AME Qudit Graph States

Absolutely Maximally Entangled (AME) Qudit Graph States

AME states are multipartite entangled states that are maximally entangled for any possible bipartition. These states can be used as a resource for various QI tasks, such as quantum secret sharing and parallel teleportation.

Definition : An $AME(n,d)$ state is a pure state $|\Phi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ and $\mathcal{H}_j \cong \mathbb{C}^d$, such that :

a) $|\Phi\rangle$ is maximally entangled for any possible bipartition

For any bipartition of $P = \{1, 2, \ldots, n\}$ into disjoint sets A and B / $A \cup B = P$ (and $m = |B| \geq |A| = n-m$) :

$$|\Phi\rangle = \frac{1}{\sqrt{d^m}} \sum_{k \in \mathbb{Z}_d^m} |k_1\rangle_{B_1} \cdots |k_m\rangle_{B_m} |\phi(k)\rangle_A$$

with $\langle \phi(k) | \phi(k') \rangle = \delta_{kk'}$

b) The ρ_A with $|A| \leq \frac{n}{2}$ is totally mixed

c) The von Neumann entropy of every A with $|A| \leq \frac{n}{2}$ is maximal, $S(A) = |A| \log d$

To check if an state is $AME(n,d)$ it suffices to check the maximal entanglement for all bipartitions with $|A| = \lfloor \frac{n}{2} \rfloor$

We want to use the graph state formalism to describe $AME(n,d)$ states

Qudit Graph States

Graph states are a special class of stabilizer states. Their graphical representation helps in visualizing entanglement. Helwig introduced 2 methods for checking bipartite entanglement in graph states. Given a graph state it is straightforward to write down a quantum circuit generating it. It consists of $(\wedge Z)$ gates

1) Generalized Pauli Operators: $Z, X \in$ unitary

$$Z\,|k\rangle = \omega^k\,|k\rangle \quad \text{and} \quad X\,|k\rangle = |k+1\rangle$$

with $\omega = e^{i\frac{2\pi}{d}}$, $k = 0, 1, \dots, d-1$, $\omega^d = 1$

$(Z = U$ and $X = V^\dagger)$ / $ZX = \omega XZ$ and $Z^d = X^d = \mathbb{1}$

$$(\wedge Z)_{ij} = \sum_{k=0}^{d-1} |k\rangle_i\langle k| \otimes Z_j^k = \sum_{k,l=0}^{d-1} \omega^{kl}\, |k\rangle_i\langle k| \otimes |l\rangle_j\langle l|$$

The Fourier gate is $F = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{kl}\,|k\rangle\langle l|$ is the generalization of the Hadamard gate for qubits. It relates $|k\rangle$ to the eigenvectors of X, $|\tilde{k}\rangle$ /

$$|\tilde{k}\rangle = F^\dagger|k\rangle = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \omega^{-kl}\,|l\rangle$$

2) Graph States: Consider a set of vertices:

$V = \{v_i\}$ connected by edges $E = \{e_{ij} = \{v_i, v_j\}\}$

Each edge carries a weight $A_{ij} \in \mathbb{Z}_d$

We consider prime dimensions $d=p$ next. The weight A_{ij} form a symmetric $n\times n$ adjacency matrix with $A_{ii}=0$.

<u>Definition</u>: For a given graph G with $|V|=n$ vertices and adjacency matrix $A\in \mathbb{Z}_p^{n\times n}$, p prime, we define the corresponding graph state $|G\rangle \in \mathcal{H}_G \cong (\mathbb{C}^p)^{\otimes n}$ as:

$$|G\rangle = \prod_{i>j} (\wedge Z)_{ij}^{A_{ij}} \, |\tilde{0}\rangle^{\otimes n} \quad ; \quad (\wedge Z)^{d=p} = \mathbb{1}$$

A labeled graph state is defined as:

$$|G_z\rangle = Z^z |G\rangle \text{ with } Z^z = Z^{z_1}\otimes Z^{z_2}\otimes \dots \otimes Z^{z_n}$$

with $z=(z_1,z_2,\dots,z_n)\in \mathbb{Z}_p^n$

A graph state can be constructed by a quantum circuit with initial state $|\tilde{0}\rangle^{\otimes n}$ and then applies $(\wedge Z)_{ij}$ between i,j according to the entries of the adjacency matrix A.

