An alternative definition for digital convexity

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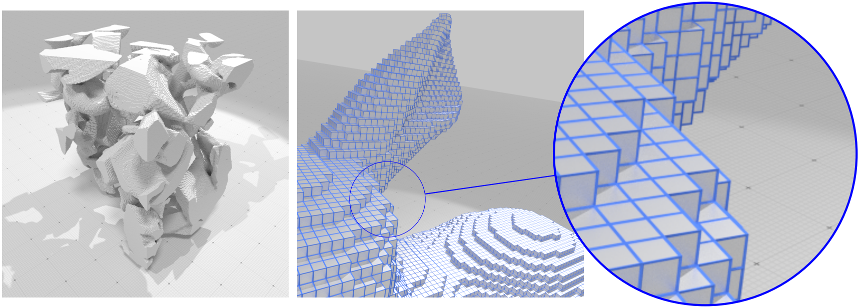
An alternative definition for digital convexity

Why digital convexity ?

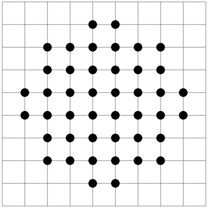


- no (infinitesimal) differential geometry for digital shapes
- convexity: a fundamental tool to analyze the geometry of shapes
- identifies convex/concave/flat/saddle regions
- gives locally its piecewise linear geometry
- facets give normal estimations

Natural digital convexity is not satisfactory

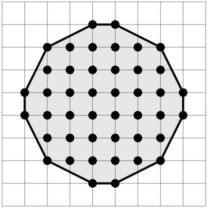
Definition (Natural digital convexity (or *H*-convexity))

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{cvxh}\,(X) \cap \mathbb{Z}^d = X$



X

$=$



$\mathrm{cvxh}\,(X) \cap \mathbb{Z}^d$

$\Rightarrow$ convex

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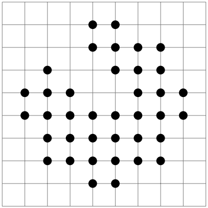
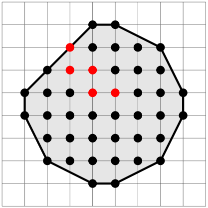


X

$\neq$



$\mathrm{cvxh}\,(X) \cap \mathbb{Z}^d$

$\Rightarrow$ not convex

Natural digital convexity is not satisfactory

Definition (Natural digital convexity (or *H*-convexity))

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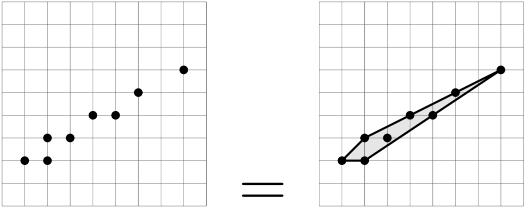
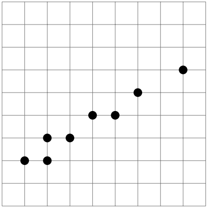


X $=$ $\mathrm{cvxh}\,(X) \cap \mathbb{Z}^d$ $\Rightarrow$ convex !

Natural digital convexity is not satisfactory

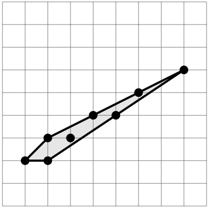
Definition (Natural digital convexity (or *H*-convexity))

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{cvxh}\,(X) \cap \mathbb{Z}^d = X$



X

$=$



$\mathrm{cvxh}\,(X) \cap \mathbb{Z}^d$

$\Rightarrow$ convex !

Digital convexity does not imply digital connectedness !

Usual digital convexity adds connectedness

Definition (Usual digital convexity)

$X \subset \mathbb{Z}^d$ is digitally convex iff $\mathrm{cvxh}(X) \cap \mathbb{Z}^d = X$ **and** X connected

- many more or less equivalent definitions **in 2D**: straight segment convexity, triangle convexity, ... [Minsky, Papert 88], [Kim, Rosenfeld 82a], [Huübler, Klette, Voss89], ...

