Classical Symbolic Retrodictive Execution of Quantum Circuits

Jacques Carette
McMaster University

Gerardo Ortiz*
Indiana University

Amr Sabry
Indiana University

February 23, 2022

1 Main

Retrodictive quantum theory [5], retrocausality [2], and the time-symmetry of physical laws [18] suggest that partial knowledge about the future can be exploited to understand the present. We demonstrate the even stronger proposition that, in concert with the computational concepts of *demand-driven lazy evaluation* [10] and *symbolic partial evaluation* [9], retrodictive reasoning can be used as a computational resource to dequantize some quantum algorithms, i.e., to provide efficient classical algorithms inspired by their quantum counterparts.

Symbolic Execution of Classical Programs Applied to Quantum Oracles. A well-established technique to simultaneously explore multiple paths that a classical program could take under different inputs is *symbolic execution* [6, 12, 11, 8, 4]. In this execution scheme, concrete values are replaced by symbols which are initially unconstrained. As the execution proceeds, the symbols interact with program constructs and this typically introduces constraints on the possible values that the symbols represent. At the end of the execution, these constraints can be solved to infer properties of the program under consideration. The idea is also applicable to quantum circuits as the following example illustrates.

Let $[\mathbf{n}]$ denote the finite set $\{0, 1, \ldots, (n-1)\}$. In Simon's problem, we are given a 2-1 (classical) function $f : [\mathbf{2^n}] \to [\mathbf{2^n}]$ with the property that there exists an a such $f(x) = f(x \oplus a)$ for all x; the goal is to determine a. The circuit in Fig. 1 implements the quantum algorithm when $n = 2$ and $a = 3$. In the circuit, the gates between barrier (1) and barrier (2) implement a quantum oracle $U_f(x, 0) = (x, f(x))$ that encapsulates the function f of interest. A direct classical simulation of the quantum circuit would need to execute the U_f block four times, once for each possible value $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ for the top two wires. Instead, let us introduce two symbols x_0 representing the top wire and x_1 representing the wire below it, and let's proceed with the execution symbolically. The state at barrier (1) is initially $|x_0x_100\rangle$. At the first CX-gate, we symbolically calculate the result of the target wire as $x_0 \oplus 0 = x_0$ evolving the state to $|x_0x_1x_00\rangle$. Going through the next three CX-gates, the state evolves as $|x_0x_1x_0x_0\rangle$, $|x_0x_1(x_0 \oplus x_1)x_0\rangle$, and $|x_0x_1(x_0 \oplus x_1)(x_0 \oplus x_1)\rangle$ at barrier (2). At that point, we have established that the bottom two wires are equal; the result of their measurement can only be 00 or 11. Since the function is promised to be 2-1 for all inputs, it is sufficient to analyze one case, say when the measurement at barrier (3) produces 00. This measurement collapses the top wires to $|x_0x_1\rangle$ subject to the constraint that $x_0 \oplus x_1 = 0$ or equivalently that $x_0 = x_1$. We have thus inferred that both $x_0 = x_1 = 0$ and $x_0 = x_1 = 1$ produce the same measurement result at barrier (3) and hence that $f(00) = f(11) = f(00 \oplus 11)$ which reveals that a is 11 in binary notation.

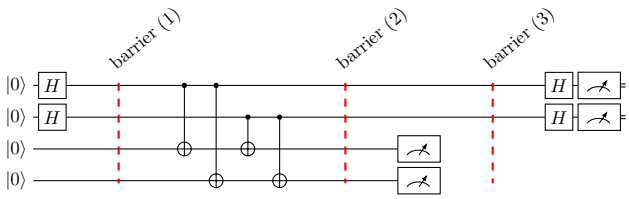
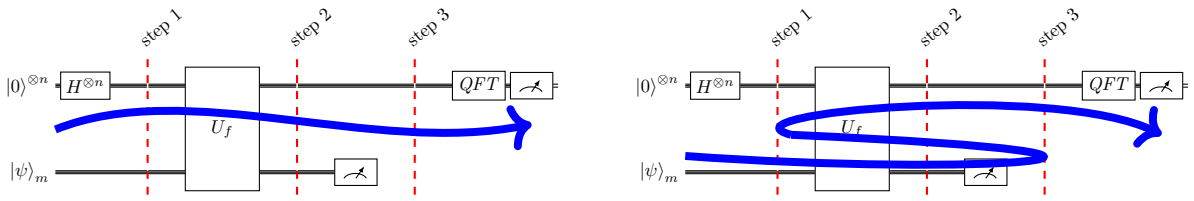


Figure 1: Circuit for Simon's Algorithm $n = 2$ and $a = 3$



(a) Conventional Flow

(b) Retrodictive Flow

Figure 2: Template quantum circuit

Since the quantum circuit between barriers (1) and (2) is reversible, we can perform the analysis above in a mixed predictive and symbolic retrodictive execution to make the flow of information conceptually clearer. We start a forward classical simulation with one arbitrary state at barrier (1), say $|0100\rangle$. This state evolves to $|0100\rangle$, then $|0100\rangle$ again, then $|0110\rangle$, and finally $|0111\rangle$. In this case, the result of measuring the bottom two wires is 11. Having produced a possible measurement at barrier (3), we start a retrodictive execution to find out what other input states might be compatible with this future measurement. To that end, we execute the circuit backwards with the symbolic state $|x_0x_111\rangle$; that execution evolves to $|x_0x_11(1 \oplus x_1)\rangle$, then $|x_0x_1(1 \oplus x_1)(1 \oplus x_1)\rangle$, then $|x_0x_1(1 \oplus x_1)(1 \oplus x_0 \oplus x_1)\rangle$, and finally $|x_0x_1(1 \oplus x_0 \oplus x_1)(1 \oplus x_0 \oplus x_1)\rangle$. Having reached the initial conditions on the bottom two wires, we reconcile them with the collected constraints to conclude that $1 \oplus x_0 \oplus x_1 = 0$ or equivalently that $x_0 \neq x_1$. The measurement of 11 at barrier (3) is consistent with not just the state $|01\rangle$ we started with but also with the state $|10\rangle$. In other words, we have $f(01) = f(10) = f(01 \oplus 11)$ and the hidden value of a is revealed to be 11.

Representing Wavefunctions Symbolically. A symbolic variable represents a boolean value that can be 0 or 1; this is similar to a qubit in a superposition $(1/\sqrt{2})(|0\rangle \pm |1\rangle)$. Thus, it appears that $H|0\rangle$ could be represented by a symbol x to denote the uncertainty. Surprisingly, this idea scales to even represent maximally entangled states. Fig. 3(left) shows a circuit to generate the Bell state $(1/\sqrt{2})(|00\rangle+|11\rangle)$. By using the symbol x for $H|0\rangle$, the input to the CX-gate is $|x0\rangle$ which evolves to $|xx\rangle$. By sharing the same symbol in two positions, the symbolic state accurately represents the entangled Bell state. Similarly, for the circuit in Fig. 3(right), the state after the Hadamard gate is $|x00\rangle$ which evolves to $|xx0\rangle$ and then to $|xxx\rangle$ again accurately capturing the entanglement correlations.

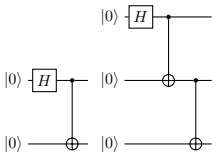


Figure 3: Bell and GHZ States

This insight allows us to symbolically execute the many quantum algorithms that match the template in Fig. 2 (including Deutsch, Deutsch-Jozsa, Bernstein-Vazirani, Simon, Grover, and Shor's algorithms). Specifically, in all these algorithms, the top collection of wires (which we will call the computational register) is prepared in a uniform superposition which can be represented using symbolic variables. Below, we report on the results of such symbolic executions. In each case, instead of the conventional execution flow depicted in Fig. 2(a), we find a possible measurement outcome w at barrier (3) and perform a retrodictive execution with a state $|xw\rangle$ going backwards to collect the constraints on x that enable us to solve the problem in question.

Deutsch. The quantum circuit in Fig. 4 determines if the function $[\mathbf{2}] \to [\mathbf{2}]$ encapsulated in the quantum oracle U_f is constant or balanced. Since 0 is always a possible measurement of the ancilla register, we start a retrodictive execution of the U_f block with state $|x0\rangle$. This execution terminates with a state $|xr\rangle$ where r is a formula expressing the dependencies of the ancilla on x. Running the experiment with different choices for f, the resulting formula always perfectly describes f. Specifically when f is the constant function that returns 0, we have $r = 0$; when f is the constant function that returns 1, we have $r = 0$; when f is the balanced function that

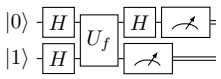


Figure 4: Deutsch

returns its input, we have $r = x$; and when f is the balanced function that returns the negation of its input, we have $r = 1 \oplus x$.

Deutsch-Jozsa. The problem is a generalization of the previous one. We are given a function $[\mathbf{2^n}] \to [\mathbf{2}]$ that is promised to be constant or balanced and we need to decide distinguish the two cases. The quantum circuit generalizes the one in Fig. 4 to use n-wires for the computation register. Similarly to before, we perform a retrodictive execution of the U_f block with the state $|x_{n-1} \cdots x_1 x_0 0\rangle$ and observe the resulting formula r. Like before, when the function is constant, the formula r is the corresponding constant and when the function is balanced, the formula r completely describes how the result is computed from the symbols $x_{n-1}, \cdots, x_1, x_0$. For example, for $n = 6$, the resulting formulae for three balanced functions were: x_0, $x_0 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5$, and $x_0x_1x_2 \oplus x_0x_1x_2x_3x_4 \oplus x_0x_1x_2x_3x_5 \oplus x_0x_1x_2x_4 \oplus x_0x_1x_2x_4x_5 \oplus x_0x_1x_3x_4 \oplus x_0x_1x_3x_5 \oplus x_0x_1x_4 \oplus x_0x_1x_4x_5 \oplus x_0x_2 \oplus x_0x_2x_3x_5 \oplus x_0x_2x_4x_5 \oplus x_0x_3 \oplus x_0x_3x_4x_5 \oplus x_0x_3x_5 \oplus x_1x_2x_3x_5 \oplus x_1x_2x_4x_5 \oplus x_1x_3x_4x_5 \oplus x_1x_3x_5 \oplus x_1x_5 \oplus x_2x_3x_4x_5 \oplus x_2x_3x_5 \oplus x_2x_4 \oplus x_3x_4x_5 \oplus x_3x_5$. In the first case, the function is balanced because its output depends on just one variable (which is 0 in half the possible inputs); in the second case the output of the function is the exclusive-or of all the input variables which is an easy instance of a balanced function. The last case is a cryptographically strong balanced function whose output pattern is, by design, difficult to discern [7]. An important insight in the case of the Deutsch-Jozsa problem is that, since we are promised the function is either constant or balanced, then any formula that refers to at least one variable must indicate a balanced function. In other words, the outcome of the algorithm can be immediately decided if the formula is anything other than 0 or 1. We confirmed this observation by running the experiment on all 12870 balanced functions from $[\mathbf{2^4}] \to [\mathbf{2}]$ and correctly identifying them as such. This is significant as some of these functions produce complicated entangled patterns during quantum evolution and could not be de-quantized using previous approaches [1]. The catch is that symbolic retrodictive execution is not consistent with "query complexity" as it operates in time proportional to the depth of the quantum oracle and the size of the formula.

Bernstein-Vazirani. We are given a function $f : [\mathbf{2^n}] \to [\mathbf{2}]$ that hides a secret number $s \in [\mathbf{2^n}]$. We are promised the function is defined using the binary representations $\sum_i^{n-1} x_i$ and $\sum_i^{n-1} s_i$ of x and s respectively as $f(x) = \sum_{i=0}^{n-1} s_i x_i \mod 2$. The goal is to determine the secret number s. The circuit in Fig. 5 solves the problem for $n = 8$ and a hidden number 92 (= 00111010 in binary notation with the rightmost bit at index 0). Retrodictive execution starting with the state $|x_0x_1x_2x_3x_4x_5x_6x_70\rangle$ terminates with the formula $x_1 \oplus x_3 \oplus x_4 \oplus x_5$. The secret string can be immediately read from the formula as the indices $\{1, 3, 4, 5\}$ of the symbols are exactly the positions at which the secret string has a 1.

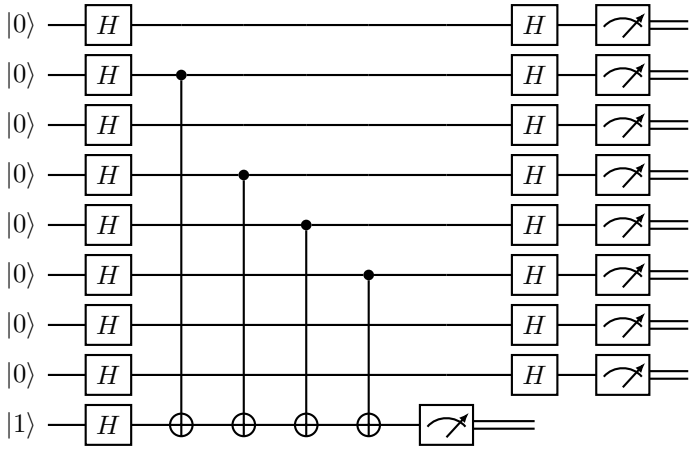


Figure 5: Circuit for Bernstein-Vazirani Algorithm ($n = 8$, $s = 92$, least significant bit is the top wire)

Grover. We are given a function $f : [\mathbf{2^n}] \to [\mathbf{2}]$ with the property that there exists only one input u such $f(u) = 1$. The goal is to find u. The conventional presentation of the quantum algorithm does not fit the template of Fig. 2. But it is still possible to construct a quantum oracle U_f from the given f and perform retrodictive execution starting from an ancilla measurement of 1 corresponding to

$u = 0$ $1 \oplus x_0 \oplus x_0x_1 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3 \oplus x_0x_1x_3 \oplus x_0x_2 \oplus x_0x_2x_3 \oplus x_0x_3 \oplus x_1 \oplus x_1x_2 \oplus x_1x_2x_3 \oplus x_1x_3 \oplus x_2 \oplus x_2x_3 \oplus x_3$

$u = 1$ $x_0 \oplus x_0x_1 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3 \oplus x_0x_1x_3 \oplus x_0x_2 \oplus x_0x_2x_3 \oplus x_0x_3$

$u = 2$ $x_0x_1 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3 \oplus x_0x_1x_3 \oplus x_1 \oplus x_1x_2 \oplus x_1x_2x_3 \oplus x_1x_3$

$u = 3$ $x_0x_1 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3 \oplus x_0x_1x_3$

$u = 4$ $x_0x_1x_2 \oplus x_0x_1x_2x_3 \oplus x_0x_2 \oplus x_0x_2x_3 \oplus x_1x_2 \oplus x_1x_2x_3 \oplus x_2 \oplus x_2x_3$

$u = 5$ $x_0x_1x_2 \oplus x_0x_1x_2x_3 \oplus x_0x_2 \oplus x_0x_2x_3$

$u = 6$ $x_0x_1x_2 \oplus x_0x_1x_2x_3 \oplus x_1x_2 \oplus x_1x_2x_3$

$u = 7$ $x_0x_1x_2 \oplus x_0x_1x_2x_3$

$u = 8$ $x_0x_1x_2x_3 \oplus x_0x_1x_3 \oplus x_0x_2x_3 \oplus x_0x_3 \oplus x_1x_2x_3 \oplus x_1x_3 \oplus x_2x_3 \oplus x_3$

$u = 9$ $x_0x_1x_2x_3 \oplus x_0x_1x_3 \oplus x_0x_2x_3 \oplus x_0x_3$

$u = 10$ $x_0x_1x_2x_3 \oplus x_0x_1x_3 \oplus x_1x_2x_3 \oplus x_1x_3$

$u = 11$ $x_0x_1x_2x_3 \oplus x_0x_1x_3$

$u = 12$ $x_0x_1x_2x_3 \oplus x_0x_2x_3 \oplus x_1x_2x_3 \oplus x_2x_3$

$u = 13$ $x_0x_1x_2x_3 \oplus x_0x_2x_3$

$u = 14$ $x_0x_1x_2x_3 \oplus x_1x_2x_3$

$u = 15$ $x_0x_1x_2x_3$

Figure 6: Result of retrodictive execution for the Grover oracle ($n = 4$, w in the range $\{0..15\}$).

the input pattern we are interested in. The resulting equations for $n = 4$ and u in the range $\{0..15\}$ are in Fig. 6. In some cases (e.g. $u = 15$) the equations immediately reveal u; in others, retrodictive executive provides no advantage since solving arbitrary equations over boolean variables is, in general, an *NP*-complete problem.

Shor 15. The circuit in Fig. 7 uses a hand-optimized implementation of the modular exponentiation $4^x \mod 15$ to factor 15 using Shor's algorithm. In a conventional forward execution, the state before the QFT block is:

$$\frac{1}{2\sqrt{2}}((|0\rangle + |2\rangle + |4\rangle + |6\rangle)|1\rangle + (|1\rangle + |3\rangle + |5\rangle + |7\rangle)|4\rangle)$$

At this point, the ancilla register is measured to either $|1\rangle$ or $|4\rangle$. In either case, the computational register snaps to a state of the form $\sum_{r=0}^{3} |a + 2r\rangle$ whose QFT has peaks at $|0\rangle$ or $|4\rangle$ making them the most likely outcomes of measurements of the computational register. If we measure $|0\rangle$, we repeat the experiment; otherwise we infer that the period is 2.

