# A Computational Reconstruction of Homotopy Type Theory for Finite Types


## Abstract

Homotopy type theory (HoTT) relates some aspects of topology, algebra, geometry, physics, logic, and type theory, in a unique novel way that promises a new and foundational perspective on mathematics and computation. The heart of HoTT is the *univalence axiom*, which informally states that isomorphic structures can be identified. One of the major open problems in HoTT is a computational interpretation of this axiom. We propose that, at least for the special case of finite types, reversible computation via type isomorphisms *is* the computational interpretation of univalence.


## 1. Introduction

***Conventional HoTT/Agda approach*** We start with a computational framework: data (pairs, etc.) and functions between them. There are computational rules (beta, etc.) that explain what a function does on a given datum.

We then have a notion of identity which we view as a process that equates two things and model as a new kind of data. Initially we only have identities between beta-equivalent things.

Then we postulate a process that identifies any two functions that are extensionally equivalent. We also postulate another process that identifies any two sets that are isomorphic. This is done by adding new kinds of data for these kinds of identities.

***Our approach*** Our approach is to start with a computational framework that has finite data and permutations as the operations between them. The computational rules apply permutations.

HoTT says id types are an inductively defined type family with refl as constructor. We say it is a family defined with pi combinators as constructors. Replace path induction with refl as base case with our induction.

***Generalization*** How would that generalize to first-class functions? Using negative and fractionals? Groupoids?

In a computational world in which the laws of physics are embraced and resources are carefully maintained (e.g., quantum computing [Abramsky and Coecke 2004; Nielsen and Chuang 2000]), programs must be reversible. Although this is apparently a limiting idea, it turns out that conventional computation can be viewed as a special case of such resource-preserving reversible programs. This thesis has been explored for many years from different perspectives [Bennett 2003, 2010, 1973; Fredkin and Toffoli 1982; Landauer 1961, 1996; Toffoli 1980]. We build on the work of James and Sabry [2012] which expresses this thesis in a type theoretic computational framework, expressing computation via type isomorphisms.



## 2. Condensed Background on HoTT

Informally, and as a first approximation, one may think of HoTT as a variation on Martin-Löf type theory in which all equalities are given *computational content*. We explain the basic ideas below.

Formally, Martin-Löf type theory, is based on the principle that every proposition, i.e., every statement that is susceptible to proof, can be viewed as a type. Indeed, if a proposition $P$ is true, the corresponding type is inhabited and it is possible to provide evidence or proof for $P$ using one of the elements of the type $P$. If, however, a proposition $P$ is false, the corresponding type is empty and it is impossible to provide a proof for $P$. The type theory is rich enough to express the standard logical propositions denoting conjunction, disjunction, implication, and existential and universal quantifications. In addition, it is clear that the question of whether two elements of a type are equal is a proposition, and hence that this proposition must correspond to a type. In Agda, one may write proofs of this proposition as shown in the two small examples below:

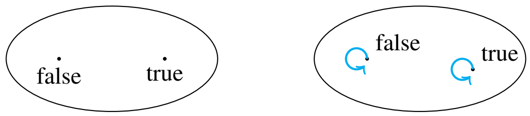
```
i0 : 3 ≡ 3
i0 = refl 3

i1 : (1 + 2) ≡ (3 * 1)
i1 = refl 3
```

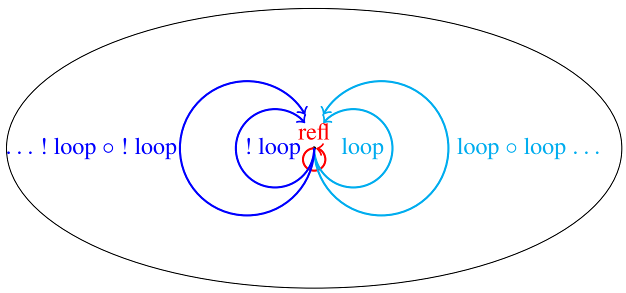
More generally, given two values $m$ and $n$ of type $\mathbb{N}$, it is possible to construct an element refl $k$ of the type $m \equiv n$ if and only if $m$, $n$, and $k$ are all "equal." As shown in example i1, this notion of *propositional equality* is not just syntactic equality but generalizes to *definitional equality*, i.e., to equality that can be established by normalizing the two values to their normal forms.

The important question from the HoTT perspective is the following: given two elements $p$ and $q$ of some type $x \equiv y$ with $x\ y : A$, what can we say about the elements of type $p \equiv q$. Or, in more familiar terms, given two proofs of some proposition $P$, are these two proofs themselves "equal." In some situations, the only interesting property of proofs is their existence, i.e., all proofs of the same proposition are considered equivalent. The twist that dates back to the paper by [Hofmann and Streicher 1996] is that proofs actually possess a structure of great combinatorial complexity. HoTT builds on this idea by interpreting types as topological spaces or weak $\infty$-groupoids, and interpreting identities between elements of a type $x \equiv y$ as *paths* from $x$ to $y$. If $x$ and $y$ are themselves paths, the elements of $x \equiv y$ become paths between paths, or homotopies in the topological language. To be explicit, we will often use $\equiv_A$ to refer to the space in which the path lives.

As a simple example, we are used to thinking of types as sets of values. So we typically view the type Bool as the figure on the left but in HoTT we should instead think about it as the figure on the right where there is a (trivial) path refl $b$ from each point $b$ to itself:




In this particular case, it makes no difference, but in general we may have a much more complicated path structure. The classical such example is the topological *circle* which is a space consisting of a point base and a *non trivial* path loop from base to itself. As stated, this does not amount to much. However, because paths carry additional structure (explained below), that space has the following non-trivial structure:




The additional structure of types is formalized as follows. Let $x$, $y$, and $z$ be elements of some space $A$:

- For every path $p : x \equiv_A y$, there exists a path $! \, p : y \equiv_A x$;
- For every pair of paths $p : x \equiv_A y$ and $q : y \equiv_A z$, there exists a path $p \circ q : x \equiv_A z$;
- Subject to the following conditions:
  - $p \circ \mathsf{refl}\ y \equiv_{(x \equiv_A y)} p$;
  - $p \equiv_{(x \equiv_A y)} \mathsf{refl}\ x \circ p$
  - $! \, p \circ p \equiv_{(y \equiv_A y)} \mathsf{refl}\ y$
  - $p \circ ! \, p \equiv_{(x \equiv_A x)} \mathsf{refl}\ x$
  - $! \, (! \, p) \equiv_{(x \equiv_A y)} p$
  - $p \circ (q \circ r) \equiv_{(x \equiv_A z)} (p \circ q) \circ r$
- This structure repeats one level up and so on ad infinitum.

