Representing, Manipulating and Optimizing Reversible Circuits

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Reversible Computing

The “obvious” intersection between quantum computing and programming languages is reversible computing.

Representing Reversible Circuits

truth table, matrix, reed muller expansion, product of cycles, decision diagram, etc.

[any easy way to reproduce Figure 4 on p.7 of Saeedi and Markov? —JC]

[important remark: these are all *Boolean* circuits! —JC]

Most important part: reversible circuits are equivalent to permutations.

A (Foundational) Syntactic Theory

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2. is easy to mechanically manipulate
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Start with a *foundational* syntactic theory on our way there:

1. easy to explain
2. clear operational rules
3. fully justified by the semantics
4. sound and complete reasoning
5. sound and complete methods of optimization

Starting Point

Typed isomorphisms. First, a universe of (finite) types

```
data U : Set where
  ZERO  : U
  ONE   : U
  PLUS  : U → U → U
  TIMES : U → U → U
```

and its interpretation $[\![\_]\!] : \mathsf{U} \to \mathsf{Set}$

$$
\begin{aligned}
[\![\ \mathsf{ZERO}\ ]\!] &= \bot \\
[\![\ \mathsf{ONE}\ ]\!] &= \top \\
[\![\ \mathsf{PLUS}\ t_1\ t_2\ ]\!] &= [\![\ t_1\ ]\!] \uplus [\![\ t_2\ ]\!] \\
[\![\ \mathsf{TIMES}\ t_1\ t_2\ ]\!] &= [\![\ t_1\ ]\!] \times [\![\ t_2\ ]\!]
\end{aligned}
$$

Equivalences and semirings

If we denote type equivalence by $\simeq$, then we can prove that

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If $A \simeq \mathrm{Fin}\, m$, $B \simeq \mathrm{Fin}\, n$ and $A \simeq B$ then $m \equiv n$.

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Theorem 3.

If $A \simeq \mathrm{Fin}\, m$ and $B \simeq \mathrm{Fin}\, n$, then the type of all equivalences $A \simeq B$ is equivalent to the type of all permutations $\mathrm{Perm}\, n$.

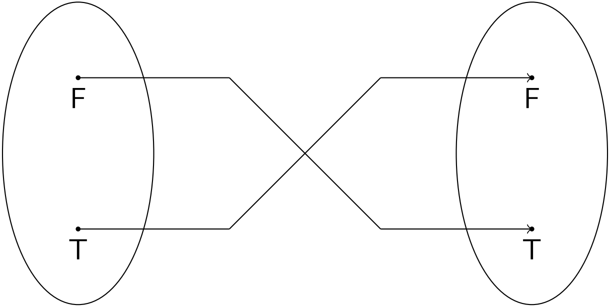
A Calculus of Permutations

In fact, we can say even more. Defining the usual disjoint union and tensor product of permutations, we can prove

Theorem 4.

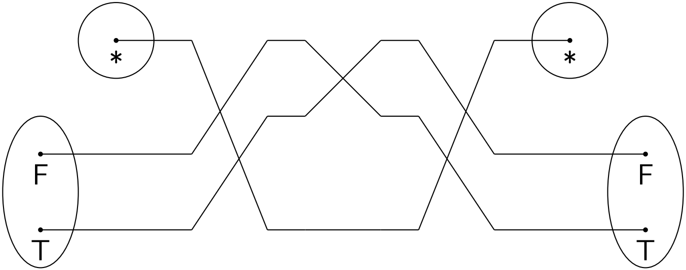
The equivalence of Theorem 3 is an isomorphism of semirings.

Example Circuit: Simple Negation



$\mathsf{n}_1 : \mathsf{BOOL} \longleftrightarrow \mathsf{BOOL}$
$\mathsf{n}_1 = \mathsf{swap}_+$

Example Circuit: Not So Simple Negation



$$
\begin{aligned}
\mathsf{n}_2 &: \mathsf{BOOL} \longleftrightarrow \mathsf{BOOL} \\
\mathsf{n}_2 &= \ \mathsf{uniti}\star \odot \\
&\quad\ \mathsf{swap}\star \odot \\
&\quad\ (\mathsf{swap}_+ \otimes \mathsf{id}\longleftrightarrow) \odot \\
&\quad\ \mathsf{swap}\star \odot \\
&\quad\ \mathsf{unite}\star
\end{aligned}
$$

Reasoning about Example Circuits

Algebraic manipulation of one circuit to the other:

