Economic Motion:

An Economic Application of the Lotka-Volterra Predator-Prey Model

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1. Introduction

The behaviors of economic systems have been viewed in three distinctly different ways in literature. The first models claimed that markets obtain a stable equilibrium. It is possible to observe random shocks, but eventually equilibrium would always be restored. Later models assumed that growth paths are cyclical and any equilibrium is the affected by past motions. Since the beginning of modern statistical methods, economists have assumed random shocks create what can essentially be viewed as chaotic behavior. Economic relations are masked in white noise, and economic motion is random without clear cycles.

Goodwin (1967) presents a simple model that describes the dynamic relationship between real wages and real employment. Modifying Goodwin's model slightly yields a model that incorporates all three, seemingly mutually exclusive, behaviors. It is possible for an economy to have stable real wages and real employment, but small changes can force the economy into a cyclical motion. Somewhat more drastic changes can cause the economy to behave rather chaotically. Although the mathematics are quite straightforward and provide ideal theoretical results, the improved model must be tested empirically to justify the modifications.

The Goodwin model, after some crucial modifications, is able to accurately predict changes in wages and employment and also estimate equipment cycles and a shorter unemployment cycle. The model is applicable for a diverse set of countries due to its ability to incorporate dynamic changes in any economy and thus describe economic motion for both centrally planned and entirely market-based countries.

Since the publication of Goodwin's model, economists have concentrated on two types of research: first, the development of more complex models by relaxing some of the original assumptions or augmenting the model, (Desai (1973), Shah-Desai (1981), Wolfstetter (1982), Pleog (1983, 1987), Mullinex-Peng (1993), and Sportelli (1995)) and second, the investigation of the stability and other mathematical properties of the model (Velupillai (1979), Flaschel (1984) and Sportelli (1995)). Very little research has focused on testing the model empirically to verify its validity (Atkinson (1969) and

Harvie (2000)) and, so far, no one has tested the Goodwin model on non-developed economies.

This paper has three objectives: first, to present Goodwin's original model; second, to present an augmented Goodwin model; and finally, to empirically test the models on a planned economy and eleven developed countries. During the development of the models, data for a planned economy, Hungary, will be tested. The model's estimates for Hungary will be helpful in understanding the somewhat complicated mathematics and will also reveal the limitations of Goodwin's original model.

2. Goodwin's Model

2.1 The Foundations of the Model

It has long been apparent that economic growth happens in cycles. By the 1920s, there were static equilibrium theories whose dynamics depended solely on exogenous shocks, while other economists merely provided descriptions of cycles. Lowe (1926) called for a coherent theoretical system to describe economic cycles. The first attempt was made by Hayek (1929, 1931) who created an economic cycle theory on the basis of an interdependent equilibrium system, which still failed to provide an entirely closed explanation. Keynes (1936) was more successful as he provided a consistent, closed, interdependent theoretical structure, which determined aggregate output and other phenomena like unemployment.

> Keynesian economics, in spite of all that has done for our understanding of business fluctuations, has beyond all doubt left at least one major thing quite unexplained; and that thing is nothing less than the business cycle itself […] For Keynes did not show us, and did not attempt to show us, save by a few hints, why it is that in the past the level of activity has fluctuated according to so definite a pattern.[1]

[1] Hicks (1950).

Keynes' simple, aggregative system needed to be extended into a more dynamic and long-run perspective. This extension of Keynes' General Theory led to the Oxbridge phase of the Keynesian Revolution. This phase consisted of three research areas: first, the development of the multiplier-accelerator theories of the cycle; second, the development of non-linear endogenous mechanisms; and third, the development of Keynesian cycle theory.

The multiplier principle implies that investment increases output, while the acceleration principle implies that output induces investment. The first economist to put together these two ideas was Harrod (1936) who eventually adapted his theory of cycles to create a theory of growth. Later, Hicks (1950) formalized Harrod's theories using the mathematical methods of Samuelson (1939) and Metzler (1941). Kalecki (1937, 1939, 1954), Kaldor (1940), and Goodwin (1951) initiated the endogenous cycle development. All of these models have heavy roots in mathematics and use a non-linear structure to address income distribution dynamics. Both the structure and the focus of these models differ from the multiplier-accelerator theories. Lastly, Keynesian cycle theory reintroduced financial variables and moved beyond output growth and income distribution. The two most notable theories are the Keynes-Wicksell models on monetary policy and Minsky's financial cycle theories.[2]

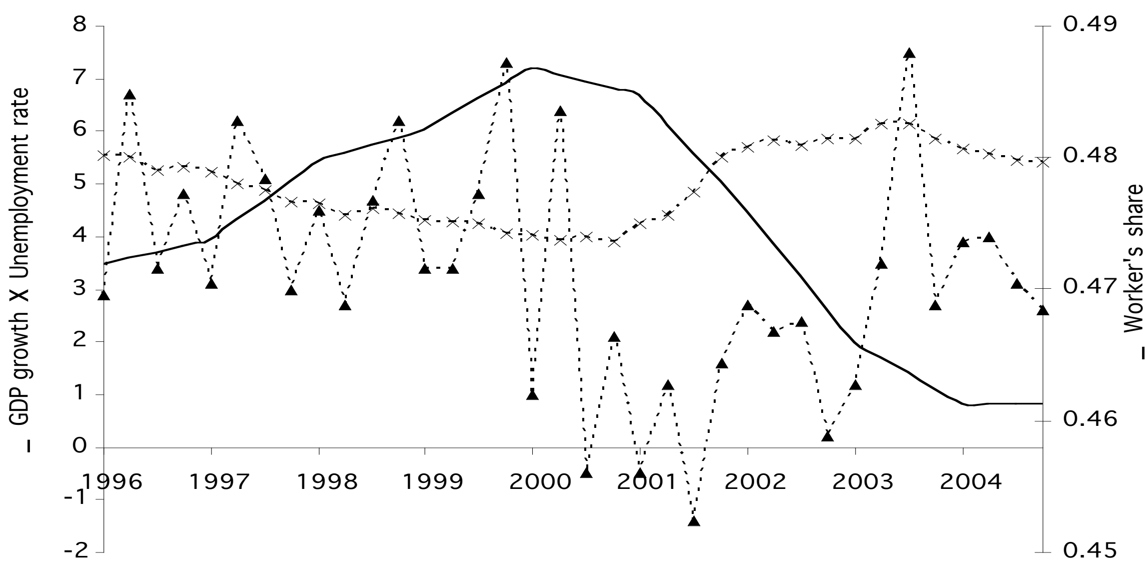
The reminder of this paper addresses one of Goodwin's many endogenous growth cycle theories and its applicability. Goodwin (1967) presented a simplistic model about wages and employment. His economic model is analogous to the Lotka-Volterra predator-prey model, where wages correspond to predators and employment to prey. At high levels of employment, the bargaining power of employed workers drives up wages, and thus shrinks profits. As profits diminish, fewer workers will be hired and employment will decrease, leading to increased profits. Afterwards, at higher profit levels more workers are hired, the employment level rises again and a cyclical pattern emerges. [3]

[2] Blaug (1986).
[3] Goodwin (1957).

