# Analytical modeling of 2D Halbach permanent magnets

Kaspar K. Nielsen & Rasmus Bjørk
rabj@dtu.dk
DTU Energy
Technical University of Denmark

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## 1 Introduction

Permanent magnets are crucial for the modern sustainable energy society. During the past 30 years, there has been an immense development within strong permanent magnets. The composition FeNdB (iron-neodynium-boron) has the best performance for any permanent magnet material in terms of the magnitude of the produced field. This material is used in wind turbines, electric cars and many other places where strong permanent magnets are needed. Throughout this exercise you will develop the theory necessary for analyzing the so-called Halbach magnets [1] in two dimensions. Such magnet configurations can create rather large and homogeneous magnetic fields and are used in many different technical applications.

Before we get to the modeling, you will need an introduction to cylindrical coordinates as these pose the natural choice of coordinate base when considering cylindrical magnets. Throughout this exercise we shall distinguish between *scalars*, written in italic (e.g. time $t$) and vector fields, which are written in bold and normal type (e.g. the current density $\mathbf{J}$).

## 2 Cylindrical coordinates

We will begin by treating vector fields from a different perspective than what we are used to in Math 1. In the eNotes a vector field at the point $(x, y, z)$ is given as a vector with the coordinates:

$$\mathbf{V} = (V_x, V_y, V_z).$$

Let us consider a different representation called cylindrical coordinates.
An orthonormal basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ is introduced at every point in space $(x, y, z)$. This basis depends on the coordinates of the point such thart the vector $\mathbf{e}_r$

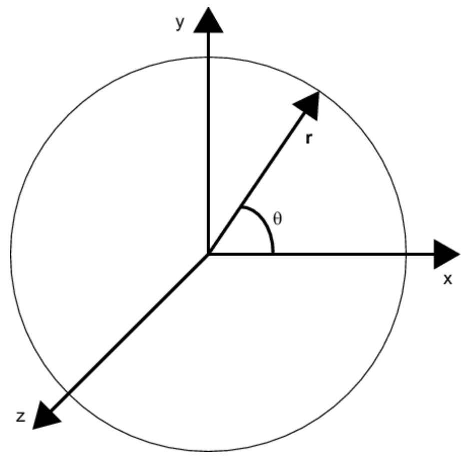



Figure 1: Definition of the coordinate system with rectangular coordinates $(x, y, z)$ and cylindrical coordinates $(r, \theta, z)$.

is parallel to the position vector to the point $(x, y, 0)$. The vector $\mathbf{e}_r$ can be considered as a plane vector and $\mathbf{e}_\theta$ is orthogonal to $\mathbf{e}_r$, i.e. parallel to the vector $(-y, x, 0)$. The vector $\mathbf{e}_z$ is identical to the usual rectangular-based vector $(0, 0, 1)$.
The vector field $\mathbf{V} = (V_x, V_y, V_z)$ may now be decomposed along these directions:

$$\mathbf{V} = V_r \mathbf{e}_r + V_\theta \mathbf{e}_\theta + V_z \mathbf{e}_z \tag{1}$$

with $V_r$, $V_\theta$ and $V_z$ denoting the projections of the vector field $\mathbf{V}$ along the new basis vectors:

$$V_r = \mathbf{V} \cdot \mathbf{e}_r \quad , \quad V_\theta = \mathbf{V} \cdot \mathbf{e}_\theta \quad , \quad V_z = \mathbf{V} \cdot \mathbf{e}_z . \tag{2}$$

**Exercise 1**

> Show that the new basis vectors may be written as a function of rectangular coordinates in the following way:
>
> $$\mathbf{e}_r = (x^2 + y^2)^{-1/2}(x, y, 0) \tag{3}$$
> $$\mathbf{e}_\theta = (x^2 + y^2)^{-1/2}(-y, x, 0) \tag{4}$$
> $$\mathbf{e}_z = (0, 0, 1) \tag{5}$$

Once again we consider the rectangular coordinate system and make a change of variables by considering polar coordinates in the plane and keeping the z-coordinate (see Fig. 1):

$$\begin{aligned} x &= r\cos\theta & (6)\\ y &= r\sin\theta & (7)\\ z &= z. & (8) \end{aligned}$$

The coordinates $(r, \theta, z)$ are known as cylindrical coordinates.

