right]_0^{\pi} + \left[\frac{1}{4}x \sin 4x \right]_0^{\pi} \right\}$

$= \frac{1}{\pi} \left\{ \left[\frac{1}{4}(\pi - \pi) \sin 4\pi \right] - \left[\frac{1}{4}\pi \sin 0 \right] \right\}$

$= \frac{1}{\pi} \left\{ 0 - 0 \right\} = 0$

\vdots

Worked Solution:

$\hat{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{(2n-1)\pi x}{\pi} + b_n \sin \frac{(2n-1)\pi x}{\pi}]$

$\hat{f}(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{2\cos[(2n-1)x]}{\pi(2n-1)^2} + \frac{\sin mx}{m} \right]$

Let $x = \pi$

$\hat{f}(\pi) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{2\cos[(2n-1)\pi]}{\pi(2n-1)^2} + \frac{\sin m\pi}{m} \right]$

$0 = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2(-1)}{(2n-1)^2} + \frac{\sin m\pi}{m}$

$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Question 21

$$f(x) = x^2, \quad x \in \mathbb{R}, \quad -\pi \leq x \leq \pi.$$

$$f(x) = f(x + 2\pi).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

(a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{2n\pi x}{T} + b_n \sin \frac{2n\pi x}{T})$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos \frac{2n\pi x}{T} dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin \frac{2n\pi x}{T} dx$$

$$T = 2\pi$$

$$\frac{2\pi}{T} = n$$

$$\frac{2\pi}{\pi} = n$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{1}{3}x^3 \right]_0^{\pi} = \frac{2}{\pi} \times \frac{1}{3}\pi^3 = \frac{2\pi^2}{3}$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos x dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos x dx \quad \text{parts...} \quad \frac{2}{\pi} \left[\left[x^2 \sin x \right]_0^{\pi} - \frac{2}{3} \int_0^{\pi} x \sin x dx \right]$$

$$= \frac{2}{\pi} \left[\frac{1}{3}x^3 \sin x \right]_0^{\pi} - \frac{2}{\pi} \left[\left[x \sin x \right]_0^{\pi} + \frac{1}{2} \int_0^{\pi} \sin x dx \right]$$

$$= -\frac{4}{3\pi} \left[\frac{1}{3}x^3 \cos x \right]_0^{\pi} = -\frac{4}{9\pi} \cos \pi = -\frac{4}{9}(-1)^3$$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin x dx = 0 \quad (\text{odd function in the interval is a symmetric domain})$$

$$\therefore Q^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos x = \frac{\pi^2}{3} + \pi^2 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$

$$= \frac{\pi^2}{3} + 4 \left[\frac{\cos x}{1} + \frac{\cos 2x}{4} - \frac{\cos 3x}{9} + \frac{\cos 4x}{16} - \dots \right]$$

(b) If $x=0$

$$Q^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$Q^2 = \frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

Question 22

$$f(x) = x^2, \quad x \in \mathbb{R}, \quad 0 \leq x \leq 2\pi.$$

$$f(x) = f(x + 2\pi).$$

a) Determine the Fourier series expansion of $f(x)$.

b) Hence determine the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

$$f(x) = \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left[\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n} \right], \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

(L = HALF PERIOD)

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

where $a_n = \frac{1}{L} \int_L f(x) dx$, $b_n = \frac{1}{L} \int_L f(x) \sin \frac{n\pi x}{L} dx$

HERE L = HALF PERIOD

$\frac{1}{2}\pi x^2 + \frac{1}{2}\pi x^2 = \pi x^2$

$f(x) = x^2$

- $a_0 = \frac{1}{\pi} \int_0^{2\pi} x^2 dx = \frac{1}{3\pi} \left[x^3 \right]_0^{2\pi} = \frac{8\pi^3}{3}$
- $a_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx dx \dots$ BY PARTS ... $\begin{aligned} &= \frac{1}{\pi} \left[\frac{x^2}{n} \sin nx \right]_0^{2\pi} + \frac{2}{\pi} \int_0^{2\pi} x \sin nx dx \\ &= -\frac{2}{\pi n} \int_0^{2\pi} x \sin nx dx \dots \text{BY PARTS AGAIN} \dots \\ &= -\frac{2}{\pi n} \left\{ -\frac{x^2}{n^2} \sin nx \Big|_0^{2\pi} + \frac{2}{n} \int_0^{2\pi} \sin nx dx \right\} \\ &= -\frac{2}{\pi n} \left\{ -\frac{4\pi^2}{n^3} \sin 2\pi + \frac{2}{n} \int_0^{2\pi} \sin nx dx \right\} \\ &= -\frac{2}{\pi n} \left(-\frac{4\pi^2}{n^3} + \frac{2}{n} \int_0^{2\pi} \sin nx dx \right) \end{aligned}$
- $b_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx dx \dots$ BY PARTS ... $\begin{aligned} &= \frac{1}{\pi} \left[\frac{-x^2}{n} \cos nx \right]_0^{2\pi} + \frac{2}{\pi} \int_0^{2\pi} x \cos nx dx \\ &= \frac{1}{n} \left(-\frac{4\pi^2}{n^2} \cos 2\pi + \frac{2}{n} \int_0^{2\pi} x \cos nx dx \right) \\ &= -\frac{4\pi^2}{n^3} + \frac{2}{n^2} \int_0^{2\pi} x \cos nx dx \\ &\dots \text{BY PARTS AGAIN} \dots \\ &= -\frac{4\pi^2}{n^3} + \frac{2}{n^2} \left\{ \frac{x^2}{n^2} \cos nx \Big|_0^{2\pi} - \frac{2}{n} \int_0^{2\pi} \cos nx dx \right\} \\ &= -\frac{4\pi^2}{n^3} - \frac{2}{n^3} \int_0^{2\pi} \cos nx dx \\ &= -\frac{4\pi^2}{n^3} + \frac{2}{n^3} \left[\sin nx \right]_0^{2\pi} \\ &= -\frac{4\pi^2}{n^3} \end{aligned}$

ANS

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{a_n}{n^2} \cos nx - \frac{b_n}{n^3} \sin nx \right]$$

b) Let $x = \pi$

$$\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\pi}{n^3} \cos n\pi$$

$$\frac{4\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^3} (-1)^n$$

$$\frac{4\pi^2}{3} = -\sum_{n=1}^{\infty} \frac{1}{n^3} (-1)^n$$

$$\frac{4\pi^2}{3} = \sum_{n=1}^{\infty} \frac{1}{n^3} (-1)^{n+1}$$

Question 23

It is given that for $x \in \mathbb{R}$, $-\pi \leq x \leq \pi$,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]}{(2n-1)^2}, \quad |x| = |x + 2\pi|.$$

- a) Use the above Fourier series expansion to deduce the Fourier series expansion of $\text{sgn}(x)$.
- b) Verify the answer of part (a) by obtaining directly the Fourier series expansion of $\text{sgn}(x)$.
- c) Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{2r-1}.$$

| | | |
|-------|------------------------------------------------------------------------------------|---------------------------------------------------------------|
| _____ | $\text{sgn}(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)x]}{(2n-1)^2}$ | $\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{2r-1} = \frac{\pi}{4}$ |
|-------|------------------------------------------------------------------------------------|---------------------------------------------------------------|

a) $|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]}{(2n-1)^2}$

$$\begin{aligned} \text{sgn}(x) &= \frac{d}{dx} [|x|] = \frac{d}{dx} \left[\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]}{(2n-1)^2} \right] \\ &= -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{-(2n-1) \sin[(2n-1)x]}{(2n-1)^2} \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)x]}{(2n-1)^2} \end{aligned}$$

b) NOW FIND THE FOURIER EXPANSION DIRECTLY

$$\begin{aligned} f(x) &= \text{Sign}(x), -\pi < x < \pi, \quad f(x+2\pi) = f(x) \\ \bullet a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{Sign}(x) dx = 0 \quad \text{ODD FUNCTION IN A SYMMETRICAL DOMAIN} \\ \bullet a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{Sign}(x) \cos(nx) dx = 0 \quad \text{ODD FUNCTION IN A SYMMETRICAL DOMAIN} \\ \bullet b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{Sign}(x) \sin(nx) dx \\ &= \frac{1}{\pi} \times 2 \int_{0}^{\pi} x \sin(nx) dx = \frac{2}{\pi} \left[-\frac{1}{n} \cos(nx) \right]_0^{\pi} \\ &= \frac{2}{\pi n} \left[\cos nx \right]_0^{\pi} = \frac{2}{\pi n} \left[1 - \cos \pi n \right] = \frac{2}{\pi n} \left[1 - (-1)^n \right] \end{aligned}$$

c) IF n IS EVEN

$$b_n = \frac{4}{\pi(2n-1)} \quad n = 2, 4, 6, \dots$$

HENCE WE REWRITE THE COEFFICIENT AS

$$b_m = \frac{4}{\pi(2m-1)}, m = 1, 2, 3, \dots$$

HENCE USE EASY SUBSTITUTION INTO THE FOURIER FORMULA

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \left(\frac{2\pi n}{\pi} x \right) + b_n \sin \left(\frac{2\pi n}{\pi} x \right) \right] \\ \text{sgn}(x) &= \sum_{n=1}^{\infty} \left[\frac{4}{\pi(2n-1)} \sin \left(\frac{2\pi n}{\pi} x \right) \right] \\ \text{sgn}(x) &= \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin \left(\frac{2\pi n}{\pi} x \right)}{(2n-1)} \right] \quad \text{REMEMBER} \end{aligned}$$

