Polarized Cubical Types

Abstract

. . .

1. Introduction

In a computational world in which the laws of physics are embraced and resources are carefully maintained (e.g., quantum computing [Abramsky and Coecke 2004; Nielsen and Chuang 2000]), programs must be reversible. Although this is apparently a limiting idea, it turns out that conventional computation can be viewed as a special case of such resource-preserving reversible programs. This thesis has been explored for many years from different perspectives [Bennett 2003, 2010, 1973; Fredkin and Toffoli 1982; Landauer 1996, 1961; Toffoli 1980]. We build on the work of James and Sabry [2012] which expresses this thesis in a type theoretic computational framework, expressing computation via type isomorphisms.

Make sure we introduce the abbreviation HoTT in the introduction [The Univalent Foundations Program 2013].

2. Computing with Type Isomorphisms

The main syntactic vehicle for the developments in this paper is a simple language called Π whose only computations are isomorphisms between finite types. The set of types τ includes the empty type 0, the unit type 1, and conventional sum and product types. The values of these types are the conventional ones: $()$ of type 1, $\mathsf{inl}\ v$ and $\mathsf{inr}\ v$ for injections into sum types, and (v_1, v_2) for product types:

$$
\begin{array}{llcl}
(\textit{Types}) & \tau & ::= & 0 \mid 1 \mid \tau_1 + \tau_2 \mid \tau_1 * \tau_2 \\
(\textit{Values}) & v & ::= & () \mid \mathsf{inl}\ v \mid \mathsf{inr}\ v \mid (v_1, v_2) \\
(\textit{Combinator types}) & & & \tau_1 \leftrightarrow \tau_2 \\
(\textit{Combinators}) & c & ::= & [\textit{see Table 1}]
\end{array}
$$

The interesting syntactic category of Π is that of *combinators* which are witnesses for type isomorphisms $\tau_1 \leftrightarrow \tau_2$. They consist of base combinators (on the left side of Table 1) and compositions (on the right side of the same table). Each line of the table on the left introduces a pair of dual constants[1] that witness the type isomorphism in the middle. This set of isomorphisms is known to be complete [Fiore 2004; Fiore et al. 2006] and the language is universal for hardware combinational circuits [James and Sabry 2012]. The *trace* operator provides a bounded iteration facility which adds no expressiveness in the current context but will be needed in Sec. 3.[2]

As simple illustrative examples of "programming" in Π, here are three useful combinators that we define here for future reference:

$$
\begin{array}{ll}
assoc_1 :: & \tau_1 + (\tau_2 + \tau_3) \leftrightarrow (\tau_2 + \tau_1) + \tau_3 \\
assoc_1 = & assocl_+ \fatsemi (swap_+ \oplus id) \\
\\
assoc_2 :: & (\tau_1 + \tau_2) + \tau_3 \leftrightarrow (\tau_2 + \tau_3) + \tau_1 \\
assoc_2 = & (swap_+ \oplus id) \fatsemi assocr_+ \fatsemi (id \oplus swap_+) \fatsemi assocl_+ \\
\\
assoc_3 :: & (\tau_1 + \tau_2) + \tau_3 \leftrightarrow \tau_1 + (\tau_3 + \tau_2) \\
assoc_3 = & assocr_+ \fatsemi (id \oplus swap_+)
\end{array}
$$

From the perspective of category theory, the language Π models what is called a traced *symmetric bimonoidal category* or a *commutative rig category*. These are categories with two binary operations $\oplus$ and $\otimes$ satisfying the axioms of a rig (i.e., a ring without negative elements also known as a semiring) up to coherent isomorphisms. And indeed the types of the Π-combinators are precisely the semiring axioms. A formal way of saying this is that Π is the *categorification* [Baez and Dolan 1998] of the natural numbers. A simple (slightly degenerate) example of such categories is the category of finite sets and permutations in which we interpret every Π-type as a finite set, the values as elements in these finite sets, and the combinators as permutations. Another common example of such categories is the category of finite dimensional vector spaces and linear maps over any field. Note that in this interpretation, the Π-type 0 maps to the 0-dimensional vector space which is *not* empty. Its unique element, the zero vector — which is present in every vector space — acts like a "bottom" everywhere-undefined element and hence the type behaves like the unit of addition and the annihilator of multiplication as desired.

