FINN 6216 Quantitative Risk Management
Volatility & Correlation Derivatives

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Outline

- The Speaker
- Carr-Madan Formula
  - Plain-vanilla options are more useful than you think.
- Volatility Derivatives
  - Exchange products: VIX and VIX derivatives
  - OTC products: Variance Swap
- Correlation Derivatives

About Me

- Ph.D. Student in Finance
- Research interests include
  - *Asset Pricing*: Anomalies, Derivatives and Options, Quantitative Finance
  - *Financial Risk Management*: Systemic Risk, Longevity Risk, Asset-Liability Management
  - *Application of Machine Learning in Finance*
- For additional info, check my homepage JacquesYL.Github.io

Carr-Madan Formula

A General Formula

- Given a C^2 function $\phi(x)$ and x_0 is one in the defined region, then

$$\phi(x) = \phi(x_0) + (x - x_0)\phi'(x_0) + \int_{x_0}^{\infty} \phi''(K)(x-K)^+ dK + \int_0^{x_0} \phi''(K)(K-x)^+ dK. \qquad (1)$$

- Replace x by S_{t+T}, the stock price at time $t+T$,
- Replace x_0 by $F_{t \to t+T}$, the futures price that matures at time $t+T$,

Carr-Madan Formula

$$
\begin{aligned}
\phi(S_{t+T}) &= \phi(F_{t\to t+T}) \text{ (a bond position)} \\
&+ \phi'(F_{t\to t+T})(S_{t+T} - F_{t\to t+T}) \text{ (a stock position)} \\
&+ \int_{F_{t\to t+T}}^{\infty} \phi''(K)(S_{t+T} - K)^{+} dK \text{ (a sequence of call options)} \\
&+ \int_{0}^{F_{t\to t+T}} \phi''(K)(K - S_{t+T})^{+} dK \text{ (a sequence of put options)}
\end{aligned} \tag{2}
$$

Carr-Madan Formula

- The idea belongs to many authors in the literature Carr, Ellis, and Gupta (1998); Carr and Madan (1999); Bakshi and Madan (2000); Bakshi, Kapadia, and Madan (2003)
    - See this link for a quick proof: Stack Exchange
- The formula allows you to replicate any European option with payoff function $\phi(.)$ using the plain-vanilla Call and Put options for all strikes
    - For instance, this formula is used in the valuation of a variance swap
    - Also, an approximation for VIX construction

VIX and VIX Derivatives

VIX Index

- VIX was introduced initially in 1993 to track the BSM implied volatility of options on S&P 100 with near-the-money strikes.

- On September 22, 2003, the CBOE uses the more actively traded S&P 500 index options.

- The old index is kept and was renamed as VXO

- In general, the VIX index and S&P 500 index tend to move in opposite directions - VIX rises when equities decline and vice versa.

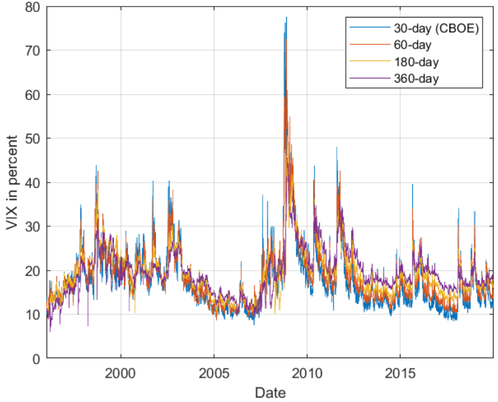
VIX Index

- Let $R_{f,t\to t+T}$ denote the gross risk-free return over $[t, t+T]$
- VIX index can be computed from

