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## IYGB - FP4 PAPER M - QUESTION 1

START BY INVESTIGATING CLOSURE

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} c & d \\ d & c \end{pmatrix} = \begin{pmatrix} ac+bd & ad+bc \\ bc+ad & bd+ac \end{pmatrix} \text{ IF OF THE SAME FORM}$$

ASSOCIATIVITY - HAS TO BE ASSUMED FROM MATRICES

If  $\underline{A}, \underline{B}, \underline{C} \subseteq$  ALL  $2 \times 2$  MATRICES THEN  $(\underline{AB})\underline{C} = \underline{A}(\underline{BC})$

EXISTENCE OF IDENTITY

$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  IS OF THE REQUIRED FORM, AND IF  $\underline{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

$$\underline{AI} = \underline{IA} = \underline{A}$$

EXISTENCE OF INVERSE

FOR ALL  $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$  AN INVERSE EXISTS AS  $\frac{1}{a^2-b^2} \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$

$$= \begin{pmatrix} \frac{a}{a^2-b^2} & \frac{b}{b^2-a^2} \\ \frac{b}{b^2-a^2} & \frac{a}{a^2-b^2} \end{pmatrix}$$

WHICH AGAIN IS OF  
THE CORRECT FORM

THEFORE INDEED A GROUP

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## IYGB - FP4 PAPER M - QUESTION 2

SETTING UP EUCLID'S ALGORITHM FOR 560 & 1169

$$1169 = 2 \times 560 + 49$$

$$560 = 11 \times 49 + 21$$

$$49 = 2 \times 21 + 7$$

$$21 = 3 \times 7 + 0$$

← H.C.F

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## IYGB - FP4 PAPER 4 - QUESTION 3

$$\begin{array}{c} t_{n+1} = 3t_n + 2 \\ t_1 = 1 \\ n \in \mathbb{N} \end{array}$$

$\iff$

$$\begin{array}{c} t_n = 2 \times 3^{n-1} - 1 \\ n \in \mathbb{N} \end{array}$$

### BASE CASE

$$t_1 = 1$$

$$t_1 = 2 \times 3^{1-1} - 1 = 2 \times 1 - 1 = 1$$

} RESULT HOLDS FOR  $n=1$

### INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR  $n=k$ ,  $k \in \mathbb{N}$

$$\Rightarrow t_k = 2 \times 3^{k-1} - 1$$

$$\Rightarrow 3t_k = 3[2 \times 3^{k-1} - 1]$$

$$\Rightarrow 3t_k = 2 \times 3^k - 3$$

$$\Rightarrow 3t_k + 2 = 2 \times 3^k - 3 + 2$$

$$\Rightarrow t_{k+1} = 2 \times 3^{(k+1)-1} - 1$$

### CONCLUSION

IF THE RESULT HOLDS FOR  $n=k$ ,  $k \in \mathbb{N}$ , THEN IT ALSO HOLDS FOR  $n=k+1$

SINCE THE RESULT HOLDS FOR  $n=1$ , THEN IT MUST HOLD FOR ALL  $n$

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## MGB - FP4 PAPER M - QUESTION 4

### PRELIMINARIES FIRST

$$\Rightarrow y = \ln(1 + \cos x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin x}{\cos x + 1}$$

$$\begin{aligned}\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(\frac{-\sin x}{\cos x + 1}\right)^2 = 1 + \frac{\sin^2 x}{(\cos x + 1)^2} = \frac{(1 + \cos x)^2 + \sin^2 x}{(\cos x + 1)^2} \\ &= \frac{1 + 2\cos x + \cos^2 x + \sin^2 x}{(\cos x + 1)^2} = \frac{2 + 2\cos x}{(\cos x + 1)^2} = \frac{2(1 + \cos x)}{(\cos x + 1)^2}\end{aligned}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{2}{1 + \cos x} \quad (\cos x \neq -1)$$

### SETTING AN ARC LENGTH INTEGRAL

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{2}{1 + \cos x}} dx = 2 \int_0^{\frac{\pi}{2}} \sqrt{\frac{2}{1 + \cos x}} dx$$

### USING DOUBLE ANGLE IDENTITIES

$$s = 2 \int_0^{\frac{\pi}{2}} \sqrt{\frac{2}{1 + (2\cos^2 \frac{x}{2} - 1)}} dx = 2 \int_0^{\frac{\pi}{2}} \sqrt{\frac{2}{2\cos^2 \frac{x}{2}}} dx$$

$$s = 2 \int_0^{\frac{\pi}{2}} \frac{1}{\cos \frac{x}{2}} dx = 2 \int_0^{\frac{\pi}{2}} \sec \frac{x}{2} dx = 2 \left[ 2 \ln \left| \sec \frac{x}{2} + \tan \frac{x}{2} \right| \right]_0^{\frac{\pi}{2}}$$

