

Polarized Cubical Types

Abstract

...

1. Introduction

In a computational world in which the laws of physics are embraced and resources are carefully maintained (e.g., quantum computing [Abramsky and Coecke 2004; Nielsen and Chuang 2000]), programs must be reversible. Although this is apparently a limiting idea, it turns out that conventional computation can be viewed as a special case of such resource-preserving reversible programs. This thesis has been explored for many years from different perspectives [Bennett 2003, 2010, 1973; Fredkin and Toffoli 1982; Landauer 1961, 1996; Toffoli 1980]. We build on the work of James and Sabry [2012] which expresses this thesis in a type theoretic computational framework, expressing computation via type isomorphisms.

Make sure we introduce the abbreviation HoTT in the introduction [The Univalent Foundations Program 2013].

2. Computing with Type Isomorphisms

The main syntactic vehicle for the developments in this paper is a simple language called Π whose only computations are isomorphisms between finite types. The set of types τ includes the empty type 0, the unit type 1, and conventional sum and product types. The values of these types are the conventional ones: $()$ of type 1, $\text{inl } v$ and $\text{inr } v$ for injections into sum types, and (v_1, v_2) for product types:

(Types)	$\tau ::= 0 \mid 1 \mid \tau_1 + \tau_2 \mid \tau_1 * \tau_2$
(Values)	$v ::= () \mid \text{inl } v \mid \text{inr } v \mid (v_1, v_2)$
(Combinator types)	$\tau_1 \leftrightarrow \tau_2$
(Combinators)	$c ::= [\text{see Table 1}]$

The interesting syntactic category of Π is that of *combinators* which are witnesses for type isomorphisms $\tau_1 \leftrightarrow \tau_2$. They consist of base combinators (on the left side of Table 1) and compositions (on the right side of the same table). Each line of the table on the left introduces a pair of dual constants¹ that witness the type isomorphism in the middle. This set of isomorphisms is known to be complete [Fiore 2004; Fiore et al. 2006] and the language is universal for hardware combinational circuits [James and Sabry 2012]. The *trace* operator provides a bounded iteration facility

¹ where swap_+ and swap_* are self-dual.

which adds no expressiveness in the current context but will be needed in Sec. 3.²

As simple illustrative examples of “programming” in Π , here are three useful combinators that we define here for future reference:

$$\begin{aligned}
 \text{assoc}_1 &: \tau_1 + (\tau_2 + \tau_3) \leftrightarrow (\tau_2 + \tau_1) + \tau_3 \\
 \text{assoc}_1 &= \text{assocl}_+ \circ (\text{swap}_+ \oplus \text{id}) \\
 \\
 \text{assoc}_2 &: (\tau_1 + \tau_2) + \tau_3 \leftrightarrow (\tau_2 + \tau_3) + \tau_1 \\
 \text{assoc}_2 &= (\text{swap}_+ \oplus \text{id}) \circ \text{assocr}_+ \circ (\text{id} \oplus \text{swap}_+) \circ \text{assocl}_+ \\
 \\
 \text{assoc}_3 &: (\tau_1 + \tau_2) + \tau_3 \leftrightarrow \tau_1 + (\tau_3 + \tau_2) \\
 \text{assoc}_3 &= \text{assocr}_+ \circ (\text{id} \oplus \text{swap}_+)
 \end{aligned}$$

From the perspective of category theory, the language Π models what is called a traced *symmetric bimonoidal category* or a *commutative rig category*. These are categories with two binary operations \oplus and \otimes satisfying the axioms of a rig (i.e., a ring without negative elements also known as a semiring) up to coherent isomorphisms. And indeed the types of the Π -combinators are precisely the semiring axioms. A formal way of saying this is that Π is the *categorification* [Baez and Dolan 1998] of the natural numbers. A simple (slightly degenerate) example of such categories is the category of finite sets and permutations in which we interpret every Π -type as a finite set, the values as elements in these finite sets, and the combinators as permutations. Another common example of such categories is the category of finite dimensional vector spaces and linear maps over any field. Note that in this interpretation, the Π -type 0 maps to the 0-dimensional vector space which is *not* empty. Its unique element, the zero vector — which is present in every vector space — acts like a “bottom” everywhere-undefined element and hence the type behaves like the unit of addition and the annihilator of multiplication as desired.

