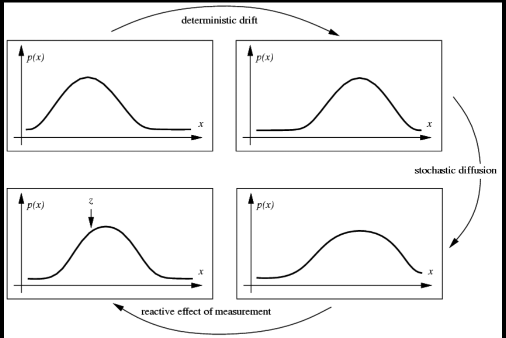
CS4495/6495
Introduction to Computer Vision

7B-L2 *The Kalman filter*



Tracking as induction

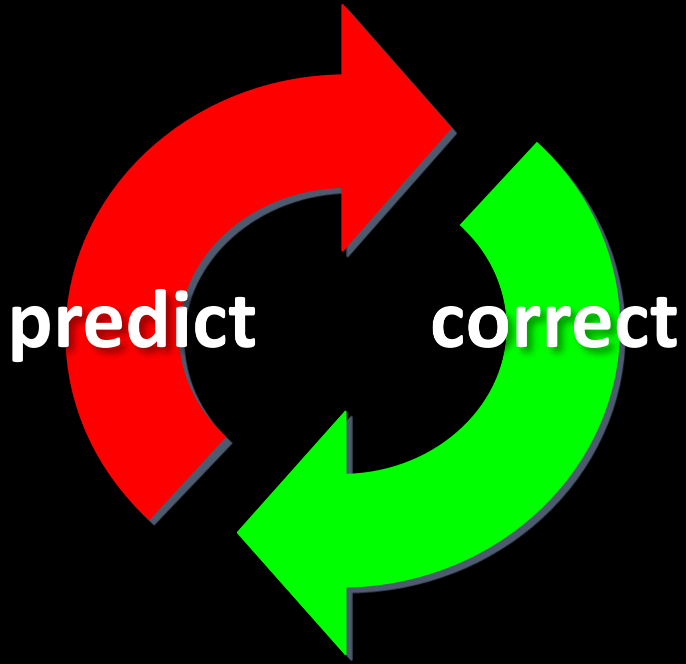
Base case:

- Assume we have some initial prior that predicts state in the absence of any evidence: $P(X_0)$
- At the first frame, *correct* this, given value of $Y_0 = y_0$

Tracking as induction

Given corrected estimate for frame t:

- Predict for frame $t+1$
- Correct for frame $t+1$



Last time: Prediction and correction

Prediction:

$$P(X_t \mid y_0, \ldots, y_{t-1}) = \int \underbrace{P(X_t \mid X_{t-1})}_{\text{dynamics model}} \underbrace{P(X_{t-1} \mid y_0, \ldots, y_{t-1})}_{\text{corrected estimate from previous step}} dX_{t-1}$$

Last time: Prediction and correction

Correction:

$$P(X_t \mid y_0, \ldots, y_{t-1}, y_t) = \frac{\overbrace{P(y_t \mid X_t)}^{\text{observation model}} \overbrace{P(X_t \mid y_0, \ldots, y_{t-1})}^{\text{predicted estimate}}}{\int P(y_t \mid X_t) P(X_t \mid y_0, \ldots, y_{t-1}) \, dX_t}$$

Linear Dynamics Model

Dynamics model: State undergoes linear transformation plus Gaussian noise

$$\mathbf{x}_t \sim N\left(D_t \mathbf{x}_{t-1}, \Sigma_{d_t}\right)$$

Linear Measurement Model

Observation model: Measurement is linearly transformed state plus Gaussian noise

$$\mathbf{y}_t \sim N\left(M_t \mathbf{x}_t, \Sigma_{m_t}\right)$$

Example: Constant velocity (1D)

State vector is position and velocity

$$x_t = \begin{bmatrix} p_t \\ v_t \end{bmatrix} \qquad \begin{aligned} p_t &= p_{t-1} + (\Delta t) v_{t-1} + \varepsilon \\ v_t &= v_{t-1} + \xi \end{aligned}$$

$$x_t = D_t x_{t-1} + noise = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{t-1} \\ v_{t-1} \end{bmatrix} + noise$$

Example: Constant velocity (1D)

Measurement is position only

$$y_t = Mx_t + noise = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_t \\ v_t \end{bmatrix} + noise$$

Example: Constant acceleration (1D)

State vector is position, velocity & acceleration

$$x_t = \begin{bmatrix} p_t \\ v_t \\ a_t \end{bmatrix} \qquad \begin{aligned} p_t &= p_{t-1} + (\Delta t) v_{t-1} + \varepsilon \\ v_t &= v_{t-1} + (\Delta t) a_{t-1} + \xi \\ a_t &= a_{t-1} + \zeta \end{aligned}$$

$$x_t = D_t x_{t-1} + noise = \begin{bmatrix} 1 & \Delta t & 0 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{t-1} \\ v_{t-1} \\ a_{t-1} \end{bmatrix} + noise$$

Example: Constant acceleration (1D)

