1.4 Existence-Uniqueness theorem

In the previous section we saw that the differential equation

$$y' = y^{\frac{1}{3}} \qquad\qquad y(0) = 0$$

does not have a unique solution. There are at least two solutions satisfying the same differential equation and the same initial condition:

$$\begin{aligned} y_1(t) &= (\frac{2}{3}t)^{\frac{3}{2}} \\ y_2(t) &= 0. \end{aligned}$$

This is not good for a physical problem. If one is calculating some physical quantity - the trajectory of a rocket, the temperature of a reactor, etc. - it is important to know that

- The problem has a solution. (Existence)
- There is **only one** solution. (Uniqueness)

This is the problem of existence and uniqueness of solutions. The example above shows us that this is something that cannot just be assumed. In this section we give a theorem that guarantees the existence and uniqueness of solutions to certain differential equations.

Let's begin by stating the following theorem, which is the fundamental existence result for ordinary differential equations

Theorem 1.4.1 (Existence and Uniqueness). *Consider the first order initial value problem*

$$\frac{dy}{dt} = f(y,t) \qquad y(t_0) = y_0$$

- *If $f(y,t)$ is continuous in a neighborhood of the point (y_0, t_0) then a solution exists in some rectangle $|t - t_0| < \delta, |y - y_0| < \epsilon$.*
- *If, in addition, $\frac{\partial f}{\partial y}(y,t)$ is continuous in a neighborhood of the point (y_0, t_0) then the solution is unique.*

The above conditions are sufficient, but not necessary. It may be that one or the other of these conditions fails and the solution still exists/is unique.

Let's consider some examples:

Example 1.4.1. *What can you say about the equation*

$$\frac{dy}{dt} = (y-5)\ln(|y-5|) + t \qquad y(0) = 5$$

the function $(y-5)\ln(|y-5|) + t$ is continuous in a neighborhood of $y = 5, t = 0$. However computing the first partial with respect to y gives

$$\frac{\partial f}{\partial y}(y,t) = 1 + \ln|y-5|$$

which is **NOT** *continuous in a neighborhood of the point $(t = 0, y = 5)$. The above theorem guarantees that there is a solution, but does not guarantee that it is unique. Again the above theorem gives a sufficient condition: The fact that $\frac{\partial f}{\partial y}$ is not continuous does not necessarily imply that there is more than one solution, only that there* **may** *be more than one solution.*

Note that if we had the initial condition $y(0) = 1$ (or anything other than $y(0) = 5$ the theorem would apply, and we would have a unique solution, at least in a neighborhood of the initial point.

The method of proof is interesting. While we won't actually prove it we will mention how it is proved. The method is called "Picard iteration". Iterative methods are important in many aspects of numerical analysis and engineering as well as in mathematics, so it is worthwhile to give some sense of how this particular iteration scheme works. In this case Picard iteration is a useful theoretical tool but is not a very good way to solve equations numerically in practice. Numerical methods will be covered later in the notes.

The idea behind Picard iteration is as follows: suppose that one want to solve

$$y' = f(y,t) \qquad y(t_0) = y_0.$$

The idea is to construct a sequence of functions $y_n(t)$ that get closer and closer to solving the differential equation. The initial guess, $y_0(t)$, will be a function satisfying the right boundary condition. The easiest thing to take is a constant $y_{(0)}(t) = y_0$. The next guess $y_{(1)}(t)$ is defined by

$$\frac{dy_{(1)}}{dt}(t) = f(y_{(0)}(t),t) \qquad y_{(1)}(t_0) = y_0$$

One continues in this way: the n^{th} iterate $y_n(t)$ satisfies

$$\frac{dy_{(n)}}{dt}(t) = f(y_{(n-1)}(t),t) \qquad y_{(n)}(t_0) = y_0.$$

which can be integrated up to give

$$y_{(n)}(t) = y_0 + \int_{t_0}^{t} f(y_{(n-1)}(t),t)dt$$

Picard's existence and uniqueness theorem guarantees that this procedure converges to a solution of the original equation.

Theorem 1.4.2. *Consider the initial value problem*

$$\frac{dy}{dt} = f(y,t) \qquad y(t_0) = y_0.$$

Suppose that $f(y,t)$ and $\frac{\partial f}{\partial y}$ are continuous in a neighborhood of the point (y_0, t_0). Then the initial value problem has a unique solution in some interval $[t_0 - \delta, t_0 + \delta]$ for $\delta > 0$, and the Picard iterates $y_n(t)$ converge to this solution as $n \to \infty$.

As an example: Lets try an easy one by hand (NOTE: In practice one would never try to do this by hand - it is primarily a theoretical tool - but this example works out nicely.)

