

# FINN 6216 Quantitative Risk Management

## Credit Risk Management

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*The slides and codes can be accessed on my website: [@JacquesYL](#)*

# Outline

- Credit Risk Overview
- Structural Model of Credit Risk
  - BSM Option Pricing Model - Revisit
  - Merton Model (Contingent Claim Approach) & KMV Model
- Other Models and Methods
  - Reduced-Form Model of Credit Risk, Hazard Model, and Altman's Z-Score

# Credit Risk

- When banks issue a loan, they expose themselves to the risk that either the interest or the principal itself may not be paid, or may not be paid on time.
- Default is the extreme case (i.e., a counterparty is unable or unwilling to fulfill its obligation), credit risk also includes changes in the credit quality of a counterparty, since this affects the value of the loan or portfolio of loans.
- Institutions are also exposed to the risk that a counterparty, which may be another institution or a government, is downgraded by a rating agency.
- Credit Risk, along with Market Risk, and Operational Risk, are three major components of risk that a bank faces
- Commercial or retail credit risk?

# Structural Model of Credit Risk

BSM Option Pricing Model - Revisit

# Assumptions

1. The underlying asset,  $S_t$ , can be bought and sold in fractions of units. The asset pays no dividends or other distributions before the option's expiration date.
2. The risk-free interest rate,  $r$ , is a constant and the same for all maturities.
3. The option is European, with exercise price  $K$  and time to expiration  $T$ .
4. There are no frictions in the market place such as taxes, transaction costs, or margin requirements. Trading in the underlying asset is continuous.
5. The asset price dynamics are described by geometric Brownian motion (GBM)

$$dS_t = \alpha S_t dt + \sigma S_t dZ_t \quad (1)$$

# Call Option Price

- Let  $C(S_t, t)$  denote the price of a European call option at time  $t$
- Ito-Doeblin formula to obtain the dynamism of the call option price

$$dC_t = \left[ \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S} dS_t \right] \quad (2)$$

where we use the *Quadratic Variation*:  $(dS)^2 = \sigma^2 S^2 dt$

- The option price and the stock price depend on the same underlying source of uncertainty,  $dZ$ .

## Hedge Portfolio

- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- Hedge portfolio by taking a long position in  $\Delta$  shares of stock and a short position in **one** call option.

$$\Pi = \frac{\partial C}{\partial S} S_t - C(S_t, t) \quad (3)$$

- The change in the portfolio value is thus,

$$d\Pi = \frac{\partial C}{\partial S} dS_t - dC = - \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt \quad (4)$$

## Hedge Portfolio is Riskless

- The riskless hedge portfolio should earn the risk-free rate of interest  $r$  per unit time.

$$\frac{d\Pi}{\Pi} = rdt \quad \text{or} \quad d\Pi = r\Pi dt \quad (5)$$

- Substituting from above for  $d\Pi$  and  $\Pi$  we have

$$-\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}\right) dt = r \left(\frac{\partial C}{\partial S} S_t - C(S_t, t)\right) dt \quad (6)$$



## Second-order PDE

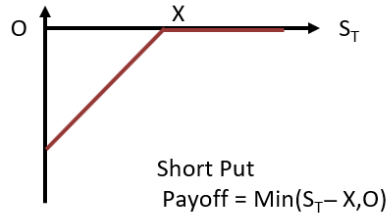
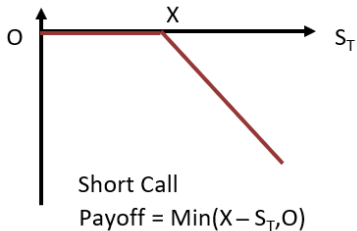
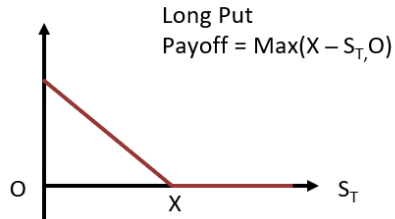
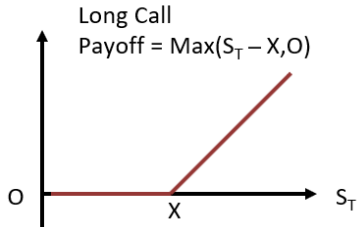
- $C(S_t, t)$ , must satisfy a second-order partial differential equation (PDE)