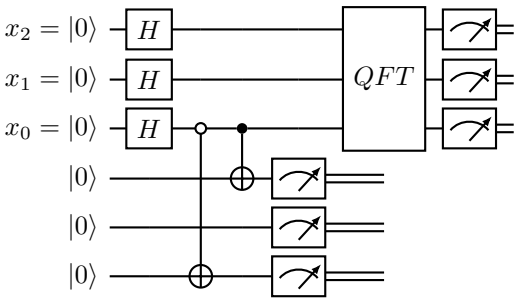


Figure 7: Finding the period of $4^x \mod 15$

In the retrodictive execution, we can start with the state $|x_2x_1x_0001\rangle$ since 1 is guaranteed to be a possible ancilla measurement. The first CX-gate changes the state to $|x_2x_1x_0x_001\rangle$ and the second CX-gate produces $|x_2x_1x_0x_00x_0\rangle$. At that point, we reconcile the retrodictive result of the ancilla register $|x_00x_0\rangle$ with the initial condition $|000\rangle$ to conclude that $x_0 = 0$. In other words, in order to observe the ancilla at 001, the computational register must be initialized to a superposition of the form $|??0\rangle$ where the least significant bit must be 0 and the other two bits are unconstrained. Expanding the possibilities, the first register needs to be in a superposition of the states $|000\rangle, |010\rangle, |100\rangle$ or $|110\rangle$ and we have just inferred using purely classical but retrodictive reasoning that the period is 2. Significantly, this approach is robust and does not require small hand-optimized circuits. Indeed, following the methods for producing quantum circuits for arithmetic operations from first principles using adders and multipliers [17], our implementation for a general circuit for $a^x \mod 15$ has 56538 generalized Toffoli gates over 9 qubits, and yet the equations resulting from the

$a = 11$	$x_0 = 0$				$\mid x_0 = 0$
$a = 4, 14$	$1 \oplus x_0 = 1$	$x_0 = 0$			$\mid x_0 = 0$
$a = 7, 13$	$1 \oplus x_0x_1 \oplus x_1 = 1$	$x_0x_1 = 0$	$x_0 \oplus x_0x_1 \oplus x_1 = 0$	$x_0 \oplus x_0x_1 = 0$	$\mid x_0 = 0, x_1 = 0$
$a = 2, 8$	$1 \oplus x_0 \oplus x_0x_1 \oplus x_1 = 1$	$x_0x_1 = 0$	$x_0x_1 \oplus x_1 = 0$	$x_0 \oplus x_0x_1 = 0$	$\mid x_0 = 0, x_1 = 0$

Figure 8: Equations generated by retrodictive execution of $a^x \mod 15$ starting from observed result 1 and unknown $x_8x_7x_6x_5x_4x_3x_2x_1x_0$. The solution for the unknown variables is given in the last column.

retrodictive execution in Fig. 8 are trivial and immediately solvable as they only involve either the least significant bit x_0 (when $a \in \{4, 11, 14\}$) or the least significant two bits x_0 and x_1 (when $a \in \{2, 7, 8, 13\}$). When the solution is $x_0 = 0$, the period is 2. When the solution is $x_0 = 0, x_1 = 0$, the period is 4.

Shor 21. The examples presented so far demonstrate that some instances of quantum algorithms can be solved via classical symbolic retrodictive execution. But as was already apparent in some examples (e.g. Grover), running retrodictive execution may produce large residual equations that are difficult to solve. To appreciate how large these equations may be, we include the full set of equations produced for a retrodictive execution of Shor's algorithm for factoring 21. Unlike the number 15 which corresponds to a rare occurrence of products of Fermat primes producing a period that is a power of 2 and hence trivial to represent by equations of binary numbers, the period of 21 is not easily representable as a system of equations over binary numbers. The equations which span about five pages in Sec. 2 glaringly show the limitations of the basic retrodictive execution approach and the need for additional insights.

Retrodictive Executions and Function Pre-images. Given finite sets A and B, a function $f : A \to B$ and an element $y \in B$, we define $\{\cdot \xleftarrow{f} y\}$, the pre-image of y under f, as the set $\{x \in A \mid f(x) = y\}$. For example, let $A = B = [\mathbf{2^4}]$ and let $f(x) = 7^x \mod 15$, then the collection of values that f maps to 4, $\{\cdot \xleftarrow{f} 4\}$, is the set $\{2, 6, 10, 14\}$ as shown in Fig. 9. Symbolic retrodictive execution can be seen as a method to generate boolean formulae that describe the pre-image of the function f under study. For the example in Fig. 9, retrodictive execution might generate the formulae $x_1 = 1$ and $x_0 = 0$. The (trivial in this case) solution for the formulae is indeed the set $\{2, 6, 10, 14\}$. The critical points to note, however, are that: (i) solving the equations describing the pre-image is in general an intractable (even for quantum computers) *NP*-complete problem, and (ii) solving the equations is not needed for the quantum algorithms in the previous section. *Only some global properties of the pre-image are needed!* Indeed, we have already seen that for solving the Deutsch-Jozsa problem, the only thing needed was whether the formula contains some variables. Also for the Bernstein-Vazirani problem, the only thing needed was the indices of the variables occurring in the formula. For Grover's algorithm, we only need to extract the singleton element in the pre-image and for Shor's algorithm we only need to extract the periodicity of the elements in the pre-image but retrodictive execution as presented so far is only able to de-quantize some rare instances of algorithms.

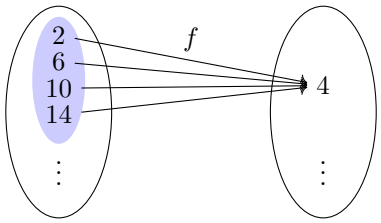


Figure 9: The pre-image of 4 under $f(x) = 7^x \mod 15$.

do communication protocols too ??

$|-\rangle$

graph state: H,H,CZ

00 00 01 01 10 10 11 -11

H control +/- distinction not important so use one class of vars H target +/- distinction important; use two classes of vars

symbolic exec H introduces uncertainty; forget +/- distinction for now; use variable safe is H wires are used as control wires but not targets run symbolically; can represent entanglement; e.g. bell state xx check if H commutes with x and cx so we only need H at beginning and end introduce vars at beginning and run symbolically if only x and cx then symbolic execution is efficient; no need for last batch of H can solve problem classically connect with Gottsman-Knill

What is have ccx sometimes fine; shor 15 example; still fine sometimes we get very complicated representation of wavefunction but if we are following up with QFT; QFT insensitive to offset, don't care variables no need to keep track of values of vars; only need to know if they are constant or not

what is H wires are used as targets; need two flavors of variables; +vars and -vars; -vars infect +vars in control gates; taint analysis with increasing precisions (more and more colors)

retrodictive? Kochen-Specker; interactive QM; observer free will; choice backtracks

values going at different speeds; intervals ideas; path types

The quantum circuit model consists of two classes of gates: (i) quantum counterparts to classical reversible gates (e.g., Toffoli gates), and (ii) genuine quantum gates with no classical counterpart (e.g., Hadamard and phase gates). We make the remarkable observation, that, for a number of well-established quantum algorithms, judicious reasoning about the classical components, ignoring all the quantum gates, is sufficient. Put differently, in those cases, the quantum gates serve no fundamental purpose and are actually distracting from an underlying efficient classical algorithm. The result relies on the ability to symbolically execute circuits, especially in a retrodictive fashion, i.e., by making partial observations at the output site and proceeding backwards to infer the implied initial conditions.

> You can't connect the dots looking forward; you can only connect them looking backwards. So you have to trust that the dots will somehow connect in your future. *Steve Jobs*

extract vars; all we need for some algos

The obvious question to ask now is whether the retrodictive execution can be tuned to only produce the required statistics instead producing the full description of pre-images.

insight 1: qft does not care about 0+2+4.... vs 1+3+5....

0 0 ? 0 1 ? 1 0 ? 1 1 ?

equiv no matter what ? is ? is used in the computation (don't care about value) others not used so we just need to keep track of which vars are used

run experiments with PEX and PEY

2. Hadamard basis: Toffoli + Hadamard is universal so we "just" need to understand how to run in X basis.

Get rid of all quantum gates and run just the reversible classical part but with different taint analyses

Essentially we have two colors and we do taint analysis

Blue and Red; when blue interacts with red it gets tainted

We have two operations +red (add red) -red (remove red)

Remember cx(+,-) = (-,-)

Some interactions (Toffoli) want to create more refined operations +/-(1/2)(red) +/-(red) The more you do these operations the more precise it wants to be +/-(1/4)(red) +/-(1/2) red +/-(red)

And so on

You can truncate at the desired level of accuracy

The taint analysis groups variables in "waves" (superpositions) of things that have the same color so the values we

propagate are "red: phase=p; frequency=f; involved variables=x1,x2,..."

Seems that naive taint analysis is just keep track of which variable is used

---

run again; refined pe; var used; if used twice then disappears

go back to that stupid paper about logic programming and xor

The equations turn out to be trivial when the period is a power of 2. This occurs when the number to factor is a product of Fermat primes: 3, 5, 17, 257, 65537, ... The equations generated for some of these cases are in ...

need stats only PEX , PEY ...

core of many quantum algos is quantum oracle uf two inputs; two outputs system; ancilla; normal eval; control ancilla; system unknown; so throw in complete superposition and eval forward

only need number of vars !!!!

solve other problems with just knowing which vars are involved

Normal quantum evolution: from present to future

Now what if I had partial knowledge about the future; what can you say about the present? (And then about the rest of the unknown future)

Can this help flow of information, complexity, etc?

In some cases, partial knowledge about the future is enough to predict the present accurately enough to then predict everything about the future; in some cases it is not enough

Possibility that collapse of wave function is information flow back from measured future to present unknown initial conditions and then back to rest of wave that was not measured

Provide a general introduction to the topic and a brief non-technical summary of your main results and their implication.

200 words ??

main text 2000-2500 words 3-4 figures 30-50 references

Methods section 3000 words more references ok

Author contributions

Code available

https://quantumalgorithmzoo.org

every quantum circuit can be written using Toffoli and Hadamard retro just go through Toffoli; ignore Had; but of course we are using symbolic eval

can H be moved past Toffoli?

universe uses lazy evaluation?

algebra of Toffoli and Hadamard ZX calculus

fourier transform classical efficient in some cases

Ewin Tang papers

kochen specker ??

2 Methods

Lazy Evaluation. Consider a program that searches for three different numbers x, y, and z each in the range $[1..n]$ and that sum to s. A well-established design principle for solving such problems is the *generate-and-test* computational paradigm. Following this principle, a simple program to solve this problem in the programming language Haskell is:

```
generate :: Int -> [(Int,Int,Int)]
generate n =  [(x,y,z) | x <- [1..n], y <- [1..n], z <- [1..n]]

test :: Int -> [(Int,Int,Int)] -> [(Int,Int,Int)]
test s nums = [(x,y,z) | (x,y,z) <- nums, x /= y, x /= z, y /= z, x+y+z == s]

find :: Int -> Int -> (Int,Int,Int)
```

```
find s =  head . test s . generate
```

The program consists of three functions: `generate` that produces all triples `(x,y,z)` from `(1,1,1)` to `(n,n,n)`; `test` that checks that the numbers are different and that their sum is equal to `s`; and `find` that composes the two functions: generating all triples, testing the ones that satisfy the condition, and returning the first solution. Running this program to find numbers in the range [1..6] that sum to 15 immediately produces $(4, 5, 6)$ as expected.

But what if the range of interest was [1..10000000] ? A naïve execution of the generate-and-test method would be prohibitively expensive as it would spend all its time generating an enormous number of triples that are un-needed. Lazy demand-driven evaluation as implemented in Haskell succeeds in a few seconds with the result $(1, 2, 12)$, however. The idea is simple: instead of eagerly generating all the triples, generate a process that, when queried, produces one triple at a time on demand. Conceptually the execution starts from the observer site which is asking for the first element of a list; this demand is propagated to the function `test` which itself propagates the demand to the function `generate`. As each triple is generated, it is tested until one triple passes the test. This triple is immediately returned without having to generate any additional values.

Partial Evaluation. Below is a Haskell program that computes a^n by repeated squaring:

```
power :: Int -> Int -> Int
power a n
  | n == 0     = 1
  | n == 1     = a
  | even n     = let r = power a (n `div` 2) in r * r
  | otherwise  = a * power a (n-1)
```

When both inputs are known, e.g., `a = 3` and `n = 5`, the program evaluates as follows:

```
   power 3 5
=  3 * power 3 4
=  3 * (let r1 = power 3 2 in r1 * r1)
=  3 * (let r1 = (let r2 = power 3 1 in r2 * r2) in r1 * r1)
=  3 * (let r1 = (let r2 = 3 in r2 * r2) in r1 * r1)
=  3 * (let r1 = 9 in r1 * r1)
=  243
```

Partial evaluation is used when we only have partial information about the inputs. Say we only know $n = 5$. A partial evaluator then attempts to evaluate `power` with symbolic input `a` and actual input `n=5`. This evaluation proceeds as follows:

```
   power a 5
=  a * power a 4
=  a * (let r1 = power a 2 in r1 * r1)
=  a * (let r1 = (let r2 = power a 1 in r2 * r2) in r1 * r1)
=  a * (let r1 = (let r2 = a in r2 * r2) in r1 * r1)
=  a * (let r1 = a * a in r1 * r1)
=  let r1 = a * a in a * r1 * r1
```

All of this evaluation, simplification, and specialization happens without knowledge of `a`. Just knowing `n` was enough to produce a residual program that is much simpler.

The evolution of a quantum system is typically understood as proceeding forwards in time — from the present to the future. As shown in Fig. 2(a),

Since the conventional execution starts with complete ignorance about the future, the initial state is prepared as a superposition that includes every possibility. In a well-designed algorithm, , by the time

the computation reaches the measurement stages, the relative phases and probability amplitudes in that enormous superposition have become biased towards states of interest which are projected to produce the final answer.

Algebraic Normal Form (ANF). circuits have generalized toffoli gates: semantics (and of controls; xor with target); ANF uses exactly those two primitives; explain

The resulting expressions are in algebraic normal form [16] where + denotes exclusive-or.

instances with no 'and' easy to solve

Function Pre-Images and *NP*-Complete Problems. To appreciate the difficulty of computing pre-images in general, note that finding the pre-image of a function subsumes several challenging computational problems such as pre-image attacks on hash functions [15], predicting environmental conditions that allow certain reactions to take place in computational biology [13, 3], and finding the pre-image of feature vectors in the space induced by a kernel in neural networks [14]. More to the point, the boolean satisfiability problem SAT is expressible as a boolean function over the input variables and solving a SAT problem is asking for the pre-image of `true`. Indeed, based on the conjectured existence of one-way functions which itself implies $P \neq NP$, all these pre-images calculations are believed to be computationally intractable in their most general setting.

Complexity Analysis. one pass over circuit BUT complexity of normalizing to ANF not trivial; be careful

Data Availability. available

Discussion. Possibility that collapse of wave function is information flow back from measured future to present unknown initial conditions and then back to rest of wave that was not measured

transactional interpretation?

Luckily, the problems of concern to us are quite special: (i) the functions are not arbitrary but have additional structure that can be exploited, and (ii) we never need access to all the elements in the pre-image; we just need to answer aggregate queries about the pre-images. Quantum algorithms somehow exploit these properties along with some physical principles to solve these problems efficiently. To understand the precise way in which this is happening, we start with the template of the quantum circuit used for solving all the problems above in Fig. 2.

The core of the circuit is the U_f block which can be assumed to be implemented using only generalized Toffoli gates. The block implements the unitary transformation: $U_f(|x\rangle\,|y\rangle) = |x\rangle\,|f(x) \oplus y\rangle$ where $\oplus$ is the (bitwise) exclusive-or operation; it defines the function of interest whose pre-image properties are to be calculated. The inputs of the U_f block are grouped in two registers: the top register contains an equal superposition of all possible inputs to f; the ancilla register is prepared in initial states that depend on the specific algorithm. Thus, the state at slice (1) in the figure is:

$$\frac{1}{\sqrt{2^n}\sqrt{2^m}} \sum_{x=0}^{2^n-1} \sum_{y=0}^{2^m-1} |x\rangle\,|y\rangle$$

This is transformed by U_f to:

$$\frac{1}{\sqrt{2^n}\sqrt{2^m}} \sum_{x=0}^{2^n-1} \sum_{y=0}^{2^m-1} |x\rangle\,|f(x) \oplus y\rangle$$

So far, nothing too interesting is happening: we have just produced a superposition of states where each state is a possible input to f, say x, tensored with $f(x) \oplus y$, the result of applying f to this particular input adjusted by the second register y. At slice (3), something remarkable occurs; the result w of measuring the second register "kicks back" information to the first register whose state becomes a superposition of

those values x that are consistent with the measurement, i.e., *the pre-image of w under f!* That pre-image representation is then analyzed using the Quantum Fourier Transform (QFT) to produce the final result.

Quantum algorithms typically operate on a *black box* holding a classical function whose properties need to be computed. The general structure of these algorithms is to (i) create a superposition of values to be passed as inputs to the black box, (ii) apply the operation inside the black box, and (iii) post-process the output of the black box. We observe that, in quite a few cases, steps (i) and (iii) are actually unnecessary and that the entire "quantum" algorithm can be executed by forward or backward, full or partial, efficient classical *symbolic execution* of the black box.

typical use: superposition, Uf, measure second register; we only care about which x has f(x) = r

By default all functions are reversible.

To make them irreversible you fix h and delete g. If you delete too much the function becomes very expensive to reverse. So one way functions emerge

simplify function has polynomial realization and we want statistics about the kernel (not necessarily compute it exactly)

collect assumptions:

important that no matter what measurement we do on w, properly we want is the same

since we say that algos related to pre-images lets do naive thing and eval backwards

assumptions we have a rev circuit efficient forward two inputs: first is full superposition; second whatever first output same as first input; but that is only at point 2; at point 3 explain kick back; misleading to think it is the same after 3 second output is result of function; measure; have element of range; go back with that elem if we knew first output as well as w then eval backwards same complexity but we only know w and we don't know first output; because we are starting at 3 not 2

we have no use for H block; it was only there for the forward exec to express our complete ignorance o the future; prepared with every x but if we have knowledge about future (w measured) we go back to find the values of x in the present that would be consistent with w so general circuit reduces to :

...

fix pics to have amplitudes with y (most general)

To what extent are the quantum algorithms above taking advantage of non-classical features. We posit that pre-image computation can be, at least for some of the some of the algorithms, be performed classically. The main insight needed for that is to perform the execution *symbolically*. We illustrate the idea with two examples.

We need to explain ideas about time-reversal, prediction and retrodiction in physics. The laws of computation and the laws of physics are intimately related. When does knowing something about the future help us unveil the structure or symmetries of the past? It is like a detective story, but one with ramifications in complexity and/or efficiency. Problems involving questions where answers demand a Many(past)-to-one(future) map are at the root of our proposal.... Difference between exploiting or not entanglement in the unitary evolution.

