Digital Calculus Frameworks and Comparative Evaluation of their Laplace-Beltrami operators

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The Laplacian

In 1D :

$$\Delta f = \frac{d^2 f}{dx^2}$$

In 2D :

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Etc...

Laplace Beltrami operator

More general definitions :

$$\Delta f = \mathrm{div}(\mathrm{grad}(f))$$

$$\Delta f = \delta d f$$

Example : Sphere Laplacian

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2}$$

A Few Properties (why do we care about the Laplacian?)

Integration by parts

$$\forall \Phi, f, \int_{\Omega} \nabla f \cdot \nabla \Phi = - \int_{\Omega} \Delta f \Phi + \int_{\partial \Omega} \Phi \langle \nabla f, \boldsymbol{n} \rangle$$

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Eigen Vectors

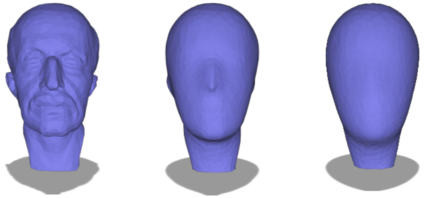
In one dimension, real eigenvectors have the form : $\cos(\omega x + \varphi)$

In 2D : $\cos(\omega_x x + \varphi_x) \cos\big(\omega_y y + \varphi_y\big)$

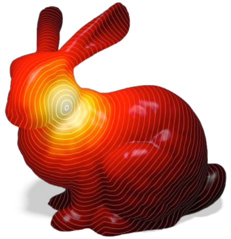
Fourier basis!

Geometry Processing usages

- Filtering **[Sorkine, O.: Laplacian mesh processing. (2005)]**



- Geodesics **[Crane, K., Weischedel, C., Wardetzky, M.: The heat method for distance computation. (2017)]**



- Shape matching… **[Lombaert, H., Grady, L., Polimeni, J.R, Cheriet, F.: FOCUSR: Feature Oriented Correspondence using Spectral Regularization–A Method for Precise Surface Matching(2013)]**

Laplacian evaluations

Bunge, A., Botsch, M.: A survey on discrete Laplacians for general polygonal meshes. (2023)

Many different methods for polygonal mesh

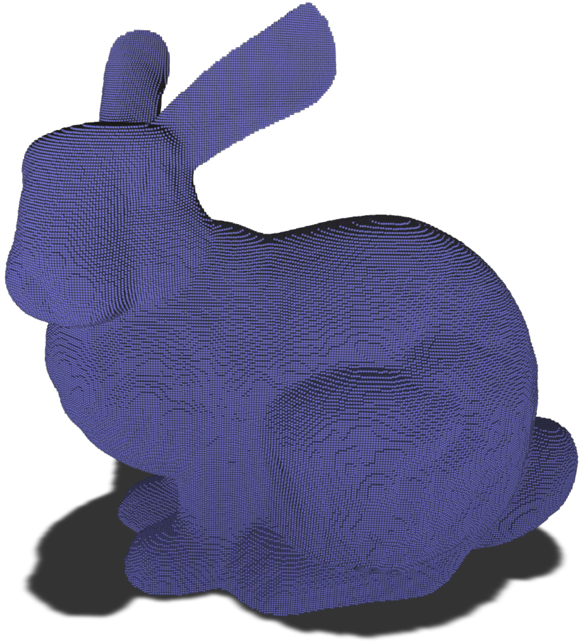
Experimental ways to evaluate them : compare expected solutions of known problems, spherical eigenvectors, spherical eigenvalues…