SUBSTITUTING $2x = \frac{2\pi}{\pi}x$ INTO THE ABOVE FORMULA GIVES

$$\begin{aligned} \text{sgn}(2x) &= \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)} \sin \left(\frac{2}{\pi} (2n-1) x \right) \right] \\ 1 &= \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)} (-1)^{2n-1} \right] \\ \sum_{n=1}^{\infty} \left[\frac{(-1)^{2n-1}}{2n-1} \right] &= \frac{\pi}{4} \\ \sum_{n=1}^{\infty} \left[\frac{(-1)^{2n}}{2n-1} \right] &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4} \end{aligned}$$

Question 24

$$f(x) = \begin{cases} -x & 0 \leq x \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = f(x + \pi).$$

Determine the Fourier series expansion of $f(x)$.

$$f(x) = -\frac{\pi}{8} + \sum_{n=1}^{\infty} \frac{\cos[(4n-2)x]}{(2n-1)^2 \pi} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n \sin 2nx}{n}$$

The derivation shows the calculation of coefficients a_n and b_n for the function $f(x) = -x$ on the interval $[0, \pi]$. The function is zero outside this interval. The period $T = \pi$ is noted, and the function is even ($f(-x) = f(x)$). The integral for a_0 is calculated as $\frac{1}{\pi} \int_0^\pi -x dx = -\frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^\pi = -\frac{\pi}{4}$. The integral for a_n is $a_n = \frac{2}{\pi} \int_0^\pi (-x) \cos(2nx) dx = -\frac{2}{\pi} \left[\frac{1}{2} x \cos(2nx) \right]_0^\pi + \frac{1}{\pi} \int_0^\pi x \sin(2nx) dx = -\frac{2}{\pi} \left[\frac{1}{2} x \cos(2nx) \right]_0^\pi + \frac{1}{\pi} \left[\frac{1}{2} \sin(2nx) \right]_0^\pi = -\frac{2}{\pi} \left[\frac{1}{2} x \cos(2n\pi) \right]_0^\pi = -\frac{2}{\pi} \left[\frac{1}{2} x \right]_0^\pi = -\frac{1}{\pi} x \Big|_0^\pi = -\frac{1}{\pi} \pi = -1$. The integral for b_n is $b_n = \frac{2}{\pi} \int_0^\pi (-x) \sin(2nx) dx = \frac{2}{\pi} \left[\frac{1}{2} x \sin(2nx) \right]_0^\pi - \frac{2}{\pi} \int_0^\pi x \cos(2nx) dx = \frac{1}{\pi} \left[x \sin(2nx) \right]_0^\pi - \frac{2}{\pi} \left[\frac{1}{2} x \cos(2nx) \right]_0^\pi = \frac{1}{\pi} \left[x \sin(2n\pi) \right]_0^\pi - \frac{2}{\pi} \left[\frac{1}{2} x \cos(2n\pi) \right]_0^\pi = \frac{1}{\pi} \left[x \cdot 0 \right]_0^\pi - \frac{2}{\pi} \left[\frac{1}{2} x \cdot 1 \right]_0^\pi = 0 - \frac{1}{\pi} x \Big|_0^\pi = 0 - \frac{1}{\pi} \pi = -1$. The final result is $f(x) = -\frac{\pi}{8} + \sum_{n=1}^{\infty} \left[(-1)^n \left(\frac{1}{2} \cos(2nx) + \frac{1}{2n} (-1)^n \sin(2nx) \right) \right]$.

Question 25

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
- b) Determine the Fourier series of

$$f(x) = e^x, -\pi \leq x \leq \pi.$$

$$e^x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n [\cos(nx) - n \sin(nx)]}{1+n^2} \right]$$

a) If $f(x)$ is piecewise continuous in $(-L, L)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

where

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n=1,2,3,\dots$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n=1,2,3,\dots$$

b) $f(x) = e^x, \quad x \in (-\pi, \pi)$

CONSIDER THE INTEGRAL

$$\frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{x+i(n\pi)} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{(1+in)x} dx =$$

$$= \frac{1-i}{\pi(1+n^2)} \int_{-\pi}^{\pi} e^{(1+in)x} - e^{-in x} dx$$

$$= \frac{1-i}{\pi(1+n^2)} \left[e^{(1+in)x} - e^{-in x} \right]_{-\pi}^{\pi}$$

$$= \frac{1-i}{\pi(1+n^2)} \left[e^{(1+in)\pi} - e^{-in\pi} - e^{(1+in)(-\pi)} + e^{-in(-\pi)} \right]$$

$$= \frac{1-i}{\pi(1+n^2)} \left[e^{(1+in)\pi} - e^{-in\pi} - e^{(1+in)(-\pi)} + e^{in(-\pi)} \right]$$

$$= \frac{1}{\pi(1+n^2)} \left[e^{(1+in)\pi} - e^{-in\pi} - e^{(1+in)(-\pi)} + e^{in(-\pi)} \right]$$

$$= \frac{1}{\pi(1+n^2)} (1-in) \left[e^{in\pi} (e^{i\pi} - e^{-i\pi}) + i e^{i\pi} \sin n\pi + i e^{-i\pi} \sin n\pi \right]$$

$$= \frac{1}{\pi(1+n^2)} (1-in) \left[2 \sinh n\pi + 2i \cosh n\pi \right]$$

$$= \frac{1}{\pi(1+n^2)} (1-in) \left[2 \sinh n\pi + 2i \cosh n\pi + 2i \cosh n\pi \right]$$

$$= \frac{1}{\pi(1+n^2)} (1-in) \left[2 \sinh n\pi + 2i \cosh n\pi + i[-2 \sinh n\pi] + 2i \cosh n\pi \right]$$

$$= \frac{2}{\pi(1+n^2)} \left[(-1)^n \sinh n\pi - i n (-1)^n \sinh n\pi \right]$$

$\therefore \frac{2(-1)^n \sinh n\pi}{\pi(1+n^2)}$

THUS

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} \left[e^x \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[e^{\pi} - e^{-\pi} \right] = \frac{1}{\pi} (2 \sinh \pi) = \frac{2}{\pi} \sinh \pi$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x (-in) \sin nx dx$$

TAKE THE REAL PART OF THE RESULT

$$a_0 = \frac{2(-1)^0 \sinh 0}{\pi(1+0^2)}$$

Similarly $b_0 = -\frac{2(-1)^0 \sinh 0}{\pi(1+0^2)}$

THUS

$$e^x = \frac{2}{\pi} \sinh \pi + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n \sinh n\pi}{\pi(1+n^2)} \cos nx - \frac{2(-1)^n \sinh n\pi}{\pi(1+n^2)} \sin nx \right]$$

$$e^x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{1+n^2} - \frac{\sinh \pi}{\pi(1+n^2)}$$

$$e^x = \frac{\sinh \pi}{\pi} \left[1 + 2 \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{1+n^2} (\cos nx - i \sin nx) \right] \right]$$

Question 26

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

- b) Show that

$$\int_{-\pi}^{\pi} e^{ax} e^{inx} dx = \frac{2(a-ni)(-1)^n}{a^2+n^2} \sinh(a\pi)$$

- c) Determine the Fourier series of

$$f(x) = e^{ax}, a > 0, -\pi \leq x \leq \pi.$$

- d) Hence find the Fourier series of $\cosh(ax)$ and $\sinh(ax)$, for $-\pi \leq x \leq \pi$.

$$e^{ax} = \frac{\sinh(a\pi)}{a\pi} + \frac{2\sinh(a\pi)}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n [a\cos(nx) - n\sin(nx)]}{a^2+n^2} \right],$$

$$\cosh(ax) = \frac{\sinh(a\pi)}{a\pi} + \frac{2a\sinh(a\pi)}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos(nx)}{a^2+n^2} \right],$$

$$\sinh(ax) = \frac{2\sinh(a\pi)}{\pi} \sum_{n=1}^{\infty} \left[\frac{n(-1)^{n+1} \sin(nx)}{a^2+n^2} \right]$$

a) If $f(x)$ is piecewise continuous on $[-L, L]$, then $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

where $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ and $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$.

b) $\int_{-\pi}^{\pi} e^{ax} e^{inx} dx = \int_{-\pi}^{\pi} e^{(a+in)x} dx = \left[\frac{e^{(a+in)x}}{a+in} \right]_{-\pi}^{\pi} = \frac{e^{-\pi(a-in)}}{a^2+n^2} \left[(a+in)e^{-\pi(a-in)} - e^{-\pi(a+in)} \right]$

$$= \frac{2(a-in)}{a^2+n^2} \sinh(a\pi) = \frac{2(a-in)}{a^2+n^2} \left[\sinh(a\pi) + i\cosh(a\pi) \right] = \frac{\pm 2(a-in)}{a^2+n^2} \left[\sinh(a\pi) \cosh(a\pi) + i\sinh(a\pi) \sinh(a\pi) \right]$$

$$= \frac{2(a-in)}{a^2+n^2} \left[\sinh^2(a\pi) \cos(a\pi) + i\sinh(a\pi) \sinh(a\pi) \right] = \frac{2(a-in)}{a^2+n^2} \left[(-1)^n \sinh^2(a\pi) \right]$$

c) $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[e^{ax} \right] dx = \frac{1}{a\pi} \int_{-\pi}^{\pi} \left[e^{ax} - e^{-ax} \right] dx = \frac{1}{a\pi} \int_{-\pi}^{\pi} \sinh(ax) dx$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[e^{ax} e^{inx} \right] dx = \dots \text{from (b)} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{2a(-1)^n \sinh(ax)}{a^2+n^2} dx = \frac{2a(-1)^n \sinh(ax)}{\pi(a^2+n^2)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[e^{ax} e^{-inx} \right] dx = \dots \text{from (b)} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{-2n(-1)^n \cosh(ax)}{a^2+n^2} dx = \frac{-2n(-1)^n \cosh(ax)}{\pi(a^2+n^2)}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{2a(-1)^n \sinh(ax)}{\pi(a^2+n^2)} - \frac{-2n(-1)^n \cosh(ax)}{\pi(a^2+n^2)} \right]$$

d) $e^{ax} = \frac{\sinh(a\pi)}{a\pi} + \frac{2\sinh(a\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} [\cosh(ax) - \sinh(ax)]$

$$e^{-ax} = \frac{\sinh(-a\pi)}{a\pi} + \frac{2\sinh(-a\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} [\cosh(ax) + \sinh(ax)]$$

$$\text{Add } e^{ax} + e^{-ax} = \frac{2\sinh(a\pi)}{a\pi} + \frac{2\sinh(-a\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} [\cosh(ax) + 2\sinh(ax)]$$

$$\frac{1}{2}(e^{ax} + e^{-ax}) = \frac{\sinh(a\pi)}{a\pi} + \frac{2\sinh(-a\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2}$$

$$\therefore \cosh(ax) = \frac{\sinh(a\pi)}{a\pi} + \frac{2\sinh(-a\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cosh(ax)}{a^2+n^2}$$