As we demonstrate, the family of quantum algorithms initiated by Deutsch's algorithm and culminating with Shor's algorithm (i) solves variants of the pre-image problem efficiently, and, in that context, (ii) answering queries about pre-images is closely related to *retrodictive quantum theory* [5], retrocausality [2], and the time-symmetry of physical laws [18].

- Retrodictive execution more efficient in some cases. What cases?
- Here are three examples: Deutsch-Jozsa, Simon, Shor when period is close to a power of 2
- Symbolic (retrodictive) evaluation as a broader perspective to classical computation
- Symbolic execution allows you to express/discover interference via shared variables
- When interference pattern is simple symbolic execution reveals solutions faster (and completely classically)
- Symbolic execution as a "classical waves" computing paradigm

to represent unequal superpositions do multiple runs with vars the first has x1 x2 etc the second has y1 2y2 etc or y2/2 etc, or with various patterns of negative weights.... And then the punchline would be to interpret the negative backwards. So instead of all forward or all retro we have some values going forward and then backwards

Start with the story about function many to one etc why superpositions because we don't know which values so we try all easy to represent by unknown vars so we can represent superpositions as vars and equations between them but at the end we want stats about superpositions slow way is to generate all equations and solve faster way is generate many sets of equations with different weights and sum to get your stats

We should use two prototypical examples to illustrate main ideas before going to the complex ones. The examples I have in mind are: Deutsch-Jozsa and Simon (precursor of Shor's). There are prior works on de-quantization of the first problem and should make contact with their resolution. Perhaps we can show that they are as efficient classically? That would justify retrodiction alone. The more complex (and important) case of factorization should be the natural follow up.

The idea of symbolic execution is not tied to forward or backward execution. We should introduce it in a way that is independent of the direction of execution. What the idea depends on however is that the wave function, at least in the cases we are considering, can be represented as equations over booleans.

Wave Functions as Equations over Booleans

in the typical scenario for using quantum oracles, we can represent wave function as equations over booleans; equations represent the wave function but the solution is unobservable just like the components of the superposition in the wave function are not observable; just like we don't directly get access to the components of the wave function; we don't directly get access to the solution of the equations; need to "observe" the equations

we can go backwards with an equation (representing a wave function sigma x where f(x) = r and go back towards the present to calculate the wave function (represented as equations again)

Musing: how to explain complementarity when wave function is represented as an equation? Kochen specker;

or contextuality

observer 1 measures wires a,b; obs2 measures wires b,c; not commuting; each obs gives partial solution to equations; but partial solutions cannot lead to a global solution

KS suggests that equations do not have unique solutions; only materialize when you measure;

can associate a probability with each variable in a equation: look at all solutions and see the contribution of each variable to these solutions.

Supplementary Information. Equations generated by retrodictive execution of $4^x \mod 21$ starting from observed result 1 and unknown x. The circuit consists of 9 qubits, 36400 CX-gates, 38200 CCX-gates, and 4000 CCCX-gates. There are only three equations but each equation is exponentially large.

$$1\oplus x_0\oplus x_0x_1x_2\oplus x_0x_1x_2x_3x_4\oplus x_0x_1x_2x_3x_4x_5x_6\oplus x_0x_1x_2x_3x_4x_5x_6x_7x_8\oplus x_0x_1x_2x_3x_4x_5x_6x_7x_9\oplus x_0x_1x_2x_3x_4x_5x_6x_8\oplus$$
$$x_0x_1x_2x_3x_4x_5x_6x_8x_9 \oplus x_0x_1x_2x_3x_4x_5x_7 \oplus x_0x_1x_2x_3x_4x_5x_7x_8x_9 \oplus x_0x_1x_2x_3x_4x_5x_7x_9 \oplus x_0x_1x_2x_3x_4x_5x_8 \oplus$$
$$x_0x_1x_2x_3x_4x_5x_9\oplus x_0x_1x_2x_3x_4x_6\oplus x_0x_1x_2x_3x_4x_6x_7\oplus x_0x_1x_2x_3x_4x_6x_7x_8\oplus x_0x_1x_2x_3x_4x_6x_7x_8x_9\oplus x_0x_1x_2x_3x_4x_6x_8x_9\oplus$$
$$x_0x_1x_2x_3x_4x_6x_9 \oplus x_0x_1x_2x_3x_4x_7x_8 \oplus x_0x_1x_2x_3x_4x_7x_9 \oplus x_0x_1x_2x_3x_4x_8 \oplus x_0x_1x_2x_3x_4x_8x_9 \oplus x_0x_1x_2x_3x_5 \oplus$$
$$x_0x_1x_2x_3x_5x_6x_7\oplus x_0x_1x_2x_3x_5x_6x_7x_8x_9\oplus x_0x_1x_2x_3x_5x_6x_7x_9\oplus x_0x_1x_2x_3x_5x_6x_8\oplus x_0x_1x_2x_3x_5x_6x_9\oplus x_0x_1x_2x_3x_5x_7\oplus$$
$$x_0x_1x_2x_3x_5x_7x_8 \oplus x_0x_1x_2x_3x_5x_7x_8x_9 \oplus x_0x_1x_2x_3x_5x_8x_9 \oplus x_0x_1x_2x_3x_5x_9 \oplus x_0x_1x_2x_3x_6 \oplus x_0x_1x_2x_3x_6x_7x_8 \oplus$$
$$x_0x_1x_2x_3x_6x_7x_9 \oplus x_0x_1x_2x_3x_6x_8 \oplus x_0x_1x_2x_3x_6x_8x_9 \oplus x_0x_1x_2x_3x_7 \oplus x_0x_1x_2x_3x_7x_8x_9 \oplus x_0x_1x_2x_3x_7x_9 \oplus$$
$$x_0x_1x_2x_3x_8\oplus x_0x_1x_2x_3x_9\oplus x_0x_1x_2x_4\oplus x_0x_1x_2x_4x_5\oplus x_0x_1x_2x_4x_5x_6\oplus x_0x_1x_2x_4x_5x_6x_7\oplus x_0x_1x_2x_4x_5x_6x_7x_8\oplus$$
$$x_0x_1x_2x_4x_5x_6x_7x_8x_9\oplus x_0x_1x_2x_4x_5x_6x_8x_9\oplus x_0x_1x_2x_4x_5x_6x_9\oplus x_0x_1x_2x_4x_5x_7x_8\oplus x_0x_1x_2x_4x_5x_7x_9\oplus x_0x_1x_2x_4x_5x_8\oplus$$
$$x_0x_1x_2x_4x_5x_8x_9 \oplus x_0x_1x_2x_4x_6x_7 \oplus x_0x_1x_2x_4x_6x_7x_8x_9 \oplus x_0x_1x_2x_4x_6x_7x_9 \oplus x_0x_1x_2x_4x_6x_8 \oplus x_0x_1x_2x_4x_6x_9 \oplus$$
$$x_0x_1x_2x_4x_7\oplus x_0x_1x_2x_4x_7x_8\oplus x_0x_1x_2x_4x_7x_8x_9\oplus x_0x_1x_2x_4x_8x_9\oplus x_0x_1x_2x_4x_9\oplus x_0x_1x_2x_5x_6\oplus x_0x_1x_2x_5x_6x_7x_8\oplus$$
$$x_0x_1x_2x_5x_6x_7x_9 \oplus x_0x_1x_2x_5x_6x_8 \oplus x_0x_1x_2x_5x_6x_8x_9 \oplus x_0x_1x_2x_5x_7 \oplus x_0x_1x_2x_5x_7x_8x_9 \oplus x_0x_1x_2x_5x_7x_9 \oplus$$
$$x_0x_1x_2x_5x_8 \oplus x_0x_1x_2x_5x_9 \oplus x_0x_1x_2x_6 \oplus x_0x_1x_2x_6x_7 \oplus x_0x_1x_2x_6x_7x_8 \oplus x_0x_1x_2x_6x_7x_8x_9 \oplus x_0x_1x_2x_6x_8x_9 \oplus$$
$$x_0x_1x_2x_6x_9\oplus x_0x_1x_2x_7x_8\oplus x_0x_1x_2x_7x_9\oplus x_0x_1x_2x_8\oplus x_0x_1x_2x_8x_9\oplus x_0x_1x_3\oplus x_0x_1x_3x_4x_5\oplus x_0x_1x_3x_4x_5x_6x_7\oplus$$

$$x_0x_1x_3x_4x_5x_6x_7x_8x_9 \oplus x_0x_1x_3x_4x_5x_6x_7x_9 \oplus x_0x_1x_3x_4x_5x_6x_8 \oplus x_0x_1x_3x_4x_5x_6x_9 \oplus x_0x_1x_3x_4x_5x_7 \oplus x_0x_1x_3x_4x_5x_7x_8 \oplus$$
$$x_0x_1x_3x_4x_5x_7x_8x_9 \oplus x_0x_1x_3x_4x_5x_8x_9 \oplus x_0x_1x_3x_4x_5x_9 \oplus x_0x_1x_3x_4x_6 \oplus x_0x_1x_3x_4x_6x_7x_8 \oplus x_0x_1x_3x_4x_6x_7x_9 \oplus$$
$$x_0x_1x_3x_4x_6x_8 \oplus x_0x_1x_3x_4x_6x_8x_9 \oplus x_0x_1x_3x_4x_7 \oplus x_0x_1x_3x_4x_7x_8x_9 \oplus x_0x_1x_3x_4x_7x_9 \oplus x_0x_1x_3x_4x_8 \oplus x_0x_1x_3x_4x_9 \oplus$$
$$x_0x_1x_3x_5 \oplus x_0x_1x_3x_5x_6 \oplus x_0x_1x_3x_5x_6x_7 \oplus x_0x_1x_3x_5x_6x_7x_8 \oplus x_0x_1x_3x_5x_6x_7x_8x_9 \oplus x_0x_1x_3x_5x_6x_8x_9 \oplus x_0x_1x_3x_5x_6x_9 \oplus$$
$$x_0x_1x_3x_5x_7x_8 \oplus x_0x_1x_3x_5x_7x_9 \oplus x_0x_1x_3x_5x_8 \oplus x_0x_1x_3x_5x_8x_9 \oplus x_0x_1x_3x_6x_7 \oplus x_0x_1x_3x_6x_7x_8x_9 \oplus x_0x_1x_3x_6x_7x_9 \oplus$$
$$x_0x_1x_3x_6x_8 \oplus x_0x_1x_3x_6x_9 \oplus x_0x_1x_3x_7 \oplus x_0x_1x_3x_7x_8 \oplus x_0x_1x_3x_7x_8x_9 \oplus x_0x_1x_3x_8x_9 \oplus x_0x_1x_3x_9 \oplus x_0x_1x_4 \oplus$$
$$x_0x_1x_4x_5x_6 \oplus x_0x_1x_4x_5x_6x_7x_8 \oplus x_0x_1x_4x_5x_6x_7x_9 \oplus x_0x_1x_4x_5x_6x_8 \oplus x_0x_1x_4x_5x_6x_8x_9 \oplus x_0x_1x_4x_5x_7 \oplus x_0x_1x_4x_5x_7x_8x_9 \oplus$$
$$x_0x_1x_4x_5x_7x_9 \oplus x_0x_1x_4x_5x_8 \oplus x_0x_1x_4x_5x_9 \oplus x_0x_1x_4x_6 \oplus x_0x_1x_4x_6x_7 \oplus x_0x_1x_4x_6x_7x_8 \oplus x_0x_1x_4x_6x_7x_8x_9 \oplus$$
$$x_0x_1x_4x_6x_8x_9 \oplus x_0x_1x_4x_6x_9 \oplus x_0x_1x_4x_7x_8 \oplus x_0x_1x_4x_7x_9 \oplus x_0x_1x_4x_8 \oplus x_0x_1x_4x_8x_9 \oplus x_0x_1x_5 \oplus x_0x_1x_5x_6x_7 \oplus$$
$$x_0x_1x_5x_6x_7x_8x_9 \oplus x_0x_1x_5x_6x_7x_9 \oplus x_0x_1x_5x_6x_8 \oplus x_0x_1x_5x_6x_9 \oplus x_0x_1x_5x_7 \oplus x_0x_1x_5x_7x_8 \oplus x_0x_1x_5x_7x_8x_9 \oplus$$
$$x_0x_1x_5x_8x_9 \oplus x_0x_1x_5x_9 \oplus x_0x_1x_6 \oplus x_0x_1x_6x_7x_8 \oplus x_0x_1x_6x_7x_9 \oplus x_0x_1x_6x_8 \oplus x_0x_1x_6x_8x_9 \oplus x_0x_1x_7 \oplus x_0x_1x_7x_8x_9 \oplus$$
$$x_0x_1x_7x_9 \oplus x_0x_1x_8 \oplus x_0x_1x_9 \oplus x_0x_2 \oplus x_0x_2x_3 \oplus x_0x_2x_3x_4 \oplus x_0x_2x_3x_4x_5 \oplus x_0x_2x_3x_4x_5x_6 \oplus x_0x_2x_3x_4x_5x_6x_7 \oplus$$
$$x_0x_2x_3x_4x_5x_6x_7x_8 \oplus x_0x_2x_3x_4x_5x_6x_7x_8x_9 \oplus x_0x_2x_3x_4x_5x_6x_8x_9 \oplus x_0x_2x_3x_4x_5x_6x_9 \oplus x_0x_2x_3x_4x_5x_7x_8 \oplus x_0x_2x_3x_4x_5x_7x_9 \oplus$$
$$x_0x_2x_3x_4x_5x_8 \oplus x_0x_2x_3x_4x_5x_8x_9 \oplus x_0x_2x_3x_4x_6x_7 \oplus x_0x_2x_3x_4x_6x_7x_8x_9 \oplus x_0x_2x_3x_4x_6x_7x_9 \oplus x_0x_2x_3x_4x_6x_8 \oplus$$
$$x_0x_2x_3x_4x_6x_9 \oplus x_0x_2x_3x_4x_7 \oplus x_0x_2x_3x_4x_7x_8 \oplus x_0x_2x_3x_4x_7x_8x_9 \oplus x_0x_2x_3x_4x_8x_9 \oplus x_0x_2x_3x_4x_9 \oplus x_0x_2x_3x_5x_6 \oplus$$
$$x_0x_2x_3x_5x_6x_7x_8 \oplus x_0x_2x_3x_5x_6x_7x_9 \oplus x_0x_2x_3x_5x_6x_8 \oplus x_0x_2x_3x_5x_6x_8x_9 \oplus x_0x_2x_3x_5x_7 \oplus x_0x_2x_3x_5x_7x_8x_9 \oplus$$
$$x_0x_2x_3x_5x_7x_9 \oplus x_0x_2x_3x_5x_8 \oplus x_0x_2x_3x_5x_9 \oplus x_0x_2x_3x_6 \oplus x_0x_2x_3x_6x_7 \oplus x_0x_2x_3x_6x_7x_8 \oplus x_0x_2x_3x_6x_7x_8x_9 \oplus$$
$$x_0x_2x_3x_6x_8x_9 \oplus x_0x_2x_3x_6x_9 \oplus x_0x_2x_3x_7x_8 \oplus x_0x_2x_3x_7x_9 \oplus x_0x_2x_3x_8 \oplus x_0x_2x_3x_8x_9 \oplus x_0x_2x_4x_5 \oplus x_0x_2x_4x_5x_6x_7 \oplus$$
$$x_0x_2x_4x_5x_6x_7x_8x_9 \oplus x_0x_2x_4x_5x_6x_7x_9 \oplus x_0x_2x_4x_5x_6x_8 \oplus x_0x_2x_4x_5x_6x_9 \oplus x_0x_2x_4x_5x_7 \oplus x_0x_2x_4x_5x_7x_8 \oplus$$
$$x_0x_2x_4x_5x_7x_8x_9 \oplus x_0x_2x_4x_5x_8x_9 \oplus x_0x_2x_4x_5x_9 \oplus x_0x_2x_4x_6 \oplus x_0x_2x_4x_6x_7x_8 \oplus x_0x_2x_4x_6x_7x_9 \oplus x_0x_2x_4x_6x_8 \oplus$$
$$x_0x_2x_4x_6x_8x_9 \oplus x_0x_2x_4x_7 \oplus x_0x_2x_4x_7x_8x_9 \oplus x_0x_2x_4x_7x_9 \oplus x_0x_2x_4x_8 \oplus x_0x_2x_4x_9 \oplus x_0x_2x_5 \oplus x_0x_2x_5x_6 \oplus$$
$$x_0x_2x_5x_6x_7 \oplus x_0x_2x_5x_6x_7x_8 \oplus x_0x_2x_5x_6x_7x_8x_9 \oplus x_0x_2x_5x_6x_8x_9 \oplus x_0x_2x_5x_6x_9 \oplus x_0x_2x_5x_7x_8 \oplus x_0x_2x_5x_7x_9 \oplus$$
$$x_0x_2x_5x_8 \oplus x_0x_2x_5x_8x_9 \oplus x_0x_2x_6x_7 \oplus x_0x_2x_6x_7x_8x_9 \oplus x_0x_2x_6x_7x_9 \oplus x_0x_2x_6x_8 \oplus x_0x_2x_6x_9 \oplus x_0x_2x_7 \oplus$$
$$x_0x_2x_7x_8 \oplus x_0x_2x_7x_8x_9 \oplus x_0x_2x_8x_9 \oplus x_0x_2x_9 \oplus x_0x_3x_4 \oplus x_0x_3x_4x_5x_6 \oplus x_0x_3x_4x_5x_6x_7x_8 \oplus x_0x_3x_4x_5x_6x_7x_9 \oplus$$
$$x_0x_3x_4x_5x_6x_8 \oplus x_0x_3x_4x_5x_6x_8x_9 \oplus x_0x_3x_4x_5x_7 \oplus x_0x_3x_4x_5x_7x_8x_9 \oplus x_0x_3x_4x_5x_7x_9 \oplus x_0x_3x_4x_5x_8 \oplus x_0x_3x_4x_5x_9 \oplus$$
$$x_0x_3x_4x_6 \oplus x_0x_3x_4x_6x_7 \oplus x_0x_3x_4x_6x_7x_8 \oplus x_0x_3x_4x_6x_7x_8x_9 \oplus x_0x_3x_4x_6x_8x_9 \oplus x_0x_3x_4x_6x_9 \oplus x_0x_3x_4x_7x_8 \oplus$$
$$x_0x_3x_4x_7x_9 \oplus x_0x_3x_4x_8 \oplus x_0x_3x_4x_8x_9 \oplus x_0x_3x_5 \oplus x_0x_3x_5x_6x_7 \oplus x_0x_3x_5x_6x_7x_8x_9 \oplus x_0x_3x_5x_6x_7x_9 \oplus x_0x_3x_5x_6x_8 \oplus$$
$$x_0x_3x_5x_6x_9 \oplus x_0x_3x_5x_7 \oplus x_0x_3x_5x_7x_8 \oplus x_0x_3x_5x_7x_8x_9 \oplus x_0x_3x_5x_8x_9 \oplus x_0x_3x_5x_9 \oplus x_0x_3x_6 \oplus x_0x_3x_6x_7x_8 \oplus$$
$$x_0x_3x_6x_7x_9 \oplus x_0x_3x_6x_8 \oplus x_0x_3x_6x_8x_9 \oplus x_0x_3x_7 \oplus x_0x_3x_7x_8x_9 \oplus x_0x_3x_7x_9 \oplus x_0x_3x_8 \oplus x_0x_3x_9 \oplus x_0x_4 \oplus x_0x_4x_5 \oplus$$
$$x_0x_4x_5x_6 \oplus x_0x_4x_5x_6x_7 \oplus x_0x_4x_5x_6x_7x_8 \oplus x_0x_4x_5x_6x_7x_8x_9 \oplus x_0x_4x_5x_6x_8x_9 \oplus x_0x_4x_5x_6x_9 \oplus x_0x_4x_5x_7x_8 \oplus$$
$$x_0x_4x_5x_7x_9 \oplus x_0x_4x_5x_8 \oplus x_0x_4x_5x_8x_9 \oplus x_0x_4x_6x_7 \oplus x_0x_4x_6x_7x_8x_9 \oplus x_0x_4x_6x_7x_9 \oplus x_0x_4x_6x_8 \oplus x_0x_4x_6x_9 \oplus$$
$$x_0x_4x_7 \oplus x_0x_4x_7x_8 \oplus x_0x_4x_7x_8x_9 \oplus x_0x_4x_8x_9 \oplus x_0x_4x_9 \oplus x_0x_5x_6 \oplus x_0x_5x_6x_7x_8 \oplus x_0x_5x_6x_7x_9 \oplus x_0x_5x_6x_8 \oplus$$
$$x_0x_5x_6x_8x_9 \oplus x_0x_5x_7 \oplus x_0x_5x_7x_8x_9 \oplus x_0x_5x_7x_9 \oplus x_0x_5x_8 \oplus x_0x_5x_9 \oplus x_0x_6 \oplus x_0x_6x_7 \oplus x_0x_6x_7x_8 \oplus x_0x_6x_7x_8x_9 \oplus$$
$$x_0x_6x_8x_9 \oplus x_0x_6x_9 \oplus x_0x_7x_8 \oplus x_0x_7x_9 \oplus x_0x_8 \oplus x_0x_8x_9 \oplus x_1 \oplus x_1x_2x_3 \oplus x_1x_2x_3x_4x_5 \oplus x_1x_2x_3x_4x_5x_6x_7 \oplus$$
$$x_1x_2x_3x_4x_5x_6x_7x_8x_9 \oplus x_1x_2x_3x_4x_5x_6x_7x_9 \oplus x_1x_2x_3x_4x_5x_6x_8 \oplus x_1x_2x_3x_4x_5x_6x_9 \oplus x_1x_2x_3x_4x_5x_7 \oplus x_1x_2x_3x_4x_5x_7x_8 \oplus$$
$$x_1x_2x_3x_4x_5x_7x_8x_9 \oplus x_1x_2x_3x_4x_5x_8x_9 \oplus x_1x_2x_3x_4x_5x_9 \oplus x_1x_2x_3x_4x_6 \oplus x_1x_2x_3x_4x_6x_7x_8 \oplus x_1x_2x_3x_4x_6x_7x_9 \oplus$$
$$x_1x_2x_3x_4x_6x_8 \oplus x_1x_2x_3x_4x_6x_8x_9 \oplus x_1x_2x_3x_4x_7 \oplus x_1x_2x_3x_4x_7x_8x_9 \oplus x_1x_2x_3x_4x_7x_9 \oplus x_1x_2x_3x_4x_8 \oplus x_1x_2x_3x_4x_9 \oplus$$
$$x_1x_2x_3x_5 \oplus x_1x_2x_3x_5x_6 \oplus x_1x_2x_3x_5x_6x_7 \oplus x_1x_2x_3x_5x_6x_7x_8 \oplus x_1x_2x_3x_5x_6x_7x_8x_9 \oplus x_1x_2x_3x_5x_6x_8x_9 \oplus x_1x_2x_3x_5x_6x_9 \oplus$$
$$x_1x_2x_3x_5x_7x_8 \oplus x_1x_2x_3x_5x_7x_9 \oplus x_1x_2x_3x_5x_8 \oplus x_1x_2x_3x_5x_8x_9 \oplus x_1x_2x_3x_6x_7 \oplus x_1x_2x_3x_6x_7x_8x_9 \oplus x_1x_2x_3x_6x_7x_9 \oplus$$
$$x_1x_2x_3x_6x_8 \oplus x_1x_2x_3x_6x_9 \oplus x_1x_2x_3x_7 \oplus x_1x_2x_3x_7x_8 \oplus x_1x_2x_3x_7x_8x_9 \oplus x_1x_2x_3x_8x_9 \oplus x_1x_2x_3x_9 \oplus x_1x_2x_4 \oplus$$
$$x_1x_2x_4x_5x_6 \oplus x_1x_2x_4x_5x_6x_7x_8 \oplus x_1x_2x_4x_5x_6x_7x_9 \oplus x_1x_2x_4x_5x_6x_8 \oplus x_1x_2x_4x_5x_6x_8x_9 \oplus x_1x_2x_4x_5x_7 \oplus x_1x_2x_4x_5x_7x_8x_9 \oplus$$
$$x_1x_2x_4x_5x_7x_9 \oplus x_1x_2x_4x_5x_8 \oplus x_1x_2x_4x_5x_9 \oplus x_1x_2x_4x_6 \oplus x_1x_2x_4x_6x_7 \oplus x_1x_2x_4x_6x_7x_8 \oplus x_1x_2x_4x_6x_7x_8x_9 \oplus$$
$$x_1x_2x_4x_6x_8x_9 \oplus x_1x_2x_4x_6x_9 \oplus x_1x_2x_4x_7x_8 \oplus x_1x_2x_4x_7x_9 \oplus x_1x_2x_4x_8 \oplus x_1x_2x_4x_8x_9 \oplus x_1x_2x_5 \oplus x_1x_2x_5x_6x_7 \oplus$$
$$x_1x_2x_5x_6x_7x_8x_9 \oplus x_1x_2x_5x_6x_7x_9 \oplus x_1x_2x_5x_6x_8 \oplus x_1x_2x_5x_6x_9 \oplus x_1x_2x_5x_7 \oplus x_1x_2x_5x_7x_8 \oplus x_1x_2x_5x_7x_8x_9 \oplus$$
$$x_1x_2x_5x_8x_9 \oplus x_1x_2x_5x_9 \oplus x_1x_2x_6 \oplus x_1x_2x_6x_7x_8 \oplus x_1x_2x_6x_7x_9 \oplus x_1x_2x_6x_8 \oplus x_1x_2x_6x_8x_9 \oplus x_1x_2x_7 \oplus x_1x_2x_7x_8x_9 \oplus$$
$$x_1x_2x_7x_9 \oplus x_1x_2x_8 \oplus x_1x_2x_9 \oplus x_1x_3 \oplus x_1x_3x_4 \oplus x_1x_3x_4x_5 \oplus x_1x_3x_4x_5x_6 \oplus x_1x_3x_4x_5x_6x_7 \oplus x_1x_3x_4x_5x_6x_7x_8 \oplus$$
$$x_1x_3x_4x_5x_6x_7x_8x_9 \oplus x_1x_3x_4x_5x_6x_8x_9 \oplus x_1x_3x_4x_5x_6x_9 \oplus x_1x_3x_4x_5x_7x_8 \oplus x_1x_3x_4x_5x_7x_9 \oplus x_1x_3x_4x_5x_8 \oplus$$
$$x_1x_3x_4x_5x_8x_9 \oplus x_1x_3x_4x_6x_7 \oplus x_1x_3x_4x_6x_7x_8x_9 \oplus x_1x_3x_4x_6x_7x_9 \oplus x_1x_3x_4x_6x_8 \oplus x_1x_3x_4x_6x_9 \oplus x_1x_3x_4x_7 \oplus$$
$$x_1x_3x_4x_7x_8 \oplus x_1x_3x_4x_7x_8x_9 \oplus x_1x_3x_4x_8x_9 \oplus x_1x_3x_4x_9 \oplus x_1x_3x_5x_6 \oplus x_1x_3x_5x_6x_7x_8 \oplus x_1x_3x_5x_6x_7x_9 \oplus x_1x_3x_5x_6x_8 \oplus$$
$$x_1x_3x_5x_6x_8x_9 \oplus x_1x_3x_5x_7 \oplus x_1x_3x_5x_7x_8x_9 \oplus x_1x_3x_5x_7x_9 \oplus x_1x_3x_5x_8 \oplus x_1x_3x_5x_9 \oplus x_1x_3x_6 \oplus x_1x_3x_6x_7 \oplus$$
$$x_1x_3x_6x_7x_8 \oplus x_1x_3x_6x_7x_8x_9 \oplus x_1x_3x_6x_8x_9 \oplus x_1x_3x_6x_9 \oplus x_1x_3x_7x_8 \oplus x_1x_3x_7x_9 \oplus x_1x_3x_8 \oplus x_1x_3x_8x_9 \oplus x_1x_4x_5 \oplus$$

