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Abstract. We are interested in algorithms that manipulate mathematical expressions in mathematically meaningful ways. Expressions are syntactic, but most logics do not allow one to discuss syntax. CTT_{qe} is a version of Church’s type theory that includes quotation and evaluation operators that are similar to quote and eval in the Lisp programming language. Since the HOL logic is also a version of Church’s type theory, we decided to add quotation and evaluation to HOL Light to demonstrate the implementability of CTT_{qe} and the benefits of having quotation and evaluation in a proof assistant. The resulting system is called HOL Light QE. Here we document the design of HOL Light QE and the challenges that needed to be overcome.

1 Introduction

A *syntax-based mathematical algorithm (SBMA)* manipulates mathematical expressions in a mathematically meaningful way. SBMAs are commonplace in mathematics. Examples include algorithms that compute arithmetic operations by manipulating numerals, linear transformations by manipulating matrices, and derivatives by manipulating functional expressions. Reasoning about the mathematical meaning of an SBMA requires reasoning about the relationship between how the expressions are manipulated by the SBMA and what the manipulations mean mathematically.

We argue in [23] that the combination of quotation and evaluation, along with some inference rules, provides the means to reason about the interplay between syntax and semantics, which is what is needed for reasoning about SBMAs. *Quotation* is an operation that maps an expression e to a special value called a *syntactic value* that represents the syntax tree of e . Quotation enables expressions to be manipulated as syntactic entities. *Evaluation* is an operation that maps a syntactic value s to the value of the expression that is represented by s . Evaluation enables meta-level reasoning via syntactic values to be reflected into object-level reasoning. Quotation and evaluation thus form an infrastructure for integrating meta-level and object-level reasoning. Quotation gives a form of *reification* of object-level values which allows introspection. Along with inference

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rules, this gives a certain amount of *logical reflection*; evaluation adds to this some aspects of *computational reflection* [20,33].

Incorporating quotation and evaluation operators — like quote and eval in the Lisp programming language — into a traditional logic like first-order logic or simple type theory is not a straightforward task. Several challenging design problems stand in the way. The three design problems that most concern us are the following. We will write the quotation and evaluation operators applied to an expression e as $\ulcorner e \urcorner$ and $\llbracket e \rrbracket$, respectively.

1. *Evaluation Problem.* An evaluation operator is applicable to syntactic values that represent formulas and thus is effectively a truth predicate. Hence, by the proof of Alfred Tarski's theorem on the undefinability of truth [48], if the evaluation operator is total in the context of a sufficiently strong theory like first-order Peano arithmetic, then it is possible to express the liar paradox using the quotation and evaluation operators. Therefore, the evaluation operator must be partial and the law of disquotation cannot hold universally (i.e., for some expressions e , $\llbracket \ulcorner e \urcorner \rrbracket \neq e$). As a result, reasoning with evaluation can be cumbersome and leads to undefined expressions.
2. *Variable Problem.* The variable x is not free in the expression $\ulcorner x + 3 \urcorner$ (or in any quotation). However, x is free in $\llbracket \ulcorner x + 3 \urcorner \rrbracket$ because $\llbracket \ulcorner x + 3 \urcorner \rrbracket = x + 3$. If the value of a constant c is $\ulcorner x + 3 \urcorner$, then x is free in $\llbracket c \rrbracket$ because $\llbracket c \rrbracket = \llbracket \ulcorner x + 3 \urcorner \rrbracket = x + 3$. Hence, in the presence of an evaluation operator, whether or not a variable is free in an expression may depend on the values of the expression's components. As a consequence, the substitution of an expression for the free occurrences of a variable in another expression depends on the semantics (as well as the syntax) of the expressions involved and must be integrated with the proof system for the logic. That is, a logic with quotation and evaluation requires a semantics-dependent form of substitution in which side conditions, like whether a variable is free in an expression, are proved within the proof system. This is a major departure from traditional logic.
3. *Double Substitution Problem.* By the semantics of evaluation, the value of $\llbracket e \rrbracket$ is the *value* of the expression whose syntax tree is represented by the *value* of e . Hence the semantics of evaluation involves a double valuation. This is most apparent when the value of a variable involves a syntax tree which refers to the name of that same variable. For example, if the value of a variable x is $\ulcorner x \urcorner$, then $\llbracket x \rrbracket = \llbracket \ulcorner x \urcorner \rrbracket = x = \ulcorner x \urcorner$. Hence the substitution of $\ulcorner x \urcorner$ for x in $\llbracket x \rrbracket$ requires one substitution inside the argument of the evaluation operator and another substitution after the evaluation operator is eliminated. This double substitution is another major departure from traditional logic.