STANDARD RESULT

$$s = 4 \left[ \ln \left( \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right) - \ln \left( \sec 0 + \tan 0 \right) \right] = 4 \left[ \ln (\sqrt{2} + 1) - \ln (1 + 0) \right]$$

$$s = 4 \ln (\sqrt{2} + 1) = 2 \ln (\sqrt{2} + 1)^2 = 2 \ln (2 + 2\sqrt{2} + 1) = 2 \ln (3 + 2\sqrt{2})$$

### REPEATING ONCE MORE

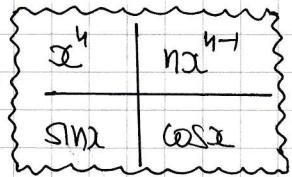
$$s = \ln (3 + 2\sqrt{2})^2 = \ln (9 + 12\sqrt{2} + 8) = \ln (17 + 12\sqrt{2})$$

~~AS REQUIRED~~

## IYGB - FP4 PAPER M - QUESTION 5

a) PROCEED BY INTEGRATION BY PARTS

$$\Rightarrow I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$$



$$\Rightarrow I_n = [x^n \sin x]_0^{\frac{\pi}{2}} - n \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

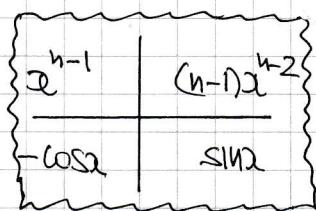
$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

INTEGRATION BY PARTS AGAIN

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n \left[ \left[ -x^{n-1} \cos x \right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx \right]$$

ZERO AT BOTH ENDS



$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) \int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx$$

$$\therefore I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) \int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx$$

∴ AS REQUIRED

b) i) REWRITE IN "I" NOTATION

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} x^4 \cos x \, dx &= I_4 \\
 &= \left(\frac{\pi}{2}\right)^4 - 4 \times 3 I_2 = \frac{\pi^4}{16} - 12 I_2 \\
 &= \frac{\pi^4}{16} - 12 \left[ \left(\frac{\pi}{2}\right)^2 - 2 \times 1 \times I_0 \right] \\
 &= \frac{\pi^4}{16} - 12 \times \frac{\pi^2}{4} + 24 I_0 \\
 &= \frac{\pi^4}{16} - 3\pi^2 + 24 \int_0^{\frac{\pi}{2}} \cos x \, dx
 \end{aligned}$$

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## IYGB - FP4 PAPER N - QUESTION 5

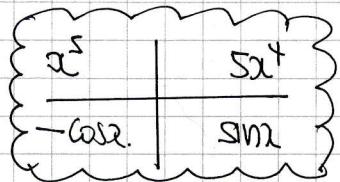
$$= \frac{\pi^4}{16} - 3\pi^2 + 2t \left[ \sin x \right]_0^{\pi/2}$$

$$= \frac{\pi^4}{16} - 3\pi^2 + 24$$

II) START BY INTEGRATION BY PARTS

$$\int_0^{\pi/2} x^5 \sin x = \left[ -x^5 \cos x \right]_0^{\pi/2} - \int -5x^4 \cos x dx$$

*ZERO AT BOTH ENDS*



$$= 5 \int x^4 \cos x dx$$

$$= 5 I_4$$

$$= 5 \left( \frac{\pi^4}{16} - 3\pi^2 + 24 \right)$$

$$= \frac{5}{16}\pi^4 - 15\pi^2 + 120$$

# + IYGB - FP4 PAPER M - QUESTION 6

PROCEED AS USUAL

$$|z + i - i| = |z - i + 2i| \quad \text{with} \quad f(z) = kz + i$$

$$w = kz + i$$

$$\frac{w-i}{k} = z$$

SUBSTITUTE INTO THE LINE & TIDY

$$\Rightarrow \left| \frac{w-i}{k} + i + i \right| = \left| \frac{w-i}{k} - i + 2i \right|$$

$$\Rightarrow \left| \frac{w-i+k+ik}{k} \right| = \left| \frac{w-i-k+2ki}{k} \right|$$

LET  $w = x+iy$

$$\Rightarrow |x+iy - i + k + ik| = |x+iy - i - k + 2ki|$$

$$\Rightarrow |(x+k) + i(y-1+k)| = |(x-k) + i(y-1+2k)|$$

$$\Rightarrow \sqrt{(x+k)^2 + (y-1+k)^2} = \sqrt{(x-k)^2 + (y-1+2k)^2}$$

$$\Rightarrow \cancel{x^2 + 2kx + k^2 + y^2 + 1 + k^2} \cancel{+ 2y - 2k + 2ky} = \cancel{x^2 - 2kx + k^2 + y^2 + 1 + 4k^2} \cancel{- 2y - 4k + 4ky}$$

$$\Rightarrow 2kx + k^2 - 2k + 2ky = -2kx + 4k^2 - 4k + 4ky$$

$$\Rightarrow 4kx - 3k^2 + 2k = 2ky$$

$$\Rightarrow y = 2x + 1 - \frac{3}{2}k$$

$$\therefore 1 - \frac{3}{2}k = -8$$

$$9 = \frac{3}{2}k$$

$$k = 6$$

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## IYGB - FP4 PAPER M - QUESTION 7

a)

