

2018. 1. 10.

$$\begin{aligned} \text{1. 原式} &= \lim_{x \rightarrow 0} \frac{\int_0^{x^2} \frac{\ln(1+t)}{t} dt}{x^2} \\ &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\frac{\ln(1+x^2)}{x^2} \cdot 2x}{2x} = 1 \end{aligned}$$

洛必达法则
+ 等价无穷小

2. 令 $t = \ln x, \quad x = e^t \quad \text{换元}$

$$\begin{aligned} I_1 &= \int \cos t \, d(e^t) = \cos t e^t - \int e^t \, d(\cos t) \\ &= \cos t e^t + \int \sin t \, d(e^t) \quad \text{多次分部积分} \\ &= (\cos t + \sin t) e^t - \int e^t \, d(\sin t) \quad \downarrow \text{反解} \\ &= (\cos t + \sin t) e^t - \int \cos t \, d(e^t) = (\cos t + \sin t) e^t - I_1 \end{aligned}$$

$$\therefore I_1 = \frac{1}{2} (\cos t + \sin t) e^t + C = \frac{1}{2} (\cos(\ln x) + \sin(\ln x)) + C$$

3. $I_2 = \int \frac{1}{x^2+2x+2} dx - \int \frac{2x+2}{(x^2+2x+2)^2} dx$

观察 \Rightarrow 微分法

$$\begin{aligned} &= \int \frac{1}{(x+1)^2+1} d(x+1) - \int \frac{d(x^2+2x+2)}{(x^2+2x+2)^2} \\ &= \arctan(x+1) + \frac{1}{x^2+2x+2} + C \end{aligned}$$

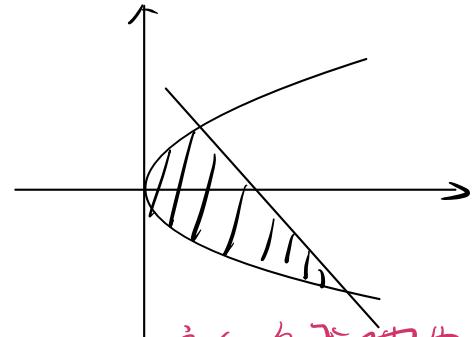
二. 1. $I_3 = \int_0^a x^2 \sqrt{a^2-x^2} dx \quad \text{令 } x = a \sin t, \quad t \in (0, \frac{\pi}{2})$

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} a^2 \sin^2 t \, a \cos t \, d(a \sin t) \\ &= a^4 \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t dt \quad \text{根式} \Rightarrow \text{三角换元} \end{aligned}$$

$$\begin{aligned}
 &= \frac{a^4}{4} \int_0^{\frac{\pi}{2}} \sin^2 2t \, dt \\
 &= \frac{a^4}{16} \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4t}{2} \, d4t \\
 &= \frac{a^4}{32} (4t - \sin 4t) \Big|_0^{\frac{\pi}{2}} = \frac{a^4}{16}\pi
 \end{aligned}$$

2. 椭圆 $y = -x + \frac{3P}{2}$ 即 $x = \frac{3P}{2} - y$

焦点 $(\frac{P}{2}, P)$, $(-\frac{9}{2}P, -3P)$



$$\begin{aligned}
 S &= \int_{-3P}^{P} \left(\frac{3P}{2} - y - \frac{1}{2P}y^2 \right) dy \\
 &= \left. \frac{3P}{2}y - \frac{1}{2}y^2 - \frac{1}{2P}y^3 \right|_{-3P}^P = \frac{16}{3}P^2
 \end{aligned}$$

定积分求面积

梯形面积

3. $y = \ln(1-x^2)$, $y' = \frac{-2x}{x^2-1}$

$$S = \int_a^b [f(x) - g(x)] dx$$

$$\begin{aligned}
 l &= \int_0^{\frac{1}{2}} \sqrt{1+y'^2} \, dx = \int_0^{\frac{1}{2}} \sqrt{\frac{1+x^2}{1-x^2}} \, dx \\
 &= \int_0^{\frac{1}{2}} \left(\frac{2}{1-x^2} - 1 \right) \, dx \\
 &= \left(\ln \left| \frac{1+x}{1-x} \right| - x \right) \Big|_0^{\frac{1}{2}} = \ln 3 - \frac{1}{2}
 \end{aligned}$$

定积分求弦长

$l = \int_a^b \sqrt{1+f'(x)^2} \, dx$

二、1. $I_4 = \int_{-\infty}^{+\infty} \frac{dx}{x^2+5} = \left. \frac{1}{\sqrt{5}} \arctan \left(\frac{x}{\sqrt{5}} \right) \right|_{-\infty}^{+\infty}$

$$= \frac{\pi}{2} \sqrt{5} \pi$$

广义积分

2. $(\vec{a} + 3\vec{b})(7\vec{a} - 5\vec{b}) = 7\vec{a}^2 + 16\vec{a}\vec{b} - 15\vec{b}^2 = 0$