These cycles should not be confused with business cycles, although recessions can affect wage-employment cycles and conversely, fluctuations in employment or wages can lead to a recession. In the United States, output growth, unemployment, and workers' share (defined as net wages as a share of national income) follow a somewhat cyclical pattern (Figure 1, next page). During the economic expansion in the 1990s, high levels of output growth complemented a rising employment level (unemployment fell substantially) and real wages increased. In 2000, output growth slowed substantially, employment fell sharply (unemployment increased) and real wages decreased. More recently, the United States has experienced high output growth and a falling unemployment rate.

Goodwin's paper does not directly deal with business cycles. Instead, changes in (un)employment and in distributions of income are dynamically modeled. The main goal of the model is to show how and why (un)employment is cyclical. Goodwin cycles can quite easily coexist with business cycles. Most notably, Goodwin cycles emerge in planned economies, such as pre-1990 Hungary, where business cycles did not exist.

Figure 1. Unemployment, GDP growth and workers' share in the US, 1996-2004



Source: Bureau of Economic Analysis; Bureau of Labor Statistics.

Goodwin's model originates from an idea in Marx's *Capital*. Marx believed that "capitalism's alternate ups and downs can be explained by the dynamic interaction of profits, wages and employment."[4] The growth rate of profits triggers production and thus excess labor demand. Hence, wages rise, squeezing the profit rate and slowly eroding the basis for accelerated accumulation. [5]

> Accumulation slackens in consequence of the rise in the price of labour, because the stimulus of the gain is blunted. The rate of accumulation lessens; but with its lessening, the primary cause of that lessening vanishes, i.e., the disproportion between capital and exploitable labour-power. The mechanism of the process of capitalist production removes the very obstacles that it temporarily creates. The price of labour falls again to a level corresponding with the needs of the self-expansion of capital, whether the level be below, the same as, or above the one which was normal before the rise of wages took place. […] To put it mathematically: the rate of accumulation is the independent, not the dependent, variable.[6]

Moreover, Marx refers to the employment rate and the workers' share as the trigger which starts these cycles.

> The law of capitalist production […] reduces itself simply to this: The correlation between accumulation of capital and rate of wages is nothing else than the correlation between the unpaid labour transformed into capital, and the additional paid labour necessary for the setting into motion of the additional capital.[7]

Goodwin (1969) formalized this concept. To express the alternating ups and downs mathematically, Goodwin turned to the predator-prey model. The following section presents the Goodwin model, with a few changes in notation.

2.2 The Original Model

If all of national income could be accumulated, then its growth rate would only be constrained by capital intensity. Capital intensity (σ) shows how many years of

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[4] Goodwin (1972) p. 442.
[5] Harvie (2000).
[6] Marx [1887] p. 341.
[7] Marx [1887] p. 342.

income have to be tied up to produce a unit of income. Its reciprocal value $(1/\sigma)$ measures capital productivity, which determines the amount of national income generated by each unit of capital invested.

The rate of growth of national income equals the rate of growth of employment unless there are changes in labor supply or productivity. Goodwin assumes that both labor supply and labor productivity are exponential processes, which means that the growth rates of both are constants: α percent per year rise in labor productivity, and β percent per year rise in labor supply. Then, the yearly rise of employment $(\dot{v}/v)$ is determined by

$$\frac{\dot{v}}{v} = \frac{1}{\sigma} - \alpha - \beta \, . \tag{1}$$

The actual growth rate of employment will be lower because part of the national income has to be spent on wages and only the remainder can be accumulated. Goodwin applies the classical notion of real wages, which was defined by Ricardo and Marx as the wages share of national income. Thus, Goodwin's real wages do not have much in common with the modern notion of real wages which measure the amount of products and services purchased by wages. This workers' share is measured by u, while $1-u$ defines savings. Thus the first equation of the model is

$$\frac{\dot{v}}{v} = \frac{1}{\sigma}(1-u) - \alpha - \beta \, . \tag{2}$$

The second equation is established even more simply by Goodwin. The change of the real wage (workers' share) depends on employment. We do not know the exact relation, but we do know that the rate of employment and real wages are positively correlated. Employment fluctuates between fairly narrow limits in real life, so it is permissible to linearize the interdependence in a small neighborhood of equilibrium and consider a linear estimate with slope ρ and intercept γ. Moreover, workers' share growth will be dampened by labor force growth. Therefore, the second equation of the model is

$$\dot{u} = [-(\alpha + \gamma) + \rho v]u . \tag{3}$$

This being said, the next few paragraphs provide a much more rigorous and mathematical account of Goodwin's model.

Goodwin begins by assuming constant technological progress, constant growth in labor force, two factors of production: labor and capital (plant and equipment), all quantities are real and net, all wages are consumed, all profits are saved and invested, the capital-output ratio is constant, and the real wage rate rises in the neighborhood of full employment. The last two assumptions, as even Goodwin points out are disputable, but softening them complicates the model and possibly makes it unsolvable.[8] Later in the paper, these assumptions are revisited and their validity is checked.

For now, following Goodwin's presentation, steady technological progress translates into steady labor productivity growth of the form

$$a = a_0 e^{\alpha t}; \alpha > 0 , \tag{4}$$

where a is labor productivity, growing at a constant rate measured by α. The equation describing labor force growth has a very similar form

$$n = n_0 e^{\beta t}; \beta > 0 , \tag{5}$$

where n is the size of the labor force, which is assumed to grow at a constant rate, β. Moreover, let us denote the constant capital-output ratio as

$$\sigma = k / q , \tag{6}$$

where k is capital and q is output. Finally, if w is the wage rate, then $u = w/a$ will measure workers' share of output, and capitalists' share of output will be the remainder, $1-u$. Since all wages are consumed, and all profits are saved and invested, the growth rate of capital will equal investment, which equals profits, such that

[8] Goodwin [1972].

