

# PARTIAL DIFFERENTIATION INTRODUCTION

**Question 1 (\*\*)**

A right circular cylinder has radius 5 cm and height 10 cm.

Use a differential approximation to find an approximate increase in the volume of this cylinder if the radius increases by 0.4 cm and its height decreases by 0.2 cm.

$$35\pi \approx 109.96\ldots \text{cm}^3$$

**Question 2 (\*\*)**

$$y = \frac{xz^3}{w^4}, \quad w \neq 0.$$

Determine an approximate percentage increase in  $y$ , if  $x$  decreases by 5%,  $z$  increases by 2% and  $w$  decreases by 10%.

$$\approx 41\%$$

**Question 3 (\*\*)**

The function  $f$  depends on  $u$  and  $v$  so that

$$f[u(x, y, z), v(x, y, z)] = uv, \quad u = x + 2y + z^2 \quad \text{and} \quad v = xyz.$$

Find simplified expressions for  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$  and  $\frac{\partial f}{\partial z}$ , in terms of  $x$ ,  $y$  and  $z$ .

$$\boxed{\frac{\partial f}{\partial x} = 2xyz + 2y^2z + yz^3}, \quad \boxed{\frac{\partial f}{\partial y} = 4xyz + x^2z + xz^3}, \quad \boxed{\frac{\partial f}{\partial z} = 3xyz^2 + x^2y + 2xy^2}$$

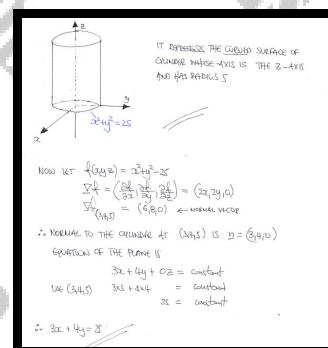
**Question 4 (\*\*)**

A surface  $S$  is defined by the Cartesian equation

$$x^2 + y^2 = 25.$$

- Draw a sketch of  $S$ , and describe it geometrically.
- Determine an equation of the tangent plane on  $S$  at the point with Cartesian coordinates  $(3, 4, 5)$ .

$$\boxed{3x + 4y = 25}$$



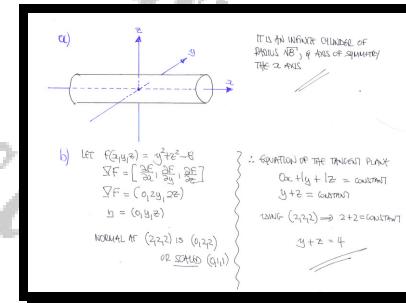
**Question 5 (\*\*)**

A surface  $S$  is defined by the Cartesian equation

$$y^2 + z^2 = 8.$$

- a) Draw a sketch of  $S$ , and describe it geometrically.
- b) Determine an equation of the tangent plane on  $S$  at the point with Cartesian coordinates  $(2, 2, 2)$ .

$$[y + z = 4]$$



**Question 6 (\*\*)**

The function  $\varphi$  depends on  $u$ ,  $v$  and  $w$  so that

$$\varphi[u(x, y, z), v(x, y, z), w(x, y, z)] = uv + w.$$

It is further given that

$$u = x + 2y, \quad v = xyz \quad \text{and} \quad w = z^2.$$

By using **the chain rule for partial differentiation** find simplified expressions for  $\frac{\partial \varphi}{\partial x}$ ,  $\frac{\partial \varphi}{\partial y}$  and  $\frac{\partial \varphi}{\partial z}$ , in terms of  $x$ ,  $y$  and  $z$ .

$$\boxed{\frac{\partial \varphi}{\partial x}}, \boxed{\frac{\partial \varphi}{\partial x} = 2yz(x+y)}, \boxed{\frac{\partial \varphi}{\partial y} = xz(x+4y)}, \boxed{\frac{\partial \varphi}{\partial z} = x^2y + 2xy^2 + 2z}$$

$$\varphi(u, v, w) = uv + w$$

$$u = x + 2y$$

$$w = z^2$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial \varphi}{\partial w} \frac{\partial w}{\partial x}$$

$$= ux + vx + 1 \times 0$$

$$= u + vx + 0$$

$$= 2yz + (x+2y)yz$$

$$= 2xyz + 2y^2z$$

$$= 2yz(x+2y)$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial \varphi}{\partial w} \frac{\partial w}{\partial y}$$

$$= vx + 2xz + 1 \times 0$$

$$= 2xyz + (x+2y)xz$$

$$= 2xyz + x^2z + 2xy^2z$$

$$= 2z(xyz + x^2 + 2xy^2)$$

$$= 2z(x^2y + 2xy^2 + 2z)$$

$$\frac{\partial \varphi}{\partial z} = \frac{\partial \varphi}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial \varphi}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial \varphi}{\partial w} \frac{\partial w}{\partial z}$$

$$= vx + 0 + 1 \times z^2$$

$$= (x+2y)yz + z^2$$

$$= 2yz + 2y^2z + z^2$$

**ALTERNATIVE APPROACH WITHOUT CHAIN RULE**

$$\varphi(u, v, w) = uv + w = (2xyz)xyz + z^2$$

$$= 2x^2y^2z + 2z^2$$

$$\bullet \frac{\partial \varphi}{\partial x} = 2xyz + 2y^2z = \underline{\underline{2yz(x+2y)}}$$

$$\bullet \frac{\partial \varphi}{\partial y} = z^2x + 4xyz = \underline{\underline{2z(x^2y + 2xy^2 + 2z)}}$$

$$\bullet \frac{\partial \varphi}{\partial z} = 2x^2y^2 + 2z = \underline{\underline{2yz + 2y^2z + z^2}}$$

**Question 7 (\*\*)**

The point  $P(1, y)$  lies on the contour with equation  $x^2y + y^2x - 6 = 0$ .

Determine the possible normal vectors at  $P$

$$\boxed{8\mathbf{i} + 5\mathbf{j}}, \boxed{3\mathbf{i} - 5\mathbf{j}}$$

$$\begin{aligned}
 & \text{Let } f(x,y) = x^2y + y^2x - 6 = 0 \\
 & \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = [2xy + y^2, x^2 + 2xy] \\
 & \nabla f \Big|_{(1,y)} = (2(1)y + y^2, 1^2 + 2(1)y) = (y, 2y+1) \leftarrow \text{NORMAL} \\
 & \nabla f \Big|_{(1,y)} = (2(1)(1)+y^2, 1^2+2(1)y) = (3, 1+y) \leftarrow \text{NORMAL} \\
 & \therefore 8\mathbf{i} + 5\mathbf{j} \quad \text{or} \quad 3\mathbf{i} - 5\mathbf{j}
 \end{aligned}$$

**Question 8 (\*\*)**

The radius of a right circular cylinder is increasing at the constant rate of  $0.2 \text{ cms}^{-1}$  and its height is decreasing at the constant rate of  $0.2 \text{ cms}^{-1}$ .

Determine the rate at which the volume of this cylinder is increasing when the radius is 5 cm and its height is 16 cm.

$$27\pi \approx 84.82 \dots \text{cm}^3 \text{s}^{-1}$$

$$\begin{aligned}
 V &= \pi r^2 h \\
 \frac{\partial V}{\partial r} &= 2\pi rh \\
 \frac{\partial V}{\partial h} &= \pi r^2
 \end{aligned}$$

$$\begin{aligned}
 \frac{dv}{dt} &= \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} \\
 \frac{dv}{dt} &= (2\pi rh) \frac{dr}{dt} + (\pi r^2) \frac{dh}{dt} \\
 \frac{dv}{dt} &= (2\pi \times 5 \times 16 \times 0.2) + (\pi \times 5^2)(-0.2) \\
 \frac{dv}{dt} &= 32\pi - 5\pi \\
 \frac{dv}{dt} &\approx 27\pi
 \end{aligned}$$

**Question 9 (\*\*)**

A curve has implicit equation

$$x^2 + 2xy + y^3 = 8.$$

Use partial differentiation to find an expression for  $\frac{dy}{dx}$ .

*No credit will be given for obtaining the answer with alternative methods*

**METHOD**, 
$$\frac{dy}{dx} = -\frac{2x+2y}{2x+3y^2}$$

Let  $f(x,y) = x^2 + 2xy + y^3 - 8$

$$\Rightarrow df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$
$$\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2xy + y^3) = 2x + 2y$$
$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2xy + y^3) = 2x + 3y^2$$

Now as  $f = 0$ ,  $\frac{df}{dx} = 0$

$$\Rightarrow 0 = (2x+2y) + (2x+3y^2) \frac{dy}{dx}$$
$$\Rightarrow (2x+3y^2) \frac{dy}{dx} = -(2x+2y)$$
$$\Rightarrow \frac{dy}{dx} = -\frac{2x+2y}{2x+3y^2}$$

**Question 10 (\*\*)**

A surface  $S$  is defined by the Cartesian equation

$$z = xy(x + y).$$

Find an equation of the tangent plane on  $S$  at the point  $(1, 2, 6)$ .

$$\boxed{\quad}, \boxed{8x + 5y - z = 12}$$

METHOD A

$$z(x,y) = 2xy(x+y) = 2xy + 2y^2$$

$$\frac{\partial z}{\partial x} = 2y + 2y^2 \quad \frac{\partial z}{\partial x}(1,2) = 2 \times 1 \times 2 + 2^2 = 8$$

$$\frac{\partial z}{\partial y} = x^2 + 2xy \quad \frac{\partial z}{\partial y}(1,2) = 1^2 + 2 \times 1 \times 2 = 5$$

EQUATION OF THE TANGENT PLANE AT  $(x_0, y_0, z_0)$ , WHERE  $(1, 2, 6)$

$$z - z_0 = \frac{\partial z}{\partial x}(1,2)(x - x_0) + \frac{\partial z}{\partial y}(1,2)(y - y_0)$$

$$z - 6 = 8(x - 1) + 5(y - 2)$$

$$z - 6 = 8x - 8 + 5y - 10$$

$$12 = 8x + 5y - 18$$

$$\therefore 8x + 5y - z = 12$$

METHOD B

LET  $\mathbf{f}(x,y,z) = z - 2xy(x+y) = z - 2xy - 2y^2$

$$\nabla \mathbf{f} = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2y - 2y^2, -2x - 4y, 1)$$

$$\mathbf{n} = \nabla \mathbf{f}(1,2) = (-8, -5, 1)$$

EQUATION OF PLANE  $\mathbf{n} \cdot \mathbf{r} = d$

$$-8x - 5y + 2 = \text{constant}$$

POINT P(1,2,6)

$$-8(1) - 5(2) + 2 = \text{constant}$$

$$\text{constant} = -12$$

$$\Rightarrow -8x - 5y + 2 = -12$$

$$\Rightarrow 8x + 5y - z = 12$$

As required.

**Question 11 (\*\*)**

A curve has implicit equation

$$e^{xy} + x + y = 1.$$

Use partial differentiation to find the value of  $\frac{dy}{dx}$  at  $(0,0)$ .

*No credit will be given for obtaining the answer with alternative methods*

$$\left. \frac{dy}{dx} \right|_{(0,0)} = -1$$

$e^{xy} + x + y = 1$   
Let  $f(x,y) = e^{xy} + x + y$   
 $f(x,y) = 1$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} e^{xy} + \frac{\partial}{\partial x} x + \frac{\partial}{\partial x} y \\ \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} e^{xy} + \frac{\partial}{\partial x} x + \frac{\partial}{\partial x} y \\ 0 &= ye^{xy} + 1 + (xe^{xy})_x + 1 \\ 0 &= ye^{xy} + 1 + xe^{xy} + 1 \\ \frac{\partial y}{\partial x} &= -\frac{ye^{xy} + 1}{xe^{xy} + 1} \\ \left. \frac{\partial y}{\partial x} \right|_{(0,0)} &= -1\end{aligned}$$

**Question 12 (\*\*+)**

The function  $f$  is defined as

$$f(x, y, z) \equiv 2x + y^2 + xz,$$

where  $x = 2t$ ,  $y = t^2$  and  $z = 3$ .

- Use partial differentiation to find an expression for  $\frac{df}{dt}$ , in terms of  $t$ .
- Verify the answer obtained in part (a) by a method **not** involving partial differentiation.

$$\boxed{\frac{df}{dt} = 4t^3 + 10}$$

4)  $f(x, y, z) = 2x + y^2 + xz$   
 $x = 2t$   
 $y = t^2$   
 $z = 3$

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}$

$\frac{\partial f}{\partial t} = (2x)2t + (2y)2t + 2z \times 0$

$\frac{\partial f}{\partial t} = 4+2t^2 + 4yt$

$\frac{\partial f}{\partial t} = 4+2t^2 + 4(t^2)t$

$\frac{\partial f}{\partial t} = 10+4t^3$

5) check my work! Partial differentiation  
 $f(x, y, z) = 2x + y^2 + xz$   
 $f(x) = 2(2t) + (t^2)^2 + 2(3)$   
 $f(x) = 4t + t^4 + 6t$   
 $f(x) = 10t + t^4$   
 $\frac{df}{dt} = 10 + 4t^3$

**Question 13 (\*\*+)**

The function  $\varphi$  is defined as

$$\varphi(x, y, z) \equiv x^2 + y^2 + tz + t, \quad t \neq 0,$$

where  $x = 3t$ ,  $y = t^2$  and  $z = \frac{1}{t}$ .