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0 \quad (7)$$

- Three boundary conditions

1. At expiration  $C(S_T, T) = \max(S_T - K, 0)$
2. At  $S_t = 0$ ,  $C(0, t) = 0$
3. As  $S_t \rightarrow \infty$ ,  $\Delta = \frac{\partial C}{\partial S} \rightarrow 1$

# Option Payoff



# BSM Option Pricing Formula

- European Call

$$C = S_0 N(d_1) - e^{-rT} K N(d_2) \quad (8)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \quad (9)$$

- How to compute European Put? Put-call Parity

# BSM Option Pricing Formula

- If the Call is exercised on the expiration date, the holder will receive  $S_T$  and pay  $K$ .
- However, this will only take place if  $S_T > K$ .
  - $e^{-rT}KN(d_2)$  is the present value of paying  $K$ , conditional on  $S_T > K$
  - $S_0N(d_1)$  is the present value of receiving  $S_T$ , conditional on  $S_T > K$

## Comments on $N(d_2)$

- $N(d_2)$  is the risk adjusted probability for call option exercise

$$Pr(S_T > K) = N(d_2) \quad (10)$$

- The present value of paying  $K$  at time  $T$  when option is in-the-money

$$K \times Pr(S_T > K) \times e^{-rT} = e^{-rT} KN(d_2)$$

- Thus,  $1 - N(d_2) = N(-d_2)$  is the probability of  $S_T < K$  for call options.

## Comments on $N(d_1)$

- $N(d_1) = \Delta > N(d_2)$ , the Delta of the option, accounts for two things:
  1. the probability of exercise as given by  $N(d_2)$ , and
  2. the **fact** that exercise or rather receipt of stock on exercise is dependent on the conditional future values that the stock price takes on the expiry date.

## Comments on $N(d_1)$

- The present value of receiving the underlying stock

$$S_0 N(d_1) = \left\{ \underbrace{e^{-rT} \mathbb{E}[S_T | S_T > K]}_{\text{PV of receiving stock}} \right\} \times \underbrace{N(d_2)}_{\text{probability}} > e^{-rT} \mathbb{E}[S_T] N(d_2) = S_0 N(d_2)$$

- Rewrite the first equality above,

$$\left( e^{-rT} \mathbb{E}[S_T] \right) N(d_1) = e^{-rT} \mathbb{E}[S_T | S_T > K] N(d_2)$$

- Thus,

$$N(d_1) = \frac{\mathbb{E}[S_T | S_T > K]}{\mathbb{E}[S_T]} N(d_2) > N(d_2) \quad (11)$$

# Structural Model of Credit Risk

Merton's Model (1974)