3) Stabilizer States: The generalized Pauli Group is

$$\mathcal{G}_1 = \{\omega^c X^a Z^b \; ; \; a,b,c \in \mathbb{Z}_p\}$$

For n qudits $\mathcal{G}_n = \underbrace{\mathcal{G}_1 \otimes \dots \otimes \mathcal{G}_1}_{n\text{-times}}$

The stabilizer subgroup $S\subset \mathcal{G}_n$ is an abelian group with generator $g_j = \omega^{c_j} X^{\vec{a}_j} Z^{\vec{b}_j}$; $S=\langle g_1 \dots g_k\rangle$

The generator matrix is defined as:

$$M = \left(\begin{array}{c|c} \vec{a}_1 & \vec{b}_1 \\ \vec{a}_2 & \vec{b}_2 \\ \vdots & \vdots \\ \vec{a}_k & \vec{b}_k \end{array} \right)$$

The stabilizer code does not depend on ω^{c_j}, and is fully specified by M. Since S is abelian $\Rightarrow \vec{a}_i \cdot \vec{b}_j = \vec{b}_i \cdot \vec{a}_j$

If the number of generators $k = n \Rightarrow$ the dimension of the code is $2^{n-k} = 2^{n-n} = 1 \Rightarrow \exists\ 1$ stabilizer state.

Graph states are a special class of stabilizer states with generators:

$$g_i = X_i \prod_j Z_j^{A_{ij}}$$

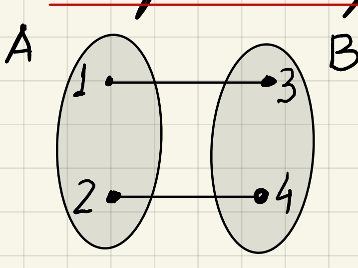
Thus, the generator matrix is: $M = (\mathbb{1} | A)$

Every stabilizer state is equivalent to a graph state under the action of the local Clifford group (generalized for qudits)

Entanglement in Graph States

Given a graph state what is the entanglement associated to $|G\rangle$? Helwig presents 2 ≠ methods for checking entanglement between bipartitions.

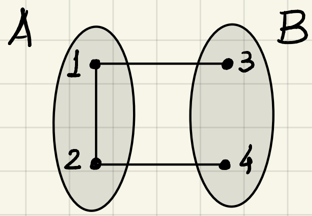
1) Graphical representation:

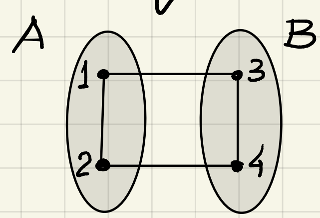


Maximal amount of entanglement between the 2 set A and B is $\min(|A|, |B|)$ "edits"

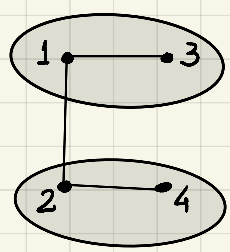
In this example $\min(2,2) = 2$ edits

Applying $(\wedge Z)_{ij}$ up to $p-1$ times also creates 1 edit of entanglement, thus we may assign any non-zero weight to the connecting edges without changing the maximal entanglement. Moreover, applying local unitaries within each set, after the entanglement between the 2 sets has been created, does not change the amount of entanglement. Thus, we may add as many edges with arbitrary weight as we like within each set and the sets A and B will remain maximally entangled:

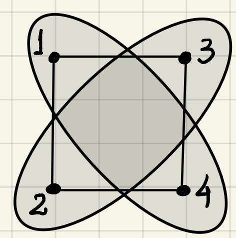




Changing bipartition:



1 edit



(a)

Not obvious

<u>Theorem</u>: 2 graph states are $\equiv$ under local Clifford operations iff one graph can be obtained from the other by a sequence of the 2 graph operations on a vertex v:

$\circ_b v$: The weight of each edge connected to v is multiplied by $b \neq 0 \in \mathbb{Z}_p$

$*_a v$: The entries of A are transformed as: $A_{jk} \to A_{jk} + a A_{vj} A_{vk}$

for $j \neq k$ and $a \in \mathbb{Z}_p$.

