Delaunay properties of digital straight segments

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Outline

Definitions: patterns and Delaunay triangulation

Observation: Delaunay triangulation of patterns?

Characterization: proof

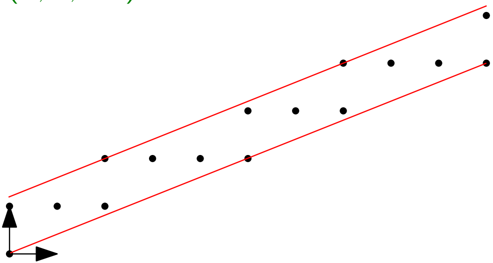
Conclusion and perspectives: new algorithms

Digital straight line (DSL)

Standard DSL

The points $(x, y) \in \mathbb{Z}^2$ verifying $\mu \leq ax - by < \mu + |a| + |b|$ belong to the standard DSL $D(a, b, \mu)$ of slope $\frac{a}{b}$ and intercept μ ($a, b, \mu \in \mathbb{Z}$ and $pgcd(a, b) = 1$).

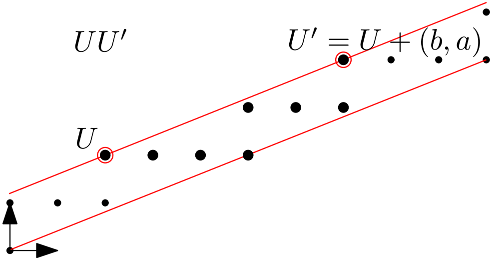
Example: $D(2, 5, -6)$



Pattern

- a pattern is a subsequence of a DSL between two consecutive upper leaning points

Example: pattern *UU′*



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- a pattern is a subsequence of a DSL between two consecutive upper leaning points
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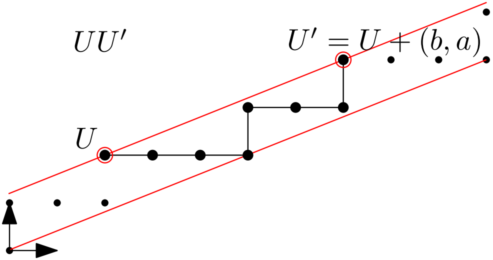
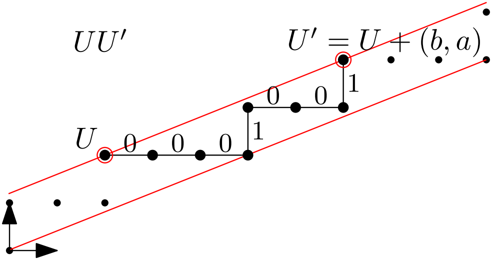


Pattern

- a pattern is a subsequence of a DSL between two consecutive upper leaning points
- its staircase representation is the polygonal line linking the points in order
- its chain code is a Christoffel word

Example: pattern *UU′*



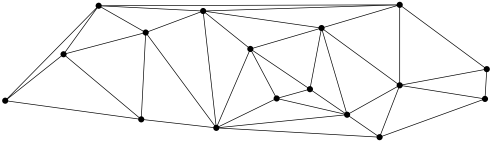
Delaunay triangulation

Triangulation of a finite set of points $\mathcal{S}$

Partition of the convex hull of $\mathcal{S}$ into triangular facets, whose vertices are points of $\mathcal{S}$.

Delaunay condition

The interior of the circumcircle of each triangular facet does not contain any set point.



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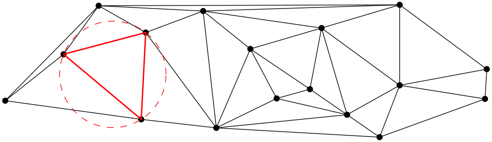

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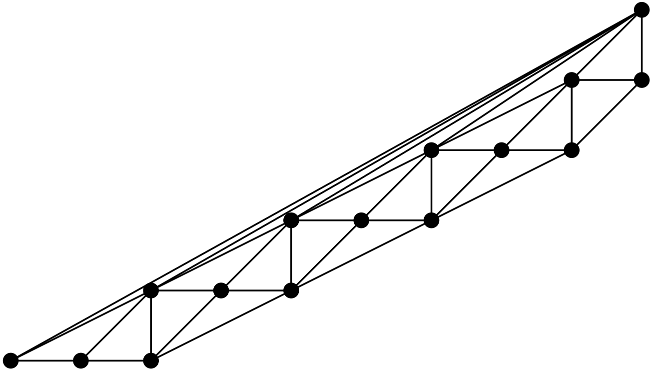
Outline

