Macroeconomics Project

Emma Terblanche, Jacques Rossouw & Samantha Scott

30/04/2022

Contents

Chapter 1

Introduction [EDIT]

According to Belongia and Ireland (2020), nominal interest rate management has been the Federal Reserve's approach to stabilize the output gap and inflation. The Taylor Rule is a good predictor of the nominal interest rate management. Interest rate management has been endorsed by literature and has shown advantages over the management of money stock, however, economists have only argued against a constant monetary growth rule but have neglected the potential benefits of a potential benefits of a flexible monetary growth rule. The Belongia and Ireland (2020) paper is a reconsideration of money growth rules in an estimated New Keynesian model.In this paper, the New Keynesian model by Belongia and Ireland (2020) is replicated. Firstly, the model is explained. The first order conditions log-linearisations are then presented.

Chapter 2

The Model

The model economy from THE PAPER includes the representative household, a representative finished goods-producing firm, "i" intermediate goods-producing firms (such that $i \in [0;1]$, is a continuum) and a central bank. For each period, a unique intermediate good is produced by each intermediate goods-producing firm i, such that the intermediate goods adopt the same indexing notation, $i \in [0;1]$. The symmetry of this model suggests that the focus can be narrowed to a representative intermediate goods-producing firm instead. This representative firm then produces good intermediate good i. The household preferences are described by their expected utility function. These preferences, along with the incomplete indexation of sticky nominal goods prices that are determined by the monopolistically competitive intermediate goods-producing firms suggests a New Keynesian IS and Phillips Curve that are both forward, and backwards-looking. Monetary Policy is assumed to follow a version of the Taylor (1993) rule, in line with the Federal Reserve behaviour over the sample period from 1983 to 2019. To allow for alternative monetary policy rules, the money demand curve used in the paper is chosen such that it may be consistent with US data ranging over the same sample period. With the basic model set up in this way, the derivations of the model's first order conditions for each representative is replicated in the following sections, followed by the complete system of linearised equations.

Chapter 3

Derivations of Model's First Order Conditions:

3.1 The Representative Household

The representative household, at the beginning of each each period $t = 0, 1, 2, ...$ holds M_{t-1} units of money and B_{t-1} units of bonds. Additionally, the household also receives a lump-sum monetary transfer T_t from the central bank at the beginning of each period t. The household's bonds mature, yielding B_{t-1} additional units of money. The household purchases B_t new bonds with a portion of its money, at an asking price $1/r_t$ units of money per bond. The gross nominal interest rate between time t and $t+1$ is therefore denoted by r_t. The household is paid $W_t h_t$ in labour income for supplying $h_t(i)$ units of labour to each intermediate goods-producing firm, given a wage rate of W_t, for each period t, where:

$$h_t = \int h_t(i)\, di$$

indicates the total hours worked at t. C_t is the household consumption at t, purchased from the final goods-producing firm at the nominal price P_t. Finally, at the end of each period t, nominal profits $D_t(i)$ from each intermediate goods-producing firm $i \in [0; 1]$ is paid out to the household. This allows the household to then carry M_t units of money into the next period, subject to the budget constraint:

$$\frac{M_{t-1} + T_t + B_{t-1} + W_t h_t + D_t}{P_t} \geq C_t + \frac{M_t + \frac{B_t}{r_t}}{P_t}$$

$\forall\ t = 0, 1, 2, ...,$ where

$$D_t = \int_0^1 D_t(i)\ di$$

is the total profits received at t.

The household's preferences are described by the expected utility function

$$EU(\cdot_t) = E_0 \sum_{t=0}^{\infty} \beta^t a_t [ln(C_t - \gamma C_{t-1}) + v(\frac{M_t}{P_t Z_t}, u_t)$$
$$-(\frac{\phi_m}{2})(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1)^2(\frac{M_t}{P_t Z_t}) - h_t]$$

where $0 < \beta < 1$ and $0 \leq \gamma \leq 1$. The preference shock a_t follows the stationary autoregressive process:

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at} \tag{2}$$

$\forall\ t = 0, 1, 2, ...,$ with $0 \leq \rho_a \leq 1$, and the serially uncorrelated innovation $\varepsilon_{at} \sim norm(0, \sigma_a)$. Utility is additively separable across consumption, real balances, and hours worked such that a specification for the model's IS curve that excludes any money and employment terms. For balanced growth, real balances $M_t = P_t$ is introduced into the utility function through the function v scaled by the aggregate productivity shock Z_t, a random walk process with drift:

$$\ln(Z_t) = \ln z + \ln(Z_{t-1}) + \varepsilon_{zt} \tag{3}$$

where $\epsilon_{zt} \sim norm(0, \sigma_z)$. The money demand shock u_t is a stationary autoregressive process:

$$\ln(u_t) = \rho_u \ln(u_{t-1}) + \varepsilon_{ut} \tag{4}$$

$\forall\ t = 0, 1, 2, ...$ with $0 \leq \rho_u < 1$, where $\varepsilon_{ut} \sim norm(0, \sigma_u)$. The magnitude of the real balances adjustment cost is $\phi_m \geq 0$. In steady state this will equal zero. The household therefore chooses Ct, ht, Bt, and $Mt\ \forall\ t = 0, 1, 2, ...,$ such that expected utility is maximised, subject to the budget constraint (1) $\forall\ t = 0, 1, 2, ...,$

Given the Household budget constraint:

$$\left[\frac{M_{t-1} + T_t + B_{t-1} + W_t h_t + D_t}{P_t} \geq C_t + \frac{M_t + \frac{B_t}{r_t}}{P_t}\right]$$

We can combine the expected utility and budget constraint above, to obtain the LaGrangian for the Household:

$$\mathcal{L}(\cdot_t) = a_t U(\cdot_t) + \beta E_t \mathcal{L}(\cdot_{t+1}) + \Lambda_t \left[\frac{M_{t-1} + T_t + B_{t-1} + W_t h_t + D_t}{P_t} - C_t - \left(\frac{M_t + \frac{B_t}{r_t}}{P_t} \right) \right]$$

where:

$$\begin{aligned} \mathcal{L}(\cdot_t) = \mathcal{L}(C_t, h_t, B_t, M_t) = & E_0 \sum_{t=0}^{\infty} \beta^t a_t \left[\ln(C_t - \gamma C_{t-1}) + v(\frac{M_t}{P_t Z_t}, u_t) - \frac{\phi_m}{2} (\frac{\frac{M_t}{P_t}}{\frac{z M_{t-1}}{P_{t-1}}} - 1)^2 (\frac{M_t}{P_t Z_t}) - h_t \right] \\ & + E_0 \sum_{t=0}^{\infty} \left[\Lambda_t \left(\frac{M_{t-1} + T_t + B_{t-1} + W_t h_t + D_t}{P_t} - C_t - \left(\frac{M_t + \frac{B_t}{r_t}}{P_t} \right) \right) \right] \end{aligned}$$

3.1.1 The FOC with respect to C_t:

$$\frac{\partial \mathcal{L}(C_t, h_t, B_t, M_t)}{\partial C_t} = a_t (\frac{1}{C_t - \gamma C_{t-1}}) - \Lambda_t + E_t \beta a_{t+1} (-\gamma) (\frac{1}{C_{t+1} - \gamma C_t})$$

Equate to zero and solve:

$$0 = a_t (\frac{1}{C_t - \gamma C_{t-1}}) - \Lambda_t + E_t \beta a_{t+1} (-\gamma) (\frac{1}{C_{t+1} - \gamma C_t})$$

$$\Lambda_t = a_t (\frac{1}{C_t - \gamma C_{t-1}}) + E_t \beta a_{t+1} (-\gamma) (\frac{1}{C_{t+1} - \gamma C_t}) \tag{5}$$

3.1.2 The FOC with respect to h_t:

$$\frac{\partial \mathcal{L}(C_t, h_t, B_t, M_t)}{\partial h_t} = (-1)(a_t) + (\frac{W_t}{P_t})(\Lambda_t)$$

Equate to zero and solve:

$$0 = (-1)(a_t) + (\frac{W_t}{P_t})(\Lambda_t)$$

$$(a_t) = (\frac{W_t}{P_t})(\Lambda_t)$$

$$a_t = (\frac{W_t}{P_t})(\Lambda_t) \tag{6}$$

3.1.3 The FOC with respect to B_t:

$$\frac{\partial \mathcal{L}(C_t, h_t, B_t, M_t)}{\partial B_t} = (\frac{-1}{r_t P_t})(\Lambda_t) + E_t \beta \Lambda_{t+1}(\frac{1}{P_{t+1}})$$

Equate to zero and solve:

$$0 = (\frac{-1}{r_t P_t})(\Lambda_t) + E_t \beta \Lambda_{t+1}(\frac{1}{P_{t+1}})$$

$$(\frac{1}{r_t P_t})(\Lambda_t) = E_t \beta \Lambda_{t+1}(\frac{1}{P_{t+1}})$$

$$\Lambda_t = (r_t P_t) E_t \beta \Lambda_{t+1}(\frac{1}{P_{t+1}})$$

$$\Lambda_t = \beta(r_t) E_t(\frac{P_t \Lambda_{t+1}}{P_{t+1}})$$

Because:

$$\pi_t = \frac{P_t}{P_{t-1}}$$

Therefore:

$$\frac{P_t}{P_{t+1}} = \frac{1}{\pi_{t+1}}$$

As such, the equation can be rewritten as:

$$\Lambda_t = \beta(r_t) E_t(\frac{\Lambda_{t+1}}{\pi_{t+1}}) \tag{7}$$

3.1.4 The FOC with respect to M_t:

$$\frac{\partial \mathcal{L}(C_t, h_t, B_t, M_t)}{\partial M_t} = a_t[v_1(\frac{M_t}{P_t Z_t}, u_t)\frac{1}{P_t Z_t} - \phi_m(\frac{\frac{M_t}{P_t}}{\frac{z_t M_{t-1}}{P_{t-1}}} - 1)(\frac{\frac{1}{P_t}}{\frac{Z M_{t-1}}{P_{t-1}}})(\frac{M_t}{P_t Z_t}) - \frac{\phi_m}{2}(\frac{\frac{M_t}{P_t}}{\frac{z_t M_{t-1}}{P_{t-1}}} - 1)^2(\frac{1}{P_t Z_t})]$$

$$+\Lambda_t[-\frac{1}{P_t}] + E_t \beta a_{t+1}\left[-\phi_m(\frac{M_{t+1}/P_{t+1}}{Z M_t/P_t} - 1)(-\frac{M_{t+1}/P_{t+1}}{z M_t^2/P_t})(\frac{M_{t+1}}{P_{t+1} Z_{t+1}})\right] + E_t \beta \Lambda_{t+1}(\frac{1}{P_{t+1}})$$

where:

$$v_1(\cdot) = \frac{\partial}{\partial M_t} v(\cdot)$$

We can now use equation (7) and the fact that $\frac{P_t}{P_{t-1}} = \pi_t$, to yield:

$$\beta E_t \Lambda_{t+1} \frac{1}{P_{t+1}} = \frac{1}{P_t r_t} \Lambda_t$$

multiply by $\frac{1}{P_t Z_t}$ throughout and set equal to zero:

$$\begin{aligned}\therefore 0 =& a_t v_1(\frac{M_t}{P_t Z_t}, u_t) - a_t \frac{\phi_m}{2}(\frac{\frac{M_t}{P_t}}{\frac{zM_{t-1}}{P_{t-1}}} - 1)^2 - a_t \phi_m(\frac{\frac{M_t}{P_t}}{\frac{zM_{t-1}}{P_{t-1}}} - 1)(\frac{\frac{M_t}{P_t}}{\frac{zM_{t-1}}{P_{t-1}}}) \\ & - \Lambda_t Z_t + \frac{Z_t}{r_t}\Lambda_t + E_t \beta a_{t+1}\left[\phi_m(\frac{M_{t+1}/P_{t+1}}{ZM_t/P_t} - 1)\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t}\right)^2 (\frac{zZ_t}{Z_{t+1}})\right]\end{aligned}$$

