Quadratic Fermionic Hamiltonians

Jacopo Surace

August 7, 2019

0.1 Quadratic Fermionic Hamiltonians

0.1.1 Representation of the generic Hamiltonian

A possible representation of a generic Quadratic Fermionic Hamiltonian (QFH) is

$$\hat{H} = \frac{1}{2}\sum_{i,j=1}^{N-1}\left(A_{i,j}a_i^\dagger a_j - \bar{A}_{i,j}a_i a_j^\dagger + B_{i,j}a_i a_j - \bar{B}_{i,j}a_i^\dagger a_j^\dagger\right), \tag{1}$$

with $A_{i,j}, B_{i,j} \in \mathbb{C}$ and $A^\dagger = A$ (hermitian) and $B^T = -B$ (skew symmetric).

This conditions on A and B come from the request for $\hat{H}$ to be Hermitian and for the fermionic creation and annihilation operators $(a_i, a_j^\dagger)$ to obey the Canonical Anticomutation Relations (CAR) (see Appendix **??**)

$$\{a_i, a_j\} = \left\{a_i^\dagger, a_j^\dagger\right\} = 0 \quad , \quad \left\{a_i^\dagger, a_j\right\} = \delta_{i,j}. \tag{2}$$

To simplify the notation we collect the fermionic operators in the vector

$$\vec{\alpha} = \begin{pmatrix} a_0^\dagger \\ \vdots \\ a_{N-1}^\dagger \\ a_0 \\ \vdots \\ a_{N-1} \end{pmatrix} \quad , \quad \vec{\alpha}^\dagger = \begin{pmatrix} a_0 & \dots & a_{N-1} & a_0^\dagger & \dots & a_{N-1}^\dagger \end{pmatrix}. \tag{3}$$

The Hamiltonian may then be written as

$$\hat{H} = \tfrac{1}{2}\vec{\alpha}^\dagger H_a \vec{\alpha} \quad , \quad H_a = \begin{pmatrix} -\bar{A} & B \\ -\bar{B} & A \end{pmatrix}. \tag{4}$$

The associated elements are:

$$H_a = \begin{pmatrix} aa^\dagger & aa \\ a^\dagger a^\dagger & a^\dagger a \end{pmatrix}. \tag{5}$$

Majorana representation

Majorana operators are defined in term of Dirac operators as:

$$x_i = \tfrac{a_i + a_i^\dagger}{\sqrt{2}} \quad , \quad p_i = \tfrac{a_i - a_i^\dagger}{i\sqrt{2}}. \tag{6}$$

The inverse transformations are:

$$a_i = \tfrac{x_i + ip_i}{\sqrt{2}} \quad , \quad a_i^\dagger = \tfrac{x_i - ip_i}{\sqrt{2}}. \tag{7}$$

Majorana operators are hermitian so $x_i^\dagger = x_i$ and $p_i^\dagger = p_i$ and they obey the Majorana CAR (MCAR)

$$\{x_i, x_j\} = \{p_i, p_j\} = \delta_{i,j} \quad , \quad \{x_i, p_j\} = 0. \tag{8}$$

We note here that in the Majorana representation the local structure is preserved. Majorana operators labelled by i correspond fermionic operators labelled by i, so we still retain the physical space information. Moving to the Majorana representation is just a local (on each site) transformation.

We define the unitary matrix

$$\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ i\mathbb{I} & -i\mathbb{I} \end{pmatrix} \quad , \quad \Omega^\dagger = \Omega^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I} & -i\mathbb{I} \\ \mathbb{I} & i\mathbb{I} \end{pmatrix} \tag{9}$$

such that $\vec{r} = \Omega\vec{\alpha}$ is the vector of the Majorana operators

$$\vec{r} = \begin{pmatrix} x_0 \\ \vdots \\ x_{N-1} \\ p_0 \\ \vdots \\ p_{N-1} \end{pmatrix}. \tag{10}$$

In the Majorana representation the MCAR can be easily written as

$$\{r_i, r_j\} = \delta_{i,j}, \tag{11}$$

this is one of the advantage of using the Majorana representation.

