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# TRIGONOMETRIC EQUATIONS & IDENTITIES INTRODUCTION

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**Question 1**

Solve each of the following trigonometric equations in the range given.

a)  $\sin x = \frac{1}{2}$

$0 \leq x < 360^\circ$

$x = 30^\circ, 150^\circ$

b)  $\cos 2x = \frac{1}{2}$

$0 \leq x < 360^\circ$

$x = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

c)  $\tan(\theta - 20^\circ) = \sqrt{3}$

$-180^\circ \leq \theta < 180^\circ$

$\theta = -100^\circ, 80^\circ$

d)  $\sin(2\theta + 30^\circ) = \frac{\sqrt{3}}{2}$

$-180^\circ \leq \theta < 180^\circ$

$\theta = -165^\circ, -135^\circ, 15^\circ, 45^\circ$

e)  $\cos(3x - 45^\circ) = -\frac{1}{2}$

$0 \leq x < 180^\circ$

$x = 55^\circ, 95^\circ, 175^\circ$

<p>(a) <math>\sin x = \frac{1}{2}, 0 \leq x &lt; 360^\circ</math>  <math>\arcsin(\frac{1}{2}) = 30^\circ</math></p> $\begin{cases} x_1 = 30^\circ \pm 360^\circ \\ x_2 = 150^\circ \pm 360^\circ \end{cases} \text{ into } 2x_1, \dots$ $\begin{cases} 2x_1 = 30^\circ \\ 2x_2 = 150^\circ \end{cases}$ $\therefore x_1 = 15^\circ, 330^\circ$ $x_2 = 75^\circ, 165^\circ$	<p>(d) <math>\sin(2x + 30^\circ) = \frac{\sqrt{3}}{2}, -180^\circ &lt; 2x &lt; 180^\circ</math>  <math>\arcsin(\frac{\sqrt{3}}{2}) = 60^\circ</math></p> $\begin{cases} 2x + 30^\circ = 60^\circ \\ 2x + 30^\circ = 120^\circ \\ 2x + 30^\circ = 240^\circ \\ 2x + 30^\circ = 300^\circ \end{cases} \text{ into } 2x_1, \dots$ $\begin{cases} 2x_1 = 30^\circ \\ 2x_2 = 90^\circ \\ 2x_3 = 150^\circ \\ 2x_4 = 210^\circ \end{cases}$ $\begin{cases} x_1 = 15^\circ \\ x_2 = 45^\circ \\ x_3 = 75^\circ \\ x_4 = 105^\circ \end{cases}$
<p>(b) <math>\cos x = \frac{1}{2}, 0 \leq x &lt; 360^\circ</math>  <math>\arccos(\frac{1}{2}) = 60^\circ</math></p> $\begin{cases} x_1 = 60^\circ \pm 360^\circ \\ x_2 = 300^\circ \pm 360^\circ \end{cases} \text{ into } 3x_1, \dots$ $\begin{cases} 3x_1 = 60^\circ \\ 3x_2 = 300^\circ \end{cases}$ $\therefore x_1 = 20^\circ, 180^\circ$ $x_2 = 120^\circ, 240^\circ$	<p>(e) <math>\cos(2x - 45^\circ) = -\frac{1}{2}, 0 \leq x &lt; 180^\circ</math>  <math>\arccos(-\frac{1}{2}) = 120^\circ</math></p> $\begin{cases} 2x - 45^\circ = 120^\circ \pm 360^\circ \\ 2x - 45^\circ = 240^\circ \pm 360^\circ \end{cases} \text{ into } 3x_1, \dots$ $\begin{cases} 2x_1 = 165^\circ \\ 2x_2 = 285^\circ \end{cases}$ $\begin{cases} x_1 = 82.5^\circ \\ x_2 = 142.5^\circ \end{cases}$
<p>(c) <math>\tan(x - 20^\circ) = \sqrt{3}, 0^\circ &lt; x &lt; 180^\circ</math>  <math>\arctan(\sqrt{3}) = 60^\circ</math></p> $\begin{cases} x - 20^\circ = 60^\circ \pm 180^\circ \end{cases} \text{ into } 3x_1, \dots$ $\begin{cases} x_1 = 80^\circ \\ x_2 = 160^\circ \end{cases}$ $\therefore x_1 = 80^\circ$ $x_2 = 160^\circ$	

**Question 2**

Solve each of the following trigonometric equations in the range given.

a)  $\sin x = -\frac{1}{2}$        $0 \leq x < 360^\circ$

$x = 210^\circ, 330^\circ$

b)  $\cos 2x = -0.1736$        $0 \leq x < 360^\circ$

$x = 50^\circ, 130^\circ, 230^\circ, 310^\circ$

c)  $\tan(x - 40^\circ) = 0.7$        $-180^\circ \leq x < 180^\circ$

$x = -105^\circ, 75^\circ$

d)  $\sin(2\theta + 40^\circ) = 0.9848$        $-180^\circ \leq \theta < 180^\circ$

$\theta = -160^\circ, -150^\circ, 20^\circ, 30^\circ$

e)  $\cos(4y - 60^\circ) = \frac{\sqrt{6} - \sqrt{2}}{4}$        $0 \leq y < 90^\circ$

$x = 33.8^\circ, 86.3^\circ$

<p>(a) <math>\sin x = -\frac{1}{2}</math>, <math>0 \leq x &lt; 360^\circ</math>  <math>\arcsin(-\frac{1}{2}) = -30^\circ</math>  <math>x = -30^\circ + 360n</math>      <math>n = 0, 1, 2, \dots</math>  <math>x_1 = 330^\circ</math>  <math>x_2 = 210^\circ</math></p>	<p>(d) <math>\sin(\theta + 40^\circ) = 0.9848</math>,  <math>-\pi/2 \leq \theta &lt; \pi/2^\circ</math>  <math>\arcsin(0.9848) = 80^\circ</math>  <math>\theta + 40^\circ = 80^\circ + 360n</math>      <math>n = 0, 1, 2, \dots</math>  <math>\theta = 40^\circ + 360n</math>  <math>\theta = 60^\circ + 360n</math>  <math>\theta = 2\theta \pm 180n</math>  <math>\theta = 30^\circ \pm 180n</math></p>
<p>(b) <math>\cos 2x = -0.1736</math>, <math>0 \leq x &lt; 360^\circ</math>  <math>\arccos(-0.1736) = 100.6^\circ</math>  <math>2x = 100.6^\circ + 360n</math>      <math>n = 0, 1, 2, \dots</math>  <math>x_1 = 200.6^\circ + 360n</math>  <math>x_2 = 360 - 200.6^\circ</math>  <math>x_3 = 59.4^\circ</math>  <math>x_4 = 180^\circ - 59.4^\circ</math>  <math>x_5 = 120.6^\circ</math>  <math>x_6 = 210.6^\circ</math></p>	<p>(e) <math>\cos(4y - 60^\circ) = \frac{\sqrt{6} - \sqrt{2}}{4}</math>,  <math>0 \leq y &lt; 90^\circ</math>  <math>\arccos(\frac{\sqrt{6} - \sqrt{2}}{4}) = 75^\circ</math>  <math>4y - 60^\circ = 75^\circ + 360n</math>      <math>n = 0, 1, 2, \dots</math>  <math>4y_1 = 115^\circ + 360n</math>  <math>y_1 = 28.8^\circ + 90n</math>  <math>y_2 = 86.3^\circ + 90n</math>  <math>y_3 = 33.8^\circ</math>  <math>y_4 = 165.8^\circ</math></p>

**Question 3**

Solve each of the following trigonometric equations in the range given.

a)  $\tan\left(\frac{3}{2}x\right) = -\frac{\sqrt{3}}{3}$

$0 \leq x < 360^\circ$

$x = 100^\circ, 220^\circ, 340^\circ$

b)  $\sin(4y - 45^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}$

$-90^\circ \leq y < 90^\circ$

$y = -75^\circ, -37.5^\circ, 15^\circ, 52.5^\circ$

c)  $5\cos 2x = 2$

$0 \leq x < 360^\circ$

$x = 33.2^\circ, 146.8^\circ, 213.2^\circ, 326.8^\circ$

d)  $3 - \tan(\theta - 10^\circ) = 1$

$0 \leq \theta < 360^\circ$

$\theta = 73.4^\circ, 253.4^\circ$

e)  $3\sin(3\theta - 30^\circ) + 4 = 5$

$-90^\circ \leq \theta < 90^\circ$

$\theta = -56.5^\circ, 16.5^\circ, 63.5^\circ$

<p>(a) <math>\tan\left(\frac{3}{2}x\right) = -\frac{\sqrt{3}}{3}</math>  <math>\arctan\left(-\frac{\sqrt{3}}{3}\right) = 30^\circ</math>  <math>\frac{3}{2}x = -30^\circ \pm 180^\circ n \quad n=0,1,2,...</math>  <math>\bullet \quad 3x = -60^\circ \pm 360^\circ n</math>  <math>\bullet \quad x = -20^\circ \pm 120^\circ n</math>  <math>\alpha_1 = 100^\circ</math>  <math>\alpha_2 = 220^\circ</math>  <math>\alpha_3 = 340^\circ</math></p>	<p>(d) <math>3 - \tan(\theta - 10^\circ) = 1</math>  <math>\Rightarrow 2 = \tan(\theta - 10^\circ)</math>  <math>\Rightarrow \tan(\theta - 10^\circ) = 2</math>  <math>\arctan(2) = 63.4^\circ</math>  <math>\bullet \quad \theta - 10^\circ = 63.4^\circ \pm 180^\circ n \quad n=0,1,2,...</math>  <math>\bullet \quad \theta = 73.4^\circ \pm 180^\circ n</math>  <math>\theta_1 = 73.4^\circ</math>  <math>\theta_2 = 253.4^\circ</math></p>
<p>(b) <math>\sin(4y - 45^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}</math>  <math>\arcsin\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) = 15^\circ</math>  <math>4y - 45^\circ = 15^\circ \pm 360^\circ n \quad n=0,1,2,...</math>  <math>\bullet \quad 4y - 45^\circ = 165^\circ \pm 360^\circ n</math>  <math>\bullet \quad 4y = 60^\circ \pm 360^\circ n</math>  <math>\bullet \quad y = 15^\circ \pm 90^\circ n</math>  <math>\bullet \quad y = 15^\circ, -75^\circ, 52.5^\circ, -37.5^\circ</math></p>	<p>(e) <math>3\sin(3\theta - 30^\circ) + 4 = 5</math>  <math>3\sin(3\theta - 30^\circ) = 1</math>  <math>\sin(3\theta - 30^\circ) = \frac{1}{3}</math>  <math>\arcsin\left(\frac{1}{3}\right) = 19.47^\circ</math>  <math>3\theta - 30^\circ = 19.47^\circ \pm 360^\circ n \quad n=0,1,2,...</math>  <math>3\theta = 49.57^\circ \pm 360^\circ n</math>  <math>\theta = 16.5^\circ \pm 120^\circ n</math>  <math>\theta_1 = 16.5^\circ</math>  <math>\theta_2 = 63.5^\circ</math>  <math>\theta_3 = -56.5^\circ</math></p>

**Question 4**

Solve each of the following trigonometric equations in the range given.

a)  $4\cos 2\theta = 3$

$0 \leq \theta < 360^\circ$

$\theta \approx 20.7^\circ, 159.3^\circ, 200.7^\circ, 339.3^\circ$

b)  $3\tan 4x - 4 = 5$

$0 \leq x < 90^\circ$

$x \approx 17.9^\circ, 62.9^\circ$

c)  $5\sin(3\psi - 70^\circ) = 4$

$-90^\circ \leq \psi < 90^\circ$

$\psi \approx -79.0^\circ, -54.4^\circ, 41.0^\circ, 65.6^\circ$

d)  $5 - 3\tan 2\phi = 1$

$0 \leq \phi < 180^\circ$

$\phi \approx 26.6^\circ, 116.6^\circ$

e)  $2\sin(y + 50^\circ) + 3 = 4$

$0 \leq y < 360^\circ$

$y = 100^\circ, 340^\circ$

<p>(a) <math>4\cos 2\theta = 3</math>   <math>0 \leq \theta &lt; 360^\circ</math></p> $\cos 2\theta = \frac{3}{4}$ $\cos(\frac{\pi}{3}) = 0.5$ $2\theta = 41.41^\circ \pm 360^\circ$ $2\theta = 38.59^\circ \pm 360^\circ$ $m=1,2,3,...$ $\theta = 20.7^\circ \pm 180^\circ$ $\theta = 159.3^\circ \pm 180^\circ$ $\theta = 20.7^\circ$ $\theta = 159.3^\circ$ $\theta = 339.3^\circ$	<p>(b) <math>3\tan 4x - 4 = 5</math>   <math>0 \leq x &lt; 90^\circ</math></p> $3\tan 4x = 9$ $\tan 4x = 3$ $\tan(\frac{\pi}{3}) = 1.732$ $4x = 71.57^\circ \pm 180^\circ$ $m=1,2,3,...$ $x = 17.9^\circ \pm 45^\circ$ $x = 62.9^\circ$	<p>(c) <math>5\sin(3\psi - 70^\circ) = 4</math>   <math>-90^\circ \leq \psi &lt; 90^\circ</math></p> $\sin(3\psi - 70^\circ) = \frac{4}{5}$ $\sin(\frac{\pi}{2}) = 1$ $3\psi - 70^\circ = 53.13^\circ$ $3\psi = 123.13^\circ \pm 360^\circ$ $m=1,2,3,...$ $\psi = 41.0^\circ \pm 120^\circ$ $\psi = 65.4^\circ \pm 120^\circ$ $\psi = 41.0^\circ, -79.0^\circ, 41.0^\circ, -54.4^\circ$
<p>(d) <math>5 - 3\tan 2\phi = 1</math>   <math>0 \leq \phi &lt; 180^\circ</math></p> $3\tan 2\phi = 4$ $\tan 2\phi = \frac{4}{3}$ $\tan(\frac{\pi}{4}) = 1$ $2\phi = 45^\circ \pm 180^\circ$ $\phi = 22.5^\circ \pm 90^\circ$ $\phi = 26.6^\circ$ $\phi = 116.6^\circ$	<p>(e) <math>2\sin(y + 50^\circ) + 3 = 4</math>   <math>0 \leq y &lt; 360^\circ</math></p> $2\sin(y + 50^\circ) = 1$ $\sin(y + 50^\circ) = \frac{1}{2}$ $\sin(\frac{\pi}{6}) = 0.5$ $y + 50^\circ = 30^\circ \pm 360^\circ$ $m=1,2,3,...$ $y = -20^\circ \pm 360^\circ$ $y_1 = 340^\circ$ $y_2 = 100^\circ$	

**Question 5**

Solve each of the following trigonometric equations in the range given.

a)  $4\cos(2\theta - 20^\circ) = 1$

$0 \leq \theta < 360^\circ$

$\theta \approx 47.8^\circ, 152.2^\circ, 227.8^\circ, 332.2^\circ$

b)  $2 + \tan 4x = 5$

$-90^\circ \leq x < 90^\circ$

$x \approx -72.1^\circ, -27.1^\circ, 17.9^\circ, 62.9^\circ$

c)  $7\sin(4t + 10^\circ) = 4$

$-90^\circ \leq t < 90^\circ$

$t \approx -83.8^\circ, -56.2^\circ, 6.2^\circ, 33.8^\circ$

d)  $5\cos(4\varphi) + 1 = 0$

$0 \leq \varphi < 90^\circ$

$\varphi \approx 25.4^\circ, 64.6^\circ$

e)  $\tan(y + 50^\circ) = 3$

$0 \leq y < 360^\circ$

$y \approx 21.6^\circ, 201.6^\circ$

<p>(a) <math>4\cos(2\theta - 20) = 1</math>, <math>0 \leq \theta &lt; 360^\circ</math>  <math>\cos(2\theta - 20) = \frac{1}{4}</math>  <math>\arccos(\frac{1}{4}) = 75.52^\circ</math>  <math>2\theta - 20 = 75.52 + 360k</math>, <math>k \in \mathbb{Z}, k \geq 0</math>,  <math>2\theta = 75.52 + 360k + 20</math>,  <math>2\theta = 345.52 + 360k</math>,  <math>\theta = 172.76 + 180k</math>,  <math>\theta = 172.76 + 180k</math>,  <math>\therefore \theta = 172.76, 352.76, 332.2^\circ</math></p>	<p>(b) <math>5\cos(4\varphi) + 1 = 0</math>, <math>0 \leq \varphi &lt; 90^\circ</math>  <math>\cos(4\varphi) = -\frac{1}{5}</math>  <math>\arccos(-\frac{1}{5}) = 101.54^\circ</math>  <math>4\varphi = 101.54 + 360k</math>, <math>k \in \mathbb{Z}, k \geq 0</math>,  <math>4\varphi = 203.52 + 360k</math>,  <math>\varphi = 50.88 + 90k</math>,  <math>\varphi = 50.88, 140.88, 230.88</math>,  <math>\therefore \varphi = 25.4^\circ, 64.6^\circ</math></p>
<p>(c) <math>2 + \tan 4x = 5</math>, <math>-90^\circ \leq x &lt; 90^\circ</math>  <math>\tan 4x = 3</math>  <math>\arctan(3) = 71.57^\circ</math>  <math>4x = 71.57 + 180k</math>, <math>k \in \mathbb{Z}, k \geq 0</math>,  <math>x = 17.89 + 45k</math>,  <math>x = 17.89, 62.89, 107.89, 152.89</math>,  <math>\therefore x = 17.89^\circ, 62.89^\circ, 107.89^\circ, 152.89^\circ</math></p>	<p>(d) <math>\tan(y + 50) = 3</math>, <math>0 \leq y &lt; 360^\circ</math>  <math>y + 50 = 71.57 + 180k</math>, <math>k \in \mathbb{Z}, k \geq 0</math>,  <math>y = 21.57 + 180k</math>,  <math>y = 21.57, 140.57, 259.57</math>,  <math>\therefore y = 21.57^\circ, 140.57^\circ</math></p>
<p>(e) <math>4\cos(4t + 10) = 1</math>, <math>-90^\circ \leq t &lt; 90^\circ</math>  <math>\cos(4t + 10) = \frac{1}{4}</math>  <math>\arccos(\frac{1}{4}) \approx 74.85^\circ</math>  <math>4t + 10 = 74.85 + 360k</math>, <math>k \in \mathbb{Z}, k \geq 0</math>,  <math>4t = 64.85 + 360k</math>,  <math>t = 16.21 + 90k</math>,  <math>t = 16.21, 116.21, 206.21</math>,  <math>\therefore t = 16.21^\circ, 116.21^\circ, 206.21^\circ</math></p>	