3. The Int Construction

Our immediate technical goal is to explore an extension of Π with a notion of higher-order functions. In the context of monoidal categories, it is known that a notion of higher-order functions emerges from having an additional degree of *symmetry*. In particular, both the **Int** construction of Joyal, Street, and Verity [1996] and the closely related \mathcal{G} construction of linear logic [Abramsky 1996] construct higher-order *linear* functions by considering a new category built on top of a given base traced monoidal category. The objects of the new category are of the form $(\tau_1 - \tau_2)$ where τ_1 and τ_2 are objects in the base category. Intuitively, the component τ_1 is viewed as a conventional type whose elements represent values flowing, as usual, from producers to consumers. The component τ_2 is viewed as a *negative type* whose elements represent demands for values or equivalently values flowing backwards. Under this interpretation,

² If recursive types are added, the trace operator provides unbounded iteration and the language becomes Turing complete [Bowman et al. 2011; James and Sabry 2012]. We will not be concerned with recursive types in this paper.

$identl_+$	$0 + \tau \leftrightarrow \tau$	$: identr_+$
$swap_+$	$\tau_1 + \tau_2 \leftrightarrow \tau_2 + \tau_1$	$: swap_+$
$assocl_+$	$\tau_1 + (\tau_2 + \tau_3) \leftrightarrow (\tau_1 + \tau_2) + \tau_3$	$: assocr_+$
$identl_*$	$1 * \tau \leftrightarrow \tau$	$: identr_*$
$swap_*$	$\tau_1 * \tau_2 \leftrightarrow \tau_2 * \tau_1$	$: swap_*$
$assocl_*$	$\tau_1 * (\tau_2 * \tau_3) \leftrightarrow (\tau_1 * \tau_2) * \tau_3$	$: assocr_*$
$dist_0$	$0 * \tau \leftrightarrow 0$	$: factor_0$
$dist$	$(\tau_1 + \tau_2) * \tau_3 \leftrightarrow (\tau_1 * \tau_3) + (\tau_2 * \tau_3)$	$: factor$

$\frac{}{\vdash id : \tau \leftrightarrow \tau}$	$\frac{\vdash c : \tau_1 \leftrightarrow \tau_2}{\vdash sym\ c : \tau_2 \leftrightarrow \tau_1}$
$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_2 \leftrightarrow \tau_3}{\vdash c_1 \circ c_2 : \tau_1 \leftrightarrow \tau_3}$	
$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \oplus c_2 : \tau_1 + \tau_3 \leftrightarrow \tau_2 + \tau_4}$	
$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \otimes c_2 : \tau_1 * \tau_3 \leftrightarrow \tau_2 * \tau_4}$	
$\frac{\vdash c : \tau + \tau_1 \leftrightarrow \tau + \tau_2}{\vdash trace\ c : \tau_1 \leftrightarrow \tau_2}$	

Table 1. Π -combinators [James and Sabry 2012]

and as we explain below, a function is nothing but an object that converts a demand for an argument into production of a result.

We begin our formal development by extending Π with a new universe of types \mathbb{T} that consists of composite types $(\tau_1 - \tau_2)$:

$$(Id\ types) \quad \mathbb{T} ::= (\tau_1 - \tau_2)$$

In anticipation of future developments, we will refer to the original types τ as 0-dimensional (0d) types and to the new types \mathbb{T} as 1-dimensional (1d) types. It turns out that, except for one case discussed below, the 1d level is a “lifted” instance of Π with its own notions of empty, unit, sum, and product types, and its corresponding notion of isomorphisms on these 1d types.

Our next step is to define lifted versions of the 0d types:

$$\begin{aligned} 0 &\triangleq (0 - 0) \\ 1 &\triangleq (1 - 0) \\ (\tau_1 - \tau_2) \boxplus (\tau_3 - \tau_4) &\triangleq (\tau_1 + \tau_3) - (\tau_2 + \tau_4) \\ (\tau_1 - \tau_2) \boxtimes (\tau_3 - \tau_4) &\triangleq ((\tau_1 * \tau_3) + (\tau_2 * \tau_4)) - ((\tau_1 * \tau_4) + (\tau_2 * \tau_3)) \end{aligned}$$