- Use partial differentiation to find an expression for  $\frac{d\varphi}{dt}$ , in terms of  $t$ .
- Verify the answer obtained in part (a) by a method not involving partial differentiation.

$$\boxed{\frac{d\varphi}{dt} = 4t^3 + 18t + 1}$$

a)  $\frac{d(\varphi(x,y,z,t))}{dt} = x^2y^2 + tz + t$

$$\begin{aligned} x &= 3t \\ y &= t^2 \\ z &= \frac{1}{t} \end{aligned}$$

$$\begin{aligned} \frac{d\varphi}{dt} &= \frac{\partial \varphi}{\partial x} \frac{dx}{dt} + \frac{\partial \varphi}{\partial y} \frac{dy}{dt} + \frac{\partial \varphi}{\partial z} \frac{dz}{dt} + \frac{\partial \varphi}{\partial t} \\ &= (2x)(3) + (2y)(2t) + (z)\left(-\frac{1}{t^2}\right) + (1) \\ \frac{d\varphi}{dt} &= 6x + 4yt - \frac{z}{t^2} + 1 \\ \frac{d\varphi}{dt} &= 6(3t) + 4(t^2)(2t) - \frac{1}{t} + \frac{1}{t^2} + 1 \\ \frac{d\varphi}{dt} &= 18t + 4t^3 - \frac{1}{t} + \frac{1}{t^2} + 1 \end{aligned}$$

b) CHECK WITHOUT PARTIAL DIFFERENTIATION

$$\begin{aligned} \frac{d(\varphi(x,y,z,t))}{dt} &= x^2y^2 + tz + t \\ \varphi(t) &= (3t)^2(t^2)^2 + t\left(\frac{1}{t}\right) + t \\ \varphi(t) &= 9t^2 + t^5 + 1 + t \\ \frac{d\varphi}{dt} &= 18t + 4t^3 + 1 \end{aligned}$$

**Question 14 (\*\*+)**

Plane Cartesian coordinates  $(x, y)$  are related to plane polar coordinates  $(r, \theta)$  by the transformation equations

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

- a) Find simplified expressions for  $\frac{\partial r}{\partial x}$ ,  $\frac{\partial r}{\partial y}$ ,  $\frac{\partial \theta}{\partial x}$  and  $\frac{\partial \theta}{\partial y}$ , in terms of  $r$  and  $\theta$ .
- b) Deduce simplified expressions for  $\frac{\partial r}{\partial x}$ ,  $\frac{\partial r}{\partial y}$ ,  $\frac{\partial \theta}{\partial x}$  and  $\frac{\partial \theta}{\partial y}$ , in terms of  $x$  and  $y$ .

$$\boxed{\frac{\partial r}{\partial x} = \cos \theta}, \quad \boxed{\frac{\partial r}{\partial y} = \sin \theta}, \quad \boxed{\frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}}, \quad \boxed{\frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}}$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}}, \quad \boxed{\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}}, \quad \boxed{\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2}}, \quad \boxed{\frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}}$$

a)  $x = r \cos \theta$        $a^2 + b^2 = r^2$   
 $y = r \sin \theta$        $\tan \theta = \frac{y}{x}$

$\bullet r^2 = x^2 + y^2$        $\bullet r^2 = x^2 + y^2$   
 $2r \frac{\partial r}{\partial x} = 2x$        $2r \frac{\partial r}{\partial y} = 2y$   
 $r \frac{\partial r}{\partial x} = x$        $r \frac{\partial r}{\partial y} = y$   
 $r \frac{\partial r}{\partial x} = r \cos \theta$        $r \frac{\partial r}{\partial y} = r \sin \theta$   
 $\frac{\partial r}{\partial x} = \cos \theta$        $\frac{\partial r}{\partial y} = \sin \theta$

$\bullet \tan \theta = \frac{y}{x}$        $\bullet \tan \theta = \frac{y}{x}$   
 $\sec^2 \theta \frac{\partial \theta}{\partial x} = -\frac{y}{x^2}$        $\sec^2 \theta \frac{\partial \theta}{\partial y} = \frac{1}{x^2 + y^2}$   
 $\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2}$        $\frac{\partial \theta}{\partial y} = \frac{1}{x^2 + y^2}$   
 $\frac{\partial \theta}{\partial x} = -\frac{y}{r^2}$        $\frac{\partial \theta}{\partial y} = \frac{y}{r^2}$   
 $\frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}$        $\frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}$

b)  $\frac{\partial r}{\partial x} = \cos \theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$   
 $\frac{\partial r}{\partial y} = \sin \theta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$   
 $\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{y}{r^2} = -\frac{y}{x^2 + y^2}$   
 $\frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r} = \frac{x}{r^2} = \frac{x}{x^2 + y^2}$

**Question 15 (\*\*+)**

The point  $P(1, y_0, z_0)$  lies on both surfaces with Cartesian equations

$$x^2 + y^2 + z^2 = 9 \quad \text{and} \quad z = x^2 + y^2 - 3.$$

At the point  $P$ , the two surfaces intersect each other at an angle  $\theta$ .

Given further that  $P$  lies in the first octant, determine the exact value of  $\cos \theta$ .

$$\boxed{\cos \theta = \frac{8}{3\sqrt{21}}}$$

Handwritten solution for Question 15:

Given surfaces:

- $x^2 + y^2 + z^2 = 9$
- $x^2 + y^2 + z^2 = 2z \Rightarrow z = x^2 + y^2 - 3$

Point  $P(1, y_0, z_0)$  lies on both surfaces.

At  $P$ :

- $x^2 + y^2 + z^2 = 9$
- $x^2 + y^2 + z^2 = 2z \Rightarrow z = x^2 + y^2 - 3$
- $(x^2 + y^2 - 3) = x^2 + y^2 - 3$
- $z = 3$
- $x^2 + y^2 = 6$
- $y_0^2 = 6 - x_0^2 = 6 - 1 = 5 \Rightarrow y_0 = \sqrt{5}$
- $P(1, \sqrt{5}, 3)$

Normal vectors at  $P$ :

- Surface 1:  $\nabla f_1 = (2x, 2y, 2z) = (2, 2\sqrt{5}, 6)$
- Surface 2:  $\nabla f_2 = (2x, 2y, 2z) = (2, 2\sqrt{5}, 6)$
- Surface 3:  $\nabla f_3 = (2x, 2y, -2) = (2, 2\sqrt{5}, -2)$
- Surface 4:  $\nabla f_4 = (2x, 2y, -2) = (2, 2\sqrt{5}, -2)$

Angle  $\theta$  between  $\nabla f_1$  and  $\nabla f_3$ :

$$\cos \theta = \frac{\nabla f_1 \cdot \nabla f_3}{|\nabla f_1| |\nabla f_3|} = \frac{(2, 2\sqrt{5}, 6) \cdot (2, 2\sqrt{5}, -2)}{\sqrt{2^2 + 2^2 + 6^2} \sqrt{2^2 + 2^2 + (-2)^2}} = \frac{4 + 20 - 12}{\sqrt{4 + 20 + 36} \sqrt{4 + 20 + 4}} = \frac{8}{\sqrt{60} \sqrt{28}} = \frac{8}{2\sqrt{15} \cdot 2\sqrt{7}} = \frac{8}{4\sqrt{105}} = \frac{2}{\sqrt{105}}$$

**Question 16 (\*\*+)**

The point  $P(1,1,2)$  lies on both surfaces with Cartesian equations

$$z(z-1) = x^2 + xy \quad \text{and} \quad z = x^2 y + xy^2.$$

At the point  $P$ , the two surfaces intersect each other at an angle  $\theta$ .

Determine the exact value of  $\theta$ .

$$\boxed{\theta = \arccos\left(\frac{15}{19}\right)}$$

$\boxed{z(z-1) = x^2 + xy \quad \text{at } (1,1,2)}$

- Let  $\vec{r}(x,y,z) = \langle x^2 + xy - z^2 + z, x, -2z + 1 \rangle$   
 $\vec{r}'_x = \langle 2x+y, 1, 0 \rangle$   
 $\vec{r}'_y = \langle x^2 + 2xy, x, 0 \rangle$
- Let  $\vec{g}(x,y,z) = \langle 2y + xy^2 - z, 2xy + y^2, x^2 + 2xy, -1 \rangle$   
 $\vec{g}'_x = \langle 2y + y^2, x^2 + 2xy, -1 \rangle$   
 $\vec{g}'_y = \langle 2x + 2y^2, x^2 + 2xy, -1 \rangle$
- BY THE DOT PRODUCT  
 $(\vec{r}_x \cdot \vec{g}_x) = |(3,1,-1)| |(3,3,-1)| \cos\theta$   
 $9 + 3 + 3 = \sqrt{9 + 1 + 9} \sqrt{9 + 9 + 1} \cos\theta$   
 $15 = 19 \cos\theta$   
 $\cos\theta = \frac{15}{19}$   
 $\theta = \arccos\left(\frac{15}{19}\right)$

**Question 17 (\*\*\*)**

The point  $P(-1,1,3)$  lies on both surfaces with Cartesian equations

$$z(z-2) = x^2 - 2xy \quad \text{and} \quad z = xy(Ax + By),$$

where  $A$  and  $B$  are non zero constants.

The two surfaces intersect each other orthogonally at the point  $P$ .

Determine the value of  $A$  and the value of  $B$ .

A = -14, B = -17

$\bullet$   $z(z-2) = x^2 - 2xy$        $\bullet$   $z = xy(Ax + By)$   $\forall z > 0$   $\forall (-1,1,3)$   
 $\bullet$   $z^2 - 2z = x^2 - 2xy$   
 $x^2 - 2xy + 2z - 2z^2 = 0$   
 $\boxed{x^2 - 2xy + 2z - 2z^2}$   
 $\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$   
 $\nabla f = [2x-2y, -2x, 2-2z]$   
 $\nabla f|_{(-1,1,3)} = (-4, -2, -4)$   
 TAKE NORMAL AT  $(-1,1,3)$   
 $\nabla f|_{(-1,1,3)} = (-4, -2, -4)$   
 $\text{NORMAL} = (-2A+B, A-2B, -1)$   
 $\text{NORMAL} = (-2A+B, A-2B, -1)$

$\bullet$  NOW  $(-1,1,3)$  MUST SATISFY  $\mathcal{N}(0,0,1)$   
 $3 = A - B$   
 $\boxed{A - B = 3}$

$\bullet$  SURFACES ARE ORTHOGONAL AT  $(-1,1,3)$ , i.e. NORMALS DOT IS ZERO  
 $(-2, -4, -1) \cdot (-2A+B, A-2B, -1) = 0$   
 $-4A+2B - 4 + 2B - 2 = 0$   
 $\boxed{-8A+4B = 2}$   
 SOLVING BY SUBSTITUTION:  $A = B+3$   
 $-8(B+3) + 4B = 2$   
 $-8B - 24 + 4B = 2$   
 $-B = 17$   
 $B = -17$   
 $\therefore A = -14$

**Question 18** (\*\*\*)

The function  $f$  depends on  $u$ ,  $v$  and  $t$  so that

$$f\{u[x(t), y(t), z(t)], v[x(t), y(t), z(t)], t\} = u^2 + v + 2t.$$

It is further given that

$$u = x + y - 2z, \quad v = 4x - 2y - z \quad \text{and} \quad x = 2t, \quad y = t^2, \quad z = 5.$$

- a) Find simplified expressions for  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$  and  $\frac{\partial f}{\partial z}$ , in terms of  $x$ ,  $y$  and  $z$ .
- b) Determine an expression for  $\frac{df}{dt}$ , in terms of  $t$ .

$$\boxed{\frac{\partial f}{\partial x} = 2x + 2y - 4z + 5}, \quad \boxed{\frac{\partial f}{\partial y} = 2x + 2y - 4z + \frac{1}{\sqrt{z}} - 2}, \quad \boxed{\frac{\partial f}{\partial z} = -x - 4y + 8z - 1},$$

$$\boxed{\frac{df}{dt} = 4t^3 + 12t^2 - 36t - 30}$$

**a)**

$f(u, v, t) = u^2 + v + 2t$

$u = 2t + t^2 - 2z$   
 $v = 4t - 2t^2 - z$   
 $z = 5$   
 $y = t^2$   
 $x = 2t$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial x}(v) + \frac{\partial}{\partial x}(2t)$   
 $= 2u \cdot 1 + 1 \times 4 + 2 \times \frac{1}{2} \leftarrow \begin{array}{l} \text{Cloud 1} \\ \frac{\partial u}{\partial x} = 1 \\ \frac{\partial v}{\partial x} = 4 \\ \frac{\partial 2t}{\partial x} = 2 \end{array}$   
 $= 2u + 4 + 1$   
 $= 2(2t + t^2 - 2z) + 4$   
 $= 2t + 2t^2 - 4z + 4$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(u^2) + \frac{\partial}{\partial y}(v) + \frac{\partial}{\partial y}(2t)$   
 $= 2u \cdot 1 + 1 \times 2 + 2 \times \frac{1}{2t} \leftarrow \begin{array}{l} \text{Cloud 2} \\ \frac{\partial u}{\partial y} = 2 \\ \frac{\partial v}{\partial y} = 2 \\ \frac{\partial 2t}{\partial y} = \frac{1}{2t} \end{array}$   
 $= 2u + 2 + \frac{1}{t}$   
 $= 2(2t + t^2 - 2z) + 2 + \frac{1}{t}$   
 $= 2t + 2t^2 - 4z - 2 + \frac{1}{t}$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(u^2) + \frac{\partial}{\partial z}(v) + \frac{\partial}{\partial z}(2t)$   
 $= 2u \cdot (-2) + 1 \times (-1) + 2 \times 0 \leftarrow \begin{array}{l} \text{Cloud 3} \\ \frac{\partial u}{\partial z} = -2 \\ \frac{\partial v}{\partial z} = -1 \\ \frac{\partial 2t}{\partial z} = 0 \end{array}$   
 $= -4u - 1$   
 $= -4(2t + t^2 - 2z) - 1$   
 $= -8t - 4t^2 + 8z - 1$

**b)**

$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} \frac{du}{dt} + \frac{\partial f}{\partial v} \frac{dv}{dt}$   
 $= (2t + 2t^2 - 4z)(2t) + (2t + 2t^2 - 4z)(4t) + (2t + 2t^2 - 4z)(0) + 2$   
 $= 4t^2 + 8t^3 + 4t^2 - 4t + 2 = 4t^3 + 8t^2 - 4t + 2$   
 $= (2t + 2t^2 - 4z)(4t) + 10 - 4t = (2t + 2t^2 - 4z)(4t) - 4t + 10$   
 $= 4t^3 + 8t^4 - 4t^2 + 4t^3 + 8t^2 - 4t - 4t + 10 = 4t^4 + 12t^3 - 34t^2 - 30$

**Question 19 (\*\*\*)**

The function  $z$  depends on  $u$  and  $v$  so that

$$z = (2x+3y)^2, \quad u = x^2 + y^2 \quad \text{and} \quad v = x+2y.$$

Find simplified expressions for  $\frac{\partial z}{\partial u}$  and  $\frac{\partial z}{\partial v}$ , in terms of  $x$  and  $y$ .

$$\boxed{\frac{\partial z}{\partial u} = \frac{2x+3y}{2x-y}, \quad \frac{\partial z}{\partial v} = \frac{2(3x-2y)(2x+3y)}{2x-y}}$$

$$\begin{aligned}
& Z = (2x+3y)^2 \quad \text{And} \quad Y = x+2y \\
& \bullet \text{Firstly, } J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 1 \\ 1 & 2y \end{vmatrix} = 4x-2y \\
& \bullet \quad \begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \\ &= \frac{2x+3y}{2x-y} \cdot 2x + \frac{2(3x-2y)(2x+3y)}{2x-y} \cdot 1 \end{aligned} \quad \rightarrow \quad \begin{aligned} \frac{\partial z}{\partial x} &= \frac{2x+3y}{2x-y} \cdot 2x + \frac{2(3x-2y)(2x+3y)}{2x-y} \\ &= -\frac{1}{2x-2y} \end{aligned} \\
& \text{Hence,} \\
& \frac{\partial z}{\partial u} = \frac{\frac{\partial z}{\partial x}}{\frac{\partial u}{\partial x}} = \frac{-\frac{1}{2x-2y}}{2x} = \frac{2x+3y}{2x-2y} + \frac{6(2x+3y)}{4x-2y} \\
&= \frac{2(2x+3y)}{2x-2y} = \frac{2x+3y}{2x-2y} // \\
& \text{And,} \\
& \frac{\partial z}{\partial v} = \frac{\frac{\partial z}{\partial x}}{\frac{\partial v}{\partial x}} = \frac{-\frac{1}{2x-2y}}{1} = 4(2x+3y) \cdot \frac{-1}{4x-2y} + 6(2x+3y) \cdot \frac{-1}{4x-2y} \\
&= -\frac{8(2x+3y)}{4x-2y} + \frac{12(2x+3y)}{4x-2y} \\
&= \frac{(12-8)(2x+3y)}{4x-2y} \\
&= \frac{4(3x-2y)(2x+3y)}{4x-2y} \\
&= \frac{2(2x-2y)(2x+3y)}{2x-2y}
\end{aligned}$$

**Question 20 (\*\*\*)**

The function  $w = \varphi[u(x, y), v(x, y)]$  satisfies

$$x = e^u \cos v \quad \text{and} \quad y = e^{-u} \sin v.$$

Determine simplified expressions for  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial u}{\partial y}$ ,  $\frac{\partial v}{\partial x}$  and  $\frac{\partial v}{\partial y}$ , in terms of  $u$  and  $v$ .