**Question 6**

Solve each of the following trigonometric equations in the range given.

a)  $\cos(2\theta + 25)^\circ = -0.454$        $0 \leq \theta < 360^\circ$

$\theta \approx 46^\circ, 109^\circ, 226^\circ, 289^\circ$

b)  $\cos(2y - 35)^\circ = 0.891$        $0 \leq y < 360^\circ$

$\theta \approx 4^\circ, 31^\circ, 184^\circ, 211^\circ$

c)  $3\cos 3x - 1 = 0.22$        $-90^\circ \leq x < 90^\circ$

$x \approx -22^\circ, 22^\circ$

d)  $\tan(5y - 35)^\circ = -2 - \sqrt{3}$        $0 \leq y < 90^\circ$

$\theta \approx 28^\circ, 64^\circ$

e)  $1 + 2\sin(\theta + 25)^\circ = 2.532$        $0 \leq \theta < 360^\circ$

$\theta \approx 25^\circ, 105^\circ$

a)  $\cos(2\theta + 25) = -0.454$   
 $\arccos(-0.454) = 117.0^\circ$   
 $2\theta + 25 = 117.0^\circ \pm 360n$   
 $2\theta + 25 = 283.0^\circ \pm 360n$   
 $n=0,1,2,3,\dots$   
 $2\theta = 92.0^\circ \pm 360n$   
 $2\theta = 218.0^\circ \pm 360n$   
 $\theta = 46.0^\circ \pm 180n$   
 $\theta = 109.0^\circ \pm 180n$   
 $\theta_1 = 46.0^\circ$   
 $\theta_2 = 226.0^\circ$   
 $\theta_3 = 109.0^\circ$   
 $\theta_4 = 289.0^\circ$

b)  $\cos(2y - 35) = 0.891$   
 $\arccos(0.891) = 21.0^\circ$   
 $2y - 35 = 21.0^\circ \pm 360n$   
 $2y - 35 = 338.0^\circ \pm 360n$   
 $n=0,1,2,3,\dots$   
 $2y = 62.0^\circ \pm 360n$   
 $2y = 368.0^\circ \pm 360n$   
 $y = 31.0^\circ \pm 180n$   
 $y = 184.0^\circ \pm 180n$   
 $y_1 = 31.0^\circ$   
 $y_2 = 211.0^\circ$   
 $y_3 = 184.0^\circ$   
 $y_4 = 4^\circ$

c)  $3\cos 3x - 1 = 0.22$   
 $3\cos 3x = 1.22$   
 $\cos 3x = 0.4066\dots$   
 $\arccos(0.4066\dots) = 66.0^\circ$   
 $3x = 66.0^\circ \pm 360n$   
 $3x = 294.0^\circ \pm 360n$   
 $n=0,1,2,3,\dots$   
 $x = 22.0^\circ \pm 120n$   
 $x = 166.0^\circ \pm 120n$   
 $x_1 = 22^\circ$   
 $x_2 = -22^\circ$

d)  $\tan(5y - 35) = -2 - \sqrt{3}$   
 $\arctan(-2 - \sqrt{3}) = -75^\circ$   
 $5y - 35 = -75^\circ \pm 180n$   
 $n=0,1,2,3,\dots$   
 $5y = -40^\circ \pm 180n$   
 $y = -8^\circ \pm 36n$   
 $y_1 = 28^\circ$   
 $y_2 = 94^\circ$

e)  $1 + 2\sin(\theta + 25) = 2.532$   
 $2\sin(\theta + 25) = 1.532$   
 $\sin(\theta + 25) = 0.766$   
 $\arcsin(0.766) = 50.0^\circ$   
 $\theta + 25 = 50.0^\circ \pm 360n$   
 $\theta + 25 = 130.0^\circ \pm 360n$   
 $n=0,1,2,3,\dots$   
 $\theta = 25^\circ$   
 $\theta_1 = 25^\circ$   
 $\theta_2 = 105^\circ$

**Question 7**

Solve each of the following trigonometric equations in the range given.

a)  $\cos 2\theta = -0.7374$

$0 \leq \theta < 2\pi$

$\theta \approx 1.20^\circ, 1.94^\circ, 4.34^\circ, 5.08^\circ$

b)  $\cos\left(x - \frac{\pi}{4}\right) = 0.61$

$0 \leq x < 2\pi$

$x \approx 1.70^\circ, 6.15^\circ$

c)  $\sqrt{2} + 4\sin x = \sqrt{6}$

$0 \leq x < 2\pi$

$x = \frac{\pi}{12}, \frac{11\pi}{12}$

d)  $2 + \tan\left(y + \frac{\pi}{3}\right) = \sqrt{3}$

$0 \leq y < 2\pi$

$y = \frac{7\pi}{12}, \frac{19\pi}{12}$

e)  $4 - 2\sin 3x = 2.5$

$0 \leq x < \pi$

$x \approx 0.28^\circ, 0.76^\circ, 2.38^\circ, 2.86^\circ$

<p>(a) <math>\cos 2\theta = -0.7374</math>  <math>\arccos(-0.7374) = 2.40^\circ</math>  <math>2\theta = 2.40^\circ \pm 2\pi n</math>      <math>n=0,1,2,3,\dots</math>  <math>\theta = 1.20^\circ \pm \pi n</math>  <math>\theta = 1.94^\circ \pm \pi n</math>  <math>\theta = 4.34^\circ \pm \pi n</math>  <math>\theta = 5.08^\circ \pm \pi n</math></p>	<p>(b) <math>\cos\left(x - \frac{\pi}{4}\right) = 0.61</math>  <math>\arccos(0.61) = 0.946^\circ</math>  <math>x - \frac{\pi}{4} = 0.946^\circ \pm 2\pi n</math>      <math>n=0,1,2,3,\dots</math>  <math>x = 1.70^\circ \pm 2\pi n</math>  <math>x = 6.15^\circ \pm 2\pi n</math>  <math>x = 1.70^\circ, 6.15^\circ</math></p>	<p>(c) <math>\sqrt{2} + 4\sin x = \sqrt{6}</math>  <math>4\sin x = -2 + \sqrt{6}</math>  <math>\arcsin\left(\frac{-2+\sqrt{6}}{4}\right) = -0.173^\circ</math>  <math>x + \frac{\pi}{3} = -\frac{\pi}{12} \pm n\pi</math>  <math>x = -\frac{5\pi}{12} \pm n\pi</math>  <math>x_1 = \frac{7\pi}{12}</math>  <math>x_2 = \frac{19\pi}{12}</math></p>	<p>(d) <math>2 + \tan\left(y + \frac{\pi}{3}\right) = \sqrt{3}</math>  <math>-2 + \tan y = -1.5</math>  <math>\tan y = 0.846^\circ</math>  <math>y = 0.846^\circ \pm 2\pi n</math>  <math>y = 2.38^\circ \pm 2\pi n</math>  <math>y = 0.28^\circ, 2.38^\circ</math></p>
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**Question 8**

Solve each of the following trigonometric equations in the range given.

a)  $\sin 2x = \sin 48^\circ$

$0 \leq x < 360^\circ$

$x = 24^\circ, 66^\circ, 204^\circ, 246^\circ$

b)  $\cos 3x = \cos 96^\circ$

$0 \leq x < 180^\circ$

$x = 32^\circ, 88^\circ, 152^\circ$

c)  $\tan(\theta + 50^\circ) = \tan 12^\circ$

$0 \leq \theta < 360^\circ$

$\theta = 142^\circ, 322^\circ$

d)  $\sin 2x = \sin 324^\circ$

$0 \leq x < 360^\circ$

$x = 108^\circ, 162^\circ, 288^\circ, 342^\circ$

e)  $\cos 4x = \cos 240^\circ$

$0 \leq x < 180^\circ$

$x = 30^\circ, 60^\circ, 120^\circ, 150^\circ$

<p>(a) <math>\sin 2x = \sin 48^\circ</math></p> $\begin{aligned} 2x &= 48^\circ \pm 360^\circ \\ 2x &= 336^\circ \pm 360^\circ \end{aligned}$ $\begin{aligned} x &= 24^\circ \pm 180^\circ \\ x &= 66^\circ \pm 180^\circ \\ x &= 24^\circ, 204^\circ, 66^\circ, 246^\circ \end{aligned}$ <p>(b) <math>\cos 3x = \cos 96^\circ</math></p> $\begin{aligned} 3x &= 96^\circ \pm 360^\circ \\ 3x &= 204^\circ \pm 360^\circ \end{aligned}$ $\begin{aligned} x &= 32^\circ \pm 120^\circ \\ x &= 88^\circ \pm 120^\circ \\ x &= 32^\circ, 152^\circ, 88^\circ, 152^\circ \end{aligned}$ <p>(c) <math>\tan(\theta + 50^\circ) = \tan 12^\circ</math></p> <ul style="list-style-type: none"> <li>• <math>\theta + 50^\circ = 12^\circ \pm 180^\circ</math></li> <li>• <math>\theta = -38^\circ \pm 180^\circ</math></li> <li>∴ <math>\theta_1 = 142^\circ</math></li> <li><math>\theta_2 = 322^\circ</math></li> </ul>	<p>(d) <math>\sin 2x = \sin 324^\circ</math></p> $\begin{aligned} 2x &= 324^\circ \pm 360^\circ \\ 2x &= -48^\circ \pm 360^\circ \end{aligned}$ $\begin{aligned} x &= 162^\circ \pm 180^\circ \\ x &= -72^\circ \pm 180^\circ \end{aligned}$ $\begin{aligned} x_1 &= 162^\circ \\ x_2 &= 324^\circ \\ x_3 &= 108^\circ \\ x_4 &= 288^\circ \end{aligned}$ <p>(e) <math>\cos 4x = \cos 240^\circ</math></p> $\begin{aligned} 4x &= 240^\circ \pm 360^\circ \\ 4x &= -120^\circ \pm 360^\circ \end{aligned}$ $\begin{aligned} x &= 60^\circ \pm 90^\circ \\ x &= 30^\circ \pm 90^\circ \end{aligned}$ $\begin{aligned} x_1 &= 60^\circ \\ x_2 &= 150^\circ \\ x_3 &= 30^\circ \\ x_4 &= 120^\circ \end{aligned}$
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**Question 9**

Solve each of the following trigonometric equations in the range given.

a)  $\sin x = \frac{1}{2}$

$0 \leq x < 2\pi$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

b)  $\cos 2x = \frac{1}{2}$

$0 \leq x < 2\pi$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

c)  $\tan(\theta - \frac{\pi}{9}) = \sqrt{3}$

$-\pi \leq \theta < \pi$

$$\theta = -\frac{5\pi}{9}, \frac{4\pi}{9}$$

d)  $\sin(2\theta + \frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$-\pi \leq \theta < \pi$

$$\theta = -\frac{11\pi}{12}, -\frac{3\pi}{4}, \frac{\pi}{12}, \frac{\pi}{4}$$

e)  $\cos(3x - \frac{\pi}{4}) = -\frac{1}{2}$

$0 \leq x < \pi$

$$x = \frac{11\pi}{36}, \frac{19\pi}{36}, \frac{35\pi}{36}$$

<p>(a) <math>\sin x = \frac{1}{2}</math>  <math>\arcsin(\frac{1}{2}) = \frac{\pi}{6}</math>  <math>x_1 = \frac{\pi}{6} \pm 2n\pi</math>  <math>x_2 = \frac{5\pi}{6} \pm 2n\pi</math>  <math>n=0,1,2,3\dots</math></p> <p>(b) <math>\cos 2x = \frac{1}{2}</math>  <math>\arccos(\frac{1}{2}) = \frac{\pi}{3}</math>  <math>2x = \frac{\pi}{3} \pm 2n\pi</math>  <math>x_1 = \frac{\pi}{6} \pm n\pi</math>  <math>x_2 = \frac{5\pi}{6} \pm n\pi</math></p> <p>(c) <math>\tan(\theta - \frac{\pi}{9}) = \sqrt{3}</math>  <math>\arctan(\sqrt{3}) = \frac{\pi}{3}</math>  <math>\theta - \frac{\pi}{9} = \frac{\pi}{3} + n\pi</math>  <math>\theta = \frac{4\pi}{9} \pm n\pi</math></p> <p>(d) <math>\sin(2\theta + \frac{\pi}{6}) = \frac{\sqrt{3}}{2}</math>  <math>\arcsin(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}</math>  <math>2\theta + \frac{\pi}{6} = \frac{\pi}{3} + 2n\pi</math>  <math>2\theta = \frac{\pi}{6} \pm 2n\pi</math>  <math>\theta = \frac{\pi}{12} \pm n\pi</math></p> <p>(e) <math>\cos(3x - \frac{\pi}{4}) = -\frac{1}{2}</math>  <math>\arccos(-\frac{1}{2}) = \frac{2\pi}{3}</math>  <math>3x - \frac{\pi}{4} = \frac{2\pi}{3} + 2n\pi</math>  <math>3x = \frac{11\pi}{12} \pm 2n\pi</math>  <math>x_1 = \frac{11\pi}{36}, x_2 = \frac{19\pi}{36}, x_3 = \frac{35\pi}{36}</math></p>	<p>(a) <math>\sin(\theta + \frac{\pi}{6}) = \frac{\sqrt{3}}{2}</math>  <math>\arcsin(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}</math>  <math>\theta + \frac{\pi}{6} = \frac{\pi}{3} + 2n\pi</math>  <math>\theta = \frac{\pi}{6} \pm 2n\pi</math></p> <p>(b) <math>\cos 2x = \frac{1}{2}</math>  <math>\arccos(\frac{1}{2}) = \frac{\pi}{3}</math>  <math>2x = \frac{\pi}{3} \pm 2n\pi</math>  <math>x_1 = \frac{\pi}{6} \pm n\pi</math></p> <p>(c) <math>\tan(\theta - \frac{\pi}{9}) = \sqrt{3}</math>  <math>\arctan(\sqrt{3}) = \frac{\pi}{3}</math>  <math>\theta - \frac{\pi}{9} = \frac{\pi}{3} + n\pi</math>  <math>\theta = \frac{4\pi}{9} \pm n\pi</math></p> <p>(d) <math>\sin(2\theta + \frac{\pi}{6}) = \frac{\sqrt{3}}{2}</math>  <math>\arcsin(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}</math>  <math>2\theta + \frac{\pi}{6} = \frac{\pi}{3} + 2n\pi</math>  <math>2\theta = \frac{\pi}{6} \pm 2n\pi</math>  <math>\theta = \frac{\pi}{12} \pm n\pi</math></p> <p>(e) <math>\cos(3x - \frac{\pi}{4}) = -\frac{1}{2}</math>  <math>\arccos(-\frac{1}{2}) = \frac{2\pi}{3}</math>  <math>3x - \frac{\pi}{4} = \frac{2\pi}{3} + 2n\pi</math>  <math>3x = \frac{11\pi}{12} \pm 2n\pi</math>  <math>x_1 = \frac{11\pi}{36}, x_2 = \frac{19\pi}{36}, x_3 = \frac{35\pi}{36}</math></p>
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**Question 10**

Solve each of the following trigonometric equations in the range given.

a)  $\tan\left(\frac{3x}{2}\right) = -\frac{\sqrt{3}}{3}$        $0 \leq x < 2\pi$

$$x = \frac{5\pi}{9}, \frac{11\pi}{9}, \frac{17\pi}{9}$$

b)  $\sin\left(4y - \frac{\pi}{4}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$        $-\frac{\pi}{2} \leq y < \frac{\pi}{2}$

$$y = -\frac{5\pi}{12}, -\frac{5\pi}{24}, \frac{\pi}{12}, \frac{7\pi}{24}$$

c)  $5\cos 2x = 2$        $0 \leq x < 2\pi$

$$x = 0.58^\circ, 2.56^\circ, 3.72^\circ, 5.70^\circ$$

d)  $3 - \tan\left(\theta - \frac{\pi}{18}\right) = 1$        $0 \leq \theta < 2\pi$

$$\theta = 1.28^\circ, 4.42^\circ$$

e)  $3\sin\left(3\theta - \frac{\pi}{6}\right) + 4 = 5$        $-\frac{\pi}{2} \leq \theta < \frac{\pi}{2}$

$$\theta = -0.99^\circ, 0.29^\circ, 1.11^\circ$$

(a)  $\tan\left(\frac{3x}{2}\right) = -\frac{\sqrt{3}}{3}$   
 $\arctan\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$   
 $\left(\frac{3x}{2}\right) = -\frac{\pi}{6} \pm 2k\pi$        $\text{mod } 3\pi$   
 $(3x) = -\frac{\pi}{3} \pm 2k\pi$   
 $x = \frac{-\pi}{9} \pm \frac{2k\pi}{3}$   
 $x_1 = \frac{\pi}{3}$   
 $x_2 = \frac{\pi}{9}$   
 $x_3 = \frac{7\pi}{9}$

(b)  $\sin\left(4y - \frac{\pi}{4}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$   
 $\arcsin\left(\frac{\sqrt{6} - \sqrt{2}}{4}\right) = \frac{\pi}{12}$   
 $\left(4y - \frac{\pi}{4}\right) = \frac{\pi}{12} \pm 2k\pi$   
 $(4y) = \frac{\pi}{12} \pm 2k\pi$   
 $y = \frac{\pi}{48} \pm \frac{k\pi}{2}$   
 $y_1 = \frac{\pi}{48}$   
 $y_2 = -\frac{\pi}{24}$   
 $y_3 = \frac{11\pi}{48}$   
 $y_4 = -\frac{13\pi}{24}$

(c)  $5\cos 2x = 2$   
 $\cos 2x = \frac{2}{5}$   
 $\arccos\left(\frac{2}{5}\right) = 1.154^\circ$   
 $(2x) = 1.154^\circ \pm 2k\pi$   
 $(2x) = 5.104^\circ \pm 2k\pi$   
 $x = 0.502^\circ \pm k\pi$   
 $x = 2.562^\circ \pm k\pi$