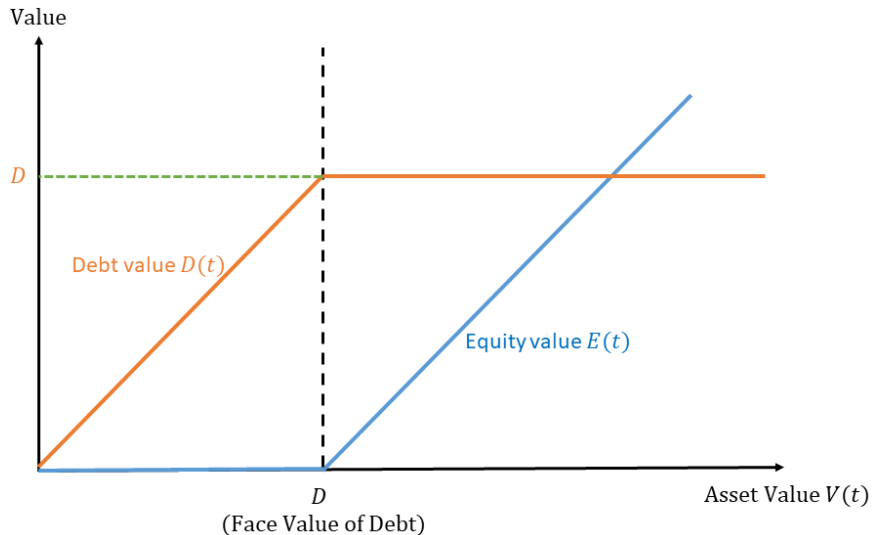
# Notations

- Structural model: use the total firm value, debt and equity level of a firm for predicting default and loss given default
- Suppose a firm is financed by equity and debt

$$V(t) = E(t) + D(t) \quad (12)$$

- $V(t)$ : total value of the firm (asset value) at time  $t$
- $E(t)$ : equity of firm at time  $t$
- $D(t)$ : the (senior) debt at time  $t$ : maturity  $T$ , face value  $D$  and zero coupon

# Convexity of Equity - Long Call & Short Put



## Relation to Option Pricing - Merton's Model

- Firm value  $V(t)$  is the underlying asset, and we can assume it follows a GBM

$$dV_t = \alpha V_t dt + \sigma V_t dZ_t \quad (13)$$

- Equity value,  $E$ , is a **long position of call options** with maturity  $T$ , and a strike price of  $D$ , the face value of debt.

$$E(T) = \max [V(T) - D, 0] \quad (14)$$

- Debt value,  $D$ , is similar to a **short position of put options** with maturity  $T$ .

$$D(T) = \min [V(T), D] \quad (15)$$

If  $V(T) < D$ , the firm goes bankrupt.

## Merton's Model - Equity Value

- The payoff to shareholders (equity holders) at time  $T$  is

$$E_T = \max(V_T - D, 0) \quad (16)$$

- Equity value today is

$$E_0 = \text{Call}(V_0, D, r, T, \sigma_V) = V_0 N(d_1) - e^{-rT} D N(d_2), \quad (17)$$

where  $\sigma_V$  denote the volatility of firm value, and

$$d_1 = \frac{\ln(V_0/D) + (\alpha + \sigma_V^2/2)T}{\sigma_V \sqrt{T}}, \quad d_2 = \frac{\ln(V_0/D) + (\alpha - \sigma_V^2/2)T}{\sigma_V \sqrt{T}} = d_1 - \sigma_V \sqrt{T}$$

## Merton's Model - Debt Value

- The payoff to debt holders at time  $T$

$$D_T = \min(V_T, D) = V_T - \min(V_T - D, 0) \quad (18)$$

- The present value of this payoff is

$$D_0 = V_0 - E_0 = V_0 - \text{Call}(V_0, D, r, T, \sigma_V) = \underbrace{e^{-rT} D - \text{Put}(V_0, D, r, T, \sigma_V)}_{\text{by put-call parity}} \quad (19)$$

Recall that  $P - C = e^{-rT} D - V_0$

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- Value of risky debt = Value of risk-free debt - Put option.
- Put option: the risk-adjusted expected losses due to default

## Merton's Model - Volatility of Equity Value

- Volatility of Equity value is

$$\sigma_E = \left( \frac{V}{E} \right) \frac{\partial E}{\partial V} \sigma_V = \left( \frac{V}{E} \right) N(d_1) \sigma_V \quad (20)$$

- Derivation? Let  $E = F(V_t, t)$ , by Ito-Doeblin formula,

$$dE = \cdots dt + \left( \underbrace{\frac{\partial E}{\partial V}}_{N(d_1)} \sigma_V V \right) dZ = \cdots dt + (\sigma_E E) dZ \quad (21)$$