CTT_{qe} [24,25] is version of Church's type theory [15] with quotation and evaluation that solves these three design problems. It is based on \mathcal{Q}_0 [2], Peter Andrews' version of Church's type theory. We believe CTT_{qe} is the first readily implementable version of simple type theory that includes *global* quotation and evaluation operators. We show in [24] that it is suitable for defining, applying, and reasoning about SBMAs.

To demonstrate that CTT_{qe} is indeed implementable, we have implemented CTT_{qe} by modifying HOL Light [34], a compact implementation of the HOL logic [31]. The resulting version of HOL Light is called HOL Light QE. Here we present its design, implementation, and the challenges to doing so.

The rest of the paper is organized as follows. Section 2 presents the key ideas underlying CTT_{qe} and explains how CTT_{qe} solves the three design problems given above. Section 3 offers a brief overview of HOL Light. The HOL Light QE implementation is described in section 4, and examples of how quotation and evaluation are used in it are discussed in section 5. Section 6 is devoted to related work. And the paper ends with some final remarks in section 7 including a brief discussion on future work.

2 CTT_{qe}

The syntax and semantics of CTT_{qe} as well as the proof system for CTT_{qe} are defined in [24]. In this section we will only introduce the definitions and results of CTT_{qe} that are key to understanding how HOL Light QE implements CTT_{qe} . The reader is encouraged to consult [24] when additional details are required.

2.1 Syntax

The syntax of CTT_{qe} has the same machinery as \mathcal{Q}_0 plus an inductive type ϵ of syntactic values, a partial quotation operator, and a typed evaluation operator.

A *type* of CTT_{qe} is defined inductively by the following formation rules:

1. *Type of individuals*: ι is a type.
2. *Type of truth values*: o is a type.
3. *Type of constructions*: ϵ is a type.
4. *Function type*: If α and β are types, then $(\alpha \rightarrow \beta)$ is a type.

Let \mathcal{T} denote the set of types of CTT_{qe} .

A *typed symbol* is a symbol with a subscript from \mathcal{T} . Let \mathcal{V} be a set of typed symbols such that, for each $\alpha \in \mathcal{T}$, \mathcal{V} contains denumerably many typed symbols with subscript α . A *variable of type* α of CTT_{qe} is a member of \mathcal{V} with subscript α . $\mathbf{x}_\alpha, \mathbf{y}_\alpha, \mathbf{z}_\alpha, \dots$ are syntactic variables ranging over variables of type α . Let \mathcal{C} be a set of typed symbols disjoint from \mathcal{V} . A *constant of type* α of CTT_{qe} is a member of \mathcal{C} with subscript α . $\mathbf{c}_\alpha, \mathbf{d}_\alpha, \dots$ are syntactic variables ranging over constants of type α . \mathcal{C} contains a set of *logical constants* that include $\mathbf{app}_{\epsilon \rightarrow \epsilon \rightarrow \epsilon}$, $\mathbf{abs}_{\epsilon \rightarrow \epsilon \rightarrow \epsilon}$, and $\mathbf{quo}_{\epsilon \rightarrow \epsilon}$.

An *expression of type* α of CTT_{qe} is defined inductively by the formation rules below. $\mathbf{A}_\alpha, \mathbf{B}_\alpha, \mathbf{C}_\alpha, \dots$ are syntactic variables ranging over expressions of type α . An expression is *eval-free* if it is constructed using just the first five formation rules.

1. *Variable*: \mathbf{x}_α is an expression of type α .
2. *Constant*: \mathbf{c}_α is an expression of type α .

3. *Function application*: $(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha)$ is an expression of type β .
4. *Function abstraction*: $(\lambda \mathbf{x}_\alpha . \mathbf{B}_\beta)$ is an expression of type $\alpha \rightarrow \beta$.
5. *Quotation*: $\lceil \mathbf{A}_\alpha \rceil$ is an expression of type ϵ if \mathbf{A}_α is eval-free.
6. *Evaluation*: $\llbracket \mathbf{A}_\epsilon \rrbracket_{\mathbf{B}_\beta}$ is an expression of type β .