**Exercise 2**

Show that the basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$, expressed by cylindrical coordinates are given by:

$$\mathbf{e}_r = (\cos\theta, \sin\theta, 0) \tag{9}$$

$$\mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0) \tag{10}$$

$$\mathbf{e}_z = (0, 0, 1) \tag{11}$$

Let us now determine the gradient of a function $f$ with respect to the basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$, expressed in cylindrical coordinates.

From math class we know that the gradient is given by:

$$\nabla f(x, y, z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y + \frac{\partial f}{\partial z}\mathbf{e}_z \tag{12}$$

The gradient may be written with respect to the new basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ and is given by:

$$\nabla f = \nabla f_r \mathbf{e}_r + \nabla f_\theta \mathbf{e}_\theta + \nabla f_z \mathbf{e}_z \tag{13}$$

where $\nabla f_r$, $\nabla f_\theta$ and $\nabla f_z$ are determined by the directional derivatives:

$$\nabla f_r = \nabla f \cdot \mathbf{e}_r \quad , \quad \nabla f_\theta = \nabla f \cdot \mathbf{e}_\theta \quad , \quad \nabla f_z = \nabla f \cdot \mathbf{e}_z \tag{14}$$

**Exercise 3**

Use the chain rule and Eq. 6-11 to find an expression for the partial derivative of the function $f$ with respect to $r$ and show that:

$$\frac{\partial f}{\partial r} = \nabla f \cdot \mathbf{e}_r \tag{15}$$

*Hint:*

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

**Exercise 4**

Use the same method as above to find the partial derivative of the function $f$ with respect to $\theta$ and show that:

$$\frac{\partial f}{\partial \theta} = r\nabla f \cdot \mathbf{e}_\theta \tag{16}$$

**Exercise 5**

> Show by using Eq. 13 and exercises 3 and 4 that the gradient is given by:
> $$\nabla f(r,\theta,z) = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z \tag{17}$$

We now seek an expression for the divergence of a vector field in cylindrical coordinates:

$$\mathbf{V} = V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_z\mathbf{e}_z \tag{18}$$

In order to find this expression we shall use the following rule

**Exercise 6**

> Let $f(x,y,z)$ be a function and $\mathbf{V}(x,y,z)$ a vector field. Show that the expression given below is true by finding the right hand side and the left hand side and equating them:
> $$\mathrm{Div}(f\mathbf{V}) = \nabla f \cdot \mathbf{V} + f\mathrm{Div}(\mathbf{V}) \tag{19}$$

*Hint:* remember the product rule for differentiation.

**Exercise 7**

> Use Eqs. 3-5 to find the divergence of the three basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$.
> Two of these are zero. Explain why this is so.
> Express the non-zero divergence in cylindrical coordinates $(r,\theta,z)$.

**Exercise 8**

> Show that the divergence of a vector field expressed in cylindrical coordinates is given by:
> $$\mathrm{Div}(\mathbf{V}) = \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} \tag{20}$$

*Hint:* use exercises 5-7.
In a similar way as above, it is possible to derive the expression for the curl of a vector field expressed in cylindrical coordinates. You are most welcome to do this and include it in your report; the result is:

$$\mathbf{Rot}(\mathbf{V}) = (\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z})\mathbf{e}_r + (\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r})\mathbf{e}_\theta + (\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta})\mathbf{e}_z. \tag{21}$$

## 3 Magnetic field around a conductor

Let us now move on by using cylindrical coordinates to do some magnetic field calculations.

Before considering permanent magnets, we will dwell a bit on magnetic fields in general. We will start by finding the magnetic field caused by an infinitely long wire with the constant current, $I$. We assume the wire to be straight and will define the direction along the wire to be the $z$-axis

**Exercise 9**

> What is the current density, $\mathbf{J}$, within a circle centered on the infinitely long wire?

*Hint:* The unit for current density is $\mathrm{A/m^2}$. The answer should be derived by thinking and not necessarily doing any calculations.
Electromagnetism is well described by the four Maxwell equations. The derivation of these, or rather, their discovery, is a story for another day. We will here settle on mere applying the equation to various problems. The fourth Maxwell equation relates the magnetic flux density to the current density $\mathbf{B}$:

$$\mathbf{Rot}(\mathbf{B}) = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{22}$$

The magnetic flux density, $\mathbf{B}$, can be understood as the generating magnetic field, as long as one considers space outside of a magnet. In Eq. 22 the electric field, $\mathbf{E}$, is also present. Finally, we encounter two fundamental constants, which are the vacuum permeability ($\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{N/A^2}$) and the vacuum permittivity ($\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C^2/(Nm^2)}$).[1]

