Representing, Manipulating and Optimizing Reversible Circuits

Jacques Carette   Amr Sabry

McMaster University

Indiana University

June 11, 2015

Quantum Computing

Quantum physics differs from classical physics in many ways:

- Superpositions
- Entanglement
- Unitary evolution
- Composition uses tensor products
- Non-unitary measurement

Quantum Computing & Programming Languages

- It is possible to adapt all at once classical programming languages to quantum programming languages.
- Some excellent examples discussed in this workshop
- This assumes that classical programming languages (and implicitly classical physics) can be smoothly adapted to the quantum world.
- There are however what appear to be fundamental differences between the classical and quantum world that make them incompatible
- Let us *re-think* classical programming foundations before jumping to the quantum world.

Resource-Aware Classical Computing

- The biggest questionable assumption of classical programming is that it is possible to freely copy and discard information
- A classical programming language which respects no-cloning and no-discarding is the right foundation for an eventual quantum extension
- We want these properties to be inherent in the language; not an afterthought filtered by a type system
- We want to program with isomorphisms or equivalences
- The simplest instance is permutations between finite types which happens to correspond to reversible circuits.

Representing Reversible Circuits

truth table, matrix, reed muller expansion, product of cycles, decision diagram, etc.

[any easy way to reproduce Figure 4 on p.7 of Saeedi and Markov? —JC]

[important remark: these are all *Boolean* circuits! —JC]

Most important part: reversible circuits are equivalent to permutations.

A (Foundational) Syntactic Theory

Ideally, want a notation that

1. is easy to write by programmers
2. is easy to mechanically manipulate
3. can be reasoned about
4. can be optimized.

A (Foundational) Syntactic Theory

Ideally, want a notation that

1. is easy to write by programmers
2. is easy to mechanically manipulate
3. can be reasoned about
4. can be optimized.

Start with a *foundational* syntactic theory on our way there:

1. easy to explain
2. clear operational rules
3. fully justified by the semantics
4. sound and complete reasoning
5. sound and complete methods of optimization

Starting Point

Typed isomorphisms. First, a universe of (finite) types

```
data U : Set where
  ZERO  : U
  ONE   : U
  PLUS  : U → U → U
  TIMES : U → U → U
```

and its interpretation

$$
\begin{aligned}
&[\![\_]\!] : \mathsf{U} \to \mathsf{Set} \\
&[\![\ \mathsf{ZERO}\ ]\!] = \bot \\
&[\![\ \mathsf{ONE}\ ]\!] = \top \\
&[\![\ \mathsf{PLUS}\ t_1\ t_2\ ]\!] = [\![\ t_1\ ]\!] \uplus [\![\ t_2\ ]\!] \\
&[\![\ \mathsf{TIMES}\ t_1\ t_2\ ]\!] = [\![\ t_1\ ]\!] \times [\![\ t_2\ ]\!]
\end{aligned}
$$

Equivalences and semirings

If we denote type equivalence by $\simeq$, then we can prove that

Theorem 1.

The collection of all types (Set) forms a commutative semiring (up to $\simeq$).

Equivalences and semirings

If we denote type equivalence by $\simeq$, then we can prove that

Theorem 1.

The collection of all types (Set) forms a commutative semiring (up to $\simeq$).

We also get

Theorem 2.

If $A \simeq \mathrm{Fin}\,m$, $B \simeq \mathrm{Fin}\,n$ and $A \simeq B$ then $m \equiv n$.

(whose *constructive* proof is quite subtle).

Theorem 3.

If $A \simeq \mathrm{Fin}\,m$ and $B \simeq \mathrm{Fin}\,n$, then the type of all equivalences $A \simeq B$ is equivalent to the type of all permutations $\mathrm{Perm}\,n$.

Equivalences and semirings II

Semiring structures abound. We can define them on:

1. equivalences (disjoint union and cartesian product)
2. permutations (disjoint union and tensor product)

Equivalences and semirings II

Semiring structures abound. We can define them on:

1. equivalences (disjoint union and cartesian product)
2. permutations (disjoint union and tensor product)

The point, of course, is that they are related:

Theorem 4.

