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An alternative definition for digital convexity

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Abstract This paper proposes *full convexity* as an alternative definition of digital convexity, which is valid in arbitrary dimension. It solves many problems related to its usual definitions, like possible non connectedness or non simple connectedness, while encompassing its desirable features. Fully convex sets are digitally convex, but are also connected and simply connected. They have a morphological characterisation, which induces a simple convexity test algorithm. Arithmetic planes are fully convex too. Full convexity implies local full convexity, hence it enables local shape analysis, with an unambiguous definition of convex, concave and planar points. As a kind of relative full convexity, we propose a natural definition of tangent subsets to a digital set. It gives rise to the tangential cover in 2D, and to consistent extensions in arbitrary dimension. Finally we present two applications of tangency: the first one is a simple algorithm for building a polygonal mesh from a set of digital points, with reversibility property, the second one is the definition and computation of shortest paths within digital sets.

Keywords Digital geometry · Digital convexity · Simple connectedness · Arithmetic planes · Tangential cover · Digital surface reconstruction.

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1 Introduction

Classical digital convexity. Classicaly a subset $X \subset \mathbb{Z}^d$ is said to be *digitally convex* whenever

$$X = \mathrm{cvxh}\,(X) \cap \mathbb{Z}^d, \tag{1}$$

where $\mathrm{cvxh}\,(\cdot)$ denotes the convex hull operator (so called H-convexity [21]). In contrast to continuous convexity, this definition does not imply digital connectedness of X starting from dimension $d \geqslant 2$ (see Figure 1abcd). Therefore, especially in 2D, many works add a connectedness constraint or propose a definition that implies it (e.g. [31] or see overviews of [21, 43]). Another definition of convexity relies on progressive intersections with half-planes [48]. Connectedness is preserved in the first steps at the price of a coarse approximation of convexity, but at the limit this definition is equivalent to H-convexity, hence connectedness may also be lost. Note that the connectedness constraint on 2D H-convex sets allows for linear convexity test algorithms [27]. When the connectedness constraint is not added, the best algorithms are only quasi-linear [16].

2D digital convexity plus connectedness. Connectedness of digital convex shapes is not only natural, it is an essential property for shape analysis. Indeed it allows their local analysis, with a possible tracking of the shape boundary. Adding connectedness to 2D H-convexity has opened the way to digital contour analysis with digital straightness [31]. They have led to the classical tangential cover of a contour [24], which can be used to decompose a contour into its convex and concave parts [20, 45]. They induce convergent tangent and length estimators [40] and even curvature estimators [14, 28]. When connected, the fastest algorithms for digital convexity caracterisation use word combinatorics to analyse the word describing the shape contour [5, 6]. Studying their asymptotic properties give rise to automatic noise detection along real image contours [29]. Hierarchical shape analysis can also be achieved through convex-concave decomposition [33]. We finally mention [32] that characterizes digital convexity and straightness by means of difference operators.

3D digital convexity and planarity. It would thus be great to have a definition of digital convexity that extends well to (at least) 3D. As already foreseen in [30], 2D definitions do not extend well to 3D. In the same paper, the authors propose a 3D digital convexity definition that relies on the triangle chordal property plus connectedness. Unfortunately, it induces a quite burdensome convexity check algorithm. But starting from $d \geqslant 3$ a connectedness constraint is not enough to build meaningful digital convex sets. For instance, when cut by a slice, digital convex sets may lose connectedness (see Figure 1e), hence tracking the boundary of the digital shape may not find the convex/concave geometry of the object. This has led many people to study instead 3D digital shapes through digital planarity (e.g. see [3]). The idea is to find digital plane segments (DPSs) that locally fit the digital shape boundary. Although there are numerous methods to check planarity, [7,9,18,23,26,41,50,51], to quote a few, the main problem is to identify which input points to gather before recognizing them as planar. In opposition to the 2D case, maximal DPS are generally not tangent, so the many existing methods rely on heuristics to determine a candidate set of points for DPS recognition: greedy decomposition [4, 13,34,47], repetitive identification of largest DPS [12] or approximately largest [42], expansion from maximal planar disks [10]. More recently, *plane-probing* algorithms have emerged as a new method to analyse the local planarity of digital shapes [36,37,38,39,44]. They perform well on planes, but they still rely on a sound connectedness on the boundary to analyse general digital shapes. Last, it was shown that optimal decomposition of a shape into planar subsets is NP-hard [46], hence DPS decomposition might not be such a great idea for 3D shape analysis.

Contributions. We present here a more consistent definition of digital convexity, which naturally entails connectedness as well as simple connectedness, and that is valid in arbitrary dimension (Section 2 and Section 3). This new definition, called *full convexity*, encompasses digital arithmetic planes or digitizations of thick enough convex shapes, and has a certain stability under intersections. In Section 4, we give a morphological characterisation of full convexity, which shares—but does not originate from—the thickening idea present in [17] for connecting 2D digital convex sets. This induces a practical full convexity check algorithm. Finally full convexity nicely addresses classical digital geometry problems (Section 5 and Section 6): it encompasses digital planarity, allows for unambiguous local caracterisation of convexity and concavity, defines a natural tangential cover in arbitrary dimension, induces a piecewise affine reversible and tight polyhedrization of digital shapes, as well as shortest paths into digital sets.

Extensions. This article is an extended version of [35], with the following differences and addenda. First we introduce stable sets, whose intersection with fully convex sets induces fully convex subsets. In particular it induces that axis-aligned slices of full convex sets are full convex, and that global full convexity implies local full convexity. We have added the proof of the theorem giving the Euler characteristic of the intersected complex of a fully convex set. We have provided two additional applications of full convexity, one related to local shape analysis, which allows a classification of points into convex/concave/planar/other classes, the other related to shortest paths and shortest paths computation, with a proof of algorithm correctness.

2 Full convexity

Cubical complex; intersection complex. Let $\mathbb{Z}^d$ be the d-dimensional digital space, $d > 0$. Let $\mathscr{C}^d$ be the *(cubical) cell complex* induced by the lattice $\mathbb{Z}^d$: its 0-cells are the points of $\mathbb{Z}^d$, its 1-cells are the open unit segments joining two 0-cells at distance 1, its 2-cells are the open unit squares, etc, and its d-cells are the d-dimensional open unit hypercubes with vertices in $\mathbb{Z}^d$. We denote $\mathscr{C}^d_k$ the set of its k-cells. In the following, a *cell* will always designate an element of $\mathscr{C}^d$, and the term *subcomplex* always designates a subset of $\mathscr{C}^d$. A cell σ is a *face* of another cell τ whenever σ is a subset of the topological closure $\bar{\tau}$ of τ, and we write $\sigma \leqslant \tau$. Given any subcomplex K of $\mathscr{C}^d$, the *closure* $\mathrm{Cl}\,(K)$ of K is the complex $\{\tau \in \mathscr{C}^d, \text{ s.t. } \exists \sigma \in K, \tau \leqslant \sigma\}$ and the *star* $\mathrm{Star}\,(K)$ of K is $\{\tau \in \mathscr{C}^d, \text{ s.t. } \exists \sigma \in K, \sigma \leqslant \tau\}$.

In combinatorial topology, a subcomplex K with $\mathrm{Star}\,(K) = K$ is *open*, while being *closed* when $\mathrm{Cl}\,(K) = K$. The *body* of a subcomplex K, i.e. the union of its cells in $\mathbb{R}^d$, is written $\|K\|$. Finally, if Y is any subset of the Euclidean space $\mathbb{R}^d$, we denote by $\bar{\mathscr{C}}^d_k[Y]$ the set of k-cells whose topological closure has a non-empty intersection with Y, i.e. $\bar{\mathscr{C}}^d_k[Y] := \{c \in \mathscr{C}^d_k, \bar{c} \cap Y \neq \varnothing\}$. The complex made of all k-cells having a non-empty intersection with Y, $0 \leqslant k \leqslant d$ is called the *intersection (cubical) complex of* Y and denoted by $\bar{\mathscr{C}}^d[Y]$.

Lemma 1 *The intersection complex of a set Y is open and its body covers Y.*

Proof If Y is the empty set, $\bar{\mathscr{C}}^d[Y]$ is empty and is open. If Y is not empty, let σ be any cell of $\bar{\mathscr{C}}^d[Y]$. Let τ be any cell of $\mathscr{C}^d$ with $\sigma \leqslant \tau$. Thus $\sigma \subset \bar{\tau} \Rightarrow \bar{\sigma} \subset \bar{\tau}$ (since topological closure is increasing and idempotent). It follows that $\sigma \in \bar{\mathscr{C}}^d[Y] \Leftrightarrow \bar{\sigma} \cap Y \neq \varnothing \Rightarrow \bar{\tau} \cap Y \neq \varnothing \Leftrightarrow \tau \in \bar{\mathscr{C}}^d[Y]$. We have just proved that $\mathrm{Star}\,(\sigma) \subset \bar{\mathscr{C}}^d[Y]$, hence $\mathrm{Star}\,\left(\bar{\mathscr{C}}^d[Y]\right) \subset \bar{\mathscr{C}}^d[Y]$. The converse inclusion being obvious, $\bar{\mathscr{C}}^d[Y]$ is open. The fact that $Y \subset \left\|\bar{\mathscr{C}}^d[Y]\right\|$ is straightforward. $\square$

The following remark is quite straightforward and tells that the 0-cells of the intersection complex of some set are exactly its digital points.

Remark 1 *For any $Y \subset \mathbb{R}^d$, we have*

$$\bar{\mathscr{C}}^d_0[Y] = \{c \in \mathscr{C}^d_0, c \cap Y \neq \varnothing\} = \{c \in \mathbb{Z}^d, c \cap Y \neq \varnothing\} = Y \cap \mathbb{Z}^d. \tag{2}$$

Full convexity. We define now our new notion of digital convexity:

Definition 1 (Full convexity) *An arbitrary subset $X \subset \mathbb{Z}^d$ is* digitally k-convex *for $0 \leqslant k \leqslant d$ whenever*

$$\bar{\mathscr{C}}^d_k[X] = \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(X)]. \tag{3}$$

Set X is fully (digitally) convex *if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.*

Equivalently, the intersection complex of a fully convex set Z covers the convex hull of Z. We can already make the following observation:

Lemma 2 *Common digital convexity is the digital 0-convexity.*

Proof From (1), $X \subset \mathbb{Z}^d$ is digitally convex iff $X = \mathrm{cvxh}\,(X) \cap \mathbb{Z}^d$, otherwise said $X \cap \mathbb{Z}^d = \mathrm{cvxh}\,(X) \cap \mathbb{Z}^d$. Applying (2) (Remark 1) on both sides for sets X and $\mathrm{cvxh}\,(X)$ respectively, it is equivalent to $\bar{\mathscr{C}}^d_0[X] = \bar{\mathscr{C}}^d_0[\mathrm{cvxh}\,(X)]$, which is exactly (3) for $k = 0$. $\square$

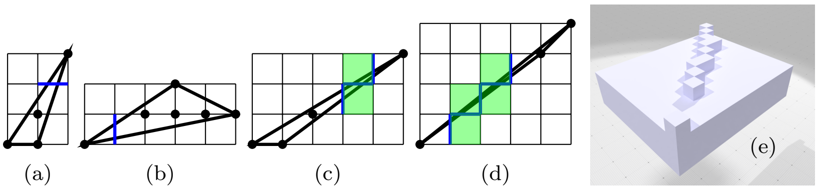


Fig. 1 (abcd) Digital triangles that are not fully convex: digital points are depicted as black disks, missing 1-cells for digital 1-convexity as blue lines, missing 2-cells for digital 2-convexity as green squares. (e) Usual digital convexity (1) plus 3D connectivity does not imply connectedness on the upper slice; it is also not fully convex.

Figure 1 shows several digitally 0-convex sets, but which are not fully convex. Clearly full convexity forbids too thin convex sets, which are typically the ones that are not connected or not simply connected in the digital sense.

Elementary properties. Denoting by $\#(X)$ the cardinal of a finite set X, the straightforward lemma below shows that is suffices to count intersected cells to check for full convexity.