The Hamiltonian in term of the Majorana operators reads as

$$\hat{H} = \frac{1}{2} \left(\vec{\alpha}^\dagger \Omega^\dagger \right) \Omega H_a \Omega^\dagger \left(\Omega\vec{\alpha} \right) = \frac{1}{2} \vec{r}^\dagger H \vec{r} \tag{12}$$

with

$$H = \Omega H_a \Omega^\dagger = i \begin{pmatrix} \Im\{A+B\} & \Re\{A+B\} \\ \Re\{B-A\} & \Im\{A-B\} \end{pmatrix} = ih. \tag{13}$$

We note that H is purely imaginary and hermitian (to see this just express the imaginary part and real part as the difference of the conjugates and use the properties of A and B).

The fact that H is hermitian and purely imaginary imply that H and h are skew-symmetric.

The associated elements are:

$$H = \begin{pmatrix} xx & xp \\ px & pp \end{pmatrix}. \tag{14}$$

The following identities can be useful:

1. $x_i^2 = p_i^2 = \frac{1}{2}$
2. $a^\dagger a = \frac{i}{2}\left(xp - px\right) + \frac{1}{2} = ixp + \frac{1}{2}$
3. $aa^\dagger = \frac{i}{2}\left(px - xp\right) + \frac{1}{2} = ipx + \frac{1}{2}$
4. $xp = -\frac{i}{2}\left(a^\dagger a - aa^\dagger\right) = -i\left(a^\dagger a - \frac{1}{2}\right)$

0.1.2 Diagonalisation

In this section we will see how to diagonalise the Hamiltonian (**??**).

To do so we will proceed by first diagonalising Hamiltonin (**??**) and then moving back to the Dirac representation with (**??**).

Diagonalising Hamiltonin (**??**) corresponds to finding a transformation O that maps the set of Majorana opeartors $\vec{r}$ to a new set of Majorana operators $\vec{s} = O\vec{r}$ such that the Hamiltonian can be written as

$$\hat{H} = \frac{i}{2} \sum_{i=0}^{N-1} \tilde{h}_i (\tilde{x}_i \tilde{p}_i - \tilde{p}_i \tilde{x}_i). \tag{15}$$

The set of transformation O that maps a set of Majorana operators to another set of Majorana operators is the set of orthogonal transformations:

$$\delta_{i,j} = \{s_i, s_j\} = \sum_{l,m} O_{i,l} O_{j,m} \{r_l, r_m\} = \sum_{l} O_{i,l} O^T_{l,j} = (OO^T)_{i,j}. \tag{16}$$

From the property of the skew-symmetric matrices [**?**, **?**] we know that exist a special orthogonal transformation O such that h of (**??**) decompose as

$$\Lambda = O^T h O = \bigoplus_{i=1}^{\lfloor N/2 \rfloor} \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}, \tag{17}$$

with $\lambda_i = \tilde{h}_i$ real and positive such that $\pm i\lambda_i$ are the eigenvalues of h.

The orthogonal transformation $\vec{s} = O\vec{r}$ defines the new collection of Majorana operators

$$\vec{s} = \begin{pmatrix} \tilde{x}_0 \\ \tilde{p}_0 \\ \tilde{x}_1 \\ \tilde{p}_1 \\ \vdots \\ \tilde{x}_{N-1} \\ \tilde{p}_{N-1} \end{pmatrix} \tag{18}$$

Thus with transformation O we can express the Hamiltonian as

$$\hat{H} = \frac{1}{2} \vec{s}^{\dagger} \Lambda \vec{s} \tag{19}$$

that is the desired form (**??**).

Considering the Dirac representation one is interested in a transformation from Dirac fermionic modes $\vec{\alpha}$ in Dirac fermionic modes $\vec{\beta}$ such that the generic quadratic Hamiltonian (**??**) is mapped to a free fermions Hamiltonian

$$\hat{H} = \frac{1}{2} \sum_{k=0}^{N-1} \epsilon_k \left(b_k^{\dagger} b_k - b_k b_k^{\dagger} \right). \tag{20}$$

To do so it is sufficient to map the Majoranas $\vec{s}$ of the last back to Fermions and one should obtain (**??**).

