一、解：1. $I_1 = -\frac{1}{3}\int\sqrt{\frac{5}{2}+\frac{3}{2}\cos 2x}\, d\left(\frac{3}{2}\cos(2x)+\frac{5}{2}\right)$

$= -\frac{1}{3}\cdot\frac{2}{3}\left(\frac{5}{2}+\frac{3}{2}\cos 2x\right)^{\frac{3}{2}}+C$ 凑微分法

$= -\frac{2}{9}\left(\frac{5}{2}+\frac{3}{2}\cos 2x\right)^{\frac{3}{2}}+C$

2、$I_2 = \int(\arcsin x)^2 dx$ 令 $t=\arcsin x$，$x=\sin t$，$dx=\cos t\, dt$

$= \int t^2\cos t\, dt = \int t^2 d\sin t$ 换元法

$= t^2\sin t - \int \sin t\, 2t\, dt = t^2\sin t + 2\int t\, d\cos t$

$= t^2\sin t + 2t\cos t - 2\int\cos t\, dt$

$= t^2\sin t + 2t\cos t - 2\sin t + C$

$= x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2x + C$

3. $I_3 = \int\frac{x^2}{(x\sin x+\cos x)^2}dx$ 凑微分 + 分部积分（放心，数据已凑好啦）

$= -\int\frac{x}{\cos x}\, d\frac{1}{x\sin x+\cos x}$

$= -\frac{x}{\cos x(x\sin x+\cos x)} + \int\frac{1}{x\sin x+\cos x}\cdot\frac{\cos x+x\sin x}{\cos^2 x}dx$

$= -\frac{x}{\cos x(x\sin x+\cos x)} + \int\sec^2 x\, dx$

$= \tan x - \frac{x}{\cos x(x\sin x+\cos x)} + C$

二、1. $I_4 = \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1+\tan\frac{x}{2}\right)^2 e^x dx$ 化半角 + 利用同角三角函数关系

$= \int_0^{\frac{\pi}{2}}\left(\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right)e^x\, d\frac{x}{2}$

$$= \int_0^{\frac{\pi}{2}} e^x \, d\tan\frac{x}{2} + \int_0^{\frac{\pi}{2}} \tan\frac{x}{2} \, de^x$$

← 数据已凑好

$$= e^x \tan\frac{x}{2}\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \tan\frac{x}{2} de^x + \int_0^{\frac{\pi}{2}} \tan\frac{x}{2} de^x$$

$$= e^{\frac{\pi}{2}}$$

2、交点 $(0, 2)$，$(0, -2)$

曲边梯形面积 ↓ 注意旋转体体积

曲线：$x = 1 - \frac{y^2}{4}$；$x = 2 - \frac{y^2}{2}$

$$\int_{-2}^{2} \left(1 - \frac{y^2}{4} - 2 + \frac{y^2}{2}\right) dy = \int_{-2}^{2} \left(\frac{y^2}{4} - 1\right) dy$$

$$= \int_0^2 \left(\frac{y^2}{2} - 2\right) dy = \frac{y^3}{6} - 2y \Big|_0^2 = -\frac{8}{3}$$

$\therefore S = \frac{8}{3}$

3、 $$l = \int_{-\pi}^{\pi} \sqrt{a^2(1-\sin\theta)^2 + a^2\cos^2\theta} \; d\theta$$

$$l = \int_\alpha^\beta \sqrt{\rho^2(\theta) + \rho'^2(\theta)} \, d\theta$$

$$= a \int_{-\pi}^{\pi} \sqrt{2 - 2\sin\theta}$$

$$= \sqrt{2} a \int_{-\pi}^{\pi} \left| \sin\frac{\theta}{2} - \cos\frac{\theta}{2} \right| d\theta$$

$$= 2\sqrt{2} a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sin\theta - \cos\theta \right| d\theta$$

$$= 4a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sin\left(\theta - \frac{\pi}{4}\right) \right| d\theta$$

$$= 4a \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \sin\left(\frac{\pi}{4} - \theta\right) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta \right)$$

$$= 4a \left(\cos\left(\frac{\pi}{4} - \theta\right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{4}} - \cos\left(\theta - \frac{\pi}{4}\right) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right)$$

$$= 8a$$

三、1. $I_1 = \int_0^{+\infty} \frac{\frac{1}{x^2}-1}{(x+\frac{1}{x})^2-2}\,dx = -\int_0^{+\infty} \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2}$ 凑微分

$= -\frac{1}{2\sqrt{2}} \ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| \Big|_0^{+\infty}$

$= \frac{1}{2\sqrt{2}} \ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right| \Big|_0^{+\infty} = 0$

2. $(\vec{a}+\vec{b}+\vec{c})^2 = \vec{a}^2+\vec{b}^2+\vec{c}^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

$= 4+9+16+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$

∴ $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = -\frac{29}{2}$

四、解：由 $\lim_{x\to 0}\frac{f(x)}{x} = A$ 可知 $f(0)=0$，$f'(0)=A$ 导数定义

∴ $g(0) = \int_0^1 f(0t)\,dt = \int_0^1 f(0)\,dt = 0$

当$x\neq 0$时，$g(x) = \frac{1}{x}\int_0^x f(t)\,dt$

则 $g'(x) = \frac{xf(x)-\int_0^x f(t)\,dt}{x^2}$

$\lim_{x\to 0} g'(x) \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{f(x)+xf'(x)-f(x)}{2x} = \lim_{x\to 0}\frac{f'(x)}{2} = \frac{A}{2}$

