An On-line Machine Learning Return Prediction

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October, 2020

IRMC 2020 - Global Virtual Conference

A preliminary version appears as "*Predict Relative Stock Index Returns by Sufficient Historical Data*"

Punchlines

1. Propose a new prediction methodology for *relative stock index returns* based on universal portfolio construction
   - *relative stock index return* $\equiv$ gross return of a stock index / (1 + interest rate)
   - An on-line strategy competitive with the best constant proportional portfolio

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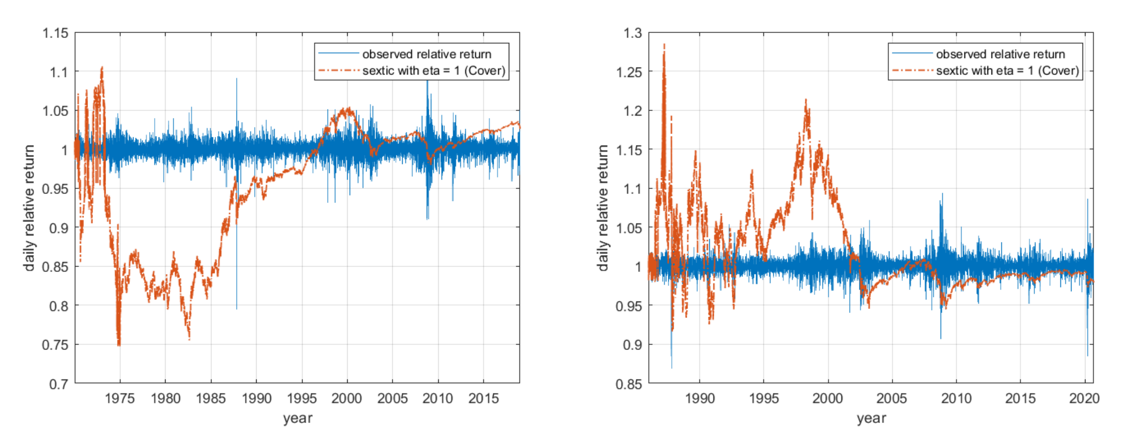
   - An on-line strategy competitive with the best constant proportional portfolio
2. Derive a closed-form 6-order (sextic) predicting formula whose coefficients are **solely** determined by historical data
   - no other model assumptions or market assumptions
3. Empirically, the daily predictive errors in the 2010-2018 period is below 2%

Data

Market	Stock Index (Asset 2)	Risk Free Rate (Asset 1)	Sample Period	# of Obs (*N*)
US	S&P 500 index	12-month T-Bill	1970 - 2018	12,228
US	CRSP value weighted index	12-month T-Bill	1970 - 2018	12,228
UK	FTSE 100 index	12-month LIBOR	1986 - 2018	8,856
Japan	NIKKEI 225	12-month Government Bond	1981 - 2018	9,753
China	CSI 300 index	12-month deposit rate	2005 - 2018	3,265

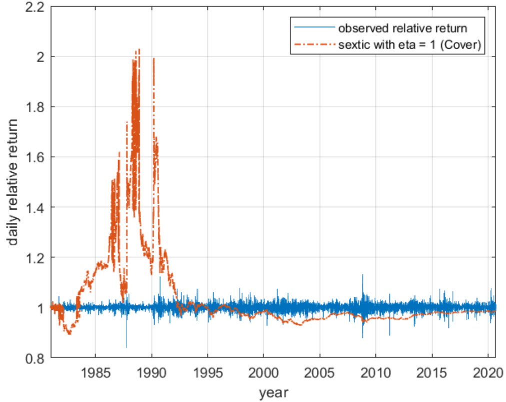
- Use daily observations to ensure *sufficient* historical data

Relative Stock Index Return Prediction

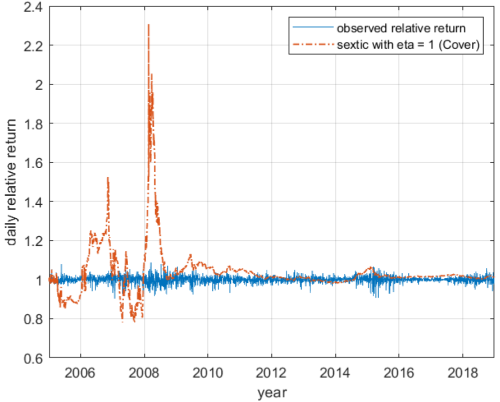


(a) S&P 500 vs 12M T-Bill (USA)   (b) FTSE 100 vs 12M LIBOR (UK)