Rearranging yields the final result:

$$\begin{aligned}& a_t v_1\left(\frac{M_t}{P_t Z_t}, u_t\right) - a_t\left(\frac{\phi_m}{2}\right)\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)^2 \\ & - a_t \phi_m\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}}\right) \\ & + \beta\phi_m E_t\left[a_{t+1}\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t} - 1\right)\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t}\right)^2\left(\frac{zZ_t}{Z_{t+1}}\right)\right] \\ & = Z_t \Lambda_t\left(1 - \frac{1}{r_t}\right)\end{aligned} \tag{8}$$

Using the fact that:

$$v_1(\frac{M_t}{P_t Z_t}, u_t) = \frac{1}{\delta}[\ln(m*) - \ln(\frac{M_t}{P_t Z_t}) + \ln(u_t)]$$

We can rewrite (8) to yield:

$$\begin{aligned}& \tfrac{a_t}{\delta}\left[\ln(m^*) - \ln\left(\tfrac{M_t}{P_t Z_t}\right) + \ln(u_t)\right] - a_t\left(\tfrac{\phi_m}{2}\right)\left(\tfrac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)^2 \\ & -a_t\phi_m\left(\tfrac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)\left(\tfrac{M_t/P_t}{zM_{t-1}/P_{t-1}}\right) \\ & +\beta\phi_m E_t\left[a_{t+1}\left(\tfrac{M_{t+1}/P_{t+1}}{zM_t/P_t} - 1\right)\left(\tfrac{M_{t+1}/P_{t+1}}{zM_t/P_t}\right)^2\left(\tfrac{zZ_t}{Z_{t+1}}\right)\right] \\ & = Z_t\Lambda_t\left(1 - \tfrac{1}{r_t}\right)\end{aligned} \tag{9}$$

3.2 The Representative Finished Goods-Producing Firm

The representative finished goods-producing firm uses $Y_t(i)$ units of each intermediate good $i \in [0, 1]$. These intermediate goods are bought at the nominal price $P_t(i)$ in order to produce Y_t units of the final good according to the technology described by

$$[\int_0^1 Y_t(i)^{\frac{(\theta_t-1)}{\theta_t}} di]^{\frac{\theta_t}{\theta_t-1}} \geq Y_t$$

where θ_t translates into a random shock to the intermediate goods-producing firms' desired markup of price over marginal cost and therefore acts like a cost push shock of the kind introduced into the New Keynesian model by Clarida, Gali, and and Gertler (1999). Here, this markup shock follows the stationary autoregressive process

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t} \tag{10}$$

$\forall\ t = 0, 1, 2, ...,$, where $\varepsilon_{\theta t} \sim norm(0, \sigma_\theta)$. The finished goods-producing firm thus faces the problem of maximising its profits for each period t, by their choice of $Y_t(i)\ \forall\ i \in [0, 1]$

$$P_t[\int_0^1 Y_t(i)^{\frac{\theta_t-1}{\theta_t}} di]^{\frac{\theta_t}{\theta_t-1}} - \int_0^1 P_t(i)Y_t(i)\ di$$

The Langrangian function from which the first order conditions are derived, is as follows:

$$\mathcal{L} = \int_0^1 P_t(i)Y_t(i)\ di + \lambda_t[Y_t - (\int_0^1 Y_t(i)^{\frac{\theta_t-1}{\theta_t}}\ di)^{\frac{\theta_t}{\theta_t-1}}]$$

3.2.1 The FOC with respect to $Y_t(i)$:

$$\frac{\partial L}{\partial Y_t(i)} = P_t(i) - \lambda_t[\frac{\theta_t}{\theta_t-1}(\int_0^1 Y_t(i)^{\frac{\theta_t-1}{\theta_t}} di)^{\frac{\theta_t}{\theta_t-1}-1} \frac{\theta_t-1}{\theta_t} Y_t(i)^{\frac{\theta_t-1}{\theta_t}-1}] = 0$$

$$P_t(i) - \lambda_t[\frac{\theta_t}{\theta_t-1}(\int_0^1 Y_t(i)^{\frac{\theta_t-1}{\theta_t}} di)^{\frac{1}{\theta_t-1}} \frac{\theta_t-1}{\theta_t} Y_t(i)^{\frac{-1}{\theta_t}}] = 0$$

$$\lambda_t[\int_0^1 Y_t(i)^{\frac{\theta_t-1}{\theta_t}} di]^{\frac{1}{\theta_t-1}} Y_t(i)^{\frac{-1}{\theta_t}} = P_t(i)$$

$$\lambda_t[\int_0^1 Y_t(i)^{\frac{\theta_t-1}{\theta_t}} di]^{\frac{1}{\theta_t-1}} = P_t(i)Y_t(i)^{\frac{1}{\theta_t}}$$

$$\lambda_t[\int_0^1 Y_t(i)^{\frac{\theta_t-1}{\theta_t}} di]^{\frac{1}{\theta_t-1}} P_t(i)^{-1} = Y_t(i)^{\frac{1}{\theta_t}}$$

$$\lambda_t^{\theta_t}[\int_0^1 Y_t(i)^{\frac{\theta_t-1}{\theta_t}} di]^{\frac{\theta_t}{\theta_t-1}} P_t(i)^{-\theta_t} = Y_t(i)$$

$$\lambda_t^{\theta_t} Y_t P_t(i)^{-\theta_t} = Y_t(i)$$

Where $Y_t = [\int_0^1 Y_t(i)^{\frac{\theta_t - 1}{\theta_t}} di]^{\frac{\theta_t}{\theta_t - 1}}$

$$(\frac{\lambda_t}{P_t(i)})^{\theta_t} Y_t = Y_t(i)$$

Sub $Y_t(i) = (\frac{\lambda_t}{P_t(i)})^{\theta_t} Y_t$ into Y_t

$$[\int_0^1 ((\frac{\lambda_t}{P_t(i)})^{\theta_t} Y_t)^{\frac{\theta_t - 1}{\theta_t}} di]^{\frac{\theta_t}{\theta_t - 1}} = Y_t$$

$$[\int_0^1 ((\frac{P_t(i)}{\lambda_t})^{-\theta_t})^{\frac{\theta_t - 1}{\theta_t}} di]^{\frac{\theta_t}{\theta_t - 1}} = 1$$

$$(\frac{1}{\lambda_t})^{-\theta_t} [\int_0^1 (P_t(i))^{-\theta_t + 1} di]^{\frac{\theta_t}{\theta_t - 1}} = 1$$

$$[\int_0^1 (P_t(i))^{1-\theta_t} di]^{\frac{\theta_t}{\theta_t - 1}} = \lambda_t{}^{-\theta_t}$$

$$[\int_0^1 (P_t(i))^{1-\theta_t} di]^{\frac{1}{1-\theta_t}} = \lambda_t$$

Therefore:

$$\lambda_t = P_t$$

Sub $Y_t = P_t$ back in:

$$(\frac{P_t}{P_t(i)})^{\theta_t} Y_t = Y_t(i)$$

Therefore:

$$Y_t(i) = (\frac{P_t(i)}{P_t})^{-\theta_t} Y_t$$

3.3 The Representative Intermediate Goods-Producing Firm

During each period $t = 0, 1, 2, ...$, the representative intermediate goods-producing firm hires $h_t(i)$ units of labour from the representative household to manufacture $Y_t(i)$ units of intermediate good i according to the technology described by:

$$Z_t h_t(i) \geq Y_t(i) \tag{11}$$

where Z_t is the aggregate productivity shock introduced in (3).

The quadratic cost of adjusting its nominal price between periods is given by:

$$\frac{\phi_p}{2}[\frac{P_t(i)}{\pi_{t-1}^a \pi^{1-a} P_{t-1}(i)} - 1]^2 Y_t$$

The firm chooses $P_t(i)$ for all $t = 0, 1, 2, ...$ to maximise its total real market value proportional to:

$$E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t [\frac{D_t(i)}{P_t}]$$

where the firm's real profits are measured by:

$$\frac{D_t(i)}{P_t} = [\frac{P_t(i)}{P_t}]^{1-\theta_t} Y_t - [\frac{P_t(i)}{P_t}]^{-\theta_t} (\frac{W_t}{P_t})(\frac{Y_t}{Z_t}) - \frac{\phi}{2}[\frac{P_t(i)}{\pi_{t-1}^a \pi^{1-a} P_{t-1}(i)}]^2 Y_t \tag{12}$$

The Value function of this optimisation problem is:

$$v = max E_t \sum_{t=0}^{\infty} \beta^t \Lambda_t \frac{D_t(i)}{P_t}$$

As such, the Bellman Equation can be constructed as:

$$v = \beta^t \Lambda_t \frac{D_t(i)}{P_t} + \beta^{t+1} E_t \Lambda_{t+1} \frac{D_{t+1}(i)}{P_{t+1}}$$

$$v = \beta^t \Lambda_t [(\frac{P_t(i)}{P_t})^{1-\theta_t} Y_t - (\frac{P_t(i)}{P_t})^{-\theta_t} (\frac{W_t}{P_t})(\frac{Y_t}{Z_t}) - \frac{\phi}{2}(\frac{P_t(i)}{\pi_{t-1}^a \pi^{1-a} P_{t-1}(i)})^2 Y_t] + \beta^{t+1} E_t \Lambda_{t+1} [(\frac{P_{t+1}(i)}{P_{t+1}})^{1-\theta_{t+1}} Y_{t+1}$$

$$-(\frac{P_{t+1}(i)}{P_{t+1}}]^{-\theta_{t+1}} (\frac{W_{t+1}}{P_{t+1}})(\frac{Y_{t+1}}{Z_{t+1}}) - \frac{\phi_p}{2}[\frac{P_{t+1}(i)}{\pi_t^a \pi^{1-a} P_t(i)}]^2 Y_{t+1}]$$

3.3.1 The FOC with respect to $P_t(i)$

$$\frac{\partial v}{\partial P_t(i)} = \beta^t \Lambda_t (1-\theta_t)(\frac{P_t(i)}{P_t})^{-\theta_t}(\frac{1}{P_t})Y_t - \beta^t\Lambda_t[(-\theta_t\frac{P_t(i)}{P_t})^{-\theta_t-1}(\frac{1}{P_t})(\frac{W_t}{P_t})(\frac{Y_t}{Z_t})]$$

$$-\beta^t\Lambda_t[\phi_p(\frac{P_t(i)}{\pi_{t-1}^a\pi^{1-a}P_{t-1}(i)}-1)(\frac{1}{\pi_{t-1}^a\pi^{1-a}P_{t-1}(i)})Y_t]-\beta^{t+1}E_t\Lambda_{t+1}\phi_p[(\frac{P_{t+1}(i)}{\pi_t^a\pi^{1-a}P_t(i)}-1)(\frac{-P_{t+1}(i)Y_{t+1}}{\pi_t^a\pi^{1-a}P_t(i)^2})]=0$$

