南京大学数学系2015/2016微积分I(A)试卷参考解答

一. 计算下列各题$(10\times 5=50$分$)$

1. 求极限$\lim\limits_{x\to\infty}\left(\sin\dfrac{1}{x^2}+\cos\dfrac{1}{x^2}\right)^{3x^2}$.

解:$\lim\limits_{x\to\infty}\left(\sin\dfrac{1}{x^2}+\cos\dfrac{1}{x^2}\right)^{3x^2}=\lim\limits_{x\to\infty}\left(1+(\sin\dfrac{1}{x^2}+\cos\dfrac{1}{x^2}-1)\right)^{\frac{1}{\sin\frac{1}{x^2}+\cos\frac{1}{x^2}-1}3x^2(\sin\frac{1}{x^2}+\cos\frac{1}{x^2}-1)}=$ e^3.

2.计算积分$\int x^2(\ln x)^2\mathrm{d}x$.

解:$\int x^2(\ln x)^2\mathrm{d}x=\dfrac{1}{3}x^3\ln^2x-\dfrac{2}{3}\int x^3\dfrac{1}{x}\ln x\mathrm{d}x=\dfrac{1}{3}x^3\ln^2x-\dfrac{2}{9}x^3\ln x+\dfrac{2}{27}x^3+C.$

3. 计算极限$\lim\limits_{x\to0}\dfrac{\int_{x^2}^{x}\frac{\sin xt}{t}\mathrm{d}t}{x^2}$.

解: $\lim\limits_{x\to0}\dfrac{\int_{x^2}^{x}\dfrac{\sin xt}{t}\mathrm{d}t}{x^2}=\lim\limits_{x\to0}\dfrac{\int_{x^2}^{x^3}\dfrac{\sin u}{u}\mathrm{d}u}{x^2}=1.$

4. 计算积分$\int_0^1\ln(x+\sqrt{x^2+1})dx$.

解:$\int_0^1\ln(x+\sqrt{x^2+1})dx=x\ln(x+\sqrt{x^2+1})|_0^1-\int_0^1 x\dfrac{1}{x+\sqrt{x^2+1}}(1+\dfrac{x}{\sqrt{x^2+1}})dx=$ $\ln2-\int_0^1\dfrac{x}{\sqrt{x^2+1}}dx=\ln2-\sqrt{x^2+1}|_0^1=\ln2-\sqrt{2}+1.$

5.求过原点且经过两平面$\begin{cases}2x-y+3z=8,\\ x+5y-z=2\end{cases}$的交线的平面方程.

解:$2x+21y-7z=0$.

6.计算广义积分$\int_0^{+\infty}\dfrac{\arctan x}{(1+x^2)^{3/2}}\mathrm{d}x$.

解:$\int_0^{+\infty}\dfrac{\arctan x}{(1+x^2)^{3/2}}\mathrm{d}x=\int_0^{\pi/2}t\cos t\mathrm{d}t=t\sin t|_0^{\pi/2}-\int_0^{\pi/2}\sin t\mathrm{d}t=\dfrac{\pi}{2}-1.$

7. 计算极限$\lim\limits_{n\to\infty}\sin\dfrac{\pi}{n}\sum\limits_{k=1}^{n}\dfrac{n}{n+k}$.

解: $\lim\limits_{n\to\infty}\sin\dfrac{\pi}{n}\sum\limits_{k=1}^{n}\dfrac{n}{n+k}=\lim\limits_{n\to\infty}\dfrac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\dfrac{\pi}{n}\sum\limits_{k=1}^{n}\dfrac{1}{1+k/n}=\pi\int_0^1\dfrac{1}{1+x}\mathrm{d}x=\pi\ln2.$

8. 求心脏线$r=a(1+\cos\theta)$ 的全长.

$s=\int_0^{2\pi}\sqrt{r(\theta)^2+r'(\theta)^2}d\theta=\int_0^{2\pi}\sqrt{a^2(1+\cos\theta)^2+(-a\sin\theta)^2}d\theta=8a.$

9.设$f(x)=\dfrac{1}{x^2-2x-8}$, 求$f^{(n)}(x)$.

解:$f(x)=\dfrac{1}{x^2-2x-8}=\dfrac{1}{6}(\dfrac{1}{x-4}-\dfrac{1}{x+2})$,

$f^{(n)}(x)=(-1)^n\dfrac{n!}{6}(\dfrac{1}{(x-4)^{n+1}}-\dfrac{1}{(x+2)^{(n+1)}})$.

