

# VECTOR INTEGRALS

**TYPE**  $\int_C \varphi \, dr$

**Question 1**

$$V(x, y, z) = 60xyz^2.$$

Evaluate the following integral along  $C$ , from  $(3,1,1)$  to  $(4,3,2)$ ,

$$\int_C V \, d\mathbf{r}, \quad d\mathbf{r} = (dx, dy, dz)^T,$$

where  $C$  is the curve with parametric equations

$$x = t + 2, \quad y = 2t - 1, \quad z = t.$$

$$1139\mathbf{i} + 2278\mathbf{j} + 1139\mathbf{k}$$

$$\begin{aligned} & V(xyz^2) = 60xyz^2 \quad \text{at} \quad \begin{array}{l} x = t+2 \\ y = 2t-1 \\ z = t \end{array} \quad \begin{array}{l} dx = dt \\ dy = 2dt \\ dz = dt \end{array} \\ \text{Thus } \int_C V \, d\mathbf{r} &= \int_{(3,1,1)}^{(4,3,2)} 60xyz^2 \, (dx, dy, dz)^T = \int_{t=1}^{t=2} 60(2t-1)t^2 \, (dt, 2dt, dt)^T \\ &= \int_{t=1}^{t=2} 60(2t^3 + 2t^2 - 2t^2) \, (2, 1, 1)^T \, dt \\ &= 60C_1(2t)^T \left[ 2t^4 + 3t^3 - 2t^2 \right] \\ &= 60C_1(2t)^T \left[ \frac{4}{5}t^5 + \frac{3}{4}t^4 - \frac{2}{3}t^3 \right]^2 \\ &= 60C_1(2t)^T \left[ \left(\frac{4}{5}t^5 + 12 - \frac{16}{3}\right) - \left(\frac{8}{5}t^8 + \frac{3}{2}t^6 - \frac{8}{9}t^5\right) \right] \\ &= 1139C_1(2t)^T \\ &\text{ie } 1139\mathbf{i} + 2278\mathbf{j} + 1139\mathbf{k}. \end{aligned}$$

**Question 2**

$$\varphi(x, y, z) \equiv 3x + 2y + z.$$

Evaluate the following integral along  $C$ , from  $(1, 0, 0)$  to  $(2, 2, 1)$ ,

$$\int_C \varphi \, d\mathbf{r}, \quad d\mathbf{r} = (dx, dy, dz)^T,$$

where  $C$  is the curve with parametric equations

$$x = t + 1, \quad y = 2t, \quad z = t^2.$$

$$\boxed{\frac{41}{6}\mathbf{i} + \frac{41}{3}\mathbf{j} + \frac{49}{6}\mathbf{k}}$$

$$\begin{aligned}
 \Phi(3x+2y+z) &= 3x+2y+z \\
 &\quad \begin{array}{l} x=t+1 \Rightarrow dx=dt \\ y=2t \Rightarrow dy=2dt \\ z=t^2 \Rightarrow dz=2t\,dt \end{array} \\
 \text{Thus } \int_C \Phi \, d\mathbf{r} &= \int_{(1,0,0)}^{(2,2,1)} (3x+2y+z) \, (dx, dy, dz) = \int_{t=0}^{t=1} [3(t+1) + 2(2t) + t^2] \, (dt, 2dt, 2t\,dt) \\
 &= \int_{t=0}^{t=1} (5t+3+4t+t^2) \, (1, 2, 2t) \, dt \\
 &= \int_{t=0}^{t=1} (5t+7t+3) \, (1, 2, 2t) \, dt \\
 &= \int_0^1 [5t^2 + 7t^2 + 3t] \, dt = \int_0^1 [12t^2 + 3t] \, dt \\
 &= \left[ \frac{4}{3}t^3 + \frac{3}{2}t^2 + 3t \right]_0^1 = \left[ \frac{4}{3} + \frac{3}{2} + 3 \right] = \left[ \frac{4}{3} + \frac{9}{6} + 3 \right] = \left[ \frac{4}{3} + \frac{15}{6} \right] = \left[ \frac{4}{3} + \frac{5}{2} \right] = \boxed{\left( \frac{41}{6}, \frac{41}{3}, \frac{49}{6} \right)}
 \end{aligned}$$

**Question 3**

$$F(x, y, z) = xyz.$$

Evaluate the following integral along  $C$ , from  $(1, 0, 0)$  to  $(0, 1, 4)$ ,

$$\int_C F \, d\mathbf{r}, \quad d\mathbf{r} = (dx, dy, dz)^T,$$

where  $C$  is the curve with parametric equations

$$x = \cos t, \quad y = \sin t, \quad z = \frac{8t}{\pi}.$$

$$\boxed{\frac{16-12\pi}{9\pi} \mathbf{i} + \frac{16}{9\pi} \mathbf{j} + \frac{8}{\pi} \mathbf{k}}$$

The handwritten solution shows the parametrization of the curve  $C$  as  $x = \cos t$ ,  $y = \sin t$ ,  $z = \frac{8t}{\pi}$ . It then sets up the line integral  $\int_C xyz \, d\mathbf{r} = \int_{(1,0,0)}^{(0,1,4)} xyz \, d\mathbf{r}$ . The differential  $d\mathbf{r} = (\cos t, \sin t, \frac{8}{\pi}) dt$  is used. The integral becomes  $\int_0^{\pi/2} (\cos t)(\sin t) \left(\frac{8}{\pi}\right) \cos t \sin t \, dt$ . This is simplified to  $\frac{8}{\pi} \int_0^{\pi/2} \cos^2 t \sin^2 t \, dt$ . The integral is evaluated using substitution  $u = \cos t$  and  $du = -\sin t \, dt$ , resulting in  $\frac{8}{\pi} \int_1^0 u^2 (1-u^2) \, du = \frac{8}{\pi} \int_0^1 (u^2 - u^4) \, du$ . This is then evaluated using integration by parts or direct substitution to get  $\frac{16-12\pi}{9\pi}$ .

**TYPE**  $\int_V \mathbf{F} dV$

**Question 1**

$$\mathbf{F}(x, y, z) \equiv xy\mathbf{i} + z\mathbf{j} - x^2\mathbf{k}.$$

Evaluate the vector integral

$$\int_V \mathbf{F} dV,$$

where  $V$  is the finite region in the first octant bounded by the planes with equations

$$x = 2, y = 3 \text{ and } z = 4.$$

$$36\mathbf{i} + 48\mathbf{j} - 32\mathbf{k}$$

$$\begin{aligned}\int_V (xy(x-y)) dV &= \int_0^4 \int_{y=0}^3 \int_{z=0}^4 (xy(x-y)) dx dy dz \\&= \int_0^4 \int_{y=0}^3 \left[ \frac{1}{2}xy^2(2z_y - \frac{1}{3}y^3) \right]_{z=0}^4 dy dz \\&= \int_0^4 \int_{y=0}^3 (20\mathbf{i} - 32\mathbf{j}) dy dz = \int_{y=0}^3 [5\mathbf{i} - 8\mathbf{j}]_{y=0}^3 dz \\&= \int_{y=0}^3 (3\mathbf{i} - 8\mathbf{j}) dz = [3\mathbf{i} - 8\mathbf{j}]_0^3 \\&= (36\mathbf{i} - 32\mathbf{j}) \quad \mathbf{i.e.} \quad 36\mathbf{i} + 48\mathbf{j} - 32\mathbf{k}.\end{aligned}$$

**Question 2**

$$\mathbf{F}(x, y, z) \equiv z\mathbf{i} + \mathbf{j} + y\mathbf{k}.$$

Evaluate the vector integral

$$\int_V \mathbf{F} dV,$$

where  $V$  is the finite region in the first octant bounded by the plane with equation

$$2x + y + z = 6.$$

$$\boxed{27\mathbf{i} + 18\mathbf{j} + 27\mathbf{k}}$$

$$\begin{aligned}
 \int_V \mathbf{F} dV &= \int_V (z\mathbf{i} + \mathbf{j} + y\mathbf{k}) dV = \int_0^3 \int_{2x}^{6-2x} \int_{2x-y}^{6-2x-y} (z\mathbf{i} + \mathbf{j} + y\mathbf{k}) dz dy dx \\
 &= \int_0^3 \int_{2x}^{6-2x} \left[ \frac{1}{2}z^2 \right]_{2x-y}^{6-2x-y} dy dx \\
 &= \int_0^3 \int_{2x}^{6-2x} \left( \frac{1}{2}(6-y-2x)^2, 6-y-2x, y-2x \right) dy dx \\
 &= \int_0^3 \left[ \left( \frac{1}{6}(6-y-2x)^3, \frac{1}{2}(6-y-2x)^2, \frac{1}{2}y(6-y-2x) \right) \right]_{2x}^{6-2x} dy \\
 &= \int_0^3 \left[ \frac{1}{6}(6-4+2x-2x)^3 + \frac{1}{2}(6-2x)^2 - \frac{1}{6}(6-4+2x-2x)^3 + \frac{1}{2}(6-2x)^2 - \frac{1}{2}(6-2x)^2 \right] dy \\
 &= \int_0^3 \left( \frac{1}{6}(6-2x)^3, \frac{1}{2}(6-2x)^2, \frac{1}{2}(6-2x)^2 - \frac{1}{3}(6-2x)^3 - \frac{1}{2}(6-2x)^2 - 3x \right) dy \\
 &= \int_0^3 \left[ \frac{1}{24}(6-2x)^4, \frac{1}{6}(6-2x)^3, \frac{1}{2}(6-2x)^2 - \frac{1}{3}(6-2x)^3 - \frac{1}{2}(6-2x)^2 - 3x \right] dy \\
 &= \left[ 0 + 27, 0 + 18, (-21 + 216 - 108) - (-108 + 54) \right] \\
 &= (27, 18, 27)
 \end{aligned}$$

**Question 3**

$$\mathbf{F}(x, y, z) \equiv \mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$$

Evaluate the vector integral

$$\int_V \mathbf{F} dV,$$

where  $V$  is the finite region enclosed by the cylinder with equation

$$x^2 + y^2 = 9, \quad 0 \leq z \leq 2.$$

$$18\pi(\mathbf{i} + 2\mathbf{j})$$

$\int_V \mathbf{F} dV = \int (1, 2z, y) dV$

SPLIT INTO CYLINDRICAL REGIONS

$$= \int_0^2 \int_0^{2\pi} \int_{-3}^3 (1, 2r\cos\theta, r\sin\theta) r dr d\theta dz$$

$$= \int_0^2 \int_0^{2\pi} \int_{-3}^3 (1, 2r\cos\theta, r^2\sin\theta) dr d\theta dz$$

$$= \int_0^2 \int_0^{2\pi} \int_{-3}^3 (\frac{1}{2} - 0, 4r\cos\theta - 0, 4r^2\sin\theta) dr d\theta dz$$

$$= \int_0^2 \int_0^{2\pi} \int_{-3}^3 (\frac{1}{2}, 4r\cos\theta, 4r^2\sin\theta) dr d\theta dz$$

$$= \int_0^2 \left[ \frac{1}{2}r^2, 4r^2\cos\theta, -4r^2\sin\theta \right]_{-3}^3 d\theta dz$$

$$= \int_0^2 (27, 144\cos\theta, -144\sin\theta) d\theta dz$$

$$= \left[ 27\theta, 144\cos\theta, 144\sin\theta \right]_0^2 = (54\pi, 384, 0)$$

$\therefore 18\pi(\mathbf{i} + 2\mathbf{j})$

**Question 4**

$$\mathbf{F}(x, y, z) \equiv \frac{1}{6\pi} \mathbf{i} + \frac{z}{18\pi} \mathbf{j} + \frac{y}{9\pi} \mathbf{k}.$$

Evaluate the vector integral

$$\int_V \mathbf{F} dV,$$

where  $V$  is the finite region enclosed by the cylinder with equation

$$x^2 + y^2 = 4, \quad 0 \leq z \leq 3.$$

**[2i + j]**

$\int_V \mathbf{F} dV = \int_V \left( \frac{1}{6\pi} \mathbf{i} + \frac{z}{18\pi} \mathbf{j} + \frac{y}{9\pi} \mathbf{k} \right) dx dy dz$

SCALAR FIELD CYLINDRICAL REGION (i, j, k)

$$= \int_0^{2\pi} \int_{-2}^2 \int_{r=0}^2 \left( \frac{1}{6\pi} \mathbf{i} + \frac{z}{18\pi} \mathbf{j} + \frac{y}{9\pi} \mathbf{k} \right) (r dr dz) d\theta$$

$$= \int_0^{2\pi} \int_{-2}^2 \int_{r=0}^2 \left( \frac{1}{6\pi} \mathbf{i} + \frac{z}{18\pi} \mathbf{j} + \frac{y}{9\pi} \mathbf{k} \right) r dr dz d\theta$$

$$= \int_{-2}^2 \int_{r=0}^2 \left[ \frac{1}{6\pi} \left( \frac{r^2}{2} \right) + \frac{z}{18\pi} r^2 + \frac{y}{9\pi} r \right]_{r=0}^{r=2} dz d\theta$$

$$= \int_{-2}^2 \int_{r=0}^2 \left[ \frac{1}{6\pi} \left( \frac{4}{2} \right) + \frac{z}{18\pi} (4) + \frac{y}{9\pi} (2) \right] dz d\theta$$

$$= \int_{-2}^2 \left[ \frac{2}{9\pi} \left( \frac{6}{5} + \frac{5z}{9} + \frac{2y}{3} \right) \right] dz$$

$$= \int_{-2}^2 \left[ \frac{2}{9\pi} \left( \frac{6}{5} + \frac{5z}{9} + \frac{2y}{3} \right) - \frac{8yz}{27\pi} \right]^{2\pi}_0 dz$$

$$= \int_{-2}^2 \left( \frac{2}{9\pi} \left( \frac{6}{5} + \frac{5z}{9} + 0 \right) \right) dz$$

$$= \left[ \frac{2}{9\pi} \left( \frac{6}{5}z + \frac{5z^2}{18} \right) \right]_{-2}^2$$

$$= (2, 1, 0) \quad L.E. \quad 21 + 1$$

$\mathbf{F} = \left( \frac{1}{6\pi} \mathbf{i} + \frac{z}{18\pi} \mathbf{j} + \frac{y}{9\pi} \mathbf{k} \right)$

$x = r \cos \theta$

$y = r \sin \theta$

$z = z$

$dV = r dr dz d\theta$

**Question 5**

$$\mathbf{F}(x, y, z) \equiv 3\mathbf{i} + -y\mathbf{j} + 6x\mathbf{k}.$$

Evaluate the vector integral

$$\int_V \mathbf{F} dV,$$

where  $V$  is the finite region enclosed by the hemisphere with equation

$$x^2 + y^2 + z^2 = 4, \quad z \geq 0.$$

**[16πi]**

$\int_V \mathbf{F} dV = \int_V (3i - yj + 6xk) dx dy dz$

Switch to Cartesian coordinates  $x^2 + y^2 + z^2$

$$= \int_{z=0}^{2} \int_{y=0}^{\sqrt{4-z^2}} \int_{x=-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} (3i - yj + 6xk) dx dy dz$$

$$= \int_{z=0}^{2} \int_{y=0}^{\sqrt{4-z^2}} \left[ 3xi - y^2 j + 3x^2 k \right]_{-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} dy dz$$

$$= \int_{z=0}^{2} \int_{y=0}^{\sqrt{4-z^2}} (6xi - 4y^2 j + 12x^2 k) dy dz$$

$$= \int_0^2 6xz^2 dy dz = 1 - 2z^3$$

$$2\pi \int_0^2 (1 - 2z^3) dz = 16\pi i$$

ANS:  $\int_V \mathbf{F} dV = 16\pi i$

... =  $\int_{z=0}^{2} \int_{y=0}^{\sqrt{4-z^2}} \int_{x=-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} (3i - yj + 6xk) dx dy dz$ 

$$= \int_{z=0}^{2} \int_{y=0}^{\sqrt{4-z^2}} \left[ -3xi + y^2 j + 6x^2 k \right]_{-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} dy dz$$

$$= \int_{z=0}^{2} \int_{y=0}^{\sqrt{4-z^2}} \left[ 0 - 2(\frac{3}{2})y^2 j + 12(\frac{1}{2})x^2 k \right]_{-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} dy dz$$

$$= \int_{z=0}^{2} \int_{y=0}^{\sqrt{4-z^2}} (6xi - 4y^2 j + 12x^2 k) dy dz$$

$$= \int_{z=0}^{2} \left[ 6x^2 z - 4y^3 j + 12x^3 k \right]_{0}^{\sqrt{4-z^2}} dz$$

$$= \left( 16\pi, 0, 0 \right) - \left( 0, 0, 0 \right)$$

$$= \left( 16\pi, 0, 0 \right)$$

**Question 6**

The finite region  $V$  in the first octant, is bounded by the surfaces with equations

$$y = 4 - x^2 \quad \text{and} \quad y = 4 - z^2.$$

Given that  $\mathbf{F} = \frac{1}{8}\mathbf{i} + 3y^2\mathbf{j} - \frac{1}{4}\mathbf{k}$  determine

$$\int_V \mathbf{F} \cdot d\mathbf{v}.$$

$$\boxed{\mathbf{i} + 64\mathbf{j} - 2\mathbf{k}}$$

$$\begin{aligned} & \int_V \mathbf{F} \cdot d\mathbf{v} \\ &= \int_{x=0}^2 \int_{y=4-x^2}^{4-z^2} \int_{z=0}^{\sqrt{4-y}} \left( \frac{1}{8}, 3y^2, -\frac{1}{4} \right) dx dy dz \\ &= \int_{x=0}^2 \int_{y=4-x^2}^{4-x^2} \left[ \frac{1}{8}x, y^2, -\frac{1}{4}z \right]_{x=0}^{\sqrt{4-y}} dy dx \\ &= \int_{x=0}^2 \int_{y=4-x^2}^{4-x^2} \left[ \frac{1}{8}(4-y)^{1/2}(4-y), -\frac{1}{4}(4-y)^{1/2} \right] dy dx \end{aligned}$$