Example 1.4.2. *Find the Picard iterates for*

$$y' = y \qquad y(0) = 1$$

The first Picard iterate is $y_{(0)}(t) = 1$. The second is the solution to

$$\frac{dy_{(1)}}{dt} = y_{(0)}(t) = 1 \qquad y_{(1)}(0) = 1$$

The solution to this is $y_{(1)}(t) = 1 + t$*. Continuing in this fashion* $y_{(2)}(t)$ *solves*

$$\frac{dy_{(2)}}{dt} = y_{(1)}(t) = 1 + t \qquad y_{(2)}(0) = 1$$

so $y_{(2)}(t) = 1 + t + \frac{t^2}{2}$*. Continuing in this fashion we find that*

$$\begin{aligned} y_{(3)}(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} \\ y_{(4)}(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} \\ y_{(5)}(t) &= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} \end{aligned}$$

It is not difficult to see by induction that the n^{th} *Picard iterate is just the* n^{th} *Taylor polynomial for the function* $y = e^t$*, which is the solution to the original differential equation.*

A second example is as follows:

Example 1.4.3. *Find the first few Picard iterates of the equation*

$$y' = t^2 + y^2 \qquad y(0) = 1$$

They are

$$\begin{aligned} y_{(0)}(t) &= 1 \\ y_{(1)}(t) &= 1 + t + t^3/3 \\ y_{(2)}(t) &= 1 + t + t^2 + \frac{2t^3}{3} + \frac{t^4}{6} + \frac{2t^5}{15} + \frac{t^7}{63} \\ y_{(3)}(t) &= 1 + t + t^2 + \frac{4}{3}t^3 + \frac{5}{6}t^6 + \frac{8}{15}t^5 + \frac{29}{90}t^6 + \frac{47t^7}{315} + \frac{41t^8}{630} \\ &\quad + \frac{299t^9}{11340} + \frac{4t^{10}}{525} + \frac{184t^{11}}{51975} + \frac{t^{12}}{2268} + \frac{4t^{13}}{12285} + \frac{t^{15}}{59535} \end{aligned}$$

The pattern for the coefficients in this example is not so clear.

We will not present the full proof here, but the basic principle behind the proof is something called the contraction mapping principle. The contraction mapping principle is the simplest example of what mathematicians call a fixed point theorem. Leaving aside technical details the contraction mapping principle says the following. Suppose we have a set Ω, and for each element $f \in \Omega$ we have a notion of the size of each element $\|f\|$. This is what is called a normed space. Now suppose that we have a map T from Ω to itself such that for any two elements $f, g \in \Omega$ we have that $\|T(f) - T(g)\| \leq \rho \|f - g\|$ for $\rho < 1$. In other words T shrinks the distances between points in the set by a fixed contraction factor ρ that is strictly less than one. Then there is a unique element h that is fixed by the map: $T(h) = h$,

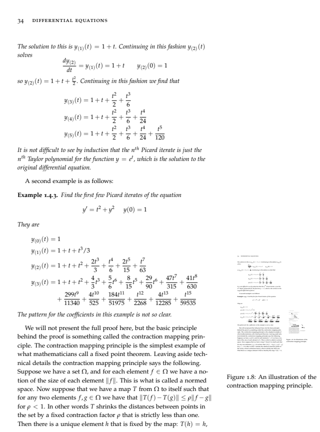

Figure 1.8: An illustration of the contraction mapping principle.

and this fixed point can be found by taking any starting point and iterating the map: $\lim_{N\to\infty} T^N(f) = h$. This theorem is illustrated by the margin figure. The proof of the Picard existence theorem amounts to showing that the Picard iterates form, for small enough neighborhood of (t_0, y_0), a contraction mapping, and thus converge to a unique fixed point.

Having some idea of how the existence and uniqueness theorem is proven we will now apply it to some examples.

Example 1.4.4. *Consider the equation*

$$\frac{dy}{dt} = \frac{ty}{t^2+y^2} \qquad y(0) = 0$$

The function $f(t,y) = \frac{ty}{t^2+y^2}$ is **NOT** *continuous in a neighborhood of the origin (Why?), so the theorem does not apply. On the other hand if we had the initial condition*

$$\frac{dy}{dt} = \frac{ty}{t^2+y^2} \qquad y(1) = 0$$

then the function $f(t,y) = \frac{ty}{t^2+y^2}$ and the partial derivative $\frac{\partial f}{\partial y}$ are continuous in a neighborhood of the point $y = 0, t = 1$, so the equation would have a unique solution.

Exercise 1.4.1

Determine if the Theorem of Existence and Uniqueness guarantees a solution and a unique solution to the following initial value problems.

a) $y' + \frac{y^2}{t^2} = ty, \quad y(0) = 1$

b) $y' + \frac{y^2}{t^2} = ty, \quad y(3) = 1$

c) $y' = \frac{y+t}{y-t}, \quad y(1) = 2$

d) $y' = \frac{y+t}{y-t}, \quad y(2) = 2$

e) $y' = \frac{y^{1/3}}{t^2}, \quad y(1) = 0$

f) $y' + \frac{ty}{\cos(y)} = 0, \quad y(0) = 0$

1.5 First Order Linear Inhomogeneous Equations:

Most of the equations considered in these notes will be linear equations, since linear equations often arise in practice, and are typically much easier to solve than nonlinear equations. A first order linear inhomogeneous initial value problem takes the form

$$\frac{dy}{dt} + P(t)y = Q(t) \qquad y(t_0) = y_0.$$