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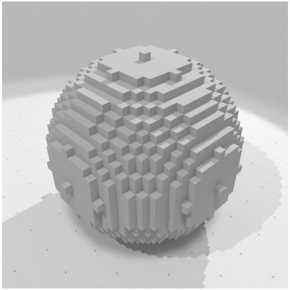
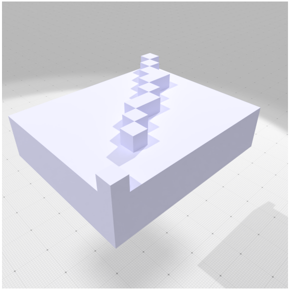
- many more or less equivalent definitions **in 2D**: straight segment convexity, triangle convexity, ... [Minsky, Papert 88], [Kim, Rosenfeld 82a], [Huübler, Klette, Voss89], ...
- **none extends well to 3D or more**



convex



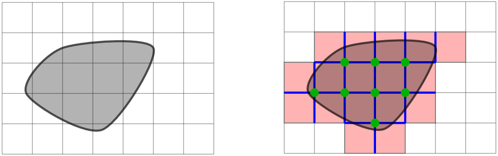
convex !

An alternative definition for digital convexity

Cubical grid, intersection complex

- cubical grid complex $\mathcal{C}^d$
  - $\mathcal{C}^d_0$ vertices or 0-cells $= \mathbb{Z}^d$
  - $\mathcal{C}^d_1$ edges or 1-cells $=$ open unit segment joining 0-cells
  - $\mathcal{C}^d_2$ faces or 2-cells $=$ open unit square joining 1-cells
  - ...
- *intersection complex* of $Y \subset \mathbb{R}^d$

$$\bar{\mathcal{C}}^d_k[Y] := \{c \in \mathcal{C}^d_k, \bar{c} \cap Y \neq \emptyset\}$$



Y cells $\bar{\mathcal{C}}^d_0[Y]$, $\bar{\mathcal{C}}^d_1[Y]$, $\bar{\mathcal{C}}^d_2[Y]$

Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}_k^d[X] = \bar{\mathcal{C}}_k^d[\mathrm{cvxh}\,(X)]. \tag{1}$$

Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.

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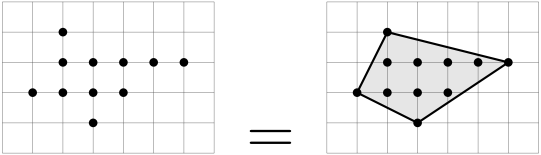

$\bar{\mathcal{C}}_0^d[X]$ $\quad = \quad$ $\bar{\mathcal{C}}_0^d[\mathrm{cvxh}\,(X)]$

X is digitally 0-convex

Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

$$\bar{\mathcal{C}}^d_k[X] = \bar{\mathcal{C}}^d_k[\mathrm{cvxh}\,(X)]. \quad (1)$$

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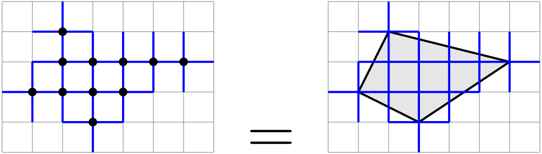

$\bar{\mathcal{C}}^d_1[X] = \bar{\mathcal{C}}^d_1[\mathrm{cvxh}\,(X)]$

X is digitally 0-convex, and 1-convex

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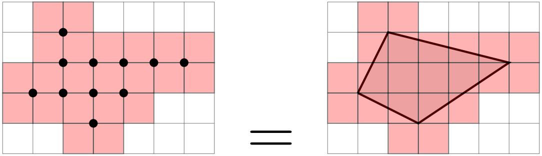

$\bar{\mathcal{C}}_2^d[X]$ $=$ $\bar{\mathcal{C}}_2^d[\mathrm{cvxh}\,(X)]$

X is digitally 0-convex, and 1-convex, and 2-convex, hence fully convex.

