A Network of Arithmetic Biform Theories

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1 Introduction

This report describes a network of arithmetic biform theories expressed in $\textsc{ctt}_{\mathrm{qe}}$ [1], a version of simple type theory with quotation and evaluation. The reader is expected to be familiar with the notation of $\textsc{ctt}_{\mathrm{qe}}$. The following are additional notes to the reader:

1. We assume that a denumerable set of unspecified base types and a definite description operator I have been added to $\textsc{ctt}_{\mathrm{qe}}$.

2. All constants that are not introduced as components of one of the biform theories listed below are logical constants of $\textsc{ctt}_{\mathrm{qe}}$, either primitive or defined. $\mathsf{is\text{-}abs}_{\epsilon \rightarrow o}$, $\mathsf{abs\text{-}body}_{\epsilon \rightarrow \epsilon}$, and $\mathsf{is\text{-}closed}_{\epsilon \rightarrow o}$ are defined logical constants not in [1]. $\mathsf{is\text{-}abs}_{\epsilon \rightarrow o}\, \mathbf{A}_\epsilon$ holds iff $\mathbf{A}_\epsilon$ represents an abstraction. If $\mathbf{A}_\epsilon$ represents an abstraction, then $\mathsf{abs\text{-}body}_{\epsilon \rightarrow \epsilon}\, \mathbf{A}_\epsilon$ represents the body of the abstraction. $\mathsf{is\text{-}closed}_{\epsilon \rightarrow o}\, \mathbf{A}_\epsilon$ holds iff $\mathbf{A}_\epsilon$ represents an expression that is closed (and eval-free).

3. The type attached to a constant may be dropped when there is no loss of meaning.

4. When it makes sense, the notation $\{\mathbf{A}^1_\alpha, \ldots, \mathbf{A}^n_\alpha\}$ denotes the predicate

$$\lambda\, \mathbf{x}_\alpha \,.\, (\mathbf{x}_\alpha = \mathbf{A}^1_\alpha \vee \cdots \vee \mathbf{x}_\alpha = \mathbf{A}^n_\alpha).$$

5. Expressions of type ϵ, i.e., expressions that denote constructions, are colored red.

2 Biform Theories

T1: Simple Theory of Successor

Base Types

1. nat.

Primitive Constants

1. 0_{nat}.
2. $S_{\mathsf{nat} \rightarrow \mathsf{nat}}$.

Defined Constants

1. $1_{\mathsf{nat}} = S\,0$.
2. $\mathsf{is\text{-}fo\text{-}T1}_{\epsilon \rightarrow \epsilon} = \lambda\, x_\epsilon \,.\, \mathbf{B}_\epsilon$ where $\mathbf{B}_\epsilon$ is a complex expression such that $(\lambda\, x_\epsilon \,.\, \mathbf{B}_\epsilon)\ulcorner \mathbf{A}_o \urcorner$ equals $\ulcorner T_o \urcorner$ [$\ulcorner F_o \urcorner$] if $\mathbf{A}_o$ is [not] a formula of first-order logic with equality whose variables are of type nat and whose nonlogical constants are members of $\{0, S\}$.

Axioms

1. $S\,x_{\mathsf{nat}} \neq 0$.
2. $S\,x_{\mathsf{nat}} = S\,y_{\mathsf{nat}} \supset x_{\mathsf{nat}} = y_{\mathsf{nat}}$.

Transformers

1. ξ_1 computes $\mathsf{is\text{-}fo\text{-}T1}_{\epsilon \rightarrow \epsilon}$ using an efficient program that accesses the data stored in the data structures that represent expressions.
2. ξ_2 computes $\mathsf{is\text{-}fo\text{-}T1}_{\epsilon \rightarrow \epsilon}$ using its definition.

T2: Simple Theory of Successor and Addition

Extended Theories

1. T1.

Primitive Constants

3. $+_{\mathsf{nat} \rightarrow \mathsf{nat} \rightarrow \mathsf{nat}}$ (infix).