$$\dot{k} = (1-u)q \,. \tag{7}$$

Formally, $\dot{k} = dk/dt$, where the 'dot' represents the time derivative; therefore, $\dot{k}/k$ is the percentage change in capital over time. Dividing both sides of (7) by k and substituting $\sigma = k/q$ yields the growth rate of capital over time

$$\dot{k}/k = \frac{(1-u)q}{k} = \frac{(1-u)}{\sigma},$$

which is identically equal to the growth rate of output over time due to the fixed capital-output ratio. Thus,

$$\dot{q}/q = \frac{(1-u)}{\sigma}.$$

Employment is set according to

$$l = q/a\,, \tag{8}$$

and so the change in employment over time is measured by

$$\frac{\dot{l}}{l} = \frac{(1-u)}{\sigma} - \alpha\,.$$

In general if a variable (f) is a ratio of two other processes ($f = g/h$), then the percentage change of the variable over time is equal to the difference of growth rates of the two processes ($\dot{f}/f = \dot{g}/g - \dot{h}/h$). Formally, using the quotient rule, $\frac{\dot{f}}{f} = \frac{1}{f}\left[\frac{g}{h}\right]' = \frac{1}{f}\frac{\dot{g}h - g\dot{h}}{h^2} = \frac{h}{g}\left[\frac{\dot{g}}{h} - \frac{g}{h}\frac{\dot{h}}{h}\right] = \frac{\dot{g}}{g} - \frac{\dot{h}}{h}$ is true for all $f = g/h$.

Finally, the real employment rate will be calculated by dividing employment by the labor force

$$v = l / n. \tag{9}$$

Thus, the growth rate of real employment over time is equal to the growth rate of employment less the growth rate of the labor force,

$$\frac{\dot{v}}{v} = \frac{(1-u)}{\sigma} - (\alpha + \beta).$$

Goodwin also assumes a real wage rate that rises in the neighborhood of full employment and uses a linear Phillips curve to describe wage growth

$$\frac{\dot{w}}{w} = -\gamma + \rho v, \tag{10}$$

where γ is the intercept, and ρ is the slope of the Phillips curve. Let us denote the real wage rate by

$$u = w / a, \tag{11}$$

then

$$\frac{\dot{u}}{u} = -(\alpha + \gamma) + \rho v.$$

Equations (4) through (11) reduce to Goodwin's real employment-real wage cycle model

$$\dot{v} = [(\frac{1}{\sigma} - (\alpha + \beta)) - \frac{u}{\sigma}]v, \tag{12}$$

$$\dot{u} = [-(\alpha + \gamma) + \rho v]u. \tag{13}$$

2.3 The Predator-Prey System

Goodwin recognized that he had arrived at the Lotka-Volterra predator-prey model. In the original Lotka-Volterra model of competing species, the predator and the prey can be distinguished by the fact that the predator population grows faster the larger the prey population is, while the prey population grows faster the smaller the predator population. Goodwin identified u with workers' share while $1-u$ is capitalists' share of output and v is the employment rate. Then, it is apparent from equation (12) that the employment rate increases more rapidly the larger the capitalists' share, and from equation (13) that the capitalists' share increases more rapidly the smaller the employment rate is. Thus, in Goodwin's model, workers are the predators and capitalists are the prey. The economic implication of this connection is quite simple. When profits are high, investment is high, and because all investment requires additional labor, high investment translates into a rapid employment growth. On the other hand, when employment is low, wages are decreasing and, in turn, profits increase.

> To some extent the similarity is purely formal, but not entirely so. It has long seemed to me that Volterra's problem of the symbiosis of two populations – partly complementary, partly hostile – is helpful in the understanding [...] of capitalism. [9]

From a mathematical standpoint, this similarity will make solving this complicated looking system immensely easier. Consider the following system:

$$\dot{x} = [\eta_1 - \theta_1 y]x \qquad (12')$$

$$\dot{y} = [-\eta_2 + \theta_2 x]y\,. \qquad (13')$$

Substituting $\eta_1 = 1/\sigma - (\alpha + \beta)$, $\theta_1 = 1/\sigma$, $\eta_2 = \alpha + \gamma$, and $\theta_2 = \rho$ yields equations (12') and (13'), which are identical to Goodwin's system given as equations (12) and (13).

[9] Goodwin [1967].

Besides $x = 0$ and $y = 0$ (i.e. no predator or prey), the predator-prey model's solution is a family of closed cycles, with each cycle sharing a common equilibrium

$$x^* = \frac{\eta_2}{\theta_2}, \tag{14'}$$

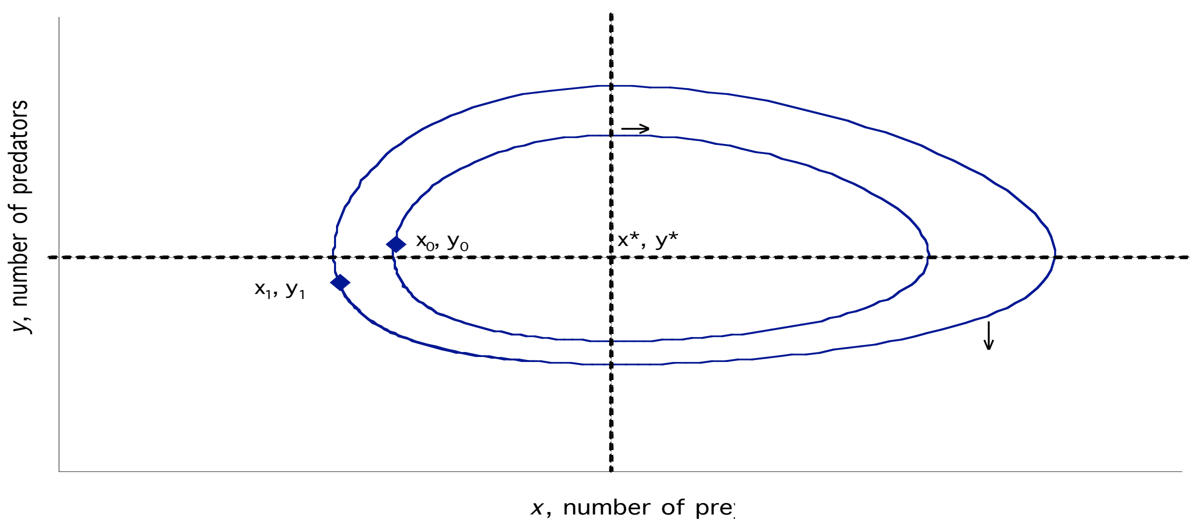
$$y^* = \frac{\eta_1}{\theta_1}, \tag{15'}$$

and where each cycle has a time length of

$$T^* = \frac{2\pi}{\sqrt{\eta_1 \eta_2}}. \tag{16'}$$

Given a set of initial conditions, (x_0, y_0), we can easily graph these cycles. There are infinitely many solutions, one for each initial condition, (x_0, y_0). All solutions oscillate around a common center point with the same cycle length (Figure 2).

