Polarized Cubical Types

Abstract

...

1. Introduction

In a computational world in which the laws of physics are embraced and resources are carefully maintained (e.g., quantum computing [Abramsky and Coecke 2004; Nielsen and Chuang 2000]), programs must be reversible. Although this is apparently a limiting idea, it turns out that conventional computation can be viewed as a special case of such resource-preserving reversible programs. This thesis has been explored for many years from different perspectives [Bennett 2003, 2010, 1973; Fredkin and Toffoli 1982; Landauer 1961, 1996; Toffoli 1980]. We build on the work of James and Sabry [2012] which expresses this thesis in a type theoretic computational framework, expressing computation via type isomorphisms.

Make sure we introduce the abbreviation HoTT in the introduction [The Univalent Foundations Program 2013].

2. Computing with Type Isomorphisms

The main syntactic vehicle for the developments in this paper is a simple language called Π whose only computations are isomorphisms between finite types. The set of types τ includes the empty type 0, the unit type 1, and conventional sum and product types. The values of these are the conventional ones: $()$ of type 1, $\mathsf{inl}\ v$ and $\mathsf{inr}\ v$ for injections into sum types, and (v_1, v_2) for product types:

(*Types*)	τ	::=	$0 \mid 1 \mid \tau_1 + \tau_2 \mid \tau_1 * \tau_2$
(*Values*)	v	::=	$() \mid \mathsf{inl}\ v \mid \mathsf{inr}\ v \mid (v_1, v_2)$
(*Combinator types*)			$\tau_1 \leftrightarrow \tau_2$
(*Combinators*)	c	::=	[*see Table 1*]

The interesting syntactic category of Π is that of *combinators* which are witnesses for type isomorphisms $\tau_1 \leftrightarrow \tau_2$. They consist of base combinators (on the left side of Table 1) and compositions (on the right side of the same table). Each line of the table on the left introduces a pair of dual constants[1] that witness the type isomorphism in the middle. This set of isomorphisms is known to be complete [Fiore 2004; Fiore et al. 2006] and the language is universal for hardware combinational circuits [James and Sabry 2012]. If recursive types and a trace operator (i.e., looping construct) are added, the language becomes Turing-complete [Bowman et al. 2011; James and Sabry 2012] but we will not be concerned with recursive types in this paper.

From the perspective of category theory, the language Π models what is called a *symmetric bimonoidal category* or a *commutative rig category*. These are categories with two binary operations $\oplus$ and $\otimes$ satisfying the axioms of a rig (i.e., a ring without negative elements also known as a semiring) up to coherent isomorphisms. And indeed the types of the Π-combinators are precisely the semiring axioms. A simple (slightly degenerate) example of such categories is the category of finite sets and permutations. Indeed, it is possible to interpret every Π-type as a finite set, the values as elements in these finite sets, and the combinators as permutations. This interpretation of Π, although valid, misses the point of taking isomorphisms seriously as *the* essence of computation. Luckily, an impressive amount of work has been happening in HoTT that builds around the computational content of equalities, equivalences, and isomorphisms, and this work will enable us to develop a more accurate model of Π that also supports higher-order functions and that additionally resolves some issues regarding the computational interpretation of functional extensionality and the univalence axiom of HoTT. For the readers familiar with the basic ideas of HoTT, a good intuition to keep in mind — to be further justified and explained in the remainder of the paper – is that the Π-combinators are syntax for *paths* in HoTT, and hence that computation in Π is nothing more than following paths in some complex combinatorial space.

3. The Int-Construction

Our immediate technical goal is to extend Π with a notion of higher-order functions. In the context of monoidal categories, it is known that a notion of higher-order functions emerges from having an additional degree of *symmetry*. In particular, the **Int**-construction of Joyal, Street, and Verity [1996] or the closely related $\mathcal{G}$ construction of linear logic [Abramsky 1996] construct higher-order *linear* functions by considering a new category built on top of a given base monoidal category. The objects of the new category are of the form $(\tau_1 - \tau_2)$ where τ_1 and τ_2 are objects in the base category. Intuitively, the component τ_1 is viewed as a conventional type whose elements represent values flowing, as usual, from producers to consumers. The component τ_2 is viewed as a *negative type* whose elements represent demands for values or equivalently values flowing backwards. Under this interpretation, a function is nothing but an object that converts a demand for an argument into the production of a result.

