Exam simulation Part B

Jacopo Ceccuti S215158

Problem 1

Question 1

Given the system of differential equations:

$$\frac{d}{dx}\left(x_1(t)\right) = x_2(t)$$

$$\frac{d}{dx}\left(x_2(t)\right) = -9\,x_1(t) - 6\,x_2(t) + e^{i\cdot t}$$

We can rewrite it into the matrix form:

$$\begin{bmatrix} \frac{d}{dt}\left(x_1(t)\right) \\ \frac{d}{dt}\left(x_2(t)\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ e^{i\cdot t} \end{bmatrix}$$

If we now consider only to homogeneous system of differential equation we have:

$$\begin{bmatrix} \frac{d}{dt}\left(x_1(t)\right) \\ \frac{d}{dt}\left(x_2(t)\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

To find the solution to the homogeneous system we have to find the eigenvalues and eigenvectors of the system matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix}$$

These are found using the command:

```
A:=<0,1;-9,-6>
```

$$A := \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix}$$

```
Eigenvectors(A,output=list)
```

$$\left[\left[-3, 2, \left\{\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right\}\right]\right]$$

It can be read that the matrix **A** has a root $\lambda_1 = -3$ with an algebraic multiplicity of $p = 2$ and the geometric multiplicity, found as shown below, $q = 1$. The first found solution is:

$$\mathbf{x}_1(t) = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} e^{-3t}$$

```
M:=unapply(<0-lambda,1;-9,-6-lambda>,lambda):
'M(lambda)'=M(lambda)
```

$$M(\lambda) = \begin{bmatrix} -\lambda & 1 \\ -9 & -6-\lambda \end{bmatrix}$$

```
q:=Rank(M(lambda__1))
```

We are now in the case where $q < p$ so we must follow the theorem 2.11 to seek a second solution of the type:

$$\mathbf{x}_2(t) = \mathbf{b}_1 e^{-3t} + \mathbf{b}_2 t e^{-3t}$$

That after differentiation becomes:

$$\frac{d}{dt}\left(\mathbf{x}_2(t)\right) = -3\mathbf{b}_1 e^{-3t} + \mathbf{b}_2 e^{-3t} - 3\,\mathbf{b}_2 t e^{-3t}$$

Inserting this into the system we see that $\mathbf{x}_2$ is a solution if:

$$-3\mathbf{b}_1 e^{-3t} + \mathbf{b}_2 e^{-3t} - 3\,\mathbf{b}_2 t e^{-3t} = \mathbf{A}\mathbf{b}_1 e^{-3t} + \mathbf{A}\mathbf{b}_2 t e^{-3t}$$

That simplifies to:

$$-3\,\mathbf{b}_1 + \mathbf{b}_2 - 3\,\mathbf{b}_2 \mathrm{t} = \mathbf{A}\mathbf{b}_1 + \mathbf{A}\mathbf{b}_2 \mathrm{t}$$

If we take $\mathrm{t} = 0$ this leads to:

$$\mathbf{A}\mathbf{b}_1 = -3\,\mathbf{b}_1 + \mathbf{b}_2$$

$$\mathbf{A}\mathbf{b}_2 = -3\,\mathbf{b}_2$$

Form the second equation we can see that $\mathbf{b}_2$ must be the eignevector corresponding to the eigenvalue

$$\lambda_1 = -3. \;\; \mathbf{b}_2 = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

The first equation can be rewritten, also substituting $\mathbf{b}_2$, as:

$$(\mathbf{A} + 3\,\mathbf{I})\mathbf{b}_1 = \mathbf{b}_2$$

Which we can solve using maple to obtain the value of $\mathbf{b}_2$.

```
D:=(A-3*IdentityMatrix(2,2))
```

$$D := \begin{bmatrix} -3 & 1 \\ -9 & -9 \end{bmatrix}$$

```
b__1:=LinearSolve(D,b__2)
```

$$b_1 := \begin{bmatrix} \frac{1}{18} \\ -\frac{1}{6} \end{bmatrix}$$

Obtaining the vector $\mathbf{b}_1 = \begin{bmatrix} \frac{1}{18} \\ -\frac{1}{6} \end{bmatrix}$ and arriving at the full second solution

$$\mathbf{x}_2(t) = \begin{bmatrix} \frac{1}{18} \\ -\frac{1}{6} \end{bmatrix} e^{-3t} + \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} t e^{-3t}$$

The two solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent. Thus the system of differential equations has the general solution:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} e^{-3t} + c_2\begin{bmatrix} \frac{1}{18} \\ -\frac{1}{6} \end{bmatrix} e^{-3t} + c_2\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} te^{-3t} \text{ with } c_1, c_2 \in \mathbb{R}$$

Question 2

To find the solution to the inhomogeneous system use the suggested solution ansatz $\mathbf{x}_\mathrm{p}(t) = \mathbf{v}_\mathrm{p} e^{it}$ and insert it into the inhomogeneous system.

