Math 231E, Lecture 32.
Parametric Equations

1 Definition

So far, all of our curves have been graphs of functions:

[width=.5]Lecture32pic1.pdf

But what about a curve that isn't?

[width=.5]Lecture32pic2.pdf

This is where **parametric curves** come in.

A **parametric curve** (in the plane) is described by a pair of functions, each a function of a common independent variable: x = f(t), y = g(t). Typically we also specify a finite interval of t values to plug into both functions, although sometimes we consider all real t as well.

Let us try $x = t^2 - 4t$, $y = 2 - t$, $t \in [-2, 6]$. We can make a table of t values:

t	-2	-1	0	1	2	3	4	5	6
x	12	5	0	-3	-4	-3	0	5	12
y	4	3	2	1	0	-1	-2	-3	-4

We can plot those points as red dots, and then use that pattern to fill in the curve:

[width=.5]Lecture32pic3.pdf

We see that this seems to make a parabola. Why should this be so? Let us solve for t in on equation and plug into the other: We have $t = 2 - y$, and plugging this in we get x = (2-y)$^2 - 4(2 - y) = 4 - 4y + y^2 + 4y - 8 = y^2 - 4$, which is clearly the equation for a parabola.

Consider x = $\cos(t)$, $\quad y = \sin(t)$, $\quad t \in [0, 2\pi]$. It is not hard to see that this is the equation for the circle of radius one (just recall the definition of $\cos, \sin$). Since $t \in [0, 2\pi]$, we go around the circle exactly once.

We could also consider

x = $\cos(t)$, $\quad y = \sin(t)$, $\quad t \in [0, 6\pi]$, and in this case we would go around the circle three times. Although the set of points traced out is the same, these two curves are different! For example, the first one has length 2π and the second one has length 6π.

If we have a curve generated by a function, we can always represent that as a parametric curve (although not vice versa, in general). For example, if $y = f(x)$, with x in some domain, then we can write this as x= t, y = f(t), and t has the domain inherited from the original function.

2 The Cycloid

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It is a place also for profound mathematical meditation. It was in the left hand try-pot of
that I was first indirectly struck by the remarkable fact, that in geometry all bodies glidi
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A nice parametric shape is the shape known as the cycloid. Let $r > 0$, and define the parametric curve

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta). \tag{1}$$

We plot this on the domain $\theta \in [0, 6\pi]$ below for $r = 2$:
[width=.75]Lecture32cycloid.pdf

Each time t goes through a 2π turn, we get one "leaf" of the cycloid. The point where the curve hits the x-axis is called a "cusp". One place this comes from is the following: consider a circle with a dot painted at one place on the circle, then roll the circle to the right. The path that the dot takes is given by the cycloid. See the Wikipedia page here for an excellent animation of this.

[width=.5]cycloid-derivation.png

To get the equations, consider the picture above. If the circle has rotated through angle θ, then the center of the circle will have moved $r\theta$ to the right. The line segment PQ in the picture has length $r\sin\theta$ and therefore the x-coordinate of P is at $r\theta - r\sin\theta$. Similarly, the length of line segment CQ is $r\cos\theta$, and the center of the circle is at height r, so the y-coordinate of the point P is $r - r\cos\theta$.

3 Fun Examples

Let us consider the parametric curves for x $= \sin t + \frac{1}{2}\sin 5t + \frac{1}{4}\cos\alpha t, \quad y = \cos t + \frac{1}{2}\cos 5t + \frac{1}{4}\sin\alpha t$ for various values of α:

[width=.9]spirograph.pdf