exercises

March 24, 2015

```
In [1]: from IPython.core.display import Image
```

1 Numpy Exercises

```
In [2]: # Note that this Document is designed to be executed from top to bottom.
        # General imports, numpy/matplotlib may not be done in all examples.
        import numpy as np

        import warnings  # Just used for some examples in the solutions....
```

1.1 Array creation and manipulation

By using miscellaneous constructors, indexing, slicing, and simple operations (+, −, *, :), large arrays with various patterns can be created. Find a way to create these arrays.

1. Create the following arrays:

a)

$$\begin{pmatrix} 1. & 1. & 1. & 1. \\ 1. & 1. & 1. & 1. \\ 1. & 1. & 1. & 2. \\ 1. & 6. & 1. & 1. \end{pmatrix}$$

b)

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

2. Form this 2-D array (without explicitly typing it):

$$\begin{pmatrix} 1 & 6 & 11 \\ 2 & 7 & 12 \\ 3 & 8 & 13 \\ 4 & 9 & 14 \\ 5 & 10 & 15 \end{pmatrix}$$

1.1.1 Solution

```
In [3]: # 1. a)
        arr = np.ones((4, 4))
        arr[2, 3] = 2
        arr[3, 1] = 6
        print(arr)
```

```
[[ 1.  1.  1.  1.]
 [ 1.  1.  1.  1.]
 [ 1.  1.  1.  2.]
 [ 1.  6.  1.  1.]]
```

In [4]:
```
# 1. b)

        print("Possible solution based on diag and slicing:")
        arr = np.diag(range(1, 7))
        # The arr now has one extra row at the start, so remove it:
        arr = arr[:,1:]
        print(arr)

        print("A possible solution based on advanced indexing:")
        arr = np.zeros((6, 5), dtype=np.int_)
        # Define the positions along first axis:
        i = np.arange(1, 6)
        # Define the positions along second axis:
        j = np.arange(5)
        # And the values to write to these positions:
        vals = np.arange(2, 7)
        arr[i, j] = vals
        print(arr)

        #print("A future solution may be this (does not work yet):")
        #arr = np.ones((6, 5), dtype=np.int_)
        #diagonal_view = arr.diagonal(-1)  # only view in the future
        #diagonal_view[:] = np.arange(2, 7)
        #print(arr)
```

```
Possible solution based on diag and slicing:
[[0 0 0 0 0]
 [2 0 0 0 0]
 [0 3 0 0 0]
 [0 0 4 0 0]
 [0 0 0 5 0]
 [0 0 0 0 6]]
A possible solution based on advanced indexing:
[[0 0 0 0 0]
 [2 0 0 0 0]
 [0 3 0 0 0]
 [0 0 4 0 0]
 [0 0 0 5 0]
 [0 0 0 0 6]]
```

In [5]:
```
# 2.
        print("We can use reshape to achieve this:")
        r = np.arange(1, 16)
        arr = r.reshape((3, 5)).T
        print(arr)

        print("ADVANCED: Understand why this is the same!:")
        arr = r.reshape((5, 3), order='F')
        print(arr)
```

```
        print("You can use broadcasting to achieve something similar:")
        r1 = np.arange(5)[:, np.newaxis]
        r2 = np.array([1, 6, 11])
        arr = r1 + r2
        print(arr)
```

```
We can use reshape to achieve this:
[[ 1  6 11]
 [ 2  7 12]
 [ 3  8 13]
 [ 4  9 14]
 [ 5 10 15]]
ADVANCED: Understand why this is the same!:
[[ 1  6 11]
 [ 2  7 12]
 [ 3  8 13]
 [ 4  9 14]
 [ 5 10 15]]
You can use broadcasting to achieve something similar:
[[ 1  6 11]
 [ 2  7 12]
 [ 3  8 13]
 [ 4  9 14]
 [ 5 10 15]]
```

1.2 A simple calculation

1. Devide each column of the array `a` elementwise by the array `b` (do you notice some problem?)

```
In [6]: a = np.arange(25).reshape(5, 5)
        b = np.array([1, 5, 10 , 15, 20])
```

1.2.1 Solution

```
In [7]: # 1.
        a = np.arange(25).reshape(5, 5)
        b = np.array([1, 5, 10 , 15, 20])

        # Warning: in python 2 with "from __future__ import division"
        # this does not work. An explicit cast is necessary!
        result = a / b[:, np.newaxis]
        print(result)

        # This does NOT work because a / b is float but a is an integer array.
        # The "with warnings.catch_warnings()" is there to show the Deprecation
        # error here. Ignore it!
        with warnings.catch_warnings():
            warnings.simplefilter("always")
            a /= b[:, np.newaxis]  # this causes the warning!
        print(a)
```

```
[[0 1 2 3 4]
 [1 1 1 1 1]
 [1 1 1 1 1]
 [1 1 1 1 1]
```

```
 [1 1 1 1 1]]
[[0 1 2 3 4]
 [1 1 1 1 1]
 [1 1 1 1 1]
 [1 1 1 1 1]
 [1 1 1 1 1]]
```

1.3 Sorting and more

a) Generate a 10×3 array of random numbers ($\in [0, 1[$). For each row, pick the number closest to 0.5. Use `abs` and `argmin` to find the column j closest for each row. Use advanced indexing to extract the numbers. (Hint: `arr[i,j]` – the array i must contain the row numbers corresponding to stuff in j.)

b) Find the two points closest using `argsort`. Advanced: Look at `argpartition` can you figure out how to improve the speed of your algorithm? Find the fastest method and test it on larger arrays!

1.3.1 Solution

In [8]:
```
# a)
data = np.random.random((10, 3))

# This approach is based on an unsorted array.
# If the array was sorted, 'searchsorted' would be an option
offset = abs(data - 0.5)
# The index j
indx = offset.argmin(axis=-1)
print(data[np.arange(len(data)), indx])
```

```
[ 0.413439    0.61147946  0.66676921  0.40563752  0.39895675  0.65050476
  0.307519    0.44918689  0.51223151  0.2958948 ]
```

In [9]:
```
# b)
data = np.random.random((10, 3))
offset = abs(data - 0.5)

# We can simply sort and slice the first two:
indx = np.argsort(offset)[:, :2]
result = data[np.arange(len(data))[:, np.newaxis], indx]

# ADVANCED PART:
# Lets find the "fastest" way:
# (Note that for more then two items, you would need to sort the result a second time)
partially_sorted = np.argpartition(offset, 1)
indx = partially_sorted[:, :2]
result2 = data[np.arange(len(data))[:, np.newaxis], indx]
print(np.array_equal(result, result2))

# Try the speed for a large array
data = np.random.random((1000, 1000))
offset = abs(data - 0.5)
%timeit fully_sorted = np.argsort(offset)
%timeit partially_sorted = np.argpartition(offset, 1)
```

```
True
10 loops, best of 3: 57.8 ms per loop
100 loops, best of 3: 3.58 ms per loop
```

2 Matplotlib Excercises

```
In [10]: # Make the IPython notebook matplotlib aware:
         %matplotlib inline
         import matplotlib
         # And fix the default savefig dpi that ipython sets very low
         matplotlib.rcParams['savefig.dpi'] = 120

         # Import the pyplot interface to matplotlib:
         from matplotlib import pyplot as plt
```