Next, in order to cancel out most Λ_t, Y_t and P_t's, multiply through by $\frac{P_t}{\Lambda_t Y_t}$:

$$\beta^t(1-\theta_t)(\frac{P_t(i)}{P_t})^{-\theta_t}+\beta^t[\theta_t(\frac{P_t(i)}{P_t})^{-\theta_t-1}(\frac{W_t}{P_t})(\frac{1}{Z_t})]-\beta^t\phi_p[(\frac{P_t(i)}{\pi_{t-1}^a\pi^{1-a}P_{t-1}(i)}-1)(\frac{P_t}{\pi_{t-1}^a\pi^{1-a}P_{t-1}(i)})]$$
$$+\beta^{t+1}\phi_p E_t(\frac{\Lambda_{t+1}}{\Lambda_t})(\frac{P_{t+1}(i)}{\pi_t^a\pi^{1-a}P_t(i)}-1)(\frac{P_{t+1}(i)}{\pi_t^a\pi^{1-a}P_t(i)})(\frac{Y_{t+1}}{Y_t})(\frac{P_t}{P_t(i)})=0$$

$$(1-\theta_t)(\frac{P_t(i)}{P_t})^{-\theta_t}+\theta_t(\frac{P_t(i)}{P_t})^{-\theta_t-1}(\frac{W_t}{P_t})(\frac{1}{Z_t})-\phi_p[(\frac{P_t(i)}{\pi_{t-1}^a\pi^{1-a}P_{t-1}(i)}-1)(\frac{P_t}{\pi_{t-1}^a\pi^{1-a}P_{t-1}(i)})]$$
$$+\beta\phi_p E_t(\frac{\Lambda_{t+1}}{\Lambda_t})(\frac{P_{t+1}(i)}{\pi_t^a\pi^{1-a}P_t(i)}-1)(\frac{P_{t+1}(i)}{\pi_t^a\pi^{1-a}P_t(i)})(\frac{Y_{t+1}}{Y_t})(\frac{P_t}{P_t(i)})=0 \qquad (13)$$

and (11) with equality for all $t = 0, 1, 2, ...,$

3.4 The Efficient Level of Output and Output Gap

A social planner for this economy who can overcome the frictions associated with monetary trade, sluggish price adjustment, and the monopolistically competitive structure of the intermediate goods-producing sector chooses Q_t and $n_t(i)$ for all $i \in [0,1]$ to maximize the social welfare function

$$E_0\sum_{t=0}^{\infty}\beta^t a_t[\ln(Q_t-\gamma Q_{t-1})-\int_0^1 n_t(i)di]$$

subject to

$$Z_t[\int_0^1 n_t(i)di]^{\frac{\theta_t}{\theta_t-1}} \geq Q_t$$

As such, the Bellman Equation of this model is:

$$v(Q_{t-1}) = a_t[\ln(Q_t-\gamma Q_{t-1})-\int_0^1 n_t(i)di]+\beta E_t v(Q_t,Q_{t+1})+\Xi[Z_t(\int_0^1 n_t(i)di)^{\frac{\theta_t}{\theta_t-1}}-Q_t]$$

3.4.1 The FOC with respect to Q_t:

$$\frac{\partial v(Q_{t-1})}{\partial Q_t} = \frac{a_t}{Q_t - \gamma Q_{t-1}} + \beta E_t \frac{\partial v(Q_t, Q_{t+1})}{\partial Q_t} - \Xi = 0$$

We know that

$$\frac{\partial v(Q_{t-1})}{\partial Q_{t-1}} = \frac{a_t}{Q_t - \gamma Q_{t-1}}(-\gamma)$$

Therefore, when iterating forward, we find that:

$$\frac{\partial v(Q_t)}{\partial Q_t} = \frac{a_{t+1}}{Q_{t+1} - \gamma Q_t}(-\gamma)$$

Subbing $\frac{\partial v(Q_t)}{\partial Q_t}$ back in gives:

$$\frac{\partial v(Q_{t-1})}{\partial Q_t} = \frac{a_t}{Q_t - \gamma Q_{t-1}} + \beta E_t \frac{a_{t+1}}{Q_{t+1} - \gamma Q_t}(-\gamma) - \Xi = 0$$

Multiply through by (-1):

$$\frac{a_t}{Q_t - \gamma Q_{t-1}} - \beta E_t \frac{a_{t+1}}{Q_{t+1} - \gamma Q_t}(\gamma) = \Xi$$

3.4.2 The FOC with respect to n_t:

$$\frac{\partial v(Q_{t-1})}{\partial n_t} = a_t(-1) + \Xi Z_t \frac{\theta_t}{\theta_t - 1}\left(\int_0^1 n_t(i)^{\frac{\theta_t - 1}{\theta_t}} di\right)^{\frac{\theta_t}{\theta_t - 1}}\left(\frac{\theta_t - 1}{\theta_t}\right) n_t(i)^{\frac{\theta_t - 1}{\theta_t} - 1} = 0$$

$$a_t + \Xi Z_t \left(\int_0^1 n_t(i)^{\frac{\theta_t - 1}{\theta_t}} di\right)^{\frac{1}{\theta_t - 1}} n_t(i)^{\frac{-1}{\theta_t}} = 0$$

$$\Xi Z_t \left(\int_0^1 n_t(i)^{\frac{\theta_t - 1}{\theta_t}} di\right)^{\frac{1}{\theta_t - 1}} n_t(i)^{\frac{-1}{\theta_t}} = a_t$$

The Feasibility Constraint of this problem is:

$$\left(\int_0^1 n_t(i)^{\frac{\theta_t - 1}{\theta_t}} di\right)^{\frac{\theta_t}{\theta_t - 1}} \geq \frac{Q_t}{Z_t}$$

Rearranging the Feasibility Constraint gives:

$$\left(\int_0^1 n_t(i)^{\frac{\theta_t - 1}{\theta_t}} di\right)^{\frac{1}{\theta_t - 1}} \geq \left(\frac{Q_t}{Z_t}\right)^{\frac{1}{\theta_t}}$$

Therefore

$$\Xi Z_t \left(\frac{Q_t}{Z_t}\right)^{\frac{1}{\theta_t}} n_t(i)^{\frac{-1}{\theta_t}} = a_t$$

Chapter 4

Log-Linearisation:

4.1 Obtaining the final system of equations:

4.1.1 Conditions:

To rewrite the equations into a usable system of equations that can be log-linearised, we make use of the following series of steady state, stationary or equilibrium equations and variables:

4.1.1.1 A:

$Y_t(i) = Y_t, h_t(i) = h_t, D_t(i) = D_t,$ and $P_t(i) = P_t$ for all $i \in [0,1]$ and $t = 0, 1, 2, \ldots$

4.1.1.2 B:

$$M_t = M_{t-1} + T_t \text{ and } B_t = B_{t-1} = 0$$

4.1.1.3 C:

$y_t = Y_t/Z_t, c_t = C_t/Z_t, m_t = (M_t/P_t)/Z_t, q_t = Q_t/Z_t, \lambda_t = Z_t\Lambda_t,$ and $z_t = Z_t/Z_{t-1}$

4.1.1.4 D:

In steady state we can write the following, per their definitions:

$$\begin{aligned}
&y_t = y,\ c_t = c,\ \pi_t = \pi,\ r_t = r,\ m_t = m,\ q_t = q,\ x_t = x, \\
&\mu_t = \mu,\ g_t = g,\ \lambda_t = \lambda,\ a_t = a = 1,\ z_t = z,\ u_t = u = 1, \\
&\text{and } \theta_t = \theta
\end{aligned}$$

4.1.2 Imposing The Coniditions:

Firstly, we combine equation (11) and (12) from Section 3.3 The Representative Intermediate Goods-Producing Firm and impose the above conditions to obtain equation (1) from the appendix of "THE PAPER":

$$Y_t = C_t + \frac{\phi_p}{2} \cdot \left[\frac{\pi_t}{{\pi_{t-1}}^{\alpha} \pi^{1-\alpha}} - 1 \right]^2 \cdot Y_t$$

Dividing both sides by Z_t we can substitute for the necessary stationarity variables and rewrite as:

$$y_t = c_t + \frac{\phi_p}{2} \left(\frac{\pi_t}{\pi_{t-1}^{\alpha} \pi^{1-\alpha}} - 1 \right)^2 y_t \tag{1}$$

Directly from Equations (2)-(4) in Section 3.1 The Representative Household:

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at} \tag{2}$$

$$\ln(Z_t) = \ln(z) + \ln(Z_{t-1}) + \varepsilon_{zt}$$

- Subtract both sides by $\ln(Z_{t-1})$

$$\ln(z_t) = \ln(z) + \varepsilon_{zt} \tag{3}$$

$$\ln(u_t) = \rho_u \ln(u_{t-1}) + \varepsilon_{ut} \tag{4}$$

From The FOC with respect to C_t:

$$\Lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta\gamma E_t \left[\frac{a_{t+1}}{C_{t+1} - \gamma C_t} \right]$$

$$\Rightarrow \lambda_t = \frac{a_t z_t}{z_t c_t - \gamma c_{t-1}} - \beta\gamma E_t \left(\frac{a_{t+1}}{z_{t+1} c_{t+1} - \gamma c_t} \right) \tag{5}$$

$$\lambda_t = \beta r_t E_t \left(\frac{\lambda_{t+1}}{z_{t+1} \pi_{t+1}} \right) \tag{7}$$

Rewrite equation (9) from The FOC with respect to M_t:

$$
\begin{aligned}
& a_t v_1\left(\frac{M_t}{P_t Z_t}, u_t\right) - a_t\left(\frac{\phi_m}{2}\right)\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)^2 \\
& - a_t\phi_m\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}}\right) \\
& + \beta\phi_m E_t\left[a_{t+1}\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t} - 1\right)\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t}\right)^2\left(\frac{zZ_t}{Z_{t+1}}\right)\right] \\
& = Z_t\Lambda_t\left(1 - \frac{1}{r_t}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{a_t}{\delta}\left[\ln\left(m^*\right) - \ln\left(m_t\right) + \ln\left(u_t\right)\right] - a_t\left(\frac{\phi_m}{2}\right)\left(\frac{z_t m_t}{zm_{t-1}} - 1\right)^2 \\
& - a_t\phi_m\left(\frac{z_t m_t}{zm_{t-1}} - 1\right)\left(\frac{z_t m_t}{zm_{t-1}}\right) \\
& + \beta\phi_m E_t\left[a_{t+1}\left(\frac{z_{t+1}m_{t+1}}{zm_t} - 1\right)\left(\frac{z_{t+1}m_{t+1}}{zm_t}\right)^2\left(\frac{z}{z_{t+1}}\right)\right] \\
& = \lambda_t\left(1 - \frac{1}{r_t}\right),
\end{aligned}
$$

4.2 Applying The Taylor Method of Linear Approximation

We apply the Taylor throughout the following section as follows:

For a function $f(x_t)$ input x_t , we apply the Taylor method such that:

$$f(x_t) = f(X^{SS}) + \frac{df(x)}{x_t}(x_t(i) - x^{SS})$$

Using the approximation:

$$
\begin{aligned}
& x_t - x^{SS} \approx x^{SS}\hat{x}_t \\
& \text{where } \hat{x}_t = \ln(x_t) - \ln(x^{SS}) \\
& \text{For short, we write: } x^{SS} \equiv x \\
& \text{(i.e. no time subscript)}
\end{aligned}
$$

4.3 Uhlig's Method of Linearisation

Applying Uhlig's method requires the following:

For the function $f(x_t)$ i.e., we use the fact that:

1.
$$\hat{x}_t = \ln(x_t) - \ln(x) \text{ where x } \equiv x^{SS}$$

2. Then, $f(^{SS})$ is used to simplify $f(x_t)$ where x_t is replaced accodring to $x_t = x \cdot e^{\hat{x}_t}$
3. Additionally, the approximation result $e^{x_t+y_t} \approx 1 + x_t + y_t$ is used for further simplification.
4. For the last step, all constants can be dropped.

For the following sections, we show the steps to reach the final log-linearised equations (19) to (31) from in Section 2.7 of "THE PAPER".

