

INTEGRATION STRUCTURED EXAM QUESTIONS PART I

Question 1 (**)

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^2 \frac{1}{\sqrt{4x+1}} dx .$

b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3x dx .$

$\boxed{}, \boxed{1}, \boxed{-\frac{1}{3}}$

$$\begin{aligned} \text{(a)} \int_0^2 \frac{1}{\sqrt{4x+1}} dx &= \int_0^2 (4x+1)^{\frac{1}{2}} dx = \left[\frac{1}{2}(4x+1)^{\frac{1}{2}} \right]_0^2 = \left[\frac{1}{2}\sqrt{4x+1} \right]_0^2 \\ &= \frac{1}{2} \times 3 - \frac{1}{2} = \frac{3}{2} - \frac{1}{2} = 1 \end{aligned}$$

$$\begin{aligned} \text{(b)} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3x dx &= \left[\frac{1}{3} \sin 3x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{3} \sin \frac{\pi}{3} - \frac{1}{3} \sin \frac{\pi}{6} = -\frac{1}{3} \end{aligned}$$

Question 2 (**)

By using the substitution $u = 4 + 3x^2$, or otherwise, find

$$\int \frac{2x}{(4+3x^2)^2} dx .$$

$\boxed{}, \boxed{\frac{1}{3(4+3x^2)} + C}$

$$\begin{aligned} \int \frac{2x}{(4+3x^2)^2} dx &= \int \frac{2x}{u^2} \frac{du}{6x} \\ &= \int \frac{1}{3} u^{-2} du = -\frac{1}{3} u^{-1} + C \\ &= -\frac{1}{3} (4+3x^2)^{-1} + C = -\frac{1}{3(4+3x^2)} + C \end{aligned}$$

$$\begin{aligned} u &= 4+3x^2 \\ \frac{du}{dx} &= 6x \\ du &= 6x dx \\ dx &= \frac{du}{6x} \end{aligned}$$

Question 3 ()**

Show clearly that

$$\int_0^{\frac{1}{3}} x e^{3x} dx = \frac{1}{9}.$$

Q47, proof

$$\begin{aligned} \int_0^{\frac{1}{3}} x e^{3x} dx &= \dots \text{by parts if ignoring limits} \\ &= \frac{1}{3} x e^{3x} - \int \frac{1}{3} e^{3x} dx \\ &= \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C \\ &\dots \text{Limits...} \\ &= \left[\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} \right]_0^{\frac{1}{3}} = \left(\frac{1}{3} \cdot \frac{1}{3} e^1 - \frac{1}{9} e^1 \right) - (0 - \frac{1}{9}) = \frac{1}{9} \end{aligned}$$


Question 4 ()**

$$\frac{3x-5}{x-1} \equiv A + \frac{B}{x-1}.$$

a) Determine the value of each of the constants A and B .

b) Hence find

$$\int \frac{3x-5}{x-1} dx.$$

A = 3, B = -2, $|3x-2\ln|x-1|+C$

(a) $\frac{3x-5}{x-1} \equiv A + \frac{B}{x-1}$

$$\begin{aligned} 3x-5 &\equiv A(x-1) + B \\ 3x-5 &\equiv Ax - A + B \\ 3x-5 &\equiv Ax + B - A \\ 3x-5 &\equiv Ax + B - A \\ A &= 3x \\ B &= -A+5 \\ A &= 3 \\ B &= 2 \end{aligned}$$

ANSWER

$$\begin{aligned} \frac{3x-5}{x-1} &= \frac{3(x-1)-2}{x-1} \\ &= \frac{3(x-1)}{x-1} - \frac{2}{x-1} \\ &= 3 - \frac{2}{x-1} \\ \therefore A &= 3, B = 2 \end{aligned}$$

(b) $\int \frac{3x-5}{x-1} dx = \int 3 - \frac{2}{x-1} dx = 3x - 2\ln|x-1| + C$

Question 5 (**)

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^4 e^{\frac{1}{2}x} dx.$

b) $\int_0^{\frac{\pi}{4}} \cos\left(3x + \frac{\pi}{4}\right) dx.$

, $2(e^2 - 1)$, $-\frac{\sqrt{2}}{6}$

(a) $\int_0^4 e^{\frac{1}{2}x} dx = \left[2e^{\frac{1}{2}x} \right]_0^4 = 2e^2 - 2 = 2(e^2 - 1) //$
(b) $\int_0^{\frac{\pi}{4}} \cos(3x + \frac{\pi}{4}) dx = \left[\frac{1}{3} \sin(3x + \frac{\pi}{4}) \right]_0^{\frac{\pi}{4}} = \frac{1}{3} \sin \frac{\pi}{4} - \frac{1}{3} \sin \frac{\pi}{4} \\ = -\frac{1}{3} \times \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{6} //$

Question 6 (**)

$$\frac{5x+13}{(2x+1)(x+4)} \equiv \frac{A}{2x+1} + \frac{B}{x+4}.$$

- a) Determine the value of each of the constants A and B .
 b) Evaluate

$$\int_0^4 \frac{5x+13}{(2x+1)(x+4)} dx,$$

giving the answer as a single simplified natural logarithm.

, $A=3$, $B=1$, $\ln 54$

(a) $\frac{5x+13}{(2x+1)(x+4)} \equiv \frac{A}{2x+1} + \frac{B}{x+4}$

$$5x+13 \equiv A(x+4) + B(2x+1)$$

- If $x=-\frac{1}{2}$, $-7 = -7B \Rightarrow B=1$
- If $x=-4$, $21 = 7A \Rightarrow A=3$

(b) $\int_0^4 \frac{5x+13}{(2x+1)(x+4)} dx = \int_0^4 \frac{3}{2x+1} + \frac{1}{x+4} dx = \left[\frac{3}{2} \ln|2x+1| + \ln|x+4| \right]_0^4$

$$= \left(\frac{3}{2} \ln 9 + \ln 8 \right) - \left(\frac{3}{2} \ln 1 + \ln 4 \right)$$

$$= \ln 9^{\frac{3}{2}} + \ln 8 - \ln 4 = \ln 27 + \ln 8 - \ln 4$$

$$= \ln \left(\frac{27 \times 8}{4} \right) = \ln 54$$

Question 7 (**)

By using the substitution $u^2 = 1 - x^2$, or otherwise, show that

$$\int_0^1 5x(1-x^2)^{\frac{3}{2}} dx = 1.$$

, proof

$$\begin{aligned} \int_0^1 5x(1-x^2)^{\frac{3}{2}} dx &= \int_0^1 5x(u)^3 \frac{-u}{2} du \\ &= \int_{-1}^0 5u^4 du = \int_0^1 5u^4 du = \left[u^5 \right]_0^1 \\ &= 1 - 0 = 1 \end{aligned}$$

AS PREDICTED

$$\begin{cases} u = \sqrt{1-x^2}, \\ u^2 = 1-x^2 \\ 2u \frac{du}{dx} = -2x \\ 2u du = -2x dx \\ dx = \frac{u du}{x} \\ x=0, u=1 \\ x=1, u=0 \end{cases}$$

Question 8 ()**

Use integration by parts to find the value of

$$\int_0^{\frac{\pi}{4}} 4x \cos 4x \, dx.$$

, $\boxed{-\frac{1}{2}}$

$\int_0^{\frac{\pi}{4}} 4x \cos 4x \, dx = \dots \text{ by parts & ignoring limits}$ $= -x \sin 4x - \int \sin 4x \, dx$ $= -x \sin 4x + \frac{1}{4} \cos 4x + C$ $\dots \text{ Limits} \dots$ $= \left[-x \sin 4x + \frac{1}{4} \cos 4x \right]_0^{\frac{\pi}{4}}$ $= \left(\frac{\pi}{4} \sin \frac{\pi}{4} + \frac{1}{4} \cos \frac{\pi}{4} \right) - (0 + \frac{1}{4} \cos 0)$ $= -\frac{1}{4} - \left(\frac{1}{4} \right) = -\frac{1}{2}$	$\begin{array}{ c c } \hline 4x & 4 \\ \hline \frac{1}{4} \cos 4x & \cos 4x \\ \hline \end{array}$
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Question 9 ()**

By using the substitution $u = 1 - x^2$, or otherwise, find

$$\int \frac{12x}{(1-x^2)^{\frac{3}{2}}} \, dx.$$

, $\boxed{\frac{12}{\sqrt{1-x^2}} + C}$

$\int \frac{12x}{(1-x^2)^{\frac{3}{2}}} \, dx = \int \frac{12x}{u^{\frac{3}{2}}} \cdot \frac{du}{-2x}$ $= \int \frac{-6}{u^{\frac{3}{2}}} \, du = \int -6u^{-\frac{3}{2}} \, du$ $= [2u^{-\frac{1}{2}} + C] = \frac{12}{\sqrt{1-x^2}} + C$	$\begin{array}{ c c } \hline u & 1-x^2 \\ \hline \frac{du}{dx} & -2x \\ \hline \frac{du}{dx} & \frac{du}{-2x} \\ \hline \end{array}$
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Question 10 ()**

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^3 \frac{4}{2x+3} dx.$

b) $\int_0^{\frac{\pi}{6}} \sin\left(4x + \frac{\pi}{6}\right) dx.$

$\boxed{}, \boxed{\ln 9}, \boxed{\frac{\sqrt{3}}{4}}$

$$\begin{aligned} \text{(a)} \quad \int_0^3 \frac{4}{2x+3} dx &= \left[2 \ln|2x+3| \right]_0^3 = 2\ln 9 - 2\ln 3 \\ &= 2\ln 9 - \ln 9 = \ln 9 \quad (\text{or } 2\ln 3) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_0^{\frac{\pi}{6}} \sin\left(4x + \frac{\pi}{6}\right) dx &= \left[\frac{1}{4} \cos\left(4x + \frac{\pi}{6}\right) \right]_0^{\frac{\pi}{6}} = \frac{1}{4} \left[\cos\left(4x + \frac{\pi}{6}\right) \right]_0^{\frac{\pi}{6}} \\ &= \frac{1}{4} \left[\cos\frac{\pi}{6} - \cos\frac{5\pi}{6} \right] = \frac{1}{4} \left[\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right) \right] \\ &= \frac{1}{4} \times \sqrt{3} = \frac{\sqrt{3}}{4} \end{aligned}$$

Question 11 ()**

Use a trigonometric identity to integrate

$$\int \frac{1}{1 + \cos 2x} dx.$$

$\boxed{\tan x}, \boxed{\frac{1}{2} \tan x + C}$

$$\begin{aligned} \int \frac{1}{1 + \cos 2x} dx &= \int \frac{1}{1 + (2\cos^2 x - 1)} dx \\ &= \int \frac{1}{2\cos^2 x} dx \\ &= \int \frac{1}{2} \sec^2 x dx \\ &= \frac{1}{2} \tan x + C \end{aligned}$$

Question 12 (**)

$$\frac{30}{(x+3)(9-2x)} \equiv \frac{A}{x+3} + \frac{B}{9-2x}.$$

a) Determine the value of each of the constants A and B .

b) Evaluate

$$\int_1^4 \frac{30}{(x+3)(9-2x)} dx,$$

giving the answer as a single simplified natural logarithm.

$$\boxed{340}, \boxed{A=2}, \boxed{B=4}, \boxed{4\ln\left(\frac{7}{2}\right) = \ln\left(\frac{2401}{16}\right)}$$

(a)

$$\begin{aligned} \frac{30}{(x+3)(9-2x)} &\equiv \frac{A}{x+3} + \frac{B}{9-2x} \\ 30 &\equiv A(9-2x) + B(x+3) \\ \end{aligned}$$

* If $x=-3$, $30 = 15A \implies A=2$

* If $x=0$, $30 = 9A + 3B$
 $30 = 18 + 3B$
 $3B = 12$
 $B=4$

(b)

$$\begin{aligned} \int_1^4 \frac{30}{(x+3)(9-2x)} dx &= \int_1^4 \frac{2}{x+3} + \frac{4}{9-2x} dx = \left[2\ln|x+3| - 2\ln|9-2x| \right]_1^4 \\ &= (2\ln 7 - 2\ln 1) - (2\ln 4 - 2\ln 7) \\ &= 2\ln 7 - 2\ln 4 + 2\ln 7 = 4\ln 7 - 2\ln 4 \\ &= 4\ln 7 - 4\ln 2 = 4(\ln 7 - \ln 2) \\ &= 4\ln\left(\frac{7}{2}\right) \checkmark \end{aligned}$$

Question 13 ()**

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^{\frac{1}{3}} e^{-3x} dx.$

b) $\int_0^{\frac{\pi}{4}} \sin\left(2x + \frac{\pi}{4}\right) dx.$

$$\boxed{\frac{1}{3}(1-e^{-1})}, \boxed{\frac{\sqrt{2}}{2}}$$

$$\begin{aligned} \text{(a)} \quad \int_0^{\frac{1}{3}} e^{-3x} dx &= \left[-\frac{1}{3} e^{-3x} \right]_0^{\frac{1}{3}} = \frac{1}{3} \left[e^{-3x} \right]_0^{\frac{1}{3}} = \frac{1}{3} \left(1 - e^{-1} \right) \\ &\quad \frac{1}{3}(1-e^{-1}) \\ \text{(b)} \quad \int_0^{\frac{\pi}{4}} \sin(2x + \frac{\pi}{4}) dx &= \left[-\frac{1}{2} \cos(2x + \frac{\pi}{4}) \right]_0^{\frac{\pi}{4}} = \frac{1}{2} \left[\cos(2x + \frac{\pi}{4}) \right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2} \left[\cos(\frac{\pi}{2}) - \cos(\frac{3\pi}{4}) \right] = \frac{1}{2} \left[\frac{\sqrt{2}}{2} - (-\frac{\sqrt{2}}{2}) \right] \\ &= \frac{1}{2} \times \sqrt{2} = \frac{\sqrt{2}}{2} \end{aligned}$$

Question 14 (+)**

By using the substitution $u^2 = 16 - 7x^2$, or otherwise, show that

$$\int_0^1 \frac{x}{\sqrt{16-7x^2}} dx = \frac{1}{7}.$$

, proof

$$\begin{aligned} \int_0^1 \frac{x}{\sqrt{16-7x^2}} dx &= \int_4^3 \frac{x}{u} \left(-\frac{1}{7} \frac{u}{x} \right) du \\ &= \int_4^3 -\frac{1}{7} du = \int_{-3}^4 \frac{1}{7} du = \left[\frac{1}{7} u \right]_3^{-3} \\ &= \frac{3}{7} - \frac{3}{7} = \frac{1}{7} \quad \text{as required} \end{aligned}$$

$\frac{u^2 - 16 + 7x^2}{2u \frac{du}{dx}} = -\frac{1}{7}x$
 $2u \frac{du}{dx} = -\frac{1}{7}x$
 $dx = -\frac{14}{7} \frac{du}{x}$
 $x = 0 \Rightarrow u = 4$
 $x = 1 \Rightarrow u = 3$

Question 15 (**+)

Determine the value of the positive constant k given further that

$$\int_k^8 \frac{4}{2x-1} dx = 1.90038.$$

Give the value of k to an appropriate degree of accuracy.

, $k \approx 3.4$

INTEGRATE FIRST

$$\int_k^8 \frac{4}{2x-1} dx = \left[4\ln|2x-1| \times \frac{1}{2} \right]_k^8 = \left[2\ln(2x-1) \right]_k^8 \\ = 2\ln(5) - 2\ln(2k-1)$$

NOW SOLVE THE EQUATION

$$\Rightarrow \int_k^8 \frac{4}{2x-1} dx = 1.90038 \\ \Rightarrow 2\ln(5) - 2\ln(2k-1) = 1.90038 \\ \Rightarrow 2\ln(5) - 1.90038 = 2\ln(2k-1) \\ \Rightarrow \ln(2k-1) = 1.75196... \\ \Rightarrow 2k-1 = e^{1.75196...} \\ \Rightarrow 2k-1 \approx 5.80003... \\ \Rightarrow k \approx 3.400006... \\ \therefore k \approx 3.4 //$$

Question 16 (**+)

By using the substitution $u = 1 + 4\ln x$, or otherwise, find

$$\int \frac{4}{x(1+4\ln x)^2} dx.$$

, $\frac{1}{1+4\ln x} + C$

$$\int \frac{4}{x(1+4\ln x)^2} dx = \int \frac{4}{xu^2} \frac{du}{dx} dx \\ = \int \frac{4}{xu^2} du = \int u^{-2} du = -u^{-1} + C \\ = -\frac{1}{u} + C = -\frac{1}{1+4\ln x} + C$$

$$u = 1 + 4\ln x$$

$$\frac{du}{dx} = \frac{4}{x}$$

$$4du = x du$$

$$du = \frac{x}{4} du$$

Question 17 (**+)

$$\frac{8x}{4x-3} \equiv A + \frac{B}{4x-3}.$$

- a) Determine the value of each of the constants A and B .
 b) Hence, or otherwise, evaluate

$$\int_1^3 \frac{8x}{4x-3} dx,$$

giving the answer in terms of natural logarithms.

A = 2, B = 6, 4 + 3ln 3

$(a) \quad \frac{8x}{4x-3} \equiv A + \frac{B}{4x-3}$ $8x \equiv A(4x-3) + B$ $8x \equiv 4Ax - 3A + B$ $\therefore 4A = 8 \quad \begin{matrix} -3A + B = 0 \\ A = 2 \end{matrix}$ $\therefore A = 2 \quad \begin{matrix} -6 + B = 0 \\ B = 6 \end{matrix}$	ALTERNATIVE $\frac{8x}{4x-3} = \frac{2(4x-3)+6}{4x-3} = 2 + \frac{6}{4x-3} \quad \therefore A = 2, B = 6$
$(b) \quad \int_1^3 \frac{8x}{4x-3} dx = \int_1^3 2 + \frac{6}{4x-3} dx = \left[2x + \frac{3}{2} \ln 4x-3 \right]_1^3$ $= \left[6 + \frac{3}{2} \ln 7 \right] - \left[2 + \frac{3}{2} \ln 1 \right] = 4 + \frac{3}{2} \ln 7$ $= 4 + \frac{3}{2} \ln 3^2$ <p style="color: green;">(NOTE THAT THE SUBSTITUTION U = 4x-3 WILL ALSO WORK)</p>	

Question 18 (**+)

Use an appropriate integration method to find

$$\int (x+1)e^{x+1} dx.$$

 , $xe^{x+1} + C$

$\int (x+1)e^{x+1} dx = \dots \text{ BY PARTS } \dots$ $= (x+1)e^{x+1} - \int e^{x+1} dx$ $= (x+1)e^{x+1} - e^{x+1} + C$ $= e^{x+1}((x+1)-1) + C$ $= xe^{x+1} + C$	$\begin{array}{c c} x+1 & 1 \\ \hline e^{x+1} & \end{array}$
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Question 19 (**+)

$$f(x) = 4x e^{2x}.$$

a) Use integration by parts to find $\int f(x) dx$.

b) Find an exact value for $\int_0^{\ln 2} f(x) dx$.

, $[2x e^{2x} - e^{2x} + C]$, $[-3 + 8 \ln 2]$

$(a) \int 4x e^{2x} dx = \dots \text{ by parts}$ $= ((4x)(e^{2x}) - \int 2e^{2x} dx$ $= 2xe^{2x} - e^{2x} + C$	
$(b) \int_0^{\ln 2} 4x e^{2x} dx = \left[2xe^{2x} - e^{2x} \right]_0^{\ln 2} =$ $= [2(\ln 2)e^{2\ln 2} - e^{2\ln 2}] - [0 - 1]$ $= 8\ln 2 - 4 + 1$ $= -3 + 8\ln 2$	

Question 20 (**+)

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^1 \frac{9}{(2x+1)^2} dx$.

b) $\int_{\pi/6}^{\pi/3} \sin\left(4x + \frac{\pi}{6}\right) dx$.

$[3]$, $-\frac{\sqrt{3}}{8}$

$(a) \int_0^1 \frac{9}{(2x+1)^2} dx = \int_0^1 9(2x+1)^{-2} dx = \left[-\frac{9}{2}(2x+1)^{-1} \right]_0^1 = \left[\frac{9}{2x+1} \right]_0^1 =$ $= \frac{9}{2} \left[\frac{1}{2x+1} \right]_1^0 = \frac{9}{2} \left(1 - \frac{1}{3} \right) = \frac{9}{2} \times \frac{2}{3} = 3$
$(b) \int_{\pi/6}^{\pi/3} \sin\left(4x + \frac{\pi}{6}\right) dx = \left[-\frac{1}{4} \cos\left(4x + \frac{\pi}{6}\right) \right]_{\pi/6}^{\pi/3} = -\frac{1}{4} \left[\cos\left(4x + \frac{\pi}{6}\right) \right]_{\pi/6}^{\pi/3}$ $= -\frac{1}{4} \left[\cos\left(\frac{8\pi}{6} + \frac{\pi}{6}\right) - \cos\left(\frac{2\pi}{6} + \frac{\pi}{6}\right) \right] = -\frac{1}{4} \left(-\frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{8}$

Question 21 (+)**

By using the substitution $u = \ln x$, or otherwise, find an exact value for

$$\int_e^3 \frac{1}{x \ln x} dx.$$

$$\boxed{\ln(\ln 3)}$$

$$\begin{aligned} \int_e^3 \frac{1}{x \ln x} dx &= \int_1^{\ln 3} \frac{1}{x u} x du = \int_1^{\ln 3} \frac{1}{u} du \\ [\ln(u)]_1^{\ln 3} &= \ln(\ln 3) - \ln 1 = \ln(\ln 3) \end{aligned}$$

$\left\{ \begin{array}{l} u = \ln x \\ \frac{du}{dx} = \frac{1}{x} \\ dx = x du \\ x=e, u=\ln 3 \end{array} \right.$

Question 22 (**+)

$$f(x) \equiv \frac{x-5}{x^2 + 5x + 4}.$$

a) Express $f(x)$ in partial fractions.

b) Find the value of

$$\int_0^2 f(x) \, dx,$$

giving the answer as a single simplified logarithm.

□	$f(x) \equiv \frac{3}{x+4} - \frac{2}{x+1}$	$\int_0^2 f(x) \, dx = \ln\left(\frac{3}{8}\right)$
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(a) $\frac{x-5}{x^2 + 5x + 4} = \frac{x-5}{(x+1)(x+4)} \equiv \frac{A}{x+1} + \frac{B}{x+4}$

$[x-5 \equiv A(x+4) + B(x+1)]$

If $x= -4$, $-9 = -3B \Rightarrow B=3$
If $x=-1$, $-6 = 3A \Rightarrow A=-2$.

$\therefore f(x) = \frac{-2}{x+4} - \frac{3}{x+1}$

(b) $\int_0^2 f(x) \, dx = \int_0^2 \frac{-2}{x+4} - \frac{3}{x+1} \, dx = [3\ln|x+4| - 2\ln|x+1|]^2$

$= (3\ln 6 - 2\ln 3) - (3\ln 4 - 2\ln 1) = 3\ln 6 - 2\ln 3 - 3\ln 4$
 $= \ln 216 - \ln 9 - \ln 64 = \ln\left(\frac{216}{9 \times 64}\right) = \ln\left(\frac{3}{8}\right)$

Question 23 (**+)

By using the substitution $u^2 = 4\cos x - 1$, or otherwise, find

$$\int \frac{\sin x}{\sqrt{4\cos x - 1}} \, dx.$$

$-\frac{1}{2}\sqrt{4\cos x - 1} + C$

$$\begin{aligned} \int \frac{\sin x}{\sqrt{4\cos x - 1}} \, dx &= \int \frac{\sin x}{\sqrt{4} \sqrt{\cos x - \frac{1}{4}}} \, dx \\ &= \int -\frac{1}{2} \frac{du}{\sqrt{u^2 - \frac{1}{4}}} = -\frac{1}{2} u + C \\ &= -\frac{1}{2} \sqrt{4\cos x - 1} + C \end{aligned}$$

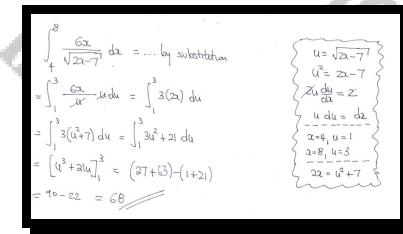
$$\begin{aligned} u &= \sqrt{4\cos x - 1} \\ u^2 &= 4\cos x - 1 \\ 2u \frac{du}{dx} &= -4\sin x \\ du &= \frac{-2\sin x}{4\cos x - 1} \, dx \\ du &= \frac{\sin x}{2\cos x - \frac{1}{2}} \, dx \end{aligned}$$

Question 24 (+)**

Use the substitution $u = \sqrt{2x-7}$ to find

$$\int_4^8 \frac{6x}{\sqrt{2x-7}} dx.$$

 [68]



Working for Question 24:

$$\begin{aligned} & \int_4^8 \frac{6x}{\sqrt{2x-7}} dx = \dots \text{by substitution} \\ & = \int_1^3 \frac{6u}{\sqrt{u^2-7}} \cdot 2u du = \int_1^3 3(2u) du \\ & = \int_1^3 3(3u^2) du = \int_1^3 9u^2 + 21 du \\ & = \left[3u^3 + 21u \right]_1^3 = (27 + 54) - (3 + 21) \\ & = 75 - 22 = 68 \end{aligned}$$

Question 25 (+)**

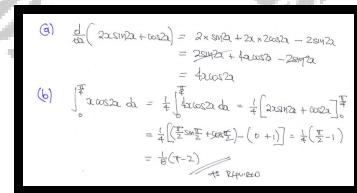
$$y = 2x \sin 2x + \cos 2x, \quad x \in \mathbb{R}$$

- a) Find an expression for $\frac{dy}{dx}$.

- b) Hence show that

$$\int_0^{\frac{\pi}{4}} x \cos 2x dx = \frac{1}{8}(\pi - 2).$$

$$\frac{dy}{dx} = 4x \cos 2x$$



Working for Question 25:

$$\begin{aligned} (a) \quad \frac{dy}{dx} (2x \sin 2x + \cos 2x) &= 2x \sin 2x + 2x \cdot 2 \cos 2x - 2 \sin 2x \\ &= 2x \sin 2x + 4x \cos 2x - 2 \sin 2x \\ &= 4x \cos 2x \\ (b) \quad \int_0^{\frac{\pi}{4}} 4x \cos 2x dx &= \frac{1}{2} \int_0^{\frac{\pi}{4}} 8x \cos 2x dx = \frac{1}{2} \left[2x \sin 2x + \cos 2x \right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2} \left[\left(\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2} \right) - (0 + 1) \right] = \frac{1}{2} \left(\frac{\pi}{2} - 1 \right) \\ &= \frac{1}{8}(\pi - 2) \end{aligned}$$

Question 26 (+)**

Determine the value of the positive constant k given further that

$$\int_k^{\frac{1}{2}} \frac{6}{e^{2-3x}} dx = 0.1998.$$

Give the value of k to an appropriate degree of accuracy.

, $k \approx 0.44$

CARRY OUT THE INTEGRATION FIRST

$$\begin{aligned} \int_k^{\frac{1}{2}} \frac{6}{e^{2-3x}} dx &= \int_k^{\frac{1}{2}} 6 \times e^{-(2-3x)} dx = \int_k^{\frac{1}{2}} 6e^{3x-2} dx \\ &= \left[2e^{3x-2} \right]_k^{\frac{1}{2}} = 2e^{-\frac{1}{2}} - 2e^{3k-2}. \end{aligned}$$

NOW SETTING UP AN EQUATION

$$\begin{aligned} \Rightarrow \int_k^{\frac{1}{2}} \frac{6}{e^{2-3x}} dx &= 0.1998 \\ \Rightarrow 2\left(e^{-\frac{1}{2}} - 2e^{3k-2}\right) &= 0.1998 \\ \Rightarrow \frac{1}{\sqrt{e}} - 2e^{3k-2} &= 0.0999 \\ \Rightarrow \frac{1}{\sqrt{e}} - 0.0999 &= 2e^{3k-2} \\ \Rightarrow e^{3k-2} &= 0.806620097... \\ \Rightarrow 3k-2 &= \ln(0.8066...) \\ \Rightarrow k &= 0.440008932... \end{aligned}$$

$\therefore k \approx 0.44$

Question 27 (***)

$$y = \frac{3x}{2+x-x^2}$$

- a) Calculate the three missing values of y in the following table.

x	0	0.25	0.5	0.75	1
y	0				1.5

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate for

$$\int_0^1 \frac{3x}{2+x-x^2} dx.$$

- c) Use a suitable method to find the exact value of

$$\int_0^1 \frac{3x}{2+x-x^2} dx.$$

, [0.3429] , [0.6667] , [1.0286] , [ln 2]

a)

x	0	0.25	0.5	0.75	1
y	0	0.3429	0.6667	1.0286	1.5

b)

$$\int_0^1 \frac{3x}{2+x-x^2} dx \approx \frac{\text{TRAPEZIUM}}{2} \left[\text{FIRST+LAST} + 2(\text{SECOND+THIRD+...}) \right]$$

$$\approx \frac{5.25}{2} [0 + 1.5 + 2(0.3429 + 0.6667 + 1.0286)]$$

$$\approx 0.697 - 0.698$$

c)

$$\int_0^1 \frac{3x}{2+x-x^2} dx = \int_1^0 \frac{3x}{x^2-3x+2} dx$$

$$= \int_1^0 \frac{3x}{(x-2)(x-1)} dx$$

PERFORM BY PARTIAL FRACTIONS

$$\frac{3x}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

$$3x = A(x-1) + B(x-2)$$

- * IF $x=1 \Rightarrow -3 = -3B \Rightarrow B=1$
- * IF $x=2 \Rightarrow 6 = 3A \Rightarrow A=2$

RETURNING TO THE INTEGRAL

$$\dots = \int_1^0 \frac{2}{x-2} + \frac{1}{x-1} dx$$

$$= \left[2\ln|x-2| + \ln|x-1| \right]_1^0$$

$$= \left[2\ln|-2| + \ln|1| \right] - \left[2\ln|-1| + \ln 2 \right]$$

$$= 2\ln 2 - \ln 2$$

$$= \ln 2 \approx 0.693$$

Question 28 (***)

Use a suitable substitution to find

$$\int \frac{30x}{\sqrt{1-2x}} dx.$$

$$5(1-2x)^{\frac{3}{2}} - 15(1-2x)^{\frac{1}{2}} + C$$

$\int \frac{30x}{\sqrt{1-2x}} dx = \dots$ by substituting

$$\begin{aligned} &= \int \frac{30x}{u} (-2du) = \int -30x du \\ &= \int -(5(2x)) du = \int -15(1-u^2) du \\ &= \int 15u^2 - 15 du = 5u^3 - 15u + C \\ &= 5(1-2x)^{\frac{3}{2}} - 15(1-2x)^{\frac{1}{2}} + C \end{aligned}$$

ALTERNATIVE SUBSTITUTION

$$\begin{aligned} &\int \frac{30x}{\sqrt{1-2x}} dx = \dots \int \frac{30x}{\sqrt{u^2}} \left(-\frac{du}{2}\right) \\ &= \int \frac{15(2x)}{u^2} \times \frac{1}{2} du = \int -\frac{15(1-u)}{2u^2} du \\ &= \int \frac{15u-15}{2u^2} du = \int \frac{15}{2} u^{\frac{1}{2}} - \frac{15}{2} u^{-\frac{1}{2}} du \\ &= \frac{15}{2} u^{\frac{3}{2}} - \frac{15}{2} u^{\frac{1}{2}} + C = 5u^{\frac{3}{2}} - 15u^{\frac{1}{2}} + C \\ &= 5(1-2x)^{\frac{3}{2}} - 15(1-2x)^{\frac{1}{2}} + C \end{aligned}$$

Question 29 (***)

$$\frac{x^2+3}{x-1} \equiv Ax+B+\frac{C}{x-1}.$$

a) Determine the value of each of the constants A , B and C .

b) Hence, or otherwise, evaluate

$$\int_2^4 \frac{x^2+3}{x-1} dx,$$

giving the answer in terms of natural logarithms.

 , $[A=1]$, $[B=1]$, $[C=4]$, $[8+4\ln 3]$

a) BY LONG DIVISION OR MANIPULATION

$$\frac{x^2+3}{x-1} = \frac{Ax-1 + (x-1) + 4}{x-1} = x+1 + \frac{4}{x-1}$$

$\cancel{A=1}$
 $\cancel{B=1}$
 $\cancel{C=4}$

ALTERNATIVE BY COMPARING COEFFICIENTS

$$\begin{aligned}\frac{x^2+3}{x-1} &\equiv Ax+B+\frac{C}{x-1} \\ \frac{x^2+3}{x-1} &\equiv \frac{Ax(x-1)+B(x-1)+C}{x-1} \\ x^2+3 &\equiv Ax^2+(B-A)x+C-C \\ A=1 & \quad B-A=0 \quad C-B=3 \\ B=1 & \quad C=1-C=3 \\ B=1 & \quad C=4 \quad \cancel{C=4}\end{aligned}$$

b) USING PART (a) WE HAVE

$$\begin{aligned}\int_2^4 \frac{x^2+3}{x-1} dx &= \int_2^4 x+1 + \frac{4}{x-1} dx = \left[\frac{1}{2}x^2+x+4\ln|x-1| \right]_2^4 \\ &= (8+4\ln 3) - (2+2+4\ln 1) \\ &= 8+4\ln 3\end{aligned}$$

Question 30 (*)**

Use the substitution $u = 1 + 2\cos x$ to find

$$\int_0^{\frac{\pi}{2}} (1+2\cos x)^3 \sin x \, dx.$$

[10]

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} (1+2\cos x)^3 \sin x \, dx \dots \text{BY THE SUBSTITUTION GIVEN} \\ &= \int_1^3 u^3 \sin x \left(-\frac{du}{2\sin x} \right) \\ &= \int_1^3 -\frac{1}{2} u^3 \, du \\ &= \left[-\frac{1}{8} u^4 \right]_1^3 = \frac{81}{8} - \frac{1}{8} = 10 \end{aligned}$$

$u = 1 + 2\cos x$
 $\frac{du}{dx} = -2\sin x$
 $dx = -\frac{du}{2\sin x}$
 $2x = \frac{\pi}{2}$
 $u = 3$
 $2x = \frac{\pi}{2}$
 $u = 1$

Question 31 (*)**

By using the substitution $u = 3x+1$, or otherwise, find

$$\int_0^5 x\sqrt{3x+1} \, dx.$$

$$\frac{204}{5} = 40.8$$

$$\begin{aligned} & \int_0^5 x\sqrt{3x+1} \, dx = \int_1^{16} x u^{\frac{1}{2}} \frac{du}{3} \\ &= \int_1^{16} \frac{1}{3} x u^{\frac{1}{2}} \, du = \int_1^{16} \frac{1}{3} (3u) u^{\frac{1}{2}} \, du \\ &= \int_1^{16} \frac{1}{3} (u-1) u^{\frac{1}{2}} \, du = \int_1^{16} \frac{1}{3} u^{\frac{3}{2}} - \frac{1}{3} u^{\frac{1}{2}} \, du \\ &= \left[\frac{2}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{3}{2}} \right]_1^{16} = \left(\frac{2048}{5} - \frac{128}{3} \right) \left(\frac{32}{5} - \frac{2}{3} \right) \\ &= \frac{204}{5} \end{aligned}$$

$u = 3x+1$
 $\frac{du}{dx} = 3$
 $dx = \frac{du}{3}$
 $2x+1 = u+1$
 $2x = u-1$
 $x = \frac{u-1}{2}$

Question 32 (*)**

Use an appropriate integration method to find an exact value for

$$\int_0^{\frac{\pi}{3}} 6x \sin 3x \, dx.$$

, $\frac{2\pi}{3}$

$$\begin{aligned}
 \int_0^{\frac{\pi}{3}} 6x \sin 3x \, dx &= \dots \text{ by parts ignoring limits} \\
 &= 6x \left(-\frac{1}{3} \cos 3x \right) - \int -2x \cos 3x \, dx \\
 &= -2x \cos 3x + \int 2x \cos 3x \, dx \\
 &= -2x \cos 3x + \frac{2}{3} \sin 3x + C \\
 &\quad \text{Introducing limits back} \\
 &= \left[-2x \cos 3x + \frac{2}{3} \sin 3x \right]_0^{\frac{\pi}{3}} \\
 &= \left[-2x \cos \pi + \frac{2}{3} \sin \pi \right] - [0 + 0] \\
 &= -\frac{2\pi}{3} + 0 \\
 &= -\frac{2\pi}{3}
 \end{aligned}$$

Question 33 (*)**

By using the substitution $u = \sec x$, or otherwise, find

$$\int \tan x \sec^4 x \, dx.$$

$\frac{1}{4} \sec^4 x + C$

$$\begin{aligned}
 \int \tan x \sec^4 x \, dx &= \dots \text{ by substitution} \\
 &\quad (\text{OR DIVIDE BY } \sec^2 x) \\
 &= \int \tan x u^4 \frac{du}{\sec^2 x} = \int u^4 \times \frac{du}{u} \\
 &= \int u^3 \, du = \frac{1}{4} u^4 + C = \frac{1}{4} \sec^4 x + C
 \end{aligned}$$

Question 34 (***)

$$\frac{3x^3 + 2x^2 - 3x + 8}{x+2} \equiv Ax^2 + Bx + C + \frac{D}{x+2}.$$

- a) Find the value of each of the constants A , B , C and D .
- b) Hence find

$$\int \frac{3x^3 + 2x^2 - 3x + 8}{x+2} dx.$$

$$[A=3], [B=-4], [C=5], [D=-2], [x^3 - 2x^2 + 5x - 2\ln|x+2| + C]$$

(a)

$$\begin{aligned} \frac{3x^3 + 2x^2 - 3x + 8}{x+2} &\equiv Ax^2 + Bx + C + \frac{D}{x+2} \\ &\equiv \frac{Ax(x+2) + B(x+2) + C(x+2) + D}{x+2} \\ &\equiv \frac{Ax^2 + 2Ax^2 + Bx + 2B + Cx + 2C + D}{x+2} \\ &\equiv \frac{Ax^2 + (2A+B)x^2 + (C+2B)x + (2C+D)}{x+2} \end{aligned}$$

$\begin{array}{l} A=3 \\ 2A+B=2 \\ 6+B=2 \\ B=-4 \end{array}$
 $\begin{array}{l} 2B+C=-3 \\ -8+C=-3 \\ C=5 \end{array}$
 $\begin{array}{l} 2C+D=8 \\ 10+D=8 \\ D=-2 \end{array}$

Answered by long division

$$\begin{array}{r} \begin{array}{c} A \quad B \quad C \\ x+2 \quad | \quad 3x^3 + 2x^2 - 3x + 8 \\ \underline{-3x^3 - 6x^2} \\ -4x^2 - 3x + 8 \\ \underline{-4x^2 - 8} \\ -3x + 16 \\ \underline{-(-3x - 6)} \\ 20 \end{array} \end{array}$$

(b)

$$\begin{aligned} \int \frac{3x^3 + 2x^2 - 3x + 8}{x+2} dx &= \int 3x^2 - 4x + 5 - \frac{2}{x+2} dx \\ &= x^3 - 2x^2 + 5x - 2\ln|x+2| + C \end{aligned}$$

Question 35 (***)

$$f(x) \equiv \frac{5}{3x^2 - 5x}.$$

a) Express $f(x)$ in partial fractions.

b) Find the value of

$$\int_3^5 f(x) \, dx,$$

giving the answer as a single simplified logarithm.

, $f(x) \equiv \frac{3}{3x-5} - \frac{1}{x}$, $\boxed{\ln\left(\frac{3}{2}\right)}$

(a) $f(x) = \frac{5}{3x^2 - 5x} = \frac{5}{x(3x-5)} \equiv \frac{A}{x} + \frac{B}{3x-5}$
 $\boxed{5 \equiv A(3x-5) + Bx}$

- If $x=0$ / $S = -5A$ • If $x=\frac{5}{3}$
- $A=-1$ $S = \frac{5}{3}B$
- $B=3$

∴ $f(x) = \frac{3}{3x-5} - \frac{1}{x}$

(b) $\int_3^5 f(x) \, dx = \int_3^5 \left(\frac{3}{3x-5} - \frac{1}{x} \right) \, dx = \left[\ln|3x-5| - \ln|x| \right]_3^5$
 $= (\ln 10 - \ln 5) - (\ln 4 - \ln 3) = \ln 2 - \ln \frac{5}{3}$
 $= \ln 2 + \ln \frac{3}{5} = \ln \left(\frac{3}{2} \right)$

Question 36 (*)**

By using the substitution $u = e^x$, or otherwise, show clearly that

$$\int_{-1}^1 \frac{e^x}{e^x + 1} dx = 1.$$

[proof]

$$\begin{aligned}
 \int_{-1}^1 \frac{e^x}{e^x + 1} dx &= \int_{e^{-1}}^e \frac{e^x}{e^x + 1} \frac{du}{e^x} = \\
 &= \int_{e^{-1}}^e \frac{1}{u+1} du = \left[\ln|u+1| \right]_{e^{-1}}^e = \\
 &= \ln(e+1) - \ln(e^{-1}+1) = \ln\left(\frac{e+1}{e^{-1}+1}\right) \\
 &= \text{MULITPLY TOP & BOTTOM BY } e^1 \dots \\
 &= \ln\left(\frac{e^2+e}{1+e}\right) = \ln\left(\frac{e(e+1)}{e+1}\right) = \ln e = 1
 \end{aligned}$$

NOTE: THIS IS EASIER BY NOTING THAT IT IS OF THE TYPE $\int \frac{f(u)}{f(u)+1} du$

Question 37 (*)**

Find an expression for the integral

$$\int \frac{3x-10}{x^2+5x-6} dx.$$

$4 \ln|x+6| - \ln|x-1| + C$

$$\begin{aligned}
 \int \frac{3x-10}{x^2+5x-6} dx &= \int \frac{3x-10}{(x-1)(x+6)} dx = \dots \text{by partial fractions} \\
 \frac{3x-10}{(x-1)(x+6)} &\equiv \frac{A}{x-1} + \frac{B}{x+6} \\
 3x-10 &\equiv A(x+6) + B(x-1) \\
 \text{if } x=1, -7=7A &\Rightarrow A=-1 \\
 \text{if } x=-6, -28=-7B &\Rightarrow B=4 \\
 &= \dots \int \frac{1}{x+6} - \frac{1}{x-1} dx = 4 \ln|x+6| - \ln|x-1| + C
 \end{aligned}$$

Question 38 (*)**

By using the substitution $u=1+x^2$, or otherwise, find

$$\int \frac{x^3}{\sqrt{1+x^2}} dx.$$

$$\boxed{\frac{1}{3}(1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}} + C}$$

$\begin{aligned} \int \frac{x^3}{\sqrt{1+x^2}} dx &= \int \frac{x^3}{\sqrt{u^2-1}} \cdot \frac{du}{2x} = \int \frac{x^2}{2\sqrt{u^2-1}} du \\ &= \int \frac{u-1}{2\sqrt{u^2-1}} du = \int \frac{u}{2\sqrt{u^2-1}} du - \frac{1}{2\sqrt{u^2-1}} du \\ &= \int \frac{1}{2} u^{\frac{1}{2}} - \frac{1}{2} u^{-\frac{1}{2}} du = \frac{1}{8} u^{\frac{3}{2}} - u^{\frac{1}{2}} + C \\ &= \frac{1}{8}(1+u^2)^{\frac{3}{2}} - (1+u^2)^{\frac{1}{2}} + C \end{aligned}$	$\begin{array}{l} u = 1+x^2 \\ \frac{du}{dx} = 2x \\ du = \frac{du}{2x} \\ 2x = du \\ u^2 = u-1 \end{array}$
--	--

Question 39 (*)**

Use integration by parts to find the value of

$$\int_1^e \ln x \, dx.$$

$$\boxed{[]}, [1]$$

$\begin{aligned} \int_1^e \ln x \, dx &= \text{by parts & ignoring limits} \\ &= \int x \ln x \, dx = x \ln x - \int 1 \, dx \\ &= x \ln x - x + C \\ &\quad \text{REVERSE LIMITS...} \\ &= [x \ln x - x]_1^e \\ &= (e \ln e - e) - (1 \ln 1 - 1) \\ &= (e - e) + 1 \\ &= 1 \end{aligned}$	$\begin{array}{ c c } \hline \ln x & \frac{d}{dx} \\ \hline x & 1 \\ \hline \end{array}$
--	--

Question 40 (*)**

Find the value of the constant k given that

$$\int_0^1 k(e^{2x} + 4x) \, dx = e^2 + 3.$$

$$k = 2$$

$$\begin{aligned} \int_0^1 k(e^{2x} + 4x) \, dx &= e^2 + 3 \\ \Rightarrow \left[k\left(\frac{1}{2}e^{2x} + 2x\right) \right]_0^1 &= e^2 + 3 \\ \Rightarrow k\left[\frac{1}{2}e^2 + 2\right] - k(0) &= e^2 + 3 \\ \Rightarrow k\left(\frac{1}{2}e^2 + 2\right) &= e^2 + 3 \end{aligned} \quad \begin{aligned} \Rightarrow \frac{1}{2}k(e^2 + 3) &= e^2 + 3 \\ \Rightarrow \frac{1}{2}k &= 1 \\ \Rightarrow k &= 2 \end{aligned}$$

Question 41 (*)**

Evaluate each of the following integrals, giving the answers in exact form.

a) $\int_0^{\ln 2} (e^x + 2e^{-x})^2 \, dx.$

$$[3 + 4\ln 2], \quad \boxed{\frac{1}{4}(\pi - 2)}$$

b) $\int_0^{\frac{\pi}{4}} 1 - \sin 4x \, dx.$

$$\begin{aligned} \text{(a)} \quad \int_0^{\ln 2} (e^x + 2e^{-x})^2 \, dx &= \int_0^{\ln 2} (e^x)^2 + 2e^x(2e^{-x}) + (2e^{-x})^2 \, dx \\ &= \int_0^{\ln 2} e^{2x} + 4x - 4e^{-2x} \Big|_0^{\ln 2} = (2 + 4\ln 2 - \frac{1}{2}) - (\frac{1}{2} + 0 - 2) \\ &= \frac{3}{2} + 4\ln 2 - \frac{1}{2} \approx 3 + 4\ln 2. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_0^{\frac{\pi}{4}} 1 - \sin 4x \, dx &= \left[x + \frac{1}{4}\cos 4x \right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} + \frac{1}{4}\cos \pi \right) - \left(0 + \frac{1}{4}\cos 0 \right) \\ &= \left(\frac{\pi}{4} - \frac{1}{4} \right) - \left(\frac{1}{4} \right) \approx \frac{\pi}{4} - \frac{1}{2} \approx \frac{1}{4}(\pi - 2). \end{aligned}$$

Question 42 (*)**

By using the substitution $u = \tan x$ and the trigonometric identity $1 + \tan^2 x = \sec^2 x$, show clearly that

$$\int_0^{\frac{\pi}{3}} \sec^4 x \, dx = 2\sqrt{3}.$$

 , 

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \sec^4 x \, dx &= \int_0^{\sqrt{3}} \sec^4 u \frac{du}{\sec u} = \int_0^{\sqrt{3}} \sec^3 u \, du \\ &= \int_0^{\sqrt{3}} (1 + \tan^2 u)^2 \, du = \int_0^{\sqrt{3}} 1 + u^2 \, du \\ &= \left[u + \frac{1}{3}u^3 \right]_0^{\sqrt{3}} = (\sqrt{3} + \sqrt{3}) - (0 + 0) = 2\sqrt{3}. \end{aligned}$$

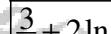
$u = \tan x$
 $\frac{du}{dx} = \sec^2 x$
 $du = \frac{du}{\sec^2 x}$
 $x = 0 \quad u = 0$
 $x = \frac{\pi}{3} \quad u = \sqrt{3}$

Question 43 (*)**

$$\frac{8x-1}{(2x-1)^2} \equiv \frac{A}{2x-1} + \frac{B}{(2x-1)^2}.$$

- a) Determine the value of each of the constants A and B .
- b) Hence find the exact value of

$$\int_1^{1.5} \frac{8x-1}{(2x-1)^2} \, dx.$$

 ,  , 

$$\begin{aligned} \text{(a)} \quad \frac{8x-1}{(2x-1)^2} &\equiv \frac{A}{2x-1} + \frac{B}{(2x-1)^2} && \text{if } x = \frac{1}{2}, \quad 3 = B + 3 \\ \frac{8x-1}{(2x-1)^2} &\equiv A/(2x-1) + B/(2x-1)^2 && \text{if } x = 0, \quad -1 = A + 0 \\ \frac{8x-1}{(2x-1)^2} &\equiv \frac{A}{2x-1} + \frac{B}{(2x-1)^2} && A = B + 3 \\ \int_1^{1.5} \frac{8x-1}{(2x-1)^2} \, dx &= \int_1^{1.5} \frac{4}{2x-1} + 3(2x-1)^{-2} \, dx && \text{if } x = \frac{1}{2}, \quad 3 = B + 3 \\ &= \left[2 \ln|2x-1| - \frac{3}{2}(2x-1)^{-1} \right]_1^{1.5} && \text{if } x = 0, \quad -1 = A + 0 \\ &= \left(2 \ln 2 - \frac{3}{4} \right) - \left(2 \ln 1 - \frac{3}{2} \right) && A = B + 3 \\ &= 2 \ln 2 - \frac{3}{4} + \frac{3}{2} = \frac{3}{4} + 2 \ln 2 \end{aligned}$$

Question 44 (***)

By using the substitution $u = x^{\frac{1}{2}}$, or otherwise, find

$$\int \frac{1}{4x^{\frac{1}{2}}\sqrt{x^2-1}} dx.$$

$$\boxed{\quad}, \boxed{\left(x^{\frac{1}{2}}-1\right)^{\frac{1}{2}}+C}$$

USING THE SUBSTITUTION GIVEN

$$\begin{aligned} \rightarrow u &= x^{\frac{1}{2}} \\ \rightarrow u^2 &= x \\ \rightarrow x &= u^2 \\ \rightarrow \frac{dx}{du} &= 2u \\ \rightarrow dx &= 2u du \end{aligned}$$

TRANSFORMING THE INTEGRAL WE OBTAIN

$$\begin{aligned} \int \frac{1}{4x^{\frac{1}{2}}\sqrt{x^2-1}} dx &= \int \frac{1}{4u\sqrt{u^2-1}} (2u du) \\ &= \int \frac{1}{2}(u-1)^{-\frac{1}{2}} du \\ &= \frac{1}{2} \cdot \frac{1}{\frac{1}{2}}(u-1)^{\frac{1}{2}} + C \\ &= (u-1)^{\frac{1}{2}} + C \\ &= \boxed{\sqrt{x^2-1} + C} \end{aligned}$$

Question 44 (*)**

- a) Use integration by parts to find

$$\int x \cos\left(\frac{1}{2}x\right) dx.$$

- b) Hence determine

$$\int x^2 \sin\left(\frac{1}{2}x\right) dx.$$

, $2x \sin\left(\frac{1}{2}x\right) + 4 \cos\left(\frac{1}{2}x\right) + C$,

$-2x^2 \cos\left(\frac{1}{2}x\right) + 8x \sin\left(\frac{1}{2}x\right) + 16 \cos\left(\frac{1}{2}x\right) + C$

a) SETTING UP INTEGRATION BY PARTS

$$\begin{aligned} \int x \cos\left(\frac{1}{2}x\right) dx &= \dots \\ &= \underline{2x \sin\frac{1}{2}x} - \int \underline{2 \cos\frac{1}{2}x} dx \\ &= \underline{2x \sin\frac{1}{2}x} + \underline{4 \sin\frac{1}{2}x} + C \end{aligned}$$

b) USING INTEGRATION BY PARTS & PART (a)

$$\begin{aligned} \int x^2 \sin\left(\frac{1}{2}x\right) dx &= \dots \\ &= \underline{-x^2 \cos\frac{1}{2}x} - \int \underline{-2x \sin\frac{1}{2}x} dx \\ &= -x^2 \cos\frac{1}{2}x + \int x \cos\frac{1}{2}x dx \\ &= -x^2 \cos\frac{1}{2}x + d \left[2x \sin\frac{1}{2}x + 4 \cos\frac{1}{2}x \right] + C \\ &= -x^2 \cos\frac{1}{2}x + \underline{8x \sin\frac{1}{2}x} + \underline{16 \cos\frac{1}{2}x} + C \end{aligned}$$

Question 45 (*)**

By using the substitution $u = x^2 + 6x$, or otherwise find

$$\int \frac{x+3}{(x^2 + 6x)^{\frac{1}{3}}} dx.$$

$$\boxed{\frac{1}{2}(x^2 + 6x)^{\frac{2}{3}} + C}$$

$$\begin{aligned} & \int \frac{x+3}{(x^2 + 6x)^{\frac{1}{3}}} dx = \dots \text{ by substitution or}\\ & \text{reverse chain rule by reworking} \\ & = \int \frac{x+3}{u^{\frac{1}{3}}} \frac{du}{2u+6} = \int \frac{2u+6}{u^{\frac{1}{3}}} \times \frac{1}{2u+6} du \\ & = \int \frac{1}{2} u^{-\frac{1}{3}} du = \frac{1}{2} u^{\frac{2}{3}} + C = \frac{3}{4} u^{\frac{2}{3}} + C \\ & = \frac{3}{4} (x^2 + 6x)^{\frac{2}{3}} + C \end{aligned}$$

Question 46 (*)**

Use the substitution $u = 10\cos x - 1$ to find

$$\int_0^{\frac{\pi}{3}} 15(10\cos x - 1)^{\frac{1}{2}} \sin x \, dx.$$

[19]

$$\begin{aligned} & \int_0^{\frac{\pi}{3}} 15(10\cos x - 1)^{\frac{1}{2}} \sin x \, dx = \\ & \dots \text{BY SUBSTITUTION} \dots \\ & = \int_0^{\frac{\pi}{3}} 15u^{\frac{1}{2}} \sin\left(\frac{du}{-10\sin x}\right) \\ & = \int_0^{\frac{\pi}{3}} \frac{3}{2} u^{\frac{1}{2}} du \\ & = \left[\frac{3}{2} u^{\frac{3}{2}} \right]_0^{\frac{\pi}{3}} = 21 - 0 \\ & = 19 \end{aligned}$$

Question 47 (***)

$$\frac{2x^2 - x + 6}{x^2(3-2x)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{3-2x}.$$

a) Determine the value of each of the constants A , B and C .

b) Evaluate

$$\int_2^3 \frac{2x^2 - x + 6}{x^2(3-2x)} dx,$$

giving the answer in the form $p - \ln q$, where p and q are constants.

$A = 1$	$B = 2$	$C = 4$	$\frac{1}{3} - \ln 6$
---------	---------	---------	-----------------------

(a)

$$\frac{2x^2 - x + 6}{x^2(3-2x)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{3-2x}$$

$$2x^2 - x + 6 \equiv Ax(3-2x) + B(3-2x) + Cx^2$$

- If $x = \frac{3}{2}$, $\frac{3}{2} - \frac{3}{2} + 6 = \frac{3}{4}C$ $\left\{ \begin{array}{l} \frac{3}{4}x=1 \\ 9= \frac{3}{4}C \\ C=4 \end{array} \right.$ $\begin{array}{l} 2-1+6=A+B+C \\ 7=4+2+4 \\ A=1 \end{array}$
- If $x=0$, $6 = 3B$ $B=2$

(b)

$$\int_2^3 \frac{2x^2 - x + 6}{x^2(3-2x)} dx = \int_2^3 \frac{\frac{1}{2}x + \frac{2}{3}x^2 + \frac{4}{3-2x}}{dx}$$

$$= \left[\left(\ln|x| - \frac{2}{3}x - 2\ln|3-2x| \right) \right]_2^3 = \left(\ln 3 - \frac{2}{3} - 2\ln 3 \right) - \left(\ln 2 - 1 - 2\ln 2 \right)$$

$$= \ln 3 - 2\ln 3 - \ln 2 + \frac{2}{3} + 1 = \frac{1}{3} - \ln 3 - \ln 2$$

$$= \frac{1}{3} - (\ln 3 + \ln 2) = \frac{1}{3} - \ln 6$$

Question 48 (***)+

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$
y	0	0.1309	0.4534	0.7854		0.6545

The table above shows tabulated values for the equation

$$y = x \sin 2x, \quad 0 \leq x \leq \frac{5\pi}{12}.$$

- a) Complete the missing value in the table.
 - b) Use the trapezium rule with all the values from the table to find an approximate value for
- $$\int_0^{\frac{5\pi}{12}} x \sin 2x \, dx.$$
- c) Use integration by parts to find an exact value for
- $$\int_0^{\frac{5\pi}{12}} x \sin 2x \, dx.$$

, , , $\frac{5\pi\sqrt{3}}{48} + \frac{1}{8}$

a) $y = \frac{\pi}{3} \times \sin(2x \cdot \frac{\pi}{3}) = \frac{\pi}{3} \times \sin \frac{2\pi}{3} \approx 0.9069$

b) $\int_0^{\frac{5\pi}{12}} x \sin 2x \, dx \approx \text{TRAPEZIUM} \left[\text{FIRST} + (\text{LAST} + 2 \times \text{REST}) \right]$
 $\approx \frac{\pi/6}{2} [0 + 0.6545 + 2(0.1309 + 0.4534 + 0.7854 + 0.6545)]$
 ≈ 0.682

c) $\int_0^{\frac{5\pi}{12}} x \sin 2x \, dx = \dots \text{INTEGRATION BY PARTS & (IGNORING UNITS)}$
 $= -\frac{1}{2}x \cos 2x - \int -\frac{1}{2} \cos 2x \, dx$
 $= -\frac{1}{2}x \cos 2x + \int \frac{1}{2} \cos 2x \, dx$
 $= -\frac{1}{2}x \cos 2x + \frac{1}{4} \sin 2x + C$
 2+ INTRODUCE UNITS
 $= \left[-\frac{1}{2}x \cos 2x + \frac{1}{4} \sin 2x \right]_0^{\frac{5\pi}{12}}$
 $= \left(-\frac{1}{2} \cdot \frac{5\pi}{12} \cdot \cos \frac{5\pi}{6} + \frac{1}{4} \cdot \frac{5\pi}{12} \cdot \sin \frac{5\pi}{6} \right) - (0 + 0)$
 $= \frac{5\pi\sqrt{3}}{48} + \frac{1}{8}$

Question 49 (***)+

By using the substitution $u = e^x + 1$, or otherwise, find

$$\int \frac{e^{2x} - 2e^x}{e^x + 1} dx.$$

, $e^x - 3\ln(e^x + 1) + C$

$$\begin{aligned}
 \int \frac{e^{2x} - 2e^x}{e^x + 1} dx &= \dots \text{by substitution} \\
 &\stackrel{u = e^x + 1}{=} \int \frac{e^{2x} - 2e^x}{u} \frac{du}{e^x} = \int \frac{e^{2x} - 2}{u} du \\
 &\stackrel{du = e^x dx}{=} \int \frac{(u-1) - 2}{u} du = \int \frac{u-3}{u} du \\
 &\stackrel{u = e^x + 1}{=} \int (1 - \frac{3}{u}) du = -u - 3\ln|u| + C \\
 &\stackrel{u = e^x + 1}{=} (e^x + 1) - 3\ln|e^x + 1| + C = e^x - 3\ln|e^x + 1| + (1+C) \\
 &= e^x - 3\ln(e^x + 1) + C
 \end{aligned}$$

Question 50 (***)+

By using the substitution $u = 1 - \cos x$, or otherwise, find

$$\int \frac{\sin x \cos x}{1 - \cos x} dx.$$

$\cos x + \ln(1 - \cos x) + C$

$$\begin{aligned}
 \int \frac{\sin x \cos x}{1 - \cos x} dx &= \dots \text{by substitution} \\
 &\stackrel{u = 1 - \cos x}{=} \int \frac{\sin x \cos x}{u} \frac{du}{\sin x} = \int \frac{1-u}{u} du = \int \frac{1}{u} - 1 du \\
 &= \ln|u| - u + C = [\ln|1-\cos x|] - (1-\cos x) + C \\
 &= \cos x + \ln|1-\cos x| + C \\
 &= \cos x + \ln(1 - \cos x) + C
 \end{aligned}$$

Question 51 (***)+

By using the substitution $u^2 = e^x - 1$, or otherwise, find

$$\int_{\ln 2}^{\ln 5} \frac{3e^{2x}}{\sqrt{e^x - 1}} dx.$$

 [20]

USING THE SUBSTITUTION METHOD

$u^2 = e^x - 1$ (in fact the substitution is $u = \sqrt{e^x - 1}$)

$$2u \frac{du}{dx} = e^x$$

$$2u du = e^x dx$$

$$du = \frac{e^x}{2u} du$$

UNITS

$$x = \ln 5 \rightarrow u = \sqrt{e^{\ln 5} - 1} = 2$$

$$x = \ln 2 \rightarrow u = \sqrt{e^{\ln 2} - 1} = 1$$

TRANSFORMING THE INTEGRAL

$$\int_{\ln 2}^{\ln 5} \frac{3e^{2x}}{\sqrt{e^x - 1}} dx = \int_1^2 \frac{3e^{2x}}{\sqrt{e^x - 1}} \left(\frac{e^x}{2u} du \right)$$

$$= \int_1^2 6e^{2x} du$$

$$= \int_1^2 6(u^2 + 1) du$$

$$= \int_1^2 6u^2 + 6 du$$

$$= \left[2u^3 + 6u \right]_1^2$$

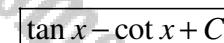
$$= (16 + 12) - (2 + 6)$$

$$= 20$$

Question 52 (***)+

Use trigonometric identities to find

$$\int \sec^2 x (1 + \cot^2 x) dx.$$



$$\int \sec^2 x (1 + \cot^2 x) dx = \int \sec^2 x + \sec^2 \cot^2 x dx$$

$$= \int \sec^2 x + \frac{1}{\sin^2 x \cos^2 x} dx$$

$$= \int \sec^2 x + \csc^2 x dx$$

$$= \tan x - \cot x + C$$

$\frac{d}{dx}(\tan x) = \sec^2 x$
 $\frac{d}{dx}(\cot x) = -\csc^2 x$

Question 53 (***)

By using the substitution $u = e^x + 1$, or otherwise, find the exact value of

$$\int_0^1 \frac{e^{2x}}{e^x + 1} dx.$$

$$\boxed{e-1+\ln\left(\frac{2}{1+e}\right)}$$

$$\begin{aligned} \int_0^1 \frac{e^{2x}}{e^x + 1} dx &= \text{using the substitution given, } \\ &\Rightarrow \int_{e^0}^{e^1} \frac{e^{2x}}{u} \frac{du}{e^x} = \int_1^{e+1} \frac{e^{2x}}{u} du \\ &= \int_1^{e+1} \frac{u-1}{u} du = \int_1^{e+1} 1 - \frac{1}{u} du \\ &= \left[u - \ln|u| \right]_1^{e+1} = \left[e+1 - \ln(e+1) \right] - \left[1 - \ln 1 \right] \\ &= e-1 + \ln 2 - \ln(e+1) = e-1 + \ln\left(\frac{2}{e+1}\right) \end{aligned}$$

Question 54 (***)

By using the substitution $u = \sin x$, or otherwise, find

$$\int \cos^3 x dx.$$

$$\boxed{\quad}, \boxed{\sin x - \frac{1}{3} \sin^3 x + C}$$

$$\begin{aligned} \int \cos^3 x dx &= \int \cos^2 x \frac{dx}{\cos x} = \int \cos^2 x du \\ &= \int 1 - \sin^2 x du \\ &= \int 1 - u^2 du \\ &= u - \frac{1}{3}u^3 + C \\ &= \sin x - \frac{1}{3}\sin^3 x + C \end{aligned}$$

Question 55 (***)+

$$I = \int (x-1)(4-x)^{\frac{1}{2}} dx, \quad x \in \mathbb{R}, \quad x \leq 4.$$

a) Use the substitution $u = (4-x)^{\frac{1}{2}}$ to find an expression for I .

b) Show that the answer of part (a) can be written as

$$I = -\frac{2}{5}(x+1)(4-x)^{\frac{3}{2}} + C.$$

c) Use integration by parts to verify the answer of part (b).

$$\boxed{\text{[]}}, \quad I = \frac{2}{5}(4-x)^{\frac{5}{2}} - 2(4-x)^{\frac{3}{2}} + C$$

$$\begin{aligned} \text{(a)} \quad & \int (x-1)(4-x)^{\frac{1}{2}} dx = \int (4-u^2)(u(-2u) du) \\ &= \int (4-u^2)(-2u^2) du = \int -2u^4 - u^2 du \\ &= \frac{2}{5}u^5 - 2u^3 + C = \frac{2}{5}(4-x)^{\frac{5}{2}} - 2(4-x)^{\frac{3}{2}} + C \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & \dots = \frac{2}{5}(4-x)^{\frac{3}{2}}[(4-x)^{\frac{1}{2}} - 5] + C \\ &= \frac{2}{5}(-1)(4-x)^{\frac{3}{2}} + C = -\frac{2}{5}(2x)(4-x)^{\frac{3}{2}} + C \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad & \int (x-1)(4-x)^{\frac{1}{2}} dx = \dots \text{ [] } \\ &= -\frac{2}{5}(x-1)(4-x)^{\frac{3}{2}} + \int \frac{2}{5}(4-x)^{\frac{1}{2}} dx \\ &= -\frac{2}{5}(x-1)(4-x)^{\frac{3}{2}} + \frac{2}{5}(4-x)^{\frac{3}{2}} + C = -\frac{16}{5}(x-1)(4-x)^{\frac{3}{2}} - \frac{2}{5}(4-x)^{\frac{3}{2}} + C \\ &= \frac{1}{5}(4-x)^{\frac{3}{2}} [4(x-1) + 4(4-x)] + C = \frac{1}{5}(4-x)^{\frac{3}{2}} (16x-16) + C \\ &= -\frac{1}{5}(4-x)^{\frac{3}{2}} (4x+16) = -\frac{1}{5}(4-x)^{\frac{3}{2}} (4x+16) + C = -\frac{2}{5}(2x)(4-x)^{\frac{3}{2}} + C \end{aligned}$$

Question 56 (***)+

Find the value of the constant k given that

$$\int_0^{\ln 4} (4k-1)e^{2.5x} + k e^{-0.5x} \, dx = 190.$$

$k = 4$

$$\begin{aligned}
 & \int_0^{\ln 4} (4k-1)e^{2.5x} + k e^{-0.5x} \, dx = 190 \\
 & \Rightarrow \int_0^{\ln 4} (4k-1)e^{2.5x} + k e^{-0.5x} \, dx = 190 \\
 & \Rightarrow \left[\frac{4k-1}{2.5} e^{2.5x} - k e^{-0.5x} \right]_0^{\ln 4} = 190 \\
 & \Rightarrow \left[\frac{2(4k-1)}{5} e^{\ln 4} - 2k e^{-\ln 4} \right] - \left[\frac{2(4k-1)}{5} e^0 - 2k e^0 \right] = 190 \\
 & \Rightarrow \frac{2(4k-1)}{5} e^{\ln 4} - 2k e^{-\ln 4} - \frac{2(4k-1)}{5} + 2k = 190 \\
 & \Rightarrow \frac{2(4k-1)}{5} e^{\ln 4} - k - \frac{2(4k-1)}{5} + 2k = 190 \\
 & \Rightarrow \frac{2(4k-1)}{5} e^{\ln 4} + k = 190 \\
 & \Rightarrow \frac{2(4k-1)}{5} + k = 190 \\
 & \Rightarrow \frac{12}{5}(4k-1) + 5k = 950 \\
 & \Rightarrow 48k - 12 + 25k = 950 \\
 & \Rightarrow 253k = 962 \\
 & \Rightarrow k = 4
 \end{aligned}$$

Question 57 (***)+

By using the substitution $u = 2x-1$, or otherwise, show that

$$\int \frac{2x}{\sqrt{2x-1}} \, dx = \frac{2}{3}(x+1)\sqrt{2x-1} + C.$$

proof

$$\begin{aligned}
 & \int \frac{2x}{\sqrt{2x-1}} \, dx = \int \frac{2x}{\sqrt{u+1}} \cdot \frac{du}{2} \\
 & = \int \frac{u+1}{u^{\frac{1}{2}}} \cdot \frac{du}{2} = \int \frac{u+1}{u^{\frac{1}{2}}} \cdot \frac{du}{2} = \int \frac{u}{u^{\frac{1}{2}}} + \frac{1}{u^{\frac{1}{2}}} \cdot \frac{du}{2} \\
 & = \int \frac{1}{2} u^{\frac{1}{2}} + \frac{1}{2} u^{-\frac{1}{2}} \cdot du = \frac{1}{3} u^{\frac{3}{2}} + u^{\frac{1}{2}} + C \\
 & = \frac{1}{3} (2x-1)^{\frac{3}{2}} + (2x-1)^{\frac{1}{2}} + C = \frac{1}{3} (2x-1)^{\frac{1}{2}} [(2x-1) + 3] + C \\
 & = \frac{1}{3} (2x-1)^{\frac{1}{2}} (2x+2) + C = \frac{2}{3} (x+1) \sqrt{2x-1} + C
 \end{aligned}$$

Question 58 (***)+

$$f(x) \equiv \frac{70}{x(x-2)(x+5)}.$$

- a) Express $f(x)$ in partial fractions.
- b) Show that $\int_3^4 f(x) dx$ can be written in the form $p \ln 3 + q \ln 2$, where p and q are integers to be found.

$$\boxed{f(x) \equiv \frac{2}{x+5} + \frac{5}{x-2} - \frac{7}{x}}, \quad \boxed{\int_3^4 f(x) dx = 11\ln 3 - 15\ln 2}$$

(a)

$$\frac{70}{x(x-2)(x+5)} \equiv \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}$$

$$70 \equiv A(x-2)(x+5) + Bx(x+5) + Cx(x-2)$$

- If $x=2$, $70 = 14B \Rightarrow B=5$
- If $x=0$, $70 = -10A \Rightarrow A=-7$
- If $x=-5$, $70 = 35C \Rightarrow C=2$

$$f(x) = -\frac{7}{x} - \frac{7}{x-2} + \frac{2}{x+5}$$

(b)

$$\int_3^4 f(x) dx = \int_3^4 \left(-\frac{7}{x} - \frac{7}{x-2} + \frac{2}{x+5} \right) dx = \left[5\ln|x-2| - 7\ln|x| + 2\ln|x+5| \right]_3^4$$

$$= (5\ln 2 - 7\ln 3 + 2\ln 7) - (5\ln 1 - 7\ln 3 + 2\ln 8)$$

$$= 5\ln 2 - 14\ln 3 + 9\ln 7 + 7\ln 3 - 6\ln 8 = 11\ln 3 - 15\ln 2$$

Question 59 (***)+

Use trigonometric identities to find

$$\int \frac{1}{\cos^2 x \tan^2 x} dx.$$

$$\boxed{-\cot x + C}$$

Method 1:

$$\int \frac{1}{\cos^2 x \tan^2 x} dx = \int \frac{\sec^2 x}{\tan^2 x} dx = \int \frac{1}{\sin^2 x} dx = \int \csc^2 x dx = -\cot x + C$$

Method 2:

$$\int \frac{1}{\cos^2 x \tan^2 x} dx = \int \sec^2(\tan x)^{-2} dx = \dots \text{by reverse chain rule}$$

since $\frac{d}{dx}(\tan x) = \sec^2 x$

$$= -(\tan x)^{-1} + C = -\frac{1}{\tan x} + C = -\cot x + C$$

Question 60 (***)+

$$f(x) \equiv \frac{2x+2}{(1-x)(1-2x)}.$$

- a) Express $f(x)$ in partial fractions.
- b) Show that $\int_{1.5}^2 f(x) dx$ can be written in the form $p \ln 2 + q \ln 3$, where p and q are integers to be found.

$$\boxed{f(x) \equiv \frac{6}{1-2x} - \frac{4}{1-x}}, \quad \boxed{\int_{1.5}^2 f(x) dx = 7 \ln 2 - 3 \ln 3}$$

(a)

$$\begin{aligned} f(x) &\equiv \frac{2x+2}{(1-x)(1-2x)} \equiv \frac{A}{1-x} + \frac{B}{1-2x} \\ 2x+2 &\equiv A(1-2x) + B(1-x) \\ \text{If } x=1, \quad 4 &= -A \Rightarrow A=-4 \\ \text{If } x=\frac{1}{2}, \quad 3 &= \frac{1}{2}B \Rightarrow B=6 \\ \therefore f(x) &\equiv \frac{6}{1-2x} - \frac{4}{1-x} \end{aligned}$$

(b)

$$\begin{aligned} \int_{1.5}^2 f(x) dx &= \left[\frac{6}{1-2x} - \frac{4}{1-x} \right]_1^2 = \left[-3\ln|1-2x| + 4\ln|1-x| \right]_1^2 \\ &= (-3\ln 3 + 4\ln 1) - (-3\ln 2 + 4\ln \frac{1}{2}) \\ &= -3\ln 3 + 3\ln 2 - 4\ln \frac{1}{2} \\ &= -3\ln 3 + 3\ln 2 + 4\ln 2 \\ &= 7\ln 2 - 3\ln 3 \end{aligned}$$

Question 61 (***)+

Use a suitable method to find

$$\int \ln\left(\frac{x}{2}\right) dx.$$

$$\boxed{x \ln\left(\frac{x}{2}\right) - x + C}$$

$$\begin{aligned} \int \ln\left(\frac{x}{2}\right) dx &= -\ln 2 \cdot \frac{1}{2}x^2 = -\int 1 \times \ln(2x) dx \\ &= x \ln \frac{x}{2} - \int x \cdot \frac{1}{2} dx \\ &= x \ln \frac{x}{2} - \frac{1}{2}x^2 + C \\ &= x \ln \frac{x}{2} - x + C \end{aligned}$$

Question 62 (***)+

$$f(x) \equiv \frac{32-17x}{(x+1)(3x-4)^2}.$$

a) Express $f(x)$ in partial fractions.

b) Show that

$$\int_0^1 f(x) \, dx,$$

can be evaluated in the form $p + \ln q$, where p and q are integers to be found.

$$f(x) \equiv \frac{4}{(3x-4)^2} - \frac{3}{(3x-4)} + \frac{1}{(x+1)}, \quad \int_0^1 f(x) \, dx = 1 + \ln 8$$

(a) $\frac{32-17x}{(3x-4)^2} \equiv \frac{A}{x+1} + \frac{B}{(3x-4)} + \frac{C}{(3x-4)^2}$

$$32-17x \equiv A(3x-4)^2 + B(3x-4) + C(x+1)(3x-4)$$

- If $x=-1$, $49=49A \Rightarrow [A=1]$
- If $x=\frac{4}{3}$, $32-\frac{4}{3}= \frac{4}{3}B \Rightarrow [B=4]$
- If $x=0$, $32=16A+B-4C \Rightarrow [C=-3]$
- ∴ $f(x) = \frac{1}{x+1} + \frac{4}{(3x-4)} - \frac{3}{(3x-4)^2}$

(b) $\int_0^1 f(x) \, dx = \int_0^1 \left(\frac{1}{x+1} + \frac{4}{(3x-4)} - \frac{3}{(3x-4)^2} \right) \, dx$

$$= \left[\ln|x+1| - \frac{4}{3}(3x-4)^{-1} - \ln|3x-4| \right]_0^1$$

$$= \left(\ln 2 + \frac{4}{3} - \ln 1 \right) - \left(\ln 1 + \frac{4}{3} - \ln 4 \right)$$

$$= \ln 2 + \ln 1 + \frac{4}{3} - \frac{1}{3} = 1 + \ln 8$$

Question 63 (***)+

Use a trigonometric identity to find the exact value of

$$\int_0^{\frac{\pi}{4}} \sin^2 2x \ dx.$$

$$\boxed{\frac{\pi}{8}}$$

$\begin{aligned} \int_0^{\frac{\pi}{4}} \sin^2 2x \ dx &= \int_0^{\frac{\pi}{4}} \frac{1}{2} - \frac{1}{2} \cos 4x \ dx \\ &= \left[\frac{1}{2}x - \frac{1}{8} \sin 4x \right]_0^{\frac{\pi}{4}} \\ &= \left(\frac{\pi}{8} - \frac{1}{8} \sin \pi \right) - (0 - 0) = \frac{\pi}{8} \end{aligned}$	$\sin^2 2x = \frac{1}{2}(1 - \cos 4x)$ $2\sin 2x = \frac{1}{2} - \frac{1}{2}\cos 4x$
--	---

Question 64 (***)+

Use integration by parts twice to find an exact value for

$$\int_0^{\frac{\pi}{2}} 4x^2 \cos x \ dx.$$

$$\boxed{}, \boxed{\pi^2 - 8}$$

$\begin{aligned} \int_0^{\frac{\pi}{2}} 4x^2 \cos x \ dx &= \text{by parts & ignoring limits} \\ &= 4x^2 \sin x - \int 8x \sin x \ dx \\ &= 4x^2 \sin x - \left[-8x \cos x - \int -8 \cos x \ dx \right] \\ &= 4x^2 \sin x + 8x \cos x - \int 8 \cos x \ dx \\ &\quad \dots \text{REVERSE LIMITS...} \\ &= \left[4x^2 \sin x + 8x \cos x - 8 \sin x \right]_0^{\frac{\pi}{2}} \\ &= \left[4 \left(\frac{\pi^2}{4} \right) \sin \frac{\pi}{2} + 8 \left(\frac{\pi}{2} \right) \cos \frac{\pi}{2} - 8 \sin 0 \right] - [0 + 0 - 8 \sin 0] \\ &= \pi^2 - 8 \end{aligned}$	$4x^2$ $\sin x$ $\cos x$ $8x$ $-8 \cos x$ $\sin x$
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Question 65 (***)+

$$\frac{2}{(u-2)(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u+2}.$$

- a) Find the value of A and B in the above identity.
 b) By using the substitution $u = \sqrt{x}$, or otherwise, find

$$\int \frac{1}{\sqrt{x}(x-4)} dx.$$

$$\boxed{A = \frac{1}{2}}, \boxed{B = -\frac{1}{2}}, \boxed{\frac{1}{2} \ln \left| \frac{\sqrt{x}-2}{\sqrt{x}+2} \right| + C}$$

$$\begin{aligned}
 \text{(a)} \quad & \frac{2}{(u-2)(u+2)} \equiv \frac{A}{u-2} + \frac{B}{u+2} \\
 & 2 \equiv A(u+2) + B(u-2) \\
 & \Rightarrow A(u+2) + B(u-2) = 2 \\
 & \Rightarrow A = \frac{1}{2}, B = -\frac{1}{2} \\
 \text{(b)} \quad & \int \frac{1}{\sqrt{x}(x-4)} dx = \int \frac{1}{u(u-4)} 2u du \\
 & = \int \frac{2}{u^2-4} du = \int \frac{2}{(u-2)(u+2)} du \\
 & = \int \frac{\frac{1}{u-2}}{u+2} du - \int \frac{\frac{1}{u+2}}{u-2} du \\
 & = \frac{1}{2} \ln |u-2| - \frac{1}{2} \ln |u+2| + C = \frac{1}{2} \left[\ln |u-2| - \ln |u+2| \right] + C \\
 & = \frac{1}{2} \ln \left| \frac{u-2}{u+2} \right| + C = \frac{1}{2} \ln \left| \frac{\sqrt{x}-2}{\sqrt{x}+2} \right| + C
 \end{aligned}$$