Lemma 3 *A finite subset* $X \subset \mathbb{Z}^d$ *is* digitally k-convex *for* $0 \leqslant k \leqslant d$ *iff* $\#\left(\bar{\mathscr{C}}^d_k[X]\right) \geqslant \#\left(\bar{\mathscr{C}}^d_k[\mathrm{cvxh}(X)]\right)$.

For example, the digital tetrahedra $T(l) = \{(0,0,0),(1,0,0),(0,1,0),(1,1,l)\}$, for positive integer l, is digitally 0-convex. However $\mathrm{cvxh}(T(l))$ intersects as many 2-cells and 3-cells as wanted above the unit square with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(1,0,0)$, just by increasing l. Meanwhile, $T(l)$ only intersects the same finite number of 2-cells and 3-cells. Hence, for $l \geqslant 2$, $T(l)$ is not fully convex.

It is not necessary to check digital d-convexity to verify if a digital set is fully convex, and this property is useful to speed up algorithms to check for full convexity. You can observe the contraposition of this lemma in Figure 1, left, where non digitally 2-convex sets in 2D are not digitally 1-convex too.

Lemma 4 *If $Z \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k < d$, it is also digitally d-convex, hence fully convex.*

Proof Let Z be such set. The conclusion of the lemma is true if Z is empty. Otherwise let $\sigma \in \bar{\mathscr{C}}^d_d[\mathrm{cvxh}(Z)]$. Let $B = \partial\bar{\sigma}$ be the topological boundary of σ. By hypothesis, we have $\bar{\sigma} \cap \mathrm{cvxh}(Z) \neq \varnothing$, hence $(B \cup \sigma) \cap \mathrm{cvxh}(Z) \neq \varnothing$.

The surface B separates $\mathbb{R}^d$ into two components, one finite equal to σ, the other infinite. Assume $B \cap \mathrm{cvxh}(Z) = \varnothing$. Since $\mathrm{cvxh}(Z)$ is arc-connected, then $\mathrm{cvxh}(Z)$ lies entirely in one component. The relation $(B \cup \sigma) \cap \mathrm{cvxh}(Z) \neq \varnothing$ then implies $\mathrm{cvxh}(Z) \subset \sigma$. This is impossible since $\mathrm{cvxh}(Z) \cap \mathbb{Z}^d = Z$ while $\sigma \cap \mathbb{Z}^d = \varnothing$.

It follows that $B \cap \mathrm{cvxh}(Z) \neq \varnothing$. But B is a union of k-cells $(b_i)_{i=0\ldots m}$ of $\mathscr{C}^d$, with $0 \leqslant k < d$. There exists at least one k-cell b_j with $b_j \cap \mathrm{cvxh}(Z) \neq$

$\emptyset$. Thus $b_j \in \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(Z)]$. But Z is digitally k-convex for $0 \leqslant k < d$, so $b_j \in \bar{\mathscr{C}}^d_k[Z]$. To conclude, $\sigma \in \mathrm{Star}\,(b_j)$ and $\bar{\mathscr{C}}^d[Z]$ is open, so σ belongs also to $\bar{\mathscr{C}}^d[Z]$. We have just shown that every d-cell of $\bar{\mathscr{C}}^d[\mathrm{cvxh}\,(Z)]$ are in $\bar{\mathscr{C}}^d[Z]$, so Z is digitally d-convex and hence fully convex. □

Other implications of digital k-convexities over digital l-convexities are unlikely. For instance in 3D, some digital sets are digitally 0-convex, 1-convex, 3-convex but are not 2-convex, like $\{(0,0,0), (1,1,2), (1,1,3), (1,2,3), (2,1,3)\}$.

Digital connectedness. We will need the definition of (digital) connectedness in the next section. Two elements x, y of $\mathbb{Z}^d$ are *k-adjacent* if $\|x-y\|_\infty \leqslant 1$ and $\|x-y\|_1 \leqslant k$. The transitive closure of this relation defines the *k-connectedness* relation. Historically, d-connectedness was called 8-connectivity in 2D, and 26-connectivity in 3D, and 1-connectedness was called 4-connectivity in 2D, and 6-connectivity in 3D.

Stability by intersection. Unfortunately, we do not have in general the stability of full convexity by intersection. As elementary example, pick $Z_1 := \{(0,0),(1,0),(2,1)\}$ and $Z_2 := \{(0,0),(1,1),(2,1)\}$, which are both fully convex. Their intersection $Z_1 \cap Z_2$ is reduced to the two points $\{(0,0),(2,1)\}$, which is not a d-connected set. By Theorem 2 below, this set is not fully convex. However, we do have a stability by intersection with the following subsets of $\mathbb{R}^d$:

Definition 2 (Stable set) *A subset Y of $\mathbb{R}^d$ is called* stable *whenever Y is convex and, for any cell c of $\mathscr{C}^d$, $Y \cap c \neq \emptyset \Rightarrow \bar{c} \subset Y$.*

Lemma 5 *If $X \subset \mathbb{Z}^d$ is fully convex and $Y \subset \mathbb{R}^d$ is stable, then $X \cap Y$ is fully convex.*

Proof Following Definition 1 and (3), we have to show, for all $k, 0 \leqslant k \leqslant d$, $\bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(X \cap Y)] = \bar{\mathscr{C}}^d_k[X \cap Y]$. Picking such a k, it is enough to show the inclusion in the above relation, since the reciprocal inclusion is obvious (the intersection complex is increasing).

Let $c \in \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(X \cap Y)]$. Since convex hull is increasing, we have $c \in \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(X) \cap \mathrm{cvxh}\,(Y)]$. Since Y is stable hence convex, it holds immediately that $c \in \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(X) \cap Y]$. So $\mathrm{cvxh}\,(X) \cap Y \cap \bar{c} \neq \emptyset$. There is thus at least one cell e included in $\bar{c}$ (which may be c itself) such that $\mathrm{cvxh}\,(X) \cap Y \cap e \neq \emptyset$.

Full convexity of X implies $\bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(X)] = \bar{\mathscr{C}}^d_k[X]$. Since $e \in \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(X)]$, so there exists a point $z \in X$, such that z belongs to $\bar{e}$. Besides, set Y being stable, $Y \cap e \neq \emptyset$ implies $\bar{e} \subset Y$. In particular, we have $z \in Y$.

Since $z \in X \cap Y$, any cell of $\mathscr{C}^d$ that has z in its boundary belongs to $\bar{\mathscr{C}}^d[X \cap Y]$. This is the case of e but also of c, since $z \in \bar{e} \subset \bar{c}$. We have just shown $c \in \bar{\mathscr{C}}^d_k[X \cap Y]$. □

Half-spaces with axis aligned normals are stable in this sense:

Lemma 6 *Any half-space of integer intercept and axis normal vector is stable.*

Proof Without losing generality, choose axis normal vector $\mathbf{e}_i$. Let $H := \{x \in \mathbb{R}^d, x \cdot \mathbf{e}_i \leqslant a\}$. Being a half-space, the set H is convex.

Now if $c \in \mathscr{C}^d$ and $H \cap c \neq \varnothing$. It means that $\exists x \in c, x \cdot \mathbf{e}_i \leqslant a$. Letting x^i be the i-th coordinate of x. We have $x^i \leqslant a$.

Let $y \in \bar{c}$. If c is closed along i-th coordinate, then $y^i = x^i$ and thus $y \in H$ also. If c is open along i-th coordinate, then x^i is a non-integer number. It follows obviously that $\lceil x^i \rceil \leqslant a$. But $\lfloor x^i \rfloor \leqslant y^i \leqslant \lceil x^i \rceil$, which implies also $y^i \leqslant a$. Hence $y \in H$, i.e. $\bar{c} \subset H$. □

It is obvious that the intersection of two stable sets is also stable. As a consequence, we get several corollaries. Since a slice is the intersection of two half-spaces, we get:

Corollary 1 *Any axis-aligned slice of a fully convex set is fully convex.*

In fact it is fully convex both in the d-dimensional space, and in the $d-1$-dimensional space spanning the slice. Looking at Figure 1e, this corollary shows that the displayed 3D digital set cannot be fully convex, since its upper slice is not fully convex (we will show in Theorem 2 that full convexity implies connectedness). Full convexity thus efficiently discards such digitally convex sets.

Besides, for any digital point z, we can define naturally its *d-dimensional cubical k-neighborhood*, $k \in \mathbb{Z}$, $k \geqslant 1$, as the set

$$N_k(z) := \{x \in \mathbb{R}^d, s.t.\ \forall i \in \mathbb{Z}, 1 \leqslant i \leqslant d, z^i - k \leqslant y^i \leqslant z^i + k\}.$$

Now any such neighborhood is obviously the intersection of 2D axis-aligned half-spaces with integer intercept, hence they form stable sets.

Corollary 2 *If X is fully convex, then for any $z \in \mathbb{Z}^d$, $X \cap N_k(z)$ is fully convex.*

This shows that the global property of full convexity implies local full convexity everywhere. We will use this result in Section 5 in order to characterize locally the geometry of digital objects.

3 Topological properties of fully convex digital sets

We give below the main topological properties of fully convex digital sets.

Theorem 1 *If the digital set $Z \subset \mathbb{Z}^d$ is fully convex, then the body of its intersection cubical complex is connected.*

Proof Let x, x' be two points of $\|\bar{\mathscr{C}}^d[Z]\|$. Since $\mathscr{C}^d$ is a partition of $\mathbb{R}^d$, there are two cells c, c' of $\bar{\mathscr{C}}^d[Z]$ such that $x \in c$, $x' \in c'$. Since Z is fully convex, then $\bar{\mathscr{C}}^d[Z] = \bar{\mathscr{C}}^d[\mathrm{cvxh}\,(Z)]$. Hence there exist $y \in \bar{c} \cap \mathrm{cvxh}\,(Z)$ and $y' \in \bar{c} \cap \mathrm{cvxh}\,(Z)$. By convexity of cells, the segment $[x, y[$ lies entirely in c hence in $\|\bar{\mathscr{C}}^d[Z]\|$. Similarly, the segment $]y', x']$ lies entirely in c' hence in $\|\bar{\mathscr{C}}^d[Z]\|$.

Now by definition of convexity, the segment $[y, y']$ lies in $\mathrm{cvxh}(Z)$. But $\mathrm{cvxh}(Z) \subset \|\bar{\mathscr{C}}^d[\mathrm{cvxh}(Z)]\| = \|\bar{\mathscr{C}}^d[Z]\|$ by cell convexity. We have just built an arc from x to x' which lies entirely in $\|\bar{\mathscr{C}}^d[Z]\|$. We conclude since arc-connectedness implies connectedness. □

Theorem 2 *If the digital set $Z \subset \mathbb{Z}^d$ is fully convex, then Z is d-connected.*

Proof We show first that 0-cells of $\bar{\mathscr{C}}^d[Z]$ are face-connected, i.e. for any points $z, z' \in Z = \bar{\mathscr{C}}_0^d[Z]$, there is a path of cells $(c_i)_{i=0..m}$ of $\bar{\mathscr{C}}^d[Z]$, such that $c_0 = \sigma$, $c_m = \tau$, and for all $i \in \mathbb{Z}, 0 \leqslant i < m$, either $c_i \preccurlyeq c_{i+1}$ or $c_{i+1} \preccurlyeq c_i$.

The straight segment $[z, z']$ is included in $\mathrm{cvxh}(Z)$, hence any one of its point belongs to a cell of $\bar{\mathscr{C}}^d[\mathrm{cvxh}(Z)]$ so a cell of $\bar{\mathscr{C}}^d[Z]$ by full convexity.