THE 1/8 IS A CONSTANT, NOT INTEGRABLE BUT THE  
3 NEEDS A SUBSTITUTION

$$\begin{aligned} & \left[ \frac{1}{8}(4-y)^{1/2} dy \right]_{y=4-x^2}^{4-x^2} \quad \begin{cases} u=4-y \\ y=4-u \\ dy=-du \end{cases} \\ &= \left[ \frac{1}{8}(4-x^2)^{1/2} du + \frac{1}{8}x^2 du \right]_{u=0}^{u=4} \\ &= \int_{x=0}^2 \left[ \frac{1}{8}(4-x^2)^{1/2} - \frac{1}{8}x^2 \right] dx \\ &= \left[ \frac{1}{8}x^{3/2} - \frac{1}{8}x^2 \right]_{x=0}^2 - \left( \frac{1}{8}x^2 - \frac{1}{8}x^2 + \frac{1}{8}x^2 \right) \end{aligned}$$

$$\begin{aligned} &= \left[ \frac{1}{8}x^{3/2} - \frac{1}{8}x^2 \right]_{x=0}^2 - \left[ \frac{1}{8}x^2 - \frac{1}{8}x^2 + \frac{1}{8}x^2 \right] \\ &= \left[ \frac{1}{8}x^{3/2} - \frac{1}{8}x^2 \right]_{x=0}^2 + 3x^2 \left[ \frac{2x^{1/2}}{3} - \frac{1}{2}x^2 + \frac{1}{2}x^2 - \frac{1}{3}x^2 \right]_{x=0}^2 + \frac{1}{8}x^2 \left[ \frac{2(2-x^2)^{1/2}}{3} \right]_{x=0}^2 \\ &= \frac{1}{8} \left[ \frac{1}{2}x^3 - 8x^2 \right]_{x=0}^2 + 3x^2 \left[ \frac{2x^{1/2}}{3} - \frac{1}{2}x^2 + \frac{1}{2}x^2 - \frac{1}{3}x^2 \right]_{x=0}^2 + \frac{1}{8}x^2 \left[ \frac{2(2-x^2)^{1/2}}{3} \right]_{x=0}^2 \\ &= \frac{1}{8} \left[ \frac{1}{2}(4-16) + 24 \times \frac{16}{3} + \frac{1}{8} \times \frac{1}{2}(4-16) \right] \\ &= \frac{1}{8} \left[ \frac{1}{2}(-12) + 24 \times \frac{16}{3} + \frac{1}{8} \times (-12) \right] \\ &= \frac{1}{8} \left[ -6 + 128 + -1.5 \right] \\ &= \frac{1}{8} \times 120.5 \\ &= 15.0625 \end{aligned}$$

**TYPE**  $\int_S F \, d\mathbf{S}$

**Question 1**

$$F(x, y, z) \equiv x + y + z.$$

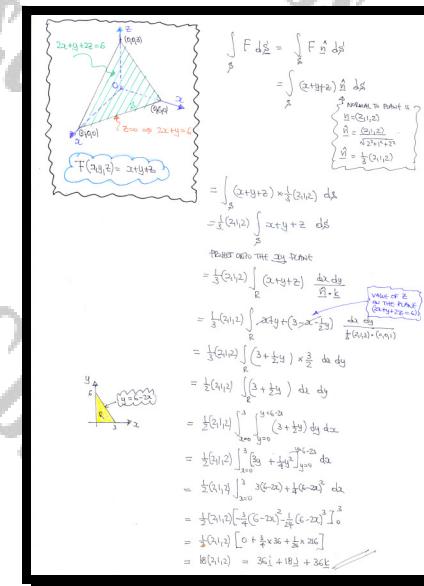
Evaluate the integral

$$\int_S F \, d\mathbf{S},$$

where  $S$  is the plane surface with equation

$$2x + y + 2z = 6, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

**36i + 18j + 36k**



**Question 2**

$$\varphi(x, y, z) \equiv \frac{3}{4}xyz.$$

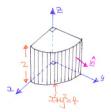
Evaluate the integral

$$\int_S \varphi \, d\mathbf{S},$$

where  $S$  is the curved surface of the cylinder with equation

$$x^2 + y^2 = 4, \quad x \geq 0, \quad y \geq 0, \quad 0 \leq z \leq 2.$$

**4i + 4j**



$\Phi(x,y,z) = \frac{3}{4}xyz$

$\int_S d\mathbf{S} = \int_S \frac{3}{4}xyz \, d\mathbf{S}$

WE NEED THE UNIT NORMAL TO THE CURVED SURFACE -

- let  $\mathbf{F}(x,y,z) = (x, y, z)$  (3D vector)
- $\nabla F = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}) = (1, 0, 1)$
- Then  $\mathbf{n} = (0, 0, 1)$
- $|\mathbf{n}| = \sqrt{0+0+1} = 1 \Rightarrow \hat{\mathbf{n}} = \frac{(0, 0, 1)}{\sqrt{0+0+1}} = \frac{(0, 0, 1)}{1} = (0, 0, 1)$

$\dots = \int_S \frac{3}{4}xyz \times \frac{1}{2}(0,0,1) \, d\mathbf{S} = \frac{3}{8} \int_S (x, y, z) \, d\mathbf{S}$

WE CAN ALSO DO THIS IN CYLINDRICAL POLARS **BECAUSE**  
(WE CAN ALSO PROJECT ON THE XY-PLANE)

$= \frac{3}{8} \int_0^{2\pi} \int_0^2 ((r\cos\theta, r\sin\theta, z), (r, 0, 1)) \cdot (r, 0, 0) \, dz \, dr$

$= \frac{3}{8} \int_0^{2\pi} \int_0^2 (rcos\theta, rsin\theta, z), (r, 0, 1) \, dz \, dr$

$= \frac{3}{8} \int_0^{2\pi} \int_0^2 \left[ \left( \frac{3}{2}z^2, \frac{3}{2}z^2, z \right) \right] \Big|_0^2 \, dr \, d\theta$

$= \frac{3}{8} \int_0^{2\pi} \int_0^2 \left[ \left( \frac{3}{2}z^2, \frac{3}{2}z^2, z \right) \right] \Big|_0^2 \, dr \, d\theta$

$= 2 \int_{2\pi}^0 \int_0^2 (z, z, 0) \, dz \, dr$

$= 2 \left[ \left( \frac{1}{2}z^2, \frac{1}{2}z^2, z \right) \right] \Big|_0^2$

$= 4\frac{1}{2} + 4\frac{1}{2}$

WE CAN OF COURSE DO THE SURFACE INTEGRAL IN CARTESIAN BY PROJECTING onto THE XY-PLANE

THE NORMAL OF THE ZZ-PLANE IS  $\mathbf{j}$

$d\mathbf{S} = dxdy = \frac{dx}{dz} dy = dz dy$

$d\mathbf{S} = \frac{2}{z} dz dy$

THE NORMAL OF THE ZZ-PLANE IS  $\mathbf{j}$

$d\mathbf{S} = dxdy = \frac{dx}{dz} dy = \frac{dx}{dz} dz dy$

$d\mathbf{S} = \frac{2}{z} dz dy$

$\dots = \frac{3}{8} \int_0^{2\pi} \int_0^2 (2z^2, 2z^2, 0) \times \frac{2}{z} dz dy$

$= \frac{3}{4} \int_0^{2\pi} \int_0^2 (2z^2, 2z^2, 0) \, dz dy$

$= \frac{3}{4} \int_0^{2\pi} \int_0^2 (2z^2, 2(4-z^2)^{\frac{1}{2}}, 0) \, dz dy$

$= \frac{3}{4} \int_0^{2\pi} \int_0^2 \left[ \left( 3z^2, 2(4-z^2)^{\frac{1}{2}}, 0 \right) \right]^2 \, dz dy$

$= \frac{3}{4} \int_0^{2\pi} \int_0^2 (z^2, \frac{3}{2}z^2, 0) \, dz dy$

$= 2 \int_{2\pi}^0 \int_0^2 (z, z, 0) \, dz dy$

$= 2 \left[ \left( \frac{1}{2}z^2, \frac{1}{2}z^2, z \right) \right] \Big|_0^2$

$= 4\frac{1}{2} + 4\frac{1}{2}$

**Question 3**

$$\varphi(x, y, z) \equiv \frac{1}{2}xyz^2.$$

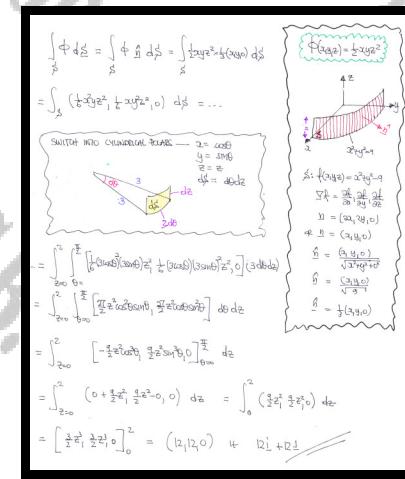
Evaluate the integral

$$\int_S \varphi \, dS,$$

where  $S$  is the curved surface of the cylinder with equation

$$x^2 + y^2 = 9, \quad x \geq 0, \quad y \geq 0, \quad 0 \leq z \leq 2.$$

**[12i + 12j]**



**Question 4**

$$\varphi(x, y, z) \equiv 2x + 2y.$$

Evaluate the integral

$$\int_S \varphi \, dS,$$

where  $S$  is the curved surface of the sphere with equation

$$x^2 + y^2 + z^2 = 1, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

$$\boxed{\frac{1}{3}(\pi+2)\mathbf{i} + (\pi+2)\mathbf{j} + 4\mathbf{k}}$$

**SIMPLIFYING THE INTEGRAL - INTEGRAL OVER A SPHERICAL SURFACE**

$$\int_S \varphi \, dS = \int_S (2x+2y) \, dS$$

$$= \int_S (2r^2 \cos\theta + 2r^2 \sin\theta) \, dS$$

$$= \int_S [2(r^2 + 2r\cos\theta + 2r\sin\theta)] \, dS$$

**SWITCH INTO SPHERICAL POLAR COORDINATES AND INTEGRATE IN COMPONENTS**

$$1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (2r^2 \cos\theta + 2r^2 \sin\theta) r^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 2r^3 \cos\theta + 2r^3 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 2r^3 \theta (\cos\theta + \sin\theta) \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 2r^3 \theta \left( \frac{1}{2} + \frac{1}{2}\cos2\theta + \frac{1}{2}\sin2\theta \right) \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left( 1 - \cos2\theta \right) \cos\theta \left( 1 + \cos2\theta + \sin2\theta \right) \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \left( \sin\theta - \cos\theta \right) \cos\theta \times \frac{r^3}{4} \left( 1 + \cos2\theta + \sin2\theta \right) \, d\theta \, d\phi$$

**NOTICE THAT THE "θ" TERM CAN BE DEALTED WITH BOTH COSINE FUNCTIONS**

$$1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 2r^3 \theta \, d\theta = 2 \int_0^{\frac{\pi}{2}} \theta \cos\theta \, d\theta = 2(\theta \sin\theta - \int_0^{\frac{\pi}{2}} \theta \sin\theta \, d\theta)$$

$$= \frac{1}{2} \times \Gamma(3) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

**NOT THE θ COMPONENT IN SPHERICAL POLAR**

$$1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (2r^2 \cos\theta + 2r^2 \sin\theta) (r^2 \sin\theta) \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 2r^4 \cos\theta \sin\theta + 2r^4 \sin\theta \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 2r^4 \sin\theta (\cos\theta + \sin\theta) \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 2r^4 \left( \frac{1}{2} \cos2\theta + \frac{1}{2} - \frac{1}{2}\cos2\theta \right) \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left( \frac{1}{2} \cos2\theta (\cos\theta + 1 - \cos2\theta) \right) \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{2} \cos\theta \sin\theta \, d\theta \int_0^{\frac{\pi}{2}} \left( \frac{1}{2} + \cos2\theta - \cos2\theta \right) \, d\phi$$

$$= \frac{1}{2} \times \frac{\pi}{2} \times \left[ \frac{1}{2} \cos2\theta - \frac{1}{2} \right]_0^{\frac{\pi}{2}}$$

**NOTICE THAT IN THE SPHERICAL POLAR COORDINATE SYSTEM, THE "θ" TERM IS THE ANGLE IN THE XY-PLANE**

**FINDING THE k COMPONENT IN POLAR**

$$1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 2r^3 \sin\theta \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} (2r^3 \cos\theta \sin\theta + 2r^3 \sin\theta \sin\theta) \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 2r^3 \sin\theta (\cos\theta + \sin\theta) \, d\theta \, d\phi$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[ \frac{2}{3} \sin^3\theta \right]^{\frac{\pi}{2}}_0 (\cos\theta + \sin\theta) \, d\theta$$

$$= \frac{2}{3} \int_0^{\frac{\pi}{2}} \cos\theta + \sin\theta \, d\theta$$

$$= \frac{2}{3} \left[ \frac{1}{2} \sin2\theta - \frac{1}{2} \cos2\theta \right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{3} \left[ (1 - 0) - (0 - 1) \right]$$

$$= \frac{4}{3}$$

$$\therefore \int_S \varphi \, dS = \frac{1}{3}(\pi+2)(\frac{4}{3}) = \frac{1}{9}(\pi+2)(4)$$

**TYPE**  $\int_S \mathbf{F} \cdot d\mathbf{S}$

**Question 1**

The Cartesian equation of a surface  $S$  is

$$z = x^2 + y^2, \quad z \leq 1.$$

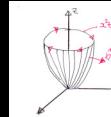
Evaluate the surface integral

$$\int_S \hat{\mathbf{n}} \wedge \nabla \phi \, dS,$$

where  $\hat{\mathbf{n}}$  is an outward normal unit vector field to  $S$ , and  $\phi$  is the function with Cartesian equation

$$\phi(x, y, z) = y.$$

■



•  $\nabla \phi(2,0,2) = 2\mathbf{i} - 2\mathbf{k}$   
 $\nabla \phi = (2x, 2y, -2)$   
 $\|\nabla \phi\| = \sqrt{4x^2 + 4y^2 + 4} = 2\sqrt{x^2 + y^2 + 1}$   
 $\hat{\mathbf{n}} = \frac{(2x, 2y, -2)}{2\sqrt{x^2 + y^2 + 1}}$   
 $\hat{\mathbf{n}} = (\frac{2x}{\sqrt{x^2 + y^2 + 1}}, \frac{2y}{\sqrt{x^2 + y^2 + 1}}, \frac{-2}{\sqrt{x^2 + y^2 + 1}})$

•  $\phi(2,0,2) = 2$   
 $\nabla \phi = (0, 0, 0)$

**NOW:**  
 $\hat{\mathbf{n}} \wedge \nabla \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2x}{\sqrt{x^2+y^2+1}} & \frac{2y}{\sqrt{x^2+y^2+1}} & \frac{-2}{\sqrt{x^2+y^2+1}} \\ 0 & 0 & 0 \end{vmatrix}$   
 $= \left[ \frac{2x}{\sqrt{x^2+y^2+1}} \mathbf{i} \right] \circ \left[ \frac{2y}{\sqrt{x^2+y^2+1}} \mathbf{k} \right]$