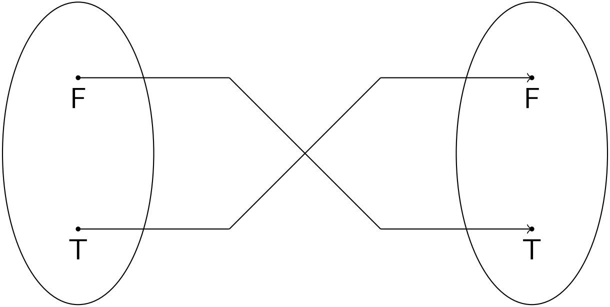
The equivalence of Theorem 3 is an isomorphism between the semirings of equivalences of finite types, and of permutations.

A Calculus of Permutations

First conclusion: it might be useful to *reify* a certain set of equivalences as combinators. We choose the fundamental "proof rules" of semirings:

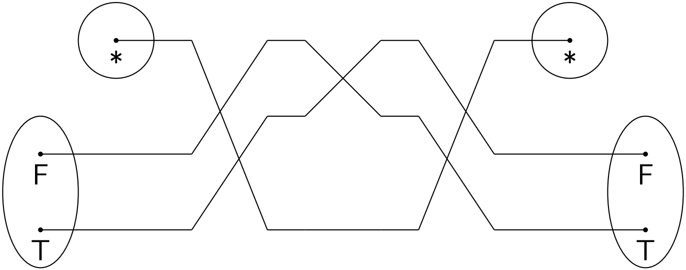
```
data _⟷_ : U → U → Set where
    unite₊   : {t : U} → PLUS ZERO t ⟷ t
    uniti₊   : {t : U} → t ⟷ PLUS ZERO t
    swap₊    : {t₁ t₂ : U} → PLUS t₁ t₂ ⟷ PLUS t₂ t₁
    assocl₊ : {t₁ t₂ t₃ : U} → PLUS t₁ (PLUS t₂ t₃) ⟷ PLUS (PLUS t₁ t₂) t₃
    assocr₊ : {t₁ t₂ t₃ : U} → PLUS (PLUS t₁ t₂) t₃ ⟷ PLUS t₁ (PLUS t₂ t₃)
    unite⋆   : {t : U} → TIMES ONE t ⟷ t
    uniti⋆   : {t : U} → t ⟷ TIMES ONE t
    swap⋆    : {t₁ t₂ : U} → TIMES t₁ t₂ ⟷ TIMES t₂ t₁
    assocl⋆ : {t₁ t₂ t₃ : U} → TIMES t₁ (TIMES t₂ t₃) ⟷ TIMES (TIMES t₁ t₂) t₃
    assocr⋆ : {t₁ t₂ t₃ : U} → TIMES (TIMES t₁ t₂) t₃ ⟷ TIMES t₁ (TIMES t₂ t₃)
    absorbr   : {t : U} → TIMES ZERO t ⟷ ZERO
    absorbl : {t : U} → TIMES t ZERO ⟷ ZERO
    factorzr : {t : U} → ZERO ⟷ TIMES t ZERO
    factorzl : {t : U} → ZERO ⟷ TIMES ZERO t
    dist      : {t₁ t₂ t₃ : U} →
        TIMES (PLUS t₁ t₂) t₃ ⟷ PLUS (TIMES t₁ t₃) (TIMES t₂ t₃)
    factor    : {t₁ t₂ t₃ : U} →
        PLUS (TIMES t₁ t₃) (TIMES t₂ t₃) ⟷ TIMES (PLUS t₁ t₂) t₃
    id⟷      : {t : U} → t ⟷ t
    _⊙_       : {t₁ t₂ t₃ : U} → (t₁ ⟷ t₂) → (t₂ ⟷ t₃) → (t₁ ⟷ t₃)
    _⊕_       : {t₁ t₂ t₃ t₄ : U} →
        (t₁ ⟷ t₃) → (t₂ ⟷ t₄) → (PLUS t₁ t₂ ⟷ PLUS t₃ t₄)
    _⊗_       : {t₁ t₂ t₃ t₄ : U} →
        (t₁ ⟷ t₃) → (t₂ ⟷ t₄) → (TIMES t₁ t₂ ⟷ TIMES t₃ t₄)
```

