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HOL Light QE

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Outline

- Syntax-based mathematical algorithms.
- Local and global reflection.
- $\mathrm{CTT}_{\mathrm{qe}}$, Church's type theory with quotation and evaluation.
- HOL Light QE, an implementation of $\mathrm{CTT}_{\mathrm{qe}}$.

A Warm-Up Example

- Problem: Are the following two functions equal?

$$\lambda\, x : \mathbb{R} \,.\, (x + 2) * (2 * x + 1) + 3 * x$$
$$\lambda\, x : \mathbb{R} \,.\, 2 * (x + 2)^2 - 6$$

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- "normalize" is an algorithm that:
  1. Manipulates mathematical expressions having the form of polynomials.
  2. Preserves the mathematical meaning of the expressions.

Syntax-Based Mathematical Algorithms

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  1. The computational behavior of A is the relationship between the input and output expressions of A.
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  2. The mathematical meaning of A is the relationship between what the input and output expressions of A mean mathematically.
- A meaning formula for A is a statement that expresses the mathematical meaning of A.

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$$\forall\, p, q : \text{Poly} \;.\; (\forall\, x : \mathbb{R} \;.\; p = \text{normalize}(p)) \wedge (\forall\, x : \mathbb{R} \;.\; p = q) \equiv \text{normalize}(p) = \text{normalize}(q)$$

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  $[\![\ \cdot\ ]\!]$ is the evaluation operator.

- The required result is obtained by instantiating p and q with

  $$\ulcorner (x + 2) * (2 * x + 1) + 3 * x \urcorner$$

  and

  $$\ulcorner 2 * (x + 2)^2 - 6 \urcorner.$$

  $\ulcorner\ \cdot\ \urcorner$ is the quotation operator.

Quotation and Evaluation

- Quotation is used to gain access to the syntax of an expression.
  - The value of $\ulcorner e \urcorner$ is a syntactic value that represents the syntactic structure of the expression e.
  - Note: $\ulcorner 2 + 3 \urcorner \neq \ulcorner 5 \urcorner$.

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  - If the value of e' is a syntactic value representing the expression e, then the value of $[\![e']\!]$ is the value of e.
- The two operators are related by the law of disquotation:

  $[\![\ulcorner e \urcorner]\!] = e$.

Formalizing SBMAs

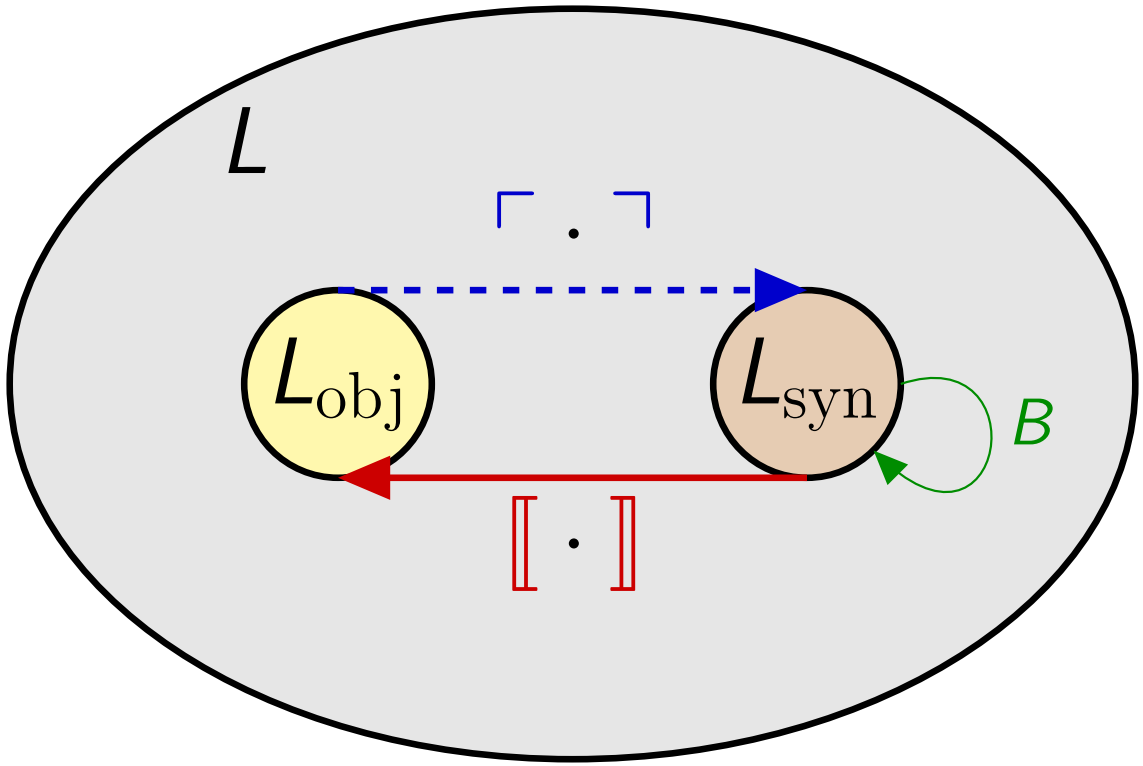
- To employ an SBMA A in a proof assistant (or other formal environment) we need to formalize A in the logic of the system.