We therefore begin our development by extending Π with a new universe of types $\mathbb{T}$ that includes the composite types $(\tau_1 - \tau_2)$. Naturally, this new layer of types should have appropriate notions of sum and product types to be computationally interesting. As discussed near the end of this section, product types will require a much more extensive construction to be developed in the remainder of the paper. Thus, for the moment, we confine our discussion to

[1] where $swap_+$ and $swap_*$ are self-dual.

$$
\begin{array}{rrcll}
identl_+ : & 0+\tau & \leftrightarrow & \tau & : identr_+ \\
swap_+ : & \tau_1+\tau_2 & \leftrightarrow & \tau_2+\tau_1 & : swap_+ \\
assocl_+ : & \tau_1+(\tau_2+\tau_3) & \leftrightarrow & (\tau_1+\tau_2)+\tau_3 & : assocr_+ \\
identl_* : & 1*\tau & \leftrightarrow & \tau & : identr_* \\
swap_* : & \tau_1*\tau_2 & \leftrightarrow & \tau_2*\tau_1 & : swap_* \\
assocl_* : & \tau_1*(\tau_2*\tau_3) & \leftrightarrow & (\tau_1*\tau_2)*\tau_3 & : assocr_* \\
dist_0 : & 0*\tau & \leftrightarrow & 0 & : factor_0 \\
dist : & (\tau_1+\tau_2)*\tau_3 & \leftrightarrow & (\tau_1*\tau_3)+(\tau_2*\tau_3) & : factor
\end{array}
$$

$$\frac{}{\vdash id : \tau \leftrightarrow \tau} \qquad \frac{\vdash c : \tau_1 \leftrightarrow \tau_2}{\vdash sym\ c : \tau_2 \leftrightarrow \tau_1}$$

$$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad c_2 : \tau_2 \leftrightarrow \tau_3}{\vdash c_1 \fatsemi c_2 : \tau_1 \leftrightarrow \tau_3}$$

$$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \oplus c_2 : \tau_1+\tau_3 \leftrightarrow \tau_2+\tau_4}$$

$$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \otimes c_2 : \tau_1*\tau_3 \leftrightarrow \tau_2*\tau_4}$$

Table 1. Π-combinators [James and Sabry 2012]

$$
\begin{array}{rrcll}
identl_+ : & (0-0) \boxplus \mathbb{T} & \Leftrightarrow & \mathbb{T} & : identr_+ \\
swap_+ : & \mathbb{T}_1 \boxplus \mathbb{T}_2 & \Leftrightarrow & \mathbb{T}_2 \boxplus \mathbb{T}_1 & : swap_+ \\
assocl_+ : & \mathbb{T}_1 \boxplus (\mathbb{T}_2 \boxplus \mathbb{T}_3) & \Leftrightarrow & (\mathbb{T}_1 \boxplus \mathbb{T}_2) \boxplus \mathbb{T}_3 & : assocr_+
\end{array}
$$

$$\frac{}{\vdash_1 id : \mathbb{T} \Leftrightarrow \mathbb{T}} \qquad \frac{\vdash_1 \mathbb{C} : \mathbb{T}_1 \Leftrightarrow \mathbb{T}_2}{\vdash_1 sym\ \mathbb{C} : \mathbb{T}_2 \Leftrightarrow \mathbb{T}_1}$$

$$\frac{\vdash_1 \mathbb{C}_1 : \mathbb{T}_1 \Leftrightarrow \mathbb{T}_2 \quad \mathbb{C}_2 : \mathbb{T}_2 \Leftrightarrow \mathbb{T}_3}{\vdash_1 \mathbb{C}_1 \fatsemi \mathbb{C}_2 : \mathbb{T}_1 \Leftrightarrow \mathbb{T}_3}$$

$$\frac{\vdash_1 \mathbb{C}_1 : \mathbb{T}_1 \Leftrightarrow \mathbb{T}_2 \quad \mathbb{C}_2 : \mathbb{T}_3 \Leftrightarrow \mathbb{T}_4}{\vdash_1 \mathbb{C}_1 \oplus \mathbb{C}_2 : \mathbb{T}_1 \boxplus \mathbb{T}_3 \Leftrightarrow \mathbb{T}_2 \boxplus \mathbb{T}_4}$$

