# Polarized Cubical Types


## Abstract

...


## 1. Introduction

In a computational world in which the laws of physics are embraced and resources are carefully maintained (e.g., quantum computing [Abramsky and Coecke 2004; Nielsen and Chuang 2000]), programs must be reversible. Although this is apparently a limiting idea, it turns out that conventional computation can be viewed as a special case of such resource-preserving reversible programs. This thesis has been explored for many years from different perspectives [Bennett 2003, 2010, 1973; Fredkin and Toffoli 1982; Landauer 1961, 1996; Toffoli 1980]. We build on the work of James and Sabry [2012] which expresses this thesis in a type theoretic computational framework, expressing computation via type isomorphisms.

Make sure we introduce the abbreviation HoTT in the introduction [The Univalent Foundations Program 2013].

## 2. Computing with Type Isomorphisms

The main syntactic vehicle for the developments in this paper is a simple language called Π whose only computations are isomorphisms between finite types. We review this language in this section.

### 2.1 Syntax and Examples

The set of types $\tau$ includes the empty type 0, the unit type 1, and conventional sum and product types. The values of these types are the conventional ones: $()$ of type 1, $\mathsf{inl}\ v$ and $\mathsf{inr}\ v$ for injections into sum types, and $(v_1, v_2)$ for product types:

| | | | |
|---|---|---|---|
| (*Types*) | $\tau$ | ::= | $0 \mid 1 \mid \tau_1 + \tau_2 \mid \tau_1 * \tau_2$ |
| (*Values*) | $v$ | ::= | $() \mid \mathsf{inl}\ v \mid \mathsf{inr}\ v \mid (v_1, v_2)$ |
| (*Combinator types*) | | | $\tau_1 \leftrightarrow \tau_2$ |
| (*Combinators*) | $c$ | ::= | [*see Table 1*] |

The interesting syntactic category of Π is that of *combinators* which are witnesses for type isomorphisms $\tau_1 \leftrightarrow \tau_2$. They consist of base combinators (on the left side of Table 1) and compositions (on the right side of the same table). Each line of the table on the left introduces a pair of dual constants[1] that witness the type isomorphism in the middle. This set of isomorphisms is known to be complete [Fiore 2004; Fiore et al. 2006] and the language is universal for hardware combinational circuits [James and Sabry 2012]. The *trace* operator provides a bounded iteration facility which adds no expressiveness in the current context but will be needed in Sec. 3.[2]

As simple illustrative examples of "programming" in Π, here are three useful combinators that we define here for future reference:

$$
\begin{array}{ll}
assoc_1 : & \tau_1 + (\tau_2 + \tau_3) \leftrightarrow (\tau_2 + \tau_1) + \tau_3 \\
assoc_1 = & assocl_+ \fatsemi (swap_+ \oplus id)
\end{array}
$$

$$
\begin{array}{ll}
assoc_2 : & (\tau_1 + \tau_2) + \tau_3 \leftrightarrow (\tau_2 + \tau_3) + \tau_1 \\
assoc_2 = & (swap_+ \oplus id) \fatsemi assocr_+ \fatsemi (id \oplus swap_+) \fatsemi assocl_+
\end{array}
$$

$$
\begin{array}{ll}
assoc_3 : & (\tau_1 + \tau_2) + \tau_3 \leftrightarrow \tau_1 + (\tau_3 + \tau_2) \\
assoc_3 = & assocr_+ \fatsemi (id \oplus swap_+)
\end{array}
$$

### 2.2 Semantics

From the perspective of category theory, the language Π models what is called a traced *symmetric bimonoidal category* or a *commutative rig category*. These are categories with two binary operations $\oplus$ and $\otimes$ satisfying the axioms of a rig (i.e., a ring without negative elements also known as a semiring) up to coherent isomorphisms. And indeed the types of the Π-combinators are precisely the semiring axioms. A formal way of saying this is that Π is the *categorification* [Baez and Dolan 1998] of the natural numbers. A simple (slightly degenerate) example of such categories is the category of finite sets and permutations in which we interpret every Π-type as a finite set, the values as elements in these finite sets, and the combinators as permutations. Another common example of such categories is the category of finite dimensional vector spaces and linear maps over any field. Note that in this interpretation, the Π-type 0 maps to the 0-dimensional vector space which is *not* empty. Its unique element, the zero vector — which is present in every vector space — acts like a "bottom" everywhere-undefined element and hence the type behaves like the unit of addition and the annihilator of multiplication as desired.

