ACPR 19 Tutorial

Digital Geometry in Pattern Recognition: Extracting Geometric Features with DGtal and Applications – Part II –

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5th Asian Conference on Pattern Recognition
26 November, 2019, Auckland, New Zealand

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Overview of the presentation - Part II -

1. Digital Surfaces

   Digital surface tracking and algebraic topology

   Duality with isosurfaces

2. Discrete Exterior Calculus

   Foundations of discrete calculus

   Discrete calculus model of Ambrosio-Tortorelli functional

   Some applications of AT functional

3. Reproducible Research

4. Practical session: Hands on the IPOL Demonstration System

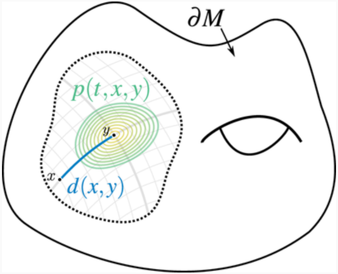
https://kerautret.github.io/ACPR19-DGPRTutorial

1. Digital Surfaces

Digital Surfaces

Sound definition for digital surfaces

- boundary of volumes must be digital surfaces
- closed connected digital surfaces have a connected interior
- valid in arbitrary dimension
- open digital surfaces are orientable
- analagous to $d-1$-manifold in $\mathbb{R}^d$: locally $d-1$ directions

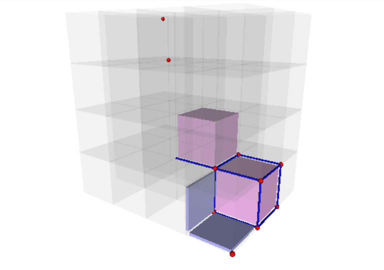


Digital Surfaces

Sound definition for digital surfaces

Digital surfaces require cell/interpixel topology

- volume elements (voxel) are d-cells
- surface elements (surfel) are $d-1$-cells



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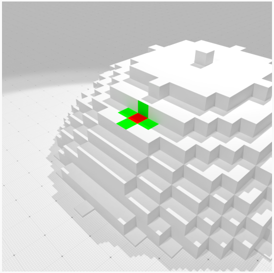
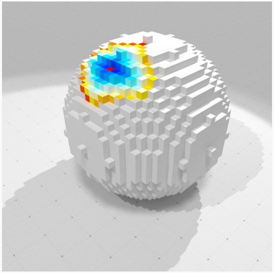
- surfels have adjacent surfels; surfels + adjacencies form a graph





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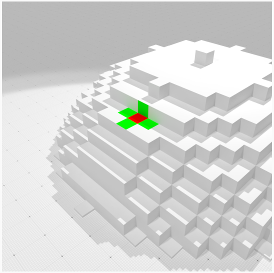

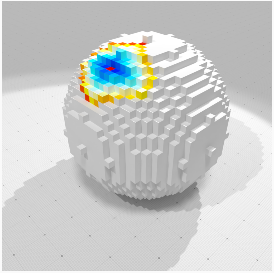

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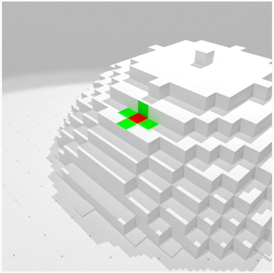
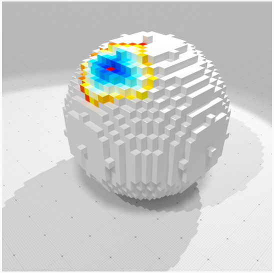
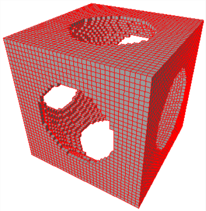
- surfel adjacency graph depends on chosen connectednesses for interior/exterior.





Topology on digital surfaces ? i



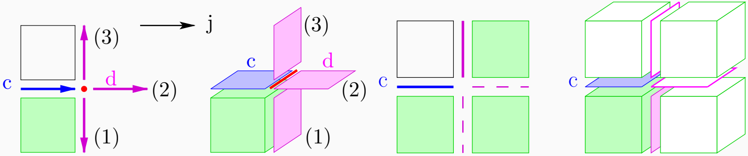
- For now, a surface is a set of surfels

Questions ?

Can we define local neighborhood relations so that

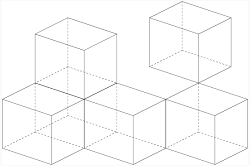
- a whole connected surface can be extracted by their tracking,
- Jordan separation property is satisfied

Bel adjacency in a picture i

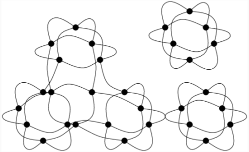


- shape / binary picture I: finite subset X of $\mathbb{Z}^n$
- boundary element or *bel* in I = surfel between X and X^c
- For each direction j ($n-1$ directions for each bel)
  - interior bel-adjacency from c along j = first follower of c which is a bel
  - exterior bel-adjacency from c along j = last follower of c which is a bel

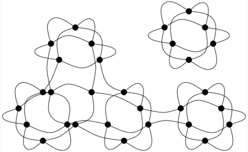
Bel adjacency graph i



bels



interior graph



exterior graph

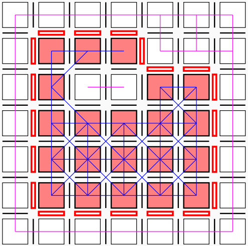
- For each direction, choose interior/exterior $\Rightarrow 2^{\frac{n(n-1)}{2}}$ bel-adjacencies

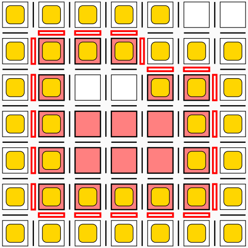
Theorem (3D [Herman,Webster83])

Let $O \subset X$ 6-connected, $Q \subset X^c$ 18-connected. c any bel.

The all-interior bel-adjacency graph component containing c is the boundary surface between O and Q.

Bel adjacency graph ii





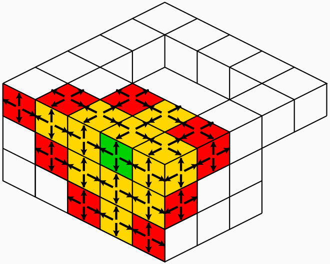
Theorem (*n*D, $n \geq 2$, [Udupa94])

Let $O \subset X$ $2n$-connected, $Q \subset X^c$ $2n^2$-connected. c a bel.

The all-interior bel-adjacency graph component containing c is the boundary surface between O and Q.

- To extract a boundary component $\Rightarrow$ track it.

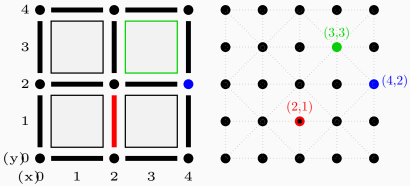
Tracking digital boundaries i



- boundary in parallepiped N^n
- number of bels is $V = O(N^{n-1})$
- degree of each vertex is $2n-2$

- breadth-first traversal of bel-adjacency graph
- each bel is visited $2n-2$ times
- time complexity $\approx (2n-2)V$

Cubical chain complex i

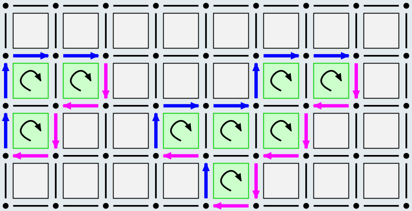


Classical trick for representing cell space

- isomorphism "grid" and "Khalimsky's space"
- cells can be represented by integer points
- a cell is an element of $\mathbb{Z}^n$, parities = topology
- pixels, voxels, n-cells have odd parities

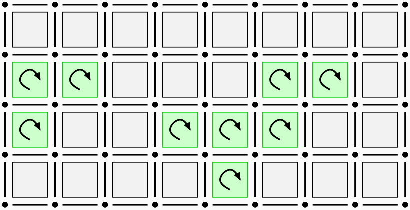
Cubical chain complex ii

Construction of a chain complex



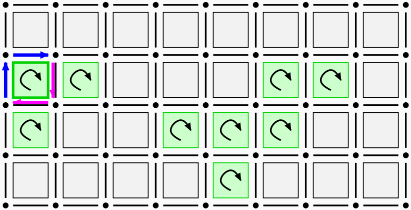
- oriented k-cells form k-dimensional bases
- k-chains are formal sums of k-cells (coefficient $\mathbb{Z}$)
  $[\sum_i +o_i^n]$ is a digital shape
  $\sum[+s_j^{n-1}] + \sum[-s_{j'}^{n-1}]$ is a digital surface
- boundary operator Δ, with $\Delta\Delta = 0$, based on cell parities

Application to digital boundaries



- digital shape is a subset X of $\mathbb{Z}^n$ (odd parities)
- its boundary $= n - 1$-chain $\Delta \sum_{x \in X} +x$
- it is a cycle since $\Delta\Delta = 0$

Application to digital boundaries



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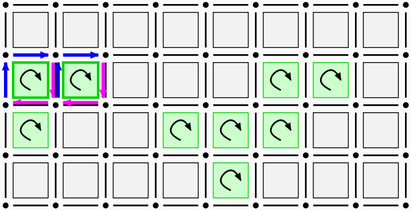

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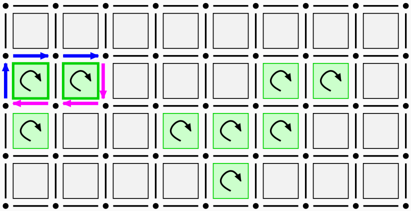

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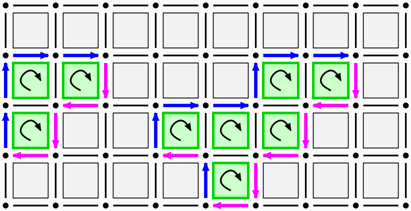

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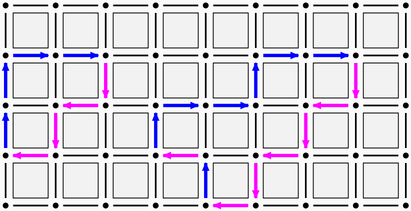

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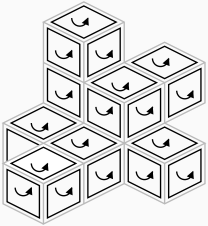
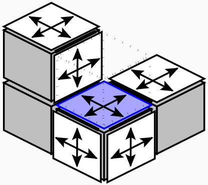
Theorem