Structure of Paths:

- What do paths in $A \times B$ look like? We can prove that $(a_1, b_1) \equiv (a_2, b_2)$ in $A \times B$ iff $a_1 \equiv a_2$ in $A$ and $b_1 \equiv b_2$ in $B$.
- What do paths in $A_1 \uplus A_2$ look like? We can prove that $inj_i\ x \equiv inj_j\ y$ in $A_1 \uplus A_2$ iff $i = j$ and $x \equiv y$ in $A_i$.
- What do paths in $A \to B$ look like? We cannot prove anything. Postulate function extensionality axiom.
- What do paths in $\mathrm{Set}_\ell$ look like? We cannot prove anything. Postulate univalence axiom.

Function Extensionality:

```
-- f ∼ g iff ∀ x. f x ≡ g x
_∼_ : ∀ {ℓ ℓ'} → {A : Set ℓ} {P : A → Set ℓ'} →
      (f g : (x : A) → P x) → Set (ℓ ⊔ ℓ')
_∼_ {ℓ} {ℓ'} {A} {P} f g = (x : A) → f x ≡ g x

-- f is an equivalence if we have g and h such that
-- the compositions with f in both ways are ∼ id
record isequiv {ℓ ℓ'} {A : Set ℓ} {B : Set ℓ'} (f : A → B) :
  Set (ℓ ⊔ ℓ') where
  constructor mkisequiv
  field
    g : B → A
    α : (f ∘ g) ∼ id
    h : B → A
    β : (h ∘ f) ∼ id
-- a path between f and g implies f ∼ g
happly : ∀ {ℓ ℓ'} {A : Set ℓ} {B : A → Set ℓ'} {f g : (a : A) → B a} →
  (f ≡ g) → (f ∼ g)
happly {ℓ} {ℓ'} {A} {B} {f} {g} p = {!!}

postulate -- that f ∼ g implies a path between f and g
  funextP :   {A : Set} {B : A → Set} {f g : (a : A) → B a} →
              isequiv {A = f ≡ g} {B = f ∼ g} happly

funext :   {A : Set} {B : A → Set} {f g : (a : A) → B a} →
           (f ∼ g) → (f ≡ g)
funext = isequiv.g funextP
```

A path between $f$ and $g$ is a collection of paths from $f(x)$ to $g(x)$. We are no longer executable!

Univalence:

```
-- Two spaces are equivalent if we have functions
-- f, g, and h that compose to id
_≃_ : ∀ {ℓ ℓ'} (A : Set ℓ) (B : Set ℓ') → Set (ℓ ⊔ ℓ')
A ≃ B = Σ (A → B) isequiv

-- A path between spaces implies their equivalence
idtoeqv : {A B : Set} → (A ≡ B) → (A ≃ B)
idtoeqv {A} {B} p = {!!}

postulate -- that equivalence of spaces implies a path
  univalence : {A B : Set} → (A ≡ B) ≃ (A ≃ B)
```

Again, we are no longer executable!

Analysis:

- We start with two different notions: paths and functions;
- We use extensional non-constructive methods to identify a particular class of functions that form isomorphisms;
- We postulate that this particular class of functions can be identified with paths.

Insight:

- Start with a constructive characterization of *reversible functions* or *isomorphisms*;
- Blur the distinction between such reversible functions and paths from the beginning.

Note that:

- Reversible functions are computationally universal (Bennett's reversible Turing Machine from 1973!)
- *First-order* reversible functions can be inductively defined in type theory (James and Sabry, POPL 2012).

# 3. Computing with Type Isomorphisms

The main syntactic vehicle for the developments in this paper is a simple language called $\Pi$ whose only computations are isomorphisms between finite types.

## 3.1 Syntax and Examples

The set of types $\tau$ includes the empty type 0, the unit type 1, and conventional sum and product types. The values classified by these types are the conventional ones: $()$ of type 1, $\mathsf{inl}\ v$ and $\mathsf{inr}\ v$ for injections into sum types, and $(v_1, v_2)$ for product types:

| | | | |
|---|---|---|---|
| (*Types*) | $\tau$ | $::=$ | $0 \mid 1 \mid \tau_1 + \tau_2 \mid \tau_1 * \tau_2$ |
| (*Values*) | $v$ | $::=$ | $() \mid \mathsf{inl}\ v \mid \mathsf{inr}\ v \mid (v_1, v_2)$ |
| (*Combinator types*) | | | $\tau_1 \leftrightarrow \tau_2$ |
| (*Combinators*) | $c$ | $::=$ | [*see Table 1*] |

The interesting syntactic category of $\Pi$ is that of *combinators* which are witnesses for type isomorphisms $\tau_1 \leftrightarrow \tau_2$. They consist of base combinators (on the left side of Table 1) and compositions (on the right side of the same table). Each line of the table on the left introduces a pair of dual constants[1] that witness the type isomorphism in the middle. This set of isomorphisms is known to be complete [Fiore 2004; Fiore et al. 2006] and the language is universal for hardware combinational circuits [James and Sabry 2012].[2]

## 3.2 Semantics

From the perspective of category theory, the language $\Pi$ models what is called a *symmetric bimonoidal category* or a *commutative rig category*. These are categories with two binary operations $\oplus$ and $\otimes$ satisfying the axioms of a rig (i.e., a ring without negative elements also known as a semiring) up to coherent isomorphisms. And indeed the types of the $\Pi$-combinators are precisely the semiring axioms. A formal way of saying this is that $\Pi$ is the *categorification* [Baez and Dolan 1998] of the natural numbers. A simple (slightly degenerate) example of such categories is the category of finite sets and permutations in which we interpret every $\Pi$-type as a finite set, the values as elements in these finite sets, and the combinators as permutations. Another common example of such categories is the category of finite dimensional vector spaces and linear maps over any field. Note that in this interpretation, the $\Pi$-type 0 maps to the 0-dimensional vector space which is *not* empty. Its unique element, the zero vector — which is present in every vector space — acts like a "bottom" everywhere-undefined element and hence the type behaves like the unit of addition and the annihilator of multiplication as desired.

Operationally, the semantics consists of a pair of mutually recursive evaluators that take a combinator and a value and propagate the value in the "forward" $\triangleright$ direction or in the "backwards" $\triangleleft$ direction. We show the complete forward evaluator; the backwards

[1] where $swap_+$ and $swap_*$ are self-dual.