$g'(0) = \lim_{x\to 0}\frac{g(x)}{x} = \lim_{x\to 0}\frac{\int_0^x f(t)\,dt}{x^2}$

$\overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{f(x)}{2x} = \frac{A}{2}$

∴ $g'(x)$在$x=0$处连续 $g'(x) = \begin{cases} \frac{A}{2}, & x=0 \\ \frac{xf(x)-\int_0^x f(t)\,dt}{x^2}, & x\neq 0 \end{cases}$

五、解：原式 $= \lim_{n\to\infty}\sqrt[n]{(1+\frac{1}{n})(1+\frac{2}{n})+\cdots+(1+\frac{n-1}{n})}$

构造定积分定义

$$= \exp \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\ln\left(1+\frac{k}{n}\right)$$

$$= \exp\int_0^1 \ln(1+x)\,dx = \exp(2\ln 2-1)$$

$$= \frac{4}{e}$$

六、解：$f(x)=\frac{\ln x}{x}$ 定义域为 $(0,+\infty)$

$f'(x)=\frac{1-\ln x}{x^2}$，$f''(x)=\frac{2\ln x-3}{x^3}$

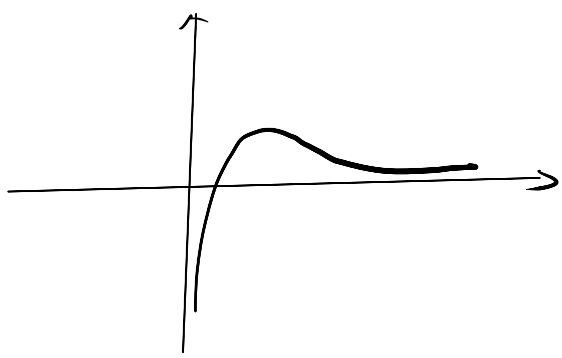
∴ $f(x)$ 单调增区间为 $(0,e)$，单调减区间为 $(e,+\infty)$

有极大值 $f(e)=\frac{1}{e}$，无极小值

在 $(0,e^{\frac{3}{2}})$ 上下凹，在 $(e^{\frac{3}{2}},+\infty)$ 上上凹

拐点为 $(e^{\frac{3}{2}}, \frac{3}{2}e^{-\frac{3}{2}})$

$x=0$ 是铅直渐近线，$y=0$ 是水平渐近线



七、解：设 l 的方向向量为 $\vec{l}=(a,b,1)$

而 π 法向量为 $\vec{n}=(2,1,-2)$，

则 $\vec{l}\cdot\vec{n}=2a+b-2=0$

设直线l上一点$Q(-2,-2,5)$，其方向向量为$\vec{n}=(-3,1,-1)$

则 $\vec{PQ}$、$\vec{l}$、$\vec{n}_2$ 共面

即 $(\vec{l},\vec{PQ},\vec{n})=\begin{vmatrix} a & b & 1\\ -3 & -5 & 1 \\ -3 & 1 & -1\end{vmatrix}=0$，即 $2a-3b=9$

$\therefore\ a=\frac{15}{8}$，$b=-\frac{7}{4}$

$\therefore\ \vec{l}=(\frac{15}{8},-\frac{7}{4},1)=\frac{1}{8}(15,-14,8)$

则 l：$\frac{x-2}{15}=\frac{y-3}{-14}=\frac{z-4}{8}$

八、证明：将$f(x)$在$\frac{a+b}{2}$处展开，

$$f(x)=f(\frac{a+b}{2})+f'(\frac{a+b}{2})(x-\frac{a+b}{2})+\frac{1}{2}f''(\eta)(x-\frac{a+b}{2})^2$$

两边积分有 $\int_a^b f(x)dx=f(\frac{a+b}{2})(b-a)+\frac{1}{2}\int_a^b f''(\eta)(x-\frac{a+b}{2})^2dx$

$\because$ $f''(x)$在$[a,b]$上连续，由最值定理，

设 $f''(x)_{max}=M$，$f''(x)_{min}=m$

则 $\int_a^b f(x)dx\le f(\frac{a+b}{2})(b-a)+\frac{1}{2}M\int_a^b(x-\frac{a+b}{2})^2dx$

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$=f(\frac{a+b}{2})(b-a)+\frac{1}{6}M(x-\frac{a+b}{2})^3\Big|_a^b$

利用介值定理

$=f(\frac{a+b}{2})(b-a)+\frac{1}{24}M(b-a)^3$

同理，$\int_a^b f(x)dx\ge f(\frac{a+b}{2})(b-a)+\frac{1}{24}m(b-a)^3$

$$\therefore \quad m \le \frac{\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a)}{\frac{(b-a)^3}{24}} \le M$$

对 $f''(x)$ 应用介值定可得 $\exists\, \xi \in [a,b]$，$f''(\xi) = \dfrac{\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a)}{\frac{(b-a)^3}{24}}$

即 $\int_a^b f(x)\,dx = f\left(\frac{a+b}{2}\right)(b-a) + \frac{(b-a)^3}{24} f''(\xi)$