The boundary of a volume is a closed digital surface

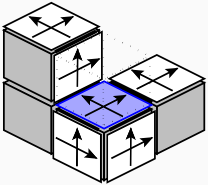
Oriented boundary tracking i



ΔX

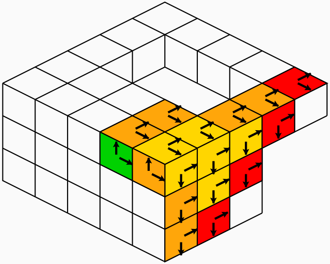


neighbors

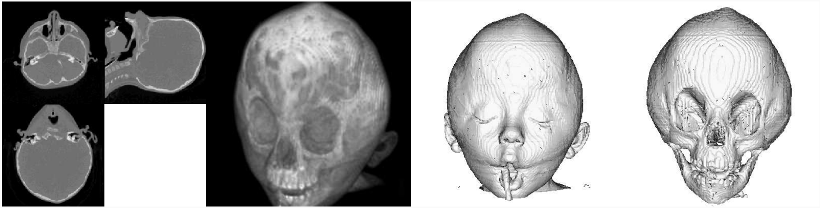


"positive" neighbors

- since $\Delta\Delta X = 0$
- breadth-first traversal of directed bel-adjacency graph
- each bel is visited $\frac{2n-2}{2}$ times
- time complexity $\approx (n-1)V$



Isosurfaces i



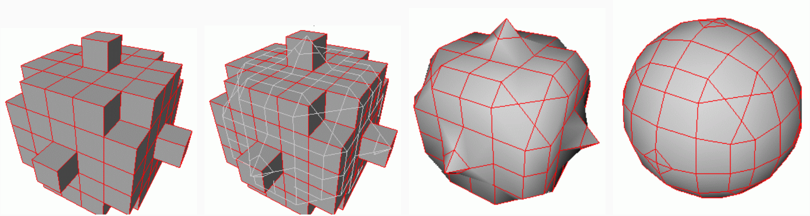
Definition (Isosurface)

Let $I : \mathbb{R}^3 \to \mathbb{R}$.

Isosurface of value s in $I = \{(x, y, z) \in \mathbb{R}^3, I(x, y, z) = s\}$.

- *marching-cubes* [Lorensen,Cline87], by scanning

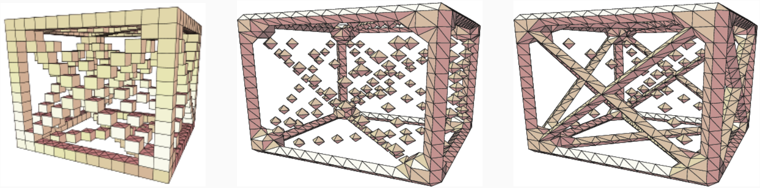
Duality isosurfaces / digital surface



$$X = \{\vec{x} \in \mathbb{Z}^3, I(\vec{x}) \geq s\}$$

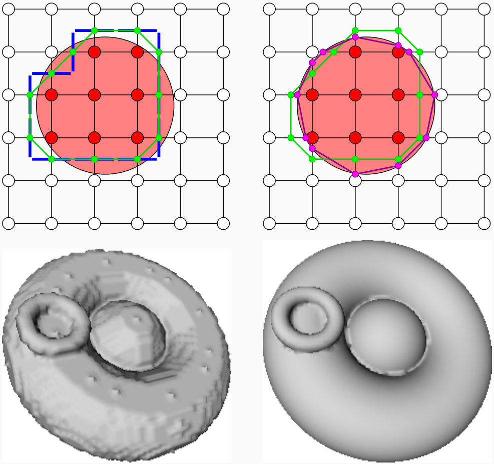
Theorem [L. Montanvert 2000]

bel-adjacency complex defines a $n-1$-pseudomanifold without boundary



shape interior exterior

Making isosurfaces nice



1. $X = \{\vec{x} \in \mathbb{Z}^3, I(\vec{x}) \geq s\}$
2. track $\Delta \sum_{x \in X} +x$
3. local triangulation
4. move vertices

Digital surfaces in DGtal

Generic representation of digital surfaces

- One interface `DigitalSurface`, many implementations
  - `DigitalSetBoundary` implicit boundary of volumes
  - `SetOfSurfels` explicit set of surfels
  - `LightImplicitDigitalSurface` implicit digital surface from interior/exterior predicate
  - `ImplicitDigitalSurface` same as above with boundary precomputation
  - `LightExplicitDigitalSurface` explicit digital surface from surfel predicate
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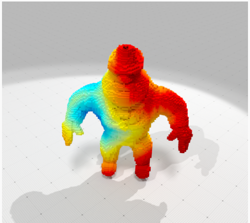
- everything generic, arbitrary dimensional
- In 3D, one interface `IndexedDigitalSurface`, same many implementations
  - all cells are numbered from 0,
  - based on half-edge data structure,
  - faster if process requires several pass, many neighborhood comput.

Digital surfaces in DGtal

Usual usages: use Shortcuts module

```
#include "DGtal/helpers/Shortcuts.h"
...
typedef Shortcuts<Z3i::KSpace> SH3;
auto params = SH3::defaultParameters();
params("surfaceTraversal", "BreadthFirst") // specifies breadth-first traversal
      ("colormap", "Jet"); // specifies the colormap
auto vol     = SH3::makeBinaryImage( "samples/Al.100.vol", params ); //load vol file
auto K       = SH3::getKSpace( vol );                            // define grid cell space
auto surface = SH3::makeLightDigitalSurface( vol, K, params );//define digital surface
auto surfels = SH3::getSurfelRange( surface, params );          //track surfels with BFT
auto cmap    = SH3::getColorMap( 0, surfels.size(), params );//paint surfels from rank
SH3::Colors colors( surfels.size() );
for ( int i = 0; i < surfels.size(); ++i ) colors[ i ] = cmap( i );
bool ok  = SH3::saveOBJ( surface, SH3::RealVectors(), colors, "al-primal-bft.obj");
```



Digital surfaces in DGtal

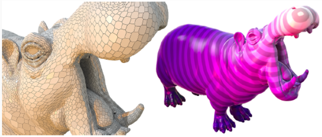
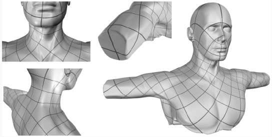
Usual usages: use Shortcuts module

- image $\rightarrow$ digital surface / indexed digital surface
- extraction of connected components / main component
- polynomial shape $\rightarrow$ digital surface / indexed surface
- digital surface $\leftrightarrow$ indexed digital surface
- (indexed) digital surface $\rightarrow$ dual triangulated surface (MC)
- (indexed) digital surface $\rightarrow$ primal polygonal surface
- (indexed) digital surface $\rightarrow$ dual polygonal surface
- traversals/visitors: breadth-first, depth-first
- many export to OBJ
- many geometric estimation: area, normals, curvatures

2. Discrete Exterior Calculus

II.2 Discrete (exterior) Calculus

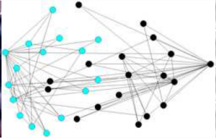
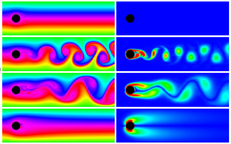
Computer graphics, geometry processing, shape optimization

(Images: Knöppel et al. 2015, Crane et al. 2013, Springborn et al. 2010)

Discrete exterior calculus [Desbrun, Hirani, Leok, . . .]
Discrete differential calculus [Polthier, Pinkall, Bobenko, . . .]
Discrete calculus [Grady, Polimeni, . . .]

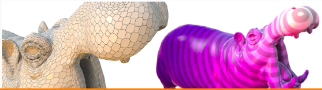
Graph and network analysis, image processing, fluid simul.

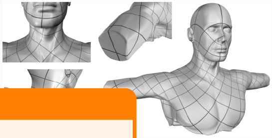




(Images: Bugeau et al. 2014, couprie et al. 2014, Elcott et al. 2006)

II.2 Discrete (exterior) Calculus

Computer graphics, geometry processing, shape optimization

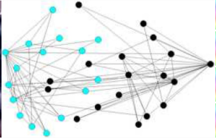


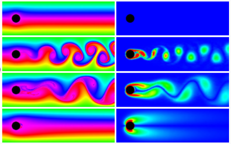


Discrete exterior calculus (DEC)

- no discretization, discrete by nature
- keep algebraic properties of calculus, exact Stokes' theorem
- reduces to matrix/vectors
- works without embedding, just metric
- "any" cell complex, arbitrary dimension

Graph and network analysis, image processing, fluid simul.





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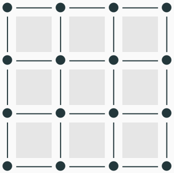
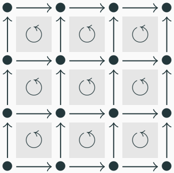
Cell complex, chains, boundary, forms



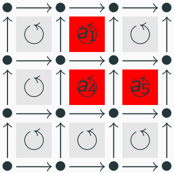
- *cell complex* K: vertices, edges, faces (pixels/surfels)

Cell complex, chains, boundary, forms



- *cell complex K*: vertices, edges, faces (pixels/surfels) with orientation

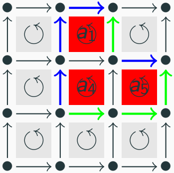
Cell complex, chains, boundary, forms



$$\sigma := a_1 + a_4 + a_5 \in C_2(K)$$

- *cell complex* K: vertices, edges, faces (pixels/surfels) with orientation
- *k-chains*: $C_k(K)$ are integral formal sums of oriented cells

Cell complex, chains, boundary, forms



$$\sigma := a_1 + a_4 + a_5 \in C_2(K)$$

$$\eta := \sum e_i - \sum e_j \in C_1(K)$$

$$\eta = \partial_2 \sigma$$

- *cell complex* K: vertices, edges, faces (pixels/surfels) with orientation
- *k-chains*: $C_k(K)$ are integral formal sums of oriented cells
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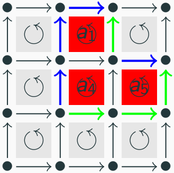
Cell complex, chains, boundary, forms



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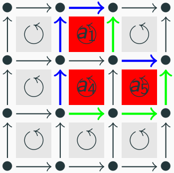

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- *Integral* $\int_\sigma \alpha$ = pairing k-form α with k-chain σ

$$\int_\sigma \alpha := \alpha(\sigma) = \sum_i a_i \alpha(c_i) \quad \text{if } \sigma = \sum_i a_i c_i$$

Exterior derivative, Stokes theorem

- *exterior derivative* defined by duality: $\mathbf{d}_k : C^k(K) \to C^{k+1}(K)$

$$(\mathbf{d}_k \alpha^k)(\sigma_{k+1}) := \alpha^k(\partial_{k+1}\sigma_{k+1})$$

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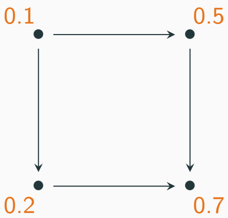

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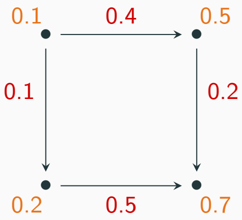