Full convexity

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A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

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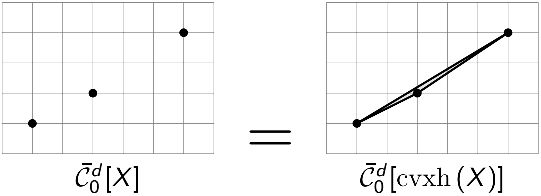

X is digitally 0-convex

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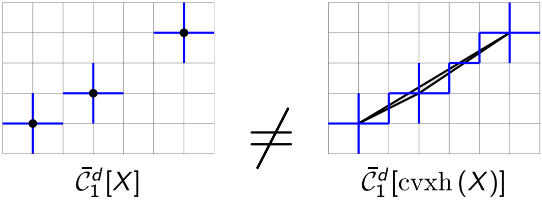

X is digitally 0-convex, but neither 1-convex

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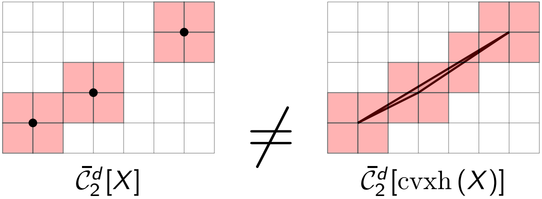

X is digitally 0-convex, but neither 1-convex, nor 2-convex.

Full convexity

Definition (Full convexity)

A non empty subset $X \subset \mathbb{Z}^d$ is *digitally k-convex* for $0 \leqslant k \leqslant d$ whenever

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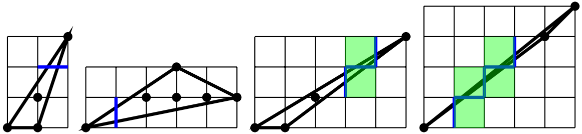
Subset X is *fully convex* if it is digitally k-convex for all $k, 0 \leqslant k \leqslant d$.

Full convexity eliminates too thin digital convex sets in arbitrary dimension.



Elementary properties

Lemma
Digital 0-convexity is classical digital convexity (H-convexity).

Lemma
A finite non-empty subset $X \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k \leqslant d$ iff $\#\left(\bar{\mathcal{C}}_k^d[X]\right) \geqslant \#\left(\bar{\mathcal{C}}_k^d[\mathrm{cvxh}\,(X)]\right)$.

Lemma
If $Z \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k < d$, it is also digitally d-convex, hence fully convex.

Proof.
Use Jordan-Brouwer surface separation theorem. □

Elementary properties

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Digital 0-convexity is classical digital convexity (H-convexity).

Lemma

A finite non-empty subset $X \subset \mathbb{Z}^d$ is digitally k-convex *for $0 \leqslant k \leqslant d$ iff $\#\left(\bar{\mathcal{C}}^d_k[X]\right) \geqslant \#\left(\bar{\mathcal{C}}^d_k[\mathrm{cvxh}\,(X)]\right)$.*

Lemma

If $Z \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k < d$, it is also digitally d-convex, hence fully convex.

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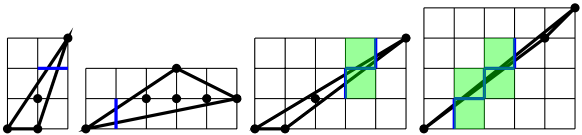
Use Jordan-Brouwer surface separation theorem. □

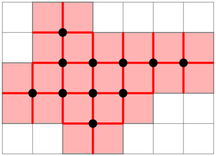


An alternative definition for digital convexity

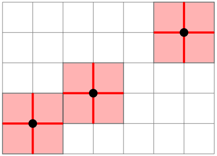
Connectedness of intersection complex

Theorem

If the digital set $X \subset \mathbb{Z}^d$ is fully convex, then the body of its intersection complex is connected.



fully convex



not fully convex

Connectedness of intersection complex

Theorem

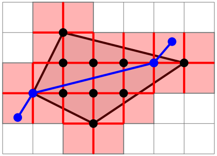
If the digital set $X \subset \mathbb{Z}^d$ is fully convex, then the body of its intersection complex is connected.

Proof.

We show that any two $x, y \in \left\|\bar{\mathcal{C}}^d[X]\right\|$ is arc-connected:

- any cell of $\bar{\mathcal{C}}^d[X]$ has a *corner* in X
- $x - x' - y' - y$, with $x', y' \in X$
- full convexity implies $\mathrm{cvxh}\,(X) \subset \left\|\bar{\mathcal{C}}^d[X]\right\|$
- $[x', y'] \subset \mathrm{cvxh}\,(X)$ which is connected.

