Voronoi-based Geometry Estimator for 3D Digital Surfaces

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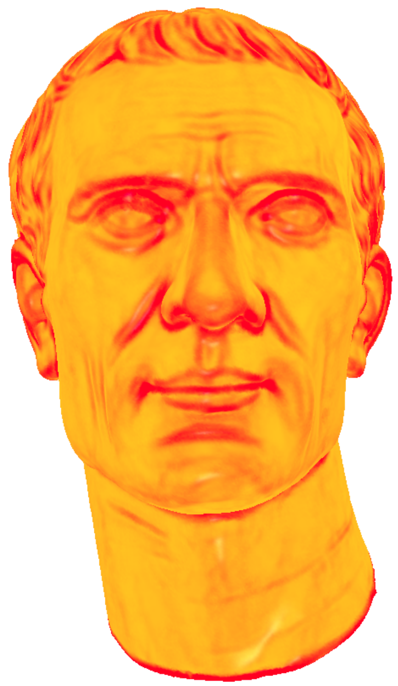
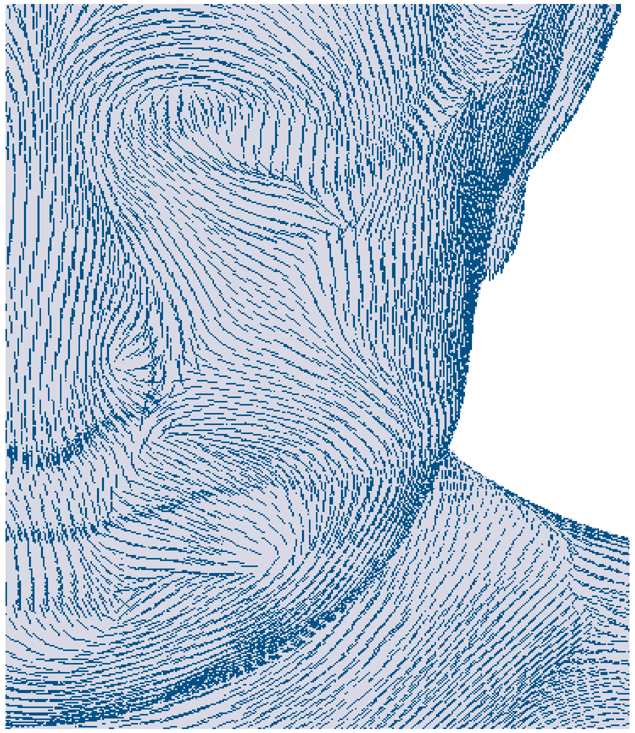
Differential quantities estimation

Normal

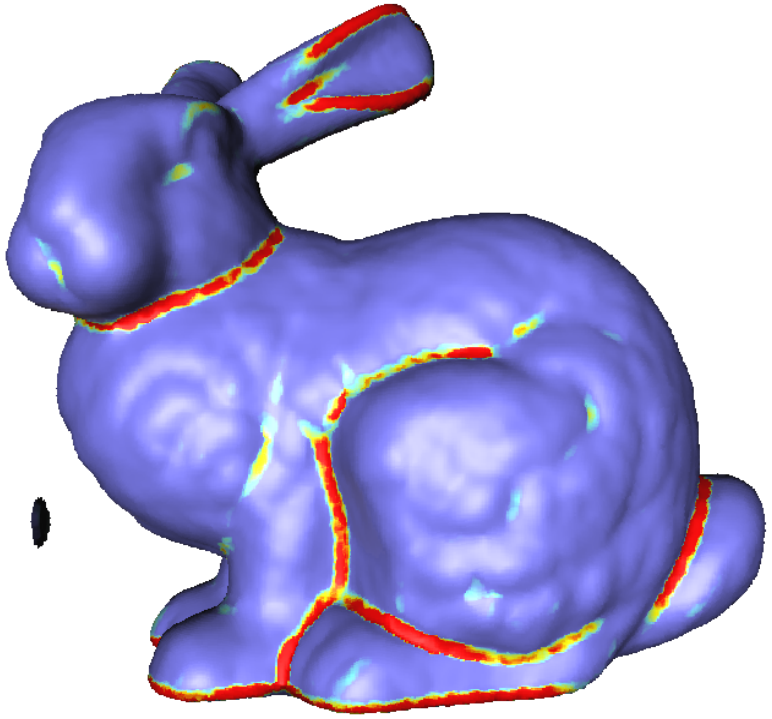
Curvature



Curvature Direction



Sharp Feature



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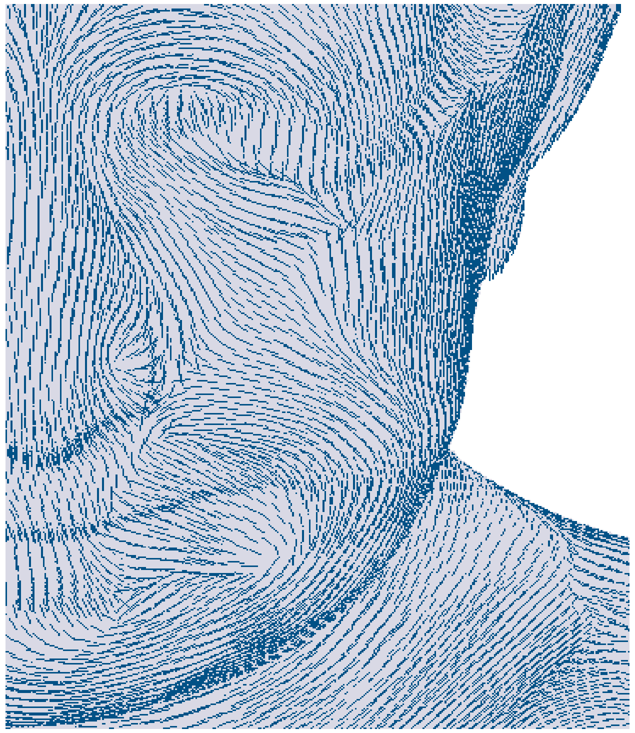

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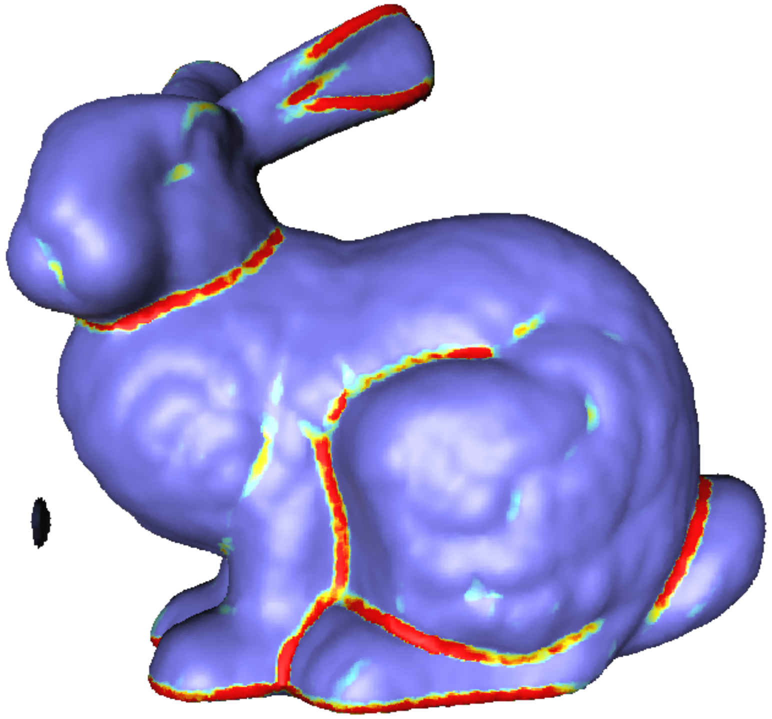


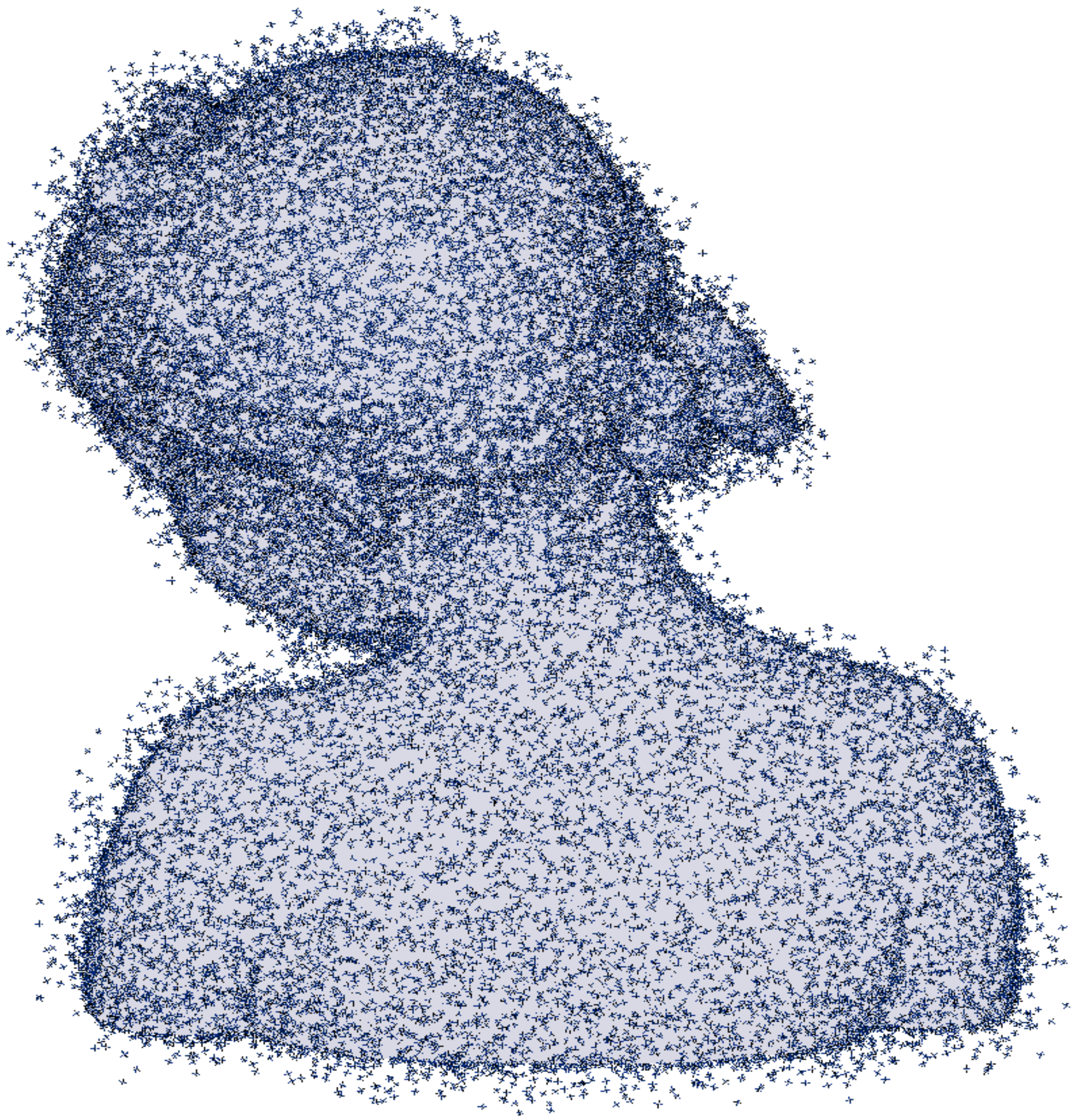
We want our method to be :

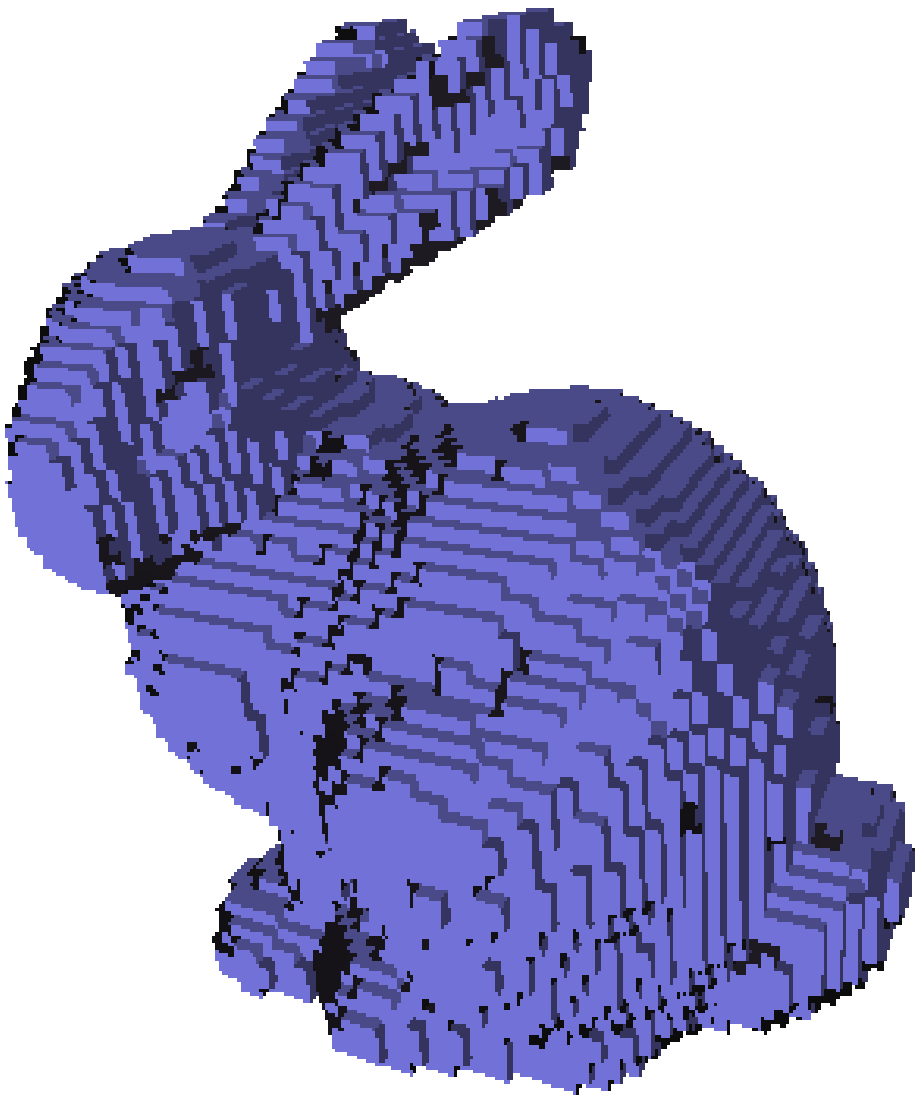
- Accurate
- Stable under noise
- Efficient
- Generic

