Math 231E, 2016. Midterm 1.

- This exam has 30 questions and is worth a total of 100 points.
- You must not communicate with other students during this test. No books, notes, **calculators**, or electronic devices allowed.
- Please fill out all of the information below. Make sure to fill out your Scantron including your name, UIN number, and NetID.

1. Fill in your information:

Full Name: ______________________________

UIN (Student Number): ______________________________

NetID: ______________________________

2. Fill out name, student number (UIN) and NetID on Scantron sheet. Then fill in the following answers on the Scantron form:

91. A
92. A
93. A
94. A
95. D
96. C

1. (3 points) If a fjarn (fj) is defined as 4 meters and a glorb (gl) is defined as 10 seconds, convert the velocity $12\,\mathrm{m/s}$ to fjarns per glorb.

(A) ★ $30\,\mathrm{fj/gl}$

(B) $3\,\mathrm{fj/gl}$

(C) $480\,\mathrm{fj/gl}$

(D) $\frac{5}{24}\,\mathrm{fj/gl}$

(E) $\frac{24}{5}\,\mathrm{fj/gl}$

Solution. We convert

$$12\frac{\mathrm{m}}{\mathrm{s}} \times \frac{1\,\mathrm{fj}}{4\,\mathrm{m}} \times \frac{10\,\mathrm{s}}{1\,\mathrm{gl}} = (12 \cdot \frac{1}{4} \cdot 10)\frac{\mathrm{fj}}{\mathrm{gl}} = 30\frac{\mathrm{fj}}{\mathrm{gl}}.$$

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2. (4 points) Let $y = f(x)$ be defined implicitly by

$$y^2 - xy = 1.$$

Find $\dfrac{dy}{dx}$ in terms of y and x.

(A) ★ $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$

(B) $\dfrac{dy}{dx} = \dfrac{x}{x + y^2}$

(C) $\dfrac{dy}{dx} = \dfrac{1}{xy^3}$

(D) $\dfrac{dy}{dx} = -\dfrac{y}{2y + x}$

(E) $\dfrac{dy}{dx} = \arctan(x/y)$

Solution. The derivative is given by implicit differentiation

$$\begin{aligned} y^2 - xy &= 1, \\ 2y\frac{dy}{dx} - y - x\frac{dy}{dx} &= 0, \\ (2y - x)\frac{dy}{dx} &= y, \\ \frac{dy}{dx} &= \frac{y}{2y - x}. \end{aligned}$$

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3. (4 points) If $f(x) = x^{14}$, then $f'(1) =$

(A) 1

(B) 13

(C) 0

(D) $14x^3$

(E) ★ 14

Solution. Using the Power Rule, we have

$$f'(x) = 14x^{13},$$

so $f'(1) = 14$.

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4. (4 points) What is the second-order Taylor series for $\frac{1}{1+3x}$ at $a = 0$?

(A) $3x + 2x^2 + O(x^3)$

(B) ★ $1 - 3x + 9x^2 + O(x^3)$

(C) $1 - \frac{1}{3x} + \frac{1}{9x^2} + O(x^3)$

(D) $1 + \frac{x^2}{9} + O(x^3)$

(E) $1 - 2x + 3x^2 + O(x^3)$

Solution. The definition of the second-order series of $f(x)$ at $a = 0$ is

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + O(x^3)$$

Noting that $f(x) = \frac{1}{1+3x}$, $f'(x) = \frac{-3}{(1+3x)^2}$, $f''(x) = \frac{18}{(1+2x)^3}$, we have

$$1 - 3x + 9x^2 + O(x^3).$$

Alternately recall that $\frac{1}{1-x}$ is the geometric series $1 + x + x^2 + x^3 + \dots$ and plug in $-3x$.

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5. (4 points) Compute the limit

$$\lim_{x \to 0+} |x|.$$

(A) ★ 0

(B) 1

(C) does not exist

(D) -1

(E) ∞

Solution. For $x > 0$, $|x| = x$ and $\lim_{x \to 0+} x = 0$.

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6. (4 points) Evaluate $\lim\limits_{x\to\infty} \dfrac{\sin(3x)}{x}$.

