

# As-rigid-as-possible modelling

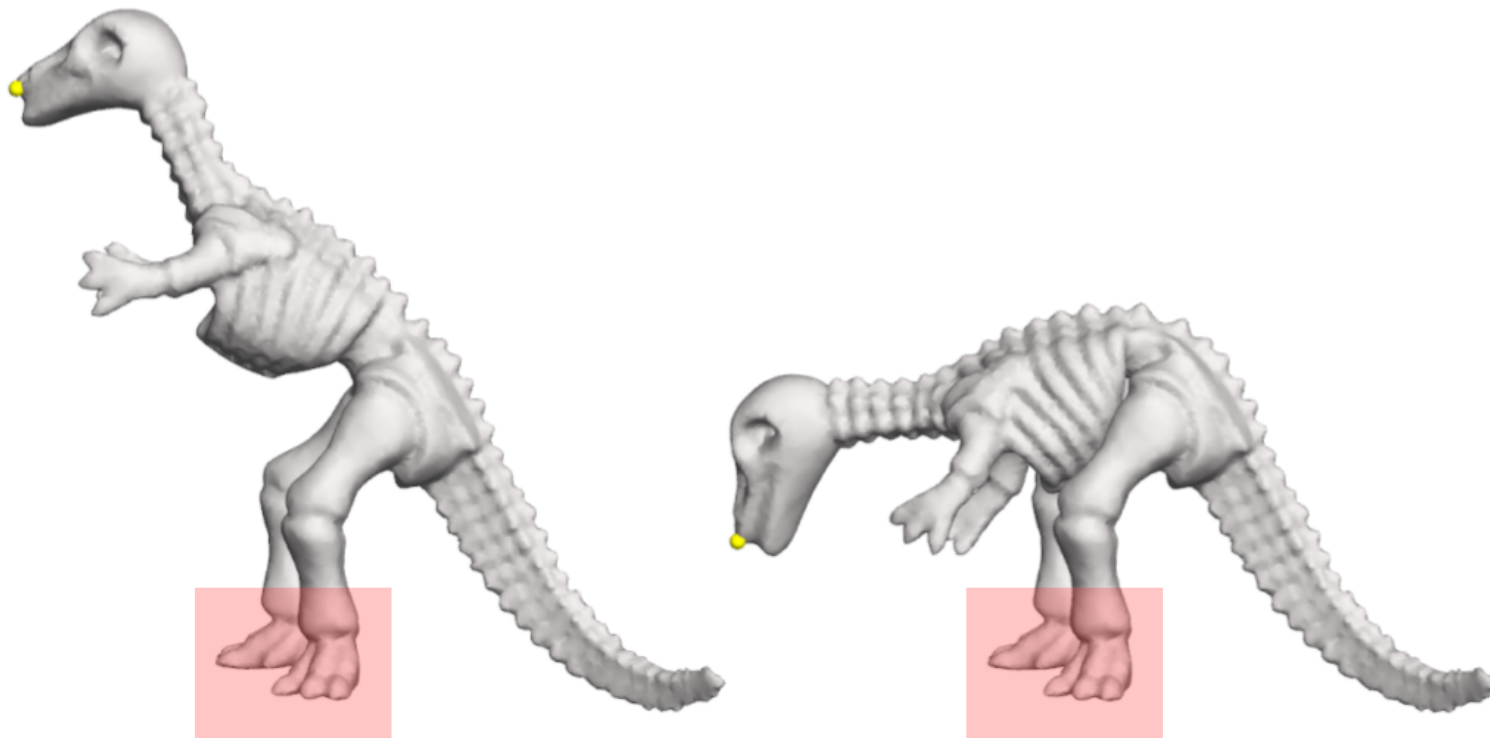
# Why ARAP for today's lesson ?

