Essay answer to HW2, Exercise 3

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The system of linear equation:

$$x-2y-z = 1$$
$$x+z = -1$$
$$-x-y+z = -1$$
$$x-y-3z = k \qquad k \in R$$

Is a linear parametric system because in the last column the parameter k appears. As a parametric system it needs to be discussed in order to find for which values of k either the system does not have solutions or has a certain/infinity solutions.

Knowing the ranks of the coefficient matrix A and the augmented matrix T the following theorems hold:

- When rank(A) < rank(T) in the reduce row echelon form the system has an inconsistent equation. Therefore the system has no solutions.
- When rank(A) = rank(T) = Number of unknowns then the system has exactly one solution, and this can be immediately read from the reduce row echelon form of T.
- When rank(A) = rank(T) < Number of unknowns then the system has $\infty^{(n-k)}$ solutions that can be written in standard parametric form.

The question A asks for the discussion of the solution for the system of linear equations.
First of all find the rank(A) using the maple function,

```
A:= <1,-2,-1;1,0,1;-1,-1,1;1,-1,-3>
```

$$A := \begin{bmatrix} 1 & -2 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -3 \end{bmatrix}$$

```
Rank(A)
```

$$3$$

then create the augmented matrix T and start applying the Gauss-Jordan Elimination to it (see appendix for row operations).

```
T:= <A|b>
```

$$T := \begin{bmatrix} 1 & -2 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & -3 & k \end{bmatrix}$$

Once it is reduced to the reduce row echelon form we see the general solutions:

$$T10 := \begin{bmatrix} 1 & 0 & 0 & \frac{15}{2} - \frac{5k}{2} \\ 0 & 1 & 0 & 3-k \\ 0 & 0 & 1 & -\frac{k}{2} + \frac{1}{2} \\ 0 & 0 & 0 & k-3 \end{bmatrix}$$

It is seen that for k=3 the rank(T)=3. Being the number of unknowns 3 that is equal to the rank(A) and rank(T) the system as exactly one solution, and this can be immediately read from the reduce row echelon form of T. The solution for k=3 is:

$$T11 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For k≠3 we have the rank(T)=4 so rank(A) < rank(T) and the system has an inconsistent equation. Therefore it has no solutions.

Question b asks to find the area of the base for the tetrahedron created using the 4 planes given with k=9 in the last one. The plane that we have to consider is the one given by the fourth equation. The coordinates of the points where the planes meet on the x-y-3z-9=0 are given by the solutions of the following linear systems of equations:

x-y-3z-9=0
x-2y-z-1=0
x+z+1=0

x-y-3z-9=0
x+z+1=0
-x-y+z+1=0

x-y-3z-9=0
x-2y-z-1=0
-x-y+z+1=0

After solving the 3 matrices we get the coordinates of the base points:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}. \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

Now calculate the vectors corresponding to two sides of the base computing the subtractions a-b and a-c to get:

```
Pab:=a-b
```

$$Pab := \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

```
Pac:=a-c
```

$$Pac := \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

To calculate the area we firstly compute the cross product PabXPac, then calculate the magnitude of the vector v and divided it by 2.

```
v:= CrossProduct(Pab,Pac)
```

$$v := \begin{bmatrix} 6 \\ -6 \\ -18 \end{bmatrix}$$

```
Area:= 1/2*sqrt(DotProduct(v,v))
```

$$Area := 3\sqrt{11}$$

To answer question c we need to find the other vector in space corresponding to the coordinates of the vertex which is not on the x-y-3z-9=0 plane. The coordinates are given by the solutions of the following linear system:

x-2y-z-1=0
x+z+1=0
-x-y+z+1=0

Coordinates:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Now calculate the vector computing the subtraction d-a:

```
Pad:= a-d
```

$$Pad := \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

To calculate the volume calculate the magnitude of the mixed product |PabXPac·Pad|. This is equal to the absolute value of the determinant of the matrix composed by the three vectors. Remember to divide the result by 1/6 because the tetrahedron's volume is one sixth of the parallelepipedon's one.

```
U:=<Pab|Pac|Pad>
```

$$U := \begin{bmatrix} 1 & 5 & 2 \\ 4 & 2 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

```
Volume:= 1/6*abs(Determinant(U))
```

$$Volume := 6$$

The other value of k for which the volume is the same as the one we found, belongs to the parrallel plane respect to the given plane x-y-3z-9=0 located on the opposite side of the top vertex at the same distance from it.
To find it we consider the plane x-y-3z+k=0 and calculate the distance between this point and the plane.

```
Distance:= abs((1*0+1*0-3*(-1)-k))/(sqrt(1+1+9))
```

$$Distance := \frac{|k-3|\sqrt{11}}{11}$$

Also find the magnitude of the projection for vector Pad on the vector v

```
Projection:= DotProduct(Pad,v)/sqrt(DotProduct(v,v))
```

$$Projection := \frac{6\sqrt{11}}{11}$$

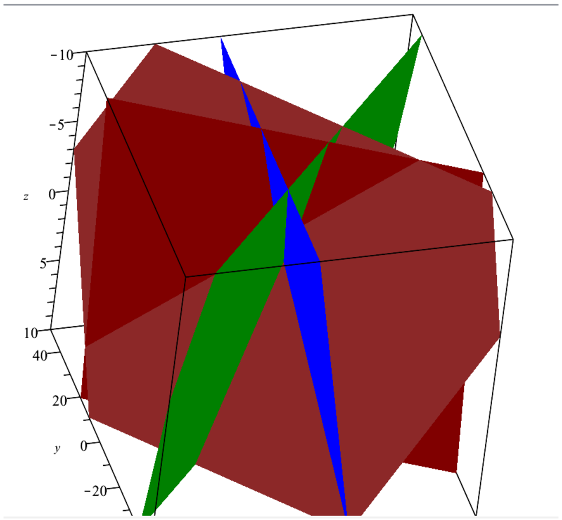
Set the projection equal to the distance and find the values of k

```
Distance = Projection
```

$$\frac{|-3+k|\sqrt{11}}{11} = \frac{6\sqrt{11}}{11}$$

From which |k-3|= 6, so k=-3 and k=9. This mean that if we consider the plane x-y-3z+3=0 the value of the volume will be the same.

