

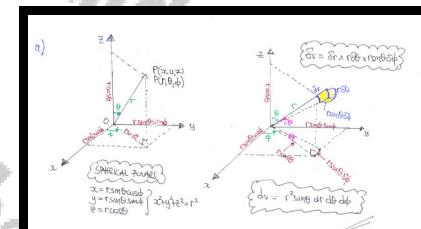
MULTIVARIABLE INTEGRATION

(SPHERICAL POLAR COORDINATES)

Question 1

- Determine with the aid of a diagram an expression for the volume element in spherical polar coordinates, (r, θ, ϕ) .
[You may not use Jacobians in this part]
- Use spherical polar coordinates to obtain the standard formula for the volume of a sphere of radius a .

$$dv = r^2 \sin \theta dr d\theta d\phi$$

a) 

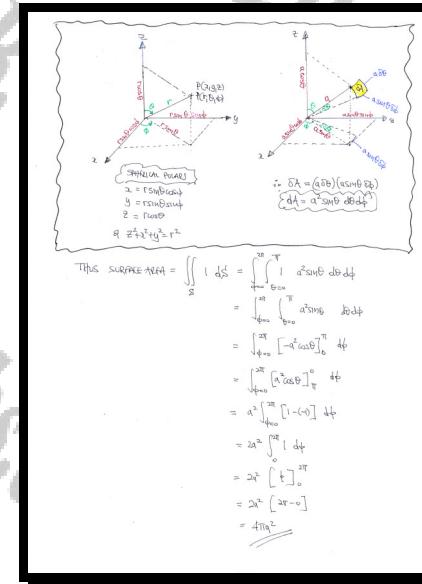
b) IN SPHERICAL POLARS

$$\begin{aligned} V &= \iiint_R 1 \, dv = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^a 1 \cdot r^2 \sin \theta \, dr \, d\theta \, d\phi \\ &= \int_{0}^{2\pi} \int_{0}^{\pi} \left[\frac{1}{3} r^3 \sin \theta \right]_0^a \, d\theta \, d\phi \\ &= \int_{0}^{2\pi} \int_{0}^{\pi} \frac{1}{3} a^3 \sin^2 \theta \, d\theta \, d\phi \\ &= \int_{0}^{2\pi} \left[-\frac{1}{3} a^3 \cos^2 \theta \right]_0^{\pi} \, d\phi = \int_{0}^{2\pi} \frac{2}{3} a^3 [\cos^2 \theta]_0^{\pi} \, d\phi \\ &= \frac{4}{3} a^3 \int_{0}^{2\pi} [1 - (-1)] \, d\phi = \frac{8}{3} a^3 \int_{0}^{2\pi} 1 \, d\phi \\ &= \frac{8}{3} a^3 \left[\frac{1}{2} \theta \right]_0^{2\pi} = \frac{8}{3} a^3 \times 2\pi = \frac{16}{3} \pi a^3 \end{aligned}$$

Question 2

Use spherical polar coordinates, (r, θ, φ) , to obtain the standard formula for the surface area of a sphere of radius a .

$$dA = a^2 \sin \theta \, d\theta \, d\varphi \Rightarrow A = 4\pi a^2$$



Question 3

Determine an exact simplified value for

$$\int_R \frac{1}{(x^2 + y^2 + z^2)^2} dx dy dz,$$

where R is the region **outside** the sphere with equation

$$x^2 + y^2 + z^2 = 1.$$

4π



Diagram showing a sphere centered at the origin with radius 1, intersecting the xy-plane at $x^2 + y^2 = 1$. A point P is shown on the sphere in the first octant.

Handwritten notes:

- $z = \text{radius}$
- $y = \text{radius}$
- $z = \text{radius}$
- $x^2 + y^2 + z^2 = 1$
- $dz dy dx = r^2 \sin \theta dr d\theta d\phi$

$$\begin{aligned}
 & \text{Thus } \iiint_R \frac{1}{(x^2 + y^2 + z^2)^2} dz dy dz \\
 & \quad \text{Using spherical} \\
 & \quad \text{coordinates} \\
 & = \int_0^{2\pi} d\phi \int_0^\pi \int_0^r \frac{1}{(r^2)^2} r^2 \sin \theta dr d\theta d\phi \\
 & = \int_0^{2\pi} d\phi \int_0^\pi \int_0^1 \frac{1}{r^2} \sin \theta dr d\theta d\phi \\
 & = \int_0^{2\pi} d\phi \int_0^\pi \left[-\frac{1}{r} \sin \theta \right]_0^1 d\theta d\phi \\
 & = \int_0^{2\pi} d\phi \int_0^\pi 0 - (-\sin \theta) d\theta d\phi \\
 & = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta d\phi \\
 & = \int_0^{2\pi} d\phi \left[-\cos \theta \right]_0^\pi d\phi \\
 & = \int_0^{2\pi} d\phi \left[-\cos \pi - (-\cos 0) \right] d\phi \\
 & = \int_0^{2\pi} d\phi \left[1 - (-1) \right] d\phi \\
 & = \int_0^{2\pi} d\phi 2 d\phi \\
 & = 2 \times 2\pi = 4\pi
 \end{aligned}$$

Question 4

The finite R region is defined as

$$1 \leq x^2 + y^2 + z^2 \leq 4.$$

Determine an exact simplified value for

$$\int_R \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} dx dy dz.$$

$$4\pi \ln 2$$

• R is a spherical annulus with radii 1 & 2.

2x = rectangular
y = rectangular
z = radial

$dxdydz = r^2 \sin\theta dr d\theta d\phi$

• THIS IS SURFACE POINTS

$$\int_R \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} dx dy dz = \int_{r=1}^{r=2} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{(r^2)^{\frac{3}{2}}} r^2 \sin\theta dr d\theta d\phi$$

$$= \int_1^2 \left[\int_0^\pi \left(\int_0^{2\pi} \frac{1}{r^3} dr \right) d\phi \right] d\theta = \int_1^2 \left[\int_0^\pi \left[\frac{1}{r^2} \right]_0^{2\pi} d\phi \right] d\theta$$

$$= \int_1^2 \left[\int_0^\pi \left[2\pi - \frac{1}{r^2} \right] d\phi \right] d\theta = \int_1^2 \left[-2\pi \ln r \right]_0^\pi d\theta$$

$$= \int_1^2 \left(2\pi \ln r \right) d\theta = \int_1^2 \ln r^2 - (-4\pi) d\theta = \int_1^2 2\ln r d\theta$$

$$= 2\ln 2 \left[\frac{1}{\theta} \right]_0^{2\pi} = 2\ln 2 [2\pi - 0] = 4\pi \ln 2$$

Question 5

Determine an exact simplified value for

$$\int_R 15z^2 \, dx \, dy \, dz,$$

where R is the region between the spheres with equations

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 4.$$

124π

Spherical coordinates: $x = r \cos \theta \sin \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$ & $r^2 = x^2 + y^2 + z^2$

$$\begin{aligned} x^2 + y^2 + z^2 = 1 &\Rightarrow r^2 = 1 \\ x^2 + y^2 + z^2 = 4 &\Rightarrow r^2 = 2 \end{aligned}$$

Thus $\iint_R 15z^2 \, dx \, dy \, dz = \dots$ switch into spherical coords

$$\begin{aligned} &= \int_0^{2\pi} \int_0^\pi \int_{r=1}^{r=2} 15(r \cos \theta)^2 (r^2 \sin \theta) \, dr \, d\theta \, d\phi \\ &= \int_0^{2\pi} \int_0^\pi \int_{r=1}^2 15r^4 \cos^2 \theta \sin \theta \, dr \, d\theta \, d\phi \\ &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^\pi \left[15r^5 \cos^2 \theta \sin \theta \right]_1^2 \, d\phi \, d\theta \\ &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^\pi 15(32 - 1) \cos^2 \theta \sin \theta \, d\phi \, d\theta \\ &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^\pi 30 \cos^2 \theta \sin \theta \, d\phi \, d\theta \\ &= \int_{\theta=0}^{2\pi} \left[30 \cos^2 \theta \right]_0^\pi \, d\theta \\ &= \int_{\theta=0}^{2\pi} 30(-1 - (-1)) \, d\theta \\ &= \int_{\theta=0}^{2\pi} 60 \, d\theta \\ &= 60 [4\theta]_0^{2\pi} \\ &= 124\pi \end{aligned}$$