- State of the art for shape direct manipulation
- Will allow me to show you how to setup a linear system (basics of numerical optimization)
- Will present the Procrustes problem (basic problem in geometry)

**[Sorkine & Alexa]** : As-Rigid-As-Possible Surface Modeling

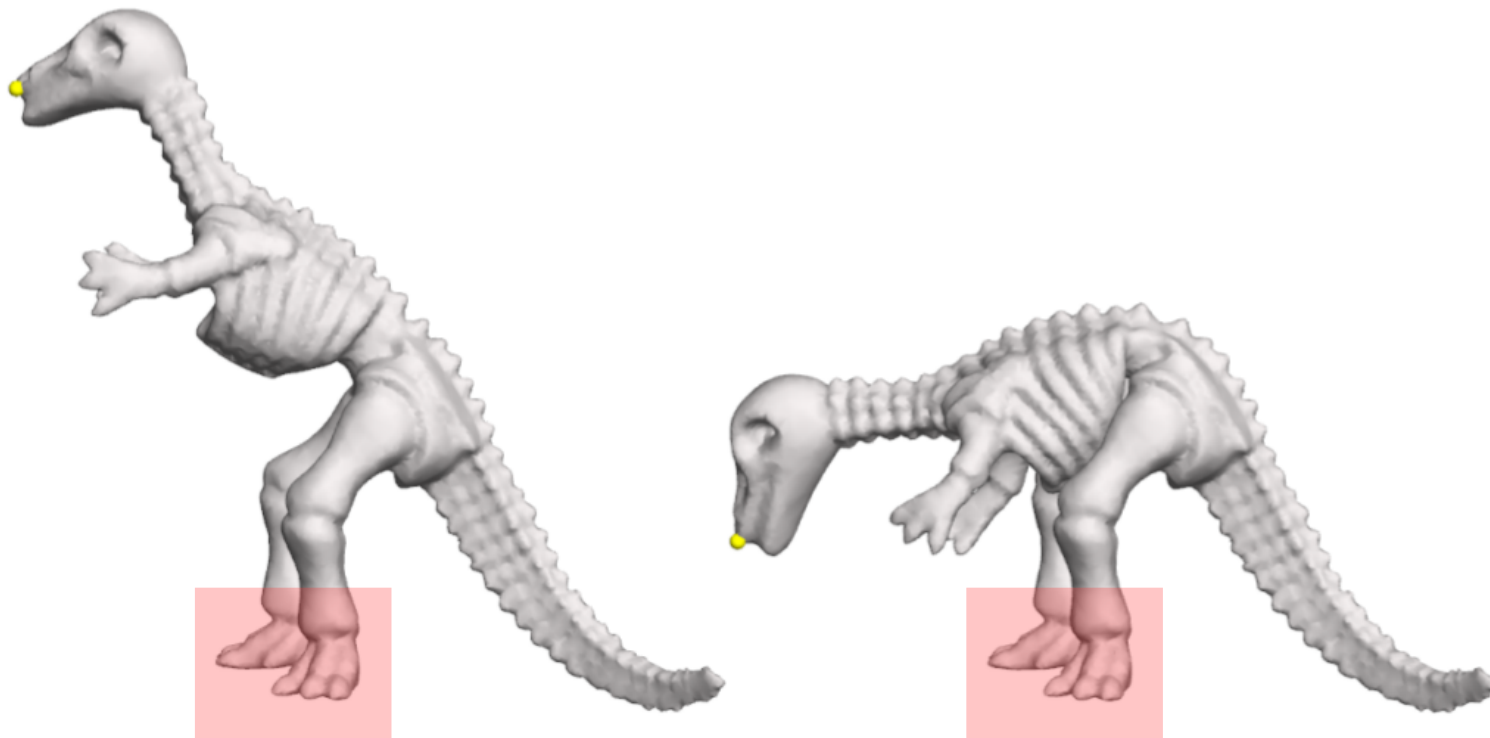
# Principle

- A few vertex positions are specified by the user
- The other vertices should be placed in « a natural fashion »



