

INTEGRATION STRUCTURED EXAM QUESTIONS

PART II

Question 1 (**)

$$\frac{4}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

- a) Find the values of A , B and C in the above identity.
 b) Hence find

$$\int \frac{4}{(x-1)(x^2+1)} dx.$$

$A = 2$	$B = -2$	$C = -2$	$2\ln x-1 - \ln(x^2+1) - 2\arctan x + C$
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(a) $\frac{4}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$$4 \equiv A(x^2+1) + (x-1)(Bx+C)$$

If $x=1$, $4 = 2A \Rightarrow A=2$

If $x=0$, $4 = A-C \Rightarrow C=-2$

If $x=2$, $4 = 5A + 2B + C$
 $4 = 10 + 2B - 2$
 $-4 = 2B$
 $B = -2$

(b) $\int \frac{4}{(x-1)(x^2+1)} dx = \int \frac{2}{x-1} dx - \frac{2x+2}{x^2+1} dx$
 $= \int \frac{2}{x-1} dx - \frac{2x}{x^2+1} dx - \frac{2}{x^2+1} dx$
 $= 2\ln|x-1| - \ln|x^2+1| - 2\arctan x + C$

Question 2 (**)

Find an exact value for

$$\int_0^{\sqrt{3}} \frac{3}{\sqrt{4-x^2}} dx.$$

□

$$\int_0^{\sqrt{3}} \frac{3}{\sqrt{4-x^2}} dx = \int_0^{\sqrt{3}} \frac{3}{\sqrt{2^2-x^2}} dx = \left[3 \arcsin \frac{x}{2} \right]_0^{\sqrt{3}} = 3 \arcsin \frac{\sqrt{3}}{2} - 3 \arcsin 0 = 3 \times \frac{\pi}{3} = \pi$$

Question 3 ()**

By using the substitution $u = \arctan x$, or otherwise, find an exact value for

$$\int_0^1 \frac{\arctan x}{1+x^2} dx.$$

$$\boxed{\frac{\pi^2}{32}}$$

$$\begin{aligned} \int_0^1 \frac{\arctan x}{1+x^2} dx &= \dots \text{ using the substitution given} \\ &= \int_0^{\frac{\pi}{4}} \frac{u}{1+\tan^2 u} (\sec^2 u) du = \int_0^{\frac{\pi}{4}} u du = \left[\frac{1}{2}u^2 \right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi^2}{32} - 0 = \frac{\pi^2}{32} \end{aligned}$$

$u = \arctan x$
 $\frac{du}{dx} = \frac{1}{1+x^2}$
 $dx = (1+x^2) du$
 $x=0, u=0$
 $x=1, u=\frac{\pi}{4}$

Question 4 ()**

Find an expression for

$$\int \frac{x+2}{\sqrt{1-4x^2}} dx.$$

$$\boxed{\arcsin 2x - \frac{1}{4}\sqrt{1-4x^2} + C}$$

$$\begin{aligned} \int \frac{x+2}{\sqrt{1-4x^2}} dx &= \int \frac{\frac{d}{dx}(x)}{\sqrt{1-4x^2}} + \frac{2}{\sqrt{1-4x^2}} dx = \int 2\sqrt{(-4x)^{-\frac{1}{2}}} + \frac{2}{\sqrt{4(1-x^2)}} dx \\ &= \int 2\sqrt{(1-x^2)^{-\frac{1}{2}}} + \frac{2}{2\sqrt{(1-x^2)^{-\frac{1}{2}}}} dx \\ &\quad \downarrow \text{EVALUATE INTEGRAL} \\ &= \frac{1}{2}(1-4x)^{\frac{1}{2}} + \arcsin(\frac{x}{2}) + C = \arcsin(2x) - \frac{1}{4}\sqrt{1-4x^2} + C \end{aligned}$$

Question 5 (**)

$$\frac{x^2+x+5}{(x+1)(x^2+4)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+4}.$$

- a) Find the values of A , B and C in the above identity.
 b) Hence find the exact value of

$$\int_0^2 \frac{x^2+x+5}{(x+1)(x^2+4)} dx.$$

$A=1$, $B=0$, $C=1$, $\frac{\pi}{8}+\ln 3$

$$(9) \quad \frac{x^2+x+5}{(x+1)(x^2+4)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$$x^2+x+5 \equiv A(x^2+4) + (Cx+B)(x+1)$$

If $x=-1 \Rightarrow S=5A \Rightarrow A=1$

If $x=0 \Rightarrow S=4+B+C \Rightarrow S=4+C \Rightarrow C=1$

If $x=1 \Rightarrow 7=5A+2(B+C)$
 $7=5+2(B+1)$
 $2=2(B+1)$
 $1=B+1$
 $B=0$

$$(10) \quad \int_0^2 \frac{x^2+x+5}{(x+1)(x^2+4)} dx = \left[\frac{1}{x+1} + \frac{1}{x^2+4} \right]_0^2 = \left[[\ln|x+1| + \frac{1}{2}\arctan\frac{x}{2}] \right]_0^2 = \left[[\ln 3 + \frac{1}{2}\arctan\frac{2}{2}] \right] - \left[[\ln 1 + \frac{1}{2}\arctan\frac{0}{2}] \right] = \ln 3 + \frac{\pi}{8}$$

Question 6 ()**

Find an exact value for

$$\int_0^{\frac{1}{2}} \frac{6x+1}{\sqrt{1-x^2}} dx.$$

$$\boxed{\frac{\pi}{6} - 3\sqrt{3} + 6}$$

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{6x+1}{\sqrt{1-x^2}} dx &= \int_0^{\frac{1}{2}} \frac{6x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} dx = \int_0^{\frac{1}{2}} 6x(-x)^{-\frac{1}{2}} + \frac{1}{(-x)^{\frac{1}{2}}} dx \\ &= \left[-6(-x)^{\frac{1}{2}} + \arcsin x \right]_0^{\frac{1}{2}} \quad \text{REVERSE CHAIN RULE} \\ &= \left[\arcsin \frac{1}{2} - 6\sqrt{1-\left(\frac{1}{2}\right)^2} \right] - \left(\arcsin 0 - 6 \right) \\ &= \frac{\pi}{6} - 3\sqrt{3} + 6 \end{aligned}$$

Question 7 ()**

Use the substitution $u = \sin x$ to find an exact value in terms of natural logarithms for

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}} dx.$$

$$\boxed{\ln(1+\sqrt{2})}$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}} dx &\quad \text{USING THE SUBSTITUTION GIVEN} \\ &= \int_0^1 \frac{\cos u}{\sqrt{1+u^2}} du \quad \text{let } u = \sin x \\ &= \left[\operatorname{arsinh} u \right]_0^1 = \operatorname{arsinh} 1 - \operatorname{arsinh} 0 \\ &= \ln(1+\sqrt{2}) \end{aligned}$$

$u = \sin x$
$du = \cos x dx$
$dx = \frac{du}{\cos x}$
$\text{when } x=0, u=0$
$x=\frac{\pi}{2}, u=1$

Question 8 ()**

The function f is defined as

$$f(x) \equiv \tanh^2 x, \quad x \in \mathbb{R}, \quad 0 \leq x \leq \ln 3.$$

Determine the mean value of f , in its entire domain.

, $\boxed{1 - \frac{4}{5\ln 3} \approx 0.272}$

USING HYPERBOLIC IDENTITIES

$$\begin{aligned} 1 + \tanh^2 x &= \operatorname{sech}^2 x \\ 1 - \tanh^2 x &= \operatorname{sinh}^2 x \\ 1 - \operatorname{sech}^2 x &= \operatorname{tanh}^2 x \end{aligned}$$

$$\begin{aligned} \int_0^{\ln 3} \tanh^2 x \, dx &= \left[\int_0^{\ln 3} 1 - \operatorname{sech}^2 x \, dx \right]_0^{\ln 3} = \left[x - \operatorname{tanh}^2 x \right]_0^{\ln 3} \\ &= \left[\ln 3 - \operatorname{tanh}(\ln 3) \right] - \left[0 - \operatorname{tanh} 0 \right] \\ &= \ln 3 - \frac{\operatorname{sinh}(\ln 3)}{\cosh^2(\ln 3)} \quad (\text{or otherwise}) \\ &= \ln 3 - \frac{g-1}{g+1} \\ &= \ln 3 - \frac{4}{5} \end{aligned}$$

Mean Value

$$\begin{aligned} \therefore \text{Mean Value} &= \frac{1}{\ln 3 - \frac{4}{5}} \int_0^{\ln 3} f(x) \, dx \\ &= \frac{1}{\ln 3 - \frac{4}{5}} \left(\ln 3 - \frac{4}{5} \right) \\ &= \frac{\ln 3 - \frac{4}{5}}{\ln 3} \\ &= \frac{5\ln 3 - 4}{5\ln 3} \quad \text{or} \quad 1 - \frac{4}{5\ln 3} // \end{aligned}$$

Question 9 (**+)

Find an exact value for

$$\int_0^{\frac{3}{4}} \frac{6}{\sqrt{3-4x^2}} dx.$$

□

$$\begin{aligned} \int_0^{\frac{3}{4}} \frac{6}{\sqrt{3-4x^2}} dx &= \int_0^{\frac{3}{4}} \frac{6}{\sqrt{4(\frac{3}{4}-x^2)}} dx = \int_0^{\frac{3}{4}} \frac{6}{2\sqrt{(\frac{3}{4}-x^2)}} dx \\ &= \int_0^{\frac{3}{4}} \frac{3}{\sqrt{(\frac{3}{4}-x^2)^2}} dx = \text{STANDARD ARCSIN INTEGRAL} = \left[3 \arcsin\left(\frac{2x}{\sqrt{3}}\right) \right]_0^{\frac{3}{4}} \\ &= 3 \left[\arcsin\left(\frac{2x}{\sqrt{3}}\right) \right]_0^{\frac{3}{4}} = 3 \left[\arcsin\left(\frac{3}{2\sqrt{3}}\right) - \arcsin(0) \right] = 3 \arcsin\left(\frac{\sqrt{3}}{2}\right) \\ &= 3 \times \frac{\pi}{3} = \pi \end{aligned}$$

Question 10 (**+)

By using a suitable substitution, find in terms of π , the value of

$$\int_0^1 \frac{1}{(x+1)\sqrt{x}} dx.$$

□ $\frac{\pi}{2}$

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{2(u+1)}} du &= \dots \text{by substitution...} \\ &= \int_0^1 \frac{1}{u(u^2+1)} (2u du) = \int_0^1 \frac{2}{u^2+1} du \\ &= \left[2 \arctan u \right]_0^1 = 2 \arctan 1 - 2 \arctan 0 \\ &= 2 \times \frac{\pi}{4} = \frac{\pi}{2} \end{aligned}$$

$u = \sqrt{x}$
 $u^2 = x$
 $2u \frac{du}{dx} = 1$
 $2u du = dx$
 $x=0 \rightarrow u=0$
 $x=1 \rightarrow u=1$

Question 11 (**+)

Show clearly that

$$\int \frac{4x+1}{\sqrt{4x^2-9}} dx = f(x) + \frac{1}{2} \ln[2x + f(x)] + C,$$

where $f(x)$ is a function to be found.

$$f(x) = \sqrt{4x^2 - 9}$$

$$\begin{aligned}
 \int \frac{4x+1}{\sqrt{4x^2-9}} dx &= \int \frac{dx}{\sqrt{4x^2-9}} + \frac{1}{\sqrt{4x^2-9}} dx \\
 &= \int dx/(2x^2-9)^{1/2} + \frac{1}{\sqrt{4x^2-9}} dx \\
 &= \int dx/(2x^2-9)^{-1/2} + \frac{1}{2\sqrt{x^2-\frac{9}{4}}} dx \\
 &\quad \text{DUKE HIGH DIVE} \quad \text{STANDARD ACCORD} \\
 &= (2x^2-9)^{-1/2} + \frac{1}{2} \operatorname{arsinh}\left(\frac{2x}{3}\right) + C \\
 &= \sqrt{4x^2-9}^{-1} + \frac{1}{2} \operatorname{arsinh}\left(\frac{2x}{3}\right) + C \\
 &= \sqrt{4x^2-9}^{-1} + \frac{1}{2} \ln\left(\frac{2x+\sqrt{4x^2-9}}{\sqrt{4x^2-9}}\right) + C \\
 &= \sqrt{4x^2-9}^{-1} + \frac{1}{2} \ln(2 + \sqrt{4x^2-9}) + C
 \end{aligned}$$

Question 12 (**+)

By using a suitable substitution, or otherwise, find

$$\int \frac{1}{(1+x^2)\arctan x} dx.$$

$$\ln|\arctan x| + C$$

$$\begin{aligned}
 \int \frac{1}{(1+x^2)\arctan x} dx &= \text{by recognizing or substitution} \\
 &= \int \frac{1}{(1+u^2)u} C(u) du = \int \frac{1}{u} du = \ln|u| + C \\
 &= \ln|\arctan x| + C
 \end{aligned}$$

Question 13 (***)

$$f(x) \equiv \frac{x^2 + 3x + 36}{(x+9)(x^2 + 9)}.$$

a) Express $f(x)$ into partial fractions.

b) Hence find

$$\int f(x) dx.$$

$$\boxed{\frac{1}{x+9} + \frac{3}{x^2+9}}, \quad \boxed{\ln|x+9| + \arctan \frac{x}{3} + C}$$

(a) $\frac{x^2 + 3x + 36}{(x+9)(x^2+9)} \equiv \frac{A}{x+9} + \frac{Bx+C}{x^2+9}$

$x^2 + 3x + 36 \equiv A(x^2+9) + (x+9)(Bx+C)$

If $x=-9$, $90 = 90A \Rightarrow A=1$

If $A=0$, $3x = 9A + 9C \Rightarrow 3x = 9C \Rightarrow C=\frac{3}{3}$

If $x=-8$, $76 = 73A - 8B + C \Rightarrow 76 = 73 - 8B + 3 \Rightarrow B=\frac{5}{8}$

$\therefore f(x) = \frac{1}{x+9} + \frac{3}{x^2+9}$

(b) $\int f(x) dx = \int \left(\frac{1}{x+9} + \frac{3}{x^2+9} \right) dx = \ln|x+9| + \frac{3}{3} \arctan \frac{x}{3} + C = \ln|x+9| + \arctan \frac{x}{3} + C$

Question 14 (*)**

Use the substitution $t = x - 8$ to find the exact value of

$$\int_8^{8.75} \frac{1}{\sqrt{x^2 - 16x + 65}} dx,$$

giving the answer as a single natural logarithm.

[ln 2]

$$\begin{aligned}
 \int_8^{8.75} \frac{1}{\sqrt{x^2 - 16x + 65}} dx &= \dots \text{USING THE SUBSTITUTION GIVEN} \\
 &= \int_0^{\frac{3}{4}} \frac{1}{\sqrt{(t+8)^2 - 16(t+8) + 65}} dt \\
 &= \int_0^{\frac{3}{4}} \frac{1}{\sqrt{t^2 + 16t + 64 - 16t - 128 + 65}} dt = \int_0^{\frac{3}{4}} \frac{1}{\sqrt{t^2 + 1}} dt \\
 &= [\arctan t]_0^{\frac{3}{4}} = \arctan \frac{3}{4} - \arctan 0 = \ln \left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1} \right) \\
 &= \ln \left(\frac{3}{4} + \sqrt{\frac{25}{16}} \right) = \ln \left(\frac{3}{4} + \frac{5}{4} \right) = \ln 2
 \end{aligned}$$

Question 15 (*)**

$$f(x) = \sinh x \cos x + \sin x \cosh x, \quad x \in \mathbb{R}.$$

- a) Find a simplified expression for $f'(x)$.
- b) Use the answer to part (a) to find

$$\int \frac{2}{\tanh x + \tan x} dx.$$

[FP2 K], $f'(x) = 2 \cosh x \cos x$, $\ln |\sinh x \cos x + \sin x \cosh x| + C$

$$\begin{aligned}
 a) \quad &f(x) = \sinh x \cos x + \sin x \cosh x \\
 &f'(x) = \cosh x \cos x + \sinh x (-\sin x) + \cos x \cosh x + \sin x \sinh x \\
 &f'(x) = 2 \cosh x \cos x \\
 b) \quad &\text{INTP WITH SAME Q FRACTION} \\
 &\int \frac{2}{\tanh x + \tan x} dx = \int \frac{2}{\frac{\sinh x}{\cosh x} + \frac{\sin x}{\cos x}} dx \\
 &\text{MULTIPLY TOP & BOTTOM OF THE FRACTION BY } \cosh x \sinh x \\
 &= \int \frac{2 \cosh x \sinh x}{\sinh^2 x + \cos^2 x} dx \\
 &\text{WHICH IS OF THE FORM } \int \frac{f'(x)}{f(x)} dx \\
 &= \ln |\sinh x \cos x + \sin x \cosh x| + C
 \end{aligned}$$

Question 16 (***)

Find the exact value of

$$\int_{2.5}^{7.5} \frac{15\sqrt{3}}{4x^2 + 75} dx.$$

$$\boxed{\frac{\pi}{4}}$$

$$\begin{aligned} & \int_{-2.5}^{7.5} \frac{15\sqrt{3}}{4x^2 + 75} dx = \text{area top of } \text{box from the x-axis by } 4 \dots \\ &= \int_{-2.5}^{7.5} \frac{15\sqrt{3}}{4x^2 + 75} dx = \left[\frac{15\sqrt{3}}{4} \arctan \left(\frac{2x}{\sqrt{75}} \right) \right]_{-2.5}^{7.5} = \dots \text{ standard outcome} \dots \\ &= \frac{15\sqrt{3}}{4} \times \frac{1}{\frac{2\sqrt{75}}{4}} \times \left[\arctan \left(\frac{2x}{\sqrt{75}} \right) \right]_{-2.5}^{7.5} = \frac{15\sqrt{3}}{4} \times \frac{2}{\sqrt{75}} \times \left[\arctan \left(\frac{2x}{\sqrt{75}} \right) \right]_{-2.5}^{7.5} \\ &\geq \frac{15\sqrt{3}}{4} \times \frac{2}{\sqrt{75}} \left[\arctan \left(\frac{15}{\sqrt{75}} \right) - \arctan \left(\frac{-5}{\sqrt{75}} \right) \right] = \frac{3}{2} \left[\arctan \sqrt{3} - \arctan \frac{\sqrt{3}}{3} \right] \\ &= \frac{3}{2} \left[\frac{\pi}{4} - \frac{\pi}{6} \right] = \frac{3}{2} \times \frac{\pi}{6} = \frac{3\pi}{12} // \end{aligned}$$

Question 17 (***)

Find the exact value of

$$\int_0^1 \frac{2\sqrt{3}}{\sqrt{4\pi^2 - 3\pi^2 x^2}} dx.$$

$$\boxed{\frac{2}{3}}$$

$$\begin{aligned} \int_0^1 \frac{2\sqrt{3}}{\sqrt{4\pi^2 - 3\pi^2 x^2}} dx &= \int_0^1 \frac{2\sqrt{3}}{\sqrt{4\pi^2(1 - \frac{3}{4}\pi^2 x^2)}} dx = \int_0^1 \frac{2\sqrt{3}}{2\sqrt{\pi^2(1 - \frac{3}{4}\pi^2 x^2)}} dx \\ &= \frac{2}{\pi} \int_0^1 \frac{1}{\sqrt{1 - \frac{3}{4}\pi^2 x^2}} dx = \left[\frac{2}{\pi} \arcsin \left(\frac{2x}{\sqrt{3}\pi} \right) \right]_0^1 \\ &= \frac{2}{\pi} \left[\arcsin \left(\frac{\sqrt{3}}{3} \right) \right]^1_0 \\ &= \frac{2}{\pi} \left[\arcsin \left(\frac{\sqrt{3}}{3} \right) - \arcsin 0 \right] \\ &= \frac{2}{\pi} \times \frac{\pi}{3} = \frac{2}{3} // \end{aligned}$$

Question 18 (*)**

Find the exact value of each of the following integrals.

a) $\int_{-3}^{-2} \frac{1}{\sqrt{-x^2 - 6x - 5}} dx.$

b) $\int_{-3}^{-2} \frac{x}{\sqrt{-x^2 - 6x - 5}} dx.$

$$\boxed{\frac{\pi}{6}}, \boxed{\sqrt{3} + \frac{\pi}{2} - 2}$$

$$\begin{aligned}
 & \text{(a)} \int_{-3}^{-2} \frac{1}{\sqrt{-x^2 - 6x - 5}} dx = \int_{-3}^{-2} \frac{1}{\sqrt{-(x^2 + 6x + 5)}} dx = \int_{-3}^{-2} \frac{1}{\sqrt{-(x+5)(x+1)}} dx \\
 & \quad \int_{-3}^{-2} \frac{1}{\sqrt{4-u^2}} du = \dots \text{ substitution } u = x+3, \frac{du}{dx} = 1, u=2 \mapsto x=-1, u=1 \mapsto x=0 \\
 & \quad = \int_0^1 \frac{1}{\sqrt{4-u^2}} du = \left[\arcsin \frac{u}{2} \right]_0^1 = \arcsin \frac{1}{2} - \arcsin 0 = \frac{\pi}{6} \checkmark \\
 & \text{(b)} \int_{-3}^{-2} \frac{x}{\sqrt{-x^2 - 6x - 5}} dx = \dots \text{ as in part (a)} = \dots \text{ substitution } u = \sqrt{4-x^2}, \frac{du}{dx} = -\frac{x}{\sqrt{4-x^2}} \\
 & \quad = \int_0^1 \frac{u-3}{\sqrt{4-u^2}} du = \int_0^1 \frac{u}{\sqrt{4-u^2}} - \frac{3}{\sqrt{4-u^2}} du \\
 & \quad = \int_0^1 u(u-u^2)^{\frac{1}{2}} du - 3 \int_0^1 \frac{1}{\sqrt{4-u^2}} du \\
 & \quad \stackrel{\text{reverse chain rule}}{=} \left[-\frac{1}{2}(4-u^2)^{\frac{3}{2}} - 3\arcsin \frac{u}{2} \right]_0^1 = \left[\sqrt{4-u^2} + 3\arcsin \frac{u}{2} \right]_0^1 \\
 & \quad = (\sqrt{3} + 3\arcsin \frac{1}{2}) - (2 + 3\arcsin 0) = \sqrt{3} + \frac{\pi}{2} - 2 \checkmark
 \end{aligned}$$

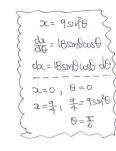
Question 19 (*)**

By using the substitution $x = 9\sin^2 \theta$, or otherwise, find the exact value of

$$\int_0^{\frac{9}{4}} \frac{1}{\sqrt{x(9-x)}} dx.$$

$$\boxed{\frac{\pi}{3}}$$

$$\begin{aligned}
 & \int_0^{\frac{9}{4}} \frac{1}{\sqrt{x(9-x)}} dx = \dots \text{ by substitution} \\
 & = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{9\sin^2 \theta(9-9\sin^2 \theta)}} (18\sin \theta \cos \theta) d\theta \\
 & \quad \frac{d}{d\theta} x = 18\sin \theta \cos \theta, \frac{d}{d\theta} dx = 18\sin \theta \cos \theta \\
 & \quad \int_0^{\frac{\pi}{2}} \frac{18\sin \theta \cos \theta}{\sqrt{81\sin^2 \theta \cos^2 \theta}} d\theta = \int_0^{\frac{\pi}{2}} \frac{18\sin \theta \cos \theta}{9\sin \theta \cos \theta} d\theta \\
 & \quad \int_0^{\frac{\pi}{2}} 2 d\theta = [2\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{3} \checkmark
 \end{aligned}$$



Question 20 (***)

Find the exact value of

$$\int_{\frac{4}{3}}^{\frac{5}{3}} \frac{x+1}{\sqrt{9x^2-16}} dx.$$

$$\boxed{\frac{1}{3}(1-\ln 2)}$$

$$\begin{aligned}
 \int_{\frac{4}{3}}^{\frac{5}{3}} \frac{x+1}{\sqrt{9x^2-16}} dx &= \int_{\frac{4}{3}}^{\frac{5}{3}} \frac{x}{\sqrt{9x^2-16}} dx - \int_{\frac{4}{3}}^{\frac{5}{3}} \frac{1}{\sqrt{9x^2-16}} dx \\
 &= \int_{\frac{4}{3}}^{\frac{5}{3}} x(9x^2-16)^{-\frac{1}{2}} dx - \int_{\frac{4}{3}}^{\frac{5}{3}} \frac{1}{3\sqrt{9x^2-16}} dx \\
 &\quad \text{BY RECOGNITION} \qquad \text{STANDARD FORM} \\
 &= \left[\frac{1}{3}(9x^2-16)^{\frac{1}{2}} - \frac{1}{3} \operatorname{arcsinh}\left(\frac{3x}{4}\right) \right]_{\frac{4}{3}}^{\frac{5}{3}} \\
 &= \left[\frac{1}{3}\sqrt{9x^2-16} \right]_{\frac{4}{3}}^{\frac{5}{3}} - \frac{1}{3} \operatorname{arcsinh}\left(\frac{3x}{4}\right) \Big|_{\frac{4}{3}}^{\frac{5}{3}} \\
 &= \left(\frac{1}{3} - \frac{1}{3} \operatorname{arcsinh}\left(\frac{5}{4}\right) \right) - \left(0 - \frac{1}{3} \operatorname{arcsinh}\left(\frac{4}{4}\right) \right) \\
 &= \frac{1}{3}(1 - \operatorname{arcsinh}\frac{5}{4}) = \frac{1}{3}[1 - \ln\left(\frac{5}{4} + \sqrt{\frac{25}{16}-1}\right)] \\
 &= \frac{1}{3}[1 - \ln\left(\frac{5}{4} + \frac{3}{4}\right)] = \frac{1}{3}[1 - \ln 2]
 \end{aligned}$$

Question 21 (***)

Find the exact value of each of the following integrals.

a) $\int_5^7 \frac{1}{x^2-10x+29} dx.$

b) $\int_5^7 \frac{x}{\sqrt{x^2-10x+29}} dx.$

$$\boxed{\frac{\pi}{8}}, \boxed{\ln(1+\sqrt{2})}$$

$$\begin{aligned}
 \text{(a)} \int_5^7 \frac{1}{x^2-10x+29} dx &= \int_5^7 \frac{1}{(x-5)^2-25+29} dx = \int_5^7 \frac{1}{(x-5)^2+4} dx \\
 &\quad \text{Let } u = x-5, \quad du = dx \\
 &\quad \text{Let } z = u-5, \quad dz = du \\
 &\quad \text{Let } y = z-4, \quad dy = dz \\
 &= \int_0^2 \frac{1}{u^2+4} du = \int_0^2 \frac{1}{u^2+2^2} du = \left[\frac{1}{2} \operatorname{arctan}\frac{u}{2} \right]_0^2 \\
 &= \frac{1}{2} \operatorname{arctan}1 - \frac{1}{2} \operatorname{arctan}0 = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8} \\
 \text{(b)} \int_5^7 \frac{1}{\sqrt{x^2-10x+29}} dx &= \dots \text{as in part (a) including the substitution} \\
 &= \int_0^2 \frac{1}{\sqrt{u^2+4}} du = \left[\operatorname{arsh}\frac{u}{2} \right]_0^2 \\
 &= \operatorname{arsh}1 - \operatorname{arsh}0 = \ln(1+\sqrt{2})
 \end{aligned}$$

Question 22 (*)**

Use the substitution $u = e^x$ to find

$$\int \frac{\sqrt{e^x}}{\sqrt{e^x + e^{-x}}} dx.$$

$\operatorname{arsinh}(e^x) + C$

$$\begin{aligned} \int \frac{\sqrt{e^x}}{\sqrt{e^x + e^{-x}}} dx &\leftarrow \dots \text{ by substitution } \dots = \int \frac{\sqrt{u+u^{-1}}}{\sqrt{u+u^{-1}}} \frac{du}{u} \\ &= \int \frac{\sqrt{u+u^{-1}}}{\sqrt{u(u+u^{-1})}} \frac{du}{u} = \int \frac{\sqrt{u+u^{-1}}}{\sqrt{u+u^{-1}}} \frac{du}{u} \\ &= \int \frac{du}{\sqrt{u+u^{-1}}} = \operatorname{arsinh} u + C = \operatorname{arsinh}(e^x) + C \end{aligned}$$

$u = e^x$
 $\frac{du}{dx} = e^x$
 $\frac{du}{u} = e^x dx$
 $du = u \frac{du}{u}$

Question 23 (*)**

Find in exact simplified form in terms of natural logarithms

$$\int_3^6 \frac{1}{2x+6} \sqrt{\frac{x+3}{x-2}} dx.$$

$\frac{1}{2} \ln(2 + \sqrt{3})$

$$\begin{aligned} \int_3^6 \frac{1}{2x+6} \sqrt{\frac{x+3}{x-2}} dx &= \frac{1}{2} \int_3^6 \frac{1}{x+3} \frac{(x+3)^{\frac{1}{2}}}{(x-2)^{\frac{1}{2}}} dx \\ &\stackrel{u = x-2}{=} \frac{1}{2} \int_3^6 \frac{1}{(x+3)(x-2)} dx = \frac{1}{2} \int_3^6 \frac{1}{\sqrt{x^2-9}} dx \\ &= \frac{1}{2} \left[\operatorname{arsinh} \frac{x}{3} \right]_3^6 = \frac{1}{2} \operatorname{arsinh} 2 - \frac{1}{2} \operatorname{arsinh} 1 \\ &= \frac{1}{2} \ln(2 + \sqrt{2^2-1}) = \frac{1}{2} \ln(2 + \sqrt{3}) \end{aligned}$$

Question 24 (*)**

Show that the exact value of the following integral

$$\int_1^2 \frac{3x^2+1}{2x^3+x} dx$$

is $\frac{1}{4} \ln 48$.

, proof

PROCEED TO TRY

$$\int_1^2 \frac{3x^2+1}{2x^3+x} dx = \int_1^2 \frac{3x^2+1}{x(2x^2+1)} dx$$

CONTINUE WITH PARTIAL FRACTION

$$\begin{aligned} \frac{3x^2+1}{x(2x^2+1)} &= \frac{A}{x} + \frac{Bx+C}{2x^2+1} \\ 3x^2+1 &= A(2x^2+1) + (Bx+C)x \\ 3x^2+1 &= 2Ax^2+A+Bx^2+Cx \\ 3x^2+1 &= (2A+B)x^2+Cx+A \end{aligned}$$

- A=1
- C=0
- 2A+B=3
- B=1
- C=1

RETURNING TO THE INTEGRAL

$$\begin{aligned} \int_1^2 \frac{\frac{1}{x} + \frac{3}{2x^2+1}}{x} dx &= \left[\ln|x| + \frac{1}{2} \ln(2x^2+1) \right]_1^2 \\ &= (\ln 2 + \frac{1}{2} \ln 9) - (\ln 1 + \frac{1}{2} \ln 3) \\ &= \ln 2 + \frac{1}{2} \ln 9 - \frac{1}{2} \ln 3 \\ &= \ln 2 + \frac{1}{2} \ln 3 \\ &= \frac{1}{2} [\ln 2 + \ln 3] \\ &= \frac{1}{2} \ln 48 \end{aligned}$$

Question 25 (***)

$$\int_0^1 \frac{x \arcsin x}{\sqrt{1-x^2}} dx.$$

Show that value of the above definite integral is 1.

[proof]

$\int_0^1 \frac{x \arcsin x}{\sqrt{1-x^2}} dx = \dots$ $= \left[(-\arcsin x) \sqrt{1-x^2} \right]_0^1 + \int_0^1 1 dx$ $= \left[-\infty \right]_0^1 + 1$	BY PARTS $\begin{array}{ c c } \hline & \frac{1}{\sqrt{1-x^2}} \\ \hline \arcsin x & \frac{2x}{(1-x^2)} \\ \hline \end{array}$
--	---

Question 26 (***)+

$$f(x) \equiv x \arctan x, \quad x \in \mathbb{R}.$$

- a) Find an expression for $f'(x)$.
- b) Use the answer to part (a) to find the exact value of

$$\int_0^1 4 \arctan x dx.$$

You may **not** use standard integration by parts to obtain the answer to part (b).

$f'(x) = \frac{x}{1+x^2} + \arctan x, \quad [\pi - \ln 4]$
--

<p>(a)</p> $f(a) = \arctan a$ $f(a) = \frac{a}{1+a^2} + \arctan a$ $f(a) = \frac{a}{1+a^2} + \arctan a$
<p>(b)</p> $\frac{d}{da}(\arctan a) = \frac{1}{1+a^2}$ $\int_0^1 \frac{1}{1+a^2} da + \int_0^1 \arctan a da$ $[\arctan a]_0^1 = [\ln(1+a^2)]_0^1 + \int_0^1 \arctan a da$ $\left(\frac{\pi}{4} - 0\right) = (\ln 2 - 0) + \int_0^1 \arctan a da$ $\therefore \int_0^1 \arctan a da = \pi - \ln 2 \quad \text{or} \quad \pi - \ln 4$

Question 27 (***)+

By using the substitution $x^2 = 3 \tan \theta$, or otherwise, find the exact value of

$$\int_0^{\sqrt{3}} \frac{x}{x^4 + 9} dx.$$

$$\boxed{\frac{\pi}{24}}$$

$\int_0^{\sqrt{3}} \frac{x}{x^4 + 9} dx = \text{by substitution or recognition}$
 $= \int_0^{\frac{\pi}{2}} \frac{2}{(3\tan\theta)^2 + 9} \cdot 3\tan^2\theta d\theta = \int_0^{\frac{\pi}{2}} \frac{3\tan^2\theta}{2(9+3\tan^2\theta)} d\theta$
 $= \int_0^{\frac{\pi}{2}} \frac{3\tan^2\theta}{18\tan^2\theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{6} d\theta = \left[\frac{1}{6}\theta \right]_0^{\frac{\pi}{2}}$
 $= \frac{\pi}{12} - 0 = \frac{\pi}{24}$

Question 28 (***)+

Use an appropriate substitution to find an exact value for the following integral.

$$\int_1^{\sqrt{e}} \frac{1}{x\sqrt{1-(\ln x)^2}} dx.$$

You may assume that the integral converges.

$$\boxed{}, \boxed{\frac{1}{6}\pi}$$

USING A SUBSTITUTION
 $u = \ln x \quad x=1 \rightarrow u=\ln 1=0$
 $\frac{du}{dx} = \frac{1}{x} \quad x=e^u \rightarrow u=\ln e^u=u$
 $dx = x du$
TRANSFORMING THE INTEGRAL
 $\int_1^{\sqrt{e}} \frac{1}{x\sqrt{1-(\ln x)^2}} dx = \int_0^{\frac{1}{2}} \frac{1}{x\sqrt{1-u^2}} (x du)$
 $= \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}} du$
 $= \left[\arcsin u \right]_0^{\frac{1}{2}}$
 $= \arcsin \frac{1}{2} - \arcsin 0$
 $= \frac{\pi}{6}$

Question 29 (*)**

The curves C_1 and C_2 have respective equations

$$y = 18 \cosh x, \quad x \in \mathbb{R} \quad \text{and} \quad y = 12 + 14 \sinh x, \quad x \in \mathbb{R}.$$

- a) Find the exact coordinates of the points of intersection between C_1 and C_2 .
- b) Sketch in the same diagram the graph of C_1 and the graph of C_2 .
- c) Show that the finite region bounded by the graphs of C_1 and C_2 has an area of

$$a \ln 2 + b,$$

where a and b are integers to be found.

$$\left[\left(\ln 2, \frac{45}{2} \right) \text{ & } \left(\ln 4, \frac{153}{4} \right) \right], [12 \ln 2 - 8]$$

(a)

$$18 \cosh x = 12 + 14 \sinh x$$

$$\Rightarrow 9e^x + 9e^{-x} = 12 + 7e^x - 7e^{-x}$$

$$\Rightarrow 2e^x + 16e^{-x} - 12 = 0$$

$$\Rightarrow e^x + 8e^{-x} - 6 = 0$$

$$\Rightarrow e^x + 8 - 6e^x = 0$$

$$\Rightarrow e^x - 6e^x + 8 = 0$$

$$\Rightarrow (e^x - 2)(e^x - 4) = 0$$

$$\Rightarrow e^x = 2 \quad \text{or} \quad e^x = 4$$

$$\Rightarrow x = \ln 2 \quad \text{or} \quad x = \ln 4$$

$$y = 18 \cosh x$$

$$y = 9e^{\ln 2} + 9e^{-\ln 2} = 18 + \frac{9}{2} = \frac{45}{2}$$

$$y = 9e^{\ln 4} + 9e^{-\ln 4} = 36 + \frac{9}{4} = \frac{153}{4}$$

$$\therefore \left(\ln 2, \frac{45}{2} \right) \text{ & } \left(\ln 4, \frac{153}{4} \right)$$

(b)

(c)

$$A_{\text{red}} = \int_{-\ln 4}^{\ln 2} (12 + 14 \sinh x) dx - \int_{-\ln 4}^{\ln 2} (18 \cosh x) dx$$

$$= \int_{-\ln 4}^{\ln 2} (12 + 14 \sinh x - 18 \cosh x) dx$$

$$= \left[12x + 14 \sinh x - 18 \cosh x \right]_{-\ln 4}^{\ln 2}$$

$$= \left[12 \ln 2 + 14 \sinh(\ln 2) - 18 \cosh(\ln 2) \right] - \left[12 \ln(-4) + 14 \sinh(-\ln 4) - 18 \cosh(-\ln 4) \right]$$

$$= (12 \ln 2 - 4) - (12 \ln 2 + \frac{25}{2} - \frac{35}{2})$$

$$= (12 \ln 2 - 4) - (12 \ln 2 + 4)$$

$$= 12 \ln 2 - 8$$

$a = 12$
 $b = -8$

Question 30 (***)+

$$f(x) \equiv \frac{4x}{1-x^4}.$$

- a) Express $f(x)$ into partial fractions.
 b) Hence find, as a single natural logarithm, the value of

$$\int_0^{\frac{1}{2}} f(x) \, dx.$$

$$\boxed{\quad}, \quad f(x) = \frac{1}{1-x} - \frac{1}{1+x} + \frac{2x}{1+x^2}, \quad \boxed{\ln \frac{5}{3}}$$

a) START BY FACTORISING THE DENOMINATOR AS FOLLOWS

$$f(x) = \frac{4x}{1-x^4} = \frac{4x}{(1-x^2)(1+x^2)} = \frac{4x}{(1-x)(1+x)(1+x^2)}$$

$$f(x) = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$$

$$\boxed{f(x) \equiv A(1+x)(1+x^2) + B(1-x)(1+x^2) + (1-x)(1+x)(Cx+D)}$$

- IF $x=1$: $A=4$
- IF $x=-1$: $B=-4$
- IF $x=0$: $C=0$
- IF $x=0$: $D=0$

COMPARING COEFFICIENTS OF x^2 IN BOTH SIDES

$$0 = A x^2 - B x^2 - C x^2$$

$$0 = 1 x^2 + 1 x^2 - C x^2$$

$$\therefore C=2$$

$$\therefore f(x) = \frac{4}{1-x} - \frac{4}{1+x} + \frac{2x}{1+x^2}$$

b) USE PART (a)

$$\int_0^{\frac{1}{2}} f(x) \, dx = \int_0^{\frac{1}{2}} \left(\frac{4}{1-x} - \frac{4}{1+x} + \frac{2x}{1+x^2} \right) \, dx$$

$$= \left[-4 \ln|1-x| - 4 \ln|1+x| + 2x \left| \frac{1}{1+x^2} \right. \right]_0^{\frac{1}{2}}$$

$$= \left(-4 \ln \frac{1}{2} - 4 \ln \frac{3}{2} + 2 \times \frac{1}{5} \right) - \left(-4 \ln 1 - 4 \ln 1 + 0 \right)$$

$$= \ln \frac{5}{3} - \ln \frac{3}{2}$$

$$= \ln \left(\frac{5}{3} \right)^{-1}$$

$$= \ln \left(\frac{3}{5} \right)$$

Question 31 (***)

$$f(x) = x \operatorname{arsinh}\left(\frac{1}{2}x\right), x \in \mathbb{R}.$$

- a) Find a simplified expression for $f'(x)$.
- b) Use the answer to part (a) to show that

$$\int_0^{\sqrt{12}} \operatorname{arsinh}\left(\frac{1}{2}x\right) dx = 2\sqrt{3} \ln(2 + \sqrt{3}) - 2.$$

$$f'(x) = \operatorname{arsinh}\left(\frac{1}{2}x\right) + \frac{x}{\sqrt{x^2 + 4}}$$

(a) $f(x) = \operatorname{arsinh}\left(\frac{1}{2}x\right)$
 $f'(x) = 1 \times \operatorname{arsinh}\left(\frac{1}{2}x\right) + x \times \frac{1}{\sqrt{\frac{1}{4}x^2 + 1}} \times \frac{1}{2}$
 $f'(x) = \operatorname{arsinh}\left(\frac{1}{2}x\right) + \frac{x}{\sqrt{4x^2 + 4}} \times \frac{1}{2}$
 $f'(x) = \operatorname{arsinh}\left(\frac{1}{2}x\right) + \frac{x}{\sqrt{x^2 + 4}}$

(b) Now: $\frac{d}{dx} [\operatorname{arsinh}(tx)] = \operatorname{arsinh}\left(\frac{1}{2}x\right) + \frac{x}{\sqrt{x^2 + 4}}$
 INTEGRATING USING SUMME
 $\int_{\sqrt{2}}^{\sqrt{12}} \frac{d}{dx} [\operatorname{arsinh}(tx)] dx = \int_{\sqrt{2}}^{\sqrt{12}} \operatorname{arsinh}(tx) dx + \int_{\sqrt{2}}^{\sqrt{12}} \frac{x(t^2+1)^{-\frac{1}{2}}}{\sqrt{t^2+x^2}} dx$
 $[\operatorname{arsinh}(tx)] \Big|_{\sqrt{2}}^{\sqrt{12}} = \int_{\sqrt{2}}^{\sqrt{12}} \operatorname{arsinh}(tx) dx + \left[1 \cdot (t^2+4)^{\frac{1}{2}} \right] \Big|_{\sqrt{2}}^{\sqrt{12}}$
 $\sqrt{12} \operatorname{arsinh}\left(\frac{1}{2}\sqrt{12}\right) = \int_{\sqrt{2}}^{\sqrt{12}} \operatorname{arsinh}(tx) dx + (4-2)$
 $2\sqrt{3} \operatorname{arsinh}\left(\frac{1}{2}\sqrt{12}\right) = \int_{\sqrt{2}}^{\sqrt{12}} \operatorname{arsinh}(tx) dx + 2$
 $2\sqrt{3} \operatorname{arsinh}\left(\frac{1}{2}\sqrt{12}\right) = \int_{\sqrt{2}}^{\sqrt{12}} \operatorname{arsinh}(tx) dx + 2$
 $\therefore \int_{\sqrt{2}}^{\sqrt{12}} \operatorname{arsinh}(tx) dx = 2\sqrt{3} \operatorname{arsinh}\left(\frac{1}{2}\sqrt{12}\right) - 2$ as required

Question 32 (***)+

$$I = \int_1^4 \frac{3}{(x+9)\sqrt{x}} dx.$$

- a) By using a suitable substitution find an exact value for I .
- b) Show clearly that the answer to part (a) can be written as $2 \arctan \frac{3}{11}$.