Inference for different datas

Unorganized Point Clouds

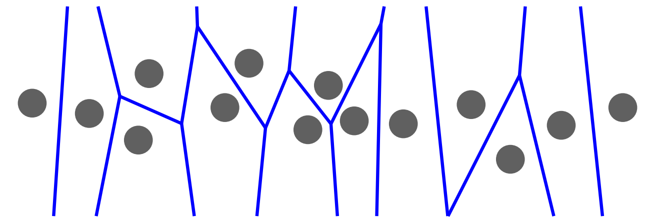
Digital sets





Contributions

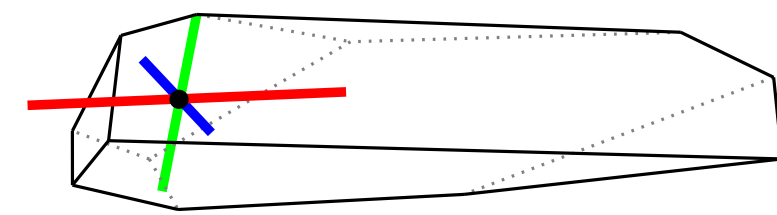
Our Approach :



–A method from computational geometry (VCM) :

- Voronoi
- Covariance

$\int_{\Omega}(x-p)(x-p)^{\mathbf{t}}\,\mathrm{d}\,x.$



–Feed it with digital points

–Show multigrid convergence for geometric quantities

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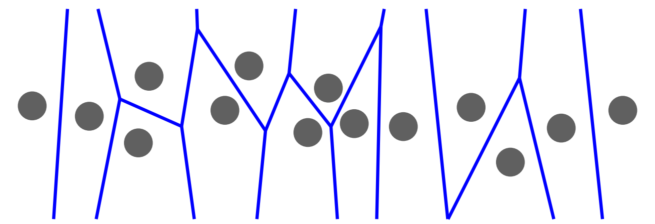

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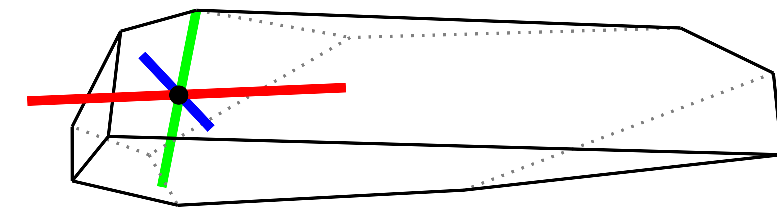

–Feed it with digital points

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Difficulties :

–Digitization of the VCM $\Rightarrow$ Digitization error

–Achieve local (even ponctual) stability $\Rightarrow$ Localization error

–Fixing parameters to achieve convergence

Plan

1) **Previous Voronoi-based Methods**

2) Voronoi Covariance Measure on Point cloud

3) Multigrid Convergence of the vcm estimators

4) Experiments

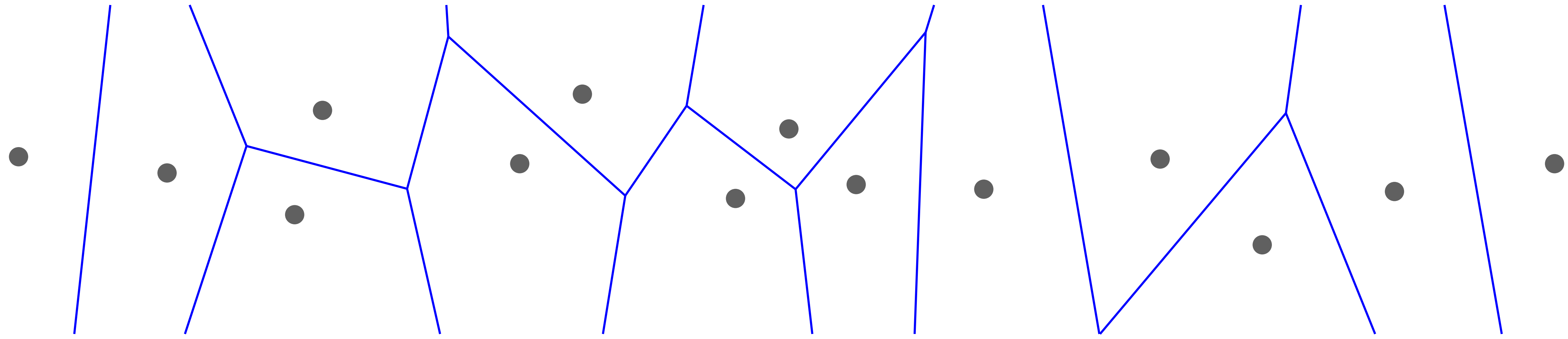
Voronoi-Based Methods

Voronoi-based algorithms :

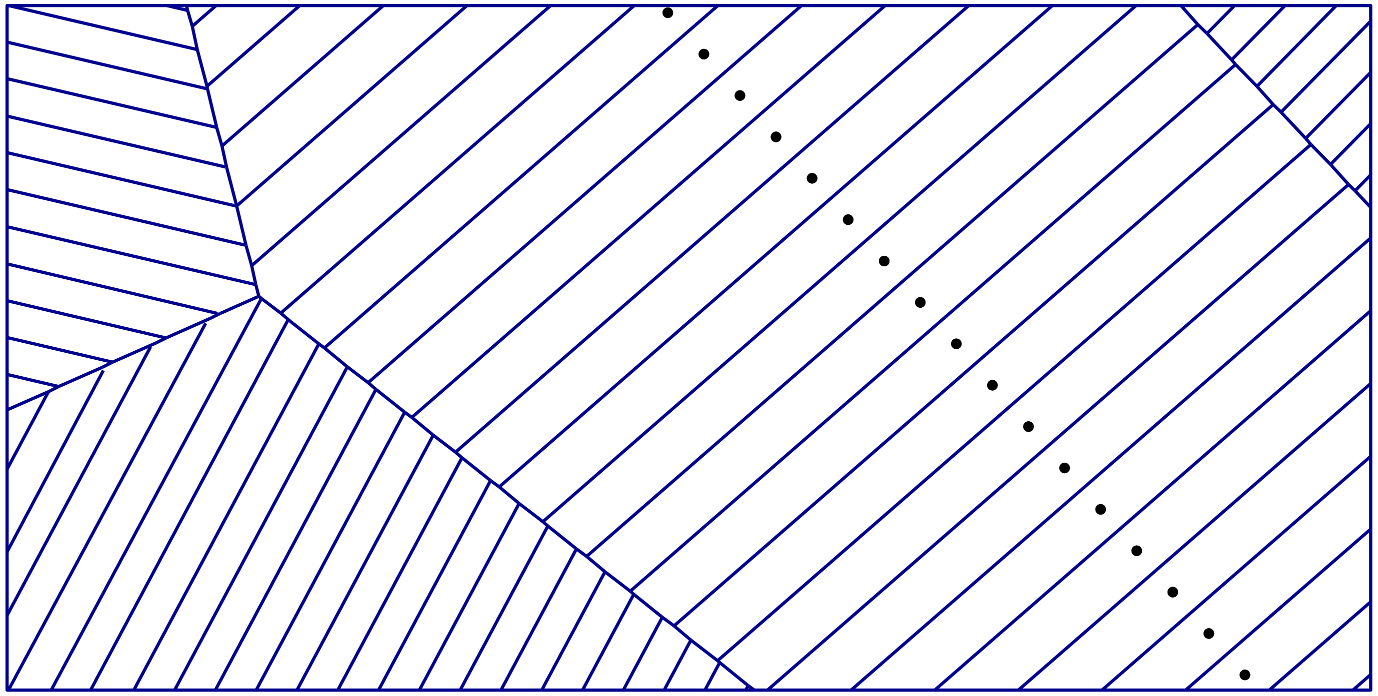
—Amenta et Bern, 1999, Surface reconstruction by Voronoi filtering

—Cohen-Steiner et al., 2007,Voronoi-based Variational Reconstruction

—G. M. O., 2009,Robust Voronoi-based Curvature and Feature Estimation



1st prvious Voronoi-based method



$$P = \{p_1, \ldots, p_N\} \subseteq \mathbb{R}^d$$

Definition :

Voronoi cell: $\mathrm{Vor}^P(q) = \{$ points whose closest point in P is $q\}$

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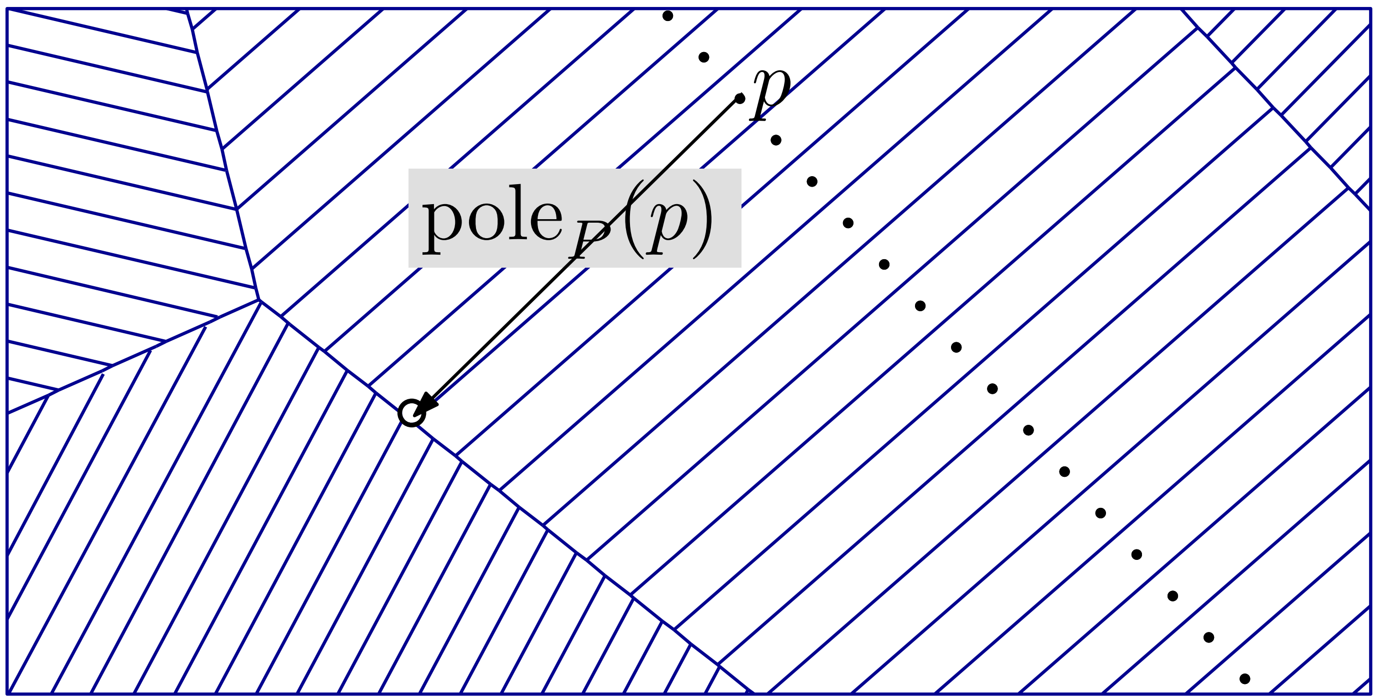
Poles method



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Amenta, Bern, *Discrete and Computational Geometry* **22** (1999)

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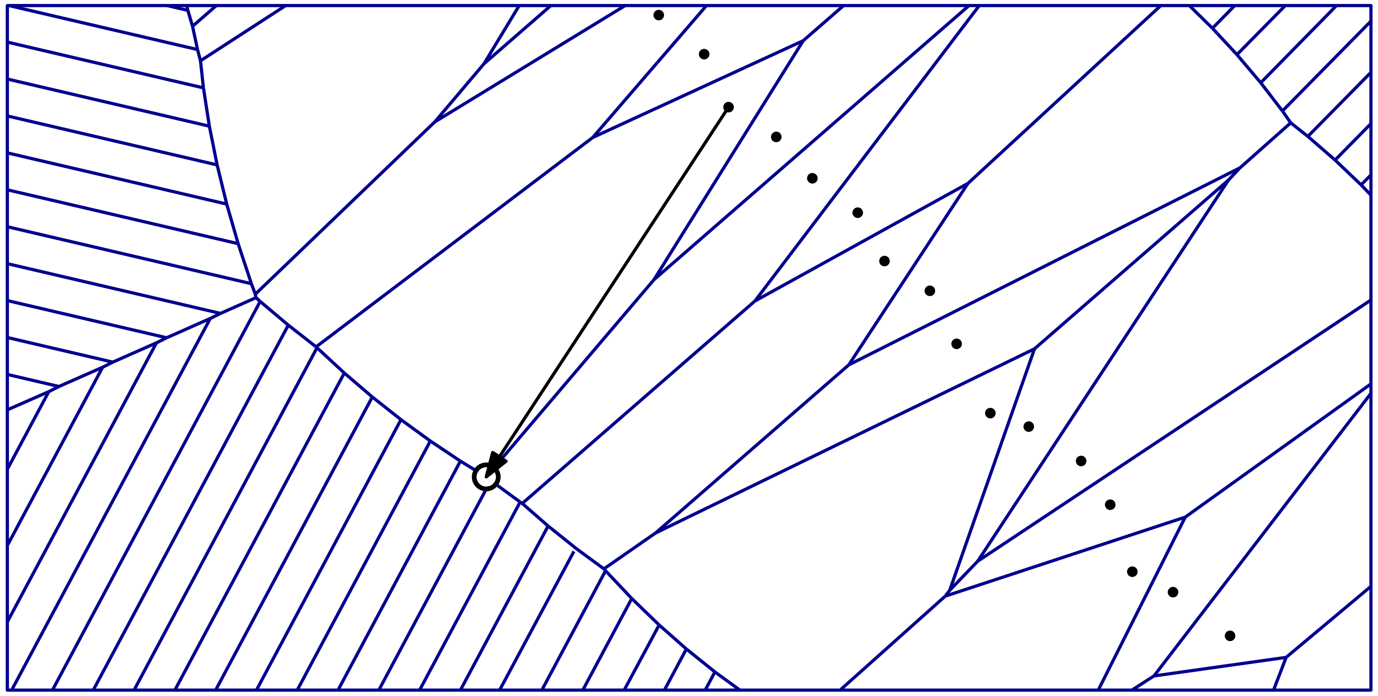