(A) $+\infty$

(B) 3

(C) ★ 0

(D) 1

(E) does not exist

Solution. The definition of the limit at ∞ means

$$\lim_{x\to\infty} \frac{\sin(3x)}{x} = \lim_{x\to 0+} \frac{\sin(3/x)}{1/x} = \lim_{x\to 0+} x\sin(3/x).$$

We can use the Squeeze Theorem. Recall that sin takes values between -1 and 1 and so:

$$-1 < \sin(3/x) < 1,$$
$$-x < x\sin(3/x) < x,$$

and

$$\lim_{x\to 0} x = \lim_{x\to 0}(-x) = 0,$$

so the limit is 0.

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7. (4 points) Which of the following are equivalent to

$$\frac{1}{3+i}?$$

(A) $\dfrac{3}{\sqrt{10}} - \dfrac{i}{\sqrt{10}}$

(B) ★ $\dfrac{3}{10} - \dfrac{i}{10}$

(C) $3 - i$

(D) $\dfrac{1}{3} + i$

(E) $i + 3$

Solution. Using the formula $|z|^2 = z\overline{z}$, we have

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

In this case, this is

$$\frac{3-i}{(3^2+1^2)} = \frac{3-i}{10} = \frac{3}{10} - \frac{i}{10}.$$

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8. (4 points) If $f(x) = xe^x$, then $f'(4) =$

(A) 0

(B) e^4

(C) ★ $5e^4$

(D) $4e^4$

(E) $4e^5$

Solution. Using the Product Rule, we have $f'(x) = e^x + xe^x$. At $x = 4$, this gives $e^4 + 4e^4 = 5e^4$.

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9. (3 points) If $f(x) = x \ln x - x$, compute $f'(2)$.

(A) $-\infty$

(B) -1

(C) 1

(D) 0

(E) ★ $\ln 2$

Solution. We use the product rule

$$\frac{d}{dx}(x \ln x - x) = \ln x + x \cdot \frac{1}{x} - 1 = \ln x,$$

and then plugging in $x = 2$ gives $\ln 2$.

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10. (4 points) Compute the limit

$$\lim_{x\to 2} \sin(3x).$$

(A) 0

(B) $\sin(2)$

(C) ★ $\sin(6)$

(D) does not exist

(E) 1

Solution. This is a continuous function, so we can just plug in. Thus

$$\lim_{x\to 2} \sin(3x) = \sin(2 \cdot 3) = \sin(6).$$

11. (4 points) Evaluate $\lim_{x \to 0} \frac{\sin(2x)}{x}$.

(A) ★ 2

(B) does not exist

(C) 1

(D) 0

(E) $+\infty$

Solution. We can use Taylor series here. Recall that the Taylor series for $\sin(x)$ is

$$x - \frac{x^3}{6} + O(x^5)$$

and plugging in $x \mapsto 2x$ gives

$$2x - \frac{4}{3}x^3 + O(x^5)$$

This means that

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = \lim_{x \to 0} \frac{2x - \frac{4}{3}x^3 + O(x^5)}{x} = \lim_{x \to 0} 2 - \frac{4}{3}x^2 + O(x^4) = 2.$$

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12. (4 points) What is the third-order Taylor polynomial for $3\sin(4x)$ at $a = 0$?

(A) $3 + 4x + 12x^2 + 36x^3$

(B) $4x + 12x^3$

(C) ★ $12x - 32x^3$

(D) $2 - 3x + 4x^2$

(E) $x - 12x^2 + 16x^3$

Solution. The definition of the third-order polynomial of $f(x)$ at $a = 0$ is

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3.$$

Noting that $f(x) = 3\sin(4x)$, $f'(x) = 12\cos(4x)$, $f''(x) = -48\sin(4x)$, and $f'''(x) = -192\cos(4x)$ we have

$$f(0) = 0, f'(0) = 12, f''(0) = 0, f'''(0) = -192,$$

so

$$T_3(x) = 12x - 32x^3.$$

Alternately, we could work out the third-order Taylor polynomial of $\sin(x)$, substitute $4x$, then multiply by 3.