Building on the idea that Π is a categorification of the natural numbers and following a long tradition that relates type isomorphisms and arithmetic identities [Di Cosmo 2005], one is tempted to think that the **Int** construction (as its name suggests) produces a categorification of the integers. Based on this hypothesis, the definitions above can be intuitively understood as arithmetic identities. The same arithmetic intuition explains the lifting of isomorphisms to 1d types:

$$(\tau_1 - \tau_2) \leftrightarrow (\tau_3 - \tau_4) \triangleq (\tau_1 + \tau_4) \leftrightarrow (\tau_2 + \tau_3)$$

In other words, an isomorphism between 1d types is really an isomorphism between “re-arranged” 0d types where the negative input τ_2 is viewed as an output and the negative output τ_4 is viewed as an input. Using these ideas, it is now a fairly standard exercise to define the lifted versions of most of the combinators in Table 1.³ There are however a few interesting cases whose appreciation is essential for the remainder of the paper that we discuss below.

³See Krishnaswami’s [2012] excellent blog post implementing this construction in OCaml.

Easy Lifting. Many of the 0d combinators lift easily to the 1d level. For example:

$$\begin{aligned} id &: \mathbb{T} \leftrightarrow \mathbb{T} \\ &: (\tau_1 - \tau_2) \leftrightarrow (\tau_1 - \tau_2) \\ &\triangleq (\tau_1 + \tau_2) \leftrightarrow (\tau_2 + \tau_1) \\ id &= swap_+ \\ identl_+ &: 0 \boxplus \mathbb{T} \leftrightarrow \mathbb{T} \\ &= assocr_+ \circ (id \oplus swap_+) \circ assocl_+ \end{aligned}$$

Composition using trace.

$$\begin{aligned} (\circ) &: (\mathbb{T}_1 \leftrightarrow \mathbb{T}_2) \rightarrow (\mathbb{T}_2 \leftrightarrow \mathbb{T}_3) \rightarrow (\mathbb{T}_1 \leftrightarrow \mathbb{T}_3) \\ f \circ g &= trace\ (assocl_+ \circ (f \oplus id) \circ assocr_+ \circ (g \oplus id) \circ assocl_+) \end{aligned}$$

New combinators curry and uncurry for higher-order functions.

$$\begin{aligned} \boxminus(\tau_1 - \tau_2) &\triangleq \tau_2 - \tau_1 \\ (\tau_1 - \tau_2) \multimap (\tau_3 - \tau_4) &\triangleq \boxminus(\tau_1 - \tau_2) \boxplus (\tau_3 - \tau_4) \\ &\triangleq (\tau_2 + \tau_3) - (\tau_1 + \tau_4) \end{aligned}$$

$$\begin{aligned} flip &: (\mathbb{T}_1 \leftrightarrow \mathbb{T}_2) \rightarrow (\boxminus \mathbb{T}_2 \leftrightarrow \boxminus \mathbb{T}_1) \\ flip\ f &= swap_+ \circ f \circ swap_+ \end{aligned}$$

$$\begin{aligned} curry &: ((\mathbb{T}_1 \boxplus \mathbb{T}_2) \leftrightarrow \mathbb{T}_3) \rightarrow (\mathbb{T}_1 \leftrightarrow (\mathbb{T}_2 \multimap \mathbb{T}_3)) \\ curry\ f &= assocl_+ \circ f \circ assocr_+ \end{aligned}$$

$$\begin{aligned} uncurry &: (\mathbb{T}_1 \leftrightarrow (\mathbb{T}_2 \multimap \mathbb{T}_3)) \rightarrow ((\mathbb{T}_1 \boxplus \mathbb{T}_2) \leftrightarrow \mathbb{T}_3) \\ uncurry\ f &= assocr_+ \circ f \circ assocl_+ \end{aligned}$$

The “phony” multiplication that is not a functor. The definition for the product of 1d types used above is:

$$\begin{aligned} (\tau_1 - \tau_2) \boxtimes (\tau_3 - \tau_4) &= \\ ((\tau_1 * \tau_3) + (\tau_2 * \tau_4)) - ((\tau_1 * \tau_4) + (\tau_2 * \tau_3)) \end{aligned}$$

That definition is “obvious” in some sense as it matches the usual understanding of types as modeling arithmetic. Using it, it is possible to lift all the 0d combinators involving products *except* the functor:

$$(\otimes) : (\mathbb{T}_1 \leftrightarrow \mathbb{T}_2) \rightarrow (\mathbb{T}_3 \leftrightarrow \mathbb{T}_4) \rightarrow ((\mathbb{T}_1 \boxtimes \mathbb{T}_3) \leftrightarrow (\mathbb{T}_2 \boxtimes \mathbb{T}_4))$$

After a few failed attempts, we suspected that this definition of multiplication is not functorial which would mean that the **Int** construction provides a limited notion of higher-order functions at the expense of losing the multiplicative structure at higher-levels.