Relative Stock Index Return Prediction



(c) NIKKEI 225 vs 12M gov bond (Japan)



(d) CSI 300 vs 12M deposit rate (China)

Motivations I

- An On-line machine learning prediction strategy for the decision maker
  - Deterministic prediction unconditionally? NO $\Longleftarrow$ Foster and Vohra (1999)
  - On-line machine learning? Promising $\Longleftarrow$ Cesa-Bianchi and Lugosi (2006)
    - To find the best on-line learning strategies for portfolio constructions against which the worst-case return data that **Nature** provides

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    - To find the best on-line learning strategies for portfolio constructions against which the worst-case return data that **Nature** provides
  - Widely studied in Computer Science, Operational Research, Game Theory, etc.
    - Cover (1991), Cover and Thomas (1991), Hazan and Kale (2015), Hazan et al. (2007), Helmbold et al. (1998), and Vovk and Watkins (1998)

Motivations II

- To predict/forecast based entirely on **sufficient historical data**
  - Without any assumptions on the market.
  - Even though efficient numerical schemes can solve our problems, we are able to derive a closed-form solution that can be easily implemented

Setup

- Consider an economy of two assets with **sufficient** discrete periods, N
- In each single period, n,
  - $x_n \equiv$ gross return of asset 1 (bond)
  - $y_n \equiv$: gross return of asset 2 (stock)
- A *relative stock return* is defined as,

$$R_n = \frac{y_n}{x_n}, \quad \Longleftrightarrow \quad r_n = \frac{1 - R_n}{R_n} = \frac{x_n}{y_n} - 1, \tag{1}$$

Constant proportional strategy I

- $b \in [0,1]$: weight in asset 1 (bond) and remains constant through time.
- The cumulative return after N periods is,

$$S_N(b) \equiv \prod_{n=1}^{N} [bx_n + (1-b)y_n] = \prod_{n=1}^{N} x_n \prod_{n=1}^{N} [b + (1-b)R_n] \tag{2}$$

- R_n is the *relative stock return*

Constant proportional strategy II

- The best proportional strategy for the first N periods is thus

$$b_N^\star = \underset{0 \le b \le 1}{\operatorname{argmax}} \, S_N(b). \tag{3}$$

which needs all information up to N (unknown).

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- Logarithm transformation and first-order condition (FOC)

$$\sum_{n=1}^{N} \frac{1 - R_n}{R_n + (1 - R_n) b_N^\star} = 0 \quad \Longleftrightarrow \quad \sum_{n=1}^{N} \frac{r_n}{1 + r_n b_N^\star} = 0. \tag{4}$$

On-line Machine Learning Strategy I

- An on-line strategy by Thomas M. Cover (Mathematical Finance, 1991)
  - μ : a distribution on $[0,1]$, either uniform or Dirchlet $(\frac{1}{2}, \frac{1}{2})$ distributed.
  - $\hat{b}_n$: the weight of asset 1 (risk-free bond),

$$\hat{b}_1 = \frac{1}{2}; \quad \hat{b}_n = \frac{\int_0^1 bS_{n-1}(b)d\mu(b)}{\int_0^1 S_{n-1}(b)d\mu(b)}, \quad n \geq 2 \tag{5}$$

- $\hat{b}_n$ is conditionally known at the time $(n-1)$

On-line Machine Learning Strategy II

- Following this strategy, the cumulative return for the first N periods is,

$$\hat{S}_N \equiv \prod_{n=1}^{N} \left[\hat{b}_n x_n + (1 - \hat{b}_n) y_n\right] = \prod_{n=1}^{N} x_n \prod_{n=1}^{N} \left[\hat{b}_n + (1 - \hat{b}_n) R_n\right]. \tag{6}$$

Intuition – Fund of Funds

Fund 1 → **Bond: b = 0.01** **Stock: 1-b = 0.99** → *(n-1) periods* → **Cumulative return** S(b) = S(0.01)

Fund 2 → **Bond: b = 0.02** **Stock: 1-b = 0.98** → *(n-1) periods* → **Cumulative return** S(b) = S(0.02)

Fund 3 → **Bond: b = 0.03** **Stock: 1-b = 0.97** → *(n-1) periods* → **Cumulative return** S(b) = S(0.03)

