Efficient Approximate Maximum Inner Product Search over Sparse Vectors

Xi Zhao[1], Zhonghan Chen[1], Kai Huang[2], Ruiyuan Zhang[1], Xiaofang Zhou[1]
[1]Hong Kong University of Science and Technology, Hong Kong
[2]Macau University of Science and Technology, Macau
{xzhaoca,zchenhj}@cse.ust.hk,kylehuangk@gmail.com,{zry,zxf}@cse.ust.hk

***Abstract*—Maximum inner product search (MIPS) in high-dimensional spaces has wide applications but is computationally expensive due to the curse of dimensionality.**

***Index Terms*—Maximum Inner Product Search, Sparse Vectors, Set Overlap Search**

I. Introduction

In recent years, there has been a significant surge in the availability of high-dimensional vector data (*i.e.,* vector representations of objects), primarily due to the proliferation of unstructured data and the remarkable success of deep neural network-based embedding models. These models have demonstrated their ability to embed various types of unstructured data, including videos [10], molecular structural information of drugs [8], and documents [27], into vectors. Subsequently, these vectors can be stored in a vector database to facilitate a wide range of applications, including personalized recommendations and information retrieval. As a result, the *Maximum Inner Product Search (MIPS) problem*, which aims to find a vector (*i.e.,* a representation of an object) from a dataset D that has the maximum inner product with a given query q, plays a crucial role in these applications. As many vectors generated by embedding models exhibit sparsity (*e.g.,* only 43 non-zero entries in the vector generated by the SPLADE model [15], [16]), the MIPS problem for sparse vectors has been widely studied [24], [35], [36], [45], [48], [50]. However, the "Dimensionality Curse" phenomenon makes exact MIPS computation in high-dimensional spaces computationally expensive. In this paper, we study the problem of *Approximate Maximum Inner Product Search (AMIPS)* over sparse vectors, which identifies the top-k "approximately" most relevant objects for the query q over the collection D consisting of sparse vectors. Formally, given a query point $q \in \mathbb{R}^d$, a positive integer k, and an approximation ratio $c \in (0, 1)$, the AMIPS over the sparse vectors D retrieves k points $x_i \in D$ $(1 \leq i \leq k)$ such that the inner product $q^\top x_i \geq c \cdot q^\top x_i^*$ where $x_i{}^*$ is the i^{th} exact MIPS point of q over D.

The AMIPS problem with sparse vectors presents a significant challenge: most vectors are near-orthogonal, which makes it difficult for MIPS algorithms designed for dense vectors to effectively operate on sparse vectors. In contrast to methods for the MIPS problem in dense datasets, existing studies [2], [6], [7], [9], [13], [14], [19], [32] tackle the AMIPS problem for sparse datasets by considering only the non-zero indices and values of each sparse data. These studies directly organize this data using inverted index-based methods [6], [7], [13]. Several evaluation strategies [6], [32], [37] have been developed to filter out certain points by estimating the inner product of two vectors, aiming to reduce the number of inner product computations. However, most of these evaluation strategies are not computationally cheaper than computing the exact inner product, and all points that share common non-zero entries with the query must be considered, resulting in a high query cost of $O(n)$ where $n = |D|$. Additionally, these evaluation strategies often rely on the assumption that the vectors follow a specific distribution, which may not be applicable in the case of dynamic datasets commonly found in modern vector databases. Another limitation is that there is no theoretical guarantee that these methods offer a better query cost than $O(n)$, which hampers their performance on large-scale datasets. As a result, designing an efficient and effective method for the AMIPS problem over large-scale and dynamic sparse datasets, without any underlying constraints or assumptions, remains a significant challenge.

To address the AMIPS problem effectively, we start by considering a simplified version of the problem on a binary-valued dataset in $\{0, 1\}^{d'}$. This simplified version is equivalent to a set overlap search problem, where the overlap of two sets A and B (denoted as $|A \cap B|$) is equivalent to the inner product of their binary representations [43]. In particular, set overlap search is a type of set similarity search problem that is related to other set search problems such as Jaccard similarity search [22], [23] and set containment search [1], [5]. These related problems have mature algorithms for approximate or exact solutions [40], [49], [52], [53], including inverted list-based methods [52] and minHash-based methods [53], which greatly improve the efficiency of set overlap search without any constraints on the dataset and enable sublinear cost solutions. With this in mind, we can explore transforming a real-valued sparse dataset into a binary-valued dataset while preserving the inner product relationships in their original space. By doing so, the AMIPS problem can be completely solved by addressing a set overlap search problem. While some sketch-based methods like BinSketch [37] attempt to transform a sparse vector in $\mathbb{R}^d$ into a binary space for addressing the MIPS problem, they fail to preserve the inner product relationship without introducing significant distortion error.

Motivated by these considerations, we propose a novel

framework called SOSIA to address the AMIPS problem in a sparse vector space. In particular, we present the SOS (Sparse Vector to Set) transformation to convert the AMIPS problem on $\mathbb{R}^d$ into the AMIPS problem on a binary space $\{0,1\}^{d'}$, which can be solved using minHash-based methods. The core of SOS is a transformation function $\mathcal{T}: \mathbb{R}^d \to \{0,1\}^{d'}$ that constructs a binary sketch for each vector in the dataset and query set. Theoretical results demonstrate that the inner product between $\mathcal{T}(o_1)$ and $\mathcal{T}(o_2)$ serves as an unbiased estimator of the inner product between o_1 and o_2 for any two points in $\mathbb{R}^d$. This property enables us to tightly control the error introduced by the transformation. Additionally, $\mathcal{T}$ is a data-independent transformation and supports dynamic datasets. Moreover, SOS offers simplicity in deployment, making it compatible with disk-based, parallel, or distributed settings. This flexibility enhances its adaptability within modern vector databases, enabling efficient processing and storage. We further develop a novel minHash-based index, which can efficiently address the AMIPS problem in the transformed space and achieve high query performance in terms of efficiency and accuracy. In summary, we make the following contributions in the paper.

- We propose a novel framework called SOSIA to address the problem of AMIPS over sparse vectors. To address the challenges posed by sparsity, we develop the SOS transformation, which converts sparse vectors into a binary space while providing an unbiased estimator of the inner product between any two vectors.
- We further develop a novel minHash-based index that enhances the efficiency of answering queries in the AMIPS problem. This indexing method ensures a high query quality with a sublinear query cost.
- We conduct extensive experiments on real-world sparse datasets to validate the effectiveness of SOSIA. The experimental results demonstrate that SOSIA outperforms competitors in terms of both query efficiency and accuracy.

The rest of the paper is organized as follows. Section II reviews related work on MIPS problem and set similarity search problem. Section III presents the problem setting and the basic concepts. Section IV shows the details of SOS sketch and SOSIA framework. Section V provides the proofs of some lemmas and the theoretical guarantee of SOSIA. Section VI presents experimental studies and Section VII concludes the paper.