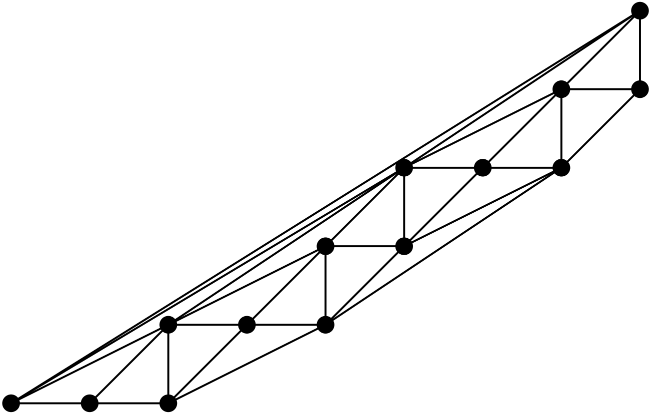
Delaunay triangulation of patterns

Pattern of slope $5/9$



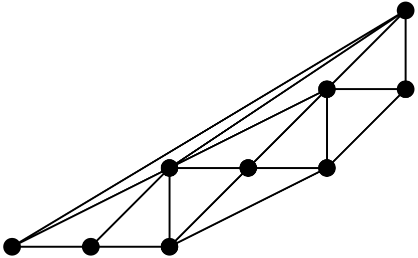
Delaunay triangulation of patterns

Pattern of slope $5/8$



Delaunay triangulation of patterns

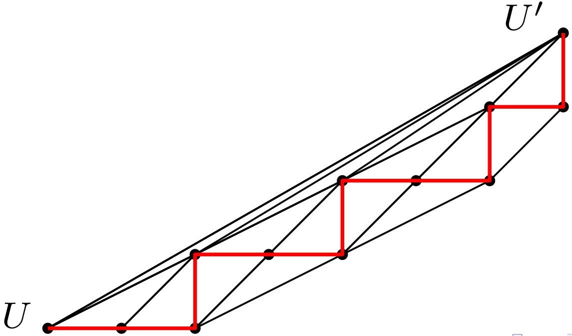
Pattern of slope $2/5$



Three remarks

1. the Delaunay triangulation of UU' contains the staircase representation of UU'.

Pattern of slope $4/7$



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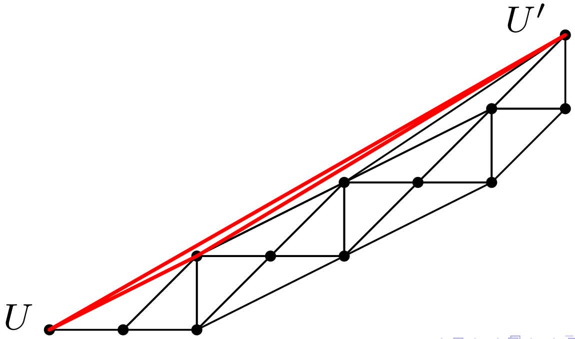
1. the Delaunay triangulation of UU' contains the staircase representation of UU'.
2. U, U' and the closest point of UU' to $[UU']$ (Bezout point) define a facet.

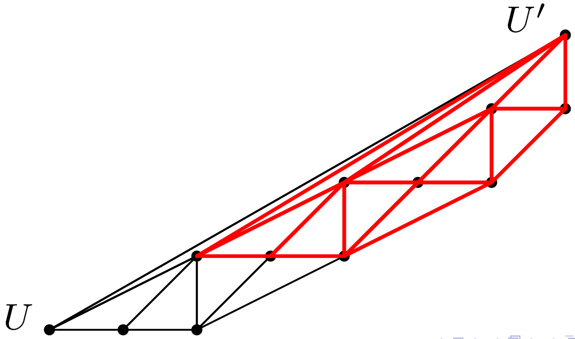
Pattern of slope $4/7$



Three remarks

1. the Delaunay triangulation of UU' contains the staircase representation of UU'.
2. U, U' and the closest point of UU' to $[UU']$ (Bezout point) define a facet.
3. the Delaunay triangulation of some patterns contains the Delaunay triangulation of subpatterns.

