Towards Specifying Symbolic Computation*

Jacques Carette and William M. Farmer

Computing and Software, McMaster University, Canada
http://www.cas.mcmaster.ca/~carette
http://imps.mcmaster.ca/wmfarmer

Abstract. ??

1 Introduction

2 Simplifying Expressions denoting Rational Functions

2.1 The Problem

Let $f = \lambda\, x : \mathbb{Q}\ .\ R(x)$ be an expression that denotes a function of type $\mathbb{Q} \to \mathbb{Q}$. Furthermore, let $R(x)$, as a syntactic expression, denote a member of the field $\mathbb{Q}(x)$ consisting of *rational expressions* of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials in x. A function that can be represented by an expression like f whose body is a rational expression is called a *rational function*.

$R(x)$ could be a very complicated expression build from the primitive components of the field $\mathbb{Q}(x)$: $0, 1, +, *, -, ^{-1}$. An obvious way to simplify f would be to simplify $R(x)$ syntactically as a rational expression in a meaning-preserving way. This operation, which we will call simpRatFun, is an example of a *syntax-based mathematical algorithm (SBMA)*. As an SBMA, simpRatFun works by manipulating syntactic expressions in a mathematically meaningful way.

SBMAs can be difficult to specify since they involve an interplay of syntax and semantics. We are interested in the following three questions:

1. What should be the specification of simpRatFun?
2. Can this specification be expressed in traditional logic?
3. How would this specification be expressed in a logic with undefinedness, quotation, and evaluation?

2.2 A Naive Specification

Let $R(x)$ be an expression that denotes a member of $\mathbb{Q}(x)$. The *normal form* of $R(x)$ is an expression $R'(x)$ of the form $\frac{P(x)}{Q(x)}$ such that $P(x)$ and $Q(x)$ are

* This research is supported by NSERC.

polynomials in standard form, $\frac{P(x)}{Q(x)}$ is in lowest terms, and $R(x)$ and $R'(x)$ both denote the same member of $\mathbb{Q}(x)$. For example, the normal form of

$$\frac{x^2+2x-1}{x^2-1} - \frac{x}{x-1}$$

is

$$\frac{1}{x+1}.$$

If we do not think too hard, we might be tempted to specify simpRatFun as follows:

> For all expressions $f = \lambda\, x : \mathbb{Q} \,.\, R(x)$ of type $\mathbb{Q} \rightarrow \mathbb{Q}$ where the expression $R(x)$ denotes a member of $\mathbb{Q}(x)$, $\mathsf{simpRatFun}(f) = \lambda\, x : \mathbb{Q} \,.\, R'(x)$ where $R'(x)$ is the normal form of $R(x)$.

Hence simpRatFun applied to

$$\lambda\, x : \mathbb{Q} \,.\, \frac{x^2+2x-1}{x^2-1} - \frac{x}{x-1}$$

should return

$$\lambda\, x : \mathbb{Q} \,.\, \frac{1}{x+1}.$$

This specification is essentially the same as Maple's normal operation that reduces the expression $\frac{x^2-2x-1}{x^2-1} - \frac{x}{x-1}$ to $\frac{1}{x+1}$.

Is this specification of simpRatFun correct? If so, f and $\mathsf{simpRatFun}(f)$ should denote the same function of type $\mathbb{Q} \rightarrow \mathbb{Q}$ for all expressions f that denote rational functions. Unfortunately, this is not the case. Let

$$f = \lambda\, x : \mathbb{Q} \,.\, \frac{x^2+2x-1}{x^2-1} - \frac{x}{x-1}.$$

Then $f(1)$ is obviously undefined, but $\mathsf{simpRatFun}(f)(1) = 1/2$. Hence simpRatFun is — as specified — not meaning preserving. What went wrong?

2.3 A Correct Specification

2.4 A Formalized Specification

3 Conclusion

Todo list