$(\vec{a} - 4\vec{b})(7\vec{a} - 2\vec{b}) = 7\vec{a}^2 - 30\vec{a}\vec{b} + 8\vec{b}^2 = 0$

从而 $46\vec{a}\vec{b} = 23\vec{b}^2$, 即 $\vec{a} \cdot \vec{b} = \frac{1}{2}\vec{b}^2$ 因此有 $|\vec{a}| = |\vec{b}|$

$$\therefore \cos \gamma = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{\frac{1}{2} \vec{b}^2}{\vec{b}^2} = \frac{1}{2}, \text{ 又 } \gamma \in (0, \pi)$$

$$\therefore r = \frac{\pi}{3}$$

利用 向量内积求夹角

$$3. \quad \vec{\eta}_1 = (1, -1, 1), \quad \vec{\eta}_2 = (5, -6, 4)$$

$$\therefore L_3 \text{ 向量 } \vec{\eta} = \vec{\eta}_1 \times \vec{\eta}_2 = (4, 1, -3)$$

验证有点 A(-\frac{4}{3}, \frac{11}{3}, 0) 在直线 L 上

$$\therefore \vec{PA} = (-\frac{7}{3}, \frac{5}{3}, -3) \quad \text{点到直线距离 } d = \frac{|\vec{PA} \times \vec{\eta}|}{|\vec{\eta}|}$$

$$\text{则距离 } d = \frac{|\vec{PA} \times \vec{\eta}|}{|\vec{\eta}|} = \frac{|(2, 19, 9)|}{|(4, 1, -3)|} = \sqrt{\frac{223}{13}}$$

$$\text{四、证明: } x_{n+1} = \frac{1}{3}(x_n + x_n + \frac{1}{x_n^2}) \geq \frac{1}{3}3\sqrt[3]{x_n \cdot x_n \cdot \frac{1}{x_n^2}} = 1$$

$$\text{而 } x_{n+1} - x_n = \frac{1}{3}(\frac{1}{x_n^2} - x_n) = \frac{1 - x_n^2}{3x_n^2} \leq 0$$

$\therefore \{x_n\}$ 单调减 $\xrightarrow{\text{数列极限}} \text{单调有界}$

由单调有界准则可知 $\{x_n\}$ 收敛.

设 $\{x_n\}$ 极限为 A, 则 $A = \frac{1}{3}(2A + \frac{1}{A^2})$, 解得 $A = 1$

$\therefore \{x_n\}$ 极限为 1

$$\text{五、解: } \text{记 } I = \sum_{k=1}^n \frac{\sin \frac{k\pi}{n}}{n + \frac{1}{k}}$$

放缩构造定积分定义形式

$$\text{则 } I < \sum_{k=1}^n \frac{1}{n} \sin \frac{k\pi}{n}$$

\Rightarrow 夹逼准则

$$\text{而} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \sin \frac{k\pi}{n} = \int_0^1 \sin x dx = \frac{1}{\pi} \int_0^\pi \sin x dx$$

$$= \frac{1}{\pi} (-\cos x) \Big|_0^\pi = \frac{2}{\pi}$$

$$I > \sum_{k=1}^n \frac{1}{n+1} \sin \frac{k\pi}{n}$$

$$\text{而} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+1} \sin \frac{k\pi}{n} = \lim_{n \rightarrow \infty} \frac{n}{n+1} \sum_{k=1}^n \frac{1}{n} \sin \frac{k\pi}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \sin \frac{k\pi}{n} = \frac{2}{\pi}$$