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13. (3 points) If

$$f(x) = ((x^2 + 1)^2 + 3x)^2,$$

compute $f'(1)$.

(A) 14

(B) 308

(C) 98

(D) 529

(E) ★ 154

Solution. We use the power and chain rules, so

$$\frac{d}{dx}((x^2 + 1)^2 + 3x)^2 = 2((x^2 + 1)^2 + 3x)(2(x^2 + 1)(2x) + 3),$$

and plugging in $x = 1$ gives $2(4 + 3)(8 + 3) = 2 \cdot 7 \cdot 11 = 154$.

14. (3 points) Simplify

$$\left(\frac{1+i}{1+3i}\right)^2.$$

(A) ★ $\frac{3}{25} - \frac{4i}{25}$

(B) $\frac{1}{10} + \frac{i}{10}$

(C) $\frac{8}{9} - \frac{2i}{3}$

(D) $1 + \frac{i}{9}$

(E) $\frac{1}{5} + \frac{4i}{25}$

Solution. There are two ways to approach this. We can square both the top and bottom, then divide; or we can divide and then square. The latter will be more efficient but either technique should give the same answer.

First, the division. We write

$$\frac{1+i}{1+3i} = \frac{(1+i)(1-3i)}{(1+3i)(1-3i)} = \frac{1-2i-3i^2}{10} = \frac{4-2i}{10} = \frac{2}{5} - \frac{i}{5}.$$

We then compute

$$\left(\frac{2}{5} - \frac{i}{5}\right)^2 = \frac{4}{25} - \frac{2i}{25} - \frac{2i}{25} - \frac{1}{5} = \frac{3}{25} - \frac{4i}{25}.$$

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15. (3 points) Find a pair α, β such that

$$\lim_{x \to 0} \frac{\sin(\alpha x)}{\beta x} = 4.$$

(A) ★ $\alpha = 8, \quad \beta = 2$

(B) $\alpha = 2, \quad \beta = 8$

(C) $\alpha = 0, \quad \beta = 4$

(D) $\alpha = 4, \quad \beta = 4$

(E) $\alpha = 4, \quad \beta = 0$

Solution. We see that if we plug in $x = 0$, we obtain $\frac{0}{0}$, which is indeterminant; moreover, we cannot use a Limit Law at this stage. There are several ways to proceed.

Let us use Taylor series at 0, writing

$$\sin(x) = x - \frac{x^3}{6} + O(x^5),$$
$$\sin(\alpha x) = \alpha x - \frac{(\alpha x)^3}{6} + O((\alpha x)^5),$$

so

$$\frac{\sin(\alpha x)}{\beta x} = \frac{\alpha x - \frac{(\alpha x)^3}{6} + O((\alpha x)^5)}{\beta x} = \frac{\alpha}{\beta} - \frac{\alpha^3 x^2}{6\beta} + O(x^4),$$

so when $x \to 0$ we obtain α/β. Therefore we need $\alpha/\beta = 4$, and only one of the choices above works, namely $\alpha = 8, \beta = 2$.

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16. (3 points) Compute the limit

$$\lim_{x\to\infty} \frac{9x^{2016} + 57x^{1776} + 128x^{1066}}{36x^{2016} + 88x^{1776} + 195x^{1066}}.$$

(A) $\dfrac{57}{88}$

(B) ★ $\dfrac{1}{4}$

(C) 0

(D) $\dfrac{128}{195}$

(E) $+\infty$

Solution. Divide both top and bottom by x^{2016}, and we obtain

$$\lim_{x\to\infty} \frac{9x^{2016} + 57x^{1776} + 128x^{1066}}{36x^{2016} + 88x^{2001} + 195x^{1066}} = \lim_{x\to\infty} \frac{9 + 57x^{-240} + 128x^{-950}}{36 + 88x^{-240} + 195x^{-950}} = \frac{9}{36} = \frac{1}{4}.$$

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17. (3 points) What is the third-order Taylor polynomial for $\sin(x)$ at $a = 9$?