We assume that the current is constant in the infinite wire and so variations in time are not present. This means that we can safely assume that $\frac{\partial \mathbf{E}}{\partial t} = 0$.[2]

Given that the wire is infinitely long we can deduce a few things about the vector-field components of the flux density. The relation between $\mathbf{B}$ and $\mathbf{J}$ in Eq. 22 dictates that $\mathbf{Rot}(\mathbf{B})$ and $\mathbf{J}$ are parallel when there is no variation in time of the electric field, $\frac{\partial \mathbf{E}}{\partial t} = \mathbf{0}$. Since the current density only has one component, in the $z$-direction, i.e. $\mathbf{J} = (0, 0, J_z)$, then $\mathbf{Rot}(\mathbf{B})$ may only have components in the $z$-direction. Furthermore, as $\mathbf{B}$ does not vary in the $z$-direction (the wire is infinitely long) we must have:

$$\frac{\partial B_r}{\partial z} = 0 \ \ , \ \ \frac{\partial B_\theta}{\partial z} = 0 \ \ , \ \ \frac{\partial B_z}{\partial z} = 0 \tag{23}$$

[1] What happens if you calculate $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$. Do you know what this new constant is?

[2] Even in the case where the current was AC, i.e. periodically varying, a large frequency would be required for this term to be of any importance. In relativity this term has a huge impact.

**Exercise 10**

Using the arguments given above and Eq. 21 to show that:

$$\frac{\partial B_z}{\partial r} = 0 \ , \ \frac{\partial B_z}{\partial \theta} = 0 \tag{24}$$

Now, since the $z$-component of $\mathbf{B}$ is constant and since $\mathbf{B}$ must go to zero infinitely far away from the wire we have that $B_z = 0$.

We can also derive that the $r$-component of $\mathbf{B}$ must be zero. We just saw that $\frac{\partial B_r}{\partial z} = 0$. Furthermore, $\frac{\partial B_r}{\partial \theta}$ must also be zero due to symmetry – the field must not vary as we rotate the wire about it's axis. This means that $B_r$ can only be a function of $r$.

**Exercise 11**

Use the arguments given above and the fact that $\mathrm{Div}(\mathbf{B}) = 0$ to write down and solve a differential equation for $B_r(r)$.

*Hint:* We demand that $B_r \to 0$ as $r \to \infty$. Show that $B_r(r) = c/r$.

Utilizing the symmetry of the wire by mirroring a cross section of the conductor we can argue that $c = 0$. The only non-zero component left is $B_\theta$.

Let us now use Stokes' theorem. This states that the surface integral over the flux of the curl of a vector field is equivalent to the tangential curve integral of the vector field along the edge of the surface. Given a vector field $\mathbf{V}$, this may be written as:

$$\int_{F_r} \mathbf{Rot}(\mathbf{V}) \cdot \mathbf{n_F} \, \mathrm{d}\mu = \int_{\partial F} \mathbf{V} \cdot \mathbf{e_F} \, \mathrm{d}\mu. \tag{25}$$

See theorem 27.3 in the eNotes.

Utilizing Stokes' theorem we may find $B_\theta$.

**Exercise 12**

Use Stokes' theorem (Eq. 25) with $\mathbf{V} = (0, B_\theta, 0)$ in conjunction with the fourth Maxwell equation (22) and find $B_\theta(r)$ about an infinitely long conductor (wire) carrying a constant current, $I$.

*Hint:* Consider a curve surrounding the wire (e.g. a circle with radius $s$ centered on the axis of the conductor).

**Exercise 13**

Plot the vector field (`fieldplot`) $\mathbf{B}$ about an infinitely long conductor (at a fixed $z$).

## 4 A Halbach magnet

Now that we have improved our understanding of how magnetic fields behave (e.g. they are divergence free) we can begin to consider permanent magnets. In this part of the exercises we will study a Halbach magnet, which essentially is a ring consisting of magnetic material with a hole (bore) in the middle and air surrounding the ring. The local magnetisation in a Halbach magnet varies with $r$ and $\theta$ as:

$$\mathbf{B}_{\text{rem}} = \begin{pmatrix} B_{\text{rem,r}} \\ B_{\text{rem},\theta} \end{pmatrix} = B_{\text{rem}} \begin{pmatrix} \cos(p\theta) \\ \sin(p\theta) \end{pmatrix} \tag{26}$$

