Problem 13.1 The algorithm is simple: We just color every node independently with one of the three colors, each with probability 1/3.

Define the random variable

$$X_e = \begin{cases} 1, & \text{edge } e \text{ is satisfied;} \\ 2, & \text{otherwise.} \end{cases}$$

For any given edge e, the probability that it is satisfied is 2/3. So

$$E[X_e] = \Pr[e \text{ is satisfied}] = \frac{2}{3}.$$

Let Y be the random variable denoting the number of satisfied edges, then by linearity of expectation,

$$E[Y] = E\left[\sum_e Y_e\right] = \sum_e E[X_e] = \frac{2}{3}m \geq \frac{2}{3}c^*.$$

To design a polynomial-time algorithm that produces a 3-coloring that satisfies at least $\frac{2}{3}c^*$ edges with constant probability, we need to lower bound $\Pr[Y \geq \frac{2}{3}m]$. Observing that $Y \geq 0$ and must be an integer, we have

$$\begin{aligned} \frac{2}{3}m &= \sum_{j<2m/3} jp_j + \sum_{j\geq 2m/3} jp_j \\ &\leq \left(\frac{2m}{3} - \frac{1}{3}\right) \sum_{j<2m/3} jp_j + m \sum_{j\geq 2m/3} p_j \\ &\leq \left(\frac{2m}{3} - \frac{1}{3}\right) \cdot 1 + m \sum_{j\geq 2m/3} p_j. \end{aligned}$$

Rearranging, we have $\Pr[Y \geq \frac{2}{3}m] \geq \frac{1}{3m}$. We repeat the algorithm $3m$ times. Then the probability of failure is at most $(1 - \frac{1}{3m})^{3m} < 1/e$.