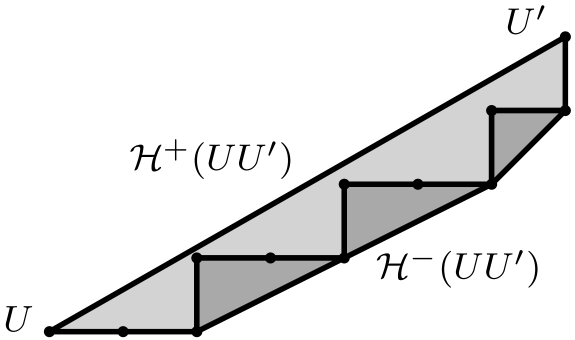
Pattern of slope $4/7$



Dividing the triangulation (remark 1)

- The convex hull is divided into a upper part $\mathcal{H}^+(UU')$ and a lower part $\mathcal{H}^-(UU')$.

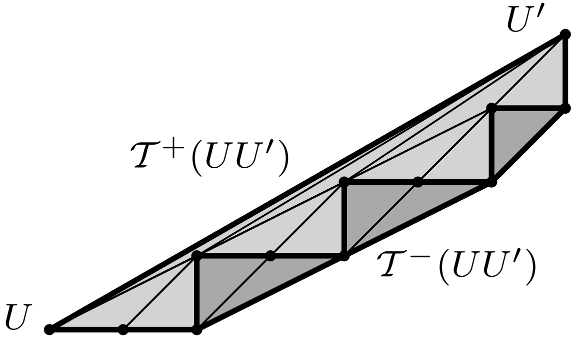
Pattern of slope $4/7$



Dividing the triangulation (remark 1)

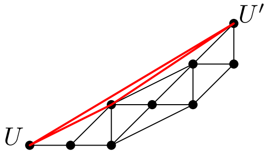
- The convex hull is divided into a upper part $\mathcal{H}^+(UU')$ and a lower part $\mathcal{H}^-(UU')$.
- The Delaunay triangulation is divided into a upper part $\mathcal{T}^+(UU')$ and a lower part $\mathcal{T}^-(UU')$.

Pattern of slope $4/7$



Facets of a pattern

- main facet (remark 2)
  - geometrical characterization (Bezout point)
  - combinatorial characterization (splitting formula)



$0 \quad 0\,1 | 0 \quad 0\,1\,0\,1$

- induction (remark 3)

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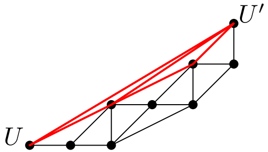

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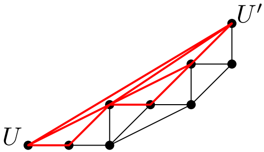

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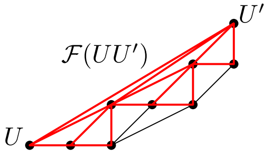

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$0 \mid 0 \mid 1 \mid 0 \mid 0 \mid 1 \mid 0 \mid 1$

- induction (remark 3)

Main result

Theorem

The facets $\mathcal{F}(UU')$ of the pattern UU' is a triangulation of $\mathcal{H}^+(UU')$ such that each facet has points of UU' as vertices and satisfies the Delaunay property, i.e. $\mathcal{F}(UU') = \mathcal{T}^+(UU')$.

the (upper part of the) Delaunay triangulation of a pattern is characterized by the continued fraction expansion of its slope

Outline

Definitions: patterns and Delaunay triangulation

Observation: Delaunay triangulation of patterns?

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Conclusion and perspectives: new algorithms

Sketch of the proof

$\sharp 1$

- no triangular facet of the Delaunay triangulation of a pattern UU' can cross its staircase representation
- the set of facets $\mathcal{F}(UU')$ is the *unique* way of triangulating $\mathcal{H}^+(UU')$

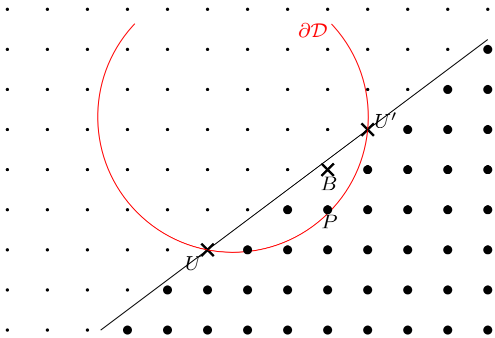
To be more constructive, we chose:

$\sharp 2$

- the set of facets $\mathcal{F}(UU')$ is a triangulation of $\mathcal{H}^+(UU')$ (easy part)
- the interior of the circumcircle of each facet of $\mathcal{F}(UU')$ does not contain any point of UU' (let us focus on that part)

Lemma 1

Let $\mathcal{D}$ be a disk whose boundary passes through U and U' and whose center is located above (UU'). Let $\partial\mathcal{D}$ be its boundary. $\mathcal{D} \setminus \partial\mathcal{D}$ contains a lattice point below or on (UU') if and only if it contains (at least) B, the lower Bezout point of $[UU']$.