Simplify $e^{ax} - e^{-ax} = \frac{2\sinh(a\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} (-2\sinh(ax))$

$$\frac{1}{2}(e^{ax} - e^{-ax}) = \frac{2\sinh(a\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} (-4\sinh(ax))$$

$$\sinh(ax) = \frac{2\sinh(a\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sinh(ax)}{a^2+n^2}$$

Question 27

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

- b) Determine the Fourier series of

$$f(x) = e^x, -\pi \leq x \leq \pi.$$

- c) Hence find the Fourier series of $\sinh x$ and $\cosh x$, for $-\pi \leq x \leq \pi$.

$$e^x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n [\cos(nx) - n \sin(nx)]}{1+n^2} \right],$$

$$\sinh x = \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{n(-1)^{n+1} \sin(nx)}{1+n^2} \right],$$

$$\cosh x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^n \cos(nx)}{1+n^2} \right],$$

a) If $f(x)$ is piecewise continuous in $(-L, L)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

$$\text{where } a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n=0, 1, 2, \dots$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n=1, 2, \dots$$

Integration:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} \left[e^x \right]_{-\pi}^{\pi} = \frac{1}{\pi} [e^\pi - e^{-\pi}]$$

$$a_0 = \frac{2}{\pi} \int_{-\pi}^{\pi} \sinh x dx$$

To find a_0 & b_0 for $n=0, 1, 2, \dots$ consider,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sinh x dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \frac{d}{dx} \sinh x dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sinh x dx$$

$$= \frac{1}{\pi} \left[\frac{1}{\pi} e^x \sinh x \right]_{-\pi}^{\pi} = \frac{1}{\pi} \left[\frac{1}{\pi} (\sinh \pi) - \frac{1}{\pi} (\sinh (-\pi)) \right]$$

$$= \frac{2(-i)^0}{\pi(\pi^2+1)} \sinh(\pi+i\pi) = \frac{2(-i)^0}{\pi(\pi^2+1)} [\sinh(\pi)\cos(i\pi) + \cosh(\pi)\sin(i\pi)]$$

$$= \frac{2(-i)^0}{\pi(\pi^2+1)} [\cos(\pi)\sinh(\pi) + i \sin(\pi)\cosh(\pi)]$$

$$= \left[\frac{2(-i)^0}{\pi(\pi^2+1)} \cos(\pi)\sinh(\pi) \right] + i \left[\frac{2(-i)^0}{\pi(\pi^2+1)} \sin(\pi)\cosh(\pi) \right]$$

$$= \frac{2(-i)^0 \sinh(\pi)}{\pi(\pi^2+1)} - i \frac{2(-i)^0 \sinh(\pi)}{\pi(\pi^2+1)}$$

$$\therefore a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos x dx = -\frac{2(-i)^0 \sinh(\pi)}{\pi(\pi^2+1)}$$

$$b_0 = -\frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin x dx = -\frac{2(-i)^0 \sinh(\pi)}{\pi(\pi^2+1)}$$

Taking $\frac{x}{\pi}$

$$a_n^* = \frac{1}{\pi} \sinh \frac{\pi}{\pi} + \sum_{n=1}^{\infty} \left[\frac{2(-i)^n \sinh \frac{\pi}{\pi}}{\pi(n^2+1)} \cos(n\pi) - \frac{2(-i)^n \sinh \frac{\pi}{\pi}}{\pi(n^2+1)} \sin(n\pi) \right]$$

$$a_0^* = \frac{1}{\pi} \sinh \frac{\pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-i)^n}{n^2+1} [\cos(n\pi) - i \sin(n\pi)] \right]$$

or $a_0^* = \sinh \pi$

$$g(x) = \frac{1}{2} \left\{ \frac{1}{\pi} \sinh \frac{\pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-i)^n}{n^2+1} [\cos(n\pi) - i \sin(n\pi)] \right] \right\}$$

$$g(0) = -\frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-i)^n}{n^2+1} \right]$$

$$g(x) = \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-i)^n \sinh \pi \cos(n\pi)}{n^2+1} \right]$$

Finally

$$(b) = \cosh x = \frac{1}{2} (e^x + e^{-x})$$

$$h(x) = \frac{1}{2} \left\{ \frac{1}{\pi} \sinh \frac{\pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-i)^n}{n^2+1} [\cos(n\pi) - i \sin(n\pi)] \right] \right\}$$

$$h(0) = \frac{1}{\pi} \sinh \frac{\pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-i)^n}{n^2+1} [\cos(n\pi) + i \sin(n\pi)] \right]$$

$$h(x) = \frac{2 \sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-i)^n}{n^2+1} \cos(n\pi) \right]$$

Question 28

A function f is defined by

$$f(t) = V|\cos \omega t|, t \in \mathbb{R},$$

where V and ω are positive constants.

Show that the Fourier series of f is given by

$$f(t) = \frac{2V}{\pi} + \frac{4V}{\pi} \left[\frac{1}{3} \cos(2\omega t) - \frac{1}{15} \cos(4\omega t) + \frac{1}{35} \cos(6\omega t) + \dots \right]$$

[proof]

$f(t) = V|\cos \omega t|$

WORKING AT THE GRAHS CROSSES
 $f(t)$ HAS PERIOD $L = \frac{\pi}{\omega}$
(HALF PERIOD $L = \frac{\pi}{2\omega}$) AND IT IS EVEN

$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L}]$

- $b_n = 0$ (As $f(t)$ is even)
- $a_0 = \frac{1}{L} \int_{-L}^L f(t) dt = \frac{1}{\frac{\pi}{\omega}} \int_0^{\frac{\pi}{\omega}} V|\cos \omega t| dt = \frac{2V}{\pi} \times 2 \int_0^{\frac{\pi}{\omega}} V|\cos \omega t| dt$
 $= \frac{4V}{\pi} \int_0^{\frac{\pi}{\omega}} |\cos \omega t| dt = \frac{4V}{\pi} \times \frac{1}{\omega} \left[\sin \omega t \right]_0^{\frac{\pi}{\omega}}$
 $= \frac{4V}{\pi} \left[\sin \frac{\pi}{2} - 0 \right] = \frac{4V}{\pi}$
- $a_1 = \frac{1}{L} \int_{-L}^L f(t) \cos \frac{n\pi t}{L} dt = \frac{1}{\frac{\pi}{\omega}} \int_0^{\frac{\pi}{\omega}} V|\cos \omega t| \cos(\omega t) dt = \frac{4V}{\pi} \int_0^{\frac{\pi}{\omega}} V|\cos \omega t| \cos(\omega t) dt$
 $= \frac{2V}{\pi} \int_0^{\frac{\pi}{\omega}} 2\cos(\omega t)\cos(\omega t) dt = \frac{2V}{\pi} \int_0^{\frac{\pi}{\omega}} [\cos(2\omega t) + \cos(0)] dt$

Now $\cos(2\omega t + \omega t) = \cos(2\omega t)\cos(\omega t) - \sin(2\omega t)\sin(\omega t)$
 $\cos(2\omega t - \omega t) = \cos(2\omega t)\cos(\omega t) + \sin(2\omega t)\sin(\omega t)$
 $\cos(2\omega t)\cos(\omega t) + \cos(2\omega t)\cos(\omega t) = 2\cos(2\omega t)\cos(\omega t)$

 $= \frac{2V}{\pi} \int_0^{\frac{\pi}{\omega}} 2\cos(2\omega t)\cos(\omega t) dt = \frac{2V}{\pi} \int_0^{\frac{\pi}{\omega}} [\cos(3\omega t) + \cos(\omega t)] dt$

$$\begin{aligned} &= \frac{2V}{\pi} \left[\frac{1}{3\omega} \sin(3\omega t) + \frac{1}{\omega} \sin(\omega t) \right]_0^{\frac{\pi}{\omega}} \\ &= \frac{2V}{\pi} \left[\left(\frac{1}{3\omega} \sin(\pi) + \frac{1}{\omega} \sin(0) \right) \right] \\ &= \frac{2V}{\pi} \left[\frac{1}{3\omega} \left(\sin \pi + \cos \omega \pi + \cos(-\omega \pi) \right) + \frac{1}{\omega} \left(\sin \omega \pi - \cos \omega \pi \right) \right] \\ &= \frac{2V}{\pi} \left[\frac{\cos \pi}{3\omega} + \frac{\cos(-\pi)}{\omega} \right] \\ &= \frac{2V}{\pi} \left[\frac{\cos \pi}{3\omega} + \frac{\cos \pi}{\omega} \right] \\ &= \frac{2V \cos \pi}{\pi} \left[\frac{1}{3\omega} + \frac{1}{\omega} \right] \\ &= \frac{2V \cos \pi}{\pi} \left[\frac{2\omega + 3}{3\omega} \right] \\ &= \frac{2V (-1)^3}{\pi(1+4/3)} \\ &= \frac{4V (-1)^3}{\pi(4/3)} \\ &\therefore f(t) = \frac{4V}{\pi} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{1-\frac{4}{n^2}} \cos(2nt) \right] \\ f(t) = \frac{2V}{\pi} + \frac{4V}{\pi} \left[\frac{1}{3} \cos(2\omega t) - \frac{1}{15} \cos(4\omega t) + \frac{1}{35} \cos(6\omega t) + \dots \right] \end{aligned}$$

PARSEVAL'S IDENTITY

Question 1

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

- b) Find the Fourier series of

$$f(x) = |x|, \quad -\pi \leq x \leq \pi.$$

- c) State Parseval's identity for the Fourier series of $f(x)$ from part (a).