	$\frac{\partial u}{\partial x} = \frac{e^{-u} \cos v}{\cos 2v}$	$\frac{\partial u}{\partial y} = \frac{e^u \sin v}{\cos 2v}$	$\frac{\partial v}{\partial x} = \frac{e^{-u} \sin v}{\cos 2v}$	$\frac{\partial v}{\partial y} = \frac{e^u \cos v}{\cos 2v}$
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**METHOD A - BY DIRECT SUBSTITUTION**

$w = \varphi(u, v) \quad u = e^u \cos v, \quad v = e^{-u} \sin v$

$$\begin{aligned} du &= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \\ dy &= \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \end{aligned}$$

**eliminate du**

$$\begin{aligned} e^u \cos v dx &= e^{-u} \sin v dx - e^{-u} \sin v du \\ e^u \cos v dx &= -e^{-u} \sin v du + e^{-u} \sin v \cos v du \end{aligned} \Rightarrow \text{cancel}$$

$$\begin{aligned} e^u \sin v dx &= \sin v \cos v du - \sin^2 v du \\ e^u \cos v dy &= -\sin v \cos v du + \cos^2 v du \end{aligned}$$

$$\begin{aligned} \Rightarrow (e^u \sin v - e^u \cos v) dx &= e^{-u} \sin v du + e^u \cos v dy \\ \Rightarrow e^u \cos v dx &= e^{-u} \sin v du + e^u \cos v dy \\ \Rightarrow du &= \frac{e^u \sin v dx + e^u \cos v dy}{e^u \cos v} \end{aligned} \quad \boxed{du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy}$$

**IN 4. SIMILAR FRACTION, ELIMINATE dy**

$$\begin{aligned} e^u \cos v dx &= e^u \cos v \cos v du - e^u \cos v \sin v du \\ e^u \cos v dy &= -e^u \sin v \sin v du + e^u \sin v \cos v du \end{aligned} \Rightarrow \text{cancel}$$

$$\begin{aligned} e^u \cos v dx &= \cos v du - \sin v \cos v du \\ e^u \cos v dy &= -\sin v du + \sin v \cos v du \end{aligned}$$

$$\begin{aligned} \Rightarrow (e^u \cos v - e^u \cos v) du &= e^u \cos v dx + e^{-u} \sin v dy \\ \Rightarrow e^u \cos v du &= e^u \cos v dx + e^{-u} \sin v dy \\ \Rightarrow du &= \frac{e^u \cos v dx + e^{-u} \sin v dy}{e^u \cos v} \end{aligned} \quad \boxed{du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy}$$

**METHOD B - USING JACOBIANS**

IF  $x = f(u, v)$  &  $y = g(u, v)$  then:

$$\begin{aligned} \frac{\partial x}{\partial u} &= \frac{\partial u}{\partial x}, \quad \frac{\partial x}{\partial v} = \frac{\partial v}{\partial x} \\ \frac{\partial y}{\partial u} &= \frac{\partial u}{\partial y}, \quad \frac{\partial y}{\partial v} = \frac{\partial v}{\partial y} \end{aligned}$$

where  $J = \frac{\partial(x, y)}{\partial(u, v)}$

- If  $x = e^u \cos v$   $y = e^{-u} \sin v$
- $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^{-u} \sin v \\ -e^{-u} \sin v & e^{-u} \cos v \end{vmatrix}$
- $= (e^u \cos v)(e^{-u} \cos v) - (-e^{-u} \sin v)(-e^{-u} \sin v) = e^{2u} \cos^2 v - e^{-2u} \sin^2 v = \cos 2v$
- $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial u}/J = \frac{e^u \cos v}{e^{2u} \cos^2 v} = \frac{e^{-u} \sin v}{e^{2u} \cos^2 v}$
- $\frac{\partial v}{\partial x} = -\frac{\partial v}{\partial u}/J = \frac{e^u \cos v}{e^{2u} \cos^2 v} = \frac{e^{-u} \sin v}{e^{2u} \cos^2 v}$
- $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}/J = \frac{e^u \cos v}{e^{2u} \cos^2 v} = \frac{e^{-u} \sin v}{e^{2u} \cos^2 v}$
- $\frac{\partial v}{\partial y} = \frac{\partial v}{\partial v}/J = \frac{e^u \cos v}{e^{2u} \cos^2 v} = \frac{e^{-u} \sin v}{e^{2u} \cos^2 v}$

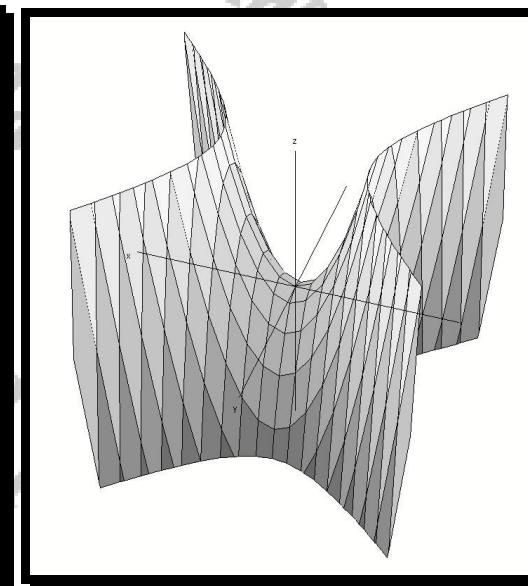
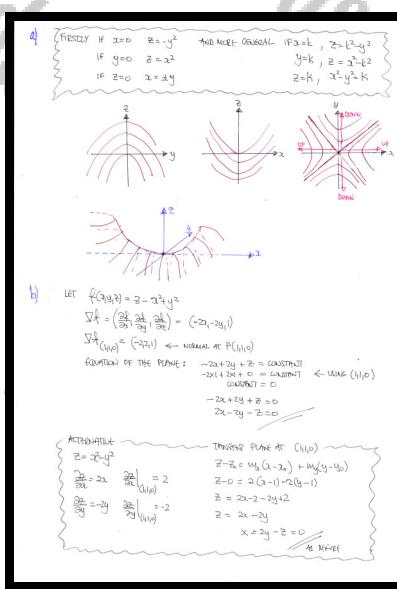
**Question 21 (\*\*\*)**

A surface  $S$  has Cartesian equation

$$z = x^2 - y^2.$$

- a) Sketch profiles of  $S$  parallel to the  $y$ - $z$  plane, parallel to the  $x$ - $z$  plane, and parallel to the  $x$ - $y$  plane.
- b) Find an equation of the tangent plane on  $S$ , at the point  $P(1,1,0)$ .

$$2x - 2y - z = 0$$



**Question 22 (\*\*\*)**

A surface  $S$  is given parametrically by

$$x = at \cosh \theta, \quad x = bt \sinh \theta, \quad z = t^2,$$

where  $t$  and  $\theta$  are real parameters, and  $a$  and  $b$  are non zero constants.

- Find a Cartesian equation for  $S$ .
- Determine an equation of the tangent plane on  $S$  at the point with Cartesian coordinates  $(x_0, y_0, z_0)$ .

$$\boxed{\quad}, \quad \boxed{z = \frac{x^2}{a^2} - \frac{y^2}{b^2}}, \quad \boxed{2b^2 x x_0 - 2a^2 y y_0 = a^2 b^2 (z + z_0)}$$

**a) ELIMINATE THE  $\theta$  BY HYPERBOLIC IDENTITIES**

$$\begin{aligned} x &= at \cosh \theta & y &= bt \sinh \theta & z &= t^2 \\ \frac{\partial}{\partial \theta} &= a \sinh \theta & \frac{\partial}{\partial \theta} &= b \cosh \theta & & \\ \Rightarrow \cosh^2 \theta - \sinh^2 \theta &= 1 & & & & \\ \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 & & & & \\ \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} &= t^2 & & & & \\ & & & \boxed{z = \frac{x^2}{a^2} - \frac{y^2}{b^2}} & & \end{aligned}$$

**b) START WITH THE NORMAL**

$$\begin{aligned} f(x, y, z) &= z - \frac{x^2}{a^2} + \frac{y^2}{b^2} \\ \mathbf{n} &= \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\ \mathbf{n} &= \left( -\frac{2x}{a^2}, \frac{2y}{b^2}, 1 \right) \\ \mathbf{n}|_{(x_0, y_0, z_0)} &= \left( -\frac{2x_0}{a^2}, \frac{2y_0}{b^2}, 1 \right) \\ \text{EQUATION OF THE TANGENT PLANE} & \\ \Rightarrow \left( -\frac{2x_0}{a^2} \right) x + \left( \frac{2y_0}{b^2} \right) y + z &= \text{constant} \\ \Rightarrow \frac{2x_0}{a^2} x - \frac{2y_0}{b^2} y - z &= \text{constant} \\ \text{USING THE POINT } (x_0, y_0, z_0) & \\ \Rightarrow \left( \frac{2x_0}{a^2} \right) x_0 - \left( \frac{2y_0}{b^2} \right) y_0 - z_0 &= \text{constant} \end{aligned}$$

$$\begin{aligned} \Rightarrow \text{constant} &= \frac{2x_0^2}{a^2} - \frac{2y_0^2}{b^2} = z_0 \\ \Rightarrow \text{constant} &= 2 \left( \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} \right) = z_0 \\ &\uparrow \\ &\text{REWRITE EQUATION OF THE SURFACE} \\ \Rightarrow \text{constant} &= 2z_0 \\ \Rightarrow \text{constant} &= z_0 \\ \text{THIS WE KNOW THAT} & \\ \Rightarrow \left( \frac{2x_0}{a^2} \right) x - \left( \frac{2y_0}{b^2} \right) y - z &= z_0 \\ \Rightarrow 2b^2 z_0 x - 2a^2 y y_0 - a^2 b^2 (z + z_0) &= 0 \\ \Rightarrow 2b^2 x x_0 - 2a^2 y y_0 &= a^2 b^2 (z + z_0) \end{aligned}$$

**Question 23 (\*\*\*)**

The function  $z$  depends on  $x$  and  $y$  so that

$$z^2(x, y) = \frac{y - x^3 - xy^2}{x}, \quad x \neq 0.$$

Show that

$$xz \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = z^2 - \frac{y}{x}.$$

proof

$z^2 = \frac{y - x^3 - xy^2}{x}$

- $2z \frac{\partial z}{\partial x} = -\frac{3x^2 - 2y^2}{x^2} - 2x$
- $2z \frac{\partial z}{\partial y} = \frac{y}{x} - 2y^2$

$\rightarrow 2z \frac{\partial z}{\partial x} + 2yz \frac{\partial z}{\partial y} = -\frac{3x^2 - 2y^2}{x^2} + \frac{y}{x} - 2y^2$

$\rightarrow 2z \frac{\partial z}{\partial x} + 2yz \frac{\partial z}{\partial y} = -2(x^2 + y^2)$

$\rightarrow 2z \frac{\partial z}{\partial x} + 2yz \frac{\partial z}{\partial y} = -(x^2 + y^2)$

BUT  $-(x^2 + y^2) = z^2 - \frac{y}{x}$  FROM ORIGINAL EXPRESSION

$2z \frac{\partial z}{\partial x} + 2yz \frac{\partial z}{\partial y} = z^2 - \frac{y}{x}$

As required

**Question 24 (\*\*\*)**

The function  $f$  depends on  $u$  and  $v$  where

$$u = 2xy \quad \text{and} \quad v = x^2 - y^2.$$

Assuming  $x \neq y$ ,  $x \neq 0$  and  $y \neq 0$ , show that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2 \left[ u \frac{\partial f}{\partial u} + v \frac{\partial f}{\partial v} \right].$$

[ ] , [ ] proof

IT IS GIVEN THAT  
 $z = f(u,v)$  WHERE  $u(x,y) = 2xy$  &  $v(x,y) = x^2 - y^2$

- DIFFERENTIATE USING THE CHAIN RULE
- $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(2x)$
- $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-2y)$
- TIDYING UP

$$\begin{aligned} \frac{\partial z}{\partial x} &= 2y \frac{\partial f}{\partial u} + 2x \frac{\partial f}{\partial v} \\ \frac{\partial z}{\partial y} &= 2x \frac{\partial f}{\partial u} - 2y \frac{\partial f}{\partial v} \end{aligned} \Rightarrow$$

$$\begin{aligned} x \frac{\partial z}{\partial x} &= 2xy \frac{\partial f}{\partial u} + 2x^2 \frac{\partial f}{\partial v} \\ y \frac{\partial z}{\partial y} &= 2xy \frac{\partial f}{\partial u} - 2y^2 \frac{\partial f}{\partial v} \end{aligned} \Rightarrow$$

- ADDING THE EQUATIONS

$$\begin{aligned} \Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= 4xy \frac{\partial f}{\partial u} + (2x^2 - 2y^2) \frac{\partial f}{\partial v} \\ \Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= 2 \left[ 2xy \frac{\partial f}{\partial u} + (x^2 - y^2) \frac{\partial f}{\partial v} \right] \\ \Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= 2 \left[ u \frac{\partial f}{\partial u} + v \frac{\partial f}{\partial v} \right] \end{aligned}$$

**Question 25 (\*\*\*)**

The functions  $F$  and  $G$  satisfy

$$G(x, y) \equiv F[u(x, y), v(x, y)],$$

where  $u$  and  $v$  satisfy the following transformation equations.

$$u = x \cos y, \quad v = x \sin y.$$

Use the chain rule for partial derivatives to show that

$$\left[ \frac{\partial G}{\partial x} \right]^2 + \left[ \frac{1}{x} \frac{\partial G}{\partial y} \right]^2 = \left[ \frac{\partial F}{\partial u} \right]^2 + \left[ \frac{\partial F}{\partial v} \right]^2.$$

[ ] , proof

$G(x, y) = F(u(x, y), v(x, y))$        $u = x \cos y$   
 $v = x \sin y$

START BY COMPUTING SOME BASIC PARTIAL DERIVATIVES

- $\frac{\partial u}{\partial x} = \cos y$       •  $\frac{\partial u}{\partial y} = -x \sin y$
- $\frac{\partial v}{\partial x} = \sin y$       •  $\frac{\partial v}{\partial y} = x \cos y$

BY THE CHAIN RULE WE HAVE

$$\begin{aligned} \frac{\partial G}{\partial x} &= \frac{\partial F}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial F}{\partial u} \cos y + \frac{\partial F}{\partial v} \sin y \\ \frac{\partial G}{\partial y} &= \frac{\partial F}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial y} = -x \frac{\partial F}{\partial u} \sin y + x \frac{\partial F}{\partial v} \cos y \end{aligned}$$

FINALLY WE OBTAIN

$$\begin{aligned} \left( \frac{\partial G}{\partial x} \right)^2 + \left( \frac{1}{x} \frac{\partial G}{\partial y} \right)^2 &= \\ &= \left[ \frac{\partial F}{\partial u} \cos y + \frac{\partial F}{\partial v} \sin y \right]^2 + \left[ \frac{1}{x} \left( -x \frac{\partial F}{\partial u} \sin y + x \frac{\partial F}{\partial v} \cos y \right) \right]^2 \\ &= \left[ \frac{\partial F}{\partial u} \cos y + \frac{\partial F}{\partial v} \sin y \right]^2 + \left[ \frac{\partial F}{\partial u} \cos y - \frac{\partial F}{\partial v} \sin y \right]^2 \\ &= \left( \frac{\partial F}{\partial u} \cos y + \frac{\partial F}{\partial v} \sin y \right)^2 + \left( \frac{\partial F}{\partial u} \cos y - \frac{\partial F}{\partial v} \sin y \right)^2 \\ &\quad \cancel{\left( \frac{\partial F}{\partial u} \cos y + \frac{\partial F}{\partial v} \sin y \right) \left( \frac{\partial F}{\partial u} \cos y - \frac{\partial F}{\partial v} \sin y \right)} \\ &= \left( \frac{\partial F}{\partial u} \right)^2 (\cos^2 y + \sin^2 y) + \left( \frac{\partial F}{\partial v} \right)^2 (\cos^2 y + \sin^2 y) \\ &= \left( \frac{\partial F}{\partial u} \right)^2 + \left( \frac{\partial F}{\partial v} \right)^2 \end{aligned}$$

**Question 26 (\*\*\*)+**

The function  $f$  is defined as

$$f(x, y, z) \equiv x^3 - 75x + 3z(y-1)^2 + z^3.$$

The point  $Q$  lies on  $f$ .

