Math 415. Exam 2. March 14, 2019

Full Name: _____

Net ID: _____

Discussion Section: _____

- Do not turn this page until instructed to.

- There are 20 problems worth 5 points each.

- Each question has only one correct answer. You can choose up to two answers. If you choose just one answer, then you will get 5 points if the answer is correct, and 0 points otherwise. However, if you choose two answers, you will get 2.5 points if one of the answers is correct, and 0 points otherwise.

- You must not communicate with other students.

- No books, notes, calculators, or electronic devices allowed.

- This is a 75 minute exam. There are several different versions of this exam.

- Fill in the answers on the scantron form provided, **and** circle your answers on the exam itself. Hand in both the exam and the scantron.

- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!

- If you have to erase something on the scantron, please make sure to do so thoroughly.

- Good luck! Have a nice Spring break!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**! On the back of the scantron bubble in the following:

95. D

96. C

1. (5 points) Consider the subspace $W := \{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \in \mathbb{R} \}$ of $\mathbb{R}^3$. Which of the following sets is a basis of W?

(A) None of the other answers.

(B) $\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$

(C) ★ $\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \}$

(D) $\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \}$

(E) $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$

Solution. Observe that
$$W = \text{span}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}.$$

Since $\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \}$ is linear independent, we only need to check that the vectors are in W and span W. It is clear that they are in W. Furthermore, they span W, since

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

2. (5 points) Let $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ be an orthonormal basis of $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Suppose we know that,

$$(T(\mathbf{v}_1)) \cdot \mathbf{v}_1 = 1, \quad (T(\mathbf{v}_1)) \cdot \mathbf{v}_2 = 2, \quad (T(\mathbf{v}_1)) \cdot \mathbf{v}_3 = 3.$$

What is the first column of the matrix $T_{\mathcal{B}\mathcal{B}}$?

(A) None of the other answers.

(B) Not enough information to determine the first column.

(C) $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

(D) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(E) ★ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solution. In Lecture 18, we saw that if we wrote a vector $\mathbf{w}$ in terms of an orthonormal basis

$$\mathbf{w} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3,$$

then the coefficients of each basis vector is precisely the dot product of $\mathbf{w}$ and that vector, i.e.

$$\mathbf{w} \cdot \mathbf{v}_1 = c_1, \quad \mathbf{w} \cdot \mathbf{v}_2 = c_2, \quad \mathbf{w} \cdot \mathbf{v}_3 = c_3.$$

Now, recall that the first column of $T_{\mathcal{B}\mathcal{B}}$ is the coordinate vector $(T\mathbf{v}_1)_{\mathcal{B}}$ and set $\mathbf{w} = T\mathbf{v}_1$.

3. (5 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Which of the following is a basis of $\mathrm{Nul}(A)$?

(A) $\left\{ \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$

(B) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

(C) ★ $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

(D) None of the other answers

(E) $\left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution. By switching the first two rows, we get

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in echelon form. It has one free variable, so the dimension of $\mathrm{Nul}(A)$ is 1. Therefore, we can rule out the options with more than 1 vectors. We check by computation that

$$A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

so this vector is not in $\mathrm{Nul}(A)$. On the other hand, we have

$$A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since we have found a nonzero vector contained in the 1-dimensional subspace $\mathrm{Nul}(A)$, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Nul}(A)$.

4. (5 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

What are the dimensions of the null space and the row space of A?

(A) $\dim \operatorname{Nul}(A) = 3, \dim \operatorname{Col}(A^T) = 0$

(B) None of the other answers.

(C) $\dim \operatorname{Nul}(A) = 2, \dim \operatorname{Col}(A^T) = 0$

(D) $\dim \operatorname{Nul}(A) = 3, \dim \operatorname{Col}(A^T) = 2$

(E) ★ $\dim \operatorname{Nul}(A) = 2, \dim \operatorname{Col}(A^T) = 3$

Solution. The dimension of a null space is the number of free variables. The matrix A has rank 3 and 5 columns, so 2 free variables. The matrix A^T has also rank 3.

5. (5 points) Let A be a 6×6 matrix and let $\mathbf{x}$ be a vector in $\mathbb{R}^6$. Which of the following statements is true?

(A) The vectors $\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, A^4\mathbf{x}, A^5\mathbf{x}, A^6\mathbf{x}$ are linear independent.

(B) The linear independence/dependence of $\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, A^4\mathbf{x}, A^5\mathbf{x}, A^6\mathbf{x}$ can not be determined from the given data.

(C) ★ The vectors $\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, A^4\mathbf{x}, A^5\mathbf{x}, A^6\mathbf{x}$ are linear dependent.

