Mini-project 1

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Section 1.2

1)
We are asked to eliminate $j(t)$ from the system (1)-(2) and show that it can be written as:
$$\frac{d^3}{dt^3}(x(t)) + (\alpha+\beta)\frac{d^2}{dt^2}(x(t)) + (1+\alpha\beta)\frac{d}{dt}(x(t)) = u(t)$$
We can proceede taking the derivative of (1) to get:
$$\frac{d^3}{dt^3}(x(t)) + \alpha\frac{d^2}{dt^2}(x(t)) = \frac{d}{dt}(j(t))$$
And substitute this result in (2) together with the initial expression for $j(t)$ to get:
$$\frac{d^3}{dt^3}(x(t)) + (\alpha+\beta)\frac{d^2}{dt^2}(x(t)) + (1+\alpha\beta)\frac{d}{dt}(x(t)) = u(t)$$
See Appendix for all the intermediate steps.

2)
Now we have to find the transfer function for (3) with $u(t) = e^{st}$. The first step is to substitute the value for $u(t)$ in (3) and then, according to Example 1.24 (and following theorems) from the textbook, in order to find the transfer function H(s) for (3) we have to set: $x(t) = H(s)e^{st}$ and evaluate the equation.
After factorinzing out $H(s)e^{st}$ and simplifying we are left with the equation for $H(s)$:
$$H(s) = \frac{1}{s^3 + (\alpha+\beta)s^2 + (1+\alpha\beta)s}$$
See Appendix for all the intermediate steps.

It is important to find the domain of this function for which its value is defined, to do so we set the denominator different from 0:

```
solve(s^3+(alpha+beta)*s^2+(alpha*beta+1)*s<>0,s)
```
$$\{s \neq 0\}, \left\{s \neq -\frac{\alpha}{2} - \frac{\beta}{2} + \frac{\sqrt{\alpha^2 - 2\alpha\beta + \beta^2 - 4}}{2}\right\}, \left\{s \neq -\frac{\alpha}{2} - \frac{\beta}{2} - \frac{\sqrt{\alpha^2 - 2\alpha\beta + \beta^2 - 4}}{2}\right\}$$

To avoid cases of division by zero.

3)
To show that the general real solution when $u(t) = \sin(2t)$ can be written in that form (4) we firstly need to find the solutions to the associated homogeneous system:
$$\frac{d^3}{dt^3}(x(t)) + (\alpha+\beta)\frac{d^2}{dt^2}(x(t)) + (1+\alpha\beta)\frac{d}{dt}(x(t)) = 0$$
See Appendix for all the intermediate steps.
To do so we follow Theorem 1.6 in the textbook and set $x(t) = e^{\lambda t}$, this allows us to find the

characteristic polynomial:
$P(\lambda) = s^3 + (\alpha+\beta)s^2 + (1+\alpha\beta)s$
And finding its roots helps us with the seeking of general real solution for the homogeneous differential equation.

```
solve(P=0,lambda)
```

$$0, -\frac{\beta}{2}-\frac{\alpha}{2}+\frac{\sqrt{\alpha^2-2\alpha\beta+\beta^2-4}}{2}, -\frac{\beta}{2}-\frac{\alpha}{2}-\frac{\sqrt{\alpha^2-2\alpha\beta+\beta^2-4}}{2}$$

These three roots are the values of lambda that we will use to compose the general real solution, but first...

From the premises of the exercise we know that $\alpha << 1$, $\beta << 1$ and $|\alpha-\beta| < 2$.
Looking at these relations it's easy to tell that in any case the argument of the square root will be always negative (e.g. The -4 at the end surely makes it become negative). Knowing so it is possible to factor out a -1 (that will be written as i^2, since $-1 = i^2$) and rewrite the solutions in the form:

$\lambda_1 = 0$

$$\lambda_{1\,\&\,2} = -\frac{\beta}{2}-\frac{\alpha}{2} \pm i\cdot\frac{\sqrt{-\alpha^2+2\,\alpha\beta-\beta^2+4}}{2}$$

