

PARAMETRIC EQUATIONS

Created by T. Madas

ALGEBRAIC ELIMINATIONS

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Question 1

Find a Cartesian equation for each of the following parametric relationships.

a) $x = t + 1, \quad y = 4 - 3t, \quad t \in \mathbb{R}$

b) $x = 2t + 1, \quad y = 3t - 2, \quad t \in \mathbb{R}$

c) $x = \frac{2}{t}, \quad y = 2t - 1, \quad t \in \mathbb{R}, \quad t \neq 0$

d) $x = 2t + 1, \quad y = t^2 - 1, \quad t \in \mathbb{R}$

$$\boxed{y = 7 - 3x}, \quad \boxed{3x - 2y = 7}, \quad \boxed{y = \frac{4}{x} - 1}, \quad \boxed{(x-1)^2 = 4(y+1)}$$

(a) $\begin{cases} x = t + 1 \\ y = 4 - 3t \end{cases} \Rightarrow t = x - 1$ [Sub into] $y = 4 - 3(x-1)$
 $y = 4 - 3x + 3$
 $y = 7 - 3x$

(b) $\begin{cases} x = 2t + 1 \\ y = 3t - 2 \end{cases} \times 3 \Rightarrow 3x = 6t + 3$ [Divide by 3] $3x - 2y = 7$
 $2y = 6t - 4$

(c) $\begin{cases} x = \frac{2}{t} \\ y = 2t - 1 \end{cases} \Rightarrow t = \frac{2}{x}$ [Sub into] $y = 2\left(\frac{2}{x}\right) - 1 \Rightarrow y = \frac{4}{x} - 1$

(d) $\begin{cases} x = 2t + 1 \\ y = t^2 - 1 \end{cases} \Rightarrow (x-1)^2 = (2t)^2 \Rightarrow 4t^2 = (x-1)^2 \Rightarrow$
 $t^2 = \frac{(x-1)^2}{4}$
 $\therefore 4(y+1) = (x-1)^2$
 $\therefore y = \frac{1}{4}(x-1)^2 - 1$
 $\therefore y = \frac{1}{4}x^2 - \frac{1}{2}x - \frac{3}{4}$

Question 2

Find a Cartesian equation for each of the following parametric relationships.

a) $x = 2t - 1, \quad y = 4 - 3t, \quad t \in \mathbb{R}$

b) $x = 2t - 1, \quad y = \frac{1}{t+1}, \quad t \in \mathbb{R}, \quad t \neq -1$

c) $x = t^2, \quad y = 2t^3, \quad t \in \mathbb{R}$

d) $x = \frac{1}{4t-1}, \quad y = \frac{t}{4t-1}, \quad t \in \mathbb{R}, \quad t \neq \frac{1}{4}$

$$\boxed{3x + 2y = 5}, \boxed{y = \frac{2}{x+3}}, \boxed{y^2 = 4x^3}, \boxed{y = \frac{1}{4}(x+1)}$$

(a) $\begin{cases} x = 2t - 1 \\ y = 4 - 3t \end{cases} \Rightarrow \begin{array}{l} \times 3 \\ \times 2 \end{array} \Rightarrow \begin{array}{l} 3x = 6t - 3 \\ 2y = 8 - 6t \end{array} \Rightarrow 3x + 2y = 5$

(b) $\begin{cases} x = 2t - 1 \\ y = \frac{1}{t+1} \end{cases} \Rightarrow \begin{array}{l} 2t = x+1 \\ t = \frac{x+1}{2} \end{array} \text{ Sub into } y = \frac{1}{t+1} \Rightarrow y = \frac{1}{\frac{x+1}{2}+1} \Rightarrow y = \frac{2}{x+3}$

(c) $\begin{cases} x = t^2 \\ y = 2t^3 \end{cases} \Rightarrow \begin{array}{l} \text{'cube'} \\ \text{'square'} \end{array} \Rightarrow \begin{array}{l} y^2 = t^6 \\ \frac{y^2}{t^4} = t^2 \end{array} \Rightarrow \begin{array}{l} y^2 = 4t^4 \\ y^2 = 4t^2 \end{array} \Rightarrow y^2 = 4x^3$

(d) $\begin{cases} x = \frac{1}{4t-1} \\ y = \frac{t}{4t-1} \end{cases} \Rightarrow \begin{array}{l} 4t-1 = \frac{1}{x} \\ 4t = \frac{1}{x} + 1 \\ t = \frac{1}{4x} + \frac{1}{4} \end{array} \text{ Sub into } y = \frac{t}{4t-1} \Rightarrow y = \frac{\frac{1}{4x} + \frac{1}{4}}{\frac{1}{x}} \Rightarrow y = \frac{1+4x}{4x} \Rightarrow y = \frac{1}{4}(x+1)$

Question 3

Find a Cartesian equation for each of the following parametric relationships.

a) $x = 1 - 4t^2$, $y = 1 + 2t$, $t \in \mathbb{R}$

b) $x = 3 - 4t, \quad y = 1 + \frac{2}{t}, \quad t \in \mathbb{R} \quad t \neq 0$

c) $x = t + 2$, $y = \ln(t - 1)$, $t \in \mathbb{R}$ $t > 1$

d) $x = e^{t-1}, \quad y = t + 7, \quad t \in \mathbb{R}$

$$\boxed{x = 2y - y^2}, \quad \boxed{y = 1 - \frac{8}{x-3}}, \quad \boxed{y = \ln(x-3)}, \quad \boxed{y = 8 + \ln x}$$

(4) $\begin{cases} x=1-4t \\ y=1+2t \end{cases} \Rightarrow \begin{cases} 4t=1-x \\ y=1+2t \end{cases} \Rightarrow \begin{cases} 4t=1-x \\ (y-1)=4t \end{cases} \Rightarrow \begin{cases} 1-x=y-1 \\ 1-x=4t \end{cases} \Rightarrow \begin{cases} x=y-2 \\ 1-x=\frac{1-x}{4} \end{cases}$
 $\therefore x=2y-3$

(5) $\begin{cases} x=3-\frac{1}{2}t \\ y=1+\frac{1}{2}t \end{cases} \Rightarrow \begin{cases} 3-x=\frac{1}{2}t \\ y=1+\frac{1}{2}t \end{cases} \Rightarrow \begin{cases} 3-x=y-1 \\ 3-x=\frac{1}{2}t \end{cases} \Rightarrow \begin{cases} y=4-x \\ y=6-2x \end{cases} \text{ or } y=1-\frac{x}{2}-\frac{1}{2}$

(6) $\begin{cases} x=t+2 \\ y=\ln(t-1) \end{cases} \Rightarrow \begin{cases} t=x-2 \\ y=\ln(x-1) \end{cases} \Rightarrow y=\ln(x-2)$

(7) $\begin{cases} x=e^{t-1} \\ y=t+7 \end{cases} \Rightarrow \begin{cases} \ln x=t-1 \\ y=t+7 \end{cases} \Rightarrow \begin{cases} \ln x=\ln(t+7) \\ y=t+7 \end{cases} \Rightarrow y=e^{\ln x}+7$

Question 4

Find a Cartesian equation for each of the following parametric expressions.

a) $x = t^2 - 1, \quad y = \frac{t^3}{5}, \quad t \in \mathbb{R}$

b) $x = \sqrt{t} + 1, \quad y = t - 1, \quad t \in \mathbb{R}, t \geq 0$

c) $x = 3t - 1, \quad y = (t - 2)(t + 1), \quad t \in \mathbb{R}$

d) $x = \frac{1}{t-2}, \quad y = t^2, \quad t \in \mathbb{R}, t \neq 2$

$$25y^2 = (x+1)^3, \quad y = x^2 - 2x, \quad 9y = (x-5)(x+4), \quad y = \left(\frac{2x+1}{x} \right)^2$$

$\textcircled{a} \quad x = t^2 - 1 \quad y = \frac{t^3}{5}$ $x+1 = t^2 \quad \cancel{y_1 = t^3}$ $(x+1)^2 = t^4 \quad (y_1)^2 = (t^3)^2$ $(x+1)^2 = t^4 \quad 25y_1^2 = t^6$ $\therefore (x+1)^2 = 25y_1^2 \quad \cancel{\cancel{}}$	$\textcircled{c} \quad x = 3t - 1 \quad y = \frac{(t-2)(t+1)}{t+1} = 3t - 1$ $y_2 = (t-2)(t+1)$ $y_2 = (t-2)(3t)$ $y_2 = (t-2)(3t+1)$ $\therefore y_2 = (x-1)(3x+3)$ $y_2 = (x-1)(3x+4) \quad \cancel{\cancel{}}$
$\textcircled{b} \quad x = \sqrt{t} + 1 \quad y = t - 1$ $x-1 = \sqrt{t} \quad \cancel{y_3 = t-1}$ $(x-1)^2 = t \quad (y_3)^2 = (t-1)^2$ $\therefore y_3 = (x-1)^2 - 1 \quad \cancel{\cancel{}}$	$\textcircled{d} \quad x = \frac{1}{t-2} \quad y = t^2$ $\frac{1}{x} = t-2 \quad \cancel{y_4 = t^2}$ $\frac{1}{x} + 2 = t \quad \cancel{\cancel{}}$ $\therefore y_4 = (\frac{1}{x} + 2)^2 \quad y_4 = (\frac{1}{x} + 2)^2$