[2] If recursive types and a trace operator are added, the language becomes Turing complete [Bowman et al. 2011; James and Sabry 2012]. We will not be concerned with this extension in the main body of this paper but it will be briefly discussed in the conclusion.

| | | | | |
|---|---|---|---|---|
| $\mathit{identl}_+ :$ | $0 + \tau$ | $\leftrightarrow$ | $\tau$ | $: \mathit{identr}_+$ |
| $\mathit{swap}_+ :$ | $\tau_1 + \tau_2$ | $\leftrightarrow$ | $\tau_2 + \tau_1$ | $: \mathit{swap}_+$ |
| $\mathit{assocl}_+ :$ | $\tau_1 + (\tau_2 + \tau_3)$ | $\leftrightarrow$ | $(\tau_1 + \tau_2) + \tau_3$ | $: \mathit{assocr}_+$ |
| $\mathit{identl}_* :$ | $1 * \tau$ | $\leftrightarrow$ | $\tau$ | $: \mathit{identr}_*$ |
| $\mathit{swap}_* :$ | $\tau_1 * \tau_2$ | $\leftrightarrow$ | $\tau_2 * \tau_1$ | $: \mathit{swap}_*$ |
| $\mathit{assocl}_* :$ | $\tau_1 * (\tau_2 * \tau_3)$ | $\leftrightarrow$ | $(\tau_1 * \tau_2) * \tau_3$ | $: \mathit{assocr}_*$ |
| $\mathit{dist}_0 :$ | $0 * \tau$ | $\leftrightarrow$ | $0$ | $: \mathit{factor}_0$ |
| $\mathit{dist} :$ | $(\tau_1 + \tau_2) * \tau_3$ | $\leftrightarrow$ | $(\tau_1 * \tau_3) + (\tau_2 * \tau_3)$ | $: \mathit{factor}$ |

$$\frac{}{\vdash \mathit{id} : \tau \leftrightarrow \tau} \qquad \frac{\vdash c : \tau_1 \leftrightarrow \tau_2}{\vdash \mathit{sym}\ c : \tau_2 \leftrightarrow \tau_1}$$

$$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_2 \leftrightarrow \tau_3}{\vdash c_1 \fatsemi c_2 : \tau_1 \leftrightarrow \tau_3}$$

$$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \oplus c_2 : \tau_1 + \tau_3 \leftrightarrow \tau_2 + \tau_4}$$

$$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \otimes c_2 : \tau_1 * \tau_3 \leftrightarrow \tau_2 * \tau_4}$$

**Table 1.** $\Pi$-combinators [James and Sabry 2012]

evaluator differs in trivial ways:

$$\begin{array}{rcl}
\mathit{identl}_+ \ \triangleright (\mathsf{inr}\ v) & = & v \\
\mathit{identr}_+ \ \triangleright v & = & \mathsf{inr}\ v \\
\mathit{swap}_+ \ \triangleright (\mathsf{inl}\ v) & = & \mathsf{inr}\ v \\
\mathit{swap}_+ \ \triangleright (\mathsf{inr}\ v) & = & \mathsf{inl}\ v \\
\mathit{assocl}_+ \ \triangleright (\mathsf{inl}\ v) & = & \mathsf{inl}\ (\mathsf{inl}\ v) \\
\mathit{assocl}_+ \ \triangleright (\mathsf{inr}\ (\mathsf{inl}\ v)) & = & \mathsf{inl}\ (\mathsf{inr}\ v) \\
\mathit{assocl}_+ \ \triangleright (\mathsf{inr}\ (\mathsf{inr}\ v)) & = & \mathsf{inr}\ v \\
\mathit{assocr}_+ \ \triangleright (\mathsf{inl}\ (\mathsf{inl}\ v)) & = & \mathsf{inl}\ v \\
\mathit{assocr}_+ \ \triangleright (\mathsf{inl}\ (\mathsf{inr}\ v)) & = & \mathsf{inr}\ (\mathsf{inl}\ v) \\
\mathit{assocr}_+ \ \triangleright (\mathsf{inr}\ v) & = & \mathsf{inr}\ (\mathsf{inr}\ v) \\
\mathit{identl}_* \ \triangleright ((), v) & = & v \\
\mathit{identr}_* \ \triangleright v & = & ((), v) \\
\mathit{swap}_* \ \triangleright (v_1, v_2) & = & (v_2, v_1) \\
\mathit{assocl}_* \ \triangleright (v_1, (v_2, v_3)) & = & ((v_1, v_2), v_3) \\
\mathit{assocr}_* \ \triangleright ((v_1, v_2), v_3) & = & (v_1, (v_2, v_3)) \\
\mathit{dist} \ \triangleright (\mathsf{inl}\ v_1, v_3) & = & \mathsf{inl}\ (v_1, v_3) \\
\mathit{dist} \ \triangleright (\mathsf{inr}\ v_2, v_3) & = & \mathsf{inr}\ (v_2, v_3) \\
\mathit{factor} \ \triangleright (\mathsf{inl}\ (v_1, v_3)) & = & (\mathsf{inl}\ v_1, v_3) \\
\mathit{factor} \ \triangleright (\mathsf{inr}\ (v_2, v_3)) & = & (\mathsf{inr}\ v_2, v_3) \\
\mathit{id} \ \triangleright v & = & v \\
(\mathit{sym}\ c) \ \triangleright v & = & c \triangleleft v \\
(c_1 \fatsemi c_2) \ \triangleright v & = & c_2 \triangleright (c_1 \triangleright v) \\
(c_1 \oplus c_2) \ \triangleright (\mathsf{inl}\ v) & = & \mathsf{inl}\ (c_1 \triangleright v) \\
(c_1 \oplus c_2) \ \triangleright (\mathsf{inr}\ v) & = & \mathsf{inr}\ (c_2 \triangleright v) \\
(c_1 \otimes c_2) \ \triangleright (v_1, v_2) & = & (c_1 \triangleright v_1, c_2 \triangleright v_2)
\end{array}$$

## 4. The Space of Types

Instead of modeling the semantics of $\Pi$ using *permutations*, which are set-theoretic functions after all, we use *paths* from the HoTT framework. More precisely, we model the universe of $\Pi$ types as a space whose points are the individual $\Pi$-types and we will consider that there is path between two points $\tau_1$ and $\tau_2$ if there is a $\Pi$ combinator $c : \tau_1 \leftrightarrow \tau_2$. If we focus on 1-paths, this is perfect as we explain next.

