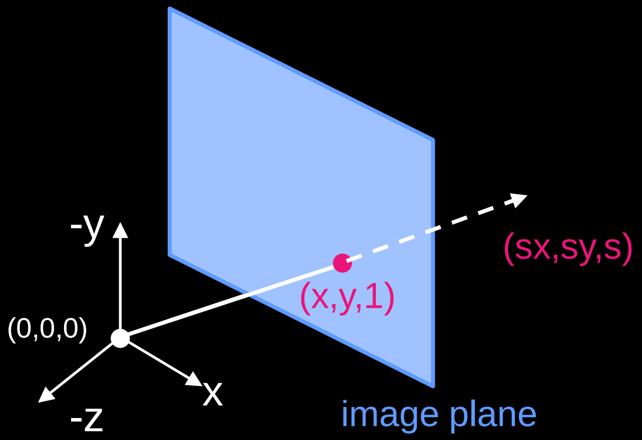
CS4495/6495
Introduction to Computer Vision

3D-L3 *Projective geometry*

Recall: The projective plane

What is the geometric intuition of using homogenous coordinates?

- A point in the image is a ray in projective space

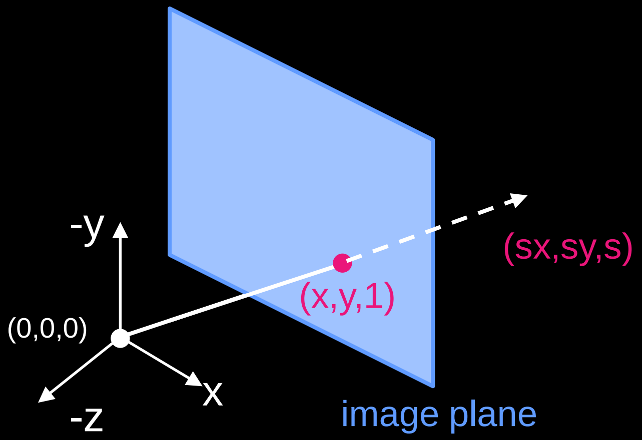


The projective plane

Each *point* (x,y) on the plane (at z=1) is represented by a *ray* (sx,sy,s)

All points on the ray are equivalent:

$$(x, y, 1) \cong (sx, sy, s)$$



Homogeneous coordinates

2D Points:

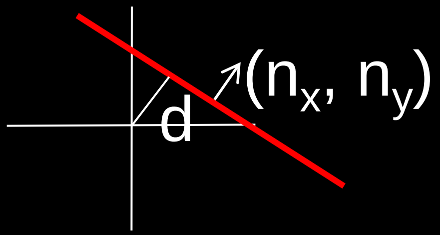
$$p = \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow p' = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad p' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \longrightarrow p = \begin{bmatrix} x'/w' \\ y'/w' \end{bmatrix}$$

Homogeneous coordinates

2D Lines: $ax + by + c = 0$

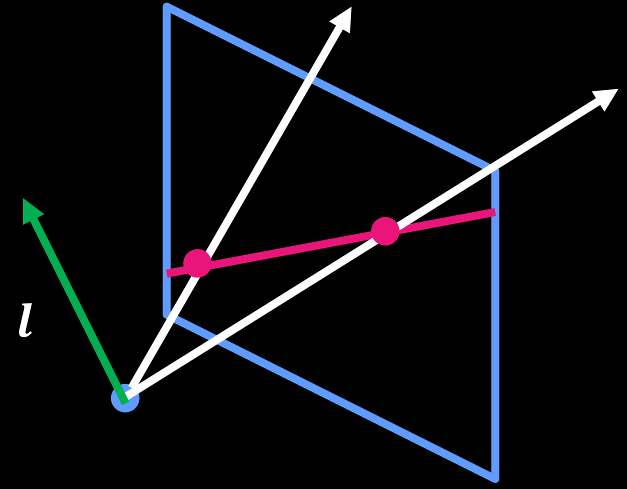
$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$l = \begin{bmatrix} a & b & c \end{bmatrix} \Rightarrow \begin{bmatrix} n_x & n_y & -d \end{bmatrix}$$



Projective lines

What does a line in the image correspond to in projective space?