Question 5

Find a Cartesian equation for each of the following parametric equations.

a) $x = 4t + 3, \quad y = \frac{1}{2t} - 1, \quad t \in \mathbb{R}, t \neq 0$

b) $x = 3 - 4t, \quad y = \frac{2}{t} + 1, \quad t \in \mathbb{R}, \quad t \neq 0$

$$\text{c) } x = \frac{1}{t-1}, \quad y = \frac{1}{t+2}, \quad t \in \mathbb{R}, t \neq 1, -2$$

$$\text{d) } x = \frac{t}{2t-1}, \quad y = \frac{t}{t+1}, \quad t \in \mathbb{R}, \quad t \neq -1, \frac{1}{2}$$

$$y = \frac{2}{x-3} - 1$$

$$y = \frac{x-11}{x-3}$$

$$y = \frac{x}{1+3x}$$

$$y = \frac{x}{3x-1}$$

(4) $\frac{xy+4x}{2x-1} \Rightarrow \frac{4t = x-2}{2t-1 = \frac{1}{x}} \left\{ \begin{array}{l} \frac{4t}{2t-1} = \frac{x-2}{\frac{1}{x}} \\ \Rightarrow y = \frac{\frac{2}{x}-1}{x-2} \\ \Rightarrow y = \frac{2-(x-2)}{x-2} \\ \Rightarrow y = \frac{4-x}{x-2} \end{array} \right.$

(5) $3x-4t=0 \Rightarrow \frac{4t = 3x}{y = \frac{2}{t} + 1} \left\{ \begin{array}{l} \frac{4t}{y} = \frac{3x}{\frac{2}{t} + 1} \\ \Rightarrow y = \frac{8+4t}{4t} \\ \Rightarrow y = \frac{1+(x-2)}{x-2} \\ \therefore y = \frac{1-\frac{2}{x}}{x-2} = \frac{x-1}{x(x-2)} \end{array} \right.$

(6) $x+\frac{1}{x-1} \Rightarrow t = \frac{1}{x-1} \Rightarrow x = \frac{1}{t} + 1$ atau $y = \frac{1}{\frac{1}{t} + 1 - 1} = \frac{1}{\frac{1}{t}}$
 $y = \frac{1}{t+1}$

(7) $x = \frac{t}{t-1} \Rightarrow 2xt - x = t \Rightarrow 2xt - t = x \Rightarrow t(2x-1) = x \Rightarrow t = \frac{x}{2x-1}$ atau $y = \frac{t}{t+1} = \frac{\frac{x}{2x-1}}{\frac{x}{2x-1} + 1} = \frac{x}{3x-1}$ MULAI TEPAT KETIADAAN Y

Question 6

Find a Cartesian equation for each of the following parametric expressions.

a) $x = 1 - 3t, \quad y = 1 + 2t^3, \quad t \in \mathbb{R}$

b) $x = \frac{1}{2t-3}, \quad y = \frac{t}{2t-3}, \quad t \in \mathbb{R}, \quad t \neq \frac{3}{2}$

c) $x = \frac{2}{2t-5}, \quad y = \frac{t}{4-t}, \quad t \in \mathbb{R}, \quad t \neq 4, \quad t \neq \frac{5}{2}$

d) $x = t + e^t, \quad y = t - e^t, \quad t \in \mathbb{R}$

$$\boxed{y = 1 + \frac{2}{27}(1-x)^3}, \boxed{2y - 3x = 1}, \boxed{y = \frac{5x+2}{3x-2}}, \boxed{x - y = 2e^{\frac{1}{2}(x+y)}}$$

<p>(a) $x = 1 - 3t \quad y = 1 + 2t^3$ $3t = 1 - x \quad y - 1 = 2t^3$ $\frac{3t^2}{3} = (1-x)^2, \quad 2t(y-1) = 5t^3$ $5t^3 = 2(1-x)^2$ $\therefore 2(y-1) = 2(1-x)^2$ $y-1 = \frac{2}{5}(1-x)^2$ $y = 1 + \frac{2}{5}(1-x)^2$</p>	<p>(c) $x = \frac{2}{2t-5}$ $\frac{1}{2} = \frac{2t-5}{2t-5}$ $\frac{2}{2} = \frac{2t-5}{2t-5}$ $\frac{2}{2} + 5 = x$</p>
<p>(b) $x = \frac{1}{2t-3}$ (eliminating fractions) $\frac{1}{2} = \frac{2t-3}{2t-3}$ $\frac{1}{2} + 3 = 2t$ $t + \frac{1}{2} + \frac{3}{2} = 2t$ $y = \frac{1}{2} + \frac{3}{2}$ $y = \frac{1}{2} + 3$ multiply top/bottom by 2 $y = \frac{1+3x}{2}$ $y = 1 + 3x$ $2y - 3x = 1$</p>	<p>(d) $x = t + e^t \quad y = t - e^t$ $x - y = 2e^t$ $\frac{2e^t}{2} = t$ $e^t - y = ze^t$ $e^t(e^t - z) = ze^t$ $e^{2t} - ze^t = ze^t$ $e^{2t} = 2ze^t$ $e^{2t}/e^t = 2z$ $e^t = 2z$ $x - y = 2e^{\frac{1}{2}(x+y)}$</p>

Question 7

Find a Cartesian equation for each of the following parametric expressions.

a) $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$, $t \in \mathbb{R}, t \neq 0$

b) $x = t^2 + \frac{1}{t}$, $y = t^2 - \frac{1}{t}$, $t \in \mathbb{R}, t \neq 0$

c) $x = 3t + \frac{1}{t^2}$, $y = 3t - \frac{1}{t^2}$, $t \in \mathbb{R}, t \neq 0$

d) $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$, $t \in \mathbb{R}$

$$\boxed{x^2 - y^2 = 4}, \quad \boxed{(x+y)(x-y)^2 = 8}, \quad \boxed{(x-y)(x+y)^2 = 72}, \quad \boxed{x^2 + y^2 = 1}$$

<p>(a)</p> $x = t + \frac{1}{t}$ $y = t - \frac{1}{t}$ $x+y = t + \frac{1}{t} + t - \frac{1}{t} = 2t$ $x-y = t + \frac{1}{t} - t + \frac{1}{t} = \frac{2}{t}$ $\therefore (x+y)(x-y) = (2t)\left(\frac{2}{t}\right)$ $(x+y)(x-y) = 4$	<p>(b)</p> $x = t^2 + \frac{1}{t}$ $y = t^2 - \frac{1}{t}$ $x+y = t^2 + \frac{1}{t} + t^2 - \frac{1}{t} = 2t^2$ $x-y = t^2 + \frac{1}{t} - t^2 + \frac{1}{t} = \frac{2}{t}$ $2t^2 = 2t$ $\therefore (x+y)(x-y)^2 = (2t^2)\left(\frac{2}{t}\right)^2$ $(x+y)(x-y)^2 = 8$	<p>(c)</p> $x = 3t + \frac{1}{t^2}$ $y = 3t - \frac{1}{t^2}$ $\begin{aligned} x+y &= 3t + \frac{1}{t^2} + 3t - \frac{1}{t^2} = 6t \\ x-y &= 3t + \frac{1}{t^2} - 3t + \frac{1}{t^2} = \frac{2}{t^2} \end{aligned}$ $\therefore (x+y)(x-y)^2 = 36t\left(\frac{2}{t^2}\right)^2$ $(x+y)(x-y)^2 = 72$	<p>(d)</p> $x = \frac{1-t^2}{1+t^2}$ $y = \frac{2t}{1+t^2}$ $\begin{aligned} x+y &= \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = \frac{1-t^2+2t}{1+t^2} \\ &= \frac{1-t^2+2t}{1+t^2} = \frac{1-2t^2+2t}{1+t^2} = \frac{1-2(t^2-1)}{1+t^2} = \frac{1-2(t-1)(t+1)}{1+t^2} \\ &\therefore \frac{1-2(t-1)(t+1)}{1+t^2} = \frac{4(1-\lambda)(1+\lambda)}{(1+\lambda)^2} \end{aligned}$ $\Rightarrow y^2 = \frac{4(1-\lambda)(1+\lambda)}{(1+\lambda)^2}$ $\Rightarrow y^2 = \frac{4(1-\lambda^2)}{(1+\lambda)^2}$ $\Rightarrow y^2 = \frac{4(1-\lambda^2)}{1+2\lambda+\lambda^2}$ $\Rightarrow y^2 = 1-2\lambda^2$ $\Rightarrow y^2 = \frac{4(1-\lambda^2)}{4(1-\lambda^2)}$ $\Rightarrow y^2 = 1$ $\therefore y^2 = 1$
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Question 8

Find a Cartesian equation for each of the following parametric equations.

