Mini-project 2

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Problem 1

In the project description, the following system of differential equations is given:

$$\frac{d}{dt}\begin{bmatrix} j \\ x \\ v \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & -\beta \end{bmatrix}\begin{bmatrix} j \\ x \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cdot u(t)$$

together with the equation:

$$u(t) = g_1[z(t) - x(t)] \text{ with } g_1 \in \mathbb{R}$$

We are asked to show how the system can be written considering the above equation. To simplify it we begin with substituting $u(t)$ in the above system.

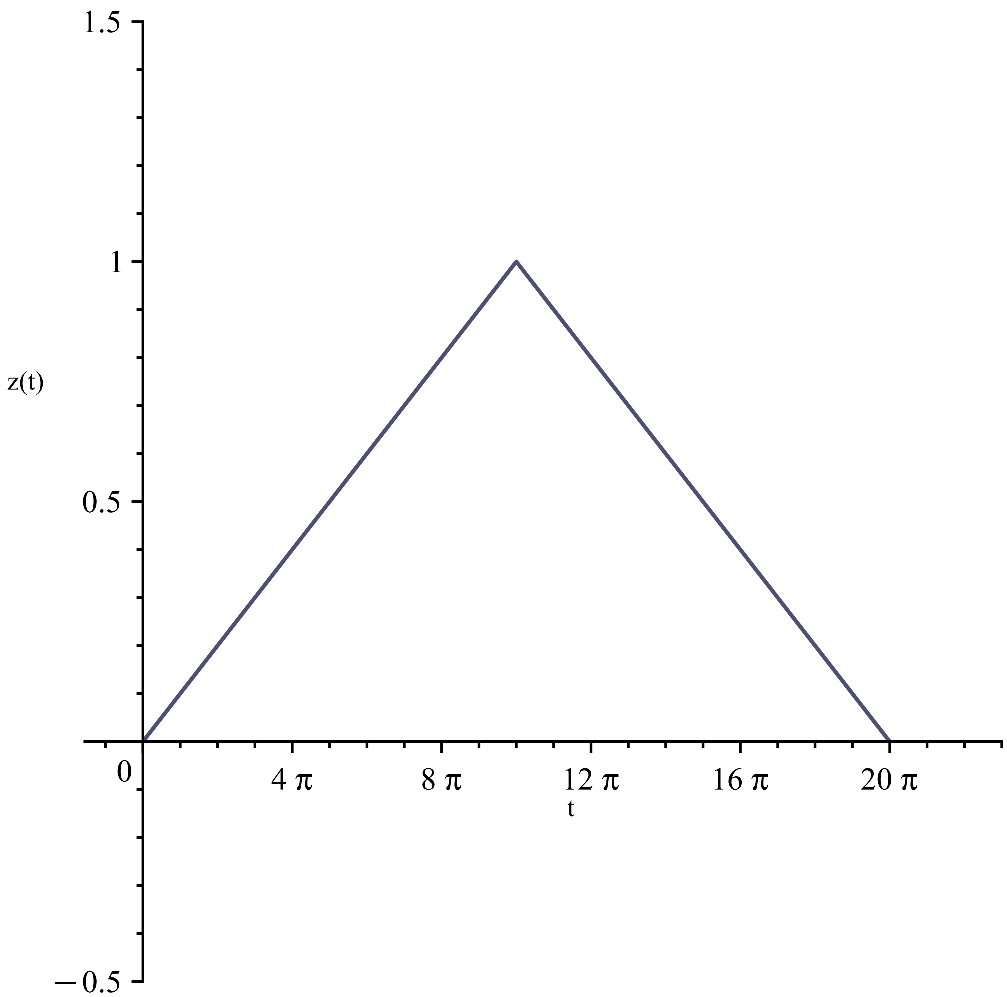
$$\frac{d}{dt}\begin{bmatrix} j \\ x \\ v \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & -\beta \end{bmatrix}\begin{bmatrix} j \\ x \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cdot\left(g_1 z(t) - g_1 x(t)\right)$$