4.4 Equation (1) from the appendix of "THE PAPER"

Recall the results from equation (11), (12), and the Budget Constraint:

$$Z_t h_t(i) \geq Y_t(i) \tag{11}$$

$$\frac{D_t(i)}{P_t} = [\frac{P_t(i)}{P_t}]^{1-\theta_t} Y_t - [\frac{P_t(i)}{P_t}]^{-\theta_t} (\frac{W_t}{P_t})(\frac{Y_t}{Z_t}) - \frac{\phi}{2}[\frac{P_t(i)}{\pi_{t-1}^a \pi^{1-a} P_{t-1}(i)}]^2 Y_t \tag{12}$$

$$\frac{M_{t-1} + T_t + B_{t-1} + W_t h_t + D_t}{P_t} \geq C_t + \frac{M_t + B_t/r_t}{P_t}$$

We can now apply the equilibrium conditions to obtain the results:

$$Z_t h_t = Y_t$$

$$\begin{aligned}
\frac{D_t}{P_t} &= [\frac{P_t}{P_t}]^{1-\theta_t} Y_t - [\frac{P_t}{P_t}]^{-\theta_t} (\frac{W_t}{P_t})(\frac{Y_t}{Z_t}) - \frac{\phi}{2}[\frac{P_t}{\pi_{t-1}^a \pi^{1-a} P_{t-1}}]^2 Y_t \\
&= Y_t - (\frac{W_t}{P_t})(\frac{Y_t}{Z_t}) - \frac{\phi}{2}[\frac{P_t}{\pi_{t-1}^a \pi^{1-a} P_{t-1}}]^2 Y_t
\end{aligned}$$

$$\begin{aligned}
C_t &= \frac{M_{t-1} + T_t + \frac{W_t Y_t}{Z_t} + D_t - M_{t-1} - T_t}{P_t} \\
&= \frac{W_t Y_t}{Z_t P_t} + \frac{D_t}{P_t}
\end{aligned}$$

$$\begin{aligned}
-Y_t &= -\left[\frac{D_t}{P_t} + (\frac{W_t}{P_t})(\frac{Y_t}{Z_t})\right] - \frac{\phi}{2}[\frac{P_t}{\pi_{t-1}^a \pi^{1-a} P_{t-1}}]^2 Y_t \\
\therefore Y_t &= C_t - \frac{\phi}{2}[\frac{P_t}{\pi_{t-1}^a \pi^{1-a} P_{t-1}}]^2 Y_t
\end{aligned}$$

Finally, we can now rewrite & linearise the final equation:

$$\begin{aligned}
y_t &= c_t + \frac{\phi_p}{2} \cdot \left[\frac{\pi_t}{{\pi_{t-1}}^{\alpha} \pi^{1-\alpha}} - 1 \right]^2 \cdot y \\
&= c_t + \frac{\phi_p}{2} \cdot \left(\frac{\pi_t}{{\pi_{t-1}}^{\alpha} \pi^{1-\alpha}} \right)^2 \cdot y_t - \phi_p \cdot \frac{\pi_t}{{\pi_{t-1}}^{\alpha} \pi^{1-\alpha}} \cdot y_t + \frac{\phi_p}{2} \cdot y_t \\
&= c_t + p_1 - p_2 + p_3
\end{aligned}$$

Now, we apply the Taylor method to each p_i, to get:

$$\begin{aligned}
p_1^{SS} &= \frac{\phi_p \cdot y}{2} \\
\left[\frac{dp_1}{dy_t} \right]^{SS} &= \frac{\phi_p}{2} \\
\left[\frac{dp_1}{d\pi_t} \right]^{SS} &= \frac{\phi_p \cdot y}{\pi} \\
\left[\frac{dp_1}{d\pi_{t-1}} \right]^{SS} &= -\frac{\phi_p \cdot y}{\pi}
\end{aligned}$$

4.4.1 p1

Therefore, we can rewrite p_1 using the Taylor method, as:

$$\begin{aligned}
p_1 &= \frac{\phi_p \cdot y}{2} + \frac{\phi_p}{2} y \cdot \hat{y}_t + \frac{\phi_p \cdot y}{\pi} \pi \cdot \hat{\pi}_t - \frac{\phi_p \cdot y}{\pi} \pi \cdot \pi_{\widehat{t-1}} \\
&= \frac{\phi_p \cdot y}{2} + \frac{\phi_p \cdot y \cdot \hat{y}_t}{2} + \phi_p \cdot y \cdot \hat{\pi}_t - \alpha \cdot \phi_p \cdot y \cdot \hat{\pi}_{t-1}
\end{aligned}$$

4.4.2 p2

and for p_2

$$\begin{aligned}
[p_2]^{SS} &= \phi_p y \\
\left[\frac{dp_2}{dy_t} \right]^{SS} &= \phi_p \\
\left[\frac{dp_2}{d\pi_t} \right]^{SS} &= \frac{\phi_p y}{\pi} \\
\left[\frac{dp_2}{d\pi_{t-1}} \right]^{SS} &= -\alpha \frac{\phi_p y}{\pi}
\end{aligned}$$

Then, writing the Taylor approximation for p_2

$$
\begin{aligned}
p_2 &= \phi_p y + \phi_p y \hat{y}_t + \frac{\phi_p y}{\pi}\pi\hat{\pi}_t - \alpha\frac{\phi_p y}{\pi}\pi\hat{\pi}_{t-1} \\
&= \phi_p y + \phi_p y \hat{y}_t + \phi_p y \hat{\pi}_t - \alpha\phi_p y \hat{\pi}_{t-1}
\end{aligned}
$$

4.4.3 p3

Lastly, repeating the same process for the last term p_3

$$
\begin{aligned}
[p_3]^{SS} &= \frac{\phi_p y}{2} \\
\left[\frac{dp_3}{dy_t}\right]^{SS} &= \frac{\phi_p}{2}
\end{aligned}
$$

Then, writing the Taylor approximation for p_3

$$
p_3 = \frac{\phi_p y}{2} + \frac{\phi_p}{2} y \hat{y}_t
$$

Putting it all together with $y_t = c + p_1 - p_2 + p_3$ we get:

$$
\begin{aligned}
y_t = c_t + &\left[\overbrace{\frac{\phi_p \cdot y}{2}}^{A} + \overbrace{\frac{\phi_p \cdot y \cdot \hat{y}_t}{2}}^{B} + \overbrace{\phi_p \cdot y \cdot \hat{\pi}_t}^{C} - \overbrace{\alpha \cdot \phi_p \cdot y \cdot \hat{\pi}_{t-1}}^{D}\right] \\
&- \left[\overbrace{\phi_p y}^{2A} + \overbrace{\phi_p y \hat{y}_t}^{2B} + \overbrace{\phi_p y \hat{\pi}_t}^{C} - \overbrace{\alpha\phi_p y \hat{\pi}_{t-1}}^{D}\right] + \left[\overbrace{\frac{\phi_p y}{2}}^{A} + \overbrace{\frac{\phi_p}{2} y \hat{y}_t}^{B}\right] \\
= c_t + A + B + C - D - 2A - 2B - C + D + A + B
\end{aligned}
$$

Now, using $y = c + \frac{\phi_p \cdot y}{2} - \phi_p y + \frac{\phi_p y}{2} = c$, we subtract y on both sides, such that

$$
\begin{aligned}
A + B + C - D - 2A - 2B - C + D + A + B = 0 &\text{ and thus,} \\
y_t - y &= c_t - c \\
\Rightarrow y \cdot \hat{y}_t &= c \cdot \hat{c}_t \\
\Rightarrow \hat{y}_t &= \hat{c}_t
\end{aligned}
$$

4.5 Equation (19) in paper:

From equation (5)

$$
\Lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta\gamma E_t\left[\frac{a_{t+1}}{C_{t+1} - \gamma C_t}\right]
$$

Using $c_t = C_t/Z_t$ $\lambda_t = Z_t \cdot \Lambda_t$, and $z_t = Z_t/Z_{t-1}$ we can rewrite the above equation using its stationary variables:

$$\lambda_t/Z_t = \frac{a_t}{Z_t c_t - \gamma Z_{t-1} c_{t-1}} - \beta\gamma E_t \left(\frac{a_{t+1}}{Z_{t+1}c_{t+1} - \gamma Z_{t-1} c_t} \right)$$

times each term with: (Z_{t-1}/Z_{t-1}) and (Z_t/Z_t) respectively

$$\lambda_t = \frac{a_t z_t}{z_t c_t - \gamma c_{t-1}} - \beta\gamma E_t \left(\frac{a_{t+1}}{z_{t+1}c_{t+1} - \gamma c_t} \right)$$

Once again, for simplicity, we separate the equation into two parts:

$$\lambda_t = \frac{a_t z_t}{z_t c_t - \gamma c_{t-1}} - \beta\gamma Et \left(\frac{a_{t+1}}{z_{t+1}c_{t+1} - \gamma c_t} \right) = a_1 + \beta\gamma E_t(a_2)$$

Now we can determine the taylor approximations individually

4.5.1 a1

$$\begin{aligned}
[a_1]^{SS} &= \frac{az}{zc - \gamma c} = \frac{az}{(z-\gamma)c} \\
\left[\frac{da_1}{da_t}\right]^{SS} &= \frac{z}{(z-\gamma)c} \\
\left[\frac{da_1}{dz_t}\right]^{SS} &= \frac{azc - \gamma ac - azc}{(zc - \gamma c)^2} \\
&= -\frac{\gamma ac}{(zc - \gamma c)^2} \\
\left[\frac{da_1}{dc_t}\right]^{SS} &= \frac{da_1}{dc_t}\left[\frac{a_t z_t}{(z_t c_t - \gamma c_{t-1})}\right]^{SS} \\
&= -a_t z_t \left(z_t c_t - \gamma c_{t-1}\right)^{-2} \cdot zt \\
&= -\frac{azz}{(zc - \gamma c)^2} \\
\left[\frac{da_1}{dc_{t-1}}\right]^{SS} &= \left[\frac{d}{dc_{t-1}}\left[-a_t z_t \left(z_t c_t - \gamma c_{t-1}\right)^{-1}\right]\right]^{SS} \\
&= \left[-a_t z_t \left(z_t c_t - \gamma c_{t-1}\right)^{-2} \cdot (-\gamma)\right]^{SS} \\
&= \frac{az\gamma}{(zc - \gamma c)^2}
\end{aligned}$$

The similarity between a_1 and a_2 means that some of the steps will yield very similar results:

4.5.2 a2

$$
\begin{aligned}
[a_2]^{SS} &= [a_1]^{SS}/z \\
&= \frac{a}{(z-\gamma)c} \\
\left[\frac{da_2}{da_{t+1}}\right]^{SS} &= \left[\frac{da_1}{da_t}\right]^{SS} \cdot 1/z \\
&= \frac{1}{(z-\gamma)c} \\
\left[\frac{da_2}{dz_{t+1}}\right]^{SS} &= \left\{\frac{d}{dz_{t+1}}\left[\frac{a_{t+1}}{z_{t+1}c_{t+1}-\gamma c_t}\right]\right\}^{SS} \\
&= \left\{-a_{t+1}\left[z_{t+1}c_{t+1}-\gamma c_t\right]^{-2}\cdot c_{t+1}\right\}^{SS} \\
&= -\frac{ac}{zc-\gamma c} \\
\left[\frac{da_2}{dc_{t+1}}\right]^{SS} &= \left[\frac{da_1}{dc_t}\right]^{SS} \cdot 1/z \\
&= -\frac{az}{(zc-\gamma c)^2} \\
\left[\frac{da_2}{dc_t}\right]^{SS} &= \left[\frac{da_1}{dc_{t-1}}\right]^{SS} \cdot 1/z \\
&= \frac{a\gamma}{(zc-\gamma c)^2}
\end{aligned}
$$

Thus, we have

$\lambda_t = a_1 - \beta\gamma E_t(a_2)$ which we can expand, and $\lambda = \dfrac{az}{(z-\gamma)c} - \beta\gamma\dfrac{a}{(z-\gamma)c}$

