一、1. $\because \frac{n}{\sqrt{n^2+n}} < I < \frac{n}{\sqrt{n^2+1}}$

而 $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$ 夹逼准则

$\therefore \lim_{n\to\infty} I = 1$

2、 $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2+2t}{(t+1)e^t} = \frac{2}{e^t}$ 参数方程求导

$\therefore \frac{dy}{dx}\Big|_{t=0} = 2$

3、 $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(1-x) - x^{\frac{2}{3}}$

$= x^{-\frac{1}{3}}\left(\frac{2}{3} - \frac{2}{3}x - x\right)$

$= \frac{1}{3}x^{-\frac{1}{3}}(2-5x)$

在 $(0, \frac{2}{5})$ 上单调增

4、 $(1-\cos x)\ln(1+x^2) \sim \frac{1}{2}x^4$

$x\sin x^n \sim x^{n+1}$ 等价无穷小

$e^{x^2}-1 \sim x^2$

$2 < n+1 < 4 \Rightarrow n = 2$

5、 $y' = e^{-\arctan^2 x}\frac{1}{1+x^2}$ 令 $x=0$，$k=1$

法线：$y=-x$

6、 $\lim_{x\to 3^-} f(x) = -1$ $\lim_{x\to 3^+} f(x) = 0$

跳跃间断点

7. $\vec{a}\cdot\vec{b}=14-12-2=0$

$\theta=\frac{\pi}{2}$

8. 原式 $=\lim\limits_{x\to0}\frac{\sin x(1-\cos x)}{x^2\cos 3x(e^x-1)}=\lim\limits_{x\to0}\frac{x\frac{1}{2}x^2}{x^2\cdot x}=\frac{1}{2}$

二、1. 原式 $=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{1}{n}\frac{1}{\sqrt{4-(\frac{i}{n})^2}}$ 构造定积分定义

$=\int_0^1\frac{1}{\sqrt{4-x^2}}dx=\arcsin\frac{x}{2}\Big|_0^1=\frac{\pi}{6}$

2. $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x+\frac{\pi}{2})\cos^6(x+\frac{\pi}{2})\sin(x+\frac{\pi}{2})\,d(x+\frac{\pi}{2})$

看到对称性 → 主动构造奇偶性

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x+\frac{\pi}{2})\sin^6 x\cos x\,dx$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x\sin^6 x\cos x\,dx+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\pi}{2}\sin^6 x\cos x\,dx$

$=\pi\int_0^{\frac{\pi}{2}}\sin^6 x\,d\sin x$

$=\pi\frac{\sin^7 x}{7}\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{7}$

3. 直线：$\frac{x-2}{-1}=\frac{y+4}{3}=\frac{z+1}{1}$ 的方向向量

为 $\vec{n}=(-1,3,1)$

平面与 $\vec{n}$ 垂直，则平面法向量也为 $\vec{n}=(-1,3,1)$

又平面过点 $M(1,2,-1)$

则 $-1(x-1)+3(y-2)+1(z+1)=0$

即 $x-3y-z+4=0$

4、原式 $=2\int \ln x \, d\sqrt{1+\ln x}$ 凑微分

$=2\int\left((\sqrt{1+\ln x})^2-1\right)d\sqrt{1+\ln x}$

$=2\frac{(\sqrt{1+\ln x})^3}{3}-2\sqrt{1+\ln x}+C$

$=\frac{2}{3}\sqrt{1+\ln x}(\ln x-2)+C$

5、原式 $=\frac{1}{e}\int_1^{+\infty}\frac{de^x}{(e^x)^2+e^2}$ 凑微分

$=\frac{1}{e}\frac{1}{e}\cdot\arctan\frac{e^x}{e}\Big|_1^{+\infty}$

$=\frac{\pi}{4e^2}$

6、原式 $=\frac{1}{3}\int_0^1 f(x)\,dx^3$

$=\frac{1}{3}f(x)x^3\Big|_0^1-\frac{1}{3}\int_0^1 x^3 f'(x)\,dx$

$=-\frac{1}{3}\int_0^1 x^3\cdot e^{-x^2}dx$

$=\frac{1}{6}\int_0^1 x^2\,de^{-x^2}$

$=\frac{1}{6}x^2e^{-x^2}\Big|_0^1+\frac{1}{6}\int_0^1 e^{-x^2}dx^2$

$= \frac{1}{6e} + \frac{1}{6} e^{-x^2}\Big|_0^1$

$= \frac{1}{3e} - \frac{1}{6}$

三、由 $\lim\limits_{x\to 0} \frac{f(x)}{x} = A$ 有 $f(0)=0$，$f'(0)=A$

$x=0$ 时 $\varphi(x)=0$

$x\neq 0$ 时，$\varphi(x) = \frac{1}{x}\int_0^x f(t)\,dt$

$\because$ $x\neq 0$ 时 $\varphi'(x) = \dfrac{xf(x) - \int_0^x f(t)\,dt}{x^2}$

$\therefore \lim\limits_{x\to 0} \varphi'(x) = \lim\limits_{x\to 0} \dfrac{xf(x) - \int_0^x f(t)\,dt}{x^2}$