$\int_S \hat{\mathbf{n}} \wedge \nabla \phi \, dS = \dots$  Project onto the  $xy$ -plane  
 onto  $x^2 + y^2 \leq 1$  (below  $z=1$ )  
 $dS = \frac{dxdy}{\sqrt{1-x^2-y^2}}$

$$= \int_S \left[ \frac{1}{\sqrt{1-x^2-y^2}} \frac{2x}{\sqrt{1-x^2-y^2}} \right] dx dy = \int_R \left[ \frac{1}{\sqrt{1-x^2-y^2}} \frac{2x}{\sqrt{1-x^2-y^2}} \right] \frac{dx dy}{\sqrt{1-x^2-y^2}} = \int_R \left[ \frac{1}{\sqrt{1-x^2-y^2}} \frac{2x}{\sqrt{1-x^2-y^2}} \right] \frac{1}{\sqrt{1-x^2-y^2}} dx dy$$

$$= \int_R \left[ \frac{2x}{1-x^2-y^2} \right] dx dy = \int_R \left( 1, 0, 2x \right) \frac{dx dy}{\sqrt{1-x^2-y^2}}$$

$= \int_R (1, 0, 2x) \, dxdy$   
 Switch into polar coords  
 $= \int_0^{2\pi} \int_{0}^1 (r \cos \theta, r \sin \theta, 2r) (r dr d\theta)$   
 $= \int_0^{2\pi} \int_{0}^1 (r \cos \theta, r \sin \theta, 2r) \, dr d\theta$  Since this is  $\theta$  integration  
 $= \int_{0}^{2\pi} \int_{0}^1 r^2 \cos^2 \theta \, dr d\theta$   
 $= \int_{0}^{2\pi} \left[ \frac{1}{3} r^3 \cos^2 \theta \right]_0^1 \, d\theta = \int_{0}^{2\pi} \frac{1}{3} \, d\theta = \frac{2\pi}{3}$

**Question 2**

The Cartesian equation of a surface  $S$  is

$$z = 1 - x^2 - y^2, \quad z \geq 0.$$

Evaluate the surface integral

$$\int_S \hat{\mathbf{n}} \wedge \nabla \varphi \, dS,$$

where  $\hat{\mathbf{n}}$  is an outward unit normal vector field to  $S$ , and  $\varphi$  is the function with Cartesian equation

$$\varphi(x, y, z) = 1 - 2x^2 y.$$

$$\frac{\pi}{2} \mathbf{i}$$

**FIND NORMAL TO THE SURFACE**  
Let  $P(x_1, y_1) = (x^2 + y^2, 1 - x^2 - y^2)$   
 $\mathbf{N} = \nabla \varphi = \left( \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right) = (2xy, -2x, 1)$   
(we need to find  $\|\mathbf{N}\|$  as we will project onto the  $xy$ -plane)

- $\varphi(x, y) = 1 - 2x^2 y$   
 $\nabla \varphi = (2xy, -2x, 1)$   
 $\|\nabla \varphi\| = (-4x^2 - 2y^2, -2x, 1)$
- $\hat{\mathbf{N}} \times \nabla \varphi = \frac{1}{\|\mathbf{N}\|} \nabla \varphi = \frac{1}{\sqrt{-4x^2 - 2y^2 + 1}} (-4x^2 - 2y^2, -2x, 1)$   
 $= \frac{1}{\|\mathbf{N}\|} [(-4x^2 - 2y^2, -2x, 1)]$

PROJECT onto the  $xy$ -Plane  
onto the  $x$ -axis  $\Rightarrow$   $y=0$ , such that  $-2x^2 + 1 = 1$

$$\begin{aligned} \iint_S \hat{\mathbf{n}} \wedge \nabla \varphi \, dS &= \iint_R \frac{1}{\|\mathbf{N}\|} [(-4x^2 - 2y^2, -2x, 1)] \, dx \, dy \\ &= \iint_R \frac{1}{\sqrt{1 - 4x^2}} [(-4x^2 - 2y^2, -2x, 1)] \, dx \, dy \\ &= \iint_R (-2x^2 - y^2, -x, 0) \, dx \, dy \quad (\text{since } \mathbf{N} \cdot \mathbf{k} = 0) \\ &= \iint_R (-2x^2 - y^2, -x, 0) \, dx \, dy \\ &\quad (\text{Note: } R \text{ is a semicircular domain in } xy\text{-plane, i.e. all powers of } x, y, z \text{ are even}) \\ &\quad (\text{So INT integrate} \rightarrow \text{use polar coordinates}) \\ &= \iint_R (x^2, 0, 0) \, dy \, dx = 2\int_{0}^{\pi/2} \int_{0}^{1/\cos \theta} r^2 \cos^2 \theta \, dr \, d\theta = 2\int_{0}^{\pi/2} \int_{0}^{1/\cos \theta} r^3 \cos^2 \theta \, dr \, d\theta \\ &= 2\int_{0}^{\pi/2} \left[ \frac{1}{4} r^4 \cos^2 \theta \right]_{0}^{1/\cos \theta} \, d\theta = 2\int_{0}^{\pi/2} \frac{1}{4} \cos^2 \theta \, d\theta = 2\int_{0}^{\pi/2} \frac{1}{4} \left( \frac{1}{2} + \frac{1}{2} \cos 2\theta \right) \, d\theta \\ &= \frac{1}{2} \int_{0}^{\pi/2} \left( \frac{1}{2} + \frac{1}{2} \cos 2\theta \right) \, d\theta \quad (\text{no contribution from } \cos 2\theta) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{8} \end{aligned}$$

**TYPE**  $\int_S \mathbf{F} \cdot d\mathbf{S}$

**Question 1**

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with equation

$$x^2 + y^2 + z^2 = a^2, \quad a > 0,$$

and  $\mathbf{F} = z^2 \mathbf{k}$ .

[0]

$x^2 + y^2 + z^2 = a^2$  is a sphere about  $\mathbf{O}$ .

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S (z^2 \mathbf{k}) \cdot \hat{n} \, dS$$

$$= \int_S (z^2 \mathbf{k}) \cdot \frac{\partial \mathbf{r}}{\partial z} \, dS = \int_S \frac{z^2}{a} \, dS$$

... convert into spherical polar co-ordinates ...

$$= \int_0^{2\pi} \int_0^\pi \frac{(a \cos \theta)^2}{a} (\rho^2 \sin \theta d\theta d\phi)$$

$$= \int_0^{2\pi} \int_0^\pi a^2 \cos^2 \theta \sin \theta \, d\theta \, d\phi$$

$$= a^2 \int_0^{2\pi} 1 \, d\phi \times \int_0^\pi \cos^2 \theta \sin \theta \, d\theta$$

$$= 2\pi a^2 \times \left[ -\frac{1}{4} \cos^2 \theta \right]_0^\pi$$

$$= 2\pi a^2 \times \frac{1}{4} \left[ \cos^2 \theta \right]_0^\pi$$

$$= \frac{1}{2} \pi a^2 [1 - 1]$$

$$= 0$$

Let  $\mathbf{r}(u, v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \cos v \\ u \sin v \\ z \end{pmatrix}$

$$\mathbf{r}_u = (u \cos v, u \sin v, 0)$$

$$|\mathbf{r}_u| = \sqrt{u^2 \cos^2 v + u^2 \sin^2 v} = u$$

$$\hat{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

$x = a \sin \theta \cos \phi$   
 $y = a \sin \theta \sin \phi$   
 $z = a \cos \theta$   
 $x^2 + y^2 + z^2 = a^2$   
 $dS = \sqrt{u^2 \cos^2 v + u^2 \sin^2 v} \, du \, dv$

**Question 2**

$$\mathbf{F}(x, y, z) \equiv x^2\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}.$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the plane surface with equation

$$2x + 2y + z = 2, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

$$-\frac{7}{6}$$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n} dS = \int_S (x^2, -2y, -2z) \cdot (2, 2, 1) dS = \int_S 2x^2 - 4y - 2z dS$$

PROJECT SURFACE onto the xy PLANE

$$= \int_R \int_{z=2-2x-2y}^{2-2x-2y} 2x^2 - 4y - 2(2-2x-2y) \frac{dxdy}{\sqrt{1+4x^2+4y^2}}$$

VALUES OF z ON S

$$= \int_0^1 \int_{y=0}^{1-x} 2x^2 - 4y - 4 + 4x + 4y \frac{dxdy}{\sqrt{1+4x^2+4y^2}} = \int_0^1 \int_{y=0}^{1-x} (2x^2 + 4x - 4) (3dy dx)$$

$$= \int_0^1 \int_{y=0}^{1-x} 2x^2 + 4x - 4 dy dx = \int_{x=0}^1 [2x^2 + 4x - 4]_{y=0}^{1-x} dx$$

$$= \int_0^1 [2x^2 + 4x - 4](1-x) dx = \int_0^1 [2x^2 + 4x - 4x^2 - 4x] dx$$

$$= \int_0^1 [-2x^2 - 2x^2 + 8x - 4] dx = \left[ -\frac{1}{3}x^3 - \frac{2}{3}x^3 + 4x^2 - 4x \right]_0^1$$

$$= -\frac{1}{3} - \frac{2}{3} + 4 - 4 = -\frac{7}{6}$$

**Question 3**

$$\mathbf{F}(x, y, z) \equiv 4y\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with equation

$$x^2 + y^2 + z^2 = 9, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

$$36 + \frac{9}{4}\pi$$

Surface:  $x^2 + y^2 + z^2 = 9$   
 $\nabla \cdot \mathbf{F} = 1$  (since  $\frac{\partial}{\partial x} 4y + \frac{\partial}{\partial y} 1 + \frac{\partial}{\partial z} 2 = 4 + 0 + 0 = 4$ )  
 $\nabla \times \mathbf{F} = \mathbf{0}$  (since  $\mathbf{F} = (4y, 1, 2)$ )  
 TAKE NORMAL AS  $(2, 0, 2)$   
 $|N| = \sqrt{x^2 + y^2 + z^2} = \sqrt{9} = 3$   
 $\therefore \frac{N}{|N|} = \frac{1}{3}(2, 0, 2) = \frac{1}{3}(2, 0, 2)$

Setup:  
 $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{N} dS = \int_S (4y, 1, 2) \cdot \frac{1}{3}(2, 0, 2) dS = \int_S \frac{1}{3}(4y + 0 + 4) dS = \int_S \frac{1}{3}(4y + 4) dS$

SWITCH THE INTEGRAL INTO SPHERICAL POLARS:

$$\begin{aligned} &= \frac{1}{3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left( \frac{1}{3} [4(\sin\theta\cos\phi)(\sin\theta\cos\phi) + 3\sin\theta\cos\phi + 2(\cos\phi)] \right) (3\sin\theta) d\theta d\phi \\ &= \frac{1}{3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 3\sin^2\theta\cos^2\phi + 3\sin^2\theta\cos\phi + 3\sin\theta\cos\phi d\theta d\phi \\ &= \frac{1}{3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 3\sin^2\theta(-\cos\phi)\cos^2\phi + \frac{3}{2}\sin^2\theta - \frac{3}{2}\sin^2\theta\cos^2\phi + (\cos\phi) d\theta d\phi \\ &= \frac{1}{3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 3\sin^2\theta(-\cos\phi)\cos^2\phi + \frac{3}{2}\sin^2\theta - \frac{3}{2}\sin^2\theta\cos^2\phi + (\cos\phi) d\theta d\phi \\ &= \frac{1}{3} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} -9\cos^2\theta\sin^2\phi\cos^2\phi + \frac{3}{2}\sin^2\theta - \frac{3}{2}\sin^2\theta\cos^2\phi + (\cos\phi) d\theta d\phi \\ &= \frac{1}{3} \int_0^{\frac{\pi}{2}} \left[ -9\cos^2\theta\sin^2\phi\cos^2\phi + \frac{3}{2}\sin^2\theta - \frac{3}{2}\sin^2\theta\cos^2\phi + (\cos\phi) \right] \Big|_0^{\frac{\pi}{2}} d\phi \\ &= \frac{1}{3} \int_0^{\frac{\pi}{2}} \left[ \frac{27}{4}\sin^4\phi + 3 \right] - \left[ -9\cos^2\theta\sin^2\phi\cos^2\phi + \frac{3}{2}\sin^2\theta \right] \Big|_0^{\frac{\pi}{2}} d\phi \\ &= \frac{1}{3} \int_0^{\frac{\pi}{2}} \left[ \frac{27}{4}\sin^4\phi + 3 + 21\cos^2\theta\sin^2\phi \right] d\phi \\ &= 3 \int_0^{\frac{\pi}{2}} \left[ -\frac{27}{4}\cos^4\theta + 12.5\cos^2\theta \right] \Big|_0^{\frac{\pi}{2}} d\phi = 3 \left[ (0 + \frac{27}{4}) - (-\frac{27}{4}) \right] = 36 + \frac{27}{2}\pi \end{aligned}$$

**Question 4**

Evaluate the surface integral

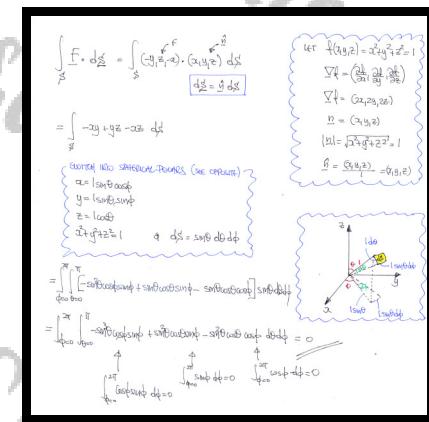
$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with equation

$$x^2 + y^2 + z^2 = 1,$$

and  $\mathbf{F} = -y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$ .

□



**Question 5**

$$\mathbf{F}(x, y, z) \equiv \mathbf{i} + \frac{1}{2}y\mathbf{j} + z^2\mathbf{k}.$$

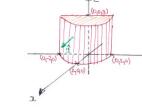
Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the curved cylindrical surface with equation

$$x^2 + y^2 = 4, \quad x \geq 0, \quad 0 \leq z \leq 3.$$

3π+12



The curved surface has equation  $x^2 + y^2 = 4$   
 Let  $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$   
 $\mathbf{n} = \nabla \mathbf{r} = \left( \frac{\partial \mathbf{r}}{\partial x}, \frac{\partial \mathbf{r}}{\partial y}, \frac{\partial \mathbf{r}}{\partial z} \right) = (2x, 2y, 1)$   
 TRACE AS NORMA  $\mathbf{n} = (2x, 2y, 1)$   
 $\hat{\mathbf{n}} = \frac{(2x, 2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}} = (2x\sqrt{4x^2 + 4y^2 + 1}, 2y\sqrt{4x^2 + 4y^2 + 1}, 1/\sqrt{4x^2 + 4y^2 + 1})$

**SURFACE AREA ELEMENT FORMULA**  
 $dA = 2\pi r dy dz$        $d\mathbf{S} = 2\pi r \hat{\mathbf{n}} dy dz$

$$\begin{aligned} \int_S \mathbf{E} \cdot d\mathbf{S} &= \int_S (\mathbf{E} \cdot \hat{\mathbf{n}}) d\mathbf{S} = \int_S ((1/2)y\mathbf{i}^2, -\frac{1}{2}x\mathbf{i}\mathbf{j}, z^2\mathbf{k}) \cdot d\mathbf{S} = \int_S \frac{1}{2}y^2 + \frac{1}{2}x^2 d\mathbf{S} \\ &= \int_0^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(2\pi r)^2 + \frac{1}{2}(2\pi r)^2 2 dy dz \\ &= \int_0^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi r d\theta + 2\pi r^2 d\theta dy dz \\ &= \int_0^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi r d\theta + 2(\frac{1}{2}r^2\sin\theta) dy dz \\ &= \int_0^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi r d\theta + (-r^2\cos\theta) dy dz \\ &= \int_0^3 \left[ 2\pi r d\theta + \frac{1}{2}r^2\sin\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy \\ &= \int_0^3 \left( 2 + \frac{\pi}{2} \right) r - \left( -2 - \frac{\pi}{2} \right) r dy \\ &= \int_0^3 4 + \pi r dy \\ &= \left[ (4 + \pi)r \right]_0^3 \\ &= 3(\pi + 4) \end{aligned}$$

**Question 6**

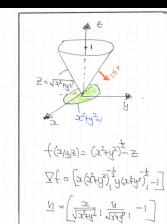
$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + z^4\mathbf{k}$$

Calculate the flux of  $\mathbf{F}$  through the open surface with equation

$$z = \sqrt{x^2 + y^2}, \quad z \leq 1,$$

in the direction of  $z$  decreasing.