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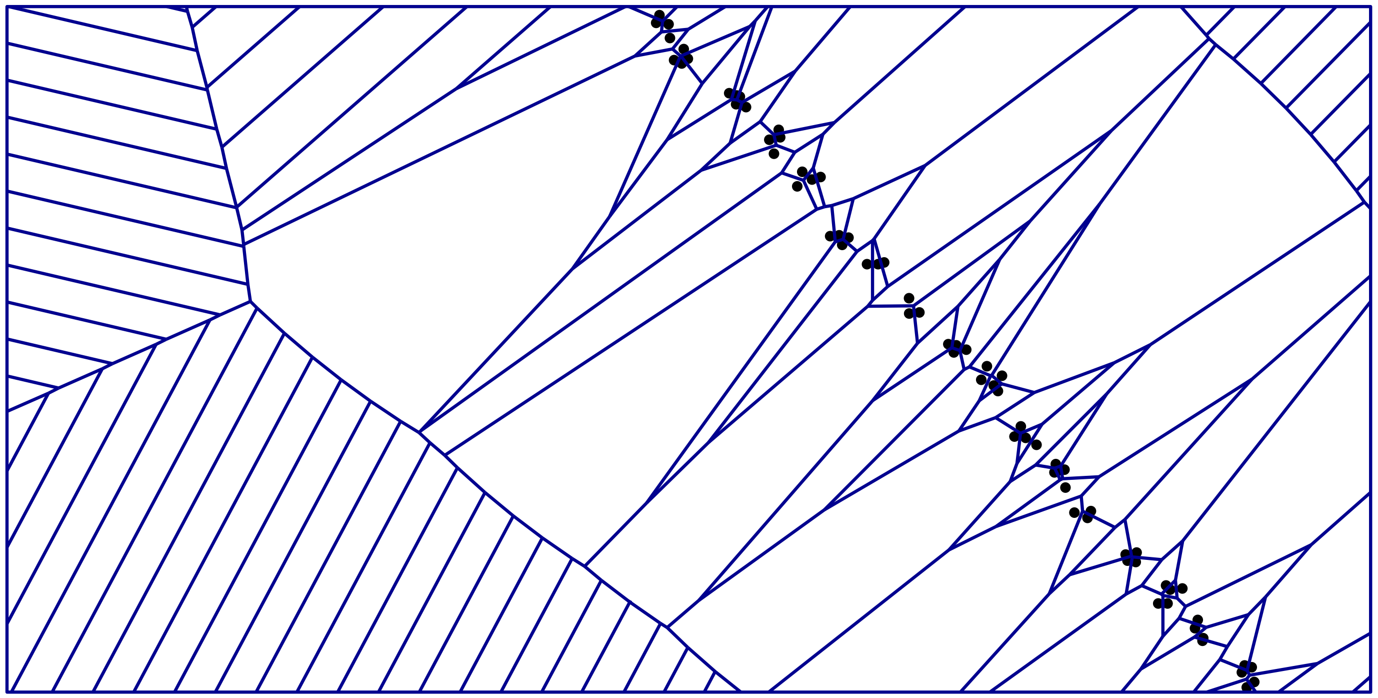

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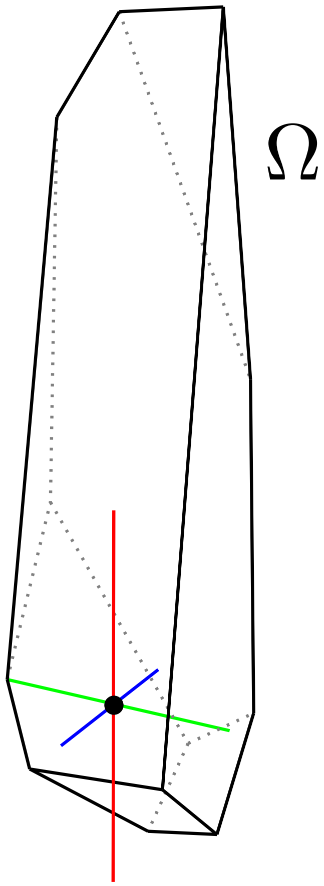
Idea : integrate to obtain stability.

2nd previous method :Voronoi Covariance

Alliez, Cohen-Steiner, Tong, Desbruns, Proc. Symposium Geometry Processing 2007

Covariance Matrix: $\mathrm{cov}_p(\Omega) := \int_\Omega (x-p)(x-p)^{\mathbf{t}} \,\mathrm{d}\, x.$

Eigenvectors of $\mathrm{cov}_p(\Omega)$ are the **principal axes** of Ω (viewed from p).



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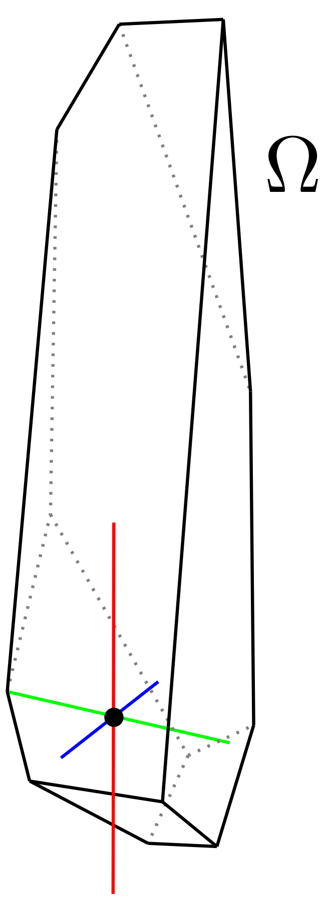
Algorithm:



- They take : $\Omega = \mathrm{Vor}_P(p_i) \cap E$
- The normal is estimated by the eigenvector corresponding to the largest eigenvalue (in red).
- Stability to noise : Sum matrices over a neighborhod.

Plan

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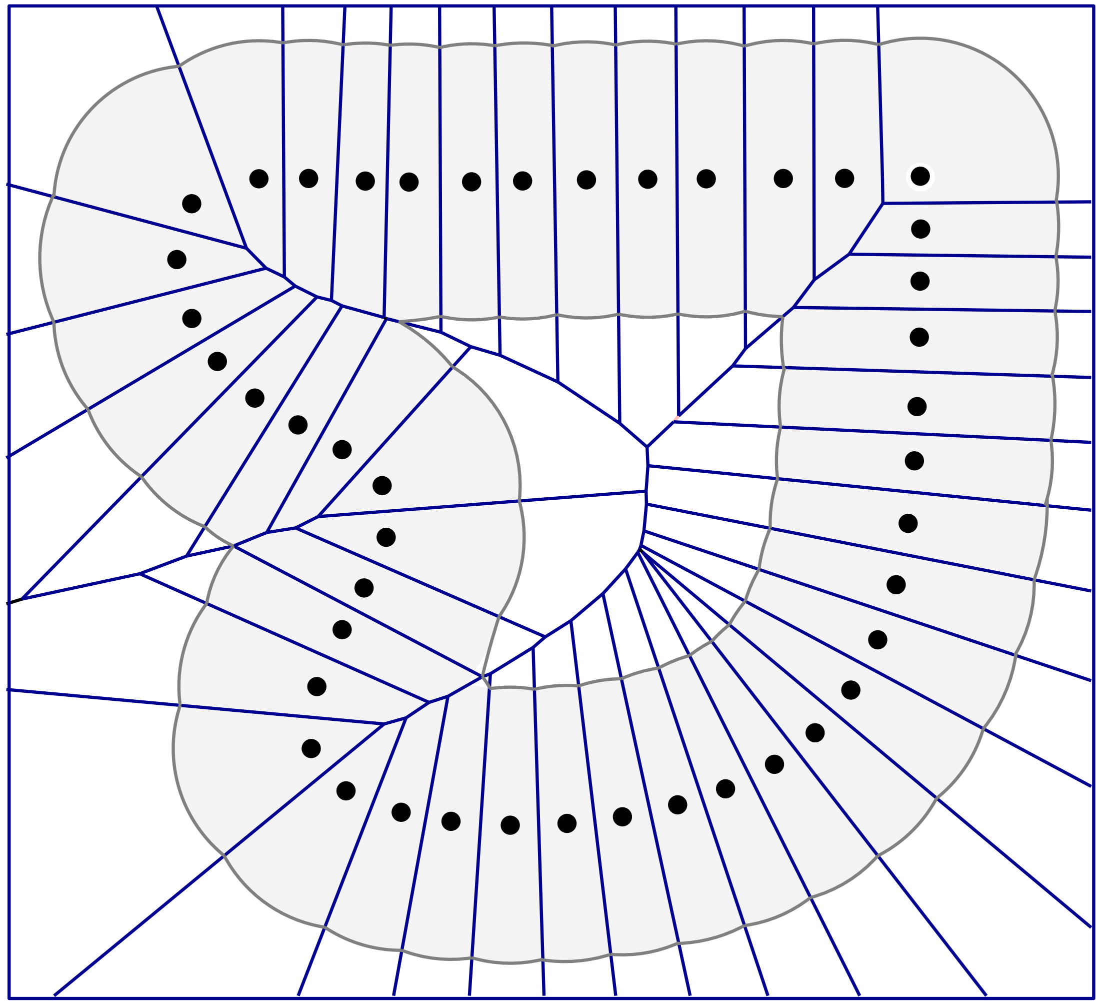
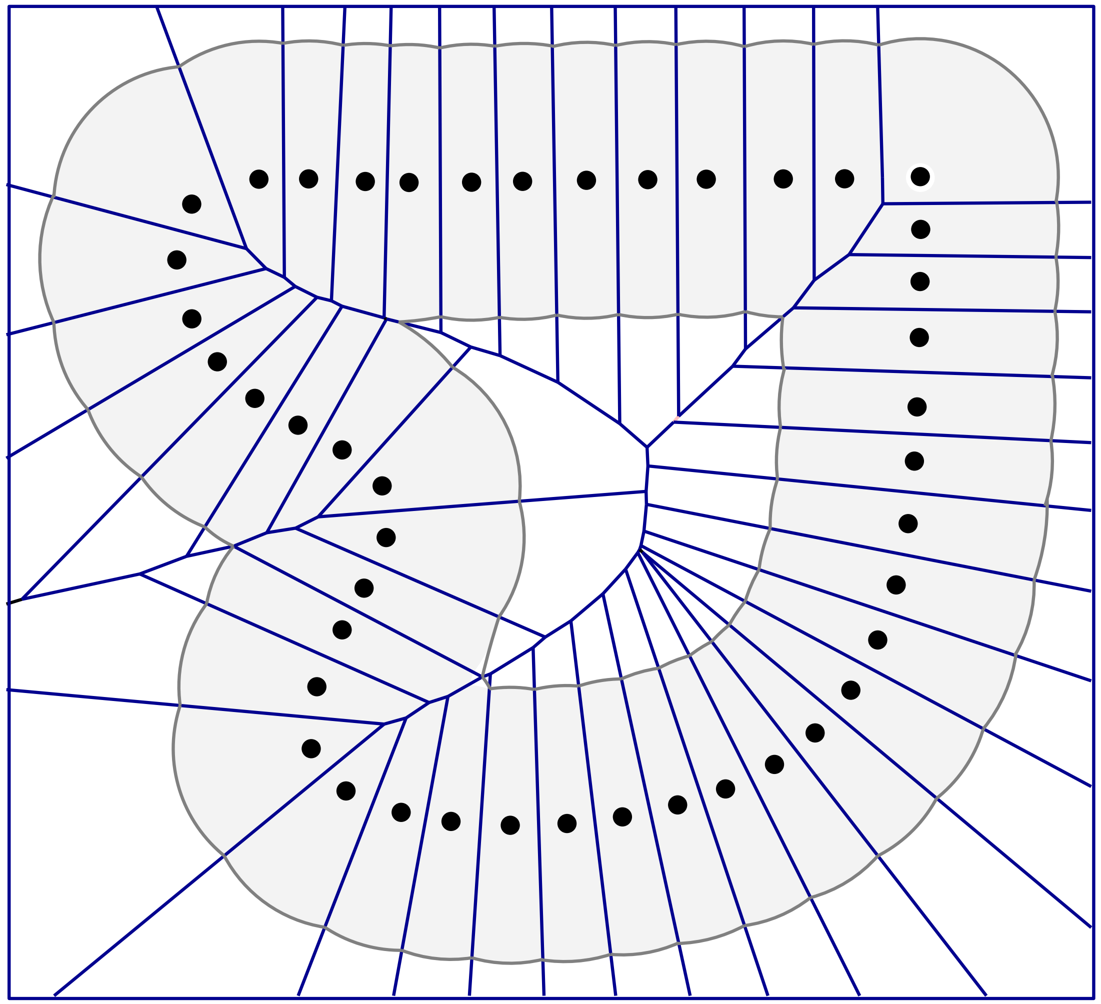
Voronoi covariance measure



Definition: Offset of P of radius R : $P^R = \cup_{p \in P} \mathrm{B}(p, R)$.