The sole purpose of the second component \mathbf{B}_β in an evaluation $\llbracket \mathbf{A}_\epsilon \rrbracket_{\mathbf{B}_\beta}$ is to establish the type of the evaluation; we will thus write $\llbracket \mathbf{A}_\epsilon \rrbracket_{\mathbf{B}_\beta}$ as $\llbracket \mathbf{A}_\epsilon \rrbracket_\beta$.

A *construction* of CTT_{qe} is an expression of type ϵ defined inductively as follows:

1. $\lceil \mathbf{x}_\alpha \rceil$ is a construction.
2. $\lceil \mathbf{c}_\alpha \rceil$ is a construction.
3. If \mathbf{A}_ϵ and \mathbf{B}_ϵ are constructions, then $\text{app}_{\epsilon \rightarrow \epsilon \rightarrow \epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon$, $\text{abs}_{\epsilon \rightarrow \epsilon \rightarrow \epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon$, and $\text{quo}_{\epsilon \rightarrow \epsilon} \mathbf{A}_\epsilon$ are constructions.

The set of constructions is thus an inductive type whose base elements are quotations of variables and constants and whose constructors are $\text{app}_{\epsilon \rightarrow \epsilon \rightarrow \epsilon}$, $\text{abs}_{\epsilon \rightarrow \epsilon \rightarrow \epsilon}$, and $\text{quo}_{\epsilon \rightarrow \epsilon}$. As we will see shortly, constructions serve as syntactic values.

Let \mathcal{E} be the function mapping eval-free expressions to constructions that is defined inductively as follows:

1. $\mathcal{E}(\mathbf{x}_\alpha) = \lceil \mathbf{x}_\alpha \rceil$.
2. $\mathcal{E}(\mathbf{c}_\alpha) = \lceil \mathbf{c}_\alpha \rceil$.
3. $\mathcal{E}(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha) = \text{app}_{\epsilon \rightarrow \epsilon \rightarrow \epsilon} \mathcal{E}(\mathbf{F}_{\alpha \rightarrow \beta}) \mathcal{E}(\mathbf{A}_\alpha)$.
4. $\mathcal{E}(\lambda \mathbf{x}_\alpha . \mathbf{B}_\beta) = \text{abs}_{\epsilon \rightarrow \epsilon \rightarrow \epsilon} \mathcal{E}(\mathbf{x}_\alpha) \mathcal{E}(\mathbf{B}_\beta)$.
5. $\mathcal{E}(\lceil \mathbf{A}_\alpha \rceil) = \text{quo}_{\epsilon \rightarrow \epsilon} \mathcal{E}(\mathbf{A}_\alpha)$.

When \mathbf{A}_α is eval-free, $\mathcal{E}(\mathbf{A}_\alpha)$ is the unique construction that represents the syntax tree of \mathbf{A}_α . That is, $\mathcal{E}(\mathbf{A}_\alpha)$ is a syntactic value that represents how \mathbf{A}_α is syntactically constructed. For every eval-free expression, there is a construction that represents its syntax tree, but not every construction represents the syntax tree of an eval-free expression. For example, $\text{app}_{\epsilon \rightarrow \epsilon \rightarrow \epsilon} \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{x}_\alpha \rceil$ represents the syntax tree of $(\mathbf{x}_\alpha \mathbf{x}_\alpha)$ which is not an expression of CTT_{qe} since the types are mismatched. A construction is *proper* if it is in the range of \mathcal{E} , i.e., it represents the syntax tree of an eval-free expression.

The purpose of \mathcal{E} is to define the semantics of quotation: the meaning of $\lceil \mathbf{A}_\alpha \rceil$ is $\mathcal{E}(\mathbf{A}_\alpha)$.

2.2 Semantics

The semantics of CTT_{qe} is based on Henkin-style general models [35]. An expression \mathbf{A}_ϵ of type ϵ denotes a construction, and when \mathbf{A}_ϵ is a construction, it denotes itself. The semantics of the quotation and evaluation operators are defined so that the following theorems hold:

Theorem 2.21 (Law of Quotation) $\lceil \mathbf{A}_\alpha \rceil = \mathcal{E}(\mathbf{A}_\alpha)$ is valid in CTT_{qe} .