The derivatives at  $Q$  in the directions  $\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $-\mathbf{i} + \mathbf{j} - \mathbf{k}$ , are equal.

- a) Show that  $Q$  must lie on the surface of a sphere  $S$ .

The point  $P(1, 3, a)$  lies on  $S$ .

- b) Find a vector equation of the normal line to  $S$  at  $P$ .

A sphere  $T$  is concentric to  $S$  and has radius three times as large as that of  $S$ .

The normal line to  $S$  at  $P$  intersects the surface of  $T$  at the points  $A$  and  $B$ .

- c) Determine the coordinates of  $A$  and  $B$ .

$$(x, y, z) = [\lambda + 1, 2\lambda + 3, 2(\lambda + 1)\sqrt{5}], A(3, 7, 6\sqrt{5}), B(-3, -5, -6\sqrt{5})$$

**Q1** •  $f(x, y, z) = x^3 - 75x + 3z(y-1)^2 + z^3$   
 $\nabla f = [3x^2 - 75, 6z(y-1), 3z^2]$

$\hat{f}_1 = \hat{G}(1, 1) \Rightarrow \hat{f}_1 = \frac{1}{\sqrt{5}} G(1, 1)$   
 $\hat{f}_2 = \hat{G}(1, -1) \Rightarrow \hat{f}_2 = \frac{1}{\sqrt{5}} G(1, -1)$

$\nabla \hat{f} \cdot \hat{f}_1 = \nabla \hat{f} \cdot \hat{f}_2 \quad \text{or} \quad \frac{1}{\sqrt{5}} \cdot \nabla \hat{f} = \frac{1}{\sqrt{5}} \cdot \nabla \hat{f}$

$\Rightarrow \cancel{\lambda} \hat{f}_1 \cdot \hat{f}_1 = \cancel{\lambda}^2 - 75 + 3(y-1)^2 + 3z^2 = \cancel{\lambda}^2 - 75 + 3(y-1)^2 + 3z^2$   
 $\Rightarrow 3\lambda^2 - 75 + 3(y-1)^2 + 3z^2 = -3\lambda^2 + 75 - 3(y-1)^2 - 3z^2$   
 $\Rightarrow 6\lambda^2 + 3(y-1)^2 + 6z^2 = 150$   
 $\Rightarrow \lambda^2 + (y-1)^2 + z^2 = 25 \quad \text{i.e. A sphere radius } 5, \text{ centre at } (0, 0, 0)$

**Q2** •  $P(1, 3, a) \Rightarrow 1^2 + 3^2 + a^2 = 25$   
 $a^2 = 20$   
 $a = \pm 2\sqrt{5}$        $\therefore P(1, 3, 2\sqrt{5})$

• Let  $G(y, z) = x^2 + (y-1)^2 + z^2 - 25$   
 $\nabla G = [2x, 2(y-1), 2z]$   
 $\hat{G} = [2, 2(y-1), 2z]$   
 $\hat{G}_p = [1, 2, 2\sqrt{5}]$

**Q3** • Draw a diagram

$\lambda = 2 \Rightarrow (3, 7, 6\sqrt{5}) \quad \text{i.e. } B(3, 7, 6\sqrt{5})$   
 $\lambda = -4 \Rightarrow (-3, -5, -6\sqrt{5}) \quad \text{i.e. } A(-3, -5, -6\sqrt{5})$

**Question 27 (\*\*\*)+**

The function  $z$  depends on  $x$  and  $y$  so that

$$z = r^2 \tan \theta, \quad x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

- a) Express  $r$  and  $\theta$  in terms of  $x$  and  $y$  and hence determine expressions for  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ , in terms of  $x$  and  $y$ .

Give each of the answers as a single simplified fraction

- b) Verify the answer to part (a) by implicit differentiation using Jacobians

$$\boxed{\frac{\partial z}{\partial x} = \frac{y(x^2 - y^2)}{x^2}}, \quad \boxed{\frac{\partial z}{\partial y} = \frac{x^2 + 3y^2}{x}}$$

a) USING STANDARD FORMS THE DIRECT IN TUTORIAL

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \Rightarrow \begin{cases} r^2 = x^2 + y^2 \\ \tan \theta = \frac{y}{x} \end{cases}$$

HENCE WE HAVE

$$z = r^2 \tan \theta = (x^2 + y^2) \left( \frac{y}{x} \right) = xy + y^2 x^{-1}$$

- $\frac{\partial z}{\partial x} = y - 2xy^{-2} = y - \frac{2y^3}{x^2} = \frac{x^2 - 2y^2}{x^2} \quad //$
- $\frac{\partial z}{\partial y} = x + 2x^{-2}y = x + \frac{2y^2}{x} = \frac{x^2 + 3y^2}{x} \quad //$

b) FINALLY COMPUTE THE JACOBIAN

$$J = \frac{\partial(z,y)}{\partial(x,\theta)} = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos \theta - 2r \sin \theta & r^2 + 3r^2 \sin^2 \theta \\ 0 & \sec^2 \theta \end{vmatrix}$$

JUST WE USE THE STANDARD RESULT

$$\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} 2y & -2x \\ 0 & \frac{1}{\cos^2 \theta} \end{bmatrix} \quad \text{NOTICE THE MATRIX INVERSE DIAMMETRICAL}$$

DRAWING WITH PENCIL OR THE EDITOR OF THE MATRIX

- $\frac{\partial z}{\partial x} = \frac{1}{J} \frac{\partial z}{\partial y} = \frac{1}{r} (\cos \theta) = \cos \theta$
- $\frac{\partial z}{\partial y} = -\frac{1}{J} \frac{\partial z}{\partial x} = -\frac{1}{r} (-r \sin \theta) = \sin \theta$
- $\frac{\partial y}{\partial x} = -\frac{1}{J} \frac{\partial y}{\partial z} = -\frac{1}{r} (\sec \theta) = -\frac{\sec \theta}{r}$
- $\frac{\partial y}{\partial z} = \frac{1}{J} \frac{\partial y}{\partial x} = \frac{1}{r} (\cos \theta) = \frac{\cos \theta}{r}$

NOW BY THE CHAIN RULE WE HAVE

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial x} = (2r \tan \theta)(\cos \theta) + (r^2 \sec^2 \theta)(-\sin \theta) \\ &= 2r \sin \theta - r \tan \theta \sec \theta \\ &= 2(r \sin \theta) - r \tan \theta \left( \frac{1}{\cos \theta} \right) \\ &= 2y - \frac{r^2}{x^2} \quad \left( \frac{1}{\cos^2 \theta} \right) \\ &= 2y - \frac{r^2}{x^2} = 2y - \frac{y^2(x^2 - y^2)}{x^2} \\ &= \frac{2y^2 - 2y - y^3}{x^2} = \frac{y(2x^2 - y^2)}{x^2} \\ &= \frac{y(x^2 - y^2)}{x^2} \quad // \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial y} = (2r \tan \theta) \sin \theta + (r^2 \sec^2 \theta) \frac{1}{r} \\ &= 2r \sin \theta (\tan \theta) + r \cos \theta \cdot \frac{1}{r} \\ &= 2 \left( \frac{y}{x} \right) y + 2 \left( \frac{x^2}{x^2 + y^2} \right) = \frac{2y^2}{x^2} + \frac{2x^2}{x^2 + y^2} \\ &= \frac{2y^2 + x^2}{x^2} = \frac{2y^2 + 3y^2}{x^2} = \frac{x^2 + 3y^2}{x} \quad // \end{aligned}$$

**Question 28 (\*\*\*)+**

The function  $\varphi$  depends on  $u$  and  $v$  so that

$$x = 2u + e^{2v} \quad \text{and} \quad y = 2v + e^{-2u}.$$

Without using standard results involving Jacobians, determine simplified expressions for  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial u}{\partial y}$ ,  $\frac{\partial v}{\partial x}$  and  $\frac{\partial v}{\partial y}$ , in terms of  $u$  and  $v$ .

	$\frac{\partial u}{\partial x} = \frac{1}{2+2e^{2(v-u)}}$	$\frac{\partial u}{\partial y} = -\frac{e^{2v}}{2+2e^{2(v-u)}}$	$\frac{\partial v}{\partial x} = \frac{e^{-2u}}{2+2e^{2(v-u)}}$
			$\frac{\partial v}{\partial y} = \frac{1}{2+2e^{2(v-u)}}$

• START BY PRODUCING DIFFERENTIALS AS FOLLOWS

$$\begin{aligned} x &= 2u + e^{2v} & y &= 2v + e^{-2u} \\ dx &= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv & dy &= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \\ dx &= 2du + e^{2v} dv & dy &= -2e^{-2u} du + 2dv \\ \cancel{e^{2u} dx} &= 2e^{2u} du + 2e^{2u} \cancel{e^{2u} dv} & \end{aligned}$$

• ADDING THE EXPRESSIONS TO ELIMINATE THE "du" TERMS

$$\begin{aligned} \Rightarrow e^{2u} dx + dy &= [2 + 2e^{2(v-u)}] dv \\ \Rightarrow dv &= \frac{e^{2u}}{2+2e^{2(v-u)}} du + \frac{1}{2+2e^{2(v-u)}} dy \end{aligned}$$

• COMBINING WITH

$$\begin{aligned} dv &= \frac{\partial v}{\partial u} du + \frac{\partial v}{\partial y} dy \\ \therefore \frac{\partial v}{\partial u} &= \frac{e^{-2u}}{2+2e^{2(v-u)}} = \frac{1}{2e^{2u}+2e^{2v}} // \\ \frac{\partial v}{\partial y} &= \frac{1}{2+2e^{2(v-u)}} = \frac{-2u}{2e^{2u}+2e^{2v}} // \end{aligned}$$

• SIMILARLY STARTING FROM THE PREVIOUSLY DRAWN EXPRESSIONS

$$\begin{aligned} (dx = 2du + 2e^{2v} dv) \times e^{2u} \\ (dy = -2e^{-2u} du + 2dv) \\ (dx = 2du + 2e^{2v} du) \\ (e^{2u} dy = -2e^{-2u} du + 2e^{2v} dv) \end{aligned}$$

• SUBTRACTING TO ELIMINATE THE "du" TERMS

$$\begin{aligned} \Rightarrow dx - e^{2u} dy &= [2+2e^{2(v-u)}] du \\ \Rightarrow du &= \frac{1}{2+2e^{2(v-u)}} du + \frac{-e^{2u}}{2+2e^{2(v-u)}} dy \\ \uparrow \frac{\partial u}{\partial x} \cdot du &+ \uparrow \frac{\partial u}{\partial y} \cdot du \\ \therefore \frac{\partial u}{\partial x} &= \frac{1}{2+2e^{2(v-u)}} = \frac{-e^{2u}}{2e^{2u}+2e^{2v}} // \\ \frac{\partial u}{\partial y} &= \frac{-e^{2u}}{2+2e^{2(v-u)}} = \frac{e^{2(v-u)}}{2e^{2u}+2e^{2v}} // \end{aligned}$$

**Question 29 (\*\*\*)+**

A hill is modelled by the equation

$$f(x, y) = 300e^{-(x^2+y^2)}, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}.$$

A railway runs along the straight line with equation

$$y = x - 2.$$

Determine the steepest slope that the train needs to climb.

$$\pm 300\sqrt{2} e^{-\frac{5}{2}}$$

$\nabla f(x, y) = \vec{z} = 300 e^{-(x^2+y^2)} \begin{pmatrix} -2x \\ -2y \end{pmatrix}$

- $\nabla z = \left( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) = \left( -600x e^{-(x^2+y^2)}, -600y e^{-(x^2+y^2)} \right)$   
 $= -600e^{-(x^2+y^2)} (x, y)$
- THE LINE  $y = x - 2$ , HAS SLOPE (EXCEPT  $(1, 1)$ ) BY INSPECTION**  
 A UNIT VECTOR IN THAT DIRECTION IS  $\frac{1}{\sqrt{2}}(1, 1)$
- THE DIRECTIONAL DERIVATIVE IN THE DIRECTION OF THIS UNIT IS**  
 $-600e^{-(x^2+y^2)} (x, y) \cdot \frac{1}{\sqrt{2}}(1, 1)$   
 $= -300\sqrt{2} e^{-(x^2+y^2)} (x, y)$
- LET**  $g(x) = -300\sqrt{2} e^{-(x^2+(x-2)^2)} (2x + x - 2)$   
 $g(x) = -300\sqrt{2} e^{-(2x-2)^2} (2x^2 - 4x + 4)$   
 $g(x) = 600\sqrt{2} (1-x) e^{-2x^2+4x-4}$
- $\begin{aligned} g'(x) &= -600\sqrt{2} e^{-2x^2+4x-4} + (600\sqrt{2} (1-x)(-4x+4)) e^{-2x^2+4x-4} \\ g'(x) &= 600\sqrt{2} e^{-2x^2+4x-4} [ -1 + (1-x)(4-4x) ] \\ g'(x) &= 600\sqrt{2} e^{-2x^2+4x-4} [ 2(1-x) - 1 ] [ 2(1-x) + 1 ] \\ g'(x) &= 600\sqrt{2} e^{-2x^2+4x-4} (1-2x)(3-2x) \end{aligned}$
- SOLVING FOR ZERO, WE GET  $x = 1 \pm \frac{1}{2}\sqrt{5}$**   $g'(x) < \frac{300\sqrt{2} e^{-\frac{5}{2}}}{-300\sqrt{2} e^{-\frac{5}{2}}}$

**Question 30 (\*\*\*)+**

The functions  $F$  and  $G$  satisfy

$$G(u, v) \equiv F[x(u, v), y(u, v)],$$

where  $x$  and  $y$  satisfy the following transformation equations.

$$x = uv, \quad y = \frac{u+v}{u-v}.$$

Use the chain rule for partial derivatives to show that

$$u \frac{\partial G}{\partial u} + v \frac{\partial G}{\partial v} = 2x \frac{\partial F}{\partial x} \quad \text{and} \quad \frac{u^2 - v^2}{2uv} \left[ v \frac{\partial G}{\partial v} - u \frac{\partial G}{\partial u} \right] = 2y \frac{\partial F}{\partial y}.$$

