

# Math 231E Engineering Calculus Module 2a: Notes on Limits

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## 1 Motivation for the definition of limits

The idea behind the expression

$$\lim_{x \rightarrow a} f(x) = L$$

is that we can make  $f(x)$  “as close to  $L$ ” as we like, by making  $x$  “close to  $a$ ”. In other words, you tell me how close you would like  $f(x)$  to be to  $L$ . I have to be able to tell you how close  $x$  needs to be to  $a$  to guarantee this. To make this precise, of course, we need to define what the quotes really mean, but first let’s start with an example.

### 1.1 Example

We are tasked at making a cube with volume  $1000 \text{ m}^3$ . We know that the volume  $V$  is related to the edge of the cube  $x$  by the relation  $V(x) = x^3$ , and we can probably see in our head that  $x = 10 \text{ m}$  will satisfy this relation.

However, there will be some errors in fabrication: let us say that we can shoot for an edge of length  $10 \text{ m}$ , but it is not reasonable to expect that we will nail this exactly. A more reasonable specification is something along the lines of the following:

Construct a cube with volume  $1000 \text{ m}^3$  with an error tolerance of  $1/2\%$ .

The question remains: how much error can we have in the edge of the cube and meet the above specification?

First, we see that our (relative) error tolerance is  $1/2\% = 0.5\%$ , and the target volume is  $1000 \text{ m}^3$ , so our absolute error tolerance is  $\pm 5 \text{ m}^3$ . So we compute:

$$\begin{aligned} |V(x) - 1000 \text{ m}^3| &< 5 \text{ m}^3 \\ -5 \text{ m}^3 &< V(x) - 1000 \text{ m}^3 < 5 \text{ m}^3 \\ 995 \text{ m}^3 &< V(x) < 1005 \text{ m}^3 \\ 995 \text{ m}^3 &< x^3 < 1005 \text{ m}^3 \\ \sqrt[3]{995 \text{ m}^3} &< x < \sqrt[3]{1005 \text{ m}^3} \\ 9.983305 \text{ m} &< x < 10.016639 \text{ m} \\ -0.016695 \text{ m} &< x - 10 \text{ m} < .016639 \text{ m}. \end{aligned}$$

Now, we see that we have both an upper and a lower constraint on how far  $x$  can deviate from  $10 \text{ m}$ . As long as we choose a number smaller than both of these constraints, we are fine, so we can choose  $0.0166 \text{ m}$  as our edge tolerance, and thus we choose

$$|x - 10 \text{ m}| < 0.0166 \text{ m} = 1.66 \text{ cm}.$$

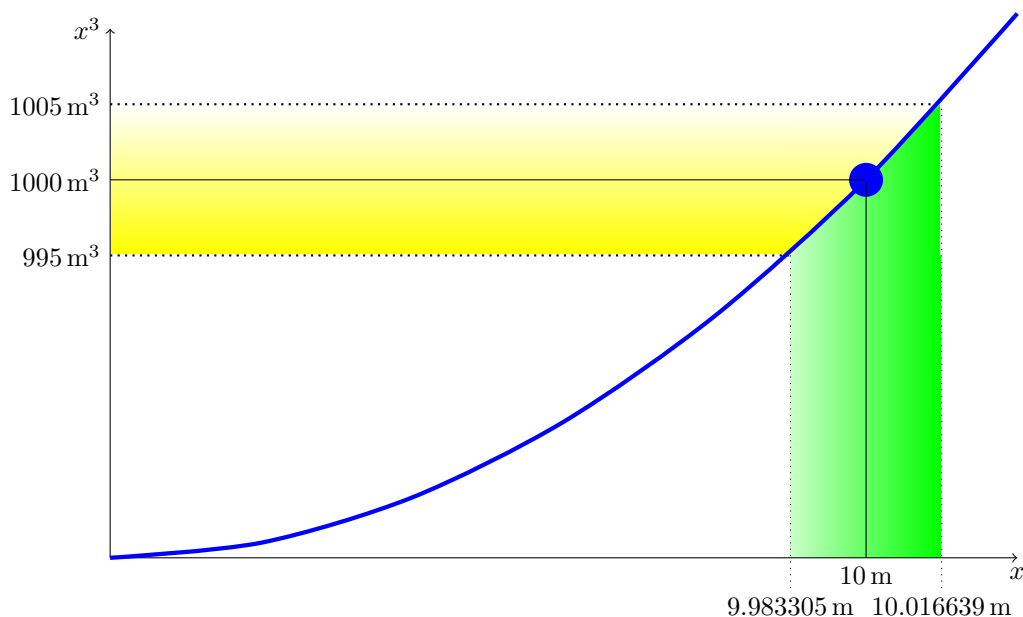


fig:cube

Figure 1: Graph of tolerance problem for cube. Anything in the yellow band is acceptable for the customer, and thus everything in the green band is where we have to be. We must choose  $\delta$  to give an interval that lives completely inside the green band, so we get  $\delta = 0.016$  m.

So, as long as we can control our process to within 1.66 cm, we are fine (see the picture in Figure [1.1](#)).

Now, this is just one specific error tolerance. Let us say that the customer comes back and says they need a relative tolerance of 1/10% instead. We would have to redo the calculations (for example, the absolute tolerance would now be  $1 \text{ m}^3$ , and these changes would percolate through, but there is no obstruction, in principle, to carrying out these calculations. If the tolerance in the volume was further changed to one part in a million (1ppm), we could again imagine redoing this whole calculation.

The fact that this calculation would **always be successful, no matter the tolerance** is equivalent to saying that the limit exists. We make the precise definition below.

