

# COMPLEX NUMBERS PRACTICE

(part 2)

# **ROOTS OF COMPLEX NUMBERS**

**Question 1**

$$z^4 = -16, z \in \mathbb{C}.$$

- Solve the above equation, giving the answers in the form  $a + bi$ , where  $a$  and  $b$  are real numbers.
- Plot the roots of the equation as points in an Argand diagram.

$$z = \sqrt{2}(\pm 1 \pm i)$$

a) This is a standard circle locus

$$\begin{aligned}|z - i| &= 4 \\ |z - (-i)| &= 4\end{aligned}$$

It centre at  $(0, 1)$ , radius 4.

b) Looking at the diagram above & noting that  $|z|$  represents the distance of a point from the origin

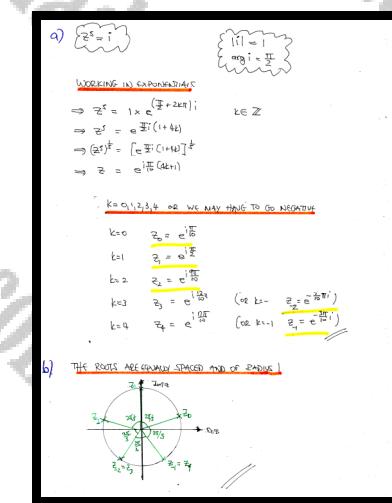
- Distance of the centre from O is  $\sqrt{2}$
- $|z|_{\max} = \text{radius} + \sqrt{2} = 4 + \sqrt{2}$  (Point P)
- $|z|_{\min} = \text{radius} - \sqrt{2} = 4 - \sqrt{2}$  (Point Q)

**Question 2**

$$z^5 = i, \quad z \in \mathbb{C}.$$

- a) Solve the equation, giving the roots in the form  $r e^{i\theta}$ ,  $r > 0$ ,  $-\pi < \theta \leq \pi$ .
- b) Plot the roots of the equation as points in an Argand diagram.

$$z = e^{i\frac{\pi}{10}}, \quad z = e^{i\frac{\pi}{2}}, \quad z = e^{i\frac{9\pi}{10}}, \quad z = e^{-i\frac{3\pi}{10}}, \quad z = e^{-i\frac{7\pi}{10}}$$



**Question 3**

$$z = 4 + 4i$$

- a) Find the fifth roots of  $z$ .

Give the answers in the form  $r e^{i\theta}$ ,  $r > 0$ ,  $-\pi < \theta \leq \pi$ .

- b) Plot the roots as points in an Argand diagram.

$$\boxed{\sqrt{2} e^{i\frac{\pi}{20}}, \sqrt{2} e^{i\frac{9\pi}{20}}, \sqrt{2} e^{i\frac{17\pi}{20}}, \sqrt{2} e^{-i\frac{7\pi}{20}}, \sqrt{2} e^{-i\frac{3\pi}{4}}}$$

(a)

$$(4+4i)^5 = \sqrt{16+16^2} = \sqrt{32^2} = 4\sqrt{2}^5 \in \mathbb{R} \times \mathbb{R}^2 = \mathbb{C}$$

$$\arg(4+4i) = \text{atan}\left(\frac{4}{4}\right) = \frac{\pi}{4}$$

$$\Rightarrow w_1^5 = 4+4i$$

$$\Rightarrow w_1^5 = 4\sqrt{2} e^{i(\frac{\pi}{4}+2k\pi)} \quad k \in \mathbb{Z}$$

$$\Rightarrow w_1^5 = 4\sqrt{2} e^{i\frac{\pi}{4}(1+4k)}$$

$$\Rightarrow (w_1^5)^{\frac{1}{5}} = \left[4\sqrt{2} e^{i\frac{\pi}{4}(1+4k)}\right]^{\frac{1}{5}}$$

$$\Rightarrow w_1 = 2 e^{i\frac{\pi}{20}(1+4k)}$$

$$\Rightarrow w_1 = \sqrt{2} e^{i\frac{\pi}{20}(1+4k)}$$

(b)

**Question 4**

$$z = 4 - 4\sqrt{3}i.$$

- a) Find the cube roots of  $z$ .

Give the answers in polar form  $r(\cos \theta + i \sin \theta)$ ,  $r > 0$ ,  $-\pi < \theta \leq \pi$ .

- b) Plot the roots as points in an Argand diagram.

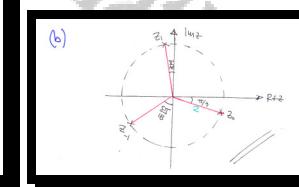
$$z = 2\left(\cos \frac{\pi}{9} - i \sin \frac{\pi}{9}\right), z = 2\left(\cos \frac{5\pi}{9} + i \sin \frac{5\pi}{9}\right), z = 2\left(\cos \frac{7\pi}{9} - i \sin \frac{7\pi}{9}\right)$$

(a)

$$\begin{aligned} 4 - 4\sqrt{3}i &= 8 e^{-i\frac{\pi}{3}} \\ \text{OR IN GENERAL: } & 4 - 4\sqrt{3}i = 8 e^{i(-\frac{\pi}{3} + 2m)} \\ \Rightarrow 4 - 4\sqrt{3}i &= 8 e^{i(-\frac{\pi}{3}(n-1))} \\ \Rightarrow 4 - 4\sqrt{3}i &= 8 e^{i(-\frac{\pi}{3}(n-1))} \\ \Rightarrow [4 - 4\sqrt{3}i]^{\frac{1}{3}} &= [8 e^{i(-\frac{\pi}{3}(n-1))}]^{\frac{1}{3}} \\ \Rightarrow [4 - 4\sqrt{3}i]^{\frac{1}{3}} &= 8^{\frac{1}{3}} e^{i(-\frac{\pi}{3}(n-1))} \\ \Rightarrow [4 - 4\sqrt{3}i]^{\frac{1}{3}} &= 2 e^{i(-\frac{\pi}{3}(n-1))} \end{aligned}$$

$\bullet |4 - 4\sqrt{3}i| = \sqrt{16 + 48} = 8$

$\bullet \arg(4 - 4\sqrt{3}i) = \operatorname{atan}(-\frac{4\sqrt{3}}{4}) = -\frac{\pi}{3}$



**Question 5**

The following complex number relationships are given

$$w = -2 + 2\sqrt{3}i, \quad z^4 = w.$$

- a) Express  $w$  in the form  $r(\cos \theta + i \sin \theta)$ , where  $r > 0$  and  $-\pi < \theta \leq \pi$ .
- b) Find the possible values of  $z$ , giving the answers in the form  $x+iy$ , where  $x$  and  $y$  are real numbers.

$$w = 2 \left[ \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right],$$

$$z_1 = \frac{1}{2}(\sqrt{6} + i\sqrt{2}), \quad z_2 = \frac{1}{2}(-\sqrt{2} + i\sqrt{6}), \quad z_3 = \frac{1}{2}(\sqrt{2} - i\sqrt{6}), \quad z_4 = \frac{1}{2}(-\sqrt{6} - i\sqrt{2})$$

(a)  $| -2 + 2\sqrt{3}i | = \sqrt{4 + 12} = 4$   
 $\arg(-2 + 2\sqrt{3}i) = \pi + \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$   
 $\therefore -2 + 2\sqrt{3}i = 4 \left[ \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right]$

(b)  $z^4 = -2 + 2\sqrt{3}i$   
 $z^4 = 4 \left[ \cos\left(\frac{2\pi}{3} + 2k\pi\right) + i \sin\left(\frac{2\pi}{3} + 2k\pi\right) \right]$   
 $z = 4^{\frac{1}{4}} \left[ \cos\left(\frac{2\pi}{3} + 2k\pi\right) + i \sin\left(\frac{2\pi}{3} + 2k\pi\right) \right]^{\frac{1}{4}}$   
 $z = \sqrt[4]{16} \left[ \cos\left(\frac{\pi}{6} + \frac{k\pi}{2}\right) + i \sin\left(\frac{\pi}{6} + \frac{k\pi}{2}\right) \right]$   
 $z_0 = \sqrt[4]{16} \left( \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i;$   
 $z_1 = \sqrt[4]{16} \left( \cos\left(\frac{7\pi}{6}\right) + i \sin\left(\frac{7\pi}{6}\right) \right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i;$   
 $z_2 = \sqrt[4]{16} \left( \cos\left(\frac{13\pi}{6}\right) + i \sin\left(\frac{13\pi}{6}\right) \right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i;$   
 $z_3 = \sqrt[4]{16} \left( \cos\left(\frac{19\pi}{6}\right) + i \sin\left(\frac{19\pi}{6}\right) \right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i;$

**Question 6**

Find the cube roots of the imaginary unit  $i$ , giving the answers in the form  $a+bi$ , where  $a$  and  $b$  are real numbers.

$$z_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad z_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad z_3 = -i$$

$\bullet$   $z^3 = i \quad (\frac{1}{2} + \frac{\sqrt{3}}{2}i), \quad k \in \mathbb{Z}$   
 $\Rightarrow z^3 = (x \in \mathbb{C})(1+ik)$   
 $\Rightarrow z^3 = e^{i\pi(1+4k)}$   
 $\Rightarrow (z^3)^{\frac{1}{3}} = \left[ e^{i\pi(1+4k)} \right]^{\frac{1}{3}}$   
 $\Rightarrow z = e^{\frac{i\pi}{3}(1+4k)}$

$\boxed{\begin{array}{l} |i| = 1 \\ \arg i = \frac{\pi}{2} \end{array}}$

$z_0 = e^{\frac{i\pi}{3}} = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{1}{2}i;$   
 $z_1 = e^{\frac{i\pi}{3}(1+4)} = \cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i;$   
 $z_2 = e^{\frac{i\pi}{3}(1+8)} = \cos\frac{9\pi}{3} + i \sin\frac{9\pi}{3} = -i$

**Question 7**

Find the cube roots of the complex number  $-8i$ , giving the answers in the form  $a+bi$ , where  $a$  and  $b$  are real numbers.

$$z_1 = \sqrt{3} - i, z_2 = -\sqrt{3} - i, z_3 = 2i$$

$$\begin{aligned} & \bullet \quad z^3 = -8i \\ & \rightarrow z^3 = 8 \times e^{i(-\frac{\pi}{2}+2k\pi)} \quad k \in \mathbb{Z} \\ & \rightarrow z^3 = 8 e^{i(-\frac{\pi}{2}+2k\pi)} \\ & \rightarrow (z^3)^{\frac{1}{3}} = \left[ 8 e^{i(-\frac{\pi}{2}+2k\pi)} \right]^{\frac{1}{3}} \\ & \rightarrow z = 2 e^{i(\frac{-\pi}{6}+\frac{2k\pi}{3})} \end{aligned}$$

$| -8i | = 8$   
 $\arg(-8i) = -\frac{\pi}{2}$

$$\begin{aligned} z_1 &= 2e^{i(-\frac{\pi}{6})} = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i \\ z_2 &= 2e^{i(\frac{5\pi}{6})} = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i \\ z_3 &= 2e^{i(\frac{11\pi}{6})} = 2\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i \end{aligned}$$

**Question 8**

$$z^4 = -8 - 8\sqrt{3}i, z \in \mathbb{C}.$$

Solve the above equation, giving the answers in the form  $a+bi$ , where  $a$  and  $b$  are real numbers.

$$z = \sqrt{3} - i, z = 1 + \sqrt{3}i, z = -\sqrt{3} + i, z = -1 - \sqrt{3}i$$

$$\begin{aligned} & \bullet \quad z^4 = -8 - 8\sqrt{3}i \\ & \rightarrow z^4 = 16 e^{i(-\frac{\pi}{3}+2k\pi)} \\ & \rightarrow z^4 = (16 e^{i(-\frac{\pi}{3})})^{\frac{1}{4}} \\ & \rightarrow (z^4)^{\frac{1}{4}} = \left[ 16 e^{i(-\frac{\pi}{3}+2k\pi)} \right]^{\frac{1}{4}} \\ & \rightarrow z = 2 e^{i(\frac{-\pi}{12}+5k\pi)} \end{aligned}$$

$| -8 - 8\sqrt{3}i | = \sqrt{64+192} = 16$   
 $\arg(-8 - 8\sqrt{3}i) = \arg\left(\frac{-8-8\sqrt{3}}{8}\right) = \arg\left(\frac{-1-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$

$$\begin{aligned} z_1 &= 2e^{i(-\frac{\pi}{12})} = 2\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right) = \sqrt{3} - i \\ z_2 &= 2e^{i(\frac{11\pi}{12})} = 2\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right) = -1 - \sqrt{3}i \\ z_3 &= 2e^{i(\frac{29\pi}{12})} = 2\left(\cos\left(\frac{29\pi}{12}\right) + i\sin\left(\frac{29\pi}{12}\right)\right) = 1 + \sqrt{3}i \\ z_4 &= 2e^{i(\frac{47\pi}{12})} = 2\left(\cos\left(\frac{47\pi}{12}\right) + i\sin\left(\frac{47\pi}{12}\right)\right) = -\sqrt{3} + i \end{aligned}$$

**Question 9**

$$z^2 = (1+i\sqrt{3})^3, \quad z \in \mathbb{C}.$$

Solve the above equation, giving the answers in the form  $a+bi$ , where  $a$  and  $b$  are real numbers.

$$z = \pm i2\sqrt{2}$$

$\bullet |1+i\sqrt{3}| = \sqrt{1+3} = 2$   
 $\bullet \arg(1+i\sqrt{3}) = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$

$$\begin{aligned} z_1^2 &= (1+i\sqrt{3})^3 \\ z_1^2 &= (2e^{i\frac{\pi}{3}})^3 \\ z_1^2 &= 8e^{i\pi} \\ z_1^2 &= 8e^{i(\pi+2m)} \\ z_1^2 &= 8e^{i\pi(1+2n)} \\ (z_1^2)^{\frac{1}{2}} &= [8e^{i\pi(1+2n)}]^{\frac{1}{2}} \\ z_1 &= \pm 2\sqrt{2} e^{i\frac{\pi}{2}(1+2n)} \end{aligned}$$

$\bullet z_2 = \sqrt{8} e^{i\frac{4\pi}{3}} = \sqrt{8} (\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = -4\sqrt{2}$   
 $\bullet z_3 = \sqrt{8} e^{i\frac{7\pi}{3}} = \sqrt{8} (\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}) = 4\sqrt{2}$   
 $\bullet z_4 = \pm 2\sqrt{2} i$

**Question 10**

$$z^3 = 32 + 32\sqrt{3}i, \quad z \in \mathbb{C}.$$

- a) Solve the above equation.

Give the answers in exponential form  $z = r e^{i\theta}$ ,  $r > 0$ ,  $-\pi < \theta \leq \pi$ .

- b) Show that these roots satisfy the equation

$$w^9 + 2^{18} = 0.$$

$$z = 4e^{i\frac{\pi}{9}}, \quad 4e^{i\frac{7\pi}{9}}, \quad 4e^{-i\frac{5\pi}{9}}$$

(a)  $z^3 = 32 + 32\sqrt{3}i$   
 $\rightarrow z^3 = 64e^{i(\frac{\pi}{3}+2k\pi)}, \quad k \in \mathbb{Z}$   
 $\rightarrow z^3 = 64e^{i\frac{7\pi}{3}(1+2k)}$   
 $\rightarrow z = 4\sqrt[3]{64} e^{i\frac{7\pi}{3}(1+2k)}$   
 $\rightarrow z = 4e^{i\frac{7\pi}{9}(1+2k)}$

$\bullet |32+32\sqrt{3}i| = 32\sqrt{1+(\sqrt{3})^2} = 32\sqrt{2} = 64$   
 $\bullet \arg(32+32\sqrt{3}i) = \tan^{-1}\frac{32\sqrt{3}}{32} = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$   
 $= \frac{3\pi}{3}$

$z_0 = 4e^{i\frac{\pi}{9}}$   
 $z_1 = 4e^{i\frac{7\pi}{9}}$   
 $z_{-1} = 4e^{-i\frac{5\pi}{9}}$

(b)  $z^9 + 2^{18} = [4e^{i\frac{7\pi}{9}(1+2k)}]^9 + 2^{18} = 4^9 e^{i9(\frac{7\pi}{9}+2k)} + 2^{18}$   
 $= 2^{18} e^{i9(\frac{7\pi}{9}+2k)} + 2^{18} = 2^{18} \left[ e^{i9(\frac{7\pi}{9}+2k)} + 1 \right]$   
 $= 2^{18} \left[ \cos(9(\frac{7\pi}{9}+2k)) + i \sin(9(\frac{7\pi}{9}+2k)) + 1 \right]$   
 $= 2^{18} \times [\cos(7\pi + 18k) + i \sin(7\pi + 18k) + 1]$   
 $= 2^{18} \times [-1 + 1] \xrightarrow{\text{cos } 7\pi = \cos \pi}$

## Question 11

$$z^7 - 1 = 0, \quad z \in \mathbb{C}$$

One of the roots of the above equation is denoted by  $\omega$ , where  $0 < \arg \omega < \frac{\pi}{3}$

- a) Find  $\omega$  in the form  $\omega = r e^{i\theta}$ ,  $r > 0$ ,  $0 < \theta \leq \frac{\pi}{3}$

b) Show clearly that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$$

c) Show further that

$$\omega^2 + \omega^5 = 2 \cos\left(\frac{4\pi}{7}\right)$$

d) Hence, using the results from the previous parts deduce that

$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$$

$$\omega = e^{i\frac{2\pi}{7}}$$

(3)  $\frac{z}{z-1} = 0 \Rightarrow z = 1$   
 $\Rightarrow \overline{z} = 1$   
 $\Rightarrow \frac{z}{\overline{z}} = 1 \Rightarrow (0+2\pi i)$   
 $\Rightarrow z = e^{2\pi i}$   
 $\Rightarrow z = \frac{w^5}{w^6}$   
 $\Rightarrow w = z_1 = e^{\frac{2\pi i}{6}}$

(4)  $w^3 + w^5 = \left(e^{\frac{2\pi i}{3}}\right)^2 + \left(e^{\frac{2\pi i}{5}}\right)^5 = e^{\frac{4\pi i}{3}} + e^{\frac{10\pi i}{5}}$   
 $= e^{\frac{4\pi i}{3}} + e^{-\frac{10\pi i}{5}} = 2\cos\left(\frac{4\pi}{3}\right) = 2\cos\frac{4\pi}{3}$  ✓ By P&Q