[ ] , [proof]

$G(u, v) = F[x(u, v), y(u, v)]$

- $x = uv$
- $y = \frac{u+v}{u-v}$

START BY COMPUTING  $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$

- $\frac{\partial x}{\partial u} = v$
- $\frac{\partial x}{\partial v} = u$
- $\frac{\partial y}{\partial u} = \frac{(u-v)-(u+v)}{(u-v)^2} = \frac{-2v}{(u-v)^2}$
- $\frac{\partial y}{\partial v} = \frac{(u-v)-(u+v)}{(u-v)^2} = \frac{-2u}{(u-v)^2}$

BY THE CHAIN RULE

$$\begin{aligned} \frac{\partial G}{\partial u} &= \frac{\partial F}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial F}{\partial x} - \frac{2v}{(u-v)^2} \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial v} &= \frac{\partial F}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial v} = u \frac{\partial F}{\partial x} + \frac{2u}{(u-v)^2} \frac{\partial F}{\partial y} \end{aligned}$$

THUS WE HAVE

$$\begin{aligned} u \frac{\partial G}{\partial u} + v \frac{\partial G}{\partial v} &= u \left[ \frac{\partial F}{\partial x} - \frac{2v}{(u-v)^2} \frac{\partial F}{\partial y} \right] + v \left[ u \frac{\partial F}{\partial x} + \frac{2u}{(u-v)^2} \frac{\partial F}{\partial y} \right] \\ &= uv \frac{\partial F}{\partial x} - \frac{2uv}{(u-v)^2} \frac{\partial F}{\partial y} + uv \frac{\partial F}{\partial x} + \frac{2uv}{(u-v)^2} \frac{\partial F}{\partial y} \\ &= 2uv \frac{\partial F}{\partial x} \\ &= 2x \frac{\partial F}{\partial x} \end{aligned}$$

AND IN SIMILAR FASHION

$$\begin{aligned} \frac{u^2 - v^2}{2uv} \left[ v \frac{\partial G}{\partial v} - u \frac{\partial G}{\partial u} \right] &= \frac{u^2 - v^2}{2uv} \left[ v \left[ u \frac{\partial F}{\partial x} + \frac{2u}{(u-v)^2} \frac{\partial F}{\partial y} \right] - u \left[ \frac{\partial F}{\partial x} - \frac{2v}{(u-v)^2} \frac{\partial F}{\partial y} \right] \right] \\ &= \frac{u^2 - v^2}{2uv} \left[ uv \frac{\partial F}{\partial x} + \frac{2uv}{(u-v)^2} \frac{\partial F}{\partial y} - uv \frac{\partial F}{\partial x} + \frac{2uv}{(u-v)^2} \frac{\partial F}{\partial y} \right] \\ &= \frac{u^2 - v^2}{2uv} \times \frac{4uv}{(u-v)^2} \frac{\partial F}{\partial y} \\ &= \frac{2(u^2 - v^2)}{(u-v)^2} \frac{\partial F}{\partial y} \\ &= \frac{2(u-v)(u+v)}{(u-v)^2} \frac{\partial F}{\partial y} \\ &= 2 \frac{u+v}{u-v} \frac{\partial F}{\partial y} \\ &= 2y \frac{\partial F}{\partial y} \end{aligned}$$

**Question 31 (\*\*\*\*)**

The function  $z$  depends on  $x$  and  $y$  so that

$$z = (u+v)^2, \quad x = u^2 - v^2 \quad \text{and} \quad y = uv.$$

Show clearly that ...

i. ...  $\frac{\partial z}{\partial x} = \frac{x}{z-2y}$ .

ii. ...  $\frac{\partial z}{\partial y} = \frac{2z}{z-2x}$ .

 , proof

$z = (uv)^2 \quad z = u^2 - v^2 \quad u = uv$

Start with the Jacobian

$$J = \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = -2u^2 + 2v^2$$

Now we have

- $\frac{\partial z}{\partial u} = \frac{1}{2}(uv)^2 = \frac{u^2v^2}{2}$
- $\frac{\partial z}{\partial v} = \frac{1}{2}(uv)^2 = \frac{u^2v^2}{2}$
- $\frac{\partial x}{\partial u} = \frac{1}{2}(2u) = u$
- $\frac{\partial x}{\partial v} = \frac{1}{2}(-2v) = -v$

Thus we obtain

- $\frac{\partial z}{\partial u} = \frac{1}{2}(uv)^2 \times u = \frac{u^3v^2}{2u^2+2v^2}$
- $\frac{\partial z}{\partial v} = \frac{1}{2}(uv)^2 \times (-v) = -\frac{u^3v^2}{2u^2+2v^2}$
- $\frac{\partial x}{\partial u} = \frac{1}{2}(2u) \times (2u) = \frac{2u^2}{2u^2+2v^2}$
- $\frac{\partial x}{\partial v} = \frac{1}{2}(-2v) \times (2u) = \frac{-2uv}{2u^2+2v^2}$

Now by the chain rule

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = 2(uv)^2 \times \frac{u}{2u^2+2v^2} + 2(uv)^2 \times \frac{-v}{2u^2+2v^2} \\ &= \frac{2u(uv)^2 - 2v(uv)^2}{2u^2+2v^2} = \frac{u(uv)^2 - v(uv)^2}{u^2+v^2} \\ &= \frac{uv(uv)^2}{u^2+v^2} = \frac{u(uv)^3}{u^2+v^2} = \frac{u(u+v)^3}{u^2+v^2} \\ &= \frac{2u(u+v)^3}{(u+v)^2-2uv} = \frac{2u(u+v)^3}{z-2xy} \end{aligned}$$

Now in an analogous fashion

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = 2(uv)^2 \times \frac{u}{2u^2+2v^2} + 2(uv)^2 \times \frac{v}{2u^2+2v^2} \\ &= \frac{2u(uv)^2 + 2v(uv)^2}{2u^2+2v^2} = \frac{2(uv)^2(u+v)}{2u^2+2v^2} = \frac{2(uv)^3}{u^2+v^2} \\ &= \frac{2(uv)^3}{(u^2+v^2)^2} = \frac{2(u+v)^3}{(u+v)^2-2uv} = \frac{2(u+v)^3}{z-2xy} \end{aligned}$$

**Question 32 (\*\*\*\*)**

The function  $z$  depends on  $x$  and  $y$  so that

$$z = f(u, v), \quad u = x + y \quad \text{and} \quad v = 2x - 2y.$$

Show clearly that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2} - 4 \frac{\partial^2 z}{\partial v^2}.$$

[proof]

$z = f(u, v) \quad u = x + y \quad v = 2x - 2y$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2} - 4 \frac{\partial^2 z}{\partial v^2}$

Now differentiate w.r.t.  $y$

$\frac{\partial^2 z}{\partial u^2} = \frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y}$

$\frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2} \left[ \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial^2 z}{\partial x \partial y} \right] \quad \text{(Chain Rule)}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x^2} \times \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x^2} \times 1 + 2 \frac{\partial^2 z}{\partial x^2} (-2)$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x^2} \quad \text{as required}$

**Question 33 (\*\*\*\*)**

The function  $z$  depends on  $x$  and  $y$  so that

$$z = \frac{x}{1+xf} \quad \text{where } f = f\left(\frac{1}{y} - \frac{1}{x}\right).$$

Show clearly that

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2.$$

proof

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\frac{\partial}{\partial x}\left(\frac{x}{1+xf}\right)}{(1+xf)^2} = \frac{1}{(1+xf)^2} \\ \bullet \frac{\partial z}{\partial x} &= \frac{(1+xf)x - x(0+x\cancel{f}) + x\cancel{f}^2 \cdot \cancel{f}}{(1+xf)^2} = \frac{1+xf - 2[\cancel{f} + \frac{x^2}{\cancel{f}}]}{(1+xf)^2} \\ &= \frac{1+xf - xf - x^2}{(1+xf)^2} = \frac{1-x^2}{(1+xf)^2} \\ \bullet \frac{\partial z}{\partial y} &= \frac{(1+xf)x\cancel{f} - 2[x\cancel{f} + x^2\cancel{f}^2 + x\cancel{f}^2 \cdot \cancel{f}]}{(1+xf)^2} = \frac{\frac{\partial z}{\partial x} \cancel{f}}{(1+xf)^2} = \frac{\frac{\partial z}{\partial x} \cancel{f}}{(1+xf)^2} \\ \text{Therefore,} \\ x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} &= x^2 \left[ \frac{1-x^2}{(1+xf)^2} \right] + y^2 \left[ \frac{\frac{\partial z}{\partial x} \cancel{f}}{(1+xf)^2} \right] \\ &= \frac{x^2 - x^4}{(1+xf)^2} + \frac{x^2 \cancel{f}}{(1+xf)^2} \\ &= \frac{x^2}{(1+xf)^2} = \left( \frac{x}{1+xf} \right)^2 = z^2 \end{aligned}$$

*As required*

**Question 34 (\*\*\*\*)**

The surface  $S$  has equation

$$z = y f\left(\frac{x}{y}\right),$$

where the function  $f\left(\frac{x}{y}\right)$  is differentiable.

Show that the tangent plane at any point on  $S$  passes through the origin  $O$

[proof]

$\boxed{z = y f\left(\frac{x}{y}\right)}$

Let  $g(x,y,z) = g\left(\frac{x}{y}\right) - z$

$$\rightarrow \nabla g = \left[ \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right] = \left[ y f'\left(\frac{x}{y}\right) \cdot \frac{1}{y}, f'\left(\frac{x}{y}\right) + y f''\left(\frac{x}{y}\right) \cdot \frac{x}{y^2}, -1 \right]$$

PRODUCT RULE

$$\rightarrow \nabla g = \left[ f'\left(\frac{x}{y}\right), f'\left(\frac{x}{y}\right) + \frac{x}{y^2} f''\left(\frac{x}{y}\right), -1 \right]$$

• Take a random point on the surface,  $P(x_0, y_0, z_0)$   
and note that  $(x_0, y_0, f\left(\frac{x_0}{y_0}\right))$ .

$$\nabla g = \nabla g|_{P} = \left[ f'\left(\frac{x_0}{y_0}\right), f'\left(\frac{x_0}{y_0}\right) + \frac{x_0}{y_0^2} f''\left(\frac{x_0}{y_0}\right), -1 \right]$$

• Equation of a plane is therefore

$$f'\left(\frac{x_0}{y_0}\right)x + \left[ f'\left(\frac{x_0}{y_0}\right) + \frac{x_0}{y_0^2} f''\left(\frac{x_0}{y_0}\right)\right]y - z = \text{constant}$$

But  $(x_0, y_0, z_0)$  lies on the plane so it must satisfy  
the above equation

$$\therefore \text{constant} = f'\left(\frac{x_0}{y_0}\right)x_0 + \left[ f'\left(\frac{x_0}{y_0}\right) + \frac{x_0}{y_0^2} f''\left(\frac{x_0}{y_0}\right)\right]y_0 - z_0$$

$$\text{constant} = x_0 f'\left(\frac{x_0}{y_0}\right) + y_0 \left[ f'\left(\frac{x_0}{y_0}\right) + \frac{x_0}{y_0^2} f''\left(\frac{x_0}{y_0}\right)\right] - z_0$$

$$\text{constant} = 0$$

• Plane is of the form  $Ax + By + Cz = 0$   
so it passes through the origin

**Question 35 (\*\*\*\*)**

The functions  $u = u(x, y)$  and  $v = v(x, y)$  satisfy

$$u + 3v^3 = 3x + y^2 \quad \text{and} \quad v - 2u^3 = x^3 - 2y.$$

Determine the value of  $\frac{\partial(u, v)}{\partial(x, y)}$  at  $(x, y) = (0, 0)$ .

$$\left| \frac{\partial(u, v)}{\partial(x, y)} \right|_{(0,0)} = -6$$

At  $x=0, y=0$

$$\begin{aligned} u + 3v^3 &= 0 \\ v - 2u^3 &= 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial x} + 9v^2 \frac{\partial v}{\partial x} &= 3 \\ \frac{\partial v}{\partial x} - 6u^2 \frac{\partial u}{\partial x} &= 0 \end{aligned}$$

At  $x=0, y=0 \quad u=v=0$

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3 & \frac{\partial v}{\partial x} &= 0 \\ \bullet \quad \frac{\partial u}{\partial y} + 9v^2 \frac{\partial v}{\partial y} &= 2y \\ \frac{\partial v}{\partial y} - 6u^2 \frac{\partial u}{\partial y} &= -2. \end{aligned}$$

At  $x=0, y=0 \quad u=v=0$

$$\begin{aligned} \frac{\partial u}{\partial y} &= 0 & \frac{\partial v}{\partial y} &= -2 \\ \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} 3 & 0 \\ 0 & -2 \end{vmatrix} = -6 \end{aligned}$$

At  $x=0, y=0$

$$\begin{aligned} u + 3v^3 &= 0 \\ v - 2u^3 &= 0 \end{aligned}$$

$$\begin{aligned} u + 3(2u)^3 &= 0 \\ u + 24u^3 &= 0 \\ u(1 + 24u^2) &= 0 \\ u=0 \text{ only} & \\ v=0 & \end{aligned}$$

**Question 36 (\*\*\*\*)**

A surface  $S$  has equation  $f(x, y, z) = 0$ , where

$$f(x, y, z) = x^2 + 2xy - 4x + 2y^2 + 2yz - 8y - z^2 + 4z.$$

- a) Show that there is no point on  $S$  where the normal to  $S$  is parallel to the  $z$  axis and hence state the geometric significance of this result with reference to the stationary points of  $S$ .

$S$  is translated to give a new surface  $T$  with equation

$$f(x, y, z) = -56.$$

The plane with equation  $x + y + z = k$ , where  $k$  is a constant, is a tangent plane to  $T$ .