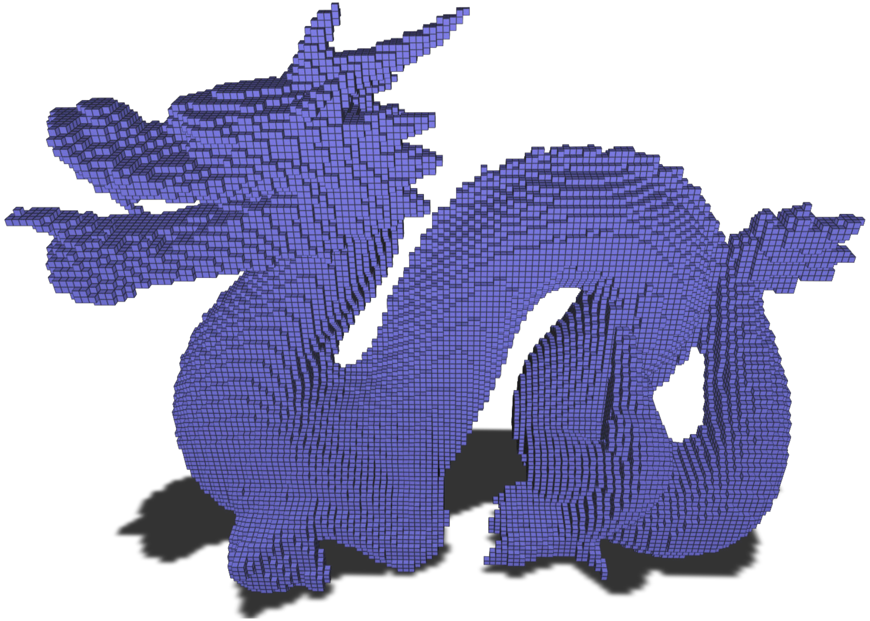
- We adapt an existing method (the Finite Element Method) on surfels to build a Laplace-Beltrami operator
- We also provide a method using interpolated normals (detailed in the paper)
- We compare previous methods to ours, with experiments suggesting convergence
  - Heat Kernel **[Caissard, T., Coeurjolly, D., Lachaud, J.O., Roussillon, T.: Laplace–Beltrami operator on digital surfaces. (2019)]**
  - Corrected PolyDEC **[Coeurjolly, D., Lachaud, J.O.: A simple discrete calculus for digital surfaces. (October 2022)]**

Corrected FEM Laplacian

Digital Surface

Surfaces of voxels





Where is our data defined?

Our mesh is defined by a set of Vertices V and a set of faces F

It depends ! Usually :

- Scalar fields on vertices
- Vector fields on faces

Reason : vertices are where the values are “sampled”

On a triangle, using linear interpolation on the values at vertices the gradient is constant

Data representation

We have a vector holding the values at all the vertices

We like to have our operators be matrices, such that : This allows us to invert these operations (to solve a Poisson problem for example)

I want to have L a Laplace Beltrami operator such that, for any vector t holding some value at vertices (for example a temperature) Lt is its laplacian.

Data representation

$$L \cdot \begin{pmatrix} t(0) \\ t(1) \\ \vdots \\ t(|V|-1) \end{pmatrix} = \begin{pmatrix} \Delta t(0) \\ \Delta t(1) \\ \vdots \\ \Delta t(|V|-1) \end{pmatrix}$$