II. Related Work

A. Maximum Inner Product Search over Dense Datasets

The MIPS problem has been extensively studied in the context of dense datasets. In low-dimensional spaces, exact solutions to the MIPS problem can be achieved using space-partitioning trees, such as the M-tree [25] and cone-tree [38]. But their performance rapidly degrades as the dimensionality d increases due to the curse of dimensionality. In high-dimensional spaces, MIPS problem is mainly solved approximately by locality-sensitive hashing (LSH) methods and graph based method [34], [50]. LSH based methods are the mainstream approximation method for solving the NNS and MIPS problems [3], [12], [18], [24], [36], [42], [50]. Since the inner product is not a metric, asymmetric transformations are applied to convert the MIPS problem into NNS ahead [3], [42], [44]. Then, the MIPS problem is reduced to an NNS problem. However, these transformations either bring distortion errors or result in an imbalance in the transformed dataset, and thus limit the query performance. FARGO [50] proposes a novel transformation named random XBOX transformation that addresses these two issues. Graph-based [31], [34], [46] and learning-based [11], [17], [41] methods are also proposed for MIPS problem. However, both of these methods incur significantly high training or construction costs, which restrict their applicability to large-scale datasets.

These methods [11], [34], [36], [50] are mainly designed for dense vectors, the case where the entries of each vector are almost surely non-zero. In this case, the data points from a real-world dataset (not a random dataset) are usually well-clustered and the local intrinsic dimensionality (LID) [20], [29], [51] of the dataset are much less than its dimensionality d, which make it possible to either partition the nearby data points into several hash buckets or organize the datasets by proximity graphs. Unfortunately, when the points are sparse with very few non-zero entries, these methods hardly port over successfully because the points are near-orthogonal in a sparse high-dimensional space.

B. Maximum Inner Product Search over Sparse Datasets

Traditional approaches for addressing the MIPS problem in sparse datasets heavily rely on inverted index [6], [7], [13], [30], [32], [33]. The inverted index are previously designed for answering set similarity search, especially set overlap search. An inverted index consists of many posting lists where each posting list represents a specific entry of a vector and stores the point IDs and their non-zero values in this entry. By considering the inner product as a weighted set overlap, the problem can be answered exactly by traversing the posting lists of query's non-zero entries because the inner product is the sum of contributions in all entries. Such a naive approach is simple but inefficient for large-scale datasets since nearly all data points are required to be computed in the worst case. Therefore, many evaluation strategies are designed for filtering some points by quickly computing their inner product upper bounds to the query. These evaluation strategies can be categorized into *Document-at-a-time* (DAAT) strategies and *Term-at-a-time* (TAAT) strategies. DAAT strategies, such as WAND, BMW, and VBMW [6], [13], [32], estimate the upper bound of the inner product of a single point appearing in the posting list of a query term before moving to the next point. The point whose upper bound is less than the current found largest inner product will be filtered out; otherwise, we compute its real inner product to the query and update the current largest inner product. The efficiency of these methods depends on how tight an upper bound is. WAND [6] adopts the maximal nonzero values in each posting list as an upper bound

TABLE I: List of Key Notations.

Notation	Description
$\mathbb{R}^d$	d-dimensional vector space
$\mathcal{D}$	The dataset
n	The cardinality of dataset
x	A data point
l	The base of SOS transformation
$x.ind$	x's non-zero indices
$x.val$	x's non-zero values
q	A query point
$q^\top x$	The inner product between q and x
$\mathcal{N}_I(q)$	The point that has the largest inner product to q in $\mathcal{D}$
$\|x\|$	The L2 norm of a point x in $\mathbb{R}^d$
$\|x\|, \|x_b\|$	The L1 norm of a point x in $\mathbb{R}^d$ or a point x_b in $\{0,1\}^d$
A, B	A set

of this posting list. Then, the inner product upper bound of a point is estimated by accumulating the sum of upper bounds in the common posting lists of this point and query point. To obtain a tighter upper bound, BMW [13] and VBMW [32] organize the terms in a posting list by several smaller-size blocks and define an upper bound for each block rather than the entire posting list, which greatly reduces the difference between the upper bound and the exact values and thus make it possible to filter out more points. In addition, some methods represent sparse vectors by the sketches such that the inner product of sketches approximates the inner product of original vectors [2], [9], [19]. Then, we estimate the upper bounds of the inner product quickly by computing the inner product between two sketches. TAAT strategies, such as Sinnamon [7], process query terms one by one and accumulate a partial inner product as the contribution of each query term is computed. To obtain a tight estimate of the exact inner product and facilitate filtering some points, Sinnamon accesses the posting lists with the decreasing order of the query term's value since a large value has a higher impact on the inner product. When accessing the posting lists, Sinnamon also adopts a low-dimensional sketch like that in BinSketch [37] for estimating the inner product of two points by accessing only a part of posting lists. Although these methods effectively reduce the number of full computations, the computational cost associated with computing these upper bounds remains substantial, often comparable to that of computing the exact inner product. Moreover, in most cases, it is necessary to compute the upper bounds for all points that share common non-zero entries with the query, resulting in a query cost that is not significantly better than $O(n)$.

III. Preliminaries

In this section, we introduce the AMIPS problem and preliminaries. Frequently used notations are in Table I.

A. Problem Definition

Definition 1 (Maximum Inner Product Search)**.** *Given a dataset $\mathcal{D} = \{x_1, x_2, \cdots, x_n\} \subseteq \mathbb{R}^d$ with $|\mathcal{D}| = n$ and a query point $q \in \mathbb{R}^d$, the maximum inner product search (MIPS) returns a point $\mathcal{N}_I(q) = x^* \in \mathcal{D}$ such that*

$$\mathcal{N}_I(q) = \arg\max_{x \in \mathcal{D}} q^\top x. \tag{1}$$

Definition 2 (Approximate Maximum Inner Product Search)**.** *Given a dataset $\mathcal{D} \subseteq \mathbb{R}^d$ with $|\mathcal{D}| = n$, a query point q, an approximation ratio $c \in (0,1)$ and a positive integer k, the approximate maximum inner product search ((c,k)-AMIPS) returns k points $x_1, \ldots, x_k$ that are sorted in descending order w.r.t. their inner products to q. If o_i^* is the i-th maximum inner product of q in $\mathcal{D}$, it satisfies $q^\top x_i \geq c \cdot q^\top x_i^*$.*

Remark 1. *We denote c-AMIPS problem as the (c,k)-AMIPS problem at $k = 1$.*

For a sparse vector $x \in \mathbb{R}^d$, we only record its non-zero information as a tuple $x = \langle x.ind, x.val \rangle$ where $x.ind = \{j \mid x[j] \neq 0\}$ is the non-zero indices and $x.val = \{x[j] \mid x[j] \neq 0\}$. For convenience, all indices begin at 0 rather than 1. We use $x.nnz$ to denote the size of $x.ind$ and $x.val$ and use $x.ind[j]$ and $x.val[j]$ to denote the j-th non-zero entry in x, *i.e.,* $x[x.ind[j]] = x.val[j]$. When x is a binary-valued data, it can be represented by only $x.ind$. Since $x.ind$ is a set, we call the vector x as the binary representation of the set $x.ind$ and call x, ind as the sketch of x. The inner product of two binary vectors is related to the set overlap.

Definition 3 (Set Overlap)**.** *The overlap of two sets A and B is the size of their intersection,* i.e., $s_O(A,B) = |A \cap B|$.

We employ the following fact to establish the connection between inner product and set overlap.