a) $x = t - \frac{1}{t^3}$, $y = \frac{1}{t} - t^3$, $t \in \mathbb{R}, t \neq 0$

b) $x = \frac{4t}{1+t^2}$, $y = \frac{1-3t^2}{1+t^2}$, $t \in \mathbb{R}$

c) $x = \frac{1}{t} + \frac{1}{t^2}$, $y = \frac{1}{t} - \frac{1}{t^2}$, $t \in \mathbb{R}, t \neq 0$

d) $x = \frac{t^2}{1+t^3}$, $y = \frac{2t}{1+t^3}$, $t \in \mathbb{R}, t \neq -1$

$$\boxed{(y^2 - x^2)^2 + x^3 y^3 = 0}, \quad \boxed{x^2 + (y+1)^2 = 4}, \quad \boxed{(x+y)^2 = 2(x-y)}, \quad \boxed{y^3 + 8x^3 = 4xy}$$

(a)

$$x = t - \frac{1}{t^3} = \frac{t^4 - 1}{t^3} \quad \left\{ \begin{array}{l} y = \frac{1}{t} - t^3 = \frac{1-t^4}{t^3} \\ \text{Now } xy = (t-\frac{1}{t})(\frac{1}{t}-t^3) \end{array} \right. \Rightarrow \frac{y}{x} = \frac{\frac{1-t^4}{t^3}}{\frac{t^4-1}{t^3}} = -t^2 \quad \text{ie } \boxed{t^2 = -\frac{y}{x}}$$

$$\begin{aligned} -2xy &= (t-\frac{1}{t})(\frac{1}{t}-t^3) \\ -2xy &= t^2 - 2 + \frac{1}{t^2} \\ -2xy &= (t^2 - \frac{1}{t^2})^2 \\ -2xy &= (\frac{y^2 - x^2}{x^2})^2 \\ -2xy &= \frac{(y^2 - x^2)^2}{x^2} \\ -2xy &= (y^2 - x^2)^2 \\ -2xy &= (x^2 - y^2)^2 \\ (xy)^2 &= (x^2 - y^2)^2 \end{aligned}$$

(b)

$$\begin{aligned} \text{Starting with } y &= \frac{1-3t^2}{1+t^2} \\ &\rightarrow y + 3t^2 = 1 - t^2 \\ &\rightarrow 3t^2 + t^2 = 1 - y \\ &\rightarrow 4t^2(1+y) = 1 - y \\ &\Rightarrow t^2 = \frac{1-y}{1+y} \end{aligned}$$

Squaring (LHS) &

$$\begin{aligned} t^4 &= \frac{(1-y)^2}{(1+y)^2} \\ &\Rightarrow x^2 = \frac{4t^2(1+y)}{(1+y)^2} \\ &\Rightarrow x^2 = \frac{4(1-y)}{(1+\frac{1-y}{1+y})^2} \\ &\Rightarrow x^2 = \frac{4(1-y)}{(2+\frac{-2y}{1+y})^2} \end{aligned}$$

(c)

$$\left. \begin{array}{l} x = \frac{1}{t} + \frac{1}{t^2} \\ y = \frac{1}{t} - \frac{1}{t^2} \end{array} \right\} \Rightarrow \begin{array}{l} xy = \frac{2}{t^3} \Rightarrow (x+y)^2 = \frac{4}{t^2} \\ x-y = \frac{2}{t^2} \Rightarrow \frac{1}{x-y} = \frac{t^2}{2} \end{array}$$

Hence $\frac{(x+y)^2}{x-y} = \frac{4}{\frac{t^2}{2}} = 8$ ie $(x+y)^2 = 8(x-y)$

(d)

$$\left. \begin{array}{l} x = \frac{t^2}{1+t^3} \\ y = \frac{2t}{1+t^3} \end{array} \right\} \Rightarrow \begin{array}{l} \frac{y}{x} = \frac{2}{1+t^3} = \frac{2}{1+t^2} = \frac{2}{(1+t)(1-t)} = \frac{1-t}{1+t} \\ \Rightarrow y = \frac{2(1-t)}{1+t} \end{array}$$

Hence $\frac{y}{1-t} = \frac{2t}{1+t}$

$$\begin{aligned} &\Rightarrow y = \frac{2t}{1+t} \\ &\Rightarrow \frac{y}{t} = \frac{2}{1+\frac{1}{t}} \\ &\Rightarrow y(1 + \frac{1}{t}) = \frac{2t}{t} \\ &\Rightarrow y(\frac{t+1}{t}) = \frac{2t}{t} \\ &\Rightarrow y = \frac{2t}{t+1} \\ &\Rightarrow \frac{1}{t} (y^2 + 2ty) = \frac{2t}{t} \\ &\Rightarrow y^2 + 2ty = 2t^2 \end{aligned}$$

TRIGONOMETRIC ELIMINATIONS

Question 1

Find a Cartesian equation for each of the following parametric relationships.

a) $x = 5 \cos t, \quad y = 5 \sin t, \quad 0 \leq t < 2\pi$

b) $x = 1 + 2 \cos \theta, \quad y = 2 + \sin \theta, \quad 0 \leq \theta < 2\pi$

c) $x = 2 \tan \theta, \quad y = \cos \theta, \quad 0 \leq \theta < 2\pi$

d) $x = \cos t, \quad y = \operatorname{cosec} t, \quad 0 \leq t < 2\pi$

e) $x = \tan \theta, \quad y = \sin \theta, \quad 0 \leq \theta < 2\pi$

f) $x = \cos \theta, \quad y = \cos 2\theta, \quad 0 \leq \theta < 2\pi$

g) $x = \frac{1}{2} \cos 2\theta, \quad y = 2 \sin \theta, \quad 0 \leq \theta < 2\pi$

h) $x = \cos t, \quad y = \sin 2t, \quad 0 \leq t < 2\pi$

$$x^2 + y^2 = 25, \quad (x-1)^2 + 4(y-2)^2 = 4, \quad y^2 = \frac{4}{4+x^2}, \quad y^2 = \frac{1}{1-x^2}, \quad y^2 = \frac{x^2}{1+x^2},$$

$$y = 2x^2 - 1, \quad y^2 = 2 - 4x, \quad y^2 = 4x^2(1-x^2)$$

<p>(a) $x = 5 \cos t \quad y = 5 \sin t$ $\frac{x}{5} = \cos t \quad \frac{y}{5} = \sin t$ $[\cos^2 t + \sin^2 t] \rightarrow$ $(\frac{x}{5})^2 + (\frac{y}{5})^2 = 1$ $\frac{x^2}{25} + \frac{y^2}{25} = 1$ $x^2 + y^2 = 25$</p>	<p>(c) $x = \tan \theta \quad y = \sin \theta$ $\frac{x}{\tan \theta} = 1 \quad \frac{y}{\sin \theta} = \cos \theta$ $[\tan^2 \theta + \cos^2 \theta] \rightarrow$ $1 + \frac{y^2}{\sin^2 \theta} = \frac{x^2}{\tan^2 \theta}$ $\frac{x^2}{\tan^2 \theta} + \frac{y^2}{\sin^2 \theta} = 1$ $y^2 = \frac{\sin^2 \theta}{\tan^2 \theta} x^2$ $y^2 = \frac{1}{\tan^2 \theta} x^2$ $y^2 = \frac{x^2}{\tan^2 \theta}$</p>
<p>(b) $x = 1 + 2 \cos \theta \quad y = 2 + \sin \theta$ $\frac{x-1}{2} = \cos \theta \quad \frac{y-2}{1} = \sin \theta$ $[\cos^2 \theta + \sin^2 \theta] \rightarrow$ $(\frac{x-1}{2})^2 + (\frac{y-2}{1})^2 = 1$ $\frac{(x-1)^2}{4} + (y-2)^2 = 1$ $(x-1)^2 + 4(y-2)^2 = 4$</p>	<p>(d) $x = \cos \theta \quad y = \cos 2\theta$ $y = 2 \cos^2 \theta - 1$ $y = 2 - 2 \cos^2 \theta$ $y = 2 - y^2$</p>
<p>(e) $x = 2 \tan \theta \quad y = \cos \theta$ $\frac{x}{2} = \tan \theta \quad \frac{y}{\cos \theta} = 1$ $[\tan^2 \theta + 1] \rightarrow$ $1 + \frac{y^2}{\cos^2 \theta} = \frac{x^2}{4}$ $\frac{4}{x^2} + \frac{y^2}{\cos^2 \theta} = 1$ $y^2 = \frac{\cos^2 \theta}{4} x^2$ $y^2 = \frac{a^2}{4} x^2$</p>	<p>(f) $x = \frac{1}{2} \cos 2\theta \quad y = 2 \sin \theta$ $x = \frac{1}{2} (1 - 2 \sin^2 \theta) \quad y = 2 \sin \theta$ $x = \frac{1}{2} (1 - \frac{y^2}{4}) \quad y = 2 \sin \theta$ $x = \frac{1}{2} - \frac{1}{2} \frac{y^2}{4}$ $2x = 1 - \frac{y^2}{4}$ $y^2 = 4x + 4$ $y^2 = 4(x + 1)$</p>
<p>(g) $x = \cos t \quad y = \operatorname{cosec} t$ $x = \cos t \quad \frac{y}{\sin t} = 1$ $[\cos^2 t + \sin^2 t] \rightarrow$ $1 + \frac{y^2}{\sin^2 t} = 1$ $\frac{y^2}{\sin^2 t} = 1 - 1$ $\frac{y^2}{\sin^2 t} = 0$ $y^2 = 0$</p>	<p>(h) $x = \cos t \quad y = \sin 2t$ $y = 2 \sin t \cos t$ $y = 4 \sin t \cos^2 t$ $y = 4(1 - \cos^2 t) \cos^2 t$ $y^2 = 4(1 - x^2)x^2$</p>