2.1 Simple Plotting

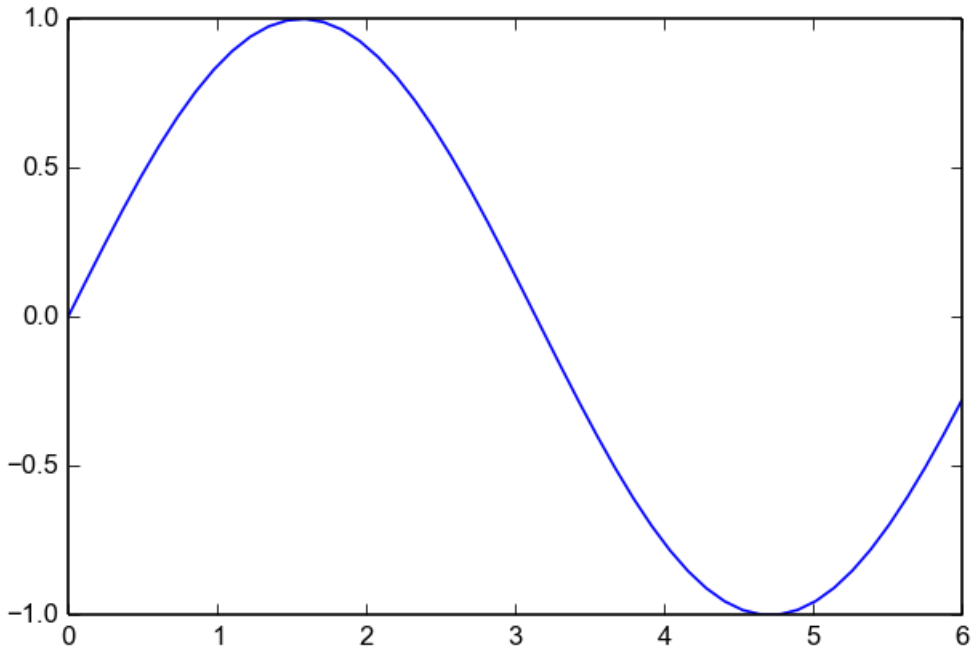
1. Plot a the sine function from 0 to 6 (inclusinve) using 50 points.
2. Plot the line again in red and with diamond shaped markers of size 5.
3. Now plot the line, but include a legend.

Helpful functions: `plt.plot`, `plt.legend`, `plt.grid`

2.1.1 Solution

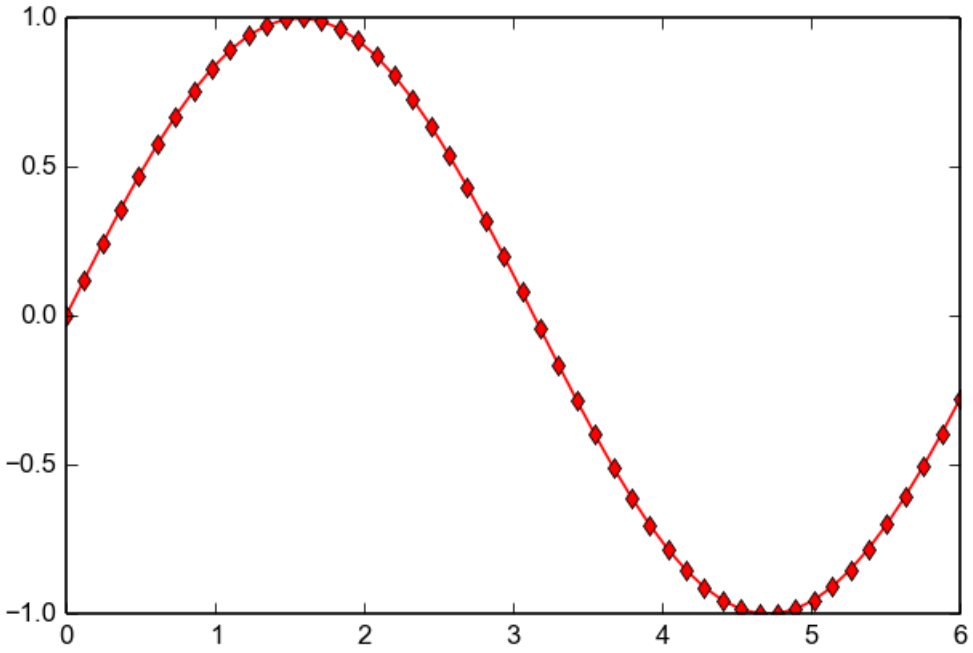
```
In [11]: x = np.linspace(0, 6, 50)
         y = np.sin(x)
         plt.plot(x, y)
```

```
Out[11]: [<matplotlib.lines.Line2D at 0x7ffd4d8f5d10>]
```

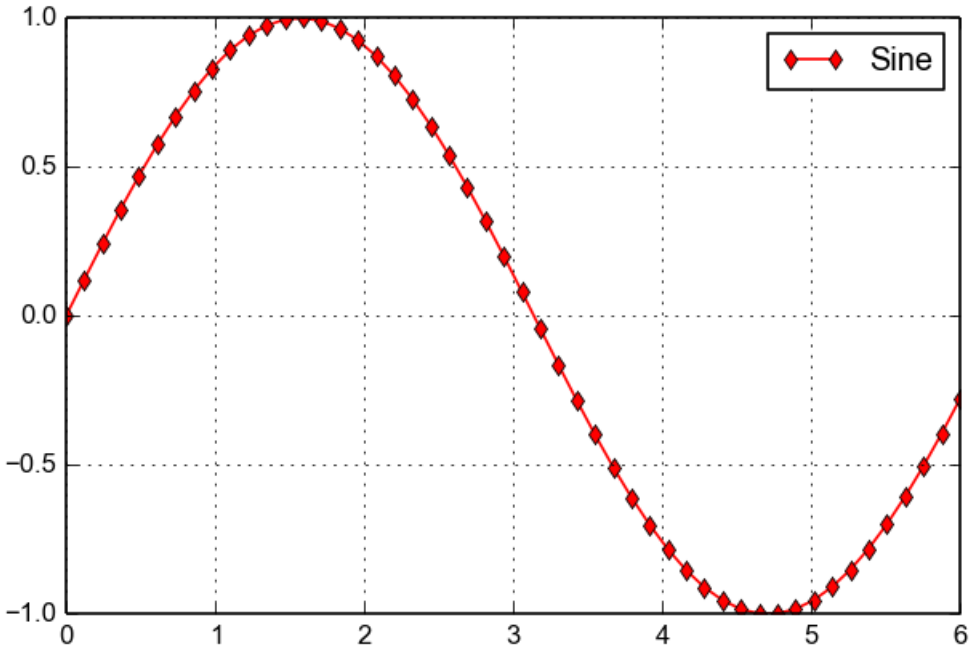



```
In [12]: plt.plot(x, y, 'r', markersize=5, marker='d')
         #plt.plot(x, y, 'r-d', markersize=5)
```

```
Out[12]: [<matplotlib.lines.Line2D at 0x7ffd4d79d910>]
```




```
In [13]: plt.plot(x, y, 'r', markersize=5, marker='d',
                  label="Sine")
         plt.legend(loc='best')
         plt.grid()
```

