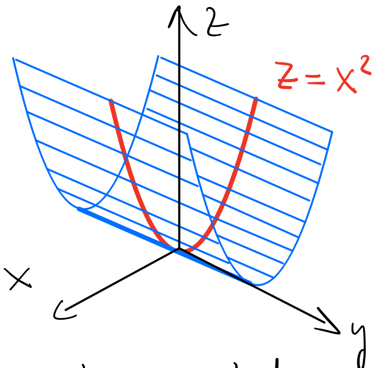
12.6 Cylinders and Quadric Surfaces

Goal: discuss various surfaces in $\mathbb{R}^3$ and their eqns

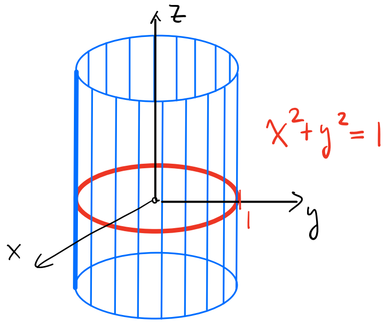
So far:
- spheres: $x^2 + y^2 + z^2 = r^2$ Section 12.1
- cylinders: $x^2 + y^2 = r^2$ Section 12.1
- planes: $ax + by + cz + d = 0$ Section 12.5

Generalized Cylinder = surface generated by sliding a line along a plane curve
= set of lines that are parallel to a given line and pass through a plane curve

RULINGS

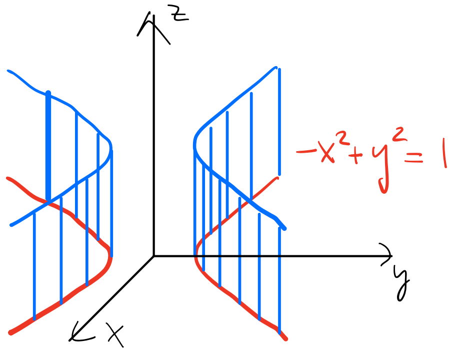


parabolic cylinder



elliptic cylinder

(special case: circular cyliner)

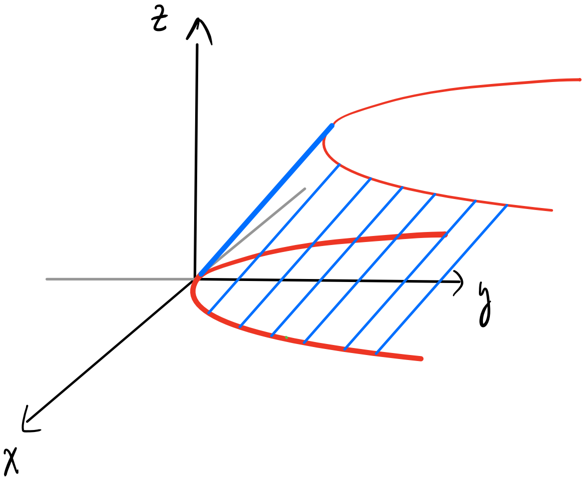


hyperbolic cylinder

z missing from eqn $\Rightarrow$ rulings parallel to z-axis

Rulings don't have to be parallel to a coordinate axis:

EX: $x^2 - y + z = 0$ is also a generalized cylider:



means: intersection of surface with the yz-plane

Trace in the yz-plane (set $x=0$):

$$-y + z = 0$$

$$\underline{z = y} \quad \text{a line}$$

Trace in the xy-plane (set $z=0$)

$$x^2 - y = 0$$

$$\underline{y = x^2} \quad \text{a parabola}$$

Quadric Surfaces

Quadric surface = graph of a second-degree eqn in three variables x, y, z.

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

$A, B, \ldots, J$ constants

- Quadric surfaces are natural extensions of 'conics'
- When a quadric surface intersects the coordinate plane the trace will be a conic section.