Operationally, the semantics consists of a pair of mutually recursive evaluators that take a combinator and a value and propagate the value in the "forward" $\triangleright$ direction or in the "backwards" $\triangleleft$ direction. We show the complete forward evaluator; the backwards

[1] where $swap_+$ and $swap_*$ are self-dual.

[2] If recursive types are added, the trace operator provides unbounded iteration and the language becomes Turing complete [Bowman et al. 2011; James and Sabry 2012]. We will not be concerned with recursive types in this paper.

$$
\begin{array}{rrcll}
identl_+ : & 0+\tau & \leftrightarrow & \tau & : identr_+ \\
swap_+ : & \tau_1+\tau_2 & \leftrightarrow & \tau_2+\tau_1 & : swap_+ \\
assocl_+ : & \tau_1+(\tau_2+\tau_3) & \leftrightarrow & (\tau_1+\tau_2)+\tau_3 & : assocr_+ \\
identl_* : & 1*\tau & \leftrightarrow & \tau & : identr_* \\
swap_* : & \tau_1*\tau_2 & \leftrightarrow & \tau_2*\tau_1 & : swap_* \\
assocl_* : & \tau_1*(\tau_2*\tau_3) & \leftrightarrow & (\tau_1*\tau_2)*\tau_3 & : assocr_* \\
dist_0 : & 0*\tau & \leftrightarrow & 0 & : factor_0 \\
dist : & (\tau_1+\tau_2)*\tau_3 & \leftrightarrow & (\tau_1*\tau_3)+(\tau_2*\tau_3) & : factor
\end{array}
$$

$$
\frac{}{\vdash id : \tau \leftrightarrow \tau} \qquad \frac{\vdash c : \tau_1 \leftrightarrow \tau_2}{\vdash sym\ c : \tau_2 \leftrightarrow \tau_1}
$$

$$
\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_2 \leftrightarrow \tau_3}{\vdash c_1 \fatsemi c_2 : \tau_1 \leftrightarrow \tau_3}
$$

$$
\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \oplus c_2 : \tau_1+\tau_3 \leftrightarrow \tau_2+\tau_4}
$$

$$
\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \otimes c_2 : \tau_1*\tau_3 \leftrightarrow \tau_2*\tau_4}
$$

$$
\frac{\vdash c : \tau+\tau_1 \leftrightarrow \tau+\tau_2}{\vdash trace\ c : \tau_1 \leftrightarrow \tau_2}
$$

**Table 1.** Π-combinators [James and Sabry 2012]

evaluator consists of trivial modifications:

$$
\begin{array}{rcl}
identl_+ \triangleright (\mathsf{inr}\ v) & = & v \\
identr_+ \triangleright v & = & \mathsf{inr}\ v \\
swap_+ \triangleright (\mathsf{inl}\ v) & = & \mathsf{inr}\ v \\
swap_+ \triangleright (\mathsf{inr}\ v) & = & \mathsf{inl}\ v \\
assocl_+ \triangleright (\mathsf{inl}\ v) & = & \mathsf{inl}\ (\mathsf{inl}\ v) \\
assocl_+ \triangleright (\mathsf{inr}\ (\mathsf{inl}\ v)) & = & \mathsf{inl}\ (\mathsf{inr}\ v) \\
assocl_+ \triangleright (\mathsf{inr}\ (\mathsf{inr}\ v)) & = & \mathsf{inr}\ v \\
assocr_+ \triangleright (\mathsf{inl}\ (\mathsf{inl}\ v)) & = & \mathsf{inl}\ v \\
assocr_+ \triangleright (\mathsf{inl}\ (\mathsf{inr}\ v)) & = & \mathsf{inr}\ (\mathsf{inl}\ v) \\
assocr_+ \triangleright (\mathsf{inr}\ v) & = & \mathsf{inr}\ (\mathsf{inr}\ v) \\
identl_* \triangleright ((), v) & = & v \\
identr_* \triangleright v & = & ((), v) \\
swap_* \triangleright (v_1, v_2) & = & (v_2, v_1) \\
assocl_* \triangleright (v_1, (v_2, v_3)) & = & ((v_1, v_2), v_3) \\
assocr_* \triangleright ((v_1, v_2), v_3) & = & (v_1, (v_2, v_3)) \\
dist \triangleright (\mathsf{inl}\ v_1, v_3) & = & \mathsf{inl}\ (v_1, v_3) \\
dist \triangleright (\mathsf{inr}\ v_2, v_3) & = & \mathsf{inr}\ (v_2, v_3) \\
factor \triangleright (\mathsf{inl}\ (v_1, v_3)) & = & (\mathsf{inl}\ v_1, v_3) \\
factor \triangleright (\mathsf{inr}\ (v_2, v_3)) & = & (\mathsf{inr}\ v_2, v_3) \\
id \triangleright v & = & v \\
(sym\ c) \triangleright v & = & c \triangleleft v \\
(c_1 \fatsemi c_2) \triangleright v & = & c_2 \triangleright (c_1 \triangleright v) \\
(c_1 \oplus c_2) \triangleright (\mathsf{inl}\ v) & = & \mathsf{inl}\ (c_1 \triangleright v) \\
(c_1 \oplus c_2) \triangleright (\mathsf{inr}\ v) & = & \mathsf{inr}\ (c_2 \triangleright v) \\
(c_1 \otimes c_2) \triangleright (v_1, v_2) & = & (c_1 \triangleright v_1, c_2 \triangleright v_2) \\
(trace\ c) \triangleright v & = & \mathsf{loop}\ (c \triangleright (\mathsf{inr}\ v)) \\
\multicolumn{3}{r}{\textbf{where}\ \mathsf{loop}\ (\mathsf{inl}\ v) = \mathsf{loop}\ (c \triangleright (\mathsf{inl}\ v))} \\
\multicolumn{3}{r}{\mathsf{loop}\ (\mathsf{inr}\ v) = v}
\end{array}
$$