Question 6

The finite R region is defined as the interior of the sphere with equation

$$x^2 + y^2 + z^2 = 1.$$

Determine an exact simplified value for

$$\int_R \sqrt{x^2 + y^2 + z^2} e^{-(x^2+y^2+z^2)} dx dy dz.$$

$$2\pi \left[1 - \frac{2}{e} \right]$$

The handwritten solution shows the evaluation of the integral $\iiint_R \sqrt{x^2 + y^2 + z^2} e^{-(x^2+y^2+z^2)} dx dy dz$. It starts by noting that the region R is a unit sphere centered at the origin. The integral is converted to spherical coordinates, resulting in $\int_0^\pi \int_0^\pi \int_0^1 r^2 e^{-r^2} r^2 \sin\theta dr d\theta d\phi$. The limits for r are from 0 to 1, for θ are from 0 to π , and for ϕ are from 0 to 2π . The integral is split into three parts: $\int_0^\pi \int_0^\pi \int_0^1 r^2 \sin\theta dr d\theta d\phi$, $\int_0^\pi \int_0^\pi \int_0^1 r^2 e^{-r^2} dr d\theta d\phi$, and $\int_0^\pi \int_0^\pi \int_0^1 e^{-r^2} dr d\theta d\phi$. The first part is evaluated as $2\pi \times [\frac{1}{2} \sin\theta]_0^\pi \times \int_0^\pi \int_0^1 r^2 \sin\theta dr d\phi = 2\pi \times [0] \times \int_0^\pi \int_0^1 r^2 \sin\theta dr d\phi = 0$. The second part is evaluated as $2\pi \times \int_0^\pi \int_0^1 r^2 dr d\phi = 2\pi \times \frac{1}{3} \times [\frac{1}{2} r^3]_0^1 \times \int_0^\pi d\phi = 2\pi \times \frac{1}{3} \times [\frac{1}{2}] \times 2\pi = \frac{2\pi^2}{3}$. The third part is evaluated as $2\pi \times \int_0^\pi \int_0^1 e^{-r^2} dr d\phi = 2\pi \times \frac{1}{2} \times [\frac{1}{2} e^{-r^2}]_0^1 \times \int_0^\pi d\phi = 2\pi \times \frac{1}{2} \times [\frac{1}{2} (e^0 - e^{-1})] \times 2\pi = 2\pi \times \frac{1}{2} \times [\frac{1}{2} (1 - \frac{1}{e})] \times 2\pi = 2\pi \times \frac{1}{4} \times [\frac{1}{2} (1 - \frac{1}{e})] \times 2\pi = 2\pi \times \frac{1}{8} (1 - \frac{1}{e}) \times 2\pi = 2\pi \left(1 - \frac{2}{e} \right)$.

Question 7

Use spherical polar coordinates to find an exact value for the following integral.

$$\iiint_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)} dx dy dz .$$

MM2F, $\pi\sqrt{\pi}$

SOLVING THE INTEGRAL

$$\begin{aligned} & \iiint_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)} dx dy dz \\ &= \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} e^{-r^2} (r^2 \sin\theta dr d\theta d\phi) \end{aligned}$$

We can split the integral into 3:

$$\begin{aligned} &= \left[\int_0^{2\pi} 1 d\phi \right] \left[\int_0^{\pi} \sin\theta d\theta \right] \left[\int_0^{\infty} r^2 e^{-r^2} dr \right] \\ &= 2\pi \times [-\cos\theta]_0^\pi \times \int_0^{\pi} r (re^{-r^2}) dr \\ &= 2\pi \times [\cos\theta]_0^\pi \times \left[-\frac{1}{2}e^{-r^2} \Big|_0^\pi + \frac{1}{2} \int_0^{\infty} e^{-r^2} dr \right] \end{aligned}$$

INTEGRATION BY PARTS

$$\begin{aligned} &= 2\pi \times (1 - (-1)) \times \frac{1}{2} \int_0^{\infty} e^{-r^2} dr \\ &= 2\pi \int_0^{\infty} e^{-r^2} dr \\ &= 2\pi \times \frac{\sqrt{\pi}}{2} \\ &= \pi\sqrt{\pi} \end{aligned}$$

Question 8

A hemispherical solid piece of glass, of radius a m, has small air bubbles within its volume.

The air bubble density $\rho(z)$, in m^{-3} , is given by

$$\rho(z) = k z,$$

where k is a positive constant, and z is a standard cartesian coordinate, whose origin is at the centre of the flat face of the solid.

Given that the solid is contained in the part of space for which $z \geq 0$, determine the total number of air bubbles in the solid.

M21 , $\frac{1}{4} \pi k a^4$

IF THE "BUBBLE DENSITY" IS GIVEN BY
 $\rho(z) = kz$, THEN

$$\begin{aligned} \text{TOTAL BUBBLES} &= \int_V \rho(z) \, dv \\ &= \int_{\text{Hemisphere}} kz \, dv \end{aligned}$$

WORKING IN SPHERICAL POLARS

$$\begin{aligned} &= \int_0^{\pi/2} \int_0^{\pi} \int_0^{r(z)} k(r \cos\theta)(r^2 \sin\theta \, dr \, d\theta \, d\phi) \\ &= \int_0^{\pi/2} \int_0^{\pi} \int_0^{r(z)} kr^3 \cos\theta \sin\theta \, dr \, d\theta \, d\phi \\ &= \left[\int_0^{\pi/2} k \, d\theta \right] \left[\int_0^{\pi} \cos\theta \sin\theta \, d\theta \right] \left[\int_0^{r(z)} r^3 \, dr \right] \\ &= \pi k \times \left[\frac{1}{2} \sin^2 \theta \right]_{0}^{\pi} \times \left[\frac{1}{4} r^4 \right]_{0}^{r(z)} \\ &= \pi k \times \frac{1}{2} \times \frac{1}{4} r(z)^4 \\ &= \frac{1}{8} \pi k a^4 \end{aligned}$$

x
 y
 z

$r \cos\theta$
 $r \sin\theta$
 z

$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$x = r \sin\theta \cos\phi$
 $y = r \sin\theta \sin\phi$
 $z = r \cos\theta$

$x^2 + y^2 + z^2 = a^2$

FOR THE HEMISPHERE

- $0 \leq r \leq a$
- $0 \leq \theta \leq \pi$
- $0 \leq \phi \leq 2\pi$

$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

Question 9

A uniform solid has equation

$$x^2 + y^2 + z^2 = a^2,$$

with $x > 0$, $y > 0$, $z > 0$, $a > 0$.

Use integration in spherical polar coordinates, (r, θ, ϕ) , to find in Cartesian form the coordinates of the centre of mass of the solid.

$$\left(\frac{3}{8}a, \frac{3}{8}a, \frac{3}{8}a\right)$$

• INFINITESIMAL VOLUME dV , THIS MASS ρdV

• NUMBER OF INFINITESIMAL ABOUT THE Oxy PLANE ($z=0$) ARE $(dV) \times 2$ (site dimension)

• SUMMING UP 3 TRADING LIMITS

$$M_{\bar{z}} = \iiint_{\text{solid}} \rho z \, dV$$

SWITCH INTO SPHERICAL POLAR

$$M_{\bar{z}} = \int_{0}^{\pi} \int_{0}^{\frac{\pi}{2}} \int_{0}^{a} \rho (r \cos \theta) r^2 \sin \theta \, dr \, d\theta \, d\phi$$