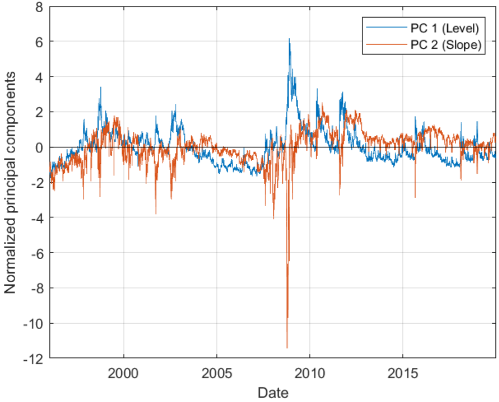
$$VIX^2_{t\to t+T} = \frac{2R_{f,t\to t+T}}{T}\left\{\int_0^{F_{t\to t+T}} \frac{1}{K^2} Put_{t,t+T}(K)dK + \int_{F_{t\to t+T}}^{\infty} \frac{1}{K^2} Call_{t,t+T}(K)dK\right\} \quad (3)$$

- In particular, CBOE VIX index is when $T = 30$ days.
- Of course, we can compute VIX for different T, which gives us a term structure of VIX.

VIX Term Structure (Johnson, 2017)

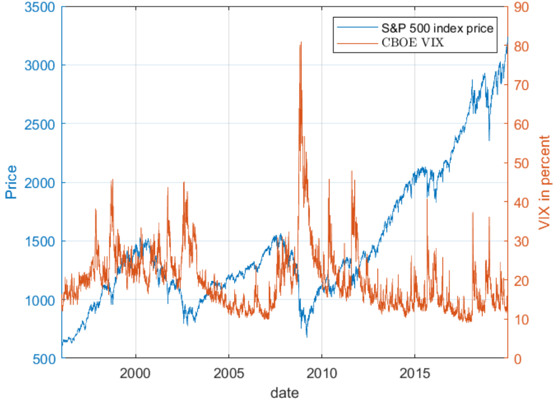


(a) VIX term structure

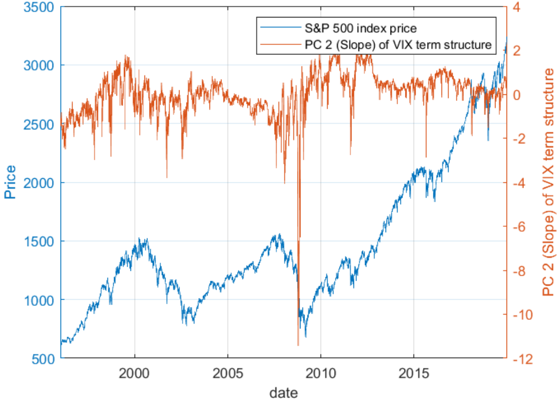


(b) Principal components

VIX Index, Term Structure, and S&P 500 Index



(c) VIX index & S&P 500 index

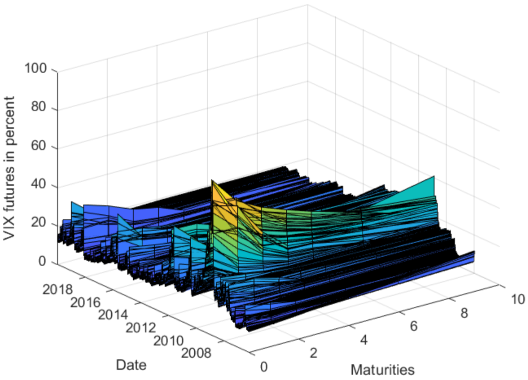


(d) VIX PC 2 & S&P 500 index

VIX Futures

- VIX futures contract were first introduced on March 26, 2004 by CBOE Futures Exchange (CFE).
- In the early days, four futures contracts were listed every day. After October 2006, 9 contracts are listed every day.
- Initially, VIX futures prices were quoted as the VIX times ten and the contract multiplier was $100.
- Starting on March 26, 2007, CFE modified the contract specification by dividing futures prices by ten and increasing the contract multiplier to $1000.

VIX Futures



- In normal times, VIX futures prices tend to have an upward-sloping term structure, suggesting a premium paid by long-time investors.

- The upward term structure of VIX futures prices to maturity is known as “contango trap”. It explains why rolling a futures contract is associated with substantial losses (Eraker and Wu, 2017).

- During market turbulence, i.e., the financial crisis in 2008/09, the futures curve tends to become inverted or hump-shaped.