Let $p(t) = (1-t)z + tz'$ for $0 \leqslant t \leqslant 1$ be a parameterization of segment $[z, z']$. The above remark implies that, for any $t \in [0,1]$, the point $p(t)$ belongs to a cell $c(t)$ of $\bar{\mathscr{C}}^d[Z]$. The sequence of intersected cells from $t = 0$ to $t = 1$ is obviously finite, and we denote it by $(c_0, c_1, \ldots, c_m)$ with $c_0 = p(0) = z$ and $c_m = p(1) = z'$. Since it corresponds to an infinitesimal change of t, two consecutive cells of this sequence are necessary in the closure of one of them, hence $c_i \preccurlyeq c_{i+1}$ or $c_{i+1} \preccurlyeq c_i$.

We use Lemma 7 given below. We associate to each cell c_i one of its Z-corner, denoted z_i. We obtain a sequence of digital points $z = z_0, z_1, \ldots, z_m = z'$. Now any two incident faces (like c_i and c_{i+1}) belong to the closure of a d-cell σ. It follows that both corner z_i and z_{i+1} are vertices of $\bar{\sigma}$, a unit hypercube. Obviously $\|z_i - z_{i+1}\|_\infty \leqslant 1$ and these two points are d-adjacent. We have just built a sequence of d-adjacent points in Z, which concludes. □

Lemma 7 *Let $Z \subset \mathbb{Z}^d$. If c is a cell of $\bar{\mathscr{C}}^d[Z]$, there there exists $z \in \bar{\mathscr{C}}_0^d[Z] = Z$ such that $z \preccurlyeq c$. We call such digital point a Z-*corner *for c. If c is a 0-cell, its only Z-corner is itself.*

Proof By definition of $\bar{\mathscr{C}}^d[Z]$ we have $\bar{c} \cap Z \neq \varnothing$. It follows that $\exists z \in Z$ such that $z \in \bar{c}$. So $z \preccurlyeq c$ and also $z \in Z = \bar{\mathscr{C}}_0^d[Z]$. □

We can show an even stronger result on fully convex sets: they present no topological holes. Indeed, we have:

Theorem 3 *If the digital set $Z \subset \mathbb{Z}^d$ is fully convex, then the body of its intersection cubical complex is* simply *connected.*

Proof Let $\mathscr{A} := \{x(t), t \in [0,1]\}$ be a closed curve in $\|\bar{\mathscr{C}}^d[Z]\|$, i.e. $x(0) = x(1)$ and $x(t) \in \|\bar{\mathscr{C}}^d[Z]\|$. We must show that there is a homotopy from $\mathscr{A}$ to a point $a \in \|\bar{\mathscr{C}}^d[Z]\|$.

The curve $x(t)$ visits cells of $\bar{\mathscr{C}}^d[Z]$. Let $c(t)$ be these cells. By finiteness of $\mathscr{A}$, $c(t)$ defines a finite sequence of cells $c_0, c_1, \ldots, c_m$ from $t = 0$ to $t = 1$, with $c_m = c_0$. We can also associate a sequence of parameters $t_0, t_1, \ldots, t_m$, such that $x(t_i) \in c_i = c(t_i)$. As in the proof of Theorem 2, two consecutive cells of this sequence are necessary in the closure of one of them. Let us set d_i to c_i or c_{i+1} such that both are in $\bar{d}_i$.

The path $x([t_i, t_{i+1}])$ lies in $c_i \cup c_{i+1}$. For each cell c_i we pick one of its Z-corner z_i. Clearly z_i and z_{i+1} belong to $\bar{d}_i$. By convexity of $\bar{d}_i$, it is in particular simply-connected and there is a homotopy in $\bar{d}_i$ between $x([t_i, t_{i+1}])$ and the segment $[z_i, z_{i+1}]$. Since $\bar{\mathscr{C}}^d[Z]$ is open and both points are in $\|\bar{\mathscr{C}}^d[Z]\|$, $[z_i, z_{i+1}] \subset \|\bar{\mathscr{C}}^d[Z]\|$ as well as the whole homotopy. Gathering all these local homotopies for every i, $0 \leqslant i < m$, we have defined a homotopy between $\mathscr{A}$ and the polyline $[z_i]_i = 0, \ldots, m$.

By full convexity, every $z_i \in Z$ is also in $\mathrm{cvxh}\,(Z)$. It follows that the vertices of the polyline $[z_i]_i = 0, \ldots, m$ belong to $\mathrm{cvxh}\,(Z)$. By convexity of $\mathrm{cvxh}\,(Z)$, the whole polyline is a subset of $\mathrm{cvxh}\,(Z)$. Being a closed curve in a simply connected set, the polyline $[z_i]_i = 0, \ldots, m$ is continuously deformable to a point of this set, say z_0, by some homotopy. Composing the two homotopies finishes the argument. □

Finally we can determine a relation between the numbers of k-cells of the intersection complex of a fully convex set. For a subcomplex K, let $\#_k(K)$ be its number of k-cells. The *Euler characteristic* of a subcomplex K is $\chi(K) := \sum_{k=0}^{d} (-1)^k \#_k(K)$. It is a famous topological invariant of CW complexes.

Theorem 4 *The Euler characteristic of the intersection cubical complex of a fully convex set is* $(-1)^d$.

Proof Let $Z \subset \mathbb{Z}^d$ be a fully convex set. Let $K = \bar{\mathscr{C}}^d[Z]$. According to Lemma 1, K is open. We build a dual (cubical) complex K_* to K in the usual way. Indeed, a natural dual $\mathscr{C}^d_*$ for $\mathscr{C}^d$ is $\mathscr{C}^d$ translated by a shift vector of $(\frac{1}{2}, \ldots, \frac{1}{2})$. Then every $d-k$-cell c_* of $\mathscr{C}^d_*$ has the same centroid as one k-cell c of $\mathscr{C}^d$, and they are defined as dual to each other. Since K is open, the complex K_* is closed. The complex K_* is thus a CW-complex.

It is not hard to see that $\|K_*\|$ is a deformation retract of $\|K\|$, simply by shrinking $\|K\| \setminus \|K_*\|$ to $\partial \|K_*\|$. Since $\|K\|$ is simply connected, $\|K_*\|$ is also simply connected. A simply connected CW-complex has Euler characteristic 1, hence $\chi(K_*) = \sum_{k=0}^{d} (-1)^k \#_k(K_*) = 1$. It follows:

$$
\begin{aligned}
\chi(K) &= \sum_{j=0}^{d} (-1)^j \#_j(K) \\
&= \sum_{j=0}^{d} (-1)^j \#_{d-j}(K_*) && \text{(by duality, } \#_j(K_*) = \#_{d-j}(K)) \\
&= (-1)^d \sum_{k=0}^{d} (-1)^k \#_k(K_*) && \text{(setting } k = d-j \text{ and using } (-1)^{-k} = (-1)^k) \\
&= (-1)^d \chi(K_*) = (-1)^d. && \text{(since } \chi(K_*) = 1)
\end{aligned}
$$

□

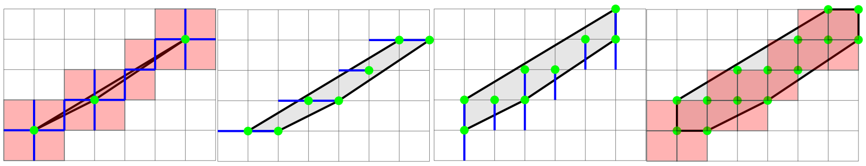

(a) $\mathcal{P}$ (b) $\mathcal{P} \oplus \mathcal{S}_{\{1\}}$ (c) $\mathcal{P} \oplus \mathcal{S}_{\{2\}}$ (d) $\mathcal{P} \oplus \mathcal{S}_{\{1,2\}}$

Fig. 2 Let $\mathcal{P} = \mathrm{cvxh}(\{(0,0),(2,1),(5,3)\})$. (a) $\bar{\mathscr{C}}^d[\mathcal{P}]$. (bcd) We can see that $\#\left(\bar{\mathscr{C}}^d_{\{1\}}[\mathcal{P}]\right) = \#\left(\bar{\mathscr{C}}^d_0[\mathcal{P} \oplus \mathcal{S}_{\{1\}}]\right) = 7$, $\#\left(\bar{\mathscr{C}}^d_{\{2\}}[\mathcal{P}]\right) = \#\left(\bar{\mathscr{C}}^d_0[\mathcal{P} \oplus \mathcal{S}_{\{2\}}]\right) = 9$, and $\#\left(\bar{\mathscr{C}}^d_{\{1,2\}}[\mathcal{P}]\right) = \#\left(\bar{\mathscr{C}}^d_0[\mathcal{P} \oplus \mathcal{S}_{\{1,2\}}]\right) = 14$. The bijections $\mathcal{Z}_{\{1\}}$, $\mathcal{Z}_{\{2\}}$, $\mathcal{Z}_{\{1,2\}}$ are made clear in (b), (c), (d) respectively.

4 Morphological properties and recognition algorithm

We provide first a morphological characterization of full convexity that will help us to design a practical algorithm for checking this property.

Morphological characterisation. Let $I^d := \{1, \dots, d\}$. The set of subsets of cardinal k of I^d is denoted by I^d_k, for $0 < k \leqslant d$. For $i \in I^d$, let $\mathcal{S}_i := \{t\mathbf{e}_i, t \in [0,1]\}$ be the unit segments aligned with axis vectors $\mathbf{e}_i$. For any point x of $\mathbb{R}^d$, we write its d coordinates with superscripts: $x^1, \dots, x^d$. Let us also denote the Minkowski sum of two sets A and B by $A \oplus B$. We further build axis-aligned unit squares, cubes, etc, by suming up the unit segments: for any $\alpha \in I^d_k$, $\mathcal{S}_\alpha := \bigoplus_{i \in \alpha} \mathcal{S}_i$. For instance, in 3D, the three unit segments are $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ (or equiv. $\mathcal{S}_{\{1\}}, \mathcal{S}_{\{2\}}, \mathcal{S}_{\{3\}}$), the three unit squares are $\mathcal{S}_{\{1,2\}}, \mathcal{S}_{\{1,3\}}, \mathcal{S}_{\{2,3\}}$, the unit cube is $\mathcal{S}_{\{1,2,3\}}$. To treat the 0-dimensional case uniformly, we set $I^d_0 = \{0\}$ and $\mathcal{S}_{\{0\}} = \{\vec{0}\}$.

We can partition the k-cells of $\mathscr{C}^d_k$ into $\#\left(I^d_k\right)$ subsets such that, for any $\alpha \in I^d_k$, each subset denoted by $\mathscr{C}^d_\alpha$ contains all the k-cells parallel to $\mathcal{S}_\alpha$. For instance, $\mathscr{C}^d_{\{1\}}$ and $\mathscr{C}^d_{\{2\}}$ partition the set $\mathscr{C}^d_1$ in dimension $d = 2$. Now let us define the mapping $\mathcal{Z} : \mathscr{C}^d \to \mathbb{Z}^d$ which associates to any cell σ, the digital vertex of $\bar{\sigma}$ with highest coordinates. Its restriction to $\mathscr{C}^d_\alpha$ is denoted by $\mathcal{Z}_\alpha$.

Lemma 8 *For any $\alpha \in I^d_k$, the mapping $\mathcal{Z}_\alpha$ is a bijection.*

Proof Clearly every digital point of $\mathbb{Z}^d$ forms the highest vertex of all possible kind of cells, so $\mathcal{Z}_\alpha$ is a surjection. Now no two cells of $\mathscr{C}^d_\alpha$ can have the same highest vertex, since all cells of $\mathscr{C}^d_\alpha$ are distinct translations of the same set. □

The intersection subcomplex of some set Y restricted to cells of $\mathscr{C}^d_\alpha$ is naturally denoted by $\bar{\mathscr{C}}^d_\alpha[Y]$. We relate k-cells intersected by set Y to digital points included in the set Y dilated in some directions, as illustrated in Figure 2.