Now the variable $x(t)$ is not multiplied by a zero vector anymore, so it is possible to write the constant term $-g_1$ inside the matrix and multiply the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ by the constant g_1 to obtain the system in the desired form:

$$\frac{d}{dt}\begin{bmatrix} j \\ x \\ v \end{bmatrix} = \begin{bmatrix} -\alpha & -g_1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & -\beta \end{bmatrix}\begin{bmatrix} j \\ x \\ v \end{bmatrix} + \begin{bmatrix} g_1 \\ 0 \\ 0 \end{bmatrix}\cdot z(t)$$

Problem 2

The plot for the piecewise given function $z(t)$ is evaluated at $T = 20\pi$ on the interval $t \in [0, T]$ and is the following:



Problem 3

We are given the trajectory $z(t)$ and its complex Fourier series, with the corresponding coefficient c_n with a $\omega = \frac{2\pi}{T}$. It is known form the previous assignment that $g_1 < (\alpha+\beta)(1+\alpha\beta)$ and now we know that $g_1 \in \mathbb{R}$, let's introduce:

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} j(t) \\ x(t) \\ v(t) \end{bmatrix} = x(t)$$

from which we can identify the vector $\mathbf{d} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$

In order to determine the transfer function $H(in\omega)$, we can use equation (2.35) and (2.39) from the textbook. Equation (2.39) tell us that the transfer function for a system of differential equations can be written as:

$H(in\omega) = -\mathbf{d}^{\mathrm{T}} \cdot (\mathbf{A} - in\omega \cdot \mathbf{I})^{-1} \cdot \mathbf{b}$

The transfer function variable, s, from equation (2.39) has been replaced with our variable, $in\omega$. The "i" in $H(in\omega)$ from the assignment has been replaced with (in the Maple input mode) "I" in order to be able to perform calculations with it. The different factors in equation (2.39) can be determined with equation (2.35):

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u(t) \\ y = \mathbf{d}^{\mathrm{T}}\mathbf{x} \end{cases}$$

Inserting all the values the transfer function $H(in\omega)$ becomes:

```
'H(I*n*omega)'=H(I*n*omega)
```

$$H(\mathrm{I}\,n\,\omega) = \frac{g_1}{\mathrm{I}\,\alpha\,\beta\,n\,\omega - \alpha\,n^2\,\omega^2 - \beta\,n^2\,\omega^2 - \mathrm{I}\,n^3\,\omega^3 + g_1 + \mathrm{I}\,n\,\omega}$$

And the denominator must be:

$$i\alpha\beta\,n\,\omega - \alpha\,n^2\,\omega^2 - \beta\,n^2\,\omega^2 - i\,n^3\,\omega^3 + g_1 + i\,n\,\omega \neq 0$$

Problem 4

The following complex Fourier series has been given for the function $z(t)$:

$$z(t) = \frac{1}{2} + \sum_{n=-\infty,\, n \neq 0}^{\infty} \left(\frac{(-1)^n - 1}{n^2 \cdot \pi^2} \cdot \mathrm{e}^{i \cdot n \cdot \omega \cdot t} \right)$$

By using Fourier's Method (Theorem 7.8), knowing that our system is asymptotically stable for $g_1 < (\alpha + \beta)(1 + \alpha\beta)$ we can conclude that the following is a solution given by the Fourier series:

$$y(t) = \sum_{n=-\infty}^{\infty} \left(c_n H(i \cdot n \cdot \omega) \cdot \mathrm{e}^{i \cdot n \cdot \omega \cdot t} \right)$$

