01019 Discrete Mathematics E22
Home assignment 2
To be handed in no later than Wednesday October 26 at 11:59pm

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Exercise A

Consider the language of predicate logic with two unary predicates S and H. Let $\mathcal{F}$ be the interpretation of this language given by:

- $\mathrm{dom}(\mathcal{F})$ = all human beings.
- $S^{\mathcal{F}} = \_$ is a student.
- $H^{\mathcal{F}} = \_$ is honest.

Translate the following sentences to predicate logic in the interpretation $\mathcal{F}$.

(a) All students are honest.

(b) All students are dishonest.

(c) Not all students are honest.

(d) Not all students are dishonest.

(e) Some students are honest.

(f) Some students are dishonest.

(g) No students are honest.

(h) No students are dishonest.

Exercise B (= problem 3, exercises 5th week)

Consider the interpretation $\mathcal{R}$ given by:

- $\mathrm{dom}(\mathcal{R})$ = the real numbers.

- $\cdot^{\mathcal{R}}$ = usual multiplication on the real numbers. Note that we use x^2 as an abbreviation for $x \cdot x$, x^3 for $x \cdot x \cdot x$, etc.
- $+^{\mathcal{R}}$ = usual addition on the real numbers.
- $=, <, >, \leq, \geq$ and $\neq$ have the usual meaning.
- $\mathbf{0}^{\mathcal{R}} = 0$.

Translate the formulas below to English in the interpretation $\mathcal{R}$ and determine which of those that become true. For each of them there must be given a short explanation *why* it is true/false.

1) $\forall x(x = x^2 \rightarrow x < \mathbf{0})$.
2) $\forall x(x > \mathbf{0} \rightarrow x^2 > x)$.
3) $\forall x(x = \mathbf{0} \vee \neg(x + x = x))$.
4) $\exists x \forall y(x > y)$.
5) $\forall x \forall y(x > y \rightarrow \exists z(x > z \wedge z > y))$.

Exercise C (= problem 3, exercises 6th week)

Let the interpretation $\mathcal{R}$ be given by, as usual:

- $\mathrm{dom}(\mathcal{R})$ = the real numbers.
- $\cdot^{\mathcal{R}}$ = usual multiplication on the real numbers. It is allowed to use x^2 as an abbreviation for $x \cdot x$, x^3 for $x \cdot x \cdot x$, etc.
- $+^{\mathcal{R}}$ = usual addition on the real numbers.
- $=, <, >, \leq, \geq$ and $\neq$ have the usual meaning.
- $\mathbf{0}^{\mathcal{R}} = 0$.
- $\mathbf{1}^{\mathcal{R}} = 1$.
- $\mathbf{I}^{\mathcal{R}} = \_$ is an integer.

Formalize the following propositions as formulas of first-order logic, and determine the truth value of each of them in $\mathcal{R}$. For each of them also provide a brief explanation *why* it is true/false (you do not have to provide an actual proof).

1) There exists a real number that is not an integer.
2) There exists a real number that is greater than all integers.
3) Every positive integer is the square of a negative real number.

Exercise D (= problem 6, exercises 6th week)

Determine which of the following formulas are valid. If a formula *is not valid*, you must provide a *countermodel*, i.e. an interpretation that makes the formula false. Usually you can construct a countermodel where the domain is the natural numbers, and where the predicate symbols then simply are interpreted fittingly (e.g. P could be interpreted as "_ is an even number" and Q as "_ is equal to _").

If the formula on the other hand *is valid*, you must provide a *linguistic argument* for it in the same way as on the slides from week 4 ("Let $\mathcal{F}$ denote an arbitrary interpretation. Then...").

a) $\exists x P(x) \to \forall x P(x)$.

b) $\forall x (P(x) \vee \neg P(x))$.

Exercise E (= problem 2, exercises 7th week)

Consider the following claims. For each of them you must use the tableau method to determine if they hold or not.

a) $\forall x (P(x) \to Q(x)), \forall x P(x) \models \forall x Q(x)$.

b) $\forall x (P(x) \vee Q(x)) \models \forall x P(x) \vee \forall x Q(x)$.

c) $\exists x \forall y P(x, y) \models \forall y \exists x P(x, y)$.

Exercise F (= problem 3, exercises 7th week)

1. A predicate logic formula is called **satisfiable**, if there exists an interpretation that makes it true. Explain how the tableau method can be used to determine if a predicate logic formula is satisfiable. *Hint:* What must the root formula of the tableau be? Then what does it mean if the tableau with this root formula closes?

2. Consider the following story about a barber:

   > Assume that there is a town with just a single male barber, and that each man in town himself makes sure to be shaved: some by shaving themselves, some by going to the barber's. It now seems reasonable to think that the barber satisfies the following rule: he shaves exactly those men that do not shave themselves.

   Briefly explain the problem of having a male barber defined by shaving exactly those men that do not shave themselves.

3. Let the be given a constant symbol b and a binary predicate symbol B. Besides, let there be given an interpretation $\mathcal{F}$ by:

- $\text{dom}(\mathcal{F})$ = all men in town.
- $b^{\mathcal{F}}$ = the barber.
- $B^{\mathcal{F}}$ is the predicate "_ shaves _".

Write a formula that in the given interpretation expresses the following:

"The barber shaves exactly those that do not shave themselves."

4. Show that the formula from the question is not satisfiable. That is, that there can be no male barber that shaves exactly those that do not shave themselves. Use the tableau method to show it (cf. the first question of the assignment).

Exercise G (= problem 6, exercises 7th week) Only for 3-person groups!!!

We consider first the interpretation over the domain of all human beings, where P is an unary predicate symbol interpreted as "_ is a girl" and K is a binary predicate symbol interpreted as $K = \{(x, y) \mid x \text{ knows } y\}$. Assume that we in this interpretation wish to formalize the proposition "All girls know someone." Here are two different takes on a formalization:

$$\forall x(P(x) \to \exists y K(x, y)).$$
$$\forall x \exists y(P(x) \to K(x, y)).$$

Use the tableau method to show that these two propositions are in fact logically equivalent, and thereby formalize the same proposition in any interpretation.

Notice: The result shows that the existential quantifier for y with no issues can be moved past the expression $P(x)$ which does not depend on y.