$$L \in R^{|V| \times |V|}$$

Finite Element Method

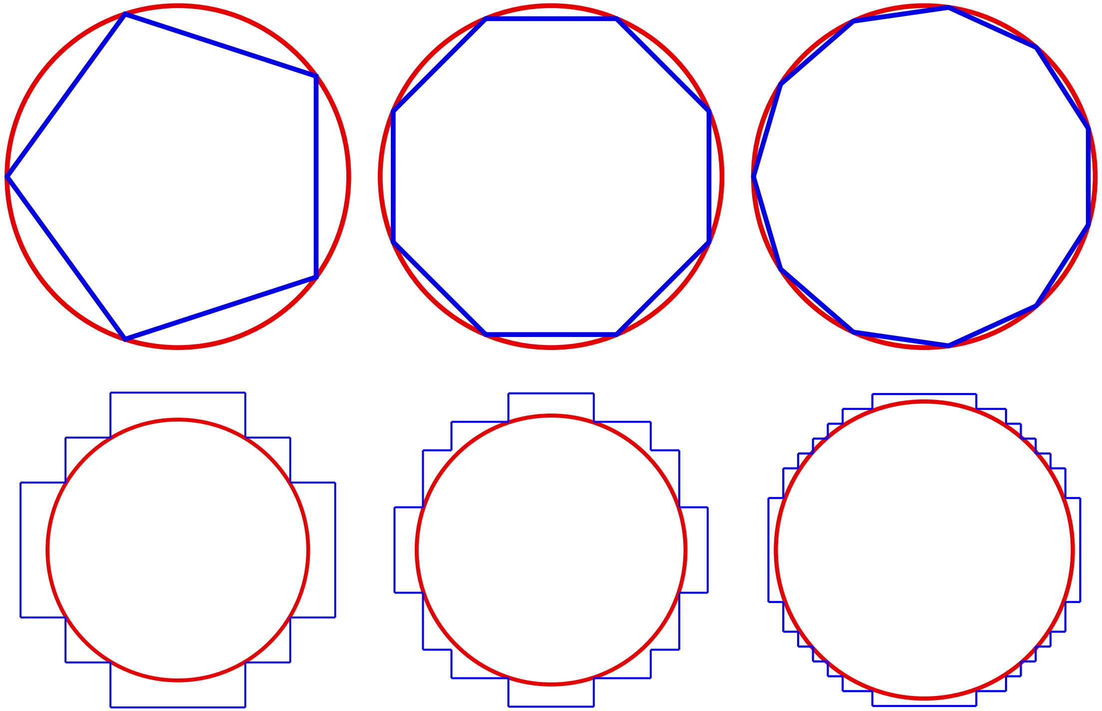
PDE approach, weak Poisson problem formulation :

$$\forall \Phi, \int_\Omega \nabla f \cdot \nabla \Phi = -\int_\Omega \Delta f \Phi + \int_{\partial\Omega} \Phi \langle \nabla f, \boldsymbol{n} \rangle$$

We want to solve $\Delta f = g$ for a given g (we assume no boundary) :

$$\forall \Phi, \int_\Omega \nabla f \cdot \nabla \Phi = -\int_\Omega g \Phi$$

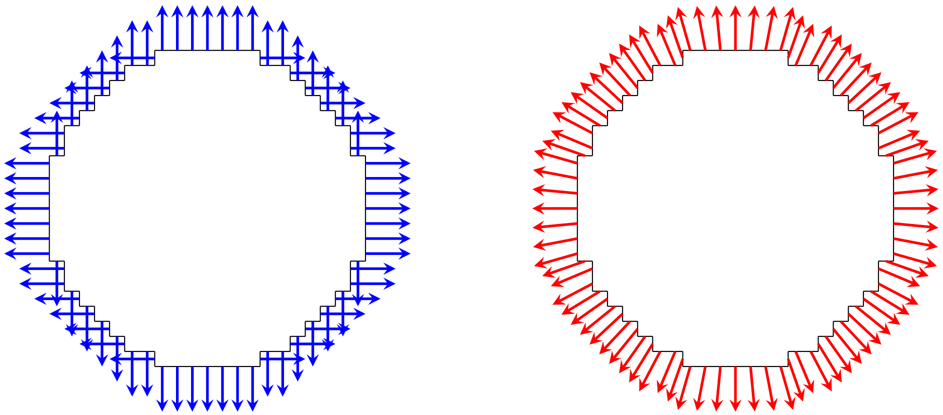
Why corrected ?



$\ell \to 2\pi$

$\ell \to 8$

How do we correct ?



$$\ell = \int_\Omega \langle {\color{blue} u_{\text{naive}}} \mid {\color{red} u_{\text{corrected}}} \rangle \ell_{\text{cell,naive}} \to 2\pi$$