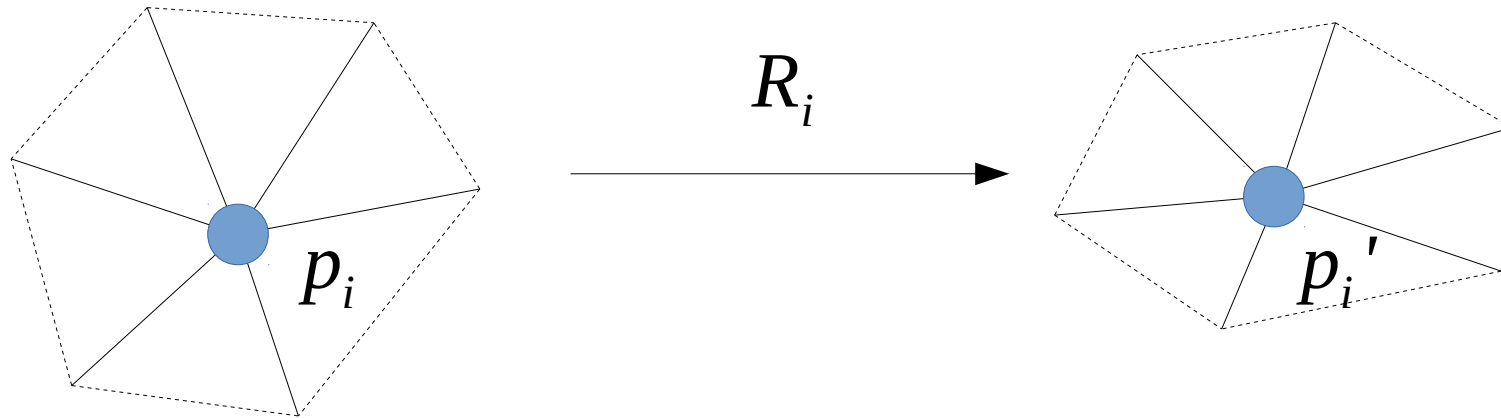
# Minimizing stretch (enforcing rigidity)

- Physically « plausible »
- Impact on textures, stretch, volume



# What is rigidity ?

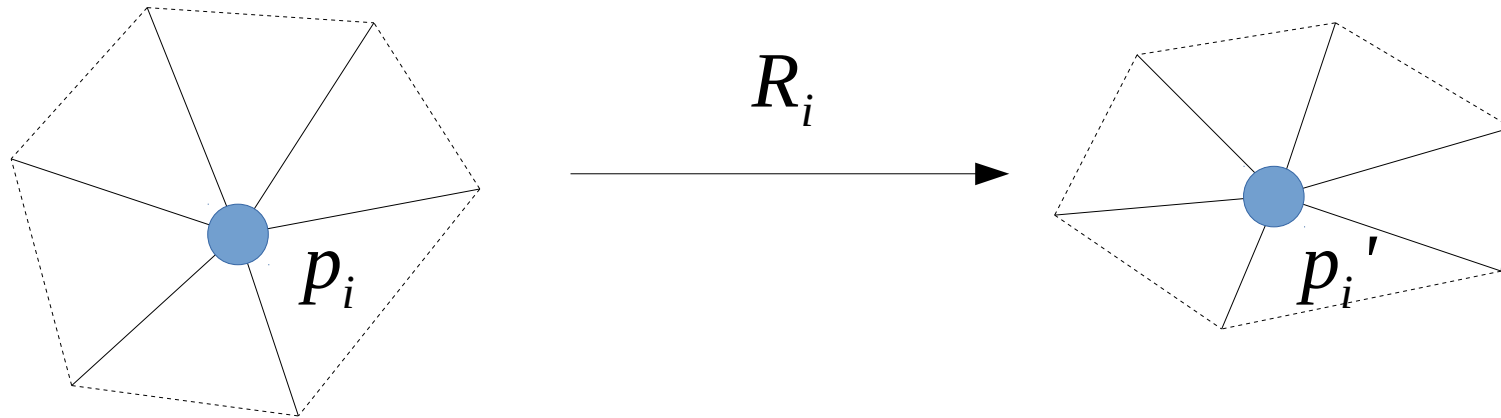
- A small piece of the shape should be globally rotated :



$$\mathbf{p}'_i - \mathbf{p}'_j = \mathbf{R}_i (\mathbf{p}_i - \mathbf{p}_j) , \quad \forall j \in \mathcal{N}(i)$$