***Note.*** But first, we note that this is a significant deviation from the HoTT framework which fundamentally includes functions, which are specialized to equivalences, which are then postulated to be paths by the univalence axiom. This axiom has no satisfactory computational interpretation, however. Instead we completely bypass the idea of extensional functions and use paths directly. Another way to understanding what is going on is the following. In the conventional HoTT framework:

- We start with two different notions: paths and functions;
- We use extensional non-constructive methods to identify a particular class of functions that form isomorphisms;
- We postulate that this particular class of functions can be identified with paths.

In our case,

- We start with a constructive characterization of *reversible functions* or *isomorphisms* built using inductively defined combinators;
- We blur the distinction between such combinators and paths from the beginning. We view computation as nothing more than *following paths*! As explained earlier, although this appears limiting, it is universal and regular computation can be viewed as a special case of that.

***Construction.*** We have a universe $U$ viewed as a groupoid whose points are the types $\Pi$-types $\tau$. The $\Pi$-combinators of Table 1 are viewed as syntax for the paths in the space $U$. We need to show that the groupoid path structure is faithfully represented. The combinator $\mathit{id}$ introduces all the $\mathsf{refl}\ \tau : \tau \equiv \tau$ paths in $U$. The adjoint $\mathit{sym}\ c$ introduces an inverse path $!p$ for each path $p$ introduced by $c$. The composition operator $\fatsemi$ introduce a path $p \circ q$ for every pair of paths whose endpoints match. In addition, we get paths like $\mathit{swap}_+$ between $\tau_1 + \tau_2$ and $\tau_2 + \tau_1$. The existence of such paths in the conventional HoTT developed is *postulated* by the univalence axiom. The $\otimes$-composition gives a path $(p, q) : (\tau_1 * \tau_2) \equiv (\tau_3 * \tau_4)$ whenever we have paths $p : \tau_1 \equiv \tau_3$ and $q : \tau_2 \equiv \tau_4$. A similar situation for the $\oplus$-composition. The structure of these paths must be discovered and these paths must be *proved* to exist using path induction in the conventional HoTT development. So far, this appears too good to be true, and it is. The problem is that paths in HoTT are subject to rules discussed at the end of Sec. 2. For example, it must be the case that if $p : \tau_1 \equiv_U \tau_2$ that $(p \circ \mathsf{refl}\ \tau_2) \equiv_{\tau_1 \equiv_U \tau_2} p$. This path lives in a higher universe: nothing in our $\Pi$-combinators would justify adding such a path as all our combinators map types to types. No combinator works one level up at the space of combinators and there is no such space in the first place. Clearly we are stuck unless we manage to express a notion of higher-order functions in $\Pi$. This would allow us to internalize the type $\tau_1 \leftrightarrow \tau_2$ as a $\Pi$-type which is then manipulated by the same combinators one level higher and so on.

To make the correspondence between $\Pi$ and the HoTT concepts more apparent we will, in the remainder of the paper, use $\mathsf{refl}$ instead of $\mathit{id}$ and $!$ instead of $\mathit{sym}$ when referring to $\Pi$ combinators when viewed as paths. Similarly we will use $\rightarrow$ instead of the $\Pi$-notation $\leftrightarrow$ or the HoTT notation $\equiv$ to refer to paths.