- d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96},$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\cos[(2n-1)x]}{(2n-1)^2} \right]$$

a) If $f(x)$ is piecewise continuous on $(-L, L)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

where

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, 3, \dots$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

b) $f(x) = |x|$, $-\pi < x < \pi$ is an even function

hence $b_n = 0$ for all n , since $|x|$ itself is odd

- $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi} \int_0^\pi 2x dx = \frac{2}{\pi} \int_0^\pi x dx$
 $= \frac{1}{\pi} \left[x^2 \right]_0^\pi = \frac{1}{\pi} (\pi^2 - 0) = \pi^2 \quad \text{L.E. } a_0, \pi^2$
- $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx dx = \frac{2}{\pi} \int_0^\pi x \cos nx dx$

INTEGRATION BY PARTS

| | |
|-------------------------|-----------|
| $\frac{d}{dx} \cos nx$ | 1 |
| $\frac{1}{n} n \sin nx$ | $\cos nx$ |

 $\therefore = \frac{2}{\pi} \left\{ \left[\frac{1}{n} x \cos nx \right]_0^\pi - \frac{2}{\pi} \int_0^\pi \sin nx dx \right\}$
 $= \frac{2}{\pi} \left[\frac{1}{n} \cos nx \right]_0^\pi = \frac{2}{\pi n^2} [\cos(n\pi) - 1] = \frac{2}{\pi n^2} [(-1)^n - 1]$
 $= \begin{cases} -\frac{4}{\pi n^2} & \text{IF } n \text{ IS ODD} \\ 0 & \text{IF } n \text{ IS EVEN} \end{cases}$
 $\therefore a_n = \frac{(-1)^n - 1}{\pi (2n-1)^2}, \quad n = 1, 2, 3, \dots$

Then $|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{4}{\pi (2n-1)} \cos[(2n-1)x] \right]$

$$\int_{-\pi}^{\pi} |x|^2 dx = \frac{\pi^2}{2} + \sum_{n=1}^{\infty} \left[\frac{4}{\pi (2n-1)^2} \right]$$

e) IF THE COEFFICIENTS OBTAINED IN PART (a) ARE SUBSTITUTED THEN PARSEVAL'S IDENTITY STATES

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 dx = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

d)

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} |x|^2 dx &= \frac{\pi^2}{2} + \sum_{n=1}^{\infty} \left[\frac{4}{\pi (2n-1)^2} \right]^2 \\ &\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{2} + \sum_{n=1}^{\infty} \frac{16}{\pi^2 (2n-1)^4} \\ &\Rightarrow \frac{1}{\pi} \int_0^\pi 2x^2 dx = \frac{\pi^2}{2} + \sum_{n=1}^{\infty} \frac{16}{\pi^2 (2n-1)^4} \\ &\Rightarrow \frac{1}{\pi} \left[\frac{2}{3} x^3 \right]_0^\pi = \frac{\pi^2}{2} + \sum_{n=1}^{\infty} \frac{16}{\pi^2 (2n-1)^4} \\ &\Rightarrow \frac{2}{3} \pi^2 = \frac{\pi^2}{2} + \sum_{n=1}^{\infty} \frac{16}{\pi^2 (2n-1)^4} \\ &\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \\ &\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96} \end{aligned}$$

Question 2

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.

- b) Find the Fourier series of

$$f(x) = \text{sign}(x), \quad -\pi \leq x \leq \pi.$$

- c) Prove Parseval's identity for the Fourier series of $f(x)$ in $(-\pi, \pi)$.

- d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8},$$

$$\text{sign}(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin[(2n-1)x]}{(2n-1)} \right]$$

a) If $f(x)$ is piecewise continuous on $(-\pi, \pi)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$\text{where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad n=1, 2, 3, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n=1, 2, 3, \dots$$

b) $f(x) = \text{Sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$
(odd function $\Rightarrow a_0 = 0$)

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \text{Sign}(x) \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} \text{Sign}(x) \sin(nx) dx \\ &= \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin(nx) dx = \frac{2}{\pi} \cdot \left(-\frac{1}{n}\right) [\cos nx]_0^{\pi} \\ &= -\frac{2}{\pi n} [\cos \pi - 1] = -\frac{2}{\pi n} [-1 - 1] \\ &= \frac{4}{\pi n} [(-1)^n + 1] \Rightarrow \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi n} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

$$= \frac{4}{(2n-1)\pi}, \quad n=1, 2, 3, \dots$$

$$\therefore \text{Sign}(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin((2n-1)x)$$

$$= -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}$$

4) **PARSEVAL'S IDENTITY**

GIVEN THAT THE SERIES OF PART (a) CONVERGES, THEN

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

PROOF

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

MULTIPLY THROUGH BY $\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0}{2} \left[\int_{-\pi}^{\pi} f(x) dx \right] + \sum_{n=1}^{\infty} \left[\int_{-\pi}^{\pi} [a_n \cos nx + b_n \sin nx] f(x) dx \right]$$

INTERSECT WITH A REAL x IN $(-\pi, \pi)$, NOTING THAT $\int \cos x dx$ AND $\int \sin x dx$ ARE IRREVERSIBLE

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0}{2} \int_{-\pi}^{\pi} f(x) dx + \sum_{n=1}^{\infty} \left[\int_{-\pi}^{\pi} f(x) \cos nx dx \right] + \sum_{n=1}^{\infty} \left[\int_{-\pi}^{\pi} f(x) \sin nx dx \right]$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0}{2} a_0 + \sum_{n=1}^{\infty} [a_n a_n] + \sum_{n=1}^{\infty} [b_n b_n]$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

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4)

$$\begin{aligned} f(x) = \text{Sign}(x) &\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx = 1 \\ &\Rightarrow a_0 = 0 \quad \forall n \\ &\Rightarrow b_n^2 = \left(\frac{4}{(2n-1)\pi} \right)^2 = \frac{16}{\pi^2 (2n-1)^2} \end{aligned}$$

Question 3

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
 - Find the Fourier series of
- $$f(x) = \frac{1}{2} + \frac{1}{2}\operatorname{sign}(x), \quad -\pi \leq x \leq \pi.$$
- Prove the validity of Parseval's identity for the Fourier series of $f(x)$ in the interval $(-L, L)$.
 - Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

□, $\frac{1}{2} + \frac{1}{2}\operatorname{sign}(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin[(2n-1)x]}{(2n-1)} \right]$

a) STATE THE FOURIER SERIES THEOREM

If $f(x)$ is piecewise continuous on $(-L, L)$, $L > 0$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \quad n=0, 1, 2, 3, \dots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx \quad n=1, 2, 3, \dots$$

b) $f(x) = \frac{1}{2} + \frac{1}{2}\operatorname{sign}(x)$

NOTE: $\operatorname{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

- $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 dx = \frac{1}{\pi} \times \pi = 1$
- $a_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 2x dx = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \times \cos 2x dx = \frac{1}{\pi} \left[\sin 2x \right]_{-\pi}^{\pi} = 0$
- $b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x dx = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \times \sin x dx = \frac{1}{\pi} \left[\cos x \right]_{-\pi}^{\pi} = -\frac{1}{\pi} [\cos \pi - 1] = -\frac{1 - \cos \pi}{\pi} = \frac{1 - (-1)}{\pi} = \frac{2}{\pi}$

c) STATE FROM THE FOURIER THEOREM

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

MULTIPLY THROUGH BY $\frac{1}{2}f(x)$ & INTEGRATE w.r.t. x , between $-L$ & L

$$\Rightarrow \frac{1}{2} \int_{-L}^{L} f(x)^2 dx = \frac{a_0^2}{2} \int_{-L}^{L} f(x) dx + \sum_{n=1}^{\infty} \int_{-L}^{L} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]^2 dx$$

INTEGRATION, EXPANSION AND EVALUATION

$$\Rightarrow \frac{1}{2} \int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} \int_{-L}^{L} f(x) dx + \sum_{n=1}^{\infty} \left[a_n^2 \int_{-L}^{L} \cos^2 \frac{n\pi x}{L} dx + b_n^2 \int_{-L}^{L} \sin^2 \frac{n\pi x}{L} dx \right]$$

$$\Rightarrow \frac{1}{2} \int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

$$\Rightarrow \frac{1}{2} \int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

WHICH IS PARSEVAL'S IDENTITY

d) USING PARSEVAL'S IDENTITY WITH $f(x) = \frac{1}{2} + \frac{1}{2}\operatorname{sign}(x)$ IN THE INTERVAL $(-\pi, \pi)$

$$\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} 1^2 dx = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi^2 (2n-1)^2}$$

$$\Rightarrow \frac{1}{\pi} \times \pi = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Rightarrow 1 = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Rightarrow \frac{1}{2} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