Here we notice that the argument of the square root have become always positive and the general real solution to the homogeneous equation can be written taking the real part of the complex roots as follows:
$\mathrm{Re}(e^{\lambda t}) = e^{at}\cos(\omega\cdot t)$ & $\mathrm{Im}(e^{\lambda t}) = e^{at}\sin(\omega\cdot t)$
Thus the general real solution to the homogeneous equation is:

$$x_{homo}(t) = c_1 e^{0\,t} + c_2 e^{\left(-\frac{\beta}{2}-\frac{\alpha}{2}\right)t}\cos\left(\frac{1}{2}\sqrt{-\alpha^2+2\,\alpha\beta-\beta^2+4}\cdot t\right) + c_3 e^{\left(-\frac{\beta}{2}-\frac{\alpha}{2}\right)t}\sin\left(\frac{1}{2}\sqrt{-\alpha^2+2\,\alpha\beta-\beta^2+4}\cdot t\right)$$

The last step is to find the particular real solution to the inhomogeneous equation when $u(t) = \sin(2\,t)$.
This can be done by using our transfer function defined in the previous exercise where s is replaced with $2\cdot i$ (from $e^{2\cdot i\cdot t} = \cos(2\,t) + i\cdot\sin(2\,t)$). With inserted values, we get:

```
evalc(Im(H(2*I)*exp(2*I*t)))
```

$$-\frac{(\alpha\beta-3)\cos(2t)}{2\left((2\alpha+2\beta)^2+(\alpha\beta-3)^2\right)}-\frac{(2\alpha+2\beta)\sin(2t)}{2\left((2\alpha+2\beta)^2+(\alpha\beta-3)^2\right)}$$

Where the values of A and B can be seen.
We finally can answer the question stating the general real solution and the we the values of:

$$a = -\frac{\beta}{2}-\frac{\alpha}{2}$$

$$\frac{1}{2}\omega = \frac{1}{2}\sqrt{-\alpha^2+2\,\alpha\beta-\beta^2+4}$$

$$A = -\frac{(\alpha\beta-3)\cos(2\,t)}{2\left((2\,\alpha+2\,\beta)^2+(\alpha\beta-3)^2\right)}$$

$$B = -\frac{(2\,\alpha + 2\,\beta)\sin(2\,t)}{2\left((2\,\alpha + 2\,\beta)^2 + (\alpha\beta - 3)^2\right)}$$

General real solution:

$$x(t) = x_{homo}(t) + x_p(t) = c_1 e^{0\,t} + c_2 e^{\left(-\frac{\beta}{2} - \frac{\alpha}{2}\right)t} \cos\left(\frac{1}{2}\sqrt{-\alpha^2 + 2\,\alpha\beta - \beta^2 + 4}\cdot t\right)$$
$$+ c_3 e^{\left(-\frac{\beta}{2} - \frac{\alpha}{2}\right)t} \sin\left(\frac{1}{2}\sqrt{-\alpha^2 + 2\,\alpha\beta - \beta^2 + 4}\cdot t\right) - \frac{(\alpha\beta - 3)\cos(2\,t)}{2\left((2\,\alpha + 2\,\beta)^2 + (\alpha\beta - 3)^2\right)}\cos(2\,t)$$
$$- -\frac{(2\,\alpha + 2\,\beta)\sin(2\,t)}{2\left((2\,\alpha + 2\,\beta)^2 + (\alpha\beta - 3)^2\right)}\sin(2\,t)$$

4)

We are asked to substitute $v(t) = \frac{d}{dt}(x)$ into the system (1)-(2) and rewrite it as 3 coupled linear differential equations in the form of (2.35) in the textbook.

The 3 coupled linear differential equations obtained substituting $v(t) = \frac{d}{dt}(x)$ and rearranging are:

$$\frac{d}{dt}(x) = v(t)$$
$$\frac{d}{dt}(v) = -\alpha\cdot v(t) + j(t)$$
$$\frac{d}{dt}(j) = -v(t) - \beta\cdot j(t) + u(t)$$

Now we can express them in the form (2.35)

$$\begin{bmatrix} \frac{d}{dt}(x) \\ \frac{d}{dt}(v) \\ \frac{d}{dt}(j) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\alpha & 1 \\ 0 & -1 & -\beta \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ v(t) \\ j(t) \end{bmatrix} + u(t)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From here we see that:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\alpha & 1 \\ 0 & -1 & -\beta \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If $y(t) = x(t)$ is the solution sought, y(t) is the first coordinate of the solution $\boldsymbol{x}(t)$. This means that:

$y(t) = x(t) = 1\cdot x(t) + 0\cdot v(t) + 0\cdot j(t) = \boldsymbol{d}^T\boldsymbol{x}(t)$ where

$$\boldsymbol{d} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Section 1.3

5)

In order to prove that $f(t)$ satisfies the differential equation (6) we use:

- The given equation for the voltage: $u(t) = g_1 \cdot f(t) + g_2 \cdot \frac{d}{dt}(f(t))$
- The function for $f(t)$ which represents the distance between $z(t)$ and $x(t)$: $f(t) = z(t) - x(t)$

The above equation can be written as $x(t) = z(t) - f(t)$, which is helpful together with the other function (3) for the voltage:

$$u(t) = \frac{d^3}{dt^3}(x(t)) + (\alpha + \beta)\frac{d^2}{dt^2}(x(t)) + (1 + \alpha\beta)\frac{d}{dt}(x(t))$$

Equation (5) will become our right-end side, while the left hand side is given by (3) after the substitution of $x(t) = z(t) - f(t)$:

Maple manipulations are not present as they are way more complex than just doing it by hand.

$$\frac{d^3}{dt^3}(z(t)) - \frac{d^3}{dt^3}(f(t)) + (\alpha + \beta)\left(\frac{d^2}{dt^2}(z(t)) - \frac{d^2}{dt^2}(f(t))\right) + (1 + \alpha\beta)\left(\frac{d}{dt}(z(t)) - \frac{d}{dt}(f(t))\right) = g_1 \cdot f(t) + g_2 \cdot \frac{d}{dt}(f(t))$$

Now we can simplify the expression by gathering all the terms with $f(t)$ on the left side and the ones with $z(t)$ on the other:

$$\frac{d^3}{dt^3}(z(t)) - \frac{d^3}{dt^3}(f(t)) + \alpha\frac{d^2}{dt^2}(z(t)) - \alpha\frac{d^2}{dt^2}(f(t)) + \beta\frac{d^2}{dt^2}(z(t)) - \beta\frac{d^2}{dt^2}(f(t)) + \frac{d}{dt}(z(t)) - \frac{d}{dt}(f(t)) + \alpha\beta\frac{d}{dt}(z(t)) - \alpha\beta\frac{d}{dt}(f(t)) = g_1 \cdot f(t) + g_2 \cdot \frac{d}{dt}(f(t))$$

Moving the terms and changing sign:

$$g_1 \cdot f(t) + g_2 \cdot \frac{d}{dt}(f(t)) + \frac{d^3}{dt^3}(f(t)) + \alpha\frac{d^2}{dt^2}(f(t)) + \beta\frac{d^2}{dt^2}(f(t)) + \frac{d}{dt}(f(t)) + \alpha\beta\frac{d}{dt}(f(t)) = \frac{d^3}{dt^3}(z(t)) + \alpha\frac{d^2}{dt^2}(z(t)) + \beta\frac{d^2}{dt^2}(z(t)) + \frac{d}{dt}(z(t)) + \alpha\beta\frac{d}{dt}(z(t))$$

Factoring out common terms:

$$\frac{d^3}{dt^3}(f(t)) + (\alpha + \beta)\frac{d^2}{dt^2}(f(t)) + (1 + \alpha\beta + g_2)\frac{d}{dt}(f(t)) + g_1 \cdot f(t) = \frac{d^3}{dt^3}(z(t)) + (\alpha$$

$$+ \beta)\frac{d^2}{dt^2}(z(t)) + (1 + \alpha\beta)\frac{d}{dt}(z(t))$$

Doing so we have proven that $f(t)$ satisfies the differential equation (6)

6)

To show that the differential equation (6) is asymptotically stable we have to follow the Routh-Hurwitz' criterion, Theorem 2.41 in the textbook. It basically state that to find out if the system is asymptotically stable we need to have all the coefficients in the characteristic equation positive and all $k \cdot k$ determinants positive (in case the characteristic equation is of degree 2 and above).