• $\frac{1}{2} \int_0^{\pi} \int_0^{\frac{\pi}{2}} \int_0^a \rho (r^3 \sin^2 \theta \cos \theta) \, dr \, d\theta \, d\phi$

$$\Rightarrow \frac{1}{2} \int_0^{\pi} \int_0^{\frac{\pi}{2}} \left[\frac{1}{4} r^4 \sin^2 \theta \cos \theta \right]_0^a \, d\theta \, d\phi$$

$$\Rightarrow \frac{1}{8} \int_0^{\pi} \int_0^{\frac{\pi}{2}} a^4 \sin^2 \theta \cos^2 \theta \, d\theta \, d\phi$$

$$\Rightarrow \frac{1}{8} \int_0^{\pi} \left[\frac{1}{8} a^4 (\sin^3 \theta \cos \theta) \right]_0^{\frac{\pi}{2}} \, d\phi$$

$$\Rightarrow \frac{1}{64} a^4 \int_0^{\pi} \left[\frac{1}{8} (\sin^3 \theta \cos \theta) \right]_0^{\frac{\pi}{2}} \, d\phi$$

$$\Rightarrow \frac{1}{64} a^4 \left[\frac{1}{8} \left(\frac{1}{4} \sin^4 \theta \right) \right]_0^{\frac{\pi}{2}} \, d\phi$$

$$\Rightarrow \frac{1}{512} a^4 \left[\frac{1}{8} \left(\frac{1}{4} \sin^4 \theta \right) \right]_0^{\frac{\pi}{2}} \, d\phi$$

$$\Rightarrow \frac{1}{512} a^4 \left[\frac{1}{8} \left(\frac{1}{4} \cdot 1 \right) \right]_0^{\frac{\pi}{2}} \, d\phi$$

$$\Rightarrow \frac{1}{512} a^4 \cdot \frac{1}{8} \cdot \frac{1}{4} \cdot \frac{\pi}{2} \, d\phi$$

$$\Rightarrow \frac{2\pi}{512} a^4 \cdot \frac{1}{8} \, d\phi$$

$$\Rightarrow \frac{2\pi}{4096} a^4 \cdot \left[\frac{1}{8} \theta \right]_0^{\frac{\pi}{2}} \, d\phi$$

$$\Rightarrow \frac{2\pi}{32768} a^4 \cdot \frac{\pi}{8} \, d\phi$$

$$\Rightarrow \bar{z} = \frac{3}{8}a$$

$$\therefore \bar{(x, y, z)} = \left(\frac{3}{8}a, \frac{3}{8}a, \frac{3}{8}a\right)$$

Question 10

A solid sphere has equation

$$x^2 + y^2 + z^2 = a^2.$$

The density, ρ , at the point of the sphere with coordinates (x_1, y_1, z_1) is given by

$$\rho = \sqrt{x_1^2 + y_1^2}.$$

Determine the **average** density of the sphere.

$$\boxed{\text{ANSWER}}, \quad \bar{\rho} = \frac{3}{16} \pi a$$

① STARTING BY THE DEFINITION OF MASS FOR VARIABLE DENSITY

$$\text{MASS} = \int_V \rho(xyz) dV = \int \sqrt{x^2+y^2+z^2} dx dy dz$$

VOLUME OF
 $x^2+y^2+z^2=a^2$

② SPLITTING INTO SPHERICAL POLARS

$$\Rightarrow \text{MASS} = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sqrt{(rsin\phi\cos\theta)^2 + (rsin\phi\sin\theta)^2 + r^2\cos^2\phi} r^2 \sin\phi dr d\theta d\phi$$

$$\Rightarrow \text{MASS} = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sqrt{r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi} r^2 \sin\phi dr d\theta d\phi$$

$$\Rightarrow \text{MASS} = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (rsin\phi)\sqrt{1+\sin^2\theta + \sin^2\phi} r^2 \sin\phi dr d\theta d\phi$$

$$\Rightarrow \text{MASS} = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^3 \sin^2\theta dr d\theta d\phi$$

③ INTEGRATING WITH RESPECT TO ϕ FIRST

$$\Rightarrow \text{MASS} = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \left[\frac{1}{4}r^4 \sin^2\theta \right]_0^{2\pi} d\theta dr$$

$$\Rightarrow \text{MASS} = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \frac{1}{4}r^4 \sin^2\theta d\theta dr$$

$$\Rightarrow \text{MASS} = \frac{1}{4}a^4 \int_{r=0}^{a} \int_{\theta=0}^{\pi} \sin^2\theta d\theta dr$$

④ INTEGRATING WITH RESPECT TO θ NEXT

$$\text{MASS} = \frac{1}{4}a^4 \times 2\pi \times \int_{r=0}^a \frac{1}{2} - \frac{1}{2}\cos 2\theta dr$$

$$\text{MASS} = \frac{1}{4}a^4 \times 2\pi \times \frac{1}{2}\pi$$

$$\text{MASS} = \frac{1}{4}a^4 \pi^2$$

⑤ NOW THE VOLUME OF A SPHERE OF RADIUS a IS $V = \frac{4}{3}\pi a^3$

$$\therefore \text{AVERAGE DENSITY} = \frac{\text{MASS}}{\text{VOLUME}} = \frac{\frac{1}{4}a^4 \pi^2}{\frac{4}{3}\pi a^3} = \frac{3}{16}\pi a$$

Question 11

A thin uniform spherical shell with equation

$$x^2 + y^2 + z^2 = a^2, \quad a > 0,$$

occupies the region in the first octant.

Use integration in spherical polar coordinates, (r, θ, ϕ) , to find in Cartesian form the coordinates of the centre of mass of the shell.

$$\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2} \right)$$

As the object is symmetrical we only need to find the position with respect to one of 3 axes
TOTAL MASS = $\int \rho \, dV = \frac{1}{2} \pi a^3$
TOTAL MASS = $\frac{1}{2} \rho V a^3$ (ρ = MASS PER UNIT VOLUME)

CONSIDER UNIFORMLY MASS $\delta m = \delta \rho \, dV$ IN SPHERICAL SHELL
IT ISN'T ABOUT THE XY PLANE IS
 $\delta m \propto (z \text{ COORD OF INFINITESIMAL})$
 $\delta m \propto r \cos \theta$

SUMMING UP A THINNING SHELL
 $M \vec{z} = \int_S r \cos \theta \, dV$

$M \vec{z} = \int_S \rho r^2 \sin \theta \cos \theta \, d\theta \, d\phi \, dr$

$\Rightarrow M \vec{z} = \int_S \rho a^2 \cos \theta \, d\theta \, d\phi \, dr$
 $\Rightarrow \frac{1}{2} \pi a^4 \vec{z} = \int_S \rho a^2 \cos \theta \, d\theta \, d\phi \, dr$
 $\Rightarrow \frac{1}{2} \pi a^4 \vec{z} = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\pi} \int_0^{2\pi} \rho a^2 \cos \theta \, (a^2 \sin \theta \, d\theta \, d\phi \, dr)$
 $\Rightarrow \frac{1}{2} \pi a^4 \vec{z} = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\pi} \frac{3}{4} a^5 \sin^2 \theta \cos \theta \, d\theta \, d\phi \, dr$
 $\Rightarrow \frac{1}{2} \pi a^4 \vec{z} = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{3}{4} a^5 \sin^2 \theta \cos \theta \right]_0^{\pi} d\phi \, dr$
 $\Rightarrow \frac{1}{2} \pi a^4 \vec{z} = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{4} a^5 \cdot \frac{1}{2} \pi \, d\phi \, dr$
 $\Rightarrow \frac{1}{2} \pi a^4 \vec{z} = \frac{3}{8} \pi a^5 \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \, dr$
 $\Rightarrow \frac{1}{2} \pi a^4 \vec{z} = \frac{3}{8} \pi a^5 \cdot \frac{1}{2} \pi \vec{z}$
 $\Rightarrow \vec{z} = \frac{1}{2} a$

By symmetry $(\vec{x}, \vec{y}, \vec{z}) = \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2} \right)$