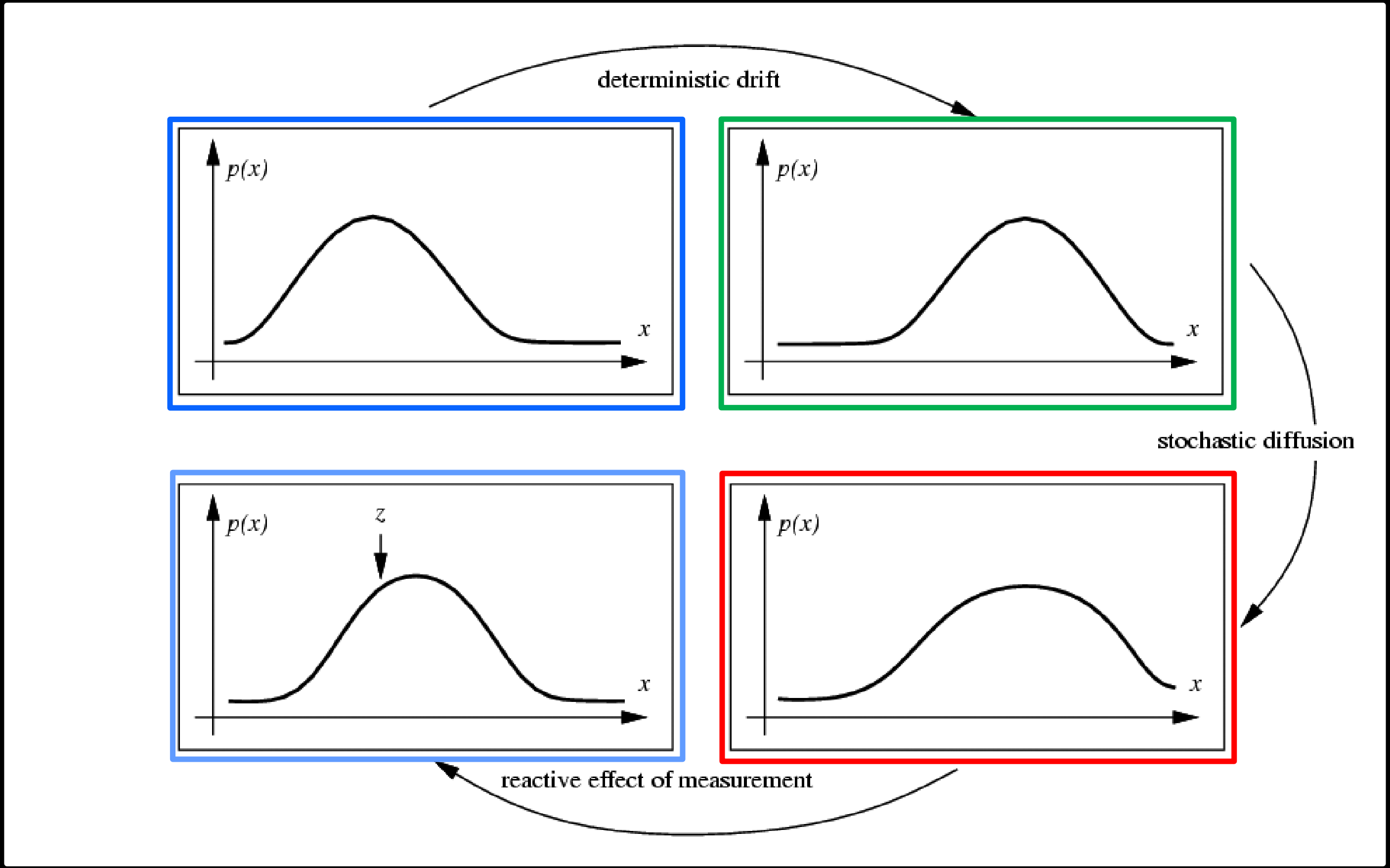
Measurement is position only

$$y_t = Mx_t + noise = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_t \\ v_t \\ a_t \end{bmatrix} + noise$$

The Kalman Filter

- A method for tracking linear dynamical models in Gaussian noise
- Predicted/corrected state distributions are Gaussian
  - You only need to maintain the mean and covariance
  - The calculations are easy (all the integrals can be done in closed form)

The Kalman Filter



The Kalman Filter: 1D state

Make measurement

$$P\left(X_t \middle| y_0, \ldots, y_{t-1}\right)$$

Mean and std. dev. of predicted state:

$$\mu_t^-, \sigma_t^-$$

Predict

Correct

$$P\left(X_t \middle| y_0, \ldots, y_t\right)$$

Mean and std. dev. of corrected state:

$$\mu_t^+, \sigma_t^+$$

Time advances (from t–1 to t)

1D Kalman Filter: Prediction

Linear dynamics model defines predicted state evolution, with noise

$$X_t \sim N\left(dx_{t-1}, \sigma_d^2\right)$$

Want to estimate distribution for next predicted state

$$P\left(X_t \middle| y_0, \ldots, y_{t-1}\right) = \int P\left(X_t \mid X_{t-1}\right) P\left(X_{t-1} \mid y_0, \ldots, y_{t-1}\right) dX_{t-1}$$

1D Kalman Filter: Prediction

Linear dynamics model defines predicted state evolution, with noise

$$X_t \sim N\left(dx_{t-1}, \sigma_d^2\right)$$

The distribution for next predicted state is also a Gaussian

$$P\left(X_t \,|\, y_0, \ldots, y_{t-1}\right) = N\left(\mu_t^-, (\sigma_t^-)^2\right)$$