$\longrightarrow$ xy-plane, xz-plane, or yz-plane

Conic sections

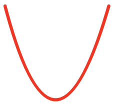
in $\mathbb{R}^2$

conic = a graph of a second-degree eqn. in two variables x, y.

$$\boxed{Ax^2 + By^2 + Cxy + Dx + Ey + F = 0}$$

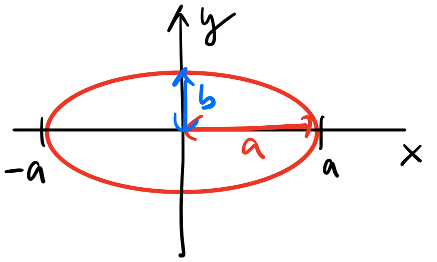
Basic types in standard form:

parabola: $y = Cx^2$



$x = Cy^2$

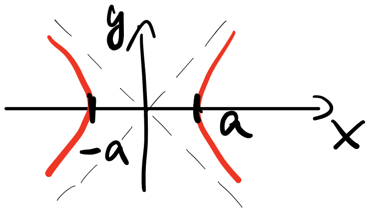
ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

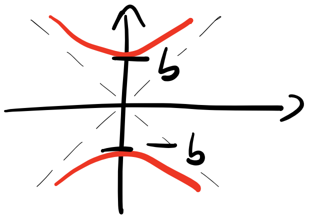


Section 10.5

hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$





2 Intersecting lines: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \longrightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} \longrightarrow \left|\frac{x}{a}\right| = \left|\frac{y}{b}\right|$

Quadric Surfaces in $\mathbb{R}^3$

$$\boxed{Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0}$$

A, B, ..., J constants

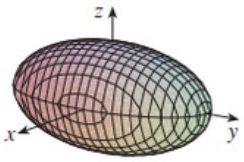
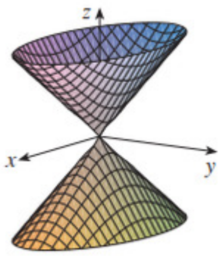
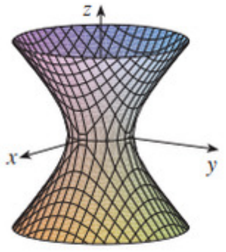
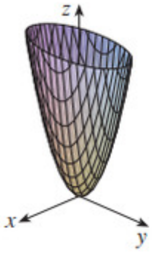
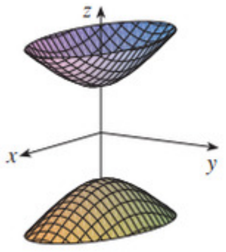
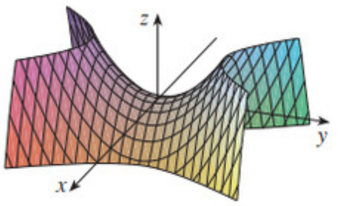
This eqn. can be reduced:

- we can eliminate mixed terms (xy, yz, xz) by rotating the surface
- we can translate the surface to center it at the origin

EX: $(x-1)^2+(y+3)^2+z^2=1 \implies x^2+y^2+z^2=1$

Basic types in standard form:

TABLE 1 Graphs of quadric surfaces

Surface	Equation	Surface	Equation
Ellipsoid  	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere. "$x^2+y^2+z^2=1$"	Cone  	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$. "$x^2+y^2-z^2=0$"
Hyperboloid of One Sheet  	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative. "$x^2+y^2-z^2=1$"	Elliptic Paraboloid  	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid. "$x^2+y^2-z=0$"
Hyperboloid of Two Sheets  	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets. "$-x^2-y^2+z^2=1$"	Hyperbolic Paraboloid  	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated. "$x^2-y^2-z=0$"

Ex: Identify the following surface.

$$-\frac{x^2}{4} + y^2 + 6y + z + 8 = 0$$

$$-\frac{x^2}{4} + (y^2+6y+9) + z + 8 = 9$$

$$-\frac{x^2}{4} + (y+3)^2 + z - 1 = 0$$

$$\frac{z-1}{1} = \frac{x^2}{4} - (y+3)^2$$

👁 No z^2.

