Representing, Manipulating and Optimizing Reversible Circuits

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Abstract

We show how a typed set of combinators for reversible computations, corresponding exactly to the semiring of permutations, is a convenient basis for representing and manipulating reversible circuits. A categorical interpretation also leads to optimization combinators, and we demonstrate their utility through an example.

1. Introduction

Amr says: Define and motivate that we are interested in defining HoTT equivalences of types, characterizing them, computing with them, etc.

Quantum Computing. Quantum physics differs from classical physics in many ways:

- Superpositions
- Entanglement
- Unitary evolution
- Composition uses tensor products
- Non-unitary measurement

Quantum Computing & Programming Languages.

- It is possible to adapt all at once classical programming languages to quantum programming languages.
- Some excellent examples discussed in this workshop
- This assumes that classical programming languages (and implicitly classical physics) can be smoothly adapted to the quantum world.
- There are however what appear to be fundamental differences between the classical and quantum world that make them incompatible
- Let us *re-think* classical programming foundations before jumping to the quantum world.

Resource-Aware Classical Computing.

- The biggest questionable assumption of classical programming is that it is possible to freely copy and discard information
- A classical programming language which respects no-cloning and no-discarding is the right foundation for an eventual quantum extension
- We want these properties to be inherent in the language; not an afterthought filtered by a type system
- We want to program with isomorphisms or equivalences
- The simplest instance is permutations between finite types which happens to correspond to reversible circuits.

Representing Reversible Circuits: truth table, matrix, reed muller expansion, product of cycles, decision diagram, etc.

any easy way to reproduce Figure 4 on p.7 of Saeedi and Markov? important remark: these are all *Boolean* circuits! Most important part: reversible circuits are equivalent to permutations.

A (Foundational) Syntactic Theory. Ideally, want a notation that

1. is easy to write by programmers
2. is easy to mechanically manipulate
3. can be reasoned about
4. can be optimized.

Start with a *foundational* syntactic theory on our way there:

1. easy to explain
2. clear operational rules
3. fully justified by the semantics
4. sound and complete reasoning
5. sound and complete methods of optimization

A Syntactic Theory. Ideally want a notation that is easy to write by programmers and that is easy to mechanically manipulate for reasoning and optimizing of circuits.

Syntactic calculi good. Popular semantics: Despite the increasing importance of formal methods to the computing industry, there has been little advance to the notion of a "popular semantics" that can be explained to *and used* effectively (for example to optimize or simplify programs) by non-specialists including programmers and first-year students. Although the issue is by no means settled, syntactic theories are one of the candidates for such a popular semantics for they require no additional background beyond knowledge of the programming language itself, and they provide a direct support for the equational reasoning underlying many program transformations.

The primary abstraction in HoTT is 'type equivalences.' If we care about resource preservation, then we are concerned with 'type equivalences'.

2. Equivalences and Commutative Semirings

Our starting point is the notion of HoTT equivalence of types. We then connect this notion to several semiring structures on finite types and on permutations with the goal of reducing the notion of finite type equivalence to a calculus of permutations.

2.1 HoTT Equivalences of Types

There are several equivalent definitions of the notion of equivalence of types. For concreteness, we use the following definition as it appears to be the most intuitive in our setting.

Definition 1 (Equivalence of types)**.** *Two types A and B are equivalent $A \simeq B$ if there exists a* bi-invertible *$f : A \to B$, i.e., if there exists an f that has both a left-inverse and a right-inverse. A function $f : A \to$ has a left-inverse if there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$. A function $f : A \to$ has a right-inverse if there exists a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$.*

Note that the function g used for the left-inverse may be different from the function g used for the right-inverse.

As the definition of equivalence is parameterized by a function f, we are concerned with, not just the fact that two types are equivalent, but with the precise way in which they are equivalent. For example, there are two equivalences between the type Bool and itself: one that uses the identity for f (and hence for g) and one uses boolean negation for f (and hence for g). These two equivalences are themselves *not* equivalent: each of them can be used to "transport" properties of Bool in a different way.

2.2 Instance I: Universe of Types

The first commutative semiring instance we examine is the universe of types (Set in Agda terminology). The additive unit is the empty type $\bot$; the multiplicative unit is the unit type $\top$; the two binary operations are disjoint union $\uplus$ and cartesian product $\times$. The axioms are satisfied up to equivalence of types $\simeq$. For example, we have equivalences such as:

$$
\begin{array}{rcl}
\bot \uplus A & \simeq & A \\
\top \times A & \simeq & A \\
A \times (B \times C) & \simeq & (A \times B) \times C \\
A \times \bot & \simeq & \bot \\
A \times (B \uplus C) & \simeq & (A \times B) \uplus (A \times C)
\end{array}
$$

Formally we have the following fact.

Theorem 1. *The collection of all types (Set) forms a commutative semiring (up to $\simeq$).*

2.3 Instance II: Finite Sets

The collection of all finite sets ($\mathsf{Fin}\ m$ for natural number m in Agda terminology) is another commutative semiring instance. In this case, the additive unit is $\mathsf{Fin}\ 0$, the multiplicative unit is $\mathsf{Fin}\ 1$, the two binary operations are still disjoint union $\uplus$ and cartesian product $\times$, and the axioms are also satisfied up to equivalence of types $\simeq$.

The reason finite sets are interesting is that each finite type A constructed from $\bot$, $\top$, $\uplus$, and $\times$ is equivalent (in $|A|$! ways) to $\mathsf{Fin}\ |A|$ where $|A|$ is the size of A defined as follows:

$$
\begin{array}{rcl}
|\bot| & = & 0 \\
|\top| & = & 1 \\
|A \uplus B| & = & |A| + |B| \\
|A \times B| & = & |A| * |B|
\end{array}
$$

Each of the $|A|$! equivalences of A with $\mathsf{Fin}\ |A|$ corresponds to a *particular* enumeration of the elements of A. For example, we have two equivalences:

$$\top \uplus \top \quad \simeq \quad \mathsf{Fin}\ 2$$

corresponding to the identity and boolean negation.