Update the mean: $\mu_t^- = d\,\mu_{t-1}^+$

Update the variance: $(\sigma_t^-)^2 = \sigma_d^2 + (d\sigma_{t-1}^+)^2$

1D Kalman Filter: Correction

Mapping of state to measurements: $Y_t \sim N\left(mx_t, \sigma_m^2\right)$

Predicted state: $P\left(X_t \middle| y_0, \ldots, y_{t-1}\right) = N\left(\mu_t^-, (\sigma_t^-)^2\right)$

Want to estimate corrected distribution:

$$P\left(X_t \middle| y_0, \ldots, y_t\right) = \frac{P\left(y_t \mid X_t\right) P\left(X_t \mid y_0, \ldots, y_{t-1}\right)}{\int P\left(y_t \mid X_t\right) P\left(X_t \mid y_0, \ldots, y_{t-1}\right) dX_t}$$

Kalman: With linear, Gaussian dynamics and measurements, the corrected distribution is:

$$P\left(X_t \,\middle|\, y_0, \ldots, y_t\right) \equiv N\left(\mu_t^+, (\sigma_t^+)^2\right)$$

Update the mean:

$$\mu_t^+ = \frac{\mu_t^- \sigma_m^2 + m y_t (\sigma_t^-)^2}{\sigma_m^2 + m^2 (\sigma_t^-)^2}$$

Update the variance:

$$(\sigma_t^+)^2 = \frac{\sigma_m^2 (\sigma_t^-)^2}{\sigma_m^2 + m^2 (\sigma_t^-)^2}$$

1D Kalman Filter: Intuition

From:

$$\mu_t^+ = \frac{\mu_t^- \sigma_m^2 + m y_t (\sigma_t^-)^2}{\sigma_m^2 + m^2 (\sigma_t^-)^2}$$

Dividing throughout by m^2...

$$\mu_t^+ = \frac{\dfrac{\mu_t^- \sigma_m^2}{m^2} + \dfrac{y_t}{m}(\sigma_t^-)^2}{\dfrac{\sigma_m^2}{m^2} + (\sigma_t^-)^2}$$

1D Kalman Filter: Intuition

From:

$$\mu_t^+ = \frac{\mu_t^- \sigma_m^2 + m y_t (\sigma_t^-)^2}{\sigma_m^2 + m^2 (\sigma_t^-)^2}$$

Prediction of x

Measurement guess of x

Variance of prediction

Variance of x computed from the measurement

$$\mu_t^+ = \frac{\frac{\mu_t^- \sigma_m^2}{m^2} + \frac{y_t}{m} (\sigma_t^-)^2}{\frac{\sigma_m^2}{m^2} + (\sigma_t^-)^2}$$

What is this?

- *The weighted average of prediction and measurement based on variances!*

Prediction vs. correction

$$\mu_t^+ = \frac{\mu_t^- \sigma_m^2 + m y_t (\sigma_t^-)^2}{\sigma_m^2 + m^2 (\sigma_t^-)^2} \qquad (\sigma_t^+)^2 = \frac{\sigma_m^2 (\sigma_t^-)^2}{\sigma_m^2 + m^2 (\sigma_t^-)^2}$$

What if there is no prediction uncertainty? $(\sigma_t^- = 0)$

$$\mu_t^+ = \mu_t^- \qquad (\sigma_t^+)^2 = 0$$

The measurement is ignored!

What if there is no measurement uncertainty? $(\sigma_m = 0)$

$$\mu_t^+ = \frac{y_t}{m} \qquad (\sigma_t^+)^2 = 0$$

The prediction is ignored!

1D Kalman Filter: Intuition

Also:

$$\mu_t^+ = \frac{\dfrac{\mu_t^- \sigma_m^2}{m^2} + \dfrac{y_t}{m}(\sigma_t^-)^2}{\dfrac{\sigma_m^2}{m^2} + (\sigma_t^-)^2}$$

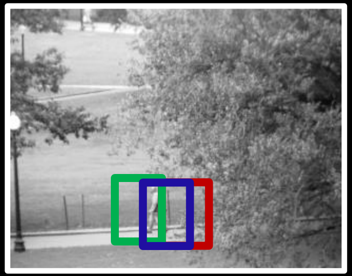
$$\mu_t^+ = \frac{a\,\mu_t^- + b\dfrac{y_t}{m}}{a+b} = \frac{(a+b)\mu_t^- + b(\dfrac{y_t}{m} - \mu_t^-)}{a+b}$$

1D Kalman Filter: Intuition

$$\mu_t^+ = \frac{a\mu_t^- + b\frac{y_t}{m}}{a+b} = \frac{(a+b)\mu_t^- + b(\frac{y_t}{m} - \mu_t^-)}{a+b}$$

