IYGB – MATHEMATICAL METHODS 4 – PAPER E – QUESTION 1

SOLVE THE P.D.E. BY "LAGRANGE'S METHOD"

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = z$$

$P \qquad Q \qquad R$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \implies \frac{dx}{2} = \frac{dy}{3} = \frac{dz}{z}$$

$① \qquad ② \qquad ③$

SOLVING ① = ②

$$\implies \frac{dx}{2} = \frac{dy}{3}$$

$$\implies 3\,dx = 2\,dy$$

$$\implies 3x = 2y + C$$

$$\implies 3x - 2y = C$$

$$u(x,y,z) = 3x - 2y$$

SOLVING ① = ③

$$\implies \frac{dx}{2} = \frac{dz}{z}$$

$$\implies \tfrac{1}{2}x = \ln z + D$$

$$\implies \tfrac{1}{2}x = \ln(Az)$$

$$\implies e^{\frac{1}{2}x} = Az$$

$$\implies z = Be^{\frac{1}{2}x}$$

$$\implies ze^{-\frac{1}{2}x} = B$$

$$\implies v(x,y,z) = ze^{-\frac{1}{2}x}$$

THE GENERAL SOLUTION IS

$$F(u,v) = 0$$

$$u = f(v) \qquad \text{OR} \qquad v = g(u)$$

$$\boxed{ze^{-\frac{1}{2}x} = g(3x-2y)}$$

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APPLY THE BOUNDARY CONDITION $z(1,y) = y$

$$\Longrightarrow \; y e^{-\frac{1}{2}} = g(3-2y)$$

LET $w = 3-2y$

$2y = 3-w$

$y = \frac{3-w}{2}$

$$\Longrightarrow \; \left(\frac{3-w}{2}\right)e^{-\frac{1}{2}} = g(w)$$

$$\Longrightarrow \; g(3x-2y) = \frac{3-(3x-2y)}{2}e^{-\frac{1}{2}}$$

$$\Longrightarrow \; g(3x-2y) = \tfrac{1}{2}e^{-\frac{1}{2}}(3-3x+2y)$$

FINALLY WE OBTAIN

$$\Longrightarrow \; z e^{-\frac{1}{2}x} = g(3x-2y)$$

$$\Longrightarrow \; z e^{-\frac{1}{2}x} = \tfrac{1}{2}e^{-\frac{1}{2}}(3-3x+2y)$$

$$\Longrightarrow \; z = \tfrac{1}{2}e^{\frac{1}{2}x}e^{-\frac{1}{2}}(3-3x+2y)$$

$$\Longrightarrow \; z(x,y) = \tfrac{1}{2}e^{\frac{1}{2}(x-1)}(3-3x+2y)$$

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OBTAIN PARTIAL DERIVATIVES FROM THE TRANSFORMATION EQUATIONS

$$\boxed{u = \frac{1}{2}\ln(x^2+y^2)} \qquad \boxed{v = \arctan\left(\frac{y}{x}\right)}$$

- $\frac{\partial u}{\partial x} = \frac{1}{2} \times \frac{1}{x^2+y^2} \times 2x = \frac{x}{x^2+y^2}$
- $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$
- $\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(-\frac{y}{x^2}\right) = -\frac{y}{\left(1+\frac{y^2}{x^2}\right)x^2} = -\frac{y}{x^2+y^2}$
- $\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x} = \frac{1}{\left(1+\frac{y^2}{x^2}\right)x} = \frac{1}{x+\frac{y^2}{x}} = \frac{x}{x^2+y^2}$

NEXT GET EXPRESSIONS FOR $x = x(u,v)$ & $y = y(u,v)$

- $2u = \ln(x^2+y^2)$

$e^{2u} = x^2+y^2$

$y^2 = e^{2u} - x^2$

- $\tan v = \frac{y}{x}$

$y = x\tan v$

$y^2 = x^2\tan^2 v$

$$e^{2u} - x^2 = x^2\tan^2 v$$
$$e^{2u} = x^2 + x^2\tan^2 v$$
$$e^{2u} = x^2(1+\tan^2 v)$$
$$e^{2u} = x^2\sec^2 v$$
$$e^{u} = x\sec v$$
$$\boxed{x = e^{u}\cos v}$$

$\Rightarrow y = x\tan v$

$\Rightarrow y = (e^{u}\cos v)\frac{\sin v}{\cos v}$

$\Rightarrow \boxed{y = e^{u}\sin v}$

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REWRITING ALL THE DERIVATIVES & EXPRESSIONS

$$\boxed{\frac{\partial x}{\partial u} = e^{u}\cos v} \quad \boxed{\frac{\partial x}{\partial v} = -e^{u}\sin v} \quad \boxed{\frac{\partial y}{\partial u} = e^{u}\sin v} \quad \boxed{\frac{\partial y}{\partial v} = e^{u}\cos v}$$

$$x+y = e^{u}\cos v + e^{u}\sin v = e^{u}(\cos v + \sin v)$$
$$y-x = e^{u}\sin v - e^{u}\cos v = e^{u}(\sin v - \cos v)$$
& $x^2+y^2 = e^{2u}$

FORMING $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ BY THE CHAIN RULE

- $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial z}{\partial v}\left(-\frac{y}{x^2+y^2}\right)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\left(\frac{e^{u}\cos v}{e^{2u}}\right) - \frac{\partial z}{\partial v}\left(\frac{e^{u}\sin v}{e^{2u}}\right)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\cos v}{e^{u}} - \frac{\partial z}{\partial v}\frac{\sin v}{e^{u}}$$

- $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial z}{\partial v}\left(\frac{x}{x^2+y^2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{e^{u}\sin v}{e^{2u}}\right) + \frac{\partial z}{\partial v}\left(\frac{e^{u}\cos v}{e^{2u}}\right)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\sin v}{e^{u}} + \frac{\partial z}{\partial v}\frac{\cos v}{e^{u}}$$