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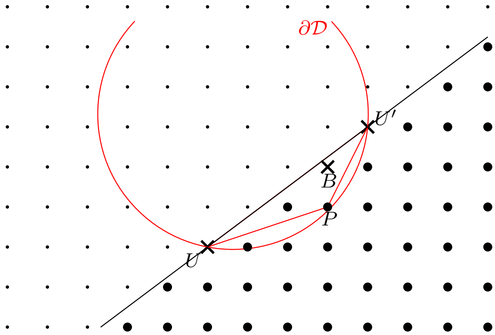

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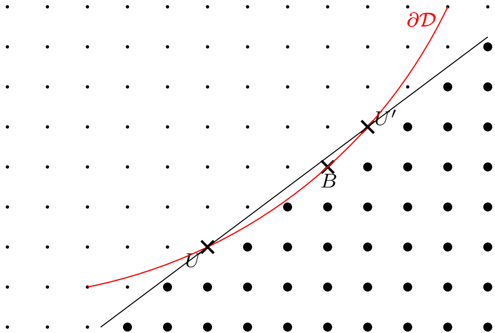
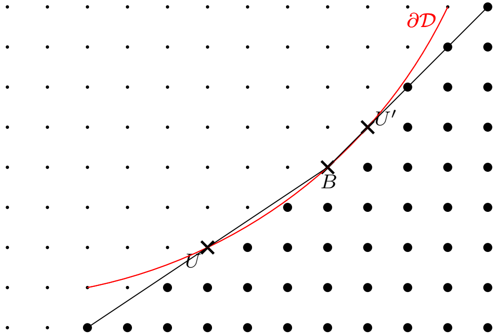


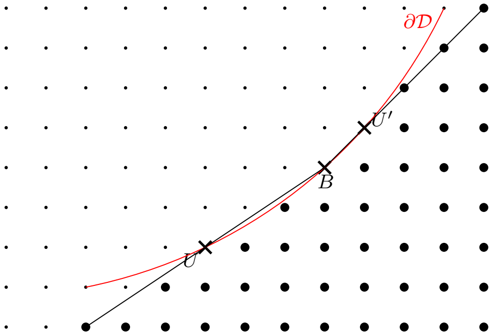
Lemma 2

Let $\mathcal{D}$ be a disk whose boundary $\partial\mathcal{D}$ is the circumcircle of *UBU′*. $\mathcal{D} \setminus \partial\mathcal{D}$ contains none of the *background points* of *UU′* (lattice points below *UB* or *BU′*).



Induction

- The circumcircle of the main facet UBU' contains none of the background points of UU' in its interior (lemma 2).
- The background points of UB and BU' contain the background points of UU', which contains the set points.



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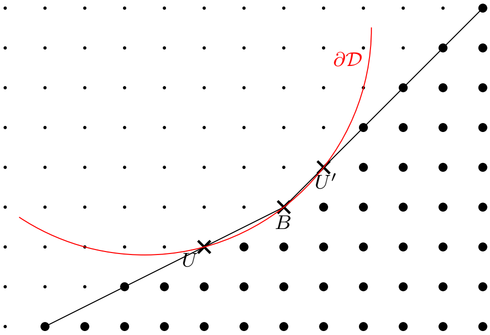

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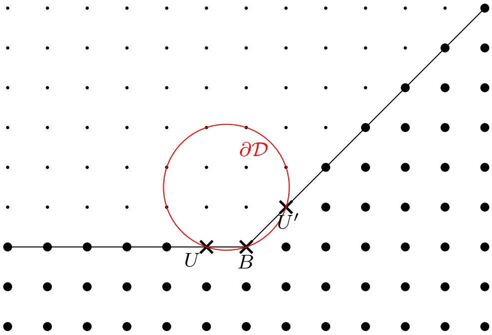

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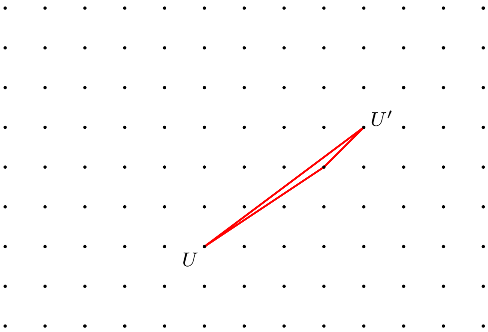