Question 2

Find a Cartesian equation for each of the following parametric relationships.

a) $x = 2 \cos t, \quad y = 2 \sin t, \quad 0 \leq t < 2\pi$

b) $x = 4 + 3 \cos t, \quad y = -2 + 3 \sin t, \quad 0 \leq t < 2\pi$

c) $x = 4 + \cos t, \quad y = 2 \sin t, \quad 0 \leq t < 2\pi$

d) $x = \sin t, \quad y = \sec t, \quad 0 \leq t < 2\pi$

$$\boxed{x^2 + y^2 = 4}, \quad \boxed{(x-4)^2 + (y+2)^2 = 9}, \quad \boxed{4(x-4)^2 + y^2 = 4}, \quad \boxed{y^2 = \frac{1}{1-x^2}}$$

(a) $\begin{cases} x = 2 \cos t \\ y = 2 \sin t \end{cases} \Rightarrow \begin{cases} \frac{x}{2} = \cos t \\ \frac{y}{2} = \sin t \end{cases} \Rightarrow \begin{aligned} \cos^2 t + \sin^2 t &= 1 \\ \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 &= 1 \\ x^2 + y^2 &= 4 \end{aligned}$

(b) $\begin{cases} x = 4 + 3 \cos t \\ y = -2 + 3 \sin t \end{cases} \Rightarrow \begin{cases} x-4 = 3 \cos t \\ \frac{y+2}{3} = \sin t \end{cases} \Rightarrow \begin{aligned} \cos^2 t + \sin^2 t &= 1 \\ \left(\frac{x-4}{3}\right)^2 + \left(\frac{y+2}{3}\right)^2 &= 1 \\ (x-4)^2 + (y+2)^2 &= 9 \end{aligned}$

(c) $\begin{cases} x = 4 + \cos t \\ y = 2 \sin t \end{cases} \Rightarrow \begin{cases} x-4 = \cos t \\ \frac{y}{2} = \sin t \end{cases} \Rightarrow \begin{aligned} \cos^2 t + \sin^2 t &= 1 \\ (x-4)^2 + \left(\frac{y}{2}\right)^2 &= 1 \\ 4(x-4)^2 + y^2 &= 16 \end{aligned}$

(d) $\begin{cases} x = \sin t \\ y = \sec t \end{cases} \Rightarrow \begin{cases} x = \sin t \\ \frac{1}{y} = \cos t \end{cases} \Rightarrow \begin{aligned} \cos^2 t + \sin^2 t &= 1 \\ \frac{1}{y^2} + x^2 &= 1 \\ y^2 &= \frac{1}{1-x^2} \end{aligned}$

Question 3

Eliminate θ to obtain a Cartesian equation for the following parametric equations.

a) $x = \tan \theta$, $y = \sec \theta$, $0 \leq \theta < 2\pi$

b) $x = 2 \sin \theta$, $y = 3 \operatorname{cosec} \theta$, $0 \leq \theta < 2\pi$

c) $x = \sin \theta$, $y = \sec^2 \theta$, $0 \leq \theta < 2\pi$

d) $x = \cos \theta$, $y = \tan^2 \theta$, $0 \leq \theta < 2\pi$

$$\boxed{y^2 = x^2 + 1}, \quad \boxed{y = \frac{6}{x}}, \quad \boxed{y = \frac{1}{1-x^2}}, \quad \boxed{y = \frac{1}{x^2} - 1}$$

The handwritten working shows the steps for each part (a, b, c, d) to eliminate θ and find the Cartesian equation.

- (a) $x = \tan \theta$, $y = \sec \theta$ leads to $y^2 = x^2 + 1$.
- (b) $x = 2 \sin \theta$, $y = 3 \operatorname{cosec} \theta$ leads to $y = \frac{6}{x}$.
- (c) $x = \sin \theta$, $y = \sec^2 \theta$ leads to $y = \frac{1}{1-x^2}$.
- (d) $x = \cos \theta$, $y = \tan^2 \theta$ leads to $y = \frac{1}{x^2} - 1$.

Question 4

Eliminate the parameter θ to obtain a Cartesian equation for each of the following parametric equations.

a) $x = \sin \theta, \quad y = \tan^2 \theta, \quad 0 \leq \theta < 2\pi$

b) $x = 2 \sec \theta, \quad y = \sin^2 \theta, \quad 0 \leq \theta < 2\pi$

c) $x = 3 \cos \theta, \quad y = 2 \cot \theta, \quad 0 \leq \theta < 2\pi$

d) $x = \frac{1}{2} \cos \theta, \quad y = 2 \cos 2\theta, \quad 0 \leq \theta < 2\pi$

$$\boxed{y = \frac{x^2}{1-x^2}}, \quad \boxed{y = 1 - \frac{4}{x^2}}, \quad \boxed{y^2 = \frac{4x^2}{9-x^2}}, \quad \boxed{y = 2(8x^2 - 1)}$$

(a) $\begin{cases} x = \sin \theta \\ y = \tan^2 \theta \end{cases} \Rightarrow \begin{cases} \frac{1}{x} = \csc \theta \\ \frac{y}{1} = \cot^2 \theta \end{cases} \Rightarrow \begin{cases} 1 + \cot^2 \theta = \csc^2 \theta \\ 1 + \frac{y}{x} = \frac{1}{x^2} \end{cases}$

$$\frac{1}{x} = \frac{1}{x^2} - 1 \Rightarrow \frac{1}{x^2} = \frac{1-x^2}{x^2}$$

$$y = \frac{1-x^2}{1-x^2} \cancel{\cancel{}}$$

(b) $\begin{cases} x = 2 \sec \theta \\ y = \sin^2 \theta \end{cases} \Rightarrow \begin{cases} \frac{x}{2} = \sec \theta \\ \frac{y}{1} = \sin^2 \theta \end{cases} \Rightarrow \begin{cases} \frac{x}{2} = \sec \theta \\ \frac{y}{1} = \tan^2 \theta \end{cases}$

$$y = \frac{1}{\sec^2 \theta} = \frac{1}{1+\tan^2 \theta} \Rightarrow \frac{1}{\sec^2 \theta} + \frac{1}{1+\tan^2 \theta} = 1 \Rightarrow \frac{1}{\sec^2 \theta} + \frac{1}{\sec^2 \theta} = 1 \Rightarrow y = 1 - \frac{1}{\sec^2 \theta} \cancel{\cancel{}}$$

(c) $\begin{cases} x = 3 \cos \theta \\ y = 2 \cot \theta \end{cases} \Rightarrow \begin{cases} \frac{x}{3} = \cos \theta \\ \frac{y}{2} = \cot \theta \end{cases} \Rightarrow \begin{cases} \frac{x}{3} = \cos \theta \\ \frac{y}{2} = \frac{1}{\tan \theta} \end{cases} \Rightarrow \begin{cases} \frac{x}{3} = \cos \theta \\ \frac{y}{2} = \frac{\cos \theta}{\sin \theta} \end{cases}$

$$1 + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1+3\cos^2 \theta}{9\sin^2 \theta} = \frac{1+3\cos^2 \theta}{9(1-\cos^2 \theta)} \Rightarrow \frac{1}{9\sin^2 \theta} = \frac{1+3\cos^2 \theta}{9(1-\cos^2 \theta)} \Rightarrow \frac{1}{\sin^2 \theta} = \frac{1+3\cos^2 \theta}{1-3\cos^2 \theta}$$

$$y = \frac{1-3\cos^2 \theta}{1+3\cos^2 \theta} \cancel{\cancel{}}$$

(d) $\begin{cases} x = \frac{1}{2} \cos \theta \\ y = 2 \cos 2\theta \end{cases} \Rightarrow \begin{cases} 2x = \cos \theta \\ y = 2(2\cos^2 \theta - 1) \end{cases} \Rightarrow y = 2[2(2x)^2 - 1] \Rightarrow y = 16x^2 - 2 \cancel{\cancel{}}$

Question 5

Find a Cartesian equation for each of the following parametric relationships.