" $-x^2 + y^2 + z = 0$ "

Hyperbolic paraboloid.

center: $(0,-3,1)$

Ellipsoid :

EX: $$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

$x=0 \Rightarrow \frac{y^2}{9} + \frac{z^2}{4} = 1$ (ellipse in the yz-plane)

$y=0 \Rightarrow x^2 + \frac{z^2}{4} = 1$ (ellipse in the xz-plane)

$z=0 \Rightarrow x^2 + \frac{y^2}{9} = 1$ (ellipse in the xy-plane)

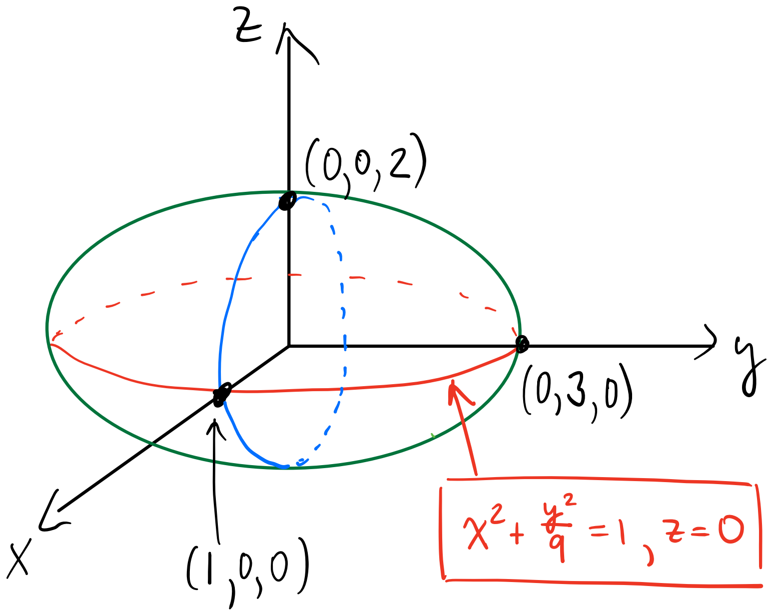
Intersection with $z=0$ (xy-plane)

set $z=0$.

$$\left.\begin{array}{l} x^2 + \frac{y^2}{9} = 1 \end{array}\right\} \text{red ellipse in the xy-plane}$$

Intersection with hor. plane $z=k$:

$$\left.\begin{array}{l} z=k \\ x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \end{array}\right\} \begin{array}{l}\text{If } k^2<4\text{, then} \\ \text{ellipse in the} \\ \text{horizontal plane} \\ z=k\end{array}$$



Horizontal traces are ellipses.

Similarly: Vertical traces in the plane $x=k$ are ellipses.

Vertical traces in the plane $y=k$ are ellipses.

Hyperboloids

of one sheet:

EX: $x^2 + \frac{y^2}{9} - z^2 = 1$

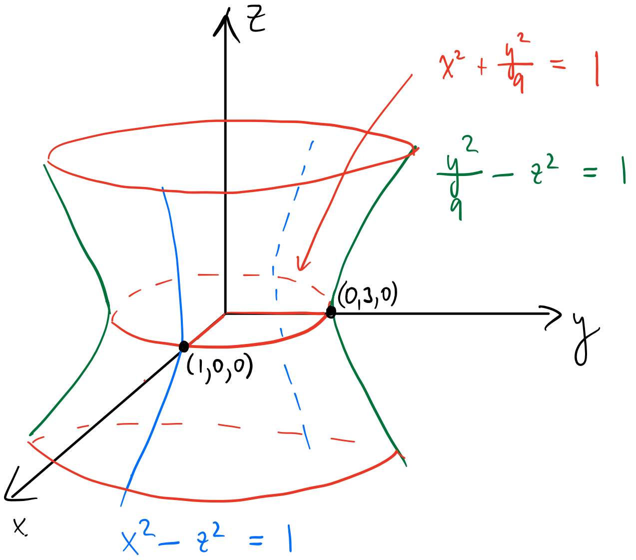
1 minus

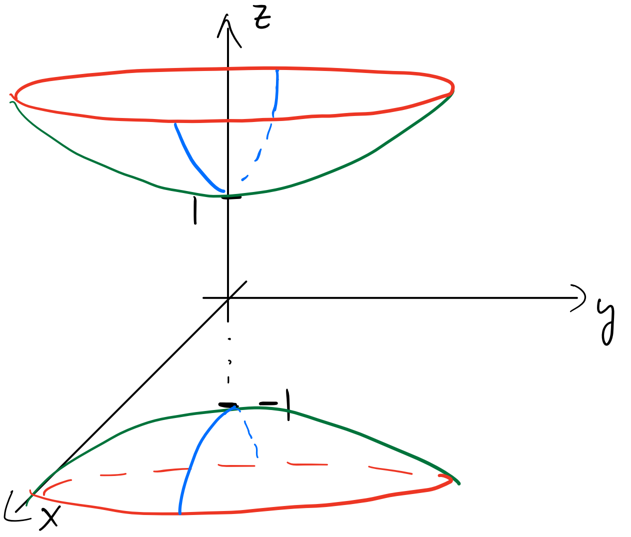
of two sheets:

EX: $-x^2 - \frac{y^2}{9} + z^2 = 1$

two minuses

If $z = 0$
→ no solution



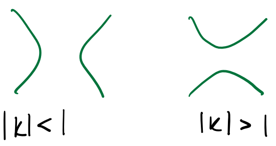


For a more detailed picture, use traces in the planes $x = k$, $y = k$, $z = k$.

EX: $x^2 + \frac{y^2}{9} - z^2 = 1$

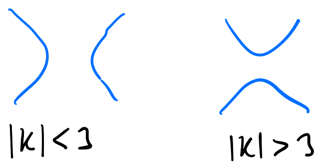
$x = k \Longrightarrow \frac{y^2}{9} - z^2 = 1 - k^2$

hyperbolas



$y = k \Longrightarrow x^2 - z^2 = 1 - \frac{k^2}{9}$

hyperbolas



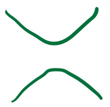
$z = k \Longrightarrow x^2 + \frac{y^2}{9} = 1 + k^2$

ellipses

EX: $-x^2 - \frac{y^2}{9} + z^2 = 1$

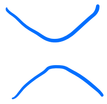
$x = k \Longrightarrow -\frac{y^2}{9} + z^2 = 1 + k^2$

hyperbolas



$y = k \Longrightarrow -x^2 + z^2 = 1 + \frac{k^2}{9}$

hyperbolas

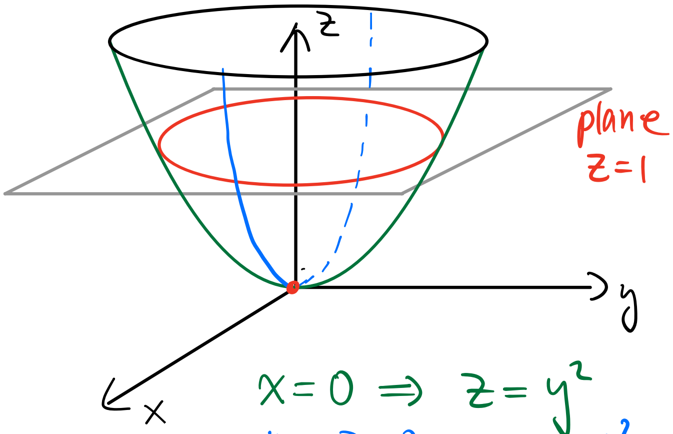


$z = k \Longrightarrow x^2 + y^2 = k^2 - 1$

ellipses if $|k| > 1$

Elliptic Paraboloid

Ex: $z = x^2 + y^2$



$x = 0 \Rightarrow z = y^2$

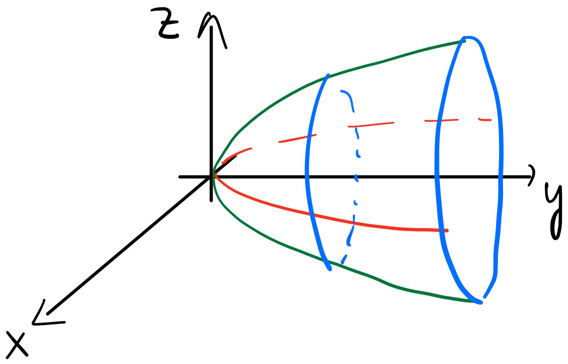
$y = 0 \Rightarrow z = x^2$

$z = 0 \Rightarrow 0 = x^2 + y^2$

$z = 1 \Rightarrow 1 = x^2 + y^2$

Ex: $x^2 - y + z^2 = 0$

$x^2 + z^2 = y$



Hyperbolic Paraboloid

Ex: $z = y^2 - x^2$

If $x = 0$: $z = y^2$

If $y = 0$: $z = -x^2$

If $z = 0$: $y^2 = x^2$

(2 intersecting lines: $y = x, y = -x$)

If $z = 1$: $y^2 - x^2 = 1$

If $z = -1$: $x^2 - y^2 = 1$