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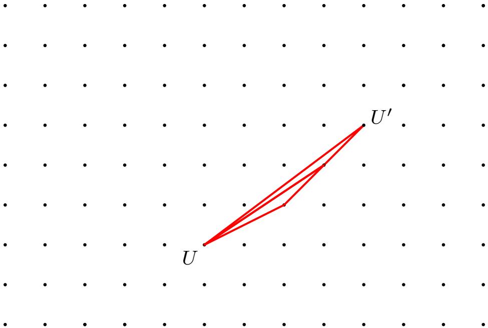

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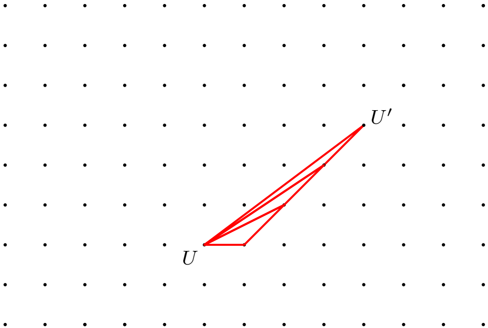

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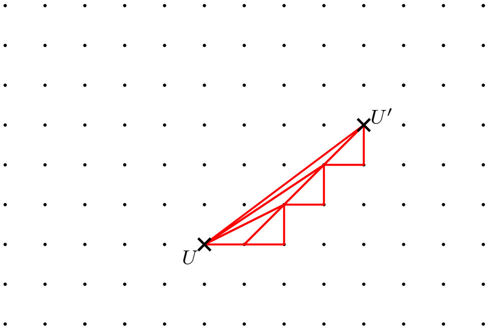
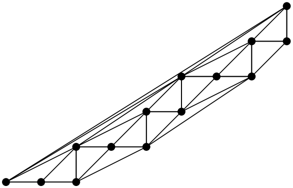


Outline

Delaunay triangulation computation

- Pattern

pattern of slope $8/5$



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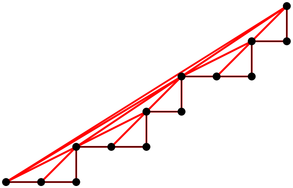

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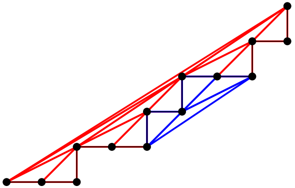

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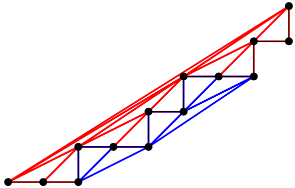

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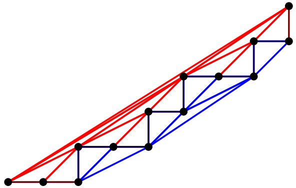
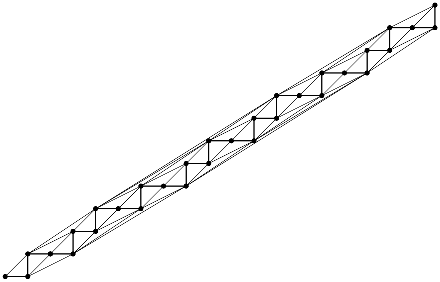


Delaunay triangulation computation

- Pattern
- DSS

DSS of slope $8/5$



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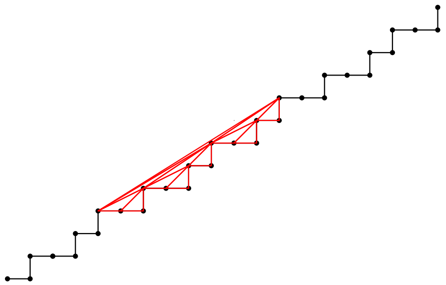

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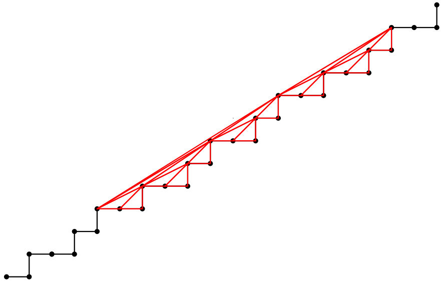