Figure 2. Sample solutions to the predator prey model



We can use the solutions of the predator-prey system to obtain the solutions of the Goodwin model. The center of the economic model is at

$$u^* = 1 - (\alpha + \beta)\sigma, \tag{14}$$

$$v^* = \frac{(\alpha + \gamma)}{\rho}, \tag{15}$$

and the system has a cycle length of

$$T^* = \frac{2\pi}{[(\alpha + \gamma)(1/\sigma - (\alpha + \beta))]^{1/2}}. \tag{16}$$

A formal derivation of the solution can be found in Appendix 1.

2.4 Testing Goodwin with Hungarian Data

To get a better understanding of the model's prediction, Hungarian data from 1985 Statistical Yearbook for the years 1955-1985 will be used.[10] Hungary had a planned economy, with a traditional planning agency; therefore, the model should be able to somehow predict the planning cycle. Planned economies are the perfect agents for the model to be tested on because they have easily identifiable cycles. Unlike developed economies, which have varying business cycle length, planned economies have a fixed planning cycle. Hungary, for example, had a 5-year planning cycle, but Brody (1990) and Ungvarszky (1986) hypothesized that the country had an inventory cycle of about 4 years (instead of the 5-year cycle) and a longer equipment cycle of 12 years, while there were also signs of a 25-year and a 50-year demographic cycles. One would expect to find cycles of similar length predicted by Goodwin's model.

Net national income was 626 billion (Bn) Forints (Fts), while the total capital stock was 2,447 Bn; therefore, capital productivity was $1/\sigma = 0.26$. Labor supply growth was very slow for the time period and always remained below half a percent per year, thus we will use $\beta \approx 0$ for practical purposes.[11] National income increased by a factor of 4.86 during the 30 year period, yielding a yearly average growth of around 5 percent. However, a significant part of this growth did not come from technological progress. During this time period, Hungary was heavily industrialized and labor was reallocated

[10] KSH (1980).

[11] Specifically $\beta = 0.00261$. Using this value for β does not change the solution significantly.

from the agricultural to the industrial sectors, which had higher prices. Thus, the five percent estimate is distorted by a severely distorted price system. Excluding the bias created by preferential pricing yields an estimate of $\alpha = 0.03$.[12]

Estimating the Phillips curve is even more difficult due to a lack of reliable employment and inflation estimators. It is impossible to create a sufficiently large data set to estimate the Phillips curve parameters. However, using data from Halpner-Molnar (1985) on employment and wages, the coefficients of equation (13) can be directly estimated. Due to uncertainty in the data, the estimates for ρ and γ provide a range of values: ρ is between 3.52 and 4.16, while γ is between 1.35 and 1.72.

The gross amount of wages and salaries was 320 Bn Fts. Dividing this amount by the national income (626 Bn) yields $\hat{u} = 0.51$ for workers' share.

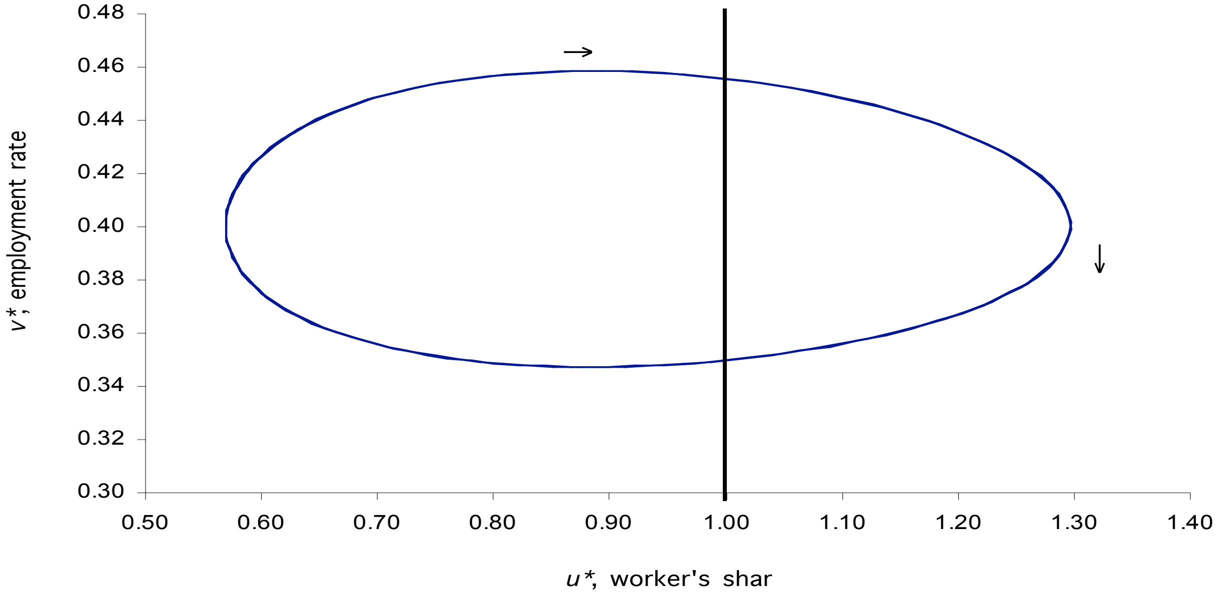
Substituting these values into (14) through (16) yields the following estimates:

- Average employment is predicted to fluctuate around $v^* = (0.38; 0.41)$ (from equation (14)). This is a fairly realistic result.
- Average wage rate is oscillating around $u^* = 0.885$ (from equation (15)), which deviates from the actual 0.51.
- Cycle length is $T^* = (9.144; 9.904)$ (from equation (16)), which is quite far from either of the suspected cycle durations.

Goodwin's original model does not fit Hungarian data well. More importantly, the predicted cycle has workers' share fluctuating between 0.57 and 1.30, meaning that there should have been times when worker's wages surpassed output (Figure 3). This result is also unacceptable.

[12] Halpern-Molnar (1985).

Figure 3. Predicted Goodwin cycle in Hungary



3. A revised model

3.1 Malthusian Growth

Goodwin, in his later works, hinted at some of the shortfall's of his model, most of which correspond to issues with the original predator-prey model.[13] Revisiting equations (12′) and (13′) we have

$$\dot{x} = [\eta_1 - \theta_1 y]x\,,$$

$$\dot{y} = [-\eta_2 + \theta_2 x]y\,.$$

The absence of predators, $y = 0$, provides the fastest growth rate for the prey, with prey levels of

$$x = x_0 e^{\eta_1 t}\,. \tag{17}$$

This means, that in the absence of predators, prey will grow in equation (12′) according to a Malthusian growth, at an exponential rate without bounds. This raises a number of issues in both the ecological and economical interpretations of Goodwin's original model. Economically this means that in the absence of wages, employment grows at an exponential rate, quickly surpasses full employment, and keeps growing without bounds. It is rather startling how, due to faulty reasoning, the model leads to such an anti-worker attitude. In reality, average employment, defined as the ratio of employed workers and working age population (people between the ages 15-65), remained below 75 percent.