Voronoi covariance measure



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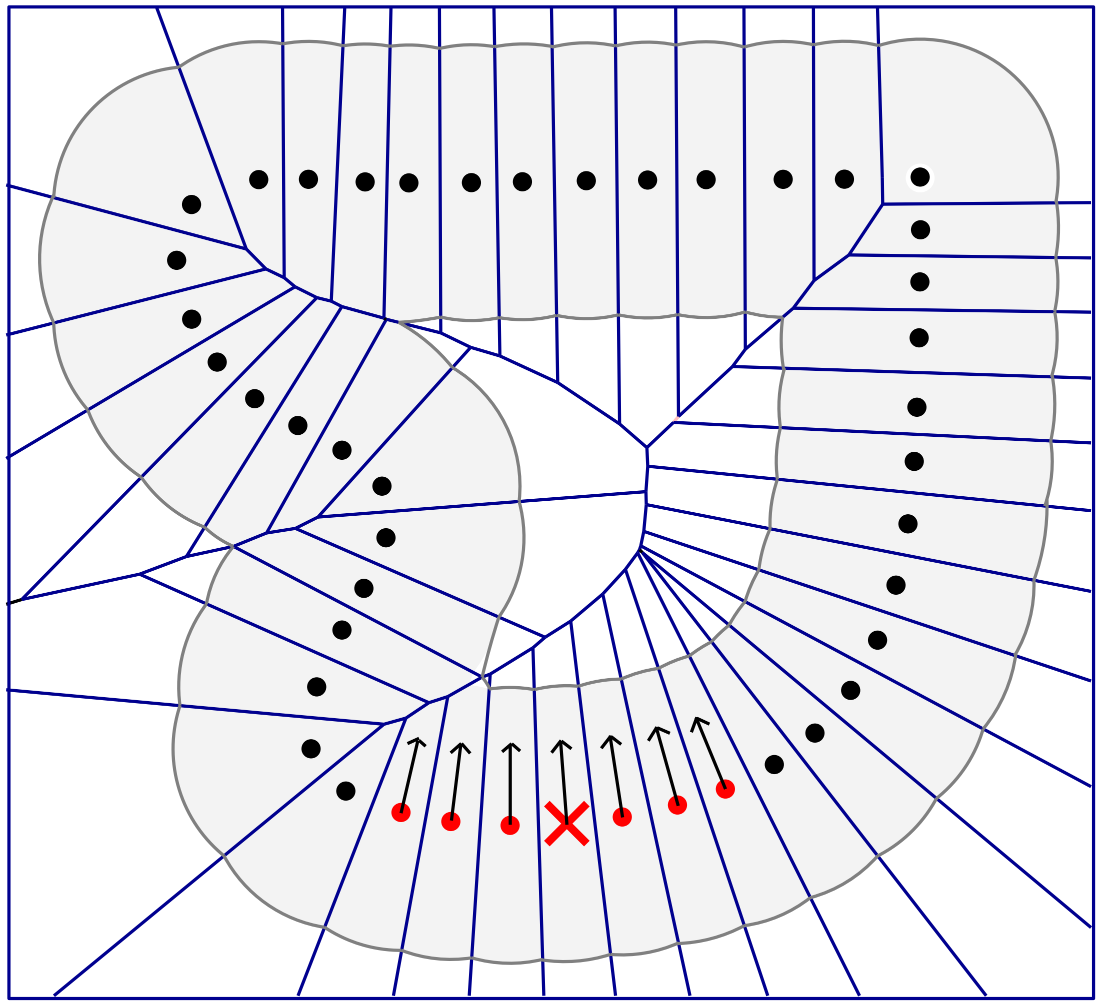
$$A(p) := \mathrm{cov}_p(\mathrm{Vor}_P(p_i) \cap P^R)$$

Definition: The *Voronoi covariance measure* of P of offset radius R is :

$$\mathcal{V}(P, R) := \sum_{i=1}^{N} A(p)\delta_p$$

Voronoi covariance measure



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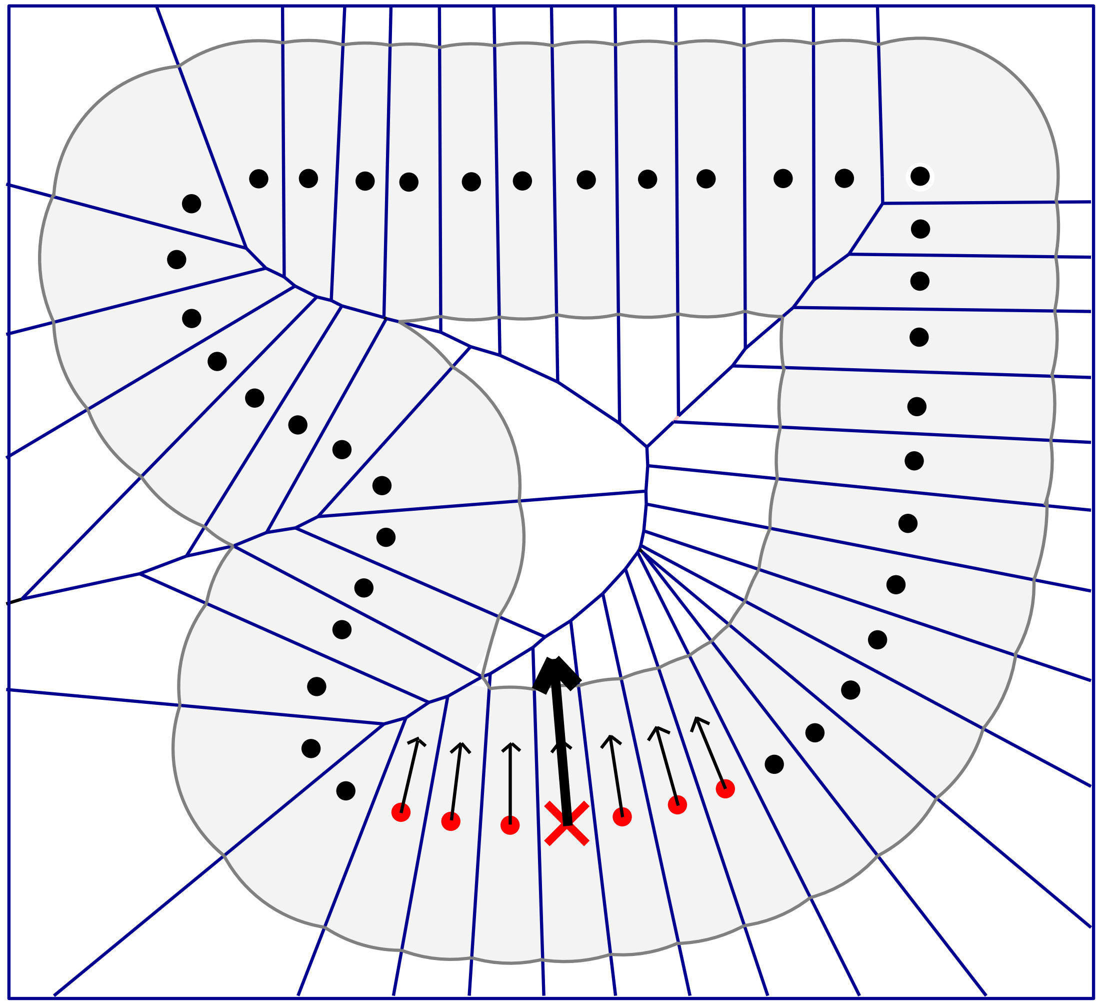
The VCM is defined for all compacts.

Voronoi covariance measure



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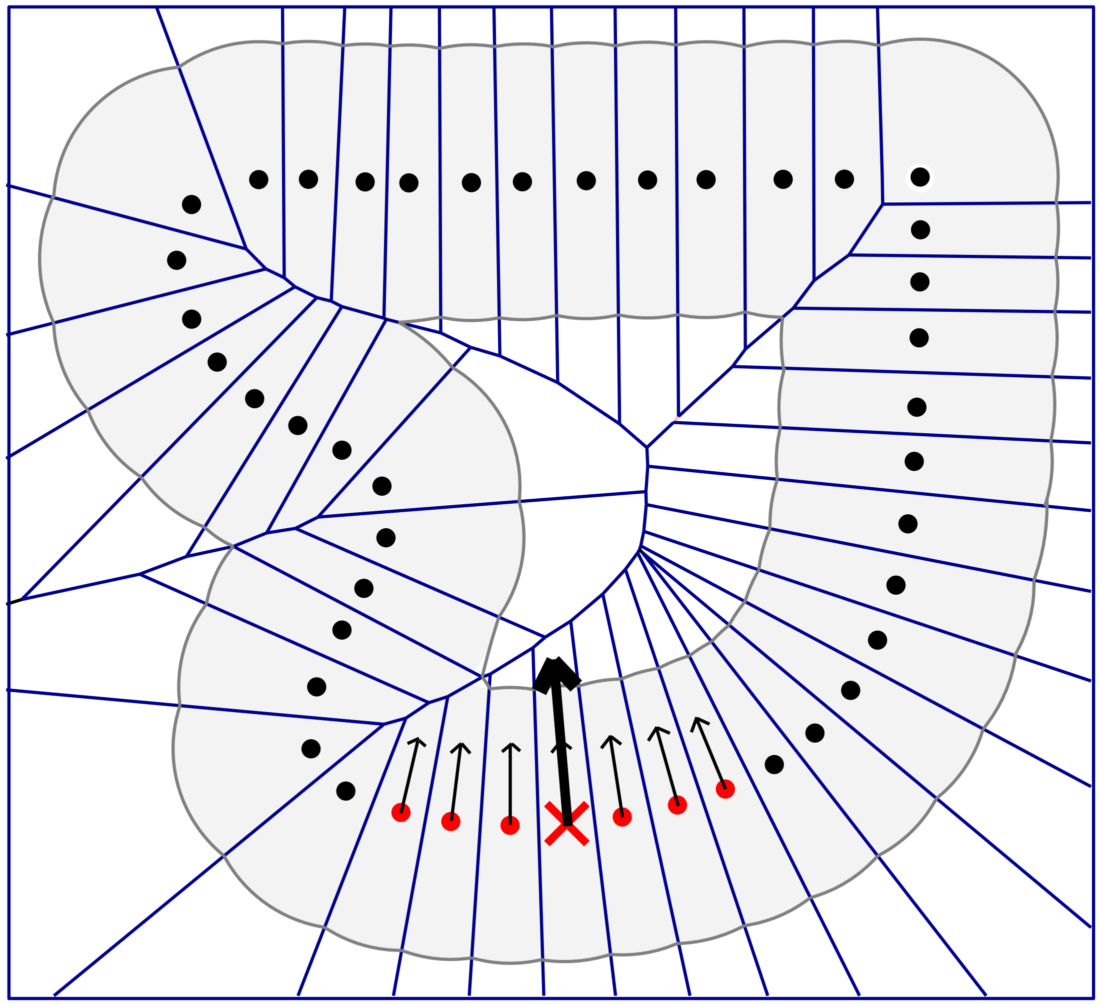

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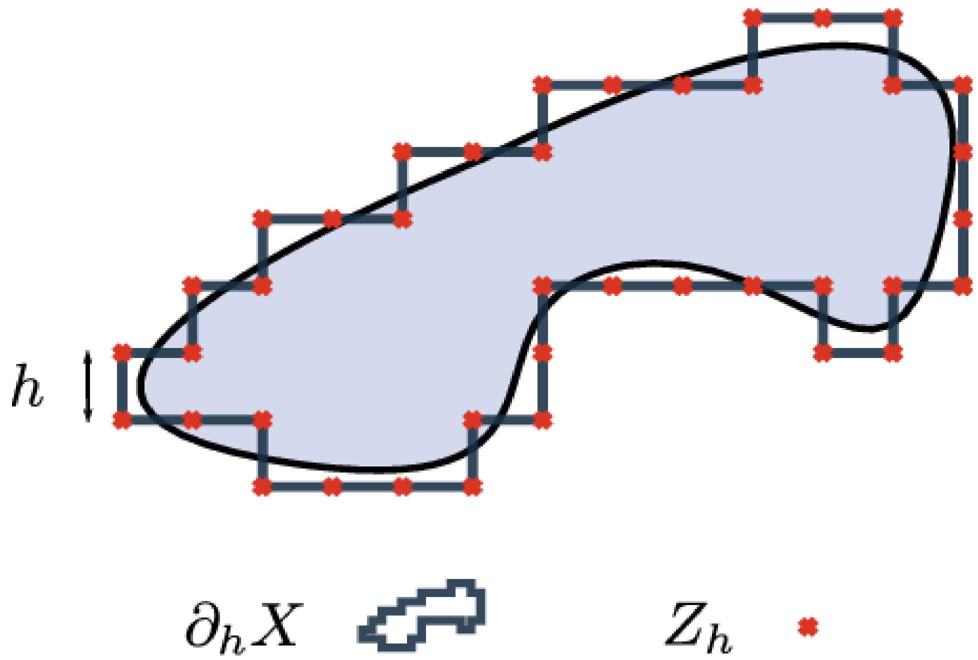
The VCM is defined for all compacts.