, $I = 2\left(\arctan \frac{2}{3} - \arctan \frac{1}{3}\right)$

a) USING THE SUBSTITUTION $u = \sqrt{x}$

$$\begin{aligned} u &= \sqrt{x} & \text{if } u=1 \\ u^2 &= x^2 & \text{if } u=1 \\ du &= 2x dx & \text{if } u=1 \\ &= 2u du & \text{if } u=1 \end{aligned}$$

$$\int_1^4 \frac{3}{(x+9)\sqrt{x}} dx = \int_1^2 \frac{3}{(u^2+9)u} (2u du) = \int_1^2 \frac{6}{u^2+9} du$$

A STANDARD INTEGRAL

$$\dots = \frac{1}{3} \times 6 \times \left[\arctan\left(\frac{u}{3}\right) \right]_1^2 = 2 \left[\arctan \frac{2}{3} - \arctan \frac{1}{3} \right]$$

b) USING THE IDENTITY $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\begin{aligned} \tan\left(\arctan \frac{2}{3} - \arctan \frac{1}{3}\right) &= \frac{\tan(\arctan \frac{2}{3}) - \tan(\arctan \frac{1}{3})}{1 + \tan(\arctan \frac{2}{3}) \tan(\arctan \frac{1}{3})} \\ &= \frac{\frac{2}{3} - \frac{1}{3}}{1 + \frac{2}{3} \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{1 + \frac{2}{9}} = \frac{3}{11} \end{aligned}$$

$$\therefore \arctan \frac{2}{3} - \arctan \frac{1}{3} = \arctan \frac{3}{11}$$

$$\therefore I = 2 \arctan \frac{3}{11}$$

As required

Question 33 (***)+

$$I = \int_0^{\frac{\pi}{3}} \frac{1}{1+8\cos^2 x} dx.$$

- a) By using the substitution $t = \tan x$, or otherwise, show clearly that

$$I = \int_0^{\sqrt{3}} \frac{1}{9+t^2} dt.$$

- b) Hence find the exact value of I .

$\frac{\pi}{18}$

(a) $\int_0^{\frac{\pi}{3}} \frac{1}{8\cos^2 x + 1} dx = \dots$ by substituting ...

$\begin{aligned} &= \int_0^{\sqrt{3}} \frac{1}{8\tan^2 t + 1} \times \frac{dt}{\sec^2 t} = \int_0^{\sqrt{3}} \frac{1}{8 + \sec^2 t} dt \\ &= \int_0^{\sqrt{3}} \frac{1}{8 + (1 + \tan^2 t)} dt = \int_0^{\sqrt{3}} \frac{1}{9 + \tan^2 t} dt \\ &= \int_0^{\sqrt{3}} \frac{1}{9 + t^2} dt \end{aligned}$

do requires

(b) $\dots = \frac{1}{3} \left[\arctan \frac{t}{3} \right]_0^{\sqrt{3}} = \frac{1}{3} \left[\arctan \frac{\sqrt{3}}{3} - \arctan 0 \right] = \frac{1}{3} \times \frac{\pi}{6} = \frac{\pi}{18}$

Question 34 (***)+

By using the substitution $u = \cosh x - 1$, or otherwise, find the value of

$$\int_{\ln 2}^{\ln 3} \frac{\cosh x + 1}{\sinh x (\cosh x - 1)} dx.$$

$\frac{5}{2}$

$\int_{\ln 2}^{\ln 3} \frac{\cosh x + 1}{\sinh x (\cosh x - 1)} dx = \dots$ by substituting ...

$\begin{aligned} &= \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{\cosh x + 1}{\sinh x \times u} du = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{\cosh x + 1}{u \sinh x} du \\ &= \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{\cosh x + 1}{u (\cosh x - 1)} du = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{\sinh x}{u (\cosh x + 1)(\cosh x - 1)} du \\ &= \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{1}{u (\cosh x - 1)} du = \int_{\frac{1}{2}}^{\frac{2}{3}} \frac{1}{u^2} du \\ &= \left[-\frac{1}{u} \right]_{\frac{1}{2}}^{\frac{2}{3}} = \left[\frac{1}{u^2} \right]_{\frac{1}{2}}^{\frac{2}{3}} = 4 - \frac{3}{2} = \frac{5}{2} \end{aligned}$

Question 35 (***)+

$$f(x) = 5 \cosh x - 4 \sinh x, \quad x \in \mathbb{R}.$$

a) Find a simplified expression for $f(x)$ in terms of e^x .

b) Hence by using the substitution $u = e^x$, or otherwise, show that

$$\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{f(x)} dx = \frac{\pi}{18}.$$

$$f(x) = \frac{1}{2}e^x + \frac{9}{2}e^{-x}$$

$$\begin{aligned}
 \text{(a)} \quad & 5\cosh x - 4\sinh x = 5\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^x - e^{-x}}{2}\right) \\
 &= \frac{5}{2}e^x + \frac{5}{2}e^{-x} - 2e^x + 2e^{-x} \\
 &= \frac{1}{2}e^x + \frac{9}{2}e^{-x} = \frac{1}{2}(e^x + 9e^{-x}) \\
 \text{(b)} \quad & \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{f(x)} dx = \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{\frac{1}{2}(e^x + 9e^{-x})} dx = \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2}{e^x + 9e^{-x}} dx \\
 &\text{Let } u = e^x \quad \left\{ \begin{array}{l} x = \ln 3 \mapsto u = 3 \\ 2 = 2e^x \mapsto u = e^2 \end{array} \right. \\
 &\frac{du}{dx} = e^x \quad \left\{ \begin{array}{l} du = e^x dx \\ \frac{du}{dx} = e^x \\ du = e^x dx \end{array} \right. \\
 &dx = \frac{du}{e^x} \quad \left\{ \begin{array}{l} dx = \frac{du}{e^x} \\ dx = \frac{du}{u} \end{array} \right. \\
 &= \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2}{u + 9u^{-1}} \frac{du}{u} = \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2}{u^2 + 9} du \\
 &= \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2}{u^2 + 9} du = \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2}{u^2 + 3^2} du \\
 &= \frac{2}{3} \left[\arctan \frac{u}{3} \right]_{\frac{1}{2}\ln 3}^{\ln 3} \\
 &= \frac{2}{3} \left[\arctan \left(\frac{3}{\sqrt{3}} \right) - \arctan \left(\frac{1}{2\sqrt{3}} \right) \right] = \frac{2}{3} \left[\frac{\pi}{4} - \frac{1}{6} \right] = \frac{\pi}{18}
 \end{aligned}$$

Question 36 (***)+

By using the substitution $x^3 = 9 \sin^2 \theta$, or otherwise, find the exact value of

$$\int_0^1 \frac{\sqrt{x}}{\sqrt{9-x^3}} dx.$$

$$\frac{2}{3} \arcsin\left(\frac{1}{3}\right)$$

$$\begin{aligned}
 \int_0^1 \frac{\sqrt{x}}{\sqrt{9-x^3}} dx &= \int_0^1 \frac{\arcsin \frac{\sqrt{x}}{3}}{\sqrt{1-\left(\frac{\sqrt{x}}{3}\right)^2}} \times \frac{1}{6\sin^2 \theta} d\theta \\
 &= \int_0^1 \frac{\arcsin \frac{\sqrt{x}}{3}}{3\cos^2 \theta} \times \frac{1}{6\sin^2 \theta} d\theta = \int_0^1 \frac{\arcsin \frac{\sqrt{x}}{3}}{18\cos^2 \theta} d\theta \\
 &= \int_0^1 \frac{\arcsin \frac{\sqrt{x}}{3}}{2\sin^2 \theta} d\theta = \int_0^1 \frac{\arcsin \frac{1}{3}}{2\sin^2 \theta} d\theta = \left[\frac{1}{2} \arcsin \frac{1}{3} \right]_0^1 \\
 &= \frac{1}{3} \arcsin \frac{1}{3} - \arcsin 0 = \frac{1}{3} \arcsin \frac{1}{3}
 \end{aligned}$$

Question 37 (***)

$$f(x) \equiv (2x^2 - 1)\arcsin x + x\sqrt{1-x^2}, -1 \leq x \leq 1.$$

a) Find a simplified expression for $f'(x)$.

b) Hence find

$$\int_0^{\frac{\sqrt{2}}{2}} x \arcsin x \, dx.$$

, $f'(x) = 4x \arcsin x$, $\frac{1}{8}$

a) DIFFERENTIATE THE SUM OF THE TWO PRODUCTS

$$\begin{aligned} f(x) &= (2x^2 - 1)\arcsin x + 2x(1-x^2)^{\frac{1}{2}} \\ f'(x) &= 4x \arcsin x + (2x^2 - 1)x \cdot \frac{1}{(1-x^2)^{\frac{1}{2}}} + 1 \cdot (1-x^2)^{\frac{1}{2}} + 2x \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) \\ &= 4x \arcsin x + \frac{2x^2-1}{(1-x^2)^{\frac{1}{2}}} + (1-x^2)^{\frac{1}{2}} - \frac{x^2}{(1-x^2)^{\frac{1}{2}}} \\ &= 4x \arcsin x + \frac{2x^2-1+(1-x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}} \\ &= 4x \arcsin x + \frac{2x^2-1+1-2x^2}{(1-x^2)^{\frac{1}{2}}} \\ &= 4x \arcsin x \end{aligned}$$

b) USING PART (a)

$$\begin{aligned} \int_0^{\frac{\sqrt{2}}{2}} 2x \arcsin x \, dx &= \frac{1}{2} \int_0^{\frac{\sqrt{2}}{2}} 4x \arcsin x \, dx \\ &= \frac{1}{2} \left[(2x^2 - 1)\arcsin x + x\sqrt{1-x^2} \right]_0^{\frac{\sqrt{2}}{2}} \\ &= \frac{1}{2} \left[\left(\frac{1}{2} + \frac{1}{2} \right) - (0 - 0) \right] \\ &= \frac{1}{8} \end{aligned}$$

Question 38 (*)+**

By using the substitution $u = \sqrt{e^x - 1}$, or otherwise, find the exact value of

$$\int_0^{\ln 2} \sqrt{e^x - 1} \, dx.$$

$$\boxed{2 - \frac{\pi}{2}}$$

$$\begin{aligned}
 & \int_0^{\ln 2} \sqrt{e^x - 1} \, dx = \dots \text{ by substitution...} \\
 & = \int_0^1 u \left(\frac{2u}{u^2+1} \right) du = \int_0^1 \frac{2u^2}{u^2+1} \, du \\
 & \quad \text{BY LONG DIVISION OR MANIPULATION} \\
 & = \int_0^1 \frac{2(u^2)-2}{u^2+1} \, du = \int_0^1 2 - \frac{2}{u^2+1} \, du \\
 & = \left[2u - 2\arctan u \right]_0^1 = (2 - 2\arctan 1) - (0) \\
 & = 2 - \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 u &= \sqrt{e^x - 1} \\
 u^2 &= e^x - 1 \\
 2u \frac{du}{dx} &= e^x \\
 2u \frac{du}{dx} &= \frac{u^2}{u^2+1} \\
 du &= \frac{u}{u^2+1} \, dx \\
 \dots & \\
 x=0 &\rightarrow u=0 \\
 x=\ln 2 &\rightarrow u=1
 \end{aligned}$$

Question 39 (*)+**

Find in exact simplified form the value of

$$\int_0^{\ln 2} \frac{e^x}{\cosh x} \, dx.$$

$$\boxed{}, \boxed{\ln \frac{5}{2}}$$

$$\begin{aligned}
 & \text{REWRITE THE INTEGRAND IN TERMS OF EXPONENTIALS} \\
 & \int_0^{\ln 2} \frac{e^x}{\cosh x} \, dx = \int_0^{\ln 2} \frac{e^x}{\frac{1}{2}(e^x + e^{-x})} \, dx = \int_0^{\ln 2} \frac{2e^x}{e^x + e^{-x}} \, dx \\
 & \text{NOW BY SUBSTITUTION WE HAVE:} \\
 & \begin{array}{ll}
 u = e^x & \text{a} \\
 \frac{du}{dx} = e^x & \text{a} \\
 \frac{du}{dx} = u & \text{a} \\
 du = \frac{du}{u} & \text{a}
 \end{array}
 \quad \begin{array}{ll}
 x=0 \rightarrow u=1 & \text{a} \\
 x=\ln 2 \rightarrow u=2 & \text{a}
 \end{array} \\
 & \text{TRANSFORMING THE INTEGRAL} \\
 & \dots = \int_1^2 \frac{2u}{u+u^{-1}} \left(\frac{du}{u} \right) = \int_1^2 \frac{2}{u+\frac{1}{u}} \, du \\
 & = \int_1^2 \frac{2u}{u^2+1} \, du = \left[\ln(u^2+1) \right]^2 \\
 & = \ln 5 - \ln 2 = \boxed{\ln \frac{5}{2}}
 \end{aligned}$$

Question 40 (***)+

$$f(x) = \frac{2}{(x-1)^2(x^2+1)}, \quad x \neq 0.$$

Use partial fractions to show that

$$\int_2^3 f(x) \, dx = \frac{1}{2}(1 - \ln 2).$$

proof

$$\begin{aligned} f(x) &= \frac{2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} \\ \therefore 2 &\equiv A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1) \\ \bullet \text{ If } x=1 \Rightarrow 2 &= 2B \Rightarrow B=1 \\ \bullet \text{ If } x=0 \Rightarrow 2 &= A+B+C \quad \Rightarrow [A+B=1] \quad (\text{I}) \\ \bullet \text{ If } x=2 \Rightarrow 2 &= SA+SB+2C+D \\ &\Rightarrow 2 = SA+5+2C+D \quad \Rightarrow [SA+2C+D=-3] \quad (\text{II}) \\ \bullet \text{ If } x=-1 \Rightarrow 2 &= -A+2B+2C+D \\ &= -4A+2+4D-4C \\ &4A+4C-4D=0 \\ &A+C-D=0 \quad (\text{III}) \end{aligned}$$

(I) $\Rightarrow D=1-A$

(II): $\begin{cases} SA+2C+(1-A)=-3 \\ SA+A=3 \end{cases} \Rightarrow \begin{cases} 6A+2C=-4 \\ SA=-6 \end{cases} \Rightarrow \begin{cases} A=-1 \\ SA=-6 \end{cases} \Rightarrow \begin{cases} A=-1 \\ D=1+A \\ D=0 \end{cases}$

$$\begin{aligned} \therefore f(x) &= \frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{x}{x^2+1} \\ &\int_2^3 \left(\frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{x}{x^2+1} \right) dx \\ &= \left[-\frac{1}{x-1} - \ln|x-1| + \frac{1}{2}\ln(x^2+1) \right]_2^3 \\ &= \left[-\frac{1}{2} - \ln 2 + \frac{1}{2}\ln 10 \right] - \left[-1 - \ln 1 + \frac{1}{2}\ln 2 \right] \\ &= -\frac{1}{2} - \ln 2 + \frac{1}{2}\ln 10 + 1 - \frac{1}{2}\ln 2 \\ &= \frac{1}{2} - \ln 2 + \frac{1}{2}\ln 10 - \ln 5 \\ &= \frac{1}{2} - \ln 2 + \frac{1}{2} \\ &= \frac{1}{2}(1 - \ln 2) \end{aligned}$$

Question 41 (***)+

Use an appropriate substitution to find an exact value for the following integral.

$$\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{1-\sqrt{\arcsin x}}{\sqrt{1-x^2} \arcsin x} dx.$$

$$\boxed{\quad}, \boxed{\ln 2 - 2\left(\sqrt{\frac{1}{3}\pi} - \sqrt{\frac{1}{6}\pi}\right)}$$

BY SUBSTITUTION

$$\begin{aligned} \int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{1-u}{\sqrt{1-u^2} \arcsin x} du &= \dots \\ u &= \sqrt{\arcsin x} \\ u^2 &= \arcsin x \\ 2u \frac{du}{dx} &= \frac{1}{\sqrt{1-x^2}} \\ du &= \frac{1}{2u\sqrt{1-x^2}} dx \\ x &= \frac{u^2}{2} \rightarrow u = \sqrt{\frac{x}{2}} \\ x &= \frac{u^2}{4} \rightarrow u = \sqrt{\frac{x}{4}} \end{aligned}$$

BY TRANSFORMING THE INTEGRAL

$$\begin{aligned} &= \int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{5}}{2}} \frac{1-u}{\sqrt{1-u^2} \sqrt{u^2}} \left(2u\sqrt{1-x^2} du \right) \\ &= \int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{5}}{2}} \frac{2-2u}{u} du &= \int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{5}}{2}} \frac{2}{u} - 2 du \\ &= \left[2\ln|u| - 2u \right]_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{5}}{2}} &= \left(2\ln\frac{\sqrt{5}}{2} - 2\sqrt{\frac{5}{2}} \right) - \left(2\ln\frac{\sqrt{3}}{2} - 2\sqrt{\frac{3}{2}} \right) \\ &= \ln\frac{\sqrt{5}}{3} - 2\sqrt{\frac{5}{2}} - \ln\frac{\sqrt{3}}{2} + 2\sqrt{\frac{3}{2}} \\ &= \left(\ln\frac{\sqrt{5}}{3} + \ln\frac{\sqrt{6}}{2} \right) - 2\left[\sqrt{\frac{5}{2}} + \sqrt{\frac{3}{2}} \right] \\ &= \ln 2 - 2\left(\sqrt{\frac{5}{2}} - \sqrt{\frac{3}{2}} \right) \end{aligned}$$

Question 42 (*****)

Find the exact value of

$$\int_0^1 \frac{1-3x^3}{\sqrt{1-x^2}} dx.$$

$$\boxed{\frac{1}{2}(\pi - 4)}$$

$$\begin{aligned}
 \int_0^1 \frac{1-3x^3}{\sqrt{1-x^2}} dx &= \int_0^1 \frac{1}{\sqrt{1-x^2}} dx - \int_0^1 \frac{3x^3}{\sqrt{1-x^2}} dx \\
 &= \left[\arcsin x \right]_0^1 - \int_1^0 \frac{3x^3}{\sqrt{1-x^2}} \left(-\frac{dx}{x} \right) \\
 &= \left(\frac{\pi}{2} - 0 \right) + \int_1^0 3x^2 dx \\
 &= \frac{\pi}{2} + \int_1^0 3(1-u^2) du \\
 &= \frac{\pi}{2} + \int_1^0 3 - 3u^2 du \\
 &= \frac{\pi}{2} + \left[3u - u^3 \right]_1^0 \\
 &= \frac{\pi}{2} + (0) - (2) \\
 &= \frac{\pi}{2} - 2 = \frac{1}{2}(\pi - 4)
 \end{aligned}$$

SUBSTITUTION
 $u = \sqrt{1-x^2}$
 $u^2 = 1-x^2$
 $2u \frac{du}{dx} = -2x$
 $dx = -\frac{u}{x} du$
 $x=1 \quad u=0$
 $x=0 \quad u=1$

Question 43 (*****)

Use a suitable trigonometric substitution to find an integrated expression for

$$\int \frac{9}{(9-x^2)^{\frac{3}{2}}} dx.$$

$$\boxed{\frac{x}{\sqrt{9-x^2}} + C}$$

$$\begin{aligned}
 \int \frac{9}{(9-x^2)^{\frac{3}{2}}} dx &= \dots \text{ by substitution} \\
 &= \int \frac{9}{[9-(3\sin\theta)^2]^{\frac{3}{2}}} (3\cos\theta d\theta) = \int \frac{27\cos\theta}{(9-9\sin^2\theta)^{\frac{3}{2}}} d\theta \\
 &= \int \frac{27\cos\theta}{[9(1-\sin^2\theta)]^{\frac{3}{2}}} d\theta = \int \frac{27\cos\theta}{(9\cos^2\theta)^{\frac{3}{2}}} d\theta \\
 &= \int \frac{27\cos\theta}{27\cos^3\theta} d\theta = \int \frac{1}{\cos^2\theta} d\theta = \int \sec^2\theta d\theta \\
 &= \tan\theta + C = \frac{x}{\sqrt{9-x^2}} + C
 \end{aligned}$$

SUBSTITUTION
 $x = 3\sin\theta$
 $\frac{dx}{d\theta} = 3\cos\theta$
 $d\theta = \frac{dx}{3\cos\theta}$
 $\frac{2}{3} = \cos\theta$
 $\theta = \frac{1}{3}\pi$
 $\frac{3}{\sqrt{9-x^2}}$
 $\frac{3}{\sqrt{9-27\sin^2\theta}}$
 $\frac{3}{\sqrt{9(1-\sin^2\theta)}}$
 $\frac{3}{\sqrt{9\cos^2\theta}}$
 $\frac{3}{9\cos\theta}$
 $\frac{1}{3\cos\theta}$
 $\frac{1}{3}\sec\theta$
 $\sec^2\theta$

Question 44 (***)**

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x} dx.$$

, $\frac{\pi}{9}$

USING THE SUBSTITUTION (CONT.)

$$t = \tan\left(\frac{x}{2}\right) \Rightarrow \frac{dt}{dx} = \frac{1}{2} \sec^2\left(\frac{x}{2}\right)$$

$$\frac{dt}{dx} = \frac{1}{2} [1 + \tan^2\left(\frac{x}{2}\right)]$$

$$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$$

$$2 \frac{dt}{dx} = 1+t^2$$

$$dx = \frac{2}{1+t^2} dt$$

ALSO USING THE COSINE DOUBLE ANGLE IDENTITY

$$\rightarrow \cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) \quad \leftarrow \cos 2x = \cos^2 x - \sin^2 x$$

$$\rightarrow \cos x = \left(\frac{1}{1+t^2}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$$

$$\rightarrow \cos x = \frac{1-t^2}{1+t^2} - \frac{t^2}{1+t^2}$$

$$\rightarrow \cos x = \frac{1-2t^2}{1+t^2}$$

$$\rightarrow 5 + 4\cos x = 5 + \frac{4(1-2t^2)}{1+t^2}$$

$$= \frac{5+4t^2-8t^2}{1+t^2}$$

$$= \frac{5-4t^2}{1+t^2}$$

$\tan \frac{x}{2} = \frac{2t}{1+t^2}$
 (PYTHAGOREAN THEOREM: A HYPOTENUSE OF $\sqrt{5}$)
 $\cos \frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$
 $\sin \frac{x}{2} = \frac{2t}{\sqrt{1+t^2}}$

FINDING THE LIMITS IF $t = \tan\left(\frac{x}{2}\right)$

$$x=0 \quad \rightarrow \quad t=0$$

$$x=\frac{2\pi}{3} \quad \rightarrow \quad t=\sqrt{3}$$

TRANSFORMING THE INTEGRAL

$$\Rightarrow \int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x} dx = \int_0^{\sqrt{3}} \frac{1}{5+\frac{5-4t^2}{1+t^2}} \times \frac{2}{1+t^2} dt$$

$$= \int_0^{\sqrt{3}} \frac{2}{9t^2+4t+6} dt$$

THIS IS A STANDARD "ARCTAN TYPE" INTEGRAL

$$= \int_0^{\sqrt{3}} \frac{2}{t^2+4/9} dt$$

$$= \left[\frac{2}{3} \arctan\left(\frac{t}{\sqrt{9/4}}\right) \right]_0^{\sqrt{3}}$$

$$= \frac{2}{3} \left[\arctan\left(\frac{\sqrt{3}}{2}\right) - \arctan(0) \right]$$

$$= \frac{2}{3} \times \frac{\pi}{6}$$

$$= \frac{\pi}{9}$$

Question 45 (****)

Show that the exact value of the following integral

$$\int_0^1 \frac{x+3}{(x+1)(x^2+4x+5)} dx$$

is $\frac{1}{2}\ln 2$.

, **proof**

START WITH PARTIAL FRACTION ANTING THAT x^2+4x+5 IS IRREDUCIBLE

$$\frac{2x+3}{(2x+1)(x^2+4x+5)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+4x+5}$$

$$2x+3 = A(2x+4x+5) + (2x+1)(Bx+C)$$

$$2x+3 = (4x+B)x^2 + (4A+B+C)x + (5A+C)$$

$$\begin{aligned} A+B &= 0 & 4A+B+C &= 1 & 5A+C &= 3 \\ A &= -B & -4B+B+C &= 1 & \Rightarrow B &= -3 \\ -3B+C &= 1 & -3B+C &= 1 & C &= 3+5B \\ C &= 1+3B & C &= 1+3B & & \end{aligned}$$

$$\begin{aligned} 1+3B &= 3+5B \\ -2B &= 2 \\ B &= -1 \\ A &= 1 \\ C &= -2 \end{aligned}$$

RETURNING TO THE INTEGRAL

$$\begin{aligned} \int_0^1 \frac{1}{2x+1} + \frac{-2x-2}{x^2+4x+5} dx &= \int_0^1 \frac{1}{2x+1} - \frac{2x+2}{x^2+4x+5} dx \\ &= \int_0^1 \frac{1}{2x+1} - \frac{1}{2} \frac{2x+4+4}{x^2+4x+5} dx = \left[\ln|2x+1| - \frac{1}{2} \ln(x^2+4x+5) \right]_0^1 \\ &= (\ln 2 - \frac{1}{2} \ln 1) - (\ln 1 - \frac{1}{2} \ln 5) = \ln 2 - \frac{1}{2} \ln 10 + \frac{1}{2} \ln 5 \\ &= \frac{1}{2} [\ln 2 - \ln 10 + \ln 5] = \frac{1}{2} [\ln 5 - \ln 10 + \ln 2] \\ &= \frac{1}{2} \ln \left(\frac{\ln 2}{\ln 10} \right) = \frac{1}{2} \ln 2. \quad \text{As Required} \end{aligned}$$

Question 46 (***)**

Use the substitution $t = \tan\left(\frac{1}{2}x\right)$ to find an exact simplified value for

$$\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x} dx.$$

Any trigonometric identities to convert $\cos x$ in terms of t must be derived.

, $\boxed{\frac{2\pi\sqrt{3}}{9}}$

SIMPLIFY MANIPULATIONS & AUXILIARIES

- $t = \tan\left(\frac{1}{2}x\right)$
- $\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{1}{2}x\right)$
- $\frac{dt}{dx} = \frac{1}{2}\left[1 + \tan^2\left(\frac{1}{2}x\right)\right]$
- $\frac{dt}{dx} = \frac{1}{2}\left[1 + t^2\right]$
- $\frac{dt}{dx} = \frac{2}{1+t^2}$
- $dx = \frac{2}{1+t^2} dt$
- $x = \frac{\pi}{2} \mapsto t=1$
- $x=0 \mapsto t=0$
- $t = \tan\left(\frac{1}{2}x\right)$
- $t^2 = \tan^2\left(\frac{1}{2}x\right)$
- $1+t^2 = 1+\tan^2\left(\frac{1}{2}x\right)$
- $1+t^2 = \sec^2\left(\frac{1}{2}x\right)$
- $\frac{1}{1+t^2} = \cos^2\left(\frac{1}{2}x\right)$
- $\frac{2}{1+t^2} = 2\cos^2\left(\frac{1}{2}x\right)$
- $\frac{2-(1-t^2)}{1+t^2} = 2\cos^2\left(\frac{1}{2}x\right)-1$
- $\frac{2(1-t^2)}{1+t^2} = \cos\left(2\frac{1}{2}x\right)$
- $\frac{1-t^2}{1+t^2} = \cos x$

MANIPULATING THE INTEGRAL

$$\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x} dx = \int_0^1 \frac{1}{2 - \frac{1-t^2}{1+t^2} \times \frac{2}{1+t^2}} dt$$

$$= \int_0^1 \frac{2}{2(1+t^2) - (1-t^2)} dt$$

$$= \int_0^1 \frac{2}{1+3t^2} dt$$

MANIPULATE INTO A STANDARD ARCTAN FORM

$$= \frac{2}{3} \int_0^1 \frac{1}{t^2 + \left(\frac{1}{\sqrt{3}}\right)^2} dt$$

$$= \frac{2}{3} \times \frac{1}{\frac{1}{\sqrt{3}}} \left[\arctan\left(\frac{1}{\sqrt{3}}t\right) \right]_0^1$$

$$= \frac{2\sqrt{3}}{3} \left[\arctan\left(\sqrt{3}t\right) \right]_0^1$$

$$= \frac{2\sqrt{3}}{3} \left[\arctan\sqrt{3} - \arctan 0 \right]$$

$$= \frac{2\sqrt{3}}{3} \times \frac{\pi}{3}$$

$$= \frac{2\pi\sqrt{3}}{9}$$

Question 47 (****)

$$I = \int \frac{18}{3\cos^2 x + \sin^2 x} dx.$$

- a) By using the substitution $t = \tan x$, or otherwise, show clearly that

$$I = 6\sqrt{3} \arctan\left(\frac{\sqrt{3}}{3} \tan x\right) + \text{constant}.$$

- b) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \frac{18}{3\cos^2 x + \sin^2 x} dx$.

$$\boxed{\pi\sqrt{3}}$$

a)

$$\begin{aligned} \int \frac{18}{3\cos^2 x + \sin^2 x} dx &= \int \frac{\frac{18}{\cos^2 x}}{3\cos^2 x + \sin^2 x} dx & t = \tan x \\ &= \int \frac{18\sec^2 x}{3 + \tan^2 x} dx = \dots \text{ by substitution} & \frac{dt}{dx} = \sec^2 x \\ &= \int \frac{18\sec^2 x}{3 + t^2} \frac{dt}{\sec^2 x} = \int \frac{18}{3+t^2} dt = \int \frac{18}{t^2+3} dt \\ &= \frac{18}{\sqrt{3}} \arctan\frac{t}{\sqrt{3}} + C = \frac{18}{\sqrt{3}} \arctan\left(\frac{\sqrt{3}}{3} \tan x\right) + C \\ &= 6\sqrt{3} \arctan\left(\frac{\sqrt{3}}{3} \tan x\right) + C \end{aligned}$$

b)

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{18}{3\cos^2 x + \sin^2 x} dx &= \left[6\sqrt{3} \arctan\left(\frac{\sqrt{3}}{3} \tan x\right) \right]_0^{\frac{\pi}{4}} \\ &= 6\sqrt{3} \left[\arctan\frac{\sqrt{3}}{3} - \arctan 0 \right] = 6\sqrt{3} \times \frac{\pi}{6} = \boxed{2\pi\sqrt{3}} \end{aligned}$$

Question 48 (****)

By using the substitution $u = \sqrt{x}$, or otherwise, find an exact value for

$$\int_0^1 \frac{\sqrt{x}}{x+1} dx.$$

$$2 - \frac{\pi}{2}$$

$$\begin{aligned} & \int_0^1 \frac{\sqrt{x}}{x+1} dx = \dots \text{by substitution} \\ &= \int_0^1 \frac{u}{u^2+1} (2u du) = \int_0^1 \frac{2u^2}{u^2+1} du = \int_0^1 \frac{2(2u^2)-2}{u^2+1} du \\ &= \int_0^1 2 - \frac{2}{u^2+1} du = [2u - 2\arctan u]_0^1 \\ &= (2 - 2\arctan \frac{1}{2}) - (0 - 0) = 2 - \frac{\pi}{2} \quad \text{✓} \end{aligned}$$

NOTE THIS CAN ALSO BE DONE BY THE SUBSTITUTION $x = \tan^2 \theta$

$$\begin{aligned} & \int_0^1 \frac{\sqrt{x}}{x+1} dx = \dots \int_0^{\frac{\pi}{4}} \frac{\tan^2 \theta}{\sec^2 \theta + 1} (2\sec^2 \theta d\theta) = \int_0^{\frac{\pi}{4}} \frac{2\tan^2 \theta}{\sec^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} 2\tan^2 \theta d\theta = \int_0^{\frac{\pi}{4}} 2(\sec^2 \theta - 1) d\theta = \int_0^{\frac{\pi}{4}} 2\sec^2 \theta - 2 d\theta \\ &= [2\tan \theta - 2\theta]_0^{\frac{\pi}{4}} = (2 - 2\arctan \frac{1}{2}) - (0 - 0) = 2 - \frac{\pi}{2} \end{aligned}$$

$\begin{array}{l} u = \tan^2 \theta \\ u^2 = \tan^2 \theta \\ 2u du = 2\sec^2 \theta d\theta \\ du = \sec^2 \theta d\theta \\ \theta = \arctan u \\ \theta = 0 \Rightarrow \theta = 0 \\ \theta = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4} \end{array}$

Question 49 (****)

By using the substitution $u = \sqrt{3 - \sec^2 x}$, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{3 - \sec^2 x}} dx.$$

$$\frac{\pi}{4}$$

$$\begin{aligned} & \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{3 - \sec^2 x}} dx \\ &= \int_0^1 \frac{\sec^2 x}{\sqrt{3 - \sec^2 x}} \left(-\frac{u}{\sec x \tan x} du \right) \quad \text{sec x tan x} \\ &= \int_1^{\sqrt{2}} \frac{1}{\sqrt{3-u^2}} du = \int_1^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}} du \\ &= \int_1^{\sqrt{2}} \frac{1}{\sqrt{(1-u^2)^2 - u^2}} du = \left[\arccos \frac{u}{\sqrt{2}} \right]_1^{\sqrt{2}} \\ &\approx 0.85 \sin 1 - 0.85 \sin \frac{1}{\sqrt{2}} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

$\begin{array}{l} u = \sqrt{3 - \sec^2 x} \\ u^2 = 3 - \sec^2 x \\ \sec^2 x = 3 - u^2 \\ 2\sec^2 x du = -2u du \\ du = -\frac{u}{\sec x \tan x} du \\ \sec x \tan x = \frac{u}{\cos x} \cdot \frac{\sin x}{\cos x} = \frac{\sin x}{\cos^2 x} = \frac{u}{\cos^2 x} \\ \cos x = \sqrt{1 - \frac{u^2}{3}} \\ u = \sqrt{3 - (1 + \tan^2 x)} \\ u = \sqrt{2 - \tan^2 x} \\ u^2 = 2 - \tan^2 x \\ \tan^2 x = 2 - u^2 \\ \tan x = \sqrt{2 - u^2} \end{array}$

Question 50 (****)

By using a suitable trigonometric substitution, show clearly that

$$\int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}} dx = \pi - 2.$$

[proof]

$$\begin{aligned}
 & \int_0^{\frac{1}{2}} \sqrt{\frac{16x}{1-x}} dx = \text{by substitution} \\
 &= \int_0^{\frac{\pi}{4}} \sqrt{\frac{16 \sin^2 \theta}{1-\sin^2 \theta}} (2\cos \theta d\theta) \\
 &= \int_0^{\frac{\pi}{4}} \sqrt{\frac{16 \sin^2 \theta}{\cos^2 \theta}} (2\sin \theta \cos \theta d\theta) \\
 &= \int_0^{\frac{\pi}{4}} \frac{4\sin \theta}{\cos \theta} (2\sin \theta \cos \theta d\theta) = \int_0^{\frac{\pi}{4}} 8\sin^2 \theta d\theta = \left[8\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta \right) \right]_0^{\frac{\pi}{4}} \\
 &= \int_0^{\frac{\pi}{4}} 4 - 4\cos 2\theta d\theta = [4\theta - 2\sin 2\theta]_0^{\frac{\pi}{4}} = (\pi - 2) - (0 - 0) = \pi - 2.
 \end{aligned}$$

Question 51 (****)

By using the substitution $u = \tan x$, or otherwise, show clearly that

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x + 25\sin^2 x} dx = \frac{1}{5} \arctan 5.$$

[proof]

$$\begin{aligned}
 & \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x + 25\sin^2 x} dx = \dots \text{ by substitution} \\
 &= \int_0^1 \frac{1}{u^2 + 25} \frac{du}{dx} dx = \int_0^1 \frac{1}{1+25u^2} du \\
 &= \int_0^1 \frac{1}{1+25u^2} du = \int_0^1 \frac{1}{1+25u^2} \frac{du}{dx} dx = \frac{1}{25} \int_0^1 \frac{du}{1+\frac{u^2}{25}} \\
 &= \frac{1}{25} \int_{\frac{1}{\sqrt{5}}}^{\frac{1}{\sqrt{5}}+1} \frac{1}{1+w^2} dw = \frac{1}{25} \left[\frac{1}{w} \arctan w \right]_{\frac{1}{\sqrt{5}}}^{\frac{1}{\sqrt{5}}+1} = \frac{1}{25} \times 5 \left[\arctan \frac{1}{\sqrt{5}} \right] \\
 &= \frac{1}{5} [\arctan 5 - \arctan 1] = \frac{1}{5} \arctan 5 \quad \text{As } \arctan 0 = 0
 \end{aligned}$$

Question 52 (*)**

By using the substitution $x = \cosh^2 u$, or otherwise, show that

$$\int \sqrt{\frac{x}{x-1}} dx = \ln(\sqrt{x} + \sqrt{x-1}) + \sqrt{x^2 - x} + \text{constant}.$$