Fact 1. *For two sets $A, B \subset U = \{e_1, e_2, \ldots, e_d\}$, the binary representation of A and B are two vectors $a, b \in \{0,1\}^d$ and $a^\top b = |A \cap B|$.*

Example 1. *Consider the dataset D shown in Fig. 1 where $D = \{x_0 = (0,0,0.7,0,0),\ x_1 = (0,0.2,0,0,0.3),\ x_2 = (0,0.5,0,0,0), x_3 = (0.6,0,0.1,0,0.3)\} \subset \mathcal{R}^5$. x_2 can be represented as $x_1 = \langle [1,4], [0.2,0.3] \rangle$, where $x_1.ind = [1,4]$ and $x_1.val = [0.2, 0.3]$. Similarly, a query point $q = (0, 0.2, 0, 0, 0.5)$ can be represented as $q = \langle [1,4], [0.2,0.5] \rangle$. The exact MIPS result for this query is x_1 ($q^\top x_1 = 0.19$). When $c = 0.5$ and $k = 2$, the (c,k)-AMIPS returns any two points in x_1, x_2, x_3 as correct results. Then, given a binary vector $x_b = \{0,1,0,0,1\}$, we can represent it as a set $\{[1,4]\}$.*

Since a binary vector x is totally equivalent to the set $A = x.ind$. We do not distinguish between a binary vector and a set longer in this paper.

B. Jaccard Similarity and MinHash

Definition 4 (Jaccard Similarity)**.** *The Jaccard Similarity of two sets A and B is defined as $s_J(A,B) = |A \cap B| / |A \cup B|$.*

A basic observation between Jaccard similarity and set overlap is

$$s_J(A,B) = \frac{s_O(A,B)}{|A| + |B| - s_O(A,B)}, \tag{2}$$

Therefore, given the Jaccard similarity between two sets, we can efficiently estimate their overlap. Since Jaccard similarity is normalized to the range [0, 1], it is easier to estimate using

techniques like MinHash, as compared to directly estimating set overlap.

Definition 5 (MinHash)**.** *Given a universal collection of sets U, a family of MinHash functions $\mathcal{H} = \{h : U \to Z\}$ maps the members of U into integers such that for any two sets $A, B \in U$, the collision probability of A and B equals to their Jaccard similarity,* i.e.,

$$\Pr[h(A) = h(B)] = s_J(A, B). \tag{3}$$

The goal of MinHash is to estimate the Jaccard similarity between a query set and the set in a collection $\mathcal{D} \subset U$ quickly without explicit computing the Jarccard. By adopting multiple MinHash functions $h_1, h_2, \cdots, h_s$ from H, we can compute an unbiased estimation of the Jaccard similarity of two sets by counting the number of their collisions among s MinHash functions.

C. Locality Sensitive Hashing index

When answering set Jaccard similarity search problem via a number of MinHash functions, a common strategy is to build Locality Sensitive Hashing (LSH) index. LSH function is a kind of hash functions based on specific metric, which is defined as:

Definition 6 (Locality Sensitive Hashing (LSH) [18], [47])**.** *Given a metric space $\mathcal{M} = (X, dist)$ where X is a collection of points and $dist$ is a function $dist : X \times X \to \mathbb{R}$, a distance r and an approximation ratio $c > 1$, a family of hash functions $\mathcal{H} = \{h : X \to Z\}$ is called (r, cr, p_1, p_2)-locality-sensitive, if for $\forall o_1, o_2 \in X$, it satisfies both conditions below:*

- *If $dist(o_1, o_2) \leq r, \Pr[h(o_1) = h(o_2)] \geq p_1$;*
- *If $dist(o_1, o_2) > cr, \Pr[h(o_1) = h(o_2)] \leq p_2$,*

where $h \in \mathcal{H}$ is chosen at random, p_1, p_2 are collision probabilities and $p_1 > p_2$.

MinHash functions are locality-sensitive based on the Jaccard distance, which is defined as:

Definition 7 (Jaccard Distance [26])**.** *The Jaccord distance of two sets A and B is $\delta_J(A, B) = 1 - s_J(A, B)$.*

Jaccard distance is a metric that satisfies the triangle inequality [26]. Under the metric space $\mathcal{M} = (U, \delta_J)$, a MinHash function is $(r, cr, 1 - r, 1 - cr)$-locality-sensitive for any $0 < r < 1/c$. So we can adopt LSH index to answer approximate Jaccard similarity search, like other nearest neighbor search (NNS), such as in Euclidean space. An LSH function is insufficient to answer NNS well since p_2 is too large in single LSH functions such that there are many false negative points, *i.e.,* the points are too far away from the query point but still have the same hash value with it. (K, L)-LSH index is a common approach to address this issue [18], [47]. In (K, L)-LSH index, to reduce the false negative ratio p_2, we usually concatenate K LSH functions $h_0, h_1, \cdots, h_{K-1} \in \mathcal{H}$ as

$$g(o) =< h_0(o), h_1(o), \cdots, h_{K-1}(o) >$$

and the points that agree on all K LSH functions are regarded as collisions and mapped into the same hash bucket. In the query phase, we only check the points that collide with the query point q. In this strategy, the false negative ratio are decreased as p_2^K, which greatly reduce the number of points to be checked. On the other hands, the true positive ratio are decreased from p_1 into p_1^K, which will also reduce the probability that a real nearest neighbor are found and degrade the query accuracy. A practical approach to address this issue is to repeat such a procedure L times independently to obtain enough candidate points in total L hash buckets. By properly choosing the values of K and L, an approximate nearest neighbor search can be solved in sublinear query cost with quality guarantee.

IV. The SOSIA framework

In this section, we present our framework called SOSIA to address the (c, k)-AMIPS problem in a sparse vector space, as depicted in Fig 1. SOSIA involves several key steps. Firstly, SOSIA employs the SOS transformation (refer to Section IV-A) to convert points in the sparse dataset into binary vectors. These binary vectors are considered as sets, which are mapped into a hash bucket in each of m hash tables using m independent MinHash functions (refer to Section IV-B). When a query q is received, it is also transformed into a binary vector q_b using the SOS transformation. Subsequently, it performs a nearest neighbor search based on Jaccard distance by utilizing the hash tables to obtain T candidates for the query q (refer to Section IV-C). During the query processing, we notice that the existing minHash LSH indexes cannot handle the issue that the collision probability varies greatly with the query points so that some queries obtain too many candidates while other queries collide with a few points. To address this issue, a novel counting&refining strategy is designed for adaptively finding no more than T candidates for different query points efficiently.

A. SOS Transformation

Given a base l, SOS transformation converts a point x in dataset $\mathcal{D} \subset \mathbb{R}^d$ into a binary vector x_b in $\{0, 1\}^{d'}$ where $d' = d \cdot l$, *i.e.,*SOS is a transform $\mathcal{T} : \mathbb{R}^{+d} \to \{0, 1\}^{d'}$. Each entry $x[i]$ in $\mathbb{R}^d$ is mapped to l entries at the range $x_b[il : il + l - 1]$ in $\{0, 1\}^{d'}$ independently. The larger $x[i]$ is, the more likely the entries of $x_b[il : il + l - 1]$ are to be 1. To be simplified, we assume the maximal entries in $\mathcal{D}$ is 1; otherwise, we normalized $\mathcal{D}$ to make its maximal entries

$$M = \max_{x \in \mathcal{D}} \max_{0 \leq i < d} x[i]$$

to be 1, which does not effect the MIPS result of any query point. The detail of SOS is shown in Alg. 1. For a non-zero entry $x[x.ind[j]] = x.val[j]$, each of l entries in $x_b[il : il + l - 1]$ will be set as 1 with the probability $x.val[j]$ (Line 5). The output of Alg. 1 is a binary vector in $x_b \in \{0, 1\}^{d \cdot l}$.