ANSWER

Question 4

$$f(x) = x, \quad x \in \mathbb{R}, \quad -\pi \leq x \leq \pi.$$

$$f(x) = f(x + 2\pi).$$

Use Parseval's identity for the Fourier coefficients of $f(x)$ to determine the exact value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{2\pi n x}{\pi} + b_n \sin \frac{2\pi n x}{\pi})$$

$$a_0 = \frac{2}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_{-\pi}^{\pi} f(x) \cos \frac{2\pi n x}{\pi} dx$$

$$b_n = \frac{2}{\pi} \int_{-\pi}^{\pi} f(x) \sin \frac{2\pi n x}{\pi} dx$$

- $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x dx = 0$ (odd integrand in a symmetric domain)
- $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx dx = 0$ (odd integrand in a symmetric domain)
- $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx = \dots$ by parts

$$\begin{aligned} &= \dots = \frac{2}{\pi} \left[-x \cos nx \right]_0^{\pi} + \frac{2}{\pi} \int_0^{\pi} \cos nx dx \\ &= -\frac{2}{\pi} \left[\tan nx \right]_0^{\pi} + \frac{2}{\pi} \int_0^{\pi} \sin nx dx = -\frac{2}{\pi} (-1)^n \end{aligned}$$

Finally

$$\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{3\pi} \left[x^3 \right]_0^{\pi} = \frac{2}{3}\pi^2$$

By Parseval's identity

$$\sum_{n=1}^{\infty} [a_n^2 + b_n^2] = \frac{2}{3}\pi^2$$

$$\sum_{n=1}^{\infty} \left[\frac{2}{\pi} (-1)^n \right]^2 = \frac{2}{3}\pi^2$$

$$4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2}{3}\pi^2$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$f(x) = x, \quad x \in [-\pi, \pi]$$

$$l = \pi, \quad \frac{2}{\pi} = \frac{1}{l}, \quad \frac{2\pi}{\pi} = 2$$

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 dx$$

(Parseval's identity)

Question 5

$$f(x) = x^2, \quad x \in \mathbb{R}, \quad -\pi \leq x \leq \pi.$$

$$f(x) = f(x + 2\pi).$$

Use Parseval's identity for the Fourier coefficients of $f(x)$ to determine the exact value of

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_n = \frac{2}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_1 = \frac{2}{\pi} \int_{-\pi}^{\pi} x^2 \cos x dx$$

$$b_1 = \frac{2}{\pi} \int_{-\pi}^{\pi} x^2 \sin x dx$$

$$f(x) = x^2, \quad x \in [-\pi, \pi]$$

$$T = \pi, \quad \frac{2\pi}{T} = \frac{1}{\pi}, \quad \frac{2\pi n}{T} = n$$

PARSEVAL'S IDENTITY

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 dx$$

• $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{3\pi} \int_0^{\pi} x^2 dx = \frac{2}{3\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{3\pi} \pi^2$

• $a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos x dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos x dx = \frac{2}{\pi} \left[\frac{1}{2} x^2 \sin x \Big|_0^{\pi} - \frac{1}{2} \int_0^{\pi} 2x \sin x dx \right]$
 (BY PARTIAL DIFF)
 $= \frac{-2}{\pi} \left[\frac{1}{2} \left[2x \sin x \right]_0^{\pi} + \frac{1}{2} \int_0^{\pi} 2x \sin x dx \right]$
 $= \frac{1}{\pi} \left[2x \sin x \Big|_0^{\pi} \right] = \frac{1}{\pi} (2\pi)^2 = 4\pi$

• $b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin x dx = 0$ (COS IS EVEN IN A SYMMETRIC DOMAIN)

FINALLY $\frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{2}{\pi} \int_0^{\pi} x^4 dx = \frac{2}{3\pi} \left[\frac{x^5}{5} \right]_0^{\pi} = \frac{2}{3\pi} \pi^5$

BY PARSEVAL'S IDENTITY: $\frac{1}{2}(a_0^2) + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{2}{3}\pi^4$
 $\frac{2}{3}\pi^4 + \sum_{n=1}^{\infty} \frac{1}{n^4} (2\pi)^4 = \frac{2}{3}\pi^4$
 $16 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{8}{3}\pi^4$
 $\therefore \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Question 6

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
 - Prove the validity of Parseval's identity for the Fourier series of $f(x)$ in the interval $(-L, L)$.
 - Find the Fourier series of
- $$f(x) = x^2, \quad -\pi \leq x \leq \pi.$$
- Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} \cos nx \right]$$

a) If $f(x)$ is piecewise continuous on $(-L, L)$, $L > 0$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

where $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ $n = 1, 2, 3, \dots$

$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$ $n = 1, 2, 3, \dots$

b) Starting from the result of part (a),

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

Multiply through by $\frac{1}{2} f(x)$ with respect to x , from $-L$ to L .

$$\Rightarrow \frac{1}{2} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \left(\int_{-L}^L a_n \cos \frac{n\pi x}{L} dx \right)^2 + \left(\int_{-L}^L b_n \sin \frac{n\pi x}{L} dx \right)^2$$

$$\Rightarrow \frac{1}{2} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \left[\sum_{n=1}^{\infty} a_n^2 \int_{-L}^L \cos^2 \frac{n\pi x}{L} dx \right] + \left[\sum_{n=1}^{\infty} b_n^2 \int_{-L}^L \sin^2 \frac{n\pi x}{L} dx \right]$$

$$\Rightarrow \frac{1}{2} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \left[\sum_{n=1}^{\infty} a_n^2 \left(\frac{1}{2} + \int_{-L}^L \cos^2 \frac{n\pi x}{L} dx \right) \right] + \left[\sum_{n=1}^{\infty} b_n^2 \left(\frac{1}{2} + \int_{-L}^L \sin^2 \frac{n\pi x}{L} dx \right) \right]$$

$$\Rightarrow \frac{1}{2} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \left[\sum_{n=1}^{\infty} a_n^2 \left(\frac{1}{2} + \int_{-L}^L \cos^2 \frac{n\pi x}{L} dx \right) \right] + \left[\sum_{n=1}^{\infty} b_n^2 \left(\frac{1}{2} + \int_{-L}^L \sin^2 \frac{n\pi x}{L} dx \right) \right]$$

$$\Rightarrow \frac{1}{2} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2$$

$$\Rightarrow \frac{1}{2} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

which is known as Parseval's identity

c) $f(x)$ is even on $(-\pi, \pi)$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0 \quad \forall n$$

Given $x = 0$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{2\pi^3}{3}$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos x dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx$$

$$+ 2 \left(\int_{-\pi}^{\pi} x^2 \cos x dx \right) - 2 \left(\int_{-\pi}^{\pi} x \cos x dx \right)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx$$

$$= -\frac{4}{\pi} \left[\left[-\frac{1}{3} \cos x \right]_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x dx \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{3} \cos x \right]_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x dx$$

$$= \frac{4}{3\pi} \times (-1) = \frac{4}{3\pi}$$

Thus

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ a_n^2 &= \frac{1}{\pi} \left(\frac{4}{3\pi} \right)^2 + \frac{2}{\pi} \left(-\frac{4}{3\pi} \right) \cos nx \\ a_n^2 &= \frac{16}{9\pi^2} + \frac{8}{3\pi} \cos nx \end{aligned}$$

d) By PARSEVAL'S IDENTITY

$$\begin{aligned} \Rightarrow \frac{1}{2} \int_{-\pi}^{\pi} [f(x)]^2 dx &= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2] \\ \Rightarrow \frac{1}{2} \int_{-\pi}^{\pi} (x^2)^2 dx &= \frac{(2\pi)^2}{2} + \sum_{n=1}^{\infty} \left[\frac{4}{9\pi^2} + \frac{8}{3\pi} \cos nx \right]^2 \\ \Rightarrow \frac{2}{\pi} \int_{-\pi}^{\pi} x^4 dx &= \frac{4\pi^4}{3} + \sum_{n=1}^{\infty} \frac{16}{9\pi^2} \\ \Rightarrow \frac{2}{\pi} \left(x^5 \right)_{-\pi}^{\pi} &= \frac{2}{3}\pi^4 + 16 \sum_{n=1}^{\infty} \frac{1}{\pi^2} \\ \Rightarrow \frac{2}{\pi} \pi^4 &= \frac{2}{3}\pi^4 + 16 \sum_{n=1}^{\infty} \frac{1}{\pi^2} \\ \Rightarrow \frac{8}{3}\pi^4 &= \sum_{n=1}^{\infty} \frac{1}{\pi^2} \\ \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{8}{45} \end{aligned}$$

Question 7

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

- State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series.
- State and prove Parseval's identity for the Fourier series of $f(x)$ in $(-L, L)$.
- By considering the Fourier series of

$$f(x) = x^3, -\pi \leq x \leq \pi,$$

show that $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

You may use without proof the following results.