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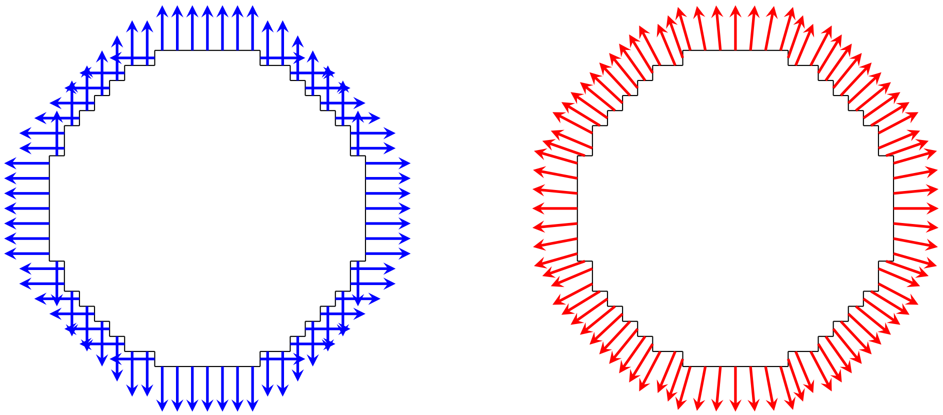


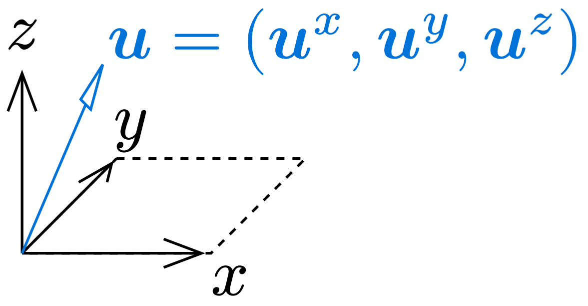
$$\ell = \int_\Omega \langle u_{\text{naive}} \mid u_{\text{corrected}} \rangle \ell_{\text{cell,naive}} \to 2\pi$$

We have convergent estimators of normals and curvature! **[Lachaud, J.O., Coeurjolly, D., Levallois, J.: Robust and convergent curvature and normal estimators with digital integral invariants. (2017)]**

How do we correct :

Metric G :

$$G = \begin{pmatrix} 1-(\boldsymbol{u}^x)^2 & -\boldsymbol{u}^x\boldsymbol{u}^y \\ -\boldsymbol{u}^x\boldsymbol{u}^y & 1-(\boldsymbol{u}^y)^2 \end{pmatrix}$$



Used in : $\langle \boldsymbol{v}, \boldsymbol{w} \rangle_G = \boldsymbol{v}^t G \boldsymbol{w}$

Penalizes dot product in a direction by how much the normal is aligned to that direction

How do we correct :

We assume an input vector field $\boldsymbol{u}$, constant over each face

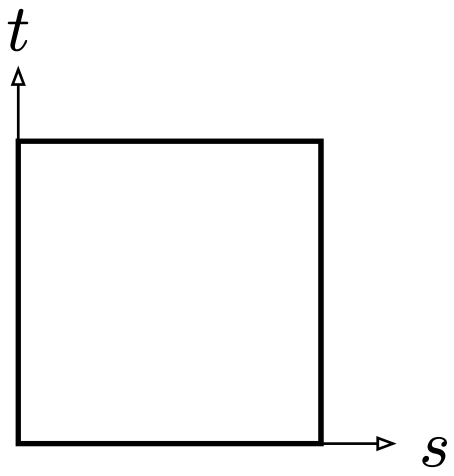
$$\nabla f = \frac{1}{\left(\boldsymbol{u}^z\right)^2} \begin{pmatrix} \left(1 - \left(\boldsymbol{u}^y\right)^2\right) \frac{\partial f}{\partial s} + \boldsymbol{u}^x \boldsymbol{u}^y \frac{\partial f}{\partial t} \\ \left(1 - \left(\boldsymbol{u}^x\right)^2\right) \frac{\partial f}{\partial t} + \boldsymbol{u}^x \boldsymbol{u}^y \frac{\partial f}{\partial s} \end{pmatrix}$$

$$\Delta f = \frac{1}{\left(\boldsymbol{u}^z\right)^2} \left(\left(1 - \left(\boldsymbol{u}^y\right)^2\right) \frac{\partial^2 f}{\partial s^2} + 2\boldsymbol{u}^x \boldsymbol{u}^y \frac{\partial^2 f}{\partial s \partial t} + \left(1 - \left(\boldsymbol{u}^x\right)^2\right) \frac{\partial^2 f}{\partial t^2} \right)$$

$$\int_{\square} f = \int_{[0,1]\times[0,1]} f(s,t) \boldsymbol{u}^z \ \mathrm{ds}\ \mathrm{dt}$$