由夹逼准则 - $\lim_{n \rightarrow \infty} I = \frac{2}{\pi}$

六：解： $f(x) = \frac{1}{(x-1)^2}$ 定义域为 $(-\infty, 1) \cup (1, +\infty)$

$$f'(x) = \frac{-(x+1)}{(x-1)^3}, \quad f''(x) = \frac{2(x+2)}{(x-1)^4}$$

$x \in (-\infty, -1)$ 时， $f'(x) < 0$, $f(x)$ 单减

$x \in (-1, 1)$ 时， $f'(x) > 0$, $f(x)$ 单增

$x \in (1, +\infty)$ 时， $f'(x) < 0$, $f(x)$ 单减

$\therefore x = -1$ 为极小值点, $f(-1) = -\frac{1}{4}$. f(x) 无极大值

$x \in (-\infty, -2)$ 时， $f''(x) < 0$, $f(x)$ 下凹,

$x \in (-2, 1), (1, +\infty)$ 时， $f''(x) > 0$, $f(x)$ 上凹。

拐点为 $(-2, -\frac{2}{9})$

$x=1$ 是 $f(x)$ 的铅直渐近线

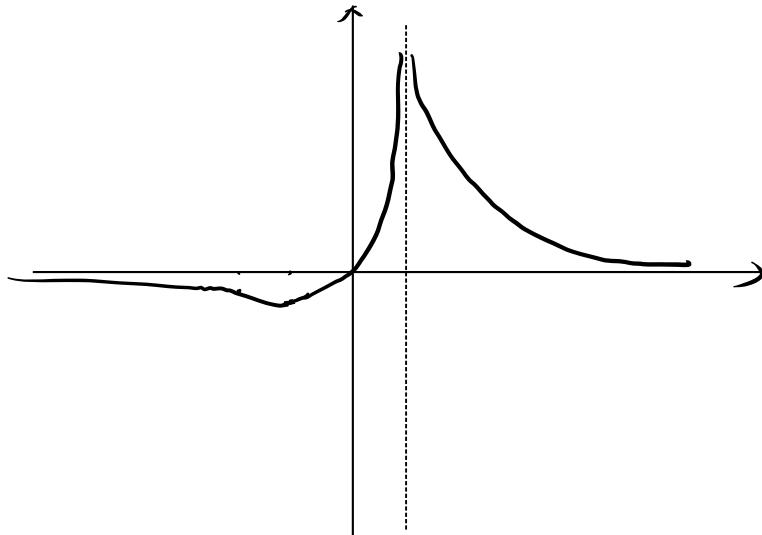
利用导数

研究函数性质

$\lim_{x \rightarrow \infty} f(x) = 0$, 则 $f(x)$ 的水平渐近线

而 $\lim_{x \rightarrow a} f(x) = \infty$, 则 $f(x)$ 无垂直渐近线

作图:



七、直线标准方程为: $L: \frac{x-\frac{1}{3}}{2} = \frac{y-\frac{5}{3}}{10} = \frac{z}{-3}$

则其参数方程为: $L: x = \frac{1}{3} + 2t, y = \frac{5}{3} + 10t, z = -3t$

$$\because L \subset \text{面}\pi \text{, 则 } \frac{\frac{1}{3}+2t}{a} + \frac{\frac{5}{3}+10t}{b} + \frac{-3t}{c} = 1 \text{ 且}$$

$$\Leftrightarrow \left(\frac{2}{a} + \frac{7}{b} \right) t + \frac{1}{3a} + \frac{5}{3b} - 1 = 0 \text{ 且}$$

$$\Leftrightarrow \frac{2}{a} + \frac{7}{b} = \frac{1}{3a} + \frac{5}{3b} - 1 = 0$$

$$\therefore a = -\frac{1}{7}, b = \frac{1}{2}$$

$$\therefore \pi: 7x - 2y - 2z + 1 = 0$$

八、证明: 不妨 $f(x) > 0$ 且, 由 $f(x)$ 连续可知 $f(x)$ 在 $(0, 1)$ 上有

最大值 M 在 x_0 处取到 - $f'(x_0) = 0$ 最值定理

由拉格朗日中值定理, 存在 $\alpha \in (0, x_0)$, $f'(\alpha) = \frac{f(x_0) - f(0)}{x_0 - 0} = \frac{M}{n}$

存在 $\beta \in (\alpha, 1)$, $f'(\beta) = \frac{f(x_0) - f(0)}{x_0 - 1} = \frac{M}{n-1}$

$$\begin{aligned} \text{而 } \int_0^1 \left| \frac{f''(x)}{f'(x)} \right| dx &\geq \frac{1}{n} \int_0^1 |f''(x)| dx \quad \leftarrow f''(x) \text{ 只需研究} \\ \text{放缩掉 } f'(x), \quad &\geq \frac{1}{n} \int_{\alpha}^{\beta} |f''(x)| dx \quad \begin{array}{l} \text{放缩} \\ f''(x) \text{ 即可} \end{array} \\ &\geq \frac{1}{n} \left| \int_{\alpha}^{\beta} f''(x) dx \right| \quad \leftarrow \text{绝对值不等式} \\ &= \frac{1}{n} |f'(\beta) - f'(\alpha)| = \frac{1}{n(n-1)} = \frac{1}{\frac{1}{4} - (\frac{1}{n-1})^2} \\ &\Rightarrow \frac{1}{n} = 4 \quad \text{基本不等式} \end{aligned}$$

九. 证明: 函数 $f(x)$ 在 $\frac{a+b}{2}$ 处的泰勒展开式为:

$$\begin{aligned} f(x) &= f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{1}{2} f''(3)(x - \frac{a+b}{2})^2 \\ &\geq f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) \quad \begin{array}{l} 3 \text{ 是与 } x \text{ 相关的量,} \\ \text{不是常量} \end{array} \end{aligned}$$

$$\begin{aligned} \text{两边取分有 } \int_a^b f(x) dx &\geq \int_a^b \left(f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) \right) dx \\ &= f\left(\frac{a+b}{2}\right)(b-a) \end{aligned}$$