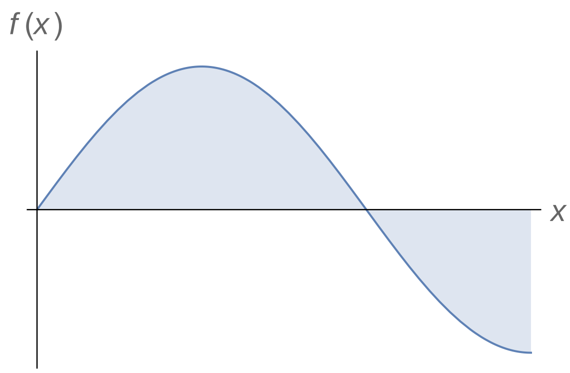
# Math 231E, Lecture 14.
# Fundamental Theorem of Calculus

## 1 Signed Area

Area is positive by definition, but how should we interpret the following area?



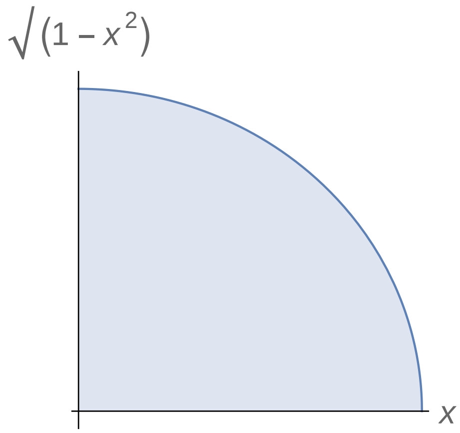

**Definition 1.** The **signed area** "below" a curve is defined to be $+1$ times the areas that occur above the curve and $-1$ times the areas that occur below the curve.

In general, we are interested in computing the signed area (but only because it is more mathematically natural!)

**Example 1.** There are some integrals that we know how to do, since we already know the area. For example,

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4},$$

and we can see this by drawing a circle.




But how do we compute

$$\int_{-1/3}^{1/2} \sqrt{1-x^2}\,dx?$$

We can also compute

$$\int_{-3}^{3} \sin(x)e^{-x^{17542}}\,dx = 0,$$

because it is an odd function, and using symmetry. But how do we compute

$$\int_{-3}^{2} \sin(x)e^{-x^{17542}}\,dx?$$

## 2 Linearity

A nice fact about definite integrals is that they are **linear**. For example, if we have a function $f(x)$ and we want to compute $\int_a^b f(x)\,dx$, then we choose some large $n$, define $\Delta x = (b-a)/n$, and $x_k^*$ in the $k$th subdomain, and write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x.$$

But what if $h(x) = f(x) + g(x)$? Then we have

$$\begin{aligned}
\int_a^b h(x)\,dx &= \lim_{n\to\infty} \sum_{k=1}^n h(x_k^*)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^n \left(f(x_k^*) + g(x_k^*)\right)\,\Delta x \\
&= \lim_{n\to\infty} \left[\sum_{k=1}^n f(x_k^*)\,\Delta x + \sum_{k=1}^n g(x_k^*)\,\Delta x\right] \\
&= \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\,\Delta x + \lim_{n\to\infty} \sum_{k=1}^n g(x_k^*)\,\Delta x \\
&= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.
\end{aligned}$$

We can do the same argument (Work it out yourself for practice!) to show that

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

**Unfortunately,** there is no nice rule to break up

$$\int_a^b f(x)g(x)\,dx.$$

# 3 Physical Interpretation and Motivation for FTC

Consider that we know the velocity of a particle for all $t \in [a, b]$, and let us denote this velocity by $v(t)$. Denote the position of the particle at time $t$ by $s(t)$. We want to compute the *total displacement* $s(b) - s(a)$.

If we consider a small time $\Delta t$, then the displacement in that small time interval is *roughly* $D_k := v(t_k^*)\,\Delta t$. The total displacement is then approximately

$$\sum_{k=1}^n D_k = \sum_{k=1}^n v(t_k^*)\,\Delta t.$$

Thus we should have

$$s(b) - s(a) = \lim_{n\to\infty} \sum_{k=1}^n v(t_k^*)\,\Delta t = \int_a^b v(t)\,dt.$$

But notice that $s'(t) = v(t)$! So we have

$$\boxed{\int_a^b s'(t)\,dt = s(b) - s(a).}$$

Similarly, let us write $x$ for $b$ in the previous equation:

$$s(x) - s(a) = \int_a^x v(t)\,dt,$$

and differentiate with respect to $x$, to obtain

$$s'(x) = \frac{d}{dx} \int_a^x v(t)\,dt,$$

or

$$\boxed{v(x) = \frac{d}{dx}\int_a^x v(t)\,dt.}$$

# 4 The Fundamental Theorem of Calculus

These two boxed equations are known as[1] **The Fundamental Theorem of Calculus**:

**Theorem 1** (FTC, Part I). If $f(x)$ is continuous on $[a,b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and $g'(x) = f(x)$.

**Theorem 2** (FTC, Part II). If $f(x)$ is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$, i.e. $F' = f$.

We can use the FTC to solve several types of problems:

- (FTC II) If we want to compute
$$\int_0^{\pi/2} \cos(t)\,dt,$$
we need to only notice that the derivative of $\sin(t)$ is $\cos(t)$, and thus
$$\int_0^{\pi/2} \cos(t)\,dt = \left(\sin(t)\right)\Big|_{t=0}^{t=\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = 1.$$

- (FTC I) If we know that
$$f(x) = \int_{-3}^x e^{-t^2}\,dt,$$
then the FTC tells us that $f'(x) = e^{-x^2}$. But we can do even more complicated problems: let us say that
$$g(x) = \int_{-2}^{\cos(x)} e^{-t^2}\,dt.$$
We see that $g(x) = f(\cos(x))$, so
$$g'(x) = f'(\cos(x))(-\sin(x)) = -\sin(x)e^{-\cos^2(x)}.$$

[1]If there was a way for me to typeset an echo, this is where I would do it.