Question 66 (***)+

By using the substitution $u = \cos x$, or otherwise, find

$$\int \frac{1+\cos x}{\sin x} dx.$$

$$\boxed{\ln |\cos x - 1| + C}$$

$$\begin{aligned}
 \int \frac{1+\cos x}{\sin x} dx &= \int \frac{1+u}{\sin u} \left(\frac{du}{-\sin u} \right) & u = \cos x, \\
 &= \int \frac{1+u}{-u^2-1} du & \frac{du}{dx} = -\sin x, \\
 &= \int \frac{1+u}{u^2-1} du = \int \frac{1+u}{u^2-1} du & du = \frac{du}{-\sin u}, \\
 &= \int \frac{1}{u^2-1} du & \int \frac{1+u}{(u-1)(u+1)} du \\
 &\approx \frac{1}{2} \ln |u-1| + C &= \ln |u-1| + C = \ln |\cos x - 1| + C
 \end{aligned}$$

Question 67 (***)+

Use integration by parts to show that

$$\int \frac{4 \ln x}{x^3} dx = -\frac{1+2 \ln x}{x^2} + C.$$

$$\boxed{\sin x - \frac{1}{3} \sin^3 x + C}$$

$$\begin{aligned} \int \frac{4 \ln x}{x^3} dx &= \int (4 \ln x) x^{-3} dx = \text{by parts} \\ &= -2x^2 \ln x - \int -2x^2 \cdot \frac{1}{x} dx \\ &= -\frac{2 \ln x}{x^2} + \int 2x^3 dx \\ &= -\frac{2 \ln x}{x^2} - x^2 + C \\ &= -\frac{2 \ln x}{x^2} - \frac{1}{x^2} + C \\ &= -\frac{1}{x^2} [1+2 \ln x] + C \quad \text{✓ b. simplified} \end{aligned}$$

Question 68 (***)+

By considering the differentiation of a product of two appropriate functions, find

$$\int e^x (\tan x + \sec^2 x) dx.$$

$$\boxed{e^x \tan x + C}$$

$$\begin{aligned} \int e^x (\tan x + \sec^2 x) dx &= \dots \text{ by inspection since } \frac{d}{dx}(\tan x) = \sec^2 x \\ &= \dots \frac{d}{dx}(e^x \tan x) = e^x \tan x + e^x \sec^2 x \\ \therefore \int e^x (\tan x + \sec^2 x) dx &= e^x \tan x + C \quad \text{✓} \end{aligned}$$

Question 69 (***)+

$$\frac{2x^2-3}{(x-1)^2} \equiv A + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

a) Determine the value of each of the constants A , B and C .

b) Evaluate

$$\int_2^3 \frac{2x^2-3}{(x-1)^2} dx,$$

giving the answer in the form $p + \ln q$, where p and q are constants.
 , $[A=2]$, $[B=4]$, $[C=-1]$, $\left[\frac{3}{2} + \ln 16\right]$

(a)

$$\begin{aligned} \frac{2x^2-3}{(x-1)^2} &\equiv A + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\ 2x^2-3 &\equiv A(x-1)^2 + B(x-1) + C \end{aligned}$$

- If $x=1$, $-3 = A$, $B+C$
- If $x=0$, $-3 = A - B + C$
- If $x=2$, $5 = A + B + C$

$$\begin{cases} A+B+C=5 \\ A-B+C=-3 \\ A= -3-B+C \end{cases}$$

$$\begin{cases} A+B+C=5 \\ A-B+C=-3 \\ A= -3-B+C \end{cases} \quad \begin{cases} A+B=6 \\ A-B=-2 \\ A= -3-B+C \end{cases}$$

$$\begin{cases} A+B=6 \\ A-B=-2 \\ A= -3-B+C \end{cases} \quad \begin{cases} A=2 \\ B=4 \\ C=-1 \end{cases}$$

(b)

$$\begin{aligned} \int_2^3 \frac{2x^2-3}{(x-1)^2} dx &= \int_2^3 2 + \frac{4}{x-1} - (x-1)^{-2} dx \\ &= \left[2x + 4\ln|x-1| + (x-1)^{-1} \right]_2^3 \\ &= \left[2x + 4\ln|x-1| + \frac{1}{x-1} \right]_2^3 \\ &= \left(6 + 4\ln 2 + \frac{1}{2} \right) - \left(4 + 4\ln 1 + 1 \right) \\ &= \frac{3}{2} + 4\ln 2 \end{aligned}$$

Question 70 (***)+

By using the substitution $u = 2x - 1$, or otherwise, find

$$\int \frac{16x^2}{2x-1} dx.$$

$$(2x-1)^2 + 4(2x-1) + 2\ln|2x-1| + C = 4x^2 + 4x + 2\ln|2x-1| + C$$

$\begin{aligned} \int \frac{16x^2}{2x-1} dx &= \dots \text{by substitution.} \\ &= \int \frac{16x^2}{u} \frac{du}{2} = \int \frac{8x^2}{u} du = \int \frac{2(4x^2)}{u} du \\ &= \int 2(u^2+2u+1) du = \int 2u+4+\frac{2}{u} du \\ &= u^2 + 4u + 2\ln u + C = (2x-1)^2 + 4(2x-1) + 2\ln 2x-1 + C \\ &= 4x^2 - 4x + 4 + 8x^2 - 8x + 2\ln 2x-1 + C \\ &= 4x^2 + 4x + 2\ln 2x-1 + C \end{aligned}$	$\begin{array}{l} u = 2x-1 \\ \frac{du}{dx} = 2 \\ du = \frac{1}{2} dx \\ 2x = u+1 \\ 4x^2 = u^2+2u+1 \end{array}$
--	--

$\begin{aligned} \text{ALTERNATIVE without substitution.} \\ \int \frac{16x^2}{2x-1} dx &= \int \frac{8x(2x-1)+4(2x-1)+4}{2x-1} dx \\ &= \int 8x+4+\frac{4}{2x-1} dx \\ &= 4x^2 + 4x + 2\ln 2x-1 + C \end{aligned}$
--

Question 71 (***)+

Use integration by parts to show that

$$\int_0^{\frac{\pi}{4}} 4x \sec^2 x \ dx = \pi - 2\ln 2.$$

proof

$\begin{aligned} \int_0^{\frac{\pi}{4}} 4x \sec^2 x \ dx &= \text{BY PARTS & KNOWING LIMITS} \\ &= 4x \tan x - \int 4 \tan x dx \\ &= 4x \tan x - 4 \ln \sec x + C \quad \text{STANDARD RESULT} \\ &= \dots \text{limits} \dots \\ &= (\cancel{4x \tan x} - 4 \ln \sec x) \Big _0^{\frac{\pi}{4}} \\ &= (\pi - 4 \ln 2) - (0 - 4 \ln 1) \\ &= \pi - 4 \ln 2 \\ &= \pi - 2 \ln 2 \quad \text{As required} \end{aligned}$	$\begin{array}{r} 4x \\ \hline \tan x \\ \hline \sec^2 x \end{array}$
---	---

Question 72 (***)+

$$\frac{18}{(3u-1)(3u+1)} \equiv \frac{A}{3u-1} + \frac{B}{3u+1}.$$

- a) Find the value of A and B in the above identity.
- b) By using the substitution $x = u^2$, or otherwise, find

$$\int \frac{9}{\sqrt{x}(9x-1)} dx.$$

$$[A=9], [B=-9], \boxed{3 \ln \left| \frac{3\sqrt{x}-1}{3\sqrt{x}+1} \right| + C}$$

$$\begin{aligned}
 \text{(a)} \quad & \frac{18}{(3u-1)(3u+1)} \equiv \frac{A}{3u-1} + \frac{B}{3u+1} \\
 & [18 \equiv A(3u+1) + B(3u-1)] \\
 & \text{if } u=\frac{1}{3} \Rightarrow 18=2A \Rightarrow A=9 \\
 & \text{if } u=-\frac{1}{3} \Rightarrow 18=-2B \Rightarrow B=-9 \\
 \text{(b)} \quad & \int \frac{9}{\sqrt{x}(9x-1)} dx = \int \frac{9}{\sqrt{u}(9u^2-1)} (2u du) \\
 & = \int \frac{18}{9u^2-1} du = \int \frac{18}{(3u-1)(3u+1)} du \\
 & = \int \frac{9}{3u-1} - \frac{9}{3u+1} du = 3 \ln |3u-1| - 3 \ln |3u+1| + C \\
 & = 3 [\ln |3u-1| - \ln |3u+1|] + C = 3 \ln \left| \frac{3u-1}{3u+1} \right| + C \\
 & = 3 \ln \left| \frac{3\sqrt{x}-1}{3\sqrt{x}+1} \right| + C
 \end{aligned}$$

Question 73 (***)+

Use an appropriate integration method to find an exact value for each of the following integrals

a) $\int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \ dx.$

b) $\int_1^e 4x \ln x \ dx.$

$\boxed{\frac{1}{2}}, \boxed{e^2+1}$

$(a) \int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \ dx = \int_0^{\frac{\pi}{4}} \cos 2x \ dx = \left[\frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}} = \frac{1}{2} \sin \frac{\pi}{2} - \frac{1}{2} \sin 0 = \frac{1}{2}$
$(b) \int_1^e 4x \ln x \ dx = \dots \text{by parts & ignoring limits}$ $= 2x^2 \ln x - \int 2x^2 dx$ $= 2x^2 \ln x - \left[\frac{2x^3}{3} \right]$ $= 2x^2 \ln x - x^3 + C$ $= \text{EVALUATE LIMITS} \dots$ $= [2x^2 \ln x - x^3]_1^e$ $= (2e^2 \ln e - e^3) - (2 \cdot 1^2 - 1)$ $= e^2 + 1$

Question 74 (***)+

Use the substitution $u = x^2$, followed by integration by parts to find

$$\int x^3 e^{x^2} dx.$$

$\boxed{\quad}, \boxed{\frac{1}{2} e^{x^2} (x^2 - 1) + C}$

$\int x^3 e^{x^2} dx = \int x^2 e^u \frac{du}{2x} = \int \frac{1}{2} x^2 e^u du$ $= \frac{1}{2} \int u e^u du$ NOW BY PARTS $\begin{cases} u = u^2 \\ du = 2u \\ du = \frac{du}{2u} \end{cases}$ $= \frac{1}{2} u e^u - \int \frac{1}{2} u e^u du$ $= \frac{1}{2} u e^u - \frac{1}{2} e^u + C$ $= \frac{1}{2} e^u (u - 1) + C$ $= \frac{1}{2} e^{x^2} (x^2 - 1) + C$	<p>SUBSTITUTION</p> $\begin{cases} u = x^2 \\ \frac{du}{dx} = 2x \\ du = \frac{du}{dx} dx \end{cases}$
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Question 75 (***)+

Use integration by parts to find the exact value of

$$\int_{\sqrt{e}}^e 16x^3 \ln x \, dx.$$

$$3e^2(e^2 - 1)$$

$$\begin{aligned}
 \int_{\sqrt{e}}^e 16x^3 \ln x \, dx &= \dots \text{ by parts & ignoring limits} \\
 &= 4x^4 \ln x - \int 4x^3 \, dx \\
 &= 4x^4 \ln x - x^4 + C \\
 &\dots \text{ RESTORE DOCE UNITS} \\
 &= [4x^4 \ln x - x^4]_{\sqrt{e}}^e = (4e^4 \ln e - e^4) - (4e^2 \ln e^2 - e^2) \\
 &= (4e^4 \ln e - e^4) - (2e^2 \ln e - e^2) = (4e^4 - e^4) - (2e^2 - e^2) \\
 &= 3e^4 - e^2 = 3e^2(e^2 - 1)
 \end{aligned}$$

Question 76 (***)+

By using the substitution $u = \sqrt{2x-3}$, or otherwise, find an expression for

$$\int (2x-1)\sqrt{2x-3} \, dx.$$

$$\frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C$$

$$\begin{aligned}
 \int (2x-1)\sqrt{2x-3} \, dx &= \dots \text{ by substitution} \\
 &= \int (2x-1) u \, (u \, du) = \int (2x-1) u^2 \, du \\
 &= \int (2^2+2) u^2 \, du = \int u^4 + 2u^2 \, du \\
 &= \frac{1}{5}u^5 + \frac{2}{3}u^3 = \frac{1}{5}(2x-3)^{\frac{5}{2}} + \frac{2}{3}(2x-3)^{\frac{3}{2}} + C
 \end{aligned}$$

Question 77 (***)+

$$\frac{3x^2 - 2x + 1}{2x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

a) Determine the value of each of the constants A , B and C .

b) Evaluate

$$\int_4^9 \frac{3x^2 - 2x + 1}{2x(x-1)^2} dx,$$

giving the answer in the form $p + q \ln 4$, where p and q are constants.

$$A = \frac{1}{2}, \quad B = 1, \quad C = 1, \quad \frac{5}{24} + \ln 4$$

(a)

$$\frac{3x^2 - 2x + 1}{2x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\frac{1}{2}(3x^2 - 2x + 1) \equiv A(x-1)^2 + Bx(x-1) + Cx$$

- If $x=1 \Rightarrow 1 = C$
- If $x=0 \Rightarrow \frac{1}{2} = A$
- If $x=2 \Rightarrow \frac{3}{2} = A+2B+2C$
 $\frac{3}{2} = \frac{1}{2} + 2B + 2$
 $2B = 2$
 $B = 1$

$A = \frac{1}{2}$
 $B = 1$
 $C = 1$

(b)

$$\int_4^9 \frac{3x^2 - 2x + 1}{2x(x-1)^2} dx = \int_4^9 \frac{\frac{1}{2}}{x} + \frac{1}{x-1} + (x-1)^2 dx$$

$$= \left[\frac{1}{2} \ln |x| + \ln|x-1| - (x-1)^3 \right]_4^9$$

$$= \left[\frac{1}{2} \ln 9 + \ln 8 - \frac{1}{8} \right] - \left[\frac{1}{2} \ln 4 + \ln 3 - \frac{1}{8} \right]$$

$$= \ln 8 + \ln 2 - \frac{1}{8} - 12 \rightarrow \ln 16 + \frac{5}{8}$$

$$= \ln 16 + \frac{5}{8}$$

Question 78 (***)+

It is given that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x.$$

- a) Prove the above trigonometric identity, by writing $\sin 3x$ as $\sin(2x+x)$.
- b) Hence, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^3 x \, dx.$$

2
3

$(a) \quad \begin{aligned} \sin 3x &= \sin(2x+x) = \sin(2x)\cos x + \cos(2x)\sin x \\ &= (2\sin x\cos x)\cos x + (1-2\cos^2 x)\sin x \\ &= 2\sin x\cos^2 x + \sin x - 2\sin x\cos^2 x \\ &= 2\sin x(1-\cos^2 x) + \sin x - 2\sin x\cos^2 x \\ &= 2\sin x - 2\sin x\cos^2 x + \sin x - 2\sin x\cos^2 x \\ &= 3\sin x - 4\sin^3 x \\ &= 4\sin^3 x \end{aligned}$
$(b) \quad \begin{aligned} \int_0^{\frac{\pi}{2}} \sin^3 x \, dx &= \dots \quad \sin 3x = 3\sin x - 4\sin^3 x \\ 4\sin^3 x &= 3\sin x - \sin^3 x \\ \sin^3 x &= \frac{3}{4}\sin x - \frac{1}{4}\sin^3 x \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{3}{4}\sin x - \frac{1}{4}\sin^3 x \right) \, dx \\ &= \left[\frac{3}{4}\cos x + \frac{1}{4}\cos^3 x \right]_0^{\frac{\pi}{2}} = \\ &= \left[\frac{3}{4}\cos \frac{\pi}{2} + \frac{1}{4}\cos \frac{3\pi}{2} \right] - \left[\frac{3}{4}\cos 0 + \frac{1}{4}\cos 0 \right] \\ &= -\left(\frac{3}{4} + \frac{1}{4} \right) = -\frac{4}{4} = -1 \end{aligned}$

Question 79 (***)+

Use integration by parts to find an exact value for

$$\int_1^{\frac{\pi}{3}} 2 \sin x \ln(\sec x) \, dx.$$

[1 - ln 2]

$$\begin{aligned} & \int_0^{\frac{\pi}{3}} 2 \sin x \ln(\sec x) \, dx = \dots \text{ by parts & ignoring limits} \\ &= -2 \cos x \ln(\sec x) - \int -2 \cos x \ln(\sec x) \, dx \\ &= 2 \cos x \ln(\sec x) + \int 2 \cos x \, dx \\ &= 2 \cos x \ln(\sec x) - 2 \cos x + C \\ &= \dots \text{ limits} \\ &= [-2 \cos x \ln(\sec x) - 2 \cos x]_0^{\frac{\pi}{3}} = [2 \cos x \ln(\sec x) + 2 \cos x]_0^{\frac{\pi}{3}} \\ &= [0 + 2] - [\sqrt{2} + 2] = 1 - \ln 2 \end{aligned}$$

Question 80 (***)+

x	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$
y	0	0.1632		0.2500

The table above shows tabulated values for the equation

$$y = \sin x \cos 2x, \quad 0 \leq x \leq \frac{\pi}{6}.$$

- a) Complete the missing value in the table. (1)
- b) Use the trapezium rule with all the values from the table to find an approximate value for (3)
- c) By using the substitution $u = \cos x$, or otherwise, find an exact value for (6)

$$\int_0^{\frac{\pi}{6}} \sin x \cos 2x \, dx.$$

, [0.2620] , [0.096] , $\boxed{\frac{1}{12}(3\sqrt{3}-4)}$

a) $y = \sin \frac{\pi}{6} \cos 2(0) = 0.2500$

b) $\int_0^{\frac{\pi}{6}} \sin x \cos 2x \, dx = \frac{1}{2} \left[-\frac{1}{2} \sin 2x + \frac{1}{2} \cos x + 2x \right]_0^{\frac{\pi}{6}}$
 $= \frac{\pi}{12} \left[0 + 0.25 + 2(0.1632 + 0.2620) \right]$
 $= 0.0802\dots$
 ≈ 0.096

c) $\int_0^{\frac{\pi}{6}} \sin x \cos 2x \, dx$
 $= \int_{\cos 0}^{\cos \frac{\pi}{6}} \sin u \cos 2u \, du$
 $= \int_{1}^{\frac{1}{2}} \sin u \cos 2u \, du$
 $= \int_{1}^{\frac{1}{2}} -\frac{1}{2} \sin 2u \, du = -\frac{1}{4} \cos 2u \Big|_1^{\frac{1}{2}}$
 $= \left[\frac{2u^2 - 1}{8} \right]_1^{\frac{1}{2}} = \left(\frac{1}{8} - \frac{1}{8} \right) \left(\frac{2}{3} \times \frac{2\pi}{3} - \frac{1}{2} \right)$
 $= -\frac{1}{3} - \left(\frac{1}{8} \sqrt{3} - \frac{1}{8} \right) = -\frac{1}{2} + \frac{1}{8} \sqrt{3}$
 $\approx \frac{1}{12}(3\sqrt{3}-4)$

Question 81 (***)+

$$\frac{12x^2+x+3}{(6x+1)(2x^2+1)} \equiv \frac{A}{6x+1} + \frac{Bx+C}{2x^2+1}.$$

- a) Determine the value of each of the constants A , B and C .
- b) Evaluate

$$\int_0^2 \frac{12x^2+x+3}{(6x+1)(2x^2+1)} dx,$$

giving the answer in the form $p \ln q$, where p and q are constants.

$A = 3$	$B = 1$	$C = 0$	$\frac{1}{2} \ln 39$
---------	---------	---------	----------------------

(a)

$$\frac{12x^2+x+3}{(6x+1)(2x^2+1)} \equiv \frac{A}{6x+1} + \frac{Bx+C}{2x^2+1}$$

$$12x^2+x+3 \equiv A(2x^2+1) + (6x+1)(Bx+C)$$

- If $x = -\frac{1}{6}$: $\frac{13}{6} = \frac{13}{6}A \Rightarrow A = 3$
- If $x = 0$: $3 = A+C \Rightarrow C = 0$
- If $x = 1$: $16 = 34 + 7(B+C)$
 $16 = 9 + 7B$
 $B = 1$

(b)

$$\int_0^2 \frac{12x^2+x+3}{(6x+1)(2x^2+1)} dx = \int_0^2 \frac{\frac{3}{6x+1} + \frac{x}{2x^2+1}}{dx} dx$$

$$= \int_0^2 \frac{\frac{3}{6x+1} + \frac{1}{4} \left(\frac{4x}{2x^2+1} \right) dx}{dx} dx$$

↑ For the term $\frac{x}{2x^2+1}$

$$= \left[\frac{1}{2} \ln(6x+1) + \frac{1}{4} \ln(2x^2+1) \right]_0^2 = \left(\frac{1}{2} \ln 13 + \frac{1}{4} \ln 9 \right) - \left(\frac{1}{2} \ln 1 + \frac{1}{4} \ln 1 \right)$$

$$= \frac{1}{2} \ln 13 + \frac{1}{2} \ln 3 = \frac{1}{2} (\ln 13 + \ln 3) = \frac{1}{2} \ln 39 //$$

Question 82 (***)+

By using the trigonometric identity

$$\cos 2\theta = 2\cos^2 \theta - 1$$

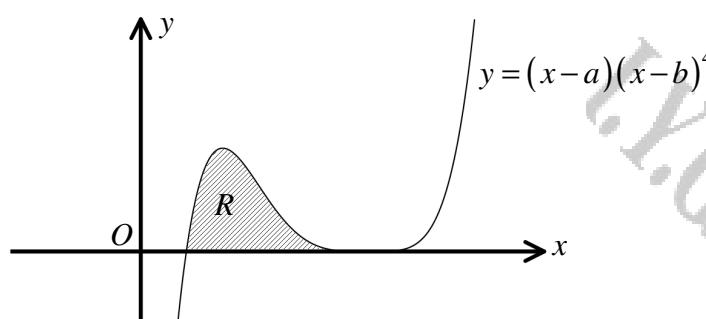
and the fact that $\frac{d}{dx}(\tan x) = \sec^2 x$, show clearly that

$$\int_0^{\frac{\pi}{3}} \frac{1}{1+\cos x} dx = \frac{\sqrt{3}}{3}.$$

proof

$$\begin{aligned} & \int_0^{\frac{\pi}{3}} \frac{1}{1+\cos x} dx = \dots \text{ IDENTITIES } \dots \\ & \quad \begin{array}{l} \cos 2\theta = 2\cos^2 \theta - 1 \\ \cos \theta = 2\cos^2 \frac{\theta}{2} - 1 \end{array} \\ & = \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{1 + 2\cos^2 \frac{x}{2} - 1}} dx = \int_0^{\frac{\pi}{3}} \frac{1}{2\cos^2 \frac{x}{2}} dx = \int_0^{\frac{\pi}{3}} \frac{1}{2} \sec^2 \frac{x}{2} dx \\ & = \left[\tan \frac{x}{2} \right]_0^{\frac{\pi}{3}} = \tan \frac{\pi}{6} - \tan 0 = \frac{\sqrt{3}}{3} \quad \text{as required} \end{aligned}$$

Question 83 (***)+



The figure above shows the graph of the curve with equation

$$y = (x-a)(x-b)^4,$$

where a and b are positive constants.

The shaded region R is bounded by the curve and the x axis.

By using integration by parts, or otherwise, show that the area of the shaded region is

$$\frac{1}{30}(a-b)^6.$$

proof

*In this part
a = critical point*

$$\begin{aligned}
 R &= \int_a^b (x-a)(x-b)^4 dx \\
 &\Rightarrow R = \left[\frac{1}{2}(x-a)(x-b)^5 \right]_a^b - \int_a^b \frac{1}{2}(x-b)^5 dx \\
 &\Rightarrow R = \left[\frac{1}{2}(a-a)(a-b)^5 - \frac{1}{12}(a-b)^6 \right]_a^b \\
 &\Rightarrow R = (0-0) - (0 - \frac{1}{12}(a-b)^6) \\
 &\Rightarrow R = \frac{1}{12}(a-b)^6
 \end{aligned}$$

By part

$$\frac{1}{2}(a-b)^6 = \frac{1}{12}(a-b)^6 + \frac{1}{12}(a-b)^6$$

As required

Question 84 (***)+

By using the substitution $u = \sqrt{x}$, or otherwise, show that

$$\int_0^{36} \frac{1}{\sqrt{x}(\sqrt{x}+2)} dx = \ln 16.$$

[proof]

Method 1 (Substitution $u = \sqrt{x}$):

$$\begin{aligned} \int_0^{36} \frac{\frac{1}{2}\sqrt{x}}{\sqrt{x}(2\sqrt{x}+2)} dx &= \int_0^{36} \frac{1}{2(2u+2)} du \quad \text{By Reckle Chain Rule} \\ &= \left[2\ln|2u+2| \right]_0^{36} = 2[\ln 8 - \ln 2] \\ &= 2(\ln 4) = \ln 16 \end{aligned}$$

Method 2 (Standard Algebraic Manipulation):

$$\begin{aligned} \int_0^{36} \frac{1}{\sqrt{x}(2\sqrt{x}+2)} dx &= \int_0^6 \frac{1}{2u(u+2)} 2u du \\ &= \int_0^6 \frac{1}{u+2} du = \left[2\ln|u+2| \right]_0^6 \\ &= 2\ln 8 - 2\ln 2 = 2(\ln 8 - \ln 2) \\ &= 2\ln 4 = \ln 16 \end{aligned}$$

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Question 85 (***)+

By using the substitution $u = 3x - 1$, or otherwise, find

$$\int \frac{9x^2}{3x-1} dx.$$

$$\boxed{\frac{1}{6}(3x-1)^2 + \frac{2}{3}(3x-1) + \frac{1}{3}\ln|3x-1| + C}$$

Method:

$$\begin{aligned} \int \frac{9x^2}{3x-1} dx &= \int \frac{2x^2}{u} \frac{du}{3} \\ &= \int \frac{(u+1)^2}{3u} du = \int \frac{u^2+2u+1}{3u} du \\ &= \int \frac{1}{3}u + \frac{2}{3} + \frac{1}{3u} + C \\ &= \frac{1}{6}u^2 + \frac{2}{3}u + \frac{1}{3}\ln|u| + C \\ &= \frac{1}{6}(3x-1)^2 + \frac{2}{3}(3x-1) + \frac{1}{3}\ln|3x-1| + C \\ &= \frac{1}{6}(3x-1)^2 + 2x + \frac{1}{3}\ln|3x-1| + C \end{aligned}$$

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Question 86 (***)+

By using the substitution $u = 2x + 3$, or otherwise, show clearly that

$$\int_{-1}^0 6 \ln(2x+3) \, dx = 9\ln 3 - 6.$$

[proof]

$ \begin{aligned} \int_{-1}^0 6 \ln(2x+3) \, dx &= \dots \text{ USING THE SUBSTITUTION (WHY)} \\ &= \int_1^3 6 \ln u \times \frac{du}{2} \\ &= \int_1^3 3 \ln u \, du \\ &= \dots \text{ by parts (ignoring limits)} \\ &= 3u \ln u - \int 3u \left(\frac{1}{u}\right) \, du \\ &= 3u \ln u - \int 3 \, du \\ &= 3u \ln u - 3u + C \\ &= \dots \text{ limits} \dots \\ &= [3u \ln u - 3u]_1^3 \\ &= (9\ln 3 - 9) - (3\ln 1 - 3) \\ &= 9\ln 3 - 6 \end{aligned} $	$ \begin{array}{l} u = 2x+3 \\ \frac{du}{dx} = 2 \\ \frac{du}{2} = \frac{du}{dx} \\ 2x+1, u=3 \\ 2x+1, u=1 \end{array} $ $ \begin{array}{c cc} \ln u & \frac{1}{u} & \\ \hline 3u & 3 & \\ \end{array} $
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Question 87 (***)+

By using the substitution $u = \tan x$, or otherwise, find

$$\int \sec^4 x \, dx.$$

$\tan x + \frac{1}{3} \tan^3 x + C$

$ \begin{aligned} \int \sec^4 x \, dx &= \int \sec^2 x \frac{\sec^2 x}{\sec^2 x} \, dx = \int \sec^2 u \, du \\ \text{BUT } \sec^2 u + \tan^2 u &= \sec^2 u \\ &= \int 1 + \tan^2 u \, du = \int 1 + u^2 \, du \\ &= u + \frac{1}{3}u^3 + C = \tan x + \frac{1}{3}\tan^3 x + C \end{aligned} $	$ \begin{array}{l} u = \tan x \\ \frac{du}{dx} = \sec^2 x \\ \frac{du}{dx} = \frac{du}{dx} \cdot \frac{1}{\sec^2 x} \end{array} $
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Question 88 (***)+

$$f(t) \equiv \frac{2}{(t-1)(t+1)} \equiv \frac{A}{(t-1)} + \frac{B}{(t+1)}.$$

- a) Find the value of each of the constants A and B in the above identity.
 b) Use the substitution $x = t^2 - 2$, $t > 0$ to show that

$$\int_4^9 \frac{1}{(x+1)\sqrt{x+2}} dx = \ln \frac{3}{2}.$$

A = 2, B = 2

a) $f(t) \equiv \frac{2}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1}$

$2 \equiv A(t+1) + B(t-1)$

If $t=1 \Rightarrow 2 = 2B \Rightarrow B=1$
 If $t=-1 \Rightarrow 2 = -2A \Rightarrow A=-1$

b) Let $2 = t^2 - 2$, $t > 0$

$\frac{dx}{dt} = 2t \Rightarrow dt = \frac{dx}{2t}$
 $t=2 \Rightarrow x=2$
 $t=7 \Rightarrow x=49$

$\int_2^9 \frac{1}{(x+1)\sqrt{x+2}} dx = \dots \text{APPLY THE SUBSTITUTION}$

$= \int_2^9 \frac{1}{(t^2+2t)\sqrt{t^2+2t+2}} (2t \, dt)$
 $= \int_2^9 \frac{2t}{(t^2+2t)t} \, dt = \int_2^9 \frac{2}{t^2+2t} \, dt$
 $= \int_2^9 \frac{2}{(t+2)(t)} \, dt = \int_2^9 \frac{1}{t+1} - \frac{1}{t} \, dt$
 $= \left[\ln|t+1| - \ln|t| \right]_2^9$
 $= (\ln 2 - \ln 4) - (\ln 9 - \ln 3) = \ln \frac{1}{2} + \ln 3$
 $= \ln \frac{3}{2}$

Question 89 (***)+

x	0	$\frac{2\pi}{5}$	$\frac{4\pi}{5}$	$\frac{6\pi}{5}$	$\frac{8\pi}{5}$	2π
y	0	0.2031	0.8602			0

The table above shows some tabulated values for the equation

$$y = \sin^3\left(\frac{1}{2}x\right), \quad 0 \leq x \leq 2\pi.$$

- a) Complete the missing values in the table.
 - b) Use the trapezium rule with all the values from the table to find an approximate value for
- $$\int_0^{2\pi} \sin^3\left(\frac{1}{2}x\right) dx.$$
- c) By using the substitution $u = \cos\left(\frac{1}{2}x\right)$, or otherwise, find the value of the integral of part (b).

, [0.8602, 0.2031] , [2.672] , $\left[\frac{8}{3}\right]$

<p>a) $y = (\sin \frac{6\pi}{5})^3 = 0.8602$ $y = (\sin \frac{8\pi}{5})^3 = 0.2031$</p> <p>b) $\int_0^{2\pi} \sin^3\left(\frac{1}{2}x\right) dx = \frac{\pi}{2} \left[\frac{1}{2} \sin^2(2x) + \frac{1}{2} \cos(2x) \right]_0^{2\pi}$ $= \frac{\pi}{2} \left[0 + 2(0.2031 + 0.8602 + \dots + 0.2031) \right]$ $\approx 2.6723\pi \dots$ ≈ 2.672</p>	<p>c) $u = \cos\left(\frac{1}{2}x\right)$ $\frac{du}{dx} = -\frac{1}{2}\sin\left(\frac{1}{2}x\right)$ $du = -\frac{1}{2}\sin\left(\frac{1}{2}x\right) dx$ $2 = u \Rightarrow u = 1$ $-2\pi = 2x \Rightarrow x = -\pi$ $0 = 2x \Rightarrow x = 0$ $\int_{-1}^1 2\sin^2(u) du$ $= \int_{-1}^1 2(1 - \cos^2(u)) du$ $= \int_{-1}^1 2(1 - u^2) du$ $= \int_{-1}^1 2 - 2u^2 du$ $= \left[2u - \frac{2}{3}u^3 \right]_{-1}^1$ $= \left(2 - \frac{2}{3} \right) - \left(-2 + \frac{2}{3} \right)$ $= \frac{4}{3} - \left(-\frac{4}{3} \right) = \frac{8}{3}$</p>
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Question 90 (*)**

Use trigonometric identities to integrate

$$\int \frac{\cos 2x}{1-\cos^2 2x} dx.$$

, $-\frac{1}{2}\operatorname{cosec} 2x + C$

$$\begin{aligned} \int \frac{\cos 2x}{1-\cos^2 2x} dx &= \int \frac{\cos 2x}{\sin^2 2x} dx = \int \frac{\cos 2x}{\sin 2x \times \sin 2x} dx \\ \int \cot 2x \operatorname{cosec} 2x dx &= -\frac{1}{2} \operatorname{cosec} 2x + C // \end{aligned}$$

Question 91 (**)**

By using the substitution $u = 2x^{\frac{5}{2}} + 1$, or otherwise, find an exact simplified value for

$$\int_0^1 \frac{10x^4}{2x^{\frac{5}{2}}+1} dx.$$

, $2 - \ln 3$

$$\begin{aligned} \int_0^1 \frac{10x^4}{2x^{\frac{5}{2}}+1} dx &= \dots \text{by substitution} \\ &= \int_1^3 \frac{10x^4}{u} \frac{du}{5u^{\frac{3}{2}}} = \int_1^3 \frac{2u^{\frac{5}{2}}}{u} du \\ &= \int_1^3 \frac{u-1}{u} du = \int_1^3 1 - \frac{1}{u} du \\ &= \left[u - \ln|u| \right]_1^3 = (3 - \ln 3) - (1 - \ln 1) \\ &= 2 - \ln 3 // \end{aligned}$$

$$\begin{aligned} u &= 2x^{\frac{5}{2}}+1 \\ \frac{du}{dx} &= 5x^{\frac{3}{2}} \\ du &= \frac{du}{5x^{\frac{3}{2}}} \\ 2x^{\frac{5}{2}} &\mapsto u=1 \\ 2x^{\frac{5}{2}}+1 &\mapsto u=3 \\ 2x^{\frac{5}{2}} &= u-1 \end{aligned}$$

Question 92 (****)

$$\frac{2u^2}{u-1} \equiv Au + B + \frac{C}{u-1}.$$

- a) Find the value of each of the constants A , B and C in the above identity.
 b) Use the substitution $u = \sqrt{x}$ to show

$$\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1} dx = 7 + 2\ln 2.$$

A = 2, B = 2, C = 2

$ \begin{aligned} \text{(a)} \quad \frac{2u^2}{u-1} &\equiv Au + B + \frac{C}{u-1} \\ 2u^2 &\equiv A(u-1) + B(u-1) + C \\ 2u^2 &\equiv Au^2 - Au + Bu - B + C \\ 2u^2 &\equiv Au^2 + (B-A)u + (C-B) \end{aligned} $	<ul style="list-style-type: none"> • $A = 2$ • $B - A = 0$ • $B = 2$ • $C - B = 0$ • $C = 2$
$ \begin{aligned} \text{(b)} \quad \int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1} dx &= \dots \text{ by substitution} \\ &= \int_2^3 \frac{u}{u-1} (2u du) \approx \int_2^3 \frac{2u^2}{u-1} du = \int_2^3 2u^2 + \frac{2}{u-1} du \\ &= \left[u^3 + 2u + 2\ln u-1 \right]_2^3 \\ &= (7+6+2\ln 2) - (4+4+2\ln 1) = 7 + 2\ln 2 \end{aligned} $	$ \begin{aligned} u &\sim \sqrt{x} \\ u^2 &\sim x \\ 2u \frac{du}{dx} &= 1 \\ du &= \frac{1}{2u} du \\ x=4, u=2 & \\ x=9, u=3 & \end{aligned} $

Question 93 (****)

$$\int_1^9 \frac{1}{2x(1+\sqrt{x})} dx.$$

- a) Show that the substitution $u = \sqrt{x}$ transforms the above integral to

$$\int_{x_1}^{x_2} \frac{1}{u(u+1)} du,$$

where x_1 and x_2 are constants to be found.

- b) Hence find an exact value for the original integral.

$$\boxed{\ln\left(\frac{3}{2}\right)}$$

$$\begin{aligned}
 & \text{(a)} \int_1^9 \frac{1}{2x(1+\sqrt{x})} dx \dots \text{USING THE SUBSTITUTION } u = \sqrt{x} \\
 &= \int_1^3 \frac{1}{2x(1+u)} 2u du = \int_1^3 \frac{u}{x(1+u)} du = \int_1^3 \frac{u}{u^2(1+u)} du \\
 &= \int_1^3 \frac{1}{u(u+1)} du \quad \text{AS REQUIRED} \\
 & \text{(b)} \text{BY PARTIAL FRACTION } \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1} \quad \text{IF } u=0, u=A \\
 & 1 = A(u+1) + Bu \quad \text{IF } u=-1, u=B \\
 & \dots = \int_1^3 \left(\frac{1}{u} - \frac{1}{u+1} \right) du = \left[\ln|u| - \ln|u+1| \right]_1^3 \\
 &= (\ln 3 - \ln 1) - (\ln 1 - \ln 2) = \ln 3 - \ln 1 + \ln 2 \\
 &= \ln \frac{3 \times 2}{1} = \ln \frac{3}{2}
 \end{aligned}$$

Question 94 (****)

$$f(x) \equiv \frac{x^3}{x^2 - 4}, \quad x \neq \pm 2.$$

- a) Use a suitable substitution to show that

$$\int_{\sqrt{6}}^{\sqrt{8}} f(x) \, dx = 1 + \ln 4.$$

- b) Express $f(x)$ in the form

$$Ax + B + \frac{C}{x-2} + \frac{D}{x+2},$$

where A , B , C and D are constants to be found.

- c) Use the result part (b) to verify the result of part (a).

A=1, B=0, C=2, D=2

(a) $\int_{\sqrt{6}}^{\sqrt{8}} \frac{x^3}{x^2 - 4} \, dx = \dots$ substitution first...

$$\begin{aligned} &= \int_{-2}^4 \frac{x^3}{u} \frac{du}{dx} \, dx = \int_{-2}^4 \frac{x^3}{2u} \, du = \int_{-2}^4 \frac{u+4}{2u} \, du \\ &= \int_{-2}^4 \frac{1}{2} + \frac{2}{u} \, du = \left[\frac{u}{2} + 2\ln|u| \right]_2^4 \\ &= (2 + 2\ln 4) - (1 + 2\ln 2) = 1 + 2\ln 4 - 2\ln 2 \\ &= 1 + 2[\ln 4 - \ln 2] = 1 + 2\ln 2 = 1 + \ln 4. \end{aligned}$$

$u = x^2 - 4$
 $\frac{du}{dx} = 2x$
 $du = \frac{du}{dx} dx$
 $2u = x^2 - 4$
 $x^2 = u+4$
 $2\ln|u|$

(b) $\frac{x^3}{x^2 - 4} = \frac{x^3}{(x-2)(x+2)} = Ax + B + \frac{C}{x-2} + \frac{D}{x+2}$

$$\begin{aligned} x^3 &\equiv (Ax+B)(x-2)(x+2) + C(x+2) + D(x-2) \\ \bullet \text{if } x=2, \quad 8 &\equiv 4A + 8 \Rightarrow A=2 \\ \bullet \text{if } x=-2, \quad -8 &\equiv -4B \Rightarrow B=2 \\ \bullet \text{if } x=0, \quad 0 &\equiv 8C + 2D \Rightarrow C=0, D=0 \\ \bullet \text{if } x=1, \quad 1 &\equiv (A+B)(-1)(3) + 3C - D \Rightarrow A=1, B=2, C=0, D=2 \end{aligned}$$

(c) $\int_{\sqrt{6}}^{\sqrt{8}} \frac{x^3}{x^2 - 4} \, dx = \int_{\sqrt{6}}^{\sqrt{8}} x + \frac{2}{x-2} + \frac{2}{x+2} \, dx = \left[\frac{1}{2}x^2 + 2\ln|x-2| + 2\ln|x+2| \right]_{\sqrt{6}}^{\sqrt{8}}$

$$\begin{aligned} &= \left[\frac{1}{2}x^2 + 2\ln|(x-2)(x+2)| \right]_{\sqrt{6}}^{\sqrt{8}} = \left[\frac{1}{2}x^2 + 2\ln|x^2-4| \right]_{\sqrt{6}}^{\sqrt{8}} \\ &= (4 + 2\ln 4) - (3 + 2\ln 2) = 1 + 2\ln 4 - 2\ln 2 \\ &= 1 + \ln 16 - \ln 4 = 1 + \ln 4. \end{aligned}$$

Question 95 (****)

By using the cosine double angle identities and the fact that $\frac{d}{dx}(\tan x) = \sec^2 x$, show clearly that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cos 2x}{1+\cos 2x} dx = \frac{1}{6}(4\sqrt{3}-\pi).$$

proof

$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cos 2x}{1+\cos 2x} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2\sin^2 x}{2\cos^2 x} dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2 x dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x - 1 dx = \left[\tan x - x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \left[\tan \frac{\pi}{3} - \frac{\pi}{3} \right] - \left[\tan \frac{\pi}{6} - \frac{\pi}{6} \right] = \left(\sqrt{3} - \frac{\pi}{3} \right) - \left(\frac{\sqrt{3}}{3} - \frac{\pi}{6} \right) \\ &= \frac{2\sqrt{3}}{3} - \frac{\pi}{6} = \frac{1}{6} [4\sqrt{3} - \pi] \quad \text{As required} \end{aligned}$$

Question 96 (****)

Use the substitution $x = \sin \theta$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}} dx.$$

 $\frac{\sqrt{3}}{3}$

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}} dx &= \int_0^{\frac{\pi}{2}} \frac{1}{(-\sin^2 \theta)^{\frac{3}{2}}} \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{(\cos^2 \theta)^{\frac{3}{2}}} d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\cos^3 \theta} d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 \theta} d\theta = \int_0^{\frac{\pi}{2}} \sec^2 \theta d\theta \\ &= \left[\tan \theta \right]_0^{\frac{\pi}{2}} = \tan \frac{\pi}{2} - \tan 0 = \frac{\sqrt{3}}{3} \end{aligned}$$

$$\begin{aligned} x &= \sin \theta \\ \frac{dx}{d\theta} &= \cos \theta \\ d\theta &= \cos \theta d\theta \\ \bullet x=0, 0=\sin \theta &\\ \theta &=0 \\ \bullet x=\frac{1}{2}, \frac{1}{2}=\sin \theta &\\ \theta &=\frac{\pi}{6} \end{aligned}$$

Question 97 (****)

$$f(x) \equiv \frac{1}{x(x^2+1)}, \quad x \neq 0.$$

- a) Use the substitution $x = \tan \theta$ to find

$$\int f(x) dx.$$

- b) Find the value of each of the constants A , B and C , so that

$$f(x) \equiv \frac{A + Bx + C}{x^2 + 1}.$$

- c) Use the result of part (b) to independently verify the answer of part (a).

$$\boxed{\ln\left(\frac{x}{\sqrt{x^2+1}}\right)}, \boxed{A=1}, \boxed{B=-1}, \boxed{C=0}$$

(a) $\int \frac{1}{x(x^2+1)} dx$ = by substitution
 $= \int \frac{\sec \theta}{\tan \theta (\sec^2 \theta - 1)} \sec \theta d\theta = \int \frac{\sec^3 \theta}{\tan \theta \sec^2 \theta} d\theta$
 $= \int \frac{\sec \theta}{\tan \theta} d\theta = \ln |\sec \theta| + C$
 $= \ln \left| \frac{x}{\sqrt{x^2+1}} \right| + C$

(b) $\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}$
 $1 \equiv A(x^2+1) + (Bx+C)x$
 $1 \equiv Ax^2 + A + Bx^2 + Cx$
 $1 \equiv (A+B)x^2 + Cx + A$

$\therefore A=1$
 $C=0$
 $B=-1$

(c) $\int \frac{1}{x(x^2+1)} dx = \int \frac{\frac{1}{x}}{x^2+1} - \frac{2}{x^2+1} dx = \int \frac{1}{x^2} - \frac{2}{x^2+1} dx$
 $= \ln |x| - \frac{1}{2} \ln (x^2+1) + C$
 $= \ln |x| - \ln \sqrt{x^2+1} + C$
 $= \ln \left| \frac{x}{\sqrt{x^2+1}} \right| + C$

Question 98 (****)

$$\frac{4x^3 - x^2 + 20x - 4}{x^2 + 4} \equiv Ax + B + \frac{Cx + D}{x^2 + 4}.$$

- a) Determine the value of each of the constants A , B , C and D .
- b) Hence show that

$$\int_0^2 \frac{4x^3 - x^2 + 20x - 4}{x^2 + 4} dx = 6 + 2\ln 2,$$

$$[A = 4], [B = -1], [C = 4], [D = 0]$$

(a) $\frac{4x^3 - x^2 + 20x - 4}{x^2 + 4} \equiv Ax + B + \frac{Cx + D}{x^2 + 4}$

$$4x^3 - x^2 + 20x - 4 \equiv (Ax+8)(x^2+4) + Cx+D$$

$$\equiv Ax^3 + Ax^2 + 8x^2 + 32x + Cx + D$$

$$\equiv Ax^3 + Bx^2 + (4A+C)x + (32+D)$$

$$\therefore A=4, B=-1 \quad 4A+C=20 \quad 4A+D=-4$$

$$16+C=20 \quad -A+D=-4$$

$$C=4 \quad D=0$$

$$4B+D=-4$$

$$-4+D=-4$$

$$D=0$$

(b) $\int_0^2 \frac{4x^3 - x^2 + 20x - 4}{x^2 + 4} dx = \int_0^2 4x - 1 + \frac{4x}{x^2 + 4} dx$

$$= \int_0^2 4x - 1 + 2\left(\frac{2x}{x^2 + 4}\right) dx$$

THIS IS THE PART

$$= \left[2x^2 - 2x + 2\ln(x^2 + 4) \right]_0^2$$

$$= (8 - 2 + 2\ln(2^2 + 4)) - (0 + 2\ln 4)$$

$$= 6 + 2\ln 8 - 2\ln 4 = 6 + 2\ln 2 - 4\ln 2 = 6 + 2\ln 2$$

A 24pt font

Question 99 (****)

Use a cosine double identity and integration by parts to find

$$\int 4x \cos^2 x dx.$$

$$x^2 + x \sin 2x + \frac{1}{2} \cos 2x + C$$

$$\int 4x \cos^2 x dx = \int 4x (\frac{1}{2} + \frac{1}{2} \cos 2x) dx$$

$$= \int 2x + 2x \cos 2x dx$$

$$= \int 2x dx + \int 2x \cos 2x dx$$

$$= \int 2x dx + \int 2x \cos 2x dx$$

$\frac{\cos 2x - 2\sin 2x}{2} + \frac{1}{2} \cos 2x = 2\sin 2x$

$\cos 2x - \frac{1}{2} + \frac{1}{2} \cos 2x$

BY PARTS

$$= x^2 + 2x \sin 2x + \frac{1}{2} \cos 2x + C$$

$\frac{2\sin 2x}{2} + \frac{1}{2} \cos 2x$

Question 100 (**)**

By using the substitution $u = 2x^2 - 8x + 3$, or otherwise, find the exact value

$$\int_4^6 \frac{x-2}{2x^2 - 8x + 3} dx.$$

$$\boxed{\frac{1}{2}\ln 3}$$

$$\begin{aligned}
 \int_4^6 \frac{x-2}{2x^2 - 8x + 3} dx &= \int_3^{27} \frac{x-2}{u} \frac{du}{4u-8} \quad \left\{ \begin{array}{l} u = 2x^2 - 8x + 3 \\ \frac{du}{dx} = 4x-8 \end{array} \right. \\
 &= \int_3^{27} \frac{x-2}{u} \times \frac{1}{4(2x-2)} du = \int_3^{27} \frac{1}{4u} du \quad \left\{ \begin{array}{l} u = 4x-8 \\ du = 4dx \\ x=4 \mapsto u=3 \\ x=6 \mapsto u=27 \end{array} \right. \\
 &= \left[\frac{1}{4} \ln|u| \right]_3^{27} = \frac{1}{4} \ln 27 - \frac{1}{4} \ln 3 \\
 &= \frac{3}{4} \ln 3 - \frac{1}{4} \ln 3 = \frac{1}{2} \ln 3
 \end{aligned}$$

Question 101 (**)**

$$\frac{6u}{(u-2)(u+1)} \equiv \frac{A}{u-2} + \frac{B}{u+1}.$$

a) Find the value of each of the constants A and B in the above identity.

b) By using the substitution $u = \sqrt{x}$, or otherwise, show that

$$\int_0^1 \frac{3}{(\sqrt{x}-2)(\sqrt{x}+1)} dx = -\ln 4.$$

$$\boxed{A=4}, \boxed{B=2}$$

$$\begin{aligned}
 \text{(a)} \quad \frac{6u}{(u-2)(u+1)} &\equiv \frac{A}{u-2} + \frac{B}{u+1} \quad \left\{ \begin{array}{l} \text{if } u=2, 12=3A \\ A=4 \\ \text{if } u=-1, -6=-3B \\ B=2 \end{array} \right. \\
 \frac{6u}{(u-2)(u+1)} &\equiv \frac{4}{u-2} + \frac{2}{u+1} \\
 \int_0^1 \frac{3}{(\sqrt{x}-2)(\sqrt{x}+1)} dx &\equiv \int_0^1 \frac{3}{(\sqrt{x}-2)(\sqrt{x}+1)} (2u dx) \quad \left\{ \begin{array}{l} u=\sqrt{x} \\ u=2 \\ 2u \frac{du}{dx}=1 \\ du=\frac{1}{2u} dx \\ x=1 \mapsto u=1 \\ x=0 \mapsto u=0 \end{array} \right. \\
 &= \int_0^1 \frac{6u}{(\sqrt{x}-2)(\sqrt{x}+1)} du = \int_0^1 \frac{4}{u-2} + \frac{2}{u+1} du \\
 &= \left[4\ln|u-2| + 2\ln|u+1| \right]_0^1 \\
 &= (4\ln 2 + 2\ln 2) - (4\ln 2 + 2\ln 1) \\
 &= 2\ln 2 - 4\ln 2 = -2\ln 2 = -\ln 4
 \end{aligned}$$

Question 102 (**)**

$$\frac{4t^2}{t-1} \equiv At + B + \frac{C}{t-1}.$$

a) Determine the value of each of the constants A , B and C .

b) Use the substitution $t = x^{\frac{1}{4}}$ to show

$$\int_{16}^{81} \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{4}}} dx = 14 + 4\ln 2.$$

_____ , $[A = 4]$, $[B = 4]$, $[C = 4]$

a) BY SUBSTITUTION

$$\frac{4t^2}{t-1} = \frac{4t(t-1) + 4(t-1) + 4}{t-1} = 4t + 4 + \frac{4}{t-1}$$

$\therefore A=B=C=4$

ALTERNATIVE BY ALGEBRAIC DIVISION

$$t-1 \overline{)4t^2}$$

$$\begin{array}{r} 4t^2 + 4t \\ -4t^2 - 4t \\ \hline 4t \\ -4t \\ \hline 4 \end{array}$$

$\therefore A=B=C=4$

ALTERNATIVE BY COMPARING

$$\frac{4t^2}{t-1} = At + B + \frac{C}{t-1}$$

$$\frac{4t^2}{t-1} = \frac{At(t-1) + B(t-1) + C}{t-1}$$

$$\Rightarrow 4t^2 = At^2 - At + Bt - B + C$$

$$\Rightarrow 4t^2 = At^2 + (B-A)t + (C-B)$$

$\therefore A=4$ $B=4$ $C=4$

b) USING THE SUBSTITUTION GIVEN

$$\int_{16}^{81} \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{4}}} dx = \dots$$

$$= \int_2^3 \frac{1}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} (4t^3 dt) = \int_2^3 \frac{4t^3}{t(t-1)} dt$$

$$= \int_2^3 \frac{4t^2}{t-1} dt$$

USING PART (a)

$$= \int_2^3 4t + 4 + \frac{4}{t-1} dt$$

$$= \left[2t^2 + 4t + 4\ln|t-1| \right]_2^3$$

$$= (18 + 12 + 4\ln 2) - (8 + 8 + 4\ln 2)$$

$$= 14 + 4\ln 2$$

AS REQUIRED

$t = x^{\frac{1}{4}}$
 $t^4 = x^{\frac{1}{2}}$
 $t^8 = x^{\frac{1}{4}}$
 $x = t^4$
 $\frac{dx}{dt} = 4t^3$
 $dt = \frac{1}{4t^3} dx$
 $x=16 \mapsto t=2$
 $x=8 \mapsto t=\sqrt[4]{8}$

Question 103 (**)**

It is given that

$$\int_k^{2k} \frac{3x-5}{x(x-1)} dx = \ln 72,$$

determine the value of k , $0 < k < 1$.

, $k = \frac{1}{4}$

SIMPLIFY BY PARTIAL FRACTIONS

$$\frac{3x-5}{2(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$3x-5 \equiv Ax-1+Bx$$

$$\text{IF } 2x-1 \Rightarrow -2 = B$$

$$\Rightarrow B = -2$$

$$\text{IF } 2x=0 \Rightarrow -5 = A$$

$$\Rightarrow A = -5$$

HENCE THE INTEGRAL BECOMES

$$\Rightarrow \int_k^{2k} \frac{3x-5}{2(x-1)} dx = \ln 72$$

$$\Rightarrow \int_k^{2k} \left(-\frac{5}{x} - \frac{2}{x-1} \right) dx = \ln 72$$

$$\Rightarrow \left[5 \ln|x| - 2 \ln|x-1| \right]_k^{2k} = \ln 72$$

$$\Rightarrow \left[5 \ln|2k| - 2 \ln|2k-1| - (5 \ln|k| - 2 \ln|k-1|) \right] = \ln 72$$

$$\Rightarrow 5 \ln|2k| - 2 \ln|2k-1| - 5 \ln|k| + 2 \ln|k-1| = \ln 72$$

$$\Rightarrow 5 \ln \frac{|2k|}{|k|} + 2 \ln \frac{|k-1|}{|2k-1|} = \ln 72$$

$$\Rightarrow 5 \ln 2 + 2 \ln \frac{k-1}{2k-1} = \ln 72$$

$$\Rightarrow 2 \ln \frac{k-1}{2k-1} = \ln 72 - 5 \ln 2$$

$$\Rightarrow 2 \ln \left| \frac{k-1}{2k-1} \right| = \ln \frac{72}{32}$$

$$\Rightarrow 2 \ln \left| \frac{k-1}{2k-1} \right| = \ln \left(\frac{9}{4} \right)^2$$

$$\Rightarrow 2 \ln \left| \frac{k-1}{2k-1} \right| = 2 \ln \frac{3}{2}$$

$$\Rightarrow \frac{k-1}{2k-1} = \frac{3}{2}$$

$$\Rightarrow 4k-3 = 2k-2$$

$$\Rightarrow 4k = 1$$

$$\Rightarrow k = \frac{1}{4}$$

Question 104 (****)

By using the substitution $u = 2x^{\frac{3}{2}} - 1$, or otherwise, find an expression for the integral

$$\int \frac{6x^2}{2x^{\frac{3}{2}} - 1} dx.$$

, $2x^{\frac{1}{2}} + \ln|2x^{\frac{3}{2}} - 1| + C$

USING THE SUBSTITUTION (METHOD)

$$\begin{aligned} \int \frac{6x^2}{2x^{\frac{3}{2}} - 1} dx &= \int \frac{6x^2}{u} \left(\frac{du}{3x^{\frac{1}{2}}} \right) \\ &= \int \frac{6x^2}{3x^{\frac{1}{2}} u} du = \int \frac{2x^{\frac{3}{2}}}{u} du = \int \frac{u+1}{u} du \\ &= \int 1 + \frac{1}{u} du = u + \ln|u| + C \\ &= (2x^{\frac{3}{2}} - 1) + \ln|2x^{\frac{3}{2}} - 1| + C \\ &= 2x^{\frac{3}{2}} + \ln|2x^{\frac{3}{2}} - 1| + C // \end{aligned}$$

$u = 2x^{\frac{3}{2}} - 1$
 $\frac{du}{dx} = 3x^{\frac{1}{2}}$
 $du = 3x^{\frac{1}{2}} dx$
 $\frac{du}{3x^{\frac{1}{2}}} = x^{\frac{1}{2}} dx$
 $\frac{1}{u} = \frac{1}{2x^{\frac{3}{2}} - 1}$

ALTERNATIVE BY MANIPULATION/DIVISION OR RECOGNITION

$$\begin{aligned} \int \frac{6x^2}{2x^{\frac{3}{2}} - 1} dx &= \int \frac{3x^{\frac{1}{2}}(2x^{\frac{3}{2}} - 1) + 3x^{\frac{1}{2}}}{2x^{\frac{3}{2}} - 1} dx \\ &= \int 3x^{\frac{1}{2}} + \frac{3x^{\frac{1}{2}}}{2x^{\frac{3}{2}} - 1} dx \\ &\quad \text{(RECOGNISE THE FORM)} \\ &= \int 3x^{\frac{1}{2}} dx + \int \frac{3x^{\frac{1}{2}}}{2x^{\frac{3}{2}} - 1} dx \\ &\quad \text{This is of the form } \int \frac{f(u)}{g(u)} du = h(u) + C \\ &= 2x^{\frac{3}{2}} + \ln|2x^{\frac{3}{2}} - 1| + C \\ &\quad \text{At Absent} \end{aligned}$$

Question 105 (**)**

Use the substitution $t = \sqrt{1-x^3}$ to show that

$$\int x^5 \sqrt{1-x^3} dx = -\frac{2}{45}(3x^3+2)(1-x^3)^{\frac{3}{2}} + C.$$

[4 marks], proof

USING THE GIVEN SUBSTITUTION

$$\begin{aligned}
 & \int x^5 \sqrt{1-x^3} dx \\
 &= \int x^5 t^{-1} \left(\frac{dt}{-3x^2} \right) dt \\
 &= \int -\frac{x^5}{3} t^{-2} dt \\
 &= \int -\frac{1}{3} t^2 (1-t^2) dt \\
 &= -\frac{1}{3} \int t^2 - t^4 dt \\
 &= -\frac{1}{3} \left[\frac{t^3}{3} - \frac{t^5}{5} \right] + C \\
 &= -\frac{1}{9} t^3 + \frac{1}{15} t^5 + C \\
 &= -\frac{1}{9} t^3 (5t^2 - 3t^2) + C \\
 &= -\frac{1}{9} t^3 (2t^2 - 3t^2) + C \\
 &= -\frac{1}{9} t^3 (2 - 3t^2) + C \\
 &= -\frac{1}{9} (1-x^3)^{\frac{3}{2}} [5 - 3(1-x^3)] + C \\
 &= -\frac{1}{9} (1-x^3)^{\frac{3}{2}} (2 + 3x^2) + C \\
 &= -\frac{2}{45} (1-x^3)^{\frac{3}{2}} (2 + 3x^2) + C
 \end{aligned}$$

AS REQUIRED

Question 106 (**)**

Use an appropriate substitution, followed by partial fractions, to show that

$$\int_{e^3}^{e^5} \frac{5}{2x[(\ln x)^2 + \ln x - 6]} dx = \ln\left(\frac{3}{2}\right).$$

You may assume that the integral converges.

, **proof**

BY SUBSTITUTION

$$u = \ln x \quad u = e^5 \rightarrow u = e^5 e^5 = e^5$$

$$\frac{du}{dx} = \frac{1}{x} \quad u = 3 \rightarrow u = e^3 \rightarrow x = e^3 \approx 3$$

$$dx = e^u du$$

TRANSFORMING THE INTEGRAL

$$\int_{e^3}^{e^5} \frac{5}{2x[(\ln x)^2 + \ln x - 6]} dx \stackrel{u = \ln x}{=} \int_3^{e^5} \frac{5}{2x(u^2 + u - 6)} du$$

$$= \int_3^{e^5} \frac{5}{2(u^2 + u - 6)} du = \frac{5}{2} \int_3^{e^5} \frac{1}{(u+3)(u-2)} du$$

BY PARTIAL FRACTIONS

$$\frac{1}{(u+3)(u-2)} = \frac{A}{u+3} + \frac{B}{u-2}$$

$$1 = A(u-2) + B(u+3)$$

$$\begin{cases} \text{IF } u=2, & 1 = 5B \\ \text{IF } u=-3, & 1 = -5A \end{cases} \quad \therefore A = -\frac{1}{5}, B = \frac{1}{5}$$

FINALLY WE HAVE

$$\dots = \frac{5}{2} \int_3^{e^5} \left(\frac{-\frac{1}{5}}{u+3} + \frac{\frac{1}{5}}{u-2} \right) du = \frac{5}{2} \left[\frac{1}{u+3} + \frac{1}{u-2} \right]_3^{e^5}$$

$$= \frac{5}{2} \left[\ln|u+3| - \ln|u+1| \right]_3^{e^5} = \frac{5}{2} [\ln(e^5 - 6) - (\ln 3 - \ln 2)]$$

$$= \frac{5}{2} [\ln(3 - 6e^5 + 6e^6)] = \frac{5}{2} \ln\left(\frac{6e^6 - 6e^5 + 3}{3}\right) = \frac{5}{2} \ln\left(\frac{3}{2}\right)^2 = \frac{1}{2} \ln\left(\frac{9}{4}\right) = \frac{1}{2} \ln\left(\frac{3}{2}\right)^2$$

Question 107 (**)**

$$\frac{2u^3}{u+1} \equiv Au^2 + Bu + C + \frac{D}{u+1}.$$

- a) Find the value of each of the constants A , B and C in the above identity.
 b) Use the substitution $u = \sqrt{x}$ to show

$$\int_0^1 \frac{x}{1+\sqrt{x}} dx = \frac{5}{3} - 2\ln 2.$$

$[A=2]$, $[B=-2]$, $[C=2]$, $[D=-2]$

(a)

$$\begin{aligned} \frac{2u^3}{u+1} &\equiv 4u^2 + Bu + C + \frac{D}{u+1} \\ 2u^3 &\equiv 4u^2(u+1) + Bu(u+1) + C(u+1) + D \\ 2u^3 &\equiv 4u^3 + 4u^2 + Bu^2 + Bu + Cu + C + D \\ 2u^3 &\equiv 4u^3 + (4+B)^2u^2 + (B+C)u + (C+D) \end{aligned}$$

$$\begin{array}{l} 4=2 \\ 4+B=0 \\ 2+B=0 \\ B=-2 \end{array} \quad \begin{array}{l} B+C=0 \\ -2+C=0 \\ C=2 \end{array} \quad \begin{array}{l} C+D=0 \\ 2+D=0 \\ D=-2 \end{array}$$

(b)

$$\begin{aligned} \int_0^1 \frac{x}{1+\sqrt{x}} dx &= \dots \text{by substitution} \\ &= \int_0^1 \frac{u^2}{1+u} (2u du) = \int_0^1 \frac{2u^3}{u+1} du \\ &= \int_0^1 2u^2 - 2u + 2 - \frac{2}{u+1} du \\ &= \left[\frac{2}{3}u^3 - u^2 + 2u - 2\ln|u+1| \right]_0^1 \\ &= \left(\frac{2}{3}(1)^3 - (1)^2 + 2(1) - 2\ln|1+1| \right) - (0) = \frac{5}{3} - 2\ln 2. \end{aligned}$$

$u = \sqrt{x}$
 $u^2 = x$
 $2u \frac{du}{dx} = 1$
 $2u du = dx$
 $\frac{2u}{2u} = \frac{dx}{2u}$
 $u = 1$

Question 109 (****)

$$\int \frac{\cos x}{1-\cos x} dx.$$

- a) Show by multiplying the numerator and denominator of the integrand by $(1+\cos x)$, that the above integral can eventually be written as

$$\int \cot x \operatorname{cosec} x + \cot^2 x dx.$$

- b) Hence show further that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{1-\cos x} dx = \frac{1}{4}(4\sqrt{2}-\pi).$$

, proof

$$\begin{aligned}
 \text{(a)} \quad & \int \frac{\cos x}{1-\cos x} dx = \int \frac{\cos x(1+\cos x)}{(1-\cos x)(1+\cos x)} dx = \int \frac{\cos x(1+2\cos x)}{1-\cos^2 x} dx \\
 &= \int \frac{\cos x + 2\cos^2 x}{\sin^2 x} dx = \int \frac{\cos x}{\sin^2 x} + \frac{2\cos^2 x}{\sin^2 x} dx = \int \frac{\cos x}{\sin^2 x} \frac{1}{\sin x} + \sin x \cos x dx \\
 &= \int \cot x \operatorname{cosec} x + \cot^2 x dx
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{1-\cos x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot x \operatorname{cosec} x + \cot^2 x) dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\operatorname{cosec} x + \cot^2 x) dx \\
 &= \left[-\operatorname{cosec} x - \cot x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \left[\alpha + \operatorname{cosec} x + \cot x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \left(\frac{\pi}{4} + \sqrt{2} + 0 \right) - \left(\frac{\pi}{4} + 0 \right) = -\frac{\pi}{4} + \sqrt{2} = \frac{1}{4}(4\sqrt{2}-\pi)
 \end{aligned}$$

Question 109 (**)**

By using the substitution $x = 9\sin^2 \theta$, or otherwise, find the exact value of

$$\int_0^{\frac{9}{4}} \frac{1}{\sqrt{x(9-x)}} dx.$$

$\boxed{\frac{\pi}{3}}$

$$\begin{aligned} \int_0^{\frac{9}{4}} \frac{1}{\sqrt{x(9-x)}} dx &= \dots \text{by substitution} \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{9\sin^2\theta(9-9\sin^2\theta)}} 18\sin\theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{18\sin\theta}{\sqrt{9\sin^2\theta(9\cos^2\theta)}} d\theta = \int_0^{\frac{\pi}{2}} \frac{18\sin\theta}{9\sin\theta\cos\theta} d\theta \\ &= \int_0^{\frac{\pi}{2}} 2 d\theta = [2\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} x &= 9\sin^2\theta \\ \frac{dx}{d\theta} &= 18\sin\theta\cos\theta \\ d\theta &= \frac{dx}{18\sin\theta\cos\theta} \\ - & \\ \alpha &= 0, \theta = 0 \\ x &= 9, \theta = \frac{\pi}{2} \\ \theta &= \frac{\pi}{2} \end{aligned}$$

Question 110 (**)**

It is given that

$$\cos^4 \theta \equiv \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta.$$

- a) Prove the validity of the above trigonometric identity.
- b) Use the substitution $u = \sin \theta$ to show

$$\int_0^1 \sqrt{(1-x^2)^3} dx = \frac{3\pi}{16}.$$

$\boxed{\text{proof}}$

$$\begin{aligned} \text{(a)} \quad \cos^4 \theta &= (\cos^2 \theta)^2 = \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)^2 = \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta \\ &= \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) = \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta \end{aligned}$$

$$\begin{aligned} u &= \sin \theta \\ \frac{du}{d\theta} &= \cos \theta \\ du &= \cos \theta d\theta \\ 2\cos \theta, \theta = 0 &= 0 \\ 2\cos \theta, \theta = \frac{\pi}{2} &= \frac{3\pi}{2} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_0^1 \sqrt{(1-x^2)^3} dx &= \dots \text{by substitution} \\ &= \int_0^{\frac{\pi}{2}} \sqrt{(1-\sin^2\theta)^3} \cos \theta d\theta = \int_0^{\frac{\pi}{2}} \cos^2 \theta \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \cos^3 \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta d\theta \\ &= \left[\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta \right]_0^{\frac{\pi}{2}} = \frac{3\pi}{16} - 0 = \frac{3\pi}{16} \end{aligned}$$

Question 111 (**)**

$$y = \frac{x^2 + 2x - 2}{x^2 - 2x + 2}.$$

- a) Find the value of each of the constants A , B and C , so that

$$y \equiv A + \frac{Bx + C}{x^2 - 2x + 2}.$$

- b) Hence, or otherwise, show that

$$\int_{\frac{1}{2}}^{\frac{3}{2}} y \, dx = 1.$$

$$[A=1], [B=4], [C=-4]$$

(a)

$$\begin{aligned} \frac{x^2 + 2x - 2}{x^2 - 2x + 2} &\equiv A + \frac{Bx + C}{x^2 - 2x + 2} \\ x^2 + 2x - 2 &\equiv A(x^2 - 2x + 2) + Bx + C \\ x^2 + 2x - 2 &\equiv Ax^2 - 2Ax + 2A + Bx + C \\ x^2 + 2x - 2 &\equiv Ax^2 + (B-2A)x + (2A+C) \end{aligned}$$

$\begin{array}{l} A=1 \\ B-2A=2 \\ B=4 \end{array}$

 $\begin{array}{l} 2A+C=2 \\ 2+C=-2 \\ C=-4 \end{array}$

(b)

$$\begin{aligned} \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{x^2 + 2x - 2}{x^2 - 2x + 2} \, dx &\stackrel{u=x-1}{=} \int_{\frac{1}{2}}^{\frac{3}{2}} 1 + \frac{B-4}{x^2 - 2x + 2} \, dx = \int_{\frac{1}{2}}^{\frac{3}{2}} 1 + 2 \left(\frac{2x-2}{x^2 - 2x + 2} \right) \, dx \\ &= \left[x + 2 \ln|x^2 - 2x + 2| \right]_{\frac{1}{2}}^{\frac{3}{2}} \\ &= \left(\frac{3}{2} + 2 \ln \frac{5}{4} \right) - \left(\frac{1}{2} + 2 \ln \frac{5}{4} \right) \\ &= 1 \end{aligned}$$

THE B IS OF THE C TYPE

Question 112 (**)**

$$f(x) \equiv \frac{2}{x + \sqrt{2x-1}}, \quad x \geq \frac{1}{2}.$$

- a) Use the substitution $u = \sqrt{2x-1}$ transforms to show

$$\int_1^5 f(x) \, dx \equiv \int_{u_1}^{u_2} \frac{4u}{(u+1)^2} \, du,$$

where u_1 and u_2 are constants to be found.