It is possible to analyze only the homogeneous part of (6) because, according to Theorem 2.47, an inhomogeneous system is asymptotically stable if and only if the associated homogeneous system is asymptotically stable.

The characteristic polynomial is easily readable for (6):

$$P(\lambda) = \lambda^3 + (\alpha + \beta)\lambda^2 + (1 + \alpha\beta + g_2)\lambda + g_1$$

According to the Theorem 2.41 for a 3rd degree polynomial we must have:

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0 \text{ and } det\left(\begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix}\right) > 0$$

That in our case become:

$$\alpha + \beta > 0,\ 1 + \alpha\beta + g_2 > 0,\ \ g_1 > 0 \ \text{ and}$$

$$det\left(\begin{bmatrix} \alpha + \beta & g_1 \\ 1 & 1 + \alpha\beta + g_2 \end{bmatrix}\right) = (\alpha + \beta)(1 + \alpha\beta + g_2) - g_1 > 0$$

This can be expressed in terms of g_2 to get an inequality of the form $y > ax + b$:

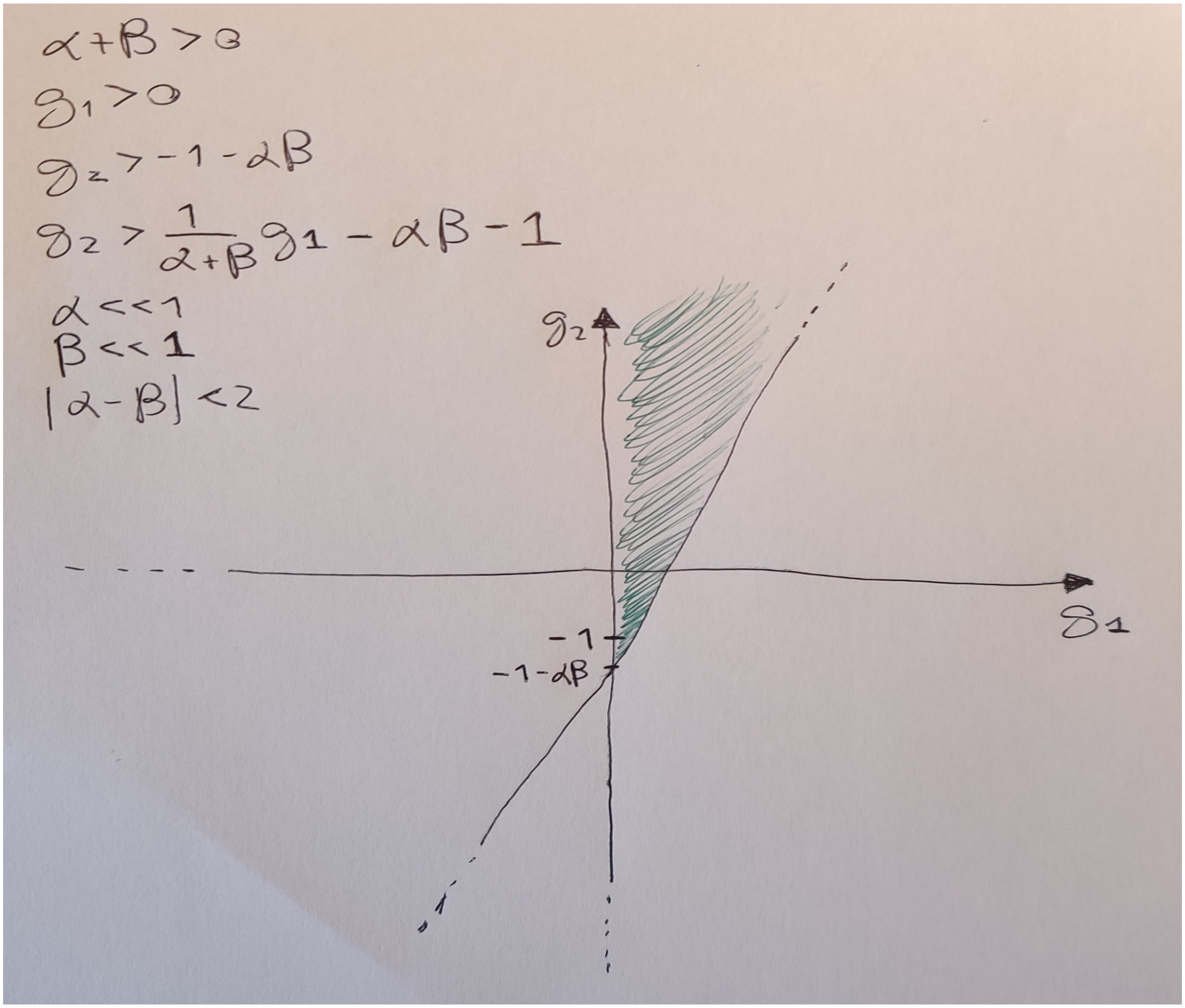
$$g_2 > \frac{1}{\alpha + \beta} g_1 - \alpha\beta - 1$$