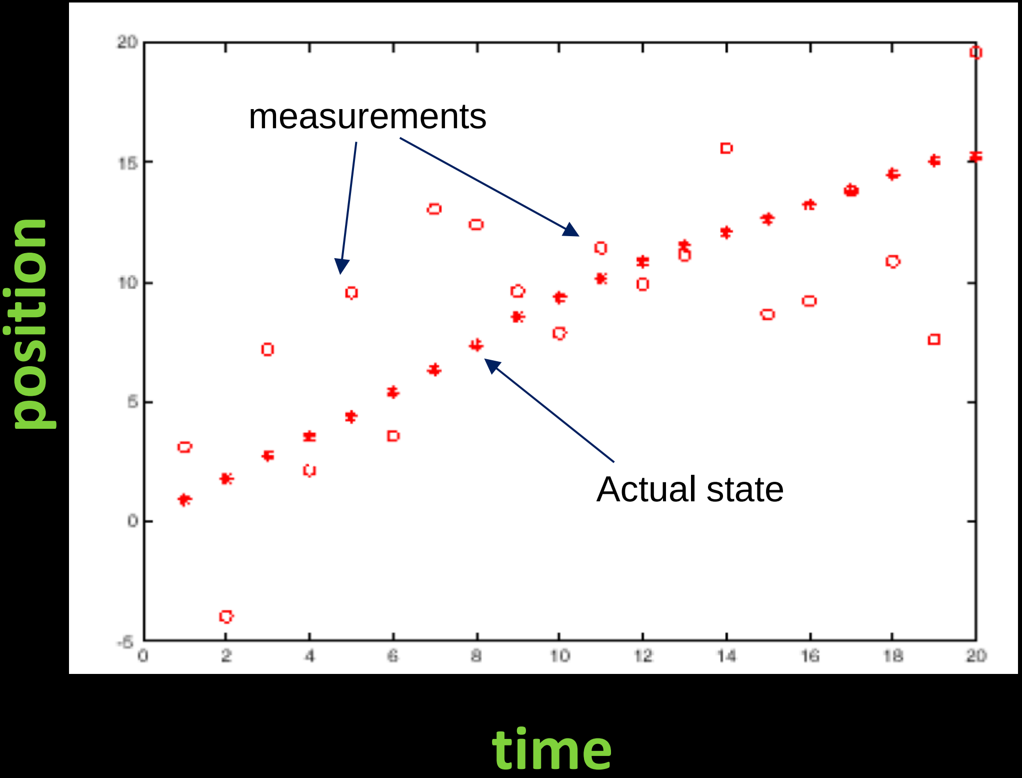
$$\mu_t^+ = \mu_t^- + \frac{b(\frac{y_t}{m} - \mu_t^-)}{a+b} = \underbrace{\mu_t^-}_{\text{Predicted}} + \underbrace{k}_{\text{Kalman Gain}}\underbrace{(\frac{y_t}{m} - \mu_t^-)}_{\text{Residual}}$$