4. $\mathsf{bplus}_{\epsilon\to\epsilon\to\epsilon}$ (infix).

Defined Constants

3. $\mathsf{bnat}_{\mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}} = \lambda\, x_{\mathsf{nat}} \,.\, \lambda\, y_{\mathsf{nat}} \,.\, ((x_{\mathsf{nat}} + x_{\mathsf{nat}}) + y_{\mathsf{nat}})$.
   Notational definition:

   $(0)_2 = \mathsf{bnat}\, 0_{\mathsf{nat}}\, 0_{\mathsf{nat}}$.
   $(1)_2 = \mathsf{bnat}\, 0_{\mathsf{nat}}\, 1_{\mathsf{nat}}$.
   $(a_1 \cdots a_n 0)_2 = \mathsf{bnat}\, (a_1 \cdots a_n)_2\, 0_{\mathsf{nat}}$ where each a_i is 0 or 1.
   $(a_1 \cdots a_n 1)_2 = \mathsf{bnat}\, (a_1 \cdots a_n)_2\, 1_{\mathsf{nat}}$ where each a_i is 0 or 1.

4. $\mathsf{is\text{-}bnum}_{\epsilon\to o} = \mathrm{I}\, f_{\epsilon\to o} \,.\, \forall\, u_\epsilon \,.\, (f_{\epsilon\to\epsilon}\, u_\epsilon \equiv$
   $\exists\, v_\epsilon \,.\, \exists\, w_\epsilon \,.\, (u_\epsilon = \ulcorner \mathsf{bnat}\, \lfloor v_\epsilon \rfloor\, \lfloor w_\epsilon \rfloor \urcorner \wedge$
   $(v_\epsilon = \ulcorner 0 \urcorner \vee f_{\epsilon\to\epsilon}\, v_\epsilon) \wedge (w_\epsilon = \ulcorner 0 \urcorner \vee w_\epsilon = \ulcorner 1 \urcorner)))$.

5. $\mathsf{is\text{-}fo\text{-}T2}_{\epsilon\to\epsilon} = \lambda\, x_\epsilon \,.\, \mathbf{B}_\epsilon$ where $\mathbf{B}_\epsilon$ is a complex expression such that $(\lambda\, x_\epsilon \,.\, \mathbf{B}_\epsilon) \ulcorner \mathbf{A}_o \urcorner$ equals $\ulcorner T_o \urcorner$ [$\ulcorner F_o \urcorner$] if $\mathbf{A}_o$ is [not] a formula of first-order logic with equality whose variables are of type nat and whose nonlogical constants are members of $\{0, S, +\}$.