(D) None of the other answers.

Solution. Seven vectors in $\mathbb{R}^6$ are always linearly dependent. Hence $\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, A^4\mathbf{x}, A^5\mathbf{x}, A^6\mathbf{x}$ are linearly dependent for all possible choices of A and $\mathbf{x}$.

6. (5 points) Let A be a 10×4 matrix with $\dim \mathrm{Nul}(A^T) = 6$. Which of the following statements is NOT true?

(A) ★ The nullspace of A has dimension 6.

(B) The matrix A has rank 4.

(C) The equation $A\mathbf{x} = 0$ has a unique solution.

(D) The columns of A are linearly independent.

(E) The row space of A has dimension 4.

Solution. The matirx A^T has rank $r = 10 - 6 = 4$. This is also the rank of A. By the Rank-Nullity Theorem, $\mathrm{Nul}(A)$ has dimension $4 - r = 0$.

7. (5 points) Consider the polynomial $p(t) = 4t^2 - 6t - 1$. Which of the following ordered sets of vectors is an ordered basis $\mathcal{B}$ for $\mathbb{P}_2$ such that the coordinate vector for $p(t)$ with respect to that basis is $p(t)_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ -8 \end{bmatrix}$?

(A) $\left(2t^2, 6, 1\right)$

(B) None of the other answers.

(C) $\left(2t^2 + t, t, 1\right)$

(D) $\left(2t^2, 6, 0\right)$

(E) ★ $\left(2t^2 + t, 1, t\right)$

Solution. Note that $\left(2t^2, 6, 0\right)$ and $\left(2t^2, 6, 1\right)$ are not bases of $\mathbb{P}_2$. So they can't be the right answer.

Since
$$2(2t^2 + t) + (-1)1 - 8t = 4t^2 - 6t - 1 = p(t),$$
$\left(2t^2 + t, 1, t\right)$ is the correct answer.

Since
$$2(2t^2 + t) + (-1)t - 8 \cdot 1 = 4t^2 + t - 8,$$
$\left(2t^2 + t, t, 1\right)$ is not the correct answer.

8. (5 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ y \\ 2x - y \end{bmatrix}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be the ordered bases $\mathcal{A} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ and $\mathcal{B} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$ of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Compute the coordinate matrix $T_{\mathcal{B}\mathcal{A}}$.

(A) ★ $\begin{bmatrix} 0 & -2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$

(E) None of the other answers

Solution. By definition, $T_{\mathcal{B}\mathcal{A}} = \left[T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)_{\mathcal{B}} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)_{\mathcal{B}}\right].$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 2(1) - 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \left(0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 2(0) - 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}_{\mathcal{B}} = \left((-2)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

9. (5 points) Recall that $M_{2\times 2}$ is the vector space of 2×2-matrices. Let T be the linear transformation from $M_{2\times 2}$ to $M_{2\times 2}$ that performs the row operation $R2 \to 2R2$. That is, $T\left(\begin{bmatrix} R1 \\ R2 \end{bmatrix}\right) = \begin{bmatrix} R1 \\ 2R2 \end{bmatrix}$. Let $\mathcal{A}$ be the ordered basis $\mathcal{A} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$ of $M_{2\times 2}$. Which one of the following equals $T_{\mathcal{A}\mathcal{A}}$?

(A) ★ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(B) None of the other answers

(C) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(E) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Solution. By definition, $T_{\mathcal{A}\mathcal{A}} = \left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)_{\mathcal{A}} \quad T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)_{\mathcal{A}} \quad T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)_{\mathcal{A}} \quad T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)_{\mathcal{A}} \right]$.

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)_{\mathcal{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{\mathcal{A}} = \left(1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)_{\mathcal{A}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)_{\mathcal{A}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{\mathcal{A}} = \left(0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)_{\mathcal{A}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)_{\mathcal{A}} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}_{\mathcal{A}} = \left(0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)_{\mathcal{A}} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)_{\mathcal{A}} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}_{\mathcal{A}} = \left(0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)_{\mathcal{A}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

10. (5 points) Let V be a vector space, and let A, B be two $m \times n$-matrices. Consider the following two statements:

(I) If B is an echelon form of A, then the pivot columns of B form a basis of the column space of A.

(II) If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and S is a set of more than p vectors in V, then S is linearly dependent.

Which of the two statements is always true?

(A) Both statement (I) and statement (II) are true.

(B) Neither statement (I) nor statement (II) is true.

(C) Only statement (I) is true.

(D) ★ Only statement (II) is true.