$$-\frac{1}{3}\pi$$



$$\int_S (x, y, z^4) \cdot \hat{n} \, dS$$
  

$$= \int_R (x, y, z^4) \cdot \frac{\hat{n}}{|n|} \, dS$$
  
 PROJECT onto the  $xy$  PLANE  
 onto the region  $R_1$ , with  
 equation  $x^2 + y^2 = z$   

$$= \int_R (x, y, z^4) \cdot \frac{\hat{n}}{\sqrt{x^2 + y^2}} \, dS$$
  

$$= \int_R (x, y, z^4) \cdot \frac{\hat{n}}{\sqrt{x^2 + y^2}} \cdot \frac{dx \, dy}{\sqrt{x^2 + y^2}}$$
  

$$= \int_R (x, y, z^4) \cdot \frac{\hat{n}}{|n|} \cdot \frac{dx \, dy}{|n|}$$
  

$$= \int_R (x, y, z^4) \cdot \frac{\hat{n}}{|n|} \cdot \frac{dx \, dy}{(x, y, z^4)}$$
  

$$= \int_R (x, y, z^4) \cdot \left[ \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, z^4 \right] \cdot \left[ \frac{dx \, dy}{\sqrt{x^2 + y^2}} \right]$$
  

$$= \int_R \left( x, y, z^4 \right) \cdot \left[ \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, z^4 \right] \cdot \frac{dx \, dy}{\sqrt{x^2 + y^2}}$$
  

$$= \int_R -\frac{x^2}{\sqrt{x^2 + y^2}} - \frac{y^2}{\sqrt{x^2 + y^2}} + z^4 \, dx \, dy$$
  
 SUBSTITUTE INTO REAR POINTS NEXT

$$\begin{aligned}
 &= \int_R (-r + r^4)(r \, dr \, d\theta) \\
 &= \int_{0}^{2\pi} \int_{r=0}^1 (-r^2 + r^5) \, dr \, d\theta \\
 &= 2\pi \left[ -\frac{1}{3}r^3 + \frac{1}{6}r^6 \right]_0^1 \\
 &= 2\pi \left[ \left(\frac{1}{6} - \frac{1}{3}\right) - 0 \right] \\
 &= -\frac{\pi}{3}
 \end{aligned}$$

**Question 7**

The surface  $S$  has Cartesian equation

$$z = 1 - x^2 - y^2, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

Evaluate the surface integral

$$\int_S 15zi \cdot d\mathbf{S}.$$

[4]

$\begin{aligned} z &= 1 - x^2 - y^2 \\ 1 - x^2 - y^2 - 1 &= 0 \\ 4x^2 + 4y^2 &= z^2 \\ \sqrt{4x^2 + 4y^2} &= |z| \\ \sqrt{4x^2 + 4y^2 + 1} &= \sqrt{z^2 + 1} \\ &= \sqrt{1 + 4x^2 + 4y^2} \\ &= \sqrt{1 + 4x^2 + 4y^2} \\ dS &= (2x, 2y, 1) \sqrt{1 + 4x^2 + 4y^2} \end{aligned}$

$$\begin{aligned} \text{Dikl} \quad \int_S 15zi \cdot d\mathbf{S} &= \int_S 15z \sqrt{1 + 4x^2 + 4y^2} dx dy \dots \text{PROJECT onto the xy PLANE} \\ &\quad \text{i.e. replace } z \text{ in terms of } x \text{ and } y \\ &= \int_S 15z \sqrt{1 + 4x^2 + 4y^2} \frac{x}{\sqrt{1 + 4x^2 + 4y^2}} dx dy = \int_S \frac{15z^2 \cdot x}{\sqrt{1 + 4x^2 + 4y^2}} dx dy \\ &= \int_S \frac{15(1 - x^2 - y^2)x}{(2x, 2y, 1) \cdot (2x, 2y, 1)} dx dy = \int_S 30xy^2 dx dy \\ &\quad \text{SINCE THIS POINT PONI ... NOTE THAT } z = 1 - x^2 - y^2 = 1 - r^2 \\ &= \int_{y=0}^{\frac{\pi}{2}} \int_{x=0}^1 30r \cos^2(r)(-r^2) (r dr) (d\theta) = \left[ \int_{y=0}^{\frac{\pi}{2}} \cos^2 \theta d\theta \right] \left[ \int_{x=0}^1 30r^3 - 30r^5 dr \right] \\ &= \left[ \sin \theta \right]_0^{\frac{\pi}{2}} \left[ 10r^4 - 10r^6 \right]_0^1 = 1 \times \left( 0 - 0 \right) = 0 \end{aligned}$$

**Question 8**

$$\mathbf{F}(x, y, z) \equiv -y\mathbf{i} + x\mathbf{j} + 3z\mathbf{k}.$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface of the hemisphere with equation

$$x^2 + y^2 + z^2 = 9, \quad z \geq 0,$$

contained inside the cylinder with equation

$$x^2 + y^2 = 4, \quad z \geq 0,$$

$$2\pi [27 - 5\sqrt{5}]$$

$\int_S (-y, x, 3z) \cdot \hat{n} \, dS$   
 $= \int_S (-y, x, 3z) \cdot \frac{\mathbf{i}}{\sqrt{x^2+y^2+z^2}} \, dA$   
 $= \int_R (-y, x, 3z) \cdot \frac{\mathbf{i}}{\sqrt{9-x^2-y^2}} \, dx \, dy$   
 $= \int_R (-y, x, 3z) \cdot \frac{\partial z}{\partial x} \, dx \, dy$   
 $= \int_R \frac{-2y + 3z^2}{\sqrt{9-x^2-y^2}} \, dx \, dy$   
 $= \int_R 3z \, dx \, dy$   
 $= \int_R 3\sqrt{9-x^2-y^2} \, dx \, dy$   
 $= \int_{\theta=0}^{2\pi} \int_{r=0}^2 3(r^2)^{\frac{1}{2}} (r \, dr \, d\theta)$   
 $= 2\pi \int_0^2 3r(9-r^2)^{\frac{1}{2}} \, dr = 2\pi \left[ -(9-r^2)^{\frac{3}{2}} \right]_0^2 = 2\pi \left[ (9-r^2)^{\frac{3}{2}} \right]_2^0$   
 $= 2\pi [27 - 5\sqrt{5}]$

Let  $\mathbf{f} = \mathbf{r}$   
 $\mathbf{f}(x, y, z) = (x, y, z)$   
 $\nabla f = (1, 0, 0)$   
 $\Omega = (2, 2, 2)$

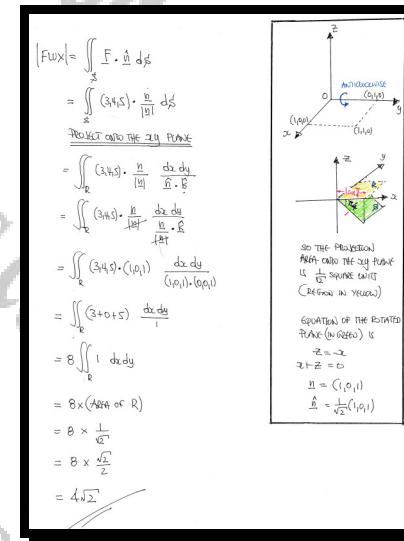
**Question 9**

Space is filled uniformly by the constant vector field  $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ .

A square lamina whose vertices are at  $(0,0,0)$ ,  $(1,0,0)$ ,  $(1,1,0)$  and  $(0,1,0)$  is rotated by  $\frac{1}{4}\pi$ , anticlockwise, about the  $y$  axis.

determine the magnitude of the flux of the field through the rotated lamina.

$$4\sqrt{2}$$



**Question 10**

The surface  $S$  has Cartesian equation

$$z = 2 - x^2 - y^2, \quad x^2 + y^2 \leq 1.$$

- a) Sketch the graph of  $S$ .
- b) Given that  $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ , evaluate the integral

$$\int_S \mathbf{F} \cdot d\mathbf{S}.$$

$$\boxed{\frac{3\pi}{2}}$$

**a)**

$z = 2 - x^2 - y^2$        $|x| \leq 1$        $|y| \leq 1$

If  $x=0$ ,  $z=2-y^2$

If  $y=0$ ,  $z=2-x^2$

$x^2 + y^2 = 1$        $z = 2 - x^2 - y^2$

Surface in "cone down" perspective

**b)** Now  $\mathbf{F} = (y, -x, z)$

Let surface be  $\hat{\mathbf{s}}(r, \theta, z) = x^2 + y^2 + z - 2$

$d\mathbf{s} = \left( \frac{\partial \hat{\mathbf{s}}}{\partial r}, \frac{\partial \hat{\mathbf{s}}}{\partial \theta}, \frac{\partial \hat{\mathbf{s}}}{\partial z} \right) = (2r, 0, 1)$

$\hat{\mathbf{F}} = (y, -x, z)$

$\hat{\mathbf{F}} \cdot d\mathbf{s} = (y, -x, z) \cdot (2r, 0, 1)$

$\int_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \hat{\mathbf{F}} \cdot \hat{\mathbf{F}} \, d\mathbf{s}$

$= \iint_S (y, -x, z) \cdot (2r, 0, 1) \, d\mathbf{s}$

$= \iint_S (2rz - 2r^2 + z) \, d\mathbf{s}$

**NOTE:** PAPER SURFACE AND THE XY PLANE GIVES THE OPPOSITE  $x^2 + y^2 = 1$ , ZERO (AND IS INWARD),  $\mathbf{R}$

$d\mathbf{s} = \sqrt{1+4r^2} \, dr \, d\theta$

Top projection onto the xy plane

$\int_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (2rz - 2r^2 + z) \sqrt{1+4r^2} \, dr \, d\theta$

**Sketch into polar coords:**

$= \int_{0}^{\pi/2} \int_{0}^{1} (2r^2 - 2r^2 + z) \sqrt{1+4r^2} \, dr \, d\theta$

$= \int_{0}^{\pi/2} \int_{0}^{1} 2r^2 \sqrt{1+4r^2} \, dr \, d\theta$

$= \int_{0}^{\pi/2} 2 \left[ \frac{1}{2} (1+4r^2)^{1/2} \right] \Big|_0^1 \, d\theta$

$= \int_{0}^{\pi/2} 2 \left[ \frac{1}{2} (1+4)^{1/2} \right] \, d\theta$

$= \pi \times \left[ 1 - \frac{1}{2} \right] = \boxed{\frac{3\pi}{2}}$

**Question 11**

The surface  $S$  has Cartesian equation

$$(z-1)^2 = x^2 + y^2, \quad 1 \leq z \leq 3.$$

- a) Sketch the graph of  $S$ .
- b) Given that  $\mathbf{F} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$ , evaluate the integral

$$\int_S \mathbf{F} \cdot d\mathbf{S}.$$

[4π]

a)  $(z-1)^2 = x^2 + y^2, \quad 2 \leq z \leq 3$

• When  $z=0$ :  
 $(z-1)^2 = y^2$   
 $z-1 = \pm y$   
 $z = 1 \pm y$

• When  $y=0$ :  
 $(z-1)^2 = x^2$   
 $z-1 = \pm x$   
 $z = 1 \pm x$

Surface  $S$  consists of two paraboloids opening upwards along the  $z$ -axis, symmetric about the  $z$ -axis, with vertices at  $(0,0,1)$  and  $(0,0,3)$ . The surface is bounded by  $1 \leq z \leq 3$ .

b)  $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_R (\hat{x}_1 \hat{x}_2 \hat{z}) \cdot \hat{n} \, dA$  ... Project onto the xy-plane

$$= \int_R (\hat{x}_1 \hat{x}_2 \hat{z}) \cdot \frac{\hat{n}}{|\hat{n}|} \, dA$$

$$= \int_R (\hat{x}_1 \hat{x}_2 \hat{z}) \cdot \frac{1}{\sqrt{1+x_1^2+x_2^2}} \, dA$$

$$= \int_R \frac{(\hat{x}_1 \hat{x}_2 \hat{z}) \cdot \frac{1}{\sqrt{1+x_1^2+x_2^2}}}{\sqrt{1+x_1^2+x_2^2}} \, dA$$

$$= \int_R \frac{(\hat{x}_1 \hat{x}_2 \hat{z}) \cdot \frac{1}{\sqrt{1+(x_1^2+y_1^2)}}}{\sqrt{1+(x_1^2+y_1^2)}} \, dA$$

Now let  $z$  be written as:  
•  $\hat{x}_1 \hat{y}_2 \hat{z} = \hat{x}_1 \hat{x}_2 - \hat{y}_1 \hat{z}$   
•  $\hat{z} = (2x_1, 2y_1, -(x_1^2+y_1^2))$   
•  $\hat{n} = (x_1, y_1, -z)$

THUS

$$\begin{aligned} \dots &= \int_R \frac{(\hat{x}_1 \hat{x}_2 \hat{z}) \cdot (x_1, y_1, -z)}{(x_1^2+y_1^2+z^2)^{1/2}} \, dA = \int_R \frac{2x_1 x_2 - y_1 z}{(x_1^2+y_1^2+z^2)^{1/2}} \, dA \\ &= \int_R \left[ \frac{2x_1^2}{1-z^2} + \frac{2x_1 y_1}{1-z^2} + y_1^2 \right] \, dA \\ &\text{Now } z = 1 + \sqrt{x_1^2+y_1^2} \quad (1 \leq z \leq 3) \\ &= \int_R \left[ \frac{2x_1^2}{1-(1+\sqrt{x_1^2+y_1^2})^2} + \frac{2x_1 y_1}{1-(1+\sqrt{x_1^2+y_1^2})^2} + y_1^2 \right] \, dA \\ &\text{Circles in } x_1 \text{ and } y_1 \text{ are symmetric about the origin.} \\ &= \int_R y_1^2 \, dA \quad \text{THUS } D^* \text{ is the top-half of } R \text{ as it is tilted in } y_1 \\ &\dots \text{Switch into polar coordinates.} \\ &= \int_{0 \pi/2}^{\pi} \int_0^2 2(r \cos \theta)^2 \, r \, dr \, d\theta = \int_{0 \pi/2}^{\pi} \int_0^2 2r^3 \cos^2 \theta \, dr \, d\theta \\ &= \int_{0 \pi/2}^{\pi} \left[ \frac{1}{2} r^4 \sin^2 \theta \right]_0^2 \, d\theta = \int_{0 \pi/2}^{\pi} 8r^3 \cos^2 \theta \, d\theta = \int_{0 \pi/2}^{\pi} 8 \left( \frac{1}{2} + \frac{1}{2} \cos 2\theta \right) \, d\theta \\ &= \int_{0 \pi/2}^{\pi} 4 + 4r^2 \cos 2\theta \, d\theta = \int_{0 \pi/2}^{\pi} 4 \, d\theta = 4\pi \end{aligned}$$

**Question 12**

$$\mathbf{F}(x, y, z) \equiv 3x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 1.$$

[4π]

$\mathbf{F} = (3x, y^2, z^2)$  ONCE THE UNIT SURFACE  $x^2 + y^2 + z^2 = 1$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS$$

$$= \int_S (3x, y^2, z^2) \cdot (x, y, z) \, dS$$

$$= \int_S 3x^2 + y^2 + z^2 \, dS$$

SWITCH INTO SPHERICAL COORDINATES

$$= \int_0^{2\pi} \int_0^\pi [3(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2] r^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^\pi 3\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^\pi \sin^2\theta(\cos^2\phi + \sin^2\phi + 1) \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^\pi \sin^2\theta \, d\theta \, d\phi$$

$$= \left[ -\frac{1}{2}\sin^2\theta \right]_0^{2\pi} \int_0^\pi d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} \left[ 3\sin^2\theta\cos^2\phi + \cos^2\theta \right]_0^\pi d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} \left[ 3\sin^2\theta(-1-\cos^2\phi) + \cos^2\theta \right]_0^\pi d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} \left[ 3\sin^2\theta(-3-\cos^2\phi) + \cos^2\theta \right]_0^\pi d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} \left[ -3\sin^2\theta\cos^2\phi + \cos^2\theta(1-\frac{1}{4}\cos^2\theta) \right]_0^\pi d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} (3\cos^2\theta - \cos^4\theta) d\phi > 2\pi \int_0^{2\pi} (-3\cos^2\theta + \cos^2\theta) d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} (2\cos^2\theta - 2\cos^4\theta) d\phi = \int_0^{2\pi} 2 + 2\cos 2\theta \, d\phi = \left[ 2\theta + \sin 2\theta \right]_0^{2\pi} = 4\pi$$

$\mathbf{F} = (3x, y^2, z^2)$  ONCE THE SURFACE OF THE SPHERE  $x^2 + y^2 + z^2 = 1$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} \, dV \quad (\text{BY GAUSS'S THEOREM})$$

$$= \int_V (2x + 2y + 2z) \cdot (3x, y^2, z^2) \, dV = \int_V 3 + 2y + 2z \, dV$$

SWITCH INTO SPHERICAL POLAR COORDINATES

$$= \int_{0 \leq \rho \leq 1} \int_{0 \leq \theta \leq \pi} \int_{0 \leq \phi \leq 2\pi} [3 + 2\sin\theta\cos\phi + 2\cos\theta\sin\phi] [\rho^2 \sin\theta] \, d\rho \, d\theta \, d\phi$$

SPHERICAL COORDINATES ( $(\rho, \theta, \phi)$ )

$$x = \rho \sin\theta \cos\phi$$

$$y = \rho \sin\theta \sin\phi$$

$$z = \rho \cos\theta$$

$$x^2 + y^2 + z^2 = 1 \Rightarrow \rho^2 = 1$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$dV = \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$$

$$d\theta = \sin\theta \, d\theta$$

$$d\phi = \sin\theta \, d\phi$$

$$d\rho = \rho^2 \sin\theta \, d\rho$$

RECALL THAT THE INTEGRATION WITH RESPECT TO  $\phi$  IS FROM  $0$  TO  $2\pi$ .  
HENCE FROM THE INTEGRATION WITH RESPECT TO  $\theta$  IS FROM  $0$  TO  $\pi$ .