Going back to figure (a), applying $(*_1 1, *_1 3, *_1 4)$ the graph is equivalent to:

$*_1 1$:

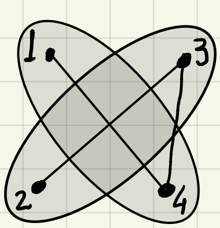
$$A_{12} \to A_{12} + A_{11} A_{12} = A_{12} = 1$$
$$A_{13} \to A_{13} + A_{11} A_{13} = A_{13} = 1$$
$$A_{14} \to A_{14} + A_{11} A_{14} = A_{14} = 0$$
$$A_{23} \to \overset{=0}{A_{23}} + A_{12} A_{13} = 1$$
$$A_{24} \to A_{24} + A_{12} A_{14} = A_{24} = 1$$
$$A_{34} \to A_{34} + A_{13} A_{14} = A_{34} = 1$$

$*_1 3$:

$$A_{12} \to A_{12} + A_{31} A_{32} = 2$$
$$A_{13} \to A_{13} + A_{31} A_{33} = A_{13} = 1$$
$$A_{14} \to A_{14} + A_{31} A_{34} = \overset{=0}{A_{14}} + A_{13} A_{34} = 1$$
$$A_{23} \to A_{23} + A_{32} A_{33} = A_{23} = 1$$
$$A_{24} \to A_{24} + A_{32} A_{34} = 2$$
$$A_{34} \to A_{34} + A_{33} A_{34} = A_{34} = 1$$

$*_1 4$:

$$A_{12} \to A_{12} + A_{41} A_{42} = 4 \cong 0$$
$$A_{13} \to A_{13} + A_{41} A_{43} = 2 \cong 0$$
$$A_{14} \to A_{14} + A_{41} A_{44} = A_{14} = 1$$
$$A_{23} \to A_{23} + A_{42} A_{43} = 3 \cong 1$$
$$A_{24} \to A_{24} + A_{42} A_{44} = A_{24} = 2 \cong 0$$
$$A_{34} \to A_{34} + A_{43} A_{44} = A_{34} = 1$$



1 edit of entanglement (It's not AME)

2) Efficient Method:

Definition: We define the truncated graph state $|G^K\rangle$, shared by $P\backslash K$, as the state corresponding to the graph G with the vertices in K (and its edges, including connecting ones) removed.

Given the $n\times n$ adjacency matrix A, the i^{th} row is denoted as:

$$A_i = (A_{i1}, A_{i2}, \dots, A_{in})$$

If $K = \{k_1, k_2, \dots, k_m\}$ ($k_j \in [1,n]$), we define the vector

$$A_i^K = A_i - \{A_{ik_1}, \dots, A_{ik_m}\}$$

For instance: $A_i^{\{2,6\}} = (A_{i1}, A_{i3}, A_{i4}, A_{i5}, A_{i7}, A_{i8}, \dots, A_{in})$

Note that a Z-projection of the k^{th} qudit of the graph state

$$|G\rangle = \prod_{i>j} (\wedge Z)_{ij}^{A_{ij}} |\tilde{0}\rangle^{\otimes n}$$

$$|G\rangle = \prod_{l\neq k} \sum_{m=0}^{P-1} |m\rangle_k\langle m| \otimes Z_l^{mA_{kl}} \prod_{\substack{i>j \\ i,j\neq k}} (\wedge Z)_{ij}^{A_{ij}} |\tilde{0}\rangle^{\otimes n}$$

with eigenvalue ω^a is:

$$_k\langle a|G\rangle = \frac{1}{\sqrt{P}} \prod_{l\neq k} Z_l^{aA_{kl}} \prod_{\substack{i>j \\ i,j\neq k}} (\wedge Z)_{ij}^{A_{ij}} |\tilde{0}\rangle^{\otimes(n-1)}$$

$$= \frac{1}{\sqrt{P}} |G^{\{k\}}_{aA_k^{\{k\}}}\rangle$$

Similarly, Z-projections of m qudits $K = \{k_1, \dots, k_m\}$

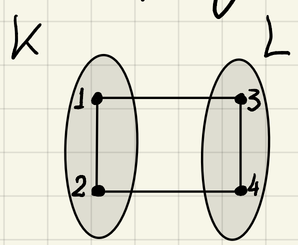
with eigenvalues $\omega^{a_1}, \omega^{a_2}, \dots, \omega^{a_m}$:

$$ {}_K\langle a_1 \dots a_m | G\rangle = \frac{1}{\sqrt{p^m}} \left| G^K_{\sum_i a_i A^K_{k_i}} \right\rangle $$