Example 2.

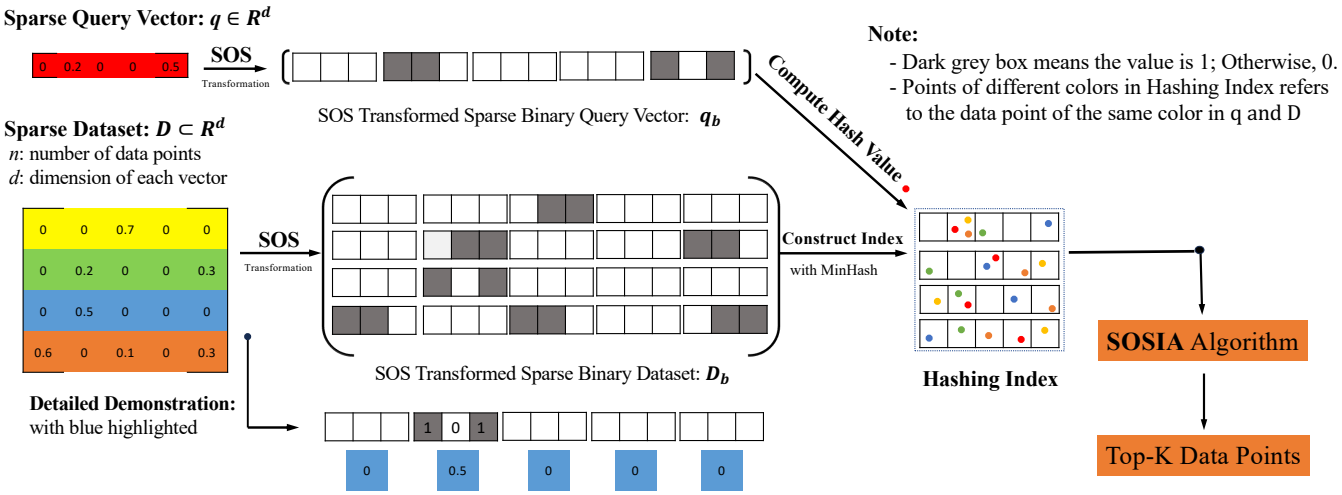

Fig. 1: SOSIA Framework

Algorithm 1: SOS Transformation

Input: A normalized sparse point $x \in \mathbb{R}^{+d}$ and a positive integer l

Output: A binary vector $x_b \in \{0,1\}^{d \cdot l}$

$S \leftarrow \varnothing$;
for $j \leftarrow 0$ *to* $x.nnz$ **do**
  for $j \leftarrow x.ind[j] \cdot l$ *to* $x.ind[j] \cdot l + l$ **do**
    Genarate a random variable r from the uniform distribution $U(0,1)$;
    if $r < x.val[j]$ **then**
      $S \leftarrow S \cup \{j\}$;
return S;

After adopting SOS to process the query point q and each data point x in $\mathcal{D}$, we can answer the AMIPS problem in $\mathcal{D}$ by considering the AMIPS of $\mathcal{T}(q)$ in a binary-valued dataset $\mathcal{D}' = \{\mathcal{T}(x)|x \in \mathcal{D}\}$ since $\mathcal{T}(q)^\top \mathcal{T}(x)$ satisfies the following lemma:

Lemma 1. *Given a query point q and a data $x \in \mathcal{D}$ where $q_b = \mathcal{T}(q)$ and $x_b = \mathcal{T}(x)$, then $q_b^\top x_b/l$ is an unbiased estimation of $q^\top x$.*

The proof of this lemma is provided in Section V-A. This lemma guarantees that SOS will not incur any distortion error in terms of the inner product relationship of points in the original space, based on which we can estimate the value of $q^\top x$ as tight as desired, *i.e.,* make $q_b^\top x_b/l$ be as close to $q^\top x$ as possible. Intuitively, enlarging l forces $q_b^\top x_b/l$ to be closer to $q^\top x$. We formulate this conclusion by computing the variance of $q_b^\top x_b/l$.

Lemma 2. *The variance of $q_b^\top x_b/l$ is*

$$\mathrm{Var}[q_b^\top x_b/l] = \frac{1}{l}\sum_{j=0}^{d-1} q[j]x[j](1 - q[j]x[j]).$$

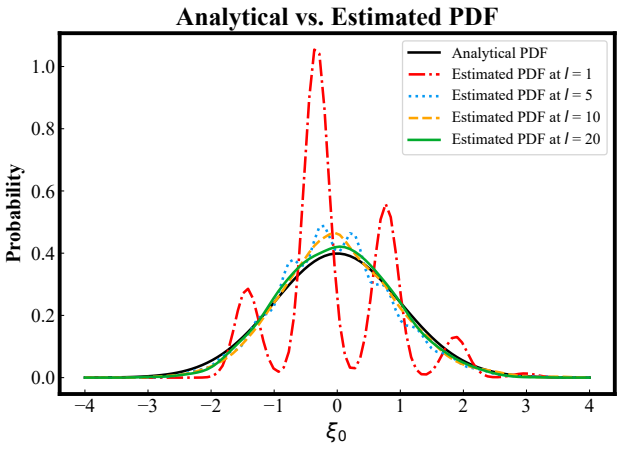

Fig. 2: The distributions of ξ_0 at different l

The proof of this lemma is provided in Section V-A. Since the value of $\sum_{j=0}^{d-1} q[j]x[j](1 - q[j]x[j])$ depends only on the q and x themselves, the variance of $q_b^\top x_b/l$ can be reduced by enlarging l, which allows us to estimate the value of $q^\top x$ more tightly by choosing a larger base l.

To further analyze $q_b^\top x_b/l$, we can find

$$\begin{aligned} q_b^\top x_b/l =& 1/l \cdot \sum_{j=0}^{dl-1} q_b[j]x_b[j] \\ =& 1/l \sum_{i=0}^{l-1} \cdot \sum_{j=0}^{d-1} q_b[jl+i]x_b[jl+i]. \end{aligned}$$

l numbers of $Y_i = \sum_{j=0}^{d-1} q_b[jl+i]x_b[jl+i]$ follows the same distribution because each $q_b[jl+i], 0 \le i < l$ is set as 1 in the same way. And $q_b^\top x_b/l$ is the average of Y_is. By the central limit theorem [1], the distribution $q_b^\top x_b/l$ tends towards

[1]https://en.wikipedia.org/wiki/Central_limit_theorem

a normal distribution when l is large enough . So, we adopt the normal distribution to decipt the distribution of $q_b^\top x_b/l$.