NEXT WE TRANSFORM THE P.D.E

$$\Rightarrow (x+y)\frac{\partial z}{\partial x} + (y-x)\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow e^{u}(\cos v+\sin v)\left[\frac{\partial z}{\partial u}\frac{\cos v}{e^{u}} - \frac{\partial z}{\partial v}\frac{\sin v}{e^{u}}\right] + e^{u}(\sin v - \cos v)\left[\frac{\partial z}{\partial u}\frac{\sin v}{e^{u}} + \frac{\partial z}{\partial v}\frac{\cos v}{e^{u}}\right] = 0$$

$$\Rightarrow (\cos v+\sin v)\left[\frac{\partial z}{\partial u}\cos v - \frac{\partial z}{\partial v}\sin v\right] + (\sin v - \cos v)\left[\frac{\partial z}{\partial u}\sin v + \frac{\partial z}{\partial v}\cos v\right] = 0$$

$$\Rightarrow \frac{\partial z}{\partial u}(\cos^2 v + \cancel{\sin v\cos v}) - \frac{\partial z}{\partial v}(\cancel{\cos v\sin v} + \sin^2 v) + \frac{\partial z}{\partial u}(\sin^2 v - \cancel{\sin v\cos v}) + \frac{\partial z}{\partial v}(\cancel{\sin v\cos v} - \cos^2 v) = 0$$

$$\Rightarrow \frac{\partial z}{\partial u}(\cos^2 v + \sin^2 v) - \frac{\partial z}{\partial v}(\sin^2 v + \cos^2 v) = 0$$

$$\Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 0$$

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ASSUME A SOLUTION OF THE FORM $u(x,t) = X(x)T(t)$

DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \quad \& \quad \frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \implies X''(x)T(t) = \frac{1}{c^2}X(x)T''(t)$$

$$\implies \frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{c^2}\frac{X(x)T''(t)}{X(x)T(t)}$$

$$\implies \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

- IF $\lambda = 0$

$$\implies \frac{X''(x)}{X(x)} = 0 \qquad\qquad \implies \frac{1}{c^2}\frac{T''(t)}{T(t)} = 0$$

$$\implies X''(x) = 0 \qquad\qquad \implies T''(t) = 0$$

$$\implies X(x) = Ax + B \qquad\qquad \implies T(t) = Ct + D$$

$$\therefore \; u(x,t) = (Ax+B)(Ct+D) \quad \text{(I)}$$

- IF $\lambda > 0$, SAY $\lambda = p^2$

$$\implies \frac{X''(x)}{X(x)} = p^2 \qquad\qquad \implies \frac{1}{c^2}\frac{T''(t)}{T(t)} = p^2$$

$$\implies X''(x) = p^2 X(x) \qquad\qquad \implies T''(t) = p^2c^2T(t)$$

$$\implies X(x) = A\cosh px + B\sinh px \qquad\qquad \implies T(t) = C\cosh pct + D\sinh pct$$

(OR EXPONENTIALS) (OR EXPONENTIALS)

$$\therefore \; u(x,t) = (A\cosh px + B\sinh px)(C\cosh pct + D\sinh pct) \quad \text{(II)}$$

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- IF $\lambda<0$, SAY $\lambda=-p^2$

$$\Rightarrow \frac{X''(x)}{X(x)} = -p^2 \qquad\qquad \Rightarrow \frac{1}{c^2}\frac{T''(t)}{T(t)} = -p^2$$

$$\Rightarrow X''(x) = -p^2 X(x) \qquad\qquad \Rightarrow T''(t) = -p^2c^2 T(t)$$

$$\Rightarrow X(x) = A\cos px + B\sin px \qquad\qquad \Rightarrow T(t) = C\cos pct + D\sin pct$$

$$\therefore\ u(x,t) = (A\cos px + B\sin px)(C\cos pct + D\sin pct) \quad \text{(III)}$$

AS WE REQUIRE A SOLUTION WITH $u(0,t)=u(1,t)$, WE PICK A PERIODIC SOLUTION (OR AT LEAST A CONSTANT SOLUTION) IN x — SO WE PICK SOLUTION (III) AND NOTE THAT THE CONSTANT PART OF (I) IS ALSO INCLUDED THERE

$u(0,t)=0$

$$\Rightarrow \quad 0 = A(C\cos pct + D\sin pct) \quad \forall t \geqslant 0$$

$$\Rightarrow \quad A=0$$

ABSORBING AND RELABELLING THE CONSTANTS

$$\therefore\ \boxed{u(x,t) = [E\cos pct + F\sin pct]\sin px}$$

DIFFERENTIATE W.R.T t AND APPLY $\frac{\partial u}{\partial t}(x,0)=0$

$$\Rightarrow \frac{\partial u}{\partial t} = [-Epc\sin pct + Fpc\cos pct]\sin px$$

$$\Rightarrow 0 = Fpc\sin px \qquad \forall x:\ 0\leqslant x\leqslant 1$$

$$\Rightarrow F=0 \quad \text{SINCE } p\neq 0,\ c\neq 0$$

$$\therefore\ \boxed{u(x,t) = E\cos cpt\sin px}$$

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APPLY $u(1,t)=0$

$$\Rightarrow \quad 0 = E \cos cpt \sin p \qquad \forall t \geqslant 0$$

$\Rightarrow$ $E \neq 0$, OTHERWISE EVERYTHING IS TRIVIALLY ZERO

$$\Rightarrow \quad \sin p = 0$$

$$\Rightarrow \quad p = n\pi \qquad n = 0,1,2,3,4,\ldots$$

$$\Rightarrow \quad u_n(x,t) = E_n \sin(n\pi x)\cos(cn\pi t)$$

$$\Rightarrow \quad \boxed{u(x,t) = \sum_{n=1}^{\infty}\left[E_n \sin(n\pi x)\cos(n\pi ct)\right]}$$