It should be noted that employment cannot be increased without limits and/or severe penalties on productivity gains. Goodwin's economic analogy erroneously assumes that additional workers will be just as productive as employed workers. This phenomenon is perfectly acceptable for the ecological model, where the prey population is homogenous. However, labor market downsizing often sheds the less

[13] Goodwin (1972).

tenured, less trained, less dexterous, and less productive workers first. In the opposite case, any upswing will have to accept whatever is available in the labor market: less knowledgeable, untrained and generally people with relatively inferior skills.

It is impossible to get reliable data about personal productivities. But, to enhance the reality of the model, instead of $\dot{x} = \eta_1 x$, a logistic saturation is considered in the system's first equation (12′) at $y = 0$, such that

$$\dot{x} = \eta_1 (1 - x/K)x . \tag{18}$$

For the ecological model, it makes sense only if the prey population remains below a fixed parameter, K. The solution to (18) is a so called logistic differential equation, with a solution at $y = 0$ of the form

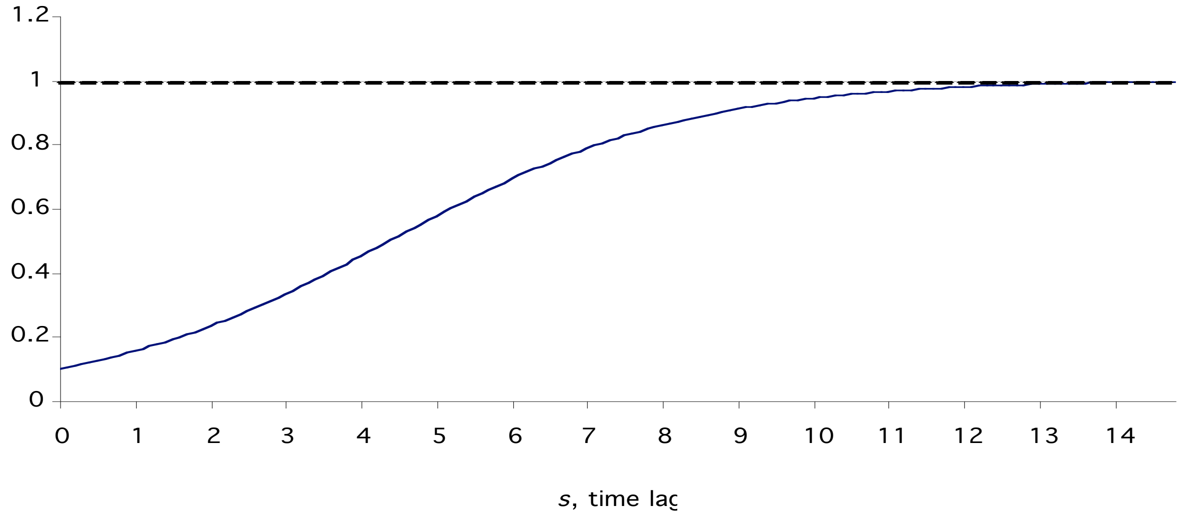
$$x = \frac{K x_0 e^{\eta_1 t}}{K + x_0 (e^{\eta_1 t} - 1)} . \tag{19}$$

For $0 < x_0 < K$ the solution is increasing, and for $K < x_0$ the solution is decreasing. In both cases, all solutions will tend to K as t tends to infinity (Figure 4, following page).

In population dynamics, $K > 0$ is the "carrying capacity" of the environment with respect to prey. The practice of introducing similar population restrictions is very common in ecological interpretations of the predator-prey model.[14] In the economic model, $K = 1$, since employment cannot surpass total population in a closed economy. Thus, marginal productivity increases until 50 percent employed. Afterwards, marginal productivity will fall, but total productivity increases until $u = 1$, which allows the theoretical possibility of full employment.

[14] Farkas (1984).

Figure 4. Logistic growth with carrying capacity $K = 1$



3.2 Wages Lag Employment

The second problem relates to the reaction of wages to employment. Any change in wages induced by changes in employment cannot be instantaneous. Wage contracts are set in advance, and rarely take into effect future changes in demand for labor. In a recession, the wage rate reacts sluggishly to mounting unemployment; real wages tend to continue to rise for a while in spite of unproductive workers being laid off. The lowest real wage rate is usually attained when the economy is already in an upswing. The length of the delay will vary from country to country. For example, in Hungary, there were cases (bus drivers, spinners, and weavers) where only years of persistent shortage on the labor market triggered adjustment. In all probability, this delay will be somewhat longer in planned economies than in market economies due to a lesser developed wage-bargaining apparatus. [15]

[15] Halpern-Molnar (1985).

This delay can be modeled by substituting into the second equation of the model, (13′), a weight function that will take into account past movements in real employment. Formally, we will replace x in equation (13′) by

$$z = \int_0^t x(\tau)G(t-\tau)d\tau , \tag{20}$$

where G is a nonnegative integrable weight function, with the property that

$$\int_{-\infty}^t G(t-\tau)d\tau = \int_0^\infty G(s)ds = 1 .$$

In population dynamics, the growth rate of predator population depends on the density of the prey in the past. The way in which the growth rate depends on past values will be determined by the choice of the weight function[16]. The function $G(s)$ will have a very similar role in the economic model.

These changes, along with the ones introduced in the previous section, will alter the predator-prey model (equations (12′)-(13′)) into the system:

$$\begin{aligned} \dot{x} &= \eta_1(1-x)x - \theta_1 xy \\ \dot{y} &= -\eta_2 y + \theta_2 y \int_0^t x(\tau)G(t-\tau)d\tau. \end{aligned} \tag{21}$$

[16] Farkas (1984).

3.3 Structural Instability

The structural instability of the system is the worst fault of the predator-prey model and in extension, of Goodwin's model as well. Goodwin hinted at this instability in his later work.[17] This instability means that very small perturbations in the initial conditions may lead to qualitatively different behavior. Specifically, solutions with initial conditions (x_0, y_0) close to the center means that (14) and (15) will eventually tend back to (x_0, y_0) but are not going to stay within a fixed distance. Mathematically, the center points of the cycles are quasi-asymptotically stable,[18] but are unstable in the Liapunov sense. [19] This instability will be remedied by strategically choosing weight functions that will stabilize the solutions.