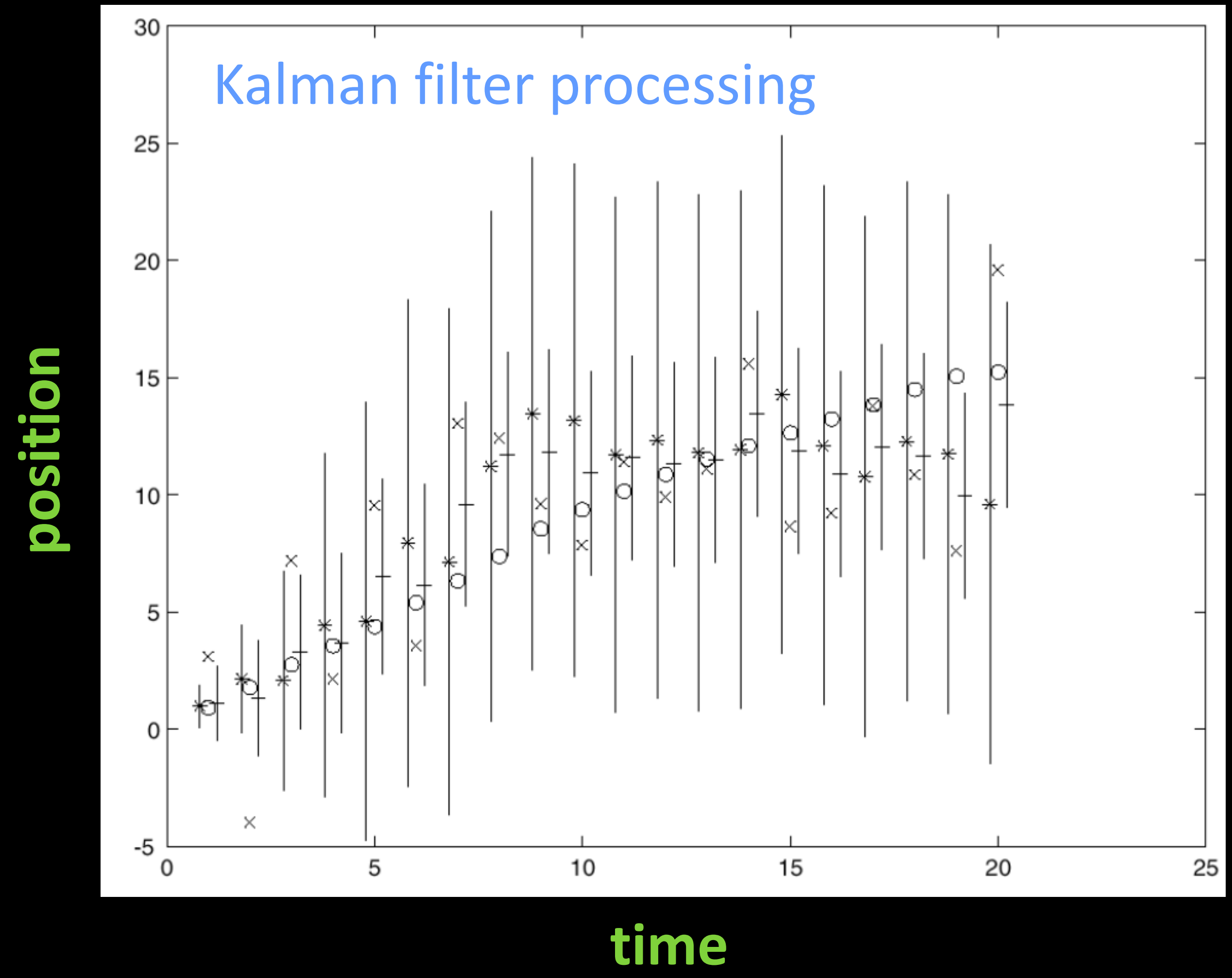
Recall: constant velocity model example



State is 2d: position + velocity

Measurement is 1d: position





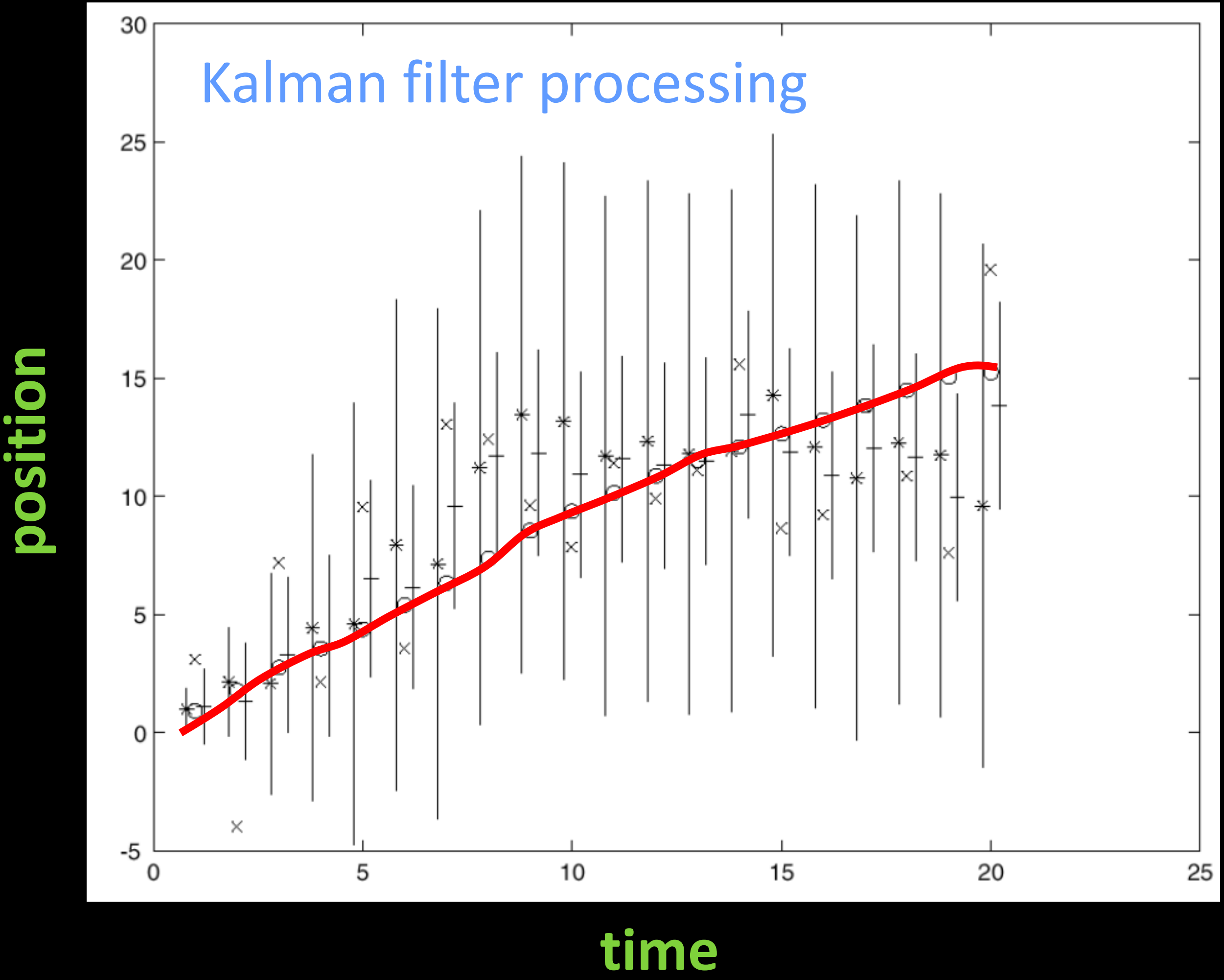
o state

x measurement

* predicted mean estimate

+ corrected mean estimate

bars: variance estimates before and after measurements



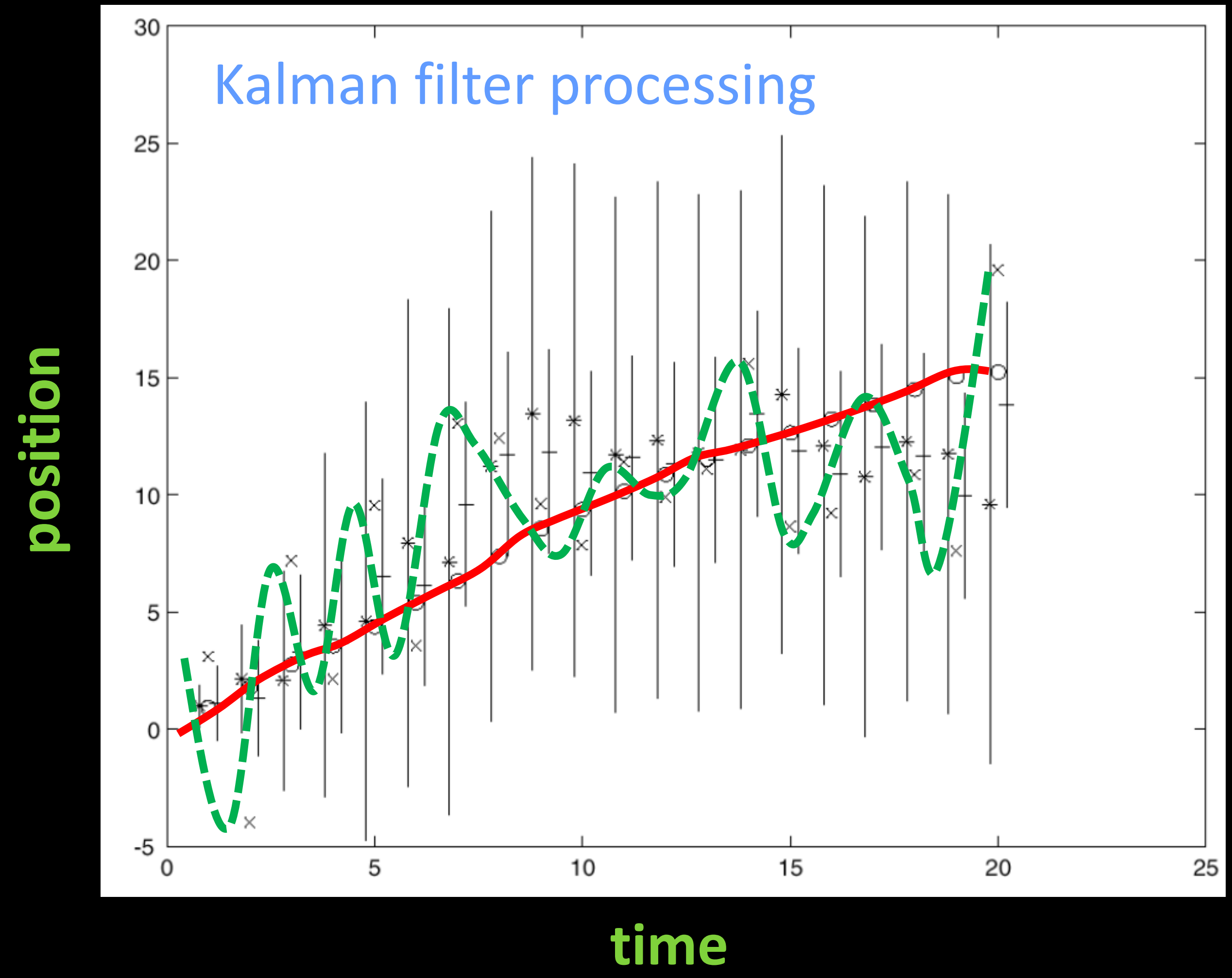
o state

x measurement

* predicted mean estimate

