Computing with Semirings and Weak Rig Groupoids

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Simple Curry-Howard

Constructors

Logic	Types
true	$\top$
false	$\bot$
$\wedge$	$\times$
$\vee$	$\uplus$
$\Rightarrow$	$\rightarrow$
$\forall$	$\prod$
$\exists$	$\sum$

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Digging deeper:

Logical Implication	Inhabiting Term
$A \wedge B \Rightarrow B \wedge A$	$\lambda\{(a, b) \rightarrow (b, a)\}$
$true \wedge A \Rightarrow A$	$\lambda\{(tt, a) \rightarrow a\}$
$A \Rightarrow true \wedge A$	$\lambda a \rightarrow tt, a$
$false \vee A \Rightarrow A$	$\bot e_+ : \{A : \mathsf{Set}\} \rightarrow \bot \uplus A \rightarrow A$ $\bot e_+ \ (\mathsf{inj}_1 \ ())$ $\bot e_+ \ (\mathsf{inj}_2 \ y) = y$
$A \Rightarrow false \vee A$	$\bot i_+ : \{A : \mathsf{Set}\} \rightarrow A \rightarrow \bot \uplus A$ $\bot i_+ \ a = \mathsf{inj}_2 \ a$

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Logical Equivalence	Type Isomorphism
$A \wedge B \Leftrightarrow B \wedge A$	$A \times B \simeq B \times A$
$true \wedge A \Leftrightarrow A$	$\top \times A \simeq A$
$false \vee A \Leftrightarrow A$	$\bot \uplus A \simeq A$

Classical Curry-Howard: Inhabitation

Logical Equivalence	Type non-Isomorphism
$A \vee A \Leftrightarrow A$	$A \uplus A \not\simeq A$
$A \wedge A \Leftrightarrow A$	$A \times A \not\simeq A$
$true \vee A \Leftrightarrow true$	$\top \uplus A \not\simeq \top$

There *are* functions which witness inhabitation in each direction, but these forget information, i.e. are not inverses.

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Desideratum: only isomorphisms in the right column.
Motivation and inspiration:

- Reversible computing; Bidirectional computing; Quantum computing.
- Conservation of resources (including information)
- Homotopy Type Theory.

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Q: What would the table look like then?

Type isomorphisms I

Definition 1 (Homotopy).

Two functions $f, g : A \to B$ are *homotopic*, written $f \sim g$, if $\forall x : A. f(x) = g(x)$.

```
_∼_ : ∀ {A : Set} {P : A → Set} → (f g : (x : A) → P x) → Set
_∼_ {A} f g = (x : A) → f x ≡ g x
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```

Definition 2 (Quasi-inverse).

For a function $f : A \to B$, a *quasi-inverse* of f is a triple (g, α, β), consisting of a function $g : B \to A$ and two homotopies $\alpha : f \circ g \sim \mathrm{id}_B$ and $\beta : g \circ f \sim \mathrm{id}_A$.

```
record isqinv {A : Set} {B : Set} (f : A → B) : Set where
  constructor qinv
  field
    g : B → A
    α : (f ∘ g) ∼ id
    β : (g ∘ f) ∼ id
```

Type isomorphisms II

Definition 3 (Equivalence of types).

Two types A and B are equivalent $A \simeq B$ if there exists a function $f : A \to B$ together with a quasi-inverse for f.

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_≃_ : Set → Set → Set
A ≃ B = Σ (A → B) isqinv
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(Tricky) Example

```
left0e : {A : Set} → ⊥ × A → ⊥
left0e = proj₁
left0i : {A : Set} → ⊥ → ⊥ × A
left0i = ⊥-elim

left0ei : {A : Set} → (left0e {A} ∘ left0i) ∼ id
left0ei ()
left0ie : {A : Set} → (left0i {A} ∘ left0e) ∼ id
left0ie (() , _)
```

```
⊥≃ : {A : Set} → (⊥ × A) ≃ ⊥
⊥≃ = left0e ,
     qinv left0i left0ei left0ie
```

Type isomorphisms III

Type isomorphisms
(Sound and complete)

$$\begin{array}{rcl} \bot \uplus A & \simeq & A \\ A \uplus B & \simeq & B \uplus A \\ A \uplus (B \uplus C) & \simeq & (A \uplus B) \uplus C \end{array}$$

$$\begin{array}{rcl} \top \times A & \simeq & A \\ A \times B & \simeq & B \times A \\ A \times (B \times C) & \simeq & (A \times B) \times C \end{array}$$

$$\begin{array}{rcl} \bot \times A & \simeq & \bot \\ (A \uplus B) \times C & \simeq & (A \times C) \uplus (B \times C) \end{array}$$

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\\
\top \times A & \simeq & A \\
A \times B & \simeq & B \times A \\
A \times (B \times C) & \simeq & (A \times B) \times C \\
\\
\bot \times A & \simeq & \bot \\
(A \uplus B) \times C & \simeq & (A \times C) \uplus (B \times C)
\end{array}$$

Semiring Equalities
(Sound and complete)

$$\begin{array}{rcl}
0 + a & = & a \\
a + b & = & b + a \\
a + (b + c) & = & (a + b) + c \\
\\
1 \cdot a & = & a \\
a \cdot b & = & b \cdot a \\
a \cdot (b \cdot c) & = & (a \cdot b) \cdot c \\
\\
0 \cdot a & = & 0 \\
(a + b) \cdot c & = & (a \cdot c) + (b \cdot c)
\end{array}$$

Observations

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   - identity
   - swap (commutativity)

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Let's make a PL out of semiring identities / type isomorphisms!

The Π Language: reifying isomorphisms

$$
\begin{array}{rrcll}
id : & \tau & \leftrightarrow & \tau & : id \\
identl_+ : & 0 + \tau & \leftrightarrow & \tau & : identr_+ \\
swap_+ : & \tau_1 + \tau_2 & \leftrightarrow & \tau_2 + \tau_1 & : swap_+ \\
assocl_+ : & \tau_1 + (\tau_2 + \tau_3) & \leftrightarrow & (\tau_1 + \tau_2) + \tau_3 & : assocr_+ \\
identl_* : & 1 * \tau & \leftrightarrow & \tau & : identr_* \\
swap_* : & \tau_1 * \tau_2 & \leftrightarrow & \tau_2 * \tau_1 & : swap_* \\
assocl_* : & \tau_1 * (\tau_2 * \tau_3) & \leftrightarrow & (\tau_1 * \tau_2) * \tau_3 & : assocr_* \\
dist_0 : & 0 * \tau & \leftrightarrow & 0 & : factorl_0 \\
dist : & (\tau_1 + \tau_2) * \tau_3 & \leftrightarrow & (\tau_1 * \tau_3) + (\tau_2 * \tau_3) & : factor
\end{array}
$$

$$
\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_2 \leftrightarrow \tau_3}{\vdash c_1 \odot c_2 : \tau_1 \leftrightarrow \tau_3}
\qquad
\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \oplus c_2 : \tau_1 + \tau_3 \leftrightarrow \tau_2 + \tau_4}
$$

$$
\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \otimes c_2 : \tau_1 * \tau_3 \leftrightarrow \tau_2 * \tau_4}
$$

Example

One possible program that corresponds to the type isomorphism:

$$
\begin{array}{rl}
 & (1+1) * ((1+1) * b) \\
= & (1+1) * ((1 * b) + (1 * b)) \\
= & (1+1) * (b+b) \\
= & (1 * (b+b)) + (1 * (b+b)) \\
= & (b+b) + (b+b)
\end{array}
$$

is:

$$
(id \otimes (dist \odot (identl_* \oplus identl_*))) \odot (dist \odot (identl_* \oplus identl_*))
$$

Syntactically

Start with a universe of (finite) types