(d)  $3 - \tan\left(\theta - \frac{\pi}{18}\right) = 1$   
 $\tan\left(\theta - \frac{\pi}{18}\right) = 2$   
 $\theta - \frac{\pi}{18} = \frac{\pi}{4}$   
 $\theta = \frac{5\pi}{36}$   
 $\theta = 1.28^\circ$

(e)  $3\sin\left(3\theta - \frac{\pi}{6}\right) + 4 = 5$   
 $3\sin\left(3\theta - \frac{\pi}{6}\right) = 1$   
 $\sin\left(3\theta - \frac{\pi}{6}\right) = \frac{1}{3}$   
 $\arcsin\left(\frac{1}{3}\right) = 0.343^\circ$   
 $3\theta - \frac{\pi}{6} = 0.343^\circ \pm 2k\pi$   
 $3\theta = 0.343^\circ \pm 2k\pi$   
 $\theta = 0.114^\circ \pm \frac{k\pi}{3}$   
 $\theta_1 = 0.114^\circ$   
 $\theta_2 = -0.86^\circ$   
 $\theta_3 = 1.08^\circ$

**Question 11**

Solve each of the following trigonometric equations in the range given.

a)  $4\cos\left(2\theta - \frac{\pi}{9}\right) = 1$        $0 \leq \theta < 2\pi$        $\theta \approx 0.83^\circ, 2.66^\circ, 3.98^\circ, 5.80^\circ$

b)  $2 + \tan 4x = 5$        $-\frac{\pi}{2} \leq x < \frac{\pi}{2}$        $x \approx -1.26^\circ, -0.47^\circ, 0.31^\circ, 1.10^\circ$

c)  $7\sin\left(4t + \frac{\pi}{18}\right) = 4$        $-\frac{\pi}{2} \leq t < \frac{\pi}{2}$        $t \approx -1.46^\circ, -0.98^\circ, 0.11^\circ, 0.59^\circ$

d)  $5\cos 4\varphi + 1 = 0$        $0 \leq \varphi < \frac{\pi}{2}$        $\varphi \approx 0.44^\circ, 1.13^\circ$

e)  $\tan\left(y + \frac{5\pi}{18}\right) = 3$        $0 \leq y < 2\pi$        $y \approx 0.38^\circ, 3.52^\circ$

$\text{(a)} \quad 4\cos\left(2\theta - \frac{\pi}{9}\right) = 1$ $\cos\left(2\theta - \frac{\pi}{9}\right) = \frac{1}{4}$ $\arccos\left(\frac{1}{4}\right) \approx 1.398^\circ$ $\left(2\theta - \frac{\pi}{9}\right) = 1.398^\circ \pm 2\pi n \quad (n \in \mathbb{Z})$ $\left(2\theta\right) = 1.667^\circ \pm 2\pi n$ $\left(2\theta\right) = 5.314^\circ \pm 2\pi n$ $\left(\theta\right) = 0.83^\circ \pm \pi n$ $\left(\theta\right) = 2.66^\circ \pm \pi n$ $\therefore \theta = 0.83^\circ, 2.66^\circ, 3.98^\circ, 5.80^\circ$	$\text{(b)} \quad 2 + \tan 4x = 5$ $\tan 4x = 3$ $\arctan(3) \approx 1.249^\circ$ $(4x) = 1.249^\circ \pm 2\pi n$ $(4x) = 0.383^\circ \pm \frac{\pi}{2}$ $\therefore x = 0.38^\circ, 3.52^\circ$	$\text{(c)} \quad 7\sin\left(4t + \frac{\pi}{18}\right) = 4$ $\sin\left(4t + \frac{\pi}{18}\right) = \frac{4}{7}$ $\arcsin\left(\frac{4}{7}\right) \approx 0.498^\circ$ $\left(4t + \frac{\pi}{18}\right) = 0.498^\circ \pm 2\pi n$ $\left(4t + \frac{\pi}{18}\right) = 2.53^\circ \pm 2\pi n$ $\left(4t\right) = 0.498^\circ \pm 2\pi n$ $\left(4t\right) = 2.53^\circ \pm 2\pi n$ $\left(t\right) = 0.124^\circ \pm \frac{\pi}{4}$ $\left(t\right) = 0.38^\circ \pm \frac{\pi}{4}$
$\text{(d)} \quad 5\cos 4\varphi + 1 = 0$ $\cos 4\varphi = -\frac{1}{5}$ $\arccos\left(-\frac{1}{5}\right) \approx 1.772^\circ$ $4\varphi = 1.772^\circ \pm 2\pi n$ $4\varphi = 4.501^\circ \pm 2\pi n$ $\left(\varphi\right) = 0.45^\circ \pm \frac{\pi}{4}$ $\left(\varphi\right) = 1.13^\circ \pm \frac{\pi}{4}$ $\therefore \varphi = 0.44^\circ, 1.13^\circ$	$\text{(e)} \quad \tan\left(y + \frac{5\pi}{18}\right) = 3$ $\arctan(3) \approx 1.249^\circ$ $\left(y + \frac{5\pi}{18}\right) = 1.249^\circ \pm 2\pi n$ $\left(y\right) = 0.316^\circ \pm \pi n$ $y = 0.316^\circ, 3.52^\circ$	

**Question 12**

Solve each of the following trigonometric equations in the range given.

a)  $2\sin 3x - 1 = 0.25$

$0 \leq \theta < \pi$

$x \approx 0.225^\circ, 0.822^\circ, 2.32^\circ, 2.92^\circ$

b)  $5\cos\left(\theta + \frac{\pi}{3}\right) - 1 = 2$

$0 \leq x < 2\pi$

$\theta = 4.31^\circ, 6.16^\circ$

c)  $1 - \tan 4y = 3$

$0 \leq y < \pi$

$y \approx 0.509^\circ, 1.29^\circ, 2.08^\circ, 2.87^\circ$

d)  $2\sin\left(2\varphi - \frac{\pi}{3}\right) = 1$

$0 \leq \varphi < 2\pi$

$\varphi = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}$

e)  $2\cos\left(3t + \frac{\pi}{3}\right) = \sqrt{3}$

$-\frac{\pi}{2} \leq t < \frac{\pi}{2}$

$t = -\frac{\pi}{6}, -\frac{\pi}{18}$

<p>(a) <math>2\sin 3x - 1 = 0.25</math>, <math>0 \leq x &lt; \pi</math>  <math>\Rightarrow \sin 3x = 0.25</math>  <math>\Rightarrow \sin 3x \approx 0.25</math>  <math>\arcsin(0.25) \approx 0.471^\circ</math>  <math>3x = 0.471^\circ + 20^\circ n</math>  <math>x = 2.446^\circ + 20^\circ n</math>      <math>n = 0, 1, 2, 3, \dots</math>  <math>x = 0.157^\circ + \frac{20^\circ}{3}</math>  <math>x = 0.157^\circ + 20^\circ</math>  <math>x = 22.2^\circ + 20^\circ n</math>, <math>244^\circ</math></p>	<p>(d) <math>2\sin\left(2\varphi - \frac{\pi}{3}\right) = 1</math>, <math>0 \leq \varphi &lt; 2\pi</math>  <math>\Rightarrow \sin\left(2\varphi - \frac{\pi}{3}\right) = \frac{1}{2}</math>  <math>\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}</math>  <math>2\varphi - \frac{\pi}{3} = \frac{\pi}{6} + 20^\circ n</math>  <math>2\varphi = \frac{\pi}{6} + \frac{\pi}{3} + 20^\circ n</math>  <math>2\varphi = \frac{5\pi}{12} + 20^\circ n</math>  <math>\varphi = \frac{5\pi}{24} + 10^\circ n</math>  <math>\varphi = \frac{5\pi}{24} + 20^\circ n</math>  <math>\varphi = \frac{5\pi}{24}, \frac{13\pi}{12}, \frac{19\pi}{12}</math></p>
<p>(b) <math>5\cos\left(\theta + \frac{\pi}{3}\right) = 1</math>, <math>0 \leq \theta &lt; \pi</math>  <math>\Rightarrow \cos\left(\theta + \frac{\pi}{3}\right) = \frac{1}{5}</math>  <math>\Rightarrow \cos\left(\theta + \frac{\pi}{3}\right) = 0.2</math>  <math>\arccos(0.2) \approx 78.5^\circ</math>  <math>\theta + \frac{\pi}{3} = 78.5^\circ + 20^\circ n</math>  <math>\theta = 4.31^\circ + 20^\circ n</math>  <math>\theta = 6.16^\circ + 20^\circ n</math></p>	<p>(e) <math>2\cos\left(3t + \frac{\pi}{3}\right) = \sqrt{3}</math>, <math>-\frac{\pi}{2} \leq t &lt; \frac{\pi}{2}</math>  <math>\Rightarrow \cos\left(3t + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}</math>  <math>\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}</math>  <math>3t + \frac{\pi}{3} = \frac{\pi}{6} + 20^\circ n</math>  <math>3t = \frac{\pi}{6} - \frac{\pi}{3} + 20^\circ n</math>  <math>t = -\frac{\pi}{18} + 20^\circ n</math>  <math>t = -\frac{\pi}{18}, -\frac{\pi}{6}</math></p>
<p>(c) <math>1 - \tan 4y = 3</math>, <math>0 \leq y &lt; \pi</math>  <math>\Rightarrow -\tan 4y = 2</math>  <math>\Rightarrow \tan 4y = -2</math>  <math>\arctan(-2) \approx -1.107^\circ</math>  <math>4y = -1.107^\circ + 20^\circ n</math>  <math>y = -0.277^\circ + \frac{20^\circ}{4}</math>  <math>y = 1.29^\circ, 2.08^\circ, 2.87^\circ, 0.509^\circ</math></p>	

**Question 13**

Solve each of the following trigonometric equations in the range given.

a)  $\sin 2x = \sin\left(\frac{4\pi}{15}\right)$        $0 \leq \theta < 2\pi$

$$x = \frac{2\pi}{15}, \frac{11\pi}{30}, \frac{17\pi}{15}, \frac{41\pi}{30}$$

b)  $\cos 3x = \cos\left(\frac{8\pi}{15}\right)$        $0 \leq x < \pi$

$$x = \frac{8\pi}{45}, \frac{22\pi}{45}, \frac{38\pi}{45}$$

c)  $\tan\left(\theta + \frac{5\pi}{18}\right) = \tan\left(\frac{\pi}{12}\right)$        $0 \leq \theta < 2\pi$

$$\theta = \frac{29\pi}{36}, \frac{65\pi}{36}$$

d)  $\sin 2x = \sin\left(\frac{29\pi}{15}\right)$        $0 \leq x < 2\pi$

$$x = \frac{8\pi}{15}, \frac{29\pi}{30}, \frac{23\pi}{15}, \frac{59\pi}{30}$$

e)  $\cos 4x = \cos\left(\frac{4\pi}{3}\right)$        $0 \leq x < \pi$

$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$$

<p>(3) <math>\sin 2x = \sin\left(\frac{4\pi}{15}\right), 0 \leq x &lt; 2\pi</math></p> $\begin{aligned} 2x &= \frac{4\pi}{15} \pm 2k\pi \\ 2x &= \frac{16\pi}{15} \pm 2k\pi \quad k \in \mathbb{Z} \dots \end{aligned}$ $\begin{aligned} x &= \frac{8\pi}{15} \pm k\pi \\ x &= \frac{8\pi}{15} \pm \frac{\pi}{2} \\ x &= \frac{2\pi}{15}, \frac{11\pi}{15}, \frac{4\pi}{3}, \frac{29\pi}{15} \end{aligned}$	<p>(4) <math>\sin 2x = \sin\left(\frac{29\pi}{15}\right), 0 \leq x &lt; 2\pi</math></p> $\begin{aligned} 2x &= \frac{29\pi}{15} \pm 2k\pi \\ 2x &= \frac{149\pi}{15} \pm 2k\pi \quad k \in \mathbb{Z} \dots \end{aligned}$ $\begin{aligned} x &= \frac{74\pi}{15} \pm k\pi \\ x &= \frac{74\pi}{15} \pm \frac{\pi}{2} \\ x &= \frac{2\pi}{15}, \frac{29\pi}{15}, \frac{8\pi}{15}, \frac{23\pi}{15} \end{aligned}$
<p>(5) <math>\cos 3x = \cos\left(\frac{4\pi}{3}\right), 0 \leq x &lt; \pi</math></p> $\begin{aligned} 3x &= \frac{4\pi}{3} \pm 2k\pi \\ 3x &= \frac{2\pi}{3} \pm 2k\pi \quad k \in \mathbb{Z} \dots \end{aligned}$ $\begin{aligned} x &= \frac{2\pi}{9} \pm \frac{2k\pi}{3} \\ x &= \frac{2\pi}{9}, \frac{8\pi}{9}, \frac{4\pi}{3} \end{aligned}$	<p>(6) <math>\cos 3x = \cos\left(\frac{4\pi}{3}\right), 0 \leq x &lt; \pi</math></p> $\begin{aligned} 3x &= \frac{4\pi}{3} \pm 2k\pi \\ 3x &= \frac{2\pi}{3} \pm 2k\pi \quad k \in \mathbb{Z} \dots \end{aligned}$ $\begin{aligned} x &= \frac{2\pi}{9} \pm \frac{2k\pi}{3} \\ x &= \frac{2\pi}{9}, \frac{8\pi}{9}, \frac{4\pi}{3} \end{aligned}$
<p>(7) <math>\tan\left(\theta + \frac{5\pi}{18}\right) = \frac{\pi}{12}, 0 \leq \theta &lt; 2\pi</math></p> $\begin{aligned} \theta + \frac{5\pi}{18} &= \frac{\pi}{12} \pm k\pi \\ \theta &= -\frac{7\pi}{36} \pm k\pi \quad k \in \mathbb{Z} \dots \end{aligned}$ $\begin{aligned} \theta &= \frac{29\pi}{36}, \frac{65\pi}{36} \end{aligned}$	<p>(8) <math>\sin 2x = \sin\left(\frac{29\pi}{15}\right), 0 \leq x &lt; 2\pi</math></p> $\begin{aligned} 2x &= \frac{29\pi}{15} \pm 2k\pi \\ 2x &= \frac{149\pi}{15} \pm 2k\pi \quad k \in \mathbb{Z} \dots \end{aligned}$ $\begin{aligned} x &= \frac{74\pi}{15} \pm k\pi \\ x &= \frac{74\pi}{15} \pm \frac{\pi}{2} \\ x &= \frac{2\pi}{15}, \frac{29\pi}{15}, \frac{8\pi}{15}, \frac{23\pi}{15} \end{aligned}$

**Question 14**

Solve each of the following trigonometric equations in the range given.

a)  $\sin \theta = 2 \cos \theta$

$0 \leq \theta < 360^\circ$

$\theta \approx 63.4^\circ, 243.4^\circ$

b)  $\sin \theta + 3 \cos \theta = 0$

$0 \leq \theta < 360^\circ$

$\theta \approx 108.4^\circ, 288.4^\circ$

c)  $4 \cos x - 3 \sin x = 0$

$0 \leq x < 360^\circ$

$x \approx 53.1^\circ, 233.1^\circ$

d)  $2 \sin x = \cos x$

$0 \leq x < 360^\circ$

$x \approx 26.6^\circ, 206.6^\circ$

e)  $\sin 2\theta + \cos 2\theta = 0$

$0 \leq \theta < 360^\circ$

$\theta = 67.5^\circ, 157.5^\circ, 247.5^\circ, 337.5^\circ$

f)  $\sqrt{3} \sin 3\theta = 3 \cos 3\theta$

$-90^\circ \leq \theta < 90^\circ$

$\theta = -40^\circ, 20^\circ, 80^\circ$

<p>1. <math>\sin \theta = 2 \cos \theta</math>  <math>\Rightarrow \frac{\sin \theta}{\cos \theta} = \frac{2 \cos \theta}{\cos \theta}</math>  <math>\Rightarrow \tan \theta = 2</math>  • <math>\arctan(2) \approx 63.4^\circ</math>  <math>(\theta = 63.4^\circ \pm 180n, n \in \mathbb{Z})</math>  <math>\theta_1 = 63.4^\circ</math>  <math>\theta_2 = 243.4^\circ</math></p>	<p>4. <math>2 \sin \theta = \cos \theta</math>  <math>\Rightarrow \frac{2 \sin \theta}{\cos \theta} = \frac{\cos \theta}{\cos \theta}</math>  <math>\Rightarrow 2 \tan \theta = 1</math>  <math>\Rightarrow \tan \theta = \frac{1}{2}</math>  • <math>\arctan(\frac{1}{2}) \approx 26.6^\circ</math>  <math>(\theta = 26.6^\circ \pm 180n, n \in \mathbb{Z})</math>  <math>\theta_1 = 26.6^\circ</math>  <math>\theta_2 = 206.6^\circ</math></p>
<p>2. <math>\sin \theta + 3 \cos \theta = 0</math>  <math>\Rightarrow \frac{\sin \theta}{\cos \theta} + \frac{3 \cos \theta}{\cos \theta} = \frac{0}{\cos \theta}</math>  <math>\Rightarrow \tan \theta + 3 = 0</math>  <math>\Rightarrow \tan \theta = -3</math>  • <math>\arctan(-3) \approx -71.6^\circ</math>  <math>(\theta = -71.6^\circ \pm 180n, n \in \mathbb{Z})</math>  <math>\theta_1 = 108.4^\circ</math>  <math>\theta_2 = 288.4^\circ</math></p>	<p>5. <math>\sin \theta + \cos \theta = 0</math>  <math>\Rightarrow \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\cos \theta} = \frac{0}{\cos \theta}</math>  <math>\Rightarrow \tan \theta + 1 = 0</math>  <math>\Rightarrow \tan \theta = -1</math>  • <math>\arctan(-1) = -45^\circ</math>  <math>(\theta = -45^\circ \pm 180n, n \in \mathbb{Z})</math>  <math>\theta_1 = -22.5^\circ</math>  <math>\theta_2 = 157.5^\circ</math></p>
<p>3. <math>4 \cos x - 3 \sin x = 0</math>  <math>\Rightarrow \frac{4 \cos x}{\cos x} - \frac{3 \sin x}{\cos x} = \frac{0}{\cos x}</math>  <math>\Rightarrow 4 - 3 \tan x = 0</math>  <math>\Rightarrow 4 = 3 \tan x</math>  <math>\Rightarrow \tan x = \frac{4}{3}</math>  • <math>\arctan(\frac{4}{3}) \approx 53.1^\circ</math>  <math>(x = 53.1^\circ \pm 180n, n \in \mathbb{Z})</math>  <math>\theta_1 = 53.1^\circ</math>  <math>\theta_2 = 233.1^\circ</math></p>	<p>6. <math>\sqrt{3} \sin 3\theta = 3 \cos 3\theta</math>  <math>\Rightarrow \frac{\sqrt{3} \sin 3\theta}{\cos 3\theta} = \frac{3 \cos 3\theta}{\cos 3\theta}</math>  <math>\Rightarrow \sqrt{3} \tan 3\theta = 3</math>  <math>\Rightarrow \tan 3\theta = \frac{3}{\sqrt{3}}</math>  • <math>\arctan(\frac{3}{\sqrt{3}}) = 60^\circ</math>  <math>(3\theta = 60^\circ \pm 180n, n \in \mathbb{Z})</math>  <math>\theta_1 = 20^\circ</math>  <math>\theta_2 = 80^\circ</math>  <math>\theta_3 = -40^\circ</math></p>

**Question 15**

Solve each of the following trigonometric equations in the range given.