## 3. The Int Construction

In the context of monoidal categories, it is known that a notion of higher-order functions emerges from having an additional degree of *symmetry*. In particular, both the **Int** construction of Joyal, Street, and Verity [1996] and the closely related $\mathcal{G}$ construction of linear logic [Abramsky 1996] construct higher-order *linear* functions by considering a new category built on top of a given base traced monoidal category. The objects of the new category are of the form $(\tau_1 - \tau_2)$ where $\tau_1$ and $\tau_2$ are objects in the base category. Intuitively, the component $\tau_1$ is viewed as a conventional type whose elements represent values flowing, as usual, from producers to consumers. The component $\tau_2$ is viewed as a *negative type* whose elements represent demands for values or equivalently values flowing backwards. Under this interpretation, and as we explain below, a function is nothing but an object that converts a demand for an argument into the production of a result.

We begin our formal development by extending Π with a new universe of types $\mathbb{T}$ that consists of composite types $(\tau_1 - \tau_2)$:

$$(\textit{1d types}) \quad \mathbb{T} \quad ::= \quad (\tau_1 - \tau_2)$$

In anticipation of future developments, we will refer to the original types $\tau$ as 0-dimensional (0d) types and to the new types $\mathbb{T}$ as 1-dimensional (1d) types. It turns out that, except for one case discussed below, the 1d level is a "lifted" instance of Π with its own notions of empty, unit, sum, and product types, and its corresponding notion of isomorphisms on these 1d types.

Our next step is to define lifted versions of the 0d types:

$$
\begin{array}{rcl}
\mathbb{0} & \triangleq & (0-0) \\
\mathbb{1} & \triangleq & (1-0) \\
(\tau_1-\tau_2) \boxplus (\tau_3-\tau_4) & \triangleq & (\tau_1+\tau_3)-(\tau_2+\tau_4) \\
(\tau_1-\tau_2) \boxtimes (\tau_3-\tau_4) & \triangleq & ((\tau_1*\tau_3)+(\tau_2*\tau_4))- \\
 & & ((\tau_1*\tau_4)+(\tau_2*\tau_3))
\end{array}
$$

Building on the idea that Π is a categorification of the natural numbers and following a long tradition that relates type isomorphisms and arithmetic identities [Di Cosmo 2005], one is tempted to think that the **Int** construction (as its name suggests) produces a categorification of the integers. Based on this hypothesis, the definitions above can be intuitively understood as arithmetic identities. The same arithmetic intuition explains the lifting of isomorphisms to 1d types:

$$(\tau_1-\tau_2) \Leftrightarrow (\tau_3-\tau_4) \quad \triangleq \quad (\tau_1+\tau_4) \leftrightarrow (\tau_2+\tau_3)$$

In other words, an isomorphism between 1d types is really an isomorphism between "re-arranged" 0d types where the negative input $\tau_2$ is viewed as an output and the negative output $\tau_4$ is viewed as an input. Using these ideas, it is now a fairly standard exercise to define the lifted versions of most of the combinators in Table 1.[3] There are however a few interesting cases whose appreciation is essential for the remainder of the paper that we discuss below.

