Math 231E, Lecture 16.
Integration by Parts

1 Integration By Parts, Formula

Just like u-substitution is "reverse chain rule", we have the technique of integration by parts, which is "reverse product rule".

Recall that

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides gives

$$\int_a^b \frac{d}{dx}(f(x)g(x))\,dx = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx,$$

and the FTC gives us that

$$f(b)g(b) - f(a)g(a) = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx,$$

or, in indefinite form:

$$\int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x).$$

If we write

$$u = f(x), \quad v = g(x), \quad du = f'(x)\,dx, \quad dv = g'(x)\,dx,$$

then we write this as

$$\int u\,dv + \int v\,du = uv,$$

or

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Good news: We got rid of the integral we wanted to solve!
Bad news: And replaced it with another integral....
So, of course, this may not make things better; in fact, making a wrong choice can make it worse!

2 All of the examples

Example 2.1. Let us consider

$$\int xe^x\,dx.$$

We want to choose $u = x$ and $dv = e^x\,dx$. It is helpful to use such a table:

$u = x$	$v = e^x$
$du = dx$	$dv = e^x\,dx$

We then have

$$\begin{aligned}\int u\,dv &= uv - \int v\,du\\ \int xe^x\,dx &= xe^x - \int e^x\,dx\\ &= xe^x - e^x + C.\end{aligned}$$

Example 2.2. Now try

$$\int x\sin(x)\,dx.$$

We want to choose

$u = x$	$v = -\cos(x)$
$du = dx$	$dv = \sin(x)\,dx$

We then have

$$\begin{aligned}\int u\,dv &= uv - \int v\,du\\ \int x\sin(x)\,dx &= -x\cos(x) - \int (-\cos(x))\,dx\\ &= -x\cos(x) + \sin(x) + C.\end{aligned}$$

Example 2.3. A tricky example. Let us consider

$$\int \ln x\,dx.$$

It's not clear what to do here because the function doesn't really look like a product. But let us choose $u = \ln x$ and $dv = dx$, then we have

$u = \ln x$	$v = x$
$du = \dfrac{dx}{x}$	$dv = dx$

We then have

$$\begin{aligned}\int u\,dv &= uv - \int v\,du\\ \int \ln(x)\,dx &= x\ln(x) - \int dx\\ &= x\ln(x) - x + C.\end{aligned}$$

Example 2.4.

$$\int e^x\sin(x)\,dx.$$

We will choose $u = \sin(x)$ and $dv = e^x\,dx$, giving

$u = \sin(x)$	$v = e^x$
$du = \cos(x)\,dx$	$dv = e^x\,dx$

We then have

$$\int u\,dv = uv - \int v\,du$$
$$\int e^x \sin(x)\,dx = e^x \sin(x) - \int e^x \cos(x)\,dx.$$

Now we see that we have an integral that we still don't know how to do, so let us try integration by parts again!

To do

$$\int e^x \cos(x)\,dx,$$

We choose $u = \cos(x)$ and $dv = e^x\,dx$, giving

$u = \cos(x)$	$v = e^x$
$du = -\sin(x)\,dx$	$dv = e^x\,dx$

We then have

$$\int u\,dv = uv - \int v\,du$$
$$\int e^x \cos(x)\,dx = e^x \cos(x) + \int e^x \sin(x)\,dx.$$

and it unfortunately looks as if we are back where we started. However, let us put it together:

$$\begin{aligned}\int e^x \sin(x)\,dx &= e^x \sin(x) - \int e^x \cos(x)\,dx \\ &= e^x \sin(x) - \left(e^x \cos(x) + \int e^x \sin(x)\,dx\right) \\ &= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)\,dx.\end{aligned}$$

We can put all of the integrals on the left-hand side to get

$$2\int e^x \sin(x)\,dx = e^x \sin(x) - e^x \cos(x)$$
$$\int e^x \sin(x)\,dx = \frac{1}{2}\left(e^x \sin(x) - e^x \cos(x)\right).$$

Example 2.5. Sometimes we need to do a little more work, but we will get there. For example, let us consider

$$\int \sin^5(x)\,dx.$$

A reasonable guess might be $u = \sin^4(x)$ and $dv = \sin(x)\,dx$.