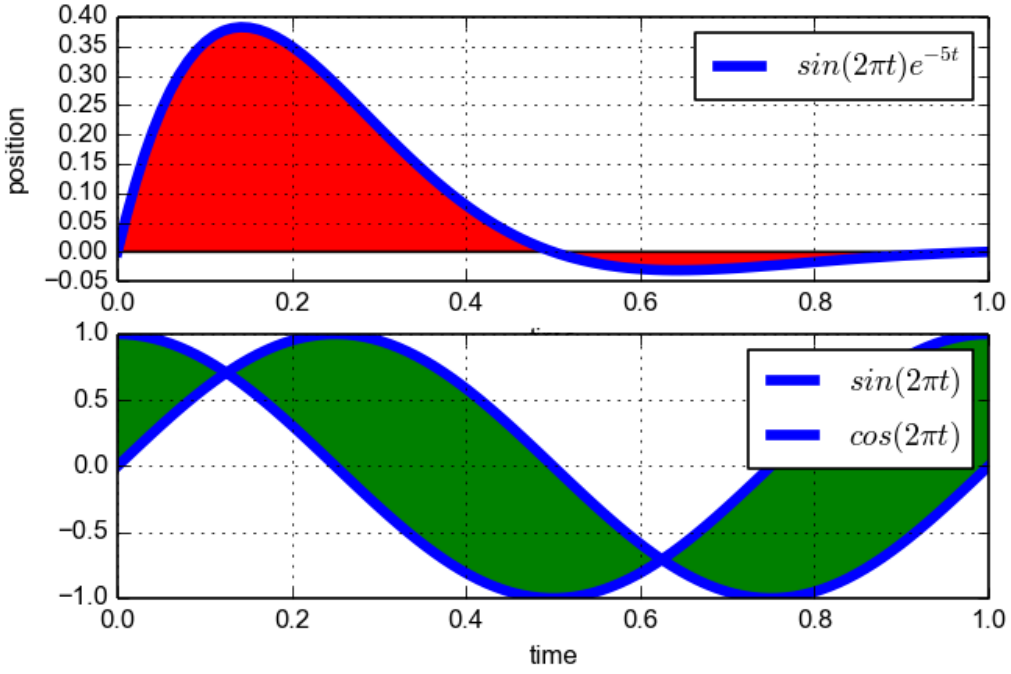


2.2 Simple Plotting II

Try to recreate the following plot:

In [14]:
```
Image("pics/simple_plotting_2.png")
```

Out[14]:

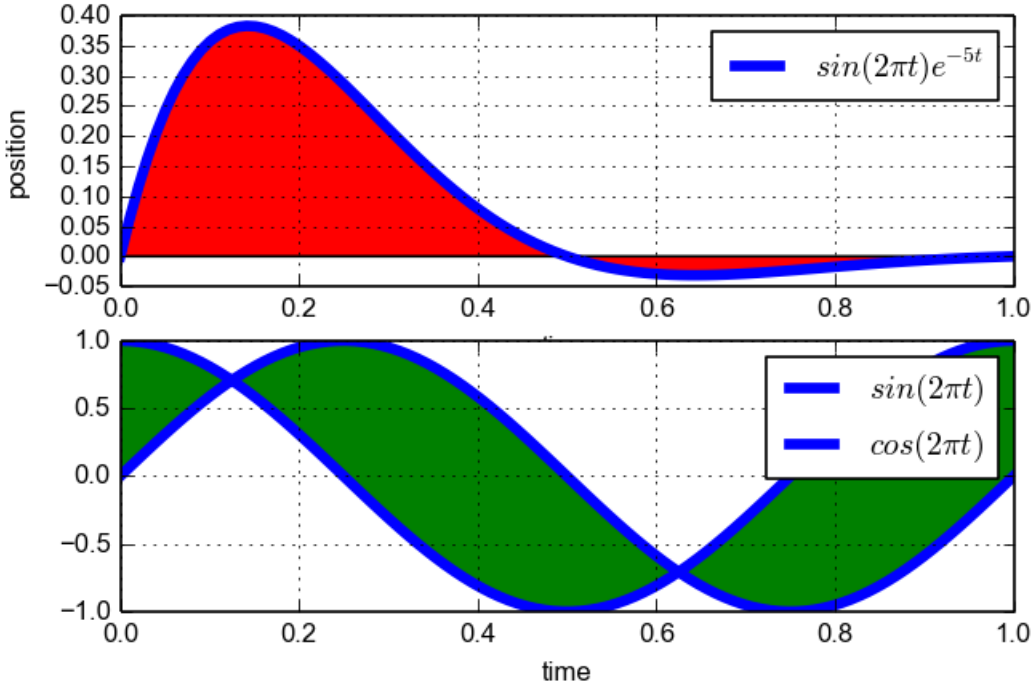


Helpful functions: `plt.fill`, `plt.fill between`

2.2.1 Solution

In [15]:
```
t = np.linspace(0.0, 1.0, 100)
s = np.sin(2*np.pi * t)

plt.subplot(2,1,1)
plt.plot(t, s * np.exp(-5*t), 'b',
         label=r'$sin (2\pi t) e^{-5 t}$', linewidth=4.)
plt.fill(t, s * np.exp(-5*t), 'r')
plt.xlabel('time')
plt.ylabel('position')
plt.legend()
plt.grid(True)

plt.subplot(2,1,2)
plt.plot(t, np.sin(2.*np.pi*t), 'b', label= r'$sin (2\pi t)$', linewidth=4.)
plt.plot(t, np.cos(2.*np.pi*t), 'b', label= r'$cos (2\pi t)$', linewidth=4.)
plt.fill_between(t, np.cos(2.*np.pi * t) , np.sin(2.*np.pi * t), color='green')
plt.xlabel('time')
```

```
plt.legend()
plt.grid(True)

plt.savefig('pics/simple_plotting_2.png')
```



3 Simple Tasks

3.1 Polynomial Fitting

1. Generate data by a polynomial function $f(x) = -3x^3 + 2x - 8$
2. Fit a polynomial (`np.polyfit()`) and recover the coefficients of the polynomial.
3. Try the same procedure with noisy data. (use np.random.randn() to add noise to the data)

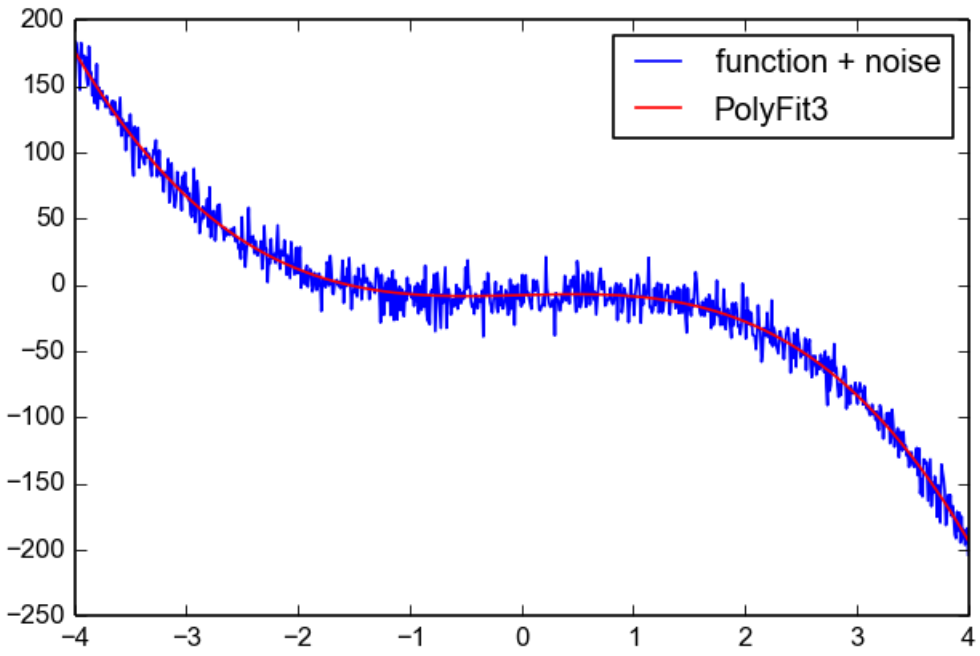
3.1.1 Solution

```
In [16]: x = np.linspace(-4, 4, 801)
         y = -3.*x**3 + 2.*x - 8

         # Fit the polynomial of order 3:
         print(np.polyfit(x, y, 3))

         # Gaussian noise of variance 10 added to the function
         noise = np.random.randn(len(y)) * 10
         y += noise

         coeff = np.polyfit(x, y, 3)
         plt.plot(x, y, label = 'function + noise')
         y_fit = coeff[0]*x**3 + coeff[1]*x**2 + coeff[2]*x + coeff[3]
```

```
plt.plot(x, y_fit, label='PolyFit3', color='red')
plt.legend()
```

```
[ -3.00000000e+00  -8.39974839e-16   2.00000000e+00  -8.00000000e+00]
```

```
Out[16]: <matplotlib.legend.Legend at 0x7ffd4d729ad0>
```



3.2 Nonlinear Fitting

1. Define the function $f(x) = e^{-ax} + b$
2. Generate noisy data from that function with parameters $a = 2$, $b = 1.4$
3. Make a simple plot of the data
4. Use `scipy.optimize.curve_fit()` to estimate a and b from the data and plot the results.