# Rigidity energy

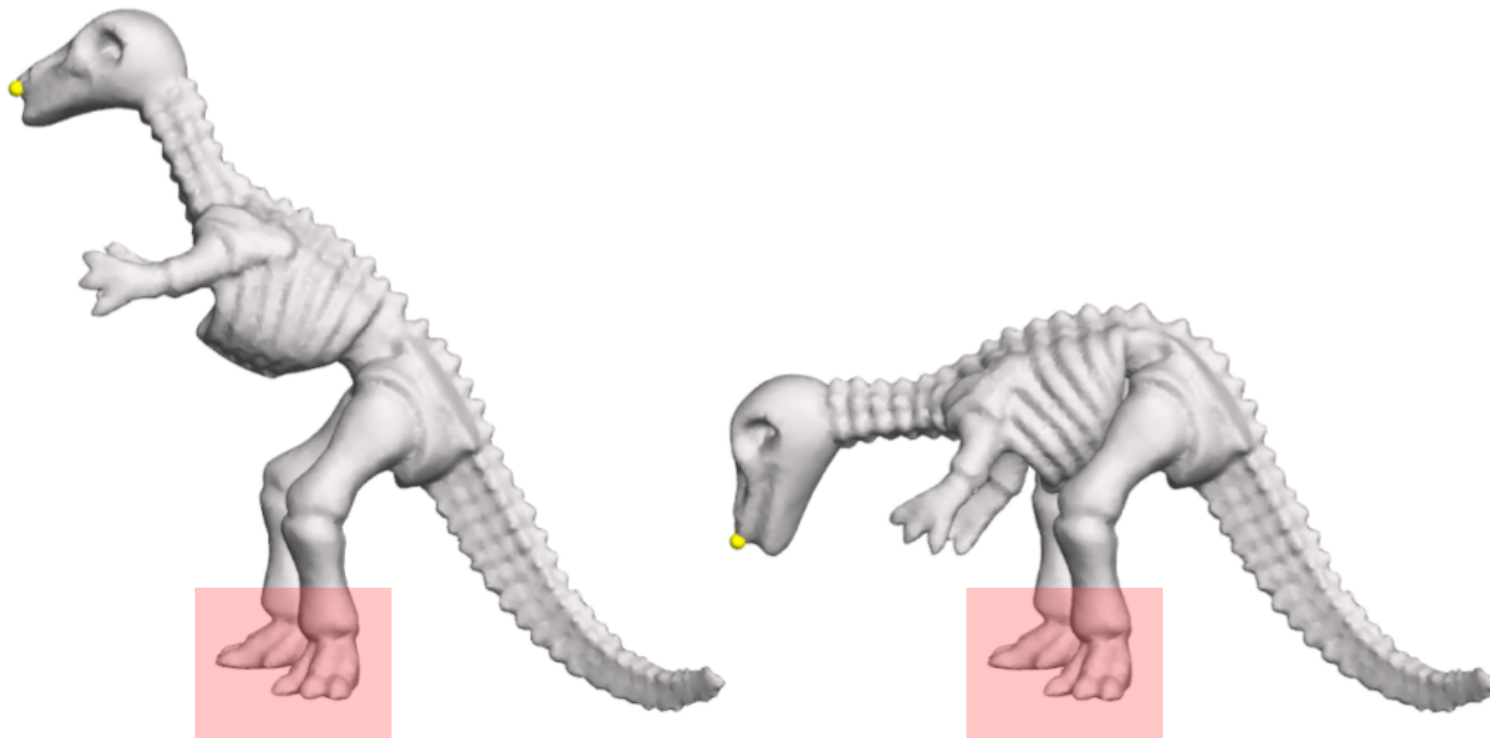
- A small piece of the shape should be globally rotated :



$$E(C_i, C_i') = \sum_{j \in \mathcal{N}(i)} w_{ij} \left\| (\mathbf{p}_i' - \mathbf{p}_j') - \mathbf{R}_i(\mathbf{p}_i - \mathbf{p}_j) \right\|^2$$

# ARAP framework

- Unknowns : New positions  $p'$  AND rotations  $R$
- Constraints : A few specified positions



# Problem

- Unknowns : New positions  $\mathbf{p}'$  AND rotations  $\mathbf{R}$
- Highly NON-LINEAR and NON-CONVEX
- Minimizing  $\mathbf{R}$  and  $\mathbf{p}'$  at the same time is not feasible
- This is the rigidity energy alone, don't forget to add the handle energies !

$$E(\mathcal{S}') = \sum_{i=1}^n w_i \sum_{j \in \mathcal{N}(i)} w_{ij} \left\| (\mathbf{p}'_i - \mathbf{p}'_j) - \mathbf{R}_i(\mathbf{p}_i - \mathbf{p}_j) \right\|^2$$