- b) By using another suitable substitution, or otherwise, show that

$$\int_1^5 f(x) \, dx = -1 + \ln 16.$$

, proof

Method 1: Using the substitution $u = \sqrt{2x-1}$

$$\begin{aligned} & \int_1^5 f(x) \, dx = \int_1^5 \frac{2}{2x+1} (2 \, dx) \\ &= \int_1^5 \frac{2u}{2x+1} \, du = \int_1^5 \frac{4u}{2x+2u} \, du \\ &= \int_1^5 \frac{4u}{(u^2+1)+2u} \, du = \int_1^5 \frac{4u}{u^2+2u+1} \, du \\ &= \int_1^5 \frac{4u}{(u+1)^2} \, du \quad \text{As required} \end{aligned}$$

Method 2: Using another substitution

$$\begin{aligned} & \dots = \int_2^4 \frac{4u}{\sqrt{2u-1}} \, du = \int_2^4 \frac{4(u-1)}{\sqrt{u}} \, du \\ &= 4 \int_2^4 \frac{u-1}{\sqrt{u}} \, du = 4 \int_2^4 \frac{\sqrt{u}-\frac{1}{\sqrt{u}}}{\sqrt{u}} \, du \\ &= 4 \int_2^4 \left(\sqrt{u} - \frac{1}{\sqrt{u}} \right) \, du = 4 \left[\frac{1}{2}u\sqrt{u} + \frac{1}{2}\sqrt{u} \right]_2^4 \\ &= 4 \left[\frac{1}{2}u\sqrt{u} + \frac{1}{2}\sqrt{u} \right]_2^4 = 4 \left[\left(\frac{1}{2}4\sqrt{4} + \frac{1}{2}\sqrt{4} \right) - \left(\frac{1}{2}2\sqrt{2} + \frac{1}{2}\sqrt{2} \right) \right] \\ &\approx 4 \left[\frac{1}{2}4\sqrt{4} + \frac{1}{2}\sqrt{4} - \frac{1}{2}2\sqrt{2} - \frac{1}{2}\sqrt{2} \right] = 4 \left[\frac{1}{2}2\sqrt{2} - \frac{1}{2}\sqrt{2} \right] \\ &= -1 + 4\ln 2 \quad \text{or} \quad -1 + \ln 16 \end{aligned}$$

Question 113 (**)**

Use integration by parts find an exact value in terms of e , for the integral

$$\int_1^e (\ln x)^2 \, dx.$$

e-2

$\begin{aligned} \int_1^e (\ln x)^2 \, dx &= \text{by parts & ignoring limits} \\ &= \int x(\ln x)^2 \, dx \\ &= x(\ln x)^2 - \int 2\ln x \, dx \\ &= \text{by parts again} \\ &= 2(\ln x)^2 - [2x\ln x - \int 2 \, dx] \\ &= 2(\ln x)^2 - 2x\ln x + \int 2 \, dx \\ &\dots \text{limits} = [(\ln x)^2 - 2x\ln x + 2x]_1^e \\ &= (e - 2e + 2e) - (0 - 0 + 2) = e - 2 \end{aligned}$	$\begin{array}{ c c } \hline (\ln x)^2 & 2(\ln x) \times \frac{1}{x} \\ \hline x & 1 \\ \hline \end{array}$ $\begin{array}{ c c } \hline \ln x & \frac{1}{2} \\ \hline x & 2 \\ \hline \end{array}$
---	--

Question 114 (**)**

Use the fact that $\frac{d}{dx}(\sec x) = \sec x \tan x$, to find

$$\int \sin x(1 + \sec^2 x) \, dx.$$

$\sec x - \cos x + C$

$\begin{aligned} \text{Since } \frac{d}{dx}(\sec x) = \sec x \tan x \text{ then} \\ \int \sin x(\sec^2 x) \, dx &= \int \sin x \sec^2 x + \sin x \, dx \\ &= \int \sin x \cdot \frac{1}{\cos^2 x} \sec x + \sin x \, dx \\ &= \int \tan x \sec x + \sin x \, dx \\ &= \sec x - \cos x + C \end{aligned}$
--

Question 115 (**)**

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}.$$

- a) Find the value of A and B in the above identity.
- b) By using the substitution $u = e^x$, or otherwise, show that

$$\int_0^{\ln 2} \frac{1}{1+e^x} dx = \ln\left(\frac{4}{3}\right).$$

$$A=1, B=-1$$

$(a) \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$ $1 = A(u+1) + Bu$	$\left\{ \begin{array}{l} \text{if } u=0, A=1 \\ \text{if } u=1, B=-1 \end{array} \right.$
$(b) \int_0^{\ln 2} \frac{1}{1+e^x} dx = \int_1^2 \frac{1}{1+u} \frac{du}{e^x}$ $= \int_1^2 \frac{1}{1+u} \times \frac{du}{u} = \int_1^2 \frac{1}{u(u+1)} du$ $= \int_1^2 \frac{1}{u} - \frac{1}{u+1} du = \left[\ln u - \ln u+1 \right]_1^2$ $= \left[\ln \frac{u}{u+1} \right]_1^2 = \ln \frac{2}{3} - \ln \frac{1}{2} = \ln \frac{2}{3} + \ln 2 = \ln \frac{4}{3}$	
$u = e^x$ $\frac{du}{dx} = e^x$ $du = e^x dx$ $x=0 \Rightarrow u=1$ $x=\ln 2 \Rightarrow u=2$	

Question 116 (**)**

By completing the square in the expression $4x^2 + 4x$, or otherwise, show that

$$\int \frac{4x^2 + 4x}{\sqrt{2x+1}} dx = A(2x+1)^{\frac{5}{2}} + B(2x+1)^{\frac{1}{2}} + C,$$

where A and B are constants to be found and C is the arbitrary constant of the integration.

$$A = \frac{1}{5}, B = -1$$

$\int \frac{4x^2 + 4x}{\sqrt{2x+1}} dx = \text{by substitution } u=2x+1 \text{ or } u=\sqrt{2x+1} \text{ or}$ $= \int \frac{(2x^2+2x+1)-1}{\sqrt{2x+1}} dx = \int \frac{(2x+1)^2-1}{(2x+1)^{\frac{1}{2}}} dx = \int (2x+1)^{\frac{3}{2}} - (2x+1)^{\frac{1}{2}} dx$ $= \frac{1}{5}(2x+1)^{\frac{5}{2}} - (2x+1)^{\frac{1}{2}} + C$
--

Question 117 (***)

$$u^3 + 1 \equiv (u+1)(u^2 + Au + 1)$$

- a) Determine the value of A in the above identity
 - b) Use the substitution $u = e^x$ to show

$$\int_0^{\ln 2} \frac{e^{3x}+1}{e^x+1} dx = \frac{1}{2} + \ln 2$$

$$A=1, B=\frac{1}{2}, C=-\frac{1}{2}$$

(b) $\int_0^{\ln 2} \frac{3x}{e^{x+1}} dx = \dots$ by substitution ...

$$\begin{aligned} &= \int_1^2 \frac{3u}{e^{u+1}} du \\ &= \int_1^2 \frac{3u^2 e^{-u-1}}{u} du \\ &= \left[-3u^2 e^{-u-1} + 6u e^{-u-1} \right]_1^2 \\ &= \left(-3(2)^2 e^{-2-1} + 6(2) e^{-2-1} \right) - \left(-3(1)^2 e^{-1-1} + 6(1) e^{-1-1} \right) \\ &= \frac{1}{e} + 12e^{-3} \end{aligned}$$

$u = e^x$
 $du = e^x dx$
 $du = u dx$
 $dx = \frac{du}{u}$
 $u = 1 \Rightarrow x = 0$
 $u = 2 \Rightarrow x = \ln 2$

Question 118 (***)

Use the substitution $u = x^{\frac{1}{4}}$ to find

$$\int_1^{16} \frac{2x^{\frac{1}{4}} + 1}{4x^{\frac{5}{4}} + 4x} dx$$

giving the final answer as an exact simplified natural logarithm

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$$\int \frac{2x+1}{\sqrt{4x^2+4x}} dx = \dots \text{ BY SUBSTITUTION } = \int \frac{2u+1}{\sqrt{4u^2+4u}} du$$

$u = x^2 + x$
 $du = (2x+1)dx$
 $\frac{du}{dx} = 2x+1$
 $4u^2 du = 4x^2 dx$
 $4u^2 du = dx$
 $2u^2 du = x dx$
 $2u^2 du = u dx$
 $2u^2 du = u = 1$
 $2u^2 du = u = 2$

$$= \int \frac{2u+1}{\sqrt{4u^2+4u}} du = \int \frac{2u+1}{\sqrt{4u(u+1)}} du \rightarrow \int \frac{1}{\sqrt{u+1}} du$$

$$= \left[\ln(u+1) \right]_1^6 = \ln 6 - \ln 3 = \ln 3$$

Question 119 (****)

$$\frac{x^3}{x^2+1} \equiv Ax+B+\frac{Cx+D}{x^2+1}.$$

- a) Determine the value of each of the constants A , B , C and D .
- b) Use integration by parts to show that

$$\int x \ln(x^2+1) dx = \frac{1}{2}(x^2+1)\ln(x^2+1) - \frac{1}{2}x^2 + C.$$

A = 1, B = 0, C = -1, D = 0

$\text{(a)} \quad \frac{x^3}{x^2+1} \equiv Ax+B+\frac{Cx+D}{x^2+1}$ $x^3 \equiv (Ax+B)(x^2+1) + Cx+D$ $x^3 \equiv Ax^3 + Ax^2 + Bx + B + Cx + D$ $x^3 \equiv Ax^3 + Bx^2 + (A+C)x + (B+D)$	$\left\{ \begin{array}{l} A=1 \\ B=0 \\ A+C=0 \\ C=-1 \\ B+D=0 \\ D=0 \end{array} \right.$
$\text{(b)} \quad \int x \ln(x^2+1) dx \dots \text{INTEGRATION BY PARTS}$ $= \frac{1}{2}x^2 \ln(x^2+1) - \int \frac{x^2}{x^2+1} dx$ $= \frac{1}{2}x^2 \ln(x^2+1) - \int x - \frac{x}{x^2+1} dx \quad \text{This is of the type } \int \frac{f(x)}{g(x)} dx$ $= \frac{1}{2}x^2 \ln(x^2+1) - \left[x - \frac{1}{2}\ln(x^2+1) \right] dx$ $= \frac{1}{2}x^2 \ln(x^2+1) - \left[\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) \right] + C$ $= \frac{1}{2}x^2 \ln(x^2+1) + \frac{1}{2}\ln(x^2+1) - \frac{1}{2}x^2 + C$ $= \frac{1}{2}(x^2+1) \ln(x^2+1) - \frac{1}{2}x^2 + C \quad \text{As required}$	

Question 120 (****)

By writing $\sec x$ as the fraction $\frac{\sec x}{1}$ and multiplying the numerator and the denominator by $(\sec x + \tan x)$, find

$$\int \sec x dx.$$

ln|sec x + tan x| + C

$\int \sec x dx = \int \frac{\sec x}{1} dx = \int \frac{\sec(x \sec x + \tan x)}{\sec x + \tan x} dx$ $= \int \frac{\sec x + \sec x \tan x}{\sec x + \tan x} dx$ $\text{let } \frac{d}{dx}(\tan x) = \sec^2 x \quad \left\{ \begin{array}{l} \frac{d}{dx}(\sec x) = \sec x \tan x \\ \frac{d}{dx}(\sec x) = \sec x \tan x \end{array} \right\} \text{ i.e. of the form } \int \frac{f'(x)}{f(x)} dx$ $= [\ln \sec x + \tan x] + C$

Question 121 (****)

$$\frac{4}{(1-u)^2(1+u)} \equiv \frac{A}{(1-u)^2} + \frac{B}{1-u} + \frac{C}{1+u}.$$

- a) Find the value of A , B and C in the above identity.
- b) Hence by using a suitable substitution find the exact value of

$$\int_0^{\frac{\pi}{6}} \frac{4}{\cos x(1-\sin x)} dx.$$

[A = 2], [B = 1], [C = 1], [2 + ln 3]

(a)

$$\frac{4}{(1-u)^2(1+u)} = \frac{A}{(1-u)^2} + \frac{B}{1-u} + \frac{C}{1+u}$$

$$4 \equiv A(1+u) + B(u-1)(1+u) + C(1-u)^2$$

- If $u=1$, $4=2A \Rightarrow A=2$
- If $u=-1$, $4=4C \Rightarrow C=1$
- If $u=0$, $4=A+B+C$
 $4=2+0+1 \quad \therefore B=1$

(b)

$$\int_0^{\frac{\pi}{6}} \frac{4}{\cos x(1-\sin x)} dx = \dots \text{by substitution}, \dots$$

$$= \int_0^{\frac{\pi}{6}} \frac{4}{\cos u(1-u)} du = \int_0^{\frac{\pi}{6}} \frac{4}{\cos u(1-u)} du$$

$$= \int_0^{\frac{\pi}{6}} \frac{4}{(1-\sin u)(1-u)} du = \int_0^{\frac{\pi}{6}} \frac{4}{(1-u^2)(1-u)} du$$

$$= \int_0^{\frac{\pi}{6}} \frac{4}{(1-u)(1-u)(1+u)} du = \int_0^{\frac{\pi}{6}} \frac{4}{(1-u)(1+u)} du \dots \text{part (a)}$$

$$= \int_0^{\frac{\pi}{6}} \left(\frac{2}{1-u} + \frac{1}{1+u} + \frac{1}{1+u} \right) du = \left[-\frac{2}{1-u} - \ln|1+u| + \ln|1-u| \right]_0^{\frac{\pi}{6}}$$

$$= \left(4 - \ln\frac{1}{2} + \ln\frac{1}{2} \right) - (2 + \ln 1 + \ln 1) = 2 + \ln 3$$

$u=\sin x$
 $du=\cos x dx$
 $du = du / \cos x$
 $\alpha = \frac{\pi}{6}, \quad u = \frac{1}{2}$
 $\alpha = 0, \quad u = 0$

Question 122 (**)**

Use the substitution $u = \ln x$, followed by integration by parts to find

$$\int \frac{1 - \ln x}{x^2} dx.$$

$$\boxed{\frac{\ln x}{x} + C}$$

$$\begin{aligned} \int \frac{1 - \ln x}{x^2} dx &= \text{using the substitution given} \\ &= \int \frac{1-u}{x^2} x du = \int \frac{1-u}{x} du = \int \frac{1-u}{e^u} du \\ &= \int (1-u) e^{-u} du \\ &\quad \text{BY PARTS} \\ &= -(1-u) e^{-u} - \int e^{-u} du \\ &= (u-1) e^{-u} + e^{-u} + C \\ &= u e^{-u} - e^{-u} + e^{-u} + C \\ &= \frac{u}{e^u} + C = \frac{\ln x}{x} + C \end{aligned}$$

Question 123 (**)**

$$y = \arctan x, \quad x \in \mathbb{R}.$$

- a) By writing the above equation as $x = \tan y$, show clearly that

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

- b) Use integration by parts and the result of part (a) to find

$$\int 2 \arctan x dx.$$

$$\boxed{2x \arctan x - \ln(1+x^2) + C}$$

$$\begin{aligned} a) \quad y &= \arctan x \\ \Rightarrow \tan y &= x \\ \Rightarrow x &= \tan y \\ \Rightarrow \frac{dx}{dy} &= \sec^2 y \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sec^2 y} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{1+\tan^2 y} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{1+x^2} \end{aligned}$$

$$\begin{aligned} b) \quad \int 2 \arctan x dx &= \dots \text{by parts} \\ &= 2x \arctan x - \int \frac{2x}{1+x^2} dx \\ &= 2x \arctan x - \int \frac{2x}{(1+x^2)^2} dx \\ &= 2x \arctan x - \ln(1+x^2) + C \end{aligned}$$

Question 124 (*****)

$$f(x) \equiv \frac{4x^2 - 23x + 21}{x^2 - 4x + 3}, \quad x \neq 1, \quad x \neq 3.$$

a) Express $f(x)$ in partial fractions.

b) Hence find an exact value for

$$\int_2^{2.5} f(x) \, dx.$$

$$f(x) \equiv 4 - \frac{1}{x-1} - \frac{6}{x-3}, \quad \left[2 - \ln\left(\frac{3}{128}\right) \right]$$

(a)

$$\frac{4x^2 - 23x + 21}{(x-1)(x-3)} \equiv \frac{4x^2 - 23x + 21}{(x-1)(x-3)} \equiv A + \frac{B}{x-1} + \frac{C}{x-3}$$

"*PROPOSE*" $A=4$ BY INSPECTION

$$4x^2 - 23x + 21 \equiv A(x-3) + B(x-3) + C(x-1)$$

- IF $x=1 \Rightarrow 4-24+21=-28$
 $2B=28$
 $B=14$
- IF $x=3 \Rightarrow 36-69+21=2C$
 $-12=2C$
 $C=-6$
- IF $x=0 \Rightarrow 34-38-C$
 $2=3A+3+C$
 $2=3A$
 $A=\frac{2}{3}$

$\therefore f(x) = 4 - \frac{1}{x-1} - \frac{6}{x-3}$

(b)

$$\int_2^{2.5} f(x) \, dx = \int_2^{2.5} \left(4 - \frac{1}{x-1} - \frac{6}{x-3} \right) \, dx = \left[4x - \ln|x-1| - 6\ln|x-3| \right]_2^{2.5}$$

$$= \left[10 - \ln\left[\frac{3}{2}\right] - 6\ln\left[\frac{1}{2}\right] \right] - \left[8 - \ln 1 - 6\ln 1 \right]$$

$$= 2 - \ln\left[\frac{3}{2}\right] - 6\ln\left[\frac{1}{2}\right] = 2 - \left[\ln\frac{3}{2} + \ln\frac{1}{64} \right]$$

$$= 2 - \ln\frac{3}{128}$$

Question 125 (**)**

By using the substitution $u = (2x+1)^{\frac{1}{2}}$, or otherwise, show that

$$\int_0^4 e^{\sqrt{2x+1}} dx = 2e^3.$$

[proof]

$$\begin{aligned} \int_0^4 e^{\sqrt{2x+1}} dx &= \int_{-1}^3 e^u (2u du) = \int_{-1}^3 ue^u du \dots \\ \dots \text{BY PARTS Q. LEAVING UNITS} \dots &\quad \begin{array}{|c|c|} \hline u & 1 \\ \hline du & e^u \\ \hline \end{array} \dots \\ &= ue^u - \int e^u du = [ue^u - e^u]_{-1}^3 \\ &= (3e^3 - e^0) - (e^{-1} - e^0) \\ &= 2e^3 \end{aligned}$$

$$\begin{array}{|c|} \hline u=(2x+1)^{\frac{1}{2}} \\ u=2x+1 \\ 2u\frac{du}{dx}=2 \\ du=u\frac{dx}{2} \\ 2x+1\rightarrow u=2 \\ x=0\rightarrow u=1 \\ x=4\rightarrow u=3 \\ \hline \end{array}$$

Question 126 (**)**

Use the substitution $u = x^2 + 2$, followed by integration by parts to show that

$$\int 2x^3 \ln(x^2 + 2) dx = 1 + 2\ln 2.$$

[proof]

$$\begin{aligned} \int_2^4 2x^3 \ln(x^2+2) dx &= \int_2^4 2x^3 \ln u \frac{du}{2x} \\ &= \int_2^4 x^2 \ln u du = \int_2^4 (u-2) \ln u du \\ \dots \text{BY PARTS Q. LEAVING UNITS} \dots &\quad \begin{array}{|c|c|} \hline u & 2 \\ \hline du & 2x \\ \hline dx & \frac{du}{2x} \\ \hline 2x+0 & u=2 \\ 2x^2+2 & u=4 \\ 2^2=4 & u=2 \\ \hline \end{array} \\ &= (\frac{1}{2}u^2 - 2u) \ln u - \int \frac{1}{2}u - 2 du \\ &= (\frac{1}{2}u^2 - 2u) \ln u - \frac{1}{4}u^2 + 2u + C \\ \dots \text{REINTRODUCING UNITS} \dots &\quad \begin{array}{|c|c|} \hline u & 1 \\ \hline du & \frac{du}{2x} \\ \hline 2x^2+2 & u=2 \\ u=2 & u=4 \\ \hline \end{array} \\ &= [\frac{1}{2}u^2 - 2u] \ln u - \frac{1}{4}u^2 + 2u \Big|_2^4 \\ &= (0 - 4 + 8) - (-2\ln 2 - 1 + 4) = 4 - (3 - 2\ln 2) = 1 + 2\ln 2 \end{aligned}$$

Question 127 (**)**

By considering the trigonometric expansions of $\sin(5x+3x)$ and $\sin(5x-3x)$, show clearly that

$$\int_0^{\frac{\pi}{4}} \cos 3x \sin 5x \, dx = \frac{1}{4}.$$

[proof]

$$\begin{aligned}
 \sin(5x+3x) &= \sin 5x \cos 3x + \cos 5x \sin 3x \\
 \sin(5x-3x) &= \sin 5x \cos 3x - \cos 5x \sin 3x
 \end{aligned}
 \quad \left. \begin{array}{l} \text{Add} \\ \hline \end{array} \right. \\
 \therefore \sin 5x \cos 3x &= \frac{1}{2} \sin(5x+3x) + \frac{1}{2} \sin(5x-3x) \\
 \int_0^{\frac{\pi}{4}} \cos 3x \sin 5x \, dx &= \int_0^{\frac{\pi}{4}} \frac{1}{2} \sin(5x+3x) + \frac{1}{2} \sin(5x-3x) \, dx \\
 &= \left[\frac{1}{10} \sin 5x - \frac{1}{6} \cos 3x \right]_0^{\frac{\pi}{4}} = \left[\frac{1}{10} \sin(5x+3x) + \frac{1}{2} \sin(5x-3x) \right]_0^{\frac{\pi}{4}} \\
 &= \left[\frac{1}{10} \sin 0 + \frac{1}{6} \cos 0 \right] - \left[\frac{1}{10} \cos 0 + \frac{1}{2} \sin 0 \right] \\
 &= \left(\frac{1}{6} + \frac{1}{4} \right) - \left(\frac{1}{10} + 0 \right) = \frac{1}{4} // \text{As required}
 \end{aligned}$$

Question 128 (**)**

By using the substitution $x = 2 + (u-1)^2$, or otherwise, find

$$\int \frac{1}{1+\sqrt{x-2}} \, dx.$$

$$2\sqrt{x-2} + 2\ln(1+\sqrt{x-2}) + C$$

$$\begin{aligned}
 &\int \frac{1}{1+\sqrt{(u-1)^2-2}} \, du \\
 &= \int \frac{1}{1+\sqrt{2(u-1)^2-2}} \cdot 2(u-1) \, du \\
 &= \int \frac{2(u-1)}{1+\sqrt{2(u-1)^2-2}} \, du \\
 &= \int \frac{2(u-1)}{1+(u-1)} \, du \quad = \int \frac{2(u-1)}{u} \, du \\
 &= \int \frac{2u-2}{u} \, du = \int 2 - \frac{2}{u} \, du = 2u - 2\ln|u| + C \\
 &= 2\left(1+\sqrt{u-1}\right) - 2\ln\left(1+\sqrt{u-1}\right) + C \\
 &\stackrel{(1)}{=} 2\sqrt{u-1} - 2\ln\left(1+\sqrt{u-1}\right) + C \\
 &= 2\sqrt{2-2} - 2\ln\left(1+\sqrt{2-2}\right) + C
 \end{aligned}$$

Question 129 (****)

$$\frac{6t^3}{t+1} \equiv At^2 + Bt + C + \frac{D}{t+1}.$$

a) Determine the value of each of the constants A , B , C and D .

b) Use the substitution $t = x^{\frac{1}{6}}$ to show

$$\int_1^{64} \frac{1}{\sqrt[3]{x} + \sqrt[3]{x}} dx = 11 + 6 \ln\left(\frac{2}{3}\right).$$

$$A = 6, B = -6, C = 6, D = -6$$

(a)

$$\begin{aligned} \frac{6t^3}{t+1} &\equiv At^2 + Bt + C + \frac{D}{t+1} \\ 6t^3 &\equiv (At^2 + Bt + C)(t+1) + D \\ 6t^3 &\equiv At^3 + At^2 + Ct + C + D \\ 6t^3 &\equiv At^3 + (At^2)t^2 + (C+D)t + (C+D) \end{aligned}$$

$\bullet A=6$ $\bullet B=-6$ $\bullet C=6$ $\bullet D=-6$
 $\bullet At^2=0$ $\bullet Bt=0$ $\bullet C+D=0$
 $\bullet C=6$ $\bullet D=0$ $\bullet B=6$
 $\bullet C=6$ $\bullet D=-6$

(b)

$$\begin{aligned} \int_1^{64} \frac{1}{\sqrt[3]{x} + \sqrt[3]{x}} dx &= \dots \text{by substitution...} \\ &= \int_1^2 \frac{1}{t^{\frac{1}{3}} + t^{\frac{1}{3}}} (6t^2 dt) = \int_1^2 \frac{6t^2}{t^{\frac{2}{3}} + t^{\frac{2}{3}}} dt \\ &= \int_1^2 \frac{6t^3}{t^{\frac{2}{3}} + t^{\frac{2}{3}}} dt = \dots \text{from part (a)} \\ &= \int_1^2 6t^3 - 6t + 6 - \frac{6}{t+1} dt \\ &= \left[2t^3 - 3t^2 + 6t - 6 \ln|t+1| \right]_1^2 \\ &= (16 - 12 + 12 - 6 \ln 3) - (2 - 3 + 6 - 6 \ln 2) \\ &= 11 + 6 \ln 2 - 6 \ln 3 = 11 + 6 \ln \frac{2}{3} \end{aligned}$$

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Question 130 (**)**

$$f(x) = 8x \ln(2x+1), \quad x > -\frac{1}{2}.$$

- a) Use the substitution $u = 2x+1$ to show that

$$\int f(x) dx = \int (2u-2) \ln u du.$$

- b) Use integration by parts to show that

$$\int f(x) dx = (4x^2-1) \ln(2x+1) - 2x^2 + 2x + C.$$

proof

$ \begin{aligned} a) \int 8x \ln(2x+1) dx &= \dots \text{by substitution} \\ &= \int 8x \ln u \frac{du}{2} = \int 4x \ln u du = \int 2(u-1) \ln u du \\ &= \int 2(u-1) \ln u du = \int (2u-2) \ln u du \end{aligned} $	$ \begin{aligned} u &= 2x+1 \\ \frac{du}{dx} &= 2 \\ du &= 2dx \\ dx &= \frac{du}{2} \\ 2x &= u-1 \end{aligned} $
$ \begin{aligned} b) &= \dots \text{by parts} \\ &= (u^2-2u) \ln u - \int u(u^2-2u) du \\ &= (u^2-2u) \ln u - \int u^3 - 2u^2 du \\ &= (u^2-2u) \ln u - \left(\frac{1}{4}u^4 - \frac{2}{3}u^3 \right) + C \\ &= [(2x+1)^2 - 2(2x+1)] \ln(2x+1) - \frac{1}{4}(2x+1)^4 + \frac{2}{3}(2x+1)^3 + 2x^2 + 2x + C \\ &= (4x^2+4x+1-4x-2) \ln(2x+1) - \frac{1}{4}(4x^4+16x^3+48x^2+32x+4) + 4x^3 + 2x^2 + 2x + C \\ &= (4x^2-1) \ln(2x+1) - 2x^4 - 2x^3 + \frac{1}{2}x^2 + 4x^3 + 2x^2 + 2x + C \\ &= (4x^2-1) \ln(2x+1) - 2x^4 + 2x^2 + 2x + C \end{aligned} $	$ \begin{array}{c c} \ln u & \frac{1}{u} \\ \hline u^2-2u & 2u-2 \end{array} $

Question 131 (*****)

$$y = \frac{(1+\sin x)^2}{\cos^2 x}.$$

- a) Calculate the two missing values of the following table.

x	$\frac{\pi}{6}$		$\frac{\pi}{4}$	$\frac{7\pi}{24}$	$\frac{\pi}{3}$
y	3		5.8284	8.6784	13.9282

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate for

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{(1+\sin x)^2}{\cos^2 x} dx.$$

- c) Use trigonometric identities to find the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{(1+\sin x)^2}{\cos^2 x} dx.$$

, $x = \frac{5\pi}{24}$, [4.112], $4 - \frac{1}{6}\pi$

a) Fill in the table

x	$\frac{\pi}{6}$	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{7\pi}{24}$	$\frac{\pi}{3}$
y	3	4.1120	5.8284	8.6784	13.9282	

By the Trapezium Rule

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{(1+\sin x)^2}{\cos^2 x} dx \approx \frac{\text{"TRAPEZIUM RULE"} }{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3) \right]$$

$$\approx \frac{7\pi}{24} \left[3 + 13.9282 + 2(4.1120 + 8.6784) \right]$$

$$\approx 3.516$$

b) Prove by direct integration

$$\begin{aligned} \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{(1+\sin x)^2}{\cos^2 x} dx &= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1+2\sin x+\sin^2 x}{\cos^2 x} dx \\ &= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} dx \\ &= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sec^2 x + 2\tan x \sec x + (\sec x - 1) dx \\ &= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2\tan x + 2\tan x \sec x - 1 dx \end{aligned}$$

Now we note that

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \text{and} \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

Hence we finally have

$$\begin{aligned} & \left[2\tan x + 2\tan x \sec x - x \right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \\ &= \left(2\tan \frac{\pi}{3} + 2\tan \frac{\pi}{3} \sec \frac{\pi}{3} - \frac{\pi}{3} \right) - \left(2\tan \frac{\pi}{6} + 2\tan \frac{\pi}{6} \sec \frac{\pi}{6} - \frac{\pi}{6} \right) \\ &= \left(2\sqrt{3} + 4 - \frac{\pi}{3} \right) - \left(\frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} - \frac{\pi}{6} \right) \\ &= \left(2\sqrt{3} + 4 - \frac{\pi}{3} \right) - \left(\frac{6}{\sqrt{3}} - \frac{\pi}{6} \right) \\ &= \cancel{\left(2\sqrt{3} + 4 - \frac{\pi}{3} \right)} - \cancel{\left(\frac{6}{\sqrt{3}} - \frac{\pi}{6} \right)} \\ &= 4 - \frac{\pi}{6} \end{aligned}$$

Question 132 (**)**

$$y = \ln(\sec x + \tan x).$$

- a) Express $\frac{dy}{dx}$ as a single trigonometric function.

- b) Hence find

$$\int x \sec x \tan x \, dx.$$

$$x \sec x - \ln|\sec x + \tan x| + C$$

(a) $y = \ln(\sec x + \tan x)$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \times (\sec x \tan x + \sec^2 x)$$

$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec(\tan x + \sec x)}{\sec x + \tan x} = \sec x$$

(b) $\int x \sec x \tan x \, dx$... by parts ...

x	1
$\sec x$	$\sec x \tan x$

$$= x \sec x - \int \sec x \, dx$$

$$= x \sec x - \ln|\sec x + \tan x| + C$$

Question 133 (**)**

Use the substitution $x = 2\sin \theta$ to find the exact value of

$$\int_0^1 \frac{12}{(4-x^2)^{\frac{3}{2}}} \, dx.$$

$$\boxed{\sqrt{3}}$$

$\int_0^1 \frac{12}{(4-x^2)^{\frac{3}{2}}} \, dx = \dots$ USING THE SUBSTITUTION GIVEN

$$= \int_0^{\frac{\pi}{2}} \frac{12}{(4-4\sin^2 \theta)^{\frac{3}{2}}} (2\cos \theta \, d\theta) = \int_0^{\frac{\pi}{2}} \frac{24\cos \theta}{[4(1-\sin^2 \theta)]^{\frac{3}{2}}} \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{24\cos \theta}{(4\cos^2 \theta)^{\frac{3}{2}}} \, d\theta = \int_0^{\frac{\pi}{2}} \frac{24\cos \theta}{8\cos^3 \theta} \, d\theta = \int_0^{\frac{\pi}{2}} \frac{3}{\cos^2 \theta} \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 3\tan^2 \theta \, d\theta = \left[3\tan \theta \theta \right]_0^{\frac{\pi}{2}} = 3\tan \frac{\pi}{2} \theta - 3\tan 0 \theta$$

$$= 3 \times \frac{\sqrt{3}}{3} = \sqrt{3}$$

$\theta = 2\sin \theta$
$\frac{d\theta}{d\theta} = 2\cos \theta$
$d\theta = 2\cos \theta \, d\theta$
$x = 2\sin \theta$
$x = 0 = 2\sin \theta$
$\sin \theta = 0$
$\theta = 0$
$x = 1 = 2\sin \theta$
$\sin \theta = \frac{1}{2}$
$\theta = \frac{\pi}{6}$

Question 134 (****)

$$\frac{u^2}{u^2 - 9} \equiv A + \frac{B}{u-3} + \frac{C}{u+3}.$$

- a) Find the value of A , B and C in the above identity.
- b) By using the substitution $u = \sqrt{x^2 + 9}$, or otherwise, find

$$\int \frac{\sqrt{x^2 + 9}}{x} dx.$$

$A = 1$	$B = \frac{3}{2}$	$C = -\frac{3}{2}$	$\sqrt{x^2 + 9} + \frac{3}{2} \ln \left \frac{\sqrt{x^2 + 9} - 3}{\sqrt{x^2 + 9} + 3} \right + C$
---------	-------------------	--------------------	---

(a) $\frac{u^2}{u^2 - 9} = \frac{(u-3)(u+3)}{(u-3)(u+3)} \equiv A + \frac{B}{u-3} + \frac{C}{u+3}$

$\boxed{u^2 = A(u-3)(u+3) + B(u+3) + C(u-3)}$

If $u=3$: $9 = 6B \Rightarrow B = \frac{3}{2}$ //

If $u=-3$: $9 = -6C \Rightarrow C = -\frac{3}{2}$ //

If $u=0$: $0 = -9A + 3B - 3C$

$9A = \frac{3}{2}(-\frac{3}{2})$
 $9A = -\frac{9}{4}$
 $A = -\frac{1}{4}$ //

(b) $u = \sqrt{x^2 + 9}$
 $u^2 = x^2 + 9$
 $2u \frac{du}{dx} = 2x$
 $\frac{du}{dx} = \frac{x}{u}$
 $du = \frac{x}{u} du$
 $u^2 = x^2 + 9$

$$\begin{aligned} \int \frac{\sqrt{x^2 + 9}}{x} dx &= \int \frac{u}{x} \frac{u}{\frac{x}{u}} du \\ &= \int \frac{u^2}{x^2} du = \int \frac{u^2}{u^2 - 9} du \\ &= \int 1 + \frac{9}{u^2 - 9} - \frac{9}{u^2 + 9} du \\ &= u + \frac{9}{2} \ln|u-3| - \frac{9}{2} \ln|u+3| + C \\ &= u + \frac{9}{2} \ln \left| \frac{u-3}{u+3} \right| + C \\ &= \sqrt{x^2 + 9} + \frac{9}{2} \ln \left| \frac{\sqrt{x^2 + 9} - 3}{\sqrt{x^2 + 9} + 3} \right| + C \end{aligned}$$

Question 135 (**)**

By using the substitution $u = 2^x$, or otherwise, find an exact value for

$$\int_0^3 \frac{2^x}{\sqrt{2^x + 1}} dx.$$

$$\boxed{\frac{2(3 - \sqrt{2})}{\ln 2}}$$

$$\begin{aligned} \int_0^3 \frac{2^x}{\sqrt{2^x + 1}} dx &= \dots = \int_1^8 \frac{u^x}{(u+1)^{\frac{1}{2}}} \frac{du}{2^x \ln 2} \\ &= \int_1^8 \frac{1}{\ln 2} \frac{(u+1)^{\frac{1}{2}}}{u^x} du = \frac{1}{\ln 2} \left[\frac{1}{2}(u+1)^{\frac{1}{2}} \right]_1^8 \\ &= \frac{2}{\ln 2} \left[\sqrt{u+1} \right]_1^8 = \frac{2}{\ln 2} [3 - \sqrt{2}] \end{aligned}$$

$\boxed{\begin{array}{l} u = 2^x \\ \frac{du}{dx} = 2^x \ln 2 \\ du = \frac{du}{dx} dx \\ du = \frac{du}{2^x \ln 2} \\ x=0, u=1 \\ x=3, u=8 \end{array}}$

Question 136 (**)**

Use the substitution $x = \tan \theta$ to show that

$$\int_1^{\sqrt{3}} \frac{2}{x(x^2 + 1)} dx = \ln\left(\frac{m}{n}\right),$$

where m and n are integers.

proof

$$\begin{aligned} \int_1^{\sqrt{3}} \frac{2}{x(x^2 + 1)} dx &= \dots = \int_{\pi/4}^{\pi/3} \frac{2}{\tan(\theta)(\tan^2(\theta) + 1)} \sec^2 \theta d\theta \\ &= \int_{\pi/4}^{\pi/3} \frac{2 \sec \theta}{\tan \theta \sec \theta} d\theta = \int_{\pi/4}^{\pi/3} 2 \csc \theta d\theta \\ &= 2 \left[\ln|\csc \theta| \right]_{\pi/4}^{\pi/3} = 2 \left(\ln\left(\csc \frac{\pi}{3}\right) - \ln\left(\csc \frac{\pi}{4}\right) \right) \\ &= 2 \left[\ln \frac{\sqrt{3}}{2} - \ln \frac{\sqrt{2}}{2} \right] = 2 \ln \left(\frac{\sqrt{3}}{\sqrt{2}} \right) \\ &= 2 \ln \left(\frac{\sqrt{6}}{2} \right) = 2 \ln \left(\sqrt{\frac{3}{2}} \right) = 2 \ln \left(\frac{3}{2}^{\frac{1}{2}} \right) = \ln \frac{3}{2} \end{aligned}$$

$\boxed{\begin{array}{l} x = \tan \theta \\ \frac{dx}{d\theta} = \sec^2 \theta \\ d\theta = \frac{dx}{\sec^2 \theta} \\ 2x = \sec^2 \theta \\ 2x = \frac{1}{\cos^2 \theta} \\ 2x = \frac{1}{1 - \sin^2 \theta} \\ 2x = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \\ x = \frac{2}{3} \end{array}}$

Question 137 (**)**

Use trigonometric identities to find

$$\int 32 \sin^2 x \cos^2 x \, dx.$$

$$4x - \sin 4x + C$$

$$\begin{aligned} \int 32 \sin^2 x \cos^2 x \, dx &= \int 32 \left(\frac{1}{2} - \frac{1}{2} \cos 2x\right) \left(\frac{1}{2} + \frac{1}{2} \cos 2x\right) \, dx \\ &= \int 32 \left(\frac{1}{4} - \frac{1}{4} \cos 4x\right) \, dx \\ &= \int 8 - 8 \cos 4x \, dx \\ &= \int 8 - 4 - 4 \cos 4x \, dx \\ &= \int 4 - 4 \cos 4x \, dx \\ &= 4x - \sin 4x + C \end{aligned}$$

ALTERNATIVE

$$\begin{aligned} \int 32 \sin^2 x \cos^2 x \, dx &= \int 32 (\sin x \cos x)^2 \, dx = \int 32 \left(\frac{1}{2} \sin 2x\right)^2 \, dx \\ &= \int 32 \left[\frac{1}{4} \sin^2 2x\right] \, dx = \int 32 \times \frac{1}{4} \sin^2 2x \, dx \\ &= \int 8 \sin^2 2x \, dx = \int 8 \left(\frac{1}{2} - \frac{1}{2} \cos 4x\right) \, dx \\ &= \int 4 - 4 \cos 4x \, dx = 4x - \sin 4x + C \end{aligned}$$

Question 138 (**)**

By using the substitution $u^2 = 1 + \tan x$, or otherwise, find

$$\int \sec^2 x \tan x \sqrt{1 + \tan x} \, dx.$$

$$\frac{2}{15}(3 \tan x - 2)(1 + \tan x)^{\frac{3}{2}}$$

$$\begin{aligned} \int \sec^2 x \tan x \sqrt{1 + \tan x} \, dx &= \dots \text{substitution} \\ &= \int \sec^2 x \tan x \times u \frac{2u}{\sec^2 x} \, du = \int 2u^2 \tan x \, du \\ &= \int 2u^2(u^2 - 1) \, du = \int 2u^4 - 2u^2 \, du \\ &= \frac{2}{5}u^5 - \frac{2}{3}u^3 = \frac{2}{5}(1 + \tan x)^{\frac{5}{2}} - \frac{2}{3}(1 + \tan x)^{\frac{3}{2}} + C \\ &= \dots \text{which could be simplified to...} \\ &= \frac{6}{15}(1 + \tan x)^{\frac{3}{2}} - \frac{10}{15}(1 + \tan x)^{\frac{5}{2}} + C = \frac{2}{15}(1 + \tan x)^{\frac{3}{2}}[3(1 + \tan x)^{-5}] + C \\ &= \frac{2}{15}(1 + \tan x)^{\frac{3}{2}}(3 \tan x - 2) + C \end{aligned}$$

Question 139 (****)

$$\frac{2u^2}{(u-1)(u+1)} \equiv A + \frac{B}{u+1} + \frac{C}{u-1}.$$

- a) Find the value of A , B and C in the above identity.
- b) By using the substitution $u^2 = x+1$, or otherwise, find an exact value for

$$\int_3^8 \frac{\sqrt{x+1}}{x} dx.$$

The table below shows some tabulated values for the equation $y = \frac{\sqrt{x+1}}{x}$, $3 \leq x \leq 8$.

x	3	4	5	6	7	8
y	0.6667	0.5590	0.4899		0.4041	0.3750

- c) Complete the missing value in the table.

[continues overleaf]

[continued from overleaf]

- d) Use the trapezium rule with all the values from the table to find an approximate value for

$$\int_3^8 \frac{\sqrt{x+1}}{x} dx.$$

- e) Calculate the difference between the exact value, found in part (b), and the trapezium rule estimate, found in part (d), and hence state whether the trapezium rule produces an overestimate or an underestimate.

, $A = 2$, $B = -1$, $C = -1$, $2 + \ln\left(\frac{3}{2}\right)$, 0.4410 , 2.4148 , 0.0093

a) $\frac{2u^2}{(3u)(u-1)} \equiv A + \frac{B}{u+1} + \frac{C}{u-1}$

 $2u^2 \equiv A(3u)(u-1) + B(u+1) + C(3u)$
 $\begin{cases} \text{if } u=1, & 2=2C \Rightarrow C=1 \\ \text{if } u=-1, & 2=-2B \Rightarrow B=-1 \\ \text{if } u=0, & 0=A-B+C \\ & A=C-B \\ & A=1-(-1) \\ & A=2. \end{cases}$
 $\therefore A=2, B=-1, C=1$

b) $\int_2^8 \frac{\sqrt{x+1}}{x} dx = \dots \text{by substitution}$

 $= \int_2^8 \frac{u}{2u} du = \int_2^8 \frac{1}{2} du$
 $= \int_2^8 \frac{u^2}{u^2-1} du = \int_2^8 \frac{2u^2}{(u+1)(u-1)} du$
 $= \int_2^8 2 - \frac{2}{u+1} + \frac{1}{u-1} du \quad \text{from part (a)}$
 $= \left[2u - \ln|u+1| + \ln|u-1| \right]_2^8 = (6 - \ln 9 + \ln 7) - (4 - \ln 3 + \ln 2)$
 $= 2 + \ln(2e^2 - \ln 4 + \ln 3) = 2 + \ln\left(\frac{2e^2}{4}\right) = 2 + \ln\frac{3}{2}$

c) AT $x=6$ $\frac{\sqrt{x+1}}{x} = \frac{1}{6}\sqrt{7} \approx 0.4410$

d) BY TRAPEZIUM RULE $\int_3^8 \frac{\sqrt{x+1}}{x} dx \approx \text{TRAPEZIUM RULE} [\text{FIRST PART} + 2(\text{MID})]$

 $= \frac{1}{2}[0.4410 + 0.536 + 2(0.5331 + 0.5468)]$
 ≈ 2.4148

e) $\therefore \text{DIFFERENCE} \approx (2 + \ln\frac{3}{2}) - 2.4148 \approx -0.0093$

$\therefore \text{UNDERESTIMATE}$

Question 140 (**)**

Use the substitution $x = \sec \theta$ to show that

$$\int_{\sqrt{2}}^2 \frac{2}{x^2 \sqrt{x^2 - 1}} dx = \sqrt{m} - \sqrt{n},$$

where m and n are integers.

$$\boxed{\sqrt{3} - \sqrt{2}}$$

$$\begin{aligned}
 & \int_{\sqrt{2}}^2 \frac{2}{x^2 \sqrt{x^2 - 1}} dx = \dots \text{using the substitution} \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2}{\sec^2 \theta \sqrt{\sec^2 \theta - 1}} (\sec \theta \tan \theta) d\theta \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2 \tan \theta}{\sec \theta \sqrt{\sec^2 \theta - 1}} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2}{\sec \theta} d\theta \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2 \cos \theta d\theta = \left[2 \sin \theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\
 &= 2 \sin \frac{\pi}{3} - 2 \sin \frac{\pi}{4} = \sqrt{3} - \sqrt{2}
 \end{aligned}$$

Question 141 (**)**

Use the substitution $x = \tan^2 \theta$ to find an exact value for

$$\int_0^1 \frac{\sqrt{x}}{x+1} dx.$$

$$\boxed{2 - \frac{\pi}{2}}$$

$$\begin{aligned}
 & \int_0^1 \frac{\sqrt{x}}{x+1} dx = \dots \text{by substitution} \\
 &= \int_0^{\frac{\pi}{4}} \frac{\tan \theta}{\tan^2 \theta + 1} (2 \sec^2 \theta d\theta) = \int_0^{\frac{\pi}{4}} \frac{2 \sec^2 \theta d\theta}{\sec^2 \theta} \\
 &= \int_0^{\frac{\pi}{4}} 2 (\sec \theta - 1) d\theta = \int_0^{\frac{\pi}{4}} 2 \cos \theta - 2 d\theta \\
 &= \left[2 \sin \theta - 2\theta \right]_0^{\frac{\pi}{4}} = (2 - 2\frac{\pi}{4}) - (0 - 0) = 2 - \frac{\pi}{2}
 \end{aligned}$$

Question 142 (****)

$$y = \frac{e^{2x}}{e^x + 1}, \quad x \in \mathbb{R}$$

- a) Calculate the missing values of x and y in the following table.

x	$\ln 2$	x_2	x_3	x_4	$\ln 8$
y	1.333	y_2	y_3	y_4	7.111

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate for

$$\int_{\ln 2}^{\ln 8} \frac{e^{2x}}{e^x + 1} dx.$$

- c) Use the substitution $u = e^x + 1$ to find an exact simplified value for

$$\int_{\ln 2}^{\ln 8} \frac{e^{2x}}{e^x + 1} dx.$$

$$\boxed{\quad}, \boxed{\frac{3}{2}\ln 2, 2\ln 2, \frac{5}{2}\ln 2, 2.090, 3.2, 4.807}, \boxed{\approx 4.96} \boxed{6 - \ln 3}$$

a) DETERMINE THE "GAPS"

$$\frac{\ln 8 - \ln 2}{4} = \frac{3\ln 2 - 1\ln 2}{4} = \frac{2\ln 2}{4} = \frac{1}{2}\ln 2.$$

$$\Rightarrow x_2 = \ln 2 + \frac{1}{2}\ln 2 = \frac{3}{2}\ln 2 \quad y_2 = 2.090$$

$$\Rightarrow x_3 = \frac{3}{2}\ln 2 + \frac{1}{2}\ln 2 = 2\ln 2 \quad y_3 = 3.2$$

$$\Rightarrow x_4 = 2\ln 2 + \frac{1}{2}\ln 2 = \frac{5}{2}\ln 2 \quad y_4 = 4.807$$

b) SIMPLIFY THE SUMMED FORMULA

$$\int_{\ln 2}^{\ln 8} \frac{e^{2x}}{e^x + 1} dx \approx \frac{\text{trapezoides}}{2} [\text{first part} + 2 \sum \text{interior}]$$

$$\approx \frac{2\ln 2}{2} [1.333 + 7.111 + 2(2.090 + 3.2 + 4.807)]$$

$$\approx \frac{1}{2} \times 28.686$$

$$\approx 4.93$$

c) USING THE SUBSTITUTION METHOD

$$\begin{aligned} & \int_{\ln 2}^{\ln 8} \frac{e^{2x}}{e^x + 1} dx = \int_3^8 \frac{e^{2u}}{e^u + 1} du \\ & \Rightarrow \int_3^8 \frac{e^u}{4} du = \int_3^8 \frac{e^u - 1}{e^u} du \\ & = \int_3^8 \left(1 - \frac{1}{e^u}\right) du = \left[u - \ln|u|\right]_3^8 \\ & = (8 - \ln 8) - (3 - \ln 3) = 6 + \ln(3 - 8) \\ & = 6 + \ln\left(\frac{1}{5}\right) = 6 - \ln 5 \end{aligned}$$

$\bullet u = e^x + 1$
 $\bullet \frac{du}{dx} = e^x$
 $\bullet du = e^x dx$
 $2\ln 2 \rightarrow u=3$
 $2\ln 8 \rightarrow u=8$
 $e^2 = u-1$

Question 143 (**)**

It is given that the value of

$$\int_0^{\frac{1}{3}\pi} (k \cos^2 x - \sec^2 x) \sin x \, dx,$$

is 2, where k is a non zero constant.

Determine the value of k .

, $k = 6$

DETERMINE THE EXPRESSION FOR THE INTEGRAL IN TERMS OF k .

$$\begin{aligned} \int_0^{\frac{\pi}{3}} (k \cos^2 x - \sec^2 x) \sin x \, dx &= \int_0^{\frac{\pi}{3}} k \cos x \sin x - \sec x \sin x \, dx \\ &= \int_0^{\frac{\pi}{3}} k \cos x \sin x - \sec x \tan x \, dx \\ \text{BY RECOGNITION, WE OBTAIN:} \\ &= \left[-\frac{k}{2} \cos^2 x - \sec x \right]_0^{\frac{\pi}{3}} = \left[\frac{k}{2} \cos^2 x + \sec x \right]_0^{\frac{\pi}{3}} \\ &= \left(\frac{k}{2} + 1 \right) - \left(\frac{k}{24} + 2 \right) = \frac{k}{2} - \frac{k}{24} - 1 \\ &= \frac{1}{24}(7k - 24) = \frac{1}{24}(7k - 24) \end{aligned}$$

Finally we have:

$$\begin{aligned} \frac{1}{24}(7k - 24) &= \frac{3}{4} \\ 7k - 24 &= 18 \\ 7k &= 42 \\ k &= 6 \end{aligned}$$

Question 144 (****)

$$f(x) = \frac{e^{\sqrt[4]{x}}}{\sqrt{x}}, \quad x \in \mathbb{R}, \quad x > 0.$$

Find the value of

$$\int_0^1 f(x) \, dx,$$

given further that the integral exists.

, [4]

$f(x) = \frac{e^{\sqrt[4]{x}}}{\sqrt{x}}$, $x > 0$

USING THE SUBSTITUTION GIVEN WE HAVE

$$\begin{aligned} u &= \sqrt[4]{x} &= x^{\frac{1}{4}} &\quad x=0 \mapsto 0 \\ u^2 &= \sqrt{x} &= x^{\frac{1}{2}} &\quad x=1 \mapsto 1 \\ u^4 &= x && \end{aligned}$$

$$\frac{du}{dx} = \frac{1}{4}x^{-\frac{3}{4}}$$

TRANSFORMING THE INTEGRAL WE HAVE:

$$\int_0^1 \frac{e^{\sqrt[4]{x}}}{\sqrt{x}} \, dx = \int_0^1 \frac{e^u}{u^2} (4u^3) \, du = \int_0^1 4ue^u \, du$$

INTEGRATION BY PARTS EQUALLY (IGNORING UNITS)

$$\begin{aligned} \frac{4u}{e^u} \Big|_0^1 &\Rightarrow \int 4ue^u \, du = 4ue^u - \int 4e^u \, du \\ &= 4ue^u - 4e^u + C \\ &= 4e^u(u-1) + C \end{aligned}$$

INSERTING THE UNITS AND EVALUATING:

$$\int_0^1 4ue^u \, du = [4e^u(u-1)]_0^1 = 4e^1(1-1) - 4e^0(0-1) = 4$$

Question 145 (****)

$$\frac{u^2}{u^2-1} \equiv A + \frac{B}{u-1} + \frac{C}{u+1}.$$

a) Find the value of A , B and C in the above identity.

b) Use the substitution $u = \sqrt{1-e^{2x}}$ to show

$$\int_0^{\ln \frac{1}{2}} \sqrt{1-e^{2x}} \, dx = \frac{\sqrt{3}}{2} + \ln(2-\sqrt{3}).$$

$$A=1, B=\frac{1}{2}, C=-\frac{1}{2}$$

(a) $\frac{u^2}{u^2-1} = A + \frac{B}{u-1} + \frac{C}{u+1}$

$u^2 \equiv A(u-1)(u+1) + B(u+1) + C(u-1)$

• If $u=1$, $2B=1$
 $B=\frac{1}{2}$

• If $u=-1$, $-2C=1$
 $C=-\frac{1}{2}$

• If $u=0$, $0=A+B+C$
 $0=A+\frac{1}{2}-\frac{1}{2}$
 $A=0$

(b) $\int_0^{\ln \frac{1}{2}} \sqrt{1-e^{2x}} \, dx = \dots$ by substitution...

$$\begin{aligned} &= \int_0^{\frac{1}{2}} u \left(\frac{u}{u^2-1} \, du \right) = \int_0^{\frac{1}{2}} \frac{u^2}{u^2-1} \, du \\ &= \int_0^{\frac{1}{2}} 1 + \frac{\frac{1}{u^2-1} - \frac{1}{u^2}}{u^2-1} \, du \\ &= \int_0^{\frac{1}{2}} \left[u + \frac{1}{2} \ln|u-1| - \frac{1}{2} \ln|u+1| \right] \, du \\ &= \left(\frac{u^2}{2} + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| \right) \Big|_0^{\frac{1}{2}} - \left(0 + \frac{1}{2} \ln \left| \frac{1-1}{1+1} \right| \right) \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2} \ln \left| \frac{\frac{\sqrt{3}}{2}-1}{\frac{\sqrt{3}}{2}+1} \right| = \frac{\sqrt{3}}{2} + \frac{1}{2} \ln \left| \frac{\sqrt{3}-2}{\sqrt{3}+2} \right| \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2} \ln \left(\frac{2-\sqrt{3}}{2+\sqrt{3}} \right) \rightarrow \frac{\sqrt{3}}{2} + \frac{1}{2} \ln \left[\frac{(2-\sqrt{3})^2}{(2+\sqrt{3})(2-\sqrt{3})} \right] \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2} \ln \frac{5-4\sqrt{3}}{4} = \frac{\sqrt{3}}{2} + \frac{1}{2} \ln (2-\sqrt{3})^2 = \frac{\sqrt{3}}{2} + \ln(2-\sqrt{3}) \end{aligned}$$

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Question 146 (****)

By using the substitution $u = \sqrt{x^3 + 1}$, or otherwise, find an expression for

$$\int \frac{9x^5}{\sqrt{x^3+1}} dx.$$

$$2(x^3+1)^{\frac{3}{2}} - 6(x^3+1)^{\frac{1}{2}} + C$$

$$\begin{aligned} \int \frac{9x^5}{\sqrt{x^3+1}} dx &= \dots \text{BY SUBSTITUTION} \dots \\ \int \frac{9x^5}{dx} \left(\frac{du}{dx} \right) &= \int 6u^3 du \\ = \int 6(u^3-1) du &= 6\left(\frac{1}{4}u^4 - u\right) + C \\ = 2u^3 - 6u + C &= 2(x^3+1)^{\frac{3}{2}} - 6(x^3+1)^{\frac{1}{2}} + C \end{aligned}$$

$u = \sqrt{x^3+1}$
 $u^3 = x^3+1$
 $2u \frac{du}{dx} = 3x^2$
 $\frac{2u}{3x^2} du = dx$
 $2^3 = u^2-1$

Question 147 (**)**

It is given that

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

- a) Use the above trigonometric identity to show that

$$\sin 3x \equiv 3 \sin x - 4 \sin^3 x.$$

- b) Hence find

$$\int \cos x (6 \sin x - 2 \sin 3x)^{\frac{2}{3}} dx.$$

, $\boxed{\frac{4}{3} \sin^3 x + C}$

a) PROOF AS PRACTICE

$$\begin{aligned}\sin 3x &= \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x \\&= (2\sin x \cos x) \cos x + (1-2\sin^2 x) \sin x \\&= 2\sin x 2\cos^2 x + \sin x - 2\sin^3 x \\&= 2\sin x (1-\sin^2 x) + \sin x - 2\sin^3 x \\&= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\&= 3\sin x - 4\sin^3 x\end{aligned}$$

// no expansion

b) FIND THE INTEGRAL OF PART (a)

$$\begin{aligned}&\int \cos x (6 \sin x - 2 \sin 3x)^{\frac{2}{3}} dx \\&= \int \cos x [6 \sin x - 2(3 \sin x - 4 \sin^3 x)]^{\frac{2}{3}} dx \\&= \int \cos x [6 \sin x - 6 \sin x + 8 \sin^3 x]^{\frac{2}{3}} dx \\&= \int \cos x (8 \sin^3 x)^{\frac{2}{3}} dx \\&= \int \cos x (4 \sin^2 x)^{\frac{1}{3}} dx \\&= \int 4 \cos x \sin^{\frac{2}{3}} x dx\end{aligned}$$

BY INTEGRATION, OR MAKING THE SUBSTITUTION, $u = \sin x$

$$= \boxed{\frac{4}{3} \sin^3 x + C}$$

Question 148 (**)**

Use the substitution $t = 3 + \sqrt{x}$ to find the value of the following integral

$$\int_1^{36} \frac{1}{\sqrt{x^{\frac{3}{2}} + 3x}} dx.$$

 [4]

• START BY THE SUBSTITUTION GIVEN

$$\begin{aligned} &\rightarrow t = \sqrt{x} + 3 \\ &\rightarrow t = x^{\frac{1}{2}} + 3 \\ &\rightarrow \frac{dt}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \\ &\rightarrow \frac{dt}{dx} = \frac{1}{2\sqrt{x}} \\ &\rightarrow 2\sqrt{x} dx = dt \\ &\rightarrow dx = \frac{1}{2\sqrt{x}} dt \end{aligned}$$

LETS
 $x=1 \mapsto t=4$
 $x=36 \mapsto t=9$

• HENCE THE INTEGRAL TRANSFORMS AS FOLLOWS

$$\begin{aligned} \int_1^{36} \frac{1}{\sqrt{x^{\frac{3}{2}} + 3x}} dx &= \int_4^9 \frac{1}{\sqrt{2t^{\frac{3}{2}} + 3t}} dt \\ &= \int_4^9 \frac{2\sqrt{x}}{\sqrt{2x^{\frac{3}{2}} + 3x}} dt \\ &= \int_4^9 \frac{2\sqrt{x}}{\sqrt{x^{\frac{3}{2}}(2 + \frac{3}{x})}} dt \\ &= \int_4^9 \frac{2}{\sqrt{2 + \frac{3}{x}}} dt \\ &= \int_4^9 2t^{-\frac{1}{2}} dt \\ &= \left[4t^{\frac{1}{2}} \right]_4^9 = (4x^{\frac{1}{2}}) - (4x^{\frac{1}{2}}) \\ &= 12 - 8 = 4 \end{aligned}$$

Question 149 (**)**

By using the substitution $u = \frac{1}{x}$, or otherwise, show that

$$\int_0^4 \left(\frac{1}{x^2} + \frac{1}{x^3} \right) e^{\frac{1}{x}} dx = -\frac{1}{x} e^{\frac{1}{x}} + C.$$



$$\begin{aligned} &\int \left(\frac{1}{x^2} + \frac{1}{x^3} \right) e^{\frac{1}{x}} dx \dots \text{by substitution} \\ &= \int \left(\frac{1}{x^2} \right) e^{\frac{1}{x}} (-u du) = \int -\frac{1}{x^2} e^{\frac{1}{x}} du \\ &= \left(-\frac{1}{x} \right) e^{\frac{1}{x}} du = \int (-u) e^u du = \dots \text{by parts} \\ &= (-1-u)e^u - \int e^u du = -(1+u)e^u + e^u + C \\ &= x^{\frac{1}{x}} - x^{\frac{1}{x}} + e^{\frac{1}{x}} + C = -x^{\frac{1}{x}} e^{\frac{1}{x}} + C = -\frac{1}{x} e^{\frac{1}{x}} + C \end{aligned}$$

$u = \frac{1}{x}$
$\frac{du}{dx} = \frac{1}{x^2}$
$dx = x^2 du$
$-1-u$
e^u

Question 150 (****)

By using the substitution $u = \cos x$, or otherwise, show clearly that

$$\int_0^{\frac{\pi}{2}} 15 \cos^5 x \ dx = 8.$$

proof

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} 15 \sin^5 x \ dx &= \int_0^{\frac{\pi}{2}} 15 \sin^5(-\frac{du}{\sin x}) \\
 &= \int_0^1 15 \sin^5 u \ du = \int_0^1 15 (\sin^2 u)^2 \ du \\
 &= \int_0^1 15 (1 - \cos^2 u)^2 \ du = \int_0^1 15 (1 - u^2)^2 \ du \\
 &= \int_0^1 15 - 30u^2 + 15u^4 \ du = \left[15u - 10u^3 + 3u^5 \right]_0^1 \\
 &= (15 - 10 + 3) - 0 = 8
 \end{aligned}$$

Question 151 (****)

Use the substitution $x = 2 \cos \theta$ to show that

$$\int_1^{\sqrt{2}} \frac{4}{x^2 \sqrt{4-x^2}} \ dx = \sqrt{m-n},$$

where m and n are integers.

proof

$$\begin{aligned}
 \int_1^{\sqrt{2}} \frac{4}{x^2 \sqrt{4-x^2}} \ dx &= \dots \text{using 1st-1st method} \\
 &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{4}{(2 \cos \theta)^2 \sqrt{4 - (2 \cos \theta)^2}} (-2 \sin \theta) d\theta \\
 &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{-8 \sin \theta}{4 \cos^2 \theta \sqrt{4 - 4 \cos^2 \theta}} d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{8 \sin \theta}{4 \cos^2 \theta \sqrt{4 \sin^2 \theta}} d\theta \\
 &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2 \sin \theta}{\cos^2 \theta \cdot 2 \sin \theta} d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\cos^2 \theta} d\theta \\
 &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sec^2 \theta d\theta = \left[\tan \theta \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \tan \frac{\pi}{2} - \tan \frac{\pi}{3} = \sqrt{3} - 1
 \end{aligned}$$

Question 152 (*****)

$$y = \frac{4x+3}{3x+4}, \quad x \neq -\frac{4}{3}.$$

- a) Calculate the five missing values of x and y in the following table.

x	0				32
y	$\frac{3}{4}$	$\frac{35}{29}$	$\frac{67}{52}$		

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate for

$$\int_0^{32} \frac{4x+3}{3x+4} dx.$$

- c) Use the substitution $u = 3x+4$ to find the exact value of

$$\int_0^{32} \frac{4x+3}{3x+4} dx.$$

$\boxed{}$, $\boxed{8, 16, 24}$
	, $\boxed{\frac{99}{76}, \frac{131}{100}}$
	, $\boxed{38.6}$, $\boxed{\frac{128}{3} - \frac{14}{9} \ln 5} = \frac{1}{9}[384 - 14 \ln 5]$

a) FILL IN THE TABLE

x	0	8	16	24	32
y	$\frac{3}{4}$	$\frac{35}{29}$	$\frac{67}{52}$	$\frac{99}{76}$	$\frac{131}{100}$

b) USING THE TRAPEZIUM RULE

$$\int_0^{32} \frac{4x+3}{3x+4} dx \approx \frac{8}{2} \left[\frac{3}{4} + \frac{35}{29} + 2 \times \frac{67}{52} \right] \approx 4 \left[6.559 \dots \right] \approx 38.623 \dots \approx 38.6$$

c) USING THE SUBSTITUTION (Method)

- $u = 3x+4$
- $du = 3dx$
- $dx = \frac{1}{3} du$

TRANSFORMING THE INTEGRAL

$$\int_0^{32} \frac{4x+3}{3x+4} dx = \int_4^{100} \frac{4u+3}{3u} \cdot \frac{1}{3} du = \int_4^{100} \frac{4u+3}{9u} du = \int_4^{100} \frac{4+3/u}{9} du = \frac{1}{9} \int_4^{100} (4u+3) du$$

Now use limits:

$$\begin{aligned} u &= 4 \Rightarrow 3x+4 \\ 4u &= 16 \\ \frac{4u-7}{9} &= \frac{125}{9} \end{aligned}$$

RETURNING TO THE INTEGRAL

$$\begin{aligned} \dots &= \int_4^{100} \frac{4u-7}{9u} du = \frac{1}{9} \int_4^{100} \frac{4u-7}{u} du = \frac{1}{9} \int_4^{100} \left(4 - \frac{7}{u} \right) du \\ &= \frac{1}{9} \left[4u - 7 \ln|u| \right]_4^{100} = \frac{1}{9} \left[(400 - 7 \ln 100) - (16 - 7 \ln 4) \right] \\ &= \frac{1}{9} [400 - 16 + 7 \ln 4 - 7 \ln 100] = \frac{1}{9} [384 + 7 \ln \frac{1}{25}] \\ &= \frac{1}{9} [384 + 7 \ln 5^{-2}] = \frac{1}{9} [384 - 14 \ln 5] \end{aligned}$$

ALTERNATIVE METHOD

$$\begin{aligned} \int_0^{32} \frac{4x+3}{3x+4} dx &= \int_0^{32} \frac{12x+9}{3x+4} dx = \int_0^{32} \frac{(3x+4)+7}{3x+4} dx = \frac{1}{3} \int_0^{32} 4 - \frac{7}{3x+4} dx \\ &= \frac{1}{3} \left[4x - \frac{7}{3} \ln|3x+4| \right]_0^{32} = \frac{1}{3} \left[(32 - \frac{7}{3} \ln 100) - (0 - \frac{7}{3} \ln 4) \right] \\ &= \frac{1}{3} \left[32 - \frac{7}{3} \ln 100 + \frac{7}{3} \ln 4 \right] = \frac{1}{3} \left[32 + \frac{7}{3} \ln \frac{4}{100} \right] \\ &= \frac{1}{3} \left[32 + \frac{7}{3} \ln \frac{1}{25} \right] = \frac{1}{3} \left[28 + \frac{7}{3} \ln 5^{-2} \right] \\ &= \frac{1}{3} \left[28 - \frac{14}{3} \ln 5 \right] = \frac{1}{3} [384 - 14 \ln 5] \end{aligned}$$

Question 153 (**)**

Determine, in terms of a , the value of the following integral.

$$\int_{\frac{2}{a}}^{\frac{17}{a}} \frac{2ax}{\sqrt{ax-1}} dx, \quad a \neq 0.$$

You may find the substitution $u^2 = ax - 1$ useful in this question.

, $\boxed{\frac{96}{a}}$

<p>Proceed by substitution: $u^2 = ax - 1$</p> $\begin{aligned} u &= +\sqrt{ax-1} & a &= \frac{u^2+1}{x} \\ 2u \frac{du}{dx} &= a & u &= \sqrt{\frac{x}{a}} \rightarrow u = \sqrt{a} \\ 2u du &= a dx & x &= \frac{u^2+1}{a} \\ dx &= \frac{2u}{a} du & \end{aligned}$	$\begin{aligned} & \boxed{\frac{96}{a}} \\ & \boxed{1} \end{aligned}$
---	---

Transforming the integral:

$$\begin{aligned} \int_{\frac{2}{a}}^{\frac{17}{a}} \frac{2ax}{\sqrt{ax-1}} dx &= \int_1^4 \frac{2ax}{\sqrt{u}} \left(\frac{2u}{a} du \right) \\ &= \int_1^4 4x du \\ &= \int_1^4 \frac{4ax}{a} du \\ &= \frac{4}{a} \int_1^4 ax du \\ &= \frac{4}{a} \int_1^4 u^2 + 1 du \\ &= \frac{4}{a} \left[\frac{1}{3}u^3 + u \right]_1^4 \\ &= \frac{4}{a} \left[\left(\frac{64}{3} + 4 \right) - \left(\frac{1}{3} + 1 \right) \right] \\ &= \frac{4}{a} \times 24 \\ &= \boxed{\frac{96}{a}} \end{aligned}$$

Question 154 (**)**

Use the substitution $x = \tan \theta$ to show that

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1-x^2}{1+x^2} dx = \frac{1}{3}(\pi - 2\sqrt{3}).$$

proof

$$\begin{aligned}
 \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1-x^2}{1+x^2} dx &= \dots \text{ by the substitution given} \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\tan^2 \theta}{1+\tan^2 \theta} (\sec^2 \theta d\theta) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\tan^2 \theta}{\sec^2 \theta} \sec^2 \theta d\theta \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 - \sec^2 \theta) d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2 - \sec^2 \theta d\theta \\
 &= [2\theta - \tan \theta] \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \left(\frac{2\pi}{3} - \sqrt{3} \right) - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{3} \right) \\
 &= \frac{\pi}{3} - \frac{2\sqrt{3}}{3} = \frac{1}{3}(\pi - 2\sqrt{3}) \quad \text{✓ EQUIVALENT}
 \end{aligned}$$

Question 155 (**)**

By using the substitution $u = \sqrt{x+2}$, or otherwise, find an expression for

$$\int \frac{1}{(x+1)\sqrt{x+2}} dx.$$

$$\ln \left| \frac{\sqrt{x+2}-1}{\sqrt{x+2}+1} \right| + C$$

$$\begin{aligned}
 \int \frac{1}{(x+1)\sqrt{x+2}} dx &= \dots \text{ by substitution} \\
 &= \int \frac{1}{(u^2+1)\sqrt{u^2+2}} 2u du = \int \frac{2}{u^2+1} du = \int \frac{2}{(u-1)(u+1)} du \\
 &= \dots \text{ BY PARTIAL FRACTIONS} \dots = \int \frac{1}{u-1} - \frac{1}{u+1} du \\
 &= [\ln|u-1| - \ln|u+1|] + C = \left[\ln \frac{|u-1|}{|u+1|} \right] + C
 \end{aligned}$$

Question 156 (**)**

Use appropriate integration techniques to show that

$$\int_0^{\frac{1}{4}\pi^2} \sin\sqrt{x} \, dx = N,$$

where N is a positive integer.

$$\boxed{\quad}, \boxed{N=2}$$

The handwritten solution shows the following steps:

$$\begin{aligned} \int_0^{\frac{1}{4}\pi^2} \sin\sqrt{x} \, dx &= \dots \text{SUBSTITUTION FIRST} \rightarrow \\ &= \int_0^{\frac{\pi^2}{4}} \sin u \cdot (2u \, du) = \int_0^{\frac{\pi^2}{4}} 2u \sin u \, du \\ &\dots \text{BY PARTS & INTEGRATION BY PARTS} \dots \\ &\int 2u \sin u \, du \quad \begin{matrix} 2u \\ \hline \text{+} \\ \sin u \end{matrix} \\ &= -2u \cos u - \int -2 \cos u \, du \\ &= -2u \cos u + \int 2 \cos u \, du \\ &= -2u \cos u + 2 \sin u + C \\ &\dots \left[-2u \cos u + 2 \sin u \right]_0^{\frac{\pi^2}{4}} = \left[(0+2) - (0+0) \right] = 2, \end{aligned}$$

ie $N=2$

Question 157 (*)**

By using the substitution $x = \tan \theta$, or otherwise, find the value of

$$\int_0^1 \frac{1-x^2}{(1-x^2)^2} dx.$$

1
2

$$\begin{aligned}
 & \int_0^1 \frac{1-x^2}{(1-x^2)^2} dx \dots \text{substitution} \\
 &= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2\theta}{(1+\tan^2\theta)^2} (\sec^2\theta d\theta) \\
 &= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2\theta}{\sec^2\theta} \times \sec^2\theta d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2\theta}{\sec^2\theta} d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta - \tan^2\theta}{\sec^2\theta} d\theta \\
 &= \int_0^{\frac{\pi}{4}} \sec^2\theta - \frac{\sin^2\theta}{\cos^2\theta} d\theta \\
 &= \int_0^{\frac{\pi}{4}} \sec^2\theta - \frac{\sin^2\theta}{\cos^2\theta} d\theta \\
 &= \int_0^{\frac{\pi}{4}} \cos^2\theta d\theta \\
 &= \left[\frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}} \\
 &= \frac{1}{2} - 0 \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 x &= \tan \theta \\
 \frac{dx}{d\theta} &= \sec^2\theta \\
 dx &= \sec^2\theta d\theta \\
 x=0 &\Rightarrow \tan\theta=0 \\
 \Rightarrow \theta &= 0 \\
 x=1 &\Rightarrow \tan\theta=1 \\
 \Rightarrow \theta &= \frac{\pi}{4}
 \end{aligned}$$

Question 158 (***)

$$y = (1 + \cot^2 x) \sec^2 x, \quad 0 < x < \frac{1}{2}\pi$$

- a) Calculate the three missing values of x in the following table.

x	$\frac{1}{6}\pi$				$\frac{1}{3}\pi$
y	$\frac{16}{3}$	$32 - 16\sqrt{3}$	4	$32 - 16\sqrt{3}$	$\frac{16}{3}$

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate for

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} (1 + \cot^2 x) \sec^2 x \, dx$$

- c) Use an appropriate integration method to find an exact simplified value for

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \left(1 + \cot^2 x\right) \sec^2 x \ dx$$

$$[\quad, \quad], \quad \left[\frac{5\pi}{24}, \frac{\pi}{4}, \frac{7\pi}{24} \right], \quad [2.34], \quad \left[\frac{4}{3}\sqrt{3} \right]$$

a) FILL IN THE TABLE

<u>x</u>	<u>$\frac{1}{2}x^2$</u>	<u>$16x^3$</u>	<u>$4x$</u>	<u>$32 - 16x^3$</u>	<u>$\frac{1}{2}$</u>
<u>y</u>	<u>$\frac{16}{3}$</u>	<u>$32 - 16x^3$</u>	<u>4</u>	<u>$32 - 16x^3$</u>	<u>$\frac{16}{3}$</u>

$$\left[\frac{\frac{16}{3} - \frac{16}{3}}{4} = \frac{\frac{16}{3}}{4} \right] \leftarrow \text{Graf}$$

b) BY THE TRAPEZOID RULE

$$\int_{\frac{1}{2}}^{\frac{1}{2}} \sec^2(x + \tan x^3) dx \approx \frac{\text{"THICKNESS"} \left[f(a) + f(b) + 2 \times \text{rest} \right]}{\Delta x}$$

$$\approx \frac{1}{2} \left[\frac{16}{3} + \frac{16}{3} + 2(2 \cdot 16x^3 + 4 + 32 - 16x^3) \right]$$

$$\approx \frac{\pi}{18} \times 35.11541 \dots$$

$$\approx 2.3441 \dots$$

2) 2.34

NOTING THE DIFFERENTIALS

$$\frac{d}{dx}(kun) = ksec^2$$

$$\frac{d}{dx}(ab) = asec^2 b$$

$$= \left[\tan x - \sec x \right] \frac{\pi}{18}$$

Question 159 (*)+**

Use appropriate integration techniques to evaluate

$$\int_{\sqrt{5}}^{\sqrt{60}} \sqrt{1 + \frac{4}{x^2}} \, dx.$$

Give the answer in the form $a + b \ln 3$, where a and b are positive integers.

5 + ln 3

The handwritten solution shows the following steps:

$$\begin{aligned} & \int_{\sqrt{5}}^{\sqrt{60}} \sqrt{1 + \frac{4}{x^2}} \, dx = \int_{\sqrt{5}}^{\sqrt{60}} \sqrt{\frac{x^2+4}{x^2}} \, dx = \int_{\sqrt{5}}^{\sqrt{60}} \frac{\sqrt{x^2+4}}{x} \, dx \\ & \dots \text{SUBSTITUTION} \dots \\ & = \int_{\sqrt{5}}^{\sqrt{60}} \frac{u}{2} \cdot \left(\frac{du}{\sqrt{u^2-4}} \right) = \int_{\sqrt{5}}^{\sqrt{60}} \frac{u^2}{2\sqrt{u^2-4}} \, du = \int_{\sqrt{5}}^{\sqrt{60}} \frac{u^2}{2(u^2-4)} \, du \\ & \dots \text{IMPROPER FRACTION} \dots \text{LONG DIVISION OR MANIPULATION} \\ & = \int_{\sqrt{5}}^{\sqrt{60}} \frac{(u^2-4)+4}{u^2-4} \, du = \int_{\sqrt{5}}^{\sqrt{60}} \frac{u^2-4}{u^2-4} + \frac{4}{u^2-4} \, du = \int_{\sqrt{5}}^{\sqrt{60}} 1 + \frac{4}{u^2-4} \, du \\ & = \int_{\sqrt{5}}^{\sqrt{60}} 1 + \frac{4}{(u-2)(u+2)} \, du \\ & \dots \text{BY PARTIAL FRACTIONS} \dots \\ & \frac{4}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2} \\ & 4 = A(u+2) + B(u-2) \\ & \frac{4}{4} \cdot u=2, \quad 4=4A \Rightarrow [A=1] \\ & \frac{4}{4} \cdot u=-2, \quad 4=4B \Rightarrow [B=-1] \\ & = \int_{\sqrt{5}}^{\sqrt{60}} 1 + \frac{1}{u-2} - \frac{1}{u+2} \, du = \left[u + \ln|u-2| - \ln|u+2| \right]_{\sqrt{5}}^{\sqrt{60}} \\ & = \left[8 + \ln 6 - \ln 10 \right] - \left[3 + \ln 1 - \ln 5 \right] = 8 + \ln 6 - \ln 10 - 3 + \ln 5 \\ & = 5 + \ln \left(\frac{6\sqrt{5}}{10} \right) = 5 + \ln 3 \end{aligned}$$

Question 160 (*)+**

Use a suitable substitution to show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} \, dx = \ln\left(\frac{9}{8}\right).$$

proof

The handwritten proof shows the following steps:

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} \, dx = \dots \text{by substitution} \\ & = \int_0^{\frac{\pi}{2}} \frac{u \cos x}{u^2+3u+2} \cdot (-\sin x) \, du = \int_0^{\frac{\pi}{2}} \frac{u}{u^2+3u+2} \, du \\ & \text{PARTIAL FRACTIONS} \\ & \frac{u}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2} \quad \bullet \text{IF } u=-1 \quad A=-1 \\ & u = A(u+2) + B(u+1) \quad \bullet \text{IF } u=-2 \quad B=2 \\ & = \int_0^{\frac{\pi}{2}} \frac{-1}{u+2} - \frac{1}{u+1} \, du = \left[2\ln|u+2| - \ln|u+1| \right]_0^{\frac{\pi}{2}} \\ & = (2\ln 3 - \ln 2) - (2\ln 2 - \ln 1) = \ln 9 - \ln 6 - \ln 4 = \ln \frac{9}{8} \end{aligned}$$

Question 161 (***)**

Use partial fractions to determine, in exact simplified form, the value of the following integral.

$$\int_0^{\frac{1}{2}} \frac{2x^3 - 5x^2 + 5}{(x^2 - 3x + 2)(x^2 - 2x + 1)} dx .$$

, $5 + \ln\left(\frac{3}{8}\right)$

PROCED BY PARTIAL FRACTIONS

$$\frac{2x^3 - 5x^2 + 5}{(x^2 - 3x + 2)(x^2 - 2x + 1)} = \frac{A}{x-2} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)} + \frac{D}{x+1}$$

$$2x^3 - 5x^2 + 5 \equiv A(x-1)^2 + B(x-2) + C(x-1)(x+1) + D(x-2)(x+1)$$

• If $x=1$	$2=-8$	• If $x=0$	$S=-A-2B+2C+2D$
$\underline{B=-2}$		$S=-A+2C+2D$	
• If $x=2$	$L=A$	$Z=2C+2D$	$C-D=1$
	$\underline{A=1}$		
SUBSTITUTING METHODS			
$\begin{array}{r} 20=3 \\ \underline{16=1} \\ 8=2 \end{array}$			

RETURNING TO THE INTEGRAL WITH THE FRACTION SPLIT

$$\dots = \int_0^{\frac{1}{2}} \frac{1}{x-2} - 2(x-1)^2 + 2(x-1)^{-2} + \frac{1}{x+1} dx$$

$$= \left[\ln|x-2| + (x-1)^{-2} - 2(x-1)^{-1} + \ln|x+1| \right]_0^{\frac{1}{2}}$$

$$= \left[\ln|\frac{1}{2}-2| + \ln|\frac{1}{2}-1| + \frac{1}{(2-1)^2} - \frac{2}{2-1} \right]^{\frac{1}{2}}$$

$$= \left(\ln\frac{1}{2} + \ln\frac{1}{2} + \frac{1}{4} - \frac{2}{1} \right) - (\ln 2 + \ln(-1+2))$$

$$= \ln\frac{1}{2} + \ln\frac{1}{2} - \ln 2 + 4 + 4 - 1 - 2$$

$$= 5 + \ln\frac{1}{2}$$

$$= 5 + \ln\frac{3}{8}$$

Question 162 (*)**

Use the substitution $u = \sin x$ to find an expression for

$$\int \frac{\cos x + \tan x}{1 + \tan^2 x} dx.$$

$\sin x - \frac{1}{3} \sin^3 x + \frac{1}{2} \sin^2 x + C$

$$\begin{aligned}
 & \int \frac{\cos x + \tan x}{1 + \tan^2 x} dx = \dots \text{ SUBSTITUTION } \dots \\
 & = \int \frac{\cos x + \tan x}{1 + \tan^2 x} \times \frac{1}{\cos x} du \\
 & = \int \frac{\cos x + \tan x}{\cos x(1 + \tan^2 x)} du \\
 & = \int \frac{\cos x + \tan x}{\cos x \sec^2 x} du = \int \frac{\cos x + \tan x}{\sec x} du \\
 & = \int \frac{\cos x}{\sec x} + \frac{\tan x}{\sec x} du = \int \cos^2 x + \tan x \sec x du \\
 & = \int (1 - \sin^2 x) + \frac{\sin x \cos x}{\cos x} du = \int 1 - \sin^2 x + \sin x du \\
 & = \int 1 - u^2 + u du = u - \frac{1}{3}u^3 + \frac{1}{2}u^2 + C \\
 & = \sin x - \frac{1}{3}\sin^3 x + \frac{1}{2}\sin^2 x + C
 \end{aligned}$$

Question 163 (**+)**

Use the substitution $u = 1 + \sqrt{x}$ to evaluate

$$\int_0^9 \frac{3x}{1+\sqrt{x}} dx.$$

45 - 12ln 2

$$\begin{aligned}
 & \int_0^9 \frac{3x}{1+\sqrt{x}} dx = \dots \text{ BY SUBSTITUTION } \\
 & = \int_1^4 \frac{3x}{4} \times 2(u-1) du = \int_1^4 \frac{3(u-1)^2 \times 2(u-1)}{4} du \\
 & = \int_1^4 \frac{6(u-1)^3}{4} du \\
 & \bullet \text{ EXPAND } \dots \\
 & = \int_1^4 \frac{6(u^3 - 3u^2 + 3u - 1)}{4} du \\
 & \bullet \text{ SPILT THE FUNCTION} \\
 & = \int_1^4 \left[2u^3 - 9u^2 + 18u - 6 \right] du = \left[2u^4 - 9u^3 + 18u^2 - 6u \right]_1^4 \\
 & = (128 - 144 + 72 - 6) - (2 - 9 + 18 - 6) \\
 & = 45 - 6u^4 \quad \text{or} \quad 45 - 12\ln 2
 \end{aligned}$$

Question 164 (****+)

$$y = \frac{x^2}{2x+1}, \quad x \neq -\frac{1}{2}$$

- a) Calculate the two missing values of y in the following table.

x	0	0.1	0.2	0.3	0.4	0.5
y	0	$\frac{1}{120}$	$\frac{1}{35}$			$\frac{1}{8}$

- b) Use the trapezium rule with all the values from the completed table of part (a) to find an estimate, correct to 4 significant figures, for the following integral.

$$\int_0^{\frac{1}{2}} \frac{x^2}{2x+1} dx.$$

- d) Use the substitution $u = 2x+1$ to find an exact simplified value for

$$\int_0^{\frac{1}{2}} \frac{x^2}{2x+1} dx.$$

- e) Hence deduce, by referring to parts (b) and (c), the approximate value of $\ln 2$ correct to 2 significant figures.