$$x_1x_4x_5x_6x_7 \oplus x_1x_4x_5x_6x_7x_8x_9 \oplus x_1x_4x_5x_6x_7x_9 \oplus x_1x_4x_5x_6x_8 \oplus x_1x_4x_5x_6x_9 \oplus x_1x_4x_5x_7 \oplus x_1x_4x_5x_7x_8 \oplus$$
$$x_1x_4x_5x_7x_8x_9 \oplus x_1x_4x_5x_8x_9 \oplus x_1x_4x_5x_9 \oplus x_1x_4x_6 \oplus x_1x_4x_6x_7x_8 \oplus x_1x_4x_6x_7x_9 \oplus x_1x_4x_6x_8 \oplus x_1x_4x_6x_8x_9 \oplus$$
$$x_1x_4x_7 \oplus x_1x_4x_7x_8x_9 \oplus x_1x_4x_7x_9 \oplus x_1x_4x_8 \oplus x_1x_4x_9 \oplus x_1x_5 \oplus x_1x_5x_6 \oplus x_1x_5x_6x_7 \oplus x_1x_5x_6x_7x_8 \oplus x_1x_5x_6x_7x_8x_9 \oplus$$
$$x_1x_5x_6x_8x_9 \oplus x_1x_5x_6x_9 \oplus x_1x_5x_7x_8 \oplus x_1x_5x_7x_9 \oplus x_1x_5x_8 \oplus x_1x_5x_8x_9 \oplus x_1x_6x_7 \oplus x_1x_6x_7x_8x_9 \oplus x_1x_6x_7x_9 \oplus$$
$$x_1x_6x_8 \oplus x_1x_6x_9 \oplus x_1x_7 \oplus x_1x_7x_8 \oplus x_1x_7x_8x_9 \oplus x_1x_8x_9 \oplus x_1x_9 \oplus x_2 \oplus x_2x_3x_4 \oplus x_2x_3x_4x_5x_6 \oplus x_2x_3x_4x_5x_6x_7x_8 \oplus$$
$$x_2x_3x_4x_5x_6x_7x_9 \oplus x_2x_3x_4x_5x_6x_8 \oplus x_2x_3x_4x_5x_6x_8x_9 \oplus x_2x_3x_4x_5x_7 \oplus x_2x_3x_4x_5x_7x_8x_9 \oplus x_2x_3x_4x_5x_7x_9 \oplus$$
$$x_2x_3x_4x_5x_8 \oplus x_2x_3x_4x_5x_9 \oplus x_2x_3x_4x_6 \oplus x_2x_3x_4x_6x_7 \oplus x_2x_3x_4x_6x_7x_8 \oplus x_2x_3x_4x_6x_7x_8x_9 \oplus x_2x_3x_4x_6x_8x_9 \oplus$$
$$x_2x_3x_4x_6x_9 \oplus x_2x_3x_4x_7x_8 \oplus x_2x_3x_4x_7x_9 \oplus x_2x_3x_4x_8 \oplus x_2x_3x_4x_8x_9 \oplus x_2x_3x_5 \oplus x_2x_3x_5x_6x_7 \oplus x_2x_3x_5x_6x_7x_8x_9 \oplus$$
$$x_2x_3x_5x_6x_7x_9 \oplus x_2x_3x_5x_6x_8 \oplus x_2x_3x_5x_6x_9 \oplus x_2x_3x_5x_7 \oplus x_2x_3x_5x_7x_8 \oplus x_2x_3x_5x_7x_8x_9 \oplus x_2x_3x_5x_8x_9 \oplus x_2x_3x_5x_9 \oplus$$
$$x_2x_3x_6 \oplus x_2x_3x_6x_7x_8 \oplus x_2x_3x_6x_7x_9 \oplus x_2x_3x_6x_8 \oplus x_2x_3x_6x_8x_9 \oplus x_2x_3x_7 \oplus x_2x_3x_7x_8x_9 \oplus x_2x_3x_7x_9 \oplus x_2x_3x_8 \oplus$$
$$x_2x_3x_9 \oplus x_2x_4 \oplus x_2x_4x_5 \oplus x_2x_4x_5x_6 \oplus x_2x_4x_5x_6x_7 \oplus x_2x_4x_5x_6x_7x_8 \oplus x_2x_4x_5x_6x_7x_8x_9 \oplus x_2x_4x_5x_6x_8x_9 \oplus$$
$$x_2x_4x_5x_6x_9 \oplus x_2x_4x_5x_7x_8 \oplus x_2x_4x_5x_7x_9 \oplus x_2x_4x_5x_8 \oplus x_2x_4x_5x_8x_9 \oplus x_2x_4x_6x_7 \oplus x_2x_4x_6x_7x_8x_9 \oplus x_2x_4x_6x_7x_9 \oplus$$
$$x_2x_4x_6x_8 \oplus x_2x_4x_6x_9 \oplus x_2x_4x_7 \oplus x_2x_4x_7x_8 \oplus x_2x_4x_7x_8x_9 \oplus x_2x_4x_8x_9 \oplus x_2x_4x_9 \oplus x_2x_5x_6 \oplus x_2x_5x_6x_7x_8 \oplus$$
$$x_2x_5x_6x_7x_9 \oplus x_2x_5x_6x_8 \oplus x_2x_5x_6x_8x_9 \oplus x_2x_5x_7 \oplus x_2x_5x_7x_8x_9 \oplus x_2x_5x_7x_9 \oplus x_2x_5x_8 \oplus x_2x_5x_9 \oplus x_2x_6 \oplus x_2x_6x_7 \oplus$$
$$x_2x_6x_7x_8 \oplus x_2x_6x_7x_8x_9 \oplus x_2x_6x_8x_9 \oplus x_2x_6x_9 \oplus x_2x_7x_8 \oplus x_2x_7x_9 \oplus x_2x_8 \oplus x_2x_8x_9 \oplus x_3 \oplus x_3x_4x_5 \oplus x_3x_4x_5x_6x_7 \oplus$$
$$x_3x_4x_5x_6x_7x_8x_9 \oplus x_3x_4x_5x_6x_7x_9 \oplus x_3x_4x_5x_6x_8 \oplus x_3x_4x_5x_6x_9 \oplus x_3x_4x_5x_7 \oplus x_3x_4x_5x_7x_8 \oplus x_3x_4x_5x_7x_8x_9 \oplus$$
$$x_3x_4x_5x_8x_9 \oplus x_3x_4x_5x_9 \oplus x_3x_4x_6 \oplus x_3x_4x_6x_7x_8 \oplus x_3x_4x_6x_7x_9 \oplus x_3x_4x_6x_8 \oplus x_3x_4x_6x_8x_9 \oplus x_3x_4x_7 \oplus x_3x_4x_7x_8x_9 \oplus$$
$$x_3x_4x_7x_9 \oplus x_3x_4x_8 \oplus x_3x_4x_9 \oplus x_3x_5 \oplus x_3x_5x_6 \oplus x_3x_5x_6x_7 \oplus x_3x_5x_6x_7x_8 \oplus x_3x_5x_6x_7x_8x_9 \oplus x_3x_5x_6x_8x_9 \oplus$$
$$x_3x_5x_6x_9 \oplus x_3x_5x_7x_8 \oplus x_3x_5x_7x_9 \oplus x_3x_5x_8 \oplus x_3x_5x_8x_9 \oplus x_3x_6x_7 \oplus x_3x_6x_7x_8x_9 \oplus x_3x_6x_7x_9 \oplus x_3x_6x_8 \oplus x_3x_6x_9 \oplus$$
$$x_3x_7 \oplus x_3x_7x_8 \oplus x_3x_7x_8x_9 \oplus x_3x_8x_9 \oplus x_3x_9 \oplus x_4 \oplus x_4x_5x_6 \oplus x_4x_5x_6x_7x_8 \oplus x_4x_5x_6x_7x_9 \oplus x_4x_5x_6x_8 \oplus x_4x_5x_6x_8x_9 \oplus$$
$$x_4x_5x_7 \oplus x_4x_5x_7x_8x_9 \oplus x_4x_5x_7x_9 \oplus x_4x_5x_8 \oplus x_4x_5x_9 \oplus x_4x_6 \oplus x_4x_6x_7 \oplus x_4x_6x_7x_8 \oplus x_4x_6x_7x_8x_9 \oplus x_4x_6x_8x_9 \oplus$$
$$x_4x_6x_9 \oplus x_4x_7x_8 \oplus x_4x_7x_9 \oplus x_4x_8 \oplus x_4x_8x_9 \oplus x_5 \oplus x_5x_6x_7 \oplus x_5x_6x_7x_8x_9 \oplus x_5x_6x_7x_9 \oplus x_5x_6x_8 \oplus x_5x_6x_9 \oplus x_5x_7 \oplus$$
$$x_5x_7x_8 \oplus x_5x_7x_8x_9 \oplus x_5x_8x_9 \oplus x_5x_9 \oplus x_6 \oplus x_6x_7x_8 \oplus x_6x_7x_9 \oplus x_6x_8 \oplus x_6x_8x_9 \oplus x_7 \oplus x_7x_8x_9 \oplus x_7x_9 \oplus x_8 \oplus x_9 = 1$$