Given our convention for the notation of the basis vectors this means that:

$$\mathbf{B}_{\text{rem}} = (B_{\text{rem}} \cos{(p\theta)})\mathbf{e}_r + (B_{\text{rem}} \sin{(p\theta)})\mathbf{e}_\theta \tag{27}$$

The 2-dimensional vector, $\mathbf{B}_{\text{rem}}$, is the so-called remanence. This is the magnet's own magnetic flux density or permanent magnetisation. $B_{\text{rem}}$ is the magnitude of this vector. Finally, $p$ is an integer parameter ($p \neq 0$) which can either a positive or a negative value. This is the key parameter for a Halbach magnet.

**Exercise 14**

> Use Maple's `fieldplot` in order to visualise the vector field given in Eq. 26. Make sure to use polar coordinates and that there is a hole in the middle. What influence does the parameter $p$ have?

### 4.1 The magnetic field exterior to the magnetic material

Under the assumption that the Halbach magnet is infinitely long we may continue to work in two dimensions. We seek a solution that gives us the magnetic flux density exterior to the magnetic material. We thus wish to find an expression for $\mathbf{B}$. During our treatment of the fourth Maxwell equation (22) we encountered the current density, $\mathbf{J}$. This is also known as the *total* current density indicating that it consists of several contributions. These are the free current density ($\mathbf{J}_f$) and the bound current density ($\mathbf{J}_b$). The free current density is what we normally consider to be electrical current − it is completely analog to the current $I$ from the first part of this assignment. We shall now emphasise that we assume no relativistic effects ($\frac{\partial \mathbf{E}}{\partial t} = \mathbf{0}$) and zero free current. We can thus write:

$$\mathbf{Rot}(\mathbf{B}) = \mu_0 \mathbf{J}_b. \tag{28}$$

How can we get a physical meaning of the bound currents? Well, a bound current is a macroscopic way of considering a collection of spins, or more generally, angular momentum of charged particles inside a solid body. This leads directly to the observation that $\mathbf{J}_b = \mathbf{0}$ in air (or vacuum). However, inside a magnetized body there is a non-zero bound current. The main contributor to this is the spin of electrons and a bound current is essentially a measure of how magnetic a body is.

Technically, when working with magnetic fields, it is an advantage to use the so-called magnetic vector potential, **A**, defined in the following way:

$$\mathbf{B} = \mathbf{Rot}(\mathbf{A}). \tag{29}$$

When there are no free currents, we furthermore have that

$$\mathbf{Rot}(\mathbf{B}) = \mathbf{Rot}(\mathbf{B}_{\text{rem}}) \tag{30}$$

Since the divergence of **B** is always zero we can choose **A** so that $\mathbf{Div}(\mathbf{A}) = 0$. In order to show this, we need to proof the following:

$$\mathbf{Rot}\,(\nabla f) = \mathbf{0}, \tag{31}$$

where $f$ is a scalr function defined on $\Re$.

**Exercise 15**

Show that $\mathbf{Rot}\,(\nabla f) = \mathbf{0}$ with $f \in \Re$.

*Hint:* Use the fact that $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ with $i, j = 1, 2, 3$.

**Exercise 16**

Show that we can freely choose **A** so that it's divergence is zero. Define $\mathbf{A} = \mathbf{A}_0 + \nabla f$, where $\mathrm{Div}(\mathbf{A_0}) \neq 0$ and $f$ is a scalar function defined on $\Re$. What is the curl of **A** then?

The result of Task 16 means that we can choose $\mathbf{A}_0$ freely relative to Eq. 29. All we need to do is choose a function $f$, so that $\mathrm{Div}(\nabla f) = -\mathrm{Div}(\mathbf{A}_0)$. One can show that this is always possible and we can therefore continue with a choice of **A** with the property that $\mathbf{Div}(\mathbf{A}) = 0$.

**Exercise 17**

We need to push a bit further with vector calculus in order to utilise Eq. 29. Find the curl of **B** from Eqs. 29–30.

*Hint:* Find another expression for $\mathbf{Rot}(\mathbf{Rot}(\mathbf{A}))$.

We now have a so-called $2^{\text{nd}}$ *partial differential equation*:

$$-\nabla^2 \mathbf{A} = \mathbf{Rot}(\mathbf{B}_{\text{rem}}) \tag{32}$$

where the independent variables are the polar coordinates, $r$ og $\theta$. We wish to find the components of the vector field **A**. In order to achieve this we will be using the curl operator in cylindrical coordinates in the following.