Example Circuit: Simple Negation



$n_1 : \mathbb{BOOL} \longleftrightarrow \mathbb{BOOL}$
$n_1 = \text{swap}_+$

Example Circuit: Not So Simple Negation



$$\begin{aligned}
n_2 &: \mathsf{BOOL} \longleftrightarrow \mathsf{BOOL} \\
n_2 &= \quad \mathsf{uniti}\star \odot \\
&\qquad \mathsf{swap}\star \odot \\
&\qquad (\mathsf{swap}_+ \otimes \mathsf{id}\!\longleftrightarrow) \odot \\
&\qquad \mathsf{swap}\star \odot \\
&\qquad \mathsf{unite}\star
\end{aligned}$$

Reasoning about Example Circuits

Algebraic manipulation of one circuit to the other:

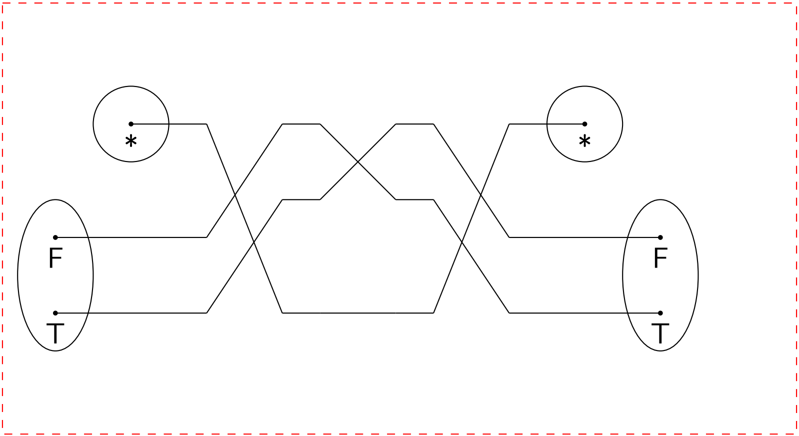
$$
\begin{array}{l}
\mathsf{negEx} : \mathsf{n}_2 \Leftrightarrow \mathsf{n}_1 \\
\mathsf{negEx} = \mathsf{uniti}\star \odot (\mathsf{swap}\star \odot ((\mathsf{swap}_+ \otimes \mathsf{id}\longleftrightarrow) \odot (\mathsf{swap}\star \odot \mathsf{unite}\star))) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot\ \mathsf{assoc}\odot\mathsf{l}\ \rangle \\
\quad \mathsf{uniti}\star \odot ((\mathsf{swap}\star \odot (\mathsf{swap}_+ \otimes \mathsf{id}\longleftrightarrow)) \odot (\mathsf{swap}\star \odot \mathsf{unite}\star)) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot\ (\mathsf{swapl}\star\Leftrightarrow \boxdot\ \mathsf{id}\Leftrightarrow)\ \rangle \\
\quad \mathsf{uniti}\star \odot (((\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+) \odot \mathsf{swap}\star) \odot (\mathsf{swap}\star \odot \mathsf{unite}\star)) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot\ \mathsf{assoc}\odot\mathsf{r}\ \rangle \\
\quad \mathsf{uniti}\star \odot ((\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+) \odot (\mathsf{swap}\star \odot (\mathsf{swap}\star \odot \mathsf{unite}\star))) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot\ (\mathsf{id}\Leftrightarrow \boxdot\ \mathsf{assoc}\odot\mathsf{l})\ \rangle \\
\quad \mathsf{uniti}\star \odot ((\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+) \odot ((\mathsf{swap}\star \odot \mathsf{swap}\star) \odot \mathsf{unite}\star)) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot\ (\mathsf{id}\Leftrightarrow \boxdot\ (\mathsf{linv}\odot\mathsf{l}\ \boxdot\ \mathsf{id}\Leftrightarrow))\ \rangle \\
\quad \mathsf{uniti}\star \odot ((\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+) \odot (\mathsf{id}\longleftrightarrow \odot \mathsf{unite}\star)) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{id}\Leftrightarrow \boxdot\ (\mathsf{id}\Leftrightarrow \boxdot\ \mathsf{idl}\odot\mathsf{l})\ \rangle \\
\quad \mathsf{uniti}\star \odot ((\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+) \odot \mathsf{unite}\star) \\
\quad\quad \Leftrightarrow\langle\ \mathsf{assoc}\odot\mathsf{l}\ \rangle \\
\quad (\mathsf{uniti}\star \odot (\mathsf{id}\longleftrightarrow \otimes \mathsf{swap}_+)) \odot \mathsf{unite}\star \\
\quad\quad \Leftrightarrow\langle\ \mathsf{unitil}\star\Leftrightarrow \boxdot\ \mathsf{id}\Leftrightarrow\ \rangle \\
\quad (\mathsf{swap}_+ \odot \mathsf{uniti}\star) \odot \mathsf{unite}\star
\end{array}
$$