Axioms

5. $x_{\mathsf{nat}} + 0 = x_{\mathsf{nat}}$.
6. $x_{\mathsf{nat}} + S\, y_{\mathsf{nat}} = S\, (x_{\mathsf{nat}} + y_{\mathsf{nat}})$.
7. $\mathsf{is\text{-}bnum}\, u_\epsilon \supset u_\epsilon\ \mathsf{bplus}\ \ulcorner (0)_2 \urcorner = u_\epsilon$.
8. $\mathsf{is\text{-}bnum}\, u_\epsilon \supset \ulcorner (0)_2 \urcorner\ \mathsf{bplus}\ u_\epsilon = u_\epsilon$.
9. $\ulcorner (1)_2 \urcorner\ \mathsf{bplus}\ \ulcorner (1)_2 \urcorner = \ulcorner (10)_2 \urcorner$.
10. $\mathsf{is\text{-}bnum}\, u_\epsilon \supset$
    $\ulcorner \mathsf{bnat}\, \lfloor u_\epsilon \rfloor\, 0 \urcorner\ \mathsf{bplus}\ \ulcorner (1)_2 \urcorner = \ulcorner \mathsf{bnat}\, \lfloor u_\epsilon \rfloor\, 1 \urcorner$.
11. $\mathsf{is\text{-}bnum}\, u_\epsilon \supset$
    $\ulcorner \mathsf{bnat}\, \lfloor u_\epsilon \rfloor\, 1 \urcorner\ \mathsf{bplus}\ \ulcorner (1)_2 \urcorner = \ulcorner \mathsf{bnat}\, \lfloor u_\epsilon\ \mathsf{bplus}\ \ulcorner (1)_2 \urcorner \rfloor\, 0 \urcorner$.
12. $\mathsf{is\text{-}bnum}\, u_\epsilon \supset$
    $\ulcorner (1)_2 \urcorner\ \mathsf{bplus}\ \ulcorner \mathsf{bnat}\, \lfloor u_\epsilon \rfloor\, 0 \urcorner = \ulcorner \mathsf{bnat}\, \lfloor u_\epsilon \rfloor\, 1 \urcorner$.
13. $\mathsf{is\text{-}bnum}\, u_\epsilon \supset$
    $\ulcorner (1)_2 \urcorner\ \mathsf{bplus}\ \ulcorner \mathsf{bnat}\, \lfloor u_\epsilon \rfloor\, 0 \urcorner = \ulcorner \mathsf{bnat}\, \lfloor u_\epsilon\ \mathsf{bplus}\ \ulcorner (1)_2 \urcorner \rfloor\, 0 \urcorner$.
14. $(\mathsf{is\text{-}bnum}\, u_\epsilon \wedge \mathsf{is\text{-}bnum}\, v_\epsilon) \supset$
    $\ulcorner \mathsf{bnat}\, \lfloor u_\epsilon \rfloor\, 0 \urcorner\ \mathsf{bplus}\ \ulcorner \mathsf{bnat}\, \lfloor v_\epsilon \rfloor 0 \urcorner = \ulcorner \mathsf{bnat}\, \lfloor u_\epsilon\ \mathsf{bplus}\ v_\epsilon \rfloor\, 0 \urcorner$.
15. $(\mathsf{is\text{-}bnum}\, u_\epsilon \wedge \mathsf{is\text{-}bnum}\, v_\epsilon) \supset$
    $\ulcorner \mathsf{bnat}\, \lfloor u_\epsilon \rfloor\, 0 \urcorner\ \mathsf{bplus}\ \ulcorner \mathsf{bnat}\, \lfloor v_\epsilon \rfloor\, 1 \urcorner = \ulcorner \mathsf{bnat}\, \lfloor u_\epsilon\ \mathsf{bplus}\ v_\epsilon \rfloor\, 1 \urcorner$.

16. $(\text{is-bnum}\, u_\epsilon \wedge \text{is-bnum}\, v_\epsilon) \supset$
    $\ulcorner \text{bnat}\, \lfloor u_\epsilon \rfloor\, 1 \urcorner\ \text{bplus}\ \ulcorner \text{bnat}\, \lfloor v_\epsilon \rfloor\, 0 \urcorner = \ulcorner \text{bnat}\, \lfloor u_\epsilon\ \text{bplus}\ v_\epsilon \rfloor\, 1 \urcorner.$
17. $(\text{is-bnum}\, u_\epsilon \wedge \text{is-bnum}\, v_\epsilon) \supset$
    $\ulcorner \text{bnat}\, \lfloor u_\epsilon \rfloor\, 1 \urcorner\ \text{bplus}\ \ulcorner \text{bnat}\, \lfloor v_\epsilon \rfloor\, 1 \urcorner =$
    $\ulcorner \text{bnat}\, \lfloor (u_\epsilon\ \text{bplus}\ v_\epsilon)\ \text{bplus}\ \ulcorner (1)_2 \urcorner \rfloor\, 0 \urcorner.$

Transformers

3. ξ_3 computes $\text{bplus}_{\epsilon\to\epsilon\to\epsilon}$ using an efficient program that satisfies Axioms 7–17.
4. ξ_4 computes $\text{bplus}_{\epsilon\to\epsilon\to\epsilon}$ using Axioms 7–17 as conditional rewrite rules.
5. ξ_5 computes $\text{is-fo-T1}_{\epsilon\to\epsilon}$ using an efficient program that accesses the data stored in the data structures that represent expressions.
6. ξ_6 computes $\text{is-fo-T2}_{\epsilon\to\epsilon}$ using its definition.