3. The Int Construction

Our immediate technical goal is to explore an extension of Π with a notion of higher-order functions. In the context of monoidal categories, it is known that a notion of higher-order functions emerges from having an additional degree of *symmetry*. In particular, both the **Int** construction of Joyal, Street, and Verity [1996] and the closely related $\mathcal{G}$ construction of linear logic [Abramsky 1996] construct higher-order *linear* functions by considering a new category built on top of a given base traced monoidal category. The objects of the new category are of the form $(\tau_1 - \tau_2)$ where τ_1 and τ_2 are objects in the base category. Intuitively, the component τ_1 is viewed as a conventional type whose elements represent values flowing, as usual, from producers to consumers. The component τ_2 is viewed as a *negative type* whose elements represent demands for values or equivalently values flowing backwards. Under this interpretation,

[1] where $swap_+$ and $swap_*$ are self-dual.

[2] If recursive types are added, the trace operator provides unbounded iteration and the language becomes Turing complete [Bowman et al. 2011; James and Sabry 2012]. We will not be concerned with recursive types in this paper.

$identl_+ :$	$0 + \tau$	$\leftrightarrow$	τ	$: identr_+$
$swap_+ :$	$\tau_1 + \tau_2$	$\leftrightarrow$	$\tau_2 + \tau_1$	$: swap_+$
$assocl_+ :$	$\tau_1 + (\tau_2 + \tau_3)$	$\leftrightarrow$	$(\tau_1 + \tau_2) + \tau_3$	$: assocr_+$
$identl_* :$	$1 * \tau$	$\leftrightarrow$	τ	$: identr_*$
$swap_* :$	$\tau_1 * \tau_2$	$\leftrightarrow$	$\tau_2 * \tau_1$	$: swap_*$
$assocl_* :$	$\tau_1 * (\tau_2 * \tau_3)$	$\leftrightarrow$	$(\tau_1 * \tau_2) * \tau_3$	$: assocr_*$
$dist_0 :$	$0 * \tau$	$\leftrightarrow$	0	$: factor_0$
$dist :$	$(\tau_1 + \tau_2) * \tau_3$	$\leftrightarrow$	$(\tau_1 * \tau_3) + (\tau_2 * \tau_3)$	$: factor$

$$\frac{}{\vdash id : \tau \leftrightarrow \tau} \qquad \frac{\vdash c : \tau_1 \leftrightarrow \tau_2}{\vdash sym\ c : \tau_2 \leftrightarrow \tau_1}$$

$$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_2 \leftrightarrow \tau_3}{\vdash c_1 \fatsemi c_2 : \tau_1 \leftrightarrow \tau_3}$$

$$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \oplus c_2 : \tau_1 + \tau_3 \leftrightarrow \tau_2 + \tau_4}$$

$$\frac{\vdash c_1 : \tau_1 \leftrightarrow \tau_2 \quad \vdash c_2 : \tau_3 \leftrightarrow \tau_4}{\vdash c_1 \otimes c_2 : \tau_1 * \tau_3 \leftrightarrow \tau_2 * \tau_4}$$

$$\frac{\vdash c : \tau + \tau_1 \leftrightarrow \tau + \tau_2}{\vdash trace\ c : \tau_1 \leftrightarrow \tau_2}$$

Table 1. Π-combinators [James and Sabry 2012]

and as we explain below, a function is nothing but an object that converts a demand for an argument into production of a result.

We begin our formal development by extending Π with a new universe of types $\mathbb{T}$ that consists of composite types $(\tau_1 - \tau_2)$:

$$(\textit{1d types}) \quad \mathbb{T} \quad ::= \quad (\tau_1 - \tau_2)$$

In anticipation of future developments, we will refer to the original types τ as 0-dimensional (0d) types and to the new types $\mathbb{T}$ as 1-dimensional (1d) types. It turns out that, except for one case discussed below, the 1d level is a "lifted" instance of Π with its own notions of empty, unit, sum, and product types, and its corresponding notion of isomorphisms on these 1d types.