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\left(v_1(t)\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(v_2(t)\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \cdot \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ e^{i \cdot t} \end{bmatrix}$$

Now it is possible to solve the system and find the components of the vector $\mathbf{v}_\mathrm{p}$:

```
dsolve(diff(x__p,t)=A.x__p+u,[v__1,v__2])
```

$$\left\{v_1 = \frac{2}{25} - \frac{3\,\mathrm{I}}{50}, v_2 = \frac{3}{50} + \frac{2\,\mathrm{I}}{25}\right\}$$

```
v__p:=<2/25-3*I/50,3/50+2*I/25>
```

$$v_p := \begin{bmatrix} \frac{2}{25} - \frac{3\,\mathrm{I}}{50} \\ \frac{3}{50} + \frac{2\,\mathrm{I}}{25} \end{bmatrix}$$

Thus the particular solution is:

$$\mathbf{x}_\mathrm{p}(t) = \begin{bmatrix} \frac{2}{25} - \frac{3\,\mathrm{I}}{50} \\ \frac{3}{50} + \frac{2\,\mathrm{I}}{25} \end{bmatrix} e^{it}$$

Question 3

The general solution now it is easily found thanks to the theorem 1.20 adding up together the homogeneous solution and the particular one:

$$\mathbf{x}(t) = \mathbf{x}_{\mathbf{homo}}(t) + \mathbf{x}_\mathbf{p}(t)$$

```
x(t):=x__hom+x__p
```

$$x(t) := \begin{bmatrix} -\frac{c_1 e^{-3t}}{3} + \frac{c_2 e^{-3t}}{18} - \frac{c_1 t e^{-3t}}{3} + \left(\frac{2}{25} - \frac{3\,\mathrm{I}}{50}\right) e^{\mathrm{I}t} \\ c_1 e^{-3t} - \frac{c_2 e^{-3t}}{6} + c_1 t e^{-3t} + \left(\frac{3}{50} + \frac{2\,\mathrm{I}}{25}\right) e^{\mathrm{I}t} \end{bmatrix}$$

Problem 2

Question 1

Given the function $f : \mathbb{R} \Rightarrow \mathbb{R}$ defined by the infinite series:

$$f(t) = \sum_{n=1}^{\infty} \left(\frac{1}{n^{3/2}} (3 \sin(2\,nt) - 2\cos(nt)) \right)$$

We are asked to find its majorant series.

It is known that $|f_n(t)| \leq k_n$ with $k_n > 0$, taking the absolute values of the argument of the series we get:

$$\left| \frac{1}{n^{3/2}} (3 \sin(2\,nt) - 2\cos(nt)) \right| = \frac{1}{n^{3/2}} \cdot |3 \cdot 2 \sin(nt)\cos(nt) - 2\cos(nt)| = \frac{2}{n^{3/2}} \cdot |3 \sin(nt)\cos(nt) - \cos(nt)|$$

Now we can apply the property for the absolute value of a difference $|A - B| \leq |A| + |B|$ to our case:

$$\frac{2}{n^{3/2}} \cdot |3 \sin(nt)\cos(nt) - \cos(nt)| \leq \frac{2}{n^{3/2}} \left(\left| 3 \sin(nt)\cos(nt) \right| + \left| \cos(nt) \right| \right)$$

Knowing that $\sin(x)\cos(x)$ has its maximum value at $\frac{\pi}{4}$ we can write a new inequality:

$$\frac{2}{n^{3/2}} (|3 \sin(nt)\cos(nt)| + |\cos(nt)|) \leq \frac{2}{n^{3/2}} \left(3 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + 1 \right) = \frac{5}{n^{3/2}}$$

That is the k_n of the majorant series:

$$\sum_{n=1}^{\infty} \left(\frac{5}{n^{3/2}} \right)$$

Question 2

According to Weierstrass' M-test (theorem 5.33) the series is uniformly convergent if:

- $f_n(t)$ is defined in the interval $t \in \mathbb{R}$ for any fixed n; which is true since $\sin(nt)$ and $\cos(nt)$ are continuous functions.
- $\sum_{n=1}^{\infty} (f_n(t))$ has a convergent majorant series.

The majorant series has been found already, we are just required to prove that it is convergent to state that the series is uniformly convergent.

This can be done in different ways (integral test, quotient test, comparison theorem) but it is already demonstrated in the book that, for the case $\sum_{n=1}^{\infty} \left(\frac{1}{n^{\alpha}} \right)$ when $\alpha > 1$, the series converges. *See example 4.34 and theorem 4.33 (the integral test).*

Thus we can state that the series $\sum_{n=1}^{\infty} \left(\frac{1}{n^{3/2}} (3 \sin(2\,nt) - 2\cos(nt)) \right)$ is uniformly convergent.