$\square$



Digital connectedness

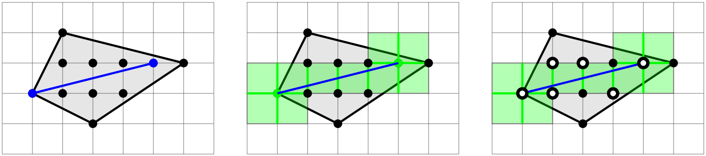
Theorem

If the digital set $X \subset \mathbb{Z}^d$ is fully convex, then X is d-connected.

Proof.

- for $x, y \in X$, segment $x - y$ intersects cells $c_0, c_1, \ldots, c_m$,
- each c_i touches at least one corner $z_i \in X$,
- each c_i is a face of c_{i+1} or inversely,
- implies z_i and z_{i+1} shares a unit cube, hence d-connected

$\square$



$[x, y]$ — intersected cells c_i — points z_i

Simple connectedness

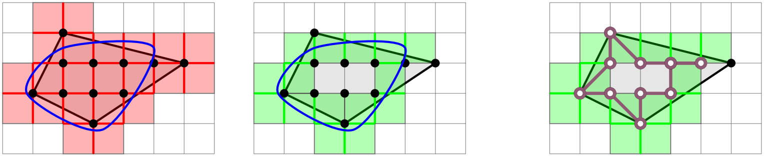
Theorem

If the digital set $X \subset \mathbb{Z}^d$ is fully convex, then the body of its intersection complex is **simply** *connected.*

Proof.

- let $\mathcal{A} := \{x(t), t \in [0,1]\}$ be a closed curve in $\left\|\bar{\mathcal{C}}^d[X]\right\|$
- sequence of intersected cells $c_i \in \bar{\mathcal{C}}^d[X]$
- sequence of associated corners $z_i \in X$
- homotopy between $\mathcal{A}$ and path $z_0 - z_1 - \cdots - z_n - z_0$
- path $z_0 - z_1 - \cdots - z_n - z_0$ subset of $\mathrm{cvxh}\,(X) \Rightarrow$ contractible

$\square$

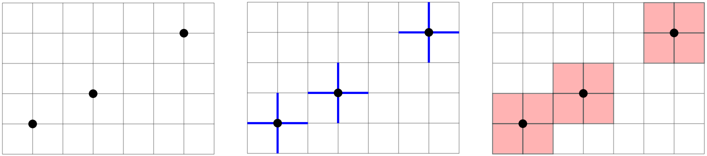


$\mathcal{A}$ — intersected cells (c_i) — path $z_0 - z_1 - \cdots - z_n - z_0$

An alternative definition for digital convexity

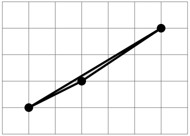
Computing cell intersections with X

$X \subset \mathbb{Z}^d$, no difficulty to compute $\bar{\mathcal{C}}_k^d[X]$, $0 \leqslant k \leqslant d$

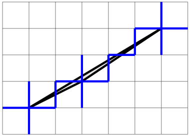


Computing cell intersections with $\mathrm{cvxh}\,(X)$

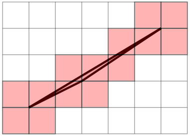
- let $\mathcal{S}_i$ be the unit segments aligned with axis i,
- for directions $\alpha \subset \{1, \ldots, d\}$, let $\mathcal{S}_\alpha = 0 \oplus (\bigoplus_{i \in \alpha} \mathcal{S}_i)$



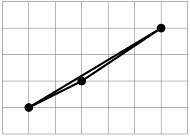
$\bar{\mathcal{C}}_0^d[\mathrm{cvxh}\,(X)]$



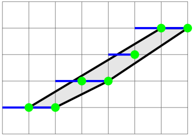
$\bar{\mathcal{C}}_1^d[\mathrm{cvxh}\,(X)]$



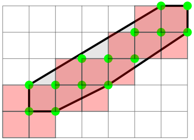
$\bar{\mathcal{C}}_2^d[\mathrm{cvxh}\,(X)]$



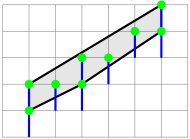
$\mathrm{cvxh}\,(X) \cap \mathbb{Z}^d$



$\mathrm{cvxh}\,(X) \oplus \mathcal{S}_{\{1\}} \cap \mathbb{Z}^d$



$\mathrm{cvxh}\,(X) \oplus \mathcal{S}_{\{1,2\}} \cap \mathbb{Z}^d$



$\mathrm{cvxh}\,(X) \oplus \mathcal{S}_{\{2\}} \cap \mathbb{Z}^d$

First morphological characterization

Theorem

A non empty subset $X \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k \leqslant d$ iff

$$\forall \alpha \text{ } k\text{-subset of } \{1, \ldots, d\},$$
$$(X \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d = (\mathrm{cvxh}\,(X) \oplus \mathcal{S}_\alpha) \cap \mathbb{Z}^d.$$

It is thus fully convex if the previous relations holds for all $k, 0 \leqslant k \leqslant d$.