- $\int x^3 \sin nx \, dx = \frac{1}{n^4} [nx(6 - n^2 x^2) \cos nx + 3(n^2 x^2 - 2) \sin nx] + C$
- $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
- $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

proof

a) If $f(x)$ is piecewise continuous on $(-L, L)$, $L > 0$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

where $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} \, dx$, $n = 0, 1, 2, 3, 4, \dots$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} \, dx$$
, $n = 1, 2, 3, 4, \dots$

b) PARSEVAL'S IDENTITY

Given the result of part (a) above, find

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

PROOF

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

Multiply through by $\frac{1}{L} \int_{-L}^L$

$$\left[\frac{1}{L} \int_{-L}^L [f(x)]^2 \, dx \right]^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left[\int_{-L}^L a_n \cos \frac{n\pi x}{L} \cos \frac{n\pi x}{L} \, dx \right]^2 + \sum_{n=1}^{\infty} \left[\int_{-L}^L b_n \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} \, dx \right]^2$$

INTERESTINGLY, $\int_{-L}^L \cos^2 \theta \, d\theta = L$, noting that $\theta = \frac{n\pi x}{L}$ is monotone

$$\frac{1}{L} \int_{-L}^L \left[\int_{-L}^L [f(x)]^2 \, dx \right]^2 \, dx = \frac{a_0^2}{2} \cdot L + \sum_{n=1}^{\infty} \left[\int_{-L}^L a_n \cos \frac{n\pi x}{L} \cos \frac{n\pi x}{L} \, dx \right]^2 + \sum_{n=1}^{\infty} \left[\int_{-L}^L b_n \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} \, dx \right]^2$$

$$\frac{1}{L} \int_{-L}^L \left[\int_{-L}^L [f(x)]^2 \, dx \right]^2 \, dx = \frac{a_0^2}{2} \cdot a_0^2 + \sum_{n=1}^{\infty} [a_n^2 + b_n^2] + \sum_{n=1}^{\infty} b_n \cdot b_n$$

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 \, dx = \frac{a_0^2}{2} \cdot a_0^2 + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

Now $f(x) = x^3$, $-\pi \leq x \leq \pi$, is odd in A symmetric domain

$\therefore \int_{-L}^L f(x)^2 \, dx = \int_{-\pi}^{\pi} x^6 \, dx = \frac{\pi^6}{6}$

Now $\int_{-L}^L x^3 \sin nx \, dx = \int_{-\pi}^{\pi} x^3 \sin nx \, dx$

$$\therefore \int_{-L}^L x^3 \sin nx \, dx = \frac{1}{n} \left[x^2 \sin nx + 2(-x^2 + 2)x \cos nx \right]_{-\pi}^{\pi}$$

$$= \frac{2}{n} \int_{-\pi}^{\pi} x^2 (-\sin nx) \cos nx + 2(-x^2 + 2)x \cos nx \, dx$$

$$= \frac{2}{n} \cdot \text{INT. } (-x^2 \sin nx) \cos nx = \frac{2}{n^3} (-x^2 \sin nx) (-1)^n$$

BY PARSEVAL'S IDENTITY

$$\Rightarrow \frac{2}{n} \int_{-\pi}^{\pi} (-x^2 \sin nx) (-1)^n \, dx = \sum_{n=1}^{\infty} \left(\frac{2}{n^3} \right)^2 (-x^2 \sin nx) (-1)^n$$

$$\Rightarrow \frac{2}{n} \int_{-\pi}^{\pi} x^2 \, dx = \sum_{n=1}^{\infty} \left(\frac{2}{n^3} \right)^2 (3x^2 - 12x^4 + 7x^6)$$

$$\Rightarrow \frac{2}{n} \left[\frac{2}{3} x^3 \right]_{-\pi}^{\pi} = \sum_{n=1}^{\infty} \left[\frac{16x^6}{n^6} - \frac{48x^4}{n^4} + \frac{24x^2}{n^2} \right]$$

$$\Rightarrow \frac{2}{n} \cdot \frac{16}{3} = 16x^6 \sum_{n=1}^{\infty} \frac{1}{n^6} - 48x^4 \sum_{n=1}^{\infty} \frac{1}{n^4} + 24x^2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow 2 \cdot \frac{16}{3} = 164 \sum_{n=1}^{\infty} \frac{1}{n^6} - 160 \cdot \frac{16}{90} + 48 \cdot \frac{2}{5}$$

$$\Rightarrow 2 \cdot \frac{16}{3} = 164 \sum_{n=1}^{\infty} \frac{1}{n^6} - \frac{16}{5} + \frac{24}{5}$$

$$\Rightarrow 164 \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{16}{5}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{16}{820}$$

FOURIER SERIES

Complex Expansions

Question 1

A periodic function $f(t)$ is defined in the interval $(-L, L)$, $L > 0$, $f(t+2L) = f(t)$.

It is further given that $f(t)$ is continuous or piecewise continuous in $(-L, L)$ and has Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right],$$

where $a_n = \frac{1}{L} \int_{-L}^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt$, $n = 0, 1, 2, 3, \dots$

and $b_n = \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt$, $n = 1, 2, 3, 4, \dots$

Show that the complex Fourier series expansion of $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} \left[c_n e^{\frac{in\pi t}{L}} \right],$$

where $c_n = \frac{1}{2L} \int_{-L}^L f(t) e^{-\frac{in\pi t}{L}} dt$, $n \in \mathbb{Z}$

[] , proof

Start with the definition of a Fourier series in t , $-L < t < L$

$$(f(t)) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)]$$

- $a_n = \frac{1}{L} \int_{-L}^L (f(t)) \cos\left(\frac{n\pi t}{L}\right) dt$ $\quad n = 0, 1, 2, 3, \dots$
- $b_n = \frac{1}{L} \int_{-L}^L (f(t)) \sin\left(\frac{n\pi t}{L}\right) dt$ $\quad n = 1, 2, 3, \dots$

By manipulating Euler's formula & substituting into the integral

- $\cos\left(\frac{n\pi t}{L}\right) = \frac{1}{2} \left[e^{i\frac{n\pi t}{L}} + e^{-i\frac{n\pi t}{L}} \right]$
- $\sin\left(\frac{n\pi t}{L}\right) = \frac{1}{2i} \left[e^{i\frac{n\pi t}{L}} - e^{-i\frac{n\pi t}{L}} \right]$

$$\Rightarrow (f(t)) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{2} \left(a_n e^{i\frac{n\pi t}{L}} + b_n e^{-i\frac{n\pi t}{L}} \right) + \frac{1}{2i} \left(a_n e^{i\frac{n\pi t}{L}} - b_n e^{-i\frac{n\pi t}{L}} \right) \right]$$

$$\Rightarrow (f(t)) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{i\frac{n\pi t}{L}} + \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-i\frac{n\pi t}{L}} \right]$$

- Let $C_0 = \frac{1}{2} a_0 = \frac{1}{2}(a_0 + ib_0)$ where $b_0 = 0$
- Let $C_n = \frac{1}{2}(a_n - ib_n)$
- Let $\bar{C}_n = \frac{1}{2}(a_n + ib_n)$, as C_n & \bar{C}_n are conjugates

$$\Rightarrow (f(t)) = C_0 + \sum_{n=1}^{\infty} \left[C_n e^{i\frac{n\pi t}{L}} + \bar{C}_n e^{-i\frac{n\pi t}{L}} \right]$$

Now for notational convenience we write the constants as follows

$$C_0 = \frac{1}{2}(a_0 - ib_0)$$

$$\Rightarrow C_0 = \frac{1}{2}(a_0 + ib_0) \Rightarrow \bar{C}_0 = \frac{1}{2}(a_0 + ib_0)$$

$$\Rightarrow (f(t)) = C_0 + \sum_{n=1}^{\infty} \left[C_n e^{i\frac{n\pi t}{L}} + \bar{C}_n e^{-i\frac{n\pi t}{L}} \right]$$

$$\Rightarrow f(t) = C_0 + \sum_{n=1}^{\infty} [C_n e^{i\frac{n\pi t}{L}}] + \sum_{n=1}^{\infty} [\bar{C}_n e^{-i\frac{n\pi t}{L}}]$$

$$\Rightarrow f(t) = C_0 + \sum_{n=1}^{\infty} [C_n e^{i\frac{n\pi t}{L}}] + \sum_{n=1}^{\infty} [C_n e^{i\frac{n\pi t}{L}}]$$

$$\Rightarrow f(t) = \sum_{n=1}^{\infty} [C_n e^{i\frac{n\pi t}{L}}] + C_0 + \sum_{n=1}^{\infty} [C_n e^{i\frac{n\pi t}{L}}]$$

$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty} [C_n e^{i\frac{n\pi t}{L}}]$$

With $C_0 = \frac{1}{2}(a_0 - ib_0)$

$$\Rightarrow C_0 = \frac{1}{2} \left[\frac{1}{L} \int_{-L}^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt - i \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt \right]$$

$$\Rightarrow C_0 = \frac{1}{2L} \int_{-L}^L f(t) \left[\cos\left(\frac{n\pi t}{L}\right) - i \sin\left(\frac{n\pi t}{L}\right) \right] dt$$

$$\Rightarrow C_0 = \frac{1}{2L} \int_{-L}^L f(t) e^{i\frac{n\pi t}{L}} dt, \quad n \in \mathbb{Z}$$

Question 2

$$f(t) = \begin{cases} 1 & -2 \leq t \leq 2 \\ 0 & 2 < t < 6 \end{cases}, \quad f(t+8) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{2} + \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[e^{\frac{1}{4}n\pi t i} \operatorname{sinc}\left(\frac{1}{2}n\pi\right) \right]$$

$f(t) = \begin{cases} 1 & -2 \leq t \leq 2 \\ 0 & 2 < t \leq 6 \end{cases}$

Start by drawing a sketch.