Lemma 3. *Given a query point q and a data $x \in \mathcal{D}$ where $q_b = \mathcal{T}(q)$ and $x_b = \mathcal{T}(x)$, we denote $\xi = q_b^\top x_b/l$, $\mu = \mathrm{E}\,\xi$ and $\sigma = \mathrm{Var}\,\xi$. When l is large enough, $\xi_0 = \frac{\xi-\mu}{\sigma} \sim \mathcal{N}(0,1)$.*

We plot the probability density functions (PDF) of ξ_0 at different l where the black line is the PDF of the standard normal distribution $\mathcal{N}(0,1)$. When $l = 1$, the variance of ξ_0 is a little large and the PDF has a high bias from the $\mathcal{N}(0,1)$. But when $l = 5, 10$ or 20, the distribution of ξ_0 has been very close to $\mathcal{N}(0,1)$. Therefore, it is reasonable to approximate the distribution of ξ_0 by $\mathcal{N}(0,1)$ at a not small l.

Algorithm 2: Indexing in SOSIA

Input: A normalized sparse dataset $\mathcal{D}$, a query point q and an approximate ratio c
Output: Hash Tables

1 Determine the value of m based on Eq. 4;
2 Generate m minHash functions $h_1, h_2, \ldots, h_m$;
3 $\mathcal{B}[1:m] \leftarrow \varnothing$;
4 **for** $x \in \mathcal{D}$ **do**
5 $x_b = \mathcal{T}(x)$;
6 $id \leftarrow x_b$'s point id;
7 **for** $j \leftarrow 0$ *to* $m-1$ **do**
8 Insert $\langle h_j(x_b), (id, |x_b|)\rangle$ into $\mathcal{B}[j]$;
9 **for** $j \leftarrow 0$ *to* $m-1$ **do**
10 **for** *a hash bucket B in $\mathcal{B}[j]$* **do**
11 Sort the tuples $\langle id, |x_b|\rangle$s by the descending order of $|x_b|$;
12 **return** $\mathcal{B}$;

B. Indexing in SOSIA

After obtaining transformed dataset $\mathcal{D}' = \{\mathcal{T}(x) | x \in \mathcal{D}\}$ and the query point $q_b = \mathcal{T}(q)$, we conduct the set overlap search on $\mathcal{D}'$ by using minHash [53] for q_b and regard the returned points as candidates of q's top-k MIPS result, which is totally equivalent to the AMIPS search on $\mathcal{D}'$. We call such a framework as SOSIA. Unlike (K, L)-LSH indexes that build L K-dimensional hash tables, SOSIA builds m one-dimensional hash tables to facilitate determining the collision conditions based on different query points adaptively and dynamically. Algorithm 2 describes the indexing phase of SOSIA. First, we need to determine the number of hash functions, m, for answering the (c, k)-MIPS problem. A larger m enables us to find higher-quality candidates but brings a higher query cost and space consumption. Based on query strategy, we choose the smallest m that guarantees a correct (c, k)-MIPS result with high probability, m is computed by

$$\begin{cases} w = \sqrt{\dfrac{4(1-\gamma)}{c(1-c\gamma)}} > 1 \\ t = \dfrac{1+cw}{1+w} \\ m = \dfrac{3c(c-\gamma)(t-\gamma)^2}{\gamma(t-c)^2} \cdot \ln\dfrac{2n}{T}, \end{cases} \tag{4}$$

where $\gamma \in (0,1)$ is a parameter to control the minimal inner product to be searched in our index and T is the maximal number of points to be checked. Such an m guarantees that SOSIA index returns a correct c-MIPS with a constant probability $1/2 - 1/e$ when $q^\top x^* \geq \gamma \|q^\top\| \|x^*\|$ where $x^* = \mathcal{N}_I(q)$ is the maximal inner product result of q in $\mathcal{D}$, as analyzed in Section V-A. The reason why introducing a γ is as following: Let $s_I^* = q^\top \mathcal{N}_I(q)$ be the maximal inner product, when s_i^* too small compared to the norm of q, Any point in the dataset will not be similar to the query point q in terms of the inner product, making it impossible to distinguish the points in the dataset via minHash. So, we only guarantee a correct (c, k)-MIPS result when Then, m minHash LSH indexes are adopted for indexing the points x_b transformed by SOS. For a term $\langle key, value\rangle$ stored in each hash table, the *key* is the hash value of the point x_b and the *value* is a tuple consisting of the x_b's identity and norm. Storing the norm $|x_b|$ is for restoring the value of $s_O(q_b, x_b)$ by $s_J(q_b, x_b)$ since minHash is designed for estimating $s_J(q_b, x_b)$ and we have:

$$s_O(q_b, x_b) = \frac{|q_b| + |x_b|}{1 + 1/s_J(q_b, x_b)}. \tag{5}$$

This equation also indicates that a larger $|x_b|$ are more likely to yield a larger $s_O(q_b, x_b)$. So, in each hash bucket *i.e.,* the hash terms with the same keys, we will sort the points by the descending order of their norms to quickly find the points with a large norm at first in the query phase, as shown in Line 11.

Example 3.

C. Search in SOSIA

When a query q comes, we also transform it into the binary vector q_b via SOS and then compute its m hash values. To find the candidate points with a large set overlap to q_b, a novel search algorithm based on collision counting is designed to quickly find enough number of high-quality candidates with a small size of indexes. The search strategy in SOSIA is quite different compared to the (K, L)-LSH indexes. In (K, L)-LSH indexes, there are L numbers of K-dimensional hash tables and the points that collide with the query point in at least one of L hash tables will be considered as a candidate and will be checked. For each data point, the collision condition of each K-dimensional hash tables is that its K hash functions are all evaluated to be the same with the hash value of query points. While such a strategy is efficient to answer an approximate nearest neighbor search problem in Euclidean space, it is inefficient and infeasible to answer an approximate Jaccard similarity problem due to the difference between the minHash function and the LSH functions under Euclidean space.

Algorithm 3: Search in SOSIA

Input: A dataset $\mathcal{D}$, the SOSIA index $\mathcal{B}$, T, c and k
Output: At most k points in $\mathcal{D}$

$q \leftarrow \frac{q}{\max q[i]}$, $q_b \leftarrow \mathcal{T}(q)$;
$l \leftarrow$ the base of SOS; (see Alg. 1)
Determine the value of t based on Eq. 4;
$I_0 \leftarrow \sqrt{\|q\|}$, $I_t \leftarrow 0$,$I \leftarrow I_0$;
Compute q_b's minHash values $h_1(q_b), \ldots, h^*_m(q_b)$;
$H \leftarrow \varnothing$,$R \leftarrow \varnothing$,$cost \leftarrow 0$;
while $I_t < cI$ *and* $cost < T + k$ **do**
  $x_b \leftarrow$ the first unseen point with largest norm in $\mathcal{B}[j].at(h_j(q_b)), (1 \le j \le m)$;
  if *no points is unseen* **then**
    Break;
  $\alpha \leftarrow |\{j | h_j(q_b) = h_j(x_b)\}|$;
  $s_O \leftarrow \frac{|q_b| + |x_b|}{1 + m/\alpha}$;
  Update(x, s_o);
while $I_t < cI$ *and* $cost < T + k$ **do**
  if H *is empty* **then**
    Break;
  $\langle x, s_O \rangle \leftarrow H.top()$;
  if $s_O / l < tI$ **then**
    $I \leftarrow cI$, **continue**;
  $H.pop()$;
  $x \leftarrow$ the point reprsented by x_b in $\mathcal{D}$;
  $I_x = q^\top x$;
  $cost \leftarrow cost + 1$;
  $e \leftarrow \{x, I_x\}$;
  $R \leftarrow R \cup \{e\}$;
  if $|R| = k + 1$ **then**
    Remove the element with the smallest I_x in R;
    $I_t \leftarrow \min\{e.I_x | e \in R\}$;
return R;