Solution. Statement (II) is always true. Since $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, we have $\dim V \leq p$. Since S contains more than p vectors of V, S has to be linearly dependent.

Statement (I) is also not always true. Consider $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and its RREF $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Clearly span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \neq$ span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

11. (5 points) What is the smallest possible dimension of $\text{Nul}(A)$ for a 9×14 matrix A?

(A) 14

(B) None of the other answers.

(C) ★ 5

(D) 0

(E) 9

Solution. Because A has only 9 rows, it can have at most 9 pivot columns. Thus A has at least 5 non-pivot columns and thus $\dim \text{Nul}(A) \geq 5$.

12. (5 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Which of the following subsets of $\mathbb{P}_2$ is linearly **dependent**?

(A) $\{t^2 + t, 1 + t, 1\}$

(B) $\{1, t, t^2\}$

(C) $\{1 + t, 1 - t, t^2\}$

(D) None of the other answers.

(E) ★ $\{t^2 + t, 1 + t, t^2 + 2t + 1\}$

Solution. Since $(t^2 + t) + (1 + t) - (t^2 + 2t + 1) = 0$, this set is linearly dependent.

13. (5 points) Let A, B be two $m \times n$-matrices that are row equivalent. Consider the following statements:

   I. $\dim(\mathrm{Nul}(A)) = \dim(\mathrm{Nul}(B))$.

  II. $\dim(\mathrm{Col}(A)) = \dim(\mathrm{Col}(B))$.

Which one of these statements is always true?

(A) ★ Statement I and Statement II.

(B) Statement II only.

(C) Neither of Statements I or II.

(D) Statement I only.

Solution. Since A and B are row equivalent, linear systems $Ax = 0$ and $Bx = 0$ have the same solution set, i.e. $\mathrm{Nul}(A) = \mathrm{Nul}(B)$. In particular, we obtain $\dim(\mathrm{Nul}(A)) = \dim(\mathrm{Nul}(B))$. Then we also have $\dim(\mathrm{Col}(A)) = n - \dim(\mathrm{Nul}(A)) = n - \dim(\mathrm{Nul}(B)) = \dim(\mathrm{Col}(B))$.

14. (5 points) Consider the following ordered basis $\mathcal{B}$ of $\mathbb{R}^3$:

$$\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right).$$

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. What is the coordinate vector $\mathbf{x}_\mathcal{B}$ of $\mathbf{x}$ with respect to the basis $\mathcal{B}$?

(A) None of the other answers

(B) $\begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(D) ★ $\begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix}$

Solution. The basis is orthonormal, therefore $x_\mathcal{B} = \begin{bmatrix} x \cdot b_1 \\ x \cdot b_2 \\ x \cdot b_3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix}$

15. (5 points) Consider the subspace $V = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^4$. Which of the following is a basis

for the orthogonal complement $V^{\perp}$?

(A) $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(B) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(C) None of the other answers.

(D) $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

(E) ★ $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Solution. You can either calculate the left null space of $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (remembering that $\text{Nul}(A^T) \perp$

$\text{Col}(A)$), or you can simply check that the vectors in the set are linearly independent and are orthogonal to every vector in V (while keeping in mind that the dimensions of V and $V^{\perp}$ must sum to 4).

16. (5 points) Recall that $M_{2\times2}$ is the vector space of 2×2-matrices. Consider the following subspaces of $M_{2\times2}$:

$$V = \{A \in M_{2\times2} \mid A^T = -A\} \qquad W = \{B \in M_{2\times2} \mid B \text{ is diagonal}\}.$$

What are the dimensions of V and W?

(A) $\dim V = 3$ and $\dim W = 2$

(B) ★ $\dim V = 1$ and $\dim W = 2$

(C) $\dim V = 4$ and $\dim W = 4$

(D) $\dim V = 2$ and $\dim W = 2$

(E) None of the other answers

Solution. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in V, then $a = -a$, $c = -b$, and $d = -d$, so $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and so $V = \mathrm{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is 1-dimensional.

A general element of W looks like

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

so $W = \mathrm{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

17. (5 points) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree up to 2. Let $T : \mathbb{R}^2 \to \mathbb{P}_2$ be a linear transformation such that $T\left(\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) = t - 1$ and $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = t^2 - 1$. What is $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$?