The best representation I can get of the 4 planes is the following one:



Appendix

```
> restart:
> with(plots):
> with(LinearAlgebra):
```

Exercise a

```
> A:= <1,-2,-1;1,0,1;-1,-1,1;1,-1,-3>
```

$$A := \begin{bmatrix} 1 & -2 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -3 \end{bmatrix} \tag{1.1}$$

```
> Rank(A)
```

$$3 \tag{1.2}$$

```
> b:= <1,-1,-1,k>
```

$$b := \begin{bmatrix} 1 \\ -1 \\ -1 \\ k \end{bmatrix} \tag{1.3}$$

```
> T:= <A|b>
```

$$T := \begin{bmatrix} 1 & -2 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & -3 & k \end{bmatrix} \tag{1.4}$$

```
> T1:=RowOperation(T,[2,3],+1):
> T2:=RowOperation(T1,[3,4],+1):
> T3:=RowOperation(T2,[4,1],-1):
> T4:=RowOperation(T3,[3,4],+2):
> T5:=RowOperation(T4,2,-1):
> T6:=RowOperation(T5,[1,2],+2):
> T7:=RowOperation(T6,[4,2],-1):
> T8:=RowOperation(T7,3,-1/6):
> T9:=RowOperation(T8,[2,3],+2):
> T10:=RowOperation(T9,[1,3],+5)
```

(1.5)

$$T10 := \begin{bmatrix} 1 & 0 & 0 & \frac{15}{2} - \frac{5k}{2} \\ 0 & 1 & 0 & 3 - k \\ 0 & 0 & 1 & \frac{1}{2} - \frac{k}{2} \\ 0 & 0 & 0 & -3 + k \end{bmatrix} \quad (1.5)$$

```
> T11:= <1,0,0,0;0,1,0,0;0,0,1,-1;0,0,0,0>
```

$$T11 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.6)$$

Exercise b

$$\begin{bmatrix} Exercise \\ -Exercise \\ -Exercise \\ Exercise\ k \end{bmatrix} \quad (1.7)$$

```
> a:= <1,-1,-3,9;1,-2,-1,1;1,0,1,-1>:
> ReducedRowEchelonForm(a)
```

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \quad (1.8)$$

```
> b:= <1,-1,-3,9;1,0,1,-1;-1,-1,1,-1>:
> ReducedRowEchelonForm(b)
```

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad (1.9)$$

```
> c:= <1,-1,-3,9;1,-2,-1,1;-1,-1,1,-1>:
> ReducedRowEchelonForm(c)
```

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix} \quad (1.10)$$

```
> a:= <2,2,-3>:
> b:= <1,-2,-2>:
> c:= <-3,0,-4>:
> Pab:=a-b
```

$$Pab := \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \quad (1.11)$$

```
> Pac:=a-c
```

$$Pac := \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \quad (1.12)$$

```
> v:= CrossProduct(Pab,Pac)
```

$$v := \begin{bmatrix} 6 \\ -6 \\ -18 \end{bmatrix} \quad (1.13)$$

```
> Area:= 1/2*sqrt(DotProduct(v,v))
```

$$Area := 3\sqrt{11} \quad (1.14)$$

Exercise c:

```
> d:= <1,-2,-1,1;1,0,1,-1;-1,-1,1,-1>:
> ReducedRowEchelonForm(d)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad (1.15)$$

```
> d:= <0,0,-1>:
> Pad:= a-d
```

$$Pad := \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \quad (1.16)$$

```
> U:=<Pab|Pac|Pad>
```

$$U := \begin{bmatrix} 1 & 5 & 2 \\ 4 & 2 & 2 \\ -1 & 1 & -2 \end{bmatrix} \quad (1.17)$$

```
> Volume:= 1/6*abs(Determinant(U))
```

$$Volume := 6 \quad (1.18)$$

```
> Distance:= abs((1*0+1*0-3*(-1)-k))/(sqrt(1+1+9))
```

$$Distance := \frac{|-3+k|\sqrt{11}}{11} \quad (1.19)$$

```
> Projection:= DotProduct(Pad,v)/sqrt(DotProduct(v,v))
```

$$Projection := \frac{6\sqrt{11}}{11} \quad (1.20)$$

```
> Distance = Projection
```

$$\frac{|-3+k|\sqrt{11}}{11} = \frac{6\sqrt{11}}{11} \tag{1.21}$$

Exercise d

```
> plot1:=implicitplot3d(x-2*y-z=1,x=-50..50,y=-50..50,z=-50..50,
  style=surface,color=red):
> plot2:=implicitplot3d(x+z=-1,x=-50..50,y=-50..50,z=-50..50,style=
  surface,color=blue):
> plot3:=implicitplot3d(-x-y+z=-1,x=-50..50,y=-50..50,z=-50..50,
  style=surface,color=green):
> plot4:=implicitplot3d(x-y-3*z=9,x=-50..50,y=-50..50,z=-50..50,
  style=surface,color=orange):
> plots[display](plot1,plot2,plot3,plot4,view=-10..10)
```