Question 12

The finite region Ω is defined as

$$x^2 + y^2 + z^2 \leq 1.$$

Use Spherical Polar Coordinates (r, θ, ϕ) , to evaluate the volume integral

$$\int_{\Omega} (r \sin \theta \cos \phi)^4 dV.$$

$\frac{4\pi}{35}$

$$\begin{aligned}
 \int_V (r \sin \theta \cos \phi)^4 dV &= \int_0^\infty \int_0^\pi \int_0^{2\pi} r^4 \sin^4 \theta \cos^4 \phi (r^2 \sin \theta dr) d\theta d\phi \\
 &= \int_0^\infty r^6 \sin^5 \theta \cos^3 \phi dr d\theta d\phi \\
 &= \left[\int_0^{2\pi} (\cos \theta)^4 d\theta \right] \left[\int_0^\pi \sin^5 \theta d\theta \right] \left[\int_0^\infty r^6 dr \right] \\
 &\quad \text{Diagram showing a spherical shell element with radius } r, \text{ height } \sin \theta, \text{ width } \cos \theta, \text{ and area element } r^2 \sin \theta dr d\theta d\phi. \\
 &= \int_0^{2\pi} 4 \cos^4 \theta d\theta \left[\int_0^\pi 2 \sin^5 \theta d\theta \right] \left[\frac{1}{7} r^7 \right]_0^\infty \\
 &= \left[2 \int_0^{\pi/2} 2 \cos^4 \theta d\theta \right] \left[\int_0^\pi 2 \sin^5 \theta d\theta \right] \times \frac{1}{7} \\
 &\approx 2 B(\tfrac{5}{2}, \tfrac{1}{2}) \times B(\tfrac{3}{2}, \tfrac{1}{2}) \times \frac{1}{7} \\
 &= 2 \frac{\Gamma(\tfrac{5}{2})\Gamma(\tfrac{1}{2})}{\Gamma(\tfrac{3}{2})} \times \frac{\Gamma(\tfrac{3}{2})\Gamma(\tfrac{1}{2})}{\Gamma(\tfrac{1}{2})} \times \frac{1}{7} \\
 &= 2 \cancel{\frac{3}{2}\Gamma(\tfrac{5}{2})\Gamma(\tfrac{1}{2})} \times \frac{2\Gamma(\tfrac{3}{2})}{\cancel{\frac{1}{2}\Gamma(\tfrac{1}{2})}} \times \frac{1}{7} \\
 &= \frac{3}{2} \left[\Gamma(\tfrac{5}{2}) \right]^2 \times \frac{16}{15} \times \frac{1}{7} \\
 &= \frac{3}{2} \cdot \frac{15}{8} \times \frac{16}{15} \times \frac{1}{7} \\
 &= \frac{4\pi}{35}
 \end{aligned}$$

Question 13

A solid sphere has equation

$$x^2 + y^2 + z^2 = 1.$$

The region defined by the double cone with Cartesian equation

$$3z^2 \geq x^2 + y^2,$$

is bored out of the sphere.

Determine the volume of the remaining solid.

$$\boxed{V = \frac{2}{3}\pi}, \quad \boxed{V = \frac{2}{3}\pi}$$

• Since by finding the intersection of the two objects

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ 3z^2 = x^2 + y^2 \end{cases} \Rightarrow$$

$$3z^2 + z^2 = 1 \Rightarrow$$

$$4z^2 = 1 \Rightarrow$$

$$z^2 = \frac{1}{4} \Rightarrow$$

$$z = \pm \frac{1}{2}$$

• Subtracting now spherical regions, the required volume, which is what remains of the sphere is given by

$$\rightarrow V = \int_{-1/2}^{1/2} 1 \, dz$$

"now"

$$\rightarrow V = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{\sqrt{1-z^2}} 1 \, dy \, dz$$

2nd order
drills
dv in SRC

$$\rightarrow V = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \, dz \left[\frac{y^2}{2} \Big|_0^{\sqrt{1-z^2}} \right]_{z=0}^{\frac{1}{2}}$$

$$\rightarrow V = 2\pi \times \left[-\cos^2 \frac{\pi}{4} \right]_{\frac{1}{2}}^{\frac{1}{2}} \times \frac{1}{3}$$

$$\rightarrow V = 2\pi \times 1 \times \frac{1}{3} = \frac{2\pi}{3}$$

Question 14

A solid sphere has radius 5 and is centred at the Cartesian origin O .

The density ρ at point $P(x_1, y_1, z_1)$ of the sphere satisfies

$$\rho = \frac{3}{85} \left[1 + |z_1| \sqrt{x_1^2 + y_1^2 + z_1^2} \right].$$

Use spherical polar coordinates, (r, θ, ϕ) , to find the mass of the sphere.

$$m = 50\pi$$

$m = \int_R \rho \, dv$

Here

$$m = 2 \int_{0}^{25} \left[\frac{3}{85} \left(1 + z \sqrt{x^2 + y^2 + z^2} \right) \right] dv$$

CONSIDER HEMISPHERE

SPLIT INTO SPHERICAL POLAR

$$\rightarrow m = \frac{3}{85} \int_{0}^{25} \int_{0}^{\pi} \int_{0}^{2\pi} \left[1 + \frac{r \cos \theta \times r}{2} \sqrt{r^2 - r^2 \sin^2 \theta} \right] r^2 \sin \theta \, d\theta \, d\phi \, dr$$

$$\rightarrow m = \frac{3}{85} \int_{0}^{25} \int_{0}^{\pi} \left(r^2 + \frac{r^3 \cos^2 \theta}{2} \right) \, d\theta \, dr$$

$$\rightarrow m = \frac{3}{85} \int_{0}^{25} \left[\frac{r^3}{3} \sin \theta + \frac{r^5}{2} \sin^3 \theta \right]_{0}^{\pi} \, dr$$

$$\rightarrow m = \frac{3}{85} \int_{0}^{25} \left[\frac{r^3}{3} \sin \theta + \frac{r^5}{2} \sin^3 \theta \right]_{0}^{\pi} \, dr$$

$$\rightarrow m = \frac{3}{85} \int_{0}^{25} \left(-\frac{r^3}{3} \cos \theta + \frac{r^5}{2} \cos^3 \theta \right)_{0}^{\pi} \, dr$$

$$\rightarrow m = \frac{3}{85} \int_{0}^{25} \left(0 + \frac{125}{2} \right) \, dr = \frac{3}{85} \int_{0}^{25} \frac{250}{2} \, dr$$

$$\rightarrow m = 25 \int_{0}^{25} dr = 50\pi$$

Question 15

A solid uniform sphere has mass M and radius a .

Use spherical polar coordinates, (r, θ, φ) , to show that the moment of inertial of this sphere about one of its diameters is $\frac{2}{5}Ma^2$.

[proof]

The derivation is contained within a black rectangular box. It includes three diagrams: a 3D coordinate system with axes x, y, z; a cross-section of the sphere in the xy-plane; and a small box labeled "Volume element". The text is handwritten in black ink.

MASS OF INERTIAL ELEMENT = $\rho dV = \rho r^2 \sin\theta dr d\theta d\varphi$

MOMENT OF INERTIA AROUND THE Z AXIS = $(\rho r^2 \sin\theta dr d\theta d\varphi) \times a^2$
 $= \rho r^2 \sin\theta dr d\theta d\varphi \times (\sin\theta)^2$
 $= \rho r^2 \sin^3\theta dr d\theta d\varphi$

SUMMING UP

$$\begin{aligned} I &= \iiint_V \rho r^2 \sin^3\theta dr d\theta d\varphi = \rho \int_0^{2\pi} \int_{\pi/2}^{\pi} \left[\frac{1}{3}r^3 \right]_0^a \sin^3\theta d\theta d\varphi \\ &= \rho \times \frac{1}{3}a^3 \int_{\pi/2}^{\pi} \int_0^{2\pi} \sin^3(1-\cos\theta) d\theta d\varphi = \frac{1}{3}\rho a^3 \int_{\pi/2}^{\pi} \int_0^{2\pi} \sin^3 - \sin^3\cos\theta d\theta d\varphi \\ &= \frac{1}{3}\rho a^3 \int_{\pi/2}^{\pi} \left[-\cos\theta + \frac{1}{3}\cos^3\theta \right]_0^\pi d\theta = \frac{1}{3}\rho a^3 \int_{\pi/2}^{\pi} (1-\frac{1}{3})(-1+\frac{1}{3}) d\theta \\ &= \frac{1}{3}\rho a^3 \int_{\pi/2}^{\pi} \frac{4}{3} d\theta = \frac{1}{3}\rho a^3 \times \frac{4}{3} \times \pi/2 = \frac{2}{9}\pi\rho a^5 = \frac{2}{9}\pi \left(\frac{M}{4\pi a^3}\right) a^5 \\ &= \frac{2}{5}Ma^2 \end{aligned}$$

Question 16

A thin uniform spherical shell has mass m and radius a .