- b) Determine the two possible values of  $k$ .

$$\boxed{\quad}, \quad k = 2 \cup k = 6$$

**Given:** First order derivatives

$$\begin{aligned} f_x &= 2x+2y-4 \\ f_y &= 2x+4y+2z-8 \\ f_z &= 2y+2z+4 \end{aligned}$$

PARALLEL TO  $\mathbf{z}$ -AXIS implies

$$\nabla f = \lambda(0,0,\mathbf{x})$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \lambda(0,0,2x+2z+4)$$

COMBINING EQUATIONS TO ZERO

$$\begin{aligned} &= (2x+2y-4=0) \text{ SUBJECT "OPENED"} \\ &\Rightarrow 2y+2z+4=0 \\ &\Rightarrow 2y+2z=4 \\ &\Rightarrow \boxed{y=2-z} \end{aligned}$$

SUBSTITUTE INTO EQUATION  $\Rightarrow 0=0$

$$\begin{aligned} &\Rightarrow (2x+2y-4=0) + (2y+2z=4) + (2x+2z+4=0) \\ &\Rightarrow 2x+4y+2z-8=0 \end{aligned}$$

NO SUCH POINTS & THEREFORE NO STATIONARY POINTS

**Now the point  $P$  has stationary & normal parallel to the  $z$ -axis**

$$\begin{aligned} &\Rightarrow \frac{\partial f}{\partial x} = 2x+2y-4 \neq 0 \\ &\Rightarrow \frac{\partial f}{\partial y} = (2x+4y+2z) \neq 0 \\ &\Rightarrow \frac{\partial f}{\partial z} = (2y+2z+4) = 0 \end{aligned}$$

**SUBSTITUTE INTO THE EQUATION  $\Rightarrow (2x+2z+4=0)$**

$$\begin{aligned} &\Rightarrow (-4)^2 + 2(4)(2) - 4(-4) + 2(2)^2 + 2(2)(-4) - 8(2) - 2^2 + 4z = -56 \\ &\Rightarrow (4x)^2 + 2x(-2)(x+4) + 4(x+4)^2 + 2(x-1)^2 - 2x(2z-2) - z^2 + 4z = -56 \\ &\Rightarrow 16x^2 + 4x^2 + 16x^2 + 16x + 16x^2 + 32x + 16 = -56 \\ &\Rightarrow 64x^2 + 48x + 16 = -56 \\ &\Rightarrow 64x^2 + 48x + 32 = 0 \\ &\Rightarrow x^2 + 3x + 1 = 0 \\ &\Rightarrow (x+4)(x-1) = 0 \end{aligned}$$

$\therefore$  POSSIBLE POINTS ARE  $(0, -4, 0)$  OR  $(4, 0, -8)$

**But**  $2x+4z+2=k$

$$\begin{aligned} &\therefore k = \boxed{0+4(-4)+2} \\ &\therefore k = \boxed{4+2(4)+2} \end{aligned}$$

**Question 37 (\*\*\*\*)**

A surface  $S$  has equation  $f(x, y, z) = 0$ , where

$$f(x, y, z) = x^2 + 3y^2 + 2z^2 + 2yz + 6xz - 4xy - 24.$$

Show that the plane with equation

$$10x - y + 2z = 6,$$

is a tangent plane to  $S$ , and find the coordinates of the point of tangency.

(-2, -6, 10)

$f(x, y, z) = x^2 + 3y^2 + 2z^2 + 2yz + 6xz - 4xy - 24$

 $\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = [2x + 6z - 4y, 6y + 2z - 4x, 4z + 2y]$ 
 $\begin{cases} 2x + 6z - 4y = 10k \\ 6y + 2z - 4x = -k \\ 4z + 2y = 2k \end{cases} \Rightarrow \begin{cases} 1 & -2 & 3 & 10k \\ -2 & 3 & 1 & -4k \\ 3 & 1 & 2 & 2k \end{cases} \xrightarrow{\text{Row Reduction}} \begin{cases} 1 & -2 & 3 & 10k \\ 0 & 1 & -7 & -9k \\ 0 & 7 & -7 & -2k \end{cases} \xrightarrow{\text{Row Reduction}} \begin{cases} 1 & -2 & 3 & 10k \\ 0 & 1 & -7 & -9k \\ 0 & 0 & 1 & -\frac{1}{2}k \end{cases} \xrightarrow{\text{Row Reduction}} \begin{cases} 1 & -2 & 3 & 10k \\ 0 & 1 & -7 & -9k \\ 0 & 0 & 1 & -\frac{1}{2}k \end{cases}$ 

NOW SCALING THE COEFFICIENT VECTOR:

 $\begin{cases} 2x + 6z - 4y = 10k \\ 6y + 2z - 4x = -k \\ 4z + 2y = 2k \end{cases} \Rightarrow \begin{cases} 1 & -2 & 3 & 10k \\ 0 & 1 & -7 & -9k \\ 0 & 0 & 1 & -\frac{1}{2}k \end{cases} \xrightarrow{\text{Row Reduction}} \begin{cases} 1 & -2 & 3 & 10k \\ 0 & 1 & -7 & -9k \\ 0 & 0 & 1 & -\frac{1}{2}k \end{cases} \xrightarrow{\text{Row Reduction}} \begin{cases} 1 & -2 & 3 & 10k \\ 0 & 1 & -7 & -9k \\ 0 & 0 & 1 & -\frac{1}{2}k \end{cases}$ 

∴  $x = 5k$   
 $y = \frac{9}{2}k$   
 $z = -\frac{1}{2}k$   
 $x - 2y + 3z = 10k$   
 $2 + 3k + \frac{9}{2}k = 10k$   
 $2 = -\frac{5}{2}k$

∴  $k = 4$

∴  $x = 20$ ,  $y = 18$ ,  $z = -2$

Verify with surface first indeed. Put  $(x, y, z) = (20, 18, -2)$  into the surface  
 $f(20, 18, -2) = 20^2 + 3(18)^2 + 2(-2)^2 + 2(18)(-2) + 6(20)(-2) - 4(20)(18) - 24 = 0$

∴  $P(20, 18, -2)$  is point of tangency!

**Question 38 (\*\*\*\*)**

It is given that  $g$  is a twice differentiable function of one variable, with domain all real numbers.

It is further given that for  $x > 0$

$$f(x, y) = g(y \ln x).$$

Show that

$$x^2 \ln x \frac{\partial^2 f}{\partial x^2} - xy \frac{\partial^2 f}{\partial x \partial y} + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 0.$$

proof

$f(u) = g(\ln u)$        $g$  IS A FUNCTION OF ONE VARIABLE

Let  $u = y \ln x \Rightarrow \frac{\partial u}{\partial x} = \frac{y}{x}$   
 $\Rightarrow \frac{\partial u}{\partial y} = \ln x$

NEXT FIND THE REQUIRED DERIVATIVES OF  $f$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(g(u)) = \frac{dg}{du} \frac{\partial u}{\partial x} = g'(u) \times \frac{y}{x} = \frac{y}{x} g'(u) \\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(g(u)) = \frac{\partial g}{\partial u} \frac{\partial u}{\partial y} = g'(u) \times \ln x = g'(u) \ln x \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left[ \frac{y}{x} g'(u) \right] = -\frac{y}{x^2} \times g'(u) + \frac{y}{x} \times g''(u) \frac{\partial u}{\partial x} \\ &= -\frac{y}{x^2} g'(u) + \frac{y^2}{x^2} g''(u) \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left[ g'(u) \ln x \right] = g'(u) \frac{\partial}{\partial y} \ln x + g'(u) \times \frac{1}{x} \\ &= \frac{1}{x} g'(u) + \frac{y \ln x}{x} g''(u)\end{aligned}$$

VERIFY THE PARTIAL DIFFERENTIAL EQUATION

$$\begin{aligned}&x^2 \ln x \frac{\partial^2 f}{\partial x^2} - xy \frac{\partial^2 f}{\partial x \partial y} + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \\ &= x^2 \ln x \left[ -\frac{y}{x^2} g'(u) + \frac{y^2}{x^2} g''(u) \right] - xy \left[ \frac{1}{x} g'(u) + \frac{y \ln x}{x} g''(u) \right] + x \left[ \frac{y}{x} g'(u) \right] + y \left[ \frac{1}{x} g'(u) \right] \\ &= -y g'(u) + g' \ln x g(u) - y^2 g''(u) - y^2 \ln x g'(u) + y g'(u) + y \ln x g'(u) \\ &= 0\end{aligned}$$

**Question 39 (\*\*\*\*)**

The function  $w$  depends on  $x$  and  $y$  so that

$$w = f(u), \quad \text{and} \quad u = (x - x_0)(y - y_0),$$

where  $x_0$  and  $y_0$  are constants.

Show clearly that

$$\frac{\partial^2 w}{\partial x \partial y} = u \frac{\partial^2 f}{\partial u^2} + \frac{\partial f}{\partial u}.$$

[proof]

LET  $w = f(u)$   
 WHERE  $u = (x - x_0)(y - y_0)$

$$\begin{aligned} \frac{\partial^2 w}{\partial x \partial y} &= \frac{\partial}{\partial x} \left[ \frac{\partial w}{\partial y} \right] \\ &= \frac{\partial}{\partial x} \left[ \frac{\partial f}{\partial u} \right] \\ &= \frac{\partial}{\partial u} \left[ \frac{\partial f}{\partial u} \right] \quad \leftarrow \text{Product Rule} \\ &= \frac{\partial^2 f}{\partial u^2} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial u} \cdot \frac{\partial}{\partial u} \left( \frac{\partial u}{\partial y} \right) \\ &= \frac{\partial^2 f}{\partial u^2} \cdot \frac{\partial}{\partial x} (y - y_0) + \frac{\partial f}{\partial u} \cdot \frac{\partial}{\partial u} \left( \frac{\partial u}{\partial y} \right) \\ &= \frac{\partial^2 f}{\partial u^2} \cdot (y - y_0)(x - x_0) + \frac{\partial f}{\partial u} \times 1 \quad \leftarrow \begin{aligned} u &= (x - x_0)(y - y_0) \\ \frac{\partial u}{\partial x} &= 1 \times (y - y_0) \\ \frac{\partial u}{\partial y} &= (x - x_0) \\ \frac{\partial}{\partial u} \left( \frac{\partial u}{\partial y} \right) &= 1 \end{aligned} \\ &= u \frac{\partial^2 f}{\partial u^2} + \frac{\partial f}{\partial u} \end{aligned}$$

**Question 40 (\*\*\*\*)**

The functions  $f$  and  $G$  satisfy

$$G(r, \theta, \varphi) \equiv f[x(r, \theta, \varphi), y(r, \theta, \varphi), z(r, \theta, \varphi)],$$

where  $x$ ,  $y$  and  $z$  satisfy the standard Spherical Polar Coordinates transformation relationships

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta.$$

Use the chain rule for partial derivatives to show that

$$\left[ \frac{\partial G}{\partial r} \right]^2 + \left[ \frac{1}{r} \frac{\partial G}{\partial \theta} \right]^2 + \left[ \frac{1}{r \sin \theta} \frac{\partial G}{\partial \varphi} \right]^2 = \left[ \frac{\partial f}{\partial x} \right]^2 + \left[ \frac{\partial f}{\partial y} \right]^2 + \left[ \frac{\partial f}{\partial z} \right]^2.$$

[ ] , [ ] proof

$G(r, \theta, \varphi) = f[x(r, \theta, \varphi), y(r, \theta, \varphi), z(r, \theta, \varphi)]$

$x = r \sin \theta \cos \varphi$   
 $y = r \sin \theta \sin \varphi$   
 $z = r \cos \theta$

START BY OBTAINING SOME BASIC PARTIAL DERIVATIVES

- $\frac{\partial x}{\partial r} = \sin \theta \cos \varphi$
- $\frac{\partial x}{\partial \theta} = r \cos \theta \cos \varphi$
- $\frac{\partial x}{\partial \varphi} = -r \sin \theta \sin \varphi$
- $\frac{\partial y}{\partial r} = \sin \theta \sin \varphi$
- $\frac{\partial y}{\partial \theta} = r \cos \theta \sin \varphi$
- $\frac{\partial y}{\partial \varphi} = r \sin \theta \cos \varphi$
- $\frac{\partial z}{\partial r} = \cos \theta$
- $\frac{\partial z}{\partial \theta} = -r \sin \theta$
- $\frac{\partial z}{\partial \varphi} = 0$

BY THE OTHER RULES ONE THEN HAS

$$\begin{aligned} \frac{\partial G}{\partial r} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial r} \\ &= \frac{\partial f}{\partial x} \sin \theta \cos \varphi + \frac{\partial f}{\partial y} r \sin \theta \sin \varphi + \frac{\partial f}{\partial z} \cos \theta \\ \frac{\partial G}{\partial \theta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \theta} \\ &= \frac{\partial f}{\partial x} r \cos \theta \cos \varphi + \frac{\partial f}{\partial y} r \cos \theta \sin \varphi + \frac{\partial f}{\partial z} -r \sin \theta \\ \frac{\partial G}{\partial \varphi} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \varphi} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \varphi} \\ &= -\frac{\partial f}{\partial x} r \sin \theta \sin \varphi + \frac{\partial f}{\partial y} r \sin \theta \cos \varphi \end{aligned}$$

HENCE WE OBTAIN

$$\begin{aligned} &\left( \frac{\partial G}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial G}{\partial \theta} \right)^2 + \left( \frac{1}{r \sin \theta} \frac{\partial G}{\partial \varphi} \right)^2 \\ &= \frac{f_x^2}{r^2} \sin^2 \theta \cos^2 \varphi + \frac{f_y^2}{r^2} \sin^2 \theta \sin^2 \varphi + f_z^2 \cos^2 \theta \\ &\quad + 2 \frac{f_x}{r^2} f_y \sin^2 \theta \cos \theta \cos \varphi + 2 \frac{f_y}{r^2} f_z \sin^2 \theta \cos \theta - 2 f_z f_x \sin^2 \theta \sin \theta \cos \varphi \\ &\quad + \frac{1}{r^2} \left[ \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 + \left( \frac{\partial f}{\partial z} \right)^2 \right] \\ &\quad + 2 \frac{f_x}{r^2} f_y \sin^2 \theta \cos \theta \sin \theta \cos \varphi - 2 f_x f_y \sin^2 \theta \cos \theta \sin \theta \sin \varphi - 2 f_z f_x \sin^2 \theta \cos \theta \sin \theta \\ &\quad + \frac{1}{r^2} \left[ \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 + \left( \frac{\partial f}{\partial z} \right)^2 \right] \end{aligned}$$

DESCOURING AFTER THE CANCELLATIONS

$$\begin{aligned} &= f_x^2 \left[ \sin^2 \theta \cos^2 \varphi + \cos^2 \theta \cos^2 \theta + \sin^2 \theta \right] \\ &\quad + f_y^2 \left[ \sin^2 \theta \cos^2 \varphi + \cos^2 \theta \cos^2 \theta + \sin^2 \theta \right] + f_z^2 \left[ \cos^2 \theta \cos^2 \theta \right] \\ &\quad + 2 f_x f_y \left[ \sin^2 \theta \cos^2 \varphi + \cos^2 \theta \cos^2 \theta - \sin^2 \theta \right] \\ &\quad + 2 f_y f_z \left[ \sin^2 \theta \cos^2 \varphi - \cos^2 \theta \cos^2 \theta \right] \\ &\quad + 2 f_x f_z \left[ \sin^2 \theta \cos^2 \varphi - \cos^2 \theta \cos^2 \theta \right] \\ &\quad + 2 \frac{f_x^2}{r^2} \left[ \sin^2 \theta \cos^2 \varphi + \cos^2 \theta \cos^2 \theta \right] \end{aligned}$$

$$\begin{aligned} &= f_x^2 \left[ \frac{(\sin^2 \theta + \cos^2 \theta) \cos^2 \varphi + \sin^2 \theta}{r^2} \right] \\ &\quad + f_y^2 \left[ \frac{(\sin^2 \theta + \cos^2 \theta) \cos^2 \varphi + \cos^2 \theta}{r^2} \right] + f_z^2 \\ &\quad + 2 f_x f_y \left[ \frac{(\sin^2 \theta + \cos^2 \theta) \cos^2 \varphi - \sin^2 \theta}{r^2} \right] \\ &\quad - f_x^2 \left[ \frac{\cos^2 \theta + \sin^2 \theta}{r^2} \right] + f_y^2 \left[ \frac{\cos^2 \theta + \sin^2 \theta}{r^2} \right] + f_z^2 \\ &\quad + 2 f_x f_y \left[ \frac{\cos^2 \theta - \sin^2 \theta}{r^2} \right] \\ &\quad - 2 f_x f_z \left[ \frac{\cos^2 \theta - \sin^2 \theta}{r^2} \right] \\ &\quad = f_x^2 + f_y^2 + f_z^2 \\ &= \frac{f_x^2}{r^2} + \frac{f_y^2}{r^2} + \frac{f_z^2}{r^2} \end{aligned}$$

**Question 41** (\*\*\*\*+)

It is given that

$$z(x, y) = f(u, v),$$

so that

$$u = x^3 + y^3 \quad \text{and} \quad v = \frac{y}{x}.$$

- a) Use the chain rule to show that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 3u \frac{\partial f}{\partial u} \quad \text{and} \quad yx^3 \frac{\partial z}{\partial y} - xy^3 \frac{\partial z}{\partial x} = uv \frac{\partial f}{\partial v}.$$

- b) Hence show further that

$$v \frac{\partial x}{\partial u} = \frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial x}{\partial v} = -v^2 \frac{\partial y}{\partial v}.$$

proof

**a)**

$z = 2x^3y^3$

 $v = \frac{y}{x}$ 
 $\frac{\partial z}{\partial x} = 6x^2y^3$ 
 $\frac{\partial z}{\partial y} = 6x^3y^2$ 
 $\frac{\partial z}{\partial u} = 3u^2$ 
 $\frac{\partial z}{\partial v} = 3u^2v^2$ 
 $\frac{\partial f}{\partial u} = 3u^2$ 
 $\frac{\partial f}{\partial v} = 3uv^2$

**b) (i)**

 $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$ 
 $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$ 
 $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$ 
 $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$

**From Part (a): Rearrange the expressions to compare**

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \\ \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \end{aligned}$$

**From Part (b): Rearrange the expressions to compare**

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \end{aligned}$$

**Question 42** (\*\*\*)+

It is given that  $f$  and  $g$  are differentiable functions of one variable, with domain all real numbers.