4.5.3 Lambda

$$
\begin{aligned}
\lambda_t = &\overbrace{\left[\frac{\mathbf{az}}{(\mathbf{z}-\gamma)\mathbf{c}} + \frac{z}{(z-\gamma)c}\cdot a\hat{a}_t - \frac{\gamma ac}{(zc-\gamma c)^2}\cdot z\hat{z}_t - \frac{azz}{(zc-\gamma c)^2}\cdot c\hat{c}_t + \frac{az\gamma}{(zc-\gamma c)^2}\cdot c\hat{c}_{t-1}\right]}^{a_1} \\
&- \beta\gamma E_t \overbrace{\left[\frac{\mathbf{a}}{(\mathbf{z}-\gamma)\mathbf{c}} + \frac{1}{(z-\gamma)c}\cdot a\hat{a}_{t+1} - \frac{ac}{zc-\gamma c}z\hat{z}_{t+1} - \frac{az}{(zc-\gamma c)^2}\cdot c\hat{c}_{t+1} + \frac{a\gamma}{(zc-\gamma c)^2}\cdot c\hat{c}_t\right]}^{a_2}
\end{aligned}
$$

subtracting λ from both sides of the equations to get $\lambda_t - \lambda$ on the left side, we get:

$$\lambda\hat{\lambda}_t = \left[\frac{z}{(z-\gamma)c}\cdot a\hat{a}_t - \frac{\gamma ac}{(zc-\gamma c)^2}\cdot z\hat{z}_t - \frac{azz}{(zc-\gamma c)^2}\cdot c\hat{c}_t + \frac{az\gamma}{(zc-\gamma c)^2}\cdot c\hat{c}_{t-1}\right]$$
$$-\beta\gamma E_t\left[\frac{1}{(z-\gamma)c}\cdot a\hat{a}_{t+1} - \frac{ac}{zc-\gamma c}z\hat{z}_{t+1} - \frac{az}{(zc-\gamma c)^2}\cdot c\hat{c}_{t+1} + \frac{a\gamma}{(zc-\gamma c)^2}\cdot c\hat{c}_t\right]$$
$$= \frac{az}{(z-\gamma)c}\cdot\hat{a}_t - \frac{\gamma acz}{(zc-\gamma c)^2}\cdot\hat{z}_t - \frac{aczz}{(zc-\gamma c)^2}\cdot\hat{c}_t + \frac{\gamma acz}{(zc-\gamma c)^2}\cdot\hat{c}_{t-1}$$
$$-\frac{\beta\gamma a}{(z-\gamma)c}\cdot E_t\hat{a}_{t+1} + \frac{acz}{zc-\gamma c}E_t\hat{z}_{t+1} + \frac{acz}{(zc-\gamma c)^2}\cdot E_t\hat{c}_{t+1} - \frac{\gamma ac}{(zc-\gamma c)^2}\cdot\hat{c}_t$$

From the paper we know that $a = 1$, $E_t\hat{a}_{t+1} = (\rho_a\hat{a}_t)$, and $E_t\hat{z}_{t+1} = 0$, and thus:

$$\lambda\hat{\lambda}_t = \frac{z}{(z-\gamma)c}\cdot\hat{a}_t - \frac{\gamma cz}{(zc-\gamma c)^2}\cdot\hat{z}_t - \frac{czz}{(zc-\gamma c)^2}\cdot\hat{c}_t + \frac{\gamma cz}{(zc-\gamma c)^2}\cdot\hat{c}_{t-1}$$
$$-\frac{\beta\gamma a}{(z-\gamma)c}\cdot\rho_a\hat{a}_t + 0 + \beta\gamma\frac{cz}{(zc-\gamma c)^2}\cdot E_t\hat{c}_{t+1} - \beta\gamma\frac{\gamma c}{(zc-\gamma c)^2}\cdot\hat{c}_t$$

The common denominator can be removed by multiplying both sides by $(zc-\gamma c)^2$ and then we divide by c

$$\lambda\hat{\lambda}_t(zc-\gamma c)^2 = z(zc-\gamma c)\hat{a}_t - \gamma cz\hat{z}_t - czz\hat{c}_t + \gamma cz\hat{c}_{t-1}$$
$$-\beta\gamma a\rho_a(zc-\gamma c)\hat{a}_t + czE_t\hat{c}_{t+1} - \gamma c\hat{c}_t$$
$$\lambda\hat{\lambda}_t(z-\gamma)^2 c = z(z-\gamma)\hat{a}_t - \gamma z\hat{z}_t - zz\hat{c}_t + \gamma z\hat{c}_{t-1}$$
$$-\beta\gamma a\rho_a(z-\gamma)\hat{a}_t + \beta\gamma zE_t\hat{c}_{t+1} - \gamma\hat{c}_t$$

We know $\hat{c}_t = \hat{y}_t$ and $c = y$

$$\lambda\hat{\lambda}_t(z-\gamma)^2 y = (z-\gamma)(z-\beta\gamma\rho_a)\hat{a}_t - \gamma z\hat{z}_t - \left(z^2+\beta\gamma^2\right)\hat{y}_t + \gamma z\hat{y}_{t-1} + \beta\gamma zE_t\hat{y}_{t+1}$$

From the Appendix, we can use the fact that

$$y\lambda = \frac{(z-\beta\gamma)}{(z-\gamma)}$$

$$(z-\gamma)(z-\beta\gamma)\hat{\lambda}t$$
$$= (z-\gamma)\left(z-\beta\gamma\rho_a\right)\hat{a}_t - \gamma z\hat{z}_t - \left(z^2+\beta\gamma^2\right)\hat{y}_t + z\gamma\hat{y}_{t-1} + \beta\gamma zE_t\hat{y}_{t+1}$$

yielding the final result:

$$(z-\gamma)(z-\beta\gamma)\hat{\lambda}_t = (z-\gamma)\left(z-\beta\gamma p_a\right)\hat{a}_t - \gamma z\hat{z}_t + z\gamma\hat{y}_{t-1} + \beta\gamma zE_t\hat{y}_{t+1} - \left(z^2+\beta\gamma^2\right)\hat{y}_t$$

4.6 Equation (20)

From Equation (7)

$$\lambda_t = \beta r_t E_t \left(\frac{\lambda_{t+1}}{z_{t+1}\pi_{t+1}} \right)$$

The steady state equation follows as

$$\lambda = \beta r \frac{\lambda}{z\pi} \Rightarrow 1 = \frac{\beta r}{z\pi}$$

Now, substituting each variable in the fashion similar to $\lambda_t = \lambda e^{\hat{\lambda}_t}$

$$\begin{aligned}
&\lambda_t = \beta r_t E_t \left[\tfrac{\lambda_{t+1}}{z_{t+1}\pi_{t+1}} \right] \\
&\Rightarrow \lambda e^{\hat{\lambda}_t} = \beta r e^{\hat{r}_t} E_t \left[\tfrac{\lambda e^{\hat{\lambda}_{t+1}}}{z e^{\hat{z}_{t+1}} \pi e^{\hat{\pi}_{t+1}}} \right] \\
&\Rightarrow \lambda \left(1 + \hat{\lambda} t \right) = \tfrac{\beta r \lambda}{z\pi} \quad E_t \left[\tfrac{\left(1 + \hat{\lambda}_{t+1} - \hat{z}_{t+} - \hat{\pi}_{t+1} + \hat{r}_t \right)}{1} \right] \\
&\Rightarrow \left(1 + \hat{\lambda}_t \right) = E_t \left[1 + \hat{\lambda}_{t+1} - \hat{z}_{t+} - \hat{\pi}_{t+1} + \hat{r}_t \right] \\
&\text{using } E_t[\hat{z}_{t+1}] = 0 \\
&\Rightarrow 1 + \hat{\lambda}_t = 1 + E_t \left[\hat{\lambda}_{t+1} - \hat{\pi}_{t+1} + \hat{r}_t \right] \\
&\therefore \hat{\lambda}_t = \hat{r}_t + E_t \left[\hat{\lambda}_{t+1} - \hat{\pi}_{t+1} \right]
\end{aligned}$$

4.7 Equation (21)

$$1 = \frac{z_t}{z_t q_t - \gamma q_{t-1}} - \beta\gamma E_t \left[\left(\frac{a_{t+1}}{a_t} \right) \left(\frac{1}{z_{t+1} q_{t+1} - \gamma q_t} \right) \right]$$

The LHS of the equation, instead of approximating, the log of 1 is zero, so that is used instead For the RHS, we continue as before, by separating the function into simpler terms

$$z_1 = \frac{z_t}{z_t q_t - \gamma q_{t-1}}$$
$$z_2 = \left(\frac{a_{t+1}}{a_t} \right) \left(\frac{1}{z_{t+1} q_{t+1} - \gamma q_t} \right)$$

4.7.1 Z1

$$
\begin{aligned}
z_1^{SS} &= \frac{z}{zq - \gamma q} \\
\left[\frac{dz_1}{dz_t}\right]^{SS} &= \left[\frac{(z_t q_t - \gamma q_{t-1}) - z_t\left(q_t\right)}{\left(z_t q_t - \gamma q_{t-1}\right)^2}\right]^{SS} \\
&= \frac{(zq - \gamma q) - zq}{(zq - \gamma q)} \\
&= -\frac{\gamma q}{(zq - \gamma q)^2} \\
\left[\frac{dz_1}{dq_t}\right]^{SS} &= \left[-z_t\left(z_t q_t - \gamma q_{t-1}\right)^{-2} z_t\right]^{SS} \\
&= -\frac{z^2}{\left(zq - \gamma q\right)^2} \\
\left[\frac{dz_1}{dq_{t-1}}\right]^{SS} &= \left[-z_t\left(z_t q_t - \gamma q_{t-1}\right)^{-2}(-\gamma)\right]^{SS} \\
&= \frac{z\gamma}{(zq - \gamma q)^2}
\end{aligned}
$$

Now we can expand the linearisation equation for z_1

$$
z_1 = \frac{z}{zq - \gamma q} - \frac{\gamma q}{(zq - \gamma q)^2} z \cdot \hat{z}_t - \frac{z^2}{(zq - \gamma q)^2} q \cdot \hat{q}_t + \frac{z\gamma}{(zq - \gamma q)^2} q \cdot \hat{q}_{t-1}
$$

4.7.2 Z2

$$
\begin{aligned}
z_2^{SS} &= \left(\frac{a}{a}\right)\left(\frac{1}{zq-\gamma q}\right) \\
&= \frac{1}{zq-\gamma q} \\
\left[\frac{dz_1}{da_{t+1}}\right]^{SS} &= \left(\frac{1}{a}\right)\left(\frac{1}{zq-\gamma q}\right) \\
\left[\frac{dz_1}{da_t}\right]^{SS} &= -\left(\frac{a}{a^2}\right)\left(\frac{1}{zq-\gamma q}\right) \\
&= -\left(\frac{1}{a}\right)\left(\frac{1}{zq-\gamma q}\right) \\
\left[\frac{dz_1}{dz_{t+1}}\right]^{SS} &= \left[-\left(\frac{a_{t+1}}{a_t}\right)(z_{t+1}q_{t+1}-\gamma q_t)^{-2}\cdot(q_{t+1})\right]^{SS} \\
&= -\left(\frac{a}{a}\right)\frac{q}{(zq-\gamma q)^2} \\
&= -\frac{q}{(zq-\gamma q)^2} \\
\left[\frac{dz_1}{dq_{t+1}}\right]^{SS} &= -\left[\left(\frac{a_{t+1}}{a_t}\right)(z_{t+1}q_{t+1}-\gamma q_t)^{-2}(z_{t+1})\right]^{SS} \\
&= -\left(\frac{a}{a}\right)\frac{z}{(zq-\gamma q)^2} \\
&= -\frac{z}{(zq-\gamma q)^2} \\
\left[\frac{dz_1}{dq_t}\right]^{SS} &= \left[\left(\frac{a_{t+1}}{a_t}\right)(z_{t+1}q_{t+1}-\gamma q_t)^{-2}(-\gamma)\right]^{SS} \\
&= \frac{\gamma}{(zq-\gamma q)^2}
\end{aligned}
$$