This observation is less well-known that it should be. Further investigation reveals that this observation is intimately related to a well-known problem in algebraic topology that was identified thirty years ago as the “phony” multiplication [Thomason 1980] in a special class categories related to ours. This problem was recently solved [Baas et al. 2012] using a technique whose fundamental ingredient is to add more dimensions. We exploit this idea in the remainder of the paper.

4. Cubes

As hinted at in the previous section, one can think of the **Int** construction as generalizing conventional 0d types to 1d types indexed by a positive or negative polarity. Generalizing this idea further, we now view types as n -dimensional cubes indexed by all 2^n combinations of positive and negative polarities.

4.1 Syntax

Our types \mathbb{T} are “cubes” defined as follows:

$$\begin{aligned}\tau &::= 0 \mid 1 \mid \tau_1 + \tau_2 \mid \tau_1 * \tau_2 \\ \mathbb{T} &::= \tau \mid \boxed{\tau_1 \mid \tau_2}\end{aligned}$$

The syntax $\boxed{\tau_1 \mid \tau_2}$ represents n -dimensional cubes as binary trees of maximum height n . The subspace τ_1 is the positive subspace along one of the dimensions and the subspace τ_2 (shaded in gray) is the negative subspace along that same dimension. Each of these subspaces is itself a cube of a lower dimension. The 0d cubes are “points” representing the conventional first-order types τ . Like in the case for the **Int** construction, a 1d cube, $\boxed{\tau_1 \mid \tau_2}$, intuitively corresponds to the difference $\tau_1 - \tau_2$ of the two types. The type can be visualized as a “line” with polarized endpoints positive τ_1 and negative τ_2 . A full 2d cube, $\boxed{\boxed{\tau_1 \mid \tau_2} \mid \boxed{\tau_3 \mid \tau_4}}$, intuitively corresponds to the iterated difference of the types $(\tau_1 - \tau_2) - (\tau_3 - \tau_4)$ where the successive “colors” from the outermost box encode the signs. The type can be visualized as a “square” with polarized corners connecting the two lines corresponding to $(\tau_1 - \tau_2)$ and $(\tau_3 - \tau_4)$. (See Fig. 1 which is further explained after we discuss multiplication below.)

There are notions of sum \boxplus , product \boxtimes , and negation \boxminus on n -dimensional types. These operations are inductively defined as follows:

$$\begin{aligned}\tau_1 \boxplus \tau_2 &= \tau_1 + \tau_2 \\ \tau \boxplus \boxed{\tau_1 \mid \tau_2} &= \boxed{\tau \boxplus \tau_1 \mid \tau \boxplus \tau_2} \\ \boxed{\tau_1 \mid \tau_2} \boxplus \tau &= \boxed{\tau_1 \boxplus \tau \mid \tau_2 \boxplus \tau} \\ \boxed{\tau_1 \mid \tau_2} \boxplus \boxed{\tau_3 \mid \tau_4} &= \boxed{\tau_1 \boxplus \tau_3 \mid \tau_2 \boxplus \tau_4} \\ \tau \boxtimes \tau_2 &= \tau_1 * \tau_2 \\ \tau \boxtimes \boxed{\tau_1 \mid \tau_2} &= \boxed{\tau \boxtimes \tau_1 \mid \tau \boxtimes \tau_2} \\ \boxed{\tau_1 \mid \tau_2} \boxtimes \tau &= \boxed{\tau_1 \boxtimes \tau \mid \tau_2 \boxtimes \tau} \\ \boxminus \tau &= \boxed{\tau} \\ \boxminus \boxed{\tau_1 \mid \tau_2} &= \boxed{\boxminus \tau_2 \mid \boxminus \tau_1}\end{aligned}$$

The sum of 0d types is the disjoint union as usual. For cubes of higher dimensions, the subspaces are recursively added. Note that the sum of 1d types reduces to the sum used in the **Int** construction. The definition of negation is natural: it recursively swaps the positive and negative subspaces. The product of 0d types is the cartesian product of sets. For cubes of higher-dimensions n and m , the result is of dimension $(n + m)$. The example in Fig. 1 illustrates the idea using the product of 1d cube (i.e., a line) with a 2d cube (i.e., a square). The result is a 3d cube as illustrated.