The values $\tilde{h}_i$ of (**??**) and ϵ_k of (**??**) are related by

$$\epsilon_k = \lambda_k = \tilde{h}_k. \tag{21}$$

0.2 Numerical Methods

0.2.1 Diagonalisation of the Hamiltonian

Block-diagonal form of real skew-symmetric matrices

In order to obtain the decomposition of h of (**??**) one can use the following algorithm

1. Compute numerically a Schur decomposition (or Schur triangularisation as called in [**?**]) of the skew-symmetric matrix h such that: $h = \tilde{O}\tilde{\Lambda}\tilde{O}^T$. The matrix $\tilde{\Lambda}$ should be a block-diagonal matrix with each block in the anti-diagonal form

$$\begin{pmatrix} 0 & \tilde{\lambda}_i \\ -\tilde{\lambda}_i & 0 \end{pmatrix}, \tag{22}$$

   it is not guaranteed that $\tilde{\lambda}_i$ is positive for each i. It is necessary to reorder it.

2. Build the ortogonal matrix $M = \bigoplus_{i=1}^{\lfloor N/2 \rfloor} m_i$ with

$$m_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{23}$$

   if $\tilde{\lambda}_i < 0$ or

$$m_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{24}$$

   if $\tilde{\lambda}_i > 0$.

3. The final orthogonal transformation is $O = \tilde{O}M$ such that $h = O\Lambda O^T$.

Schur decomposition are usually standard decomposition already implemented in standard libraries. In order to avoid numerical errors it is a good practice to forcefully skew-symmetryse matrix passed to the Schur decomposition routine, thus Schur decompose $\frac{h-h^T}{2}$.

In `F-utilities.jl` this is implemented with the function `Diag_Real_Skew(M)` $\rightarrow M_f, O_f$.

Dirac representation

In order to obtain the decomposition of H_a of (**??**) one can use the following algorithm:

1. Transform H_a in H of (**??**) with the matrix Ω and consider only h.

2. Apply the Majorana diagonalisation of section **??** to h to obtain its block diagonal form and the orthogonal transformation O such that $\vec{s} = O\vec{r}$

3. In order to move back to the Dirac representation one has to pay attention to how the Majorana operators are ordered in $\vec{s}$. In fact, in $\vec{s}$ the order is xp (see (**??**)), so we cannot transform back to the Dirac representation simply applying Ω again. To reorder $\vec{s}$ to order

xx (see (**??**)) we define the $2N \times 2N$ matrix

$$F_{xp\to xx} = \begin{matrix} i=0 \\ i=1 \\ \vdots \\ \vdots \\ i=N \\ i=N+1 \\ \vdots \\ i=2N+1 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & \vdots & 1 & \vdots & & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & 0 & \vdots \\ \vdots & 0 & 0 & 0 & & & 1 & \vdots \\ \vdots & 1 & 0 & 0 & & & 0 & \vdots \\ \vdots & 0 & \vdots & 1 & & & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & & & \vdots & 0 \\ 0 & 0 & 0 & \vdots & \dots & \dots & 0 & 1 \end{pmatrix} \tag{25}$$

so that $F_{xp\to xx}\vec{s}$ has the order xx as $\vec{r}$ and we get

$$\Lambda^M = F_{xp\to xx}\Lambda F^T_{xp\to xx} = \begin{pmatrix} 0 & \dots & 0 & \lambda_0 & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & 0 & 0 & \lambda_{\lfloor N/2 \rfloor} \\ -\lambda_0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & -\lambda_{\lfloor N/2 \rfloor} & 0 & \dots & 0 \end{pmatrix} \tag{26}$$

4. . Move from Majoranas to Dirac representation

$$-i\epsilon = \Omega^\dagger \Lambda^M \Omega = \begin{pmatrix} i\lambda_0 & & & & & \\ & \ddots & & & & \\ & & i\lambda_{\lfloor N/2 \rfloor} & & & \\ & & & -i\lambda_0 & & \\ & & & & \ddots & \\ & & & & & -i\lambda_{\lfloor N/2 \rfloor} \end{pmatrix} \tag{27}$$

5. . The final unitary transformation $\vec{\alpha} = U\vec{\beta}$ read as

$$U = \Omega^\dagger \cdot O \cdot F^\dagger_{xp\to xx} \cdot \Omega \tag{28}$$

6. . Finally we have $H_a = U \cdot H_D \cdot U^\dagger$, that is

$$\hat{H} = \frac{1}{2}\vec{\beta}^\dagger H_D \vec{\beta} = \frac{1}{2}\vec{\beta}^\dagger \begin{pmatrix} \epsilon_1 & 0 & \dots & & \dots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \epsilon_N & & & \\ & & & -\epsilon_1 & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \dots & & & \dots 0 & -\epsilon_N \end{pmatrix} \vec{\beta} \tag{29}$$

In `F-utilities.jl` this is implemented with the function `Diag_ferm(M)`$\to M_f, U_f$.