Therefore, we have z_2

{Recall a=1}

$$
\begin{aligned}
z_2 =& \frac{1}{zq-\gamma q} + \left(\frac{1}{a}\right)\left(\frac{1}{zq-\gamma q}\right)\cdot a\hat{a}_{t+1} - \left(\frac{1}{a}\right)\left(\frac{1}{zq-\gamma q}\right)\cdot a\hat{a}_t \\
&- \left(\frac{1}{a}\right)\left(\frac{1}{zq-\gamma q}\right)\cdot z\hat{z}_t - \frac{q}{(zq-\gamma q)^2}\cdot q\hat{q}_{t+1} + \frac{\gamma}{(zq-\gamma q)^2}\cdot q\hat{q}_t \\
=& \frac{1}{zq-\gamma q} + \left(\frac{1}{zq-\gamma q}\right)\cdot \hat{a}_{t+1} - \left(\frac{1}{zq-\gamma q}\right)\cdot \hat{a}_t \\
&- \left(\frac{1}{zq-\gamma q}\right)\cdot z\hat{z}_{t+1} - \frac{q^2}{(zq-\gamma q)^2}\cdot \hat{q}_{t+1} + \frac{\gamma}{(zq-\gamma q)^2}\cdot q\hat{q}_t
\end{aligned}
$$

Now for $\beta\gamma E_t z_2$, using $E_t\hat{a}_{t+1} = \rho_a \hat{a}_t$

$$\begin{aligned}
\beta\gamma E_t z_2 &= \beta\gamma \left[\frac{1}{zq-\gamma q} + \left(\frac{1}{zq-\gamma q}\right)\cdot E_t\hat{a}_{t+1} - \left(\frac{1}{zq-\gamma q}\right)\cdot E_t\hat{a}_t - \left(\frac{1}{zq-\gamma q}\right)\cdot zE_t\hat{z}_{t+1} - \frac{q^2}{(zq-\gamma q)^2}\cdot E_t\hat{q}_{t+}\right. \\
&= \beta\gamma \left[\frac{1}{zq-\gamma q} + \left(\frac{1}{zq-\gamma q}\right)\cdot \rho_a\hat{a}_t - \left(\frac{1}{zq-\gamma q}\right)\cdot \hat{a}_t - 0 - \frac{q^2}{(zq-\gamma q)^2}\cdot E_t\hat{q}_{t+1} + \frac{\gamma}{(zq-\gamma q)^2}\cdot q\hat{q}_t\right] \\
&= \frac{\beta\gamma}{zq-\gamma q} + \left(\frac{\beta\gamma}{zq-\gamma q}\right)\cdot \rho_a\hat{a}_t - \left(\frac{\beta\gamma}{zq-\gamma q}\right)\cdot \hat{a}_t - \frac{\beta\gamma q^2}{(zq-\gamma q)^2}\cdot E_t\hat{q}_{t+1} + \frac{\beta\gamma^2}{(zq-\gamma q)^2}\cdot q\hat{q}_t
\end{aligned}$$

4.7.3 Together

Now, we can write out $0 = z_1 - \beta\gamma E_t z_2$

Using

$$z_1 - z_1^{SS} - \beta\gamma(z_2 - z_2^{SS}) \approx z_1\hat{z_1} - z_2\hat{z_2} \Rightarrow \text{subtract from the RHS: } \frac{z}{zq-\gamma q} - \beta\gamma\frac{a}{azq-\gamma aq}$$

$$\begin{aligned}
0 =& \frac{z}{zq-\gamma q} - \frac{\gamma q}{(zq-\gamma q)^2} z\cdot\hat{z}_t - \frac{z^2}{(zq-\gamma q)^2} q\cdot\hat{q}_t + \frac{z\gamma}{(zq-\gamma q)^2} q\cdot\hat{q}_{t-1} \\
&- \left[\frac{\beta\gamma}{zq-\gamma q} + \left(\frac{\beta\gamma}{zq-\gamma q}\right)\cdot\rho_a\hat{a}_t - \left(\frac{\beta\gamma}{zq-\gamma q}\right)\cdot\hat{a}_t - \frac{\beta\gamma q^2}{(zq-\gamma q)^2}\cdot E_t\hat{q}_{t+1} + \frac{\beta\gamma^2}{(zq-\gamma q)^2}\cdot q\hat{q}_t\right]
\end{aligned}$$

Multiply both sides by $(zq-\gamma q)^2\frac{1}{q}$

$$\begin{aligned}
0 =& z\cdot(z-\gamma) - \gamma z\cdot\hat{z}_t - z^2\cdot\hat{q}_t + z\gamma\cdot\hat{q}_{t-1} \\
&- \left[\beta\gamma(z-\gamma) + \beta\gamma\cdot\rho_a\hat{a}_t(z-\gamma) - \beta\gamma(z-\gamma)\cdot\hat{a}_t - \beta\gamma q\cdot E_t\hat{q}_{t+1} + \beta\gamma^2\cdot\hat{q}_t\right] \\
0 =& z^2 - \gamma z - \gamma z\cdot\hat{z}_t - z^2\cdot\hat{q}_t + z\gamma\cdot\hat{q}_{t-1} \\
&- \beta\gamma z + \beta\gamma^2 - \beta\gamma\rho_a z\hat{a}_t + \beta\gamma^2\rho_a\hat{a}_t \\
&+ \beta\gamma z\hat{a}_t - \beta\gamma^2\hat{a}_t + \beta\gamma q E_t\hat{q}_{t+1} - \beta\gamma^2\hat{q}_t
\end{aligned}$$

Finally, we subtract:

$$\left[\frac{z}{zq-\gamma q} - \beta\gamma\frac{a}{azq-\gamma aq}\right]\cdot(zq-\gamma q)^2\frac{1}{q} = z(z-\gamma) - \beta\gamma(z-\gamma) = z^2 - \gamma z - \beta\gamma z + \beta\gamma^2$$

to yield the final solution:

$$
\begin{aligned}
0 =& z^2 - \gamma z - \gamma z \cdot \hat{z}_t - z^2 \cdot \hat{q}_t + z\gamma \cdot \hat{q}_{t-1} \\
& - \beta\gamma z + \beta\gamma^2 - \beta\gamma\rho_a z\hat{a}_t + \beta\gamma^2\rho_a\hat{a}_t \\
& + \beta\gamma z\hat{a}_t - \beta\gamma^2\hat{a}_t + \beta\gamma q E_t\hat{q}_{t+1} - \beta\gamma^2\hat{q}_t \\
& - z^2 + \gamma z + \beta\gamma z - \beta\gamma^2 \\
=& - \gamma z \cdot \hat{z}_t - z^2 \cdot \hat{q}_t + z\gamma \cdot \hat{q}_{t-1} \\
& - \beta\gamma\rho_a z\hat{a}_t + \beta\gamma^2\rho_a\hat{a}_t \\
& + \beta\gamma z\hat{a}_t - \beta\gamma^2\hat{a}_t + \beta\gamma q E_t\hat{q}_{t+1} - \beta\gamma^2\hat{q}_t \\
=& \hat{z}_t(-\gamma z) + \hat{q}_t\left(-z^2 - \beta\gamma^2\right) + \hat{q}_{t-1}(z\gamma) \\
& + \hat{a}_t\left(-\beta\gamma\rho_a z + \beta\gamma^2 p_a + \beta\gamma z - \beta\gamma^2\right) \\
& + \beta\gamma q E_t\hat{q}_{t+1} \\
=& \gamma z\hat{q}_{t-1} - \hat{q}_t\left(z^2 + \beta\gamma^2\right) + \beta\gamma q E_t\hat{q}_{t+1} + \hat{a}_t\beta\gamma\left(1-\rho_a\right)\left(z-\gamma\right) - \gamma z\hat{z}_t
\end{aligned}
$$

4.8 Equation (22)

$$
\begin{aligned}
x_t &= y_t/q_t \\
\text{Steady State: } x &= y/p
\end{aligned}
$$

Now,

$$
\begin{aligned}
x_t &= y_t/q_t \\
\Rightarrow xe^{\hat{x}t} &= \frac{ye^{\hat{y}t}}{qe^{\hat{q}t}} \\
\Rightarrow (1+\hat{x}_t) &= (1+\hat{y}_t - \hat{q}_t) \\
\therefore \quad \hat{x}_t &= \hat{y}_t - \hat{q}_t
\end{aligned}
$$

4.9 Equation (23)

Starting from equation (13)

$$
\begin{aligned}
&(1-\theta_t)(\tfrac{P_t(i)}{P_t})^{-\theta_t} + \theta_t(\tfrac{P_t(i)}{P_t})^{-\theta_t-1}(\tfrac{W_t}{P_t})(\tfrac{1}{Z_t}) - \phi_p[(\tfrac{P_t(i)}{\pi_{t-1}^a\pi^{1-a}P_{t-1}(i)} - 1)(\tfrac{P_t}{\pi_{t-1}^a\pi^{1-a}P_{t-1}(i)})] \\
&+\beta\phi_p E_t(\tfrac{\Lambda_{t+1}}{\Lambda_t})(\tfrac{P_{t+1}(i)}{\pi_t^a\pi^{1-a}P_t(i)} - 1)(\tfrac{P_{t+1}(i)}{\pi_t^a\pi^{1-a}P_t(i)})(\tfrac{Y_{t+1}}{Y_t})(\tfrac{P_t}{P_t(i)}) = 0
\end{aligned}
$$

Substituting for the equilibrium Conditions and using (6):

$$
\begin{aligned}
(1-\theta_t)+\theta_t(\frac{W_t}{P_t})(\frac{1}{Z_t})-\phi_p[(\frac{P_t}{\pi_{t-1}^a\pi^{1-a}P_{t-1}}-1)(\frac{P_t}{\pi_{t-1}^a\pi^{1-a}P_{t-1}})] \\
+\beta\phi_p E_t(\frac{\Lambda_{t+1}}{\Lambda_t})(\frac{P_{t+1}}{\pi_t^a\pi^{1-a}P_t}-1)(\frac{P_{t+1}}{\pi_t^a\pi^{1-a}P_t})(\frac{Y_{t+1}}{Y_t})=0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \theta_t-1 =& \theta_t\frac{a_t}{\Lambda_t Z_t}-\phi_p[(\frac{\pi_t}{\pi_{t-1}^a\pi^{1-a}}-1)(\frac{\pi_t}{\pi_{t-1}^a\pi^{1-a}})] \\
&+\beta\phi_p E_t(\frac{\Lambda_{t+1}Y_{t+1}}{\Lambda_t Y_t})(\frac{\pi_{t+1}}{\pi_t^a\pi^{1-a}}-1)(\frac{\pi_{t+1}}{\pi_t^a\pi^{1-a}})
\end{aligned}
$$

$$
\begin{aligned}
\therefore \theta_t-1 =& \theta_t\left(\frac{a_t}{\lambda_t}\right)-\phi_p\left(\frac{\pi_t}{\pi_{t-1}^\alpha\pi^{1-\alpha}}-1\right)\left(\frac{\pi_t}{\pi_{t-1}^\alpha\pi^{1-\alpha}}\right) \\
&+\beta\phi_p E_t\left[\left(\frac{\lambda_{t+1}y_{t+1}}{\lambda_t y_t}\right)\left(\frac{\pi_{t+1}}{\pi_t^\alpha\pi^{1-\alpha}}-1\right)\left(\frac{\pi_{t+1}}{\pi_t^\alpha\pi^{1-\alpha}}\right)\right]
\end{aligned}
$$

