Representing, Manipulating and Optimizing Reversible Circuits

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Quantum Computing

Quantum physics differs from classical physics in many ways:

- Superpositions
- Entanglement
- Unitary evolution
- Composition uses tensor products
- Non-unitary measurement

Quantum Computing & Programming Languages

- It is possible to adapt all at once classical programming languages to quantum programming languages.
- Some excellent examples discussed in this workshop
- This assumes that classical programming languages (and implicitly classical physics) can be smoothly adapted to the quantum world.
- There are however what appear to be fundamental differences between the classical and quantum world that make them incompatible
- Let us *re-think* classical programming foundations before jumping to the quantum world.

Resource-Aware Classical Computing

- The biggest questionable assumption of classical programming is that it is possible to freely copy and discard information
- A classical programming language which respects no-cloning and no-discarding is the right foundation for an eventual quantum extension
- We want these properties to be inherent in the language; not an afterthought filtered by a type system
- We want to program with isomorphisms or equivalences
- The simplest instance is permutations between finite types which happens to correspond to reversible circuits.

A (Foundational) Syntactic Theory

Ideally, want a notation that

1. is easy to write by programmers
2. is easy to mechanically manipulate
3. can be reasoned about
4. can be optimized.

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Start with a *foundational* syntactic theory on our way there:

1. easy to explain
2. clear operational rules
3. fully justified by the semantics
4. sound and complete reasoning
5. sound and complete methods of optimization

Starting Point

Typed isomorphisms. First, a universe of (finite) types

```
data U : Set where
  ZERO  : U
  ONE   : U
  PLUS  : U → U → U
  TIMES : U → U → U
```

and its interpretation

$$
\begin{aligned}
&[\![\_]\!] : \mathsf{U} \to \mathsf{Set} \\
&[\![\mathsf{ZERO}]\!] = \bot \\
&[\![\mathsf{ONE}]\!] = \top \\
&[\![\mathsf{PLUS}\ t_1\ t_2]\!] = [\![t_1]\!] \uplus [\![t_2]\!] \\
&[\![\mathsf{TIMES}\ t_1\ t_2]\!] = [\![t_1]\!] \times [\![t_2]\!]
\end{aligned}
$$

Equivalences and semirings

If we denote type equivalence by $\simeq$, then we can prove that

Theorem 1.

The collection of all types (Set) forms a commutative semiring (up to $\simeq$).

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We also get

Theorem 2.

If $A \simeq \mathrm{Fin}\, m$, $B \simeq \mathrm{Fin}\, n$ and $A \simeq B$ then $m \equiv n$.

(whose *constructive* proof is quite subtle).

Theorem 3.

If $A \simeq \mathrm{Fin}\, m$ and $B \simeq \mathrm{Fin}\, n$, then the type of all equivalences $A \simeq B$ is equivalent to the type of all permutations $\mathrm{Perm}\, n$.

Equivalences and semirings II

Semiring structures abound. We can define them on:

1. equivalences (disjoint union and cartesian product)
2. permutations (disjoint union and tensor product)

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The equivalence of Theorem 3 is an isomorphism between the semirings of equivalences of finite types, and of permutations.

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A more evocative phrasing might be:

Theorem 5.

$$(A \simeq B) \simeq \mathrm{Perm}|A|$$

A Calculus of Permutations

First conclusion: it might be useful to *reify* a (sound and complete) set of equivalences as combinators, such as the fundamental "proof rules" of semirings:

