Problem §2.3: 2: Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$ if

(a) $f(n) = \pm n$

(b) $f(n) = \sqrt{n^2+1}$

(c) $f(n) = \frac{1}{n^2-4}$

Solution.

(a) $f(n) = \pm n$: not even a valid function, as one input leads to multiple outputs

(b) $f(n) = \sqrt{n^2+1}$: it is

(c) $f(n) = \frac{1}{n^2-4}$: it is not, as the function is not defined for ± 2 which $\in \mathbb{Z}$

$\square$

Problem §2.3: 12: Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is one-to-one.

(a) $f(n) = n - 1$.

(b) $f(n) = n^2 + 1$.

(c) $f(n) = n^3$.

(d) $f(n) = \lceil n/2 \rceil$.

Solution.

(a) $f(n) = n - 1$: it is

(b) $f(n) = n^2 + 1$: it is not, as $\forall a \in \mathbb{Z}, \pm a$ has the same output

(c) $f(n) = n^3$: it is

(d) $f(n) = \lceil n/2 \rceil$: n and $n + k, 0 \leq k < 1$ hit the same value

$\square$

Problem §2.3: 14(a,b,c,d): Determine whether $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if

(a) $f(m, n) = 2m - n$.

(b) $f(m, n) = m^2 - n^2$.

(c) $f(m, n) = m + n + 1$.

(d) $f(m, n) = |m| - |n|$.

Solution.

(a) $f(m, n) = 2m - n$: onto, let $m = n, f(m, n) = m \in \mathbb{Z}$

(b) $f(m, n) = m^2 - n^2$: not onto, 2 cannot be represented by difference between square.

(c) $f(m, n) = m + n + 1$: onto, let $m = k - 1, n = 0, k \in \mathbb{Z}$, then $f(m, n) = k \in \mathbb{Z}$

(d) $f(m, n) = |m| - |n|$: not onto, let $n = 0, f(m, n) = |m|$, all positive integers hit. Let $m = 0, f(m, n) = -|n|$, all negative integers hit. Let $m = n = 0, f(m, n) = 0$, 0 hit.

$\square$

Problem §2.3: 20: Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is

(a) one-to-one but not onto.

(b) onto but not one-to-one.

(c) both onto and one-to-one (but not the identity function).

(d) neither one-to-one nor onto.

Solution.

(a) one-to-one but not onto: $f(x) = x + 1$

(b) onto but not one-to-one. $f(x) = \lceil \frac{x}{2} \rceil$

(c) both onto and one-to-one (but not the identity function):

$$f(x) = \begin{cases} 2, & x = 1 \\ 1, & x = 2 \\ x, & x \geq 3 \end{cases}$$

(d) neither one-to-one nor onto. $f(x) = 1$

□

Problem §2.3: 22(a,b): Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

(a) $f(x) = -3x + 4$.

(b) $f(x) = -3x^2 + 7$.

Solution.

(a) $f(x) = -3x + 4$: bijection

(b) $f(x) = -3x^2 + 7$: not bijection; $f(x) = f(-x), f(x) \leq 7$ as $-3x^2 \leq 0$

□

Problem §2.3: 36: Find $f \circ g$ and $g \circ f$ where $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from $\mathbb{R}$ to $\mathbb{R}$.

Solution.

$$\begin{aligned} f \circ g &= f(g(x)) \\ &= (x + 2)^2 + 1 \end{aligned}$$

$$\begin{aligned} g \circ f &= g(f(x))a \\ &= x^2 + 1 + 2 \end{aligned}$$

□

Problem §2.3: 39: Show that the function $f(x) = ax + b$ from $\mathbb{R}$ to $\mathbb{R}$ is invertible, where a and b are constants, with $a \neq 0$, and find the inverse of f.

Solution. To prove an invertible function, we can show that it is both an injection and a surjection, in another word, a bijection.