[proof]

The handwritten proof shows the following steps:

$$\begin{aligned} & \int \sqrt{\frac{x}{x-1}} dx = \int \sqrt{\cosh^2 u / (\cosh^2 u - 1)} 2\cosh u \sinh u du \\ &= \int \frac{\cosh u}{\sinh u} 2\cosh u \sinh u du = \int 2\cosh^2 u du \\ &= \int 2(1 + \tanh^2 u) du = \int 1 + \tanh^2 u du \\ &= u + \frac{1}{2} \sinh 2u + C = u + \sinh u \cosh u + C \\ &= \cosh u \sinh u + \sqrt{2\cosh^2 u - 1} + C \\ &= \cosh u \sinh u + \sqrt{x^2 - x} + C \\ &= \ln(\sqrt{x} + \sqrt{x-1}) + \sqrt{x^2 - x} + C \end{aligned}$$

Notes on the right side of the box:

- $x = \cosh^2 u$
- $du = 2\cosh u \sinh u du$
- $\cosh u = \sqrt{x}$
- $\sinh u = \sqrt{x-1}$
- $\sinh^2 u = \cosh^2 u - 1$
- $\cosh^2 u = 1 + \tanh^2 u$
- $\cosh u \sinh u = \sqrt{2\cosh^2 u - 1}$

Question 53 (***)

$$\sin 2x \equiv \frac{2 \tan x}{1 + \tan^2 x}$$

- a) Prove the validity of the above trigonometric identity

b) Express $\frac{8}{(3t+1)(t+3)}$ into partial fractions.

c) Hence use the substitution $t = \tan x$ to show that

$$\int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x} dx = \ln 3$$

$$\frac{8}{(3t+1)(t+3)} = \frac{3}{3t+1} - \frac{1}{t+3}$$

(4) $\int_{\ln 5}^{\ln 15} \frac{dt}{1+t^2} = \frac{2t}{\ln 5 - \ln 3} = \frac{2\ln 15 - 2\ln 5}{\ln 5 - \ln 3} = \frac{2\ln 3}{\ln 5 - \ln 3}$

(5) $\int_{A(4t+3)}^{B(4t+3)} \frac{dt}{(3t+1)(t+3)} = \frac{B}{3t+1} + \frac{A}{t+3}$
 $[B \equiv A(4t+3) + B(3t+1)]$

If $t=0$, $B=B$ $\Rightarrow B=-1$
If $t=\frac{1}{3}$, $B=\frac{B}{3} \Rightarrow A=3$ $\therefore \int_{A(4t+3)}^{B(4t+3)} \frac{dt}{(3t+1)(t+3)} = \frac{3}{3t+1} - \frac{1}{t+3}$

(6) $\int_0^{\frac{\pi}{4}} \frac{B}{3+5\sin 2x} dx = \dots$ BY SUBSTITUTION
 $= \int_0^1 \frac{B}{3+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{dt}{1+t^2} = \frac{B}{3+5\frac{1-t^2}{1+t^2}}$
 $= \int_0^1 \frac{B}{3+5\frac{1-(1-t^2)}{1+t^2}} \times \frac{dt}{1+t^2} = \int_0^1 \frac{B}{3(1+t^2)+5(1-t^2)} dt$
 $= \int_0^1 \frac{B}{3(1+t^2)+5(1-t^2)} dt = \int_0^1 \frac{B}{3t^2+5-5t^2+3} dt = \int_0^1 \frac{B}{8-2t^2} dt$
 $= \int_0^1 \frac{B}{8(1-\frac{t^2}{4})} dt = \int_0^1 \frac{B}{8(1-\frac{t^2}{4})} \cdot \frac{dt}{1+\frac{t^2}{4}} = \left[B \ln \left| \frac{1+\frac{t^2}{4}}{1} \right| \right]_0^1 = \left[B \ln \left| \frac{5}{4} \right| \right]_0^1 = B \ln \frac{5}{4}$
 $= \left(\frac{1}{4} - \frac{1}{4} \right) \cdot \left(\ln 5 - \ln 3 \right) = \ln \frac{5}{3}$ as required

Question 54 (****)

$$\frac{2t}{(t+1)(t^2+1)} \equiv \frac{A}{t+1} + \frac{Bt+C}{t^2+1}.$$

a) Determine the values of A , B and C in the above identity.

b) Hence find an value for

$$\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\cos x}{1+\sin x}} dx.$$

 , A = -1, B = 1, C = 1, $\frac{\sqrt{2}}{2}(\pi - 2\ln 2)$

(a) $\frac{2t}{(t+1)(t^2+1)} \equiv \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$

$2t \equiv A(t^2+1) + (t+1)(Bt+C)$

- If $t=1$, $2=2A+2(B+C)$
- If $t=0$, $0=A+C$
- If $t=-1$, $-2=A-B-C$

$A=1$, $C=1$, $B=-2$

$I = A + B + C$
 $I = -1 + B + C$
 $I = -1 + (-2) + 1$
 $I = -2$

(b)
$$\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\cos x}{1+\sin x}} dx$$

$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1-t^2)-(1-t^2)}{(t+1)^2+t^2}} \times \frac{2}{1+t^2} dt$ (using $\cos x = 1 - 2\sin^2 x$ and $\sin x = \frac{t}{1+t^2}$)

$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{2t^2}{(t+1)^2+2t^2}} \times \frac{2}{1+t^2} dt = \int_0^{\frac{\pi}{2}} \sqrt{\frac{2t}{t^2+2t+1}} \times \frac{2}{1+t^2} dt$

$= \sqrt{2} \int_0^{\frac{\pi}{2}} \sqrt{\frac{t^2}{(t+1)^2}} \times \frac{2}{1+t^2} dt = \sqrt{2} \int_0^{\frac{\pi}{2}} \frac{t}{(t+1)^2} dt$

$= \sqrt{2} \int_0^{\frac{\pi}{2}} \left(-\frac{1}{t+1} + \frac{t+1}{t+1} \right) dt$

$= \sqrt{2} \left[\frac{t}{t+1} + \ln|t+1| - \ln|t+1| \right]_0^{\frac{\pi}{2}}$

$= \sqrt{2} \left[\left(\frac{\pi}{2} + \frac{\pi}{4} - \ln 2 \right) - 0 \right]$

$= \sqrt{2} \left[\frac{3\pi}{4} - \frac{1}{2}\ln 2 \right]$

$= \frac{\sqrt{2}}{2} \left[\pi - 2\ln 2 \right]$

Question 55 (**)**

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} dx.$$

[1]

DOING THE SUBSTITUTION GIVEN

$$\begin{aligned} t &= \tan \frac{x}{2} \\ \frac{dt}{dx} &= \frac{1}{2} \sec^2 \frac{x}{2} \\ \frac{dt}{dx} &= \frac{1}{2}(1 + \tan^2 \frac{x}{2}) \\ \frac{dt}{dx} &= \frac{1}{2}(1+t^2) \\ dt &= \frac{dt}{dx} dx \\ dt &= \frac{1}{2(1+t^2)} dt \end{aligned}$$

CHANGING THE LIMITS

$$\begin{aligned} x=0 &\mapsto t=0 \\ x=\frac{\pi}{2} &\mapsto t=1 \end{aligned}$$

REWRITING THE INTEGRAL IN TERMS OF t , BY ANY MEANS

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} dx &= \dots \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2} dt = \int_0^1 \frac{2}{1+2t+t^2} dt \\ &= \int_0^1 \frac{2}{(t+1)^2} dt = \int_0^1 \frac{2}{(t+1)^2} dt \\ &= \left[-\frac{2}{t+1} \right]_0^1 = \left[\frac{2}{t+1} \right]_0^1 = 2-1 = 1 \end{aligned}$$

Question 56 (**)**

Use suitable substitution to find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{4-\sin^4 x}} dx.$$

$\boxed{\frac{\pi}{6}}$

SUBSTITUTION...

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{4-\sin^4 x}} dx &= \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{4-\sin^2 u}} du \\ &= \int_0^1 \frac{\sin 2u}{\sqrt{4-u^2}} du = \int_0^1 \frac{1}{\sqrt{4-u^2}} du \\ &= \left[\arcsin \frac{u}{2} \right]_0^1 = \arcsin \frac{1}{2} - \arcsin 0 \\ &= \frac{\pi}{6} \end{aligned}$$

LET'S TRY IT

$M = \sin^2 x$
 $\frac{dM}{dx} = 2 \sin x \cos x$
 $du = \sin x dx$
 $dx = \frac{du}{\sin x}$
 $2=0 \quad u=0$
 $2=\frac{\pi}{2} \quad u=1$

Question 57 (****)

$$I = \int \sqrt{\frac{x}{1-x}} dx$$

- a) Use the substitution $\sqrt{x} = \sin \theta$ to show that

$$I = \int 2\sin^2 \theta d\theta.$$

- b) Hence show further that

$$I = \arcsin \sqrt{x} - \sqrt{x-x^2} + \text{constant}$$

proof

$\begin{aligned} \text{(a)} \quad & \int \sqrt{\frac{x}{1-x}} dx = \int \frac{\sqrt{x}}{\sqrt{1-x}} dx = \dots \text{by the substitution} \\ & = \int \frac{\sin \theta}{\sqrt{1-\sin^2 \theta}} (2\sin \theta \cos \theta) d\theta = \int \frac{2\sin^2 \theta \cos \theta}{\cos^2 \theta} d\theta \\ & = \int 2\sin^2 \theta d\theta \quad \text{as } \cancel{\cos \theta} \\ & = \dots \text{evaluate} \end{aligned}$	$\begin{aligned} \sqrt{x} &= \sin \theta \\ x &= \sin^2 \theta \\ \frac{dx}{d\theta} &= 2\sin \theta \cos \theta \\ dx &= 2\sin \theta \cos \theta d\theta \\ \theta &= \arcsin \sqrt{x} \end{aligned}$
$\begin{aligned} \text{(b)} \quad & \int 2\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta = \int 1 - \cos 2\theta d\theta \\ & = \theta - \frac{1}{2}\sin 2\theta + C = \theta - \frac{1}{2}(2\sin \theta \cos \theta) + C \\ & = \theta - \sin \theta \cos \theta + C = \arcsin \sqrt{x} - \sqrt{x-x^2} + C \\ & = \arcsin \sqrt{x} - \sqrt{x-x^2} + C \quad \text{as } \cancel{\cos \theta} \end{aligned}$	<p>BY PYTHAGORAS $\therefore \cos \theta = \sqrt{1-x^2}$</p>

Question 58 (**)**

The curve with the following equation is defined in the largest real domain.

$$y = (4x-3)\sqrt{-8(2x^2-3x+1)} + \arcsin(4x-3).$$

- a) Show that

$$\frac{dy}{dx} = k\sqrt{-2x^2+3x-1},$$

where k is an exact constant to be found.

- b) Hence find the exact value of the following integral.

$$\int_{\frac{1}{2}}^1 \sqrt{-2x^2+3x-1} dx.$$

$$\boxed{\quad}, \boxed{k=16\sqrt{2}}, \boxed{\frac{\pi}{16\sqrt{2}}}$$

a)

$$\begin{aligned} y &= \arcsin(4x-3) + (4x-3)\left[-8(2x^2-3x+1)\right]^{\frac{1}{2}} \\ \rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1-(4x-3)^2}} \times 4 + 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} \cdot (4x-3) + \frac{1}{2}\left[8(-2x^2+3x-1)\right]^{-\frac{1}{2}}(2x-3) \\ \rightarrow \frac{dy}{dx} &= \frac{4}{\sqrt{1-(4x-3)^2}} + 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} - (4x-3)\left[8(-2x^2+3x-1)\right]^{-\frac{1}{2}} \\ \rightarrow \frac{dy}{dx} &= \frac{4}{\sqrt{1-(4x-3)^2}} + 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} - \frac{4(4x-3)}{\sqrt{8(-2x^2+3x-1)}} \\ \rightarrow \frac{dy}{dx} &= \frac{4}{\sqrt{1-16x^2+32x-9}} + 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} - 4(4x-3)\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} \\ \rightarrow \frac{dy}{dx} &= 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} + 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} - 4(4x-3)\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} \\ \rightarrow \frac{dy}{dx} &= 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} + 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} - 4(4x-3)\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} \\ \rightarrow \frac{dy}{dx} &= 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} \left[1 + \left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} - (4x-3) \right] \\ \rightarrow \frac{dy}{dx} &= 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} \left[1 - 16x^2 + 24x - 8 - (16x^2 - 24x + 9) \right] \\ \rightarrow \frac{dy}{dx} &= 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} \left[-32x^2 + 48x - 16 \right] \\ \rightarrow \frac{dy}{dx} &= 4\left[8(-2x^2+3x-1)\right]^{\frac{1}{2}} \left[16(-2x^2+3x-1) \right] \\ \rightarrow \frac{dy}{dx} &= 4 \times 8^{\frac{1}{2}} \times 16 \times (-2x^2+3x-1)^{\frac{1}{2}} (-2x^2+3x-1) \end{aligned}$$

b)

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= 4 \times \frac{1}{\sqrt{8}} \times 16 \times (-2x^2+3x-1)^{\frac{1}{2}} \\ \rightarrow \frac{dy}{dx} &= \frac{4\sqrt{8}}{8} (-2x^2+3x-1)^{\frac{1}{2}} \\ \rightarrow \frac{dy}{dx} &= 16\sqrt{2} (-2x^2+3x-1)^{\frac{1}{2}} \quad \text{Hence } k = 16\sqrt{2} \\ \text{USING PART (a)} \quad \int_{\frac{1}{2}}^1 \sqrt{-2x^2+3x-1} dx &= \frac{1}{16\sqrt{2}} \int_{\frac{1}{2}}^1 4\sqrt{2} (-2x^2+3x-1)^{\frac{1}{2}} dx \\ \text{Thus we have} \quad &= \frac{1}{16\sqrt{2}} \left[\arcsin(4x-3) + (4x-3)\left[-8(-2x^2+3x-1)\right]^{\frac{1}{2}} \right]_0^1 \\ &= \frac{1}{16\sqrt{2}} \left[\arcsin(1) + \sqrt{8(-2+3-1)} \right] - \left[\arcsin(0) - \sqrt{8(-\frac{1}{2}+3-\frac{9}{4})} \right] \\ &= \frac{1}{16\sqrt{2}} \left\{ \frac{\pi}{2} + \frac{\pi}{2} \right\} \\ &= \frac{\pi}{16\sqrt{2}} \end{aligned}$$

Question 59 (****)

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{3\sqrt{3}}{2-\cos x} dx.$$

[2π]

USING THE SUBSTITUTION FORM $\rightarrow t = \tan\frac{x}{2}$ $\rightarrow \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ $\Rightarrow dt = \frac{1}{2}(1+t^2)dx$ $\Rightarrow \frac{dt}{dx} = \frac{1}{2}(1+t^2)$ $\Rightarrow dx = \frac{2}{2(1+t^2)}dt$ $\Rightarrow dx = \frac{1}{1+t^2}dt$ CHANGE THE UNITS $x=0 \rightarrow t=0$ $x=\frac{\pi}{2} \rightarrow t=1$	GETTING AN EXPRESSION FOR $\cos x$ IN TERMS OF t $\cos x = 2\cot^2\frac{x}{2} - 1$ $\cos x = \frac{2}{\sec^2\frac{x}{2}} - 1$ $\cos x = \frac{2}{1+4t^2} - 1$ $\cos x = \frac{2-1-4t^2}{1+4t^2} - 1$ $\cos x = \frac{1-4t^2}{1+4t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$
TRANSFORMING THE GIVEN INTEGRAL USING THE ABOVE PRODUCTS $\int_0^{\frac{\pi}{2}} \frac{3\sqrt{3}}{2-\cos x} dx = \dots = \int_0^1 \frac{3\sqrt{3}}{2-\frac{1-4t^2}{1+t^2}} \times \frac{2}{1+t^2} dt$ $= \int_0^1 \frac{6\sqrt{3}}{2(1+t^2)-(1-4t^2)} dt = \int_0^1 \frac{6\sqrt{3}}{1+3t^2} dt = \int_0^1 \frac{2\sqrt{3}}{1+\left(\frac{t}{\sqrt{3}}\right)^2} dt$ <p style="text-align: center;"><small>THIS IS A "STANDARD ARCTAN" INTEGRAL</small></p> $= \left[2\sqrt{3} \arctan\left(\frac{t}{\sqrt{3}}\right) \right]_0^1 = \left[6 \arctan\left(\frac{t}{\sqrt{3}}\right) \right]_0^1$ $= 6\arctan\left(\sqrt{3}\right) - 6\arctan(0) = 6 \times \frac{\pi}{3} = 2\pi$	

Question 60 (****)

Use the substitution $u = \frac{1}{x}$ to find

$$\int \frac{1}{x\sqrt{3x^2+2x-1}} dx.$$

$$-\arcsin\left(\frac{1-x}{2x}\right) + C$$

$$\begin{aligned}
 & \int \frac{1}{x\sqrt{3x^2+2x-1}} dx = \dots \text{by substitution} \dots \\
 &= \int \frac{1}{\frac{1}{u} \cdot \left(\frac{1}{u^2} \cdot \frac{-2}{u} - 1\right)^{\frac{1}{2}}} \left(-\frac{1}{u^2} du\right) = \int \frac{u}{(3+2u-u^2)^{\frac{1}{2}}} \left(-\frac{1}{u^2} du\right) \\
 &= -\int \frac{u}{(3+2u-u^2)^{\frac{1}{2}}} \left(\frac{1}{u^2} du\right) = -\int \frac{u^{1/2}}{(3+2u-u^2)^{\frac{1}{2}}} \left(\frac{1}{u^2} du\right) \\
 &= -\int \frac{1}{(u^2-2u-3)^{\frac{1}{2}}} du = -\int \frac{1}{(u-1)^2-4^2} du = -\int \frac{1}{(4-(u-1)^2)} du \\
 &\quad \text{let } u-1 = 2\sin\theta \quad \dots = -\int \frac{1}{4\cos^2\theta} d\theta = -\arcsin\frac{u-1}{4} + C \\
 &\quad \left\{ \begin{array}{l} \frac{du}{d\theta} = 2\cos\theta \\ \frac{d\theta}{du} = \frac{1}{2\cos\theta} \end{array} \right. \\
 &= -\arcsin\left(\frac{u-1}{4}\right) + C = -\arcsin\left(\frac{\frac{1}{x}-1}{4}\right) + C \\
 &= -\arcsin\left(\frac{1-x}{4x}\right) + C
 \end{aligned}$$

Question 61 (***)+

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the value of

$$\int_0^{\frac{\pi}{2}} \frac{1}{5 + 3\sin x + 4\cos x} dx.$$

All relevant results used in this evaluation must be carefully derived.

, $\boxed{\frac{1}{6}}$

USING THE SUBSTITUTION GIVEN

$\bullet t = \tan\frac{x}{2}$	$\bullet \cos x = \frac{2t^2+1}{t^2+2}$	$\bullet \sin x = 2\frac{t^2+1}{t^2+2}$
$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$	$= \frac{2}{t^2+2} - 1$	$\sec x = \frac{2\sqrt{t^2+1}}{t^2+2}$
$\frac{dt}{dx} = \frac{1}{2}(1 + \tan^2\frac{x}{2})$	$= \frac{2}{1+t^2} - 1$	$\sin x = 2t\frac{\sqrt{t^2+1}}{t^2+2}$
$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$	$= \frac{1-t^2}{1+t^2} - 1$	$\tan x = 2t\left(\frac{1}{\sqrt{t^2+1}}\right)$
$\frac{dt}{dx} = \frac{2}{(1+t^2)}$	$= \frac{-t^2}{1+t^2}$	$\csc x = \frac{2}{1+t^2}$
$dx = \frac{dt}{1+t^2}$	$= \frac{-t^2}{1+t^2}$	$\sec x = \frac{2t}{1+t^2}$

NOW TRANSFORM THE INTEGRAL USING THE CHANGE OF LIMITS

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{1}{5 + 3\sin x + 4\cos x} dx &= \int_0^{\frac{\pi}{2}} \frac{1}{5 + 3\left(2\frac{t^2+1}{t^2+2}\right) + 4\left(\frac{2t^2+1}{t^2+2}\right)} \left(\frac{2}{1+t^2}\right) dt \\ &= \int_0^1 \frac{1}{5 + \frac{5t^2}{1+t^2} + \frac{4-4t^2}{1+t^2}} \left(\frac{2}{1+t^2}\right) dt = \int_0^1 \frac{2}{5(t^2+1) + 6t^2 + 4t^2} dt \\ &= \int_0^1 \frac{2}{11t^2 + 6t^2 + 5} dt = \int_0^1 \frac{2}{(11t^2 + 5)^{1/2}} dt \\ &= \left[-\frac{2}{t\sqrt{11t^2 + 5}} \right]_0^1 = \left[\frac{2}{t+1} \right]_1^0 = -\frac{2}{3} - \frac{1}{2} = \frac{1}{6} // \end{aligned}$$

Question 62 (****+)

Use the substitution $t = \tan\left(\frac{1}{2}x\right)$ to find the exact value for the integral

$$\int_0^{\frac{1}{2}\pi} \frac{2}{1 + \sin x + 2 \cos x} dx$$

All relevant results used in this evaluation must be carefully derived.

, $\ln 3$

SOLVE BY DRAWING INFORMATION BASED ON THE GIVEN SUBSTITUTION

$\bullet t = \tan\left(\frac{1}{2}x\right)$
 $\bullet \sin x = \frac{2t\sqrt{1+t^2}}{1+t^2} = \frac{2\sin\frac{1}{2}x \cos\frac{1}{2}x}{\cos^2\frac{1}{2}x}$
 $\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{1}{2}x\right)$
 $\frac{dx}{dt} = \frac{2}{1+t^2}$
 $\frac{dt}{dx} = \frac{1}{2}(1+t^2)$
 $\bullet \cos x = \frac{1-t^2}{1+t^2} = \frac{2}{\sec^2\frac{1}{2}x} - 1$
 $\frac{dt}{dx} = \frac{2}{1+t^2}$
 $\frac{dx}{dt} = \frac{1-t^2}{1+t^2}$
 $\frac{dx}{dt} = \frac{2(1+t^2)}{1+t^2}$

SWAP THE LIMITS

$x=0 \mapsto t=0$
 $x=\frac{\pi}{2} \mapsto t=1$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{\pi}{2}} \frac{2}{1 + \sin x + 2 \cos x} dx = \int_0^1 \frac{2}{1 + \frac{2t}{1+t^2} + 2\frac{1-t^2}{1+t^2}} \left(\frac{2}{1+t^2} dt\right)$$

$$= \int_0^1 \frac{4}{1+t^2 + 2t + 2(1-t^2)} dt$$

$$= \int_0^1 \frac{4}{1+t^2 + 2t + 2 - 2t^2} dt$$

$$= \int_0^1 \frac{4}{-t^2 + 2t + 3} dt$$

$$= \int_1^0 \frac{4}{t^2 - 2t - 3} dt$$

PROCEED BY PARTIAL FRACTION (BY INSPECTION)

$$\int_1^0 \frac{4}{t^2 - 2t - 3} dt = \int_1^0 \frac{4}{(t+1)(t-3)} dt$$

$$= \int_1^0 \frac{1}{t-3} - \frac{1}{t+1} dt$$

$$= \left[\ln|t-3| - \ln|t+1| \right]_1^0$$

$$= \left[\ln|-3| - \ln 1 \right] - \left[\ln|-2| - \ln 1 \right]$$

$$= \ln 3 - \ln 2 + \ln 1$$

$$= \ln 3$$

Question 63 (***)+

$$y = \arcsin x, -1 \leq x \leq 1.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

b) Use the substitution $x = \sin \theta$ to find

$$\int \frac{2x^2}{\sqrt{1-x^2}} dx.$$

c) Hence find an exact value for

$$\int_0^1 4x \arcsin x dx$$

$$\boxed{\arcsin x - x\sqrt{1-x^2} + C}, \quad \boxed{\left[\frac{\pi}{2} \right]}$$

<p>a) $y = \arcsin x$</p> $\Rightarrow \sin y = x$ $\Rightarrow x = \sin y$ $\Rightarrow \frac{dx}{dy} = \cos y$ $\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$ <p>Now $\cos^2 y + \sin^2 y = 1$ $\cos y = \pm \sqrt{1-\sin^2 y}$ So if $y = \arcsin x$, $-\frac{\pi}{2} < y \leq \frac{\pi}{2}$ $0 \leq \cos y \leq 1$ $\cos y = +\sqrt{1-\sin^2 y}$</p> $\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ $\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$	<p>b) $\int \frac{2x^2}{\sqrt{1-x^2}} dx$ substitution $= \int \frac{2\sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta$</p> $= \int 2\sin^2 \theta \times \cos \theta d\theta = \int 2\sin^2 \theta d\theta$ $= \int (2 - 2\cos 2\theta) d\theta = \int 1 - \cos 2\theta d\theta$ $= \theta - \frac{1}{2}\sin 2\theta + C = \theta - \sin \theta \cos \theta + C = 0 - \sin \theta \sqrt{1-\sin^2 \theta} + C$ $= \arcsin x - x\sqrt{1-x^2} + C$ <p>$2 = \cos 0$ $0 = \arcsin 0$ $\frac{d}{d\theta} = \cos \theta$ $d\theta = \cos \theta d\theta$ $\cos \theta = \sqrt{1-\sin^2 \theta}$</p>
<p>c) $\int_0^1 4x \arcsin x dx$... by parts ... (choose $u(x)$)</p> $= 2x^2 \arcsin x - \int \frac{2x^2}{\sqrt{1-x^2}} dx$ $= 2x^2 \arcsin x - \left[\arcsin x - x\sqrt{1-x^2} \right] + C$ $= (2x^2 - 1) \arcsin x + 2x\sqrt{1-x^2} + C$ <p>RE-introduce units</p> $\int_0^1 4x \arcsin x dx = \left[(2x^2 - 1) \arcsin x + 2x\sqrt{1-x^2} \right]_0^1$ $= \left(\frac{\pi}{2} - 0 \right) - (0 + 0) = \frac{\pi}{2}$	<p>$\arcsin x = \frac{1}{2x^2} \int \frac{1}{\sqrt{1-x^2}} dx$</p>

Question 64 (***)+

It is given that

$$x = -2 + \sqrt{3} \cosh \theta, \quad \theta \geq 0.$$

- a) Show clearly that ...

i. ... $\sinh \theta = \frac{\sqrt{x^2 + 4x + 1}}{\sqrt{3}}$.

ii. ... $\int \frac{x+2}{(x^2+4x+1)^{\frac{3}{2}}} dx = \frac{\sqrt{3}}{3} \int \frac{\cosh \theta}{\sinh^2 \theta} d\theta$.

- b) By considering the derivative of $\operatorname{cosech} \theta$ find

$$\int \frac{x+2}{(x^2+4x+1)^{\frac{3}{2}}} dx$$

$$-\left(x^2+4x+1\right)^{-\frac{1}{2}} + C$$

(a) $x = -2 + \sqrt{3} \cosh \theta$
 $x+2 = \sqrt{3} \cosh \theta$
 $(x+2)^2 = 3 \cosh^2 \theta$
 $\frac{x^2+4x+4}{3} = \cosh^2 \theta$
 $\frac{x^2+4x+4-1}{3} = \cosh^2 \theta - 1$
 $\frac{x^2+4x+1}{3} = \sinh^2 \theta$
 $\sinh^2 \theta = \frac{x^2+4x+1}{3}$

(b) NOW
 $\int \frac{x+2}{(x^2+4x+1)^{\frac{3}{2}}} dx$
Let $x = -2 + \sqrt{3} \cosh \theta$
 $\frac{dx}{d\theta} = \sqrt{3} \sinh \theta$
 $d\theta = \frac{1}{\sqrt{3} \sinh \theta} dx$
 $\therefore (x^2+4x+1)^{\frac{1}{2}} = \sqrt{3} \sinh \theta$
 $\Rightarrow \frac{2x+4}{(x^2+4x+1)^{\frac{1}{2}}} dx = \frac{2(-2+\sqrt{3} \cosh \theta)+4}{(\sqrt{3} \sinh \theta)^2} dx$
 $= \int \frac{2\sqrt{3} \cosh \theta}{(\sqrt{3} \sinh \theta)^2} dx = \int \frac{2\sqrt{3} \cosh \theta}{3 \sinh^2 \theta} dx$
 $= \frac{2\sqrt{3}}{3} \int \frac{\cosh \theta}{\sinh^2 \theta} d\theta$
 $\Rightarrow \frac{2\sqrt{3}}{3} \operatorname{cosech} \theta + C$

(c) ... $= \frac{2\sqrt{3}}{3} \int \frac{\cosh \theta}{\sinh^2 \theta} d\theta = \frac{2\sqrt{3}}{3} \operatorname{cosech} \theta$
BUT $\frac{d}{d\theta} (\operatorname{cosech} \theta) = -\operatorname{cosech} \theta \operatorname{coth} \theta$
 $\therefore -\frac{\sqrt{3}}{3} \operatorname{cosech} \theta + C = -\frac{\sqrt{3}}{3} \times \frac{1}{\sinh \theta} + C = -\frac{\sqrt{3}}{3} \times \frac{1}{\sqrt{3} \sinh \theta} + C$
 $= -\frac{1}{\sqrt{3} \sinh \theta} + C$
NOTE - THIS REQUIRES NO SUBSTITUTION SINCE
 $\int \frac{x+2}{(x^2+4x+1)^{\frac{3}{2}}} dx = \int (2\sqrt{3}) \operatorname{cosech} \theta d\theta = \dots = -(x^2+4x+1)^{-\frac{1}{2}} + C$
 \therefore This is the derivative of $(x^2+4x+1)^{-\frac{1}{2}}$

Question 65 (***)+

Use the substitution $u = \sqrt{x}$ to find

$$\int \frac{1}{(x^2-1)\sqrt{x}} dx.$$

$$\boxed{\frac{1}{2} \ln \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| - \arctan \sqrt{x} + C}$$

$$\begin{aligned} \int \frac{1}{(x^2-1)\sqrt{x}} dx &= \dots \text{substitution} = \int \frac{1}{(u^2-1)\sqrt{u}} du = \int \frac{2}{u^2-1} du \\ u = \sqrt{x} & \\ u^2 = x & \\ 2u du = dx & \\ & \end{aligned} \quad \begin{aligned} &= \int \frac{2}{(u^2-1)(u^2+1)} du = \dots \text{partial fractions} \\ &\quad \text{by inspection} \\ &= \int \frac{1}{u^2-1} - \frac{1}{u^2+1} du = \int \frac{1}{(u-1)u(u+1)} - \frac{1}{u^2+1} du \\ &= \dots \text{partial fractions again by inspection} \\ &= \int \frac{1}{u-1} - \frac{1}{u+1} - \frac{1}{u^2+1} du \\ &= \frac{1}{2} \ln |u-1| - \frac{1}{2} \ln |u+1| - \arctan u + C \\ &= \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| - \arctan u + C = \frac{1}{2} \ln \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| - \arctan \sqrt{x} + C \end{aligned}$$

Question 66 (***)+

a) Find a simplified expression for

$$\frac{d}{dx} \left[\arctan \frac{2}{x} \right].$$

b) Hence show that

$$\int_{\frac{2\sqrt{3}}{3}}^2 9x \arctan \left(\frac{2}{x} \right) dx = \pi + 18 - 6\sqrt{3}$$

$$\boxed{\frac{d}{dx} \left[\arctan \frac{2}{x} \right] = -\frac{2}{x^2 + 4}}$$

(a) $\frac{d}{dx} \left[\arctan \left(\frac{2}{x} \right) \right] = -\frac{1}{1 + \left(\frac{2}{x} \right)^2} \times -\frac{2}{x^2} = \frac{1}{1 + \frac{4}{x^2}} \times \left(-\frac{2}{x^2} \right) = \frac{-2}{x^2 + 4} \left(-\frac{2}{x^2} \right) = \frac{2}{x^2 + 4} //$

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$\frac{d}{dx} \left[\arctan \left(\frac{2}{x} \right) \right] = \frac{d}{dx} \left[\frac{\pi}{2} - \arctan \frac{x}{2} \right] = -\frac{1}{1 + \frac{x^2}{4}} \times \frac{1}{2} = -\frac{4}{4+x^2} \times \frac{1}{2} = -\frac{2}{x^2+4} //$ ~~錯~~

(b)
$$\begin{aligned} \int_0^2 9x \arctan \left(\frac{2}{x} \right) dx &= \dots \text{ by parts } \dots \\ &= \left[\frac{9}{2} x^2 \arctan \frac{2}{x} \right]_0^2 - \int_0^2 \frac{9x^2}{x^2 + 4} dx \\ &= \left[18x \arctan \frac{2}{x} \right]_0^2 - \left[9 \int_0^2 \frac{x^2 + 4 - 4}{x^2 + 4} dx \right] \\ &= \left[18 \left(\frac{\pi}{4} \right) - 6 \left(\frac{\pi}{3} \right) \right] + 9 \left[\int_0^2 1 - \frac{4}{x^2 + 4} dx \right] \\ &= \frac{9}{2}\pi - 2\pi + 9 \left[x - \frac{4}{2} \arctan \frac{2}{x} \right]_0^2 \\ &= \frac{5}{2}\pi + 9 \left[\left(2 - 2\arctan 1 \right) - \left(\frac{2}{3}\sqrt{3} - 2\arctan \frac{\sqrt{3}}{3} \right) \right] \\ &= \frac{5}{2}\pi + 9 \left[2 - 2\frac{\pi}{4} - \frac{2}{3}\sqrt{3} + \frac{\pi}{3} \right] \\ &= \frac{5}{2}\pi + 18 - \frac{9\pi}{4} - 6\sqrt{3} + 3\pi \\ &= 18 - 6\sqrt{3} + \pi \end{aligned}$$

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Question 67 (***)+

$$\int_0^4 \frac{16}{3(3x^2+16)^{\frac{5}{2}}} dx.$$

- a) By using a suitable trigonometric substitution in terms of θ , show that the above integral can be transformed to

$$\frac{\sqrt{3}}{144} \int_0^{\frac{\pi}{3}} \cos^3 \theta d\theta.$$

- b) Hence evaluate the original integral.