$$x_0x_1 \oplus x_0x_1x_2x_3 \oplus x_0x_1x_2x_3x_4x_5 \oplus x_0x_1x_2x_3x_4x_5x_6x_7 \oplus x_0x_1x_2x_3x_4x_5x_6x_7x_8x_9 \oplus x_0x_1x_2x_3x_4x_5x_6x_7x_9 \oplus$$
$$x_0x_1x_2x_3x_4x_5x_6x_8 \oplus x_0x_1x_2x_3x_4x_5x_6x_9 \oplus x_0x_1x_2x_3x_4x_5x_7 \oplus x_0x_1x_2x_3x_4x_5x_7x_8 \oplus x_0x_1x_2x_3x_4x_5x_7x_8x_9 \oplus$$
$$x_0x_1x_2x_3x_4x_5x_8x_9 \oplus x_0x_1x_2x_3x_4x_5x_9 \oplus x_0x_1x_2x_3x_4x_6 \oplus x_0x_1x_2x_3x_4x_6x_7x_8 \oplus x_0x_1x_2x_3x_4x_6x_7x_9 \oplus x_0x_1x_2x_3x_4x_6x_8 \oplus$$
$$x_0x_1x_2x_3x_4x_6x_8x_9 \oplus x_0x_1x_2x_3x_4x_7 \oplus x_0x_1x_2x_3x_4x_7x_8x_9 \oplus x_0x_1x_2x_3x_4x_7x_9 \oplus x_0x_1x_2x_3x_4x_8 \oplus x_0x_1x_2x_3x_4x_9 \oplus$$
$$x_0x_1x_2x_3x_5 \oplus x_0x_1x_2x_3x_5x_6 \oplus x_0x_1x_2x_3x_5x_6x_7 \oplus x_0x_1x_2x_3x_5x_6x_7x_8 \oplus x_0x_1x_2x_3x_5x_6x_7x_8x_9 \oplus x_0x_1x_2x_3x_5x_6x_8x_9 \oplus$$
$$x_0x_1x_2x_3x_5x_6x_9 \oplus x_0x_1x_2x_3x_5x_7x_8 \oplus x_0x_1x_2x_3x_5x_7x_9 \oplus x_0x_1x_2x_3x_5x_8 \oplus x_0x_1x_2x_3x_5x_8x_9 \oplus x_0x_1x_2x_3x_6x_7 \oplus$$
$$x_0x_1x_2x_3x_6x_7x_8x_9 \oplus x_0x_1x_2x_3x_6x_7x_9 \oplus x_0x_1x_2x_3x_6x_8 \oplus x_0x_1x_2x_3x_6x_9 \oplus x_0x_1x_2x_3x_7 \oplus x_0x_1x_2x_3x_7x_8 \oplus$$
$$x_0x_1x_2x_3x_7x_8x_9 \oplus x_0x_1x_2x_3x_8x_9 \oplus x_0x_1x_2x_3x_9 \oplus x_0x_1x_2x_4 \oplus x_0x_1x_2x_4x_5x_6 \oplus x_0x_1x_2x_4x_5x_6x_7x_8 \oplus x_0x_1x_2x_4x_5x_6x_7x_9 \oplus$$
$$x_0x_1x_2x_4x_5x_6x_8 \oplus x_0x_1x_2x_4x_5x_6x_8x_9 \oplus x_0x_1x_2x_4x_5x_7 \oplus x_0x_1x_2x_4x_5x_7x_8x_9 \oplus x_0x_1x_2x_4x_5x_7x_9 \oplus x_0x_1x_2x_4x_5x_8 \oplus$$
$$x_0x_1x_2x_4x_5x_9 \oplus x_0x_1x_2x_4x_6 \oplus x_0x_1x_2x_4x_6x_7 \oplus x_0x_1x_2x_4x_6x_7x_8 \oplus x_0x_1x_2x_4x_6x_7x_8x_9 \oplus x_0x_1x_2x_4x_6x_8x_9 \oplus$$
$$x_0x_1x_2x_4x_6x_9 \oplus x_0x_1x_2x_4x_7x_8 \oplus x_0x_1x_2x_4x_7x_9 \oplus x_0x_1x_2x_4x_8 \oplus x_0x_1x_2x_4x_8x_9 \oplus x_0x_1x_2x_5 \oplus x_0x_1x_2x_5x_6x_7 \oplus$$
$$x_0x_1x_2x_5x_6x_7x_8x_9 \oplus x_0x_1x_2x_5x_6x_7x_9 \oplus x_0x_1x_2x_5x_6x_8 \oplus x_0x_1x_2x_5x_6x_9 \oplus x_0x_1x_2x_5x_7 \oplus x_0x_1x_2x_5x_7x_8 \oplus$$
$$x_0x_1x_2x_5x_7x_8x_9 \oplus x_0x_1x_2x_5x_8x_9 \oplus x_0x_1x_2x_5x_9 \oplus x_0x_1x_2x_6 \oplus x_0x_1x_2x_6x_7x_8 \oplus x_0x_1x_2x_6x_7x_9 \oplus x_0x_1x_2x_6x_8 \oplus$$
$$x_0x_1x_2x_6x_8x_9 \oplus x_0x_1x_2x_7 \oplus x_0x_1x_2x_7x_8x_9 \oplus x_0x_1x_2x_7x_9 \oplus x_0x_1x_2x_8 \oplus x_0x_1x_2x_9 \oplus x_0x_1x_3 \oplus x_0x_1x_3x_4 \oplus$$
$$x_0x_1x_3x_4x_5 \oplus x_0x_1x_3x_4x_5x_6 \oplus x_0x_1x_3x_4x_5x_6x_7 \oplus x_0x_1x_3x_4x_5x_6x_7x_8 \oplus x_0x_1x_3x_4x_5x_6x_7x_8x_9 \oplus x_0x_1x_3x_4x_5x_6x_8x_9 \oplus$$
$$x_0x_1x_3x_4x_5x_6x_9 \oplus x_0x_1x_3x_4x_5x_7x_8 \oplus x_0x_1x_3x_4x_5x_7x_9 \oplus x_0x_1x_3x_4x_5x_8 \oplus x_0x_1x_3x_4x_5x_8x_9 \oplus x_0x_1x_3x_4x_6x_7 \oplus$$
$$x_0x_1x_3x_4x_6x_7x_8x_9 \oplus x_0x_1x_3x_4x_6x_7x_9 \oplus x_0x_1x_3x_4x_6x_8 \oplus x_0x_1x_3x_4x_6x_9 \oplus x_0x_1x_3x_4x_7 \oplus x_0x_1x_3x_4x_7x_8 \oplus$$
$$x_0x_1x_3x_4x_7x_8x_9 \oplus x_0x_1x_3x_4x_8x_9 \oplus x_0x_1x_3x_4x_9 \oplus x_0x_1x_3x_5x_6 \oplus x_0x_1x_3x_5x_6x_7x_8 \oplus x_0x_1x_3x_5x_6x_7x_9 \oplus x_0x_1x_3x_5x_6x_8 \oplus$$
$$x_0x_1x_3x_5x_6x_8x_9 \oplus x_0x_1x_3x_5x_7 \oplus x_0x_1x_3x_5x_7x_8x_9 \oplus x_0x_1x_3x_5x_7x_9 \oplus x_0x_1x_3x_5x_8 \oplus x_0x_1x_3x_5x_9 \oplus x_0x_1x_3x_6 \oplus$$
$$x_0x_1x_3x_6x_7 \oplus x_0x_1x_3x_6x_7x_8 \oplus x_0x_1x_3x_6x_7x_8x_9 \oplus x_0x_1x_3x_6x_8x_9 \oplus x_0x_1x_3x_6x_9 \oplus x_0x_1x_3x_7x_8 \oplus x_0x_1x_3x_7x_9 \oplus$$
$$x_0x_1x_3x_8 \oplus x_0x_1x_3x_8x_9 \oplus x_0x_1x_4x_5 \oplus x_0x_1x_4x_5x_6x_7 \oplus x_0x_1x_4x_5x_6x_7x_8x_9 \oplus x_0x_1x_4x_5x_6x_7x_9 \oplus x_0x_1x_4x_5x_6x_8 \oplus$$
$$x_0x_1x_4x_5x_6x_9 \oplus x_0x_1x_4x_5x_7 \oplus x_0x_1x_4x_5x_7x_8 \oplus x_0x_1x_4x_5x_7x_8x_9 \oplus x_0x_1x_4x_5x_8x_9 \oplus x_0x_1x_4x_5x_9 \oplus x_0x_1x_4x_6 \oplus$$
$$x_0x_1x_4x_6x_7x_8 \oplus x_0x_1x_4x_6x_7x_9 \oplus x_0x_1x_4x_6x_8 \oplus x_0x_1x_4x_6x_8x_9 \oplus x_0x_1x_4x_7 \oplus x_0x_1x_4x_7x_8x_9 \oplus x_0x_1x_4x_7x_9 \oplus$$
$$x_0x_1x_4x_8 \oplus x_0x_1x_4x_9 \oplus x_0x_1x_5 \oplus x_0x_1x_5x_6 \oplus x_0x_1x_5x_6x_7 \oplus x_0x_1x_5x_6x_7x_8 \oplus x_0x_1x_5x_6x_7x_8x_9 \oplus x_0x_1x_5x_6x_8x_9 \oplus$$
$$x_0x_1x_5x_6x_9 \oplus x_0x_1x_5x_7x_8 \oplus x_0x_1x_5x_7x_9 \oplus x_0x_1x_5x_8 \oplus x_0x_1x_5x_8x_9 \oplus x_0x_1x_6x_7 \oplus x_0x_1x_6x_7x_8x_9 \oplus x_0x_1x_6x_7x_9 \oplus$$
$$x_0x_1x_6x_8 \oplus x_0x_1x_6x_9 \oplus x_0x_1x_7 \oplus x_0x_1x_7x_8 \oplus x_0x_1x_7x_8x_9 \oplus x_0x_1x_8x_9 \oplus x_0x_1x_9 \oplus x_0x_2 \oplus x_0x_2x_3x_4 \oplus x_0x_2x_3x_4x_5x_6 \oplus$$
$$x_0x_2x_3x_4x_5x_6x_7x_8 \oplus x_0x_2x_3x_4x_5x_6x_7x_9 \oplus x_0x_2x_3x_4x_5x_6x_8 \oplus x_0x_2x_3x_4x_5x_6x_8x_9 \oplus x_0x_2x_3x_4x_5x_7 \oplus x_0x_2x_3x_4x_5x_7x_8x_9 \oplus$$
$$x_0x_2x_3x_4x_5x_7x_9 \oplus x_0x_2x_3x_4x_5x_8 \oplus x_0x_2x_3x_4x_5x_9 \oplus x_0x_2x_3x_4x_6 \oplus x_0x_2x_3x_4x_6x_7 \oplus x_0x_2x_3x_4x_6x_7x_8 \oplus x_0x_2x_3x_4x_6x_7x_8x_9 \oplus$$

