

# POLYNOMIALS

**Question 1 (\*\*)**

When  $f(x)$  is divided by  $(x^2 + 1)$  the quotient is  $(3x - 1)$  and the remainder is  $(2x - 1)$ .

Determine an expression for  $f(x)$ .

$$f(x) = 3x^3 - x^2 + 5x - 2$$

$$\begin{aligned} \frac{f(x)}{x^2+1} &= (3x-1) + \frac{2x-1}{x^2+1} \\ \therefore f(x) &= (3x-1)(x^2+1) + (2x-1) \\ f(x) &= 3x^3 + 3x - x^2 - 1 + 2x - 1 \\ f(x) &= 3x^3 - x^2 + 5x - 2 \end{aligned}$$

**Question 2 (\*\*\*)**

Find the three solutions of the cubic equation

$$2x^3 - x^2 = 7x - 6.$$

$$x = -2, 1, \frac{3}{2}$$

$$\begin{aligned} 2x^3 - x^2 &= 7x - 6 \\ 2x^3 - x^2 - 7x + 6 &= 0 \\ \text{Let } f(x) &= 2x^3 - x^2 - 7x + 6 \\ \text{look for factors} \\ f(x) &= 2 - 1 - 7 + 6 = 0 \\ \therefore (x-1) &\text{ is a factor} \\ x-1 &\mid \overline{2x^3 - x^2 - 7x + 6} \\ &\quad \overline{-2x^3 + 2x^2} \\ &\quad \overline{x^2 - 7x + 6} \\ &\quad \overline{-x^2 + 2x} \\ &\quad \overline{-5x + 6} \\ &\quad \overline{-5x + 10} \\ &\quad \overline{-4} \end{aligned}$$

THUS

$$\begin{aligned} 2x^3 - x^2 - 7x + 6 &= 0 \\ (x-1)(2x^2 + 2x - 6) &= 0 \\ (x-1)(2x+6)(x-3) &= 0 \\ x-1 &= -2 \\ x &= -1 \\ x-3 &= 0 \\ x &= 3 \end{aligned}$$

**Question 3    (\*\*\*)**

Find the quotient of the division of

$$2x^6 - 3x^5 - 2x^4 + 2x^2 - 88x + 168 \quad \text{by} \quad x^2 - 4x + 4.$$

$$\boxed{2x^4 + 5x^3 + 10x^2 + 20x + 42}$$

$$\begin{array}{r}
 2x^4 + 5x^3 + 10x^2 + 20x + 42 \\
 \hline
 x^2 - 4x + 4 \overline{)2x^6 - 3x^5 - 2x^4 + 2x^2 - 88x + 168} \\
 2x^6 - 8x^5 + 16x^4 \\
 \hline
 5x^5 - 20x^4 + 2x^2 - 88x + 168 \\
 5x^5 - 20x^4 + 20x^3 \\
 \hline
 20x^3 - 88x + 168 \\
 20x^3 - 80x^2 + 16x \\
 \hline
 80x^2 - 88x + 168 \\
 80x^2 - 64x + 16 \\
 \hline
 -24x + 168 \\
 -24x + 16 \\
 \hline
 168
 \end{array}$$

∴ QUOTIENT IS  
 $2x^4 + 5x^3 + 10x^2 + 20x + 42$ .

**Question 4    (\*\*\*)**

$$\frac{x^4 + 1}{x^2 + 1} \equiv Ax^2 + B + \frac{C}{x^2 + 1}.$$

Find the value of each of the constants  $A$ ,  $B$  and  $C$ .

$$\boxed{A=1}, \boxed{B=-1}, \boxed{C=2}$$

$$\begin{aligned}
 \frac{x^4 + 1}{x^2 + 1} &\equiv (Ax^2 + B + C)(x^2 + 1) \\
 x^4 + 1 &\equiv (Ax^2)(x^2 + 1) + B(x^2 + 1) + C(x^2 + 1) \\
 x^4 + 1 &\equiv Ax^4 + Bx^2 + Ax^2 + B + C \\
 x^4 + 1 &\equiv Ax^4 + (A+B)x^2 + B + C
 \end{aligned}$$

$\therefore A=1$   
 $A+B=0$   
 $B=-1$   
 $B+C=1$   
 $\therefore C=2$

$$\begin{aligned}
 \frac{x^4 + 1}{x^2 + 1} &= \frac{x^2(x^2 + 1) - 2x^2}{x^2 + 1} \\
 &= \frac{x^2(x^2 + 1) - 2x^2}{x^2 + 1} \\
 &= x^2 - 1 + \frac{2}{x^2 + 1}
 \end{aligned}$$

$\therefore A=1$   
 $B=-1$   
 $C=2$

**Question 5    (\*\*\*)+**

A quintic polynomial is defined, in terms of the constants  $a$  and  $b$ , by

$$f(x) = x^5 + ax^4 + bx^3 - x^2 + 4x - 3.$$

When  $f(x)$  is divided by  $(x-2)$  the remainder is  $-7$ .

When  $f(x)$  is divided by  $(x+1)$  the remainder is  $-16$ .

- Determine in any order the value of  $a$  and the value of  $b$ .
- Find the remainder when  $f(x)$  is divided by  $(x-2)(x+1)$ .

,  $a = -4$  ,  $b = 3$  ,  $3x - 13$

<p><b>(a)</b> <math>f(x) = x^5 + ax^4 + bx^3 - x^2 + 4x - 3</math></p> $\begin{aligned} f(2) &= -7 \Rightarrow 2^5 + 16a + 8b - 4 + 8 - 3 = -7 \\ f(-1) &= 4 \Rightarrow -1 + a - b - 1 - 4 - 3 = -16 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 16a + 8b &= -40 \\ a - b &= -7 \end{aligned}$ <p><math>\begin{array}{l} 2a + b = -5 \\ a - b = -7 \end{array} \quad \left\{ \begin{array}{l} 3a = -12 \\ a = -4 \end{array} \right. \quad \begin{array}{l} 2a + b = -5 \\ -6 + b = -5 \\ b = 1 \end{array}</math></p>
<p><b>(b)</b> <math>f(x) = (x-2)(x+1)g(x) + Ax + B</math></p> $\begin{aligned} f(0) &= -7 \Rightarrow 2A + B = -7 \\ f(-1) &= 4 \Rightarrow -A + B = 16 \end{aligned} \quad \text{SUBTRACT } \begin{aligned} 3A &= 9 \\ A &= 3 \end{aligned} \quad \begin{aligned} 2A + B &= -7 \\ 6 + B &= -7 \\ B &= -13 \end{aligned}$ <p><math>\therefore \text{REMAINDER } 3x - 13</math></p>

**Question 6   (\*\*\*)+**

A polynomial  $p(x)$  is defined, in terms of a constant  $a$ , by

$$p(x) = x^4 + 2x^3 + 9x + a.$$

When  $p(x)$  is divided by  $x^2 - x + 2$  the quotient is  $x^2 + bx + 1$  and the remainder is  $cx + 5$ , where  $b$  and  $c$  are constants.

Find the value of  $a$ ,  $b$  and  $c$ .

$$a = 7, \quad b = 3, \quad c = 4$$

$$\begin{aligned} x^4 + 2x^3 + 9x + a &\equiv (x^2 - x + 2)(x^2 + bx + 1) + cx + 5 \\ x^4 + 2x^3 + 9x + a &= x^4 + bx^3 + x^2 - x^3 - bx^2 - x \\ &\quad -2x^2 + 2x + 2 \\ &\quad cx + 5 \end{aligned}$$

$\cancel{x^4} + \cancel{2x^3} + \cancel{9x} + a \equiv x^4 + (b-1)x^3 + (1-b)x^2 + (2+b)x + (2b+c-1)x + 7$

$\therefore a = 7 \quad b-1=2 \quad \text{FINALY } 2b+c-1=9$   
 $b=3 \quad c=3-b=0 \quad c+1=9$   
 $(\cancel{a=3-b=0}) \quad c=4$

## Question 7 (\*\*\*)+

$$x^3 + \left(2 - \frac{1}{5}k\right)x^2 + (2k+1)x + 20 = 0.$$

- a) Determine the value of the real constant  $k$ , if the above equation is to have  $x=1$  as one of its roots.
- b) Solve the equation for the value of  $k$ , found in part (a).

$$k=10, x=-5, 4, 1$$

(a)  $x^3 + \left(2 - \frac{1}{5}k\right)x^2 + (2k+1)x + 20 = 0$   
 If  $x=1$  is a root  
 $1 + \left(2 - \frac{1}{5}k\right) - (2k+1) + 20 = 0$   
 $3 - \frac{1}{5}k - 2k - 1 + 20 = 0$   
 $22 = \frac{11}{5}k$   
 $11k = 110$   
 $k = 10$

(b) Now  $x^3 + 10x^2 - 2bx + 20 = 0$   
 $\Rightarrow x^3 - 2bx + 20 = 0$  By basic division or manipulation  
 $\Rightarrow x^2(x-1) + 2x(x-1) - 2x(x-1) = 0$   
 $\Rightarrow (x-1)(x^2 + 2x - 20) = 0$   
 $\Rightarrow (x-1)(x-4)(x+5) = 0$   
 $x = 1, -5 //$

**Question 5** (\*\*\*)+

The following information is given for a polynomial  $f(x)$ .