⋮

Fund M → **Bond: b = 0.99** **Stock: 1-b = 0.01** → *(n-1) periods* → **Cumulative return** S(b) = S(0.99)

$$\hat{b}_n = \frac{\sum b_i S_i}{\sum S_i}$$

An Example of Implementation

Algorithm 1 Pseudo algorithm for Cover (1991) on-line strategy

```
Initialization: b = [0.01 : 0.01 : 0.99]^T
for n = 2 to N do
    for i = 1 to length(b) do
        b_i ← b(i)
        S_i ← ∏_{t=1}^{n-1} [b_i x_t + (1 − b_i) y_t]        ▷ cumulative return of N − 1 periods given b_i
    ω_i ← S_i / ∑_{i=1} S_i        ▷ the contribution of i-th "fund", b_i, to the on-line, b̂_n
    b̂_n ← ∑_i ω_i ∗ b_i        ▷ the on-line strategy, b̂_n, for the period n
```

$$b = [0.01 : 0.01 : 0.99]^T$$

$$S_i \leftarrow \prod_{t=1}^{n-1} \left[b_i x_t + (1 - b_i) y_t\right]$$

$$\omega_i \leftarrow \frac{S_i}{\sum_{i=1} S_i}$$

$$\hat{b}_n \leftarrow \sum_i \omega_i * b_i$$

A Rough Comparison

Optimal constant strategy

$$S_N^\star = \prod_{n=1}^{N} x_n \prod_{n=1}^{N} \left[b_N^\star + (1 - b_N^\star) R_n \right]$$

- $b_N^\star = \underset{0 \leq b \leq 1}{\operatorname{argmax}} \, S_N(b)$
- needs the information in all N time periods

On-line ML strategy

$$\hat{S}_N = \prod_{n=1}^{N} x_n \prod_{n=1}^{N} \left[\hat{b}_n + (1 - \hat{b}_n) R_n \right]$$

- $\hat{b}_n = \dfrac{\int_0^1 b S_{n-1}(b) d\mu(b)}{\int_0^1 S_{n-1}(b) d\mu(b)}$
- needs the information in $(N-1)$ time periods

Universal Portfolio as *Input*

- Cover (1991) proves that

$$1 \leq \frac{S_N^\star}{\hat{S}_N} \leq 2\sqrt{N+1}. \tag{7}$$

- A universal portfolio is the on-line strategy with the property such that

$$\lim_{N\to\infty} \frac{1}{N} \log\left(\frac{S_N^\star}{\hat{S}_N}\right) = 0. \tag{8}$$

Universal Portfolio as *Input*

- Rewrite Equation (7) with a constant, $c_N \in [0, 1]$

$$\log\left(\frac{S_N^\star}{\hat{S}_N}\right) = \log(2c_N) + \frac{\log(N+1)}{2}, \quad 0 \leq c_N \leq 1 \tag{9}$$

- Drop $\log(2c_N)$ and express the left side above with r_n such that,

$$r_n = \frac{1 - R_n}{R_n} = \frac{x_n}{y_n} - 1$$

Solve $\{b_N^\star, \hat{r}_N\}$ Simultaneously

1. Function from universal portfolio construction

$$\left[\log(1 + b_N^\star \hat{r}_N) - \log(1 + \hat{b}_n \hat{r}_N)\right] + \sum_{n=1}^{N-1} \left[\log(1 + b_N^\star r_n) - \log(1 + \hat{b}_n r_n)\right] = \frac{\log(N+1)}{2}. \quad (10)$$

Solve $\{b_N^\star, \hat{r}_N\}$ Simultaneously

1. Function from universal portfolio construction

$$\left[\log(1+b_N^\star\hat{r}_N)-\log(1+\hat{b}_n\hat{r}_N)\right]+\sum_{n=1}^{N-1}\left[\log(1+b_N^\star r_n)-\log(1+\hat{b}_n r_n)\right]=\frac{\log(N+1)}{2}. \tag{10}$$

2. Function from first-order condition

$$\frac{\hat{r}_N}{1+\hat{r}_N b_N^\star}+\sum_{n=1}^{N-1}\frac{r_n}{1+r_n b_N^\star}=0. \tag{11}$$

How Close between r_N and $\hat{r}_N$?

Proposition 1

- Assume that $0 < m \leq r_n \leq M, \forall n$ for two fixed positive numbers m and M.
- By viewing b_N^* as a function of $\{r_1, \cdots, r_N\}$, its partial derivative to r_N is uniformly bounded from below by a positive constant c.
- Then the solution of the Equation (10) and (11), $\hat{r}_N$, approximates to r_N as close as possible when N is large enough.

How Close between r_N and $\hat{r}_N$?

- As a function of the data $\{r_1, \cdots, r_N\}$, we can show that,

$$\frac{\partial b_N^*}{\partial r_N} > 0. \tag{12}$$

- When the relative return of the asset 1 to the asset 2 is increased, the proportion on the asset 1 is increased as well following the strategy b_N^*

How Close between r_N and $\hat{r}_N$?