$$x_0x_2x_3x_4x_6x_8x_9 \oplus x_0x_2x_3x_4x_6x_9 \oplus x_0x_2x_3x_4x_7x_8 \oplus x_0x_2x_3x_4x_7x_9 \oplus x_0x_2x_3x_4x_8 \oplus x_0x_2x_3x_4x_8x_9 \oplus x_0x_2x_3x_5 \oplus$$
$$x_0x_2x_3x_5x_6x_7 \oplus x_0x_2x_3x_5x_6x_7x_8x_9 \oplus x_0x_2x_3x_5x_6x_7x_9 \oplus x_0x_2x_3x_5x_6x_8 \oplus x_0x_2x_3x_5x_6x_9 \oplus x_0x_2x_3x_5x_7 \oplus$$
$$x_0x_2x_3x_5x_7x_8 \oplus x_0x_2x_3x_5x_7x_8x_9 \oplus x_0x_2x_3x_5x_8x_9 \oplus x_0x_2x_3x_5x_9 \oplus x_0x_2x_3x_6 \oplus x_0x_2x_3x_6x_7x_8 \oplus x_0x_2x_3x_6x_7x_9 \oplus$$
$$x_0x_2x_3x_6x_8 \oplus x_0x_2x_3x_6x_8x_9 \oplus x_0x_2x_3x_7 \oplus x_0x_2x_3x_7x_8x_9 \oplus x_0x_2x_3x_7x_9 \oplus x_0x_2x_3x_8 \oplus x_0x_2x_3x_9 \oplus x_0x_2x_4 \oplus$$
$$x_0x_2x_4x_5 \oplus x_0x_2x_4x_5x_6 \oplus x_0x_2x_4x_5x_6x_7 \oplus x_0x_2x_4x_5x_6x_7x_8 \oplus x_0x_2x_4x_5x_6x_7x_8x_9 \oplus x_0x_2x_4x_5x_6x_8x_9 \oplus x_0x_2x_4x_5x_6x_9 \oplus$$
$$x_0x_2x_4x_5x_7x_8 \oplus x_0x_2x_4x_5x_7x_9 \oplus x_0x_2x_4x_5x_8 \oplus x_0x_2x_4x_5x_8x_9 \oplus x_0x_2x_4x_6x_7 \oplus x_0x_2x_4x_6x_7x_8x_9 \oplus x_0x_2x_4x_6x_7x_9 \oplus$$
$$x_0x_2x_4x_6x_8 \oplus x_0x_2x_4x_6x_9 \oplus x_0x_2x_4x_7 \oplus x_0x_2x_4x_7x_8 \oplus x_0x_2x_4x_7x_8x_9 \oplus x_0x_2x_4x_8x_9 \oplus x_0x_2x_4x_9 \oplus x_0x_2x_5x_6 \oplus$$
$$x_0x_2x_5x_6x_7x_8 \oplus x_0x_2x_5x_6x_7x_9 \oplus x_0x_2x_5x_6x_8 \oplus x_0x_2x_5x_6x_8x_9 \oplus x_0x_2x_5x_7 \oplus x_0x_2x_5x_7x_8x_9 \oplus x_0x_2x_5x_7x_9 \oplus$$
$$x_0x_2x_5x_8 \oplus x_0x_2x_5x_9 \oplus x_0x_2x_6 \oplus x_0x_2x_6x_7 \oplus x_0x_2x_6x_7x_8 \oplus x_0x_2x_6x_7x_8x_9 \oplus x_0x_2x_6x_8x_9 \oplus x_0x_2x_6x_9 \oplus x_0x_2x_7x_8 \oplus$$
$$x_0x_2x_7x_9 \oplus x_0x_2x_8 \oplus x_0x_2x_8x_9 \oplus x_0x_3 \oplus x_0x_3x_4x_5 \oplus x_0x_3x_4x_5x_6x_7 \oplus x_0x_3x_4x_5x_6x_7x_8x_9 \oplus x_0x_3x_4x_5x_6x_7x_9 \oplus$$
$$x_0x_3x_4x_5x_6x_8 \oplus x_0x_3x_4x_5x_6x_9 \oplus x_0x_3x_4x_5x_7 \oplus x_0x_3x_4x_5x_7x_8 \oplus x_0x_3x_4x_5x_7x_8x_9 \oplus x_0x_3x_4x_5x_8x_9 \oplus x_0x_3x_4x_5x_9 \oplus$$
$$x_0x_3x_4x_6 \oplus x_0x_3x_4x_6x_7x_8 \oplus x_0x_3x_4x_6x_7x_9 \oplus x_0x_3x_4x_6x_8 \oplus x_0x_3x_4x_6x_8x_9 \oplus x_0x_3x_4x_7 \oplus x_0x_3x_4x_7x_8x_9 \oplus$$
$$x_0x_3x_4x_7x_9 \oplus x_0x_3x_4x_8 \oplus x_0x_3x_4x_9 \oplus x_0x_3x_5 \oplus x_0x_3x_5x_6 \oplus x_0x_3x_5x_6x_7 \oplus x_0x_3x_5x_6x_7x_8 \oplus x_0x_3x_5x_6x_7x_8x_9 \oplus$$
$$x_0x_3x_5x_6x_8x_9 \oplus x_0x_3x_5x_6x_9 \oplus x_0x_3x_5x_7x_8 \oplus x_0x_3x_5x_7x_9 \oplus x_0x_3x_5x_8 \oplus x_0x_3x_5x_8x_9 \oplus x_0x_3x_6x_7 \oplus x_0x_3x_6x_7x_8x_9 \oplus$$
$$x_0x_3x_6x_7x_9 \oplus x_0x_3x_6x_8 \oplus x_0x_3x_6x_9 \oplus x_0x_3x_7 \oplus x_0x_3x_7x_8 \oplus x_0x_3x_7x_8x_9 \oplus x_0x_3x_8x_9 \oplus x_0x_3x_9 \oplus x_0x_4 \oplus x_0x_4x_5x_6 \oplus$$
$$x_0x_4x_5x_6x_7x_8 \oplus x_0x_4x_5x_6x_7x_9 \oplus x_0x_4x_5x_6x_8 \oplus x_0x_4x_5x_6x_8x_9 \oplus x_0x_4x_5x_7 \oplus x_0x_4x_5x_7x_8x_9 \oplus x_0x_4x_5x_7x_9 \oplus$$
$$x_0x_4x_5x_8 \oplus x_0x_4x_5x_9 \oplus x_0x_4x_6 \oplus x_0x_4x_6x_7 \oplus x_0x_4x_6x_7x_8 \oplus x_0x_4x_6x_7x_8x_9 \oplus x_0x_4x_6x_8x_9 \oplus x_0x_4x_6x_9 \oplus x_0x_4x_7x_8 \oplus$$
$$x_0x_4x_7x_9 \oplus x_0x_4x_8 \oplus x_0x_4x_8x_9 \oplus x_0x_5 \oplus x_0x_5x_6x_7 \oplus x_0x_5x_6x_7x_8x_9 \oplus x_0x_5x_6x_7x_9 \oplus x_0x_5x_6x_8 \oplus x_0x_5x_6x_9 \oplus$$
$$x_0x_5x_7 \oplus x_0x_5x_7x_8 \oplus x_0x_5x_7x_8x_9 \oplus x_0x_5x_8x_9 \oplus x_0x_5x_9 \oplus x_0x_6 \oplus x_0x_6x_7x_8 \oplus x_0x_6x_7x_9 \oplus x_0x_6x_8 \oplus x_0x_6x_8x_9 \oplus$$
$$x_0x_7 \oplus x_0x_7x_8x_9 \oplus x_0x_7x_9 \oplus x_0x_8 \oplus x_0x_9 \oplus x_1 \oplus x_1x_2 \oplus x_1x_2x_3 \oplus x_1x_2x_3x_4 \oplus x_1x_2x_3x_4x_5 \oplus x_1x_2x_3x_4x_5x_6 \oplus$$
$$x_1x_2x_3x_4x_5x_6x_7 \oplus x_1x_2x_3x_4x_5x_6x_7x_8 \oplus x_1x_2x_3x_4x_5x_6x_7x_8x_9 \oplus x_1x_2x_3x_4x_5x_6x_8x_9 \oplus x_1x_2x_3x_4x_5x_6x_9 \oplus x_1x_2x_3x_4x_5x_7x_8 \oplus$$
$$x_1x_2x_3x_4x_5x_7x_9 \oplus x_1x_2x_3x_4x_5x_8 \oplus x_1x_2x_3x_4x_5x_8x_9 \oplus x_1x_2x_3x_4x_6x_7 \oplus x_1x_2x_3x_4x_6x_7x_8x_9 \oplus x_1x_2x_3x_4x_6x_7x_9 \oplus$$
$$x_1x_2x_3x_4x_6x_8 \oplus x_1x_2x_3x_4x_6x_9 \oplus x_1x_2x_3x_4x_7 \oplus x_1x_2x_3x_4x_7x_8 \oplus x_1x_2x_3x_4x_7x_8x_9 \oplus x_1x_2x_3x_4x_8x_9 \oplus x_1x_2x_3x_4x_9 \oplus$$
$$x_1x_2x_3x_5x_6 \oplus x_1x_2x_3x_5x_6x_7x_8 \oplus x_1x_2x_3x_5x_6x_7x_9 \oplus x_1x_2x_3x_5x_6x_8 \oplus x_1x_2x_3x_5x_6x_8x_9 \oplus x_1x_2x_3x_5x_7 \oplus x_1x_2x_3x_5x_7x_8x_9 \oplus$$
$$x_1x_2x_3x_5x_7x_9 \oplus x_1x_2x_3x_5x_8 \oplus x_1x_2x_3x_5x_9 \oplus x_1x_2x_3x_6 \oplus x_1x_2x_3x_6x_7 \oplus x_1x_2x_3x_6x_7x_8 \oplus x_1x_2x_3x_6x_7x_8x_9 \oplus$$
$$x_1x_2x_3x_6x_8x_9 \oplus x_1x_2x_3x_6x_9 \oplus x_1x_2x_3x_7x_8 \oplus x_1x_2x_3x_7x_9 \oplus x_1x_2x_3x_8 \oplus x_1x_2x_3x_8x_9 \oplus x_1x_2x_4x_5 \oplus x_1x_2x_4x_5x_6x_7 \oplus$$
$$x_1x_2x_4x_5x_6x_7x_8x_9 \oplus x_1x_2x_4x_5x_6x_7x_9 \oplus x_1x_2x_4x_5x_6x_8 \oplus x_1x_2x_4x_5x_6x_9 \oplus x_1x_2x_4x_5x_7 \oplus x_1x_2x_4x_5x_7x_8 \oplus$$
$$x_1x_2x_4x_5x_7x_8x_9 \oplus x_1x_2x_4x_5x_8x_9 \oplus x_1x_2x_4x_5x_9 \oplus x_1x_2x_4x_6 \oplus x_1x_2x_4x_6x_7x_8 \oplus x_1x_2x_4x_6x_7x_9 \oplus x_1x_2x_4x_6x_8 \oplus$$
$$x_1x_2x_4x_6x_8x_9 \oplus x_1x_2x_4x_7 \oplus x_1x_2x_4x_7x_8x_9 \oplus x_1x_2x_4x_7x_9 \oplus x_1x_2x_4x_8 \oplus x_1x_2x_4x_9 \oplus x_1x_2x_5 \oplus x_1x_2x_5x_6 \oplus$$
$$x_1x_2x_5x_6x_7 \oplus x_1x_2x_5x_6x_7x_8 \oplus x_1x_2x_5x_6x_7x_8x_9 \oplus x_1x_2x_5x_6x_8x_9 \oplus x_1x_2x_5x_6x_9 \oplus x_1x_2x_5x_7x_8 \oplus x_1x_2x_5x_7x_9 \oplus$$
$$x_1x_2x_5x_8 \oplus x_1x_2x_5x_8x_9 \oplus x_1x_2x_6x_7 \oplus x_1x_2x_6x_7x_8x_9 \oplus x_1x_2x_6x_7x_9 \oplus x_1x_2x_6x_8 \oplus x_1x_2x_6x_9 \oplus x_1x_2x_7 \oplus$$
$$x_1x_2x_7x_8 \oplus x_1x_2x_7x_8x_9 \oplus x_1x_2x_8x_9 \oplus x_1x_2x_9 \oplus x_1x_3x_4 \oplus x_1x_3x_4x_5x_6 \oplus x_1x_3x_4x_5x_6x_7x_8 \oplus x_1x_3x_4x_5x_6x_7x_9 \oplus$$
$$x_1x_3x_4x_5x_6x_8 \oplus x_1x_3x_4x_5x_6x_8x_9 \oplus x_1x_3x_4x_5x_7 \oplus x_1x_3x_4x_5x_7x_8x_9 \oplus x_1x_3x_4x_5x_7x_9 \oplus x_1x_3x_4x_5x_8 \oplus x_1x_3x_4x_5x_9 \oplus$$
$$x_1x_3x_4x_6 \oplus x_1x_3x_4x_6x_7 \oplus x_1x_3x_4x_6x_7x_8 \oplus x_1x_3x_4x_6x_7x_8x_9 \oplus x_1x_3x_4x_6x_8x_9 \oplus x_1x_3x_4x_6x_9 \oplus x_1x_3x_4x_7x_8 \oplus$$
$$x_1x_3x_4x_7x_9 \oplus x_1x_3x_4x_8 \oplus x_1x_3x_4x_8x_9 \oplus x_1x_3x_5 \oplus x_1x_3x_5x_6x_7 \oplus x_1x_3x_5x_6x_7x_8x_9 \oplus x_1x_3x_5x_6x_7x_9 \oplus x_1x_3x_5x_6x_8 \oplus$$
$$x_1x_3x_5x_6x_9 \oplus x_1x_3x_5x_7 \oplus x_1x_3x_5x_7x_8 \oplus x_1x_3x_5x_7x_8x_9 \oplus x_1x_3x_5x_8x_9 \oplus x_1x_3x_5x_9 \oplus x_1x_3x_6 \oplus x_1x_3x_6x_7x_8 \oplus$$
$$x_1x_3x_6x_7x_9 \oplus x_1x_3x_6x_8 \oplus x_1x_3x_6x_8x_9 \oplus x_1x_3x_7 \oplus x_1x_3x_7x_8x_9 \oplus x_1x_3x_7x_9 \oplus x_1x_3x_8 \oplus x_1x_3x_9 \oplus x_1x_4 \oplus x_1x_4x_5 \oplus$$
$$x_1x_4x_5x_6 \oplus x_1x_4x_5x_6x_7 \oplus x_1x_4x_5x_6x_7x_8 \oplus x_1x_4x_5x_6x_7x_8x_9 \oplus x_1x_4x_5x_6x_8x_9 \oplus x_1x_4x_5x_6x_9 \oplus x_1x_4x_5x_7x_8 \oplus$$
$$x_1x_4x_5x_7x_9 \oplus x_1x_4x_5x_8 \oplus x_1x_4x_5x_8x_9 \oplus x_1x_4x_6x_7 \oplus x_1x_4x_6x_7x_8x_9 \oplus x_1x_4x_6x_7x_9 \oplus x_1x_4x_6x_8 \oplus x_1x_4x_6x_9 \oplus$$
$$x_1x_4x_7 \oplus x_1x_4x_7x_8 \oplus x_1x_4x_7x_8x_9 \oplus x_1x_4x_8x_9 \oplus x_1x_4x_9 \oplus x_1x_5x_6 \oplus x_1x_5x_6x_7x_8 \oplus x_1x_5x_6x_7x_9 \oplus x_1x_5x_6x_8 \oplus$$
$$x_1x_5x_6x_8x_9 \oplus x_1x_5x_7 \oplus x_1x_5x_7x_8x_9 \oplus x_1x_5x_7x_9 \oplus x_1x_5x_8 \oplus x_1x_5x_9 \oplus x_1x_6 \oplus x_1x_6x_7 \oplus x_1x_6x_7x_8 \oplus x_1x_6x_7x_8x_9 \oplus$$
$$x_1x_6x_8x_9 \oplus x_1x_6x_9 \oplus x_1x_7x_8 \oplus x_1x_7x_9 \oplus x_1x_8 \oplus x_1x_8x_9 \oplus x_2x_3 \oplus x_2x_3x_4x_5 \oplus x_2x_3x_4x_5x_6x_7 \oplus x_2x_3x_4x_5x_6x_7x_8x_9 \oplus$$
$$x_2x_3x_4x_5x_6x_7x_9 \oplus x_2x_3x_4x_5x_6x_8 \oplus x_2x_3x_4x_5x_6x_9 \oplus x_2x_3x_4x_5x_7 \oplus x_2x_3x_4x_5x_7x_8 \oplus x_2x_3x_4x_5x_7x_8x_9 \oplus x_2x_3x_4x_5x_8x_9 \oplus$$
$$x_2x_3x_4x_5x_9 \oplus x_2x_3x_4x_6 \oplus x_2x_3x_4x_6x_7x_8 \oplus x_2x_3x_4x_6x_7x_9 \oplus x_2x_3x_4x_6x_8 \oplus x_2x_3x_4x_6x_8x_9 \oplus x_2x_3x_4x_7 \oplus x_2x_3x_4x_7x_8x_9 \oplus$$
$$x_2x_3x_4x_7x_9 \oplus x_2x_3x_4x_8 \oplus x_2x_3x_4x_9 \oplus x_2x_3x_5 \oplus x_2x_3x_5x_6 \oplus x_2x_3x_5x_6x_7 \oplus x_2x_3x_5x_6x_7x_8 \oplus x_2x_3x_5x_6x_7x_8x_9 \oplus$$
$$x_2x_3x_5x_6x_8x_9 \oplus x_2x_3x_5x_6x_9 \oplus x_2x_3x_5x_7x_8 \oplus x_2x_3x_5x_7x_9 \oplus x_2x_3x_5x_8 \oplus x_2x_3x_5x_8x_9 \oplus x_2x_3x_6x_7 \oplus x_2x_3x_6x_7x_8x_9 \oplus$$
$$x_2x_3x_6x_7x_9 \oplus x_2x_3x_6x_8 \oplus x_2x_3x_6x_9 \oplus x_2x_3x_7 \oplus x_2x_3x_7x_8 \oplus x_2x_3x_7x_8x_9 \oplus x_2x_3x_8x_9 \oplus x_2x_3x_9 \oplus x_2x_4 \oplus x_2x_4x_5x_6 \oplus$$
$$x_2x_4x_5x_6x_7x_8 \oplus x_2x_4x_5x_6x_7x_9 \oplus x_2x_4x_5x_6x_8 \oplus x_2x_4x_5x_6x_8x_9 \oplus x_2x_4x_5x_7 \oplus x_2x_4x_5x_7x_8x_9 \oplus x_2x_4x_5x_7x_9 \oplus$$
$$x_2x_4x_5x_8 \oplus x_2x_4x_5x_9 \oplus x_2x_4x_6 \oplus x_2x_4x_6x_7 \oplus x_2x_4x_6x_7x_8 \oplus x_2x_4x_6x_7x_8x_9 \oplus x_2x_4x_6x_8x_9 \oplus x_2x_4x_6x_9 \oplus x_2x_4x_7x_8 \oplus$$
$$x_2x_4x_7x_9 \oplus x_2x_4x_8 \oplus x_2x_4x_8x_9 \oplus x_2x_5 \oplus x_2x_5x_6x_7 \oplus x_2x_5x_6x_7x_8x_9 \oplus x_2x_5x_6x_7x_9 \oplus x_2x_5x_6x_8 \oplus x_2x_5x_6x_9 \oplus$$
$$x_2x_5x_7 \oplus x_2x_5x_7x_8 \oplus x_2x_5x_7x_8x_9 \oplus x_2x_5x_8x_9 \oplus x_2x_5x_9 \oplus x_2x_6 \oplus x_2x_6x_7x_8 \oplus x_2x_6x_7x_9 \oplus x_2x_6x_8 \oplus x_2x_6x_8x_9 \oplus$$

$$x_2x_7 \oplus x_2x_7x_8x_9 \oplus x_2x_7x_9 \oplus x_2x_8 \oplus x_2x_9 \oplus x_3 \oplus x_3x_4 \oplus x_3x_4x_5 \oplus x_3x_4x_5x_6 \oplus x_3x_4x_5x_6x_7 \oplus x_3x_4x_5x_6x_7x_8 \oplus$$
$$x_3x_4x_5x_6x_7x_8x_9 \oplus x_3x_4x_5x_6x_8x_9 \oplus x_3x_4x_5x_6x_9 \oplus x_3x_4x_5x_7x_8 \oplus x_3x_4x_5x_7x_9 \oplus x_3x_4x_5x_8 \oplus x_3x_4x_5x_8x_9 \oplus$$
$$x_3x_4x_6x_7 \oplus x_3x_4x_6x_7x_8x_9 \oplus x_3x_4x_6x_7x_9 \oplus x_3x_4x_6x_8 \oplus x_3x_4x_6x_9 \oplus x_3x_4x_7 \oplus x_3x_4x_7x_8 \oplus x_3x_4x_7x_8x_9 \oplus x_3x_4x_8x_9 \oplus$$
$$x_3x_4x_9 \oplus x_3x_5x_6 \oplus x_3x_5x_6x_7x_8 \oplus x_3x_5x_6x_7x_9 \oplus x_3x_5x_6x_8 \oplus x_3x_5x_6x_8x_9 \oplus x_3x_5x_7 \oplus x_3x_5x_7x_8x_9 \oplus x_3x_5x_7x_9 \oplus$$
$$x_3x_5x_8 \oplus x_3x_5x_9 \oplus x_3x_6 \oplus x_3x_6x_7 \oplus x_3x_6x_7x_8 \oplus x_3x_6x_7x_8x_9 \oplus x_3x_6x_8x_9 \oplus x_3x_6x_9 \oplus x_3x_7x_8 \oplus x_3x_7x_9 \oplus x_3x_8 \oplus$$
$$x_3x_8x_9 \oplus x_4x_5 \oplus x_4x_5x_6x_7 \oplus x_4x_5x_6x_7x_8x_9 \oplus x_4x_5x_6x_7x_9 \oplus x_4x_5x_6x_8 \oplus x_4x_5x_6x_9 \oplus x_4x_5x_7 \oplus x_4x_5x_7x_8 \oplus$$
$$x_4x_5x_7x_8x_9 \oplus x_4x_5x_8x_9 \oplus x_4x_5x_9 \oplus x_4x_6 \oplus x_4x_6x_7x_8 \oplus x_4x_6x_7x_9 \oplus x_4x_6x_8 \oplus x_4x_6x_8x_9 \oplus x_4x_7 \oplus x_4x_7x_8x_9 \oplus$$
$$x_4x_7x_9 \oplus x_4x_8 \oplus x_4x_9 \oplus x_5 \oplus x_5x_6 \oplus x_5x_6x_7 \oplus x_5x_6x_7x_8 \oplus x_5x_6x_7x_8x_9 \oplus x_5x_6x_8x_9 \oplus x_5x_6x_9 \oplus x_5x_7x_8 \oplus$$
$$x_5x_7x_9 \oplus x_5x_8 \oplus x_5x_8x_9 \oplus x_6x_7 \oplus x_6x_7x_8x_9 \oplus x_6x_7x_9 \oplus x_6x_8 \oplus x_6x_9 \oplus x_7 \oplus x_7x_8 \oplus x_7x_8x_9 \oplus x_8x_9 \oplus x_9 = 0$$