, $\frac{9}{160}, \frac{4}{45}$, 0.02445 , $\frac{1}{16}[-1+2\ln 2]$, $\ln 2 \approx 0.70$

<p>a) <u>FILL IN THE TABLE</u></p> <table border="1"> <thead> <tr> <th>x</th> <th>u</th> <th>0.1</th> <th>0.2</th> <th>0.3</th> <th>0.4</th> <th>0.5</th> </tr> </thead> <tbody> <tr> <td>y</td> <td>0</td> <td>$\frac{1}{120}$</td> <td>$\frac{1}{35}$</td> <td>$\frac{2}{25}$</td> <td>$\frac{8}{25}$</td> <td>$\frac{1}{8}$</td> </tr> </tbody> </table> <p>b) <u>APPROXIMATING BY THE TRAPEZIUM RULE</u></p> $\int_0^{\frac{1}{2}} \frac{x^2}{2x+1} dx \approx \frac{1}{2} \left[f(0) + 2f(0.1) + 2f(0.2) + f(0.3) \right] \approx 0.02445$ <p>c) <u>BY THE SUBSTITUTION FORMULA WE HAVE</u></p> <ul style="list-style-type: none"> $u = 2x+1 \Rightarrow 2x = u-1 \Rightarrow x = \frac{u-1}{2}$ 	x	u	0.1	0.2	0.3	0.4	0.5	y	0	$\frac{1}{120}$	$\frac{1}{35}$	$\frac{2}{25}$	$\frac{8}{25}$	$\frac{1}{8}$	<p>$\bullet \frac{du}{dx} = 2$</p> $\frac{dx}{du} = \frac{1}{2} du$ <p>$\bullet x=0 \rightarrow u=1$</p> $x=\frac{u-1}{2} \rightarrow u=2$ <p><u>TRANSFORMING THE INTEGRAL</u></p> $\int_0^{\frac{1}{2}} \frac{x^2}{2x+1} dx = \int_1^2 \frac{\left(\frac{u-1}{2}\right)^2}{u} \cdot \frac{1}{2} du = \frac{1}{8} \int_1^2 (u^2 - 2u + 1) du = \frac{1}{8} \left[\frac{u^3}{3} - u^2 + u \right]_1^2 = \frac{1}{8} \left[\frac{8}{3} - 4 + 2 \right] = \frac{1}{8} \left(-\frac{4}{3} \right) = -\frac{1}{6}$	<p>$\dots = \frac{1}{8} \left[(2 - 4 + \ln 2) - \left(\frac{1}{2} - \frac{1}{2} + \ln 2 \right) \right] = \frac{1}{8} (\ln 2 - \frac{1}{2}) = \frac{1}{16} (-1 + 2\ln 2)$</p> <p>d) <u>FROM PART (b)</u> $\int_0^{\frac{1}{2}} \frac{x^2}{2x+1} dx \approx 0.02445$</p> <p><u>FROM PART (c)</u> $\int_0^{\frac{1}{2}} \frac{x^2}{2x+1} dx = \frac{1}{16} (-1 + 2\ln 2)$</p> <p>$\Rightarrow \frac{1}{16} (-1 + 2\ln 2) \approx 0.02445$</p> <p>$\Rightarrow -1 + 2\ln 2 \approx 0.32$</p> <p>$\Rightarrow 2\ln 2 \approx 1.32$</p> <p>$\Rightarrow \ln 2 \approx 0.70$</p>
x	u	0.1	0.2	0.3	0.4	0.5										
y	0	$\frac{1}{120}$	$\frac{1}{35}$	$\frac{2}{25}$	$\frac{8}{25}$	$\frac{1}{8}$										

Question 165 (***)+

$$f(x) = -x^2 + 4x - 3, \quad 1 \leq x \leq 3.$$

a) Show clearly that $f(2 + \sin \theta) = \cos^2 \theta$.

b) Hence find the exact value of

$$\int_2^3 \sqrt{f(x)} \, dx.$$

$\boxed{\frac{\pi}{4}}$

(a) $x = 2 + \sin \theta, \quad 14x - x^2 - 3 = 4(2 + \sin \theta) - (2 + \sin \theta)^2 - 3$
 $= 8 + 4\sin \theta - (4 + 4\sin \theta + \sin^2 \theta) - 3$
 $= 8 + 4\sin \theta - 4 - 4\sin \theta - 3\sin^2 \theta - 3$
 $= 1 - \sin^2 \theta$
 $= \cos^2 \theta$ ✓ $\boxed{\cos^2 \theta}$

(b) $\int_2^3 \sqrt{4x - x^2 - 3} \, dx \dots \text{by substituting}$
 $= \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 \theta - \cos \theta} \, d\theta = \int_0^{\frac{\pi}{2}} |\cos \theta| \, d\theta$
 $= \int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos 2\theta \, d\theta = \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_0^{\frac{\pi}{2}}$
 $= \left(\frac{\pi}{4} + 0 \right) - (0) = \frac{\pi}{4}$

$\begin{array}{l} x = 2 + \sin \theta \\ \frac{dx}{d\theta} = \cos \theta \\ dx = \cos \theta \, d\theta \\ \bullet x=2 \quad \sin \theta=0 \\ \theta=0 \\ \bullet x=3 \quad \sin \theta=1 \\ \theta=\frac{\pi}{2} \end{array}$

Question 166 (***)+

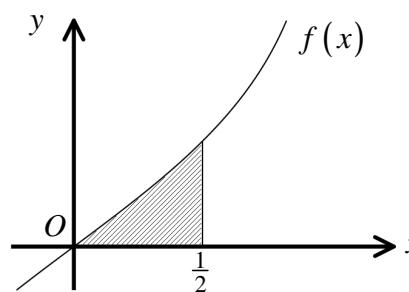
$$f(x) = \ln\left(\frac{1+x}{1-x}\right), |x| < 1.$$

- Show that $f(x)$ is an odd function.
- Find an expression for $f'(x)$ as a single simplified fraction, showing further that $f'(x)$ is an even function.
- Determine an expression for $f^{-1}(x)$.

[continues overleaf]

[continued from overleaf]

The figure below shows part of the graph of $f(x)$.



- d) Use the substitution $u = e^x + 1$ to find the exact value of

$$\int_0^{\ln 3} f^{-1}(x) \, dx.$$

- e) Hence find an exact value for the area of the shaded region, bounded by $f(x)$, the coordinate axes and the line $x = \frac{1}{2}$.

$$f'(x) = \frac{2}{1-x^2}, \quad f'(x) = \frac{e^x - 1}{e^x + 1}, \quad \ln\left(\frac{4}{3}\right), \quad \text{area} = \frac{1}{2} \ln 3 - 2 \ln 2 \approx 0.262$$

(a) $\lim_{x \rightarrow 0^+} f(-x) = \lim_{x \rightarrow 0^+} \left[\frac{1+x}{1-(e^{-x})} \right] = \lim_{x \rightarrow 0^+} \left[\frac{1-x}{1+e^{-x}} \right] = \lim_{x \rightarrow 0^+} \left[\frac{(1-x)^{-1}}{(1+e^{-x})^{-1}} \right] = -\ln(1-e^{-x}) = -f(0)$
 $\therefore f(0)$ is ∞

(b) $f(x) = \ln\left(\frac{1+x}{1-e^{-x}}\right) = \ln(1+x) - \ln(1-e^{-x})$
 $\therefore f'(x) = \frac{1}{1+x} + \frac{1}{1-e^{-x}} = \frac{(1-x)+(1+x)}{(1+x)(1-e^{-x})} = \frac{2}{1-x^2}$
 Now, $f'(x) = \frac{2}{1-x^2} = \frac{2}{1-x^2} = f'(x)$
 $\therefore f'(0)$ is 0 (by symmetry)

(c) $y = \ln\left(\frac{1+x}{1-e^{-x}}\right)$
 $e^y = \frac{1+x}{1-e^{-x}}$
 $e^y - e^{-y} = 1+x$
 $e^{2y} - 1 = (1+x)e^{2y}$
 $e^{2y} - 1 = x(1+e^{2y})$
 $x = \frac{e^{2y}-1}{e^{2y}+1}$
 $\therefore f'(x) = \frac{e^{2x}}{e^{2x}+1}$

(d) $\int_0^{\ln 3} f'(x) \, dx$
 $= \int_0^{\ln 3} \frac{e^{2x}}{e^{2x}+1} \, dx$
 $= \int_0^{\ln 3} \frac{e^{2x}-1+1}{e^{2x}+1} \, dx$
 $= \int_0^{\ln 3} \frac{e^{2x}-1}{e^{2x}+1} \, dx + \int_0^{\ln 3} \frac{1}{e^{2x}+1} \, dx$
 $= \int_0^{\ln 3} \frac{e^{2x}-1}{e^{2x}+1} \, dx + \int_0^{\ln 3} \frac{e^{-2x}}{e^{-2x}+1} \, dx$
 $= \int_0^{\ln 3} \frac{e^{2x}-1}{e^{2x}+1} \, dx + \int_0^{\ln 3} \frac{u-2}{u+1} \, du$
 $\text{... by partial fractions}$
 $\frac{u-2}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$
 $u-2 = A(u+1) + Bu$
 $u=0 \Rightarrow -2 = A \Rightarrow A = -2$
 $u=1 \Rightarrow -1 = B$
 $= \int_0^{\ln 3} \frac{2}{u} - \frac{1}{u+1} \, du$
 Continued

$$= \int_0^{\ln 3} \frac{2}{u} \, du - \int_0^{\ln 3} \frac{1}{u+1} \, du = (2\ln u - \ln(u+1)) \Big|_0^{\ln 3} = (2\ln 3 - \ln 4) - (2\ln 1 - \ln 2) = \ln 16 - \ln 4 = \ln 4 = \ln \frac{4}{3}$$

(e) $f(x) = \ln\left(\frac{1+x}{1-e^{-x}}\right) = \ln 3$
 $\bullet \text{Area of shaded region} = \int_0^{\ln 3} f(x) \, dx = \int_0^{\ln 3} \ln\left(\frac{1+x}{1-e^{-x}}\right) \, dx$
 $\therefore \text{Required area (in green)} = \frac{1}{2} \times \ln 3 - \ln \frac{4}{3} = \frac{1}{2} \ln 3 - \ln 2 = \frac{1}{2} \ln \frac{3}{4}$

Question 167 (***)+

By using the substitution $u = \sqrt{x}$, find

$$\int_1^4 \frac{1}{x(2+\sqrt{x})} dx$$

giving the answer as an exact single natural logarithm.

$$\ln\left(\frac{3}{2}\right)$$

Question 168 (*)+**

Use the substitution $x = \sqrt{2} \sin \theta$ to show that

$$\int_0^{\sqrt{2}} \sqrt{2-x^2} dx = \frac{\pi}{2}$$

proof

$$\begin{aligned}
 & \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \sqrt{2 - 2\cos^2 \theta} d\theta = \dots = \int_0^{\frac{\pi}{4}} \sqrt{2 - (\sqrt{2}\sin\theta)^2} \sqrt{2}\sin\theta d\theta \\
 & = \int_0^{\frac{\pi}{4}} \sqrt{2 - 2\sin^2 \theta} \sqrt{2}\sin\theta d\theta = \int_0^{\frac{\pi}{4}} \sqrt{2(1 - \sin^2 \theta)} \sqrt{2}\sin\theta d\theta \\
 & = \int_0^{\frac{\pi}{4}} \sqrt{2}\cos^2 \theta \sqrt{2}\sin\theta d\theta = \left[\frac{2}{3}\cos^3 \theta \right]_0^{\frac{\pi}{4}} = \dots \\
 & \text{TEILKONTROLLE: } \text{WURZEL} \\
 & = \left[\frac{2}{3}(2(\frac{1}{2})^2 + \cos 0) \right]_0^{\frac{\pi}{4}} = \left[\frac{2}{3} + 14\cos^2 0 \sin 0 \right]_0^{\frac{\pi}{4}} = \left[0 + \frac{1}{2}\sin^2 0 \right]_0^{\frac{\pi}{4}} \\
 & = \left(\frac{1}{2} + \frac{1}{2}\sin^2 \frac{\pi}{4} \right) - \left(0 - \frac{1}{2}\sin^2 0 \right) = \frac{\pi}{8}
 \end{aligned}$$

Question 169 (****+)

$$\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

- a) Find the value of each of the constants A , B and C .
- b) Use the substitution $x = \cos \theta$ to show

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2 \sin \theta}{\cos \theta + \cos^3 \theta} d\theta = \ln\left(\frac{5}{3}\right).$$

$$[A=1], [B=-1], [C=0]$$

(a) $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$$\begin{aligned} 1 &\equiv A(x^2+1) + C(Bx+C) \\ &\equiv Ax^2 + A + Bx^2 + Cx \\ &\equiv (A+B)x^2 + Cx + A \end{aligned}$$

$\therefore A=1$
 $C=0$
 $A+B=0$
 $B=-1$

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2 \sin \theta}{\cos \theta + \cos^3 \theta} d\theta = \dots$ by substitution

$$\begin{aligned} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{-2 \sin^2 \theta}{\cos \theta + \cos^3 \theta} d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{-2}{\cos \theta + \cos^3 \theta} d\theta = 2 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos \theta + \cos^3 \theta} d\theta \\ &= 2 \left[\ln|\sec \theta + \tan \theta| \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= 2 \left[\ln|\sec \frac{\pi}{3} + \tan \frac{\pi}{3}| - \ln|\sec \frac{\pi}{4} + \tan \frac{\pi}{4}| \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= 2 \left[\ln 2^2 - \ln(\sqrt{2}+1)^2 \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \left[\ln\left(\frac{2^2}{\sqrt{2}+1}\right)^2 \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \ln\left(\frac{4}{3}\right) - \ln\left(\frac{2}{\sqrt{2}}\right) \\ &= \ln\left(\frac{4}{3}\right) - \ln\left(\frac{2}{\sqrt{2}}\right) = -\ln 3 + \ln 5 = \ln\frac{5}{3} \end{aligned}$$

$\therefore 2 \ln(5/3)$

Question 170 (****+)

By using the substitution $u = 1 - \tan^2 x$, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{6}} \tan x \sec 2x \, dx.$$

$$\boxed{\frac{1}{2} \ln\left(\frac{3}{2}\right)}$$

$$\begin{aligned}
 \int_0^{\frac{\pi}{6}} \tan x \sec 2x \, dx &= \dots \text{by substitution} \\
 &= \int_0^{\frac{\pi}{6}} (\tan x \sec 2x) \left(-\frac{du}{2 \tan^2 x} \right) \\
 &= \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{\sec 2x}{\sec^2 x} \, du \\
 &= \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{1}{\cos^2 x} \, du \\
 &= \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{1}{(\cos^2 x - \sin^2 x) \sec x} \, du = \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{1}{1 - \sin^2 x} \, du \\
 &= \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{1}{1 - \frac{\sin^2 x}{\cos^2 x}} \, du = \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{1}{1 - \tan^2 x} \, du = \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{1}{u} \, du \\
 &= \left[\frac{1}{2} \ln|u| \right]_{\frac{2}{3}}^1 = \frac{1}{2} \ln 1 - \frac{1}{2} = -\frac{1}{2} \ln \frac{3}{2} = \frac{1}{2} \ln \frac{2}{3}
 \end{aligned}$$

OR
 ALTERNATIVE FROM
 $\dots = \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{\sec 2x}{\sec^2 x} \, du = \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{\cos 2x}{\cos 2x} \, du = \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{\cos^2 x}{\cos^2 x - \sin^2 x} \, du$
 DIVIDE TOP & BOTTOM OF THE INTEGRAL BY $\cos^2 x$ WHICH
 $= \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{1}{1 - \frac{\sin^2 x}{\cos^2 x}} \, du = \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{1}{1 - \tan^2 x} \, du = \frac{1}{2} \int_{\frac{2}{3}}^1 \frac{1}{u} \, du$

Question 171 (***)+

a) Show clearly that $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

b) Use trigonometric identities to find

$$\int \frac{1}{\sin^2 x \cos^2 x} dx.$$

$$-2\cot 2x + C$$

(a) $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin(-\sin x) - \cos(\cos x)}{\sin^2 x}$

$$= -\frac{\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

(b) $\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{1}{(\frac{1}{4} - \frac{1}{2}\cot 2x)(\frac{1}{4} + \frac{1}{2}\cot 2x)} dx$

$$= \int \frac{1}{\frac{1}{4} - \frac{1}{2}\cot 2x} dx \quad \text{FACTORY METHOD BY 4}$$

$$= \int \frac{4}{1 - 2\cot 2x} dx = \int \frac{4}{\sin^2 2x} dx$$

$$= \int 4\operatorname{cosec}^2 2x dx = \dots \text{by part(a)}$$

$$= -2\cot 2x + C$$

ALTERNATIVE

$$\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{1}{(\sin x \cos x)^2} dx = \int \frac{1}{(\frac{1}{2}\sin 2x)^2} dx$$

$$= \int \frac{1}{(\frac{1}{4}\sin^2 2x)} dx = \int \frac{4}{\sin^2 2x} dx$$

$$= \int \frac{4}{\sin^2 2x} dx = \int 4\operatorname{cosec}^2 2x dx$$

$$= \dots \text{by part(a)} \dots = -2\cot 2x + C$$

ALTERNATIVE

$$\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\cos x + \sin x}{\cos x \sin x} dx = \int \frac{\cos x}{\cos x \sin x} + \frac{\sin x}{\cos x \sin x} dx$$

$$= \int \operatorname{cosec} x + \operatorname{sec} x dx = -\operatorname{cosec} x - \operatorname{cosec} x + C$$

Question 172 (***)+

Use the substitution $x = \operatorname{cosec} \theta$ to find the exact value of

$$\int_{\sqrt{2}}^2 \frac{\sqrt{x^2-1}}{x} dx.$$

$$\sqrt{3}-1-\frac{\pi}{12}$$

$\alpha = \operatorname{cosec} \theta$

$$\frac{d\alpha}{d\theta} = -\operatorname{cosec} \theta \operatorname{cot} \theta$$

$$d\theta = -\frac{1}{\operatorname{cosec} \theta \operatorname{cot} \theta} d\alpha$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$\frac{1}{\sin \theta} = \frac{1}{\sqrt{1-x^2}}$$

$$\sin \theta = \frac{1}{\sqrt{1-x^2}}$$

$$\theta = \arcsin \frac{1}{\sqrt{1-x^2}}$$

$$\theta = \arcsin \frac{1}{\sqrt{1-\frac{1}{4}}} = \arcsin \frac{1}{\sqrt{\frac{3}{4}}} = \arcsin \frac{1}{\frac{\sqrt{3}}{2}} = \frac{\pi}{6}$$

$$\theta = \frac{\pi}{6}$$

$$\int_{\sqrt{2}}^2 \frac{\sqrt{x^2-1}}{x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sqrt{\operatorname{cosec}^2 \theta - 1}}{\operatorname{cosec} \theta} (-\operatorname{cosec} \theta \operatorname{cot} \theta) d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\operatorname{cot}^2 \theta} (\operatorname{cot} \theta) d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cot}^2 \theta d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^2 \theta - 1 d\theta$$

$$= \left[\operatorname{cot} \theta - \theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \left[\operatorname{cot} \theta + \theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \left(\frac{1}{2}\pi + \frac{\pi}{6} \right) - \left(\operatorname{cot} \frac{\pi}{6} + \frac{\pi}{6} \right)$$

$$= \sqrt{3} + \frac{\pi}{6} - 1 - \frac{\pi}{6} = \sqrt{3} - 1 - \frac{\pi}{12}$$

Question 173 (*)+**

a) Write down an expression for $\frac{d}{dx}(e^{\cos x})$.

b) By using integration by parts, or otherwise, show that

$$\int e^{\cos x} \cos x \sin x \, dx = e^x (1 - \cos x) + \text{constant}.$$

$$\boxed{\frac{d}{dx}(e^{\cos x}) = -e^{\cos x} \sin x}$$

(a)

$$\int \sqrt{\frac{x}{1-x}} \, dx = \int \frac{\sqrt{x}}{\sqrt{1-x}} \, dx = \dots \text{by the substitution}$$

$$= \int \frac{\sin \theta}{\sqrt{1-\sin^2 \theta}} (2\cos \theta d\theta) = \int \frac{2\cos^2 \theta d\theta}{\cos \theta} =$$

$$= \int 2\cos \theta \, d\theta \quad \cancel{\text{from } u = \sin \theta}$$

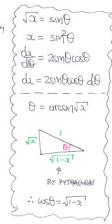
$$(b)$$

$$= \int 2(1 - \cos 2\theta) \, d\theta = \int 1 - \cos 2\theta \, d\theta$$

$$= \theta - \frac{1}{2}\sin 2\theta + C = \theta - \frac{1}{2}(2\sin \theta \cos \theta) + C$$

$$= \theta - \sin \theta \cos \theta + C = \arcsin(\sqrt{x}) - \sqrt{x}\sqrt{1-x} + C$$

$$= \arcsin(\sqrt{x}) - \sqrt{x}\sqrt{1-x} + C \quad \cancel{\text{from } u = \sin \theta}$$

$\sqrt{x} = \sin \theta$
 $x = \sin^2 \theta$
 $\frac{dx}{d\theta} = 2\sin \theta \cos \theta$
 $dx = 2\sin \theta \cos \theta \, d\theta$
 $\theta = \arcsin(\sqrt{x})$


By Pythagoras
 $\therefore \cos \theta = \sqrt{1-x}$

Question 174 (**+)**

By using trigonometric identities, show that

$$\int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx = \frac{3\pi}{16}.$$

proof

$$\begin{aligned}
 & \int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx = \int_0^{\frac{\pi}{4}} (\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x \cos^2 x \, dx \\
 &= \int_0^{\frac{\pi}{4}} (\sin^2 x + \cos^2 x)^2 - \frac{1}{2}(4\sin^2 x \cos^2 x) \, dx = \int_0^{\frac{\pi}{4}} 1 - \frac{1}{2}(\sin 2x)^2 \, dx \\
 &= \int_0^{\frac{\pi}{4}} 1 - \frac{1}{2}\sin^2 2x \, dx = \int_0^{\frac{\pi}{4}} 1 - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) \, dx \\
 &= \int_0^{\frac{\pi}{4}} \frac{3}{4} + \frac{3}{8}\cos 4x \, dx = \left[\frac{3}{4}x + \frac{3}{8}\sin 4x \right]_0^{\frac{\pi}{4}} \\
 &= \left(\frac{3\pi}{16} - 0 \right) - (0 - 0) = \frac{3\pi}{16}.
 \end{aligned}$$

ALTERNATIVE VARIATION

$$\begin{aligned}
 & \int_0^{\frac{\pi}{4}} \sin^4 x + \cos^4 x \, dx = \int_0^{\frac{\pi}{4}} (\sin^2 x)^2 + (\cos^2 x)^2 \, dx \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 + \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^2 \, dx \\
 &= \int_0^{\frac{\pi}{4}} \frac{1}{2} - \frac{1}{2}\cos 2x + \frac{1}{2}\cos^2 2x + \frac{1}{2} + \frac{1}{2}\cos 2x + \frac{1}{2}\cos^2 2x \, dx \\
 &= \int_0^{\frac{\pi}{4}} \frac{1}{2} + \frac{1}{2}\cos^2 2x \, dx = \int_0^{\frac{\pi}{4}} \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) \, dx \\
 &= \int_0^{\frac{\pi}{4}} \frac{3}{4} + \frac{1}{8}\cos 4x \, dx = \dots = \frac{3\pi}{16} \text{ at } 480 \text{ sv}
 \end{aligned}$$

NOTE: $\cos^2 \theta \equiv \frac{1}{2} + \frac{1}{2}\cos 2\theta$
 $\sin^2 \theta \equiv \frac{1}{2} - \frac{1}{2}\cos 2\theta$

Question 175 (*)+**

By using the substitution $u = \sin 2x$, or otherwise, find an exact simplified value for the following trigonometric integral.

$$\int_0^{\frac{1}{4}\pi} \frac{1 - \tan^2 x}{\sec^2 x + 2 \tan x} dx .$$

, $\frac{1}{2} \ln 2$

(USING THE SUBSTITUTION GIVEN)

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1 - \tan^2 x}{\sec^2 x + 2 \tan x} dx &= \int_0^1 \frac{1 - \tan^2 x}{\sec^2 x + 2 \tan x} \left(\frac{du}{2\cos^2 x} \right) \\ &= \frac{1}{2} \int_0^1 \frac{1 - \tan^2 x}{(\sec^2 x + 2 \tan x) \cos^2 x} du \\ \text{SWITCH SIGNATURES INTO SINES & COSINES} \\ &= \frac{1}{2} \int_0^1 \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{\left(\frac{1}{\cos^2 x} + \frac{2 \sin x}{\cos x} \right) (\cos^2 x - \sin^2 x)} du \\ \text{MULTIPLY TOP & BOTTOM OF THE DOUBLE FRACTION BY } \cos^2 x \\ &= \frac{1}{2} \int_0^1 \frac{\cos^2 x - \sin^2 x}{(1 + 2 \sin x \cos x)(\cos^2 x - \sin^2 x)} du = \frac{1}{2} \int_0^1 \frac{1}{1 + 2 \sin x \cos x} du \\ &= \frac{1}{2} \int_0^1 \frac{1}{1 + \sin 2x} du = \frac{1}{2} \int_0^1 \frac{1}{1 + u} du \\ &= \frac{1}{2} \left[\ln|1+u| \right]_0^1 = \frac{1}{2} [\ln 2 - \ln 1] = \underline{\underline{\frac{1}{2} \ln 2}} \end{aligned}$$

Question 176 (*)+**

By using a suitable substitution, or otherwise, find the value of

$$\int_0^1 \frac{x \arcsin x}{\sqrt{1-x^2}} dx .$$

BY SUBSTITUTION

$$\begin{aligned} \int_0^1 \frac{x \arcsin x}{\sqrt{1-x^2}} dx &= \dots \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin \theta \arcsin(\sin \theta)}{\sqrt{1-\sin^2 \theta}} (\cos \theta d\theta) \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin \theta \times \theta}{\sqrt{\cos^2 \theta}} \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \theta \sin \theta d\theta \\ \text{BY FACTS} \quad \begin{array}{|c|c|} \hline \theta & 1 \\ \hline -\cos \theta & \sin \theta \\ \hline \end{array} \\ &= \left[-\theta \cos \theta \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos \theta d\theta \\ &= \left[\sin \theta \right]_0^{\frac{\pi}{2}} = 1 \end{aligned}$$

Question 177 (*)+**

- a) Use the substitution $u = 2x - 1$ to show that

$$\int_1^5 \frac{x+1}{(2x-1)^{\frac{3}{2}}} dx = 2.$$

- b) By using integration by parts and the result of part (a), find the value of

$$\int_1^5 \frac{(x+1)^2}{(2x-1)^{\frac{5}{2}}} dx.$$

20
9

(a) $\int_1^5 \frac{x+1}{(2x-1)^{\frac{3}{2}}} dx = \dots$ substitution

$$= \int_1^5 \frac{x+1}{u^{\frac{3}{2}}} \left(\frac{du}{2} \right) = \int_1^5 \frac{\frac{u+1}{u^{\frac{1}{2}}}}{u^{\frac{3}{2}}} \left(\frac{du}{2} \right)$$

WANT TO GET RID OF $u^{\frac{1}{2}}$ BY 2

$$= \int_1^5 \frac{u+1+2}{2u^{\frac{3}{2}}} \frac{du}{2} = \int_1^5 \frac{u+3}{4u^{\frac{3}{2}}} du$$

$$= \frac{1}{4} \int_1^5 \frac{u+3}{u^{\frac{3}{2}}} du = \frac{1}{2} \int_1^5 \frac{u}{u^{\frac{3}{2}}} + \frac{3}{u^{\frac{3}{2}}} du = \frac{1}{4} \int_1^5 u^{-\frac{1}{2}} + 3u^{-\frac{1}{2}} du$$

$$= \frac{1}{4} \left[\frac{u^{\frac{1}{2}}}{2} - 3u^{\frac{1}{2}} \right]_1^5 = \frac{1}{4} \left[(6-2) - (2-6) \right] = \frac{1}{4} [4-(-4)] = 2.$$

as required

$u = 2x-1$
 $\frac{du}{dx} = 2$
 $du = \frac{du}{2}$
 $x=1, u=1$
 $x=5, u=9$
 $u = \frac{u+1}{2}$

(b) $\int_1^5 \frac{(2x-1)^2}{(2x+1)^{\frac{5}{2}}} dx = \int_1^5 (2x)^2 (2x-1)^{-\frac{3}{2}} dx = \dots$ part

$$= \int_1^5 \frac{1}{3} (2x-1)^{\frac{3}{2}} (2x)^2 dx = \int_1^5 \frac{1}{3} (2x)(2x-1)^{\frac{3}{2}} dx$$

$$= \int_1^5 \frac{(2x)^2}{3(2x-1)^{\frac{1}{2}}} dx + \frac{2}{3} \int_1^5 \frac{2x-1}{(2x-1)^{\frac{5}{2}}} dx$$

$$= \left(\frac{4}{3} - \frac{4}{5} \right) + \frac{2}{3} \times 2. \quad \text{Ans (b)}$$

$$= \frac{20}{9}$$

Question 178 (*)+**

Use the substitution $u = \ln x$ to show that

$$\int 3^{\ln x} dx = \frac{x(3^{\ln x})}{1+\ln 3} + \text{constant}.$$

V, proof

SOLVE WITH AN OBVIOUS SUBSTITUTION

- $u = \ln x$
- $e^u = x$
- $\frac{du}{dx} = \frac{1}{x}$
- $dx = x du$

$$\int 3^{\ln x} dx = \int 3^u e^u du < \int (3e)^u du = \int a^u du$$

where $a = 3e$

NOW WE KNOW THAT

$$\frac{d}{dx}(a^x) = a^x \ln a \Rightarrow a^x = \int a^x \ln a dx$$

$$\Rightarrow \frac{1}{\ln a} a^x = \int a^x dx$$

RETURNING TO OUR INTEGRAL IN x

$$\begin{aligned} \int a^x dx &= \frac{1}{\ln a} a^x + C = \frac{1}{\ln(3e)} (3e)^x + C \\ &= \frac{3^x e^x}{\ln 3 + \ln e} + C = \frac{3^x x}{\ln 3 + 1} + C \\ &= \frac{3^x x}{1 + \ln 3} + C = \frac{x(3^{\ln x})}{1 + \ln 3} + C \end{aligned}$$

Question 179 (***)+

$$J = \int_{-1}^1 \frac{1}{1+e^{-x}} dx.$$

- a) Show that the substitution $u = 1 + e^{-x}$ transforms J into

$$\int_{1/e}^{1+e^{-1}} \frac{1}{u(1-u)} du.$$

- b) By expressing $\frac{1}{u(1-u)}$ into partial fractions show clearly that $J = 1$.

proof

(a)

$$\begin{aligned} I &= \int_{-1}^1 \frac{1}{1+e^{-x}} dx = \int_{1/e}^{1+e^{-1}} \left(-\frac{1}{e^{-x}} \right) du \\ &= \int_{1/e}^{1+e^{-1}} -\frac{1}{u} du = \int_{1/e}^{1+e^{-1}} -\frac{1}{u(u-1)} du \\ &= \int_{1/e}^{1+e^{-1}} \frac{1}{u(u-1)} du \end{aligned}$$

$$\begin{aligned} u &= 1 + e^{-x} \\ \frac{du}{dx} &= -e^{-x} \\ dx &= -\frac{du}{e^{-x}} \\ 2x+1 &\rightarrow u = 1 + e^x \\ 2x+1 &\rightarrow u = 1 + e^x \\ e^x &= u-1 \end{aligned}$$

(b) BY PARTIAL FRACTIONAL

$$\begin{aligned} \frac{1}{u(u-1)} &\equiv \frac{A}{u} + \frac{B}{u-1} \quad \left\{ \begin{array}{l} u=0 \Rightarrow A=-1 \\ u=1 \Rightarrow B=1 \end{array} \right. \quad A=-1 \\ 1 &\equiv A(u-1) + Bu \\ 1 &\equiv (A+B)u - A \\ 1 &\equiv (1+0)u + 1 \\ 1 &\equiv u + 1 \end{aligned}$$

\square

Question 180 (*)+**

Use the substitution $x = \tan \theta$ to find the exact value of

$$\int_0^1 \frac{8}{(1+x^2)^2} dx.$$

, $\pi + 2$

$$\begin{aligned}
 \int_0^1 \frac{8}{(1+x^2)^2} dx &= \dots \text{ USE THE SUBSTITUTION FROM} \\
 &= \int_0^{\frac{\pi}{4}} \frac{8}{(1+\tan^2 \theta)^2} \sec^2 \theta d\theta = \int_0^{\frac{\pi}{4}} \frac{8}{(\sec^2 \theta)^2} \sec^2 \theta d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{8}{\sec^4 \theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{8}{\sec^2 \theta} d\theta = \int_0^{\frac{\pi}{4}} 8 \cos^2 \theta d\theta \\
 &\quad \text{STANDARD INTEGRATION IDENTITY} \dots \\
 &= \int_0^{\frac{\pi}{4}} 8 \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta = \int_0^{\frac{\pi}{4}} 4 + 4 \cos 2\theta d\theta \\
 &= [4\theta + 2 \sin 2\theta]_0^{\frac{\pi}{4}} = (\pi + 2 \sin \frac{\pi}{2}) - (0 + 2 \sin 0) = \pi + 2
 \end{aligned}$$

Question 181 (*)+**

Use the substitution $u = 400 - 20\sqrt{x}$ to show that

$$\int_0^{100} \frac{1}{400 - 20\sqrt{x}} dx = -1 + 2 \ln 2.$$

proof

$$\begin{aligned}
 \int_0^{100} \frac{1}{400 - 20\sqrt{x}} dx &= \text{USE THE SUBSTITUTION} \\
 &= \int_{400}^{200} \frac{1}{u} \left(-\frac{1}{10} \left(20 - \frac{1}{20} u \right) \right) du \\
 &\quad \text{ALIVE ARRIVES TO SWAP UNITS OF RADICALS} \quad \frac{1}{10} \\
 &= \frac{1}{10} \int_{400}^{200} \frac{1}{u} \left(20 - \frac{1}{20} u \right) du \\
 &= \frac{1}{10} \int_{400}^{200} \frac{20}{u} - \frac{1}{20} du = \frac{1}{10} \left[20 \ln |u| - \frac{1}{20} u \right]_{400}^{200} \\
 &= \frac{1}{10} \left[(20 \ln 400 - 20) - (20 \ln 200 - 20) \right] = \frac{1}{10} [20 \ln 400 - 20 - 20 \ln 200 + 20] \\
 &= \frac{1}{10} [20 \ln 2 - 10] = 2 \ln 2 - 1
 \end{aligned}$$

Question 182 (*)+**

It is given that

$$\frac{(2x^2 - 10x + 7)(x^2 - 3x - 3)}{(x-4)^2} \equiv Ax^2 + Bx + C + \frac{D}{x-4} + \frac{E}{(x-4)^2}.$$

- a) Find the value of A , B , C , D and E in the above identity.
- b) Hence find the exact value of

$$\int_0^3 f(x) \, dx.$$

$$A = 2, B = 0, C = -1, D = 1, E = -1, \frac{57}{4} - \ln 4$$

(a)

$$\frac{(2x^2 - 10x + 7)(x^2 - 3x - 3)}{(x-4)^2} \equiv Ax^2 + Bx + C + \frac{D}{x-4} + \frac{E}{(x-4)^2}$$

$$(2x^2 - 10x + 7)(x^2 - 3x - 3) \equiv (Ax^2 + Bx + C)(x^2 - 3x - 3) + D(x-4) + E$$

- If $x=4$, $C=1$ $\Rightarrow E = -1$

Simplify

$$\begin{aligned} 2x^2 - 6x^2 - 12x^2 \\ - 10x^2 + 30x^2 + 30x \\ 7x^2 - 21x - 21 \end{aligned} \equiv (Ax^2 + Bx + C)(x^2 - 3x - 3) + Dx - 4D - 1$$

$$\Rightarrow 2x^4 - 16x^3 + 30x^2 + 9x - 21 \equiv Ax^4 - 3Ax^3 + (Ax^2 - Bx^2 - 8Bx + 16C) \\ - 16Cx + 48C + 4Dx - 4D - 1$$

$$\Rightarrow 2x^4 - 16x^3 + 30x^2 + 9x - 20 \equiv Ax^4 + (B-8A)x^3 + (C-16C+4D)x^2 \\ + (16C-4D-1)x + (48C+4D)$$

$$\Rightarrow A = 2, \quad \begin{aligned} B-8A &= -16 \\ B-16 &= -16 \\ B &= 0 \end{aligned}, \quad \begin{aligned} 16A - 8B + C &= 31 \\ 32 + C &= 31 \\ C &= -1 \end{aligned}, \quad \begin{aligned} 16C - 4D &= -20 \\ -16 - 4D &= -20 \\ 4 &= 4D \\ D &= 1 \end{aligned}$$

(b)

$$\begin{aligned} \int_0^3 f(x) \, dx &= \int_0^3 2x^2 - 1 + \frac{1}{x-4} - \frac{1}{(x-4)^2} \, dx \\ &= \left[\frac{2}{3}x^3 - x + \ln|x-4| + \frac{1}{x-4} \right]_0^3 \\ &= \left(\frac{18}{3} - 3 + \ln|1| - 1 \right) - \left(0 - 0 + \ln|0-4| - \frac{1}{4} \right) \\ &= 14 - \ln 4 + \frac{1}{4} \\ &\approx \frac{57}{4} - \ln 4 \end{aligned}$$

Question 183 (*)+**

$$x = \frac{1}{2}(-1 + 4 \tan \theta), \quad -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi.$$

- a) Use trigonometric identities to show that

$$4x^2 + 4x + 17 = 16 \sec^2 \theta.$$

- b) Hence find the exact value of

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4x^2 + 4x + 17} dx.$$

$$\boxed{\frac{\sqrt{3}}{2}}, \quad \boxed{\frac{\pi}{32}}$$

a) SUBSTITUTE, EXPAND & SIMPLIFY

$$\begin{aligned} 4x^2 + 4x + 17 &= 4 \left[\frac{1}{2}(1 + 4 \tan \theta) \right]^2 + 4 \left[\frac{1}{2}(-1 + 4 \tan \theta) \right] + 17 \\ &= 4 \cancel{\left(1 + 4 \tan \theta \right)^2} + 2 (-1 + 4 \tan \theta) + 17 \\ &= 16 \cancel{\tan^2 \theta} + 16 \tan \theta - 2 + 17 \\ &= 16(1 + \tan^2 \theta) \\ &= 16 \sec^2 \theta \end{aligned}$$

AS REQUIRED

b) BY SUBSTITUTION FROM PART (a)

$$\begin{aligned} \rightarrow x &= \frac{1}{2}(1 + 4 \tan \theta) = -\frac{1}{2} + 2 \tan \theta \\ \rightarrow \frac{dx}{d\theta} &= 2 \sec^2 \theta \\ \rightarrow dx &= 2 \sec^2 \theta d\theta \end{aligned}$$

- when $x = -\frac{1}{2}$
- $-\frac{1}{2} = -\frac{1}{2} + 2 \tan \theta$
- $0 = 2 \tan \theta$
- $\theta = 0$
- when $x = \frac{3}{2}$
- $\frac{3}{2} = -\frac{1}{2} + 2 \tan \theta$
- $2 = 2 \tan \theta$
- $\tan \theta = 1$
- $\theta = \frac{\pi}{4}$

TRANSFORMING THE INTEGRAL VARIOUS

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4x^2 + 4x + 17} dx &= \int_0^{\frac{\pi}{4}} \frac{1}{16 \sec^2 \theta} (2 \sec^2 \theta) d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{8} d\theta \\ &= \left[\frac{1}{8} \theta \right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{32} - 0 \\ &= \frac{\pi}{32} \end{aligned}$$

Question 184 (***)+

It is given that

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

Use the above trigonometric identity to show that

$$\sin 3x \equiv 3 \sin x - 4 \sin^3 x,$$

and hence find

$$\int \sqrt[3]{\sin 2x - 2 \sin 3x \cos x} dx.$$

$$-\frac{3}{2} \sin^{\frac{4}{3}} x + C$$

$$\begin{aligned}\sin 3x &= \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x \\&= (\sin 2x \cos x) \cancel{\cos x} + (1 - 2\sin^2 x) \sin x \\&= 2\sin x \cos^2 x + \sin x - 2\sin^3 x \\&= 2\sin x (1 - \sin^2 x) + \sin x - 2\sin^3 x \\&= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\&= 3\sin x - 4\sin^3 x\end{aligned}$$

$$\begin{aligned}&\int (3\sin x - 2\sin^3 x)^{\frac{1}{3}} dx \quad \text{Let } u = 3\sin x - 2\sin^3 x \\&= \int (\sin x \cancel{u}) - 2(\sin^2 x \cdot \frac{1}{3}u^{\frac{2}{3}}) \cancel{du}^{\frac{1}{3}} dx \\&= \int (\sin x u^{\frac{1}{3}} - \sin x u^{\frac{2}{3}} + \frac{2}{3}u^{\frac{1}{3}}) dx \\&= \int 2\sin x u^{\frac{1}{3}} dx \\&= -\frac{2}{4} (\cos x)^{\frac{4}{3}} + C \\&= -\frac{3}{2} (\cos x)^{\frac{4}{3}} + C = \left(-\frac{3}{2} \sin^{\frac{4}{3}} x + C\right)\end{aligned}$$

Question 185 (*)+**

Use the substitution $u = x + \frac{\pi}{4}$ to show that

$$\int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \frac{\sin x}{\sin\left(x + \frac{\pi}{4}\right)} dx = \frac{\sqrt{2}}{6}(\pi + \ln 8).$$

proof

$$\begin{aligned}
 & \int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \frac{\sin x}{\sin\left(x + \frac{\pi}{4}\right)} dx = \dots \text{ by substitution} \\
 &= \int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \frac{\sin(u - \frac{\pi}{4})}{\sin u} du = \int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \frac{\sin u \cos \frac{\pi}{4} - \cos u \sin \frac{\pi}{4}}{\sin u} du \\
 &= \int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} (\cos \frac{\pi}{4} - \sin \frac{\pi}{4} \cot u) du = \int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cot u \right) du \\
 &= \frac{\sqrt{2}}{2} \int_{\frac{\pi}{4}}^{\frac{7\pi}{12}} 1 - \cot u du = \frac{\sqrt{2}}{2} \left[u - \ln|\sin u| \right]_{\frac{\pi}{4}}^{\frac{7\pi}{12}} \\
 &= \frac{\sqrt{2}}{2} \left[\left(\frac{7\pi}{12} - \ln \frac{1}{\sqrt{2}} \right) - \left(\frac{\pi}{4} - \ln 1 \right) \right] = \frac{\sqrt{2}}{2} \left[\frac{\pi}{3} - \ln \frac{1}{\sqrt{2}} \right] \\
 &= \frac{\sqrt{2}}{2} \left[\frac{\pi}{3} + |\ln 2| \right] = \frac{\sqrt{2}}{2} [\pi + 3\ln 2] = \frac{\sqrt{2}}{6} [\pi + \ln 8]
 \end{aligned}$$

Question 186 (*)+**

By using the substitution $u = \sqrt[3]{x}$, or otherwise, show that

$$\int_0^{\sqrt[3]{27}} \frac{2}{x + \sqrt[3]{x}} dx = 6\ln 2.$$

proof

<p><u>USING THE SUBSTITUTION</u> Given</p> $\int_0^{\sqrt[3]{27}} \frac{2}{x + \sqrt[3]{x}} dx = \int_0^{\sqrt[3]{27}} \frac{2}{u + u^{1/3}} (3u^2 du)$ $= \int_0^{\sqrt[3]{27}} \frac{6u^2}{u(1+u^{1/3})} du = \int_0^{\sqrt[3]{27}} \frac{6u}{u^{1/3} + 1} du$ $= 3 \int_0^{\sqrt[3]{27}} \frac{2u}{u^{1/3} + 1} du$ <p>$\int f(u) du = \ln f(u) + C$</p>	<ul style="list-style-type: none"> • $u = \sqrt[3]{x}$ • $u^3 = x$ • $u = u^3$ • $\frac{dx}{du} = 3u^2$ • $du = 3u^2 du$ <ul style="list-style-type: none"> • $u = \sqrt[3]{27}$ • $27^{\frac{1}{3}} = 3$ • $3^{\frac{1}{3}} = u^3$ • $3^{\frac{1}{3}} = u$ • $u = 0, u = 0$
<p><u>USING THE ABOVE RESULT (OR ANOTHER SUBSTITUTION)</u></p> $\dots = 3 \left[\ln u^{1/3} + 1 \right]_0^{\sqrt[3]{27}} = 3 \left[\ln 4 - \ln 1 \right]$ $= 3 \cdot 2\ln 2$ $= 6\ln 2$ <p style="text-align: right;">As required</p>	

Question 187 (***)+

$$I = \int_0^1 \frac{3}{(1+8x^2)^{\frac{3}{2}}} dx$$

- a) Use the substitution $x = \frac{1}{\sqrt{8}} \tan \theta$ to show that

$$I = \frac{3}{\sqrt{8}} \sin(\arctan \sqrt{8}).$$

- b) Show, presenting detailed calculations, that $I = 1$.

proof

$$\begin{aligned}
 \text{(a)} \quad & \int_0^1 \frac{3}{(1+8x^2)^{\frac{3}{2}}} dx = \dots \int_0^{\arctan \sqrt{8}} \frac{3}{(1+(\tan^2 \theta)^2)^{\frac{3}{2}}} (\sec^2 \theta d\theta) \\
 &= \frac{3 \sec \theta}{\sqrt{1+\tan^2 \theta}} \Big|_0^{\arctan \sqrt{8}} = \frac{3 \sec \theta}{\sqrt{1+(\tan \theta)^2}} (\sec^2 \theta d\theta) \\
 &= \frac{3 \sec \theta}{\sqrt{1+\tan^2 \theta}} \times \frac{1}{\sqrt{8}} \sec^2 \theta d\theta = \int_0^{\arctan \sqrt{8}} \frac{3}{\sqrt{8}} \sec^3 \theta d\theta \\
 &= \left[\frac{3}{\sqrt{8}} \sec \theta \tan \theta \right]_0^{\arctan \sqrt{8}} = \frac{3}{\sqrt{8}} (\sec(\arctan \sqrt{8}) - \sec 0) \\
 &= \frac{3}{\sqrt{8}} \sin(\arctan \sqrt{8}) \quad \text{as } \sec(\arctan \sqrt{8}) = 1 \\
 \text{(b)} \quad & \sin(\arctan \sqrt{8}) \\
 &= \sin \theta \quad \text{where } \theta = \arctan \sqrt{8} \\
 &\quad \tan \theta = \sqrt{8} \\
 &\quad \sin^2 \theta + \cos^2 \theta = 1 \\
 &\therefore \frac{3}{\sqrt{8}} \sin(\arctan \sqrt{8}) = \frac{3}{\sqrt{8}} \times \frac{\sqrt{8}}{3} = 1
 \end{aligned}$$

Question 188 (***)+

$$I = \int_1^2 \frac{1}{x^2 - x\sqrt{x^2 - 1}} dx.$$

- a) Show that the substitution $x = \sec \theta$ transforms I to

$$I = \int_0^{\frac{1}{3}\pi} \frac{\tan \theta}{\sec \theta - \tan \theta} d\theta.$$

- b) Hence use trigonometric identities to show that

$$I = 1 + \sqrt{3} - \frac{1}{3}\pi.$$

 , proof

a) WITH THE SUBSTITUTION GIVEN

$$\begin{aligned} x &= \sec \theta & x=1 &\mapsto \theta=0 \\ \frac{dx}{d\theta} &= \sec \theta \tan \theta & x=2 &\mapsto \theta=\frac{\pi}{3} \\ dx &= \sec \theta \tan \theta d\theta \end{aligned}$$

TRANSFORMING THE INTEGRAL

$$\begin{aligned} \int_1^2 \frac{1}{x^2 - x\sqrt{x^2 - 1}} dx &= \int_0^{\frac{\pi}{3}} \frac{\sec \theta \tan \theta}{\sec^2 \theta - \sec \theta \sqrt{\sec^2 \theta - 1}} d\theta \\ &= \int_0^{\frac{\pi}{3}} \frac{\sec \theta \tan \theta}{\sec \theta - \sec \theta \sqrt{1 - \tan^2 \theta}} d\theta = \int_0^{\frac{\pi}{3}} \frac{\sec \theta \tan \theta}{\sec \theta - \sec \theta (\tan \theta)} d\theta \\ &= \int_0^{\frac{\pi}{3}} \frac{\tan \theta}{\sec \theta - \tan \theta} d\theta \end{aligned}$$

As required

b) USING $(\sec \theta - \tan \theta)(\sec \theta + \tan \theta) = \sec^2 \theta - \tan^2 \theta = 1$

$$\begin{aligned} \dots &= \int_0^{\frac{\pi}{3}} \frac{\tan \theta (\sec \theta + \tan \theta)}{(\sec \theta - \tan \theta)(\sec \theta + \tan \theta)} d\theta \\ &= \int_0^{\frac{\pi}{3}} \frac{\tan^2 \theta + \tan \theta \sec \theta}{1} d\theta \\ &= \int \tan^2 \theta + \tan \theta \sec \theta d\theta \\ &= \int \sec^2 \theta - 1 + \tan \theta \sec \theta d\theta \end{aligned}$$

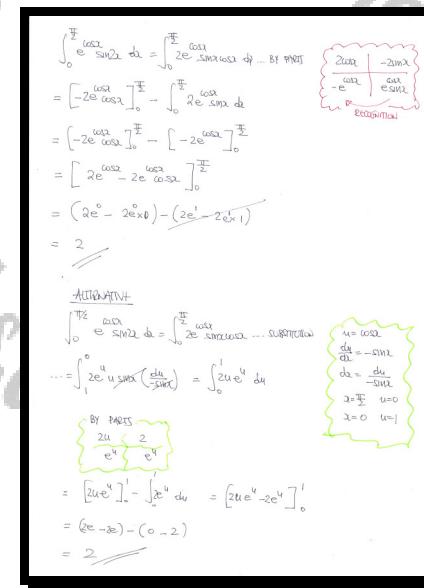
$$\begin{aligned} &= [\tan \theta - \theta + \sec \theta]_0^{\frac{\pi}{3}} \\ &= \left(\sqrt{3} - \frac{\pi}{3} + 2 \right) - (0 - 0 + 1) \\ &= \underline{\underline{\sqrt{3} - \frac{\pi}{3} + 1}} \quad \text{As required} \end{aligned}$$

Question 189 (*)+**

Use integration by parts to find the value of

$$\int_0^{\frac{\pi}{2}} e^{\cos x} \sin 2x \, dx.$$

[2]



Standard Method:

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} e^{\cos x} \sin 2x \, dx = \int_0^{\frac{\pi}{2}} 2e^{\cos x} \sin x \, dx \dots \text{BY PARTS} \\ &= \left[-2e^{\cos x} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2e^{\cos x} \sin x \, dx \\ &= \left[-2e^{\cos x} \right]_0^{\frac{\pi}{2}} - \left[-2e^{\cos x} \right]_0^{\frac{\pi}{2}} \\ &= \left[2e^{\cos x} - 2e^{\cos x} \right]_0^{\frac{\pi}{2}} \\ &= (2e^0 - 2e^0) - (2e^1 - 2e^1) \\ &= 2 \end{aligned}$$

Alternative Method:

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} e^{\cos x} \sin 2x \, dx = \int_0^{\frac{\pi}{2}} 2e^{\cos x} \sin x \, dx \dots \text{SUBSTITUTION} \\ &= \int_1^0 2e^u \sin(\frac{\pi}{2} - u) \, du = \int_0^1 2e^u \sin(u) \, du \\ & \quad \text{BY PARTS} \\ &= \left[2ue^u \right]_0^1 - \int_0^1 2e^u \, du = \left[2ue^u - 2e^u \right]_0^1 \\ &= (2e - 2e) - (0 - 2) \\ &= 2 \end{aligned}$$

Notes: $u = \cos x$, $\frac{du}{dx} = -\sin x$, $du = -\sin x \, dx$, $x = \frac{\pi}{2}$, $u = 0$, $x = 0$, $u = 1$.

Question 190 (****+)

$$\sin 2x \equiv \frac{2 \tan x}{1 + \tan^2 x}.$$

a) Prove the validity of the above trigonometric identity.

b) Express $\frac{8}{(3t+1)(t+3)}$ into partial fractions.

c) Hence use the substitution $t = \tan x$ to show that

$$\int_0^{\frac{\pi}{4}} \frac{8}{3 + 5\sin 2x} dx = \ln 3.$$

SOLN

$$\frac{8}{(3t+1)(t+3)} = \frac{3}{3t+1} - \frac{1}{t+3}$$

a) STARTING FROM THE R.H.S.

$$\begin{aligned} \text{R.H.S.} &= \frac{2\tan x}{1 + \tan^2 x} = \frac{2\tan x}{\sec^2 x} = 2\tan x \cos^2 x \\ &= 2 \times \frac{\sin x}{\cos x} \times \cos^2 x = 2\sin x \cos x = 2\sin 2x \\ &= \text{L.H.S.} \quad \text{AS REQUIRED} \end{aligned}$$

b) THE PARTIAL FRACTIONS NEXT

$$\begin{aligned} \frac{8}{(3t+1)(t+3)} &\equiv \frac{A}{3t+1} + \frac{B}{t+3} \\ 8 &\equiv A(t+3) + B(3t+1) \\ \text{IF } t=-3 &\quad \text{IF } t=0 \\ 8=-8B &\quad B=\frac{8}{3} \\ B=1 &\quad A=3 \\ \therefore \frac{8}{(3t+1)(t+3)} &\equiv \frac{3}{3t+1} - \frac{1}{t+3} \end{aligned}$$

c) USING THE SUBSTITUTION GIVEN IN THE PREVIOUS PARTS

$$\begin{aligned} \Rightarrow t &= \tan x \\ \Rightarrow \frac{dt}{dx} &= \sec^2 x \end{aligned}$$

$$dx = \frac{dt}{\sec^2 x}$$

$$dx = \frac{dt}{\frac{1}{\tan^2 x} dt}$$

$$dx = \frac{dt}{1+t^2}$$

• WHEN $x=0, t=0$

• WHEN $x=\frac{\pi}{4}, t=1$

TRANSFORMING THE INTEGRAL

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{8}{3 + 5\sin 2x} dx &= \int_0^{\frac{\pi}{4}} \frac{8}{3 + 5\left(\frac{2t}{1+t^2}\right)} dt \quad \text{← BY PART (a)} \\ &= \int_0^1 \frac{8}{3 + 5\left(\frac{2t}{1+t^2}\right)} \left(\frac{dt}{1+t^2} \right) dt \quad \text{← BY THE SUBSTITUTION} \\ &\approx \int_0^1 \frac{8}{3(1+t^2) + 10t} dt = \int_0^1 \frac{8}{3t^2 + 10t + 3} dt \\ &= \int_0^1 \frac{8}{(3t+1)(t+3)} dt = \int_0^1 \frac{3}{3t+1} - \frac{1}{t+3} dt \quad \text{← BY (b)} \\ &= \left[\ln|3t+1| - \ln|t+3| \right]_0^1 = (\ln 4 - \ln 1) - (\ln 1 - \ln 3) \\ &= \ln 3 \end{aligned}$$

AS REQUIRED

Question 191 (***)+

It is given that for some constants A and B

$$6\sin x \equiv A(\cos x + \sin x) + B(\cos x - \sin x).$$

- a) Find the value of A and the value of B .

- b) Hence find

$$\int \frac{6\sin x}{\cos x + \sin x} dx.$$

, $A = 3$, $B = -3$, $3x - 3\ln|\cos x + \sin x| + C$

a) EXPAND & COMBINE

$$\begin{aligned} 6\sin x &\Rightarrow A(\cos x + \sin x) + B(\cos x - \sin x) \\ &\Rightarrow 6\sin x = (A+B)\cos x + (A-B)\sin x \\ &\Rightarrow \begin{cases} A+B=0 \\ A-B=6 \end{cases} \quad \text{ADDING & SUBTRACTING EQUES} \\ &\Rightarrow \begin{cases} A=3 \\ B=-3 \end{cases} // \end{aligned}$$

b) USING PART (a)

$$\begin{aligned} \int \frac{6\sin x}{\cos x + \sin x} dx &= \int \frac{3(\cos x + \sin x) - 3(\cos x - \sin x)}{\cos x + \sin x} dx \\ &= \int \frac{3(\cos x + \sin x)}{\cos x + \sin x} - \frac{3(\cos x - \sin x)}{\cos x + \sin x} dx \\ &= \int 3 - 3 \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right) dx \\ &\quad \uparrow \text{THIS IS OF THE FORM } \int \frac{f(x)}{f'(x)} dx = \ln|f(x)| + C \\ &= 3x - 3 \ln|\cos x + \sin x| + C // \end{aligned}$$

Question 192 (***)+

Use the substitution $x = 2\sin \theta$ to show that

$$\int_0^2 \sqrt{4-x^2} dx = \pi.$$

proof

$$\begin{aligned} \int_0^2 \sqrt{4-x^2} dx &= \dots = \int_{\frac{\pi}{2}}^0 \sqrt{4-4\sin^2 \theta} (-2\cos \theta d\theta) \\ &= \int_{\frac{\pi}{2}}^0 -2\cos \theta \sqrt{4(1-\sin^2 \theta)} d\theta = \int_{\frac{\pi}{2}}^0 -2\cos \theta \sqrt{4\cos^2 \theta} d\theta \\ &= \int_{\frac{\pi}{2}}^0 4\cos^2 \theta d\theta = 4\int_{\frac{\pi}{2}}^0 (\frac{1}{2}(1+\cos 2\theta)) d\theta \\ &= \int_{\frac{\pi}{2}}^0 2 + 2\cos 2\theta d\theta = [2\theta + \sin 2\theta]_{\frac{\pi}{2}}^0 \\ &= (\pi - \sin \pi) - (0 - \sin 0) = \pi // \end{aligned}$$

$x = 2\sin \theta$
 $\frac{dx}{d\theta} = 2\cos \theta$
 $d\theta = -\frac{1}{2\cos \theta} dx$
 $2\cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$
 $2\sin \theta = 2 \Rightarrow \theta = \frac{\pi}{2}$
 $2\cos \theta = 1 \Rightarrow \theta = 0$

Question 193 (*)+**

Use the substitution $u = 1 + x^2 e^{-3x}$ to find an expression for

$$\int \frac{x(2-3x)}{e^{3x}+x^2} dx.$$

, $\boxed{\ln(1+x^2 e^{-3x})+C}$

USING THE SUBSTITUTION GIVEN

$$\begin{aligned}\Rightarrow u &= 1 + x^2 e^{-3x} \\ \Rightarrow \frac{du}{dx} &= 2x e^{-3x} + x^2 (-3e^{-3x}) \\ \Rightarrow \frac{du}{dx} &= 2x e^{-3x} - 3x^2 e^{-3x} \\ \Rightarrow \frac{du}{dx} &= x e^{-3x} (2 - 3x) \\ \Rightarrow dx &= \frac{du}{x(2-3x)e^{-3x}}\end{aligned}$$

TRANSFORMING THE INTEGRAL

$$\begin{aligned}\int \frac{x(2-3x)}{e^{3x}+x^2} dx &= \int \frac{x(2-3x)}{e^{3x}+x^2} \left(\frac{du}{x(2-3x)e^{-3x}} \right) \\ &= \int \frac{1}{e^{3x}+x^2} \times \frac{1}{e^{3x}} du \\ &= \int \frac{1}{1+x^2 e^{3x}} du \\ &= \int \frac{1}{u} du \\ &= \ln|u| + C \\ &= \boxed{\ln(1+x^2 e^{-3x}) + C}\end{aligned}$$

Question 194 (*)+**

Use the substitution $u = \frac{1}{x} + xe^x$ to find an expression for

$$\int \frac{x^3 + x^2 - e^{-x}}{x^3 + xe^{-x}} dx.$$

,
 $\ln \left| \frac{1}{x} + xe^x \right| + C$

USING THE SUBSTITUTION GIVEN

$$\begin{aligned} u &= \frac{1}{x} + xe^x \\ \frac{du}{dx} &= -\frac{1}{x^2} + e^x + xe^x \\ dx &= \frac{du}{-\frac{1}{x^2} + e^x + xe^x} \quad \text{MULTIPLY TOP BOTTOM BY } x^2 \\ du &= \frac{x^2}{2e^x + xe^x - 1} dx \end{aligned}$$

SUBSTITUTE INTO THE INTEGRAL

$$\begin{aligned} \int \frac{x^3 + x^2 - e^{-x}}{x^3 + xe^{-x}} dx &= \int \frac{2x^3 - e^{-x}}{x^3 + xe^{-x}} \times \frac{x^2}{2e^x + xe^x - 1} dx \\ &= \int \frac{2x^3 - e^{-x}}{x^3 + xe^{-x}} \times \frac{x^2 e^x}{2e^x + xe^x - 1} dx \\ &= \int \frac{2x^3 - e^{-x}}{x^3 + xe^{-x}} \times \frac{x^2 e^x}{2x^3 + x^2 e^x - e^x} dx \\ &= \int \frac{x^2 e^x}{-x^3 + xe^{-x}} dx \\ &= \int \frac{e^x}{x + \frac{1}{x} e^{-x}} dx \\ &= \int \frac{1}{xe^x - 1} dx \\ &= \int \frac{1}{u} du \\ &= \ln|u| + C = \ln \left| \frac{1}{x} + xe^x \right| + C \end{aligned}$$

Question 195 (***)+

$$\int \frac{1}{\sqrt{x^2 + x^n}} dx, n \neq 2, x \geq 0.$$

- a) Show that the substitution $u^2 = 1 + x^{n-2}$ transforms above integral into

$$\frac{1}{n-2} \int \frac{2}{(u-1)(u+1)} du.$$

- b) Use partial fractions to find, in terms of x and n , an integrated expression for the original integral.

$$[] , \boxed{\frac{1}{n-2} \ln \left| \frac{\sqrt{1+x^{n-2}} - 1}{\sqrt{1+x^{n-2}} + 1} \right| + C}$$

a) USING THE SUBSTITUTION GIVEN $u = (1+x^{n-2})^{\frac{1}{2}}$

$$\Rightarrow u^2 = 1 + x^{n-2} \iff x^{n-2} = u^2 - 1$$

$$\Rightarrow 2u \frac{du}{dx} = (n-2)x^{n-3}$$

$$\Rightarrow 2u du = (n-2)x^{n-3} dx$$

$$\Rightarrow dx = \frac{2u}{(n-2)x^{n-3}} du$$

TRANSFORMING THE INTEGRAL

$$\int \frac{1}{\sqrt{x^2 + x^n}} dx = \int \frac{1}{x \sqrt{1-x^{n-2}}} dx \quad (x \geq 0)$$

$$= \int \frac{1}{x \sqrt{u^2}} \frac{2u}{(n-2)x^{n-3}} du = \int \frac{2}{(n-2)x^{n-2}} du$$

$$= \int \frac{2}{(n-2)(u^{n-1})} du = \frac{2}{n-2} \int \frac{1}{(u-1)(u+1)} du \quad // \text{AS EQUIV}$$

b) PROVED BY PARTIAL FRACTIONS

$$\frac{2}{(u-1)(u+1)} \equiv \frac{A}{u-1} + \frac{B}{u+1}$$

$$2 = A(u+1) + B(u-1)$$

- IF $u=1$ • IF $u=-1$
- $2 = 2A$ $2 = -2B$
- $\underline{A=1}$ $\underline{B=-1}$

$$\dots = \frac{1}{n-2} \int \frac{1}{u-1} - \frac{1}{u+1} du$$

$$= \frac{1}{n-2} \left[\ln|u-1| - \ln|u+1| \right] + C$$

$$= \frac{1}{n-2} \ln \left| \frac{u-1}{u+1} \right| + C$$

$$= \frac{1}{n-2} \ln \left| \frac{\sqrt{1+x^{n-2}} - 1}{\sqrt{1+x^{n-2}} + 1} \right| + C \quad //$$

Question 196 (***)+

$$f(x) \equiv 2 - \sqrt{x-1}, \quad x \geq 1.$$

a) Find a simplified expression for $g(x)$ so that $f(x)g(x) = 1$.

b) Hence, or otherwise, find

$$\int \frac{5-x}{2-\sqrt{x-1}} dx.$$

(M1P)	$g(x) \equiv \frac{2+\sqrt{x-1}}{5-x}$	$\left[2x + \frac{1}{2}(x-1)^{\frac{3}{2}} + C \right]$
-------	--	--

a) $f(x) = 2 - \sqrt{x-1}, \quad x \geq 1$
 USING THE FACT GIVEN
 $\Rightarrow f(x)g(x) = 1$
 $\Rightarrow g(x) = \frac{1}{f(x)} = \frac{1}{2 - \sqrt{x-1}}$
 $\Rightarrow \frac{1}{f(x)} = \frac{(2 + \sqrt{x-1})}{(2 - \sqrt{x-1})(2 + \sqrt{x-1})}$
 $\Rightarrow \frac{1}{f(x)} = \frac{2 + \sqrt{x-1}}{4 - (x-1)}$
 $\Rightarrow \frac{1}{f(x)} = \frac{2 + \sqrt{x-1}}{5-x}$

b) USING PART (a)
 $\int \frac{5-x}{2-\sqrt{x-1}} dx = \int \frac{5-x}{4(x)} dx = \int (5-x) \cdot \frac{1}{4x} dx$
 $= \int (5-x) \cdot \frac{2+\sqrt{x-1}}{5-x} dx = \int 2 + \sqrt{x-1} dx$
 $= \int 2 + (x-1)^{\frac{1}{2}} dx$
 $= 2x + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$

Question 197 (*)+**

By using the substitution $u = 1 + \sin^2 x$, or otherwise, show clearly that

$$\int_0^{\frac{\pi}{4}} \frac{4 \tan x}{1 + \sin^2 x} dx = \ln 3.$$

[proof]

$$\begin{aligned}
 & \int_0^{\frac{\pi}{4}} \frac{4 \tan x}{1 + \sin^2 x} dx = \dots \text{ by substitution ...} \\
 & = \int_1^{\frac{3}{2}} \frac{4 \tan x}{u} \frac{du}{2 \sin x \cos x} \\
 & = \int_1^{\frac{3}{2}} \frac{2 \tan x}{u \sin x \cos x} du = \int_1^{\frac{3}{2}} \frac{2}{u} \times \frac{\sin x}{\cos x} \times \frac{1}{\sin x \cos x} du \\
 & = \int_1^{\frac{3}{2}} \frac{2}{u(u-1)} du = \int_1^{\frac{3}{2}} \frac{2}{u(1-\sin^2 x)} du \\
 & = \int_1^{\frac{3}{2}} \frac{2}{u(1-(u-1))} du = \int_1^{\frac{3}{2}} \frac{2}{u(2-u)} du \\
 & \text{BY PARTIAL FRACTIONS} \\
 & \frac{2}{u(2-u)} \equiv \frac{A}{u} + \frac{B}{2-u} \quad \begin{cases} u=0 \Rightarrow 2A=2 \Rightarrow A=1 \\ 2 \equiv A(2-u) + Bu \end{cases} \\
 & \frac{2}{u(2-u)} \equiv \frac{1}{u} + \frac{1}{2-u} \quad \begin{cases} u=2 \Rightarrow B=2 \Rightarrow B=1 \\ 2 \equiv A(2-u) + Bu \end{cases} \\
 & = \int_1^{\frac{3}{2}} \left(\frac{1}{u} + \frac{1}{2-u} \right) du = \left[\ln|2-u| + \ln|u| \right]_1^{\frac{3}{2}} = \left[(\ln|u| - \ln|2-u|) \right]_1^{\frac{3}{2}} \\
 & = \left[\ln \frac{3}{2} - \ln \frac{1}{2} \right] - \left[\ln 1 - \ln 2 \right] = \ln \frac{3}{2} + \ln 2 = \ln 3
 \end{aligned}$$

Question 198 (***)+

$$\sec x \equiv \frac{\cos x}{1 - \sin^2 x}.$$

- a) Prove the validity of the above trigonometric identity.
- b) Use the substitution $u = \sin x$ to show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \, dx = \frac{1}{2} \ln \left(\frac{7+4\sqrt{3}}{3} \right).$$

- c) Show clearly that

$$\frac{1}{2} \ln \left(\frac{7+4\sqrt{3}}{3} \right) = \ln \left(1 + \frac{2}{3}\sqrt{3} \right)$$

proof

<p>(a) LHS = $\sec x = \frac{1}{\cos x} = \frac{\cos x}{1 - \sin^2 x} = \frac{\cos x}{1 - u^2}$ to RHS</p>
<p>(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x}{1 - \sin^2 x} \, dx = \dots$ substitution $= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\cos x \, du}{1 - u^2} = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{1 - u^2} \, du$ $= \dots$ by partial fractions $= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\frac{1}{u} + \frac{1}{1-u}}{1-u^2} \, du = \left[\frac{1}{2} \ln 1+u - \frac{1}{2} \ln 1-u \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$ $= \left[\frac{1}{2} \ln \frac{ 1+\sqrt{3}/2 }{ 1-\sqrt{3}/2 } \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{1}{2} \left[\ln \left(\frac{1+\sqrt{3}/2}{1-\sqrt{3}/2} \right) - \ln \left(\frac{1}{\sqrt{3}/2} \right) \right]$ $= \frac{1}{2} \left[\ln \left(\frac{2+\sqrt{3}}{2-\sqrt{3}} \right) - \ln 3 \right] = \frac{1}{2} \left[\ln \left(1+4\sqrt{3} \right) - \ln 3 \right]$ $= \frac{1}{2} \ln \left(\frac{7+4\sqrt{3}}{3} \right)$ to RHS</p>
<p>(c) $\frac{1}{2} \ln \left(\frac{7+4\sqrt{3}}{3} \right) = \frac{1}{2} \ln \left(\frac{21+12\sqrt{3}}{9} \right) = \frac{1}{2} \ln \left[\frac{9 + 2 \times 3 \times 2\sqrt{3} + 12}{9} \right]$ $= \frac{1}{2} \ln \left[\frac{3^2 + 2 \times 3 \times 2\sqrt{3} + (2\sqrt{3})^2}{9} \right]$ $= \frac{1}{2} \ln \left[\frac{(3+2\sqrt{3})^2}{9} \right] = \ln \left(\frac{3+2\sqrt{3}}{3} \right)$ $= \ln \left(1 + \frac{2}{3}\sqrt{3} \right)$ to RHS</p>

Question 199 (**+)**

Use the substitution $x = 2\sec\theta$, to find the exact value of

$$\int_{\frac{4}{\sqrt{3}}}^4 \frac{6}{(x^2 - 4)^{\frac{3}{2}}} dx.$$

3 - $\sqrt{3}$

$$\begin{aligned}
 & \int_{\frac{4}{\sqrt{3}}}^4 \frac{6}{(x^2 - 4)^{\frac{3}{2}}} dx = \dots \text{BY SUBSTITUTION} \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6}{(4\sec^2\theta - 4)^{\frac{3}{2}}} \times 2\sec\theta\tan\theta d\theta \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{12\sec\theta\tan\theta}{[4(\sec^2\theta - 1)]^{\frac{3}{2}}} d\theta \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{12\sec\theta\tan\theta}{(4\tan^2\theta)^{\frac{3}{2}}} d\theta \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{12\sec\theta\tan^2\theta}{8\tan^3\theta} d\theta \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{3\sec\theta}{2\tan^2\theta} d\theta \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{3}{2\sin^2\theta} d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{3}{2}\csc^2\theta + 1}{\sin^2\theta} d\theta \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 3(\cot^2\theta + \csc^2\theta) d\theta = \left[-\frac{3}{2}\cot\theta - \csc\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
 &= \frac{3}{2} \left[(\cot\theta) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{3}{2} \left[2 - \frac{2}{\sqrt{3}} \right] \\
 &= 3 - \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 x &= 2\sec\theta \\
 \frac{dx}{d\theta} &= 2\sec\theta\tan\theta \\
 dx &= 2\sec\theta\tan\theta d\theta \\
 2 &= 2\sec\theta \\
 \frac{4}{\sqrt{3}} &= 2\sec\theta \\
 \frac{2}{\sqrt{3}} &= \sec\theta \\
 \frac{4}{3} &= \cos\theta \\
 \theta &= \frac{\pi}{6} \\
 2 &= 2\sec\theta \\
 2 &= \sec\theta \\
 \cos\theta &= \frac{1}{2} \\
 \theta &= \frac{\pi}{3}
 \end{aligned}$$

Question 200 (*)+**

Use the substitution $u = 1 + xe^{\sin x}$ to find an exact simplified value for the following definite integral.

$$\int_0^\pi \frac{1+x\cos x}{x+e^{-\sin x}} dx .$$

, $\ln(1+\pi)$

USING THE SUBSTITUTION

$$\Rightarrow u = 1 + xe^{\sin x}$$

$$\Rightarrow \frac{du}{dx} = 1 + e^{\sin x} + x \cdot e^{\sin x}(\cos x)$$

$$\Rightarrow \frac{du}{dx} = e^{\sin x} + x e^{\sin x} \cos x$$

$$\Rightarrow \frac{du}{dx} = e^{\sin x}(1 + x \cos x)$$

$$\Rightarrow dx = \frac{du}{e^{\sin x}(1 + x \cos x)}$$

CHANGING THE LIMITS

$$x=0 \quad \mapsto \quad u=1$$

$$x=\pi \quad \mapsto \quad u=1+\pi$$

TRANSFORMING THE INTEGRAL

$$\int_0^\pi \frac{1+x\cos x}{x+e^{-\sin x}} dx = \int_1^{1+\pi} \frac{1+xe^{\sin x}}{x+e^{-\sin x}} \times \frac{e^{\sin x} du}{e^{\sin x}(1+x\cos x)}$$

$$= \int_1^{1+\pi} \frac{1}{x+e^{-\sin x}} du$$

$$= \int_1^{1+\pi} \frac{1}{x+e^{-\sin x}} du$$

$$= \left[\ln|x| \right]_1^{1+\pi}$$

$$= \ln(1+\pi) - \ln 1$$

$$= \ln(1+\pi)$$

Question 201 (***)+

$$I = \int \frac{1}{x^2 \sqrt{4-x^2}} dx.$$

- a) Use the substitution $x = 2\sin\theta$ to show clearly that

$$I = -\frac{\sqrt{4-x^2}}{4x} + C.$$

- b) Verify the answer to part (a) by using the substitution $u = \frac{2}{x}$.