We work on each element separately first

Our element here is :



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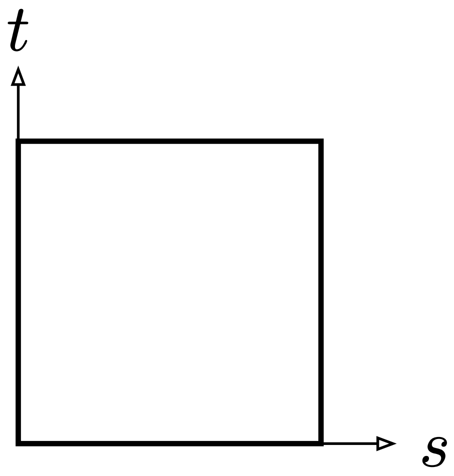

Basis functions

- $\Phi_0 = (1-s)(1-t)$
- $\Phi_1 = s(1-t)$
- $\Phi_2 = st$
- $\Phi_3 = (1-s)t$

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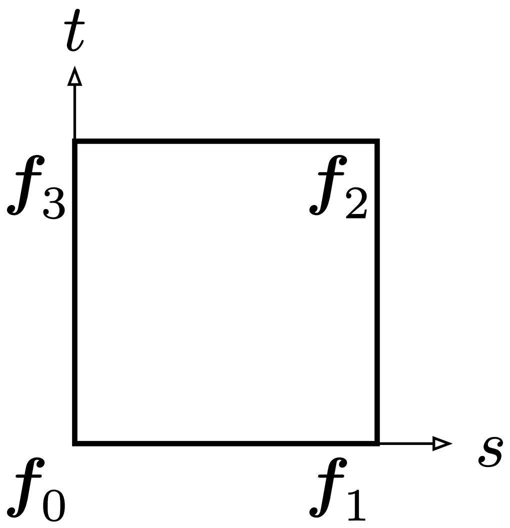

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$$\boldsymbol{f} = [\boldsymbol{f}_0, \boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_2]$$

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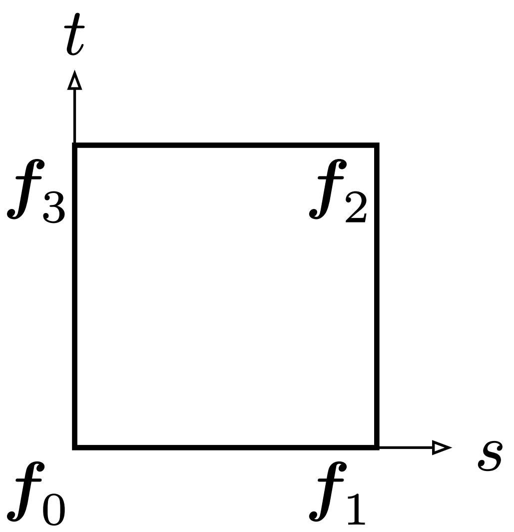

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$$\boldsymbol{f} = [\boldsymbol{f}_0, \boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_2]$$

$$f = \boldsymbol{f}_0\Phi_0 + \boldsymbol{f}_1\Phi_1 + \boldsymbol{f}_2\Phi_2 + \boldsymbol{f}_3\Phi_3$$

Find L_M and M_M such that $\forall f, p$ bilinear, $\int_\Box \langle \nabla f, \nabla p \rangle = \boldsymbol{f}^T L_M \boldsymbol{p}$ and $\int_\Box fp = \boldsymbol{f}^T M_M \boldsymbol{p}$

$$L_{Mi,j} = \int_\Box \langle \nabla \Phi_i, \nabla \Phi_j \rangle$$

$$M_{Mi,j} = \int_\Box \Phi_i \Phi_j$$

Without metric

$$L_M = \frac{1}{6}\begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix}$$

$$M_M = \frac{1}{36}\begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix}$$