Use spherical polar coordinates, (r, θ, φ) , to show that the moment of inertial of this spherical shell about one of its diameters is $\frac{2}{3}ma^2$.

proof

SUPER ELEMENT IN SPHERICAL POLARIS IS
 $dV = r^2 \sin\theta dr d\theta d\phi$

- SURFACE AREA = $4\pi r^2$
- MASS PER UNIT AREA
 $\rho = \frac{m}{4\pi r^2}$

WITHOUT LOSS OF GENERALITY TAKE THE Z-AXIS AS THE DIAHETER OF THE SHELL

- MASS OF INFINESIMAL IS ρdV
- MOMENT OF INERTIA ABOUT THE Z-AXIS IS
 $(\rho dV) \times r^2 = \rho dV \times (a \sin\theta)^2$
 $= \frac{m}{4\pi r^2} \times a^2 \sin^2\theta dV$
 $= \frac{m}{4\pi} \sin^2\theta dV$
 $- \frac{m}{4\pi} \sin^2\theta (a^2 \sin^2\theta dV)$
 $= \frac{m a^2}{4\pi} \sin^2\theta dV$

SUMMING UP AND TAKING LIMITS

$$I = \iiint \frac{m a^2}{4\pi} \sin^2\theta dV$$

$$I = \left[\frac{m a^2}{4\pi} \right] \int_0^\pi \int_0^\pi \int_0^{2\pi} \sin^2\theta d\phi d\theta d\phi$$

$$I = \frac{m a^2}{4\pi} \int_0^\pi \int_0^\pi \int_0^{2\pi} \sin^2\theta \sin\theta d\phi d\theta d\phi$$

$$I = \frac{m a^2}{4\pi} \int_0^\pi \int_0^\pi \int_0^{2\pi} \sin^3\theta (-\cos\theta) d\phi d\theta d\phi$$

$$I = \frac{m a^2}{4\pi} \int_0^\pi \int_0^\pi \left[-\cos\theta + \frac{1}{3}\cos^3\theta \right]_0^{2\pi} d\phi d\theta$$

$$I = \frac{m a^2}{4\pi} \int_0^\pi \left[(1-\frac{1}{3})(-1+1) \right] d\phi$$

$$I = \frac{m a^2}{4\pi} \int_0^\pi \frac{4}{3} d\phi$$

$$I = \frac{m a^2}{4\pi} \left[\frac{4}{3}\phi \right]_0^\pi$$

$$I = \frac{m a^2}{4\pi} \left[\frac{4}{3}\pi - 0 \right]$$

$$I = \frac{2}{3}ma^2$$

Question 17

The finite region R is defined as the region enclosed by the ellipsoid with Cartesian equation

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1.$$

By first transforming the Cartesian coordinates into a new Cartesian coordinate system, use spherical polar coordinates, (r, θ, φ) , find the value of

$$\iiint_R (x^2 + y^2 + z^2) \, dx \, dy \, dz.$$

800π

Given the substitution:

$$x = 3r \sin \theta \cos \phi, \quad -\pi/2 \leq \theta \leq \pi, \quad -1 \leq \cos \phi \leq 1$$

$$y = 4r \sin \theta \sin \phi, \quad -\pi/2 \leq \phi \leq \pi/2$$

$$z = 5r \cos \theta, \quad -5 \leq z \leq 5$$

$$dx \, dy \, dz = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} \, dr \, d\theta \, d\phi$$

$$dr \, d\theta \, d\phi = r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$\text{WHERE THE VOLUME ELEMENT IS THE CUBE OF } r^2 \sin \theta$$

$$x^2 + y^2 + z^2 = 1$$

SUMMING OVER SPHERICAL POLARIS

$$dx \, dy \, dz = 60 \, dr \, d\theta \, d\phi$$

$$= 60 \int_0^5 \left[(r^2 \sin^2 \theta \cos^2 \phi)^2 + 16(r^2 \sin^2 \theta \sin^2 \phi)^2 + 25(r^2 \cos^2 \theta)^2 \right] r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$= 60 \int_0^5 \left[r^4 \sin^4 \theta \cos^2 \phi + 16r^4 \sin^4 \theta \sin^2 \phi + 25r^4 \cos^2 \theta \right] r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$= 60 \int_0^5 r^6 \left[\sin^4 \theta \cos^2 \phi + 16 \sin^4 \theta \sin^2 \phi + 25 \cos^2 \theta \right] \, dr \, d\theta \, d\phi$$

$$= 60 \int_0^5 \left[\frac{1}{5} r^7 \right]_0^5 \times \left[\sin^4 \theta (\cos^2 \phi + 16 \sin^2 \phi) + 25 \cos^2 \theta \right] \, d\theta \, d\phi$$

$$= 12 \int_{-\pi/2}^{\pi/2} \left[\frac{1}{5} (9 + 7 \sin^2 \phi) + 25 \cos^2 \theta \right] \, d\theta \, d\phi$$

$$= 12 \int_{-\pi/2}^{\pi/2} \left[\sin^2 \theta (1 - \cos^2 \theta) (9 + 7 \sin^2 \phi) + 25 \cos^2 \theta \sin^2 \theta \right] \, d\theta \, d\phi$$

$$= 12 \int_{-\pi/2}^{\pi/2} \left[(\sin^2 \theta - \sin^4 \theta)(9 + 7 \sin^2 \phi) + 25 \cos^2 \theta \sin^2 \theta \right] \, d\theta \, d\phi$$

$$= 12 \int_{-\pi/2}^{\pi/2} \left[(-\cos^2 \theta + \frac{1}{2} \cos^4 \theta)(9 + 7 \sin^2 \phi) - \frac{25}{2} \cos^2 \theta \sin^2 \theta \right] \, d\theta \, d\phi$$

$$= 12 \int_{-\pi/2}^{\pi/2} \left[(-\cos \theta - \frac{1}{2} \cos^2 \theta)(9 + 7 \sin^2 \phi) + \frac{25}{2} \cos \theta \sin^2 \theta \right] \Big|_{\theta=-\pi/2}^{\theta=\pi/2} \, d\phi$$

$$= 12 \int_{-\pi/2}^{\pi/2} \left[\left(1 - \frac{1}{2}\right)(9 + 7 \sin^2 \phi) + \frac{25}{2} \right] - \left[\left(-1 + \frac{1}{2}\right)(9 + 7 \sin^2 \phi) - \frac{25}{2} \right] \, d\phi$$

$$= 12 \int_{-\pi/2}^{\pi/2} \frac{25}{2} (1 + 7 \sin^2 \phi) + \frac{25}{2} + \frac{2}{3} (9 + 7 \sin^2 \phi) + \frac{25}{3} \, d\phi$$

$$= 12 \int_{-\pi/2}^{\pi/2} \frac{4}{3} \left[9 + 7 \left(\frac{1}{2} + \cos 2\phi \right) \right] + \frac{50}{3} \, d\phi$$

$$= 12 \int_{-\pi/2}^{\pi/2} 12 + \frac{14}{3} - \frac{14}{3} \cos 2\phi + \frac{50}{3} \, d\phi$$

$$= 12 \int_{-\pi/2}^{\pi/2} \frac{100}{3} - \frac{14}{3} \cos 2\phi \, d\phi$$

$$= 12 \int_{-\pi/2}^{\pi/2} \frac{100}{3} - \frac{7}{3} \sin 2\phi \, d\phi$$

$$= 12 \times \frac{100}{3} \times 2\pi$$

$$= 800\pi$$

Question 18

A solid uniform sphere of radius a , has variable density $\rho(r) = r$, where r is the radial distance of a given point from the centre of the sphere.

- Use spherical polar coordinates, (r, θ, ϕ) , to find the moment of inertia of this sphere I , about one of its diameters.
- Given that the total mass of the sphere is m , show that

$$I = \frac{4}{9}ma^2.$$

$$I = \frac{4}{9}\pi a^6$$

a) Work in spherical polar coordinates

- With less or greater time the 25 will be the rotation. Integrate.
- $\rho(r) = r$
- MASS OF INHABITANT VOLUME IN SPHERICAL POLAR COORDINATES
is given by
 $dV = r^2 \sin\theta dr d\theta d\phi$
- MOMENT OF INERTIA OF THE INHABITANT IS
 $(r^2 \sin\theta dr d\theta d\phi) \times r^2 = (r^4 \sin\theta dr d\theta d\phi) (r^2)$
 $= r^6 \sin\theta dr d\theta d\phi$

$$\text{Hence } I = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^a r^6 \sin\theta dr d\theta d\phi = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^a r^6 \sin\theta (1 - \cos\theta) dr d\theta d\phi$$

$$= \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^a \frac{1}{6} r^6 (1 - \cos\theta)^2 dr d\theta d\phi = \frac{1}{6} \int_{0}^{2\pi} \int_{0}^{\pi} \left[-\frac{1}{5} r^6 + \frac{1}{3} r^6 \cos^2\theta \right]_0^a d\theta d\phi$$

$$= \frac{1}{6} \int_{0}^{2\pi} \int_{0}^{\pi} \left(\frac{1}{5} a^6 - \left(\frac{1}{5} a^6 \right) \right) d\theta d\phi = \frac{1}{6} \int_{0}^{2\pi} \frac{4}{5} a^6 d\theta = \frac{1}{6} \cdot \frac{4}{5} a^6 \times \frac{2\pi}{3} = \frac{4\pi a^6}{9}$$

b) Mass Total Mass

$$m = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^a r^2 \rho(r) \sin\theta dr d\theta d\phi = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^a r^2 r^2 \sin\theta dr d\theta d\phi$$

$$= \int_{0}^{2\pi} \int_{0}^{\pi} \left[\frac{1}{3} r^4 \sin\theta \right]_{0}^a d\theta d\phi = \int_{0}^{2\pi} \int_{0}^{\pi} \frac{1}{3} a^4 \sin\theta d\theta d\phi$$

$$= \frac{1}{3} a^4 \int_{0}^{2\pi} \left[-\cos\theta \right]_0^{\pi} d\phi = \frac{1}{3} a^4 \int_{0}^{2\pi} (2\cos 0) d\phi$$

$$= \frac{1}{3} a^4 \int_{0}^{2\pi} (2) d\phi = \frac{1}{3} a^4 \left[2\phi \right]_0^{2\pi} = \frac{1}{3} a^4 \cdot 4\pi = \frac{4\pi a^4}{3}$$

$$\therefore I = \frac{4\pi a^6}{9} = \frac{4\pi a^4 a^2}{3} = \frac{4\pi a^2}{3} \quad \text{as required}$$