```
data U : Set where
  ZERO  : U
  ONE   : U
  PLUS  : U → U → U
  TIMES : U → U → U
```

and its interpretation

```
⟦_⟧ : U → Set
⟦ ZERO ⟧        = ⊥
⟦ ONE ⟧         = ⊤
⟦ PLUS t₁ t₂ ⟧  = ⟦ t₁ ⟧ ⊎ ⟦ t₂ ⟧
⟦ TIMES t₁ t₂ ⟧ = ⟦ t₁ ⟧ × ⟦ t₂ ⟧
```

For notational brevity, we will denote these $0, 1, +$, and $*$ respectively.

and the terms and combinators:

```
data _↔_ : U → U → Set where
unite₊  : {t : U} → PLUS ZERO t ↔ t
uniti₊  : {t : U} → t ↔ PLUS ZERO t
swap₊   : {t₁ t₂ : U} → PLUS t₁ t₂ ↔ PLUS t₂ t₁
assocl₊ : {t₁ t₂ t₃ : U} → PLUS t₁ (PLUS t₂ t₃) ↔ PLUS (PLUS t₁ t₂) t₃
assocr₊ : {t₁ t₂ t₃ : U} → PLUS (PLUS t₁ t₂) t₃ ↔ PLUS t₁ (PLUS t₂ t₃)
unite⋆  : {t : U} → TIMES ONE t ↔ t
uniti⋆  : {t : U} → t ↔ TIMES ONE t
swap⋆   : {t₁ t₂ : U} → TIMES t₁ t₂ ↔ TIMES t₂ t₁
assocl⋆ : {t₁ t₂ t₃ : U} → TIMES t₁ (TIMES t₂ t₃) ↔ TIMES (TIMES t₁ t₂) t₃
assocr⋆ : {t₁ t₂ t₃ : U} → TIMES (TIMES t₁ t₂) t₃ ↔ TIMES t₁ (TIMES t₂ t₃)
absorbr  : {t : U} → TIMES ZERO t ↔ ZERO
absorbl : {t : U} → TIMES t ZERO ↔ ZERO
factorzr : {t : U} → ZERO ↔ TIMES t ZERO
factorzl : {t : U} → ZERO ↔ TIMES ZERO t
dist    : {t₁ t₂ t₃ : U} → TIMES (PLUS t₁ t₂) t₃ ↔ PLUS (TIMES t₁ t₃) (TIMES t₂ t₃)
factor  : {t₁ t₂ t₃ : U} → PLUS (TIMES t₁ t₃) (TIMES t₂ t₃) ↔ TIMES (PLUS t₁ t₂) t₃
id↔  : {t : U} → t ↔ t
_⊙_   : {t₁ t₂ t₃ : U}  → (t₁ ↔ t₂) → (t₂ ↔ t₃) → (t₁ ↔ t₃)
_⊕_   : {t₁ t₂ t₃ t₄ : U} → (t₁ ↔ t₃) → (t₂ ↔ t₄) → (PLUS t₁ t₂ ↔ PLUS t₃ t₄)
_⊗_   : {t₁ t₂ t₃ t₄ : U} → (t₁ ↔ t₃) → (t₂ ↔ t₄) → TIMES t₁ t₂ ↔ TIMES t₃ t₄
```

Semantics I: Operational

It really is a PL:

$$
\begin{array}{l}
\mathsf{eval} \ : \{t_1 \ t_2 : \mathsf{U}\} \to (t_1 \leftrightarrow t_2) \to [\![\ t_1\ ]\!] \to [\![\ t_2\ ]\!] \\
\mathsf{eval}\ \mathsf{unite}_+\ (\mathsf{inj}_1\ ()) \\
\mathsf{eval}\ \mathsf{unite}_+\ (\mathsf{inj}_2\ y) = y \\
\mathsf{eval}\ \mathsf{uniti}_+\ x = \mathsf{inj}_2\ x \\
\mathsf{eval}\ \mathsf{swap}_+\ (\mathsf{inj}_1\ x) = \mathsf{inj}_2\ x \\
\mathsf{eval}\ \mathsf{swap}_+\ (\mathsf{inj}_2\ y) = \mathsf{inj}_1\ y