Thus, as we prove next, up to equivalence, the only interesting property of a finite type is its size. In other words, given two equivalent types A and B of completely different structure, e.g., $A = (\top \uplus \top) \times (\top \uplus (\top \uplus \top))$ and $B = \top \uplus (\top \uplus (\top \uplus (\top \uplus (\top \uplus (\top \uplus \bot)))))$, we can find equivalences from either type to the finite set $\mathsf{Fin}\ 6$ and use the latter for further reasoning. Indeed, as the next section demonstrate, this result allows us to characterize equivalences between finite types in a canonical way as permutations between finite sets.

The following theorem precisely characterizes the relationship between finite types and finite sets.

Theorem 2. *If $A \simeq \mathsf{Fin}\ m$, $B \simeq \mathsf{Fin}\ n$ and $A \simeq B$ then $m = n$.*

Proof. We proceed by cases on the possible values for m and n. If they are different, we quickly get a contradiction. If they are both 0 we are done. The interesting situation is when $m = suc\ m'$ and $n = suc\ n'$. The result follows in this case by induction assuming we can establish that the equivalence between A and B, i.e., the equivalence between $\mathsf{Fin}\ (suc\ m')$ and $\mathsf{Fin}\ (suc\ n')$, implies an equivalence between $\mathsf{Fin}\ m'$ and $\mathsf{Fin}\ n'$. In our setting, we actually need to construct a particular equivalence between the smaller sets given the equivalence of the larger sets with one additional element. This lemma is quite tedious as it requires us to isolate one element of $\mathsf{Fin}\ (suc\ m')$ and analyze every position this element could be mapped to by the larger equivalence and in each case construct an equivalence that excludes this element. □

2.4 Permutations on Finite Sets

Given the correspondence between finite types and finite sets, we will prove that equivalences on finite types are equivalent to permutations on finite sets. Formalizing the notion of permutations is delicate however: straightforward attempts turn out not to capture enough of the properties of permutations for our purposes. We therefore formalize a permutation using two sizes: m for the size of the input finite set and n for the size of the resulting finite set. Naturally in any well-formed permutations, these two sizes are equal but the presence of both types allows us to conveniently define permutations as follows. A permutation $\mathsf{CPerm}\ m\ n$ consists of four components. The first two components are:

- a vector of size n containing elements drawn from the finite set $\mathsf{Fin}\ m$;
- a dual vector of size m containing elements drawn from the finite set $\mathsf{Fin}\ n$;

Each of the above vectors is viewed as a map f that acts on the incoming finite set sending the element at index i to position $f!!i$ in the resulting finite set. To guarantee that these maps define an actual permutation, the last two components are proofs that the sequential composition of the maps in both direction produce the identity.

2.5 Equivalences of Equivalences

The main result of this section is that the type of type equivalences is equivalent to the type of permutations.

Type of All Equivalences between Finite Types.

Type of All Permutations between Finite Sets.

Theorem 3. *If $A \simeq \mathsf{Fin}\ m$ and $B \simeq \mathsf{Fin}\ n$, then the type of all equivalences $A \simeq B$ is equivalent to the type of all permutations* $\mathsf{Perm}\ n$.

In fact we have the following stronger theorem.

Theorem 4. *The equivalence of Theorem 3 is an* isomorphism *between the semirings of equivalences of finite types, and of permutations.*

A more evocative phrasing might be:

Theorem 5.

$$(A \simeq B) \simeq \mathsf{Perm}|A|$$

Amr says:

- types are a commutative semiring
- type equivalences are a commutative semiring
- permutations on finite sets are another commutative semiring
- these two structures are themselves equivalent

SO if we are interested in studying type equivalences, we can study permutations on finite sets; the latter can be axiomatized which is nice

3. A Calculus of Permutations

A Calculus of Permutations. Syntactic theories only rely on transforming source programs to other programs, much like algebraic calculation. Since only the *syntax* of the programming language is relevant to the syntactic theory, the theory is accessible to non-specialists like programmers or students.

In more detail, it is a general problem that, despite its fundamental value, formal semantics of programming languages is generally inaccessible to the computing public. As Schmidt argues in a recent position statement on strategic directions for research on programming languages [**?**]:

> . . . formal semantics has fed upon increasing complexity of concepts and notation at the expense of calculational clarity. A newcomer to the area is expected to specialize in one or more of domain theory, intuitionistic type theory, category theory, linear logic, process algebra, continuation-passing style, or whatever. These specializations have generated more experts but fewer general users.

Typed Isomorphisms
First, a universe of (finite) types

```
data U : Set where
  ZERO  : U
  ONE   : U
  PLUS  : U → U → U
  TIMES : U → U → U
```

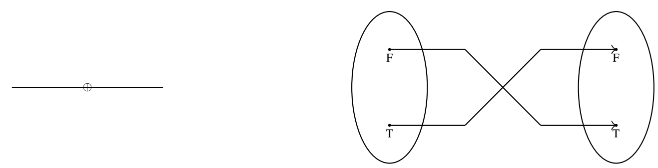
and its interpretation

```
⟦_⟧ : U → Set
⟦ ZERO ⟧          = ⊥
⟦ ONE ⟧           = ⊤
⟦ PLUS t₁ t₂ ⟧    = ⟦ t₁ ⟧ ⊎ ⟦ t₂ ⟧
⟦ TIMES t₁ t₂ ⟧ = ⟦ t₁ ⟧ × ⟦ t₂ ⟧
```

