Representing, Manipulating and Optimizing Reversible Circuits

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Reversible Computing

The "obvious" intersection between quantum computing and programming languages is reversible computing.

Representing Reversible Circuits

truth table, matrix, reed muller expansion, product of cycles, decision diagram, etc.

[any easy way to reproduce Figure 4 on p.7 of Saeedi and Markov? —JC]

[important remark: these are all *Boolean* circuits! —JC]

Most important part: reversible circuits are equivalent to permutations.

A (Foundational) Syntactic Theory

Ideally, want a notation that

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2. is easy to mechanically manipulate
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Start with a *foundational* syntactic theory on our way there:

1. easy to explain
2. clear operational rules
3. fully justified by the semantics
4. sound and complete reasoning
5. sound and complete methods of optimization

Starting Point

Typed isomorphisms. First, a universe of (finite) types

$$
\begin{array}{l}
\mathsf{data}\ \mathsf{U} : \mathsf{Set}\ \mathsf{where} \\
\quad \mathsf{ZERO} \ : \mathsf{U} \\
\quad \mathsf{ONE} \ \ \ : \mathsf{U} \\
\quad \mathsf{PLUS} \ \ : \mathsf{U} \to \mathsf{U} \to \mathsf{U} \\
\quad \mathsf{TIMES} : \mathsf{U} \to \mathsf{U} \to \mathsf{U}
\end{array}
$$

and its interpretation

$$
\begin{array}{ll}
[\![\_]\!] : \mathsf{U} \to \mathsf{Set} & \\
[\![\ \mathsf{ZERO}\ ]\!] & = \bot \\
[\![\ \mathsf{ONE}\ ]\!] & = \top \\
[\![\ \mathsf{PLUS}\ t_1\ t_2\ ]\!] & = [\![\ t_1\ ]\!] \uplus [\![\ t_2\ ]\!] \\
[\![\ \mathsf{TIMES}\ t_1\ t_2\ ]\!] & = [\![\ t_1\ ]\!] \times [\![\ t_2\ ]\!]
\end{array}
$$

Equivalences and semirings

If we denote type equivalence by $\simeq$, then we can prove that

Theorem 1.

The collection of all types (Set) forms a commutative semiring (up to $\simeq$).

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We also get

Theorem 2.

If $A \simeq \mathsf{Fin}\, m$, $B \simeq \mathsf{Fin}\, n$ and $A \simeq B$ then $m \equiv n$.

(whose *constructive* proof is quite subtle).

Theorem 3.

If $A \simeq \mathsf{Fin}\, m$ and $B \simeq \mathsf{Fin}\, n$, then the type of all equivalences $A \simeq B$ is equivalent to the type of all permutations $\mathsf{Perm}\, n$.

Equivalences and semirings II

Semiring structures abound. We can define them on:

1. equivalences (disjoint union and cartesian product)
2. permutations (disjoint union and tensor product)

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The point, of course, is that they are related:

Theorem 4.

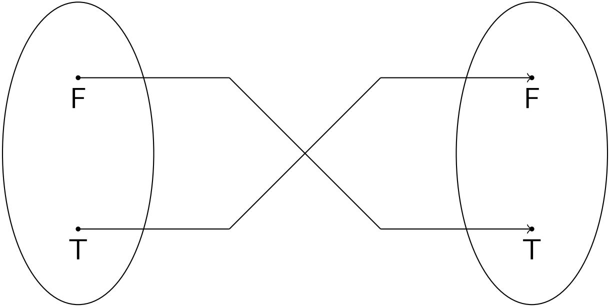
The equivalence of Theorem 3 is an isomorphism between the semirings of equivalences of finite types, and of permutations.

A Calculus of Permutations

First conclusion: it might be useful to *reify* a certain set of equivalences as combinators. We choose the fundamental "proof rules" of semirings:

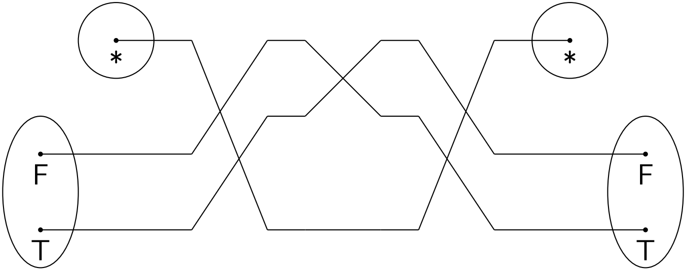
```
data _⟷_ : U → U → Set where
  unite₊  : {t : U} → PLUS ZERO t ⟷ t
  uniti₊  : {t : U} → t ⟷ PLUS ZERO t
  swap₊   : {t₁ t₂ : U} → PLUS t₁ t₂ ⟷ PLUS t₂ t₁
  assocl₊ : {t₁ t₂ t₃ : U} → PLUS t₁ (PLUS t₂ t₃) ⟷ PLUS (PLUS t₁ t₂) t₃
  assocr₊ : {t₁ t₂ t₃ : U} → PLUS (PLUS t₁ t₂) t₃ ⟷ PLUS t₁ (PLUS t₂ t₃)
  unite⋆  : {t : U} → TIMES ONE t ⟷ t
  uniti⋆  : {t : U} → t ⟷ TIMES ONE t
  swap⋆   : {t₁ t₂ : U} → TIMES t₁ t₂ ⟷ TIMES t₂ t₁
  assocl⋆ : {t₁ t₂ t₃ : U} → TIMES t₁ (TIMES t₂ t₃) ⟷ TIMES (TIMES t₁ t₂) t₃
  assocr⋆ : {t₁ t₂ t₃ : U} → TIMES (TIMES t₁ t₂) t₃ ⟷ TIMES t₁ (TIMES t₂ t₃)
  absorbr  : {t : U} → TIMES ZERO t ⟷ ZERO
  absorbl : {t : U} → TIMES t ZERO ⟷ ZERO
  factorzr : {t : U} → ZERO ⟷ TIMES t ZERO
  factorzl : {t : U} → ZERO ⟷ TIMES ZERO t
  dist     : {t₁ t₂ t₃ : U} →
    TIMES (PLUS t₁ t₂) t₃ ⟷ PLUS (TIMES t₁ t₃) (TIMES t₂ t₃)
  factor   : {t₁ t₂ t₃ : U} →
    PLUS (TIMES t₁ t₃) (TIMES t₂ t₃) ⟷ TIMES (PLUS t₁ t₂) t₃
  id⟷     : {t : U} → t ⟷ t
  _⊙_     : {t₁ t₂ t₃ : U} → (t₁ ⟷ t₂) → (t₂ ⟷ t₃) → (t₁ ⟷ t₃)
  _⊕_     : {t₁ t₂ t₃ t₄ : U} →
    (t₁ ⟷ t₃) → (t₂ ⟷ t₄) → (PLUS t₁ t₂ ⟷ PLUS t₃ t₄)
  _⊗_     : {t₁ t₂ t₃ t₄ : U} →
    (t₁ ⟷ t₃) → (t₂ ⟷ t₄) → (TIMES t₁ t₂ ⟷ TIMES t₃ t₄)
```

Example Circuit: Simple Negation



$$n_1 : \mathsf{BOOL} \longleftrightarrow \mathsf{BOOL}$$
$$n_1 = \mathsf{swap}_+$$

Example Circuit: Not So Simple Negation



$$
\begin{aligned}
n_2 &: \mathsf{BOOL} \longleftrightarrow \mathsf{BOOL} \\
n_2 &= \; \mathsf{uniti}\star \odot \\
&\quad\; \mathsf{swap}\star \odot \\
&\quad\; (\mathsf{swap}_+ \otimes \mathsf{id}\!\longleftrightarrow) \odot \\
&\quad\; \mathsf{swap}\star \odot \\
&\quad\; \mathsf{unite}\star
\end{aligned}
$$

Reasoning about Example Circuits

Algebraic manipulation of one circuit to the other:

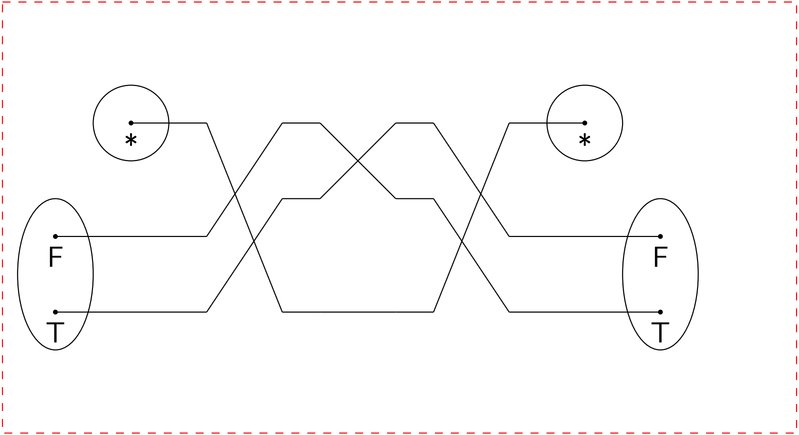
$$
\begin{array}{l}
\mathsf{negEx} : \mathsf{n}_2 \Leftrightarrow \mathsf{n}_1 \\
\mathsf{negEx} = \mathsf{uniti}\star \odot (\mathsf{swap}\star \odot ((\mathsf{swap}_+ \otimes \mathsf{id}\longleftrightarrow) \odot (\mathsf{swap}\star \odot \mathsf{unite}\star))) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot \mathsf{assoc}\odot\mathsf{l}\ \rangle \\
\quad \mathsf{uniti}\star \odot ((\mathsf{swap}\star \odot (\mathsf{swap}_+ \otimes \mathsf{id}\longleftrightarrow)) \odot (\mathsf{swap}\star \odot \mathsf{unite}\star)) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot (\mathsf{swapl}\star\Leftrightarrow \boxdot \mathsf{id}\Leftrightarrow)\ \rangle \\
\quad \mathsf{uniti}\star \odot (((\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+) \odot \mathsf{swap}\star) \odot (\mathsf{swap}\star \odot \mathsf{unite}\star)) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot \mathsf{assoc}\odot\mathsf{r}\ \rangle \\
\quad \mathsf{uniti}\star \odot ((\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+) \odot (\mathsf{swap}\star \odot (\mathsf{swap}\star \odot \mathsf{unite}\star))) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot (\mathsf{id}\Leftrightarrow \boxdot \mathsf{assoc}\odot\mathsf{l})\ \rangle \\
\quad \mathsf{uniti}\star \odot ((\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+) \odot ((\mathsf{swap}\star \odot \mathsf{swap}\star) \odot \mathsf{unite}\star)) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot (\mathsf{id}\Leftrightarrow \boxdot (\mathsf{linv}\odot\mathsf{l} \boxdot \mathsf{id}\Leftrightarrow))\ \rangle \\
\quad \mathsf{uniti}\star \odot ((\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+) \odot (\mathsf{id}\longleftrightarrow \odot \mathsf{unite}\star)) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot (\mathsf{id}\Leftrightarrow \boxdot \mathsf{idl}\odot\mathsf{l})\ \rangle \\
\quad \mathsf{uniti}\star \odot ((\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+) \odot \mathsf{unite}\star) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{assoc}\odot\mathsf{l}\ \rangle \\
\quad (\mathsf{uniti}\star \odot (\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+)) \odot \mathsf{unite}\star \\
\quad\quad \Leftrightarrow\langle\ \mathsf{unitil}\star\Leftrightarrow \boxdot \mathsf{id}\Leftrightarrow\ \rangle \\
\quad (\mathsf{swap}_+ \odot \mathsf{uniti}\star) \odot \mathsf{unite}\star
\end{array}
$$

Reasoning about Example Circuits

foo

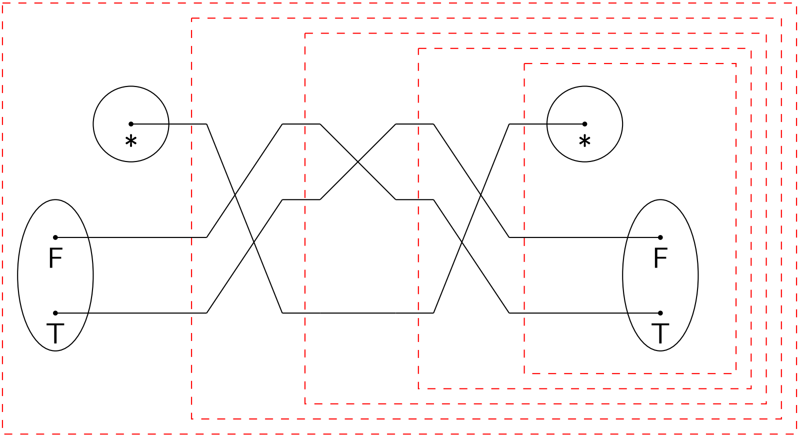
Visually

Original circuit:



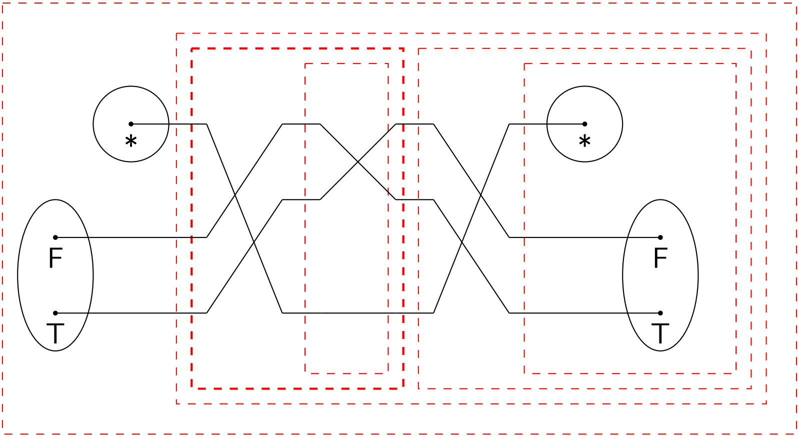
Visually

Making grouping explicit:



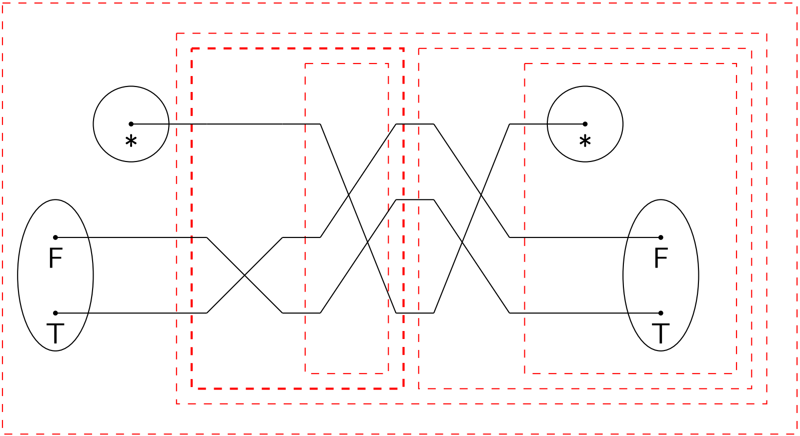
Visually

By associativity:



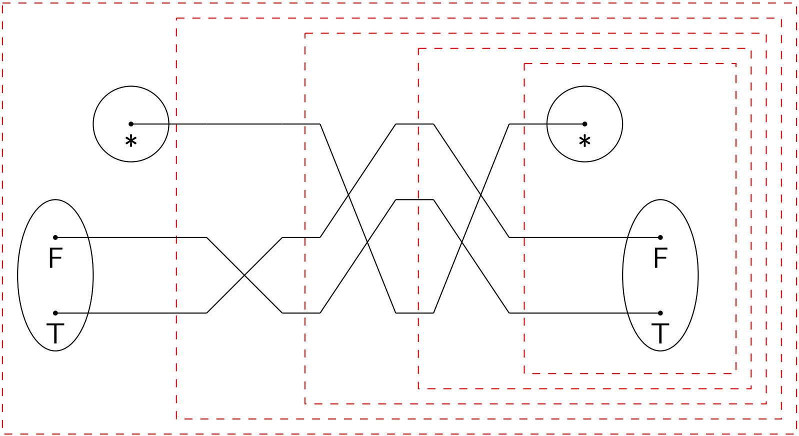
Visually

By pre-post-swap:



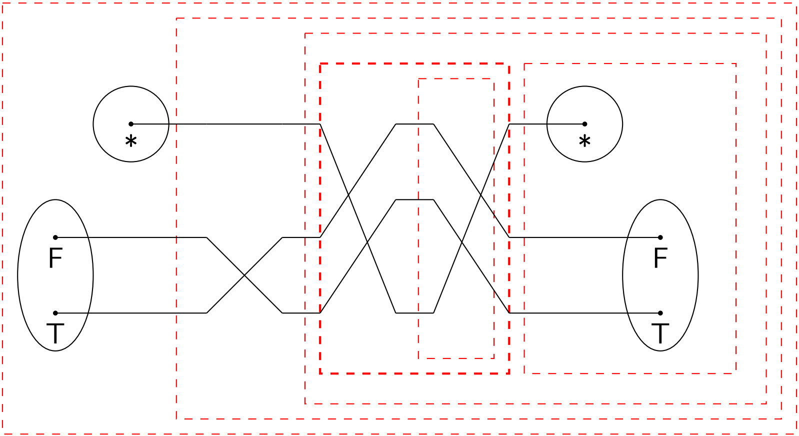
Visually

By associativity:



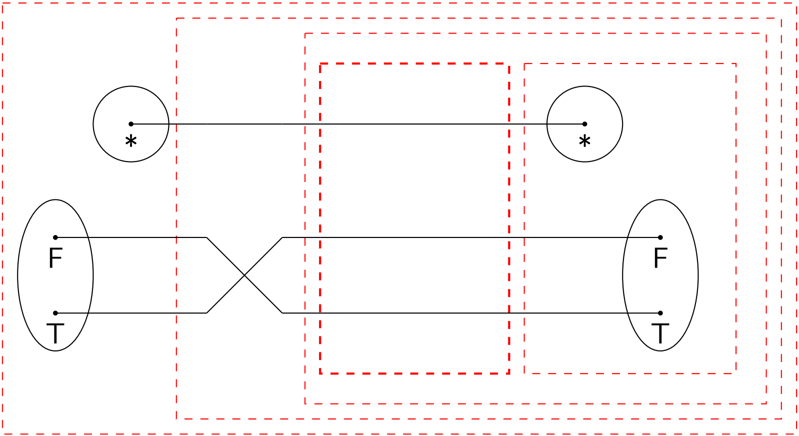
Visually

By associativity:



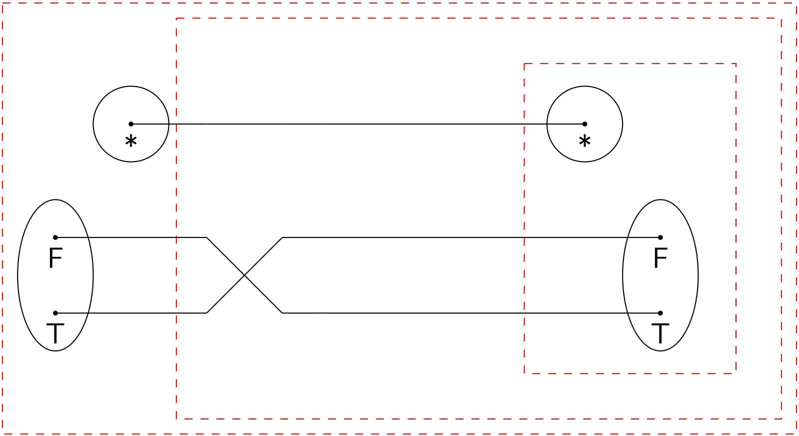
Visually

By swap-swap:



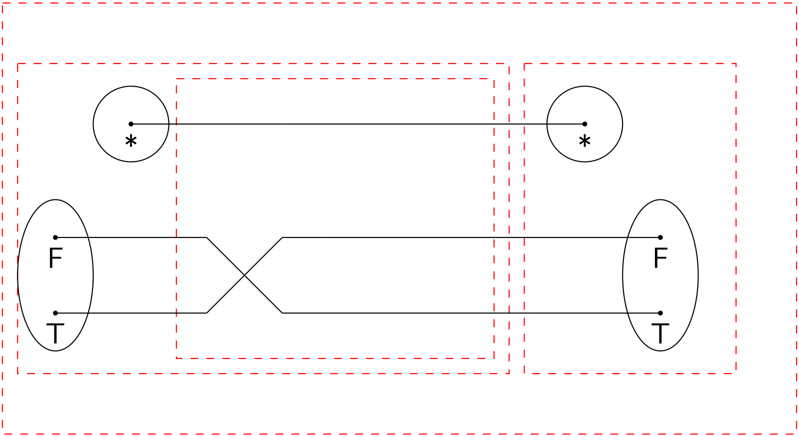
Visually

By id-compose-left:



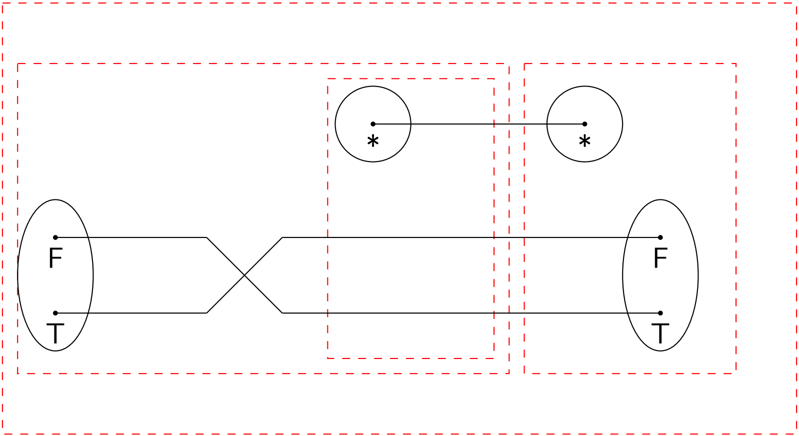
Visually

By associativity:



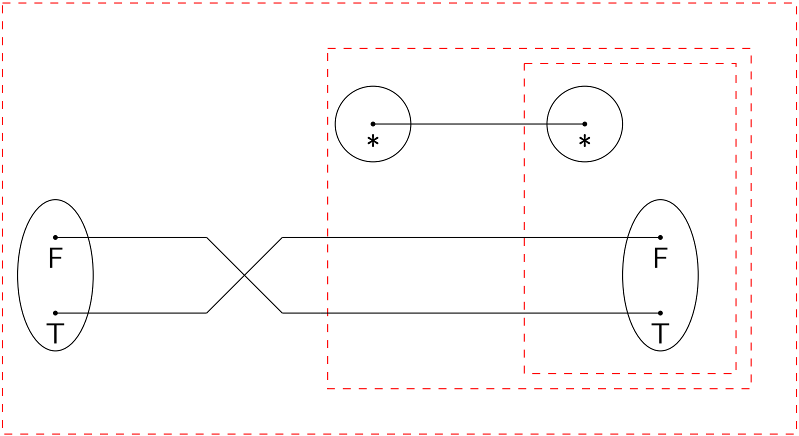
Visually

By swap-unit:



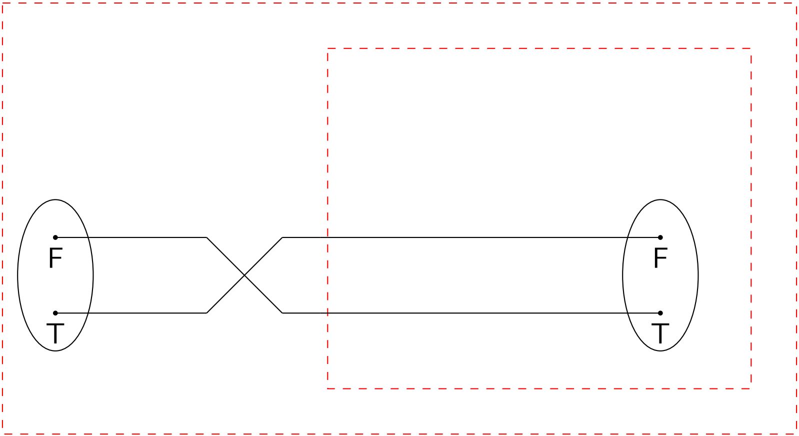
Visually

By associativity:



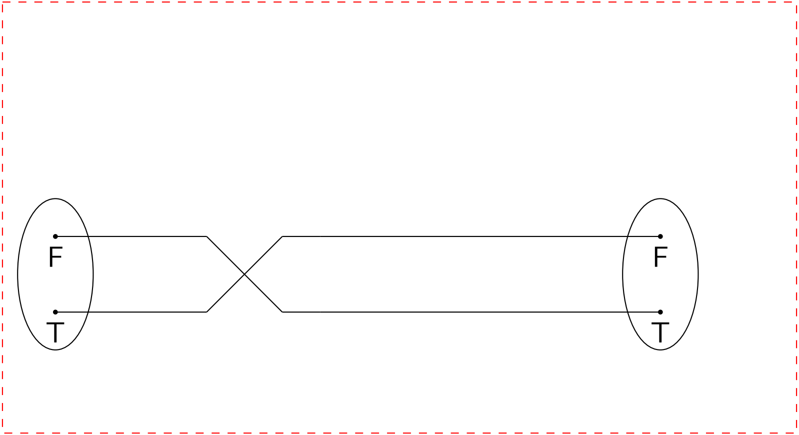
Visually

By unit-unit:



Visually

By id-unit-right:



Questions

- We don't want an ad hoc notation with ad hoc rewriting rules
- Notions of soundness; completeness; canonicity in some sense; what can we say?

1-paths vs. 2-paths

1-paths are between isomorphic types, e.g., $A * B$ and $B * A$. List them all.

1-paths vs. 2-paths

2-paths are between 1-paths, e.g.,
%endcode

1-paths vs. 2-paths