$\overset{洛}{=} \lim\limits_{x\to 0} \dfrac{f(x) + xf'(x) - f(x)}{2x}$

$= \lim\limits_{x\to 0} \dfrac{f'(x)}{2} = \dfrac{A}{2}$

而 $\varphi'(0) = \lim\limits_{x\to 0} \dfrac{\varphi(x)}{x} = \lim\limits_{x\to 0} \dfrac{\int_0^x f(t)\,dt}{x^2} \overset{洛}{=} \lim\limits_{x\to 0} \dfrac{f(x)}{2x} = \dfrac{A}{2}$

$\therefore \lim\limits_{x\to 0} \varphi'(x) = \varphi'(0)$，即 $\varphi'(x)$ 在 $x=0$ 处连续

$\therefore \varphi'(x) = \begin{cases} 0 & x=0 \\ \dfrac{xf(x) - \int_0^x f(t)\,dt}{x^2}, & x\neq 0 \end{cases}$ 在 $x=0$ 处连续

四、$f'(x) = 2x - \frac{1}{x^2} = \frac{2x^3-1}{x^2}$，$f''(x) = \frac{2(x^3+1)}{x^3}$

$f(x)$ 定义域为 $(-\infty, 0)\cup(0,+\infty)$

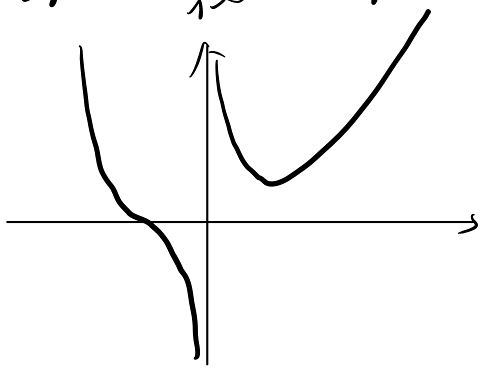
单调增区间为 $(\frac{1}{\sqrt[3]{2}}, +\infty)$，单调减区间为 $(-\infty, 0)$，$(0, \frac{1}{\sqrt[3]{2}})$

上凹区间为 $(-\infty,-1)$，$(0,+\infty)$

下凹区间为 $(-1,0)$，拐点为 $(-1,0)$

$x=0$ 是斜直渐近线

$\lim\limits_{x\to\infty}\frac{f(x)}{x}=\lim\limits_{x\to\infty}x+\frac{1}{x^2}\to+\infty$，无水平渐近线与斜渐近线



五、设切点为 x_0，切线方程 $y=f'(x_0)(x-x_0)+f(x_0)$

面积 $S(x_0)=\int_0^1\left(f'(x_0)(x-x_0)+f(x_0)-f(x)\right)dx$

$=f'(x_0)\int_0^1 x\,dx-f'(x_0)x_0+f(x_0)$

$-\int_0^1 f(x)\,dx$

$=\frac{1}{2}f'(x_0)-f'(x_0)x_0+f(x_0)-\int_0^1 f(x)\,dx$

$S'(x_0)=\frac{1}{2}f''(x_0)-f''(x_0)x_0-f'(x_0)+f'(x_0)$

$=f''(x_0)(\frac{1}{2}-x_0)$ ……

$S(x_0)_{min}=S(\frac{1}{2})=f(\frac{1}{2})-\int_0^1 f(x)\,dx$

六、(1) 即 $\exists\, x_0\in(0,1)$ $\quad x_0 f(x_0)=\int_{x_0}^1 f(x)\,dx$

$\Leftrightarrow \quad xf(x) + \int_0^x f(x)\,dx = \int_0^1 f(x)\,dx \geq 0$

设 $F(x) = xf(x) + \int_0^x f(t)\,dt$ 连续

而 $F(0) = 0 \leq \int_0^1 f(x)\,dx$

$F(1) = f(1) + \int_0^1 f(x)dx \geq \int_0^1 f(x)\,dx$

用介值定理，$\exists\ x \in (0,1)$，使得 $F(x) = \int_0^1 f(x)\,dx$

(2) $F'(x) = f(x) + xf'(x) + f(x)$

$= xf'(x) + 2f(x) > 0$

即 $F(x)$ 单增

$\therefore x$ 唯一

七、(1) 设 $F(x) = f(x) + x - 1$，

$F(0) = -1 < 0$，$F(1) = 1 > 0$

$\therefore \exists\ \xi \in (0,1)$，$F(\xi) = 0 \Leftrightarrow f(\xi) = 1 - \xi$

(2) $f'(\eta_1) = \dfrac{f(\xi) - f(0)}{\xi - 0} = \dfrac{1-\xi}{\xi}$

$f'(\eta_2) = \dfrac{f(1) - f(\xi)}{1 - \xi} = \dfrac{\xi}{1-\xi}$

$\therefore f'(\eta_1) \cdot f'(\eta_2) =$