[proof]

$(a) \int \frac{1}{x^2 \sqrt{4-x^2}} dx = \dots \text{ by substitution}$ $= \int \frac{1}{4x^2 b \sqrt{4-4x^2 b^2}} (2axb d\theta) \quad \begin{array}{l} x = 2\sin\theta \\ \frac{dx}{d\theta} = 2\cos\theta \\ dx = 2\cos\theta d\theta \end{array}$ $= \int \frac{2axb d\theta}{4x^2 b \sqrt{4(1-\sin^2\theta)}} = \int \frac{2axb d\theta}{4x^2 b \sqrt{4\cos^2\theta}} = \int \frac{2axb d\theta}{4x^2 b \cdot 2\cos\theta} = -\frac{1}{4}axb + C$ $= \frac{2axb d\theta}{4x^2 b \cdot 2\cos\theta} = \int \frac{1}{4}ax\sec^2\theta d\theta = -\frac{1}{4}axb + C$ $\text{(Now } x = 2\sin\theta \text{)}$ $\begin{array}{l} \frac{x}{2} = \sin\theta \\ \frac{x^2}{4} = \sin^2\theta \\ 1 - \frac{x^2}{4} = 1 - \sin^2\theta = \cos^2\theta \end{array}$ $\therefore \sec^2\theta = \frac{2}{x^2} \quad \therefore axb = \frac{2\sin\theta}{x} = \frac{2\sin\theta}{2\cos\theta} = \frac{1}{\cos\theta}$ $= -\frac{1}{4} \cdot \frac{\sqrt{4-x^2}}{x} + C$	$(b) \int \frac{1}{x^2 \sqrt{4-x^2}} dx = \dots \text{ by substitution}$ $u = \frac{2}{x}, \quad x = \frac{2}{u}$ $\frac{dx}{d\theta} = -\frac{2}{u^2}, \quad dx = -\frac{2}{u^2} du$ $= \int \frac{1}{\frac{4}{u^2} \sqrt{4-\frac{4}{u^2}}} \left(-\frac{2}{u^2} du \right) = \int \frac{u^2}{4\sqrt{4-\frac{4}{u^2}}} \left(-\frac{2}{u^2} du \right)$ $= \int \frac{-\frac{1}{2}\sqrt{4-\frac{4}{u^2}}}{u} du = \int -\frac{u}{4\sqrt{u^2-1}} du$ $\text{BY RECOGNITION, OR USEFUL SUBSTITUTION ...}$ $\text{AFTER DIFFERENTIATING TO } 2u$ $= \int -\frac{1}{2}u \left(\frac{4-2u^2}{u}\right)^{-\frac{1}{2}} du = -\frac{1}{4}(2u^2-1)^{-\frac{1}{2}} + C$ $= -\frac{1}{4}\sqrt{\frac{4-2u^2}{u^2}-1} + C = -\frac{1}{4}\sqrt{\frac{4-2x^2}{x^2}} + C$ $= -\frac{1}{4}\frac{\sqrt{4-2x^2}}{x} + C = -\frac{\sqrt{4-2x^2}}{4x} + C$ <p style="text-align: right;">As Before</p>
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Question 202 (*)+**

By using the substitution $x = -\frac{1}{2} + \frac{1}{2}\sin\theta$, or otherwise, find the exact value of

$$\int_{-\frac{1}{4}}^0 \frac{3}{\sqrt{-x(x+1)}} dx.$$

, π

USING THE SUBSTITUTION (METHOD)

$$x = -\frac{1}{2} + \frac{1}{2}\sin\theta$$

$$dx = \frac{1}{2}\cos\theta d\theta$$

CHANGING THE LIMITS

- $x=0 \Rightarrow 0 = -\frac{1}{2} + \frac{1}{2}\sin\theta$
 $\frac{1}{2} = \frac{1}{2}\sin\theta$
 $\sin\theta = 1$
 $\theta = \frac{\pi}{2}$
- $x=-\frac{1}{4} \Rightarrow -\frac{1}{4} = -\frac{1}{2} + \frac{1}{2}\sin\theta$
 $\frac{1}{4} = \frac{1}{2}\sin\theta$
 $\sin\theta = \frac{1}{2}$
 $\theta = \frac{\pi}{6}$

TRANSFORMING THE DENOMINATOR OF THE INTEGRAND

$$\sqrt{-x(x+1)} = \sqrt{(-\frac{1}{2} + \frac{1}{2}\sin\theta)(-\frac{1}{2} + \frac{1}{2}\sin\theta + 1)}$$

$$= \sqrt{(\frac{1}{2} - \frac{1}{2}\sin\theta)(\frac{1}{2} + \frac{1}{2}\sin\theta)}$$

$$= \sqrt{\frac{1}{4} - \frac{1}{4}\sin^2\theta}$$

$$= \sqrt{\frac{1}{4}(1 - \sin^2\theta)}$$

$$= \sqrt{\frac{1}{4}\cos^2\theta}$$

$$= \frac{1}{2}\cos\theta$$

MAKING THE INTEGRAL NOW EASIER

$$\int_{-\frac{1}{4}}^0 \frac{3}{\sqrt{-x(x+1)}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{3}{\sqrt{\frac{1}{4}\cos^2\theta}} (\frac{1}{2}\cos\theta d\theta)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3 d\theta$$

$$= \left[3\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{3\pi}{2} - \frac{3\pi}{6}$$

$$= \frac{\pi}{2}$$

Question 203 (*)+**

By using multiplying the numerator and denominator of the integrand by $(\sec x + 1)$, and manipulating it further by various trigonometric identities, show clearly that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6}{\sec x - 1} dx = 12 - \pi.$$

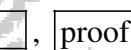
proof

$$\begin{aligned}
 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6}{\sec x - 1} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6(\sec x + 1)}{(\sec x - 1)(\sec x + 1)} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6(\sec x + 1)}{\sec^2 x - 1} dx \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{6(\sec x + 1)}{\tan^2 x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6(\sec x + 1) \csc^2 x dx \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6 \left(\frac{1}{\cos x} + 1 \right) \frac{\csc^2 x}{\sin^2 x} dx \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6 \left(\frac{\csc x}{\sin x} + \frac{\csc^2 x}{\sin^2 x} \right) dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6 \left(\frac{\csc x}{\sin x} \cdot \frac{1}{\sin x} + \csc^2 x \right) dx \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 6 \left(\csc x \cot x + \csc^2 x - 1 \right) dx \\
 &= \left[-6 \csc x - 6 \cot x - 6x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
 &= \left[6 \csc x + 6 \cot x + 6x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
 &= \left(6 \csc \frac{\pi}{3} + 6 \cot \frac{\pi}{3} + \pi \right) - \left(6 \csc \frac{\pi}{6} + 6 \cot \frac{\pi}{6} + 2\pi \right) \\
 &= (12 - 6\sqrt{3} + 6\sqrt{3} + \pi) - (6\sqrt{3} + 6\sqrt{3} + 2\pi) \\
 &= (12 - 4\pi) \quad \text{as required}
 \end{aligned}$$

Question 204 (***)+

By changing the base of the logarithmic integrand into base e and further using integration by parts, show that

$$\int_1^e \log_{10} x \, dx = \frac{1}{\ln 10}.$$

CHANGE INTO BASE e

$$\begin{aligned} \int_1^e \log_{10} x \, dx &= \int_1^e \frac{\log_e x}{\log_e 10} \, dx \\ &\quad \boxed{\log_a b = \frac{\log_e b}{\log_e a}} \\ &= \int_1^e \frac{\ln x}{\ln 10} \, dx \\ &= \int_1^e \frac{1}{\ln 10} (\ln x) \, dx \end{aligned}$$

INTEGRATION BY PARTS (EVALUATE INTEGRAL $\frac{1}{\ln 10}$ IS A CONSTANT)

$$\begin{aligned} \left[\frac{\ln x}{\ln 10} \right]_1^e - \int_1^e \frac{1}{\ln 10} x \times \frac{1}{x} \, dx &= \left[\frac{x \ln x}{\ln 10} \right]_1^e - \int_1^e \frac{1}{\ln 10} \, dx \\ &= \left[\frac{x \ln x}{\ln 10} \right]_1^e - \int_1^e \frac{1}{\ln 10} \, dx \\ &= \left[\frac{x \ln x}{\ln 10} - \frac{1}{\ln 10} x \right]_1^e \\ &= \left(\frac{e \ln e}{\ln 10} - \frac{e}{\ln 10} \right) - \left(\frac{\ln 1}{\ln 10} - \frac{1}{\ln 10} \right) \\ &= \left(\frac{e}{\ln 10} - \frac{e}{\ln 10} \right) + \frac{1}{\ln 10} \\ &= \frac{1}{\ln 10} \end{aligned}$$

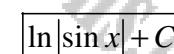
\checkmark AS REQUIRED

Question 205 (***)+

Use trigonometric identities to find

$$\int \frac{1}{\operatorname{cosec} 2x - \cot 2x} \, dx,$$

giving the answer in the form $\ln|f(x)|$



$$\begin{aligned} \int \frac{1}{\operatorname{cosec} 2x - \cot 2x} \, dx &= \int \frac{1}{\frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x}} \, dx = \dots \quad \text{(MULTIPLY TOP AND BOTTOM BY } \sin 2x) \\ &= \int \frac{\sin 2x}{1 - \cos 2x} \, dx = \int \frac{2 \sin x \cos x}{1 - (1 - 2 \sin^2 x)} \, dx = \int \frac{2 \sin x \cos x}{2 \sin^2 x} \, dx \\ &= \int \frac{\cos x}{\sin x} \, dx = \dots \text{ of the type } \int \frac{f'(x)}{f(x)} \, dx = \dots = \ln|\sin x| + C \end{aligned}$$

Question 206 (*)+**

By using the substitution $u = \tan x$, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{4}} \frac{1}{(\cos x + 2 \sin x)^2} dx.$$

$\boxed{\frac{1}{3}}$

$$\begin{aligned}
 & \int_0^{\frac{\pi}{4}} \frac{1}{(\cos x + 2 \sin x)^2} dx = \text{by substitution} \\
 & = \int_0^1 \frac{1}{(1+2u)^2} \frac{du}{\sec^2 x} = \int_0^1 \frac{1}{(1+2u)^2} du \\
 & = \int_0^1 \frac{1}{(1+2u)^2} du = \int_0^1 \frac{1}{(1+2u)^2} du \\
 & = \int_0^1 (1+2u)^{-2} du = \left[-\frac{1}{2}(1+2u)^{-1} \right]_0^1 = \frac{1}{2} \left[\frac{1}{1+2u} \right]_0^1 \\
 & = \frac{1}{2} \left[1 - \frac{1}{3} \right] = \frac{1}{3} \quad \checkmark
 \end{aligned}$$

$u = \tan x$
 $\frac{du}{dx} = \sec^2 x$
 $\frac{du}{dx} = \frac{du}{\sec^2 x}$
 $2u = \frac{1}{\sec^2 x}$
 $2u = \frac{1}{1+2u}$
 $2u = u+1$
 $u = 1$
 $2u = 2$
 $u = 1$

$$\begin{aligned}
 & \text{ALTERNATIVE BY REVERSING THE ORDER OF INTEGRATION} \\
 & \int_0^{\frac{\pi}{4}} \frac{1}{(\cos x + 2 \sin x)^2} dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x (1 + \frac{2 \sin x}{\cos x})^2} dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x (1 + 2 \tan x)^2} dx \\
 & = \int_0^{\frac{\pi}{4}} \frac{\sec^2 x (1 + 2 \tan x)^2}{(1 + 2 \tan x)^2} dx = \int_0^{\frac{\pi}{4}} \sec^2 x dx = \left[\frac{1}{2} (\sec^2 x)^{-1} \right]_0^{\frac{\pi}{4}} \\
 & = \frac{1}{2} \left[\frac{1}{1+2\tan x} \right]_0^{\frac{\pi}{4}} = \frac{1}{2} \left[1 - \frac{1}{3} \right] = \frac{1}{3} \quad \checkmark
 \end{aligned}$$

Primes

Question 207 (***)+

Use trigonometric identities and integration by parts to find an exact value for

$$\int_0^{\frac{\pi}{2}} 9x \sin x \sin 2x \, dx.$$

$, 3\pi - 4$

SOLVE BY "BREAKING THE DOUBLE ANGLE" IN THE MIDDLE:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} 9x \sin x \sin 2x \, dx &= \int_0^{\frac{\pi}{2}} 9x \sin x (2 \sin x \cos x) \, dx \\ &= \int_0^{\frac{\pi}{2}} 18x \sin^2 x \cos x \, dx \\ &= \int_0^{\frac{\pi}{2}} (6x) (3 \sin^2 x \cos x) \, dx \end{aligned}$$

NEXT: INTEGRATION BY PARTS

$\frac{d}{dx}$	6	$= [6x \sin^2 x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 6 \sin^2 x \, dx$
$\frac{du}{dx}$	$3 \sin x \cos x$	$= 3\pi - 0 - \int_0^{\frac{\pi}{2}} 6 \sin^2 x \, dx$

BY REARRANGEMENT

NOW MANIPULATE INTO "RECOGNISABLE" FORM OR USE THE SUBSTITUTION $u = \cos x$, $du = -\sin x \, dx$

$$\begin{aligned} \dots &= 3\pi - \int_0^{\frac{\pi}{2}} 6 \sin^2 x \, dx \\ &= 3\pi - \int_0^{\frac{\pi}{2}} 6 \sin^2 (1 - \cos^2 x) \, dx \\ &= 3\pi - \int_0^{\frac{\pi}{2}} 6 \sin^2 (-(\sin^2 x)) \, dx \end{aligned}$$

BY REARRANGEMENT

$$\begin{aligned} &= 3\pi - [-6 \cos x + 2 \cos^3 x]_0^{\frac{\pi}{2}} \\ &= 3\pi + [6 \cos x - 2 \cos^3 x]_0^{\frac{\pi}{2}} \\ &= 3\pi + [(0 - 0) - (6 - 2)] \\ &= 3\pi - 4 \end{aligned}$$

Question 208 (***)+

$$I \equiv \int \frac{1}{1+\sin 2x} dx.$$

- a) Integrate I by multiplying the numerator and denominator of the integrand by $(1-\sin 2x)$.

- b) Hence evaluate

$$\int_0^{\frac{\pi}{8}} \frac{1}{1+\sin 2x} dx.$$

- c) Use the substitution $t = \tan x$ to integrate I .

- d) Hence evaluate

$$\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin 2x} dx.$$

$$\boxed{\frac{1}{2}\tan 2x - \frac{1}{2}\sec 2x + C}, \boxed{\frac{1}{2}(2-\sqrt{2})}, \boxed{-\frac{1}{1+\tan x} + C}, \boxed{\frac{1}{2}}$$

$$\begin{aligned}
 \text{(a)} \quad & \int \frac{1}{1+\sin 2x} dx = \int \frac{(1-\sin 2x)}{(1+\sin 2x)(1-\sin 2x)} dx = \int \frac{1-\sin 2x}{1-\sin^2 2x} dx = \int \frac{1-\sin 2x}{\cos^2 2x} dx \\
 &= \int \frac{1}{\cos^2 2x} \frac{\sin 2x}{\cos 2x} dx = \int \frac{1}{\cos^2 2x} \frac{1}{\cos 2x} \frac{\sin 2x}{\cos 2x} dx = \int \sec^2 2x - \tan 2x \sec 2x dx = \frac{1}{2}\tan 2x - \frac{1}{2}\sec 2x + C \\
 \text{(b)} \quad & \int_0^{\frac{\pi}{8}} \frac{1}{1+\sin 2x} dx = \frac{1}{2} \left[\tan 2x - \sec 2x \right]_0^{\frac{\pi}{8}} = \frac{1}{2} \left[(1-\sqrt{2}) - (0-1) \right] = \frac{1}{2}(2-\sqrt{2}) \\
 \text{(c)} \quad & I = \int \frac{1}{1+2\tan x \sec x} dx = \int \frac{\sec^2 x}{\sec x + 2\tan x} dx = \int \frac{1+\tan^2 x}{1+\tan x + 2\tan x} dx = \int \frac{1+\tan^2 x}{(1+\tan x)^2} dx \\
 & \text{Now let } t = \tan x \quad \Rightarrow \quad dt = \sec^2 x dx = \frac{dt}{(1+t^2)} \quad \Rightarrow \quad \frac{dt}{dt} = \frac{dt}{1+t^2} \quad \Rightarrow \quad \frac{dt}{1+t^2} = dt \\
 & \int \frac{1}{(1+t)^2} dt = \int \frac{1}{1+t^2} dt = -\frac{1}{1+t} + C = -\frac{1}{1+\tan x} + C \\
 \text{(d)} \quad & \text{Finally, } \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin 2x} dx = \left[-\frac{1}{1+\tan x} \right]_0^{\frac{\pi}{4}} = \left[-\frac{1}{1+\tan x} \right]_{\frac{\pi}{4}}^0 = 1 - \frac{1}{2} = \frac{1}{2}
 \end{aligned}$$

Question 209 (***)+

Show clearly that

$$\int_0^1 \sqrt{x^2 - x^4} \, dx = \frac{1}{3}.$$

[, proof]

PROCEED AS FOLLOWS

$$\begin{aligned} \int_0^1 \sqrt{x^2 - x^4} \, dx &= \int_0^1 \sqrt{x^2(1-x^2)} \, dx \\ &= \int_0^1 x\sqrt{1-x^2} \, dx \\ &= \int_0^1 |x|\sqrt{1-x^2} \, dx \\ &= \int_0^1 x\sqrt{1-x^2} \, dx \end{aligned}$$

USING THE SUBSTITUTION $x = \sin\theta$

$$\begin{aligned} x &= \sin\theta & x=0 &\mapsto \theta=0 \\ dx &= \cos\theta \, d\theta & x=1 &\mapsto \theta=\frac{\pi}{2} \end{aligned}$$

TRANSFORMING THE INTEGRAL

$$-\int_0^{\frac{\pi}{2}} \sin\theta \sqrt{1-\sin^2\theta} (\cos\theta \, d\theta) = \int_0^{\frac{\pi}{2}} \sin\theta \cos^2\theta \, d\theta$$

BY RECOGNITION

$$\begin{aligned} \left[-\frac{1}{3} \cos^3\theta \right]_0^{\frac{\pi}{2}} &= \frac{1}{3} \left[\cos^3\theta \right]_0^{\frac{\pi}{2}} = \frac{1}{3} [1-0] \\ &= \frac{1}{3} \end{aligned}$$

 As required

Question 210 (*)+**

Use the substitution $u = 1 + x^2 \operatorname{cosec} x$ to find an expression for

$$\int \frac{2x - x^2 \cot x}{x^2 + \sin x} dx.$$

$$\boxed{\quad}, \boxed{\ln|1+x^2 \operatorname{cosec} x| + C}$$

USING THE SUBSTITUTION, GIVEN

$$\begin{aligned}\rightarrow u &= 1 + x^2 \operatorname{cosec} x \\ \rightarrow \frac{du}{dx} &= 2x \operatorname{cosec} x - x^2 \operatorname{cosec} x \cot x \\ \rightarrow \frac{du}{dx} &= x^2 \operatorname{cosec} x (2 - \cot x) \\ \rightarrow du &= \frac{du}{(2-x\cot x)(\operatorname{cosec} x)}\end{aligned}$$

SUBSTITUTE INTO THE INTEGRAL

$$\begin{aligned}\int \frac{2x - x^2 \cot x}{x^2 + \sin x} dx &= \int \frac{x(2-\cot x)}{x^2 + \sin x} \times \frac{du}{(2-x\cot x)\operatorname{cosec} x} \\ &= \int \frac{x}{(\sin x + x^2)\operatorname{cosec} x} du \\ &= \int \frac{1}{x\operatorname{cosec} x + x} du \\ &= \int \frac{1}{u} du \\ &= \ln|u| + C \\ &= \ln|1+x^2 \operatorname{cosec} x| + C\end{aligned}$$

Question 211 (***)+

Use a suitable trigonometric substitution to find an exact simplified value for

$$\int_0^a x^{\frac{1}{2}} \sqrt{a-x} dx,$$

where a is a positive constant.

$$\boxed{\frac{1}{8}\pi a^2}$$

$$\begin{aligned}
 & \int_0^a x^{\frac{1}{2}}(a-x)^{\frac{1}{2}} dx \quad \dots \text{ by substitution} \\
 &= \int_0^{\frac{\pi}{2}} \sqrt{a} \sin\theta (a - a\sin^2\theta)^{\frac{1}{2}} (2a\sin\theta\cos\theta) d\theta \\
 &= \left[\frac{\pi}{2} \sqrt{a} \sin\theta \sqrt{a}(1-\sin^2\theta)^{\frac{1}{2}} \times 2a\sin\theta\cos\theta \right]_0^{\frac{\pi}{2}} \\
 &= \left[\frac{\pi}{2} 2a^2 \sin^2\theta \times \sin\theta\cos\theta \right]_0^{\frac{\pi}{2}} \\
 &= \left[\frac{\pi}{2} \frac{1}{2}a^2 (2\sin\theta\cos\theta)(2\sin\theta\cos\theta) \right]_0^{\frac{\pi}{2}} \\
 &= \left[\frac{\pi}{2} \frac{1}{2}a^2 (\sin^2 2\theta) \right]_0^{\frac{\pi}{2}} = \frac{1}{2}a^2 \left[\frac{\pi}{2} - \frac{1}{2}\cos 4\theta \right]_0^{\frac{\pi}{2}} \\
 &= \frac{1}{2}a^2 \left[\frac{\pi}{2} - 0 \right] = \frac{1}{2}a^2 \left[\frac{\pi}{2} - \frac{1}{2}\sin 4\theta \right]_0^{\frac{\pi}{2}} \\
 &= \frac{1}{2}a^2 \left[\left(\frac{\pi}{2} - 0 \right) - (0 - 0) \right] = \frac{\pi a^2}{8}
 \end{aligned}$$

Question 212 (****+)

$$f(u) \equiv \frac{1}{u^2 + 5u + 6}.$$

a) Express $f(u)$ into partial fractions.

$$I = \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{(\sin x + 2\cos x)(\sin x + 3\cos x)} dx.$$

b) Express I in the form

$$I = \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{\sec^2 x}{g(\tan x)} dx,$$

where g is a function to be found.

c) Hence show that

$$I = \ln\left(\frac{a}{b}\right),$$

where a and b are positive integers to be found.

MP2	$f(u) \equiv \frac{1}{u+2} - \frac{1}{u+3}$	$g(\tan x) \equiv (2+\tan x)(3+\tan x)$	$I = \ln\left(\frac{150}{143}\right)$
-----	---	---	---------------------------------------

a) $\frac{1}{u^2 + 5u + 6} \equiv \frac{1}{(u+2)(u+3)} = \frac{A}{u+2} + \frac{B}{u+3}$

$$1 \equiv A(u+3) + B(u+2)$$

$$\text{If } u=-3 \Rightarrow 1 = -8 \quad \text{If } u=-2 \Rightarrow 1 = A$$

$$\therefore \frac{1}{u^2 + 5u + 6} = \frac{1}{u+2} - \frac{1}{u+3} //$$

b) $\int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{(\sin x + 2\cos x)(\sin x + 3\cos x)} dx$
 divide for a bottom of the integrand by $\sec^2 x$
 $= \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{\frac{1}{\cos^2 x}}{(\tan x + 2)(\tan x + 3)} dx$
 $= \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{\sec^2 x}{(\tan x + 2)(\tan x + 3)} dx$

c) USING THE COORDINATE SYSTEM
 $u = \tan x$
 $\frac{du}{dx} = \sec^2 x$
 $dx = \frac{du}{\sec^2 x}$

THE UNIT CIRCLE
 $\arccos \frac{3}{5} = \arccos \frac{4}{5}$
 $\sec \frac{3}{5} = \sec \frac{4}{5}$
 $\bullet \theta = \arccos \frac{3}{5} \rightarrow u = \frac{3}{5}$
 $\bullet \theta = \arccos \frac{4}{5} \rightarrow u = \frac{4}{5}$

HERE WE NOW HAVE

$$\int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{\sec^2 x}{(\tan x + 2)(\tan x + 3)} dx = \int_{\frac{3}{5}}^{\frac{4}{5}} \frac{\sec^2 x}{(\tan x + 2)(\tan x + 3)} \frac{du}{\sec^2 x} dx$$

$$= \int_{\frac{3}{5}}^{\frac{4}{5}} \frac{1}{(\tan x + 2)(\tan x + 3)} du = \dots \text{part (a)} \dots = \int_{\frac{3}{5}}^{\frac{4}{5}} \frac{1}{u+2} - \frac{1}{u+3} du$$

$$= \left[\ln|u+2| - \ln|u+3| \right]_{\frac{3}{5}}^{\frac{4}{5}} = \left(\ln \frac{10}{3} - \ln \frac{11}{3} \right) - \left(\ln \frac{10}{4} - \ln \frac{11}{4} \right)$$

$$= \ln \frac{10}{3} \frac{11}{11} - \ln \frac{11}{4} \frac{10}{10} = \ln \frac{10}{13} - \ln \frac{11}{15}$$

$$= \ln \frac{10}{11} \frac{15}{15} = \ln \frac{150}{143} //$$

Question 213 (*)+**

Find in exact simplified form an expression for

$$\int \frac{3x}{x - \sqrt{x^2 - 1}} dx.$$

, $x^3 + (x^2 - 1)^{\frac{3}{2}} + C$

ALTERNATIVE BY HYPERBOLIC SUBSTITUTION

$$\begin{aligned} \int \frac{3x}{x - \sqrt{x^2 - 1}} dx &= \int \frac{3x(x + \sqrt{x^2 - 1})}{(x - \sqrt{x^2 - 1})(x + \sqrt{x^2 - 1})} dx \\ &= \int \frac{3x^2 + 3x(x^2 - 1)^{\frac{1}{2}}}{x^2 - (x^2 - 1)} dx = \int \frac{3x^2 + 3x(x^2 - 1)^{\frac{1}{2}}}{1} dx \\ &= \int 3x^2 + 3x(x^2 - 1)^{\frac{1}{2}} dx = x^3 + (x^2 - 1)^{\frac{3}{2}} + C \quad \text{By Integration} \end{aligned}$$

ALTERNATIVE BY TRIGONOMETRIC SUBSTITUTION

$$\begin{aligned} \int \frac{3x}{x - \sqrt{x^2 - 1}} dx &= \int \frac{3\sec\theta (\sec\theta + \tan\theta) d\theta}{\sec\theta - \tan\theta} \quad \begin{array}{l} \boxed{d\theta = \sec^2\theta d\theta} \\ \boxed{\sec\theta = \sec\theta + \tan\theta d\theta} \end{array} \\ &= \int \frac{3\sec^2\theta \tan\theta}{\sec\theta - \tan\theta} d\theta = \int \frac{3\sec^2\theta \tan\theta (\sec\theta + \tan\theta)}{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)} d\theta \\ &= \int \frac{3\sec^2\theta \tan\theta + 3\sec^2\theta \tan^2\theta}{\sec\theta - \tan\theta} d\theta = \int \frac{3\sec^2\theta \tan\theta + 3\sec^2\theta \tan^2\theta}{\sec\theta - \tan\theta} d\theta \\ &\quad \boxed{(\sec^2\theta - \tan^2\theta = 1)} \\ &\text{Now by Integration: } \begin{aligned} \frac{d}{d\theta}(\sec\theta) &= 3\sec^2\theta (\sec\theta + \tan\theta) = 3\sec^2\theta \tan\theta \\ \frac{d}{d\theta}(\tan\theta) &= 3\sec^2\theta \times \sec\theta = 3\sec^2\theta \tan\theta \end{aligned} \end{aligned}$$

$$\begin{aligned} \dots &= \sec^2\theta + \tan^2\theta + C = \sec^2\theta [\sqrt{\sec^2\theta - 1}]^2 + C \\ &= 3x^3 + (x^2 - 1)^{\frac{3}{2}} + C \quad \text{By Integration} \end{aligned}$$

ALTERNATIVE BY HYPERBOLIC SUBSTITUTION

$$\begin{aligned} \int \frac{3x}{x - \sqrt{x^2 - 1}} dx &= \int \frac{3\cosh\theta (\sinh\theta d\theta)}{\cosh\theta - \sqrt{\cosh^2\theta - 1}} \\ &= \int \frac{3\cosh\theta \sinh\theta}{\cosh\theta - \sinh\theta} d\theta \\ &= \int \frac{3\cosh\theta \sinh\theta (\cosh\theta + \sinh\theta)}{(\cosh\theta - \sinh\theta)(\cosh\theta + \sinh\theta)} d\theta \\ &= \int \frac{3\cosh\theta \sinh\theta + 3\cosh^2\theta \sinh\theta}{\cosh\theta - \sinh\theta} d\theta \\ &\quad \boxed{(\cosh^2\theta + \sinh^2\theta = 1)} \\ &= \int 3\cosh\theta \sinh\theta + 3\cosh^2\theta \sinh\theta d\theta \\ &\quad \text{... By Integration ...} \\ &= \cosh^2\theta + \sinh^2\theta + C \\ &= \cosh^2\theta + (\sqrt{\cosh^2\theta - 1})^2 + C \\ &= x^3 + (x^2 - 1)^{\frac{3}{2}} + C \end{aligned}$$

Question 214 (*)+**

Use the substitution $u = \sin x + x \tan x$ to find an expression for

$$\int \frac{2x + \sin 2x + 2\cos^3 x}{(x + \cos x)\sin 2x} dx.$$

, $\ln|\sin x + x \tan x| + C$

Using the substitution given

$$u = \sin x + x \tan x$$

$$\frac{du}{dx} = \cos x + \tan x + x \sec^2 x$$

$$du = (\cos x + \tan x + x \sec^2 x) dx$$

To evaluate the given integral

$$\int \frac{2x + \sin 2x + 2\cos^3 x}{(x + \cos x)\sin 2x} dx$$

$$= \int \frac{2x + 2\sin x \cos x + 2\cos^3 x}{(x + \cos x)(\sin x + x \tan x)} \times \frac{1}{\cos x + \tan x + x \sec^2 x} du$$

$$= \int \frac{2 + 2\sin x \cos x + \cos^2 x}{(\sin x + x \tan x) \sin x \cos x} \times \frac{1}{\cos x + \frac{\sin x}{\cos x} + \frac{x}{\cos^2 x}} du$$

$$= \int \frac{2 + 2\sin x \cos x + \cos^2 x}{(\sin x + x \tan x) \sin x \cos x} \times \frac{\cos^2 x}{\cos^2 x + \sin x \cos x} du$$

$$= \int \frac{\cos^2 x}{x^2 \sin^2 x + \sin x \cos x} du$$

$$= \int \frac{1}{x^2 \sin^2 x + \sin x \cos x} du$$

$$= \int \frac{1}{u} du$$

$$= \ln|u| + C$$

$$= \ln|\sin x + x \tan x| + C$$

Question 215 (***)+

Find, in exact simplified form, the value of

$$\int_{\ln(\ln 2)}^{\ln[2\ln(e+1)]} e^{x+e^x} dx.$$

, $e^2 + 2e - 1$

Start by rewriting the integral using the rules of indices.
 $\int e^{x+e^x} dx = \int e^{x+\ln(u)} dx$
 $= \int u \ln(u) e^x dx$

By substitution (or inspection),
 $u = e^x \quad \text{units reduction to}$
 $\frac{du}{dx} = e^x, \quad u = e^{\ln(ux)}, \quad du = e^{\ln(ux)} \cdot \ln(ux)^2$

Translating the integral yields
 $= \int \frac{(ue^x)^2}{u^2} \cdot u \cdot e^x \left(\frac{du}{e^x} \right) = \int \frac{ue^{x+2x}}{e^x} du$
 $= \left[ue^{3x} \right]_{\ln 2} = e^{3\ln(2)} - e^{\ln 2}$
 $= (e+1)^2 - 2$
 $= e^2 + 2e - 2$
 $= e^2 + 2e - 1$

Question 216 (*****)

By using the substitution $u = 1 + \cos^4 x$, or otherwise, find the exact value of

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4 \cot^3 x}{1 + 2 \cot^2 x + 2 \cot^4 x} dx.$$

, $\ln\left(\frac{5}{4}\right)$

(USING THE SUBSTITUTION) Given

$$u = 1 + \cos^4 x \quad \begin{cases} x = \frac{\pi}{4} \rightarrow u = \frac{5}{4} \\ x = \frac{\pi}{2} \rightarrow u = 1 \end{cases}$$

$$\frac{du}{dx} = -4 \cos^2 x \sin x \quad \begin{cases} u = 1 + \cos^4 x \\ u = 1 + \cos^2 x \cdot \cos^2 x \end{cases}$$

TRANSFORMING THE INTEGRAL

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4 \cot^3 x}{1 + 2 \cot^2 x + 2 \cot^4 x} dx = \int_1^{\frac{5}{4}} \frac{-4 \cot^3 x}{1 + 2 \cot^2 x + 2 \cot^4 x} \left(-\frac{du}{4 \cos^2 x \sin x} \right)$$

$$= \int_1^{\frac{5}{4}} \frac{4 \cot^3 x}{1 + 2 \cot^2 x + 2 \cot^4 x} \times \frac{du}{4 \cos^2 x \sin x}$$

$$= \int_1^{\frac{5}{4}} \frac{\frac{\cot^3 x}{\sin^2 x}}{1 + \frac{2 \cot^2 x}{\sin^2 x} + \frac{2 \cot^4 x}{\sin^2 x}} \times \frac{1}{\cot^2 x \sin x} du$$

MULTIPLY TOP & BOTTOM OF THIS FRACTION BY $\sin^2 x$

$$= \int_1^{\frac{5}{4}} \frac{\frac{\cot^3 x}{\sin^2 x}}{\frac{\sin^2 x + 2 \cot^2 x + 2 \cot^4 x}{\sin^2 x}} \times \frac{1}{\cot^2 x \sin x} du$$

$$= \int_1^{\frac{5}{4}} \frac{1}{\sin^2 x + 2 \cot^2 x + 2 \cot^4 x} du$$

$$= \int_1^{\frac{5}{4}} \frac{1}{(1 - \cos^2 x)^2 + 2 \cos^2 x (1 - \cos^2 x) + 2 \cos^4 x} du$$

$$= \int_1^{\frac{5}{4}} \frac{1}{1 - 2 \cos^2 x + \cos^2 x + 2 \cos^2 x - 2 \cos^4 x + 2 \cos^4 x} du$$

$$= \int_1^{\frac{5}{4}} \frac{1}{1 + \cos^2 x} du \quad \text{SINCE } u = 1 + \cos^2 x$$

$$= \int_1^{\frac{5}{4}} \frac{1}{u} du$$

$$= \left[\ln|u| \right]_1^{\frac{5}{4}}$$

$$= \ln\frac{5}{4} - \ln 1$$

$$= \ln\frac{5}{4}$$

Question 217 (*****)

$$\frac{2}{u(u-2)} = \frac{A}{u-2} + \frac{B}{u}.$$

- a) Find the value of each of the constants A and B .
- b) By using the substitution $u = 1 + \cos^2 x$, or otherwise, show clearly that

$$\int \frac{4 \cot x}{1 + \cos^2 x} dx = -\ln(\operatorname{cosec}^2 x + \cot^2 x) + C.$$

A=1, B=1

(a) $\frac{2}{u(u-2)} \equiv \frac{A}{u-2} + \frac{B}{u}$ { * If $u=0$, $2=-2B \Rightarrow B=1$
 * If $u=2$, $2=2A \Rightarrow A=1$

(b) $\int \frac{4 \cot x}{1 + \cos^2 x} dx = \text{using the substitution given}$

$$\begin{aligned} &= \int \frac{4 \cot x}{u} \times \frac{du}{-2 \cos^2 x} \\ &= \int -\frac{2(\cot x)}{u} \times \frac{1}{\cos^2 x} du \\ &= \int -\frac{2}{u} \frac{\cot x}{\sin u} du = \int -\frac{2}{u} \times \frac{1}{\sin^2 u} du \\ &= \int -\frac{2}{u} \frac{1}{1-\cos^2 u} du = \int -\frac{2}{u} \times \frac{1}{1-(u-1)} du \\ &= \int -\frac{2}{u(2-u)} du = \int -\frac{2}{u(2-u)} du = \dots \text{part a...} \\ &= \int \frac{1}{u-2} - \frac{1}{u} du = \ln|u-2| - \ln|u| = \ln|\frac{u-2}{u}| + C \\ &= \ln\left|\frac{1+\cos^2 x-2}{1+\cos^2 x}\right| + C = \ln\left|\frac{\cos^2 x-1}{1+\cos^2 x}\right| + C = \ln\left(\frac{1-\cos^2 x}{1+\cos^2 x}\right) + C \\ &= \ln\left(\frac{\sin^2 x}{1+\cos^2 x}\right) + C = -\ln\left(\frac{1+\cos^2 x}{\sin^2 x}\right) + C \\ &= -\ln\left(\frac{1+\cos^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right) + C = -\ln(\operatorname{cosec}^2 x + \cot^2 x) + C \end{aligned}$$

As required

Question 218 (*****)

By using the substitution $\tan \theta = \sqrt{x^3 - 1}$, or otherwise, find an exact value for the following integral.

$$\int_1^{\sqrt[3]{2}} \frac{\sqrt{x^3 - 1}}{\frac{1}{6}x} dx.$$

, 4π

By substitution
 $\tan \theta = \sqrt{x^3 - 1}$
 $\tan^2 \theta = x^3 - 1$
 $x^3 = 1 + \tan^2 \theta$
 $x^3 = \sec^2 \theta$
 $3x^2 dx = 2\sec^2 \theta \tan \theta d\theta$
 $dx = \frac{2\sec^2 \theta \tan \theta}{3x^2} d\theta$
 $-----$
 $x=1 \rightarrow \theta=0$
 $x=\sqrt[3]{2} \rightarrow \theta=\frac{\pi}{4}$

$$\begin{aligned}
 \int_1^{\sqrt[3]{2}} \frac{\sqrt{x^3 - 1}}{\frac{1}{6}x} dx &= \dots \\
 &= \int_0^{\frac{\pi}{4}} \frac{4\tan \theta}{\frac{1}{6}\sec^2 \theta} \frac{2\sec^2 \theta \tan \theta}{3\sec^2 \theta} d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{24\tan^2 \theta}{3} d\theta \\
 &= \int_0^{\frac{\pi}{4}} 8\tan^2 \theta d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{4\sec^2 \theta - 4\sec^2 \theta \tan^2 \theta}{1 + \tan^2 \theta} d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{4\sec^2 \theta \tan^2 \theta}{\sec^2 \theta} d\theta \\
 &= \int_0^{\frac{\pi}{4}} 4\tan^2 \theta d\theta \\
 &= \int_0^{\frac{\pi}{4}} 4(\sec^2 \theta - 1) d\theta \\
 &= \left[4(\tan \theta - \theta) \right]_0^{\frac{\pi}{4}} \\
 &= 4 \left(1 - \frac{\pi}{4} \right) - 0 \\
 &= 4 - \pi
 \end{aligned}$$

Question 219 (*****)

Use appropriate integration methods to find an exact simplified value for

$$\int_0^{\frac{1}{2}} \cos(5 \arcsin x) dx.$$

V, **□**, $\frac{\sqrt{3}}{16}$

PROCEED BY A SUBSTITUTION

$$\begin{aligned} \theta &= \arcsin x & \text{d}\theta &= x \quad \text{d}x \quad x=0 \implies \theta=0 \\ \sin\theta &= x & \text{d}x &= \cos\theta \text{d}\theta \quad x=\frac{1}{2} \implies \theta=\frac{\pi}{6} \end{aligned}$$

TRANSFORM THE INTEGRAL

$$\int_0^{\frac{1}{2}} \cos(5 \arcsin x) dx = \int_0^{\frac{\pi}{6}} (\cos 5\theta)(\cos\theta d\theta) = \int_0^{\frac{\pi}{6}} \cos 5\theta \cos\theta d\theta$$

USING SUM TWO IDENTITIES (OR EXPAND WITH DOUBLE INTEGRATION BY PARTS)

$$\begin{aligned} \{\cos(5\theta)\} &\equiv \{\cos 5\theta - \sin 5\theta\} \quad \text{Hence} \\ \{\cos(5\theta)\} &= \{\cos 5\theta + \sin 5\theta\} \\ \rightarrow \cos 5\theta + \sin 5\theta &= 2\cos 5\theta \cos\theta \\ \rightarrow \cos 5\theta \cos\theta &= \frac{1}{2}(\cos 5\theta + \sin 5\theta) \end{aligned}$$

RETURNING TO THE INITIAL

$$\begin{aligned} &= \int_0^{\frac{\pi}{6}} \frac{1}{2}(\cos 5\theta + \sin 5\theta) d\theta = \left[\frac{1}{2}(\sin 5\theta + \cos 5\theta) \right]_0^{\frac{\pi}{6}} \\ &= \left(0 + \frac{\sqrt{3}}{16} \right) - (0+0) \\ &= \frac{\sqrt{3}}{16} \end{aligned}$$

Question 220 (*****)

By using the substitution $e^x = \frac{1}{u}$, or otherwise, show clearly that

$$\int \frac{9}{e^x \sqrt{e^{2x}-9}} dx = \frac{\sqrt{e^{2x}-9}}{e^x} + C.$$

proof

$$\begin{aligned}
 & \int \frac{9}{e^x \sqrt{e^{2x}-9}} dx = \dots \text{by substitution} \\
 &= \int \frac{9}{\frac{1}{u} \sqrt{\frac{1}{u^2}-9}} \left(-\frac{du}{u^2} \right) = \int \frac{9u'}{\sqrt{\frac{1}{u^2}-9}} \left(-\frac{du}{u^2} \right) \\
 &= \int \frac{-9}{\sqrt{1-9u^2}} du = \int \frac{-9}{\sqrt{1-9u^2}} \frac{du}{u} \\
 &= \int \frac{-9u}{\sqrt{1-9u^2}} du = \int -9u(1-9u^2)^{-\frac{1}{2}} du \\
 &\quad \text{BY PARTIAL FRACTION OR ALTERNATE SUBSTITUTION} \\
 &= (1-9u^2)^{\frac{1}{2}} + C \\
 &= (1-\frac{u^2}{e^2})^{\frac{1}{2}} + C = \sqrt{\frac{e^2}{e^2-u^2}} + C \\
 &= \frac{\sqrt{e^2-9}}{e^2} + C \quad \text{AS REQUIRED}
 \end{aligned}$$

Question 221 (*****)

By using a reciprocal substitution, or otherwise, find the value of the following integral.

$$\int_1^2 \frac{x^2-1}{x^3 \sqrt{2x^4-2x^2+1}} dx.$$

□, $\frac{1}{8}$

$$\begin{aligned}
 & \int_1^2 \frac{x^2-1}{x^3 \sqrt{2x^4-2x^2+1}} dx \dots \text{SUBSTITUTION} \\
 &= \int_1^2 \frac{\frac{1}{u^2}-1}{\frac{1}{u^3} \sqrt{\frac{1}{u^4}+\frac{2}{u^2}+1}} \left(\frac{1}{u^2} du \right) \\
 &= \int_1^2 \frac{\frac{1}{u^2}-1}{\frac{1}{u^2} \sqrt{\frac{1+2u^2}{u^4}}} du \\
 &= \int_1^2 \frac{\frac{1}{u^2}-1}{\frac{1}{u^2} \sqrt{u^2+2u^2+1}} du \\
 &\quad \text{MULTIPLY TOP & BOTTOM OF THE INTEGRAND BY } u^5 \\
 &= \int_1^2 \frac{u-u^3}{\sqrt{u^2-2u^2+2}} du \\
 &\quad \text{NOTICE THAT} \\
 &\quad \frac{d}{du} (u^4-2u^2+2) = 4u^3-4u = 4(u-u^2)
 \end{aligned}$$

$$\begin{aligned}
 & \text{EVALUATE AS} \\
 &= \int_1^2 (u-u^3) (u^4-2u^2+2)^{\frac{1}{2}} du \\
 &= -\frac{1}{4} \int_1^2 (u-u^3) (u^4-2u^2+2)^{\frac{1}{2}} du \\
 &\quad \text{BY EVALUATION} \\
 &= -\frac{1}{4} \left[2(u^4-2u^2+2)^{\frac{1}{2}} \right]_1^2 \\
 &\quad - \frac{1}{2} \left[(u^4-2u^2+2)^{\frac{1}{2}} \right]_1^2 \\
 &= \frac{1}{2} \sqrt{\frac{35}{16}} - \frac{1}{2} \sqrt{1} \\
 &= \frac{5}{8} - \frac{1}{2} \\
 &= \frac{1}{8}
 \end{aligned}$$

Question 222 (***** non calculator

Show clearly that

$$\int_0^{\frac{\pi}{2} - \arctan \frac{12}{5}} 5 \cos x - 12 \sin x \, dx = 1.$$

proof

$$\begin{aligned} \int_0^{\frac{\pi}{2} - \arctan \frac{12}{5}} 5 \cos x - 12 \sin x \, dx &= \left[5 \sin x + 12 \cos x \right]_0^{\frac{\pi}{2} - \arctan \frac{12}{5}} \\ &= \left[5 \sin \left(\frac{\pi}{2} - \arctan \frac{12}{5} \right) + 12 \cos \left(\frac{\pi}{2} - \arctan \frac{12}{5} \right) \right] - \left[0 + 12 \right] \\ &= \frac{5 \sin \pi}{2} \cos x - \frac{5 \cos \pi}{2} \sin x + 12 \cos \frac{\pi}{2} \cos x + 12 \sin \frac{\pi}{2} \sin x - 12 \\ &= 5 \sin \frac{\pi}{2} \cos x + 12 \sin \frac{\pi}{2} \sin x - 12 \\ &= 5 \times 1 \times \frac{5}{13} + 12 \times 1 \times \frac{12}{13} - 12 \\ &= \frac{25}{13} + \frac{144}{13} - 12 \\ &\approx \frac{169}{13} - 12 \\ &\approx 13 - 12 \\ &\approx 1 \end{aligned}$$

$\alpha = \arctan \frac{12}{5}$
 $\tan \alpha = \frac{12}{5}$
 $\sin \alpha = \frac{12}{13}$
 $\cos \alpha = \frac{5}{13}$
 $\alpha = \arcsin \frac{12}{13}$
 $\alpha = \arccos \frac{5}{13}$

Question 223 (*****)

Use trigonometric identities to find

$$\int \frac{1}{\cos x \sin^2 x} \, dx.$$

$\boxed{\ln |\sec x + \tan x| - \operatorname{cosec} x + C}$

$$\begin{aligned} \int \frac{1}{\cos x \sin^2 x} \, dx &= \int \frac{\sec^2 x}{\sec x} \, dx = \int \frac{1 + \cot^2 x}{\sec x} \, dx \\ &= \int \frac{1}{\sec x} + \frac{\cot^2 x}{\sec x} \, dx = \int \sec x + \frac{\csc^2 x}{\sec x} \, dx \\ &= \int \sec x + \frac{\csc^2 x}{\sec x} \, dx = \int \sec x + \csc x \operatorname{cosec} x \, dx \\ &= \ln |\sec x + \tan x| - \operatorname{cosec} x + C \end{aligned}$$

↑ STANDARD INTEGRALS

ALTERNATIVE MANIPULATION FOR THE INTEGRAND

$$\frac{1}{\operatorname{cosec} x} = \frac{\cos x + \sin x}{\cos x \sin x} = \frac{\cos^2 x + \sin^2 x}{\cos x \sin x} = \frac{1}{\sin x} + \frac{\cos x}{\sin x} = \dots \text{ with } \sec x + \tan x \dots \text{ etc}$$

Question 224 (*****)

$$I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cot^3 x}{\operatorname{cosec} x} dx.$$

Use appropriate integration techniques to show that

$$I = \frac{1}{6} [a + b\sqrt{3}],$$

where a and b are integers to be found.

, $I = \frac{1}{6} [15 - 7\sqrt{3}]$

FIRSTLY: WRITE THE INTEGRAND IN TERMS OF SINES AND COSECANTS

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cot^3 x}{\operatorname{cosec} x} dx = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\frac{\cos^2 x}{\sin^2 x} \times \sin x}{\operatorname{cosec} x} dx = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos^2 x}{\sin^3 x} dx$$

BY SUBSTITUTION

$$\begin{aligned} u &= \sin x \\ \frac{du}{dx} &= \cos x \\ du &= \cos x dx \\ 2\pi \frac{du}{2} &\rightarrow u = \frac{1}{2} \\ 2\pi \frac{du}{3} &\rightarrow u = \frac{2\pi}{3} \end{aligned}$$

WE NOW HAVE

$$\begin{aligned} \dots &= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos^2 x}{u^3} \frac{du}{\cos x} &= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos^2 x}{u^3} du &= \int_{\frac{1}{2}}^{\frac{2\pi}{3}} \frac{1 - \cos^2 u}{u^3} du \\ &= \int_{\frac{1}{2}}^{\frac{2\pi}{3}} \frac{1 - u^2}{u^3} du &= \int_{\frac{1}{2}}^{\frac{2\pi}{3}} \frac{1}{u^3} - 1 du &= \left[-\frac{1}{2u^2} - u \right]_{\frac{1}{2}}^{\frac{2\pi}{3}} \\ &= \left[\frac{1}{u} + u \right]_{\frac{1}{2}}^{\frac{2\pi}{3}} &= \left[\frac{1+u^2}{u} \right]_{\frac{1}{2}}^{\frac{2\pi}{3}} \\ &= \frac{1 + \frac{4}{9}}{\frac{2\pi}{3}} - \frac{1 + \frac{1}{4}}{\frac{1}{2}} &= \frac{4+1}{2} - \frac{4+3}{2\sqrt{3}} \\ &= \frac{5}{2\pi} - \frac{7}{2\sqrt{3}} &= \frac{5}{2} - \frac{7\sqrt{3}}{6} \\ &= \frac{1}{6} [15 - 7\sqrt{3}] \end{aligned}$$

Question 225 (*****)

$$I = \int_1^a \frac{1}{\left(x^{\frac{4}{3}} + 7x\right)^{\frac{2}{3}}} dx.$$

Given that $I = 9$, determine the value a .

, $a = 8000$

IGNORING THE UNITS & THE EQUATION, WE HAVE

$$\int \frac{1}{(x^{\frac{4}{3}} + 7x)^{\frac{2}{3}}} dx = \int \frac{1}{(2x(x^{\frac{1}{3}} + 7))^{\frac{2}{3}}} dx$$

$$= \int \frac{1}{2x(x^{\frac{1}{3}} + 7)^{\frac{2}{3}}} dx = \int x^{-\frac{1}{3}}(x^{\frac{1}{3}} + 7)^{-\frac{2}{3}} dx$$

BY RECOGNITION AS $\frac{d}{dx}(x^{\frac{1}{3}}) = \frac{1}{3}x^{-\frac{2}{3}}$ WE OBTAIN

$$\cdot \frac{d}{dx} \left[(2x(x^{\frac{1}{3}} + 7)^{\frac{1}{3}}) \right] = \frac{1}{3}(x^{\frac{1}{3}} + 7)^{-\frac{2}{3}} \times \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{9}x^{-\frac{2}{3}}(x^{\frac{1}{3}} + 7)^{-2}$$

HENCE WE HAVE

$$\int \frac{1}{(x^{\frac{4}{3}} + 7)^{\frac{2}{3}}} dx = 9(x^{\frac{1}{3}} + 7)^{\frac{1}{3}}$$

FINALLY THE UNITS & EQUATION

$$\Rightarrow \int_1^a \frac{1}{(x^{\frac{4}{3}} + 7)^{\frac{2}{3}}} dx = 9$$

$$\Rightarrow \left[9(x^{\frac{1}{3}} + 7)^{\frac{1}{3}} \right]_1^a = 9$$

$$\Rightarrow (a^{\frac{1}{3}} + 7)^{\frac{1}{3}} - 8^{\frac{1}{3}} = 1$$

$$\Rightarrow (a^{\frac{1}{3}} + 7)^{\frac{1}{3}} = 9$$

$$\Rightarrow a^{\frac{1}{3}} + 7 = 27$$

$$\Rightarrow a^{\frac{1}{3}} = 20$$

$$\Rightarrow a = 20^3 = 8000$$

Question 226 (*****)

By using a suitable trigonometric substitution, show clearly that

$$\int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}} dx = \pi - 2.$$

proof

$\int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}} dx =$ by substitution

$$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{16\sin^2\theta}{1-\sin^2\theta}} (2\sin\theta\cos\theta d\theta)$$

$$= \int_0^{\frac{\pi}{2}} \frac{16\sin^2\theta}{\cos^2\theta} (2\sin\theta\cos\theta d\theta)$$

$$= \int_0^{\frac{\pi}{2}} \frac{4\sin^2\theta}{\cos^2\theta} (2\sin\theta\cos\theta d\theta) = \int_0^{\frac{\pi}{2}} 2\sin^2\theta d\theta = \int_0^{\frac{\pi}{2}} 2\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4 - 4\cos 2\theta d\theta = \left[4\theta - 2\sin 2\theta \right]_0^{\frac{\pi}{2}} = (\pi - 2) - (0 - 0) = \pi - 2.$$

$x = \sin^2\theta$
$\frac{dx}{d\theta} = 2\sin\theta\cos\theta$
$d\theta = \frac{1}{2\sin\theta\cos\theta} dx$
$2x = 2\sin^2\theta$
$x = \frac{1}{2}$
$\theta = \frac{\pi}{2}$

Question 227 (*****)

Find an exact simplified value for

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x \sin 2x} dx.$$

$$\boxed{\frac{1}{2} \ln \left[\frac{2+\sqrt{3}}{\sqrt{3}} \right] + 1 - \frac{\sqrt{3}}{3}}$$

$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x \sin 2x} dx &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x + \sin x}{2 \sin x \cos x} dx \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x \cos x} + \sec x dx = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x} + \sec x dx \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc x + \sec x dx = \frac{1}{2} \left[-\ln |\csc x + \cot x| \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \frac{1}{2} \left[\left(\ln(2+\sqrt{3}) - \frac{2}{\sqrt{3}} \right) - \left(-2 + \ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) \right) \right] \\ &= \frac{1}{2} \left[\ln(2+\sqrt{3}) - \frac{2}{\sqrt{3}} + 2 - \ln\sqrt{3} \right] \\ &= \frac{1}{2} \left[\ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right) + 2\left(1 - \frac{1}{\sqrt{3}}\right) \right] \\ &= \frac{1}{2} \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right) + 1 - \frac{\sqrt{3}}{3} \end{aligned}$$

Question 228 (*****)

Evaluate the following definite integral.

$$\int_0^1 e^{\arccos x} dx.$$

Give the answer in exact simplified form.

, proof

Start with A substitution:

$$\int_0^1 e^{\arccos x} dx = \int_{\frac{\pi}{2}}^0 e^{\theta} (-\sin \theta) d\theta$$

$$= \left[-e^{\theta} \sin \theta \right]_0^{\frac{\pi}{2}}$$

Note by complex numbers - or by double integration by parts - or integration by substitution:

$$\frac{d}{ds} \left[e^s (A \cos s + B \sin s) \right] \equiv e^s (A \cos s + B \sin s) + e^s (A \cos s - B \sin s) = 2A \cos s$$

$$(A-B) \cos s + (A+B) \sin s \equiv \sin s$$

$$A = \frac{1}{2}, B = -\frac{1}{2}$$

This we have

$$= \left[e^s \left(\frac{1}{2} \cos s - \frac{1}{2} \sin s \right) \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left[e^{\frac{\pi}{2}} (1 - 0) - e^0 (0 - 1) \right]$$

$$= \frac{1}{2} [e^{\frac{\pi}{2}} + 1]$$

Question 229 (*****)

Use a suitable trigonometric substitution to find the exact value of

$$\int_{-1}^5 \sqrt{(1+x)(5-x)} \, dx.$$

$$\boxed{\frac{9\pi}{2}}$$

$$\begin{aligned}
 \int_{-1}^5 \sqrt{(1+x)(5-x)} \, dx &= \int_{-1}^5 \sqrt{5-x+1-x^2} \, dx = \int_{-1}^5 \sqrt{5+2x-x^2} \, dx \\
 &= \int_{-1}^5 \sqrt{-[x^2-2x-5]} \, dx = \int_{-1}^5 \sqrt{-(x-2)^2-9} \, dx = \int_{-1}^5 \sqrt{9-(x-2)^2} \, dx \\
 &\text{(Now by trigonometric substitution)} \\
 &\quad x-2 = 3\sin\theta \\
 &\quad x = 2 + 3\sin\theta \\
 &\quad dx = 3\cos\theta \, d\theta \\
 &\quad \text{I}=1 \Rightarrow \sin\theta=-1 \Rightarrow \theta = -\frac{\pi}{2} \\
 &\quad x=5 \Rightarrow 3\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{6} \\
 \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \sqrt{9-9\sin^2\theta} (3\cos\theta \, d\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \sqrt{9(1-\sin^2\theta)} (3\cos\theta \, d\theta) \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \sqrt{9\cos^2\theta} (3\cos\theta \, d\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} 3(\cos\theta)(3\cos\theta) \, d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} 9\cos^2\theta \, d\theta = \int_0^{\frac{\pi}{6}} 18\cos^2\theta \, d\theta = \int_0^{\frac{\pi}{6}} 18\left(\frac{1}{2}(1+\cos 2\theta)\right) \, d\theta \\
 &\quad (\text{Easier integration}) \\
 &= \int_0^{\frac{\pi}{6}} 9+9\cos 2\theta \, d\theta = \left[9\theta + \frac{9}{2}\sin 2\theta \right]_0^{\frac{\pi}{6}} \\
 &= \left(\frac{9\pi}{2} + 0 \right) - (0+0) = \frac{9\pi}{2}
 \end{aligned}$$

Question 230 (*****)

By using trigonometric identities, show that

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \frac{1}{8}(16 - 3\pi).$$

proof

PROCEED BY SPLITTING THE FRACTION

$$\begin{aligned} & \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x}{\sin^2 x \cos^2 x} + \frac{\cos^6 x}{\sin^2 x \cos^2 x} dx \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos^2 x} + \frac{\cos^4 x}{\sin^2 x} dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{(\sin^2 x)^2 + (\cos^2 x)^2}{\sin^2 x \cos^2 x} dx \\ & \text{EXPANDING & SPLIT THE FRACTION TERM} \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1 - 2\sin^2 x + \cos^2 x}{\cos^2 x} + \frac{1 - 2\sin^2 x + \sin^2 x}{\sin^2 x} dx \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x - 2 + \csc^2 x + \csc^2 x - 2 + \sec^2 x dx \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x + \csc^2 x + (\csc^2 x + \sec^2 x) - 4 dx \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x + \csc^2 x - 3 dx \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \tan x - \cot x - 3 dx \end{aligned}$$

Tidy before integrating

$$\begin{aligned} &= \left[-\tan x - \frac{1}{\tan x} - 3x \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} \\ &= \left[-\tan^2 x - 1 - 3x \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} \\ &= \left[3x + \frac{1 - \tan^2 x}{\tan x} \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} \\ &= \left[3x + 2\left(\frac{1 - \tan^2 x}{2\tan x}\right) \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} \\ &= \left[3x + 2x + 2x - 2 \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} \\ &= \left[3x + \frac{2}{\tan x} \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} \\ &= \left(\frac{3\pi}{8} + \frac{2}{\tan \frac{\pi}{4}} \right) - \left(\frac{3\pi}{8} + \frac{2}{\tan \frac{\pi}{8}} \right) \\ &= -\frac{3\pi}{8} + 2 \\ &= \frac{1}{8}(16 - 3\pi) // \end{aligned}$$

Question 231 (*****)

Find, in exact simplified form, the value of the following integral.

$$\int_0^{\frac{1}{2}\pi} \sqrt{1+4\cos^2 2x - 4\cos 2x} \ dx.$$

$$\boxed{\quad}, \quad \boxed{\frac{1}{6}\pi + \sqrt{3}}$$

MANIPULATING AS FOLLOWS

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \sqrt{1+4\cos^2 2x - 4\cos 2x} \ dx = \int_0^{\frac{\pi}{2}} \sqrt{(2\cos 2x - 1)^2} \ dx \\ &= \int_0^{\frac{\pi}{2}} |2\cos 2x - 1| \ dx \end{aligned}$$

NOW THE CRITICAL VALUE(S) FOR THE INTEGRATION

$$\begin{aligned} 2\cos 2x - 1 &= 0 \\ \cos 2x &= \frac{1}{2} \\ 2x &= \pm \frac{\pi}{3}, \pm \frac{5\pi}{3}, \dots \\ x &= \pm \frac{\pi}{6}, \pm \frac{5\pi}{6}, \dots \end{aligned}$$

$x = \frac{\pi}{6}$ IS A CRITICAL VALUE FOR THE INTEGRATION INTERVAL

$$\begin{aligned} 2\cos 2x - 1 &> 0 \quad 0 < x < \frac{\pi}{6} \\ 2\cos 2x - 1 &< 0 \quad \frac{\pi}{6} < x < \frac{\pi}{2} \end{aligned}$$

RETURNING TO THE INITIAL

$$\begin{aligned} & \dots = \int_0^{\frac{\pi}{6}} 2\cos 2x \ dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} |1 - 2\cos 2x| \ dx \\ &= [\sin 2x]_0^{\frac{\pi}{6}} + [x - \sin 2x]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) - (0) + \left(\frac{\pi}{6} - 0\right) - \left(\frac{\pi}{2} - \frac{\sqrt{3}}{2}\right) \\ &= -\frac{\sqrt{3}}{2} \times 2 + \frac{\pi}{6} - \frac{\pi}{3} = \boxed{\sqrt{3} + \frac{\pi}{6}} \end{aligned}$$

Question 232 (*****)

By using a suitable cosine double angle trigonometric identity find

$$\int \frac{6}{(1+\cos x)^2} dx.$$

$$\boxed{3\tan \frac{x}{2} + \tan^3 \frac{x}{2} + C}$$

$$\begin{aligned} \int \frac{6}{(1+\cos x)^2} dx &= \int \frac{6}{(1+2\cos^2 \frac{x}{2} - 1)^2} dx = \int \frac{6}{4\cos^2 \frac{x}{2}} dx \quad \text{cos}^2 \theta = 2\cos^2 \frac{\theta}{2} - 1 \\ &= \int \frac{3}{2} \sec^2 \frac{x}{2} dx = \int \frac{3}{2} \cdot 2 \sec^2 \frac{x}{2} \tan \frac{x}{2} dx = \int \frac{3}{2} \sec^2 \frac{x}{2} (1 + \tan^2 \frac{x}{2}) dx \\ &= \int \frac{3}{2} \sec^2 \frac{x}{2} + \frac{3}{2} \sec^2 \frac{x}{2} \tan^2 \frac{x}{2} dx = \dots \text{by recognition} \\ &= \boxed{3 \tan \frac{x}{2} + \tan^3 \frac{x}{2} + C} \end{aligned}$$

Question 233 (*****)

By expressing the integrand in the form $\sec^2 x f(\tan x)$, or otherwise, find a simplified expression for the following integral.

$$\int \frac{3\sin^2 x \cos^2 x}{(\cos^3 x - \sin^3 x)^2} dx.$$

, $\frac{1}{1-\tan^3 x} + C$

PROCEED AS FOLLOWS

$$\begin{aligned} \int \frac{3\sin^2 x \cos^2 x}{(\cos^3 x - \sin^3 x)^2} dx &= \int 3 \left(\frac{\sin x \cos x}{\cos^2 x - \sin^2 x} \right)^2 dx \\ &= \int 3 \left[\frac{\sin x \cos x}{\cos^2 x - \frac{\sin^2 x}{\cos^2 x}} \right]^2 dx = \int \frac{3 (\tan x \sec x)^2}{(1 - \tan^2 x)^2} dx \\ &= \int 3 \tan^2 x \sec^2 x (1 - \tan^2 x)^{-2} dx \end{aligned}$$

BY REVERSE CHAIN RULE OR CORDON

$$\dots = (1 - \tan^2 x)^{-1} + C = \frac{1}{1 - \tan^2 x} + C$$

ALTERNATIVE BY SUBSTITUTION $u = \tan x$ or $u = 1 - \tan^2 x$

$$\begin{aligned} \dots &= \int 3 \tan^2 x \sec^2 x (1 - \tan^2 x)^{-2} dx \\ &= \int 3 u^2 \sec^2 u^{-2} \left(\frac{du}{3 \sec^2 u} \right) \\ &= \int u^{-2} du \\ &= u^{-1} + C \\ &= \frac{1}{1 - u} + C \quad \text{AS BEFORE} \end{aligned}$$

$u = 1 - \tan^2 x$
 $\frac{du}{dx} = -2 \tan x \sec^2 x$
 $du = -2 \tan x \sec^2 x dx$

Question 234 (*****)

$$\sec x \equiv \frac{1 + \tan^2\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}.$$

- a) Prove the validity of the above trigonometric identity.
- b) Express $\frac{2}{1-t^2}$ into partial fractions.
- c) Hence use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$\int \sec x \, dx = \ln \left| \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \right| + C.$$

$$\frac{2}{1-t^2} = \frac{1}{1+t} + \frac{1}{1-t}$$

<p>(a) LHS = $\frac{1 + \tan^2\frac{x}{2}}{1 - \tan^2\frac{x}{2}} = \frac{1 + \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{1 - \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \dots$ MULTIPLY THE TOP BY $\cos^2\frac{x}{2}$</p> $= \frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} = \frac{1}{\cos(2\frac{x}{2})} = \frac{1}{\cos x} = \text{RHS}$ <p><i>check: $\sin^2 A + \cos^2 A = 1$</i></p>
<p>(b) $\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)} = \frac{A}{1-t} + \frac{B}{1+t}$</p> $2 = A(1+t) + B(1-t)$ <p>$\begin{cases} t=1 \Rightarrow 2=2B, B=1 \\ t=-1 \Rightarrow 2=2A, A=1 \end{cases}$</p> <p>THUS $\frac{2}{1-t^2} = \frac{1}{1-t} + \frac{1}{1+t}$</p>
<p>(c) $\int \sec x \, dx = \dots$ BY PART (a)</p> $= \int \frac{1 + \tan^2\frac{x}{2}}{1 - \tan^2\frac{x}{2}} \, dx = \dots$ <p>... BY SUBSTITUTION ...</p> $= \int \frac{1+t^2}{1-t^2} \left(\frac{2}{1+t} dt \right) = \int \frac{2}{1-t^2} dt$ $= \int \frac{1}{1-t} + \frac{1}{1+t} dt = \ln 1+t - \ln 1-t + C$ $= \ln \frac{1+t}{1-t} + C = \ln\left \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right + C$ <p><i>check: $\tan(\frac{\pi}{4} + \frac{x}{2}) = \frac{\tan\frac{x}{2} + \tan\frac{\pi}{4}}{1 - \tan\frac{x}{2}\tan\frac{\pi}{4}}$</i></p> $= \ln\left \frac{\tan\frac{x}{2} + 1}{1 - \tan\frac{x}{2}}\right = \ln\left \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right + C$
$\begin{aligned} t &= \tan\frac{x}{2} \\ \frac{dt}{dx} &= \frac{1}{2}\sec^2\frac{x}{2} \\ dx &= \frac{dt}{\frac{1}{2}\sec^2\frac{x}{2}} \\ dx &= \frac{2}{1+\tan^2\frac{x}{2}} dt \\ dx &= \frac{2}{1+t^2} dt \end{aligned}$

Question 235 (*****)

By using the substitution $e^x = \frac{1}{t}$, or otherwise, show clearly that

$$\int \frac{4}{e^x \sqrt{e^{2x}+4}} dx = -\frac{\sqrt{e^{2x}+4}}{e^x} + C.$$

proof

$$\begin{aligned}
 & \int \frac{4}{e^x \sqrt{e^{2x}+4}} dx \quad \text{BY SUBSTITUTION} \\
 &= \int \frac{4}{\frac{1}{t} \sqrt{\frac{1}{t^2} + 4}} \left(-\frac{dt}{t^2}\right) = \int \frac{4t}{\sqrt{\frac{1}{t^2} + 4}} \left(-\frac{dt}{t^2}\right) \\
 &= \int \frac{-4}{\sqrt{1+\frac{4}{t^2}}} dt = \int \frac{-4}{\sqrt{1+\frac{4}{t^2}}} \frac{dt}{t^2} \\
 &= \int \frac{-4t}{t^2 \sqrt{1+\frac{4}{t^2}}} dt = \int -4t(1+\frac{4}{t^2})^{-\frac{1}{2}} dt \\
 &\quad \text{BY REVERSE CHAIN RULE OR ANOTHER SUBSTITUTION} \\
 &= -(1+\frac{4}{t^2})^{\frac{1}{2}} + C = -(1+\frac{4}{e^{2x}})^{\frac{1}{2}} + C \\
 &= -\left(\frac{e^{2x}+4}{e^{2x}}\right)^{\frac{1}{2}} + C = -\frac{\sqrt{e^{2x}+4}}{e^x} + C \quad // \text{AS REQUIRED}
 \end{aligned}$$

Question 236 (*****)

Use the fact that $\sin A \equiv \cos\left(\frac{\pi}{2} - A\right)$ and other trigonometric identities to show that

$$\int_{\frac{5\pi}{12}}^{\frac{\pi}{4}} \frac{4}{\sqrt{1-\sin 2x}} dx = 2\ln 3.$$

proof

$$\begin{aligned}
 & \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4}{\sqrt{1-\sin 2x}} dx = \dots \quad \sin A \equiv \cos\left(\frac{\pi}{2} - A\right) \Rightarrow \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4}{\sqrt{1-\cos\left(\frac{\pi}{2}-2x\right)}} dx \\
 & \quad (\text{Now } \cos 2A = 1 - 2\sin^2 A) \quad \dots = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4}{\sqrt{1-(1-2\sin^2(2x))}} dx \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4}{\sqrt{2\sin^2(2x)}} dx = \frac{4}{\sqrt{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin(2x)} dx \\
 &= 2\sqrt{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos(2x)}{\sin(2x)} dx = \dots \quad \text{standard result } \int \frac{\cos u}{\sin u} du = \ln|\sin u| + C \\
 &= 2\sqrt{2} \left[\ln|\sin(\frac{\pi}{2})| - \ln|\sin(\frac{\pi}{4})| \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -2\sqrt{2} \left[\ln|\sin(\frac{\pi}{2})| - \ln|\sin(\frac{\pi}{4})| \right] \\
 &= -2\sqrt{2} \left[\ln 1 - \ln\sqrt{2} \right] = 2\sqrt{2} \ln\sqrt{2} = 2\sqrt{2} \times \frac{1}{2}\ln 2 \\
 &= 2\ln 3
 \end{aligned}$$

Question 237 (*****)

Use the substitution $x = \frac{1}{u}$ to find the value of

$$\int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx.$$

0

$$\begin{aligned}
 \int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx &= \int_2^{\frac{1}{2}} \frac{\ln \frac{1}{u}}{1+\frac{1}{u^2}} \left(-\frac{1}{u^2} du \right) \\
 &= \int_2^{\frac{1}{2}} \frac{-\ln u}{u^2+1} \left(-\frac{1}{u^2} du \right) = \int_2^{\frac{1}{2}} \frac{u^2 \ln u}{u^2+1} \left(\frac{1}{u^2} du \right) \\
 &= \int_2^{\frac{1}{2}} \frac{\ln u}{u^2+1} du = - \int_{\frac{1}{2}}^2 \frac{\ln u}{u^2+1} du \\
 \therefore \int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx &= - \int_{\frac{1}{2}}^2 \frac{\ln u}{u^2+1} du \\
 \therefore \int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2} dx &= 0 \quad \text{---} \\
 \end{aligned}$$

$$\begin{aligned}
 u &= \frac{1}{x}, \quad x = \frac{1}{u} \\
 \frac{du}{dx} &= -\frac{1}{x^2} \\
 -dx &= u^2 du \\
 dx &= -u^2 du \\
 du &= -\frac{1}{u^2} du \\
 \alpha &= \frac{1}{x}, \quad u=2 \\
 \alpha &= 2, \quad u=\frac{1}{2}
 \end{aligned}$$

Question 238 (*****)

Use the substitution $x = \frac{1}{u^2+1}$ to show that

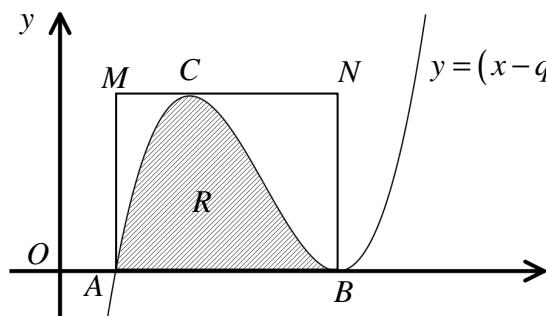
$$\int_{0.2}^{0.5} \frac{\sqrt{x-x^2}}{x^4} dx = \frac{256}{15}.$$

proof

$$\begin{aligned}
 \int_{0.2}^{0.5} \frac{\sqrt{\frac{x-x^2}{x^4}}}{x^4} dx &= \int_{0.2}^{0.5} \frac{\sqrt{\frac{1}{x^2} - \frac{1}{(x^2+1)^2}}}{x^4} \left(\frac{2x}{(x^2+1)^2} \right) dx \\
 &= \int_1^{\frac{1}{0.25}} \frac{\sqrt{\frac{(x^2+1)^2 - 1}{(x^2+1)^4}}}{x^4} \left(\frac{2x}{(x^2+1)^2} \right) dx = \int_1^{\frac{1}{0.25}} \frac{\sqrt{\frac{u^2}{(u^2+1)^2}} \times \frac{2u}{(u^2+1)^2}}{u^4} du \\
 &= \int_1^{\frac{1}{0.25}} \frac{\frac{u}{(u^2+1)^{\frac{1}{2}}} \times \frac{2u}{(u^2+1)^2} du}{u^4} = \int_1^{\frac{1}{0.25}} \frac{u(u^2+1)^{-\frac{1}{2}} \times \frac{2u}{(u^2+1)^2}}{u^4} du \\
 &= \int_1^{\frac{1}{0.25}} 2u^2(u^2+1)^{-\frac{1}{2}} du = \int_1^{\frac{1}{0.25}} 2u^4 + 2u^2 du \\
 &= \left[\frac{2}{5}u^5 + \frac{2}{3}u^3 \right]_1^{\frac{1}{0.25}} = \left(\frac{512}{5} + \frac{16}{3} \right) - \left(\frac{2}{5} + \frac{2}{3} \right) \\
 &= \frac{256}{15}
 \end{aligned}$$

$$\begin{aligned}
 \alpha &= \frac{1}{x^2+1} \\
 x &= (\alpha^2+1)^{-\frac{1}{2}} \\
 \frac{dx}{d\alpha} &= -2\alpha(\alpha^2+1)^{-\frac{3}{2}} \\
 dx &= -\frac{2\alpha}{(\alpha^2+1)^{\frac{3}{2}}} d\alpha \\
 \alpha &= \frac{1}{x} = \frac{1}{\frac{1}{u^2+1}} = u^2+1 \\
 u^2 &= \frac{1}{\alpha} - 1 \\
 u &= \pm \sqrt{\frac{1}{\alpha} - 1} \\
 u &= 2, \quad \alpha = 0.25 \\
 u &= 0.5, \quad \alpha = 1
 \end{aligned}$$

Question 239 (*****)



The figure above shows the graph of the curve with equation

$$y = (x-q)(x-p)^2,$$

where p and q are positive constants.

The curve meets the x axis at the points A and B . The region R , shown shaded in the figure, is bounded by the curve and the x axis.

- a) Show that the area of the shaded region is

$$\frac{1}{12}(p-q)^4.$$

The point C is the local maximum of the curve. The rectangle $AMCN$ is such so that MCN is parallel to the x axis and both AM and BN are parallel to the y axis.

- b) Show that the area of the rectangle $AMCN$ is $\frac{16}{9}$ times as large as the area of R , regardless of the values of p and q .

proof

QUESTION

$$R = \int_q^p (x-q)(x-p)^2 dx$$

P = TANGENT POINT
C = CRITICAL POINT

BY PARTS

$$\begin{aligned} & \int_q^p (x-q)(x-p)^2 dx \\ &= \left[\frac{1}{2}(x-q)(x-p)^3 \right]_q^p - \int_q^p \frac{1}{2}(x-p)^3 dx \\ &= \left(\frac{1}{2}(p-q)(p-p)^3 \right) - \left[\frac{1}{8}(x-p)^4 \right]_q^p \\ &= (0-0) - (0 - \frac{1}{8}(q-p)^4) = \frac{1}{8}(q-p)^4 = \frac{1}{8}(p-q)^4 \end{aligned}$$

NOTE $(q-p) \equiv -(p-q)$ if n is even

ANSWER

$$y = \frac{1}{3}(x-q)(x-p)^2$$

$$y = \frac{1}{3}(x+2q-3p)(x+2q-2p)^2$$

$$y = \frac{1}{3}(x+2q-2p)^3$$

SCALE FOR ZERO GRAD

$$x \approx \sqrt{\frac{p-q}{3}}$$

AREA = $\frac{1}{3}(p-q)$

$$\text{So } \frac{\frac{1}{3}(p-q)}{\frac{1}{8}(p-q)^4} = \frac{\frac{1}{3}}{\frac{1}{8}} = \frac{8}{3} = \frac{16}{9}$$

Question 240 (*****)

$$f(x) = \frac{\sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2}, \quad x \in \mathbb{R}.$$

Use trigonometric identities to find the exact value of

$$\int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} f(x) \, dx.$$

$\boxed{\frac{2-\sqrt{2}}{12}}$

EXPAND THE DENOMINATOR & TIDY UP

$$\begin{aligned}
 & \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{(\cos 7x + \cos x)^2 + (\sin 7x + \sin x)^2} \, dx \\
 &= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{\cos^2 7x + 2\cos 7x \cos x + \cos^2 x + \sin^2 7x + 2\sin 7x \sin x + \sin^2 x} \, dx \\
 &= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{1 + 2(\cos 7x \cos x + \sin 7x \sin x)} \, dx \\
 &= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{2 + 2\cos(7x - x)} \, dx \\
 &= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{2 + 2\cos 6x} \, dx \\
 &= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\sin 3x}{4\cos^2 3x} \, dx \\
 &= \int_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \frac{\frac{1}{4} \cdot \frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\cos 3x} \, dx}{\frac{1}{2} \sec^2 3x \, dx} \\
 &= \left[\frac{1}{12} \sec 3x \right]_{\frac{1}{12}\pi}^{\frac{1}{9}\pi} \\
 &= \frac{1}{12} \left[\frac{1}{\cos 3\pi} \right]^{\frac{1}{9}\pi}_{\frac{1}{12}\pi} \\
 &= \frac{1}{12} \left[\frac{1}{\frac{1}{2}} - \frac{1}{\frac{1}{2}} \right] = \boxed{\frac{2-\sqrt{2}}{12}}
 \end{aligned}$$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\cos 2\theta = 2\cos^2 \theta - 1$
 $\cos 6x = 2\cos^2 3x - 1$
 $2\cos 6x = 4\cos^2 3x - 2$
 $2 + 2\cos 6x = 4\cos^2 3x$

Question 241 (*****)

Use a suitable trigonometric manipulation to find an exact simplified answer for the following integral.

$$\int_0^{\frac{\pi}{3}} \frac{1}{(\cos x + \sqrt{3} \sin x)^2} dx.$$

, $\frac{1}{4}\sqrt{3}$

START WITH THE DENOMINATOR, BY AN "U-T-TRANSFORMATION
OR A MANIPULATION

$$\begin{aligned} \cos x + \sqrt{3} \sin x &= 2 \left[\frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x \right] \\ &= 2 \left[\cos \frac{\pi}{6} \cos x + \sin \frac{\pi}{6} \sin x \right] \\ &= 2 \cos \left(\frac{\pi}{6} - x \right) \\ &= 2 \cos \left(x - \frac{\pi}{6} \right) \end{aligned}$$

HENCE THE INTEGRAL BECOMES

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \frac{1}{(\cos x + \sqrt{3} \sin x)^2} dx &= \int_0^{\frac{\pi}{3}} \frac{1}{[2 \cos(x - \frac{\pi}{6})]^2} dx \\ &= \int_0^{\frac{\pi}{3}} \frac{1}{4 \cos^2(x - \frac{\pi}{6})} dx = \int_0^{\frac{\pi}{3}} \frac{1}{4 \cos^2(u)} du \\ &= \left[\frac{1}{4} \tan(u) \right]_0^{\frac{\pi}{3}} = \frac{1}{4} \left[\tan 0 - \tan \left(-\frac{\pi}{6} \right) \right] \\ &= \frac{1}{4} \left(0 + \tan \frac{\pi}{6} \right) = \frac{1}{4} \cdot \frac{1}{\sqrt{3}} \end{aligned}$$

Question 242 (*****)

$$f(x) = 3\sin x - \cos x + 3, \quad x \in \mathbb{R}.$$

$$g(x) = \sin x + \cos x, \quad x \in \mathbb{R}.$$

- a) Express $f(x)$ in the form

$$A \times g(x) + B \times g'(x) + 3,$$

where A and B are constants.

- b) Express $g(x)$ in the form

$$R \cos(x - \varphi),$$

where R and φ are positive constants.