- To formalize an SBMA A in a logic **Log** we need to be able to:
  1. Define in **Log** a function B on syntactic values representing A.
  2. State and prove in **Log** the meaning formula for B from the definition of B.
  3. Apply B to mathematical expressions in **Log** by instantiating the meaning formula for B and then simplifying.

Standard Approach: Local Reflection

- Let A be an SBMA on expressions in a language L_{obj} of some logic **Log**.
- We build a metareasoning infrastructure in **Log** consisting of:
  1. An inductive type L_{syn} of syntactic values representing the expressions in L_{obj}.
  2. A quotation operator $\ulcorner \cdot \urcorner$ mapping expressions in L_{obj} to syntactic values of L_{syn}.
  3. An evaluation operator $[\![\cdot]\!]$ mapping syntactic values of L_{syn} to values of L_{obj}.
- We define a function B in **Log** from syntactic values representing inputs of A to syntactic values representing outputs of A.
- The infrastructure is local in the sense that L_{obj} is not the whole language L of Log.

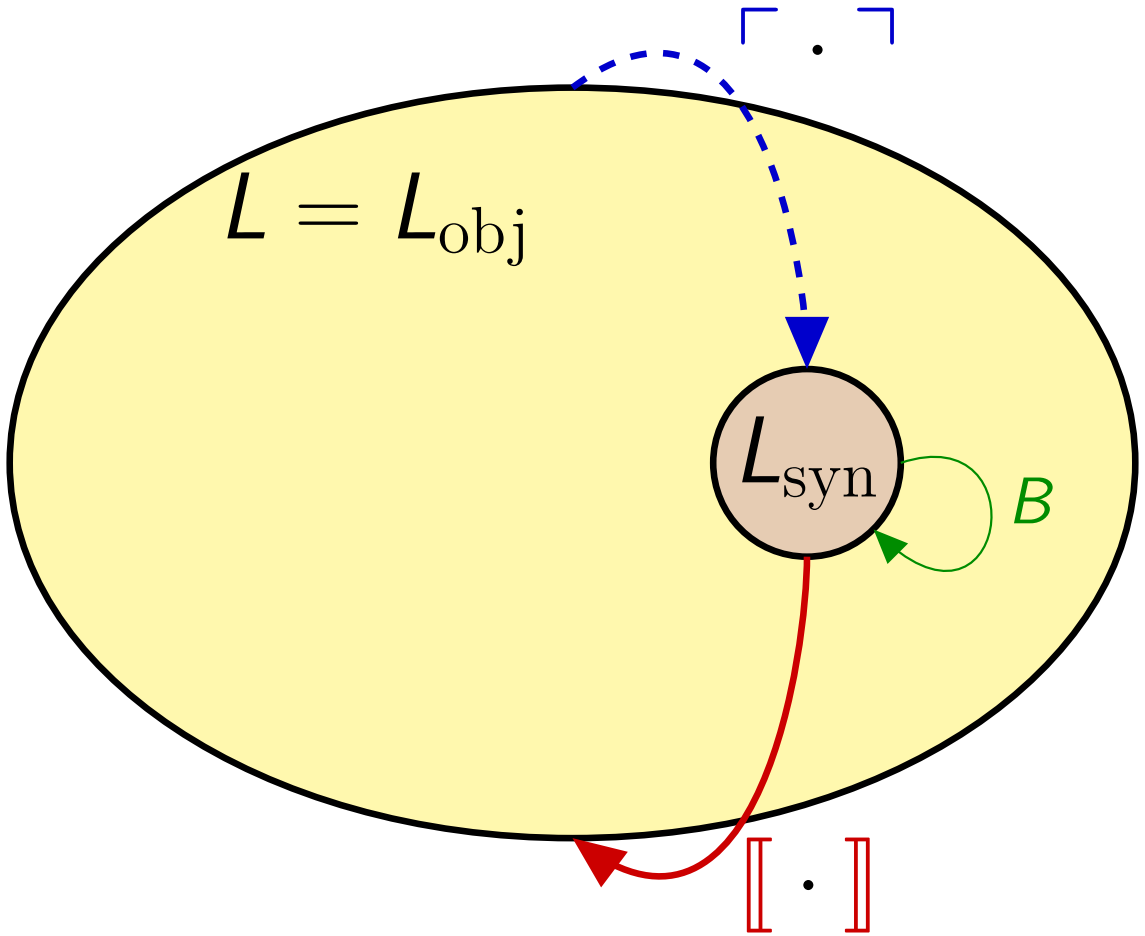
Local Reflection



An Alternate Approach: Global Reflection

- Local reflection does not scale up well:
  - Each collection of SBMAs requires a separate infrastructure.
  - Extending an SBMA to a new domain requires a new infrastructure.
- Global reflection employs a single infrastructure for all SBMAs:
  1. An inductive type representing the entire set of expressions.
  2. A global quotation operator $\ulcorner \cdot \urcorner$.
  3. A global evaluation operator $[\![\cdot]\!]$.
- Global reflection requires a logic with global quotation and evaluation operators.
- It is an open problem whether global reflection is viable!
- To test the viability of global reflection, we want to incorporate global quotation and evaluation into a traditional logic.

Global Reflection



Design Problems

Several challenging design problems face the logic engineer who seeks to incorporate global quotation and evaluation into a traditional logic:

1. Evaluation Problem. In a sufficiently strong theory, it is possible to express the liar paradox as an expression LIAR that equals $\ulcorner \neg [\![\mathsf{LIAR}]\!] \urcorner$ so that

   $$[\![\mathsf{LIAR}]\!] = [\![\ulcorner \neg [\![\mathsf{LIAR}]\!] \urcorner]\!] = \neg [\![\mathsf{LIAR}]\!]$$

   by the law of disquotation.
2. Variable Problem. If $c = \ulcorner x \urcorner$, then x is free in $[\![c]\!]$ since

   $$[\![c]\!] = [\![\ulcorner x \urcorner]\!] = x$$