(5) SIMILARLY  $w + w^6 = e^{\frac{2\pi i}{6}} + \left(e^{\frac{2\pi i}{6}}\right)^6 = e^{\frac{2\pi i}{6}} + e^{\frac{12\pi i}{6}} = e^{\frac{2\pi i}{6}} + e^{\frac{2\pi i}{3}}$   
 $= 2\cos\frac{2\pi}{3} = 2\cos\frac{2\pi}{3}$   
 $w^3 + w^4 = \left(e^{\frac{2\pi i}{3}}\right)^3 + \left(e^{\frac{2\pi i}{4}}\right)^4 = e^{\frac{6\pi i}{3}} + e^{\frac{8\pi i}{4}} = e^{\frac{6\pi i}{3}} + e^{\frac{2\pi i}{2}}$   
 $= 2\cos\frac{6\pi}{3} = 2\cos\frac{2\pi}{2}$

So  $1 + (w + w^6)^2 + (w^3 + w^4)^2 + (w^3 + w^5)^2 = 0$   
 $1 + 2\cos\frac{2\pi}{3} + 2\cos\frac{2\pi}{3} + 2\cos\frac{2\pi}{2} =$   
 $6\cos\frac{2\pi}{3} + 6\cos\frac{2\pi}{3} + 6\cos\frac{2\pi}{2} = -\frac{1}{2}$  ✓ By P&Q

**Question 12**

$$z^3 = (1+i\sqrt{3})^8 (1-i)^5, \quad z \in \mathbb{C}.$$

Find the three roots of the above equation, giving the answers in the form  $k\sqrt{2}e^{i\theta}$ , where  $-\pi < \theta \leq \pi$ ,  $k \in \mathbb{Z}$ .

$$z = 8\sqrt{2}e^{i\theta}, \quad \theta = -\frac{31\pi}{36}, -\frac{7\pi}{36}, \frac{17\pi}{36}$$

$$\begin{aligned}
 z^3 &= (1+i\sqrt{3})^8 (1-i)^5 \\
 &\Rightarrow z^3 = (2e^{i\frac{\pi}{3}})^8 \times (\sqrt{2}e^{-i\frac{\pi}{4}})^5 \\
 &\Rightarrow z^3 = 256e^{i\frac{8\pi}{3}} \times 4\sqrt{2}e^{-i\frac{5\pi}{4}} \\
 &\Rightarrow z^3 = 1024\sqrt{2}e^{i\frac{17\pi}{3}} \\
 &\Rightarrow z^3 = 2^3 \times 2^{\frac{17}{3}} e^{i(\frac{17\pi}{3} + 2k\pi)} \quad k \in \mathbb{Z} \\
 &\Rightarrow (z^3)^{\frac{1}{3}} = (2^3 \times 2^{\frac{17}{3}} e^{i(\frac{17\pi}{3} + 2k\pi)})^{\frac{1}{3}} \\
 &\Rightarrow z = 2^{\frac{3}{3}} \times e^{\frac{i}{3}(\frac{17\pi}{3} + 2k\pi)} \\
 &\text{Thus } z_1 = 8\sqrt[3]{2}e^{i\frac{17\pi}{9}} \\
 &z_2 = 8\sqrt[3]{2}e^{i\frac{17\pi}{9}} \\
 &z_3 = 8\sqrt[3]{2}e^{i\frac{17\pi}{9}}
 \end{aligned}$$

# TRIGONOMETRIC IDENTITIES QUESTIONS

## Question 1

If  $z = \cos\theta + i\sin\theta$ , show clearly that ..

$$\text{a) } \dots z^n + \frac{1}{z^n} \equiv 2 \cos n\theta.$$

b) ...  $16\cos^5 \theta \equiv \cos 5\theta + 5\cos 3\theta + 10\cos \theta$

proof

(b)  $Z = \cos\theta + i\sin\theta$

$$\begin{aligned} Z^2 &= (\cos\theta + i\sin\theta)^2 = \cos^2\theta + i2\cos\theta\sin\theta \\ Z^3 &= (\cos\theta + i\sin\theta)^3 = \cos^3\theta + i3\cos^2\theta\sin\theta \\ Z^4 &= (\cos\theta + i\sin\theta)^4 = \cos^4\theta + i4\cos^3\theta\sin\theta \\ \vdots & \quad Z^n = (\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n = 2\cos n\theta \end{aligned}$$

Answere  
R 1 3 5 7  
I 4 6 8 10  
M 5 10 15 20 1

**(b)**  $Z^2 + \frac{1}{Z} = 2\cos 2\theta$

$\bullet$   $Z = 1$

$$\begin{aligned} Z^2 + \frac{1}{Z} &= 2\cos 0^\circ \\ (2\cos 0^\circ)^2 &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 \\ 32\cos^2 0^\circ &= \frac{1}{4} + 5\frac{\sqrt{3}}{2}i + 10\frac{1}{2} + 10\frac{\sqrt{3}}{2}i + 5\frac{1}{2} + \frac{1}{4} + \frac{\sqrt{3}}{2}i \\ 32\cos^2 0^\circ &= Z^2 + \frac{1}{Z^2} + 10z + \frac{10}{z} + \frac{\sqrt{3}}{2}i + \frac{\sqrt{3}}{2}i \\ 32\cos^2 0^\circ &= \left(Z + \frac{1}{Z}\right)^2 + 5\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 10\left(1 + \frac{1}{2}i\right) \\ 32\cos^2 0^\circ &= (2\cos 30^\circ)^2 + 5(2\cos 30^\circ)(1 + \frac{1}{2}i) \\ 16\cos^2 0^\circ &= 6\cos 30^\circ + 5\cos 30^\circ + 5\cos 30^\circ \end{aligned}$$

Answere  
R 1 3 5 7  
I 4 6 8 10  
M 5 10 15 20 1

## Question 2

It is given that

$$\sin 5\theta \equiv 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

- a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

It is further given that

$$\sin 3\theta \equiv 3\sin \theta - 4\sin^3 \theta$$

- b) Solve the equation

$$\sin 5\theta = 5 \sin 3\theta \quad \text{for } 0 \leq \theta < \pi$$

giving the solutions correct to 3 decimal places

$$\theta = 0, 1.095^\circ, 2.046^\circ$$

(b)  $\begin{cases} \cos^2\theta + \sin^2\theta = 1 \\ (\cos\theta + i\sin\theta)^5 = (-1+i\sqrt{3})^5 \end{cases}$

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ (\cos\theta + i\sin\theta)^5 &= |C|^5 + 5|C|^3\theta^2 - 10|C|^2\theta^4 + 5\cos\theta + i\sin\theta^5 \end{aligned}$$

Simplify RHS as a Imaginary; Pick imaginary

$$\begin{aligned} \Rightarrow \sin 5\theta &= 5|C|^3\theta^2 - 10|C|^2\theta^4 + \theta^5 \\ \Rightarrow \sin 5\theta &= 5S\left(-1 - \frac{\theta^2}{3}\right)^2 - 10S^2\left(\frac{\theta^4}{9}\right) + \theta^5 \\ \Rightarrow \sin 5\theta &= \frac{5}{9}\left((-2S^2 + S^4) - 10S^2 + 81S^5\right) + \theta^5 \\ \Rightarrow \sin 5\theta &= \frac{5}{9}S - 10S^3 + 81S^5 - 10S^5 + 81S^5 + \theta^5 \\ \Rightarrow \sin 5\theta &= 160S^5 - 20S^3 + \theta^5 \end{aligned}$$

as required

(b)  $\begin{cases} \sin 5\theta = 5\sin\theta \cos^4\theta \\ 160S^5 - 20S^3 + \sin\theta^5 = 5(3\sin\theta + 4\sin^3\theta) \\ 160S^5 - 20S^3 + 5\sin\theta^5 = 15\sin\theta + 20\sin^3\theta \\ 160S^5 - 10\sin\theta^5 = 0 \\ 2\sin\theta(80S^4 - 5) = 0 \end{cases}$

$\bullet \sin\theta = 0$

$\bullet \theta = \frac{\pi}{2}$

$\bullet \theta = \frac{\pi}{2} + n\pi$

$\bullet \sin^4\theta = \frac{1}{16}$

$\sin\theta = \pm \frac{1}{4}$

$\theta = \arcsin\left(\pm \frac{1}{4}\right)$

No Solution in first quadrant

$\theta = 0, 1.0477, 2.0462$

**Question 3**

The complex number  $z$  is given by

$$z = e^{i\theta}, \quad -\pi < \theta \leq \pi.$$

- a) Show clearly that

$$z^n + \frac{1}{z^n} \equiv 2 \cos n\theta.$$

- b) Hence show further that

$$\cos^4 \theta \equiv \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}.$$

- c) Solve the equation

$$2 \cos 4\theta + 8 \cos 2\theta + 5 = 0, \quad 0 \leq \theta < 2\pi.$$

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

(a)

$$\begin{aligned} z &= e^{i\theta} \\ \bar{z} &= (\bar{e}^{i\theta})^n = e^{-in\theta} \\ \bar{z}^n &= (\bar{e}^{i\theta})^n = e^{-in\theta} \end{aligned} \quad \left. \begin{aligned} z^n + \bar{z}^n &= e^{in\theta} + e^{-in\theta} \\ &\rightarrow z^n + \bar{z}^n = \frac{\cos n\theta + i \sin n\theta}{\cos(-n\theta) + i \sin(-n\theta)} \end{aligned} \right\} \begin{aligned} z^n + \bar{z}^n &= \frac{\cos n\theta + i \sin n\theta}{\cos n\theta - i \sin n\theta} \\ \Rightarrow z^n + \frac{1}{z^n} &= \frac{2 \cos n\theta}{\cos n\theta} \end{aligned}$$

(b)

$$z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$\begin{aligned} \text{If } n=1 \\ z + \frac{1}{z} &= 2 \cos \theta \\ (z + \frac{1}{z})^4 &= (2 \cos \theta)^4 \\ (z + \frac{1}{z})^4 &= z^4 + 4z^3 \cdot \frac{1}{z} + 6z^2 \cdot \frac{1}{z^2} + 4z \cdot \frac{1}{z^3} + \frac{1}{z^4} \\ 16 \cos^4 \theta &= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} \\ 16 \cos^4 \theta &= (z^2 + \frac{1}{z^2})^2 + 4(z^2 + \frac{1}{z^2}) + 6 \\ 16 \cos^4 \theta &= 2 \cos 4\theta + 4(2 \cos 2\theta) + 6 \\ \cos^4 \theta &= \frac{1}{16} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8} \end{aligned}$$

(c)

$$\begin{aligned} 2 \cos 4\theta + 8 \cos 2\theta + 5 &= 0 \\ \frac{1}{16} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8} &= 0 \\ \frac{1}{16} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8} &= -\frac{5}{16} \cdot \frac{2}{5} \\ \cos 4\theta &= \frac{1}{16} \\ \cos 2\theta &= -\frac{1}{2} \\ \cos 2\theta &= -\frac{1}{2} \\ \cos 2\theta &= \frac{1}{2} \\ \arccos(\frac{1}{2}) &= \frac{\pi}{3} \\ \arccos(\frac{1}{2}) &= \frac{2\pi}{3} \end{aligned}$$

$$\therefore \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

**Question 4**

The complex number  $z$  is given by

$$z = e^{i\theta}, -\pi < \theta \leq \pi.$$

- a) Show clearly that

$$z^n + \frac{1}{z^n} \equiv 2 \cos n\theta.$$

- b) Hence show further that

$$16 \cos^5 \theta \equiv \cos 5\theta + 5 \cos 3\theta + 10 \cos \theta.$$

- c) Use the results of part (a) and (b) to solve the equation

$$\cos 5\theta + 5 \cos 3\theta + 6 \cos \theta = 0, 0 \leq \theta < \pi.$$

$$\boxed{\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}}$$

(a) Let  $z = \cos \theta + i \sin \theta$  ∴  $\bar{z} = \cos \theta - i \sin \theta$ ,  $\bar{z}^2 = (\cos \theta + i \sin \theta)^2 = (\cos^2 \theta - \sin^2 \theta) + i(2 \cos \theta \sin \theta)$   
 $\bar{z}^n = (\cos \theta + i \sin \theta)^n$   $\therefore z^n + \frac{1}{z^n} = 2 \cos n\theta$   
 $\bar{z}^{-n} = (\cos \theta - i \sin \theta)^n$

(b) Let  $n=1$  in (a)  
 $\Rightarrow 2 \cos \theta = 2 + \frac{1}{z}$   
 $\Rightarrow (2 \cos \theta)^2 = (2 + \frac{1}{z})^2$   
 $\Rightarrow 32 \cos^2 \theta = 2^2 + 5z^2 + 10z + \frac{10}{z} + \frac{5}{z^2} + \frac{1}{z^2}$   
 $\Rightarrow 32 \cos^2 \theta = \left(2 + \frac{1}{z}\right)^2 + \left(\frac{1}{z} + \frac{1}{z^2}\right)^2 + 10\left(z + \frac{1}{z}\right)$   
 $\Rightarrow 32 \cos^2 \theta = (2 \cos \theta)^2 + 10(2 \cos \theta)$   
 $\Rightarrow 16 \cos^2 \theta = \cos^2 \theta + 10 \cos \theta$  AS REQUIRED

(c)  $\cos 5\theta + 5 \cos 3\theta + 6 \cos \theta = 0$   
 $\Rightarrow \cos 5\theta + 5 \cos 3\theta + 10 \cos \theta = 4 \cos \theta$   
 $\Rightarrow 4 \cos \theta = 4 \cos \theta$   
 $\Rightarrow 4 \cos^2 \theta = \cos^2 \theta$   
 $\Rightarrow 4 \cos^2 \theta - \cos^2 \theta = 0$   
 $\Rightarrow \cos(\theta)(4 \cos \theta - 1) = 0$   
 $\Rightarrow \cos \theta = 0 \text{ or } \cos \theta = \frac{1}{4}$   $\cos \theta = \frac{1}{4}$ ,  $\cos \theta = -\frac{1}{4}$   
 For  $0 \leq \theta < \pi$   
 $\theta = \frac{\pi}{2}, \frac{3\pi}{4}, \frac{11\pi}{12}$

**Question 5**

De Moivre's theorem asserts that

$$(\cos \theta + i \sin \theta)^n \equiv \cos n\theta + i \sin n\theta, \quad \theta \in \mathbb{R}, \quad n \in \mathbb{Q}.$$

- a) Use the theorem to prove the validity of the following trigonometric identity.

$$\cos 6\theta \equiv 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1.$$

- b) Use the result of part (a) to find, in exact form, the largest positive root of the equation

$$64x^6 - 96x^4 + 36x^2 - 1 = 0.$$

$$x = \cos\left(\frac{\pi}{9}\right)$$

(a) Let  $\cos \theta + i \sin \theta = c + is$

thus

$$(\cos \theta + i \sin \theta)^6 = (c + is)^6$$

$$\cos 6\theta + i \sin 6\theta = c^6 + 6ic^5s - 15c^4s^2 - 20ic^3s^3 + 15c^2s^4 + 6ic^1s^5 - s^6$$

EXPAND IDEAL PATH

$$\Rightarrow \cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$$

$$\Rightarrow \cos 6\theta = c^6 - 15c^4(c - s) + 15c^2(c - s)^2 - (c - s)^3$$

$$\Rightarrow \cos 6\theta = c^6 - 15c^4 + 15c^2 - 20c^3 + (c - s)^3$$

$$\Rightarrow \cos 6\theta = c^6 - 15c^4 + 15c^2 - 20c^3 + 15c^6 + 4c^3 - 3c^5 + c^6$$

$$\Rightarrow \cos 6\theta = 32c^6 - 48c^4 + 18c^2 - 1$$

$$\therefore \cos 6\theta = 32\cos^6 \theta - 48\cos^4 \theta + 18\cos^2 \theta - 1 \quad \text{AS REQUIRED}$$

(b)  $\cos^6 \theta - 48\cos^4 \theta + 18\cos^2 \theta - 1 = 0$

$$\Rightarrow 32x^6 - 48x^4 + 18x^2 - \frac{1}{2} = 0$$

$$\Rightarrow 32x^6 - 48x^4 + 18x^2 - 1 = -\frac{1}{2}$$

LAT  $\alpha = \cos \theta$

$$\Rightarrow 32\cos^6 \theta - 48\cos^4 \theta + 18\cos^2 \theta - 1 = -\frac{1}{2}$$

$$\Rightarrow \cos 6\theta = -\frac{1}{2}$$

- $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$

$$(G) = \frac{2\pi}{3} + 2k\pi$$

$$(G) = \frac{4\pi}{3} + 2k\pi$$

$$(G) = \frac{4\pi}{3} + 2\pi k$$

$$(G) = \frac{4\pi}{3} + 2\pi k$$

$\therefore \alpha = \cos \frac{4\pi}{3}$  IS THE LARGEST POSITIVE ROOT OF THE EQUATION

**Question 6**

Euler's identity states

$$e^{i\theta} \equiv \cos \theta + i \sin \theta, \quad \theta \in \mathbb{R}.$$

- a) Use the identity to show that

$$e^{in\theta} + e^{-in\theta} \equiv 2 \cos n\theta.$$

- b) Hence show further that

$$32 \cos^6 \theta \equiv \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10.$$

- c) Use the fact that  $\cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin \theta$  to find a similar expression for  $32 \sin^6 \theta$ .

