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Lightweight Curvature Estimation on Point Clouds with Randomized Corrected Curvature Measures

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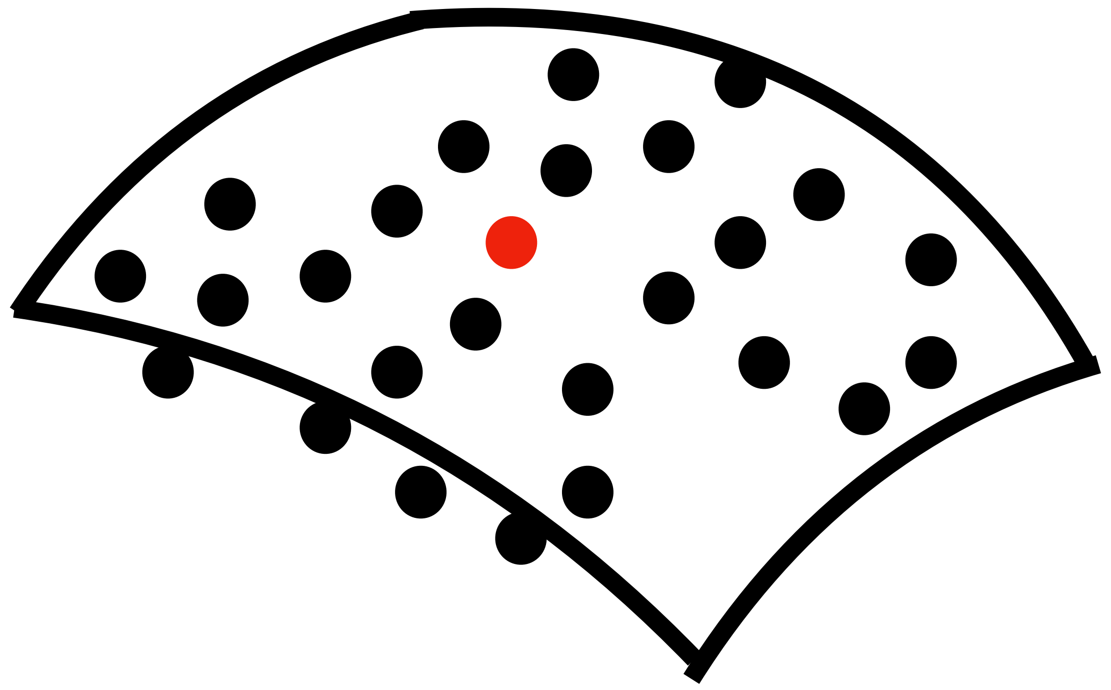
Our contribution in a nutshell

- New estimators of **curvatures** for **oriented point clouds**
  - Use theory of corrected curvature measures
  - Local and independent computations per point
  - More accurate and faster than state-of-the-art
- **Stability theorem** in case of **positions** and **normals perturbations**
  - Error bounded by $O(\delta)$, for δ the computation window
  - Convergence if variances of perturbations are lower than $O(\delta^2)$

Curvature estimations for point clouds

Usual approach:



Smooth surface fitting

+

Classical **differential** geometry

Efficient algorithms, but challenging to have guarantees in case of noise in data

- Polynom gne, Chaîne 18]
- Point se
  - moving least squares [Alexa et al 01],
  - algebraic sphere fitting [Guennebaud, Gross 07][Mellado et al 12]
  - whole curvature tensor through differentiation [Lejemble, Coeurjolly, Barthe, Mellado 21]
- Integral invariants + kernel functions [Pottmann et al. 07 and 09] [Digne, Morel 14]
- Deep learning methods : PCPNet [Guerrero, Kleiman, Ovsjanikov, Mitra 18]

Curvature estimations: theories with stability

- Embed discrete and smooth objects in the same framework
- Define geometric information as integral measures

Stability in normal and position
⇒ stability in features/curvatures

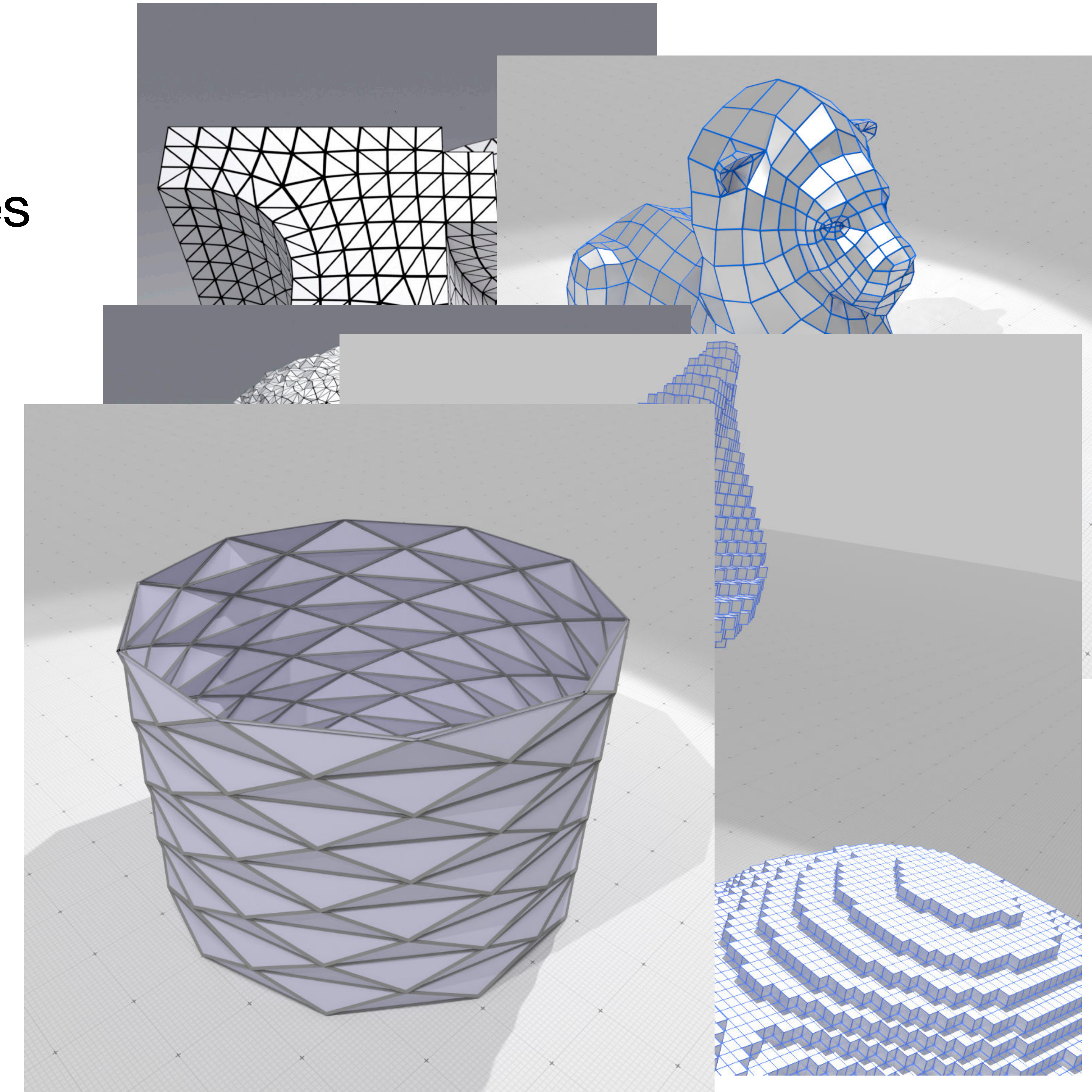
Nice theories, but lack of efficient algorithms for curvature estimation

- Voronoi C
    - Stal
    - Stable to ouliers with distance to a measure [Cuel, L., Mérigot, Thibert 14]
- Curvature measures for piecewise smooth surfaces :
    - Normal cycle [Wintgen 82][Cohen-Steiner, Morvan 03 and 06],
    - Point clouds through double offsets [Chazal, Cohen-Steiner, Lieutier, Thibert 09]
- Varifolds : [Almgren 66] [Buet, Leonardi, Masnou 17, 18, and 19]
    - Unsigned variants of curvatures

Corrected curvatures measures for discrete surfaces [L., Romon, Thibert 22]

Curvature measures for discrete surfaces

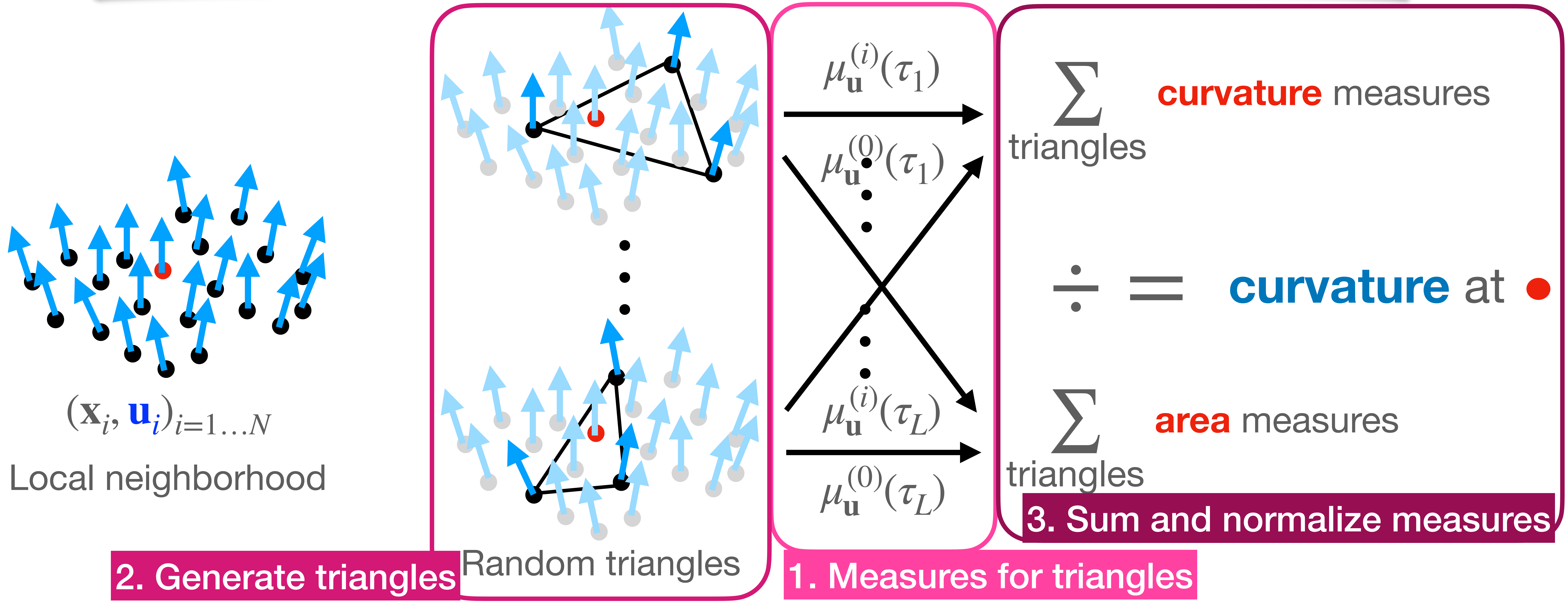
- triangulated
- quadrangulated
- noisy positions or normals
- digital surfaces
- Schwarz lantern