   by the law of disquotation.
3. Double Substitution Problem.
4. Constant Interpretation Problem.

$\mathrm{CTT}_{\mathrm{qe}}$, a Version of Church's Type Theory

- Based on Andrews' $\mathcal{Q}_0$, $\mathrm{CTT}_{\mathrm{qe}}$ is a version of Church's type theory with a built-in global metareasoning infrastructure:
  1. An inductive type ϵ of syntactic values that represent all the "eval-free" expressions of the logic.
  2. A partial global quotation operator $\ulcorner \cdot \urcorner$.
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- The proof system for $\mathrm{CTT}_{\mathrm{qe}}$ provides the ability to express, prove, and instantiate schemas and meaning formulas.
- We believe $\mathrm{CTT}_{\mathrm{qe}}$ is the first readily implementable version of simple type theory with global quotation and evaluation.

$\mathrm{CTT}_{\mathrm{qe}}$: Syntax

- A expression of type α is inductively defined by the following formation rules:
  1. Variable: A variable $\mathbf{x}_\alpha$ is an expression of type α.
  2. Constant: A constant $\mathbf{c}_\alpha$ is an expression of type α.
  3. Function application: $(\mathbf{F}_{\alpha\to\beta}\,\mathbf{A}_\alpha)$ is an expression of type β.
  4. Function abstraction: $(\lambda\,\mathbf{x}_\alpha\,.\,\mathbf{B}_\beta)$ is an expression of type $\alpha \to \beta$.
  5. Quotation: $\ulcorner \mathbf{A}_\alpha \urcorner$ is an expression of type ϵ if $\mathbf{A}_\alpha$ is eval-free.
  6. Evaluation: $[\![\mathbf{A}_\epsilon]\!]_\beta$ is an expression of type β.

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- The restriction on quotation resolves the Evaluation Problem!

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- We could introduce a logical constant sub in $\mathrm{CTT}_{\mathrm{qe}}$ such that

  $$\mathsf{sub}\ \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner = \ulcorner \mathbf{C}_\beta \urcorner$$

  holds iff "$\mathbf{C}_\beta$ is the result of substituting $\mathbf{A}_\alpha$ for each free occurrence of $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$ without any variable captures"
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- We implement substitution using Andrews' beta-reduction rules:
  - In the function abstraction rule we replace the syntactic notion of "a variable is free in an expression" with the more restrictive semantic notion of "a variable is effective in an expression".
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- This approach resolves the Variable Problem!