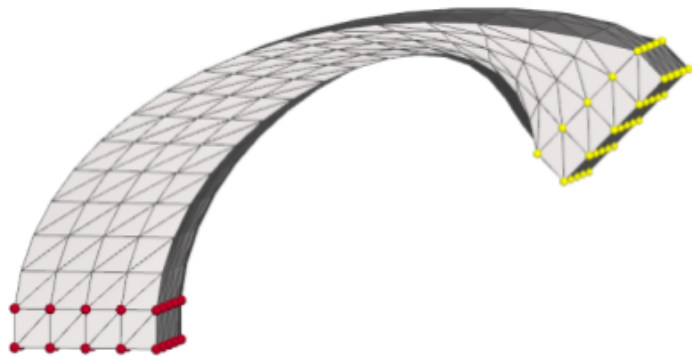
# Ad hoc solution

- Fix  $R$ , optimize  $p'$
- Fix  $p'$ , optimize  $R$
- Fix  $R$ , optimize  $p'$
- ...

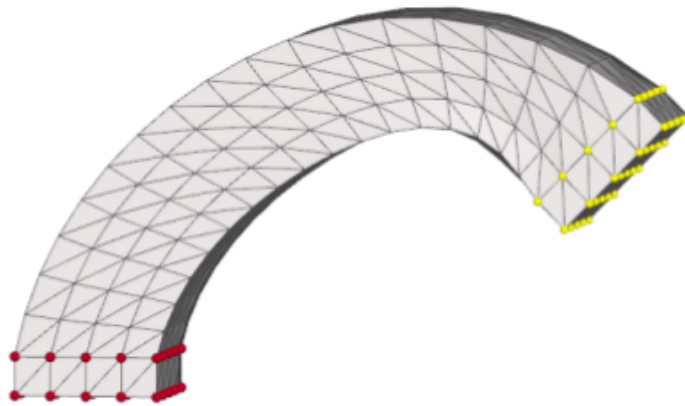
$$E(\mathcal{S}') = \sum_{i=1}^n w_i \sum_{j \in \mathcal{N}(i)} w_{ij} \left\| (\mathbf{p}'_i - \mathbf{p}'_j) - \mathbf{R}_i(\mathbf{p}_i - \mathbf{p}_j) \right\|^2$$

# Ad hoc solution

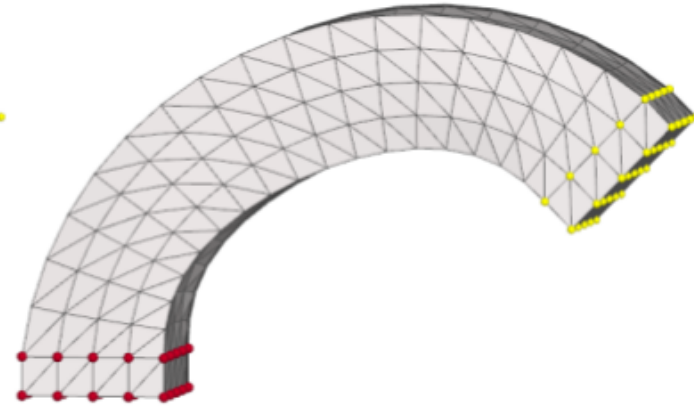
- Fix  $R$ , optimize  $p'$
- Fix  $p'$ , optimize  $R$
- Fix  $R$ , optimize  $p'$
- ...