Lemma 9 *For any $Y \subset \mathbb{R}^d$, for any $\alpha \in I^d_k$, $\mathcal{Z}_\alpha(\bar{\mathscr{C}}^d_\alpha[Y]) = \bar{\mathscr{C}}^d_0[Y \oplus \mathcal{S}_\alpha]$.*

Proof We proceed by equivalences (the logical "and", symbol $\wedge$, has higher priority than "if and only if", symbol $\Leftrightarrow$, but less than any other operations):

$$
\begin{aligned}
& z \in \mathcal{Z}_\alpha(\bar{\mathscr{C}}^d_\alpha[Y]) \\
\Leftrightarrow\ & \sigma \in \bar{\mathscr{C}}^d_\alpha[Y] \wedge \sigma = \mathcal{Z}_\alpha^{-1}(z) \qquad (\mathcal{Z}_\alpha \text{ is a bijection, Lemma 8}) \\
\Leftrightarrow\ & \exists y \in Y, y \in \bar{\sigma} \wedge \sigma = \mathcal{Z}_\alpha^{-1}(z) \\
\Leftrightarrow\ & \exists y \in Y, (\forall i \in \alpha, z^i - 1 \leqslant y^i \leqslant z^i \wedge \forall j \in I^d \backslash \alpha, z^j = y^i) \\
\Leftrightarrow\ & \exists y \in Y, (\forall i \in \alpha, 0 \leqslant x^i \leqslant 1 \wedge \forall j \in I^d \backslash \alpha, x^j = 0) \wedge x = z - y \\
\Leftrightarrow\ & \exists y \in Y, x \in \mathcal{S}_\alpha \wedge z = x + y \in \mathbb{Z}^d \\
\Leftrightarrow\ & z \in (Y \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d.
\end{aligned}
$$

We conclude since $(Y \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = \bar{\mathscr{C}}^d_0[Y \oplus \mathcal{S}_\alpha]$. □

We arrive to our morphological characterization of full convexity: full convexity can thus be checked with common algorithms for checking digital convexity. We denote by $\mathbf{x}(Z)$ the set Z translated by some lattice vector $\mathbf{x}$. We proceed in two steps. First we provide a morphological characterization using Minkowski sums with unit lines, squares, cubes, etc (Theorem 5). Second, Minkwoski sums are replaced by equivalent operations involving solely digital points (Theorem 6). This simplifies the writing of the algorithm for checking digital convexity in arbitrary dimension.

Theorem 5 *A subset $X \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k \leqslant d$ iff*

$$\forall \alpha \in I^d_k, (X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = (\mathrm{cvxh}\,(X) \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d, \tag{4}$$

$$\text{or } (X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = (\mathrm{cvxh}\,(X \oplus \mathcal{S}_\alpha)) \cap \mathbb{Z}^d. \tag{5}$$

It is thus fully convex if the previous relations hold for all $k, 0 \leqslant k \leqslant d$.

Proof We proceed by equivalence for (4):

$$
\begin{aligned}
& \bar{\mathscr{C}}^d_k[X] = \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(X)] \\
\Leftrightarrow\ & \forall \alpha \in I^d_k, \bar{\mathscr{C}}^d_\alpha[X] = \bar{\mathscr{C}}^d_\alpha[\mathrm{cvxh}\,(X)] \\
\Leftrightarrow\ & \forall \alpha \in I^d_k, \mathcal{Z}_\alpha(\bar{\mathscr{C}}^d_\alpha[X]) = \mathcal{Z}_\alpha(\bar{\mathscr{C}}^d_\alpha[\mathrm{cvxh}\,(X)]) \qquad (\mathcal{Z}_\alpha \text{ is a bijection}) \\
\Leftrightarrow\ & \forall \alpha \in I^d_k, \bar{\mathscr{C}}^d_0[X \oplus \mathcal{S}_\alpha] = \bar{\mathscr{C}}^d_0[\mathrm{cvxh}\,(X) \oplus \mathcal{S}_\alpha] \qquad (\text{Lemma 9}) \\
\Leftrightarrow\ & \forall \alpha \in I^d_k, (X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = (\mathrm{cvxh}\,(X) \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d.
\end{aligned}
$$

(5) follows since the convex hull operation commutes with Minkowski sum. □

We introduce a discrete analog of Minkowski sums of unit axis-aligned edges, squares, cubes, etc. Let $U_\varnothing(Z) := Z$, and, for $\alpha \subset I^d$ and $i \in \alpha$, we define recursively $U_\alpha(Z) := U_{\alpha\backslash i}(Z) \cup \mathbf{e}_i(U_{\alpha\backslash i}(Z))$. The previous definition is consistent since it does not depend on the order of the sequence $i \in \alpha$.

First, Lemma 10 establishes that the operation $U_\alpha(\cdot)$ is indeed equivalent to specific Minkowski sums for digital sets. Then Lemma 11 asserts that this operation also commutes with convex hull operation.

Lemma 10 *For any $X \subset \mathbb{Z}^d$, for any $\alpha \subset I^d$, $(X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = U_\alpha(X)$.*

Proof Let $\alpha = \{i_1, \dots, i_k\} \subset I^d$, non empty. Note that $\mathcal{S}_\alpha = \bigoplus_{j=1}^{k} \mathcal{S}_j$ and that it does not depend on the chosen order. For conciseness, we write $X^\gamma := X \oplus \mathcal{S}_\gamma$ for any subset γ of I^d. Let $\beta = \{i_1, \dots, i_{k-1}\}$ and let us first show that:

$$(X^\alpha) \cap \mathbb{Z}^d = (X^\beta \cap \mathbb{Z}^d) \cup \mathbf{e}_{i_k}(X^\beta \cap \mathbb{Z}^d). \tag{6}$$

$\boxed{\supset}$ $(X^\beta \cap \mathbb{Z}^d) \cup \mathbf{e}_{i_k}(X^\beta \cap \mathbb{Z}^d) = (X^\beta \cup \mathbf{e}_{i_k}(X^\beta)) \cap \mathbb{Z}^d \subset (X^\beta \oplus \mathcal{S}_{i_k}) \cap \mathbb{Z}^d = (X^\alpha) \cap \mathbb{Z}^d$.

$\boxed{\subset}$ Let $z \in X^\alpha \cap \mathbb{Z}^d$. We can write z as $z = x + t_{i_1}\mathbf{e}_{i_1} + \dots + t_{i_k}\mathbf{e}_{i_k}$, with $x \in X$ and every $t_{i_j} \in [0,1]$. More precisely, since $z \in \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ and e_{i_j} is a unit vector, every $t_{i_j} \in \{0,1\}$. Clearly $z' = z + \sum_{j=1}^{k-1} t_{i_j}\mathbf{e}_{i_j}$ belongs to $X^\beta \cap \mathbb{Z}^d$. If $t_{i_k} = 0$ then $z' = z$ and we are done. Otherwise, $t_{i_k} = 1$ then $z' = z + \mathbf{e}_{i_k}$, which belongs to $\mathbf{e}_{i_k}(X^\beta \cap \mathbb{Z}^d)$.

We prove the lemma by induction on the cardinal of α. For $\alpha = \varnothing$, $(X \oplus \mathcal{S}_\varnothing) \cap \mathbb{Z}^d = X = U_\varnothing(X)$. Assume the lemma is true for any β of cardinal $k - 1 \geqslant 0$, and let us show it for $\alpha = \beta \cup \{i\}$.

$$\begin{aligned}(X^\alpha) \cap \mathbb{Z}^d &= (X^\beta \cap \mathbb{Z}^d) \cup \mathbf{e}_i(X^\beta \cap \mathbb{Z}^d) && \text{(Using (6))}\\ &= U_\beta(X) \cup \mathbf{e}_i(U_\beta(X)) && \text{(Induction)}\\ &= U_\alpha(X) && \text{(Definition).}\end{aligned}$$

□

Lemma 11 *For any $X \subset \mathbb{Z}^d$, for any $\alpha \subset I^d$, $\mathrm{cvxh}\,(X) \oplus \mathcal{S}_\alpha = \mathrm{cvxh}\,(U_\alpha(X))$.*

Proof We prove it by induction on the cardinal k of α. It holds obviously for $k = 0$. Otherwise let $\alpha = \beta \cup \{i\}$ of cardinal k.

$\boxed{\supset}$ We have $U_\alpha(X) = U_\beta(X) \cup \mathbf{e}_i(U_\beta(X)) \subset U_\beta(X) \oplus \mathcal{S}_i$. Since convex hull is increasing, $\mathrm{cvxh}\,(U_\alpha(X)) \subset \mathrm{cvxh}\,(U_\beta(X) \oplus \mathcal{S}_i)$ holds. But convex hull commutes with Minkowski sum, so $\mathrm{cvxh}\,(U_\beta(X) \oplus \mathcal{S}_i) = \mathrm{cvxh}\,(U_\beta(X)) \oplus \mathcal{S}_i = \mathrm{cvxh}\,(X) \oplus \mathcal{S}_\beta \oplus \mathcal{S}_i$ by induction hypothesis. We conclude with $\mathcal{S}_\beta \oplus \mathcal{S}_i = \mathcal{S}_\alpha$.

$\boxed{\subset}$ Let $y \in \mathrm{cvxh}\,(X) \oplus \mathcal{S}_\alpha = \mathrm{cvxh}\,(X) \oplus \mathcal{S}_\beta \oplus \mathcal{S}_i = \mathrm{cvxh}\,(U_\beta(X)) \oplus \mathcal{S}_i$. Denoting by z_j, $j \in B$ the points of $U_\beta(X)$, it follows that y can be written as a convex linear combination of these points plus a point of $\mathcal{S}_i$, i.e. $y = (\sum_{j \in B} \mu_j z_j) + t\mathbf{e}_i$, $\sum_{j \in B} \mu_j = 1$, $\forall j \in B, \mu_j \geqslant 0$ and $t \in [0,1]$. All following sums are taken over $j \in B$. Since $\sum \mu_j = 1$, we rewrite y as

$$\begin{aligned}y &= \left(\sum \mu_j z_j\right) + t\left(\sum \mu_j\right)\mathbf{e}_i\\ &= \sum (1-t)\mu_j z_j + t\mu_j z_j + t\mu_j \mathbf{e}_i\\ &= \left(\sum (1-t)\mu_j z_j\right) + \left(\sum t\mu_j (z_j + \mathbf{e}_i)\right)\\ &\in \mathrm{cvxh}\,(U_\beta(X) \cup \mathbf{e}_i(U_\beta(X))),\end{aligned}$$

which shows that $y \in \mathrm{cvxh}\,(U_\alpha(X))$. □

Algorithm 1: IsFullyConvex: given the dimension d of the space and a subset Z of the digital space $\mathbb{Z}^d$, returns true iff Z is fully convex. IsConvex: given a subset Z of the digital space $\mathbb{Z}^d$, returns true iff Z is digitally convex (0-convexity).

```
Function IsConvex( In Z: subset of Z^d ) : boolean;
begin
    Polytope P ← ConvexHull(Z);
    return Cardinal(Z) = CountLatticePoint(P);

Function IsFullyConvex( In d : integer, In Z: subset of Z^d ) : boolean;
Var C : array[0 ... d-1] of lists of subsets of I^d;
Var X : map associating subsets of I^d → sets of digital points ;
begin
    if Cardinal(Z) = 0 or ¬IsDConnected(Z, d) then return false;
    C[0] ← ({0}) ;   X[0] ← Z ;
    if ¬IsConvex(Z) then return false;
    for k ← 1 to d − 1 do
        C[k] ← ∅;
        foreach β ∈ C[k − 1] do
            for j ← 1 to d do
                α ← Append(β, j) ;
                if IsStrictlyIncreasing(α) then
                    C[k] ← Append(C[k], α);
                    X[α] ← Union(X[β], e_j(X[β])) ;
                    if ¬IsConvex(X[α]) then return false;
    return true
```

Theorem 6 *A subset $X \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k \leqslant d$ iff*

$$U_\alpha(X) = \mathrm{cvxh}\left(U_\alpha(X)\right) \cap \mathbb{Z}^d. \tag{7}$$

It is thus fully convex if the previous relation holds for all $k, 0 \leqslant k \leqslant d$.