Stable notions of area, mean, Gaussian, principal curvatures

Extend a surface theory to oriented point clouds

Key idea:
measures do not need consistent mesh topology



1. Interpolated corrected curvature measures on a triangle

[L., Romon, Thibert, Coeurjolly SGP2020]

Lipschitz-Killing differential form (area)

lives in

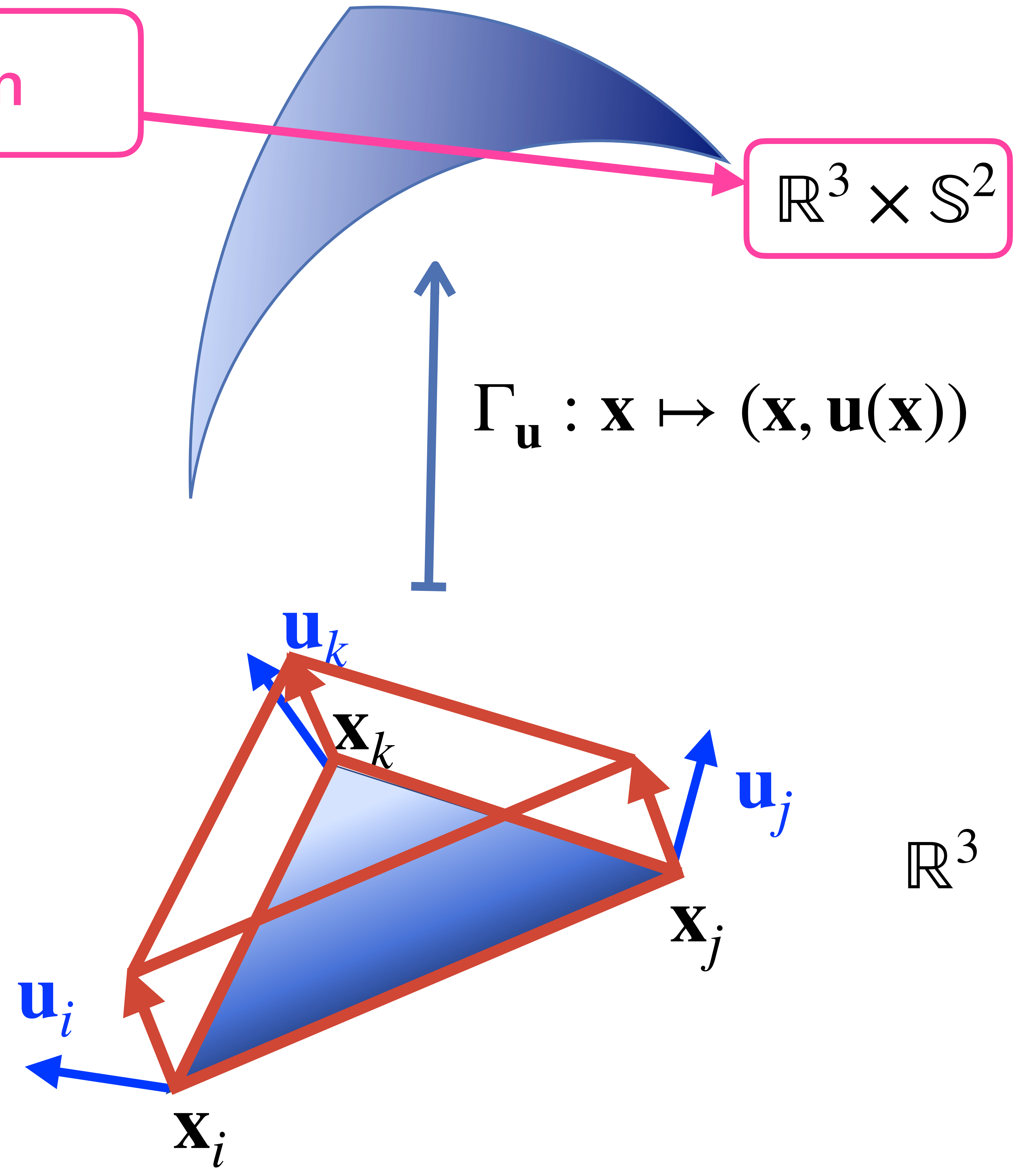
Grassmannian

Area measure $\mu_{\mathbf{u}}^{(0)}(\tau) = \int_\tau \Gamma_{\mathbf{u}}^* \boxed{\omega^{(0)}}$

$$= \int_0^1 \int_0^{1-t} \det\left(\mathbf{u}, \frac{\partial \mathbf{x}}{\partial s}, \frac{\partial \mathbf{x}}{\partial t}\right) \mathrm{d}s\mathrm{d}t$$

$$= \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i)\rangle$$

with $\bar{\mathbf{u}} := (\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k)/3$



1. Interpolated corrected curvature measures on a triangle

[L., Romon, Thibert, Coeurjolly SGP2020]

Mean curvature measure

$$\mu_{\mathbf{u}}^{(1)}(\tau) = \int_\tau \Gamma_{\mathbf{u}}^* \boxed{\omega^{(1)}}$$

$$= \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{u}_k - \mathbf{u}_j) \times \mathbf{x}_i + (\mathbf{u}_i - \mathbf{u}_k) \times \mathbf{x}_j + (\mathbf{u}_j - \mathbf{u}_i) \times \mathbf{x}_k \rangle$$

Lipschitz-Killing
differential forms
(Mean and Gaussian forms)

Gaussian curvature measure

$$\mu_{\mathbf{u}}^{(2)}(\tau) = \int_\tau \Gamma_{\mathbf{u}}^* \boxed{\omega^{(2)}}$$

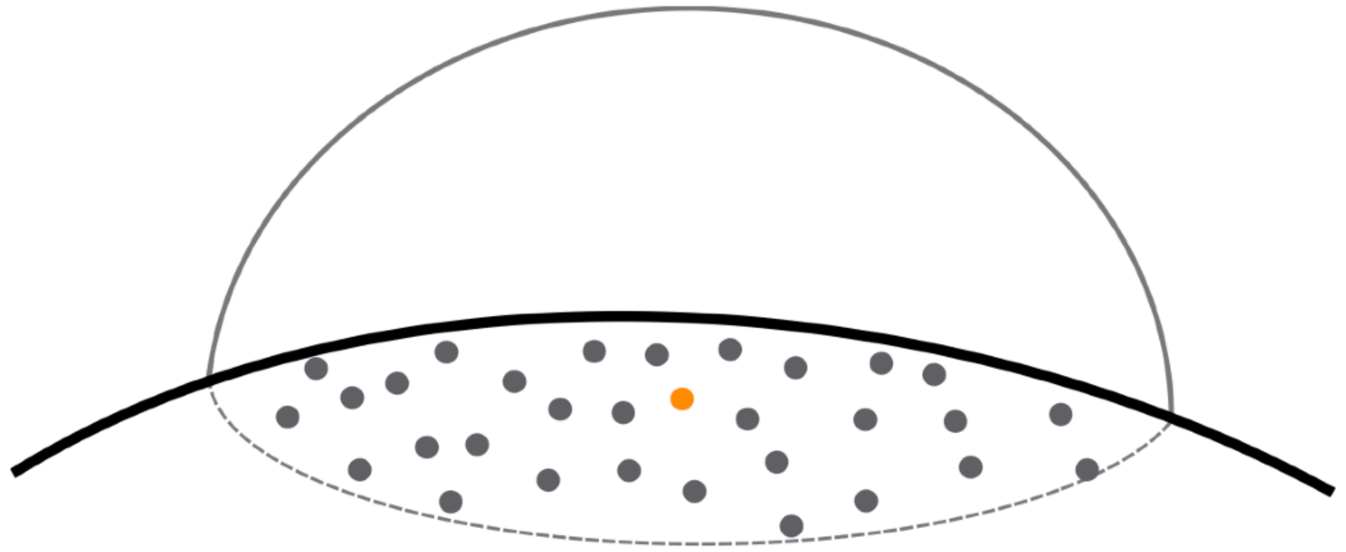
$$= \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{u}_j - \mathbf{u}$$