VIX Options

- CBOE launched European options on the VIX index on February 24, 2006.
- Like VIX futures,they are cash settled according to the difference between the value of the VIX at expiration and their strike price
- VIX derivative positions can be used to hedge the risks of investments in the S&P 500 index.
    - By holding VIX derivatives, investors can achieve exposure to S&P 500 volatility without having to delta-hedge the option positions with index itself.
    - It is cheaper to buy OTM VIX call options than to buy OTM index put options.
    - VIX options are the ONLY asset in which open interest is highest for OTM call strikes

Variance Swap & Simple Variance Swap

Building Blocks: Volatility Products

- Realized variance:

$$RV = \frac{1}{T}\sum_{t=1}^{T}\left(\log\frac{S_t}{S_{t-1}}\right)^2 \tag{4}$$

- OTC products to trade realized variance:
  - Delta-hedged options (straddles)
  - Volatility swap
  - Variance swap

Variance Swap

- A variance swap (conventional) is an agreement at time 0 to exchange

$$\left(\log \frac{S_\Delta}{S_0}\right)^2 + \left(\log \frac{S_{2\Delta}}{S_\Delta}\right)^2 + \cdots + \left(\log \frac{S_T}{S_{T-\Delta}}\right)^2 \tag{5}$$

  for some fixed "strikes", $\tilde{V}$. Typically, $\Delta = 1$ day.

- For derivative pricing, we want to find a $\tilde{V}$ such that the contract is of zero-value at inception for two counterparties

$$\tilde{V} = \mathbb{E}_0^Q \left[\left(\log \frac{S_\Delta}{S_0}\right)^2 + \left(\log \frac{S_{2\Delta}}{S_\Delta}\right)^2 + \cdots + \left(\log \frac{S_T}{S_{T-\Delta}}\right)^2\right] \tag{6}$$

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- Question: what if S_t hits zero at sometime before T?

Variance Swap

- We assume that

  1. The continuously compounded interest rate is constant, at r
  2. The underlying asset does not pay dividends
  3. The underlying follows an Ito process (no jumps) in risk-neutral: $dS_t = rS_t dt + \sigma_t S_t dZ_t$

- The strike on a variance swap is thus

$$\tilde{V} = 2R_{f,0\to T}\left\{\int_0^{F_{0\to T}} \frac{1}{K^2} Put_{0,T}(K)dK + \int_{F_{0\to T}}^{\infty} \frac{1}{K^2} Call_{0,T}(K)dK\right\} \tag{7}$$

- In the limit as $\Delta \to 0$,

$$\tilde{V} = \mathbb{E}_0^Q\left[\int_0^T \sigma_t^2 dt\right] = \underbrace{T * \sigma^2}_{\text{if assuming a GBM}} \tag{8}$$

How to Hedge?

- The variance swap can be hedged by holding

  1. a static position in $\left(\frac{2}{K^2}dK\right)$ PUT options expiring at time T with strike K, for each $K \leq F_{0,t}$
  2. a static position in $\left(\frac{2}{K^2}dK\right)$ CALL options expiring at time T with strike K, for each $K \geq F_{0,t}$, and
  3. a dynamic position in $\left[\frac{2\left(\frac{F_{0,t}}{S_t}-1\right)}{F_{0,T}}dK\right]$ units of the underlying asset at time t, financed by borrowing.

An Issue in Variance Swap

- Equation (7) has a similar form to VIX^2 defined in Equation (3). Theoretically, it is tempting to have

$$\tilde{V} = T * VIX^2_{0 \to T} \tag{9}$$

- However,, in reality, the market does not follow an Ito process
    - Jumps and other extreme events in the market
- Thus, the square of VIX does not correspond to the fair value strike on a variance swap, $\tilde{V}$
- Variance swaps cannot be hedged at times of jumps
    - Variance swap market collapsed during the events of 2008

Simple Variance Swap

- A fundamental problem of conventional variance swap is that, if the underlying asset, i.e., an individual stock, goes bankrupt, S_t hits zero at some point before expiry T, then the payoff in Equation (5) is *infinite*