With metric

$$L_M = \frac{1}{6\boldsymbol{u}^z}\begin{pmatrix} 3\boldsymbol{u}^x\boldsymbol{u}^y + 2 + 2(\boldsymbol{u}^z)^2 & 2(\boldsymbol{u}^y)^2 - 1 - (\boldsymbol{u}^x)^2 & -1 - 3\boldsymbol{u}^x\boldsymbol{u}^y - (\boldsymbol{u}^z)^2 & 2(\boldsymbol{u}^x)^2 - 1 - (\boldsymbol{u}^y)^2 \\ 2(\boldsymbol{u}^y)^2 - 1 - (\boldsymbol{u}^x)^2 & -3\boldsymbol{u}^x\boldsymbol{u}^y + 2 + 2(\boldsymbol{u}^z)^2 & 2(\boldsymbol{u}^x)^2 - 1 - (\boldsymbol{u}^y)^2 & -1 + 3\boldsymbol{u}^x\boldsymbol{u}^y - (\boldsymbol{u}^z)^2 \\ -1 - 3\boldsymbol{u}^x\boldsymbol{u}^y - (\boldsymbol{u}^z)^2 & 2(\boldsymbol{u}^x)^2 - 1 - (\boldsymbol{u}^y)^2 & 3\boldsymbol{u}^x\boldsymbol{u}^y + 2 + 2(\boldsymbol{u}^z)^2 & 2(\boldsymbol{u}^y)^2 - 1 - (\boldsymbol{u}^x)^2 \\ 2(\boldsymbol{u}^x)^2 - 1 - (\boldsymbol{u}^y)^2 & -1 + 3\boldsymbol{u}^x\boldsymbol{u}^y - (\boldsymbol{u}^z)^2 & 2(\boldsymbol{u}^y)^2 - 1 - (\boldsymbol{u}^x)^2 & -3\boldsymbol{u}^x\boldsymbol{u}^y + 2 + 2(\boldsymbol{u}^z)^2 \end{pmatrix}$$

$$M_M = \frac{\boldsymbol{u}^z}{36}\begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix}$$

Back to global

We sum our local L_M and M_M into a big L and M and consider $\mathbf{\Phi}$ and $\boldsymbol{f}$ global now

We define $\mathbf{\Phi}_i$ as $[0, ..., 0, 1, 0, ..., 0]$, bilinear inside each face

Our problem

$$\forall \Phi, \int_\Omega \nabla f \cdot \nabla \Phi = -\int_\Omega g \Phi$$

Becomes : $\forall \mathbf{\Phi}, \mathbf{\Phi}^T L \boldsymbol{f} = \mathbf{\Phi}^T M \boldsymbol{g}$

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Becomes : $\forall \boldsymbol{\Phi}, \boldsymbol{\Phi}^T L\boldsymbol{f} = \boldsymbol{\Phi}^T M\boldsymbol{g} \;\Leftrightarrow L\boldsymbol{f} = M\boldsymbol{g}$

Poisson problem

$$f = L^{-1} M g$$

- L is SPD, which is nice to invert
- Has a kernel of dimension 1, made of constant functions
- Problem can be fixed to a unique solution using boundary conditions

Forward evaluation

Why not use it the other way? Knowing $\boldsymbol{f}$, find $\boldsymbol{g} = \Delta \boldsymbol{f}$

$$M^{-1} L \boldsymbol{f} = \boldsymbol{g}$$

Results

The laplacian has a known expression on the sphere as well as known eigenvalues

We can compare our results to the expected one

We average results over random discretization of the sphere (random centers)

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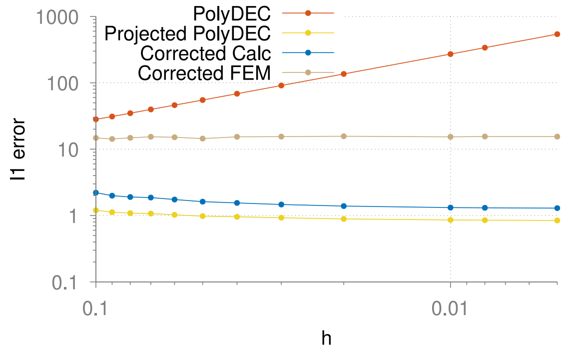
As we decrease the gridstep h, error should decrease too (and eventually converge to 0)