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$$

...

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...

And it can also be run backwards:

$$\mathsf{evalB}\ :\ \{t_1\ t_2 : \mathsf{U}\} \to (t_1 \leftrightarrow t_2) \to [\![\ t_2\ ]\!] \to [\![\ t_1\ ]\!]$$

...

$$\mathsf{evalB}\ (c_0 \odot c_1)\ x = \mathsf{evalB}\ c_0\ (\mathsf{evalB}\ c_1\ x)$$
$$\mathsf{evalB}\ (c_0 \oplus c_1)\ (\mathsf{inj}_1\ x) = \mathsf{inj}_1\ (\mathsf{evalB}\ c_0\ x)$$
$$\mathsf{evalB}\ (c_0 \oplus c_1)\ (\mathsf{inj}_2\ y) = \mathsf{inj}_2\ (\mathsf{evalB}\ c_1\ y)$$
$$\mathsf{evalB}\ (c_0 \otimes c_1)\ (x\ ,\ y) = \mathsf{evalB}\ c_0\ x\ ,\ \mathsf{evalB}\ c_1\ y$$

Good properties I

Forward and backward are inverses of each other:

$\mathsf{fwd{\circ}bwd{\sim}id} : \forall\ \{t_1\ t_2\} \to (c : t_1 \leftrightarrow t_2) \to (\mathsf{eval}\ c \circ \mathsf{evalB}\ c) \sim \mathsf{id}$

$\mathsf{bwd{\circ}fwd{\sim}id} : \forall\ \{t_1\ t_2\} \to (c : t_1 \leftrightarrow t_2) \to (\mathsf{evalB}\ c \circ \mathsf{eval}\ c) \sim \mathsf{id}$

We have a syntactic reverse operator:

$! : \{t_1\ t_2 : \mathsf{U}\} \to (t_1 \leftrightarrow t_2) \to (t_2 \leftrightarrow t_1)$

$!\ \mathsf{unite}_+ \quad = \mathsf{uniti}_+$

$!\ \mathsf{uniti}_+ \quad = \mathsf{unite}_+$

$!\ \mathsf{swap}_+ \quad = \mathsf{swap}_+$

$!\ \mathsf{assocl}_+ \quad = \mathsf{assocr}_+$

$!\ \mathsf{assocr}_+ \quad = \mathsf{assocl}_+$

...

which behaves as expected:

$!{\sim}\mathsf{B} : \forall\ \{t_1\ t_2\} \to (c : t_1 \leftrightarrow t_2) \to \mathsf{eval}\ (!\ c) \sim \mathsf{evalB}\ c$

Good properties II

Every syntactic combinator induces a type equivalence:
c2equiv : $\{t_1\ t_2 : \mathsf{U}\} \to (c : t_1 \leftrightarrow t_2) \to [\![\ t_1\ ]\!] \simeq [\![\ t_2\ ]\!]$

which also behaves nicely:

lemma0 : $\{t_1\ t_2 : \mathsf{U}\} \to (c : t_1 \leftrightarrow t_2) \to$
eval $c \sim \mathsf{proj}_1$ (c2equiv c)

lemma1 : $\{t_1\ t_2 : \mathsf{U}\} \to (c : t_1 \leftrightarrow t_2) \to$
evalB $c \sim \mathsf{proj}_1$ (sym≃ (c2equiv c))