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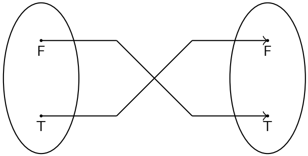
```
data _⟷_ : U → U → Set where
  unite₊   : {t : U} → PLUS ZERO t ⟷ t
  uniti₊   : {t : U} → t ⟷ PLUS ZERO t
  swap₊    : {t₁ t₂ : U} → PLUS t₁ t₂ ⟷ PLUS t₂ t₁
  assocl₊  : {t₁ t₂ t₃ : U} → PLUS t₁ (PLUS t₂ t₃) ⟷ PLUS (PLUS t₁ t₂) t₃
  assocr₊  : {t₁ t₂ t₃ : U} → PLUS (PLUS t₁ t₂) t₃ ⟷ PLUS t₁ (PLUS t₂ t₃)
  unite⋆   : {t : U} → TIMES ONE t ⟷ t
  uniti⋆   : {t : U} → t ⟷ TIMES ONE t
  swap⋆    : {t₁ t₂ : U} → TIMES t₁ t₂ ⟷ TIMES t₂ t₁
  assocl⋆  : {t₁ t₂ t₃ : U} → TIMES t₁ (TIMES t₂ t₃) ⟷ TIMES (TIMES t₁ t₂) t₃
  assocr⋆  : {t₁ t₂ t₃ : U} → TIMES (TIMES t₁ t₂) t₃ ⟷ TIMES t₁ (TIMES t₂ t₃)
  absorbr  : {t : U} → TIMES ZERO t ⟷ ZERO
  absorbl  : {t : U} → TIMES t ZERO ⟷ ZERO
  factorzr : {t : U} → ZERO ⟷ TIMES t ZERO
  factorzl : {t : U} → ZERO ⟷ TIMES ZERO t
  dist     : {t₁ t₂ t₃ : U} → TIMES (PLUS t₁ t₂) t₃ ⟷ PLUS (TIMES t₁ t₃) (TIMES t₂ t₃)
  factor   : {t₁ t₂ t₃ : U} → PLUS (TIMES t₁ t₃) (TIMES t₂ t₃) ⟷ TIMES (PLUS t₁ t₂) t₃
  id⟷      : {t : U} → t ⟷ t
  _⊙_      : {t₁ t₂ t₃ : U} → (t₁ ⟷ t₂) → (t₂ ⟷ t₃) → (t₁ ⟷ t₃)
  _⊕_      : {t₁ t₂ t₃ t₄ : U} → (t₁ ⟷ t₃) → (t₂ ⟷ t₄) → (PLUS t₁ t₂ ⟷ PLUS t₃ t₄)
  _⊗_      : {t₁ t₂ t₃ t₄ : U} → (t₁ ⟷ t₃) → (t₂ ⟷ t₄) → (TIMES t₁ t₂ ⟷ TIMES t₃ t₄)
```

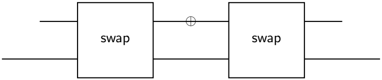
Example Circuit: Simple Negation

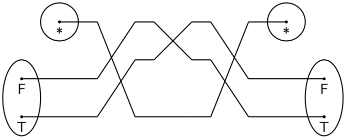


$\mathsf{BOOL} : \mathsf{U}$
$\mathsf{BOOL} = \mathsf{PLUS\ ONE\ ONE}$

$\mathsf{n}_1 : \mathsf{BOOL} \longleftrightarrow \mathsf{BOOL}$
$\mathsf{n}_1 = \mathsf{swap}_+$

Example Circuit: Not So Simple Negation





$\mathsf{n}_2 : \mathsf{BOOL} \longleftrightarrow \mathsf{BOOL}$
$\mathsf{n}_2 = \mathsf{uniti}\star \odot$
$\quad \mathsf{swap}\star \odot$
$\quad (\mathsf{swap}_+ \otimes \mathsf{id}{\longleftrightarrow}) \odot$
$\quad \mathsf{swap}\star \odot$
$\quad \mathsf{unite}\star$