Theorems

1. Meaning formula schema for $\text{bplus}_{\epsilon\to\epsilon\to\epsilon}$
   $((\text{is-bnum}\, \mathbf{A}_\epsilon \wedge \text{is-bnum}\, \mathbf{B}_\epsilon) \supset$
   $(\text{is-bnum}\, (\mathbf{A}_\epsilon\ \text{bplus}\ \mathbf{B}_\epsilon) \wedge$
   $([\![\mathbf{A}_\epsilon\ \text{bplus}\ \mathbf{B}_\epsilon]\!]_{\text{nat}} = [\![\mathbf{A}_\epsilon]\!]_{\text{nat}} + [\![\mathbf{B}_\epsilon]\!]_{\text{nat}}))).$

T3: Simple Theory of Successor, Addition, and Multiplication

Extended Theories

2. T2.

Primitive Constants

5. $*_{\text{nat}\to\text{nat}\to\text{nat}}$ (infix).
6. $\text{btimes}_{\epsilon\to\epsilon\to\epsilon}$ (infix).

Defined Constants

4. $\text{is-fo-T3}_{\epsilon\to\epsilon} = \lambda\, x_\epsilon\, .\, \mathbf{B}_\epsilon$ where $\mathbf{B}_\epsilon$ is a complex expression such that $(\lambda\, x_\epsilon\, .\, \mathbf{B}_\epsilon)\ulcorner \mathbf{A}_o \urcorner$ equals $\ulcorner T_o \urcorner$ [$\ulcorner F_o \urcorner$] if $\mathbf{A}_o$ is [not] a formula of first-order logic with equality whose variables are of type nat and whose nonlogical constants are members of $\{0, S, +, *\}$.

Axioms

18. $x_{\mathsf{nat}} * 0 = 0$.
19. $x_{\mathsf{nat}} * S\, y_{\mathsf{nat}} = (x_{\mathsf{nat}} * y_{\mathsf{nat}}) + x_{\mathsf{nat}}$.
20. $\mathsf{is\text{-}bnum}\; u_\epsilon \supset u_\epsilon \;\mathsf{btimes}\; \ulcorner(0)_2\urcorner = \ulcorner(0)_2\urcorner$.
21. $\mathsf{is\text{-}bnum}\; u_\epsilon \supset \ulcorner(0)_2\urcorner \;\mathsf{btimes}\; u_\epsilon = \ulcorner(0)_2\urcorner$.
22. $\mathsf{is\text{-}bnum}\; u_\epsilon \supset u_\epsilon \;\mathsf{btimes}\; \ulcorner(1)_2\urcorner = u_\epsilon$.
23. $\mathsf{is\text{-}bnum}\; u_\epsilon \supset \ulcorner(1)_2\urcorner \;\mathsf{btimes}\; u_\epsilon = u_\epsilon$.
24. $(\mathsf{is\text{-}bnum}\; u_\epsilon \wedge \mathsf{is\text{-}bnum}\; v_\epsilon) \supset$
    $\ulcorner\mathsf{bnat}\; \lfloor u_\epsilon\rfloor\; 0\urcorner \;\mathsf{btimes}\; v_\epsilon = \ulcorner\mathsf{bnat}\; \lfloor u_\epsilon \;\mathsf{btimes}\; v_\epsilon\rfloor\; 0\urcorner$.
25. $(\mathsf{is\text{-}bnum}\; u_\epsilon \wedge \mathsf{is\text{-}bnum}\; v_\epsilon) \supset$
    $\ulcorner\mathsf{bnat}\; \lfloor u_\epsilon\rfloor\; 1\urcorner \;\mathsf{btimes}\; v_\epsilon = \ulcorner\mathsf{bnat}\; \lfloor u_\epsilon \;\mathsf{btimes}\; v_\epsilon\rfloor\; 0\urcorner \;\mathsf{badd}\; v_\epsilon$.
26. $(\mathsf{is\text{-}bnum}\; u_\epsilon \wedge \mathsf{is\text{-}bnum}\; v_\epsilon) \supset$
    $v_\epsilon \;\mathsf{btimes}\; \ulcorner\mathsf{bnat}\; \lfloor u_\epsilon\rfloor\; 0\urcorner = \ulcorner\mathsf{bnat}\; \lfloor u_\epsilon \;\mathsf{btimes}\; v_\epsilon\rfloor\; 0\urcorner$.
27. $(\mathsf{is\text{-}bnum}\; u_\epsilon \wedge \mathsf{is\text{-}bnum}\; v_\epsilon) \supset$
    $v_\epsilon \;\mathsf{btimes}\; \ulcorner\mathsf{bnat}\; \lfloor u_\epsilon\rfloor\; 1\urcorner = \ulcorner\mathsf{bnat}\; \lfloor u_\epsilon \;\mathsf{btimes}\; v_\epsilon\rfloor\; 0\urcorner \;\mathsf{badd}\; v_\epsilon$.