(A) $\cos(9) - \sin(9)(x-9) + \frac{-\cos(9)}{2}(x-9)^2 + \frac{\sin(9)}{6}(x-9)^3$

(B) $(x-9) - \frac{(x-9)^3}{6}$

(C) ★ $\sin(9) + \cos(9)(x-9) + \frac{-\sin(9)}{2}(x-9)^2 + \frac{-\cos(9)}{6}(x-9)^3$

(D) $\cos(9) + \sin(9)(x-9) + \frac{-\cos(9)}{2}(x-9)^2 + \frac{\sin(9)}{6}(x-9)^3$

(E) $1 - \frac{(x-9)^2}{2} + \frac{(x-9)^4}{24}$

Solution. The definition of the third-order polynomial of $f(x)$ at $a = 9$ is

$$T_3(x) = f(9) + f'(9)(x-9) + \frac{f''(9)}{2}(x-9)^2 + \frac{f'''(9)}{6}(x-9)^3.$$

Noting that $f(x) = \sin(x)$; $f'(x) = \cos(x)$; $f''(x) = -\sin(x)$; $f'''(x) = -\cos(x)$, we have

$$\begin{aligned} f(9) &= \sin(9) \\ f'(9) &= \cos(9) \\ f''(9) &= -\sin(9) \\ f'''(9) &= -\cos(9), \end{aligned}$$

so

$$\sin(9) + \cos(9)(x-9) + \frac{-\sin(9)}{2}(x-9)^2 + \frac{-\cos(9)}{6}(x-9)^3$$

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18. (3 points) Compute $f'(x)$, where

$$f(x) = x \sin(x^3).$$

(A) $\sin(x^3) - 3x^3 \cos(x^3)$

(B) $3x \cos(x^3) + \sin(x^3)$

(C) $-3x \cos(x^3) + \sin(x^3)$

(D) $-x \cos(x^3)$

(E) ★ $\sin(x^3) + 3x^3 \cos(x^3)$

Solution. We use the product and chain rules, so

$$\frac{d}{dx}(x \sin(x^3)) = \sin(x^3) + x\frac{d}{dx}(\sin(x^3)) = \sin(x^3) + x(3x^2)(\cos(x^3))$$

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19. (3 points) Compute $f'(e^e)$, where

$$f(x) = \ln \ln \ln x.$$

(A) $\dfrac{1}{e^e + 1}$

(B) e^{-e}

(C) $\ln \ln \ln 1$

(D) ★ $\dfrac{1}{e^{e+1}}$

(E) $\ln \ln \ln e$

Solution. We use the Chain Rule over and over:

$$f'(x) = \frac{1}{\ln \ln x} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}.$$

Plugging in $x = e^e$ gives

$$1 \cdot \frac{1}{e} \cdot \frac{1}{e^e} = \frac{1}{ee^e} = \frac{1}{e^{e+1}}.$$

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20. (3 points) Compute the first three nonzero terms of the Taylor series at $a = 0$ for

$$f(x) = \frac{\sin x - x + x^3/6}{x^4}$$

(A) $f(x) = -\frac{x}{5!} + \frac{x^3}{7!} - \frac{x^5}{9!} + O(x^7)$

(B) ★ $f(x) = \frac{x}{5!} - \frac{x^3}{7!} + \frac{x^5}{9!} + O(x^9)$

(C) $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)$

(D) $f(x) = \frac{x}{5!} - \frac{x^3}{7!} + \frac{x^5}{9!} + O(x^7)$

(E) $f(x) = \frac{1}{5!} - \frac{x^2}{7!} + \frac{x^3}{9!} + O(x^5)$

Solution. Use the known series for $\sin(x)$ and algebraic manipulations.

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + O(x^{11}) \\
\sin x - x + \frac{x^3}{6} &= \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + O(x^{11}) \\
\frac{\sin x - x + \frac{1}{6}x^3}{x^2} &= \frac{x^3}{5!} - \frac{x^5}{7!} + \frac{x^7}{9!} + O(x^9)
\end{aligned}$$

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21. (3 points) Compute the limit

$$L = \lim_{x \to 0} \sin\left(\frac{\pi \sin(2x)}{8x}\right)$$

if it exists.