To prove injection, we can show that the function is strictly increasing or decreasing. For the function $f(x) = ax + b$, its derivative $f'(x) = a$, which means the gradient of the function is a constant a. And it is given that $a \neq 0$, so the function is either strictly increasing ($a > 0$) or decreasing ($a < 0$). Hence it is proven that the function is injection.

To prove surjection, we can prove that for every arbitrary $y \in \mathbb{R}$, there exists an $x \in \mathbb{R}$. Let $f(x) = y$, we have:

$$\begin{aligned} f(x) &= y \\ ax + b &= y \\ ax &= y - b \\ x &= \frac{y-b}{a} \end{aligned}$$

Since $a \neq 0$, for all $y \in \mathbb{R}$, there exists a $x \in \mathbb{R}$ so that $f(x) = y$. Hence proven surjection.

And during our process to prove surjection, we have already known that $x = \frac{y-b}{a}$, hence the inverse of $f(x), f^{-1}(x) = \frac{x-b}{a}$ □

Problem §2.3: 40(a): Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that $f(S \cup T) = f(S) \cup f(T)$.

Solution. To prove equivalence, we can prove that both right-hand-side and left-hand-side are subsets to each other, which means $f(S \cup T) \subseteq f(S) \cup f(T)$ and $f(S) \cup f(T) \subseteq f(S \cup T)$.

To prove $f(S \cup T) \subseteq f(S) \cup f(T)$, left-hand-side can be written as $\{f(x)|x \in \ S \cup T\}$. Since $x \in \ S \cup T$ then $x \in S$ or $x \in T$, or both.

If $x \in S$, then $f(x) \in f(S)$. As $f(S) \subseteq f(S) \cup f(T), f(x) \in f(S) \cup f(T)$.

If $x \in T$, then $f(x) \in f(T) \subseteq f(S) \cup f(T)$, as $f(T) \subseteq f(S) \cup f(T)$.

Hence, under any circumstance, $f(x) \in f(S) \cup f(T)$, where $x \in S \cup T$. Then $f(S \cup T) = \{f(x)|x \in S \cup T\} \subseteq f(S) \cup f(T)$

To show that $f(S) \cup f(T) \subseteq f(S \cup T)$, let $y \in f(S) \cup f(T)$. Then $y \in f(S)$ or $y \in f(T)$.

If $y \in f(S)$, then $y = f(x)$ for some $x \in S$. Since $S \subseteq S \cup T$, we have $x \in S \cup T$, and hence $y = f(x) \in f(S \cup T)$.

If $y \in f(T)$, then $y = f(x)$ for some $x \in T$. Since $T \subseteq S \cup T$, we again have $x \in S \cup T$, and hence $y = f(x) \in f(S \cup T)$.

Hence $f(S) \cup f(T) \subseteq f(S \cup T)$.

As $f(S \cup T) \subseteq f(S) \cup f(T)$ and $f(S) \cup f(T) \subseteq f(S \cup T)$, $f(S \cup T) = f(S) \cup f(T)$

□

Problem §2.3: 44(b): Let f be a function from A to B. Let S and T be subsets of B. Show that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

Solution. To prove equivalence, we can prove that both right-hand-side and left-hand-side are subsets to each other, which means $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$ and $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.

To prove $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$, left-hand-side can be written as $\{x \mid f(x) \in S \cap T\}$. Since $f(x) \in S \cap T$, then $f(x) \in S$ and $f(x) \in T$.

Since $f(x) \in S$, then by definition $x \in f^{-1}(S)$. Since $f(x) \in T$, then by definition $x \in f^{-1}(T)$.

Hence $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$, where $x \in f^{-1}(S \cap T)$. So $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.

To show that $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$, let $x \in f^{-1}(S) \cap f^{-1}(T)$. Then $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$.

Since $x \in f^{-1}(S)$, then $f(x) \in S$. Since $x \in f^{-1}(T)$, then $f(x) \in T$.

Hence $f(x) \in S$ and $f(x) \in T$, which means $f(x) \in S \cap T$. Then by definition $x \in f^{-1}(S \cap T)$, so $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.

As $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$ and $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$, we have $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$. $\square$