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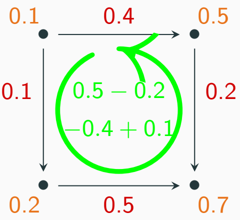

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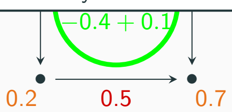
thu

(discrete) Stokes theorem is trivial by definition

$$\int_\sigma \mathbf{d}\alpha = \int_{\partial\sigma} \alpha$$

for σ any k-chain and α any $k-1$-form



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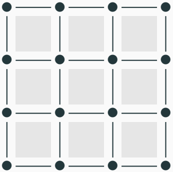
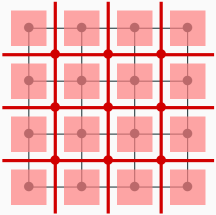
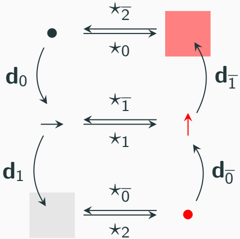
Dual cell complex, Hodge star, calculus



complex K



dual complex $\overline{K}$



primal dual

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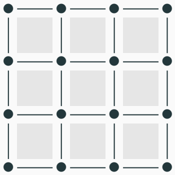

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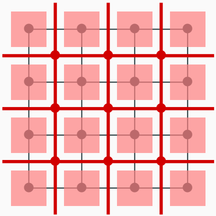

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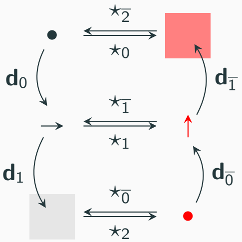

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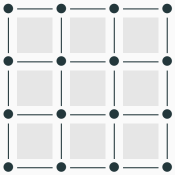

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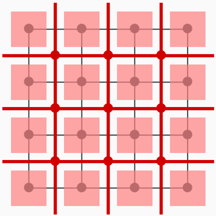

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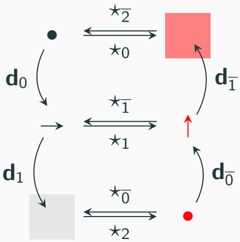

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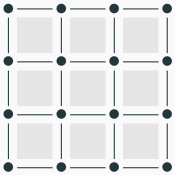

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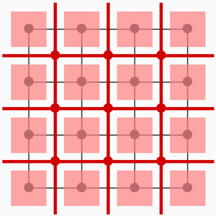

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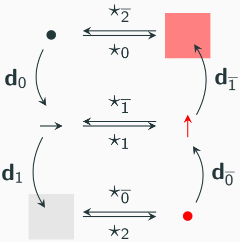

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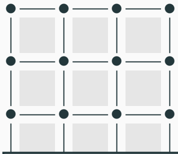
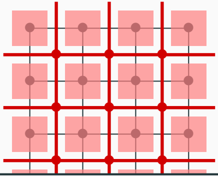
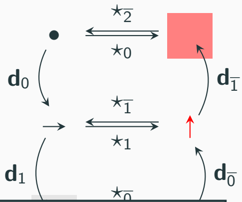
  - e.g. $\star_k \mathbf{1} = \alpha$ is the area 2-form $\mathrm{d}A$
- wedge products satisfy algebraic properties (Leibniz rules ...)
  - $\alpha \wedge \beta := \mathrm{diag}(\alpha)\beta$, for $\alpha \in C^k(K), \beta \in C^{2-k}(\overline{K})$,
  - $f \wedge \gamma := \mathrm{diag}(\mathbf{M}_{01} f)\gamma$, for $f \in C^0(K), \gamma \in C^1(K)$ ...

Dual cell complex, Hodge star, calculus







Almost all the calculus is built from the previous operators

- codifferentials $\delta_1 := -\star_{\bar{2}}\mathbf{d}_{\bar{1}}\star_1$, $\delta_2 := -\star_{\bar{1}}\mathbf{d}_{\bar{0}}\star_2$,
- Laplacian $\Delta := \delta_1 \mathbf{d}_0$
- Edge Laplacian $\Delta_1 := \mathbf{d}_0\delta_1 + \delta_2\mathbf{d}_1$,
- musical ops : Vector field $\xrightarrow{\flat}$ 1-form $\xrightarrow{\sharp}$ Vector field
- gradient $\nabla f := (\mathbf{d}_0 f)^\sharp$
- divergence $\mathrm{div}\mathbf{V} := \delta_1 \mathbf{V}^\flat$
- L^2 inner-product $(\alpha, \beta)_{\Omega,k} := \int_\Omega \alpha \wedge \star_k \beta$, for α, β k-forms
- $f \wedge \gamma := \mathrm{diag}(\mathbf{M}_{01} f)\gamma$, for $f \in C^0(K), \gamma \in C^1(K)$ ...

Formulate PDE/variational problems with digital calculus

Anisotropic diffusion on image I

- $\frac{\partial u}{\partial t} = \mathrm{div}(f(\|\nabla u\|)\nabla u)$ with $u_0 = I$, f decreasing from 1 to 0.
- $\frac{u^{t+dt} - u^t}{dt} = \delta_1(F \wedge \mathbf{d}_0 u)$, with function F defined for all edge e as $F(e) = f(\|u(v_j) - u(v_i)\|)$, with $\partial_1 e = v_j - v_i$.
- in matrix form, with matrices $\mathbf{A} := \mathbf{d}_0$, $\mathbf{G}_k := \star_k$:

$$\mathbf{u}^{t+dt} = \left(\mathbf{Id} + dt\mathbf{G}_{\bar{2}}\mathbf{A}^\intercal\mathbf{G}_1 \mathrm{diag}\left(\mathbf{f}\right)\mathbf{A}\right)\mathbf{u}^t,$$

  with $\mathbf{f}$ computed per edge from value of $\mathbf{u}^t$.
- exactly the “strangely stable” algorithm of Perona-Malik for rectangular domains
- more general formulation in digital calculus (any complex)

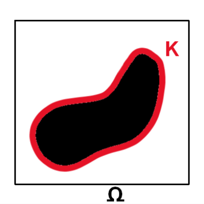
An important example: Mumford-Shah functional

Mumford-Shah functional for image restoration

We minimize

$$\mathcal{MS}(K,u) = \alpha \underbrace{\int_{\Omega\setminus K} |u-g|^2 \, \mathrm{dx}}_{\text{fidelity term}} + \underbrace{\int_{\Omega\setminus K} |\nabla u|^2 \, \mathrm{dx}}_{\text{smoothness term}} + \lambda \underbrace{\mathcal{H}^1(K\cap\Omega)}_{\text{discontinuities length}}$$

- Ω the image domain
- g the input image
- u a piecewise smooth approximation of g
- K the set of discontinuities
- $\mathcal{H}^1$ the Hausdorff measure



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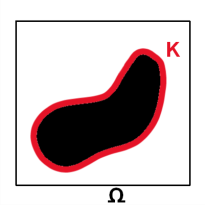

Mumford-Shah functional

Notably difficult to minimize

Many relaxations and convexifications have been proposed.

- Total Variation [Rudin et al., 1992] and its variants
- Multi-phase level sets [Vese and Chan, 2002] and follow-ups
- Discrete graph approaches [Boykov et al., 2001, Boykov and Funka-Lea, 2006]
- Calibration method [Alberti et al., 2003] and associated algorithms [Pock et al., 2009, Chambolle and Pock, 2011]
- Ambrosio-Tortorelli functional [Ambrosio and Tortorelli, 1992]
- convex relaxations of AT [Kee and Kim, 2014]

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Ambrosio-Tortorelli functional

$$AT_\epsilon(u, v) = \alpha \int_\Omega |u - g|^2 \, \mathrm{dx} + \int_\Omega v^2 |\nabla u|^2 \, \mathrm{dx} + \lambda \int_\Omega \epsilon |\nabla v|^2 + \frac{1}{\epsilon} \frac{(1 - v)^2}{4} \, \mathrm{dx}$$

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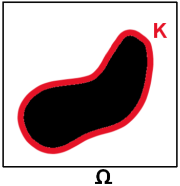
- v a smooth approximation of $1 - \chi_K$

✓ whole domain integration

✓ no Hausdorff measure



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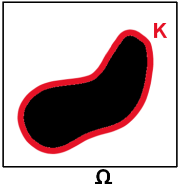
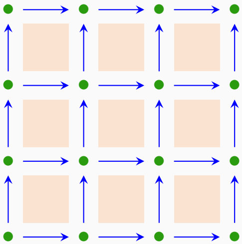


$$\Gamma\text{-convergence:} \quad AT_\epsilon \xrightarrow[\epsilon \to 0]{\Gamma} \mathcal{MS}$$

Discrete formulation of AT

$$AT_\epsilon(u,v) = \alpha \int_\Omega |u-g|^2 \,\mathrm{dx} + \int_\Omega v^2 |\nabla u|^2 \,\mathrm{dx} + \lambda \int_\Omega \epsilon |\nabla v|^2 + \frac{1}{4\epsilon}(1-v)^2 \,\mathrm{dx}$$



We choose :

- functions u, g to live on faces
  - u, g are 2-forms
  - equivalently dual 0-forms
- function v to live on vertices
  - v is a 0-form

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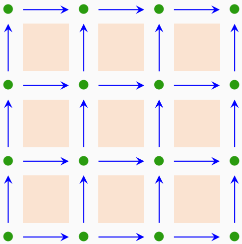

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$$\mathrm{AT}_\epsilon^{2,0}(\mathrm{u},\mathrm{v}) = \ \alpha\,(\mathrm{u}-\mathrm{g},\mathrm{u}-\mathrm{g})_{\Omega,2}$$

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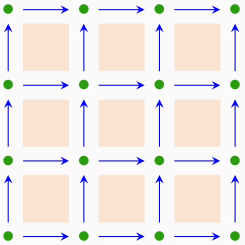

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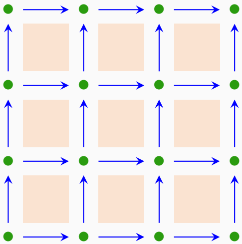

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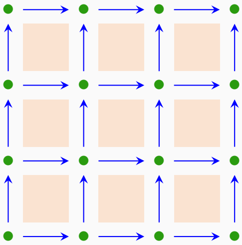

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$$+ \lambda\epsilon\,(\mathbf{d}_0\mathrm{v},\mathbf{d}_0\mathrm{v})_{\Omega,1} + \frac{\lambda}{4\epsilon}(1-\mathrm{v},1-\mathrm{v})_{\Omega,0}$$

Discrete formulation of AT

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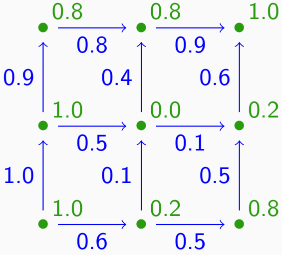
0.8	0.8	1.0
1.0	0.0	0.2
1.0	0.2	0.8