That is more restrictive respect to $1 + \alpha\beta + g_2 > 0$ and we can see that:

$$a = \frac{1}{\alpha + \beta}$$

$$b = -\alpha\beta - 1$$

We can now make a sketch:



The painted area is where the differential equation (6) is asymptotically stable.

Appendix

1.2

1)

```
> restart:with(LinearAlgebra):with(plots):with(student):with
  (PDEtools): with(SolveTools:-Inequality):
> eq1:=diff(x(t),t,t)+alpha*diff(x(t),t)=j(t)
```

$$eq1 := \frac{d^2}{dt^2}\, x(t) + \alpha \left(\frac{d}{dt}\, x(t)\right) = j(t) \tag{1.1}$$

```
> eq2:=diff(j(t),t)=-beta*j(t)-diff(x(t),t)+u(t)
```

$$eq2 := \frac{d}{dt}\, j(t) = -\beta\, j(t) - \frac{d}{dt}\, x(t) + u(t) \tag{1.2}$$

```
> j(t):=diff(x(t),t,t)+alpha*diff(x(t),t)
```

$$j(t) := \frac{d^2}{dt^2}\, x(t) + \alpha \left(\frac{d}{dt}\, x(t)\right) \tag{1.3}$$

```
> L:=simplify(isolate(eq2,x(t)))
```

$$L := \frac{d^3}{dt^3}\, x(t) + (\beta + \alpha) \left(\frac{d^2}{dt^2}\, x(t)\right) + (\alpha\,\beta + 1) \left(\frac{d}{dt}\, x(t)\right) = u(t) \tag{1.4}$$

```
> l:=diff(x(t),t,t,t)+(beta+alpha)*diff(x(t),t,t)+(alpha*beta+1)*
  diff(x(t),t)=0
```

$$l := \frac{d^3}{dt^3}\, x(t) + (\beta + \alpha) \left(\frac{d^2}{dt^2}\, x(t)\right) + (\alpha\,\beta + 1) \left(\frac{d}{dt}\, x(t)\right) = 0 \tag{1.5}$$

2)

```
> u(t):=exp(s*t)
```

$$u(t) := e^{s\,t} \tag{1.6}$$

```
> L
```

$$\frac{d^3}{dt^3}\, x(t) + (\beta + \alpha) \left(\frac{d^2}{dt^2}\, x(t)\right) + (\alpha\,\beta + 1) \left(\frac{d}{dt}\, x(t)\right) = e^{s\,t} \tag{1.7}$$

```
> m:=H*exp(s*t)*(s^3+(beta+alpha)*s^2+(alpha*beta+1)*s)=exp(s*t)
```

$$m := H\, e^{s\,t} \left(s^3 + (\beta + \alpha)\, s^2 + (\alpha\,\beta + 1)\, s\right) = e^{s\,t} \tag{1.8}$$

```
> solve(m,H):
  H:=unapply(1/(s*(alpha*beta + alpha*s + beta*s + s^2 + 1)),s):
  'H(s)'=H(s)
```

$$H(s) = \frac{1}{s\,(\alpha\,\beta + \alpha\,s + \beta\,s + s^2 + 1)} \tag{1.9}$$

```
> solve(s^3+(alpha+beta)*s^2+(alpha*beta+1)*s<>0,s)
```