3.3 Solution

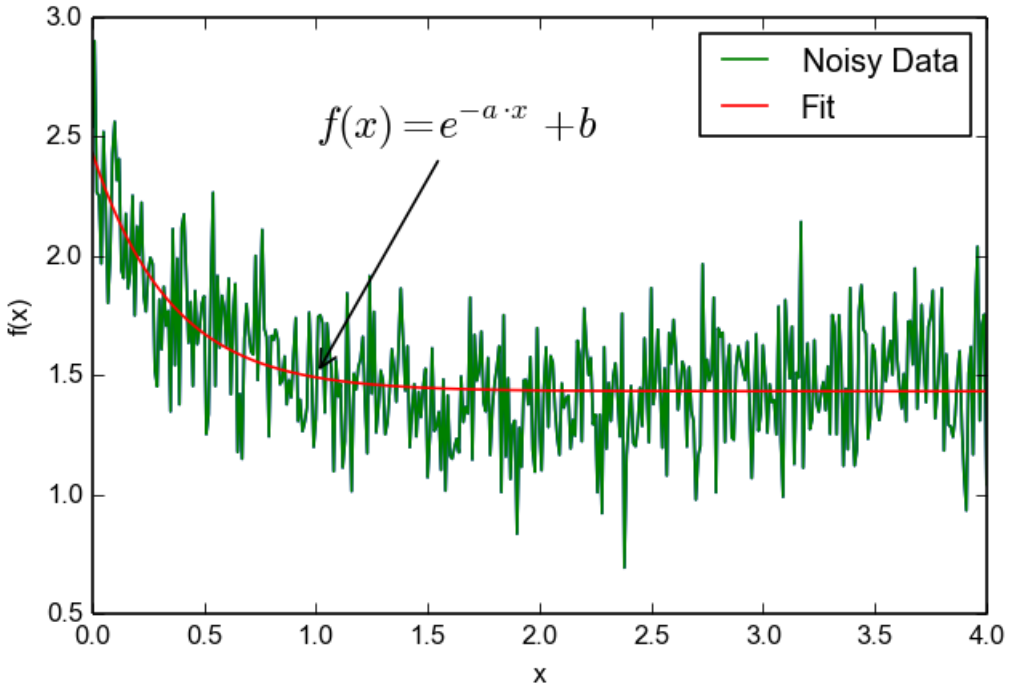
```
In [17]: from scipy import optimize as opt
```

```
# define a lambda function with three arguments
f = lambda x, a, b: np.exp(-a*x) + b

x = np.linspace(0, 4, 401)
# generate y values from the lambda function
y = f(x, 2.5, 1.4)
# Add Gaussian noise to them
y += np.random.normal(scale=0.25, size=y.shape)

plt.plot(x, y)
```

```
        # Note that providing an initial guess `p0` can often be
        # vital for fast and correct convergence!
        # Also providing the Jacobian can improve the convergence
        # dramatically (`Dfun` keyword argument). And sometimes just
        # reformulating the parameters, i.e. scaling them or making
        # sure that they are as orthogonal as possible helps a lot.
        popt, pcov = opt.curve_fit(f, x, y)
        print(popt)

        plt.plot(x, y, label='Noisy Data')

        plt.plot(x, f(x, *popt), label='Fit', color='r')

        plt.annotate(
            '$f(x) = e^{-a \cdot x} + b$', size=16,
            xy=(1, 1.5), xycoords='data', xytext=(1, 2.5),
            textcoords='data', arrowprops=dict(arrowstyle="->"))

        plt.xlabel('x')
        plt.ylabel('f(x)')
        plt.legend()
```

```
[ 2.83916897  1.43238053]
```

Out[17]: <matplotlib.legend.Legend at 0x7ffd446f8b10>



4 Integration

1. Define the function $f(x) = e^{-(x+3)^2}$. Make sure the function also accepts lists/array-likes.

2. Calculate the integral $\int_{-20}^{20} f(x)\,dx$ by hand using Riemann sums.
3. Use `scipy.integrate.quad()` to evaluate the integral.
4. Try to calculate $\int_1^\infty \frac{1}{x^2}\,dx$ and $\int_1^\infty \frac{1}{x}\,dx$

4.0.1 Solution

```
In [18]: def f(x):
             # Make sure x is an array. Do *not* use np.vectorize.
             x = np.asarray(x)
             return np.exp(-(x + 3)**2)

         def riemann(func, start, stop, steps):
             """Calculates the Riemann sum for a given function

             Parameters
             ----------
             func : callable
                 Function taking a single array of values
             start : float
                 Integral lower bound
             stop : float
                 Integral upper bound
             steps : integer
                 Number of steps to use (precision)

             Returns
             -------
             riemann_sum : float
             """
             x, step = np.linspace(start, stop, steps, retstep=True)

             y = func(x)
             integral = step * y.sum()
             return integral

         # Calculate the Riemann sum for various number of steps for
         # the function "f"
         for steps in [41, 401, 4001]:
             integral = riemann(f, -20, 20, steps)
             print("Rieman sum using {} steps {:1.12g}:".format(steps, integral))


         # Now Calculate it using scipy.integrate.quad
         from scipy.integrate import quad

         integral, estimated_error = quad(f, -20, 20)
         print()
         print("Result using quad: {:1.12g}".format(integral))

         # And integrate the other two examples:
         print()
         with warnings.catch_warnings():
             # This warnings code, is just to show the warning every time..
             warnings.simplefilter("always")
```

```
        integral, estimated_error = quad(lambda x: 1/x**2, 1, np.inf)
    print("Integrate 1/x^2 from 1 to infinity: {:1.12g}".format(integral))

    print()
    with warnings.catch_warnings():
        # This warnings code, is just to show the warning every time....
        warnings.simplefilter("always")
        integral, estimated_error = quad(lambda x: 1/x, 1, np.inf)

    print("Integrate 1/x from 1 to infinity: {:1.12g}".format(integral))
    print("Note the warning:")
```

```
Rieman sum using 41 steps 1.77263720483:
Rieman sum using 401 steps 1.77245385091:
Rieman sum using 4001 steps 1.77245385091:
()
Result using quad: 1.77245385091
()
Integrate 1/x^2 from 1 to infinity: 1
()
Integrate 1/x from 1 to infinity: 40.9960128192
Note the warning:

/usr/lib/python2.7/dist-packages/scipy/integrate/quadpack.py:321: IntegrationWarning: The maximum numbe
  If increasing the limit yields no improvement it is advised to analyze
  the integrand in order to determine the difficulties.  If the position of a
  local difficulty can be determined (singularity, discontinuity) one will
  probably gain from splitting up the interval and calling the integrator
  on the subranges.  Perhaps a special-purpose integrator should be used.
  warnings.warn(msg, IntegrationWarning)
```

4.1 Solving Systems of Linear Equations

You find three shopping bags with following content:

- bag A: 10 kg apples, 5 kg pears, 1 kg oranges (35.35 EUR)
- bag B: 1 kg apples, 8 kg pears, 1 kg oranges (24.91 EUR)
- bag C: 9 kg apples, 3 kg pears, 5 kg oranges (40.38 EUR)

Determine the price of apples, oranges and pears: 1. Formulate a linear system of equations describing the shopping bags. 2. Use `scipy.linalg.solve()` (or `numpy.linalg`) to solve the system of equations. 3. Verify the results by using `np.dot()`.