Transformers

7. ξ_7 computes $\mathsf{btimes}_{\epsilon\to\epsilon\to\epsilon}$ using an efficient program that satisfies Axioms 20–27.
8. ξ_8 computes $\mathsf{btimes}_{\epsilon\to\epsilon\to\epsilon}$ using Axioms 20–27 as conditional rewrite rules.
9. ξ_9 computes $\mathsf{is\text{-}fo\text{-}T3}_{\epsilon\to\epsilon}$ using an efficient program that accesses the data stored in the data structures that represent expressions.
10. ξ_{10} computes $\mathsf{is\text{-}fo\text{-}T3}_{\epsilon\to\epsilon}$ using its definition.

Theorems

2. Meaning formula schema for $\mathsf{btimes}_{\epsilon\to\epsilon\to\epsilon}$
   $((\mathsf{is\text{-}bnum}\; \mathbf{A}_\epsilon \wedge \mathsf{is\text{-}bnum}\; \mathbf{B}_\epsilon) \supset$
   $(\mathsf{is\text{-}bnum}\; (\mathbf{A}_\epsilon \;\mathsf{btimes}\; \mathbf{B}_\epsilon) \wedge$
   $(\llbracket \mathbf{A}_\epsilon \;\mathsf{btimes}\; \mathbf{B}_\epsilon\rrbracket_{\mathsf{nat}} = \llbracket \mathbf{A}_\epsilon\rrbracket_{\mathsf{nat}} + \llbracket \mathbf{B}_\epsilon\rrbracket_{\mathsf{nat}})))$.

T4: Robinson Arithmetic (Q)

Extended Theories

3. T3.

Axioms

28. $x_{\mathsf{nat}} = 0 \vee \exists\, y_{\mathsf{nat}} \;.\; S\, y_{\mathsf{nat}} = x_{\mathsf{nat}}$.

T5: Complete Theory of Successor

Extended Theories

1. T1.

Primitive Constants

7. $\mathsf{T5\text{-}dec\text{-}proc}_{\epsilon\to\epsilon}$.

Defined Constants

6. $\mathsf{is\text{-}fo\text{-}T1\text{-}abs}_{\epsilon\to\epsilon} =$
$\lambda\, x_\epsilon \;.\; (\mathsf{if}\; (\mathsf{is\text{-}abs}_{\epsilon\to o}\, x_\epsilon)\; (\mathsf{is\text{-}fo\text{-}T1}_{\epsilon\to\epsilon}\, (\mathsf{abs\text{-}body}_{\epsilon\to\epsilon}\, x_\epsilon))\; \ulcorner F_o \urcorner)$.