$$x_0 \oplus x_0x_1 \oplus x_0x_1x_2 \oplus x_0x_1x_2x_3 \oplus x_0x_1x_2x_3x_4 \oplus x_0x_1x_2x_3x_4x_5 \oplus x_0x_1x_2x_3x_4x_5x_6 \oplus x_0x_1x_2x_3x_4x_5x_6x_7 \oplus$$
$$x_0x_1x_2x_3x_4x_5x_6x_7x_8 \oplus x_0x_1x_2x_3x_4x_5x_6x_7x_8x_9 \oplus x_0x_1x_2x_3x_4x_5x_6x_8x_9 \oplus x_0x_1x_2x_3x_4x_5x_6x_9 \oplus x_0x_1x_2x_3x_4x_5x_7x_8 \oplus$$
$$x_0x_1x_2x_3x_4x_5x_7x_9 \oplus x_0x_1x_2x_3x_4x_5x_8 \oplus x_0x_1x_2x_3x_4x_5x_8x_9 \oplus x_0x_1x_2x_3x_4x_6x_7 \oplus x_0x_1x_2x_3x_4x_6x_7x_8x_9 \oplus x_0x_1x_2x_3x_4x_6x_7x_9 \oplus$$
$$x_0x_1x_2x_3x_4x_6x_8 \oplus x_0x_1x_2x_3x_4x_6x_9 \oplus x_0x_1x_2x_3x_4x_7 \oplus x_0x_1x_2x_3x_4x_7x_8 \oplus x_0x_1x_2x_3x_4x_7x_8x_9 \oplus x_0x_1x_2x_3x_4x_8x_9 \oplus$$
$$x_0x_1x_2x_3x_4x_9 \oplus x_0x_1x_2x_3x_5x_6 \oplus x_0x_1x_2x_3x_5x_6x_7x_8 \oplus x_0x_1x_2x_3x_5x_6x_7x_9 \oplus x_0x_1x_2x_3x_5x_6x_8 \oplus x_0x_1x_2x_3x_5x_6x_8x_9 \oplus$$
$$x_0x_1x_2x_3x_5x_7 \oplus x_0x_1x_2x_3x_5x_7x_8x_9 \oplus x_0x_1x_2x_3x_5x_7x_9 \oplus x_0x_1x_2x_3x_5x_8 \oplus x_0x_1x_2x_3x_5x_9 \oplus x_0x_1x_2x_3x_6 \oplus$$
$$x_0x_1x_2x_3x_6x_7 \oplus x_0x_1x_2x_3x_6x_7x_8 \oplus x_0x_1x_2x_3x_6x_7x_8x_9 \oplus x_0x_1x_2x_3x_6x_8x_9 \oplus x_0x_1x_2x_3x_6x_9 \oplus x_0x_1x_2x_3x_7x_8 \oplus$$
$$x_0x_1x_2x_3x_7x_9 \oplus x_0x_1x_2x_3x_8 \oplus x_0x_1x_2x_3x_8x_9 \oplus x_0x_1x_2x_4x_5 \oplus x_0x_1x_2x_4x_5x_6x_7 \oplus x_0x_1x_2x_4x_5x_6x_7x_8x_9 \oplus$$
$$x_0x_1x_2x_4x_5x_6x_7x_9 \oplus x_0x_1x_2x_4x_5x_6x_8 \oplus x_0x_1x_2x_4x_5x_6x_9 \oplus x_0x_1x_2x_4x_5x_7 \oplus x_0x_1x_2x_4x_5x_7x_8 \oplus x_0x_1x_2x_4x_5x_7x_8x_9 \oplus$$
$$x_0x_1x_2x_4x_5x_8x_9 \oplus x_0x_1x_2x_4x_5x_9 \oplus x_0x_1x_2x_4x_6 \oplus x_0x_1x_2x_4x_6x_7x_8 \oplus x_0x_1x_2x_4x_6x_7x_9 \oplus x_0x_1x_2x_4x_6x_8 \oplus$$
$$x_0x_1x_2x_4x_6x_8x_9 \oplus x_0x_1x_2x_4x_7 \oplus x_0x_1x_2x_4x_7x_8x_9 \oplus x_0x_1x_2x_4x_7x_9 \oplus x_0x_1x_2x_4x_8 \oplus x_0x_1x_2x_4x_9 \oplus x_0x_1x_2x_5 \oplus$$
$$x_0x_1x_2x_5x_6 \oplus x_0x_1x_2x_5x_6x_7 \oplus x_0x_1x_2x_5x_6x_7x_8 \oplus x_0x_1x_2x_5x_6x_7x_8x_9 \oplus x_0x_1x_2x_5x_6x_8x_9 \oplus x_0x_1x_2x_5x_6x_9 \oplus$$
$$x_0x_1x_2x_5x_7x_8 \oplus x_0x_1x_2x_5x_7x_9 \oplus x_0x_1x_2x_5x_8 \oplus x_0x_1x_2x_5x_8x_9 \oplus x_0x_1x_2x_6x_7 \oplus x_0x_1x_2x_6x_7x_8x_9 \oplus x_0x_1x_2x_6x_7x_9 \oplus$$
$$x_0x_1x_2x_6x_8 \oplus x_0x_1x_2x_6x_9 \oplus x_0x_1x_2x_7 \oplus x_0x_1x_2x_7x_8 \oplus x_0x_1x_2x_7x_8x_9 \oplus x_0x_1x_2x_8x_9 \oplus x_0x_1x_2x_9 \oplus x_0x_1x_3x_4 \oplus$$
$$x_0x_1x_3x_4x_5x_6 \oplus x_0x_1x_3x_4x_5x_6x_7x_8 \oplus x_0x_1x_3x_4x_5x_6x_7x_9 \oplus x_0x_1x_3x_4x_5x_6x_8 \oplus x_0x_1x_3x_4x_5x_6x_8x_9 \oplus x_0x_1x_3x_4x_5x_7 \oplus$$
$$x_0x_1x_3x_4x_5x_7x_8x_9 \oplus x_0x_1x_3x_4x_5x_7x_9 \oplus x_0x_1x_3x_4x_5x_8 \oplus x_0x_1x_3x_4x_5x_9 \oplus x_0x_1x_3x_4x_6 \oplus x_0x_1x_3x_4x_6x_7 \oplus$$
$$x_0x_1x_3x_4x_6x_7x_8 \oplus x_0x_1x_3x_4x_6x_7x_8x_9 \oplus x_0x_1x_3x_4x_6x_8x_9 \oplus x_0x_1x_3x_4x_6x_9 \oplus x_0x_1x_3x_4x_7x_8 \oplus x_0x_1x_3x_4x_7x_9 \oplus$$
$$x_0x_1x_3x_4x_8 \oplus x_0x_1x_3x_4x_8x_9 \oplus x_0x_1x_3x_5 \oplus x_0x_1x_3x_5x_6x_7 \oplus x_0x_1x_3x_5x_6x_7x_8x_9 \oplus x_0x_1x_3x_5x_6x_7x_9 \oplus x_0x_1x_3x_5x_6x_8 \oplus$$
$$x_0x_1x_3x_5x_6x_9 \oplus x_0x_1x_3x_5x_7 \oplus x_0x_1x_3x_5x_7x_8 \oplus x_0x_1x_3x_5x_7x_8x_9 \oplus x_0x_1x_3x_5x_8x_9 \oplus x_0x_1x_3x_5x_9 \oplus x_0x_1x_3x_6 \oplus$$
$$x_0x_1x_3x_6x_7x_8 \oplus x_0x_1x_3x_6x_7x_9 \oplus x_0x_1x_3x_6x_8 \oplus x_0x_1x_3x_6x_8x_9 \oplus x_0x_1x_3x_7 \oplus x_0x_1x_3x_7x_8x_9 \oplus x_0x_1x_3x_7x_9 \oplus$$
$$x_0x_1x_3x_8 \oplus x_0x_1x_3x_9 \oplus x_0x_1x_4 \oplus x_0x_1x_4x_5 \oplus x_0x_1x_4x_5x_6 \oplus x_0x_1x_4x_5x_6x_7 \oplus x_0x_1x_4x_5x_6x_7x_8 \oplus x_0x_1x_4x_5x_6x_7x_8x_9 \oplus$$
$$x_0x_1x_4x_5x_6x_8x_9 \oplus x_0x_1x_4x_5x_6x_9 \oplus x_0x_1x_4x_5x_7x_8 \oplus x_0x_1x_4x_5x_7x_9 \oplus x_0x_1x_4x_5x_8 \oplus x_0x_1x_4x_5x_8x_9 \oplus x_0x_1x_4x_6x_7 \oplus$$
$$x_0x_1x_4x_6x_7x_8x_9 \oplus x_0x_1x_4x_6x_7x_9 \oplus x_0x_1x_4x_6x_8 \oplus x_0x_1x_4x_6x_9 \oplus x_0x_1x_4x_7 \oplus x_0x_1x_4x_7x_8 \oplus x_0x_1x_4x_7x_8x_9 \oplus$$
$$x_0x_1x_4x_8x_9 \oplus x_0x_1x_4x_9 \oplus x_0x_1x_5x_6 \oplus x_0x_1x_5x_6x_7x_8 \oplus x_0x_1x_5x_6x_7x_9 \oplus x_0x_1x_5x_6x_8 \oplus x_0x_1x_5x_6x_8x_9 \oplus x_0x_1x_5x_7 \oplus$$
$$x_0x_1x_5x_7x_8x_9 \oplus x_0x_1x_5x_7x_9 \oplus x_0x_1x_5x_8 \oplus x_0x_1x_5x_9 \oplus x_0x_1x_6 \oplus x_0x_1x_6x_7 \oplus x_0x_1x_6x_7x_8 \oplus x_0x_1x_6x_7x_8x_9 \oplus$$
$$x_0x_1x_6x_8x_9 \oplus x_0x_1x_6x_9 \oplus x_0x_1x_7x_8 \oplus x_0x_1x_7x_9 \oplus x_0x_1x_8 \oplus x_0x_1x_8x_9 \oplus x_0x_2x_3 \oplus x_0x_2x_3x_4x_5 \oplus x_0x_2x_3x_4x_5x_6x_7 \oplus$$
$$x_0x_2x_3x_4x_5x_6x_7x_8x_9 \oplus x_0x_2x_3x_4x_5x_6x_7x_9 \oplus x_0x_2x_3x_4x_5x_6x_8 \oplus x_0x_2x_3x_4x_5x_6x_9 \oplus x_0x_2x_3x_4x_5x_7 \oplus x_0x_2x_3x_4x_5x_7x_8 \oplus$$
$$x_0x_2x_3x_4x_5x_7x_8x_9 \oplus x_0x_2x_3x_4x_5x_8x_9 \oplus x_0x_2x_3x_4x_5x_9 \oplus x_0x_2x_3x_4x_6 \oplus x_0x_2x_3x_4x_6x_7x_8 \oplus x_0x_2x_3x_4x_6x_7x_9 \oplus$$
$$x_0x_2x_3x_4x_6x_8 \oplus x_0x_2x_3x_4x_6x_8x_9 \oplus x_0x_2x_3x_4x_7 \oplus x_0x_2x_3x_4x_7x_8x_9 \oplus x_0x_2x_3x_4x_7x_9 \oplus x_0x_2x_3x_4x_8 \oplus x_0x_2x_3x_4x_9 \oplus$$
$$x_0x_2x_3x_5 \oplus x_0x_2x_3x_5x_6 \oplus x_0x_2x_3x_5x_6x_7 \oplus x_0x_2x_3x_5x_6x_7x_8 \oplus x_0x_2x_3x_5x_6x_7x_8x_9 \oplus x_0x_2x_3x_5x_6x_8x_9 \oplus x_0x_2x_3x_5x_6x_9 \oplus$$
$$x_0x_2x_3x_5x_7x_8 \oplus x_0x_2x_3x_5x_7x_9 \oplus x_0x_2x_3x_5x_8 \oplus x_0x_2x_3x_5x_8x_9 \oplus x_0x_2x_3x_6x_7 \oplus x_0x_2x_3x_6x_7x_8x_9 \oplus x_0x_2x_3x_6x_7x_9 \oplus$$
$$x_0x_2x_3x_6x_8 \oplus x_0x_2x_3x_6x_9 \oplus x_0x_2x_3x_7 \oplus x_0x_2x_3x_7x_8 \oplus x_0x_2x_3x_7x_8x_9 \oplus x_0x_2x_3x_8x_9 \oplus x_0x_2x_3x_9 \oplus x_0x_2x_4 \oplus$$
$$x_0x_2x_4x_5x_6 \oplus x_0x_2x_4x_5x_6x_7x_8 \oplus x_0x_2x_4x_5x_6x_7x_9 \oplus x_0x_2x_4x_5x_6x_8 \oplus x_0x_2x_4x_5x_6x_8x_9 \oplus x_0x_2x_4x_5x_7 \oplus x_0x_2x_4x_5x_7x_8x_9 \oplus$$
$$x_0x_2x_4x_5x_7x_9 \oplus x_0x_2x_4x_5x_8 \oplus x_0x_2x_4x_5x_9 \oplus x_0x_2x_4x_6 \oplus x_0x_2x_4x_6x_7 \oplus x_0x_2x_4x_6x_7x_8 \oplus x_0x_2x_4x_6x_7x_8x_9 \oplus$$
$$x_0x_2x_4x_6x_8x_9 \oplus x_0x_2x_4x_6x_9 \oplus x_0x_2x_4x_7x_8 \oplus x_0x_2x_4x_7x_9 \oplus x_0x_2x_4x_8 \oplus x_0x_2x_4x_8x_9 \oplus x_0x_2x_5 \oplus x_0x_2x_5x_6x_7 \oplus$$
$$x_0x_2x_5x_6x_7x_8x_9 \oplus x_0x_2x_5x_6x_7x_9 \oplus x_0x_2x_5x_6x_8 \oplus x_0x_2x_5x_6x_9 \oplus x_0x_2x_5x_7 \oplus x_0x_2x_5x_7x_8 \oplus x_0x_2x_5x_7x_8x_9 \oplus$$
$$x_0x_2x_5x_8x_9 \oplus x_0x_2x_5x_9 \oplus x_0x_2x_6 \oplus x_0x_2x_6x_7x_8 \oplus x_0x_2x_6x_7x_9 \oplus x_0x_2x_6x_8 \oplus x_0x_2x_6x_8x_9 \oplus x_0x_2x_7 \oplus x_0x_2x_7x_8x_9 \oplus$$
$$x_0x_2x_7x_9 \oplus x_0x_2x_8 \oplus x_0x_2x_9 \oplus x_0x_3 \oplus x_0x_3x_4 \oplus x_0x_3x_4x_5 \oplus x_0x_3x_4x_5x_6 \oplus x_0x_3x_4x_5x_6x_7 \oplus x_0x_3x_4x_5x_6x_7x_8 \oplus$$
$$x_0x_3x_4x_5x_6x_7x_8x_9 \oplus x_0x_3x_4x_5x_6x_8x_9 \oplus x_0x_3x_4x_5x_6x_9 \oplus x_0x_3x_4x_5x_7x_8 \oplus x_0x_3x_4x_5x_7x_9 \oplus x_0x_3x_4x_5x_8 \oplus$$
$$x_0x_3x_4x_5x_8x_9 \oplus x_0x_3x_4x_6x_7 \oplus x_0x_3x_4x_6x_7x_8x_9 \oplus x_0x_3x_4x_6x_7x_9 \oplus x_0x_3x_4x_6x_8 \oplus x_0x_3x_4x_6x_9 \oplus x_0x_3x_4x_7 \oplus$$
$$x_0x_3x_4x_7x_8 \oplus x_0x_3x_4x_7x_8x_9 \oplus x_0x_3x_4x_8x_9 \oplus x_0x_3x_4x_9 \oplus x_0x_3x_5x_6 \oplus x_0x_3x_5x_6x_7x_8 \oplus x_0x_3x_5x_6x_7x_9 \oplus x_0x_3x_5x_6x_8 \oplus$$
$$x_0x_3x_5x_6x_8x_9 \oplus x_0x_3x_5x_7 \oplus x_0x_3x_5x_7x_8x_9 \oplus x_0x_3x_5x_7x_9 \oplus x_0x_3x_5x_8 \oplus x_0x_3x_5x_9 \oplus x_0x_3x_6 \oplus x_0x_3x_6x_7 \oplus$$