Theorem 2.22 (Law of Disquotation) $\llbracket \lceil \mathbf{A}_\alpha \rceil \rrbracket_\alpha = \mathbf{A}_\alpha$ is valid in CTT_{qe} .

The definition below seems trivial, it just maps from one syntax to another. On the flip-side, if a reader is expecting a λ -calculus, then \mathcal{E} in the app case is ill-defined as you can't pattern match on an application.

Some comments that = is not up to $\alpha\beta$, otherwise the Law of Quotation does not hold – $\lceil 2 + 3 \rceil$ is not the same as $\mathcal{E}(5)$.

2.3 Proof System

The proof system for CTT_{qe} consists of the axioms for \mathcal{Q}_0 , the single rule of inference for \mathcal{Q}_0 , and additional axioms that extend the rules for beta-reduction, define the logical constants of CTT_{qe} , specify ϵ as an inductive type, and state the properties of quotation and evaluation. We prove in [24] that this proof system is sound for all formulas and complete for eval-free formulas.

Substitution is performed in the proof system for CTT_{qe} using the properties of beta-reduction as Andrews does in the proof system for \mathcal{Q}_0 [2, p. 213]. The syntactic notion of “a variable is free in an expression” is replaced in Andrews’ beta-reduction axioms by the semantic notion of “a variable is effective in an expression” when the expression is not necessarily eval-free, and beta-reduction axioms for quotation and evaluation are added to Andrews’ beta-reduction axioms. “ \mathbf{x}_α is effective in \mathbf{B}_β ” means the value of \mathbf{B}_β depends on the value of \mathbf{x}_α . Clearly, if \mathbf{B}_β is eval-free, “ \mathbf{x}_α is effective in \mathbf{B}_β ” implies “ \mathbf{x}_α is free in \mathbf{B}_β ”. However, “ \mathbf{x}_α is effective in \mathbf{B}_β ” is a refinement of “ \mathbf{x}_α is free in \mathbf{B}_β ” on eval-free expressions since \mathbf{x}_α is free in $\mathbf{x}_\alpha = \mathbf{x}_\alpha$, but \mathbf{x}_α is not effective in $\mathbf{x}_\alpha = \mathbf{x}_\alpha$.

I think that some, if not most, of these rules should be shown. They either need to be here or in the implementation section, with a forward pointer.

Because substitution is so important, I think more will be needed – in the same vein as above.

2.4 Design Problems

CTT_{qe} solves the three design problems given in section 1. The Evaluation Problem is completely avoided by restricting the quotation operator to eval-free expressions and thus making it impossible to express the liar paradox. The Variable Problem is overcome by modifying Andrews’ beta-reduction axioms as described above. The Double Substitution Problem is eluded by using a beta-reduction axiom for evaluations that excludes beta-reductions that embody a double substitution.

3 HOL Light

HOL Light [34] is an open-source proof assistant developed by John Harrison. It implements a logic (HOL) which is a version of Church’s type theory. It is a simple implementation of the HOL proof assistant [31] written in OCaml and hosted on GitHub at <https://github.com/jrh13/hol-light/>. Although it is a relatively small system, it has been used to formalize many kinds of mathematics and to check many proofs including the lion’s share of Tom Hale’s proof of the Kepler conjecture [22].

HOL Light is very well suited to serve as a foundation on which to build an implementation of CTT_{qe} : First, it is an open-source system that can be freely modified as long as certain very minimal conditions are satisfied. Second, it is an implementation of a version of simple type theory that is essentially \mathcal{Q}_0 , the version of Church’s type theory underlying CTT_{qe} , plus (1) polymorphic type variables, (2) an axiom of choice expressed by asserting that the Hilbert ϵ operator is a choice (indefinite description) operator, and (3) an axiom of infinity that asserts that ind , the type of individuals, is infinite [34]. The type

variables in the implemented logic are not a hindrance; they actually facilitate the implementation of CTT_{qe} . The presence of the axioms of choice and infinity in HOL Light alter the semantics of CTT_{qe} without compromising in any way the semantics of quotation and evaluation. And third, HOL Light supports the definition of inductive types so that ϵ can be straightforwardly defined.