In our case, c_n can be seen in $z(t)$ as:

$$c_n = \frac{(-1)^n - 1}{n^2 \cdot \pi^2}$$

Putting all together, with the previously found transfer function, we can say that the solution $x(t) = y(t)$ of the system using the Fourier series for can be written as:

$$y(t) = x(t) = \frac{1}{2} + \sum_{n=-\infty,\, n \neq 0}^{\infty} \left(\frac{(-1)^n - 1}{n^2 \cdot \pi^2} \cdot \frac{g_1}{i\alpha\beta\,n\,\omega - \alpha\,n^2\,\omega^2 - \beta\,n^2\,\omega^2 - i\,n^3\,\omega^3 + g_1 + i\,n\,\omega} \cdot \mathrm{e}^{in\omega t} \right), t \in \mathbb{R}$$

Problem 5

We have been given the following parameter values from problem 5:

$T = 20\pi \quad \alpha = 0.1 \quad \beta = 0.1 \quad g_1 = 0.2$

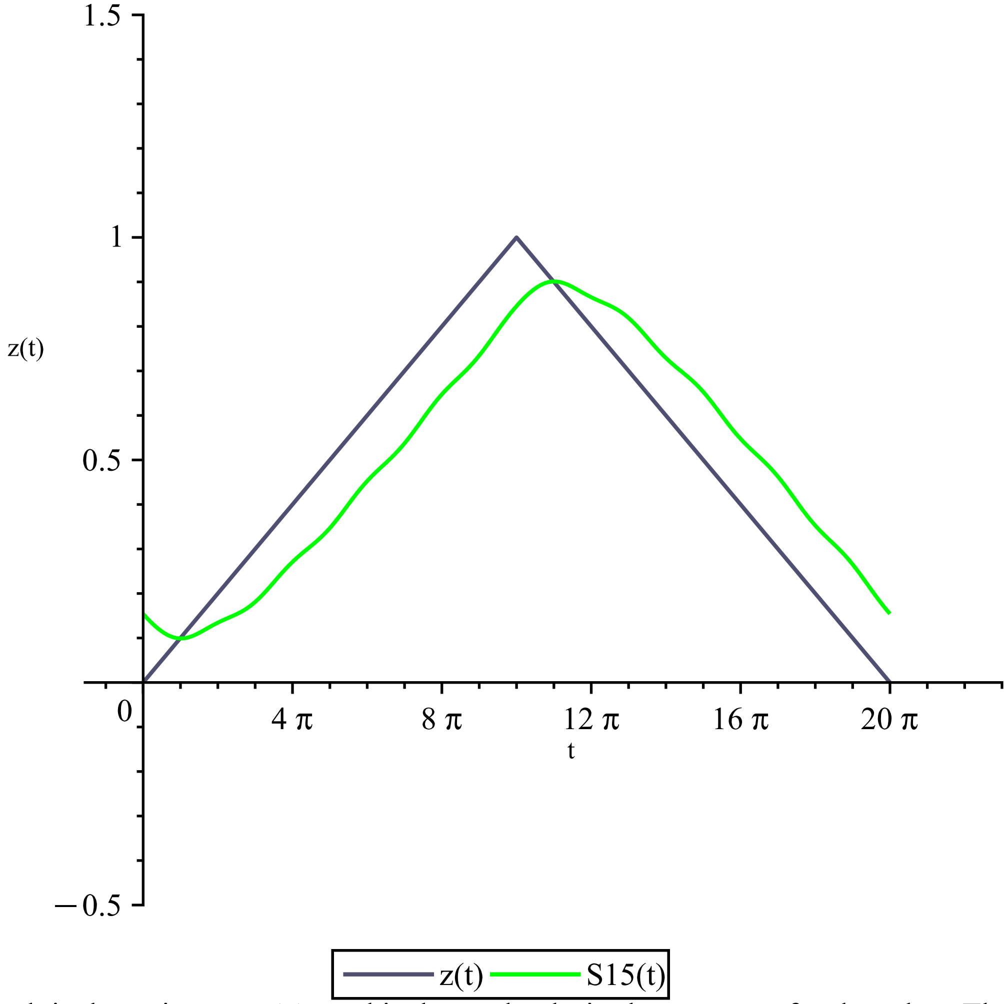
With the function $z(t)$:

$$z(t) = \begin{cases} \frac{2}{T} \cdot t & 0 \leq t < \frac{T}{2} \\ \frac{2}{T} \cdot (T - t) & \frac{T}{2} \leq t < T \end{cases}$$