0.3 Fermionic Gaussian States

0.3.1 Representations

Hilbert space representation

Fermionic Gaussian states are represented by density operators that are exponentials of a QFH. A general Gaussian state is of the form

$$\rho = \frac{e^{-\frac{1}{2}\vec{\alpha}^\dagger H_\rho \vec{\alpha}}}{Z} \tag{30}$$

where Z is a normalisation constant, and H_ρ is a $2N \times 2N$ matrix of the same form of (**??**) in which are encoded all the information about the state and $\vec{\alpha}$ is a collection of creation and annihilation operators as in (**??**).

Correlation matrix

Gaussian states are Gibbs states of QFH. Since the matrix at the exponent can be diagonalised as in (**??**), a Gaussian state has a normal-mode decomposition in terms of N single-mode thermal states of the form $\sim e^{-\beta_i b_i^\dagger b_i}$. From this one can see that the state is fully determined by the expectation values of quadratic operators –Kraus–Peschel2002b (see Appendix subsec:correlation-and-parent).

Thus, instead of using the $2^N \times 2^N$ density matrix to describe the state, we are allowed to just consider the correlation matrix

$$\Gamma_{i,j} \equiv Tr\left[\rho \vec{\alpha}_i \vec{\alpha}_j^\dagger\right] = \begin{pmatrix} a^\dagger a & a^\dagger a^\dagger \\ aa & aa^\dagger \end{pmatrix} = \begin{pmatrix} \Gamma^{UL} & \Gamma^{UR} \\ \Gamma^{BL} & \Gamma^{BR} \end{pmatrix}. \tag{31}$$

Where Γ is Hermitian, matrices Γ^{BL} and Γ^{UR} are skew-symmetric (note that their diagonal is always 0) and thus also $\Gamma^{BL} = -\overline{\Gamma^{UR}}$ and Γ^{UL} and Γ^{BR} are Hermitian.

The matrix $\tilde{\Gamma} = \Gamma - \frac{1}{2}\mathbb{I}$ has the same form of (**??**), thus is it possible to diagonalise it with the same method used for the Hamiltonian and then add again $\frac{1}{2}\mathbb{I}$ to obtain the diagonal form of Γ

$$\Gamma^D = \begin{pmatrix} \nu_1 & 0 & \dots & & \dots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \nu_N & & & \\ & & & 1-\nu_1 & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \dots & & & \dots 0 & 1-\nu_N \end{pmatrix} \tag{32}$$

with $\nu_i \in [0,1]$ (the diagonal elements of $\tilde{\Gamma}$ are $\tilde{\nu}_i \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$).

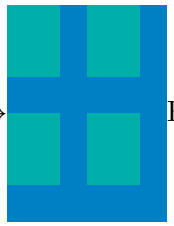
To obtain this diagonal form with `F-utilities.jl` from Γ it's sufficient to write:

Appendix A

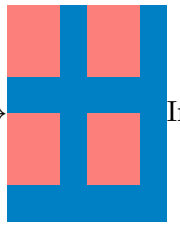
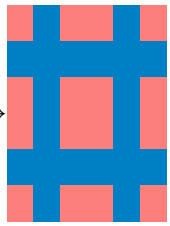
F-utilities

