Math 231E, Week 2c
Limits at ∞, and Squeeze Theorem

1 Limits at ∞ and horizontal asymptotes

We now would like to define limits as $x \to \infty$, and we can do so in the following manner:

We say that

$$\boxed{\lim_{x\to\infty} f(x) = L, \text{ if } \lim_{x\to 0+} f\left(\frac{1}{x}\right) = L.}$$

In short, we first plug in $1/x$ for x, then take the limit $x \to 0$ from the right.

Example 1.1. We want to compute

$$\lim_{x\to\infty} \frac{3x^2+2x-7}{x^2-4x+1}.$$

We plug in $1/x$ for x to obtain

$$\frac{3(1/x)^2+2(1/x)-7}{(1/x)^2-4(1/x)+1} = \frac{\dfrac{3}{x^2}+\dfrac{2}{x}-7}{\dfrac{1}{x^2}-\dfrac{4}{x}+1} = \frac{1/x^2}{1/x^2}\cdot\frac{3+x-7x^2}{1-4x+x^2},$$

and taking the limit $x \to 0+$, we obtain 3.

We also say that

$$\boxed{\lim_{x\to-\infty} f(x) = L, \text{ if } \lim_{x\to 0-} f\left(\frac{1}{x}\right) = L.}$$

Example 1.2. Let us compute

$$\lim_{x\to\infty} \frac{e^x}{x^n},$$

for some fixed $n > 0$. We know that $e^x > x^{n+1}/(n+1)!$. (Why is this?) Then we have

$$\frac{e^x}{x^n} > \frac{x^{n+1}}{x^n(n+1)!} = \frac{x}{(n+1)!}.$$

Now,

$$\lim_{x\to\infty} \frac{x}{(n+1)!} = \lim_{x\to 0+} \frac{1/x}{(n+1)!} = \frac{1}{(n+1)!}\lim_{x\to 0+}\frac{1}{x} = \infty.$$

Therefore

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty.$$

Example 1.3. Similarly, we would like to compute

$$\lim_{x\to\infty} x^n e^{-x}.$$

From the arguments above, we have that

$$\frac{x^n}{e^x} < \frac{(n+1)!}{x}.$$

Also, for $x > 0$, we have $x^n e^{-x} > 0$, so

$$0 < \frac{x^n}{e^x} < \frac{(n+1)!}{x}.$$

We compute

$$\lim_{x\to\infty} \frac{(n+1)!}{x} = \lim_{x\to 0+} \frac{(n+1)!}{1/x} = (n+1)! \lim_{x\to 0+} x = 0,$$

so by the Squeeze Theorem we have

$$\lim_{x\to\infty} x^n e^{-x} = 0.$$

This is why we would say that "e^{-x} decays to zero faster than **any** polynomial as $x \to \infty$".

2 Squeeze Theorem

Theorem 2.1. If $f(x) \leq g(x) \leq h(x)$ for x near a, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} g(x) = L$.

Example 2.2. We compute

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right).$$

We cannot use a Limit Law, since $\lim_{x\to 0} \sin(1/x)$ does not exist. However, let us Squeeze it:

$$-1 < \sin(1/x) < 1$$
$$-x^2 < x^2 \sin(1/x) < x^2,$$

and we know

$$\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0,$$

so

$$\lim_{x\to 0} x^2 \sin(1/x) = 0.$$

See Figure 1.

3 Rule of Thumb

When considering rational functions[1] with limits at infinity, there is a good rule of thumb to follow. We'll work through three examples of this first.

[1] A rational function is a quotient of two polynomials

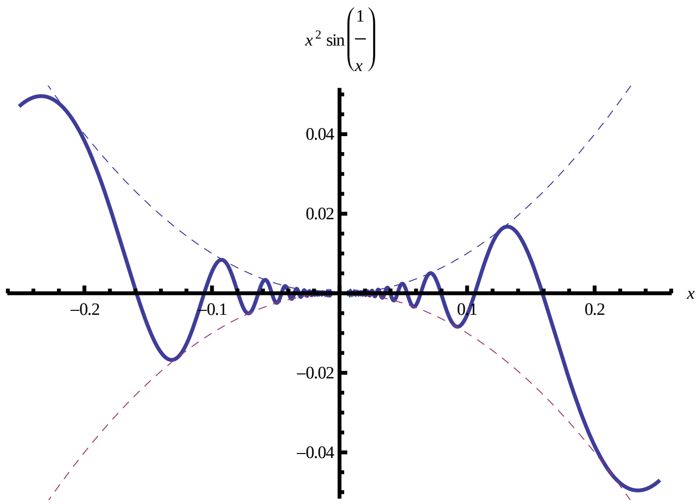

Figure 1: Plot of $x^2 \sin(1/x)$, and the envelopes x^2, $-x^2$

Example 3.1. Consider the limit

$$\lim_{x\to\infty} \frac{3x^3+2x-7}{4x^3+x^2-x-1}.$$

Plug in $1/x$ and take the limit as $x \to 0+$:

$$\lim_{x\to 0+} \frac{\dfrac{3}{x^3}+\dfrac{2}{x}-7}{\dfrac{4}{x^3}+\dfrac{1}{x^2}-\dfrac{1}{x}-1} = \lim_{x\to 0+} \frac{1/x^3}{1/x^3}\frac{3+2x^2-7x^3}{4+x-x^2-x^3} = \lim_{x\to 0+} \frac{3+2x^2-7x^3}{4+x-x^2-x^3} = \frac{3}{4}.$$

Rule 1: if the top and bottom are **the same degree**, then the limit is the ratio of the leading coefficients. In this example, we see that both top and bottom are exactly cubic, and the leading coefficients are 3 and 4.

Example 3.2. Now consider

$$\lim_{x\to\infty} \frac{-3x^3+2x-7}{x^2-x-1}.$$

Plug in $1/x$ and take the limit as $x \to 0+$:

$$\lim_{x\to 0+} \frac{\dfrac{-3}{x^3}+\dfrac{2}{x}-7}{\dfrac{1}{x^2}-\dfrac{1}{x}-1} = \lim_{x\to 0+} \frac{1/x^3}{1/x^3}\frac{-3+2x^2-7x^3}{x-x^2-x^3} = \lim_{x\to 0+} \frac{-3+2x^2-7x^3}{x-x^2-x^3}.$$

We see here that the numerator goes to -3 and the denominator to $0+$, so the limit is $-\infty$.

Rule 2: If the numerator is of a higher degree than the denominator, then the limit is $\pm\infty$, where the sign is the same as the sign of the leading-order coefficient of the numerator.

Example 3.3. Consider the limit

$$\lim_{x\to\infty} \frac{2x-7}{4x^3+x^2-x-1}.$$

Plug in $1/x$ and take the limit as $x \to 0+$:

$$\lim_{x\to 0+} \frac{\dfrac{2}{x}-7}{\dfrac{4}{x^3}+\dfrac{1}{x^2}-\dfrac{1}{x}-1} = \lim_{x\to 0+} \frac{1/x^3}{1/x^3}\frac{2x^2-7x^3}{4+x-x^2-x^3} = \lim_{x\to 0+} \frac{2x^2-7x^3}{4+x-x^2-x^3}.$$

We see here that the numerator goes to 0 while the denominator goes to 4, so the limit is zero.

Rule 3: if the denominator is of a higher degree than the numerator, then the limit is zero.

Exercise: Which of these rules change, and how, if we take the limit $x \to -\infty$?