We recall that the dimension of a type $\boxed{\tau_1 \mid \tau_2}$ is the maximum height of its tree representation with 0d types at the leaves. The sum, product, and negation operations affect this maximum height as follows.

Proposition 4.1 (Dimensions). *The dimensions of sums, products, and negations are:*

$$\begin{aligned}\dim(\tau_1 \boxplus \tau_2) &= \max(\dim(\tau_1), \dim(\tau_2)) \\ \dim(\tau_1 \boxtimes \tau_2) &= \dim(\tau_1) + \dim(\tau_2) \\ \dim(\boxminus \tau) &= \max(1, \dim(\tau))\end{aligned}$$

The base types have dimension 0. If negative types are not used, all dimensions remain at 0. If negative types are used but no products of negative types appear anywhere, the dimension is raised to 1. This is the situation with the **Int** or \mathcal{G} construction. Once negative and product types are freely used, the dimension can increase without bounds.

4.2 Isomorphisms

The main point of the generalization to arbitrary dimensions beyond the **Int** construction is that all higher-dimensions would have the full computational power of Π . In other words, all type isomorphisms (including the ones involving products) should lift to the higher dimensions. The lifting is surprisingly simple: everything is defined pointwise. For example, there is an isomorphism between

$\boxed{\tau_1 \mid \tau_2} \boxplus \boxed{\tau_3 \mid \tau_4}$ and $\boxed{\tau'_1 \mid \tau'_2} \boxplus \boxed{\tau'_3 \mid \tau'_4}$ if there are Π -isomorphisms between each of the corresponding τ_i and τ'_i .

Table 2 replicates all the Π -isomorphisms for higher-dimensional cubical types. Because everything is defined pointwise, most of the new combinators are just families of base combinators. As an example, consider:

$$\begin{aligned}\text{swap}_+ : & \boxed{\tau_1 \mid \tau_2} \boxplus \boxed{\tau_3 \mid \tau_4} \boxplus \boxed{\tau'_1 \mid \tau'_2} \boxplus \boxed{\tau'_3 \mid \tau'_4} \\ \Leftrightarrow & \boxed{\tau'_1 \mid \tau'_2} \boxplus \boxed{\tau'_3 \mid \tau'_4} \boxplus \boxed{\tau_1 \mid \tau_2} \boxplus \boxed{\tau_3 \mid \tau_4} \\ : & \boxed{\tau_1 + \tau'_1 \mid \tau_2 + \tau'_2} \boxplus \boxed{\tau_3 + \tau'_3 \mid \tau_4 + \tau'_4} \\ \Leftrightarrow & \boxed{\tau'_1 + \tau_1 \mid \tau'_2 + \tau_2} \boxplus \boxed{\tau'_3 + \tau_3 \mid \tau'_4 + \tau_4} \\ = & \boxed{\boxed{\text{swap}_+ \mid \text{swap}_+} \boxplus \boxed{\text{swap}_+ \mid \text{swap}_+}}\end{aligned}$$

In other words, the definition of swap_+ at a 2d dimensional type reduces to a family of swap_+ combinators at each of the vertices of the 2d cube.

Start with table. 0 boxplus cubt is the same as t for all high dimensions; when you get 0d you need the explicit iso. So express it using regular + and tau instead boxplus and cubt.

The Π isomorphisms are, in spirit, isomorphisms of n -dimensional cubical types. For example, we still expect addition \boxplus to be associative, commutative, etc. The situation is however more involved and the negative types introduce new isomorphisms.

how to formalize that n-dim comb are families of pi-comb

subtle point: need 0 and 1 at all dimensions. Can add new iso 0 = [0|0] but not sure is the best idea