$\frac{1}{128}$

(a)

$$\begin{aligned} & \int_0^4 \frac{16}{3(3x^2+16)^{\frac{5}{2}}} dx = \dots \\ &= \int_0^{\frac{\pi}{3}} \frac{16}{3(\sec^2 \theta)^{\frac{5}{2}}} \cdot \frac{4}{\sqrt{3}} \sec \theta d\theta \\ &= \int_0^{\frac{\pi}{3}} \frac{16}{3 \times 16 \tan^5 \theta} \times \frac{4}{\sqrt{3}} \sec \theta d\theta \\ &= \int_0^{\frac{\pi}{3}} \frac{4\sqrt{3}}{48 \tan^5 \theta} d\theta = \int_0^{\frac{\pi}{3}} \frac{\sqrt{3}}{144} \sec^3 \theta d\theta \\ &= \frac{\sqrt{3}}{144} \int_0^{\frac{\pi}{3}} \sec^3 \theta d\theta \quad / \\ (b) & \dots = \frac{\sqrt{3}}{144} \int_0^{\frac{\pi}{3}} \sec^3 \theta (-\sin^2 \theta) d\theta = \frac{\sqrt{3}}{144} \int_0^{\frac{\pi}{3}} \cos \theta - \cos^3 \theta \sin^2 \theta d\theta \\ &= \frac{\sqrt{3}}{144} \left[\sin \theta - \frac{1}{3} \sin^3 \theta \right]_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{144} \left[\frac{\sqrt{3}}{2} - \frac{1}{3} \cdot \frac{3\sqrt{3}}{8} \right] \\ &= \frac{\sqrt{3}}{144} \left[\frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{8} \right] = \frac{\sqrt{3}}{144} \times \frac{5\sqrt{3}}{8} = \frac{9}{1152} = \frac{1}{128} \end{aligned}$$

Let $\sqrt{3}x = \tan \theta$
 $\frac{\sqrt{3}}{3}dx = \sec^2 \theta d\theta$
 $d\theta = \frac{1}{\sqrt{3}} \sec \theta d\theta$
 $x=0, \theta=0$
 $x=4, \tan \theta=\sqrt{3}$
 $\theta=\frac{\pi}{3}$

Question 68 (***)+

$$I = \int_0^{\frac{\pi}{8}} \frac{\sqrt{3}}{2 + \sin 4x} dx.$$

a) Show that the substitution $u = \tan 2x$ transforms I into

$$J = \int_0^1 \frac{\sqrt{3}}{(2u+1)^2 + 3} du.$$

b) Hence find the exact value of I , giving the answer in terms of π .

$$\boxed{I = \frac{\pi}{12}}$$

\textcircled{a} $\int_0^{\frac{\pi}{8}} \frac{\sqrt{3}}{2 + \sin 4x} dx = \dots \int_0^1 \frac{\sqrt{3}}{2 + \tan^2 2x} \frac{du}{2 \sec^2 2x}$ $= \int_0^1 \frac{\sqrt{3}}{2 + 2 \tan^2 2x + 2 \sec^2 2x} \times \frac{du}{2 \sec^2 2x}$ $= \int_0^1 \frac{\sqrt{3}}{4 \sec^2 2x + 2 \tan^2 2x + 2} du$ $= \int_0^1 \frac{\sqrt{3}}{4 \sec^2 2x + 4 \tan^2 2x + 4} du = \int_0^1 \frac{\sqrt{3}}{4(1 + \tan^2 2x) + 4} du$ $= \int_0^1 \frac{\sqrt{3}}{4 \tan^2 2x + 4} du = \int_0^1 \frac{\sqrt{3}}{4u^2 + 4} du$ $= \int_0^1 \frac{\sqrt{3}}{(2u+1)^2 + 3} du$ <p style="text-align: right;">$\cancel{\text{AS 24/09/2020}}$</p>	$u = \tan 2x$ $\frac{du}{dx} = 2u^2 + 2$ $dx = \frac{du}{2u^2 + 2}$ $2 = \frac{du}{2u^2 + 2} \Rightarrow u = 1$ $2 = \frac{du}{2u^2 + 2} \Rightarrow u = 0$
(b) ANOTHER SUBSTITUTION $\text{LET } V = 2u+1$ $\frac{dv}{du} = 2$ $du = \frac{dv}{2}$ $u=0 \Rightarrow v=1$ $u=1 \Rightarrow v=3$	
$\dots = \int_1^3 \frac{\sqrt{3}}{v^2 + 3} \frac{dv}{2} = \frac{\sqrt{3}}{2} \int_1^3 \frac{1}{v^2 + (\sqrt{3})^2} dv$ $\dots \text{ STANDARD ARCSIN INTERVAL} \dots$ $\dots = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}} \left[\arctan \left(\frac{v}{\sqrt{3}} \right) \right]_1^3$ $= \frac{1}{2} \left[\arctan \frac{3}{\sqrt{3}} - \arctan \frac{1}{\sqrt{3}} \right]$ $= \frac{1}{2} \left(\arctan \sqrt{3} - \arctan \frac{1}{\sqrt{3}} \right)$ $= \frac{1}{2} \left(\frac{\pi}{3} - \frac{\pi}{6} \right)$ $= \frac{\pi}{12}$ <p style="text-align: right;">$\cancel{\text{AS 24/09/2020}}$</p>	

Question 69 (***)+

$$\sec x \equiv \frac{1 + \tan^2\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}.$$

- a) Prove the validity of the above trigonometric identity.
- b) Express $\frac{2}{1-t^2}$ into partial fractions.
- c) Hence use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$\int \sec x \, dx = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C.$$

$$\boxed{\quad}, \quad \boxed{\frac{2}{1-t^2} = \frac{1}{1+t} + \frac{1}{1-t}}$$

a) WORKING AS FOLLOWS

$$\begin{aligned} LHS &= \sec x = \frac{1}{\cos x} = \frac{1}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} = \frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} \\ &= \frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} + \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1 + \tan^2\frac{x}{2}}{1 - \tan^2\frac{x}{2}} = RHS \end{aligned}$$

[OR USE THE IDENTITIES FROM THE R.H.S TO L.H.S.]

b) BY INSPECTION / COUNT UP OR ANY SIMPLE METHOD

$$\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)} = \frac{1}{1-t} + \frac{1}{1+t}$$

c) USING THE SUBSTITUTION METHOD

$$\begin{aligned} t &= \tan\frac{x}{2} \rightarrow \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} \\ &\Rightarrow \frac{dt}{dx} = \frac{1}{2}(1 + \tan^2\frac{x}{2}) \\ &\Rightarrow \frac{dt}{dx} = \frac{1}{2}(1 + t^2) \\ &\Rightarrow \frac{dt}{dx} = \frac{1}{1+t^2} \\ &\Rightarrow dx = \frac{2}{1+t^2} dt \end{aligned}$$

(a) SIMPLIFYING

$$\begin{aligned} \int \sec x \, dx &= \int \frac{1 + \tan^2\frac{x}{2}}{1 - \tan^2\frac{x}{2}} \, dx = \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2} \, dt \\ &= \int \frac{2}{1-t^2} \, dt = \int \frac{1+t}{1+t} + \frac{1-t}{1-t} \, dt \\ &= \ln|1+t| - \ln|1-t| + C \\ &= \ln\left|\frac{1+t}{1-t}\right| + C \end{aligned}$$

KNOWING THAT $\tan\frac{x}{2} = t \Rightarrow \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = 1$

$$\begin{aligned} &= \ln\left|\frac{\tan\frac{x}{2} + \tan\frac{\pi}{4}}{1 - \tan\frac{x}{2}\tan\frac{\pi}{4}}\right| + C \quad \text{as } \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C \quad \text{as required} \end{aligned}$$

Question 70 (***)+

Find an exact value for

$$\int_0^\infty \frac{16}{(1+x^2)^3} dx.$$

[3π]

$$\begin{aligned} \int_0^\infty \frac{16}{(1+x^2)^3} dx &= \dots = \int_0^{\frac{\pi}{2}} \frac{16}{(1+\tan^2 \theta)^3} \sec^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{16 \sec^2 \theta}{(\sec^2 \theta)^3} d\theta = \int_0^{\frac{\pi}{2}} \frac{16 \sec^2 \theta}{\sec^6 \theta} d\theta = \int_0^{\frac{\pi}{2}} 16 \cos^4 \theta d\theta = \int_0^{\frac{\pi}{2}} 16(1 + \cos 2\theta)^2 d\theta \\ &= \int_0^{\frac{\pi}{2}} 16 + 32 \cos 2\theta + 16 \cos^2 2\theta d\theta = \int_0^{\frac{\pi}{2}} 16 + 8 \cos 2\theta + 4(1 + \cos 4\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} 16 + 8 \cos 2\theta + 2 \cos 4\theta d\theta = \left[16 \theta + 8 \sin 2\theta + \frac{1}{2} \sin 4\theta \right]_0^{\frac{\pi}{2}} = (3\pi + 0 + 0) - (0) = 3\pi. \end{aligned}$$

Question 71 (***)+

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x + \cos x} dx.$$

By using the substitution $t = \tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$I = \ln 2.$$

proof

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x + \cos x} dx \\ &= \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2} dt \\ &\quad \text{Multiply top and bottom of each double fraction by } 1+t^2 \\ &= \int_0^1 \frac{1+t^4}{(1+t^2+2t+1-t^2)(1+t^2)} \times \frac{2}{1+t^2} dt \\ &= \int_0^1 \frac{2}{2(1+t^2)} dt = \int_0^1 \frac{1}{1+t^2} dt \\ &= \left[\ln|1+t^2| \right]_0^1 = \ln 2 - \ln 1 \\ &= \ln 2. \end{aligned}$$

$t = \tan \frac{x}{2}$
 $\frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{x}{2}$
 $\frac{dx}{dt} = \frac{1}{2}(1+\tan^2 \frac{x}{2})$
 $\frac{dt}{dx} = \frac{1}{2}(1+t^2)$
 $dx = \frac{2}{1+t^2} dt$

 $\sin x = \frac{2t}{1+t^2}$
 $\cos x = \frac{1-t^2}{1+t^2}$
 $\sin x = \left(\frac{2t}{1+t^2}\right)^2 - \left(\frac{1-t^2}{1+t^2}\right)^2$
 $\cos x = \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}$
 $\cos x = \frac{1-t^2}{1+t^2}$

Question 72 (***)+

Use the substitution $x = 2 \cosh \theta$, to find a simplified expression for

$$\int \frac{6}{(x^2 - 4)^{\frac{3}{2}}} dx.$$

$$-\frac{x}{\sqrt{x^2 - 4}} + C$$

$$\begin{aligned}
 & \int \frac{4}{(x^2 - 4)^{\frac{3}{2}}} dx = \dots \text{ hyperbolic substitution} \\
 &= \int \frac{4}{(2 \sinh \theta)^{\frac{3}{2}}} (2 \sinh \theta d\theta) \\
 &= \int \frac{8 \sinh \theta}{(2 \sinh \theta)^{\frac{3}{2}}} d\theta \\
 &= \int \frac{8 \sinh \theta}{(4 \sinh^3 \theta)^{\frac{1}{2}}} d\theta \\
 &= \int \frac{8 \sinh \theta}{4 \sinh^{\frac{3}{2}} \theta} d\theta \\
 &= \int \frac{2}{\sinh^{\frac{1}{2}} \theta} d\theta \\
 &= \int \operatorname{csch} \theta d\theta \\
 &= -\operatorname{coth} \theta + C \\
 &= -\frac{\cosh \theta}{\sinh \theta} + C \\
 &= -\frac{x}{\sqrt{x^2 - 4}} + C
 \end{aligned}$$

$x = 2 \cosh \theta$
 $d\theta = 2 \sinh \theta d\theta$
 $\cosh^2 \theta = \frac{x^2}{4}$
 $\cosh^2 \theta - 1 = \frac{x^2}{4} - 1$
 $\sinh^2 \theta = \frac{x^2 - 4}{4}$
 $\operatorname{sinh} \theta = \sqrt{\frac{x^2 - 4}{4}}$
 ACUTE ANGLE
 $\theta = \operatorname{arccosh} \frac{x}{2}$
 $\sinh \theta = \sqrt{\frac{x^2 - 4}{4}}$
 $\frac{1}{\sinh \theta} = \frac{2}{\sqrt{x^2 - 4}}$
 $\operatorname{cosec} \theta = \frac{2(\frac{x}{2})}{\sqrt{x^2 - 4}}$
 $\operatorname{cosec} \theta = \frac{x}{\sqrt{x^2 - 4}}$

Question 73 (***)+

Use the substitution $u = \frac{1}{x+1}$ to find

$$\int \frac{1}{(x+1)\sqrt{x^2+4x+2}} dx.$$

$$\arcsin\left[\frac{x}{(x+1)\sqrt{2}}\right] + C$$

$$\begin{aligned}
 \int \frac{1}{(x+1)\sqrt{x^2+4x+2}} dx &= \dots \text{ by substitution } \\
 &= \int \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}+\frac{2}{u}-1}} \left(-\frac{1}{u^2} du\right) \\
 &= \int \frac{-u^2}{\sqrt{1+2u-u^2}} \left(-\frac{1}{u^2} du\right) \\
 &= \int \frac{-1}{\sqrt{1+2u-u^2}} \times \frac{1}{u^2} du \\
 &= \int \frac{-1}{\sqrt{2-(u-1)^2}} du \quad \text{Now } v=u-1 \\
 &= \int \frac{-1}{\sqrt{2-v^2}} dv \\
 &= \int \frac{-1}{\sqrt{(1-\frac{v}{\sqrt{2}})^2}} dv = -\arcsin\left(\frac{v}{\sqrt{2}}\right) + C \\
 &= -\arcsin\left(\frac{u-1}{\sqrt{2}}\right) + C = -\arcsin\left(\frac{u-1}{\sqrt{2u^2}}\right) + C \quad \text{by (3ii)} \\
 &= -\arcsin\left(\frac{1-(x+1)}{\sqrt{2(x+1)}}\right) + C = -\arcsin\left(\frac{-x^2}{\sqrt{2(x+1)}}\right) + C = \arcsin\left[\frac{x^2}{\sqrt{2(x+1)}}\right] + C //
 \end{aligned}$$

Question 74 (***)+

$$I = \int_0^{\frac{\pi}{2}} \frac{2\cos x}{1+\cos x} dx.$$

By using the substitution $t = \tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$I = \pi - 2.$$

proof

$$\begin{aligned}
 & \int_0^{\frac{\pi}{2}} \frac{2\cos x}{1+\cos x} dx = \text{by using t substitution} \\
 & = \int_0^1 \frac{2(1-t^2)}{1+t^2} \times \frac{2}{1+t^2} dt \\
 & = \int_0^1 \frac{2(1-t^2)}{(1+t^2)(1-t^2)} \times \frac{2}{1+t^2} dt \\
 & = \int_0^1 \frac{2(1-t^2)}{2t^2} \times \frac{2}{1+t^2} dt = 2 \int_0^1 \frac{1-t^2}{1+t^2} dt \\
 & = 2 \int_0^1 \frac{2(-1+t^2)}{1+t^2} dt = 2 \int_0^1 \frac{2}{1+t^2} - 1 dt \\
 & = 2 \left[2 \arctan t - \frac{2}{1+t^2} \right]_0^1 = 2[\arctan 1 - 1] - 0 \\
 & = 2 \left[2 \times \frac{\pi}{4} - 1 \right] = \pi - 2
 \end{aligned}$$

$$\begin{aligned}
 t &= \tan\frac{x}{2} \\
 \frac{dt}{dx} &= \frac{1}{2}x\sec^2\frac{x}{2} \\
 dx &= \frac{dt}{\frac{1}{2}x\sec^2\frac{x}{2}} \\
 dx &= \frac{2dt}{x\sec^2\frac{x}{2}} \\
 dx &= \frac{2}{1+t^2} dt \\
 \sec^2 x &= \frac{1-t^2}{1+t^2} \\
 x=0, t=0 & \\
 x=\frac{\pi}{2}, t=1 &
 \end{aligned}$$

Question 75 (***)+

Find an exact value for

$$\int_{-1.5}^{1.5} 8x \arcsin\left(\frac{1}{3}x\right) dx.$$

, $9\sqrt{3} - 3\pi$

AS THE INTEGRAND IS EVEN, WE MAY REWRITE AS FOLLOWS

$$\begin{aligned} \int_{-1.5}^{1.5} 8x \arcsin\left(\frac{1}{3}x\right) dx &= 2 \int_0^{1.5} 8x \arcsin\left(\frac{1}{3}x\right) dx \\ &= 16 \int_0^{1.5} x \arcsin\left(\frac{1}{3}x\right) dx \end{aligned}$$

USING A SUBSTITUTION

$$\begin{aligned} &= 16 \int_0^{\frac{\pi}{6}} \left(3\sin\theta\right) \theta \left(3\cos\theta d\theta\right) \\ &= 144 \int_0^{\frac{\pi}{6}} \theta \sin\theta \cos\theta d\theta \\ &= 72 \int_0^{\frac{\pi}{6}} \theta \sin\theta d\theta \end{aligned}$$

PROCEEDED BY INTEGRATION BY PARTS

$$\begin{aligned} &= 72 \left[-\frac{1}{2}\theta \cos 2\theta \Big|_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} -\frac{1}{2} \cos 2\theta d\theta \right] \\ &= 72 \left[-\frac{1}{2}\theta \cos 2\theta \Big|_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} \frac{1}{2} \cos 2\theta d\theta \right] \\ &= 72 \left[-\frac{1}{2}\theta \cos 2\theta + \frac{1}{4} \sin 2\theta \Big|_0^{\frac{\pi}{6}} \right] \\ &= 72 \left[\left(-\frac{1}{2} \times \frac{\pi}{6} \times \frac{1}{2} + \frac{1}{4} \times \frac{\sqrt{3}}{2}\right) - (0) \right] \\ &= 72 \left[\frac{\sqrt{3}}{8} - \frac{\pi}{24} \right] \\ &= 9\sqrt{3} - 3\pi \end{aligned}$$

$\theta = \arcsin\left(\frac{1}{3}x\right)$
$\sin\theta = \frac{1}{3}x$
$x = 3\sin\theta$
$\frac{dx}{d\theta} = 3\cos\theta$
$d\theta = \frac{1}{3}\cos\theta d\theta$
$2\pi/3 \rightarrow 0 \rightarrow \pi/6$
$3\sin\theta \rightarrow 0 \rightarrow \frac{1}{3}$

Question 76 (***)+

$$I = \int \frac{\operatorname{sech} x}{\cosh x - \sinh x} dx.$$

- a) By multiplying the numerator and denominator of the integrand by $\operatorname{sech} x$, show that

$$I = -\ln(1 - \tanh x) + C,$$

where C in an arbitrary constant.

- b) By multiplying the numerator and denominator of the integrand by $(\cosh x - \sinh x)$, show that

$$I = x + \ln(\cosh x) + K,$$

where K in an arbitrary constant.

- c) Show clearly that $C = K$.

proof

(a) $\int \frac{\operatorname{sech} x}{\cosh x - \sinh x} dx = \int \frac{\operatorname{sech} x \operatorname{sech} x}{(\cosh x - \sinh x) \operatorname{sech} x} dx = \int \frac{\operatorname{sech}^2 x}{1 - \tanh^2 x} dx$

$$= -\ln(1 - \tanh x) + C$$

as required

for the rule
 $\frac{d}{dx} \ln(u) = \frac{u'}{u}$

(b) $\int \frac{\operatorname{sech} x}{\cosh x - \sinh x} dx = \int \frac{\operatorname{sech}(x + \ln(\cosh x))}{(\cosh(x + \ln(\cosh x)) - \sinh(x + \ln(\cosh x)))} dx = \int \frac{1 + \tanh(x + \ln(\cosh x))}{\cosh(x + \ln(\cosh x)) - \sinh(x + \ln(\cosh x))} dx$

$$= \int 1 + \tanh x dx = x + \ln(\cosh x) + C$$

as required

(c) $I = -\ln[1 - \tanh x] = \ln\left[\frac{1}{1 - \tanh x}\right] = \ln\left[\frac{1}{1 - \frac{\sinh x}{\cosh x}}\right]$

$$= \ln\left[\frac{\cosh x}{\cosh x - \sinh x}\right] = \ln\left[\frac{1}{2}(\cosh x + \sinh x)\right] = \ln\left[\frac{1}{2}e^{2x}(e^2 + e^{-2})\right]$$

$$= \ln\left[e^{2x}\left(\frac{1}{2}e^2 + \frac{1}{2}e^{-2}\right)\right] = \ln(e^{2x}) + \ln\left[\frac{1}{2}e^2 + \frac{1}{2}e^{-2}\right]$$

$$= 2x + \ln(\cosh x) = RHQ$$

$\therefore C = k$
as required

Question 77 (***)+

$$\sec x \equiv \frac{\cos x}{1 - \sin^2 x}.$$

- a) Prove the validity of the above trigonometric identity.
- b) Use the substitution $u = \sin x$ to show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \, dx = \frac{1}{2} \ln \left(\frac{7+4\sqrt{3}}{3} \right).$$

- c) Show clearly that

$$\frac{1}{2} \ln \left(\frac{7+4\sqrt{3}}{3} \right) = \ln \left(1 + \frac{2}{3}\sqrt{3} \right)$$

proof

<p>(a) LHS = $\sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x} = \text{RHS}$</p>
<p>(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x}{1 - \sin^2 x} \, dx = \dots \text{substitution}$</p>
<p>$u = \sin x$ $du = \cos x \, dx$ $dx = \frac{du}{\cos x}$ $1 - u^2 = 1 - \cos^2 x$ $2u = 2\sin x$ $u = \frac{1}{2}\sin x$</p>
$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\cos x \, du}{1 - \cos^2 x} = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{1 - u^2} \, du$
$= \dots \text{by partial fractions}$
$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\frac{1}{2u} + \frac{1}{2(1-u)}}{1-u^2} \, du = \left[\frac{1}{2} \ln 1+u - \frac{1}{2} \ln 1-u \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$
$= \left[\frac{1}{2} \ln \frac{ 1+\frac{\sqrt{3}}{2} }{ 1-\frac{\sqrt{3}}{2} } \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{1}{2} \left[\ln \left(\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}} \right) - \ln \left(\frac{1}{2} \right) \right]$
$= \frac{1}{2} \left[\ln \left(\frac{2+\sqrt{3}}{2-\sqrt{3}} \right) - \ln 3 \right] = \frac{1}{2} \left[\ln(1+4\sqrt{3}) - \ln 3 \right]$
$= \frac{1}{2} \ln \left(\frac{7+4\sqrt{3}}{3} \right)$
<p style="text-align: right;">to R.H.S</p>
<p>(c) $\frac{1}{2} \ln \left(\frac{7+4\sqrt{3}}{3} \right) = \frac{1}{2} \ln \left(\frac{2(1+2\sqrt{3})}{3} \right) = \frac{1}{2} \ln \left[\frac{2 + 2 \times 3 \times 2\sqrt{3} + (2\sqrt{3})^2}{3} \right]$</p>
$= \frac{1}{2} \ln \left[\frac{2^2 + 2 \times 3 \times 2\sqrt{3} + (2\sqrt{3})^2}{3} \right]$
$= \frac{1}{2} \ln \left[\frac{(3+2\sqrt{3})^2}{3} \right] = \ln \left(\frac{3+2\sqrt{3}}{3} \right)$
$= \ln \left(1 + \frac{2}{3}\sqrt{3} \right)$
<p style="text-align: right;">to R.H.S</p>

Question 78 (***)+

$$\frac{9}{x^3+1} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

- a) Find the value of each of the constants A , B and C in the above identity.
 b) Hence find the exact value of

$$\int_0^1 \frac{9}{x^3-1} dx.$$

$$A=3, B=-3, C=6, 3\ln 2 + \pi\sqrt{3}$$

(a) $\frac{9}{x^3+1} \equiv \frac{9}{(x+1)(x^2-x+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

$9 \equiv A(x^2-x+1) + (Ax+B)(x+1)$

$9 \equiv Ax^2 - Ax + 1 + Ax^2 + Bx + Ax + B$

$9 \equiv (A+3)x^2 + (B+C-A)x + (A+B)$

\bullet If $x=-1$, $9=3A$, $A=3$

\bullet $A+B=0$, $B=-3$

\bullet $A+C=9$, $C=6$

(b) $\int_0^1 \frac{9}{x^3+1} dx = \int_0^1 \frac{3}{x+1} + \frac{-3x+6}{x^2-x+1} dx = \int_0^1 \frac{3}{x+1} - \frac{3}{2} \left(\frac{2x-4}{x^2-x+1} \right) dx$

$= \int_0^1 \frac{3}{x+1} - \frac{3}{2} \left(\frac{2x-1-3}{x^2-x+1} \right) dx = \int_0^1 \frac{3}{x+1} - \frac{3}{2} \left(\frac{2x-1}{x^2-x+1} \right) + \frac{9}{2} \left(\frac{1}{x^2-x+1} \right) dx$

$= \int_0^1 \frac{3}{x+1} - \frac{3}{2} \left(\frac{2x-1}{(x-\frac{1}{2})^2 + \frac{3}{4}} \right) dx = \int_0^1 \frac{3}{x+1} - \frac{3}{2} \left(\frac{2x-1}{(x-\frac{1}{2})^2 + \frac{3}{4}} \right) dx$

$= \int_0^1 \frac{3}{x+1} - \frac{3}{2} \left(\frac{2x-1}{(x-\frac{1}{2})^2 + \frac{3}{4}} \right) dx + \frac{9}{2} \int_0^1 \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}} dx$

$= \int_0^1 \frac{3}{x+1} - \frac{3}{2} \left(\frac{2x-1}{(x-\frac{1}{2})^2 + \frac{3}{4}} \right) dx + \frac{9}{2} \int_0^1 \frac{1}{u^2 + \frac{3}{4}} du$ $u=x-\frac{1}{2}$

$= \left[3\ln|x+1| - \frac{3}{2}\ln|x^2+x+1| \right]_0^1 + \frac{9}{2} \cdot \frac{1}{\sqrt{3}} \left[\arctan\left(\frac{u}{\sqrt{3}}\right) \right]_{-\frac{1}{2}}^{\frac{1}{2}}$

$= \left(3\ln 2 - \frac{3}{2}\ln 3 \right) - \left(3\ln 1 - \frac{3}{2}\ln 2 \right) + \frac{9}{2} \left[\arctan\left(\frac{1}{\sqrt{3}}\right) \right]_{-\frac{1}{2}}^{\frac{1}{2}}$

$= 3\ln 2 + 3\sqrt{3} \left[\arctan\left(\frac{1}{\sqrt{3}}\right) - \arctan\left(-\frac{1}{\sqrt{3}}\right) \right]$

$= 3\ln 2 + 3\sqrt{3} \left[\frac{\pi}{6} - \left(-\frac{\pi}{6} \right) \right]$

$= 3\ln 2 + 3\sqrt{3} \times \frac{\pi}{3}$

$= 3\ln 2 + \pi\sqrt{3}$

Question 79 (****+)

Show that

$$\int_0^{\frac{1}{3}\pi} \sec^2 x \operatorname{artanh}(\sin x) dx = -1 + \sqrt{3} \ln(2 + \sqrt{3}).$$

proof

... BY PARTS, SINCE $\frac{d}{dx}(\operatorname{artanh} u) = \frac{1}{1-u^2}$

$\operatorname{artanh}(u)$	$\frac{1}{1-u^2} \times \sec x$
$\tan x$	$\sec^2 x$

$$\begin{aligned}
 &= [\operatorname{artanh}(\sin x) \times \tan x]_0^{\frac{1}{3}\pi} - \int_0^{\frac{1}{3}\pi} \frac{\cos x}{1-\sin^2 x} \tan x dx \\
 &= [\tan x \operatorname{artanh}(\sin x)]_0^{\frac{1}{3}\pi} - \int_0^{\frac{1}{3}\pi} \frac{\tan x}{\sec x} dx \\
 &= [\tan x \operatorname{artanh}(\sin x)]_0^{\frac{1}{3}\pi} - \int_0^{\frac{1}{3}\pi} \sec x dx \\
 &= [\tan x \operatorname{artanh}(\sin x) - \sec x]_0^{\frac{1}{3}\pi} \\
 &= \left[\sqrt{3} \operatorname{artanh}\left(\frac{\sqrt{3}}{2}\right) - 2 \right] - [0 - 1] \\
 &= \sqrt{3} \operatorname{artanh}\left(\frac{\sqrt{3}}{2}\right) - 1 \\
 &= \frac{\sqrt{3}}{2} \ln \left[\frac{1 + \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}} \right] - 1 = \frac{\sqrt{3}}{2} \ln \left[\frac{2 + \sqrt{3}}{2 - \sqrt{3}} \right] - 1 \\
 &= \frac{\sqrt{3}}{2} \left[\frac{(2 + \sqrt{3})^2}{4 - 3} \right] - 1 = \frac{\sqrt{3}}{2} \ln(2 + \sqrt{3})^2 - 1 \\
 &= \sqrt{3} \ln(2 + \sqrt{3}) - 1
 \end{aligned}$$

Question 80 (****+)

Use appropriate integration techniques to find an exact simplified value for

$$\int_0^{\frac{1}{4}\pi} \frac{10}{2 - \tan x} dx.$$

, $\pi + 3\ln 2$

• SIMPLY A SUBSTITUTION

$$\int_0^{\frac{\pi}{4}} \frac{10}{2 - \tan x} dx = \int_0^1 \frac{10}{2-u} \left(\frac{1}{1+u^2} du \right)$$

$$= \int_0^1 \frac{10}{(2-u)(1+u^2)} du$$

• BY PARTIAL FRACTIONS

$$\frac{10}{(2-u)(1+u^2)} = \frac{A_1 + B_1}{u^2 + 1} + \frac{C_1}{2-u}$$

$$10 = (2-u)(A_1 u + B_1) + C_1(u^2 + 1)$$

- If $u=2$: $10 = 5C$, $C=2$.
- If $u=0$: $10 = 2B + C$, $10 = 2B + 2$, $B=4$.
- If $A_1=1$: $10 = A_1 + B + 2C$, $10 = 1 + 4 + 4$, $A_1=2$.

• REDUCING TO THE INTERVAL

$$\dots = \int_0^1 \frac{\frac{2u+4}{u^2+1} + \frac{2}{2-u}}{du} = \int_0^1 \frac{\frac{4}{u^2+1} + \frac{2u}{u^2+1} + \frac{2}{2-u}}{du}$$

$$= \left[4 \arctan u + \ln(1+u^2) - 2\ln|2-u| \right]_0^1$$

$$= (4\arctan 1 + \ln 2 - 2\ln 1) - (0 + \ln 1 - 2\ln 2)$$

$$= 4 \times \frac{\pi}{4} + 3\ln 2,$$

$$= \pi + 3\ln 2.$$

Question 81 (*****)

Use appropriate integration techniques to find an exact simplified value for

$$\int_0^\infty \frac{1}{(x+x^{-1})^2} dx.$$

<input type="text"/>	,	$\frac{\pi}{4}$
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TRY UP FIRST BEFORE A SUBSTITUTION

$$\begin{aligned} \int_0^\infty \frac{1}{(x+\frac{1}{x})^2} dx &= \int_0^\infty \frac{1}{(\frac{x^2+1}{x})^2} dx = \int_0^\infty \frac{1}{(x^2+1)^2} \cdot \frac{1}{x^2} dx \\ &= \int_0^\infty \frac{x^2}{(x^2+1)^2} dx \end{aligned}$$

Now make a trigonometric substitution

$$\begin{aligned} x &= \tan\theta \quad (\theta = \arctan x) \\ dx &= \sec^2\theta d\theta \\ x=0 &\mapsto \theta=0 \\ x=\infty &\mapsto \theta=\frac{\pi}{2} \end{aligned}$$

TRANSLATE THE INTEGRAL

$$\begin{aligned} \int_0^\infty \frac{\tan\theta}{(\tan\theta+1)^2} (\sec^2\theta) d\theta &= \int_0^{\pi/2} \frac{\tan\theta}{(\tan\theta+1)^2} d\theta \\ &= \int_0^{\pi/2} \frac{\frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos^2\theta}}{\left(\frac{\sin\theta}{\cos\theta} + 1\right)^2} d\theta \\ &= \int_0^{\pi/2} \frac{\frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos^2\theta}}{\frac{\sin^2\theta + \cos^2\theta + \sin\theta\cos\theta}{\cos^2\theta}} d\theta \\ &= \int_0^{\pi/2} \frac{\sin\theta}{\cos^2\theta + \sin\theta\cos\theta} d\theta \end{aligned}$$

$$\begin{aligned} &= \int_0^{\pi/2} \frac{1}{\cos^2\theta - \frac{1}{2}\sin2\theta} d\theta \\ &= \left[\frac{1}{2}\theta - \frac{1}{4}\sin2\theta \right]_0^{\pi/2} \\ &= \left(\frac{\pi}{4} - 0 \right) - (0 - 0) \\ &= \frac{\pi}{4} \end{aligned}$$

Question 82 (*****)

Use the substitution $t = \tan\left(\frac{3}{2}x\right)$ to find, in terms of π , the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \frac{1}{2+\cos 3x} dx.$$

$$\boxed{\frac{\pi\sqrt{3}}{54}}$$

The handwritten solution shows the following steps:

$$\begin{aligned} & \int_{\frac{\pi}{6}}^{\frac{2\pi}{9}} \frac{1}{2+\cos 3x} dx \\ &= \int_1^{\sqrt{3}} \frac{1}{2+\frac{1-t^2}{1+t^2}} \times \frac{2}{3(1+t^2)} dt \quad (\text{using } t = \tan(\frac{3}{2}x) \Rightarrow dt = \frac{2}{3(1+t^2)} dx) \\ &= \int_1^{\sqrt{3}} \frac{1+t^2}{2(1+t^2)(1-t^2)} \times \frac{2}{3(1+t^2)} dt \\ &= \frac{2}{3} \int_1^{\sqrt{3}} \frac{1}{t^2+3} dt \\ & \quad (\text{which is a standard arctan type integral}) \\ &= \frac{2}{3} \int_1^{\sqrt{3}} \frac{1}{t^2+3\sqrt{3}^2} dt \\ &= \frac{2}{3} \times \frac{1}{\sqrt{3}} \left[\arctan\left(\frac{t}{\sqrt{3}}\right) \right]_1^{\sqrt{3}} \\ &= \frac{2\sqrt{3}}{3} \left[\arctan(1) - \arctan\left(\frac{1}{\sqrt{3}}\right) \right] \\ &= \frac{2\sqrt{3}}{3} \left[\frac{\pi}{4} - \frac{\pi}{6} \right] \\ &= \frac{2\sqrt{3}}{3} \times \frac{\pi}{12} = \frac{\pi\sqrt{3}}{54} \end{aligned}$$

Diagram illustrating the substitution: $t = \tan(\frac{3}{2}x)$. A right-angled triangle is shown with the angle $\frac{3}{2}x$ at the top vertex. The adjacent side is labeled 1 and the opposite side is labeled t . The hypotenuse is labeled $\sqrt{1+t^2}$.

Further steps on the right:

$$\begin{aligned} & \text{Let } t = \tan(\frac{3}{2}x) \\ & \frac{dt}{dx} = \frac{3}{2}\sec^2(\frac{3}{2}x) \\ & dt = \frac{3}{2}\sec^2(\frac{3}{2}x) dx \\ & dx = \frac{2}{3(1+t^2)} dt \\ & x = \frac{2}{3}\arctan(t) \\ & x = \frac{2}{3}\arctan(\sqrt{3}) \end{aligned}$$

Question 83 (*****)

Use the substitution $u = \sqrt{x+2}$ to find

$$\int \frac{16}{(x+6)(x-2)\sqrt{x+2}} dx.$$

$$\boxed{\ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right| - 2\arctan\left(\frac{\sqrt{x+2}}{2}\right) + C}$$

$$\begin{aligned}
 & \int \frac{16}{(x+6)(x-2)\sqrt{x+2}} dx = \dots \text{ by substitution } \dots \\
 &= \int \frac{16}{(u^2+4)(u^2-2)} du \quad (2u du) = \int \frac{32}{(u^2+4)(u^2-4)} du \\
 &= \dots \text{ PARTIAL FRACTIONS BY INSPECTION} \\
 &= \int \frac{4}{u^2+4} - \frac{4}{u^2-4} du = \int \frac{4}{(u-2)(u+2)} - \frac{4}{u^2+4} du \\
 &= \dots \text{ PARTIAL FRACTIONS BY INSPECTION} \\
 &= \int \frac{1}{u+2} - \frac{1}{u-2} - \frac{4}{u^2+4} du \\
 &= \ln|u+2| - \ln|u-2| - 4 \times \frac{1}{2} \arctan\left(\frac{u}{2}\right) + C \\
 &= \ln\left|\frac{u+2}{u-2}\right| - 2\arctan\left(\frac{u}{2}\right) + C \\
 &= \ln\left|\frac{\sqrt{x+2}+2}{\sqrt{x+2}-2}\right| - 2\arctan\left(\frac{\sqrt{x+2}}{2}\right) + C
 \end{aligned}$$

Question 84 (*****)

By using the substitution $t = \tan\left(\frac{x}{2}\right)$, or otherwise, show that

$$\int \frac{5}{4\cos x + 3\sin x} dx = \ln \left| \frac{2 + \sin x - 2\cos x}{2\sin x + \cos x - 1} \right| + C.$$

proof

$$\begin{aligned} \int \frac{5}{4\cos x + 3\sin x} dx &= \dots \text{by little t identities} \quad [t = \frac{x}{2}, \sin x = \frac{1-t^2}{1+t^2}, \cos x = \frac{2t}{1+t^2}] \\ \text{#use } t &= \int \frac{5}{4\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right)} \times \frac{2}{1+t^2} dt \\ &= \int \frac{5(1+t^2)}{4(1-t^2) + 3(2t)} \times \frac{2}{1+t^2} dt \\ &= \int \frac{10}{4-4t^2+6t} dt = \int \frac{10}{2(2-t^2-3t)} dt \\ &= \int \frac{5}{(2t+1)(2-t)} dt = \int \frac{5}{(2t+1)(2-t)} dt \\ \text{BY PARTIAL FRACTION} \\ &= \int \frac{1}{2-t} + \frac{2}{2t+1} dt = \ln|2t+1| - \ln|2-t| + C \\ &= \ln\left|\frac{2t+1}{2-t}\right| + C = \ln\left|\frac{2t+1}{2-t}\right| + C, \quad \cancel{\text{#}} \\ \text{#} &= \ln\left|\frac{\frac{2t+1}{2-t} + 1}{2 - \frac{2t+1}{2-t}}\right| + C = \ln\left|\frac{2t+1+2t-1}{2(2t+1)-(2-t)}\right| + C \\ &= \ln\left|\frac{2(2t+1)}{2(2t+1)-2+2t}\right| + C = \ln\left|\frac{2(2t+1)}{2(2t+1)+2t}\right| + C \\ &= \ln\left|\frac{2(2t+1)}{2(2t+1)+2t}\right| + C = \ln\left|\frac{1-\cos x + \frac{1}{2}\sin x}{\sin x - \frac{1}{2} + \cos x}\right| + C \\ &= \ln\left|\frac{2-2\cos x + \sin x}{2\sin x + \cos x - 1}\right| + C \end{aligned}$$

$$\begin{aligned} \text{ALTERNATIVE} \\ \int \frac{5}{4\cos x + 3\sin x} dx &= \int \frac{5}{\sec(x-\pi)} dx \\ &= \int \sec(x-\pi) dx \\ &= \ln|\sec(x-\pi) + \tan(x-\pi)| + C \\ &= \ln\left|\frac{1 + \tan(x-\pi)}{\cos(x-\pi)}\right| + C = \ln\left|\frac{1 + \tan(x-\pi) - \sec(x-\pi)}{\cos(x-\pi) + \sin(x-\pi)}\right| + C \\ &= \ln\left|\frac{1 + \frac{1}{2}\sin(2x-\pi) - \frac{3}{2}\cos(2x-\pi)}{\frac{1}{2}\cos(2x-\pi) + \frac{3}{2}\sin(2x-\pi)}\right| + C = \ln\left|\frac{5 + 4\cos x - 3\cos x}{4\cos x + 3\sin x}\right| + C \end{aligned}$$

$\cos(2x-\pi) = \cos 2x \cos \pi - \sin 2x \sin \pi$
This
 $4\cos x + 3\sin x = 5 \cos(2x-\pi) \equiv 5\cos 2x + 3\sin 2x$
 $\therefore \sec x = \frac{5}{3}$
 $\sin x = \frac{3}{5}$
 $\tan x = \frac{3}{4}$

Question 85 (*****)

By using the substitution $x = e^{-\frac{1}{2}t}$, or otherwise, find a simplified expression for

$$\int \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx.$$

$$\boxed{\frac{\sqrt{x^4 + 1}}{x} + C}$$

$$\begin{aligned} \int \frac{x^4 - 1}{x^2 \sqrt{x^4 + 1}} dx &= \dots \text{by substitution} \dots \\ &= \int \frac{e^{-4t} - 1}{e^{-2t} \sqrt{e^{-4t} + 1}} (-\frac{1}{2}e^{-t} dt) = \int \frac{e^{4t}(e^{-4t} - 1)(-\frac{1}{2}e^{-t})}{e^{2t} \sqrt{e^{4t}(e^{-4t} + 1)}} dt \\ &= \int \frac{-2e^{3t} + e^{4t}}{e^{2t} \sqrt{e^{4t}(e^{-4t} + 1)}} dt = \int \frac{2e^{3t}}{\sqrt{e^{4t}(e^{-4t} + 1)}} dt \\ &= \int \frac{1}{e^2} \sinh(e^{2t})^{\frac{1}{2}} dt = \frac{1}{e^2} 2(e^{2t})^{\frac{1}{2}} + C = \frac{1}{e^2} \sqrt{e^{4t} + 1} + C \\ &= \sqrt{2\cosh^2 t} + C = \sqrt{e^{4t} + e^{4t}} + C = \sqrt{\frac{1+e^{4t}}{e^4}} + C \\ &= \sqrt{\frac{1+e^{4t}}{e^4}} + C = \frac{\sqrt{1+e^{4t}}}{e^2} + C \end{aligned}$$

$x = e^{-\frac{1}{2}t}$
 $dx = -\frac{1}{2}e^{-t} dt$

Question 86 (*****)

Use a suitable hyperbolic substitution to find the exact value of

$$\int_2^{3.75} \sqrt{(2x+5)(2x-3)} \ dx.$$

$$\frac{195}{16} - 4\ln 2$$

$$\begin{aligned}
 & \int_2^{3.75} \sqrt{(2x+5)(2x-3)} \ dx = \int_2^{3.75} \sqrt{4x^2 + 4x - 15} \ dx = \int_2^{3.75} \sqrt{4(x^2 + x) - 15} \ dx \\
 & = \int_2^{3.75} \sqrt{4(x+1)^2 - 16} \ dx \\
 & \text{Let the integrand to a hyperbolic substitution i.e } 2x+1 = 4\cosh\theta \\
 & \text{Then } 2x+1 = 4\cosh\theta \quad \theta=2 \Rightarrow \cosh\theta = \frac{1}{2} \\
 & 2dx = 4\sinh\theta \ d\theta \quad \theta=3.75 \Rightarrow \cosh\theta = \frac{1}{16} \Rightarrow \sinh\theta = \frac{1}{8} \\
 & dx = 2\sinh\theta \ d\theta \quad \theta=\cosh^{-1}\frac{1}{2} = 0 \\
 & \dots \int_{\theta=0}^{\theta=3.75} \sqrt{4\cosh^2\theta - 16} (2\sinh\theta \ d\theta) = \int_{\theta=0}^{\theta=3.75} \sqrt{4(\cosh^2\theta - 4)} (2\sinh\theta \ d\theta) \\
 & = \int_{\theta=0}^{\theta=3.75} \sqrt{4\sinh^2\theta} (2\sinh\theta \ d\theta) = \int_{\theta=0}^{\theta=3.75} 8\sinh^2\theta \ d\theta \\
 & = \int_{\theta=0}^{\theta=3.75} 8(\frac{1}{2}\cosh 2\theta - \frac{1}{2}) \ d\theta = \int_{\theta=0}^{\theta=3.75} 4\cosh 2\theta - 4 \ d\theta \\
 & = [2\sinh 2\theta - 4\theta]_{\theta=0}^{\theta=3.75} = [4\sinh^2\theta \cosh\theta - 4\theta]_{\theta=0}^{\theta=3.75} \\
 & = [4 \times \frac{15}{16} \cdot \frac{17}{8} - 4\cosh^{-1}\frac{1}{2}] - [4 \times \frac{3}{2} \cdot \frac{5}{2} - 4\cosh^{-1}\frac{1}{2}] \\
 & = \frac{255}{16} - 4\ln\left[\frac{5}{3} + \sqrt{\left(\frac{5}{3}\right)^2 - 1}\right] - \frac{15}{4} + 4\ln\left[\frac{3}{5} + \sqrt{\left(\frac{3}{5}\right)^2 - 1}\right] \\
 & = \frac{195}{16} - 4\ln\left[\frac{5}{3} + \sqrt{\frac{16}{9}}\right] + 4\ln\left[\frac{3}{5} + \sqrt{\frac{16}{25}}\right] \\
 & = \frac{195}{16} + 4\ln 2 - 4\ln \frac{4}{3} = \frac{195}{16} + 4\ln\left(\frac{1}{2}\right) \\
 & = \frac{195}{16} - 4\ln 2
 \end{aligned}$$

$$\begin{aligned}
 \bullet \cosh\theta &= \frac{1}{2} \\
 \cosh\theta &= \frac{1}{16} \\
 \cosh^2\theta - 1 &= \frac{15}{16} \\
 \sinh^2\theta &= \frac{15}{16} \\
 \sinh\theta &= \frac{\sqrt{15}}{4} \\
 \text{9 similarly} \\
 \bullet \cosh\theta &= \frac{1}{16} \\
 \sinh\theta &= \frac{1}{8}
 \end{aligned}$$

