3D Rigid Body Analysis for Physics Engine

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1 Introduction

Consider a body of mass m kg and isotropic moment of inertia I kg m^2, subject to gravitational acceleration g m s^{-2}. The body collides with a horizontal surface and rebounds with a coefficient of restitution e and coefficient of friction μ between the point of contact and the surface.

We shall work in terms of impulse to determine the motion of the body immediately after the collision, assuming that time of contact is instantaneous.

The motion immediately before contact is denoted by state 1 and that immediately after by state 2.

The impulses acting on the point of contact A, linear and angular velocities of/about the centre of mass B, and position of A relative to B are as shown in Figure 1.

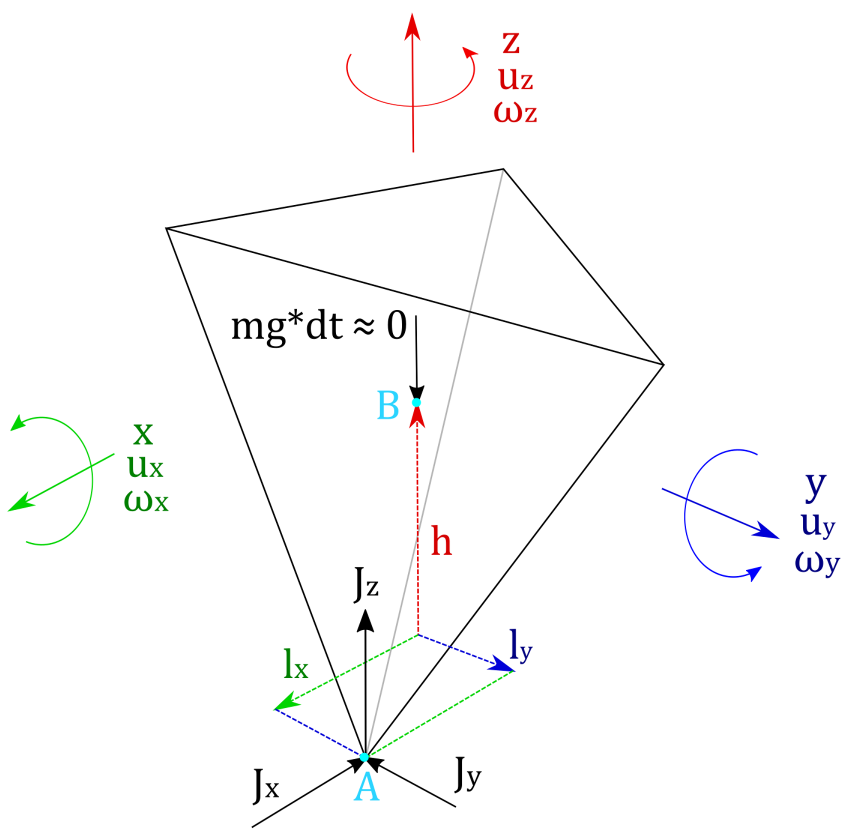

Figure 1

2 Finding Expressions for the Impulses

The linear velocities at the point of contact at state 1 are given by

$$u_{x1T} = u_{x1} - \omega_{y1}h - \omega_{z1}l_y \tag{1}$$
$$u_{y1T} = u_{y1} + \omega_{x1}h + \omega_{z1}l_x \tag{2}$$
$$u_{z1T} = u_{z1} - \omega_{y1}l_x + \omega_{x1}l_y \tag{3}$$

with equivalent expressions for those at state 2 (simply replace '1' with '2').

By Conservation of Linear Momentum,

$$u_{x2} = u_{x1} - \frac{J_x}{m} \tag{4}$$
$$u_{y2} = u_{y1} - \frac{J_y}{m} \tag{5}$$
$$u_{z2} = u_{z1} + \frac{J_z}{m} \tag{6}$$

and by Conservation of Angular Momentum,

$$\omega_{x2} = \omega_{x1} + \frac{J_z l_y - J_y h}{I} \tag{7}$$
$$\omega_{y2} = \omega_{y1} + \frac{-J_z l_x + J_x h}{I} \tag{8}$$
$$\omega_{z2} = \omega_{z1} + \frac{J_x l_y - J_y l_x}{I} \tag{9}$$

We will assume (for now) that the point of contact does not slip, $\therefore u_{x2T} = 0$ and $u_{y2T} = 0$.

By Newton's Law of Restitution, $u_{z2T} = -eu_{z1T}$.

By substituting, rearranging and multiplying by I, we can now produce the following system of linear equations:

$$J_x(-\frac{I}{m} - h^2 - {l_y}^2) + J_y(l_x l_y) + J_z(hl_x) = -u_{x1T}I \tag{10}$$
$$J_x(l_x l_y) + J_y(-\frac{I}{m} - h^2 - {l_x}^2) + J_z(hl_y) = -u_{y1T}I \tag{11}$$
$$J_x(-hl_x) + J_y(-hl_y) + J_z(\frac{I}{m} + l_x^2 + l_y^2) = -(1+e)u_{z1T}I \tag{12}$$

Solving these equations gives us the value of each impulse. We must now compare the total horizontal impulse to the maximum possible impulse caused by friction to check whether our assumption of no slip was correct.

3 Case A: No Slip

If $\sqrt{J_x^2 + J_y^2} \leqslant |\mu J_z|$, then friction can provide the impulse necessary for no slip and our assumption was correct. Use expressions (4)-(9) to determine the motion after the collision.

4 Case B: Slip Occurs

If $\sqrt{J_x^2 + J_y^2} > |\mu J_z|$, then friction cannot provide the impulse necessary for no slip and our assumption was incorrect. We must now recalculate each impulse such that $\sqrt{J_x^2 + J_y^2} = |\mu J_z|$.

Direction of impulse caused by friction:

$$\text{If } u_{y1T} > 0 \text{ and } u_{x1T} < 0,\ \theta = \pi + tan^{-1}(\frac{u_{y1T}}{u_{x1T}}); \tag{13}$$

$$\text{If } u_{y1T} < 0 \text{ and } u_{x1T} < 0,\ \theta = -\pi + tan^{-1}(\frac{u_{y1T}}{u_{x1T}}); \tag{14}$$

$$\text{Otherwise, } \theta = tan^{-1}(\frac{u_{y1T}}{u_{x1T}}) \tag{15}$$

To find each impulse:

$$J_x = \mu J_z cos(\theta) \tag{16}$$

$$J_y = \mu J_z sin(\theta) \tag{17}$$

Substituting these into (12),

$$J_z(l_x^2 + l_y^2 + \frac{I}{m}) = \mu h J_z(l_x cos(\theta) + l_y sin(\theta)) - (1+e)u_{z1T} I \tag{18}$$

$$\therefore J_z = \frac{-(1+e)u_{z1T} I}{l_x^2 + l_y^2 + \frac{I}{m} - \mu h J_z(l_x cos(\theta) + l_y sin(\theta))} \tag{19}$$

We can now use (13)-(17) and (19) to find each impulse, and then we can use expressions (4)-(9) to determine the motion after the collision.