AUGMENTATION

$$= \int_0^1 \int_0^\pi \int_0^{2\pi} [3 + 2\sin\theta\cos\phi + 2\cos\theta\sin\phi] \rho^2 \sin\theta \, d\phi \, d\theta \, d\rho$$

$$= \int_0^1 \int_0^\pi \left[ \int_0^{2\pi} [3 + 2\sin\theta\cos\phi + 2\cos\theta\sin\phi] \, d\phi \right] \rho^2 \sin\theta \, d\theta \, d\rho$$

$$= \int_0^1 \left[ \frac{1}{2} \int_0^\pi [3 + 2\sin\theta\cos\phi + 2\cos\theta\sin\phi]^2 \, d\phi \right] \rho^2 \sin\theta \, d\theta \, d\rho$$

$$= \int_0^1 \left[ \frac{1}{2} \int_0^\pi [3 + 2\sin\theta\cos\phi + 2\cos\theta\sin\phi]^2 \, d\phi \right] \rho^2 \, d\rho$$

$$= \int_0^1 \left[ \frac{1}{2} \int_0^\pi [3 + 2\sin\theta\cos\phi + 2\cos\theta\sin\phi]^2 \, d\phi \right] \rho^2 \, d\rho$$

$$= 3 \times \text{VOLUME OF A UNIT SPHERE}$$

$$= 3 \times \frac{4}{3}\pi = 4\pi$$

**Question 13**

$$\mathbf{F}(x, y, z) \equiv (x+y)\mathbf{i} + (x-y)\mathbf{j} + (x+z)\mathbf{k}.$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with Cartesian equation

$$z = 1 - x^2 - y^2, \quad z \geq 0.$$

$$\boxed{\frac{\pi}{2}}$$

$\mathbf{F} = (x+y, x-y, x+z)$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{n} \, dS$$

$$= \int_S (x+y, x-y, x+z) \cdot \frac{\mathbf{i}}{|v|} \, dS$$

Project onto the  $xy$ -plane, onto the region  $R: z = 1 - x^2 - y^2 \leq 0$

$$dS = \frac{dx \, dy}{\sqrt{1-x^2-y^2}}$$

$$= \int_R (x+y, x-y, x+z) \cdot \frac{\mathbf{i}}{\sqrt{1-x^2-y^2}} \, dx \, dy$$

$$= \int_R (x+y, x-y, x+z) \cdot \frac{\mathbf{i}}{\sqrt{1-x^2-y^2}} \, dx \, dy$$

$$= \int_R (x^2+2xy+2y^2+x^2+y^2+1) \, dx \, dy$$

Note:  $R$  is a symmetric double cone in  $x$ ,  $y$ , so the integral of  $x$  or  $y$  will be zero

$$= \int_R 2x^2+2y^2+1 \, dx \, dy$$

SURFACE IN PIANO ROUND

$$= \int_0^\infty \left[ 2\left( r\cos\theta - r^2\sin^2\theta \right) + 1 - r^2 \right] (r \, dr \, d\theta)$$

$$= \int_{0^\circ}^{2\pi} \int_{r=0}^1 \left[ 2r^3(\cos\theta - \sin^2\theta) + r - r^3 \right] \, dr \, d\theta$$

$$= \int_{0^\circ}^{2\pi} \int_{r=0}^1 \left[ 2r^3\cos\theta + r - r^3 \right] \, dr \, d\theta$$

Raw to answer

$$= \int_{0^\circ}^{2\pi} \int_{r=0}^1 r - r^3 \, dr \, d\theta$$

$$= \int_{0^\circ}^{2\pi} \left[ \frac{1}{2}r^2 - \frac{1}{4}r^4 \right]_0^1 \, d\theta$$

$$= \int_{0^\circ}^{2\pi} \frac{1}{4} \, d\theta$$

$$= \frac{1}{4} \times 2\pi = \frac{\pi}{2}$$

**Question 14**

$$\mathbf{F}(x, y, z) = (x+z+xy)\mathbf{i} + (z^2 - 2xz - y)\mathbf{j} + \mathbf{k}$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 4, \quad z \geq 0.$$

[4π]

$\mathbf{F} \cdot d\mathbf{S} = (x+z+xy)\mathbf{i} + (z^2 - 2xz - y)\mathbf{j} + \mathbf{k}$

$\nabla \Phi = \left( \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} \right)$

$\nabla \Phi = (2x+2y, 2x, 2z)$

$|\nabla \Phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2$

$dS = \frac{1}{2} |\nabla \Phi| dxdy = dxdy$

$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{S \cap \{z=0\}} (x+xy, z^2 - 2xz - y, 1) \cdot (2x, 2x, 2z) dxdy$

$= \frac{1}{2} \int_{S \cap \{z=0\}} (x^2 + xy + x^2 + y^2 + z^2 - 2xz - y) dxdy$

Switch to polar coordinates:

$dS = \frac{r dr}{\sqrt{1+r^2}}$

$dS = \frac{r dr}{\sqrt{4-r^2}}$

$= \int_0^2 \int_{-\pi}^{\pi} (r^2 + r^2 \cos^2 \theta + r^2 \sin^2 \theta - 2r^2 \cos \theta - r) r dr d\theta$

$= \int_0^2 \int_{-\pi}^{\pi} (r^3 - 2r^3 \cos \theta) r dr d\theta + 4\pi$

$= \int_0^2 \int_{-\pi}^{\pi} r^4 (\cos \theta - 2) r dr d\theta + 4\pi$

$= 4\pi$

NOTE:  $R$  IS A SEMI-CIRCLE LOCATED IN THE PLANE  $z=0$ , SO ALL PARTS OF  $\mathbf{F}$  ARE  $0$  HERE (NO CONDUCTION)

$= \int_R (2x^2 + 2y^2 + 2z^2 + 2xz - 2y - \frac{y^2}{2} + 1) dxdy \quad (z=\sqrt{4-x^2-y^2})$

$= \int_R \frac{3x^2}{2} + \frac{3y^2}{2} + 2z^2 + 2xz - \frac{y^2}{2} + 1 dxdy = \int_R \frac{3x^2}{2} dxdy + \int_R 1 dxdy$

SWITCH TO POLAR COORDINATES:

$= \int_0^2 \frac{r^2 + r^2}{\sqrt{4-r^2}} d\theta dr + (4\pi \times 2^2)$

$= \int_0^2 \frac{(r^2 \cos^2 \theta + r^2 \sin^2 \theta)}{\sqrt{4-r^2}} d\theta dr + 4\pi$

$= \int_0^2 \frac{r^2 (\cos^2 \theta + \sin^2 \theta)}{\sqrt{4-r^2}} d\theta dr + 4\pi$

$= \int_0^2 r^2 d\theta dr + 4\pi$

$= 4\pi$

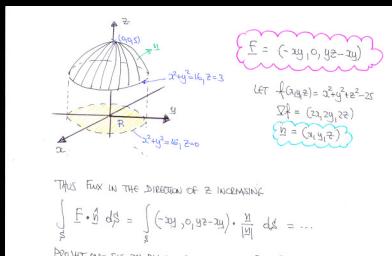
## Question 15

$$\mathbf{F}(x, y, z) \equiv -xy\mathbf{i} + (yz - xy)\mathbf{k}.$$

Show that there is zero net flux of  $\mathbf{F}$  through the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 25, \quad z \geq 3.$$

proof



$\mathbf{F} = (-xy, 0, yz - xy)$   
 Let  $f(x, y, z) = x^2 + y^2 + z^2 - 25$   
 $\nabla f = (2x, 2y, 2z)$   
 $\mathbf{n} = (x, y, z)$

THIS FLUX IN THE DIRECTION OF  $\mathbf{z}$  INCLINING

$$\int_S \mathbf{F} \cdot \mathbf{\hat{n}} \, dS = \int_S (-xy, 0, yz - xy) \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \, dS = \dots$$

PROJECT onto the  $xy$  plane onto the region  $R$ :  $x^2 + y^2 \leq 16$

$$dS = \frac{dx}{dz} dy$$

$$\dots = \int_R (-xy, 0, yz - xy) \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \frac{dx}{dz} dy$$

$$= \int_R (-xy, 0, yz - xy) \cdot (x, y, z) \frac{dx}{dz} dy$$

$$= \int_R -xy + yz^2 - xyz \left( \frac{dx}{dz} dy \right)$$

$$= \int_R -\frac{x^2 y}{z} + yz^2 - xyz \, dx \, dy$$

Now  $x^2 + y^2 + z^2 = 25$   
 $z^2 = 25 - x^2 - y^2$   
 $z = \pm \sqrt{25 - x^2 - y^2}$

$$\dots = \int_R \left[ -\frac{x^2 y}{\sqrt{25 - x^2 - y^2}} + y \sqrt{25 - x^2 - y^2} - xyz \right] dx \, dy$$

BUT  $R$  IS A SYMMETRIC DOMAIN IN  $x$  &  $y$ , SO  $\cancel{xyz}$  TERMS  
IN A HAVING  $y$  WHICH NO CONTRIBUTION

$$= 0$$

∴ ZERO FLUX

**Question 16**

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

- a) Given that  $S$  is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 1, z \geq 0,$$

show that

$$\int_S \mathbf{F} \cdot d\mathbf{S} = 4 \int_R \left[ \frac{x^2}{\sqrt{1-x^2-y^2}} + 1 - x^2 - y^2 \right] dx dy,$$

where  $R$  is the region in the first quadrant with Cartesian equation

$$x^2 + y^2 \leq 1, x \geq 0, y \geq 0.$$

- b) Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S}.$$

$$\boxed{\frac{7}{6}\pi}$$

**a)**  $\int_S \mathbf{F} \cdot d\mathbf{S}$  over the surface  $x^2 + y^2 + z^2 = 1, z \geq 0$

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S \mathbf{F} \cdot \hat{n} \, dS = \int_S (\alpha(x, y, z)^T \cdot (\alpha(x, y, z))^T) \, dS \\ &= \int_S x^2 + y^2 + z^2 \, dS \end{aligned}$$

PROJECTION onto the  $xy$  plane, onto the circle (and its interior)

unit normal vector  $\hat{n} = \langle x, y, z \rangle$ , length  $r$

$$\begin{aligned} \hat{n} \cdot \hat{F} &= (\alpha(x, y, z)) \cdot (\alpha(x, y, z)) = z \\ dS &= \frac{dxdy}{r} \end{aligned}$$

$$\begin{aligned} &= \int_R (x^2 + y^2 + z^2) \frac{dxdy}{r} \quad \text{where } z = \sqrt{1-x^2-y^2} \\ &= \int_R \frac{z^2}{r} + \frac{z^2}{r} + z^2 \, dxdy \\ &= \int_R \frac{z^2}{\sqrt{1-x^2-y^2}} + \frac{z^2}{\sqrt{1-x^2-y^2}} + (1-x^2-y^2) \, dxdy \\ &\quad \uparrow \quad \uparrow \quad \uparrow \quad \text{note } z = \sqrt{1-x^2-y^2} \\ &= 4 \int_R \frac{z^2}{\sqrt{1-x^2-y^2}} + (1-x^2-y^2) \, dxdy \\ &\quad \text{where } R \text{ is the unit circle } x^2+y^2=1 \text{ in the first quadrant} \end{aligned}$$

**b)** Switch into polar coordinates

$$\begin{aligned} 4 \int_R \frac{z^2}{\sqrt{1-x^2-y^2}} + (1-x^2-y^2) \, dxdy &= 4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^1 \frac{r^2}{\sqrt{1-r^2}} w \hat{S}_\theta + r - r^3 \, dr d\theta \\ &= 4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^1 r^2(1-r^2)^{\frac{1}{2}} w \hat{S}_\theta \, dr d\theta + 4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^1 r - r^3 \, dr d\theta \\ &\quad \uparrow \quad \uparrow \quad \uparrow \quad \text{note } w = r \\ &\quad \text{where } r = \sqrt{1-r^2} \\ &\quad \text{and } dr = \frac{1}{2\sqrt{1-r^2}} dr \\ &\quad \text{and } r^2 = 1-r^2 \\ &\quad \text{and } r^3 = r^2 r \\ &\quad \text{and } r^2 = 1-r^2 \\ &\quad \text{and } r^2 = 1-r^2 \end{aligned}$$

$$\begin{aligned} &= 4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^1 r^2 \sin^2 \theta \left( \frac{1}{2\sqrt{1-r^2}} \right) dr d\theta + 4 \int_{\theta=0}^{\frac{\pi}{2}} \left( \frac{1}{2}r^2 - \frac{1}{4}r^4 \right) \Big|_0^1 d\theta \\ &= 4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^1 r^2 \sin^2 \theta \, dr d\theta + 4 \int_{\theta=0}^{\frac{\pi}{2}} \frac{1}{4} \, d\theta \\ &= 4 \int_{\theta=0}^{\frac{\pi}{2}} \left[ \frac{1}{3}r^3 \sin^2 \theta \right]_0^1 d\theta + \frac{\pi}{2} \\ &= 4 \int_{\theta=0}^{\frac{\pi}{2}} \left[ \frac{1}{3} - \frac{1}{3} \sin^2 \theta \right] d\theta + \frac{\pi}{2} \\ &= 4 \int_{\theta=0}^{\frac{\pi}{2}} \frac{2}{3} \cos^2 \theta \, d\theta + \frac{\pi}{2} = \frac{8}{3} \int_{\theta=0}^{\frac{\pi}{2}} \frac{1}{2} \cos 2\theta \, d\theta + \frac{\pi}{2} \\ &= \frac{8}{3} \left[ \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_0^{\frac{\pi}{2}} + \frac{\pi}{2} = \frac{8}{3} \left[ \frac{\pi}{4} \right] + \frac{\pi}{2} \\ &= \frac{2}{3}\pi + \frac{1}{2}\pi = \frac{7}{6}\pi \end{aligned}$$

**Question 17**

$$\mathbf{F} = x^2 \mathbf{y}^3 \mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

Show by direct evaluation that

$$\int_S \nabla \wedge \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = 0,$$

where  $S$  is the sphere with equation

$$x^2 + y^2 + z^2 = 1,$$

and  $\hat{\mathbf{n}}$  is an outward unit normal to  $S$ .