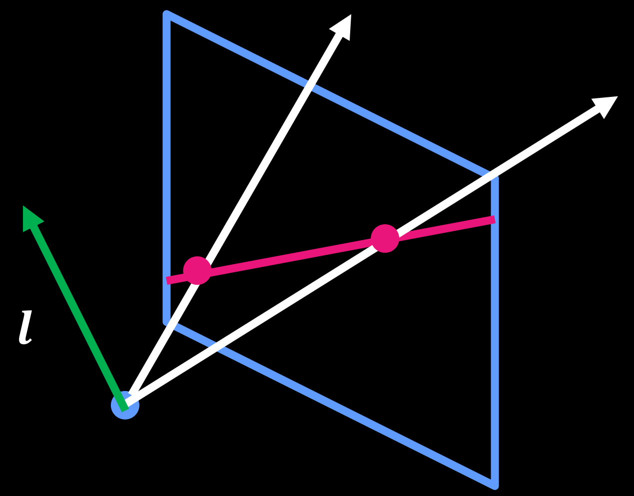


Projective lines

A line is a *plane* of rays through origin define by the normal $l = (a, b, c)$

All rays (x,y,z) satisfying:

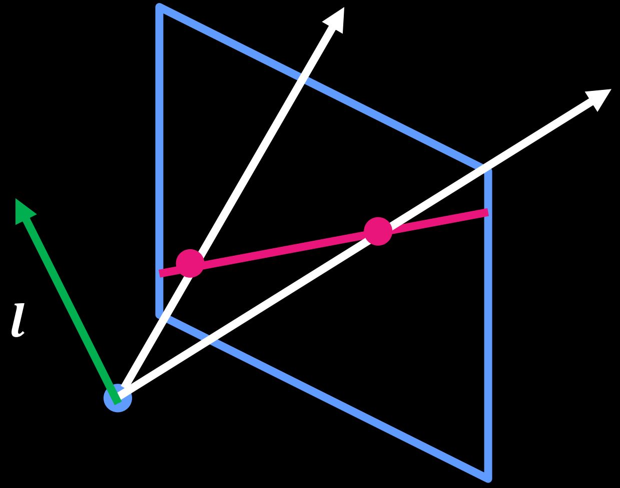
$$ax + by + cz = 0$$



Projective lines

In vector notation:

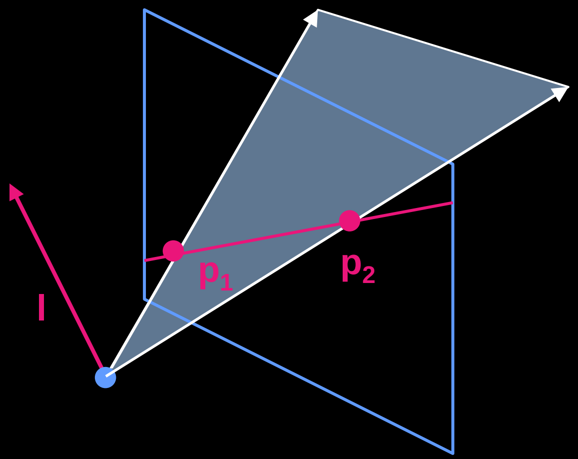
$$0 = \underbrace{\begin{bmatrix} a & b & c \end{bmatrix}}_{l} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{p}$$



A line is also represented as a homogeneous 3-vector!

