SKI (a) You can just rent until day n. If you still feel like skiing on day n, you buy one. Let m be the number of days you have skied. If $m < n$, you achieve OPT. If $m \geq n$, you pay $n - 1 + n = 2n - 1$, while $\mathsf{OPT} = n$ (buy skis on day 1).

(b) The adversary would just break your leg right after you have bought the skis. Suppose you buy the skis on day m. Your cost is always $m - 1 + n$ for any m. Now consider OPT. If $m < n$, $\mathsf{OPT} = m$, and $(m - 1 + n)/m \geq 2$. If $m \geq n$, $\mathsf{OPT} = n$, and $(m - 1 + n)/n \geq 2 - 1/n$.

(c) There are many ways to achieve this. One strategy is the following. You rent until day $n/2$ (assuming n is even). If you are still up for skiing, with probability p you buy the skis (the value of p will be decided later); with probability $1-p$ you continue to rent until day n, on which you must buy.

Let m be the number of days you have skied. If $m < n/2$, you achieve OPT. If $n/2 \leq m < n$, you pay m with probability $1 - p$ (didn't buy on day $n/2$), and $n/2 - 1 + n = \frac{3}{2}n - 1$ with probability p (bought on day $n/2$). The expected cost is $(1 - p)m + p(\frac{3}{2}n - 1) < (1 - p)m + p\frac{3}{2}2m = (1 + 2p)m$. In this case, $\mathsf{OPT} = m$. So the competitive ratio $< 1 + 2p$. If $m \geq n$, then $\mathsf{OPT} = n$, and your expected cost is $(1 - p)(2n - 1) + p(\frac{3}{2}n - 1) < (2 - p/2)n$. So the competitive ratio in this case $< 2 - p/2$. Balancing this two cases with $1 + 2p = 2 - p/2$, we get $p = 0.4$. And the expected competitive ratio is always less than 1.8 with this value of p no matter how many days you would ski.

BTW, there is a more complicated randomized algorithm that achieves an expected competitive ratio of $e/(e - 1) \approx 1.58$.

MG (a) As shown in the lecture notes, the error is at most the number of decrement operations. Each decrement operation deletes $k + 1$ items. The stream has N items, and M of them have not been deleted. So the number of decrement operations is at most $(N - M)/(k + 1)$.

(b) Let Δ be the largest error in estimating the frequency of any item. From the result above, we have that $\Delta \leq (N - M)/(k + 1)$, and so

$$M \leq N - \Delta(k + 1). \tag{1}$$

The counter value corresponding to f_i is at least $f_i - \Delta$ (it is considered as 0 if the counter is not maintained). Considering just the t most frequent items, we have

$$\sum_{i=1}^{k} (f_i - \Delta) \leq M. \tag{2}$$

Combining (1) and (2), we have

$$\sum_{i=1}^{t} (f_i - \Delta) \leq N - \Delta(k + 1).$$

Rearranging, we obtain

$$\Delta \leq N^{\mathrm{res}(t)}/(k + 1 - t).$$

Parallel Run the parallel prefix sums algorithm on the array, but replace $+$ with $\vee$ (logical "or"). Assume the results are stored in array D. Then for each $i = 1, .., n$ in parallel, check if $D[i - 1] = 0$ ($D[0]$ is considered to be 0) and $D[i] = 1$. This is true only for one i. We just return that i.