- |   |                                |   |
|---|--------------------------------|---|
| a) $\cos x = 3 \sin x$                      | $0 \leq x < 360^\circ$         | $x \approx 18.4^\circ, 198.4^\circ$                   |
| b) $4 \cos x - 7 \sin x = 0$                | $0 \leq x < 360^\circ$         | $x \approx 29.7^\circ, 209.7^\circ$                   |
| c) $5 \sin x + 6 \cos x = 0$                | $0 \leq x < 360^\circ$         | $x \approx 129.8^\circ, 309.8^\circ$                  |
| d) $3 \sin 2\theta = \sqrt{3} \cos 2\theta$ | $-180 \leq \theta < 180^\circ$ | $\theta = -165^\circ, -75^\circ, 15^\circ, 105^\circ$ |
| e) $\sin 3\theta = \cos 3\theta$            | $0 \leq \theta < 180^\circ$    | $\theta = 15^\circ, 75^\circ, 135^\circ$              |
| f) $5 \cos 2x + 9 \sin 2x = 0$              | $0 \leq x < 180^\circ$         | $x \approx 75.5^\circ, 165.5^\circ$                   |

<p>(a) <math>\cos x = 3 \sin x</math></p> $\frac{\cos x}{\sin x} = \frac{3 \sin x}{\cos x}$ $1 = 3 \tan x$ $\tan x = \frac{1}{3}$ $\arctan(\frac{1}{3}) = 18.4^\circ$ $(x = 18.4^\circ + 180n, n \in \mathbb{Z}, n \neq 0)$ $\therefore x = 18.4^\circ$ $x = 18.4^\circ + 198.4^\circ$	<p>(d) <math>3 \sin 2\theta = \sqrt{3} \cos 2\theta</math></p> $\frac{3 \sin 2\theta}{\cos 2\theta} = \frac{\sqrt{3} \cos 2\theta}{\cos 2\theta}$ $3 \tan 2\theta = \sqrt{3}$ $\tan 2\theta = \frac{\sqrt{3}}{3}$ $\arctan(\frac{\sqrt{3}}{3}) = 30^\circ$ $(2\theta = 30^\circ + 180n, n \in \mathbb{Z}, n \neq 0)$ $\therefore \theta = 15^\circ + 90n$ $\theta = 15^\circ - 165^\circ$
<p>(b) <math>4 \cos x - 7 \sin x = 0</math></p> $\frac{4 \cos x}{7 \sin x} = \frac{7 \sin x}{7 \sin x}$ $4 = 7 \tan x$ $\tan x = \frac{4}{7}$ $\arctan(\frac{4}{7}) = 29.7^\circ$ $(x = 29.7^\circ + 180n, n \in \mathbb{Z}, n \neq 0)$ $\therefore x = 29.7^\circ$ $x = 29.7^\circ + 180^\circ$	<p>(e) <math>\sin 3\theta = \cos 3\theta</math></p> $\frac{\sin 3\theta}{\cos 3\theta} = \frac{\cos 3\theta}{\cos 3\theta}$ $\tan 3\theta = 1$ $\arctan(1) = 45^\circ$ $(3\theta = 45^\circ + 180n, n \in \mathbb{Z}, n \neq 0)$ $\therefore \theta = 15^\circ + 60n$ $\theta = 15^\circ - 135^\circ$
<p>(c) <math>5 \sin x + 6 \cos x = 0</math></p> $\frac{5 \sin x}{6 \cos x} + \frac{6 \cos x}{6 \cos x} = \frac{0}{6 \cos x}$ $\frac{5 \sin x}{6 \cos x} + 1 = 0$ $\tan x = -\frac{6}{5}$ $\arctan(-\frac{6}{5}) = -53.17^\circ$ $(x = -53.17^\circ + 180n, n \in \mathbb{Z}, n \neq 0)$ $\therefore x = 126.82^\circ$ $x = 126.82^\circ + 180^\circ$	<p>(f) <math>5 \cos 2x + 9 \sin 2x = 0</math></p> $\frac{5 \cos 2x}{9 \sin 2x} + \frac{9 \sin 2x}{9 \sin 2x} = \frac{0}{9 \sin 2x}$ $\frac{5 \cos 2x}{9 \sin 2x} + 1 = 0$ $\tan 2x = -\frac{5}{9}$ $\arctan(-\frac{5}{9}) = -24.05^\circ$ $(2x = -24.05^\circ + 180n, n \in \mathbb{Z}, n \neq 0)$ $\therefore x = -12.05^\circ$ $x = -12.05^\circ + 90^\circ$

**Question 16**

Solve each of the following trigonometric equations in the range given.

a)  $\sin \theta = 2\cos \theta$

$0 \leq \theta < 2\pi$

$\theta \approx 1.11^\circ, 4.25^\circ$

b)  $\sin \theta + 3\cos \theta = 0$

$0 \leq \theta < 2\pi$

$\theta \approx 1.89^\circ, 5.03^\circ$

c)  $4\cos x - 3\sin x = 0$

$0 \leq x < 2\pi$

$x \approx 0.927^\circ, 4.07^\circ$

d)  $2\sin x = \cos x$

$0 \leq x < 2\pi$

$x \approx 0.464^\circ, 3.61^\circ$

e)  $\sin 2\theta + \cos 2\theta = 0$

$0 \leq \theta < 2\pi$

$\theta = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

f)  $\sqrt{3}\sin 3\theta = 3\cos 3\theta$

$-\frac{\pi}{2} \leq \theta < \frac{\pi}{2}$

$\theta = -\frac{2\pi}{9}, \frac{\pi}{9}, \frac{4\pi}{9}$

<p>(a) <math>\sin \theta = 2\cos \theta, 0 \leq \theta &lt; 2\pi</math></p> $\Rightarrow \frac{\sin \theta}{\cos \theta} = \frac{2\cos \theta}{\cos \theta}$ $\Rightarrow \tan \theta = 2$ $\arctan(2) \approx 1.107^\circ$ $\therefore \theta_1 \approx 1.107^\circ$ $\theta_2 = 4.25^\circ$	<p>(b) <math>2\sin 2\theta = \cos 2\theta, 0 \leq 2\theta &lt; 2\pi</math></p> $\Rightarrow \frac{2\sin 2\theta}{\cos 2\theta} = \frac{\cos 2\theta}{\cos 2\theta}$ $\Rightarrow \tan 2\theta = 1$ $\arctan(1) = \frac{\pi}{4}$ $\arctan(1) = 0.464^\circ$ $\therefore \theta_1 \approx 0.464^\circ$ $\theta_2 = 3.61^\circ$
<p>(c) <math>\sin \theta + 3\cos \theta = 0, 0 \leq \theta &lt; 2\pi</math></p> $\Rightarrow \frac{\sin \theta}{\cos \theta} + \frac{3\cos \theta}{\cos \theta} = \frac{0}{\cos \theta}$ $\Rightarrow \tan \theta + 3 = 0$ $\Rightarrow \tan \theta = -3$ $\arctan(-3) \approx -12.4^\circ$ $\theta = -12.4^\circ \pm 180^\circ \quad n=0,1,2,3, \dots$ $\therefore \theta_1 \approx 167.5^\circ$ $\theta_2 = 5.03^\circ$	<p>(d) <math>\sin 2\theta + \cos 2\theta = 0, 0 \leq \theta &lt; 2\pi</math></p> $\Rightarrow \frac{\sin 2\theta}{\cos 2\theta} + \frac{\cos 2\theta}{\cos 2\theta} = \frac{0}{\cos 2\theta}$ $\Rightarrow \tan 2\theta + 1 = 0$ $\Rightarrow \tan 2\theta = -1$ $\arctan(-1) = -\frac{\pi}{4}$ $2\theta = -\frac{\pi}{4} \pm 180^\circ \quad n=0,1,2,3, \dots$ $\theta = -\frac{\pi}{8} \pm 90^\circ \quad n=0,1,2,3, \dots$ $\therefore \theta_1 \approx -3.5^\circ$ $\theta_2 = 86.5^\circ$
<p>(e) <math>4\cos x - 3\sin x = 0, 0 \leq x &lt; 2\pi</math></p> $\Rightarrow \frac{4\cos x}{\sin x} - \frac{3\sin x}{\sin x} = \frac{0}{\sin x}$ $\Rightarrow 4 - 3\tan x = 0$ $\Rightarrow \tan x = \frac{4}{3}$ $\arctan(\frac{4}{3}) \approx 0.927^\circ$ $\theta = 0.927^\circ \pm 180^\circ \quad n=0,1,2,3, \dots$ $\therefore \theta_1 \approx 0.927^\circ$ $\theta_2 = 4.07^\circ$	<p>(f) <math>\sqrt{3}\sin 3\theta = 3\cos 3\theta, -\frac{\pi}{2} \leq \theta &lt; \frac{\pi}{2}</math></p> $\Rightarrow \frac{\sqrt{3}\sin 3\theta}{3\cos 3\theta} = \frac{3\cos 3\theta}{3\cos 3\theta}$ $\Rightarrow \sqrt{3}\tan 3\theta = 3$ $\Rightarrow \tan 3\theta = \frac{3}{\sqrt{3}}$ $\arctan(\frac{3}{\sqrt{3}}) = \frac{\pi}{3}$ $3\theta = \frac{\pi}{3} \pm 180^\circ \quad n=0,1,2,3, \dots$ $\theta = \frac{\pi}{9} \pm 60^\circ \quad n=0,1,2,3, \dots$ $\therefore \theta_1 \approx -5.2^\circ$ $\theta_2 = 4.25^\circ$

**Question 17**

Solve each of the following trigonometric equations in the range given.

a)  $\cos x = 3 \sin x$

$0 \leq x < 2\pi$

$x \approx 0.32^\circ, 3.46^\circ$

b)  $4 \cos x - 7 \sin x = 0$

$0 \leq x < 2\pi$

$x \approx 0.52^\circ, 3.66^\circ$

c)  $5 \sin x + 6 \cos x = 0$

$0 \leq x < 2\pi$

$x \approx 2.27^\circ, 5.41^\circ$

d)  $3 \sin 2\theta = \sqrt{3} \cos 2\theta$

$-\pi \leq \theta < \pi$

$\theta = -\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$

e)  $\sin 3\theta = \cos 3\theta$

$0 \leq \theta < \pi$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$

f)  $5 \cos 2\theta + 9 \sin 2\theta = 0$

$0 \leq \theta < \pi$

$\theta \approx 1.32^\circ, 2.89^\circ$

<p>(a) <math>\cos x = 3 \sin x</math>  <math>\Rightarrow \frac{\cos x}{\sin x} = \frac{3 \sin x}{\cos x}</math>  <math>\Rightarrow 1 = 3 \tan x</math>  <math>\Rightarrow \tan x = \frac{1}{3}</math>  <math>\arctan(\frac{1}{3}) \approx 0.322^\circ</math>  <math>(x = 0.322^\circ + n\pi, n \in \mathbb{Z}, \dots)</math>  <math>\therefore x \approx 0.32^\circ, 3.46^\circ</math></p>	<p>(d) <math>3 \sin 2\theta = \sqrt{3} \cos 2\theta</math>  <math>\Rightarrow \frac{3 \sin 2\theta}{\cos 2\theta} = \frac{\sqrt{3} \cos 2\theta}{\cos 2\theta}</math>  <math>\Rightarrow 3 \tan 2\theta = \sqrt{3}</math>  <math>\Rightarrow \tan 2\theta = \frac{\sqrt{3}}{3}</math>  <math>\arctan(\frac{\sqrt{3}}{3}) = \frac{\pi}{6}</math>  <math>\text{Let } 2\theta = m\pi + \frac{\pi}{2}, m \in \mathbb{Z}, \dots</math>  <math>\Rightarrow \theta = \frac{m\pi}{2} + \frac{\pi}{12}</math>  <math>\therefore \theta = \frac{\pi}{12}, \frac{5\pi}{12}, -\frac{5\pi}{12}, \frac{7\pi}{12}</math></p>
<p>(b) <math>4 \cos x - 7 \sin x = 0</math>  <math>\Rightarrow \frac{4 \cos x}{\sin x} - \frac{7 \sin x}{\sin x} = \frac{0}{\sin x}</math>  <math>\Rightarrow 4 - 7 \tan x = 0</math>  <math>\Rightarrow 4 = 7 \tan x</math>  <math>\Rightarrow \tan x = \frac{4}{7}</math>  <math>\arctan(\frac{4}{7}) \approx 0.519</math>  <math>(x = 0.519^\circ + n\pi, n \in \mathbb{Z}, \dots)</math>  <math>\therefore x \approx 0.52^\circ, 3.66^\circ</math></p>	<p>(e) <math>\sin 3\theta = \cos 3\theta</math>  <math>\Rightarrow \frac{\sin 3\theta}{\cos 3\theta} = \frac{\cos 3\theta}{\cos 3\theta}</math>  <math>\Rightarrow \tan 3\theta = 1</math>  <math>\arctan(1) = \frac{\pi}{4}</math>  <math>(3\theta = \frac{\pi}{4} + n\pi, n \in \mathbb{Z}, \dots)</math>  <math>\theta = \frac{n\pi}{3} + \frac{\pi}{12}</math>  <math>\therefore \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}</math></p>
<p>(c) <math>5 \sin x + 6 \cos x = 0</math>  <math>\Rightarrow \frac{5 \sin x}{\cos x} + \frac{6 \cos x}{\cos x} = \frac{0}{\cos x}</math>  <math>\Rightarrow 5 \tan x + 6 = 0</math>  <math>\Rightarrow 5 \tan x = -6</math>  <math>\Rightarrow \tan x = -\frac{6}{5}</math>  <math>\arctan(-\frac{6}{5}) \approx -0.574</math>  <math>(x = -0.574^\circ + n\pi, n \in \mathbb{Z}, \dots)</math>  <math>\therefore x \approx 2.27^\circ, 5.41^\circ</math></p>	<p>(f) <math>5 \cos 2\theta + 9 \sin 2\theta = 0</math>  <math>\Rightarrow \frac{5 \cos 2\theta}{\sin 2\theta} + \frac{9 \sin 2\theta}{\sin 2\theta} = \frac{0}{\sin 2\theta}</math>  <math>\Rightarrow 5 + 9 \tan 2\theta = 0</math>  <math>\Rightarrow \tan 2\theta = -\frac{5}{9}</math>  <math>\arctan(-\frac{5}{9}) \approx -0.527^\circ</math>  <math>(2\theta = -0.527^\circ + n\pi, n \in \mathbb{Z}, \dots)</math>  <math>\theta = -0.26^\circ + \frac{n\pi}{2}</math>  <math>\therefore \theta \approx 1.32^\circ, 2.89^\circ</math></p>

**Question 18**

Solve each of the following trigonometric equations in the range given.

a)  $2\cos^2 \theta + 3\cos \theta - 2 = 0$        $0^\circ \leq \theta < 360^\circ$

$\theta = 60^\circ, 300^\circ$

b)  $6\sin^2 x + 5\sin x + 1 = 0$        $0^\circ \leq x < 360^\circ$

$x = 210^\circ, 330^\circ, x \approx 340.5^\circ, 199.5^\circ$

c)  $2\tan^2 \theta - \tan \theta - 6 = 0$        $0^\circ \leq \theta < 360^\circ$

$\theta \approx 63.4^\circ, 123.7^\circ, 243.4^\circ, 303.7^\circ$

(a)  $2\cos^2 \theta + 3\cos \theta - 2 = 0$   
 $\Rightarrow (2\cos \theta - 1)(\cos \theta + 2) = 0$   
 $\therefore \cos \theta = \frac{1}{2}$    
 $\arccos(\frac{1}{2}) \approx 60^\circ$

(b)  $6\sin^2 x + 5\sin x + 1 = 0$   
 $\Rightarrow (2\sin x + 1)(3\sin x + 1) = 0$   
 $\therefore \sin x = -\frac{1}{2}$    
 $\arcsin(-\frac{1}{2}) \approx -30^\circ$   
 $\therefore x = -30^\circ \pm 360^\circ$   
 $x = 330^\circ, 210^\circ, 340.5^\circ, 199.5^\circ$

(c)  $2\tan^2 \theta - \tan \theta - 6 = 0$   
 $\Rightarrow (2\tan \theta + 3)(\tan \theta - 2) = 0$   
 $\therefore \tan \theta = -\frac{3}{2}$    
 $\arctan(-\frac{3}{2}) \approx -49.47^\circ$   
 $\therefore \theta = -49.47^\circ \pm 180^\circ$   
 $\theta = 130.53^\circ, 130.53^\circ + 360^\circ$

**Question 19**

Solve each of the following trigonometric equations in the range given.

