Danmarks Tekniske Universitet

DTU

Home assignment 2

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1 Exercise A

(a) All students are honest:
$\forall x(S(x) \to H(x))$

(b) All students are dishonest:
$\forall x(S(x) \to \neg H(x))$

(c) Not all students are honest:
$\neg\forall x(S(x) \to H(x))$

(d) Not all students are dishonest:
$\neg\forall x(S(x) \to \neg H(x))$

(e) Some students are honest:
$\exists x(S(x) \land H(x))$

(f) Some students are dishonest:
$\exists x(S(x) \land \neg H(x))$

(g) No students are honest:
$\neg\exists x(S(x) \land H(x))$

(h) No students are dishonest:
$\neg\exists x(S(x) \land \neg H(x))$

2 Exercise B

1. $\forall x(x = x^2 \to x < 0)$
   The above formula can be translated into the following:
   For all real numbers, if x is equal to itself raised to the power of 2 then x is negative.

   The formula is false because x can be equal to 1, which fulfills $x = x^2$, but 1 is not a negative number.

2. $\forall x(x > 0 \to x^2 > x)$
   The above formula can be translated into the following:
   For all real numbers, if x is positive then x to the power of 2 is always larger than x.

   The formula is false since x can be equal to either 1 or a given fraction. These numbers will never get bigger than themselves when they're raised to power of 2.

3. $\forall x(x = 0 \lor \neg(x + x = x))$
   The above formula can be translated into the following:
   For all real numbers, x is equal to zero or x is not equal to itself after being added together twice.

   The formula is true due to x either being 0 or a number that's not equal to itself twice. However, 0 is the only number that's equal to itself twice. This formula is also true because $x = 0$ is the only number that's equal to itself twice.

4. $\exists x \forall y(x > y)$
   The above formula can be translated into the following:
   There exists a real number x that's always larger than all other real numbers y.

   This formula is false because the x-value will always be chosen first. Therefore, it's always possible to choose a y-value that's larger than the chosen x-value.

   In essence, regardless of what x you choose, because the number line is infinite, then there will always exist a y, such that y = x + 1.

5. $\forall x \forall y(x > y \to \exists x(x > z \land z > y))$
   The above formula can be translated into the following:
   For all real numbers x and y, there exists a real number z which is in-between x and y if x is larger than y.

   This formula is true since it's always possible to find a number (fraction) that's in-between two given numbers x and y, when x is bigger than y. For example: $z = \frac{x+y}{2}$

3 Exercise C

Using the given interpretation **R** we have:

1. There exists a real number that is not an integer.
   The above sentence can be translated into the following:

   $$\exists x(\neg I(x))$$

   The formula is true because integers are only a subsets of real numbers. Hence, real numbers also consists of fractions. Thus a counter-model would be: $\frac{2}{3}$

2. There exists a real number that is greater than all integers.

The above sentence can be translated into the following:

$$\exists x \forall y (I(y) \rightarrow (x > y))$$

The formula is false because there is always going to be an integer grater than a certain real number. For instance, let $y = \lceil x \rceil + 1$, this value will always be bigger than any chosen value for x and it is a part of the domain.

3. Every positive integer is the square of a negative real number.
The above sentence can be translated into the following:

$$\forall x \exists y ((I(x) \wedge (x > 0)) \rightarrow (x = y^2 \wedge y < 0))$$

The formula is true because every positive integer can be expressed as the square of a negative real number; e.g. 1 can be expressed as $(-1)^2$, 3 as $(-\sqrt{3})^2$ and so on.

4 Exercise D

In order to determine which of the following formulas are valid we do the following:

1. $\exists x P(x) \rightarrow \forall x P(x)$

Validity:
The above formula is not valid.

Explanation:
Taking for example the set of real numbers **R** where $P(x) = x$ is an integer. It is easy to see that, if there exists a real number x that is an integer that does not mean that all the real number x are also integers.

Basically, for some proposition P(x), if it contains both evaluations that can yield true or false, then the statement becomes false.

2. $\forall x (P(x) \vee \neg P(x))$

Validity:
The above formula valid.

Explanation:
In every domain and for every instance of $P(\_)$ the formula can be translated to propositional logic:
$A \vee \neg A$
Which is known to be always true for every value assigned to A.