Axioms

29. Induction Schema for Successor
$\forall\, f_\epsilon \;.\; ((\mathsf{is\text{-}expr}^{\mathsf{nat}\to o}_{\epsilon\to o}\, f_\epsilon \wedge [\![\mathsf{is\text{-}fo\text{-}T1\text{-}abs}_{\epsilon\to\epsilon}\, f_\epsilon]\!]_o) \supset$
$(([\![f_\epsilon]\!]_{\mathsf{nat}\to o}\, 0 \wedge (\forall\, x_{\mathsf{nat}} \;.\; [\![f_\epsilon]\!]_{\mathsf{nat}\to o}\, x_{\mathsf{nat}} \supset [\![f_\epsilon]\!]_{\mathsf{nat}\to o}\, (\mathsf{S}\, x_{\mathsf{nat}}))) \supset$
$\forall\, x_{\mathsf{nat}} \;.\; [\![f_\epsilon]\!]_{\mathsf{nat}\to o}\, x_{\mathsf{nat}}))$.

30. Meaning Formula for $\mathsf{T5\text{-}dec\text{-}proc}_{\epsilon\to\epsilon}$
$\forall\, u_\epsilon \;.\; ((\mathsf{is\text{-}expr}^{o}_{\epsilon\to o}\, u_\epsilon \wedge \mathsf{is\text{-}closed}_{\epsilon\to o}\, u_\epsilon \wedge [\![\mathsf{is\text{-}fo\text{-}T1}_{\epsilon\to\epsilon}\, u_\epsilon]\!]_o) \supset$
$((\mathsf{T5\text{-}dec\text{-}proc}_{\epsilon\to\epsilon}\, u_\epsilon = \ulcorner T_o \urcorner \vee \mathsf{T5\text{-}dec\text{-}proc}_{\epsilon\to\epsilon}\, u_\epsilon = \ulcorner F_o \urcorner) \wedge$
$[\![\mathsf{T5\text{-}dec\text{-}proc}_{\epsilon\to\epsilon}\, u_\epsilon]\!]_o = [\![u_\epsilon]\!]_o))$.

Transformers

11. ξ_{11} computes the decision procedure $\mathsf{T5\text{-}dec\text{-}proc}_{\epsilon\to\epsilon\to\epsilon}$ using an efficient program that satisfies Axiom 30.

12. ξ_{12} computes $\mathsf{is\text{-}fo\text{-}T1\text{-}abs}_{\epsilon\to\epsilon}$ using an efficient program that accesses the data stored in the data structures that represent expressions.

13. ξ_{13} computes $\mathsf{is\text{-}fo\text{-}T1\text{-}abs}_{\epsilon\to\epsilon}$ using its definition.

T6: Presburger Arithmetic

Extended Theories

2. T2.

Primitive Constants

8. $\mathsf{T6\text{-}dec\text{-}proc}_{\epsilon\to\epsilon}$.

Defined Constants

7. $\mathsf{is\text{-}fo\text{-}T2\text{-}abs}_{\epsilon\to\epsilon} =$
   $\lambda\, x_\epsilon\, .\, (\mathsf{if}\ (\mathsf{is\text{-}abs}_{\epsilon\to o}\, x_\epsilon)\ (\mathsf{is\text{-}fo\text{-}T2}_{\epsilon\to\epsilon}\, (\mathsf{abs\text{-}body}_{\epsilon\to\epsilon}\, x_\epsilon))\ \ulcorner F_o \urcorner)$.