1. `Print_matrix(title, M)`: Print a graphical representation of matrix `M` in a figure called title.
2. `Build_Omega(N)`$\rightarrow \Omega$: This function return the $2N \times 2N$ matrix Ω (**??**).
3. `Build_FxxTxp(N)`$\rightarrow F_{xx\rightarrow xp}$: This function return the $2N \times 2N$ matrix $F_{xx\rightarrow xp}$ (**??**).
4. `Build_FxpTxx(N)`$\rightarrow F_{xp\rightarrow xx}$: This function return the $2N \times 2N$ matrix $F^T_{xx\rightarrow xp}$.
5. `Diag_Real_Skew(M)`$\rightarrow M_f, O_f$: This function implement the Majorana algorithm of section **??** with $M = h$ a generic skew-symmetric real matrix. M_f is the matrix we called Λ in (**??**) and has the following property: it is in the block diagonal form, each 2×2 block is skew-symmetric with the upper-right element positive and real and M_f is in ascending order for the upper diagonal. O_f is the orthogonal matrix that we called O and it is an orthogonal matrix such that: $M = O_f M_f O_f^T$.
6. `Diag_ferm(M)`$\rightarrow M_f, U_f$:This function implement the fermionic algorithm of section **??** with $M = H_a$. M_f is the matrix we called H_D and it is in diagonal form with the first half diagonal negative and the second one positive. U_f is the orthogonal matrix that we called U and it is a unitary matrix such that: $M = U_f M_f U_f^\dagger$.
7. `Purity(M)`$\rightarrow p$: This function takes as input the correlation matrix Γ and return a Float p from 0 to 1 that is the purity.
8. `Evolve_gamma(M,D,U,t)`$\rightarrow M_t$: This function evolve for a time `t` (last argument) the correlation matrix Γ (first argument), `D` is the Dirac Hamiltonian diagonalised, U is the unitary transformation that change basis from the diagonal one to the original one. e.g. If I want to evolve the correlation matrix M with H_a for a time t I would write `M_t=Evolve_gamma(M,Diag_ferm(`H_a`),`t`)`.
9. `Evolve_gamma_imag(M,D,U,t)`$\rightarrow M_t$: This function evolve for a time `t` (last argument) the pure state correlation matrix Γ (first argument) with the evolution **??**, `D` is the Dirac Hamiltonian diagonalised, U is the unitary transformation that change basis from the diagonal one to the original one .N.B. `M` must be a pure state (Purity$(M) = 1$), `Evolve_gamma_imag(M,D,U,t)` is a pure state.
10. `Energy_fermion(M,D,U)`$\rightarrow e$: This function return the energy of the correlation matrix M with respect of the Hamiltonian in the diagonal form M where U is the change of basis from the diagonal one to the space one. e.g. If I want to compute the energy of the correlation matrix Γ with respect to the generic quadratic Hamiltonian represented by H I would write `Energy_fermion(`Γ`,Diag_ferm(`H`))`.

11. `Reduce_gamma(M,N_partition,first_index)` $\rightarrow M_r$: This function return the correlation matrix of a subsystem with `N_partition` size starting from the site at `first_index`. e.g. `Reduce_gamma()` return the green the element of the matrix $M_{6\times6}$

Reduce_gamma($M_{6\times6}$,2,1)→  Reduce_gamma($M_{6\times6}$,2,3)→  (A.1)

12. `Inject_gamma(gamma, injection, first_index)`$\rightarrow M_T$:This function overwrite the subsystem of gamma starting at `first_index` with the system with correlation matrix injection. The system of injection has to be smaller then the one of gamma. The returned system has same dimension of gamma. e.g. `Inject_gamma()` return the red and blue matrix where the elements in red are the one of the matrix injection.

Inject_gamma($M_{6\times6}$,2,1)→  Inject_gamma($M_{6\times6}$,2,3)→  (A.2)

13. `Eigenvalues_of_rho(M)`$\rightarrow \vec{e}$: This function take as input a correlation matrix Γ and return the vector with all its eigenvalues. N.B. The number of eigenvalues grows exponentially with the size of Γ.

14. `VN_entropy(M)` $\rightarrow s$: This function return the Von Neumann Entropy of the system described by the correlation matrix M in the form of Γ.

A.1 Useful relations

A.1.1 Pauli Matrices

1. $\sigma^+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \sigma^- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \sigma^z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \sigma^y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \sigma^x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, |+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}, |-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}, |+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}, |-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}, |0_-\rangle_z = \begin{pmatrix}0\\ 1\end{pmatrix}, |1_+\rangle_z = \begin{pmatrix}1\\ 0\end{pmatrix}$
2. $\sigma^z\sigma^- = -\sigma^-$
3. $\sigma^z\sigma^+ = \sigma^+$
4. $\sigma^-\sigma^z = \sigma^-$
5. $\sigma^+\sigma^z = -\sigma^+$
6. $\sigma^+\sigma^- = \frac{\sigma^z+\mathbb{I}}{2}$
7. $\sigma^-\sigma^+ = \frac{\mathbb{I}-\sigma^z}{2}$