***Easy Lifting.*** Many of the 0d combinators lift easily to the 1d level. For example:

$$
\begin{array}{rcl}
id & : & \mathbb{T} \Leftrightarrow \mathbb{T} \\
 & : & (\tau_1-\tau_2) \Leftrightarrow (\tau_1-\tau_2) \\
 & \triangleq & (\tau_1+\tau_2) \leftrightarrow (\tau_2+\tau_1) \\
id & = & swap_+
\end{array}
$$

$$
\begin{array}{rcl}
identl_+ & : & \mathbb{0} \boxplus \mathbb{T} \Leftrightarrow \mathbb{T} \\
 & = & assocr_+ \fatsemi (id \oplus swap_+) \fatsemi assocl_+
\end{array}
$$

[3] See Krishnaswami's [2012] excellent blog post implementing this construction in OCaml.

***Composition using*** $trace$***.***

$$\begin{array}{rl}(\fatsemi): & (\mathbb{T}_1 \Leftrightarrow \mathbb{T}_2) \to (\mathbb{T}_2 \Leftrightarrow \mathbb{T}_3) \to (\mathbb{T}_1 \Leftrightarrow \mathbb{T}_3) \\ f \fatsemi g = & trace\ (assoc_1 \fatsemi (f \oplus id) \fatsemi assoc_2 \fatsemi (g \oplus id) \fatsemi assoc_3)\end{array}$$

***New combinators*** $curry$ ***and*** $uncurry$ ***for higher-order functions.***

$$\begin{array}{rcl}\boxminus(\tau_1 - \tau_2) & \triangleq & \tau_2 - \tau_1 \\ (\tau_1 - \tau_2) \multimap (\tau_3 - \tau_4) & \triangleq & \boxminus(\tau_1 - \tau_2) \boxplus (\tau_3 - \tau_4) \\ & \triangleq & (\tau_2 + \tau_3) - (\tau_1 + \tau_4)\end{array}$$

$$\begin{array}{rcl}flip & : & (\mathbb{T}_1 \Leftrightarrow \mathbb{T}_2) \to (\boxminus\mathbb{T}_2 \Leftrightarrow \boxminus\mathbb{T}_1) \\ flip\ f & = & swap_+ \fatsemi f \fatsemi swap_+\end{array}$$

$$\begin{array}{rcl}curry & : & ((\mathbb{T}_1 \boxplus \mathbb{T}_2) \Leftrightarrow \mathbb{T}_3) \to (\mathbb{T}_1 \Leftrightarrow (\mathbb{T}_2 \multimap \mathbb{T}_3)) \\ curry\ f & = & assocl_+ \fatsemi f \fatsemi assocr_+\end{array}$$

$$\begin{array}{rcl}uncurry & : & (\mathbb{T}_1 \Leftrightarrow (\mathbb{T}_2 \multimap \mathbb{T}_3)) \to ((\mathbb{T}_1 \boxplus \mathbb{T}_2) \Leftrightarrow \mathbb{T}_3) \\ uncurry\ f & = & assocr_+ \fatsemi f \fatsemi assocl_+\end{array}$$

***The "phony" multiplication that is not a functor.*** The definition for the product of 1d types used above is:

$$\begin{array}{l}(\tau_1 - \tau_2) \boxtimes (\tau_3 - \tau_4) = \\ ((\tau_1 * \tau_3) + (\tau_2 * \tau_4)) - ((\tau_1 * \tau_4) + (\tau_2 * \tau_3))\end{array}$$

That definition is "obvious" in some sense as it matches the usual understanding of types as modeling arithmetic identities. Using this definition, it is possible to lift all the 0d combinators involving products *except* the functor:

$$(\otimes) : (\mathbb{T}_1 \Leftrightarrow \mathbb{T}_2) \to (\mathbb{T}_3 \Leftrightarrow \mathbb{T}_4) \to ((\mathbb{T}_1 \boxtimes \mathbb{T}_3) \Leftrightarrow (\mathbb{T}_2 \boxtimes \mathbb{T}_4))$$

After a few failed attempts, we suspected that this definition of multiplication is not functorial which would mean that the **Int** construction only provides a limited notion of higher-order functions at the cost of losing the multiplicative structure at higher-levels. This observation is less well-known that it should be. Further investigation reveals that this observation is intimately related to a well-known problem in algebraic topology and homotopy theory that was identified thirty years ago as the "phony" multiplication [Thomason 1980] in a special class categories related to ours. This problem was recently solved [Baas et al. 2012] using a technique whose fundamental ingredients are to add more dimensions and then take various homotopy colimits. We exploit this idea in the remainder of the paper.