**Exercise 18**

> Use Eq. 26 and the curl in cylindrical coordinates to show that the only non-zero component of $\mathbf{Rot}(\mathbf{B}_{\text{rem}})$ is $\mathbf{Rot}(\mathbf{B}_{\text{rem}}) \cdot \mathbf{e}_z$.

We now have the equation:

$$-\nabla^2 A_z(r,\theta) = \mathbf{Rot}(\mathbf{B}_{\text{rem}}) \cdot \mathbf{e}_z. \tag{33}$$

Note that we are only considering the $z$-component of $\mathbf{Rot}(\mathbf{B}_{\text{rem}})$. The solution to Eq. 33 is outside the scope of the curriculum of Math 1. The idea is, however, to find a solution to the homogeneous part of the equation and a particular solution. The solution to Eq. 33 furthermore depends on the parameter $p$ in the sense that one solution is valid when $p = 1$ while another solution is valid when $p \neq 0$. In the following we shall only consider the case with $p = 1$. In this case the linear combination of the solution to the homogeneous and the particular solutions is:

$$A_z(r,\theta) = \left(Cr + Dr^{-1}\right)\sin(\theta) - B_{\text{rem}} r \ln(r) \sin(\theta)\ . \tag{34}$$

**Exercise 19**

> Show that this is indeed a solution and that $A_z$ satisfies the differential equation 33.

It is important to emphasise that $C$ and $D$ are constants depending on the boundary conditions. It is possible to find these constants by defining a specific problem with appropriate boundary conditions.

## 4.2 Solution of the partial differential equation

Consider the drawing in Fig. 2, which defines a region in the interval $R_i \leq r \leq R_o$ where the permanent magnetic material is situated and follows the expression in Eq. 26. Air (or rather, vacuum) is assumed to fill the domains where $R_c \leq r \leq R_i$ and $R_o \leq r \leq R_e$. This means that there is no magnetisation (remanence) in these domains. This can be mathematically stated by letting the relative permeability be $\mu_r = 1$. The permeability is assumed to be $\mu_r = \infty$ on the domains defined by $0 \leq r \leq R_c$ and $R_e \leq r$. This means that a very small field is sufficient to align the magnetic spins.

Let us now introduce the so-called magnetic vector field $\mathbf{H}$ through:

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H} + \mathbf{B}_{\text{rem}}. \tag{35}$$

Using electromagnetic theory it is possible to derive boundary conditions that are generally valid for $\mathbf{B}$ and $\mathbf{H}$. These may be written in the following way:

$$B_r^I = B_r^{II} | r = R_i \tag{36}$$

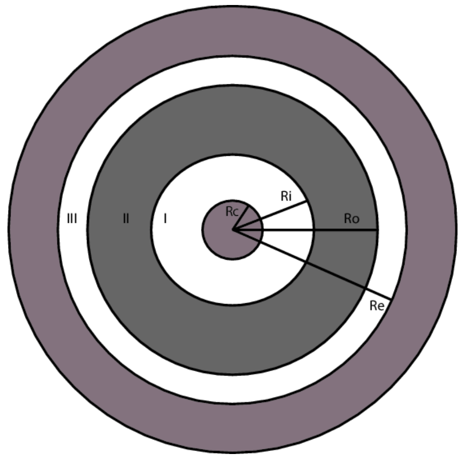



Figure 2: Drawing of the relevant domain for the solution. The Halbach magnet is placed in the grey area between $r = R_i$ and $r = R_o$.

$$
\begin{aligned}
B_r^{II} &= B_r^{III} | r = R_o & (37)\\
H_\theta^{I} &= 0 | r = R_c & (38)\\
H_\theta^{I} &= H_\theta^{II} | r = R_i & (39)\\
H_\theta^{II} &= H_\theta^{III} | r = R_o & (40)\\
H_\theta^{III} &= 0 | r = R_e & (41)
\end{aligned}
$$

The Roman numerals $I$, $II$ and $III$ refer to the three domains as shown in Fig. 2.

**Exercise 20**

> Find the components of the vector field for **B** and **H**, i.e. $B_r$, $B_\theta$, $H_r$ and $H_\theta$..

*Hint 1:* We know the $z$-component of the vector field **A**. Use this in combination with Eq. 29, 34 and 35 . *Hint 2:* First find the components of $\frac{\partial A_z}{\partial r}$ and $\frac{\partial A_z}{\partial \theta}$.