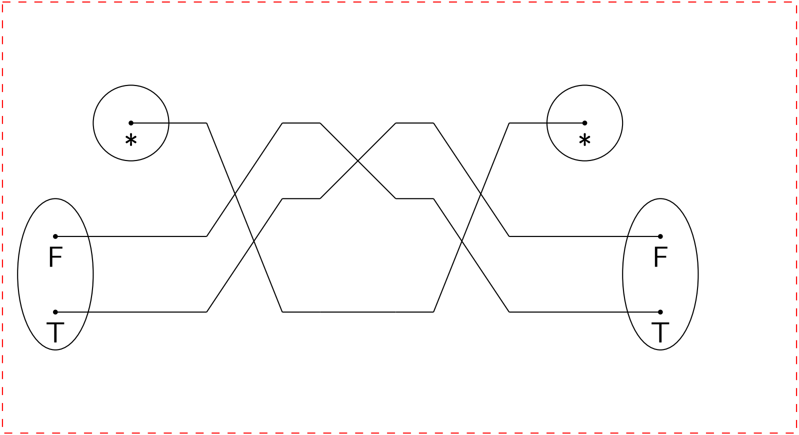
```
open import PiLevel1 hiding (negEx)

negEx : n₂ ⇔ n₁
negEx = uniti⋆ ⊙ (swap⋆ ⊙ ((swap₊ ⊗ id⟷) ⊙ (swap⋆ ⊙ unite⋆)))
    ⇔⟨ id⇔ ⊡ assoc⊙l ⟩
  uniti⋆ ⊙ ((swap⋆ ⊙ (swap₊ ⊗ id⟷)) ⊙ (swap⋆ ⊙ unite⋆))
    ⇔⟨ id⇔ ⊡ (swapl⋆⇔ ⊡ id⇔) ⟩
  uniti⋆ ⊙ (((id⟷ ⊗ swap₊) ⊙ swap⋆) ⊙ (swap⋆ ⊙ unite⋆))
    ⇔⟨ id⇔ ⊡ assoc⊙r ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ (swap⋆ ⊙ (swap⋆ ⊙ unite⋆)))
    ⇔⟨ id⇔ ⊡ (id⇔ ⊡ assoc⊙l) ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ ((swap⋆ ⊙ swap⋆) ⊙ unite⋆))
    ⇔⟨ id⇔ ⊡ (id⇔ ⊡ (linv⊙l ⊡ id⇔)) ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ (id⟷ ⊙ unite⋆))
    ⇔⟨ id⇔ ⊡ (id⇔ ⊡ idl⊙l) ⟩
  uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ unite⋆)
    ⇔⟨ assoc⊙l ⟩
  (uniti⋆ ⊙ (id⟷ ⊗ swap₊)) ⊙ unite⋆
    ⇔⟨ unitil⋆⇔ ⊡ id⇔ ⟩
```

Reasoning about Example Circuits

foo

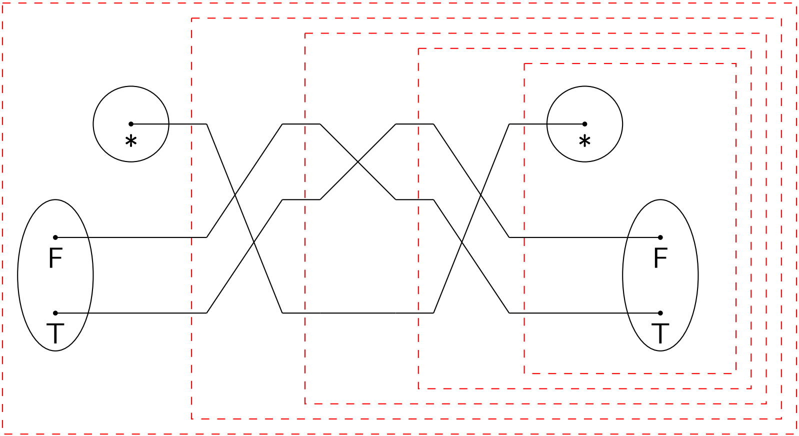
Visually

Original circuit:



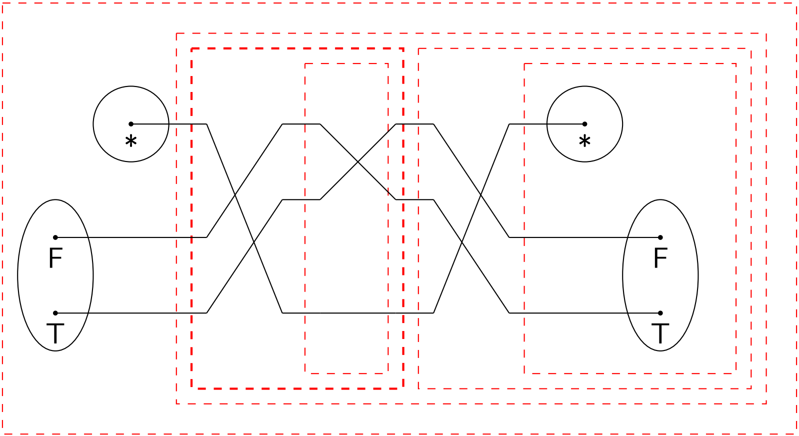
Visually

Making grouping explicit:



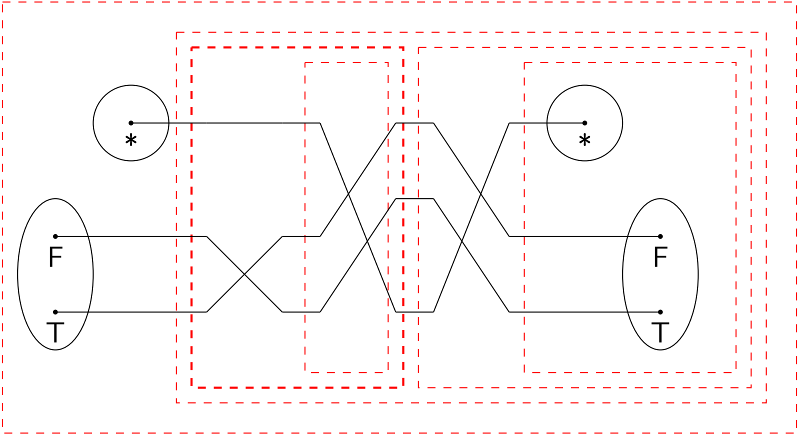
Visually

By associativity:



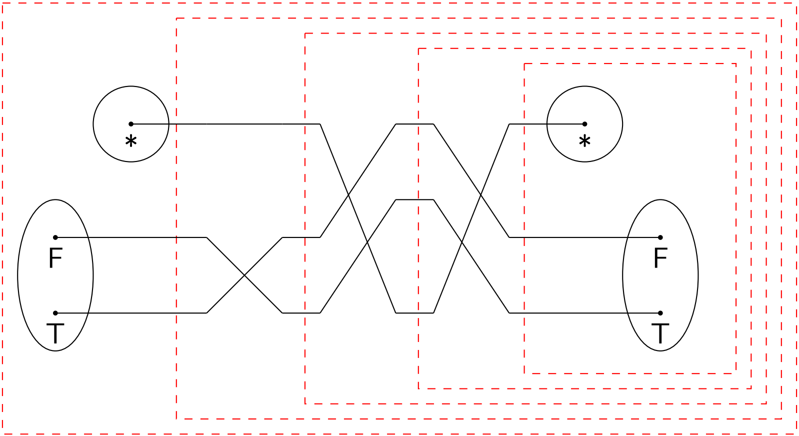
Visually

By pre-post-swap:



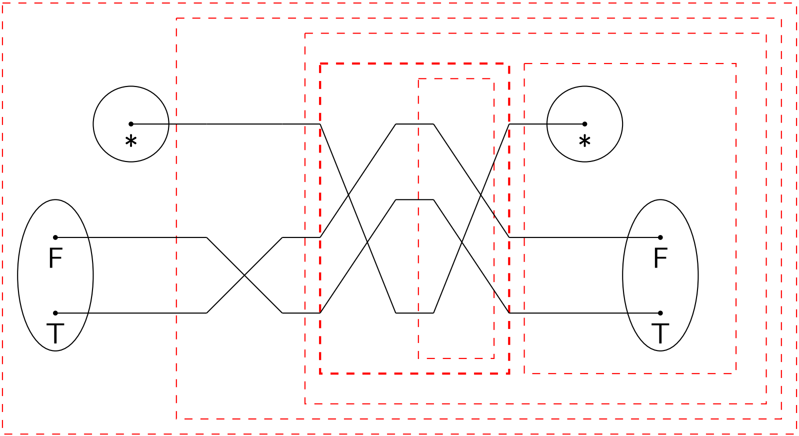
Visually

By associativity:



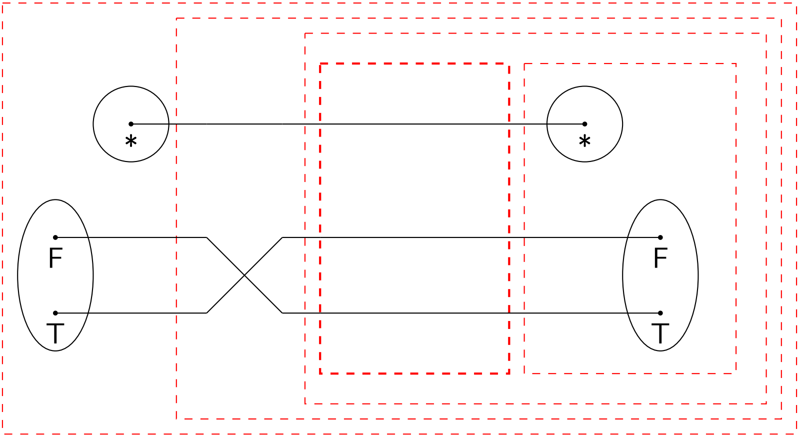
Visually

By associativity:



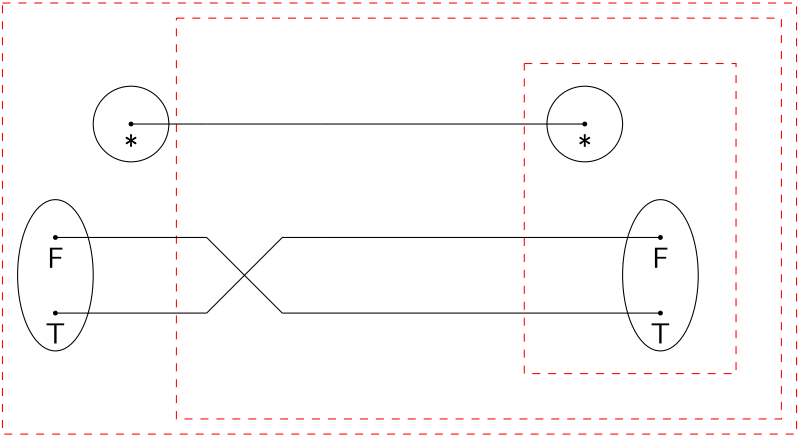
Visually

By swap-swap:



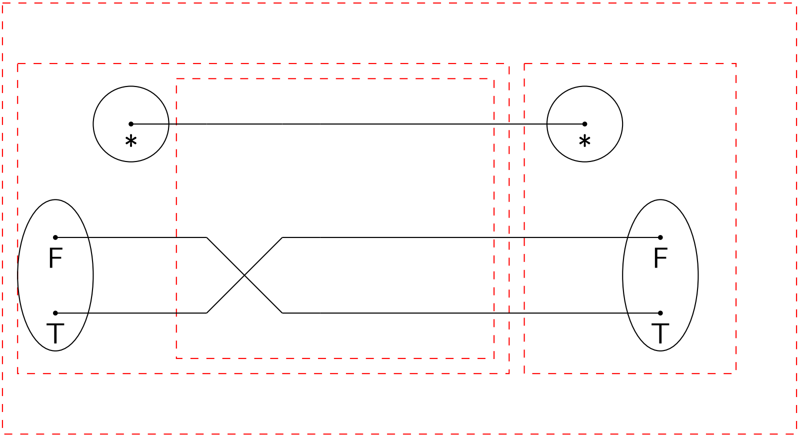
Visually

By id-compose-left:



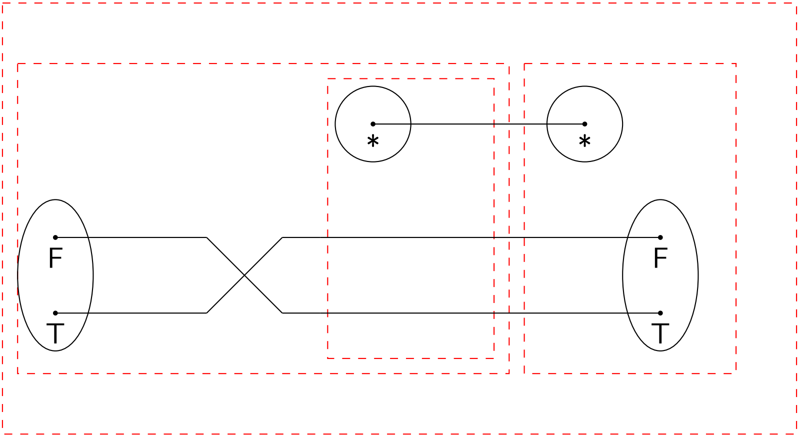
Visually

By associativity:



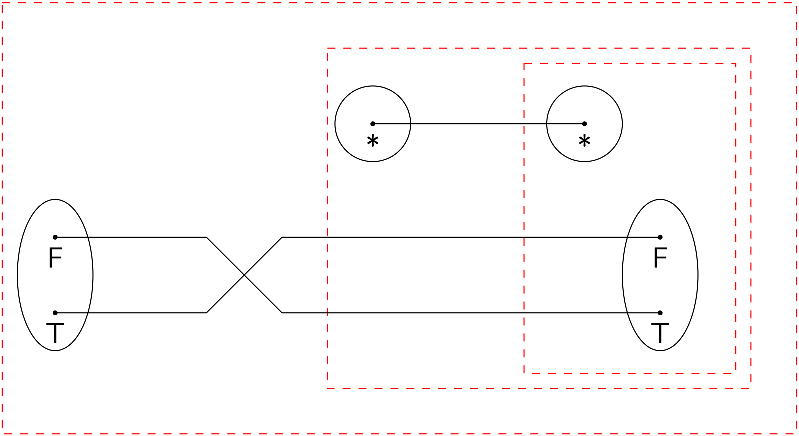
Visually

By swap-unit:



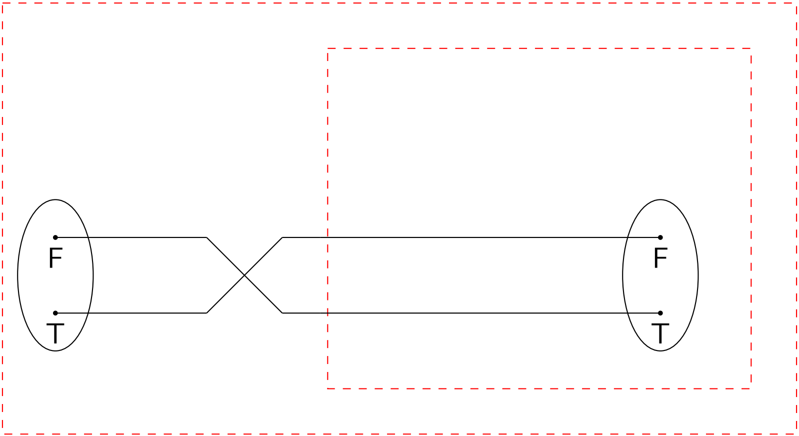
Visually

By associativity:



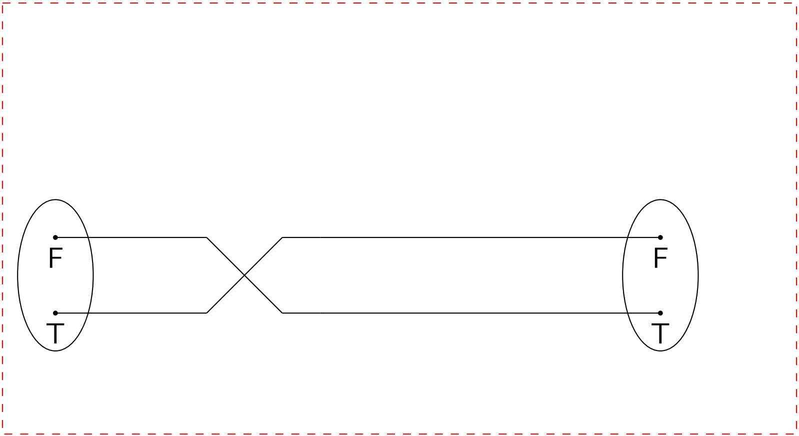
Visually

By unit-unit:



Visually

By id-unit-right:



Questions

- We don't want an ad hoc notation with ad hoc rewriting rules
- Notions of soundness; completeness; canonicity in some sense; what can we say?

1-paths vs. 2-paths

1-paths are between isomorphic types, e.g., $A * B$ and $B * A$. List them all.

1-paths vs. 2-paths

2-paths are between 1-paths, e.g.,
%endcode

1-paths vs. 2-paths