- Thus,

$$N(d_1) \sigma_V V = \sigma_E E \quad (22)$$

# Structural Model of Credit Risk

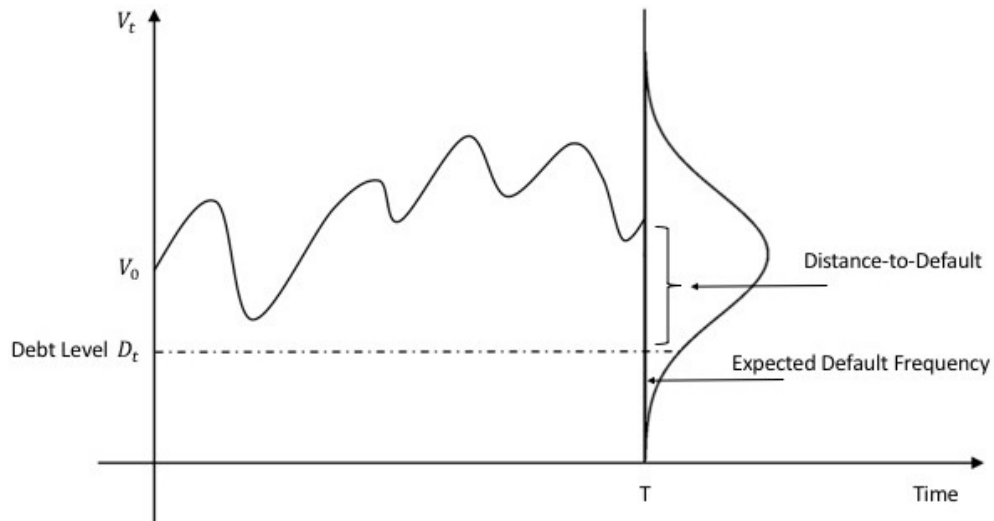
Credit Risk Measurement - KMV Model



# KMV Model

- Kealhofer, McQuown, and Vasicek (KMV) argues that credit ratings did not tell the whole story
  - Bonds with same rating show different risks of default
- Use Merton model to compute the probabilities of default - Expected Default Frequency
  - EDF is a forward-looking measure of probability of default and firm specific
- Credit risk is driven by the firm value process: incorporate stock price and balance sheet information into implied default probabilities.
- This can be extended to multivariate firms and systemic risk of financial system (see e.g. Gupta, Lu, and Wang, 2021)

# KMV Model



## KMV model

- Distance-to-default

$$DD = \frac{\ln(V/D) + (\alpha - \sigma_v^2/2)(T - t)}{\sigma_v \sqrt{T - t}} \quad (23)$$

DD is essentially the  $d_2$

- Expected Default Frequency (the probability of default)

$$EDF = Pr(V_T < D) = 1 - N(DD) = N(-DD) \quad (24)$$

EDF is  $1 - N(d_2)$

- Replace  $\alpha$  with risk-free rate,  $r$ , we obtain the risk-neutral probability

## Example

- The value of a company's equity is 3 million and the volatility of the equity is 80 percent.  
 $E_0 = 3, \sigma_E = 0.8$
- The debt that will have to be paid in one year  $T = 1$  is \$10 million.  $D = 10$
- The risk-free rate is 5% per annum
- What is the firm's risk-neutral probability of default in one year?

## Solution: Two Equations and Two Unknowns

- To unknowns: initial firm value  $V_0$  and firm volatility  $\sigma_V$
- Equity value by Merton model

$$E_0 = \text{Call}(V_0, D, r, T, \sigma_V) = V_0 N(d_1) - e^{-rT} D N(d_2) \quad (25)$$

- Equity volatility

$$N(d_1) \sigma_V V_0 = \sigma_E E_0 \quad (26)$$

$$d_1 = \frac{\ln(V_0/D) + (r + 0.5\sigma_V^2)T}{\sigma_V\sqrt{T}}, \quad d_2 = \frac{\ln(V_0/D) + (r - 0.5\sigma_V^2)T}{\sigma_V\sqrt{T}} = d_1 - \sigma_V\sqrt{T}$$