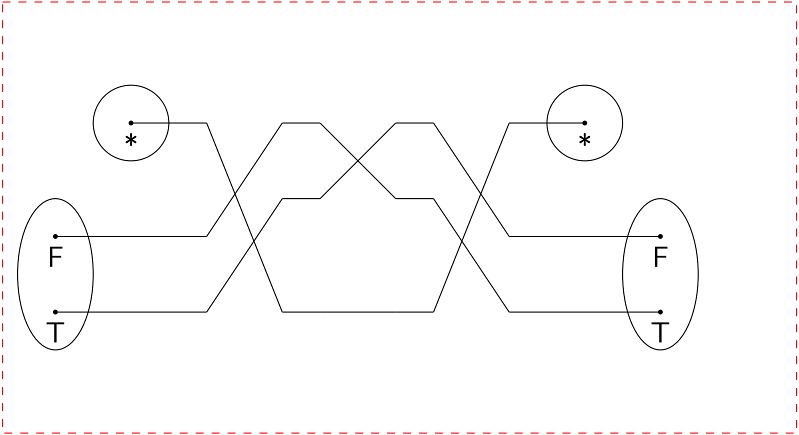
Reasoning about Example Circuits

Algebraic manipulation of one circuit to the other:

```
negEx : n₂ ⇔ n₁
negEx = uniti⋆ ⊙ (swap⋆ ⊙ ((swap₊ ⊗ id⟷) ⊙ (swap⋆ ⊙ unite⋆)))
      ⇔⟨ id⇔ ⊡ assoc⊙l ⟩
    uniti⋆ ⊙ ((swap⋆ ⊙ (swap₊ ⊗ id⟷)) ⊙ (swap⋆ ⊙ unite⋆))
      ⇔⟨ id⇔ ⊡ (swapl⋆⇔ ⊡ id⇔) ⟩
    uniti⋆ ⊙ (((id⟷ ⊗ swap₊) ⊙ swap⋆) ⊙ (swap⋆ ⊙ unite⋆))
      ⇔⟨ id⇔ ⊡ assoc⊙r ⟩
    uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ (swap⋆ ⊙ (swap⋆ ⊙ unite⋆)))
      ⇔⟨ id⇔ ⊡ (id⇔ ⊡ assoc⊙l) ⟩
    uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ ((swap⋆ ⊙ swap⋆) ⊙ unite⋆))
      ⇔⟨ id⇔ ⊡ (id⇔ ⊡ (linv⊙l ⊡ id⇔)) ⟩
    uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ (id⟷ ⊙ unite⋆))
      ⇔⟨ id⇔ ⊡ (id⇔ ⊡ idl⊙l) ⟩
    uniti⋆ ⊙ ((id⟷ ⊗ swap₊) ⊙ unite⋆)
      ⇔⟨ assoc⊙l ⟩
    (uniti⋆ ⊙ (id⟷ ⊗ swap₊)) ⊙ unite⋆
      ⇔⟨ unitil⋆⇔ ⊡ id⇔ ⟩
    (swap₊ ⊙ uniti⋆) ⊙ unite⋆
      ⇔⟨ assoc⊙r ⟩
    swap₊ ⊙ (uniti⋆ ⊙ unite⋆)
      ⇔⟨ id⇔ ⊡ linv⊙l ⟩
    swap₊ ⊙ id⟷
      ⇔⟨ idr⊙l ⟩
    swap₊ □
```

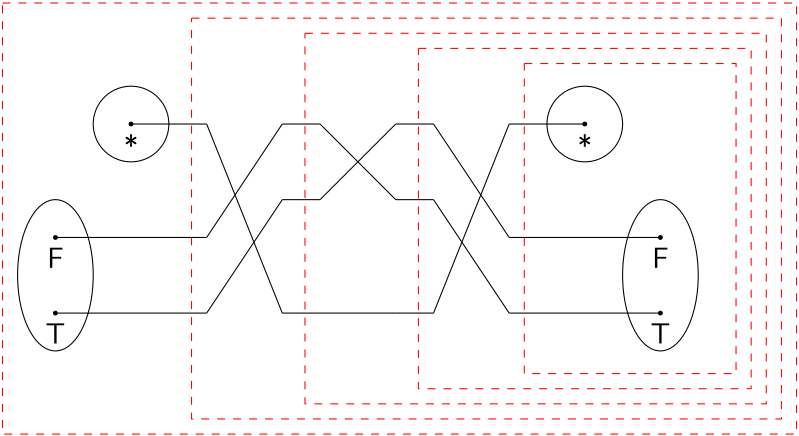
Visually

Original circuit:



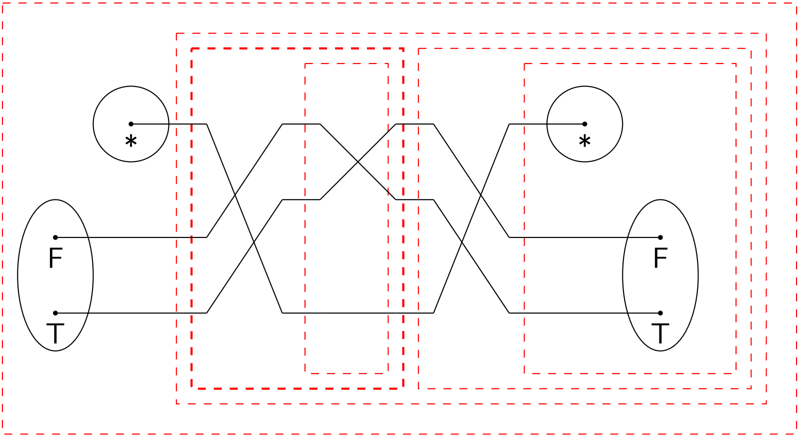
Visually

Making grouping explicit:



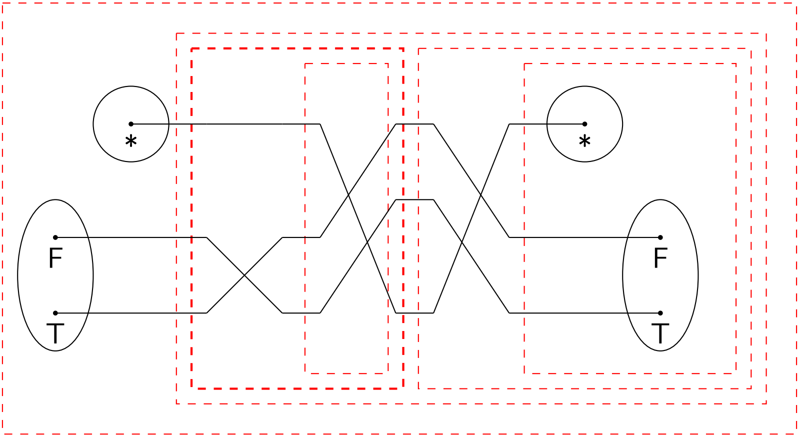
Visually

By associativity:



Visually

By pre-post-swap:



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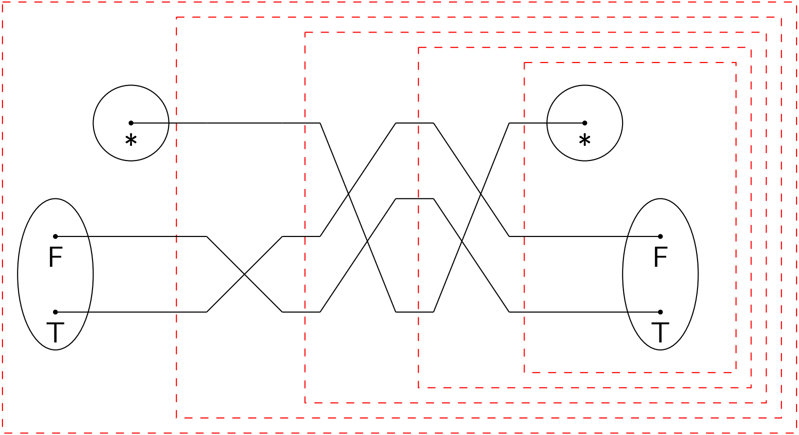
By associativity:



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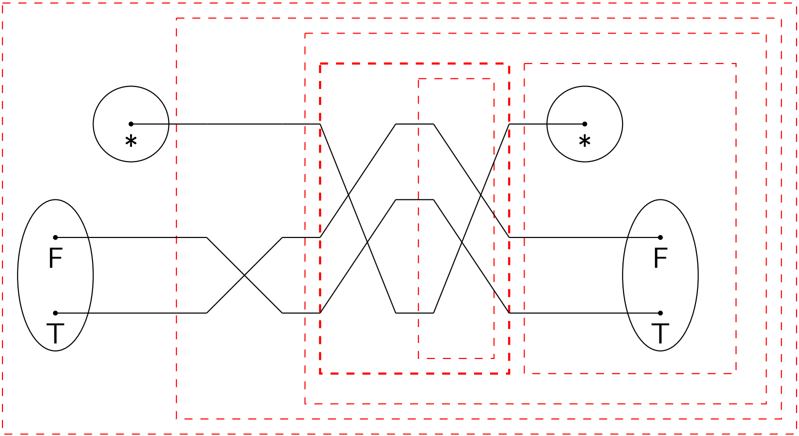
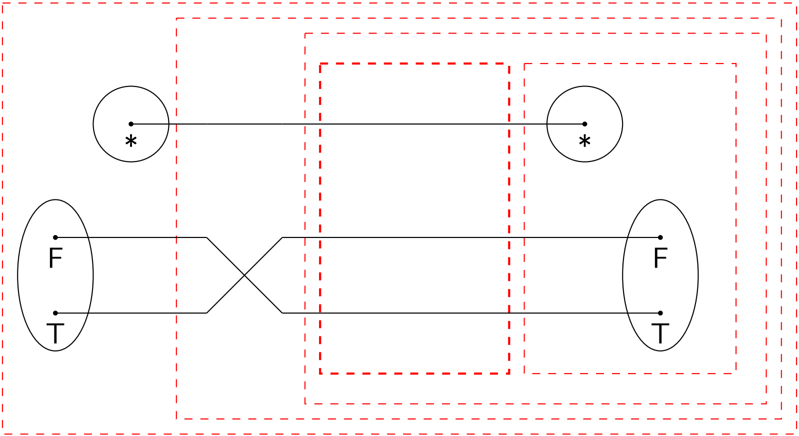
By associativity:



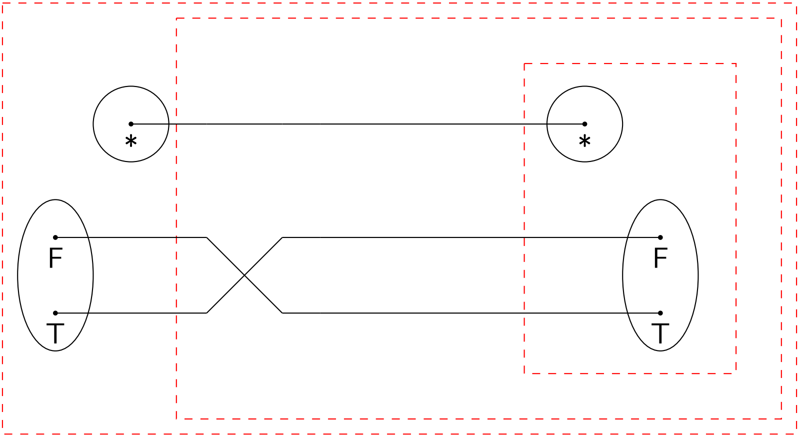
Visually

By swap-swap:



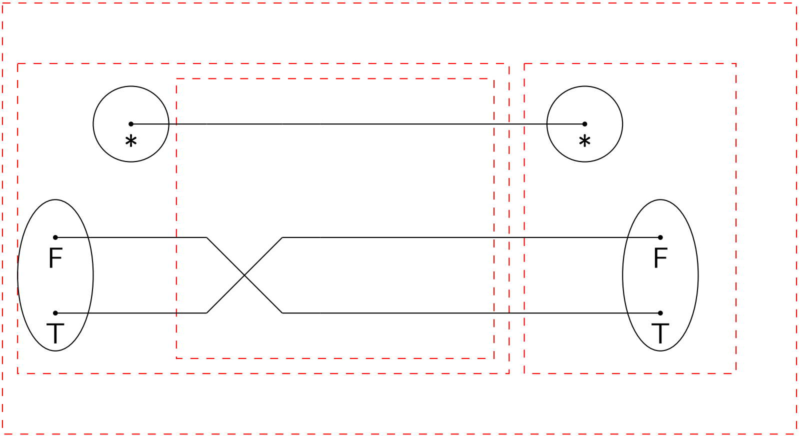
Visually

By id-compose-left:



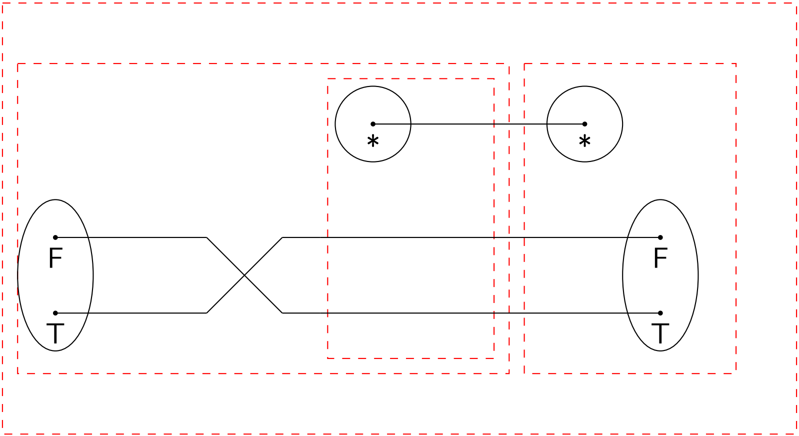
Visually

By associativity:



Visually

By swap-unit:



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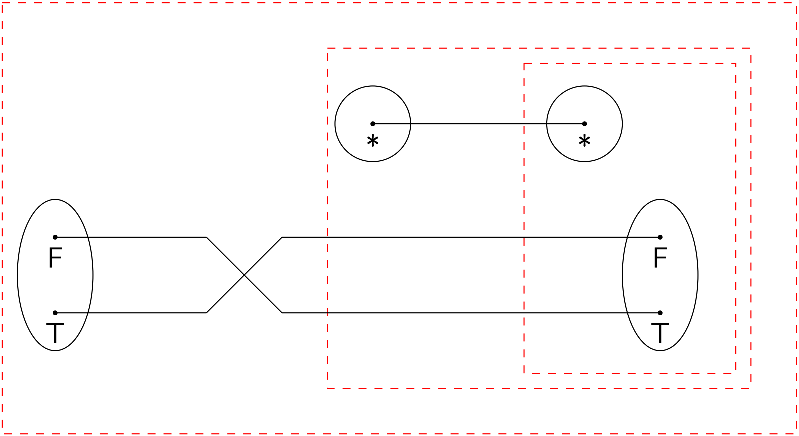
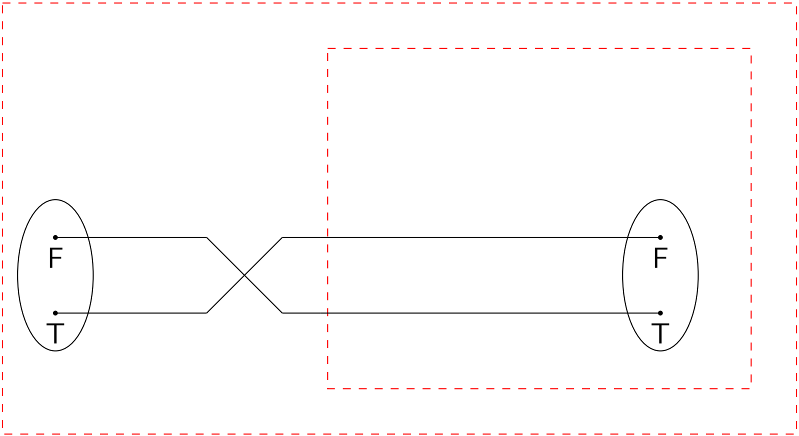
By associativity:



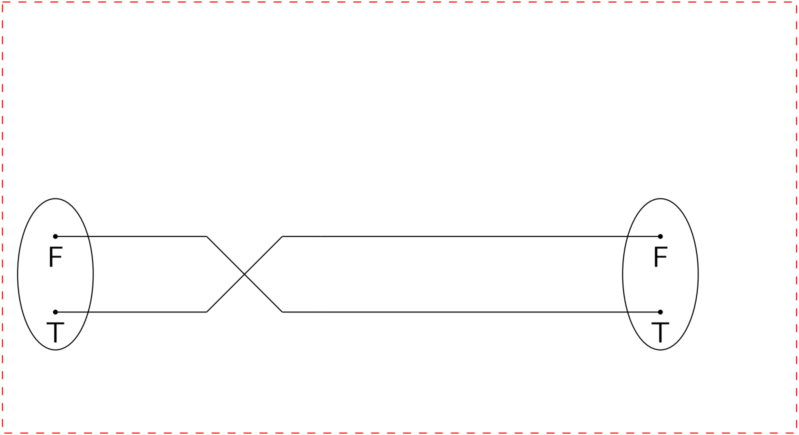
Visually

By unit-unit:



Visually

By id-unit-right:



But is this a programming language?