- d) Determine the exact value of

$$\int_0^{\frac{\pi}{4}} \sin^6 \theta + \cos^6 \theta \, d\theta.$$

$$32 \sin^6 \theta = -\cos 6\theta + 6 \cos 4\theta - 15 \cos 2\theta + 10, \quad \boxed{\frac{5\pi}{32}}$$

**(a)**  $e^{i\theta} = \cos \theta + i \sin \theta$   
 $(e^{i\theta})^n = \cos n\theta + i \sin n\theta$   
 $(e^{i\theta})^{-n} = \cos(-n\theta) - i \sin(-n\theta)$  } adding  $[e^{in\theta} + e^{-in\theta} = 2 \cos n\theta]$   
 At Board

**(b)** If  $n=1$   
 $\Rightarrow 2 \cos \theta = 2^{1/2} e^{i\theta}$   
 $\Rightarrow (2 \cos \theta)^2 = (2^{1/2} e^{i\theta})^2$   
 $\Rightarrow 64 \cos^2 \theta = e^{i2\theta} + e^{i2\theta} + 15e^{i2\theta} + 20 + 15e^{-i2\theta} + 15e^{-i2\theta} + e^{-i2\theta}$   
 $\Rightarrow 64 \cos^2 \theta = (e^{i2\theta} + e^{i2\theta}) + (6e^{i2\theta} + e^{-i2\theta}) + 15(e^{i2\theta} + e^{-i2\theta}) + 20$   
 $\Rightarrow 64 \cos^2 \theta = 2 \cos 4\theta + 6(2 \cos 2\theta) + 15(2 \cos 2\theta) + 20$   
 $\Rightarrow 32 \cos^2 \theta = \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10$  At Board

**(c)**  $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$   
 $\cos(2(\frac{\pi}{2} - \theta)) = \cos(\pi - 2\theta) = (\cos(\pi)\cos(2\theta) - \sin(\pi)\sin(2\theta)) = -\cos 2\theta$   
 $\cos(4(\frac{\pi}{2} - \theta)) = \cos(2\pi - 4\theta) = (\cos(2\pi)\cos(4\theta) - \sin(2\pi)\sin(4\theta)) = \cos 4\theta$   
 $\cos(2(\frac{\pi}{2} - \theta)) = \cos(\pi - 2\theta) = (\cos(\pi)\cos(2\theta) - \sin(\pi)\sin(2\theta)) = -\cos 2\theta$   
 $\therefore 32 \sin^6 \theta = -\cos 6\theta + 6 \cos 4\theta - 15 \cos 2\theta + 10$

**(d)** 
$$\int_0^{\frac{\pi}{4}} \sin^6 \theta + \cos^6 \theta \, d\theta = \int_0^{\frac{\pi}{4}} 32 \sin^6 \theta + 32 \cos^6 \theta \, d\theta$$
  
 $= \frac{1}{32} \int_0^{\frac{\pi}{4}} 32 \sin^6 \theta + 20 \, d\theta = \frac{1}{32} \left[ 32 \sin^6 \theta + 20\theta \right]_0^{\frac{\pi}{4}}$   
 $= \frac{1}{32} \left[ (0 + 5\pi) - (0) \right] = \frac{5\pi}{32}$

**Question 7**

De Moivre's theorem asserts that

$$(\cos \theta + i \sin \theta)^n \equiv \cos n\theta + i \sin n\theta, \quad \theta \in \mathbb{R}, \quad n \in \mathbb{Q}.$$

- a) Use the theorem to prove validity of the following trigonometric identity

$$\sin 5\theta = \sin \theta (16 \cos^4 \theta - 12 \cos^2 \theta + 1).$$

- b) Hence, or otherwise, solve the equation

$$\sin 5\theta = 10 \cos \theta \sin 2\theta - 11 \sin \theta, \quad 0 < \theta < \pi.$$

$$\boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}}$$

(a) Let  $\cos \theta + i \sin \theta = C + iS$

$$(C + iS)^5 = (C + iS)^2$$

$$\cos 5\theta + i \sin 5\theta = C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$$

EQUATE IMAGINARY PARTS

$$\Rightarrow \sin 5\theta = 5C^4S - 10C^2S^3 + S^5$$

$$\Rightarrow \sin 5\theta = S [5C^4 - 10C^2S^2 + S^4]$$

$$\Rightarrow \sin 5\theta = S [5C^4 - 10C^2 + 1 - 2C^2 + C^4]$$

$$\Rightarrow \sin 5\theta = S [5C^4 - 12C^2 + 1]$$

LE.  $\sin 5\theta = S [10\cos^4 \theta - 12\cos^2 \theta + 1]$  as required

(b)  $\sin 5\theta = 10 \cos \theta \sin 2\theta - 11 \sin \theta$

$$\sin \theta [10\cos^4 \theta - 12\cos^2 \theta + 1] = 10\cos \theta (2\sin \theta \cos \theta) - 11 \sin \theta$$

As  $0 < \theta < \pi$   $\sin \theta \neq 0$  hence divide it

$$10\cos^4 \theta - 12\cos^2 \theta + 1 = 20\cos^2 \theta - 11$$

$$10\cos^2 \theta - 20\cos^2 \theta + 12 = 0$$

$$4\cos^2 \theta - 8\cos^2 \theta + 3 = 0$$

$$(2\cos \theta - 1)(2\cos \theta - 3) = 0$$

$$\cos \theta = \frac{1}{2}$$

~~$\cos \theta = \frac{\sqrt{2}}{2}$~~

$$\cos \theta = \frac{1}{2} \quad \dots \dots \theta = \frac{\pi}{3} \text{ only}$$

$$\cos \theta = -\frac{1}{2} \quad \dots \dots \theta = \frac{2\pi}{3} \text{ only}$$

**Question 8**

It is given that

$$\sin 5\theta \equiv \sin \theta (16\cos^4 \theta - 12\cos^2 \theta + 1).$$

- a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

Consider the general solution of the trigonometric equation

$$\sin 5\theta = 0.$$

- b) Find exact simplified expressions for

$$\cos^2\left(\frac{\pi}{5}\right) \text{ and } \cos^2\left(\frac{2\pi}{5}\right),$$

fully justifying each step in the workings.

$$\boxed{\cos^2\left(\frac{\pi}{5}\right) = \frac{3+\sqrt{5}}{8}}, \quad \boxed{\cos^2\left(\frac{2\pi}{5}\right) = \frac{3-\sqrt{5}}{8}}$$

(a)  $\cos \theta + i \sin \theta = e^{i\theta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{\frac{1}{5}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \dots$

$$\begin{aligned} (\cos \theta + i \sin \theta)^5 &= (e^{i\theta})^5 \\ \Rightarrow \cos 5\theta + i \sin 5\theta &= e^{5i\theta} = e^{i(5\theta)} = 10e^{i\theta} - 10e^{3i\theta} + 10e^{5i\theta} - 10e^{7i\theta} + 10e^{9i\theta} \\ \Rightarrow \cos 5\theta + i \sin 5\theta &= (e^{i\theta} - 10e^{3i\theta} + 15e^{5i\theta} - 10e^{7i\theta} + e^{9i\theta}) + i(10e^{3i\theta} - 10e^{5i\theta} + e^{7i\theta}) \\ \therefore \cos 5\theta &= \cos \theta - 10\cos^3 \theta + \cos^5 \theta \\ \Rightarrow \sin 5\theta &= i[\cos^2 \theta - 10\cos^4 \theta + \cos^6 \theta] \\ \Rightarrow \sin 5\theta &= i[\cos^2 \theta - 10\cos^4 \theta + (\cos^2 \theta)^3] \\ \Rightarrow \sin 5\theta &= i[\cos^2 \theta - 10\cos^4 \theta + 1 - 2\cos^2 \theta + \cos^4 \theta] \\ \Rightarrow \sin 5\theta &= i[16\cos^4 \theta - 12\cos^2 \theta + 1] \\ \Rightarrow \sin 5\theta &= \sin \theta [16\cos^4 \theta - 12\cos^2 \theta + 1] \end{aligned}$$

At Biquadratic

(b)

- $\bullet \sin \theta = 0 \Rightarrow \theta = n\pi, n \in \mathbb{Z}$
- $\bullet \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$
- $\bullet 16\cos^4 \theta - 12\cos^2 \theta + 1 = 0$

$$\begin{aligned} \cos^2 \theta &= \frac{12 \pm \sqrt{144-64}}{32} \\ \cos^2 \theta &= \frac{12 \pm \sqrt{80}}{32} \\ \cos^2 \theta &= \frac{3 \pm \sqrt{5}}{8} < 0.95 \end{aligned}$$

$\cancel{\cos^2 \theta < \frac{3}{4} < \frac{3}{2}}$

$\cos^2 \theta < \frac{3}{4}$ ,  $\cos \theta$  is definitely

$$\begin{aligned} \cos^2 \theta &< \cos^2 \frac{\pi}{3} \\ \frac{1}{4} &< \cos^2 \frac{\pi}{3} \\ \therefore \cos^2 \frac{\pi}{3} &= \frac{3-\sqrt{5}}{8} \end{aligned}$$

SIMILARLY

$$\begin{aligned} \cos^2 \frac{2\pi}{5} &> \frac{3}{4} \\ \cos^2 \frac{2\pi}{5} &< \cos^2 \frac{\pi}{3} \quad \text{Cos becomes 0 at vertices} \\ \cos^2 \frac{2\pi}{5} &< \cos^2 \frac{\pi}{2} \\ \cos^2 \frac{2\pi}{5} &< \frac{1}{4} \\ \therefore \cos^2 \frac{2\pi}{5} &= \frac{1-\sqrt{5}}{8} \\ \therefore \cos^2 \frac{2\pi}{5} &= \frac{3+\sqrt{5}}{8} \end{aligned}$$

**Question 9**

By considering the binomial expansion of  $(\cos \theta + i \sin \theta)^4$  show that

$$\tan 4\theta \equiv \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}.$$

proof

Let  $\cos \theta + i \sin \theta = C + iS$

$(C + iS)^4 = (C + iS)^2 \cdot (C + iS)^2$

$= (C^2 - S^2 + 2iCS) \cdot (C^2 - S^2 + 2iCS)$

$= C^4 - 2C^2S^2 + S^4 + 4iC^3S - 4iCS^3 + 4i^2C^2S^2$

$= C^4 - 6C^2S^2 + S^4 + i(4C^3S - 4CS^3)$

$\therefore \tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4iC^3S - 4CS^3}{C^4 - 6C^2S^2 + S^4}$

DIVIDE TOP & BOTTOM BY  $C^4$

$\tan 4\theta = \frac{4iC^3S/C^4 - 4CS^3/C^4}{C^4/C^4 - 6C^2S^2/C^4 + S^4/C^4}$

$= \frac{4i\tan 3\theta - 4\tan \theta}{1 - 6\tan^2 \theta + \tan^4 \theta}$  // as required

**Question 10**

By using de Moivre's theorem followed by a suitable trigonometric identity, show clearly that ...

a) ...  $\cos 3\theta \equiv 4\cos^3 \theta - 3\cos \theta$ .

b) ...  $\cos 6\theta \equiv (2\cos^2 \theta - 1)(16\cos^4 \theta - 16\cos^2 \theta + 1)$

Consider the solutions of the equation.

$$\cos 6\theta = 0, 0 \leq \theta \leq \pi.$$

c) By fully justifying each step in the workings, find the exact value of

$$\cos \frac{\pi}{12} \cos \frac{5\pi}{12} \cos \frac{7\pi}{12} \cos \frac{11\pi}{12}.$$

1  
16

**(a)**  $(\cos \theta + i \sin \theta)^3 \equiv C + iS$

$$(\cos \theta + i \sin \theta)^3 = (C + iS)^3$$

$$\cos 3\theta + i \sin 3\theta = C^3 + 3C^2iS - 3C(S^2) - iS^3$$

D.I.R. PART  $\Rightarrow \cos 3\theta = C^3 - 3CS^2$

$$\cos 3\theta = C^3 - 3(C^2 - S^2)$$

$$\cos 3\theta = C^3 - 3C^2 + 3S^2$$

$$\cos 3\theta = 4C^3 - 3\cos \theta$$

As required

**(b)**  $\cos 6\theta = (\cos(2 \times 3\theta)) = 2(\cos^2 3\theta - 1)$   

$$= 2(4\cos^2 \theta - 3\cos^2 \theta)^2 - 1$$
  

$$= 2[4\cos^2 \theta - 2\cos^2 \theta(4\cos^2 \theta)] - 1$$
  

$$= 32\cos^6 \theta - 48\cos^4 \theta + 16\cos^2 \theta - 1$$
  

Let  $\cos \theta = x$

$$\cos 6\theta = 32x^6 - 48x^4 + 16x^2 - 1$$

$$\cos 6\theta = (2x^2 - 1)(16x^4 - 16x^2 + 1)$$

verified by multiplication

$$\therefore \cos 6\theta = (\cos 6\theta - 1)(\cos 6\theta + 1)$$

As required

**(c)**  $\cos 6\theta = 0$

$$0 = \cos(6 \cdot \frac{\pi}{12}) \quad \text{or} \quad 0 = \cos(6 \cdot \frac{5\pi}{12}) \quad \text{or} \quad 0 = \cos(6 \cdot \frac{7\pi}{12}) \quad \text{or} \quad 0 = \cos(6 \cdot \frac{11\pi}{12})$$

Now  $2\cos 6\theta - 1 = 0$

$$2\cos 6\theta = 2$$

$$\cos 6\theta = 1$$

$$\cos 6\theta = \pm \frac{1}{\sqrt{2}}$$

$$(\cos 6\theta - 1)^2 = \frac{1}{2}$$

$$2\cos^2 6\theta - 1 = \frac{1}{2}$$

$$\cos^2 6\theta = \frac{3}{4}$$

$$\cos 6\theta = \pm \frac{\sqrt{3}}{2}$$

**Final Answer**

$$\cos \frac{\pi}{12} \cos \frac{5\pi}{12} \cos \frac{7\pi}{12} \cos \frac{11\pi}{12} = \cos \frac{\pi}{12} \cos \frac{5\pi}{12} (-\cos \frac{7\pi}{12})(-\cos \frac{11\pi}{12})$$

$$= (-\cos \frac{\pi}{12})(-\cos \frac{5\pi}{12})$$

$$= \cos^2 \frac{\pi}{12} \cos^2 \frac{5\pi}{12}$$

$$= \frac{2\sqrt{3}}{4} \times \frac{2\sqrt{3}}{4} = \frac{3}{4}$$

$$= \frac{3}{16}$$

**ALTERNATIVE**

All the solutions of  $\cos 6\theta = 0$  are the same in pairs!

So product of roots in pairs =  $\cos \frac{\pi}{12} \cos \frac{5\pi}{12} \cos \frac{7\pi}{12} \cos \frac{11\pi}{12} = \frac{3}{16}$

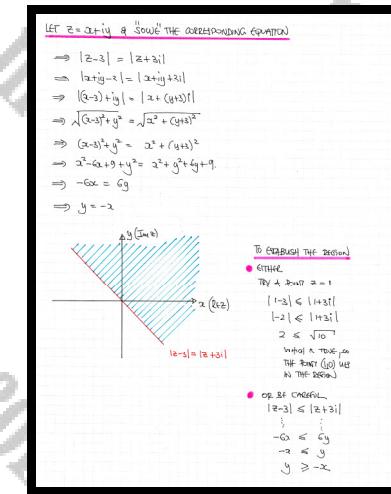
# COMPLEX LOCI

**Question 1**

By finding a suitable Cartesian locus in the complex  $z$  plane, shade the region  $R$  that satisfies the inequality

$$|z - 3| \leq |z + 3i|.$$

$$x + y \geq 0$$



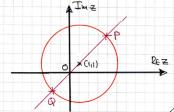
**Question 2**

$$|z - 1 - i| = 4, z \in \mathbb{C}.$$

- a) Sketch the locus of the points that satisfy the above equation in a standard Argand diagram.
- b) Find the minimum and maximum values of  $|z|$  for points that lie on this locus.

$$|z_{\min}| = 4 - \sqrt{2}, |z_{\max}| = 4 + \sqrt{2}$$

a) This is a standard circle locus  
 $|z - 1 - i| = 4$   
 $|z - (1+i)| = 4$   
It centre at  $C(1, 1)$ , radius 4



b) Looking at the diagram above & noting that  $|z|$  represents the distance of a point from the origin

- DISTANCE OF THE CENTRE FROM O IS  $\sqrt{2}$
- $|z|_{\max} = \text{RADIUS} + \sqrt{2} = 4 + \sqrt{2}$  (Point P)
- $|z|_{\min} = \text{RADIUS} - \sqrt{2} = 4 - \sqrt{2}$  (Point Q)

**Question 3**

The complex number  $z$  represents the point  $P(x, y)$  in the Argand diagram.

Given that

$$|z - 1| = 2|z + 2|,$$

show that the locus of  $P$  is given by

$$(x + 3)^2 + y^2 = 4.$$

proof

$$\begin{aligned}
 |z - 1| &= 2|z + 2| \\
 \text{Let } z &= x + iy \\
 \Rightarrow |(x - 1) + iy| &= 2|(x + 2) + iy| \\
 \Rightarrow |(x - 1) + iy| &= 2|(x + 2) + iy| \\
 \Rightarrow \sqrt{(x-1)^2 + y^2} &= 2\sqrt{(x+2)^2 + y^2} \\
 \Rightarrow (x-1)^2 + y^2 &= 4((x+2)^2 + y^2) \\
 \Rightarrow (x-1)^2 + y^2 &= 4(x^2 + 4x + 4) \\
 \Rightarrow x^2 - 2x + 1 + y^2 &= 4x^2 + 16x + 16 \\
 \Rightarrow 0 &= 3x^2 + 18x + 15 \\
 \Rightarrow 0 &= x^2 + y^2 + 6x + 5
 \end{aligned}
 \quad \Rightarrow (x+3)^2 + y^2 = 4$$

**Question 4**

The complex number  $z = x + iy$  represents the point  $P$  in the complex plane.

Given that

$$\bar{z} = \frac{1}{z}, \quad z \neq 0$$

determine a Cartesian equation for the locus of  $P$ .

$$x^2 + y^2 = 1$$

$$\begin{aligned}
 \bar{z} &= \frac{1}{z} \quad \text{Let } z = x + iy \\
 (x - iy) &= \frac{1}{(x + iy)} \\
 (x - iy)(x + iy) &= 1 \\
 x^2 + y^2 &= 1
 \end{aligned}$$

$\therefore$  A UNIT CIRCLE  
CENTRE AT (0,0)

**Question 5**

Sketch, on the same Argand diagram, the locus of the points satisfying each of the following equations.

a)  $|z - 3 + i| = 3$ .

b)  $|z| = |z - 2i|$ .

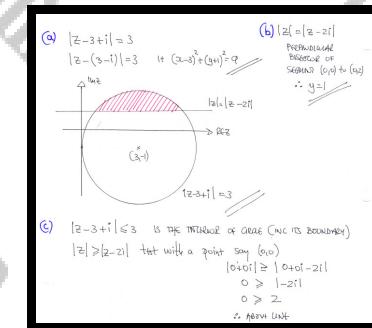
Give in each case a Cartesian equation for the locus.

- c) Shade in the sketch the region that is satisfied by both these inequalities

$$|z - 3 + i| \leq 3$$

$$|z| \geq |z - 2i|$$

$$(x - 3)^2 + (y + 1)^2 = 9, \quad y = 1$$



**Question 6**

- a) Sketch on the same Argand diagram the locus of the points satisfying each of the following equations.

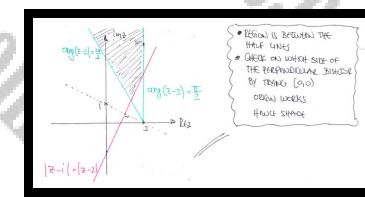
i.  $|z - i| = |z - 2|$ .

ii.  $\arg(z - 2) = \frac{\pi}{2}$ .

- b) Shade in the sketch the region that is satisfied by both these inequalities

$$|z - i| \leq |z - 2| \quad \text{and} \quad \frac{\pi}{2} \leq \arg(z - 2) \leq \frac{2\pi}{3}.$$

sketch



**Question 7**

The complex number  $z$  represents the point  $P(x, y)$  in the Argand diagram.