- When  $f(x)$  is divided by  $(x-2)$  the remainder is 5.
- When  $f(x)$  is divided by  $(x+2)$  the remainder is -11.
- When  $f(x)$  is divided by  $(x+2)(x-2)$  the remainder is  $ax+b$ , and the quotient is  $g(x)$ , where  $a$  and  $b$  are constants.

- a) Determine the value of  $a$  and the value of  $b$ .

It is further given that

$$f(x) = 3x^4 + px + q,$$

where  $p$  and  $q$  are constants.

- b) Find a simplified expression for  $g(x)$ .

$\boxed{\phantom{0}}$	$\boxed{a = -4}$	$\boxed{b = 3}$	$\boxed{g(x) = 3(x^2 + 4)}$
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a) FROM THE "THIRD" (7TH ON THE INSTRUCTION SHEET)

$$f(x) \equiv (x-2)(x+2)g(x) + ax+b$$

NOW  $f(2) = 5$  AND  $f(-2) = -11$

$$\begin{aligned} 5 &\equiv 0 + 2a + b & -11 &\equiv 0 - 2a + b \\ 5 &\equiv 2a + b & -11 &\equiv -2a + b \\ 2a + b &\equiv 5 & -2a + b &\equiv -11 \end{aligned}$$

ADDING & SUBTRACTING VITALS

$$\begin{aligned} b &\equiv -3 & a &\equiv 4 \\ \hline \end{aligned}$$

b)  $f(2) = 32^4 + p2 + q$

$$\begin{aligned} f(2) &\equiv 5 & f(-2) &\equiv -11 \\ 32^4 + 2p + q &\equiv 5 & 3(-2)^4 - 2p + q &\equiv -11 \\ 2p + q &\equiv -43 & -2p + q &\equiv -59 \end{aligned}$$

FINDING:

$$\begin{aligned} 2q &\equiv -102 & q &\equiv -51 \\ q &\equiv -51 & \hline \end{aligned}$$

$$\begin{aligned} q &\equiv -51 & 2p + q &\equiv -43 \\ 2p - 51 &\equiv 43 & 2p &\equiv 8 \\ 2p &\equiv 8 & p &\equiv 4 \end{aligned}$$

Therefore we have

$$\begin{aligned} f(x) &\equiv (x-2)(x+2)g(x) + ax+b \\ 3x^4 + px + q &\equiv (x^2 - 4)g(x) + bx - 3 \\ 3x^4 + bx - 3 &\equiv (x^2 - 4)g(x) + bx - 3 \\ 3x^4 - 48 &\equiv (x^2 - 4)g(x) \\ 3(x^2 - 16)(x^2 + 4) &\equiv (x^2 - 4)g(x) \\ 3(x^2 - 4)(x^2 + 4) &\equiv (x^2 - 4)g(x) \end{aligned}$$

BY COMPARISON  $g(x) = 3(x^2 + 4)$

THE ABOVE CROSSES HAVE TO BE DONE JUST AS GOOD BY LONG DIVISION

**Question 9 (\*\*\*\*)**

$$f(x) = x^3 + (a+2)x^2 - 2x + b,$$

where  $a$  and  $b$  are non zero constants.

It is given that  $(x-2)$  and  $(x+a)$  are factors of  $f(x)$ ,  $a > 0$ .

- By forming two equations show that  $a = 3$  and find the value of  $b$ .
- Solve the equation  $f(x) = 0$ .

$$b = -24, \quad x = -4, -3, 2$$

**(a)**  $f(x) = x^3 + (a+2)x^2 - 2x + b$

$\bullet f(2) = 0$   
 $8 + 4(a+2) - 4 + b = 0$   
 $8 + 4a + 8 - 4 + b = 0$   
 $\boxed{4a + b = -12}$

$\downarrow$   
 $b = -a - 12$

**(b)**  $f(-a) = 0$   
 $(-a)^3 + (a+2)(-a)^2 - 2(-a) + b = 0$   
 $-a^3 + (a+2)a^2 + 2a + b = 0$   
 $-a^3 + a^3 + 2a^2 + 2a + b = 0$   
 $\boxed{2a^2 + 2a + b = 0}$

$\downarrow$   
 $b = -2a^2 - 2a$

$-12 - 4a = -2a^2 - 2a$   
 $2a^2 - 2a - 12 = 0$   
 $a^2 - a - 6 = 0$   
 $(a-3)(a+2) = 0$   
 $a = -3 \quad \text{or} \quad a = 3$   
 $\therefore b = -12 - 4a$   
 $b = -2a^2 - 2a$

**(b)** Hence  $f(x) = x^3 + 5x^2 - 2x - 24$   
Hence  $x^3 + 5x^2 - 2x - 24 = 0$   
 $\Rightarrow (x+4)(x+3)(x-2) = 0$   
 $(x+4)(x-2)(x+3) = 0$   
 $4\text{Hence } x = \frac{-2}{-4} = -3$

**Question 10 (\*\*\*\*)**

When the polynomial  $f(x)$  is divided by  $(x-2)$  the remainder is 7.

When  $f(x)$  is divided by  $(x-3)^2$  the remainder is  $(4x+17)$ .

Find the remainder when  $f(x)$  is divided by  $(x-2)(x-3)$ .

$$22x - 37$$

$$\begin{aligned} f(2) &= (x-2)g(x) + 7 & f(2) &= 7 \\ f(3) &= (x-3)^2 h(x) + 4x + 17 & f(3) &= 29 \\ \text{Now } & \\ f(x) &= (x-2)(x-3)k(x) + Ax + B \end{aligned}$$
$$\begin{aligned} f(2) = 7 &\Rightarrow 7 = 2A + B \\ f(3) = 29 &\Rightarrow 29 = 3A + B \end{aligned} \quad \begin{cases} A = 22, \\ B = -37 \end{cases}$$

$\therefore \text{REMAINDER is } 22x - 37$  ✓

**Question 11** (\*\*\*\*)

A polynomial  $p(x)$  is given by

$$p(x) = 4x^3 - 2x^2 + x + 5.$$

- a) Find the remainder and the quotient when  $p(x)$  is divided by  $x^2 + 2x - 5$ .

A different polynomial  $q(x)$  is defined as

$$q(x) = 4x^3 - 2x^2 + ax + b.$$

- b) Find the value of each of the constants  $a$  and  $b$  so that when  $q(x)$  is divided by  $x^2 + 2x - 5$  there is no remainder.

$$R = 41x - 45, \quad Q = 4x - 10, \quad a = -40, \quad b = 50$$

(a)  $q(x) = (x^2 + 2x - 5) \frac{4x - 10}{-40x^2 - 20x + 20}$

Quotient:  $4x - 10$   
Divisor:  $x^2 + 2x - 5$

(b) Now  $q(x) = 4x^3 - 2x^2 + ax + b \equiv (x^2 + 2x - 5)(ax + c)$

$\begin{cases} q(0) = b = -5c \\ q(1) = 4 - 2 + a + b = -2(4 + c) \\ q(2) = 32 - 8 + 2a + b = 24(4 + c) \end{cases} \Rightarrow \begin{cases} b = -5c \\ 2a + b = -8 - 2c \\ 24a + 24b = 96 + 24c \end{cases} \Rightarrow \begin{cases} a = 4c \\ b = -5c \\ 24a + 24b = 96 + 24c \end{cases}$

$\begin{cases} 2a + b = -8 - 2c \\ 24a + 24b = 96 + 24c \end{cases} \Rightarrow \begin{cases} a - 4c = -10 \\ a - 4c = 4c \end{cases} \Rightarrow \begin{cases} a = -10 \\ a = 4c \end{cases} \Rightarrow a = -10$

∴  $a = -10, \quad b = 50$

**Question 12** (\*\*\*)<sup>(+)</sup>

$$f(x) \equiv x^4 - 9x^3 + 30x^2 - 44x.$$

The polynomial  $f(x)$  satisfies the relationship

$$f(x) \equiv (x-3)(x-A)^3 + B,$$

where  $A$  and  $B$  are constants.

- a) Find the value of  $A$  and the value of  $B$ .

