COMP 5711: Advanced Algorithms
Fall 2018 Midterm Exam

Name: _____ Student ID: _____

Instructions

1. This is an open-book, open-notes exam.
2. Time limit: 80 minutes.
3. You can write on the back of the paper if you run out of space. Please let us know if you need more scratch paper.

1. (20 pts) Design a data structure to support the following two operations on a dynamic set S of elements:

   - `INSERT`(S, x) inserts x into S.
   - `DELETE-LARGER-HALF`(S) deletes the largest $\lceil |S|/2 \rceil$ elements from S.

   Explain how to implement this data structure so that each operation takes $O(1)$ time amortized. [Hint: The median of n elements can be found in $O(n)$ time.]

2. (30 pts) We want to implement a queue Q with two stacks S_1 and S_2, which support `POP` and `PUSH` operations in $O(1)$ time. You can also check if the stack is empty in $O(1)$ time.

   (a) How would you implement the queue? Specifically, how should the `ENQUEUE`(Q, x) and `DEQUEUE`(Q) operations of the queue be implemented with the two stacks? Note that you are not allowed to use any other data structures.

   (b) Show that the amortized cost of each operation on the queue is $O(1)$.

3. (30 pts) Let S be a set of n students in a class. There are m study groups, denoted as $G_1, \ldots, G_m$. Each student can be in more than one group, and each study group has at least δn students for some $\delta \in (0, 1)$. We would like to select a committee $C \subseteq S$ of small size such that each study group has at least one of its members in the committee, i.e., $C \cap G_i \neq \emptyset$ for all i. Finding such a committee with the smallest size is an NP-hard problem, but can you design a polynomial-time randomized algorithm that, with probability at least $1 - \frac{1}{m}$, finds such a committee of size at most $\frac{2}{\delta} \ln m$?

4. (20pts) Show that the randomized selection algorithm runs in $O(\log n)$ iterations in expectation. One iteration is defined as selecting a pivot, partitioning the array into two parts, and reducing the problem to one part.