NOTE THAT NEGATIVE VALUES OF n CAN BE ABSORBED INTO E_n & $n=0$ YIELDS $u(x,t)=0$ SO WE MAY OMIT

APPLY $u(x,0) = \sin(5\pi x) + 2\sin(7\pi x)$

$$\Rightarrow \quad \sin(5\pi x) + 2\sin(7\pi x) = \sum_{n=1}^{\infty}\left[E_n \sin(n\pi x)\right]$$

$$\Rightarrow \quad E_5 = 1, \quad E_7 = 2 \qquad \text{(THE REST ARE ZERO)}$$

$$\therefore \quad \boxed{u(x,t) = 1\sin(5\pi x)\cos(5\pi ct) + 2\sin(7\pi x)\cos(7\pi ct)}$$

$$u(x,t) = \sin 5\pi x \cos 5\pi ct + 2\sin 7\pi x \cos 7\pi ct$$

IYGB - MATHEMATICAL METHODS 4 - PAPER E - QUESTION 4

a) SOLVING THE P.D.E

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$ HAS AUXILIARY EQUATION $\lambda^2 = \frac{1}{c^2}$

$\lambda = \pm \frac{1}{c}$

GENERAL SOLUTION COULD BE

$$z(x,t) = f\left(-\tfrac{1}{c}x + t\right) + g\left(\tfrac{1}{c}x + t\right)$$

$$z(x,t) = f(x-ct) + g(x+ct)$$

APPLY CONDITIONS $z(x,0) = F(x)$ & $\frac{\partial z}{\partial t}(x,0) = G(x)$

- $z(x,0) = f(x)$

$\Rightarrow f(x) + g(x) = F(x)$

$\Rightarrow \frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}\left[F(x)\right]$

$\Rightarrow \boxed{f'(x) + g'(x) = F'(x)}$

- $\frac{\partial z}{\partial t} = -cf'(x-ct) + cg'(x+ct)$

$\Rightarrow \frac{\partial z}{\partial t}(x,0) = -cf'(x) + cg'(x)$

$\Rightarrow \boxed{-f'(x) + g'(x) = \frac{1}{c}G(x)}$

MANIPULATING AS FOLLOWS, BY ADDING & SUBTRACTING THE ABOVE TWO EQUATIONS

$$\left.\begin{aligned} 2f'(x) &= F'(x) - \tfrac{1}{c}G(x) \\ 2g'(x) &= F'(x) + \tfrac{1}{c}G(x) \end{aligned}\right\} \Rightarrow \begin{aligned} f'(x) &= \tfrac{1}{2}F'(x) - \tfrac{1}{2c}G(x) \\ g'(x) &= \tfrac{1}{2}F'(x) + \tfrac{1}{2c}G(x) \end{aligned}$$

INTEGRATE EACH OF THE TWO EQUATIONS W.R.T x (LEIBNITZ RULE)

$$f(x) = \tfrac{1}{2}F(x) - \frac{1}{2c}\int_0^x G(\xi)\,d\xi$$

$$g(x) = \tfrac{1}{2}F(x) + \frac{1}{2c}\int_0^x G(\xi)\,d\xi$$

AS THE ABOVE EQUATIONS HOLD FOR ALL x THEY WILL HOLD FOR "OUR GENERAL SOLUTION"

$\Rightarrow z(x,t) = f(x-ct) + g(x+ct)$

$$\Rightarrow z(x,t) = \tfrac{1}{2}F(x-ct) - \frac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi + \tfrac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$$

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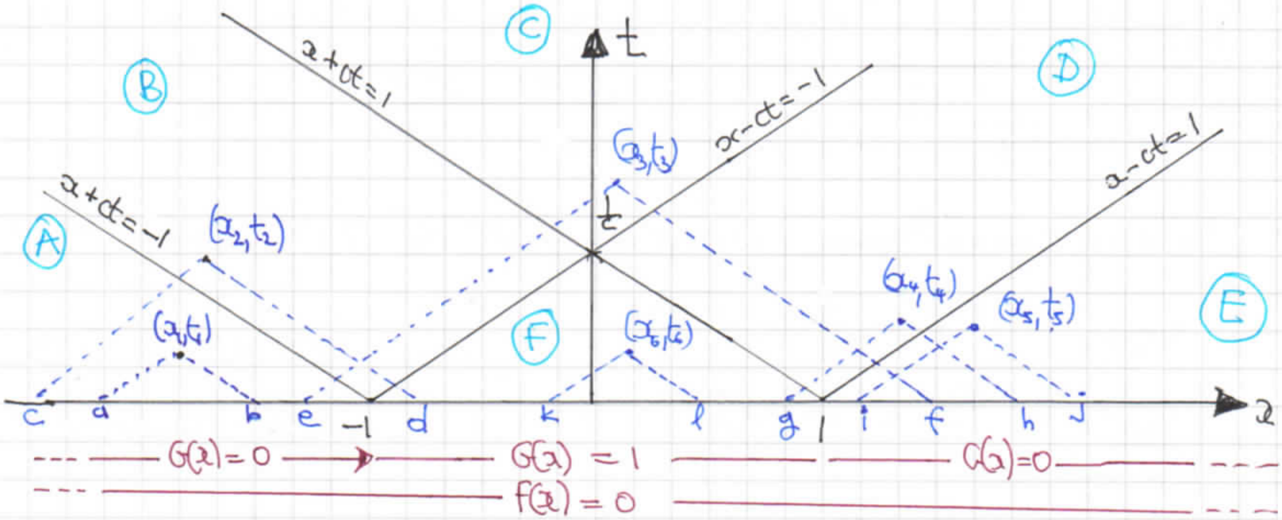
$$\Rightarrow z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi - \frac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi$$

$$\Rightarrow z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi + \frac{1}{2c}\int_{x-ct}^{0} G(\xi)\,d\xi$$

$$\Rightarrow z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

b) DRAWING AN x-t PLANE

- Place initial conditions on the x axis
- Draw the characteristics, lines of gradient $\pm\frac{1}{c}$, through the critical values ± 1, the values of discontinuity of $G(x)$
- Draw a point in each of the six regions – draw lines parallel to the characteristics & find the x intercepts