$u = \sin^4(x)$	$v = -\cos(x)$
$du = 4\sin^3(x)\cos(x)\,dx$	$dv = \sin(x)\,dx$

We then have

$$\begin{aligned}\int u\,dv &= uv - \int v\,du\\ \int \sin^5(x)\,dx &= -\sin^4\cos(x) - \int 4\sin^3(x)\cos(x)(-\cos(x))\,dx\\ &= -\sin^4\cos(x) + \int 4\sin^3(x)\cos^2(x)\,dx.\end{aligned}$$

This doesn't necessarily look like an improvement, but of course we can use the fact that

$$\cos^2(x) = 1 - \sin^2(x),$$

so the last line becomes

$$\begin{aligned}\int \sin^5(x)\,dx &= -\sin^4\cos(x) + \int 4\sin^3(x)\cos^2(x)\,dx\\ &= -\sin^4\cos(x) + \int 4\sin^3(x)(1-\sin^2(x))\,dx\\ &= -\sin^4\cos(x) + \int 4\sin^3(x)\,dx - \int 4\sin^5(x)\,dx.\end{aligned}$$

Again, collecting like terms on the left-hand side gives

$$5\int \sin^5(x)\,dx = -\sin^4\cos(x) + \int 4\sin^3(x)\,dx \tag{1}$$

$$\int \sin^5(x)\,dx = -\frac{1}{5}\sin^4\cos(x) + \frac{4}{5}\int \sin^3(x)\,dx. \tag{2}$$

There is still some work to do, but the last integral that we obtained is better. So we try

$$\int \sin^3(x)\,dx$$

A reasonable guess might be $u = \sin^2(x)$ and $dv = \sin(x)\,dx$.

$u = \sin^2(x)$	$v = -\cos(x)$
$du = 2\sin(x)\cos(x)\,dx$	$dv = \sin(x)\,dx$

We then have

$$\begin{aligned}\int u\,dv &= uv - \int v\,du\\ \int \sin^3(x)\,dx &= -\sin^2\cos(x) - \int 2\sin(x)\cos(x)(-\cos(x))\,dx\\ &= -\sin^2\cos(x) + \int 2\sin(x)\cos^2(x)\,dx.\end{aligned}$$

Using the same trick gives

$$\begin{aligned}\int \sin^3(x)\,dx &= -\sin^2\cos(x) + \int 2\sin(x)\cos^2(x)\,dx\\ &= -\sin^2\cos(x) + 2\int \sin(x)(1-\sin^2(x))\,dx\\ &= -\sin^2\cos(x) + 2\int \sin(x)\,dx - 2\int \sin^3(x)\,dx\end{aligned}$$

and this gives

$$\int \sin^3(x)\,dx = -\frac{1}{3}\sin^2(x)\cos(x) + \frac{2}{3}\int \sin(x)\,dx = -\frac{1}{3}\sin^2(x)\cos(x) - \frac{2}{3}\cos(x).$$

Adding this to (1) gives

$$\begin{aligned}\int \sin^5(x)\,dx &= -\frac{1}{5}\sin^4\cos(x) + \frac{4}{5}\int \sin^3(x)\,dx \\ &= -\frac{1}{5}\sin^4\cos(x) + \frac{4}{5}\left(-\frac{1}{3}\sin^2(x)\cos(x) - \frac{2}{3}\cos(x)\right) \\ &= -\frac{1}{5}\sin^4\cos(x) - \frac{4}{15}\sin^2(x)\cos(x) - \frac{8}{15}\cos(x).\end{aligned}$$

3 Rule of Thumb: The ladder

It is not always so clear which function should be integrated, and which function should be differentiated. There is no hard-and-fast rule as to what will always work but there is a rule of thumb that frequently works, and is probably the first thing that you should try. Draw the following ladder

Trigonometric and exponential functions.
Powers.
Inverse trigonometric and logarithms.

Rule of Thumb: *Differentiate the term that is lower on the ladder and integrate the term that is higher on the ladder.*

This will not always work (it is a rule of thumb, not a theorem), but will work in most of the examples that you encounter in practice. For example, if you were given the integral $\int x^2\sin(3x)dx$ you would want to differentiate the x^2 and integrate the $\sin(3x)$ since x^2 is a power and is lower on the table than $\sin(3x)$, a trig function. So you would take $u = x^2, dv = \sin(3x)dx$. On the other hand if you were given the integral $\int x^2\arcsin(3x)$ you would want to differentiate the $\arcsin$ and integrate the x^2, since inverse trig functions are lower on the ladder. So you would want to take $u = \arcsin(3x), dv = x^2dx$

If the functions are on the same rung of the ladder it usually won't matter which one you differentiate and which one you integrate. For instance if you are trying to compute $\int e^{2x}\sin(3x)dx$ either setting $u = e^{2x}, dv = \sin(3x)dx$ or setting $u = \sin(3x), dv = e^{2x}dx$ will work.

Example 3.1. Compute

$$\int \arctan(x)dx$$

Inverse trig functions are at the bottom of the ladder, so we take $u = \arctan(x), dv = dx$ gives $du = \frac{dx}{1+x^2}, v = x$. Thus we have

$$\int \arctan(x)dx = x\arctan(x) - \int \frac{xdx}{1+x^2}dx.$$

The integral on the left yields to the u-substitution $u = 1+x^2; du = 2xdx$ to give

$$\int \arctan(x)dx = x\arctan(x) - \int \frac{xdx}{1+x^2}dx = x\arctan(x) - \frac{1}{2}\ln|1+x^2| + C$$