Question 87 (*****)

$$\tan 3\theta \equiv \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}.$$

- a) Prove the validity of the above trigonometric identity by writing $\tan 3\theta$ as $\tan(2\theta + \theta)$.
- b) Hence, show clearly that

$$\int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{1-2x^2-3x^4} dx = \ln 2.$$

proof

(a) $\tan 3\theta = \tan(2\theta + \theta) = \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta}$

$$= \frac{\frac{2 \tan \theta}{1 - \tan^2 \theta} + \tan \theta}{1 - \frac{2 \tan \theta}{1 - \tan^2 \theta} \tan \theta} = \dots$$

REVERSE TAN & ROTATE
OF THE FRACTION BY $1 - \tan^2 \theta$

$$= \frac{3 \tan \theta + \tan^2 \theta (1 - \tan^2 \theta)}{1 - \tan^2 \theta - 2 \tan^2 \theta} = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \quad \checkmark$$

REVERSE

(b)
$$\int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{1-2x^2-3x^4} dx = \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{(3x^2+1)(x^2-1)} dx = \dots$$

FRACTION

$$= \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{(-3x^2)(x^2+1)} dx = \int_0^{2-\sqrt{3}} \frac{6x(3-x^2)}{(-3x^2)(x^2+1)} dx = \dots$$

PUT IN

by substitution

$$\dots = \int_0^{\frac{\pi}{4}} \frac{6 \tan \theta (3 - \sec^2 \theta)}{(-3 \sec^2 \theta) (\sec \theta + 1)} d\theta$$

$$= \int_0^{\frac{\pi}{4}} 6 \tan \theta \sec \theta d\theta = \left[2 \ln |\sec \theta| \right]_0^{\frac{\pi}{4}}$$

$$= 2 \ln(\sec \frac{\pi}{4}) - 2 \ln(\sec 0) \quad \checkmark$$

$$= 2 \ln(\sqrt{2}) = \ln 2. \quad \checkmark$$

Question 88 (*****)

$$J = \int_0^{\frac{\pi}{3}} \frac{6\sqrt{3} \cos x}{4 + \sin 2x \tan\left(\frac{1}{2}x\right)} dx.$$

Use the substitution $t = \tan\left(\frac{1}{2}x\right)$ to show that

$$J = \pi - 1.$$

, proof

(a) $\int_0^{\frac{\pi}{3}} \frac{6\sqrt{3} \cos x}{4 + \sin 2x \tan\left(\frac{1}{2}x\right)} dx$ = METHOD: TRIG

$$\begin{aligned} &= \int_0^{\frac{\pi}{3}} \frac{6\sqrt{3} \cos x}{4 + 2\sin x \cdot 2\tan\left(\frac{1}{2}x\right)} dx \\ &= \int_0^{\frac{\pi}{3}} \frac{6\sqrt{3} \cos x}{4 + 4\tan^2 x + 4\tan x} \times \frac{2}{1+t^2} dt \\ &\quad \text{MULITPLY TOP & BOTTOM BY } (1+t^2)^2 \\ &= \int_0^{\frac{\pi}{3}} \frac{6\sqrt{3} (1+t^2)(1-t^2)}{4(1+t^2)^2 + 4t^2(1-t^2)} \times \frac{2}{1+t^2} dt \\ &= \int_0^{\frac{\pi}{3}} \frac{6\sqrt{3} (1-t^2)}{4(1+2t^2+1-t^2)} dt \\ &= \int_0^{\frac{\pi}{3}} \frac{3\sqrt{3} (1-t^2)}{4t^2+2t^2+1-t^2} dt = \int_0^{\frac{\pi}{3}} \frac{3\sqrt{3} (1-t^2)}{3t^2+1} dt \\ &\quad \text{As required} \end{aligned}$$

Let $t = \tan\left(\frac{1}{2}x\right)$
 $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{1}{2}x$
 $\frac{dt}{dx} = \frac{1}{2}(1+t^2)\frac{1}{2}$
 $\frac{dt}{dx} = \frac{1}{2}(1+t^2)$
 $dx = \frac{2}{1+t^2} dt$

$x = 0, t = 0$
 $x = \frac{\pi}{3}, t = \sqrt{3}$

STANDARD INTEGRAL
 $\int t^n dt = \frac{t^{n+1}}{n+1} + C$

$\sin x = \frac{2t}{1+t^2}$
 $\cos x = \frac{1-t^2}{1+t^2}$

(b) $\dots = \int_0^{\frac{\pi}{3}} \frac{3\sqrt{3} (1-t^2)}{3t^2+1} dt = \sqrt{3} \int_0^{\frac{\pi}{3}} \frac{3t^2-3}{3t^2+1} dt = \sqrt{3} \int_0^{\frac{\pi}{3}} \frac{(3t^2+4)-4}{3t^2+1} dt$
 $= \sqrt{3} \int_0^{\frac{\pi}{3}} \left(1 - \frac{4}{3t^2+1}\right) dt = \sqrt{3} \int_0^{\frac{\pi}{3}} \frac{4}{3t^2+1} dt - 1 dt$
 $= \sqrt{3} \left[\frac{4}{3} \arctan\left(\frac{t}{\sqrt{3}}\right) - t \right]_0^{\frac{\pi}{3}} = \sqrt{3} \left[\frac{4\sqrt{3}}{3} \arctan\left(\frac{\pi}{3}\right) - \frac{\pi}{3} \right]$
 $= \sqrt{3} \left[\left(\frac{4\sqrt{3}}{3} \arctan\left(\frac{\pi}{3}\right)\right) - \left(\frac{\pi}{3}\right) \right] - [0 - 0] = 4\pi\arctan\left(\frac{\pi}{3}\right) - \frac{\pi^2}{3}$
 $= 4\pi \frac{\pi}{4} - 1 = \pi - 1$ As required

Question 89 (*****)

By considering the derivatives of $e^x \sin x$ and $e^x \cos x$, find

$$\int e^x (2\cos x - 3\sin x) dx.$$

$$, \frac{1}{2}e^x (5\cos x - \sin x) + C$$

$$\begin{aligned} \frac{d}{dx}(e^x \sin x) &= e^x \sin x + e^x \cos x && \text{(Add & subtract terms)} \\ \frac{d}{dx}(e^x \cos x) &= e^x \cos x - e^x \sin x \\ \left. \begin{aligned} \frac{d}{dx}(e^x \sin x + e^x \cos x) &= 2e^x \cos x \\ \frac{d}{dx}(e^x \sin x - e^x \cos x) &= 2e^x \sin x \end{aligned} \right\} &\rightarrow \frac{d}{dx}(e^x(5\cos x - \sin x)) = 2e^x \cos x \\ \frac{d}{dx}(2e^x \sin x - 3e^x \cos x) &= 2e^x \sin x - 3e^x \cos x && \text{(Differentiate)} \\ 2e^x \cos x - 3e^x \sin x &= 2 \frac{d}{dx}(e^x \sin x) - 3 \frac{d}{dx}(e^x \cos x) \\ 2e^x \cos x - 3e^x \sin x &= \frac{d}{dx}[e^x(\sin x + 2\cos x - 3\sin x + 3\cos x)] \\ 2e^x \cos x - 3e^x \sin x &= \frac{d}{dx}[e^x(5\cos x - \sin x)] \\ \therefore \int e^x(2\cos x - 3\sin x) dx &= \frac{1}{2}e^x(5\cos x - \sin x) + C \end{aligned}$$

Question 90 (*****)

Use integration by parts and suitable trigonometric identities to find

$$\int \sec^3 x dx.$$

$$\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$$

$$\begin{aligned} \int \sec^3 x dx &= \int \sec x \sec^2 x dx \dots \text{by parts} && \text{sec } | \quad \text{sec tan } x \\ \sec x dx &= \sec \tan x - \int \sec \tan x dx \\ \sec x dx &= \sec \tan x - \int (\sec^2 - 1) dx \\ \sec x dx &= \sec \tan x - \int \sec^2 x dx - \sec x dx \\ \sec x dx &= \sec \tan x - \sec x dx + \sec x dx \\ 2 \int \sec x dx &= \sec \tan x + \int \sec x dx && \text{shaded result} \\ 2 \int \sec x dx &= \sec \tan x + \ln |\sec x + \tan x| + C \\ \int \sec x dx &= \frac{1}{2} \sec \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C \end{aligned}$$

Question 91 (*****)

Show that the exact value of

$$\int_0^1 \frac{(1-x)e^x}{x^2 + e^{2x}} dx,$$

can be written as

$$\arccot(e).$$

proof

THE FORM OF THE ANSWER SUGGESTS AN ARCTAN FORM
MAY NEED TO BE "CREATED"

$$\begin{aligned} \int_0^1 \frac{e^x(1-x)}{x^2 + e^{2x}} dx &= \int_0^1 \frac{e^x e^x(1-x)}{x^2 e^{2x} + e^{2x} e^{2x}} dx \\ &= \int_0^1 \frac{e^x(1-x)}{x^2 e^{2x} + 1} dx \\ &= \int_0^1 \frac{e^x(1-x)}{(xe^x)^2 + 1} dx \end{aligned}$$

REMOVED BY SUBSTITUTION

$$\begin{aligned} u &= xe^x & q &= 0 \quad \mapsto u=0 \\ \frac{du}{dx} &= 1xe^x + x(-e^x) & x &= 1 \quad \mapsto u=e \\ \frac{du}{dx} &= e^x - xe^x \\ \frac{du}{dx} &= e^x(1-x) \\ du &= e^x(1-x) dx \end{aligned}$$

TRANSFORMING THE INTEGRAL

$$\begin{aligned} \int_0^1 \frac{e^x(1-x)}{(xe^x)^2 + 1} dx &= \int_0^e \frac{e^x(1-x)}{u^2 + 1} \frac{du}{e^x(1-x)} \\ &= \int_0^e \frac{1}{u^2 + 1} du \\ &= \left[\arctan u \right]_0^e \\ &= \arctan \frac{1}{e} - \arctan 0 \\ &= \arctan \frac{1}{e} \quad // \text{REVIEWED} \end{aligned}$$

Question 92 (*****)

By using the substitution $u = 1 + e^{-x} \tan x$, or otherwise, show that the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{2 - \sin 2x}{(1 + \cos 2x)e^x + \sin 2x} dx,$$

can be written as

$$\ln\left[2e^{-\frac{1}{8}\pi} \cosh\left(\frac{1}{8}\pi\right)\right].$$

, [proof]

USING THE SUBSTITUTION $u = 1 + e^{-x} \tan x$

$$\begin{aligned} \frac{du}{dx} &= -e^{-x} \sec^2 x + e^{-x} \tan^2 x \\ dx &= \frac{du}{-e^{-x} \sec^2 x + e^{-x} \tan^2 x} \\ du &= \frac{e^x}{\sec^2 x - \tan^2 x} du \\ \text{with } x=0 &\mapsto u=1 \\ x=\frac{\pi}{4} &\mapsto u=1+e^{-\frac{\pi}{4}}=k \end{aligned}$$

TRANSFORMING THE INTEGRAL

$$\begin{aligned} &\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2 - \sin 2x}{e^x(1 + \cos 2x) + \sin 2x} dx \\ &= \int_1^{\infty} \frac{2 - 2\sin u \cos u}{e^u(1 + 2\sin u \cos u) + 2\sin u \cos u} du \\ &= \int_1^{\infty} \frac{2 - 2\sin u \cos u}{2e^u \cos^2 u + 2\sin u \cos u} du \\ &= \int_1^{\infty} \frac{1 - \sin u \cos u}{e^u \cos^2 u + \sin u \cos u} du \\ &= \int_1^{\infty} \frac{1 - \sin u \cos u}{e^u \cos^2 u + \sin u \cos u} \times \frac{\sec^2 u}{\sec^2 u} du \end{aligned}$$

NEXT DIVIDE "TOP & BOTTOM" OF THE FIRST FRACTION IN THE NUMERATOR BY $\cos^2 u$

$$\begin{aligned} &= \int_1^{\infty} \frac{\frac{1}{\cos^2 u} - \frac{\sin u \cos u}{\cos^2 u}}{e^u \cos^2 u + \sin u \cos u} \times \frac{e^u}{\sin u - \cos u} du \\ &= \int_1^{\infty} \frac{\sec^2 u - \tan u}{e^u - \tan u} \times \frac{e^u}{\sin u - \cos u} du \\ &= \int_1^{\infty} \frac{e^u}{e^u - \tan u} du \\ &= \int_1^{\infty} \frac{e^u e^u}{e^u e^u - e^u \tan u} du \\ &= \int_1^{\infty} \frac{1}{1 - e^{-u} \tan u} du \\ &= \int_1^{\infty} \frac{1}{u} du \\ &= \left[\ln|u| \right]_1^{1+e^{-\frac{\pi}{4}}} \\ &= \ln(1+e^{-\frac{\pi}{4}}) - \ln 1 = \ln\left[e^{\frac{\pi}{4}}(e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}})\right] \\ &= \ln\left[2e^{-\frac{\pi}{8}}\cosh\frac{\pi}{8}\right] \end{aligned}$$

+ BEGUN

Question 93 (*****)

The function f is a continuous function and a is a real constant.

$$\int_0^a f(x) \, dx \equiv \int_0^a f(a-x) \, dx.$$

a) Prove the validity of the above identity.

b) Hence show clearly that

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx = \frac{1}{4}\pi^2.$$

[proof]

(a) $\int_0^a f(x) \, dx$

Let $x = a-y \Rightarrow y = a-x$ \bullet Limits
 $\frac{dx}{dy} = -1$
 $dx = -dy$

$= \int_0^a f(a-y) (-dy) = \int_0^a f(a-y) \, dy = \int_0^a f(a-x) \, dx$

Moving straight away to step 3

(b) $\int_0^\pi \frac{2x \sin x}{1 + \cos^2 x} \, dx = \text{using part (a)} = \int_0^\pi \frac{(2\pi-2x) \sin(\pi-x)}{1 + \cos^2(\pi-x)} \, dx$

NOW $\sin(\pi-x) = \sin x$ and $\cos(\pi-x) = -\cos x$
 $\Rightarrow \sin(\pi-x) = \sin x$ and $\cos(\pi-x) = -\cos x$

$\int_0^\pi \frac{2x \sin x}{1 + \cos^2 x} \, dx = \int_0^\pi \frac{(2\pi-2x) \sin x}{1 + \cos^2 x} \, dx = \int_0^\pi \frac{2\sin x}{1 + \cos^2 x} \, dx$

$\int_0^\pi \frac{2 \sin x}{1 + \cos^2 x} \, dx = \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} \, dx - \int_0^\pi \frac{2 \sin x}{1 + \cos^2 x} \, dx$

$\int_0^\pi \frac{2 \sin x}{1 + \cos^2 x} \, dx = \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} \, dx$ $\Rightarrow \frac{d}{dx} \arctan(\cos x)$

$2 \int_0^\pi \frac{2 \sin x}{1 + \cos^2 x} \, dx = \pi \left[-\arctan(\cos x) \right]_0^\pi$ $= \frac{-\sin x}{1 + \cos^2 x}$
 $\text{OBWB } u = \cos x$

$2 \int_0^\pi \frac{2 \sin x}{1 + \cos^2 x} \, dx = \pi \left[\arctan(\cos x) \right]_0^\pi$

$2 \int_0^\pi \frac{2 \sin x}{1 + \cos^2 x} \, dx = \pi \left[\arctan(1) - \arctan(-1) \right] = \pi \left[\arctan 1 + \arctan 1 \right]$

$2 \int_0^\pi \frac{2 \sin x}{1 + \cos^2 x} \, dx = \frac{\pi^2}{2}$

$\int_0^\pi \frac{2x \sin x}{1 + \cos^2 x} \, dx = \frac{1}{2}\pi^2$ \therefore as required

Question 94 (*****)

By using a suitable substitution, find the exact value of

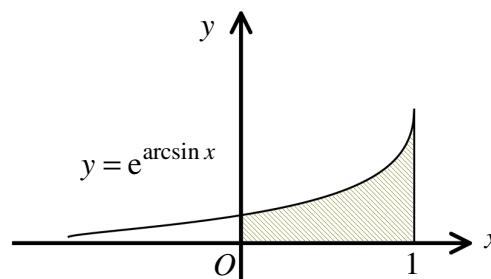
$$\int_1^{\sqrt[4]{17}} \frac{2x}{\sqrt{x^4 - 1}} dx.$$

, $\ln(4 + \sqrt{17})$

$$\begin{aligned} & \int_1^{\sqrt[4]{17}} \frac{2x}{\sqrt{x^4 - 1}} dx = \dots \text{substituting ...} \\ & \dots = \int_0^{\frac{\pi}{2}} \frac{2\sin\theta}{\sqrt{16\sin^4\theta - 1}} (2\cos\theta d\theta) \\ & = \int_0^{\frac{\pi}{2}} \frac{\ln(4 + \sqrt{17})}{\sqrt{16\sin^4\theta}} d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta} d\theta \\ & = \left[\theta \right]_0^{\frac{\pi}{2}} = \ln(4 + \sqrt{17}) // \end{aligned}$$

$$\begin{aligned} x^2 &= \cos^2\theta \\ 2x\,dx &= -\sin^2\theta\,d\theta \\ d\theta &= -\frac{\sin^2\theta}{2x}\,dx \\ \bullet 2=1 &\quad \rightarrow \cos^2\theta \\ \bullet \theta=0 &\rightarrow 0=0 \\ \bullet x=\sqrt[4]{17} &\rightarrow \sqrt[4]{17}=\cos\theta \\ \rightarrow \theta &= \ln\left(\sqrt[4]{17}+\sqrt{17+4}\right) \\ \theta &= \ln\left(\sqrt{17+4}\right) \end{aligned}$$

Question 95 (*****)



The figure above shows the curve with equation

$$y = e^{\arcsin x}, \quad x \in \mathbb{R}, \quad |x| \leq 1.$$

The finite region, shown shaded in the figure, bounded by the curve, the coordinate axes and the straight line with equation $x=1$, is fully revolved about the x axis.

Find, an exact simplified value, for the volume of the solid of revolution formed.

, $\frac{1}{5}\pi(e^\pi - 2)$

<p><u>SETTING UP A VOLUME INTEGRAL</u></p> $V = \pi \int_{-\alpha}^{\alpha} (y(x))^2 dx = \pi \int_0^1 (e^{\arcsin x})^2 dx$ $= \pi \int_0^1 e^{2\arcsin x} dx$ <p><u>BY SUBSTITUTION</u></p> $\theta = \arcsin x \quad \begin{cases} x=0 \rightarrow \theta=0 \\ x=1 \rightarrow \theta=\frac{\pi}{2} \end{cases}$ $\frac{dx}{d\theta} = \cos\theta$ $dx = \cos\theta d\theta$ <p><u>TRANSFORMING THE INTEGRAL</u></p> $\Rightarrow V = \pi \int_0^{\frac{\pi}{2}} e^{2\theta} (\cos\theta d\theta) = \pi \int_0^{\frac{\pi}{2}} e^{2\theta} \cos\theta d\theta$ <p><u>BY PARTS TWICE OR COMPLEX NUMBERS</u></p> $\Rightarrow V = \pi \operatorname{Re} \left\{ \int_0^{\frac{\pi}{2}} e^{2\theta} e^{i0} d\theta \right\}$ $\Rightarrow V = \pi \operatorname{Re} \left\{ \int_0^{\frac{\pi}{2}} e^{(2+i)0} d\theta \right\}$ $\Rightarrow V = \pi \operatorname{Re} \left\{ \left[\frac{1}{2+i} e^{(2+i)\theta} \right]_0^{\frac{\pi}{2}} \right\}$	$\Rightarrow V = \pi \operatorname{Re} \left\{ \frac{1}{2-i} [e^{(2+i)\frac{\pi}{2}} - 1] \right\}$ $\Rightarrow V = \pi \operatorname{Re} \left\{ \frac{2-i}{(2+i)(2-i)} \left[e^{\frac{\pi}{2}(2+i)} - 1 \right] \right\}$ $\Rightarrow V = \pi \operatorname{Re} \left\{ \frac{2-i}{5} \left[e^{\frac{\pi}{2}(2+i)} - 1 \right] \right\}$ $\Rightarrow V = \pi \operatorname{Re} \left\{ \frac{2-i}{5} (ie^{\frac{\pi}{2}} - 1) \right\}$ $\Rightarrow V = \pi \times \left(\frac{1}{5}\right) \operatorname{Re} \left\{ (2-i)(ie^{\frac{\pi}{2}} - 1) \right\}$ $\Rightarrow V = \frac{\pi}{5} \operatorname{Re} \left\{ 2ie^{\frac{\pi}{2}} - 2 + e^{\frac{\pi}{2}} + i \right\}$ $\Rightarrow V = \frac{\pi}{5} (e^{\frac{\pi}{2}} - 2)$
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Question 96 (*****)

$$I = \int_0^1 \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-4)} dx.$$

Use appropriate integration techniques to show that

$$I = 1 + \frac{2}{7} \left[\frac{\pi}{6\sqrt{3}} - 5\ln 3 \right].$$

, proof

SPLIT BY PARTIAL FRACTIONS

$$\begin{aligned} \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-4)} &= \frac{x^4+5x^2+4}{x^4-x^2-12} = \frac{(x^4-x^2-12)+6x^2+16}{(x^4-x^2-12)} \\ &= 1 + \frac{6x^2+16}{x^4-x^2-12} = 1 + \frac{6x^2+16}{(x^2+3)(x^2-4)}. \end{aligned}$$

NOW LET $t = x^2$

$$\begin{aligned} \dots &= 1 + \frac{6t+16}{(t+3)(t-4)} = 1 + \frac{\frac{-2}{t-4}}{t+3} + \frac{\frac{40}{7}}{t-4} \quad (\text{BY CROSS-OF METHOD}) \\ &= 1 + \frac{2}{7} \left[\frac{1}{t+3} + \frac{20}{t-4} \right] \\ &= 1 + \frac{2}{7} \left[\frac{1}{x^2+3} + \frac{20}{x^2-4} \right] \\ &= 1 + \frac{2}{7} \left[\frac{1}{x^2+3} + \frac{20}{(x-2)(x+2)} \right] \\ &= 1 + \frac{2}{7} \left[\frac{1}{x^2+3} + \frac{5}{x-2} + \frac{-5}{x+2} \right] \quad (\text{BY CROSS-OF AGAIN}) \\ &= 1 + \frac{2}{7} \left[\frac{1}{x^2+3} + \frac{5}{x-2} - \frac{5}{x+2} \right] \end{aligned}$$

RETURNING TO THE INTEGRAL

$$\begin{aligned} \int_0^1 \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-4)} dx &= \int_0^1 1 + \frac{2}{7} \left[\frac{1}{x^2+3} + \frac{5}{x-2} - \frac{5}{x+2} \right] dx \\ &= \left[x + \frac{2}{7} \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + 5\ln|x-2| - 5\ln|x+2| \right] \right]_0^1 \\ &= \left[1 + \frac{2}{7} \left[\frac{1}{\sqrt{3}} \times \frac{\pi}{6} + 5\ln 1 - 5\ln 3 \right] - \frac{2}{7} \left[5\ln 2 - 5\ln 2 \right] \right] \\ &= 1 + \frac{2}{7} \left[\frac{\pi}{6\sqrt{3}} - 5\ln 3 \right] \\ &\approx 1 + \frac{1}{7} \left[\frac{\pi}{3\sqrt{3}} - 10\ln 3 \right] \end{aligned}$$

Question 97 (*****)

By using a suitable substitution, find the exact value of

$$\int_{\sqrt[4]{3}}^{\sqrt[4]{8}} \frac{2}{x\sqrt{x^4+1}} dx.$$

, $\ln\left(\frac{3}{2}\right)$

The handwritten solution shows the following steps:

$$\begin{aligned} & \int_{\sqrt[4]{3}}^{\sqrt[4]{8}} \frac{4}{x\sqrt{x^4+1}} dx = \dots \text{ by substitution...} \\ & \dots = \int_{\arcsin(\sqrt[4]{3})}^{\arcsin(\sqrt[4]{8})} \frac{4}{2\sqrt{\sin^2\theta}} (\cos\theta d\theta) \\ & = \int_{\arcsin(\sqrt[4]{3})}^{\arcsin(\sqrt[4]{8})} \frac{2\cos\theta}{\sqrt{1-\cos^2\theta}} d\theta = \int_{\arcsin(\sqrt[4]{3})}^{\arcsin(\sqrt[4]{8})} \frac{\cos\theta}{\sin\theta} d\theta \\ & = \int_{\arcsin(\sqrt[4]{3})}^{\arcsin(\sqrt[4]{8})} \frac{2\sin\theta d\theta}{\sin^2\theta} = \int_{\arcsin(\sqrt[4]{3})}^{\arcsin(\sqrt[4]{8})} \frac{2}{\sin\theta} d\theta \\ & \text{HENCE SUBSTITUTION...} \\ & \dots = \int_2^3 \frac{2}{u^2-1} \left(\frac{du}{\sqrt{1-u^2}} \right) = \int_2^3 \frac{2}{u^2-1} du \\ & = \int_2^3 \frac{2}{(u-1)(u+1)} du \\ & \text{PARTIAL FRACTIONS...} \\ & \dots = \int_2^3 \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du \\ & = \left[\ln|u-1| - \ln|u+1| \right]_2^3 \\ & = (\ln 2 - \ln 4) - (\ln 3 - \ln 5) \\ & = \ln 2 + \ln 5 - \ln 4 \\ & = \ln \left(\frac{2 \cdot 5}{4} \right) \\ & = \ln \frac{5}{2} \end{aligned}$$

Notes on the right side of the box:

- $x = \sqrt[4]{8}$
- $\theta = \arcsin\theta$
- $2x dx = 2\cos\theta d\theta$
- $d\theta = \frac{\cos\theta}{\sin\theta} du$
- $2 = \sqrt[4]{8}$
- $\sin\theta = \sqrt{8}$
- $\theta = \arcsin\sqrt[4]{8}$
- $\theta = \arcsin\sqrt[4]{3}$
- $\theta = \arcsin\sqrt[4]{5}$
- $u = \cos\theta$
- $du = -\sin\theta d\theta$
- $du = -\frac{du}{\sqrt{1-u^2}}$
- $6 = \arcsin\sqrt[4]{8}$
- $\sin\theta = \sqrt{8}$
- $\sin\theta = \sqrt{3}$
- $\sin\theta = 9$
- $\cos\theta = 9$
- $\cos\theta = 3$
- $u = 3$
- Q. SIMILARITY
- $6 = \arcsin\sqrt[4]{5}$
- $u = 2$

Question 98 (*****)

$$I = \int_0^{\arctan(\tanh(\ln 2))} \frac{\sec^2 x \tan 2x}{\tan x - \tan^3 x} dx$$

Use appropriate integration techniques to show that

$$I = k + \ln 2,$$

where k is a rational constant to be found.

You may assume that the limit of the integrand, as x tends to zero, exists.

$$\boxed{\quad}, \quad k = \frac{15}{16}$$

Proceed by a substitution after rewriting the $\tan 2x$ in terms of $\tan x$.

$$\begin{aligned} \int_{0}^{\arctan(\tanh(\ln 2))} \frac{\tan 2x \sec^2 x}{\tan x - \tan^3 x} dx &= \int_{0}^{\arctan(\tanh(\ln 2))} \frac{\sec^2(\tan(2x)) \sec^2 x}{\tan x - \tan^3 x} dx \\ &= \int_{0}^{\arctan(\tanh(\ln 2))} \frac{\sec^2(\tan(2x)) \sec^2 x}{1 - \tan^2 x - \tan^2 x(1 - \tan^2 x)} dx \\ &= \int_{0}^{\arctan(\tanh(\ln 2))} \frac{2 \sec^2(\tan(2x)) \sec^2 x}{(1 - \tan^2 x)^2} dx \\ &= \int_{0}^{\arctan(\tanh(\ln 2))} \frac{\tan(2x)}{(1 - u^2)^2} \left(\frac{du}{\sec^2 x} \right) \\ &= \int_{0}^{\arctan(\tanh(\ln 2))} \frac{\tan(2x)}{(1 - u^2)^2} du \end{aligned}$$

Another substitution is needed.

$$\begin{aligned} &= \int_{0}^{\arctan(\tanh(\ln 2))} \frac{2}{(1 - u^2)^2} (\sec^2 \theta) du \\ &= \int_{0}^{\arctan(\tanh(\ln 2))} \frac{2 \sec^2 \theta}{(1 - u^2)^2} du \end{aligned}$$

$$\begin{aligned} u &= \tan \theta \\ \frac{du}{d\theta} &= \sec^2 \theta \\ du &= \frac{du}{\sec^2 \theta} \\ 2 = \sec^2(\tan(\ln 2)) &\mapsto \tan(\ln 2) \\ u=0 &\mapsto \theta=0 \end{aligned}$$

Subs in and eval.

$$\begin{aligned} &= \int_0^{\ln 2} 2 \sec^2 \theta d\theta = \int_0^{\ln 2} 2 \left(\frac{1}{2} + \frac{1}{2} \sinh(2\theta) \right) d\theta \\ &= \int_0^{\ln 2} 1 + \cosh(2\theta) d\theta = \left[\theta + \frac{1}{2} \sinh(2\theta) \right]_0^{\ln 2} \\ &= \left[\theta + \cosh(\theta) \sinh(\theta) \right]_0^{\ln 2} - \left[\ln 2 + \frac{1}{2} \sinh(2\ln 2) \right] - 0 \\ \text{Now } e^{\ln 2} &= 2 \quad \therefore e^{-\ln 2} = \frac{1}{2} \\ &= \ln 2 + \frac{1}{2} \left[e^{\ln 2} - e^{-\ln 2} \right] \times \frac{1}{2} \left[e^{\ln 2} + e^{-\ln 2} \right] \\ &= \ln 2 + \frac{1}{4} (2 - \frac{1}{2})(2 + \frac{1}{2}) \\ &= \ln 2 + \frac{1}{2} \times \frac{3}{2} \times \frac{5}{2} \\ &= \frac{15}{16} + \ln 2 \end{aligned}$$

Question 99 (*****)

Use a suitable hyperbolic substitution to find a simplified expression for

$$\int \sqrt{(2x+5)(2x-3)} \, dx.$$

$$\boxed{\quad}, \frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4\ln\left[2x+1+\sqrt{(2x+5)(2x-3)}\right] + C$$

$$\begin{aligned}
 \int \sqrt{(2x+5)(2x-3)} \, dx &= \int \sqrt{4x^2 + 4x - 15} \, dx = \int \sqrt{4(x^2 + x + 1) - 16} \, dx \\
 &= \int \sqrt{(2x+1)^2 - 16} \, dx \\
 &\quad \text{• By a hyperbolic substitution } \Rightarrow \text{ let } 2x+1 = 4\sinh\theta \quad \text{This is because } (2x+1)^2 - 16 \text{ is a difference of squares} \\
 &\quad \text{let } 2x+1 = 4\sinh\theta \quad \text{so } 2dx = 4\sinh\theta d\theta \\
 &\quad 2dx = 4\sinh\theta d\theta \quad \text{so } d\theta = \frac{2dx}{4\sinh\theta} = \frac{dx}{2\sinh\theta} \\
 &\quad \dots = \int \sqrt{(4\sinh\theta)^2 - 16} \quad (2\sinh\theta d\theta) = \int \sqrt{16(\cosh^2\theta - 1)} (2\sinh\theta d\theta) \\
 &\quad = \int \sqrt{16\sinh^2\theta} (2\sinh\theta d\theta) = \int 4\sinh\theta \cdot 2\sinh\theta d\theta \\
 &\quad \quad \quad \begin{array}{l} \sinh\theta = \frac{1}{2} - \frac{1}{2}\cosh 2\theta \\ -\sinh^2\theta = \frac{1}{2} - \frac{1}{2}\cosh 2\theta \end{array} \\
 &\quad = \int 8(-\frac{1}{2}\cosh 2\theta - \frac{1}{2}) d\theta = \int 4\sinh\theta - 4 \, d\theta = 2\sinh\theta - 4\theta + C \\
 &\quad - 4\sinh\theta \cosh\theta - 4\theta + C = \frac{1}{2}(4\sinh\theta)(\cosh\theta) - 4\theta + C \\
 &\quad = \frac{1}{2} \left[4\sinh\theta \sqrt{(\cosh\theta)^2 - 1} \right] - 4\theta \cosh\theta + C \\
 &\quad = \frac{1}{2} \left[(2x+1) \sqrt{(2x+4)^2 - 16} \right] - 4\ln\left[\frac{2x+1}{4} + \sqrt{\frac{(2x+1)^2 - 16}{16}}\right] + C \\
 &\quad = \frac{1}{2} \left[(2x+1) \sqrt{(2x+5)(2x-3)} \right] - 4\ln\left[2x+1 + \sqrt{(2x+5)(2x-3)}\right] + C \\
 &\quad = \frac{1}{4}(2x+1)\sqrt{(2x+5)(2x-3)} - 4\ln\left[2x+1 + \sqrt{(2x+5)(2x-3)}\right] + C
 \end{aligned}$$

Question 100 (*****)

It is given that the following integral converges to a finite value L .

$$\int_0^1 \frac{\ln x}{x-1} dx.$$

Show, with details workings, that

$$L = \sum_{r=1}^{\infty} \left[\frac{1}{r^2} \right].$$

You may further assume that integration and summation commute.

[SPN X], [proof]

The image shows handwritten mathematical work on a grid background. It starts with a substitution $u = 1-x$, followed by a series expansion of $\ln(1-u)$ and $\ln(1+u)$. The Riemann zeta function is then introduced, and the integral is shown to be equivalent to the sum of the first two terms of the Riemann zeta function for even integers.

START WITH A SUBSTITUTION:

$$\begin{cases} u = 1-x \\ du = -dx \\ x = 1-u \\ 1 \mapsto 0 \\ 0 \mapsto 1 \end{cases}$$

$$\int_0^1 \frac{\ln x}{x-1} dx = \int_{-1}^0 \frac{\ln(1-u)}{u} du$$

NOW RECALL RIEGMAN SERIES:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\ln(1+u) = \frac{u}{1} - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots$$

$$\ln(1-u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} u^n$$

RETURNING TO THE INTEGRAL:

$$\dots = \int_{-1}^0 \frac{1}{u} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} u^n \right) du = \int_{-1}^0 \frac{1}{u} \sum_{n=1}^{\infty} (-1)^{n+1} u^{n-1} du$$

DIMINISHING THE ORDER OF INTEGRATION AND SUMMATION, CARRYING INDEXES:

$$\dots = \sum_{n=1}^{\infty} \left[(-1)^{n+1} \int_{-1}^0 \frac{1}{u} u^{n-1} du \right] = \sum_{n=1}^{\infty} \left[(-1)^{n+1} \int_0^{-1} u^{n-1} du \right]$$

$$= \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n} \left[\frac{1}{n} u^n \right]_0^{-1} \right] = \sum_{n=1}^{\infty} \left[(-1)^n \times \frac{1}{n} (-1)^n \right]$$

$$= \sum_{n=1}^{\infty} \left[(-1)^{2n} \times (-1)^n \right] = \left[\sum_{n=1}^{\infty} (-1)^{2n} \times (-1)^n \right]$$

$$= \sum_{n=1}^{\infty} \left[\frac{(-1)^{2n}}{n^2} \right] = \left\{ (-1)^{2n+2} = [(-1)^2]^n = 1^n = 1 \right\}$$

$$\therefore \int_0^1 \frac{\ln x}{x-1} dx = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

A.B.P. 100

Question 101 (*****)

a) If $p \in (0, \infty)$, show that

$$\lim_{x \rightarrow 0^+} [x^p \ln x] = 0, \quad x \in (0, \infty).$$

b) Hence find a simplified expression for

$$\int_0^1 x^n \ln x \, dx, \quad n \in \mathbb{N}.$$

c) Hence, showing a detailed method, evaluate

$$\int_0^1 [\ln(1-x)] \ln x \, dx.$$

You may assume without proof that

- the integral converges.

- integration and summation commute.