Question 19

Evaluate the triple integral

$$\int_V 5x^2 \, dxdydz .$$

where V is the finite region contained within the closed surface with equation

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

8π

Question 20

The finite R region is defined as

$$x^2 + y^2 + z^2 \leq 2z$$

Determine an exact simplified value for

$$\int_R \frac{z}{\sqrt{x^2 + y^2 + z^2}} dx dy dz$$

$$\frac{16\pi}{15}$$

Question 21

The finite R region is defined as

$$4z \leq x^2 + y^2 + z^2 \leq 16z.$$

Determine an exact simplified value for

$$\int_R \left(\frac{z}{8}\right)^3 dx dy dz.$$

_____ , 1092π

THE INTEGRATION REGION SURFED SPHERICAL POLARS SINCE

- $x^2 + y^2 + z^2 = r^2$
- $2r^2 - 2z^2 = 16z$
- $2r^2 - 2z^2 - 16z = 0$
- $2r^2 - (z-8)^2 = 64$

SURFACE OF RADII 2,
CENTRE AT (0,0,2),
CENTRE AT (0,0,8)

USING SPHERICAL POLARS

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

TRANSFORM THE EQUATIONS OF SPHERES

$$\begin{aligned} x^2 + y^2 + z^2 &= r^2 & x^2 + y^2 + z^2 &= r^2 \\ r^2 &= 16r \cos\theta & r^2 &= 16r \cos\theta \\ 1 &= 16 \cos\theta & r &= 16 \cos\theta \end{aligned}$$

TRANSFORM THE LIMITS

$$\begin{aligned} r > 0 \Rightarrow 0 &\leq \theta \leq \frac{\pi}{2} \\ 0 &\leq \phi \leq 2\pi \end{aligned}$$

TRANSFORM THE VOLUME ELEMENT IN SPHERICAL POLARS

$$dV = r^2 \sin\theta dr d\theta d\phi$$

TRANSFORM THE INTEGRAL TERMS

$$\iiint \left(\frac{z^3}{8}\right) dx dy dz = \int_{0}^{\pi} \int_{0}^{\frac{\pi}{2}} \int_{0}^{16\cos\theta} \left(\frac{r^3 \cos^3\theta}{8}\right)^3 (r^2 \sin\theta dr) d\theta d\phi$$

$$\begin{aligned} &= \int_0^{\pi} \int_0^{\frac{\pi}{2}} \int_0^{16\cos\theta} \frac{r^9 \cos^9\theta \sin\theta}{512} dr d\theta d\phi \\ &= \int_0^{\pi} \int_0^{\frac{\pi}{2}} \left[\frac{r^{10} \cos^{10}\theta \sin\theta}{512 \cdot 10} \right]_{r=0}^{r=16\cos\theta} d\theta d\phi \\ &= \int_0^{\pi} \int_0^{\frac{\pi}{2}} \frac{16384 \cos^{10}\theta \sin\theta}{5120} - \frac{1}{5} \cos^8\theta \sin\theta d\theta d\phi \\ &= \int_0^{\pi} \int_0^{\frac{\pi}{2}} 512 \cos^8\theta \sin\theta d\theta d\phi \\ &= \int_0^{\pi} \left[512 \cos^9\theta \right]_0^{\frac{\pi}{2}} d\theta \\ &= \int_0^{\pi} 512 d\theta \\ &= \left[512\theta \right]_0^{\pi} \\ &= 1024\pi \end{aligned}$$

Question 22

A solid sphere has equation

$$x^2 + y^2 + z^2 = a^2, \quad a > 0.$$

The sphere has variable density ρ , given by

$$\rho = k(a - z), \quad k > 0.$$

Use integration in spherical polar coordinates, (r, θ, φ) , to find in Cartesian form the coordinates of the centre of mass of the sphere.

$$\boxed{(0, 0, -\frac{1}{5}a)}$$

AS THE DENSITY FUNCTION ONLY DEPENDS ON Z THE COORDINATE OF WHICH LIES ON THE Z-AXIS

WE WANT FIND THE CENTRE OF MASS OF THE SPHERE

$$M = \int_V \rho \, dv = \int_V k(a-z) \, dv$$

SUM OF TWO SPHERICAL POINTS IN ORDINATES

$$M_1 = \int_{0 \leq r \leq a} \int_0^{\pi} \int_0^{2\pi} k(a-r\cos\theta) r^2 \sin\theta \, d\theta \, d\phi \, dr \quad (\text{ON } "y" \text{ PLANE})$$

$$M_2 = \int_{0 \leq r \leq a} \int_0^{\pi} \int_0^{2\pi} k(a r^2 \sin\theta - r^2 \sin^2\theta) r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$M = 2\pi k \int_{0 \leq r \leq a} \int_0^{\pi} \left[ar^2 \sin\theta - \frac{1}{3} r^4 \sin^3\theta \right]_0^{\pi} \, dr$$

$$M = \frac{4}{3}\pi k a^3$$

NEXT CONSIDER THE WEIGHT OF AN INFINITE ELEMENT VOLUME dv AT THE Z-PLANE POINT (z = a cos\theta)

$$(\rho \, dv)z = k(a-z)z = k(a z - z^2)$$

$$= k(a z - z^2) \quad (\text{IN SPHERICAL POLAR})$$

SUMMING UP A THOUSANDS

$$M\bar{z} = \iiint_V k(a z - z^2) \, dv$$

$$M = \int_{0 \leq r \leq a} \int_0^{\pi} \int_0^{2\pi} k \left[ar^2 \sin\theta - \frac{1}{3} r^4 \sin^3\theta \right] \, d\theta \, d\phi \, dr$$

$$\frac{4}{3}\pi k a^3 \bar{z} = k \int_{0 \leq r \leq a} \int_0^{\pi} \left[\frac{1}{2}ar^2 \sin^2\theta - \frac{1}{3}r^4 \sin^3\theta \right]_0^{\pi} \, dr$$

$$\frac{4}{3}\pi k a^3 \bar{z} = \int_{0 \leq r \leq a} \int_0^{\pi} \frac{1}{2}ar^2 \sin^2\theta - \frac{1}{3}r^4 \sin^3\theta \, dr \, d\theta$$

$$\frac{4}{3}\pi k a^3 \bar{z} = \int_{0 \leq r \leq a} \int_{\frac{\pi}{2}}^0 \frac{1}{2}ar^2 \sin^2\theta - \frac{1}{3}r^4 \sin^3\theta \, dr \, d\theta$$

$$\frac{4}{3}\pi k a^3 \bar{z} = \int_{0 \leq r \leq a} \left[\frac{1}{2}ar^2 \theta + \frac{1}{12}r^4 \theta \right]_0^0 \, dr$$

$$\frac{4}{3}\pi k a^3 \bar{z} = \int_{0 \leq r \leq a} \left(0 - \frac{1}{12}r^4 \right) - \left(0 + \frac{1}{12}r^4 \right) \, dr$$

$$\frac{4}{3}\pi k a^3 \bar{z} = \int_{0 \leq r \leq a} -\frac{1}{6}r^4 \, dr$$

$$\frac{4}{3}\pi k a^3 \bar{z} = -\frac{1}{6}a^5$$

$$\bar{z} = -\frac{1}{3}a$$

ie SECURE THE Z-PLANE AT $(0, 0, -\frac{1}{3}a)$

Question 23

A solid is defined in a Cartesian system of coordinates by

$$x^2 + y^2 = xz, \quad 0 \leq z \leq 2.$$

- Describe the solid with the aid of a sketch.
- Use standard elementary formulas to find the volume of the solid.
- Use spherical polar coordinates to verify the answer to part (b)

$$V = \frac{2}{3}\pi$$

a)