Table 2. 1d combinators

sum types in the new universe of types:

$$(\textit{1d types}) \quad \mathbb{T} \quad ::= \quad (\tau_1 - \tau_2) \mid \mathbb{T}_1 \boxplus \mathbb{T}_2$$

In anticipation of future developments, we will refer to the original types τ as 0-dimensional (0d) types and to the new types $\mathbb{T}$ as 1-dimensional (1d) types. The combinators of Table 1 all work on 0d types. These 0d combinators will be the base *values* of 1d types and will be manipulated by a new layer of 1d combinators:

$$
\begin{array}{lrcl}
(\textit{1d values}) & \mathbb{V} & ::= & c \mid \mathsf{inl}\ \mathbb{V} \mid \mathsf{inr}\ \mathbb{V} \\
(\textit{1d Combinator types}) & & & \mathbb{T}_1 \Leftrightarrow \mathbb{T}_2 \\
(\textit{1d combinators}) & \mathbb{C} & ::= & [\textit{see Table 2}]
\end{array}
$$

We use the same names for the 0d and 1d combinators which should not lead to confusion as the types will always be clear from context. The new layer of 1d combinators consists of lifted versions of all the 0d combinators that do not refer to product types. The lifting of the empty type 0 is the type $(0-0)$ which is no longer empty since we have (at least) $id : 0 \leftrightarrow 0$. We return to this subtle point below.

The 1d values have the following type rules:

$$\frac{\vdash c : \tau_1 \leftrightarrow \tau_2}{\vdash_1 c : \tau_2 - \tau_1}$$

$$\frac{\vdash_1 \mathbb{V} : \mathbb{T}_1}{\vdash_1 \mathsf{inl}\ \mathbb{V} : \mathbb{T}_1 \boxplus \mathbb{T}_2} \qquad \frac{\vdash_1 \mathbb{V} : \mathbb{T}_2}{\vdash_1 \mathsf{inr}\ \mathbb{V} : \mathbb{T}_1 \boxplus \mathbb{T}_2}$$

A 0d-combinator $\tau_1 \leftrightarrow \tau_2$ is viewed as a function demanding τ_1 and producing τ_2 which is encoded by putting τ_1 in the negative position. The values $\mathsf{inl}\ \mathbb{V}$ and $\mathsf{inr}\ \mathbb{V}$ are the usual injection into the sum type. Before presenting the formal semantics of the 1d level of Π, we consider a small example showing we can reason about equivalence of 0d combinators. Consider a 0d-combinator $c : \tau_1 + \tau_2 \leftrightarrow \tau_3 + \tau_4$. In the 1d world, this combinator has type $(\tau_3 + \tau_4) - (\tau_1 + \tau_2)$.

in the int construction the problem with 0 is avoided because boxplus expands to plus in the base category where the 0 cancels. we don't want to expand in the base category and lose the structure of paths. instead we add a new 2path that eliminates the 0-0

we need combinator that use neg and that show that we have firstclass function (at least 2nd order). we need to show some identities on paths that are validated by the semantics

The semantics consist of a pair of mutually recursive evaluators that take a 1d combinator and a 1d value and either propagate the value in the "forward" $\triangleright$ direction or in the "backwards" $\triangleleft$

direction:

$$
\begin{array}{rcll}
\mathit{identl}_+ \triangleright & (\mathsf{inl}\ \mathbb{V}) & = & ?? \\
\mathit{identl}_+ \triangleright & (\mathsf{inr}\ \mathbb{V}) & = & \mathbb{V} \\
\mathit{identr}_+ \triangleright & \mathbb{V} & = & \mathsf{inr}\ \mathbb{V} \\
\mathit{swap}_+ \triangleright & (\mathsf{inl}\ \mathbb{V}) & = & \mathsf{inr}\ \mathbb{V} \\
\mathit{swap}_+ \triangleright & (\mathsf{inr}\ \mathbb{V}) & = & \mathsf{inl}\ \mathbb{V} \\
\mathit{assocl}_+ \triangleright & (\mathsf{inl}\ \mathbb{V}) & = & \mathsf{inl}\ (\mathsf{inl}\ \mathbb{V}) \\
\mathit{assocl}_+ \triangleright & (\mathsf{inr}\ (\mathsf{inl}\ \mathbb{V})) & = & \mathsf{inl}\ (\mathsf{inr}\ \mathbb{V}) \\
\mathit{assocl}_+ \triangleright & (\mathsf{inr}\ (\mathsf{inr}\ \mathbb{V})) & = & \mathsf{inr}\ \mathbb{V} \\
\mathit{assocr}_+ \triangleright & (\mathsf{inl}\ (\mathsf{inl}\ \mathbb{V})) & = & \mathsf{inl}\ \mathbb{V} \\
\mathit{assocr}_+ \triangleright & (\mathsf{inl}\ (\mathsf{inr}\ \mathbb{V})) & = & \mathsf{inr}\ (\mathsf{inl}\ \mathbb{V}) \\
\mathit{assocr}_+ \triangleright & (\mathsf{inr}\ \mathbb{V}) & = & \mathsf{inr}\ (\mathsf{inr}\ \mathbb{V}) \\
\mathit{id} \triangleright & \mathbb{V} & = & \mathbb{V} \\
(\mathit{sym}\ \mathbb{C}) \triangleright & \mathbb{V} & = & \mathbb{C} \triangleleft \mathbb{V} \\
(\mathbb{C}_1 \fatsemi \mathbb{C}_2) \triangleright & \mathbb{V} & = & \mathbb{C}_2 \triangleright (\mathbb{C}_1 \triangleright \mathbb{V}) \\
(\mathbb{C}_1 \oplus \mathbb{C}_2) \triangleright & (\mathsf{inl}\ \mathbb{V}) & = & \mathsf{inl}\ (\mathbb{C}_1 \triangleright \mathbb{V}) \\
(\mathbb{C}_1 \oplus \mathbb{C}_2) \triangleright & (\mathsf{inr}\ \mathbb{V}) & = & \mathsf{inr}\ (\mathbb{C}_2 \triangleright \mathbb{V}) \\
\\
\mathit{identl}_+ \triangleleft & \mathbb{V} & = & \mathsf{inr}\ \mathbb{V} \\
\mathit{identr}_+ \triangleleft & (\mathsf{inl}\ \mathbb{V}) & = & ?? \\
\mathit{identr}_+ \triangleleft & (\mathsf{inr}\ \mathbb{V}) & = & \mathbb{V} \\
\mathit{swap}_+ \triangleleft & (\mathsf{inl}\ \mathbb{V}) & = & \mathsf{inr}\ \mathbb{V} \\
\mathit{swap}_+ \triangleleft & (\mathsf{inr}\ \mathbb{V}) & = & \mathsf{inl}\ \mathbb{V} \\
\mathit{assocl}_+ \triangleleft & (\mathsf{inl}\ (\mathsf{inl}\ \mathbb{V})) & = & \mathsf{inl}\ \mathbb{V} \\
\mathit{assocl}_+ \triangleleft & (\mathsf{inl}\ (\mathsf{inr}\ \mathbb{V})) & = & \mathsf{inr}\ (\mathsf{inl}\ \mathbb{V}) \\
\mathit{assocl}_+ \triangleleft & (\mathsf{inr}\ \mathbb{V}) & = & \mathsf{inr}\ (\mathsf{inr}\ \mathbb{V}) \\
\mathit{assocr}_+ \triangleleft & (\mathsf{inl}\ \mathbb{V}) & = & \mathsf{inl}\ (\mathsf{inl}\ \mathbb{V}) \\
\mathit{assocr}_+ \triangleleft & (\mathsf{inr}\ (\mathsf{inl}\ \mathbb{V})) & = & \mathsf{inl}\ (\mathsf{inr}\ \mathbb{V}) \\
\mathit{assocr}_+ \triangleleft & (\mathsf{inr}\ (\mathsf{inr}\ \mathbb{V})) & = & \mathsf{inr}\ \mathbb{V} \\
\mathit{id} \triangleleft & \mathbb{V} & = & \mathbb{V} \\
(\mathit{sym}\ \mathbb{C}) \triangleleft & \mathbb{V} & = & \mathbb{C} \triangleright \mathbb{V} \\
(\mathbb{C}_1 \fatsemi \mathbb{C}_2) \triangleleft & \mathbb{V} & = & \mathbb{C}_1 \triangleleft (\mathbb{C}_2 \triangleleft \mathbb{V}) \\
(\mathbb{C}_1 \oplus \mathbb{C}_2) \triangleleft & (\mathsf{inl}\ \mathbb{V}) & = & \mathsf{inl}\ (\mathbb{C}_1 \triangleleft \mathbb{V}) \\
(\mathbb{C}_1 \oplus \mathbb{C}_2) \triangleleft & (\mathsf{inr}\ \mathbb{V}) & = & \mathsf{inr}\ (\mathbb{C}_2 \triangleleft \mathbb{V})
\end{array}
$$