Given that

$$|z - 1| = \sqrt{2}|z - i|,$$

show that the locus of  $P$  is a circle, stating its centre and radius.

$$\boxed{(x+1)^2 + (y-2)^2 = 4}, \boxed{(-1, 2), r=2}$$

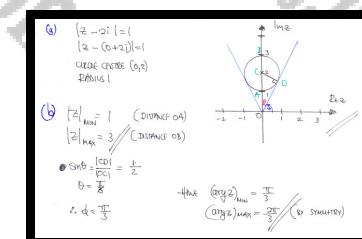
$$\begin{aligned}
 |z - 1| &= \sqrt{2}|z - i| \\
 \text{Let } z &= x+iy \\
 \Rightarrow |x+iy-1| &= \sqrt{2}|x+iy-i| \\
 \Rightarrow |(x-1)+iy| &= \sqrt{2} \sqrt{x^2 + (y-1)^2} \\
 \Rightarrow \sqrt{(x-1)^2+y^2} &= \sqrt{2} \sqrt{x^2 + (y-1)^2} \\
 \Rightarrow (x-1)^2+y^2 &= 2(x^2 + (y-1)^2) \\
 \Rightarrow x^2-2x+1+y^2 &= 2(x^2+2y^2-2y+1) \\
 \Rightarrow 0 &= x^2+2y^2+2x-4y+1
 \end{aligned}
 \quad
 \begin{aligned}
 \Rightarrow x^2+2x+y^2-4y+1 &= 0 \\
 \Rightarrow (x+1)^2+(y-2)^2-1-4+1 &= 0 \\
 \Rightarrow (x+1)^2+(y-2)^2 &= 4 \\
 \text{Centre} &(-1, 2) \\
 \text{Radius} &2
 \end{aligned}$$

**Question 8**

$$|z - 2i| = 1, z \in \mathbb{C}.$$

- a) In the Argand diagram, sketch the locus of the points that satisfy the above equation.
- b) Find the minimum value and the maximum value of  $|z|$ , and the minimum value and the maximum of  $\arg z$ , for points that lie on this locus.

$$\boxed{|z|_{\min} = 1}, \boxed{|z|_{\max} = 3}, \boxed{\arg z_{\min} = \frac{\pi}{3}}, \boxed{\arg z_{\max} = \frac{2\pi}{3}}$$



**Question 9**

The complex number  $z$  represents the point  $P(x, y)$  in the Argand diagram.

Given that

$$|z+1|=2|z-2i|,$$

show that the locus of  $P$  is a circle and state its radius and the coordinates of its centre.

$$\left(\frac{1}{3}, \frac{8}{3}\right), \quad r = \frac{2}{3}\sqrt{5}$$

$$\begin{aligned} & |z+1|=2|z-2i| \\ \Rightarrow & |x+iy+1|=2|x+iy-2i| \\ \Rightarrow & |(x+1)+iy|=2|(x-2)+iy| \\ \Rightarrow & \sqrt{(x+1)^2+y^2}=2\sqrt{x^2+(y-2)^2} \\ \Rightarrow & (x+1)^2+y^2=4(x^2+y^2-4y+4) \\ \Rightarrow & x^2+2x+1+y^2=4x^2+4y^2-16y+16 \\ \Rightarrow & 0=3x^2-2x+3y^2-16y+15 \\ \Rightarrow & x^2-\frac{2}{3}x+y^2-\frac{16}{3}y+5=0 \\ \Rightarrow & (x-\frac{1}{3})^2+(y-\frac{8}{3})^2-\frac{1}{3}-\frac{64}{9}+5=0 \\ \Rightarrow & (x-\frac{1}{3})^2+(y-\frac{8}{3})^2=\frac{28}{9} \\ \text{CENTRE } & (\frac{1}{3}, \frac{8}{3}), \quad \text{RADIUS } \frac{2}{3}\sqrt{5} \end{aligned}$$

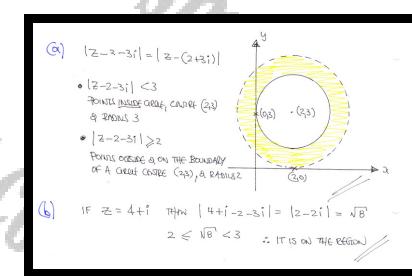
**Question 10**

The complex number  $z = x + iy$  satisfies the relationship

$$2 \leq |z - 2 - 3i| < 3.$$

- Shade accurately in an Argand diagram the region represented by the above relationship.
- Determine algebraically whether the point that represents the number  $4+i$  lies inside or outside this region.

inside the region



**Question 11**

Two sets of loci in the Argand diagram are given by the following equations

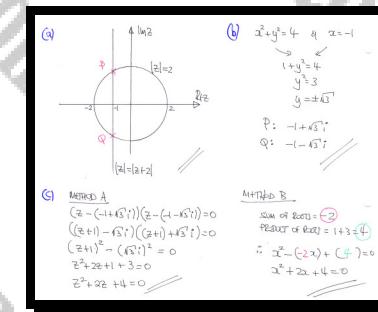
$$|z| = |z+2| \quad \text{and} \quad |z| = 2, \quad z \in \mathbb{C}.$$

- a) Sketch both these loci in the same Argand diagram.

The points  $P$  and  $Q$  in the Argand diagram satisfy both loci equations.

- b) Write the complex numbers represented by  $P$  and  $Q$ , in the form  $a+ib$ , where  $a$  and  $b$  are real numbers.  
 c) Find a quadratic equation with real coefficients, whose solutions are the complex numbers represented by the points  $P$  and  $Q$ .

$$z = -1 \pm \sqrt{3}, \quad z^2 + 2z + 4 = 0$$



**Question 12**

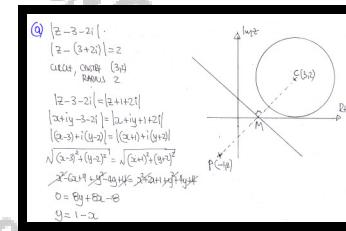
- a) Sketch in the same Argand diagram the locus of the points satisfying each of the following equations

i.  $|z - 3 - 2i| = 2$ .

ii.  $|z - 3 - 2i| = |z + 1 + 2i|$ .

- b) Show by a **geometric** calculation that no points lie on both loci.

proof



$$\begin{aligned} \textcircled{Q} \quad & |z - 3 - 2i| \\ & |z - (3+2i)| = 2 \\ & \text{CIRCLE, CENTER } (3,2) \\ & \text{RADII } 2 \\ & |z - 3 - 2i| = |z - (2+1+2i)| \\ & |z - 3 - 2i| = |z - (-1-2i)| \\ & |z - 3 - 2i| = |z - (-1+2i)| \\ & |z - 3 - 2i| = \sqrt{(x-3)^2 + (y-2)^2} \\ & \sqrt{(x-3)^2 + (y-2)^2} = \sqrt{(x+1)^2 + (y-2)^2} \\ & (x-3)^2 + (y-2)^2 = (x+1)^2 + (y-2)^2 \\ & x^2 - 6x + 9 + y^2 - 4y + 4 = x^2 + 2x + 1 + y^2 - 4y + 4 \\ & 0 = 8x - 8 \\ & x = 1 - 2 \\ & y = 1 - 2 \end{aligned}$$

• Let the midpoint be  $M(1, 0)$

- IF  $|MC| < 2 \rightarrow$  INSIDE
- 2 INTERSECTING
- IF  $|MC| = 2 \rightarrow$  TANGENT
- IF  $|MC| > 2 \rightarrow$  NO INTERSECTIONS

$$|MC| = \sqrt{2^2 + 2^2} = \sqrt{8} > 2,$$

∴ NO INTERSECTIONS

**Question 13**

The point  $A$  represents the complex number on the  $z$  plane such that

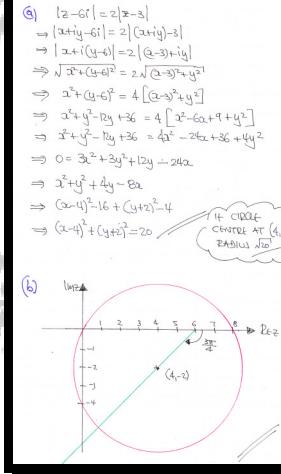
$$|z - 6i| = 2|z - 3|,$$

and the point  $B$  represents the complex number on the  $z$  plane such that

$$\arg(z - 6) = -\frac{3\pi}{4}.$$

- a) Show that the locus of  $A$  as  $z$  varies is a circle, stating its radius and the coordinates of its centre.
- b) Sketch, on the same  $z$  plane, the locus of  $A$  and  $B$  as  $z$  varies.
- c) Find the complex number  $z$ , so that the point  $A$  coincides with the point  $B$ .

$$C(4, -2), r = \sqrt{20}, z = (4 - \sqrt{10}) + i(-2 - \sqrt{10})$$



GENERAL OF THIS EQUATION IS 1. PASSING THROUGH  $(6, 0)$

$$\begin{aligned} y - 0 &= 1(x - 6) \\ y &= x - 6 \quad (x \leq 6) \\ (y+2)^2 + (x-4)^2 &= 20 \\ (x-4)^2 + (x-4)^2 &= 20 \\ 2(x-4)^2 &= 20 \\ (x-4)^2 &= 10 \\ x-4 &= \pm \sqrt{10} \\ x &= 4 \pm \sqrt{10} > 6 \\ \therefore x &= 4 + \sqrt{10} \\ y &= (4 - \sqrt{10}) - 6 = -2 - \sqrt{10} \\ \therefore (4 - \sqrt{10}) + i(-2 - \sqrt{10}) & \end{aligned}$$

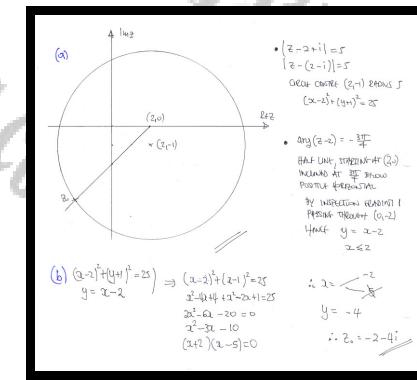
**Question 14**

$$|z - 2 + i| = 5.$$

$$\arg(z - 2) = -\frac{3\pi}{4}.$$

- a) Sketch each of the above complex loci in the same Argand diagram.
- b) Determine, in the form  $x+iy$ , the complex number  $z_0$  represented by the intersection of the two loci of part (a).

$$z_0 = -2 - 4i$$



## Question 15

The locus of the point  $z$  in the Argand diagram, satisfy the equation

$$|z - 2 + i| = \sqrt{3}$$

- a) Sketch the locus represented by the above equation

The half line  $L$  with equation

$$y = mx - 1, \quad x \geq 0, \quad m > 0$$

**touches** the locus described in part (a) at the point  $P$

- b) Find the value of  $m$ .  
 c) Write the equation of  $L$ , in the form

$$\arg(z - z_0) = \theta, \quad z_0 \in \mathbb{C}, \quad -\pi < \theta \leq \pi$$

- d) Find the complex number  $w$ , represented by the point  $P$

$$m = \sqrt{3}, \quad \arg(z + i) = \frac{\pi}{3}, \quad w = \frac{1}{2} + i \left( \frac{\sqrt{3}}{2} - 1 \right)$$

(4)  $\begin{cases} z-2+i = \sqrt{3} \\ |z-(2-i)| = \sqrt{3} \end{cases}$

$y = mx + b$

(5)  $\arg(z - (0-i)) = \frac{\pi}{3}$

$\therefore \arg(z \pm i) = \frac{\pi}{3}$

(6) If we use ARITHMETIC MEANING  
in (5)  
 $m = \sqrt{3}$   
 $\therefore (m^2)^2 - 4x_1x_2 + 1 = 0$   
 $4x_1^2 - 4x_1x_2 + 1 = 0$   
 $(2x_1 - 1)^2 = 0$   
 $2x_1 - 1 = 0$   
 $x_1 = \frac{1}{2}$   
 $y_1 = \sqrt{3} - 1$   
 $y_1 = \frac{\sqrt{3}}{2} - 1$   
 $\therefore m = \frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)$

QUESTION

(7)  $\begin{cases} (x-2)^3 + (y+1)^3 = 3 \\ y = mx + b \end{cases}$

$(2-x)^3 + (y_1+1)^3 = 3$

$x_1^3 - 8x_1^2 + 14 + 1^3 = 3$

$x_1^3 - 8x_1^2 + 14 + 1 = 3$

$(x_1-1)^3 = 8x_1^2 - 12x_1 + 1 = 0$

SOL: RUMUS  $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$b^2 = 4x_1$   
 $b = \sqrt{4x_1}$   
 $1 = 4(x_1)$   
 $4 = x_1^2 + 1$   
 $3 = x_1^2$   
 $x_1 = \sqrt{3}$   $(x_1 > 0)$

QUESTION

(8)  $\begin{cases} \text{DE OF GEOMETRY} \\ |\cos \theta| = \frac{1}{2} \\ |\sin \theta| = \frac{\sqrt{3}}{2} \\ \theta = \frac{\pi}{3} \\ \text{COMPLEX} = b + \frac{\pi}{3}i = k^3 \end{cases}$

$\theta = \text{POLE}(ae \frac{\pi}{3}, \frac{1}{2})$

$1 = -1 + i\sqrt{3} = -1 + \frac{\sqrt{3}}{2}i$

$\therefore k = 1 + i\left(-1 + \frac{\sqrt{3}}{2}i\right)$

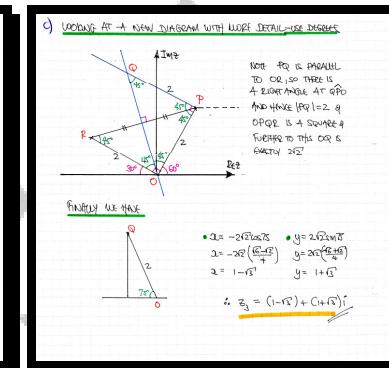
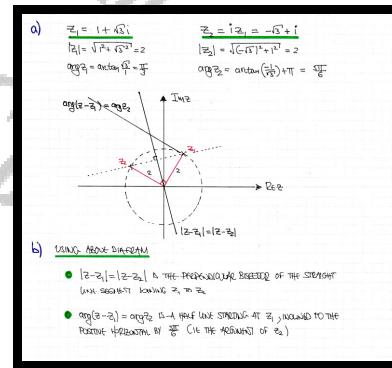
**Question 16**

The complex numbers  $z_1$  and  $z_2$  are given by

$$z_1 = 1 + i\sqrt{3} \quad \text{and} \quad z_2 = iz_1.$$

- a) Label accurately the points representing  $z_1$  and  $z_2$ , in an Argand diagram.
- b) On the same Argand diagram, sketch the locus of the points  $z$  satisfying ...
- i. ...  $|z - z_1| = |z - z_2|$ .
  - ii. ...  $\arg(z - z_1) = \arg z_2$ .
- c) Determine, in the form  $x + iy$ , the complex number  $z_3$  represented by the intersection of the two loci of part (b).

$$z_3 = (1 - \sqrt{3}) + i(1 + \sqrt{3})$$



**Question 17**

The complex number  $z$  lies in the region  $R$  of an Argand diagram, defined by the inequalities

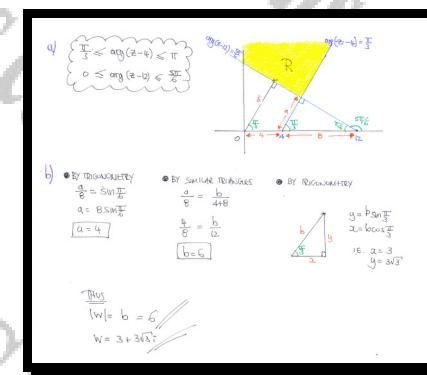
$$\frac{\pi}{3} \leq \arg(z-4) \leq \pi \quad \text{and} \quad 0 \leq \arg(z-12) \leq \frac{5\pi}{6}$$

- a) Sketch the region  $R$ , indicating clearly all the relevant details.

The complex number  $w$  lies in  $R$ , so that  $|w|$  is minimum.

- b) Find  $|w|$ , further giving  $w$  in the form  $u+iv$ , where  $u$  and  $v$  are real numbers.

$$|w|=3, \quad w=3+3\sqrt{3}i$$



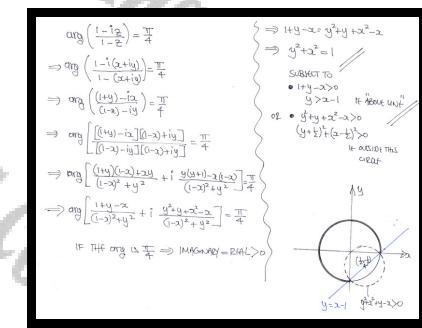
**Question 18**

The point  $P$  represents the number  $z = x + iy$  in an Argand diagram and further satisfies the equation

$$\arg\left(\frac{1-iz}{1-z}\right) = \frac{\pi}{4}, \quad z \neq -i.$$

Use an algebraic method to find an equation of the locus of  $P$  and sketch this locus accurately in an Argand diagram.

$$x^2 + y^2 = 1, \quad \text{such that } y > x - 1$$



**Question 19**

The complex number  $x+iy$  in the  $z$  plane of an Argand diagram satisfies the inequality

$$x^2 + y^2 + x > 0.$$

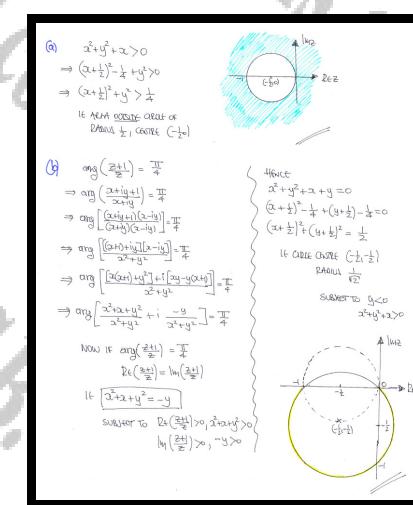
- a) Sketch the region represented by this inequality.

A locus in the  $z$  plane of an Argand diagram is given by the equation

$$\arg\left(\frac{z+1}{z}\right) = \frac{\pi}{4}.$$

- b) Sketch the locus represented by this equation.

sketch



**Question 20**

The complex number  $z$  satisfies the relationship

$$\arg(z-2) - \arg(z+2) = \frac{\pi}{4}.$$

Show that the locus of  $z$  is a circular arc, stating ....