Our next step is to define lifted versions of the 0d types:

$$\begin{array}{rcl}
\mathbb{0} & \triangleq & (0 - 0) \\
\mathbb{1} & \triangleq & (1 - 0) \\
(\tau_1 - \tau_2) \boxplus (\tau_3 - \tau_4) & \triangleq & (\tau_1 + \tau_3) - (\tau_2 + \tau_4) \\
(\tau_1 - \tau_2) \boxtimes (\tau_3 - \tau_4) & \triangleq & ((\tau_1 * \tau_3) + (\tau_2 * \tau_4)) - \\
 & & ((\tau_1 * \tau_4) + (\tau_2 * \tau_3))
\end{array}$$

Building on the idea that Π is a categorification of the natural numbers and following a long tradition that relates type isomorphisms and arithmetic identities [Di Cosmo 2005], one is tempted to think that the **Int** construction (as its name suggests) produces a categorification of the integers. Based on this hypothesis, the definitions above can be intuitively understood as arithmetic identities. The same arithmetic intuition explains the lifting of isomorphisms to 1d types:

$$(\tau_1 - \tau_2) \Leftrightarrow (\tau_3 - \tau_4) \quad \triangleq \quad (\tau_1 + \tau_4) \leftrightarrow (\tau_2 + \tau_3)$$

In other words, an isomorphism between 1d types is really an isomorphism between "re-arranged" 0d types where the negative input τ_2 is viewed as an output and the negative output τ_4 is viewed as an input. Using these ideas, it is now a fairly standard exercise to define the lifted versions of most of the combinators in Table 1.[3] There are however a few interesting cases whose appreciation is essential for the remainder of the paper that we discuss below.

[3] See Krishnaswami's [2012] excellent blog post implementing this construction in OCaml.

Easy Lifting. Many of the 0d combinators lift easily to the 1d level. For example:

$$\begin{array}{rcl}
id & :: & \mathbb{T} \Leftrightarrow \mathbb{T} \\
 & :: & (\tau_1 - \tau_2) \Leftrightarrow (\tau_1 - \tau_2) \\
 & \triangleq & (\tau_1 + \tau_2) \leftrightarrow (\tau_2 + \tau_1) \\
id & = & swap_+ \\
 & & \\
identl_+ & :: & \mathbb{0} \boxplus \mathbb{T} \Leftrightarrow \mathbb{T} \\
 & = & assocr_+ \fatsemi (id \oplus swap_+) \fatsemi assocl_+
\end{array}$$

Composition using $trace$***.***

$$\begin{array}{rl}
(\fatsemi) :: & (\mathbb{T}_1 \Leftrightarrow \mathbb{T}_2) \to (\mathbb{T}_2 \Leftrightarrow \mathbb{T}_3) \to (\mathbb{T}_1 \Leftrightarrow \mathbb{T}_3) \\
f \fatsemi g = & trace\ (assoc_1 \fatsemi (f \oplus id) \fatsemi assoc_2 \fatsemi (g \oplus id) \fatsemi assoc_3)
\end{array}$$

New combinators $curry$ ***and*** $uncurry$ ***for higher-order functions.***

$$\begin{array}{rcl}
\boxminus(\tau_1 - \tau_2) & \triangleq & \tau_2 - \tau_1 \\
(\tau_1 - \tau_2) \multimap (\tau_3 - \tau_4) & \triangleq & \boxminus(\tau_1 - \tau_2) \boxplus (\tau_3 - \tau_4) \\
 & \triangleq & (\tau_2 + \tau_3) - (\tau_1 + \tau_4)
\end{array}$$

$$\begin{array}{rcl}
flip & :: & (\mathbb{T}_1 \Leftrightarrow \mathbb{T}_2) \to (\boxminus\mathbb{T}_2 \Leftrightarrow \boxminus\mathbb{T}_1) \\
flip\ f & = & swap_+ \fatsemi f \fatsemi swap_+ \\
 & & \\
curry & :: & ((\mathbb{T}_1 \boxplus \mathbb{T}_2) \Leftrightarrow \mathbb{T}_3) \to (\mathbb{T}_1 \Leftrightarrow (\mathbb{T}_2 \multimap \mathbb{T}_3)) \\
curry\ f & = & assocl_+ \fatsemi f \fatsemi assocr_+ \\
 & & \\
uncurry & :: & (\mathbb{T}_1 \Leftrightarrow (\mathbb{T}_2 \multimap \mathbb{T}_3)) \to ((\mathbb{T}_1 \boxplus \mathbb{T}_2) \Leftrightarrow \mathbb{T}_3) \\
uncurry\ f & = & assocr_+ \fatsemi f \fatsemi assocl_+
\end{array}$$