- $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2, \quad n \in \mathbb{N}.$

$\boxed{-\frac{1}{(n+1)^2}}$, $\boxed{2 - \frac{1}{6}\pi^2}$

a) THE LIMIT IS OF THE TYPE " $\frac{0}{0}$ " OR " $\frac{\infty}{\infty}$ ". MANIPULATE IT FURTHER

$$\lim_{x \rightarrow 0^+} [x^p \ln x] = \lim_{x \rightarrow 0^+} \left[\frac{\ln x}{x^{-p}} \right] \quad \text{OF THE TYPE } \frac{\infty}{\infty}$$

APPLY L'HOSPITAL RULE

$$= \lim_{x \rightarrow 0^+} \left[\frac{\frac{1}{x}}{-px^{-p-1}} \right] = \lim_{x \rightarrow 0^+} \left[\frac{\frac{1}{x}}{-\frac{p}{x^{p+1}}} \right] = \lim_{x \rightarrow 0^+} \left[\frac{x^{p+1}}{-p} \right] = -\frac{1}{p} \lim_{x \rightarrow 0^+} [x^{p+1}] = 0$$

b) INTEGRATION BY PARTS

$$\int_0^1 x^n \ln x \, dx = \left[\frac{x^{n+1}}{n+1} \right]_0^1 - \frac{1}{n+1} \int_0^1 x^n \, dx$$

AT $x=1$ THE TERM IS ZERO AS $n>0$.
AT $x=0^+$ THE TERM IS ZERO AS THERE IS NO TERM x^n .

$$\int_0^1 x^n \ln x \, dx = -\frac{1}{n+1} \int_0^1 x^n \, dx = -\frac{1}{n+1} \left[\frac{x^{n+1}}{n+1} \right]_0^1 = -\frac{1}{(n+1)^2}$$

c) APPROXIMATE THE INTEGRAL USING RADIICAL SERIES. SINCE

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \dots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$$

$$\int_0^1 \ln(1-x) \ln x \, dx = \int_0^1 (\ln x) \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n \right] dx$$

$$= \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n} \int_0^1 x^n \ln x \, dx \right]$$

$$= -\sum_{n=1}^{\infty} \left[\frac{1}{n} \left(-\frac{1}{(n+1)^2} \right) \right] \rightarrow \text{ANSWER}$$

NEXT PROCEED BY PARTIAL FRACTION

$$\frac{1}{n(n+1)^2} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{(n+1)^2}$$

$$\boxed{1 = A(n+1)^2 + Bn(n+1) + Cn}$$

• IF $n=0$ • IF $n=1$
 $A = 1$ $B = -1$
 $C = -1$ $A = 2A + B + C$
 $= 2A - 1$ $1 + 2A + 2B - 1$
 $= 2B$ $B = -1$

RETURNING TO THE SUM

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} \right] - \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} \right] - \left[1 + \frac{1}{2} \left(\frac{1}{(n+1)^2} \right) \right] + 1$$

$$= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) - \dots - \frac{1}{6} + 1$$

$$= 1 - \frac{1}{6} + 1$$

$$= 2 - \frac{1}{6}\pi^2$$

$$= \frac{1}{3}(12 - \pi^2)$$

Question 102 (*****)

$$I = \int \cos(\ln x) dx \quad \text{and} \quad J = \int \sin(\ln x) dx$$

- a) Use an appropriate method to find expressions for I and J .
- b) Use the integral $\int x^i dx$, where i is the imaginary unit, to verify the answers given in part (a).
- c) Find an exact simplified value for

$$\int_1^{e^{\frac{\pi}{2}}} 2x^i dx.$$

 , $I = \frac{1}{2}x[\sin(\ln x) + \cos(\ln x)]$, $J = \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)]$,

$$\int_1^{e^{\frac{\pi}{2}}} 2x^i dx = \left(e^{\frac{1}{2}\pi} - 1 \right) + \left(e^{\frac{1}{2}\pi} + 1 \right)i$$

a) STARTING WITH A SUBSTITUTION

$$\begin{aligned} u &= \ln x & I &= \int \cos(u) du = \int \cos(e^u) du \\ u &= e^u & &= \int e^u \cos(u) du \end{aligned}$$

NOW DOUBLE INTEGRATION BY PARTS, COMPLEX EXPONENTIALS, OR INSPIRATION

$$\begin{aligned} \frac{d}{du} [e^u(P+Qiu)] &= \frac{d}{du}(P\cos(u) + Q\sin(u)) + \frac{d}{du}(-P\sin(u) + Q\cos(u)) \\ &= e^u[P\cos(u) + Q\sin(u)] + (Q-P)\sin(u) \\ P+Q &= 1 \quad Q-P=0 \\ P &= \frac{1}{2}, \quad Q = \frac{1}{2} \\ \Rightarrow I &= \frac{1}{2}e^u(\cos(u) + \sin(u)) \\ \Rightarrow I &= \frac{1}{2}x[\cos(\ln x) + \sin(\ln x)] \end{aligned}$$

USING THE SAME SUBSTITUTION AND APPROXIMATION

$$\begin{aligned} J &= \int \sin(u) du = \dots \int e^u \sin(u) du \dots \text{BUT NOW} \\ &\quad \begin{matrix} P+Q=0 \\ Q-P=1 \end{matrix} \\ &\quad \begin{matrix} Q=\frac{1}{2} & \text{ & } P=-\frac{1}{2} \end{matrix} \\ \Rightarrow J &= \frac{1}{2}e^u(\sin(u) - \cos(u)) \\ \Rightarrow J &= \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] \end{aligned}$$

b) SIMPLY BY CONSIDERING x^i

$$\begin{aligned} x^i &= e^{i\ln x} = e^{i\ln x} = \cos(\ln x) + i\sin(\ln x) \\ &\quad \{x^i = \cos(\ln x) + i\sin(\ln x)\} \\ \int x^i dx &= \frac{1}{1+i} x^{1+i} + C \\ \int \cos(\ln x) + i\sin(\ln x) dx &= \frac{-i}{2} x^{1+i} + C \\ \int \cos(\ln x) dx + i \int \sin(\ln x) dx &= \frac{1}{2}(-i)^2 x^{1+i} + C \\ I + iJ &= \frac{1}{2}(-i)[\cos(\ln x) + i\sin(\ln x)] + C \\ I + iJ &= \frac{1}{2}[\cos(\ln x) + i\sin(\ln x)] + \frac{1}{2}[\cos(\ln x) + i\sin(\ln x)]i \\ I + iJ &= \frac{1}{2}x[\cos(\ln x) + i\sin(\ln x)] + \frac{1}{2}x[\sin(\ln x) - i\cos(\ln x)] \\ \therefore I - \frac{1}{2}x[\cos(\ln x) + i\sin(\ln x)] &\quad \text{ & } J = \frac{1}{2}x[\sin(\ln x) - i\cos(\ln x)] \end{aligned}$$

c) FINDING (AND, PART (b))

$$\begin{aligned} \int x^i dx &= x^{\frac{i}{1+i}} = x^{\frac{1}{1+i}} \\ &= 2 \left[\frac{1}{2}x[\cos(\ln x) + i\sin(\ln x)] + \frac{1}{2}[\sin(\ln x) - i\cos(\ln x)]i \right]_{-1}^{\frac{\pi}{2}} \\ &= \left[2 \left[\cos(\ln x) + i\sin(\ln x) + i[\sin(\ln x) - i\cos(\ln x)] \right] \right]_{-1}^{\frac{\pi}{2}} \\ &= e^{\frac{\pi}{2}}[(i+1) + i(-i)] - i[(i+0) + (0-i)] \\ &= e^{\frac{\pi}{2}}(i+1) - i^2 \\ &= (e^{\frac{\pi}{2}} - 1) + i(e^{\frac{\pi}{2}}) \end{aligned}$$

Question 103 (*****)

$$I(\alpha) = \int_0^\pi \frac{1}{\alpha - \cos x} dx, |\alpha| > 1.$$

Use an appropriate method to show that

$$I(\alpha) = \frac{\pi}{\sqrt{\alpha^2 - 1}}.$$

, proof

BY LIETTE T SUBSTITUTION

$$\begin{aligned} & \int_0^\pi \frac{dx}{\alpha - \cos x} \\ & \rightarrow \int_0^\infty \frac{1}{\alpha - \frac{1-t^2}{1+t^2}} \times \frac{2 dt}{1+t^2} = \int_0^\infty \frac{2}{\alpha(1+t^2) - (1-t^2)} dt \\ & = \int_0^\infty \frac{2}{(x+1)^2 + (\alpha-1)} dt = -\frac{1}{\alpha+1} \int_0^\infty \frac{2}{t^2 + \frac{\alpha-1}{\alpha+1}} dt \\ & = \frac{2}{\alpha+1} \int_0^\infty \frac{1}{t^2 + \left(\frac{\sqrt{\alpha-1}}{\sqrt{\alpha+1}}\right)^2} dt \end{aligned}$$

STANDARD INTEGRAL TO REACTON

$$\begin{aligned} & = \frac{2}{\alpha+1} \times \frac{1}{\sqrt{\alpha+1}} \left[\arctan \left[\frac{t}{\sqrt{\alpha+1}} \right] \right]_0^\infty \\ & = \frac{2}{\alpha+1} \times \frac{\sqrt{\alpha+1}}{\sqrt{\alpha+1}} \left[\frac{\pi}{2} - 0 \right] \\ & = \frac{2}{\sqrt{\alpha+1}} \times \frac{\pi}{2} \\ & = \frac{\pi}{\sqrt{\alpha^2 - 1}} \end{aligned}$$

AS REQUIRED

LET $t = \tan \frac{x}{2}$

$dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$

$dx = \frac{2}{\sec^2 \frac{x}{2}} dt$

$dx = \frac{2}{1+2 \frac{1}{t^2}} dt$

$dx = \frac{2 t^2}{1+t^2} dt$

• 

$\tan \frac{\theta}{2} = t \Rightarrow \frac{t}{1} = \frac{\sqrt{\alpha-1}}{\sqrt{\alpha+1}}$

$\sin \frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$

$\cos \frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$

THIS

$\cos x = \cos \theta \cos \frac{\theta}{2}$

$\cos x = \frac{1}{1+t^2} \times \frac{1}{\sqrt{1+t^2}}$

• UNLESS

$3(t^2) = 1-t^2 \Rightarrow 4t^2 = 1 \Rightarrow t=0$

Question 104 (*****)

Use appropriate integration techniques to show that

$$\int_0^{\frac{1}{2}} \frac{\arcsin \sqrt{x} - \arccos \sqrt{x}}{\arcsin \sqrt{x} + \arccos \sqrt{x}} dx = \frac{1}{\pi} - \frac{1}{2}.$$

[P.37], proof

$$\begin{aligned}
 & \int_0^{\frac{1}{2}} \frac{\arcsin \sqrt{x} - \arccos \sqrt{x}}{\arcsin \sqrt{x} + \arccos \sqrt{x}} dx = \dots \text{ BY SUBSTITUTION} \\
 & = \int_0^{\frac{1}{2}} \frac{\arcsin y - \arccos y}{\arcsin y + \arccos y} (2y dy) = \int_0^{\frac{1}{2}} \frac{\arcsin y - (\frac{\pi}{2} - \arccos y)}{\arcsin y + \arccos y} (2y dy) \\
 & = \int_0^{\frac{1}{2}} \frac{2\arccos y - \frac{\pi}{2}}{\arcsin y + \arccos y} (2y dy) = \int_0^{\frac{1}{2}} \frac{2y \arccos y - 2y}{\arcsin y + \arccos y} dy \\
 & = \int_0^{\frac{1}{2}} 2y \arccos y dy - \int_0^{\frac{1}{2}} 2y dy = \frac{1}{2} \int_0^{\frac{1}{2}} 2y \arccos y dy - \int_0^{\frac{1}{2}} 2y dy \\
 & \quad \downarrow \text{BY ANOTHER SUBSTITUTION} \\
 & \quad \begin{cases} y = \sin \theta \\ dy = \cos \theta d\theta \\ y=0 \rightarrow \theta=0 \\ y=\frac{1}{2} \rightarrow \theta=\frac{\pi}{6} \end{cases} \\
 & = \frac{1}{2} \int_0^{\frac{\pi}{6}} 2 \sin \theta \arccos(\cos \theta) \cos \theta d\theta - \frac{1}{2} \\
 & = \frac{1}{2} \int_0^{\frac{\pi}{6}} 2 \theta \sin \theta \cos \theta d\theta - \frac{1}{2} = -\frac{1}{2} + \frac{1}{2} \int_0^{\frac{\pi}{6}} 4 \theta \sin \theta d\theta \\
 & \quad \downarrow \\
 & \quad \text{BY PARTS} \\
 & \quad \begin{cases} u = \theta \\ v = 4 \sin \theta \\ du = d\theta \\ dv = 4 \cos \theta d\theta \end{cases} \\
 & = \frac{1}{2} \left[(-2 \theta \cos \theta) \Big|_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} 2 \cos \theta d\theta \right] - \frac{1}{2} \\
 & = \frac{1}{2} \left[\sin 2\theta \Big|_0^{\frac{\pi}{6}} \right] - \frac{1}{2} \\
 & = \frac{1}{12} - \frac{1}{2}
 \end{aligned}$$

Question 105 (*****)

If $0 < k < \sqrt{2} - 1$ prove that

$$\int_k^{\frac{1-k}{1+k}} \frac{\ln x}{x^2-1} dx = \int_k^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x} dx.$$

You need not evaluate these integrals.

, proof

Starting on the LHS and use integration by parts

$$\begin{aligned} \int_k^{\frac{1-k}{1+k}} \frac{\ln x}{x^2-1} dx &= \int_k^{\frac{1-k}{1+k}} (\ln x) \frac{1}{x^2-1} dx \\ &= \left[-(\ln x) \operatorname{artanh} x \right]_k^{\frac{1-k}{1+k}} - \int_k^{\frac{1-k}{1+k}} \frac{1}{x} \operatorname{artanh} x dx \\ &= \left[(\ln x) \operatorname{artanh} x \right]_{\frac{1-k}{1+k}}^k + \int_k^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x} dx \end{aligned}$$

Now it suffices to show that $\left[(\ln x) \operatorname{artanh} x \right]_{\frac{1-k}{1+k}}^k = 0$

$$\begin{aligned} \left[(\ln x) \operatorname{artanh} x \right]_{\frac{1-k}{1+k}}^k &= \left[\ln x \times \frac{1}{2} \operatorname{artanh} \frac{1-k}{1+k} \right]_{\frac{1-k}{1+k}}^k \\ &= \frac{1}{2} \left[(\ln k) \operatorname{artanh} \left(\frac{1-k}{1+k} \right) - \left(\ln \left(\frac{1-k}{1+k} \right) \operatorname{artanh} \left(\frac{1+k}{1-k} \right) \right) \right] \\ &= \frac{1}{2} \left[(\ln k) \operatorname{artanh} \left(\frac{1-k}{1+k} \right) - \operatorname{artanh} \left(\frac{1-k}{1+k} \right) \ln \left(\frac{1+k}{1-k} \right) \right] \\ &= \frac{1}{2} \left[(\ln k) \operatorname{artanh} \left(\frac{1-k}{1+k} \right) - \ln \left(\frac{1-k}{1+k} \right) \operatorname{artanh} \left(\frac{1}{k} \right) \right] \\ &= \frac{1}{2} \left[(\ln k) \operatorname{artanh} \left(\frac{1-k}{1+k} \right) - \operatorname{artanh} \left(\frac{1-k}{1+k} \right) \ln \left(\frac{1}{k} \right) \right] \\ &= \frac{1}{2} \left[(\ln k) \operatorname{artanh} \left(\frac{1-k}{1+k} \right) - \operatorname{artanh} \left(\frac{1-k}{1+k} \right) \ln \left(\frac{1}{k} \right) \right] \quad \ln \left(\frac{1}{k} \right) = -\ln \left(\frac{1}{k} \right) \\ &\therefore \int_k^{\frac{1-k}{1+k}} \frac{\ln x}{x^2-1} dx = \int_k^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x} dx \end{aligned}$$

Question 106 (***)**

Use integration by parts and trigonometric identities to find the exact value of

$$\int_0^{\frac{\pi}{6}} 12 \sec^3 x \ dx.$$

$$4 + 3\ln 3$$

$$\begin{aligned} \int_0^{\frac{\pi}{6}} 12 \sec^3 x \ dx &= \int_0^{\frac{\pi}{6}} 12 \sec x \sec^2 x \ dx \dots \text{by part} \\ &= \left[12 \sec x \tan x \right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} 12 \sec x \tan x \ dx \\ &= \left[12 \sec x \tan x \right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} 12 \sec x (\sec^2 x - 1) \ dx \\ &= 12 \sqrt{3} \sqrt{3} - \int_0^{\frac{\pi}{6}} 12 \sec x - 12 \sec x \ dx \\ &= 8 - \int_0^{\frac{\pi}{6}} 12 \sec^2 x \ dx + \int_0^{\frac{\pi}{6}} 12 \sec x \ dx \\ &= 8 - \int_0^{\frac{\pi}{6}} 12 \sec^2 x \ dx + \left[12 \ln |\sec x + \tan x| \right]_0^{\frac{\pi}{6}} \\ &= 8 - \int_0^{\frac{\pi}{6}} 12 \sec^2 x \ dx + 12 \ln \left| \frac{\sqrt{10}}{\sqrt{3}} - 1 \right| - 12 \ln 1 \\ &= 8 - \int_0^{\frac{\pi}{6}} 12 \sec^2 x \ dx + 12 \ln \frac{\sqrt{10}}{\sqrt{3}} \leftarrow 12 \ln \frac{\sqrt{10}}{\sqrt{3}} = 6 \ln 3 \\ \text{Thus so far...} \\ \int_0^{\frac{\pi}{6}} 12 \sec^2 x \ dx &= 8 + 6 \ln 3 - \int_0^{\frac{\pi}{6}} 12 \sec^2 x \ dx \\ 2 \int_0^{\frac{\pi}{6}} 12 \sec^2 x \ dx &= 8 + 6 \ln 3 \\ \therefore \int_0^{\frac{\pi}{6}} 12 \sec^2 x \ dx &= 4 + 3 \ln 3 \end{aligned}$$

Question 107 (*****)

Determine, as an exact simplified fraction, the value of the following integral.

$$\int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2 - 16x + 15)^4 \, dx .$$

	$\frac{128}{315}$
--	-------------------

PROCESSED BY TRIAL & ERROR

$$\begin{aligned} \int_{\frac{3}{2}}^{\frac{5}{2}} (4x^2 - 16x + 15)^4 \, dx &= \int_{\frac{3}{2}}^{\frac{5}{2}} [(2x-3)(2x-5)]^4 \, dx \\ &= \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^4 \, dx \\ &\text{INTEGRATE, BY PARTS} \\ &= \left[\frac{1}{10} (2x-3)^5 (2x-5)^5 \right]_{\frac{3}{2}}^{\frac{5}{2}} - \frac{1}{5} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^3 (2x-5)^5 \, dx \\ &\quad \boxed{(2x-3)^5} \quad \boxed{8(2x-5)^5} \\ &\quad \boxed{\frac{1}{5}(2x-5)^4} \quad \boxed{(2x-5)^4} \\ &\text{INTEGRATE, BY PARTS, FOR A SECOND TIME} \\ &= \left[\frac{1}{50} \left[(2x-3)^6 (2x-5)^5 - \frac{1}{2} (2x-3)^4 (2x-5)^6 \right] \right]_{\frac{3}{2}}^{\frac{5}{2}} \\ &\quad \boxed{(2x-3)^6} \quad \boxed{(2x-5)^5} \\ &\quad \boxed{\frac{1}{2}(2x-3)^4} \quad \boxed{(2x-5)^6} \\ &\text{BY PARTS FOR A THIRD TIME} \\ &= \frac{1}{250} \left[\left[\frac{1}{2} (2x-3)^7 (2x-5)^5 - \frac{1}{2} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^5 (2x-5)^7 \, dx \right] \right] \\ &= -\frac{1}{250} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^5 (2x-5)^7 \, dx \end{aligned}$$

SIMPLIFY THE LAST INTEGRATION BY PARTS

$$\begin{aligned} &= \frac{1}{250} \left[\left[\frac{1}{10} (2x-3)^6 (2x-5)^6 - \frac{1}{6} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-3)^4 (2x-5)^8 \, dx \right] \right] \\ &= \frac{1}{250} \int_{\frac{3}{2}}^{\frac{5}{2}} (2x-5)^8 \, dx \\ &= \frac{1}{1250} \left[\frac{1}{9} (2x-5)^9 \right]_{\frac{3}{2}}^{\frac{5}{2}} \\ &= \frac{1}{1250} \left[0 - (-2)^9 \right] \\ &= \frac{512}{1250} \\ &= \frac{128}{3125} \end{aligned}$$

Question 108 (*****)

Use the substitution $u = \sqrt{\frac{1+x}{1-x}}$, to evaluate the following integral.

$$\int_0^{\frac{1}{4}} \frac{3}{(4x+5)\sqrt{1-x^2} - 3(1-x^2)} dx.$$

Give the answer in the form $\frac{1}{7}(a + \sqrt{b})$, where a and b are integers.

 , $\frac{1}{7}(6 - \sqrt{15})$

SIMPLY BY READING THE SUBSTITUTION Given

$u = \sqrt{\frac{1+x}{1-x}}$ $\frac{du}{dx} = \frac{1}{2\sqrt{\frac{1+x}{1-x}}} \cdot \frac{1}{(1-x)^2} = \frac{1}{2\sqrt{\frac{1+x}{1-x}}} \cdot \frac{2x}{(1-x)^2}$ $1-x^2 = 1 - \frac{(1-u^2)(1+u^2)}{(1+u^2)(1-u^2)}$
 $u^2 = \frac{1+x}{1-x}$ $\frac{du}{dx} = \frac{2x}{(1-x)^2} = \frac{2x}{(1-x)^2} \cdot 2x + 2u$ $1-x^2 = \frac{(1+u^2)(1+u^2) - (1-u^2)(1+u^2)}{(1+u^2)(1-u^2)}$
 $1-x^2 = 1+2x$ $\frac{du}{dx} = \frac{4u}{(1-x)^2}$ $1-x^2 = \frac{4u^2}{(1+u^2)(1-u^2)}$
 $1-x^2 = u^2(1-x^2)$ $\frac{du}{dx} = \frac{4u}{(1+u^2)(1-u^2)}$ $1-x^2 = \frac{4u^2}{(1+u^2)(1-u^2)}$
 $1-x^2 = u^2(1-x^2)$ $\times \frac{du}{dx} = \frac{4u}{(1+u^2)(1-u^2)}$ $1-x^2 = \frac{4u^2}{(1+u^2)(1-u^2)}$
 $\therefore 2 = \frac{u^2-1}{u^2+1}$ $\therefore \frac{du}{dx} = \frac{4u}{(1+u^2)(1-u^2)}$ $\therefore 1-x^2 = \frac{4u^2}{(1+u^2)(1-u^2)}$
 $\therefore 2 = \frac{u^2-1}{u^2+1}$
 $\therefore 4(x+5) = 4 \left(\frac{u^2-1}{u^2+1} \right) + 5 = \frac{4u^2 - 4 + 5u^2 + 5}{u^2+1} = \frac{9u^2 + 1}{u^2+1}$
 $\therefore 4x+5 = 9u^2 + 1$
 $\therefore 4x+5 = 9u^2 + 1$
 $\therefore 2x+\frac{5}{2} \rightarrow u = \sqrt{\frac{1+u^2}{1-u^2}} = \infty$

BEGIN THE TRANSFORMATION

$$\int_0^{\frac{1}{4}} \frac{3}{(4x+5)\sqrt{1-x^2} - 3(1-x^2)} dx = \int_1^{\infty} \frac{3}{\frac{9u^2+1}{u^2+1} \cdot \frac{2u}{(1-u^2)^2} - 3 \cdot \frac{4u^2}{(1-u^2)^2}} \times \frac{4u}{(1+u^2)(1-u^2)} du$$

$$= \int_1^{\infty} \frac{12u}{2u(1+u^2) - \frac{12u^2}{(1+u^2)^2}} du$$

$$= \int_1^{\infty} \frac{12u}{2u(1+u^2) - 12u^2} du$$

$$= \int_1^{\infty} \frac{6}{u^2 + 1 - 6u} du$$

THIS IS NOW A STRAIGHT FORWARD INTEGRATION

$$= \int_1^{\infty} \frac{6}{(2u-3)^2} du = \left[-\frac{2}{2u-3} \right]_1^{\infty} = \left[\frac{2}{1-2u} \right]_1^{\infty} = \left[\frac{2}{1-2u} \right]_1^{\frac{1}{4}}$$

EVALUATING

$$= \frac{2}{1-4\sqrt{2}} - \frac{2}{1-3} = \frac{2}{1-\sqrt{8}} + 1$$

$$= \frac{2(1+\sqrt{8})}{1-8} + 1 = -\frac{(1+\sqrt{8})}{3} + 1$$

$$= \frac{1+\sqrt{8}}{-7} + 1 = -\frac{1}{7} - \frac{1}{7}\sqrt{8} + 1$$

$$= \frac{6}{7} - \frac{1}{7}\sqrt{8}$$

$$= \frac{1}{7}(6 - \sqrt{15})$$

NOTE THAT THE SUBSTITUTION $2=x$ OR $2=\sqrt{1-x^2}$ IS EASIER

By the "little 'c'" identifier is far more natural; you have to change it in its manipulations

Question 109 (*****)

Use the substitution $x = \frac{ab}{t}$ to find the exact value of

$$\int_0^\infty \frac{\ln x}{(x+a)(x+b)} dx,$$

where a and b are real positive constants with $a > b$.

$$\boxed{\quad}, \boxed{\frac{\ln(ab)}{2(a-b)} \ln\left(\frac{a}{b}\right) = \frac{(\ln a)^2 - (\ln b)^2}{2(a-b)}}$$

Start with the substitution

$$x = \frac{ab}{t} \quad t = \frac{ab}{x} dt = -\frac{ab}{t^2} dt$$

$$\begin{aligned} &\int_0^\infty \frac{\ln x}{(x+a)(x+b)} dx = \frac{\ln(ab)}{2(a-b)} \ln\left(\frac{a}{b}\right) \\ &\dots = \int_0^\infty \frac{\ln\left(\frac{ab}{t}\right) - \ln t}{\left(\frac{ab}{t}+a\right)\left(\frac{ab}{t}+b\right)} \left(-\frac{ab}{t^2} dt\right) \\ &\quad t=0 \mapsto t=\infty \\ &\quad t=\infty \mapsto t=0 \\ &= \int_0^\infty \frac{\ln(ab) - \ln t}{a^2(b+\frac{a}{t})(b+\frac{b}{t})} dt \\ &= \int_0^\infty \frac{\ln(ab) - \ln t}{ab^2(b+\frac{a}{t})(a+\frac{b}{t})} \frac{ab}{t^2} dt \\ &= \int_0^\infty \frac{\ln(ab) - \ln t}{\frac{ab}{t}(b+t)(a+t)^2} dt \\ &= \int_0^\infty \frac{\ln(ab) - \ln t}{(t+a)(t+b)} dt \\ &\bullet \text{ Summing up, we find:} \\ &I = \int_0^\infty \frac{\ln x}{(x+a)(x+b)} dx = \int_0^\infty \frac{\ln(ab)}{(t+a)(t+b)} dt - \int_0^\infty \frac{\ln t}{(t+a)(t+b)} dt \\ &I = \int_0^\infty \frac{\ln(ab)}{(t+a)(t+b)} dt - I \\ &2I = \ln(ab) \int_0^\infty \frac{1}{(t+a)(t+b)} dt \end{aligned}$$

Now by partial fractions

$$\frac{1}{(t+a)(t+b)} = \frac{P}{t+a} + \frac{Q}{t+b}$$

$$\begin{aligned} 1 &\equiv P(t+b) + Q(t+a) \\ \bullet t=b &\quad \bullet t=-a \\ 1 &\equiv Q(b-a) \quad 1 \equiv P(b-a) \\ Q &= \frac{1}{a-b} \quad P = \frac{1}{b-a} \end{aligned}$$

Returning to the integral

$$\begin{aligned} 2I &= \ln(ab) \int_0^\infty \frac{\frac{1}{b-a}}{t+a} + \frac{\frac{1}{a-b}}{t+b} dt \\ 2I &\approx \frac{\ln(ab)}{a-b} \int_0^\infty \frac{1}{t+b} - \frac{1}{t+a} dt \\ 2I &\approx \frac{\ln(ab)}{a-b} \left[\ln\frac{t+b}{t+a} \right]_0^\infty \\ 2I &= \frac{\ln(ab)}{a-b} \left[\ln\frac{b}{a} - \ln\frac{1}{a} \right] \\ 2I &= \frac{\ln(ab)}{a-b} \left[\ln\frac{a}{b} \right] \\ 2I &= \frac{1}{a-b} \ln(ab) \ln\left(\frac{a}{b}\right) \\ I &= \frac{1}{2(a-b)} \ln(ab) \ln\left(\frac{a}{b}\right) \end{aligned}$$

Question 110 (*****)

Use appropriate integration methods to show that

$$\int_0^1 12x^2 \arctan x \, dx = \pi - 2 + \ln 4.$$

[PSS], proof

CONSIDER THE DIFFERENTIATION

$$\frac{d}{dx} [4^x \arctan x] = 4^x \arctan x + 4^x \times \frac{1}{1+x^2}$$
$$\Rightarrow \frac{d}{dx} [4^x \arctan x] = 12x^2 \arctan x + \frac{4x^3}{1+x^2}$$

INTEGRATE w.r.t. x

$$\Rightarrow 4^x \arctan x = \int 12x^2 \arctan x \, dx + \int \frac{4x^3}{1+x^2} \, dx$$
$$\Rightarrow 4^x \arctan x = \int 12x^2 \arctan x \, dx + \int \frac{4x(3x^2+1) - 4x}{x^2+1} \, dx$$
$$\Rightarrow 4^x \arctan x = \int 12x^2 \arctan x \, dx + \int 4x \, dx - \int \frac{4x}{x^2+1} \, dx$$

THIS REARRANGES GIVES

$$\Rightarrow \int 12x^2 \arctan x \, dx = 4^x \arctan x + \int \frac{4x}{x^2+1} \, dx - \int 4x \, dx$$
$$\Rightarrow \int 12x^2 \arctan x \, dx = 4^x \arctan x + 2\ln(x^2+1) - 2x^2 + C$$

APPLY LIMITS

$$\Rightarrow \int_0^1 12x^2 \arctan x \, dx = \left[4^x \arctan x + 2\ln(x^2+1) - 2x^2 \right]_0^1$$
$$= \left[4x^{\frac{11}{4}} + 2\ln 2 - 2 \right] - \left[0 + 2\ln 1 - 0 \right]$$
$$= \pi - 2 + \ln 4$$

Question 111 (*****)

Use appropriate integration methods to find, in terms of k , a simplified expression for

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2 \tan^2 x} dx, \quad |k| \neq 1.$$

, $\frac{\pi}{2(k+1)}$

• **SOLVE BY A SUBSTITUTION**

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \frac{1}{1+k^2 \tan^2 x} dx = \dots \int_0^{\infty} \frac{1}{u^2+1} \frac{du}{u \cot x} \\ &= \int_0^{\infty} \frac{1}{1+u^2} \times \frac{1}{k(1+\cot^2 x)} du \\ & \quad u= \frac{1}{\cot x} \rightarrow u=0 \quad u=\infty \\ &= \int_0^{\infty} \frac{1}{1+u^2} \frac{1}{k(1+\frac{u^2}{k^2})} du \\ &= \int_0^{\infty} \frac{1}{1+u^2} \frac{k^2}{k^2+u^2} du = \int_0^{\infty} \frac{k}{(u^2+k^2)} du \end{aligned}$$

• **INTEGRATE BY PARTIAL FRACTION**

$$\begin{aligned} \frac{k}{(u^2+k^2)(u^2+1)} &\equiv \frac{A+Bi}{u^2+1} + \frac{C+Di}{u^2+k^2} \\ \Rightarrow k &\equiv (Au+B)(u^2+1) + (Cu+D)(u^2+k^2) \\ \Rightarrow k &\equiv \left\{ \begin{array}{l} Au^3 + Bu^2 + Au^2 + B \\ Cu^3 + Du^2 + Cu + D \end{array} \right\} \\ \Rightarrow k &\equiv (A+C)u^3 + (B+D)u^2 + (A+D)u + (C+D) \\ \bullet A+C=0 \\ \bullet A^2+AC=0 \Rightarrow A(A-1)=0 & \quad B+D=0 \Rightarrow B(-1)=0 \\ \Rightarrow A=C=0 & \quad B=0 \\ & \Rightarrow B=\frac{k}{k^2-1}, D=-\frac{k}{k^2-1} \end{aligned}$$

• **RETURNING TO THE INTEGRAL WE NOW HAVE**

$$\begin{aligned} & \int_0^{\infty} \frac{k}{(u^2+k^2)(u^2+1)} du = \int_0^{\infty} \frac{k}{u^2+1} - \frac{k}{u^2+k^2} du \\ &= \frac{k}{k^2-1} \int_0^{\infty} \frac{1}{u^2+1} - \frac{1}{u^2+k^2} du = \frac{k}{k^2-1} \left[\arctan u - \frac{1}{k} \arctan \frac{u}{k} \right]_0^{\infty} \\ &= \frac{k}{k^2-1} \left[\left(\frac{\pi}{2} - \frac{\pi}{2k} \right) - 0 \right] = \frac{k}{k^2-1} \times \frac{\pi}{2} \times \left(1 - \frac{1}{k} \right) \\ &= \frac{k}{(k-1)(k+1)} \times \frac{\pi}{2} = \frac{k\pi}{2(k^2-1)} = -\frac{\pi}{2(k+1)} // \end{aligned}$$

Question 112 (*****)

$$I = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} x \, dx$$

- a) Use the substitution $u = e^x$ to show that $I = \frac{\pi}{k}$, where k is a positive integer.

- b) Given that $t = \tanh\left(\frac{1}{2}x\right)$ show that ...

$$\text{i. } \dots \frac{dt}{dx} = \frac{1}{2}(1-t^2).$$

ii. ... if $x = \frac{1}{2} \ln 3$, then $t = 2 - \sqrt{3}$

- c) Use the results of part (b) to find again the exact value of I .
 - d) Show that I can be written as

$$\int_0^{\frac{1}{2}\ln 3} \frac{\cosh x}{1+\sinh^2 x} dx$$

and hence obtain the exact value of I for a third time.

[] , proof

$$\begin{aligned}
 & \text{(4)} \int_0^{\frac{1}{2}\ln 3} \sec^2 x \, dx = \int_0^{\frac{1}{2}\ln 3} \frac{1}{\cos^2 x} \, dx = \dots \\
 &= \int_0^{\frac{1}{2}\ln 3} \frac{z}{u - u^2 + z^2} \, du = \dots \text{ by substitution...} \\
 &= \int_0^{\sqrt{3}} \frac{z}{u + \sqrt{u^2 + z^2}} \, du = \int_0^{\sqrt{3}} \frac{z}{u + \sqrt{u^2 + 1}} \, du \\
 &= \left[2 \arctan(u\sqrt{u^2 + 1}) \right]_0^{\sqrt{3}} = 2 \arctan(\sqrt{3}) - 2 \arctan(0) \\
 &= \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}
 \end{aligned}$$

$$\begin{aligned} \text{(b) (i)} \quad & t = \tan \frac{\theta}{2} \\ & \frac{dt}{d\theta} = \frac{1}{2} \sec^2 \frac{\theta}{2} \\ & \frac{dt}{d\theta} = \frac{1}{2} \left(1 - \tan^2 \frac{\theta}{2} \right)^{-\frac{1}{2}} \\ & \frac{dt}{d\theta} = \frac{1}{2} \left(1 - t^2 \right)^{-\frac{1}{2}} \quad \swarrow \\ \text{(ii)} \quad & t^2 = 2 - \frac{1}{1-t^2} \\ & t^2 + t^2 - 1 = 2 - 1 \\ & t^2 = 1 \quad \text{now } \tan \frac{\theta}{2} = \frac{x}{2} \\ & \therefore t = \pm \frac{\sqrt{1-x^2}}{x} = \frac{\sqrt{1-x^2}}{\sqrt{x^2+1}} = \frac{\sqrt{(1-x^2)(x^2+1)}}{\sqrt{x^2+1}(x^2+1)} \\ & t = \pm \frac{\sqrt{1-x^2}}{\sqrt{x^2+1}} = \frac{\sqrt{1-x^2}}{x^2+1} = \frac{\sqrt{1-x^2}}{x^2+1} \end{aligned}$$

$$\begin{aligned}
 & \text{(C)} \quad \int_{\infty}^{\frac{1}{2}\sqrt{3}} \sec^2 da = \int_0^{\frac{1}{2}\sqrt{3}} \frac{1}{\cos^2 da} da = \dots \text{by little t identities} \\
 & = \int_0^{\frac{1}{2}\sqrt{3}} \frac{1-t^2}{1+t^2} \cdot \frac{2}{1+t^2} dt \\
 & = \int_0^{\frac{1}{2}\sqrt{3}} \frac{2-2t^2}{1+2t^2+1} dt = \left[2 \arctan t \right]_0^{\frac{1}{2}\sqrt{3}} \\
 & = 2 \arctan \left(\frac{1}{2}\sqrt{3} \right) - 0 \\
 & = 2 \arctan \frac{\sqrt{3}}{2} \\
 & = 2 \times \frac{\pi}{12} = \frac{\pi}{6}
 \end{aligned}$$

$$\int \frac{dx}{1+x^2} = \int \frac{\sec^2 u}{\tan u} du = \int \frac{\sec u}{\sin u} du = \int \frac{\csc u}{\cos u} du = \int \frac{\csc u}{1-\sin^2 u} du$$

By Substitution

$\begin{aligned} u &= \tan x \\ du &= \sec^2 x dx \\ dx &= \frac{du}{\sec^2 x} \\ du &= \csc^2 u dx \end{aligned}$

$$\begin{aligned} \int \frac{dx}{1+x^2} &= \int \frac{\sec^2 u}{\tan u} du \\ &= \int \frac{\sec u}{\sin u} du \\ &= \int \frac{\csc u}{\cos u} du \\ &= \int \frac{\csc u}{1-\sin^2 u} du \\ &= \int \frac{1}{1+\sin^2 u} du \\ &= \int \frac{1}{\csc^2 u} du \\ &= \int \csc^2 u du \\ &= -\cot u + C \\ &= -\cot(\tan x) + C \\ &= -\cot u + C \\ &= -\cot \left(\frac{\pi}{2} - \frac{1}{2}\ln|1-\sqrt{1-x^2}| \right) + C \end{aligned}$$

Question 113 (*****)

$$I = \int_0^1 2 \operatorname{arsinh} \sqrt{x} \, dx.$$

The value of I is to be found using two methods.

- a) Use the substitution $x = \sinh^2 \theta$ to show that

$$I = 3 \ln(1 + \sqrt{2}) - \sqrt{2}.$$

A different approach is to be used to find the value of I .

- b) Use the substitution $u = \sqrt{x}$, followed by a suitable hyperbolic substitution to verify the answer of part (a).

, proof

<p>(a)</p> <p>Let $\alpha = \operatorname{arsinh} 1$</p> $\sinh \alpha = 1$ $\sinh^2 \alpha = 1$ $1 + \sinh^2 \alpha = 2$ $\cosh^2 \alpha = 2$ $\cosh \alpha = \sqrt{2}$	<p>$\sinh \alpha = \operatorname{arsinh} 1$</p> $\therefore \operatorname{arsinh}(\operatorname{arsinh} 1) = \sqrt{2}$ <p>$\Rightarrow I = 3 \ln(1 + \sqrt{2}) - \sqrt{2}$</p>	
<p>(b)</p> $\begin{aligned} I &= \int_0^1 2 \operatorname{arsinh} \sqrt{x} \, dx \\ &= \int_0^1 2 \operatorname{arsinh}(\operatorname{arsinh} 1) \times \sinh 2\theta \, d\theta \\ &= \dots \text{by parts} \quad \left[\begin{array}{l l} u & 1 \\ \operatorname{arsinh} 1 & 2\sinh 2\theta \end{array} \right] \\ &= \dots \left[\operatorname{arsinh} 2\theta - \int \operatorname{arsinh} 2\theta \, d\theta \right] \\ &\quad \left[\begin{array}{l l} u & 1 \\ \operatorname{arsinh} 1 & 2\sinh 2\theta \end{array} \right] \\ &= \dots \left[\operatorname{arsinh} 2\theta - \frac{1}{2} \operatorname{arsinh} 2\theta \right] \\ &= \dots \left[\frac{1}{2} (1 + 2\operatorname{arsinh}^2 1) - \operatorname{arsinh} 2\theta \right] \\ &= \dots \left[(\operatorname{arsinh} 1)^2 \times 3 - 2 \operatorname{arsinh}(\operatorname{arsinh} 1) \right] - [0 - 0] \\ &= 3 \operatorname{arsinh} 1 - \sqrt{2} = 3 \ln(1 + \sqrt{2}) - \sqrt{2} \end{aligned}$	<p>$u = \sqrt{x}$</p> $u^2 = x$ $2u \cdot \frac{1}{2} = 1$ $du = 2u \, dx$ $2u = 2 \Rightarrow u = 1$	
		<p>(c)</p> $\begin{aligned} I &= \int_0^1 2 \operatorname{arsinh} \sqrt{x} \, dx = \dots \int_0^1 2 \operatorname{arsinh}(u) \times 2u \, du \\ &= \int_0^1 4u \operatorname{arsinh} u \, du = \dots \text{by parts} \quad \left[\begin{array}{l l} u & 1 \\ 2u & \frac{1}{\sqrt{u^2+1}} \end{array} \right] \\ &= 2u^2 \operatorname{arsinh} u \Big _0^1 - \int_0^1 \frac{2u^2}{\sqrt{u^2+1}} \, du \quad \text{if } \operatorname{arsinh} u = \frac{1}{\sqrt{u^2+1}} \\ &= 2u^2 \operatorname{arsinh} u \Big _0^1 - \int_0^1 \frac{2u^2}{\sqrt{u^2+1}} \, du \\ &= \dots \text{by parts} \quad \left[\begin{array}{l l} u & 1 \\ 2u^2 & \operatorname{arsinh} u \end{array} \right] \\ &= 2u^2 \operatorname{arsinh} u \Big _0^1 - \int_0^1 2 (2u^2 - 1) \operatorname{arsinh} u \, du \\ &= \int_0^1 2 \operatorname{arsinh} u \, du = \int_0^1 2 (2u^2 - 1) \operatorname{arsinh} u \, du \\ &= \int_0^1 \operatorname{arsinh} u - 1 \, du = \left[\frac{1}{2} \operatorname{arsinh} u - u \right]_0^1 \\ &= \left[\frac{1}{2} \operatorname{arsinh} u - u \right]_0^1 = \left(\frac{1}{2} \operatorname{arsinh}(\operatorname{arsinh} 1) - \operatorname{arsinh} 1 \right) - (0 - 0) \\ &= \frac{\sqrt{2}}{2} - \ln(1 + \sqrt{2}) \end{aligned}$
		<p>$u = \operatorname{arsinh} 1$</p> $du = \operatorname{arsinh} 1 \, dx$ $dx = \operatorname{arsinh} 1 \, du$ $u = 1, \operatorname{arsinh} 1 = 0$ $u = 0, \operatorname{arsinh} 1 = 0$

Question 114 (*****)

By considering the differentiation of products of appropriate functions, find

$$\int e^x (3\sec^2 x + 2\sec^2 x \tan x + 2 \tan x) dx.$$

, $e^x (2 \tan x + \sec^2 x) + C$

$\int e^x (3\sec^2 x + 2\sec^2 x \tan x + 2 \tan x) dx = \text{BY RECOGNISING DIFFERENTIATION}$

DIFFERENTIATE OF EACH TERM
IN PRODUCT
 $\frac{d}{dx}(e^x) = e^x$
 $\frac{d}{dx}(\sec^2 x) = 2\sec x \cdot \sec x \tan x$
 $\frac{d}{dx}(\tan x) = \sec^2 x$

PENAL DIFFERENTIATION OF
 $(3\sec^2 x + 2\sec^2 x \tan x + 2 \tan x)$
BY INSPECTION OF THE BRAINS GUFFAWD

$\therefore \int e^x (3\sec^2 x + 2\sec^2 x \tan x + 2 \tan x) dx = 2\sec^2 x e^x + e^x \tan x + C$
 $= e^x (2 \tan x + \sec^2 x) + C$

Question 115 (*****)

By using a trigonometric substitution or otherwise, find an exact simplified value for the following integral.

