COMP 5711: Advanced Algorithm
2019 Fall Semester Midterm Exam Solutions

1. (a) $O(n \log^2 n)$, $O(n \log n)$; (b) $k = \log\log n + O(1)$; (c) Consider any machine. The probability that it receives exactly one job is $k \cdot \frac{1}{k}(1 - \frac{1}{k})^{k-1} \to 1/e$. This is also the expected fraction of machines with exactly one job.

2. When the array is $A = [0, 1, ..., n-k+i-1, n-k+i+1, n-k+i+2, \ldots, n-1]$ for some i, the algorithm will spend $O(n-i)$ time and the potential increases by 1, and the amortized cost cannot be bounded.

   To fix the algorithm, we introduce a global variable m that remembers the largest i such that $A[i] < n-k+i$. Initially, $m = k-1$. The modified algorithm is like this:

Algorithm 1: EnumerateNext

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1  A[m] ← A[m] + 1;
2  if A[m] = n − k + m then
3  |   m ← m − 1;
4  |   enumerate A;
5  |   if m < 0 then report "no more combinations!";
6  else
7  |   for j ← m + 1 to k − 1 do
8  |   |   A[j] ← A[j − 1] + 1;
9  |   m ← k − 1;
10 |   enumerate A;
```

   Define the potential as $\Phi = k - m$. If lines 3–5 are executed, the algorithm spends $O(1)$ time and the potential increases by 1. If lines 7–10 are executed, the algorithm spends $O(k-m)$ time and the potential change is $1 - (k-m)$, and the amortized cost is $O(1)$.

3. Let p^* be the true shortest path. We know that p^* has at most n edges. Consider a particular edge $uv \in p^*$. Let $\bar{u}, \bar{v}$ be the locations of the two points after rounding. Then we have

$$dist(\bar{u}, \bar{v}) \le dist(\bar{u}, u) + dist(u, v) + dist(v, \bar{v}) \le dist(u, v) + 2\sqrt{2}\theta.$$

   So $dist(\bar{p}^*) \le dist(p^*) + 2\sqrt{2}n\theta$.

   Similarly, let p be the path found by Dijkstra's algorithm. We have $dist(p) \le dist(\bar{p}) + 2\sqrt{2}n\theta$. Thus,

$$dist(p) \le dist(\bar{p}) + 2\sqrt{2}n\theta \le dist(\bar{p}^*) + 2\sqrt{2}n\theta \le dist(p^*) + 4\sqrt{2}n\theta.$$

   We want $4\sqrt{2}n\theta \le \epsilon \cdot dist(p^*)$. Since we know $dist(p^*) \ge \sqrt{2}$, so it suffices to have $4\sqrt{2}n\theta \le \epsilon\sqrt{2}$, i.e., $\theta \le \frac{\epsilon}{4n}$.

4. The skip list has at most $\log n$ levels if and only if no element survives at level $\log n$. This probability is

$$\left(1 - \frac{1}{2^{\log n}}\right)^n = \left(1 - \frac{1}{n}\right)^n \le \frac{1}{e}.$$

   So the probability that it has more than $\log n$ levels is at least $1 - \frac{1}{e}$. Thus, the expectation is at least $(1 - \frac{1}{e}) \log n = \Omega(\log n)$.