a)  $3\cos^2 \theta - 13\cos \theta + 4 = 0$        $0 \leq \theta < 360^\circ$

$\theta \approx 70.5^\circ, 289.5^\circ$

b)  $8\sin^2 x + 6\sin x + 1 = 0$        $0 \leq x < 360^\circ$

$x = 210^\circ, 330^\circ, x \approx 194.5^\circ, 345.5^\circ$

c)  $3\tan^2 \theta + 8\tan \theta - 3 = 0$        $0 \leq \theta < 360^\circ$

$\theta \approx 18.4^\circ, 108.4^\circ, 198.4^\circ, 288.4^\circ$

<p>(a) <math>3\cos^2 \theta - 13\cos \theta + 4 = 0</math></p> $(3\cos \theta - 1)(\cos \theta - 4) = 0$ $\cos \theta = \frac{1}{3}$ $\arccos(\frac{1}{3}) = 70.53^\circ$ $\theta = 70.53^\circ \pm 360^\circ n = 70.53^\circ, 360^\circ + 70.53^\circ$ $\theta_1 = 70.53^\circ$ $\theta_2 = 289.53^\circ$	<p>(b) <math>8\sin^2 x + 6\sin x + 1 = 0</math></p> $(4\sin x + 1)(2\sin x + 1) = 0$ $\sin x = -\frac{1}{4}$ $\arcsin(-\frac{1}{4}) = -14.48^\circ$ $x = -14.48^\circ + 360^\circ n$ $x = 345.52^\circ + 360^\circ n$ $(x = 210^\circ, 330^\circ)$ $\therefore x = 210^\circ, 330^\circ, 194.5^\circ, 345.5^\circ$
<p>(c) <math>3\tan^2 \theta + 8\tan \theta - 3 = 0</math></p> $(3\tan \theta - 1)(\tan \theta + 3) = 0$ $\tan \theta = -3$ $\arctan(-3) = -16.43^\circ$ $\theta = -16.43^\circ + 180^\circ n$ $\theta = -17.57^\circ + 180^\circ n$ $(\theta = 172.43^\circ, 180^\circ, 187.57^\circ)$ $\therefore \theta = 18.4^\circ, 108.4^\circ, 198.4^\circ, 288.4^\circ$	

**Question 20**

Solve each of the following trigonometric equations in the range given.

a)  $2\cos^2 x - \sin x = 1$

$0 \leq x < 360^\circ$

$x = 30^\circ, 150^\circ, 270^\circ$

b)  $2\cos^2 x + 1 = 5\sin x$

$0 \leq x < 360^\circ$

$x = 30^\circ, 150^\circ$

c)  $2\sin^2 x - 5\cos x + 1 = 0$

$0 \leq x < 360^\circ$

$x = 60^\circ, 300^\circ$

(a)  $2\cos^2 x - \sin x = 1$   
 $\Rightarrow 2(1 - \sin^2 x) - \sin x = 1$   
 $\Rightarrow 2 - 2\sin^2 x - \sin x = 1$   
 $\Rightarrow 0 \approx 2\sin^2 x + \sin x - 1$   
 $\Rightarrow 0 = (2\sin x - 1)(\sin x + 1)$

$\sin x < -1$

$\arcsin\left(-\frac{1}{2}\right) = -30^\circ$   
 $(x = 30^\circ \pm 360^\circ) \quad n=0,1,3$   
 $x = 150^\circ \pm 360^\circ$

$\arcsin(-1) = -90^\circ$   
 $(x = -10^\circ \pm 360^\circ)$   
 $x = 350^\circ \pm 360^\circ$

$x = 30^\circ, 150^\circ, 270^\circ$

(b)  $2\cos^2 x + 1 = 5\sin x$   
 $\Rightarrow 2(1 - \sin^2 x) + 1 = 5\sin x$   
 $\Rightarrow 2 - 2\sin^2 x + 1 = 5\sin x$   
 $\Rightarrow 0 = 2\sin^2 x + 5\sin x - 3$   
 $\Rightarrow (2\sin x + 3)(\sin x - 1) = 0$

$\sin x < 1$

$\arcsin\left(\frac{1}{2}\right) = 30^\circ$   
 $(x = 30^\circ \pm 360^\circ) \quad n=0,1,3$   
 $x = 150^\circ \pm 360^\circ$

$\arcsin(1) = 90^\circ$   
 $(x = 60^\circ \pm 360^\circ)$   
 $x = 30^\circ, 150^\circ$

(c)  $2\sin^2 x - \sin x + 1 = 0$   
 $\Rightarrow 2(1 - \cos^2 x) - \sin x + 1 = 0$   
 $\Rightarrow 2 - 2\cos^2 x - \sin x + 1 = 0$   
 $\Rightarrow 2\cos^2 x + \sin x - 3 = 0$   
 $\Rightarrow (2\cos x - 1)(\cos x + 3) = 0$

$\cos x < -1$

$\arccos\left(-\frac{1}{2}\right) = 120^\circ$   
 $(x = 60^\circ \pm 360^\circ) \quad n=0,1,3$   
 $x = 30^\circ \pm 360^\circ$

$\arccos(1) = 0^\circ$   
 $(x = 60^\circ)$   
 $x = 60^\circ, 300^\circ$

**Question 21**

Solve each of the following trigonometric equations in the range given.

a)  $2\cos^2 x + 7\sin x + 2 = 0 \quad 0 \leq x < 360^\circ$

$x = 210^\circ, 330^\circ$

b)  $5\cos^2 \theta - \sin \theta = 1 \quad 0 \leq \theta < 360^\circ$

$\theta \approx 53.1^\circ, 126.9^\circ, \theta = 270^\circ$

c)  $5\sin^2 y = 19\cos y + 1 \quad 0 \leq y < 360^\circ$

$y \approx 78.5^\circ, 281.5^\circ$

(a)  $2\cos^2 x + 7\sin x + 2 = 0$   
 $\Rightarrow 2(1 - \sin^2 x) + 7\sin x + 2 = 0$   
 $\Rightarrow 2 - 2\sin^2 x + 7\sin x + 2 = 0$   
 $\Rightarrow 0 = 2\sin^2 x - 7\sin x - 4$   
 $\Rightarrow 0 = (2\sin x + 1)(\sin x - 4)$   
 $\sin x = -\frac{1}{2}$   
 $\arcsin(-\frac{1}{2}) = -30^\circ$   
 $(x = -30^\circ + 360^\circ k, k=0,1,2,3, \dots)$   
 $x = 210^\circ, 330^\circ$

(b)  $5\cos^2 \theta - \sin \theta = 1$   
 $\Rightarrow 5(1 - \sin^2 \theta) - \sin \theta = 1$   
 $\Rightarrow 5 - 5\sin^2 \theta - \sin \theta = 1$   
 $\Rightarrow 0 = 5\sin^2 \theta + \sin \theta - 4$   
 $\Rightarrow 0 = (\sin \theta + 4)(\sin \theta + 1)$   
 $\sin \theta = -\frac{1}{2}$   
 $\arcsin(-\frac{1}{2}) = -30^\circ$   
 $(\theta = -30^\circ + 360^\circ k, k=0,1,2,3, \dots)$   
 $\theta = 210^\circ, 330^\circ$

(c)  $5\sin^2 y = 19\cos y + 1$   
 $\Rightarrow 5(1 - \cos^2 y) = 19\cos y + 1$   
 $\Rightarrow 5 - 5\cos^2 y = 19\cos y + 1$   
 $\Rightarrow 0 = 5\cos^2 y + 19\cos y - 4$   
 $\Rightarrow 0 = (\cos y - 1)(\cos y + 4)$   
 $\cos y = -\frac{1}{4}$   
 $\arccos(-\frac{1}{4}) = 78.46^\circ$   
 $(y = 78.46^\circ + 360^\circ k, k=0,1,2,3, \dots)$   
 $y = 281.54^\circ, 218.54^\circ$

**Question 22**

Solve each of the following trigonometric equations in the range given.

a)  $3\sin^2 \theta = 8 + 16\cos \theta$        $0^\circ \leq \theta < 360^\circ$

$\theta \approx 109.5^\circ, 250.5^\circ$

b)  $3\cos^2 y + 11\sin y = 9$        $0^\circ \leq y < 360^\circ$

$y \approx 48.2^\circ, 138.2^\circ$

c)  $5\sin^2 x + 8\cos x = 8$        $0^\circ \leq x < 360^\circ$

$x = 0^\circ, x \approx 53.1^\circ, 306.9^\circ$

<p>(a) <math>3\sin^2 \theta = 8 + 16\cos \theta</math>  <math>\Rightarrow 3(1 - \cos^2 \theta) = 8 + 16\cos \theta</math>  <math>\Rightarrow 3 - 3\cos^2 \theta = 8 + 16\cos \theta</math>  <math>\Rightarrow 0 = 3\cos^2 \theta + 16\cos \theta + 5</math>  <math>\Rightarrow 0 = (3\cos \theta + 1)(\cos \theta + 5)</math>  <math>\Rightarrow \cos \theta = -\frac{1}{3}</math></p>	<p><math>\arccos(-\frac{1}{3}) = 109.47^\circ</math>  <math>\theta = 109.47^\circ \pm 360^\circ</math>  <math>\theta = 260.53^\circ \pm 360^\circ</math>  <math>\therefore \theta_1 = 109.5^\circ</math>  <math>\theta_2 = 250.5^\circ</math></p>
<p>(b) <math>3\cos^2 y + 11\sin y = 9</math>  <math>\Rightarrow 3(1 - \sin^2 y) + 11\sin y = 9</math>  <math>\Rightarrow 3 - 3\sin^2 y + 11\sin y = 9</math>  <math>\Rightarrow 0 = 3\sin^2 y - 11\sin y + 6</math>  <math>\Rightarrow (3\sin y - 2)(3\sin y - 3) = 0</math>  <math>\Rightarrow \sin y = \frac{2}{3}</math></p>	<p><math>\arcsin(\frac{2}{3}) = 41.17^\circ</math>  <math>y = 41.17^\circ \pm 360^\circ</math>  <math>y = 386.17^\circ \pm 360^\circ</math>  <math>\therefore y_1 = 41.17^\circ</math>  <math>y_2 = 386.17^\circ</math></p>
<p>(c) <math>5\sin^2 x + 8\cos x = 8</math>  <math>\Rightarrow 5(1 - \cos^2 x) + 8\cos x = 8</math>  <math>\Rightarrow 5 - 5\cos^2 x + 8\cos x = 8</math>  <math>\Rightarrow 0 = 5\cos^2 x - 8\cos x + 3</math>  <math>\Rightarrow (5\cos x - 3)(\cos x - 1) = 0</math>  <math>\Rightarrow \cos x = \frac{3}{5}</math></p>	<p><math>\arccos(\frac{3}{5}) = 53.13^\circ</math>  <math>\arccos(1) = 0^\circ</math>  <math>x = 53.13^\circ \pm 360^\circ</math>  <math>x = 386.87^\circ \pm 360^\circ</math>  <math>\therefore x_1 = 53.1^\circ</math>  <math>x_2 = 386.87^\circ</math>  <math>x_3 = 0^\circ</math></p>

**Question 23**

Solve each of the following trigonometric equations in the range given.

a)  $6\cos^2 x + \sin x = 4$        $0 \leq x < 360^\circ$

$x \approx 41.8^\circ, 138.2^\circ, x = 210^\circ, 330^\circ$

b)  $\cos^2 \theta = 7\sin^2 \theta - 2\sin \theta$        $0 \leq \theta < 360^\circ$

$\theta \approx 194.5^\circ, 345.5^\circ, \theta = 30^\circ, 150^\circ$

c)  $\sin y + 5 = 6\cos^2 y$        $0 \leq \theta < 360^\circ$

$y \approx 19.5^\circ, 160.5^\circ, y = 210^\circ, 330^\circ$

<p>(a) <math>6\cos^2 x + \sin x = 4</math>  <math>\Rightarrow 6(1-\sin^2 x) + \sin x = 4</math>  <math>\Rightarrow 6 - 6\sin^2 x + \sin x = 4</math>  <math>\Rightarrow 0 = 6\sin^2 x - \sin x - 2</math>  <math>\Rightarrow (2\sin x + 1)(3\sin x - 2) = 0</math>  <math>\Rightarrow \sin x = -\frac{1}{2}</math>  <math>\Rightarrow \sin x = \left\langle \begin{array}{l} -\frac{1}{2} \\ \frac{1}{2} \end{array} \right\rangle</math></p>	<p>• <math>\arcsin(-\frac{1}{2}) \approx -30^\circ</math>  <math>(x = 30^\circ \pm 360^\circ)</math>  <math>(x = 210^\circ \pm 360^\circ)</math>  <math>\therefore x_1 = 330^\circ</math>  <math>x_2 = 210^\circ</math>  <math>x_3 = 41.8^\circ</math>  <math>x_4 = 138.2^\circ</math></p>	<p>• <math>\arcsin(\frac{1}{2}) \approx 41.8^\circ</math>  <math>(x = 41.8^\circ \pm 360^\circ)</math>  <math>(x = 138.2^\circ \pm 360^\circ)</math>  <math>\therefore x_5 = 194.5^\circ</math>  <math>x_6 = 345.5^\circ</math></p>
<p>(b) <math>\cos^2 \theta = 7\sin^2 \theta - 2\sin \theta</math>  <math>\Rightarrow (1-\sin^2 \theta) = 7\sin^2 \theta - 2\sin \theta</math>  <math>\Rightarrow 0 = 8\sin^2 \theta - 2\sin \theta - 1</math>  <math>\Rightarrow (4\sin \theta + 1)(2\sin \theta - 1) = 0</math>  <math>\Rightarrow \sin \theta = \left\langle \begin{array}{l} -\frac{1}{4} \\ \frac{1}{2} \end{array} \right\rangle</math></p>	<p>• <math>\arcsin(\frac{1}{2}) \approx 30^\circ</math>  <math>(\theta = 30^\circ \pm 360^\circ)</math>  <math>(\theta = 150^\circ \pm 360^\circ)</math>  <math>\therefore \theta_1 = 30^\circ</math>  <math>\theta_2 = 150^\circ</math>  <math>\theta_3 = 345.5^\circ</math>  <math>\theta_4 = 194.5^\circ</math></p>	<p>• <math>\arcsin(-\frac{1}{4}) \approx -14.46^\circ</math>  <math>(\theta = -14.46^\circ \pm 360^\circ)</math>  <math>(\theta = 345.5^\circ \pm 360^\circ)</math>  <math>\therefore \theta_5 = 345.5^\circ</math>  <math>\theta_6 = 194.5^\circ</math></p>
<p>(c) <math>\sin y + 5 = 6\cos^2 y</math>  <math>\Rightarrow \sin y + 5 = 6(1 - \sin^2 y)</math>  <math>\Rightarrow \sin y + 5 = 6 - 6\sin^2 y</math>  <math>\Rightarrow 6\sin^2 y + \sin y - 1 = 0</math>  <math>\Rightarrow (3\sin y - 1)(2\sin y + 1) = 0</math>  <math>\Rightarrow \sin y = \left\langle \begin{array}{l} -\frac{1}{2} \\ \frac{1}{3} \end{array} \right\rangle</math></p>	<p>• <math>\arcsin(-\frac{1}{2}) \approx -30^\circ</math>  <math>(y = -30^\circ \pm 360^\circ)</math>  <math>(y = 210^\circ \pm 360^\circ)</math>  <math>\therefore y_1 = 330^\circ</math>  <math>y_2 = 210^\circ</math>  <math>y_3 = 19.5^\circ</math>  <math>y_4 = 160.5^\circ</math></p>	<p>• <math>\arcsin(\frac{1}{3}) \approx 19.47^\circ</math>  <math>(y = 19.47^\circ \pm 360^\circ)</math>  <math>(y = 180.5^\circ \pm 360^\circ)</math>  <math>\therefore y_5 = 320^\circ</math>  <math>y_6 = 210^\circ</math>  <math>y_7 = 19.5^\circ</math>  <math>y_8 = 160.5^\circ</math></p>

**Question 24**

Solve each of the following trigonometric equations in the range given.

a)  $4\cos^2 \theta + 9\sin \theta - 6 = 0$        $0 \leq \theta < 360^\circ$

$\theta \approx 14.5^\circ, 165.5^\circ$

b)  $8\cos^2 x - 6\sin x - 9 = 0$        $0 \leq x < 360^\circ$

$x \approx 194.5^\circ, 345.5^\circ, x = 210^\circ, 330^\circ$

c)  $6\sin^2 y + \cos y - 5 = 0$        $0 \leq y < 360^\circ$

$y \approx 109.5^\circ, 250.5^\circ, y = 60^\circ, 300^\circ$

<p>(a) <math>4\cos^2 \theta + 9\sin \theta - 6 = 0</math>  <math>\Rightarrow 4(1-\sin^2 \theta) + 9\sin \theta - 6 = 0</math>  <math>\Rightarrow 4 - 4\sin^2 \theta + 9\sin \theta - 6 = 0</math>  <math>\Rightarrow 4\sin^2 \theta - 9\sin \theta + 2 = 0</math>  <math>\Rightarrow 0 = (3\sin \theta - 2)(3\sin \theta - 1)</math>  <math>\Rightarrow \sin \theta = \frac{2}{3} \quad \text{or} \quad \sin \theta = \frac{1}{3}</math></p>	<p><math>\bullet \arcsin(\frac{2}{3}) = 42.4^\circ</math>  <math>\bullet \arcsin(\frac{1}{3}) = 19.5^\circ</math></p>
<p><math>\Rightarrow \theta = 42.4^\circ + 360^\circ n</math>  <math>\Rightarrow \theta = 19.5^\circ + 360^\circ n</math>  <math>\Rightarrow \theta_1 = 42.4^\circ</math>  <math>\Rightarrow \theta_2 = 19.5^\circ</math></p>	<p><math>\bullet \arcsin(\frac{1}{2}) = 14.5^\circ</math>  <math>\bullet \arcsin(-\frac{1}{2}) = -14.5^\circ</math></p>
<p><math>\Rightarrow \theta = 14.5^\circ + 360^\circ n</math>  <math>\Rightarrow \theta = 210^\circ + 360^\circ n</math>  <math>\Rightarrow \theta_3 = 14.5^\circ</math>  <math>\Rightarrow \theta_4 = 210^\circ</math></p>	<p><math>\bullet \arcsin(\frac{1}{2}) = 60^\circ</math>  <math>\bullet \arcsin(-\frac{1}{2}) = 120^\circ</math></p>
<p><math>\Rightarrow \theta = 60^\circ + 360^\circ n</math>  <math>\Rightarrow \theta = 240^\circ + 360^\circ n</math>  <math>\Rightarrow \theta_5 = 60^\circ</math>  <math>\Rightarrow \theta_6 = 240^\circ</math></p>	<p><math>\bullet \arcsin(\frac{1}{2}) = 109.5^\circ</math>  <math>\bullet \arcsin(-\frac{1}{2}) = 250.5^\circ</math></p>
<p><math>\Rightarrow \theta = 109.5^\circ + 360^\circ n</math>  <math>\Rightarrow \theta = 250.5^\circ + 360^\circ n</math>  <math>\Rightarrow \theta_7 = 109.5^\circ</math>  <math>\Rightarrow \theta_8 = 250.5^\circ</math></p>	<p><math>\bullet \arcsin(\frac{1}{2}) = 30^\circ</math>  <math>\bullet \arcsin(-\frac{1}{2}) = 210^\circ</math></p>
<p><math>\Rightarrow \theta = 30^\circ + 360^\circ n</math>  <math>\Rightarrow \theta = 210^\circ + 360^\circ n</math>  <math>\Rightarrow \theta_9 = 30^\circ</math>  <math>\Rightarrow \theta_{10} = 210^\circ</math></p>	<p><math>\bullet \arcsin(\frac{1}{2}) = 60^\circ</math>  <math>\bullet \arcsin(-\frac{1}{2}) = 120^\circ</math></p>