- ... the coordinates of its endpoints.
- ... the coordinates of its centre.
- ... the length of its radius.

$$(-2,0), (2,0), [0,2], r = 2\sqrt{2}$$

**GEOMETRIC APPROACH**

$\theta - \phi = \frac{\pi}{4}$   
 $(\theta = \frac{\pi}{4} + \phi)$

- So  $z$  lies on the arc of a circle whose chord is  $BC$ , where  $B(-2,0)$  and  $C(2,0)$  and inside the major segment.
- Centre must lie on the  $y$ -axis (perpendicular bisector of the chord).
- By geometry the centre is at  $(0,2)$  & radius  $2\sqrt{2}$ .

**ALGEBRAIC APPROACH**

$$\begin{aligned} \arg(z-2) - \arg(z+2) &= \frac{\pi}{4} \\ \arg\left(\frac{z-2}{z+2}\right) &= \frac{\pi}{4} \quad \text{& } \operatorname{Re}\left(\frac{z-2}{z+2}\right) > 0 \quad \& \quad \ln\left(\frac{z-2}{z+2}\right) > 0 \\ \arg\left(\frac{z+2-z-2}{z+2+z-2}\right) &= \frac{\pi}{4} \\ \arg\left(\frac{-4i}{2z} \right) &= \frac{\pi}{4} \\ \arg\left[\frac{(z-2)+i(z+2)-iz}{(z-2)+i(z+2)-iz}\right] &= \frac{\pi}{4} \\ \arg\left[\frac{(z-2)(1-i)+(z+2)(1+i)-iz(z+2)}{(z-2)^2+1}\right] &= \frac{\pi}{4} \\ \arg\left[\frac{2z^2+2i-4z-4+2z^2+2i+4z-2z^2-2z}{(z-2)^2+1}\right] &= \frac{\pi}{4} \\ \arg\left[\frac{2z^2+2i-4z+4}{(z-2)^2+1}\right] &= \frac{\pi}{4} \end{aligned}$$

SINCE THE  $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4} \rightarrow \operatorname{Im}(z) > 0$

$$\begin{aligned} z^2 + 4z + 4 &= 0 \\ z^2 + 4z &= -4 \\ z^2 + 4z + 4 &= 0 \\ (z+2)^2 &= 0 \quad \text{SOLVED} \\ z+2 &> 0 \\ z &> -2 \\ &\& y > 0 \end{aligned}$$

$\therefore$  CIRCULAR ARC FROM THE CIRCLE CENTRE  $(0,2)$  RADIUS  $2\sqrt{2}$  WHICH HAS POSITIVE  $y$  AND HAS POINTS LIE ON THE  $x+y=4$  AS EQUATION

# COMPLEX FUNCTIONS

**Question 1**

A transformation from the  $z$  plane to the  $w$  plane is defined by the complex function

$$w = \frac{3-z}{z+1}, \quad z \neq -1.$$

The locus of the points represented by the complex number  $z = x + iy$  is transformed to the circle with equation  $|w|=1$  in the  $w$  plane.

Find, in Cartesian form, an equation of the locus of the points represented by the complex number  $z$ .

$$x=1$$

$$\begin{aligned} w &= \frac{3-z}{z+1} \\ \Rightarrow |w| &= \left| \frac{3-z}{z+1} \right| \\ \Rightarrow 1 &= \frac{|3-z|}{|z+1|} \\ \Rightarrow 1 &= \frac{|z-3|}{|z+1|} \\ \Rightarrow |z+1| &= |z-3| \end{aligned}$$

Let  $z = x+iy$   
 $\Rightarrow |x+iy+1| = |(x-3)+iy|$   
 $\Rightarrow |(x+1)+iy| = |(x-3)-iy|$   
 $\Rightarrow \sqrt{(x+1)^2+y^2} = \sqrt{(x-3)^2+y^2}$   
 $\Rightarrow (x+1)^2+y^2 = (x-3)^2+y^2$   
 $\Rightarrow x^2+2x+1 = x^2-6x+9$   
 $\Rightarrow 8x = 8$   
 $\Rightarrow x = 1$

**Question 2**

Find an equation of the locus of the points which lie on the half line with equation

$$\arg z = \frac{\pi}{4}, \quad z \neq 0$$

after it has been transformed by the complex function

$$w = \frac{1}{z}.$$

$$\arg w = -\frac{\pi}{4}$$

$$\begin{aligned} w &= \frac{1}{z} \Rightarrow z = \frac{1}{w} \\ \Rightarrow \arg z &= \arg\left(\frac{1}{w}\right) \\ \Rightarrow \frac{\pi}{4} &= \arg w - \arg w \\ \Rightarrow \arg w &= \frac{\pi}{4} \end{aligned}$$

if  $y = -\infty, x > 0$

**Question 3**

The complex function

$$w = \frac{1}{z-1}, \quad z \neq 1, z \in \mathbb{C}, \quad z \neq 1$$

transforms the point represented by  $z = x + iy$  in the  $z$  plane into the point represented by  $w = u + iv$  in the  $w$  plane.

Given that  $z$  satisfies the equation  $|z| = 1$ , find a Cartesian locus for  $w$ .

$$u = -\frac{1}{2}$$

$\begin{aligned} w &= \frac{1}{z-1} \\ \Rightarrow z-1 &= \frac{1}{w} \\ \Rightarrow z &= \frac{1}{w} + 1 \\ \Rightarrow z &= \frac{w+1}{w} \\ \Rightarrow  z  &= \left  \frac{w+1}{w} \right  \\ \Rightarrow 1 &= \frac{ w+1 }{ w } \\ \Rightarrow  w  &=  w+1  \end{aligned}$	$\begin{aligned} \Rightarrow  u+iv  &=  u+iv+1  \\ \Rightarrow  u+iv  &=  (u+1)+iv  \\ \Rightarrow \sqrt{u^2+v^2} &= \sqrt{(u+1)^2+v^2} \\ \Rightarrow u^2+v^2 &= (u+1)^2+v^2 \\ \Rightarrow u^2+v^2 &= u^2+2u+1+v^2 \\ \Rightarrow 2u &= -1 \\ \Rightarrow u &= -\frac{1}{2} \end{aligned}$	$\Rightarrow  u+iv  =  u+iv+1 $ $\Rightarrow  u+iv  =  (u+1)+iv $ $\Rightarrow \sqrt{u^2+v^2} = \sqrt{(u+1)^2+v^2}$ $\Rightarrow u^2+v^2 = (u+1)^2+v^2$ $\Rightarrow u^2+v^2 = u^2+2u+1+v^2$ $\Rightarrow 2u = -1$ $\Rightarrow u = -\frac{1}{2}$ // (if the line $2u = -1$ )
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**Question 4**

The complex function  $w = f(z)$  is given by

$$w = \frac{3-z}{z+1} \text{ where } z \in \mathbb{C}, z \neq -1.$$

A point  $P$  in the  $z$  plane gets mapped onto a point  $Q$  in the  $w$  plane.

The point  $Q$  traces the circle with equation  $|w| = 3$ .

Show that the locus of  $P$  in the  $z$  plane is also a circle, stating its centre and its radius.

centre  $\left(-\frac{3}{2}, 0\right)$ , radius  $= \frac{3}{2}$

$$\bullet w = \frac{3-z}{z+1}$$

$$\Rightarrow |w| = \left| \frac{3-z}{z+1} \right|$$

$$\Rightarrow 3 = \left| \frac{3-z}{z+1} \right|$$

$$\Rightarrow 3|z+1| = |3-z|$$

$$\Rightarrow 3|z+1| = |z-3|$$

$$\Rightarrow 3|z+iy| = |z-(3+0i)|$$

$$\Rightarrow 3(z+iy) = (z-3) - iy$$

$$\Rightarrow 3\sqrt{(z+iy)^2 + y^2} = \sqrt{(z-3)^2 + y^2}$$

$$\Rightarrow 3\sqrt{z^2 + 2zy + y^2 + y^2} = \sqrt{z^2 - 6z + 9 + y^2}$$

$$\Rightarrow 9(z^2 + 2zy + 2y^2) = (z^2 - 6z + 9 + y^2)$$

$$\Rightarrow 9z^2 + 18zy + 18y^2 = z^2 - 6z + 9 + y^2$$

$$\Rightarrow 8z^2 + 18zy + 17y^2 + 6z = 0$$

$$\Rightarrow z^2 + 2zy + 4y^2 = 0$$

$$\Rightarrow (z+2y)^2 = 0$$

$$\Rightarrow (z+2y)^2 - \frac{9}{4} + \frac{9}{4} = 0$$

$$\Rightarrow (z+\frac{3}{2})^2 + y^2 = \frac{9}{4}$$

Indeed a circle, centre  $(-\frac{3}{2}, 0)$ , radius  $\frac{3}{2}$

**Question 5**

The general point  $P(x, y)$  which is represent by the complex number  $z = x + iy$  in the  $z$  plane, lies on the locus of

$$|z| = 1.$$

A transformation from the  $z$  plane to the  $w$  plane is defined by

$$w = \frac{z+3}{z+1}, \quad z \neq -1,$$

and maps the point  $P(x, y)$  onto the point  $Q(u, v)$ .

Find, in Cartesian form, the equation of the locus of the point  $Q$  in the  $w$  plane.

$$u = 2$$

$\bullet \quad w = \frac{z+3}{z+1}$ $\Rightarrow wz + w = z + 3$ $\Rightarrow wz - z = 3 - w$ $\Rightarrow z(w-1) = (3-w)$ $\Rightarrow z = \frac{3-w}{w-1}$ $\Rightarrow  z  = \sqrt{\left \frac{3-w}{w-1}\right ^2}$ $\Rightarrow  z  = \sqrt{\frac{(3-w)^2}{(w-1)^2}}$ $\Rightarrow  z  = \boxed{\sqrt{ w-3 } =  w-1 }$	$\bullet \quad \text{Let } w = u+iv$ $\Rightarrow  u+iv-1  =  u+iv-3 $ $\Rightarrow \sqrt{(u-1)^2+v^2} = \sqrt{(u-3)^2+v^2}$ $\Rightarrow \sqrt{(u-1)^2+v^2} = \sqrt{(u-3)^2+v^2}$ $\Rightarrow (u-1)^2+v^2 = (u-3)^2+v^2$ $\Rightarrow u^2-2u+1 = u^2-6u+9$ $\Rightarrow 4u = 8$ $\Rightarrow u = 2$ <span style="border: 1px solid black; padding: 2px;">(Hence <math>w=2</math>)</span>
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**Question 6**

The point  $P$  represented by  $z = x + iy$  in the  $z$  plane is transformed into the point  $Q$  represented by  $w = u + iv$  in the  $w$  plane, by the complex transformation

$$w = \frac{2z}{z-1}, z \neq 1.$$

The point  $P$  traces a circle of radius 2, centred at the origin  $O$ .

Find a Cartesian equation of the locus of the point  $Q$ .

$$\left(u - \frac{8}{3}\right)^2 + v^2 = \frac{16}{9}$$

$$\begin{aligned}
 & \text{CIRCLE CENTRE } (0,0) \Rightarrow |z|=2, \quad \Rightarrow z = \frac{|u+iv|}{|(u-2)+iv|} \\
 & \Rightarrow |w| = \frac{2|z|}{2-1} \quad \Rightarrow z = \frac{|u+iv|}{\sqrt{(u-2)^2+v^2}} \\
 & \Rightarrow w=2z \quad \Rightarrow 4 = \frac{u^2+v^2}{(u-2)^2+v^2} \\
 & \Rightarrow w^2=4z^2 \quad \Rightarrow 4u^2-16u+16+4v^2=u^2+v^2 \\
 & \Rightarrow w^2-4z^2=16 \quad \Rightarrow 3u^2-16u+3v^2+16=0 \\
 & \Rightarrow 2(w-2)=w \quad \Rightarrow u^2-\frac{16}{3}u+v^2+\frac{16}{3}=0 \\
 & \Rightarrow w^2-4w+4=w^2 \quad \Rightarrow \left(u-\frac{8}{3}\right)^2+v^2+\frac{16}{3}=0 \\
 & \Rightarrow 2(w-2)=w \quad \Rightarrow \left(u-\frac{8}{3}\right)^2+v^2=\frac{16}{3} \\
 & \Rightarrow w=2 \quad \text{IF CIRCLE CENTRE } (\frac{8}{3}, 0) \\
 & \Rightarrow |w|=2 \quad \text{RADIUS } \frac{4}{3}
 \end{aligned}$$

**Question 7**

The complex numbers  $z = x + iy$  and  $w = u + iv$  are represented by the points  $P$  and  $Q$ , respectively, in separate Argand diagrams.

The two numbers are related by the equation

$$w = \frac{1}{z+1}, \quad z \neq -1.$$

If  $P$  is moving along the circle with equation

$$(x+1)^2 + y^2 = 4,$$

find in Cartesian form an equation of the locus of the point  $Q$ .

$$u^2 + v^2 = \frac{1}{4}$$

$\bullet (x+1)^2 + y^2 = 4$ $\Rightarrow  z - (-1)  = 2$ $\Rightarrow  z+1  = 2$ Hence $\Rightarrow w = \frac{1}{z+1}$ $\Rightarrow z+1 = \frac{1}{w}$ $\Rightarrow  z+1  = \left  \frac{1}{w} \right $	Given: $(-1, 0)$ Radius 2 $\Rightarrow  w  = \frac{1}{2}$ $\Rightarrow  u+iv  = \frac{1}{2}$ $\Rightarrow \sqrt{u^2+v^2} = \frac{1}{2}$ $\Rightarrow u^2 + v^2 = \frac{1}{4}$
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### Question 8

A transformation from the  $z$  plane to the  $w$  plane is defined by the equation

$$w = \frac{z+2i}{z-2}, \quad z \neq 2.$$

Find in the  $w$  plane, in Cartesian form, the equation of the image of the circle with equation  $|z|=1$ ,  $z \in \mathbb{C}$ .

$$\left(u + \frac{1}{3}\right)^2 + \left(v + \frac{4}{3}\right)^2 = \frac{8}{9}$$

$$\begin{aligned} w &= \frac{z+2i}{z-2} \\ \Rightarrow wz - 2w &= z + 2i \\ \Rightarrow wz - z &= 2w + 2i \\ \Rightarrow z(w-1) &= 2(w+1) \\ \Rightarrow z &= \frac{2(w+1)}{w-1} \\ \Rightarrow |z| &= \sqrt{\frac{|2(w+1)|}{|w-1|}} \\ \Rightarrow 1 &= \frac{2|w+1|}{|w-1|} \\ \Rightarrow |w-1| &= 2|w+1| \\ \text{Let } w=u+iv & \end{aligned} \quad \begin{aligned} \Rightarrow |u+iv-1| &= 2|u+iv+1| \\ \Rightarrow |(u-1)+iv| &\leq 2|u+(v+1)i| \\ \Rightarrow \sqrt{(u-1)^2+v^2} &\leq 2\sqrt{u^2+(v+1)^2} \\ \Rightarrow u^2-2u+1+v^2 &\leq 4(u^2+v^2+2v+1) \\ \Rightarrow u^2-2u+v^2 &= 4u^2+4v^2+8v+4 \\ \Rightarrow 0 &= 3u^2-3v^2-2u+8v+3 \\ \Rightarrow u^2-\frac{3}{4}v^2+8v+2u+\frac{1}{3} &= 0 \\ \Rightarrow (u+\frac{1}{3})^2 + (v+\frac{4}{3})^2 &= \frac{8}{9} \end{aligned}$$

### Question 9

A transformation from the  $z$  plane to the  $w$  plane is given by the equation

$$w = \frac{1+2z}{3-z}, \quad z \neq 3.$$

Show that in the  $w$  plane, the image of the circle with equation  $|z|=1$ ,  $z \in \mathbb{C}$ , is another circle, stating its centre and its radius.

$$\left(u - \frac{5}{8}\right)^2 + v^2 = \frac{49}{64}, \quad \text{centre } \left(\frac{5}{8}, 0\right), \quad r = \frac{7}{8}$$

$$\begin{aligned} w &= \frac{1+2z}{3-z} \\ \Rightarrow 3w &- 2w = 1+2z \\ \Rightarrow 3w-1 &= 2w+2z \\ \Rightarrow 3w-1 &= 2(w+1) \\ \Rightarrow z &= \frac{3w-1}{w+2} \\ \Rightarrow |z| &= \sqrt{\frac{|3w-1|}{|w+2|}} \\ \Rightarrow 1 &= \frac{|3w-1|}{|w+2|} \\ \Rightarrow |w+2| &= |3w-1| \\ \text{Let } w=u+iv & \end{aligned} \quad \begin{aligned} \Rightarrow \sqrt{(u+2)^2+v^2} &= \sqrt{3(u-v)^2+9v^2} \\ \Rightarrow u^2+4u+4+v^2 &= 9u^2-6uv+9v^2 \\ \Rightarrow 0 &= 8u^2-10uv+8v^2-3 \\ \Rightarrow u^2-\frac{5}{4}v^2+uv-\frac{3}{8} &= 0 \\ \Rightarrow \left(u-\frac{5}{8}v\right)^2+v^2-\frac{25}{64}-\frac{3}{8} &= 0 \\ \Rightarrow \left(u-\frac{5}{8}v\right)^2+v^2-\frac{41}{64} &= 0 \\ \text{4 AND THE Q.E.D.} \\ \text{CENTRE } \left(\frac{5}{8}, 0\right) \\ \text{RADIUS } \frac{7}{8} & \end{aligned}$$

**Question 10**

The complex numbers  $z = x + iy$  and  $w = u + iv$  are represented by the points  $P$  and  $Q$ , respectively, in separate Argand diagrams.

The two numbers are related by the equation

$$w = \frac{1}{z}, \quad z \neq 0.$$

If  $P$  is moving along the circle with equation

$$x^2 + y^2 = 2,$$

find in Cartesian form an equation for the locus of the point  $Q$ .

$$u^2 + v^2 = \frac{1}{2}$$

$\bullet \quad x^2 + y^2 = 2 \Leftrightarrow  z  = \sqrt{2}$ $\Rightarrow w = \frac{1}{z}$ $\Rightarrow  w  = \frac{1}{ z }$ $\Rightarrow  w  = \frac{1}{\sqrt{2}}$ $\Rightarrow  w  = \frac{\sqrt{2}}{2}$ $\Rightarrow  u+iv  = \frac{\sqrt{2}}{2}$ $\Rightarrow \sqrt{u^2+v^2} = \frac{\sqrt{2}}{2}$ $\Rightarrow u^2+v^2 = \frac{1}{2}$ $( x ^2 +  y ^2 = \frac{1}{2})$	Alternative $w = \frac{1}{z}$ $\Rightarrow u+iv = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$ $\Rightarrow u+iv = \frac{x-iy}{x^2+y^2}$ $\Rightarrow u+iv = \frac{x-iy}{2}$ $\Rightarrow u+iv = \frac{x}{2} + i\left(\frac{-y}{2}\right)$ $\text{Therefore } u = \frac{x}{2} \Rightarrow u^2 = \frac{x^2}{4}$ $v = \frac{-y}{2} \Rightarrow v^2 = \frac{y^2}{4}$ $u^2 + v^2 = \frac{x^2}{4} + \frac{y^2}{4}$ $\Rightarrow 4u^2 + 4v^2 = x^2 + y^2$ $\Rightarrow 4u^2 + 4v^2 = 2$ $\Rightarrow u^2 + v^2 = \frac{1}{2}$
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**Question 11**

The complex numbers  $z = x + iy$  and  $w = u + iv$  are represented by the points  $P$  and  $Q$  on separate Argand diagrams.