## 5. Agda Model

```
------------------------------------------------------
- Level 0:
- Types at this level are just plain sets with no interest
- The path structure is defined at levels 1 and beyond.

data U : Set where
   ZERO    : U
   ONE     : U
```

```
  PLUS  : U → U → U
  TIMES : U → U → U

⟦_⟧ : U → Set
⟦ ZERO ⟧        = ⊥
⟦ ONE ⟧         = ⊤
⟦ PLUS t1 t2 ⟧  = ⟦ t1 ⟧ ⊎ ⟦ t2 ⟧
⟦ TIMES t1 t2 ⟧ = ⟦ t1 ⟧ × ⟦ t2 ⟧

-- Programs
-- We use pointed types; programs map a pointed type to another
-- In other words, each program takes one particular value to another; if we
-- want to work on another value, we generally use another program

record U• : Set where
  constructor •[_,_]
  field
    |_| : U
    • : ⟦ |_| ⟧

open U•

Space : (t• : U•) → Set
Space •[ t , v ] = ⟦ t ⟧

point : (t• : U•) → Space t•
point •[ t , v ] = v

-- examples of plain types, values, and pointed types

ONE• : U•
ONE• = •[ ONE , tt ]

BOOL : U
BOOL = PLUS ONE ONE

BOOL^2 : U
BOOL^2 = TIMES BOOL BOOL

TRUE : ⟦ BOOL ⟧
TRUE = inj1 tt

FALSE : ⟦ BOOL ⟧
FALSE = inj2 tt

BOOL•F : U•
BOOL•F = •[ BOOL , FALSE ]

BOOL•T : U•
BOOL•T = •[ BOOL , TRUE ]

-- The actual programs are the commutative semiring isomorphisms between
-- pointed types.

data _⟷_ : U• → U• → Set where
  unite+   : ∀ {t v} → •[ PLUS ZERO t , inj2 v ] ⟷ •[ t , v ]
  uniti+   : ∀ {t v} → •[ t , v ] ⟷ •[ PLUS ZERO t , inj2 v ]
  swap1+   : ∀ {t1 t2 v1} → •[ PLUS t1 t2 , inj1 v1 ] ⟷ •[ PLUS t2 t1 , inj2 v1 ]
  swap2+   : ∀ {t1 t2 v2} → •[ PLUS t1 t2 , inj2 v2 ] ⟷ •[ PLUS t2 t1 , inj1 v2 ]
  assocl1+ : ∀ {t1 t2 t3 v1} →
    •[ PLUS t1 (PLUS t2 t3) , inj1 v1 ] ⟷
    •[ PLUS (PLUS t1 t2) t3 , inj1 (inj1 v1) ]
  assocl2+ : ∀ {t1 t2 t3 v2} →
    •[ PLUS t1 (PLUS t2 t3) , inj2 (inj1 v2) ] ⟷
    •[ PLUS (PLUS t1 t2) t3 , inj1 (inj2 v2) ]
  assocl3+ : ∀ {t1 t2 t3 v3} →
    •[ PLUS t1 (PLUS t2 t3) , inj2 (inj2 v3) ] ⟷
    •[ PLUS (PLUS t1 t2) t3 , inj2 v3 ]
```

```
  assocr1+ : ∀ {t1 t2 t3 v1} →
    •[ PLUS (PLUS t1 t2) t3 , inj1 (inj1 v1) ] ⟷
    •[ PLUS t1 (PLUS t2 t3) , inj1 v1 ]
  assocr2+ : ∀ {t1 t2 t3 v2} →
    •[ PLUS (PLUS t1 t2) t3 , inj1 (inj2 v2) ] ⟷
    •[ PLUS t1 (PLUS t2 t3) , inj2 (inj1 v2) ]
  assocr3+ : ∀ {t1 t2 t3 v3} →
    •[ PLUS (PLUS t1 t2) t3 , inj2 v3 ] ⟷
    •[ PLUS t1 (PLUS t2 t3) , inj2 (inj2 v3) ]
  unite⋆   : ∀ {t v} → •[ TIMES ONE t , (tt , v) ] ⟷ •[ t , v ]
  uniti⋆   : ∀ {t v} → •[ t , v ] ⟷ •[ TIMES ONE t , (tt , v) ]
  swap⋆    : ∀ {t1 t2 v1 v2} →
    •[ TIMES t1 t2 , (v1 , v2) ] ⟷ •[ TIMES t2 t1 , (v2 , v1) ]
  assocl⋆ : ∀ {t1 t2 t3 v1 v2 v3} →
          •[ TIMES t1 (TIMES t2 t3) , (v1 , (v2 , v3)) ] ⟷
          •[ TIMES (TIMES t1 t2) t3 , ((v1 , v2) , v3) ]
  assocr⋆ : ∀ {t1 t2 t3 v1 v2 v3} →
          •[ TIMES (TIMES t1 t2) t3 , ((v1 , v2) , v3) ] ⟷
          •[ TIMES t1 (TIMES t2 t3) , (v1 , (v2 , v3)) ]
  distz : ∀ {t v absurd} →
          •[ TIMES ZERO t , (absurd , v) ] ⟷ •[ ZERO , absurd ]
  factorz : ∀ {t v absurd} →
          •[ ZERO , absurd ] ⟷ •[ TIMES ZERO t , (absurd , v) ]
  dist1   : ∀ {t1 t2 t3 v1 v3} →
          •[ TIMES (PLUS t1 t2) t3 , (inj1 v1 , v3) ] ⟷
          •[ PLUS (TIMES t1 t3) (TIMES t2 t3) , inj1 (v1 , v3) ]
  dist2   : ∀ {t1 t2 t3 v2 v3} →
          •[ TIMES (PLUS t1 t2) t3 , (inj2 v2 , v3) ] ⟷
          •[ PLUS (TIMES t1 t3) (TIMES t2 t3) , inj2 (v2 , v3) ]
  factor1  : ∀ {t1 t2 t3 v1 v3} →
          •[ PLUS (TIMES t1 t3) (TIMES t2 t3) , inj1 (v1 , v3) ] ⟷
          •[ TIMES (PLUS t1 t2) t3 , (inj1 v1 , v3) ]
  factor2  : ∀ {t1 t2 t3 v2 v3} →
          •[ PLUS (TIMES t1 t3) (TIMES t2 t3) , inj2 (v2 , v3) ] ⟷
          •[ TIMES (PLUS t1 t2) t3 , (inj2 v2 , v3) ]
  id⟷    : ∀ {t v} → •[ t , v ] ⟷ •[ t , v ]
  sym⟷   : ∀ {t1 t2 v1 v2} → (•[ t1 , v1 ] ⟷ •[ t2 , v2 ]) →
    (•[ t2 , v2 ] ⟷ •[ t1 , v1 ])
  _◎_    : ∀ {t1 t2 t3 v1 v2 v3} → (•[ t1 , v1 ] ⟷ •[ t2 , v2 ]) →
    (•[ t2 , v2 ] ⟷ •[ t3 , v3 ]) →
    (•[ t1 , v1 ] ⟷ •[ t3 , v3 ])
  _⊕1_   : ∀ {t1 t2 t3 t4 v1 v2 v3 v4} →
    (•[ t1 , v1 ] ⟷ •[ t3 , v3 ]) → (•[ t2 , v2 ] ⟷ •[ t4 , v4 ]) →
    (•[ PLUS t1 t2 , inj1 v1 ] ⟷ •[ PLUS t3 t4 , inj1 v3 ])
  _⊕2_   : ∀ {t1 t2 t3 t4 v1 v2 v3 v4} →
    (•[ t1 , v1 ] ⟷ •[ t3 , v3 ]) → (•[ t2 , v2 ] ⟷ •[ t4 , v4 ]) →
    (•[ PLUS t1 t2 , inj2 v2 ] ⟷ •[ PLUS t3 t4 , inj2 v4 ])
  _⊗_     : ∀ {t1 t2 t3 t4 v1 v2 v3 v4} →
    (•[ t1 , v1 ] ⟷ •[ t3 , v3 ]) → (•[ t2 , v2 ] ⟷ •[ t4 , v4 ]) →
          (•[ TIMES t1 t2 , (v1 , v2) ] ⟷ •[ TIMES t3 t4 , (v3 , v4) ])

-- example programs

NOT•T : •[ BOOL , TRUE ] ⟷ •[ BOOL , FALSE ]
NOT•T = swap1+

NOT•F : •[ BOOL , FALSE ] ⟷ •[ BOOL , TRUE ]
NOT•F = swap2+

CNOT•Fx : {b : ⟦ BOOL ⟧} →
          •[ BOOL^2 , (FALSE , b) ] ⟷ •[ BOOL^2 , (FALSE , b) ]
CNOT•Fx = dist2 ◎ ((id⟷ ⊗ NOT•F) ⊕2 id⟷) ◎ factor2

CNOT•TF : •[ BOOL^2 , (TRUE , FALSE) ] ⟷ •[ BOOL^2 , (TRUE , TRUE) ]
CNOT•TF = dist1 ◎
          ((id⟷ ⊗ NOT•F) ⊕1 (id⟷ {TIMES ONE BOOL} {(tt , TRUE)})) ◎
          factor1
```

CNOT•TT : •[ BOOL$^2$ , (TRUE , TRUE) ] $\leftrightarrow$ •[ BOOL$^2$ , (TRUE , FALSE) ]
CNOT•TT = dist1 ◎
((id$\leftrightarrow$ $\otimes$ NOT•T) $\oplus$1 (id$\leftrightarrow$ {TIMES ONE BOOL} {(tt , TRUE)})) ◎
factor1

```
- The evaluation of a program is not done in order to figure out the output
- value. Both the input and output values are encoded in the type of the
- program; what the evaluation does is follow the path to constructively
- reach the output value from the input value. Even though programs of the
- same pointed types are, by definition, observationally equivalent, they
- may follow different paths. At this point, we simply declare that all such
- programs are "the same." At the next level, we will weaken this "proof-
- irrelevant" equivalence and reason about which paths are equated to
- other paths via 2paths etc.

- Even though individual types are sets, the universe of types is a
- groupoid. The objects of this groupoid are the pointed types; the
- morphisms are the programs; and the equivalence of programs is the
- degenerate observational equivalence that equates every two programs that
- are extensionally equivalent.
```