A typical LSH defined in the Euclidean space $\mathbb{R}^d$ is as follows [12]:

$$h^{L2}(x) = \left\lfloor \frac{a^\top x + b}{w} \right\rfloor, \tag{6}$$

where a is a d-dimensional vector whose entries are chosen independently from the standard normal distribution, w is a pre-defined integer and b is a real number chosen uniformly from $[0, w)$. The flexibility of such a hash function is that we can tune the collision probability for any two points by varying the value of w. A larger w enables a more number of points agreeing on a hash value and enlarging the collision probability for any two points. Moreover, no matter how far away are two point from each other, they are able to be mapped into the same hash bucket by choose a large enough w. This enables LSH functions in Euclidean space to solve the approximate nearest neighbor search problem with varied distance between query points and their exact nearest neighbor.

Algorithm 4: ***Update***(x, s_o)

Input: x_b, q, l, t, I, $cost$, c, k, H, and R
Output: k points in $\mathcal{D}$

if $s_O / l \ge tI$ **then**
  $x \leftarrow$ the point reprsented by x_b in $\mathcal{D}$;
  $I_x = q^\top x$;
  $cost \leftarrow cost + 1$;
  $e \leftarrow \{x, I_x\}$;
  $R \leftarrow R \cup \{e\}$;
else
  $x \leftarrow$ the point reprsented by x_b in $\mathcal{D}$;
  $H \leftarrow H \cup \{\langle x, s_o \rangle\}$;
if $|R| = k + 1$ **then**
  Remove the element with the smallest I_x in R;
  $I_t \leftarrow \min\{e.I_x | e \in R\}$;

However, these conditions are not satisfied by a minHash function. A minHash function h, defined for the dataset $\mathcal{D}$, is usually based on a random permutation of the elements of the set $U = \cup_{A \in \mathcal{D}} A$. Let $P = \{e_1, e_2, \ldots, e_d\}$ be a random permutation of U, $h(A)$ is the minimal position of A's elements in P, *i.e.*, $h(A) = \min\{j \mid P[j] \in A\}$. Since no hyper-parameter like w in h^{L2} is used for tuning the collision probability of two points, we are unable to enlarge the collision probability of two sets when their Jaccard similarity is fixed. So, it becomes extremely difficult to choose a static K and L for varied query points. Specifically, denote $\mathcal{N}_J(q)$ as the set that has the maximal Jaccard similarity to the query q in the dataset $\mathcal{D}$, a large K enables us to find the good results for those queries q having a large $s_J(q, \mathcal{N}_J(q))$ but make the queries q having a small $s_J(q, \mathcal{N}_J(q))$ hard to find any candidate. While a small K enables us to find the good results for those queries q having a small $s_J(q, \mathcal{N}_J(q))$ but make the queries q having a small $s_J(q, \mathcal{N}_J(q))$ find too many candidates and increase the query cost. To address this dilemma, we adopt a dynamic counting based strategy for adaptively addresssing varied query point.

Alg. 3 describes the search processing in SOSIA. Although the LSH method cannot solve a c-MIPS problem directly, it can solve the problem by resolving a sequence of threshold-based (I, c)-MIPS problem by decreasing the inner product threshold I [21], [50]. Therefore, to answer a c-MIPS problem, we execute a sequence of (I, c)-MIPS problems as defined below.

Definition 8 ((I, c)-MIPS [28])**.** *Given a threshold I and an approximation factor $c < 1$, the (I, c)-MIPS for a query point q over dataset $\mathcal{D}$ returns the following result:*

1) *If at least one point x exists in $\mathcal{D}$ such that $q^\top x > I$, it returns a point x' in $\mathcal{D}$ such that $q^\top x' > c \cdot I$;*
2) *If no point x exists in $\mathcal{D}$ such that $q^\top x > c \cdot I$, it returns nothing.*

So, we execute a sequence of (I, c)-MIPS problems by

beginning with $I = \sqrt{\|q\|}$ (Line 4), which is the maximal value of $q^\top \mathcal{N}_I(q)$ since all entries have been normalized into $[0, 1]$ in $\mathcal{D}$. I_t is used for recording the current found k-th largest inner produce value. We initialize a max-heap H to store the all counted points and a min-heap R with size no more than k to store the found best k results so far. We divide the remaining search processing into two stages – the *counting stage* and *refining stage*.

Counting stage (Lines 7-13). In this stage, we count the number of collisions between $q_b = \mathcal{T}(q)$ and the points in m hash buckets where q_b falls by the decreasing order of the points' norm (Line 7). Since the points in a hash bucket has been sorted based on their norms, it only takes $O(\log n)$ time to obtain the first unseen point x_b that has the largest norm in q_b's m hash buckets by maintaining a max-heap that stores the first unseen point with largest norm in q_b's each hash bucket. For such a point x_b that collides α times with q_b, its estimated Jaccard similarity to q_b is α/m and we will compute its set overlap s_o to q_b based on Eq. 5. Next, we updating the results by inserting x_b into either H or R based on the condition $s_o/l > tI$ where t is a threshold computed by Eq. 4. As shown in Lemma 4 defined below, s_o/l is an unbiased estimator of $q^\top x$. The proof of Lemma 4 is provided in Section V-A.

Lemma 4. *Given a set space X and a family of MinHash functions $\mathcal{H} = \{h : X \to Z\}$, $h_0, h_1, \ldots, h_{m-1}$ are m min-Hash functions that are drawn independently from $\mathcal{H}$. For two points $q, x \in \mathbb{R}^d$, denote $\alpha = |\{j | h_j(\mathcal{T}(q)) = h_j(\mathcal{T}(x))\}|$. If $\alpha > 0$, then*

$$s_o/l = \frac{|\mathcal{T}(q)| + |\mathcal{T}(x)|}{(1 + m/\alpha)l}, \tag{7}$$

is an unbiased estimator of $q^\top x$ where $\mathcal{T}$ is the SOS transformation with the base l.

Alg. 4 describes the details of updating the results: When $s_o/l > tI$, $q^\top x$ is believed to be larger than cI with a high probability. We then compute the value of $q^\top x$ and put it in the result set R. Otherwise, $q^\top x$ is believed to be smaller than cI with a high probability and we insert it into H without computing $q^\top x$ in this stage. We keep record the k-th largest found result so far as I_t when R has more than k points. The counting stage terminates when one of the following three conditions satisfies:

T1. All points in q_b's m hash buckets are counted (Line 9).
T2. I_t is no smaller than cI (Line 7).
T3. There are $T + k$ points x that $q^\top x$ is computed (Line 7).