In the  $z$  plane, the point  $P$  is tracing the line with equation  $y = x$ .

The complex numbers  $z$  and  $w$  are related by

$$w = z - z^2.$$

- a) Find, in Cartesian form, the equation of the locus of  $Q$  in the  $w$  plane.
- b) Sketch the locus traced by  $Q$ .

$$v = u - 2u^2 \text{ or } y = x - 2x^2$$

(a)  $w = z - z^2$   
 $u+iv = (x+iy) - (x+iy)^2$   
 $u+iv = (x+iy) - (x^2+2xy-i^2y^2)$   
 $u+iv = (x-x^2+y^2) + i(y-2xy)$   
Now  $y=x$   
 $u+iv = t + i(t-2t^2)$   
 $i(t-2t^2) = v = u - 2u^2$

(b)  $v = u(1-2u)$

**Question 12**

The complex numbers  $z = x + iy$  and  $w = u + iv$  are represented by the points  $P$  and  $Q$  on separate Argand diagrams.

In the  $z$  plane, the point  $P$  is tracing the line with equation  $y = 2x$ .

Given that the complex numbers  $z$  and  $w$  are related by

$$w = z^2 + 1$$

find, in Cartesian form, the locus of  $Q$  in the  $w$  plane.

$$4u + 3v = 4 \quad \text{or} \quad 4x + 3y = 4$$

$$\begin{aligned} w &= z^2 + 1 \\ \Rightarrow u + iv &= (x + iy)^2 + 1 \\ \Rightarrow u + iv &= 2x^2 - y^2 + 1 \\ \Rightarrow u + iv &= (2x^2 - y^2 + 1) + i(2xy) \\ \text{Now } y = 2x, \\ \Rightarrow u + iv &= (2x^2 - 4x^2) + i(4x^2) \\ \Rightarrow u + iv &= (1 - 3x^2) + 4xi \\ \text{Let } u &= 1 - 3t^2 \\ v &= 4t^2 \end{aligned}$$

$$\left\{ \begin{array}{l} 2t^2 = 1 - u \\ 4t^2 = v \end{array} \right. \times 4$$

$$(2t^2)^2 = 4 - 4u$$

$$4t^4 = 4 - 4u$$

$$\therefore 3v = 4 - 4u$$

$$3v + 4u = 4$$

$$4t^4 + 4u = 4$$

## Question 13

A transformation of the  $z$  plane to the  $w$  plane is given by

$$w = \frac{1+3z}{1-z}, \quad z \in \mathbb{C}, \quad z \neq 1$$

where  $z = x + iy$  and  $w = u + iv$ .

The set of points that lie on the  $y$  axis of the  $z$  plane, are mapped in the  $w$  plane onto a curve  $C$ .

Show that a Cartesian equation of  $C$  is

$$(u+1)^2 + v^2 = 4$$

proof

$$\begin{aligned}
 W &= \frac{1+3i}{1-2i} \\
 \Rightarrow W - W_2 &= 1-3i \\
 \Rightarrow W - 1 &= W_2 + 3i \\
 \Rightarrow W - 1 &= 2(i+3) \\
 \Rightarrow z &= \frac{W-1}{W+3} \\
 \Rightarrow z &= \frac{W_1+i-1}{W_2+i+3} \\
 \Rightarrow z &= \frac{(W_1-i)+i-1}{(W_2+3)+i+3} \\
 \Rightarrow z &= \frac{[(W_1-1)+iV] - iV}{[(W_2+3)+iV] + (2+i)V}
 \end{aligned}$$

So  $(W_1-1)(W_2+3) + V^2 = 0$   
 $\Rightarrow W_1W_2 - 3 + V^2 = 0$   
 $\Rightarrow (W_1V)^2 - 3 + V^2 = 0$   
 $\Rightarrow (W_1V)^2 + V^2 = 4$

But the y axis is the line  $x=0$

It is a circle centre  $(-1, 0)$   
 radius 2.

**Question 14**

The complex function  $w = f(z)$  is given by

$$w = \frac{1}{z}, z \in \mathbb{C}, z \neq 0.$$

This function maps a general point  $P(x, y)$  in the  $z$  plane onto the point  $Q(u, v)$  in the  $w$  plane.

Given that  $P$  lies on the line with Cartesian equation  $y=1$ , show that the locus of  $Q$  is given by

$$\left|w + \frac{1}{2}i\right| = \frac{1}{2}.$$

proof

ALTERNATIVE

• PROVE  $y=1$   
 $\therefore z = x+i$   
 $\Rightarrow w = \frac{1}{x+i}$  (CONJUGATE)  
 $\Rightarrow \bar{w} = \frac{1}{x-i}$  (CONJUGATE)  
 $\Rightarrow u+iv = \frac{1}{x-i}$   
 $\Rightarrow u+iv = \frac{x+i}{x^2+1}$   
 $\Rightarrow u+iv = \frac{x}{x^2+1} + i \frac{1}{x^2+1}$   
 BUT  $GIVEN$   $y=1$   
 $\therefore u+iv = \frac{x}{x^2+1} + i \frac{1}{x^2+1}$   
 $\therefore |w - (0 - \frac{1}{2}i)| = \frac{1}{2}$   
 $|w + \frac{1}{2}i| = \frac{1}{2}$  //  
 AS REQUIRED

ALTERNATIVE

• PROVE  $y=1$   
 $\therefore z = x+i$   
 $\Rightarrow w = \frac{1}{x+i}$  (CONJUGATE)  
 $\Rightarrow \bar{w} = \frac{x-i}{x^2+1}$   
 $\Rightarrow u+iv = \frac{x}{x^2+1} - i \frac{1}{x^2+1}$   
 If  $\begin{cases} u = \frac{x}{x^2+1} \\ v = -\frac{1}{x^2+1} \end{cases}$  SWAP CONJUGATES  
SIDE BY SIDE  
TO ELIMINATE  
 $\Rightarrow \frac{u}{v} = -t$   
 THIS  $v = -\frac{1}{x^2+1}$   
 $\Rightarrow v = -\frac{1}{u^2+1}$  SWAP CONJUGATES  
BY  $x^2+1 = u^2+v^2$   
 $\Rightarrow v = -\frac{v^2}{u^2+v^2}$  SWAP CONJUGATES  
 $v^2 = -v^2$   
 $\Rightarrow 1 = -\frac{v}{u^2+v^2}$   
 $\Rightarrow u^2+v^2 = -v$   
 $\Rightarrow u^2+v^2+v = 0$   
 $\Rightarrow u^2+(v+\frac{1}{2})^2 - \frac{1}{4} = 0$   
 $\Rightarrow u^2+(v+\frac{1}{2})^2 = \frac{1}{4}$   
 • CIRCLE, CENTER  $(0, -\frac{1}{2})$  RADIOUS  $\frac{1}{2}$   
 $\therefore |w - (0 - \frac{1}{2}i)| = \frac{1}{2}$   
 $\Rightarrow |w + \frac{1}{2}i| = \frac{1}{2}$  SWAP CONJUGATES

**Question 15**

A transformation of the  $z$  plane onto the  $w$  plane is given by

$$w = \frac{az + b}{z + c}, z \in \mathbb{C}, z \neq -c$$

where  $a$ ,  $b$  and  $c$  are real constants.

Under this transformation the point represented by the number  $1+2i$  gets mapped to its complex conjugate and the origin remains invariant.

- Find the value of  $a$ , the value of  $b$  and the value of  $c$ .
- Find the number, other than the number represented by the origin, which remains invariant under this transformation.

$$\boxed{a = \frac{5}{2}}, \boxed{b = 0}, \boxed{c = -\frac{5}{2}}, \boxed{z = 5}$$

**(a)**

$\bullet z=0 \rightarrow w=0 \Rightarrow 0 = \frac{b}{c} \Rightarrow b=0$

$w = \frac{az}{z+c}$

$\bullet z=1+2i \rightarrow w=1-2i$

$$\begin{aligned} &\Rightarrow 1-2i = \frac{a(1+2i)}{(1+2i)+c} \\ &\Rightarrow (1-2i)(1+2i) + c(1+2i) = a(1+2i) \\ &\Rightarrow 5 + c + 2ci = a + 2ai \\ &\Rightarrow \begin{cases} 5+c=a \\ 2c=2a \end{cases} \quad \Delta \quad a=c \\ &\Rightarrow 5+c=c \quad \cancel{2c=2a} \\ &\Rightarrow c=5 \quad \cancel{a=c} \end{aligned}$$

**(b)**

$w = \frac{\frac{5}{2}z}{z-\frac{5}{2}} \Rightarrow w = \frac{5z}{2z-5}$

Invariant  $\Rightarrow z \mapsto z$

$z = \frac{5z}{2z-5}$

$$\begin{aligned} 1 &= \frac{5}{2z-5} \quad (z \neq 0, \text{ REINFORCE DOMAIN}) \\ 2z-5 &= 5 \\ 2z &= 10 \\ z &= 5 \end{aligned}$$

**Question 16**

A transformation of the  $z$  plane to the  $w$  plane is given by

$$w = \frac{1}{z-2}, z \in \mathbb{C}, z \neq 2$$

where  $z = x + iy$  and  $w = u + iv$ .

The line with equation

$$2x + y = 3$$

is mapped in the  $w$  plane onto a curve  $C$ .

- a) Show that  $C$  represents a circle and determine the coordinates of its centre and the size of its radius.

The points of a region  $R$  in the  $z$  plane are mapped onto the points which lie inside  $C$  in the  $w$  plane.

- b) Sketch and shade  $R$  in a suitable labelled Argand diagram, fully justifying the choice of region.

centre at  $(-1, \frac{1}{2})$ , radius  $= \frac{\sqrt{5}}{2}$

a)

$$w = \frac{1}{z-2}$$

$$\Rightarrow w(z-2) = 1$$

$$\Rightarrow wz - 2w = 1$$

$$\Rightarrow z = \frac{2w+1}{w}$$

$$\Rightarrow z+iy = \frac{2(w+iy)+1}{w+iy}$$

$$\Rightarrow z+iy = \frac{(2w+1)(w+iy)}{w^2+iy^2}$$

$$\Rightarrow z+iy = \frac{[2w^2+2iy+1]w+iy^2-ix}{w^2+iy^2}$$

$$\Rightarrow z+iy = \frac{[2w^3+w^2y+2w+iy^2-ix]}{w^2+iy^2}$$

$$\Rightarrow z+iy = \frac{2w^3+2w^2iy+2w+iy^2-ix}{w^2+iy^2}$$

$$\Rightarrow 2\left(\frac{2w^3+2w^2iy+2w}{w^2+iy^2}\right) + \left(\frac{-ix+iy^2}{w^2+iy^2}\right) = 3$$

$$\Rightarrow \frac{4w^3+4w^2iy+4w}{w^2+iy^2} - \frac{ix+iy^2}{w^2+iy^2} = 3$$

$$\Rightarrow 4w^3+4w^2iy+4w - ix - iy^2 = 3w^2+3y^2$$

$$\Rightarrow w^3 + w^2 + 2w - y^2 = 0$$

$$\Rightarrow (w+1)^2 + (w-1)^2 - y^2 = 0$$

$$\Rightarrow (w+1)^2 + (v-\frac{1}{2})^2 = \frac{5}{4}$$

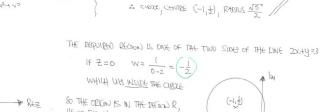
$\Delta$  centre, centre  $(-1, \frac{1}{2})$ , radius  $\sqrt{\frac{5}{4}}$

b)



The required region is one of the two sides of the line  $2w+1=0$ . If  $z \neq 0$ ,  $w = \frac{1}{z-2} \neq \frac{1}{2}$ . Hence the region is in the region  $R$ , hence the choice of suitable region.



**Question 17**

A transformation of the  $z$  plane to the  $w$  plane is given by

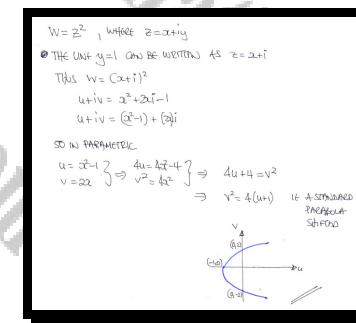
$$w = z^2, \quad z \in \mathbb{C},$$

where  $z = x + iy$  and  $w = u + iv$ .

The line with equation  $y=1$  is mapped in the  $w$  plane onto a curve  $C$ .

Sketch the graph of  $C$ , marking clearly the coordinates of all points where the graph of  $C$  meets the coordinate axes.

**sketch**



**Question 18**

A transformation of points from the  $z$  plane onto points in the  $w$  plane is given by the complex relationship

$$w = z^2, \quad z \in \mathbb{C},$$

where  $z = x + iy$  and  $w = u + iv$ .

Show that if the point  $P$  in the  $z$  plane lies on the line with equation

$$y = x - 1,$$

the locus of this point in the  $w$  plane satisfies the equation

$$v = \frac{1}{2}(u^2 - 1),$$

proof

$\begin{aligned} \text{Let } z &= x + iy \\ \Rightarrow w &= z^2 \\ \Rightarrow u + iv &= (x + iy)^2 \\ \Rightarrow u + iv &= x^2 + 2xyi - y^2 \\ (u &= x^2 - y^2) \\ (v &= 2xy) \end{aligned}$ <p style="border: 1px solid black; padding: 2px; margin-top: 5px;">Now <math>v = 2x(u - 1)</math></p> $\begin{aligned} (u &= x^2 - (2x - 1)^2) \\ (v &= 2x(2x - 1)) \end{aligned}$	$\begin{cases} u = 2x - 1 \\ v = 2x^2 - 2x \end{cases} \quad (x \neq 0)$ $\begin{aligned} 2x &= u + 1 \\ 2x &= 2x^2 - 4x \end{aligned}$ <p style="margin-left: 20px;">Hence eliminate <math>x</math></p> $\begin{aligned} 2u &= (2x^2 - 2)(2x) \\ \Rightarrow 2u &= (u + 1)^2 - 2(u + 1) \\ \Rightarrow 2u &= u^2 + 2u + 1 - 2u - 2 \\ \Rightarrow 2u &= u^2 - 1 \\ \Rightarrow v &= \pm\sqrt{u^2 - 1} \end{aligned}$ <p style="margin-left: 20px;">As required</p>
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## Question 19

A complex transformation from the  $z$  plane to the  $w$  plane is defined by

$$w = \frac{z+i}{3+iz}, \quad z \in \mathbb{C}, \quad z \neq 3i$$

The point  $P(x, y)$  is mapped by this transformation into the point  $Q(u, v)$

It is further given that  $Q$  lies on the real axis for all the possible positions of  $P$ .

Show that the  $P$  traces the curve with equation

$$|z-i|=2$$

proof

$$\begin{aligned}
 W &= \frac{z+i}{3+iz} \\
 \Rightarrow u+iv &= \frac{2+iy+i^2}{3+2iy} \\
 \Rightarrow u+iv &= \frac{2+iy(1+i)}{(3-y)+iz} \\
 &\quad \text{CONJUGATE RHTS} \\
 \Rightarrow u+iv &= \frac{[2-i](1+iy)(3-y)-12}{(3-y)^2+i^2} \\
 \Rightarrow u+iv &= \frac{[2(3-y)+15i+2y^2-2y]}{(3-y)^2+1} \\
 &\quad \text{NOW Q. NOVEMBER RHIC 4ANS} \\
 &\quad \therefore v=0 \\
 \text{THIS } &(3y+4)(3-y)-x^2=0 \\
 \Rightarrow &3y^2+4y-3y^2-x^2=0 \\
 \Rightarrow &0+y^2-2y+x^2-3 \\
 \Rightarrow &0=x^2+(y-1)^2-4 \\
 \Rightarrow &x^2+(y-1)^2=4 \\
 &\text{it create, USE } (y_1) \\
 &\text{PARABOL 2} \\
 |z-(0,1)| &= 2 \\
 |z-i| &= 2 \quad \text{to review}
 \end{aligned}$$

HARD ALGEBRA

- $w = \frac{2+i}{3+2i}$
- ⇒  $3w + 12w = 2+i$
- ⇒  $3w - i = 2 - 12w$
- ⇒  $3w - i = z - (j-w)$
- ⇒  $z = \frac{3w-i}{1-jw}$

Now we need to find  $jw$

$$\begin{aligned} w &= u+jv \\ w &= t+0i \\ w &= t \\ \Rightarrow z &= \frac{3t-i}{1-it} \\ \Rightarrow z &= \frac{(3t-i)(1+it)}{(1-it)(1+it)} \\ \Rightarrow z &= \frac{3t+3it^2-i+t^2}{1+t^2} \\ \Rightarrow x+iy &= \frac{4t}{1+t^2} + i \frac{3t^2-1}{1+t^2} \end{aligned}$$

$z = \frac{4t}{1+t^2}$

$y = \frac{3t^2-1}{1+t^2}$

$\begin{array}{l} \bullet y + 4t^2 = 3t^2 - 1 \\ \quad y + i = 3t^2 - i^2 \\ \quad y + i = t^2(3 - i^2) \\ \quad \frac{t^2}{t^2} = \frac{3 - i^2}{3 - 3} \end{array}$

$\begin{array}{l} \text{Hence} \\ \Rightarrow z^2 = \frac{16(y+1)}{(1-\frac{4}{3})^{1/2}} \\ \Rightarrow z^2 = \frac{16(y+1)}{\left(\frac{2}{3}\right)^{1/2}} \\ \Rightarrow z^2 = \frac{16(y+1)}{\left(\frac{4}{9}\right)^{1/2}} \\ \Rightarrow z^2 = \frac{16(y+1)}{\frac{2}{3}} \end{array}$

AUXILIARY EQUATION BY  $(3-y)^2$

$$\begin{aligned} \Rightarrow z^2 &= \frac{16(y+1)}{16} \\ \Rightarrow z^2 &= y^2 + 2y + 1 \\ \Rightarrow z^2 &= 3y^2 - 3y + 3 \\ \Rightarrow z^2 &= -y^2 + 2y + 3 \\ \Rightarrow z^2 &= y^2 + 2y = 3 \\ \Rightarrow z^2 &= (y+1)^2 = 1 - 3 \\ \Rightarrow z &= (y+1)^2 = 4 \end{aligned}$$

(AS 3RD PAPER)

**Question 20**

A transformation of the  $z$  plane to the  $w$  plane is given by

$$w = \frac{2z+1}{z}, z \in \mathbb{C}, z \neq 0$$

where  $z = x + iy$  and  $w = u + iv$ .