- 0-form v

Discrete formulation of AT

$$\int_\Omega v^2 |\nabla u|^2 \ \mathrm{dx} \quad = (\mathsf{v}\delta_2\mathsf{u}, \mathsf{v}\delta_2\mathsf{u})_{\Omega,1}$$

- $\mathsf{v}\delta_2\mathsf{u} = \mathsf{v} \wedge \delta_2\mathsf{u} = \mathrm{diag}(\mathbf{M}_{01}\mathsf{v})\delta_2\mathsf{u}$



- 0-form v
- 1-form $\mathbf{M}_{01}\mathsf{v}$

Discrete formulation of AT

$$\int_\Omega v^2 |\nabla u|^2 \, \mathrm{dx} \quad = (\mathrm{v}\delta_2 \mathrm{u}, \mathrm{v}\delta_2 \mathrm{u})_{\Omega,1}$$

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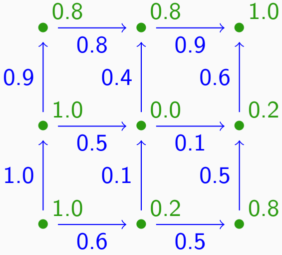
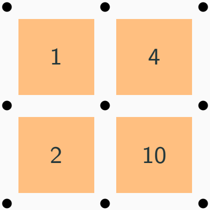




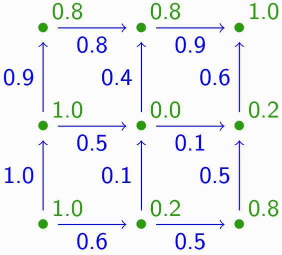
- 2-form u

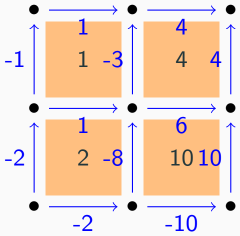
- 0-form v
- 1-form $\mathbf{M}_{01}\mathrm{v}$

Discrete formulation of AT

$$\int_\Omega v^2 |\nabla u|^2 \ \mathrm{dx} \quad = (\mathrm{v}\delta_2\mathrm{u}, \mathrm{v}\delta_2\mathrm{u})_{\Omega,1}$$

- $\mathrm{v}\delta_2\mathrm{u} = \mathrm{v} \wedge \delta_2\mathrm{u} = \mathrm{diag}(\mathbf{M}_{01}\mathrm{v})\delta_2\mathrm{u}$





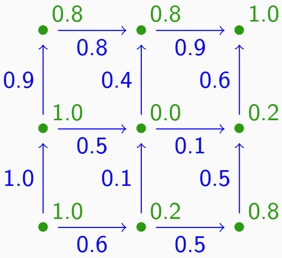
- 0-form v
- 1-form $\mathbf{M}_{01}\mathrm{v}$

- 2-form u
- 1-form $\delta_2\mathrm{u}$

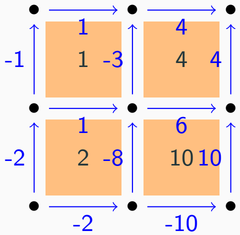
Discrete formulation of AT

$$\int_{\Omega} v^2 |\nabla u|^2 \ \mathrm{dx} \quad = (\mathsf{v}\delta_2\mathsf{u}, \mathsf{v}\delta_2\mathsf{u})_{\Omega,1}$$

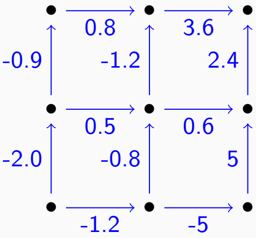
- $\mathsf{v}\delta_2\mathsf{u} = \mathsf{v} \wedge \delta_2\mathsf{u} = \mathrm{diag}(\mathbf{M}_{01}\mathsf{v})\delta_2\mathsf{u}$



- 0-form v
- 1-form $\mathbf{M}_{01}\mathsf{v}$



- 2-form u
- 1-form $\delta_2\mathsf{u}$



- 1-form $\mathrm{diag}(\mathbf{M}_{01}\mathsf{v})\delta_2\mathsf{u}$

Discrete formulation of AT

$$\mathrm{AT}_{\epsilon}^{2,0}(\mathsf{u},\mathsf{v}) = \alpha\,(\mathsf{u}-\mathsf{g},\mathsf{u}-\mathsf{g})_{\Omega,2} + (\mathrm{diag}(\mathbf{M}_{01}\mathsf{v})\delta_2\mathsf{u},\mathrm{diag}(\mathbf{M}_{01}\mathsf{v})\delta_2\mathsf{u})_{\Omega,1}$$
$$+\,\lambda\epsilon\,(\mathbf{d}_0\mathsf{v},\mathbf{d}_0\mathsf{v})_{\Omega,1} + \frac{\lambda}{4\epsilon}\,(1-\mathsf{v},1-\mathsf{v})_{\Omega,0}$$

Discrete formulation of AT

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- with matrices $\mathbf{A} := \mathbf{d}_0$, $\mathbf{B}' := \delta_2$, $\mathbf{G}_k := \star_k$.

$$\mathrm{AT}_\epsilon^{2,0}(\mathsf{u},\mathsf{v}) = \alpha(\mathsf{u}-\mathsf{g})^\intercal\mathbf{G}_2(\mathsf{u}-\mathsf{g}) + \mathsf{u}^\intercal\mathbf{B}'^\intercal\mathrm{diag}(\mathbf{M}_{01}\mathsf{v})^2\mathbf{G}_1\mathbf{B}'\mathsf{u}$$
$$+ \lambda\epsilon\mathsf{v}^\intercal\mathbf{A}^\intercal\mathbf{G}_1\mathbf{A}\mathsf{v} + \frac{\lambda}{4\epsilon}(1-\mathsf{v})^\intercal\mathbf{G}_0(1-\mathsf{v})$$

Discrete formulation of AT

$$\mathrm{AT}_\epsilon^{2,0}(\mathsf{u},\mathsf{v}) = \alpha\,(\mathsf{u}-\mathsf{g},\mathsf{u}-\mathsf{g})_{\Omega,2} + (\mathrm{diag}(\mathbf{M}_{01}\mathsf{v})\delta_2\mathsf{u}, \mathrm{diag}(\mathbf{M}_{01}\mathsf{v})\delta_2\mathsf{u})_{\Omega,1}$$
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- Euler-Lagrange: $\min_{\mathsf{u},\mathsf{v}} \mathrm{AT}_\epsilon^{2,0} \Rightarrow \dfrac{\mathrm{dAT}_\epsilon^{2,0}}{\mathrm{du}} = 0$ and $\dfrac{\mathrm{dAT}_\epsilon^{2,0}}{\mathrm{dv}} = 0$
- $\mathrm{AT}_\epsilon^{2,0}$ is quadratic in u and in v

Discrete formulation of AT

$$\mathrm{AT}_\epsilon^{2,0}(u,v) = \alpha\,(u-g,u-g)_{\Omega,2} + (\mathrm{diag}(\mathbf{M}_{01}v)\delta_2 u, \mathrm{diag}(\mathbf{M}_{01}v)\delta_2 u)_{\Omega,1}$$
$$+\,\lambda\epsilon\,(\mathbf{d}_0 v, \mathbf{d}_0 v)_{\Omega,1} + \frac{\lambda}{4\epsilon}\,(1-v,1-v)_{\Omega,0}$$

- with matrices $\mathbf{A} := \mathbf{d}_0$, $\mathbf{B}' := \delta_2$, $\mathbf{G}_k := \star_k$.

$$\mathrm{AT}_\epsilon^{2,0}(u,v) = \alpha(u-g)^\intercal \mathbf{G}_2(u-g) + u^\intercal \mathbf{B}'^\intercal \mathrm{diag}(\mathbf{M}_{01}v)^2 \mathbf{G}_1 \mathbf{B}' u$$
$$+\,\lambda\epsilon v^\intercal \mathbf{A}^\intercal \mathbf{G}_1 \mathbf{A} v + \frac{\lambda}{4\epsilon}(1-v)^\intercal \mathbf{G}_0 (1-v)$$

- Euler-Lagrange: $\min\limits_{u,v} \mathrm{AT}_\epsilon^{2,0} \Rightarrow \dfrac{d\mathrm{AT}_\epsilon^{2,0}}{du} = 0$ and $\dfrac{d\mathrm{AT}_\epsilon^{2,0}}{dv} = 0$
- $\mathrm{AT}_\epsilon^{2,0}$ is quadratic in u and in v
- We solve alternatively for u and v the sparse linear systems:

$$\left\{\begin{array}{rcl} \left[\alpha\mathbf{G}_2 - \mathbf{B}'^\intercal \mathrm{diag}\left(\mathbf{M}_{01}v\right)^2 \mathbf{G}_1\mathbf{B}'\right] u &=& \alpha\mathbf{G}_2 g, \\ \left[\frac{\lambda}{4\varepsilon}\mathbf{G}_0 + \lambda\varepsilon\mathbf{A}^\intercal\mathbf{G}_1\mathbf{A} + \mathbf{M}_{01}{}^\intercal \mathrm{diag}\left(\mathbf{B}'u\right)^2 \mathbf{G}_1\mathbf{M}_{01}\right] v &=& \frac{\lambda}{4\varepsilon}\mathbf{G}_0 1. \end{array}\right.$$

Discrete formulation of AT: vectorial data

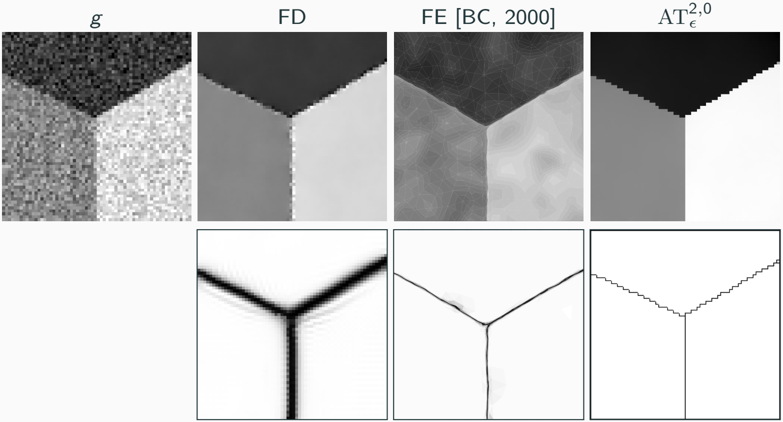
$$
\begin{aligned}
\mathrm{AT}_{\epsilon}^{2,0}(u_1,\ldots,u_n,\mathrm{v}) = \alpha \sum_i (u_i - g_i, u_i - g_i)_{\Omega,2} \\
+ \sum_i (\mathrm{diag}(\mathbf{M}_{01}\mathrm{v})\delta_2 u_i, \mathrm{diag}(\mathbf{M}_{01}\mathrm{v})\delta_2 u_i)_{\Omega,1} \\
+ \lambda\epsilon\,(\mathbf{d}_0\mathrm{v}, \mathbf{d}_0\mathrm{v})_{\Omega,1} + \frac{\lambda}{4\epsilon}(1-\mathrm{v}, 1-\mathrm{v})_{\Omega,0}
\end{aligned}
$$

