8. (5 points) Let A be an $n \times n$ matrix, let D be a diagonal matrix and let P be an invertible matrix such that $A = PDP^{-1}$. Which of the following statements is always true?

(A) $\operatorname{Nul}(A) = \operatorname{Nul}(D)$

(B) $\dim \operatorname{Nul}(A) \neq \dim \operatorname{Nul}(D)$

(C) None of the other answers

(D) ★ $\dim \operatorname{Nul}(A) = \dim \operatorname{Nul}(D)$

Solution. We check that $\dim \operatorname{Nul}(A) = \dim \operatorname{Nul}(D)$. Let $\mathbf{v}$ in $\mathbb{R}^n$. Since P is invertible, $A\mathbf{v} = 0$ if and only if $DP^{-1}\mathbf{v} = 0$. Thus $\mathbf{v}$ is in $\operatorname{Nul}(A)$ if and only if $P^{-1}\mathbf{v}$ is in $\operatorname{Nul}(D)$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ be a basis of $\operatorname{Nul}(A)$. It is left to check that $P^{-1}\mathbf{v}_1, \ldots, P^{-1}\mathbf{v}_d$ is a basis of $\operatorname{Nul}(D)$.

First observe that $v_i = PP^{-1}\mathbf{v}_i$. Thus $P^{-1}\mathbf{v}_1, \ldots, P^{-1}\mathbf{v}_d$ are linearly independent by Worksheet 7 Problem 7(3). Thus it is left to check that $P^{-1}\mathbf{v}_1, \ldots, P^{-1}\mathbf{v}_d$ span $\operatorname{Nul}(D)$.

Let $\mathbf{z}$ in $\operatorname{Nul}(D)$. Then $A(P\mathbf{z}) = PDP^{-1}P\mathbf{z} = PD\mathbf{z} = P\mathbf{0} = \mathbf{0}$. Thus $P\mathbf{z}$ is in $\operatorname{Nul}(A)$. Therefore there are scalars $c_1, \ldots, c_d$ such that

$$P\mathbf{z} = c_1\mathbf{v}_1 + \cdots + c_d\mathbf{v}_d.$$

Then

$$\mathbf{z} = P^{-1}P\mathbf{z} = P^{-1}(c_1\mathbf{v}_1 + \cdots + c_d\mathbf{v}_d) = c_1P^{-1}\mathbf{v}_1 + \cdots + c_dP^{-1}\mathbf{v}_d.$$

Thus $P^{-1}\mathbf{v}_1, \ldots, P^{-1}\mathbf{v}_d$ span $\operatorname{Nul}(D)$.

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17. (5 points) Let A be 2×2-matrix with eigenvalues $\frac{1}{2}$ and $\frac{1}{3}$. What can you say about $\lim_{k\to\infty} A^k$?

(A) $\lim_{k\to\infty} A^k$ does not exist.

(B) Not enough information to say anything

(C) $\lim_{k\to\infty} A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(D) ★ $\lim_{k\to\infty} A^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(E) $\lim_{k\to\infty} A^k = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

Solution.

$$A^k = V \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}^k V^{-1} \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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18. (5 points) Let A be a diagonalizable 3×3 matrix with only two distinct eigenvalues. Which of the following statements is FALSE?

(A) The matrix $5A$ is diagonalizable.

(B) The matrix A has an eigenbasis.

(C) ★ There are no more than two linearly independent eigenvectors of A.

(D) There is an eigenvalue of A for which the corresponding eigenspace is spanned by two linearly independent eigenvectors.

Solution. Since A is diagonalizable, it has an eigenbasis. Since we know that the matrix has an eigenbasis but only two distinct eigenvalues, it follows that one of those eigenvalues is a repeated eigenvalue and its corresponding eigenspace must be spanned by two eigenvectors. The vectors in an eigenbasis are always linearly independent eigenvectors. Lastly, since an eigenbasis exists, the matrix A is diagonalizable and so $5A$ should be diagonalizable too.

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19. (5 points) Let A be a 4×4 matrix with eigenvectors

$$\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

corresponding to the eigenvalues $\lambda = -2, 0, 1, 3$. What is A?

(A) $\begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$

(B) ★ $\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix} \begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3\end{bmatrix} \begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}^{-1}$

(C) $\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}^{-1} \begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3\end{bmatrix} \begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}$

(D) Not enough information to determine A

(E) $\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}$

Solution. If P is the eigenvector matrix and D is the eigenvalues diagonal matrix, then $A = PDP^{-1}$.