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+3\cos 3x} dx.$$

, $\frac{\sqrt{2}}{6} \ln(\sqrt{2}-1)$

$\int_0^{\frac{\pi}{2}} \frac{1}{1+3\cos 3x} dx = \dots \text{by letting } t = \tan \frac{3x}{2}$

$\frac{dt}{dx} = \frac{3}{2} \sec^2 \frac{3x}{2}$

$\frac{dt}{dx} = \frac{3}{2} (1 + \tan^2 \frac{3x}{2})$

$\frac{dt}{dx} = \frac{3}{2} (1 + k^2)$

$dt = \frac{3}{2} (1 + k^2) dx$

$t=0 \quad x=0$

$t=\frac{\sqrt{2}}{2} \quad x=\frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} \frac{1}{1+3\cos 3x} dx = \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{1+3(1-k^2)} dt$

$= \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{4-2k^2} dt = \frac{1}{2} \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{2-k^2} dt$

$= \dots \text{BY PARTIAL FRACTIONS}$

$= \frac{1}{2} \int_0^{\frac{\sqrt{2}}{2}} \left(\frac{1}{\sqrt{2}+\sqrt{2-k^2}} - \frac{1}{\sqrt{2}-\sqrt{2-k^2}} \right) dt$

$= \frac{1}{2\sqrt{2}} \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{2}+\sqrt{2-k^2}} dt + \frac{1}{2\sqrt{2}} \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{2}-\sqrt{2-k^2}} dt$

$= \frac{\sqrt{2}}{12} \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{2}+\sqrt{2-k^2}} dt + \frac{1}{12} \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{2}-\sqrt{2-k^2}} dt$

$= \frac{\sqrt{2}}{12} \left[\ln \left| \frac{\sqrt{2}+\sqrt{2-k^2}}{\sqrt{2}-\sqrt{2-k^2}} \right| \right]_0^{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{12} \ln \left| \frac{(\sqrt{2})^2}{2-1} \right| = \frac{\sqrt{2}}{12} \ln (\sqrt{2}-1)^2$

REARRANGING

$= \frac{\sqrt{2}}{6} \ln (\sqrt{2}-1)$

ANSWER

Question 116 (*****)

Find the value of the following definite integral.

$$\int_0^{\frac{1}{2}} \frac{12x-1}{(6x^2-x-1)(6x^2-x-5)+10} dx.$$

Give the answer in the form $\arctan\left(\frac{1}{n}\right)$, where n is a positive integer.

, $n=7$

The image shows a handwritten mathematical derivation for the integral. It starts with the integral $\int_0^{\frac{1}{2}} \frac{12x-1}{(6x^2-x-1)(6x^2-x-5)+10} dx$. A note says "use a substitution". The substitution is $u = 6x^2 - x - 1$, $du = (12x-1)dx$, $dx = du/(12x-1)$. The limits change from $x=0$ to $u=5$ and from $x=\frac{1}{2}$ to $u=0$. The integral becomes $\int_5^0 \frac{1}{(u+10)(u+6)} du$. This is simplified to $\int_{-1}^0 \frac{1}{(u^2+8u+9)+1} du$. Then, it is converted to an arctan form: $\left[\arctan(u+4) \right]_{-1}^0 = \arctan 3 - \arctan 2$. Using the tangent addition formula, $\tan(\arctan 3 - \arctan 2) = \frac{\tan(\arctan 3) - \tan(\arctan 2)}{1 + \tan(\arctan 3)\tan(\arctan 2)} = \frac{3-2}{1+3\times 2} = \frac{1}{7}$. Therefore, $\arctan 3 - \arctan 2 = \arctan \frac{1}{7}$, so $n=7$.

Question 117 (*****)

$$I = \int_{\frac{1}{\sqrt{3}}}^1 \frac{\sqrt{1+x^2}}{x^4} dx.$$

- a) Use a trigonometric substitution to show that

$$I = \frac{2}{3}(a + b\sqrt{2}),$$

where a and b are integers to be found.

- b) Use a hyperbolic substitution to verify the answer of part (a).

, $I = \frac{2}{3}(4 - \sqrt{2})$

a) STARTING WITH A TANGENT SUBSTITUTION DUE TO THE FORM OF THE NUMERATOR

$$\begin{aligned} \int_{\frac{1}{\sqrt{3}}}^1 \frac{\sqrt{1+x^2}}{x^4} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{1+\tan^2 \theta}}{\sec^2 \theta} (\sec^2 \theta d\theta) \quad x = \tan \theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\sec^2 \theta}}{\sec^2 \theta} (\sec^2 \theta d\theta) \quad d\theta = \sec \theta d\theta \\ &\quad x = 1 \rightarrow \theta = \frac{\pi}{4} \\ &\quad x = \frac{1}{\sqrt{3}} \rightarrow \theta = \frac{\pi}{6} \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec \theta d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cos^2 \theta} \times \frac{\cos \theta}{\sin^2 \theta} d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\csc \theta}{\sin^2 \theta} d\theta \end{aligned}$$

NOW BY RECOGNITION (OR HYPERBOLIC SUBSTITUTION $u = \sin \theta$)

$$\begin{aligned} &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc \theta (\sin \theta)^{-4} d\theta = \left[-\frac{1}{3} (\sin \theta)^{-3} \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\ &= \frac{1}{3} \left[\frac{1}{\sin^3 \theta} \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{1}{3} \left[\frac{1}{(\sqrt{2})^3} - \frac{1}{(\sqrt{3})^3} \right] \\ &= \frac{1}{3} \left[\frac{1}{8} - \frac{1}{2\sqrt{2}} \right] = \frac{1}{3} \left[8 - 2\sqrt{2} \right] \\ &= \frac{2}{3} (4 - \sqrt{2}) \end{aligned}$$

b) NOW BY A HYPERBOLIC SUBSTITUTION

$$\begin{aligned} \int_{\frac{1}{\sqrt{3}}}^1 \frac{\sqrt{1+x^2}}{x^4} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{1+\tanh^2 \theta}}{\operatorname{sech}^2 \theta} (\operatorname{sech}^2 \theta d\theta) \\ &= \int_{\operatorname{sech}^2 \theta=2}^{\operatorname{sech}^2 \theta=1} \frac{\operatorname{cosech} \theta}{\operatorname{sech}^2 \theta} (\operatorname{sech}^2 \theta d\theta) = \int_{\operatorname{sech}^2 \theta=2}^{\operatorname{sech}^2 \theta=1} \frac{\operatorname{cosech} \theta}{\operatorname{sech}^2 \theta} d\theta \\ &= \int_{\operatorname{sech}^2 \theta=2}^{\operatorname{sech}^2 \theta=1} \frac{\operatorname{cosech} \theta}{\operatorname{sech}^2 \theta} \times \frac{1}{\operatorname{sech}^2 \theta} \operatorname{sech}^2 \theta d\theta = \int_{\operatorname{sech}^2 \theta=2}^{\operatorname{sech}^2 \theta=1} \operatorname{cosech} \theta d\theta \\ &= \left[-\frac{1}{2} \operatorname{coth} \theta \right]_{\operatorname{sech}^2 \theta=2}^{\operatorname{sech}^2 \theta=1} = \frac{1}{2} \left[\operatorname{coth} \theta \right]_{\operatorname{sech}^2 \theta=2}^{\operatorname{sech}^2 \theta=1} \\ &= \frac{1}{2} \left[2 - (\sqrt{2})^2 \right] = \frac{1}{2} [8 - 2\sqrt{2}] \\ &= \frac{2}{3} [4 - \sqrt{2}] \end{aligned}$$

AS REQUIRED

$x = \sin \theta$
$dx = \cos \theta d\theta$
$\theta = 1$
$\sin \theta = 1$
$\sin^2 \theta = 1$
$\operatorname{sech}^2 \theta = 1$
$\operatorname{sech}^2 \theta + 1 = 2$
$\operatorname{cosech} \theta = 2$
$\operatorname{cosech} \theta = \frac{1}{2}$
$\operatorname{coth} \theta = \frac{1}{2}$
$\theta = \frac{\pi}{4}$
$\sin \theta = \frac{1}{\sqrt{2}}$
$\sin^2 \theta = \frac{1}{2}$
$\operatorname{cosech} \theta = \frac{1}{2}$
$\operatorname{cosech}^2 \theta = \frac{1}{4}$
$\operatorname{cosech}^2 \theta + 1 = 5$
$\operatorname{coth} \theta = 4$
$\operatorname{coth}^2 \theta = 2$

Question 118 (*****)

The function f is defined in the largest real domain by the equation

$$f(x) \equiv \arccos|2x-1|.$$

Determine the area of the finite region bounded by f and the coordinate axes.

, area = 1

START WITH A SKETCH

HENCE WE HAVE $y = \arccos|2x-1|$

HENCE THE REQUIRED AREA CAN BE FOUND BY SYMMETRY

AREA = $2 \int_0^{\frac{1}{2}} \arccos(1-2x) dx$

$$= 2 \int_0^{\frac{\pi}{2}} \theta (\frac{1}{2} \sin\theta) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \theta \sin\theta d\theta$$

θ	0	$\frac{1}{2}$
$\sin\theta$	0	$\frac{1}{2}$
$\theta \sin\theta$	0	$\frac{1}{2}$

INTEGRATION BY PARTS GIVES

$$= \left[-\theta \cos\theta \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos\theta d\theta$$

$$= \left[\sin\theta \right]_0^{\frac{\pi}{2}}$$

$$= 1$$

ALTERNATIVE LOOKING AT THE DIAGRAM

REQUIRED AREA = $2 \left[\frac{x}{2} - \frac{1}{2} \arccos(1-2x) \right]$

SQUARE SIMPLIFY AREA

$$= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} 1-\cos\theta d\theta$$

$$= \frac{\pi}{2} - \left[y - \sin y \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - \left[(\frac{\pi}{2} - 1) - 0 \right]$$

$$= 1$$

$y = \arccos(1-2x)$
 $\cos y = 1-2x$
 $2x = 1-\cos y$
 $2dx = \sin y dx$
 $dy = \frac{1}{2} \sin y dx$

Question 119

By considering

(*****)

$$\frac{\sin[(2m+1)x]}{\sin x} - \frac{\sin[(2m-1)x]}{\sin x}, \quad m \in \mathbb{N},$$

determine the exact value of

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 7x}{\sin x} dx.$$

, $\frac{1}{2}\pi$

CONSIDER THE FOLLOWING TRIGONOMETRIC EXPRESSION, FOR $m \in \mathbb{N}$:

$$\frac{\sin[(2m+1)x]}{\sin x} - \frac{\sin[(2m-1)x]}{\sin x} = \frac{2\sin[(2m+1)x]\cos[(2m+1)x] - 2\sin[(2m-1)x]\cos[(2m-1)x]}{\sin^2 x} = 2\cos(2mx) \quad [\sin(2x) = 2\sin(x)\cos(x)]$$

NOW CONSIDER THE INTEGRAL OF $2\cos(2mx)$ IN $[0, \frac{\pi}{2}]$:

$$\int_0^{\frac{\pi}{2}} 2\cos(2mx) dx = \left[\frac{1}{m} \sin(2mx) \right]_0^{\frac{\pi}{2}} = \frac{1}{m} [\sin(m\pi) - \sin(0)] = 0$$

HENCE

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin[(2m+1)x]}{\sin x} dx &= \int_0^{\frac{\pi}{2}} \frac{\sin[(2m-1)x]}{\sin x} dx = \int_0^{\frac{\pi}{2}} \cos(2mx) dx \\ \int_0^{\frac{\pi}{2}} \frac{\sin[(2m+1)x]}{\sin x} dx &- \int_0^{\frac{\pi}{2}} \frac{\sin[(2m-1)x]}{\sin x} dx = 0 \\ \int_0^{\frac{\pi}{2}} \frac{\sin(2mx)}{\sin x} dx &= \int_0^{\frac{\pi}{2}} \frac{\sin(2mx)}{\sin x} dx \end{aligned}$$

THUS WE HAVE

$$\int_0^{\frac{\pi}{2}} \frac{\sin 7x}{\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin 5x}{\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin 7x}{\sin x} dx = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}$$

Question 120 (*****)

Find in exact simplified form the value of the following definite integral.

$$\int_{3^{-\frac{1}{6}}}^{3^{\frac{1}{6}}} \left(x^2 + \frac{1}{x^4}\right)^{-2} dx .$$

, $\frac{\pi}{36}$

SPLIT BY AN INITIAL TRY OR - LET $u = x^2 \Leftrightarrow \delta = 2t$

$$\int_{\sqrt{x}}^{\sqrt[3]{x}} (x^2 + \frac{1}{x^4})^{-2} dx = \int_x^{\sqrt[3]{x}} (\frac{2x+1}{x^2})^{-2} dx = \int_x^{\sqrt[3]{x}} (\frac{2^{\frac{1}{2}}}{x^{\frac{3}{2}}+1})^2 dx$$

$$= \int_x^{\sqrt[3]{x}} \frac{2^2}{(x^{\frac{3}{2}}+1)^2} dx$$

NOV THE SUBSTITUTION $x^{\frac{3}{2}} = 4u^2$ WHICHES NOT IT IS SOE MUSY - EASILY $x^{\frac{3}{2}}$ SHULD EASILY ADD BE OK - WE PROCEED AS FOLLOWS

$$\dots = \int_x^{\sqrt[3]{x}} \frac{2^2}{(x^{\frac{3}{2}}+1)^2} x^2 dx = \dots \text{ BY PARTS}$$

INTEGRATE

$$\int \frac{2^2}{(x^{\frac{3}{2}}+1)^2} dx = \int x^2 dx = -\frac{1}{3}(x^{\frac{3}{2}}) + C$$

$$= \left[-\frac{1}{3}(x^{\frac{3}{2}}) \right]_x^{\sqrt[3]{x}} - \int_x^{\sqrt[3]{x}} \frac{3}{2}x^{\frac{1}{2}} dx$$

$$= \left[-\frac{2^2}{3(x^{\frac{3}{2}}+1)} \right]_x^{\sqrt[3]{x}} + \frac{1}{2} \int_x^{\sqrt[3]{x}} \frac{3}{2}x^{\frac{1}{2}} dx$$

BY RECOGNITION THIS IS THE DIFFERENTIAL OF $\arctan(x^{\frac{3}{2}})$, SINCE

$$\frac{d}{dx}(\arctan x^{\frac{3}{2}}) = \frac{1}{1+(x^{\frac{3}{2}})^2} \times 2x^{\frac{1}{2}} = \frac{2x^{\frac{1}{2}}}{1+x^3}$$

$$\therefore \frac{d}{dx}(\arctan x^{\frac{3}{2}}) = \frac{2^2}{1+x^3}$$

FINALLY COLLECTING ALL THE RESULTS FOR THE EVALUATION

$$\dots = \left[-\frac{2^2}{3(x^{\frac{3}{2}}+1)} + \frac{1}{6}\arctan(x^{\frac{3}{2}}) \right]_x^{\sqrt[3]{x}}$$

$$= \frac{1}{6} \left[\arctan(x^{\frac{3}{2}}) - \frac{2^2}{x^{\frac{3}{2}}+1} \right]_x^{\sqrt[3]{x}}$$

$$= \frac{1}{6} \left[(\arctan(x^{\frac{3}{2}}) - \frac{2^2}{3+1}) - (\arctan(\sqrt[3]{x}) - \frac{2^2}{3\sqrt[3]{x}+1}) \right]$$

$$= \frac{1}{6} \left[\arctan(\sqrt[3]{x}) - \frac{1}{4}\sqrt[3]{x} - \arctan(\sqrt[3]{x}) + \frac{2^2}{3\sqrt[3]{x}} \right]$$

$$= \frac{1}{6} \left[\frac{2^2}{3} - \frac{1}{4}\sqrt[3]{x} - \frac{2^2}{6} + \frac{2^2}{4\sqrt[3]{x}} \right]$$

$$= \frac{1}{6} \left[\frac{2^2}{3} - \frac{1}{4}\sqrt[3]{x} + \frac{2^2}{6\sqrt[3]{x}} \right]$$

$$= \frac{1}{6} \left[\frac{2^2}{3} - \frac{1}{4}\sqrt[3]{x} + \frac{1}{3}\sqrt[3]{x} \right]$$

$$= \frac{2^2}{36}$$

Question 121 (*****)

Determine a simplified expression, in the form $\ln[f(n)]$, for the following sum.

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right].$$

, $\ln \left[\frac{2 \times 3^{N-1}}{N(N+1)} \right]$

• START BY PARTIAL FRACTIONS IN THE INTEGRAND (BY INSPECTION)

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right] = \sum_{r=2}^N \left[\int_2^r \frac{2}{(x-1)(x+1)} dx \right]$$

$$= \sum_{r=2}^N \left[\int_2^r \frac{1}{x-1} - \frac{1}{x+1} dx \right] = \sum_{r=2}^N \left[\ln|x-1| - \ln|x+1| \right]_{x=2}^{x=r}$$

• WRITING THE TERMS EXPLICITLY, LOOKING FOR PATTERNS

$$= \sum_{r=2}^N \left[\ln(r-1) - \ln(r+1) \right] - \left[\ln(1) - \ln(3) \right]$$

$$= \sum_{r=2}^N \left[\ln(r-1) - \ln(r+1) + \ln 3 \right]$$

$$= \begin{aligned} & \ln 1 - \ln 3 + \ln 3 && \leftarrow r=2 \\ & \cancel{\ln 2} - \ln 4 + \ln 3 && \leftarrow r=3 \\ & \ln 3 - \cancel{\ln 5} + \ln 3 && \leftarrow r=4 \\ & \ln 4 - \cancel{\ln 6} + \ln 3 && \leftarrow r=5 \\ & \vdots && \end{aligned} \quad \left\{ \begin{array}{l} (N-1) \text{ TERMS} \\ \ln(1 \cdot 2) - \ln N + \ln 3 & \leftarrow r=N-1 \\ \ln(N!) - \ln(N!) + \ln 3 & \leftarrow r=N \end{array} \right.$$

• ADDING

$$\sum_{r=2}^N \left[\int_2^r \frac{2}{x^2-1} dx \right] = \ln 2 - \ln N - \ln(N+1) + (N-1)\ln 3$$

$$= \ln 2 + (N-1)\ln 3 - (\ln N + \ln(N+1))$$

$$= \ln \left[\frac{2 \cdot 3^{N-1}}{N(N+1)} \right]$$

Question 122 (*****)

Use appropriate integration methods to find a simplified expression for

$$\int x \arccos \left[\frac{1-x^2}{1+x^2} \right] dx.$$

\boxed{x}	$-x + (1+x^2) \arctan x + \text{constant}$
-------------	--

USING THE SUBSTITUTION METHOD

$x = \tan(\frac{\theta}{2})$
 $dx = \sec^2(\frac{\theta}{2}) d\theta$
 OR $[dx = \sqrt{1 + \tan^2(\frac{\theta}{2})} d\theta]$
 OR $[dx = \frac{1}{2}(1+x^2) d\theta]$

(we shall see which form is better, in the question)

$$\begin{aligned} \frac{1-x^2}{1+x^2} &= \frac{(1-\tan^2(\frac{\theta}{2}))}{(1+\tan^2(\frac{\theta}{2}))} \\ &= \frac{(1-\tan^2(\frac{\theta}{2}))}{\sec^2(\frac{\theta}{2})} \\ &= \frac{1}{\sec^2(\frac{\theta}{2})} - \frac{\tan^2(\frac{\theta}{2})}{\sec^2(\frac{\theta}{2})} \\ &= \cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2}) \cos^2(\frac{\theta}{2}) \\ &= \cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2}) \\ &= \cos(2\theta) \end{aligned}$$

TRANSFORMING THE INTEGRAL WE HAVE

$$\begin{aligned} \int 2x \arccos \left(\frac{1-x^2}{1+x^2} \right) dx &= \int 2x \tan(\frac{\theta}{2}) \arccos(\cos(2\theta)) \frac{1}{\sec^2(\frac{\theta}{2})} d\theta \\ &= \int 2x \tan(\frac{\theta}{2}) \sec^2(\frac{\theta}{2}) d\theta \end{aligned}$$

INTEGRATION BY PARTS

$\frac{1}{2}\theta$	$\frac{1}{2}$
$\tan^2(\frac{\theta}{2})$	$\tan(\frac{\theta}{2}) \sec^2(\frac{\theta}{2})$

$$\begin{aligned} &= \frac{1}{2}\theta \tan^2(\frac{\theta}{2}) - \int \frac{1}{2} \tan^2(\frac{\theta}{2}) d\theta \\ &= \frac{1}{2}\theta \tan^2(\frac{\theta}{2}) - \frac{1}{2} \int \sec^2 \frac{\theta}{2} - 1 d\theta \end{aligned}$$

Question 123 (***)**

Find the exact value of

$$\int_0^1 \left[\sum_{n=1}^{\infty} \frac{(n+1)x^n}{(n+2)!} \right] dx.$$

$$\boxed{}, \boxed{\frac{1}{2}(2e - 5)}$$

$$\begin{aligned} & \int_0^1 \left[\sum_{n=1}^{\infty} \frac{(n+1)x^n}{(n+2)!} \right] dx = \dots \text{ (REVERSE INTEGRATION & SUMMATION)} \\ &= \sum_{n=1}^{\infty} \left[\frac{(n+1)}{(n+2)!} \int_0^1 x^n dx \right] = \sum_{n=1}^{\infty} \left[\frac{(n+1)}{(n+2)!} \left[\frac{x^{n+1}}{n+1} \right]_0^1 \right] \\ &= \sum_{n=1}^{\infty} \frac{1}{(n+2)!} \\ &= \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots \\ &= \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots \right) - \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} \right) \\ &= e - \left(1 + 1 + \frac{1}{2} \right) = e - \frac{5}{2} = \frac{1}{2}(2e - 5) \quad // \end{aligned}$$

Question 124 (*****)

- a) Use an appropriate integration method to evaluate the following integral.

$$\int_0^1 x^3 \arctan x \, dx.$$

- b) Obtain an infinite series expansion for $\arctan x$ and use this series expansion to verify the answer obtained for the above integral in part (a).

[you may assume that integration and summation commute]

, $\frac{1}{6}$

a) Solve by integration by parts

$$\begin{aligned} \int_0^1 x^3 \arctan x \, dx &= [\arctan x]_0^1 - \int_0^1 \frac{x^3}{1+x^2} \, dx \\ &= \frac{1}{2}x^2 - 0 - \frac{1}{4} \int_0^1 \frac{2x^3}{2x^2+1} \, dx \\ &= \frac{1}{6} - \frac{1}{4} \int_0^1 \frac{2x(2x^2+1)-2x^2-1}{2x^2+1} \, dx \\ &= \frac{1}{6} - \frac{1}{4} \int_0^1 (x^2-1+\frac{1}{2x^2+1}) \, dx \\ &= \frac{1}{6} - \frac{1}{4} \left[\frac{1}{3}x^3 - x + \arctan x \right]_0^1 \\ &= \frac{1}{6} - \frac{1}{4} \left[\frac{1}{3} - 1 + \frac{\pi}{4} \right] \\ &= \frac{1}{6} - \frac{1}{4} \left(-\frac{2}{3} + \frac{\pi}{4} \right) \\ &= \frac{1}{6} + \frac{1}{6} - \frac{\pi}{16} \\ &= \frac{1}{6} \end{aligned}$$

b) Need the expansion of $\arctan x$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1-x^2+x^4-x^6+\dots$$

INTEGRATE BOTH SIDES GIVES

$$\begin{aligned} \arctan x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \dots + C \\ \arctan 0 &= 0 \Rightarrow C = 0 \\ \therefore \arctan x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \\ \therefore \arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)2^n} \end{aligned}$$

This we know that

$$\begin{aligned} \int_0^1 x^3 \arctan x \, dx &= \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)2^n} \, dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \int_0^1 x^{2n+4} \, dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \left[\frac{x^{2n+5}}{2n+5} \right]_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+5)} \left[\frac{1}{2n+5} \right] \end{aligned}$$

NEED TO SUM THIS SERIES BY PARTIAL FRACTIONS

$$\frac{1}{(2m+1)(2n+5)} = \frac{1}{2n+1} - \frac{1}{2n+5} \quad (\text{By inspection})$$

THUS WE NOW HAVE

$$\int_0^1 x^3 \arctan x \, dx = \frac{1}{2} \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n+5} \right]$$

WHERE ARE THE PATTERNS?

• $n=0$	$\frac{1}{1} - \frac{1}{5}$
• $n=1$	$\frac{1}{3} - \frac{1}{7}$
• $n=2$	$\frac{1}{5} - \frac{1}{9}$
• $n=3$	$\frac{1}{7} - \frac{1}{11}$
• $n=4$	$\cancel{\frac{1}{9} - \frac{1}{13}}$
• $n=5$	$\cancel{\frac{1}{11} - \frac{1}{15}}$

FINALLY WE HAVE THE RESULT

$$\begin{aligned} \int_0^1 x^3 \arctan x \, dx &= \frac{1}{2} \lim_{n \rightarrow \infty} \sum_{k=0}^n \left[\frac{(-1)^k}{2k+1} - \frac{(-1)^k}{2k+5} \right] \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \left[1 - \frac{1}{3} - \frac{(-1)^n}{2n+3} - \frac{(-1)^n}{2n+5} \right] \\ &\uparrow \qquad \uparrow \\ &\text{THIS TENDS TO ZERO} \\ &= \frac{1}{2} \times \left(1 - \frac{1}{3} \right) \\ &= \frac{1}{2} \times \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

ANSWER

Question 125 (*****)

Find the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos x + \sin x}{\sqrt{\sin 2x}} dx.$$

, $2 \arcsin \left[\frac{\sqrt{3}-1}{2} \right]$

BY SUBSTITUTION

$$\begin{aligned}
 & \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos x + \sin x}{\sqrt{\sin 2x}} dx \\
 &= \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos x + \sin x}{\sqrt{2\sin x \cos x}} dx \\
 &= \int_{-\pi}^{\pi} \frac{\cos x + \sin x}{\sqrt{1-u^2}} \frac{du}{\cos x - \sin x} \\
 &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{1-u^2}} du \\
 &= 2 \int_0^\pi \frac{1}{\sqrt{1-u^2}} du \\
 &= 2 \left[\arcsin u \right]_0^\pi \\
 &= 2 \arcsin \pi = 2 \arcsin \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) \\
 &\approx 2 \arcsin \left(\frac{\sqrt{3}-1}{2} \right)
 \end{aligned}$$

Question 126 (*****)

$$I = \int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \frac{\sqrt{3}(1+\pi x^3)}{2-\cos(|x|+\frac{1}{3}\pi)} dx.$$

Show that

$$I = 4 \arctan \frac{1}{2}.$$

, proof

$$\begin{aligned} & \int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \frac{\sqrt{3}(1+\pi x^3)}{2-\cos(|x|+\frac{1}{3}\pi)} dx \\ &= \int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \frac{\sqrt{3}}{2-\cos((|x|+\frac{1}{3}\pi))} dx + \pi \int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \frac{x^3}{2-\cos((|x|+\frac{1}{3}\pi))} dx \quad (\text{extra}) \\ &= 2 \int_0^{\frac{1}{3}\pi} \frac{\sqrt{3}}{2-\cos(x+\frac{\pi}{3})} dx = 2\sqrt{3} \int_0^{\frac{1}{3}\pi} \frac{1}{2-\cos(x+\frac{\pi}{3})} dx \\ &\quad \boxed{\text{BY SUBSTITUTION}} \\ &\quad u = x + \frac{\pi}{3} \\ &\quad du = dx \\ &\quad x=0 \mapsto \frac{\pi}{3} \\ &\quad x=\frac{\pi}{3} \mapsto 2\frac{\pi}{3} \\ &= 2\sqrt{3} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{2-\cos u} du \\ &\quad \boxed{\text{BY ANOTHER SUBSTITUTION}} \\ &\quad t = \tan \frac{u}{2} \\ &\quad \frac{dt}{du} = \frac{1}{2} \sec^2 \frac{u}{2} \\ &\quad \frac{dt}{du} = \frac{1}{2} (1 + \tan^2 \frac{u}{2}) \\ &\quad \frac{dt}{du} = \frac{1}{2} (1+t^2) \\ &\quad du = \frac{2}{1+t^2} dt \\ &\quad \boxed{\text{A UNIT}} \\ &\quad u = 2\frac{\pi}{3} \mapsto t = \sqrt{3} \\ &\quad u = \frac{\pi}{3} \mapsto t = \sqrt{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} &= 2\sqrt{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2 - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\ &= 4\sqrt{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{2+2t^2-1+t^2} dt \\ &= 4\sqrt{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+3t^2} dt \\ &= \frac{4\sqrt{3}}{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+t^2} dt \\ &= \frac{4}{3}\sqrt{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{(\frac{1}{\sqrt{3}})^2+t^2} dt \\ &= \frac{4}{3}\sqrt{3} \times \frac{1}{\frac{1}{\sqrt{3}}} \left[\arctan \left(\frac{t}{\frac{1}{\sqrt{3}}} \right) \right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \\ &= 4 \left[\arctan \left(\sqrt{3}t \right) \right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \\ &= 4 \left[\arctan 3 - \arctan 1 \right] \\ &= 4 \arctan \frac{1}{2} \end{aligned}$$

$$\begin{aligned} &\text{arctan} 3 - \arctan 1 = \pi/2 \\ &\theta - \phi = \pi/2 \\ &\tan(\theta - \phi) = \tan \pi/2 \\ &\frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi} = \tan \pi/2 \\ &\frac{3 - 1}{1 + 3 \times 1} = \tan \pi/2 \\ &\tan \pi/2 = \frac{1}{2} \\ &\pi/2 = \arctan 3 - \arctan 1 = \arctan \frac{1}{2} \end{aligned}$$

Question 127 (***)**

By expressing the integrand in the form $\operatorname{sech}^2 x f(\tanh x)$, or otherwise, find the value of the following integral.

$$\int_0^{\frac{1}{2} \ln 3} \frac{\sqrt{2 \operatorname{sech} x}}{\sqrt[4]{\sinh 2x \cosh x} - \sqrt[4]{2 \sinh^3 x}} dx.$$

[2]

• USING / ATTEMPTING WITH THE SUGGESTION GIVEN

$$\begin{aligned} I &= \int_0^{\frac{1}{2} \ln 3} \frac{\sqrt{2 \operatorname{sech} x}}{\sqrt[4]{\sinh 2x \cosh x} - \sqrt[4]{2 \sinh^3 x}} dx \\ &\Rightarrow I = \int_0^{\frac{1}{2} \ln 3} \frac{(2 \operatorname{sech} x)^{\frac{1}{2}}}{(\sqrt{\sinh 2x \cosh x} - \sqrt[4]{2 \sinh^3 x})^2} dx \\ &\Rightarrow I = \int_0^{\frac{1}{2} \ln 3} \frac{2^{\frac{1}{2}} (\operatorname{sech} x)^{\frac{1}{2}}}{(2 \sinh x \cosh x)^{\frac{1}{2}} - (2 \sinh x)^{\frac{3}{2}}} dx \\ &\Rightarrow I = \int_0^{\frac{1}{2} \ln 3} \frac{2^{\frac{1}{2}} (\operatorname{sech} x)^{\frac{1}{2}}}{2^{\frac{1}{2}} (\tanh x)^{\frac{1}{2}} (\cosh x)^{\frac{1}{2}} - 2^{\frac{3}{2}} (\sinh x)^{\frac{3}{2}}} dx \\ &\Rightarrow I = \int_0^{\frac{1}{2} \ln 3} \frac{(\operatorname{sech} x)^{\frac{1}{2}}}{(\tanh x)^{\frac{1}{2}} (\cosh x)^{\frac{1}{2}} - (\sinh x)^{\frac{3}{2}}} dx \\ &\Rightarrow I = \int_0^{\frac{1}{2} \ln 3} \frac{\operatorname{sech} x}{(\tanh x)^{\frac{1}{2}} - (\sinh x)^{\frac{3}{2}}} dx \\ &\Rightarrow I = \int_0^{\frac{1}{2} \ln 3} \frac{\operatorname{sech} x}{[(\tanh x)^{\frac{1}{2}} - (\tanh x)^{\frac{3}{2}}]^2} dx \end{aligned}$$

• KNOW BY REVERSE CHAIN RULE (REGRESSION) OR BY USING THE SUBSTITUTION $u = 1 - (\tanh x)^{\frac{1}{2}}$, we obtain

$$\begin{aligned} &= \left[2 \left[1 - (\tanh x)^{\frac{1}{2}} \right]^{-1} \right]_0^{\frac{1}{2} \ln 3} \\ &\quad \left\{ \frac{d}{dx} \left[\left[1 - (\tanh x)^{\frac{1}{2}} \right]^{-1} \right] = -\left[1 - (\tanh x)^{\frac{1}{2}} \right]^{-2} \times \left[\frac{1}{2} (\tanh x) \times \operatorname{sech}^2 x \right] \right\} \\ &= +\frac{1}{2} \operatorname{sech}^2 (\tanh x)^{\frac{1}{2}} \left[1 - (\tanh x)^{\frac{1}{2}} \right]^2 \\ &\bullet \text{SIMPLIFY } \tanh \left[\frac{1}{2} \ln \frac{3}{2} \right] \text{ FIRST} \\ \tanh x &= \frac{e^x - 1}{e^x + 1} \Rightarrow \tanh \left(\frac{1}{2} \ln \frac{3}{2} \right) = \frac{e^{\frac{1}{2} \ln \frac{3}{2}} - 1}{e^{\frac{1}{2} \ln \frac{3}{2}} + 1} \\ &\Rightarrow \tanh \left(\frac{1}{2} \ln \frac{3}{2} \right) = \frac{\frac{3}{2} - 1}{\frac{3}{2} + 1} = \frac{1}{2} \\ &\Rightarrow \tanh \left(\frac{1}{2} \ln \frac{3}{2} \right) = \frac{1}{2} \\ &\bullet \text{RETURNING TO THE INTEGRAL EVALUATION} \\ ... &= 2 \left[\frac{1}{1 - \sqrt{\tanh x}} \right]_0^{\frac{1}{2} \ln 3} = 2 \left[\frac{1}{1 - \frac{1}{2}} - \frac{1}{1 - 0} \right] \\ &= 2 \left[\frac{1}{\frac{1}{2}} - 1 \right] = 2 \end{aligned}$$

Question 128 (*****)

Use appropriate integration techniques to show that

$$\int_{\operatorname{arsinh} \frac{1}{\sqrt{3}}}^{\operatorname{arsinh} \sqrt{3}} \operatorname{sech} x (1 - \operatorname{sech} x) \, dx = \frac{\pi}{12}.$$

, proof

METHOD ONE

Given $\int \operatorname{sech} x - \operatorname{sech}^2 x \, dx$

$$= \int \frac{1 - \operatorname{tanh}^2 x}{\operatorname{cosh} x} - \frac{(1 - \operatorname{tanh}^2 x)^2}{\operatorname{cosh}^2 x} \, dx$$

$$= \int \frac{\operatorname{tanh}^2 x}{\operatorname{cosh} x} \left(\frac{1}{1 - \operatorname{tanh}^2 x} - \frac{1}{\operatorname{cosh} x} \right) \, dx$$

$$= \int \frac{\operatorname{tanh}^2 x}{\operatorname{cosh} x} \left(\frac{\operatorname{cosh} x}{\operatorname{cosh}^2 x} - \frac{1}{\operatorname{cosh} x} \right) \, dx$$

$$= \int \frac{\operatorname{tanh}^2 x}{\operatorname{cosh} x} \left(\frac{1 - \operatorname{cosh} x}{\operatorname{cosh}^2 x} \right) \, dx$$

$$= \int \frac{\operatorname{tanh}^2 x}{\operatorname{cosh} x} \left(\frac{\operatorname{sinh} x}{\operatorname{cosh} x} \right) \, dx$$

$$= \int \operatorname{tanh}^2 x \operatorname{sinh} x \, dx$$

$$= \int \frac{1 - \operatorname{cosh}^2 x}{\operatorname{cosh} x} \operatorname{sinh} x \, dx$$

$$= \int (1 - \operatorname{cosh}^2 x)^{\frac{1}{2}} \operatorname{sinh} x \, dx$$

$$= \int (1 - \operatorname{cosh}^2 x)^{\frac{1}{2}} \operatorname{sinh} x \, dx$$

BY SUBSTITUTION

$\operatorname{sech} x = \operatorname{cosec} \theta$
 $-\operatorname{sech} x \operatorname{tanh} x \, dx = -\operatorname{cosec} \theta \operatorname{sec} \theta \, d\theta$
 $d\theta = \frac{\operatorname{cosec} \theta \operatorname{cot} \theta}{\operatorname{cosec}^2 \theta} \, d\theta$
 FOR THE LIMITS
 $\Rightarrow \operatorname{sech} x = \operatorname{cosec} \theta$
 $\Rightarrow \operatorname{cosec}^2 x = \operatorname{cosec}^2 \theta$
 $\Rightarrow \operatorname{cosec}^2 x = \operatorname{cosec}^2 \theta$
 $\Rightarrow \operatorname{cosec}^2 x - 1 = \operatorname{cosec}^2 \theta - 1$
 $\Rightarrow \operatorname{cosec}^2 x = \operatorname{cosec}^2 \theta$
 $\Rightarrow \operatorname{cosec}^2 x = 1 + \operatorname{tan}^2 \theta$
 $\Rightarrow \operatorname{cosec}^2 x = 1 + \operatorname{tan}^2 \theta$
 $\bullet x = \operatorname{arcsinh} \operatorname{cosec} \theta$
 $\Rightarrow \operatorname{cosec} \theta = \sqrt{x}$
 $\Rightarrow \operatorname{tan} \theta = \sqrt{x}$
 $\Rightarrow \theta = \frac{1}{2} \operatorname{arctan} x$
 $\bullet x = \operatorname{arcsinh} \frac{1}{\sqrt{3}}$
 $\Rightarrow \operatorname{cosec} \theta = \frac{1}{\sqrt{3}}$
 $\Rightarrow \operatorname{tan} \theta = \frac{1}{\sqrt{3}}$
 $\Rightarrow \theta = \frac{\pi}{6}$

METHOD TWO

INCIDE EACH INDEFINITE INTEGRAL SEPARATELY

$I_1 = \int \operatorname{sech}^2 x \, dx = \int \operatorname{sech} x \operatorname{sech} x \, dx$

BY PARTS

$\operatorname{sech} x \quad \operatorname{sech} x \operatorname{tanh} x$
 $\operatorname{tanh} x \quad \operatorname{sech} x$

$$I_1 = \operatorname{sech} x \operatorname{tanh} x + \int \operatorname{sech} x \operatorname{tanh} x \, dx$$

$$= \operatorname{sech} x \operatorname{tanh} x + \int \operatorname{sech} x (1 - \operatorname{sech}^2 x) \, dx$$

$$= \operatorname{sech} x \operatorname{tanh} x + \int \operatorname{sech} x \, dx - \int \operatorname{sech}^3 x \, dx$$

$$I_1 = \operatorname{sech} x \operatorname{tanh} x + \int \operatorname{sech} x \, dx - I_1$$

RETURNING TO THE REQUIRED INTEGRAL

$\int \operatorname{sech} x - \operatorname{sech}^2 x \, dx = \frac{1}{2} \operatorname{artanh}(\operatorname{sech} x) - \frac{1}{2} \operatorname{sech} x \operatorname{tanh} x + C$