10.已知$|\boldsymbol{a}|$=4, $|\boldsymbol{b}|$=1. $\langle \boldsymbol{a},\ \boldsymbol{b}\rangle=\dfrac{\pi}{3}$. 求$\boldsymbol{A}=2\,\boldsymbol{a}+\boldsymbol{b}$, $\boldsymbol{B}=-\boldsymbol{a}+3\,\boldsymbol{b}$的夹角.

解: $|\boldsymbol{A}|=\sqrt{73}$, $|\boldsymbol{B}|=\sqrt{13}$, $\boldsymbol{A}\cdot\boldsymbol{B}=-19$,

$\langle \boldsymbol{a},\ \boldsymbol{b}\rangle=\arccos\dfrac{-19}{\sqrt{73\times 13}}$.

二. 设$f(\ln x)=\dfrac{\ln(1+x)}{x}$, 计算$\int f(x)\mathrm{d}x$.(10分)

解: $f(t)=\dfrac{\ln(1+e^t)}{e^t}$.

$\int f(x)\mathrm{d}x=\int\dfrac{\ln(1+e^x)}{e^x}\mathrm{d}x=-e^{-x}\ln(1+e^x)+\int\dfrac{1}{1+e^x}\mathrm{d}x=-e^{-x}\ln(1+e^x)+\int(1-\dfrac{e^x}{1+e^x})\mathrm{d}x=x-(1+e^{-x})\ln(1+e^x)+C$.

三. (本题满分10分) 已知当$x\to 0$时, $e^x-\dfrac{1+ax}{1+bx}$是关于x的3阶无穷小, 求常数a,b之值.

解: $e^x-\dfrac{1+ax}{1+bx}=(1-a+b)x+(\dfrac{1}{2}+ab-b^2)x^2+(\dfrac{1}{6}+b^3-ab^2)x^3+o(x^3)$.

$1-a+b=0, \dfrac{1}{2}+ab-b^2=0, \dfrac{1}{6}+b^3-ab^2\neq 0$.

$a=\dfrac{1}{2}, b=-\dfrac{1}{2}$.

四. (本题满分14分)讨论函数$f(x)=\dfrac{x^3}{(x-1)^2}$的定义域单调区间,极值,凹向与拐点,求函数的渐近线,并作出草图.

解:定义域为$(-\infty,1)\cup(1,+\infty)$. $f'(x)=\frac{x^2(x-3)}{(x-1)^3}$; $f''(x)=\frac{6x}{(x-1)^4}$.

令$f'(x)=0$ 解得$x=0$ 及$x=3$. 令$f''(x)=0$ 解得$x=0$

x	$(-\infty,0)$	0	$(0,1)$	$(1,3)$	3	$(3,+\infty)$
y'	+	0	+	−	0	+
y''	−	0	+	+	+	+
$y=f(x)$	上升,上凸	拐点	下降下凸	下降,下凸	极小值	上升,下凸

铅直渐近线$x=1$. 斜渐近线$y=x+2$.

五. (本题满分10分) 设$S(x)=\int_0^x|\cos t|\mathrm{d}t$,

1, 当n为正整数, 且$n\pi\leq x<(n+1)\pi$时证明不等式$2n\leq S(x)<2(n+1)$.

2,求 $\lim\limits_{x\to+\infty}\dfrac{S(x)}{x}$.

1, 证明: $|\cos x|\geq 0, n\pi\leq x<(n+1)\pi, \int_0^{n\pi}|\cos t|\mathrm{d}t\leq\int_0^{x}|\cos t|\mathrm{d}t<\int_0^{(n+1)\pi}|\cos t|\mathrm{d}t,$

$\displaystyle\int_0^{n\pi}|\cos t|\mathrm{d}t=n\int_0^{\pi}|\cos t|\mathrm{d}t, \int_0^{\pi}|\cos t|\mathrm{d}t=2$

$2n\leq S(x)<2(n+1)$.

2, 解：当 $n\pi\leq x<(n+1)\pi$ 时, $\dfrac{2n}{(n+1)\pi}<\dfrac{S(x)}{x}<\dfrac{2(n+1)}{n\pi}$.

$\lim\limits_{x\to+\infty}\dfrac{S(x)}{x}=\dfrac{2}{\pi}$.

六. (本题满分6分）设函数$f(x)$在$[0,1]$上连续, 在$(0,1)$内可导,并且存在$M>0$使得$|f'(x)|\leq M$. 设n是正整数.

证明: $\left|\displaystyle\sum_{k=0}^{n-1}\frac{f(k/n)}{n}-\int_0^1 f(x)\mathrm{d}x\right|\leq\frac{M}{2n}$.