Theorem: Let P, K be two compacts and $p \in \mathbb{R}^d$.

$$\|\mathcal{V}(P, R) * \chi_r(p) - \mathcal{V}(K, R) * \chi_r(p)\| = \mathrm{O}(\mathrm{d_H}(K, P)^{\frac{1}{2}})$$

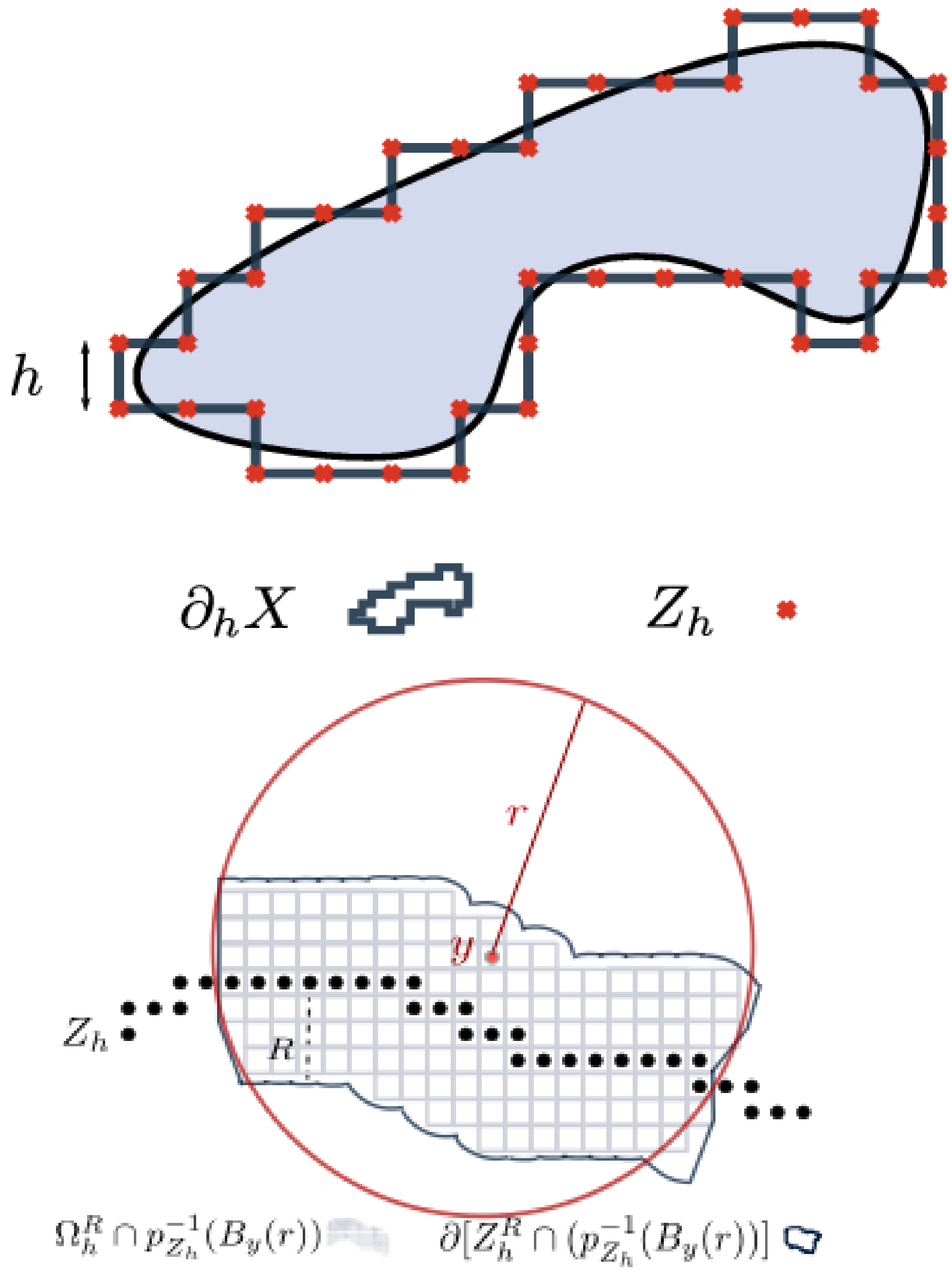
Guibas, Mérigot, Ovsjanikov, 2009

Digital Voronoi covariance measure



- X : an object
- $Dig_h(X)$: his Gauss digitization
- $Z_h = \partial_h X \cap h(\mathbb{Z} + \frac{1}{2})^3$

Digital Voronoi covariance measure



- X : an object
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We approach the VCM of the point cloud Z_h :

$$\Omega_h^R = \{x \in Z_h^R \cap h(\mathbb{Z} + \tfrac{1}{2})^3, vox(x) \subset Z_h^R\}$$

Definition: The VCM digital estimator

$$\widehat{\mathcal{V}}_{Z_h,R}(\chi) := \sum_{x \in \Omega_h^R} h^3 (x - p_{Z_h}(x))(x - p_{Z_h}(x))^t \chi(p_{Z_h}(x)),$$

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3) **Multigrid Convergence**

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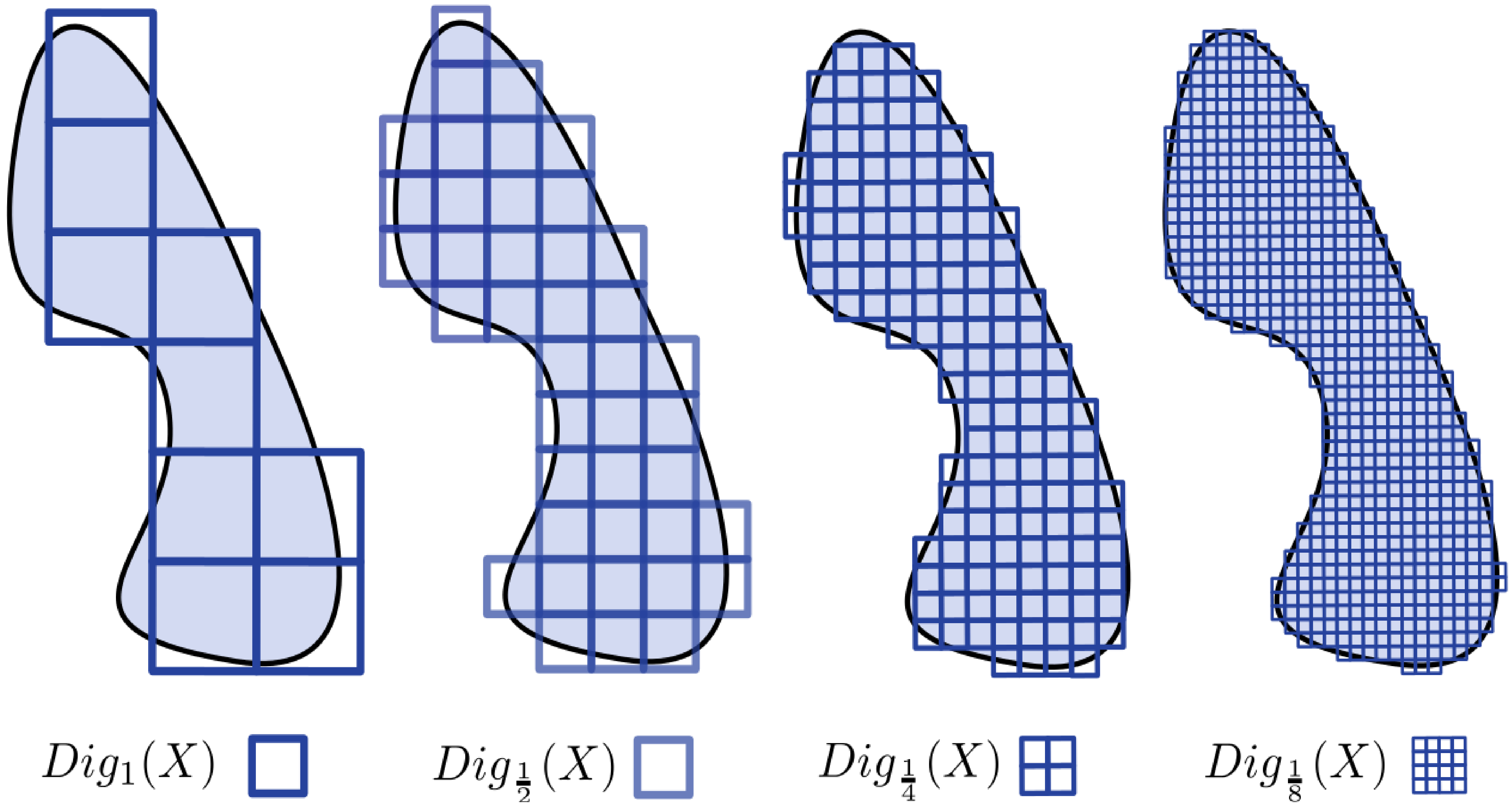
Multigrid Convergence



Multigrid Convergence : $\|\hat{E}(h) - E\| \to 0$ when $h \to 0$

VCM and Normal Multigrid Convergence

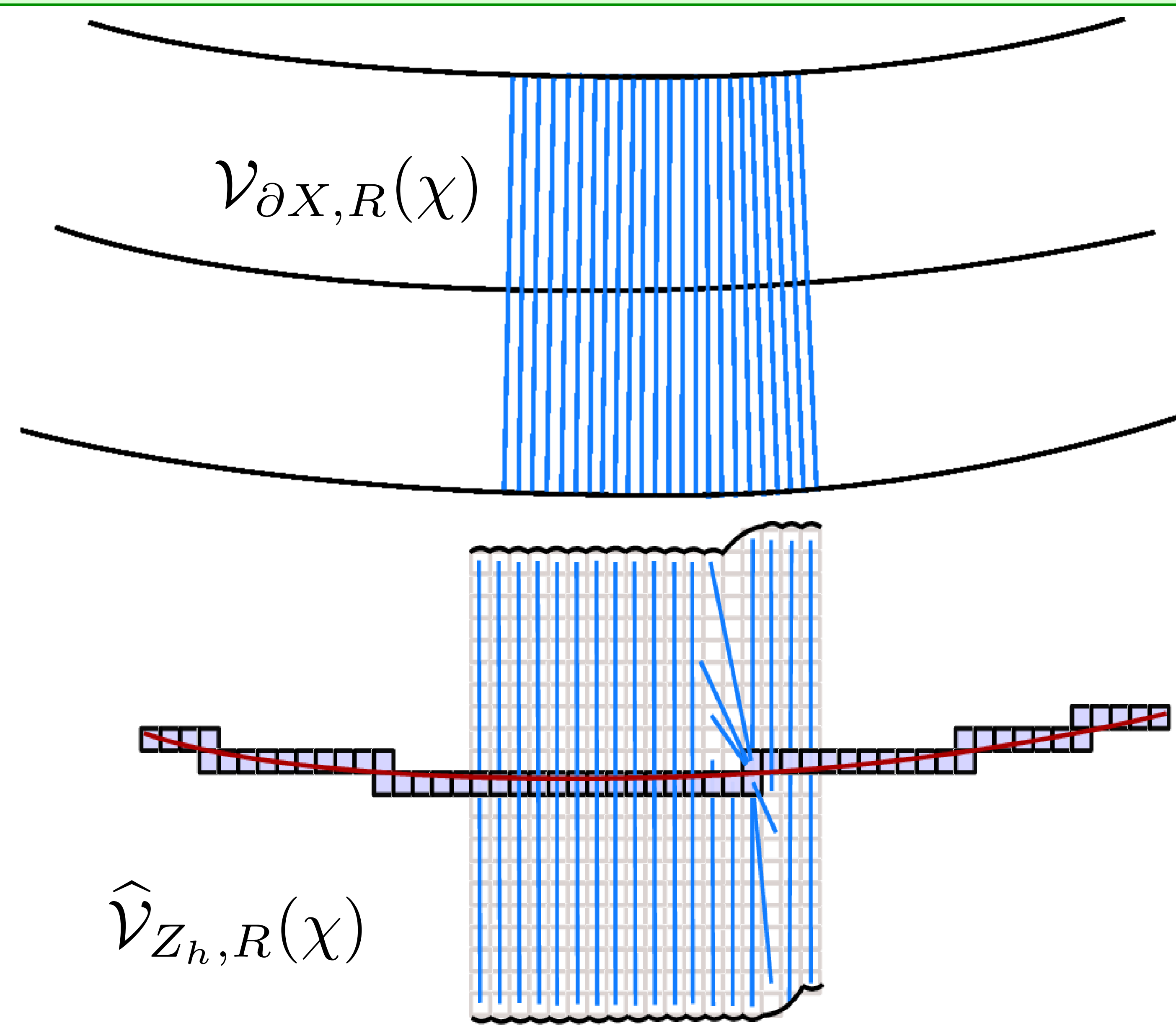
Theorem: If ∂X is C^2, $R < \frac{\rho}{2}$, $diam(supp(\chi)) < r$, $h \leq \min\left(R, \frac{r^2}{32\rho}\right)$

$$||\mathcal{V}_{\partial X,R}(\chi) - \widehat{\mathcal{V}}_{Z_h,R}(\chi)||_{op} = O\Big(\mathrm{Lip}(\chi) \times [(r^3R^{\frac{5}{2}} + r^2R^3 + rR^{\frac{9}{2}})h^{\frac{1}{2}}] + \|\chi\|_\infty \times [(r^3R^{\frac{3}{2}} + r^2R^2 + rR^{\frac{7}{2}})h^{\frac{1}{2}} + r^2Rh]\Big)$$

Quantitative
and
localized
bound

$\Downarrow$

Parameters
choice



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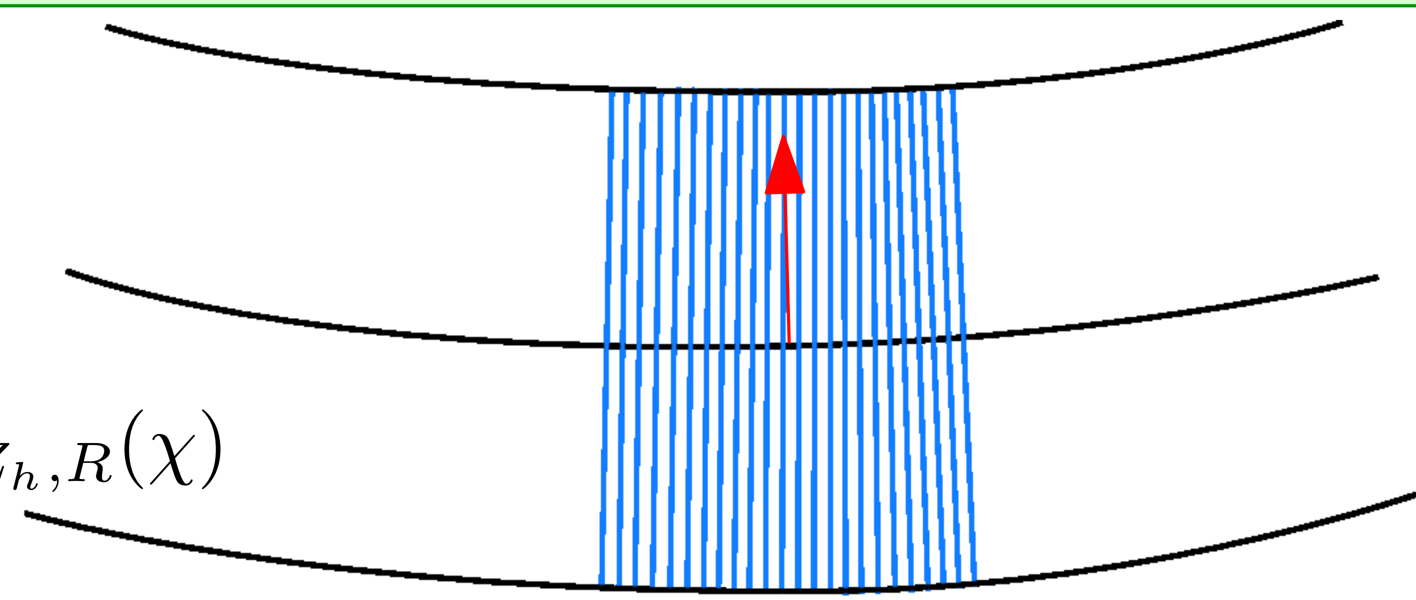
p_0 a point of ∂X