## 4. Polarized Cubes

As hinted at in the previous section, one can think of the **Int** construction as generalizing conventional 0d types to 1d types indexed by a positive or negative polarity. We will generalize this idea further by considering types of arbitrary dimensions $n$ indexed by sequences of length $n$ of positive and negative polarities. The extension is somewhat tedious but the idea is fundamentally simple as everything is defined pointwise.

### 4.1 Syntax

The set of types is now indexed by a dimension $n$:

$$\begin{array}{rcl}\tau, \mathbb{T}^0 & ::= & 0 \mid 1 \mid \tau_1 + \tau_2 \mid \tau_1 * \tau_2 \\ \mathbb{T}^{n+1} & ::= & \boxed{\mathbb{T}_1^n \mid \mathbb{T}_2^n}\end{array}$$

At dimension 0, we have the usual first-order types $\tau$ representing "points." At dimension $n+1$, a type is a pair of subspaces $\boxed{\mathbb{T}_1^n \mid \mathbb{T}_2^n}$, each of a lower dimension $n$. The subspace $\mathbb{T}_1^n$ is the positive subspace along the $(n+1)$st dimension and the subspace $\mathbb{T}_2^n$, shaded in gray, is the negative subspace along that same dimension. Like in the case for the **Int** construction, a 1d cube, $\boxed{\tau_1 \mid \tau_2}$, intuitively corresponds to the difference $\tau_1 - \tau_2$ of the two types. The type can be visualized as a "line." A 2d cube, $\boxed{(\boxed{\tau_1 \mid \tau_2}) \mid (\boxed{\tau_3 \mid \tau_4})}$, intuitively corresponds to the iterated difference of the types $(\tau_1 - \tau_2) - (\tau_3 - \tau_4)$ where the successive shades from the outermost box encode the signs. The type can be visualized as a "square" connecting the two lines corresponding to $(\tau_1 - \tau_2)$ and $(\tau_3 - \tau_4)$. (See Fig. 1 which is further explained after we discuss multiplication below.)

Even though the type constants 0 and 1 and the sums and products operations are only defined at dimension 0, cubes of all dimensions inherit this structure. We have constants $\mathbb{0}^n$ and $\mathbb{1}^n$ at every dimension and we also have families of sum ${}^n\boxplus^n$ and product ${}^m\boxtimes^n$ operations on higher dimensional cubes. The sum operation ${}^n\boxplus^n$ takes two $n$-dimensional cubes and produces another $n$-dimensional cube. Note that for the case of 1d types, the definition coincides with the one used in the **Int** construction. The product operation ${}^m\boxtimes^n$ takes two cubes of dimensions $m$ and $n$ respectively and as confirmed in Prop. 4.1 below and illustrated with a small example in Fig. 1, produces a cube of dimension $m+n$.

$$\begin{array}{rcl}\mathbb{0}^0 & = & 0 \\ \mathbb{0}^{n+1} & = & \boxed{\mathbb{0}^n \mid \mathbb{0}^n}\end{array}$$

$$\begin{array}{rcl}\mathbb{1}^0 & = & 1 \\ \mathbb{1}^{n+1} & = & \boxed{\mathbb{1}^n \mid \mathbb{0}^n}\end{array}$$

$$\begin{array}{rcl}\tau_1\ {}^0\boxplus^0\ \tau_2 & = & \tau_1 + \tau_2 \\ (\boxed{\mathbb{T}_1^n \mid \mathbb{T}_2^n})\ {}^{n+1}\boxplus^{n+1}\ (\boxed{\mathbb{T}_3^n \mid \mathbb{T}_4^n}) & = & \\ & & \boxed{\mathbb{T}_1^n\ {}^n\boxplus^n\ \mathbb{T}_3^n \mid \mathbb{T}_2^n\ {}^n\boxplus^n\ \mathbb{T}_4^n}\end{array}$$

$$\begin{array}{rcl}\tau_1\ {}^0\boxtimes^0\ \tau_2 & = & \tau_1 * \tau_2 \\ \tau\ {}^0\boxtimes^{n+1}\ (\boxed{\mathbb{T}_1^n \mid \mathbb{T}_2^n}) & = & \\ & & \boxed{\tau\ {}^0\boxtimes^n\ \mathbb{T}_1^n \mid \tau\ {}^0\boxtimes^n\ \mathbb{T}_2^n} \\ (\boxed{\mathbb{T}_1^m \mid \mathbb{T}_2^m})\ {}^{m+1}\boxtimes^n\ \mathbb{T}^n & = & \\ & & \boxed{\mathbb{T}_1^m\ {}^m\boxtimes^n\ \mathbb{T}^n \mid \mathbb{T}_2^m\ {}^m\boxtimes^n\ \mathbb{T}^n}\end{array}$$