In Steady State this becomes

$$
\begin{aligned}
\theta-1 =& \theta\left(\frac{a}{\lambda}\right)-\phi_p\left(\frac{\pi}{\pi^\alpha\pi^{1-\alpha}}-1\right)\left(\frac{\pi}{\pi^\alpha\pi^{1-\alpha}}\right) \\
&+\beta\phi_p E\left[\left(\frac{\lambda y}{\lambda y}\right)\left(\frac{\pi}{\pi^\alpha\pi^{1-\alpha}}-1\right)\left(\frac{\pi}{\pi^\alpha\pi^{1-\alpha}}\right)\right] \\
\Rightarrow \theta-1 =& \theta\left(\frac{1}{\lambda}\right)-0+0 \\
\therefore \quad \lambda =& \frac{\theta}{\theta-1}
\end{aligned}
$$

Using the above result we can obtain and simplify further. Then, log-linearise using Uhlig's Method:

First we separate the four terms, for simplicity:

4.9.1 Term 1:

$$
\begin{aligned}
\theta_t-1 &= \theta e^{\hat{\theta}_t} \\
&= \theta\left(1+\hat{\theta}_t\right)
\end{aligned}
$$

4.9.2 Term 2:

$$
\begin{aligned}
\theta_t\left(\frac{a_t}{\lambda_t}\right) &= \theta e^{\hat{\theta}_t}\left(\frac{ae^{\hat{a}_t}}{\lambda\hat{e}^{\hat{\lambda}_t}}\right)\\
&= \frac{\theta a}{\lambda}\left(1+\hat{\theta}_t+\hat{a}_t-\hat{\lambda}_t\right)\\
&= \frac{\theta}{\lambda}\left(1+\hat{\theta}_t\right)+\frac{\theta}{\lambda}\left(\hat{a}_t-\hat{\lambda}_t\right)
\end{aligned}
$$

4.9.3 Term 3:

$$
\begin{aligned}
-\phi_p\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}}-1\right)\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}}\right) &= -\phi_p\left(\frac{\pi e^{\pi_t}}{\pi^{\alpha}e^{\alpha\hat{\pi}_{t-1}}\pi^{1-\alpha}}-1\right)\left(\frac{\pi e^{\hat{\pi}_t}}{\pi^{\alpha}e^{\alpha\hat{\pi}_{t-1}}\pi^{1-\alpha}}\right)\\
&= -\phi_p\left[\exp\left[2\hat{\pi}_t-2\alpha\hat{\pi}_{t-1}\right]-\exp\left[\hat{\pi}_t-\hat{\pi}_{t-1}\right]\right]\\
&= -\phi_p\left[\hat{\pi}_t-\alpha\hat{\pi}_{t-1}\right]
\end{aligned}
$$

4.9.4 Term 4:

$$
\begin{aligned}
&\beta\phi_p E_t\left[\left(\frac{\lambda_{t+1}y_{t+1}}{\lambda_t y_t}\right)\left(\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}}-1\right)\left(\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}}\right)\right]\\
=&\beta\phi_p E_t\left[\left(\frac{\lambda e^{\hat{\lambda}_{t+1}}\cdot ye^{\hat{y}_{t+1}}}{\lambda e^{\hat{\lambda}_t}ye^{\hat{y}_t}}\right)\right]\left(\frac{\pi e^{\pi_{t+1}}}{\pi^{\alpha}e^{\alpha\hat{\pi}_t}\pi^{1-\alpha}}-1\right)\left(\frac{\pi e^{\hat{\pi}_{t+1}}}{\pi^{\alpha}e^{\alpha\hat{\pi}_t}\pi^{1-\alpha}}\right)\\
=&\beta\phi_p E_t\Big[\exp\left(\hat{\lambda}_{t+1}+\hat{y}_{t+1}-\hat{\lambda}_t-\hat{y}_t+2\cdot\hat{\pi}_{t+1}-2\alpha\hat{\pi}_t\right)\\
&-\exp\left(\hat{\lambda}_{t+1}+\hat{y}_{t+1}-\hat{\lambda}_t-\hat{y}_t+\hat{\pi}_{t+1}-\alpha\hat{\pi}_t\right)\Big]\\
=&\beta\phi_p E_t\left[\hat{\pi}_{t+1}-\alpha\hat{\pi}_t\right]
\end{aligned}
$$

Putting all the components together then yields:

$$
\theta\left(1+\hat{\theta}_t\right)=\frac{\theta}{\lambda}\left(1+\hat{\theta}_t\right)+\frac{\theta}{\lambda}\left(\hat{a}_t-\hat{\lambda}_t\right)-\phi_p\left[\hat{\pi}_t-\alpha\hat{\pi}_{t-1}\right]+\beta\phi_p E_t\left[\hat{\pi}_{t+1}-\alpha\hat{\pi}_t\right]
$$

Simplify using

$$
\lambda=\frac{\theta}{\theta-1},\quad \hat{e}_t=-\left(\frac{1}{\phi_p}\right)\hat{\theta}_t,\text{ and }\psi=\frac{(\theta-1)}{\phi_p}
$$

i.e.

$$
1-\frac{1}{\lambda}=1/\theta,\frac{\theta}{\lambda}=\theta-1
$$

$$
\begin{aligned}
\theta\left(1+\hat{\theta}_t\right)\cdot\left[1-\frac{1}{\lambda}\right] =&\frac{\theta}{\lambda}\left(\hat{a}_t-\hat{\lambda}_t\right)-\phi_p\left[\hat{\pi}_t-\alpha\hat{\pi}_{t-1}\right]+\beta\phi_p E_t\left[\hat{\pi}_{t+1}-\alpha\hat{\pi}_t\right]\\
\frac{\left(1+\hat{\theta}_t\right)}{\phi_p} =&\frac{\theta-1}{\phi_p}\left(\hat{a}_t-\hat{\lambda}_t\right)-\left[\hat{\pi}_t-\alpha\hat{\pi}_{t-1}\right]+\beta E_t\left[\hat{\pi}_{t+1}-\alpha\hat{\pi}_t\right]
\end{aligned}
$$

Ignoring the constant parameter $-1/\phi_p$, and rearranging the terms yields the final result:

$$
\begin{aligned}
&-\hat{e}_t = \psi\left(\hat{a}_t - \hat{\lambda}_t\right) - (1+\beta\alpha)\hat{\pi}_t + \alpha\hat{\pi}_{t-1} + \beta E_t\left[\hat{\pi}_{t+1}\right] \\
\Rightarrow &(1+\beta\alpha)\hat{\pi}_t = \alpha\hat{\pi}_{t-1} + \beta E_t\hat{\pi}_{t+1} - \psi\hat{\lambda}_t + \psi\hat{a}_t + \hat{e}_t
\end{aligned}
$$

4.10 Equation (24)

$$
\ln\left(r_t/r\right) = \rho_r \ln\left(r_{t-1}/r\right) + \rho_\pi \ln\left(\pi_{t-1}/\pi\right) + \rho_x \ln\left(x_{t-1}/x\right) + \varepsilon_{rt}
$$

Directly applying Uhlig's method using $r_t = r \cdot e^{\hat{r}_t}$ for each variable of the equation yields the required result:

$$
\begin{aligned}
&\ln\left[\frac{re^{\hat{r}t}}{r}\right] = \rho_{\mathrm{r}}\ln\left[\frac{re^{\hat{r}_{t-1}}}{r}\right] + \rho\pi\ln\left[\frac{\pi e^{\hat{\pi}_{t-1}}}{n}\right) + \rho_x\ln\left(\frac{xe^{\hat{x}_{t+1}}}{x}\right) + \varepsilon_{rt} \\
&\therefore \quad \hat{r}_t = \rho_r\hat{r}_{t-1} + \rho_\pi\hat{\pi}_{t-1} + \rho x\hat{x}_{t-1} + \varepsilon_{rt}
\end{aligned}
$$

4.11 Equation (25)

Starting Form Equation (9) and using the conditions from section C:

$$
\begin{aligned}
&\frac{a_t}{\delta}\left[\ln\left(m^*\right) - \ln\left(m_t\right) + \ln\left(u_t\right)\right] - a_t\left(\frac{\phi_m}{2}\right)\left(\frac{Z_t m_t}{zZ_{t-1}m_{t-1}} - 1\right)^2 \\
&- a_t\phi_m\left(\frac{Z_t m_t}{zZ_{t-1}m_{t-1}} - 1\right)\left(\frac{Z_t m_t}{zZ_{t-1}m_{t-1}}\right) \\
&+ \beta\phi_m E_t\left[a_{t+1}\left(\frac{Z_{t+1}m_{t+1}}{zZ_t m_t} - 1\right)\left(\frac{Z_{t+1}m_{t+1}}{zZ_t m_t}\right)^2\left(\frac{z}{z_{t+1}}\right)\right] \\
&= Z_t\Lambda_t\left(1 - \frac{1}{r_t}\right) \\
\Rightarrow &\frac{a_t}{\delta}\left[\ln\left(m^*\right) - \ln\left(m_t\right) + \ln\left(u_t\right)\right] - a_t\left(\frac{\phi_m}{2}\right)\left(\frac{z_t m_t}{zm_{t-1}} - 1\right)^2 \\
&- a_t\phi_m\left(\frac{z_t m_t}{zm_{t-1}} - 1\right)\left(\frac{z_t m_t}{zm_{t-1}}\right) \\
&+ \beta\phi_m E_t\left[a_{t+1}\left(\frac{z_{t+1}m_{t+1}}{zm_t} - 1\right)\left(\frac{z_{t+1}m_{t+1}}{zm_t}\right)^2\left(\frac{z}{z_{t+1}}\right)\right] \\
&= \lambda_t\left(1 - \frac{1}{r_t}\right)
\end{aligned}
$$

As indicated in the Appendix of the paper, substituting for the steady state conditions in equation (9) yields

$$\begin{aligned}
&\frac{a}{\delta}\left[\ln\left(m^{*}\right)-\ln\left(m\right)+\ln\left(u\right)\right]-a\left(\frac{\phi_m}{2}\right)\left(\frac{zm}{zm}-1\right)^{2}\\
&-a\phi_m\left(\frac{zm}{zm_{t-1}}-1\right)\left(\frac{zm}{zm_{t-1}}\right)\\
&+\beta\phi_m E\left[a\left(\frac{zm}{zm}-1\right)\left(\frac{zm}{zm}\right)^{2}\left(\frac{z}{z}\right)\right]\\
&=\lambda\left(1-\frac{1}{r}\right)\\
\Rightarrow&\frac{1}{\delta}\left[\ln\left(m^{*}\right)-\ln\left(m\right)\right]-0-0+0=\lambda/r\left(r-1\right)\\
\therefore\quad&\ln(m)=\ln\left(m^{*}\right)-\delta_r(r-1)
\end{aligned}$$

where

$$\delta_r=\left(\frac{\delta}{r}\right)\left(\frac{\theta}{\theta-1}\right)$$

Now, to simplify the linearisation process, we consider the equation (9) in parts:

4.11.1 Term 1:

$$\begin{aligned}
\frac{a_t}{\delta}\left[\ln\left(m^{*}\right)-\ln\left(m_t\right)+\ln\left(u_t\right)\right]=&\frac{a_t}{\delta}\left\{\ln\left(m^{*}\right)-\ln(m)-\hat{m}_t+\hat{u}_t\right\}\\
=&\frac{a_t}{\delta}\left[\delta_r(r-1)-\hat{m}_t+\hat{u}_t\right]\\
=&\frac{a_t}{\delta}\delta_r(r-1)+\frac{a_t}{\delta}\left(\hat{u}_t-\hat{m}_t\right)
\end{aligned}$$