Consider a summa trial domain of period $T = B$; from -4 to 4 ($L = 4$ = Half Period)

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi t}{L}}$$

with $c_n = \frac{1}{L} \int_{-L}^L f(t) e^{-i \frac{n\pi t}{L}} dt$

Evaluate the complex coefficients

$$c_0 = \frac{1}{8} \int_{-4}^4 f(t) e^{-i \frac{n\pi t}{4}} dt$$

$$c_n = \frac{1}{8} \int_{-4}^4 f(t) e^{-i \frac{n\pi t}{4}} dt$$

$$c_n = \frac{1}{8} \int_{-4}^4 \left[\cos \frac{n\pi t}{4} - \sin \frac{n\pi t}{4} \right] dt$$

$$c_n = \frac{1}{4} \int_0^4 \cos \frac{n\pi t}{4} dt$$

$$c_n = \frac{1}{4} \times \frac{4}{n\pi} \left[\sin \frac{n\pi t}{4} \right]_0^4, \quad n \neq 0$$

$\Rightarrow c_0 = \frac{1}{8} \left[\sin \frac{n\pi t}{4} \Big|_0^4 \right], \quad n \neq 0$

$\Rightarrow c_0 = \frac{1}{8} \sin \frac{n\pi t}{2}, \quad n \neq 0$

The special one where we calculate separately

$$c_0 = c_0 = \frac{1}{8} \int_{-4}^4 f(t) e^{i \frac{0\pi t}{4}} dt = \frac{1}{8} \int_{-4}^4 1 dt = \frac{1}{2}$$

Thus we can form the Fourier expansion for $f(t)$

$$f(t) = c_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} c_n e^{i \frac{n\pi t}{4}}$$

$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{\sin \frac{n\pi t}{2}}{n\pi} e^{i \frac{n\pi t}{4}} \right]$$

$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{2} \left(\frac{\sin \frac{n\pi t}{2}}{n\pi} \right) e^{i \frac{n\pi t}{4}}$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{2} \operatorname{sinc}\left(\frac{n\pi t}{4}\right) e^{i \frac{n\pi t}{4}}$$

Question 3

$$f(t) = \begin{cases} 1 & 0 \leq t \leq a \\ 0 & a < t < T \end{cases}, \quad a < \frac{1}{2}T, \quad f(t+T) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{a}{T} + \frac{a}{T} \sum_{n=-\infty, n \neq 0}^{\infty} \left[\exp\left(\frac{n\pi(2t-a)}{T}\right) \operatorname{sinc}\left(\frac{n\pi a}{T}\right) \right]$$

$f(t) = \begin{cases} 1 & 0 \leq t \leq a \\ 0 & a < t < T, a < \frac{T}{2} \\ f(t+T) = f(t) \end{cases}$

- SIMILAR WITH A QUICKE REACT
- THIS FUNCTION IS NEITHER ODD, NOR EVEN — PERIOD IS T — HALF PERIOD $\frac{T}{2}$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi t}{T}}$$

$$\text{where } c_n = \frac{1}{2T} \int_{-L}^{L} f(t) e^{-\frac{i n \pi t}{T}} dt$$

- EVALUATE THE COMPLEX COEFFICIENTS

$$c_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i \frac{0 \pi t}{T}} dt$$

$$c_1 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \times e^{-i \frac{1 \pi t}{T}} dt$$

$$c_1 = \frac{1}{T} \times \left[\frac{1}{2\pi i} e^{-i \frac{1 \pi t}{T}} \right]_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{1}{2\pi i} \left[e^{-i \frac{1 \pi t}{T}} \right]_{-\frac{T}{2}}^{\frac{T}{2}}, \text{ info}$$

$$c_1 = \frac{1}{2\pi i} \left[e^{-i \frac{1 \pi t}{T}} \right]_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{1}{2\pi i} \left[e^{-i \frac{1 \pi t}{T}} \right]_{-\frac{T}{2}}^{\frac{T}{2}}, \text{ info}$$

$$\Rightarrow c_1 = \frac{1}{2\pi i} \left[e^{-i \frac{1 \pi t}{T}} - e^{-i \frac{1 \pi (-t)}{T}} \right], \text{ info}$$

$$\Rightarrow c_1 = \frac{-i \frac{1 \pi}{T}}{2\pi i} \times \frac{1}{2i} \left[e^{\frac{i \pi t}{T}} - e^{-\frac{i \pi t}{T}} \right], \text{ info}$$

$$\Rightarrow c_1 = e^{-\frac{i \pi t}{T}} \sin\left(\frac{\pi t}{T}\right) \times \frac{1}{2i}, \text{ info}$$

$$\Rightarrow c_1 = e^{-\frac{i \pi t}{T}} \sin\left(\frac{\pi t}{T}\right) \times \frac{1}{2i} \times \frac{a}{\frac{T}{2}}, \text{ info}$$

$$\Rightarrow c_1 = \frac{a}{T} e^{-\frac{i \pi t}{T}} \sin\left(\frac{\pi t}{T}\right), \text{ info}$$

- THE SPECIAL CASE WHERE $t=0$, WE CONSIDER SEPARATELY

$$c_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \times e^0 dt = \frac{1}{T} \int_{0}^a 1 dt = \frac{a}{T}$$

- FINALLY WE HAVE A COMPLEX FOURIER SERIES

$$f(t) = c_0 + \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi t}{T}}$$

$$f(t) = \frac{a}{T} + \sum_{n=-\infty}^{\infty} \left[\frac{a}{T} e^{-\frac{i \pi t}{T}} \sin\left(\frac{\pi t}{T}\right) \right] e^{\frac{i n \pi t}{T}}$$

$$f(t) = \frac{a}{T} + \frac{a}{T} \sum_{n=-\infty}^{\infty} \left[e^{\frac{i \pi t}{T}} \sin\left(\frac{\pi t}{T}\right) \right]$$

Question 4

$$f(t) = t, \quad 0 \leq t < 1, \quad f(t+1) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{1}{2} + \frac{i}{2\pi} \sum_{n=-\infty, n \neq 0}^{\infty} \left[\frac{e^{2n\pi t i}}{n} \right]$$

$f(t) = \begin{cases} t & 0 \leq t < 1 \\ f(t+1) = f(t) \end{cases}$

- START WITH A QUICK SKETCH
- THE FUNCTION IS NEITHER ODD NOR EVEN WITH $T=1$, $L=\frac{1}{2}$
$$f(t) = \sum_{n=-\infty}^{\infty} [C_n e^{int}] \text{ where } C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

$$\text{OR } C_n = \frac{1}{2} \int_0^1 f(t) e^{-int} dt$$
- CALCULATE THE COEFFICIENTS C_0 AND C_1
$$C_0 = \frac{1}{2} \int_0^1 t e^{-i2\pi t} dt = \int_0^1 t e^{-i2\pi t} dt$$

BY PARTS

| | |
|-----------------|-----------------|
| t | 1 |
| $\frac{d}{dt}t$ | $\frac{d}{dt}1$ |

$$\Rightarrow C_0 = \left[-\frac{te^{-i2\pi t}}{2\pi i} \right]_0^1 + \frac{1}{2\pi i} \int_0^1 e^{-i2\pi t} dt$$

$$\Rightarrow C_0 = \left[-\frac{te^{-i2\pi t}}{2\pi i} \right]_0^1 + \frac{1}{2\pi i^2} e^{-i2\pi t} \Big|_0^1$$

$$\Rightarrow C_n = \left[\frac{1}{4\pi^2 n^2} e^{-2n\pi t i} - \frac{t}{2\pi n i} e^{2n\pi t i} \right]_0^1$$

$$\Rightarrow C_n = \left[\frac{1}{4\pi^2 n^2} e^{-2n\pi i} - \frac{1}{2\pi n i} e^{2n\pi i} \right] - \frac{1}{4\pi^2 n^2}$$

$$\Rightarrow C_0 = \frac{1}{4\pi^2} [\cos(2\pi) - i\sin(2\pi)] - \frac{1}{4\pi^2}$$

$$\Rightarrow C_0 = \frac{1}{4\pi^2} - \frac{1}{4\pi^2} = -\frac{1}{4\pi^2}$$

$$\Rightarrow C_1 = \frac{i}{2\pi i}$$

FOR $n=0$, C_0

$$C_0 = \frac{1}{2} \int_0^1 t e^{-i2\pi t} dt = \int_0^1 t e^{-i2\pi t} dt = \left[\frac{1}{2} t^2 e^{-i2\pi t} \right]_0^1 = \frac{1}{2}$$

FINALLY THE COMPLEX FOURIER SERIES

$$f(t) = C_0 + \sum_{n=-\infty, n \neq 0}^{\infty} C_n e^{int}$$

$$f(t) = \frac{1}{2} + \sum_{n=-\infty, n \neq 0}^{\infty} \frac{i}{2\pi n} e^{2n\pi t i}$$

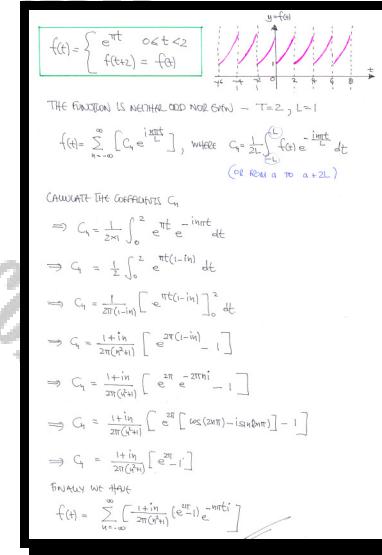
$$f(t) = \frac{1}{2} + \frac{i}{2\pi} \sum_{n=-\infty, n \neq 0}^{\infty} \frac{e^{2n\pi t i}}{n}$$

Question 5

$$f(t) = e^{\pi t}, \quad 0 \leq t < 2, \quad f(t+2) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{(e^{2\pi} - 1)}{2\pi} \sum_{n=-\infty}^{\infty} \left[\frac{(1+in)e^{2n\pi t i}}{(n^2 + 1)} \right]$$



Question 6

$$f(t) = \cos(\pi t), \quad -\frac{1}{2} \leq t < \frac{1}{2}, \quad f(t+1) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^{n+1} e^{2n\pi t i}}{4n^2 - 1} \right]$$

• THE FUNCTION IS EVEN IN A SYMMETRICAL DOMAIN, $T=1$, $L=\frac{1}{2}$

$$f(t) = \sum_{n=-\infty}^{\infty} [C_n e^{\frac{i n \pi t}{L}}] \text{ where } C_n = \frac{1}{2L} \int_L^L f(t) e^{-\frac{i n \pi t}{L}} dt$$