4.1.1 Solution

In [19]:
```
from scipy.linalg import solve

fruits = np.array([[10, 5, 1], [1, 8, 1], [9, 3, 5]])
total_prices = np.array([35.35, 24.91, 40.38])

# Solve the system of linear equations
# total_prices = fruits * single_prices for single_prices
single_prices = solve(fruits, total_prices)
print("The single prices are:")
print("apples: {}".format(single_prices[0]))
```

```
print("pears: {}".format(single_prices[1]))
print("oranges: {}".format(single_prices[2]))

# Make sure the result is correct. Note that due to how
# floating points works, the result is only almost correct.
print("The found prices fit to the real prices:")
print(np.allclose(np.dot(fruits, single_prices), total_prices))
```

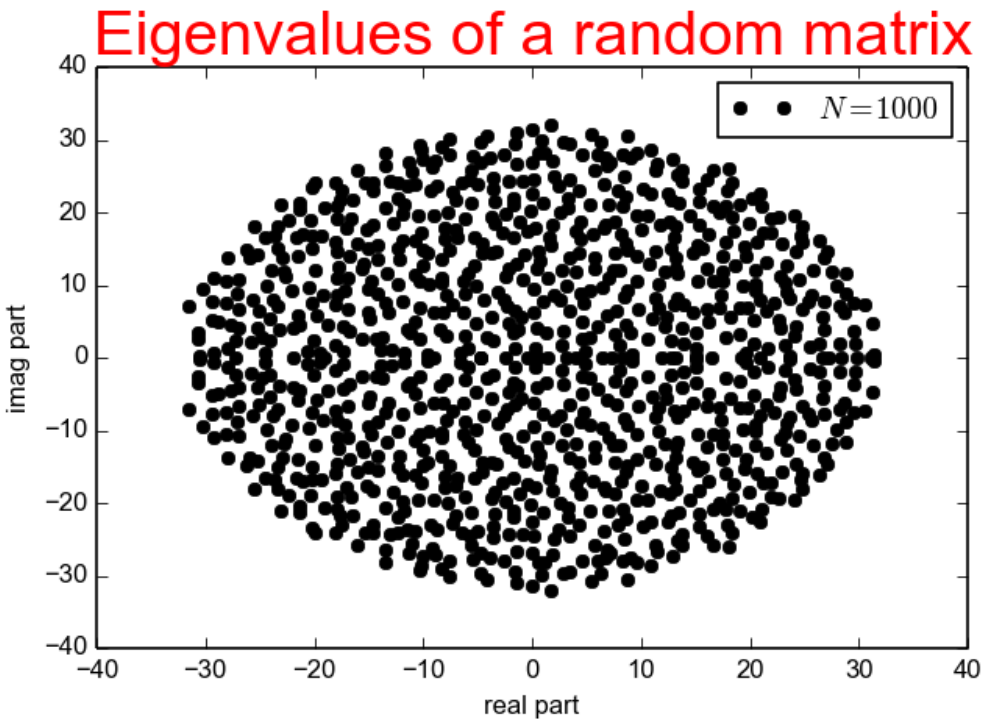
```
The single prices are:
apples: 1.99
pears: 2.49
oranges: 3.0
The found prices fit to the real prices:
True
```

4.2 Some Linear Algebra

1. Create a random matrix, where the entries are randomly chosen from a standard normal (Gaussian) distribution.
2. Plot the (complex!) eigenvalues of that matrix, add labels to the axis and a legend.
3. Add a title for your plot and make it red and increase the size.
4. Store your graph as an `.pdf`. It should look something like this:

```
In [20]: Image("pics/matrix1.png")
```

```
Out[20]:
```



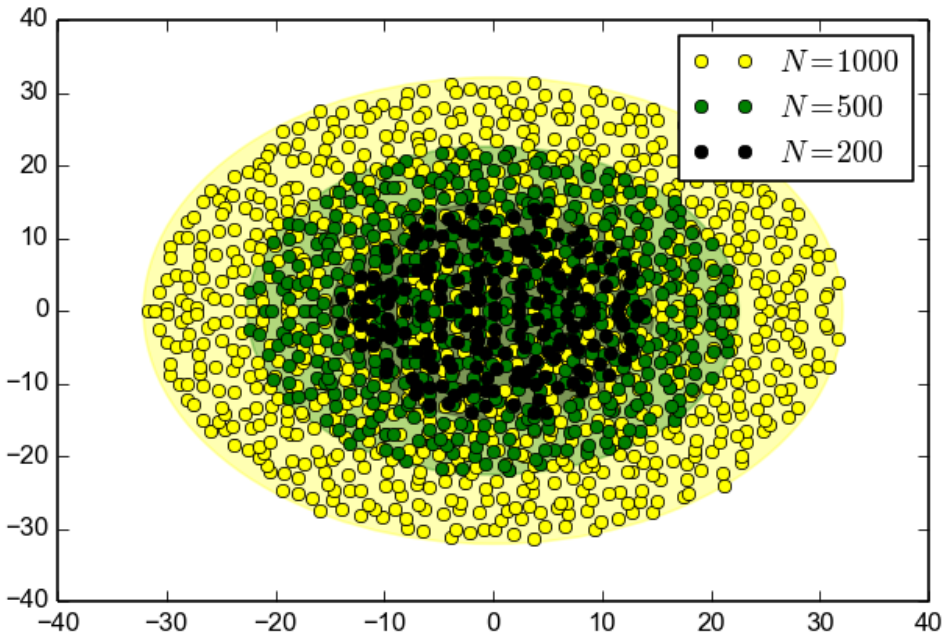
5. Define a function that returns True if a point (x, y) is contained in a circle of radius r around the origin. Make sure it works for array inputs.

6. Define a function to find the smallest circle containing all the eigenvalues of your matrix.

7. Plot the eigenvalues for a matrix of size N = 50, 100, 200, 500 together with the smallest disc containing the eigenvalues. Therefore first import the matplotlib module `matplotlib.patch` and have a look at the classes contained in that mod- ule. Having identified a suitable class create a patch (a circle) and try to add it to your plot. (*Hint:* Patches can be added to axes objects)

```
In [21]: Image("pics/matrix2.png")
```

Out[21]:



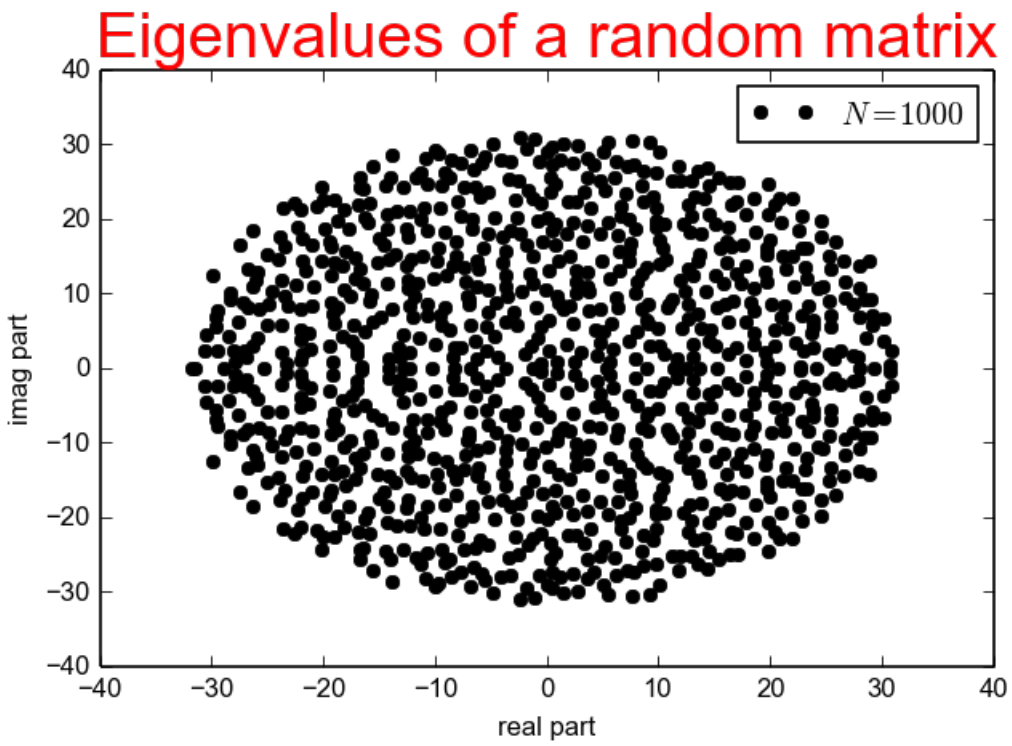
4.2.1 Solution

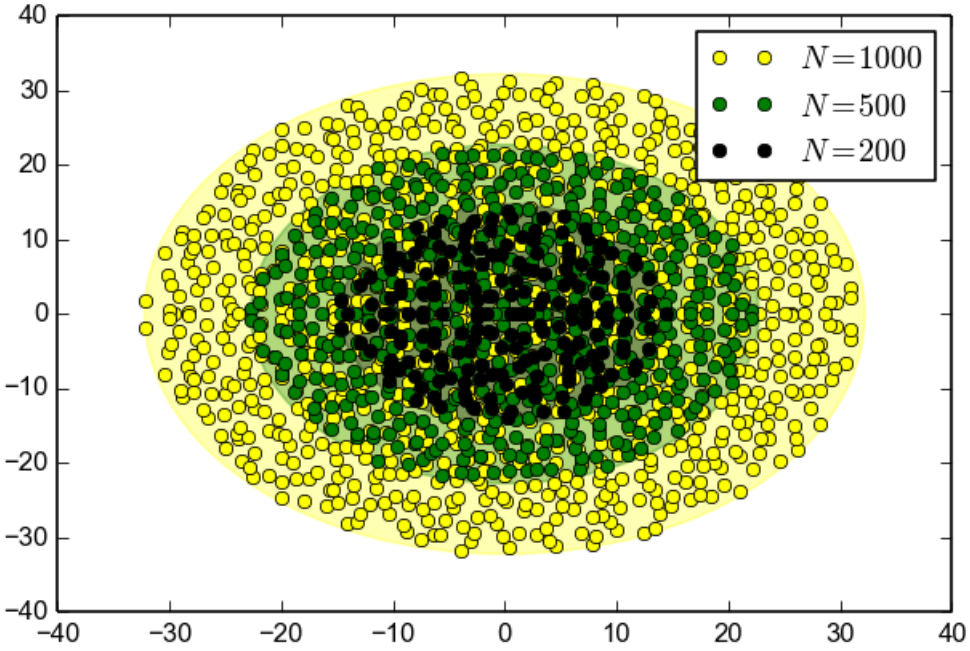
```
In [22]: from scipy.linalg import eigvals
```

```
def geteig(n):
    """Get eigenvalues for a random nxn matrix.
    """
    m = np.random.normal(size=(n, n))
    return eigvals(m)

M = geteig(1000)

plt.figure()
plt.plot(M.real, M.imag, 'ko', markersize=5, label=r'$N = 1000$')
title_text = plt.title('Eigenvalues of a random matrix')
title_text.set_fontsize(24)
title_text.set_color('red')
```

```
plt.xlabel('real part')
plt.ylabel('imag part')
plt.legend()
#plt.savefig('pics/matrix1.png')

def is_inside_circle(x, y, radius):
    # Squaring the radius instead of the x and y.
    return x**2 + y**2 < r**2

def minimum_radius(m):
    """Get the minimum "radius" including all complex values
    inside m.
    """
    # We could also calculate m.real()**2 + m.imag()**2
    # other versions may be faster.
    return abs(m).max()


# Create a clean figure:
fig = plt.figure()

graph = fig.add_subplot(1, 1, 1)

Ns = [200, 500, 1000][::-1]
colors = ['yellow', 'green', 'black', 'yellow', 'black']
for i in range(len(Ns)):
    M = geteig(Ns[i])
    r = minimum_radius(M)
    graph.plot(M.real, M.imag, 'o', color = colors[i] , markersize=5,
               label='$N = {}$'.format(Ns[i]))
    circ = plt.Circle((0,0), r, color=colors[i], alpha=0.3)
    graph.add_patch(circ)

plt.legend()
#plt.savefig('pics/matrix2.png')
```

Out[22]: <matplotlib.legend.Legend at 0x7ffd44244e90>


Eigenvalues of a random matrix
$N=1000$
imag part
real part
40
30
20
10
0
−10
−20
−30
−40
−40
−30
−20
−10
0
10
20
30
40


$N=1000$
$N=500$
$N=200$
40
30
20
10
0
−10
−20
−30
−40
−40
−30
−20
−10
0
10
20
30
40

4.3 Basic Statistics

1. Define a 1D Gaussian probability density function with $\mu = 10$ and $\sigma = 5$ using the formula

$$\frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

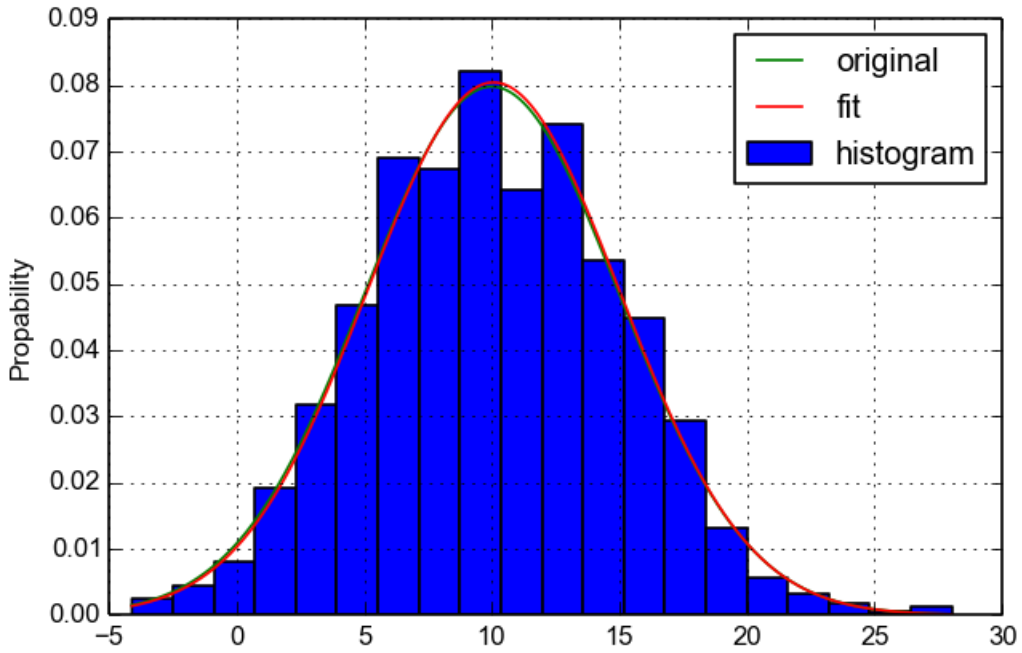
2. Confirm that the integral of this function is approximatly 1.
3. Instead of defining the distribution yourself, now use *only* `scipy.stats.norm` to:
4. Draw 1000 samples. And plot a histogram using matplotlib.
5. Do a maximum likelyhood estimation to find the parameters of the gaussian from the random samples. (For a gaussian these are just the mean and standard deviation, but use `scipy.stats.norm` instead of calculating these yourself)
6. Add both the estimated and the real Gaussian to the plot. If the histogram is not normed correctly already, norm it to fit the lines (see the histogram documentation).
7. Find the points that include 90% around the mean in the original distribution. And confirm that these include 90% using both the comulative distribution function (`cdf`) and the original integration method.