**proof**

$\nabla \cdot \mathbf{F} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (x^2 y^3 \mathbf{i} + z\mathbf{j} + x\mathbf{k}) = [0, -1, 0] \cdot [x^2 y^3, 0, 1] = [-1, 0, 1]$

- $\mathbf{n} = (x, y, z)$
- $\hat{\mathbf{n}} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} = (xy, z)$

Integrating over the surface of the sphere  $x^2 + y^2 + z^2 = 1$

$$\int_S \nabla \cdot \mathbf{E} \, dS = \int_S \nabla \cdot \mathbf{F} \, dS = \int_S (-1, 0, 1) \cdot (xy, z) \, dS$$

$$= \int_S -x - y - z \, dS = - \int_S x + y + z \, dS$$

Switch into spherical polar coords

$x = r \sin \theta \cos \phi$	$dS = r^2 \sin \theta \, d\theta \, d\phi$
$y = r \sin \theta \sin \phi$	$0 \leq \theta \leq \pi$
$z = r \cos \theta$	$0 \leq \phi \leq 2\pi$
$x^2 + y^2 + z^2 = 1$	

$$= - \int_0^\pi \int_0^{2\pi} [(r \sin \theta \cos \phi) + (r \sin \theta \sin \phi) + 3(r \cos \theta)] r^2 \sin \theta \, d\theta \, d\phi$$

$$= - \int_0^\pi \int_0^{2\pi} r^3 (\sin \theta \cos \phi + \sin \theta \sin \phi + 3 \cos \theta) \sin \theta \, d\theta \, d\phi$$

**Question 18**

$$\mathbf{F}(x, y, z) = (x + yz)\mathbf{i} + (y^3 z + x)\mathbf{j} + (z + xyz)\mathbf{k}$$

Calculate the magnitude of the flux of  $\mathbf{F}$  through the open cylindrical surface with equation

$$x^2 + y^2 = 1, \quad 0 \leq z \leq 4.$$

**[10π]**

$\mathbf{F} = (x + yz)\mathbf{i} + (y^3 z + x)\mathbf{j} + (z + xyz)\mathbf{k}$

flux =  $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{n} \, dS$

switch into parametric

$= \int_S \mathbf{F} \cdot \left( \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) \cdot \left( \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du dv$

$\frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} \cos u \\ \sin u \\ 0 \end{pmatrix}$

$\frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= \int_S \mathbf{F} \cdot \left( \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) du dv$

$= \int_0^{2\pi} \int_0^4 \left( (x + yz) \frac{\partial}{\partial u} (yz) + (y^3 z + x) \frac{\partial}{\partial v} (yz) \right) (yz, \sin u, 0) du dv$

parametrize fully

$= \int_0^{2\pi} \int_0^4 \left[ (yz + yz \cos^2 u + yz \sin^2 u + yz \cos u \sin u) + (y^3 z + x) \right] (yz, \sin u, 0) du dv$

$= \int_0^{2\pi} \int_0^4 \left[ yz + yz \cos^2 u + yz \sin^2 u + yz \cos u \sin u \right] (yz, \sin u, 0) du dv$

NO CONTRIBUTION FROM  $\theta$ ,  $0 \leq \theta \leq 2\pi$

$= \int_{y=0}^{2\pi} \int_{z=0}^4 \left[ yz + \frac{1}{2} yz \sin 2u \right] (yz, \sin u, 0) du dv$

$= \int_{y=0}^{2\pi} \int_{z=0}^4 4yz^2 + 8yz^2 \sin u \, dv$

$\int_{y=0}^{2\pi} \int_{z=0}^4 4yz^2 \, dv + \int_{y=0}^{2\pi} \int_{z=0}^4 8yz^2 \sin u \, dv$

switch into beta & gamma functions

$= 8 \int_0^{2\pi} u^2 \cos^2 u (8u^3) \, du + 16 \int_0^{2\pi} u (8u^3) \sin u \, du$

$= 8B(\frac{3}{2}, \frac{1}{2}) + 16B(\frac{5}{2}, \frac{1}{2}) = 8 \left[ \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)} \right] + 16 \left[ \frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{\Gamma(4)} \right]$

$= 8 \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{1!} + 16 \times \frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{2!}$

$= 8 \Gamma(\frac{5}{2})\Gamma(\frac{1}{2}) + 8 \Gamma(\frac{7}{2})\Gamma(\frac{1}{2})$

$= 8 \Gamma(\frac{5}{2})\Gamma(\frac{1}{2}) + 8 \times \frac{3}{2}\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})$

$= 8 \Gamma(\frac{5}{2})\Gamma(\frac{1}{2}) + 12 \Gamma(\frac{5}{2})\Gamma(\frac{1}{2})$

$= 20 \Gamma(\frac{5}{2})\Gamma(\frac{1}{2})$

$= 20 \times \frac{1}{2} \Gamma(\frac{7}{2})\Gamma(\frac{1}{2})$

$= 10 (\sqrt{\pi})^2 = 10\pi$

ALTERNATIVE BY CYLINDRICAL SHELLS (DIRECT)

$d\mathbf{r} = \begin{pmatrix} \cos u \\ \sin u \\ z \end{pmatrix} \, du \, dz \, dv$

$\mathbf{F} = (x + yz)\mathbf{i} + (y^3 z + x)\mathbf{j} + (z + xyz)\mathbf{k}$

$\mathbf{r} = (x, y, z) \quad \Rightarrow \quad \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \Rightarrow \quad \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$

$r \in [0, 1], \theta \in [0, 2\pi], z \in [0, 4]$

THEN LET THE CYLINDRICAL SURFACE BE  $\mathbf{g}(u, v, z) = z^2 \mathbf{i} - z \mathbf{j} + \mathbf{k}$

$\mathbf{g} = (z, 0, 1) \quad \Rightarrow \quad \mathbf{g} = (z, 0, 1)$

$|g| = \sqrt{z^2 + 0^2 + 1^2} = \sqrt{z^2 + 1} \quad \Rightarrow \quad \mathbf{g} = (z, 0, 1)$

Flux =  $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S (\mathbf{F} \cdot \mathbf{g}) \cdot d\mathbf{S}$

$= \int_0^{2\pi} \int_0^4 (x + yz) \cdot (z, 0, 1) \cdot (z, 0, 1) \, dz \, dv$

switch into cylindrical form

$= \int_0^{2\pi} \int_0^4 [(r \cos \theta + r \sin \theta \cos z) + zr^2 \cos^2 \theta + r^2 \cos \theta \sin^2 \theta] \, dz \, dv$

NO CONTRIBUTION FROM  $\theta$ ,  $0 \leq \theta \leq 2\pi$

$= \int_{z=0}^{2\pi} \int_{r=0}^1 (r \cos \theta + r^2 \sin^2 \theta) \, dz \, dv$

$= \int_{z=0}^{2\pi} \int_{r=0}^1 (r \cos \theta + r^2 \sin^2 \theta) \, dv \, dr$

which matches with previous method

**Question 19**

$$\mathbf{F}(x, y, z) \equiv y\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$$

Find the magnitude of the flux through the surface with parametric equations

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u+v)\mathbf{k}, \quad 0 \leq u \leq 1, \quad 1 \leq v \leq 4.$$

*All integrations must be carried out in parametric.*

,  $\boxed{\frac{1}{2}}$

$\boxed{\mathbf{F}(uv) = \begin{pmatrix} u \\ v \\ uv \end{pmatrix} \quad \mathbf{f}(uv) = \begin{pmatrix} u \\ v \\ u+v \end{pmatrix} \quad 0 \leq u \leq 1 \quad 1 \leq v \leq 4}$

FIND AN EXPRESSION FOR THE "ARM FLUX ELEMENT"  $d\mathbf{s}$ .

- $\frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
- $\frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
- NORMAL =  $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \langle -1, 1, 1 \rangle$
- UNIT NORMAL  $\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

COLLECTING THESE RESULTS

$$d\mathbf{s} = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv$$

$$\hat{\mathbf{n}} \cdot d\mathbf{s} = \frac{1}{\sqrt{3}} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv$$

$$d\mathbf{s} = \frac{\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|}{\sqrt{3}} du dv$$

$$d\mathbf{s} = \left( \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du dv$$

$$d\mathbf{s} = \langle -1, 1, 1 \rangle du dv$$

FINALLY THE FLUX CAN BE CALCULATED

$$\begin{aligned} \text{FLUX} &= \int_S \mathbf{F} \cdot d\mathbf{s} = \int \mathbf{f}(uv) \cdot d\mathbf{s} \\ &= \int_{v=1}^4 \int_{u=0}^1 \langle v^2, uv, -(u+v) \rangle \cdot \langle -1, 1, 1 \rangle du dv \\ &= \int_{v=1}^4 \int_{u=0}^1 (-v - u^2 + u + v) du dv \\ &= \int_{v=1}^4 \int_{u=0}^1 (u - u^2) du dv \\ &= \int_{v=1}^4 \left[ \frac{1}{2}u^2 - \frac{1}{3}u^3 \right]_0^1 du \\ &= \int_1^4 \left( \frac{1}{2} - \frac{1}{3} \right) du \\ &= \int_1^4 \frac{1}{6} du \\ &= \left[ \frac{1}{6}u \right]_1^4 \\ &= \frac{2}{3} \\ &= \boxed{\frac{1}{2}} \end{aligned}$$

**Question 20**

Evaluate the surface integral

$$\int_S z \mathbf{k} \cdot d\mathbf{S},$$

where  $S$  is the surface represented parametrically by

$$\mathbf{r}(\theta, \varphi) = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}, \quad 0 \leq \theta \leq \frac{1}{2}\pi, \quad 0 \leq \varphi \leq \frac{1}{2}\pi.$$

,  $\frac{1}{6}\pi$

$\mathbf{r}(\theta, \varphi) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix} \quad 0 \leq \theta \leq \frac{\pi}{2} \\ 0 \leq \varphi \leq \frac{\pi}{2}$

$$\int_S z \mathbf{k} \cdot d\mathbf{S} = \int_S z \mathbf{k} \cdot \hat{n} dS$$

FIND THE OUT NORMAL TO THE PARAMETRIC SURFACE & SWITCH THE INTEGRAND INTO PARAMETRIC

$$\frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix} + \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{bmatrix} -\sin \theta \sin \varphi \\ \sin \theta \cos \varphi \\ 0 \end{bmatrix}$$

$$\therefore \mathbf{n} = \left| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right| = \begin{vmatrix} 1 & 1 & b \\ \cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta \\ -\sin \theta \sin \varphi & \sin \theta \cos \varphi & 0 \end{vmatrix}$$

$$= \begin{bmatrix} \sin^2 \theta \cos^2 \varphi, \sin^2 \theta \sin^2 \varphi, \cos \theta (-\cos^2 \varphi - \sin^2 \varphi) \\ \sin^2 \theta \cos^2 \varphi, \sin^2 \theta \sin^2 \varphi, \cos \theta (-\cos^2 \varphi - \sin^2 \varphi) \\ \sin^2 \theta \cos^2 \varphi, \sin^2 \theta \sin^2 \varphi, \cos \theta (-\cos^2 \varphi - \sin^2 \varphi) \end{bmatrix}$$

STRIPPING WITH A DIAMETER

SPHERE:  $x^2 + y^2 + z^2 = a^2$   
CYLINDER:  $x^2 + y^2 = b^2$  ( $a > b$ )

AREA OF THE INNER CYLINDRICAL SURFACE IS GIVEN BY

$$2\pi r H = 2\pi b(2h) = 4\pi b h = 4\pi b (a^2 - b^2)^{1/2}$$

NEXT WE FIND THE AREA OF ONE OF THE SEMIHALVES, SHOWN IN YELLOW - PROJECT THE "TOP" (z>0) ONTO THE xy PLANE

$$z = +(\alpha^2 - x^2 - y^2)^{1/2}$$

$$\frac{\partial z}{\partial x} = -x(\alpha^2 - x^2 - y^2)^{-1/2} \quad \frac{\partial z}{\partial y} = -y(\alpha^2 - x^2 - y^2)^{-1/2}$$

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dx dy$$

$$\Rightarrow dS = \sqrt{\frac{x^2}{\alpha^2 - x^2 - y^2} + \frac{y^2}{\alpha^2 - x^2 - y^2} + 1} dx dy$$

ALTERNATIVELY BY SPHERICAL POLARS & PROJECTION ONTO THE xy PLANE

$$\int_S z \mathbf{k} \cdot d\mathbf{S} = \int_S (a \rho \sin \varphi) \mathbf{k} \cdot \hat{n} d\rho d\varphi d\theta$$

$$= \int_S (a \rho \sin \varphi) \cdot (a \rho \sin \varphi) d\rho d\varphi d\theta \quad (\text{SEE ABOVE, BOTTOM RIGHT})$$

$$= \int_S a^2 d\rho d\varphi d\theta$$

NO NEED TO SWICH INTO SPHERICAL POLARS OR PROJECT

$$\int_S \frac{a^2}{\rho^2} d\rho d\varphi d\theta = \int_S a^2 \frac{d\rho}{\rho} d\varphi d\theta$$

$$= \int_0^\pi \left[ -\frac{1}{\rho} \ln \rho \right]_0^a d\varphi d\theta$$

$$= \int_0^\pi \left[ -\frac{1}{a} \ln a \right]_0^a d\varphi d\theta$$

$$= \int_0^\pi \frac{1}{a} d\varphi d\theta$$

$$= \int_0^\pi \frac{1}{2} d\theta$$

$$= \frac{\pi}{6}$$

LET  $f(\rho) = a^2/\rho^2 - 1$   
 $\Delta f = (a^2, 2a^2/a)$   
 $|f| = (a^2, a^2/a^2) = 1$   
 $\delta = |f|/|f'| = (2a^2, 2)$

**Question 21**

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface represented parametrically by

$$\mathbf{r}(u, v) = \begin{bmatrix} u+v \\ u-v \\ u \end{bmatrix}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 3,$$

and  $\mathbf{F}$  is the vector field

$$x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}.$$

All integrations must be carried out in parametric.

 , [36]

• PREPARE ALL THE AUXILIARY ITEMS

- $\mathbf{F}(x, y, z) = (x, y, z^2)$
- $\mathbf{I}(u, v) = (u+v, u-v, u), \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 3$   
(THIS IS IN FACT A PLANE THROUGH O)
- $\frac{\partial \mathbf{r}}{\partial u} = (1, 1, 1) \quad \frac{\partial \mathbf{r}}{\partial v} = (1, -1, 0)$
- $\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = (1, 1, -2)$
- $\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = (1, 1, -2)$
- $d\mathbf{S} = \left\| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right\| du dv$

• HENCE WE ALSO HAVE IN PARAMETRIC

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S \mathbf{F} \cdot \mathbf{n} d\mathbf{S} \\ &= \int_S \mathbf{F} \cdot \left( \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) d\mathbf{S} \\ &= \int_S \mathbf{F} \cdot \left( \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) du dv \end{aligned}$$

• SUBSTITUTING FULLY INTO THE REQUIRED SURFACE INTEGRAL

$$\begin{aligned} \Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_{v=0}^3 \int_{u=0}^2 \left[ (u+v)^2 (u-v)^2 u^2 \right] \cdot [(1, 1, -2)] du dv \\ &= \int_{v=0}^3 \int_{u=0}^2 (u+v)^2 (u-v)^2 - 2u^2 du dv \\ &= \int_{v=0}^3 \int_{u=0}^2 (u^2 + v^2 + 2uv)^2 (u^2 - v^2)^2 - 2u^4 du dv \\ &= \int_{v=0}^3 \int_{u=0}^2 2u^2 du dv \\ &= \int_{v=0}^3 \left[ 2u^3 \right]_{u=0}^2 dv \\ &= \int_{v=0}^3 4v^3 dv \\ &= \left[ \frac{4}{4}v^4 \right]_0^3 \\ &= \frac{4}{4} \times 81 \\ &= 36 \end{aligned}$$

**Question 22**

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with parametric equations

$$\mathbf{r}(u, v) = (u \cos v)\mathbf{i} + (u \sin v)\mathbf{j} + u\mathbf{k},$$

such that  $0 \leq u \leq 1$ ,  $0 \leq v \leq 2\pi$ .

*All integrations must be carried out in parametric.*

$$\boxed{\frac{2\pi}{3}}$$

$\mathbf{F}(u, v) = \begin{bmatrix} u \cos v \\ u \sin v \\ 2u \end{bmatrix} \quad 0 \leq u \leq 1 \\ 0 \leq v \leq 2\pi$

$\mathbf{F}(u, v, z) = (u, u, 2z)$

• **Firstly find the "Jacobian" and the normal.**

- $\frac{\partial \mathbf{r}}{\partial u} = \begin{bmatrix} \cos v \\ \sin v \\ 1 \end{bmatrix}$
- $\frac{\partial \mathbf{r}}{\partial v} = \begin{bmatrix} -u \sin v \\ u \cos v \\ 0 \end{bmatrix}$
- $\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = \begin{bmatrix} u \cos^2 v - u \sin^2 v \\ u \sin v \cos v + u \cos v \sin v \\ u \sin v \end{bmatrix} = \begin{bmatrix} u \\ u \\ u \end{bmatrix} \leftarrow \text{NORMAL}$
- $\left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| = \sqrt{u^2 \cos^2 v + u^2 \sin^2 v + u^2} = u\sqrt{2} \leftarrow \text{"JACOBIAN"}$
- $\hat{n} = \frac{\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}}{\left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right|}$

• **NOW THE FLUX INTEGRAL ON  $R$  IS COMPUTED**

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_R \mathbf{F} \cdot \hat{n} \, du \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 \left( u \cos v, u \sin v, 2u \right) \cdot \left( \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) \, du \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 \left( -u^2 \cos^2 v - u^2 \sin^2 v + 2u^2 \right) \, du \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 -u^2 (1 + \tan^2 v) + 2u^2 \, du \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 u^2 \, du \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \left[ \frac{1}{3} u^3 \right]_0^1 \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \frac{1}{3} \, dv$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} \times 2\pi$$

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \boxed{\frac{2\pi}{3}}$$

**Question 23**

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with parametric equations

$$\mathbf{r}(u, v) = (1 + \sin u \cos v)\mathbf{i} + (\sin u \sin v)\mathbf{j} + (\cos u)\mathbf{k},$$

such that  $0 \leq u \leq \pi$ ,  $0 \leq v \leq 2\pi$ .