5 Exercise E

The following tableau are made:

1. $\forall x(P(x) \rightarrow Q(x)), \forall x P(x) \models \forall x Q(x)$

$$1\ \forall x(P(x) \rightarrow Q(x)) : \mathbf{T}$$
$$2\ \forall x P(x) : \mathbf{T}$$
$$3\ \forall x Q(x) : \mathbf{F}$$

$\forall F\ on\ 3$

$$4\ Q(c_0) : \mathbf{F}$$

$\forall T\ on\ 2$

$$5\ P(x) : \mathbf{T}$$
$$6\ P(c_0) : \mathbf{T}$$

$\forall T\ on\ 1$

$$7\ P(x) \rightarrow Q(x) : \mathbf{T}$$
$$8\ P(c_0) \rightarrow Q(c_0) : \mathbf{T}$$

$\rightarrow T\ on\ 7$

$$9\ P(x) : \mathbf{F} \quad \times \qquad\qquad 10\ Q(x) : \mathbf{T}$$

$\rightarrow T\ on\ 8$

$$11\ P(c_0) : \mathbf{F} \quad \times \qquad\qquad 12\ Q(c_0) : \mathbf{T} \quad \times$$

It can be seen that the claim holds since all branches in the tableau are closed.

2. $\forall x(P(x) \lor Q(x)) \models \forall x P(x), \forall x Q(x)$

$$
\begin{array}{c}
1\ \forall x(P(x) \lor Q(x)) : \mathbf{T} \\
2\ \forall x P(x) \lor \forall x Q(x) : \mathbf{F} \\
\quad | \ \lor F \ on\ 2 \\
3\ \forall x P(x) : \mathbf{F} \\
4\ \forall x Q(x) : \mathbf{F} \\
\quad | \ \forall F \ on\ 3 \\
5\ P(c_0) : \mathbf{F} \\
\quad | \ \forall F \ on\ 4 \\
6\ Q(c_1) : \mathbf{F} \\
\quad | \ \forall T \ on\ 1 \\
7\ P(c_0) \lor Q(c_0) : \mathbf{T} \\
8\ P(c_1) \lor Q(c_1) : \mathbf{T}
\end{array}
$$

$\lor T \ on\ 7$

$$
\begin{array}{cc}
9\ P(c_0) : \mathbf{T} & 10\ Q(c_0) : \mathbf{T} \\
\times &
\end{array}
$$

$\lor T \ on\ 8$

$$
\begin{array}{cc}
11\ Q(c_1) : \mathbf{T} & 12\ P(c_1) : \mathbf{T} \\
\times & \circ
\end{array}
$$

As it is shown in the tableau method, the claim does not hold because there's an open branch.

3. $\exists x \forall y P(x,y) \models \forall y \exists x P(x,y)$

$$
\begin{array}{c}
1\ \exists x \forall y P(x,y) : \mathbf{T} \\
2\ \forall y \exists x P(x,y) : \mathbf{F} \\
\quad | \ \exists T \ on\ 1 \\
3\ \forall y P(c_0,y) : \mathbf{T} \\
\quad | \ \forall F \ on\ 2 \\
4\ \exists x P(x,c_1) : \mathbf{F} \\
\quad | \ \forall T \ on\ 3 \\
5\ P(c_0,y) : \mathbf{T} \\
6\ P(c_0,x) : \mathbf{T} \\
7\ P(c_0,c_1) : \mathbf{T} \\
\quad | \ \exists F \ on\ 4 \\
8\ P(x,c_1) : \mathbf{F} \\
9\ P(y,c_1) : \mathbf{F} \\
10\ P(c_0,c_1) : \mathbf{F} \\
\times
\end{array}
$$

As it is shown in the tableau method, the claim holds because of the contradiction between the terms in line 10 and in line 8 which closes the branch.

6 Exercise F

1. In order to determine if a predicate logic formula is satisfiable, the tableau method can be used. By setting the root formula equal to true, it allows us to investigate the satisfiability of the formula. If all branches are closed, it means that there exists no interpretation of the formula that makes it true. Thus, the formula is not satisfiable. However, if a branch is open then there exists an interpretation of the formula that makes it true. Thereby making the formula satisfiable.

2. This leaves the question of who shaves the barber? If he only shaves those who do not shave themselves, then he cannot shave himself, as then he, the barber, shaves someone who does shave himself breaking the fundamental rule. Furthermore, if he does not shave himself, then there is no one else to shave him, hence he cannot remain freshly shaved, breaking the fundamental rule that all the men in the town are shaved.