**Question 25**

Solve each of the following trigonometric equations in the range given.

a)  $4\cos^2 \theta - \cos \theta = 2\sin^2 \theta$

$0 \leq \theta < 360^\circ$

$\theta \approx 48.2^\circ, 311.8^\circ, \theta = 120^\circ, 240^\circ$

b)  $3\cos^2 y - \sin^2 y - 6\cos y = 3$

$0 \leq y < 360^\circ$

$y = 120^\circ, 240^\circ$

c)  $10\sin^2 x - 3\sin x = 4\cos^2 x + 1$

$0 \leq x < 360^\circ$

$x \approx 45.6^\circ, 134.4^\circ, x = 210^\circ, 330^\circ$

$  \begin{aligned}  \text{(a)} \quad & 4\cos^2 \theta - \cos \theta = 2\sin^2 \theta \\  \rightarrow & 4\cos^2 \theta - \cos \theta = 2(1 - \sin^2 \theta) \\  \rightarrow & 4\cos^2 \theta - \cos \theta = 2 - 2\sin^2 \theta \\  \rightarrow & 6\cos^2 \theta - \cos \theta - 2 = 0 \\  \rightarrow & (2\cos \theta + 1)(3\cos \theta - 2) = 0 \\  \Rightarrow & \cos \theta = -\frac{1}{2} \quad \text{or} \quad \cos \theta = \frac{2}{3} \\  \bullet \arccos(-\frac{1}{2}) = 240^\circ & \quad \bullet \arccos(\frac{2}{3}) = 48.2^\circ \\  (\theta = 120^\circ \pm 360^\circ) & \quad (\theta = 48.2^\circ \pm 360^\circ) \\  (\theta = 210^\circ \pm 360^\circ) & \quad (\theta = 311.8^\circ \pm 360^\circ) \\  (\theta = 48.2^\circ, 134.4^\circ, 210^\circ, 330^\circ) &  \end{aligned}  $	$  \begin{aligned}  \text{(b)} \quad & 3\cos^2 y - \sin^2 y - 6\cos y - 3 = 0 \\  \rightarrow & 3\cos^2 y - (1 - \sin^2 y) - 6\cos y - 3 \\  \rightarrow & 3\cos^2 y + \sin^2 y - 6\cos y - 3 = 0 \\  \rightarrow & 4\cos^2 y - 6\cos y - 3 = 0 \\  \rightarrow & 2\cos y - 3\cos y - 2 = 0 \\  \rightarrow & (2\cos y + 1)(\cos y - 2) = 0 \\  \cos y < -\frac{1}{2} & \quad \cos y = 2 \\  \bullet \arccos(-\frac{1}{2}) = 120^\circ & \quad \bullet \arccos(2) \text{ is undefined} \\  (\theta = 120^\circ \pm 360^\circ) & \quad (\theta = 240^\circ \pm 360^\circ) \\  (\theta = 120^\circ, 240^\circ) & \quad (\theta = 240^\circ, 360^\circ)  \end{aligned}  $	$  \begin{aligned}  \text{(c)} \quad & 10\sin^2 x - 3\sin x = 4\cos^2 x + 1 \\  \rightarrow & 10\sin^2 x - 3\sin x = 4(1 - \sin^2 x) + 1 \\  \rightarrow & 10\sin^2 x - 3\sin x = 4 - 4\sin^2 x + 1 \\  \rightarrow & 14\sin^2 x - 3\sin x - 5 = 0 \\  \rightarrow & (7\sin x - 5)(2\sin x + 1) = 0 \\  \Rightarrow & \sin x = -\frac{1}{2} \quad \text{or} \quad \sin x = \frac{5}{7} \\  \bullet \arcsin(-\frac{1}{2}) = -30^\circ & \quad \bullet \arcsin(\frac{5}{7}) = 45.6^\circ \\  (\theta = -30^\circ \pm 360^\circ) & \quad (\theta = 45.6^\circ \pm 360^\circ) \\  (\theta = 210^\circ \pm 360^\circ) & \quad (\theta = 134.4^\circ \pm 360^\circ) \\  (\theta = 45.6^\circ, 134.4^\circ, 210^\circ, 330^\circ) &  \end{aligned}  $
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**Question 26**

Solve each of the following trigonometric equations in the range given.

a)  $2\sin^2 x - 2\cos x - \cos^2 x = 1$        $0 \leq x < 360^\circ$        $x \approx 70.5^\circ, 289.5^\circ, x = 180^\circ$

b)  $4\sin^2 \theta - \cos^2 \theta = 8\sin \theta + 3$        $0 \leq \theta < 360^\circ$        $\theta \approx 203.6^\circ, 336.4^\circ$

c)  $3\cos^2 2\phi - 4\sin^2 2\phi = 15\cos 2\phi - 6$        $0 \leq \phi < 360^\circ$        $\phi \approx 40.9^\circ, 139.1^\circ, 220.9^\circ, 319.1^\circ$

a)  $2\sin^2 x - 2\cos x - \cos^2 x = 1$   
 $\Rightarrow 2(1-\cos^2 x) - 2\cos x - \cos^2 x = 1$   
 $\Rightarrow 2 - 4\cos^2 x - 2\cos x - \cos^2 x = 1$   
 $\Rightarrow 0 = 3\cos^2 x + 2\cos x - 1$   
 $\Rightarrow 0 = (3\cos x - 1)(\cos x + 1)$   
 $\cos x = \begin{cases} -1 \\ \frac{1}{3} \end{cases}$   
 $\arccos(-1) = 180^\circ$        $\arccos(\frac{1}{3}) = 70.55^\circ$   
 $(\theta = 180^\circ \pm 360n)$        $(\theta = 70.55^\circ \pm 360n)$   
 $n=0,1,2,3,\dots$   
 $\theta_1 = 180^\circ$   
 $\theta_2 = 70.55^\circ$   
 $\theta_3 = 289.5^\circ$

b)  $4\sin^2 \theta - \cos^2 \theta = 8\sin \theta + 3$   
 $= 4\sin^2 \theta - (1-\sin^2 \theta) = 8\sin \theta + 3$   
 $\Rightarrow 4\sin^2 \theta - 1 + \sin^2 \theta = 8\sin \theta + 3$   
 $\Rightarrow 5\sin^2 \theta - 8\sin \theta - 4 = 0$   
 $\Rightarrow (5\sin \theta + 2)(\sin \theta - 2) = 0$   
 $\sin \theta = \begin{cases} -\frac{2}{5} \\ 2 \end{cases}$   
 $\arcsin(-\frac{2}{5}) = -23.56^\circ$   
 $(\theta = -23.56^\circ \pm 360n)$        $(\theta = 203.56^\circ \pm 360n)$   
 $n=0,1,2,3,\dots$   
 $\theta_1 = -23.56^\circ$   
 $\theta_2 = 203.56^\circ$

c)  $3\cos^2 2\phi - 4\sin^2 2\phi = 15\cos 2\phi - 6$ ,  $0 \leq \phi < 360^\circ$   
 $\Rightarrow 3\cos^2 2\phi - 4(1-\cos^2 2\phi) = 15\cos 2\phi - 6$   
 $\Rightarrow 3\cos^2 2\phi - 4 + 4\cos^2 2\phi = 15\cos 2\phi - 6$   
 $\Rightarrow 7\cos^2 2\phi - 15\cos 2\phi + 2 = 0$   
 $\Rightarrow (7\cos 2\phi - 1)(\cos 2\phi - 2) = 0$   
 $\cos 2\phi = \begin{cases} \frac{1}{7} \\ 2 \end{cases}$   
 $\arccos(\frac{1}{7}) = 80.7867^\circ$   
 $(\phi = 80.7867^\circ \pm 360n)$        $(\phi = 218.213^\circ \pm 360n)$   
 $n=0,1,2,3,\dots$   
 $\phi \approx 40.89^\circ \pm 180n$   
 $\phi \approx 139.11^\circ \pm 180n$   
 $\therefore \phi = 40.89^\circ, 220.89^\circ, 139.11^\circ, 319.11^\circ$

**Question 27**

Solve each of the following trigonometric equations in the range given.

a)  $12\sin^2 \beta - 5\cos \beta = 9$

$0 \leq \beta < 360^\circ$

$\beta \approx 70.5^\circ, 138.6^\circ, 221.4^\circ, 289.5^\circ$

b)  $5\cos^2 \psi + 2 = 3\sin^2 \psi - 2\cos \psi$

$0 \leq \psi < 360^\circ$

$\psi \approx 75.5^\circ, 284.5^\circ, \psi = 120^\circ, 240^\circ$

c)  $3 - 7\cos 3\theta = 6\sin^2 3\theta$

$0 \leq \theta < 180^\circ$

$\theta \approx 36.5^\circ, 83.5^\circ, 156.5^\circ$

a)  $12\sin^2 \beta - 5\cos \beta = 9$   
 $\Rightarrow 12(1-\cos^2 \beta) - 5\cos \beta = 9$   
 $\Rightarrow 12 - 12\cos^2 \beta - 5\cos \beta = 9$   
 $\Rightarrow 12 = 12\cos^2 \beta + 5\cos \beta - 3$   
 $\Rightarrow 0 = (4\cos \beta + 3)(3\cos \beta - 1)$   
 $\Rightarrow \cos \beta = -\frac{3}{4}$   
 $\arccos\left(-\frac{3}{4}\right) = 138.6^\circ$   
 $\beta_1 = 138.6^\circ \pm 360^\circ$   
 $\beta_2 = 221.4^\circ \pm 360^\circ$   
 $\beta_3 = 70.5^\circ$   
 $\beta_4 = 284.5^\circ$

b)  $5\cos^2 \psi + 2 = 3\sin^2 \psi - 2\cos \psi$   
 $\Rightarrow 5\cos^2 \psi + 2 = 3(1-\cos^2 \psi) - 2\cos \psi$   
 $\Rightarrow 5\cos^2 \psi + 2 = 3 - 3\cos^2 \psi - 2\cos \psi$   
 $\Rightarrow 8\cos^2 \psi + 2\cos \psi - 1 = 0$   
 $\Rightarrow (4\cos \psi - 1)(2\cos \psi + 1) = 0$   
 $\cos \psi = \frac{1}{4}$   
 $\arccos\left(\frac{1}{4}\right) = 70.55^\circ$   
 $\psi_1 = 70.55^\circ \pm 360^\circ$   
 $\psi_2 = 209.55^\circ \pm 360^\circ$   
 $\psi_3 = 75.5^\circ$   
 $\psi_4 = 284.5^\circ \pm 360^\circ$

c)  $3 - 7\cos 3\theta = 6\sin^2 3\theta$   
 $\Rightarrow 3 - 7\cos 3\theta = 6(1 - \cos^2 3\theta)$   
 $\Rightarrow 3 - 7\cos 3\theta = 6 - 6\cos^2 3\theta$   
 $\Rightarrow 6\cos^2 3\theta - 7\cos 3\theta - 3 = 0$   
 $\Rightarrow (2\cos 3\theta - 3)(3\cos 3\theta + 1) = 0$   
 $\cos 3\theta = \frac{3}{2}$   
 $\arccos\left(-\frac{1}{2}\right) = 120^\circ$   
 $\theta_1 = 36.5^\circ$   
 $\theta_2 = 156.5^\circ$   
 $\theta_3 = 83.5^\circ$

**Question 28**

Solve each of the following trigonometric equations in the range given.

a)  $12\cos^2 \theta - 5\sin \theta = 10$

$0 \leq \theta < 360^\circ$

$\theta \approx 14.5^\circ, 165.5^\circ, 221.8^\circ, 318.2^\circ$

b)  $6\sin^2 \theta + 19\cos \theta = 16$

$0 \leq \theta < 360^\circ$

$\theta \approx 48.2^\circ, 311.8^\circ$

c)  $\cos^2 x - 9\sin^2 x = \sin x - 1$

$0 \leq x < 360^\circ$

$x \approx 23.6^\circ, 156.4^\circ, x = 210^\circ, 330^\circ$

(a)  $12\cos^2 \theta - 5\sin \theta = 10$   
 $\rightarrow 12(1 - \sin^2 \theta) - 5\sin \theta = 10$   
 $\rightarrow 12 - 12\sin^2 \theta - 5\sin \theta = 10$   
 $\rightarrow 0 = 12\sin^2 \theta + 5\sin \theta - 2$   
 $\Rightarrow 0 = (3\sin \theta + 2)(4\sin \theta - 1)$   
 $\therefore \sin \theta = \begin{cases} -\frac{2}{3} \\ \frac{1}{4} \end{cases}$

$\bullet \cos(\frac{\pi}{3}) = \frac{1}{2}$   
 $\cos(\frac{2\pi}{3}) = -\frac{1}{2}$   
 $\left\{ \begin{array}{l} \theta = 45^\circ \pm 360^\circ \\ \theta = 165.5^\circ \pm 360^\circ \\ \theta = 14.5^\circ \pm 360^\circ \end{array} \right.$

(b)  $6\sin^2 \theta + 19\cos \theta = 16$   
 $\rightarrow 6(-\cos^2 \theta) + 19\cos \theta = 16$   
 $\rightarrow 6 - 6\cos^2 \theta + 19\cos \theta = 16$   
 $\Rightarrow 0 = 6\cos^2 \theta - 19\cos \theta + 10$   
 $\Rightarrow 0 = (3\cos \theta - 2)(2\cos \theta - 5)$   
 $\therefore \cos \theta = \begin{cases} \frac{2}{3} \\ \frac{5}{2} \end{cases}$

$\cos(\frac{\pi}{3}) = \frac{1}{2}$   
 $\cos(\frac{2\pi}{3}) = -\frac{1}{2}$   
 $\left\{ \begin{array}{l} \theta = 45^\circ \pm 360^\circ \\ \theta = 315^\circ \pm 360^\circ \end{array} \right.$

$\therefore \theta = 48.2^\circ, 311.8^\circ$

$\cos^2 x - 9\sin^2 x = \sin x - 1$   
 $\rightarrow (-\sin x)^2 - 9\sin^2 x = \sin x - 1$   
 $\rightarrow 1 - 10\sin^2 x = \sin x - 1$   
 $\rightarrow 0 = 10\sin^2 x + \sin x - 2$   
 $\Rightarrow 0 = (5\sin x - 2)(2\sin x + 1)$   
 $\therefore \sin x = \begin{cases} \frac{2}{5} \\ -\frac{1}{2} \end{cases}$

$\bullet \cos(\frac{\pi}{3}) = \frac{1}{2}$   
 $\cos(\frac{2\pi}{3}) = -\frac{1}{2}$   
 $\left\{ \begin{array}{l} x = 23.6^\circ \pm 360^\circ \\ x = 156.4^\circ \pm 360^\circ \\ x = 330^\circ, 210^\circ, 23.6^\circ, 156.4^\circ \end{array} \right.$

**Question 29**

Solve each of the following trigonometric equations in the range given.

a)  $4\sin^2 \psi = 15\cos \psi$

$0 \leq \psi < 2\pi$

$\psi \approx 1.32^\circ, 4.97^\circ$

b)  $6\cos^2 x + \sin x = 4$

$0 \leq x < 2\pi$

$x \approx 0.73^\circ, 2.41^\circ, 3.67^\circ, 5.76^\circ$

c)  $4\sin^2 2\varphi - \cos^2 2\varphi = 3 + 8\sin 2\varphi$

$0 \leq \varphi < 2\pi$

$\varphi \approx 1.78^\circ, 2.94^\circ, 4.92^\circ, 6.08^\circ$

a)  $4\sin^2 \psi = 15\cos \psi, \quad 0 \leq \psi < 2\pi$

$$\begin{aligned} 4\sin^2 \psi &= 15\cos \psi \\ \Rightarrow 4(1 - \cos^2 \psi) &= 15\cos \psi \\ \Rightarrow 4 - 4\cos^2 \psi &= 15\cos \psi \\ \Rightarrow 0 &= 4\cos^2 \psi + 15\cos \psi - 4 \\ \Rightarrow 0 &= (4\cos \psi - 1)(\cos \psi + 4) \\ \Rightarrow \cos \psi &= \end{aligned}$$

$\arccos\left(\frac{1}{4}\right) = 1.318^\circ$

$\psi = 1.318^\circ \pm 2\pi n, \quad n = 0, 1, 2, \dots$

$\psi_1 = 1.32^\circ$

$\psi_2 = 4.97^\circ$

b)  $6\cos^2 x + \sin x = 4, \quad 0 \leq x < 2\pi$

$$\begin{aligned} 6\cos^2 x + \sin x &= 4 \\ \Rightarrow 6(1 - \sin^2 x) + \sin x &= 4 \\ \Rightarrow 6 - 6\sin^2 x + \sin x &= 4 \\ \Rightarrow 0 &= 6\sin^2 x - \sin x - 2 \\ \Rightarrow (2\sin x + 1)(3\sin x - 2) &= 0 \\ \sin x &= \end{aligned}$$

$$\begin{aligned} \bullet \arcsin\left(-\frac{1}{2}\right) &= -\frac{\pi}{6} & \bullet \arcsin\left(\frac{2}{3}\right) &= 0.727^\circ \\ (x = -\frac{\pi}{6} \pm 2\pi n) & & (x = 0.727^\circ \pm 2\pi n) & \\ (x = \frac{11\pi}{6} \pm 2\pi n) & & (x = 2.419^\circ \pm 2\pi n) & \\ n = 0, 1, 2, \dots & & n = 0, 1, 2, \dots & \end{aligned}$$