Proof Recalling full convexity characterization (5) of Theorem 5, we have:

$$(X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = (\mathrm{cvxh}\left(X \oplus \mathcal{S}_\alpha\right)) \cap \mathbb{Z}^d.$$

Now Lemma 10 states that $(X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = U_\alpha(X)$, which we apply on the left-hand side of the previous characterization. And Lemma 11 states that $\mathrm{cvxh}\left(X\right) \oplus \mathcal{S}_\alpha = \mathrm{cvxh}\left(U_\alpha(X)\right)$, which we apply on the right-hand side of the same characterization, while intersecting it with $\mathbb{Z}^d$. This gives (7). □

Finally we can remark that, if $X \subset \mathbb{Z}^d$ is d-connected, then necessarily all $U_\alpha(X)$ are by construction d-connected.

Recognition algorithm. Algorithm 1 checks the full convexity of a digital set $Z \subset \mathbb{Z}^d$. Due to the bijections $\mathcal{Z}_\alpha$, all the processed sets are subsets of $\mathbb{Z}^d$.

Theorem 7 *Algorithm 1 correctly checks if a digital set Z is fully convex.*

Proof First of all, IsConvex checks the classical digital convexity of any digital set S by counting lattice points within $\mathrm{cvxh}\,(S)$ (Lemma 3).

Looking now at IsFullyConvex, line 3 checks the d-connectedness of Z and outputs false if Z is not connected (valid since Lemma 2).

Line 4 checks for digital 0-convexity (i.e. usual digital convexity). The loop starting at line 5 builds, for each dimension k from 1 to $d-1$ the possible $\alpha \in I^d_k$, and stores them in $C[k]$. Line 6 guarantees that each possible subsets of I^d_k is used exactly once.

Line 7 builds $X[\alpha] = U_\alpha(Z)$. Indeed, by induction assume $X[\beta] = U_\beta(Z)$. Then $X[\alpha] = X[\beta] \cup \mathbf{e}_j(X[\beta])$ which is exactly the definition of $U_\alpha(Z)$.

Finally line 8 verifies $U_\alpha(Z) = \mathrm{cvxh}\,(U_\alpha(Z)) \cap \mathbb{Z}^d$. Since it does this check for every $\alpha \in I^d_k$, it checks digital k-convexity according to Theorem 6, (7). Now Lemma 4 tells that it is not necessary to check digital d-convexity if all other digital k-convexities are satisfied. This establishes the correctness. □

Then the complexity of the algorithm can be determined as follows. First, letting $n = \#\,(Z)$, function IsDConnected takes $O(n)$ operations by depth first algorithm and bounded number of adjacent neighbors. There are less than $d2^d$ calls to IsStrictlyIncreasing, which takes $O(d)$ time complexity. However the total number of calls to IsConvex is exactly $2^d - 1$, and its time complexity dominates (by far) the previous $d^2 2^d$ in practical uses. The overall complexity $T(n)$ of this algorithm is thus governed by (1) the complexity $T_1(n)$ for computing the convex hull and (2) the complexity $T_2(n)$ for counting the lattice points within. For $T_1(n)$, it is: $O(n)$ for $d = 2$ (since we know S is d-connected, it can be done in linear time [27] or [6] for a fast practical algorithm), $O(n \log n)$ in 3D [8] and $n^{\lfloor d/2 \rfloor}/\lfloor d/2 \rfloor!$ in dD [11,1]. For $T_2(n)$, it is $O(n)$ in 2D using Pick's formula, and in general dimension it is related to Ehrhart's theory [22], where best algorithms run in $O(n^{O(d)})$ [2].

In practice, when the number of facets of the convex hull is low, counting lattice points within the hull is done efficiently by visiting all lattice points within the bounding box, which is not too big since all digital sets are connected.

5 Digital planarity and local shape analysis

We explore the link of full convexity with digital planarity, and we show that thick enough arithmetic planes are indeed fully convex (Theorem 8). Combined with our results about stable sets, this leads us to propose new geometric tools for digital shape local analysis.

5.1 Arithmetic planes are fully convex

An arithmetic plane of intercept $\mu \in \mathbb{Z}$, positive thickness $\omega \in \mathbb{Z}, \omega > 0$, and irreducible normal vector $N \in \mathbb{Z}^d$ is defined as the digital set $P(\mu, N, \omega) := \{x \in \mathbb{Z}^d, \mu \leqslant x \cdot N < \mu + \omega\}$.

Theorem 8 *Arithmetic planes are digitally 0-convex for arbitrary thickness, and fully convex for thickness $\omega \geqslant \|N\|_\infty$.*

Proof Let $Q := P(\mu, N, \omega)$ be some arithmetic plane. Let $Y^- := \{x \in \mathbb{R}^d, \mu \leqslant x \cdot N\}$, $Y^+ := \{x \in \mathbb{R}^d, x \cdot N < \mu + \omega\}$, and $Y := Y^- \cap Y^+$. We have $Q = Y \cap \mathbb{Z}^d$, with Y convex, so Q is digitally 0-convex (non emptyness comes from $\omega > 0$).

Now let c be any k-cell of $\bar{\mathscr{C}}^d[\mathrm{cvxh}\,(Q)]$, $1 \leqslant k \leqslant d$. Let us show that at least one vertex of c is in $\bar{\mathscr{C}}^d[Q]$. There exists $x \in c$ with $x \in \mathrm{cvxh}\,(Q) = \mathrm{cvxh}\left(Y \cap \mathbb{Z}^d\right) \subset \mathrm{cvxh}\,(Y) = Y$. It follows that $\mu \leqslant x \cdot N < \mu + \omega$.

Let $(z_i)_{i=1\ldots 2^k}$ be the vertices of $\bar{c}$ ordered from lowest to highest scalar product with N. There exists $j \in \{1, 2^k\}$ such that $\forall i \in \{1, j\}, z_i \cdot N \leqslant x \cdot N$ and $\forall i \in \{j+1, 2^k\}, x \cdot N < z_i \cdot N$. Should no z_i belong to Y, since x belongs to Y, we have :

$$\forall i \in \{1, j\}, z_i \cdot N < \mu, \quad \text{and} \quad \forall i \in \{j+1, 2^k\}, \mu + \omega \leqslant z_i.$$

It follows that $(z_{j+1} - z_j) \cdot N > \omega$. Now the (z_i) are vertices of a hypercube of side one, and ordered according to their projection along vector N. It is easy to see that any $(z_{i+1} - z_i) \cdot N \leqslant \max_{i=1\ldots d}(|N_i|)$ (for instance by constructing a subsequence moving along axes in order, it achieves the bound, and then the actual sequence (z_i) is much finer than this one), so it holds in particular for $i = j$. To sum up, should no z_i belong to Y, then $\omega < \max_{i=1\ldots d}(|N_i|)$.

Otherwise, for $\omega \geqslant \max_{i=1\ldots d}(|N_i|)$, either z_j or z_{j+1} or both belong to Y, and thus to Q. It follows that $c \in \bar{\mathscr{C}}_k^d[Q]$, which concludes. □

The classic 2D and 3D machinery of digital straight lines and planes, very rich in results and applications, thus belongs to the fully convex framework.

5.2 Local analysis of digital shapes

Corollary 2 tells that the intersection of a fully convex set with an axis-aligned parallelepiped is also fully convex. Reciprocally of course, non locally fully convex subsets tells the set is not fully convex. Furthermore the full convexity of the local complement of the set is also an indicator of the local geometry.

For any digital set X, the predicate "X is fully convex" is denoted by $F(X)$. We define the following local sets, for any positive integer k and any $z \in X \subset \mathbb{Z}^d$:

$$X_k(z) := N_k(z) \cap X \qquad X_k^*(z) := X_k(z) \backslash \{z\} \qquad \bar{X}_k(z) := N_k(z) \cap (\mathbb{Z}^d \backslash X)$$

These sets allow us to study the local shape geometry at fixed k.

Definition 3 *Let $j \in \mathbb{Z}$, $j \geqslant 0$. X is* j-convex *at z iff $F(X_j(z))$. X is* j-concave *at z iff $F(\bar{X}_j(z))$. X is* j-planar *at z iff it is j-convex and j-concave at z. Last, X is* j-atypical *at z if it is neither j-convex nor j-concave.*

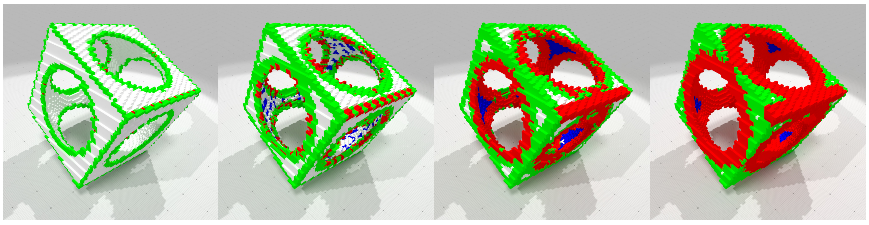


Fig. 3 Local shape geometry analysis at fixed scale (green: k-convex, blue: k-concave, red: k-atypical, white: k-planar). One easily see that k-atypical implies $(k+1)$-atypical, while k-convex implies $(k-1)$-convex and k-concave implies $(k-1)$-concave.

Figure 3 illustrates the relevance of these definitions for capturing the shape geometry at a given scale. Planarity, convexity and concavity are correctly identified at their respective scale.

Note that arithmetic half-spaces (whose border are arithmetic planes) are k-convex and k-concave for arbitrary k, hence k-planar.

This study of shape geometry can also be carried out in a multiscale fashion. Indeed, we have the following relations:

Lemma 12 *Let $j \in \mathbb{Z}, j \geqslant 0$, $z \in X$. If X is $(j+1)$-convex at z, then X is j-convex at z. If X is $(j+1)$-concave at z, then X is j-concave at z. If X is j-atypical at z, then X is $(j+1)$-atypical at z.*

Proof Assuming X $(j+1)$-convex means $X_{j+1}(z)$ is fully convex. But $X_j(z) := X \cap N_j(z) = X_{j+1}(z) \cap N_j(z)$. Now, $N_j(z)$ is the intersection of $2d$ axis-aligned half-spaces with integer intercept, hence it is stable (Lemma 6). Applying Lemma 5 concludes that $X_j(z)$ is fully convex. The reasoning is similar for concavity. The result for atypicality is simply the contraposition of the two previous properties. □

Denoting by $a_X(z)$, resp. by $b_X(z)$, the maximum j for which X is j-convex at z, resp. j-concave at z, we can classify each point z of X as:

- *atypical* if $a_X(z) = b_X(z) = 0$,
- *planar* if $a_X(z) = b_X(z) > 0$,
- *convex* if $a_X(z) > b_X(z)$,
- *concave* if $a_X(z) < b_X(z)$.

Therefore local convexity, concavity, planarity or neither are definable in an unambiguous way. Figure 4 illustrates this approach to digital shape local geometric classification. If one needs a more progressive classification, values $a_X(z)$ and $b_X(z)$ are of course useful. Figure 5 shows a smooth classification of convex, concave, and planar parts of the same shapes.

It is worth noting that locii of noisy digitizations are detected by atypical configurations, often surrounded by small and varying convex and concave zones. Finally, if we consider a finer digitization of the same shape, we observe a consistency between the two classifications, with of course more details in the finest digitization.

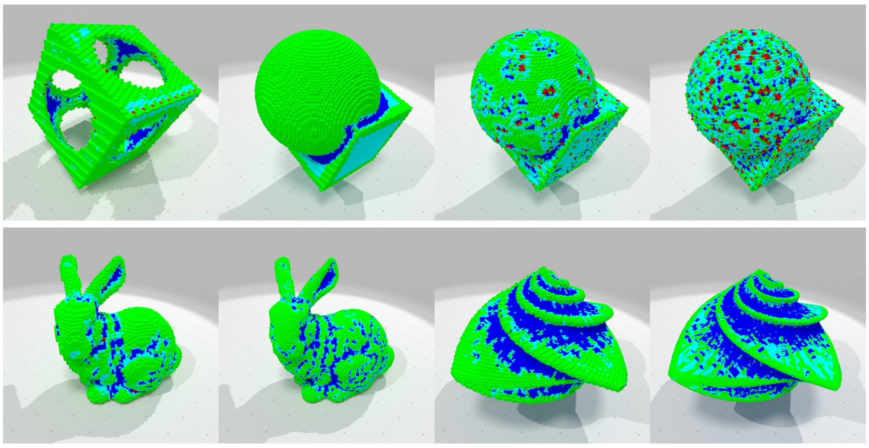


Fig. 4 Raw multiscale local shape geometry analysis (green: convex, blue: concave, red: atypical, cyan: planar), with a maximal scale of 5.