Point and line duality

- A line $\mathbf{l}$ is a homogeneous 3-vector
- It is $\perp$ to every point (ray) **p** on the line: $\mathbf{l}^T\mathbf{p}=0$



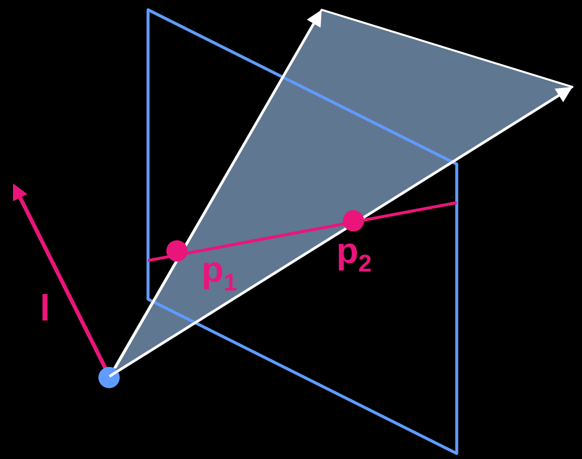
Point and line duality

What is the line $\mathbf{l}$ spanned by rays $\mathbf{p}_1$ and $\mathbf{p}_2$?

$\mathbf{l}$ is $\perp$ to $\mathbf{p}_1$ and $\mathbf{p}_2$

$$\Rightarrow \quad \mathbf{l} = \mathbf{p}_1 \times \mathbf{p}_2$$

$\mathbf{l}$ is the plane normal

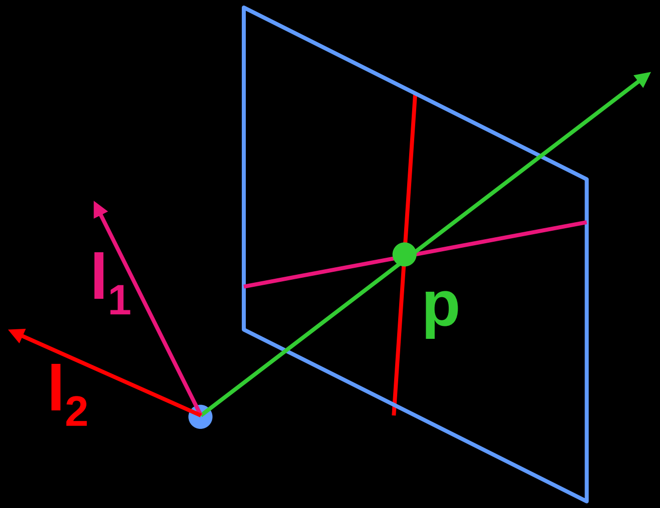


Point and line duality

What is the intersection of two lines $\mathbf{l}_1$ and $\mathbf{l}_2$?

$\mathbf{p}$ is $\perp$ to $\mathbf{l}_1$ and $\mathbf{l}_2$ $\Rightarrow$

$\mathbf{p} = \mathbf{l}_1 \times \mathbf{l}_2$

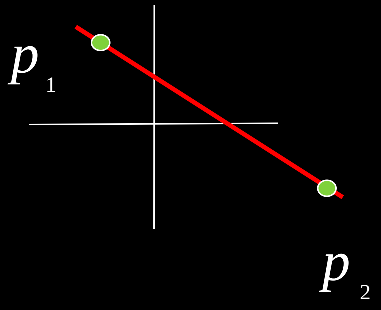


Points and lines are *dual* in projective space

- Given any formula, can switch the meanings of points and lines to get another formula

Homogeneous coordinates

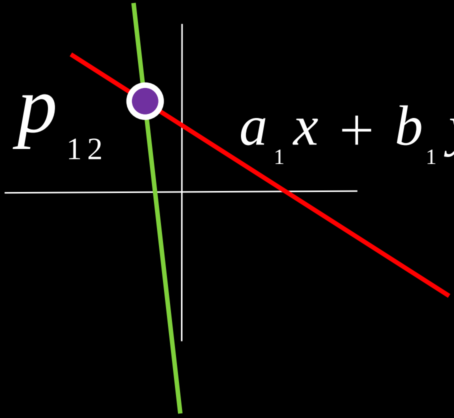
Line joining two points:



$$\left.\begin{aligned} p_1 &= \begin{bmatrix} x_1 & y_1 & 1 \end{bmatrix} \\ p_2 &= \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \end{aligned}\right\} \quad l = p_1 \times p_2$$