The polynomial  $f(x)$  also satisfies the relationship

$$f(x) \equiv (x+3)^2 g(x) + Px + Q, \text{ where } P \text{ and } Q \text{ are constants.}$$

- b) Find the value of each of the constants  $P$  and  $Q$ .

$$A = 2, B = -24, P = -575, Q = -999$$

**(a)**

$$\begin{aligned} f(x) &\equiv x^4 - 9x^3 + 30x^2 - 44x \equiv (x-3)(x-A)^3 + B \\ \bullet f(3) &= 81 - 243 + 270 - 132 = B \quad \therefore B = -24 \\ \bullet f(3) &= 25x - 575 + 480 - 114 = (4-3)(4-A)^3 + B \\ -16 &= (4-A)^3 - 24 \\ B &= (4-A)^3 \\ 2 &= 4-A \\ A &= 2. \end{aligned}$$

**(b)**

$$\begin{aligned} f(x) &\equiv x^4 - 9x^3 + 30x^2 - 44x \equiv (x+3)^2 g(x) + Px + Q \\ f(x) &\equiv x^4 - 2x^3 + 6x^2 - 44 \equiv 2(x+3)g(x) + (x+3)^2 g(x) + P \\ \text{Now} \\ f(-3) &= 81 + 243 + 270 + 132 = 0 - 3P + Q \quad \left. \begin{array}{l} \end{array} \right\} \Rightarrow P = -575 \\ f(-3) &= -108 - 343 - 180 - 44 = 0 + P \quad \left. \begin{array}{l} \end{array} \right\} \Rightarrow Q = -999 \end{aligned}$$

**Question 13** (\*\*\*)<sup>+</sup>

$$f(x) \equiv 9x^4 + 24x^3 - 32x - 16.$$

The polynomial  $f(x)$  has linear factors

$$(x+2) \quad \text{and} \quad (3x+2)$$

- a) Show that the roots of the equation  $f(x)=0$  are

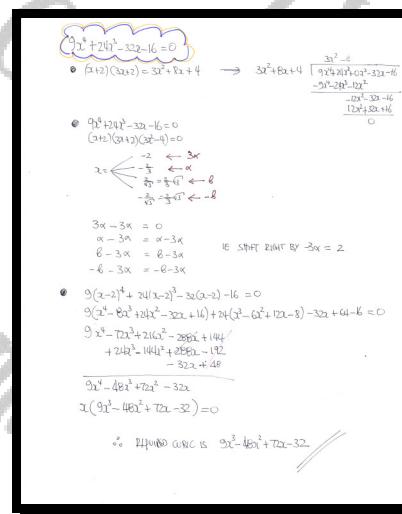
$$\alpha, 3\alpha, \beta \text{ and } -\beta,$$

where  $\alpha$  and  $\beta$  must be stated in exact form if appropriate.

- b) Hence determine a cubic equation with integer coefficients with roots

$$-2\alpha, \beta - 3\alpha \text{ and } -\beta - 3\alpha.$$

$$\boxed{\alpha = -\frac{2}{3}}, \boxed{\beta = -\frac{2}{3}\sqrt{3}}, \boxed{9x^3 - 48x^2 + 72x - 32 = 0}$$



Given  $9x^4 + 24x^3 - 32x - 16 = 0$ , we divide by 3 to get  $3x^4 + 8x^3 - 32x - 16 = 0$ . Then we factor by grouping:

$$3x^4 + 8x^3 - 32x - 16 = (3x^4 + 8x^3 + 4x^2) - (4x^2 + 32x + 16) = x^2(3x^2 + 8x + 4) - 4(x^2 + 8x + 4) = (x^2 + 4)(3x^2 + 8x + 4) = 0$$

So,  $x^2 + 4 = 0$  or  $3x^2 + 8x + 4 = 0$ .

Solving  $x^2 + 4 = 0$  gives  $x = \pm 2i$ .

Solving  $3x^2 + 8x + 4 = 0$  using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{64 - 48}}{6} = \frac{-8 \pm 4\sqrt{10}}{6} = \frac{-4 \pm 2\sqrt{10}}{3}$$

Let's choose  $\alpha = -\frac{2}{3}\sqrt{10}$  and  $\beta = \frac{2}{3}\sqrt{10}$ .

Now, we find  $\alpha - 3\alpha, \beta - 3\alpha, -\beta - 3\alpha$ :

- $\alpha - 3\alpha = -2\alpha = -2(-\frac{2}{3}\sqrt{10}) = \frac{4}{3}\sqrt{10}$
- $\beta - 3\alpha = -2\alpha = -2(-\frac{2}{3}\sqrt{10}) = \frac{4}{3}\sqrt{10}$
- $-\beta - 3\alpha = -2\alpha = -2(-\frac{2}{3}\sqrt{10}) = \frac{4}{3}\sqrt{10}$

So, the roots are  $-2\alpha, \beta - 3\alpha, -\beta - 3\alpha$ .

ALTERNATIVE ONCE THE ROOTS OF THE QUADRATIC ARE DETERMINED

$\alpha = -\frac{2}{3}\sqrt{10}, \beta = \frac{2}{3}\sqrt{10}$

$(x+2)(x - (\alpha - 3\alpha))(x - (-\beta - 3\alpha)) = 0$

$(x+2)(x - (\alpha - 3\alpha))(x + \beta + 3\alpha) = 0$

$x \mapsto (x-2)$

$x, (x-6+\alpha)(x+4+\beta) = 0$

$x, [x^2 + (6\alpha + 4\beta + 16)x + (4\alpha\beta + 24)] = 0$

$x, [x^2 + 2x\alpha + x^2 + \beta^2] = 0$

$x, [x^2 - \frac{4}{3}\alpha x + \frac{4}{9}\alpha^2] = 0$

$x, [x^2 - \frac{4}{3}\alpha x - \frac{4}{9}\alpha^2] = 0$

$x, [9x^2 - 12x - 8] = 0$

SWITCH BACK

$x \mapsto (x+2) = (x - \frac{2}{3}\sqrt{10})$

$(x - \frac{2}{3}\sqrt{10})[9(x - \frac{2}{3}\sqrt{10})^2 - 12 - 8] = 0$

$(x - \frac{2}{3}\sqrt{10})(9x^2 - 12x + 16 - 12 - 8) = 0$

$(3x - 4)(9x^2 - 24x + 16) = 0$

$(3x - 4)(3x^2 - 12x + 8) = 0$

$9x^3 - 36x^2 + 24x - 32 = 0$

$9x^3 - 48x^2 + 72x - 32 = 0$

AS REQUIRED

**Question 14** (\*\*\*\*+)

A polynomial  $f(x)$  is defined by

$$f(x) \equiv 2x^6 + ax^5 + bx^4 + 2x^2,$$

where  $a$  and  $b$  are constants.

When  $f(x)$  is divided by  $(x-2)(2x+1)$  the remainder is  $(3x+2)$ .

- a) Determine the value of  $a$  and the value of  $b$ .

When  $f(x)$  is divided by  $(x-2)^2$  the quotient is  $h(x)$  and the remainder is  $(Ax+B)$ , where  $A$  and  $B$  are constants..

- b) Find ...

- i. ... the value of  $A$  and the value of  $B$ .

- ii. ... an expression for  $h(x)$ .

$a = -3$	$b = -2$	$A = 88$	$B = -168$	$h(x) = 2x^4 + 5x^3 + 10x^2 + 20x + 42$
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(a)  $f(x) = 2x^6 + ax^5 + bx^4 + 2x^2 \equiv (x-2)(2x+1)h(x) + 3x+2$

- $f(2) = 128 + 32a + 16b + 8 = 35 \Rightarrow \frac{32a + 16b}{2a+b} = -128 \Rightarrow 2a+b = -8$
- $f'(2) = \frac{1}{2} - \frac{1}{32}a + \frac{1}{16}b + \frac{1}{2} = \frac{1}{2} \Rightarrow 1-a+2b+4 = 3 \Rightarrow -a+2b = -1 \Rightarrow a = 2b+1$

$\therefore \begin{cases} 2(2b+1)+b = -8 \\ 2b = -10 \\ b = -5 \end{cases} \quad \therefore a = -3$

(b) Now  $f(x) = 2x^6 - 3x^5 - 2x^4 + 2x^2 = (x-2)^2 h(x) + \frac{Ax+B}{2(x-2)}$

- $f(2) = 128 - 96 - 32 + 8 = 2A+B \Rightarrow A+B = 8$
- $f'(2) = 12x^5 - 15x^4 - 8x^3 + 4x = 2(x-2)h(x) + (2x-3)h'(x) + A$
- $f'(2) = 384 - 240 - 64 + 8 = A \Rightarrow A = 88$

$\Rightarrow B = -168$

$\therefore 2x^6 - 3x^5 - 2x^4 + 2x^2 = (x^2 - 4x + 4)h(x) + 88x - 168$

$2x^6 - 3x^5 - 2x^4 + 2x^2 - 88x + 168 = h(x)(2x^2 - 4x + 4)$

$2^2 - 4x + 4 \quad 2x^2 - 4x + 4 \quad 2x^2 - 4x + 4$

$\frac{2^2 - 4x + 4}{2^2 - 4x + 4} \cdot \frac{2x^2 - 4x + 4}{2x^2 - 4x + 4} \cdot \frac{2x^2 - 4x + 4}{2x^2 - 4x + 4}$

$\frac{5x^6 - 10x^5 + 10x^4 - 8x^3 + 16x + 168}{20x^6 - 40x^5 + 40x^4 - 20x^3 + 16x + 168}$

$\frac{10x^6 - 20x^5 + 20x^4 - 16x^3 + 16x + 168}{20x^6 - 40x^5 + 40x^4 - 20x^3 + 16x + 168}$

$\frac{20x^6 - 36x^5 + 36x^4 - 20x^3 + 16x + 168}{20x^6 - 40x^5 + 40x^4 - 20x^3 + 16x + 168}$

$\frac{4x^6 - 6x^5 + 6x^4 - 4x^3 + 8x + 88}{20x^6 - 40x^5 + 40x^4 - 20x^3 + 16x + 168}$

$\frac{-4x^6 + 8x^5 - 8x^4 + 8x^3 - 8x - 88}{20x^6 - 40x^5 + 40x^4 - 20x^3 + 16x + 168}$

**Question 15 (\*\*\*\*\*)**

A polynomial in  $x$  satisfies the relationship

$$f(x) = (x^2 - 4)g(x) + Ax + B,$$

where  $A$  and  $B$  are constants.