+ corrected mean estimate

bars: variance estimates before and after measurements



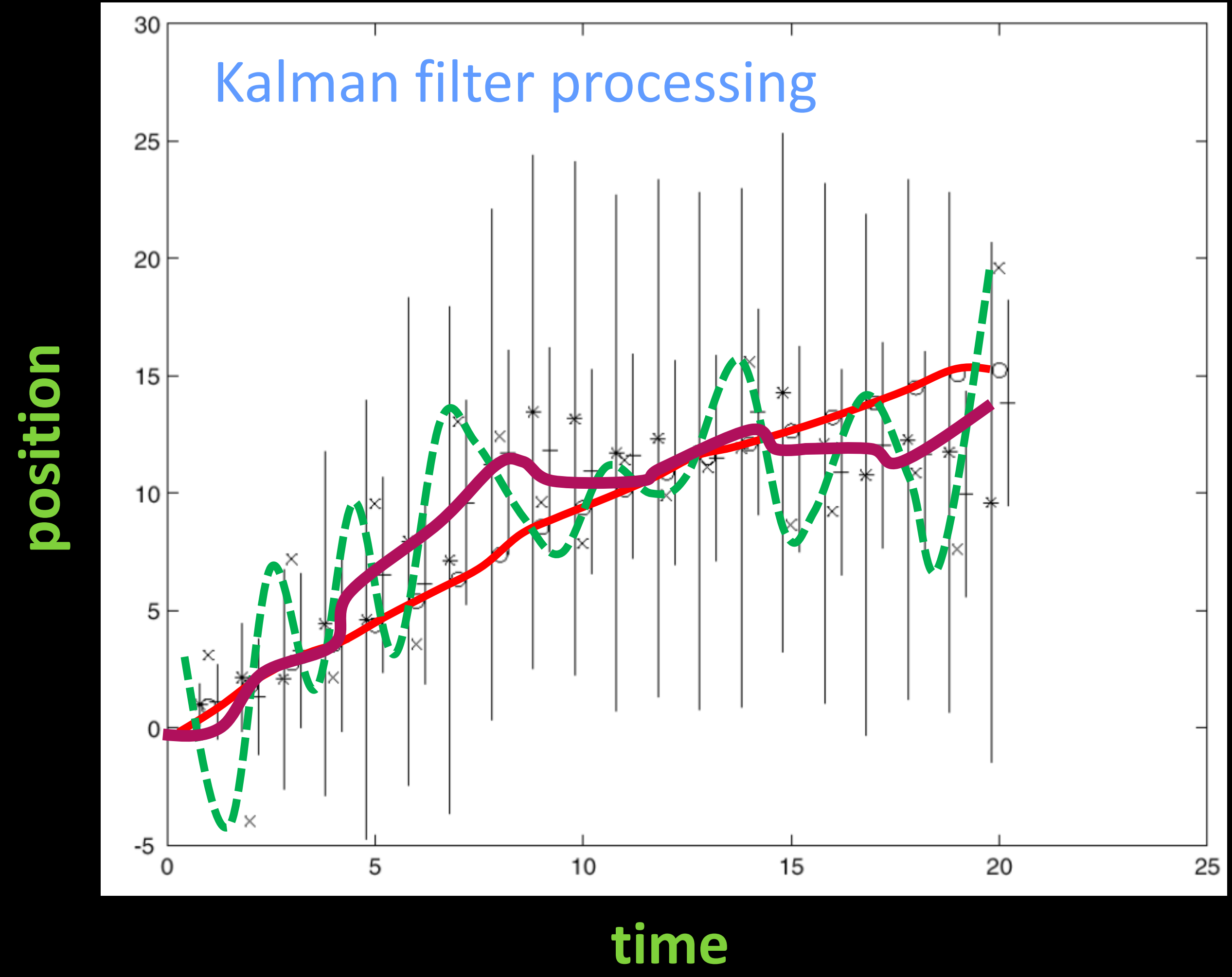
o state

x measurement

* predicted mean estimate

+ corrected mean estimate

bars: variance estimates before and after measurements



o state

x measurement

* predicted mean estimate

+ corrected mean estimate

bars: variance estimates before and after measurements

N-dimensional

PREDICT

$$x_t^- = D_t x_{t-1}^+$$

$$\Sigma_t^- = D_t \Sigma_{t-1}^+ D_t^T + \Sigma_{d_t}$$

CORRECT

$$K_t = \Sigma_t^- M_t^T \left(M_t \Sigma_t^- M_t^T + \Sigma_{m_t} \right)^{-1}$$

$$x_t^+ = x_t^- + K_t \left(y_t - M_t x_t^- \right)$$

$$\Sigma_t^+ = \left(I - K_t M_t \right) \Sigma_t^-$$

More weight on residual when measurement error covariance approaches zero.

Less weight on residual as a priori estimate error covariance approaches zero.