_obs≅_ : $\{t_1\ t_2$ : U•$\} \to (c_1\ c_2 : t_1 \leftrightarrow t_2) \to$ Set
$c_1$ obs≅ $c_2 = \top$

UG : 1Groupoid
UG = record
{ set = U•
; _⇝_ = _$\leftrightarrow$_
; _≈_ = _obs≅_
; id = id$\leftrightarrow$
; _∘_ = λ $y \leftrightarrow z\ x \leftrightarrow y \to x \leftrightarrow y$ ◎ $y \leftrightarrow z$
; _$^{-1}$ = sym$\leftrightarrow$
; lneutr = λ _ → tt
; rneutr = λ _ → tt
; assoc = λ _ _ _ → tt
; equiv = record { refl = tt
; sym = λ _ → tt
; trans = λ _ _ → tt
}
; linv = λ _ → tt
; rinv = λ _ → tt
; ∘-resp-≈ = λ _ _ → tt
}

```
------------------------------------------------------------------------
- Simplify various compositions
```

simplifySym : $\{t_1\ t_2$ : U•$\} \to (c_1 : t_1 \leftrightarrow t_2) \to (t_2 \leftrightarrow t_1)$
simplifySym unite$_+$ = uniti$_+$
simplifySym uniti$_+$ = unite$_+$
simplifySym swap1$_+$ = swap2$_+$
simplifySym swap2$_+$ = swap1$_+$
simplifySym assocl1$_+$ = assocr1$_+$
simplifySym assocl2$_+$ = assocr2$_+$
simplifySym assocl3$_+$ = assocr3$_+$
simplifySym assocr1$_+$ = assocl1$_+$
simplifySym assocr2$_+$ = assocl2$_+$
simplifySym assocr3$_+$ = assocl3$_+$
simplifySym unite⋆ = uniti⋆
simplifySym uniti⋆ = unite⋆
simplifySym swap⋆ = swap⋆
simplifySym assocl⋆ = assocr⋆
simplifySym assocr⋆ = assocl⋆
simplifySym distz = factorz
simplifySym factorz = distz
simplifySym dist1 = factor1
simplifySym dist2 = factor2
simplifySym factor1 = dist1
simplifySym factor2 = dist2
simplifySym id$\leftrightarrow$ = id$\leftrightarrow$
simplifySym (sym$\leftrightarrow$ $c$) = $c$
simplifySym ($c_1$ ◎ $c_2$) = simplifySym $c_2$ ◎ simplifySym $c_1$
simplifySym ($c_1$ $\oplus$1 $c_2$) = simplifySym $c_1$ $\oplus$1 simplifySym $c_2$
simplifySym ($c_1$ $\oplus$2 $c_2$) = simplifySym $c_1$ $\oplus$2 simplifySym $c_2$
simplifySym ($c_1$ $\otimes$ $c_2$) = simplifySym $c_1$ $\otimes$ simplifySym $c_2$

simplifyl◎ : $\{t_1\ t_2\ t_3$ : U•$\} \to (c_1 : t_1 \leftrightarrow t_2) \to (c_2 : t_2 \leftrightarrow t_3) \to (t_1 \leftrightarrow t_3)$
simplifyl◎ id$\leftrightarrow$ $c$ = $c$
simplifyl◎ unite$_+$ uniti$_+$ = id$\leftrightarrow$
simplifyl◎ uniti$_+$ unite$_+$ = id$\leftrightarrow$
simplifyl◎ swap1$_+$ swap2$_+$ = id$\leftrightarrow$
simplifyl◎ swap2$_+$ swap1$_+$ = id$\leftrightarrow$
simplifyl◎ assocl1$_+$ assocr1$_+$ = id$\leftrightarrow$
simplifyl◎ assocl2$_+$ assocr2$_+$ = id$\leftrightarrow$
simplifyl◎ assocl3$_+$ assocr3$_+$ = id$\leftrightarrow$
simplifyl◎ assocr1$_+$ assocl1$_+$ = id$\leftrightarrow$
simplifyl◎ assocr2$_+$ assocl2$_+$ = id$\leftrightarrow$
simplifyl◎ assocr3$_+$ assocl3$_+$ = id$\leftrightarrow$
simplifyl◎ unite⋆ uniti⋆ = id$\leftrightarrow$
simplifyl◎ uniti⋆ unite⋆ = id$\leftrightarrow$
simplifyl◎ swap⋆ swap⋆ = id$\leftrightarrow$
simplifyl◎ assocl⋆ assocr⋆ = id$\leftrightarrow$
simplifyl◎ assocr⋆ assocl⋆ = id$\leftrightarrow$
simplifyl◎ factorz distz = id$\leftrightarrow$
simplifyl◎ dist1 factor1 = id$\leftrightarrow$
simplifyl◎ dist2 factor2 = id$\leftrightarrow$
simplifyl◎ factor1 dist1 = id$\leftrightarrow$
simplifyl◎ factor2 dist2 = id$\leftrightarrow$
simplifyl◎ ($c_1$ ◎ $c_2$) $c_3$ = $c_1$ ◎ ($c_2$ ◎ $c_3$)
simplifyl◎ ($c_1$ $\oplus$1 $c_2$) swap1$_+$ = swap1$_+$ ◎ ($c_2$ $\oplus$2 $c_1$)
simplifyl◎ ($c_1$ $\oplus$2 $c_2$) swap2$_+$ = swap2$_+$ ◎ ($c_2$ $\oplus$1 $c_1$)
simplifyl◎ (_$\otimes$_ {ONE} {ONE} $c_1$ $c_2$) unite⋆ = unite⋆ ◎ $c_2$
simplifyl◎ ($c_1$ $\otimes$ $c_2$) swap⋆ = swap⋆ ◎ ($c_2$ $\otimes$ $c_1$)
simplifyl◎ ($c_1$ $\otimes$ $c_2$) ($c_3$ $\otimes$ $c_4$) = ($c_1$ ◎ $c_3$) $\otimes$ ($c_2$ ◎ $c_4$)
simplifyl◎ $c_1$ $c_2$ = $c_1$ ◎ $c_2$