- We solve alternatively for the u_i and v the sparse linear systems:

$$
\left\{
\begin{array}{rcl}
\forall i \in \{1,\ldots,n\}, \left[\alpha\mathbf{G}_2 - \mathbf{B}'^{\mathsf{T}}\mathrm{diag}\left(\mathbf{M}_{01}\mathrm{v}\right)^2 \mathbf{G}_1\mathbf{B}'\right] u_i & = & \alpha\mathbf{G}_2 g_i, \\
\left[\frac{\lambda}{4\varepsilon}\mathbf{G}_0 + \lambda\varepsilon\mathbf{A}^{\mathsf{T}}\mathbf{G}_1\mathbf{A} + \mathbf{M}_{01}{}^{\mathsf{T}}(\sum_i \mathrm{diag}\left(\mathbf{B}' u_i\right)^2)\mathbf{G}_1\mathbf{M}_{01}\right]\mathrm{v} & = & \frac{\lambda}{4\varepsilon}\mathbf{G}_0 1.
\end{array}
\right.
$$

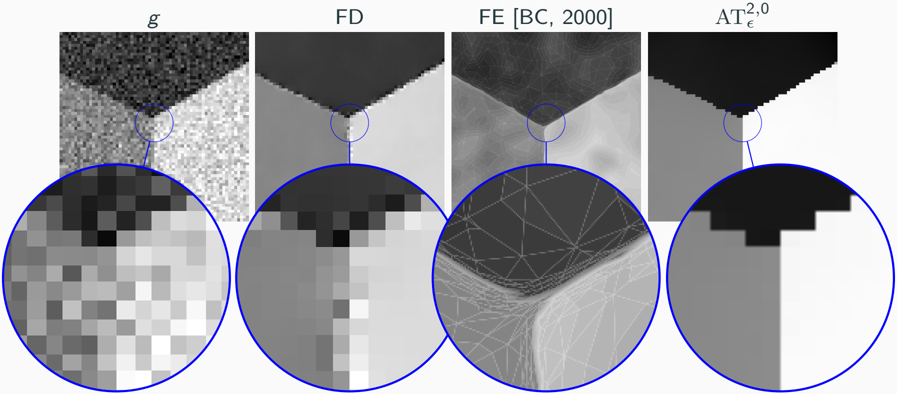
- Our algorithm progressively decreases ϵ to get a better chance of capturing the optimum
  - ϵ follows typically sequence $2, 1, 0.5, 0.25$ (for $h = 1$ sampling)
  - results on u and v are starting point for next ϵ

Image restoration on toy examples



- systems are solved using Cholesky decomposition (Eigen)
- ϵ takes the successive values $2, 1, 0.5, 0.25,$ for sampling step $h = 1$.

Image restoration on toy examples



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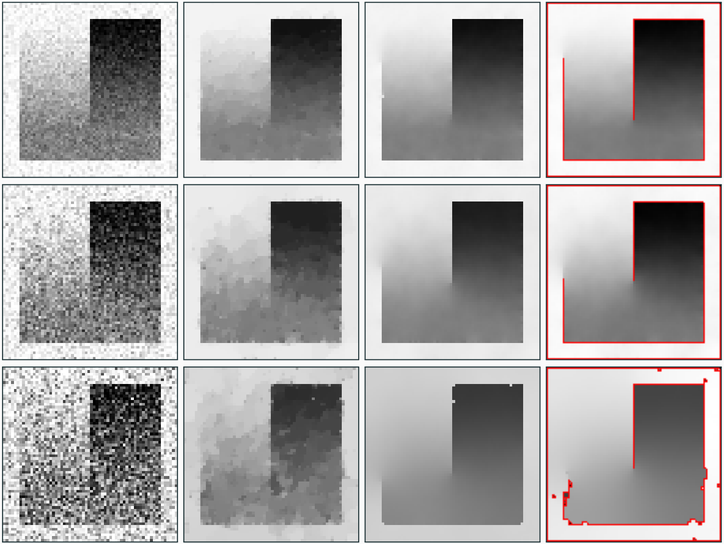
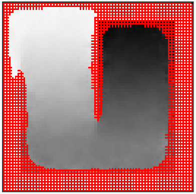
g	TV [Duran et al. 2013]	[Strek. et al. 2014]	$\mathrm{AT}_{\epsilon}^{2,0}$



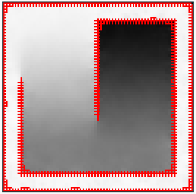
Influence of parameter ϵ

$$AT_\epsilon(u,v) = \alpha \int_\Omega |u-g|^2 \, \mathrm{dx} + \int_\Omega v^2 |\nabla u|^2 \, \mathrm{dx} + \lambda \int_\Omega \epsilon |\nabla v|^2 + \frac{1}{\epsilon} \frac{(1-v)^2}{4} \, \mathrm{dx}$$

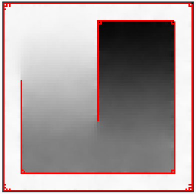
- Γ-convergence parameter
- Controls the thickness of the contours
  - large ϵ convexifies AT and helps to detect the discontinuities;
  - as ϵ goes to 0, the discontinuities become thinner and thinner.



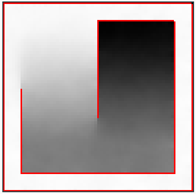
$\epsilon = 2 \searrow 2$



$\epsilon = 2 \searrow 1$



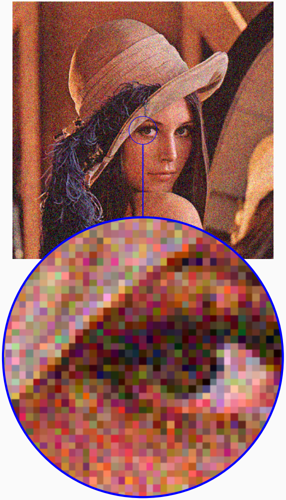
$\epsilon = 2 \searrow 0.5$



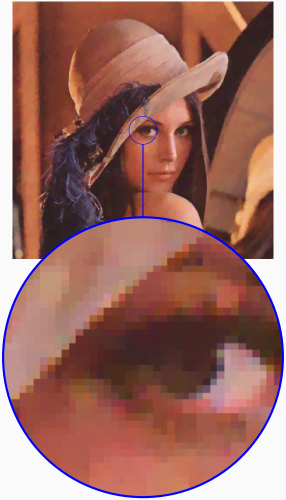
$\epsilon = 2 \searrow 0.25$

Image restoration / denoising

g
(PSNR = 20.23 dB)



TV
(PSNR = 29.36 dB)



$\mathrm{AT}_{\epsilon}^{2,0}$
(PSNR = 29.03 dB)

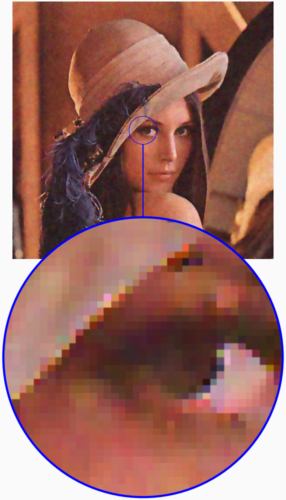
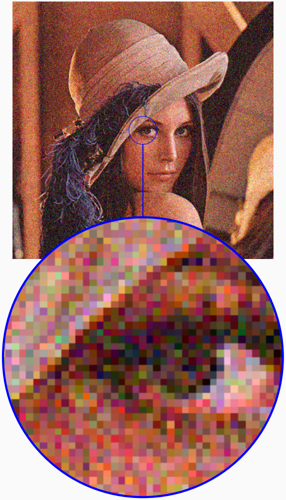


Image restoration / denoising

g
(PSNR $=$ 20.23 dB)



TV
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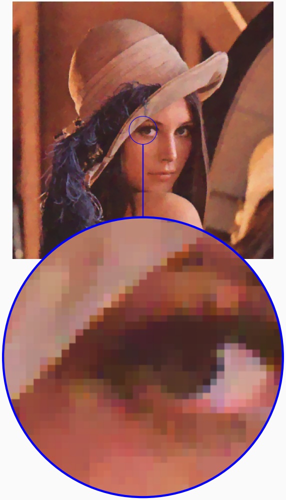
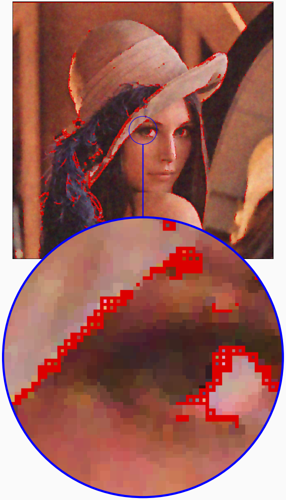


$\mathrm{AT}_{\epsilon}^{2,0}$
(PSNR $=$ 29.03 dB)

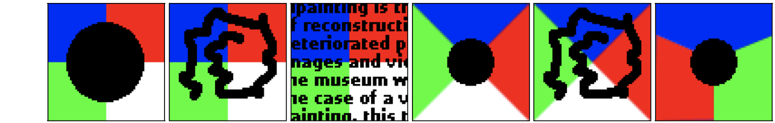


Scale-space given by α and λ and image segmentation

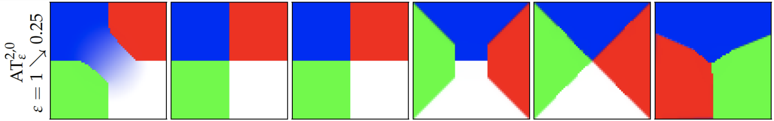
for decreasing λ

Image inpainting (on toy example)

- mask (in black) : domain M where data g (in color) is unknown
- $\alpha(x) := \{\alpha \in \Omega \setminus M, 0 \text{ elswhere}\}$
- initialization: u random in M, $= g$ in $\Omega \setminus M$