GETTING A GENERAL EXPRESSION FOR THE x INTERCEPTS FOR LINES THROUGH (x_i, t_i)

$$t - t_i = \frac{1}{c}(x - x_i) \qquad \& \qquad t - t_i = -\frac{1}{c}(x - x_i)$$

$$0 - t_i = \frac{1}{c}(x - x_i) \qquad\qquad 0 - t_i = -\frac{1}{c}(x - x_i)$$

$$-ct_i = x - x_i \qquad\qquad ct_i = x - x_i$$

$$x = x_i - ct_i \qquad\qquad x = x_i + ct_i$$

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REGION A: $z = \frac{1}{2c}\int_a^b G(\xi)\,d\xi = \frac{1}{2c}\int_{x_1-ct_1}^{x+ct_1} 0 \; d\xi = 0$

REGION B: $z = \frac{1}{2c}\int_c^d G(\xi)\,d\xi = \frac{1}{2c}\int_{x_2-ct_2}^{x_2+ct_2} G(\xi)\,d\xi = \frac{1}{2c}\int_{-1}^{x_2+ct_2} 1 \; d\xi$

$= \frac{1}{2c}\Big[\xi\Big]_{-1}^{x_2+ct_2} = \frac{1}{2c}\big[x_2+ct_2+1\big]$

REGION C: $z = \frac{1}{2c}\int_e^f G(\xi)\,d\xi = \frac{1}{2c}\int_{x_3-ct_3}^{x_3+ct_3} G(\xi)\,d\xi = \frac{1}{2c}\int_{-1}^{1} 1 \; d\xi$

$= \frac{1}{2c}\Big[\xi\Big]_{-1}^{1} = \frac{1}{2c}\big[1+1\big] = \frac{1}{c}$

REGION D: $z = \frac{1}{2c}\int_g^h G(\xi)\,d\xi = \frac{1}{2c}\int_{x_4-ct_4}^{x_4+ct_4} G(\xi)\,d\xi = \frac{1}{2c}\int_{x_4-ct_4}^{1} 1 \; d\xi$

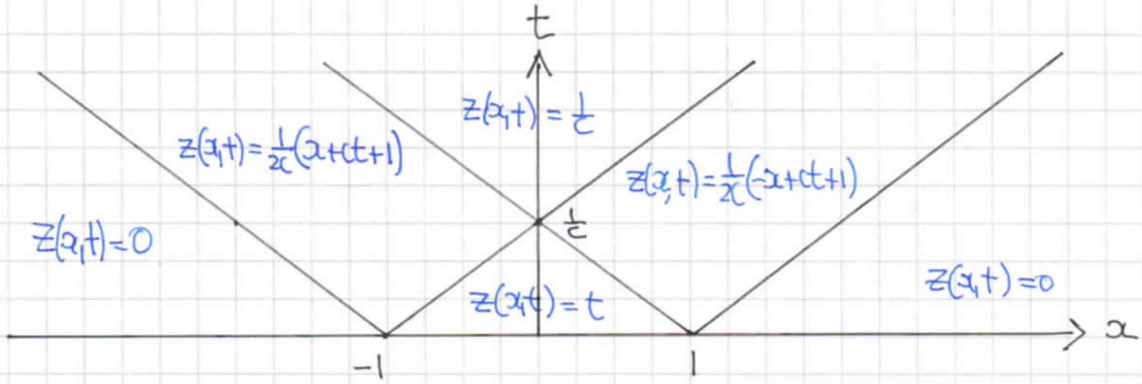
$= \frac{1}{2c}\Big[\xi\Big]_{x_4-ct_4}^{1} = \frac{1}{2c}\big[1-x_4+ct_4\big]$

REGION E: $z = \frac{1}{2c}\int_i^j G(\xi)\,d\xi = \frac{1}{2c}\int_{x_5-ct_5}^{x_5+ct_5} G(\xi)\,d\xi = \frac{1}{2c}\int_{x_5-ct_5}^{x_5+ct_5} 0 \; d\xi = 0$

REGION F: $z = \frac{1}{2c}\int_k^l G(\xi)\,d\xi = \frac{1}{2c}\int_{x_6-ct_6}^{x_6+ct_6} G(\xi)\,d\xi = \frac{1}{2c}\int_{x_6-ct_6}^{x_6-ct_6} 1 \; d\xi$

$= \frac{1}{2c}\Big[\xi\Big]_{x_6-ct_6}^{x_6+ct_6} = \frac{1}{2c}\big[x_6+ct_6-x_6+ct_6\big] = \frac{2ct_6}{2c} = t_6$

DROPPING THE SUBSCRIPTS



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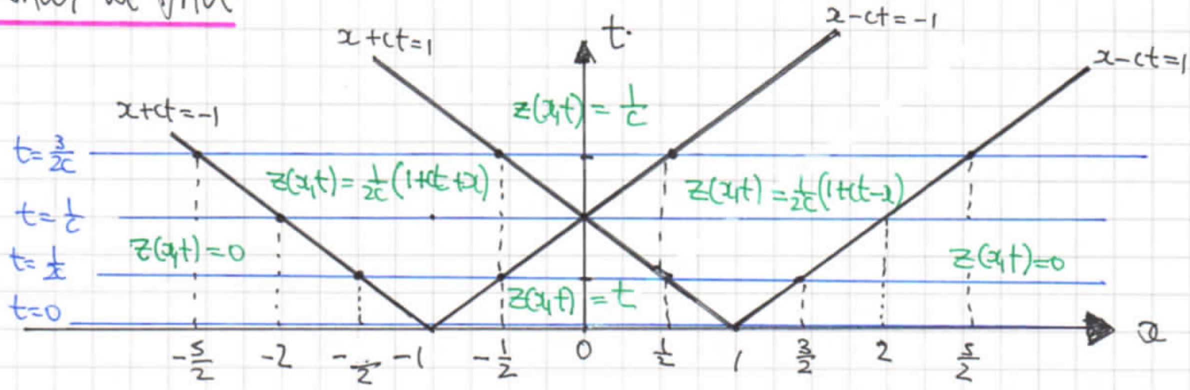
FINALLY WE HAVE