• CALCULATE THE COEFFICIENTS C_n ← (NOTE: C_0 IS PART OF THIS)

$$\Rightarrow C_n = \frac{1}{2 \cdot \frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(\pi t) e^{-\pi n t} dt$$

$$\Rightarrow C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(\pi t) [\cos(-\pi n t) - i \sin(-\pi n t)] dt$$

$$\Rightarrow C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} 2 \cos(\pi t) \cos(\pi n t) dt$$

$$\cos(\pi t + \pi n t) \equiv \cos(\pi t) \cos(\pi n t) - \sin(\pi t) \sin(\pi n t)$$

$$\cos(\pi t - \pi n t) \equiv \cos(\pi t) \cos(\pi n t) + \sin(\pi t) \sin(\pi n t)$$

ADD: $\cos((2n+1)\pi t) + \cos((2n-1)\pi t) = 2 \cos(\pi t) \cos(2nt)$

$$\Rightarrow C_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\cos((2n+1)\pi t) + \cos((2n-1)\pi t)) dt$$

$$\Rightarrow C_n = \left[\frac{1}{(2n+1)\pi} \sin((2n+1)\pi t) + \frac{1}{(2n-1)\pi} \sin((2n-1)\pi t) \right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$\Rightarrow C_n = \frac{1}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi}{2}\right) + \frac{1}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi}{2}\right)$$

$$\Rightarrow C_n = \frac{1}{(2n+1)\pi} \sin\left(\pi n \frac{1}{2}\right) + \frac{1}{(2n-1)\pi} \sin\left(\pi n \frac{-1}{2}\right)$$

• TO GET OP WE NOTE THAT

$$\sin\left[(2n+1)\frac{\pi}{2}\right] = (-1)^{n+1}$$

$$\sin\left[(2n-1)\frac{\pi}{2}\right] = (-1)^{n+1}$$

$$\Rightarrow C_n = \frac{1}{(2n+1)\pi} (-1)^{n+1} + \frac{1}{(2n-1)\pi} (-1)^{n+1}$$

$$\Rightarrow C_n = \frac{(-1)^{n+1}}{\pi} \left[\frac{1}{2n+1} + \frac{1}{2n-1} \right]$$

$$\Rightarrow C_n = \frac{(-1)^{n+1}}{\pi} \left[\frac{2n-1 - (2n+1)}{(2n+1)(2n-1)} \right]$$

$$\Rightarrow C_n = \frac{(-1)^{n+1}}{\pi} \frac{-2}{4n^2 - 1}$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{i n \pi t}{L}}$$

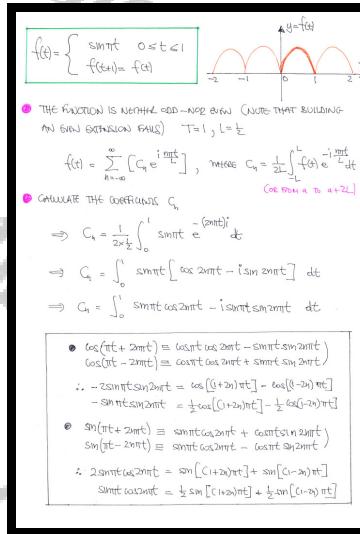
$$f(t) = \sum_{n=-\infty}^{\infty} \frac{-2(-1)^n}{\pi(4n^2 - 1)} e^{2n\pi t i}$$

Question 7

$$f(t) = \sin(\pi t), \quad 0 \leq t < 1, \quad f(t+1) = f(t).$$

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = -\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \left[\frac{e^{2n\pi t i}}{4n^2 - 1} \right]$$



INTEGRATING THE NEW EXPRESSIONS

$$\Rightarrow C_n = \int_0^1 \left[\frac{1}{2} \sin((1+2n)\pi t) + \frac{1}{2} \cos((1+2n)\pi t) + \frac{1}{2} \sin((1-2n)\pi t) - \frac{1}{2} \cos((1-2n)\pi t) \right] dt$$

$$\Rightarrow C_n = \frac{1}{2} \left[\frac{-1}{(2n+1)\pi} \sin((1+2n)\pi t) + \frac{1}{(1+2n)\pi} \cos((1+2n)\pi t) + \frac{1}{(2n-1)\pi} \sin((1-2n)\pi t) - \frac{1}{(1-2n)\pi} \cos((1-2n)\pi t) \right]_0^1$$

$$\Rightarrow C_n = \frac{1}{2} \left[\frac{-\cos((2n+1)\pi)}{(2n+1)^2\pi} + \frac{\cos((1-2n)\pi)}{(1-2n)^2\pi} \right] - \frac{1}{2} \left[\frac{-1}{(1+2n)\pi} - \frac{1}{(1-2n)\pi} \right]$$

$$\Rightarrow C_n = \frac{1}{2} \left[\frac{1}{(2n+1)^2\pi} + \frac{1}{(1-2n)^2\pi} \right] + \frac{1}{2} \left[\frac{1}{(1+2n)\pi} + \frac{1}{(1-2n)\pi} \right]$$

$$\Rightarrow C_n = \frac{1}{2\pi} \times 2 \left[\frac{1}{(1+2n)} + \frac{1}{(1-2n)} \right]$$

$$\Rightarrow C_n = \frac{1}{\pi} \left[\frac{1-2n+1+2n}{(1+2n)(1-2n)} \right]$$

$$\Rightarrow C_n = \frac{1}{\pi} \left[\frac{2}{(-4n^2)} \right] = -\frac{2}{\pi(1-4n^2)}$$

FINALLY WE CAN OBTAIN A COMPLEX FOURIER SERIES

$$f(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{int} \quad (\text{Note } C_0 \text{ is } 0)$$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{int}$$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{2}{\pi(1-4n^2)} e^{int}$$

$$f(t) = -\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{2n\pi t i}}{4n^2 - 1}$$

Question 8

The function f is defined as

$$f(t) = V \cos\left(\frac{\pi t}{T}\right), \quad -\frac{1}{2}T \leq t < \frac{1}{2}T, \quad f(t) = f(t+T),$$

where V and T are positive constants.

Determine the complex Fourier series expansion of $f(t)$.

$$f(t) = \frac{2V}{\pi} \sum_{n=-\infty}^{\infty} \left[\frac{e^{2n\pi t i}}{(1-4n^2)} \right]$$

$f(t) = V \cos\left(\frac{\pi t}{T}\right) \quad -\frac{T}{2} \leq t < \frac{T}{2}$

$f(t) = f(t+T)$

For a "complex Fourier"

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} [C_n e^{\frac{i2\pi nt}{T}}] \quad \text{where } C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-\frac{i2\pi nt}{T}} dt$$

$$\Rightarrow C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} V \cos\left(\frac{\pi t}{T}\right) e^{-\frac{i2\pi nt}{T}} dt$$

$$\Rightarrow C_n = \frac{V}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{\pi t}{T}\right) \left[\cos\left(\frac{2\pi nt}{T}\right) - i \sin\left(\frac{2\pi nt}{T}\right) \right] dt$$

$$\Rightarrow C_n = \frac{V}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} 2 \cos\frac{\pi t}{T} \cos\frac{2\pi nt}{T} dt$$

Now drawing an identity

$$\cos\left(2\pi nt + \frac{\pi t}{T}\right) = \cos\frac{2\pi nt}{T} \cos\frac{\pi t}{T} - \sin\frac{2\pi nt}{T} \sin\frac{\pi t}{T}$$

$$\cos\left(2\pi nt - \frac{\pi t}{T}\right) = \cos\frac{2\pi nt}{T} \cos\frac{\pi t}{T} + \sin\frac{2\pi nt}{T} \sin\frac{\pi t}{T}$$

Adding the two

$$\cos\left(2\pi nt + \frac{\pi t}{T}\right) + \cos\left(2\pi nt - \frac{\pi t}{T}\right) = 2 \cos\frac{2\pi nt}{T} \cos\frac{\pi t}{T}$$

$$\Rightarrow C_n = \frac{V}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\cos\left(2\pi nt + \frac{\pi t}{T}\right) + \cos\left(2\pi nt - \frac{\pi t}{T}\right) \right] dt$$

$$\Rightarrow C_n = \frac{V}{T} \left[\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2} \cos\left(2\pi nt + \frac{\pi t}{T}\right) dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2} \cos\left(2\pi nt - \frac{\pi t}{T}\right) dt \right]$$

$$\Rightarrow C_0 = \frac{V}{T} \left[\frac{1}{2\pi n} \sin\left(2\pi nt + \frac{\pi t}{T}\right) \Big|_{-\frac{T}{2}}^{\frac{T}{2}} + \frac{1}{2\pi n} \sin\left(2\pi nt - \frac{\pi t}{T}\right) \Big|_{-\frac{T}{2}}^{\frac{T}{2}} \right]$$

$$\Rightarrow C_0 = \frac{V}{T} \left[\frac{1}{2\pi n} \left[\sin\pi nt \cos\frac{\pi t}{T} + \cos\pi nt \sin\frac{\pi t}{T} \right] + \frac{1}{2\pi n} \left[\sin\pi nt \cos\frac{\pi t}{T} - \cos\pi nt \sin\frac{\pi t}{T} \right] \right]$$

$$\Rightarrow C_0 = \frac{V}{T} \left[\frac{\cos\pi nt}{2n+1} - \frac{\cos\pi nt}{2n-1} \right]$$

$$\Rightarrow C_0 = \frac{\sqrt{2}V\pi n}{\pi} \left[\frac{1}{2n+1} - \frac{1}{2n-1} \right]$$

$$\Rightarrow C_0 = \frac{\sqrt{-1}(-1)^n}{\pi} \left[\frac{2n-1-2n+1}{2n-1} \right]$$

$$\Rightarrow C_0 = \frac{2V(-1)^n}{\pi(C_0-4C_0)}$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{2V(-1)^n}{\pi(C_0-4C_0)} e^{\frac{i2\pi nt}{T}} \right] //$$