- c) Hence find a simplified expression for

$$\int \frac{f(x)}{g(x)} dx.$$

$$\boxed{\quad}, \boxed{A=1}, \boxed{B=-2}, \boxed{R=\sqrt{2}}, \boxed{\varphi=\frac{1}{4}\pi},$$

$$\boxed{x - 2 \ln|\sin x + \cos x| + \frac{3}{2} \sqrt{2} \ln|\sec\left(x - \frac{1}{4}\pi\right) + \tan\left(x - \frac{1}{4}\pi\right)| + C}$$

a) DIFFERENTIATE $f(x)$

$$f'(x) = \cos x + \sin x$$

EQUATE & COMPLETE COMPARISON

$$\begin{aligned} \rightarrow f(x) &= A \times g(x) + B \times g'(x) + 3 \\ \rightarrow 3\sin x - \cos x + 3 &\equiv A(\sin x + \cos x) + B(\cos x - \sin x) + 3 \\ \rightarrow 3\sin x - \cos x &\equiv (A-1)\sin x + (A+B)\cos x \\ \left\{ \begin{array}{l} A-1=3 \\ A+B=-1 \end{array} \right. &\therefore \begin{array}{l} A=2 \\ A=-1 \end{array} \quad \begin{array}{l} B=-2 \\ B=1 \end{array} \end{aligned}$$

$$\rightarrow f(x) = g(x) - 2g'(x) + 3$$

b) $\boxed{g(x) = \sin x + \cos x}$

$$\begin{aligned} \rightarrow g(x) &= \sqrt{2} \left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right) \\ \rightarrow g(x) &= \sqrt{2} \left(\sin \frac{\pi}{4} x + \cos \frac{\pi}{4} x \right) \\ \rightarrow g(x) &= \sqrt{2} \cos \left(x - \frac{\pi}{4} \right) \end{aligned}$$

IF $R = \sqrt{2}$, $\varphi = \frac{\pi}{4}$

CAN BE DONE ALSO BY COMPARING
 $\sin x + \cos x \equiv R \cos(x - \frac{\pi}{4})$

c)
$$\begin{aligned} \int \frac{f(x)}{g(x)} dx &= \int \frac{g(x) - 2g'(x) + 3}{g(x)} dx \\ &= \int 1 + \frac{2g'(x)}{g(x)} + \frac{3}{g(x)} dx \\ &= \int 1 dx + 2 \int \frac{g'(x)}{g(x)} dx + \int \frac{3}{g(x)} dx \\ &= x + 2 \ln|g(x)| + \int \frac{3}{\sqrt{2} \cos(x - \frac{\pi}{4})} dx \\ &= x + 2 \ln|g(x)| + \frac{3}{\sqrt{2}} \int \sec(x - \frac{\pi}{4}) dx \end{aligned}$$

NOTING THAT $\int \sec x dx = \ln|\sec x + \tan x| + C$

$$\begin{aligned} \rightarrow \int \frac{f(x)}{g(x)} dx &= x + 2 \ln|\sin x + \cos x| + \frac{3}{\sqrt{2}} \ln|\sec(x - \frac{\pi}{4}) + \tan(x - \frac{\pi}{4})| + C \end{aligned}$$

Question 243 (*****)

Use the substitution $x = 2\sec\theta$, to find a simplified expression for

$$\int \frac{6}{(x^2 - 4)^{\frac{3}{2}}} dx.$$

$$-\frac{x}{\sqrt{x^2 - 4}} + C$$

The handwritten solution shows the following steps:

$$\begin{aligned} \int \frac{4}{(x^2 - 4)^{\frac{3}{2}}} dx &= \dots \text{by substitution} \dots \\ &= \int \frac{4}{(4\sec^2\theta - 4)^{\frac{3}{2}}} (2\sec\theta \tan\theta d\theta) \\ &= \int \frac{8\sec\theta \tan\theta}{(4(\sec^2\theta - 1))^{\frac{3}{2}}} d\theta \\ &= \int \frac{8\sec\theta \tan\theta}{(4\tan^2\theta)^{\frac{3}{2}}} d\theta \\ &= \int \frac{8\sec\theta \tan\theta}{8\tan^3\theta} d\theta \\ &= \int \frac{\sec\theta}{\tan^2\theta} d\theta \\ &= \int \frac{\sec\theta}{\tan\theta \cdot \sec\theta} d\theta \\ &= \int \frac{\cos\theta}{\sin\theta \cos^2\theta} d\theta \\ &= \int \frac{\cos\theta}{\sin\theta \cos^2\theta} d\theta \\ &\stackrel{\text{INTRODUCE } u = \sin\theta}{=} \int \frac{\cos\theta}{u \cos^2\theta} du \\ &\stackrel{\text{RECOGNISE } \cos^2\theta = 1 - \sin^2\theta}{=} \int \frac{\cos\theta}{u(1 - u^2)} du \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{\sin\theta} + C \\ &= -\frac{1}{\sqrt{1 - \cos^2\theta}} + C \\ &= -\frac{1}{\sqrt{1 - \frac{x^2 - 4}{x^2}}} + C \\ &= -\frac{x}{\sqrt{x^2 - 4}} + C \end{aligned}$$

Annotations on the right side of the box include:

- $x = 2\sec\theta$
- $\frac{dx}{d\theta} = 2\sec\theta \tan\theta$
- $d\theta = \frac{dx}{2\sec\theta \tan\theta}$
- $\frac{x}{2} = \sec\theta$
- $\cos\theta = \frac{2}{x}$
- $\sin\theta = \sqrt{\frac{x^2 - 4}{x^2}}$
- $\tan\theta = \frac{x}{\sqrt{x^2 - 4}}$
- Above the diagram: $\frac{2}{\sqrt{x^2 - 4}} = \frac{x}{2}$
- Diagram: A right-angled triangle with hypotenuse x , adjacent side 2 , and opposite side $\sqrt{x^2 - 4}$.
- Annotation: $\therefore \sin\theta = \sqrt{\frac{x^2 - 4}{x^2}}$
- Annotation: $\cos\theta = \frac{2}{x}$
- Annotation: $\tan\theta = \frac{x}{\sqrt{x^2 - 4}}$
- Annotation: ADDITIVE
- Annotation: $= \int \frac{\cos\theta}{\sin\theta \cos^2\theta} d\theta$
- Annotation: $= \int \frac{1}{\sin\theta} \cos\theta d\theta$
- Annotation: $= -\csc\theta + C$

Question 244 (*****)

Find an exact simplified value for

$$\int_0^1 \frac{x^2 - 3x + 1}{\sqrt{1-x^2}} dx.$$

$$\boxed{\frac{3}{4}[\pi - 4]}$$

$$\begin{aligned}
 & \int_0^1 \frac{x^2 - 3x + 1}{\sqrt{1-x^2}} dx = \dots \text{ BY SUBSTITUTION} \\
 & = \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta - 3\sin \theta + 1}{\sqrt{1-\sin^2 \theta}} (\cos \theta d\theta) \\
 & = \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta - 3\sin \theta + 1}{\cos \theta} \times \cos \theta d\theta \\
 & = \int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos 2\theta - 3\sin \theta + 1 d\theta \\
 & = \int_0^{\frac{\pi}{2}} \frac{3}{2} + \frac{1}{2}\cos 2\theta - 3\sin \theta d\theta \\
 & = \left[\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta + 3\cos \theta \right]_0^{\frac{\pi}{2}} \\
 & = \left(\frac{3\pi}{4} + 0 + 0 \right) - \left(0 + 0 + 3 \right) \\
 & = \frac{3\pi}{4} - 3 \\
 & = \frac{3}{4}[\pi - 4]
 \end{aligned}$$

$\alpha = \sin \theta$
 $d\theta = \cos \theta d\theta$
 $\theta = \arcsin x$
 $x=0 \mapsto \theta=0$
 $x=1 \mapsto \theta=\frac{\pi}{2}$

Question 245 (*****)

Given that a and b are integers, evaluate

$$\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx.$$

$$\boxed{2\pi}$$

$$\begin{aligned}
 \int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx &= \int_{-\pi}^{\pi} \frac{\cos^2 ax - 2\cos ax \sin bx + \sin^2 bx}{\cos^2 ax + \sin^2 bx} dx \\
 &= 2 \int_0^{\pi} \cos^2 ax + \sin^2 bx dx = 2 \int_0^{\pi} \frac{1}{2} + \frac{1}{2}\cos 2ax + \frac{1}{2} - \frac{1}{2}\sin 2bx dx \\
 &= \int_0^{\pi} 2 + (\cos 2ax - \sin 2bx) dx = \left[2a + \frac{1}{2b} \sin 2ax - \frac{1}{2b} \cos 2bx \right]_0^{\pi} \\
 &= \left[2\pi + \frac{1}{2b} \sin(2\pi a) - \frac{1}{2b} \cos(2\pi b) \right] - \left[0 + 0 + 0 \right] \\
 &= 2\pi
 \end{aligned}$$

BUT IF a, b NOT INTEGERS $\sin(2\pi a) = \sin(2\pi b) = 0$

Question 246 (*****)

$$I = \int_{1.5}^2 \frac{(x-2)(2x^2-5x-1)}{(x-1)(x-3)} dx.$$

Use appropriate integrations techniques to show that

$$I = \frac{5}{4} - \ln k,$$

where k is a positive integer.

, $k=6$

$\int_{1.5}^2 \frac{(x-2)(2x^2-5x-1)}{(x-1)(x-3)} dx = \frac{5}{4} - \ln k$

• FIRSTLY THE INTEGRAND IS IMPROPER – SO WE EXPAND IT IN ORDER TO DIVIDE IT OUT

$$(x-2)(2x^2-5x-1) = \frac{2x^3-2x^2-2x}{-4x^2+10x+2}$$

$$\frac{-4x^2+10x+2}{2x^3-9x^2+9x+2}$$

$$(x-1)(x-3) = x^2-4x+3$$

$$\frac{x^2-4x+3}{2x^3-9x^2+9x+2}$$

$$\frac{-8x^2+16x-12}{-8x^2+16x+2}$$

$$\frac{2x^3-9x^2+9x+2}{2x^3-9x^2+9x+2}$$

$$\frac{-2x+5}{-2x+5}$$

• Hence we have so far

$$\frac{(x-2)(2x^2-5x-1)}{(x-1)(x-3)} = 2x-1 + \frac{5-2x}{(x-1)(x-3)}$$

$$= 2x-1 + \frac{\frac{5}{2x}}{x-1} + \frac{\frac{5}{2}}{x-3}$$

$$= 2x-1 - \frac{2}{x-1} + \frac{1}{x-3}$$

$$\begin{aligned} & \int_{1.5}^2 2x-1 - \frac{2}{x-1} + \frac{1}{x-3} dx \\ &= \left[x^2 - 2\ln|x-1| + \ln|x-3| \right]_{1.5}^2 \\ &= \left[4-2 - 2\ln 1 + \ln 1 \right] - \left[\frac{9}{4}-\frac{3}{2} - 2\ln \frac{1}{2} + \ln \left(-\frac{1}{2}\right) \right] \\ &= 2 - \frac{9}{4} + \frac{3}{2} + 2\ln \frac{1}{2} - \ln \frac{1}{2} \\ &= \frac{8-9+6}{4} + \ln \frac{1}{4} - \ln \frac{1}{2} \\ &= \frac{5}{4} + \ln \frac{1}{4} + \ln \frac{3}{2} \\ &= \frac{5}{4} + \ln \frac{1}{6} \\ &= \frac{5}{4} - \ln k \end{aligned}$$

$\therefore k=6$

Question 247 (*****)

$$I = \int_0^1 \left(x^{\frac{7}{6}} + 4x^{\frac{2}{3}} \right)^{-\frac{3}{4}} dx.$$

Use appropriate integration techniques to show that

$$I = 8 \left[\sqrt[4]{5} - \sqrt{2} \right].$$

 [] proof

$$\int_0^1 \frac{1}{(x^{\frac{7}{6}} + 4x^{\frac{2}{3}})^{\frac{3}{4}}} dx = 8 \left[\sqrt[4]{5} - \sqrt{2} \right]$$

• SPLIT BY FACTORISING OUT OF THE RADICAL IN THE DENOMINATOR

$$\int_0^1 \frac{1}{(2^{\frac{3}{4}}(x^{\frac{1}{6}} + 4)^{\frac{3}{4}})^{\frac{1}{4}}} dx = \int_0^1 \frac{1}{(2^{\frac{3}{4}})^{\frac{1}{4}}(x^{\frac{1}{6}} + 4)^{\frac{3}{4}}} dx$$

$$= \int_0^1 \frac{1}{2^{\frac{3}{4}}(x^{\frac{1}{6}} + 4)^{\frac{3}{4}}} dx$$

• BY SUBSTITUTION NOW (OR RECOGNITION) AS $(2^{\frac{3}{4}})^{\frac{1}{4}} = 2^{\frac{1}{4}}$

$u = x^{\frac{1}{6}}$
$u^2 = x^{\frac{1}{3}}$
$du = \frac{1}{3}x^{-\frac{2}{3}}dx$
LIMITS ARE CHANGED AS

$$\dots = \int_0^1 \frac{1}{2(u+4)^{\frac{3}{4}}} (2^{\frac{1}{4}} du) = \int_0^1 \frac{2}{(u+4)^{\frac{3}{4}}} du$$

$$= \int_0^1 2(u+4)^{-\frac{3}{4}} du = \left[8(u+4)^{-\frac{1}{4}} \right]_0^1$$

$$= 8 \left[5^{\frac{1}{4}} - 4^{\frac{1}{4}} \right] = 8 \left[\sqrt[4]{5} - \sqrt{2} \right]$$

Question 248 (*****)

$$I = \int_0^{\frac{1}{4}\pi} \frac{1}{9\cos^2 x - \sin^2 x} dx.$$

By using a tangent substitution, or otherwise, show that

$$I = \frac{1}{6} \ln 2.$$

, proof

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1}{9\cos^2 x - \sin^2 x} dx &= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x(9 - \tan^2 x)} dx \\ &= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{9 - \tan^2 x} dx \\ \text{NOW USE THE SUBSTITUTION } u &= \tan x \\ \frac{du}{dx} &= \sec^2 x \\ du &= \sec^2 x \\ 2=0 \rightarrow u=0 & \\ 2=\frac{\pi}{4} \rightarrow u=1 & \\ \text{THE INTEGRAL NOW TRANSFORMS TO} \\ \int_0^1 \frac{\sec^2 x}{9 - u^2} \frac{du}{\sec^2 x} &= \int_0^1 \frac{1}{9-u^2} du \\ &= \int_0^1 \frac{1}{(3-u)(3+u)} du \\ \text{BY PARTIAL FRACTIONS (COVER UP OR FULL METHOD)} \\ \dots &= \int_0^1 \left[\frac{1}{3-u} + \frac{1}{3+u} \right] du = \frac{1}{6} \left[\ln|3+u| - \ln|3-u| \right]_0^1 \\ &= \frac{1}{6} \left[(\ln 4 - \ln 2) - (\ln 3 - \ln 3) \right] = \frac{1}{6} \ln 2. \end{aligned}$$

Question 249 (*****)

Find an exact simplified value for the following definite integral.

$$\int_0^\infty \frac{e^{8x} - e^{2x}}{(e^{8x}+3)(e^{2x}+3)} dx.$$

You may assume without proof that the integral converges.

$\boxed{\frac{1}{4}\ln 2}$

Start by partial fractions (by inspection)

$$\frac{e^{8x} - e^{2x}}{(e^{8x}+3)(e^{2x}+3)} = -\frac{1}{e^{8x}+3} + \frac{1}{e^{2x}+3}$$

Thus we have

$$\int_0^\infty \frac{e^{8x} - e^{2x}}{(e^{8x}+3)(e^{2x}+3)} dx = \int_0^\infty \frac{1}{e^{8x}+3} dx - \int_0^\infty \frac{1}{e^{2x}+3} dx$$

$$= \int_0^\infty \frac{1}{e^{2x}+3} dx - \int_0^\infty \frac{1}{e^{8x}+3} dx$$

Integrate one of the two, as they are identical in structure:

$$\int_0^\infty \frac{1}{e^{2x}+3} dx = \int_0^\infty \frac{-e^{-2x}}{1+3e^{-2x}} dx = -\frac{1}{2} \int_0^\infty \frac{-2e^{-2x}}{1+3e^{-2x}} dx$$

$$= \left[-\frac{1}{6} \ln(1+3e^{-2x}) \right]_0^\infty = -\frac{1}{6} \ln 1 + \frac{1}{6} \ln 4 = \frac{1}{6} \ln 4$$

The other one can be treated as

$$-\int_0^\infty \frac{1}{e^{8x}+3} dx = - \int_0^\infty \frac{-e^{-8x}}{1+3e^{-8x}} dx = -\frac{1}{8} \int_0^\infty \frac{-8e^{-8x}}{1+3e^{-8x}} dx = \left[-\frac{1}{24} \ln(1+3e^{-8x}) \right]_0^\infty$$

$$= \frac{1}{24} \ln 1 - \frac{1}{24} \ln 4$$

Combining results

$$\int_0^\infty \frac{e^{8x} - e^{2x}}{(e^{8x}+3)(e^{2x}+3)} dx = -\frac{1}{8} \ln 4 - \frac{1}{24} \ln 4 = \frac{3}{24} \ln 4 = \frac{1}{8} \ln 4 = \frac{1}{4} \ln 2$$

Question 250 (*****)

$$I = \int_{-\frac{5}{2}}^{\frac{7}{2}} \frac{4x+1}{\sqrt{35+4x-4x^2}} dx.$$

By writing $35+4x-4x^2$ in completed the square form, followed by a suitable trigonometric substitution, show that

$$I = \frac{3}{2}\pi.$$

, proof

• **SOLVE BY COMPLETING THE SQUARE IN THE DENOMINATOR.**

$$35+4x-4x^2 = -[(4x^2-4x-35)] = -(4(x-\frac{1}{2})^2 - 36) \\ = 36(2x-\frac{1}{2})^2 = 6^2(2x-\frac{1}{2})^2$$

↑
6sinθ (or 6cosθ)

• **USE THE SUBSTITUTION:** $2x-\frac{1}{2} = 6\sin\theta$
 $2\frac{d\theta}{dx} = 6\cos\theta$
 $d\theta = 3\cos\theta d\theta$
 $2x-\frac{1}{2} \Rightarrow \theta = 6\sin\theta$
 $\theta = \frac{\pi}{2}$ $\Rightarrow \theta = 6\sin\theta$
 $\Rightarrow \theta = -\frac{\pi}{2}$

• **HENCE WE MAY TRANSFORM THE INTEGRAL:**

$$\int_{-\frac{5}{2}}^{\frac{7}{2}} \frac{4x+1}{\sqrt{35+4x-4x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{12\sin\theta + 3}{\sqrt{36-36\sin^2\theta}} \times (6\cos\theta d\theta) \\ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(3+4\sin\theta)(2\cos\theta)}{6(1-\sin^2\theta)^{\frac{1}{2}}} d\theta = \frac{3}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+4\sin\theta)\cos^2\theta}{\cos^2\theta} d\theta \\ = \frac{3}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+4\sin\theta) d\theta = 2 \times \frac{3}{2} \int_0^{\frac{\pi}{2}} 1 d\theta \\ = 3 \times [\theta]_0^{\frac{\pi}{2}} = \frac{3\pi}{2}$$

Question 251 (*****)

$$I = \int_{-1}^1 (x+3) \sqrt{7-6x-x^2} \, dx$$

- a) Use a suitable trigonometric substitution to show that $I = 8\sqrt{3}$.
- b) Verify the answer of part (a) by an alternative method.

, **proof**

a)

$$\begin{aligned} & \int_{-1}^1 (2x+3) \sqrt{7-6x-x^2} \, dx = \int_{-1}^1 (2x+3) \sqrt{-[x^2+6x-7]} \, dx \\ &= \int_{-1}^1 (2x+3) \sqrt{-(x+3)^2 + 9-7} \, dx = \int_{-1}^1 (2x+3) \sqrt{16 - (2x+3)^2} \, dx \\ & \text{MATCH WITH A TRIGONOMETRIC SUBSTITUTION: } 16 \left[1 - \frac{(2x+3)^2}{16} \right] \\ & \quad \text{Thus } \frac{2x+3}{4} = 2\sin\theta \quad \text{or} \quad \cos\theta = \frac{2x+3}{4} \\ & \quad 2x+3 = 4\sin\theta \\ & \quad x = -3 + 2\sin\theta \\ & \quad dx = 2\cos\theta \, d\theta \\ &= \int_{-1/2}^{1/2} 4\sin\theta \sqrt{16 - 16\sin^2\theta} \, (4\cos\theta \, d\theta) \\ &= \int_{-1/2}^{1/2} 4\sin\theta \sqrt{16(1-\sin^2\theta)} \, (4\cos\theta \, d\theta) \\ &= \int_{-1/2}^{1/2} 4\sin\theta \times 4\cos\theta \times 4\cos\theta \, d\theta \\ &= \int_{-1/2}^{1/2} 64\sin\theta\cos^2\theta \, d\theta \\ &= \left[-\frac{64}{3} \cos^3\theta \right]_{-1/2}^{1/2} = \frac{64}{3} \left[\cos^3\theta \right]_{-1/2}^{1/2} \\ &= \frac{64}{3} \left[\left(\frac{\sqrt{3}}{2}\right)^3 - 0^3 \right] = \frac{64}{3} \times \frac{2\sqrt{3}}{8} = 8\sqrt{3} \end{aligned}$$

As required

b) BY AN ALGEBRAIC SUBSTITUTION

$$\begin{aligned} & \int_{-1}^1 (2x+3) \sqrt{7-6x-x^2} \, dx = \dots \\ &= \int_{-1}^1 (2x+3) \, u \left(\frac{-u}{2x+3} \, du \right) \\ &= \int_{-1}^1 -u^2 \, du \\ &= \int_0^{12} u^2 \, du \\ &= \left[\frac{1}{3} u^3 \right]_0^{12} \\ &= \frac{1}{3} (\sqrt{12})^3 = \frac{1}{3} \times 12 \times \sqrt{12} = 4\sqrt{12} = 8\sqrt{3} \end{aligned}$$

As before

VARIATION WITHOUT SUBSTITUTION SINCE THE ARGUMENT OF THE RADICAL DISINTEGRATES TO $-6-2x = -2(3+x)$

$$\begin{aligned} & \int_{-1}^1 (2x+3) (7-x^2)^{\frac{1}{2}} \, dx \dots \text{by recognition} \\ &= \left[(7-x^2)^{\frac{1}{2}} \times \left(\frac{1}{2}\right) \right]_{-1}^1 = \frac{1}{2} \left[(7-x^2)^{\frac{3}{2}} \right]_{-1}^1 \\ &= \frac{1}{2} \left[12^{\frac{3}{2}} - 0^{\frac{3}{2}} \right] = \frac{1}{2} \times 12 \times 12^{\frac{1}{2}} = 8\sqrt{12} \end{aligned}$$

As above.

Question 252 (*****)

Use trigonometric identities to find the value of

$$\int_0^{\frac{\pi}{3}} 32 \sin x \sin 2x \sin 3x \ dx.$$

[MP] , [9]

(WORKING AT THE "CALCULATOR" STATION, WE WORK AS FOLLOWS)

$$\begin{aligned} \cos(3x+2x) &= \cos 5x = -\sin 5x \\ \cos(3x-x) &= \cos 2x = 1 + 2\sin^2 x \end{aligned}$$

SUBSTITUTING THESE INTO

$$\begin{aligned} \cos(3x-x) - \cos(3x+2x) &= 2\sin x \sin 2x \\ \cos 2x - \cos 5x &= 2\sin x \sin 2x \end{aligned}$$

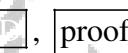
RETURNING TO THE INTEGRAL

$$\begin{aligned} \int_0^{\frac{\pi}{3}} 32 \sin x \sin 2x \sin 3x \ dx &= \int_0^{\frac{\pi}{3}} 16 \sin^2 x [2\sin x \sin 2x] \ dx \\ &= \int_0^{\frac{\pi}{3}} 16 \sin x [\cos 2x - \cos 5x] \ dx = \int_0^{\frac{\pi}{3}} 16 \sin x \cos 2x - 16 \sin x \cos 5x \ dx \\ &= \int_0^{\frac{\pi}{3}} 8(\sin 2x \cos 2x) - 16 \cos 5x (\sin 2x \cos 2x - 1) \ dx \\ &= \int_0^{\frac{\pi}{3}} 8\sin 4x - 32x^2 \sin 2x + 16\sin 2x \ dx \\ &= \left[-2\cos 4x + \frac{16}{3}x^3 \sin 2x - 8\cos 2x \right]_0^{\frac{\pi}{3}} \\ &= \left[-2(-\frac{1}{2}) + \frac{16}{3}(\frac{\pi}{3})^3 \sin 2(\frac{\pi}{3}) - 8(-1) \right] - [-2 + \frac{16}{3} - 0] \\ &= 1 - \frac{2}{3} + \frac{4}{3} + 2 = \frac{16}{3} = 0 \\ &= \underline{\underline{0}} \end{aligned}$$

Question 253 (*****)

Use suitable integration techniques to show that

$$\int_0^1 \frac{x^2}{(x^2+1)^3} dx = \frac{\pi}{32}.$$

START WITH A TRIGONOMETRIC SUBSTITUTION

$$\begin{aligned} & \int_0^1 \frac{x^2}{(x^2+1)^3} dx = \dots \\ &= \int_0^{\frac{\pi}{2}} \frac{\tan^2 \theta}{(\sec^2 \theta)^3} (\sec^2 \theta d\theta) \quad \begin{array}{l} x = \tan \theta \\ dx = \sec^2 \theta d\theta \\ x=0 \mapsto \theta=0 \\ x=1 \mapsto \theta=\frac{\pi}{2} \end{array} \\ &= \int_0^{\frac{\pi}{2}} \frac{\tan^2 \theta \sec^2 \theta}{(\sec^2 \theta)^3} d\theta = \int_0^{\frac{\pi}{2}} \frac{\tan^2 \theta}{\sec^4 \theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta}{\cos^6 \theta} d\theta \end{aligned}$$

SCOTCH IN SINCE Q. USING AND APPARENT SIMPLIFICATION

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \frac{\tan^2 \theta \cos^4 \theta}{\cos^6 \theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta}{\cos^2 \theta} \times \cos^4 \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^2 \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta \right) \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta \quad \begin{array}{l} \cos 2\theta = \cos(\theta) - 1 \\ \cos 2\theta = 1 - 2\sin^2 \theta \end{array} \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{4} - \frac{1}{4} \cos^2 2\theta d\theta \quad \begin{array}{l} \cos 2\theta = \cos(2\theta) + 2\cos(2\theta) - 1 \\ \cos 2\theta = \cos(2\theta) + 2\cos(2\theta) - 1 \end{array} \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{4} - \frac{1}{4} \left(\frac{1}{2} + \frac{1}{2} \cos(2\theta) \right) d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{8} - \frac{1}{8} \cos(2\theta) d\theta \\ &= \left[\frac{1}{8}\theta - \frac{1}{16} \sin(2\theta) \right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{\pi}{16} - 0 \right) - (0 - 0) = \frac{\pi}{16}. \end{aligned}$$

Question 254 (*****)

Use suitable integration techniques to show that

$$\int_{-\frac{1}{6}\ln 3}^{\frac{1}{6}\ln 3} 6e^{-3x} \arctan(e^{3x}) dx = \ln 3 + \frac{\pi\sqrt{3}}{9}.$$

 proof

Solved by A. Substitution as follows

$$\begin{aligned} & \int_{-\frac{1}{6}\ln 3}^{\frac{1}{6}\ln 3} 6e^{-3x} \arctan(e^{3x}) dx \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 6e^{-3x} \arctan(e^{3x}) dx \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2e^{-3x} \theta \sec^2 \theta dx \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{-2\theta \sec^2 \theta}{(e^{3x})'} dx \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{-2\theta \sec^2 \theta}{(e^{3x})^2} dx \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} -2\theta \sec^2 \theta d\theta \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} -2\theta \times \frac{d(\tan \theta)}{d\theta} d\theta \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} -2\theta d(\tan \theta) \quad \leftarrow \text{INTEGRATION BY PARTS} \\ &= \left[-2\theta \tan \theta + \int \tan \theta d\theta \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \end{aligned}$$

$\theta = \arctan(e^{3x})$
 $\tan \theta = e^{3x}$
 $\sec^2 \theta d\theta = 3e^{3x} dx$
 $dx = \frac{e^{-3x}}{3e^{3x}} d\theta$

$\theta = \arctan(e^{-3x})$
 $\theta = \arctan(e^{3x})$
 $\theta = \arctan(\frac{1}{e^{3x}}) = -\frac{\pi}{2}$
 $2x = \frac{1}{3}\ln 3$
 $\theta = \arctan(\sqrt{3}) = \frac{\pi}{3}$

$$\begin{aligned} &= \left[-2\theta \tan \theta + 2\ln|\sin \theta| \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\ &= \left[2\ln(\sin \frac{\pi}{3}) - 2\theta \tan \frac{\pi}{3} \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\ &= \left[2\ln(\sin \frac{\pi}{3}) - \frac{2\pi}{3} \cot \frac{\pi}{3} \right] - \left[2\ln(\sin \frac{\pi}{3}) - \frac{2\pi}{3} \cot \frac{\pi}{3} \right] \\ &= 2\ln \frac{\sqrt{3}}{2} - \frac{2\pi}{3} \times \frac{\sqrt{3}}{3} - 2\ln \frac{1}{2} + \frac{2\pi}{3} \times \frac{\sqrt{3}}{3} \\ &= \ln \frac{3}{4} - \frac{2\pi\sqrt{3}}{9} + \ln 4 + \frac{2\pi\sqrt{3}}{9} \\ &= \ln \left(\frac{3}{4} \times 4 \right) + \left(\frac{2}{9} + \frac{1}{3} \right) \pi \sqrt{3} \\ &= \ln 3 + \frac{\pi\sqrt{3}}{9} \end{aligned}$$

Question 255 (*****)

Use suitable integration techniques to show that

$$\int_0^{\frac{1}{2}\pi} \frac{1+\cos x + \sin x - \tan x}{1+\tan x} dx = 1.$$

You may assume that the above integral converges.

, **proof**

SUBSTITUTION AND SIMPLIFYING

$$\int_0^{\frac{1}{2}\pi} \frac{1+\cos x + \sin x - \tan x}{1+\tan x} dx$$

$$= \int_0^{\frac{1}{2}\pi} \frac{1+\cos x + \sin x - \frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}} dx$$

$$= \int_0^{\frac{1}{2}\pi} \frac{(\cos x + \sin x) - \sin x}{(\cos x + \sin x) \cos x} dx$$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY $\cos x$

RECOGNISING THE TRIGS

$$= \int_0^{\frac{1}{2}\pi} \frac{\cos x - \sin x + (\cos x \sin x + \cos^2 x)}{\cos x + \sin x} dx$$

$$= \int_0^{\frac{1}{2}\pi} \frac{\cos x - \sin x}{\cos x + \sin x} + \frac{\cos x \sin x + \cos^2 x}{\cos x + \sin x} dx$$

$$= \int_0^{\frac{1}{2}\pi} \frac{-\sin x + \cos x}{\cos x + \sin x} + \frac{\cos x (\sin x + \cos x)}{\cos x + \sin x} dx$$

OF THE FORM $\int \frac{f(x)}{g(x)} dx = \ln|g(x)| + C$

INTEGRATING YIELDS

$$= \left[\ln|\cos x + \sin x| \right]_0^{\frac{1}{2}\pi} + \sin x \Big|_0^{\frac{1}{2}\pi}$$

$$= [\ln(0+0) + 1] - [\ln(1+0) + 0]$$

$$= 1$$

Question 256 (*****)

Use a suitable trigonometric substitution to find a simplified expression for

$$\int \sqrt{(1+x)(5-x)} dx.$$

$$\boxed{\frac{9}{2} \arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{(1+x)(5-x)} + C}$$

$$\int \sqrt{(1+x)(5-x)} dx = \int \sqrt{5-x+5-x^2} dx = \int \sqrt{5+4x-x^2} dx$$

$$= \int \sqrt{-x^2+4x+5} dx = \int \sqrt{-(x-2)^2+9} dx = \int \sqrt{9-(x-2)^2} dx$$

USE A TRIGONOMETRIC SUBSTITUTION

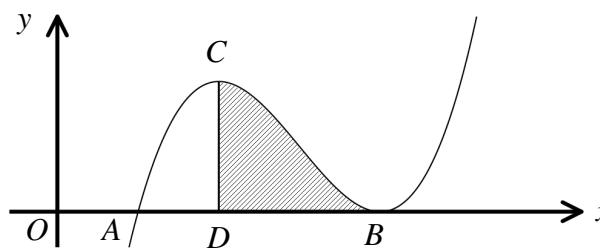
NEED TO MAP $(x-2) = 3\sin\theta$

$$\begin{aligned} x &= 2 + 3\sin\theta \\ \theta &= \arcsin\left(\frac{x-2}{3}\right) \\ \frac{dx}{d\theta} &= 3\cos\theta \\ dx &= 3\cos\theta d\theta \end{aligned}$$

$$\begin{aligned} \sin\theta &= \frac{3\sin\theta}{3} \\ \sin\theta &= \frac{x-2}{3} \end{aligned}$$

$$\begin{aligned} &= \dots \int \sqrt{9-9\sin^2\theta} (3\cos\theta) d\theta = \int \sqrt{9(1-\sin^2\theta)} (3\cos\theta) d\theta \\ &= \int \sqrt{9\cos^2\theta} (3\cos\theta) d\theta = \int (3\cos\theta)(3\cos\theta) d\theta = \int 9\cos^2\theta d\theta \\ &= \int \frac{9}{2} + \frac{9}{2}\cos 2\theta d\theta = -\frac{9}{2}\theta + \frac{9}{4}\sin 2\theta + C \\ &= \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C \\ &= \frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{9}{2}\left(\frac{x-2}{3}\right)\left(\frac{\sqrt{9-(x-2)^2}}{3}\right) + C \\ &= \frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{(1+x)(5-x)} + C \end{aligned}$$

Question 257 (*****)



The figure above shows a cubic curve that crosses the x axis at $A(a,0)$ and touches the x axis at $B(b,0)$, where a and b are positive constants. The point C is a local maximum of the curve.

- a) Find the x coordinate of C , in terms of a and b .

The point D lies on the x axis so that CD is parallel to the y axis.

- b) Show that $|AB| = 3|AD|$.

The region R is bounded by the curve, the line segment CD and the x axis.

- c) Use integration by parts to show that the area of R is $\frac{4}{81}(b-a)^4$.

$$x = \frac{1}{3}(2a+b)$$

(a) $y = (x-a)(x-b)^2$

$$\begin{aligned} \frac{dy}{dx} &= 1 \times (x-b)^2 + (x-a) \times 2(x-b) \\ &= (x-b)^2 + 2(x-b)(x-a) \\ &= (x-b)(x-b+2(x-a)) \\ &= (x-b)(3x-b-2a) \end{aligned}$$

SHADING REGION
YIELDING
 $x = b$
 $\frac{2a+b}{3} \leftarrow C$

(b) Now $|AB| = b-a$
 $|AD| = \frac{2a+b}{3}-a = \frac{2a+b-3a}{3} = \frac{b-a}{3}$
 $\therefore 3|AD| = 3 \cdot \frac{b-a}{3} = b-a = |AB|$ // At Equivento

(c) $A(R) = \int_{\frac{2a+b}{3}}^{x=b} (x-a)(x-b)^2 dx = \dots$ by parts

$\frac{2a+b}{3}$	1
$\frac{2a+b}{3}(x-b)^2$	$\frac{1}{3}(x-b)^3$

$$\begin{aligned} &= \left[\frac{1}{2}(x-a)(x-b)^2 \right]_{\frac{2a+b}{3}}^{x=b} - \left[\frac{1}{3}(x-b)^3 \right]_{\frac{2a+b}{3}}^{x=b} \\ &= \left[\frac{1}{2}(x-a)(x-b)^2 \right]_{\frac{2a+b}{3}}^{x=b} - \frac{1}{12}(x-b)^3 \Big|_{\frac{2a+b}{3}}^{x=b} \\ &= \frac{1}{12} \left[(x-a)^2 [4(x-a)-(x-b)] \right]_{\frac{2a+b}{3}}^{x=b} \\ &= \frac{1}{12} \left[(x-b)^2 (3x-b-2a) \right]_{\frac{2a+b}{3}}^{x=b} \\ &= \frac{1}{12} \left[0 - \left[\frac{2a+b-3a}{3} \right]^2 (2b-2a) \right] \\ &= -\frac{1}{12} \left[\frac{2a+b-3a}{3} \right]^2 \times 2(b-a) \\ &= -\frac{1}{12} \times \left(\frac{b-a}{3} \right)^2 \times 2 \times (b-a) = -\frac{4}{81}(b-a)^4 \end{aligned}$$

ANSWERED

Question 258 (*****)

By considering the derivatives of $e^x \sin x$ and $e^x \cos x$, find

$$\int e^x (2\cos x - 3\sin x) dx.$$

$$\boxed{\frac{1}{2}e^x(5\cos x - \sin x) + C}$$

$$\begin{aligned} \frac{d}{dx}(e^x \sin x) &= e^2 \sin x + e^x \cos x \\ \frac{d}{dx}(e^x \cos x) &= e^x \cos x - e^x \sin x \end{aligned} \quad \text{(Add & subtract)} \\ \left. \begin{aligned} \frac{d}{dx}(e^x \sin x + e^x \cos x) &= 2e^x \sin x \\ \frac{d}{dx}(e^x \sin x - e^x \cos x) &= 2e^x \cos x \end{aligned} \right\} \Rightarrow \left. \begin{aligned} \frac{d}{dx}(e^x(\sin x + \cos x)) &= e^x \sin x \\ \frac{d}{dx}(e^x(\sin x - \cos x)) &= e^x \cos x \end{aligned} \right. \\ \text{Hence } 2e^x \sin x - 2e^x \cos x &= 2 \frac{d}{dx}(e^x(\sin x + \cos x)) - 2 \frac{d}{dx}(e^x(\sin x - \cos x)) \\ 2e^x \sin x - 2e^x \cos x &= \frac{d}{dx}[e^x(\sin x + \cos x) - e^x(\sin x - \cos x)] \\ 2e^x \sin x - 2e^x \cos x &= \frac{d}{dx}[e^x(2\sin x - 2\cos x + 2\cos x - 2\sin x)] \\ 2e^x \sin x - 2e^x \cos x &= \frac{d}{dx}[e^x(4\cos x - 2\sin x)] \\ \therefore \int e^x(2\cos x - 3\sin x) dx &= \frac{1}{2}e^x(8\cos x - 4\sin x) + C \end{aligned}$$

Question 259 (*****)

Use integration by parts and suitable trigonometric identities to find

$$\int \sec^3 x dx.$$

$$\boxed{\quad, \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C}$$

• $\int \sec^2 x dx = \int \sec x \sec x dx \dots$	BY PARTS
$\dots = \sec x \tan x - \int \sec x \tan^2 x dx$ $= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$ $= \sec x \tan x - \int \sec^3 x dx - \sec x dx$ $= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$ $= \sec x \tan x - \int \sec^3 x dx + \ln \sec x + \tan x $	$\begin{array}{c c} \sec & \sec x \\ \tan & \tan x \\ \hline \sec & \sec x \\ \tan & \tan x \end{array}$
COLLECTING THE RESULTS	
$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \ln \sec x + \tan x $ $2 \int \sec^3 x dx = \sec x \tan x + \ln \sec x + \tan x + C$ $\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln \sec x + \tan x + C //$	

Question 260 (*****)

By using the substitution $\sqrt{x} = \tan \theta$, or otherwise, find a simplified expression for the following integral.

$$\int \frac{1-x}{\sqrt{x}(x+1)^2} dx .$$

, $\frac{2\sqrt{x}}{x+1} + C$

$\int \frac{1-x}{\sqrt{x}(x+1)^2} dx = \dots$

USING THE GIVEN SUBSTITUTION

$\begin{aligned} \sqrt{x} &= \tan \theta \\ x &= \tan^2 \theta \\ dx &= 2\tan \theta \sec^2 \theta d\theta \end{aligned}$

$\dots = \int \frac{1 - (\tan^2 \theta)}{\sqrt{\tan^2 \theta}(\tan^2 \theta + 1)} (2\tan \theta \sec^2 \theta d\theta)$

$= \int \frac{2\tan \theta (1 - \tan^2 \theta)}{(\sec^2 \theta)^2} d\theta$

$= \int \frac{2\tan \theta (1 - \tan^2 \theta)}{\sec^4 \theta} d\theta$

$= \int \frac{2(1 - \tan^2 \theta)}{\sec^2 \theta} d\theta$

CONVERT THE INTEGRAL INTO SEC²θ, SO IT MAY BE SUIT

$= \int \frac{2[1 - (\sec^2 \theta - 1)]}{\sec^2 \theta} d\theta$

$= \int \frac{2(2 - \sec^2 \theta)}{\sec^2 \theta} d\theta$

$= \int 2 \left(\frac{2}{\sec^2 \theta} - 1 \right) d\theta$

$= \int 2 \left[2\cos^2 \theta - 1 \right] d\theta$

$= \int 2\cos 2\theta d\theta$

$= \sin 2\theta + C$

$= 2\sin \theta \cos \theta + C$

$= \frac{2\sin \theta \times \cos^2 \theta}{\cos^2 \theta} + C$

$= 2\tan \theta \times \frac{1}{\sec^2 \theta} + C$

$= \frac{2\tan \theta}{1 + \tan^2 \theta} + C$

$= \frac{2\sqrt{x}}{1+x} + C$

Question 261 (***)**

Find the value of the following definite integral.

$$\int_{\frac{1}{2}}^2 \frac{1}{x+x^4} dx .$$

Give the answer in the form $\ln k$, where k is a positive integer.

, $k = 2$

THE "INSPIRATION" IS $x+x^4 = x(1+x^3) = x(x+2)(x-x+2)$, WHICH LEADS TO PAINFUL/FOOLISH PARTIAL FRACTION (STUFF).

$$\int_{\frac{1}{2}}^2 \frac{1}{x+x^4} dx = \int_{\frac{1}{2}}^2 \frac{1}{x(x^3+1)} dx = \int_{\frac{1}{2}}^2 \frac{x^2}{x^3+1} dx$$

BUT THIS IS OF THE FORM

$$\int \frac{f(x)}{g(x)} dx = \ln|f(x)| + C$$

THAT IT CAN BE EVALUATED EASILY

$$\begin{aligned} &= \int_{\frac{1}{2}}^2 \frac{x^2}{x^3+1} dx = -\frac{1}{3} \int_{\frac{1}{2}}^2 \frac{-3x^2}{x^3+1} dx = \left[-\frac{1}{3} \ln|x^3+1| \right]_{\frac{1}{2}}^2 \\ &= \frac{1}{3} \left[\ln|x^3+1| \right]_{\frac{1}{2}}^2 = \frac{1}{3} \left[\ln\left(\frac{9}{8}\right) + \frac{1}{2} \ln\left(\frac{9}{8}\right) \right]^{\frac{1}{2}}_2 \\ &= \frac{1}{3} \left[\ln\left(1+\frac{1}{8}\right) - \ln\left(1+\frac{1}{8}\right) \right]_2^{\frac{1}{2}} = \frac{1}{3} \left[\ln\left(9 \times \frac{9}{8}\right) \right]_2^{\frac{1}{2}} \\ &\approx \frac{1}{3} \left[\ln 9 + \ln \frac{9}{8} \right] = \frac{1}{3} \ln(9 \times \frac{9}{8}) = \frac{1}{3} \ln 81 \\ &= \ln 2 \end{aligned}$$

Question 262 (*****)

$$I = \int_{-2}^2 \frac{1}{\sqrt{1-ax+a^2}} dx, \quad a > 0, \quad a \neq 0.$$

Find the two possible values of I , giving the answer in terms of a where appropriate.

<input type="text"/>	$I = 4 \quad 0 < a < 1$
$I = \frac{4}{a} \quad a > 1$	

$I = \int_{-2}^2 \frac{1}{\sqrt{1-ax+a^2}} dx \quad a > 0, \quad a \neq 1$

BY INSPECTION (OR SUBSTITUTION)

 $\Rightarrow I = \int_{-2}^2 (1-ax+a^2)^{-\frac{1}{2}} dx = \left[\frac{2}{a} (1-ax+a^2)^{\frac{1}{2}} \right]_{-2}^2$
 $\Rightarrow I = \frac{2}{a} \left[(1-ax+a^2)^{\frac{1}{2}} \right]_{-2}^2 = \frac{2}{a} \left[(1+2a+a^2)^{\frac{1}{2}} - (1-2a+a^2)^{\frac{1}{2}} \right]$

NOW THERE ARE TWO POSSIBILITIES

<p>IF $0 < a < 1$</p> $I = \frac{2}{a} \left[(a+1)^{\frac{1}{2}} - \sqrt{(a-1)^2} \right]$ $I = \frac{2}{a} \left[a+1 - a-1 \right]$ $I = \frac{2}{a} \left[(a+1) - (1-a) \right]$ $I = \frac{2}{a} \times 2a$ $I = 4$	<p>IF $a > 1$</p> $I = \frac{2}{a} \left[\sqrt{(a+1)^2} - \sqrt{(a-1)^2} \right]$ $I = \frac{2}{a} \left[a+1 - a-1 \right]$ $I = \frac{2}{a} \left[(a+1) - (a-1) \right]$ $I = \frac{2}{a} \times 2$ $I = \frac{4}{a}$
--	---

Question 263 (*****)

$$I = \int \frac{\cos^3 x}{(1+\sin^2 x)\sin x} dx.$$

By using the substitution $u = \sin x + \operatorname{cosec} x$, or otherwise, show that

$$I = \ln \left| \frac{\sin x}{1+\sin^2 x} \right| + \text{constant}$$

, proof

USING THE SUBSTITUTION (SUM)

$$\begin{aligned} u &= \sin x + \operatorname{cosec} x \\ \frac{du}{dx} &= \cos x - \operatorname{cosec} x \operatorname{csc} x \\ du &= (\cos x - \operatorname{cosec} x \operatorname{csc} x) dx \end{aligned}$$

TRANSFORMING THE INTEGRAL WE OBTAIN

$$\int \frac{\cos^3 x}{(1+\sin^2 x)\sin x} dx = \int \frac{\cos x}{(1+\sin^2 x)\sin x} \times \frac{1}{\cos x - \operatorname{cosec} x \operatorname{csc} x} du$$

SIMPLIFY AND WORK IT

$$\begin{aligned} &\sim \int \frac{\cos x}{(1+\sin^2 x)\sin x} \times \frac{1}{\cos x - \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x}} du \\ &= \int \frac{\cos x}{(1+\sin^2 x)\sin x} \times \frac{1}{\cos x \left(1 - \frac{1}{\sin^2 x}\right)} du \quad \text{NOTING "TOP A BOTTOM" OF THIS FRACTION} \\ &< \int \frac{\cos x}{(1+\sin^2 x)\sin x} \times \frac{\sin^2 x}{\cos x (\sin^2 x - 1)} du \\ &= \int \frac{\cos x}{(1+\sin^2 x)\sin x} \times \frac{-\sin^2 x}{\cos x \sin^2 x} du \\ &= \int -\frac{\sin x}{1+\sin^2 x} du \\ &= \int -\frac{\sin x \cos x}{(\sin x + \frac{1}{\sin x}) \cos x} du \end{aligned}$$

DENOMINATE THE SUBSTITUTION

$$\begin{aligned} &= \int -\frac{1}{\sin x + \operatorname{cosec} x} du \\ &= \int -\frac{1}{u} du \\ &= -\ln |u| + C \\ &= -\ln \left| \frac{1}{\sin x} \right| + C \\ &= \ln \left| \frac{1}{\sin x} \right| + C \end{aligned}$$

AS REQUIRED

Question 264 (*****)

$$I = \int_0^{\frac{1}{2}\pi} x \cot x \, dx.$$

Use appropriate integration techniques to show that

$$I = \frac{1}{2}\pi \ln 2.$$

 , proof

Start by integration by parts

$$\int_0^{\frac{1}{2}\pi} x \cot x \, dx = \left[x \sqrt{\sin x} \right]_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} \ln(\sin x) \, dx$$

BECAUSE AS $x \rightarrow 0$ FASTER THAN
 $\ln x \rightarrow -\infty$

$$\int_0^{\frac{1}{2}\pi} x \cot x \, dx = - \int_0^{\frac{1}{2}\pi} \ln(\sin x) \, dx$$

Now proceed as follows

$$\Rightarrow I = \int_0^{\frac{1}{2}\pi} \ln(\sin x) \, dx \quad \dots \text{substitution}$$

$$\Rightarrow I = \int_{\frac{\pi}{2}}^0 \ln[\sin(\frac{\pi}{2}-x)] (-dx)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln[\sin(\frac{\pi}{2}-x)] \, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\cos x) \, dx$$

This regarding the journey back into I , we find

$$\Rightarrow I = \int_0^{\frac{1}{2}\pi} \ln(\sin x) \, dx = \int_0^{\frac{1}{2}\pi} \ln(\cos x) \, dx$$

Then we find

$$\Rightarrow I + I = \int_0^{\frac{1}{2}\pi} \ln(\sin x) \, dx + \int_0^{\frac{1}{2}\pi} \ln(\cos x) \, dx$$

$$\Rightarrow 2I = \int_0^{\frac{1}{2}\pi} \ln(\sin x) + \ln(\cos x) \, dx = \int_0^{\frac{1}{2}\pi} \ln(\cos 2x) \, dx$$

$$\Rightarrow 2I = \int_0^{\frac{1}{2}\pi} \ln(1/\sin 2x) \, dx = \int_0^{\frac{1}{2}\pi} \ln(\frac{1}{2}) + \ln(\sin 2x) \, dx$$

$$\Rightarrow 2I = \int_0^{\frac{1}{2}\pi} \ln 2 \, dx + \int_0^{\frac{1}{2}\pi} \ln(\sin 2x) \, dx$$

$u = 2x$
 $2 = \frac{du}{dx}$
 $dx = \frac{1}{2}du$
 $2x = \frac{1}{2}u$
 $dx = \frac{1}{4}du$

$$\Rightarrow 2I = -\ln 2 \left(\frac{\pi}{2} \right) + \int_0^{\frac{\pi}{2}} \ln(\sin u) (\frac{1}{2}du)$$

$$\Rightarrow 2I = -\frac{\pi}{2} \ln 2 + \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin u) \, du$$

THE SAME FUNCTION IS EVEN ABOUT $\frac{\pi}{2}$
SO WE CAN USE IT

$$\Rightarrow 2I = -\frac{\pi}{2} \ln 2 + \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \ln(\sin u) \, du$$

$$\Rightarrow 2I = -\frac{\pi}{2} \ln 2 + I$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\sin x) \, dx = -\frac{\pi}{2} \ln 2$$

And finally we find

$$\int_0^{\frac{1}{2}\pi} x \cot x \, dx = - \int_0^{\frac{1}{2}\pi} \ln(\sin x) \, dx = \frac{\pi}{2} \ln 2$$

Question 265 (*****)

Use the substitution $x = \frac{1}{u}$ to find the value of

$$\int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx.$$

0

$$\begin{aligned}
 & \int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx = \dots \text{substitution} = \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{\frac{1}{u^4} - 1}{\frac{1}{u^2} \sqrt{\frac{1+u^4}{u^4}}} \left(-\frac{1}{u^2} du \right) \\
 &= \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{\frac{1-u^4}{u^4}}{\frac{1}{u^2} \sqrt{\frac{1+u^4}{u^4}}} \left(-\frac{1}{u^2} du \right) = \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{1-u^4}{u^2 \sqrt{1+u^4}} \left(-\frac{1}{u^2} du \right) \\
 &= \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{1-u^4}{u^2 \sqrt{1+u^4}} \left(\frac{1}{u^2} du \right) = \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{1-u^4}{u^4 \sqrt{1+u^4}} du = - \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{u^4 - 1}{u^4 \sqrt{1+u^4}} du \\
 &\text{Thus } \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} du = - \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{u^4 - 1}{u^4 \sqrt{1+u^4}} du \\
 &\therefore \int_{\frac{1}{2}}^2 \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} du = 0
 \end{aligned}$$

Question 266 (*****)

Use a suitable substitution to find the value of

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} dx.$$

□, 1

$$\begin{aligned}
 & \text{LET } I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} dx \\
 & \bullet \text{USING THE SUBSTITUTION RESULT} \\
 & \quad \int_a^b f(a) dx = \int_a^b f(a+b-x) dx \\
 & \rightarrow I = \int_2^4 \frac{\sqrt{\ln(9-(4-x))}}{\sqrt{\ln(9-(4-x))} + \sqrt{\ln(3-(4-x))}} dx \\
 & \rightarrow I = \int_2^4 \frac{\sqrt{\ln(x+5)}}{\sqrt{\ln(2x+3)} + \sqrt{\ln(9-x)}} dx \\
 & \bullet \text{ADDING THE EXPRESSIONS} \\
 & \rightarrow 2I = \int_2^4 \frac{\sqrt{\ln(9-x)} + \sqrt{\ln(x+5)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+5)}} dx + \int_2^4 \frac{\sqrt{\ln(x+5)} - \sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+5)}} dx \\
 & \rightarrow 2I = \int_2^4 \frac{\sqrt{\ln(9-x)} + \sqrt{\ln(x+5)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+5)}} dx \\
 & \rightarrow 2I = \int_2^4 1 dx \\
 & \rightarrow 2I = 2 \\
 & \rightarrow I = 1
 \end{aligned}$$

Question 267 (*****)

Use an appropriate substitution followed by integration by parts to find a simplified expression for

$$\int \frac{[\ln(x^2+1) - 2\ln x]\sqrt{x^2+1}}{x^4} dx.$$

_____	$\boxed{\frac{2}{9x^3}(x^2+1)^{\frac{3}{2}} \left[1 - 3\ln\left(\frac{x^2+1}{x^2}\right) \right] + C}$
-------	---

Start by manipulating the integral as follows

$$\begin{aligned} & \int \frac{[\ln(x^2+1) - 2\ln x]\sqrt{x^2+1}}{x^4} dx \\ &= \int \ln\left(\frac{x^2+1}{x^2}\right) \times \frac{\sqrt{x^2+1}}{x^3} dx \\ &= \int \ln\left(1 + \frac{1}{x^2}\right) \times \frac{\sqrt{x^2+1}}{\sqrt{x^2}} \times \frac{1}{x^3} dx \\ &= \int \sqrt{\frac{x^2+1}{x^2}} \ln\left(1 + \frac{1}{x^2}\right) \times \frac{1}{x^3} dx \\ &= \int \sqrt{1 + \frac{1}{x^2}} \ln\left(1 + \frac{1}{x^2}\right) \times \frac{1}{x^3} dx \end{aligned}$$

Now we have "A SUBSTITUTION"

$$\begin{aligned} u &= \sqrt{1 + \frac{1}{x^2}} \\ u^2 &= 1 + \frac{1}{x^2} \\ 2u du &= -\frac{2}{x^3} dx \\ \therefore \frac{du}{x^3} &= -u du \end{aligned}$$

TRANSFORM THE INTEGRAL

$$\begin{aligned} &= \int u \ln u^2 (-u du) \\ &= \int -u^2 \ln u^2 du \\ &= \int -2u^2 \ln u du \end{aligned}$$

PROCEED BY INTEGRATION BY PARTS

$$\begin{aligned} & \int \frac{\ln u}{-3u^3} \left| \begin{array}{l} \frac{1}{u} \\ -u^2 \end{array} \right. \\ &= -\frac{1}{3} u^3 \ln u - \int \frac{2}{3} u^2 \left(\frac{1}{u} du \right) \\ &= -\frac{2}{3} u^3 \ln u + \frac{2}{3} u^2 + C \\ &= -\frac{2}{3} u^3 \ln u + \frac{2}{3} u^3 + C \\ &= \frac{2}{3} u^3 \left[1 - \ln u \right] + C \\ &= \frac{2}{3} u^3 \left[1 - \ln u^2 \right] + C \\ &= \frac{2}{3} \left(1 + \frac{1}{x^2} \right)^{\frac{3}{2}} \left[1 - 3\ln\left(\frac{1}{x^2}\right) \right] + C \\ &= \frac{2}{3} \left(\frac{x^2+1}{x^2} \right)^{\frac{3}{2}} \left[1 - 3\ln\left(\frac{x^2+1}{x^2}\right) \right] + C \\ &= \frac{2}{3} \sqrt{x^2+1} \left[1 - 3\ln\left(\frac{x^2+1}{x^2}\right) \right] + C \end{aligned}$$

Question 268 (*****)

Use appropriate integration techniques to show that

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \ln(1+\sqrt{2}).$$

SOLN, proof

LET $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$

BY SUBSTITUTION
 $x = \frac{\pi}{2} - y \Rightarrow dx = -dy$ $\alpha = \frac{\pi}{2} \Rightarrow y=0$
 $y = \frac{\pi}{2} - x \Rightarrow x=0 \Rightarrow y=\frac{\pi}{2}$

 $\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^2(\frac{\pi}{2}-y)}{\sin(\frac{\pi}{2}-y) + \cos(\frac{\pi}{2}-y)} (-dy)$

$\begin{aligned} \sin(\frac{\pi}{2}-x) &= \cos x \\ \cos(\frac{\pi}{2}-x) &= \sin x \end{aligned}$

 $\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 y}{\cos y + \sin y} dy$
 $\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x} dx$

ADDING WT. OBTAIN

 $\Rightarrow I + I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} + \frac{\cos^2 x}{\sin x + \cos x} dx$
 $\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx$
 $\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x]} dx$
 $\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}[\cos(\frac{\pi}{4})\sin x + \sin(\frac{\pi}{4})\cos x]} dx$
 $\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}\cos(x-\frac{\pi}{4})} dx$
 $\Rightarrow 2I = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\sec(x-\frac{\pi}{4})} dx$

STANDARD REARR

 $\int \sec \theta d\theta = \ln|\sec \theta + \tan \theta| + C$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \left[\ln|\sec(x-\frac{\pi}{4}) + \tan(x-\frac{\pi}{4})| \right]_0^{\frac{\pi}{2}}$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \left[\ln[\sec(\frac{\pi}{4}) + \tan(\frac{\pi}{4})] - \ln[\sec(-\frac{\pi}{4}) + \tan(-\frac{\pi}{4})] \right]$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \left[\ln(\sqrt{2}+1) - \ln(\sqrt{2}-1) \right]$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right)$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \ln \left[\frac{(\sqrt{2}+1)(\sqrt{2}-1)}{(\sqrt{2}-1)(\sqrt{2}+1)} \right]$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \ln \left[\frac{2}{2-1} \right]$
 $\Rightarrow I = \frac{1}{2\sqrt{2}} \ln(\sqrt{2}+1)$

AS REQUIRED

ALTERNATIVE FROM THE POINT WHERE THE INTEGRAL IS OBTAINED

 $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + \sin x} dx$

BY LITTLE + IDENTITIES

- \bullet $\tan \frac{\pi}{4} = t$
 $\frac{dt}{dx} = \sec^2 \frac{\pi}{4} \times \frac{1}{2}$
 $\frac{dt}{dx} = \frac{1}{2}(1+4t^2)$
 $\frac{dt}{dx} = \frac{1}{2}(1+t^2)$
 $dx = \frac{dt}{\frac{1}{2}(1+t^2)}$
 $dx = \frac{2}{1+t^2} dt$
 $\bullet \sin x = \frac{\sqrt{1-t^2}}{\sqrt{1+t^2}}$
 $= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$
 $= \frac{2t}{1+t^2}$
 $\bullet \cos x = \frac{\sqrt{1-t^2}}{\sqrt{1+t^2}}$
 $= \frac{(1-t^2)^{1/2}}{\sqrt{1+t^2}} = \frac{1-t^2}{1+t^2}$

 $\begin{aligned} &= \int_0^1 \frac{1}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}} \left(\frac{2}{1+t^2} dt \right) = \int_0^1 \frac{2}{1-t^2+2t} dt \\ &= \int_0^1 \frac{1}{-(t^2-2t-1)} dt = \int_1^0 \frac{1}{(t-1)^2-2} dt \\ &= \int_1^0 \frac{1}{(t-1-\sqrt{2})(t-1+\sqrt{2})} dt \end{aligned}$

NOW BY PARTIAL FRACTION (BY CROSS-OUT)

$$\begin{aligned} &= \int_1^0 \frac{1}{(t-1-\sqrt{2})(t-1+\sqrt{2})} + \frac{1}{t-1+\sqrt{2}} dt \\ &= \int_1^0 \frac{\frac{1}{2\sqrt{2}}}{t-1-\sqrt{2}} - \frac{\frac{1}{2\sqrt{2}}}{t-1+\sqrt{2}} dt \\ &= \frac{1}{2\sqrt{2}} \int_1^0 \frac{1}{t-1-\sqrt{2}} - \frac{1}{t-1+\sqrt{2}} dt \\ &\approx \frac{1}{2\sqrt{2}} \left[\ln \left| \frac{t-1-\sqrt{2}}{t-1+\sqrt{2}} \right| \right]_1^0 \\ &= \frac{1}{2\sqrt{2}} \left[\ln \left| \frac{-1-\sqrt{2}}{-1+\sqrt{2}} \right| - \ln \left| \frac{-\sqrt{2}}{\sqrt{2}} \right| \right] \\ &= \frac{1}{2\sqrt{2}} \ln \left[\frac{-1-\sqrt{2}}{-1+\sqrt{2}} \times -1 \right] \\ &= \frac{1}{2\sqrt{2}} \ln \left[\frac{\sqrt{2}+1}{\sqrt{2}-1} \right] \\ &= \frac{1}{2\sqrt{2}} \ln \left[\frac{(\sqrt{2}+1)(\sqrt{2}-1)}{(\sqrt{2}-1)(\sqrt{2}+1)} \right] \\ &= \frac{1}{2\sqrt{2}} \ln \left(\frac{(\sqrt{2}+1)^2}{2-1} \right) \\ &= \frac{1}{2\sqrt{2}} \ln (\sqrt{2}+1)^2 \\ &= \frac{1}{\sqrt{2}} \ln (\sqrt{2}+1) \end{aligned}$$

AS REQUIRED

Question 269 (*****)

Use appropriate integration techniques to show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1+\sqrt{\cot x}} dx = \frac{\pi}{12}.$$

proof

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}} dx = \frac{\pi}{12}$

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}} dx$... use the substitution

$x = \frac{\pi}{3} - y$
$dx = -dy$
$\frac{\pi}{3} \mapsto \frac{\pi}{3}$
$\frac{\pi}{6} \mapsto \frac{\pi}{6}$

$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot(\frac{\pi}{3}-y)}} (-dy) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan y}} dy$

$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}} dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}} dx$

$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}} + \frac{1}{1+\sqrt{\tan x}} dx$

$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}} + \frac{1}{1+\sqrt{\tan x}} \frac{\sqrt{\cot x}}{\sqrt{\cot x}} dx$

Multiply top & bottom of the first fraction by $\sqrt{\tan x}$, top & bottom of the second fraction by $\sqrt{\cot x}$

$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\cos x}^{-1}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$

$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 dx = \frac{\pi}{6}$

$\therefore = \frac{\pi}{12}$ as required

Question 270 (*****)

Use appropriate integration techniques to find an exact answer for the following definite integral.

$$\int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} \sqrt[3]{3\sin 2x - 2\sin 3x \cos x} \ dx$$

, $-\frac{3}{32}$

Start by drawing an expression for $\sin 3x$ by identity:

$$\begin{aligned}\sin 3x &= \sin(2x + x) = 2\sin 2x \cos x + \cos 2x \sin x \\&= (2\sin 2x \cos x) \cos x + (1 - 2\sin^2 x) \sin x \\&= 2\sin 2x \cos^2 x + \sin x - 2\sin^3 x \\&= 2\sin 2x(-\sin^2 x) + \sin x - 2\sin^3 x \\&= 3\sin 2x - 4\sin^3 x\end{aligned}$$

Hence the integral can now be simplified:

$$\begin{aligned}&\int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} 3\sin 2x - 2\sin 3x \cos x \ dx \\&= \int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} (6\sin 2x \cos x - 2\cos x(3\sin 2x - 4\sin^3 x))^{\frac{1}{3}} \ dx \\&= \int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} (6\sin 2x \cos x - 6\cos^2 x \sin 2x + 8\sin^3 x \cos x)^{\frac{1}{3}} \ dx \\&= \int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} (8\cos^3 x \sin^3 x)^{\frac{1}{3}} \ dx \\&= \int_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}} 2\sin^2 x (\cos x)^{\frac{1}{3}} \ dx \\&= -2 \times \frac{3}{4} \left[(\cos x)^{\frac{4}{3}} \right]_{\frac{1}{2}\pi}^{2\pi - \arccos \frac{1}{8}}\end{aligned}$$

$$\begin{aligned}&= -\frac{3}{2} \left[[\cos(\frac{1}{2}\pi - \arccos \frac{1}{8})]^{\frac{4}{3}} - [\cos(\frac{2\pi}{3} - \arccos \frac{1}{8})]^{\frac{4}{3}} \right] \\&= -\frac{3}{2} \left[[\cos \arcsin(\arccos \frac{1}{8}) + \sin \arcsin(\arccos \frac{1}{8})]^{\frac{4}{3}} \right] \\&= -\frac{3}{2} \left[1 \times \frac{1}{8} \right]^{\frac{4}{3}} \\&= -\frac{3}{2} \left(\frac{1}{8} \right)^{\frac{1}{3}} \\&= -\frac{3}{2} \times \frac{1}{16} \\&= -\frac{3}{32}\end{aligned}$$

Question 271 (*****)

- a) Use the compound angle identity $\cos(A+B)$ to show that

$$\cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

- b) Use a suitable trigonometric substitution to find the exact value of

$$\int_{\sqrt{2}}^{\sqrt{6+\sqrt{2}}} \frac{2}{x\sqrt{x^4-1}} dx.$$

$$\boxed{\frac{\pi}{24}}$$

a) $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4} //$$

b)
$$\int_{\sqrt{2}}^{\sqrt{6+\sqrt{2}}} \frac{1}{x\sqrt{x^4-1}} dx = \dots$$

THE INTEGRAL TRANSFORMS TO

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{12}} \frac{1}{x\sqrt{\sec^2\theta-1}} \times \frac{\sec\theta\tan\theta}{2x} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{12}} \frac{\sec\theta\tan\theta}{2x^2\sqrt{\tan^2\theta}} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{12}} \frac{\sec\theta\tan\theta}{2x^2\sec^2\theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{12}} \frac{1}{2x^2} d\theta$$

$$= \left[\frac{1}{2x^2} \theta \right]_{\frac{\pi}{4}}^{\frac{5\pi}{12}} = \frac{1}{2} \left[\frac{5\pi}{12} - \frac{\pi}{4} \right] = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$$

BY SUBSTITUTION

- $x^2 = \sec\theta$
- $2x \frac{dx}{d\theta} = \sec\theta\tan\theta$
- $dx = \frac{\sec\theta\tan\theta}{2x} d\theta$
- $d\theta = \sqrt{x^2-1} d\theta$
- $\theta = \text{arcsec } x$

- $\theta_1 = \text{arcsec } \sqrt{2}$
- $\theta^2 = 2$
- $\theta = \sec\theta$
- $\theta = \frac{5\pi}{12}$

- $\theta_2 = \text{arcsec } \sqrt{6+\sqrt{2}}$
- $\theta^2 = \sqrt{6+\sqrt{2}}$
- $\sec\theta = \sqrt{6+\sqrt{2}}$
- $\cos\theta = \frac{1}{\sqrt{6+\sqrt{2}}}$
- $\cos\theta = \frac{\sqrt{6-\sqrt{2}}}{\sqrt{6+\sqrt{2}}(\sqrt{6-\sqrt{2}})}$
- $\cos\theta = \frac{\sqrt{6-\sqrt{2}}}{6-2}$
- $\cos\theta = \frac{\sqrt{6-\sqrt{2}}}{4}$
- $\theta = \frac{5\pi}{12}$

Question 272 (*****)

It is given that the functions of x , $u(x)$ and $v(x)$ satisfy

$$\int u(x)v(x)dx = \left[\int u(x)dx \right] \times \left[\int v(x)dx \right], \text{ for } x \in \mathbb{R}, x \neq 0, x \neq 1.$$

- a) Show clearly that

$$\frac{\int u(x)dx}{u(x)} + \frac{\int v(x)dx}{v(x)} = 1.$$

- b) Given further that

$$\frac{\int u(x)dx}{u(x)} = \frac{1}{x},$$

show that

$$u(x) = Ax e^{\frac{1}{2}x^2}, \text{ where } A \text{ is an arbitrary constant.}$$

- c) Determine a similar expression for $v(x)$.

$$v(x) = Bx e^{\frac{1}{2}x^2}$$

<p>(a) $\int uv dx = \int u dx \times \int v dx$</p> <p>DIFF w.r.t x. FOLLOWING THE PRODUCT RULE $uv = u \int v dx + \int u dx \times v$</p> <p>Divide by uv</p> $1 = \frac{\int u dx}{u} + \frac{\int v dx}{v}$ <p style="text-align: right;">At step (b)</p>	<p>(b) $\frac{\int u dx}{u} = \frac{1}{x}$</p> <p>$\Rightarrow \int u dx = \frac{u}{x}$</p> <p>DIFF w.r.t x</p> $\Rightarrow u = \frac{d}{dx} \left(\frac{u}{x} \right)$ $\Rightarrow u = \frac{x \frac{du}{dx} - u}{x^2}$ $\Rightarrow x \frac{du}{dx} - u = \frac{u}{x^2}$ $\Rightarrow x \frac{du}{dx} = u + \frac{u}{x^2}$ $\Rightarrow u(x^2+1) = x \frac{du}{dx}$ $\Rightarrow \frac{x^2+1}{x} du = \int \frac{u}{x} dx$ $\Rightarrow \int x + \frac{1}{x} dx = \int \frac{u}{x} dx$ $\Rightarrow \frac{1}{2}x^2 + \ln x + C = \ln u $ $\Rightarrow u = e^{\frac{1}{2}x^2 + \ln x + C}$ $\Rightarrow u = e^{\frac{1}{2}x^2} e^{\ln x } \cdot e^C$ $\Rightarrow u = A x e^{\frac{1}{2}x^2}$	<p>(c) $1 = \frac{\int v dx}{v} + \frac{1}{x}$</p> <p>$\Rightarrow \int v dx = 1 - \frac{1}{x}$</p> <p>$\Rightarrow \int v dx = V - \frac{1}{x}$</p> <p>DIFFERENTIATE w.r.t x</p> $\Rightarrow V = \frac{dV}{dx} = -\frac{1}{x^2}$ $\Rightarrow Vx^2 = \frac{dV}{dx} = -\frac{1}{x^2} + V$ $\Rightarrow Vx^2 - V = \left(\frac{1}{x^2} - 1 \right) \frac{dV}{dx}$ $\Rightarrow V(x^2 - 1) = \left(\frac{1}{x^2} - 1 \right) \frac{dV}{dx}$ $\Rightarrow V(x^2 + 1) \frac{dV}{dx} = x^2 \left(\frac{1}{x^2} - 1 \right) \frac{dV}{dx}$ $\Rightarrow \frac{x^2+1}{x^2} dV = \frac{1}{x^2} - 1$ $\Rightarrow \int \frac{dV}{x^2} = \int \frac{1}{x^2} - 1 dx$ $\Rightarrow \ln V = \frac{1}{x} + \ln x + D$ $\Rightarrow V = e^{\frac{1}{x} + \ln x + D}$ $\Rightarrow V = \frac{e^{\frac{1}{x}}}{x} e^{\ln x } e^D$ $\Rightarrow V = B x e^{\frac{1}{x}}$
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Question 273 (*****)

Use the substitution $\tan x = \frac{1}{2}(-1 + \sqrt{3} \tan \theta)$ to find the exact value of

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{3}}{2 + \sin 2x} dx.$$

, $\frac{\pi}{6}$

$$\begin{aligned}
 & \int_0^{\frac{\pi}{4}} \frac{\sqrt{3}}{2 + \sin 2x} dx = \dots \text{USING THE SUBSTITUTION} \\
 & = \int_0^{\frac{\pi}{4}} \frac{\sqrt{3}}{2 + 2\sin \theta \cos \theta} \frac{\sqrt{3} \sec^2 \theta}{2\sec^2 \theta} d\theta \\
 & = \int_0^{\frac{\pi}{4}} \frac{3\sec^2 \theta}{2 + 2\sin \theta \cos \theta} \frac{1}{2\sec^2 \theta} d\theta \\
 & = \frac{3}{4} \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{(1 + \sin \theta \cos \theta)^2} d\theta \\
 & = \frac{3}{4} \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{\sec^2 \theta + \tan \theta} d\theta \\
 & = \frac{3}{4} \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{1 + \tan^2 \theta + \tan \theta} d\theta \\
 & = \frac{3}{4} \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{1 + \frac{1}{2}(5\tan \theta - 1) + \frac{1}{2}(\tan^2 \theta - 1)} d\theta \\
 & = \frac{3}{4} \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{1 + \frac{1}{2}(5\tan \theta - 1) + \frac{1}{2}(\tan^2 \theta - 2\tan \theta + 1)} d\theta \\
 & = \frac{3}{4} \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{1 + \frac{5}{2}\tan \theta - \frac{1}{2} + \frac{1}{2}(\tan^2 \theta - 2\tan \theta + 1)} d\theta \\
 & = \frac{3}{4} \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{\frac{5}{2}\tan \theta + \frac{1}{2} + \frac{1}{2}\tan^2 \theta - \frac{1}{2}\tan \theta + \frac{1}{2}} d\theta \\
 & = \frac{3}{4} \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{\frac{1}{2}\tan \theta + \frac{1}{2} + \frac{1}{2}\tan^2 \theta - \frac{1}{2}\tan \theta + \frac{1}{2}} d\theta \\
 & = \frac{3}{4} \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{\frac{1}{2}(\tan \theta + 1) + \frac{1}{2}(\tan^2 \theta - \tan \theta + 1)} d\theta = \frac{3}{4} \int_0^{\frac{\pi}{4}} \frac{1 + \tan \theta}{\frac{1}{2}(\tan \theta + 1)} d\theta = \frac{3}{4} \int_0^{\frac{\pi}{4}} \frac{1}{\frac{1}{2}(\tan \theta + 1)} d\theta \\
 & = \left[\frac{3}{4} \right] \frac{1}{\frac{1}{2}} = \frac{3}{4} \cdot \frac{2}{1} = \frac{3}{2}
 \end{aligned}$$

Question 274 (*****)

$$I = \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{(\sin x + 2\cos x)(\sin x + 3\cos x)} dx.$$

Use appropriate integration techniques to show that

$$I = \ln \left(\frac{a}{b} \right),$$

where a and b are positive integers to be found.

$$\boxed{A}, \boxed{B}, I = \ln \left(\frac{150}{143} \right)$$

Start manipulating the integrand as follows

$$\begin{aligned} & \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{(\sin x + 2\cos x)(\sin x + 3\cos x)} dx \\ &= \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{\sin^2 x + 5\sin x \cos x + 6\cos^2 x} dx \end{aligned}$$

Divide "top & bottom" by $\cos^2 x$ to create tangents

$$\begin{aligned} &= \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{\sec^2 x + \tan x \sec x + 6} dx \\ &= \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{\sec^2 x}{\sec^2 x + \tan x \sec x + 6} dx \end{aligned}$$

By substitution as tanx differentiates to sec^2 x

<ul style="list-style-type: none"> $u = \tan x$ $\frac{du}{dx} = \sec^2 x$ $dx = \frac{du}{\sec^2 x}$ 	<p>To change the limits</p> <ul style="list-style-type: none"> $\theta = \arcsin \frac{3}{5} = \tan^{-1} \frac{3}{4}$ $\theta = \arccos \frac{3}{5} = \tan^{-1} \frac{4}{3}$
---	--

$$\begin{aligned} &= \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{\sec^2 x}{u^2 + 5u + 6} du \\ &= \int_{\arcsin \frac{3}{5}}^{\arccos \frac{3}{5}} \frac{1}{u^2 + 5u + 6} du \end{aligned}$$

BY PARTIAL FRACTIONS

$$\frac{1}{(u+2)(u+3)} = \frac{A}{u+2} + \frac{B}{u+3}$$

$$\begin{cases} u+2=0 \Rightarrow u=-2 \\ u+3=0 \Rightarrow u=-3 \end{cases}$$

RETURNING TO THE INTEGRAL

$$\begin{aligned} &= \int_{-3}^{-2} \frac{1}{u+2} - \frac{1}{u+3} du \\ &= \left[\ln|u+2| - \ln|u+3| \right]_{-3}^{-2} \\ &= \left(\ln \frac{10}{3} - \ln \frac{1}{3} \right) - \left(\ln \frac{1}{4} - \ln \frac{1}{1} \right) \\ &= \left(\ln \frac{10}{3} + \ln \frac{1}{3} \right) - \left(\ln \frac{1}{4} + \ln \frac{1}{1} \right) \\ &= \ln \frac{10}{12} - \ln \frac{1}{4} \\ &= \ln \frac{10}{12} + \ln \frac{1}{4} \\ &= \ln \frac{150}{143} \end{aligned}$$

Question 275 (***)**

Use appropriate integration techniques to show that

$$\int_0^1 \frac{1}{x + \sqrt{1-x^2}} dx = \frac{\pi}{4}.$$

 proof

USING A TRIGONOMETRIC SUBSTITUTION

$u = \sin\theta$	$du = \cos\theta d\theta$
$u = 0 \rightarrow \theta = 0$	$u = 1 \rightarrow \theta = \frac{\pi}{2}$

TRANSFORMING THE INTEGRAL

$$\begin{aligned} \int_0^1 \frac{1}{x + \sqrt{1-x^2}} dx &= \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta + \sqrt{1-\sin^2\theta}} (\cos\theta d\theta) \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta + \cos\theta} d\theta \end{aligned}$$

PROCEEDED AS FOLLOWS

LET $I = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta + \cos\theta} d\theta$

using $\int_a^b f(x) dx \equiv \int_a^b f(a+b-x) dx$

$$\begin{aligned} \Rightarrow I &= \int_0^{\frac{\pi}{2}} \frac{\cos(\frac{\pi}{2}-\theta)}{\sin(\frac{\pi}{2}-\theta) + \cos(\frac{\pi}{2}-\theta)} d\theta \\ \Rightarrow I &= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta + \sin\theta} d\theta \end{aligned}$$

THIS WE NOW HAVE

$$\begin{aligned} \Rightarrow 2I &= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta + \cos\theta} d\theta + \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta + \sin\theta} d\theta \\ \Rightarrow 2I &= \int_0^{\frac{\pi}{2}} \frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta} d\theta \\ \Rightarrow 2I &= \int_0^{\frac{\pi}{2}} 1 d\theta \\ \Rightarrow 2I &= \frac{\pi}{2} \end{aligned}$$

$\Rightarrow I = \frac{\pi}{4}$

ALTERNATIVE AFTER THE TRIGONOMETRIC SUBSTITUTION

$$\begin{aligned} \int_0^1 \frac{1}{x + \sqrt{1-x^2}} dx &= \dots = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta + \cos\theta} d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{2\cos\theta}{\sin\theta + \cos\theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos\theta - \sin\theta + \sin\theta + \cos\theta}{\sin\theta + \cos\theta} d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} d\theta + \frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta} d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} d\theta + 1 d\theta \\ &\quad \uparrow \text{THIS IS OF THE FORM } \int \frac{f'(u)}{f(u)} du = \ln|f(u)| + C \\ &= \frac{1}{2} \left[\ln|\sin\theta + \cos\theta| + C \right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{2} \left[(\ln 1 + \frac{\pi}{2}) - (\ln 1 + 0) \right] \\ &= \frac{\pi}{4} \end{aligned}$$

As Before

Question 276 (*****)

Use polynomial division to find the exact value of

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1} dx.$$

You may assume that

$$\int \frac{1}{1+x^2} dx = \arctan x + \text{constant}.$$

, $\frac{22}{7} - \pi$

• **SOLVE BY FULL EXPANSION** THE NUMERATOR OF THE INTEGRAND

$$\int_0^1 \frac{x^8(1-x)^4}{x^2+1} dx = \int_0^1 \frac{x^8(1-4x+6x^2-4x^3+x^4)}{x^2+1} dx$$

$$= \int_0^1 \frac{x^8 - 4x^9 + 6x^{10} - 4x^{11} + x^{12}}{x^2+1} dx$$

• **BY LONG DIVISION NEXT**

$$\begin{array}{r} \frac{x^8 - 4x^9 + 6x^{10} - 4x^{11} + x^{12}}{x^2+1} \\ \hline -2x^6 - 4x^5 + 6x^4 - 4x^3 + x^2 \\ \hline -4x^8 + 4x^7 + 4x^6 - 4x^5 + 4x^4 \\ \hline 4x^7 + 4x^6 \\ \hline -5x^8 + 5x^7 + 5x^6 - 5x^5 + 5x^4 \\ \hline -4x^9 + 4x^8 + 4x^7 \\ \hline -4x^9 + 4x^8 \\ \hline -4x^{10} + 4x^9 \\ \hline -4x^{10} \end{array}$$

$$\therefore \frac{x^8 - 4x^9 + 6x^{10} - 4x^{11} + x^{12}}{x^2+1} = x^8 - 4x^9 + 5x^{10} - 4x^{11} + \frac{-4}{x^2+1}$$

• **REDUCING TO THE INTEGRAL**

$$\dots = \int_0^1 x^8 - 4x^9 + 5x^{10} - 4x^{11} + \frac{-4}{x^2+1} dx$$

$$= \left[\frac{1}{9}x^9 - \frac{4}{10}x^{10} + 5x^{11} - \frac{4}{12}x^{12} - 4 \arctan x \right]_0^1$$

$$= \left(\frac{1}{9} - \frac{4}{10} + 5 - \frac{4}{12} - 4 \arctan 1 \right) - \left(0 - \arctan 0 \right)$$

$$= \frac{1}{9} - \frac{2}{5} + 5 - 4 \times \frac{\pi}{4}$$

$$= \frac{1}{9} - \frac{2}{5} + 5 - \pi$$

$$= \frac{1}{9} - 2 + 5 - \pi$$

$$= 3 + \frac{1}{9} - \pi$$

$$= \frac{28}{9} - \pi$$

Question 277 (*****)

Use integration by parts to find a simplified exact value for

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\cos 2x + \sin 2x)(\ln \cos x + \ln \sin x) \, dx.$$

You may assume that

$$\int \cosec x \, dx = \ln \left| \tan \left(\frac{1}{2}x \right) \right| + \text{constant}.$$

, $\frac{1}{2} \ln 2$

$\begin{aligned} \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\cos 2x + \sin 2x)(\ln \cos x + \ln \sin x) \, dx &= \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\cos 2x + \sin 2x) \ln(\cos 2x) \, dx \\ &= \left[\frac{1}{2}(-\sin 2x - \cos 2x) \ln(\cos 2x) \right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} - \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\sin 2x - \cos 2x) \frac{\cos 2x}{\sin 2x} \, dx \\ &= \left[\frac{1}{2}(-\sin 2x - \cos 2x) \ln(\cos 2x) \right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} - \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cos 2x - \frac{\cos^2 2x}{\sin 2x} \, dx \\ &\quad - \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{1}{2}(\sin 2x - \cos 2x) \ln(\cos 2x) \, dx - \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cos 2x - \frac{1 - \sin^2 2x}{\sin 2x} \, dx \\ &\quad - \left[\frac{1}{4}(\sin 2x - \cos 2x) \ln(\cos 2x) \right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} - \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cos 2x + \sin 2x - (\cos 2x)^2 \, dx \\ &\quad - \left[\frac{1}{2}(\sin 2x - \cos 2x) \ln(\cos 2x) - \frac{1}{2} \sin 2x + \frac{1}{2} \cos 2x + \frac{1}{2} \ln(\cos 2x) \right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \\ &\quad - \left[\frac{1}{2} \ln(\frac{1}{2} \sin 2x) - 0 - \frac{1}{2} + \frac{1}{2} \ln(\cos \frac{\pi}{4}) \right] - \left[\frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} + 0 + \frac{1}{2} \ln 2 \right] \\ &= \frac{1}{2} \ln \left(\sin \frac{\pi}{4} \cos \frac{\pi}{4} \right) - \frac{1}{2} + \frac{1}{2} \ln \left(\frac{\sin \frac{\pi}{4}}{\cos \frac{\pi}{4}} \right) - \frac{1}{2} \ln \frac{1}{2} + \frac{1}{2} \ln 2 \\ &= \frac{1}{2} \ln \left[\sin \frac{\pi}{4} \cos \frac{\pi}{4} \times \frac{\sin \frac{\pi}{4}}{\cos \frac{\pi}{4}} \right] + \frac{1}{2} \ln 2 \\ &= \frac{1}{2} \ln 1 + \frac{1}{2} \ln 2 \\ &= \frac{1}{2} \ln 2 \end{aligned}$	<small>INTEGRATION BY PARTS</small> <table border="1" style="width: 100%; border-collapse: collapse;"> <tr> <td style="padding: 5px;">$\ln(\cos 2x)$</td> <td style="padding: 5px;">$\frac{2\cos 2x}{\sin 2x}$</td> </tr> <tr> <td style="padding: 5px;">$\frac{1}{2}(\sin 2x - \cos 2x)$</td> <td style="padding: 5px;">$\ln(\cos 2x) + C$</td> </tr> </table>	$\ln(\cos 2x)$	$\frac{2\cos 2x}{\sin 2x}$	$\frac{1}{2}(\sin 2x - \cos 2x)$	$\ln(\cos 2x) + C$
$\ln(\cos 2x)$	$\frac{2\cos 2x}{\sin 2x}$				
$\frac{1}{2}(\sin 2x - \cos 2x)$	$\ln(\cos 2x) + C$				

Question 278 (*****)

It is given that

$$x^2 + x + 2 = (u - x)^2.$$

a) Show clearly that ...

i. ... $x = \frac{u^2 - 2}{2u + 1}$.

ii. ... $\frac{dx}{du} = \frac{2(u^2 + u + 2)}{(2u + 1)^2}$.

b) Find a simplified expression for

$$\int \frac{1}{x\sqrt{x^2 + x + 2}} dx.$$

$\frac{1}{\sqrt{2}} \ln \left| \frac{x + \sqrt{x^2 + x + 2} - \sqrt{2}}{x + \sqrt{x^2 + x + 2} + \sqrt{2}} \right| + C$

(a) $x^2 + x + 2 = (u - x)^2$
 $x^2 + x + 2 = u^2 - 2ux + x^2$
 $2ux + 2 = u^2 - 2u$
 $2ux + 2 = u^2 - 2u$
 $2(2u+1) = u^2 - 2u$
 $2(2u+1) = u^2 - 2u$
 $2 = \frac{u^2 - 2u}{2u+1}$ ~~cross multiply~~
 $2 = \frac{u^2 - 2u}{2u+1}$

(b) $\frac{du}{dx} = \frac{(2u+1)(2u+2) - (u^2 - 2u)(2)}{(2u+1)^2}$
 $\frac{du}{dx} = \frac{4u^2 + 4u + 2 - 2u^2 + 4u}{(2u+1)^2}$
 $\frac{du}{dx} = \frac{2u^2 + 8u + 2}{(2u+1)^2}$
 $\frac{du}{dx} = \frac{2(u^2 + 4u + 1)}{(2u+1)^2}$ ~~cancel~~
 $\frac{du}{dx} = \frac{2(u+1)^2}{(2u+1)^2}$ ~~cancel~~

$\int \frac{1}{x\sqrt{x^2 + x + 2}} dx = \int \frac{1}{x} \frac{1}{\sqrt{(2u+1)^2}} \times \frac{2(u+1)^2}{(2u+1)^2} du$

$= \int \frac{1}{\frac{u^2 - 2}{2u+1}} \times \frac{2(u+1)^2}{(2u+1)^2} du$
 $= \int \frac{1}{\frac{u^2 - 2}{2u+1}} \times \frac{2(u^2 + 4u + 1)}{(2u+1)^2} du$
 $= \int \frac{\frac{2(u^2 + 4u + 1)}{2u+1}}{\frac{u^2 - 2}{2u+1}} du = \int \frac{2(u^2 + 4u + 1)}{(u^2 - 2)(2u+1)} du$
 $= \int \frac{2}{u^2 - 2} du$...
Now by partial fractions or using standard result $\int \frac{1}{u^2 - a^2} du = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| + C$
 $= -\frac{2}{2u^2 - 4} \ln \left| \frac{u-\sqrt{u^2-2}}{u+\sqrt{u^2-2}} \right| + C$
Now $u-x = \sqrt{u^2-2}$
 $u = x + \sqrt{x^2+x+2}$
 $= \frac{1}{\sqrt{2}} \ln \left| \frac{2 + \sqrt{2x^2+2x+2} - \sqrt{2}}{2 + \sqrt{2x^2+2x+2} + \sqrt{2}} \right| + C$

Question 279 (*****)

It is given that a and b are distinct real constants and λ is a real parameter.