$x_1 = \frac{11\pi}{6} \approx -5.76^\circ$

$x_2 = \frac{11\pi}{6} \approx 3.67^\circ$

$x_3 \approx 0.73^\circ$

$x_4 \approx 2.41^\circ$

c)  $4\sin^2 2\varphi - \cos^2 2\varphi = 3 + 8\sin 2\varphi, \quad 0 \leq \varphi < 2\pi$

$$\begin{aligned} 4\sin^2 2\varphi - \cos^2 2\varphi &= 3 + 8\sin 2\varphi + 3 \\ \Rightarrow 4\sin^2 2\varphi - (1 - \sin^2 2\varphi) &= 3 + 8\sin 2\varphi + 3 \\ \Rightarrow 4\sin^2 2\varphi - 1 + \sin^2 2\varphi &= 3 + 8\sin 2\varphi + 3 \\ \Rightarrow 5\sin^2 2\varphi - 8\sin 2\varphi - 4 &= 0 \\ \Rightarrow (5\sin 2\varphi + 2)(\sin 2\varphi - 2) &= 0 \\ \Rightarrow \sin 2\varphi &= \end{aligned}$$

$\arcsin\left(-\frac{2}{5}\right) = -0.4115^\circ$

$(x = -0.4115^\circ \pm 2\pi n, \quad n = 0, 1, 2, \dots)$

$(x = 3.588^\circ \pm 2\pi n)$

$(\varphi = -0.209^\circ \pm \pi n)$

$(\varphi = 1.790^\circ \pm \pi n)$

$\varphi_1 \approx 2.94^\circ$

$\varphi_2 \approx 6.08^\circ$

$\varphi_3 \approx 1.78^\circ$

$\varphi_4 \approx 4.92^\circ$

**Question 30**

Solve each of the following trigonometric equations in the range given.

a)  $3 - 7\cos\theta = 6\sin^2\theta$

$0 \leq \theta < 2\pi$

$\theta \approx 1.91^\circ, 4.37^\circ$

b)  $4\cos^2 x - \cos x = 2\sin^2 x$

$0 \leq x < 2\pi$

$x \approx 0.84^\circ, 2.09^\circ, 4.19^\circ, 5.44^\circ$

c)  $5\cos^2 y + 2 = 3\sin^2 y - 2\cos y$

$0 \leq y < 2\pi$

$y \approx 1.32^\circ, 2.09^\circ, 4.19^\circ, 4.19^\circ$

a)  $3 - 7\cos\theta = 6\sin^2\theta$   
 $\Rightarrow 3 - 7\cos\theta = 6(1 - \cos^2\theta)$   
 $\Rightarrow 3 - 7\cos\theta = 6 - 6\cos^2\theta$   
 $\Rightarrow 6\cos^2\theta - 7\cos\theta - 3 = 0$   
 $\Rightarrow (2\cos\theta - 3)(3\cos\theta + 1) = 0$   
 $\Rightarrow \cos\theta = \frac{3}{2} \text{ or } -\frac{1}{3}$   
 $\bullet \arccos\left(-\frac{1}{3}\right) = 1.91^\circ, \dots$   
 $\begin{cases} \theta = 1.91^\circ \pm 2m \\ \theta = 4.37^\circ \pm 2m \end{cases} \quad m=0,1,2,3, \dots$   
 $\theta_1 \approx 1.91^\circ$   
 $\theta_2 \approx 4.37^\circ$

b)  $4\cos^2 x - \cos x = 2\sin^2 x$   
 $\Rightarrow 4\cos^2 x - \cos x = 2(1 - \cos^2 x)$   
 $\Rightarrow 4\cos^2 x - \cos x = 2 - 2\cos^2 x$   
 $\Rightarrow 6\cos^2 x - \cos x - 2 = 0$   
 $\Rightarrow (3\cos x - 2)(2\cos x + 1) = 0$   
 $\cos x = \frac{2}{3} \text{ or } -\frac{1}{2}$   
 $\bullet \arccos\left(\frac{2}{3}\right) = 0.84^\circ, \dots$   
 $\begin{cases} x = \frac{\pi}{3} \pm 2m \\ x = \frac{4\pi}{3} \pm 2m \end{cases} \quad m=0,1,2,3, \dots$   
 $x_1 \approx 0.84^\circ$   
 $x_2 \approx 4.19^\circ$   
 $x_3 \approx 0.84^\circ$   
 $x_4 \approx 5.44^\circ$

c)  $5\cos^2 y + 2 = 3\sin^2 y - 2\cos y$   
 $\Rightarrow 5\cos^2 y + 2 = 3(1 - \cos^2 y) - 2\cos y$   
 $\Rightarrow 5\cos^2 y + 2 = 3 - 3\cos^2 y - 2\cos y$   
 $\Rightarrow 8\cos^2 y + 2\cos y - 1 = 0$   
 $\Rightarrow (4\cos y - 1)(2\cos y + 1) = 0$   
 $\cos y = -\frac{1}{2}$   
 $\bullet \arccos\left(-\frac{1}{2}\right) = 1.80^\circ, \dots$   
 $\begin{cases} y = 1.80^\circ \pm 2m \\ y = 4.965^\circ \pm 2m \end{cases} \quad m=0,1,2,3, \dots$   
 $y_1 \approx 1.32^\circ$   
 $y_2 \approx 4.37^\circ$   
 $y_3 \approx 2.09^\circ$   
 $y_4 \approx 4.19^\circ$

**Question 31**

Solve each of the following trigonometric equations in the range given.

a)  $2\cos^2 x - \sin x = 1$

$$0 \leq x < 2\pi$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

b)  $2\sin^2 \theta + 1 = 5\cos \theta$

$$0 \leq \theta < 2\pi$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

c)  $3\cos^2 2y - \sin^2 2y - 6\cos 2y = 3$

$$0 \leq y < 2\pi$$

$$y = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

(a)  $2\cos^2 x - \sin x = 1$

$$\Rightarrow 2(1 - \sin^2 x) - \sin x = 1$$

$$\Rightarrow 2 - 2\sin^2 x - \sin x = 1$$

$$\Rightarrow 0 = 2\sin^2 x + \sin x - 1$$

$$\Rightarrow 0 = (2\sin x - 1)(\sin x + 1)$$

$$\sin x = \begin{cases} \frac{1}{2} \\ -1 \end{cases}$$

$$\arcsin(-1) = -\frac{\pi}{2}$$

$$\left( \begin{array}{l} x = -\frac{\pi}{2} + 2m\pi \\ x = \frac{3\pi}{2} + 2m\pi \end{array} \right)$$

$$\text{or } \arcsin(\frac{1}{2}) = \frac{\pi}{6}$$

$$\left( \begin{array}{l} x = \frac{\pi}{6} + 2m\pi \\ x = \frac{5\pi}{6} + 2m\pi \end{array} \right)$$

$$n = 0, 1, 2, \dots$$

$$x_1 = \frac{\pi}{6}$$

$$x_2 = \frac{5\pi}{6}$$

$$x_3 = -\frac{\pi}{2}$$

$$x_4 = \frac{3\pi}{2}$$

(b)  $2\sin^2 \theta + 1 = 5\cos \theta$

$$\Rightarrow 2(1 - \cos^2 \theta) + 1 = 5\cos \theta$$

$$\Rightarrow 2 - 2\cos^2 \theta + 1 = 5\cos \theta$$

$$\Rightarrow 0 = 2\cos^2 \theta + 5\cos \theta - 3$$

$$\Rightarrow 0 = (2\cos \theta - 1)(5\cos \theta + 3)$$

$$\cos \theta = \begin{cases} \frac{1}{2} \\ -\frac{3}{5} \end{cases}$$

$$\arccos(-\frac{3}{5}) = \frac{4\pi}{3}$$

$$\left( \begin{array}{l} \theta = \frac{\pi}{3} + 2m\pi \\ \theta = \frac{4\pi}{3} + 2m\pi \end{array} \right)$$

$$n = 0, 1, 2, \dots$$

$$\theta_1 = \frac{\pi}{3}$$

$$\theta_2 = \frac{4\pi}{3}$$

(c)  $3\cos^2 2y - \sin^2 2y - 6\cos 2y = 3$

$$\Rightarrow 3\cos^2 2y - (1 - \sin^2 2y) - 6\cos 2y = 3$$

$$\Rightarrow 3\cos^2 2y - 1 + \sin^2 2y - 6\cos 2y = 3$$

$$\Rightarrow 4\cos^2 2y - 6\cos 2y - 4 = 0$$

$$\Rightarrow 2\cos^2 2y - 3\cos 2y - 2 = 0$$

$$\Rightarrow (2\cos 2y + 1)(\cos 2y - 2) = 0$$

$$\cos 2y = \begin{cases} -\frac{1}{2} \\ 2 \end{cases}$$

$$\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$$

$$\left( \begin{array}{l} 2y = \frac{2\pi}{3} + 2m\pi \\ 2y = \frac{4\pi}{3} + 2m\pi \end{array} \right)$$

$$\left( \begin{array}{l} y = \frac{\pi}{3} + m\pi \\ y = \frac{2\pi}{3} + m\pi \end{array} \right)$$

$$\therefore y = \frac{\pi}{3} + \frac{m\pi}{2}$$

**Question 32**

Solve each of the following trigonometric equations in the range given.

a)  $12\sin^2 x - 5\cos x = 9$

$0 \leq x < 2\pi$

$x \approx 1.23^\circ, 2.42^\circ, 3.86^\circ, 5.05^\circ$

b)  $10\sin^2 \theta - 3\sin \theta = 4\cos^2 \theta + 1$

$0 \leq \theta < 2\pi$

$\theta \approx 0.52^\circ, 0.80^\circ, 2.62^\circ, 2.36^\circ$

c)  $\cos^2 \varphi = 7\sin^2 \varphi - 2\sin \varphi$

$0 \leq y < 2\pi$

$\varphi \approx 0.52^\circ, 2.62^\circ, 3.39^\circ, 6.03^\circ$

a)  $12\sin^2 x - 5\cos x = 9, \quad 0 \leq x < 2\pi$

$$\begin{aligned} &\Rightarrow 12\sin^2 x - 5\cos x = 9 \\ &\Rightarrow 12(1-\cos^2 x) - 5\cos x = 9 \\ &\Rightarrow 12 - 12\cos^2 x - 5\cos x = 9 \\ &\Rightarrow 0 = 12\cos^2 x + 5\cos x - 3 \\ &\Rightarrow 0 = (3\cos x - 1)(2\cos x + 3) \\ &\cos x = \frac{-1}{2} \\ &\cos x = \frac{1}{3} \\ &\arccos(\frac{1}{3}) = 72.5^\circ \quad \left\{ \begin{array}{l} \cos x = -\frac{1}{2} \\ \arccos(-\frac{1}{2}) = 120^\circ \end{array} \right. \\ &x = 72.5^\circ + 2n\pi \quad \left\{ \begin{array}{l} x = 2.419^\circ + 2n\pi \\ x = 3.864^\circ + 2n\pi \end{array} \right. \\ &n=0,1,2,3, \dots \\ &x = 1.23^\circ, 5.05^\circ, 2.42^\circ, 3.86^\circ \end{aligned}$$

b)  $10\sin^2 \theta - 3\sin \theta = 4\cos^2 \theta + 1, \quad 0 \leq \theta < 2\pi$

$$\begin{aligned} &\Rightarrow 10\sin^2 \theta - 3\sin \theta = 4\cos^2 \theta + 1 \\ &\Rightarrow 10\sin^2 \theta - 3\sin \theta = 4(1 - \sin^2 \theta) + 1 \\ &\Rightarrow 10\sin^2 \theta - 3\sin \theta = 4 - 4\sin^2 \theta + 1 \\ &\Rightarrow 14\sin^2 \theta - 3\sin \theta - 5 = 0 \\ &\Rightarrow (7\sin \theta - 5)(2\sin \theta + 1) = 0 \\ &\sin \theta = \frac{5}{7} \\ &\sin \theta = \frac{1}{2} \\ &\arcsin(\frac{1}{2}) = 30^\circ \quad \left\{ \begin{array}{l} \sin \theta = \frac{5}{7} \\ \arcsin(\frac{5}{7}) = 54.6^\circ \end{array} \right. \\ &\theta = 30^\circ \pm 2n\pi \quad \left\{ \begin{array}{l} \theta = 0.52^\circ \pm 2n\pi \\ \theta = 2.62^\circ \pm 2n\pi \end{array} \right. \\ &n=0,1,2,3, \dots \\ &\theta_1 = 30^\circ \quad \theta_2 = 150^\circ \\ &\theta_3 = 54.6^\circ \quad \theta_4 = 2.36^\circ \end{aligned}$$

c)  $\cos^2 \varphi = 7\sin^2 \varphi - 2\sin \varphi, \quad 0 < \varphi < 2\pi$

$$\begin{aligned} &\Rightarrow \cos^2 \varphi = 7\sin^2 \varphi - 2\sin \varphi \\ &\Rightarrow (1 - \sin^2 \varphi) = 7\sin^2 \varphi - 2\sin \varphi \\ &\Rightarrow (-\sin^2 \varphi) = 7\sin^2 \varphi - 2\sin \varphi \\ &\Rightarrow 0 = 8\sin^2 \varphi - 2\sin \varphi - 1 \\ &\Rightarrow 0 = (4\sin \varphi + 1)(2\sin \varphi - 1) \\ &\sin \varphi = -\frac{1}{4} \\ &\sin \varphi = \frac{1}{2} \\ &\arcsin(\frac{1}{2}) = 30^\circ \quad \left\{ \begin{array}{l} \sin \varphi = -\frac{1}{4} \\ \arcsin(-\frac{1}{4}) = -0.253^\circ \end{array} \right. \\ &\varphi = 30^\circ \pm 2n\pi \quad \left\{ \begin{array}{l} \varphi = -0.253^\circ \pm 2n\pi \\ \varphi = 3.39^\circ \pm 2n\pi \end{array} \right. \\ &n=0,1,2,3, \dots \\ &\varphi_1 = 0.52^\circ \quad \varphi_2 = 2.62^\circ \\ &\varphi_3 = 6.03^\circ \quad \varphi_4 = 3.39^\circ \end{aligned}$$

**Question 33**

Solve each of the following trigonometric equations in the range given.

a)  $4 \tan^2 \theta \cos \theta = 15$        $0 \leq \theta < 360^\circ$

$\theta \approx 75.5^\circ, 284.5^\circ$

b)  $\frac{3 + \sin^2 \theta}{\cos \theta - 2} = 3 \cos \theta$        $0 \leq \theta < 360^\circ$

$\theta = 120^\circ, 240^\circ$

c)  $2 \sin x = \tan x$        $0 \leq x < 360^\circ$

$x = 0^\circ, 60^\circ, 180^\circ, 300^\circ$

(a)  $4 \tan^2 \theta \cos \theta = 15$   
 $\Rightarrow 4 \frac{\sin^2 \theta}{\cos^2 \theta} \cos \theta = 15$   
 $\Rightarrow 4 \frac{\sin^2 \theta}{\cos \theta} = 15$   
 $\Rightarrow 4 \sin^2 \theta = 15 \cos \theta$   
 $\Rightarrow 4(1 - \cos^2 \theta) = 15 \cos \theta$   
 $\Rightarrow 4 - 4 \cos^2 \theta = 15 \cos \theta$   
 $\Rightarrow 0 = 4 \cos^2 \theta + 15 \cos \theta - 4$   
 $\Rightarrow (4 \cos \theta - 1)(\cos \theta + 4) = 0$   
 $\Rightarrow \cos \theta = \frac{1}{4}$   
 $\arccos(\frac{1}{4}) = 75.5^\circ$   
 $\theta = 75.5^\circ \pm 360^\circ$   
 $\theta = 284.5^\circ \pm 360^\circ$   
 $\theta_1 = 75.5^\circ$   
 $\theta_2 = 284.5^\circ$

(b)  $\frac{3 + \sin^2 \theta}{\cos \theta - 2} = 3 \cos \theta$   
 $\Rightarrow 3 + \sin^2 \theta = 3 \cos \theta (\cos \theta - 2)$   
 $\Rightarrow 3 + \sin^2 \theta = 3 \cos^2 \theta - 6 \cos \theta$   
 $\Rightarrow 3 + (1 - \cos^2 \theta) = 3 \cos^2 \theta - 6 \cos \theta$   
 $\Rightarrow 4 - \cos^2 \theta = 3 \cos^2 \theta - 6 \cos \theta$   
 $\Rightarrow 0 = 4 \cos^2 \theta - 6 \cos \theta - 4$   
 $\Rightarrow 0 = 2 \cos^2 \theta - 3 \cos \theta - 2$   
 $\Rightarrow 0 = (2 \cos \theta + 1)(\cos \theta - 2)$   
 $\cos \theta = -\frac{1}{2}$   
 $\arccos(-\frac{1}{2}) =$   
 $\theta = 120^\circ \pm 360^\circ$   
 $\theta = 240^\circ \pm 360^\circ$   
 $\theta_1 = 120^\circ$   
 $\theta_2 = 240^\circ$

(c)  $2 \sin x = \tan x$   
 $\Rightarrow 2 \sin x = \frac{\sin x}{\cos x}$   
 $\Rightarrow 2 \sin x \cos x = \sin x$   
 $\Rightarrow 2 \sin x \cos x - \sin x = 0$   
 $\Rightarrow \sin x(2 \cos x - 1) = 0$   
 $\sin x = 0 \quad \cos x = \frac{1}{2}$   
 $x = 0^\circ \pm 360^\circ$   
 $x = 180^\circ \pm 360^\circ$   
 $x = 60^\circ \pm 360^\circ$   
 $x = 300^\circ \pm 360^\circ$   
 $x = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$

**Question 34**

Solve each of the following trigonometric equations in the range given.