χ the hat function centered in p_0 with radius r

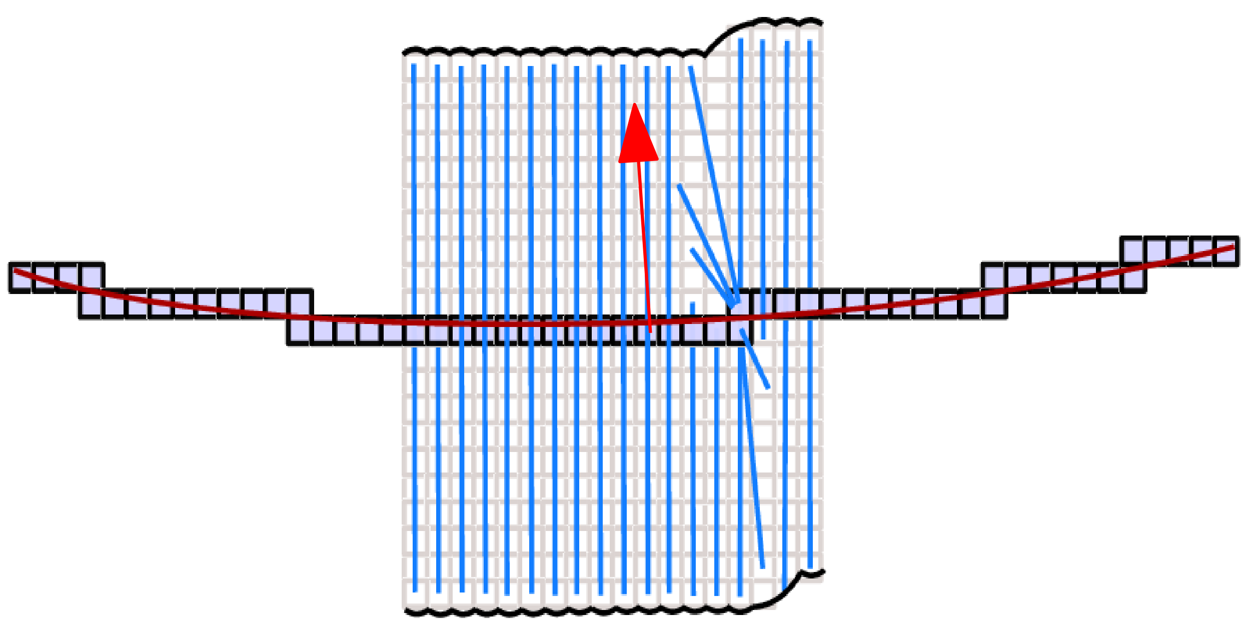
$\widehat{n}_{r,R}(\widehat{p_0}) = first\ eigenvecor\ of\ \widehat{\mathcal{V}}_{Z_h,R}(\chi)$



Corollary: normal estimator

If $r = R = ah^{\frac{1}{3}}$. Then

$\langle \widehat{n}_{r,R}(\widehat{p_0}), n(p_0)\rangle = O\left(h^{\frac{1}{6}}\right).$



Theorem, sketch of proof

$$||\mathcal{V}_{\partial X,R}(\chi) - \widehat{\mathcal{V}}_{Z_h,R}(\chi)||_{op} \leq$$
$$||\mathcal{V}_{\partial X,R}(\chi) - \mathcal{V}_{Z_h,R}(\chi)||_{op} + ||\mathcal{V}_{Z_h,R}(\chi) - \widehat{\mathcal{V}}_{Z_h,R}(\chi)||_{op}.$$

Localization error　　　　　　Digitization error

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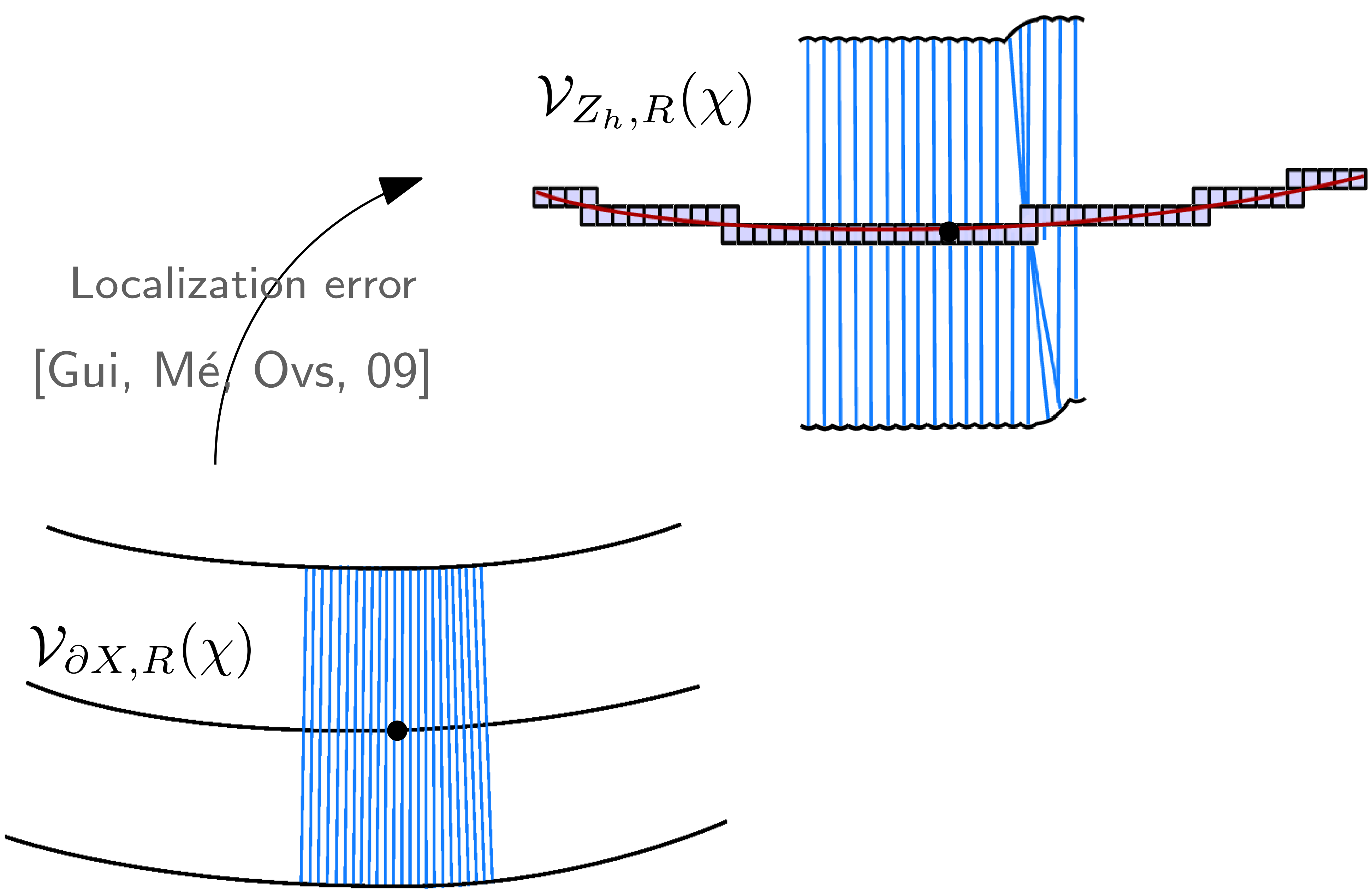

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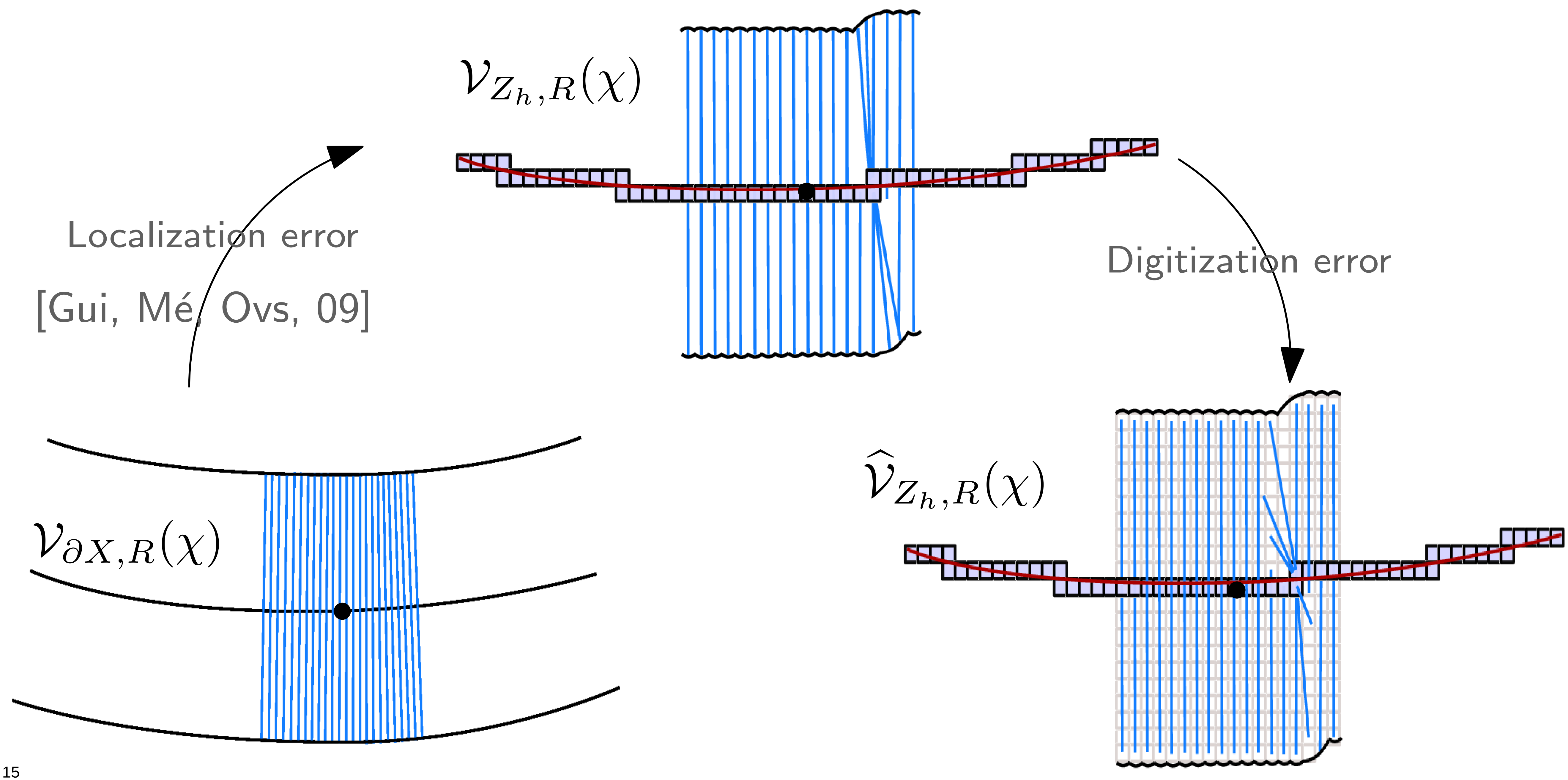

Plan

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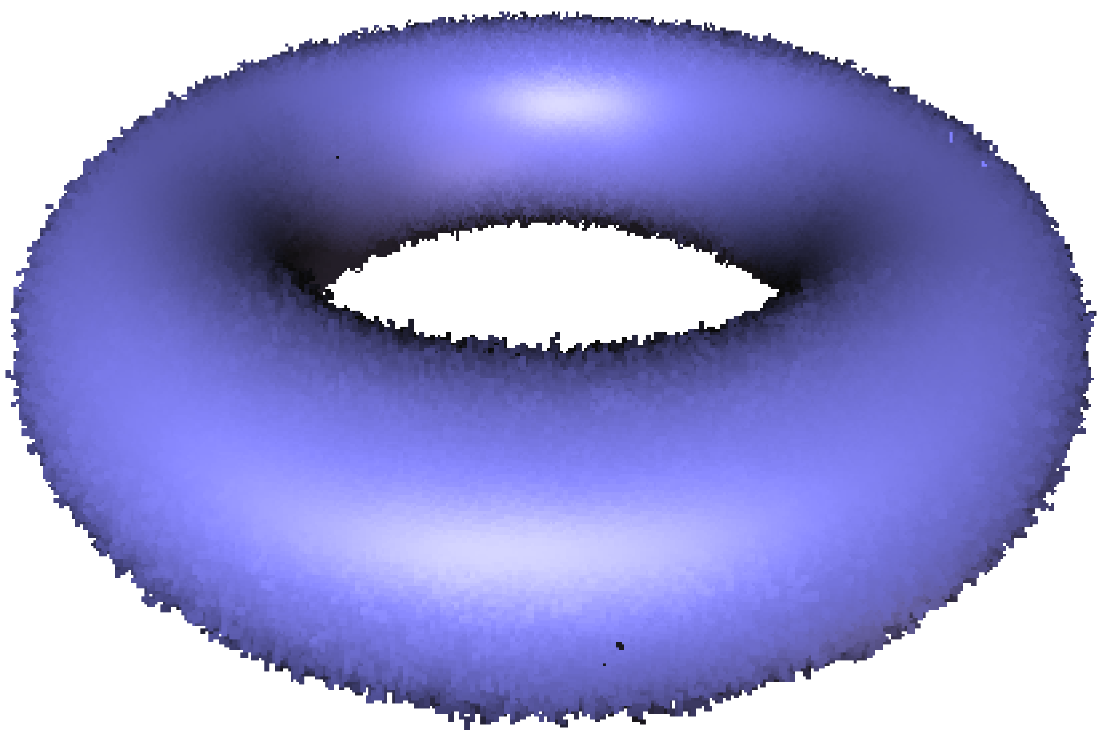
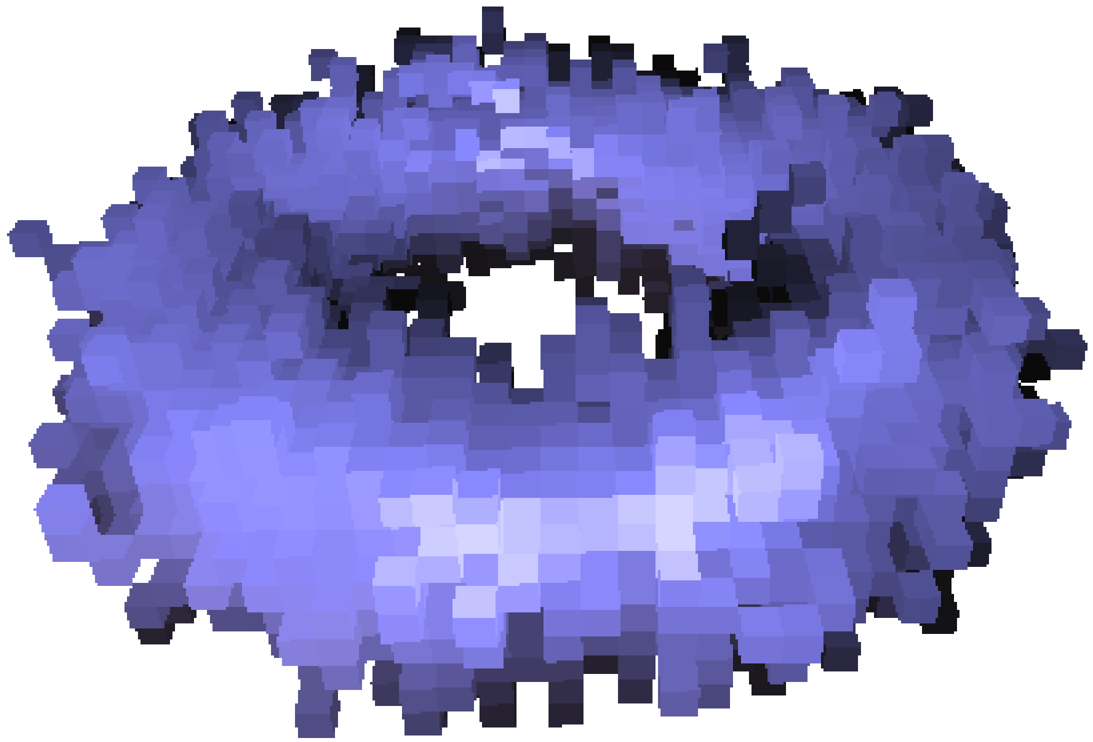
3) Multigrid Convergence