We now have four expressions, one for each of the non-zero components of **B** and **H**. Note that there is only a non-zero remanence in domain II, i.e. inside the magnetic material.

By considering the boundary conditions given in Eq. 36 and the expression for $B_r$ from Eq. 20 we obtain:

$$
\begin{aligned}
& \cos(\theta)\left(C^I + D^I R_i^{-2}\right) = \cos(\theta)\left(C^{II} + D^{II} R_i^{-2} - B_{\text{rem}} \ln(R_i)\right) \\
\Rightarrow\quad & C^I + D^I R_i^{-2} - C^{II} - D^{II} R_i^{-2} = -B_{\text{rem}} \ln(R_i) \qquad (42)
\end{aligned}
$$

**Exercise 21**

> Write a similar expression by considering the boundary conditions for **B** (Eq. 37).

**Exercise 22**

> Use the components of the vector field from Task 20 and the boundary conditions for $H_\theta$ to write similar expressions for $C^I$ and $C^{III}$.

*Hint:* Note that $\mu_r \neq 1$ on domain II and $\mu_r = 1$ on domains I and III.

**Exercise 23**

> Use the boundary conditions for $H_\theta$ at $r = R_c$ and $r = R_e$ to obtain two expressions (where $C^I$ and $D^I$ appear in one while $C^{III}$ and $D^{III}$ appear in the other).

We now have six equations with six unknowns: $C^I$, $C^{II}$, $C^{III}$, $D^I$, $D^{II}$, $D^{III}$. This can be written in a matrix:

$$\begin{pmatrix} 1 & -1 & 0 & R_i^{-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ & & & & \\ & & & & \\ & & & & \end{pmatrix} \begin{pmatrix} C^I \\ C^{II} \\ C^{III} \\ D^I \\ D^{II} \\ D^{III} \end{pmatrix} = \begin{pmatrix} -B_{\text{rem}} \ln(R_i) \\ \vdots \\ \\ \\ \\ \end{pmatrix} \tag{43}$$

**Exercise 24**

> Find the six constants. It is a great help to use Maple, e.g. matrix inversion or solve.

Maple's output can be quite messy. It is therefore imperative to check it. We postulate that the solution may be written as (for $p = 1$):

$$a = \frac{R_e^2 - R_o^2}{R_e^2 + R_o^2} \quad , \quad b = -\frac{R_i^2 - R_c^2}{R_i^2 + R_c^2} \tag{44}$$

$$D^{II} = -\left(\frac{a\mu_r - 1}{a\mu_r + 1} R_o^{-2} - \frac{\mu_r b - 1}{\mu_r b + 1} R_i^{-2}\right)^{-1} B_{\text{rem}} \ln\left(\frac{R_i}{R_o}\right) \tag{45}$$

$$C^I = \frac{D^{II}}{R_i^2 + R_c^2}\left(1 - \frac{\mu_r b - 1}{\mu_r b + 1}\right) \tag{46}$$

$$D^I = C^I R_c^2 \tag{47}$$

$$C^{II} = -D^{II} \frac{\mu_r a - 1}{\mu_r a + 1} R_o^{-2} + B_{\text{rem}} \ln(R_o) \tag{48}$$

$$C^{III} = \frac{D^{II}}{R_o^2 + R_e^2}\left(1 - \frac{\mu_r a - 1}{\mu_r a + 1}\right) \tag{49}$$

$$D^{III} = C^{III} R_e^2 \tag{50}$$

**Exercise 25**

Show that the solution you have found in Maple is identical to Eqs. 45-50.

## 4.3 A single Halbach cylinder in air

Considering a single Halbach cylinder in air/vacuum we need to modify the boundary conditions in the central bore ($r = R_c$) and on the outside of the magnet (when $r \to \infty$). In order to derive the expressions for the six constants above we had to assume an infinite permeability at the centre and outside the magnet. We will now let the radius of the central infinitely permeable cylinder go to zero and the outer cylinder's radius towards infinity, which in effect creates a Halbach magnet in air.

**Exercise 26**

Show that

$$\lim_{R_e \to \infty} a = 1 \tag{51}$$

and

$$\lim_{R_c \to 0} b = -1. \tag{52}$$

We can now find the magnetic field given $a$ and $b$ for a single Halbach magnet in air. Considering only permanent magnets with a small relative permeability, $\mu_r \approx 1$, then the expressions for the six constants may be simplified. Physically, this is a good approximation as the best permanent magnets based on iron (Fe), neodynium (Nd) and boron (B) have a relative permeability of $\mu_r = 1.05$.