Reasoning about Example Circuits

foo

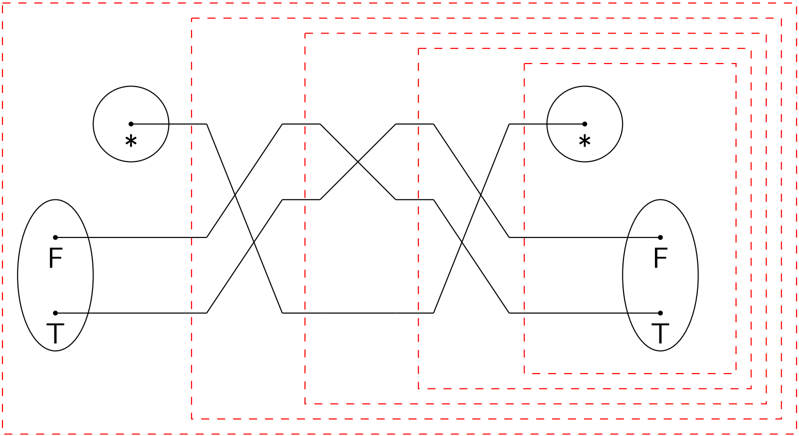
Visually

Original circuit:



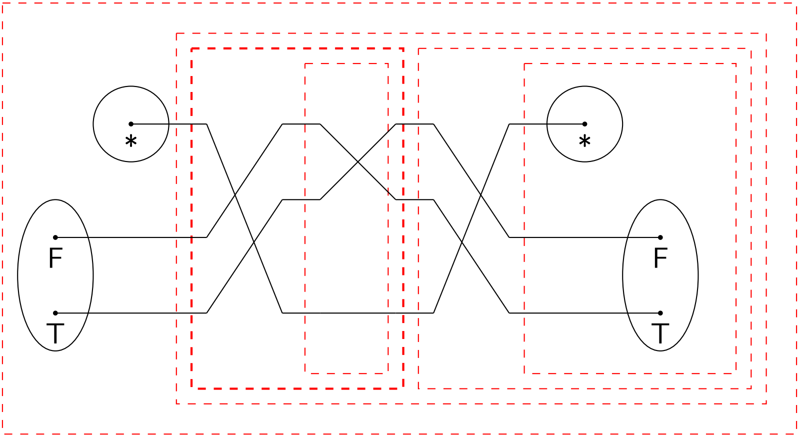
Visually

Making grouping explicit:



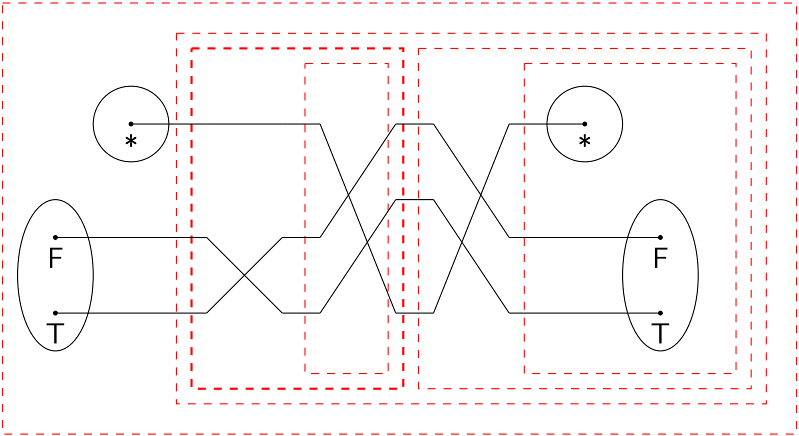
Visually

By associativity:



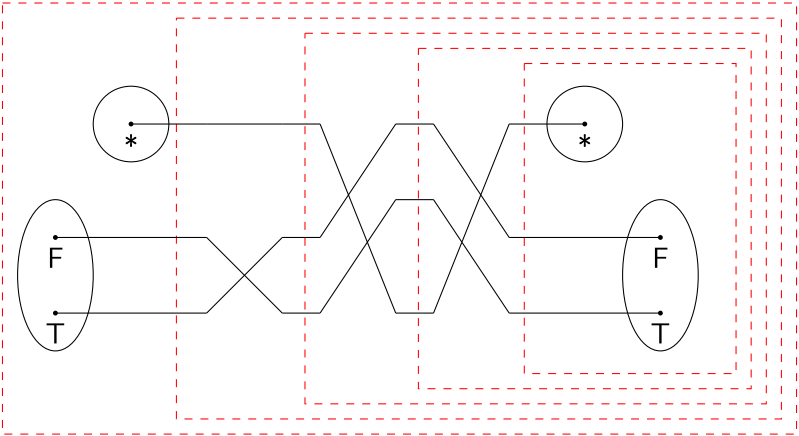
Visually

By pre-post-swap:



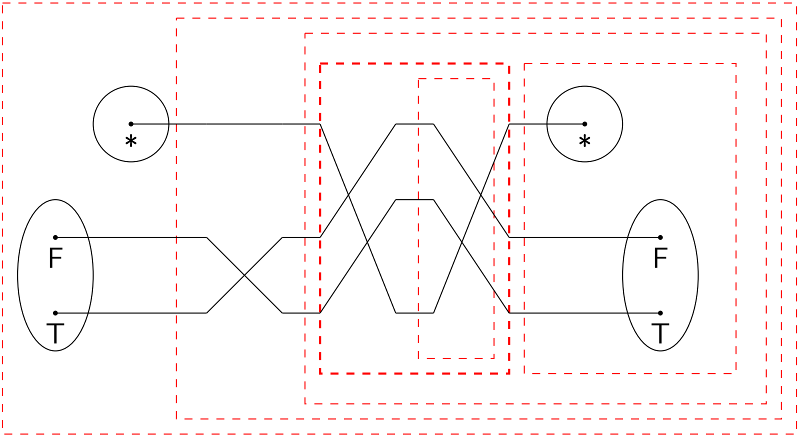
Visually

By associativity:



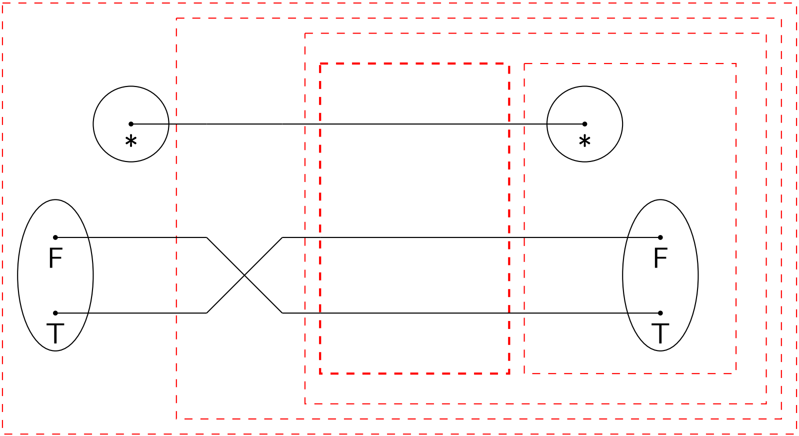
Visually

By associativity:



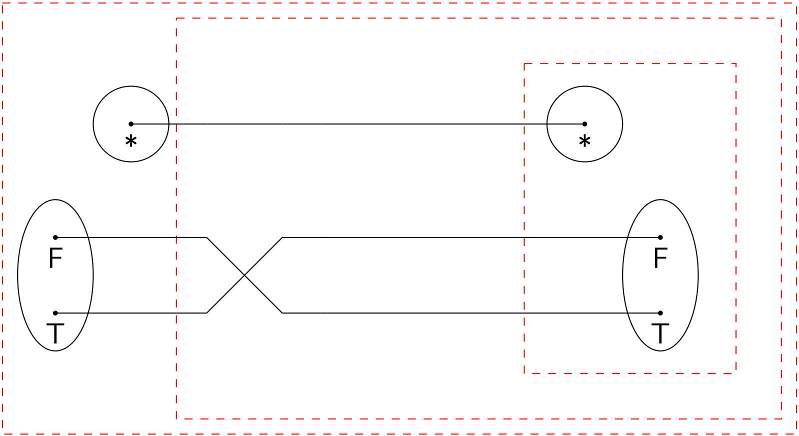
Visually

By swap-swap:



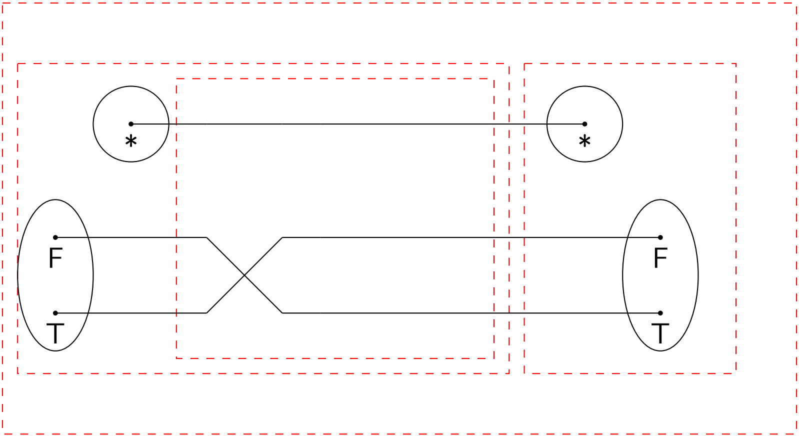
Visually

By id-compose-left:



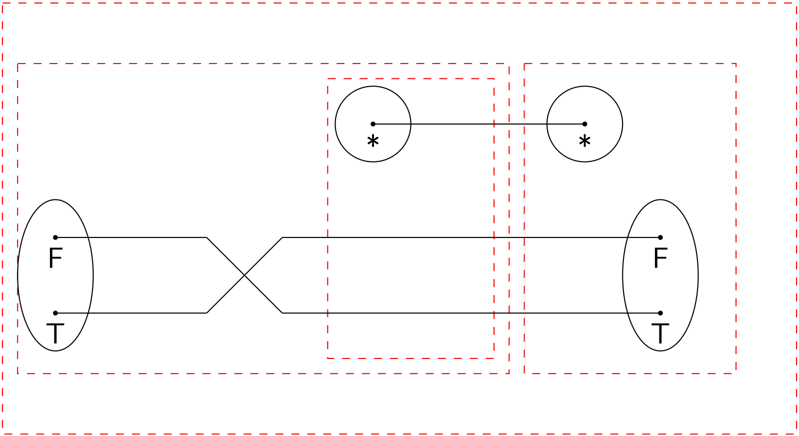
Visually

By associativity:



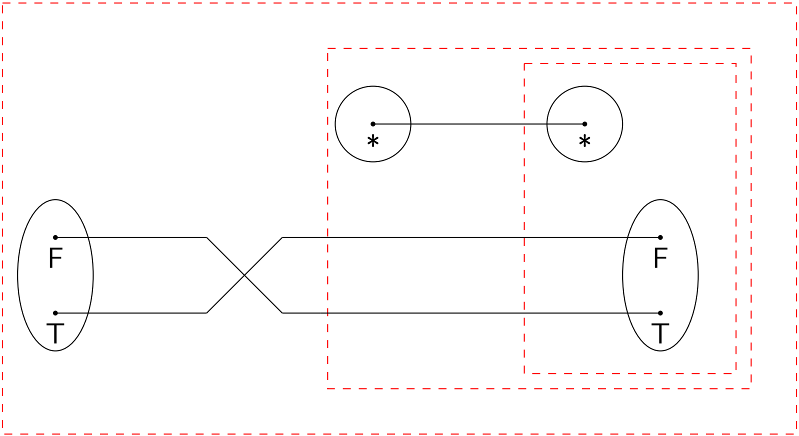
Visually

By swap-unit:



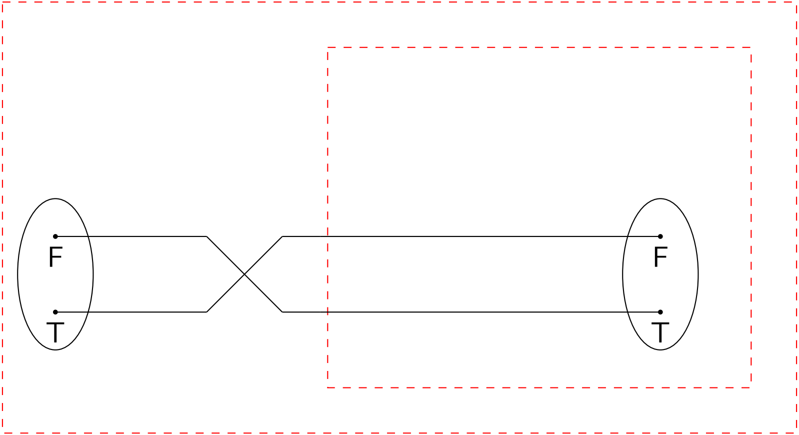
Visually

By associativity:



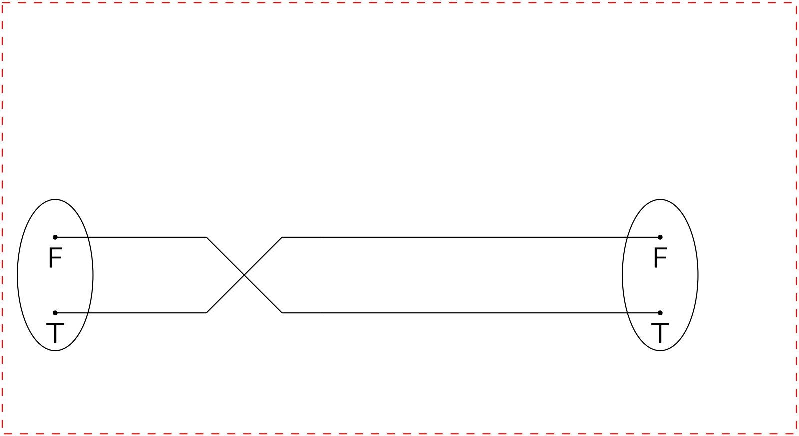
Visually

By unit-unit:



Visually

By id-unit-right:



Questions

- We don't want an ad hoc notation with ad hoc rewriting rules
- Notions of soundness; completeness; canonicity in some sense; what can we say?

1-paths vs. 2-paths

1-paths are between isomorphic types, e.g., $A * B$ and $B * A$. List them all.

1-paths vs. 2-paths

2-paths are between 1-paths, e.g.,
%endcode

1-paths vs. 2-paths