A Calculus of Permutations. First conclusion: it might be useful to *reify* a (sound and complete) set of equivalences as combinators, such as the fundamental “proof rules” of semirings:

```
data _⟷_ : U → U → Set where
  unite₊   : {t : U} → PLUS ZERO t ⟷ t
  uniti₊   : {t : U} → t ⟷ PLUS ZERO t
  swap₊    : {t₁ t₂ : U} → PLUS t₁ t₂ ⟷ PLUS t₂ t₁
  assocl₊ : {t₁ t₂ t₃ : U} → PLUS t₁ (PLUS t₂ t₃) ⟷ PLUS (PLUS t₁ t₂) t₃
  assocr₊ : {t₁ t₂ t₃ : U} → PLUS (PLUS t₁ t₂) t₃ ⟷ PLUS t₁ (PLUS t₂ t₃)
  unite⋆   : {t : U} → TIMES ONE t ⟷ t
  uniti⋆   : {t : U} → t ⟷ TIMES ONE t
  swap⋆    : {t₁ t₂ : U} → TIMES t₁ t₂ ⟷ TIMES t₂ t₁
  assocl⋆ : {t₁ t₂ t₃ : U} → TIMES t₁ (TIMES t₂ t₃) ⟷ TIMES (TIMES t₁ t₂) t₃
  assocr⋆ : {t₁ t₂ t₃ : U} → TIMES (TIMES t₁ t₂) t₃ ⟷ TIMES t₁ (TIMES t₂
  absorbr   : {t : U} → TIMES ZERO t ⟷ ZERO
  absorbl : {t : U} → TIMES t ZERO ⟷ ZERO
  factorzr : {t : U} → ZERO ⟷ TIMES t ZERO
  factorzl : {t : U} → ZERO ⟷ TIMES ZERO t
  dist      : {t₁ t₂ t₃ : U} → TIMES (PLUS t₁ t₂) t₃ ⟷ PLUS (TIMES t₁ t₃)
  factor    : {t₁ t₂ t₃ : U} → PLUS (TIMES t₁ t₃) (TIMES t₂ t₃) ⟷ TIMES (
  id⟷      : {t : U} → t ⟷ t
  _⊙_       : {t₁ t₂ t₃ : U}    → (t₁ ⟷ t₂) → (t₂ ⟷ t₃) → (t₁ ⟷ t₃)
  _⊕_       : {t₁ t₂ t₃ t₄ : U} → (t₁ ⟷ t₃) → (t₂ ⟷ t₄) → (PLUS t₁ t₂ ⟷ P
  _⊗_       : {t₁ t₂ t₃ t₄ : U} → (t₁ ⟷ t₃) → (t₂ ⟷ t₄) → (TIMES t₁ t₂ ⟷
```

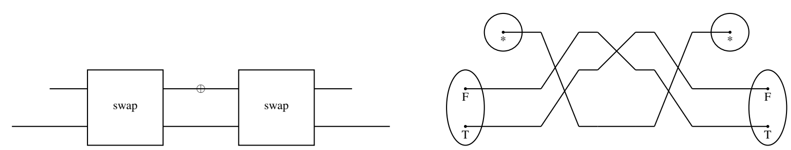
4. Example Circuit: Simple Negation




```
BOOL : U
BOOL = PLUS ONE ONE

n₁ : BOOL ⟷ BOOL
n₁ = swap₊
```

Example Circuit: Not So Simple Negation.

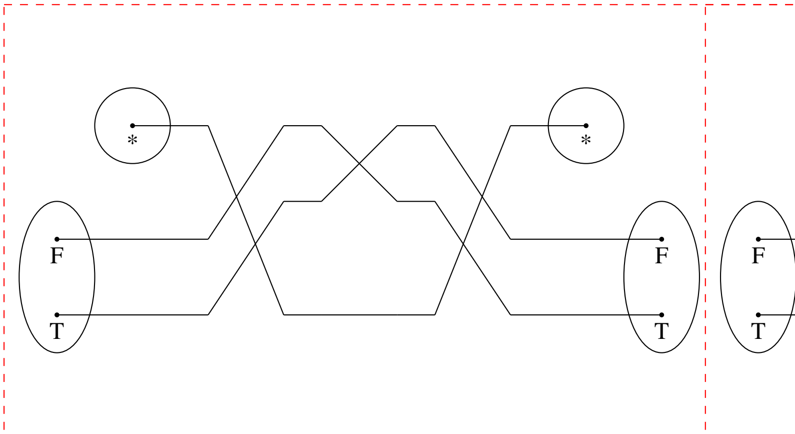



```
n₂ : BOOL ⟷ BOOL
n₂ =   uniti⋆ ⊙
       swap⋆ ⊙
       (swap₊ ⊗ id⟷) ⊙
       swap⋆ ⊙
       unite⋆
```

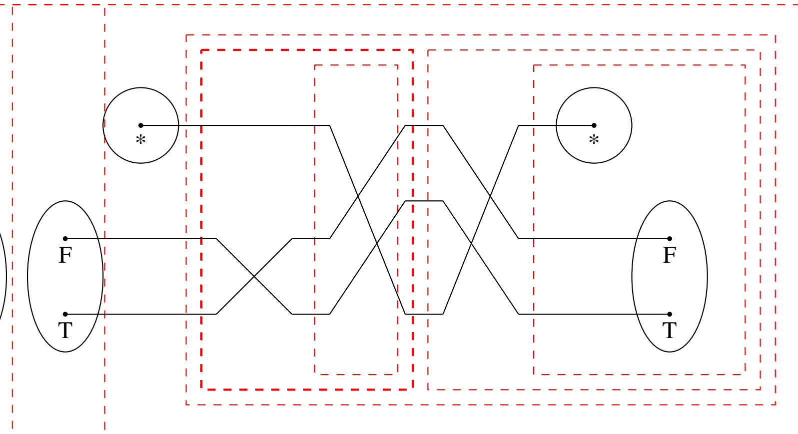
Reasoning about Example Circuits. Algebraic manipulation of one circuit to the other:

```
negEx : n₂ ⇔ n₁
negEx = uniti⋆ ⊙ (swap⋆ ⊙ ((swap₊ ⊗ id⟷) ⊙ (swap⋆ ⊙ unite⋆)))
      ⇔⟨ id⇔ ⊡ assoc⊙l ⟩
  uniti⋆ ⊙ ((swap⋆ ⊙ (swap₊ ⊗ id⟷)) ⊙ (swap⋆ ⊙ unite⋆))
      ⇔⟨ id⇔ ⊡ (swapl⋆⇔ ⊡ id⇔) ⟩
  uniti⋆ ⊙ (((id⟷ ⊗ swap₊) ⊙ swap⋆) ⊙ (swap⋆ ⊙ unite⋆))
      ⇔⟨ id⇔ ⊡ assoc⊙r ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ (swap⋆ ⊙ (swap⋆ ⊙ unite⋆)))
      ⇔⟨ id⇔ ⊡ (id⇔ ⊡ assoc⊙l) ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ ((swap⋆ ⊙ swap⋆) ⊙ unite⋆))
      ⇔⟨ id⇔ ⊡ (id⇔ ⊡ (linv⊙l ⊡ id⇔)) ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ (id⟷ ⊙ unite⋆))
      ⇔⟨ id⇔ ⊡ (id⇔ ⊡ idl⊙l) ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ unite⋆)
      ⇔⟨ assoc⊙l ⟩
  (uniti⋆ ⊙ (id⟷ ⊗ swap₊)) ⊙ unite⋆
      ⇔⟨ unitil⋆⇔ ⊡ id⇔ ⟩
  (swap₊ ⊙ uniti⋆) ⊙ unite⋆
      ⇔⟨ assoc⊙r ⟩
  swap₊ ⊙ (uniti⋆ ⊙ unite⋆)
      ⇔⟨ id⇔ ⊡ linv⊙l ⟩
  swap₊ ⊙ id⟷
      ⇔⟨ idr⊙l ⟩
  swap₊ □
```