- a) $x = 2 \sin^2 \theta, \quad y = \cot \theta, \quad 0 \leq \theta < 2\pi$
- b) $x = 2 \sin \theta, \quad y = \cos 2\theta, \quad 0 \leq \theta < 2\pi$
- c) $x = 2 \cos \theta, \quad y = 6 \cos 2\theta, \quad 0 \leq \theta < 2\pi$
- d) $x = 2 \cos \theta, \quad y = 6 \sin 2\theta, \quad 0 \leq \theta < 2\pi$

$$\boxed{y^2 = \frac{2-x}{x}}, \quad \boxed{y = 1 - \frac{1}{2}x^2}, \quad \boxed{y = 3x^2 - 6}, \quad \boxed{y^2 = 9x^2(4-x^2)}$$

(a) $\begin{cases} x = 2 \sin^2 \theta \\ y = \cot \theta \end{cases} \Rightarrow \begin{cases} \frac{x}{2} = \sin^2 \theta \\ y = \cot \theta \end{cases} \Rightarrow \begin{cases} \frac{x}{2} = \cos^2 \theta \\ y = \cot \theta \end{cases} \Rightarrow \begin{cases} 1 + \frac{x}{2} = 1 \\ y = \frac{\cos \theta}{\sin \theta} \end{cases} \Rightarrow \begin{cases} \frac{x}{2} = 0 \\ y^2 = \frac{1}{\sin^2 \theta} - 1 \end{cases} \Rightarrow y^2 = \frac{2-x}{x}$

(b) $\begin{cases} x = 2 \sin \theta \\ y = \cos 2\theta \end{cases} \Rightarrow \begin{cases} \frac{x}{2} = \sin \theta \\ y = 1 - 2 \sin^2 \theta \end{cases} \Rightarrow y = 1 - 2(\frac{x}{2})^2 \Rightarrow y = 1 - \frac{1}{2}x^2$

(c) $\begin{cases} x = 2 \cos \theta \\ y = 6 \cos 2\theta \end{cases} \Rightarrow \begin{cases} \frac{x}{2} = \cos \theta \\ y = 6 \cos 2\theta \end{cases} \Rightarrow \begin{cases} \frac{x}{2} = \cos^2 \theta - \sin^2 \theta \\ y = 6(2 \cos^2 \theta - 1) \end{cases} \Rightarrow y = 6[\frac{1}{2}(2(\frac{x}{2})^2 - 1)] \Rightarrow y = 3x^2 - 6$

(d) $\begin{cases} x = 2 \cos \theta \\ y = 6 \sin 2\theta \end{cases} \Rightarrow \begin{cases} \frac{x}{2} = \cos \theta \\ y = 12 \sin \theta \cos \theta \end{cases} \Rightarrow y = 12 \sin \theta \cos \theta \Rightarrow y^2 = 144 \sin^2 \theta \cos^2 \theta$

$$y^2 = 144 \cdot (\frac{1}{4}x^2)(1 - \frac{x^2}{4})$$

$$y^2 = 144 \cdot \frac{1}{4}x^2 \cdot (1 - \frac{x^2}{4})$$

$$y^2 = 144 \cdot \frac{1}{4}x^2 \cdot \frac{4-x^2}{4}$$

$$y^2 = 9x^2(4-x^2)$$

Question 6

Eliminate the parameter θ to obtain a Cartesian equation for each of the following parametric equations.

a) $x = \sin^2 \theta, \quad y = \sin 2\theta, \quad 0 \leq \theta < 2\pi$

b) $x = \sin \theta + \cos \theta, \quad y = \sin \theta - \cos \theta, \quad 0 \leq \theta < 2\pi$

c) $x = \cos 2\theta, \quad y = \tan \theta, \quad 0 \leq \theta < 2\pi$

d) $x = \tan \theta, \quad y = 2 \sin 2\theta, \quad 0 \leq \theta < 2\pi$

$$\boxed{y^2 = 4x(1-x)}, \quad \boxed{x^2 + y^2 = 2}, \quad \boxed{y^2 = \frac{1-x}{1+x}}, \quad \boxed{y = \frac{4x}{1+x^2}}$$

(c) $\begin{aligned} x &= \sin \theta \\ y &= \sin 2\theta \end{aligned} \Rightarrow \begin{aligned} x &= \sin \theta \\ y &= 2 \sin \theta \cos \theta \end{aligned} \Rightarrow \begin{aligned} x^2 &= \sin^2 \theta \\ y^2 &= 4 \sin^2 \theta (1 - \sin^2 \theta) \end{aligned} \Rightarrow y^2 = 4x(1-x)$

differentiate implicitly

$\begin{aligned} x &= \frac{1}{2} + \frac{1}{2} \cos 2\theta \\ y &= \sin 2\theta \end{aligned} \Rightarrow \begin{aligned} \cos 2\theta &= 1-2x \\ \sin 2\theta &= y \end{aligned} \Rightarrow \begin{aligned} \cos^2 2\theta + \sin^2 2\theta &= 1 \\ (1-2x)^2 + y^2 &= 1 \\ 4x^2 - 4x + 1 + y^2 &= 1 \\ y^2 &= 4x - 4x^2 \\ y^2 &= 4x(1-x)$

(d) $\begin{aligned} x &= \sin \theta \cos \theta \\ y &= \sin \theta - \cos \theta \end{aligned} \Rightarrow \begin{aligned} x^2+y^2 &= (\sin \theta \cos \theta)^2 + (\sin \theta - \cos \theta)^2 \\ &= \sin^2 \theta \cos^2 \theta + \sin^2 \theta - 2 \sin \theta \cos \theta + \cos^2 \theta \\ &= \sin^2 \theta (\cos^2 \theta + 1) + \cos^2 \theta - 2 \sin \theta \cos \theta \\ &= \sin^2 \theta (1 + \sin^2 \theta) + \cos^2 \theta - 2 \sin \theta \cos \theta \\ &= \sin^2 \theta + \sin^4 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta \\ &= 1 - 2 \sin \theta \cos \theta \\ &= 1 - \sin 2\theta \end{aligned}$

(c) $\omega = 2\sin 2\theta$

$$\omega = 2\cos^2 \theta - 1$$

$$2\cos^2 \theta = 2\sin^2 \theta$$

$$\frac{\omega+1}{2} = \sin^2 \theta$$

$$\left[\frac{\omega}{2+1} = \sin^2 \theta \right]$$

(d) $y = 2\sin 2\theta$

$$y = 4\sin(\theta + \theta)$$

$$2 = \tan \theta$$

$$x^2 = \tan^2 \theta$$

$$x^2 + 1 = \tan^2 \theta + 1$$

$$x^2 + 1 = \sec^2 \theta$$

$$\left[\frac{1}{x^2+1} = \cos^2 \theta \right]$$

$\omega = 2\sin 2\theta$

$$\omega = 2\cos^2 \theta - 1$$

$$2\cos^2 \theta = 2\sin^2 \theta$$

$$\frac{\omega+1}{2} = \sin^2 \theta$$

$$\left[\frac{\omega}{2+1} = \sin^2 \theta \right]$$

$y = 4\sin(\theta + \theta)$

$$y = 16\sin(\theta + \theta)$$

$$y = 16 \cdot \frac{-3^2}{(1+3^2)} = \frac{1}{1+3^2}$$

$$y = \frac{48}{1+9} = \frac{48}{10} = 4.8$$

Question 7

Eliminate the parameter θ to obtain a Cartesian equation for each of the following parametric expressions.

a) $x = \sin \theta \cos \theta, \quad y = 4 \cos^2 \theta, \quad 0 \leq \theta < 2\pi$

b) $x = \sin 2\theta, \quad y = \cot \theta, \quad 0 \leq \theta < 2\pi$

c) $x = \sin^2 \theta, \quad y = \tan 2\theta, \quad 0 \leq \theta < 2\pi$

d) $x = \operatorname{cosec} \theta - \sin \theta, \quad y = \sec \theta - \cos \theta, \quad 0 \leq \theta < 2\pi$

$$16x^2 = y(4-y), \quad y(2-xy) = x, \quad y^2 = \frac{4x(1-x)}{(1-2x)^2}, \quad y^2 x^2 \left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^3 = 1$$

(a) $x = \sin \theta \cos \theta$ $y = 4 \cos^2 \theta$

$$\begin{aligned} \Rightarrow x^2 &= \sin^2 \theta \cos^2 \theta \\ \Rightarrow x^2 &= (-\cos^2 \theta)(\cos^2 \theta) \\ \Rightarrow 16x^2 &= 4\cos^2 \theta (4 - 4\cos^2 \theta) \\ \Rightarrow 16x^2 &= 4(4-y) \end{aligned}$$

(b) $x = \sin 2\theta$ $y = \cot \theta$

$$\begin{aligned} \Rightarrow x &= 2\sin \theta \cos \theta \\ \Rightarrow x^2 &= 4\sin^2 \theta \cos^2 \theta \\ \text{Now } y^2 \cdot \cot^2 \theta &= \cos^2 \theta - 1 \\ y^2 \cot^2 \theta &= \cos^2 \theta - 1 \\ \frac{y^2}{\sin^2 \theta} &= \frac{\cos^2 \theta - 1}{\cos^2 \theta} \\ \text{Thus, } x^2 &= 4 \times \frac{1}{\sin^2 \theta} \times \left(1 - \frac{1}{\sin^2 \theta}\right) \\ \Rightarrow x^2 &= \frac{4}{\sin^2 \theta} \times \frac{y^2 + 1}{y^2 + 1} \\ \Rightarrow x^2 &= \frac{4y^2}{(y^2 + 1)^2} \\ \Rightarrow x &= \frac{2y}{y^2 + 1} \end{aligned}$$

$$\begin{aligned} \Rightarrow x^2 + y^2 &= 2y \\ \Rightarrow x &= y(2-y) \\ \text{Hence, } x &= y(2-y) \end{aligned}$$