The "phony" multiplication. The "obvious" definition for the product of 1d types is:

$$
\begin{array}{l}
(\tau_1 - \tau_2) \boxtimes (\tau_3 - \tau_4) = \\
((\tau_1 * \tau_3) + (\tau_2 * \tau_4)) - ((\tau_1 * \tau_4) + (\tau_2 * \tau_3))
\end{array}
$$

This definition of multiplication is not functorial which means that we have built a limited notion of higher-order functions at the expense of losing the multiplicative structure at higher-levels. This problem turns out to be intimately related to a well-known problem in algebraic topology that goes back at least thirty years [Thomason 1980]. This problem was recently solved [Baas et al. 2012] using a technique whose fundamental ingredient is to add more dimensions. We exploit this idea in the remainder of the paper.

4. Cubes

We first define the syntax and then present a simple semantic model of types which is then refined.

4.1 Negative and Cubical Types

Our types τ include the empty type 0, the unit type 1, conventional sum and product types, as well as *negative* types:

$$\tau \quad ::= \quad 0 \mid 1 \mid \tau_1 + \tau_2 \mid \tau_1 * \tau_2 \mid \ -\tau$$

We use $\tau_1 - \tau_2$ to abbreviate $\tau_1 + (-\tau_2)$ and more interestingly $\tau_1 \multimap \tau_2$ to abbreviate $(-\tau_1) + \tau_2$. The *dimension* of a type is defined as follows:

$$
\begin{array}{rcl}
\mathsf{dim}(\cdot) & :: & \tau \to \mathbb{N} \\
\mathsf{dim}(0) & = & 0 \\
\mathsf{dim}(1) & = & 0 \\
\mathsf{dim}(\tau_1 + \tau_2) & = & \max(\mathsf{dim}(\tau_1), \mathsf{dim}(\tau_2)) \\
\mathsf{dim}(\tau_1 * \tau_2) & = & \mathsf{dim}(\tau_1) + \mathsf{dim}(\tau_2) \\
\mathsf{dim}(-\tau) & = & \max(1, \mathsf{dim}(\tau))
\end{array}
$$

The base types have dimension 0. If negative types are not used, all dimensions remain at 0. If negative types are used but no products of negative types appear anywhere, the dimension is raised to 1. This is the situation with the **Int** or $\mathcal{G}$ construction. Once negative and product types are freely used, the dimension can increase without bounds.

This point is made precise in the following tentative denotation of types (to be refined in Sec. **??**) which maps a type of dimension n to an n-dimensional cube. We represent such a cube syntactically as a binary tree of maximum depth n with nodes of the form $\boxed{\mathbb{T}_1 \mid \mathbb{T}_2}$. In such a node, $\mathbb{T}_1$ is the positive subspace and $\mathbb{T}_2$ (shaded in gray) is the negative subspace along the first dimension. Each of these subspaces is itself a cube of a lower dimension. The 0-dimensional cubes are plain sets representing the denotation of conventional first-order types. We use S to denote the denotations of these plain types. A 1-dimensional cube, $\boxed{S_1 \mid S_2}$, intuitively corresponds to the difference $\tau_1 - \tau_2$ of the two types whose denotations are S_1 and S_2 respectively. The type can be visualized as a "line" with polarized endpoints connecting the two points S_1 and S_2. A full 2-dimensional cube, $\boxed{(\boxed{S_1 \mid S_2}) \mid (\boxed{S_3 \mid S_4})}$, intuitively corresponds to the iterated difference of the appropriate types $(\tau_1 - \tau_2) - (\tau_3 - \tau_4)$ where the successive "colors" from the outermost box encode the sign. The type can be visualized as a "square" with polarized corners connecting the two lines corresponding to $(\tau_1 - \tau_2)$ and $(\tau_3 - \tau_4)$. (See Fig. 1 which is further explained after we discuss multiplication below.)