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A foundational PL design method:

1. figure out a good denotational semantics,
2. derive a syntax for it,
3. then make sure you can implement it!

Semantics II: $\mathbb{B}$ as a Model

A simple and adequate model: finite sets and bijections (and its decategorification):

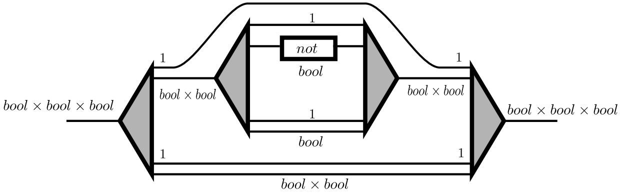
- A type is interpreted as a finite set (i.e., a natural number)
- 0 is the empty set (the number 0)
- 1 is a singleton set (the number 1)
- $+$ is disjoint union (addition of natural numbers)
- $\times$ is cartesian product (multiplication of natural numbers)
- Combinators are permutations (algebraic identities of natural numbers)

Fredkin and Toffoli Gates

- Universal gates for reversible combinational circuits
- Toffoli gate:

  BOOL BOOL^2 : U
  BOOL = PLUS ONE ONE
  BOOL^2 = TIMES BOOL BOOL

  Toffoli : TIMES BOOL $\text{BOOL}^2 \leftrightarrow$ TIMES BOOL BOOL^2
  Toffoli = dist $\odot$ (id$\leftrightarrow$ $\oplus$ (id$\leftrightarrow$ $\otimes$ cnot)) $\odot$ factor
  where
  cnot : $\text{BOOL}^2 \leftrightarrow \text{BOOL}^2$
  cnot = dist $\odot$ (id$\leftrightarrow$ $\oplus$ (id$\leftrightarrow$ $\otimes$ swap_+)) $\odot$ factor



Proofs

Now we can write down the two proofs that $A + A = A + A$:

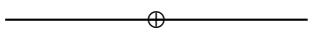
$\mathrm{pf}_1\pi\ \mathrm{pf}_2\pi : \{A : \mathrm{U}\} \rightarrow \mathrm{PLUS}\ A\ A \leftrightarrow \mathrm{PLUS}\ A\ A$
$\mathrm{pf}_1\pi = \mathrm{id}{\leftrightarrow}$
$\mathrm{pf}_2\pi = \mathrm{swap}_+$

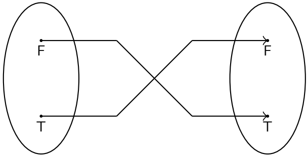
But also other proofs of semiring equalities:

$\mathrm{pf}_3\pi\ \mathrm{pf}_4\pi : \{A\ B : \mathrm{U}\} \rightarrow \mathrm{PLUS}\ (\mathrm{PLUS}\ A\ \mathrm{ZERO})\ B \leftrightarrow \mathrm{PLUS}\ A\ B$
$\mathrm{pf}_3\pi = (\mathrm{swap}_+ \oplus \mathrm{id}{\leftrightarrow}) \odot (\mathrm{unite}_+ \oplus \mathrm{id}{\leftrightarrow})$
$\mathrm{pf}_4\pi = \mathrm{assocr}_+ \odot (\mathrm{id}{\leftrightarrow} \oplus \mathrm{unite}_+)$