a)  $3\cos x = 8 \tan x$

$0 \leq x < 360^\circ$

$x \approx 19.5^\circ, 160.5^\circ$

b)  $2 \tan x \sin x = 3$

$0 \leq x < 360^\circ$

$x = 60^\circ, 300^\circ$

c)  $2 \cos x = 3 \tan x$

$0 \leq x < 360^\circ$

$x = 30^\circ, 150^\circ$

<p>(a) <math>3\cos x = 8 \tan x</math>  <math>\Rightarrow 3\cos x = 8 \left( \frac{\sin x}{\cos x} \right)</math>  <math>\Rightarrow 3\cos^2 x = 8\sin x</math>  <math>\Rightarrow 8(\cos^2 x) = 8\sin x</math>  <math>\Rightarrow 8 - 8\sin^2 x = 8\sin x</math>  <math>\Rightarrow 0 = 8\sin^2 x + 8\sin x - 8</math>  <math>\Rightarrow 0 = (\sin x - 1)(\sin x + 8)</math></p>	<p><math>\Rightarrow \sin x = \frac{1}{2}</math>  <math>\arcsin \left( \frac{1}{2} \right) = 19.47^\circ</math>  <math>x = 19.47^\circ + 360^\circ n</math>  <math>n = 0, 1, 2, \dots</math>  <math>\therefore x_1 = 19.5^\circ</math>  <math>x_2 = 160.5^\circ</math></p>
<p>(b) <math>2\tan x \sin x = 3</math>  <math>\Rightarrow 2 \left( \frac{\sin x}{\cos x} \right) \sin x = 3</math>  <math>\Rightarrow \frac{2\sin^2 x}{\cos x} = 3</math>  <math>\Rightarrow 2\sin^2 x = 3\cos x</math>  <math>\Rightarrow 2(1 - \cos^2 x) = 3\cos x</math>  <math>\Rightarrow 2 - 2\cos^2 x = 3\cos x</math>  <math>\Rightarrow 0 = 2\cos^2 x + 3\cos x - 2</math></p>	<p><math>\Rightarrow (2\cos x - 1)(\cos x + 2) = 0</math>  <math>\Rightarrow \cos x = \frac{1}{2}</math>  <math>\arccos \left( \frac{1}{2} \right) = 60^\circ</math>  <math>x = 60^\circ + 360^\circ n</math>  <math>n = 0, 1, 2, \dots</math>  <math>\therefore x_1 = 60^\circ</math>  <math>x_2 = 300^\circ</math></p>
<p>(c) <math>2\cos x = 3 \tan x</math>  <math>\Rightarrow 2\cos x = 3 \left( \frac{\sin x}{\cos x} \right)</math>  <math>\Rightarrow 2\cos^2 x = 3\sin x</math>  <math>\Rightarrow 2(1 - \sin^2 x) = 3\sin x</math>  <math>\Rightarrow 2 - 2\sin^2 x = 3\sin x</math>  <math>\Rightarrow 0 = 2\sin^2 x + 3\sin x - 2</math></p>	<p><math>\Rightarrow (\sin x - 1)(\sin x + 2) = 0</math>  <math>\Rightarrow \sin x = \frac{1}{2}</math>  <math>\arcsin \left( \frac{1}{2} \right) = 30^\circ</math>  <math>x = 30^\circ + 360^\circ n</math>  <math>n = 0, 1, 2, \dots</math>  <math>\therefore x_1 = 30^\circ</math>  <math>x_2 = 150^\circ</math></p>

**Question 35**

Solve each of the following trigonometric equations in the range given.

a)  $4 \tan x \sin x = 15$

$0 \leq x < 2\pi$

$x \approx 1.32^\circ, 4.97^\circ$

b)  $6 \cos x = 5 \tan x$

$0 \leq x < 2\pi$

$x \approx 0.73^\circ, 2.41^\circ$

c)  $\tan x \sin x = 2$

$0 \leq x < 2\pi$

$x \approx 2.00^\circ, 4.29^\circ$

$$\begin{aligned}
 \text{(a)} \quad & 4 \tan x \sin x = 15 \\
 & \Rightarrow 4 \left( \frac{\sin x}{\cos x} \right) \sin x = 15 \\
 & \Rightarrow \frac{4 \sin^2 x}{\cos x} = 15 \\
 & \Rightarrow 4 \sin^2 x = 15 \cos x \\
 & \Rightarrow 4(1 - \cos^2 x) = 15 \cos x \\
 & \Rightarrow 4 - 4 \cos^2 x = 15 \cos x \\
 & \Rightarrow 0 = 4 \cos^2 x + 15 \cos x - 4 \\
 & \Rightarrow (4 \cos x - 1)(\cos x + 4) = 0 \\
 & \Rightarrow \cos x = \cancel{-4} \\
 & \bullet \cos x = \cancel{1} \\
 & \cos x = 1.32^\circ \pm 2\pi n \quad n=0,1,2,3 \\
 & x_1 = 1.32^\circ \\
 & x_2 = 4.97^\circ \\
 \text{(b)} \quad & 6 \cos x = 5 \tan x \\
 & \Rightarrow 6 \cos x = 5 \left( \frac{\sin x}{\cos x} \right) \\
 & \Rightarrow 6 \cos^2 x = 5 \sin x \\
 & \Rightarrow 6(1 - \sin^2 x) = 5 \sin x \\
 & \Rightarrow 6 - 6 \sin^2 x = 5 \sin x \\
 & \Rightarrow 0 = 6 \sin^2 x + 5 \sin x - 6 \\
 & \Rightarrow (6 \sin x - 5)(\sin x + 1) = 0 \\
 & \Rightarrow (\sin x - \cancel{1})(\sin x + \cancel{6}) = 0 \\
 & \sin x = \cancel{-6} \\
 & \bullet \sin x = \cancel{1} \\
 & \sin x = 0.73^\circ \\
 & x = 0.73^\circ \pm 2\pi n \quad n=0,1,2,3 \\
 & x_1 = 0.73^\circ \\
 & x_2 = 24^\circ \\
 \text{(c)} \quad & \tan x \sin x = 2 \\
 & \Rightarrow \frac{\sin x}{\cos x} \sin x = 2 \\
 & \Rightarrow \frac{\sin^2 x}{\cos x} = 2 \\
 & \Rightarrow \sin^2 x = 2 \cos x \\
 & \Rightarrow 1 - \cos^2 x = 2 \cos x \\
 & \Rightarrow 0 = \cos^2 x + 2 \cos x - 1 \\
 & \text{By (WORKING THE SQUARE)} \\
 & \Rightarrow 0 = (\cos x + 1)^2 - 1 - 1 \\
 & \Rightarrow (\cos x + 1)^2 = 2 \\
 & \Rightarrow \cos x + 1 = \pm \sqrt{2} \\
 & \Rightarrow \cos x = \cancel{-1} \pm \sqrt{2} \\
 & \bullet \cos x = 1 - \sqrt{2} \\
 & \cos x = 1.32^\circ \pm 2\pi n \quad n=0,1,2,3 \\
 & x_1 = 2.00^\circ \\
 & x_2 = 4.29^\circ
 \end{aligned}$$

**Question 36**

Solve each of the following trigonometric equations in the range given.

a)  $2\cos x - 3\tan x = 0$

$0 \leq x < 2\pi$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

b)  $3\tan x \sin x = 8$

$0 \leq x < 2\pi$

$$x \approx 1.23^\circ, 5.05^\circ$$

c)  $3\tan x + 2\cos x = 0$

$0 \leq x < 2\pi$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\begin{aligned}
 \text{(a)} \quad & 2\cos x - 3\tan x = 0 \\
 & \Rightarrow 2\cos x - \frac{3\sin x}{\cos x} = 0 \\
 & \Rightarrow 2\cos^2 x = 3\sin x \\
 & \Rightarrow 2(1-\sin^2 x) = 3\sin x \\
 & \Rightarrow 2 - 2\sin^2 x = 3\sin x \\
 & \Rightarrow 0 = 2\sin^2 x + 3\sin x - 2 \\
 & \Rightarrow (2\sin x - 1)(\sin x + 2) = 0 \\
 & \Rightarrow \sin x = \frac{1}{2} \quad \cancel{\sin x = -2} \\
 & \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \\
 & \left(x = \frac{\pi}{6} \pm 2n\pi\right) \quad n=0,1,2,\dots \\
 & \therefore x = \frac{\pi}{6} \pm 2n\pi \\
 \text{(b)} \quad & 3\tan x \sin x = 8 \\
 & \Rightarrow 3\left(\frac{\sin x}{\cos x}\right) \sin x = 8 \\
 & \Rightarrow \frac{3\sin^2 x}{\cos x} = 8 \\
 & \Rightarrow 3\sin^2 x = 8\cos x \\
 & \Rightarrow 3(1-\cos^2 x) = 8\cos x \\
 & \Rightarrow 3 - 3\cos^2 x = 8\cos x \\
 & \Rightarrow 0 = 3\cos^2 x + 8\cos x - 3 \\
 & \Rightarrow (3\cos x - 1)(\cos x + 3) = 0 \\
 & \Rightarrow \cos x = \frac{1}{3} \quad \cancel{\cos x = -3} \\
 & \arccos\left(\frac{1}{3}\right) = 1.23^\circ \\
 & \left(x = 1.23^\circ \pm 2n\pi\right) \quad n=0,1,2,\dots \\
 & \therefore x = 1.23^\circ, 5.05^\circ \\
 \text{(c)} \quad & 3\tan x + 2\cos x = 0 \\
 & \Rightarrow 3\left(\frac{\sin x}{\cos x}\right) + 2\cos x = 0 \\
 & \Rightarrow \frac{3\sin x}{\cos x} + 2\cos x = 0 \\
 & \Rightarrow 3\sin x + 2\cos^2 x = 0 \\
 & \Rightarrow 3\sin x + 2(1-\sin^2 x) = 0 \\
 & \Rightarrow 3\sin x + 2 - 2\sin^2 x = 0 \\
 & \Rightarrow 0 = 2\sin^2 x - 3\sin x - 2 \\
 & \Rightarrow (2\sin x + 1)(\sin x - 2) = 0 \\
 & \Rightarrow \sin x = -\frac{1}{2} \quad \cancel{\sin x = 2} \\
 & \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \\
 & \left(x = -\frac{\pi}{6} \pm 2n\pi\right) \quad n=0,1,2,\dots \\
 & \therefore x_1 = \frac{11\pi}{6} \\
 & \quad x_2 = \frac{5\pi}{6}
 \end{aligned}$$

**Question 37**

Solve each of the following trigonometric equations in the range given.

a)  $\frac{1}{2}\tan x - \sin x = 0$

$0 \leq x < 360^\circ$

$x = 0^\circ, 60^\circ, 180^\circ, 300^\circ$

b)  $\frac{\sin x - \cos x}{\cos x} = 2$

$0 \leq x < 360^\circ$

$x \approx 71.6^\circ, 251.6^\circ$

c)  $4\tan y \sin y \cos y + 4\tan y \cos y + 1 = 0$

$0 \leq y < 360^\circ$

$y = 210^\circ, 330^\circ$

<p>(a) <math>\frac{1}{2}\tan x - \sin x = 0</math>  <math>\Rightarrow \tan x - 2\sin x = 0</math>  <math>\Rightarrow \tan x = 2\sin x</math>  <math>\Rightarrow \frac{\sin x}{\cos x} = 2\sin x</math>  <math>\Rightarrow \sin x = 2\sin x \cos x</math>  <math>\Rightarrow \sin x - 2\sin x \cos x = 0</math>  <math>\Rightarrow \sin x(1 - 2\cos x) = 0</math></p>	<p><math>\bullet \sin x = 0</math>  <math>\arcsin(0) = 0</math>  <math>\therefore x = 0^\circ \pm 360^\circ</math>  <math>\therefore x = 180^\circ, 360^\circ</math>  <math>\therefore x = 0^\circ, 180^\circ, 360^\circ</math></p>	<p><math>\bullet \cos x = \frac{1}{2}</math>  <math>\arccos(\frac{1}{2}) = 60^\circ</math>  <math>\therefore x = 0^\circ \pm 360^\circ</math>  <math>\therefore x = 120^\circ, 240^\circ</math>  <math>\therefore x = 120^\circ, 240^\circ, 360^\circ</math></p>
<p>(b) <math>\frac{\sin x - \cos x}{\cos x} = 2</math>  <math>\Rightarrow \sin x - \cos x = 2\cos x</math>  <math>\Rightarrow \sin x = 3\cos x</math>  <math>\Rightarrow \frac{\sin x}{\cos x} = \frac{3\cos x}{\cos x}</math>  <math>\Rightarrow \tan x = 3</math></p>	<p><math>\arctan 3 \approx 71.6^\circ</math>  <math>\therefore x = 71.6^\circ \pm 180^\circ</math>  <math>\therefore x_1 = 71.6^\circ</math>  <math>\therefore x_2 = 261.6^\circ</math></p>	<p><math>\bullet \arctan(-3) \approx -30^\circ</math>  <math>\therefore x = -30^\circ \pm 180^\circ</math>  <math>\therefore y = 150^\circ, 330^\circ</math></p>
<p>(c) <math>4\tan y \sin y \cos y + 4\tan y \cos y + 1 = 0</math>  <math>\Rightarrow 4\tan y \cos y (\sin y + 1) + 1 = 0</math>  <math>\Rightarrow 4\frac{\sin y}{\cos y} \cos y (\sin y + 1) + 1 = 0</math>  <math>\Rightarrow 4\sin y (\sin y + 1) + 1 = 0</math>  <math>\Rightarrow 4\sin^2 y + 4\sin y + 1 = 0</math>  <math>\Rightarrow (2\sin y + 1)^2 = 0</math>  <math>\therefore \sin y = -\frac{1}{2}</math></p>	<p><math>\therefore y = 210^\circ, 330^\circ</math></p>	<p><math>\bullet \arcsin(-\frac{1}{2}) \approx 210^\circ</math>  <math>\therefore y = 210^\circ</math></p>

**Question 38**

Prove the validity of each of the following trigonometric identities.

a)  $(2\cos x + \sin x)^2 + (\cos x - 2\sin x)^2 \equiv 5$ .

b)  $\cos x + \sin x \tan x \equiv \frac{1}{\cos x}$ .

c)  $\frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} \equiv \frac{2}{\sin \theta}$ .

(a) LHS =  $(2\cos x + \sin x)^2 + (\cos x - 2\sin x)^2$   
 $= 4\cos^2 x + 4\cos x \sin x + \sin^2 x + \cos^2 x - 4\cos x \sin x + 4\sin^2 x$   
 $= 5\cos^2 x + 5\sin^2 x = 5(\cos^2 x + \sin^2 x) = 5 \times 1 = 5 = \text{RHS}$

(b) LHS =  $\cos x + \sin x \tan x = \cos x + \sin x \left( \frac{\sin x}{\cos x} \right)$   
 $= \cos x + \frac{\sin^2 x}{\cos x} = \frac{\cos x + \sin^2 x}{\cos x} = \frac{1}{\cos x} = \text{RHS}$

(c) LHS =  $\frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} = \frac{\sin^2 \theta + (1 + \cos \theta)(1 + \cos \theta)}{(1 + \cos \theta)\sin \theta}$   
 $= \frac{\sin^2 \theta + 1 + 2\cos \theta + \cos^2 \theta}{(1 + \cos \theta)\sin \theta} = \frac{2 + 2\cos \theta}{\sin \theta (1 + \cos \theta)}$   
 $= \frac{2(1 + \cos \theta)}{\sin \theta (1 + \cos \theta)} = \frac{2}{\sin \theta} = \text{RHS}$

ALTERNATIVE  
 LHS =  $\frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} = \frac{\sin \theta(-1 - \cos \theta)}{(1 + \cos \theta)(-1 - \cos \theta)} + \frac{1 + \cos \theta}{\sin \theta}$   
 $= \frac{\sin \theta(-1 - \cos \theta)}{1 - \cos^2 \theta} + \frac{1 + \cos \theta}{\sin \theta} = \frac{\sin \theta(-1 - \cos \theta)}{\sin^2 \theta} + \frac{1 + \cos \theta}{\sin \theta}$   
 $= \frac{-1 - \cos \theta}{\sin \theta} + \frac{1 + \cos \theta}{\sin \theta} = \frac{2}{\sin \theta} = \text{RHS}$

**Question 39**

Prove the validity of each of the following trigonometric identities.

$$\text{a) } \frac{\sin x}{1-\sin x} - \frac{\sin x}{1+\sin x} \equiv 2 \tan^2 x.$$

$$\text{b) } \frac{\cos x}{1-\sin x} + \frac{1-\sin x}{\cos x} \equiv \frac{2}{\cos x}.$$

$$\text{c) } \frac{1+\sin x}{\cos x} \equiv \frac{\cos x}{1-\sin x}.$$

$$\begin{aligned}
 \text{(a) LHS} &= \frac{\sin x}{1-\sin x} - \frac{\sin x}{1+\sin x} = \frac{\sin x(1+\sin x) - \sin x(1-\sin x)}{(1-\sin x)(1+\sin x)} \\
 &= \frac{\sin x + \sin^2 x - \sin x + \sin^2 x}{1 - \sin^2 x} = \frac{2\sin^2 x}{\cos^2 x} = 2\tan^2 x \equiv \text{RHS} \\
 \text{(b) LHS} &= \frac{\cos x}{1-\sin x} + \frac{1-\sin x}{\cos x} = \frac{\cos x + (1-\sin x)(1-\sin x)}{(1-\sin x)\cos x} \\
 &= \frac{\cos x + 1 - 2\sin x + \sin^2 x}{(1-\sin x)\cos x} = \frac{2 - 2\sin x}{(1-\sin x)\cos x} = \frac{2(1-\sin x)}{(1-\sin x)\cos x} \\
 &= \frac{2}{\cos x} = \text{RHS} \\
 \text{(c) LHS} &= \frac{1+\sin x}{\cos x} = \frac{(1+\sin x)(1-\sin x)}{\cos x(1-\sin x)} = \frac{1-\sin^2 x}{\cos x(1-\sin x)} \\
 &= \frac{\cos x}{\cos x(1-\sin x)} = \frac{\cos x}{1-\sin x} = \text{RHS}
 \end{aligned}$$

(c) *Alternative for (b) using result from part (c)*

$$\begin{aligned}
 \text{LHS} &= \frac{\cos x}{1-\sin x} + \frac{1-\sin x}{\cos x} = \frac{\cos x(1+\sin x)}{(1-\sin x)(1+\sin x)} + \frac{1-\sin x}{\cos x} \\
 &= \frac{\cos x(1+\sin x)}{1-\sin^2 x} + \frac{1-\sin x}{\cos x} = \frac{\cos x(1+\sin x)}{\cos^2 x} + \frac{1-\sin x}{\cos x} \\
 &= \frac{1+\sin x}{\cos x} + \frac{1-\sin x}{\cos x} = \frac{2}{\cos x} = \text{RHS}
 \end{aligned}$$