And the N'th partial sum:

$$S_N^x(t) = \frac{1}{2} + \sum_{n=-N,\, n \neq 0}^{N} \left(\frac{(-1)^n - 1}{n^2 \cdot \pi^2} \cdot \frac{g_1}{i\alpha\beta n \omega - \alpha n^2 \omega^2 - \beta n^2 \omega^2 - i n^3 \omega^3 + g_1 + i n \omega} \cdot e^{in\omega t} \right)$$

Which can be plotted together in the interval $t \in [0, T]$.



The violet graph is the trajectory $z(t)$, and it shows the desired movement for the robot. The green graph is the partial sum $S_N^x(t)$ with $N = 15$, and it is an approximation of the actual movement of the robot. It can be notice a small delay and deviation in the actual movement of the robot but it is not that much of an error for such small *N*.

To determine if our system of differential equations is BIBO-stable, we can follow (Theorem 2.49) and (Theorem 2.38) from the textbook. Form (2.49) we know that if the associated homogeneous system to the inhomogeneous system of differential equations is asymptotically stable then the inhomogeneous system is BIBO-stable; while form (2.38) we know that a system is asymptotically stable if all of the eigenvalues for the system matrix **A** have a negative real parts.

Our homogeneous system is:

$$\frac{d}{dt}\begin{bmatrix} j \\ x \\ v \end{bmatrix} = \begin{bmatrix} -\alpha & -g_1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & -\beta \end{bmatrix} \begin{bmatrix} j \\ x \\ v \end{bmatrix}$$

After inserting the values we get the following system matrix A:

$$\mathbf{A} = \begin{bmatrix} -0.1 & -0.2 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & -0.1 \end{bmatrix}$$

Now we can use Maple to find the real part of its eigenvalues:

```
Re(Eigenvalues(A))
```

$$\begin{bmatrix} -0.000953069724444838 \\ -0.000953069724444838 \\ -0.198093860551110 \end{bmatrix}$$

Which can be seen to have all negative real part. This means that the our homogeneous system is asymptotically stable and our inhomogeneous system is BIBO-stable for the chosen values of α β and g_1.