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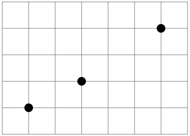
Algorithm in low dimension

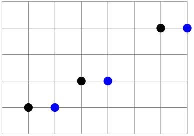
Relatively easily to implement Minkowski sum with $\mathcal{S}_\alpha$ for H-polytopes in low dimension. Slowest part is lattice point enumeration in convex hull.

Discrete Minkowski sum U_α

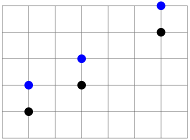
- let $X \subset \mathbb{Z}^d$, denote $\mathrm{e}_i(X)$ the translation of X with axis vector e_i
- let $U_\emptyset(X) := X$, and, for $\alpha \subset I^d$ and $i \in \alpha$, recursively $U_\alpha(X) := U_{\alpha\setminus i}(X) \cup \mathrm{e}_i(U_{\alpha\setminus i}(X))$.



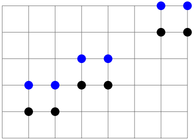
$U_\emptyset(X) = X$



$U_{\{1\}}(X) = U_\emptyset(X) \cup \mathrm{e}_1(U_\emptyset(X))$



$U_{\{2\}}(X) = U_\emptyset(X) \cup \mathrm{e}_2(U_\emptyset(X))$



$U_{\{1,2\}}(X) = U_{\{1\}}(X) \cup \mathrm{e}_1(U_{\{1\}}(X))$

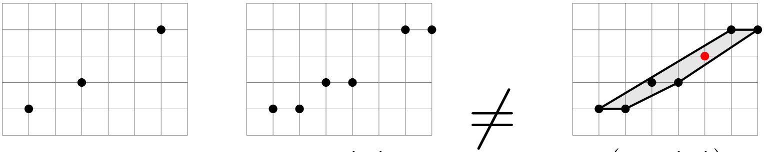
Second morphological characterization

Theorem

A non empty subset $X \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k \leqslant d$ iff

$$\forall \alpha \in I_k^d, U_\alpha(X) = \mathrm{cvxh}\left(U_\alpha(X)\right) \cap \mathbb{Z}^d. \tag{2}$$

It is thus fully convex if the previous relations holds for all $k, 0 \leqslant k \leqslant d$.



X $\quad$ $U_{\{1\}}(X)$ $\quad\neq\quad$ $\mathrm{cvxh}\left(U_{\{1\}}(X)\right) \cap \mathbb{Z}^d$

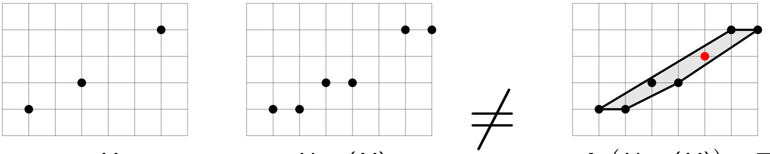
Second morphological characterization

Theorem

A non empty subset $X \subset \mathbb{Z}^d$ is digitally k-convex for $0 \leqslant k \leqslant d$ iff

$$\forall \alpha \in I_k^d, U_\alpha(X) = \mathrm{cvxh}\left(U_\alpha(X)\right) \cap \mathbb{Z}^d. \tag{2}$$

It is thus fully convex if the previous relations holds for all $k, 0 \leqslant k \leqslant d$.