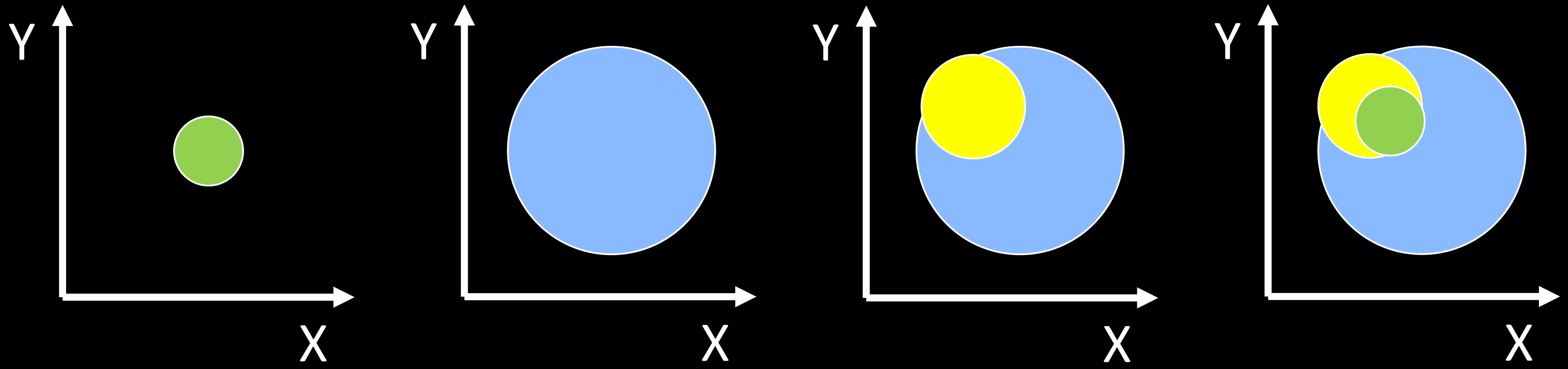
Tracking with KFs: Gaussians

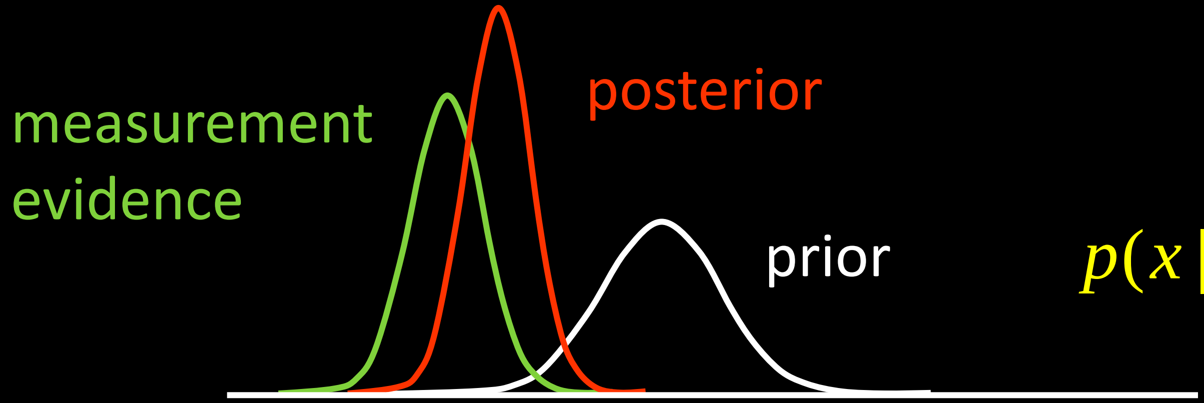
Initial (prior) estimate

prediction

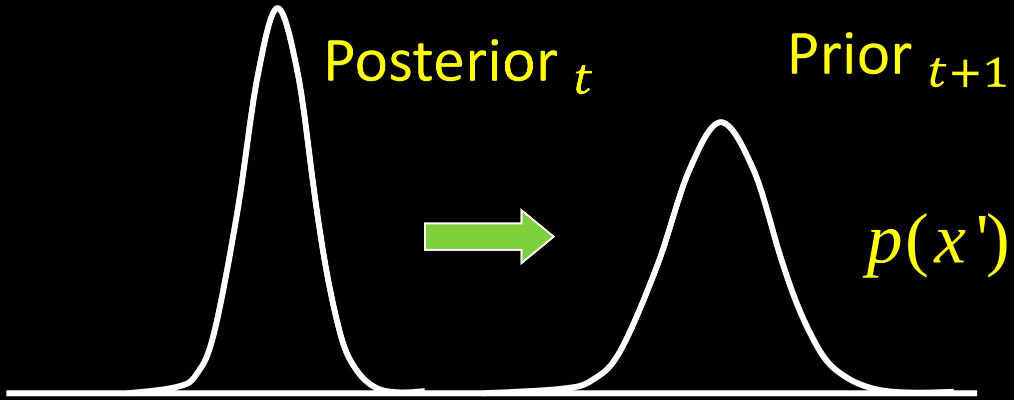
measurement

update





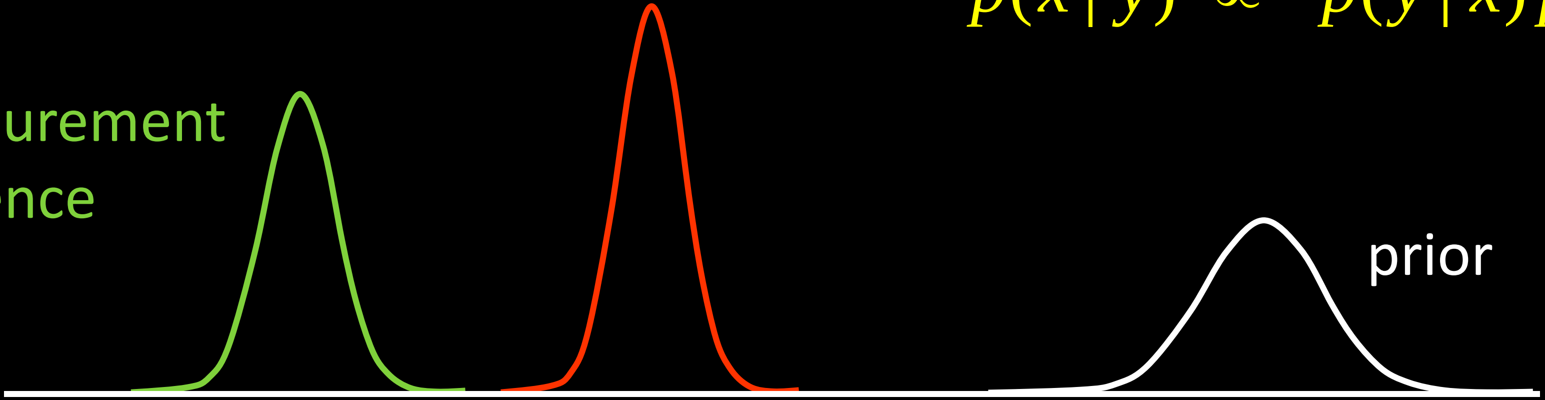
$$p(x \mid y) \propto p(y \mid x)\, p(x)$$



$$p(x') = \int p(x' \mid x)\; p(x)\; dx$$

A Quiz

$$p(x \mid y) \propto p(y \mid x)\, p(x)$$



Does this agree with your intuition?

Kalman filter pros and cons

- Pros
  - Simple updates, compact and efficient

Kalman filter pros and cons

- Cons
  - Unimodal distribution, only single hypothesis
  - Restricted class of motions defined by linear model
    - Extensions call “Extended Kalman Filtering”

- So what might we do if not Gaussian? Or even unimodal?