- Solution: to modify the payoff function. A simple variance swap is an agreement to exchange

$$\left(\frac{S_\Delta - S_0}{F_{0,0}}\right)^2 + \left(\frac{S_{2\Delta} - S_\Delta}{F_{0,\Delta}}\right)^2 + \cdots + \left(\frac{S_T - S_{T-\Delta}}{F_{0,T-\Delta}}\right)^2 \tag{10}$$

- We no longer need to assume a diffusion process. Instead,

    1. The continuously compounded interest rate is constant, at r
    2. The underlying asset pays dividends continuously at rate δS_t per unit time

Simple Variance Swap

- The strike on a simple variance swap is thus

$$\tilde{V} = \frac{2R_{f,0\to T}}{F^2_{0\to T}} \left\{ \int_0^{F_{0\to T}} Put_{0,T}(K)dK + \int_{F_{0\to T}}^{\infty} Call_{0,T}(K)dK \right\} \tag{11}$$

- In the limit as $\Delta \to 0$ and $\delta = 0$

$$\tilde{V} = T * SVIX^2_{0\to T} \tag{12}$$

where SVIX is a volatility index defined in Martin (2017) such as

$$SVIX^2_{t\to t+T} = \frac{2}{TR_{f,t\to t+T}S_t^2} \left\{ \int_0^{F_{t\to t+T}} Put_{t,t+T}(K)dK + \int_{F_{t\to t+T}}^{\infty} Call_{t,t+T}(K)dK \right\} \tag{13}$$

How to Hedge?

- The simple variance swap can be hedged by holding

  1. a static position in $\left(\frac{2}{F_{0,T}^2}dK\right)$ PUT options expiring at time T with strike K, for each $K \leq F_{0,t}$

  2. a static position in $\left(\frac{2}{F_{0,T}^2}dK\right)$ CALL options expiring at time T with strike K, for each $K \geq F_{0,t}$, and

  3. a dynamic position in $\left[2e^{-\delta(T-t)}\frac{\left(1-\frac{S_t}{F_{0,t}}\right)}{F_{0,T}}dK\right]$ units of the underlying asset at time t, financed by borrowing.

What Does VIX Really Measure?

- Assume the underlying asset does not pay dividends, so that $R_{t\to t+T} = S_{t+T}/S_t$
- Then the VIX measures the risk-neutral entropy of the simple return

$$VIX^2_{t\to t+T} = \frac{2}{T}\mathbb{L}_t^Q\left(\frac{R_{t\to t+T}}{R_{f,t\to t+T}}\right) \tag{14}$$

  where the entropy is defined by

$$\mathbb{L}_t^Q(X) \equiv \log\left(\mathbb{E}_t^Q X\right) - \mathbb{E}_t^Q\left(\log X\right) \tag{15}$$

- Entropy is more sensitive to the left tail of returns, whereas variance is more sensitive to the right tail
  - VIX places more weight on OTM puts and less weight on OTM calls, and hence places more weight on left-tail events.

Correlation Derivatives

Implied Correlation

- The variance of a portfolio consisting of N assets is

$$\sigma_{p,t}^2 = \sum_{i=1}^{N} \omega_{i,t}^2 \sigma_{i,t}^2 + 2 \sum_{i=1}^{N-1} \sum_{j>i} \omega_{i,t} \omega_{j,t} \rho_{ij,t} \sigma_{i,t} \sigma_{j,t} \tag{16}$$

where $\rho_{ij,t}$ is the pairwise correlation between asset i and j.

Implied Correlation

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where $\rho_{ij,t}$ is the pairwise correlation between asset i and j.

- The Implied Correlation Index (IC) is defined as the correlation, ρ_t, that gives the same portfolio variance,

$$\sigma_{p,t}^2 = \sum_{i=1}^{N} \omega_{i,t}^2 \sigma_{i,t}^2 + 2 \sum_{i=1}^{N-1} \sum_{j>i} \omega_{i,t} \omega_{j,t} \rho_{t} \sigma_{i,t} \sigma_{j,t} \tag{17}$$

- This is similar to YTM (yield-to-maturity) that uses one number to describe the relation between bond price and coupon rate.