need to understand and then add

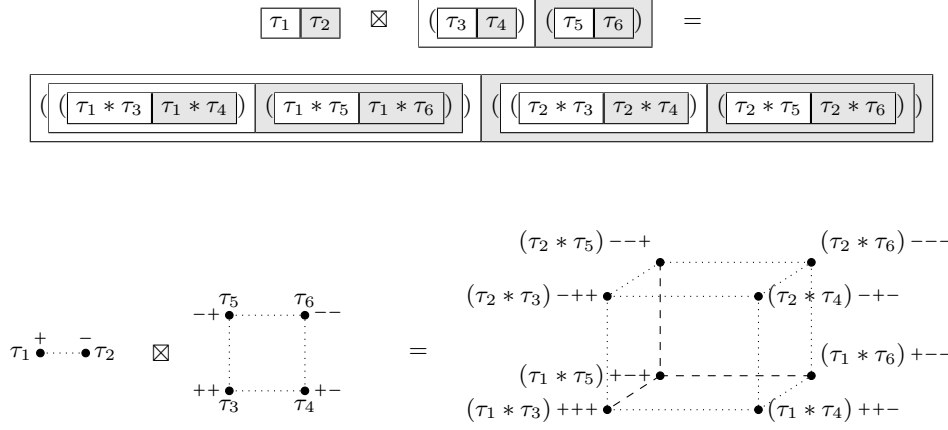


Figure 1. Example of multiplication of two cubical types.

$identl_+ :$	$0 \boxplus \mathbb{T} \Leftrightarrow \mathbb{T}$	$: identr_+$	$\frac{}{\vdash id : \mathbb{T} \Leftrightarrow \mathbb{T}}$	$\frac{\vdash c : \mathbb{T}_1 \Leftrightarrow \mathbb{T}_2}{\vdash sym\ c : \mathbb{T}_2 \Leftrightarrow \mathbb{T}_1}$
$swap_+ :$	$\mathbb{T}_1 \boxplus \mathbb{T}_2 \Leftrightarrow \mathbb{T}_2 \boxplus \mathbb{T}_1$	$: swap_+$	$\frac{\vdash c_1 : \mathbb{T}_1 \Leftrightarrow \mathbb{T}_2 \quad \vdash c_2 : \mathbb{T}_2 \Leftrightarrow \mathbb{T}_3}{\vdash c_1 \circ c_2 : \mathbb{T}_1 \Leftrightarrow \mathbb{T}_3}$	
$assocl_+ :$	$\mathbb{T}_1 \boxplus (\mathbb{T}_2 \boxplus \mathbb{T}_3) \Leftrightarrow (\mathbb{T}_1 \boxplus \mathbb{T}_2) \boxplus \mathbb{T}_3$	$: assocr_+$	$\frac{\vdash c_1 : \mathbb{T}_1 \Leftrightarrow \mathbb{T}_2 \quad \vdash c_2 : \mathbb{T}_3 \Leftrightarrow \mathbb{T}_4}{\vdash c_1 \oplus c_2 : \mathbb{T}_1 \boxplus \mathbb{T}_3 \Leftrightarrow \mathbb{T}_2 \boxplus \mathbb{T}_4}$	
$identl_* :$	$1 \boxtimes \mathbb{T} \Leftrightarrow \mathbb{T}$	$: identr_*$	$\frac{\vdash c_1 : \mathbb{T}_1 \Leftrightarrow \mathbb{T}_2 \quad \vdash c_2 : \mathbb{T}_3 \Leftrightarrow \mathbb{T}_4}{\vdash c_1 \otimes c_2 : \mathbb{T}_1 \boxtimes \mathbb{T}_3 \Leftrightarrow \mathbb{T}_2 \boxtimes \mathbb{T}_4}$	
$swap_* :$	$\mathbb{T}_1 \boxtimes \mathbb{T}_2 \Leftrightarrow \mathbb{T}_2 \boxtimes \mathbb{T}_1$	$: swap_*$		
$assocl_* :$	$\mathbb{T}_1 \boxtimes (\mathbb{T}_2 \boxtimes \mathbb{T}_3) \Leftrightarrow (\mathbb{T}_1 \boxtimes \mathbb{T}_2) \boxtimes \mathbb{T}_3$	$: assocr_*$		
$dist_0 :$	$0 \boxtimes \mathbb{T} \Leftrightarrow 0$	$: factor_0$		
$dist :$	$(\mathbb{T}_1 \boxplus \mathbb{T}_2) \boxtimes \mathbb{T}_3 \Leftrightarrow (\mathbb{T}_1 \boxtimes \mathbb{T}_3) \boxplus (\mathbb{T}_2 \boxtimes \mathbb{T}_3)$	$: factor$		

Table 2. Combinators on cubical types

the $t=t=0$ equations

need to make sure there are no other isos implied by the development in secs 2 and 3.

finally need to move on to sec. 4 and beyond and understand what they add in terms of isos

5. Related Work and Context

A ton of stuff here.

6. Conclusion

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