It is further given that for  $x > 0$

$$F(x, y) = f\left[x^2 + y^2 + g(3x - 2y)\right].$$

If the function  $y = y(x)$  is a rearrangement of  $F(x, y) = 0$ , show that

$$\frac{dy}{dx} = \frac{\frac{3}{2} \frac{dg}{du} + 2x}{2 \frac{dg}{du} - 2y},$$

where  $u = 3x - 2y$

proof

F(u) =  $f(x^2 + y^2 + g(3x - 2y))$        $f$  is a function of 1 variable  
 $g$  is a function of 1 variable

Let  $F(x, y) = f(v)$ , where  $v = x^2 + y^2 + g(3x - 2y)$

$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial f}{\partial v} \left[ 2x + \frac{\partial g}{\partial u} \frac{\partial u}{\partial x} \right] = \frac{\partial f}{\partial v} \left[ 2x + \frac{3}{2} \frac{\partial g}{\partial u} \right]$

$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial f}{\partial v} \left[ 2y + \frac{\partial g}{\partial u} \frac{\partial u}{\partial y} \right] = \frac{\partial f}{\partial v} \left[ 2y - 2 \frac{1}{2} \frac{\partial g}{\partial u} \right]$

Now if  $F(x, y) = 0$ , then  $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\frac{\partial f}{\partial v} \left[ 2x + \frac{3}{2} \frac{\partial g}{\partial u} \right]}{\frac{\partial f}{\partial v} \left[ 2y - 2 \frac{1}{2} \frac{\partial g}{\partial u} \right]} \\ &= -\frac{\frac{3}{2} \frac{\partial g}{\partial u} + 2x}{2y - \frac{1}{2} \frac{\partial g}{\partial u}} \end{aligned}$$

**Question 43 (\*\*\*\*+)**

The surface  $S$  has Cartesian equation

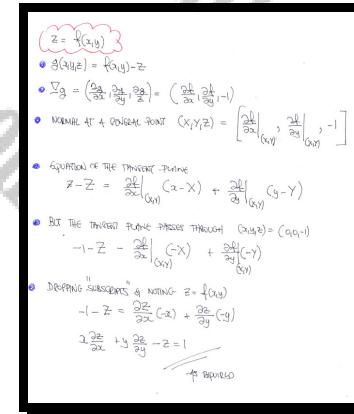
$$z = f(x, y).$$

The tangent plane at any point on  $S$  passes through the point  $(0, 0, -1)$ .

Show that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - z = 1.$$

proof



**Question 44 (\*\*\*\*\*)**

It is given that the function  $f$  depends on  $x$  and  $y$ , and the function  $g$  depends on  $u$  and  $v$ , so that

$$f(x, y) = g(u, v), \quad u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$$

a) Show that

$$\frac{\partial^2 f}{\partial x^2} = 2 \frac{\partial g}{\partial u} + 4x^2 \frac{\partial^2 g}{\partial u^2} + 8xy \frac{\partial^2 g}{\partial u \partial v} + 4y^2 \frac{\partial^2 g}{\partial v^2},$$

and find a similar expression for  $\frac{\partial^2 f}{\partial y^2}$ .

b) Deduce that if  $f(x, y) = x + y$

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0.$$

proof

a)  $f(x, y) = g(u, v)$  with  $u = x^2 - y^2$   
 $v = 2xy$

• START BY COMPUTING FIRST DERIVATIVES

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial g}{\partial u} = \frac{\partial g}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \frac{\partial v}{\partial x} = 2x \frac{\partial g}{\partial u} + 2y \frac{\partial g}{\partial v} \\ \frac{\partial f}{\partial y} &= \frac{\partial g}{\partial v} = \frac{\partial g}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \frac{\partial v}{\partial y} = -2y \frac{\partial g}{\partial u} + 2x \frac{\partial g}{\partial v}.\end{aligned}$$

• WRITE THESE ALSO IN OPERATOR FORM, AS WE WILL NEED THEM FOR THE SECOND DERIVATIVES

$$\begin{aligned}\frac{\partial f}{\partial x} &\equiv 2x \frac{\partial g}{\partial u} + 2y \frac{\partial g}{\partial v} \\ \frac{\partial f}{\partial y} &\equiv -2y \frac{\partial g}{\partial u} + 2x \frac{\partial g}{\partial v}.\end{aligned}$$

• NEXT THE SECOND DERIVATIVES

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left[ 2x \frac{\partial g}{\partial u} + 2y \frac{\partial g}{\partial v} \right] \\ &= 2 \frac{\partial}{\partial x} \left[ 2x \frac{\partial g}{\partial u} \right] + 2y \frac{\partial}{\partial x} \left[ \frac{\partial g}{\partial v} \right] \\ &= 2 \frac{\partial}{\partial x} \left[ 2x \frac{\partial g}{\partial u} + 2x \frac{\partial g}{\partial u} \right] + 2y \left[ 2 \frac{\partial g}{\partial u} + 2y \frac{\partial g}{\partial v} \right] \\ &= 2 \frac{\partial}{\partial x} \left[ 4x \frac{\partial g}{\partial u} \right] + 2y \frac{\partial g}{\partial u} + 2xy \frac{\partial g}{\partial v} + 4y^2 \frac{\partial g}{\partial v} \\ &= 2 \frac{\partial^2 g}{\partial u^2} + 4x^2 \frac{\partial^2 g}{\partial u^2} + 8xy \frac{\partial^2 g}{\partial u \partial v} + 4y^2 \frac{\partial^2 g}{\partial v^2}.\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial^2 g}{\partial v^2} = \frac{\partial^2 g}{\partial v^2} \left[ -2y \frac{\partial g}{\partial u} + 2x \frac{\partial g}{\partial v} \right] \\ &= -2 \frac{\partial}{\partial y} \left[ -2y \frac{\partial g}{\partial u} + 2x \frac{\partial g}{\partial v} \right] \\ &= -2 \frac{\partial}{\partial y} \left[ -2y \frac{\partial g}{\partial u} + 2x \frac{\partial g}{\partial v} \right] \left[ \frac{\partial g}{\partial u} + 2v \frac{\partial g}{\partial v} \right] \\ &= -2 \frac{\partial}{\partial y} \left[ -2y \frac{\partial g}{\partial u} + 2x \frac{\partial g}{\partial v} \right] \left[ -2y \frac{\partial g}{\partial u} + 4x \frac{\partial g}{\partial v} \right] \\ &= -2 \frac{\partial}{\partial y} \left[ -2y \frac{\partial g}{\partial u} + 2x \frac{\partial g}{\partial v} \right] \left[ -2y \frac{\partial g}{\partial u} + 4x \frac{\partial g}{\partial v} \right] \\ &= -2 \frac{\partial^2 g}{\partial u^2} - 8xy \frac{\partial^2 g}{\partial u \partial v} + 4x^2 \frac{\partial^2 g}{\partial v^2}.\end{aligned}$$

$$\begin{aligned}\bullet b) \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= \frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \\ &= 2 \frac{\partial^2 g}{\partial u^2} + 4x^2 \frac{\partial^2 g}{\partial u^2} + 8xy \frac{\partial^2 g}{\partial u \partial v} + 4y^2 \frac{\partial^2 g}{\partial v^2} \\ &\quad - 2 \frac{\partial^2 g}{\partial u^2} + 4y^2 \frac{\partial^2 g}{\partial v^2} - 8xy \frac{\partial^2 g}{\partial u \partial v} + 4x^2 \frac{\partial^2 g}{\partial v^2} \\ &= (4x^2 + 4y^2) \frac{\partial^2 g}{\partial u^2} + (4x^2 + 4y^2) \frac{\partial^2 g}{\partial v^2} \\ &= (4x^2 + 4y^2) \left[ \frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \right].\end{aligned}$$

• NOW IF  $f(x, y) = x + y \Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$

• HENCE

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= (4x^2 + 4y^2) \left[ \frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \right] \\ &\Rightarrow (4x^2 + 4y^2) \left[ \frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \right] \\ &\Rightarrow 0 = (4x^2 + 4y^2) \left[ \frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \right] \\ &\Rightarrow \frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0 \quad \text{if } 4x^2 + 4y^2 = 0\end{aligned}$$

**Question 45 (\*\*\*\*\*)**

The function  $z$  depends on  $x$  and  $y$  so that

$$z = f(u, v), \quad u = x - 2\sqrt{y} \quad \text{and} \quad v = x + 2\sqrt{y}.$$

Show that the partial differential equation

$$2 \frac{\partial^2 z}{\partial x^2} - 2y \frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = 0,$$

can be simplified to

$$\frac{\partial^2 z}{\partial u \partial v} = 0.$$

 , proof

<p><u>Start by preparing all the required expressions in order to substitution into the given PDE</u></p> <p>• <math>u = x - 2\sqrt{y}</math>  • <math>v = x + 2\sqrt{y}</math></p> $\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = -\frac{1}{\sqrt{y}}$ $\frac{\partial v}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = \frac{1}{\sqrt{y}}$ <p><u>Find <math>\frac{\partial z}{\partial u}</math> and <math>\frac{\partial z}{\partial v}</math> by differentiating <math>z = f(u, v)</math> with respect to <math>u</math> and <math>v</math> respectively.</u></p> $\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial u} \cdot 1 + \frac{\partial z}{\partial v} \cdot 0 = \frac{\partial z}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial u} \cdot 0 + \frac{\partial z}{\partial v} \cdot 1 = \frac{\partial z}{\partial v} \end{aligned}$ <p><u>Start with the first order derivatives</u></p> $\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \times 1 + \frac{\partial z}{\partial v} \times 1 = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} \left( -\frac{1}{\sqrt{y}} \right) + \frac{\partial z}{\partial v} \left( \frac{1}{\sqrt{y}} \right) = -\frac{1}{\sqrt{y}} \frac{\partial z}{\partial u} + \frac{1}{\sqrt{y}} \frac{\partial z}{\partial v} \end{aligned}$ $\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{\partial z}{\partial x} - \frac{1}{\sqrt{y}} \frac{\partial z}{\partial v} + \frac{1}{\sqrt{y}} \frac{\partial z}{\partial v} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial x} + \frac{1}{\sqrt{y}} \frac{\partial z}{\partial v} - \frac{1}{\sqrt{y}} \frac{\partial z}{\partial v} \end{aligned}$ <p><u>Next the second derivatives</u></p> $\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \left[ \frac{\partial}{\partial u} \frac{\partial z}{\partial u} + \frac{\partial}{\partial v} \frac{\partial z}{\partial v} \right] \left( \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \\ \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial u} \left( \frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial u} \left( \frac{\partial z}{\partial v} \right) + \frac{\partial}{\partial v} \left( \frac{\partial z}{\partial v} \right) \\ \frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial u} \left( \frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial u} \left( \frac{\partial z}{\partial v} \right) + \frac{\partial}{\partial v} \left( \frac{\partial z}{\partial v} \right) \\ \frac{\partial^2 z}{\partial v^2} &= \frac{\partial}{\partial v} \left( \frac{\partial z}{\partial v} \right) = \frac{\partial}{\partial u} \left( \frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{\partial z}{\partial v} \right) + \frac{\partial}{\partial u} \left( \frac{\partial z}{\partial v} \right) + \frac{\partial}{\partial v} \left( \frac{\partial z}{\partial v} \right) \end{aligned}$	<p><u>FINISH THE OTHER SECOND DERIVATIVES - OPERATOR FORM TO AVOID MISSING OUT TERMS AND PRODUCTS</u></p> $\begin{aligned} \frac{\partial^2 z}{\partial u^2} &= \frac{\partial}{\partial u} \left( \frac{\partial z}{\partial u} \right) = \left[ \frac{\partial}{\partial u} \frac{\partial z}{\partial u} + \frac{\partial}{\partial v} \frac{\partial z}{\partial u} \right] \left( -\frac{1}{\sqrt{y}} \frac{\partial z}{\partial u} + \frac{1}{\sqrt{y}} \frac{\partial z}{\partial v} \right) \\ &= \frac{1}{\sqrt{y}} \left[ \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} \right] + \frac{1}{\sqrt{y}} \left[ -\frac{1}{\sqrt{y}} \frac{\partial^2 z}{\partial u \partial v} + \frac{1}{\sqrt{y}} \frac{\partial^2 z}{\partial v^2} \right] \\ &+ \frac{1}{\sqrt{y}} \left[ \frac{\partial^2 z}{\partial u^2} \frac{\partial z}{\partial u} - \frac{1}{\sqrt{y}} \frac{\partial^2 z}{\partial u \partial v} \frac{\partial z}{\partial v} + \frac{1}{\sqrt{y}} \frac{\partial^2 z}{\partial v^2} \frac{\partial z}{\partial v} \right] \end{aligned}$ <p><u>FINISH AND REARRANGE</u></p> $\begin{aligned} \frac{\partial^2 z}{\partial u^2} &= -\frac{1}{\sqrt{y}} \frac{\partial^2 z}{\partial u \partial v} \frac{\partial z}{\partial u} + \frac{1}{\sqrt{y}} \frac{\partial^2 z}{\partial v^2} \frac{\partial z}{\partial v} + \frac{1}{\sqrt{y}} \frac{\partial^2 z}{\partial u^2} \frac{\partial z}{\partial u} - \frac{1}{\sqrt{y}} \frac{\partial^2 z}{\partial u \partial v} \frac{\partial z}{\partial v} + \frac{1}{\sqrt{y}} \frac{\partial^2 z}{\partial v^2} \frac{\partial z}{\partial v} \\ &= \frac{1}{\sqrt{y}} \left[ \frac{\partial^2 z}{\partial u^2} \right] \frac{\partial z}{\partial u} + \frac{1}{\sqrt{y}} \left[ \frac{\partial^2 z}{\partial v^2} \right] \frac{\partial z}{\partial v} + \frac{1}{\sqrt{y}} \left[ \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} \right] \frac{\partial z}{\partial u} - \frac{1}{\sqrt{y}} \left[ \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v^2} \right] \frac{\partial z}{\partial v} \end{aligned}$ <p><u>NEXT SUBSTITUTION THESE RESULTS INTO THE PDE</u></p> $\begin{aligned} 2 \frac{\partial^2 z}{\partial x^2} - 2y \frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} &= 0 \\ 2 \left[ \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right] - 2y \left[ \frac{\partial^2 z}{\partial u^2} \frac{\partial z}{\partial u} + \frac{\partial^2 z}{\partial v^2} \frac{\partial z}{\partial v} + \frac{1}{\sqrt{y}} \left( \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} \right) \frac{\partial z}{\partial u} - \frac{1}{\sqrt{y}} \left( \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v^2} \right) \frac{\partial z}{\partial v} \right] &= 0 \\ -2 \left[ \frac{\partial^2 z}{\partial u^2} \frac{\partial z}{\partial u} + \frac{\partial^2 z}{\partial v^2} \frac{\partial z}{\partial v} + \frac{1}{\sqrt{y}} \left( \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} \right) \frac{\partial z}{\partial u} - \frac{1}{\sqrt{y}} \left( \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v^2} \right) \frac{\partial z}{\partial v} \right] &= 0 \end{aligned}$ $\begin{aligned} \Rightarrow 2 \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{1}{\sqrt{y}} \left( \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} \right) \frac{\partial z}{\partial u} - \frac{2}{\sqrt{y}} \frac{\partial^2 z}{\partial u \partial v} \frac{\partial z}{\partial u} - \frac{2}{\sqrt{y}} \frac{\partial^2 z}{\partial v^2} \frac{\partial z}{\partial v} + \frac{1}{\sqrt{y}} \frac{\partial^2 z}{\partial u^2} \frac{\partial z}{\partial u} - \frac{1}{\sqrt{y}} \frac{\partial^2 z}{\partial v^2} \frac{\partial z}{\partial v} &= 0 \\ \Rightarrow 5 \frac{\partial^2 z}{\partial u^2} + \frac{1}{\sqrt{y}} \left( \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} \right) \frac{\partial z}{\partial u} - \frac{1}{\sqrt{y}} \left( \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} \right) \frac{\partial z}{\partial u} + \frac{1}{\sqrt{y}} \left( \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} \right) \frac{\partial z}{\partial u} &= 0 \\ \Rightarrow 5 \frac{\partial^2 z}{\partial u^2} + \frac{1}{\sqrt{y}} \left( \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} \right) \left( \frac{\partial z}{\partial u} - \frac{\partial z}{\partial u} \right) &= 0 \\ \Rightarrow 5 \frac{\partial^2 z}{\partial u^2} &= 0 \end{aligned}$ <p style="text-align: right;"><small>∴ <math>\frac{\partial^2 z}{\partial u^2} = 0</math> AS DESIRED</small></p>
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**Question 46 (\*\*\*\*\*)**