$\mathrm{AT}_\epsilon^{2,0}$ with ϵ from **1** to 0.25



$\mathrm{AT}_\epsilon^{2,0}$ with ϵ from **4** to 0.25

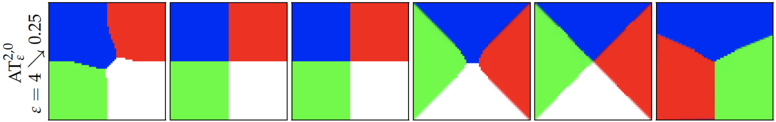
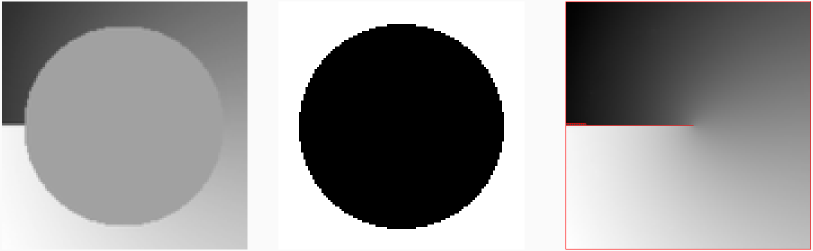


Image inpainting (on classical crack-tip example)

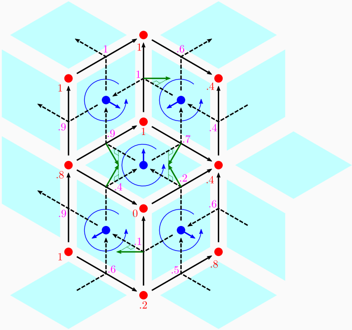


g — mask M — $\mathrm{AT}_{\epsilon}^{2,0}$, $\alpha = 1$, $\lambda = 0.0024$

- Decreasing sequence of λ (irreversibility !?)
- same result as [Pock, Bishof, Cremers, Pock 2009], based on MS relaxation of [Alberti, Bouchitté, Dal Maso 2003]
- result independent of initialization as long as first ϵ is big enough (ϵ from 4 to 0.25 here, for image of size 110×110).

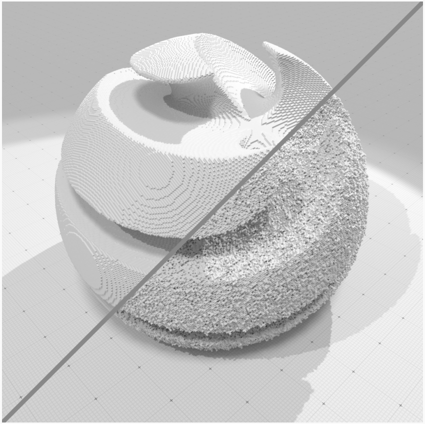
Feature delineation on digital surfaces

digital surface = boundary of set of voxels



same discrete calculus

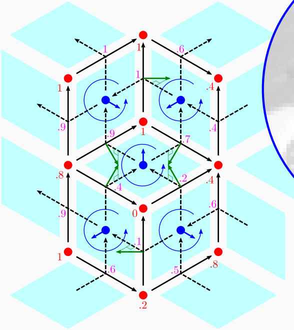
same $\mathrm{AT}_{\epsilon}^{2,0}$

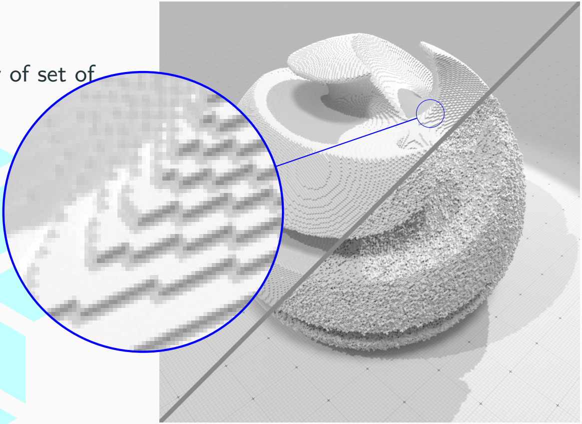


Input: normal vector field g estimated by Integral Invariant digital normal estimator.

Feature delineation on digital surfaces

digital surface = boundary of set of voxels





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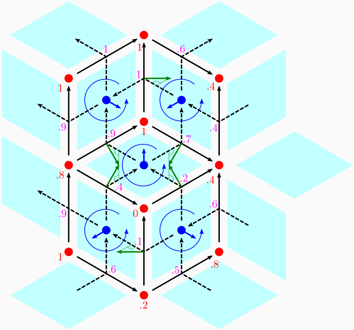

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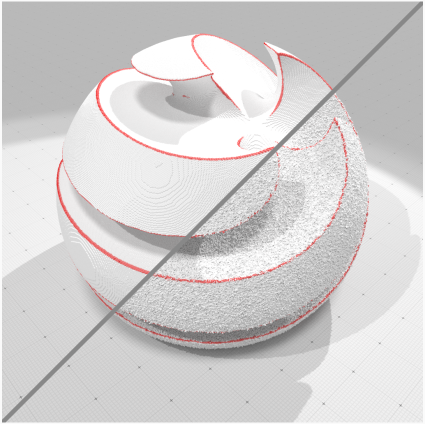

Input: normal vector field g estimated by Integral Invariant digital normal estimator.

Output: piecewise smooth normals $(u_i)_{i=1,2,3}$ and features v

Feature delineation on digital surfaces

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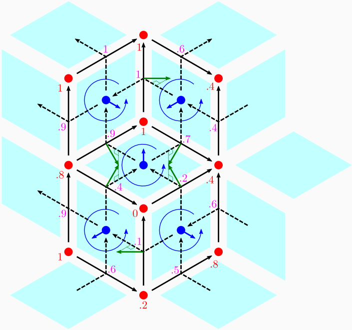

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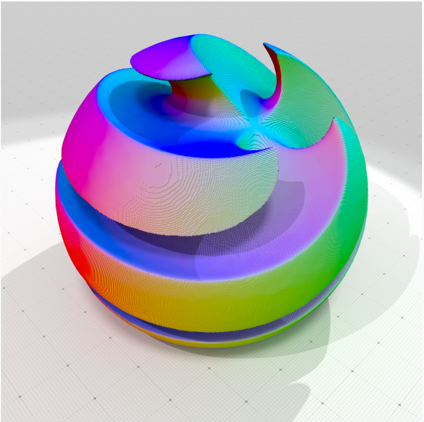

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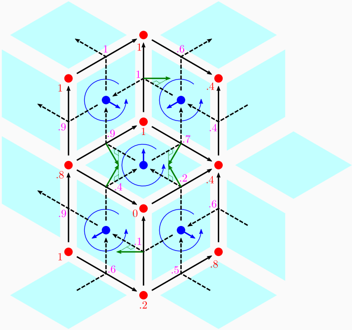
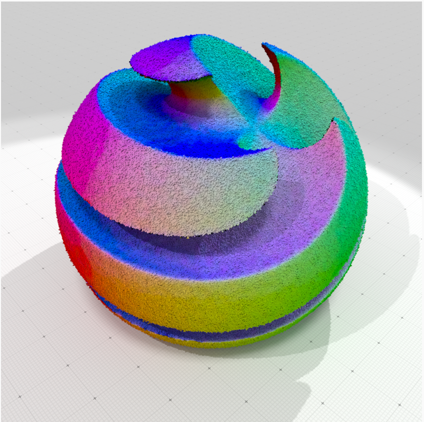


same discrete calculus
same $\mathrm{AT}_{\epsilon}^{2,0}$



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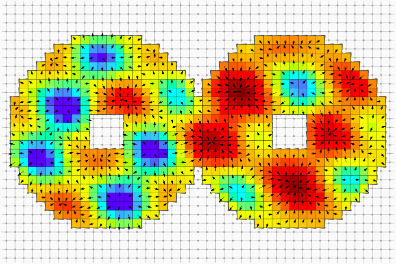
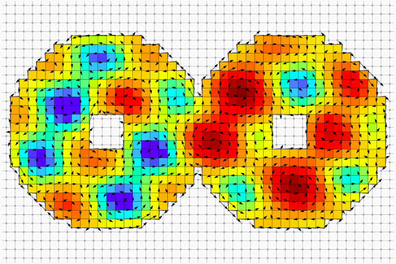
Discrete Exterior Calculus: conclusion



curl-free $\mathbf{d}_1 \mathbf{v} = 0$



div-free $\delta_1 \mathbf{v} = 0$

- a sound framework for variational modeling / PDE
- well suited to digital domains: images, digital surfaces
- reduces to linear algebra computations
- visit DEC package on `dgtal.org` documentation

3. Reproducible Research

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Overview

- 3.1 (short) Introduction to Reproducible Research.
- 3.2 Rapid overview of the RR platforms.
- 3.3 New ways of publications.

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Reproducible research in sciences:

- *Theoretical scientists* share demonstrations;

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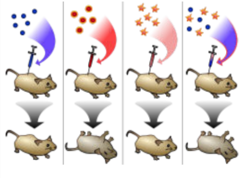
Reproducible research in sciences:

- *Theoretical scientists* share demonstrations;
- *Experimental scientists* share procedures;



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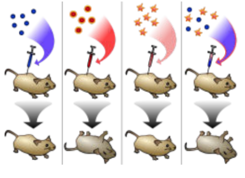
Overview

- 3.1 (short) Introduction to Reproducible Research.
- 3.2 Rapid overview of the RR platforms.
- 3.3 New ways of publications.

Reproducible research in sciences:

- *Theoretical scientists* share demonstrations;
- *Experimental scientists* share procedures;
- *Computational scientists... ?*



3. Reproducible Research

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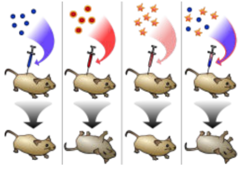


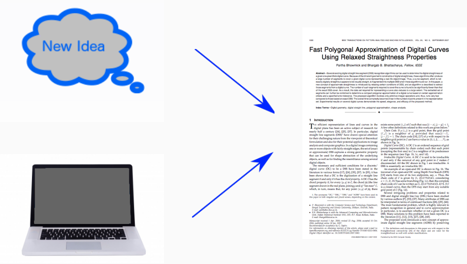
Computer Science:

- Description of methods/algorithms;
- description often limited (constraints on page limits);
- parameters not given or not well described;
- steps of pre/post processing missing.

3.1 (short) Introduction to Reproducible Research (1)

Research in Computer Science:

1. New idea;
2. demonstration, implementation;
3. article publication.



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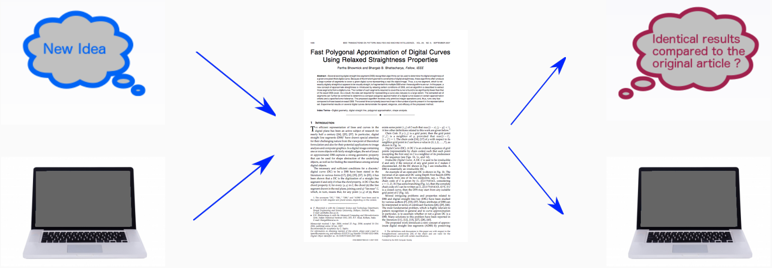
Reusable Research:

1. Article which seems interesting;
2. re-implement the algorithm;
3. result conformity with the original.



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$\Rightarrow$ more than the half affirm to also fail to **reproduce their own results**.