4.3.1 Solution

```
In [23]: from scipy.stats import norm
         from scipy.integrate import quad

         def gaussian_pdf(x, mu=10, sigma=5):
             return 1/(np.sqrt(2*np.pi)*sigma) * np.exp(-(x-mu)**2/(2*sigma**2))

         integral, _ = quad(gaussian_pdf, -np.inf, np.inf)
         print("The overall integral is about:", integral)

         # Actuall do this with scipy.stats.norm:
         orig_gaussian = norm(loc=10, scale=5)
         random_draws = orig_gaussian.rvs(1000)
         # Or:
         # random_draws = norm.rvs(loc=10, scale=5, size=1000)

         plt.hist(random_draws, bins=20, label='histogram', normed=True)

         # Try to find the parameters:
         mu, sigma = norm.fit(random_draws)
         print("The estimated parameters are mu={}, sigma={}".format(mu, sigma))

         x = np.linspace(random_draws.min(), random_draws.max(), 400)
         plt.plot(x, orig_gaussian.pdf(x), label='original')
         plt.plot(x, norm.pdf(x, loc=mu, scale=sigma), label='fit')
         plt.ylabel("Propability")
         plt.grid()
         plt.legend()

         # Find the 90th percentile:
         lower_bound, upper_bound = orig_gaussian.interval(0.9)

         print()
         print("The lower and upper bounds are: {} and {}".format(lower_bound, upper_bound))
         cdf_bounds = orig_gaussian.cdf(upper_bound) - orig_gaussian.cdf(lower_bound)
         print("The cdf says {} are between these bounds.".format(cdf_bounds))
```

```
        cdf_integral, _ = quad(gaussian_pdf, lower_bound, upper_bound)
        print("The first integral method finds: {}".format(cdf_integral))
```

```
('The overall integral is about:', 1.0000000000000002)
The estimated parameters are mu=10.0680005235, sigma=4.96007347425
()
The lower and upper bounds are: 1.77573186524 and 18.2242681348
The cdf says 0.9 are between these bounds.
The first integral method finds: 0.9
```



4.4 Power Spectrum Analaysis

A way to look at a signal is to view its spectral density (i.e., the Fourier transform of the signal). The Fourier transform views the signal as a whole. It swaps the dimension of time with the dimension of frequency. One can think of the Fourier transform as a combination of slow and fast oscillations with different amplitude. A very strong and slow component in the frequency domain implies that there is a high correlation between the large-scale pieces of the signal in time (macro-structures), while a very strong and fast oscillation implies correlation in the micro-structures. Therefore, if our signal $f(t)$ represents values in every single moment of time, its Fourier transform $F(\omega)$ represents the strength of every oscillation in a holistic way in that chunk of Z time. These two signals are related to each other by the following formula:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}$$

For simplicity, in the following we will consider a signal that is constructed by summing different sine waves and try to get information about the underlying frequencies.

1. Construct a 20 seconds long signal samples at 1kHz by adding 20 sine waves with a frequency randomly choosen from the interval [1Hz, 300Hz] (no phase shift is necessary). Add some gaussian noise to the signal.

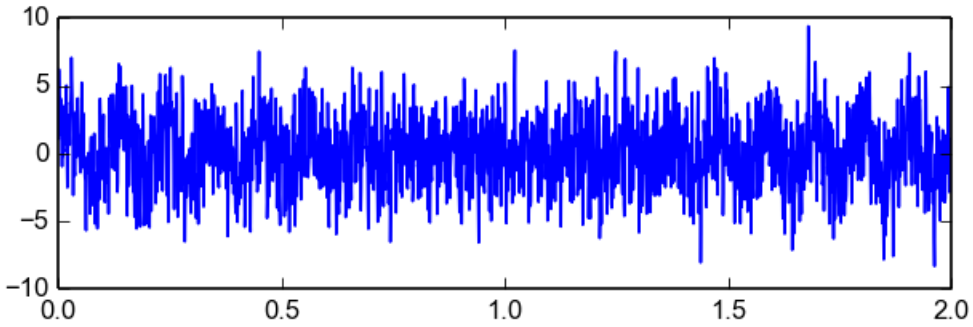
2. Plot the signal in the interval t = 0, . . . , 2s.
3. Use numpy to calculate the one-dimensional discrete Fourier Transform.
4. The square of the absolute value of the fourier transform gives you the power carried by each frequency (power spectrum). To calculate the Fourier transform sample frequencies you can also use a numpy function - have a look at the documentation of the `numpy.fft` module. Plot the power spectrum logarithmically and also indicate the frequencies that were used for generating the data.
5. *** Add a plot of the original time series to the plot of the power spectrum. To generate the subplot you can add an axes object directly to the figure. If you don't know how to do that have a look at the documentation of matplotlib or the examples in the gallery (http://matplotlib.sourceforge.net/gallery.html).***

4.4.1 Solution

In [24]:
```
sampleing_freq = 1/1000

times = np.linspace(0, 20, 20*1000+1)

freqs = np.random.uniform(1, 300, size=10)

# Construct the signal
y = np.zeros_like(times)
# Could use broadcasting, but lets not overdo it:
for freq in freqs:
    y += np.sin(2*np.pi * freq * times)

y += np.random.normal(size=y.shape)

# Plot the data on top:
plt.figure()
plt.subplot(2, 1, 1)
plt.plot(times[:2000], y[:2000], label='Signal')

# Find the power spectrum. We can use rfft since the data is purely real.
# (For those who have ever need speed critical FFT. First you can use
# padding, and also there are much faster FFTs that numpy cannot wrap
# easily for legal reasons.)
power = abs(np.fft.rfft(y)**2)
frequencies = np.fft.rfftfreq(len(y), d=sampleing_freq)

plt.subplot(2, 1, 2)
plt.semilogy(frequencies, power)
plt.xlim(0, 300)  # Only the interesting range

plt.xlabel('Frequency [Hz]')
plt.ylabel('Power')

plt.vlines(freqs, plt.ylim()[0], plt.ylim()[1], color='r', zorder=3, alpha=0.5)
```

```
---------------------------------------------------------------------------
ZeroDivisionError                         Traceback (most recent call last)

<ipython-input-24-ad1964dbdf23> in <module>()
 23 # easily for legal reasons.)
```

```
     24 power = abs(np.fft.rfft(y)**2)
---> 25 frequencies = np.fft.rfftfreq(len(y), d=sampleing_freq)
     26
     27 plt.subplot(2, 1, 2)


/usr/lib/python2.7/dist-packages/numpy/fft/helper.pyc in rfftfreq(n, d)
    218     if not isinstance(n, integer_types):
    219         raise ValueError("n should be an integer")
--> 220     val = 1.0/(n*d)
    221     N = n//2 + 1
    222     results = arange(0, N, dtype=int)


ZeroDivisionError: float division by zero
```



4.5 Vectorization of a Monte-Carlo Simulation

Recall the dice-simulation from the basic-exercise sheet. The goal of the current exercise is to optimize this program using vectorization.