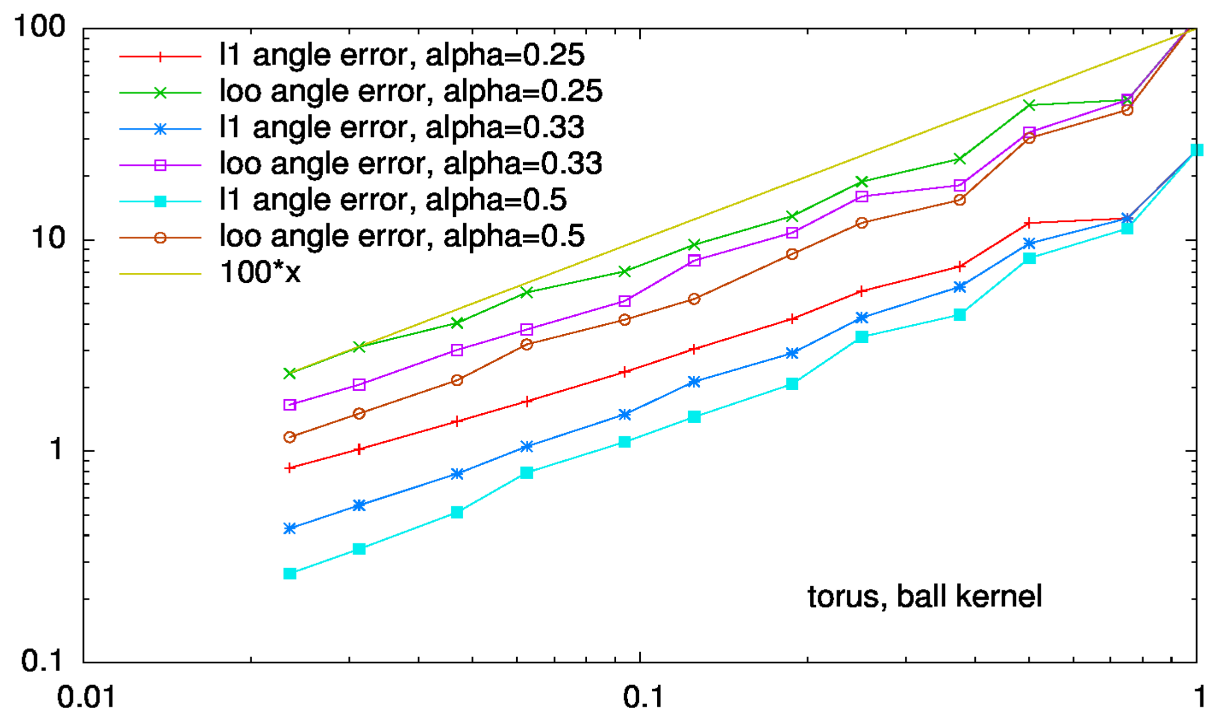
4) **Experiments**

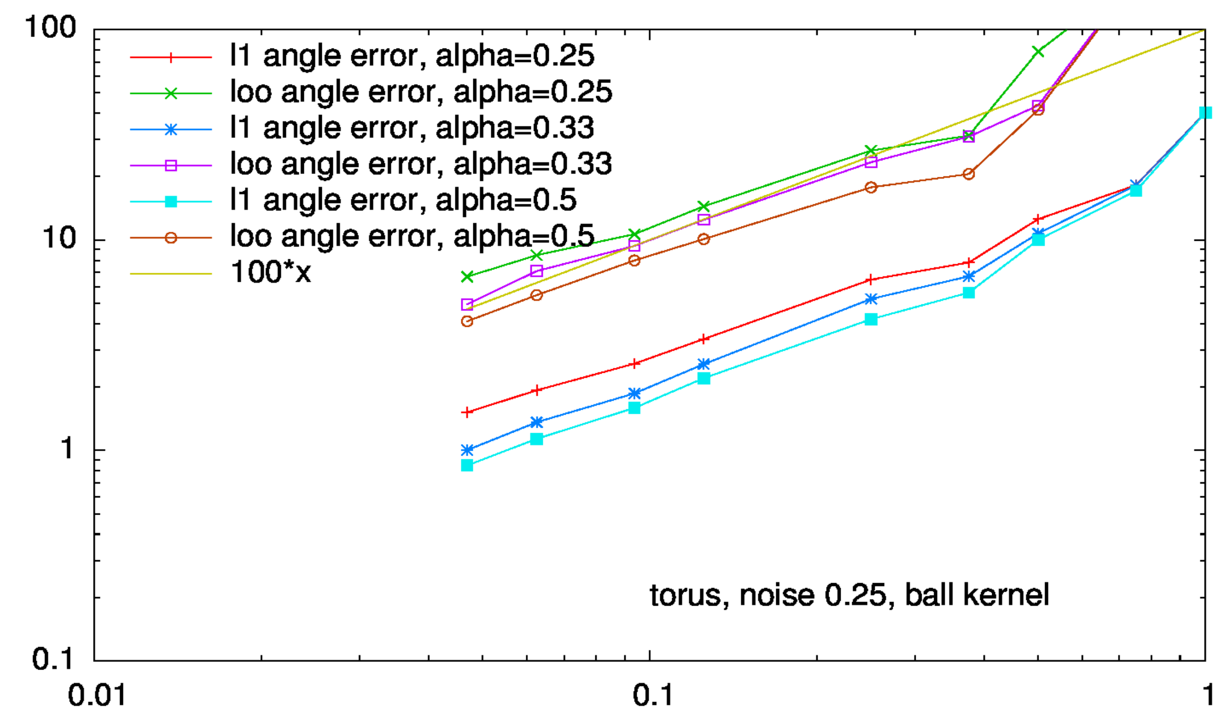
Experiments, Convergence Validation

Implementatated on DGtal library :

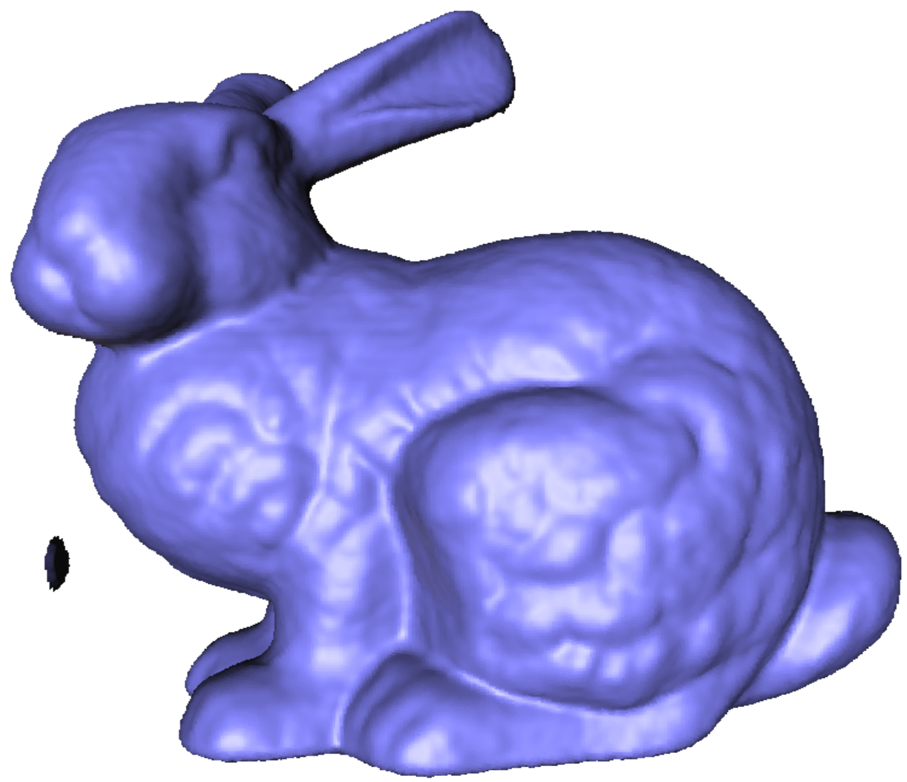




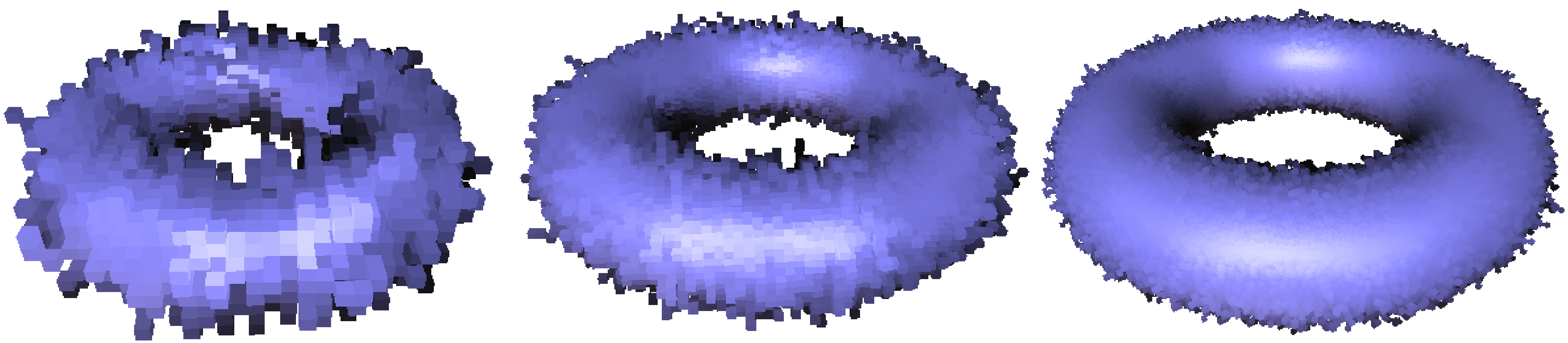




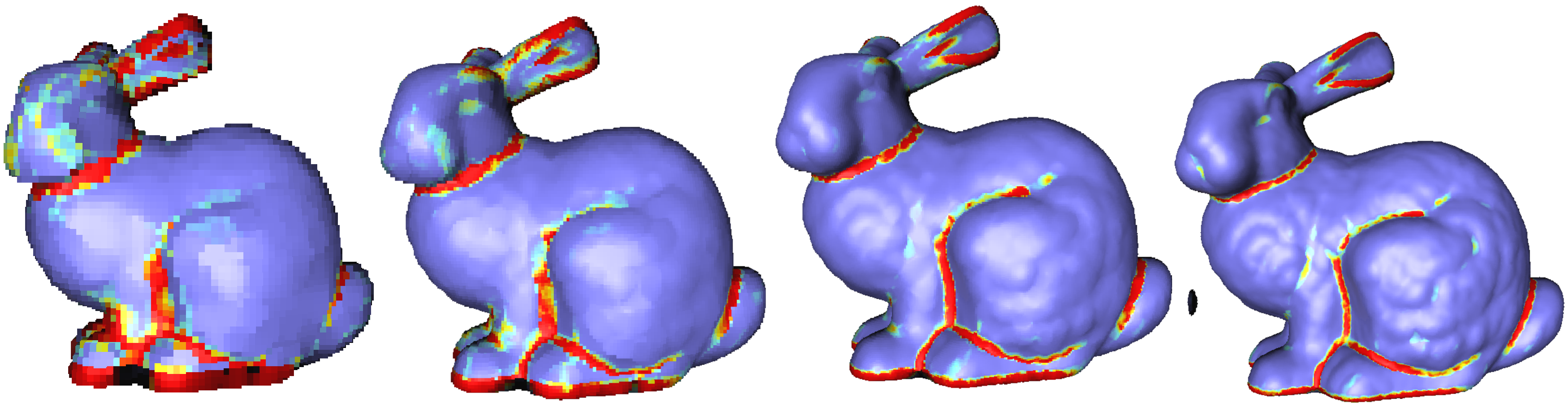
Experiments, Normal Visualization



$$r = R = 3h^{0.5}$$



Experiments, Sharp Feature Estimation



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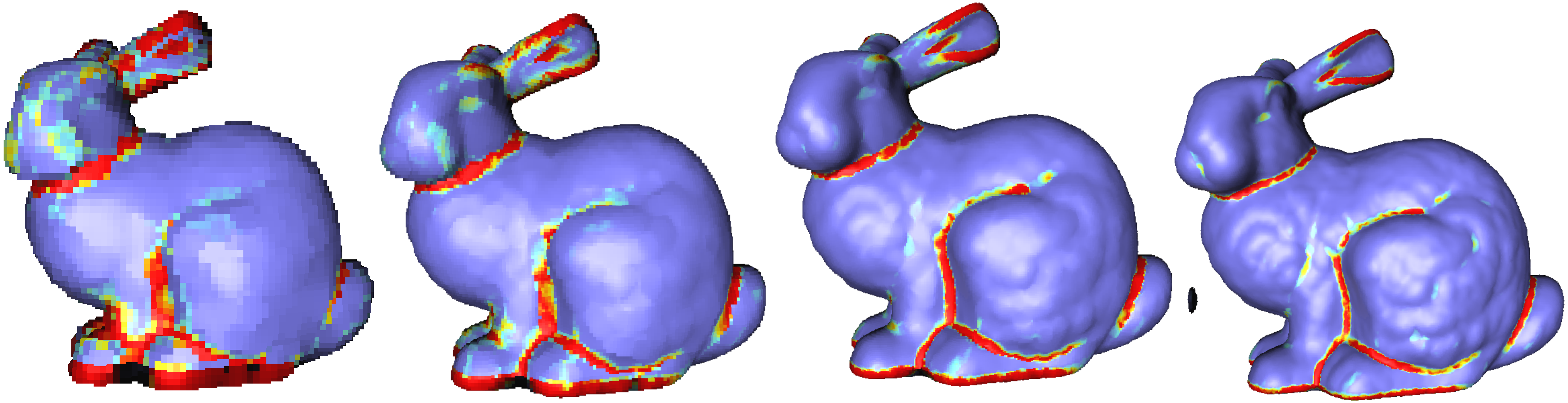