PROOF

$\int \operatorname{sech} x - \operatorname{sech}^2 x \, dx = \frac{1}{2} \left[\operatorname{artanh}(\operatorname{sech} x) - \frac{\operatorname{sech} x}{\operatorname{cosh} x} \right] + C$

$$= \frac{1}{2} \left[\operatorname{artanh}(\operatorname{sech} x) - \frac{\operatorname{sech} x}{1 + \operatorname{sech}^2 x} \right] + C$$

EVALUATING

$$= \frac{1}{2} \left[\operatorname{artanh} \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{1 + \frac{1}{16}} \right] - \frac{1}{2} \left[\operatorname{artanh} \frac{1}{\sqrt{3}} - \frac{1}{1 + \frac{1}{3}} \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right] - \frac{1}{2} \left[\frac{\pi}{6} - \frac{3}{4\sqrt{3}} \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{3}{4\sqrt{3}} \right]$$

$$= \frac{1}{2} \times \frac{\pi}{6}$$

$$= \frac{\pi}{12}$$

AS REQUIRED

$2I_1 = \operatorname{sech} x \operatorname{tanh} x + \int \operatorname{sech} x \, dx$

$I_1 = \operatorname{sech} x \operatorname{tanh} x + \frac{1}{2} \int \operatorname{sech} x \, dx$

RETURNING TO THE REQUIRED INTEGRAL, WITHOUT LIMITS WE OBTAIN

$\int \operatorname{sech} x - \operatorname{sech}^2 x \, dx = \int \operatorname{sech} x \, dx - \int \operatorname{sech}^2 x \, dx$

$$= \int \operatorname{sech} x \, dx - \left[\frac{1}{2} \operatorname{sech} x \operatorname{tanh} x + \frac{1}{2} \int \operatorname{sech} x \, dx \right]$$

$$= \frac{1}{2} \int \operatorname{sech} x \, dx - \frac{1}{2} \operatorname{sech} x \operatorname{tanh} x$$

WE GET NOW $\int \operatorname{sech} x \, dx$

$I_2 = \int \operatorname{sech} x \, dx + \int \frac{1}{\operatorname{cosec} x} \, dx = \int \frac{\operatorname{cosec} x}{\operatorname{cosec}^2 x} \, dx$

$$= \int \frac{\operatorname{cosec} x}{1 + \operatorname{cosec}^2 x} \, dx = \dots$$

NOW BY INSPECTION AS $\frac{d}{dx}(\operatorname{arctan}(\operatorname{sech} x)) = \frac{1}{1 + \operatorname{sech}^2 x} \times \operatorname{sech} x$
 OR A SUBSTITUTION $u = \operatorname{sech} x$

$$\dots \operatorname{arctan}(\operatorname{sech} x) + C$$

$I_2 = \int \operatorname{sech} x \, dx = \operatorname{arctan}(\operatorname{sech} x) + C$

Question 129 (*****)

The function f is defined as.

$$f(x) = \arctan x, \quad x \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right).$$

- a) Find a simplified expression for $\int f(x) dx$.
- b) By considering the tangent compound angle identity, or otherwise, find an exact simplified value for

$$\int_1^2 \arctan \left[\frac{1}{x^2 - 3x + 3} \right] dx.$$

$\boxed{}$	$\boxed{x \arctan x - \frac{1}{2} \ln(x^2 + 1)}$	$\boxed{\frac{1}{2}\pi - \ln 2}$
-----------------------	--	----------------------------------

a) $\int \arctan x dx$ BY SUBSTITUTION

$$u = \arctan x \quad u' = \frac{1}{1+x^2}$$

$$du = \frac{1}{1+x^2} dx \quad dx = (1+x^2) du$$

$$\int \arctan x \cdot \frac{1}{1+x^2} dx = \int u du$$

NOW PROCEED BY INTEGRATION BY PARTS

$$\begin{aligned} &= u \cdot \frac{1}{2}u^2 - \int u \cdot \frac{1}{2}u^2 du \\ &= \frac{1}{2}u^2 \arctan x - \frac{1}{2} \int u^2 \arctan x du + C \\ &= \frac{1}{2}u^2 \arctan x + \ln|1+u^2| + C \\ &= \arctan x + \ln\left(\frac{1}{1+x^2} + 1\right) + C \\ &= \arctan x + \frac{1}{2} \ln(2+1) + C \end{aligned}$$

ALTERNATIVE BY PARTS OR RECOGNITION

$$\begin{aligned} \int \arctan x dx &= \int x \cdot \arctan x dx \\ &= \arctan x \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot \frac{1}{1+x^2} dx \\ &= \arctan x \cdot \frac{1}{2}x^2 - \frac{1}{2} \int \frac{2x^2}{1+x^2} dx \\ &\quad \uparrow \text{RECOGNITION} \\ &= \arctan x \cdot \frac{1}{2}x^2 - \frac{1}{2} \ln(1+x^2) + C \end{aligned}$$

b) USING THE TAN IDENTITY

$$\begin{aligned} \frac{1}{x^2 - 3x + 3} &= \frac{1}{(x-3)(x+1)} \quad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= \frac{1}{1+(x-3)(x+1)} = \frac{(x-1)-(x+2)}{1+(x-1)(x+1)} \quad \tan A = 2-1 \\ &= \frac{1}{x^2 - 2x - 2} \quad \tan B = x-2 \end{aligned}$$

THUS WE KNOW THAT

$$\begin{aligned} \int_1^2 \arctan \left(\frac{1}{x^2 - 2x - 2} \right) dx &= \int_1^2 \arctan \left[\frac{(x-1)-(x+2)}{1+(x-1)(x+1)} \right] dx \\ &= \int_1^2 \arctan \left[\frac{\tan x - \tan 2}{1 + \tan x \tan 2} \right] dx = \int_1^2 \arctan \left[\tan(x-2) \right] dx \\ &= \int_1^2 A - B dx = \int_1^2 \arctan(x-2) - \arctan(x-2) dx \end{aligned}$$

BY SWAPPING THE INTEGRAL AND SUBSTITUTION / INTEGRATION

$$\begin{aligned} &= \int_1^2 \arctan(x-2) dx - \int_1^2 \arctan(x-2) dx \\ &= \int_0^1 \arctan u du - \int_{-1}^0 \arctan v dv \\ &= \left[u \arctan u - \frac{1}{2} \ln(u^2 + 1) \right]_0^1 - \left[v \arctan v - \frac{1}{2} \ln(v^2 + 1) \right]_{-1}^0 \\ &= \left[\frac{\pi}{4} - \frac{1}{2} \ln 2 \right] - \left[0 - \frac{1}{2} \ln 1 \right] - \left[\left(0 - \frac{1}{2} \ln 1 \right)^2 - \left(-\frac{\pi}{4} - \frac{1}{2} \ln 2 \right)^2 \right] \\ &= \frac{\pi}{4} - \frac{1}{2} \ln 2 - \left[\frac{\pi}{4} + \frac{1}{2} \ln 2 \right] \\ &= \frac{\pi}{2} - \ln 2 \end{aligned}$$

Question 130 (*****)

Use appropriate integration techniques to find an exact simplified value for the following improper integral.

$$\int_0^\infty \frac{x}{e^x - 1} dx.$$

You may assume without proof that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}.$$

ANSWER, $\boxed{\frac{\pi^2}{6}}$

Start by multiplying "top & bottom" of the integrand by e^{-x}

$$\int_0^\infty \frac{x}{e^x - 1} dx = \int_0^\infty \frac{xe^{-x}}{1 - e^{-x}} dx = \int_0^\infty xe^{-x} \left(\frac{1}{1 - e^{-x}} \right) dx$$

Now for $|e^{-x}| < 1$ or simply that $e^{-x} < 1$ we can expand binomially (or equivalently use the sum to infinity formula for a geometric progression)

i.e. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

Thus we have

$$\dots = \int_0^\infty xe^{-x} \left[1 + e^{-x} + e^{-2x} + e^{-3x} + \dots \right] dx$$

$$= \int_0^\infty x \left[e^0 + e^{-1} + e^{-2} + e^{-3} + \dots \right] dx$$

$$= \int_0^\infty x \left[\sum_{n=0}^{\infty} e^{-nx} \right] dx$$

Integrating by substitution & integrating

$$= \sum_{n=1}^{\infty} \left[\int_0^\infty x e^{-nx} dx \right]$$

Produced by gamma functions, differentiation under the integral sign, change of variables, or simply integration by parts

$$\dots = \sum_{n=1}^{\infty} \left[\frac{-x e^{-nx}}{n} \Big|_0^\infty + \int_0^\infty \frac{1}{n} e^{-nx} dx \right]$$

$\frac{x}{n}$	1
$-e^{-nx}$	n

ALTERNATIVE INTEGRATION METHODS (SECTION)

- $\dots = \int_0^\infty x e^{-tx} dx = \int_0^\infty t e^{-xt} dt = \Gamma(t)$

$$= \frac{1}{t^2} = \frac{1}{\pi^2} \text{ etc...}$$

• $\dots = \int_0^\infty x e^{-tx} dx = \text{BY SUBSTITUTION} \dots$

$$= \int_0^\infty \frac{1}{t} e^{-t^2} dt = \frac{1}{t^2} \int_0^\infty t e^{-t^2} dt$$

$$= \frac{1}{t^2} \Gamma(2) = \frac{1}{t^2} \times 1! = \frac{1}{\pi^2} \text{ etc...}$$

$t = rx$
 $dt = rdx$
 $dx = \frac{1}{r} dt$
 (VALUES EXCHANGED)

Question 131 (*****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ , and is commonly known as the golden section or golden number.

This implies that $\phi^2 - \phi - 1 = 0$, $\phi = \frac{1}{2}(1 + \sqrt{5}) \approx 1.62$.

a) Show, with a detailed method, that

$$\frac{d}{dx} \left[x(x^\phi + 1)^{1-\phi} \right] = (x^\phi + 1)^{-\phi}.$$

b) Show, with full justification, that

$$\lim_{x \rightarrow \infty} \left[x(x^\phi + 1)^{1-\phi} \right] = 1.$$

c) Show further that

$$1 - \frac{1}{\sqrt[\phi]{2}} = \int_1^\infty \frac{1}{(x^\phi + 1)^\phi} dx.$$

 , proof

a) DIFFERENTIATE THE PRODUCT AND USE "CHAINS"

$$\begin{aligned} \frac{d}{dx} \left[x(x^\phi + 1)^{1-\phi} \right] &= 1 \times (\phi x^{\phi-1})^{-\phi} + 2(1-\phi)(x^{\phi-1})^{\phi-2} \\ &= (x^\phi + 1)^{1-\phi} + (1-\phi)x^{\phi-2}(x^\phi + 1)^{-\phi} \\ &\quad \text{FACTORISE } (x^\phi + 1)^{-\phi}, \text{ AS IN THE QSNW} \\ &= (x^\phi + 1)^{1-\phi} \left[(x^\phi + 1)^{\phi-1} + \phi(1-\phi)x^{\phi-2} \right] \\ &= (x^\phi + 1)^{1-\phi} \left[(x^\phi + 1) + \phi x^{\phi-2} - \phi^2 x^{\phi-2} \right] \\ &= (x^\phi + 1)^{1-\phi} \left[1 + x^{\phi-2} + \phi x^{\phi-2} - \phi^2 x^{\phi-2} \right] \\ &\quad \text{NOTE TO SHOW THIS IS 0:} \\ &= (x^\phi + 1)^{1-\phi} \left[1 + \phi(1 + \phi - \phi^2) \right] \\ \text{BUT } x^2 - x - 1 = 0 \quad \text{(C. } \phi^2 - \phi - 1 = 0 \text{ C. } -\phi^2 + \phi + 1 = 0\text{)} \\ &= (x^\phi + 1)^{1-\phi} \left[1 + \phi^2 - \phi^2 \right] \\ &= (x^\phi + 1)^{1-\phi} \\ &\quad \text{AS REQUIRED} \end{aligned}$$

b) OTHER OF EVALUATING THE LIMIT

$$\lim_{x \rightarrow \infty} \left[x(x^\phi + 1)^{1-\phi} \right] = \lim_{x \rightarrow \infty} \left[\frac{x}{(x^\phi + 1)^{\phi-1}} \right] \leftarrow \frac{\infty}{\infty}$$

TESTING L'HOSPITAL

$$\lim_{x \rightarrow \infty} \left[\frac{1}{(x-1)(x^{\phi-1})^{\phi-2} + x^{\phi-2}} \right] \leftarrow \frac{1}{\infty}$$

ANSWER: FAILURE

PROCEED AS FOLLOWS

$$\begin{aligned} \dots \lim_{x \rightarrow \infty} \left[\frac{x}{(x^\phi + 1)^{\phi-1}} \right] &= \lim_{x \rightarrow \infty} \left[\frac{2}{(x^\phi + 1)^{\phi-1}} \right] \\ &= \lim_{x \rightarrow \infty} \left[\frac{2}{x^{\phi-1}(1 + x^{-\phi})^{\phi-1}} \right] = \lim_{x \rightarrow \infty} \left[\frac{\frac{2}{x^{\phi-1}}}{(1 + x^{-\phi})^{\phi-1}} \right] \\ &= \lim_{x \rightarrow \infty} \left[\frac{\frac{2}{x^{\phi-1}}(1-x^{-\phi})}{(1+x^{-\phi})^{(\phi-1)+1}} \right] \sim 1 \quad \text{(C. } \frac{1}{x^{\phi-1}} \rightarrow 0 \text{)} \\ &\quad \text{AS REQUIRED} \end{aligned}$$

c) FINALLY THE INTEGRAL

$$\begin{aligned} \int_1^\infty \frac{1}{(x^\phi + 1)^\phi} dx &= \dots \text{INT. (a)} = \left[\frac{x(x^\phi + 1)^{1-\phi}}{1-\phi} \right]_1^\infty \\ &= \frac{x(x^\phi + 1)^{1-\phi}}{1-\phi} \Big|_{x=1}^\infty = \frac{1}{1-\phi} (1 + 1)^{1-\phi} \\ &= 1 - (1+1)^{1-\phi} = 1 - (1+1)^{-\phi} = 1 - 2^{-\phi} \\ \text{NOW SINCE } \phi^2 - \phi - 1 = 0 &\Rightarrow \phi^2 - \phi = 1 \Rightarrow \phi = \frac{1}{\phi-1} \\ &\Rightarrow \phi - 1 = \frac{1}{\phi} \\ &\Rightarrow 1 - \phi = -\frac{1}{\phi} \\ \text{HENCE WE HAVE} &\\ \dots = 1 - 2^{-\phi} &= 1 - 2^{-\frac{1}{\phi-1}} = 1 - \frac{1}{2^{\frac{1}{\phi-1}}} = 1 - \frac{1}{2^{\frac{1}{\frac{1}{\phi-1}}}} \quad \text{AS REQUIRED} \end{aligned}$$

Question 132 (*****)

It is given that

- ♦ $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{1}{4}\pi$
- ♦ $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots = \frac{1}{12}\pi^2$
- ♦ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln 2$

Assuming the following integral converges find its exact value.

$$\int_0^1 (\ln x)(\arctan x) dx .$$

[you may assume that integration and summation commute]

, $\frac{1}{48}[\pi^2 - 12\pi + 24\ln 2]$

IT IS UNLIKELY THAT THIS INTEGRAL HAS A CLOSED FORM IN TERMS OF ELEMENTARY FUNCTIONS IN INDEFINITE FORM—USE SERIES EXPANSION

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

INTEGRATING WITH RESPECT TO x

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + C$$

$$\text{where } C = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)} x^{2n+1}$$

YOU REFERRING TO THE INTEGRAL & SUMMATION AND SUMMATION

$$\int_0^1 (\arctan x)(\ln x) dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)} x^{2n+1} \ln x dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}}{(2n+1)} \int_0^1 x^{2n+1} \ln x dx \right]$$

INTEGRATION BY PARTS INSIDE THE SUM

$$\int_0^1 x^{2n+1} \ln x dx = \left[\frac{x^{2n+2}}{2n+2} \right]_0^1 - \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)} \int_0^1 x^{2k+2} dx$$

Let $t = 2k+2$, then $dt = 2x dx$, $x = t/2$, $x^2 = t^2/4$, $x^{2k+2} = t^{2k+2}/4^k$, $x^{2k+2} dx = t^{2k+2}/4^k dt$

$$= \sum_{k=0}^{\infty} \left[\frac{(-1)^{k+1}}{(2k+1)} \frac{t^{2k+2}}{2(2k+2)} \right] dt$$

$$= \sum_{k=0}^{\infty} \left[\frac{(-1)^{k+1}}{(2k+1)} \frac{t^{2k+2}}{4(2k+2)} \right] dt$$

SUMMING UP SO FAR

$$\int_0^1 (\arctan x)(\ln x) dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}}{(2n+1)} \frac{x^{2n+2}}{4(2n+2)} \right] = \frac{1}{4} \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}}{(2n+1)(2n+2)} \right]$$

OBVIOUS SCALE PARTIAL FRACTION

$$\frac{1}{(2n+1)(2n+2)} = \frac{A}{2n+1} + \frac{B}{2n+2} + \frac{C}{(2n+1)(2n+2)}$$

• $16 = 4n+4$
 $A = 1$
 $B = -A = -1$
 $C = \frac{1}{2}C$
 $A = 1$
 $B = -1$
 $C = \frac{1}{2}$

THIS WE KNOW HAVE

$$\int_0^1 (\arctan x)(\ln x) dx = -\frac{1}{2} \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}}{(2n+1)^2} + \frac{(-1)^{n+1}}{(2n+2)^2} \right] + \frac{(-1)^{n+1}}{2(2n+1)}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^{2n+2}}{(2n+1)^2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^{2n+3}}{(2n+2)^2} - \frac{(-1)^1}{2(2n+1)}$$

LOOKING AT THE RESULT (CONT.)

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)^2} = \frac{1}{4}\pi^2$$

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^{2n+2}}{(2n+2)^2} = \frac{\pi^2}{12}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)} = \ln 2$$

Finalised (we have)

$$\int_0^1 (\arctan x)(\ln x) dx = \frac{1}{4} \left(\frac{\pi^2}{12} \right) + \frac{1}{2} \ln 2 - \frac{1}{4}\pi^2$$

$$= \frac{\pi^2}{48} - \frac{1}{4}\pi^2 + \frac{1}{2} \ln 2$$

$$= \boxed{\frac{\pi^2}{48} - \frac{1}{4}\pi^2 + \frac{1}{2} \ln 2}$$

Question 133 (*****)

Find the value of

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin \frac{9}{2}x}{\sin \frac{1}{2}x} dx.$$

You may assume that the integrand is continuous at $x = 0$.

[2]

SIMPLIFYING THE INTEGRAND

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin \frac{9}{2}x}{\sin \frac{1}{2}x} dx = \frac{2}{\pi} \int_0^\pi \frac{\sin \frac{9}{2}x}{\sin \frac{1}{2}x} dx$$

NOW LET I BE THE REQUIRED INTEGRAL AND PROCEED BY A SUBSTITUTION

$$\begin{aligned} \theta &= \pi - x \\ d\theta &= -dx \\ 0 &\mapsto \pi \\ \pi &\mapsto 0 \end{aligned}$$

$$\Rightarrow I = \frac{2}{\pi} \int_0^\pi \frac{\sin(3(\pi-x))}{\sin(\frac{1}{2}\pi-x)} (-d\theta)$$

$$\Rightarrow I = \frac{2}{\pi} \int_0^\pi \frac{\sin(\frac{9}{2}\pi - \frac{9}{2}x)}{\sin(\frac{1}{2}\pi - \frac{1}{2}x)} d\theta \quad \text{TAKES TWO PERIODS OF } 2\pi$$

$$\Rightarrow I = \frac{2}{\pi} \int_0^\pi \frac{\sin(\frac{9}{2}\pi - \frac{9}{2}x)}{\sin(\frac{1}{2}\pi - \frac{1}{2}x)} d\theta$$

$$\Rightarrow I = \frac{2}{\pi} \int_0^\pi \frac{\cos \frac{9}{2}x}{\sin \frac{1}{2}x} d\theta = \frac{2}{\pi} \int_0^\pi \frac{\cos \frac{9}{2}x}{\sin \frac{1}{2}x} d\theta = I$$

REWRITING THE ABOVE EQUATION AS FOLLOWS

$$\begin{aligned} \Rightarrow I + I &= \frac{2}{\pi} \int_0^\pi \frac{\sin \frac{1}{2}x}{\sin \frac{1}{2}x} dx + \frac{2}{\pi} \int_0^\pi \frac{\cos \frac{9}{2}x}{\sin \frac{1}{2}x} dx \\ \Rightarrow 2I &= \frac{2}{\pi} \int_0^\pi \frac{\sin \frac{1}{2}x + \cos \frac{9}{2}x}{\sin \frac{1}{2}x} dx \end{aligned}$$

ADDING THE ITEM IN THE INTEGRAND USING THE COMPOUND ANGLE IDENTITIES AFTERWARD

$$\Rightarrow I = \frac{1}{\pi} \int_0^\pi \frac{\sin \frac{9}{2}x \cos \frac{1}{2}x + \cos \frac{9}{2}x \sin \frac{1}{2}x}{\sin \frac{1}{2}x} dx$$

$$\Rightarrow I = \frac{1}{\pi} \int_0^\pi \frac{\sin(\frac{9}{2}x + \frac{1}{2}x)}{\sin \frac{1}{2}x} dx$$

$$\Rightarrow I = \frac{1}{\pi} \int_0^\pi \frac{\sin \frac{5}{2}x}{\sin \frac{1}{2}x} dx$$

$$\Rightarrow I = \frac{2}{\pi} \int_0^\pi \frac{\sin \frac{5}{2}x}{\sin \frac{1}{2}x} dx$$

NEXT BY COMPLEX NUMBERS (OR + INDUCTION FORMULA)

$$\begin{aligned} \rightarrow \cos \theta + i \sin \theta &= C + iS \\ \rightarrow (\cos \theta + i \sin \theta)^2 &= (C+iS)^2 \\ \rightarrow \cos 2\theta + i \sin 2\theta &= C^2 - S^2 + 2iCS - 10i^2CS^2 + 10i^2S^2 + i^2S^2 \\ \Rightarrow \sin 5\theta &= 5C^4 - 10C^2S^2 + S^4 \\ &= 5S^4 - 10S^2C^2 - 10S^4C^2 + 10S^2C^4 + S^4 \\ &= 5S^4 - 10S^2C^2 + 10S^2C^4 + S^4 \\ &= 10S^4 - 20S^2C^2 + 5S^4 \\ &= 15S^4 - 20S^2C^2 + 5S^4 \end{aligned}$$

RETURNING TO THE INTERVAL WE OBTAIN

$$I = \frac{2}{\pi} \int_0^\pi \frac{15S^4x - 20S^2x^2 + 5S^4}{\sin x} dx$$

$$I = \frac{2}{\pi} \int_0^\pi 15S^3x - 20S^2x + 5 dx$$

$$I = \frac{2}{\pi} \int_0^\pi 16\left(\frac{1}{2}\sin 2x\right)^2 - 20\left(\frac{1}{2}\sin 2x\right) + 5 dx$$

$$I = \frac{2}{\pi} \int_0^\pi 4 - 8\sin 2x + 16\sin^2 2x - 10\sin 2x + 5 dx$$

$$I = \frac{2}{\pi} \int_0^\pi -1 + 2\sin 2x + 16\left(\frac{1}{2}\sin 2x\right)^2 dx$$

NO CONTRIBUTION SINCE THESE UNITS

$$I = \frac{2}{\pi} \int_0^\pi 1 dx$$

$$I = \frac{2}{\pi} \times \pi$$

$$I = 2$$

Question 134 (*****)

Use appropriate integration techniques to find an exact simplified value for the following improper integral.

$$\int_0^\infty \frac{x^3}{e^x - 1} dx.$$

You may assume without proof that

$$\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{\pi^4}{90}.$$

$\boxed{6}$	\boxed{X}	$\boxed{\frac{\pi^4}{15}}$
-------------	-------------	----------------------------

Start by multiplying "top & bottom" of the integrand by e^x

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \int_0^\infty \frac{x^3 e^x}{e^x - e^0} dx = \int_0^\infty x^3 e^x \left(\frac{1}{e^x - e^0} \right) dx$$

For $e^x \geq 1$ (or simply here $0 < e^x - 1$), we can expand binomially or equivalently see the sum to infinity expansion for a geometric series

$$1e^{-\frac{1}{e^x}} = (1 - e^x + e^{2x} - e^{3x} + \dots)$$

Thus we now have

$$\dots = \int_0^\infty x^3 e^x [1 + e^x + e^{2x} + e^{3x} + \dots] dx$$

$$= \int_0^\infty x^3 [e^0 + e^x + e^{2x} + e^{3x} + \dots] dx$$

$$= \int_0^\infty x^3 \left[\sum_{k=0}^{\infty} e^{kx} \right] dx$$

Differentiating, summing & integrating

$$= \sum_{k=1}^{\infty} \left[\int_0^\infty x^3 e^{kx} dx \right]$$

Proceed to evaluate the integral by parts

$$\int_0^\infty x^3 e^{kx} dx = \left[\frac{x^3 e^{kx}}{k} \right]_0^\infty + \int_0^\infty \frac{3x^2 e^{kx}}{k} dx$$

$$= \frac{3}{k} \int_0^\infty x^2 e^{kx} dx$$

$$= \frac{3}{k} \left\{ \left[\frac{x^2 e^{kx}}{k} \right]_0^\infty + \int_0^\infty \frac{2x e^{kx}}{k} dx \right\}$$

REGARDING TO THE "MAIN LINE" OF THE PROBLEM

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \dots \sum_{k=1}^{\infty} \left[\int_0^\infty x^3 e^{kx} dx \right] = \dots \sum_{k=1}^{\infty} \frac{c}{k^4}$$

$$= 6 \sum_{k=1}^{\infty} \frac{1}{k^4} = 6 \times \frac{\pi^4}{90} = \frac{\pi^4}{15}$$

ALTERNATIVE APPROACHES. RE: $\int_0^\infty x^3 e^{-kx} dx$

- \bullet $\int_0^\infty x^3 e^{-kx} dx = \dots \frac{x^3}{k} \Big|_0^\infty = \int_0^\infty t^3 e^{-kt} dt = \int_0^\infty t^3 dt$

$$= \frac{3!}{k^4} = \frac{6}{k^4}$$
 AS ABOVE
- \bullet $\int_0^\infty x^3 e^{-kx} dx = \dots$ BY SUBSTITUTION ...

$t = kx$
$x = \frac{t}{k}$
$dx = \frac{1}{k} dt$

$$= \int_0^\infty \left(\frac{t^3}{k^3} \right) e^{-t} \left(\frac{1}{k} dt \right)$$

$$= \frac{1}{k^4} t^3 e^{-t} dt$$

$$\begin{aligned} &= \frac{1}{k^4} \Gamma(4) \\ &= \frac{1}{k^4} \cdot 3! \\ &= \frac{6}{k^4} \quad \text{AS ABOVE} \\ \bullet \quad &\int_0^\infty x^3 e^{-kx} dx = -\frac{\partial^3}{\partial k^3} \left[\int_0^\infty e^{-kx} dx \right] \\ &= -\frac{\partial^3}{\partial k^3} \left[-\frac{1}{k} e^{-kx} \Big|_0^\infty \right] \\ &= -\frac{2!}{k^3} \left[0 + \frac{1}{k} \right] \\ &= -\frac{2!}{k^3} \left(\frac{1}{k} \right) \\ &= -\frac{2!}{k^3} \left(-\frac{1}{k^2} \right) \\ &= -\frac{2!}{k^3} \left(+\frac{2}{k^2} \right) \\ &= -\left(-\frac{6}{k^3} \right) \\ &= \frac{6}{k^3} \quad \text{AS DIRECT} \end{aligned}$$

Question 135 (*****)

The function f is defined as

$$f(x) = \arctan\left(\frac{1}{2x^2}\right), \quad x \in (-\infty, \infty).$$

a) Find a simplified expression for $f'(x)$.

b) Show that $\lim_{x \rightarrow \pm\infty} [xf(x)] = 0$.

c) Determine the value of $\lim_{x \rightarrow \pm\infty} \left[\ln\left(\frac{2x^2 - 2x + 1}{2x^2 + 2x + 1}\right) \right]$.

d) Hence find the value of $\int_{-\infty}^{\infty} f(x) dx$.

$$\square, \boxed{f'(x) = -\frac{4x}{4x^4 + 1}}, \boxed{\lim_{x \rightarrow \pm\infty} \left[\frac{2x^2 - 2x + 1}{2x^2 + 2x + 1} \right] = 0}, \boxed{\int_0^{\infty} f(x) dx = \frac{1}{2}\pi}$$

a) DIFFERENTIATE AND TRY

$$\begin{aligned} \frac{d}{dx} \left[\arctan\left(\frac{1}{2x^2}\right) \right] &= \frac{1}{1 + (\frac{1}{2x^2})^2} \times \frac{d}{dx}(\frac{1}{2x^2}) = \frac{1}{1 + \frac{1}{4x^4}} \times \frac{-4x}{x^3} \\ &= \frac{-4x}{4x^4 + 1} \times \frac{1}{x^3} = -\frac{4x}{4x^4 + 1} \end{aligned}$$

b) INSERT THE FIRST LIMIT

$$\lim_{x \rightarrow \infty} [xf(x)] = \lim_{x \rightarrow \infty} \left[x \cdot \arctan\left(\frac{1}{2x^2}\right) \right]$$

THIS IS AN INDETERMINATE FORM $(\infty)(0)$ - REWRITE TO USE L'HOSPITAL

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \left[\frac{\arctan\left(\frac{1}{2x^2}\right)}{\frac{1}{x}} \right] \quad \leftarrow \text{NOW REWRITTEN AS A QUOTIENT, URGENTLY USE L'HOSPITAL} \\ &= \lim_{x \rightarrow \infty} \left[-\frac{\frac{4x}{4x^4 + 1}}{-\frac{1}{2x^3}} \right] \quad \leftarrow \text{DERIVATIVE} = \lim_{x \rightarrow \infty} \left[\frac{4x^2}{4x^4 + 1} \right] \\ &= \lim_{x \rightarrow \infty} \left[\frac{\frac{4}{4 + \frac{1}{x^2}}}{1} \right] = \frac{0}{1} = 0 \end{aligned}$$

c) THE NEXT LIMIT IS VERY EASY

$$\lim_{x \rightarrow \infty} \left[\ln\left(\frac{2x^2 - 2x + 1}{2x^2 + 2x + 1}\right) \right] = \lim_{x \rightarrow \infty} \left[\ln\left(\frac{2 - \frac{2}{x} + \frac{1}{x^2}}{2 + \frac{2}{x} + \frac{1}{x^2}}\right) \right]$$

$$= \ln\left[\frac{2}{2}\right] = \ln[1] = 0$$

d) PROCEEDED BY INTEGRATION BY PARTS

$$\int_{-\infty}^{\infty} \arctan\left(\frac{1}{2x^2}\right) dx$$

NOTE THAT AT $x = 0$, $f(x) = \frac{\pi}{4}$ (REALLY SEE SKETCH)

$$\begin{aligned} &= \left[x \arctan\left(\frac{1}{2x^2}\right) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{x^2}{4x^4 + 1} dx \\ &\stackrel{\text{HINT b}}{=} \int_{-\infty}^{\infty} \frac{x^2}{4x^4 + 1} dx \end{aligned}$$

NOW BY THE SEPARATE GRAMMEN'S IDENTITY

$$a^2 + b^2 \leq (a^2 + b^2)(c^2 + d^2) = (a^2 + c^2 + b^2 + d^2)$$

OR BY COMPUTING THE SQUARE

$$\begin{aligned} a^2 + b^2 &= ((a^2 + b^2) + (c^2 + d^2)) - (c^2 + d^2) \\ &= (a^2 + 1 - 2a) + (b^2 + 1 + 2b) \\ &\leq (a^2 + 1 - 2a) + (b^2 + 1 + 2b) \end{aligned}$$

OR BY LOOKING AT THE EXPONENTIAL UNIT OF PART (c) & SUBSTITUTING THAT $(2x^2 - 2x + 1)(2x^2 + 2x + 1) = e^{2x^2 + 1}$

$$\begin{aligned} &\dots = \int_{-\infty}^{\infty} \frac{x^2}{4x^4 + 1} dx = \int_{-\infty}^{\infty} \frac{a^2}{(2x^2 - 2x + 1)(2x^2 + 2x + 1)} dx \\ &\quad \uparrow \quad \uparrow \\ &\quad \text{BOTH ARE IRREDUCIBLE} \end{aligned}$$

PARTIAL FRACTION DECOMPOSITION

$$\frac{a^2}{(2x^2 - 2x + 1)(2x^2 + 2x + 1)} = \frac{Ax + B}{2x^2 - 2x + 1} + \frac{Cx + D}{2x^2 + 2x + 1}$$

$$\begin{aligned} a^2 &\equiv (Ax + B)(2x^2 + 2x + 1) + (Cx + D)(2x^2 - 2x + 1) \\ Ax^3 &\equiv 2Ax^3 + 2Ax^2 + Ax \\ 2x^3 &\equiv -2Ax^3 + 2Ax^2 + Ax \\ 4x^2 &\equiv 2(Ax^2 + Cx^2) + (A + B + C - 2D)x + (B + D) \\ Ax + B &\equiv Ax^2 + Cx^2 + Bx + D \\ \Rightarrow Ax^2 + Bx &= 0 \\ \Rightarrow B + D &= 0 \\ \Rightarrow B &= -D \end{aligned}$$

Thus $2(A + B - C + D) = 0$

$$\begin{aligned} 2(A + B - C + D) &= 0 \\ 2(A + B) &= 0 \\ A + B &= 0 \\ A &= -B \end{aligned}$$

DEFINING $A = 1$ & $B = -1$

$$\begin{aligned} &\dots = \int_{-\infty}^{\infty} \frac{x}{2x^2 - 2x + 1} dx - \int_{-\infty}^{\infty} \frac{x}{2x^2 + 2x + 1} dx \\ &= \int_{-\infty}^{\infty} \frac{2}{2x^2 - 2x + 1} dx - \int_{-\infty}^{\infty} \frac{2}{2x^2 + 2x + 1} dx \end{aligned}$$

MANIPULATE INTO LOG & ARCTAN AS FOLLOWS

$$\begin{aligned} &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{2x - 2}{2x^2 - 2x + 1} dx = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{2x + 2}{2x^2 + 2x + 1} dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{2x - 2}{2x^2 - 2x + 1} dx - \frac{1}{2} \left[\int_{-\infty}^{\infty} \frac{2x + 2}{2x^2 + 2x + 1} dx + \int_{-\infty}^{\infty} \frac{2}{2x^2 + 2x + 1} dx \right] \\ &= \frac{1}{2} \left[\ln(2x^2 - 2x + 1) - \frac{1}{2} \ln(2x^2 + 2x + 1) \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{2x^2 - 2x + 1} dx \\ &\quad + \int_{-\infty}^{\infty} \frac{1}{2x^2 + 2x + 1} dx \\ &= \left[\frac{1}{2} \ln\left(\frac{2x^2 - 2x + 1}{2x^2 + 2x + 1}\right) \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{(2x^2 + 2x + 1)^2} dx + \int_{-\infty}^{\infty} \frac{1}{2x^2 - 2x + 1} dx \\ &\quad \text{PART C} \end{aligned}$$

THESE ARE STANDARD ARCTAN INTEGRALS, $\int \frac{1}{1 + u^2} du = \frac{1}{2} \arctan(u) + C$

$$\begin{aligned} &= \left[\frac{1}{2} \arctan(2x + 1) + \frac{1}{2} \arctan(2x - 1) \right]_{-\infty}^{\infty} \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\arctan(2x + 1) + \arctan(2x - 1) - \arctan(-b - 1) - \arctan(b + 1) \right] \\ &= \frac{1}{2} \left[\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} \right] \\ &= \pi \end{aligned}$$

Question 136 (*****)

A family of functions $f_n(x)$, where $n = 0, 1, 2, 3, 4, \dots$, satisfies the equation

$$\sum_{n=0}^{\infty} [t^n f_n(x)] = (1 - 2xt + t^2)^{-\frac{1}{2}}.$$

By integrating both sides of the above equation with respect to t , from 0 to 1, show that

$$\sum_{n=0}^{\infty} \left[\frac{f_n(\cos \theta)}{n+1} \right] = \ln \left[1 + \operatorname{cosec} \left(\frac{1}{2} \theta \right) \right].$$

You may assume in this question that integration and summation commute.

proof

$$\sum_{n=0}^{\infty} t^n f_n(x) = (1 - 2xt + t^2)^{-\frac{1}{2}}$$

Let $u = \cos \theta$

$$\sum_{n=0}^{\infty} t^n f_n(u) = \frac{1}{\sqrt{1 - 2u\cos \theta + t^2}}$$

Integrate both sides of the equation w.r.t. t , from 0 to 1

$$\Rightarrow \int_0^1 \left[\sum_{n=0}^{\infty} t^n f_n(u) \right] dt = \int_0^1 \frac{1}{\sqrt{1 - 2u\cos \theta + t^2}} dt$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[f_n(u) \int_0^1 t^n dt \right] = \int_0^1 \frac{1}{\sqrt{(1-u\cos \theta)^2 + u^2}} dt$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[f_n(u) \left[\frac{t^{n+1}}{n+1} \right] \right]_0^1 = \int_0^1 \frac{1}{\sqrt{(1-u\cos \theta)^2 + u^2}} dt$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \int_0^1 \frac{1}{\sqrt{(1-u\cos \theta)^2 + u^2}} dt$$

Let $u = t - u\cos \theta$
 $du = dt$
 $t \rightarrow u + u\cos \theta$
 $t = 1 \rightarrow u = 1 - u\cos \theta$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \int_{1-u\cos \theta}^{1} \frac{1}{\sqrt{u^2 + u^2 \sin^2 \theta}} du$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \left[\operatorname{cosec} \left(\frac{u}{\sin \theta} \right) \right]_{1-u\cos \theta}^1$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \left[\ln \left[\frac{u}{\sin \theta} + \sqrt{\frac{u^2 + u^2 \sin^2 \theta}{\sin^2 \theta}} \right] \right]_{1-u\cos \theta}^1$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \left[\ln \left[\frac{u}{\sin \theta} + \sqrt{\frac{u^2 + u^2 \sin^2 \theta}{\sin^2 \theta}} \right] \right]_{u=1-u\cos \theta}^{u=1}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \left[\ln \left[\frac{u + \sqrt{u^2 + u^2 \sin^2 \theta}}{\sin \theta} \right] \right]_{u=1-u\cos \theta}^{u=1}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \ln \left[\frac{1 - u\cos \theta + \sqrt{(1-u\cos \theta)^2 + u^2}}{\sin \theta} \right] - \ln \left[\frac{1 - u\cos \theta}{\sin \theta} \right]$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \ln \left[\frac{1 - u\cos \theta + \sqrt{1 - 2u\cos \theta + u^2}}{\sin \theta} \right] + \ln \left[\frac{\sin \theta}{1 - u\cos \theta} \right]$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \ln \left[\frac{1 - u\cos \theta + \sqrt{1 - 2u\cos \theta}}{1 - u\cos \theta} \right]$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \ln \left[\frac{(1 - u\cos \theta) + \sqrt{1 - 2u\cos \theta}}{1 - u\cos \theta} \right]$$

NOW $1 - u\cos \theta = 1 - (1 - 2\sin^2 \frac{\theta}{2}) = 2\sin^2 \frac{\theta}{2}$
 $\sqrt{1 - u\cos \theta} = \sqrt{2} \sin \frac{\theta}{2}$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \ln \left[\frac{2\sin^2 \frac{\theta}{2} + \sqrt{2} \sin \frac{\theta}{2} \sin \frac{\theta}{2}}{2\sin^2 \frac{\theta}{2}} \right]$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \ln \left[\frac{\sin \frac{\theta}{2} + 1}{\sin \frac{\theta}{2}} \right]$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f_n(u)}{n+1} = \ln \left[1 + \operatorname{cosec} \frac{\theta}{2} \right]$$

as required