X $\quad$ $U_{\{1\}}(X)$ $\quad \neq \quad$ $\mathrm{cvxh}\left(U_{\{1\}}(X)\right) \cap \mathbb{Z}^d$

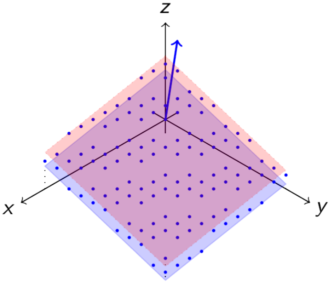
Algorithm in arbitrary dimension

$U_\alpha(X)$ easily computed while convex hull algorithms exist in arbitrary dimension. Slowest part is lattice point enumeration in convex hull.

An alternative definition for digital convexity

- Context and objectives
- Full convexity
- Topological properties
- Morphological characterization and algorithm
- Planarity, tangency and reversible reconstruction

Thick enough arithmetic planes are full convex

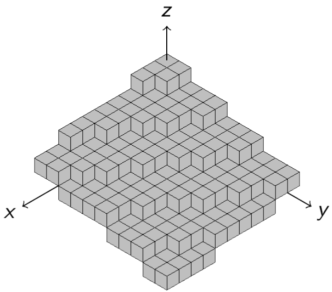


Arithmetic plane

- irreducible normal vector $N \in \mathbb{Z}^d$
- intercept $\mu \in \mathbb{Z}$
- positive thickness $\omega \in \mathbb{Z}, \omega > 0$

$$P(\mu, N, \omega) := \{x \in \mathbb{Z}^d, \mu \leqslant x \cdot N < \mu + \omega\}$$

Thick enough arithmetic planes are full convex



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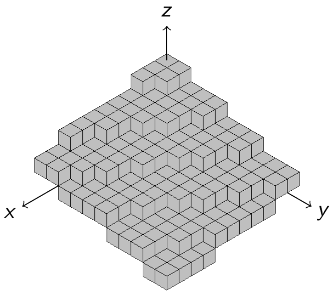

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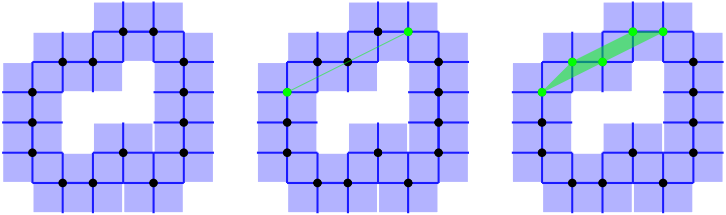
Theorem

Arithmetic planes are fully convex for thickness $\omega \geqslant \|N\|_\infty$.

Tangency

Definition

The digital set $A \subset X \subset \mathbb{Z}^d$ is said to be *k-tangent to X* for $0 \leqslant k \leqslant d$ whenever $\bar{\mathcal{C}}^d_k[\mathrm{cvxh}\,(A)] \subset \bar{\mathcal{C}}^d_k[X]$. It is *tangent to X* if the relation holds for all such k. Elements of A are called *cotangent*.



X and $\bar{\mathcal{C}}^d[X]$ — tangent — tangent

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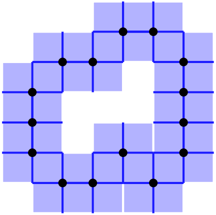

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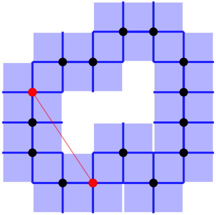
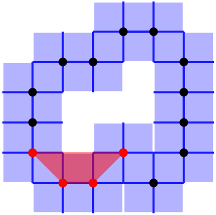


not tangent



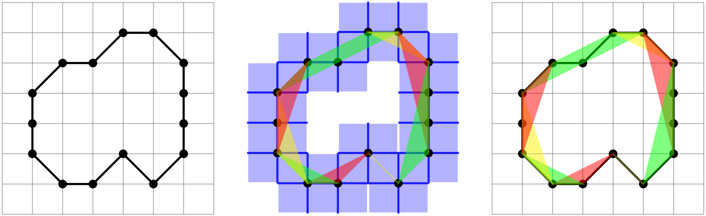
not tangent

Tangential cover

In 2D, maximal fully convex tangent subsets form the classical tangential cover of [Feschet, Tougne 99]

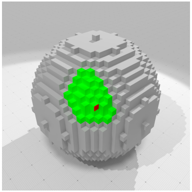
Theorem

When $d = 2$, if C is a simple 2-connected digital contour (i.e. 8-connected in Rosenfeld's terminology), then the fully convex subsets of C that are maximal and tangent are the classical maximal naive digital straight segments.