Homogeneous coordinates

Intersection between two lines:



p_{12}

$$a_1 x + b_1 y + c_1 = 0$$

$$a_2 x + b_2 y + c_2 = 0$$

$$\left.\begin{aligned} l_1 &= \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \\ l_2 &= \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} \end{aligned}\right\} \quad p_{12} = l_1 \times l_2$$

Quiz

How can I tell whether a point p is on a line L in an image?

a) Check if $p \times L$ is zero.

b) Check if $p \bullet L$ is zero.

c) Check if the magnitude of the sum is greater than 1.

Quiz – answer

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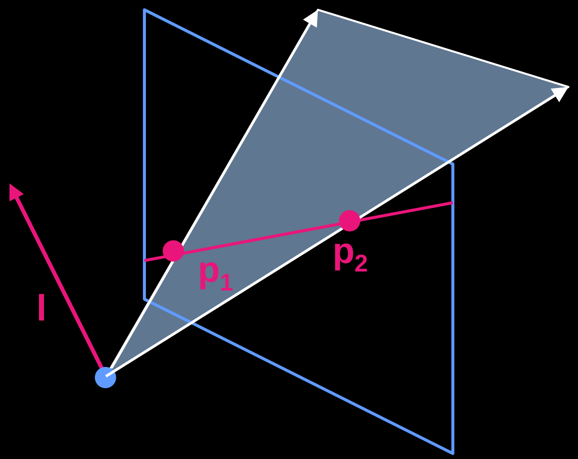
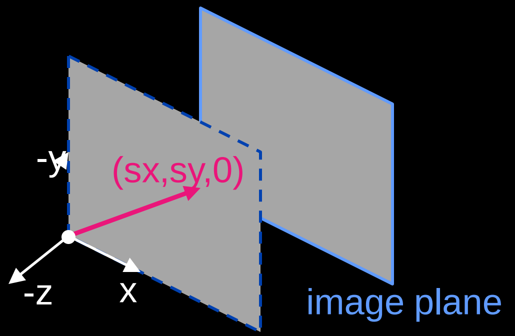


Ideal points and lines

Ideal point ("point at infinity")

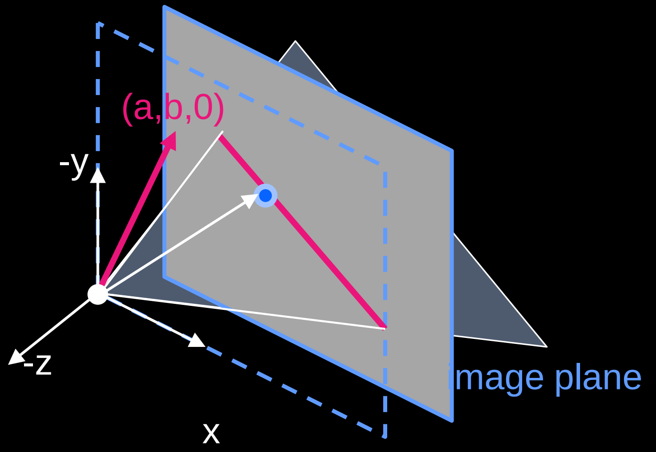
- $p \cong (x, y, 0)$ – parallel to image plane
- It has infinite image coordinates



Ideal points and lines

Ideal line

- $l \cong (a, b, 0)$ – normal is parallel to image plane
- Corresponds to a line in the image (finite coordinates)
  - –goes through image origin (*principle point*)



3D projective geometry

- These concepts generalize naturally to 3D
- Recall the equation of a plane:

$$aX + bY + cZ + d = 0$$

- Homogeneous coordinates

  Projective 3D points have four coords: p = (wX,wY,wZ,w)

3D projective geometry

- Duality
  - A plane N is also represented by a 4-vector
    N = (a,b,c,d)
  - Points and planes are dual in 3D: $N^T p = 0$
- Projective transformations
  - Represented by 4x4 matrices T: P' = TP