**Exercise 27**

Find the limit of the constant $D^{II}$ when $\mu_r \to 1$.

**Exercise 28**

Find the limit of the constants $C^{II}$, $D^{III}$ and $C^{III}$ in the same limit.

**Exercise 29**

Find the limits of the constants $C^I$ and $D^I$ when $\mu_r \to 1$.

With the six constants simplified in the limit $\mu_r \to 1$ we can now obtain the components of the magnetic field. We encourage you do this on your own. We get:

$$
\begin{aligned}
B_r^I &= B_{\text{rem}} \ln\left(\frac{R_o}{R_i}\right)\cos(\theta) && (53)\\
B_\theta^I &= -B_{\text{rem}} \ln\left(\frac{R_o}{R_i}\right)\sin(\theta) && (54)\\
B_r^{II} &= B_{\text{rem}}\ln\left(\frac{R_o}{r}\right)\cos(\theta) && (55)\\
B_\theta^{II} &= -B_{\text{rem}}\left(\ln\left(\frac{R_o}{r}\right)-1\right)\sin(\theta). && (56)
\end{aligned}
$$

**Exercise 30**

Show graphically how the norm $B = \sqrt{B_r^2 + B_\theta^2}$ behaves. Assume that $B_{\text{rem}} = 1.4$ T. Try different values for $R_i$ and $R_o$.

*Hint:* Try out the `contour` function in Maple in cylindrical coordinates.

**Exercise 31**

Find the norm $B = \sqrt{B_r^2 + B_\theta^2}$ in the cylinder bore, i.e. domain I.

# 5 Halbach magnets and their applications

Let us consider actual applications of Halbach cylinders. We will first dimension a Halbach magnet for magnetic resonance imaging (MRI) and subsequently we will consider a magnetic coupling / force transfer between two Halbach magnets.

## 5.1 Halbach magnet for MRI

A Magnetic Resonance Imaging (MRI) scanner is a device that has medical and biological applications. The principle behind MRI is that the spins of the atomic nuclei precess with a certain frequency (the Larmor frequency), which is a function of field. In this way the various constituents of an (internal) organic body can be imaged. A large magnetic field is required for this operation.

Let us consider an MRI magnet made with a coil (a solenoid) and a Halbach cylinder for comparison. A coil is essentially a structure consisting of a wound wire that is electrically conducting.

**Exercise 32**

> Assume that a coil is a collection of stacked ring-shaped wires (Fig. 4). There are $N$ windings. In what direction does the magnetic field point inside the coil?

It turns out that the field generated inside the coil is constant and we denote its magnitude $B_0$. By using Eq. 22 we may find the field outside the coil. Consider this equation on integral form and that the coil is long and narrow. This gives, what is also known as Ampere's law:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \tag{57}$$

The current, $I_{\text{enc}}$, is the total current within the region bound by the integral.

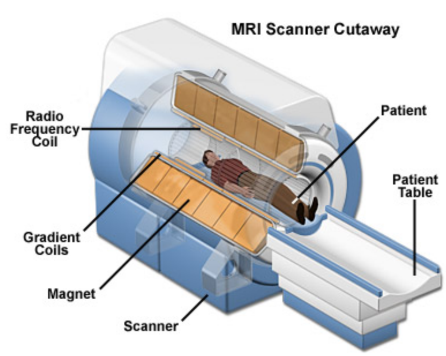
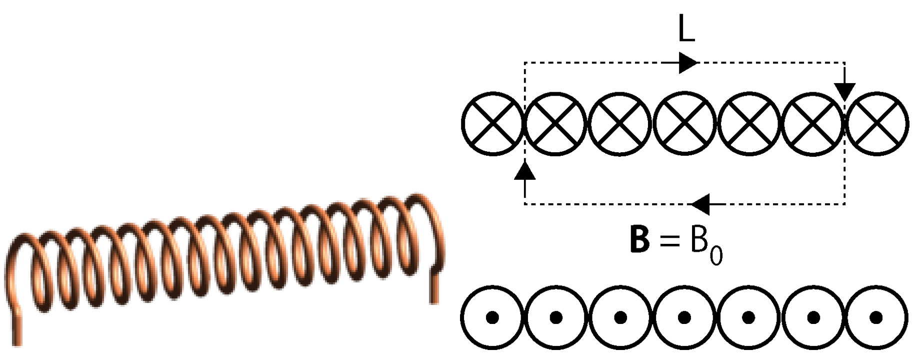



Figure 3: MR scanners are used by hospitals and biological scientific researc.