$x^2 + y^2 = xz, \quad 0 \leq z \leq 2.$

- z is now negative
This is $xz < 0 \rightarrow x^2 + y^2 < 0$
which is impossible
- $z \geq 0$
- However y can be negative
- Solid's projection onto the xy -plane can sit in the first/fourth quadrant
- Now if $y \neq 0$ ($x=0$ plane)
$$x^2 = xz$$

$$x = z$$
- However if $z = 0$ = diagonal constraint
$$x^2 + y^2 = kz$$

$$x^2 - kz + y^2 = 0$$

$$(x - \frac{k}{2})^2 + y^2 = \frac{k^2}{4}$$

If circles as the cross section below

b)

$x^2 + y^2 = xz$

BASE OF THE CONE

$$x^2 + y^2 = z^2$$

$$x^2 - 2xz + z^2 = 0$$

$$(x-z)^2 = 0$$

$$x = z$$

\therefore BASE AREA = $\pi r^2 = \pi$
HEIGHT = 2
Volume = $\frac{1}{3}\pi r^2 h = \frac{2}{3}\pi$

c)

Differential volumes

$$dV = r^2 \sin\theta dr d\theta d\phi$$

$$y = r \cos\theta \sin\phi$$

$$z = r \cos\theta$$

$$x^2 + y^2 + z^2 = r^2$$

\therefore $x^2 + y^2 = z^2$

$r^2 \sin^2\theta + r^2 \cos^2\theta \sin^2\phi = r^2 \cos^2\theta$

$$\sin^2\theta + \cos^2\theta \sin^2\phi = \cos^2\theta$$

$$\sin^2\phi = \cos^2\theta$$

$$\phi = \cot^{-1}\theta$$

$$\theta = \operatorname{arctan}(\cot\phi)$$

$\therefore 0 \leq \theta \leq \operatorname{arctan}(\cot\phi)$

NOW $z = 2$
 $r \cos\theta = 2$
 $r = \frac{2}{\cos\theta}$

$\therefore 0 \leq r \leq \frac{2}{\cos\theta}$

$\therefore -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$
(Cut & 4th quadrant)

THU

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{2}{\cos\theta}} \int_0^{\operatorname{arctan}(\cot\phi)} r^2 \sin\theta dr d\theta d\phi$$

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{2}{\cos\theta}} \operatorname{arctan}(\cot\phi) \left[\frac{r^3}{3} \sin\theta \right]_0^{\operatorname{arctan}(\cot\phi)} d\phi$$

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{2}{\cos\theta}} \frac{8}{3} \operatorname{arctan}(\cot\phi) \sin\theta d\phi$$

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3} \left(\frac{4}{3} + \frac{2}{3} \sin^2\theta \right) d\phi$$

$$V = \left(\frac{32}{9} + \frac{16}{9} \sin^2\theta \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$V = \frac{2\pi}{3}$$

As required

Question 24

A solid sphere has radius a and mass m .

The density ρ at any point in the sphere is inversely proportional to the distance of this point from the centre of the sphere

Show that the moment of inertia of this sphere about one of its diameters is $\frac{1}{3}ma^2$

proof

FIRSTLY WE NEED THE MASS - USE SPHERICAL POLARIES (r, θ, ϕ)

$$\rho(r, \theta, \phi) = \frac{k}{r}$$

$$\text{MASS} = \int_V \rho \, dv = \int_0^{2\pi} \int_0^{\pi} \int_{r=0}^a \frac{k}{r} (r^2 \sin \theta \, dr \, d\theta \, d\phi)$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_{r=a}^0 k r \sin \theta \, dr \, d\theta \, d\phi = \int_{r=a}^0 \int_0^{\pi} \int_0^{2\pi} \left[\frac{1}{2} k r^2 \sin \theta \right]_a^0 \, d\theta \, d\phi$$

$$= \int_{r=a}^0 \int_0^{\pi} \frac{1}{2} k a^2 \sin \theta \, d\theta \, d\phi = \int_{r=a}^0 \int_0^{\pi} \left[-\frac{1}{2} k a^2 \cos \theta \right]_0^\pi \, d\theta \, d\phi$$

$$= \int_{r=a}^0 \frac{1}{2} k a^2 - (-\frac{1}{2} k a^2) \, d\phi = k a^2 \int_{r=a}^0 \frac{1}{2} \, d\phi = \frac{1}{2} k \pi a^2$$

NOW THE MOMENT OF INERTIA - TAKE THE DIAMETER TO BE THE Z-AXIS.

$$dv = (\rho \, dr) \, d\theta \, d\phi$$

$$I = \int_V r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$I = \int_{r=a}^0 \int_0^{\pi} \int_0^{2\pi} k r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$I = \int_{r=a}^0 \int_0^{\pi} \left[\frac{1}{3} k r^3 \sin \theta \right]_a^0 \, d\theta \, d\phi$$

$$= \int_{r=a}^0 \int_0^{\pi} \int_{\theta=0}^{\pi} \frac{1}{3} k a^3 \sin^2 \theta \, d\theta \, d\phi$$

$$= \int_{r=a}^0 \int_0^{\pi} \int_{\theta=0}^{\pi} k a^3 (\sin^2 \theta - \sin^2(180^\circ - \theta)) \, d\theta \, d\phi = \int_{r=a}^0 \int_0^{\pi} \int_{\theta=0}^{\pi} k a^3 [1 + \cos(\theta) - \cos(180^\circ - \theta)] \, d\theta \, d\phi$$

$$= \int_{r=a}^0 \int_0^{\pi} k a^3 [\cos \theta - (\cos \theta)] \, d\theta \, d\phi = \int_{r=a}^0 \int_0^{\pi} k a^3 \, d\theta \, d\phi = \frac{2 \pi k a^3}{3} a^2$$

$\therefore I = \frac{2 \pi k a^3}{3} a^2 = \frac{m a^2}{3} = \frac{1}{3} m a^2$

As required

Question 25

The finite R region is defined as

$$1 \leq x^2 + y^2 + z^2 \leq 2z$$

Determine an exact simplified value for

$$\int_R z \ dx dy dz$$

$$\boxed{\frac{9\pi}{8}}$$

SURFACE IN SPHERICAL COORDINATES

- $\rho^2 = x^2 + y^2 + z^2$
- $\theta = \tan^{-1} \frac{y}{x}$
- $\phi = \text{azimuthal angle}$
- $z = \rho \cos \phi$

$d\sigma = d\rho \sin \phi d\theta d\phi$

$$\int_{-1}^{1} \int_{-\pi}^{\pi} \int_{0}^{\sqrt{1-x^2-y^2}} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \int_{-1}^{1} \int_{-\pi}^{\pi} \left[\frac{\rho^3}{3} \right]_0^{\sqrt{1-x^2-y^2}} \cos \phi \, d\theta \, d\phi$$

$$= \int_{-1}^{1} \int_{-\pi}^{\pi} \left[\frac{\rho^3}{3} \cos^2 \phi \right]_0^{\sqrt{1-x^2-y^2}} \, d\theta \, d\phi$$

$$= \int_{-1}^{1} \int_{-\pi}^{\pi} \left[\frac{\rho^3}{3} \cos^2 \phi - \frac{1}{3} \right]_{x^2+y^2}^{\sqrt{1-x^2-y^2}} \, d\theta \, d\phi$$

$$= \int_{-1}^{1} \left[\left[-\frac{1}{3} \cos^2 \phi - \frac{1}{3} \rho^3 \right]_{x^2+y^2}^{\sqrt{1-x^2-y^2}} \right]^{\pi}_0 \, d\phi$$

$$= \int_{-1}^{1} \left[\left(\frac{1}{3} \cos^2 \phi + \frac{1}{3} \rho^3 \right) \Big|_{x^2+y^2}^{\sqrt{1-x^2-y^2}} \right]^{\pi}_0 \, d\phi$$

$$= \int_{-1}^{1} \left(0 + \frac{3}{32} \right) - \left(\frac{3}{32} + \frac{1}{32} \right) \, d\phi$$

$$= \frac{q}{16} \int_{-1}^{1} 1 \, d\phi$$

$$= \frac{q}{16} \times 2\pi$$

$$= \frac{q\pi}{8}$$

TOP SURFACE

$1 \leq r \leq 2\cos \theta$
 $0 \leq \theta \leq \frac{\pi}{3}$ (or $2\pi/3$)
 $0 \leq \phi \leq \frac{\pi}{2}$

Question 26

The finite R region is defined as

$$x^2 + y^2 + z^2 \leq 4z \quad \text{and} \quad z \geq 2.$$

Determine an exact simplified value for

$$\int_R \frac{z^2}{\sqrt{x^2 + y^2 + z^2}} dx dy dz.$$

$$\boxed{\frac{44\pi}{3}}$$

PART

- $x^2 + y^2 + z^2 = 4z$
- $x^2 + y^2 + z^2 - 4z = 0$
- $x^2 + y^2 + (z-2)^2 = 4$
- Sphere centre $(0,0,2)$ & radius $2\sqrt{3}$ or 2.