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

The above partial differential equation is Laplace's equation in a two dimensional Cartesian system of coordinates.

Show clearly that Laplace's equation in the standard two dimensional Polar system of coordinates is given by

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0.$$

proof

$\nabla^2 \phi = 0$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$\bullet \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial \theta} \frac{\partial \theta}{\partial x}$

$\bullet \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial \theta} \frac{\partial \theta}{\partial y}$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial \theta} \frac{\partial \theta}{\partial x} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial \theta} \frac{\partial \theta}{\partial x} \right) = \frac{\partial^2 \phi}{\partial r^2} \frac{\partial r}{\partial x} + \left( -\frac{\partial \phi}{\partial r} \times \frac{\partial^2 r}{\partial x^2} + \left( -\frac{\partial \phi}{\partial \theta} \times \frac{\partial^2 \theta}{\partial x^2} \right) \right) \frac{\partial \theta}{\partial x}$$

$$= \frac{\partial^2 \phi}{\partial r^2} \frac{\partial r}{\partial x} - \frac{\partial \phi}{\partial r} \frac{\partial^2 \theta}{\partial x^2}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial \theta} \frac{\partial \theta}{\partial y} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial \theta} \frac{\partial \theta}{\partial y} \right) = \frac{\partial^2 \phi}{\partial r^2} \frac{\partial r}{\partial y} + \left( -\frac{\partial \phi}{\partial r} \times \frac{\partial^2 r}{\partial y^2} + \left( -\frac{\partial \phi}{\partial \theta} \times \frac{\partial^2 \theta}{\partial y^2} \right) \right) \frac{\partial \theta}{\partial y}$$

$$= \frac{\partial^2 \phi}{\partial r^2} \frac{\partial r}{\partial y} - \frac{\partial \phi}{\partial r} \frac{\partial^2 \theta}{\partial y^2}$$

Now the second insulation

$\bullet \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial r} \right) = \left( \cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta} \right) \left( \cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta} \right)$

$$= \cos^2 \theta \left( \cos^2 \frac{\partial}{\partial r} \right) + \cos \theta \frac{\partial}{\partial r} \left( -\sin \theta \frac{\partial}{\partial \theta} \right) - \sin \theta \frac{\partial}{\partial r} \left( \cos \theta \frac{\partial}{\partial r} \right) - \sin^2 \theta \left( \sin^2 \frac{\partial}{\partial \theta} \right)$$

$$= \cos^2 \theta \frac{\partial^2 \phi}{\partial r^2} - \cos \theta \sin \theta \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial \theta} \right) - \sin \theta \cos \theta \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial r} \right) + \sin^2 \theta \frac{\partial^2 \phi}{\partial \theta^2}$$

Product rule      Product rule      Product rule

$$\begin{aligned} &= \cos^2 \theta \frac{\partial^2 \phi}{\partial r^2} - \cos \theta \sin \theta \frac{\partial}{\partial r} \left[ -\frac{1}{r} \frac{\partial \phi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] - \frac{\partial \phi}{\partial r} \left[ -\sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta} \right] \\ &\quad + \frac{\sin \theta}{r^2} \left[ \cos \theta \frac{\partial}{\partial r} + \sin \theta \frac{\partial}{\partial \theta} \right] \\ &= \cos^2 \theta \frac{\partial^2 \phi}{\partial r^2} + \frac{\cos \theta \sin \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} - \frac{-\cos \theta \sin \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} \\ &= \cos^2 \theta \frac{\partial^2 \phi}{\partial r^2} + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} \\ &\text{Now} \\ &\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial r} \right) = \left( \cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta} \right) \left( \cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta} \right) \\ &= \sin^2 \theta \left( \cos^2 \frac{\partial}{\partial r} \right) + \sin \theta \frac{\partial}{\partial r} \left( -\sin \theta \frac{\partial}{\partial \theta} \right) + \cos \theta \frac{\partial}{\partial r} \left( \cos \theta \frac{\partial}{\partial r} \right) + \cos^2 \theta \left( \sin^2 \frac{\partial}{\partial \theta} \right) \\ &= \sin^2 \theta \frac{\partial^2 \phi}{\partial r^2} + \sin \theta \cos \theta \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial \theta} \right) + \cos \theta \sin \theta \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial r} \right) + \cos^2 \theta \frac{\partial^2 \phi}{\partial \theta^2} \\ &\quad \swarrow \text{Product rule} \\ &= \sin^2 \theta \frac{\partial^2 \phi}{\partial r^2} + \sin \theta \cos \theta \frac{\partial}{\partial r} \left[ -\frac{1}{r} \frac{\partial \phi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] + \frac{\cos \theta}{r} \left[ \cos \theta \frac{\partial}{\partial r} + \sin \theta \frac{\partial}{\partial \theta} \right] \\ &\quad + \frac{\cos^2 \theta}{r^2} \left[ -\sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta} \right] \\ &= \sin^2 \theta \frac{\partial^2 \phi}{\partial r^2} - \sin \theta \cos \theta \frac{\partial}{\partial r} \left[ \frac{\partial \phi}{\partial \theta} \right] + \frac{\cos^2 \theta \sin \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} + \frac{\cos \theta \sin \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \\ &\quad - \frac{\cos^2 \theta \sin \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \\ &= \sin^2 \theta \frac{\partial^2 \phi}{\partial r^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{2 \cos \theta \sin \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} - \frac{2 \cos \theta \sin \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} \end{aligned}$$

$$\begin{aligned} &= \sin^2 \theta \frac{\partial^2 \phi}{\partial r^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{2 \cos \theta \sin \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} - \frac{-2 \cos \theta \sin \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} \\ &\text{Hence} \\ &\frac{\partial^2 \phi}{\partial y^2} = \cos^2 \theta \frac{\partial^2 \phi}{\partial r^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} - \frac{-\sin^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} \\ &\frac{\partial^2 \phi}{\partial y^2} = \sin^2 \theta \frac{\partial^2 \phi}{\partial r^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{-\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 \phi}{\partial r^2} \\ &\text{Add} \\ &\frac{\partial^2 \phi}{\partial y^2} = \left( \cos^2 \theta + \sin^2 \theta \right) \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \left( \cos^2 \theta + \sin^2 \theta \right) \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r^2} \left( \cos^2 \theta + \sin^2 \theta \right) \frac{\partial^2 \phi}{\partial r^2} \\ &\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial r^2} \end{aligned}$$

$$\begin{aligned} &\text{as to operator} \\ &\boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2}{\partial r^2}} \end{aligned}$$

**Question 47 (\*\*\*\*\*)**

It is given that for  $\varphi = \varphi(x, y)$  and  $\psi = \psi(x, y)$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Show that the above pair of coupled partial differential equations transform in plane polar coordinates to

$$\frac{\partial \varphi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad \text{and} \quad \frac{\partial \psi}{\partial r} = -\frac{1}{r} \frac{\partial \varphi}{\partial \theta}.$$

[proof]

Starting from the polar transformation equations

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \Rightarrow \begin{cases} x^2 + y^2 = r^2 \\ \tan \theta = \frac{y}{x} \end{cases} \Rightarrow r = \sqrt{x^2 + y^2} \quad \theta = \arctan(\frac{y}{x})$$

• EXPANDING THE PARTIAL DERIVATIVES

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{2}\sqrt{x^2+y^2} \frac{1}{2}x = \frac{x}{r^2} = \frac{r \cos \theta}{r^2} = \cos \theta \\ \frac{\partial}{\partial y} &= \frac{1}{2}\sqrt{x^2+y^2} \frac{1}{2}y = \frac{y}{r^2} = \frac{r \sin \theta}{r^2} = \sin \theta \end{aligned}$$

$$\theta = \arctan(\frac{y}{x})$$

$$\begin{aligned} \frac{\partial}{\partial x} &= -\frac{y}{x^2} \times \frac{1}{1+\frac{y^2}{x^2}} = -\frac{y}{x^2} \times \frac{x^2}{x^2+y^2} = -\frac{y}{x^2+y^2} = -\frac{y \cos \theta}{r^2} = -\frac{y \sin \theta}{r} \\ \frac{\partial}{\partial y} &= \frac{1}{x} \times \frac{1}{1+\frac{y^2}{x^2}} = \frac{1}{x} \times \frac{x^2}{x^2+y^2} = \frac{x}{r^2} = \frac{r \cos \theta}{r^2} = \cos \theta \end{aligned}$$

$$\begin{aligned} \bullet \quad \frac{\partial \varphi}{\partial x} &= \frac{\partial \varphi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \varphi}{\partial \theta} \frac{\partial \theta}{\partial x} = \frac{\partial \varphi}{\partial r} \cos \theta - \frac{\partial \varphi}{\partial \theta} \frac{\sin \theta}{r} \\ \frac{\partial \varphi}{\partial y} &= \frac{\partial \varphi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \varphi}{\partial \theta} \frac{\partial \theta}{\partial y} = \frac{\partial \varphi}{\partial r} \sin \theta + \frac{\partial \varphi}{\partial \theta} \frac{\cos \theta}{r} \\ \frac{\partial \psi}{\partial x} &= \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \psi}{\partial \theta} \frac{\partial \theta}{\partial x} = \frac{\partial \psi}{\partial r} \cos \theta - \frac{\partial \psi}{\partial \theta} \frac{\sin \theta}{r} \\ \frac{\partial \psi}{\partial y} &= \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \psi}{\partial \theta} \frac{\partial \theta}{\partial y} = \frac{\partial \psi}{\partial r} \sin \theta + \frac{\partial \psi}{\partial \theta} \frac{\cos \theta}{r} \end{aligned}$$

• BY THE "FIRST" GAUSS-RIEMANN EQUATION WE OBTAIN

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \varphi}{\partial r} \cos \theta - \frac{\partial \varphi}{\partial \theta} \frac{\sin \theta}{r} = \frac{\partial \psi}{\partial r} \sin \theta + \frac{\partial \psi}{\partial \theta} \frac{\cos \theta}{r}$$

• EXPANDING

$$\left[ \frac{\partial \varphi}{\partial r} \cos \theta - \frac{\partial \varphi}{\partial \theta} \frac{\sin \theta}{r} \right] - \left[ \frac{\partial \psi}{\partial r} \sin \theta + \frac{\partial \psi}{\partial \theta} \frac{\cos \theta}{r} \right] = 0$$

$$(\cos \theta) \left[ \frac{\partial \varphi}{\partial r} - \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right] - \sin \theta \left[ \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right] = 0$$

• FOR THE SECOND GAUSS-RIEMANN EQUATION, WE OBTAIN IN ANALOGY

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \psi}{\partial \theta} \frac{\partial \theta}{\partial x} = - \left[ \frac{\partial \psi}{\partial r} \cos \theta - \frac{\partial \psi}{\partial \theta} \frac{\sin \theta}{r} \right] \\ \frac{\partial \psi}{\partial y} &= \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \psi}{\partial \theta} \frac{\partial \theta}{\partial y} = \frac{\partial \psi}{\partial r} \sin \theta + \frac{\partial \psi}{\partial \theta} \frac{\cos \theta}{r} = 0 \\ \sin \theta \left[ \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right] + \cos \theta \left[ \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right] &= 0 \end{aligned}$$

• ELIMINATING  $\psi$  FROM

$$\begin{aligned} \cos \theta \left[ \frac{\partial \varphi}{\partial r} - \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right] - \sin \theta \left[ \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right] &= 0 \\ \sin \theta \left[ \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right] + \cos \theta \left[ \frac{\partial \varphi}{\partial r} - \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right] &= 0 \\ (\cos \theta + \sin \theta) \left[ \frac{\partial \varphi}{\partial r} - \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right] &= 0 \end{aligned}$$

AND

$$-\sin \theta \left[ \frac{\partial \varphi}{\partial r} - \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right] + \sin \theta \left[ \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right] = 0$$

$$\cos \theta \sin \theta \left[ \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right] + \cos \theta \left[ \frac{\partial \varphi}{\partial r} - \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right] = 0$$

$$\left[ \sin \theta + \cos \theta \right] \left[ \frac{\partial \varphi}{\partial r} - \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right] = 0$$

$$\frac{\partial \varphi}{\partial r} = -\frac{1}{r} \frac{\partial \varphi}{\partial \theta}$$