4 Implementation

4.1 Overview

HOL Light QE was implemented in four stages:

1. The set of HOL Light terms was extended so that CTT_{qe} expressions could be mapped to HOL Light terms. This required the introduction of ϵ , the type of construction, and term constructors for quotations and evaluations. See subsection 4.2.
2. The HOL Light proof system was modified to include the machinery in CTT_{qe} for reasoning about quotations and evaluations. This required adding new rules of inference to HOL Light and modifying the HOL Light **INST** rule of inference that simultaneously substitutes terms t_1, \dots, t_n for the free variables x_1, \dots, x_n in a sequent. See subsection 4.3.
3. Machinery, consisting of HOL functions definitions, tactics, and theorems, was created for supporting reasoning about quotations and evaluations in the new system. See subsection 4.4.
4. Examples were developed in the new system to test the implementation and to demonstrate the benefits of having quotation and evaluation in higher-order logic. See section 5.

The first and second stages have been completed; both stages involved modifying the kernel of HOL Light. The third stage is sufficiently complete that our examples in 5 work well, and did not involve any further changes to the HOL Light kernel. We do expect that adding further examples, which is ongoing, will require the addition of machinery but no changes to the kernel.

The HOL Light QE system was developed by the third author under the supervision of the first two authors on an undergraduate NSERC USRA research project at McMaster University. HOL Light QE is available at

<https://github.com/JacquesCarette/hol-light>,

It should be further remarked that our fork, from late April 2017, is not fully up-to-date with respect to HOL Light. In particular, this means that it is best to compile it with OCAML 4.03.0 and CAMLP5 6.16, both available from OPAM.

4.2 Mapping of CTT_{qe} Expressions to HOL Terms

Tables 1 and 2 illustrates how the CTT_{qe} types and expressions are mapped to the HOL types and terms, respectively. The HOL types and terms are written in the

the internal representation form employed in HOL Light QE. The type `epsilon` and the term constructors `Quote` and `Eval` are additions to HOL Light explained below. Since CTT_{qe} does not have type variables, there is a logical constant $=_{\alpha \rightarrow \alpha \rightarrow o}$ representing equality for each $\alpha \in \mathcal{T}$. The members of this family of constants are all mapped to a single HOL constant with the polymorphic type `a_ty_var -> a_ty_var -> bool` where `a_ty_var` is any chosen HOL type variable.

The other logical constants of CTT_{qe} [24, Table 1] are not mapped to primitive HOL constants. $\text{app}_{\epsilon \rightarrow \epsilon \rightarrow \epsilon}$, $\text{abs}_{\epsilon \rightarrow \epsilon \rightarrow \epsilon}$, and $\text{quo}_{\epsilon \rightarrow \epsilon}$ are implemented by `App`, `Abs`, and `Quo`, constructors for the inductive type `epsilon` given below. The remaining logical constants are predicates on constructions that are implemented by HOL functions.

| CTT_{qe} Type α | HOL Type $\mu(\alpha)$ | Abbreviation for $\mu(\alpha)$ |
|--|--|--------------------------------------|
| o | <code>Tyapp("bool", [])</code> | <code>bool</code> |
| ι | <code>Tyapp("ind", [])</code> | <code>ind</code> |
| ϵ | <code>Tyapp("epsilon", [])</code> | <code>epsilon</code> |
| $\beta \rightarrow \gamma$ | <code>Tyapp("fun", [$\mu(\beta)$, $\mu(\gamma)$])</code> | $\mu(\beta) \rightarrow \mu(\gamma)$ |

Table 1. Mapping of CTT_{qe} Types to HOL Types

| CTT_{qe} Expression e | HOL Term $\nu(e)$ |
|--|---|
| \mathbf{x}_α | <code>Var("x", $\mu(\alpha)$)</code> |
| \mathbf{c}_α | <code>Const("c", $\mu(\alpha)$)</code> |
| $=_{\alpha \rightarrow \alpha \rightarrow o}$ | <code>Const("=", a_ty_var -> a_ty_var -> bool)</code> |
| $(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha)$ | <code>Comb($\nu(\mathbf{F}_{\alpha \rightarrow \beta})$, $\nu(\mathbf{A}_\alpha)$)</code> |
| $(\lambda \mathbf{x}_\alpha. \mathbf{B}_\beta)$ | <code>Abs(Var("x", $\mu(\alpha)$), $\nu(\mathbf{B}_\beta)$)</code> |
| $\ulcorner \mathbf{A}_\alpha \urcorner$ | <code>Quote($\nu(\mathbf{A}_\alpha)$, $\mu(\alpha)$)</code> |
| $\llbracket \mathbf{A}_\epsilon \rrbracket_{\mathbf{B}_\beta}$ | <code>Eval($\nu(\mathbf{A}_\epsilon)$, $\mu(\beta)$)</code> |