(A) None of the other answers

(B) $3t + 2$

(C) ★ $3t^2 - t - 2$

(D) $-3t^2 + t - 2$

(E) $3t^2 + 2$

Solution. Writing $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we obtain

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = T\left(3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) = 3T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) = 3(t^2 - 1) - (t - 1) = 3t^2 - t - 2.$$

18. (5 points) Suppose A is a 3×6 matrix

$$A = \begin{bmatrix} 1 & * & * & 0 & * & * \\ 0 & * & * & 1 & * & * \\ 1 & * & * & -2 & * & * \end{bmatrix}$$

whose column space is

$$\mathrm{Col}(A) = \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

Which of the following is a basis for $\mathrm{Nul}(A^T)$?

(A) ★ $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

(B) There is not enough information to determine the answer.

(C) None of the other answers.

(D) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

(E) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Solution. By the FTLA, $\mathrm{Nul}(A^T)$ is the orthogonal complement of $\mathrm{Col}(A)$, thus $\mathrm{Nul}(A^T)$ is one dimensional and spanned by the vector $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ since the orthogonality conditions give the equations $x_1 = -x_3$ and $x_2 = 2x_3$.

19. (5 points) Let V be a vector space of dimension at least 3. Let $\mathcal{B} = \left(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \right)$, $\mathcal{C} = \left(\mathbf{c}_1, \mathbf{c}_2, \ldots \mathbf{c}_n \right)$ be two different ordered bases of V. Which of the following statements are always true?

(I) If the second entry of $\mathbf{v}_\mathcal{B}$ is zero, then at least one entry of $\mathbf{v}_\mathcal{C}$ is zero.

(II) If $\mathbf{v}_\mathcal{B}$ is not the zero vector, then $\mathbf{v}_\mathcal{C}$ is not the zero vector.

(III) If $\mathbf{c}_1 = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, then $\{\mathbf{c_1}, \mathbf{b_2}, \mathbf{b_3}, \ldots, \mathbf{b_n}\}$ is a basis for V. That is, if $\mathbf{c}_1 = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, then substituting $\mathbf{c_1}$ for $\mathbf{b_1}$ in $\mathcal{B}$ creates a basis for V.

(A) All three statements.

(B) Only statements (I) and (II).

(C) Only statements (I) and (III).

(D) None of the other answers.

(E) ★ Only statements (II) and (III).

Solution.

- Consider $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, a vector in $\mathbb{R}^3$ written in terms of the standard basis. Consider the basis

  $\mathcal{C} = (\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix})$. Then $\mathbf{v}_\mathcal{C} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The second component of $\mathbf{v}$ is zero, but every

  component of $\mathbf{v}_\mathcal{C}$ is nonzero.

- The coordinate vector for $\mathbf{v}$ with respect to a basis $\mathcal{B}$ is given by $\mathbf{v}_\mathcal{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ where $\mathbf{v} =$

  $k_1\mathbf{c_1} + k_2\mathbf{c_2} + \cdots + k_n\mathbf{c_n}$. So $\mathbf{v}_\mathcal{B} = \mathbf{0}$ if and only if $\mathbf{v}_\mathcal{C} = \mathbf{0}$.

- If $\mathbf{c}_1 = \mathbf{b_1} + \mathbf{b_2} + \mathbf{b_3}$, then $\mathcal{B}' = \{\mathbf{c_1}, \mathbf{b_2}, \mathbf{b_3} \ldots \mathbf{b_n}\}$ is a basis for V.

  Any vector v that can be written as a linear combination of vectors in $\mathcal{B}$ can be rewritten in terms of a linear combination of vectors in $\mathcal{B}'$.

  The vectors in $\mathcal{B}'$ must be linearly independent: If $k_1\mathbf{c_1} + k_2\mathbf{b_2} + \cdots + k_n\mathbf{b_n} = 0$, then $k_1(\mathbf{b_1} + \mathbf{b_2} + \mathbf{b_3}) + k_2\mathbf{b_2} + \cdots + k_n\mathbf{b_n} = 0$. By linear independence of $\mathcal{B}$, we know that all k_i are zero, and so $\mathcal{B}'$ is also linearly independent.

  So $\mathcal{B}'$ is a basis for V.

20. (5 points) Let A be a 7×5 matrix and B be a 5×6 matrix such that the rank of AB is 5. What is the rank of A?

(A) None of the other answers.

(B) ★ 5

(C) 7

(D) 8

(E) Not enough information to determine the rank of A.

Solution. Recall that each column of AB is a linear combinations of the columns of A. Hence $\mathrm{Col}(AB) \subseteq \mathrm{Col}(A)$. Now, the rank of AB is 5 implies that the dimension of $\mathrm{Col}(AB)$ is 5. So the previous inclusion implies that the dimension of $\mathrm{Col}(A)$ is at least 5, so the rank of A is at least 5. On the other hand, since the dimension of A is 5×7, the rank of A can be at most 5.