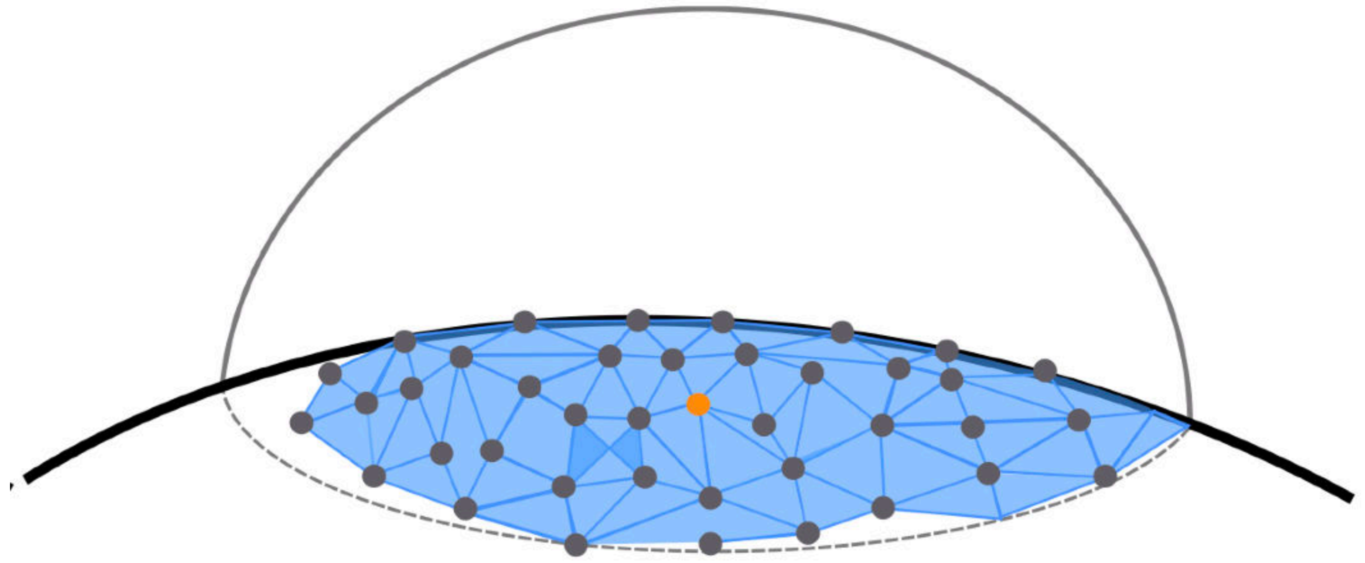
Anisotropic form ≈ curvature tensor measure

$$\mu_{\mathbf{u}}^{(\mathbf{X},\mathbf{Y})}(\tau_{ijk}) = \frac{1}{2}\langle \bar{\mathbf{u}} \mid \langle \mathbf{Y} \mid \mathbf{u}_k - \mathbf{u}_i \rangle \mathbf{X} \times (\mathbf{x}_j - \mathbf{x}_i) \rangle - \frac{1}{2}\langle \bar{\mathbf{u}} \mid \langle \mathbf{Y} \mid \mathbf{u}_j - \mathbf{u}_i \rangle \mathbf{X} \times (\mathbf{x}_k - \mathbf{x}_i) \rangle$$

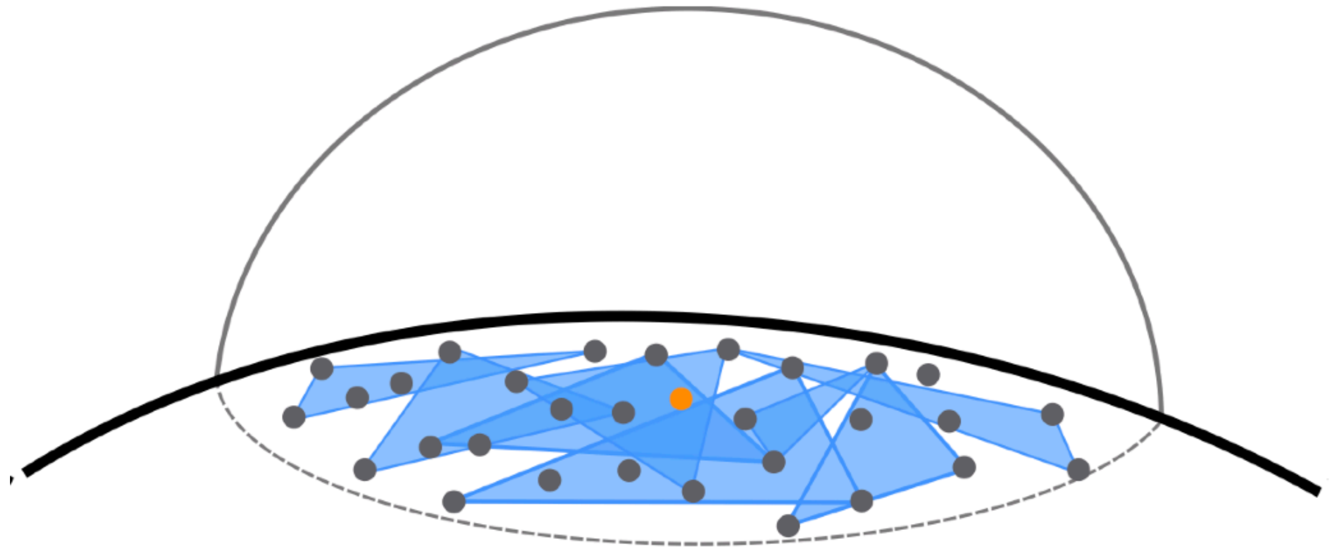
2. Per point $\mathbf{x}$, generate locally random triangles

- Neighbours of $\mathbf{x}$: either K nearest or within Ball($\mathbf{x}$, δ)
- Choose a strategy to build L triangles within

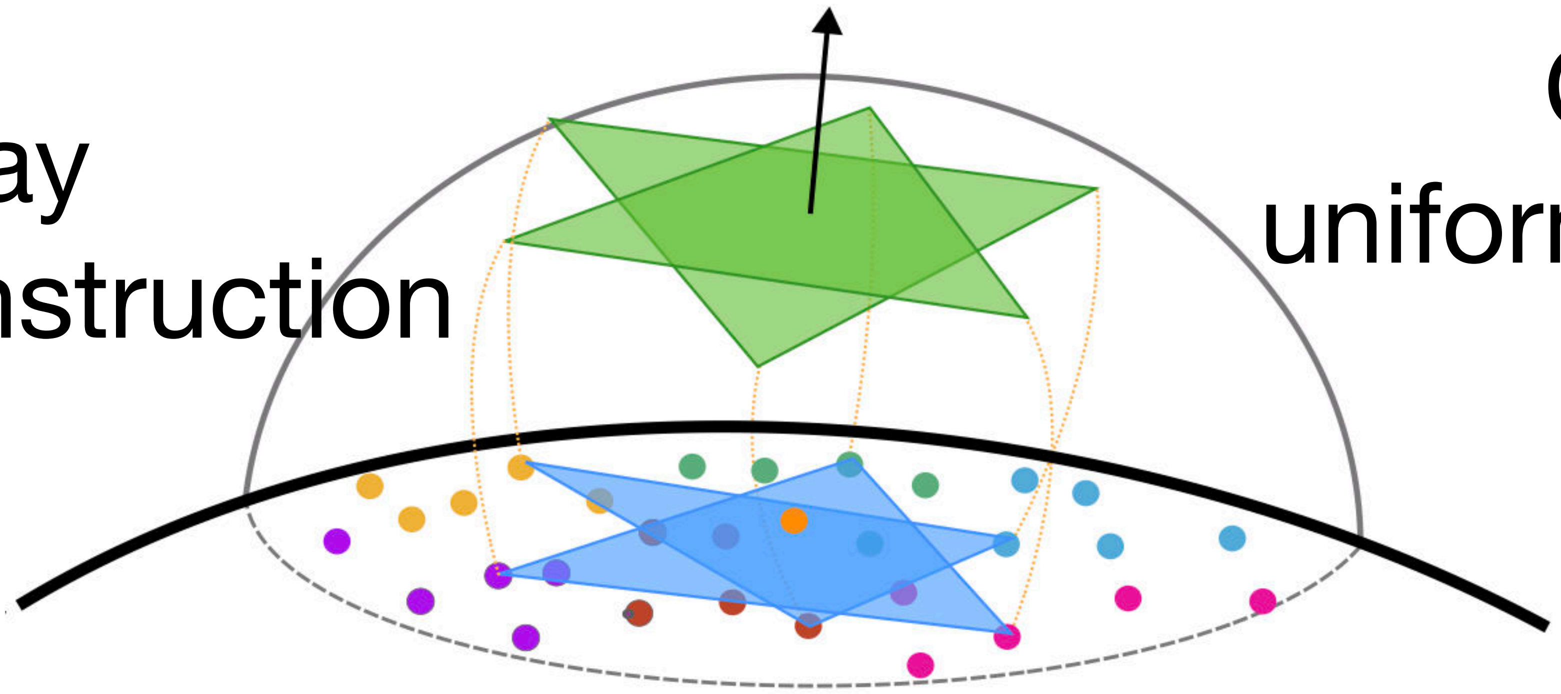




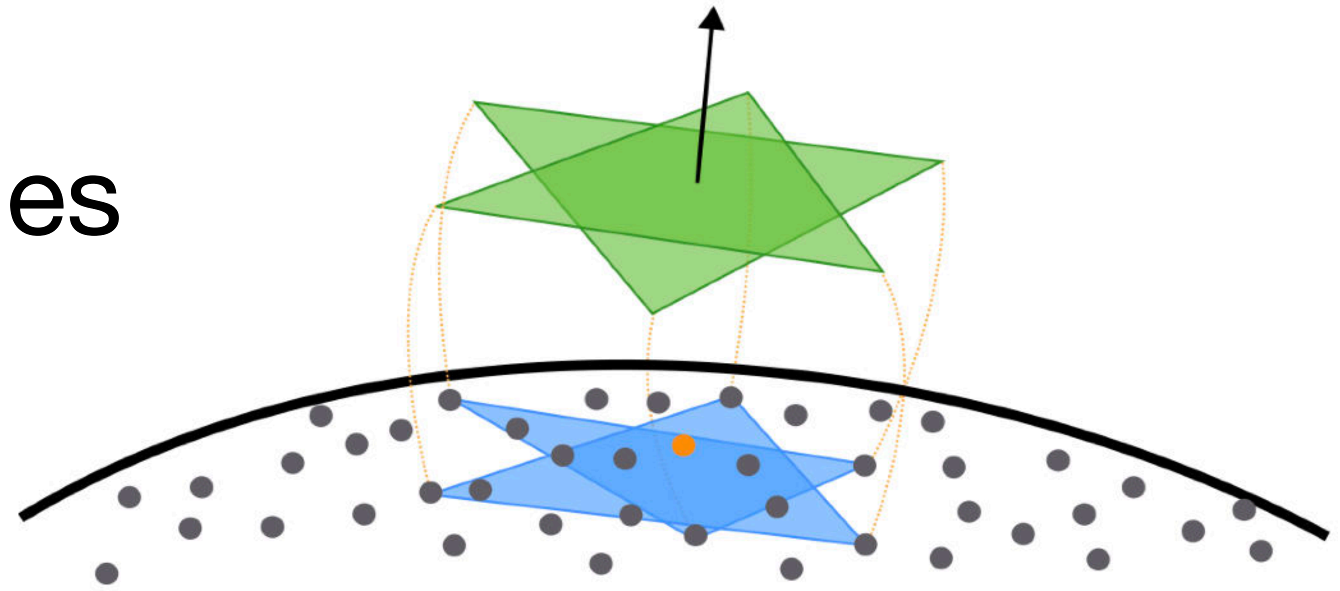
CNC-Delaunay
local Delaunay reconstruction