Table 2. Mapping of CTT_{qe} Expressions to HOL Terms

The CTT_{qe} type ϵ is the type of constructions, the syntactic values that represent the syntax trees of eval-free expressions. ϵ is defined as an inductive type `epsilon`. `type` and `epsilon` are defined as the following inductive types

```

define_type "type" = TyVar string
                  | TyBase string
                  | TyMonoCons string type
                  | TyBiCons string type type

define_type "epsilon" = QuoVar string type
                  | QuoConst string type
                  | App epsilon epsilon

```

```

| Abs epsilon epsilon
| Quo epsilon"

```

Terms of type `type` denote the syntax trees of HOL Light QE types (which are the same as HOL types). The terms of type `epsilon` denote the syntax trees of HOL Light QE terms that are eval-free (i.e., do not contain evaluations).

The OCaml type of HOL types in HOL Light QE

```

type hol_type = Tyvar of string
               | Tyapp of string * hol_type list

```

is the same as in HOL Light, but the OCaml type of HOL terms in HOL Light QE

```

type term = Var of string * hol_type
           | Const of string * hol_type
           | Comb of term * term
           | Abs of term * term
           | Quote of term * hol_type
           | Hole of term * hol_type
           | Eval of term * hol_type

```

has three new constructors — `Quote`, `Hole`, and `Eval` — which are new.

`Quote` constructs a quotation of type `epsilon` with components t and α from a term t of type α that is eval-free. `Eval` constructs an evaluation of type α with components t and α from a term t of type `epsilon` and a type α . `Hole` is used to construct “holes” of type `epsilon` in an quasiquotation as described in [24]. A HOL Light QE quotation that contains holes is a quasiquotation, while a quotation without any holes is a normal quotation. The construction of terms has been modified to allow a hole (of type `epsilon`) to be used where a term of some other type is expected.

The external representation of a quotation `Quote(t,ty)` is Q_t_Q . Similarly, the external representation of a hole `Hole(t,ty)` is H_t_H . The external representation of an evaluation `Quote(t,ty)` is

```
eval t to ty.
```

4.3 Modification of the HOL Light Proof System

4.4 Creation of Support Machinery

The HOL Light QE contains a number of HOL functions, tactics, and theorems that are useful for reasoning about constructions, quotations, and evaluations. An important example is the HOL function `isExprType` that implements the CTT_{qe} family of logical constants $\text{is-expr}_{\epsilon \rightarrow o}^\alpha$ where α ranges over members of \mathcal{T} . This function takes terms s_1 and s_2 of type `epsilon` and `type`, respectively, and returns true iff s_1 represents the syntax tree of a term t , s_2 represents the syntax tree of a type α , and t is of type α .

5 Examples

5.1 Law of Excluded Middle

5.2 Induction Schema

6 Related Work

Quotation, evaluation, reflection, reification, issues of intensionality versus extensionality, metaprogramming and metareasoning each have extensive literature – sometimes in more than one field. For example, one can find a vast literature on reflection in logic, programming languages and theorem proving. Due to space restrictions, we cannot do justice to the full breadth of issues. For a full discussion, please see the related work section in [26]. The surveys of Costantini [20], Harrison [33] are excellent. From a programming perspective, the discussion and extensive bibliography of Kavvos’ D.Phil. thesis [41] are well worth reading.