(A) $L = 0$

(B) $L = 1/2$

(C) ★ $L = \sqrt{2}/2$

(D) $L = \sqrt{2}$

(E) $L = \infty$

Solution. Let us first look at the "inside" part of the function, namely $\sin(2x)/8x$. Using L'Hôpital:

$$\lim_{x \to 0} \frac{\sin(2x)}{8x} = \lim_{x \to 0} \frac{2\cos(2x)}{8} = \frac{1}{4}.$$

Since $\sin(x)$ is continuous,

$$\lim_{x \to 0} \sin\left(\frac{\pi \sin(4x)}{16x}\right) = \sin\left(\lim_{x \to 0} \pi \frac{\sin(4x)}{16x}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

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22. (3 points) Compute the following limit: $\lim_{x \to 0^-} \frac{|3x|}{1 - e^{3x}}$

(A) $+\infty$

(B) 0

(C) ★ 1

(D) does not exist

(E) -1

Solution. For $x < 0$ we have $|3x| = -3x$ and apply l'Hôpital. Alternately, after replacing $|3x| = -3x$, we can use the Taylor series $e^{3x} = 1 + 3x + O(x^2)$ to find

$$\lim_{x \to 0^-} \frac{|3x|}{1 - e^{3x}} = \lim_{x \to 0^-} \frac{-3x}{-3x + O(x^2)} = 1.$$

23. (3 points) Compute

$$L = \lim_{x\to 1-} \frac{d}{dx}(\cos^{-1}(x)).$$

(A) $L = \infty$

(B) ★ $L = -\infty$

(C) $L = 1$

(D) limit does not exist

(E) $L = 0$

Solution. We first use implicit differentiation to compute the derivative. If $y = \sin^{-1}(x)$, then

$$x = \cos(y)$$

and differentiating both sides gives

$$1 = -\sin(y)\frac{dy}{dx},$$

or $y'(x) = -1/\sin(y) = -1/\sin(\cos^{-1}(x))$. Drawing the applicable triangle gives

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

Now, we see that the limit as $x \to 1-$ is a bit complicated, since we cannot plug in. Notice that this function has a vertical asymptote at $x = 1$ when approaching from the left, and is undefined to the right of $x = 1$. For x less than 1, but close to 1, we have a very small positive denominator, so the fraction is large and negative. From this we see that the limit from the left is $-\infty$.

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24. (3 points) Compute the limit

$$\lim_{x\to 0} \frac{\cos(x^9)-1}{\sin(x^6)-x^6}.$$

(A) 0

(B) $\frac{1}{3}$

(C) ★ 3

(D) $\frac{2}{3}$

(E) $\frac{3}{2}$

Solution. Plugging in gives 0/0, so we need some technique. l'Hôpital's Rule is applicable in theory, but will take forever. So we use Taylor Series. Expanding the numerator gives

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{6} + O(x^5),\\
\sin(x^6) &= x^6 - \frac{x^{18}}{6} + O(x^{30}),\\
\sin(x^6) - x^6 &= -\frac{x^{18}}{6} + O(x^{30}).
\end{aligned}$$

Expanding the denominator gives

$$\begin{aligned}
\cos(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6),\\
\cos(x^9) &= 1 - \frac{x^{18}}{2} + \frac{x^{36}}{24} + O(x^{54}),\\
\cos(x^9) - 1 &= -\frac{x^{18}}{2} + \frac{x^{36}}{24} + O(x^{54}),
\end{aligned}$$

so

$$\frac{\cos(x^9)-1}{\sin(x^6)-x^6} = \frac{\frac{x^{18}}{2} + \frac{x^{36}}{24} + O(x^{54})}{-\frac{x^{18}}{6} + O(x^{30})} = \frac{x^{18}}{x^{18}} \frac{-\frac{1}{2} + O(x^{18})}{-\frac{1}{6} + O(x^{12})},$$

and as $x \to 0$ this limit is clearly $(-1/2)/(-1/6) = 3$.

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25. (3 points) Consider the expression

$$L = \lim_{x\to 0} \frac{\cos(\sin(x)) - q}{\cos^{-1}(\sin^{-1}(x^2)) - \pi/2}.$$

First determine the value of q so that the limit exists and is finite, and then compute L.