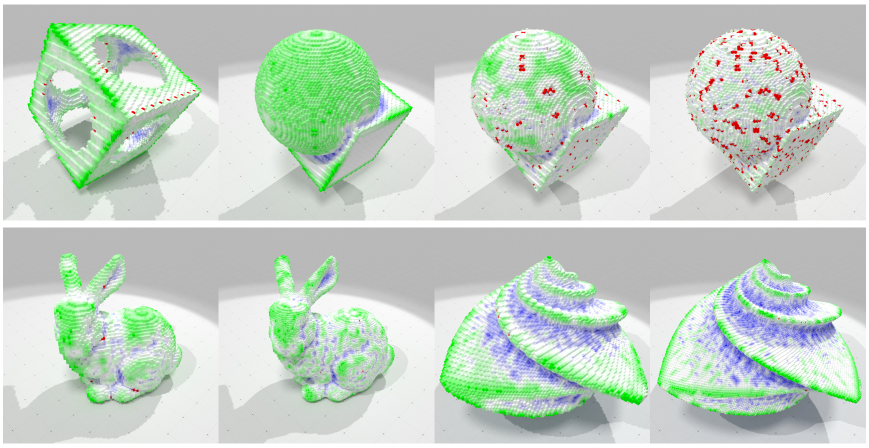


Fig. 5 Smooth multiscale local shape geometry analysis (green: convex, blue: concave, red: atypical, white: planar), with a maximal scale of 5. For a convex zone, the saturation of green is the ratio $(a_X(z) - b_X(z))/5$. For a concave zone, the saturation of blue is the ratio $(b_X(z) - a_X(z))/5$.

6 Tangent subsets to a digital shape

This section proposes the new concept of *tangency* to a digital set X, which are subsets whose convex hull stays close to X, and which are related to full convexity. In 2D, this notion provides another definition of the maximal digital straight segments along digital contours and induces the classical tangential cover. However it is a much more generic and powerful tool since it induces *tangent* maximal digital planes in arbitrary dimension.

This approach to tangency appears thus to be very fruitful, and we show after two direct applications of this concept of tangency: the first one is an algorithm to build a reversible piecewise linear reconstruction of a digital shapes, the second one is the computation of shortest paths onto digital shapes.

Definition 4 *A digital set $A \subset X \subset \mathbb{Z}^d$ is said to be* k-tangent *to X for $0 \leqslant k \leqslant d$ whenever $\bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(A)] \subset \bar{\mathscr{C}}^d_k[X]$. It is* tangent *to X if the relation holds for all such k. Elements of A are called* cotangent *(in X).*

It is immediate that any subset of a tangent set to X is also a tangent set to X. We prove first that tangent sets to X are indeed close to X.

Lemma 13 *Let A be digital points and $\tau = \mathrm{cvxh}\,(A)$ be the convex hull of A. If A is tangent to X, then $A \subset X$ and the L_∞-distance of τ to X is upper-bounded by 1.*

Proof Assume A tangent to X. Then in particular $\bar{\mathscr{C}}^d_0[\mathrm{cvxh}\,(A)] \subset \bar{\mathscr{C}}^d_0[X]$, otherwise said $\mathrm{cvxh}\,(A) \cap \mathbb{Z}^d \subset X$ which implies $A \subset X$. Now, let $y \in \tau$. By tangency, for any k, $0 \leqslant k \leqslant d$, $\bar{\mathscr{C}}^d_k[\tau] = \bar{\mathscr{C}}^d_k[\mathrm{cvxh}\,(A)] \subset \bar{\mathscr{C}}^d_k[X]$. Hence the point y belongs to some cell σ of $\bar{\mathscr{C}}^d_k[X]$ and is at most at L_∞-distance 1 of any one of its X-corner. □

As immediate examples of tangency, in arbitrary dimension, two cotangent points of a digital surface X delineates a straight segment that stays close to the surface, i.e. a tangent vector. A simplex made of d cotangent points to X lies close to the surface, and so defines a (local) tangent plane. The link with full convexity is established by the following lemma:

Lemma 14 *If X is fully convex, any subset $A \subset X$ is tangent to X.*

Proof We have $\mathrm{cvxh}\,(A) \subset \mathrm{cvxh}\,(X)$ since convex hull is increasing, so:

$$\begin{aligned} \bar{\mathscr{C}}^d[\mathrm{cvxh}\,(A)] \subset & \bar{\mathscr{C}}^d[\mathrm{cvxh}\,(X)] && \text{(Intersection complex is increasing)} \\ = & \bar{\mathscr{C}}^d[X] && \text{(by full convexity).} \end{aligned}$$

This shows that A is tangent to X. □

6.1 Maximal tangent subsets

As often in digital geometry, small objects are not precise enough. We are thus more interested in "big" tangent subsets: a set A, tangent to X, that is not included in any other tangent set to X is said *maximal in X*. In 2D, they give rise to the classical tangential cover of a contour [24]:

Theorem 9 *When $d = 2$, if C is a simple 2-connected digital contour (i.e. 8-connected in Rosenfeld's terminology), then the fully convex subsets of C that are maximal and tangent are the classical maximal naive digital straight segments.*

Proof Let M be a fully convex subset of C, both maximal and tangent. Full convexity implies that M is 2-connected (Theorem 2). Full convexity implies H-convexity, and connected convex subsets of simple 2-connected contours are digital straight segments. Maximality implies that they are inextensible. The converse is obvious from Theorem 8. □

Maximal fully convex tangent subsets to X seem a good candidate for a sound definition of maximal digital plane segments. Our definition avoids the classical problem of 3D planes that are not tangent to the surface (as noted in [10]) as well as the many heuristics to cope with this issue [47,42].

We leave the exploration of maximal tangent planes for future works and we present here two quite straightforward applications of tangency: one related to surface reconstruction, the other related to shortest paths and visibility.

6.2 Elementary polyhedrization algorithm

Let X be a finite subset of $\mathbb{Z}^d$. Let $\mathrm{Vor}\,(X)$ be its Voronoi diagram: it is the cellular complex made of convex cells, where each maximal d-cell σ_p, $p \in X$, is the open region of the space that gathers all points of the space closer to p than any other point of X; its low dimensional cells are defined naturally as their boundaries with incidence relations. Then the *Delaunay complex of* X, denoted by $\mathrm{Del}\,(X)$, is the cellular complex dual to $\mathrm{Vor}\,(X)$ (combinatorially and orthogonally). It is well known that the vertices of every d-cell of $\mathrm{Del}\,(X)$ form hyper-cospherical subsets of X. Note that, when X are points in general position, the Delaunay complex is a simplicial complex called the Delaunay triangulation. We prefer to use Delaunay complexes here since the elements of digital sets are usually not in general position, with many cosphericities.

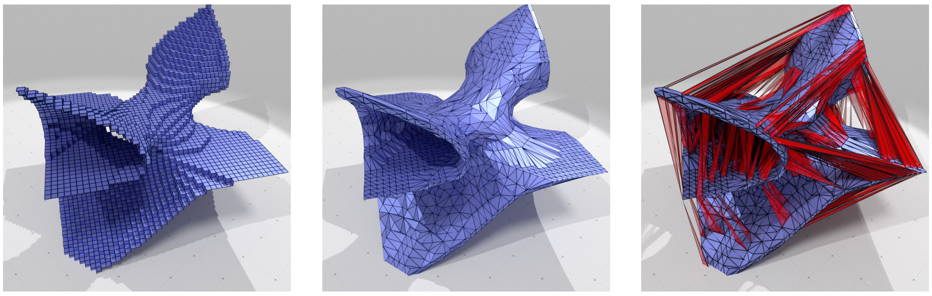
Definition 5 *The* tangent Delaunay complex $\mathrm{Del}_{\mathrm{T}}\,(X)$ *to* X *is the complex made of the cells* τ *of the Delaunay complex* $\mathrm{Del}\,(X)$*, such that the vertices of* τ *are tangent to* X*.*

The boundary of a tangent Delaunay complex is a piecewise linear reconstruction of voxel shapes, and it is the boundary of the convex hull for fully convex shapes:

Lemma 15 *If* X *is fully convex, then* $\mathrm{Del}_{\mathrm{T}}\,(X) = \mathrm{Del}\,(X)$*. Hence the boundary of* $\mathrm{Del}_{\mathrm{T}}\,(X)$ *is the boundary of the convex hull of* X*, i.e.* $\partial\,\|\mathrm{Del}_{\mathrm{T}}\,(X)\| = \partial \mathrm{cvxh}\,(X)$*.*

Proof Let $\tau \in \mathrm{Del}\,(X)$. Let $A \subset X$ be the vertices of cell τ (i.e. $\tau = \mathrm{cvxh}\,(A)$). Since X is fully convex, Lemma 14 implies that A is tangent to X, hence $\tau \in \mathrm{Del}_{\mathrm{T}}\,(X)$. So $\mathrm{Del}_{\mathrm{T}}\,(X) = \mathrm{Del}\,(X)$. It follows that $\partial\,\|\mathrm{Del}_{\mathrm{T}}\,(X)\| = \partial\,\|\mathrm{Del}\,(X)\| = \partial \mathrm{cvxh}\,(X)$, since the boundary of the Delaunay complex of a set of points is the boundary of its convex hull. □

From Theorem 8, the tangent Delaunay complex of an arithmetic plane is the boundary of its convex hull, hence its facets have exactly the same normal



Input digital shape X Our reconstruction $\mathrm{Del}_{\mathrm{T}}(X)$ Bad simplices of $\mathrm{Del}(X)$

Fig. 6 The tangent Delaunay complex $\mathrm{Del}_{\mathrm{T}}(X)$ (middle) is a piecewise linear reconstruction of the input digital surface X (left). On (right), we display in red simplices of $\mathrm{Del}(X)$ which avoid lattice points of $\mathbb{Z}^3 \setminus X$ but are not tangent to X. Tangency thus eliminates the "sliver" simplices of $\mathrm{Del}(X)$ that are not geometrically informative. ($\#(X) = 6013$, $\mathrm{Del}(X)$ has 36361 tetrahedra, $\mathrm{Del}_{\mathrm{T}}(X)$ has 25745 tetrahedra, computing $\mathrm{Del}(X)$ takes 70ms, computing $\mathrm{Del}_{\mathrm{T}}(X)$ takes 773ms.)

as the arithmetic plane. Furthermore, since the tangent Delaunay complex is built with local geometric considerations, it is able to capture the geometry of local pieces of planes on digital objects, and nicely reconstructs convex and concave parts (see Figure 6). We prove now that it is also a tight and reversible reconstruction of X:

Theorem 10 *The Hausdorff L_∞-distance between the set X and the body of $\mathrm{Del}_{\mathrm{T}}(X)$ is at most 1. Furthermore $\mathrm{Del}_{\mathrm{T}}(X)$ is a reversible polyhedrization, i.e. $\|\mathrm{Del}_{\mathrm{T}}(X)\| \cap \mathbb{Z}^d = X$.*

Proof First the distance of any point of X to $\|\mathrm{Del}_{\mathrm{T}}(X)\|$ is zero, since any point of X is tangent to X and is also a 0-cell of $\mathrm{Del}(X)$. Second, any point y of $\|\mathrm{Del}_{\mathrm{T}}(X)\|$ belongs to a simplex $\tau \in \mathrm{Del}_{\mathrm{T}}(X)$. Let A be the vertices of τ. By definition, τ is tangent to X, so by Lemma 13, τ is at L_∞-distance at most 1 to X. This proves the first assertion of the theorem. Finally, for the reversibility property, by tangency, $\|\mathrm{Del}_{\mathrm{T}}(X)\| \subset \bar{\mathscr{C}}^d[X]$, so $\|\mathrm{Del}_{\mathrm{T}}(X)\| \cap \mathbb{Z}^d \subset \bar{\mathscr{C}}^d[X] \cap \mathbb{Z}^d = X$. The statement $X \subset \|\mathrm{Del}_{\mathrm{T}}(X)\| \cap \mathbb{Z}^d$ is obvious. □

Although very simple to define and compute, our approach improves the existing greedy reversible polyhedrization methods of [25, 47].