CNC-Uniform
uniform random triangles

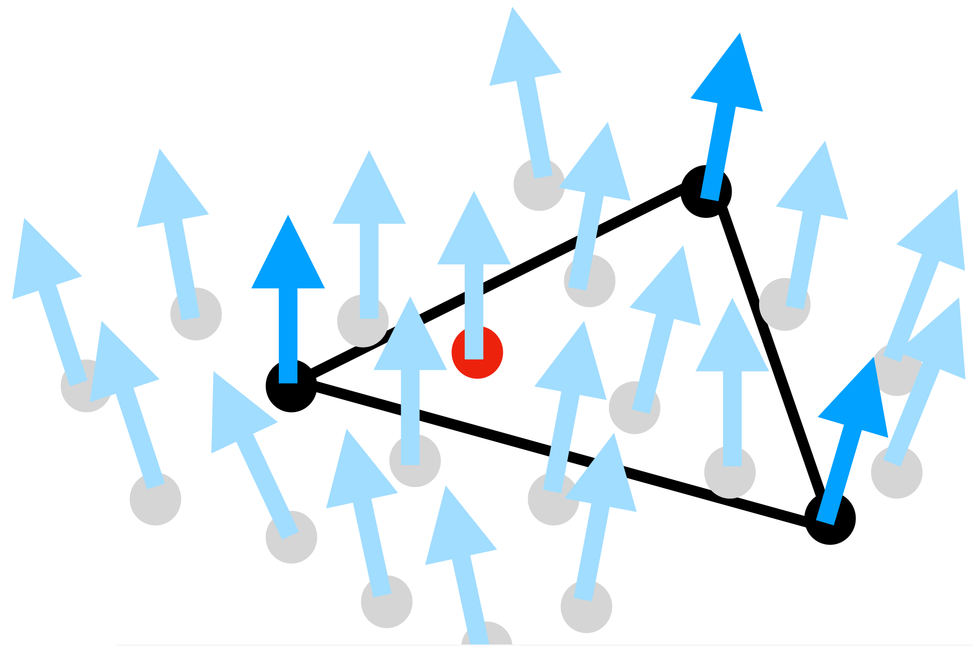


CNC-AvgHexagram
2 triangles with average nearest points



CNC-Hexagram
2 triangles with nearest points

3. Sum up results and normalise curvature measures



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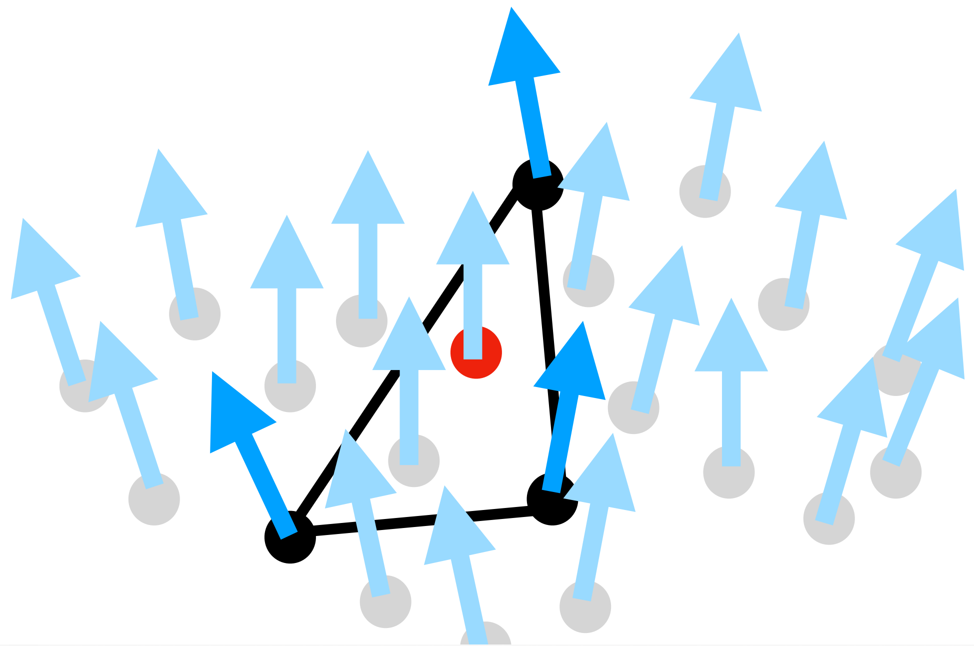


Triangles oriented such that $\mu_{\mathbf{u}}^{(0)}(\hat{\tau}_l) \geq 0$

$\mu_{\mathbf{u}}^{(0)}(\hat{\tau}_1)$

Lightweight computations:

- either K-NN
- or 6-NN for CNC-Hexagram
- sums of $\langle \cdot \mid \cdot \rangle$ and $\cdot \times \cdot$ formulas per triangle

- Sum area ... urvature meas.

$\hat{A}^{(0)} = \sum_{l=1} \ldots$ $\sum_{l=1} \ldots$ $\sum_{l=1} \ldots \hat{\tau}_l)$

Mean curvature $\hat{H}(\mathbf{x}) = \hat{A}^{(1)}/\hat{A}^{(0)}$

Gaussian curvature $\hat{G}(\mathbf{x}) = \hat{A}^{(2)}/\hat{A}^{(0)}$

Example: mean curvature with Avg-Hexagram

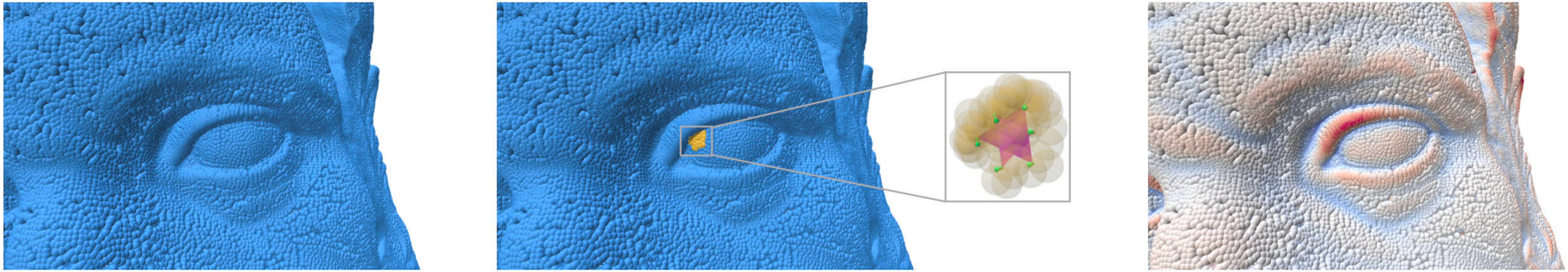
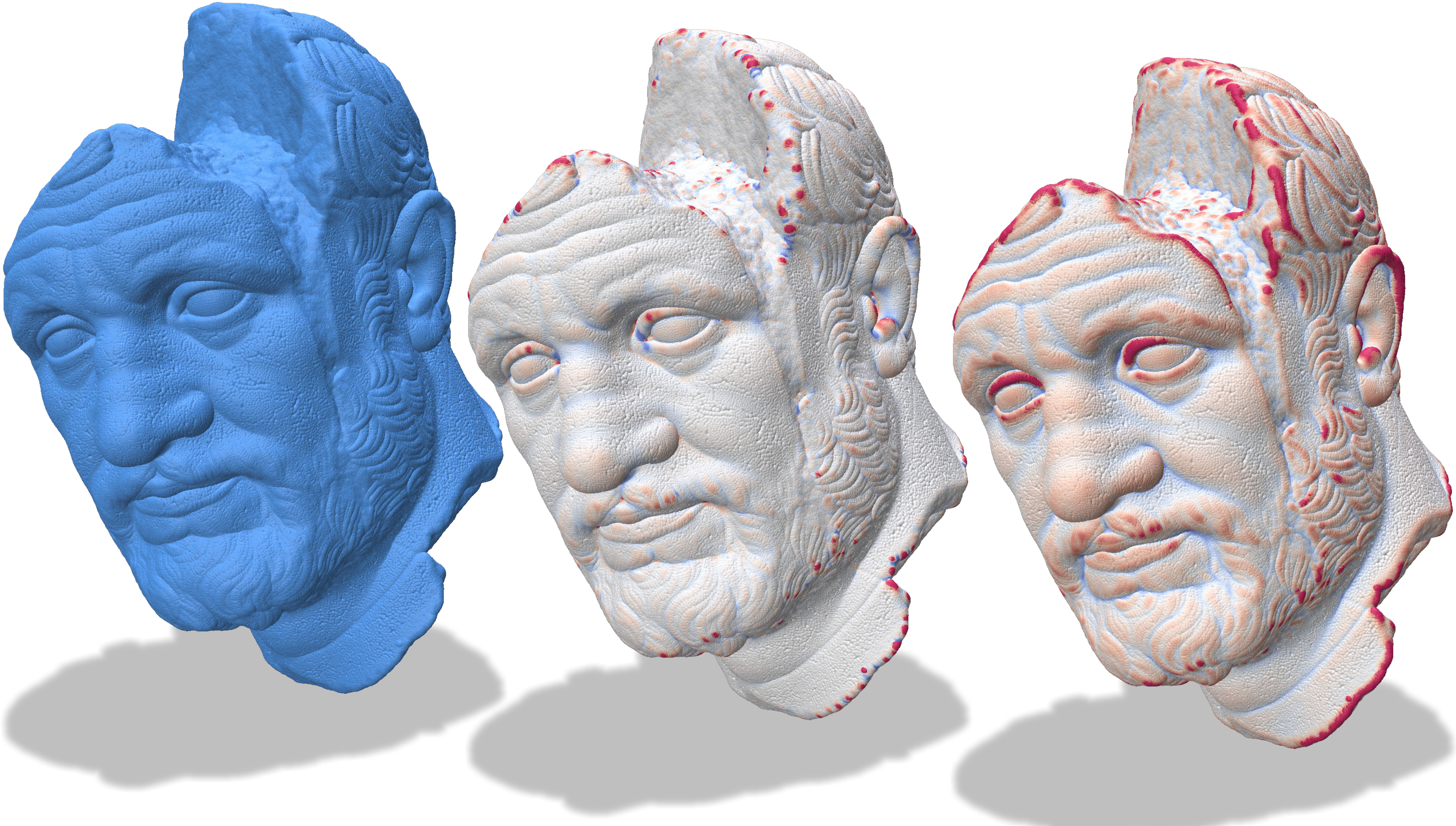


Figure 1: *Our new technique uses corrected curvature measures on (quasi-)random triangles to estimate differential quantities on point clouds: stable and accurate estimations (mean curvature here) are achieved with few neighbors (50) and triangles (2).*




Works well in practice
What tells the theory ?