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Benefit for authors

- possibility of publishing research of high quality.
- increase of the visibility of the publications

3.2 Rapid overview of the RR platforms

- **Galaxy** - https://galaxyproject.org
- IPython - https://ipython.org
- **Jupyter** - http://jupyter.org
- **Code Ocean** - https://codeocean.com
- Research Compendia - http://researchcompendia.science
- RunMyCode - http://www.runmycode.org
- **DAE** - http://dae.cse.lehigh.edu/DAE
- **IPOL** - https://www.ipol.im

3.2 Rapid overview of the RR platforms: Galaxy

Description:

- Platform for genomic research.
- It makes available tools which can be used by non-expert users too.
- Galaxy defines a workflow as a reusable templates which contains different algorithms applied to the input data.
- In order to achieve reproducibility the system stores:
  - the input dataset,
  - the tools and algorithms which were applied to the data along the chain,
  - the parameters,
  - the output result.

https://galaxyproject.org

3.2 Rapid overview of the RR platforms: Jupyter

Description:

- Spin-off of IPython in 2014.
- Main goal: separate the Python language used in IPython from all the other functionalities needed to run the notebooks (for example, the notebook format, the web framework, or the message protocols).
- Languages: execution kernels in Jupyter.
- Nowadays it supports more than 40 languages that can be used as kernels.

http://jupyter.org

3.2 Rapid overview of the RR platforms: Code Ocean

Description:

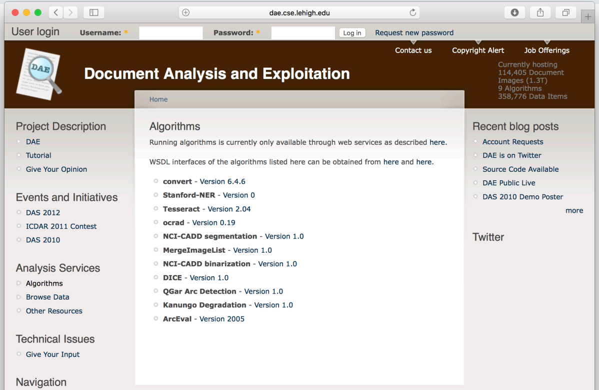
- Stated in 2014 as part *Runway Startup Postdoc Program* at the Jacobs Technion Cornell Institute.
- Sponsored by IEEE.
- Defined by themselves as a *computational reproducibility platform*.
- Not a journal itself. It only runs code, but not publishes articles.
- Assigns a DOI to each source code.
- Several languages accepted: Python, R, Julia, Matlab, Octave, C++, Fortran, Perl, Java
- They claim *to view and download for everyone for free*. The free plan is limited and the other are paid.
- Plans based on CPU time and storage usage. For example, the *researcher* plan allows 1h CPU and 5GB of storage per month.
- No statistics on usage. Seems low by website inspection.

https://codeocean.com/

3.2 Rapid overview of the RR platforms: DAE

Description: [Lamiroy & Lopresti 16]

- Platform for **D**ocument **A**nalysis and **E**xploitation.
- Allows to run document analysis algorithms and apply comparisons.



`http://dae.cse.lehigh.edu`

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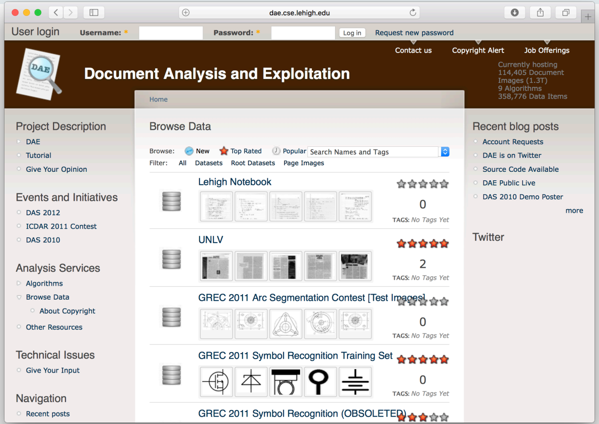
- RR framework for document analysis with image data base.



3.2 Rapid overview of the RR platforms: IPOL Journal

Description: [Arevalo *et al.* 16]

- A complete peer-reviewed journal can be considered as a platforms.
- Image Processing domain.
- Focused on mathematical rigorness. Detailed descriptions.
- Fast to create new demos for editors: automatic system.
- Accepted languages: C/C++, Python, MATLAB, Octave.
- Free to use/submit.
- Next move: machine learning applications. Servers with GPU.

https://www.ipol.im

3.3 New Ways of Publications

Recent original journals

- IPOL

  https://www.ipol.im

- ReScience

  http://rescience.github.io

- JOSS

  https://joss.theoj.org

3.3 New Ways of Publications: Image Processing On Line

Origin: `http://www.ipol.im`

- Journal started in October 2009.
- Initiative of Nicolas Limare and Jean-Michel Morel (CMLA).
- First article published in 2010.
- Domain of Image Processing.

IPOL
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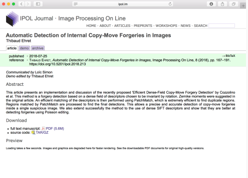
Motivations

- Reproducible Research.
- New way to publish research results.
- Allows everybody to test the algorithms (with their own images).
- Free online demonstration (user-platform independant) and source code.

3.3 New Ways of Publications: Image Processing On Line

Short overview

- Journal publishing algorithm description, source code, online demonstration with experiment archives.



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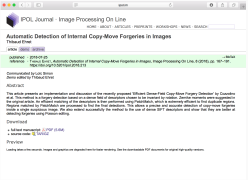

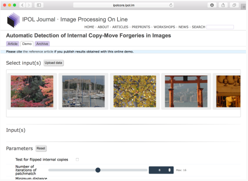

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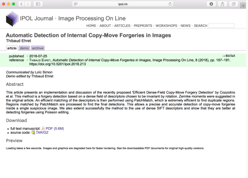

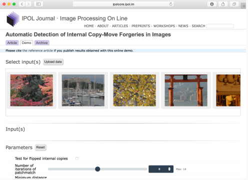

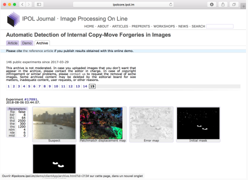

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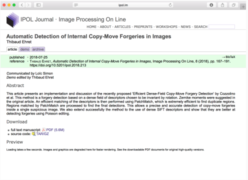

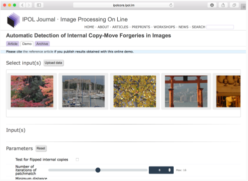

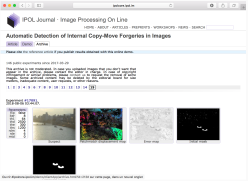

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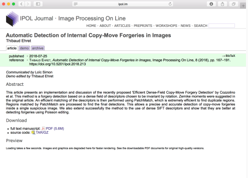

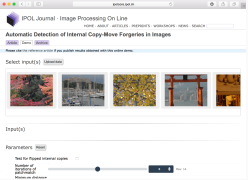

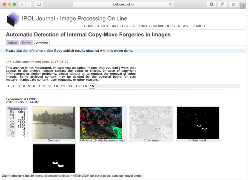

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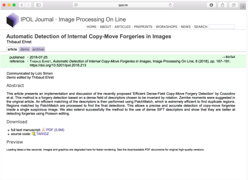
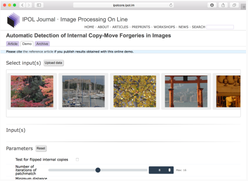
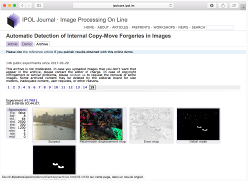
- Like classic journal: ISSN, DOI, indexed by: *SCOPUS, DBLP, Scirus, Google Scholar, DOAJ, SHERPA/RoMEO, Héloïse, WorldCat, CrossRef, Ulrich, Index Copernicus, PBN, JGate, VisionBib, CVonline, JournalSeek and NewJour.*







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Recent evolution

- New system to automatically create our own online demontrations (see next session).
- Extended topics to sound and video processing with 3D processing.

3.2 ReScience Journal

Philosophy (http://rescience.github.io)

- Context of Reproducible research [Buckheit & Donoho 95].
- Explicit replication: propose a new implementation of an existing work.
- Motivated from replication problems in computational science [Hinsen 15], [Topalidou *et al.* 15], [Hinsen 14].

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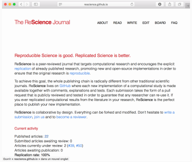
Details:

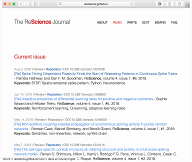
- Origin: first volume in 2015.
- Editorial Board:
  - Editors-in-Chief:
    - Konrad Hinsen (Molecular Biophysics - Python, C, Racket, Clojure).
    - Nicolas P. Rougier (Comp. Neuroscience, Computer Science - Python, C/C++).
  - 11 Associate Editors with roles in: Bioinformatics; Cognitive Modelling; Computational Ecology; Computational Physics; Image processing; Ecology, High-Performance Computing; Physics; Robotics; Signal Processing

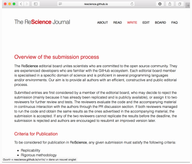
3.2 ReScience Journal: short overview

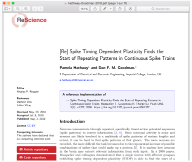
Characteristics

- Same presentation as in a “classic” journal.









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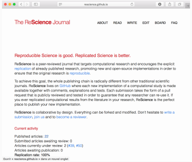
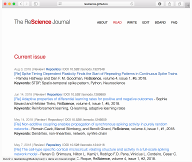
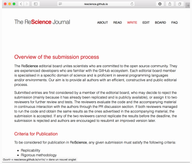
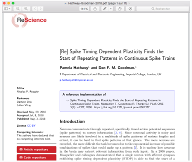
- Original submission process through *Pull Request* on *GitHub*.









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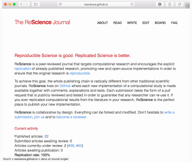

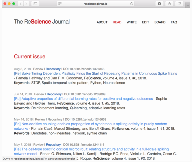

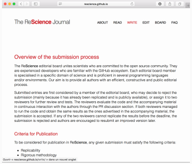

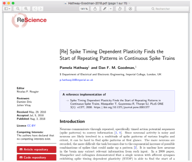

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Journal of Open Source Software (https://joss.theoj.org)

- Origin: founded by Arfon M. Smith in May 2016.
- Free and Open Access.
- Peer reviewing.
- Motivated by the fact that [Smith *et al.* 17]: *"Current publishing and citation do not acknowledge software as a first-class research output"*.