simplifyr◎ : $\{t_1\ t_2\ t_3$ : U•$\} \to (c_1 : t_1 \leftrightarrow t_2) \to (c_2 : t_2 \leftrightarrow t_3) \to (t_1 \leftrightarrow t_3)$
simplifyr◎ $c$ id$\leftrightarrow$ = $c$
simplifyr◎ unite$_+$ uniti$_+$ = id$\leftrightarrow$
simplifyr◎ uniti$_+$ unite$_+$ = id$\leftrightarrow$
simplifyr◎ swap1$_+$ swap2$_+$ = id$\leftrightarrow$
simplifyr◎ swap2$_+$ swap1$_+$ = id$\leftrightarrow$
simplifyr◎ assocl1$_+$ assocr1$_+$ = id$\leftrightarrow$
simplifyr◎ assocl2$_+$ assocr2$_+$ = id$\leftrightarrow$
simplifyr◎ assocl3$_+$ assocr3$_+$ = id$\leftrightarrow$
simplifyr◎ assocr1$_+$ assocl1$_+$ = id$\leftrightarrow$
simplifyr◎ assocr2$_+$ assocl2$_+$ = id$\leftrightarrow$
simplifyr◎ assocr3$_+$ assocl3$_+$ = id$\leftrightarrow$
simplifyr◎ unite⋆ uniti⋆ = id$\leftrightarrow$
simplifyr◎ uniti⋆ unite⋆ = id$\leftrightarrow$
simplifyr◎ swap⋆ swap⋆ = id$\leftrightarrow$
simplifyr◎ assocl⋆ assocr⋆ = id$\leftrightarrow$
simplifyr◎ assocr⋆ assocl⋆ = id$\leftrightarrow$
simplifyr◎ factorz distz = id$\leftrightarrow$
simplifyr◎ dist1 factor1 = id$\leftrightarrow$
simplifyr◎ dist2 factor2 = id$\leftrightarrow$
simplifyr◎ factor1 dist1 = id$\leftrightarrow$
simplifyr◎ factor2 dist2 = id$\leftrightarrow$
simplifyr◎ ($c_1$ ◎ $c_2$) $c_3$ = $c_1$ ◎ ($c_2$ ◎ $c_3$)
simplifyr◎ ($c_1$ $\oplus$1 $c_2$) swap1$_+$ = swap1$_+$ ◎ ($c_2$ $\oplus$2 $c_1$)
simplifyr◎ ($c_1$ $\oplus$2 $c_2$) swap2$_+$ = swap2$_+$ ◎ ($c_2$ $\oplus$1 $c_1$)
simplifyr◎ (_$\otimes$_ {ONE} {ONE} $c_1$ $c_2$) unite⋆ = unite⋆ ◎ $c_2$
simplifyr◎ ($c_1$ $\otimes$ $c_2$) swap⋆ = swap⋆ ◎ ($c_2$ $\otimes$ $c_1$)
simplifyr◎ ($c_1$ $\otimes$ $c_2$) ($c_3$ $\otimes$ $c_4$) = ($c_1$ ◎ $c_3$) $\otimes$ ($c_2$ ◎ $c_4$)
simplifyr◎ $c_1$ $c_2$ = $c_1$ ◎ $c_2$

## 6. Examples

Let's start with a few simple types built from the empty type, the unit type, sums, and products, and let's study the paths postulated by HoTT.

For every value in a type (point in a space) we have a trivial path from the value to itself:

In addition to all these trivial paths, there are structured paths. In particular, paths in product spaces can be viewed as pair of paths. So in addition to the path above, we also have:

## 7. Theory

## 8. Pi

### 8.1 Base isomorphisms

$$
\begin{array}{rrcll}
identl_+ : & 0+b & \leftrightarrow & b & : identr_+ \\
swap_+ : & b_1+b_2 & \leftrightarrow & b_2+b_1 & : swap_+ \\
assocl_+ : & b_1+(b_2+b_3) & \leftrightarrow & (b_1+b_2)+b_3 & : assocr_+ \\
identl_* : & 1*b & \leftrightarrow & b & : identr_* \\
swap_* : & b_1*b_2 & \leftrightarrow & b_2*b_1 & : swap_* \\
assocl_* : & b_1*(b_2*b_3) & \leftrightarrow & (b_1*b_2)*b_3 & : assocr_* \\
dist_0 : & 0*b & \leftrightarrow & 0 & : factor_0 \\
dist : & (b_1+b_2)*b_3 & \leftrightarrow & (b_1*b_3)+(b_2*b_3) & : factor
\end{array}
$$

$$
\frac{}{\vdash id : b \leftrightarrow b} \qquad \frac{\vdash c : b_1 \leftrightarrow b_2}{\vdash sym\ c : b_2 \leftrightarrow b_1}
$$

$$
\frac{\vdash c_1 : b_1 \leftrightarrow b_2 \quad c_2 : b_2 \leftrightarrow b_3}{\vdash c_1 \fatsemi c_2 : b_1 \leftrightarrow b_3}
$$

$$
\frac{\vdash c_1 : b_1 \leftrightarrow b_2 \quad c_2 : b_3 \leftrightarrow b_4}{\vdash c_1 \oplus c_2 : b_1 + b_3 \leftrightarrow b_2 + b_4}
$$

$$
\frac{\vdash c_1 : b_1 \leftrightarrow b_2 \quad c_2 : b_3 \leftrightarrow b_4}{\vdash c_1 \otimes c_2 : b_1 * b_3 \leftrightarrow b_2 * b_4}
$$

These isomorphisms:

- Form an inductive type
- Identify each isomorphism with a collection of paths
- For example:

$$swap_+ : \quad b_1 + b_2 \quad \leftrightarrow \quad b_2 + b_1$$

becomes:

$$
\begin{array}{rccc}
swap_+^1 : & inj_1 v & \equiv & inj_2 v \\
swap_+^2 : & inj_2 v & \equiv & inj_1 v
\end{array}
$$