Stability of curvature estimates

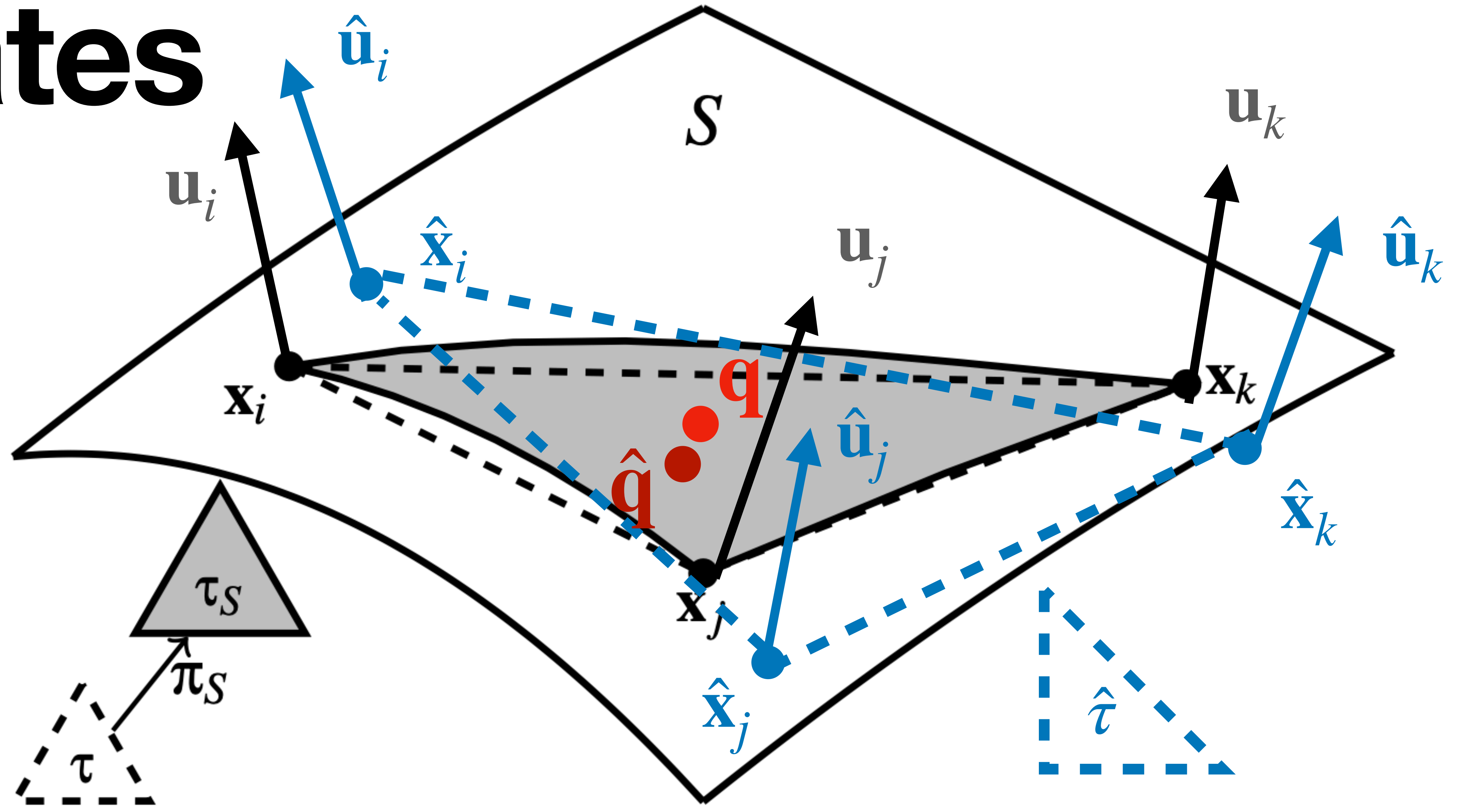
Hypotheses and notations

- Perturbated sampling of smooth surface S
- $\hat{\mathbf{q}}$ point of computation, $\mathbf{q} = \pi_S(\hat{\mathbf{q}})$
- δ a computation window
- $(\hat{\mathbf{x}}_i, \hat{\mathbf{u}}_i)$ input points/normals

  $\hat{\mathbf{x}}_i = \mathbf{x}_i + \epsilon_i$ with ϵ_i i.i.d. random variables of null exp. and variance $\sigma_\epsilon^2 \mathrm{Id}$

  $\hat{\mathbf{u}}_i = \mathbf{u}_i + \xi_i$ with ξ_i i.i.d. random variables of null exp. and variance $\sigma_\xi^2 \mathrm{Id}$

- Focus here on mean curvature



Input triangles

$$\hat{A}^{(0)} = \sum_{l=1}^{L} \mu_{\hat{\mathbf{u}}}^{(0)}(\hat{\tau}_l)$$

$$\hat{A}^{(1)} = \sum_{l=1}^{L} \mu_{\hat{\mathbf{u}}}^{(1)}(\hat{\tau}_l)$$

$\vdots$

Ideal triangles

$$A^{(0)} = \sum_{l=1}^{L} \mu_{\mathbf{u}}^{(0)}(\tau_l)$$

$$A^{(1)} = \sum_{l=1}^{L} \mu_{\mathbf{u}}^{(1)}(\tau_l)$$

$\vdots$

Stability of curvature estimates (perfect data)

Estimated curvature

True curvature on S

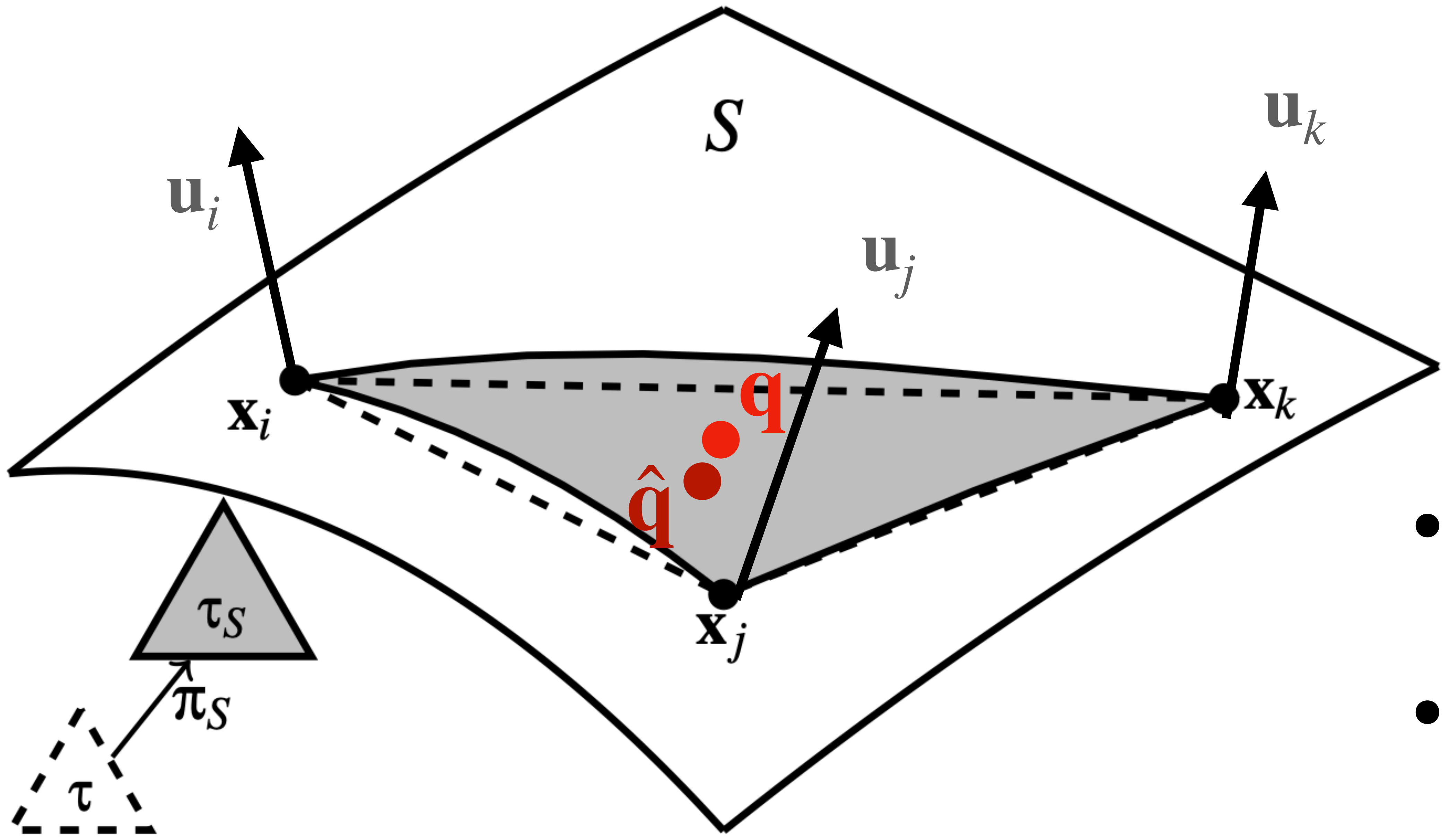
Theorem If $\hat{A}^{(0)} = A^{(0)}$ and $\hat{A}^{(1)} = A^{(1)}$ (perfect data), then

$$\left| \boxed{\hat{H}(\hat{\mathbf{q}})} - \boxed{H(\mathbf{q})} \right| \leq \boxed{O(\delta)}$$