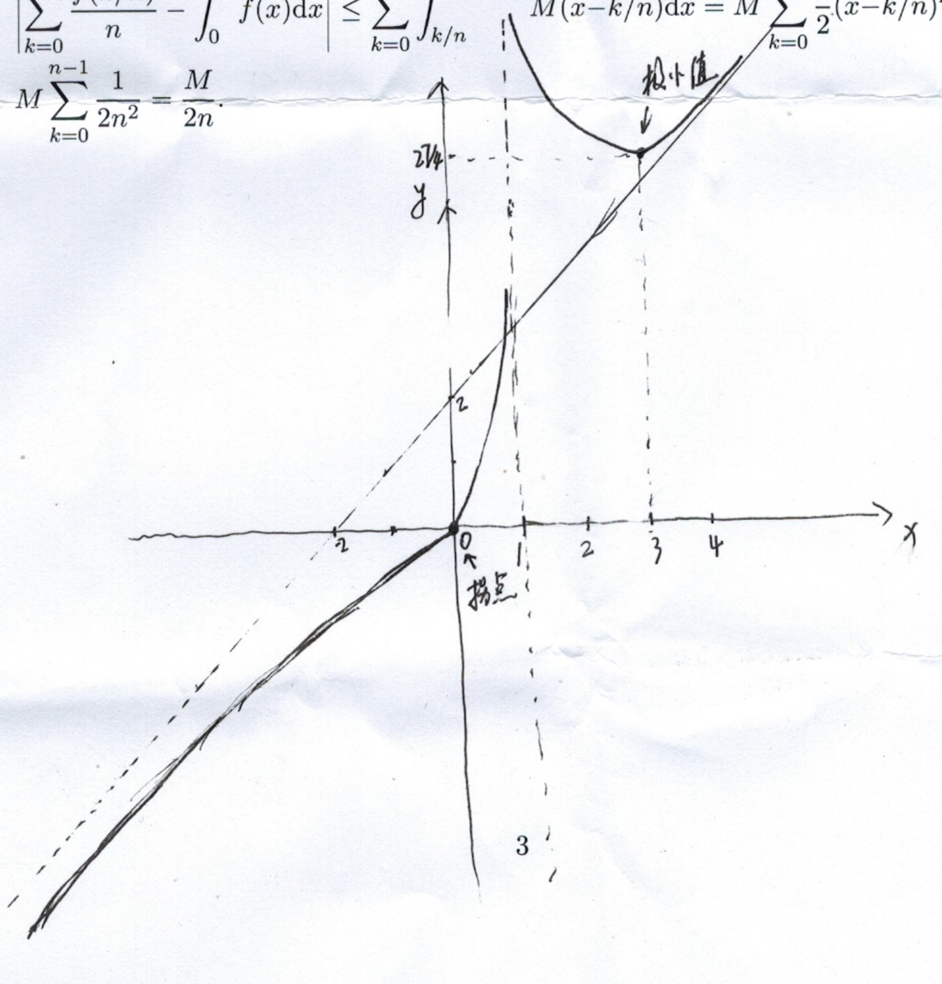
证明: $\left|\displaystyle\sum_{k=0}^{n-1}\frac{f(k/n)}{n}-\int_0^1 f(x)\mathrm{d}x\right|=\left|\sum_{k=0}^{n-1}\left(\frac{f(k/n)}{n}-\int_{k/n}^{(k+1)/n}f(x)\mathrm{d}x\right)\right|\leq\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}|\frac{f(k/n)}{n}-f(x)|\mathrm{d}x.$

存在$\zeta_k\in(k/n,(k+1)/n)$使得$f(x)-f(k/n)=f'(\zeta_n)(x-k/n)$,

$\left|\displaystyle\sum_{k=0}^{n-1}\frac{f(k/n)}{n}-\int_0^1 f(x)\mathrm{d}x\right|\leq\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}M(x-k/n)\mathrm{d}x=M\sum_{k=0}^{n-1}\frac{1}{2}(x-k/n)^2\Big|_{k/n}^{(k+1)/n}=$

$M\displaystyle\sum_{k=0}^{n-1}\frac{1}{2n^2}=\frac{M}{2n}$.

四 图



$\dfrac{f(x)}{x}\to 1=a$

$f(x)-ax\to 2=b$

∴有斜 $y=x+2$

$\lim\limits_{x\to1^+}f(x)=+\infty$

$x=3\quad y=\dfrac{3^3}{2^2}=\dfrac{27}{4}$