- $z(x,0) = 0$

- $z\left(x,\frac{1}{2c}\right) = \begin{cases} 0 & x < -\frac{3}{2} \\ \frac{1}{2c}\left(x+\frac{3}{2}\right) & -\frac{3}{2} \leqslant x \leqslant -\frac{1}{2} & \quad \frac{1}{2c}\left(1+c\times\frac{1}{2c}+x\right) \\ \frac{1}{2c} & -\frac{1}{2} < x < \frac{1}{2} \\ \frac{1}{2c}\left(\frac{3}{2}-x\right) & \frac{1}{2} \leqslant x \leqslant \frac{3}{2} & \quad \frac{1}{2c}\left(1+c\times\frac{1}{2c}-x\right) \\ 0 & x > \frac{3}{2} \end{cases}$

- $z\left(x,\frac{1}{c}\right) \begin{cases} 0 & x < -2 \\ \frac{1}{2c}(x+2) & -2 \leqslant x \leqslant 0 & \quad \frac{1}{2c}\left(1+c\times\frac{1}{c}+x\right) \\ \frac{1}{2c}(2-x) & 0 < x \leqslant 2 & \quad \frac{1}{2c}\left(1+c\times\frac{1}{c}-x\right) \\ 0 & x > 2 \end{cases}$

- $z\left(x,\frac{3}{2c}\right) = \begin{cases} 0 & x < -\frac{5}{2} \\ \frac{1}{2c}\left(x+\frac{5}{2}\right) & -\frac{5}{2} \leqslant x \leqslant -\frac{1}{2} & \quad \frac{1}{2c}\left(1+c\times\frac{3}{2c}+x\right) \\ \frac{1}{c} & -\frac{1}{2} < x < \frac{1}{2} \\ \frac{1}{2c}\left(\frac{5}{2}-x\right) & \frac{1}{2} \leqslant x \leqslant \frac{5}{2} & \quad \frac{1}{2c}\left(1+c\times\frac{3}{2c}-x\right) \\ 0 & x > \frac{5}{2} \end{cases}$

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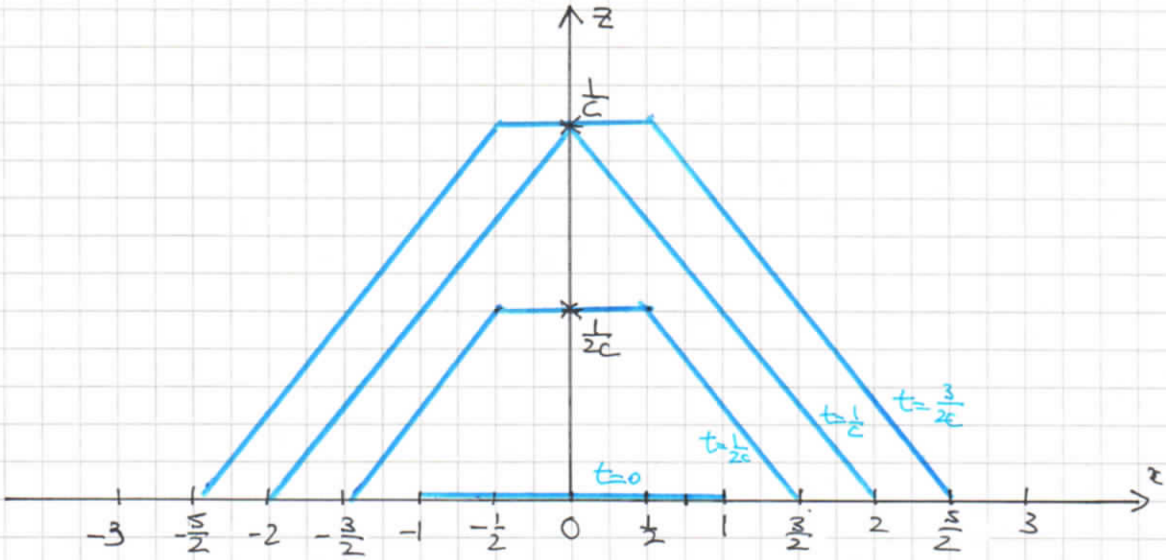
c) FINALLY DRAWING THE WAVE PROFILES



IYGB - MATHEMATICAL METHODS 4 - PAPER E QUESTION 5

START BY TAKING LAPLACE TRANSFORM OF THE P.D.E W.R.T. t

$$\Rightarrow \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}$$

$$\Rightarrow \mathcal{L}\left[\frac{\partial^2 \theta}{\partial x^2}\right] = \mathcal{L}\left[\frac{\partial \theta}{\partial t}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\left[\mathcal{L}[\theta]\right] = s\mathcal{L}[\theta] - \theta(x,0)$$

$$\Rightarrow \frac{\partial^2 \bar{\theta}}{\partial x^2} = s\bar{\theta} - 8\sin(2\pi x)$$

$$\boxed{\theta(x,0) = 8\sin(2\pi x)}$$

THIS IS A STANDARD 2^{ND} ORDER O.D.E FOR $\bar{\theta}(x,s)$, WHERE s IS TREATED AS A POSITIVE CONSTANT

$$\Rightarrow \frac{\partial^2 \bar{\theta}}{\partial x^2} - s\bar{\theta} = -8\sin(2\pi x)$$

$$\Rightarrow \bar{\theta}(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} + \text{PARTICULAR INTEGRAL}$$