**Proposition 4.1** (Dimensions)**.** *The type* $(\mathbb{T}^m\ {}^m\boxtimes^n\ \mathbb{T}^n)$ *has dimension* $m+n$*.*

*Proof.* By a simple induction on $m$. □

### 4.2 Isomorphisms

The main point of the generalization to arbitrary dimensions beyond the **Int** construction is that all higher-dimensions would have the full computational power of Π. In other words, all type isomorphisms (including the ones involving products) should lift to the higher dimensions. The lifting is surprisingly simple: everything is defined pointwise. Formally, the combinators of type $\overset{n}{\Longleftrightarrow}$ on $n$-dimensional cubes are defined as follows:

$$\frac{\vdash c : \tau_1 \leftrightarrow \tau_2}{\vdash c : \tau_1 \overset{0}{\Longleftrightarrow} \tau_2}$$

$$\frac{\vdash c_1 : \mathbb{T}_1^n \overset{n}{\Longleftrightarrow} \mathbb{T}_2^n \qquad \vdash c_2 : \mathbb{T}_3^n \overset{n}{\Longleftrightarrow} \mathbb{T}_4^n}{\vdash \boxed{c_1 \mid c_2} : \boxed{\mathbb{T}_1^n \mid \mathbb{T}_3^n} \overset{n+1}{\Longleftrightarrow} \boxed{\mathbb{T}_2^n \mid \mathbb{T}_4^n}}$$

In other words, a combinator at dimension $n$ is defined by induction on $n$ and consists of a family of $2^n$ 0d combinators. As an example, the combinator $swap_+$ at dimension 2 of type:

$$\boxed{(\boxed{\tau_1 \mid \tau_2}) \mid (\boxed{\tau_3 \mid \tau_4})} \overset{2}{\Longleftrightarrow} \boxed{(\boxed{\tau_1' \mid \tau_2'}) \mid (\boxed{\tau_3' \mid \tau_4'})}$$