Formally, the denotation of types discussed so far is as follows:

$$
\begin{array}{rcl}
[\![0]\!] & = & \emptyset \\
[\![1]\!] & = & \{\star\} \\
[\![\tau_1 + \tau_2]\!] & = & [\![\tau_1]\!] \oplus [\![\tau_2]\!] \\
[\![\tau_1 * \tau_2]\!] & = & [\![\tau_1]\!] \otimes [\![\tau_2]\!] \\
[\![-\tau]\!] & = & \ominus [\![\tau]\!]
\end{array}
$$

where:

$$
\begin{array}{rcl}
S_1 \oplus S_2 & = & S_1 \uplus S_2 \\
S \oplus \boxed{\mathbb{T}_1 \mid \mathbb{T}_2} & = & \boxed{S \oplus \mathbb{T}_1 \mid \mathbb{T}_2} \\
\boxed{\mathbb{T}_1 \mid \mathbb{T}_2} \oplus S & = & \boxed{\mathbb{T}_1 \oplus S \mid \mathbb{T}_2} \\
\boxed{\mathbb{T}_1 \mid \mathbb{T}_2} \oplus \boxed{\mathbb{T}_3 \mid \mathbb{T}_4} & = & \boxed{\mathbb{T}_1 \oplus \mathbb{T}_3 \mid \mathbb{T}_2 \oplus \mathbb{T}_4}
\end{array}
$$

$$
\begin{array}{rcl}
S_1 \otimes S_2 & = & S_1 \times S_2 \\
S \otimes \boxed{\mathbb{T}_1 \mid \mathbb{T}_2} & = & \boxed{S \otimes \mathbb{T}_1 \mid S \otimes \mathbb{T}_2} \\
\boxed{\mathbb{T}_1 \mid \mathbb{T}_2} \otimes \mathbb{T} & = & \boxed{\mathbb{T}_1 \otimes \mathbb{T} \mid \mathbb{T}_2 \otimes \mathbb{T}}
\end{array}
$$

$$
\begin{array}{rcl}
\ominus S & = & \boxed{\ \mid S} \\
\ominus \boxed{\mathbb{T}_1 \mid \mathbb{T}_2} & = & \boxed{\ominus \mathbb{T}_2 \mid \ominus \mathbb{T}_1}
\end{array}
$$

The type 0 maps to the empty set. The type 1 maps to a singleton set. The sum of 0-dimensional types is the disjoint union as usual. For cubes of higher dimensions, the subspaces are recursively added. Note that the sum of 1-dimensional types reduces to the sum used in the **Int** construction. The definition of negation is natural: it recursively swaps the positive and negative subspaces. The product of 0-dimensional types is the cartesian product of sets.

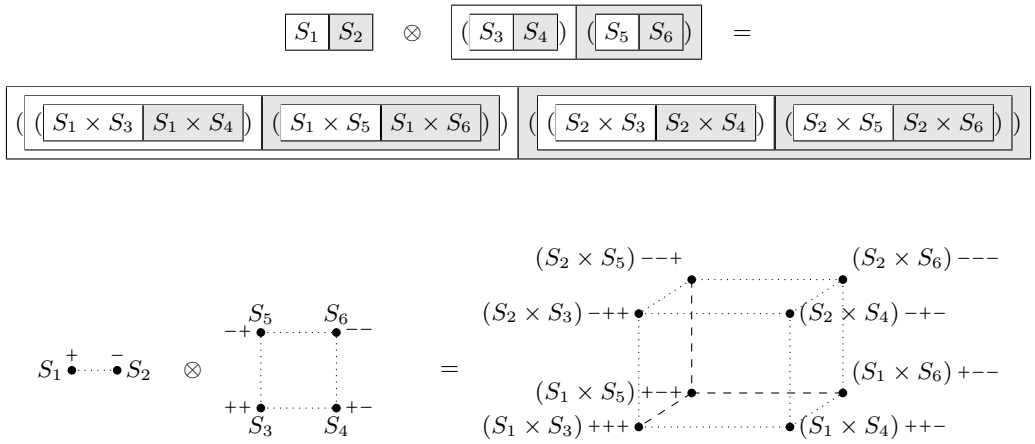


Figure 1. Example of multiplication of two cubical types.

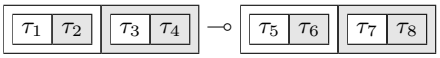
For cubes of higher-dimensions n and m, the result is of dimension $(n+m)$. The example in Fig. 1 illustrates the idea using the product of 1-dimensional cube (i.e., a line) with a 2-dimensional cube (i.e., a square). The result is a 3-dimensional cube as illustrated.