(c) $x = \sin \theta \cos \theta$ $y = \tan 2\theta$

$$\begin{aligned} \Rightarrow y &= \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2 \tan \theta}{\frac{1 - \tan^2 \theta}{\tan^2 \theta}} = \frac{2 \tan \theta}{\tan^2 \theta - 1} \\ \Rightarrow y &= \frac{2 \tan \theta}{\tan^2 \theta - 1} \\ \Rightarrow y^2 &= \frac{4 \tan^2 \theta}{(\tan^2 \theta - 1)^2} = \frac{4 (\cos^2 \theta - 1)}{(\cos^2 \theta - 1)^2} \\ \text{But } \sin^2 \theta &\geq 0, \text{ so } \cos^2 \theta = 1 - \sin^2 \theta \\ \Rightarrow y^2 &= \frac{4 \left(\frac{1 - \sin^2 \theta}{1 - \sin^2 \theta}\right)}{\left(\frac{1 - \sin^2 \theta}{1 - \sin^2 \theta}\right)^2} = \frac{4 \left(\frac{1 - \sin^2 \theta}{1 - \sin^2 \theta}\right)}{\left(\frac{1 - 2\sin^2 \theta + \sin^4 \theta}{1 - \sin^2 \theta}\right)} \\ \Rightarrow y^2 &= \frac{4(1 - \sin^2 \theta)}{(1 - 2\sin^2 \theta + \sin^4 \theta)} \end{aligned}$$

(d) $x = \cos \theta - \sin \theta$ $y = \sec \theta - \cos \theta$

$$\begin{aligned} \Rightarrow \frac{x}{y} &= \frac{\cos \theta - \sin \theta}{\sec \theta - \cos \theta} = \frac{1 - \sin \theta}{\sec \theta - 1} = \frac{1 - \sin \theta}{\frac{1}{\cos \theta} - 1} = \frac{\cos \theta}{1 - \cos \theta} = \frac{\cos \theta}{\sin \theta} = \tan \theta \sin \theta \\ \frac{x}{y} &= \frac{\tan \theta \sin \theta}{\sin \theta} = \tan \theta \quad \therefore \tan \theta = \left(\frac{x}{y}\right)^{\frac{1}{2}} \\ y &= \tan \theta \sin \theta \\ y &= \frac{\tan \theta}{1 + \tan^2 \theta} \sin \theta \\ y &= \tan \theta \left(1 - \frac{1}{1 + \tan^2 \theta}\right) \\ y &= \tan \theta \left(1 - \frac{1}{1 + \frac{y^2}{x^2}}\right) \\ y &= \tan \theta \left(\frac{x^2 - 1}{x^2 + y^2}\right) \\ y &= \frac{\tan \theta}{1 + \tan^2 \theta} \left(\frac{x^2 - 1}{x^2 + y^2}\right) \\ y^2 &= \frac{\tan^2 \theta}{1 + \tan^2 \theta} \left(\frac{x^2 - 1}{x^2 + y^2}\right)^2 \end{aligned}$$

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$$\begin{aligned} y^2 &= \frac{\left(\frac{y^2}{x^2}\right)^2}{1 + \left(\frac{y^2}{x^2}\right)^2} = \frac{\frac{y^4}{x^4}}{1 + \frac{y^4}{x^4}} \\ \Rightarrow y^2 &= \frac{y^4}{x^4 + y^4} \\ \Rightarrow y^2 &= \frac{1}{x^2 + y^2} \\ \Rightarrow y^2(x^2 + y^2) &= 1 \\ \Rightarrow y^2 x^2 + y^4 &= 1 \end{aligned}$$

PARAMETRIC ALGEBRA

Question 1

Find the x and y intercepts for each pair of parametric equations.

a) $x = 2t + 1$, $y = 2t + 6$, $t \in \mathbb{R}$

b) $x = t^2$, $y = (t+1)(t+2)$, $t \in \mathbb{R}$

c) $x = \frac{t-1}{t+1}$, $y = \frac{2t}{t^2+1}$, $t \in \mathbb{R}$, $t \neq -1$

$\boxed{(0,5) \text{ & } (-5,0)}$, $\boxed{(0,2) \text{ & } (4,0), (1,0)}$, $\boxed{(0,1) \text{ & } (-1,0)}$

(a) $x = 2t + 1$
 $y = 2t + 6$

- when $x = 0$
 $0 = 2t + 1$
 $t = -\frac{1}{2}$
 $\therefore y = 2(-\frac{1}{2}) + 6$
 $y = 5$
 $\therefore (x, y) = (0, 5)$
- when $y = 0$
 $0 = 2t + 6$
 $t = -3$
 $\therefore x = 2(-3) + 1$
 $x = -5$
 $\therefore (-5, 0)$

(b) $x = t^2$
 $y = (t+1)(t+2)$

- when $x = 0$
 $0 = t^2$
 $t = 0$
 $\therefore y = (0+1)(0+2)$
 $y = 2$
 $\therefore (0, 2)$
- when $y = 0$
 $0 = (t+1)(t+2)$
 $t = -1, -2$
 $\therefore x = (-1)^2 = 1$
 $x = 4$
 $\therefore (1, 0) \text{ & } (4, 0)$

(c) $x = \frac{t-1}{t+1}$
 $y = \frac{2t}{t^2+1}$

- when $x = 0$
 $0 = \frac{t-1}{t+1}$
 $t = 1$
 $\therefore y = \frac{2(1)}{1^2+1} = 1$
 $\therefore (1, 1)$
- when $y = 0$
 $0 = \frac{2t}{t^2+1}$
 $t = 0$
 $\therefore x = \frac{0-1}{0+1} = -1$
 $\therefore (-1, 0)$

Question 2

A curve is defined by the following parametric equations

$$x = 4at^2, \quad y = a(2t+1), \quad t \in \mathbb{R}.$$

where a is non zero constant.

Given the curves passes through the point $A(4, 0)$, find the value of a .

$\boxed{a = 4}$

$$\begin{aligned} \begin{cases} x = 4at^2 \\ y = a(2t+1) \end{cases} \Rightarrow \begin{cases} 4at^2 = 4 \\ a(2t+1) = 0 \end{cases} \Rightarrow \begin{cases} at^2 = 1 \\ 2t+1 = 0 \end{cases} \Rightarrow \begin{cases} t = \pm \frac{1}{\sqrt{a}} \\ t = -\frac{1}{2} \end{cases} \end{aligned}$$

$a \neq 0$

$\therefore a(\frac{-1}{2})^2 = 1$
 $\frac{1}{4}a = 1$
 $a = 4$

Question 3

A curve is given by the parametric equations

$$x = 2t^2 - 1, \quad y = 3(t+1), \quad t \in \mathbb{R}.$$

Find the coordinates of the points of intersection of this curve and the line with equation

$$3x - 4y = 3.$$

$(17, 12) \text{ & } (1, 0)$

$\begin{cases} x = 2t^2 - 1 \\ y = 3(t+1) \end{cases}$ SOLVING SIMULTANEOUSLY
 $\Rightarrow 3(2t^2 - 1) - 4(3(t+1)) = 3$
 $\Rightarrow 6t^2 - 3 - 12(t+1) = 3$
 $\Rightarrow 6t^2 - 3 - 12t - 12 = 3$
 $\Rightarrow 6t^2 - 12t - 18 = 0$
 $\Rightarrow t^2 - 2t - 3 = 0$
 $\Rightarrow (t+1)(t-3) = 0$
 $\Rightarrow t = -1 \quad \Rightarrow x = 1, \quad y = 0$
 $\therefore (1, 0) \text{ & } (17, 12)$

Question 4

The curve C_1 has Cartesian equation

$$x^2 + y^2 = 9x - 4.$$

The curve C_2 has parametric equations

$$x = t^2, \quad y = 2t, \quad t \in \mathbb{R}.$$

Find the coordinates of the points of intersection of C_1 and C_2 .

$(4, 4), (4, -4), (1, 2), (1, -2)$

$\begin{cases} x^2 + y^2 = 9x - 4 \\ x = t^2 \\ y = 2t \end{cases}$ SOLVING SIMULTANEOUSLY
 $(t^2)^2 + (2t)^2 = 9t^2 - 4$
 $t^4 + 4t^2 = 9t^2 - 4$
 $t^4 - 5t^2 - 4 = 0$
 $(t^2 - 1)(t^2 + 4) = 0$
 $t^2 = 1 \quad \Rightarrow t = \pm 1$
 $\therefore (1, 2), (1, -2), (4, 4), (4, -4)$

Question 5

The curve with Cartesian equation $xy = 3$ is also traced by the following parametric equations

$$x = \frac{4tp}{t+p}, \quad y = \frac{4}{t+p}, \quad t, p \in \mathbb{R}, \quad t \neq -p$$

where t and p are parameters.