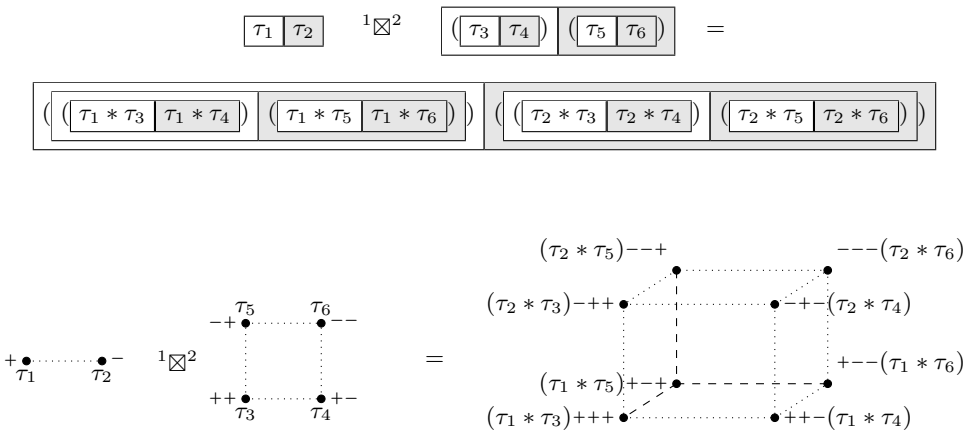


**Figure 1.** Example of multiplication of two cubical types.

$$
\begin{array}{rrcl}
identl_+ : & \mathbb{0}^n \ {}^n\boxplus^n \ \mathbb{T}^n & \overset{n}{\Longleftrightarrow} & \mathbb{T}^n \quad : identr_+ \\
swap_+ : & \mathbb{T}_1^n \ {}^n\boxplus^n \ \mathbb{T}_2^n & \overset{n}{\Longleftrightarrow} & \mathbb{T}_2^n \ {}^n\boxplus^n \ \mathbb{T}_1^n \quad : swap_+ \\
assocl_+ : & \mathbb{T}_1^n \ {}^n\boxplus^n \ (\mathbb{T}_2^n \ {}^n\boxplus^n \ \mathbb{T}_3^n) & \overset{n}{\Longleftrightarrow} & (\mathbb{T}_1^n \ {}^n\boxplus^n \ \mathbb{T}_2^n) \ {}^n\boxplus^n \ \mathbb{T}_3^n \quad : assocr_+ \\
identl_* : & \mathbb{1}^0 \ {}^0\boxtimes^n \ \mathbb{T}^n & \overset{n}{\Longleftrightarrow} & \mathbb{T}^n \quad : identr_* \\
swap_* : & \mathbb{T}_1^m \ {}^m\boxtimes^n \ \mathbb{T}_2^n & \overset{m+n}{\Longleftrightarrow} & \mathbb{T}_2^n \ {}^n\boxtimes^m \ \mathbb{T}_1^m \quad : swap_* \\
assocl_* : & \mathbb{T}_1^m \ {}^m\boxtimes^{n+k} \ (\mathbb{T}_2^n \ {}^n\boxtimes^k \ \mathbb{T}_3^k) & \overset{m+n+k}{\Longleftrightarrow} & (\mathbb{T}_1^m \ {}^m\boxtimes^n \ \mathbb{T}_2^n) \ {}^{m+n}\boxtimes^k \ \mathbb{T}_3^k \quad : assocr_* \\
dist_0 : & \mathbb{0}^m \ {}^m\boxtimes^n \ \mathbb{T}^n & \overset{m+n}{\Longleftrightarrow} & \mathbb{0}^{m+n} \quad : factor_0 \\
dist : & (\mathbb{T}_1^m \ {}^m\boxplus^m \ \mathbb{T}_2^m) \ {}^m\boxtimes^n \ \mathbb{T}_3^n & \overset{m+n}{\Longleftrightarrow} & (\mathbb{T}_1^m \ {}^m\boxtimes^n \ \mathbb{T}_3^n) \ {}^{m+n}\boxplus^{m+n} \ (\mathbb{T}_2^m \ {}^m\boxtimes^n \ \mathbb{T}_3^n) \quad : factor
\end{array}
$$

$$
\frac{}{\vdash id : \mathbb{T}^n \overset{n}{\Longleftrightarrow} \mathbb{T}^n}
\qquad
\frac{\vdash c : \mathbb{T}_1^n \overset{n}{\Longleftrightarrow} \mathbb{T}_2^n}{\vdash sym\ c : \mathbb{T}_2^n \overset{n}{\Longleftrightarrow} \mathbb{T}_1^n}
\qquad
\frac{\vdash c_1 : \mathbb{T}_1^n \overset{n}{\Longleftrightarrow} \mathbb{T}_2^n \quad \vdash c_2 : \mathbb{T}_2^n \overset{n}{\Longleftrightarrow} \mathbb{T}_3^n}{\vdash c_1 \fatsemi c_2 : \mathbb{T}_1^n \overset{n}{\Longleftrightarrow} \mathbb{T}_3^n}
$$

$$
\frac{\vdash c_1 : \mathbb{T}_1^n \overset{n}{\Longleftrightarrow} \mathbb{T}_2^n \quad \vdash c_2 : \mathbb{T}_3^n \overset{n}{\Longleftrightarrow} \mathbb{T}_4^n}{\vdash c_1 \oplus c_2 : \mathbb{T}_1^n \ {}^n\boxplus^n \ \mathbb{T}_3^n \overset{n}{\Longleftrightarrow} \mathbb{T}_2^n \ {}^n\boxplus^n \ \mathbb{T}_4^n}
\qquad
\frac{\vdash c_1 : \mathbb{T}_1^m \overset{m}{\Longleftrightarrow} \mathbb{T}_2^m \quad \vdash c_2 : \mathbb{T}_3^n \overset{n}{\Longleftrightarrow} \mathbb{T}_4^n}{\vdash c_1 \otimes c_2 : \mathbb{T}_1^m \ {}^m\boxtimes^n \ \mathbb{T}_3^n \overset{m+n}{\Longleftrightarrow} \mathbb{T}_2^m \ {}^m\boxtimes^n \ \mathbb{T}_4^n}
$$

**Table 2.** Combinators on cubical types

is defined as $\left(\boxed{swap_+ \mid swap_+}\right) \mid \left(\boxed{swap_+ \mid swap_+}\right)$ where each of the internal 0d $swap_+$ combinators is of type $\tau_i \leftrightarrow \tau_i'$. For completeness, Table 2 shows the types of the lifted versions of all the $\Pi$ combinators.