A.1.2 Fermionic operators

1. $\{a_i, a_j\} = \left\{a_i^\dagger, a_j^\dagger\right\} = 0$
2. $a_i a_j = -a_j a_i$; $a_i^\dagger a_j^\dagger = -a_j^\dagger a_i^\dagger$
3. $\left\{a_i, a_j^\dagger\right\} = \delta_{i,j}$
4. $a_i a_j^\dagger = \delta_{i,j} - a_j^\dagger a_i$
5. $a_i a_j = \frac{a_i a_j - a_j a_i}{2}$
6. $a_i a_j^\dagger = \frac{a_i a_j^\dagger - a_j^\dagger a_i}{2} + \frac{\delta_{i,j}}{2}$
7. $a_i^\dagger a_j = \frac{a_i^\dagger a_j - a_j a_i^\dagger}{2} + \frac{\delta_{i,j}}{2}$

The action on a state in the occupation basis (defined such that the Canonical Anticomutation Relations (relations 1 and 3) are respected)

1. $a_i|\vec{\alpha}\rangle = -(-1)^{S_\alpha^j}|\alpha'\rangle$ if $\alpha_j = 1$ with $\alpha'_j = 0$ and $S_\alpha^j = \sum_{k=1}^{j-1}\alpha_k$ (if $\alpha_j = 0$ then is 0)
2. $a_i^\dagger|\vec{\alpha}\rangle = -(-1)^{S_\alpha^j}|\alpha'\rangle$ if $\alpha_j = 0$ with $\alpha'_j = 1$ and $S_\alpha^j = \sum_{k=1}^{j-1}\alpha_k$ (if $\alpha_j = 1$ then is 0)

Commutators

1. $[a_i^\dagger, a_j] = \delta_{i,j} - 2a_j a_i^\dagger = a_i^\dagger a_j - \delta_{i.j}$
2. $[a_i, a_j^\dagger] = \delta_{i,j} - 2a_j^\dagger a_i = a_i a_j^\dagger - \delta_{i,j}$
3. $[a_i, a_j] = 2a_i a_j$
4. $[a_i^\dagger, a_j^\dagger] = 2a_i^\dagger a_j^\dagger$

A.1.3 Jordan-Wigner Transformations

spinless fermions → spins

1. $a_j = -\bigotimes_{k=1}^{j-1}\sigma_k^z \otimes \sigma_j^- \bigotimes_{k=j+1}^{N}\mathbb{I}_k$
2. $a_j^\dagger = -\bigotimes_{k=1}^{j-1}\sigma_k^z \otimes \sigma_j^+ \bigotimes_{k=j+1}^{N}\mathbb{I}_k$
3. $a_j^\dagger a_j = \bigotimes_{k=1}^{j-1}\otimes\frac{\sigma_j^z + \mathbb{I}_j}{2} \bigotimes_{k=j+1}^{N}\mathbb{I}_k$

spins → spinless fermions

1. $\sigma_j^z = a_j^\dagger a_j - a_j a_j^\dagger$
2. $\sigma_j^x = -\bigotimes_{k=1}^{j-1}\sigma_j^z \otimes (a_j + a_j^\dagger)\bigotimes_{k=j+1}^{N}\mathbb{I}_j$
3. $\sigma_j^x = i\bigotimes_{k=1}^{j-1}\sigma_j^z \otimes (a_j^\dagger - a_j)\bigotimes_{k=j+1}^{N}\mathbb{I}_j$
4. $\sigma_j^x\sigma_{j+1}^x = (a_j^\dagger - a_j)(a_{j+1} + a_{j+1}^\dagger)$

5. $\sigma_j^y \sigma_{j+1}^y = -(a_j^\dagger + a_j)(a_{j+1}^\dagger - a_{j+1})$
6. $\sigma_j^x \sigma_{j+1}^y = i(a_j^\dagger - a_j)(a_{j+1}^\dagger + a_{j+1})$
7. $\sigma_j^y \sigma_{j+1}^x = i(a_j^\dagger + a_j)(a_{j+1}^\dagger + a_{j+1})$

Bibliography

[1] B. Zumino, "Normal Forms of Complex Matrices," *Journal of Mathematical Physics*, vol. 3, pp. 1055–1057, Sept. 1962.

[2] R. A. Horn and C. R. Johnson, "Matrix Analysis by Roger A. Horn." /core/books/matrix-analysis/9CF2CB491C9E97948B15FAD835EF9A8B, Dec. 1985.