Find the relationship between the two parameters t and p in the form $p = f(t)$.

$$p = 3t \quad \text{or} \quad p = \frac{1}{3}t$$

$$\begin{aligned} \left. \begin{aligned} x &= \frac{4tp}{t+p} \\ y &= \frac{4}{t+p} \end{aligned} \right\} \Rightarrow xy = 3 \Rightarrow \frac{16tp}{(t+p)^2} = 3 \\ \Rightarrow 16tp = 3(t+p)^2 \\ \Rightarrow 16tp = 3t^2 + 6tp + 3p^2 \\ \Rightarrow 0 = 3t^2 - 10tp + 3p^2 \\ \Rightarrow 0 = (3t - p)(t - 3p) \\ \Rightarrow t = \frac{1}{3}p \quad \text{or} \quad p = \frac{3t}{3-t} \end{aligned}$$

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PARAMETRIC DIFFERENTIATION

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Question 1

A curve is given parametrically by the equations

$$x = 1 - \cos 2\theta, \quad y = \sin 2\theta, \quad 0 \leq \theta < 2\pi.$$

The point P lies on this curve, and the value of θ at that point is $\frac{\pi}{6}$.

Show that an equation of the normal at the point P is given by

$$y + \sqrt{3}x = \sqrt{3}.$$

proof

$$\begin{aligned}
 \left. \begin{aligned} x &= 1 - \cos 2\theta \\ y &= \sin 2\theta \end{aligned} \right\} &\Rightarrow \frac{dx}{d\theta} = 2\sin 2\theta \\
 &\Rightarrow \frac{dy}{d\theta} = 2\cos 2\theta \\
 \bullet \frac{1}{dx} \times \frac{dy}{d\theta} &= \frac{2\cos 2\theta}{2\sin 2\theta} = \frac{\cos 2\theta}{\sin 2\theta} \Rightarrow \frac{dy}{dx} = \frac{\cos 2\theta}{2\sin 2\theta} = \frac{\sqrt{3}}{3} \\
 \bullet \text{when } \theta = \frac{\pi}{6} &\Rightarrow x = 1 - \cos \frac{\pi}{3} = \frac{1}{2} \\
 &\Rightarrow y = \sin \frac{\pi}{6} = \frac{1}{2} \\
 \bullet \text{normal: } m &= -\frac{1}{\sqrt{3}} = -\sqrt{3} \quad \text{a } (\frac{1}{2}, \frac{1}{2}) \\
 y - \frac{1}{2} &= -\sqrt{3}(x - \frac{1}{2}) \\
 y - \frac{1}{2} &= \sqrt{3}x + \frac{\sqrt{3}}{2} \\
 y + \sqrt{3}x &= \sqrt{3}
 \end{aligned}$$

Question 2

A curve is given parametrically by the equations

$$x = \frac{2}{t}, \quad y = t^2 - 1, \quad t \in \mathbb{R}, \quad t \neq 0.$$

The point $P(4, y)$ lies on this curve.

Show that an equation of the tangent at the point P is given by

$$x + 8y + 2 = 0.$$

proof

$$\begin{aligned} & \left\{ \begin{array}{l} x = \frac{2}{t} = 2t \\ y = t^2 - 1 \end{array} \right. \quad \left\{ \begin{array}{l} \frac{dx}{dt} = \frac{d}{dt}(2t) = 2 \\ \frac{dy}{dt} = \frac{d}{dt}(t^2 - 1) = 2t^2 \end{array} \right. \\ & \text{When } x=4 \quad t=\frac{1}{2} \\ & \therefore y = \left(\frac{1}{2}\right)^2 - 1 = -\frac{3}{4} \\ & \therefore P\left(4, -\frac{3}{4}\right) \text{ at } t=\frac{1}{2} \quad \left. \begin{array}{l} \frac{du}{dx} = \frac{du}{dt} \Big|_{t=\frac{1}{2}} = -\left(\frac{1}{2}\right)^3 = -\frac{1}{8} \\ y - y_0 = m(x - x_0) \\ y + \frac{3}{4} = -\frac{1}{8}(x - 4) \\ 8y + 6 = -x + 4 \\ x + 8y + 2 = 0 \end{array} \right. \end{aligned}$$

Question 3

A curve is given parametrically by the equations

$$x = 3t - 2\sin t, \quad y = t^2 + t \cos t, \quad 0 \leq t < 2\pi.$$

Show that an equation of the tangent at the point on the curve where $t = \frac{\pi}{2}$ is given by

$$y = \frac{\pi}{6}(x+2).$$

proof

$$\begin{aligned}
 x &= 3t - 2\sin t \Rightarrow \frac{dx}{dt} = 3 - 2\cos t \\
 y &= t^2 + t \cos t \Rightarrow \frac{dy}{dt} = 2t + \cos t - t \sin t
 \end{aligned}$$

$\bullet \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t + \cos t - t \sin t}{3 - 2\cos t}$
 $\bullet \left. \frac{dy}{dx} \right|_{t=\frac{\pi}{2}} = \frac{2 \cdot \frac{\pi}{2} + \cos \frac{\pi}{2} - \frac{\pi}{2} \sin \frac{\pi}{2}}{3 - 2 \cos \frac{\pi}{2}} = \frac{\frac{\pi}{2} - \frac{\pi}{2}}{3} = \frac{\pi}{6}$
 $\bullet \text{when } t = \frac{\pi}{2}, \quad x = 3 \left(\frac{\pi}{2} \right) - 2 \sin \frac{\pi}{2} = \frac{3\pi}{2} - 2$
 $y = \left(\frac{\pi}{2} \right)^2 + \frac{\pi}{2} \cos \frac{\pi}{2} = \frac{\pi^2}{4} - 2 \quad \left(\frac{3\pi}{2} - 2, \frac{\pi^2}{4} \right)$
 $\bullet \text{EQUATION OF TANGENT: } y - \frac{\pi^2}{4} = \frac{\pi}{6} \left(x - \frac{3\pi}{2} + 2 \right)$
 $y - \frac{\pi^2}{4} = \frac{\pi}{6}x - \frac{\pi^2}{4} + \frac{\pi}{3}$
 $y = \frac{\pi}{6}x + \frac{\pi}{3}$
 $y = \frac{\pi}{6}(x+2)$

Question 4

Find the turning points of the curve given parametrically by the equations

$$x = 1 - \cos 2t, \quad y = \sin 2t, \quad 0 \leq t < 2\pi.$$

Determine the nature of these turning points.

max (1,1), min (1,-1)

$$\begin{aligned}
 &\text{Given: } \begin{cases} x = 1 - \cos 2t \\ y = \sin 2t \end{cases} \quad \frac{dx}{dt} = \frac{dy}{dt} = \frac{2\sin 2t}{2\cos 2t} = \frac{\sin 2t}{\cos 2t} \\
 &\text{FOR MIN/MAX } \frac{dx}{dt} = 0 \Rightarrow \cos 2t = 0 \Rightarrow 2t = \frac{\pi}{2} + n\pi \quad n \in \mathbb{Z} \\
 &t = \frac{\pi}{4} + \frac{n\pi}{2} \quad n \in \mathbb{Z} \\
 &\therefore A \left(\frac{1}{2}, 1 \right) \quad B \left(\frac{1}{2}, -1 \right) \quad C \left(1, 1 \right) \quad D \left(1, -1 \right) \\
 &t = \frac{\pi}{4} \quad t = \frac{3\pi}{4} \quad t = \frac{\pi}{2} \quad t = \frac{5\pi}{4} \quad \text{at } t = \frac{\pi}{2} \text{ curve is flat} \\
 &\frac{d^2x}{dt^2} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d}{dt} \left(\frac{\sin 2t}{\cos 2t} \right) = \frac{d}{dt} \left(\cot 2t \right) = -\csc^2 2t \times 2 \times \frac{dt}{dt} \\
 &= -2\csc^2 2t \times \frac{1}{2\cos 2t} = -\frac{1}{\sin^2 2t} \\
 &\therefore \left(\frac{1}{2}, 1 \right) \quad t = \frac{\pi}{4} \quad \left. \frac{d^2x}{dt^2} \right|_{t=\frac{\pi}{4}} = -1 < 0 \quad \therefore \left(\frac{1}{2}, 1 \right) \text{ is MAX} \\
 &\left(\frac{1}{2}, -1 \right) \quad t = \frac{3\pi}{4} \quad \left. \frac{d^2x}{dt^2} \right|_{t=\frac{3\pi}{4}} = 1 > 0 \quad \therefore \left(\frac{1}{2}, -1 \right) \text{ is MIN}
 \end{aligned}$$

Question 5

A curve is given parametrically by the equations

$$x = 3 \sin 2\theta, \quad y = 4 \cos 2\theta, \quad 0 \leq \theta \leq 2\pi.$$

The point P is such so that $\cos \theta = \frac{3}{5}$ with $0 \leq \theta \leq \frac{\pi}{2}$.