Implied Correlation

- The IC can be solved as

$$
\begin{aligned}
IC_t &= \frac{\sum_{i=1}^{N-1} \sum_{j>i} \omega_{i,t}\omega_{j,t}\rho_{ij,t}\sigma_{i,t}\sigma_{j,t}}{\sum_{i=1}^{N-1} \sum_{j>i} \omega_{i,t}\omega_{j,t}\sigma_{i,t}\sigma_{j,t}} \\
&= \sum_{i=1}^{N-1} \sum_{j>i} C_{ij,t}\rho_{ij,t} \quad \text{where} \quad C_{ij,t} \frac{\omega_{i,t}\omega_{j,t}\sigma_{i,t}\sigma_{j,t}}{\sum_{i=1}^{N-1} \sum_{j>i} \omega_{i,t}\omega_{j,t}\sigma_{i,t}\sigma_{j,t}}
\end{aligned} \tag{18}
$$

- Alternatively, we express the IC as a function of σ_t of portfolio and $\sigma_{i,t}$ of individual assets

$$
IC_t = \frac{\sigma_{p,t}^2 - \sum_{i=1}^{N} \omega_{i,t}^2\sigma_{i,t}^2}{2\sum_{i=1}^{N-1} \sum_{j>i} \omega_{i,t}\omega_{j,t}\sigma_{i,t}\sigma_{j,t}} \tag{19}
$$

Implied Correlation Interpretation

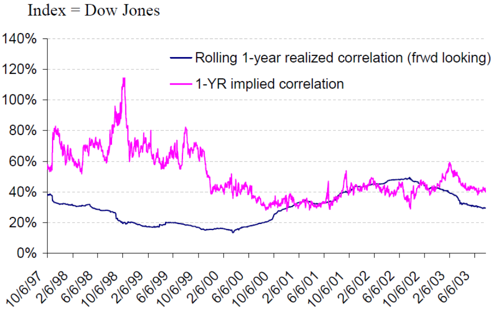
- An interesting property: $0 < IC < 1$
- $IC = 0 \implies \sigma^2_{p,min,t} = \sum_{i=1}^{N} \omega^2_{i,t}\sigma^2_{i,t}$
- $IC = 1 \implies \sigma^2_{p,max,t} = \sum_{i=1}^{N} \omega^2_{i,t}\sigma^2_{i,t} + 2\sum_{i=1}^{N-1}\sum_{j>i}\omega_{i,t}\omega_{j,t}\sigma_{i,t}\sigma_{j,t} = (\sum_{i=1}^{N}\omega_i\sqrt{\sigma_{i,t}})^2$
- The IC index is thus an linear interpolation between the maximum (perfect correlation) and the minimum (zero correlation) portfolio variance

$$IC_t = \frac{\sigma^2_{p,t} - \sigma^2_{p,min,t}}{\sigma^2_{p,max,t} - \sigma^2_{p,min,t}} \tag{20}$$

Implied Correlation and Index Options

- We option-implied volatility for both individual assets and portfolios to compute IC
- Thus, we first need the options written on both individual assets and portfolios
- How to construct the portfolio - which ω_i to choose?
- A natural application is to use index options: S&P 500, S&P 100, and Dow Jones
- IC calculated from an index provides a measure of the market portfolio diversification in the specific market represented by the underlying index.
- How to compute implied volatility - model risk or model-free ?

Correlation Market “Anomaly”-Foresi, Vesval, and Sachs (2006)



- Input as either implied volatility or realized volatility in Equation (19)
- This is essentially due to some stylized facts and empirical puzzles on index options (Christoffersen, Heston, and Jacobs, 2013)
- Implied volatility exceeds realized volatility
- Selling straddles is profitable on average
- Long-term options tend to overreact to changes in short-term volatility.

Correlation Trading Products

- Correlation swaps: pay the difference between an IC strike and the average pairwise correlation in a basket of stocks

- Delta-hedged straddles: sell index straddle, and buy single-stock straddles

- Index variance-swaps against single-stock-variance-swaps

Questions?

- Email: ylu28@uncc.edu
- Webpage: JacquesYL.github.io

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