The circle  $C_1$  with centre at  $(1, -\frac{1}{2})$  and radius  $\frac{\sqrt{5}}{2}$  in the  $z$  plane is mapped in the  $w$  plane onto another curve  $C_2$ .

- a) Show that  $C_2$  is also a circle and determine the coordinates of its centre and the size of its radius.

The points inside  $C_1$  in the  $z$  plane are mapped onto points of a region  $R$  in the  $w$  plane.

- b) Sketch and shade  $R$  in a suitably labelled Argand diagram, fully justifying the choice of the region.

centre at  $(\frac{3}{2}, 0)$ , radius =  $\frac{1}{\sqrt{2}}$

(a)

$$w = \frac{2z+1}{z} = 2 + \frac{1}{z}$$

CREATE CENTRE  $(1, -\frac{1}{2})$ ; RADIUS  $\frac{\sqrt{5}}{2} \Rightarrow |z - (1, -\frac{1}{2})| = \frac{\sqrt{5}}{2}$

REARRANGE

$$\begin{aligned} \Rightarrow w-2 &= \frac{1}{z} \\ \Rightarrow z &= \frac{1}{w-2} \\ \Rightarrow z-1+\frac{1}{2} &= \frac{1}{w-2}-1+\frac{1}{2} \\ \Rightarrow z-1+\frac{1}{2} &= \frac{1-(w-2)+\frac{1}{w-2}}{w-2} \\ \Rightarrow z-1+\frac{1}{2} &= \frac{1-w+2+\frac{1}{w-2}-1}{w-2} \\ \Rightarrow z-1+\frac{1}{2} &= \frac{2-\frac{w}{w-2}}{w-2} \\ \Rightarrow z-1+\frac{1}{2} &= \frac{4-w}{2w-4} \\ \Rightarrow |z-1+\frac{1}{2}| &= \left| \frac{4-w}{2w-4} \right| \\ \Rightarrow \frac{\sqrt{5}}{2} &= \frac{|4-w|}{|2w-4|} \\ \Rightarrow \frac{\sqrt{5}}{2} &= \frac{|4-w|}{2|w-2|} \end{aligned}$$

$\Rightarrow 4\sqrt{5} = \frac{(4-w)^2}{|w-2|}$

$\Rightarrow \sqrt{5} = \frac{|4-(w+2)|}{|(w+2)-2|}$

$\Rightarrow \sqrt{5} = \frac{|(w+2)-4|}{|(w+2)+1|}$

$\Rightarrow \sqrt{5} = \frac{\sqrt{(w+2)^2+1^2}}{\sqrt{(w+2)+1}}$

$\Rightarrow 5 = \frac{(w+2)^2+1^2}{(w+2)+1}$

$\Rightarrow 5 = \frac{w^2+4w+4+1}{w+3}$

$\Rightarrow 5w^2+20w+20 = w^2+4w+5$

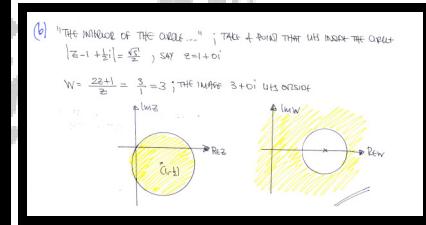
$\Rightarrow 4w^2+16w+15 = 0$

$\Rightarrow (w+3)^2+1^2=0$

$\Rightarrow (w+3)^2+1^2-1=0$

$\Rightarrow (w+3)^2=1$

16 marks (part (a))  $|w-1|=|\frac{3}{2}|$   $\frac{1}{\sqrt{2}}$



**Question 21**

A transformation of the  $z$  plane to the  $w$  plane is given by

$$w = z + \frac{1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0,$$

where  $z = x + iy$  and  $w = u + iv$ .

The locus of the points in the  $z$  plane that satisfy the equation  $|z| = 2$  are mapped in the  $w$  plane onto a curve  $C$ .

By considering the equation of the locus  $|z| = 2$  in exponential form, or otherwise, show that a Cartesian equation of  $C$  is

$$36u^2 + 100v^2 = 225.$$

**[proof]**

$$\begin{aligned} |z|=2 &\text{ CAN BE WRITTEN AS } z=2e^{i\theta} \text{ IN EXPONENTIAL FORM} \\ \text{So} \\ w &= z + \frac{1}{z} = 2e^{i\theta} + \frac{1}{2e^{i\theta}} = 2e^{i\theta} + \frac{1}{2}e^{-i\theta} \\ &= 2(\cos\theta + i\sin\theta) + \frac{1}{2}(\cos\theta - i\sin\theta) = \frac{5}{2}\cos\theta + \frac{3}{2}i\sin\theta \\ \text{So } u+iv &= \frac{5}{2}\cos\theta + \frac{3}{2}i\sin\theta \\ \left. \begin{aligned} u &= \frac{5}{2}\cos\theta \\ v &= \frac{3}{2}\sin\theta \end{aligned} \right\} &\Rightarrow \left. \begin{aligned} \frac{5}{2}u &= 5\cos\theta \\ \frac{3}{2}v &= 3\sin\theta \end{aligned} \right\} \Rightarrow \frac{5}{2}u^2 + \frac{3}{2}v^2 = 1 \\ &\Rightarrow \frac{25}{4}u^2 + \frac{9}{4}v^2 = 1 \\ &\Rightarrow 36u^2 + 100v^2 = 225 \\ &\text{--- REASON} \end{aligned}$$

**Question 22**

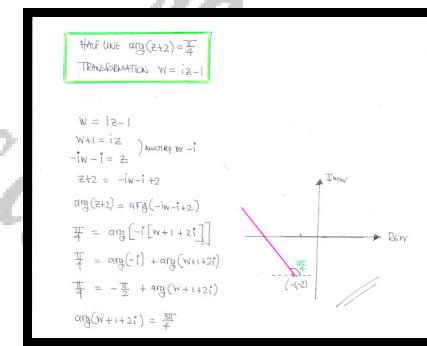
A transformation from the  $z$  plane to the  $w$  plane is defined by the equation

$$w = iz - 1, \quad z \in \mathbb{C}.$$

Sketch in the  $w$  plane, in Cartesian form, the equation of the image of the half line with equation

$$\arg(z+2) = \frac{\pi}{4}, \quad z \in \mathbb{C}.$$

[graph](#)



**Question 23**

A transformation from the  $z$  plane to the  $w$  plane is defined by the equation

$$f(z) = \frac{iz}{z-i}, \quad z \in \mathbb{C}.$$

Find, in Cartesian form, the equation of the image of straight line with equation

$$|z - i| = |z - 2|, \quad z \in \mathbb{C}.$$

$$\left(u + \frac{2}{5}\right)^2 + \left(v - \frac{4}{5}\right)^2 = \frac{1}{5}$$

Given  $|z - i| = |z - 2|$  or  $f(z) = w = \frac{iz}{z-i}$

$w = \frac{iz}{z-i}$

$wz - iw = iz$

$wz - iz = iw$

$z(w-i) = iw$

$z = \frac{iw}{w-i}$

Now  $|z - i| = |z - 2|$

$\left|\frac{iw}{w-i} - i\right| = \left|\frac{w}{w-i} - 2\right|$

$\left|\frac{iw - iw + i^2}{w-i}\right| = \left|\frac{(w-2w+2i)}{w-i}\right|$

$\left|\frac{-i}{w-i}\right| = \left|\frac{(w-2w+2i)}{w-i}\right|$

$| -i | = |(w-2w+2i)|$

$|w - 2w + 2i| = 1$

$|w(i-2) + 2i| = 1$

Now proceed by letting  $w = u+iv$  in the above equation

$\left|(-2+1)(u+iv) + 2i\right| = 1$

$|-2u - 2v + (u - v + 2i)| = 1$

$\left|(-2u - v) + (-2v + u + 2)\right| = 1$

$\sqrt{(-2u - v)^2 + (-2v + u + 2)^2} = 1$

$4u^2 + v^2 + 4uv + 4v^2 + u^2 + 4u - 4uv - 8v + 4u = 1$

$5u^2 + 5v^2 - 8v + 4u = -3$

$u^2 + v^2 - \frac{8}{5}v + \frac{4}{5}u = -\frac{3}{5}$

$(u + \frac{2}{5})^2 + (v - \frac{4}{5})^2 = \frac{3}{5} - \frac{3}{25} + \frac{16}{25}$

$(u + \frac{2}{5})^2 + (v - \frac{4}{5})^2 = \frac{1}{5}$

∴ circle centre  $(-\frac{2}{5}, \frac{4}{5})$  radius  $\frac{1}{\sqrt{5}}$

**Question 24**

The complex function  $w = f(z)$  is given by

$$w = \frac{1}{1-z}, \quad z \neq 1.$$

The point  $P(x, y)$  in the  $z$  plane traces the line with Cartesian equation

$$y + x = 1.$$

Show that the locus of the image of  $P$  in the  $w$  plane traces the line with equation

$$v = u.$$

**[proof]**

$\begin{aligned} & W = \frac{1}{1-z} \\ & \Rightarrow 1 - z = \frac{1}{W} \\ & \Rightarrow 1 - \frac{1}{W} = z \\ & \Rightarrow z = \frac{W-1}{W} \\ & \Rightarrow z = \frac{u+iv-1}{u+iv} = \frac{(u-1)+iv}{u+iv} \end{aligned}$ <p style="text-align: center;">CONVERGE THIS</p> $\begin{aligned} & \Rightarrow z = \frac{[(u-1)+iv][(u-iv)]}{(u+iv)(u-iv)} \\ & \Rightarrow z = \frac{u(u-1)+iv^2+i(u-v(u-1))}{u^2+v^2} \\ & \Rightarrow u+iv = \frac{u^2+v^2+iu}{u^2+v^2} + i\frac{v}{u^2+v^2} \end{aligned}$	$\begin{aligned} & \text{Now } y+z=1 \\ & \text{Thus} \\ & \frac{u^2+v^2+iu}{u^2+v^2} + i\frac{v}{u^2+v^2} = 1 \\ & \cancel{u^2+v^2} + v = \cancel{u^2+v^2} \\ & v = u \end{aligned}$ <p style="text-align: center;">As Required</p>
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**Question 25**

The complex function  $w = f(z)$  satisfies

$$w = \frac{1}{z}, z \in \mathbb{C}, z \neq 0.$$

This function maps the point  $P(x, y)$  in the  $z$  plane onto the point  $Q(u, v)$  in the  $w$  plane.

It is further given that  $P$  traces the curve with equation

$$\left| z + \frac{1}{2}i \right| = \frac{1}{2}.$$

Find, in Cartesian form, the equation of the locus of  $Q$ .

$$v = 1$$

WORK AS FOLLOWS

$$w = \frac{1}{z} \rightarrow z = \frac{1}{w}$$

$$\Rightarrow z + \frac{1}{2}i = \frac{1}{w} + \frac{1}{2}i$$

$$\Rightarrow z + \frac{1}{2}i = \frac{2 - w}{2w}$$

TRACING MODULI ON BOTH SIDES

$$\Rightarrow |z + \frac{1}{2}i| = \left| \frac{2-w}{2w} \right|$$

$$\Rightarrow \frac{1}{2} = \frac{|2-w|}{|2w|}$$

$$\Rightarrow |w| = |2-w|$$

LET  $w = u+iv$

$$\Rightarrow |u+iv| = |2+(-u+iv)|$$

$$\Rightarrow |u+iv| = |2-u-iw|$$

$$\Rightarrow |u+iv| = |(2-u)+iw|$$

$$\Rightarrow \sqrt{u^2+v^2} = \sqrt{(2-u)^2+w^2}$$

$$\Rightarrow u^2 + v^2 = 4 - 4u + u^2 + w^2$$

$$\Rightarrow v^2 = 4 - 4u$$

$$\Rightarrow u = 1$$

$\boxed{\text{or } y=1}$

**Question 26**

$$z = \cos \theta + i \sin \theta, -\pi < \theta \leq \pi.$$

a) Show clearly that

$$\frac{2}{1+z} = 1 - i \tan \frac{\theta}{2}.$$

The complex function  $w = f(z)$  is defined by

$$w = \frac{2}{1+z}, z \in \mathbb{C}, z \neq -1.$$

The circular arc  $|z|=1$ , for which  $0 \leq \arg z < \frac{\pi}{2}$ , is transformed by this function.

b) Sketch the image of this circular arc in a suitably labelled Argand diagram.

proof/sketch

(a)  $\frac{2}{1+z} = \frac{2}{1+(\cos \theta + i \sin \theta)} = \frac{2}{(\cos \theta + 1) + i \sin \theta}$

$$= \frac{2[(\cos \theta + 1) - i \sin \theta]}{[(\cos \theta + 1) + i \sin \theta][(\cos \theta + 1) - i \sin \theta]} = \frac{2[\cos \theta + 1 - 2i \sin \theta]}{(\cos \theta + 1)^2 + \sin^2 \theta}$$

$$= \frac{2[\cos \theta + 1 - 2i \sin \theta]}{\cos^2 \theta + 2\cos \theta + 1 + \sin^2 \theta} = \frac{2[\cos \theta + 1 - 2i \sin \theta]}{2 + 2\cos \theta}$$

$$= \frac{2\cos \theta + 2}{2 + 2\cos \theta} - \frac{2i \sin \theta}{2 + 2\cos \theta} = 1 - i \frac{\sin \theta}{1 + \cos \theta}$$

$$= 1 - i \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{1 + (2 \cos^2 \frac{\theta}{2} - 1)} = 1 - i \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} = 1 - i \tan \frac{\theta}{2} //$$

(b)  $|z|=1, 0 \leq \arg z < \frac{\pi}{2}$   
 $Z = \cos \theta + i \sin \theta, 0 \leq \theta < \frac{\pi}{2}$   
 $\therefore V = 1 - i \tan \frac{\theta}{2}$   
 $\begin{cases} U=1 \\ V=\tan \frac{\theta}{2} \end{cases} \text{ i.e. PARAMETRIC EQUATIONS } 0 < \theta < \frac{\pi}{2}$

## Question 27

The complex function with equation

$$f(z) = \frac{1}{z^2}, z \in \mathbb{C}, z \neq 0$$

maps the complex number  $x+iy$  from the  $z$  plane onto the complex number  $u+iv$  in the  $w$  plane.

The line with equation

$$y = mx, \ x \neq 0$$

is mapped onto the line with equation

$$v = Mu$$

where  $m$  and  $M$  are the respective gradients of the two lines.

Given that  $m = M$ , determine the three possible values of  $m$ .

$$m = 0, \pm\sqrt{3}$$

$$\begin{aligned}
 & \Rightarrow W = \frac{1}{Z^2} \\
 & \Rightarrow u+iv = \frac{1}{(x+iy)^2} \\
 & \Rightarrow u+iv = \frac{1}{x^2+2xyi-y^2} \\
 & \Rightarrow u+iv = \frac{(x^2-y^2)-2xyi}{(x^2+y^2)^2} \\
 & \Rightarrow u+iv = \frac{(x^2-y^2)+2xyi}{(x^2+y^2)^2} \\
 & \Rightarrow u+iv = \frac{(x^2-y^2)-2xyi}{(x^2+y^2)^2} \\
 & \Rightarrow u+iv = \frac{2x^2-y^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} \\
 \\ 
 \bullet \quad & \text{Now } y=mx \\
 & U = \frac{x^2-y^2+2xy}{(x^2+y^2)^2} = \frac{x^2(1-m^2)}{x^2(1+m^2)^2} = \frac{1-m^2}{(1+m^2)^2} \\
 & V = \frac{-2x(2m)}{(x^2+y^2)^2} = \frac{-2m x^2}{x^2(1+m^2)^2} = \frac{-2m}{(1+m^2)^2} \\
 \\ 
 \bullet \quad & UxV = \frac{\frac{1-m^2}{(1+m^2)^2} \times \frac{-2m}{(1+m^2)^2}}{-2m} \Rightarrow \frac{U}{V} = \frac{1-m^2}{-2m} \\
 & \Rightarrow \frac{U}{V} = \frac{m^2-1}{2m} \\
 & \Rightarrow V = \frac{2m}{(m^2-1)} \\
 & \quad \text{4P} \\
 & \quad \text{GIVEN}
 \end{aligned}$$

**Question 28**

A complex transformation of points from the  $z$  plane onto points in the  $w$  plane is defined by the equation

$$w = z^2, \quad z \in \mathbb{C}.$$

The point represented by  $z = x + iy$  is mapped onto the point represented by  $w = u + iv$ .

Show that if  $z$  traces the curve with Cartesian equation

$$y^2 = 2x^2 - 1,$$

the locus of  $w$  satisfies the equation

$$v^2 = 4(u-1)(2u-1).$$

proof

$$\begin{aligned} w &= z^2 \\ &\Rightarrow (u+iv) = (x+iy)^2 \\ &\Rightarrow 4+iv = x^2+2xyi-y^2 \\ &\Rightarrow (u = x^2-y^2), \quad \text{subject to } y^2 = 2x^2 \\ &\Rightarrow (u = x^2-(2x^2-1)) \\ &\Rightarrow v^2 = 4x^2y^2 \end{aligned}$$

$$\begin{aligned} &\Rightarrow \begin{cases} u = 1-x^2 \\ v^2 = 4x^2(2x^2-1) \end{cases} \\ &\bullet \text{ Thus } x^2 = 1-u \\ &\Rightarrow v^2 = 4(1-u)(2(1-u)) \\ &\Rightarrow v^2 = 4(1-u)(2-2u) \\ &\Rightarrow v^2 = 4(u-1)(2u-1) \end{aligned}$$

$\therefore$  ~~unproven~~

**Question 29**

The complex function  $w = f(z)$  is defined by

$$w = \frac{1}{z-1}, \quad z \in \mathbb{C}, \quad z \neq 1.$$

The half line with equation  $\arg z = \frac{\pi}{4}$  is transformed by this function.