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Details

- Design: defined in the current merit system of science.
- Aim: can be considered as a "journal for research software packages".
- Editorial Board:
  - Arfon Smith (@arfon), Editor-in-Chief.
  - 19 Topic Editors: representing: Astronomy; Biodiversity Informatics; Bioinformatics; Computational Science; Data Science; Engineering, Computational Combustion; Computational Social Science; Fluid Dynamics; Energy Engineering; Geophysics; Geoscience; High Perf. Computing; Image; Information Sciences; Machine Learning; Neuroimaging; Nuclear Engineering; Open Science; Psychology; Semantic Web; Social Sciences; Software Deployment; Reproducible Research.

3.3 JOSS Journal: motivations

Motivation of JOSS Editor in chief [Smith *et al.* 17]:

- Software more and more present in numerous disciplines:
  $\Rightarrow$ from a 2014 survey 90% mention to use software and 70% indicates that they were obligatory [Hettrick *et al.* 14].
- Software leak of scholarity support: no echosystem of publication, citation, acknowledge.
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As mentionned by Buckheit and Dohono [Buckheit & Donoho 95]:
"An article about computational science in a scientic publication is not the scholarship itself it is merely advertising of the scholarship"

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Specific form:

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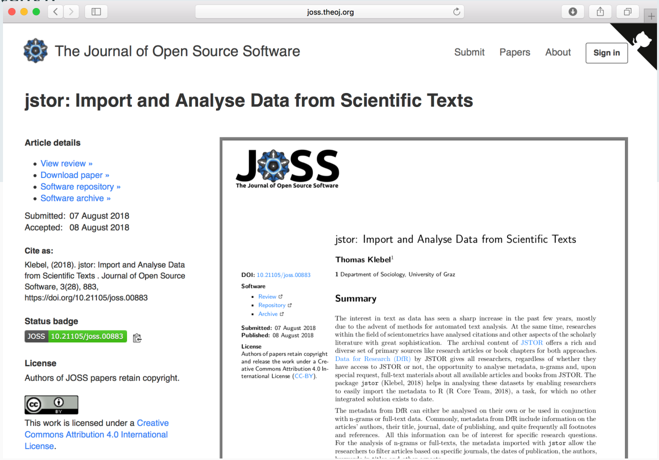

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Content requested:

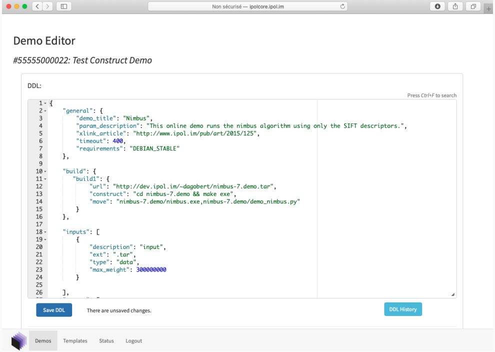
- Software need to be open source.
- Research application.
- Submitter needs to be main software contributor.
- Significant new contribution.
- Feature-complete (not partial).

4. Practical Session: Hands on IPOL Demos

4. Practical session: hands on the IPOL Demonstration System

Focus on online demonstration

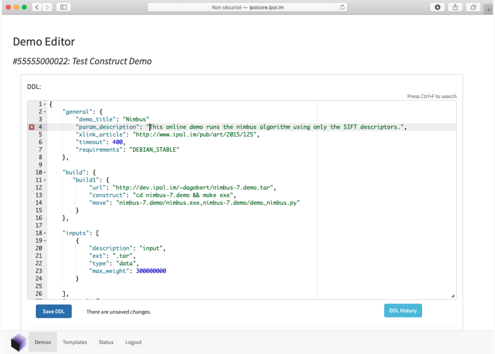
- Construction of demonstration from your own code.
- New Demo Editor panel:
  → to construct your demonstration without hard coding the demo.



4. Practical session: hands on the IPOL Demonstration System

Focus on online demonstration

- Construction of demonstration from your own code.
- New Demo Editor panel:
  $\rightarrow$ to construct your demonstration without hard coding the demo.
- Interpreted in live from the online editor.

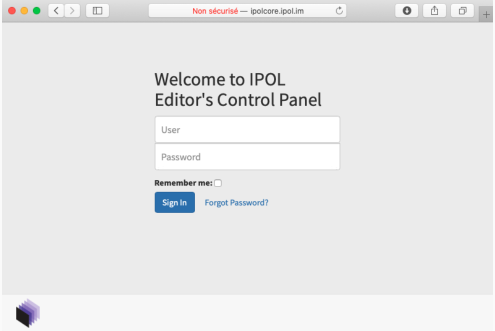


4. Practical session: hands on the IPOL Demonstration System

What you need to construct your demonstration

1. Request a demo editor account (when submitting a paper to IPOL)
   → use temporary account to experiment the construction.
2. Image based source code archive (to be available online).
3. Typical command line/parameters.

http://ipolcore.ipol.im/cp/login/



4. Practical session: first step

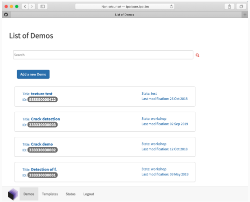
Step 1: main demo repository construction

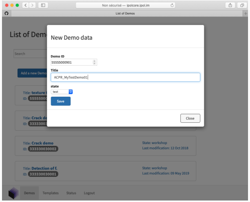
1. Login on : `http://ipolcore.ipol.im/cp/login`

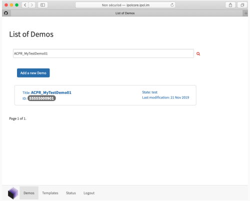
   (login: acpr2019 password: 222hhh888k)

2. Add a new Demo.

3. Select an number ID and titles using this convention:
   DemoID: 555559000XX Title: ACPRMyTestDemoXX





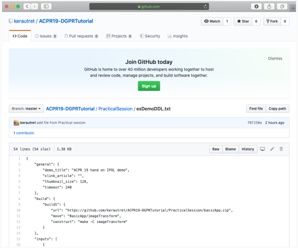


4. Practical session: second step

Step 2: Edit configuration file

1. Start from the base example available from the tutorial repository:

   `https://github.com/kerautret/ACPR19-DGPRTutorial` file: `PracticalSession/exDemoDDL.txt`



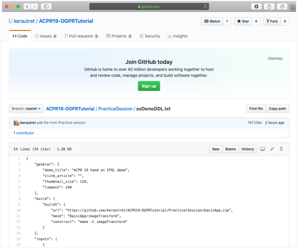
4. Practical session: second step

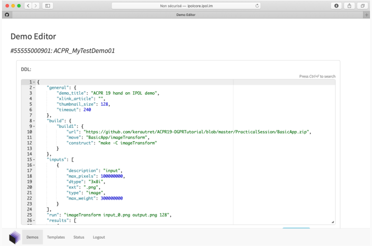
Step 2: Edit configuration file

1. Start from the base example available from the tutorial repository:

   `https://github.com/kerautret/ACPR19-DGPRTutorial` file: `PracticalSession/exDemoDDL.txt`

2. Copy the content in the demo editor panel.





4. Practical session: second step

Step 2: Edit configuration file

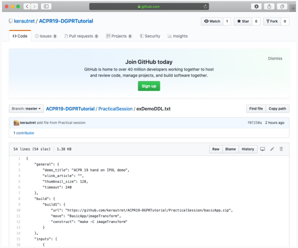
1. Start from the base example available from the tutorial repository:

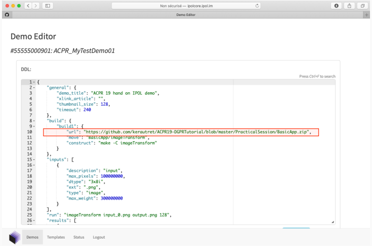
   `https://github.com/kerautret/ACPR19-DGPRTutorial` file: `PracticalSession/exDemoDDL.txt`

2. Copy the content in the demo editor panel.

3. Ajust parameters and your source code link: (or this toy example)

   `https://github.com/kerautret/ACPR19-DGPRTutorial` file: `PracticalSession/BasicApp.zip`

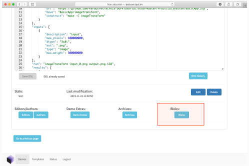


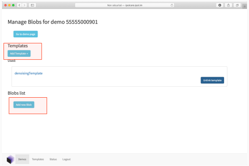


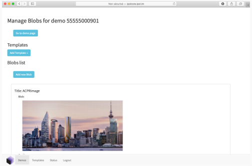
4. Practical session: last step

Step 3: Select Blobs and start!

1. Need to select the default images to test your demo: use the "Blobs".
2. Customize (or not default blob).





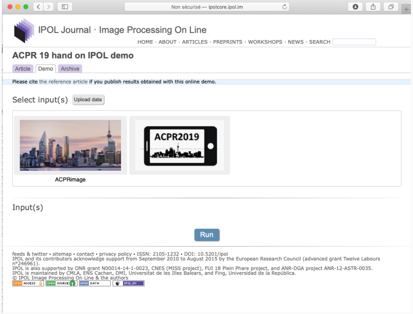


4. Practical session: last step

Step 3: Select Blobs and start!

1. Need to select the default images to test your demo: use the "Blobs".
2. Customize (or not default blob).
3. Check the resulting online demo! (using following url)

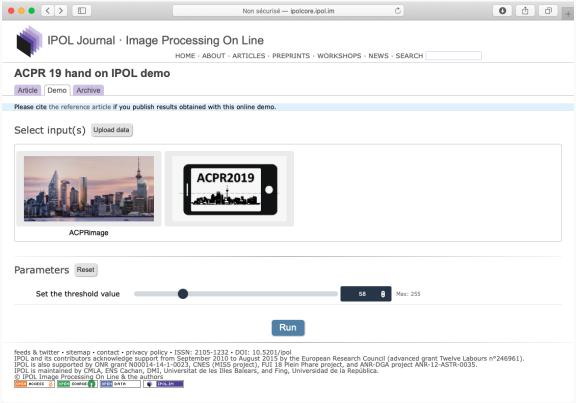
`http://ipolcore.ipol.im/demo/clientApp/demo.html?id=555559000XX`



4. Practical session: customize parameters

Exercice

1. Add a new parameter on the demonstration.
   → instead using the default parameter, add a 'params' section in the DDL.
   → use $ to transmit the parameter.



4. Practical session: customize parameters

Exercice

1. Add a new parameter on the demonstration.
   → instead using the default parameter, add a 'params' section in the DDL.
   → use $ to transmit the parameter.

```
"params": [
    {
        "id": "parameterID",
        "label": "Label description...",
        "type": "range",
        "values": {
            "default": 50,
            "max": 100,
            "min": 0,
            "step": 1
        }
    },
```

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Thank you for your attention!

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