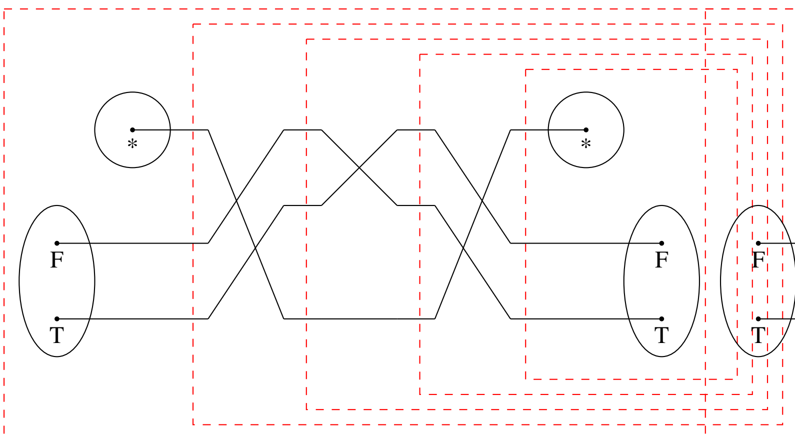
Visually.
Original circuit:



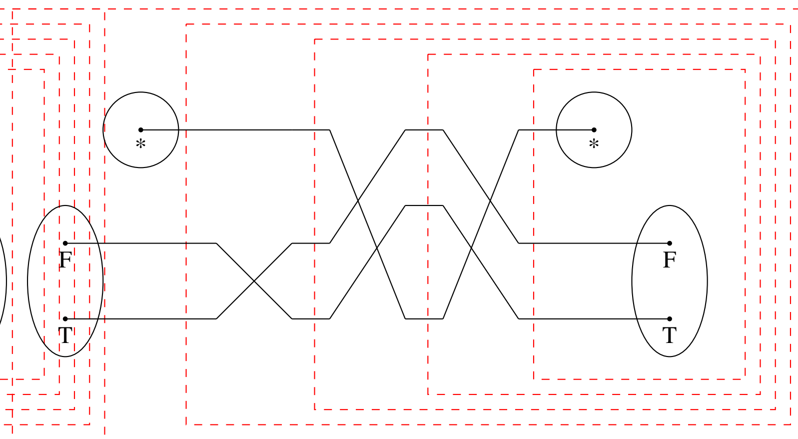
Making grouping explicit:



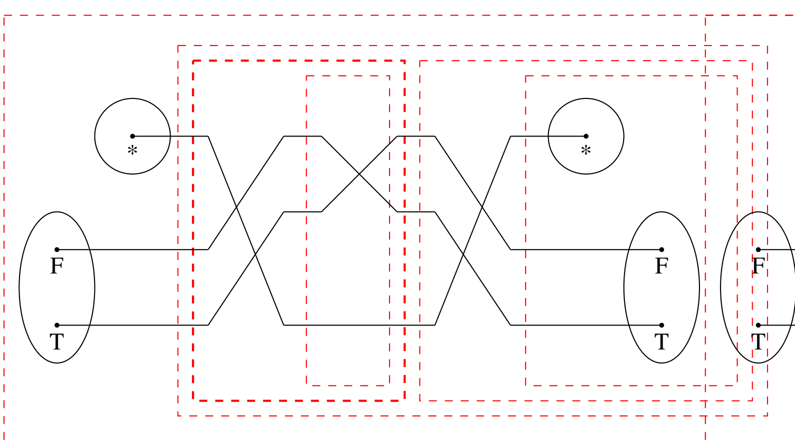
By associativity:



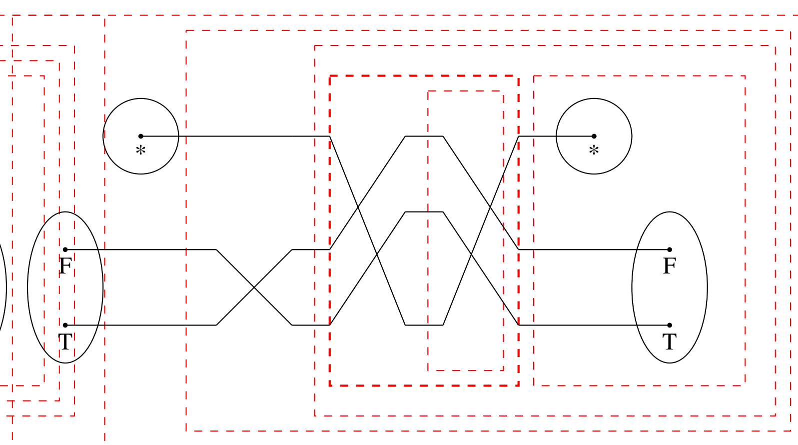
By associativity:



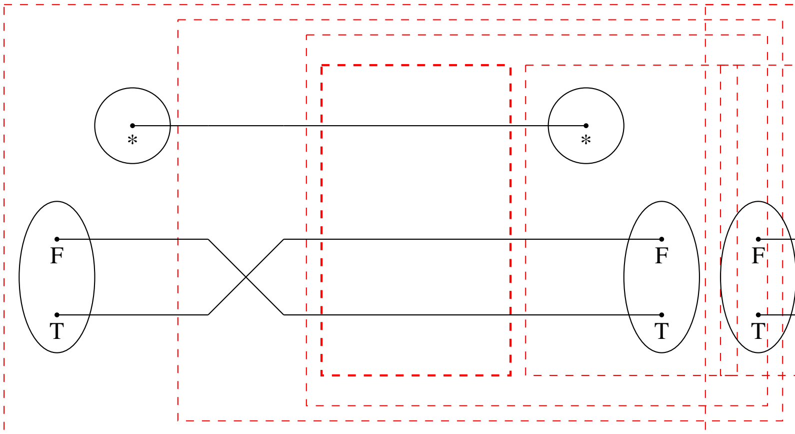
By associativity:



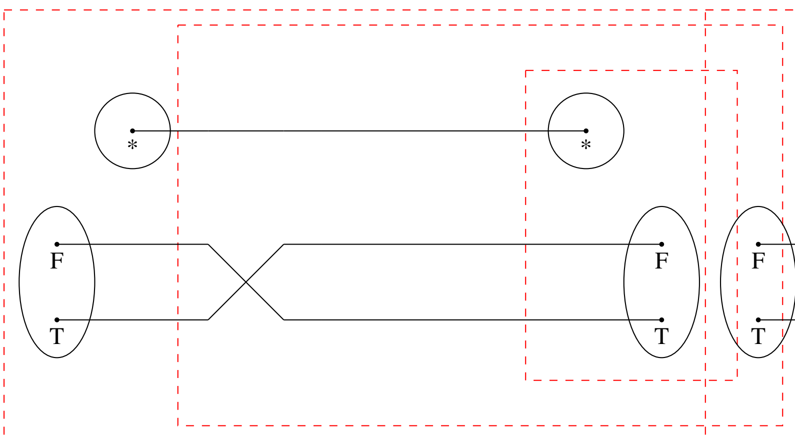
By pre-post-swap:



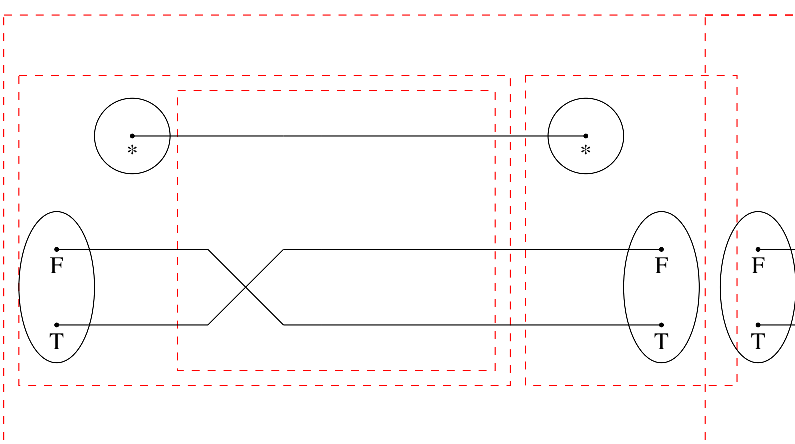
By swap-swap:



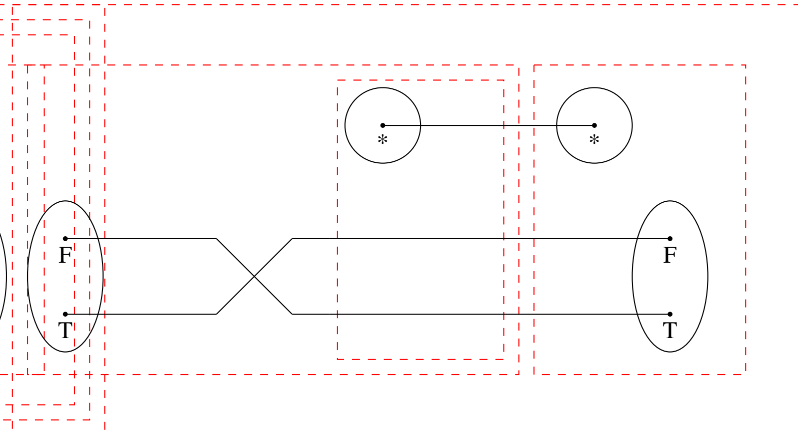
By id-compose-left:



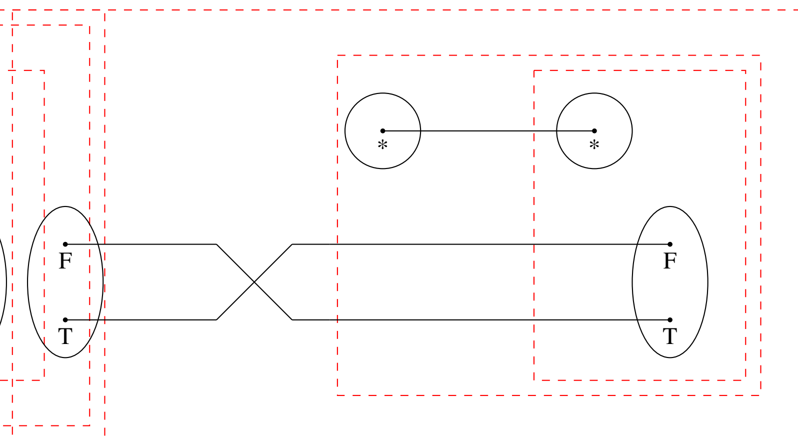
By associativity:



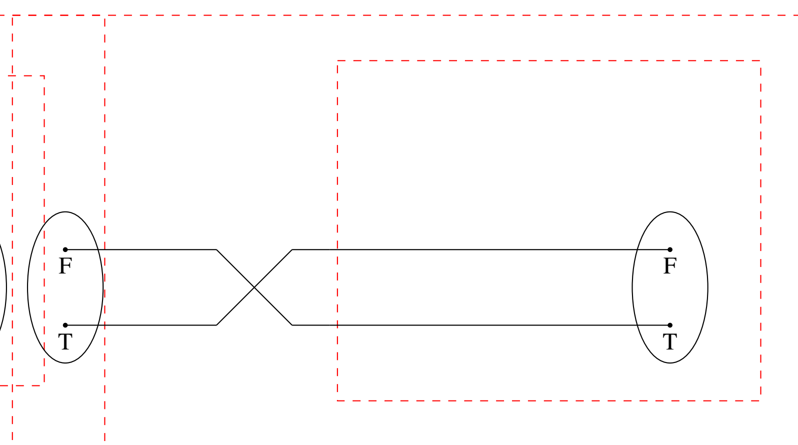
By swap-unit:



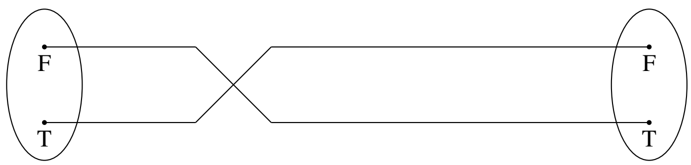
By associativity:



By unit-unit:



By id-unit-right:



5. But is this a programming language?

We get forward and backward evaluators $\mathsf{eval} : \{t_1\ t_2 : \mathsf{U}\} \to (t_1 \longleftrightarrow t_2) \to [\![\ t_1\ ]\!] \to [\![\ t_2\ ]\!]$
$\mathsf{evalB} : \{t_1\ t_2 : \mathsf{U}\} \to (t_1 \longleftrightarrow t_2) \to [\![\ t_2\ ]\!] \to [\![\ t_1\ ]\!]$
which really do behave as expected $\mathsf{c2equiv} : \{t_1\ t_2 : \mathsf{U}\} \to (c : t_1 \longleftrightarrow t_2) \to [\![\ t_1\ ]\!] \simeq [\![\ t_2\ ]\!]$
Manipulating circuits. Nice framework, but:

- We don't want ad hoc rewriting rules.
  - Our current set has 76 rules!
- Notions of soundness; completeness; canonicity in some sense.
  - Are all the rules valid? (yes)
  - Are they enough? (next topic)
  - Are there canonical representations of circuits? (open)

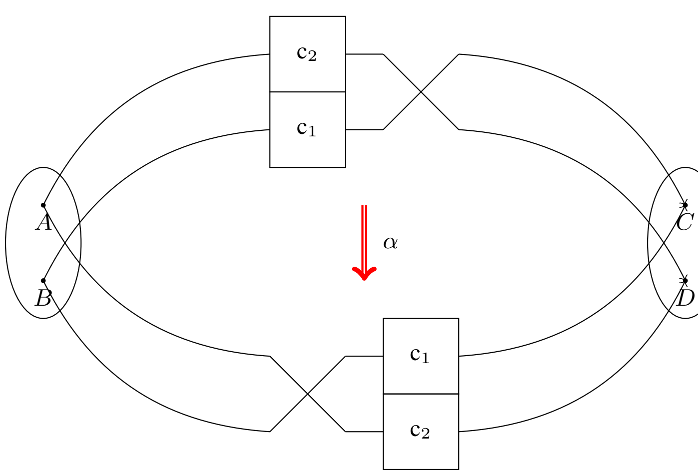
6. Categorification I

Type equivalences (such as between $A \times B$ and $B \times A$) are Functors.
Equivalences between Functors are Natural Isomorphisms. At the value-level, they induce 2-morphisms:

```
postulate
    c₁ : {B C : U} → B ⟷ C
    c₂ : {A D : U} → A ⟷ D

p₁ p₂ : {A B C D : U} → PLUS A B ⟷ PLUS C D
p₁ = swap₊ ⊙ (c₁ ⊕ c₂)
p₂ = (c₂ ⊕ c₁) ⊙ swap₊
```

2-morphism of circuits



Categorification II. The categorification of a semiring is called a Rig Category. As with a semiring, there are two monoidal structures, which interact through some distributivity laws.

Theorem 6. *The following are Symmetric Bimonoidal Groupoids:*

- *The class of all types (Set)*
- *The set of all finite types*
- *The set of permutations*
- *The set of equivalences between finite types*
- *Our syntactic combinators*

The coherence rules for Symmetric Bimonoidal groupoids give us 58 rules.
Categorification III.

Conjecture 1. *The following are Symmetric Rig Groupoids:*

- *The class of all types (Set)*
- *The set of all finite types, of permutations, of equivalences between finite types*
- *Our syntactic combinators*

and of course the punchline:

Theorem 7 (Laplaza 1972)**.** *There is a sound and complete set of coherence rules for Symmetric Rig Categories.*

Conjecture 2. *The set of coherence rules for Symmetric Rig Groupoids are a sound and complete set for circuit equivalence.*

7. Emails

```
Reminder of
http://mathoverflow.net/questions/106070/int-constructi

Also,
http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1
seems relevant

Indeed, this does not seem to be in the library.

On 2015-04-10 10:52 AM, Amr Sabry wrote:
I had checked and found no traced categories or Int con

The story without trace and without the Int constructio

On 04/10/2015 09:06 AM, Jacques Carette wrote:
I don't know, that a "symmetric rig" (never mind higher
programming language, even if only for "straight line p
interesting! ;)

But it really does depend on the venue you'd like to se
POPL, then I agree, we need the Int construction.  The
can be made, the better.

It might be in 'categories' already!  Have you looked?

In the meantime, I will try to finish the Rig part.  Th
conditions are non-trivial.
Jacques

On 2015-04-10, 06:06 , Sabry, Amr A. wrote:
I am thinking that our story can only be compelling if
that h.o. functions might work. We can make that case b
implementing the Int Construction and showing that a li
h.o. functions emerges and leave the big open problem o
the multiplication etc. for later work. I can start wor
will require adding traced categories and then a generi
```

Construction in the categories library. What do you think? --Amr

On Apr 9, 2015, at 10:59 PM, Jacques Carette <carette@mcmaster.ca>
wrote:

I have the braiding, and symmetric structures done. Most of the
RigCategory as well, but very close.