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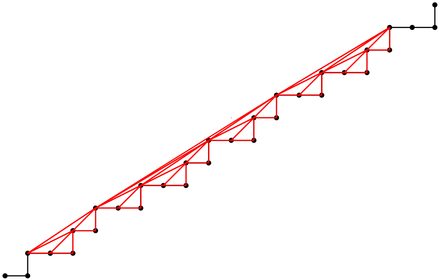

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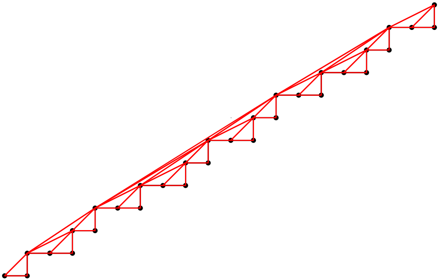

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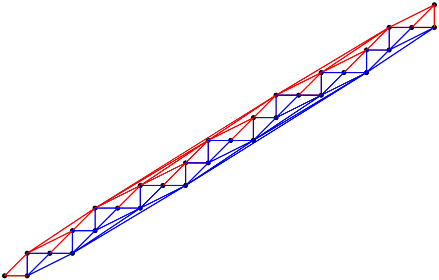
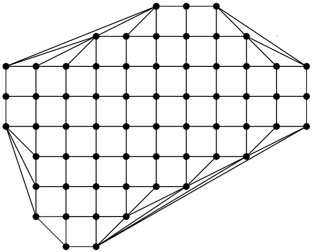


Delaunay triangulation computation

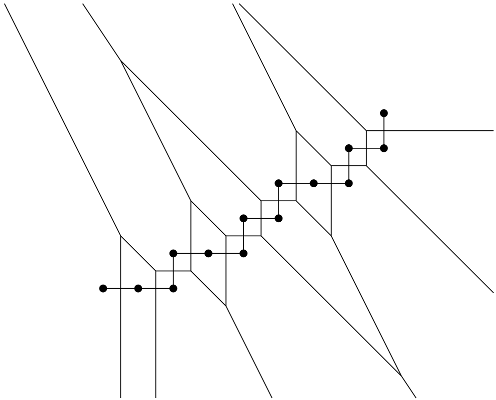
- Pattern
- DSS
- Convex digital object

Convex digital object



Voronoï diagram computation

Pattern



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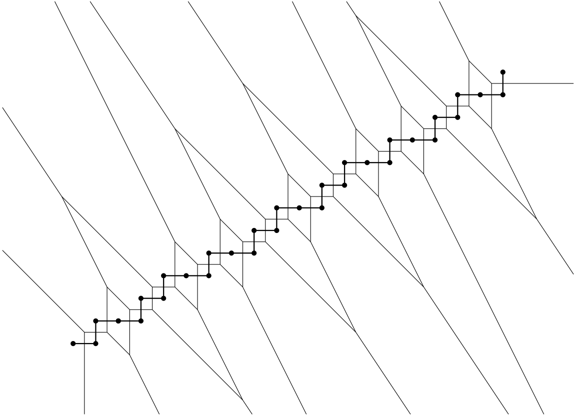

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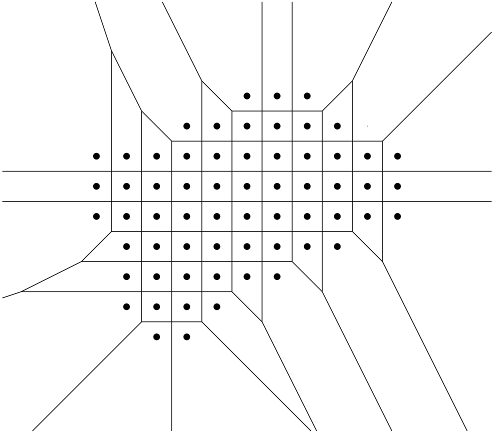

Output-sensitive algorithm for α-hull computation

Definition

For all $\alpha \in [0, 1]$, the α-hull of a set *S* is defined as the intersection of all closed complements of discs of radius $1/\alpha$ that contain all the points of *S*.

figures pour diffrents α de 0 1.

Number of vertices of α-hulls: question

Let X be the Gauss digitization of a convex body of diameter δ.
Let $\sharp V(\alpha)$ be the number of the vertices of the α-hull of X.

- $\sharp V(1) = O(\delta)$ is trivial.
- it is known that $\sharp V(0) = O(\delta^{\frac{2}{3}})$ (convex hull).
- is there a generic formula for all $\alpha \in [0, 1]$ such that: $\sharp V(\alpha) = O(f(\delta, \alpha))$?