Focusing just on interactive proof assistants, we find that Robert Boyer and J Moore developed a global infrastructure [5], for incorporating symbolic algorithms into the Nqthm [6] theorem prover. This approach is also used in ACL2 [40], the successor to Nqthm; see [39]. Over the last 30 years, the Nuprl group lead by Robert Constable has produced a large body of work on metareasoning and reflection for theorem proving [1,3,17,37,42,43,52] that has been implemented in the Nuprl [18] and MetaPRL [36] systems. Proof by reflection has become a mainstream technique in the Coq [19] proof assistant with the development of tactics based on symbolic computations like the Coq ring tactic [4,32] and the formalizations of the *four color theorem* [28] and the *Feit-Thompson odd-order theorem* [29] led by Georges Gonthier. See [4,7,11,30,32,38,45] for a selection of the work done on using reflection in Coq. Many other systems also support metareasoning and reflection: Agda [44,49,50], Idris [13,12,14] Isabelle/HOL [10], Lean [21], Maude [16], PVS [51], and Theorema [27,8].

The semantics of the quotation operator $\ulcorner \cdot \urcorner$ is based on the *disquotational theory of quotation* [9]. According to this theory, a quotation of an expression e is an expression that denotes e itself. In CTT_{qe} , $\ulcorner \mathbf{A}_\alpha \urcorner$ denotes a value that represents the syntactic structure of \mathbf{A}_α . Polonsky [46] presents a set of axioms for quotation operators of this kind. Other theories of quotation have been proposed – see [9] for an overview.. For instance, quotation can be viewed as an operation that constructs literals for syntactic values [47].

It is worth quoting Boyer and Moore [5] here:

The basic premise of all work on extensible theorem-provers is that it should be possible to add new proof techniques to a system without endangering the soundness of the system. It seems possible to divide current work into two broad camps. In the first camp are those systems that allow the introduction of arbitrary new procedures, coded in the implementation language, but require that each application of such a procedure produce a formal proof of the correctness of the transformation performed. In the second camp are those systems that contain a

formal notion of what it means for a proof technique to be sound and require a machine-checked proof of the soundness of each new proof technique. Once proved, the new proof technique can be used without further justification.

This remains true to this day. The systems in the LCF tradition (Isabelle/HOL, Coq, HOL Light) are in the “first camp”, while Nqthm, ACL2, Nuprl, MetaPRL, Agda, Idris, Lean, Maude and Theorema, as well as our approach broadly fall in the “second camp”. However, all systems in the first camp have started to offer some reflection capabilities on top of their tactic facilities. Below we give some additional details for each system, leveraging information from the papers already cited above as well as the documentation of each system¹.

In more detail, the Coq extension for reflection, SSReflect [30], which stands for **small scale reflection**, works by locally reflecting the syntax of particular kinds of objects – such as decidable predicates and finite structures. It is the pervasive use of decidability and computability which gives small scale reflection its power, and at the same time, its limitation. An extension to PVS allows reasoning much in the style of SSReflect. Isabelle/HOL offers a non-logical **reify** function (aka quotation), while its **interpret** function is in the logic; it uses global datatypes to represent HOL Lightterms.

The approach for the second list of systems also varies quite a bit. Nqthm, ACL2, Theorema (as well as now HOL Light QE) have global quotation and evaluation operators in the logic, as well as careful restrictions on their use to avoid paradoxes. Idris also have global quotation and evaluation, but it is unclear what features are in place to avoid paradoxes. MetaPRL have evaluation but no global quotation. Agda has global quotation and evaluation, but their use are mediated by a built-in **TC** (TypeChecking) monad which ensures soundness. Lean works similarly: all reflection must happen in the **tactic** monad, from which one cannot escape. Maude appears to offer a global quotation operator, but it is unclear if there is a global evaluation operator; quotations are offered by a built-in module, and those are extra-logical.

7 Conclusion

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¹ And some personal communication with some of system authors

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Todo list

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|---|--|---|
| ■ | Changed HOL Light from sc to rm as used by J. Harrison. | 1 |
| ■ | The definition below seems trivial, it just maps from one syntax to another. On the flip-side, if a reader is expecting a λ -calculus, then \mathcal{E} in the app case is ill-defined as you can't pattern match on an application. | 4 |
| ■ | Some comments that $=$ is not up to $\alpha\beta$, otherwise the Law of Quotation does not hold – $\ulcorner 2 + 3 \urcorner$ is not the same as $\mathcal{E}(5)$ | 4 |
| ■ | I think that some, if not most, of these rules should be shown. They either need to be here or in the implementation section, with a forward pointer. | 5 |
| ■ | Because substitution is so important, I think more will be needed – in the same vein as above. | 5 |