Of course, we're still missing the coherence conditions for Rig.

Jacques

On 2015-04-09 11:41 AM, Sabry, Amr A. wrote:
Can you make sense of how this relates to us?

https://pigworker.wordpress.com/2015/04/01/warming-up-to-homotopy-type-theory/

Unfortunately not. Yes, there is a general feeling of relatedness, but I can't pin it down.

I do believe that all our terms have computational rules, so we can't get stuck.

Note that at level 1, we have equivalences between Perm(A,B) and Perm(A,B), [illegible] Perm(C,D) [illegible]

Yes, we should dig into the Licata/Harper work and adapt to our setting.

Though I think we have some short-term work that we simply need to do [illegible]

Jacques

On 2015-04-09 12:05 PM, Amr Sabry wrote:
Trying to get a handle on what we can transport or more precisely if we can transport something that HoTT [illegible]

(I use permutation for level 0 to avoid too many uses of 'equivalence' which gets confusing.)

Level 0: Given two types A and B, if we have a permutation between them then we can transport something

For example: take P = . + C; we can build a permutation between A+C and B+C from the given permutation b

--

Level 1: Given types A, B, C, and D. let Perm(A,B) be the type of permutations between A and B and Perm(

This is more interesting. What's a good example though of a property P that we can implement?

In think that in HoTT the only way to do this transport is via univalence. First you find an equivalence

In HoTT this is exhibited by the failure of canonicity [illegible]

Perhaps we can adapt the discussion/example in http://homotopytypetheory.org/2011/07/27/canonicity-for-2

--Amr

I hope not! [only partly joking]

Actually, there is a fair bit about this that I dislike: it seems to over-simplify by arbitrarily saying

On 2015-04-09 12:36 PM, Amr Sabry wrote:
This came up in a different context but looks like it might be useful to us too.

http://arxiv.org/pdf/gr-qc/9905020

Separate. The Grothendieck construction in this case is about fibrations, and is not actually related t

Jacques

On 2015-04-10 11:56 AM, Sabry, Amr A. wrote:
Yes. The categories library has a Grothendieck construc

On Apr 10, 2015, at 11:04 AM, Jacques Carette <carette@

Reminder of the
http://mathoverflow.net/questions/106070/int-constructi

Also,
http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1
seems relevant

Indeed, this does not seem to be in the library.

On 2015-04-10 10:52 AM, Amr Sabry wrote:
I had checked and found no traced categories or Int con

The story without trace and without the Int constructio

On 04/10/2015 09:06 AM, Jacques Carette wrote:
I don't know, that a "symmetric rig" (never mind higher
programming language, [illegible]
interesting! ;)
But it really does depend on the venue you'd like to se
POPL, then I agree, we need the Int construction. The [illegible]
can be made, the better.

It might be in 'categories' already! Have you looked?

In the meantime, I will try to finish the Rig part. The
conditions are non-trivial.
Jacques

On 2015-04-10 08:06 AM, Sabry, Amr A. wrote:
I am thinking that our story can only be compelling if
that h.o. functions might work. We can make that case b
implementing the Int Construction and showing that a li
h.o. functions emerges and leave the big open problem o
the multiplication etc. for later work. I can start wor
will require defining traced categories and then a generic
Construction in the categories library. What do you thi
On Apr 9, 2015, at 10:59 PM, Jacques Carette <carette@m
wrote:

I have the braiding, and symmetric structures done. Mo
RigCategory as well, but very close.

Of course, we're still missing the coherence conditions

Jacques

solutions to quintic equations proof by arnold is all a
I thought we'd gotten at least one version, but could n

On 2015-04-25 8:37 AM, Sabry, Amr A. wrote:
Didn't we get stuck in the reverse direction. We never

On Apr 25, 2015, at 8:27 AM, Jacques Carette <carette@m
Right. We have one direction, from Pi combinators to F

Note that quite a bit of the code has (already!!) bit-r

We do not have the other direction currently in the code. That may not be too bad, as we do have LeftCa

Jacques

On 2015-04-25 7:28 AM, Sabry, Amr A. wrote:
That's obsolete for now.

By the way, do we have a complement to thm2 that connects to Pi. Ideally what we want to say is what I s

On Apr 24, 2015, at 5:25 PM, Jacques Carette <carette@mcmaster.ca> wrote:

Is that going somewhere, or is it an experiment that should be put into Obsolete/?
Jacques

Thanks. I like that idea ;).

I have a bunch of things I need to do, so I won't really put more head into this until the weekend.

I understand the desire to not want to rely on the full coherence conditions. I also don't know how to

As I was trying really hard to come up with a single story, I am a little confused as to what "my" story

On 2015-04-23 9:07 PM, Sabry, Amr A. wrote:
Instead of discussing this over and over, I think it is clear that thm2 will be an important part of any

On Apr 23, 2015, at 6:07 PM, Amr Sabry <sabry@indiana.edu> wrote:

I wasn't too worried about the symmetric vs. non-symmetric notion of equivalence. The HoTT book has more

I do recall the other discussion about extensionality. That is related story which is what I started

I just really want to avoid the full reliance on the coherence conditions. I also noted you have a diffe

--Amr

On 04/23/2015 12:23 PM, Jacques Carette wrote:
Did you see my "HoTT-agda" question on the Agda mailing list on March
11th, and Dan Licata's reply?

What you wrote reduces to our definition of *equivalence*, not
permutation. To prove that equivalence, we would need funext -- see my
question of February 18th on the Agda mailing list.

Another way to think about it is that this is EXACTLY what thm2
provides: a proof that for finite A and B, equivalence between A and B
(as below) is equivalent to permutations implemented as (Vec, Vec, pf,
pf).

Now, we may want another representation of permutations which uses
functions (qua bijections) internally instead of vectors. Then the
answer to your question would be "yes", modulo the question/idea about
which encoding of equivalence to use.

Jacques

On 2015-04-23 10:32 AM, Sabry, Amr A. wrote:
Thought a bit more about this. We need a little bridge from HoTT to
our code and we're good to go I think.