TO FIND THE PARTICULAR INTEGRAL TRY $\bar{\theta}(x,s) = P(s)\sin(2\pi x)$, AS NO COSINE TERM IS NEEDED DUE TO THE ABSENCE OF THE FIRST DERIVATIVE

$$\Rightarrow \frac{\partial^2 \bar{\theta}}{\partial x^2} = -4\pi^2 P(s)\sin(2\pi x)$$

SUBSTITUTE INTO THE O.D.E.

$$\Rightarrow -4\pi^2 P(s)\sin(2\pi x) - sP(s)\sin(2\pi x) \equiv -8\sin(2\pi x)$$

$$\Rightarrow (4\pi^2 - s)P(s) = -8$$

$$\Rightarrow P(s) = \frac{8}{4\pi^2 + s}$$

∴ THE GENERAL SOLUTION OF THE O.D.E IS

$$\bar{\theta}(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} + \frac{8}{4\pi^2+s}\sin(2\pi x)$$

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NEXT WE NEED TO TAKE THE LAPLACE TRANSFORM OF THE BOUNDARY CONDITIONS WHICH INVOLVE t

- $\theta(0,t)=0$

$$\mathcal{L}[\theta(0,t)] = \mathcal{L}[0]$$
$$\bar{\theta}(0,s)=0$$

- $\theta(1,t)=0$

$$\mathcal{L}[\theta(1,t)] = \mathcal{L}[0]$$
$$\bar{\theta}(1,s)=0$$

APPLYING THE TO THE SOLUTION

$$\bar{\theta}(0,s)=0 \implies 0 = A(s)+B(s)+0$$
$$\implies A(s) = -B(s)$$

$$\bar{\theta}(1,s)=0 \implies 0 = A(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}} + 0$$
$$\implies 0 = -B(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}}$$
$$\implies 0 = B(s)\left[e^{-\sqrt{s}} - e^{\sqrt{s}}\right]$$
$$\implies 0 = -B(s)\left[e^{\sqrt{s}} - e^{-\sqrt{s}}\right]$$
$$\implies 0 = -2B(s)\sinh\sqrt{s}$$
$$\implies B(s) = 0 \qquad (\sinh\sqrt{s} \neq 0, \text{ as } s\neq 0)$$
$$\implies A(s) = 0$$
$$\implies \boxed{\bar{\theta}(x,s) = \frac{8}{4\pi^2+s}\sin(2\pi x)}$$

INVERTING THE TRANSFORM, NOTING THAT x IS A CONSTANT

$$\boxed{\theta(x,t) = 8e^{-4\pi^2 t}\sin(2\pi x)}$$

IYGB - MATHEMATICAL METHODS 4 - PAPER E - QUESTION 6

ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM — DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$$u(x,t) = X(x)T(t) \implies \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

$$\implies \frac{\partial u}{\partial t} = X(x)T'(t)$$

$$\implies \frac{\partial^2 u}{\partial x^2} = \frac{1}{9}\frac{\partial u}{\partial t}$$

$$\implies X''(x)T(t) = \frac{1}{9}X(x)T'(t)$$

$$\implies \frac{X''(x)T(t)}{X(x)T(t)} = \frac{X(x)T'(t)}{9X(x)T(t)}$$

$$\implies \frac{X''(x)}{X(x)} = \frac{T'(t)}{9T(t)}$$

AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

IF $\lambda = 0$

- $\frac{X''(x)}{X(x)} = 0$

  $X''(x) = 0$

  $X(x) = Ax + B$

- $\frac{T'(t)}{9T(t)} = 0$

  $T'(t) = 0$

  $T(t) = C$

$$\therefore \ u(x,t) = (A'x + B')C$$

$$\boxed{u(x,t) = Ax + B} \ \text{—(I)}$$

IF $\lambda > 0$, SAY $\lambda = p^2$

- $\frac{X''(x)}{X(x)} = p^2$

  $X''(x) = p^2 X(x)$

  $X(x) = A\cosh px + B\sinh px$

  (OR EXPONENTIALS)

- $\frac{T'(t)}{9T(t)} = p^2$

  $T'(t) = 9p^2 T(t)$

  $T(t) = Ce^{9p^2 t}$

$$\therefore \ u(x,t) = (A'\cosh px + B'\sinh px)(Ce^{9p^2 t})$$

$$\boxed{u(x,t) = e^{9p^2 t}(A\cosh px + B\sinh px)} \ \text{—(II)}$$

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IF $\lambda < 0$, SAY $\lambda = -p^2$

- $\frac{X''(x)}{X(x)} = -p^2$

$X''(x) = -p^2 X(x)$

$X(x) = A\cos px + B\sin px$

- $\frac{T'(t)}{9T(t)} = -p^2$

$T'(t) = -9p^2 T(t)$

$T(t) = Ce^{-9p^2 t}$

$$\therefore u(x,t) = (A'\cos px + B'\sin px)(Ce^{-9p^2 t})$$

$$\boxed{u(x,t) = e^{-9p^2 t}(A\cos px + B\sin px)} \quad \text{—(III)}$$

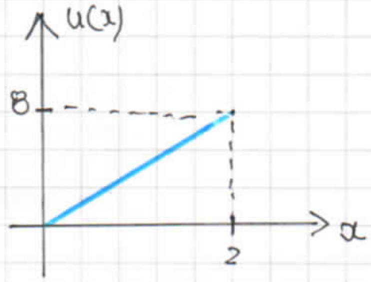
NOW WE LOOK AT THE BOUNDARY CONDITIONS, IN ORDER TO DECIDE THE NATURE OF THE SOLUTION