- a) Starting by the relationship between two functions of x , $f(x)$ and $g(x)$

$$[\lambda f(x) + g(x)]^2 \geq 0,$$

show clearly that

$$\lambda^2 \int_a^b [f(x)]^2 dx + 2\lambda \int_a^b f(x)g(x) dx + \int_a^b [g(x)]^2 dx \geq 0.$$

- b) Deduce the Cauchy Schwarz inequality for integrals

$$\left[\int_a^b f(x)g(x) dx \right]^2 \leq \left[\int_a^b [f(x)]^2 dx \right] \left[\int_a^b [g(x)]^2 dx \right].$$

[continues overleaf]

[continued from overleaf]

c) By letting $f(x) = \sqrt{\sin x}$ and $g(x) = 1$, show that

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx \leq \sqrt{\frac{\pi}{2}}.$$

d) By letting $f(x) = \sqrt{\sqrt{\sin x}}$ and $g(x) = \cos x$, show that

$$\int_0^{\frac{\pi}{2}} \sqrt{\sqrt{\sin x}} dx \geq \frac{64}{25\pi}.$$

proof

(a)

$$\begin{aligned} & [f(a) + g(b)]^2 > 0 \\ & f(a)^2 + 2f(a)g(b) + g(b)^2 > 0 \\ & \int_a^b [f(a)]^2 dx + \int_a^b 2f(a)g(b) dx + \int_a^b [g(b)]^2 dx \geq [c]^b \\ & 2^2 \int_a^b [g(b)]^2 dx + 2 \times \int_a^b f(a)g(b) dx + \int_a^b [f(a)]^2 dx \geq c - c \\ & 2^2 \int_a^b [g(b)]^2 dx + 2 \times \int_a^b f(a)g(b) dx + \int_a^b [f(a)]^2 dx > 0 \end{aligned}$$

(b) IF THE QUADRATIC INEQUALITY FOR λ , SINCE THE DEFINITE INTEGRALS ARE CONSTANT NUMBERS, THE DISCRIMINANT MUST BE NEGATIVE OR ZERO

THUS

$$\begin{aligned} & 4 \left[\int_a^b f(a)g(b) dx \right]^2 - 4 \int_a^b [f(a)]^2 dx \int_a^b [g(b)]^2 dx \leq 0 \\ & \left[\int_a^b f(a)g(b) dx \right]^2 \leq \int_a^b [f(a)]^2 dx \int_a^b [g(b)]^2 dx \end{aligned}$$

(c)

$$\begin{aligned} & f(x) = \sqrt{\sin x} \\ & g(x) = \cos x \\ & f(a) = \sqrt{\sin a} \\ & g(b) = \cos b \end{aligned}$$

$$\begin{aligned} & \left[\int_a^b f(a)g(b) dx \right]^2 \leq \int_a^b (\sin x)^{\frac{1}{2}} dx \int_a^b \cos x dx \\ & \left[\frac{4}{5} (\sin x)^{\frac{5}{2}} \right]^{\frac{2}{5}} \leq \int_a^b \sqrt{\sin x} dx \int_a^b \frac{1}{2} \cos x dx \\ & \left(\frac{16}{25} \right)^{\frac{2}{5}} \leq \int_a^b \sqrt{\sin x} dx \times \left[\frac{1}{2} \cos x \right]_a^b \\ & \frac{16}{25} \leq \int_a^b \sqrt{\sin x} dx \times \frac{1}{2} \\ & \int_a^b \sqrt{\sin x} dx > \frac{16}{25} \end{aligned}$$

Question 280 (*****)

By using the substitution $\sqrt{x} = \tan \theta$, or otherwise, find

$$\int \frac{(x+3)\sqrt{x}}{(x+1)^2} dx.$$

, $\frac{2x^{\frac{3}{2}}}{x+1} + C$

Using the substitution \sqrt{x}

$$\begin{aligned} \sqrt{x} &= \tan \theta \quad [i.e. \theta = \arctan \sqrt{x}] \\ x &= \tan^2 \theta \\ dx &= 2\tan \theta \sec^2 \theta d\theta \end{aligned}$$

TRANSFORMING THE INTEGRAL

$$\begin{aligned} \int \frac{(x+3)\sqrt{x}}{(x+1)^2} dx &= \int \frac{(3+\tan^2 \theta) \sec^2 \theta}{(\tan^2 \theta + 1)^2} \times 2\tan \theta \sec^2 \theta d\theta \\ &= \int \frac{2\tan \theta \sec^2 \theta (3+\tan^2 \theta)}{(\sec^2 \theta)^2} d\theta \\ &= \int \frac{2\tan^2 \theta (3+\tan^2 \theta)}{\sec^2 \theta} d\theta \end{aligned}$$

Simplifying all terms

$$\begin{aligned} &= \int \frac{2(\sec^2 \theta - 1)(3 + \sec^2 \theta - 1)}{\sec^2 \theta} d\theta \\ &= \int \frac{2(\sec^2 \theta - 1)(\sec^2 \theta + 2)}{\sec^2 \theta} d\theta \\ &= \int \frac{2\sec^2 \theta + 2\sec^2 \theta - 4}{\sec^2 \theta} d\theta \\ &= \int 2\sec^2 \theta + 2 - \frac{4}{\sec^2 \theta} d\theta \end{aligned}$$

$$\begin{aligned} &= \int 2\tan^2 \theta + 2 - 4\cot^2 \theta d\theta \\ &= \int 2\tan^2 \theta + 2 - 4\left(\frac{1}{1+\tan^2 \theta}\right) d\theta \\ &= \int 2\tan^2 \theta + 2 - \frac{4}{1+\tan^2 \theta} d\theta \\ &\quad - 2\tan^2 \theta - 2\cot^2 \theta + C \\ &= 2\tan^2 \theta - 2\cot^2 \theta + C \\ &= 2\tan^2 \theta - \frac{2\sec^2 \theta}{\tan^2 \theta} \times \sec^2 \theta + C \\ &= 2\tan^2 \theta - \frac{2\sec^2 \theta}{1+\tan^2 \theta} + C \\ &= 2\tan^2 \theta - \frac{2\frac{1}{\cos^2 \theta}}{1+\frac{\sin^2 \theta}{\cos^2 \theta}} + C \\ &= 2\tan^2 \theta - \frac{2}{1+\tan^2 \theta} + C \\ &= 2\tan^2 \theta \left[1 - \frac{1}{1+\tan^2 \theta} \right] + C \\ &= 2\tan^2 \theta \left[\frac{2\tan^2 \theta}{1+\tan^2 \theta} \right] + C \\ &= 2\tan^2 \theta \left(\frac{2}{1+\tan^2 \theta} \right) + C \\ &= \frac{2\tan^4 \theta}{1+\tan^2 \theta} + C \end{aligned}$$

$\frac{2x^{\frac{3}{2}}}{x+1} + C$

Question 281 (*****)

By using the substitution $u = \sec x + \sqrt{\tan x}$, or otherwise, find

$$\int \frac{1+2\sin x\sqrt{\tan x}}{2[1+\cos x\sqrt{\tan x}]\cos x\sqrt{\tan x}} dx.$$

, $\ln|\sec x + \sqrt{\tan x}| + C$

<p><u>USING THE SUBSTITUTION GIVEN</u></p> $u = \sec x + \sqrt{\tan x} = \sec x + (\tan x)^{\frac{1}{2}}$ $\frac{du}{dx} = \sec x \tan x + \frac{1}{2}(\tan x)^{-\frac{1}{2}} \sec^2 x$ $\frac{du}{dx} = \frac{1}{\cos x} \frac{\sin x}{\cos x} + \frac{(\cos x)^{\frac{1}{2}}}{2(\sin x)^{\frac{1}{2}} \cos^2 x}$ $\frac{du}{dx} = \frac{\sin x}{\cos x} + \frac{(\cos x)^{\frac{1}{2}}}{2(\sin x)^{\frac{1}{2}} \cos^2 x}$ $\frac{du}{dx} = \frac{1}{\cos x} \left[\sin x + \frac{(\cos x)^{\frac{1}{2}}}{2(\sin x)^{\frac{1}{2}}} \right]$ $\frac{du}{dx} = \frac{1}{\cos x} \left[\frac{2\sin x \cos x + \cos^2 x}{2\sin x \cos x} \right]$ $\frac{du}{dx} = \frac{2\sin x + \cos^2 x}{2\sin x \cos x}$ $du = \frac{2\sin x + \cos^2 x}{2\sin x \cos x} dx$ <p><u>TRANSFORMING THE INTEGRAL</u></p> $\int \frac{2\sin x \cos x + 1}{2\sin x \cos x (\cos x + \sqrt{\tan x})} dx$ $= \int \frac{2\sin x \left(\frac{\cos x}{\sin x} + 1 \right)}{2\sin x \left(\frac{\cos x}{\sin x} \right) \left[\cos \left(\frac{\cos x}{\sin x} + 1 \right) \right]} \times \frac{2\sin x \cos x}{2\sin x \cos x} dx$ $= \int \frac{2\sin x \sec^2 x + 1}{2\sin x \sec x \left[\cos \left(\frac{\cos x}{\sin x} + 1 \right) + 1 \right]} \times \frac{2\sin x \cos x}{2\sin x \cos x} dx$	$= \int \frac{[\cos^2 x \sec^2 x + 1][\cos^2 x]}{[\cos^2 x + 1][2\sin x \cos x]} dx$ $= \int \frac{2\cos^2 x \cos x}{[\cos^2 x + 1][\cos x + \sqrt{\tan x}]} dx$ $= \int \frac{\cos x [\cos x + \sqrt{\tan x}]}{[\cos^2 x + 1][\cos x + \sqrt{\tan x}]} dx$ $= \int \frac{\cos x}{\cos^2 x + \cos x + 1} dx$ <p><u>Multiplying Top & Bottom by $\cos x$</u></p> $= \int \frac{\sec x \cos x}{\sec x \cos x + \sec x} dx$ $= \int \frac{1}{\sec x + \tan x} dx$ $= \int \frac{1}{\sqrt{1 + \tan^2 x}} dx$ $= \int \frac{1}{\sec x} dx$ $= \ln \sec x + C$ $= \ln \sqrt{1 + \tan^2 x} + \sec x + C$
--	--

Question 282 (*****)

It is given that

$$\int \frac{\cot x \operatorname{cosec} x + 2 \cot x}{1 + \operatorname{cosec} x} dx \equiv \ln [1 + f(x)] f(x) + \text{constant}.$$

Using integration techniques, determine an expression for $f(x)$.

, $f(x) = \sin x$

LOOKING AT THE INTEGRAND & NOTING THAT $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec}^2 x$

$$\begin{aligned} \int \frac{\cot x \operatorname{cosec} x + 2 \cot x}{1 + \operatorname{cosec} x} dx &= \int \frac{\operatorname{cosec} x \cot x}{\operatorname{cosec} x + 1} + \frac{2 \cot x}{\operatorname{cosec} x + 1} dx \\ &= \int \frac{\operatorname{cosec} x \cot x}{\operatorname{cosec} x + 1} dx + \int \frac{2 \cot x}{\operatorname{cosec} x + 1} dx \\ &= -\ln |\operatorname{cosec} x + 1| + 2 \int \frac{\cot x}{\operatorname{cosec} x + 1} dx \\ &= -\ln |\operatorname{cosec} x + 1| + 2 \int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\sin x} + 1} dx \quad \leftarrow \text{NOTING "TOP" & "BOTTOM" OF FRACTION}\right. \\ &= -\ln |\operatorname{cosec} x + 1| + 2 \int \frac{\sin x}{1 + \sin x} dx \\ \text{NOW THE INTEGRAL IS OF THE FORM "BOTTOM" DIFFERENTIATE TO TOP} \\ &= -\ln |\operatorname{cosec} x + 1| + 2 \ln |1 + \sin x| + C \\ &= \ln |(1 + \sin x)^2| - \ln |\operatorname{cosec} x + 1| + C \\ &= \ln (1 + \sin x)^2 - \ln \left| \frac{1}{\sin x} + 1 \right| + C \\ &= \ln (1 + \sin x)^2 - \ln \left| \frac{1 + \sin x}{\sin x} \right| + C \\ &= \ln (1 + \sin x)^2 + \ln \left| \frac{\sin x}{1 + \sin x} \right| + C \\ &= \ln \left| \frac{(\sin x)^2 \times \frac{\sin x}{1 + \sin x}}{1 + \sin x} \right| + C \\ &= \ln \left| \frac{(\sin x)^3}{1 + \sin x} \right| + C \quad \leftarrow \text{I.E. } f(x) = \sin x \end{aligned}$$

Question 283 (*****)

$$I = \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{6}{\sin x + \sin 2x} dx$$

Use appropriate integration techniques to show that

$$I = A \ln N + B \ln M ,$$

where A , B , N and M are integers to be found.

SOLN, $I = 8 \ln 2 - 3 \ln 3$

$$\begin{aligned} \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{6}{\sin x + \sin 2x} dx &= \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{6}{\sin x + 2\sin x \cos x} dx = \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{6}{\sin x(1+2\cos x)} dx \\ \text{BY SUBSTITUTION} \\ u &= \cos x & x = \frac{\pi}{2} & \rightarrow u = \frac{1}{2} \\ \frac{du}{dx} &= -\sin x & x = \frac{\pi}{3} & \rightarrow u = 0 \\ du &= -\frac{du}{\sin x} & x = 0 & \rightarrow u = 0 \\ &= \int_0^{\frac{1}{2}} \frac{6}{\sin x(1+2u)} \left(-\frac{du}{\sin x}\right) = \int_0^{\frac{1}{2}} \frac{6}{\sin^2 x(1+2u)} du \\ &= \int_0^{\frac{1}{2}} \frac{6}{(1-\cos^2 x)(1+2u)} du = \int_0^{\frac{1}{2}} \frac{6}{(1-u^2)(1+2u)} du \\ &= \int_0^{\frac{1}{2}} \frac{6}{(1-u)(1+u)(1+2u)} du \\ \text{BY PARTIAL FRACTIONS} \\ \frac{6}{(1-u)(1+u)(1+2u)} &= \frac{A}{2u+1} + \frac{B}{1-u} + \frac{C}{1+u} \\ 6 &= A(1-u)(1+u) + B(2u+1)(1+u) + C(2u+1)(1-u) \\ \bullet \text{ if } u=1 \rightarrow 6 = 6B \Rightarrow B=1 \\ \bullet \text{ if } u=-1 \rightarrow 6 = -2C \Rightarrow C=-3 \\ \bullet \text{ if } u=\frac{1}{2} \rightarrow 6 = \frac{3}{2}A \Rightarrow A=8 \end{aligned}$$

$$\begin{aligned} &\dots = \int_0^{\frac{1}{2}} \frac{8}{2u+1} + \frac{1}{1-u} - \frac{3}{1+u} du \\ &= \int_0^{\frac{1}{2}} \frac{8}{2u+1} - \frac{1}{u-1} - \frac{3}{u+1} du \\ &= \left[4 \ln|2u+1| - \ln|u-1| - 3 \ln|u+1| \right]_0^{\frac{1}{2}} \\ &= (4 \ln 2 - \ln \frac{1}{2} - 3 \ln \frac{1}{2}) - (4 \ln 1 - \ln 1 - 3 \ln 1) \\ &= 4 \ln 2 + \ln 2 - 3 \ln 3 + 3 \ln 2 \\ &= 8 \ln 2 - 3 \ln 3 \end{aligned}$$

Question 284 (*****)

Use the substitution $x = \tan\left(\frac{1}{2}\theta\right)$, to find a simplified expression for

$$\int x \arccos\left[\frac{1-x^2}{1+x^2}\right] dx.$$

$-x + (1+x^2)\arctan x + \text{constant}$

<p>USING THE SUBSTITUTION OWN</p> <p>$x = \tan\left(\frac{1}{2}\theta\right)$ $dx = \sec^2\left(\frac{1}{2}\theta\right) d\theta$</p> <p>or $[dx = \frac{1}{2}(1+\tan^2\left(\frac{1}{2}\theta\right)) d\theta]$ or $[dx = \frac{1}{2}(1+x^2) d\theta]$</p> <p>(we shall see which form is better for this question)</p> <p>... ...</p> <p>TRANSFORMING THE INTEGRAL WE HAVE</p> $\begin{aligned} \int 2x \arccos\left(\frac{1-x^2}{1+x^2}\right) dx &= \int \tan\left(\frac{1}{2}\theta\right) \arccos\left(\frac{1-\tan^2\left(\frac{1}{2}\theta\right)}{1+\tan^2\left(\frac{1}{2}\theta\right)}\right) \sec^2\left(\frac{1}{2}\theta\right) d\theta \\ &= \int \frac{1}{2} \tan\left(\frac{1}{2}\theta\right) \sec^2\left(\frac{1}{2}\theta\right) d\theta \end{aligned}$ <p>INTEGRATION BY PARTS</p> $\begin{array}{c c} \frac{1}{2}\theta & \frac{1}{2} \\ \hline \tan^2\left(\frac{1}{2}\theta\right) & \sec^2\left(\frac{1}{2}\theta\right) \end{array}$ $\begin{aligned} \dots &= \frac{1}{2}\theta \tan^2\left(\frac{1}{2}\theta\right) - \int \frac{1}{2} \sec^2\left(\frac{1}{2}\theta\right) d\theta \\ &= \frac{1}{2}\theta \tan^2\left(\frac{1}{2}\theta\right) - \frac{1}{2} \int \sec^2\frac{\theta}{2} - 1 d\theta \end{aligned}$	$\begin{aligned} \frac{1-x^2}{1+x^2} &= \frac{1-\tan^2\left(\frac{1}{2}\theta\right)}{1+\tan^2\left(\frac{1}{2}\theta\right)} \\ &= \frac{1-\tan^2\left(\frac{1}{2}\theta\right)}{\sec^2\left(\frac{1}{2}\theta\right)} \\ &= \frac{1}{\sec^2\left(\frac{1}{2}\theta\right)} - \frac{\tan^2\left(\frac{1}{2}\theta\right)}{\sec^2\left(\frac{1}{2}\theta\right)} \\ &= (\cos^2\left(\frac{1}{2}\theta\right)) - \frac{\sin^2\left(\frac{1}{2}\theta\right)}{\cos^2\left(\frac{1}{2}\theta\right)} \\ &= (\cos^2\left(\frac{1}{2}\theta\right)) - \frac{\sin^2\left(\frac{1}{2}\theta\right)}{\cos^2\left(\frac{1}{2}\theta\right)} \\ &= (\cos^2\left(\frac{1}{2}\theta\right)) - \frac{1-\cos^2\left(\frac{1}{2}\theta\right)}{\cos^2\left(\frac{1}{2}\theta\right)} \\ &= (\cos^2\left(\frac{1}{2}\theta\right)) - \frac{1}{\cos^2\left(\frac{1}{2}\theta\right)} \\ &= (\cos^2\left(\frac{1}{2}\theta\right)) - \frac{1}{\sec^2\left(\frac{1}{2}\theta\right)} \end{aligned}$ <p>REVERSE SUBSTITUTION</p> <p>$\boxed{x = \tan\left(\frac{1}{2}\theta\right)}$</p> $\begin{aligned} &= \frac{1}{2}\theta \ln^2\left(\frac{1}{2}\theta\right) - \frac{1}{2} [\tan\left(\frac{1}{2}\theta\right) - 1] + C \\ &= \frac{1}{2}\theta \ln^2\left(\frac{1}{2}\theta\right) - \tan\left(\frac{1}{2}\theta\right) + \frac{1}{2}\theta + C \\ &= \frac{1}{2}\theta \left(1 + \tan^2\left(\frac{1}{2}\theta\right)\right) - \tan\left(\frac{1}{2}\theta\right) + C \\ &= \boxed{-x + (1+x^2)\arctan x + C} \end{aligned}$
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Question 285 (*****)

By using a suitable trigonometric substitution, or otherwise, find

$$\int \frac{(3x^2 + 5x)\sqrt{x}}{(x+1)^2} dx.$$

, $\frac{2x^{\frac{5}{2}}}{x+1} + C$

Start by the substitution $\sqrt{x} = \tan\theta$.

$$\begin{aligned} & \int \frac{(3x^2 + 5x)\sqrt{x}}{(x+1)^2} dx \\ &= \int \frac{(3\tan^4\theta + 5\tan^2\theta)\sec^2\theta}{(\tan^2\theta + 1)^2} (\sec^2\theta d\theta) \\ &= \int \frac{2\tan^2\theta\sec^2\theta (3\tan^2\theta + 5\tan\theta)}{\sec^4\theta} d\theta \\ &= \int \frac{6\tan^4\theta\sec^2\theta + 10\tan^3\theta\sec^2\theta}{\sec^4\theta} d\theta \\ &= \int \frac{6\tan^4\theta + 10\tan^3\theta}{\sec^2\theta} d\theta = \int 6\tan^2\theta\cos^2\theta + 10\tan^3\theta\cos^2\theta d\theta \\ &= \int \frac{6\tan^2\theta \times \cos^2\theta + 10\tan^3\theta \times \cos^2\theta}{\cos^2\theta} d\theta \\ &= \int \frac{6\tan^2\theta}{\cos^2\theta} + \frac{10\tan^3\theta}{\cos^2\theta} d\theta \\ &= \int \frac{6(1 - \tan^2\theta)^2 + 10(1 - \tan^2\theta)^3}{\cos^2\theta} d\theta \\ &= \int \frac{6(1 - 3\tan^2\theta + 3\tan^4\theta - \tan^6\theta) + 10(1 - 2\tan^2\theta + \tan^4\theta)}{\cos^2\theta} d\theta \\ &= \int \frac{6}{\cos^2\theta} - \frac{18}{\cos^2\theta} + 18 - 6\tan^2\theta + \frac{10}{\cos^2\theta} - 20 + 10\tan^2\theta d\theta \\ &= \int 6\sec^2\theta - 8\sec^2\theta + 4\tan^2\theta - 2 d\theta \end{aligned}$$

Reducing the integrand as follows

$$\begin{aligned} &= \int 6\sec^2\theta(1 + \tan^2\theta) - 8\sec^2\theta + 4\left(\frac{1}{2} + \frac{1}{2}\tan^2\theta\right) - 2 d\theta \\ &= \int (6\sec^2\theta + 6\tan^2\theta\sec^2\theta) - 8\sec^2\theta + \sqrt{2 + 2\tan^2\theta} d\theta \\ &= \int 6\sec^2\theta\sec^2\theta - 2\sec^2\theta + 2\tan^2\theta d\theta \\ &= 2\tan^3\theta - 2\tan\theta + \sin 2\theta + C \\ &= 2\tan^2\theta - 2\tan\theta + 2\tan\theta\cot\theta + C \\ &= 2\tan^2\theta - 2\tan\theta + \frac{2\tan\theta\cot\theta}{\cot\theta} + C \\ &= 2\tan^2\theta - 2\tan\theta + 2\tan\theta\cot\theta + C \\ &= 2\tan^2\theta - 2\tan\theta + \frac{2\tan\theta}{1 + \tan^2\theta} + C \\ &= 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^{\frac{1}{2}} + \frac{2\left(\frac{1}{2}\right)}{1 + \left(\frac{1}{2}\right)^2} + C \\ &= \frac{2\left(\frac{1}{2}\right)^{\frac{1}{2}} + 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^{\frac{1}{2}} + 2\left(\frac{1}{2}\right)}{1 + \left(\frac{1}{2}\right)^2} + C \\ &= \frac{2\left(\frac{1}{2}\right)^{\frac{1}{2}}}{1 + \left(\frac{1}{2}\right)^2} + C \end{aligned}$$

Question 286 (*****)

The function f is defined as

$$f(x) \equiv 2^{\ln x}, \quad x \in [1, \infty).$$

Show, with details workings, that

$$\int_1^e f(x) \, dx = \frac{2e-1}{1+\ln 2}.$$

, proof

TRANSPOSE INTO EXPONENTIALS AS FOLLOWS:

$$\int_1^e 2^{\ln x} \, dx = \int_1^e e^{\ln(2^{\ln x})} \, dx = \int_1^e e^{(\ln 2)(\ln x)} \, dx$$

MANIPULATE FURTHER

$$= \int_1^e (e^{\ln x})^{\ln 2} \, dx = \int_1^e 2^{\ln x} \, dx$$

INTEGRATE AND SIMPLIFY

$$= \left[\frac{1}{1+\ln 2} x^{1+\ln 2} \right]_1^e = \frac{1}{1+\ln 2} \left[e^{1+\ln 2} - 1^{1+\ln 2} \right]$$

$$= \frac{1}{1+\ln 2} \left[e^{1+\ln 2} - 1 \right] = \frac{1}{1+\ln 2} [2e-1]$$

$$\therefore \int_1^e 2^{\ln x} \, dx = \frac{2e-1}{1+\ln 2}$$

Question 287 (***)**

A function F is defined by the integral

$$F(x) \equiv \int_1^x \frac{e^t}{t} dt, \quad x \geq 1.$$

Find a simplified expression, in terms of F , for

$$\int_1^x \frac{e^t}{t+a} dt,$$

where a is a positive constant.

	$\int_1^x \frac{e^t}{t+a} dt = e^{-a} [F(x+a) - F(a+1)]$
--	--

$F(x) = \int_1^\infty \frac{e^t}{t} dt$

$\int_1^x \frac{e^t}{t+a} dt = \dots \text{ BY SUBSTITUTION}$

$u = t+a$
 $du = dt$
 $t=1 \mapsto a+1$
 $t=x \mapsto x+a$

$$\dots = \int_{a+1}^{a+x} \frac{e^u}{u-a} du = \int_{a+1}^{a+x} \frac{e^u \times e^{-a}}{u} du$$

$$= e^{-a} \int_{a+1}^{a+x} \frac{e^u}{u} du = e^{-a} \left[\int_1^{a+x} \frac{e^u}{u} du - \int_1^{a+1} \frac{e^u}{u} du \right]$$

$$= e^{-a} [F(a+x) - F(a+1)]$$

Question 288 (*****)

It is given that

$$u^2 = \frac{1-x^2}{(1-x)^2}, \quad x \neq \pm 1.$$

a) Show clearly that ...

i. ... $x = \frac{u^2 - 1}{u^2 + 1}$.

ii. ... $1 - x^2 = \frac{4u^2}{(u^2 + 1)^2}$

iii. ... $\frac{dx}{du} = \frac{4u}{(u^2 + 1)^2}$.

b) Hence show further that

$$\int \frac{3}{(4x+5)\sqrt{1-x^2} - 3(1-x^2)} dx = \frac{2\sqrt{1-x}}{\sqrt{1-x} - 3\sqrt{1+x}} + \text{constant}.$$

□, proof

a) Transforming to u

$$u^2 = \frac{1-x^2}{(1-x)^2} = \frac{(1-x)(1+x)}{(1-x)^2} = \frac{1+x}{1-x}$$

$$\Rightarrow u^2(1-x) = 1+x$$

$$\Rightarrow u^2 - xu^2 = 1+x$$

$$\Rightarrow u^2(1-x) = 2u^2 + x$$

$$\Rightarrow u^2 - 1 = x(u^2 + 1)$$

$$\Rightarrow 2u^2 - 2 = x(u^2 + 1)$$

// AS REPO

using the above result

$$1-x^2 = 1 - \left(\frac{u^2+1}{u^2+u}\right)^2 = 1 - \frac{u^4 - 2u^2 + 1}{u^4 + 2u^2 + 1}$$

$$= \frac{u^4 + 2u^2 + 1 - (u^4 - 2u^2 + 1)}{u^4 + 2u^2 + 1}$$

$$= \frac{4u^2}{(u^2+1)^2}$$

// AS REPO

(b) differentiate (a) with respect to u

$$2u = \frac{u^2 - 1}{u^2 + 1} = \frac{(u^2 + 1) - 2}{u^2 + 1} = 1 - \frac{2}{u^2 + 1} = (-2(u^2 + 1)^{-2})$$

$$\frac{du}{dx} = 0 + 2(u^2 + 1)^2 \times (2u)$$

$$\frac{du}{dx} = \frac{4u}{(u^2 + 1)^2}$$

b) using the results from part (a)

$$\int \frac{3}{(2u+5)\sqrt{1-u^2} - 3(1-u^2)} du$$

$$= \int \frac{3}{\left[4\left(\frac{u^2-1}{u^2+1}\right)u + 5\right]\sqrt{1-u^2} - 3(1-u^2)} \times \frac{4u}{(u^2+1)^2} du$$

$$= \int \frac{3}{\left[\frac{4(u^2-1)u^2+5u^2+5}{u^2+1}\right]\sqrt{1-u^2} - 3(1-u^2)} du$$

$$= \int \frac{3}{\frac{4u^4+12u^2+5}{u^2+1}\sqrt{1-u^2} - 3(1-u^2)} du$$

$$= \int \frac{3u}{(4u^4+12u^2+5)\sqrt{1-u^2} - 6u^2} du = \int \frac{3}{(4u^4+12u^2+5) - 6u^2} du$$

$$= \int \frac{3}{4u^4+6u^2+5} du = \int \frac{3}{(2u^2+1)^2} du = \int \frac{3}{(2u^2+1)} du$$

$$= -\frac{3}{2} \left(\frac{1}{2u^2+1}\right)^{-1} + C = \frac{-3}{2u^2+1} + C = \frac{2}{(1-x)} + C$$

$$= \frac{2}{1-3\frac{(1-x)^2}{(1-x)}} + C = \frac{2(1-x)}{1-x-3(1-x)} + C$$

$$= \frac{2(1-x)}{(1-x)-3(1-x)\sqrt{1-x}} + C = \frac{2\sqrt{1-x}}{\sqrt{1-x}-3\sqrt{1+x}} + C$$

Question 289 (*****)

By using the substitution $x = 2 \tan^2 \theta$, or otherwise, find

$$\int \frac{2-x}{\sqrt{x(x+2)^2}} dx.$$

$$\boxed{\quad}, \boxed{\frac{2\sqrt{x}}{x+2} + C}$$

$$\int \frac{2-x}{\sqrt{x(x+2)^2}} dx$$

BY SUBSTITUTION

$$u = 2 \tan^2 \theta$$

$$\frac{du}{d\theta} = 4 \tan \theta \sec^2 \theta$$

$$du = 4 \tan \theta \sec^2 \theta d\theta$$

$$\frac{u}{2} = \tan^2 \theta$$

$$\tan \theta = (\frac{u}{2})^{1/2}$$

$$= \int \frac{2 - 2 \tan^2 \theta}{\sqrt{2 \tan^2 \theta (2 \tan^2 \theta + 2)^2}} (4 \tan \theta \sec^2 \theta d\theta)$$

$$= \int \frac{2(1 - \tan^2 \theta) \sec^2 \theta}{\sqrt{2} \tan^2 \theta \times 4 (\tan^2 \theta + 1)^2} d\theta = \frac{2}{\sqrt{2}} \int \frac{(1 - \tan^2 \theta) \sec^2 \theta}{(\sec^2 \theta)^2} d\theta$$

$$= \sqrt{2} \int \frac{(1 - \tan^2 \theta) \sec^2 \theta}{\sec^4 \theta} d\theta = \sqrt{2} \int \frac{1 - \tan^2 \theta}{\sec^2 \theta} d\theta$$

$$= \sqrt{2} \int \frac{1}{\sec^2 \theta} - \frac{\tan^2 \theta}{\sec^2 \theta} d\theta = \sqrt{2} \int \cos^2 \theta - \frac{\sin^2 \theta}{\cos^2 \theta} \cos^2 \theta d\theta$$

$$= \sqrt{2} \int \cos^2 \theta - \sin^2 \theta d\theta = \sqrt{2} \int \cos 2\theta d\theta = \frac{1}{2} \sin 2\theta + C$$

$$= \frac{\sqrt{2}}{2} (2 \sin \theta \cos \theta) + C = \sqrt{2} \sin \theta \cos \theta + C = \sqrt{2} \frac{\cos \theta \sin \theta}{\cos^2 \theta} + C$$

$$= \sqrt{2} \frac{\cos \theta \tan \theta}{\sec^2 \theta} + C = \sqrt{2} \frac{\tan \theta}{\sec^2 \theta} + C = \sqrt{2} \frac{\tan \theta}{1 + \tan^2 \theta} + C$$

$$= \sqrt{2} \frac{\frac{\sqrt{x}}{2}}{1 + \frac{x}{2}} + C = \frac{\sqrt{2}}{1 + \frac{x}{2}} + C = \frac{2\sqrt{x}}{2+x} + C$$

Question 290 (*****)

It is given that

$$\sqrt{5-4x-x^2} = (1-x)u, \quad x \neq 1, \quad x \neq -5.$$

a) Show clearly that ...

i. ... $x = \frac{u^2 - 5}{u^2 + 1}$.

ii. ... $dx = \frac{12u}{(u^2 + 1)^2} du$.

b) Hence show further that

$$\int \frac{x}{(5-4x-x^2)^{\frac{3}{2}}} dx = \int \frac{u^2 - 5}{18u^2} du.$$

c) Find a simplified expression for

$$\int \frac{x}{(5-4x-x^2)^{\frac{3}{2}}} dx.$$

 , $\frac{5-2x}{9\sqrt{5-4x-x^2}} + C$

Q12 SWAPPING & TRANSFORMING

$$\begin{aligned} \sqrt{5-4x-x^2} &\sim (1-x)u \\ 5-4x-x^2 &= (1-x)u^2 \\ -(5-4x-x^2) &= u^2(2-x)^2 \\ -5+4x+x^2 &= u^2(2-x)^2 \\ +4x &= u^2(2-x)^2 \\ 4x &= u^2(2-x)^2 \\ 4x &= u^2(3-x) \end{aligned}$$

Q12 Differentiate the quotient with respect to u

$$\begin{aligned} \frac{du}{dx} &= \frac{(1-x)(2u) - (2-x)(1)}{(2u)^2} = \frac{2u^2 + 2u - 2x^2 + 10x}{(2u)^2} \\ \frac{du}{dx} &= \frac{2u}{(2u)^2} \\ du &= \frac{2u}{(2u)^2} du \end{aligned}$$

Q13 ISOLATE THE PLANE FROM PART (a)

$$\begin{aligned} \int \frac{x}{(5-4x-x^2)^{\frac{3}{2}}} dx &= \int \frac{x}{(1-x)u^2} \times \frac{12u}{(1+u^2)^2} du \\ &= \int \frac{\frac{u^2-5}{(u^2+1)^2}}{(1-\frac{u^2-5}{(u^2+1)})^3} \times \frac{12u}{(1+u^2)^2} du \\ &= \int \frac{1}{(u^2+1)^4} \times \frac{u^2-5}{(u^2+1)} \times \frac{12u}{(1+u^2)^2} du \\ &= \int \frac{1}{(u^2+1)^4} \times \frac{12u(u^2-5)}{(u^2+1)^3} du \\ &= \int \frac{12u(u^2-5)}{(u^2+1)^7} du \\ &= \int \frac{12u(u^2-5)}{2u^4u^3} du \\ &= \int \frac{u^2-5}{2u^4} du \end{aligned}$$

Q13 SPLIT THE GENERAL AND INTEGRATE

$$\begin{aligned} &= \int \frac{1}{16} \times \frac{u^2-5}{u^4} du \\ &= \frac{1}{16} u^{-2} + C \end{aligned}$$

$$= \frac{1}{16} u + \frac{5}{16u} + C$$

$$= \frac{1}{16u} [u^2 + 5] + C$$

NOW REIN PART (a)

$$(2)(1-u) = -2(1-u)$$

$$u = -\frac{2(1-u)}{2(1-u)} = -\frac{1}{1-u}$$

$$u^2 = \frac{1}{(1-u)^2}$$

TIDYING UP Finally

$$\begin{aligned} &= \dots \frac{1}{16} \times \frac{1}{(1-u)^2} \times \left[\frac{2+5}{1-u} + 5 \right] + C \\ &= \dots \frac{1}{16} \times \frac{1}{(1-u)^2} \times \frac{2x+5-5x}{1-u} + C \\ &= \frac{1}{16} \times \frac{1}{(1-u)^2} \times \frac{10-3x}{1-u} + C \\ &= \frac{1}{16} \times \frac{1}{(1-u)^2} \times \frac{10-4x}{1-u} + C \\ &= \frac{1}{16} \times \frac{2(5-x)}{4(1-u)(1-x)} + C \\ &= \frac{5-2x}{9(4x-4u-x^2)} + C \end{aligned}$$

Question 291 (*****)

By using an appropriate trigonometric substitution, or otherwise, find an exact value for the following integral.

$$\int_7^9 \sqrt{\frac{x-7}{11-x}} dx.$$

, $\pi - 2$

$\int_7^9 \sqrt{\frac{x-7}{11-x}} dx = \pi - 2$

THE INTEGRAND IS OF THE FORM $\int (\sqrt{ax+b}, \sqrt{c-x}) dx$.
STANDARD SUBSTITUTION IS $x = a\cos^2\theta + b\sin^2\theta$

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{1+4\sin^2\theta-1}{11-(1+4\sin^2\theta)}} (8\cos\theta\sin\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{4\sin^2\theta}{10-4\sin^2\theta}} (8\cos\theta\sin\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{4\sin^2\theta}{4(1-\sin^2\theta)}} (8\cos\theta\sin\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} (8\cos\theta\sin\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta} (8\cos\theta\sin\theta) d\theta \\ &\quad - \int_0^{\frac{\pi}{2}} 8\sin^2\theta d\theta = \int_0^{\frac{\pi}{2}} 8\left(\frac{1}{2}(1-\cos 2\theta)\right) d\theta - \int_0^{\frac{\pi}{2}} 4+4\cos 2\theta d\theta \\ &= [4\theta - 2\sin 2\theta]_0^{\frac{\pi}{2}} = (\pi - 2) - (0) = \pi - 2. \end{aligned}$$

ALTERNATIVE SUBSTITUTION

$$\begin{aligned} &x = 9 - 2\sin\theta \\ &dx = -2\cos\theta d\theta \\ &x=7 \quad \theta=\frac{\pi}{2} \\ &x=9 \quad \theta=0 \\ &9-7 = -2\cos\theta \\ &2\cos\theta = 2 \\ &\cos\theta = 1 \\ &\theta = 0 \\ &\theta = \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} &\int_7^9 \sqrt{\frac{x-7}{11-x}} dx \\ &= \int_{\frac{\pi}{2}}^0 \sqrt{\frac{9-2\sin\theta-7}{11-(9-2\sin\theta)}} (-2\cos\theta) d\theta \\ &= \int_{\frac{\pi}{2}}^0 \sqrt{\frac{2-2\sin\theta}{2+2\sin\theta}} (-2\cos\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{1-2\sin\theta}{1+2\sin\theta}} (2\cos\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1-2\sin\theta)(1+2\sin\theta)}{1+4\sin^2\theta}} (2\cos\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1-4\sin^2\theta)(1+2\sin\theta)^2}{1+4\sin^2\theta}} (2\cos\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1-4\sin^2\theta}{\cos^2\theta}\right) (2\cos\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} (2\theta + 2\cos\theta) d\theta \\ &= (\pi + 0) - (0 + 2) \\ &= \pi - 2 \end{aligned}$$

Question 292 (*****)

$$I = \int_0^\infty \frac{1}{(x + \sqrt{x^2 + 1})^2} dx.$$

a) Use the substitution $u = x + \sqrt{x^2 + 1}$ to find the value of I .

b) Verify the answer to part (a) by a trigonometric substitution.

, $\boxed{\frac{3}{8}}$

a) START WITH THE SUBSTITUTION $u = x + \sqrt{x^2 + 1}$

$$\begin{aligned} u &= \sqrt{x^2 + 1} + x \\ u - x &= \sqrt{x^2 + 1} \\ u^2 - 2ux + x^2 &= x^2 + 1 \\ u^2 - 2ux &= 2 \\ u^2 - 1 &= 2ux \\ u &= \frac{u^2 - 1}{2u} = \frac{1}{2}u - \frac{1}{2u} \\ \frac{du}{dx} &= \frac{1}{2} + \frac{1}{2u^2} \\ du &= \frac{1}{2}\left(1 + \frac{1}{u^2}\right) du \\ dx &= \frac{1}{2}\left(\frac{u^2 - 1}{u^2}\right) du \end{aligned}$$

UNITS

$$\begin{aligned} x=0 &\mapsto u=1 \\ x=\infty &\mapsto u=\infty \end{aligned}$$

HENCE USE THIS

$$\begin{aligned} \int_0^\infty \frac{1}{(x + \sqrt{x^2 + 1})^2} dx &= \int_1^\infty \frac{1}{u^2} \times \frac{1}{2}\left(\frac{u^2 - 1}{u^2}\right) du \\ &= \int_1^\infty \frac{1}{2} \cdot \frac{u^2 - 1}{u^4} du = \frac{1}{2} \int_1^\infty \frac{1}{u^2} + \frac{1}{u^4} du \\ &= \frac{1}{2} \left[-\frac{1}{2u^2} - \frac{1}{4u^4} \right]_1^\infty = -\frac{1}{2} \left[\frac{1}{2u^2} + \frac{1}{4u^4} \right]_1^\infty \\ &= \frac{1}{2} \left[\left(\frac{1}{2} + \frac{1}{4} \right) - 0 \right] = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8} // \end{aligned}$$

b) $\int_0^\infty \frac{1}{(x + \sqrt{x^2 + 1})^2} dx = \dots$ BY A TRIGONOMETRIC SUBSTITUTION

$$\begin{aligned} x &= t \tan \theta & dx = \sec^2 \theta d\theta \\ x=0 &\mapsto \theta=0 & x=\infty \mapsto \theta=\frac{\pi}{2} \\ x &= t \tan \theta & \theta=0 \mapsto 0=\frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} \int_0^\infty \frac{1}{(t \tan \theta + \sqrt{t^2 \tan^2 \theta + 1})^2} \sec^2 \theta d\theta &= \int_0^\frac{\pi}{2} \frac{\sec^2 \theta}{(\sec \theta + \tan \theta)^2} d\theta \\ &= \int_0^\frac{\pi}{2} \frac{\sec \theta}{(\cos \theta + \sin \theta)^2} d\theta = \int_0^\frac{\pi}{2} \frac{\sec^2 \theta}{\cos^2 \theta + \sin^2 \theta} d\theta \\ &= \int_0^\frac{\pi}{2} \frac{\cos \theta}{(1 + \sin \theta)^2} d\theta = \int_0^\frac{\pi}{2} \cos \theta (1 + \sin \theta)^{-2} d\theta \\ & \text{BY RECOGNITION} \\ &= \left[\frac{1}{2} (1 + \sin \theta)^{-2} \right]_0^\frac{\pi}{2} = \left[\frac{1}{2(1 + \sin \theta)^2} \right]_0^\frac{\pi}{2} \\ &= \frac{1}{2} - \frac{1}{8} = \frac{3}{8} // \end{aligned}$$

Question 293 (*****)

Find an exact value for the following integral.

$$\int_0^\pi x \sin^3 x \, dx.$$

F, $\frac{2\pi}{3}$

$$\begin{aligned} & \bullet \text{ LET } I = \int_0^\pi x \sin^3 x \, dx \\ & \bullet \text{ LET } \alpha = \pi - x \\ & \quad d\alpha = -dx \\ & \quad x=0 \mapsto X=\pi \\ & \quad x=\pi \mapsto X=0 \\ & \bullet \text{ Hence we have} \\ & \Rightarrow I = \int_0^\pi (\pi-x) (\sin(\pi-x))^3 (-dx) \\ & \Rightarrow I = \int_0^\pi (\pi-x) [\sin^3(\pi-x) - \cos^3(\pi-x)]^3 dx \quad \text{SWAP } x \text{ AND } \pi-x \\ & \Rightarrow I = \int_0^\pi (\pi-x) \sin^3 x \, dx \\ & \Rightarrow I = \pi \int_0^\pi \sin^3 x \, dx - \int_0^\pi x \sin^3 x \, dx \\ & \Rightarrow I = \pi \int_0^\pi \sin x \sin^2 x \, dx - I \\ & \Rightarrow 2I = \pi \int_0^\pi \sin x (1-\cos^2 x) \, dx \\ & \Rightarrow 2I = \pi \int_0^\pi \sin x - \sin x \cos^2 x \, dx \\ & \Rightarrow 2I = \pi \left[-\cos x + \frac{1}{3}\cos^3 x \right]_0^\pi \\ & \Rightarrow I = \frac{\pi}{2} \left[(1 - \frac{1}{3}) - (-1 + \frac{1}{3}) \right] \\ & \Rightarrow \int_0^\pi x \sin^3 x \, dx = \frac{\pi}{2} \times \frac{4}{3} = \frac{2\pi}{3} \end{aligned}$$

Question 294 (*****)

Determine, as an exact simplified fraction, the value of the following integral.

$$\int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2 - 16x + 15)^4 \, dx .$$

	, $\frac{128}{315}$
--	---------------------

PROCESSED BY FRACDEVEL

$$\begin{aligned} \int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2 - 16x + 15)^4 \, dx &= \int_{\frac{3}{2}}^{\frac{5}{2}} [(2x-3)(2x-5)]^4 \, dx \\ &= \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 \, dx \\ \text{INTEGRATE BY PARTS} \\ \dots &= \left[\frac{1}{10} (2x-3)^5 (2x-5)^5 \right]_{\frac{3}{2}}^{\frac{5}{2}} - \int_{\frac{3}{2}}^{\frac{5}{2}} \frac{1}{10} (2x-3)^4 (2x-5)^5 \, dx \\ \text{INTEGRATE BY PARTS FOR A SECOND TIME} \\ &= \frac{1}{50} \left[\frac{1}{2} (2x-3)^6 (2x-5)^5 - \frac{1}{2} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^5 (2x-5)^6 \, dx \right] \\ &\quad - \int_{\frac{3}{2}}^{\frac{5}{2}} \frac{1}{2} (2x-3)^5 (2x-5)^6 \, dx \\ \text{BY PARTS FOR A THIRD TIME} \\ &= \frac{1}{250} \left[\frac{1}{3} (2x-3)^7 (2x-5)^5 - \frac{7}{3} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^6 (2x-5)^7 \, dx \right] \\ &= -\frac{1}{350} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^7 (2x-5)^7 \, dx \end{aligned}$$

FINALLY THE LAST INTEGRATION BY PARTS

$$\begin{aligned} &= -\frac{1}{350} \left[\frac{1}{8} (2x-3)^8 (2x-5)^7 - \frac{7}{8} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^7 (2x-5)^8 \, dx \right] \\ &= \frac{1}{70} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^8 (2x-5)^7 \, dx \\ &= \frac{1}{1400} \left[\frac{1}{9} (2x-3)^9 (2x-5)^7 \right]_{\frac{3}{2}}^{\frac{5}{2}} \\ &= \frac{1}{1400} \left[0 - (-5)^7 \right] \\ &= \frac{512}{1400} \\ &= \frac{128}{350} \\ &= \frac{128}{315} \end{aligned}$$

Question 295 (*****)

Find an exact value for

$$\int_0^\pi \frac{x \sin x}{\sqrt{4 - \cos^2 x}} dx.$$

You may assume without proof that

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + \text{constant}.$$

$\boxed{\frac{\pi^2}{6}}$

• Let $I = \int_0^\pi \frac{x \sin x}{\sqrt{4 - \cos^2 x}} dx$

• USE THE SUBSTITUTION

$x = \pi - y \Leftrightarrow y = \pi - x$
$dx = -dy$
WHICH
$x=0 \mapsto y=\pi$
$x=\pi \mapsto y=0$

$\sin x = \sin(\pi-y) = \sin(\pi-y) = -\cos y = \cos y$

$\cos^2 x = [\cos(\pi-x)]^2 = [\cos(\pi-y)-\sin(\pi-y)]^2 = \cos^2 y$

• Hence we have obtained

$$I = \int_0^\pi \frac{-(\pi-y) \sin y}{\sqrt{4 - \cos^2 y}} (-dy) = \int_0^\pi \frac{(\pi-y) \sin y}{\sqrt{4 - \cos^2 y}} dy$$

$$I = \int_0^\pi \frac{\pi \sin y}{\sqrt{4 - \cos^2 y}} dy - \int_0^\pi \frac{y \sin y}{\sqrt{4 - \cos^2 y}} dy$$

$$I = \pi \int_0^\pi \sin y (4 - \cos^2 y)^{-\frac{1}{2}} dy - I$$

$$2I = \pi \int_0^\pi \sin y (4 - \cos^2 y)^{-\frac{1}{2}} dy$$

• SUBSTITUTION AGAIN

$y = \cos y$
$\frac{dy}{dy} = -\sin y$
$dy = -\sin y dy$
WHICH
$y=0 \mapsto v=1$
$y=\pi \mapsto v=-1$

$$\Rightarrow 2I = \pi \int_{-1}^1 \frac{\sin y}{\sqrt{4 - \cos^2 y}} (-\sin y) dy$$

$$\Rightarrow I = \frac{\pi}{2} \int_{-1}^1 \frac{1}{\sqrt{4 - v^2}} dv$$

(Simplify integrand in substitution bounds)

$$\Rightarrow I = \frac{\pi}{2} \times 2 \int_0^1 \frac{1}{\sqrt{4 - v^2}} dv$$

$$\Rightarrow I = \pi \int_0^1 \frac{1}{\sqrt{4 - v^2}} dv$$

$$\Rightarrow I = \pi \left[\arcsin \frac{v}{2} \right]_0^1$$

$$\Rightarrow I = \pi \left[\arcsin \frac{1}{2} - \arcsin 0 \right]$$

$$\Rightarrow I = \pi \times \frac{\pi}{6}$$

$$\Rightarrow \int_0^\pi \frac{x \sin x}{\sqrt{4 - \cos^2 x}} dx = \frac{\pi^2}{6}$$

Question 296 (*****)

A family of functions, known as the Chebyshev polynomials of the first kind $T_n(x)$, is defined as

$$T_n(x) = \cos(n \arccos x), -1 \leq x \leq 1, n \in \mathbb{N}.$$

Evaluate the following integral

$$\int_{-1}^1 \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx.$$

□, 0

Start by the definition of $T_n(x)$

$$\int_{-1}^1 \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx = \int_{-1}^1 \frac{\cos(n \arccos x) \cos(m \arccos x)}{\sqrt{1-x^2}} dx$$

Process by a substitution

$$x = \cos \theta / \sqrt{1-x^2}$$

$$dx = -\sin \theta d\theta$$

$$\theta = \arccos x$$

$$\theta = \pi - \theta$$

$$x = 1 \rightarrow \theta = 0$$

$$x = -1 \rightarrow \theta = \pi$$

$$= \int_0^\pi \frac{\cos(n \arccos(\cos \theta)) \cos(m \arccos(\cos \theta))}{\sin \theta} d\theta$$

$$= \int_0^\pi \cos(n \theta) \cos(m \theta) d\theta$$

Using the compound angle identities

$$\cos(n\theta + m\theta) = \cos n \theta \cos m \theta - \sin n \theta \sin m \theta$$

$$\cos(n\theta - m\theta) = \cos n \theta \cos m \theta + \sin n \theta \sin m \theta$$

$$\cos((n+m)\theta) + \cos((n-m)\theta) = 2 \cos n \theta \cos m \theta$$

Returning to the integral

$$= \int_0^\pi [\cos((n+m)\theta) + \cos((n-m)\theta)] d\theta$$

$$= \left[\frac{1}{2(n+m)} \sin((n+m)\theta) + \frac{1}{2(n-m)} \sin((n-m)\theta) \right]_0^\pi = 0$$

[As $\sin k\theta = 0$, $k \in \mathbb{Z}$ & $n+m, n-m \in \mathbb{Z}$]

Question 297 (*****)

The function $y = f(x)$ is defined in the largest possible real domain by

$$f(x) \equiv \ln[x^2 - 2x + 2].$$

Sketch the graph of $f(x)$ and determine an exact simplified value for the area of the finite region bounded by the graph of $f(x)$ and the coordinate axes.

, $\frac{1}{2}\pi - 2 + \ln 2$

$y = \ln(x^2 - 2x + 2)$

- START BY COMPUTING THE SQUARE INSIDE THE ARGUMENT OF THE LOG.
 $y = \ln((x-1)^2 + 1)$
- HENCE THE DOMAIN OF THE CURVE IS ALL THE REAL NUMBERS AND IT IS EVEN ABOUT $x=1$.
- CROSS ZERO: $y_1 = \ln(2) \rightarrow x_1 = 1 + \sqrt{e}$
 $y_2 = \ln(-2) \rightarrow x_2 = 1 - \sqrt{e}$
- HENCE WE PRODUCE THE GRAPH BELOW.
- THIS THE REQUIRED AREA IS GIVEN BY

$$\int_{-2}^1 \ln[(x-1)^2 + 1] dx \quad \dots \text{BY SUBSTITUTIONS} \dots$$

$$= \int_{-2}^1 \left(\ln u \right) \left(-\frac{1}{2} (u-1)^{-\frac{1}{2}} du \right) = \frac{1}{2} \int_{-2}^1 \frac{\ln u}{(u-1)^{\frac{1}{2}}} du$$

$$= \frac{1}{2} \int_{-2}^1 \frac{(u-1)^{\frac{1}{2}} \ln u}{u} du \quad \dots \text{BY PARTS} \dots$$

$$= \frac{1}{2} \left\{ \left[2(u-1)^{\frac{1}{2}} \ln u \right]_{-2}^1 - 2 \int_{-2}^1 \frac{(u-1)^{\frac{1}{2}}}{u} du \right\}$$

$\ln u$	$\frac{1}{u}$
$2(u-1)^{\frac{1}{2}}$	$(u-1)^{\frac{1}{2}}$

$$= \left[(u-1)^{\frac{1}{2}} \ln u \right]_{-2}^1 - \int_{-2}^1 \frac{(u-1)^{\frac{1}{2}}}{u} du \quad \dots \text{BY SUBSTITUTIONS} \dots$$

$\sqrt{u^2 - 2u}$	u
$du = 2u\sqrt{u^2 - 2u} du$	$du = 2(u-1)^{\frac{1}{2}} du$
$u=1 \rightarrow 0=0$	$u=-2 \rightarrow 0=\frac{4}{3}$

$$= \ln 2 - \int_0^{\frac{4}{3}} \frac{(sec\theta)^{\frac{1}{2}}}{sec\theta} (2sec\theta\tan\theta d\theta)$$

$$= \ln 2 - \int_0^{\frac{\pi}{4}} 2\tan^2\theta d\theta$$

$$= \ln 2 - \left[2(\sec\theta - 1) \right]_0^{\frac{\pi}{4}}$$

$$= \ln 2 - \left[2(1 - \frac{1}{\sqrt{2}}) - 0 \right]$$

$$= \ln 2 - \left[2 - \frac{2}{\sqrt{2}} \right]$$

$$= \frac{\pi}{2} + \ln 2 - 2$$

Question 298 (*****)

$$I = \int_1^3 (3-x)^7 (x-1)^7 \, dx.$$

Show that

$$I = \frac{(7!)^2 \times 2^{15}}{15!}.$$

, proof

• Let $I(7,7) = \int_1^3 (3-x)^7 (x-1)^7 \, dx$

• Proceed by integration by parts

$\frac{(3-x)^8}{8(x-1)^8}$	$-7(3-x)^6$
----------------------------	-------------

$$\Rightarrow I(7,7) = \left[\frac{1}{8}(3-x)^8(x-1)^8 \right]_1^3 + \frac{7}{8} \int_1^3 (3-x)^8 (x-1)^6 \, dx$$

$$\Rightarrow I(7,7) = \frac{7}{8} I(6,8) = \frac{7}{8} \int_1^3 (3-x)^6 (x-1)^8 \, dx$$

• By parts again

$\frac{(3-x)^6}{6(x-1)^9}$	$-6(3-x)^5$
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$$\Rightarrow I(7,7) = \frac{7}{8} \left\{ \left[\frac{1}{6}(3-x)^9(x-1)^9 \right]_1^3 + \frac{5}{6} \int_1^3 (3-x)^9 (x-1)^8 \, dx \right\}$$

$$\Rightarrow I(7,7) = \frac{7}{8} \times \frac{6}{9} \times I(5,9)$$

• Following the pattern by parts

$$\Rightarrow I(7,7) = \frac{1}{8} \times \frac{6}{7} \times \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{14} I(0,14)$$

$$\Rightarrow I(7,7) = \frac{7!}{14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8} I(0,14)$$

$$\Rightarrow I(7,7) = \frac{7! \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8} I(0,14)$$

$$\Rightarrow I(7,7) = \frac{7! \times 7!}{14!} I(0,14)$$

$$\Rightarrow I(7,7) = \frac{(7!)^2}{14!} \int_1^3 (3-x)^9 (x-1)^6 \, dx$$

$$\Rightarrow I(7,7) = \frac{(7!)^2}{14!} \int_1^3 (x-1)^{15} \, dx$$

$$\Rightarrow I(7,7) = \frac{(7!)^2}{14!} \left[\frac{1}{16} (x-1)^{16} \right]_1^3$$

$$\Rightarrow I(7,7) = \frac{(7!)^2}{14!} \times \frac{1}{16} \times 2^{15}$$

$$\Rightarrow I(7,7) = \frac{(7!)^2}{14!} \times 2^{15}$$

$$\therefore \int_1^3 (3-x)^7 (x-1)^7 \, dx = \frac{(7!)^2 \times 2^{15}}{14!}$$

Question 299 (*****)

Use integration by parts to find a simplified expression for

$$\int \left(1+x-\frac{1}{x}\right) e^{x+\frac{1}{x}} dx.$$

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$$, xe^{x+\frac{1}{x}} + C$$

• LOOKING AT THE INTEGRAND WE SUSPECT THAT A TERM CONTAINING CONTAINING THE EXPONENTIAL MUST BE INTEGRATED

• SEEK AN EASY DIFFERENTIATE

$$\frac{d}{dx} \left[e^{x+\frac{1}{x}} \right] = \left(e^{x+\frac{1}{x}} \right) \times \left(1 - \frac{1}{x^2} \right)$$

• REWRITE THE INTEGRAL AS FOLLOWS

$$\begin{aligned} \int \left(1+x-\frac{1}{x}\right) e^{x+\frac{1}{x}} dx &= \int \left[1 + x \left(1 - \frac{1}{x^2} \right) \right] e^{x+\frac{1}{x}} dx \\ &= \int e^{x+\frac{1}{x}} + x \left(1 - \frac{1}{x^2} \right) e^{x+\frac{1}{x}} dx \\ &= \int e^{x+\frac{1}{x}} dx + \int x \left(1 - \frac{1}{x^2} \right) e^{x+\frac{1}{x}} dx \end{aligned}$$

BY PARTS

--	--	--

$$\begin{aligned} &= \int e^{x+\frac{1}{x}} dx + \left[x e^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}} dx \right] \\ &= x e^{x+\frac{1}{x}} + C \end{aligned}$$

Question 300 (*****)

Use trigonometric identities to find a simplified expression for

$$\int \frac{\sin^8 x - \cos^8 x}{1 - \frac{1}{2} \sin^2 2x} dx.$$

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$$, -\frac{1}{2} \sin 2x + C$$

$$\int \frac{\sin^8 x - \cos^8 x}{1 - \frac{1}{2} \sin^2 2x} dx$$

• STARTING FROM THE DIFFERENCE OF SQUARES IN THE NUMERATOR & THE SINE DOUBLE ANGLE IN THE DENOMINATOR

$$\dots = \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - \frac{1}{2}(2 \sin x \cos x)^2} dx = \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - 2 \sin^2 x \cos^2 x} dx$$

• NEXT CREATE A PERFECT SQUARE IN THE DENOMINATOR AS FOLLOWS

$$\dots = \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1^2 - 2 \sin^2 x \cos^2 x} dx = \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x} dx$$

• EXPAND THE DIFFERENCE OF SQUARES IN THE MULTIFACITOR & THE BRACKET IN THE DENOMINATOR

$$\begin{aligned} &= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{\sin^2 x + \cos^2 x + 2 \sin^2 x \cos^2 x - 2 \sin^2 x \cos^2 x} dx \\ &= \int \frac{-\cos 2x (\sin^4 x + \cos^4 x)}{\sin^2 x + \cos^2 x} dx = \int -\cos 2x dx \\ &= -\frac{1}{2} \sin 2x + C \end{aligned}$$

Question 301 (*****)

By using an appropriate substitution followed by trigonometric identities, show that

$$\int_0^\pi \frac{x \tan x}{\tan x + \sec x} dx = \frac{1}{2}\pi(\pi - 2).$$

S.E., proof

• START BY A SUBSTITUTION

$\theta = \pi - x$	$\tan(\pi - \theta) = \frac{\tan\theta - \tan\theta}{1 + \tan\theta \tan\theta} = -\tan\theta$
$d\theta = -dx$	
$x \mapsto 0 \mapsto \theta = \pi$	$\sec(\pi - \theta) = \frac{1}{\cos(\pi - \theta)} = \frac{1}{-\cos\theta}$
$x \mapsto \pi \mapsto \theta = 0$	$= -\frac{1}{\cos\theta} = -\sec\theta$

• THEN THE INTEGRAL CAN BE TRANSFORMED TO

$$\begin{aligned} \int_0^\pi \frac{x \tan x}{\tan x + \sec x} dx &= \int_\pi^0 (\pi - \theta) \frac{(-\tan\theta)}{-\sec\theta - \tan\theta} (-d\theta) \\ &= \int_0^\pi \frac{\theta \tan\theta - \pi \tan\theta}{\sec\theta + \tan\theta} d\theta \\ &= \int_0^\pi \frac{\pi \tan\theta - \theta \tan\theta}{\sec\theta + \tan\theta} d\theta \\ &= \int_0^\pi \frac{\pi \tan\theta}{\sec\theta + \tan\theta} d\theta - \int_0^\pi \frac{\theta \tan\theta}{\sec\theta + \tan\theta} d\theta \end{aligned}$$

• COLLECTING THE RESULTS SO FAR WE HAVE

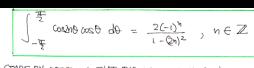
$$\begin{aligned} I &= \pi \int_0^\pi \frac{\theta \tan\theta}{\sec\theta + \tan\theta} d\theta - I \quad \text{with } I = \int_0^\pi \frac{\pi \tan\theta}{\sec\theta + \tan\theta} d\theta \\ \Rightarrow 2I &= \pi \int_0^\pi \frac{\theta \tan\theta}{\sec\theta + \tan\theta} d\theta \\ \Rightarrow 2I &= \pi \int_0^\pi \frac{\theta \tan\theta (\sec\theta - \tan\theta)}{(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)} d\theta \end{aligned}$$

$$\begin{aligned} &\Rightarrow 2I = \pi \int_0^\pi \frac{\sec\theta \tan\theta - \tan^2\theta}{\sec^2\theta - \tan^2\theta} d\theta \\ &\quad \boxed{1 + \tan^2\theta \equiv \sec^2\theta} \\ &\quad \boxed{\sec^2\theta - \tan^2\theta \equiv 1} \\ &\Rightarrow 2I = \pi \int_0^\pi \sec\theta \tan\theta - \tan^2\theta d\theta \\ &\Rightarrow 2I = \pi \int_0^\pi \sec\theta \tan\theta - (\sec^2\theta - 1) d\theta \\ &\Rightarrow 2I = \pi \left[\sec\theta \tan\theta - \sec\theta + 1 \right]_0^\pi \\ &\Rightarrow 2I = \pi \left[(\sec\theta - \tan\theta + 1) - (1 - 0 + 1) \right] \\ &\Rightarrow 2I = \pi [-1] \\ &\Rightarrow I = \frac{1}{2}\pi(\pi - 2) \\ &\therefore \int_0^\pi \frac{x \tan x}{\tan x + \sec x} dx = \frac{1}{2}\pi(\pi - 2) \end{aligned}$$

Question 302 (*****)Show that if n is an integer, then

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos(2n\theta) \cos \theta \, d\theta = \frac{2(-1)^n}{1-(2n)^2}.$$

 , proof

 START BY OBSERVING THAT THE INTEGRAND IS EVEN

$$I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2n\theta) \cos \theta \, d\theta = \int_0^{\frac{\pi}{2}} 2\cos 2n\theta \cos \theta \, d\theta$$

 NEXT BY THE COSINE COMPOUND ANGLE IDENTITIES

$$\begin{aligned} \cos(2n\theta + \theta) &= \cos 2n\theta \cos \theta - \sin 2n\theta \sin \theta \\ \cos(2n\theta - \theta) &= \cos 2n\theta \cos \theta + \sin 2n\theta \sin \theta \\ \cos(2n\theta + \theta) + \cos(2n\theta - \theta) &= 2\cos 2n\theta \cos \theta \\ 2(\cos 2n\theta \cos \theta) &= \cos[(2n+1)\theta] + \cos[(2n-1)\theta] \end{aligned}$$

$$\Rightarrow I_n = \int_0^{\frac{\pi}{2}} \cos[(2n+1)\theta] + \cos[(2n-1)\theta] \, d\theta$$

$$\Rightarrow I_n = \left[\frac{1}{2n+1} \sin[(2n+1)\theta] + \frac{1}{2n-1} \sin[(2n-1)\theta] \right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow I_n = \frac{1}{2n+1} \sin\left[\frac{(2n+1)\pi}{2}\right] + \frac{1}{2n-1} \sin\left[\frac{(2n-1)\pi}{2}\right]$$

$$\Rightarrow I_n = \frac{1}{2n+1} (-1)^n + \frac{1}{2n-1} (-1)^{n+1}$$

$$\Rightarrow I_n = (-1)^n \left[\frac{1}{2n+1} + \frac{(-1)}{2n-1} \right]$$

$$\Rightarrow I_n = (-1)^n \left[\frac{1}{2n+1} - \frac{1}{2n-1} \right]$$

$$\begin{aligned} \Rightarrow I_n &= (-1)^n \left[\frac{(2n-1) - (2n+1)}{(2n+1)(2n-1)} \right] \\ \Rightarrow I_n &= (-1)^n \left[\frac{-2}{4n^2 - 1} \right] \\ \Rightarrow I_n &= (-1)^n \times \frac{2}{1-4n^2} \\ \Rightarrow I_n &= \frac{2(-1)^n}{1-4n^2} \end{aligned}$$

Question 303 (*****)

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1 + \tan^n x} dx, \quad n \in \mathbb{Q}.$$

Find the value of the above integral, for all values of n

$$\boxed{\frac{\pi}{4}}$$

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^2 x} dx &= \int_0^{\frac{\pi}{2}} \frac{1}{1 + \frac{\sin^2 x}{\cos^2 x}} dx \\
 &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + \sin^2 x} dx \\
 &\quad \text{Now by substitution } x = \frac{\pi}{2} - X \\
 &\quad dx = -dX \\
 &\quad x=0 \mapsto X=\frac{\pi}{2} \\
 &\quad x=\frac{\pi}{2} \mapsto X=0 \\
 &\text{Also } \sin^2 x = \sin^2(\frac{\pi}{2}-x) = \cos^2(x) \\
 &\quad \cos^2 x = \cos^2(\frac{\pi}{2}-x) = \sin^2(x) \\
 &\cdots = \int_{\frac{\pi}{2}}^0 \frac{\sin^2 X}{\sin^2 X + \cos^2 X} (-dX) = \int_0^{\frac{\pi}{2}} \frac{\sin^2 X}{\sin^2 X + \cos^2 X} dx \\
 \text{Thus } I &= \int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^2 x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + \sin^2 x} dx \\
 I &\approx \dots \text{substitution} = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin^2 x + \cos^2 x} dx \\
 2I &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin^2 x + \cos^2 x} dx + \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin^2 x + \cos^2 x} dx \\
 2I &= \int_0^{\frac{\pi}{2}} 1 dx \\
 2I &= \frac{\pi}{2} \\
 I &= \frac{\pi}{4}
 \end{aligned}$$

Question 304 (*****)

By suitably rewriting the numerator of the integrand, find a simplified expression for the following integral.

$$\int \frac{12\sin x - 5\cos x}{2\sin x - 3\cos x} dx.$$

L.P.E., $[3x + 2\ln|2\sin x - 3\cos x| + C]$

$\int \frac{12\sin x - 5\cos x}{2\sin x - 3\cos x} dx = \dots$ MANIPULATE AS FOLLOWS

• $\frac{d}{dx}(2\sin x - 3\cos x) = 2\cos x + 3\sin x$

• Thus $12\sin x - 5\cos x \equiv A(2\sin x - 3\cos x) + B(2\cos x + 3\sin x)$

SO IT'S EASIER
TO INTEGRATE
IF WE
MANIPULATE
THE FORM
INTO
 $\int f(x) dx$

$\Rightarrow 12\sin x - 5\cos x \equiv (2A + 3B)\sin x + (2B - 3A)\cos x$

$\Rightarrow \begin{cases} 2A + 3B = 12 \\ -3A + 2B = -5 \end{cases} \Rightarrow \begin{cases} 6A + 9B = 36 \\ -6A + 4B = -10 \end{cases} \Rightarrow \begin{cases} 13B = 26 \\ B = 2 \end{cases} \Rightarrow \boxed{A = 3}$

$\dots = \int \frac{(2\sin x - 3\cos x)}{2\sin x - 3\cos x} dx = \int \frac{3(2\sin x - 3\cos x)}{2\sin x - 3\cos x} + \frac{2(\cos x + 3\sin x)}{2\sin x - 3\cos x} dx$

$= \int 3 + \left(\frac{2\cos x + 2\sin x}{2\sin x - 3\cos x} \right) dx$

$= 3x + 2\ln|2\sin x - 3\cos x| + C$

Question 305 (*****)

Find an exact value for the following integral

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+(\tan x)^{\sqrt{2}} e^{-x}} dx.$$

$\boxed{\frac{1}{4}\pi}$

THE SPANNING WIRE COULD POSSIBLY BE USEFUL!! (GO ON)

LET $x = \sqrt{2}y$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan(\sqrt{2}y))^2} dy = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^2 y} dy = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{\sin^2 y}{\cos^2 y}} dy$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 y}{\cos^2 y + \sin^2 y} dy$$

Now use a substitution

$y = \frac{\pi}{2} - z$
 $dy = -dz$
 $2z \mapsto y \mapsto 0$
 $2z \mapsto y \mapsto \pi$

$$\dots = \int_{\frac{\pi}{2}}^0 \frac{\cos^2(\frac{\pi}{2}-z)}{\cos^2(\frac{\pi}{2}-z) + \sin^2(\frac{\pi}{2}-z)} (-dz)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 z}{\sin^2 z + \cos^2 z} dz$$

THIS IS ALL VERY STANDARD

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 z}{\cos^2 z + \sin^2 z} dz = \int_0^{\frac{\pi}{2}} \frac{\cos^2 z}{\cos^2 z + \cos^2 z} dz \\ J &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 z}{\cos^2 z + \sin^2 z} dz \\ QI &= \int_0^{\frac{\pi}{2}} 1 dz \\ DI &= \frac{\pi}{2} \\ I &= \frac{\pi}{2} \end{aligned}$$

$$\therefore \boxed{\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\sqrt{2}} e^{-x}} dx = \frac{\pi}{2}}$$

Question 306 (*****)

$$I = \int_{-\infty}^{\infty} \left| x^3 (2^{-x^2}) \right| dx$$

It is given that $I \approx 2$.

Use this fact to estimate the value of $\ln 2$ correct to 1 significant figure.

, $\ln 2 \approx 0.7$

• As $x^3 \times 2^{-x^2}$ is odd, the modulus in the integrand will be even

E.g.