*All integrations must be carried out in parametric.*

**[4π]**

$\mathbf{F}(u, v) = [1 + \sin u \cos v, \sin u \sin v, \cos u] \quad 0 \leq u \leq \pi \quad 0 \leq v \leq 2\pi$

$\mathbf{F}(u, v) = (x, y, z)$

- $\frac{\partial \mathbf{F}}{\partial u} = [0, \sin u \cos v, -\sin u]$
- $\frac{\partial \mathbf{F}}{\partial v} = [-\sin u \cos v, \sin u \cos v, 0]$
- $\frac{\partial \mathbf{F}}{\partial u} \times \frac{\partial \mathbf{F}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u \cos v & \sin u \cos v & -\sin u \\ -\sin u \cos v & \sin u \cos v & 0 \end{vmatrix}$
- $= [1 + \sin u \cos v, \sin u \sin v - \sin u \cos v, \sin u \cos v + \sin u \sin v]$
- $= [\sin u \cos v, \sin u \sin v, \sin u \cos v]$
- $= [\sin u \cos v, \sin u \sin v, \sin u \cos v] \quad \leftarrow \text{REALLY N}$
- $\hat{L} = \sqrt{\frac{\partial \mathbf{F}}{\partial u} \cdot \frac{\partial \mathbf{F}}{\partial v}} = \sqrt{12} \quad \text{& } d\mathbf{S} = \left[ \frac{\partial \mathbf{F}}{\partial u}, \frac{\partial \mathbf{F}}{\partial v} \right] du dv$
- $\text{NO NEED TO EXPAND } \left[ \frac{\partial \mathbf{F}}{\partial u}, \frac{\partial \mathbf{F}}{\partial v} \right] \text{ SINCE AN IT WILL CANCEL}$

Hence

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S \mathbf{F} \cdot \hat{L} \, d\mathbf{S} = \int_S \mathbf{F}(u, v) \cdot \left( \frac{\partial \mathbf{F}}{\partial u}, \frac{\partial \mathbf{F}}{\partial v} \right) du dv \\ &= \int_0^{\pi} \int_0^{2\pi} [1 + \sin u \cos v, \sin u \sin v, \cos u] \cdot [\sin u \cos v, \sin u \sin v, \sin u \cos v] du dv \\ &= \int_0^{\pi} \int_0^{2\pi} \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \sin^2 u \cos^2 v + \sin u \cos u \sin u \cos v du dv \\ &= \int_0^{\pi} \int_0^{2\pi} \sin^2 u (\cos^2 v + \sin^2 v) + \sin u \cos u \sin u \cos v du dv \\ &= \int_0^{\pi} \int_0^{2\pi} \sin^2 u + \sin u \cos u \sin u \cos v du dv \\ &= \int_0^{\pi} \int_0^{2\pi} \sin u [\sin u + \cos u \sin v] du dv \\ &= 2\pi \int_0^{\pi} [-\cos u]_0^\pi du \\ &= 2\pi \left[ \cos u \right]_0^\pi \pi \\ &= \pi \left[ 1 - (-1) \right] \pi \\ &= 4\pi \end{aligned}$$

**Question 24**

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with parametric equations

$$\mathbf{r}(u, v) = (u \cos v)\mathbf{i} + (1+u \sin v)\mathbf{j} + (u-1)\mathbf{k},$$

such that  $0 \leq u \leq 1$ ,  $0 \leq v \leq 2\pi$ .

*All integrations must be carried out in parametric.*

$$\boxed{\frac{1}{3}\pi}$$

The handwritten solution shows the following steps:

- Parametric equations:  
 $\Gamma(u, v) = [u \cos v, 1+u \sin v, u-1]$  for  $0 \leq u \leq 1$ ,  $0 \leq v \leq 2\pi$ .  
 $\mathbf{F}(u, v) = (x, y, z)$
- Normal vector calculation:  
 $\frac{\partial \mathbf{r}}{\partial u} = [\cos v, \sin v, 1]$   
 $\frac{\partial \mathbf{r}}{\partial v} = [-u \sin v, u \cos v, 0]$   
 $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = [u \cos^2 v - u \sin^2 v, -u \sin v, -u]$  (Normal vector)
- Surface integral setup:  
 $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 \mathbf{F} \cdot (\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}) du dv$
- Evaluation:  
 $= \int_0^{2\pi} \int_0^1 (u \cos v, 1+u \sin v, u-1) \cdot [u \cos^2 v - u \sin^2 v, -u \sin v, -u] du dv$   
 $= \int_0^{2\pi} \int_0^1 (u \cos v - u \sin v - u^2 \cos^2 v + u^2 \sin^2 v + u \sin v - u) du dv$   
 $= \int_0^{2\pi} \int_0^1 (u \cos v - u^2 \cos^2 v + u^2 \sin^2 v) du dv$   
 $= \int_0^{2\pi} \int_0^1 u^2 \cos v du dv$   
 $= 2\pi \left[ \frac{1}{3}u^3 \cos v \right]_0^{2\pi} = 2\pi \times \left( \frac{8\pi}{3} \right) = \frac{16\pi^2}{3}$

**Question 25**

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with parametric equations

$$\mathbf{r}(\theta, \varphi) = [(4 + \cos \theta)\cos \varphi]\mathbf{i} + [(4 + \cos \theta)\sin \varphi]\mathbf{j} + (\sin \theta)\mathbf{k},$$

such that  $0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq 2\pi$ .

*All integrations must be carried out in parametric.*

$$24\pi^2$$

$\mathbf{F}(\theta, \varphi) = [4 + \cos \theta]\cos \varphi, [4 + \cos \theta]\sin \varphi, \sin \theta$        $\mathbf{F}(x, y, z) = (x, y, z)$

•  $\frac{\partial \mathbf{r}}{\partial \theta} = [-\sin \theta \cos \varphi, -\sin \theta \sin \varphi, \cos \theta]$        $\frac{\partial \mathbf{r}}{\partial \varphi} = [-4\cos \theta \cos \varphi, -4\cos \theta \sin \varphi, 0]$

•  $\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta \cos \varphi & -\sin \theta \sin \varphi & \cos \theta \\ -4\cos \theta \cos \varphi & -4\cos \theta \sin \varphi & 0 \end{vmatrix}$

=  $[0 - 4\sin^2 \theta \cos \varphi, -4\sin^2 \theta \sin \varphi, -4\cos^2 \theta]$

=  $[-4\sin^2 \theta \cos \varphi, -4\sin^2 \theta \sin \varphi, -4\cos^2 \theta]$

=  $[-4\sin^2 \theta \cos \varphi, -4\sin^2 \theta \sin \varphi, -4\cos^2 \theta]$

=  $(4 + \cos \theta)[-\sin^2 \theta \cos \varphi, -\sin^2 \theta \sin \varphi]$

=  $(4 + \cos \theta)[-\sin^2 \theta \cos \varphi, -\sin^2 \theta \sin \varphi]$  ← normal vector  $\mathbf{n}$

•  $\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{2\hat{\mathbf{i}}}{\sqrt{36}}, \frac{2\hat{\mathbf{j}}}{\sqrt{36}}, \frac{2\hat{\mathbf{k}}}{\sqrt{36}}$        $d\mathbf{S} = |\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi}| d\theta d\varphi$  ← magnitude of  $\mathbf{n}$

NOTE THAT EVALUATION OF  $|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi}|$  IS NOT NEEDED IN THIS TYPE OF QUESTION, AS IT WILL CANCEL

NOW

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_S \mathbf{F} \cdot \hat{\mathbf{n}} d\mathbf{S} = \int_S \mathbf{F}(\theta, \varphi) \cdot \left( \frac{2\hat{\mathbf{i}}}{\sqrt{36}}, \frac{2\hat{\mathbf{j}}}{\sqrt{36}}, \frac{2\hat{\mathbf{k}}}{\sqrt{36}} \right) d\theta d\varphi \\ &= \int_S [(4 + \cos \theta) \cos \varphi, (4 + \cos \theta) \sin \varphi, \sin \theta] \cdot \left[ \frac{2\hat{\mathbf{i}}}{\sqrt{36}}, \frac{2\hat{\mathbf{j}}}{\sqrt{36}}, \frac{2\hat{\mathbf{k}}}{\sqrt{36}} \right] d\theta d\varphi \\ &= \int_S -(4 + \cos \theta) \left[ (4 + \cos \theta) \cos \varphi \cos \theta + (4 + \cos \theta) \sin \varphi \cos \theta + \sin \theta \right] d\theta d\varphi \\ &= \int_S -(4 + \cos \theta) \left[ (4 + \cos \theta) \cos \theta \cos \varphi + \sin^2 \theta \right] d\theta d\varphi \\ &= \int_S -(4 + \cos \theta) \left[ 4\cos \theta + \cos^2 \theta + \sin^2 \theta \right] d\theta d\varphi \\ &= \int_S -(4 + \cos \theta)(4\cos \theta + 1) d\theta d\varphi = - \int_0^{2\pi} \int_0^{2\pi} 16\cos^2 \theta + 4 + 4\cos \theta + \cos^2 \theta d\theta d\varphi \quad \text{NO CONTRACTION IN } \theta \\ &= - \int_{0 \leq \theta \leq 2\pi} \int_{0 \leq \varphi \leq 2\pi} 6 d\theta d\varphi \quad \text{NO CONTRACTION IN } \theta \\ &= -6 \times 2\pi \times 2\pi = -24\pi^2 \\ \therefore \left| \int_S \mathbf{F} \cdot d\mathbf{S} \right| &= 24\pi^2 \end{aligned}$$

### Question 26

It is given that the vector field  $\mathbf{F}$  satisfies

$$\mathbf{F} = 2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0,$$

cut off by the cylinder with cartesian equation

$$x^2 + y^2 = x.$$

You **must** find a suitable parameterization for  $S$ , and carry out the **integration in parametric**, without using any integral theorems.

$\frac{\pi}{4}$

**PARAMETRIZING THE REGION R — LEFT** — LOOK AT THE VERTICAL SLICE ABOVE

$$\Rightarrow x^2 + y^2 = 2$$

$$\Rightarrow x^2 + y^2 = 0$$

$$\Rightarrow (x-1)^2 + y^2 = \frac{1}{4}$$

$$\Rightarrow 4(x-\frac{1}{2})^2 + y^2 = 1$$

$$\Rightarrow (2x-1)^2 + (2y)^2 = 1$$

$$\Rightarrow (2x-1)^2 + (2y)^2 = 1$$

Hence the curve would parameterise as

$$\begin{cases} 2x-1 = \cos\theta \\ 2y = \sin\theta \end{cases} \Rightarrow$$

$$\begin{cases} x = \frac{1}{2}(1+\cos\theta) \\ y = \frac{1}{2}\sin\theta \end{cases} \Rightarrow$$

NOTES TO ADD A  $\sqrt{1-\cos^2\theta}$  TERM SO IT COULD BE THE RADIAL DISTANCE FROM THE ORIGIN, SAY  $r$

$$r = \sqrt{1-\cos^2\theta}$$

$$r = \sqrt{1-\cos^2\theta}$$

WE WILL TRACE THE CIRCLE AT (3/2, 0) RADIUS 1/2, INCLUSIVE OF BORDER

**E.G.**  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 2$

$$\frac{\partial z}{\partial x} = 1 - \frac{1}{2}\tau(1+\cos\theta)$$

$$\frac{\partial z}{\partial y} = (-\frac{1}{2}\tau(1+\cos\theta))^{\frac{1}{2}}$$

Hence

$$z = \left[ \frac{1}{2}\tau(1+\cos\theta), \frac{1}{2}\tau\sin\theta, \left[ 1 - \frac{1}{2}\tau(1+\cos\theta) \right]^{\frac{1}{2}} \right]$$

$$0 < \theta < 1$$

$$0 < r < 1$$

**NOW**  $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F}(r\theta, \phi) \cdot \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} d\theta d\phi$

$$\mathbf{F}(r\theta, \phi) = (\tau\sin\theta, -\tau\cos\theta, 1)$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{2}\tau^2\sin\theta \left[ 1 - \frac{1}{2}\tau(1+\cos\theta) \right]^{\frac{1}{2}}$$

$$+ \frac{1}{2}\tau^2\cos\theta\sin\theta(1+\cos\theta) \left[ 1 - \frac{1}{2}\tau(1+\cos\theta) \right]^{\frac{1}{2}}$$

$$+ \frac{1}{2}\tau\cos^2\theta(1+\cos\theta) + \frac{1}{2}\tau\sin^2\theta$$

SIMPLIFY

$$= \frac{1}{2}\tau^2\sin\theta \left[ 1 - \frac{1}{2}\tau(1+\cos\theta) \right]^{\frac{1}{2}} \left[ \sin\theta + \cos\theta(1+\cos\theta) \right]$$

$$+ \frac{1}{2}\tau \left[ \cos^2\theta + \sin^2\theta + \sin^2\theta \right]$$

$$= \frac{1}{2}\tau^2\sin\theta \left[ 1 - \frac{1}{2}\tau(1+\cos\theta) \right]^{\frac{1}{2}} \left[ \sin\theta + \cos\theta + \cos^2\theta \right] + \frac{1}{2}\tau(1+\cos\theta)$$

$$= \frac{1}{2}\tau^2\sin\theta \left[ 1 - \frac{1}{2}\tau(1+\cos\theta) \right]^{\frac{1}{2}} \left[ \sin\theta + \cos\theta + \cos^2\theta \right] + \frac{1}{2}\tau(1+\cos\theta)$$

SIMPLIFY

$$= \frac{1}{2}\tau^2\sin\theta \left[ 1 - \frac{1}{2}\tau(1+\cos\theta) \right]^{\frac{1}{2}} + \frac{1}{2}\tau\sin\theta\cos\theta \left[ 1 - \frac{1}{2}\tau(1+\cos\theta) \right]^{\frac{1}{2}} + \frac{1}{2}\tau\cos^2\theta$$

NOTICE LOOKING AT THE ABOVE INTEGRAND, UNDER THE  $\theta$  INTEGRATION,  $\theta \in [0, \pi/2]$

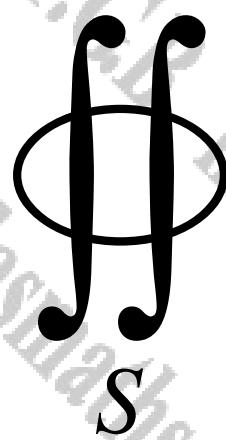
$$\int_0^{\pi/2} \frac{1}{2}\tau^2\sin\theta d\theta = 0$$

$$\int_0^{\pi/2} \frac{1}{2}\tau^2\sin\theta \left[ 1 - \frac{1}{2}\tau(1+\cos\theta) \right]^{\frac{1}{2}} d\theta = \int_0^{\pi/2} \frac{1}{2}(\cos\theta) d\theta = 0$$

$$\int_0^{\pi/2} \frac{1}{2}\tau^2\sin\theta\cos\theta \left[ 1 - \frac{1}{2}\tau(1+\cos\theta) \right]^{\frac{1}{2}} d\theta = \int_0^{\pi/2} \frac{1}{2}(\cos^2\theta) d\theta = 0$$

$$\int_0^{\pi/2} \frac{1}{2}\tau^2\cos^2\theta d\theta = \int_0^{\pi/2} \left[ \frac{1}{2}\tau^2 \right] d\theta = \int_0^{\pi/2} \frac{1}{2} d\theta = \frac{\pi}{4}$$

**TYPE**



**F · dS**

**Question 1**

$$\mathbf{F}(x, y, z) \equiv xy\mathbf{i} + y\mathbf{j} + 4\mathbf{k}$$

Evaluate the integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the **closed** surface enclosing the finite region  $V$ , defined by

$$x^2 + y^2 \leq 9, \quad x \geq 0, \quad y \geq 0, \quad 0 \leq z \leq 4.$$

**[9π + 36]**

$\int \mathbf{F} \cdot d\mathbf{S} = \dots$  DIVERGENCE THEOREM  
 $\int \nabla \cdot \mathbf{F} dV = \int_V (\frac{\partial}{\partial x} 3x^2, \frac{\partial}{\partial y} 3y^2, 0) \cdot (2x, 2y, 0) dV$   
 $= \int_V (9x^2 + 9y^2) dV = \int_V 9r^2 r dr d\theta dz$   
 SWITCH TO CYLINDRICAL POLARS ( $r, \theta, z$ )  
 $= \int_0^4 \int_0^{\frac{\pi}{2}} \int_0^3 (9r^2 + r) (r dr d\theta dz)$   
 $= \int_0^4 \int_0^{\frac{\pi}{2}} \int_0^3 (9r^3 + r^2) dr d\theta dz = \int_0^4 \int_0^{\frac{\pi}{2}} \left[ \frac{9}{4}r^4 + \frac{1}{2}r^2 \right]_0^3 d\theta dz$   
 $= \int_0^4 \int_0^{\frac{\pi}{2}} \left( \frac{9}{4}r^4 + \frac{9}{2}r^2 \right) d\theta dz = \int_0^4 \left[ -\frac{9}{16}\cos\theta + \frac{9}{4}r^2 \right]_0^{\frac{\pi}{2}} d\theta dz$   
 $= \int_0^4 (0 + \frac{9\pi}{4}) - (-\frac{9}{16}) d\theta dz = \int_{\frac{9\pi}{4}}^4 d\theta dz$   
 $= \left[ \frac{9}{4}z + \frac{9\pi}{4} \right]_0^4 = (36 + 9\pi) - 0 = 36 + 9\pi \blacksquare$

**Question 2**

$$\mathbf{F}(x, y, z) \equiv (x + y^2)\mathbf{i} + (2y + xz)\mathbf{j} + (3z + xyz)\mathbf{k}.$$

Evaluate the integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with Cartesian equation

$$4x^2 + 4y^2 + 4z^2 = 1.$$

You may not use the Divergence Theorem in this question.