Show that an equation of the tangent at the point P is

$$32x - 7y = 100.$$

[proof]

$$\begin{aligned}
 x &= 3 \sin 2\theta = -6 \sin \theta \cos \theta \\
 y &= 4 \cos 2\theta = 4(2\cos^2 \theta - 1) = 8\cos^2 \theta - 4
 \end{aligned}$$

$\cos \theta = \frac{3}{5}$, $\sin \theta = \sqrt{1 - \cos^2 \theta} = \frac{4}{5}$, $x = 3 \times \frac{4}{5} \times \frac{3}{5} = \frac{36}{25}$, $y = 8 \times \left(\frac{3}{5}\right)^2 - 4 = \frac{72}{25} - 4 = -\frac{28}{25}$
 $\therefore P \left(\frac{36}{25}, -\frac{28}{25} \right)$

$$\frac{dx}{d\theta} = \frac{dy}{d\theta} = \frac{d}{d\theta}(3 \sin 2\theta) = \frac{d}{d\theta}(6 \sin \theta \cos \theta) = 6 \cos^2 \theta - 6 \sin^2 \theta = 6(\cos 2\theta)$$

$$\left. \frac{dy}{d\theta} \right|_P = \frac{-16 \times \frac{3}{5} \times \frac{3}{5}}{6 \left[2 \left(\frac{3}{5} \right)^2 - 1 \right]} = \frac{-16 \times \frac{3}{5} \times \frac{3}{5}}{6 \times \frac{28}{25}} = -\frac{3}{7}$$

EQUATION OF TANGENT \Rightarrow

$$y - y_1 = m(x - x_1)$$

$$y + \frac{28}{25} = \frac{3}{7}(x - \frac{36}{25})$$

$$\times 75 \quad \Rightarrow 175y + 210 = 150(x - \frac{36}{25})$$

$$\Rightarrow 175y + 210 = 150x - 216$$

$$\Rightarrow 175y = 600x - 426$$

$$\Rightarrow 7y = 32x - 100$$

$$\Rightarrow 32x - 7y = 100$$

Question 6

For the curve given parametrically by

$$x = \frac{t}{1-t}, \quad y = \frac{t^2}{1-t}, \quad t \in \mathbb{R}, \quad t \neq 1,$$

find the coordinates of the turning points and determine their nature.

$$\boxed{\max(-2, -4), \min(0, 0)}$$

$\bullet \quad x = \frac{t}{1-t}$ $\frac{dx}{dt} = \frac{(1-t) - t(-1)}{(1-t)^2} = \frac{1+t}{(1-t)^2}$ $\frac{d^2x}{dt^2} = \frac{1+2t}{(1-t)^3}$ $\bullet \quad y = \frac{t^2}{1-t}$ $\frac{dy}{dt} = \frac{(1-t)(2t) - t^2(-1)}{(1-t)^2} = \frac{2t - t^2 + t^2}{(1-t)^2} = \frac{2t}{(1-t)^2}$ $\frac{d^2y}{dt^2} = \frac{2(1-t)^2 - 2t(2)(-1)}{(1-t)^4} = \frac{2t - 4t^2 + 4t}{(1-t)^3} = \frac{2t - 4t^2}{(1-t)^3} = \frac{2t(1-2t)}{(1-t)^3} = 2t - \frac{4t^2}{(1-t)^2}$ $\bullet \quad \text{For TP: } \frac{dy}{dx} = 0 \Rightarrow \frac{dy/dt}{dx/dt} = 0 \Rightarrow \frac{2t}{1+t} = 0 \Rightarrow t = 0 \Rightarrow x = 0, y = 0$ $\text{Now: } \frac{d^2y}{dx^2} = 2t - \frac{4t^2}{(1-t)^2}$ $\Rightarrow \frac{d}{dt} \left[\frac{d^2y}{dx^2} \right] = \frac{d}{dt} (2t - \frac{4t^2}{(1-t)^2})$ $\Rightarrow \frac{d^3y}{dx^3} = (2-2t) \frac{d}{dt} \left(\frac{1}{(1-t)^2} \right) = (2-2t) \frac{2}{(1-t)^3} = \frac{4(1-t)}{(1-t)^3} = \frac{4}{(1-t)^2}$ $\therefore \frac{d^3y}{dx^3} \Big _{t=0} = 2 > 0 \Rightarrow (0, 0) \text{ is A min}$ $\frac{d^3y}{dx^3} \Big _{t=2} = (-2)(-1) = -2 < 0 \Rightarrow (-2, 0) \text{ is A max}$	$\bullet \quad y = \frac{t^2}{1-t}$ $\frac{dy}{dt} = \frac{(1-t)(2t) - t^2(-1)}{(1-t)^2} = \frac{2t - t^2 + t^2}{(1-t)^2} = \frac{2t}{(1-t)^2}$ $\frac{d^2y}{dt^2} = \frac{2(1-t)^2 - 2t(2)(-1)}{(1-t)^4} = \frac{2t - 4t^2 + 4t}{(1-t)^3} = \frac{2t - 4t^2}{(1-t)^3} = \frac{2t(1-2t)}{(1-t)^3} = 2t - \frac{4t^2}{(1-t)^2}$
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Question 7

A curve is given parametrically by the equations

$$x = \frac{2t}{1+t^2}, \quad y = \frac{1-t^2}{1+t^2}, \quad t \in \mathbb{R}.$$

The point $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ lies on this curve.

Show that an equation of the tangent at the point P is given by

$$x + y = \sqrt{2}.$$

proof

$\bullet x_1 = \frac{2t}{1+t^2}$ $\bullet y_1 = \frac{1-t^2}{1+t^2}$
 $\frac{dx_1}{dt} = \frac{(1+t^2) \cdot 2 - 2t \cdot 2t}{(1+t^2)^2} = \frac{(1+t^2) \cdot (2-2t) - (1-t^2) \cdot 2t}{(1+t^2)^2}$
 $\frac{dx_1}{dt} = \frac{2+2t^2 - 4t^2}{(1+t^2)^2} = \frac{-2t^2 + 2}{(1+t^2)^2}$
 $\frac{dy_1}{dt} = \frac{2-2t^2}{(1+t^2)^2} = \frac{-4t^2}{(1+t^2)^2} = -\frac{4t^2}{(1+t^2)^2}$
 $\bullet \text{Hinzt: } \frac{dy_1}{dx_1} = \frac{\frac{dy_1}{dt}}{\frac{dx_1}{dt}} = \frac{\frac{-4t^2}{(1+t^2)^2}}{\frac{2-2t^2}{(1+t^2)^2}} = -\frac{4t^2}{2-2t^2} = \frac{2t^2}{1-t^2}$

 $\bullet \frac{x_1^2}{y_1^2} = \frac{2t^2}{1-t^2} \Leftrightarrow \text{2.GERÜNTION}$ $y_1 \text{ EQUATION: } \frac{1-t^2}{1+t^2} = \frac{\sqrt{2t^2}}{\sqrt{1-t^2}} \Rightarrow$
 $\Rightarrow (t^2+1) \cdot t^2 = 2t^2 \cdot (1-t^2)$ $\Rightarrow 2-2t^2 = 2t^2 + t^2 \cdot 4$
 $\Rightarrow t^4 + 1 = 2t^2 \cdot t$ $\Rightarrow 2-4t^2 = (2+t^2) \cdot t^2$
 $\Rightarrow t^4 - 2t^3 + 1 = 0$ $\Rightarrow 2+t^2 = t^2$
 $\Rightarrow (t^2 - t^2)^2 = 1$ $\Rightarrow t^2 = 3-2t^2$
 $\Rightarrow t^2 - t^2 = 1$ $\Rightarrow t^2 = \frac{(1-2t^2)}{(1+2t^2)}$
 $\Rightarrow t = \sqrt{1-t^2}$ $\Rightarrow t^2 = (\beta-1)^2$
 $\Rightarrow t = \sqrt{1-\frac{1}{t^2}}$ $\Rightarrow t = \pm \sqrt{\beta-1}$
 \rightarrow AT P $t = -\sqrt{\beta-1}$ \swarrow \searrow
 $\frac{dy_1}{dx_1} \Big|_{t=-\sqrt{\beta-1}} = \frac{2(-1+\sqrt{\frac{1}{\beta}})}{(-1+\sqrt{\frac{1}{\beta}})^2-1} = \frac{2(-1+\sqrt{\frac{1}{\beta}})}{3-2\sqrt{\frac{1}{\beta}}-1} = \frac{2(-1+\sqrt{\frac{1}{\beta}})}{2-2\sqrt{\frac{1}{\beta}}} = \frac{-(-1+\sqrt{\frac{1}{\beta}})}{-2(\sqrt{\frac{1}{\beta}}-1)}$
 $= -1$

 \bullet $y - y_1 = M \cdot (x - x_1)$
 $y - \frac{y_1}{\sqrt{2}} = -(\sqrt{\frac{1}{\beta}}) \cdot (x - \frac{2t}{1+t^2})$
 $y - \frac{y_1}{\sqrt{2}} = -\sqrt{\frac{1}{\beta}} \cdot x + \frac{2t}{\sqrt{2}} \quad ; \quad \exists x y = \sqrt{2}$