Variation of H in ball of radius δ

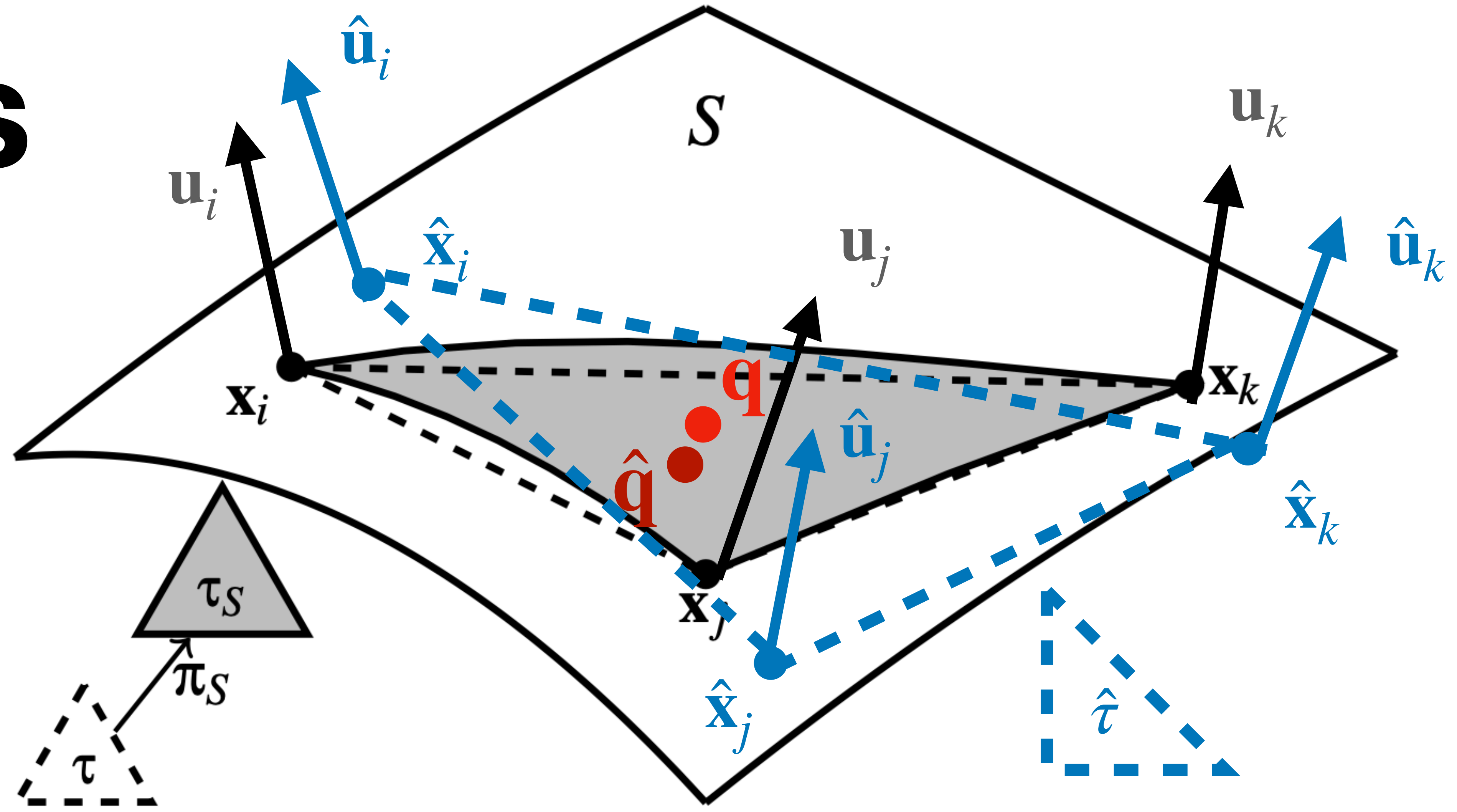
+

Measure errors between τ and τ_S



- Requires $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k \in \text{Ball}(\mathbf{q}, \delta)$
- One triangle is enough, the smaller the better

Stability of curvature estimates (noisy data)



Estimated curvature

True curvature on S

Theorem If $\hat{A}^{(0)}/L = \Theta(\delta^2)$ then

$$\left| \boxed{\hat{H}(\hat{\mathbf{q}})} - \boxed{H(\mathbf{q})} \right| \leq \left| \boxed{O(\delta)} + \boxed{\Theta(\delta^{-2})\left(\bar{Z}_L^{(1)} - \bar{Z}_L^{(0)} H(\mathbf{q})\right)} \right| \Big/ \left| 1 + \boxed{\Theta(\delta^{-2})\bar{Z}_L^{(0)}} \right|$$

Variation of H + …

Sum of mean curvature measure errors

Sum of area measure errors

$\hat{A}^{(0)}/L$ = average area of triangles

$\bar{Z}_L^{(0)} = \dfrac{1}{L}(\hat{A}^{(0)} - A^{(0)})$ area error law

$\bar{Z}_L^{(1)} = \dfrac{1}{L}(\hat{A}^{(1)} - A^{(1)})$ mean curvature error law

Property of error laws

Convergence if $\bar{Z}_L^{(0)}$ and $\bar{Z}_L^{(1)}$ are below $O(\delta^2)$

Property 2 The error laws $\bar{Z}_L^{(0)}$ and $\bar{Z}_L^{(1)}$ have both null expectations. Their variance follows, for C and C' some constants:

$$\mathbb{V}\left[\bar{Z}_L^{(0)}\right] \leq \frac{C}{L}\left((\sigma_\xi^2\delta^2 + \sigma_\varepsilon^2)\delta^2 + \sigma_\varepsilon^2\sigma_\xi^2\delta^2 + \sigma_\varepsilon^4(1+\sigma_\xi^2)\right)$$

$$\mathbb{V}\left[\bar{Z}_L^{(1)}\right] \leq \frac{C'}{L}\left(\sigma_\xi^2\delta^2 + \sigma_\varepsilon^2 + \sigma_\varepsilon^2\sigma_\xi^2 + \sigma_\xi^4\delta^2 + \sigma_\varepsilon^2\sigma_\xi^4\right).$$

Area error is very low (order 4)

Mean curvature error is essentially σ_ϵ^2

Increasing $L =$ #triangles decreases both errors !

Proof (sketch)

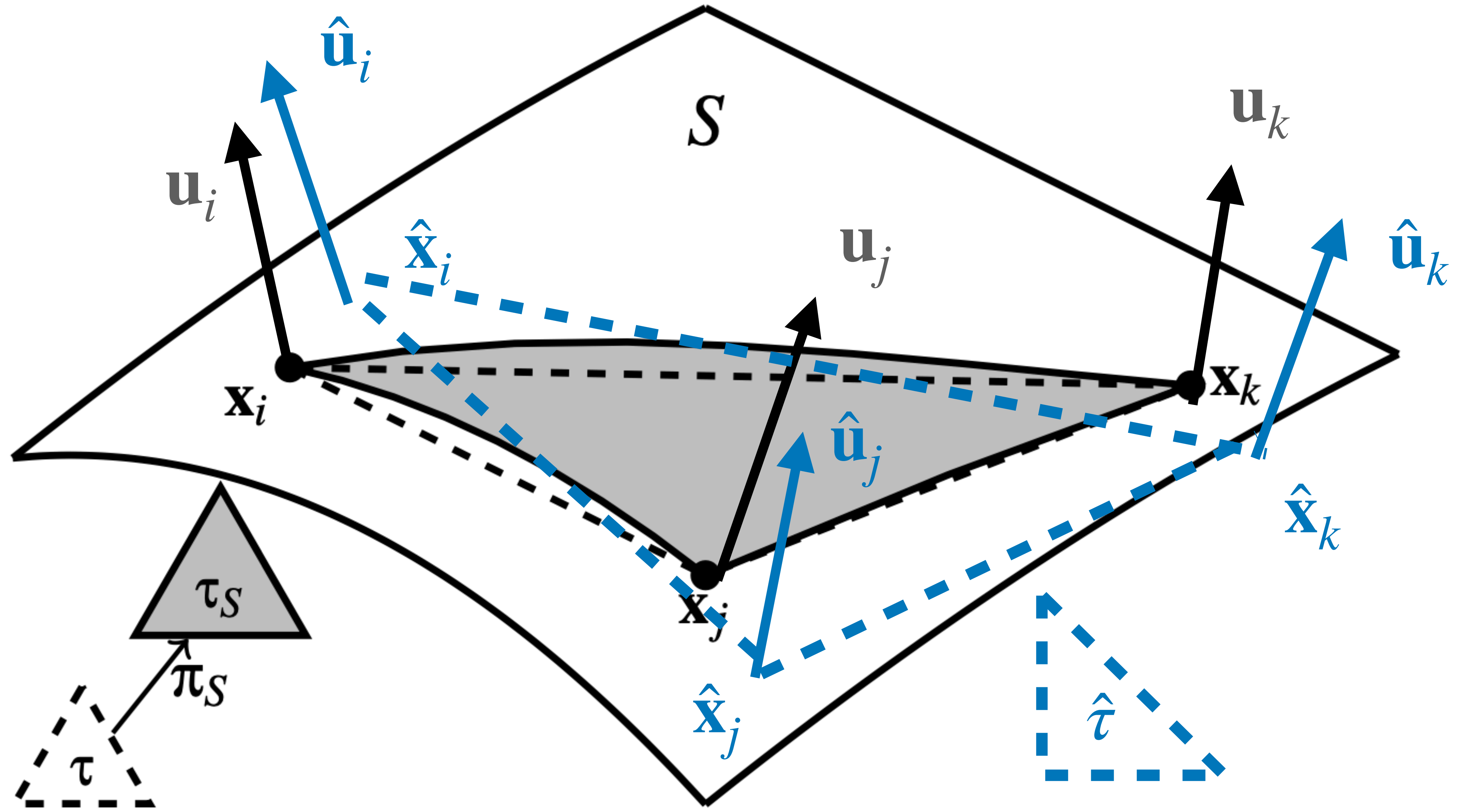
1. Area($\triangle$) ≈ Area($\blacktriangle$)

$$\left| \mu_{\mathbf{u}}^{(0)}(\tau) - \text{Area}(\tau_S) \right| \leq \text{Area}(\tau) O(\delta^2)$$