Several authors[20] have studied the system with the weight function

$$G(s) = G_1(s) = ae^{-as}, a > 0 . \tag{22}$$

This means that we consider a $1/a$ discount in the reaction of wages. In this case, employers take into account changes in capital when setting wage contracts. Because the capital output ratio is fixed and output is directly related to employment, capital depreciation will cause employers to discount past employment levels when setting wages by a discount rate, a. Past employment levels that happened further away in the past (s is large) will have a smaller affect on wages than will more recent movement. Eventually, the influence of the past is practically nil (Figure 5).[21]

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[17] Goodwin (1972).

[18] A point is quasi-asymptotically stable iff there exists $\delta > 0$ such that if $|x - y| < \delta$ then $\lim_{t \to \infty} |\varphi(x,t) - \varphi(y,t)| = 0$, Glendinning (1994).

[19] A point is Liapunov stable iff for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - y| < \delta$ then $|\varphi(x,t) - \varphi(y,t)| < \varepsilon$ for all $t \geq 0$, where φ is a solution, Glendinning (1994).

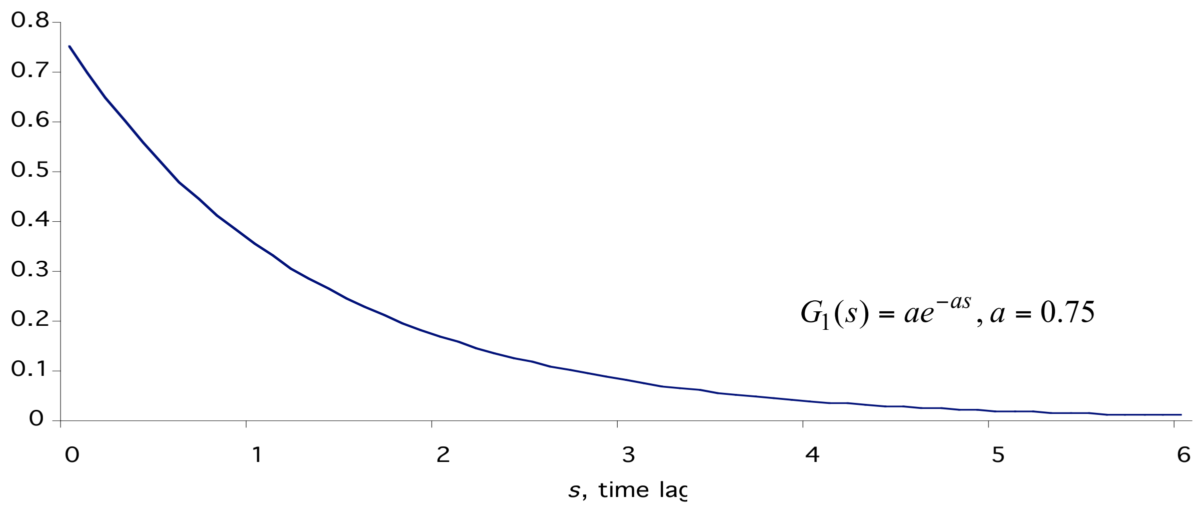
[20] See Cushing (1977), Dai (1981), MacDonald (1977), Farkas (1984), Farkas-Kajtar-Farkas (1987).

[21] Formally, $\lim_{t \to \infty} G_1(s) = 0$.

Adding this result to (21) will transform the model into a system of integro-differential equations of the form

$$\dot{x} = \eta_1(1-x)x - \theta_1 xy$$
$$\dot{y} = -\eta_2 y + \theta_2 y \int_0^t x(\tau) a e^{-a(t-\tau)} d\tau \, .$$

Figure 5. The weight function $G_1(s)$, (22)



$a = 0.75 \;\; and \;\; 0 < s < 6$

4. An improved model

4.1 Solution and Stability

This seemingly complicated system has been investigated as an ecological model by a number of authors.[22] It can be shown that the system is equivalent to the three dimensional system

$$\begin{aligned} \dot{x} &= \eta_1(1-x)x - \theta_1 xy \\ \dot{y} &= -\eta_2 y + \theta_2 yz \\ \dot{z} &= a(x-z) \end{aligned} \tag{23}$$

This system allows a straightforward economic interpretation. In the labor market, wages will be set not according to actual employment x, but according to expectations of future employment levels that are based on past employment levels. Expectations will change and continuously correct themselves as expressed by the last equation in (20).

System (20) has three equilibria: the trivial and unstable $S_1 = (0,0,0)$, $S_2 = (1,0,1)$ representing the absence of predators, and

$$S_3 = \left(\frac{\eta_2}{\theta_2}, \left(1-\frac{\eta_2}{\theta_2}\right)\frac{\eta_1}{\theta_1}, \frac{\eta_2}{\theta_2}\right).$$

S_2 is asymptotically stable when $\eta_2/\theta_2 > 1$ and when $\eta_2/\theta_2 < 1$ the stability of S_3 will depend on the delay $\mu = 1/a$.

In its economic interpretation, $S_2 = (1,0,1)$ represents the case where zero real wages correspond with full employment. This solution is asymptotically stable when $\eta_2 > \theta_2$, which is unrealistic because it means that the real wage Phillips curve indicates a wage decrease even at full employment. Practically, therefore $\eta_2 < \theta_2$ and

[22] Farkas (1984), Szabo (1987), Farkas-Kajtar-Farkas (1987), and Farkas-Farkas (1987).

$$S_3 = \left(\frac{\eta_2}{\theta_2}, \left(1 - \frac{\eta_2}{\theta_2}\right) \frac{\eta_1}{\theta_1}, \frac{\eta_2}{\theta_2} \right). \tag{24}$$

Clearly, the position of the equilibrium point does not depend on the delay $\mu = 1/a$, but its stability will be decisively influenced. Namely, if

$$\theta_2(\theta_2 - \eta_2) - \eta_1\eta_2 < 0,$$

then the equilibrium described by equation (24) will be asymptotically stable, no matter how large the delay is. This will be the case not only when $\theta_2 - \eta_2 > 0$, but also when η_1, capital productivity less labor productivity growth less population growth, is very high. Thus, this is a case seldom encountered in real economic life. If, on the other hand,

$$\theta_2(\theta_2 - \eta_2) - \eta_1\eta_2 > 0, \tag{25}$$

which also implies $\theta_2 - \eta_2 > 0$, then the equilibrium will be stable with a small delay, and unstable with large delays. Specifically, if besides equation (25) we also have

$$\mu\left(\theta_2 - \eta_2 - \frac{\eta_1\eta_2}{\theta_2}\right) < 1, \tag{26}$$

then the equilibrium is asymptotically stable. However, if

$$\mu\left(\theta_2 - \eta_2 - \frac{\eta_1\eta_2}{\theta_2}\right) > 1, \tag{27}$$

then the equilibrium will be unstable.