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & -3 & 1 \\ 3 & -5 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \therefore \lambda = 2$$

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b)

$$\left. \begin{array}{l} x - y + z = -2x \\ 3x - 3y + z = -2y \\ 3x - 5y + 3z = -2z \end{array} \right\} \Rightarrow \left. \begin{array}{l} 3x - y + z = 0 \\ 3x - y + z = 0 \\ 3x - 5y + 5z = 0 \end{array} \right\} \Rightarrow$$

$$\left. \begin{array}{l} y = 3x + z \\ 3x - 5(3x+z) + 5z = 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} y = 3x + z \\ -12x = 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} y = z \\ x = 0 \end{array} \right.$$

$$\therefore \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

c)

$$\underline{A} \underline{w} = \underline{A} (\underline{u} + \underline{v}) = \underline{A}^7 \underline{u} + \underline{A}^7 \underline{v}$$

$$= \underline{A}^6 [\underline{A} \underline{u} + \underline{A} \underline{v}] = \underline{A}^6 [2\underline{u} - 2\underline{v}]$$

$$= \underline{A}^5 [\underline{A} 2\underline{u} - \underline{A} 2\underline{v}] = \underline{A}^5 [4\underline{u} + 4\underline{v}]$$

$$= \underline{A}^4 [\underline{A} 4\underline{u} + \underline{A} 4\underline{v}] = \underline{A}^4 [8\underline{u} - 8\underline{v}]$$

⋮

⋮

$$= \underline{A} [\underline{A} 2^5 \underline{u} + \underline{A} (-2)^5 \underline{v}] = \underline{A} [2^6 \underline{u} + (-2)^6 \underline{v}]$$

$$= 2^7 \underline{u} + (-2)^7 \underline{v} = 128 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 128 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 128 \\ 128 \\ 256 \end{pmatrix} - \begin{pmatrix} 0 \\ 128 \\ 128 \end{pmatrix} = \begin{pmatrix} 128 \\ 0 \\ 128 \end{pmatrix}$$

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## IYGB - FP4 PAPER M - QUESTION 8

a) REQUIRED NUMBER IS GIVEN BY

$$\frac{7!}{3! 2!} = 420 \cancel{/}$$

↑      ↗  
TRIPLE "A"  
(REPEAT)      DOUBLE "N"  
(REPEAT)

b) TREATING THE VOWELS "AAA", AS A SINGLE LETTER

AAA	B	N	N	S
1	2	3	4	5

$$\frac{5!}{2!} = 60 \cancel{/}$$

↗  
DOUBLE "N" REPEAT

c) BLOCKING THE VOWELS & CONSONANTS TOGETHER

AAA	B N N S
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1 WAY       $\frac{4!}{2!}$  WAYS = 12

$$\text{Hence } (1 \times 12) \times 2 = 24 \cancel{/}$$

↑  
3 VOWELS - 4 CONSONANTS OR 4 CONSONANTS - 3 VOWELS

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## IYGB - FP4 PAPER M - QUESTION 8

d) SPLITTING IN SEPARATE CASES & NOT IN THIS PART ORDER

DOES NOT MATTER

I A A A WITH ONE OF B, N, S 3 WAYS

II A A WITH TWO OF B, N, S 3 WAYS

III N N WITH THREE OF A, B, S 3 WAYS

IV A A N N 1 WAY

V ALL DIFFERENT B A N S 1 WAY

11 WAYS

e) USING PART (d)

$$\text{CASE I, SAY } \underline{\underline{A}} \underline{\underline{A}} \underline{\underline{A}} \underline{B} \quad 3 \times \frac{4!}{3!} = 12$$

$$\text{CASE II, SAY } \underline{\underline{A}} \underline{\underline{A}} \underline{B} \underline{N} \quad 3 \times \frac{4!}{2!} = 36$$

$$\text{CASE III, SAY } \underline{\underline{N}} \underline{\underline{N}} \underline{A} \underline{B} \quad 3 \times \frac{4!}{2!} = 36$$

$$\text{CASE IV, } \underline{\underline{A}} \underline{\underline{A}} \underline{N} \underline{N} \quad 1 \times \frac{4!}{2! 2!} = 6$$

$$\text{CASE V } \underline{\underline{B}} \underline{\underline{A}} \underline{\underline{N}} \underline{\underline{S}} \quad 1 \times 4! = 24$$

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