Appendix

Problem 1

No calculations needed

Problem 2

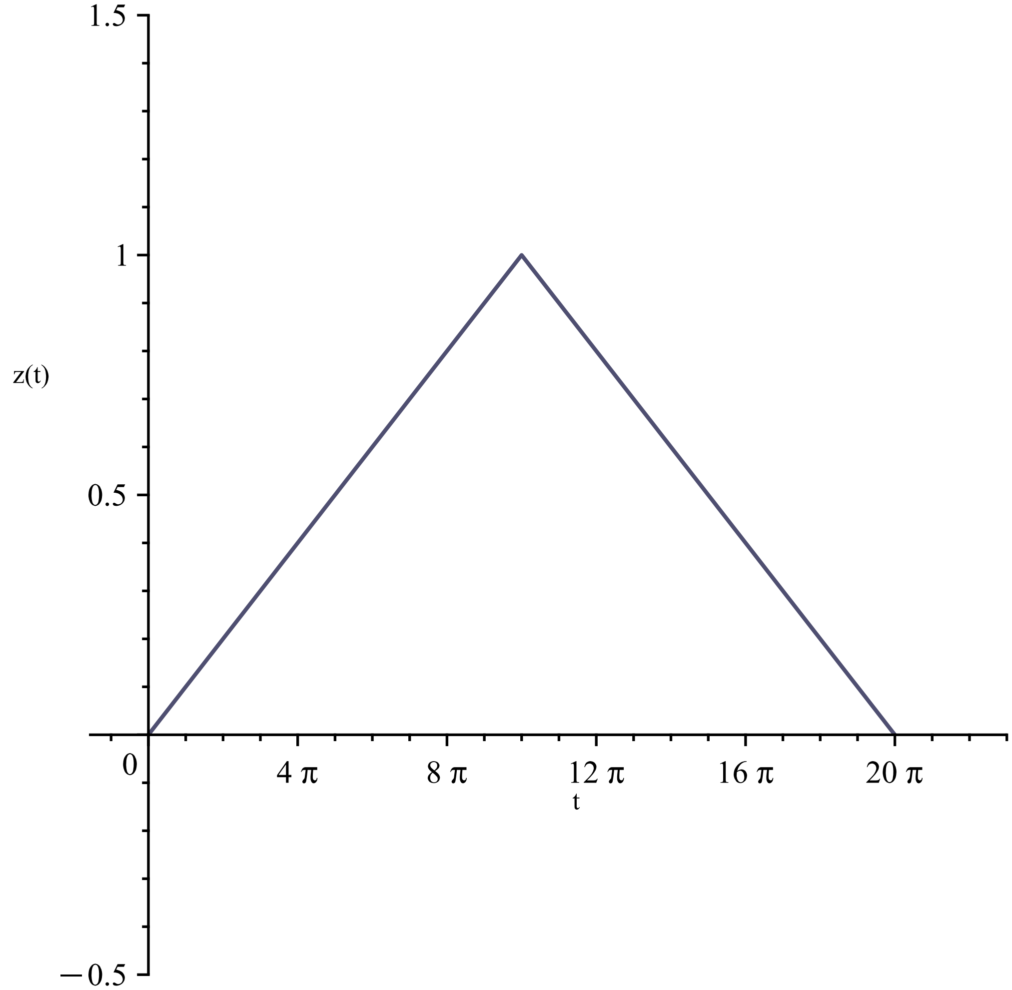
```
> restart:with(LinearAlgebra):with(plots):with(student):with
  (PDEtools):with(SolveTools:-Inequality):
> T:=20*Pi
```

$$T := 20\,\pi \tag{2.1}$$

```
> z:=t->piecewise(And(t>=0,t<T/2),2*t/T,And(t>=T/2,t<T),2/T*(T-t))
```

$$z := t \mapsto \begin{cases} \dfrac{2 \cdot t}{T} & 0 \leq t \wedge t < \dfrac{T}{2} \\ \dfrac{2 \cdot (T - t)}{T} & \dfrac{T}{2} \leq t \wedge t < T \end{cases} \tag{2.2}$$

```
> plot(z(t),t=0..T,labels=["t","z(t)"],view=[-5..23*Pi,-0.5..1.5],
  color="Nautical GrayViolet")
```



Problem 3

```
> assume(g__1,real)
> A:=<-alpha,-g__1,-1;0,0,1;1,0,-beta>
```

$$A := \begin{bmatrix} -\alpha & -g_1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & -\beta \end{bmatrix} \qquad (3.1)$$

```
> d:=<0,1,0>
```

$$d := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (3.2)$$

```
> b:=<g__1,0,0>
```

$$b := \begin{bmatrix} g_1 \\ 0 \\ 0 \end{bmatrix} \qquad (3.3)$$

```
> H:=unapply(-Transpose(d).MatrixInverse((A-s*IdentityMatrix(3))).
  b,s)
```

$$H := s \mapsto \frac{g_1}{\alpha \cdot \beta \cdot s + \alpha \cdot s^2 + \beta \cdot s^2 + s^3 + g_1 + s} \qquad (3.4)$$

```
> solve(alpha*beta*s+alpha*s^2+beta*s^2+s^3+g__1+s<>0,s) assuming
  g__1<(alpha+beta)*(1+alpha*beta), alpha and beta in real:
> 'H(I*n*omega)'=H(I*n*omega)
```

$$H(\mathrm{I}\, n\, \omega) = \frac{g_1}{\mathrm{I}\, \alpha\, \beta\, n\, \omega - \alpha\, n^2\, \omega^2 - \beta\, n^2\, \omega^2 - \mathrm{I}\, n^3\, \omega^3 + g_1 + \mathrm{I}\, n\, \omega} \qquad (3.5)$$