### 4.3 Operational Semantics

A value $\bullet v$ of an $n$-dimensional type is a 0d value located at one of the $2^n$ vertices. To keep the correspondence between values and types evident, and to prepare for the generalization in the next section, we denote $n$-dimensional values $\bullet v$ using a (possibly empty) sequence of polarities $\bullet$ that ends with a 0d value $v$. For example, the values of type $\left(\boxed{1 \mid 1}\right) \mid \left(\boxed{1 \mid 1+1}\right)$ are $++()$, $+-()$, $-+()$, $--\mathsf{inl}\ ()$, and $--\mathsf{inr}\ ()$. Generalizing the operational semantics of Sec. 2.2 is straightforward:

$$
\begin{array}{rcl}
c \triangleright^0 v & = & c \triangleright v \\
\boxed{c_1 \mid c_2} \ \triangleright^{n+1} (+ \bullet v) & = & +(c_1 \triangleright^n \bullet v) \\
\boxed{c_1 \mid c_2} \ \triangleright^{n+1} (- \bullet v) & = & -(c_2 \triangleright^n \bullet v)
\end{array}
$$

The evaluator is essentially a 0d evaluator operating in one fixed dimension and ignoring all others. Note that values never change their polarities.

```
TODO

say that the labels on isos
implicitly say that
m+n = n+m etc

explain why identl*
only works on 0dim unit

do we need embedding of
n-dim types into
n+1-dim types ??

Finish implement op. sem. in Agda.
Basically a version of pi indexed
by dimensions!

need to make sure there are
no other isos implied by
the development in secs 2
and 3; and that we don't have
any isos that are not justified
by the math
```

```
check all the axioms and commuting
diagrams in Sec. 2
```

## 5. Homotopies

The development in the previous section resulted in an $n$-dimensional version of $\Pi$ where everything is defined pointwise. For example, an interpreter starting with a 3d value indexed by $+-+$ will always result in a value indexed by the same polarities $+-+$. It is not possible for the computation in one dimension to migrate to or to interact in any way with another dimension. In some sense, the polarities are not interpreted yet. More precisely, if the polarities are to indicate "forward" and "backwards" flow of information, it should be the case that identical positive and negative flows cancel each other. Technically, we want the *ring completion* of our set of isomorphisms which means having combinators witnessing isomorphisms such as $\boxed{\tau}\boxed{\tau} \overset{1}{\Longleftrightarrow} \mathbb{0}^1$. Recalling the connection to elementary algebra, this just means that we are now categorifying identities such as $\tau - \tau = 0$. Operationally, this would, for example, allow an interpreter manipulating a value indexed by $+-+$ to reverse its flow among one of the three dimensions, i.e., to migrate to the vertex indexed by $--+$, $+++$, or $+--$, and be processed by the code indexed by that new sequence of polarities instead of the original code indexed by $+-+$.

### 5.1 Recovering the Int Construction

As motivated in the previous paragraph, we begin by adding a new way to create 1d combinators that witnesses the interpretation of negative types as additive inverses to conventional positive types:

$$\frac{\vdash c : \tau \leftrightarrow \tau}{\vdash \mathit{promote}\ c : \boxed{\tau}\boxed{\tau} \overset{1}{\Longleftrightarrow} \mathbb{0}^1 : \mathit{demote}\ c}$$

Unlike the case in the previous section, 1d combinators are no longer exclusively a family of $2^1$ combinators of dimension 0. There are now some inherently 1d combinators that mediate between $\boxed{\tau}\boxed{\tau}$ and $\mathbb{0}^1$. As a concrete example, let us abbreviate $1+1$ as $\mathsf{bool}$, the type of booleans. There are several isomorphisms $\mathsf{bool} \leftrightarrow \mathsf{bool}$ including the trivial one witnessed by the combinator $id$ and the boolean negation witnessed by the combinator $swap_+$. Each of these isomorphisms gives rise to a *different* 1d isomorphism between $\boxed{\mathsf{bool}}\boxed{\mathsf{bool}}$ and $\mathbb{0}^1$. Before presenting the formal evaluation rules, we show the intuition of how the new combinators behave operationally:

$$\begin{array}{rcl} (\mathit{promote}\ id) \triangleright^1 (+\mathsf{inl}\,()) & = & -\mathsf{inl}\,() \\ (\mathit{promote}\ swap_+) \triangleright^1 (+\mathsf{inl}\,()) & = & -\mathsf{inr}\,() \end{array}$$

The evaluation of the new combinators does not simply keep recurring until dimension 0; instead the combinators act as "bridges" that transfer values from one vertex in dimension 1 to another vertex.

```
Need to have abort or
something and need to transfer control to the evaluator in the other
dimension? How do I get hold of that evaluator; keep all the n-dim code
around and index into it?

evaluator
curry...
```

## 6. Related Work and Context

A ton of stuff here.

## 7. Conclusion

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