## 2 Precise definition

The definition of the statement

$$\lim_{x \rightarrow a} f(x) = L$$

is:

For any  $\epsilon > 0$ , there is a  $\delta > 0$  such that if  $0 < |x - a| < \delta$ , then  $|f(x) - L| < \epsilon$ .

In the previous case,  $L = 1000 \text{ m}^3$  is the target we are shooting for, and  $\epsilon = 5 \text{ m}^3$  is the tolerance that we allow in hitting that target. We saw that  $a = 10 \text{ m}$  is the exactly correct answer, and  $\delta = 0.0166 \text{ m}$  is how close we need to be on the inputs.

The statement of limits is then equivalent to saying “no matter what tolerance you give me on the output of the function, if I am close enough to the correct input, I will be close enough to the correct output”.

### 3 Examples

You will also work on some examples in Worksheet 02B.

**Example 3.1.** *Prove that*

$$\lim_{x \rightarrow 2} (3x + 1) = 7.$$

*To prove this, we need to show that for any  $\epsilon > 0$ , we can find a  $\delta > 0$  that means that  $|x - 2| < \delta$  will imply that  $|f(x) - 7| < \epsilon$ . In many of these cases, we find it convenient to work backwards:*

$$\begin{aligned} |f(x) - 7| &< \epsilon \\ |(3x + 1) - 7| &< \epsilon \\ |3x - 6| &< \epsilon \\ 3|x - 2| &< \epsilon \\ |x - 2| &< \frac{\epsilon}{3} = \frac{\epsilon}{3}. \end{aligned}$$

*Therefore, we can choose  $\delta = \epsilon/3$ , and we are guaranteed to make this work.*

**Question:** In the previous example, is it ok to choose  $\delta = \epsilon/5$  instead?

**Example 3.2.** *Prove that  $\lim_{x \rightarrow 3} x^2 = 9$ .*

*We proceed as before, with  $f(x) = x^2$  and  $L = 9$ . Again working backwards,*

$$\begin{aligned} |x^2 - 9| &< \epsilon \\ -\epsilon &< x^2 - 9 < \epsilon \\ 9 - \epsilon &< x^2 < 9 + \epsilon \\ \sqrt{9 - \epsilon} &< x < \sqrt{9 + \epsilon}. \end{aligned}$$

*Now, the pattern that we are looking for is to see something like*

$$3 - \text{something} < x < 3 + \text{something},$$

*but that is not quite what we have. But we can approximate each side of this equation with a Taylor series!*

**Side calculation!**

If we think of  $\epsilon$  as small, this is the same as saying it is near zero, so we have

$$g(\epsilon) = g(0) + \epsilon g'(0) + \epsilon^2 \frac{g''(0)}{2} + O(\epsilon^3).$$

If we think of  $g(x) = \sqrt{9+x}$ , then we have

$$\begin{aligned} g'(x) &= \frac{1}{2}(9+x)^{-1/2}, & g'(0) &= \frac{1}{2} \cdot 9^{-1/2} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}, \\ g''(x) &= -\frac{1}{4}(9+x)^{-3/2}, & g''(0) &= -\frac{1}{4} \cdot 9^{-3/2} = -\frac{1}{4} \cdot \frac{1}{27} = -\frac{1}{108}, \end{aligned}$$

so

$$\sqrt{9+\epsilon} = 3 + \frac{1}{6}\epsilon - \frac{1}{216}\epsilon^2 + O(\epsilon^3). \quad (3.1)$$

eq:sqrtTaylor

Now **and here is the crucial, but subtle point:** If  $\epsilon$  is small, then  $\epsilon^2$  is much smaller, and  $O(\epsilon^3)$  is smaller still. So, let us pick any number less than  $1/6$  (say  $1/10$ ), and we claim that  $3 + \epsilon/10$  is less than everything on the right-hand side of (3.1). To see this, we have

$$\begin{aligned} 3 + \frac{\epsilon}{10} &< 3 + \frac{1}{6}\epsilon - \frac{1}{216}\epsilon^2 + O(\epsilon^3) \\ \frac{\epsilon}{10} - \frac{\epsilon}{6} &< -\frac{1}{216}\epsilon^2 + O(\epsilon^3) \\ -\frac{\epsilon}{15} &< -\frac{1}{216}\epsilon^2 + O(\epsilon^3) \\ -\frac{1}{15} &< -\frac{1}{216}\epsilon + O(\epsilon^2). \end{aligned}$$

The last line is clearly correct for  $\epsilon$  sufficiently small, so just reverse the order and this proves it. In general, if we ever have a constraint of the form

$$x < a + A\epsilon + B\epsilon^2 + O(\epsilon^3),$$

then choose **any**  $\hat{A} < A$ , and the constraint  $x < a + \hat{A}\epsilon$  is always good enough for  $\epsilon$  sufficiently small, because by the same argument above,  $a + \hat{A}\epsilon < a + A\epsilon + B\epsilon^2 + O(\epsilon^3)$ , but only when  $\epsilon$  is **sufficiently small**.

Now let's get back to the problem. We need to have

$$x < \sqrt{9+\epsilon} = 3 + \frac{1}{6}\epsilon - \frac{1}{216}\epsilon^2 + O(\epsilon^3),$$

and the side calculation above tells us that  $x < 3 + \epsilon/10$  will suffice.

We also had to match the constraint

$$\sqrt{9-\epsilon} < x$$

and it is not hard to see (Why??) that

$$\sqrt{9-\epsilon} = 3 - \frac{1}{6}\epsilon - \frac{1}{216}\epsilon^2 + O(\epsilon^3),$$

and by the same argument requiring that  $x > 3 - \epsilon/10$  will suffice.

Therefore we choose  $\delta = \epsilon/10$  and we are done!

**Question:** Would  $\delta = \epsilon/20$  work?