Generally speaking, large delays destabilize the system, which is a common, but not always true phenomenon often encountered in stability theory. Stability is lost if, while equation (24) holds, the solution crosses the surface

$$\mu\left(\theta_2 - \eta_2 - \frac{\eta_1\eta_2}{\theta_2}\right) = 1$$

in a four dimensional space. This loss of stability is an Andronov-Hopf bifurcation, meaning loosely that, in the neighborhood of the critical parameters, periodic solutions appear with small but increasing amplitudes. The bifurcation will be "soft"[23] if this periodic solution appears for unstable parameter values - that is, if equation (27) holds. In this case, the bifurcating periodic solutions are orbitally asymptotically stable. If, on the other hand, these small amplitude oscillations appear in stable region (26), then the bifurcation is "hard"[24] and the oscillations are not stable. It is still possible in the unstable domain (26) that stable periodic solutions appear, but their amplitude will be large and the fluctuations will be hard.

For the functioning of a real economy the soft loss of stability must be preferred. Explicitly, if

$$2\theta_2 - \eta_2\left(\eta_2 \frac{8\eta_1 + 9\eta_1\eta_2 + 2\eta_2{}^2}{\eta_1 + 2\eta_2}\right)^{1/2} > 0\,, \tag{28}$$

then the loss of stability will be smooth, and it will be hard if the inequality (28) is less than zero.

Economically, all parameters may change, and from a purely mathematical standpoint, destabilize the system, meaning that changes in θ_1, θ_2, η_1, and η_2 will cause a loss of stability.

4.2 Results with Hungarian Data

Similar to section 2.4, the numeric results provide greater insight to the proposed changes. We calculated the parameters necessary for this process in section 2.4. Recall, that we substituted $\eta_1 = 1/\sigma - (\alpha + \beta)$, $\theta_1 = 1/\sigma$, $\eta_2 = \alpha + \gamma$, and $\theta_2 = \rho$; therefore, the solutions of system (20) transform into

[23] Specifically: supercritical.
[24] Specifically: subcritical.

$$u^{**} = \left[1 - \frac{(\alpha+\gamma)}{\rho}\right]\left[1 - (\alpha+\beta)\sigma\right]$$
$$v^{**} = \frac{(\alpha+\gamma)}{\rho} \quad . \tag{29}$$

Average employment $v^{**} = (0.38,\ 0.41)$, which is unchanged. Average wages are predicted to fluctuate around $u^{**} = (0.502,\ 0.549)$, which is a significant improvement. Moreover, inequality (25) holds, since $\theta_2(\theta_2 - \eta_2) - \eta_1\eta_2 = (7.39,\ 9.62) > 0$. Therefore, we have a stable equilibrium if, from inequality (26), the delay is $\mu < 0.344$. This roughly means, that if wages react to employment within a four month period, then we cannot expect cycles at all. If reaction time increases beyond four months, then we should expect movement in employment and wages. Moreover, since inequality (28) is now $(4.24,\ 4.37) > 0$, meaning that stable cycles appear with small amplitudes and an approximate cycle length of

$$T^{**} \approx \frac{2\pi}{[(\alpha+\gamma)(1/\sigma - (\alpha+\beta))(1-(1+(\alpha+\gamma)\sigma)(1-(\alpha+\beta)\sigma))]^{1/2}} \,. \tag{30}$$

This cycle is approximately $T^{**} = (11.68,\ 12.35)$, which is fairly close to the so called equipment cycle. This cycle refers to the average depreciation of equipment and machinery, which is estimated to be roughly 12 years.[25] Not surprisingly, using a weight function that incorporates capital depreciation allows the model to estimate the length of the equipment cycle.

4.3 Further Improvements

Although the changes introduced in Section 3 significantly improved the solutions estimated by the model, the system still fails to predict the strongest 4-year cycle. The weight function (22) established in Section 3.3 introduced a delay. However, it does not solve or introduce the real life phenomenon of rising real wages during a

[25] Brody (1990).

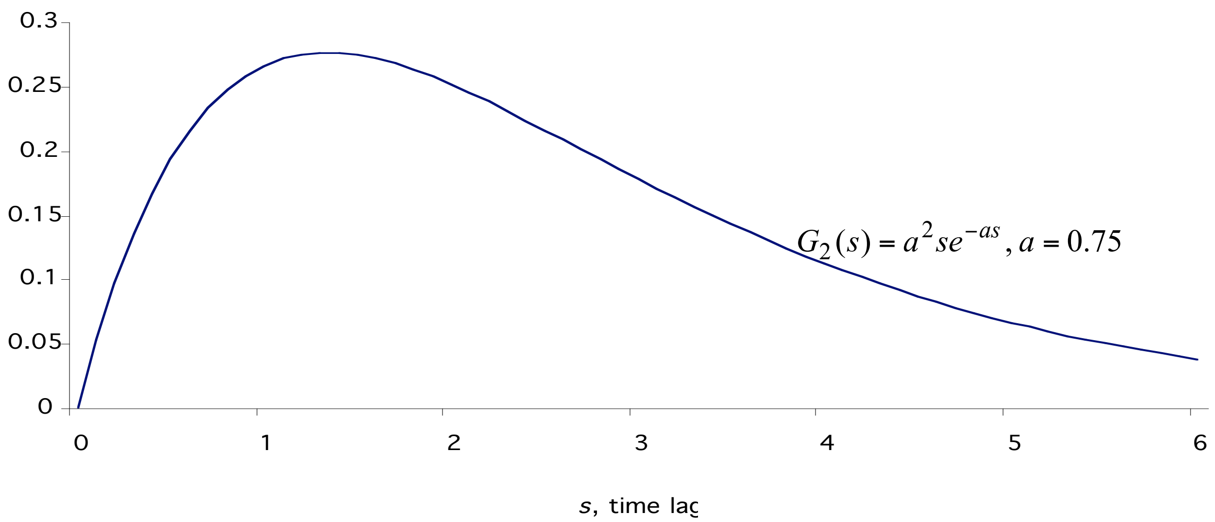
recession. Instead, weight function (22) slows the wage growth the moment employment levels start falling.

As an alternative to the weight function given as equation (22) we may assume another weight function in system (21) that is of the form

$$G(s) = G_2(s) = a^2 s e^{-as}, a > 0 \tag{31}$$

to get a more realistic model. This weight function introduces a true lag in how wages react to employment. The function assumes its maximum at $s = 1/a$ such that $G_2(1/a) = a/e$ (Figure 6). Accordingly, for $a = 0.75$, the largest weight will be assigned to events that happened at $s = 1.33$ (with a weight of $G_2(1/a) = 0.276$). In this example, employment levels 9 months ago have the largest influence on current wages. Once more, employment levels very far back in the past have practically no effect on current wages.