In HoTT we have several notions of equivalence that are equivalent (in
the technical sense). The one that seems easiest to work with is the
following:

A ≃ B if exists f : A → B such that:
  (exists g : B → A with g o f ~ idA) X
  (exists h : B → A with f o h ~ idB)

Does this definition reduce to our semantic notion of p
and B are finite sets?

--Amr

On Apr 21, 2015, at 11:03 AM, Jacques Carette <carette@
wrote:

I'm ok with the HoTT bias, but concerned that our code do
match that. But since we have no specific deadline, I
bit more time isn't too bad.

Since propositional equivalence is really HoTT equivale
I'm not too concerned about that side of things -- our
permutations should be the same whether in HoTT or in r
with various notions of equivalence, especially since m
code was lifted from a previous HoTT-based attempt at t
I would certainly agree with the not-not-statement: usi
equivalence known to be incompatible with HoTT is not a
Jacques

On 2015-04-21 10:38 AM, Sabry, Amr A. wrote:
I think that I should start trying to write down a mor
story so that we can see how things fit together. I am
towards the HoTT-based story which is what I started. I
we should have a different initial bias let me know.
What is there is just one paragraph for now but it alre
question: if we are pursuing that HoTT story we should
prove that the HoTT notion of equivalence when speciali
types reduces to permutations. That should be a strong
and the precise notion of permutation we get (
by enumerations or not should help quite a bit).

More generally always keeping our notions of equivalenc
levels in-sync with the HoTT definitions seems to
thing to do. --Amr

and if these coherence conditions are really comple
Set up, we could get a nice language for expressin

--Amr

On 04/27/2015 6:16 AM, Sabry, Amr A. wrote:
Here is an idea: we need a canonical form for every

Indeed! Good idea.

However, it may not give us a normal form. This is bec

In other words, because we have associativity and commu

However, I think it is not that bad: we can use the obj

Here is another thought:
1. think of the combinators as polynomials in 3 operato
2. expand things out, with + being outer, * middle, . i

3. within each . term, use combinators to re-orde[illegible]
4. show this terminates

On 2015-06-02 7:53 PM, Sabry, Amr A. wrote:

the issue is that the re-ordering could produce ne[illegible]

Jacques

There are some slightly different approaches to impleme

On 2015-04-27 6:16 AM, Sabry, Amr A. wrote:

A category can be formalized as a kind of elementary ax

Here is a nice idea: we need a canonical form for every pi-combinator. Our previous approach gave us som

f:X to Y equiv Domain(f) = X and Range(

I've been thinking about this some more. I can't help but think that, somehow, Laplaza has already work

is used for the three place predicate.

Pi-combinators might be simpler, I don't know.

The operations such as the binary composition of maps a

Another place to look is in Fiore (et al?)'s proof of completeness of a similar case. Again, in their c

f:Z to Y, g:Y to X implies g(f):Z to X

On 2015-04-26 6:34 AM, Sabry, Amr A. wrote:

What's the proof strategy for establishing that a[illegible]

Well enough. Last talk on the last day, so peopl[illegible]

I think the idea that (reversible circuits == pro[illegible]

If we had a similar story for Caley+T (as they li[illegible]

Note that I've pushed quite a few things forward [illegible]

Yes, I think this can make a full paper -- especi[illegible]

I think the details are fine. A little bit of po[illegible]

Writing it up actually forced me to add PiEquiv.a[illegible]

Firstly, thanks Spencer for setting this up.

In any symmetric monoidal 2-category, we have a notion

This is partly a response to Amr, and partly my o[illegible]

One of the key ingredients to getting diagrammati[illegible]

If you ignore these theorems and insist on workin[illegible]

Of course, when it comes to computing with diagra[illegible]

Jacques

(1: combinatoric) its a graph with some extra bells and whistles

(2: syntactic) its a convenient way of writing do[illegible]

On 2015-05-17 [illegible] AM, Sabry, Amr A. wrote:

(3: "lego" style) its a collection of tiles, conn[illegible] http://www.informatik.uni-

-Amr

Point of view (1) is basically what Quantomatic is built on. "String graphs" aka "open-graphs" give a co

Naiively, point of view (2) is that a diagram rep[illegible]

Point of view (3) is the one espoused by the 2D/h[illegible] Yves Lafont an

http://iml.univ-mrs.fr/~lafont/pub/diagrams.pdf

This eliminates the need for the interchange law, but keeps pretty much everything else "rigid". This be

A Homotopical Completion Procedure with Applications to

This is a very good example of CCT. As I am sure [illegible]

My primary CCT interest, so far, has been with wh[illegible]

http://www.lix.polytechnique.fr/Labo/Samuel.Mimram/docs

There's also the perspective that string diagrams of various flavors are morphisms in some operad (the c

I think there is something very important going on in s

From that perspective, the string diagrams for tr[illegible] just bije

which I also attach. [I googled 'Knuth Bendix coherenc

Yes, I am sure this observation has been made before. We'd have to verify it for all the 2-paths before

There are also seems to be relevant stuff buried (very

Also, Tarmo Uustalu’s "Coherence for skew-monoidal

[Apparently I could have saved myself some of that

Somehow, at the end of the day, it seems we’re look

A. Commutative Semirings

Given that the structure of commutative semirings is central to this paper, we recall the formal algebraic definition.

Definition 2. *A* commutative semiring *consists of a set R, two distinguished elements of R named 0 and 1, and two binary operations $+$ and $\cdot$, satisfying the following relations for any $a, b, c \in R$:*

$$
\begin{array}{rcl}
0 + a & = & a \\
a + b & = & b + a \\
a + (b + c) & = & (a + b) + c \\
\\
1 \cdot a & = & a \\
a \cdot b & = & b \cdot a \\
a \cdot (b \cdot c) & = & (a \cdot b) \cdot c \\
\\
0 \cdot a & = & 0 \\
(a + b) \cdot c & = & (a \cdot c) + (b \cdot c)
\end{array}
$$

In the paper, we are interested into various commutative semiring structures up to some congruence relation instead of strict equality $=$.