Question 3

To determine if the function is continuous we need to use theorem (5.35), it requires:

- $f_n(t)$ to be continuous on the interval $t \in \mathbb{R}$ for any fixed n; which is true since the $\cos(nt)$ and $\sin(nt)$ are continuous functions.
- $\sum_{n=1}^{\infty} (f_n(t))$ has a convergent majorant series or it is uniformly convergent.

Se we now have all the informations needed to state that the function

$f(t) = \sum_{n=1}^{\infty} \left(\frac{1}{n^{3/2}} (3 \sin(2\, nt) - 2 \cos(nt)) \right)$ is continuous.

Question 4

In order to determine the value $N \in \mathbb{N}$ such that for all $t \in \mathbb{R}$

$$\left| f(t) - S_N(t) \right| \le 2 \cdot 10^{-3}$$

We need to use method II shown in corollary 4.35.

As we previously showed:

$$\frac{2}{n^{3/2}} \cdot |3 \sin(nt)\cos(nt) - \cos(nt)| \le \frac{2}{n^{3/2}} \left(\left| 3 \sin(nt)\cos(nt) \right| + \left| \cos(nt) \right| \right) \le \frac{5}{n^{3/2}}$$

$\forall\, t \in \mathbb{R}, \forall\, n \in \mathbb{N}$

so this is true for every term in the series.

Knowing this it is possible to state that:

$$\left| f(t) - S_N(t) \right| = \left| \sum_{n=1}^{\infty} \left(\frac{1}{n^{3/2}} (3 \sin(2\, nt) - 2 \cos(nt)) \right) - \sum_{n=1}^{N} \left(\frac{1}{n^{3/2}} (3 \sin(2\, nt) - 2 \cos(nt)) \right) \right|$$

$$= \left| \sum_{n=N+1}^{\infty} \left(\frac{1}{n^{3/2}} (3 \sin(2\, nt) - 2 \cos(nt)) \right) \right| \le \sum_{n=N+1}^{\infty} \left(\frac{5}{n^{3/2}} \right)$$

Now if we find an N that makes $\sum_{n=N+1}^{\infty} \left(\frac{5}{n^{3/2}} \right) \le 2 \cdot 10^{-3}$ true, the same N will also fulfill the required condition $\left| f(t) - S_N(t) \right| \le 2 \cdot 10^{-3}$.

Method 2 says:

$$S - \left(S_N + \int_{N+1}^{\infty} f(x)\, dx \right) \le f(N+1) \le \epsilon$$

Using maple we evaluate $f(N+1) \le \epsilon$ and find N:

```
evalf(solve(f(N+1)<=epsilon))
```

$$[183.2015749, \infty)$$

Thus $N = 184$. Now we evaluate the sum of $\sum_{n=1}^{N} \left(\frac{5}{n^{3/2}} \right) + \int_{N+1}^{\infty} \frac{5}{x^{3/2}}\, dx$ and find the approximate value for $\sum_{n=1}^{\infty} \left(\frac{5}{n^{3/2}} \right)$.

```
evalf(sum(f(n),n=1..N)+int(f(x),x=N+1..infinity))
```

$$13.06088188$$

Which we can check that $\sum_{n=1}^{N} \left(\frac{5}{n^{3/2}} \right) + \int_{N+1}^{\infty} \frac{5}{x^{3/2}}\, dx \le \epsilon$:

Check

```
> evalf(sum(5/n^(3/2),n=1..infinity))-evalf(sum(f(n),n=1..N)+int(f(x),x=N+1..infinity))<evalf(2*10^(-3))
```

$$0.00099486 < 0.002000000000$$

Appendix

Appendix

```
> restart:with(LinearAlgebra):with(plots):with(student)
  :restart:with(LinearAlgebra):with(plots):with(student):
```

Problem 1

```
> restart:with(LinearAlgebra):with(plots):with(student)
  :restart:with(LinearAlgebra):with(plots):with(student):
```