- a) Find the value of  $A$  and the value of  $B$ , given that  $f(2) = 5$  and  $f(-2) = -7$ .

It is now given that the polynomial in  $x$  also satisfies the relationship

$$f(x) = (x-2)^2 h(x) + Cx + D.$$

- b) Find the value of each of the constants  $C$  and  $D$ , given that  $f'(2) = 31$ .

- c) Given further that  $g(x) = 3x+1$ , find  $h(x)$ .

$$\boxed{A=3}, \boxed{B=-1}, \boxed{C=31}, \boxed{D=-57}, \boxed{h(x)=3x+13}$$

**a)**  $f(x) = (x^2 - 4)g(x) + Ax + B$

$$\begin{aligned} f(2) &= (2^2 - 4)g(2) + 2A + B \\ 5 &= 2A + B \end{aligned}$$

$\bullet f(-2) = ((-2)^2 - 4)g(-2) - 2A + B$

$$\begin{aligned} -7 &= -2A + B \\ \hline 4B &= -2 \\ B &= -1 \end{aligned}$$

$\therefore A = 3, B = -1$

$\bullet f(x) = (x^2 - 4)g(x) + Ax + B$

$$\begin{aligned} f(x) &= (x-2)^2 h(x) + Cx + D \\ f(x) &= 2(x-2) h(x) + (x-2)^2 h'(x) + C \end{aligned}$$

$\bullet f'(x) = 5$

$$\begin{aligned} 5 &= (x-2)^2 h(x) + 2C + D \\ 2C + D &= 5 \end{aligned}$$

$\bullet f'(2) = 31$

$$\begin{aligned} 31 &= 2(2-2) h(2) + (2-2)^2 h'(2) + C \\ C &= C \end{aligned}$$

$\therefore C = 31, D = -57$

**c)**  $f(x) = (x^2 - 4)(3x + 1) + 3x - 1$

$$\begin{aligned} \Rightarrow f(x) &= 3x^3 + x^2 - 12x - 4 + 3x - 1 \\ \Rightarrow f(x) &= 3x^3 + x^2 - 9x - 5 \\ \Rightarrow (x-2)^2 h(x) + 3x - 5 &\equiv 3x^3 + x^2 - 9x - 5 \\ \Rightarrow (x-2)^2 h(x) &\equiv 3x^3 + x^2 - 4x^2 + 52 \\ \Rightarrow h(x) &= \frac{3x^3 + x^2 - 4x^2 + 52}{(x-2)^2} \\ \Rightarrow h(x) &= \frac{3x^3 + x^2 - 4x^2 + 52}{x^2 - 4x + 4} \end{aligned}$$

By LONG DIVISION OR  
SUBSTITUTION

$$\begin{array}{r} 3x^2 + 13 \\ \hline x^2 - 4x + 4 \quad | \quad 3x^3 + x^2 - 4x^2 + 52 \\ -3x^3 + 12x^2 - 12x \\ \hline 13x^2 + 52 \\ -13x^2 + 52 \\ \hline 0 \end{array}$$

$\therefore h(x) = 3x^2 + 13$

**Question 16 (\*\*\*\*\*)**

A polynomial in  $x$  is given by

$$f(x) = x^8 + kx^5 - 27x^2 - 13, \text{ where } k \text{ is a constant.}$$

The polynomial also satisfies the relationship

$$f(x) = (x-1)^2 g(x) + Ax + B,$$

where  $A$  and  $B$  are constants.

Find the value of  $A$  and the value of  $B$ , given that  $f(2) = 7$

$$\boxed{A = -26}, \boxed{B = -9}$$

$$\begin{aligned}
 f(x) &= x^8 + kx^5 - 27x^2 - 13 \\
 \bullet \quad f(2) = 7 &\Rightarrow 7 = (2)^8 + k(2)^5 - 27(2)^2 - 13 \\
 &\Rightarrow 7 = 256 + 32k - 108 - 13 \\
 &\Rightarrow 32k = 26 \\
 &\Rightarrow k = \frac{13}{16} \\
 \text{So } f(x) &= x^8 + \frac{13}{16}x^5 - 27x^2 - 13 & f'(x) &= 8x^7 + 20x^4 - 54x^3 \\
 f(1) &= 1 + 1 + 27 - 13 = -35 & f'(1) &= 8 + 20 - 54 \\
 \boxed{f(1) = -35} & & \boxed{f'(1) = -26} & \\
 \text{Now } f(x) &= (x-1)^2 g(x) + Ax + B & f(x) &= 2(x-1)g(x) + (x-1)^2 g'(x) + A \\
 f(1) &= 2(1-1)g(1) + 0 + A & f'(1) &= 2(g(1) + (1-1)g'(1)) + 2(1-1)g'(1) + A \\
 \text{THU } -35 &= A + B & \text{SINCE } f(1) = -35 & \text{SINCE } f'(1) = -26 \\
 -26 &= A & & \\
 \therefore B &= -35 - A & \therefore A &= -26 \\
 B &= -35 + 26 & B &= -9 \\
 \end{aligned}$$

**Question 17 (\*\*\*\*\*)**

$$f(x) \equiv Ax^5 + Bx^4 + 8x^2,$$

where  $A$  and  $B$  are non zero constants.

The polynomial  $f(x)$  satisfies the relationship

$$f(x) \equiv (2x-1)(x-2)g(x) + 169x - 82.$$

- Find the value of  $A$  and the value of  $B$ .
- Determine the polynomial  $g(x)$ .

The polynomial  $f(x)$  also satisfies the relationship

$$f(x) \equiv (x+2)^2 h(x) + Px + Q,$$

where  $P$  and  $Q$  are constants.

- Find the value of each of the constants  $P$  and  $Q$ .

$$\boxed{\quad}, \boxed{A=4}, \boxed{B=6}, \boxed{g(x) \equiv 2x^3 + 8x^2 + 18x + 41}, \boxed{P=96}, \boxed{Q=192}$$

a) AS THE TWO EXPRESSIONS OF  $f(x)$  ARE IDENTICAL, WE MAY TRY DIFFERENT SIMPLIFIED VALUES OF  $x$  TO SUBTRACT

$$f(6) \equiv Ax^5 + Bx^4 + 8x^2 \equiv (2x-1)(x-2)g(x) + 169x - 82$$

$$(6) \equiv 32A + 6B + 32 = 0 + 338 - 82$$

$$(6) \equiv \frac{1}{2}A + \frac{1}{2}B + 2 = 0 + \frac{151}{2} - 82$$

TRY THE EQUATIONS & SOLVE

$$32A + 6B = 224 \quad \left\{ \begin{array}{l} 2A + B = 14 \\ \frac{1}{2}A + \frac{1}{2}B + 2 = \frac{151}{2} - 82 \end{array} \right. \Rightarrow \begin{array}{l} 2A + B = 14 \\ A + 2B = 16 \end{array}$$

$$(B = 14 - 2A)$$

$$\Rightarrow A + 2(14 - 2A) = 16$$

$$\Rightarrow A + 28 - 4A = 16$$

$$\Rightarrow -3A = -12$$

$$\Rightarrow A = 4 \quad \text{and} \quad B = 6$$

b) USING THE ANSWERS FROM PART (a)

$$7x^5 + 6x^4 + 8x^2 = (2x-1)(x-2)g(x) + 169x - 82$$

$$7x^5 + 6x^4 + 8x^2 - 169x + 82 \equiv (2x-1)(x-2)g(x)$$

$$\Rightarrow 4x^5 + 6x^4 + 8x^2 - 169x + 82 \equiv (2x^5 - 5x^4 + 8x^2)g(x)$$

BY LONG DIVISION

$$\begin{array}{r} 2x^2 + 6x^3 + 16x + 82 \\ \overline{4x^5 + 6x^4 + 8x^2 + 169x - 82} \\ - (4x^5 - 4x^4 + 8x^2) \\ \hline 10x^4 + 16x^2 + 169x - 82 \\ - (10x^4 + 40x^3 + 16x^2) \\ \hline 40x^3 + 8x^2 + 169x - 82 \\ - (40x^3 + 80x^2 + 32x) \\ \hline 82x^2 + 169x - 82 \\ - (82x^2 + 20x^3 + 16x^2) \\ \hline - 20x^3 + 169x - 82 \\ - (-20x^3 - 40x^2 - 32x) \\ \hline 0 \end{array}$$

$$\therefore f(x) = 2x^5 + 6x^4 + 8x^2 + 169x - 82$$

c) PROCEED AS FOLLOWING

$$\begin{cases} f(2) = 16x^5 + 6x^4 + 8x^2 = (2x^5 - 5x^4 + 8x^2)g(x) + P \\ f(2) = 2x^5 + 2x^4 + 16x^2 \equiv 2(2x^2)h(x) + (2x^2)^2 k(x) + P \end{cases}$$

$$\begin{cases} f(2) = -128 + 96 + 32 = 0 - 2P + P \\ f(2) = 320 - 192 - 32 = P \end{cases}$$

$$\therefore P = 96$$

$$\begin{cases} Q - 2P = 0 \\ Q = 2P \\ Q = 192 \end{cases}$$

Question 18 (\*\*\*\*\*)

$$ax^3 + ax^2 + ax + b = 0,$$

where  $a$  and  $b$  are non zero real constants.