Problem 4

```
> c__n:=((-1)^n-1)/(n^2*Pi^2)
```

$$c_n := \frac{(-1)^n - 1}{n^2\, \pi^2} \qquad (4.1)$$

```
> 'xy(t)'=1/2+Sum(c__n*H(I*n*omega)*exp(I*n*omega*t),n=-infinity..
  infinity) assuming n<>0
```

$$xy(t) = \frac{1}{2} + \left(\sum_{n=-\infty}^{\infty} \frac{\left((-1)^n - 1 \right) g_1\, \mathrm{e}^{\mathrm{I}\, n\, \omega\, t}}{n^2\, \pi^2 \left(\mathrm{I}\, \alpha\, \beta\, n\, \omega - \alpha\, n^2\, \omega^2 - \beta\, n^2\, \omega^2 - \mathrm{I}\, n^3\, \omega^3 + g_1 + \mathrm{I}\, n\, \omega \right)} \right) \qquad (4.2)$$

Problem 5

```
> omega:=2*Pi/T
```

$$\omega := \frac{1}{10} \qquad (5.1)$$

```
> alpha:=0.1
```

$$\alpha := 0.1 \tag{5.2}$$

```
> beta:=0.1
```

$$\beta := 0.1 \tag{5.3}$$

```
> g__1:=0.2
```

$$g_1 := 0.2 \tag{5.4}$$

```
> N:=15
```

$$N := 15 \tag{5.5}$$

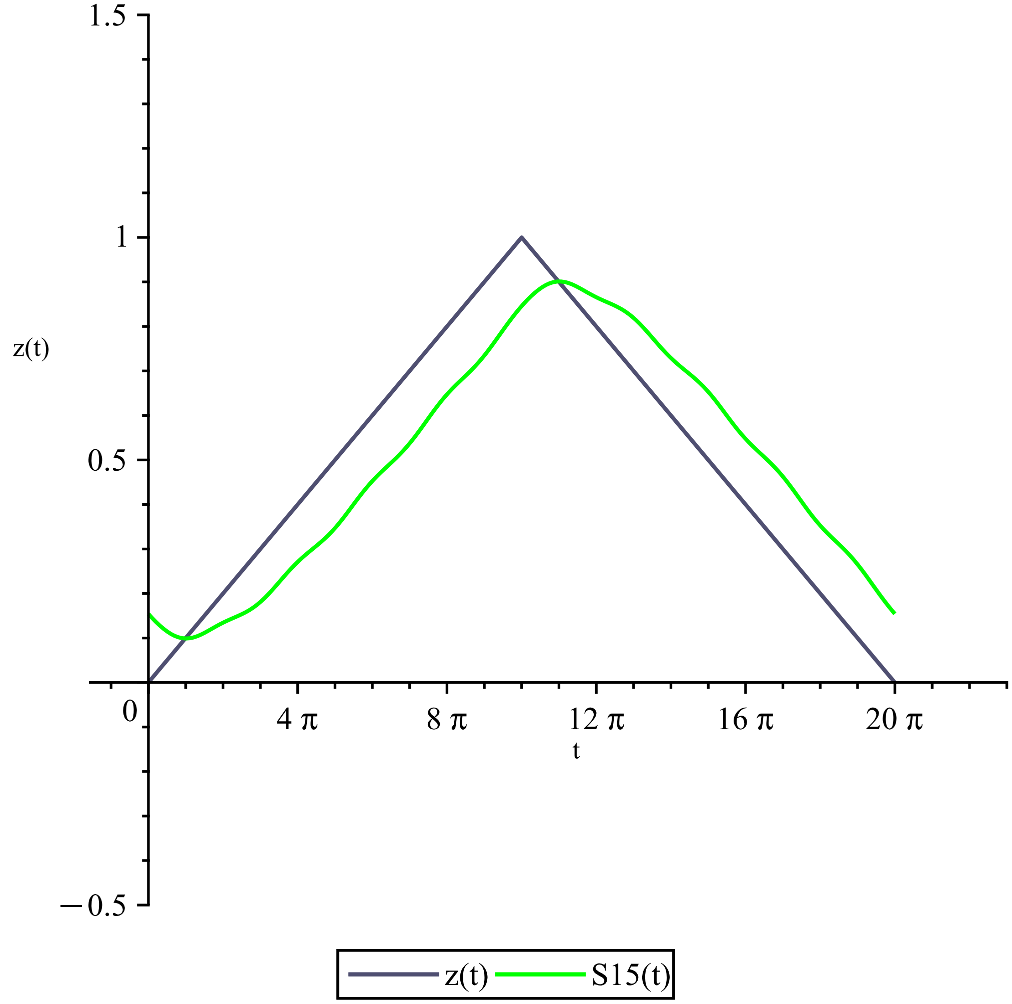
```
> S:=(n,t)->piecewise(n=0,1/2,((-1)^n - 1)*g__1*exp(n*omega*t*I)/
  (n^2*Pi^2*(alpha*beta*n*omega*I - alpha*n^2*omega^2 - beta*n^2*
  omega^2 - n^3*omega^3*I + g__1 + n*omega*I)))
```