4.2 Higher-Order Functions

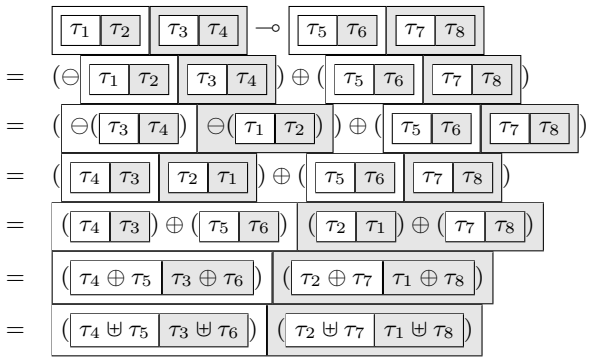
In the **Int** construction a function from $T_1 = (t_1 - t_2)$ to $T_2 = (t_3 - t_4)$ is represented as an object of type $-T_1 + T_2$. Expanding the definitions, we get:

$$\begin{array}{rcl} -T_1 + T_2 & = & -(t_1 - t_2) + (t_3 - t_4) \\ & = & (t_2 - t_1) + (t_3 - t_4) \\ & = & (t_2 + t_3) - (t_1 + t_4) \end{array}$$

The above calculation is consistent with our definitions specialized to 1-dimensional types. Note that the function is represented as an object of the same dimension as its input and output types. The situation generalizes to higher-dimensions. For example, consider a function of type



This function is represent by an object of dimension 2 as the calculation below shows:

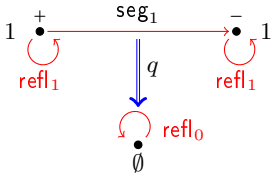


This may be better understood by visualizing each of the argument type and result types as two squares. The square representing the argument type is flipped in both dimensions effectively swapping the labels on both diagonals. The resulting square is then superimposed on the square for the result type to give the representation of the function as a first-class object.

4.3 Type Isomorphisms: Paths to the Rescue

Our proposed semantics of types identifies several structurally different types such as $(1+(1+1))$ and $((1+1)+1)$. In some sense, this is innocent as the types are isomorphic. However, in the operational semantics discussed in Sec. **??**, we make the computational content of such type isomorphisms explicit. Some other isomorphic types like $(\tau_1 * \tau_2)$ and $(\tau_2 * \tau_1)$ map to different cubes and are *not* identified: explicit isomorphisms are needed to mediate between them. We therefore need to enrich our model of types with isomorphisms connecting types we deem equivalent. So far, our types are modeled as cubes which are really sets indexed by polarities. An isomorphism between $(\tau_1 * \tau_2)$ and $(\tau_2 * \tau_1)$ requires nothing more than a pair of set-theoretic functions between the spaces, and that compose to the identity. What is much more interesting are the isomorphisms involving the empty type 0. In particular, if negative types are to be interpreted as their name suggests, we must have an isomorphism between $(t - t)$ and the empty type 0. Semantically the former denotes the "line" $\boxed{\mathbb{T}\ \mathbb{T}}$ and the latter denotes the empty set. Their denotations are different and there is no way, in the world of plain sets, to express the fact that these two spaces should be identified. What is needed is the ability to *contract* the *path* between the endpoints of the line to the trivial path on the empty type. This is, of course, where the ideas of homotopy (type) theory enter the development.

Consider the situation above in which we want to identify the spaces corresponding to the types $(1-1)$ and the empty type:



The top of the figure is the 1-dimensional cube representing the type $(1-1)$ as before except that we now add a path seg_1 to connect the two endpoints. This path identifies the two occurrences of 1. (Note that previously, the dotted lines in the figures were a visualization aid and were *not* meant to represent paths.) We also make explicit the trivial identity paths from every space to itself. The bottom of the figure is the 0-dimensional cube representing the empty type. To express the equivalence of $(1-1)$ and 0, we add a 2-path q, i.e. a path between paths, that connects the

path seg_1 to the trivial path refl_0. That effectively makes the two points "disappear." Surprisingly, that is everything that we need. The extension to higher dimensions just "works" because paths in HoTT have a rich structure. We explain the details after we include a short introduction of the necessary concepts from HoTT.

5. Related Work and Context

A ton of stuff here.

Connection to our work on univalence for finite types. We didn't have to rely on sets for 0-dimensional types. We could have used groupoids again.

6. Conclusion

References

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