• **Plane equation:** (x,y,z)

$$x^2 + y^2 + z^2 = 4z$$

$\Rightarrow x^2 + y^2 + z^2 - 4z = 0$

$\Rightarrow x^2 + y^2 + (z-2)^2 = 4$

\Rightarrow Sphere centre $(0,0,2)$ & radius $2\sqrt{3}$ or 2.

$\Rightarrow z = 2$ or $z = 2\sqrt{3}$

\Rightarrow $z = 2$ or $z = 2\sqrt{3}$

\Rightarrow $0 \leq z \leq r \leq 4z$

$\Rightarrow 0 \leq z \leq \frac{r}{2}$ (use reason)

$\Rightarrow 0 \leq \phi \leq \pi$

\bullet $dxdydz = r^2 \sin\theta dr d\theta d\phi$

Question 27

A non right circular cone has Cartesian equation

$$x^2 + y^2 \equiv xz, \quad 0 \leq z \leq 2$$

Use spherical polar coordinates to find an exact simplified value for

$$\int_R xz \ dV,$$

where R is the interior of the cone.

$$\frac{4\pi}{5}$$

- Need to stretch the cat: $x^2 + y^2 = 2x$
- If $x \geq 0$, if $2 < x$, then $x^2 + y^2 < 0$
 $\Rightarrow 2 > x, x \geq 0$
- If you, $x = 2$
- At $x = k$: $(k, 0)$ (different constant)
- $x^2 + y^2 = k^2$
 $x^2 - k^2 + y^2 = 0$
 $(x - k)^2 + y^2 = k^2$
- So these are the concentric circles
The xy plane

(IN SPHERICAL POLAR)
 $z = r \sin \theta \cos \phi$
 $y = r \sin \theta \sin \phi$
 $x = r \cos \theta$
 $x^2 + y^2 + z^2 = r^2$

- $x^2 + y^2 = 2x$
 $r^2 \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) = (r \cos \theta)^2$
 $r^2 \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) = r^2 \cos^2 \theta \sin^2 \theta \cos^2 \phi$
 $\sin^2 \theta = \sin^2 \theta \cos^2 \phi$
 $\sin^2 \theta = \cos^2 \theta \cos^2 \phi$
 $\tan^2 \theta = \cos^2 \phi$
 $\theta = \arctan(\cos \phi)$
- $0 \leq \phi \leq \pi$
 $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ (1st & 4th quadrant for y)

Question 28

A solid uniform sphere has mass M and radius a .

Use spherical polar coordinates, (r, θ, φ) , and direct calculus methods, to show that the moment of inertial of this sphere about one of its tangents is $\frac{7}{5}Ma^2$.

You may **not** use any standard rules or standard results about moments of inertia in this question apart from the definition of moment of inertia.

[proof]

● IN ORDER TO USE DIRECT INTEGRATION DUE TO THE SPHERE HAVING EQUATION $x^2+y^2+z^2=a^2$ "ROTATE" BY a , SO THAT THE xy PLANE IS A TANGENT PLANE AND THE xz -PLANE IS THE ROTATION AXES.

$$x^2+y^2+(a-z)^2=a^2$$

$$x^2+y^2+z^2-2az+a^2=a^2$$

$$x^2+y^2=2az$$

● USE SPHERICAL POLARICS (r, θ, φ)

- LET $r = \frac{a}{\cos \theta} = \frac{a}{\cos \theta}$ = RADIUS PER UNIT VOLUME
- AN INFINITESIMAL VOLUME ELEMENT HAS MASS

$$dm = r^2 \sin \theta dr d\theta d\varphi$$

- THE SPHERICAL POLARIC EQUATIONS OF THE SPHERE ARE

$$x^2+y^2+z^2=a^2$$

$$r^2=a^2 \cos^2 \theta$$

$$r^2=a^2 \cos^2 \theta$$

$$r^2=2az$$

● THE DISTANCE OF THE INFINITESIMAL FROM THE z -AXIS IS $\sqrt{r^2+z^2}$, SO THE MOMENT OF INERTIA OF THE INFINITESIMAL AROUND THE z -AXIS IS

$$I_z = mr^2 d^2 = (r^2 \sin \theta dr d\theta) \times (r^2 d\theta d\varphi)$$

$$[r^2+z^2] \sin^2 \theta d\theta d\varphi + r^2 d\theta d\varphi$$

REMEMBER:

$$mr^2 = (r^2 \sin \theta dr d\theta) (r^2 \sin^2 \theta d\theta d\varphi + r^2 d\theta d\varphi)$$

$$= r^4 \sin^2 \theta (dr d\theta d\theta d\varphi) dr d\theta d\varphi$$

SUMMARISE AS:

$$\int_0^{2\pi} \int_0^\pi \int_0^a r^4 \sin^2 \theta (dr d\theta d\varphi) dm$$

two times due to symmetry (note that $0 \leq \theta \leq \frac{\pi}{2}$ as sphere is rotating)

$$I_z = \frac{1}{5} \int_{\text{sphere}} \int_0^\pi \int_0^{2\pi} [r^2]^{2a \cos \theta} (\text{constant} + \text{surface}) dr d\theta d\varphi$$

$$I_z = \frac{1}{5} \int_{\text{sphere}} \int_0^\pi \int_0^{2\pi} 2a^2 \cos^2 \theta \sin^2 \theta r^2 dr d\theta d\varphi + 2a^2 \cos^2 \theta r^2 dr d\theta d\varphi$$

$$I_z = \frac{1}{5} \int_{\text{sphere}} \int_0^\pi \left\{ 16a^5 \cos^4 \theta \int_0^a 2a^2 \cos^2 \theta dr + \left[-4a^3 \cos^2 \theta \right]_0^a \right\} d\theta d\varphi$$

$$\uparrow$$

$$8(\theta, \varphi) = \frac{16a^5 \cos^4 \theta}{15(2)} - \frac{2a^3(1)}{15(1)} = \frac{1}{15}$$

$$I_z = \frac{1}{5} \int_{\text{sphere}} \int_0^\pi \left\{ \frac{4}{3} a^4 \cos^4 \theta + [0 + 4a^2] \right\} d\theta d\varphi$$

$$I_z = \frac{1}{5} \int_{\text{sphere}} \int_0^\pi \frac{4}{3} a^4 \cos^4 \theta + 4a^2 d\theta d\varphi = \frac{1}{5} a^4 \int_{\text{sphere}} \frac{4}{3} \cos^4 \theta + 4 d\theta d\varphi$$

$$I_z = \frac{1}{5} a^4 \int_0^\pi \frac{4}{3} \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) + 4 d\theta d\varphi$$

$$I_z = \frac{1}{5} a^4 \int_0^\pi \left[\frac{2}{3} - \frac{2}{3} \cos 2\theta \right] + 4 d\theta d\varphi$$

$$I_z = \frac{1}{5} a^4 \times \int_0^\pi \frac{16}{3} d\theta d\varphi$$

$$I_z = \frac{16}{5} a^4 \int_0^\pi 1 d\theta d\varphi$$

$$I_z = \frac{16}{5} a^4 \times 2\pi$$

$$I_z = \frac{32}{5} a^4 \pi$$

$$I_z = \frac{28}{15} \left(\frac{3}{4}\pi a^2 \right)^2$$

$$I_z = \frac{7}{5} Ma^2$$

As required