- a) Find a Cartesian equation of the locus of the **image** of the half line.
- b) Sketch the **image** of the locus in an Argand diagram.

$$\left(u + \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}, \quad v < 0, \quad u^2 + v^2 + u > 0$$

Given  $w = \frac{1}{z-1}$

$$\Rightarrow z-1 = \frac{1}{w}$$

$$\Rightarrow z = \frac{1}{w} + 1$$

$$\Rightarrow z = \frac{w+1}{w}$$

$$\Rightarrow z = \frac{(w+1)(w-1)}{w(w-1)}$$

$$\Rightarrow z = \frac{(w+1)(w-1)}{(w-1)(w-1)}$$

$$\Rightarrow z = \frac{(w+1)^2 - 1}{(w-1)^2}$$

$$\Rightarrow z = \frac{(w+1)^2 - 1}{w^2 - 2w + 1}$$

$$\Rightarrow z = \frac{\frac{w^2 + 2w + 1}{w^2 - 2w + 1} - 1}{w^2 - 2w + 1}$$

$$\Rightarrow z = \frac{\frac{w^2 + 2w + 1}{w^2 - 2w + 1} - \frac{w^2 - 2w + 1}{w^2 - 2w + 1}}{w^2 - 2w + 1}$$

$$\Rightarrow z = \frac{4w}{w^2 - 2w + 1}$$

$$\Rightarrow w(z) = \frac{4w}{w^2 - 2w + 1} + i \frac{-v}{w^2 - 2w + 1}$$

$$\Rightarrow w(z) = \frac{4w^2 + 4w - 4w^2 + 4w - v}{w^2 - 2w + 1} + i \frac{-v}{w^2 - 2w + 1}$$

$$\Rightarrow w(z) = \frac{4w^2 + 4w - v}{w^2 - 2w + 1} + i \frac{-v}{w^2 - 2w + 1}$$

Therefore  $u^2 + v^2 + u = -v$

$$u^2 + 2uv + v^2 + u = 0$$

$$(u+v)^2 + u = 0$$

$$(u+v)^2 = -u$$

$$u+v = \pm \sqrt{-u}$$

$$u+v = \pm \sqrt{u^2 + v^2}$$

$$u+v = \pm \sqrt{\frac{u^2 + v^2}{u^2 + v^2}}$$

$$u+v = \pm \frac{\sqrt{u^2 + v^2}}{\sqrt{u^2 + v^2}}$$

$$u+v = \pm \frac{\sqrt{u^2 + v^2}}{1}$$

$$u+v = \pm \sqrt{u^2 + v^2}$$

Squaring to  $uv > 0 \Rightarrow u^2v^2 > 0$

$$u^2v^2 > 0$$

$$u^2 > 0$$

$$u > 0$$

It locus is circle centre  $(-\frac{1}{2}, 0)$  radius  $\frac{1}{2}$

- For  $v < 0$  it outside the circle centre  $(-\frac{1}{2}, 0)$  radius  $\frac{1}{2}$  (shown shaded)

**Question 30**

The complex function  $w = f(z)$  is defined by

$$w = \frac{3z+i}{1-z}, z \in \mathbb{C}, z \neq 1.$$

The half line with equation  $\arg z = \frac{3\pi}{4}$  is transformed by this function.

- a) Find a Cartesian equation of the locus of the **image** of the half line.
- b) Sketch the **image** of the locus in an Argand diagram.

$$(u+1)^2 + (v+1)^2 = 5, v > \frac{1}{3}u + 1$$

**Given:**  $w = \frac{3z+i}{1-z}$ ,  $\arg z = \frac{3\pi}{4}$

**Find:**  $w - 1 = \frac{3z-1}{1-z}$

$w - 1 = \frac{3z-1}{1-z} = \frac{3z-1}{(1-z)(1+\bar{z})} = \frac{3z-1}{1-z-\bar{z}} = \frac{3z-1}{1-2z+z^2}$

$\Rightarrow \arg(w-1) = \arg\left(\frac{3z-1}{1-2z+z^2}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{3z-1}{\sqrt{3}z^2-3z+1} = \text{cis}\left(\frac{\pi}{4}\right) = i$

$\Rightarrow 3z-1 = i(\sqrt{3}z^2-3z+1)$

$\Rightarrow 3z^2-3z+1 = -i\sqrt{3}z^2+3iz-i$

$\Rightarrow (3+i\sqrt{3})z^2+(3i-3)z+i-1=0$

**Let  $z = u+iv$**

$\Rightarrow (3+i\sqrt{3})(u+iv)^2 + (3i-3)(u+iv) + i-1 = 0$

$\Rightarrow (3u^2-3v^2+uv\sqrt{3}+iu\sqrt{3}) + i(3uv+3u-3v+1) = 0$

$\Rightarrow (3u^2-3v^2+uv\sqrt{3}) + i(3uv+3u-3v+1) = 0$

**Now**  $\arg(w-1) = \frac{\pi}{4}$

$\Rightarrow \frac{3u-1}{\sqrt{3}u^2-3u+1} = \frac{\pi}{4}$

$\Rightarrow 3u-1 = \frac{\pi}{4}(\sqrt{3}u^2-3u+1)$

$\Rightarrow 3u^2-3u+1 = \frac{4\pi}{3}u-1$

$\Rightarrow 3u^2-(3+4\pi/3)u+2=0$

**Check subject to the following conditions:**

- $3u-1 > 0 \Rightarrow u > \frac{1}{3}$
- $u^2v^2+3u-1 < 0$
- $v > 0$
- $(u-1)^2 + (v-1)^2 < 5$

**Thus,**  $(u-1)^2 + (v-1)^2 = 5$

The required locus is in the first quadrant.

# COMPLEX SERIES

**Question 1**

The following convergent series  $C$  and  $S$  are given by

$$C = 1 + \frac{1}{2} \cos \theta + \frac{1}{4} \cos 2\theta + \frac{1}{8} \cos 3\theta \dots$$

$$S = -\frac{1}{2} \sin \theta + \frac{1}{4} \sin 2\theta + \frac{1}{8} \sin 3\theta \dots$$

- a) Show clearly that

$$C + iS = \frac{2}{2 - e^{i\theta}}.$$

- b) Hence show further that

$$C = \frac{4 - 2 \cos \theta}{5 - 4 \cos \theta},$$

and find a similar expression for  $S$ .

$$S = \frac{2 \sin \theta}{5 - 4 \cos \theta}$$

$$\begin{aligned} \text{(a)} \quad C + iS &= 1 + \frac{1}{2}( \cos \theta + i \sin \theta ) + \frac{1}{4}( \cos 2\theta + i \sin 2\theta ) + \frac{1}{8}( \cos 3\theta + i \sin 3\theta ) + \dots \\ &= 1 + \frac{1}{2} e^{i\theta} + \frac{1}{4} e^{i2\theta} + \frac{1}{8} e^{i3\theta} + \dots \\ &= \underbrace{\left( 1 + \frac{e^{i\theta}}{2} + \left( \frac{e^{i\theta}}{2} \right)^2 + \left( \frac{e^{i\theta}}{2} \right)^3 + \dots \right)}_{\text{G.P. with } q = \frac{e^{i\theta}}{2}} \\ &= \frac{1}{1 - \frac{e^{i\theta}}{2}} = \frac{2}{2 - e^{i\theta}} \quad \text{using } \frac{a}{1-r} \end{aligned}$$
  

$$\begin{aligned} \text{(b)} \quad C + iS &= \frac{2}{2 - e^{i\theta}} = \frac{2(z - e^{-i\theta})}{(z - e^{i\theta})(z - e^{-i\theta})} = \frac{2(z - \cos \theta - i \sin \theta)}{4 - 2z^2 + 2e^{-2i\theta}} \\ &= \frac{2(z - \cos \theta - i \sin \theta)}{5 - 2(z \cos \theta + i \sin \theta)} = \frac{4 - 2z \cos \theta + 2i \sin \theta}{5 - 4z \cos \theta} \\ &= \frac{(4 - 2z \cos \theta)(1 + 2i \sin \theta)}{5 - 4z \cos \theta} \\ \therefore C &= \frac{4 - 2z \cos \theta}{5 - 4z \cos \theta} \quad \text{and } S = \frac{2i \sin \theta}{5 - 4z \cos \theta} \end{aligned}$$

**Question 2**

The following finite sums,  $C$  and  $S$ , are given by

$$C = 1 + 5\cos 2\theta + 10\cos 4\theta + 10\cos 6\theta + 5\cos 8\theta + \cos 10\theta$$

$$S = 5\sin 2\theta + 10\sin 4\theta + 10\sin 6\theta + 5\sin 8\theta + \sin 10\theta$$

By considering the binomial expansion of  $(1+A)^5$ , show clearly that

$$C = 32\cos^5 \theta \cos 5\theta,$$

and find a similar expression for  $S$

$$S = 32\cos^5 \theta \sin 5\theta$$

$$C = 1 + 5\cos 2\theta + 10\cos 4\theta + 10\cos 6\theta + 5\cos 8\theta + \cos 10\theta$$

$$S = 5\sin 2\theta + 10\sin 4\theta + 10\sin 6\theta + 5\sin 8\theta + \sin 10\theta$$

Thus

$$C + iS = 1 + e^{i2\theta} + 10e^{i4\theta} + 10e^{i6\theta} + e^{i8\theta} + e^{i10\theta}$$

which is THE BINOMIAL EXPANSION .

$$\begin{aligned} &= (1 + e^{i2\theta})^5 \\ &= (1 + (\cos 2\theta + i\sin 2\theta))^5 \\ &= (1 + 2i\cos 2\theta - 1 + 2i\sin 2\theta)^5 \\ &= ((2i\cos 2\theta) + i(2\sin 2\theta))^5 \\ &= [2i\cos 2\theta(\cos 2\theta + i\sin 2\theta)]^5 \\ &= 32i\cos^5 2\theta(\cos 2\theta + i\sin 2\theta)^5 \\ &= 32\cos^5 2\theta(\cos 5\theta + i\sin 5\theta) \\ &= (32\cos^5 2\theta\cos 5\theta) + i(32\cos^5 2\theta\sin 5\theta) \\ \therefore C &= 32\cos^5 2\theta\cos 5\theta \\ S &= 32\cos^5 2\theta\sin 5\theta \end{aligned}$$

**Question 3**

The following convergent series  $S$  is given below

$$S = \sin \theta - \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta \dots$$

By considering the sum to infinity of a suitable geometric series involving the complex exponential function, show that

$$S = \frac{9 \sin \theta}{10 + 6 \cos \theta}$$

**proof**

$\sin \theta = \frac{1}{2} i \sin 2\theta + \frac{1}{4} i \sin 3\theta - \frac{1}{8} i \sin 4\theta + \frac{1}{16} i \sin 5\theta - \dots$

❷  $C = \cos \theta = \frac{1}{2} \cos 2\theta + \frac{1}{4} \cos 3\theta - \frac{1}{8} \cos 4\theta + \dots$   
 $S = \sin \theta - \frac{1}{3} \sin 2\theta + \frac{1}{9} \sin 3\theta - \frac{1}{27} \sin 4\theta + \dots$

❸ THA  
 $C + iS = [\cos \theta + i \sin \theta] = \frac{1}{2} [\cos 2\theta + i \sin 2\theta] + \frac{1}{4} [\cos 3\theta + i \sin 3\theta] - \frac{1}{8} [\cos 4\theta + i \sin 4\theta] + \dots$   
 $C + iS = \underbrace{\cos \theta}_{1} + \underbrace{\frac{1}{2} \cos 2\theta}_{i \sin 2\theta} + \underbrace{\frac{1}{4} \cos 3\theta}_{-i \sin 3\theta} - \underbrace{\frac{1}{8} \cos 4\theta}_{+i \sin 4\theta} + \dots$

THIS IS A GEOMETRIC PROGRESSION WITH FIRST TERM  $e^{i\theta}$  & COMMON RATIO  $(-\frac{1}{2}e^{i\theta})$

❹ Sum to infinity =  $\frac{0}{1 - r} = \frac{0}{1 - (-\frac{1}{2}e^{i\theta})} = \frac{3e^{i\theta}}{3 + e^{i\theta}} = \frac{3e^{i\theta}(3 - e^{-i\theta})}{(3 + e^{i\theta})(3 - e^{-i\theta})} = \frac{9e^{i\theta} + 3}{9 + 3e^{i\theta} + 3e^{-i\theta} + 1} = \frac{9[\cos \theta + i \sin \theta] + 3}{10 + 6 \cos \theta + i[6 \sin \theta]}$

❺ THE REQUIRED PART IS THE IMAGINARY PART OF THE EXPRESSION, I.E.  $\sum_{n=1}^{\infty} (C + iS)^n = \frac{9 \sin \theta}{10 + 6 \cos \theta}$

**Question 4**

The sum  $C$  is given below

$$C = 1 - \binom{n}{1} \cos \theta \cos \theta + \binom{n}{2} \cos^2 \theta \cos 2\theta - \binom{n}{3} \cos^3 \theta \cos 3\theta + \dots + (-1)^n \cos^n \theta \cos n\theta$$

Given that  $n \in \mathbb{N}$  determine the 4 possible expressions for  $C$ .

Give the answers in exact simplified form.

$$n = 4k, k \in \mathbb{N} : C = \cos n\theta \sin^n \theta , \quad n = 4k+1, k \in \mathbb{N} : C = \sin n\theta \sin^n \theta$$

$$n = 4k+2, k \in \mathbb{N} : C = -\cos n\theta \sin^n \theta , \quad n = 4k+3, k \in \mathbb{N} : C = -\sin n\theta \sin^n \theta$$

$$1 - \binom{n}{1} \cos \theta \cos \theta + \binom{n}{2} \cos^2 \theta \cos 2\theta - \binom{n}{3} \cos^3 \theta \cos 3\theta + \dots + (-1)^n \cos^n \theta \cos n\theta$$

$$C = 1 - \binom{n}{1} \cos \theta \cos \theta + \binom{n}{2} \cos^2 \theta \cos 2\theta - \binom{n}{3} \cos^3 \theta \cos 3\theta + \dots + (-1)^n \cos^n \theta \cos n\theta$$

$$S = -\binom{n}{1} \cos \theta \cos \theta + \binom{n}{2} \cos^2 \theta \cos 2\theta - \binom{n}{3} \cos^3 \theta \cos 3\theta + \dots + (-1)^n \cos^n \theta \cos n\theta$$

$$C + iS = (1 - \binom{n}{1} \cos \theta \cos \theta + \binom{n}{2} \cos^2 \theta \cos 2\theta - \binom{n}{3} \cos^3 \theta \cos 3\theta + \dots + (-1)^n \cos^n \theta \cos n\theta) + i(-\binom{n}{1} \cos \theta \cos \theta + \binom{n}{2} \cos^2 \theta \cos 2\theta - \binom{n}{3} \cos^3 \theta \cos 3\theta + \dots + (-1)^n \cos^n \theta \cos n\theta)$$

$$= (1 - \cos \theta)^n e^{i\theta} + (\cos^2 \theta - \cos \theta)^n e^{2i\theta} + \dots + (-1)^n \cos^n \theta e^{ni\theta}$$

which is a binomial expansion " $(1 - \cos \theta)^n$ "

$$= (1 - e^{i\theta})^n = (1 - \cos(\theta + i\sin\theta))^n = (1 - \cos\theta - i\sin\theta\sin\theta)^n$$

$$= (\cos\theta - i\sin\theta\sin\theta)^n = \sin^n \theta [ \sin^2 \theta - i\cos\theta\sin\theta ]^n = (-1)^n \sin^n \theta [ \cos\theta + i\sin\theta ]^n$$

$$= (-1)^n \sin^n \theta (e^{i\theta})^n = (-1)^n \sin^n \theta (e^{i\theta}) = (-1)^n \sin^n \theta [\cos n\theta + i\sin n\theta]$$

- If  $n = 4k, k \in \mathbb{N}$   $(-1)^n = 1 \Rightarrow C + iS = \cos n\theta \sin^n \theta + i\sin n\theta \sin^n \theta \Rightarrow C = \cos n\theta \sin^n \theta$
- If  $n = 4k+1, k \in \mathbb{N}$   $(-1)^n = -1 \Rightarrow C + iS = \sin n\theta \sin^n \theta - i\cos n\theta \sin^n \theta \Rightarrow C = \sin n\theta \sin^n \theta$
- If  $n = 4k+2, k \in \mathbb{N}$   $(-1)^n = -1 \Rightarrow C + iS = -\cos n\theta \sin^n \theta - i\sin n\theta \sin^n \theta \Rightarrow C = -\cos n\theta \sin^n \theta$
- If  $n = 4k+3, k \in \mathbb{N}$   $(-1)^n = 1 \Rightarrow C + iS = -\sin n\theta \sin^n \theta + i\cos n\theta \sin^n \theta \Rightarrow C = -\sin n\theta \sin^n \theta$

**Question 5**

The following convergent series  $S$  is given below

$$S = \frac{\sin \theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \dots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S = e^{-\cos \theta} \sin(\sin \theta).$$

proof

$$\begin{aligned} S &= \frac{\sin \theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \dots \\ C &= \cos \theta = \frac{\cos \theta}{1!} - \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} - \frac{\cos 4\theta}{4!} + \dots \\ S &= \frac{\sin \theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \dots \\ \bullet C + iS &= \frac{1}{1!}( \cos \theta + i \sin \theta ) - \frac{1}{2!}( \cos 2\theta + i \sin 2\theta ) + \frac{1}{3!}( \cos 3\theta + i \sin 3\theta ) - \frac{1}{4!}( \cos 4\theta + i \sin 4\theta ) + \dots \\ &= \frac{1}{1!} e^{i\theta} - \frac{1}{2!} e^{i2\theta} + \frac{1}{3!} e^{i3\theta} - \frac{1}{4!} e^{i4\theta} + \dots \\ \text{NOW } e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \dots \\ -e^{i\theta} &= -1 - i\theta - \frac{i^2\theta^2}{2!} - \frac{i^3\theta^3}{3!} - \dots \\ 1 - e^{i\theta} &= \frac{\theta}{1!} - \frac{\theta^2}{2!} + \frac{\theta^3}{3!} - \frac{\theta^4}{4!} \dots \\ 1 - e^{-i\theta} &= \left( -e^{-i\theta} \right) = \left( -e^{-i\cos \theta} e^{i\sin \theta} \right) = \left( -e^{-i\cos \theta} e^{i\sin \theta} \right) \\ &= \left( -e^{-i\cos \theta} \left[ \cos(\sin \theta) + i \sin(\sin \theta) \right] \right) = \left[ \left( -e^{-i\cos \theta} \cos(\sin \theta) \right) + i \left( e^{-i\cos \theta} \sin(\sin \theta) \right) \right] \\ \therefore \text{ AS WE REQUIRE THE IMAGINARY PART, THE ANSWER IS } &= \frac{-e^{-i\cos \theta}}{\sin(\sin \theta)} \end{aligned}$$