When **T2** is satisfied, the algorithm successfully returns k correct (I, c)-MIPS results. Since the current $I = \sqrt{\|q\|}$ is the maximal possible inner product between q and any point in $\mathcal{D}$, the found results are correct (c, k)-MIPS results. When $T3$ is satisfied, the algorithm has checked $T + k$ points but less than k satisfied (I, c)-MIPS result are found. In this case, let I_k^* be the k-th exact largest inner product between q and the point in $\mathcal{D}$, I_k^* is believed to be less than I_t/c^2 with high probability, *i.e.,* the algorithm has found correct (c^2, k)-MIPS results. We will prove this statement in Section V-A. When **T1** is satisfied, the algorithm neither checks enough number of candidates, *i.e.,* $T + k$ candidates, nor has found k correct (I, c)-MIPS result. In this case, we requires to conduct another (cI, c)-MIPS result with decreasing the threshold from I to cI in the next *refine stage*.

Refining stage (Lines 14-28). In this stage, we attempt to extract more candidates x from H whose estimated inner product to q, s_o/l, exceed cI. If no such point x exists but the termination conditions **T2** and **T3** are not satisfied, we keep decreasing the inner product threshold from I to cI. The counting stage terminates when **T2**, **T3** and **T4** satisfies. **T2** and **T3** are same as that in the counting stage and **T4** is:

T4. The max-heap H becomes empty (Line 15).

When the *refining stage* is terminated by **T4**, it indicates we fail to find enough number of candidates from m minHash indexes. In this case, it is uncertainty whether we have found correct (c^2, k)-MIPS results. But by setting m based on Eq. 4, we ensure this case will happen only when $I_k^* < \gamma\sqrt{\|I_0\|}$ where γ is a parameter defined in Eq. 4 with a high probability. In other word, when I_k^* is larger than this threshold, we would have found it with a high probability in one of two stages. When the *refining stage* is terminated by **T2** or **T3**, we guarantee a correct (c^2, k)-MIPS result as analyzed in the *counting stage*. The correctness of these statements will be proven in Section V-A.

Example 4.

V. Theoretical Analysis

In this section, we first provide the proofs of Lemmas 1, 2 and 4. Then, we demonstrate the quality guarantee and query cost of SOSIA.

Lemma 5. *Given $\mu > 0$ and $\sigma > 0$, the variable $p \sim \mathcal{N}(\mu, \sigma^2)$ (to be simplified, you can think $0 < p < t$, (t is a constant) always satisfies) and the variable x follows the 0-1 distribution $\mathcal{B}(1, p/t, 1 - p/t)$, what is the distribution of x with parameters t, μ and σ?*

A. Some Proofs

We first prove Lemma 1 that reveals $q_b^\top x_b/l$ is an unbiased estimator of $q^\top x$.

Proof. The expectation of $q_b^\top x_b/l$ is

$$\begin{aligned}\mathrm{E}[q_b^\top x_b/l] =& 1/l \cdot \mathrm{E}\left[\sum_{j=0}^{dl-1} q_b[j]x_b[j]\right] \\ =& 1/l \cdot \mathrm{E}\left[\sum_{j=0}^{d-1}\sum_{m=0}^{l-1} q_b[jl+m]x_b[jl+m]\right].\end{aligned}$$

Since all the entries of q_b and x_b are set as 1 independently, we have:

$$\mathrm{E}[q_b^\top x_b/l] = 1/l \cdot \sum_{j=0}^{d-1}\sum_{m=0}^{l-1} \mathrm{E}[q_b[jl+m]]\,\mathrm{E}[x_b[jl+m]].$$

$q_b[jl+m]=1$ incurs when the random number $r \sim U(0,1)$ is less than $q[j]$. Since $0 \le q[j] \le 1$, $\mathrm{E}[q_b[jl+m]] = \Pr[q_b[jl+m]=1] = q[j]$. Similarly, $\mathrm{E}[x_b[jl+m]] = x[j]$. So,

$$\mathrm{E}[q_b^\top x_b/l] = 1/l \cdot \sum_{j=0}^{d-1}\sum_{m=0}^{l-1} q[j]x[j] = \sum_{j=0}^{d-1} q[j]x[j] = q^\top x,$$

which indicates $q_b^\top x_b/l$ is an unbiased estimation of $q^\top x$. □

Then, we compute the variance of $q_b^\top x_b/l$ for proving Lemma 2.

Proof. The variance of $q_b^\top x_b/l$ is

$$\begin{aligned}\mathrm{Var}[q_b^\top x_b/l] =& 1/l^2 \cdot \mathrm{Var}\left[\sum_{j=0}^{dl-1} q_b[j]x_b[j]\right] \\ =& 1/l^2 \cdot \sum_{j=0}^{d-1}\sum_{m=0}^{l-1} \mathrm{Var}\left[q_b[jl+m]x_b[jl+m]\right].\end{aligned}$$

The variable $q_b[jl+m]x_b[jl+m]$ follows a binomial distribution and $\Pr[q_b[jl+m]x_b[jl+m]=1] = \Pr[q_b[jl+m]=1] \cdot \Pr[x_b[jl+m]=1] = q[j]x[j]$. So, $\mathrm{Var}\left[q_b[jl+m]x_b[jl+m]\right] = q[j]x[j](1-q[j]x[j])$ and

$$\begin{aligned}\mathrm{Var}[q_b^\top x_b/l] =& 1/l^2 \cdot \sum_{j=0}^{d-1}\sum_{m=0}^{l-1} q[j]x[j](1-q[j]x[j]) \\ =& 1/l \cdot \sum_{j=0}^{d-1} q[j]x[j](1-q[j]x[j]).\end{aligned}$$

□

Next, we prove Lemma 4 to demonstrate that we can obtain an unbiased estimator of $q^\top x$ by using minHash.

Proof. Let ν_j be a 0-1 random variable and $\nu_i = 1$ indicates $h_j(\mathcal{T}(q)) = h_j(\mathcal{T}(x))$. Then, we have $\mathrm{E}\,\nu_j = \Pr[\nu_j=1] = \Pr[h_j(\mathcal{T}(q)) = h_j(\mathcal{T}(x))] = s_J(\mathcal{T}(q),\mathcal{T}(x))$. Since $\alpha = \sum_{j=0}^{m-1}\nu_j$ and $\nu_1,\cdots,\nu_j$ are independent, $\mathrm{E}\,\alpha = \sum_{j=0}^{m-1} E\nu_j = m \cdot s_J(\mathcal{T}(q),\mathcal{T}(x))$. $\mathrm{E}\,\frac{1}{(1+m/\alpha)l} = \frac{1}{(1+m/\mathrm{E}\,\alpha)l} = (1+1/s_J(\mathcal{T}(q),\mathcal{T}(x)))/l$. So,

$$\mathrm{E}\,s_o/l = \mathrm{E}\,\frac{|\mathcal{T}(q)|+|\mathcal{T}(x)|}{\mathrm{E}(1+m/\alpha)l} = 1/l \cdot \frac{|\mathcal{T}(q)|+|\mathcal{T}(x)|}{1+1/s_J(\mathcal{T}(q),\mathcal{T}(x))}$$

$$\begin{aligned}\mathrm{E}\,s_o/l =& \mathrm{E}\,\frac{|\mathcal{T}(q)|+|\mathcal{T}(x)|}{\mathrm{E}(1+m/\alpha)l} \\ =& 1/l \cdot \mathrm{E}\,\frac{|\mathcal{T}(q)|+|\mathcal{T}(x)|}{1+1/s_J(\mathcal{T}(q),\mathcal{T}(x))} \\ =& \mathrm{E}\,\frac{s_O(\mathcal{T}(q),\mathcal{T}(x))}{l},\end{aligned}$$

which equals to $q^\top x$ based on Lemma 1. □

TABLE II: Summary of Datasets

Datasets	n	φ_d	φ_q	d	Size (GB)
SPLADE-Full	8,841,823	126.8	49.1	30,109	8.42
SPLADE1M	1,000,000	126.3	49.1	30,109	0.97
BM-Full	8,841,823	42.6	5.93	2,387,460	2.87
BM1M	1,000,000	42.2	5.93	593,037	0.33