$$\left\{s \neq -\frac{\beta}{2} - \frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - 2\,\alpha\,\beta + \beta^2 - 4}}{2}\right\}, \left\{s \neq -\frac{\beta}{2} - \frac{\alpha}{2} \tag{1.10}\right.$$

$$-\frac{\sqrt{\alpha^2-2\alpha\beta+\beta^2-4}}{2}\Bigg\}, \{s \neq 0\}$$

3)

```
> x(t):=exp(lambda*t)
```

$$x(t) := e^{\lambda t} \tag{1.11}$$

```
> factor(l)
```

$$\lambda e^{\lambda t}\left(\alpha\beta+\alpha\lambda+\beta\lambda+\lambda^2+1\right)=0 \tag{1.12}$$

```
> P:=simplify(expand(lambda*(alpha*beta+alpha*lambda+beta*lambda+
lambda^2+1)))
```

$$P := \left(\lambda^2+(\beta+\alpha)\lambda+\alpha\beta+1\right)\lambda \tag{1.13}$$

```
> solve(P=0,lambda)
```

$$0, -\frac{\beta}{2}-\frac{\alpha}{2}+\frac{\sqrt{\alpha^2-2\alpha\beta+\beta^2-4}}{2}, -\frac{\beta}{2}-\frac{\alpha}{2}-\frac{\sqrt{\alpha^2-2\alpha\beta+\beta^2-4}}{2} \tag{1.14}$$

```
> lambda__1:=0;
lambda__2:=-beta/2-alpha/2+I*sqrt(-alpha^2+2*alpha*beta-beta^2+4)
/2;
lambda__3:=-beta/2-alpha/2-I*sqrt(-alpha^2+2*alpha*beta-beta^2+4)
/2;
```

$$\lambda_1 := 0$$

$$\lambda_2 := -\frac{\beta}{2}-\frac{\alpha}{2}+\frac{I\sqrt{-\alpha^2+2\alpha\beta-\beta^2+4}}{2}$$

$$\lambda_3 := -\frac{\beta}{2}-\frac{\alpha}{2}-\frac{I\sqrt{-\alpha^2+2\alpha\beta-\beta^2+4}}{2} \tag{1.15}$$

```
> evalc(exp(2*I*t))
```

$$\cos(2t)+I\sin(2t) \tag{1.16}$$

```
> evalc(Im(H(2*I)*exp(2*I*t)))
```

$$-\frac{(\alpha\beta-3)\cos(2t)}{2\left((2\alpha+2\beta)^2+(\alpha\beta-3)^2\right)}-\frac{(2\alpha+2\beta)\sin(2t)}{2\left((2\alpha+2\beta)^2+(\alpha\beta-3)^2\right)} \tag{1.17}$$

4)

Only reordering the equations, no calculations have been done.

1.3

5)

Only reordering the equations and substitutions, no calculations have been done.

6)

```
> restart:with(LinearAlgebra):with(plots):with(student):with
(PDEtools): with(SolveTools:-Inequality):
> a__1:=alpha+beta;
a__2:=1+alpha*beta+g__2;
a__3:=g__1;
```

$$a_1 := \alpha + \beta$$
$$a_2 := \alpha\beta + g_2 + 1$$
$$a_3 := g_1 \tag{2.1}$$

```
> a__1>0
```

$$0 < \alpha + \beta \tag{2.2}$$

```
> a__2>0
```

$$0 < \alpha\beta + g_2 + 1 \tag{2.3}$$

```
> a__3>0
```

$$0 < g_1 \tag{2.4}$$

```
> Determinant(<a__1,a__3;1,a__2>)>0
```

$$0 < \alpha^2\beta + \alpha\beta^2 + \alpha g_2 + \beta g_2 + \alpha + \beta - g_1 \tag{2.5}$$

```
> G__2:=simplify(solve(Determinant(<a__1,a__3;1,a__2>)=0,g__2))
```

$$G_2 := \frac{-\alpha^2\beta - \alpha\beta^2 - \alpha - \beta + g_1}{\alpha + \beta} \tag{2.6}$$

```
> g__2>1/(alpha+beta)*g__1-1-alpha*beta
```

$$\frac{g_1}{\alpha + \beta} - 1 - \alpha\beta < g_2 \tag{2.7}$$