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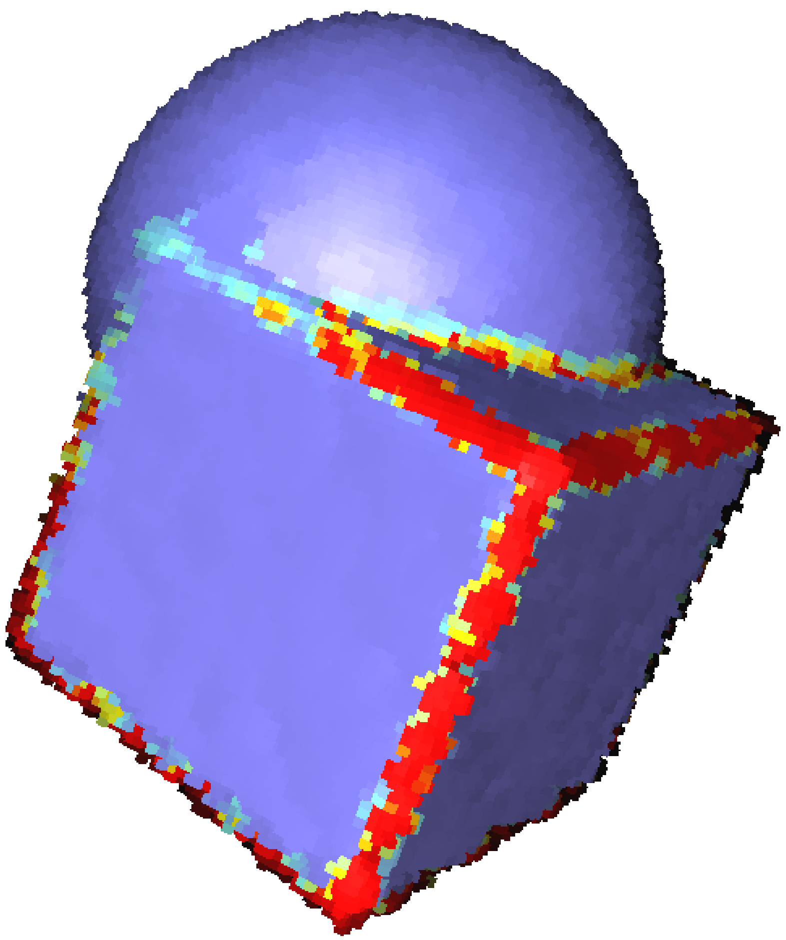
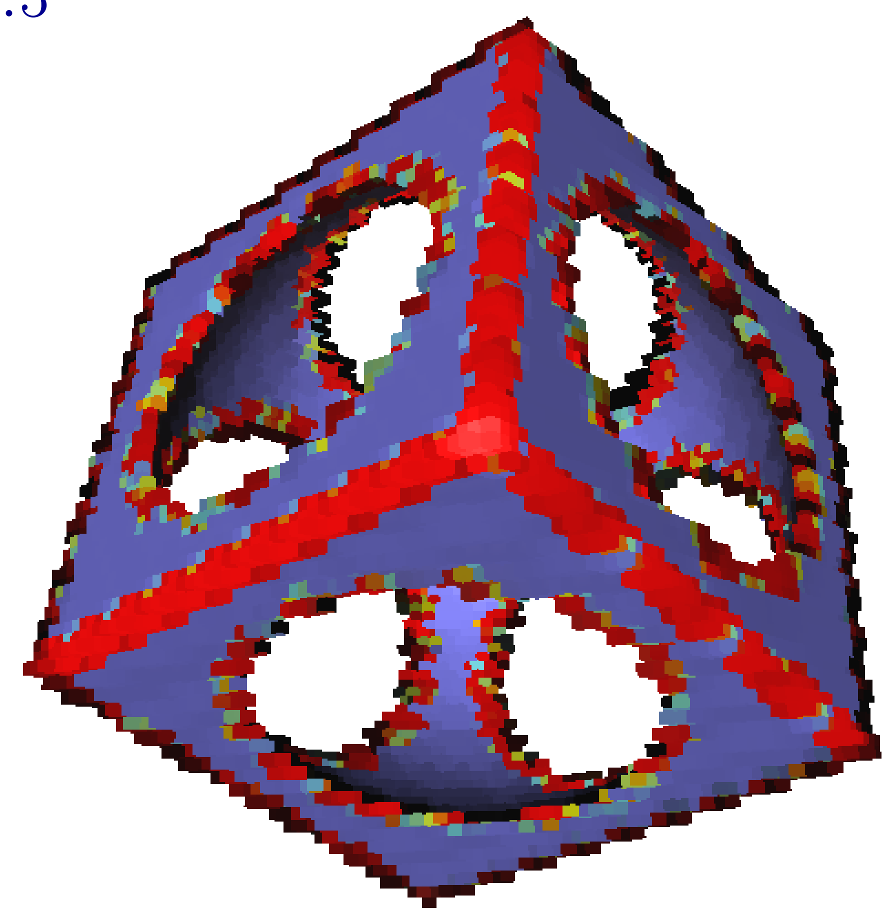
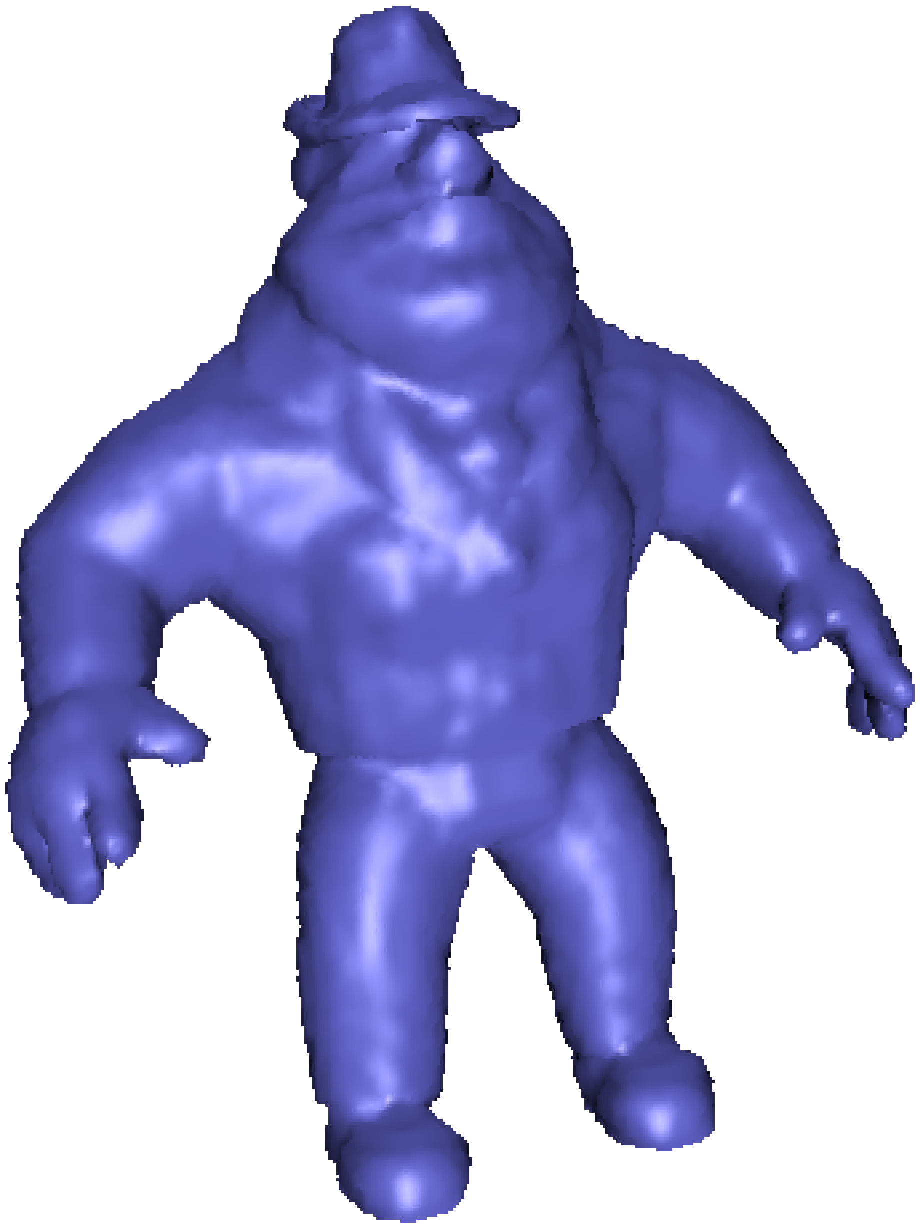




Conclusion

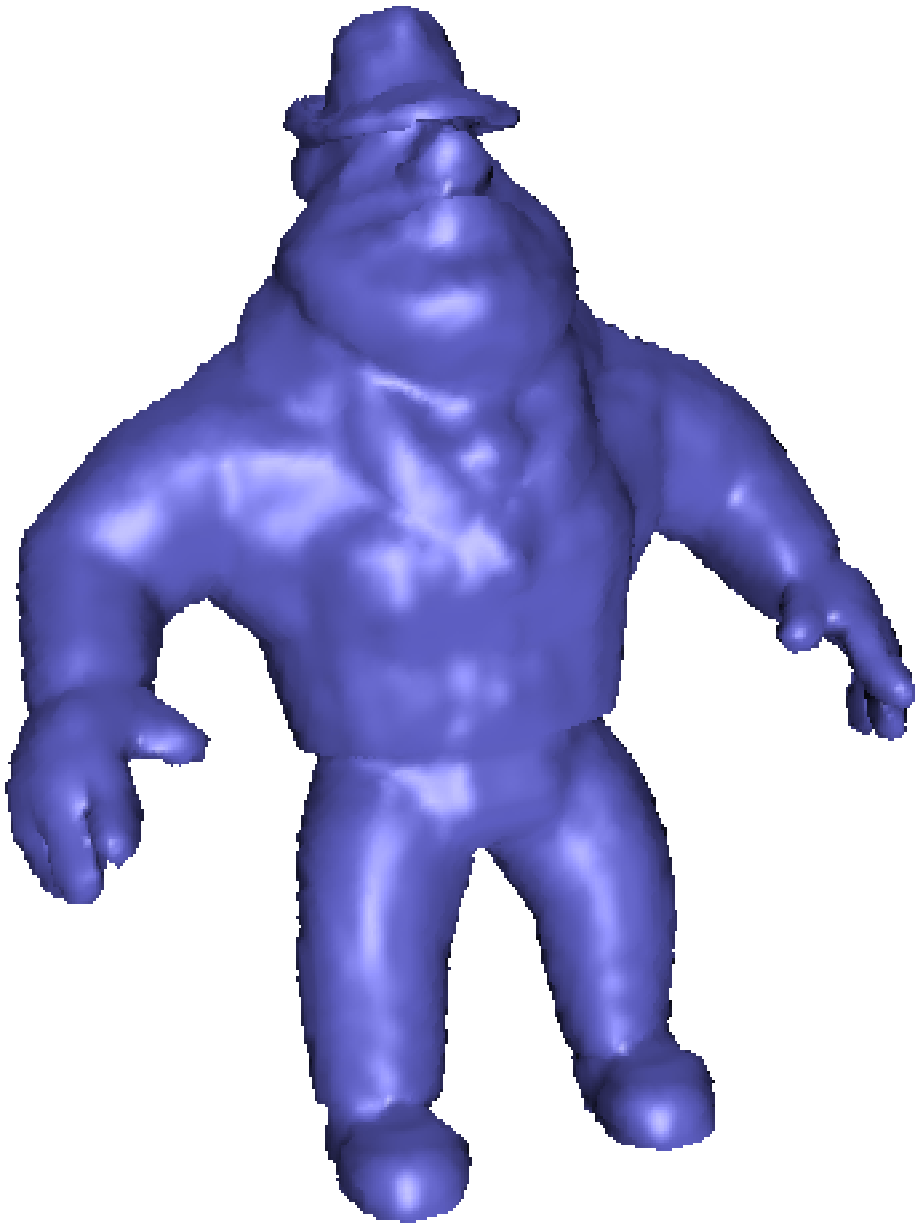


Summary:

We have several geometric estimators

- For normals, curvatures and sharp features :
- Theorically stable
- Efficient
- Convergent and accurate in practice

Conclusion



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Perspectives:

- Distance to a measure + Voronoi Covariance

$= \delta$-VCM robust to outliers

Even with outliers