Claim: $\mathrm{pf}_1\pi$ and $\mathrm{pf}_2\pi$ are different, while $\mathrm{pf}_3\pi$ and $\mathrm{pf}_4\pi$ are equivalent.

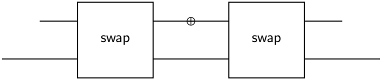
Visual example: Simple Negation

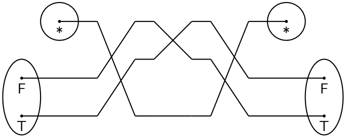




$n_1 : \mathrm{BOOL} \leftrightarrow \mathrm{BOOL}$
$n_1 = \mathrm{swap}_+$

Visual example: Not So Simple Negation





$n_2 : \text{BOOL} \leftrightarrow \text{BOOL}$

$n_2 = \;\text{uniti}\star \odot$
$\quad\;\text{swap}\star \odot$
$\quad\;(\text{swap}_+ \otimes \text{id}\leftrightarrow) \odot$
$\quad\;\text{swap}\star \odot$
$\quad\;\text{unite}\star$

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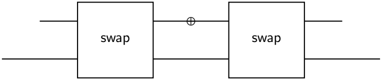

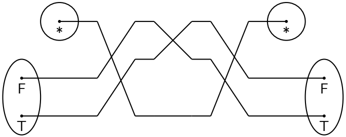

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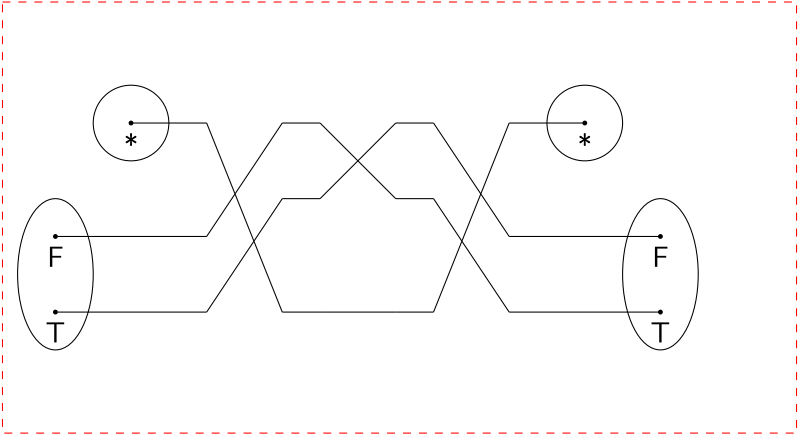
$\quad \mathsf{swap}\star \odot$
$\quad (\mathsf{swap}_+ \otimes \mathsf{id}\!\leftrightarrow) \odot$
$\quad \mathsf{swap}\star \odot$
$\quad \mathsf{unite}\star$

Question: are these equivalent combinators?

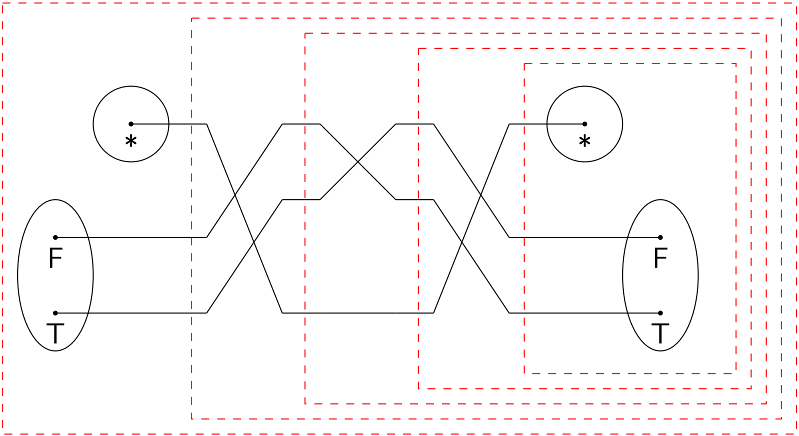
Visual “proof”

Original circuit:



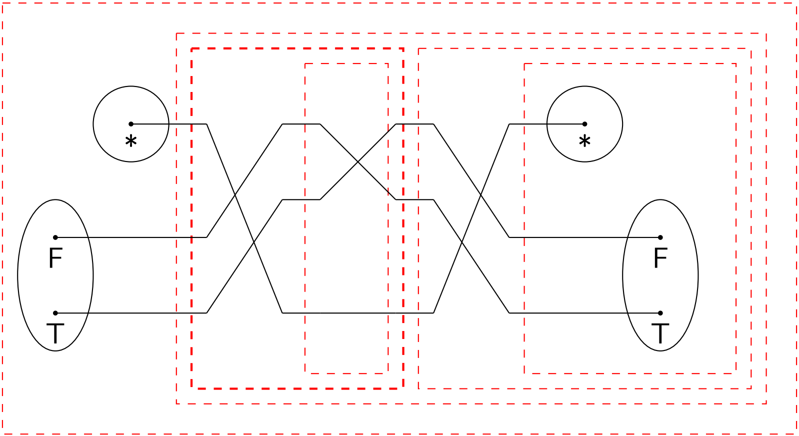
Visual "proof"

Making grouping explicit:



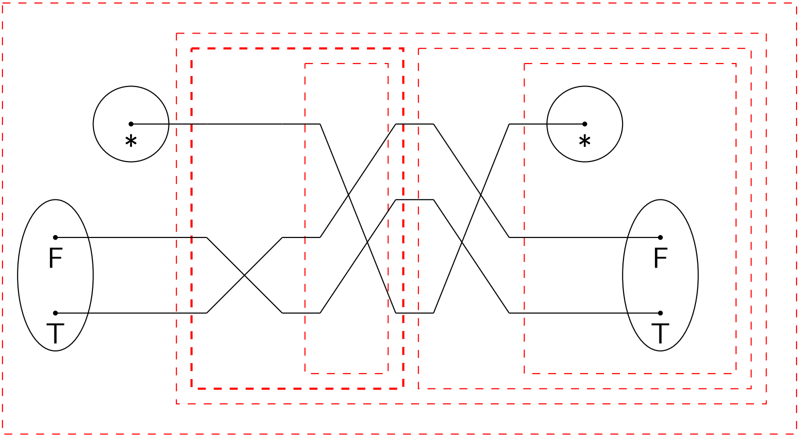
Visual "proof"

By associativity:



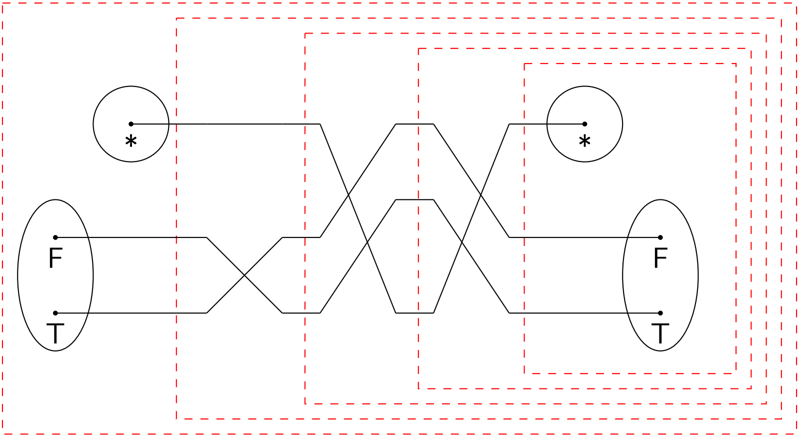
Visual "proof"

By pre-post-swap:



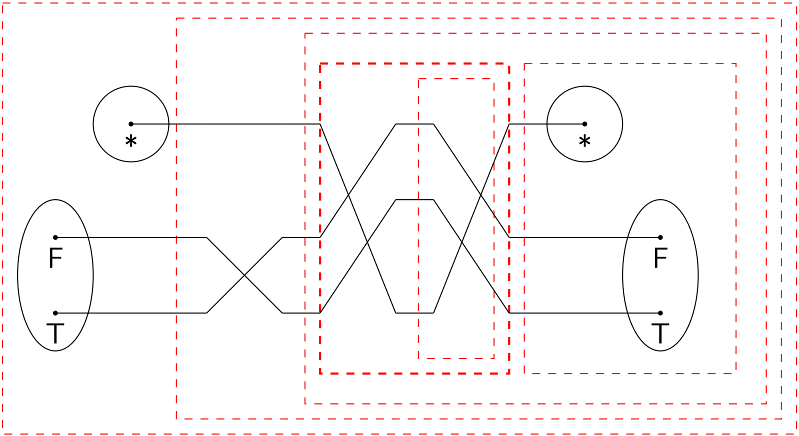
Visual "proof"

By associativity:



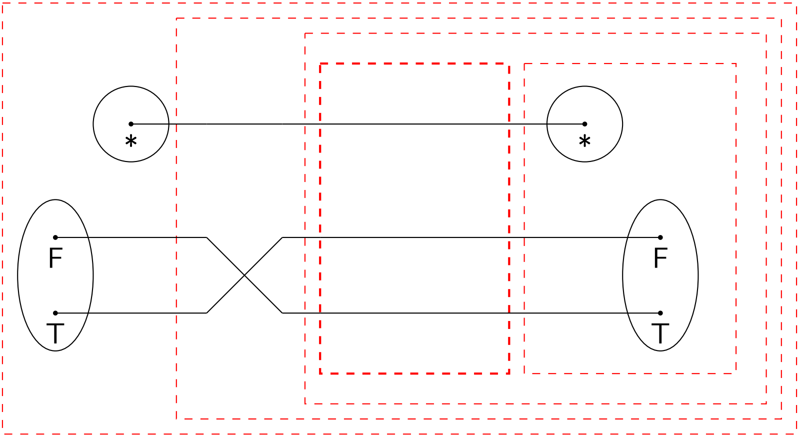
Visual “proof”

By associativity:



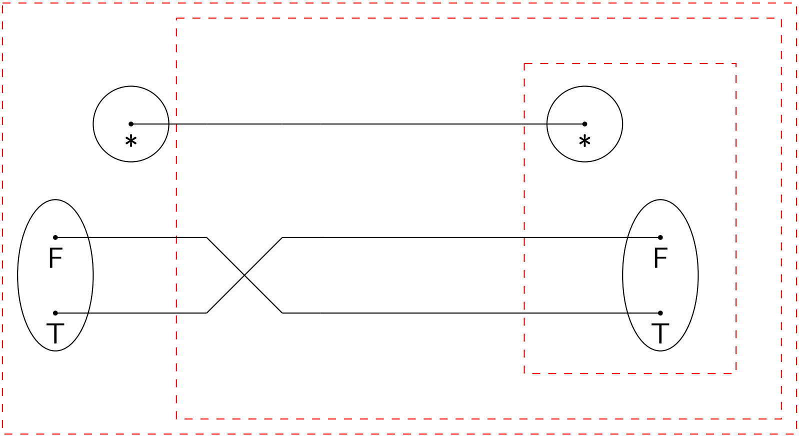
Visual "proof"

By swap-swap:



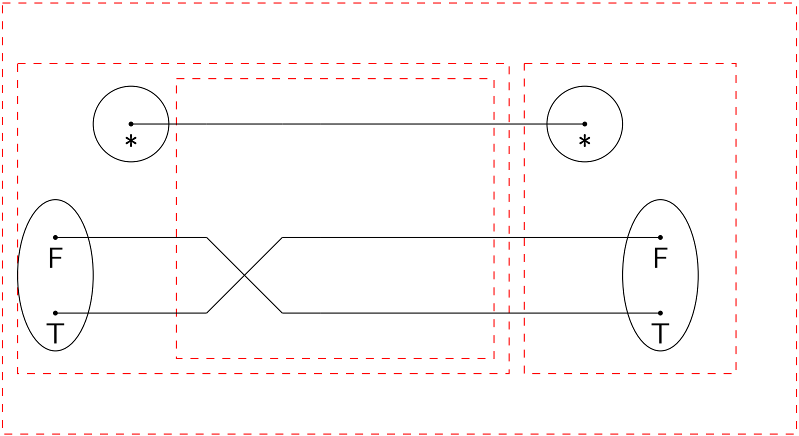
Visual "proof"

By id-compose-left:



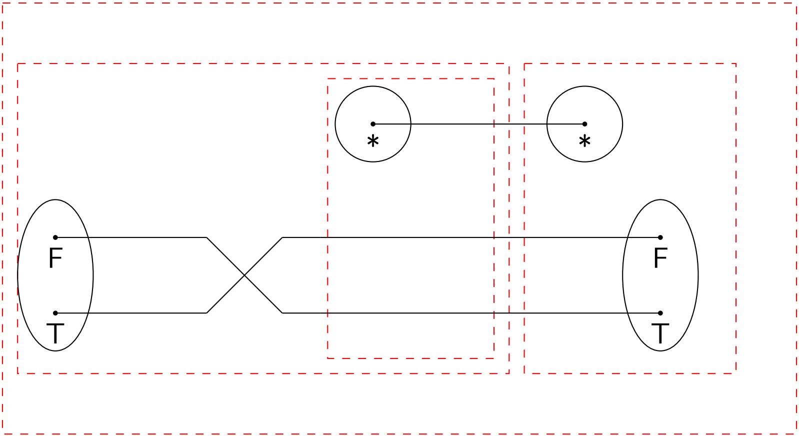
Visual "proof"

By associativity:



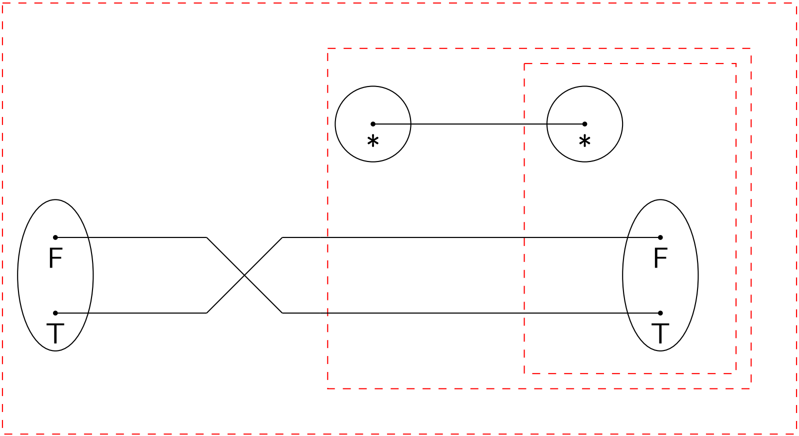
Visual “proof”

By swap-unit:



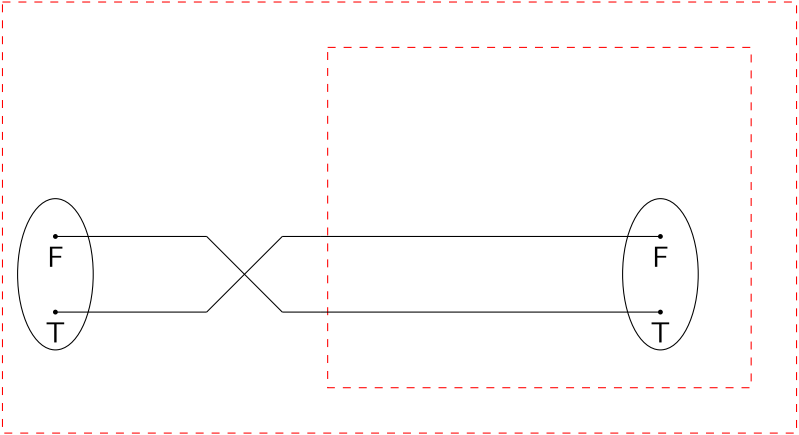
Visual "proof"

By associativity:



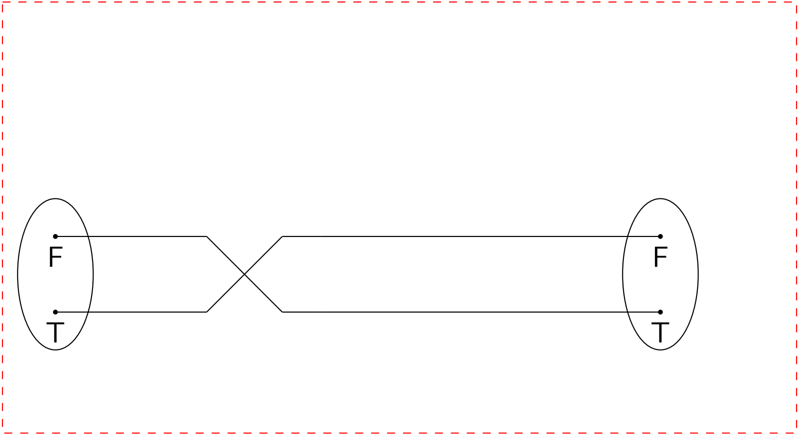
Visual “proof”

By unit-unit:



Visual "proof"

By id-unit-right:



Algebraic reasoning