We get forward and backward evaluators

$\mathsf{eval} : \{t_1\ t_2 : \mathsf{U}\} \to (t_1 \longleftrightarrow t_2) \to [\![\ t_1\ ]\!] \to [\![\ t_2\ ]\!]$

$\mathsf{evalB} : \{t_1\ t_2 : \mathsf{U}\} \to (t_1 \longleftrightarrow t_2) \to [\![\ t_2\ ]\!] \to [\![\ t_1\ ]\!]$

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which really do behave as expected

$\mathsf{c2equiv} : \{t_1\ t_2 : \mathsf{U}\} \rightarrow (c : t_1 \longleftrightarrow t_2) \rightarrow [\![\ t_1\ ]\!] \simeq [\![\ t_2\ ]\!]$

Manipulating circuits

Nice framework, but:

- We don't want ad hoc rewriting rules.
  - Our current set has 76 rules!
- Notions of soundness; completeness; canonicity in some sense.
  - Are all the rules valid? (yes)
  - Are they enough? (next topic)
  - Are there canonical representations of circuits? (open)

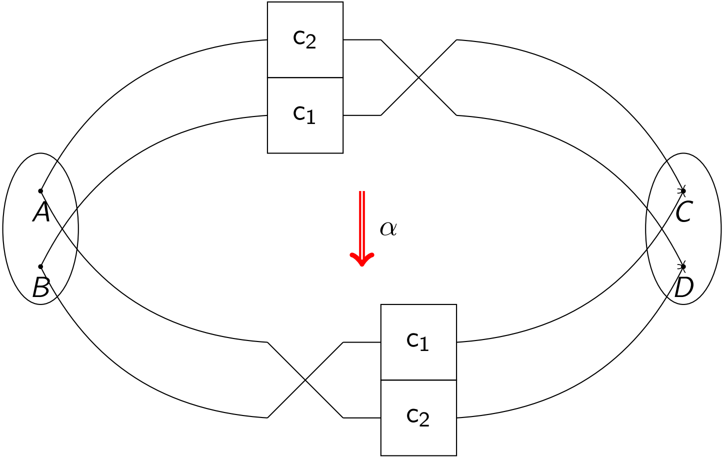
Categorification I

Type equivalences (such as between $A \times B$ and $B \times A$) are Functors. Equivalences between Functors are Natural Isomorphisms. At the value-level, they induce 2-morphisms:

$$
\begin{aligned}
&\mathsf{postulate}\\
&\quad c_1 : \{B\ C : \mathsf{U}\} \to B \longleftrightarrow C\\
&\quad c_2 : \{A\ D : \mathsf{U}\} \to A \longleftrightarrow D
\end{aligned}
$$

$$
\begin{aligned}
&p_1\ p_2 : \{A\ B\ C\ D : \mathsf{U}\} \to \mathsf{PLUS}\ A\ B \longleftrightarrow \mathsf{PLUS}\ C\ D\\
&p_1 = \mathsf{swap}_+ \odot (c_1 \oplus c_2)\\
&p_2 = (c_2 \oplus c_1) \odot \mathsf{swap}_+
\end{aligned}
$$

2-morphism of circuits



Categorification II

The categorification of a semiring is called a Rig Category. As with a semiring, there are two monoidal structures, which interact through some distributivity laws.

Theorem 6.

The following are Symmetric Bimonoidal Groupoids:

- *The class of all types (Set)*
- *The set of all finite types*
- *The set of permutations*
- *The set of equivalences between finite types*

The coherence rules for Symmetric Bimonoidal groupoids give us 58 rules.

Categorification III

Conjecture 1.

The following are Symmetric Rig Groupoids:

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Categorification III

Conjecture 1.

The following are Symmetric Rig Groupoids:

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- *The set of permutations*
- *The set of equivalences between finite types*

and of course the punchline:

Theorem 7 (Laplaza 1972).

There is a sound and complete set of coherence rules for Symmetric Rig Categories.

Conjecture 2.

The set of coherence rules for Symmetric Rig Groupoids are a sound and complete set for circuits.