- Proposition 1 states that if the contribution of r_N to b_N^* is *non-degenerate* in the sense that, for a positive number c,

$$\frac{\partial b_N^*}{\partial r_N} \geq c, N = 1, 2, \cdots, \tag{13}$$

- then

$$r_N - \hat{r}_N = o(1). \tag{14}$$

Approximation

- Introduce the following notations by historical data only

$$P_1 = \sum_{n=1}^{N-1} r_n, \quad P_2 = \sum_{n=1}^{N-1} r_n^2, \quad Q_1 = \sum_{n=1}^{N-1} \hat{b}_n r_n, \quad Q_2 = \sum_{n=1}^{N-1} \hat{b}_n^2 r_n^2.$$

- To derive a analytical formula for $\hat{r}_N$ with Taylor expansion
  - 2nd-order for log(x) $\implies$ sextic formula
  - 1st-order for log(x) $\implies$ cubic formula

A Sextic Predicting Formula

- $\hat{r}_N$ can be solved from the following formula,

$$f_6(x) \equiv \quad \hat{b}_N^2 x^6 - 2\hat{b}_N x^5 + \left[Q_2 - 2Q_1 + 2P_2\hat{b}_N - \log(N+1) + 1\right] x^4$$

$$+ (2P_1 - 2P_2\hat{b}_N)x^3 + \left[P_1^2 - 2P_2Q_1 + P_2 + P_2^2\hat{b}_N^2 - 2P_2^2\log(N+1)\right] x^2$$

$$+ (2P_1P_2 - 2P_2^2\hat{b}_N)x + \left[P_1^2P_2 - 2P_2^2Q_1 + P_2^2Q_2 - P_2^2\log(N+1)\right],$$

- whose coefficients are **solely** determined by historical data.

Alternative Construction of $\hat{b}_n$

- Vovk and Watkins (1998)

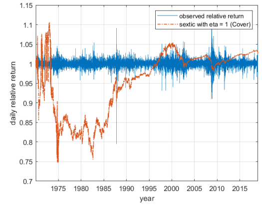
$$\hat{b}_n^{VW} = \frac{\int_0^1 b S_{n-1}(b)^{\eta} db}{\int_0^1 S_{n-1}^{\eta} db}, \quad 0 < \eta \leq 1, n \geq 2 \tag{15}$$

- When $\eta = 1$,

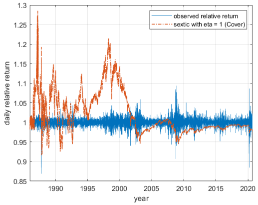
$$\hat{b}_n^{VW} \implies \hat{b}_n$$

Predicted Relative Stock Index Return

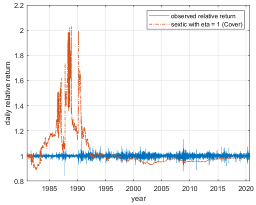
- red line: sextic formula predicted *relative stock index return*
- blue line: market observed *relative stock index return*



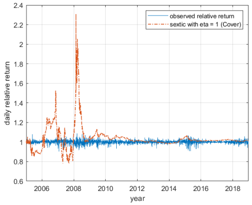
(e) S&P 500 vs 12M T-Bill (USA)



(f) FTSE 100 vs 12M LIBOR (UK)



(g) NIKKEI 225 vs 12M gov bond (Japan)



(h) CSI 300 vs 12M deposit rate (China)

Predictive Errors I

- We evaluate the prediction performance during the period 2010 - 2018

$$\text{predictive error} \equiv \frac{\hat{R}_N - R_N}{R_N}. \tag{16}$$

Predictive Errors II

- Consider 4 statistics for the time series of predictive errors suggested by Cremers (2002)

  1. *MAD*: the average of the absolute predictive errors
  2. *Bias*: the average of the predictive errors
  3. *Deviation*: the average of the squared difference between the bias and the predictive errors
  4. *RMSE*: the square root of the average of the sum of squared predictive errors.