4.11.2 Term 5:

$$\begin{aligned}
\lambda_t\left(1-\frac{1}{r_t}\right)&=\lambda e^{\hat{\lambda}_t}\left(1-\frac{e^{-\hat{r}_t}}{r}\right)\\
&=\lambda\left[e^{\hat{\lambda}_t}-\frac{e^{\lambda_t-\hat{r}_t}}{r}\right]\\
&=\lambda\left[1+\hat{\lambda}_t-\frac{1+\hat{\lambda}_t-\hat{r}_t}{r}\right]\\
&=\lambda+\hat{\lambda\lambda}_t-\lambda/r+\lambda\cdot\hat{\lambda}_t/r-\lambda\cdot\hat{r}_t/r
\end{aligned}$$

Using $\lambda-\lambda/r=\frac{\delta_r(r-1)}{\delta}$ to add the previous two solutions together and changing the sign for carrying the term to the other side of the equality, we get:

$$
\begin{aligned}
&\lambda + \lambda\hat{\lambda}_t - \lambda/r + \lambda \cdot \hat{\lambda}_t/r - \lambda \cdot \hat{r}_t/r - \frac{a_t}{\delta}\delta_r(r-1) \\
=& -\frac{e^{\hat{a}_t}}{\delta}\delta_r(r-1) + \lambda - \lambda/r + \hat{\lambda}_t \cdot \lambda/r \cdot (r-1) + \lambda \cdot \hat{r}_t/r \\
=& -\frac{\delta_r(r-1)\hat{a}_t}{\delta} + \hat{\lambda}\frac{\delta_r(r-1)}{\delta} + \hat{r}_t\frac{\delta_r}{\delta}
\end{aligned}
$$

4.11.3 Term 2:

$$
\begin{aligned}
& a_t\left(\frac{\phi_m}{2}\right)\left(\frac{Z_t m_t}{z Z_{t-1} m_{t-1}} - 1\right)^2 \\
=& \left(\frac{\phi_m}{2}\right)\left[\frac{a_t z_t^2 m_t^2}{z^2 m_{t-1}^2} - 2\frac{a_t z_t m_t}{z m_{t-1}} + a_t\right] \\
=& \left(\frac{\phi_m}{2}\right)[\exp(\hat{a}_t + 2\hat{z}_t + 2\hat{m}_t - 2\hat{m}_{t-1}) - 2\exp(\hat{a}_t + \hat{m}_t + \hat{z}_t - \hat{m}_{t-1}) + \exp(\hat{a}_t)] \\
=& \left(\frac{\phi_m}{2}\right)[1 + \hat{a}_t + 2\hat{z}_t + 2\hat{m}_t - 2\hat{m}_{t-1} - 2(1 + \hat{a}_t + \hat{m}_t + \hat{z}_t - \hat{m}_{t-1}) + (1 + \hat{a}_t)] \\
=& \left(\frac{\phi_m}{2}\right)[-1 - \hat{a}_t + 1 + \hat{a}_t] \\
=& 0
\end{aligned}
$$

4.11.4 Term 3:

$$
\begin{aligned}
& -a_t\phi_m\left(\frac{Z_t m_t}{z Z_{t-1} m_{t-1}} - 1\right)\left(\frac{Z_t m_t}{z Z_{t-1} m_{t-1}}\right) \\
=& e^{\hat{a}_t}\phi_m\left[\frac{zme^{\hat{z}_t + \hat{m}_t}}{zme^{\hat{m}_{t-1}}} - 1\right]\left[\frac{zme^{\hat{z}_t + \hat{m}_t}}{zme^{\hat{m}_{t-1}}}\right] \\
=& \phi_m\left[\exp\left(2\hat{z}_t + 2\hat{m}_t - 2\hat{m}_{t-1} + \hat{a}_t\right) - \exp\left(\hat{z}_t + \hat{m}_t - \hat{m}_{t-1} + \hat{a}_t\right)\right] \\
=& \phi_m\left[\hat{z}_t + \hat{m}_t - \hat{m}_{t-1}\right]
\end{aligned}
$$

4.11.5 Term 4:

$$
\begin{aligned}
& \beta\phi_m E_t\left[e^{\hat{a}_{t+1}}\left(\frac{zme^{\hat{z}_{t+1} + \hat{m}_{t+1}}}{zme^{\hat{m}_t}} - 1\right)\left(\frac{zme^{\hat{z}_{t+1} + \hat{m}_{t+1}}}{zme^{\hat{m}_t}}\right)^2\left(\frac{z}{ze^{\hat{z}_{t+1}}}\right)\right] \\
=& \beta\phi_m E_t\left\{\exp\left(\hat{z}_{t+1} + \hat{m}_{t+1} + \hat{a}_{t+1} - \hat{m}_t + 2\hat{z}_{t+1} + 2\hat{m}_{t+1} - 2\hat{m}_t - \hat{z}_{t+1}\right)\right. \\
& \left. - \exp\left(\hat{a}_{t+1} + 2\hat{z}_{t+1} + 2\hat{m}_{t+1} - 2\hat{m}_t - \hat{z}_{t+1}\right)\right\} \\
=& \beta\phi_m E_t\left\{\exp\left(2\hat{z}_{t+1} + 3\hat{m}_{t+1} - 3\hat{m}_t + \hat{a}_{t+1}\right) - \exp\left(\hat{z}_{t+1} + 2\hat{m}_{t+1} - 2\hat{m}_t + \hat{a}_{t+1}\right)\right\} \\
=& \beta\phi_m E_t\left\{\hat{z}_{t+1} + \hat{m}_{t+1} - \hat{m}_t\right\} \\
=& \beta\phi_m E_t\left\{\hat{m}_{t+1}\right\} - \beta\phi_m\hat{m}_t
\end{aligned}
$$

4.11.6 Adding The Terms Together for the Final Solution:

$$\begin{aligned}&\frac{a_t}{\delta}\left(\hat{u}_t-\hat{m}_t\right)-\phi_m\left[\hat{z}_t+\hat{m}_t-\hat{m}_{t-1}\right]+\beta\phi_m E_t\left\{\hat{m}_{t+1}\right\}-\beta\phi_m\hat{m}_t\\=&-\frac{\delta_r(r-1)\hat{a}_t}{\delta}+\hat{\lambda}\frac{\delta_r(r-1)}{\delta}+\hat{r}_t\frac{\delta_r}{\delta}\end{aligned}$$

Multiply by δ throughout:

$$\begin{aligned}&a_t\left(\hat{u}_t-\hat{m}_t\right)-\phi\left[\hat{z}_t+\hat{m}_t-\hat{m}_{t-1}\right]+\beta\phi E_t\left\{\hat{m}_{t+1}\right\}-\beta\phi\hat{m}_t\\=&-\delta_r(r-1)\hat{a}_t+\hat{\lambda}\delta_r(r-1)+\hat{r}_t\delta_r\end{aligned}$$

Assuming $a_t=1$ the final solution follows:

$$\therefore\quad \delta_r(r-1)\hat{\lambda}-\delta_r(r-1)\hat{a}_t-\hat{u}_t+\phi\hat{z}_t=\phi\hat{m}_{t-1}-[1+(1+\beta)\phi]\hat{m}_t+\beta\phi\hat{m}_{t+1}-\delta_r\hat{r}_t$$

4.12 Equation (26)

$$\mu_t=z_t\left(m_t/m_{t-1}\right)\pi_t$$

Applying uhlig's Method:

$$\begin{aligned}\mu_t&=z_t\left[\frac{m_t}{m_{t-1}}\right]\pi_t\\\mu e^{\hat{\mu}_t}&=ze^{\hat{z}_t}\left[\frac{me^{\hat{m}_t}}{me^{\hat{m}_{t-1}}}\right]\pi e^{\hat{\pi}_t}\\\Rightarrow(1+\hat{\mu}_t)&=(1+\hat{z}_t+\hat{m}_t-\hat{m}_{t-1}+\hat{\pi}_t)\\\therefore\quad\hat{\mu}_t&=\hat{z}_t+\hat{m}_t-\hat{m}_{t-1}+\pi_t\end{aligned}$$

4.13 Equation (27)

This equation

$$g_t=\left(y_t/y_{t-1}\right)z_t$$

Using the Steady State solution

$$g=z$$

Yields the following solution, using Uhlig's method:

$$
\begin{aligned}
g_t &= \left(\frac{y_t}{y_{t-1}}\right) z_t \\
\Rightarrow g e^{g_t} &= \left[\frac{y e^{\hat{y}_t}}{y e^{\hat{y}_{t-1}}}\right] z e^{\hat{z}_t} \\
\Rightarrow (1+\hat{g}_t) &= (1+\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t) \\
\therefore \hat{g}_t &= \hat{y}_t - \hat{y}_{t-1} + \hat{z}_t
\end{aligned}
$$

4.14 Equation (28)

$$
\begin{aligned}
&\ln(a_t) = p_a \ln(a_{t-1}) + \varepsilon_{at} \\
&\ln\left(a e^{\hat{a}_t}\right) = p_a \ln\left(a e^{\hat{a}_{t-1}}\right) + \varepsilon_{at} \\
\Rightarrow &\ln(a) + \hat{a}_t = \rho_a \ln(a) + p_a \hat{a}_{t-1} + \varepsilon_{at} \\
&[a = 1] \\
\therefore \quad &\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}
\end{aligned}
$$

4.15 Equation (29)

$$
\begin{aligned}
\ln(z_t) &= \ln(z) + \varepsilon_{zt} \\
\Rightarrow \ln\left(z e^{\hat{z}_t}\right) &= \ln(z) + \varepsilon_{zt} \\
\Rightarrow \ln(z) + \hat{z}_t &= \ln(z) + \varepsilon_{zt} \\
\therefore \hat{z}_t &= \varepsilon_{zt}
\end{aligned}
$$

4.16 Equation (30)

$$
\begin{aligned}
&\ln(u_t) = p_u \ln(u_{t-1}) + \varepsilon_{ut} \\
&\ln\left(u e^{\hat{u}_t}\right) = p_u \ln\left(u e^{\hat{u}_{t-1}}\right) + \varepsilon_{ut} \\
\Rightarrow &\ln(u) + \hat{u}_t = \rho_u \ln(u) + p_u \hat{u}_{t-1} + \varepsilon_{ut} \\
&[u = 1] \\
\therefore \quad &\hat{u}_t = \rho_u \hat{u}_{t-1} + \varepsilon_{ut}
\end{aligned}
$$

4.17 Equation (31)

$$
\begin{aligned}
&\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t} \\
\Rightarrow &\ln\left(\theta e^{\hat{\theta}_t}\right) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln\left(\theta e^{\hat{\theta} t-1}\right) + \varepsilon_{\theta t} \\
\Rightarrow &\ln(\theta) + \hat{\theta}_t = \ln(\theta) - \rho_\theta \ln(\theta) + \rho_\theta \ln(\theta) + \rho_\theta \hat{\theta}_{t-1} + \varepsilon_{\theta t} \\
\therefore \quad &\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \varepsilon_{\theta t}
\end{aligned}
$$

using the normalisation

$$\hat{e}_t = -\left(1/\phi_p\right)\hat{\theta}_t$$

and

$$\rho_e = \rho_\theta$$

We can rewrite:

$$\begin{aligned}
&\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \varepsilon_{\theta t} \\
\Rightarrow &-\left(1/\phi_p\right) \cdot \hat{\theta}_t = \hat{e}_t = -\left(1/\phi_p\right)\left[\rho_\theta \hat{\theta}_{t-1} + \varepsilon_{\theta t}\right] \\
\Rightarrow \quad &\hat{e}_t = \rho_e \hat{e}_{t-1} + \varepsilon_{et}
\end{aligned}$$