1)

```
> A:=<0,1;-9,-6>
```

$$A := \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \tag{1.1.1}$$

```
> Eigenvectors(A,output=list)
```

$$\left[\left[-3, 2, \left\{\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right\}\right]\right] \tag{1.1.2}$$

```
> lambda__1:=-3
```

$$\lambda_1 := -3 \tag{1.1.3}$$

```
> M:=unapply(<0-lambda,1;-9,-6-lambda>,lambda):
  'M(lambda)'=M(lambda)
```

$$M(\lambda) = \begin{bmatrix} -\lambda & 1 \\ -9 & -6-\lambda \end{bmatrix} \tag{1.1.4}$$

```
> q:=Rank(M(lambda__1))
```

$$q := 1 \tag{1.1.5}$$

```
> b__2:=<-1/3,1>
```

$$b_2 := \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} \tag{1.1.6}$$

```
> D:=(A-3*IdentityMatrix(2,2))
```

$$\mathrm{D} := \begin{bmatrix} -3 & 1 \\ -9 & -9 \end{bmatrix} \tag{1.1.7}$$

```
> b__1:=LinearSolve(D,b__2)
```

$$b_1 := \begin{bmatrix} \frac{1}{18} \\ -\frac{1}{6} \end{bmatrix} \tag{1.1.8}$$

```
> x__hom:=c__1*<-1/3,1>*exp(-3*t)+c__2*<1/18,-1/6>*exp(-3*t)+c__1*
  <-1/3,1>*t*exp(-3*t)
```

$$x_{hom} := \begin{bmatrix} -\frac{c_1 e^{-3t}}{3} + \frac{c_2 e^{-3t}}{18} - \frac{c_1 t e^{-3t}}{3} \\ c_1 e^{-3t} - \frac{c_2 e^{-3t}}{6} + c_1 t e^{-3t} \end{bmatrix} \quad (1.1.9)$$

2)

```
> u:=<0,exp(I*t)>
```

$$u := \begin{bmatrix} 0 \\ e^{It} \end{bmatrix} \quad (1.1.10)$$

```
> x__p:=<v__1,v__2>*exp(I*t)
```

$$x_p := \begin{bmatrix} e^{It} v_1 \\ e^{It} v_2 \end{bmatrix} \quad (1.1.11)$$

```
> diff(x__p,t)=A.x__p+u
```

$$\begin{bmatrix} I e^{It} v_1 \\ I e^{It} v_2 \end{bmatrix} = \begin{bmatrix} e^{It} v_2 \\ -9 e^{It} v_1 - 6 e^{It} v_2 + e^{It} \end{bmatrix} \quad (1.1.12)$$

```
> dsolve(diff(x__p,t)=A.x__p+u,[v__1,v__2])
```

$$\left\{ v_1 = \frac{2}{25} - \frac{3\,I}{50}, v_2 = \frac{3}{50} + \frac{2\,I}{25} \right\} \quad (1.1.13)$$

```
> v__p:=<2/25-3*I/50,3/50+2*I/25>
```

$$v_p := \begin{bmatrix} \frac{2}{25} - \frac{3\,I}{50} \\ \frac{3}{50} + \frac{2\,I}{25} \end{bmatrix} \quad (1.1.14)$$

```
> x__p:=v__p*exp(I*t)
```

$$x_p := \begin{bmatrix} \left(\frac{2}{25} - \frac{3\,I}{50}\right) e^{It} \\ \left(\frac{3}{50} + \frac{2\,I}{25}\right) e^{It} \end{bmatrix} \quad (1.1.15)$$

3)

```
> x(t):=x__hom+x__p
```

$$x(t) := \begin{bmatrix} -\frac{c_1 e^{-3t}}{3} + \frac{c_2 e^{-3t}}{18} - \frac{c_1 t e^{-3t}}{3} + \left(\frac{2}{25} - \frac{3\,I}{50}\right) e^{It} \\ c_1 e^{-3t} - \frac{c_2 e^{-3t}}{6} + c_1 t e^{-3t} + \left(\frac{3}{50} + \frac{2\,I}{25}\right) e^{It} \end{bmatrix} \quad (1.1.16)$$

Problem 2

```
> restart:with(LinearAlgebra):with(plots):with(student)
  :restart:with(LinearAlgebra):with(plots):with(student):
```

1)

2)

3)

4)

```
> f:=unapply(5/n^(3/2),n):
  'f(n)'=f(n)
```

$$f(n) = \frac{5}{n^{3/2}} \qquad \textbf{(1.2.1)}$$

```
> f:=unapply(5/x^(3/2),x):
  'f(x)'=f(x)
```

$$f(x) = \frac{5}{x^{3/2}} \qquad \textbf{(1.2.2)}$$

```
> epsilon:=2*10^(-3)
```

$$\epsilon := \frac{1}{500} \qquad \textbf{(1.2.3)}$$

```
> evalf(solve(f(N+1)<=epsilon))
> N:=184
```

$$N := 184 \qquad \textbf{(1.2.4)}$$

```
> evalf(sum(f(n),n=1..N)+int(f(x),x=N+1..infinity))
```

$$13.06088188 \qquad \textbf{(1.2.5)}$$

Check

```
> evalf(sum(5/n^(3/2),n=1..infinity))-evalf(sum(f(n),n=1..N)+int(f
  (x),x=N+1..infinity))<evalf(2*10^(-3))
```

$$0.00099486 < 0.002000000000 \qquad \textbf{(1.2.6)}$$