6.3 Shortest paths onto digital sets

The concept of tangency to a digital shape X allows us to define unambiguously shortest paths between pairs of points of X, such that the path stays "in" X. Our approach is similar to the visibility method of [15]. However our method does not assume a particular model of digital straight lines, and is valid in arbitrary dimension. For instance, if you pick any two points along a digital straight line or plane, their shortest path is indeed the Euclidean straight line joining them.

Definition 6 (path) *Let $\gamma = (x_i)_{i=0,\ldots,n}$, $n \geqslant 0$, be a sequence of points in some digital set X. The sequence γ is a* path from point a to point b in X, *if and only if, $x_0 = a$, $x_n = b$, and every two consecutive points of γ are co-tangent in X. The* embedding of γ into $\mathbb{R}^n$ *is the embedding of the straight segments joining consecutive points, i.e. $\overline{\overline{\gamma}} := \bigcup_{i=0}^{n-1} \mathrm{cvxh}\left(\{x_i, x_{i+1}\}\right)$. When a sequence $(x_i)_{i=0,\ldots,n}$ is a path, we will denote it by $[\![x_0, \ldots, x_n]\!]$ or $[\![x_i]\!]_{i=0,\ldots,n}$.*

It is straightforward to check that any d-connected sequence of points in X is a path in X. Therefore there is at least one path between a and b in X if and only if a and b are d-connected in X. The set of all paths between a and b in X is denoted by $\mathscr{P}_X(a, b)$ (which is empty when a and b are not in the same d-connected component of X). The set of every path between a and b in X such that its points are pairwise distinct is denoted by $\mathscr{P}^*_X(a, b)$, and their elements are called *simple paths* from a to b.

Any path in the digital set X stays in the vicinity of X:

Lemma 16 *If γ is a path in X, then $\overline{\overline{\gamma}} \subset \left\|\bar{\mathscr{C}}^d[\overline{\overline{\gamma}}]\right\| \subset \left\|\bar{\mathscr{C}}^d[X]\right\|$. Hence the L_∞-distance of any point of $\overline{\overline{\gamma}}$ to X is at most 1.*

Proof Since any consecutive points x_i, x_{i+1} of γ are co-tangent in X, then $\bar{\mathscr{C}}^d_k[\mathrm{cvxh}\left(\{x_i, x_{i+1}\}\right)] \subset \bar{\mathscr{C}}^d_k[X]$ for all $0 \leqslant k \leqslant d$. It follows that $\bigcup_{i=0}^{n-1} \mathrm{cvxh}\left(\{x_i, x_{i+1}\}\right) \subset \bigcup_{k=0}^{d} \bar{\mathscr{C}}^d_k[\overline{\overline{\gamma}}] \subset \bigcup_{k=0}^{d} \bar{\mathscr{C}}^d_k[X] = \left\|\bar{\mathscr{C}}^d[X]\right\|$. Finally, Lemma 13 concludes for the distance by tangency of consecutive points. □

Definition 7 (path length; shortest path) *The* length of γ *is* $\mathrm{length}(\gamma) := \sum_{i=0}^{n-1} \|x_{i+1} - x_i\|$. *The path γ from a to b is a* shortest path *from a to b if there exists no other path from a to b with a smaller length. The set of shortest paths from a to b is denoted by $\mathit{\Gamma}_X(a, b)$.*

A first observation is that, if $[\![x_i]\!]_{i=0,\ldots,n}$ is a shortest path from a to b, then $[\![x_{n-i}]\!]_{i=0,\ldots,n}$ is a shortest path from b to a. Then we can speak of a shortest path between a and b. A second observation is that if a and b are cotangent in X, then the path $[\![a, b]\!]$ is a shortest path between a and b. This is because the triangle inequality holds for the Euclidean distance. Last, the length of γ is the same as the Euclidean length of $\overline{\overline{\gamma}}$.

Definition 8 (digital distance) *The* digital distance d_X in $X \subset \mathbb{Z}^d$ *is*

$$\forall x, y \in X, \mathrm{d}_X(x, y) := \begin{cases} +\infty \text{ if } \mathscr{P}_X(x, y) \text{is empty}, \\ \inf_{\gamma \in \mathscr{P}_X(x,y)} \mathrm{length}(\gamma) \text{ otherwise.} \end{cases}$$

Lemma 17 *If X is d-connected then $\forall a, b \in X$, $\mathit{\Gamma}_X(a, b)$ is non empty and finite, so the infimum above is a minimum, i.e. $\mathrm{d}_X(a, b) = \mathrm{length}(\gamma)$ with γ any shortest path of $\mathit{\Gamma}_X(a, b)$.*

Proof Since X is d-connected, there exists a sequence $P = (x_i)_{i=0,\ldots,n}$ of points in X, $x_0 = a$, $x_n = b$, such that x_i and x_{i+1} are d-adjacent. Clearly x_i and x_{i+1} are co-tangent in X since they lie on the boundary of some cell. So P is a path and $\mathscr{P}_X(a, b)$ is not empty and let $\alpha := \mathrm{length}(P)$. Since the length of a path is the sum of the lengths of its subsegments, any path of length no greater than α is included in the Euclidean ball B of center a and radius α. So every path from a to b in X of length no greater than α belong to $B \cap X$. Hence it holds that

$$\inf_{\gamma \in \mathscr{P}_X(x,y)} \mathrm{length}(\gamma) = \inf_{\gamma \in \mathscr{P}_{B\cap X}(x,y)} \mathrm{length}(\gamma).$$

But $B \cap X$ has a finite number of elements since $X \subset \mathbb{Z}^d$. Furthermore if $\gamma = [\![y_i]\!]_{i=0,\dots,m} \in \mathscr{P}_{B\cap X}(x,y)$ traverses a point several times, for instance $y_i = y_j$, $i < j$, then the path $\bar{\gamma} = [\![y_i]\!]_{i=0,\dots,i,j+1,\dots m}$ also belongs to $\mathscr{P}_{B\cap X}(x,y)$ and is shorter. It follows that

$$\inf_{\gamma \in \mathscr{P}_{B\cap X}(x,y)} \mathrm{length}(\gamma). = \inf_{\gamma \in \mathscr{P}^*_{B\cap X}(x,y)} \mathrm{length}(\gamma).$$

But $\mathscr{P}^*_{B\cap X}(x,y)$ is a finite set, so the infimum above is an element of $\mathscr{P}^*_{B\cap X}(x,y)$, and $\mathit{\Gamma}_X(a,b)$ is not empty. □

Theorem 11 *If $X \subset \mathbb{Z}^d$ is d-connected and non-empty, then (X, d_X) is a metric space.*

Proof Let $x, y, z \in X$. First it is obvious that $\mathrm{d}_X(x,x) = 0$. If $\mathrm{d}_X(x,y) = 0$, then there exists a path $\gamma \in \mathscr{P}_X(x,y)$ between x and y with a length 0. Since γ is a sequence $[\![x_i]\!]_{i=0,\dots,n}$, $n \geqslant 0$, its length is the sum of the length of its straight subsegments, which is zero only if all its straight segments are of length zero. This implies that $x_i = x_{i+1}$, hence $x = y$. The symmetry of d_X comes from the fact that a shortest path from a to b is also a shortest path from b to a.

Finally let us show the triangle inequality. Let $\gamma \in \mathit{\Gamma}_X(x,y)$, $\gamma' \in \mathit{\Gamma}_X(y,z)$, $\gamma_2 \in \mathit{\Gamma}_X(x,z)$ be shortest paths (they exist since X is d-connected). Then if $\alpha := \mathrm{length}(\gamma) + \mathrm{length}(\gamma') < \mathrm{length}(\gamma_2)$, the path γ'' obtained by concatenating the paths γ and γ' has also a length $\alpha < \mathrm{length}(\gamma_2)$. Hence γ_2 cannot be a shortest path between x and y. So

$$\mathrm{length}(\gamma) + \mathrm{length}(\gamma') \geqslant \mathrm{length}(\gamma_2) \quad \Leftrightarrow \quad \mathrm{d}_X(x,y) + \mathrm{d}_X(y,z) \geqslant \mathrm{d}_X(x,z),$$

since these paths are shortest paths. □

Consequently tangency induces a sound notion of distance and shortest paths between elements of a digital set. Furthermore, shortest paths in fully convex sets are the usual straight lines as shown below.

Theorem 12 *If X is a non-empty fully convex set, then for any pair of points $x, y \in X$, $[\![x,y]\!]$ is a shortest path between x and y and $\mathrm{d}_X = \|x - y\|$.*

Proof By Theorem 2, the set X is d-connected. By Lemma 17, the set of shortest paths $\mathit{\Gamma}_X(x,y)$ is non empty. Now x and y are co-tangent in X. Indeed $\mathrm{cvxh}\,(\{x,y\}) \subset \mathrm{cvxh}\,(X)$. So $\bar{\mathscr{C}}^d[\mathrm{cvxh}\,(\{x,y\})] \subset \bar{\mathscr{C}}^d[\mathrm{cvxh}\,(X)]$. But X being fully convex, $\bar{\mathscr{C}}^d[\mathrm{cvxh}\,(X)] = \bar{\mathscr{C}}^d[X]$. We have just shown that x and y are co-tangent in X, so $[\![x,y]\!]$ is a valid path in X. Its length is exactly $\|y - x\|$, which is the length of the Euclidean shortest path between x and y. So no path can be shorter and $[\![x,y]\!] \in \mathit{\Gamma}_X(x,y)$. □

We thus recover the fact that shortest paths on any arithmetic plane are straight segments, and their distance is simply the Euclidean distance.

Algorithm 2 computes all the shortest paths to a given point $a \in X$ (and hence all the distances $\mathrm{d}_X(a, \cdot)$). It is a Dijkstra shortest path algorithm [19], where neighbors are computed on the fly, with a few specific optimizations.

Theorem 13 *Let* $X \subset \mathbb{Z}^d$ *such that* X *is d-connected. Then* ShortestPaths(X, a, A, D) *(Algorithm 2) correctly computes all the shortest paths from* $x \in X$ *to* a *(as the sequence* $[\![x, A[x], A[A[x]], \ldots]\!]$*), and with distance* $D[x]$.

Proof First assume that CotangentPoints(X, b, V, D) computes all the cotangent points to b in X (whatever V and D). Then Algorithm 2 is a Dijkstra shortest path algorithm [19], where the weighted graph has vertices X, edges that are all the cotangent pairs of points in X, and edge weights that are the Euclidean distance between cotangent points. The slight differences with a classical shortest path algorithm at line 1 and line 2 are just optimizations related to the priority queue: since the binary heap underneath authorizes duplicates, we only push into the queue routes that improve the distance (line 2) and popped routes are considered only if they correspond to the best route (line 1).

Now CotangentPoints(X, b, V, D) (Algorithm 3) computes in general only a subset of the cotangent points to b in X. Let us show that it returns the ones that may improve the best route. First CotangentPoints finds cotangent points to b by breadth-first traversal of the d-adjacency graph of X. Lemma 18 below shows that any path (b, x) has a d-connected path in its vicinity, so traversing the d-adjacency graph of X from b will indeed reach all the possible cotangent points to b.

However the function does not visit all neighbors. Line 1 discards all neighbors in the direction of the local start point b. This is legitimate since they have already been visited. Line 2 discards the points that belong to V and that belong to M. Points belonging to M have already been visited by the breadth-first traversal in CotangentPoints and are already in the output. Points belonging to V have already been visited in Dijkstra's shortest path loop, so all shortest routes passing through them have already been considered and no new route may originate from them.