B. Theoretical Guarantee

To reveal why it is reasonable to adopt $s_O/l \ge tI$ as the condition for generating the candidates (Line 1 in Alg. 4), we provide the following lemma:

Lemma 6. *Given a query q and a point x, a threshold $0<\gamma<1$, m minHash functions, an inner product threshold $I \ge \gamma\|q\|$, an approximation ratio $0<c<1$ and a parameter $c<t<1$, let $q_b = \mathcal{T}(q)$ and $x_b = \mathcal{T}(x)$. we define $s_O(q_b,x_b) = \frac{|q_b|+|x_b|}{1+m/\alpha}$ where α is the counting numbers between q_b and x_b in m minHash functions. For the following two events:*

- ***E1:** If $q^\top x \ge I$, then $s_O(q_b,x_b) > ltI$.*
- ***E2:** There are fewer than T points x that $s_O(q_b,x_b) > ltI$ but $q^\top x < cI$.*

the probability that E1 occurs is at least $1-1/e$, and the probability that E2 occurs is at least $1/2$ by setting t and w based on Eq. 4.

VI. Experimental Study

In this section, we conduct extensive experiments on real world data and synthesized datasets for a comprehensive evaluation and analysis on SOSIA. We implement the SOSIA [2] and competitors in C++ and compiled with g++ using -Ofast optimization and openMP for parallelism. All experiments are conducted on a Ubuntu server with 4 Intel(R) Xeon(R) Gold 6218 CPUs (160 threads) and 1.5 TB RAM.

A. Experimental Settings

Datasets. The sparse datasets and query sets in our experiments are obtaining by embedding MS Marco Passage Retrieval v1 dataset [4], a widely used dataset for document retrieval tasks [3]. We use a learning based methods SPLADE model [15], [16] and a traditional model BM25 [39] to embed the MS Marco Passage Retrieval v1 dataset. Among the four datasets listed in Table II, SPLADE-Full and SPLADE1M are embedded by the SPLADE model where SPLADE1M contains 1M points chosen randomly from SPLADEFull; BM-Full and BM1M are embedded by the SPLADE model where BM1M contains 1M points chosen randomly from BMFull. In Table II, φ_d and φ_q are the average number of non-zero values in the dataset and query set, respectively.

Competitors. To demonstrate the indexing and query performance of our SOSIA Framework, we compare it with the WAND [6] and the Sinnamon [7]. WAND is the most representative inverted index based method using DAAT (Document-at-a-time) query strategy. For the Sinnamon [7] proposed by

[2]https://github.com/Jacyhust/SOSIA

[3]https://microsoft.github.io/msmarco/

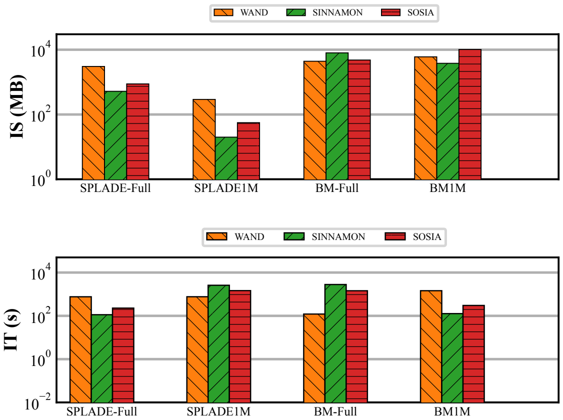

Fig. 3: Indexing Performance in All Datasets

Pinecone, it is a good representative for TAAT (Term-at-a-time) strategy-based method, where it utilizes BinSketch [37] to derive the approximation of inner products. Therefore, we choose WAND and Sinnamon as our competitors.

Parameter Settings. TBD

Evaluation Metrics. We evaluate the performance of our SOSIA framework with respect to four aspects: index size (GB / MB), indexing efficiency, query efficiency, and query accuracy. Specifically, we use the running time to evaluate indexing time and query time; and the recall is used to measure the query accuracy. For a query q, we denote the result of a (c, k)-MIPS query returned by a algorithm as $R = \{x_1, x_2, \cdots, x_k\}$. Let $R^* = \{x_1^*, x_2^*, \cdots, x_k^*\}$ be the exact k MIPS results of q. The recall are computed as follows.

$$Recall = \frac{|R \cap R^*|}{|R^*|} \tag{8}$$

B. Evaluation of Indexing Performance

C. Evaluation of Query Performance

1. effect of K 2. effect of n (carnality of the data) 3. recall time (trade off)

VII. Conclusion

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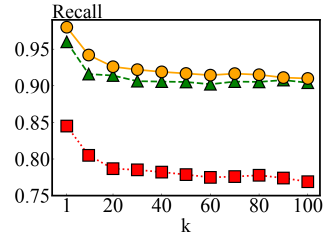

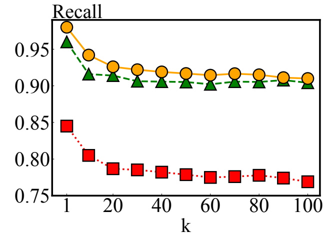

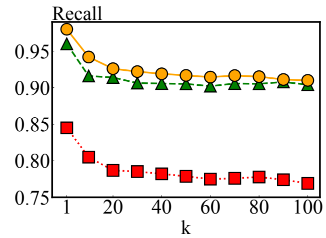

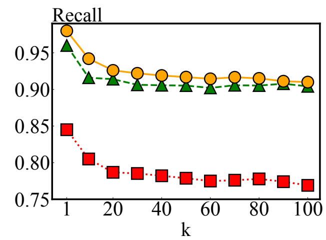

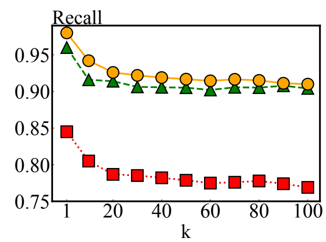

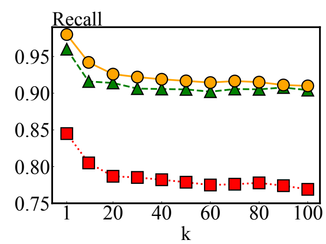

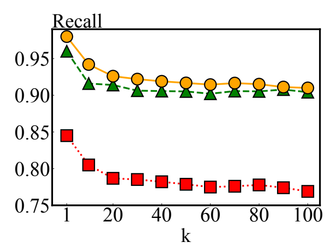

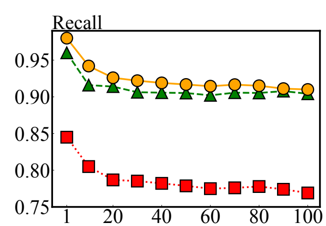

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