$A^K_{k_i}$ is $A_{k_i} = (A_{k_i 1}, \dots, A_{k_i n})$ minus $\{A_{k_i k_1}, \dots, A_{k_i k_m}\}$

Theorem: A graph state is AME iff $\forall$ sets $K = \{k_1, \dots, k_m\}$ of size $m = \lfloor \frac{n}{2} \rfloor$, the vectors $A^K_{k_i}$ are linearly independent in $\mathbb{Z}_p^{n-m}$;

Let's apply this thm. to



$K = \{1,2\}$ and $L = \{3,4\}$

$A^K_1 = (A_{13}, A_{14}) = (1,0)$; $A^K_2 = (A_{23}, A_{24}) = (0,1)$

These are independent $\Rightarrow$ max entanglement between K and L

For the bipartion $K = \{1,3\}$ and $L = \{2,4\}$ we also get max entang.

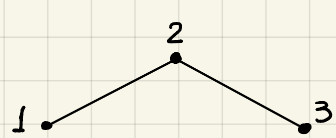
How about $K = \{1,4\}$ and $L = \{2,3\}$? In this case,

$A^K_1 = (A_{12}, A_{13}) = (1,1)$ and $A^K_4 = (A_{42}, A_{43}) = (1,1)$. These are linearly dependent vectors $\Rightarrow$ we do not have max entang. for this bipartition.

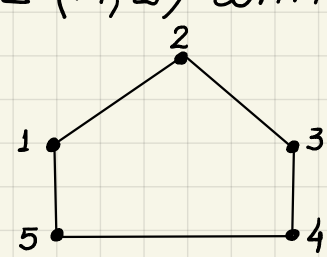
AME Graph States

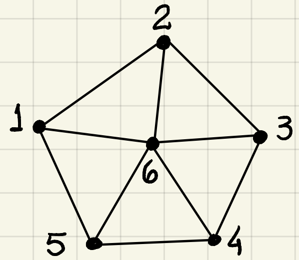
For $d=2$, we know it exist $AME(n,2)$ with $n = 2,3,5,6$

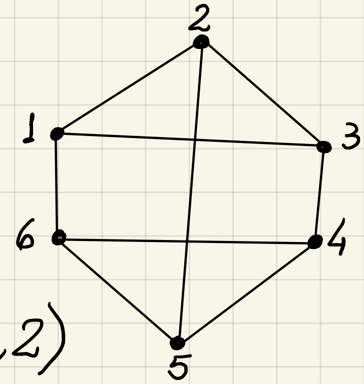
EPR



GHZ





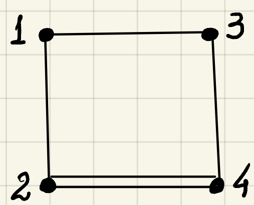


These are 2 locally $\equiv$ AME(6,2)

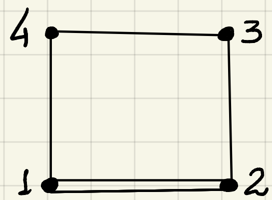
AME(4,2) does not exist, and AME($n \geqslant 8$, 2) do not exist either.

Consider now the case of $n=4$ qudits with $d \geqslant 3$. To have AME graph states we need to consider weights larger than 1.

Consider AME(4,3):



$$\xrightarrow{(*_1 1, *_1 3, *_1 2)}$$



max entang. for $\{1,2\}/\{3,4\}$ and $\{1,3\}/\{2,4\}$

max entang. for $\{1,4\}/\{2,3\}$

$*_1 1$: $A_{12} \to 1$; $A_{13} \to 1$; $A_{14} \to 0$; $A_{23} \to 1$; $A_{24} \to 2$; $A_{34} \to 1$

$*_1 3$: $A_{12} \to 2$; $A_{13} \to 1$; $A_{14} \to 1$; $A_{23} \to 1$; $A_{24} \to 3$; $A_{34} \to 1$

$*_1 2$:
$$A_{12} \to A_{12} + A_{21} A_{22} = A_{12} = 2$$
$$A_{13} \to A_{13} + A_{21} A_{23} = 3 \cong 0$$
$$A_{14} \to A_{14} + A_{21} A_{24} = 7 \cong 1$$
$$A_{23} \to A_{23} + A_{22} A_{23} = A_{23} = 1$$
$$A_{24} \to A_{24} + A_{22} A_{24} = A_{24} = 3 \cong 0$$
$$A_{34} \to A_{34} + A_{23} A_{24} = 4 \cong 1$$