(A) $q = \pi/2, \quad L = -1/2$

(B) $q = \pi/2, \quad L = 2$

(C) $q = 0, \quad L = -1$

(D) $q = 1, \quad L = 1$

(E) ★ $q = 1, \quad L = 1/2$

Solution. The answer is $q = 1$ and $L = 1/2$.

First let us see why q must be 1. Notice that as $x \to 0$, the denominator goes to 0: $\sin^{-1}(0) = 0$ and $\cos^{-1}(0) = \pi/2$. If the numerator does not go to zero, then the limit will not exist and be finite. Therefore we need the numerator to go to zero, and since $\cos(\sin(0)) = 1$, we need to choose $q = 1$.

Now, we compute the limit using Taylor series. First recall

$$\cos(x) = 1 - \frac{x^2}{2} + O(x^5),$$
$$\sin(x) = x - \frac{x^3}{6} + O(x^5),$$

so plugging the second into the first gives

$$\cos(\sin(x)) = 1 - \frac{1}{2}\left(x - \frac{x^3}{6} + O(x^4)\right)^2 = 1 - \frac{x^2}{2} + O(x^4).$$

We don't have the Taylor series for $\sin^{-1}(x)$ and $\cos^{-1}(x)$ memorized but we can derive them. Note that if $f(x) = \sin^{-1}(x)$, then

$$f'(x) = (1-x^2)^{-1/2}, \quad f''(x) = x(1-x^2)^{-3/2}, \quad f'''(x) = (2x^2+1)(1-x^2)^{-5/2},$$

and thus

$$f(0) = 0, \quad , f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = 1.$$

Therefore

$$\sin^{-1}(x) = x + \frac{x^3}{6} + O(x^4).$$

Similarly, if $g(x) = \cos^{-1}(x)$, then

$$g'(x) = -(1-x^2)^{-1/2}, \quad g''(x) = -x(1-x^2)^{-3/2}, \quad g'''(x) - (2x^2+1)(1-x^2)^{-5/2},$$

and thus

$$g(0) = \frac{\pi}{2}, \quad , f'(0) = -1, \quad f''(0) = 0, \quad f'''(0) = -1.$$

Therefore

$$\cos^{-1}(x) = \frac{\pi}{2} - x - \frac{x^3}{6} + O(x^4).$$

Finally, plugging in we have

$$\cos^{-1}(\sin^{-1}(x^2)) = \frac{\pi}{2} - \left(x^2 + \frac{x^6}{6} + O(x^8)\right) - \frac{1}{6}\left(x^2 + \frac{x^6}{6} + O(x^8)\right)^3$$

and so

$$\cos^{-1}(\sin^{-1}(x^2)) - \frac{\pi}{2} = -x^2 + O(x^4).$$

Then we have

$$\frac{\cos(\sin(x)) - 1}{\cos^{-1}(\sin^{-1}(x^2)) - \pi/2} = \frac{-x^2/2 + O(x^4)}{-x^2 + O(x^4)} = \frac{1}{2} + O(x^2),$$

so as $x \to 0$ we obtain $1/2$.

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26. (3 points) Compute the third-order Taylor series for $f(x) = 6e^x \cos(2x)$ at $a = 0$.

(A) ★ $6 + 6x - 9x^2 - 11x^3 + O(x^4)$

(B) $2 + 4x - 9x^2 + 7x^3 + O(x^4)$

(C) $6 + 6x - x^2 - 7x^3 + O(x^4)$

(D) $3 + x - 5x^2 + 6x^3 + O(x^4)$

(E) $6 + 2x - 9x^2 - 11x^3 + O(x^4)$

Solution. We expand out each piece of the product then multiply together. Since we want a final answer that involves the third-order term, we must expand each term to third order. So:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \mathcal{O}(x^4), \\ \cos(2x) &= 1 - \frac{(2x)^2}{2} + O(x^4). \end{aligned}$$

Multiplying the two polynomials gives

$$\begin{aligned} 6\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \mathcal{O}(x^4)\right)\left(1 - \frac{(2x)^2}{2} + O(x^4)\right) \\ &= 6\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} - 2x^2 - 2x^3 + O(x^4)\right) \\ &= 6 + 6x - 9x^2 - 11x^3 + O(x^4). \end{aligned}$$