(I) IS DISCARDED AS IT PRODUCES A NON-TIME DEPENDENT SOLUTION (I.E STEADY STATE)

(II) IS DISCARDED AS IT PRODUCES UNBOUNDED SOLUTIONS AS t INCREASES

(III) IS THE ONLY VIABLE SOLUTION FOR THIS PROBLEM

NEXT WE OBSERVE THAT AS $t \to +\infty$, THE TEMPERATURE DISTRIBUTION OF THE ROD WILL BE A LINEAR FUNCTION IN x, SINCE THE TEMPERATURE AT $x=0$ & AT $x=2$ IS MAINTAINED AT 0 & AT 8 RESPECTIVELY



BY INSPECTION, AS $t \to \infty$

$$u(x,t) = u(x) = 4x \quad \text{(BY INSPECTION)}$$

WE NEED TO "BUILD" THIS FEATURE INTO THE SOLUTION

$$\boxed{u(x,t) = 4x + v(x,t)}$$ WHERE $v(x,t)$ IS THE SOLUTION OF

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{9}\frac{\partial v}{\partial t}$$

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NEXT WE TRANSFORM THE BOUNDARY & INITIAL CONDITIONS

$u(0,t)=0 \implies 0 = 4\times 0 + U(0,t) \implies \boxed{U(0,t)=0}$

$u(2,t)=8 \implies 8 = 4\times 2 + U(2,t) \implies \boxed{U(2,t)=0}$

$u(x,0)=x^2 \implies 2x^2 = 4x + U(x,0) \implies \boxed{U(x,0)=2x^2-4x}$

APPLY $U(0,t)=0$ INTO THE GENERAL SOLUTION (III), FROM EARLIER

$$\implies U(x,t) = e^{-9p^2t}(A\cos px + B\sin px)$$
$$\implies 0 = e^{-9p^2t}A$$
$$\implies A=0$$

$$\therefore \boxed{U(x,t) = Be^{-9p^2t}\sin px}$$

APPLY $U(2,t)=0$ INTO THE SOLUTION ABOVE

$$0 = Be^{-9p^2t}\sin 2p$$

$\sin 2p = 0$ $\quad$ ($B \neq 0$, OTHERWISE TRIVIAL SOLUTION)

$2p = n\pi \quad n=0,1,2,3,\dots$

$p = \frac{n\pi}{2}$

$$\therefore U_n(x,t) = B_n e^{-\frac{9n^2\pi^2t}{4}}\sin\left(\frac{n\pi x}{2}\right)$$

$$\boxed{U(x,t) = \sum_{n=1}^{\infty}\left[B_n e^{-\frac{9n^2\pi^2t}{4}}\sin\left(\frac{n\pi x}{2}\right)\right]}$$

NOTE THAT $n=0$, YIELDS ZERO, SO WE MAY OMIT FROM THE SOLUTION

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APPLY $U(x,0) = 2x^2 - 4x$ INTO THE "LATEST" VERSION OF THE SOLUTION

$$2x^2 - 4x = \sum_{n=1}^{\infty}\left[B_n \sin\left(\frac{n\pi x}{2}\right)\right]$$

$$B_n = \frac{1}{1}\int_0^2 (2x^2-4x)\sin\left(\frac{n\pi x}{2}\right)\,dx$$

[HALF PERIOD — PERIOD OF FOURIER IN x IS 2 HERE]

INTEGRATE BY PARTS

$2x^2-4x$	$4x-4$
$-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$	$\sin\left(\frac{n\pi x}{2}\right)$

$$\Rightarrow B_n = \cancel{\left[-\frac{2}{n\pi}(2x^2-4x)\cos\left(\frac{n\pi x}{2}\right)\right]_0^2} + \frac{8}{n\pi}\int_0^2 (x-1)\cos\left(\frac{n\pi x}{2}\right)\,dx$$

$$\Rightarrow B_n = \frac{8}{n\pi}\int_0^2 (x-1)\cos\left(\frac{n\pi x}{2}\right)\,dx$$

BY PARTS AGAIN

$x-1$	1
$\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$	$\cos\left(\frac{n\pi x}{2}\right)$

$$\Rightarrow B_n = \frac{8}{n\pi}\left\{\cancel{\left[\frac{2}{n\pi}(x-1)\sin\left(\frac{n\pi x}{2}\right)\right]_0^2} - \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)\,dx\right\}$$

$$\Rightarrow B_n = -\frac{16}{n^2\pi^2}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)\,dx$$

$$\Rightarrow B_n = \frac{32}{n^3\pi^3}\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$\Rightarrow B_n = \frac{32}{n^3\pi^3}\left[\cos n\pi - \cos 0\right]$$

$$\Rightarrow B_n = \frac{32}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$\Rightarrow B_n = \frac{32}{n^3\pi^3} \times \begin{cases} -2 & \text{IF } n \text{ IS ODD} \\ 0 & \text{IF } n \text{ IS EVEN} \end{cases}$$

$$\Rightarrow B_m = \frac{-64}{(2m-1)\pi^3}$$

THUS THE SPECIFIC SOLUTION TO THIS PROBLEM CAN BE FORMED

$$U(x,t) = \sum_{m=1}^{\infty} \left[\frac{-64}{(2m-1)^3 \pi^3} e^{-\frac{9(2m-1)^2 \pi^2}{4} t} \sin\left[\frac{(2m-1)\pi x}{2}\right] \right]$$

OR $$U(x,t) = \sum_{k=0}^{\infty} \left[\frac{-64}{(2k+1)^3 \pi^3} e^{-\frac{9(2k+1)^2 \pi^2}{4} t} \sin\left[\frac{(2k+1)\pi x}{2}\right] \right]$$