Last line 3 considers if the point q neighbor of p offers a better direct route to a than the current point p could offer by going through point b. It is equivalent to the expression:

$$\begin{aligned} & D[q] + \|r - q\| \leqslant D[b] + \|p - b\| + \|r - p\| \\ \Leftarrow\ & D[q] + \|p - q\| \leqslant D[b] + \|p - b\| \qquad \text{(since } \|r - q\| \leqslant \|p - q\| + \|r - p\|\text{).} \end{aligned}$$

If this test is true, then a route going through b will always be longer so the point may safely be discarded. This concludes for the correctness of the algorithm. $\square$

Algorithm 2: ShortestPaths: given a digital set X and a point $a \in X$, computes the shortest paths from any point of X to a.

```
  Procedure ShortestPaths( In X, In a, Out A, Out D );
  In X : subset of ℤ^d ;                        // any non-empty subset of ℤ^d
  In a : Point ;                                // source point in X
  Out A : map<Point,Point> ;                    // ancestor in the shortest path
  Out D : map<Point,Real> ;                     // distance to a
  Type Node = tuple<Point,Point,Real>;
  Var V : set<Point> ;                          // visited points
  Var Q : priority_queue<Node> ;                // top is smallest distance
  begin
      foreach p ∈ X do D[p] ← +∞;
      Q.push((a, a, 0.0)) ;                     // starting point
      while ¬Q.empty() do
          (q, r, d) ← Q.pop(); ;    // pop top node (point, ancestor,distance)
1         if d > D[q] then continue;            // Not best route for q
          A[q] ← r ;          // set ancestor, distance is already updated
          V.insert(q) ;                         // point is now visited
          N ← CotangentPoints(X, q, V, D) ;
          foreach p ∈ N do
              d' ← D[q] + ‖p − q‖ ;
2             if d' < D[p] then
                  D[p] ← d' ;
                  Q.push((p, q, d')) ;
```

Figure 7 illustrates some results of geodesic computations. To give an idea of computation times, geodesics onto bunny datasets (15028 points) takes about 38s on a macbook pro (2.7 GHz, Intel Core i7, 16 Gb).

The algorithm is thus quite slow as it is written. This is because it computes almost for each point all its cotangent points. It could certainly be optimized. For instance, the test at line 3 in Algorithm 3 is very conservative. Replacing the expression $D[b] + \|p - b\| > D[q] + \|p - q\|$ by the expression $D[b] + \|p - b\| > D[q] + K$, where K is a user-chosen constant less than $\sqrt{d}$ considerably improves the speed of the algorithm, with almost no change in the output. Note that choosing $K = \sqrt{d}$ guarantees the correct output (since $\|p - q\| \leqslant \sqrt{d}$), but smaller values speed up the algorithm (up to a factor 86) with almost the same result, as shown by computations summed up in Table 1, which computes the shortest paths onto a 3D sphere of radius 1, digitized at gridstep h.

To conclude the section, we give below the lemma and corollary that tells us that any path can be approximated by a digital d-connected path, properties that are used for establishing the correctness of our shortest path algorithm.

Lemma 18 *If $[\![a, b]\!]$ is a path in X, then there exists a d-connected path of pixels $P = [\![y_i]\!]_{i=0,\ldots,m}$ in X, $y_0 = a$, $y_m = b$, such that it stays close to the straight segment $[a, b]$, i.e. $P \subset \mathrm{Cl}\left(\|\bar{\mathscr{C}}^d[\mathrm{cvxh}\,(\{a, b\})]\|\right)$.*

Proof $[\![a, b]\!]$ being a path in X, a and b are thus cotangent in X, and $\bar{\mathscr{C}}^d[\mathrm{cvxh}\,(\{a, b\})] \subset \bar{\mathscr{C}}^d[X]$ (Definition 4). Now the straight segment $[a, b] :=$

Algorithm 3: CotangentPoints: given a digital set X and a point $b \in X$, computes the cotangent point to x that are not in set V; uses distance D to prune the traversal.

```
Function CotangentPoints( In X, In b, In V, In D ) : vector<Point>;
In X : subset of Z^d ;                              // any non-empty subset of Z^d
In b : Point ;                                                   // a point in X
In V : set<Point> ;                               // the set of points to discard
In D : map<Point,Real> ;                      // a map giving its distance to b
Var R : vector<Point> ;                   // the non-visited cotangent points to b
Var Q : queue<Point> ;             // the queue for breadth-first traversal from b
Var M : set<Point> ;                 // the set of visited points or in the queue
begin
    Q.push(b) ; M.insert(b) ;                                   // starting point
    while ¬Q.empty() do
        p ← Q.pop() ;                                          // pop top of queue
        if {b, p} are cotangent in X then
            if b ≠ p then R.push_back(p);
            foreach q ∈ X_1^*(p) do                     // for each neighbor of p
1               if (b − p) · (q − p) < 0 then continue;
2               if q ∈ V or q ∈ M then continue;
3               if D[b] + ‖p − b‖ > D[q] + ‖p − q‖ then
                    continue ;                      // do not visit closest points
                Q.push(q) ; M.insert(q) ;                     // add point to queue
```

Table 1 Computation times of shortest paths to a given source and maximal error onto a 3D sphere digitized with gridstep h. The chosen K indicates the value used to replace $\|p-q\|$ in the expression $D[b] + \|p-b\| > D[q] + \|p-q\|$, at line 3 of Algorithm 3. Choosing $K = \|p-q\|$ or $K = \sqrt{3}$ guarantees the correctness of the output. However, decreasing K to 0 speeds up the algorithm, while the maximal relative error in the distance estimation stays very low.

		chosen K: time in s (error in %)			
gridstep h	$\#(X)$	$\|p-q\|$	$\sqrt{3}/4$	$\sqrt{3}/16$	0
0.25	296	0.13	0.05 (0.000%)	0.02 (0.000%)	0.01 (0.000%)
0.125	1184	1.08	0.33 (0.000%)	0.11 (0.000%)	0.04 (0.226%)
0.0625	4784	8.40	2.84 (0.000%)	0.74 (0.000%)	0.23 (0.538%)
0.03125	19256	69.67	28.83 (0.000%)	6.40 (0.005%)	1.27 (0.205%)
0.015625	77120	579.49	258.48 (0.000%)	52.82 (0.037%)	6.70 (0.292%)

$\mathrm{cvxh}(\{a,b\})$ traverses in sequence m cells of $\bar{\mathscr{C}}^d[\mathrm{cvxh}(\{a,b\})]$, which are thus also in $\bar{\mathscr{C}}^d[X]$. We denote these cells by $(c_i)_{i=0,\ldots,m}$, and we have $c_0 = a$ and $c_m = b$. Each cell c_i has at least an X-corner y_i (Lemma 7), with $y_0 = a$ and $y_m = b$. Since each $y_i \in \mathrm{Cl}(c_i)$, we have $P \subset \mathrm{Cl}\left(\left\|\bar{\mathscr{C}}^d[\mathrm{cvxh}(\{a,b\})]\right\|\right)$. □

Corollary 3 *If γ is a path in X between a and b, then there exists a d-connected path P of points in X between a and b such that $P \subset \mathrm{Cl}\left(\left\|\bar{\mathscr{C}}^d[\overline{\gamma}]\right\|\right)$ (it stays* close *to γ), and P visits the digital points of γ in the same order.*

Among all d-connected path that stays close to γ, the shortest ones are said to be *d-approximating* γ. It is straightforward to see that a d-approximating

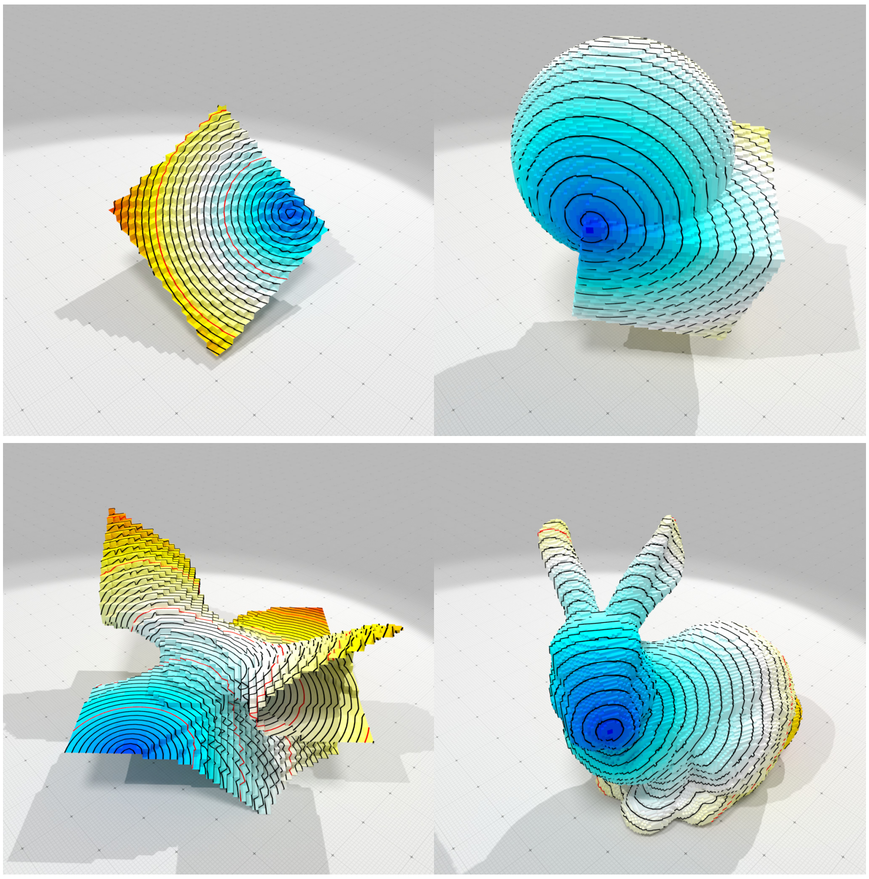


Fig. 7 Illustration of geodesics onto digital sets (source in deep blue).

path (y_i) to a geodesic cannot have three consecutive points such that y_{i-1} is d-adjacent to y_{i+1}. This could be used to optimize slightly the preceding algorithm, at the price of memorizing the direct ancestor.

7 Conclusion and perspectives

We have proposed an original definition for digital convexity in arbitrary dimension, called full convexity, which possesses topological and geometric properties that are more akin to continuous convexity. We exhibited an algorithm to check full convexity, which relies on standard algorithms. We illustrated the potential of full convexity for addressing classical discrete geometry problems like building a tangential cover, analyzing the local shape geometry, reconstructing a reversible first-order polygonal surface approximation, or computing geodesics on digital sets. We believe that full convexity opens the path to d-dimensional digital shape geometry analysis. Note that full convexity is available in the open-source library DGtal [49], module digital convexity.

This work opens many perspectives. On a fundamental level, we work on a variant of full convexity that keeps the intersection property of continuous convexity. Advances along this path would induce natural definitions of digital convex hulls.

Another fundamental question is the number of cells intersected by dilations t of a polytope. Our morphological characterization implies that it is a degree d polynomial in t like the number of lattice points intersected by dilations of a polytope (see Ehrhart theory [22]). We can wonder if fully convex polytopes induce specific polynomials, and what are the relations between polynomials for different cell dimensions.

On a more algorithmic level, we also wish to improve the convexity check algorithm, especially in 3D, for instance when $\mathrm{cvxh}\,(X)$ has few facets. Dedicated enumerating lattice points algorithms could also be explored.

We also wish to explore the properties of a tangential cover made of the maximal fully convex tangent subsets that are included in some arithmetic plane. In order to restrict their number, plane probing algorithms [37,38,39, 44] could provide significant points (like local upper or lower leaning points).

Finally, tangency also opens up a theory of digital tangent vector fields and shortest paths, being geodesics, might provide parallel transport of vector fields. This would give another approach to defining covariant derivatives, and perhaps a digital calculus.

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