• Hence we have

$$\int_{-\infty}^{\infty} |x^3 (2^{-x^2})| dx = 2 \int_0^{\infty} |x^3 (2^{-x^2})| dx = \int_0^{\infty} 2x^3 (2^{-x^2}) dx$$

• By substitution first

$$\begin{aligned} &= \int_0^{\infty} 2x^3 (2^u) \left(\frac{du}{-2x} \right) \\ &= \int_{-\infty}^0 u^2 (2^u) du \\ &= \int_{-\infty}^0 -u (2^u) du \\ &= \int_0^{\infty} u (2^u) du \end{aligned}$$

• By parts next

$$\begin{aligned} &= \left[\frac{u (2^u)}{2} \right]_0^{-\infty} - \int_0^{\infty} \frac{2^u}{\ln 2} du \\ &= 0 - 0 - \frac{1}{\ln 2} \int_0^{\infty} 2^u du \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{\ln 2} \left[\frac{2^u}{\ln 2} \right]_0^{-\infty} \\ &= -\frac{1}{\ln 2} \left[0 - \frac{1}{\ln 2} \right] \\ &= \frac{1}{(\ln 2)^2} \end{aligned}$$

• Finally we have

$$\begin{aligned} \frac{1}{(\ln 2)^2} &\approx 2 \quad (1 \text{ s.f.}) \\ (\ln 2)^2 &\approx \frac{1}{2} \approx 0.49 \\ \ln 2 &\approx 0.7 \quad (1 \text{ s.f.}) \end{aligned}$$

Question 307 (*****)

The definite integral I is defined in terms of the constant k , where $k \neq 0$, $k \neq \pm 1$.

$$I = \int_0^{\frac{1}{2}\pi} \frac{1}{1+k^2 \tan^2 x} dx.$$

Use appropriate integration techniques to show that

$$I = \frac{\pi}{2(k+1)}.$$

, proof

• **SOLVE BY A SUBSTITUTION**

$$\begin{aligned} \int_0^{\frac{1}{2}\pi} \frac{1}{1+k^2 \tan^2 x} dx &= \int_0^{\infty} \frac{1}{1+k^2 u^2} \frac{du}{k^2 u^2} \\ &= \int_0^{\infty} \frac{1}{1+u^2} \times \frac{1}{k(1+k^2 u^2)} du \\ &= \int_0^{\infty} \frac{1}{1+u^2} \times \frac{1}{k \left(1 + \frac{u^2}{k^2}\right)} du \\ &= \int_0^{\infty} \frac{1}{1+u^2} \frac{k^2}{k^2 + u^2} du = \int_0^{\infty} \frac{k}{(u^2 + k^2)} du \end{aligned}$$

• **DO THIS BY PARTIAL FRACTIONS**

$$\begin{aligned} \frac{k}{(u^2 + k^2)} &\equiv \frac{A u + B}{u^2 + 1} + \frac{C u + D}{u^2 + k^2} \\ \Rightarrow k &\equiv (A u + B)(u^2 + 1) + (C u + D)(u^2 + k^2) \\ \Rightarrow k &\equiv \begin{cases} A u^3 + B u^2 + A u + B \\ C u^3 + D u^2 + C u + D \end{cases} \\ \Rightarrow k &\equiv (A+C)u^3 + (B+D)u^2 + (A+B)u + (C+D) \\ \bullet A+C=0 \\ \bullet A^2+BC=0 \end{aligned}$$

$$\begin{aligned} \bullet A+C=0 &\Rightarrow A(u^2-1)=0 \\ \Rightarrow A=C=0 &\quad B+D=0 \quad \{ \Rightarrow B(u^2-1)=k \\ \Rightarrow B=\frac{k}{u^2-1}, D=-\frac{k}{u^2-1} &\quad B(u^2-1)=k \end{aligned}$$

• **RETURNING TO THE INTEGRAL WE NOW HAVE**

$$\begin{aligned} \int_0^{\infty} \frac{k}{(u^2+1)(u^2+k^2)} du &= \int_0^{\infty} \frac{\frac{k}{u^2+1}}{u^2+k^2} du = \frac{k}{k^2-1} \left[\arctan u - \frac{1}{k} \arctan \frac{u}{k} \right]_0^{\infty} \\ &= \frac{k}{k^2-1} \left[\left(\frac{\pi}{2} - \frac{\pi}{2k} \right) - 0 \right] = \frac{k}{k^2-1} \times \frac{\pi}{2} \times \left(1 - \frac{1}{k} \right) \\ &= \frac{k}{(k-1)(k+1)} \times \frac{\pi}{2} \times \frac{k+1}{k-1} = \frac{\pi}{2(k+1)} // \end{aligned}$$

Question 308 (*****)

By suitably rewriting the numerator of the integrand, find a simplified expression for the following integral.

$$\int \frac{3\cos x + 2\sin x}{2\cos x + 3\sin x} dx .$$

$$\boxed{\frac{12}{13}x + \frac{5}{13}\ln|2\cos x + 3\sin x| + C}$$

$\int \frac{3\cos x + 2\sin x}{2\cos x + 3\sin x} dx = ?$

- MANIPULATE AS REQUIRES
 $\frac{1}{dx}[2\cos x + 3\sin x] = -2\sin x + 3\cos x$
- REWRITE THE NUMERATOR AS
 $3\cos x + 2\sin x \equiv A(2\cos x + 3\sin x) + B(3\cos x - 2\sin x)$

SO IT CAN BE DIVIDED BY THE DENOMINATOR
ON IT'S OWN
- SOLVE
 $\begin{cases} 2A + 3B = 3 \\ 3A - 2B = 2 \end{cases} \times 2 \rightarrow 4A + 6B = 6 \quad \begin{cases} A + B = 1 \\ 4A + 6B = 6 \end{cases} \Rightarrow B = 1/2$
 $A = 1/2$
- RETURNING TO THE INTEGRAL

$$\begin{aligned} &= \int \frac{1}{2} \left(\frac{2\cos x + 3\sin x}{2\cos x + 3\sin x} \right) + \frac{1}{2} \left(\frac{3\cos x - 2\sin x}{2\cos x + 3\sin x} \right) dx \\ &= \int \frac{1}{2} dx + \frac{1}{2} \int \frac{3\cos x - 2\sin x}{2\cos x + 3\sin x} dx \\ &= \frac{12}{13}x + \frac{5}{13}\ln|2\cos x + 3\sin x| + C \end{aligned}$$

Question 309 (*****)

Find the value of the following definite integral.

$$\int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2(1+x^4)^{\frac{3}{4}}} dx .$$

, 1

AS THE DOMAIN OF INTEGRATION IS POSITIVE WE MAY FRACTIONATE OUT OF THE RADICAL

$$\begin{aligned} \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2(1+2x^4)^{\frac{3}{4}}} dx &= \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2(2^{\frac{3}{4}}(2x^4)^{\frac{1}{4}})^{\frac{3}{4}}} dx \\ &= \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{2^{\frac{3}{4}}x^2(2x^4)^{\frac{1}{4}}} dx = \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2(1+2x^4)^{\frac{3}{4}}} dx \\ &= \int_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \frac{1}{x^2(1+2x^4)^{\frac{3}{4}}} dx \end{aligned}$$

BY RECOGNITION OR + SUBSTITUTION: $u = 2x^4$

$$\frac{du}{dx} = 8x^3 \Rightarrow du = 8x^3 dx \Rightarrow \frac{1}{8x^3} du = \frac{1}{2}x^2 dx$$

INTRODUCING THE MIND AND EVALUATING:

$$\begin{aligned} &= \left[\frac{1}{8} \left(1+2x^4 \right)^{-\frac{1}{4}} \right]_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} = \left[\frac{1}{8} \left(1+2x^4 \right)^{-\frac{1}{4}} \times (-4x^3) \right]_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} = -\frac{1}{2} \left(1+2x^4 \right)^{-\frac{1}{4}} \Big|_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} \\ &= -\left[\left(1+2x^4 \right)^{-\frac{1}{4}} \right]_{80^{-\frac{1}{4}}}^{15^{-\frac{1}{4}}} = -\left[\left(1+2 \cdot 80^{-\frac{1}{4}} \right)^{-\frac{1}{4}} - \left(1+2 \cdot 15^{-\frac{1}{4}} \right)^{-\frac{1}{4}} \right] \\ &= -\left(16^{-\frac{1}{4}} - 8^{-\frac{1}{4}} \right) = -(2-3) = 1 \end{aligned}$$

Question 310 (*****)

Find in exact simplified form the value of

$$\int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}} dx.$$

You may assume that the integral converges.

V, **□**, $\frac{1}{2}(\pi+4)$

WORKING

USING THE LIMITS OF ABSOLUTE CONVERGENCE, START BY "PARTITIONING"

THE INTEGRAL

$$\int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}} dx = \int_0^1 \frac{\sqrt{1-x}(1+\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})} dx = \int_0^1 \frac{\sqrt{1-x}(1+\sqrt{x})}{1-x} dx$$

$$= \int_0^1 \frac{1+\sqrt{x}}{\sqrt{1-x}} dx$$

Now use L'hopital's rule on the denominator

$$= \int_0^1 \frac{1+\sqrt{1-u^2}}{1-u^2} du$$

$$= \int_0^1 2 - 2\sqrt{1-u^2} du$$

$$= \int_0^1 2 + 2\sqrt{1-u^2} du$$

$$< [2u]_0^1 + 2 \int_0^1 \sqrt{1-u^2} du$$

$$= 2 + 2 \int_0^1 \sqrt{1-u^2} du$$

By a trigonometric substitution

$$= 2 + 2 \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 \theta} (\cos \theta d\theta)$$

$$= 2 + 2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

$$= 2 + 2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

$$= 2 + 2 \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta) d\theta$$

ANSWER

$u = \sqrt{1-x}$
 $u^2 = 1-x$
 $du = -dx$
 $2x = 1-2u$
 $2x = 1-2u$
 $x = 1-2u$
 $x = 1-2u$

$\theta = \arcsin u$
 $\sin \theta = u$
 $d\theta = \cos \theta d\theta$
 $\cos \theta = \sqrt{1-u^2}$
 $\cos \theta = \sqrt{1-(1-x)}$
 $\cos \theta = \sqrt{x}$
 $u = \cos \theta$
 $u = \cos \theta$

Question 311 (*****)

Find an exact simplified value for

$$\int_{\sqrt{e}}^e \ln(\ln x) + \frac{1}{(\ln x)^2} dx.$$

	$\boxed{}$
--	------------------------

 $e^{\frac{1}{2}}(2 + \ln 2) - e$

• ATTAIN INTEGRATION BY PARTS ON THE FIRST INTEGRAL

$\ln(\ln x)$	$\frac{1}{(\ln x)^2}$
x	1

$$\begin{aligned} \int_{\sqrt{e}}^e \ln(\ln x) dx &= \left[x \ln(\ln x) \right]_{\sqrt{e}}^e - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx \\ &= 0 - \sqrt{e} \ln(\tfrac{1}{2}) - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx \\ &= e^{\frac{1}{2}} \ln 2 - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx \\ &\quad \text{Zero this --- might be able to cancel this integral} \end{aligned}$$

• PROCEED WITH THE NEXT INTERVAL, ALSO BY PARTS

$(\ln x)^2$	$-2(\ln x) \times \frac{1}{x}$
x	1

$$\int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx = \left[\frac{2}{(\ln x)^3} \right]_{\sqrt{e}}^e + 2 \int_{\sqrt{e}}^e \frac{1}{(\ln x)^3} dx$$

THIS FINISHES OFF IF WE NOTICE THE PATTERN OF THE POWERS, WE MAY ATTAIN THE INTERVALS BY PARTS BY STARTING WITH A POWER THREE.

• SO DO THE PARTS "IN REVERSE" AS FOLLOWS

$(\ln x)^4$	$-(\ln x)^2 \times \frac{1}{x}$
x	1

PART OF OUR INTEGRAL

$$\begin{aligned} \int_{\sqrt{e}}^e (\ln x)^4 x dx &= \left[\frac{x}{\ln x} \right]_{\sqrt{e}}^e + \int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx \\ \Rightarrow \int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx &= \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - \left[\frac{x}{\ln x} \right]_{\sqrt{e}}^e \\ \Rightarrow \int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx &= \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - \left[\frac{e}{\ln e} - \frac{e^{\frac{1}{2}}}{\ln \sqrt{e}} \right] \\ \Rightarrow \int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx &= \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - e + 2e^{\frac{1}{2}} \end{aligned}$$

• COLLECTING THE RESULTS

$$\begin{aligned} \int_{\sqrt{e}}^e \ln(\ln x) dx &= e^{\frac{1}{2}} \ln 2 - \int_{\sqrt{e}}^e \frac{1}{\ln x} dx \\ \int_{\sqrt{e}}^e \frac{1}{(\ln x)^2} dx &= \int_{\sqrt{e}}^e \frac{1}{\ln x} dx - e + 2e^{\frac{1}{2}} \\ \text{ADDING THE RESULTS} \\ \int_{\sqrt{e}}^e \ln(\ln x) + \frac{1}{(\ln x)^2} dx &= e^{\frac{1}{2}} \ln 2 - e + 2e^{\frac{1}{2}} \\ &= e^{\frac{1}{2}}(2 + \ln 2) - e \end{aligned}$$

Question 312 (*****)

By using appropriate substitutions, or otherwise, show that

$$\int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx = \frac{\pi \ln 2}{16}.$$

proof

THE ANSWER SUGGESTS A TRIGONOMETRIC SUBSTITUTION

$$1 + 4x^2 = 1 + (\tan\theta)^2 = \sec^2\theta$$

- $2x = 1 \tan\theta$ [θ - acute, $\tan\theta > 0$ is the condition]
- $2dx = \sec^2\theta d\theta$
- $x = \frac{1}{2}\tan\theta$
- $x = \frac{1}{2} \rightarrow \theta = \frac{\pi}{4}$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{\sec^2\theta} \left(\frac{1}{2}\sec^2\theta d\theta\right)$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta) d\theta$$

ANOTHER SUBSTITUTION

- $\theta = \frac{\pi}{4} - \phi \Leftrightarrow \frac{\pi}{4} - \theta$
- $d\theta = -d\phi \Leftrightarrow d\phi = -d\theta$
- $\theta = 0 \rightarrow \phi = \frac{\pi}{4}$
- $\theta = \frac{\pi}{4} \rightarrow \phi = 0$

THUS WE HAVE THAT

$$I = \frac{1}{2} \int_{\frac{\pi}{4}}^0 \ln(1 + \tan(\frac{\pi}{4} - \phi)) (-d\phi) = \frac{1}{2} \int_{\frac{\pi}{4}}^0 \ln \left[1 + \frac{\tan(\frac{\pi}{4} - \phi)}{1 + \tan(\frac{\pi}{4} - \phi)} \right] d\phi$$

$$= \frac{1}{2} \int_{\frac{\pi}{4}}^0 \ln \left[1 + \frac{1 - \tan\phi}{1 + \tan\phi} \right] d\phi = \frac{1}{2} \int_{\frac{\pi}{4}}^0 \ln \left[\frac{1 + \tan\phi + 1 - \tan\phi}{1 + \tan\phi} \right] d\phi$$

$$= \frac{1}{2} \int_{\frac{\pi}{4}}^0 \ln \left[\frac{2}{1 + \tan\phi} \right] d\phi$$

SPLIT THE LOG, OBSERVING THE DEFINITION OF I

$$\Rightarrow I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta) d\theta$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln 2 d\theta - \ln(1+\tan\theta) d\theta$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln 2 d\theta - \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta) d\theta$$

$$\Rightarrow I = \frac{1}{2} \ln 2 \times \frac{\pi}{4} - I$$

$$\Rightarrow 2I = \frac{\pi \ln 2}{8}$$

$$\Rightarrow I = \frac{\pi \ln 2}{16}$$

A BRIEF WORD

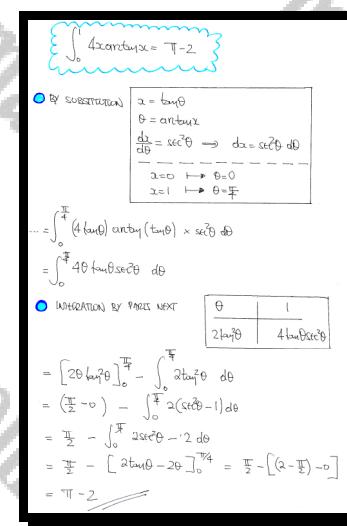
$$\therefore \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx = \frac{\pi \ln 2}{16}$$

Question 313 (*****)

Use appropriate integration techniques to show that

$$\int_0^1 4x \arctan x \, dx = \pi - 2.$$



The handwritten working shows two methods for solving the integral $\int_0^1 4x \arctan x \, dx$.

Method 1: Substitution

Let $x = \tan \theta$, then $\theta = \arctan x$.
 $\frac{dx}{d\theta} = \sec^2 \theta \Rightarrow dx = \sec^2 \theta \, d\theta$

When $x=0 \rightarrow \theta=0$
 $x=1 \rightarrow \theta=\frac{\pi}{4}$

$$\begin{aligned} \int_0^1 4x \arctan x \, dx &= \int_0^{\frac{\pi}{4}} 4 \tan \theta \arctan(\tan \theta) \times \sec^2 \theta \, d\theta \\ &= \int_0^{\frac{\pi}{4}} 4\theta \tan \theta \sec^2 \theta \, d\theta \\ &= \left[2\theta \ln(2\theta) \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2\ln(2\theta) \, d\theta \\ &= \left(\frac{\pi}{2} - 0 \right) - \int_0^{\frac{\pi}{4}} 2(\ln 2\theta - 1) \, d\theta \\ &= \frac{\pi}{2} - \int_0^{\frac{\pi}{4}} 2\ln 2\theta - 2 \, d\theta \\ &= \frac{\pi}{2} - \left[2\ln 2\theta - 2\theta \right]_0^{\frac{\pi}{4}} = \frac{\pi}{2} - \left[(2 - \frac{\pi}{2}) - 0 \right] \\ &= \pi - 2 \end{aligned}$$

Question 314 (*****)

$$I = \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx.$$

a) Use an appropriate trigonometric substitution to show that

$$I = \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln \sqrt{2 + \frac{1}{2} \ln \left[\frac{\cos(\theta - \frac{1}{4}\pi)}{\cos \theta} \right]} d\theta.$$

b) Show further that

$$I = \frac{\pi \ln 2}{16} + \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{1}{2} \ln \left[\frac{\cos(\varphi - \frac{1}{8}\pi)}{\cos(\varphi + \frac{1}{8}\pi)} \right] d\varphi.$$

c) Deduce that

$$I = \frac{\pi \ln 2}{16}.$$

V, , proof

a) LOOKING AT THE INTEGRAL

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx &= \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan \theta)}{1+\tan^2 \theta} \left(\frac{1}{2} \sec^2 \theta \right) d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan \theta)}{\sec^2 \theta} \left(\frac{1}{2} \sec^2 \theta \right) d\theta = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(1+\tan \theta) d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left(1 + \frac{\sin \theta}{\cos \theta} \right) d\theta = \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left(\frac{\cos \theta + \sin \theta}{\cos \theta} \right) d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(\cos \theta + \sin \theta) d\theta - \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln(\cos \theta) d\theta \\ &\text{MULTIPLY OUT TO SIMPLIFY EASY BY INTEGRATION} \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln \left[\sqrt{2} \left(\frac{1}{\sqrt{2}} (\cos \theta + \frac{1}{\sqrt{2}} \sin \theta) \right) \right] - \frac{1}{2} \ln(\cos \theta) d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln \left[\sqrt{2} \left(\cos \left(\theta + \frac{\pi}{4} \right) + \sin \left(\theta + \frac{\pi}{4} \right) \right) \right] - \frac{1}{2} \ln(\cos \theta) d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln \left[\sqrt{2} \cos \left(\theta + \frac{\pi}{4} \right) \right] - \frac{1}{2} \ln(\cos \theta) d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln \left[\sqrt{2} \right] + \frac{1}{2} \ln \left[\cos \left(\theta + \frac{\pi}{4} \right) \right] - \frac{1}{2} \ln(\cos \theta) d\theta \\ &= \frac{1}{2} \ln \sqrt{2} + \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left[\frac{\cos(\theta + \frac{\pi}{4})}{\cos \theta} \right] d\theta \quad \text{As Required} \end{aligned}$$

b) SPLIT THE INTEGRAL & USE A FURTHER SUBSTITUTION

$$\begin{aligned} &= \int_0^{\frac{\pi}{4}} \frac{1}{2} \ln \sqrt{2} d\theta + \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left[\frac{\cos(\theta + \frac{\pi}{4})}{\cos \theta} \right] d\theta \\ &= \frac{1}{2} \ln 2 \int_0^{\frac{\pi}{4}} d\theta + \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln \left[\frac{\cos(\theta + \frac{\pi}{4})}{\cos \theta} \right] d\theta \end{aligned}$$

ANOTHER SUBSTITUTION

- $\frac{\pi}{4} - \theta \rightarrow \theta \Leftrightarrow \theta = \frac{\pi}{4} - \theta$
- $d\theta = d\theta$
- $\theta = 0 \mapsto \frac{\pi}{4} \mapsto \frac{\pi}{4} - 0 = \frac{\pi}{4}$
- $\theta = \frac{\pi}{4} \mapsto \frac{\pi}{4} - \frac{\pi}{4} = 0$

$$= \frac{\pi \ln 2}{16} + \frac{1}{2} \int_{\frac{\pi}{4}}^{0} \ln \left[\frac{\cos(\frac{\pi}{4} - \theta)}{\cos(\frac{\pi}{4} + \theta)} \right] d\theta \quad \text{As Required}$$

c) LOOKING AT THE FINAL ANSWER GIVEN WE SUSPECT WE HAVE TWO INTEGRALS IN THE SIMPLIFIED FORMULA - LET $f(\theta)$ BE THE INTEGRAND

$$\begin{aligned} f(\theta) &= \ln \left[\frac{\cos(\frac{\pi}{4} - \theta)}{\cos(\frac{\pi}{4} + \theta)} \right] + \ln \left[\frac{\cos(-(\frac{\pi}{4} - \theta))}{\cos(-(\frac{\pi}{4} + \theta))} \right] = \dots \cos(\theta) \sin(\theta) \\ &= \ln \left[\frac{\cos(\frac{\pi}{4} - \theta)}{\cos(\frac{\pi}{4} + \theta)} \right] = -\ln \left[\frac{\cos(\frac{\pi}{4} - \theta)}{\cos(\frac{\pi}{4} + \theta)} \right] = -f(\theta) \\ &\therefore \int_{\frac{\pi}{4}}^{0} \ln \left[\frac{\cos(\frac{\pi}{4} - \theta)}{\cos(\frac{\pi}{4} + \theta)} \right] d\theta = 0 \\ \therefore \quad I &= \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} dx = \frac{\pi \ln 2}{16} \quad \text{As Required} \end{aligned}$$

Question 315 (*****)

$$I = \int \frac{2x+1}{\sqrt{x-1}} dx.$$

a) Show that

$$I = \frac{4}{3}(x-1)^{\frac{3}{2}} + 6(x-1)^{\frac{1}{2}} + \text{constant}.$$

You may not use any substitution or integration by parts.

b) Determine the value of a , given that

$$\int_2^a \frac{2x+1}{\sqrt{x-1}} dx = 102.$$

, $a = 17$

a) Given that we may not use substitution or integration by parts,
use only manipulation as shown:

$$\begin{aligned} \frac{2x+1}{\sqrt{x-1}} &= \frac{2x+1}{(x-1)^{\frac{1}{2}}} = \frac{2(x-1)+3}{(x-1)^{\frac{1}{2}}} = \frac{2(x-1)}{(x-1)^{\frac{1}{2}}} + \frac{3}{(x-1)^{\frac{1}{2}}} \\ &= 2(x-1)^{-\frac{1}{2}} + 3(x-1)^{-\frac{1}{2}} \end{aligned}$$

$$\therefore \int \frac{2x+1}{\sqrt{x-1}} dx = \int [2(x-1)^{-\frac{1}{2}} + 3(x-1)^{-\frac{1}{2}}] dx = \frac{2}{3}(x-1)^{\frac{1}{2}} + C_1(x-1)^{\frac{1}{2}} + C$$

$$= \frac{2}{3}(x-1)^{\frac{1}{2}} + 3(x-1)^{\frac{1}{2}} + C$$

b) Now proceed with the limits:

$$\begin{aligned} \int_2^a \frac{2x+1}{\sqrt{x-1}} dx &= \left[\frac{2}{3}(x-1)^{\frac{1}{2}} + 3(x-1)^{\frac{1}{2}} \right]_2^a = 102 \\ &= \left[\frac{2}{3}(a-1)^{\frac{1}{2}} + 3(a-1)^{\frac{1}{2}} \right] - \left[\frac{2}{3}(1-1)^{\frac{1}{2}} + 3(1-1)^{\frac{1}{2}} \right] = 102 \\ &\quad \frac{2}{3}(a-1)^{\frac{1}{2}} + 3(a-1)^{\frac{1}{2}} - \frac{2}{3} = 102 \\ &\quad 4(a-1)^{\frac{1}{2}} + 9(a-1)^{\frac{1}{2}} - 10 = 306 \\ &\quad 2(a-1)^{\frac{1}{2}} + 9(a-1)^{\frac{1}{2}} - 2 = 153 \\ &\quad 2(a-1)^{\frac{1}{2}} + 9(a-1)^{\frac{1}{2}} - 161 = 0 \end{aligned}$$

Now write $A = (a-1)^{\frac{1}{2}}$ for simplicity, $A^2 > 0 \Rightarrow A > 0$

$$\Rightarrow 2A^2 + 9A - 161 = 0$$

Try $A=1$ $2(1)^2 + 9(1) - 161 \neq 0$
 $A=2$ $2(2)^2 + 9(2) - 161 \neq 0$
 $A=4$ $2(4)^2 + 9(4) - 161 = 0$
 $(A-4)(2A+21) = 0$

BY LONG DIVISION OR FURTHER MANIPULATION:

$$\begin{aligned} 2A^2 + 9A - 161 &= 0 \\ 2A(A-4) + 8A(A-4) + 41(A-4) &= 0 \\ (A-4)(2A^2 + 8A + 41) &= 0 \\ \uparrow b^2 - 4ac = 8^2 - 4 \times 2 \times 41 &= 0 \\ \text{ONLY SOLUTION IS } A=4, \text{ i.e. } (a-1)^{\frac{1}{2}}=4 & \\ a-1=16 & \\ a=17 & \end{aligned}$$

Question 316 (*****)

$$J = \int_0^1 \frac{(x^2+1)e^x}{(x+1)^2} dx.$$

Show that $J = 1$, proof

FINDING PARTIAL FRACTIONS ON THE
INTERVAL (0, 1) (IGNORING e^x)

$\frac{x^2+1}{(x+1)^2} \equiv A + \frac{B}{x+1} + \frac{C}{(x+1)^2}$	$A=1$	$B=0$	$C=1$
$x^2+1 \equiv A(x+1)^2 + B(x+1) + C$	$A=1$	$B=0$	$C=1$
$x^2+1 \equiv Ax^2+2Ax+A+Bx+B+C$	$A=1$	$B=0$	$C=1$
$x^2+1 \equiv Ax^2+(2A+B)x+(A+B+C)$	$A=1$	$B=0$	$C=1$

Hence

$$\Rightarrow J = \int_0^1 e^x - \frac{2e^x}{x+1} + \frac{e^x}{(x+1)^2} dx$$

$$\Rightarrow J = \int_0^1 e^x dx - \int_0^1 \frac{2e^x}{x+1} dx + \int_0^1 \frac{e^x}{(x+1)^2} dx$$

(CHOOSE)
(BY PARTS)
(LEAVE IT AS IT IS)

$$\begin{aligned} &\Rightarrow J = \left[e^x \right]_0^1 - \left\{ \left[\frac{2e^x}{x+1} \right]_0^1 + \int_0^1 \frac{2e^x}{(x+1)^2} dx \right\} + \int_0^1 \frac{2e^x}{(x+1)^2} dx \\ &\Rightarrow J = \left[e^x \right]_0^1 - \left[\frac{2e^x}{x+1} \right]_0^1 - \int_0^1 \frac{2e^x}{(x+1)^2} dx + \int_0^1 \frac{2e^x}{(x+1)^2} dx \\ &\Rightarrow J = (e-1) - (e-1) \\ &\Rightarrow J = 1 \end{aligned}$$

Question 317 (***)**

By using an appropriate substitution or substitutions, show that

$$\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{4x \sin(x^2)}{\sin(x^2) + \sin(\ln 6 - x^2)} dx = \ln\left(\frac{3}{2}\right).$$

[[solution](#)] , [[proof](#)]

$$\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{4x \sin(x^2)}{\sin(x^2) + \sin(\ln 6 - x^2)} dx = \ln\left(\frac{3}{2}\right)$$

SOLVE BY A SUBSTITUTION

$$\begin{aligned} u &= x^2 \\ du &= 2x \\ dx &= \frac{du}{2x} \\ x = \sqrt{\ln 2} &\mapsto u = \ln 2 \\ x = \sqrt{\ln 3} &\mapsto u = \ln 3 \end{aligned}$$

$$\begin{aligned} &= \int_{\ln 2}^{\ln 3} \frac{2 \sin(u)}{\sin(u) + \sin(\ln 6 - u)} \left(\frac{du}{2x}\right) \\ &= \int_{\ln 2}^{\ln 3} \frac{2 \sin(u)}{\sin(u) + \sin(\ln 6 - u)} du \\ &= \int_{\ln 2}^{\ln 3} \frac{-2 \sin(\ln 6 - u)}{\sin(u) + \sin(\ln 6 - u)} (-du) \\ &= \int_{\ln 2}^{\ln 3} \frac{2 \sin(\ln 6 - u)}{\sin(u) + \sin(\ln 6 - u)} du \\ &= \int_{\ln 2}^{\ln 3} \frac{2 \sin(\ln 6 - u)}{2 \sin(u) + \sin(\ln 6 - u)} du \end{aligned}$$

ANOTHER SUBSTITUTION

$$\begin{aligned} v &= \ln 6 - u \\ u &= \ln 6 - v \\ du &= -dv \\ u = \ln 2 &\mapsto v = \ln 6 - \ln 2 = \ln 4 \\ u = \ln 3 &\mapsto v = \ln 6 - \ln 3 = \ln 3 \end{aligned}$$

$$\begin{aligned} &= \int_{\ln 4}^{\ln 3} \frac{2 \sin(v)}{\sin(v) + \sin(\ln 6 - v)} dv \\ &= \int_{\ln 4}^{\ln 3} \frac{2 \sin(v) - 2 \sin(\ln 6 - v)}{\sin(v) + \sin(\ln 6 - v)} dv \\ &\Rightarrow I = \int_{\ln 4}^{\ln 3} \frac{2 \sin(v) - 2 \sin(\ln 6 - v)}{\sin(v) + \sin(\ln 6 - v)} dv \\ &\Rightarrow 2I = \int_{\ln 4}^{\ln 3} \frac{2 \sin(v) - 2 \sin(\ln 6 - v)}{\sin(v) + \sin(\ln 6 - v)} dv \\ &\Rightarrow I = \int_{\ln 4}^{\ln 3} 1 dv = [u]_{\ln 4}^{\ln 3} = \ln 3 - \ln 4 \\ &\Rightarrow \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{4x \sin(x^2)}{\sin(x^2) + \sin(\ln 6 - x^2)} dx = \ln\left(\frac{3}{2}\right) \end{aligned}$$

COLLECTING THE RESULTS

Question 318 (*****)

$$f(x) = \begin{cases} x - [x] & x \in \mathbb{R}, [x] = 2k+1, k \in \mathbb{Z} \\ -x + [x] + 1 & x \in \mathbb{R}, [x] = 2k, k \in \mathbb{Z} \end{cases}$$

where $[x]$ is defined as the greatest integer less or equal to x .

Find the value of

$$\frac{\pi^2}{8} \int_{-8}^8 f(x) \cos(\pi x) dx.$$

 , 4

$f(x) = \begin{cases} x - [x] & \text{IF } [x] \text{ IS ODD} \\ -x + [x] + 1 & \text{IF } [x] \text{ IS EVEN} \end{cases}$

• Firstly we produce a quick sketch for the graph of $f(x)$ by trying some values for x .

• We notice from the graph that: $f(x)$ is even
 $f(x)$ is periodic (period=2).

Thus $\frac{\pi^2}{8} \int_{-8}^8 f(x) \cos(\pi x) dx = \frac{\pi^2}{4} \int_0^8 f(x) \cos(\pi x) dx$

• Next consider the function " $\frac{x}{2}$ " = $\frac{\pi}{\pi}x = 2x$, so the two functions in the product share periods.

$$\begin{aligned} \dots &= \frac{\pi^2}{4} \int_0^8 f(2x) \cos(\pi x) dx = \frac{\pi^2}{4} \int_0^8 f(u) \cos(\pi u/2) du \\ &= \pi^2 \int_0^1 (-2x) \cos(\pi u/2) du + \pi^2 \int_1^2 (2x-1) \cos(\pi u/2) du \end{aligned}$$

• Proceed by parts of use & "substitution" in the second integral.

$u = 2x-1$	$du = 2dx$	$u = 0 \rightarrow 1$	$u = 1 \rightarrow 2$	$\text{Also } \cos(\pi u/2) = \cos[\pi(u+1)/2]$
$x = 1 \rightarrow 0$				$= \cos(\pi(u+1))$
$x = 2 \rightarrow 1$				$= \cos(\pi(u+1)) - \cos(\pi u/2)$

$$\begin{aligned} &= -\frac{\pi^2}{4} \int_0^1 u \sin(\pi u/2) du + \frac{\pi^2}{4} \int_1^2 (2u-1) \sin(\pi u/2) du \\ &= 2\pi \int_0^1 u \sin(\pi u/2) du \\ &= 2 \left[-\frac{1}{\pi} \cos(\pi u/2) \right]_0^1 \\ &= 2 \left[1 - \cos(\pi) \right] \\ &= 4 \end{aligned}$$

• Finally integration by parts

$u = 1-2x$	$v = -2$
$du = -2dx$	

$$\begin{aligned} &\dots = \pi^2 \int_0^1 (1-2x) \cos(\pi u/2) du + \pi^2 \int_0^1 -u \cos(\pi u/2) du \\ &\quad \downarrow \text{write back in } x \\ &= \pi^2 \int_0^1 (1-2x) \cos(\pi x) dx + \pi^2 \int_0^1 -x \cos(\pi x) dx \\ &= \pi^2 \int_0^1 (1-2x) \cos(\pi x) dx \end{aligned}$$

• Finally integration by parts

$u = 1-2x$	$v = -2$
$du = -2dx$	

$$\begin{aligned} &= \pi^2 \left\{ \left[\frac{1-2x}{\pi} \sin(\pi x) \right]_0^1 + \frac{2}{\pi} \int_0^1 \sin(\pi x) dx \right\} \\ &= 2\pi \int_0^1 \sin(\pi x) dx \\ &= 2\pi \left[-\frac{1}{\pi} \cos(\pi x) \right]_0^1 \\ &= 2 \left[\cos(\pi) \right]_0^1 \\ &= 2 \left[1 - \cos(\pi) \right] \\ &= 4 \end{aligned}$$

Question 319 (*****)

By using symmetry arguments, find the exact value of the following integral

$$\int_0^\pi e^{|\cos x|} [\sin(\cos x) + \cos(\cos x)] \sin x \, dx.$$

, $e(\cos 1 + \sin 1) - 1$

$$\begin{aligned}
 & \int_0^\pi e^{\cos x} [\sin(\cos x) + \cos(\cos x)] \sin x \, dx \\
 &= \int_0^\pi e^{\cos x} \sin x \sin(\cos x) \, dx + \int_0^\pi e^{\cos x} \cos x \cos(\cos x) \, dx \\
 &\quad \text{↑ } \text{↑ } \text{↑ } \text{↑ } \text{↑ } \text{↑ } \\
 &\quad \text{GIVEN } \sin x = \frac{du}{dx} \quad \text{GIVEN } \cos x = \frac{du}{dx} \\
 &\quad \text{GIVEN } \cos x = \frac{du}{dx} \quad \text{GIVEN } \cos x = \frac{du}{dx} \\
 &= 2 \int_0^{\frac{\pi}{2}} e^{\cos x} \sin x \cos(\cos x) \, dx \\
 &\quad \boxed{\text{BY SUBSTITUTION } u = \cos x \quad u = 0 \mapsto u = 1 \\
 &\quad \frac{du}{dx} = -\sin x \quad u = \frac{\pi}{2} \mapsto u = 0 \\
 &\quad du = -\frac{du}{\sin x}}
 \end{aligned}$$

$$\begin{aligned}
 & \text{RETURNING TO THE INDEFINITE INTEGRAL} \\
 &= 2 \int e^u \cos u \, du = [e^u (\cos u + \sin u)]_0^1 \\
 &= e^1 [\cos 1 + \sin 1] - e^0 (\cos 0 + \sin 0) \\
 &= e \cos 1 + e \sin 1 - 1
 \end{aligned}$$

Question 320 (*****)

$$I = \int_0^1 \left[\prod_{r=1}^{10} (x+r) \right] \left[\sum_{r=1}^{10} \left(\frac{1}{x+r} \right) \right] dx.$$

Show by a detailed method that

$$I = a \times b!,$$

where a and b are positive integers to be found.

$$\boxed{\quad}, \quad a = b = 10$$

QUESTION

$$I = \int_0^1 \left[\prod_{r=1}^{10} (x+r) \right] \left[\sum_{r=1}^{10} \frac{1}{x+r} \right] dx$$

SOLN

Let $u = \prod_{r=1}^{10} (x+r) = (x+1)(x+2)(x+3)\dots(x+10)$

 $\Rightarrow \ln u = \ln[(x+1)(x+2)(x+3)\dots(x+10)]$
 $\Rightarrow \ln u = \ln(x+1) + \ln(x+2) + \ln(x+3) + \dots + \ln(x+10)$

Differentiate with respect to x ,

 $\Rightarrow \frac{1}{u} \frac{du}{dx} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \dots + \frac{1}{x+10}$
 $\Rightarrow \frac{1}{u} \frac{du}{dx} = \frac{10}{\prod_{r=1}^{10} (x+r)}$
 $\Rightarrow \frac{du}{dx} = u \frac{10}{\prod_{r=1}^{10} (x+r)}$
 $\Rightarrow \frac{du}{dx} = \left[\prod_{r=1}^{10} (x+r) \right] \left[\sum_{r=1}^{10} \frac{1}{x+r} \right]$

Now L.E.T $\frac{du}{dx} = \sum_{r=1}^{10} \frac{1}{x+r} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \dots + \frac{1}{x+10}$

 $\Rightarrow V = \ln(x+1) + \ln(x+2) + \ln(x+3) + \dots + \ln(x+10)$
 $\Rightarrow V = \ln \left[\prod_{r=1}^{10} (x+r) \right]$
 $\Rightarrow V = \ln \left[\prod_{r=1}^{10} (x+r) \right]$

INTEGRATE BY PARTS

$$\int_0^1 \left[\prod_{r=1}^{10} (x+r) \right] \left[\sum_{r=1}^{10} \frac{1}{x+r} \right] dx = \left[\prod_{r=1}^{10} (x+r) \right] \ln \left[\prod_{r=1}^{10} (x+r) \right] \Big|_0^1$$

$$- \int_0^1 \left[\prod_{r=1}^{10} (x+r) \right] \sum_{r=1}^{10} \frac{1}{x+r} \left[\ln \left[\prod_{r=1}^{10} (x+r) \right] \right] dx$$

ABBREVIATING FOR SIMPLICITY

$$\Rightarrow \int_0^1 \left[\prod_{r=1}^{10} (x+r) \right] \left[\sum_{r=1}^{10} \frac{1}{x+r} \right] dx = [uv]_0^1 - \int_0^1 \frac{du}{dx} v dx$$

BUT FROM earlier $V = \ln \left[\prod_{r=1}^{10} (x+r) \right] = \ln u$

$$\Rightarrow \int_0^1 \left[\prod_{r=1}^{10} (x+r) \right] \left[\sum_{r=1}^{10} \frac{1}{x+r} \right] dx = [uv]_0^1 - \int_0^1 \frac{du}{dx} \ln u dx$$
 $= [uv]_0^1 - \int_0^1 \ln u du$

STANDARD RESULT OR BY PAGES $\int \ln u du = u \ln u - u + C$

$$= [uv]_0^1 - \left[u \ln u - u \right]_0^1$$
 $= [uv - u \ln u + u]_0^1$
 $= (ux - ux \ln u + u)_0^1$
 $= \left[\prod_{r=1}^{10} (x+r) \right]_{x=1}^{x=10}$
 $= \prod_{r=1}^{10} (1+r) = \prod_{r=1}^{10} r$
 $= 10! - 10!$
 $= (10 \times 9!) - 10!$
 $= 10 \times 9!$

$a-b=10$

Question 321 (*****)

$$I = \int \sqrt{\tan x} \, dx.$$

- a) Use a suitable substitution to show that

$$I = \int \frac{1 + \frac{1}{u^2}}{\left(u - \frac{1}{u}\right)^2 + 2} \, du + I = \int \frac{1 - \frac{1}{u^2}}{\left(u + \frac{1}{u}\right)^2 - 2} \, du.$$

- b) By using a further substitution in each of the integrals of part (a) find a simplified expression for I , in terms of x .

You may assume without proof that

$$\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan\left[\frac{x}{a}\right] + \text{constant}.$$

, $\frac{1}{\sqrt{2}} \arctan\left[\frac{\tan x - 1}{\sqrt{2 \tan x}}\right] + \frac{1}{2\sqrt{2}} \ln\left[\frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1}\right] + C$

a) $\int \sqrt{\tan x} \, dx \dots \text{BY SUBSTITUTION}$

$$\begin{aligned} &= \int u \left(\frac{2u}{1+u^2} \right) du = \int \frac{2u^2}{1+u^2} du \\ &= \int \frac{2}{u^2+1} du = \int \frac{1+\frac{1}{u^2} + \frac{1}{u^2}}{u^2+1} du \\ &= \int \frac{1+\frac{1}{u^2}}{u^2+1} du + \int \frac{1-\frac{1}{u^2}}{u^2+1} du \\ &= \int \frac{1+\frac{1}{u^2}}{(u-\frac{1}{u})^2+2} du + \int \frac{1-\frac{1}{u^2}}{(u+\frac{1}{u})^2-2} du \end{aligned}$$

$u = \sqrt{\tan x}$
 $u^2 = \tan x$
 $2u \, du = \sec^2 x \, dx$
 $2u \, du = (1+u^2) \, dx$
 $2u \, du = (1+u^2) \, dx$
 $dx = \frac{2u}{1+u^2} \, du$

b) NOW ANOTHER SUBSTITUTION FOR EACH INTEGRAL

$v = u - \frac{1}{u}$	$w = u + \frac{1}{u}$
$dv = \left(1 + \frac{1}{u^2}\right) du$	$dw = \left(1 - \frac{1}{u^2}\right) du$

$$\begin{aligned} &= \int \frac{1+\frac{1}{u^2}}{v^2+2} \cdot \frac{dv}{1-u^{-2}} + \int \frac{1-\frac{1}{u^2}}{w^2-2} \cdot \frac{dw}{1-u^{-2}} \\ &= \int \frac{1}{v^2+2} \, dv + \int \frac{1}{w^2-2} \, dw \\ &= \frac{1}{\sqrt{2}} \arctan\left(\frac{v}{\sqrt{2}}\right) + \int \frac{1}{(w-2)(w+2)} \, dw \quad \text{PARTIAL FRACTIONs} \\ &= \frac{1}{\sqrt{2}} \arctan\left(\frac{v}{\sqrt{2}}\right) + \int \frac{\frac{1}{2\sqrt{2}}}{w-2} - \frac{\frac{1}{2\sqrt{2}}}{w+2} \, dw \\ &= \frac{1}{\sqrt{2}} \arctan\left(\frac{v}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}} \int \frac{1}{w-2} - \frac{1}{w+2} \, dw \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sqrt{2}} \arctan\left(\frac{v^2-1}{\sqrt{2}v}\right) + \frac{1}{2\sqrt{2}} \ln\left[\frac{w-2}{w+2}\right] + C \\ &= \frac{1}{\sqrt{2}} \arctan\left(\frac{u^2-1}{\sqrt{2}u}\right) + \frac{1}{2\sqrt{2}} \ln\left[\frac{u+\frac{1}{u}-\sqrt{2}}{u+\frac{1}{u}+\sqrt{2}}\right] + C \\ &= \frac{1}{\sqrt{2}} \arctan\left(\frac{u^2-1}{\sqrt{2}u}\right) + \frac{1}{2\sqrt{2}} \ln\left[\frac{u+1-\sqrt{2}u^{-1}}{u+1+\sqrt{2}u^{-1}}\right] + C \\ &\text{Finally reversing the final substitution we obtain} \\ &= \frac{1}{\sqrt{2}} \arctan\left(\frac{\tan x - 1}{\sqrt{2 \tan x}}\right) + \frac{1}{2\sqrt{2}} \ln\left[\frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1}\right] + C \end{aligned}$$

Question 322 (*****)

By using an appropriate substitution or substitutions, show that

$$\int_0^\pi \ln(\sin x) dx = -\pi \ln 2.$$

S.E., proof

• If $I = \int_0^\pi \ln(\sin x) dx = -\pi \ln 2$

$$\Rightarrow I = 2 \int_0^{\frac{\pi}{2}} \ln(\sin x) dx$$

(SINE IS EVEN ABOUT $\frac{\pi}{2}$)

$$\Rightarrow I = 2 \int_0^{\frac{\pi}{2}} \ln[\sin(\frac{\pi}{2}-x)] (-dx)$$

($x = \frac{\pi}{2}-X$
 $dx = -dX$
 CHANGES SIGN)

$$\Rightarrow I = 2 \int_0^{\frac{\pi}{2}} \ln(\cos x) dx$$

($\sin(\frac{\pi}{2}-0) = \cos 0$)

• Now ADDING EXPRESSIONS AS FOLLOWS, BY SWAPPING THE DUMMY VARIABLES BACK TO x

$$\Rightarrow I + I = 2 \int_0^{\frac{\pi}{2}} \ln(\sin x) dx + 2 \int_0^{\frac{\pi}{2}} \ln(\cos x) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\sin x) + \ln(\cos x) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln(\frac{1}{2} \sin 2x) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln \frac{1}{2} dx + \int_0^{\frac{\pi}{2}} \ln \sin 2x dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ln \frac{1}{2} dx + \int_0^{\frac{\pi}{2}} \ln 2 \sin x dx$$

$$\Rightarrow I = \left[2 \ln \frac{1}{2} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \ln(\sin 2x) dx$$

↓

($u = 2x$
 $2 = \frac{1}{2}u$
 $du = \frac{1}{2}dx$
 $2x = u \rightarrow x = \frac{u}{2}$
 $2 = \frac{u}{2} \rightarrow u = 4$)

$$\Rightarrow I = \frac{1}{2} \ln \frac{1}{2} + \int_0^\pi \ln(\sin u) du$$

$$\Rightarrow I = -\frac{\pi}{2} \ln 2 + \frac{1}{2} \int_0^\pi \ln(\sin u) du$$

$$\Rightarrow I = -\frac{\pi}{2} \ln 2 + \frac{1}{2} I$$

$$\Rightarrow \frac{1}{2} I = -\frac{\pi}{2} \ln 2$$

$$\Rightarrow I = -\pi \ln 2$$

$\therefore \int_0^\pi \ln(\sin x) dx = -\pi \ln 2$

Question 323 (*****)

It is given that

$$I = \int_{\frac{1}{2}\pi}^{\pi} \frac{3 + \cos x}{13 + 3\cos x + 2\sin x} dx \quad \text{and} \quad J = \int_{\frac{1}{2}\pi}^{\pi} \frac{2 + \sin x}{13 + 3\cos x + 2\sin x} dx.$$

By considering two linear combinations in I and J , show that

$$I = \frac{1}{26} \left[3\pi - \ln\left(\frac{81}{16}\right) \right],$$

and find a similar expression for J .

$$\boxed{\quad}, \quad I = \frac{1}{13} \left[\pi + \ln\left(\frac{27}{8}\right) \right]$$

CREATE A NEW INTEGRAL AS FOLLOWS

$$3I + 2J = \int_{\frac{1}{2}\pi}^{\pi} \frac{9 + 3\cos x}{13 + 3\cos x + 2\sin x} dx + \int_{\frac{1}{2}\pi}^{\pi} \frac{4 + 2\sin x}{13 + 3\cos x + 2\sin x} dx$$

$$= \int_{\frac{1}{2}\pi}^{\pi} \frac{13 + 3\cos x + 2\sin x}{13 + 3\cos x + 2\sin x} dx$$

$$= \int_{\frac{1}{2}\pi}^{\pi} 1 dx$$

$$= \pi$$

CREATE ALSO ANOTHER NEW INTEGRAL

$$2I - 3J = \int_{\frac{1}{2}\pi}^{\pi} \frac{6 + 2\cos x}{13 + 3\cos x + 2\sin x} dx - \int_{\frac{1}{2}\pi}^{\pi} \frac{5 + 3\sin x}{13 + 3\cos x + 2\sin x} dx$$

$$= \int_{\frac{1}{2}\pi}^{\pi} \frac{2\cos x - 3\sin x}{13 + 3\cos x + 2\sin x} dx$$

$$= \left[\ln \left| \sqrt{13 + 3\cos x + 2\sin x} \right| \right]_{\frac{1}{2}\pi}^{\pi}$$

$$= \ln 10 - \ln 15$$

$$= \ln \frac{2}{3}$$

THENCE WE HAVE

$$3I + 2J = \pi \quad \times 2$$

$$2I - 3J = \ln \frac{2}{3} \quad \times 3$$

$$9I + 6J = 3\pi \quad \underline{-----}$$

$$4I - 6J = 2\ln \frac{2}{3}$$

$$13I = \frac{39\pi}{2} + 2\ln \frac{2}{3}$$

$$I = \frac{3}{13} \left[\frac{39}{2} + 2\ln \frac{2}{3} \right]$$

$$I = \frac{1}{26} \left[3\pi - 4\ln \frac{2}{3} \right]$$

$$I = \frac{1}{26} \left[3\pi - \ln \frac{81}{16} \right]$$

AND SIMILARLY

$$3I + 2J = \pi \quad \times 2$$

$$2I - 3J = \ln \frac{2}{3} \quad \times 3$$

$$6I + 4J = \pi \quad \underline{-----}$$

$$6I - 9J = 3\ln \frac{2}{3}$$

$$13J = \pi - 3\ln \frac{2}{3}$$

$$J = \frac{1}{13} \left[\pi + 3\ln \frac{2}{3} \right]$$

$$J = \frac{1}{13} \left[\pi + \ln \frac{27}{8} \right]$$

Question 324 (*****)

By using an appropriate substitution or substitutions, show that

$$\int_0^1 \frac{\ln(x+1)}{1+x^2} dx = \frac{\pi \ln 2}{8}.$$

S.G., proof

$\int_0^1 \frac{\ln(x+1)}{1+x^2} dx = \frac{\pi \ln 2}{8}$

- USE $I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$
- BY SUBSTITUTION

$x = \tan\theta$
$dx = \sec^2\theta d\theta$
$x=0 \rightarrow \theta=0$
$x=1 \rightarrow \theta=\frac{\pi}{4}$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta} (\sec^2\theta d\theta)$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{\sec^2\theta} (\sec^2\theta d\theta)$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta) d\theta$

 - ANOTHER SUBSTITUTION

$\theta = \frac{\pi}{4} - \phi \quad \text{or} \quad d\theta = -d\phi$
$d\theta = -d\phi$
$\theta=0 \rightarrow \phi=\frac{\pi}{4}$
$\theta=\frac{\pi}{4} \rightarrow \phi=0$

$\Rightarrow I = \int_{\frac{\pi}{4}}^0 \ln\left[1 + \tan\left(\frac{\pi}{4}-\phi\right)\right] (-d\phi)$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln\left[1 + \tan\left(\frac{\pi}{4}-\phi\right)\right] d\phi$

KNOW EXPANDING THE ARGUMENT OF THE LOG BY THE TANGENT COMPOUND FORMULA

$$\tan\left(\frac{\pi}{4}-\phi\right) = \frac{\tan\frac{\pi}{4} - \tan\phi}{1 + \tan\frac{\pi}{4}\tan\phi} = \frac{1-\tan\phi}{1+\tan\phi}$$

TRYING THE ARGUMENT FURTHER

$$1 + \tan\left(\frac{\pi}{4}-\phi\right) = 1 + \frac{1-\tan\phi}{1+\tan\phi} = \frac{1+\tan\phi+1-\tan\phi}{1+\tan\phi} = \frac{2}{1+\tan\phi}$$

$\therefore I = \int_0^{\frac{\pi}{4}} \frac{\ln(1+x)}{1+x^2} dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta) d\theta = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$

THUS USING THE LAST TWO TERMS IN A COMMON DENOMINATOR

$\Rightarrow 2I = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta) d\theta + \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$

$\Rightarrow 2I = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta) + \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$

$\Rightarrow 2I = \int_0^{\frac{\pi}{4}} \ln\left[\left(1+\tan\theta\right) \cdot \frac{2}{1+\tan\theta}\right] d\theta$

$\Rightarrow 2 \int_0^{\frac{\pi}{4}} \frac{\ln(1+x)}{1+x^2} dx = \int_0^{\frac{\pi}{4}} \ln 2 d\theta$

$\Rightarrow 2 \int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{4} \ln 2$

$\Rightarrow \int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi \ln 2}{8}$

Question 325 (*****)

It is given that

$$I = \int_0^{\frac{1}{3}} \frac{32x^2}{(x^2 - 1)(x+1)^3} dx.$$

Show that $I = \frac{7}{6} - 2\ln 2$.

, proof

$\int_0^{\frac{1}{3}} \frac{32x^2}{(x^2 - 1)(x+1)^3} dx = \int_0^{\frac{1}{3}} \frac{32x^2}{(x-1)(x+1)^4} dx$

CONTINUE WITH A SIMPLE SUBSTITUTION

$u = 2x+1 \Rightarrow x = u - \frac{1}{2}$
 $du = dx$
 $x=0 \rightarrow u=1$
 $x=\frac{1}{3} \rightarrow u=\frac{5}{6}$

$\int_1^{\frac{5}{6}} \frac{32(u-1)^2}{(u-2)u^4} du = \int_1^{\frac{5}{6}} \frac{32u^2}{u^4} \times \frac{u^2 - 2u + 1}{u-2} du$

MANIPULATE BY DIVIDING "BACKGROUND" AS SHOWN BELOW

$\frac{-\frac{1}{2} + \frac{3}{2}u - \frac{3}{2}u^2 - \frac{1}{2}u^3}{1 - 2u + u^2}$
 $-2 + u$
 $-1 + \frac{3}{2}u$
 $-\frac{3}{2}u + \frac{3}{2}u^2$
 $+\frac{3}{2}u^2 - \frac{3}{2}u^3$
 $-\frac{1}{2}u^3 + \frac{1}{2}u^4$
 $-\frac{1}{2}u^4$
 $-\frac{1}{2}u^4 + \frac{1}{2}u^4$
 $\frac{1}{2}u^4$

(Note: Observe that "keep" the 2x with $u^2 - 2u + 1$)

REDUCING TO THE INTEGRAL WE WANT

$\dots = \int_1^{\frac{5}{6}} \frac{32}{u^4} \left[-\frac{1}{2} + \frac{3}{2}u - \frac{3}{2}u^2 - \frac{1}{2}u^3 + \frac{1}{2}u^4 \right] du$
 $= \int_1^{\frac{5}{6}} \left[-\frac{16}{u^2} + \frac{24}{u^3} - \frac{12}{u^4} + \frac{2}{u-2} + \frac{2}{u-1} \right] du$
 $= \left[\frac{16}{3u^3} - \frac{12}{u^2} + \frac{4}{u} - 2\ln|u| + 2\ln|u-2| \right]_1^{\frac{5}{6}}$
 $= \left[\frac{16}{3 \times \frac{125}{216}} - \frac{12}{\frac{25}{36}} + \frac{4}{\frac{5}{6}} - 2\ln\frac{5}{3} + 2\ln\left|\frac{5}{3} - 2\right| \right] - \dots$

$\dots = \left[\frac{16}{3} - 12 + 4 - 2\ln\frac{5}{3} + 2\ln\left|\frac{5}{3} - 2\right| \right]$
 $= \frac{16}{3} - \frac{12 \times \frac{9}{16}}{16} + \frac{4 \times \frac{3}{2}}{4} - 2\ln\frac{5}{3} + 2\ln\frac{2}{3} - \frac{16}{3} + 12 - 4$
 $= \frac{16 \times \frac{9}{16}}{24} - \frac{3 \times \frac{9}{4}}{4} + 3 - 2\ln\frac{5}{3} - 2\ln\frac{2}{3} - \frac{16}{3} + 8$
 $= \frac{9}{4} - \frac{27}{4} - \frac{16}{3} + 11 - 2\ln\left(\frac{5}{3} \times \frac{2}{3}\right)$
 $= 11 - \frac{16}{3} - \frac{18}{4} - 2\ln 2$
 $= 11 - \frac{16}{3} - \frac{9}{2} - 2\ln 2$
 $= \frac{66 - 32 - 27}{6} - 2\ln 2 = \frac{7}{6} - 2\ln 2$

Question 326 (*****)

$$I = \int_0^{\frac{1}{2}\pi} 4 \sin x \sqrt{\cos 2x} \ dx.$$

By using an appropriate substitution or substitutions, show that

$$I = 2 - \sqrt{2} \ln(1 + \sqrt{2}).$$

, proof

$I = \int_0^{\frac{\pi}{2}} 4 \sin x \sqrt{\cos 2x} \ dx = 2 - \sqrt{2} \ln(1 + \sqrt{2})$ <p><u>START WITH TRIGONOMETRIC IDENTITIES</u></p> $\Rightarrow I = \int_0^{\frac{\pi}{2}} 4 \sin x \sqrt{2\sin^2 x - 1} \ dx \quad \leftarrow \text{SUBSTITUTION NOTE}$ $\Rightarrow I = \int_0^{\frac{\pi}{2}} 4 \sin x \sqrt{2u^2 - 1} \frac{du}{\sqrt{u^2 - 1}}$ $\Rightarrow I = \int_{\frac{\pi}{2}}^1 4 \sqrt{2u^2 - 1} du$ <p><u>NEXT CONTINUE WITH A TRIGONOMETRIC (OR HYPERBOLIC) SUBSTITUTION:</u></p> $\Rightarrow I = \int_{\frac{\pi}{2}}^1 4 \sqrt{\sec^2 u - 1} \frac{\sec u \tan u}{\sqrt{u^2 - 1}} du$ $\Rightarrow I = \int_{\frac{\pi}{2}}^1 \frac{4}{\sec u} \sec u (\sec u \tan u) du$ $\Rightarrow I = \int_{\frac{\pi}{2}}^1 \frac{4}{\sec u} \tan^2 u \sec u du$ <p>AS THERE IS NO OBVIOUS IDENTITY OR SUBSTITUTION PROPOSED BY PAPER</p> $\Rightarrow I = \left[\frac{4}{\sec u} \tan u \right]_0^{\frac{\pi}{2}} - \frac{1}{\sec u} \int_0^{\frac{\pi}{2}} \sec^2 u du$	$\Rightarrow I = \left[\frac{4}{\sqrt{u^2 - 1}} \right]_0^{\frac{\pi}{2}} - \frac{4}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \sec u \tan u du$ $\Rightarrow I = 4 - \frac{4}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \sec u du - \frac{4}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \sec u \tan u du$ $\rightarrow I = 4 - \frac{4}{\sqrt{2}} [\ln(\sec u + \tan u)]_0^{\frac{\pi}{2}} - I$ $\rightarrow I = 4 - \frac{4}{\sqrt{2}} [\ln(\sqrt{2} + 1) - \ln 1] - I$ $\Rightarrow 2I = 4 - \frac{4}{\sqrt{2}} \ln(\sqrt{2} + 1) \quad \leftarrow \text{THIS}$ $\rightarrow I = 2 - \frac{2}{\sqrt{2}} \ln(\sqrt{2} + 1)$ $\Rightarrow I = 2 - \sqrt{2} \ln(1 + \sqrt{2}) \quad //$
---	--

Question 327 (*****)Find as an exact fraction the value of I ,

$$I = \frac{\int_0^1 (1-x^{20})^{50} dx}{\int_0^1 (1-x^{20})^{51} dx}.$$

, $\frac{1021}{1020}$

• Let $I_1 = \int_0^1 (1-x^{20})^{50} dx$ & $I_2 = \int_0^1 (1-x^{20})^{51} dx$

$$\begin{aligned} I_1 - \int_0^1 (1-x^{20})^{51} dx &= \int_0^1 (1-x^{20})(1-x^{20})^{50} dx \\ &= \int_0^1 (1-x^{20})^{50} - x^{20}(1-x^{20})^{50} dx = \int_0^1 (1-x^{20})^{50} dx - \int_0^1 x^{20}(1-x^{20})^{50} dx \\ &= I_{20} - \int_0^1 x^{20} [x^{20}(1-x^{20})^{50}] dx. \end{aligned}$$

• THIS IS WHERE WE RECOGNISE SOURCE ONE USE IT IN INTEGRATION BY PARTS

$$\int u dv = uv - \int v du$$

$$= I_{20} - \left\{ \left[-\frac{x(1-x^{20})^{51}}{20} \right]_0^1 + \frac{1}{20} \int_0^1 (1-x^{20})^{51} dx \right\}$$

$$= I_{20} - \frac{1}{20} I_{21}$$

• FINALLY REARRANGE THE BETTER ESTIMATE

$$\begin{aligned} I_{21} &= I_{20} - \frac{1}{20} I_{21} \\ 1020 I_{21} &= 1020 I_{20} - I_{21} \\ 1021 I_{21} &= 1020 I_{20} \\ \frac{I_{20}}{I_{21}} &= \frac{1021}{1020} \end{aligned}$$

Question 328 (*****)

The integral I is defined as

$$I = \int_0^{\pi} \frac{\sin^2 x}{1 + \cos^2 x} dx$$

- a) Show by a detailed method that

$$I + \pi = \int_{-\pi}^{\frac{1}{2}\pi} \frac{4}{1+\cos^2 x} dx$$

- b) Hence, find the value of I in exact simplified form.

- c) Verify the answer obtained in part (b) by an alternative method by first writing the integrand of I as a function of $\cot^2 x$.

$$\boxed{}, \quad I = \pi(\sqrt{2} - 1)$$

PROCEED AS FOLLOWS

$$\Rightarrow \int_0^{\pi} \frac{\sin x}{1+\cos x} dx + \int_0^{\pi} \frac{\cos x}{1+\cos x} dx = \int_0^{\pi} \frac{\sin x + \cos x}{1+\cos x} dx$$

$$\Rightarrow \int_0^{\pi} \frac{\sin x}{1+\cos x} dx + \int_0^{\pi} \frac{(1+\cos x)-1}{1+\cos x} dx = \int_0^{\pi} \frac{1}{1+\cos x} dx$$

$$\Rightarrow \int_0^{\pi} \frac{\sin x}{1+\cos x} dx + \int_0^{\pi} 1 - \frac{1}{1+\cos x} dx = \int_0^{\pi} \frac{1}{1+\cos x} dx$$

$$\Rightarrow \int_0^{\pi} \frac{\sin x}{1+\cos x} dx + \int_0^{\pi} 1 - \int_0^{\pi} \frac{1}{1+\cos x} dx = \int_0^{\pi} \frac{1}{1+\cos x} dx$$

$$\Rightarrow \int_0^{\pi} \frac{\sin x}{1+\cos x} dx - \int_0^{\pi} 1 dx - \int_0^{\pi} \frac{1}{1+\cos x} dx = \int_0^{\pi} \frac{1}{1+\cos x} dx$$

$$\Rightarrow \int_0^{\pi} \frac{\sin x}{1+\cos x} dx + [x]_0^{\pi} = 2 \int_0^{\pi} \frac{1}{1+\cos x} dx$$

$$\Rightarrow \int_0^{\pi} \frac{\sin x}{1+\cos x} dx + \pi = 4 \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} dx$$

$$\Rightarrow I + \pi = \int_0^{\frac{\pi}{2}} \frac{4}{1+\cos x} dx$$

as required

CONTINUE BY INTEGRATING 'OVER A SECTION' OF THE INTEGRATION BY PARTS (HAVE YOU NOTICED THE DISCONTINUITY NOW AT TJS?)

$$\Rightarrow I + \pi = \int_0^{\frac{\pi}{2}} \frac{4 \cos x}{\sec x + 1} dx$$

$$\Rightarrow I + \pi = \int_0^{\frac{\pi}{2}} \frac{-4 \cos x}{(1+\cos x)+1} dx$$

NEXT CREATE AN ARCTAN DERIVATIVE (BY INSPECTION) & USE A SUBSTITUTION

$$\Rightarrow I + \pi = \int_0^{\frac{\pi}{2}} \frac{4\cos^2 x}{2 + 3\sin^2 x} dx$$

$$\Rightarrow I + \pi = \int_0^{\frac{\pi}{2}} \frac{4\cos^2 x}{(5\sin^2 x + 4\cos^2 x)} dx$$

$$\Rightarrow I + \pi = \left[\frac{4}{\sqrt{5}} \arctan\left(\frac{2\cos x}{\sqrt{5}\sin x}\right) \right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow I + \pi = \frac{4\sqrt{5}}{\sqrt{5+4}} \left[\arctan\left(\frac{2\cos 0}{\sqrt{5}\sin 0}\right) - \arctan\left(\frac{2\cos \frac{\pi}{2}}{\sqrt{5}\sin \frac{\pi}{2}}\right) \right]$$

$$\Rightarrow I + \pi = 2\sqrt{5} \left[\frac{\pi}{2} - 0 \right]$$

$$\Rightarrow I + \pi = \pi\sqrt{5}$$

$$\Rightarrow I = \pi\sqrt{5} - \pi$$

$$\Rightarrow I = \pi(\sqrt{5} - 1)$$

DO BY PARTS

C) WE NEED A TRIGONOMETRIC SUBSTITUTION - START BY CLEAVING $\sqrt{1 - u^2}$ BY DIVIDING TOP & BOTTOM BY $\sin x$.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \tan^2 x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sec^2 x + \tan^2 x} dx$$

(NOTE THAT THERE IS NO INDEFINITE INTEGRAL AT $\frac{\pi}{2}$)

$$\begin{aligned}
 &= \int_0^{\pi} \frac{1}{(1+2\sin u)^2 + (1+2\cos u)} du = \int_0^{\pi} \frac{1}{1+2\sin u} du \\
 \text{By substitution next:} \\
 &\bullet u = \arctan x \\
 \Rightarrow du = -\sec^2 u du \\
 \Rightarrow du = -\frac{du}{\sec^2 u} \\
 \Rightarrow du = -\frac{du}{1+u^2} \\
 \Rightarrow du = -\frac{du}{1+u^2} \\
 \Rightarrow du = -\frac{du}{1+u^2} \\
 \text{TRANSFORMING THE INTEGRAL:} \\
 &\dots = \int_{-\infty}^{\infty} \frac{1}{1+2u^2} \left(\frac{-du}{1+u^2} \right) = \int_{-\infty}^{\infty} \frac{1}{(1+2u^2)(1+u^2)} du \\
 \text{PARTIAL FRACTIONS (FULL METHOD OR INSPECTION)} \\
 &\dots = \frac{2}{(1+2u^2)^2} - \frac{1}{(1+u^2)^2} du \\
 &= \int_{-\infty}^{\infty} \frac{1}{1^2 + \frac{1}{u^2}} - \frac{1}{1+u^2} du
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} \frac{1}{u^2 + (\frac{1}{\sqrt{2}})^2} - \frac{1}{u^2 + 1} du \\
 &\text{SINQ. INTEGRANDO IN A SYMMETRIC DOMAIN} \\
 &= 2 \int_0^{\infty} \frac{1}{u^2 + (\frac{1}{\sqrt{2}})^2} - \frac{1}{u^2 + 1} du \\
 &= 2 \left[\frac{1}{\frac{1}{\sqrt{2}}} \arctan\left(\frac{u}{\frac{1}{\sqrt{2}}}\right) - \arctan u \right]_0^{\infty} \\
 &= 2 \left[\sqrt{2} \arctan\left(\sqrt{2}u\right) - \arctan u \right]_0^{\infty} \\
 &= 2 \left[\sqrt{2} \times \frac{\pi}{2} - \frac{\pi}{2} \right] = (0 - 0) \\
 &= \pi\sqrt{2} - \pi \\
 &= \pi(\sqrt{2} - 1) \quad \text{As before}
 \end{aligned}$$

Question 329 (*****)

By using an appropriate substitution or substitutions, followed by partial fractions show that

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx = \frac{\ln 3}{20}.$$

 , proof

• $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx = \text{BY SUBSTITUTION} \dots$

let $u = \sin 2x - \cos 2x$
 $\frac{du}{dx} = \cos 2x + \sin 2x$
 $du = (\cos 2x + \sin 2x) dx$
 $dx = \frac{du}{\cos 2x + \sin 2x}$
 $2x = 0 \rightarrow u = -1$
 $2x = \frac{\pi}{2} \rightarrow u = 0$

ALSO $u^2 = (\sin 2x - \cos 2x)^2 = 1 - \sin 2x$
 $\Rightarrow u^2 = 1 - \sin 2x$
 $\Rightarrow \sin 2x = 1 - u^2$
 $\Rightarrow 16 \sin 2x = 16 - 16u^2$
 $\Rightarrow 16 \sin 2x = 25 - 16u^2$

$\dots \int_{-1}^0 \frac{\sin x + \cos x}{25 - 16u^2} \left(\frac{du}{\cos 2x + \sin 2x} \right) = \int_{-1}^0 \frac{1}{(5-u)(5+u)} du$

• BY PARTIAL FRACTIONS (COVER UP METHOD)

 $= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{10}}{5+4u} + \frac{\frac{1}{10}}{5-4u} du = \frac{1}{40} \int_0^{\frac{\pi}{4}} \frac{1}{5+4u} + \frac{1}{5-4u} du \dots$
 $= \frac{1}{40} \left[\frac{1}{4} \ln |5+4u| - \frac{1}{4} \ln |5-4u| \right]_0^{\frac{\pi}{4}} = \frac{1}{40} \left[\ln |5+4u| - \ln |5-4u| \right]_0^{\frac{\pi}{4}}$
 $= \frac{1}{40} \left[(\ln 5 - \ln 5) - (\ln 1 - \ln 9) \right] = \frac{1}{40} \ln 9 = \frac{1}{20} \ln 3 \quad \text{✓ PLS REOPENED}$

ALTERNATIVE METHOD

• USING $\int_a^b f(x) dx \equiv \int_a^b f(a+b-x) dx$

 $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx = \int_0^{\frac{\pi}{4}} \frac{\sin(\frac{\pi}{4}-x) + \cos(\frac{\pi}{4}-x)}{9 + 16 \sin(\frac{\pi}{4}-2x)} dx$

$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

$\boxed{\sin \frac{x}{2} + \cos \frac{x}{2} = \frac{\sqrt{3}}{2}}$

$= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{2}(\cos x + i \sin x)}{9 + 16(1 - 2\sin^2 x)} dx = \int_0^{\frac{\pi}{4}} \frac{\frac{1}{2}\cos x}{25 - 32\sin^2 x} dx$

• BY SUBSTITUTION

$u = \sin 2x$
 $\frac{du}{dx} = \cos 2x$
 $du = \frac{du}{\cos 2x}$
 $u=0 \rightarrow u=0$
 $2x=\frac{\pi}{4} \rightarrow u=\frac{\pi}{2}$

$\dots = \int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\cos x}{25 - 32u^2} \frac{du}{\cos 2x} = \sqrt{\frac{1}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{(5+\sqrt{5}u)(5-\sqrt{5}u)} du$
 $= \sqrt{\frac{1}{2}} \left[\frac{1}{\sqrt{5}} \ln |5+\sqrt{5}u| - \frac{1}{\sqrt{5}} \ln |5-\sqrt{5}u| \right]_0^{\frac{\pi}{2}}$
 $= \frac{1}{40} \left[\ln |5+\sqrt{5}u| - \ln |5-\sqrt{5}u| \right]_0^{\frac{\pi}{2}} = \frac{1}{40} [\ln 9 - \ln 1]$
 $= \frac{1}{40} \ln 9 = \frac{1}{20} \ln 3$

Question 330 (*****)

$$I = \int \frac{\sec^2 x}{\sqrt{\sec x + \tan x}} dx.$$

Without using a verification approach, show that

$$I = (\sec x + \tan x)^{\frac{1}{2}} - \frac{1}{3}(\sec x + \tan x)^{-\frac{3}{2}} + \text{constant}.$$

You may consider the substitution $u = \sec x + \tan x$ useful at some stages in the manipulation of the integrand.

[proof]

LOOKING AT THE SUBSTITUTION SUGGESTS $\frac{d}{dx}(\sec x + \tan x) = \sec^2 x$

$$\int \frac{\sec^2 x}{\sqrt{\sec x + \tan x}} dx = \frac{1}{2} \int \frac{\sec^3 x + \sec x \tan x - \sec x \tan x + \sec^2 x}{\sqrt{\sec x + \tan x}} dx$$

$$= \frac{1}{2} \int \underbrace{\frac{\sec^3 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx}_{\text{MANIPULATE}} + \frac{1}{2} \int \underbrace{\frac{\sec^2 x}{\sqrt{\sec x + \tan x}} dx}_{\text{MANIPULATE}}$$

MANIPULATE THE SECOND INTEGRAL AS FOLLOWS, @ NOTE THAT $1 + \tan^2 x = \sec^2 x$

$$= \frac{1}{2} \int \frac{\sec^3 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec(x - \tan x)(\sec x + \tan x)}{(\sec x + \tan x)^2(\sec x + \tan x)} dx$$

$$= \frac{1}{2} \int \frac{\sec^3 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec(x - \tan x)}{(\sec x + \tan x)^2} dx$$

$$= \frac{1}{2} \int \frac{\sec^3 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec x}{(\sec x + \tan x)^2} dx$$

$$= \frac{1}{2} \int \frac{\sec^3 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec(x - \tan x)}{(\sec x + \tan x)^2} dx$$

$$= \frac{1}{2} \int \frac{\sec^3 x + \sec x \tan x}{\sqrt{\sec x + \tan x}} dx + \frac{1}{2} \int \frac{\sec x}{(\sec x + \tan x)^2} dx$$

THE NOW ALSO INTEGRATES

TRY BY SUBSTITUTION OR INTEGRATION

$$= \frac{1}{2} \int (\sec x + \tan x)(\sec x + \tan x)^{\frac{1}{2}} dx + \frac{1}{2} \int (\sec x + \tan x)(\sec x + \tan x)^{\frac{1}{2}} dx$$

$$= \frac{1}{2} \cdot \frac{1}{2} (\sec x + \tan x)^{\frac{3}{2}} + \frac{1}{2} \cdot \frac{1}{2} (\sec x + \tan x)^{\frac{3}{2}} + C$$

$$= (\sec x + \tan x)^{\frac{3}{2}} - \frac{1}{2} (\sec x + \tan x)^{\frac{3}{2}} + C$$

AS REQUIRED

Question 331 (*****)

Use an appropriate integration method to determine an antiderivative for the following indefinite integral.

$$\int \frac{x^2(x^4+1)}{\sqrt[4]{x^4+1}} dx .$$

V, **□**, $\frac{1}{6}(x^8+2x^4)^{\frac{1}{2}} + C$

MANIPULATE AS FOLLOWS

$$\begin{aligned} \int \frac{x^2(x^4+1)}{\sqrt[4]{x^4+1}} dx &= \int x^2 x^{1/4} (x^3+2)^{-1/4} dx = \int x^2 (x^4+1)^{1/4} (x^3+2)^{-1/4} dx \\ &= \int (x^2 x^4) (x^4+1)^{1/4} (x^3+2)^{-1/4} dx = \int (x^2 x^4) [x^3 (x^3+2)]^{-1/4} dx \\ &= (x^2 x^4) (x^3+2)^{-1/4} \end{aligned}$$

NOTE THAT $\frac{d}{dx}(x^3+2^4) = 3x^2 + 8x^3 = 8(x^3+2)$

$$\begin{aligned} &= \frac{1}{8} \int 8(x^3+2) (x^3+2)^{-1/4} dx \\ &= \frac{1}{8} \times \frac{1}{\frac{1}{4}} (x^3+2)^{3/4} + C \\ &= \frac{1}{6} (x^3+2)^{3/4} + C \end{aligned}$$

Question 332 (*****)

Use partial fractions followed by integration by parts to show that

$$\int_0^\infty \left[\frac{x^2+3x+3}{(x+1)^3} \right] e^{-x} \sin x \, dx = \frac{1}{2}.$$

,

• **SPLIT BY PARTIAL FRACTIONS FIRST**

$$\frac{x^2+3x+3}{(x+1)^3} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$x^2+3x+3 \equiv Ax(x)^2 + Bx(x) + C$$

$$x^2+3x+3 \equiv Ax^2 + 2Ax + A + Bx + B + C$$

$$x^2+3x+3 \equiv Ax^2 + (2A+B)x + (A+B+C)$$

$$\text{Hence } A+B+C=1$$

• **NEXT USE THE INTEGRAL OF $e^x \sin x$, BY PARTS TWICE**

$$\int e^x \sin x \, dx = -e^x \cos x - \int e^x \cos x \, dx$$

$$= -e^x \cos x - e^x \sin x$$

$$\int e^x \sin x \, dx = -e^x \cos x - \left[e^x \cos x + \int e^x \cos x \, dx \right]$$

$$\int e^x \sin x \, dx = -e^x \cos x - e^x \sin x - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = -e^x (\cos x + \sin x) + C$$

$$\int e^x \sin x \, dx = -\frac{1}{2}e^x (\cos x + \sin x) + C$$

• **NEXT SPILT THE INTEGRAL INTO 3. A CANCELLATION IN THE FIRST & THIRD INTEGRAL BUT NOT IN THE SECOND (LEFT FOR CANCELLING)**

$$\int_0^\infty \left[\frac{x^2+3x+3}{(x+1)^3} \right] e^{-x} \sin x \, dx = \int_0^\infty \frac{e^{-x}}{x+1} \, dx + \boxed{\int_0^\infty \frac{e^{-x}}{(x+1)^2} \, dx} + \boxed{\int_0^\infty \frac{e^{-x}}{(x+1)^3} \, dx}$$

$$\begin{aligned} & \int_0^\infty \frac{e^{-x}}{x+1} \, dx + \int_0^\infty \frac{e^{-x} \sin x}{(x+1)^2} \, dx \\ & \quad - \frac{1}{2} \int_0^\infty \frac{e^{-x} \sin x}{(x+1)^3} \, dx \\ & \quad = e^{-x} (x+1) - \frac{e^{-x} \sin x}{(x+1)^2} \Big|_0^\infty \\ & \quad = \left[-\frac{1}{2} \frac{e^{-x}}{x+1} (x+1) + \int_0^\infty \frac{e^{-x} (x+1)}{(x+1)^2} \, dx \right] \\ & \quad + \int_0^\infty \frac{e^{-x} \cos x}{(x+1)^2} \, dx \\ & \quad = \left[-\frac{1}{2} \frac{e^{-x}}{(x+1)^2} \right]_0^\infty + \int_0^\infty \frac{e^{-x} \cos x}{(x+1)^2} \, dx \\ & \quad = \left[0 - \left(-\frac{1}{2} \right) \right] + \int_0^\infty \frac{e^{-x} \cos x}{(x+1)^2} \, dx \\ & \quad + \int_0^\infty \frac{e^{-x} \sin x}{(x+1)^2} \, dx \\ & \quad = \left[0 - \left(0 \right) \right] + \int_0^\infty \frac{e^{-x} \cos x}{(x+1)^2} \, dx - \int_0^\infty \frac{e^{-x} \sin x}{(x+1)^2} \, dx \\ & \quad = \frac{1}{2} \end{aligned}$$