Naive Forward evaluation

We try computing $\boldsymbol{g} = M^{-1} L \boldsymbol{f}$

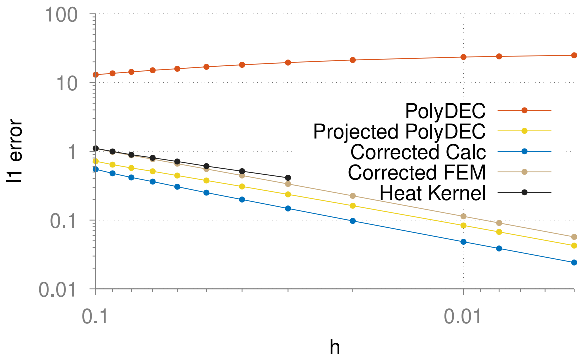


$$f(x) = e^x$$

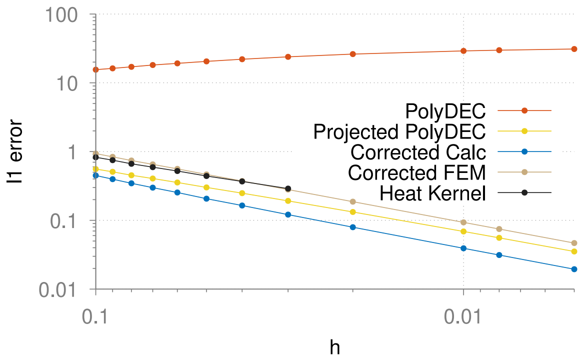
Fixed Forward evaluation

Apply some diffusion:

$\boldsymbol{g} = M^{-1}L\boldsymbol{f}$ becomes $\boldsymbol{g} = (M - dtL)^{-1}L\boldsymbol{f}$ with $dt = 0.035h$



$f(x) = x^2$



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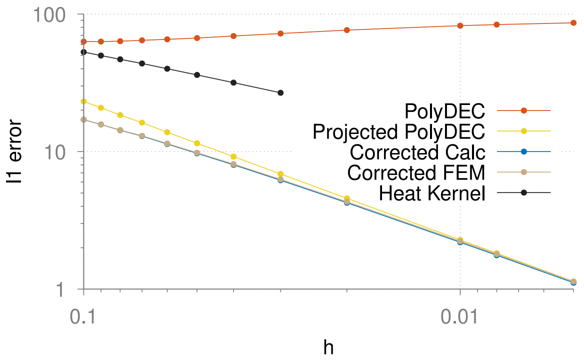
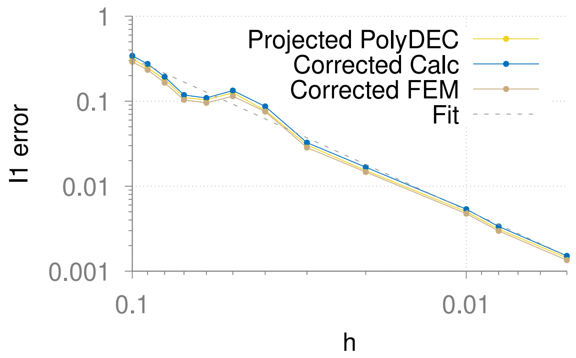


$f : \mathrm{SH}(6,6)$

Backward evaluation



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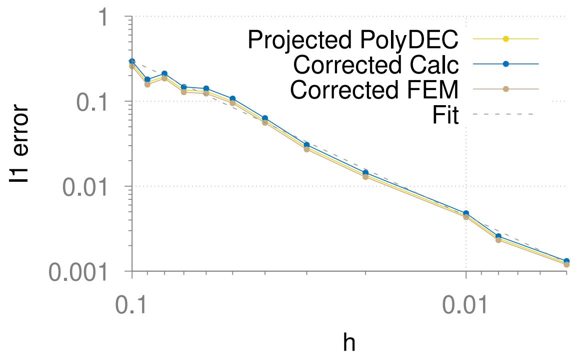