$$x_0x_3x_6x_7x_8 \oplus x_0x_3x_6x_7x_8x_9 \oplus x_0x_3x_6x_8x_9 \oplus x_0x_3x_6x_9 \oplus x_0x_3x_7x_8 \oplus x_0x_3x_7x_9 \oplus x_0x_3x_8 \oplus x_0x_3x_8x_9 \oplus x_0x_4x_5 \oplus$$
$$x_0x_4x_5x_6x_7 \oplus x_0x_4x_5x_6x_7x_8x_9 \oplus x_0x_4x_5x_6x_7x_9 \oplus x_0x_4x_5x_6x_8 \oplus x_0x_4x_5x_6x_9 \oplus x_0x_4x_5x_7 \oplus x_0x_4x_5x_7x_8 \oplus$$
$$x_0x_4x_5x_7x_8x_9 \oplus x_0x_4x_5x_8x_9 \oplus x_0x_4x_5x_9 \oplus x_0x_4x_6 \oplus x_0x_4x_6x_7x_8 \oplus x_0x_4x_6x_7x_9 \oplus x_0x_4x_6x_8 \oplus x_0x_4x_6x_8x_9 \oplus$$
$$x_0x_4x_7 \oplus x_0x_4x_7x_8x_9 \oplus x_0x_4x_7x_9 \oplus x_0x_4x_8 \oplus x_0x_4x_9 \oplus x_0x_5 \oplus x_0x_5x_6 \oplus x_0x_5x_6x_7 \oplus x_0x_5x_6x_7x_8 \oplus x_0x_5x_6x_7x_8x_9 \oplus$$
$$x_0x_5x_6x_8x_9 \oplus x_0x_5x_6x_9 \oplus x_0x_5x_7x_8 \oplus x_0x_5x_7x_9 \oplus x_0x_5x_8 \oplus x_0x_5x_8x_9 \oplus x_0x_6x_7 \oplus x_0x_6x_7x_8x_9 \oplus x_0x_6x_7x_9 \oplus$$
$$x_0x_6x_8 \oplus x_0x_6x_9 \oplus x_0x_7 \oplus x_0x_7x_8 \oplus x_0x_7x_8x_9 \oplus x_0x_8x_9 \oplus x_0x_9 \oplus x_1x_2 \oplus x_1x_2x_3x_4 \oplus x_1x_2x_3x_4x_5x_6 \oplus$$
$$x_1x_2x_3x_4x_5x_6x_7x_8 \oplus x_1x_2x_3x_4x_5x_6x_7x_9 \oplus x_1x_2x_3x_4x_5x_6x_8 \oplus x_1x_2x_3x_4x_5x_6x_8x_9 \oplus x_1x_2x_3x_4x_5x_7 \oplus x_1x_2x_3x_4x_5x_7x_8x_9 \oplus$$
$$x_1x_2x_3x_4x_5x_7x_9 \oplus x_1x_2x_3x_4x_5x_8 \oplus x_1x_2x_3x_4x_5x_9 \oplus x_1x_2x_3x_4x_6 \oplus x_1x_2x_3x_4x_6x_7 \oplus x_1x_2x_3x_4x_6x_7x_8 \oplus x_1x_2x_3x_4x_6x_7x_8x_9 \oplus$$
$$x_1x_2x_3x_4x_6x_8x_9 \oplus x_1x_2x_3x_4x_6x_9 \oplus x_1x_2x_3x_4x_7x_8 \oplus x_1x_2x_3x_4x_7x_9 \oplus x_1x_2x_3x_4x_8 \oplus x_1x_2x_3x_4x_8x_9 \oplus x_1x_2x_3x_5 \oplus$$
$$x_1x_2x_3x_5x_6x_7 \oplus x_1x_2x_3x_5x_6x_7x_8x_9 \oplus x_1x_2x_3x_5x_6x_7x_9 \oplus x_1x_2x_3x_5x_6x_8 \oplus x_1x_2x_3x_5x_6x_9 \oplus x_1x_2x_3x_5x_7 \oplus$$
$$x_1x_2x_3x_5x_7x_8 \oplus x_1x_2x_3x_5x_7x_8x_9 \oplus x_1x_2x_3x_5x_8x_9 \oplus x_1x_2x_3x_5x_9 \oplus x_1x_2x_3x_6 \oplus x_1x_2x_3x_6x_7x_8 \oplus x_1x_2x_3x_6x_7x_9 \oplus$$
$$x_1x_2x_3x_6x_8 \oplus x_1x_2x_3x_6x_8x_9 \oplus x_1x_2x_3x_7 \oplus x_1x_2x_3x_7x_8x_9 \oplus x_1x_2x_3x_7x_9 \oplus x_1x_2x_3x_8 \oplus x_1x_2x_3x_9 \oplus x_1x_2x_4 \oplus$$
$$x_1x_2x_4x_5 \oplus x_1x_2x_4x_5x_6 \oplus x_1x_2x_4x_5x_6x_7 \oplus x_1x_2x_4x_5x_6x_7x_8 \oplus x_1x_2x_4x_5x_6x_7x_8x_9 \oplus x_1x_2x_4x_5x_6x_8x_9 \oplus x_1x_2x_4x_5x_6x_9 \oplus$$
$$x_1x_2x_4x_5x_7x_8 \oplus x_1x_2x_4x_5x_7x_9 \oplus x_1x_2x_4x_5x_8 \oplus x_1x_2x_4x_5x_8x_9 \oplus x_1x_2x_4x_6x_7 \oplus x_1x_2x_4x_6x_7x_8x_9 \oplus x_1x_2x_4x_6x_7x_9 \oplus$$
$$x_1x_2x_4x_6x_8 \oplus x_1x_2x_4x_6x_9 \oplus x_1x_2x_4x_7 \oplus x_1x_2x_4x_7x_8 \oplus x_1x_2x_4x_7x_8x_9 \oplus x_1x_2x_4x_8x_9 \oplus x_1x_2x_4x_9 \oplus x_1x_2x_5x_6 \oplus$$
$$x_1x_2x_5x_6x_7x_8 \oplus x_1x_2x_5x_6x_7x_9 \oplus x_1x_2x_5x_6x_8 \oplus x_1x_2x_5x_6x_8x_9 \oplus x_1x_2x_5x_7 \oplus x_1x_2x_5x_7x_8x_9 \oplus x_1x_2x_5x_7x_9 \oplus$$
$$x_1x_2x_5x_8 \oplus x_1x_2x_5x_9 \oplus x_1x_2x_6 \oplus x_1x_2x_6x_7 \oplus x_1x_2x_6x_7x_8 \oplus x_1x_2x_6x_7x_8x_9 \oplus x_1x_2x_6x_8x_9 \oplus x_1x_2x_6x_9 \oplus x_1x_2x_7x_8 \oplus$$
$$x_1x_2x_7x_9 \oplus x_1x_2x_8 \oplus x_1x_2x_8x_9 \oplus x_1x_3 \oplus x_1x_3x_4x_5 \oplus x_1x_3x_4x_5x_6x_7 \oplus x_1x_3x_4x_5x_6x_7x_8x_9 \oplus x_1x_3x_4x_5x_6x_7x_9 \oplus$$
$$x_1x_3x_4x_5x_6x_8 \oplus x_1x_3x_4x_5x_6x_9 \oplus x_1x_3x_4x_5x_7 \oplus x_1x_3x_4x_5x_7x_8 \oplus x_1x_3x_4x_5x_7x_8x_9 \oplus x_1x_3x_4x_5x_8x_9 \oplus x_1x_3x_4x_5x_9 \oplus$$
$$x_1x_3x_4x_6 \oplus x_1x_3x_4x_6x_7x_8 \oplus x_1x_3x_4x_6x_7x_9 \oplus x_1x_3x_4x_6x_8 \oplus x_1x_3x_4x_6x_8x_9 \oplus x_1x_3x_4x_7 \oplus x_1x_3x_4x_7x_8x_9 \oplus$$
$$x_1x_3x_4x_7x_9 \oplus x_1x_3x_4x_8 \oplus x_1x_3x_4x_9 \oplus x_1x_3x_5 \oplus x_1x_3x_5x_6 \oplus x_1x_3x_5x_6x_7 \oplus x_1x_3x_5x_6x_7x_8 \oplus x_1x_3x_5x_6x_7x_8x_9 \oplus$$
$$x_1x_3x_5x_6x_8x_9 \oplus x_1x_3x_5x_6x_9 \oplus x_1x_3x_5x_7x_8 \oplus x_1x_3x_5x_7x_9 \oplus x_1x_3x_5x_8 \oplus x_1x_3x_5x_8x_9 \oplus x_1x_3x_6x_7 \oplus x_1x_3x_6x_7x_8x_9 \oplus$$
$$x_1x_3x_6x_7x_9 \oplus x_1x_3x_6x_8 \oplus x_1x_3x_6x_9 \oplus x_1x_3x_7 \oplus x_1x_3x_7x_8 \oplus x_1x_3x_7x_8x_9 \oplus x_1x_3x_8x_9 \oplus x_1x_3x_9 \oplus x_1x_4 \oplus$$
$$x_1x_4x_5x_6 \oplus x_1x_4x_5x_6x_7x_8 \oplus x_1x_4x_5x_6x_7x_9 \oplus x_1x_4x_5x_6x_8 \oplus x_1x_4x_5x_6x_8x_9 \oplus x_1x_4x_5x_7 \oplus x_1x_4x_5x_7x_8x_9 \oplus$$
$$x_1x_4x_5x_7x_9 \oplus x_1x_4x_5x_8 \oplus x_1x_4x_5x_9 \oplus x_1x_4x_6 \oplus x_1x_4x_6x_7 \oplus x_1x_4x_6x_7x_8 \oplus x_1x_4x_6x_7x_8x_9 \oplus x_1x_4x_6x_8x_9 \oplus$$
$$x_1x_4x_6x_9 \oplus x_1x_4x_7x_8 \oplus x_1x_4x_7x_9 \oplus x_1x_4x_8 \oplus x_1x_4x_8x_9 \oplus x_1x_5 \oplus x_1x_5x_6x_7 \oplus x_1x_5x_6x_7x_8x_9 \oplus x_1x_5x_6x_7x_9 \oplus$$
$$x_1x_5x_6x_8 \oplus x_1x_5x_6x_9 \oplus x_1x_5x_7 \oplus x_1x_5x_7x_8 \oplus x_1x_5x_7x_8x_9 \oplus x_1x_5x_8x_9 \oplus x_1x_5x_9 \oplus x_1x_6 \oplus x_1x_6x_7x_8 \oplus x_1x_6x_7x_9 \oplus$$
$$x_1x_6x_8 \oplus x_1x_6x_8x_9 \oplus x_1x_7 \oplus x_1x_7x_8x_9 \oplus x_1x_7x_9 \oplus x_1x_8 \oplus x_1x_9 \oplus x_2 \oplus x_2x_3 \oplus x_2x_3x_4 \oplus x_2x_3x_4x_5 \oplus x_2x_3x_4x_5x_6 \oplus$$
$$x_2x_3x_4x_5x_6x_7 \oplus x_2x_3x_4x_5x_6x_7x_8 \oplus x_2x_3x_4x_5x_6x_7x_8x_9 \oplus x_2x_3x_4x_5x_6x_8x_9 \oplus x_2x_3x_4x_5x_6x_9 \oplus x_2x_3x_4x_5x_7x_8 \oplus$$
$$x_2x_3x_4x_5x_7x_9 \oplus x_2x_3x_4x_5x_8 \oplus x_2x_3x_4x_5x_8x_9 \oplus x_2x_3x_4x_6x_7 \oplus x_2x_3x_4x_6x_7x_8x_9 \oplus x_2x_3x_4x_6x_7x_9 \oplus x_2x_3x_4x_6x_8 \oplus$$
$$x_2x_3x_4x_6x_9 \oplus x_2x_3x_4x_7 \oplus x_2x_3x_4x_7x_8 \oplus x_2x_3x_4x_7x_8x_9 \oplus x_2x_3x_4x_8x_9 \oplus x_2x_3x_4x_9 \oplus x_2x_3x_5x_6 \oplus x_2x_3x_5x_6x_7x_8 \oplus$$
$$x_2x_3x_5x_6x_7x_9 \oplus x_2x_3x_5x_6x_8 \oplus x_2x_3x_5x_6x_8x_9 \oplus x_2x_3x_5x_7 \oplus x_2x_3x_5x_7x_8x_9 \oplus x_2x_3x_5x_7x_9 \oplus x_2x_3x_5x_8 \oplus x_2x_3x_5x_9 \oplus$$
$$x_2x_3x_6 \oplus x_2x_3x_6x_7 \oplus x_2x_3x_6x_7x_8 \oplus x_2x_3x_6x_7x_8x_9 \oplus x_2x_3x_6x_8x_9 \oplus x_2x_3x_6x_9 \oplus x_2x_3x_7x_8 \oplus x_2x_3x_7x_9 \oplus$$
$$x_2x_3x_8 \oplus x_2x_3x_8x_9 \oplus x_2x_4x_5 \oplus x_2x_4x_5x_6x_7 \oplus x_2x_4x_5x_6x_7x_8x_9 \oplus x_2x_4x_5x_6x_7x_9 \oplus x_2x_4x_5x_6x_8 \oplus x_2x_4x_5x_6x_9 \oplus$$
$$x_2x_4x_5x_7 \oplus x_2x_4x_5x_7x_8 \oplus x_2x_4x_5x_7x_8x_9 \oplus x_2x_4x_5x_8x_9 \oplus x_2x_4x_5x_9 \oplus x_2x_4x_6 \oplus x_2x_4x_6x_7x_8 \oplus x_2x_4x_6x_7x_9 \oplus$$
$$x_2x_4x_6x_8 \oplus x_2x_4x_6x_8x_9 \oplus x_2x_4x_7 \oplus x_2x_4x_7x_8x_9 \oplus x_2x_4x_7x_9 \oplus x_2x_4x_8 \oplus x_2x_4x_9 \oplus x_2x_5 \oplus x_2x_5x_6 \oplus x_2x_5x_6x_7 \oplus$$
$$x_2x_5x_6x_7x_8 \oplus x_2x_5x_6x_7x_8x_9 \oplus x_2x_5x_6x_8x_9 \oplus x_2x_5x_6x_9 \oplus x_2x_5x_7x_8 \oplus x_2x_5x_7x_9 \oplus x_2x_5x_8 \oplus x_2x_5x_8x_9 \oplus x_2x_6x_7 \oplus$$
$$x_2x_6x_7x_8x_9 \oplus x_2x_6x_7x_9 \oplus x_2x_6x_8 \oplus x_2x_6x_9 \oplus x_2x_7 \oplus x_2x_7x_8 \oplus x_2x_7x_8x_9 \oplus x_2x_8x_9 \oplus x_2x_9 \oplus x_3x_4 \oplus x_3x_4x_5x_6 \oplus$$
$$x_3x_4x_5x_6x_7x_8 \oplus x_3x_4x_5x_6x_7x_9 \oplus x_3x_4x_5x_6x_8 \oplus x_3x_4x_5x_6x_8x_9 \oplus x_3x_4x_5x_7 \oplus x_3x_4x_5x_7x_8x_9 \oplus x_3x_4x_5x_7x_9 \oplus$$
$$x_3x_4x_5x_8 \oplus x_3x_4x_5x_9 \oplus x_3x_4x_6 \oplus x_3x_4x_6x_7 \oplus x_3x_4x_6x_7x_8 \oplus x_3x_4x_6x_7x_8x_9 \oplus x_3x_4x_6x_8x_9 \oplus x_3x_4x_6x_9 \oplus x_3x_4x_7x_8 \oplus$$
$$x_3x_4x_7x_9 \oplus x_3x_4x_8 \oplus x_3x_4x_8x_9 \oplus x_3x_5 \oplus x_3x_5x_6x_7 \oplus x_3x_5x_6x_7x_8x_9 \oplus x_3x_5x_6x_7x_9 \oplus x_3x_5x_6x_8 \oplus x_3x_5x_6x_9 \oplus$$
$$x_3x_5x_7 \oplus x_3x_5x_7x_8 \oplus x_3x_5x_7x_8x_9 \oplus x_3x_5x_8x_9 \oplus x_3x_5x_9 \oplus x_3x_6 \oplus x_3x_6x_7x_8 \oplus x_3x_6x_7x_9 \oplus x_3x_6x_8 \oplus x_3x_6x_8x_9 \oplus$$
$$x_3x_7 \oplus x_3x_7x_8x_9 \oplus x_3x_7x_9 \oplus x_3x_8 \oplus x_3x_9 \oplus x_4 \oplus x_4x_5 \oplus x_4x_5x_6 \oplus x_4x_5x_6x_7 \oplus x_4x_5x_6x_7x_8 \oplus x_4x_5x_6x_7x_8x_9 \oplus$$
$$x_4x_5x_6x_8x_9 \oplus x_4x_5x_6x_9 \oplus x_4x_5x_7x_8 \oplus x_4x_5x_7x_9 \oplus x_4x_5x_8 \oplus x_4x_5x_8x_9 \oplus x_4x_6x_7 \oplus x_4x_6x_7x_8x_9 \oplus x_4x_6x_7x_9 \oplus$$
$$x_4x_6x_8 \oplus x_4x_6x_9 \oplus x_4x_7 \oplus x_4x_7x_8 \oplus x_4x_7x_8x_9 \oplus x_4x_8x_9 \oplus x_4x_9 \oplus x_5x_6 \oplus x_5x_6x_7x_8 \oplus x_5x_6x_7x_9 \oplus x_5x_6x_8 \oplus$$
$$x_5x_6x_8x_9 \oplus x_5x_7 \oplus x_5x_7x_8x_9 \oplus x_5x_7x_9 \oplus x_5x_8 \oplus x_5x_9 \oplus x_6 \oplus x_6x_7 \oplus x_6x_7x_8 \oplus x_6x_7x_8x_9 \oplus x_6x_8x_9 \oplus x_6x_9 \oplus$$
$$x_7x_8 \oplus x_7x_9 \oplus x_8 \oplus x_8x_9 = 0$$

Author Contributions.

Competing Interests.

Materials & Correspondence.

References

[1] Alastair A. Abbott. The Deutsch–Jozsa problem: de-quantisation and entanglement. *Natural Computing*, 11, 2012.

[2] Yakir Aharonov and Lev Vaidman. *The Two-State Vector Formalism: An Updated Review*, pages 399–447. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.

[3] Tatsuya Akutsu, Morihiro Hayashida, Shu-Qin Zhang, Wai-Ki Ching, and Michael K Ng. Analyses and algorithms for predecessor and control problems for boolean networks of bounded indegree. *Information and Media Technologies*, 4(2):338–349, 2009.

[4] Roberto Baldoni, Emilio Coppa, Daniele Cono D'elia, Camil Demetrescu, and Irene Finocchi. A survey of symbolic execution techniques. *ACM Comput. Surv.*, 51(3), may 2018.

[5] Stephen M. Barnett, John Jeffers, and David T. Pegg. Quantum retrodiction: Foundations and controversies. *Symmetry*, 13(4), 2021.

[6] Robert S. Boyer, Bernard Elspas, and Karl N. Levitt. Select—a formal system for testing and debugging programs by symbolic execution. *SIGPLAN Not.*, 10(6):234–245, apr 1975.

[7] Linda Burnett, William Millan, Edward Dawson, and Andrew Clark. Simpler methods for generating better boolean functions with good cryptographic properties. *Australasian Journal of Combinatorics*, 29:231–247, 2004.

[8] Lori A. Clarke. A program testing system. In *Proceedings of the 1976 Annual Conference*, ACM '76, page 488–491, New York, NY, USA, 1976. Association for Computing Machinery.

[9] Yoshihiko Futamura. Partial computation of programs. In Eiichi Goto, Koichi Furukawa, Reiji Nakajima, Ikuo Nakata, and Akinori Yonezawa, editors, *RIMS Symposia on Software Science and Engineering*, pages 1–35, Berlin, Heidelberg, 1983. Springer Berlin Heidelberg.

[10] Peter Henderson and James H. Morris. A lazy evaluator. In *Proceedings of the 3rd ACM SIGACT-SIGPLAN Symposium on Principles on Programming Languages*, POPL '76, page 95–103, New York, NY, USA, 1976. Association for Computing Machinery.

[11] William E. Howden. Experiments with a symbolic evaluation system. In *Proceedings of the National Computer Conference*, 1976.

[12] James C. King. Symbolic execution and program testing. *Commun. ACM*, 19(7):385–394, jul 1976.

[13] Johannes Georg Klotz, Martin Bossert, and Steffen Schober. Computing preimages of boolean networks. *BMC Bioinformatics*, 14(10):S4, Aug 2013.

[14] J.T.-Y. Kwok and I.W.-H. Tsang. The pre-image problem in kernel methods. *IEEE Transactions on Neural Networks*, 15(6):1517–1525, 2004.

[15] Phillip Rogaway and Thomas Shrimpton. Cryptographic hash-function basics: Definitions, implications, and separations for preimage resistance, second-preimage resistance, and collision resistance. In Bimal Roy and Willi Meier, editors, *Fast Software Encryption*, pages 371–388, Berlin, Heidelberg, 2004. Springer Berlin Heidelberg.

[16] Natalia Tokareva. Chapter 1 - Boolean functions. In Natalia Tokareva, editor, *Bent Functions*, pages 1–15. Academic Press, Boston, 2015.

[17] Vlatko Vedral, Adriano Barenco, and Artur Ekert. Quantum networks for elementary arithmetic operations. *Phys. Rev. A*, 54:147–153, Jul 1996.

[18] Satosi Watanabe. Symmetry of physical laws. Part III. prediction and retrodiction. *Rev. Mod. Phys.*, 27:179–186, Apr 1955.