Axioms

31. Induction Schema for Successor and Addition
    $\forall\, f_\epsilon\, .\, ((\mathsf{is\text{-}expr}^{\mathsf{nat}\to o}_{\epsilon\to o}\, f_\epsilon \wedge [\![\mathsf{is\text{-}fo\text{-}T2\text{-}abs}_{\epsilon\to\epsilon}\, f_\epsilon]\!]_o) \supset$
    $(([\![f_\epsilon]\!]_{\mathsf{nat}\to o}\, 0 \wedge (\forall\, x_{\mathsf{nat}}\, .\, [\![f_\epsilon]\!]_{\mathsf{nat}\to o}\, x_{\mathsf{nat}} \supset [\![f_\epsilon]\!]_{\mathsf{nat}\to o}\, (\mathsf{S}\, x_{\mathsf{nat}}))) \supset$
    $\forall\, x_{\mathsf{nat}}\, .\, [\![f_\epsilon]\!]_{\mathsf{nat}\to o}\, x_{\mathsf{nat}}))$.
32. Meaning formula for $\mathsf{T6\text{-}dec\text{-}proc}_{\epsilon\to\epsilon}$.
    $\forall\, u_\epsilon\, .\, ((\mathsf{is\text{-}expr}^{o}_{\epsilon\to o}\, u_\epsilon \wedge \mathsf{is\text{-}closed}_{\epsilon\to o}\, u_\epsilon \wedge [\![\mathsf{is\text{-}fo\text{-}T2}_{\epsilon\to\epsilon}\, u_\epsilon]\!]_o) \supset$
    $((\mathsf{T6\text{-}dec\text{-}proc}_{\epsilon\to\epsilon}\, u_\epsilon = \ulcorner T_o \urcorner \vee \mathsf{T6\text{-}dec\text{-}proc}_{\epsilon\to\epsilon}\, u_\epsilon = \ulcorner F_o \urcorner) \wedge$
    $[\![\mathsf{T6\text{-}dec\text{-}proc}_{\epsilon\to\epsilon}\, u_\epsilon]\!]_o = [\![u_\epsilon]\!]_o))$.

Transformers

14. ξ_{14} computes the decision procedure $\mathsf{T6\text{-}dec\text{-}proc}_{\epsilon\to\epsilon\to\epsilon}$ using an efficient program that satisfies Axiom 32.
15. ξ_{15} computes $\mathsf{is\text{-}fo\text{-}T2\text{-}abs}_{\epsilon\to\epsilon}$ using an efficient program that accesses the data stored in the data structures that represent expressions.
16. ξ_{16} computes $\mathsf{is\text{-}fo\text{-}T2\text{-}abs}_{\epsilon\to\epsilon}$ using its definition.

Theorems

3. Meaning formula for $\mathsf{bplus}_{\epsilon\to\epsilon\to\epsilon}$
   $\forall\, u_\epsilon\, .\, \forall\, v_\epsilon\, .\, ((\mathsf{is\text{-}bnum}\, u_\epsilon \wedge \mathsf{is\text{-}bnum}\, v_\epsilon) \supset$
   $(\mathsf{is\text{-}bnum}\, (u_\epsilon\ \mathsf{bplus}\ v_\epsilon) \wedge$
   $([\![u_\epsilon\ \mathsf{bplus}\ v_\epsilon]\!]_{\mathsf{nat}} = [\![u_\epsilon]\!]_{\mathsf{nat}} + [\![v_\epsilon]\!]_{\mathsf{nat}})))$.

T7: First-Order Peano Arithmetic

Extended Theories

3. T3.

Defined Constants

8. $\mathsf{is\text{-}fo\text{-}T3\text{-}abs}_{\epsilon \to \epsilon} =$
   $\lambda\, x_\epsilon \,.\, (\mathsf{if}\ (\mathsf{is\text{-}abs}_{\epsilon \to o}\, x_\epsilon)\ (\mathsf{is\text{-}fo\text{-}T3}_{\epsilon \to \epsilon}\, (\mathsf{abs\text{-}body}_{\epsilon \to \epsilon}\, x_\epsilon))\ \ulcorner F_o \urcorner)$.

Axioms

33. Induction Schema for Successor, Addition, and Multiplication
    $\forall\, f_\epsilon \,.\, ((\mathsf{is\text{-}expr}^{\mathsf{nat} \to o}_{\epsilon \to o}\, f_\epsilon \wedge [\![\mathsf{is\text{-}fo\text{-}T3\text{-}abs}_{\epsilon \to \epsilon}\, f_\epsilon]\!]_o) \supset$
    $(([\![f_\epsilon]\!]_{\mathsf{nat} \to o}\, 0 \wedge (\forall\, x_{\mathsf{nat}} \,.\, [\![f_\epsilon]\!]_{\mathsf{nat} \to o}\, x_{\mathsf{nat}} \supset [\![f_\epsilon]\!]_{\mathsf{nat} \to o}\, (\mathsf{S}\, x_{\mathsf{nat}}))) \supset$
    $\forall\, x_{\mathsf{nat}} \,.\, [\![f_\epsilon]\!]_{\mathsf{nat} \to o}\, x_{\mathsf{nat}}))$.