3. The above situation can be translated into the following formula:
   $\forall x (B(b,x) \leftrightarrow \neg B(x,x))$

4. Using the tableau method on the above formula:

$$1\ \forall x(B(b,x) \leftrightarrow \neg B(x,x)) : \mathbf{T}$$

$$\forall T \text{ on } 1$$

$$2\ B(b,b) \leftrightarrow \neg B(b,b) : \mathbf{T}$$

$$\leftrightarrow T \text{ on } 2$$

$3\ B(b,b) : \mathbf{T}$	$5\ B(b,b) : \mathbf{F}$
$4\ \neg B(b,b) : \mathbf{T}$	$6\ \neg B(b,b) : \mathbf{F}$
$\neg T \text{ on } 4$	$\neg F \text{ on } 6$
$7\ B(b,b) : \mathbf{F}$	$8\ B(b,b) : \mathbf{T}$
$\times$	$\times$

It can be seen that the formula is not satisfiable because it does not have any open branches that could make it true.

7 Exercise G

We have to show that $\forall x(P(x) \rightarrow \exists y K(x,y))$ is logically equivalent to $\forall x \exists y(P(x) \rightarrow K(x,y))$, thus:

$$\forall x(P(x) \rightarrow \exists y K(x,y)) \equiv \forall x \exists y(P(x) \rightarrow K(x,y))$$

Using the tableau method we can analyze the equivalent formula:

$$\forall x(P(x) \rightarrow \exists y K(x,y)) \leftrightarrow \forall x \exists y(P(x) \rightarrow K(x,y))$$

Tableau method on next page.

$1\ \forall x(P(x) \rightarrow \exists y K(x,y)) \leftrightarrow \forall x \exists y(P(x) \rightarrow K(x,y)) : \mathbf{F}$

$\leftrightarrow F$ *on* 1

$2\ \forall x(P(x) \rightarrow \exists y K(x,y)) : \mathbf{F}$
$3\ \forall x \exists y(P(x) \rightarrow K(x,y)) : \mathbf{T}$

$\forall F$ *on* 2

$6\ P(c_1) \rightarrow \exists y K(c_1,y) : \mathbf{F}$

$\rightarrow F$ *on* 6

$7\ P(c_1) : \mathbf{T}$
$8\ \exists y K(c_1,y) : \mathbf{F}$

$\exists F$ *on* 8

$9\ K(c_1,c_1) : \mathbf{F}$

$\forall T$ *on* 3

$10\ \exists y(P(c_1) \rightarrow K(c_1,y)) : \mathbf{T}$

$\exists T$ *on* 10

$11\ P(c_1) \rightarrow K(c_1,c_2) : \mathbf{T}$
$12\ K(c_1,c_2) : \mathbf{F}$ (from 8)

$\rightarrow T$ *on* 11

$13\ P(c_1) : \mathbf{F}$ $\times$ | $14\ K(c_1,c_2) : \mathbf{T}$ $\times$

$4\ \forall x(P(x) \rightarrow \exists y K(x,y)) : \mathbf{T}$
$5\ \forall x \exists y(P(x) \rightarrow K(x,y)) : \mathbf{F}$

$\forall F$ *on* 5

$15\ \exists y(P(c_1) \rightarrow K(c_1,y)) : \mathbf{F}$

$\exists F$ *on* 15

$16\ P(c_1) \rightarrow K(c_1,c_1) : \mathbf{F}$

$\rightarrow F$ *on* 16

$17\ P(c_1) : \mathbf{T}$
$18\ K(c_1,c_1) : \mathbf{F}$

$\forall T$ *on* 4

$19\ P(c_1) \rightarrow \exists y K(c_1,y) : \mathbf{T}$

$\rightarrow T$ *on* 19

$20\ P(c_1) : \mathbf{F}$ $\times$ | $21\ \exists y K(c_1,y) : \mathbf{T}$

$\exists T$ *on* 21

$22\ K(c_1,c_2) : \mathbf{T}$
$23\ P(c_1) \rightarrow K(c_1,c_2) : \mathbf{F}$ (from 15)

$\rightarrow F$ *on* 23

$24\ P(c_1) : \mathbf{T}$
$25\ K(c_1,c_2) : \mathbf{F}$
$\times$

As can be seen, all branches of the tableau closes, and for that reason there is no falsifying assignment for the statement. Hence, the statement is valid, and as such, the two formulae are equivalent.