S&P 500 index *vs* 12-month T-Bill

$$\hat{b}_1^{VW} = 0.5, \quad \hat{b}_n^{VW} = \frac{\int_0^1 b S_{n-1}(b)^{\eta} db}{\int_0^1 S_{n-1}^{\eta} db}, \quad 0 < \eta \leq 1, n \geq 2$$

	η = 1 (Cover)	η = 0.75	η = 0.5	η = 0.15	η = 0.05
MAD	1.96%	2.00%	2.06%	2.13%	2.15%
Bias	1.78%	1.83%	1.88%	1.96%	1.98%
Deviation	0.02%	0.02%	0.02%	0.02%	0.02%
RMSE	2.29%	2.34%	2.40%	2.48%	2.51%

CRSP value index *vs* 12-month T-Bill

$$\hat{b}_1^{VW} = 0.5, \quad \hat{b}_n^{VW} = \frac{\int_0^1 b S_{n-1}(b)^\eta db}{\int_0^1 S_{n-1}^\eta db}, \quad 0 < \eta \leq 1, n \geq 2$$

	η = 1 (Cover)	η = 0.75	η = 0.5	η = 0.15	η = 0.05
MAD	1.14%	1.16%	1.18%	1.20%	1.21%
Bias	0.23%	0.24%	0.25%	0.26%	0.26%
Deviation	0.02%	0.02%	0.02%	0.02%	0.02%
RMSE	1.42%	1.44%	1.46%	1.49%	1.50%

FTSE 100 index *vs* 12-month LIBOR

$$\hat{b}_1^{VW} = 0.5, \quad \hat{b}_n^{VW} = \frac{\int_0^1 b S_{n-1}(b)^{\eta} db}{\int_0^1 S_{n-1}^{\eta} db}, \quad 0 < \eta \leq 1, n \geq 2$$

	η = 1 (Cover)	η = 0.75	η = 0.5	η = 0.15	η = 0.05
MAD	1.46%	1.48%	1.51%	1.54%	1.54%
Bias	-1.35%	-1.37%	-1.40%	-1.43%	-1.44%
Deviation	0.01%	0.01%	0.01%	0.01%	0.01%
RMSE	1.80%	1.82%	1.84%	1.87%	1.88%

NIKKEI 225 index *vs* 12-month Gov-bonds

$$\hat{b}_1^{VW} = 0.5, \quad \hat{b}_n^{VW} = \frac{\int_0^1 b S_{n-1}(b)^{\eta} db}{\int_0^1 S_{n-1}^{\eta} db}, \quad 0 < \eta \leq 1, n \geq 2$$

	η = 1 (Cover)	η = 0.75	η = 0.5	η = 0.15	η = 0.05
MAD	2.81%	2.89%	2.97%	3.10%	3.13%
Bias	-2.70%	-2.79%	-2.88%	-3.01%	-3.04%
Deviation	0.03%	0.03%	0.03%	0.03%	0.03%
RMSE	3.23%	3.31%	3.39%	3.52%	3.55%

CSI 300 index *vs* 12-month Deposit Rate

$$\hat{b}_1^{VW} = 0.5, \quad \hat{b}_n^{VW} = \frac{\int_0^1 b S_{n-1}(b)^{\eta} db}{\int_0^1 S_{n-1}^{\eta} db}, \quad 0 < \eta \leq 1, n \geq 2$$

	η = 1 (Cover)	η = 0.75	η = 0.5	η = 0.15	η = 0.05
MAD	4.27%	4.33%	4.40%	4.49%	4.52%
Bias	4.21%	4.28%	4.35%	4.45%	4.47%
Deviation	0.11%	0.11%	0.11%	0.11%	0.12%
RMSE	5.33%	5.40%	5.48%	5.59%	5.62%

Future Work

- Economic meanings behind the on-line machine learning strategy
  - Two time series: any similarities between them? Patterns?
- Option strategy based on the predicted relative return
  - If we have available (sufficient) option data, we can construct an option trading strategy based on our 2% error bounded predicted value

References I

Cesa-Bianchi, N. and Lugosi, G. (2006). *Prediction, learning, and games*. Cambridge university press.

Cover, T. M. (1991). Universal portfolios. *Mathematical Finance*, 1(1):1–29.

Cover, T. M. and Thomas, J. A. (1991). Elements of information theory. *New York: Wiley*, 3:37–38.

Cremers, K. M. (2002). Stock return predictability: A bayesian model selection perspective. *The Review of Financial Studies*, 15(4):1223–1249.

Foster, D. P. and Vohra, R. (1999). Regret in the on-line decision problem. *Games and Economic Behavior*, 29(1-2):7–35.

Hazan, E., Agarwal, A., and Kale, S. (2007). Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2-3):169–192.

Hazan, E. and Kale, S. (2015). An online portfolio selection algorithm with regret logarithmic in price variation. *Mathematical Finance*, 25(2):288–310.

Helmbold, D. P., Schapire, R. E., Singer, Y., and Warmuth, M. K. (1998). On-line portfolio selection using multiplicative updates. *Mathematical Finance*, 8(4):325–347.

Vovk, V. and Watkins, C. (1998). Universal portfolio selection. In *Proceedings of the eleventh annual conference on Computational learning theory*, pages 12–23.