initial guess



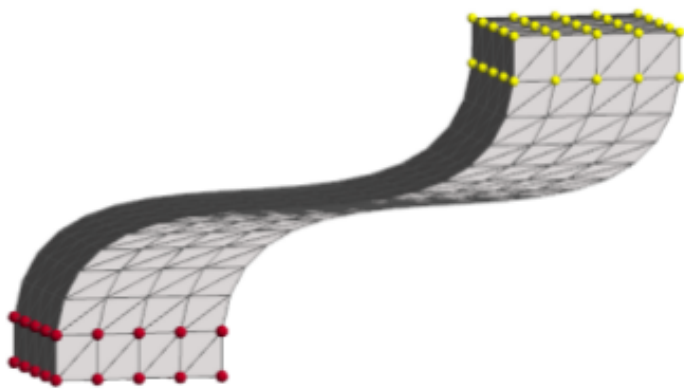
1 iteration



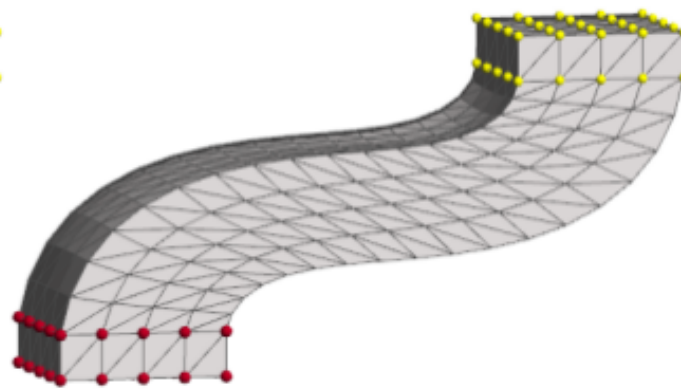
4 iterations

# Ad hoc solution

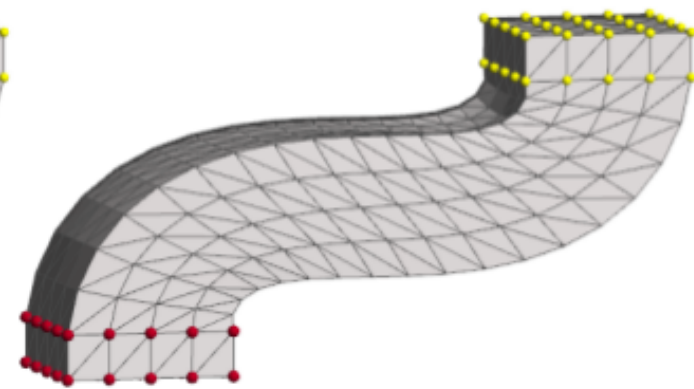
- Fix  $R$ , optimize  $p'$
- Fix  $p'$ , optimize  $R$
- Fix  $R$ , optimize  $p'$
- ...



initial guess



1 iteration



2 iterations

# 1) Fix $\mathbf{R}$ , optimize $\mathbf{p}'$

- Linear system with  $\mathbf{p}'$  as unknowns
- Example on black board
- C++/Matlab : Cholmod, Eigen, ...
- Recall that ! Solving a linear system is the ABC of numerical optimization !

$$E(\mathcal{S}') = \sum_{i=1}^n w_i \sum_{j \in \mathcal{N}(i)} w_{ij} \left\| (\mathbf{p}'_i - \mathbf{p}'_j) - \mathbf{R}_i(\mathbf{p}_i - \mathbf{p}_j) \right\|^2$$

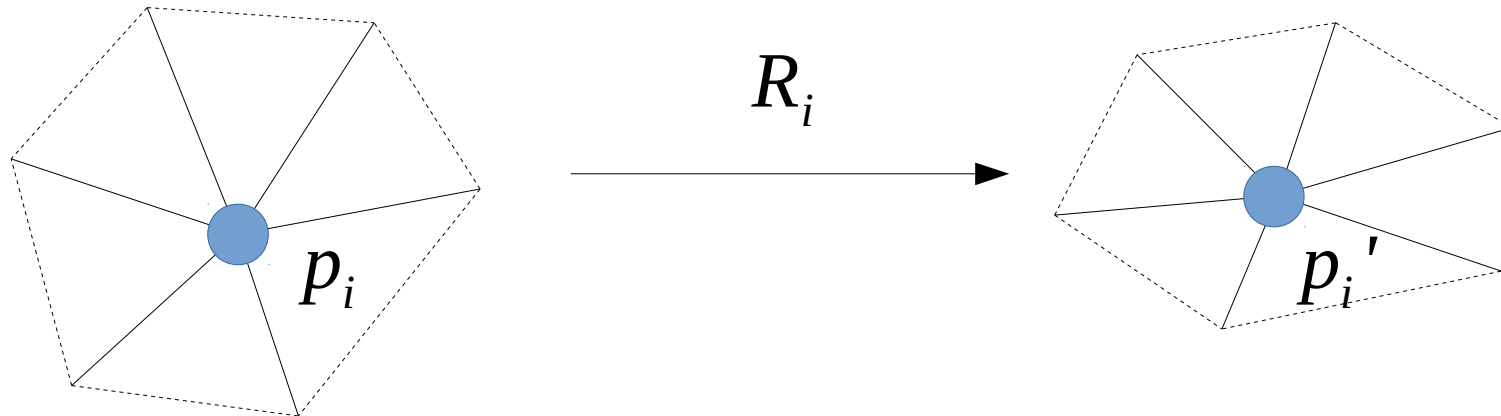
## 2) Fix $\mathbf{p}'$ , optimize $R$

- Can be done per vertex  $i$

$$E(\mathcal{S}') = \sum_{i=1}^n w_i \sum_{j \in \mathcal{N}(i)} w_{ij} \left\| (\mathbf{p}'_i - \mathbf{p}'_j) - \mathbf{R}_i(\mathbf{p}_i - \mathbf{p}_j) \right\|^2$$