The "phony" multiplication that is not a functor. The definition for the product of 1d types used above is:

$$\begin{array}{l}
(\tau_1 - \tau_2) \boxtimes (\tau_3 - \tau_4) = \\
((\tau_1 * \tau_3) + (\tau_2 * \tau_4)) - ((\tau_1 * \tau_4) + (\tau_2 * \tau_3))
\end{array}$$

That definition is "obvious" in some sense as it matches the usual understanding of types as modeling arithmetic. Using it, it is possible to lift all the 0d combinators involving products *except* the functor:

$$(\otimes) :: (\mathbb{T}_1 \Leftrightarrow \mathbb{T}_2) \to (\mathbb{T}_3 \Leftrightarrow \mathbb{T}_4) \to ((\mathbb{T}_1 \boxtimes \mathbb{T}_3) \Leftrightarrow (\mathbb{T}_2 \boxtimes \mathbb{T}_4))$$

After a few failed attempts, we suspected that this definition of multiplication is not functorial which would mean that the **Int** construction provides a limited notion of higher-order functions at the expense of losing the multiplicative structure at higher-levels.

This observation is less well-known that it should be. Further investigation reveals that this observation is intimately related to a well-known problem in algebraic topology that was identified thirty years ago as the "phony" multiplication [Thomason 1980] in a special class categories related to ours. This problem was recently solved [Baas et al. 2012] using a technique whose fundamental ingredient is to add more dimensions. We exploit this idea in the remainder of the paper.

4. Cubes

As hinted at in the previous section, one can think of the **Int** construction as generalizing conventional (0-dimensional) types to 1-dimensional types indexed by a positive or negative polarity. Generalizing this idea further, we now view types as n-dimensional cubes polarized by all 2^n combinations of positive and negative polarities.

4.1 Syntax

Our types $\mathbb{T}$ are "cubes" defined as follows:

$$\begin{array}{rcl} \tau & ::= & 0 \mid 1 \mid \tau_1 + \tau_2 \mid \tau_1 * \tau_2 \\ \mathbb{T} & ::= & \tau \mid \boxed{\mathbb{T}_1}\boxed{\mathbb{T}_2} \end{array}$$

The syntax $\boxed{\mathbb{T}_1}\boxed{\mathbb{T}_2}$ represents n-dimensional cubes as binary trees of maximum depth n. The subspace $\mathbb{T}_1$ is the positive subspace along the first dimension and the subspace $\mathbb{T}_2$ (shaded in gray) is the negative subspace along that same dimension. Each of these subspaces is itself a cube of a lower dimension. The 0-dimensional cubes are the conventional first-order types τ. A 1-dimensional cube, $\boxed{\tau_1}\boxed{\tau_2}$, intuitively corresponds to the difference $\tau_1 - \tau_2$ of the two types. The type can be visualized as a "line" with polarized endpoints connecting the two points τ_1 and τ_2. A full 2-dimensional cube, $\boxed{(\boxed{\tau_1}\boxed{\tau_2})}\boxed{(\boxed{\tau_3}\boxed{\tau_4})}$, intuitively corresponds to the iterated difference of the types $(\tau_1 - \tau_2) - (\tau_3 - \tau_4)$ where the successive "colors" from the outermost box encode the sign. The type can be visualized as a "square" with polarized corners connecting the two lines corresponding to $(\tau_1 - \tau_2)$ and $(\tau_3 - \tau_4)$. (See Fig. 1 which is further explained after we discuss multiplication below.)

There are notions of sum $\boxplus$, product $\boxtimes$, and negation $\boxminus$ on n-dimensional types. These operations are inductively defined as follows:

$$\begin{array}{rcl} \tau_1 \boxplus \tau_2 & = & \tau_1 + \tau_2 \\ \tau \boxplus (\boxed{\mathbb{T}_1}\boxed{\mathbb{T}_2}) & = & \boxed{\tau \boxplus \mathbb{T}_1}\boxed{\mathbb{T}_2} \\ (\boxed{\mathbb{T}_1}\boxed{\mathbb{T}_2}) \boxplus \tau & = & \boxed{\mathbb{T}_1 \boxplus S}\boxed{\mathbb{T}_2} \\ (\boxed{\mathbb{T}_1}\boxed{\mathbb{T}_2}) \boxplus (\boxed{\mathbb{T}_3}\boxed{\mathbb{T}_4}) & = & \boxed{\mathbb{T}_1 \boxplus \mathbb{T}_3}\boxed{\mathbb{T}_2 \boxplus \mathbb{T}_4} \\ \\ \tau_1 \boxtimes \tau_2 & = & \tau_1 \times \tau_2 \\ \tau \boxtimes (\boxed{\mathbb{T}_1}\boxed{\mathbb{T}_2}) & = & \boxed{\tau \boxtimes \mathbb{T}_1}\boxed{S \boxtimes \mathbb{T}_2} \\ (\boxed{\mathbb{T}_1}\boxed{\mathbb{T}_2}) \boxtimes \mathbb{T} & = & \boxed{\mathbb{T}_1 \boxtimes \mathbb{T}}\boxed{\mathbb{T}_2 \boxtimes \mathbb{T}} \\ \\ \boxminus \tau & = & \boxed{}\boxed{\tau} \\ \boxminus \boxed{\mathbb{T}_1}\boxed{\mathbb{T}_2} & = & \boxed{\boxminus \mathbb{T}_2}\boxed{\boxminus \mathbb{T}_1} \end{array}$$

The sum of 0-dimensional types is the disjoint union as usual. For cubes of higher dimensions, the subspaces are recursively added. Note that the sum of 1-dimensional types reduces to the sum used in the **Int** construction. The definition of negation is natural: it recursively swaps the positive and negative subspaces. The product of 0-dimensional types is the cartesian product of sets. For cubes of higher-dimensions n and m, the result is of dimension $(n+m)$. The example in Fig. 1 illustrates the idea using the product of 1-dimensional cube (i.e., a line) with a 2-dimensional cube (i.e., a square). The result is a 3-dimensional cube as illustrated.

```
write a proposition that the dimensions
are as follows:

dim(tau) = 0

\bdim{\cubt_1 + \cubt_2} &=& \max(\bdim{\cubt_1},\bdim{
\bdim{\cubt_1 * \cubt_2} &=& \bdim{\cubt_1} + \bdim{\cu
\bdim{- \cubt} &=& \max(1,\bdim{\cubt})

Prove it
The base types have dimension 0. If negative types are
dimensions remain at 0. If negative types are used but
negative types appear anywhere, the dimension is raised
situation with the \textbf{Int} or $\mathcal{G}$ constr
and product types are freely used, the dimension can in
```

4.2 Isomorphisms

The Π isomorphisms are, in spirit, isomorphisms of n-dimentional cubical types. For example, we still expect addition $\boxplus$ to be associative, commutative, etc. The situation is however more involved and the negative types introduce new isomorphisms.

```
need obvious things to get started: id, seq-comp

now go to sec. 2.3.

need multiplication functor:
t1 (dim n1) x t2 (dim n2) => (t1 x t2) (dim n1+n2)
```

5. Related Work and Context

A ton of stuff here.

6. Conclusion

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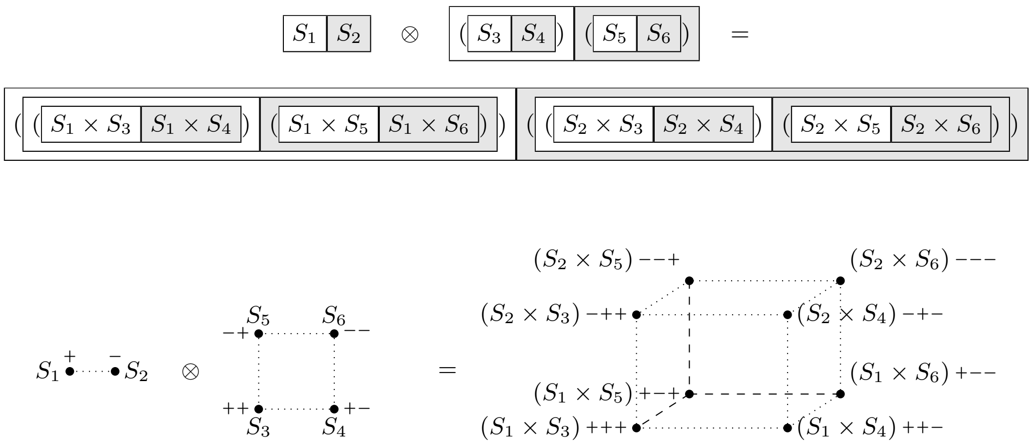

Figure 1. Example of multiplication of two cubical types.

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