## Solution: Two Equations and Two Unknowns

- The numerical solution is

$$V_0 = 12.40, \quad \sigma_V = 0.2123 \quad (27)$$

- The risk-neutral default probability (EDF) is

$$EDF = N(-d_2) = 1 - N(d_2) = 12.7\% \quad (28)$$

## Implied Credit Spread of Debt?

- The market value of (zero-coupon) debt with face value of 10 in one year is

$$D_0 = V_0 - E_0 = 12.40 - 3 = 9.40 \quad (29)$$

- By debt pricing, the yield-to-maturity,  $y$ , satisfies,

$$9.4 = 10 \times e^{-y*1} \implies y = 0.0619 \quad (30)$$

- The credit spread is thus,

$$CS = y - r = 0.0619 - 0.05 = 1.19\% \quad (31)$$

## Relation to Corporate Finance & Capital Structure

- Optimal capital structure of a firm
  - Coupon interests on debt are paid before the taxable income. This serves as the tax shield. In contrast, dividend earned from equity is paid after the tax. Together, it gives the firm an incentive to issue **more** debt.
  - The bankruptcy risk from debt issuing gives the firm an incentive to issue **less** debt.



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- Agent-Principal problem
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  - Equity is like buying call options, the higher the volatility, the higher the option value.
  - Equity claims are **residual claims**.

# Other Models and Methods on Credit Risk

## Weakness of Structural Model

- It requires some subjective estimation of the input parameters.
- It is difficult to construct theoretical EDF's without the assumption of normality of asset returns
- Private firms' EDFs can be calculated only by using some comparability analysis based on accounting data.
- It does not distinguish among different types of long-term bonds according to their seniority, collateral, covenants, or convertibility

# Reduced-Form Model of Credit Risk

- A reduced-form model of credit risk uses prices of traded debt instruments of the firm to elicit estimates of credit risk of the firm.
- Specifically, the difference in prices of risk-free bonds and risky-bonds issued by firms are used to assess the likelihood of the firms to default
- Assumption: the credit spread is solely due to increased default probability

# Hazard Rate Model

- Define a Default Intensity (or hazard rate):  $\lambda(t)$
- Default probability in a period of time  $\Delta t$  is thus  $\lambda(t)\Delta t$
- Estimate the default intensity using
  - Historical data
  - Bonds price (reduced-form)
  - Credit derivatives (CDS)
- Since the hazard rate is calibrated using price data, the calibration is in the risk-neutral measure, not the real-world measure

## Altman's Z-Score Method

- Z-score combines five measures based on reported accounting and stock market variables

$$Z = 1.2X_1 + 1.4X_2 + 3.3X_3 + 0.6X_4 + 0.999X_5 \quad (32)$$

where

- $X_1$ : the ratio of Working Capital to Total Assets of the firm (liquidity)
- $X_2$ : the ratio of Retained Earnings to Total Assets
- $X_3$ : the Earnings Before Interest and Taxes (EBIT) to Total Assets ratio (profitability)
- $X_4$ : the ratio of Market Value of Equity to Book Value of Total Liabilities of (leverage)
- $X_5$ , the Sales to Total Assets ratio (efficiency)

## Altman's Z-Score Method

- The higher the Z-score, the stronger a firm's financial health is.
  - If  $Z > 3.0$ , the company is considered unlikely to default
  - $3.0 > Z > 2.7$ , we should be 'on alert'
  - $2.7 > Z > 1.8$ , there is a good chance of default.
  - $Z < 1.8$ , the probability of financial embarrassment is very high
- Type 1 error: a firm that was predicted to not go bankrupt, actually did go bankrupt
- Type 2 error: the firm should have very high probability of default, and it actually ends up surviving the financial turbulence.



# Questions?

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- Webpage: [JacquesYL.github.io](https://JacquesYL.github.io)

# References I

Aparna Gupta, Yueliang Lu, and Runzu Wang. Addressing systemic risk using contingent convertible debt—a network analysis. *European Journal of Operational Research*, 290(1):263–277, 2021.