## 2) Fix $p'$ , optimize $R$

- Can be done per vertex  $i$
- Minimize  $E(C_i, C_i')$



$$E(C_i, C_i') = \sum_{j \in \mathcal{N}(i)} w_{ij} \left\| (\mathbf{p}'_i - \mathbf{p}'_j) - \mathbf{R}_i(\mathbf{p}_i - \mathbf{p}_j) \right\|^2$$

## 2) Fix $p'$ , optimize $R$

$$\sum_j w_j \|e_j' - R.e_j\|^2 \longrightarrow \text{To minimize}$$

$$\sum_j w_j \|e_j' - R.e_j\|^2 = \sum_j w_j \|e_j'\|^2 + \sum_j w_j \|R.e_j\|^2 - 2 \sum_j w_j (R.e_j)^T . e_j'$$

$$\sum_j w_j \|e_j' - R.e_j\|^2 = \text{const} - 2 \sum_j w_j \text{Trace}(R.e_j.e_j'^T)$$

$$\text{Trace}(R.\sum_j w_j e_j.e_j'^T) \longrightarrow \text{To maximize}$$

1) Build  $S := \sum_j w_j e_j'.e_j^T$

2) Compute SVD :  $S := U . \Sigma . V^T$

3) Solution :  $R := U . \text{diag}(1, 1, \det(U.V^T)) . V^T$

# At this point

- You know how to setup a linear system
- You know how to solve the Procrustes problem
- You can implement ARAP in c++ with a few lines of code (something like 20 in Matlab)



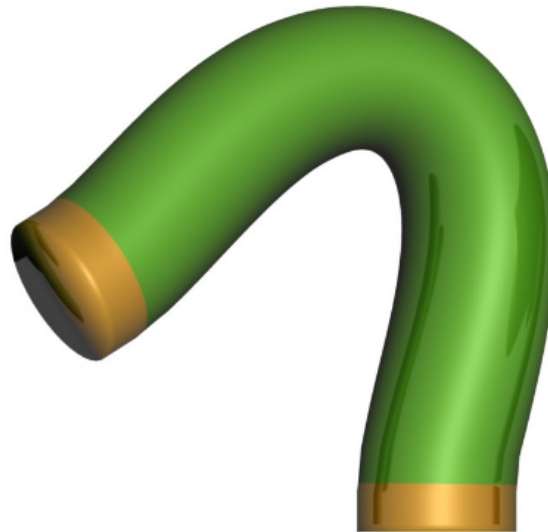
# Different flavours of ARAP



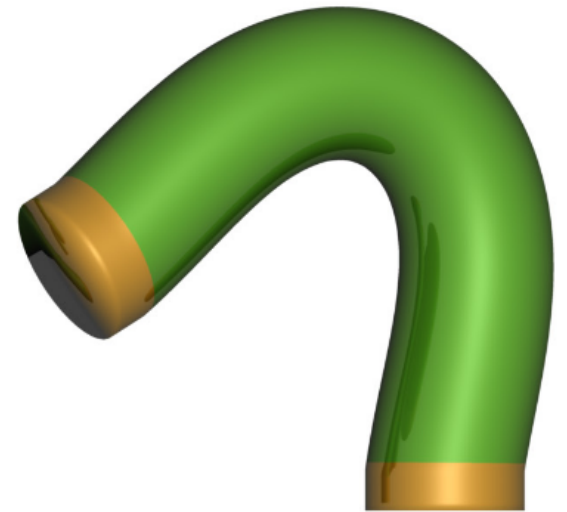
Surface ARAP



Volume ARAP



Surface ARAP, with  
« smooth » rotations



**[Levi & Gotsman]** : Smooth Rotation Enhanced As-Rigid-As-Possible Mesh Animation