2a. $\int_{\blacktriangle} H(\mathbf{x}) \mathrm{d}\mathbf{x} \approx \text{Area}(\blacktriangle) H(\mathbf{q})$

2b. $\mu_{\mathbf{u}}^{(1)}(\triangle) \approx \int_{\blacktriangle} H(\mathbf{x}) \mathrm{d}\mathbf{x}$

$$\left| \mu_{\mathbf{u}}^{(1)}(\tau) - \text{Area}(\tau_S) H(\mathbf{q}) \right| \leq O(\delta^3)$$



3. $\mu_{\mathbf{u}}^{(i)}(\,\cdot\,)$ are linear formulas

$$\left| \mu_{\hat{\mathbf{u}}}^{(0)}(\triangle) - \mu_{\mathbf{u}}^{(0)}(\triangle) \right|$$

$$= \frac{1}{2} \langle \mathbf{u} \mid \epsilon_i \times (\mathbf{x}_j - \mathbf{x}_k) \rangle + \cdots$$

⇒ Null exp., variance easily bounded

Comparative evaluation

ELSEVIER

Co [SGP 2003] sign

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Estimating differential quantities using polynomial fitting of osculating jets ☆

F. Cazals, M. Pouget

Jet Fitting

- classic method
- accurate without noise

Eurographics Symposium on Geometry Processing
K. Crane and J. Digne
(Guest Editors)

Volume 40 (2021), Number 5

[SGP 2021]

Stable and efficient differential estimators on oriented point clouds

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[2]Université de Lyon, CNRS, LIRIS

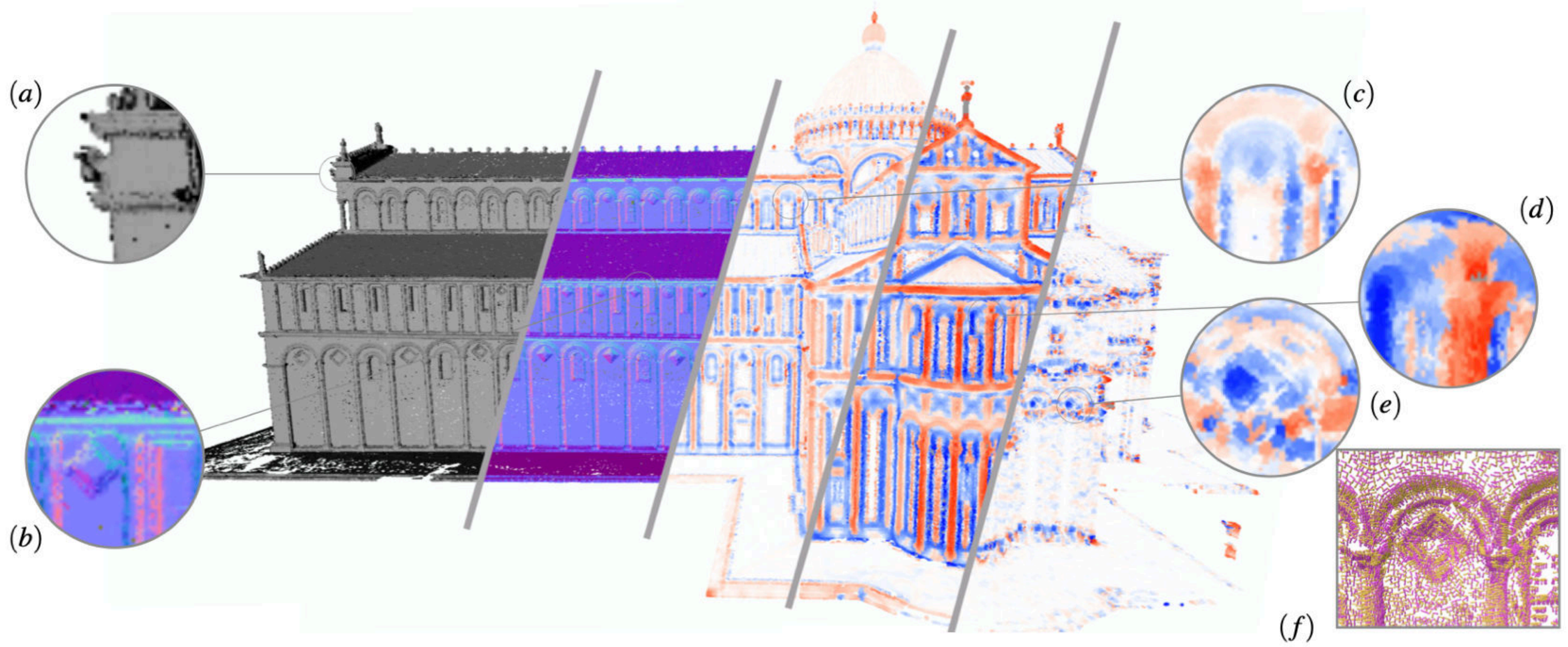
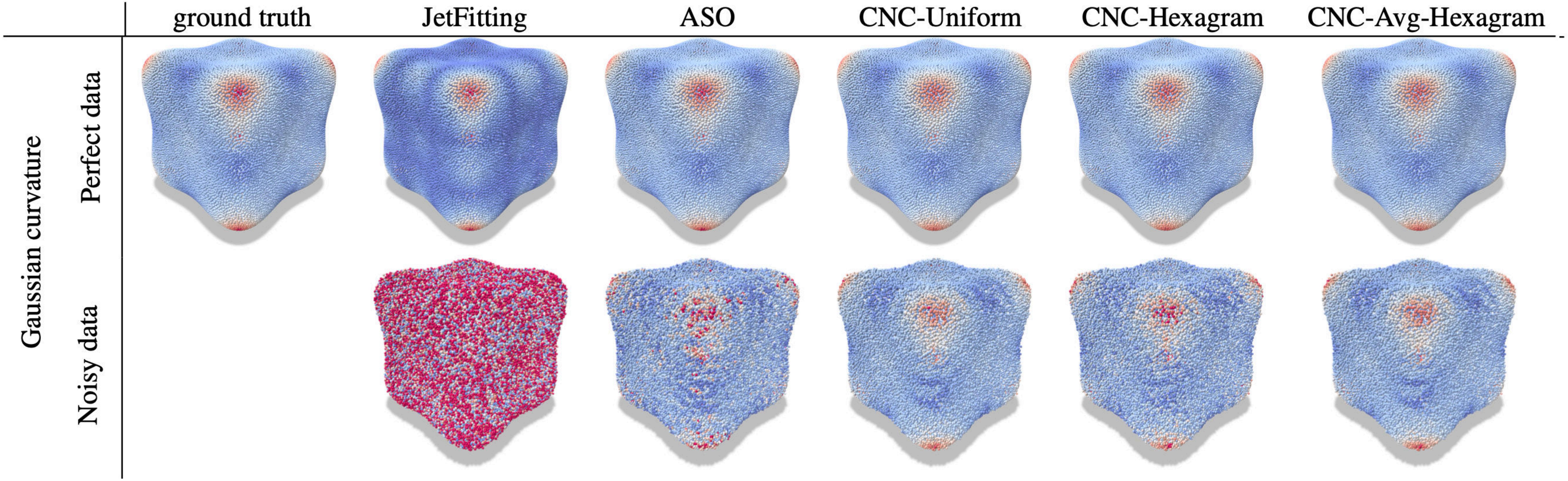


Figure 1: *Differential estimations computed with our stable estimators on a large point cloud with normals (2.5M points). Zoom on: (a) the initial point cloud, (b) our corrected normal vectors, (c) mean curvature, (d,e) principal curvatures, and (f) principal curvature directions.*