Given that  $x = b$  is a root of the above cubic equation, determine the range of possible values of  $a$ .

$$\boxed{-\frac{4}{3} \leq a \leq 0}$$

$$\begin{aligned} & ax^3 + ax^2 + ax + b = 0 \\ & 2 = b + 4, \text{ so } 2a = 0 \\ & ab^2 + ab + a + b = 0 \quad b \neq 0 \\ & ab^2 + ab + a + 1 = 0 \\ & ab^2 + ab + (a+1) = 0 \\ & b^2 \text{ and } a \\ & b^2 - 4ac > 0 \end{aligned} \quad \left. \begin{aligned} & a^2 - 4xa + (a+1) \geq 0 \\ & a^2 - 4a \geq 0 \\ & -3a^2 - 4a \geq 0 \\ & 3a + 4 \leq 0 \\ & a(3a+4) \leq 0 \\ & \therefore -\frac{4}{3} \leq a \leq 0 \end{aligned} \right\}$$

**Question 19** (\*\*\*\*\*)

$$f(x) \equiv x^5 + 3x^4 - 40x^3 - 47x^2.$$

The polynomial  $f(x)$  satisfies the relationship

$$f(x) \equiv (x-2)(x+A)(x+B)(x^2+3x-1) - 249x + 70,$$

where  $A$  and  $B$  are integer constants.

- a) Find the value of  $A$  and the value of  $B$ .

The polynomial  $f(x)$  also satisfies the relationship

$$f(x) \equiv (x-2)^2 h(x) + Px + Q,$$

where  $P$  and  $Q$  are constants.

- b) Find the value of each of the constants  $P$  and  $Q$ .

$$\boxed{A=7}, \boxed{B=-5}, \boxed{P=-492}, \boxed{Q=556}$$

**a)**

$$f(x) \equiv x^5 + 3x^4 - 40x^3 - 47x^2 \equiv (x-2)(x+4)(x+B)(x^2+3x-1) - 249x + 70$$

• EXPAND THE QUADRATIC AS FOLLOWS

$$x^5 + 3x^4 - 40x^3 - 47x^2 - 249x + 70 \equiv (x-2)(x+4)(x+B)(x^2+3x-1) - 249x + 70$$

- IF  $x=1 \Rightarrow 1+3-40-47+249-70 = -141 \neq 0$
- IF  $x=2 \Rightarrow -1+3-40-47+249-70 = -3(A-1)(B-1) \neq 0$

• ADD THE EQUATIONS

$$\begin{cases} 235-157 = -3(AB+4+B+1) \\ 43-3C = 9(A-1-B+1) \end{cases} \Rightarrow \begin{cases} 98 = -3(AB+4+B+1) \\ 2C = 9(A-1-B+1) \end{cases}$$

$$\Rightarrow \begin{cases} AB+A+B+1 = -32 \\ AB-A-B+1 = -36 \end{cases} \Rightarrow \begin{cases} AB+A+B = -32 \\ AB-A-B = -37 \end{cases}$$

• ADDING THE EQUATIONS

$$\Rightarrow 2AB = -70$$

• DIVIDE ONE OF THE TWO EQUATIONS

$$\Rightarrow AB + A + B = -33$$

$$\Rightarrow -3S + A + B = -33$$

$$\boxed{A + B = 21}$$

• BY INSPECTION  $A=7, B=-5$  (OR THE OTHER WAY)

**b)**

$$f(x) \equiv (x-2)^2 h(x) + Px + Q$$

• DIFFERENTIATE W.R.T.  $x$

$$\Rightarrow x^4 + 3x^3 - 40x^2 - 47x^2 = (x-2)^2 h(x) + Px + Q$$

$$\Rightarrow 5x^3 + 12x^2 - 120x^2 - 47x = 2(x-2)h(x) + (x-2)^2 h'(x) + Px + Q$$

• SUBSTITUTE  $x=2$  INTO THE LAST TWO EXPRESSIONS

$$\begin{cases} 2^5 + 3(2^4) - 40(2^3) - 47(2^2) = 2^5 + Q \\ 5(2^3) + 12(2^2) - 120(2^2) - 47(2) = P \end{cases} \Rightarrow$$

$$\begin{cases} 32 + 48 - 320 - 188 = 2^5 + Q \\ 80 + 96 - 480 - 188 = P \end{cases} \Rightarrow$$

$$\begin{cases} 80 - 508 = 2^5 + Q \\ 176 - 468 = P \end{cases} \Rightarrow$$

$$\begin{cases} P = -12 - 480 \\ Q = -492 \end{cases}$$

$$\begin{aligned} 80 - 508 &= (-492) \times 2 + Q \\ -428 &= -984 + Q \\ Q &= 984 - 428 \\ Q &= 556 \end{aligned}$$

**Question 20** (\*\*\*\*\*)

$$f(x, y) \equiv 4x^4 - 4x^3y - 7x^2y^2 + 4xy^3 + 3y^4.$$

Express  $f(x, y)$  as a product of 4 linear factors.

$$\boxed{\quad}, f(x, y) \equiv (x+y)(x-y)(2x+y)(2x-3y)$$

$\fbox{ } f(2,y) = 4x^4 - 4x^3y - 7x^2y^2 + 4xy^3 + 3y^4$

• TREAT THE POLYNOMIAL AS A POLYNOMIAL IN  $x$ , WHERE  $y$  IS A CONSTANT AND LOOK FOR FACTORS

$$f(2,y) = 4y^4 - 4y^3 - 7y^2 + 4y^3 + 3y^4$$

$\therefore x=2$  PRODUCES ZERO  
 $(x-2)$  IS A FACTOR

$$f(2,y) = 4(y^4 - 4y^3 - 7y^2 + 4y^3 + 3y^4)$$

$$f(2,y) = 4y^4 + 4y^3 - 7y^4 - 4y^4 + 3y^4 = 0$$

$\therefore x=-y$  PRODUCES ZERO  
 $(x+y)$  IS A FACTOR

• WE DEDUCE THAT  
 $(x+y)(x-y) = x^2 - y^2$  IS A FACTOR OF  $f(x,y)$

• BY INSPECTION (OR LONG DIVISION)

$$\begin{aligned} 4x^4 - 4x^3y - 7x^2y^2 + 4xy^3 + 3y^4 &\equiv (x^2 - y^2)(4x^2 + 4xy - 3y^2) \\ &\equiv 4x^4 + 4x^3y - 4x^3y - 4x^2y^2 + 3y^4 \\ &\equiv 4x^4 + 4x^3y - 4x^2y^2 - 4xy^3 + 3y^4 \\ &\equiv 4x^4 + 4x^3y - 4x^2y^2 - 4xy^3 + 3y^4 \\ &\therefore \boxed{A = -4} \end{aligned}$$

• THIS WE REDUCE  $f(x,y)$  INTO 2 LINEAR FACTORS & A QUADRATIC FACTOR

$$f(x,y) = (x+y)(x-y)(4x^2 - 4xy - 3y^2)$$

$$f(x,y) = (x+y)(x-y)(2x+3y)(2x-y)$$

→ QUADRATIC FORMULA IN  $x$  FOR  $4x^2 - 4xy - 3y^2 = 0$

$$x = \frac{-(-4y) \pm \sqrt{(-4y)^2 - 4 \times 4 \times (-3y^2)}}{2 \times 4}$$

$$x = \frac{4y \pm \sqrt{16y^2}}{8}$$

$$x = \frac{4y \pm 4y}{8}$$

$$x = \pm \frac{3y}{2}$$

$$\therefore x = \frac{3}{2}y \text{ or } x = -\frac{1}{2}y$$

$$\therefore x - \frac{3}{2}y = 0 \text{ or } x + \frac{1}{2}y = 0$$

$$2x - 3y = 0 \text{ or } 2x + y = 0$$

**Question 21** (\*\*\*\*\*)

Solve the cubic equation

$$4x^3 - 4(1+\sqrt{3})x^2 + (9+4\sqrt{3})x - 9 = 0, \quad x \in \mathbb{R}.$$

$$\boxed{\quad}, \quad x=1$$

$$4x^3 - 4(1+\sqrt{3})x^2 + (9+4\sqrt{3})x - 9 = 0$$

Firstly look for integer rational fractions by inspection, trying  $\pm 1, \pm 2, \pm 9$  to start with.

$$x=1 \Rightarrow 4 - 4(1+\sqrt{3}) + 9 + 4\sqrt{3} - 9 = 4 - 4\sqrt{3} + 9 + 4\sqrt{3} - 9 = 0$$

$\therefore (x-1)$  is a factor.