```
negEx : n₂ ⇔ n₁
negEx = uniti⋆ ⊙ (swap⋆ ⊙ ((swap₊ ⊗ id⟷) ⊙ (swap⋆ ⊙ unite⋆)))
    ⇔⟨ id⇔ ⊡ assoc⊙l ⟩
  uniti⋆ ⊙ ((swap⋆ ⊙ (swap₊ ⊗ id⟷)) ⊙ (swap⋆ ⊙ unite⋆))
    ⇔⟨ id⇔ ⊡ (swapl⋆⇔ ⊡ id⇔) ⟩
  uniti⋆ ⊙ (((id⟷ ⊗ swap₊) ⊙ swap⋆) ⊙ (swap⋆ ⊙ unite⋆))
    ⇔⟨ id⇔ ⊡ assoc⊙r ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ (swap⋆ ⊙ (swap⋆ ⊙ unite⋆)))
    ⇔⟨ id⇔ ⊡ (id⇔ ⊡ assoc⊙l) ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ ((swap⋆ ⊙ swap⋆) ⊙ unite⋆))
    ⇔⟨ id⇔ ⊡ (id⇔ ⊡ (linv⊙l ⊡ id⇔)) ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ (id⟷ ⊙ unite⋆))
    ⇔⟨ id⇔ ⊡ (id⇔ ⊡ idl⊙l) ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ unite⋆)
    ⇔⟨ assoc⊙l ⟩
  (uniti⋆ ⊙ (id⟷ ⊗ swap₊)) ⊙ unite⋆
    ⇔⟨ unitil⋆⇔ ⊡ id⇔ ⟩
  (swap₊ ⊙ uniti⋆) ⊙ unite⋆
    ⇔⟨ assoc⊙r ⟩
  swap₊ ⊙ (uniti⋆ ⊙ unite⋆)
    ⇔⟨ id⇔ ⊡ linv⊙l ⟩
  swap₊ ⊙ id⟷
    ⇔⟨ idr⊙l ⟩
  swap₊ □
```

Equivalence of Equivalences

Definition 4 (Equivalence of equivalences).

Two equivalences $eq_1, eq_2 : A \simeq B$ are themselves equivalent, written $eq_2 \approxeq eq_2$, if the forward and backward functions underlying eq_1 and eq_2 are homotopic.