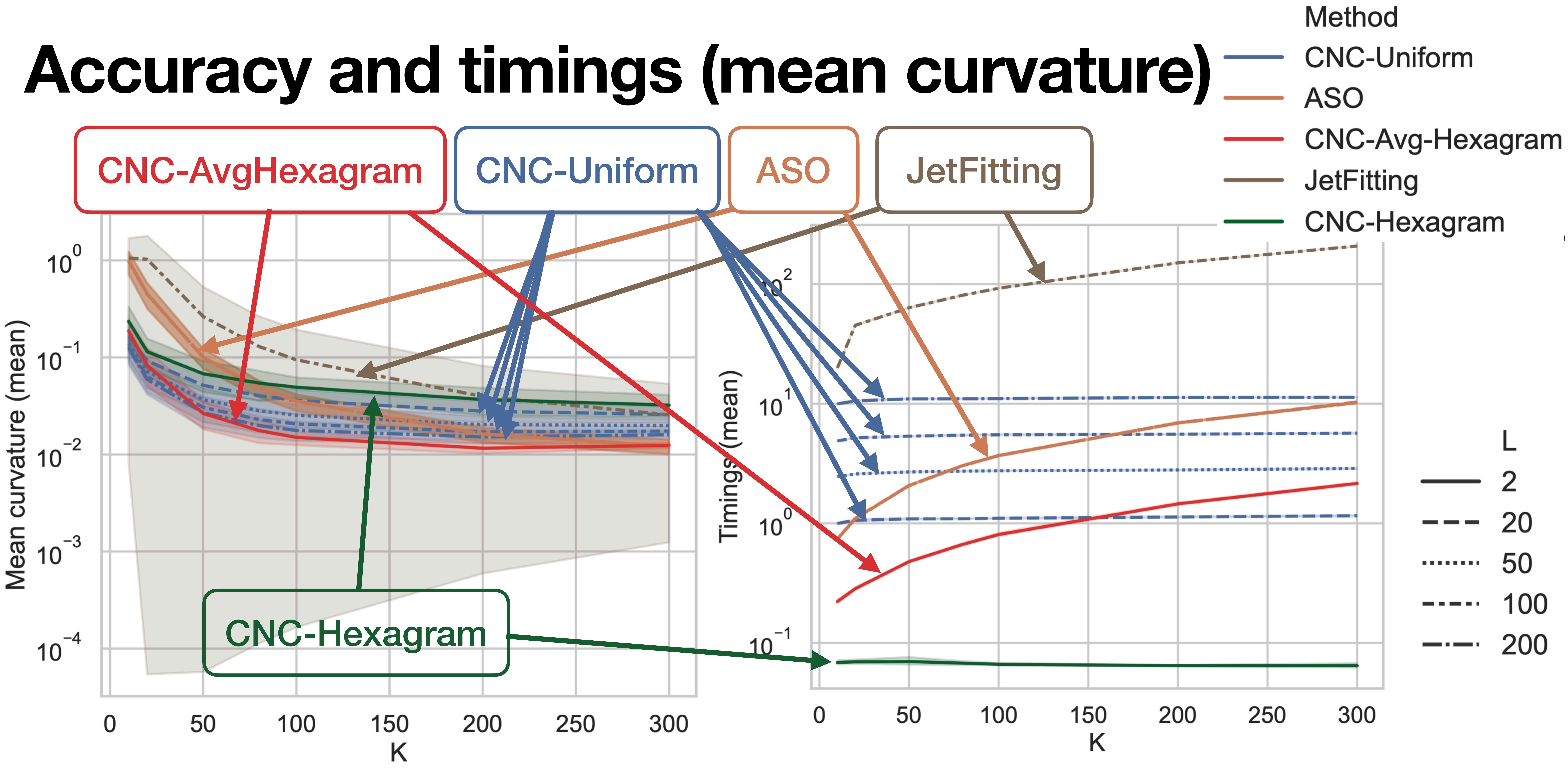
Algebraic Sphere Operator (ASO)

- theoretical stability wrt position perturbations
- accurate and fast

Accuracy on « Goursat » shape (visual comparison)

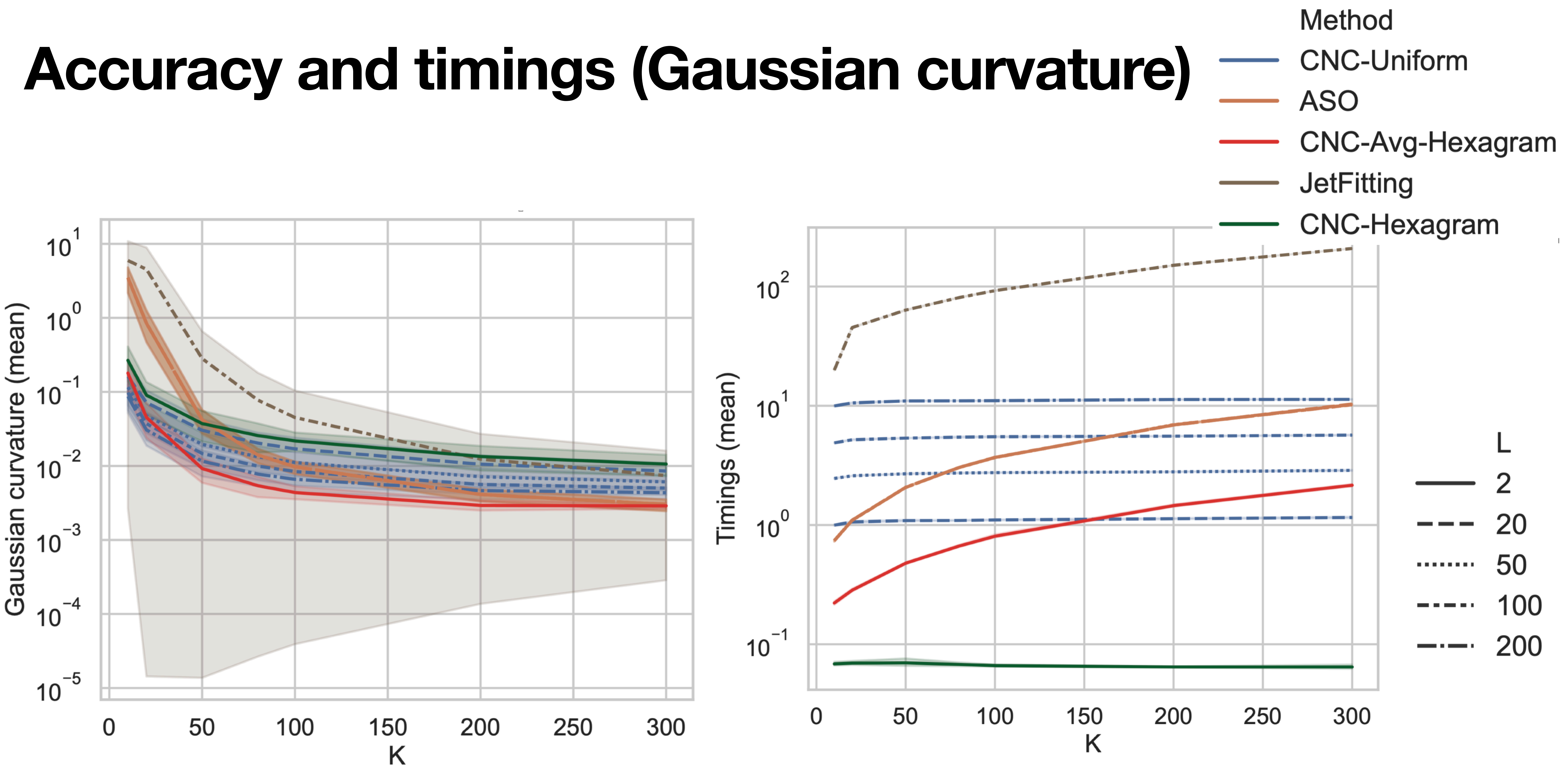


Accuracy and timings (mean curvature)



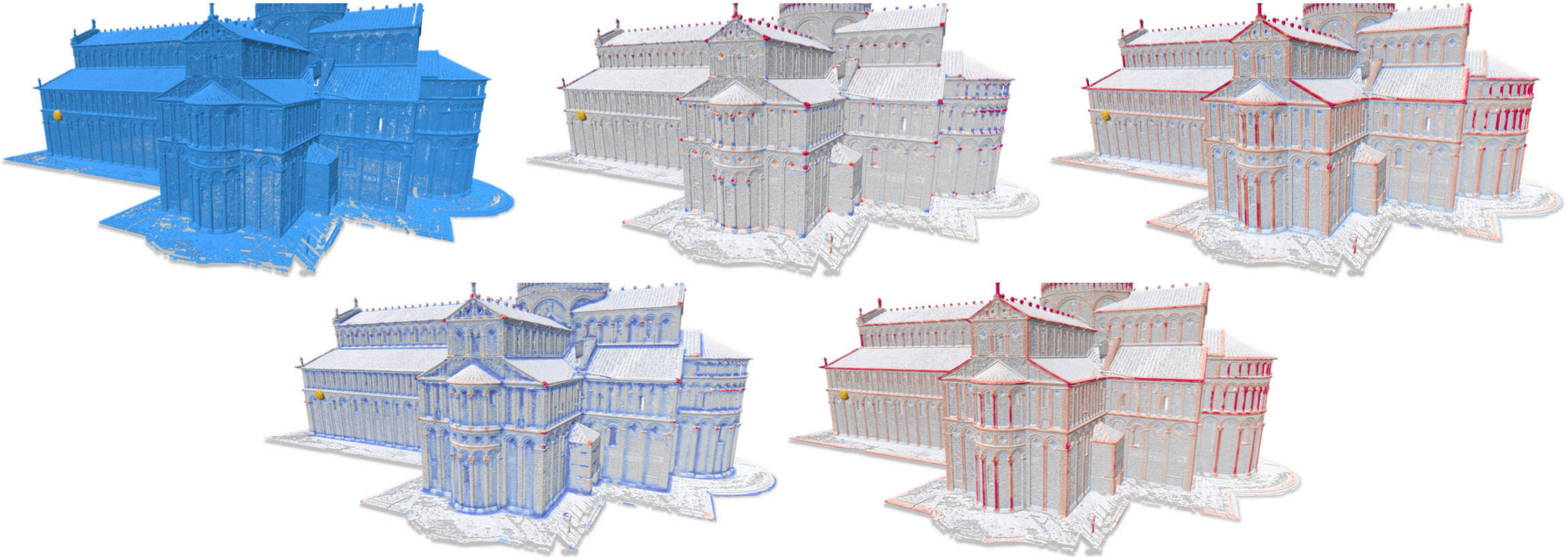
- **Goursat** shape : $N \in \{10000, 25000, 50000, 75000, 100000\}$, $\sigma_\epsilon, \sigma_\xi \in \{0, 0.1, 0.2\}$

Accuracy and timings (Gaussian curvature)



- **Goursat** shape : $N \in \{10000, 25000, 50000, 75000, 100000\}$, $\sigma_\epsilon, \sigma_\xi \in \{0, 0.1, 0.2\}$

Results on filtered LiDAR data



- 2,5M of points, K=300

Simplest Hexagram gives multiscale geometric information

More results / comparisons in paper and supplementary

(Other shapes, simulated LiDAR samples, etc)

- 1,8M of points, 4s for all $H, G, \kappa_1, \kappa_2, \mathbf{v}_1, \mathbf{v}_2$ curvatures, 81s for NN

Conclusion

- new method for curvature estimations on oriented point clouds
- local computations without surface reconstruction, fully parallelizable
- theoretical stability in the presence of noise on positions **and** normals
- smaller computation window than state-of-the-art for better accuracy, faster
- https://github.com/JacquesOlivierLachaud/PointCloudCurvCNC

Future works

- Extend this approach to unoriented point clouds
- Stability results for non iid noise perturbations
- Specific randomization strategies for data with outliers
- Automatic global/local tuning of window δ or K
- Higher-order differential quantities using an extended Grassmannian