Tangential cover in 3D ? dD ?

- can we define facets of X as inextensible connected pieces of arithmetic planes standard planes along X ?



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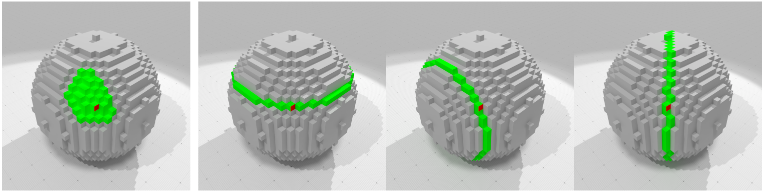

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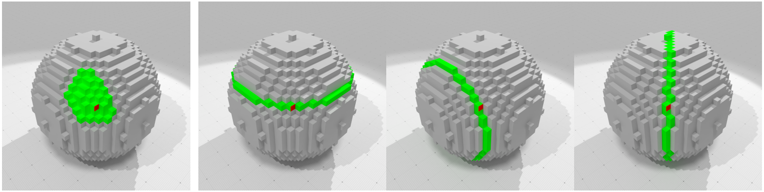

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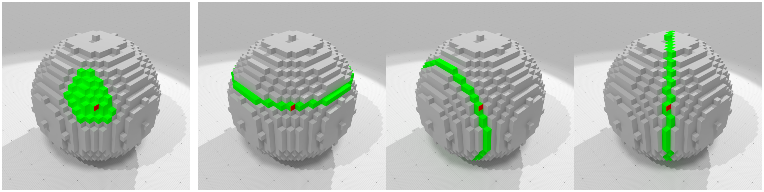

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Tangency extends to *d*D!

Tangent subsets in our sense are indeed tangent to X since their convex hull must lie close to X.

Piecewise linear reversible reconstruction in dD

Let $\mathrm{Del}(X)$ be the Delaunay complex of X.

Definition

The *tangent Delaunay complex* $\mathrm{Del}_{\mathrm{T}}(X)$ to X is the complex made of the cells τ of $\mathrm{Del}(X)$ such that the vertices of τ are tangent to X.

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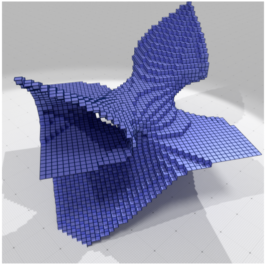
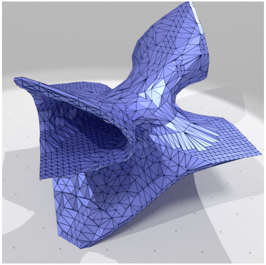
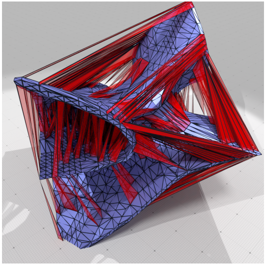


Input digital shape X



Reconstruction $\mathrm{Del}_{\mathrm{T}}\,(X)$



Bad simplices of $\mathrm{Del}\,(X)$

Theorem

The body of $\mathrm{Del}_{\mathrm{T}}\,(X)$ *is at Hausdorff* L_∞*-distance* 1 *to* X. $\mathrm{Del}_{\mathrm{T}}\,(X)$ *is a reversible polyhedrization, i.e.* $\|\mathrm{Del}_{\mathrm{T}}\,(X)\| \cap \mathbb{Z}^d = X$.

Conclusion

Pros of full convexity

- natural definition in arbitrary dimension that uses $\mathbb{Z}^d \subset \mathcal{C}^d$
- guarantees connectedness and simple connectedness
- morphological characterization that allows simple convexity check
- thick enough arithmetic planes are fully convex
- entails a consistent definition of tangency
- simple tight and reversible polyhedrization

Cons of full convexity

- $(2^d - 1)$ times slower to check convexity

Future works

Theoretical side

- extend this definition to cell complexes in order to have the intersection property
- convergence of maximal tangent planes for normal estimation ?

Algorithmic side

- fast lattice point enumeration within polytopes
- fast computation of maximal tangent planes (link with plane probing)

Explore its natural applications

- other polyhedrization algorithms using tangency