Figure 6. The weight function $G_2(s)$, (30)



$a = 0.75$ *and* $0 < s < 6$

Because of these changes, system (21) takes on the form

$$\dot{x} = \eta_1(1-x)x - \theta_1 xy$$
$$\dot{y} = -\eta_2 y + \theta_2 y \int_0^t x(\tau) a^2 (t-\tau) e^{-a(t-\tau)} d\tau \; ,$$

which can be simplified similarly to the previous integro-differential equation system to

$$\begin{aligned} \dot{x} &= \eta_1(1-x)x - \theta_1 xy \\ \dot{y} &= -\eta_2 y + \theta_2 y z_1 \\ \dot{z}_1 &= a(z_2 - z_1) \\ \dot{z}_2 &= a(x - z_2) \quad . \end{aligned} \tag{32}$$

The straightforward economic interpretation of this system proves more difficult. Nevertheless, real wages in the second equation will be set according to some rule, which still takes into account past levels of employment, specified by the third and fourth equations.

System (32) has three equilibria. The trivial $S_1 = (0,0,0,0)$ remains unstable. $S_2 = (1,0,1,1)$ represents the absence of predators, which is asymptotically stable if $\eta_2 / \theta_2 > 1$ and unstable when the Phillips curve restriction $\eta_2 / \theta_2 < 1$ holds. Finally,

$$S_3 = (\eta_2/\theta_2, (1-\eta_2/\theta_2)\eta_1/\theta_1, \eta_2/\theta_2, \eta_2/\theta_2) \, . \tag{33}$$

Similar to the previous case, $S_2 = (1,0,1,1)$ represents the unrealistic case of zero wages and full employment, and will once more only prove to be a stable solution under very special conditions. The first two coordinates of S_3 did not change compared to the previous model. Therefore, we still have average employment at $v^{**} = (0.38,\ 0.41)$, and average wages at $u^{**} = (0.502,\ 0.549)$. The other two coordinates are unimportant, since they are only help functions.

Applying the results of Farkas[26] to system (32) we get that stable periodic solutions appear if $\mu_0 < \mu < \mu_0 + \delta$ holds with some positive δ, where the critical delay $\mu_0 = 1/a_0$ is approximately 0.344 year (4 months) with cycle time

$$T^{***} \approx \frac{2\pi}{\dfrac{\eta_2}{\theta_2} \dfrac{\sqrt{2a} - \sqrt{\theta_2 - \eta_2}}{(2\sqrt{a} - (\sqrt{2(\theta_2 - \eta_2)} - \sqrt{a}))^{1/2}}}. \tag{34'}$$

In the economic model (34′) is equivalent to

$$T^{***} \approx \frac{2\pi}{\dfrac{\alpha + \gamma}{\rho} \dfrac{\sqrt{2a} - \sqrt{\rho - (\alpha + \gamma)}}{(2\sqrt{a} - (\sqrt{2(\rho - (\alpha + \gamma))} - \sqrt{a}))^{1/2}}}. \tag{34}$$

Evaluating this rather large expression yields $T^{***} \approx (3.87,\ 4.12)$, which is approximately equal to the inventory cycle time that was present and observed in the Hungarian economy.

5. Developed OECD Countries

5.1 Choice of Data

The previous sections used Hungarian data to show the validity of the improvements to the model. Unfortunately, it is not possible to create a time series from Hungarian data and therefore we are unable to construct real life workers' share-employment rate graphs (uv-graphs). Thus, it is impossible to see if clockwise closed orbits existed in Hungary. Nevertheless, the augmented Goodwin models accurately predicted long-run macroeconomic dynamics in Hungary. Using a weight function based on capital depreciation allows the model to estimate equipment cycles, while

[26] Farkas, Kajtar, Farkas (1987).

using a time lag in the reaction of wages to changes in employment allows the model to predict smaller cycles.

The remaining sections of this paper will use OECD data to test the validity of the models for eleven developed countries: Austria, Belgium, Canada, Finland, France, Italy, Netherlands, Norway, Spain, United Kingdom, and United States. All data is available from the OECD's Main Economic Indicators, National Accounts, Labor Force Statistics, and Flows and Stocks of Fixed Capital databases. The list of countries was restricted by data availability; specifically, historic capital level estimates and wage statistics are not available for every OECD member. All data are annual, for the time period 1970 to 2004.

The two state variables are the workers' share of national income (real wages) and the employment rate (real employment). Workers' share is calculated by letting

$$\hat{u} = \frac{\text{wages and salaries}}{\text{gross national income}},$$

while the employment rate is defined as

$$\hat{v} = \frac{\text{total employment}}{\text{working age population}}.$$

Population estimates for the age group 15 to 65 was used as the working age population. Table 1 summarizes the mean values of workers' share and employment rate for the eleven countries along with standard deviations.

Table 1. Average employment and wages in OECD countries

	$\hat{u}$	*StDev* $\hat{u}$	$\hat{v}$	StDev $\hat{v}$
Austria	0.4326	*0.0150*	0.6587	*0.0241*
Belgium	0.3995	*0.0274*	0.5718	*0.0233*
Canada	0.5045	*0.0202*	0.6618	*0.0330*
Finland	0.4352	*0.0299*	0.6978	*0.0391*
France	0.3845	*0.0141*	0.6126	*0.0186*

Italy	0.3327	*0.0265*	0.5488	*0.0148*
Netherlands	0.4390	*0.0169*	0.6059	*0.0724*
Norway	0.4163	*0.0258*	0.7370	*0.0387*
Spain	0.3936	*0.0134*	0.5096	*0.0463*
United Kingdom	0.4988	*0.0270*	0.6929	*0.0211*
United States	0.4879	*0.0179*	0.6921	*0.0365*

Source: OECD.

5.2 Parameter estimation

Labor productivity is assumed to grow at an exponential rate according to (4),

$$a = a_0 e^{\alpha t}; \alpha > 0.$$

The ratio of gross national income (GNI) to total net capital stock ratio measures labor productivity. Ordinary Least Squares (OLS) was used to estimate α, the growth rate of productivity. Similarly, labor supply growth is assumed to be exponential as well. Consistent with equation (5),

$$n = n_0 e^{\beta t}; \beta > 0,$$

OLS estimates the growth of the 15-65 years of age population segment, β. In line with equation (6) and a constant capital-output ratio (inverse of capital productivity) is assumed,

$$\sigma = k / q.$$

To estimate this ratio, the mean value of the total net capital to gross national product is used. Finally, the linear estimators of the real wage Phillips curve are γ and ρ, in line with (10),

$$\frac{\dot{w}}{w} = -\gamma + \rho v.$$

An OLS estimation was applied to employment rate and inflation data. Inflation was measured by the implicit price deflator. The results of these estimates are summarized in Table 2 (following page).