```
record _≋_ {A B : Set} (eq₁ eq₂ : A ≃ B) : Set where
  constructor eq
  open isqinv
  field
    f≡ : proj₁ eq₁ ∼ proj₁ eq₂
    g≡ : g (proj₂ eq₁) ∼ g (proj₂ eq₂)
```

Categorification

Theorem 5.

Taking Set as objects, $\simeq$ for homomorphisms and $\approx$ as equivalence of $\simeq$ gives a Groupoid.

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- identity equivalence exists $id_{\simeq}$
- equivalences compose $\boxdot$
- Coherence Laws require the identity equivalence to be a left and right unit for composition and composition to be associative

$$
\begin{array}{rcl}
id_{\simeq} \boxdot E & \approx & E \\
E \boxdot id_{\simeq} & \approx & E \\
E_1 \boxdot (E_2 \boxdot E_3) & \approx & (E_1 \boxdot E_2) \boxdot E_3
\end{array}
$$

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E_1 \boxdot (E_2 \boxdot E_3) & \approx & (E_1 \boxdot E_2) \boxdot E_3
\end{array}
$$

And that is the source of our next level combinators

Categorification II

The category of finite types and equivalences is much richer than a plain category:

- every morphism has an inverse; it is a groupoid
- $\oplus, \bot$ is a commutative monoid; category is symmetric monoidal
- $\otimes, \top$ is another commutative monoid; category is symmetric bi-monoidal
- the multiplicative monoid distributes over the additive one; it is a rig groupoid
- equivalence of morphisms is up to $\approx$; category is a weak rig groupoid
- in fact, a weak rig groupoid bicategory !

Categorification III

Each coherence law is an equivalence of equivalences, which is an equivalence of combinational circuits, which must be an equivalence of Π programs.

- The full set of coherence laws is huge ($56 + 1$ laws)
- Add them to Π, level 2
- Double them to give syntactic reversibility ($2 \cdot 56 + 1 = 113$ laws)
- Can write Π programs (now level 1) and can write level 2 programs that manipulate level 1 programs in semantically-preserving ways!

Π level 2

Let $c_1 : t_1 \leftrightarrow t_2$, $c_2 : t_2 \leftrightarrow t_3$, and $c_3 : t_3 \leftrightarrow t_4$:

$$c_1 \odot (c_2 \odot c_3) \Leftrightarrow (c_1 \odot c_2) \odot c_3$$
$$(c_1 \oplus (c_2 \oplus c_3)) \odot \mathit{assocl}_+ \Leftrightarrow \mathit{assocl}_+ \odot ((c_1 \oplus c_2) \oplus c_3)$$
$$(c_1 \otimes (c_2 \otimes c_3)) \odot \mathit{assocl}_* \Leftrightarrow \mathit{assocl}_* \odot ((c_1 \otimes c_2) \otimes c_3)$$
$$((c_1 \oplus c_2) \oplus c_3) \odot \mathit{assocr}_+ \Leftrightarrow \mathit{assocr}_+ \odot (c_1 \oplus (c_2 \oplus c_3))$$
$$((c_1 \otimes c_2) \otimes c_3) \odot \mathit{assocr}_* \Leftrightarrow \mathit{assocr}_* \odot (c_1 \otimes (c_2 \otimes c_3))$$
$$\mathit{assocr}_+ \odot \mathit{assocr}_+ \Leftrightarrow ((\mathit{assocr}_+ \oplus \mathit{id}) \odot \mathit{assocr}_+) \odot (\mathit{id} \oplus \mathit{assocr}_+)$$
$$\mathit{assocr}_* \odot \mathit{assocr}_* \Leftrightarrow ((\mathit{assocr}_* \otimes \mathit{id}) \odot \mathit{assocr}_*) \odot (\mathit{id} \otimes \mathit{assocr}_*)$$

$$((a \oplus b) \otimes c) \odot \mathit{dist} \Leftrightarrow \mathit{dist} \odot ((a \otimes c) \oplus (b \otimes c))$$
$$(a \otimes (b \oplus c)) \odot \mathit{distl} \Leftrightarrow \mathit{distl} \odot ((a \otimes b) \oplus (a \otimes c))$$
$$((a \otimes c) \oplus (b \otimes c)) \odot \mathit{factor} \Leftrightarrow \mathit{factor} \odot ((a \oplus b) \otimes c)$$
$$((a \otimes b) \oplus (a \otimes c)) \odot \mathit{factorl} \Leftrightarrow \mathit{factorl} \odot (a \otimes (b \oplus c))$$

Let $c, c_1, c_2, c_3 : t_1 \leftrightarrow t_2$ and $c', c'' : t_3 \leftrightarrow t_4$:

$$\mathit{id} \odot c \Leftrightarrow c \qquad c \odot \mathit{id} \Leftrightarrow c \qquad c \odot !c \Leftrightarrow \mathit{id} \qquad !c \odot c \Leftrightarrow \mathit{id}$$
$$c \Leftrightarrow c \qquad \dfrac{c_1 \Leftrightarrow c_2 \quad c_2 \Leftrightarrow c_3}{c_1 \Leftrightarrow c_3} \qquad \dfrac{c_1 \Leftrightarrow c' \quad c_2 \Leftrightarrow c''}{c_1 \odot c_2 \Leftrightarrow c' \odot c''}$$

Let $c_0 : 0 \leftrightarrow 0$, $c_1 : 1 \leftrightarrow 1$, and $c : t_1 \leftrightarrow t_2$:

$$\mathit{identl}_+ \odot c \Leftrightarrow (c_0 \oplus c) \odot \mathit{identl}_+ \qquad \mathit{identr}_+ \odot (c_0 \oplus c) \Leftrightarrow c \odot \mathit{identr}_+$$
$$\mathit{unite}_+ r \odot c \Leftrightarrow (c \oplus c_0) \odot \mathit{unite}_+ r \qquad \mathit{uniti}_+ r \odot (c \oplus c_0) \Leftrightarrow c \odot \mathit{uniti}_+ r$$
$$\mathit{identl}_* \odot c \Leftrightarrow (c_1 \otimes c) \odot \mathit{identl}_* \qquad \mathit{identr}_* \odot (c_1 \otimes c) \Leftrightarrow c \odot \mathit{identr}_+$$
$$\mathit{unite}_* r \odot c \Leftrightarrow (c \otimes c_1) \odot \mathit{unite}_* r \qquad \mathit{uniti}_* r \odot (c \otimes c_1) \Leftrightarrow c \odot \mathit{uniti}_* r$$
$$\mathit{identl}_* \Leftrightarrow \mathit{distl} \odot (\mathit{identl}_* \oplus \mathit{identl}_*)$$
$$\mathit{identl}_+ \Leftrightarrow \mathit{swap}_+ \odot \mathit{uniti}_+ r \qquad \mathit{identl}_* \Leftrightarrow \mathit{swap}_* \odot \mathit{uniti}_* r$$