(a) (b)

Figure 4: A coil. The schematic of the coil is seen from the side, The cross indicates that the current runs into the page while a dot indicates that it runs out of the page.

**Exercise 33**

> Find the field produced by the coil using Ampere's law written as a function of the current in the coil, $I$, the number of windings, $N$ and the length of the coil, $L$.

*Hint*: The field outside the coil may be considered to be zero. How much current is contained by the curve defined in the line integral?

Consider an MRI scanner with a hole diameter of 60 cm and producing a field of $B_0 = 1.5$ T and has a length of 50 cm. Furthermore, let us assume that the remanence of a permanent magnet is $B_{\text{rem}} = 1.4$ T.

**Exercise 34**

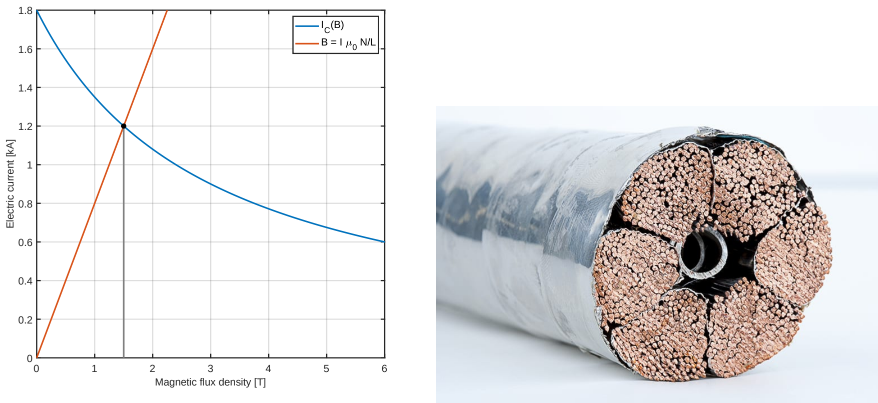
> How large should the Halbach cylinder be in order to produce this field?

We want to compare the Halbach cylinder to the solenoid.

**Exercise 35**

> Assuming that we make a solenoid of 5000 windings, how much current do we have to run through it to produce the desired field? Is this realistic?

We will now try to estimate if using the Halbach cylinder for an MRI-scanner is also an environmentally more friendly solution than the solenoid. We assume that we will make the solenoid of copper wire with a 1 mm diameter.




(a) (b)

Figure 5: Left panel: function $I_C(B)$ (blue curve) and load line (red curve); the operating point is indicated by the black dot. Right panel: superconducting wire.

**Exercise 36**

> Assuming again the 5000 windings, how long a copper wire do we need for the solenoid?

Unfortunately conducting a current through a wire results in heating of the wire due to its resistivity, an effect known as Joule heating. The resistivity of copper is $\rho = 1.68 * 10^{-8}\Omega m$. The total resistance of the copper wire is given as $R = \rho\frac{l}{A}$, where $l$ is the length of the wire and $A$ is its cross-sectional area. The Joule heating is given as $P = I^2 R$.

**Exercise 37**

> How much heating can we expect in the copper wire? Is this negligible?

Now we consider a superconducting solenoid made of superconducting wire to avoid the problem of resistance in the wire. In a superconductor below the critical current, there is zero resistance. The maximum current $I_C$ that we can run through a superconducting wire is a function of the magnetic flux density $B$ experienced by the wire: $I_C(B) = I_0/(1 + B/B_0)$, where $B_0 = 3$ T, and $I_0 = 1.8$ kA. We assume $B$ to be equal to the magnetic flux density in the bore of the solenoid. The $I_C(B)$ curve is shown in Fig: 5.

**Exercise 38**

> Using the result from exercise 33, express the magnetic field $B$ and the current $I_C$ as function of the variables $N$, $L$, $B_0$, $I_0$. If we

require $B = 1.5$ T, how many windings $N$ are necessary? (Find the expression of $N$ as function of the required flux density $B$, the length $L$ and the parameters $B_0$, and $I_0$. Then plug the values into the expression to find the numerical value)

# References


[1] K. Halbach, Design of permanent multipole magnets with oriented rare earth cobalt material, Nucl. Instrum Methods 169.