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27. (3 points) Compute the limit

$$\lim_{x\to\infty} \frac{4x+\sin(x)}{x}$$

(A) does not exist

(B) ★ 4

(C) ∞

(D) 0

(E) 1/6

Solution. First write

$$\lim_{x\to\infty} \frac{4x+\sin(x)}{x} = \lim_{x\to\infty} \left(4 + \frac{\sin(x)}{x}\right) = 4 + \lim_{x\to\infty} \frac{\sin(x)}{x}.$$

The remaining limit can be handled by a Squeeze Theorem, namely:

$$\lim_{x\to\infty} \frac{\sin(x)}{x} = \lim_{x\to 0+} x\sin(1/x),$$

and for $x > 0$,

$$\begin{aligned} -1 &< \sin(1/x) < 1, \\ -x &< x\sin(1/x) < x, \end{aligned}$$

and since $\lim_{x\to 0+} x = 0$, the Squeeze Theorem tells us that the limit is zero. And of course $4 + 0 = 4$.

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28. (3 points) Determine for which p it is true that

$$\lim_{x\to\infty} x^p = \infty.$$

(A) $p \leq 1$

(B) $p < 0$

(C) $p > 1$

(D) $p \geq 0$

(E) ★ $p > 0$

Solution. By definition, we have

$$\lim_{x\to\infty} x^p = \lim_{x\to 0+} \left(\frac{1}{x}\right)^p = \lim_{x\to 0+} x^{-p}.$$

Now, if $-p > 0$, then the limit is zero. If $-p = 0$, then the limit is one. However, if $-p < 0$ then the limit is indeed infinity. We can use an "M–δ proof" to prove this:

$$\begin{aligned} x^{-p} &> M, \\ \frac{1}{M} &> x^p \\ \frac{1}{\sqrt[p]{M}} &> x \end{aligned}$$

and since $p > 0, M > 0$ all of these are equivalent. In short, if we choose $0 < x < 1/\sqrt[p]{M}$, then $x^{-p} > M$. Since this works for any $M > 0$, this means that the limit is ∞.

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29. (3 points) Compute the limit $\lim_{x\to\infty} \dfrac{x^{23}}{e^{3x}}$.

(A) $-\infty$

(B) 23

(C) $\dfrac{23}{3}$

(D) ∞

(E) ★ 0

Solution. There are many ways to approach this problem. One of them is to use the result proved in class, namely that $\lim_{x\to\infty} x^n/e^x = 0$ for any integer n, which means $(3x)^n/e^{3x} \to 0$ as well, and this function is just a constant times the function we care about.

Another way is to use the ideas behind that result, and notice that

$$e^x > \frac{x^{24}}{24!},$$

since for $x > 0$ the exponential is larger than any of its Taylor polynomials. So we have

$$\frac{x^{23}}{e^x} < \frac{24!x^{24}}{x^{23}} = \frac{24!}{x}.$$

Since $\lim_{x\to\infty} \dfrac{24!}{x} = 0$, and since $\dfrac{x^{23}}{e^x} > 0$, this means by the Squeeze Theorem that our limit is 0 as well.

Finally, one could apply l'Hôpital's Rule. However, it would take 42 applications to do this, but it would again give the correct answer.

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30. (3 points) Compute the limit

$$L = \lim_{x \to 0} \frac{\cos(1/x)}{x}$$

if it exists.

(A) $L = 1$

(B) $L = \frac{\sqrt{2}}{2}$

(C) $L = \infty$

(D) ★ limit does not exist

(E) $L = 0$

Solution. Let $f(x) = \cos(1/x)/x$.To see that this function does not have a limit as $x \to 0$, first notice that if x is in $\{\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \ldots\}$ then $f(x) = 0$ which shows that the only possible limit as $x \to 0$ is $L = 0$. However, if x is in $\{\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots\}$ then $f(x) = \frac{1}{x}$ gets larger and larger in absolute value which shows that the limit is not equal to zero. It follows that the limit does not exist.