$$(\mathit{id} \otimes \mathit{swap}_+) \odot \mathit{distl} \quad \Leftrightarrow \quad \mathit{distl} \odot \mathit{swap}_+$$
$$\mathit{dist} \odot (\mathit{swap}_* \oplus \mathit{swap}_*) \quad \Leftrightarrow \quad \mathit{swap}_* \odot \mathit{distl}$$

Let $c_1 : t_1 \leftrightarrow t_2$ and $c_2 : t_3 \leftrightarrow t_4$:

$$\mathit{swap}_+ \odot (c_1 \oplus c_2) \Leftrightarrow (c_2 \oplus c_1) \odot \mathit{swap}_+ \qquad \mathit{swap}_* \odot (c_1 \otimes c_2) \Leftrightarrow (c_2 \otimes c_1) \odot \mathit{swap}_*$$
$$(\mathit{assocr}_+ \odot \mathit{swap}_+) \odot \mathit{assocr}_+ \Leftrightarrow ((\mathit{swap}_+ \oplus \mathit{id}) \odot \mathit{assocr}_+) \odot (\mathit{id} \oplus \mathit{swap}_+)$$
$$(\mathit{assocl}_+ \odot \mathit{swap}_+) \odot \mathit{assocl}_+ \Leftrightarrow ((\mathit{id} \oplus \mathit{swap}_+) \odot \mathit{assocl}_+) \odot (\mathit{swap}_+ \oplus \mathit{id})$$
$$(\mathit{assocr}_* \odot \mathit{swap}_*) \odot \mathit{assocr}_* \Leftrightarrow ((\mathit{swap}_* \otimes \mathit{id}) \odot \mathit{assocr}_*) \odot (\mathit{id} \otimes \mathit{swap}_*)$$
$$(\mathit{assocl}_* \odot \mathit{swap}_*) \odot \mathit{assocl}_* \Leftrightarrow ((\mathit{id} \otimes \mathit{swap}_*) \odot \mathit{assocl}_*) \odot (\mathit{swap}_* \otimes \mathit{id})$$

Let $c_1 : t_1 \leftrightarrow t_2$, $c_2 : t_3 \leftrightarrow t_4$, $c_3 : t_1 \leftrightarrow t_2$, and $c_4 : t_3 \leftrightarrow t_4$.
Let $a_1 : t_5 \leftrightarrow t_1$, $a_2 : t_6 \leftrightarrow t_2$, $a_3 : t_1 \leftrightarrow t_3$, and $a_4 : t_2 \leftrightarrow t_4$.

$$\dfrac{c_1 \Leftrightarrow c_3 \quad c_2 \Leftrightarrow c_4}{c_1 \oplus c_2 \Leftrightarrow c_3 \oplus c_4} \qquad \dfrac{c_1 \Leftrightarrow c_3 \quad c_2 \Leftrightarrow c_4}{c_1 \otimes c_2 \Leftrightarrow c_3 \otimes c_4}$$
$$\mathit{id} \oplus \mathit{id} \Leftrightarrow \mathit{id} \qquad \mathit{id} \otimes \mathit{id} \Leftrightarrow \mathit{id}$$
$$(a_1 \odot a_3) \oplus (a_2 \odot a_4) \Leftrightarrow (a_1 \oplus a_2) \odot (a_3 \oplus a_4)$$
$$(a_1 \odot a_3) \otimes (a_2 \odot a_4) \Leftrightarrow (a_1 \otimes a_2) \odot (a_3 \otimes a_4)$$
$$\mathit{unite}_+ r \oplus \mathit{id} \quad \Leftrightarrow \quad \mathit{assocr}_+ \odot (\mathit{id} \oplus \mathit{identl}_+)$$
$$\mathit{unite}_* r \otimes \mathit{id} \quad \Leftrightarrow \quad \mathit{assocr}_* \odot (\mathit{id} \otimes \mathit{identl}_*)$$

Let $c : t_1 \leftrightarrow t_2$:

$$(c \otimes \mathit{id}) \odot \mathit{absorbl} \Leftrightarrow \mathit{absorbl} \odot \mathit{id} \qquad (\mathit{id} \otimes c) \odot \mathit{absorbr} \Leftrightarrow \mathit{absorbr} \odot \mathit{id}$$
$$\mathit{id} \odot \mathit{factorl}_0 \Leftrightarrow \mathit{factorl}_0 \odot (\mathit{id} \otimes c) \qquad \mathit{id} \odot \mathit{factorzr} \Leftrightarrow \mathit{factorzr} \odot (c \otimes \mathit{id})$$
$$\mathit{absorbr} \Leftrightarrow \mathit{absorbl}$$
$$\mathit{absorbr} \Leftrightarrow (\mathit{distl} \odot (\mathit{absorbr} \oplus \mathit{absorbr})) \odot \mathit{identl}_+$$
$$\mathit{unite}_* r \Leftrightarrow \mathit{absorbr} \qquad \mathit{absorbl} \Leftrightarrow \mathit{swap}_* \odot \mathit{absorbr}$$
$$\mathit{absorbr} \Leftrightarrow (\mathit{assocl}_* \odot (\mathit{absorbr} \otimes \mathit{id})) \odot \mathit{absorbr}$$
$$(\mathit{id} \otimes \mathit{absorbr}) \odot \mathit{absorbl} \Leftrightarrow (\mathit{assocl}_* \odot (\mathit{absorbl} \otimes \mathit{id})) \odot \mathit{absorbr}$$
$$\mathit{id} \otimes \mathit{identl}_+ \Leftrightarrow (\mathit{distl} \odot (\mathit{absorbl} \oplus \mathit{id})) \odot \mathit{identl}_+$$
$$((\mathit{assocl}_+ \otimes \mathit{id}) \odot \mathit{dist}) \odot (\mathit{dist} \oplus \mathit{id}) \Leftrightarrow (\mathit{dist} \odot (\mathit{id} \oplus \mathit{dist})) \odot \mathit{assocl}_+$$
$$\mathit{assocl}_* \odot \mathit{distl} \Leftrightarrow ((\mathit{id} \otimes \mathit{distl}) \odot \mathit{distl}) \odot (\mathit{assocl}_* \oplus \mathit{assocl}_*)$$
$$\begin{aligned}(\mathit{distl} \odot (\mathit{dist} \oplus \mathit{dist})) \odot \mathit{assocl}_+ \quad \Leftrightarrow \quad & \mathit{dist} \odot (\mathit{distl} \oplus \mathit{distl}) \odot \mathit{assocl}_+ \odot \\ & (\mathit{assocr}_+ \oplus \mathit{id}) \odot ((\mathit{id} \oplus \mathit{swap}_+) \oplus \mathit{id}) \odot \\ & (\mathit{assocl}_+ \oplus \mathit{id})\end{aligned}$$

Revised Curry-Howard (here)

Rig of natural numbers with 0, 1, +, and * (1+1)*(1+0)	Syntax of finite types $(\top \uplus \top) \times (\top \uplus \bot)$
Semiring identities $a + 0 = a$ Proofs	Type isomorphisms $A \uplus \bot \simeq A$ Programs (reversible)
Rig category $\Rightarrow$ Coherence laws	Programs $\Rightarrow$ Program equivalences and transformation

Revised Curry-Howard (speculation)

Algebra	Syntactic classifier "type"
Equations	Isomorphisms
Proofs	Programs (reversible)
Categorification $\Rightarrow$ Coherence laws	Programs $\Rightarrow$ Program equivalences and transformation