By long division

$$\begin{array}{r} 4x^2 - 4\sqrt{3}x + 9 \\ x-1 \overline{)4x^3 + (-4-4\sqrt{3})x^2 + (9+4\sqrt{3})x - 9} \\ \underline{-4x^3 + (-4-\sqrt{3})x^2} \\ -4\sqrt{3}x^2 + (9+4\sqrt{3})x - 9 \\ \underline{-4\sqrt{3}x^2 + (-4\sqrt{3})x} \\ 9x - 9 \\ \underline{-9x + 9} \\ 0 \end{array}$$

Now

$$(x-1)(4x^2 - 4\sqrt{3}x + 9) = 0$$

$\therefore x-1 = 0 \Rightarrow x=1$

$$4x^2 - 4\sqrt{3}x + 9 = 0$$

$\therefore b^2 - 4ac = (-4\sqrt{3})^2 - 4 \cdot 4 \cdot 9 = 48 - 144 < 0$

$\therefore$  ONLY REAL SOLUTION IS  $x=1$

**Question 22** (\*\*\*\*\*)

$$f(x) \equiv x^4 - 16x^3 + 68x^2 - 32x + 3, \quad x \in \mathbb{R}.$$

Factorize  $f(x)$  into a product of 4 linear factors.

$$\boxed{\quad}, \quad f(x) \equiv (x - 4 - \sqrt{15})(x - 4 + \sqrt{15})(x - 4 - \sqrt{13})(x - 4 + \sqrt{13})$$

$f(x) = x^4 - 16x^3 + 68x^2 - 32x + 3$

• FIRST look for "INTEGER LINEAR FACTORS"

- $f(1) = 1 - 16 + 68 - 32 + 3 \neq 0 \quad (> 0)$
- $f(-1) = -1 + 16 + 68 + 32 + 3 \neq 0 \quad (> 0)$
- $f(0) = 0! - 16 \cdot 0!^2 + 68 \cdot 0!^2 - 32 \cdot 0! + 3 \neq 0 \quad (> 0)$
- $f(-2) = 8! - 16 \cdot (-2)!^2 + 68 \cdot (-2)!^2 - 32 \cdot (-2)! + 3 \neq 0 \quad (> 0)$

∴ TRYING FOR "NICE FRACTIONS"

• WE TRY TO FACTORIZE IT BY SQUARING A SUITABLE EXPRESSION, FOLLOWED BY DIFFERENCE OF SQUARES

$$(A+B+C)^2 = A^2 + B^2 + C^2 + 2AB + 2BC + 2CA$$

TRYING WITH TRYING

$$(x^2 + ax + b)^2 = x^4 + 2ax^3 + b^2 + 2a^2x^2 + 2abx + 2b^2x^2$$

$$= x^4 + 2ax^3 + a^2x^2 + 2abx + b^2$$

SPLITTING COEFFICIENTS

$$x^4 - 16x^3 + 68x^2 - 32x + 3 \equiv x^4 + 2ax^3 + (a+2b)^2x^2 + 2abx + b^2$$

$\bullet 2a = -16$	$\bullet a+2b = 68$	$\bullet 2ab = -32$	$\bullet b^2 = 3$
$\boxed{a = -8}$	$\boxed{a+2b = 68}$	$\boxed{-16b = -32}$	$\boxed{b = \pm\sqrt{3}}$
$\boxed{b = 2}$	$\boxed{b = 2}$	$\boxed{b = -2}$	$\boxed{b = \pm\sqrt{3}}$

$\therefore (x^2 - 8x + 2)^2 \equiv x^4 - 16x^3 + 68x^2 - 32x + 4$

• WE CAN REARRANGE  $f(x)$  AS FOLLOWS

$$f(x) = x^4 - 16x^3 + 68x^2 - 32x + 3$$

$$f(x) = x^4 - 16x^3 + (68 - 4)x^2 + 4 - 1$$

$$f(x) = (x^2 - 8x + 2)^2 - 1^2$$

$$f(x) = (x^2 - 8x + 2 - 1)(x^2 - 8x + 2 + 1)$$

$$f(x) = (x^2 - 8x + 1)(x^2 - 8x + 3)$$

$$f(x) = [(x-4)^2 - 16 + 1][(x-4)^2 - 16 + 3]$$

$$f(x) = [(x-4)^2 - 15][[(x-4)^2 - 13]$$

$$f(x) = [(x-4)^2 - 4\sqrt{3}^2][[(x-4)^2 - 4\sqrt{3}^2]$$

$$f(x) = (x-4-\sqrt{15})(x-4+\sqrt{15})(x-4-\sqrt{13})(x-4+\sqrt{13})$$

**Question 23** (\*\*\*\*\*)

$$f(a,b,c) \equiv a^4(b-c) + b^4(c-a) + c^4(a-b).$$

Factorize  $f(a,b,c)$  into a product of 3 linear factors and 1 quadratic factor.

$$\boxed{\quad}, \quad f(a,b,c) \equiv -(a-b)(b-c)(c-a)(a^2 + b^2 + c^2 + ab + bc + ca)$$

① FIRSTLY WRITE THE EXPRESSION AS A POLYNOMIAL IN  $a, b$  &  $c$  AND LOOK FOR FACTORS

$$(a_1, b_1, c) = a^4(b-c) + b^4(c-a) + c^4(a-b)$$

LET  $a=b \Rightarrow f(a_1, b_1, c) = a^4(c-a) + a^4(c-a) + 0 = 0 \quad \therefore (a-b) \text{ IS A FACTOR}$   
 LET  $b=c \Rightarrow f(a_1, b_1, c) = 0 + b^4(b-a) + b^4(c-a) = 0 \quad \therefore (b-c) \text{ IS A FACTOR}$   
 LET  $c=a \Rightarrow f(a_1, b_1, c) = c^4(b-a) + 0 + c^4(c-b) = 0 \quad \therefore (c-a) \text{ IS A FACTOR}$

② AS THE POLYNOMIAL IS SYMMETRIC IN  $a, b$  &  $c$ , THE REMAINING QUADRATIC MUST ALSO BE SYMMETRIC

$$f(a_1, b_1, c) \equiv (a-b)(b-c)(c-a) \quad (\text{SYMMETRIC QUADRATIC IN } a, b \text{ & } c)$$

$$f(a_1, b_1, c) = (a-b)(b-c)(c-a) [k(a^2 + b^2 + c^2) + h(ab + bc + ca)]$$

③ COMBINING COEFFICIENTS TO FIND THE VALUE OF  $k$  &  $h$  — SAY SOURCE OF A

$$\text{LHS} = [a^4(b-c) + b^4(c-a) + c^4(a-b)]$$

$$\text{RHS} = [a^4(b-c)(a) + a^4(b-c)(c-a) - b^4(b-c)(c-a) - b(b-c)^2] [ka^2 + k(b^2 + c^2) + h(ab + bc + ca)]$$

$$= [(c-b)a^3 + (bc - c^2 + b^2 - bc)a + (b^2 - bc^2)] [ka^2 + h(b+ca)a + t(b+ca) + kbc]$$

$$= \boxed{(c-b)a^3 + (b^2 - bc^2)a + (b^2 - bc^2)a} \quad \boxed{ka^2 + (b+ca)a + k(b^2 + c^2) + kbc}$$

④ EQUALISING POWERS OF  $a^4$  (IN RHS)

$$k(c-b)a^4 \equiv (b-c)a^4 \Rightarrow [k=-1]$$

⑤ NEXT LOOKING AT POWERS OF  $a^3$  (IN RHS)

$$\Rightarrow 0a^3 \equiv [k(c-b)(bc) + k(b^2 - c^2)] a^3$$

$$\Rightarrow 0 \equiv k(c^2 - b^2) - b^2 + c^2$$

$$\Rightarrow 0 \equiv k(c^2 - b^2) + (c^2 - b^2)$$

$$\Rightarrow 0 \equiv (c^2 - b^2)(1 + k)$$

$$\Rightarrow \boxed{k = -1}$$

$$\therefore f(a_1, b_1, c) = (a-b)(b-c)(c-a) [- (a^2 + b^2 + c^2) - (ab + bc + ca)]$$

$$\therefore f(a_1, b_1, c) = -(a-b)(b-c)(c-a)(a^2 + b^2 + c^2 + ab + bc + ca)$$

**Question 24 (\*\*\*\*\*)**

A quartic curve  $C$  has the following equation.

$$y = x(x-4)(x+2)(x-6), \quad x \in \mathbb{R}.$$

By considering suitable transformations, show that  $C$  is even about the straight line with equation  $x = 2$ .