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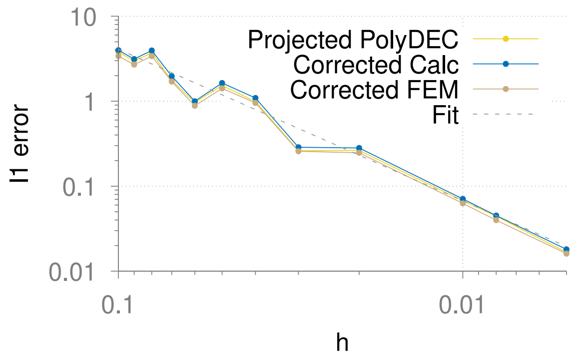

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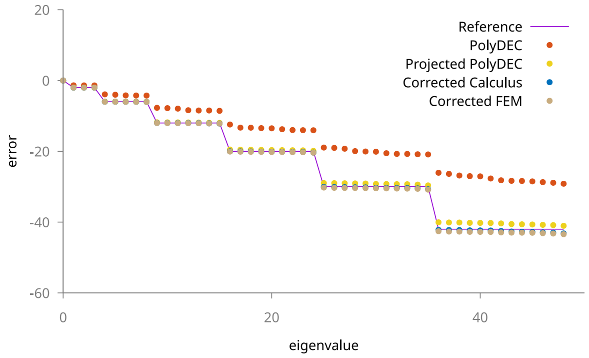
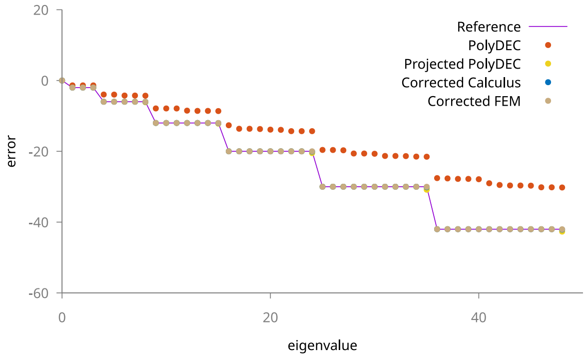
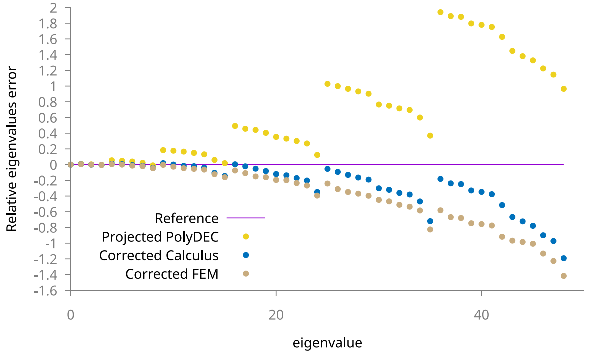
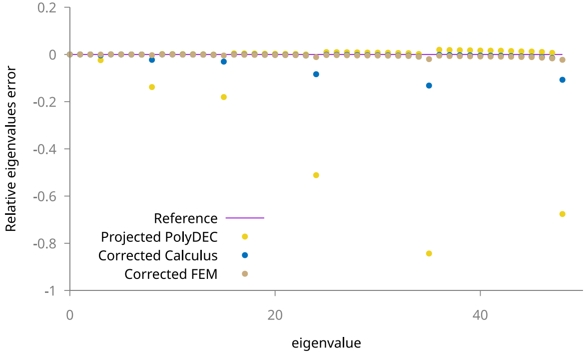
Eigenvalues









Perspectives

- Proof of convergence
- Play with curvature : better metric? recover position from normal and mean curvature $\Delta \boldsymbol{p} = -2H\boldsymbol{n}$
- Explore use for other geometries (triangle mesh with normal maps)