Transformers

17. ξ_{17} computes $\mathsf{is\text{-}fo\text{-}T3\text{-}abs}_{\epsilon \to \epsilon}$ using an efficient program that accesses the data stored in the data structures that represent expressions.
18. ξ_{18} computes $\mathsf{is\text{-}fo\text{-}T3\text{-}abs}_{\epsilon \to \epsilon}$ using its definition.

Theorems

4. Meaning formula $\mathsf{btimes}_{\epsilon \to \epsilon \to \epsilon}$
   $\forall\, u_\epsilon \,.\, \forall\, v_\epsilon \,.\, ((\mathsf{is\text{-}bnum}\, u_\epsilon \wedge \mathsf{is\text{-}bnum}\, v_\epsilon) \supset$
   $(\mathsf{is\text{-}bnum}\, (u_\epsilon\ \mathsf{btimes}\ v_\epsilon) \wedge$
   $([\![u_\epsilon\ \mathsf{btimes}\ v_\epsilon]\!]_{\mathsf{nat}} = [\![u_\epsilon]\!]_{\mathsf{nat}} * [\![v_\epsilon]\!]_{\mathsf{nat}})))$.

T8: Higher-Order Peano Arithmetic

Extended Theories

1. T1.

Defined Constants

9. $+_{\mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}} = \mathrm{I}\, f_{\mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}} \mathrel{.} \forall\, x_{\mathsf{nat}} \mathrel{.} \forall\, y_{\mathsf{nat}} \mathrel{.}$
   $(f_{\mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}}\, x_{\mathsf{nat}}\, 0 = x_{\mathsf{nat}} \wedge$
   $f_{\mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}}\, x_{\mathsf{nat}}\, (S\, y_{\mathsf{nat}}) = S\, (f_{\mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}}\, x_{\mathsf{nat}}\, y_{\mathsf{nat}}))$.
10. $*_{\mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}} = \mathrm{I}\, f_{\mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}} \mathrel{.} \forall\, x_{\mathsf{nat}} \mathrel{.} \forall\, y_{\mathsf{nat}} \mathrel{.}$
   $(f_{\mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}}\, x_{\mathsf{nat}}\, 0 = 0 \wedge$
   $f_{\mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}}\, x_{\mathsf{nat}}\, (S\, y_{\mathsf{nat}}) = (f_{\mathsf{nat}\to\mathsf{nat}\to\mathsf{nat}}\, x_{\mathsf{nat}}\, y_{\mathsf{nat}}) + x_{\mathsf{nat}})$.

Axioms

34. Induction Axiom for the Natural Numbers
    $\forall\, p_{\mathsf{nat}\to o} \mathrel{.} ((p_{\mathsf{nat}\to o}\, 0 \wedge (\forall\, x_{\mathsf{nat}} \mathrel{.} (p_{\mathsf{nat}\to o}\, x_{\mathsf{nat}} \supset p_{\mathsf{nat}\to o}\, (S\, x_{\mathsf{nat}})))) \supset$
    $\forall\, x_{\mathsf{nat}} \mathrel{.} p_{\mathsf{nat}\to o}\, x_{\mathsf{nat}})$.

Theorems

5. Induction Schema for Successor.
6. Induction Schema for Successor and Addition.
7. Induction Schema for Successor, Addition, and Multiplication.

References

[1] W. M. Farmer. Incorporating quotation and evaluation into church's type theory. *Computing Research Repository (CoRR)*, abs/1612.02785 (72 pp.), 2016.