, proof

$y = x(x-4)(x+2)(x-6), \quad x \in \mathbb{R}$

- If shift about  $x=2$ , then translating "left" by 2 units, then reflecting in the  $y$  axis & finally translating "right" by 2 units will give the curve invariant.

- OR translating by 4 units to the "left" then reflecting in the  $y$  axis

- Hence if  $f(x) = f(4-x)$  then the curve is given about the line  $x=2$

$$\begin{aligned}f(4-x) &= (4-x)[(4-x)-4][(4-x)+2][(4-x)-6] \\&= (4-x)(-x)(6-x)(2-x) \\&= (-1)^4(x-4)(x-2)(x+2) \\&= x(x-4)(x+2)(x-6) = f(x)\end{aligned}$$

ALSO NOTE ON PARTIAL MULTIPLICATION AND  $x-2 \mapsto 2-x$

$$f(x) = (x^2-16)(x^2-4x+12) = [(x-2)^2-4][(x-2)^2+16]$$

**Question 25 (\*\*\*\*\*)**

A cubic curve with equation

$$y = x^3 - 3x^2 - 9x + 3, \quad x \in \mathbb{R},$$

is odd about some point  $P$ .

Find the coordinates of  $P$  and use transformation arguments to justify the assertion that the curve is odd about  $P$ .

,  $P(1, -8)$

Given the cubic curve  $y = x^3 - 3x^2 - 9x + 3$ , we find its first and second derivatives:

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

$$\frac{d^2y}{dx^2} = 6x - 6$$

By inspection, the curve has a point of inflection at  $x = 1$ :

$$\therefore y = 1 - 3(1)^2 - 9(1) + 3 = -8 \quad \therefore P(1, -8)$$

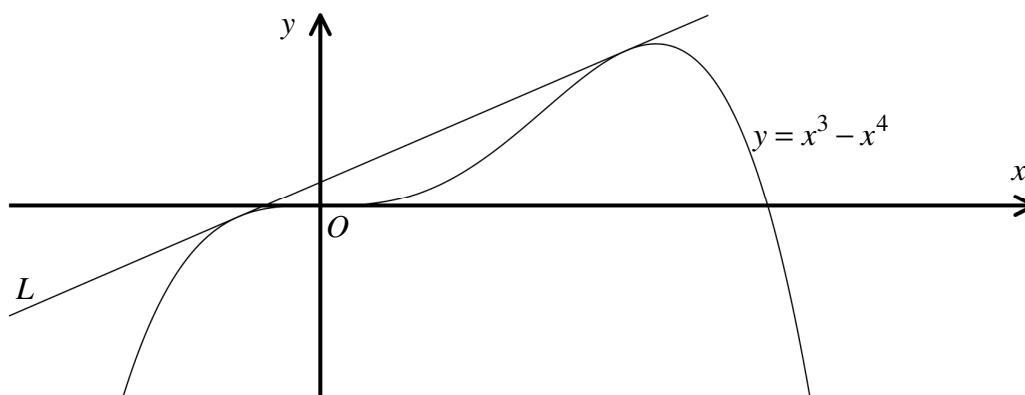
To justify the oddity about  $P$ , translate the curve to the origin. To investigate oddity about  $O$ :

- "UP BY 8"  $\Rightarrow y = (x^3 - 3x^2 - 9x + 3) + 8$   
 $\Rightarrow y = x^3 - 3x^2 - 9x + 11$
- "LEFT BY 1"  $\Rightarrow y = ((x+1)^3 - 3(x+1)^2 - 9(x+1)) + 11$   
 $\Rightarrow y = x^3 + 3x^2 + 3x + 1 - 3x^2 - 6x - 3 - 9x - 9 + 11$   
 $\Rightarrow y = x^3 - 12x$

Given that this is odd, as  $f(x) = x^3 - 12x$   
 $f(-x) = (-x)^3 - 12(-x)$   
 $= -x^3 + 12x$   
 $= -f(x)$

Consequently "our result" is odd about  $P(1, -8)$

## Question 26 (\*\*\*\*\*)



The figure above shows the curve  $C$  with equation

$$y = x^3 - x^4.$$

The straight line  $L$  is a tangent to the  $C$ , at two distinct points.

Determine an equation of  $L$ .

,  $y = \frac{1}{8}x + \frac{1}{64}$

LET THE TANGENT LINE EQUATION  $y = mx + c$  & SOLVE THIS EQUATION SIMULTANEOUSLY WITH  $y = x^3 - x^4$

$$\begin{aligned} \Rightarrow x^3 - x^4 &= mx + c \\ \Rightarrow -x^3 + x^4 - mx - c &= 0 \\ \Rightarrow x^4 - x^3 + mx + c &= 0 \end{aligned}$$

Now the equation has three & DOUBLE ROOTS FOR

$$\begin{aligned} \Rightarrow (x+1)^2(x+8)^2 &= x^4 - x^3 + mx + c \\ \Rightarrow (x^2+2x+1)(x^2+8x+64) &= x^4 - x^3 + mx + c \\ \Rightarrow \left\{ \begin{array}{l} x^4 + 2x^3 + 18x^2 + 64x \\ + 2x^2 + 4x^3 + 8x^2 + 64x \end{array} \right\} &= x^4 - x^3 + mx + c \\ A^2x^2 + 2Bx^3 + 2Ax^2 + 2Abx^2 &= x^4 - x^3 + mx + c \end{aligned}$$

COMPARING COEFFICIENTS

$$\begin{aligned} [x^4]: \quad 2A+2B &= -1 \rightarrow A+B = -\frac{1}{2} \\ [x^3]: \quad B^2+4AB+A^2 &= 0 \\ (A+B)^2+2AB &= 0 \\ (-\frac{1}{2})^2+2AB &= 0 \\ \frac{1}{4}+2AB &= 0 \\ 2AB &= -\frac{1}{4} \\ AB &= -\frac{1}{8} \end{aligned}$$

FINALLY LOOKING AT THE COEFFICIENT OF UNKNOWN  $m$  &  $c$

$$\begin{aligned} [x^2]: \quad m &= 2AB^2 + 2A^2B \\ m_1 &= 2AB(A+B) \\ m_2 &= 2(-\frac{1}{8})(-\frac{1}{2}) \\ m &= \frac{1}{8} \end{aligned}$$

$$\begin{aligned} [x^0]: \quad c &= A^2B^2 \\ c &= (AB)^2 \\ c &= (-\frac{1}{8})^2 \\ c &= \frac{1}{64} \\ \therefore y &= \frac{1}{8}x + \frac{1}{64} \end{aligned}$$

**Question 27** (\*\*\*\*\*)

Solve the cubic equation

$$16x^3 - 48x^2 + 60x - 31 = 0, \quad x \in \mathbb{R}.$$

You may assume that this cubic equation only has one real root.

$$\boxed{\quad}, \quad x = 1 + 2^{-\frac{2}{3}} - 2^{-\frac{4}{3}}$$

**1. SIMPLIFY THE CUBIC IN REDUCED FORM**

$$16x^3 - 48x^2 + 60x - 31 = 0$$

$$x^3 - 3x^2 + \frac{15}{4}x - \frac{31}{16} = 0$$

Let  $x = y - \frac{3}{8}$   $\Rightarrow y = \frac{-x}{3}$   $\Rightarrow \boxed{x = y + 1}$

**2. SUBSTITUTE BACK INTO THE CUBIC**

$$\Rightarrow 16(y+1)^3 - 48(y+1)^2 + 60(y+1) - 31 = 0$$

$$\Rightarrow 16(y^3 + 3y^2 + 3y + 1) - 48(y^2 + 2y + 1) + 60(y + 1) - 31 = 0$$

$$\Rightarrow 16y^3 + 48y^2 + 12y +$$

$$- 48y^2 - 96y - 48$$

$$60y + 60 - 31 = 0$$

$$\Rightarrow \frac{16y^3 + 12y - 3}{16} = 0$$

$$\Rightarrow 16y^3 + 12y = 3$$

**3. WE NOW USE THE IDENTITY OF SINEHAT AS THE COEFFICIENT OF  $y^3$  IS POSITIVE.**

$$\sinh 3t = 3\sinh t + 4\sinh^3 t$$

$$\Rightarrow 4y^3 + 3y = \frac{3}{4}$$

$$\Rightarrow 4\sinh^3 t + 3\sinh t = \frac{3}{4}$$

$$\Rightarrow \sinh t = \frac{3}{4}$$

$$\Rightarrow 3t = \operatorname{arsinh} \frac{3}{4}$$

$$\Rightarrow t = \frac{1}{3} \operatorname{ln} \left[ \frac{3}{4} + \sqrt{\frac{3}{16} + 1} \right]$$