$\mathrm{CTT}_{\mathrm{qe}}$: New Beta-Reduction Rules

- Function Abstraction Rule

  $$(\neg\text{IS-EFFECTIVE-IN}(\mathbf{y}_\beta, \mathbf{A}_\alpha) \vee \neg\text{IS-EFFECTIVE-IN}(\mathbf{x}_\alpha, \mathbf{B}_\gamma)) \supset (\lambda\, \mathbf{x}_\alpha\, .\, \lambda\, \mathbf{y}_\beta\, .\, \mathbf{B}_\gamma)\, \mathbf{A}_\alpha = \lambda\, \mathbf{y}_\beta\, .\, ((\lambda\, \mathbf{x}_\alpha\, .\, \mathbf{B}_\gamma)\, \mathbf{A}_\alpha)$$

  where $\mathbf{x}_\alpha$ and $\mathbf{y}_\beta$ are distinct.

- Quotation Rule

  $$(\lambda\, \mathbf{x}_\alpha\, .\, \ulcorner \mathbf{B}_\beta \urcorner)\, \mathbf{A}_\alpha = \ulcorner \mathbf{B}_\beta \urcorner.$$

- Evaluation Rule

  $$(\text{is-expr}^{\beta}_{\epsilon\to o}\, ((\lambda\, \mathbf{x}_\alpha\, .\, \mathbf{B}_\epsilon)\, \mathbf{A}_\alpha) \wedge \neg(\text{is-free-in}_{\epsilon\to\epsilon\to o}\, \ulcorner \mathbf{x}_\alpha \urcorner\, ((\lambda\, \mathbf{x}_\alpha\, .\, \mathbf{B}_\epsilon)\, \mathbf{A}_\alpha))) \supset (\lambda\, \mathbf{x}_\alpha\, .\, [\![\mathbf{B}_\epsilon]\!]_\beta)\, \mathbf{A}_\alpha = [\![(\lambda\, \mathbf{x}_\alpha\, .\, \mathbf{B}_\epsilon)\, \mathbf{A}_\alpha]\!]_\beta.$$

HOL Light QE

- HOL Light QE is an experimental implementation of $\mathrm{CTT}_{\mathrm{qe}}$ obtained by modifying the HOL Light proof assistant.
  - HOL Light QE is, roughly speaking, HOL Light plus global quasiquotation and evaluation operators.
  - Developed by Patrick Laskowski under the supervision of Jacques Carette and William Farmer on a summer undergraduate research project.
  - It is available at

    https://github.com/JacquesCarette/hol-light-qe.
- Why we are using HOL Light:
  1. It is open source software.
  2. Its kernel is small.
  3. Its underlying logic is a version of Church's type theory.
  4. It provides support for defining inductive types.

Implementation Stages

1. HOL term structure is extended.
   - Constructors are added to the OCaml type: Quote, Hole, and Eval for building quasiquotations, antiquotations, and evaluations.
   - Inductive types of syntactic values are added for HOL Light QE types and eval-free terms.
2. HOL Light proof system is modified.
   - Rules for reasoning about quotations and evaluations.
   - INST rule is modified to work like substitution in $\mathrm{CTT}_{\mathrm{qe}}$.
3. Machinery for quotations and evaluations is created.
   - New functions, tactics, and theorems.
4. Examples are developed in the new system.
   - Law of Excluded Middle as a schema.
   - Induction schema for Peano arithmetic.

Future Work

- Continue the development of HOL Light QE.
- Formalize simple SBMAs such as:
  - isEven on unary natural number numerals.
  - Tautology checker on boolean expressions.
  - Addition and multiplication on binary natural number numerals.
- Formalize symbolic differentiation algorithms.
- Formalize a graph of biform theories encoding natural number arithmetic.

Conclusion

1. SBMAs can be formalized using reflection.
2. Global reflection requires only a single metareasoning infrastructure unlike local reflection.
3. $\mathrm{CTT}_{\mathrm{qe}}$ is a version of Church's type theory with a built-in global metareasoning infrastructure.
4. HOL Light QE demonstrates that is $\mathrm{CTT}_{\mathrm{qe}}$ is implementable.
5. Further development of HOL Light QE is needed to demonstrate the advantages global reflection has over local reflection.

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Thank You!