$$S := (n, t) \mapsto \begin{cases} \frac{1}{2} & n = 0 \\ \frac{\left((-1)^n - 1\right) \cdot g_1 \cdot e^{I \cdot n \cdot \omega \cdot t}}{n^2 \cdot \pi^2 \cdot \left(I \cdot \alpha \cdot \beta \cdot n \cdot \omega - \alpha \cdot n^2 \cdot \omega^2 - \beta \cdot n^2 \cdot \omega^2 - I \cdot n^3 \cdot \omega^3 + g_1 + I \cdot n \cdot \omega\right)} & \mathit{otherwise} \end{cases} \tag{5.6}$$

```
> S15:=t->simplify(sum(S(n,t),n=-N..N))
```

$$S15 := t \mapsto \mathit{simplify}\left(\sum_{n=-N}^{N} S(n, t)\right) \tag{5.7}$$

```
> plot1:=plot(z(t),t=0..T,color="Nautical GrayViolet",legend="z(t)
  "):
  plot2:=plot(S15(t),t=0..T,color="green",legend="S15(t)"):
  display(plot1,plot2,labels=["t","z(t)"],view=[-5..23*Pi,-0.5.
  .1.5])
```




```
> restart:with(LinearAlgebra):with(plots):with(student):with
  (PDEtools):with(SolveTools:-Inequality):
> alpha:=0.1
```

$$\alpha := 0.1 \quad (5.8)$$

```
> beta:=0.1
```

$$\beta := 0.1 \quad (5.9)$$

```
> g__1:=0.2
```

$$g_1 := 0.2 \quad (5.10)$$

```
> A:=<-alpha,-g__1,-1;0,0,1;1,0,-beta>
```

$$A := \begin{bmatrix} -0.1 & -0.2 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & -0.1 \end{bmatrix} \quad (5.11)$$

```
> Re(Eigenvalues(A))
```

$$\begin{bmatrix} -0.000953069724444838 \\ -0.000953069724444838 \\ -0.198093860551110 \end{bmatrix} \quad (5.12)$$