1. Create a function that draws N-times two uniform random integers from 1 to 6 and counts how many times X at least one of the two takes the value 6. Do this in a single N $\times$ 2 matrix.
2. Run this simulation for growing N and plot the estimated probability as a function of N. Try to use linear and logarithmic axis. Include the exact value of 11/36.
3. Repeat the dice throwing Z times for a fixed value of N and plot a histogram of the number of throws with at least one six (e.g. for N = 1000, Z = 10000). The data are integers, so choose a bin width of 1 or use integer functions such as `np.bincount`. Further print out the mean value and the standard deviation.
4. As above repeat the dice throwing for a fixed number N of dice throws Z times and calculate the probability to get at least one six with the two dices. From this data you can calculate the empirical average and standard deviation for every pair of N and Z. Make an error-bar plot for N = 1, . . . , 20 and Z = 5000.

4.5.1 Solution

```
In []: # 1.
       def count_dice(N):
           dice = np.random.randint(1, 7, size=(N, 2))
```

```
    equal_6 = dice == 6
    one_is_6 = equal_6.any(-1)
    count = np.count_nonzero(one_is_6)
    return count

# 2.
Ns = np.arange(5, 30000, 100)
# We do not vectorize this part. Since N varies, it is not easy/possible.
# If N is small the whole thing will be much faster implemented i.e. in C
# or Cython. Tricks could be used in this example, but I consider them
# highly non-basic.
count = [count_dice(N) for N in Ns]
count = np.asarray(count)
probs = count / Ns

plt.semilogx(Ns, probs, label='estimation')
plt.hlines(11/36, Ns[0], Ns[-1], label='correct', zorder=3, alpha=0.5)
plt.xlim(Ns[0], Ns[-1])
plt.legend(loc='best')
plt.ylabel('Probability')
plt.xlabel('Number of rolls')

# 3.
def repeat_count_dice(Z, N):
    """Repeat dice throwing Z times.
    Note that if Z and N is very large it may be better
    to loop in python to avoid high memory usage.
    If N is very large (probably above 10000 but not I did
    not time) the difference will be neglegible.
    """
    dice = np.random.randint(1, 7, size=(Z, N, 2))
    equal_6 = dice == 6
    one_is_6 = equal_6.any(-1)
    # Unfortunatly count_nonzero does not support the axis paramter
    # at this time (but it can be much faster in new versions)
    count = np.sum(one_is_6, axis=-1)
    return count

counts = repeat_count_dice(10000, 1000)
# This function counts how often each result occurred
# specifically for integers:
binned = np.bincount(counts)
plt.figure()
plt.bar(np.arange(len(binned)), binned)

print("Mean:", counts.mean())
print("Standard deviation:", counts.std(ddof=1))

# 4.
Ns = np.arange(1, 21)
means = np.empty(len(Ns))
stds = np.empty(len(Ns))

for i, N in enumerate(Ns):
```

```
        counts = repeat_count_dice(5000, N) / N
        means[i] = counts.mean()
        stds[i] = counts.std(ddof=1)

    plt.figure()
    plt.errorbar(Ns, means, stds, marker='o')
```

5 Integration

5.1 Lotka-Volterra System

Consider the Lotka-Volterra equation of predator-prey interactions

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = N_1 \cdot (\varepsilon_1 - \gamma_1 N_2), \frac{\mathrm{d}N_2}{\mathrm{d}t} = -N_2 \cdot (\varepsilon_2 - \gamma_2 N_1)$$

This is a system of ordinary differential equations, where $N_1(t)$ is the number of preys and $N_2(t)$ the number of predators. $\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2$ are parameters representing the growth and interaction between preys and predators.

1. Find the fixed points of the system for the parameters $\varepsilon_1 = 1.0, \varepsilon_2 = 1.5, \gamma_1 = 0.1, \gamma_2 = 0.075$.
   - Define a function that returns the growing rates $\frac{\mathrm{d}N_1}{\mathrm{d}t}$ and $\frac{\mathrm{d}N_2}{\mathrm{d}t}$ in an array. In the second part of the exercise, we will use `scipy.integrate.odeint()` to obtain the solution of the ODE system. Therefore, choose the parameters of your function accordingly, $f = f(N, t, \dots)$.
   - Use `scipy.optimize.fsolve` to find the fixed points of the system.
2. Solve the system of equations for the following initial conditions $N_1(0) = 10,\ N_2(0) = 5$.
   - Define a vector containing the time steps of the integration and one for the initial conditions.
   - Solve the system using `scipy.integrate.odeint()`
   - Lower the precision of the integrator until you see a difference after many periods. The integrator has multiple options for precision control. Also the number of steps can have an effect on the result.
3. *Bonus (no solution)*: Also try `scipy.integrate.ode` which provides different integrators but requires manual looping (this is not a big speed issue, integrating in python is pretty slow in any case).

5.1.1 Solution

```
In []: import scipy.optimize as opt
       from scipy import integrate


       def dN_dt(N, t0, eps1, gamma1, eps2, gamma2):
           """Growth rates of the Lotka-Volterra equations:
           """
           dN1_dt = N[0] * (eps1 - gamma1*N[1])
           dN2_dt = -N[1] * (eps2 - gamma2*N[0])
           return np.array([dN1_dt, dN2_dt])

       # Return the growth rates for the initial conditions
       print('Growth rates for the initial values N1 = 10 and N2 = 5')
       print('Parameters are eps1 = 1, gamma1 = 0.1, eps2 = 1.5, gamma2 = 0.075')
       print(dN_dt([10., 5.], 0., 1., 0.1, 1.5, 0.075))

       # Find the fixed points by searching for stable solutions.
       # Try different inital values of N1 and N2 (x0 = [N1, N2]).
```

```
print()
print('Fixed points of the system:')
print(opt.fsolve(dN_dt, x0=[0, 0], args=(0, 1.0, 0.1, 1.5, 0.075)))
print(opt.fsolve(dN_dt, x0=[10, 10], args=(0, 1.0, 0.1, 1.5, 0.075)))

# Check the fixed points. Zero growth rate means that the solution is stable.
print()
print('Growth rates of the fixed points')
print(dN_dt([0., 0.], 0., 1.0, 0.1, 1.5, 0.075))
print(dN_dt([20., 10.], 0., 1.0, 0.1, 1.5, 0.075))

# Solve the system of ODEs by integrating with scipy.integrate.odeint
# First determine time resolution and start values.
times = np.linspace(0, 100, 801)
N_0 = np.array([10, 5])

# Then solve the system for different time resolution.
sol1, info1 = integrate.odeint(dN_dt, N_0, times, args=(1.0, 0.1, 1.5, 0.075),
                               full_output=True)
# Decrease atol and rtol a lot. For more complex systems (or even chaotic ones)
# you will see differences much earlier. The time steps themselves also have
# some influence on the integration precision.
sol2, info2 = integrate.odeint(dN_dt, N_0, times, args=(1.0, 0.1, 1.5, 0.075),
                               atol=0.025, rtol=0.025, full_output=True)

print()
print('Total number of function evaluations')
print('Normal precision:', info1['nfe'][-1])
print('Low precision:', info2['nfe'][-1])

# Compare the two solutions in a plot.
plt.plot(times, sol1[:, 0], label='Prey', color=[0, 0.5, 1], linewidth=2)
plt.plot(times, sol2[:, 0], label='Prey, imprecise', color='b', linewidth=2, alpha=0.4)
plt.plot(times, sol1[:, 1], label='Predator', color=[1, 0.5, 0], linewidth=2)
plt.plot(times, sol2[:, 1], label='Predator, imprecise', color='r', linewidth=2, alpha=0.4)
# Start at time=50
plt.axis([50, 100, 0, 60])
plt.xlabel('Time')
plt.ylabel('Population')
plt.title('Predator-Prey Evolution', size=14)
plt.legend(loc='best', prop={'size':10}, labelspacing=0)
```