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$$\frac{}{() : 1} \qquad \frac{v_1 : t_1}{\mathsf{inl}\ v_1 : t_1 + t_2} \qquad \frac{v_2 : t_2}{\mathsf{inr}\ v_2 : t_1 + t_2} \qquad \frac{v_1 : t_1 \quad v_2 : t_2}{(v_1, v_2) : t_1 * t_2} \qquad \frac{}{\mathsf{inr}\ v \xrightarrow{identl_+} v : \mathsf{inr}\ v \equiv_{identl_+} v}$$

$$\frac{}{v \xrightarrow{identr_+} \mathsf{inr}\ v : v \equiv_{identr_+} \mathsf{inr}\ v} \qquad \frac{}{\mathsf{inl}\ v \xrightarrow{swap_+} \mathsf{inr}\ v : \mathsf{inl}\ v \equiv_{swap_+} \mathsf{inr}\ v} \qquad \frac{}{\mathsf{inr}\ v \xrightarrow{swap_+} \mathsf{inl}\ v : \mathsf{inr}\ v \equiv_{swap_+} \mathsf{inl}\ v}$$

$$\frac{}{\mathsf{inl}\ v \xrightarrow{assocl_+} \mathsf{inl}\ (\mathsf{inl}\ v) : \mathsf{inl}\ v \equiv_{assocl_+} \mathsf{inl}\ (\mathsf{inl}\ v)} \qquad \frac{}{\mathsf{inr}\ (\mathsf{inl}\ v) \xrightarrow{assocl_+} \mathsf{inl}\ (\mathsf{inr}\ v) : \mathsf{inr}\ (\mathsf{inl}\ v) \equiv_{assocl_+} \mathsf{inl}\ (\mathsf{inr}\ v)}$$

$$\frac{}{\mathsf{inr}\ (\mathsf{inr}\ v) \xrightarrow{assocl_+} \mathsf{inr}\ v : \mathsf{inr}\ (\mathsf{inr}\ v) \equiv_{assocl_+} \mathsf{inr}\ v} \qquad \frac{}{\mathsf{inl}\ (\mathsf{inl}\ v) \xrightarrow{assocr_+} \mathsf{inl}\ v : \mathsf{inl}\ (\mathsf{inl}\ v) \equiv_{assocr_+} \mathsf{inl}\ v}$$

$$\frac{}{\mathsf{inl}\ (\mathsf{inr}\ v) \xrightarrow{assocr_+} \mathsf{inr}\ (\mathsf{inl}\ v) : \mathsf{inl}\ (\mathsf{inr}\ v) \equiv_{assocr_+} \mathsf{inr}\ (\mathsf{inl}\ v)} \qquad \frac{}{\mathsf{inr}\ v \xrightarrow{assocr_+} \mathsf{inr}\ (\mathsf{inr}\ v) : \mathsf{inr}\ v \equiv_{assocr_+} \mathsf{inr}\ (\mathsf{inr}\ v)}$$

$$\frac{}{((), v) \xrightarrow{identl_*} v : ((), v) \equiv_{identl_*} v} \qquad \frac{}{v \xrightarrow{identr_*} ((), v) : v \equiv_{identr_*} ((), v)} \qquad \frac{}{((v_1, v_2) \xrightarrow{swap_*} (v_2, v_1) : (v_1, v_2) \equiv_{swap_*} (v_2, v_1)}$$

$$\frac{}{(v_1, (v_2, v_3)) \xrightarrow{assocl_*} ((v_1, v_2), v_3) : (v_1, (v_2, v_3)) \equiv_{assocl_*} ((v_1, v_2), v_3)} \qquad \frac{}{((v_1, v_2), v_3) \xrightarrow{assocr_*} (v_1, (v_2, v_3)) : ((v_1, v_2), v_3) \equiv_{assocr_*} (v_1,}$$

$$\frac{}{(\mathsf{inl}\ v_1, v_2) \xrightarrow{dist} \mathsf{inl}\ (v_1, v_2) : (\mathsf{inl}\ v_1, v_2) \equiv_{dist} \mathsf{inl}\ (v_1, v_2)} \qquad \frac{}{(\mathsf{inr}\ v_1, v_2) \xrightarrow{dist} \mathsf{inr}\ (v_1, v_2) : (\mathsf{inr}\ v_1, v_2) \equiv_{dist} \mathsf{inr}\ (v_1, v_2)}$$

$$\frac{}{\mathsf{inl}\ (v_1, v_2) \xrightarrow{factor} (\mathsf{inl}\ v_1, v_2) : \mathsf{inl}\ (v_1, v_2) \equiv_{factor} (\mathsf{inl}\ v_1, v_2)} \qquad \frac{}{\mathsf{inr}\ (v_1, v_2) \xrightarrow{factor} (\mathsf{inr}\ v_1, v_2) : \mathsf{inr}\ (v_1, v_2) \equiv_{factor} (\mathsf{inr}\ v_1, v_2)}$$

$$\frac{}{v \xrightarrow{id} v : v \equiv_{id} v} \qquad \frac{p : v_2 \equiv_c v_1}{!p : v_1 \equiv_{sym\ c} v_2} \qquad \frac{p : v_1 \equiv_{c_1} v_2 \quad q : v_2 \equiv_{c_2} v_3}{p \overset{v_2}{\bullet} q : v_1 \equiv_{c_1 ⨾ c_2} v_3} \qquad \frac{p : v \equiv_{c_1} v'}{\mathsf{inl}\ p : \mathsf{inl}\ v \equiv_{c_1 \oplus c_2} \mathsf{inl}\ v'}$$

$$\frac{p : v \equiv_{c_2} v'}{\mathsf{inr}\ p : \mathsf{inr}\ v \equiv_{c_1 \oplus c_2} \mathsf{inr}\ v'} \qquad \frac{p : v_1 \equiv_{c_1} v_1' \quad q : v_2 \equiv_{c_2} v_2'}{(p, q) : (v_1, v_2) \equiv_{c_1 \otimes c_2} (v_1', v_2')} \qquad \frac{p : v \equiv_c v'}{(v \xrightarrow{id} v) \overset{v}{\bullet} p \xrightarrow{\mathsf{lid}} p : (v \xrightarrow{id} v) \overset{v}{\bullet} p \equiv_{\mathsf{lid}} p}$$

$$\frac{p : v' \equiv_c v}{p \overset{v}{\bullet} (v \xrightarrow{id} v) \xrightarrow{\mathsf{rid}} p : p \overset{v}{\bullet} (v \xrightarrow{id} v) \equiv_{\mathsf{rid}} p} \qquad \frac{}{!(\mathsf{inr}\ v \xrightarrow{identl_+} v) \xrightarrow{!1} v \xrightarrow{identr_+} \mathsf{inr}\ v} \qquad \frac{p : v' \equiv_c v}{(!p \overset{v'}{\bullet} p) \xrightarrow{l!} (v \xrightarrow{id} v) : (!p \overset{v'}{\bullet} p) \equiv_{l!} (v \xrightarrow{id} v)}$$

$$\frac{}{? :? \equiv_{r!} ?} \qquad \frac{}{? :? \equiv_{!!} ?} \qquad \frac{}{? :? \equiv_{\circ} ?}$$