*y = sinh t*

**4. FINISH THE REQUIRED SOLUTION CAREFULLY**

$$\Rightarrow x = y + 1$$

$$\Rightarrow x = 1 + \sinh t$$

$$\Rightarrow x = 1 + \sinh(\operatorname{ln} \frac{3}{4})$$

$$\Rightarrow x = 1 + \frac{1}{2} e^{\operatorname{ln} \frac{3}{4}} - \frac{1}{2} e^{-\operatorname{ln} \frac{3}{4}}$$

$$\Rightarrow x = 1 + 2^{\frac{1}{2}} \left[ 2^{\frac{1}{2}} - 2^{-\frac{1}{2}} \right]$$

$$\Rightarrow x = 1 + 2^{\frac{1}{2}} - 2^{-\frac{1}{2}}$$

**Question 28** (\*\*\*\*\*)

Solve the cubic equation

$$16x^3 + 96x^2 + 180x + 99 = 0, \quad x \in \mathbb{R}.$$

You may assume that this cubic equation only has one real root.

$$\boxed{\quad}, \quad x = -2 + 2^{-\frac{2}{3}} - 2^{-\frac{4}{3}}$$

**Start by writing the cubic in required form.**

$$16x^3 + 96x^2 + 180x + 99 = 0$$

$$x^3 + 6x^2 + \frac{45}{4}x + \frac{99}{16} = 0$$

LET  $x = y - \frac{3}{3} = y - \frac{5}{3} \Rightarrow \begin{cases} x = y - 2 \\ y = x + 2 \end{cases}$

**Substituting into the cubic yields:**

$$\begin{aligned} &\rightarrow 16(y-2)^3 + 96(y-2)^2 + 180(y-2) + 99 = 0 \\ &\rightarrow 16(y^3 - 6y^2 + 12y - 8) + 96(y^2 - 4y + 4) + 180(y-2) + 99 = 0 \\ &\rightarrow \left\{ \begin{aligned} &16y^3 - 96y^2 + 192y - 128 \\ &+ 96y^2 - 384y + 384 \\ &+ 180y - 360 \end{aligned} \right\} + 99 = 0 \\ &\Rightarrow 16y^3 - 12y - 5 = 0 \\ &\Rightarrow 16y^3 - 12y - 5 = 0 \end{aligned}$$

**Now we use the identity** (coefficients of  $y$  normal)

$$\begin{aligned} \cos 3t &\equiv 4\cos^3 t - 3\cos t \\ \cosh 3t &\equiv 4\cosh^3 t - 3\cosh t \end{aligned}$$

$$\begin{aligned} &\Rightarrow 16y^3 - 12y = 5 \\ &\Rightarrow 4y^3 - 3y = \frac{5}{4} \\ &\Rightarrow 4\cosh^3 t - 3\cosh t = \frac{5}{4} \\ &\Rightarrow \cosh 3t = \frac{5}{4} \end{aligned}$$

**Finally we find a real solution.**

$$\begin{aligned} &\Rightarrow 3t = \pm \operatorname{arccosh}\left(\frac{5}{4}\right) = \pm \ln\left[\frac{1}{4} + \sqrt{\frac{25}{16} - 1}\right] \\ &\Rightarrow 3t = \pm \ln\left(\frac{5}{4} \sqrt{\frac{9}{16}}\right) = \pm \ln\left(\frac{5}{4} \cdot \frac{3}{4}\right) = \pm \ln\frac{15}{16} \\ &\Rightarrow t = \pm \frac{1}{3} \ln\frac{15}{16} \end{aligned}$$

$$\Rightarrow 3t = \pm \operatorname{arccosh}\left(\frac{5}{4}\right) = \pm \ln\left[\frac{1}{4} + \sqrt{\frac{25}{16} - 1}\right]$$

$$\Rightarrow 3t = \pm \ln\left(\frac{5}{4} \sqrt{\frac{9}{16}}\right) = \pm \ln\left(\frac{5}{4} \cdot \frac{3}{4}\right) = \pm \ln\frac{15}{16}$$

$$\Rightarrow t = \pm \frac{1}{3} \ln\frac{15}{16}$$

**Finally we find a real solution.**

$$\begin{aligned} &\Rightarrow x = y - 2 \\ &\Rightarrow x = \cosh t - 2 \\ &\Rightarrow x = \cosh\left(\pm \frac{1}{3} \ln\frac{15}{16}\right) - 2 \\ &\Rightarrow x = \cosh\left(\frac{1}{3} \ln\frac{15}{16}\right) - 2 \\ &\Rightarrow x = \cosh\left(\ln\frac{5}{4}\right) - 2 \\ &\Rightarrow x = \frac{1}{2} \left[ e^{\ln\frac{5}{4}} + e^{-\ln\frac{5}{4}} \right] - 2 \\ &\Rightarrow x = 2^{\frac{1}{2}} \left[ 2^{\frac{1}{2}} + 2^{-\frac{1}{2}} \right] - 2 \\ &\Rightarrow x = 2^{\frac{1}{2}} + 2^{-\frac{1}{2}} - 2 \end{aligned}$$

**Question 29** (\*\*\*\*\*)

Solve the cubic equation

$$x^3 - 9x^2 + 3x - 3 = 0, \quad x \in \mathbb{R}.$$

You may assume that this cubic equation only has one real root.

$$\boxed{\quad}, \quad x = 3 + 2^{\frac{5}{3}} + 2^{\frac{4}{3}}$$

• START BY WRITING THE CUBIC IN REDUCED FORM

$$x^3 - 9x^2 + 3x - 3 = 0$$

LET  $z = y - \frac{a}{3} = y - \frac{-9}{3} \Rightarrow z = y + 3$

• SUBSTITUTE INTO THE CUBIC

$$\Rightarrow (y+3)^3 - 9(y+3)^2 + 3(y+3) - 3 = 0$$

$$\Rightarrow y^3 + 9y^2 + 27y + 27 - 9(y^2 + 6y + 9) + 3y + 9 - 3 = 0$$

$$\Rightarrow \begin{bmatrix} y^3 + 9y^2 + 27y + 27 \\ -9y^2 - 54y - 81 \\ 3y + 9 \end{bmatrix} = 0$$

$$\Rightarrow y^3 - 24y - 48 = 0$$

$$\Rightarrow y^3 - 24y = 48$$

• WE USE THE IDENTITY  $\boxed{\cos 3t = 4\cos^3 t - 3\cos t}$   
(coefficient of  $y$  negative)

$$\boxed{\cos 3t = 4\cos^3 t - 3\cos t}$$

• Let  $y = 2\cos t, z \neq 0$

$$(4\cos^3 t - 24\cos t) = 48$$

$$(4\cos^3 t - 3\cos t) = \cos 3t$$

$$\frac{3}{4} = \frac{-24}{z} = \frac{48}{\sin 3t}$$

• FROM THE FIRST TWO WE OBTAIN

$$\frac{24}{z} = 8$$

• THIS WE KNOW THAT

$$\Rightarrow \frac{ab}{\cos 3t} = 8t = \pm 32\sqrt{2}$$

$$\Rightarrow \cos 3t = \pm \frac{4b}{32\sqrt{2}} = \pm \frac{3}{4\sqrt{2}} = \pm \frac{3\sqrt{2}}{4}$$

$$\Rightarrow 3t = \pm \arccos\left(\pm \frac{3}{4}\sqrt{2}\right)$$

$$\Rightarrow 3t = \pm \arccos\left(\frac{3}{4}\sqrt{2}\right)$$

$$\Rightarrow t = \pm \frac{1}{3}\ln\left[\frac{3\sqrt{2} + \sqrt{31-1}}{3\sqrt{2} - \sqrt{31-1}}\right]$$

$$\Rightarrow t = \pm \frac{1}{3}\ln\left[\frac{3\sqrt{2} + \sqrt{15}}{3\sqrt{2} - \sqrt{15}}\right]$$

$$\Rightarrow t = \pm \frac{1}{3}\ln\left[\frac{4\sqrt{2} + \sqrt{15}}{4\sqrt{2} - \sqrt{15}}\right]$$

$$\Rightarrow t = \pm \frac{1}{3}\ln\sqrt{2}$$

• FINALLY WE HAVE

$$x = 3 + y = 3 + 2\cos t = 3 + 4\sqrt{2}\cos\left[\pm \frac{1}{3}\ln\sqrt{2}\right]$$

$$x = 3 + 4\sqrt{2}\cos\left(\frac{1}{3}\ln\sqrt{2}\right) = 3 + 4\sqrt{2}\lambda^{\frac{1}{3}}\left[e^{\frac{i\pi}{3}\ln\sqrt{2}} + e^{-\frac{i\pi}{3}\ln\sqrt{2}}\right]$$

$$x = 3 + 2\sqrt{2}\times\left[2^{\frac{1}{3}} + 2^{\frac{1}{3}}\right] = 3 + 2^{\frac{2}{3}}\left[2^{\frac{1}{3}} + 2^{\frac{1}{3}}\right]$$

$$x = 3 + 2^{\frac{5}{3}} + 2^{\frac{4}{3}}$$