π

$\mathbf{F} = (x+y^2, 2y+xz, 3z+xyz)$

$$\begin{aligned} S: \quad & 4x^2 + 4y^2 + 4z^2 = 1 \\ & x^2 + y^2 + z^2 = \frac{1}{4} \\ & \text{in } A \text{ SPHERE} \end{aligned}$$

Let  $\hat{n}(\theta, \phi) = \langle \cos\theta, \sin\theta, 0 \rangle$

$$\begin{aligned} \nabla \phi &= (\partial_\theta, \partial_\phi, \partial_z) \\ \hat{n} &= (\cos\theta, \sin\theta, 0) \\ |\hat{n}| &= \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1 \\ \hat{n} &= \frac{\partial}{\partial \phi} = \frac{\partial(x+y^2)}{\partial \phi} = 2\cos\theta \hat{\mathbf{e}}_\phi \end{aligned}$$

Then

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \hat{n} d\phi = \iint_S (x+y^2, 2y+xz, 3z+xyz) \cdot \hat{n} d\phi \\ &= 2 \iint_S x^2 + y^2 + z^2 + xyz^2 d\phi = 2 \iint_S x^2 + y^2 + z^2 + xyz^2 d\phi \end{aligned}$$

As THE DOMAIN OF A SURFACE IS SPHERICAL, ANY SPD SURFACE OF SPHERE ARE WITH ZERO NC CONTRACTION

$$\begin{aligned} 2 \iint_S x^2 + y^2 + z^2 d\phi &= 2 \iint_S (x^2 + y^2 + z^2) + 0^2 d\phi \\ \text{SINCE NO SPHERICAL ROADS} \quad d\phi &= \left(\frac{\pi}{2}\right) \sin\theta d\theta d\phi \\ 2\pi &\cdot \frac{1}{2} \sin\theta d\theta \\ y &= \frac{1}{2} \sin\theta \cos\phi \\ z &= \frac{1}{2} \sin\theta \\ x^2 + y^2 + z^2 &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} &= 2 \iint_{\theta=0}^{\pi/2} \iint_{\phi=0}^{\pi} \left[ \left( \frac{1}{4} + \left( \frac{1}{2} \sin\theta \cos\phi \right)^2 + 2\left( \frac{1}{2} \sin\theta \right)^2 \right) \right] \sin\theta d\theta d\phi \\ &= 2 \iint_{\theta=0}^{\pi/2} \left[ \frac{1}{4} \sin\theta + \frac{1}{4} \sin^3\theta + \frac{1}{4} \cos^2\theta \sin^2\theta \right] \sin\theta d\theta d\phi \quad \text{REAS OF NOMERATION} \\ &= 2\pi \int_{\theta=0}^{\pi/2} \left[ \frac{1}{16} \sin^2\theta + \frac{1}{16} \sin^4\theta + \frac{1}{4} \cos^2\theta \sin^2\theta \right] d\theta \\ &= 2\pi \left[ -\frac{3}{16} \cos\theta + \frac{1}{16} \cos^3\theta \right]_0^{\pi/2} \\ &= 2\pi \left[ \frac{3}{16} \cos 0 + \frac{1}{16} \cos^3 0 \right]_{\pi/2} \\ &= 2\pi \left[ \left( \frac{3}{16} + \frac{1}{16} \right) - \left( -\frac{3}{16} - \frac{1}{16} \right) \right] \\ &= 2\pi \left[ \frac{1}{4} + \frac{1}{4} \right] \\ &= \pi \end{aligned}$$

Created by T. Madas

**Question 3**

It is given that

$$\mathbf{F}(x, y, z) \equiv \mathbf{k} \wedge \mathbf{r}, \text{ where } \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show by direct integration that

$$\iint_S \nabla \wedge \mathbf{F} \cdot d\mathbf{S} = 0,$$

where  $S$  is the **closed** surface enclosing the finite region  $V$ , defined by

$$x^2 + y^2 + z^2 \leq 1, \quad z \geq 0, \quad \text{and} \quad x^2 + y^2 \leq 1.$$

You may not use any Integral Theorems in this question.

proof

LET  $\mathbf{f}(x,y,z) = \mathbf{k} \wedge (\mathbf{x}) = \mathbf{k} \wedge (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$   
 $\mathbf{k} = (0, 0, 1)$   
 $\mathbf{x} = (x, y, z)$   
 $|x| = \sqrt{x^2 + y^2 + z^2} = 1$   
 $\mathbf{k} = \mathbf{x}/|x| = (x, y, z)$

$\bullet \mathbf{f} = \mathbf{k} \wedge \mathbf{x} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = (-y, x, 0)$   
 $\bullet \nabla_x \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0, 0, 2)$   
 $\bullet \iint_S \nabla_x \mathbf{f} \cdot d\mathbf{S} = \int_{S_1} (\mathbf{f}(x, y, z) \cdot \mathbf{n}) ds + \int_{S_2} (\mathbf{f}(x, y, z) \cdot \mathbf{n}) ds$   
 $= \int_{S_1} 2z ds + \int_{S_2} -2 ds$   
SURFACE ELEMENT  
 $\begin{cases} x = \sin\theta \cos\phi \\ y = \sin\theta \sin\phi \\ z = \cos\theta \end{cases}$  For  $0 < \theta < \pi/2$   
 $0 \leq \theta \leq 2\pi$   
 $ds^2 = 1^2 \sin^2\theta d\phi d\theta$   
 $d\mathbf{S} = \sin\theta d\phi d\theta$

$$\begin{aligned}
&= \int_{0}^{2\pi} \int_{0}^{\pi/2} (2\cos\theta)(\sin\theta d\theta d\phi) - 2 \int_{S_2} 1 ds \\
&= \int_{0}^{2\pi} \int_{0}^{\pi/2} \sin 2\theta d\theta d\phi - 2 \times (\text{Area of } S_2) \\
&= \int_{0}^{2\pi} \left[ \frac{1}{2} \cos 2\theta \right]_0^{\pi/2} d\phi - 2 \times (\pi \times 1^2) \\
&= \int_{0}^{2\pi} \frac{1}{2} - (-\frac{1}{2}) d\phi = -2\pi \\
&= 1 \times 2\pi = 2\pi \\
&= 0
\end{aligned}$$

**Question 4**

$$\mathbf{F}(x, y, z) = (4yz)\mathbf{i} + (2y^2)\mathbf{j} + (5xyz + 6z^2 + 3z)\mathbf{k}.$$

Evaluate the integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where  $S$  is the surface with Cartesian equation

$$x^2 + y^2 + 4z^2 = 1.$$

You may not use the Divergence Theorem in this question.

[2π]

$\mathbf{E} = 4yz\mathbf{i} + 2y^2\mathbf{j} + (5xyz + 6z^2 + 3z)\mathbf{k}$

- Let the ellipsoid have equation  
 $\frac{x^2}{R^2} + \frac{y^2}{R^2} + 4z^2 = 1$   
 $\nabla \mathbf{F} = (2y, 2y, 5xz + 12z + 3)$   
 $\mathbf{n} = (2y, 2y, 4z)$
- (Note: As we will project onto the  $xy$ -plane)
- Split the ellipsoid into 2 halves
- 
- Hence  

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_R (4yz, 2y^2, 5xyz + 6z^2 + 3z) \cdot \frac{\mathbf{n}}{R} dS$$
- To this, project onto the  $xy$ -plane
- $I_1 = \iint_R (4yz, 2y^2, 5xyz + 6z^2 + 3z) \cdot \frac{\mathbf{n}}{R} dS$   
 $I_2 = \iint_R (4yz, 2y^2, 5xyz + 6z^2 + 3z) \cdot \frac{\mathbf{n}}{R} dS$   
 $I_3 = \iint_R (4yz, 2y^2, 5xyz + 6z^2 + 3z) \cdot \frac{\mathbf{n}}{R} dS$

$I_1 = \iint_R 2yz + \frac{y^2}{R} + 5xyz + 6z^2 + 3z \, dxdy$   
 $I_2 = \iint_R 2yz + \frac{y^2}{R} + 5xyz + \frac{1}{R} (1-x^2-y^2)^{\frac{1}{2}} + 6z^2 + 3z \, dxdy$   
 $I_3 = \iint_R 2yz + \frac{y^2}{R} + 2y^2 + 2y(1-x^2-y^2)^{\frac{1}{2}} + \frac{5}{2}(1-x^2-y^2)^{\frac{1}{2}} + \frac{3}{2}(1-x^2-y^2)^{\frac{1}{2}} \, dxdy$

NOTICE: SPLITTING ELLIPSOID, GET AN EXPRESSION FOR  $I_2$ , IS THE INTEGRAL OVER THE BOTTOM HALF OF THE ELLIPSOID AND NOTE THAT

$\bar{z} = -\frac{1}{2}(1-x^2-y^2)^{\frac{1}{2}}$

$dS = \frac{dxdy}{R(\sqrt{1-x^2-y^2})}$

Each of these will contain angles, so some terms will cancel.

$I_2 = \iint_R -2yz - \frac{y^2}{(1-x^2-y^2)^{\frac{1}{2}}} - \frac{5}{2}xy\left[(1-x^2-y^2)^{\frac{1}{2}}\right] - \frac{3}{2}\left[(1-x^2-y^2)^{\frac{1}{2}}\right]^2 + \frac{3}{2}\left[(1-x^2-y^2)^{\frac{1}{2}}\right]^3 \, dxdy$

$I_3 = \iint_R -2yz + \frac{y^2}{(1-x^2-y^2)^{\frac{1}{2}}} + \frac{5}{2}xy\left[(1-x^2-y^2)^{\frac{1}{2}}\right] - \frac{3}{2}\left[(1-x^2-y^2)^{\frac{1}{2}}\right]^2 + \frac{3}{2}\left[(1-x^2-y^2)^{\frac{1}{2}}\right]^3 \, dxdy$

ADDING  $I_2 + I_3$

$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_R \frac{2y^2}{(1-x^2-y^2)^{\frac{1}{2}}} + 5xy(1-x^2-y^2)^{\frac{1}{2}} + 3(1-x^2-y^2)^{\frac{1}{2}} \, dxdy$

BUT WE KNOW WE HAVE THAT  $R((x^2+y^2)^{\frac{1}{2}}) < 1$  & SYMMETRIC DOMAIN IN  $x^2 + y^2 \leq 1$  SO ALL POWERS OF  $x$  &  $y$  HAVE NO CONTRIBUTION

Thus

$$\dots = \iint_R \frac{2y^2}{(1-x^2-y^2)^{\frac{1}{2}}} + 5xy(1-x^2-y^2)^{\frac{1}{2}} + 3(1-x^2-y^2)^{\frac{1}{2}} \, dxdy$$

$$\begin{aligned} &= \iint_R 3(1-x^2-y^2)^{\frac{1}{2}} \, dxdy \\ &\text{SWITCH INTO POLAR COORDINATES} \\ &= \int_0^{2\pi} \int_0^1 3(1-r^2)^{\frac{1}{2}} (rdrd\theta) \\ &= \int_0^{2\pi} \int_{r=0}^1 3r(1-r^2)^{\frac{1}{2}} \, dr \, d\theta \\ &= \left[ \int_0^{2\pi} 1 \, d\theta \right] \left[ \int_0^1 3r(1-r^2)^{\frac{1}{2}} \, dr \right] \\ &= 2\pi \times \left[ - (1-r^2)^{\frac{3}{2}} \right]_0^1 \\ &= 2\pi \times \left[ (1-r^2)^{\frac{3}{2}} \right]_0^1 \\ &= 2\pi \end{aligned}$$

**Question 5**

The surface  $\Omega$  is the sphere with Cartesian equation

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$$

Evaluate the surface integral

$$\oint\int_{\Omega} \left[ (x+y)\mathbf{i} + (x^2+xy)\mathbf{j} + z^2\mathbf{k} \right] \cdot d\mathbf{S},$$

where  $d\mathbf{S}$  is a unit surface element on  $\Omega$ .

You may not use the Divergence Theorem in this question.

$$\boxed{\frac{16}{3}\pi}$$

$$\int \mathbf{F} \cdot d\mathbf{S} = \int_{\Omega} (x+y, x^2+xy, z^2) \cdot \hat{n} dS = \dots$$

MOVING THE ORIGIN DOES NOT AFFECT THE ANSWER, SO TRANSLATE THE ORIGIN AT  $(1,1,1)$ .

THIS  $(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$  BECAUSE  $x^2 + y^2 + z^2 = 1$

$(x+y, x^2+xy, z^2)$  BECOMES  $(x+1+y+1, x^2+1+xy+1, z^2)$

THUS  $\mathbf{F} = [x+1+y+1, x^2+1+xy+1, z^2]$

$$= [x+1+y+1, x^2+1+xy+1, z^2]$$

$$= [x+1+y+1, x^2+1+xy+1, z^2]$$

LET  $\mathbf{f}(x,y,z) = x^2 + y^2 + z^2 - 1$

$$\nabla \mathbf{f} = (2x, 2y, 2z)$$

$$\mathbf{n} = (x, y, z)$$

$$|\mathbf{n}| = \sqrt{x^2 + y^2 + z^2} = 1$$

$$\mathbf{n} = \hat{\mathbf{n}} = (x, y, z)$$

NOW THE DOMAIN (SURFACE) IS SYMMETRIC IN  $x$ ,  $y$  AND IN  $z$  (Hence no circular cross sections) – so ALL COORDINATES IN ANY VARIABLE WILL HAVE NO CONTRIBUTION

$$= \int_{\Omega} x^2 + y^2 + z^2 + 1 + xy + 2x + 2y + 2z + 2 + 1 dS$$

$$= \int_{\Omega} (x^2 + y^2 + z^2) dS$$

SPLIT INTO SPHERICAL POLARIS

$$dS = r \sin\theta dr d\theta d\phi$$

$$x = r \cos\theta \sin\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$dS = r^2 \sin\theta dr d\theta d\phi$$

$$= \int_0^{2\pi} \int_0^\pi \int_0^r (1 + r^2) r^2 \sin\theta dr d\theta d\phi$$

$$= \int_0^{2\pi} \int_0^\pi [r^2 + \frac{1}{3}r^3] \sin\theta d\theta d\phi$$

$$= 2\pi \times \left[ -\cos\theta - \frac{1}{3}\cos^3\theta \right]_0^\pi$$

$$= 2\pi \times \left[ \cos\theta + \frac{1}{3}\cos^3\theta \right]_0^\pi$$

$$= 2\pi \times \left[ (1 + \frac{1}{3}) - (-1 - \frac{1}{3}) \right]$$

$$= 2\pi \times \frac{8}{3}$$

$$= \frac{16}{3}\pi$$