AND HENCE OBTAINING $u(x,t)$ AS

$$\boxed{u(x,t) = 4x - \frac{64}{\pi^3} \sum_{k=0}^{\infty} \left[\frac{1}{(2k+1)^3} \exp\left[-\frac{9}{4}\pi^2 (2k+1)^2 t\right] \sin\left[\frac{(2k+1)\pi x}{2}\right] \right]}$$

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$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \ , \ y \geq 0$$

SUBJECT TO THE CONDITIONS

- $\phi(x,0) = f(x)$
- $\phi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$

TAKING THE FOURIER TRANSFORM OF THE P.D.E. IN x

$$\Rightarrow \mathcal{F}\left[\frac{\partial^2 \phi}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 \phi}{\partial y^2}\right] = \mathcal{F}[0]$$

$$\Rightarrow (ik)^2 \hat{\phi}(k,y) + \frac{\partial^2}{\partial y^2}\hat{\phi}(k,y) = 0$$

$$\Rightarrow \frac{\partial^2 \hat{\phi}}{\partial y^2} - k^2 \hat{\phi} = 0$$

THIS IS A STANDARD 2ND ORDER, AS k IS TREATED AS A CONSTANT

$$\therefore \hat{\phi}(k,y) = A(k)e^{|k|y} + B(k)e^{-|k|y} \ ,$$

ASSUMING HERE $k \in \mathbb{R}$, AS $\sqrt{k^2} \neq |k|$ IN GENERAL

AS $\phi(x,y) \to 0$ AS $\sqrt{x^2+y^2} \to \infty$, SO WILL $\hat{\phi}(k,y)$ AS $\sqrt{k^2+y^2} \to \infty$,

SO $A(k) = 0$

$$\therefore \boxed{\hat{\phi}(k,y) = B(k)e^{-|k|y}}$$

APPLY THE BOUNDARY CONDITION $\phi(x,0) = f(x) \Rightarrow \hat{\phi}(k,0) = \hat{f}(k)$

$$\Rightarrow \hat{f}(k) = B(k)e^0$$

$$\Rightarrow B(k) = \hat{f}(k)$$

$$\therefore \boxed{\hat{\phi}(k,y) = \hat{f}(k)e^{-|k|y}}$$

TO INVERT WE LOOK AT THE CONVOLUTION THEOREM

$$\boxed{\mathcal{F}[f*g] = \sqrt{2\pi}\,\mathcal{F}[f]\,\mathcal{F}[g]}$$

$$\Rightarrow \mathcal{F}[\phi(x,y)] = \mathcal{F}[f(x,0)] \times e^{-|k|y}$$

$$\Rightarrow \sqrt{2\pi}\,\mathcal{F}[\phi(x,y)] = \sqrt{2\pi}\,\mathcal{F}[f(x,0)] \times e^{-|k|y}$$

$$\Rightarrow \sqrt{2\pi}\,\hat{\phi}(k,y) = \sqrt{2\pi}\,\hat{f}(k)\,e^{-|k|y}$$

$$\Rightarrow \sqrt{2\pi}\,\hat{\phi}(k,y) = \sqrt{2\pi}\,\mathcal{F}[f]\,\mathcal{F}[g]$$

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$$\Rightarrow \boxed{\sqrt{2\pi}\,\hat{\phi}(k,y) = \mathcal{F}[f*g]}$$ (BY THE CONVOLUTION THEOREM)

$f(x)$ IS A "GIVEN" FUNCTION

$\hat{g}(k) = e^{-|k|y}$

INVERTING $\hat{g}(k) = e^{-|k|y}$

$$g(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|k|y}\, e^{ikx}\, dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|k|y}(\cos kx + i\sin kx)\, dk$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ky}\cos kx\, dk = \sqrt{\frac{2}{\pi}}\,\text{Re}\left\{\int_0^{\infty} e^{-ky}\, e^{ikx}\, dk\right\}$$

$$= \sqrt{\frac{2}{\pi}}\,\text{Re}\left\{\int_0^{\infty} e^{(-y+ix)k}\, dk\right\} = \sqrt{\frac{2}{\pi}}\,\text{Re}\left\{\frac{1}{-y+ix}\left[e^{(-y+ix)k}\right]_0^{\infty}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\,\text{Re}\left\{\frac{-y-ix}{y^2+x^2}\left[e^{-ky}\, e^{ikx}\right]_0^{\infty}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\,\text{Re}\left\{\frac{-y-ix}{y^2+x^2}\left[e^{-ky}(\cos kx + i\sin kx)\right]_0^{\infty}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\,\text{Re}\left\{\frac{-y-ix}{y^2+x^2}(0-1)\right\} = \sqrt{\frac{2}{\pi}}\,\text{Re}\left\{\frac{y+ix}{y^2+x^2}\right\} = \sqrt{\frac{2}{\pi}}\,\frac{y}{y^2+x^2}$$

FINALLY RETURNING TO THE "CONVOLUTION INVERSION"

$$\sqrt{2\pi}\,\hat{\phi}(k,y) = \mathcal{F}[f*g]$$

$$\Rightarrow \sqrt{2\pi}\,\phi(x,y) = f*g = \int_{-\infty}^{\infty} f(x-u)\, g(u)\, du$$

CANNOT USE y AS PER USUAL IN THE FORMULA AS THERE IS ALREADY A y IN THE PROBLEM

$$\Rightarrow \sqrt{2\pi}\,\phi(x,y) = \int_{-\infty}^{\infty} f(x-u)\left[\sqrt{\frac{2}{\pi}}\,\frac{y}{y^2+u^2}\right] du$$

y IS A CONSTANT IN THIS EXPRESSION

$$\Rightarrow \sqrt{2\pi}\,\phi(x,y) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x-u)\,\frac{y}{y^2+u^2}\, du$$

$$\Rightarrow \phi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x-u)}{y^2+u^2}\, du$$

AS REQUIRED