Using the other efficient method: Consider $K = \{1,4\}$

$$A_1^K = (A_{12}, A_{13}) = (1,1) \text{ and } A_4^K = (A_{42}, A_{43}) = (2,1)$$

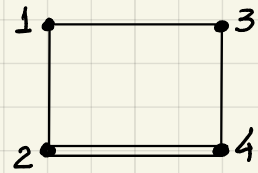
and $\{(1,1),(2,1)\}$ are linearly independent in $\mathbb{Z}_3$

Are $\{(1,1),(2,1)\}$ linearly independent in $\mathbb{Z}_6$? $\Rightarrow$

$$r_1\,(1,1) + r_2\,(2,1) = 0 \quad \text{with } r_1, r_2 \in \mathbb{Z}_6$$

$\Rightarrow \begin{cases} r_1 + 2r_2 = 0 \\ r_1 + r_2 = 0 \end{cases}$. The only solution in $\mathbb{Z}_6$ is $r_1 = 0 = r_2$

Therefore, they are also linearly independent in $\mathbb{Z}_6$. Then, with respect to the bipartition $K=\{1,4\}$, $L=\{2,3\}$



is maximally entangled. How about $K=\{1,2\}$, $L=\{3,4\}$?

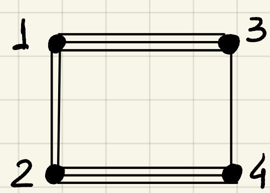
$A_1^K = (A_{13}, A_{14}) = (1,0)$; $A_2^K = (A_{23}, A_{24}) = (0,2)$, also l.i

and $K=\{1,3\}$, $L=(2,4)$?

$A_1^K = (A_{12}, A_{14}) = (1,0)$; $A_3^K = (A_{32}, A_{34}) = (0,1)$, also l.i.

That implies that we have found an AME(4,6)? Not so fast. The methods we derived to check for entanglement work for $d=p$, i.e., prime dimensions , and 6 is not prime.

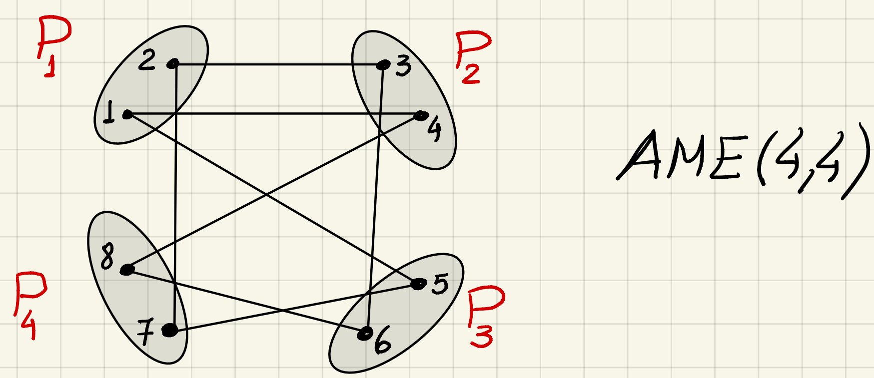
Many times a graph for an AME(n,p) is also a graph for an AME$(n, p' \geq p)$ but not always. For instance :



corresponds to AME(4,5) but not AME(4,7) !

1) Non-prime dimensions d: In those cases we consider the prime factorization $d = P_1 P_2 \cdots P_m$ and look for AME states for $P_1, \ldots, P_m$ independently, and if an AME state exists for each of the prime factors, then we can just construct an AME state for d by taking the tensor product of the m AME states and assigning one qudit of each AME state to each of the parties.
This means we cannot construct an AME(4,6) in this way since AME(4,2) does not exist.

If 2 or more prime factors are the same, e.g. $d = 4 = 2\times 2$, one can apply ($\wedge Z$) operations between qubits that may lead to an AME. For instance, for AME(4,4):



This state is max entang. with 4 ebits (2 edits) of entanglement for the bipartitions $\{P_1 P_2\}/\{P_3 P_4\}$; $\{P_1 P_3\}/\{P_2 P_4\}$ and $\{P_1, P_4\}/\{P_2, P_3\}$.

Question: If an AME(n,d) exists, can one always find an AME(n,d) graph state?

2) AME graph states from classical codes: