Math 231E, 2014. Midterm 3.

- This exam has 24 questions.
- You must not communicate with other students during this test. No books, notes, **calculators**, or electronic devices allowed.
- Please fill out all of the information below. Make sure to fill out your Scantron form as directed in class; fill in name, UIN number, and NetID.
- Please draw a picture of an unusual fruit somewhere on this exam booklet.

1. Fill in your information:

Full Name: ______________________________

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2. Fill out name, student number (UIN) and NetID on Scantron sheet. Then fill in the following answers on the Scantron form:

Zone 1

1/1. (5 points) Which of the following expressions best represents the volume of the solid of revolution found by rotating the curve $y = 1 + x - 2x^3$ between $x = 0$ and $x = 1$ around the y axis

A. ★ $2\pi \int_0^1 (x + x^2 - 2x^4)\ dx$

B. $2\pi \int_0^1 (1 + x - 2x^3)\ dx$

C. $2\pi \int_0^1 x\sqrt{1 + (1 - 6x^2)^2}\ dx$

D. $2\pi \int_0^1 \sqrt{1 + (1 - 6x^2)^2}\ dx$

E. $2\pi \int_0^1 y\sqrt{1 + (1 - 6x^2)^2}\ dx$

Solution. This is most easily done by spherical shells, giving

$$\int_0^1 2\pi xy\ dx = 2\pi \int_0^1 (x + x^2 - 2x^4)\ dx$$

1/2. (5 points) Which of the following expressions best represents the volume of the solid of revolution found by rotating the curve $y = 1 + x - 2x^2$ between $x = 0$ and $x = 1$ around the y axis

A. ★ $2\pi \int_0^1 (x + x^2 - 2x^3)\ dx$

B. $2\pi \int_0^1 (1 + x - 2x^2)\ dx$

C. $2\pi \int_0^1 x\sqrt{1 + (1 - 4x)^2}\ dx$

D. $2\pi \int_0^1 \sqrt{1 + (1 - 4x)^2}\ dx$

E. $2\pi \int_0^1 y\sqrt{1 + (1 - 4x)^2}\ dx$

Solution. This is most easily done by spherical shells, giving

$$\int_0^1 2\pi xy\ dx = 2\pi \int_0^1 (x + x^2 - 2x^3)\ dx$$

2/1. (5 points) Determine the type of the integral (proper, improper type I, or improper type II), and point that makes the integral improper

$$\int_0^1 \frac{e^x}{x^2 - 3x + 2}\, dx$$

A. ★ Type II — integrand undefined at $x = 1$

B. Proper

C. Type II — integrand undefined at $x = 0$

D. Type I — Infinite domain

E. Type II — integrand undefined at $x = \frac{1}{2}$

Solution. The denominator vanishes at $x = 1$ and $x = 2$. The former is in the domain of integration and causes the integral to be improper.

3/1. (5 points) Which of the following integrals best represents the length of the curve $y = \sin(x)$ between $x = 0$ and $x = \pi$.

A. ★ $\int_0^\pi \sqrt{1+\cos^2(x)}dx$

B. $2\pi \int_0^\pi \sqrt{1+\cos^2(x)}dx$

C. $2\pi \int_0^\pi x\sqrt{1+\cos^2(x)}dx$

D. $2\pi \int_0^\pi x\sin(x)dx$

E. $\int_0^\pi \sqrt{1+\sin^2(x)}dx$

Solution. The element of arc length is $ds = \sqrt{dx^2+dy^2}$ giving

$$\int_0^\pi \sqrt{1+\cos^2(x)}dx$$

3/2. (5 points) Which of the following integrals best represents the length of the curve $y = \cos(x)$ between $x = 0$ and $x = \pi$.

A. ★ $\int_0^\pi \sqrt{1+\sin^2(x)}dx$

B. $2\pi \int_0^\pi \sqrt{1+\sin^2(x)}dx$

C. $2\pi \int_0^\pi x\sqrt{1+\sin^2(x)}dx$

D. $2\pi \int_0^\pi x\cos(x)dx$

E. $\int_0^\pi \sqrt{1+\cos^2(x)}dx$

Solution. The element of arc length is $ds = \sqrt{dx^2+dy^2}$ giving

$$\int_0^\pi \sqrt{1+\sin^2(x)}dx$$

4/1. (5 points) Use polynomial division to express the following rational function

$$\frac{x^4 + 11x - 13}{x^2 + 1}$$

as a polynomial plus a rational function where the degree of the numerator is less than the degree of the denominator.

A. ★ $x^2 - 1 + \dfrac{11x - 12}{x^2 + 1}$

B. $x^2 - 1 + \dfrac{12x - 11}{x^2 + 1}$

C. $x^2 - x + \dfrac{12x - 11}{x^2 + 1}$

D. $x^2 - 1 + \dfrac{11}{x^2 + 1}$

E. $x^2 - x + 2 + \dfrac{11x - 12}{x^2 + 1}$

Solution. We see that

$$(x^2 + 1)(x^2 - 1) + 11x - 12 = x^4 - 1 + 11x - 12 = x^4 + 11x - 13$$

So, polynomial division gives a quotient of $x^2 - 1$ and a remainder of $11x - 12$.

4/2. (5 points) Use polynomial division to express the following rational function

$$\frac{x^4 + 11x - 13}{x^2 - 1}$$

as a polynomial plus a rational function where the degree of the numerator is less than the degree of the denominator.

A. ★ $x^2 + 1 + \dfrac{11x - 12}{x^2 - 1}$

B. $x^2 - 1 + \dfrac{12x - 11}{x^2 - 1}$

C. $x^2 - x + \dfrac{12x - 11}{x^2 - 1}$

D. $x^2 - 1 + \dfrac{11}{x^2 - 1}$

E. $x^2 - x + 2 + \dfrac{11x - 12}{x^2 - 1}$

Solution. We see that

$$(x^2 + 1)(x^2 - 1) + 11x - 12 = x^4 - 1 + 11x - 12 = x^4 + 11x - 13$$

So, polynomial division gives a quotient of $x^2 + 1$ and a remainder of $11x - 12$.

5/1. (5 points) Evaluate the following improper integral for convergence. If it converges find the value.

$$\int_0^\infty e^{-(8/5)x} dx$$

A. ★ Converges, value is 5/8

B. Diverges

C. Converges, value is $-5/8$

D. Converges, value is 8/5

E. Converges, value is $-8/5$

Solution.

$$\begin{aligned}
\lim_{R\to\infty} \int_0^R e^{-(8/5)x}\, dx &= \lim_{R\to\infty} -\frac{5}{8} e^{-(8/5)x}\Big|_{x=0}^{x=R} \\
&= \lim_{R\to\infty} \frac{5}{8}(1 - e^{-(8/5)R}) \\
&= \frac{5}{8}.
\end{aligned}$$

5/2. (5 points) Evaluate the following improper integral for convergence. If it converges find the value.

$$\int_0^\infty e^{-(5/8)x} dx$$

A. ★ Converges, value is 8/5

B. Diverges

C. Converges, value is $-5/8$

D. Converges, value is 5/8

E. Converges, value is $-8/5$

Solution.

$$\begin{aligned}
\lim_{R\to\infty} \int_0^R e^{-(5/8)x}\, dx &= \lim_{R\to\infty} -\frac{8}{5} e^{-(5/8)x}\Big|_{x=0}^{x=R} \\
&= \lim_{R\to\infty} \frac{8}{5}(1 - e^{-(5/8)R}) \\
&= \frac{8}{5}.
\end{aligned}$$

6/1. (5 points) For what values of p does the following integral converge?

$$\int_1^2 \frac{dx}{x^p}$$

A. ★ All values of p

B. All $p > 1$.

C. All $p \geq 1$.

D. All $p < 1$.

E. All $p \leq 1$.

Solution. This is a proper integral - it is the integral of a continuous function over a bounded interval - and thus it always converges.

7/1. (5 points) Find the integral that is equivalent to the integral

$$\int \frac{x^4}{(1+x^2)^{3/2}}\,dx.$$

A. ★ $\displaystyle\int \frac{\sin^4(\theta)}{\cos^3(\theta)}\,d\theta$

B. $\displaystyle\int \frac{\sin^4(\theta)}{\cos^2(\theta)}\,d\theta$

C. $\displaystyle\int \sin^4(\theta)\,d\theta$

D. $\displaystyle\int \frac{\sec^5(\theta)}{\tan^2(\theta)}\,d\theta$

E. $\displaystyle\int \frac{\sin^5(\theta)}{\cos^3(\theta)}\,d\theta$

Solution. We make a substitution $x = \tan(\theta)$, which gives $dx = \sec^2(\theta)\,d\theta$, so we have

$$\int \frac{x^4}{(1+x^2)^{3/2}}\,dx = \int \frac{\tan^4(\theta)\sec^2(\theta)}{\sec^3(\theta)}\,d\theta = \int \frac{\sin^4(\theta)}{\cos^3(\theta)}\,d\theta.$$

8/1. (5 points) Let $\{a_n\}_{n=1}^{\infty}$ denote the sequence $a_n = \dfrac{1}{n^{\frac{2}{3}}}$ and S denote the series $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$. Which of the following statements is true?

A. The sequence and the series both converge.

B. ★ The sequence converges and the series diverges.

C. the sequence diverges and the series converges.

D. The sequence and the series both diverge.

E. None of the other statements are true.

Solution. The p-test tells us that the series diverges; any limit technique tells us that $a_n \to 0$.

8/2. (5 points) Let $\{a_n\}_{n=1}^{\infty}$ denote the sequence $a_n = \dfrac{1}{n^{\frac{3}{4}}}$ and S denote the series $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{4}}}$. Which of the following statements is true?

A. The sequence and the series both converge.

B. ★ The sequence converges and the series diverges.

C. the sequence diverges and the series converges.

D. The sequence and the series both diverge.

E. None of the other statements are true.

Solution. The p-test tells us that the series diverges; any limit technique tells us that $a_n \to 0$.

Zone 2

9/1. (5 points) Evaluate the following improper integral

$$\int_1^\infty \frac{dx}{x^2+3x+2}$$

A. ★ Converges to $\ln|3| - \ln|2|$

B. Diverges

C. Converges to $\ln|3| + \ln|2|$

D. Converges to $\frac{1}{2}\ln|3| - \frac{1}{2}\ln|2|$

E. Converges to $\frac{1}{3}\ln|3| + \frac{2}{3}\ln|2|$

Solution. We start by factoring $x^2+3x+2 = (x+1)(x+2)$ and then set up partial fractions

$$\frac{1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}$$

which has solution $A = 1$, $B = -1$. Thus

$$\begin{aligned}\int_1^\infty \frac{dx}{x^2+3x+2} = \lim_{t\to\infty}\int_1^t \left(\frac{1}{x+1} - \frac{1}{x+2}\right)\,dx &= \lim_{t\to\infty}(\ln|t+1| - \ln|t+2| - \ln 2 + \ln 3)\\ &= \lim_{t\to\infty}\left(\ln\frac{|t+1|}{|t+2|} - \ln 2 + \ln 3\right) = \ln 3 - \ln 2\end{aligned}$$

10/1. (5 points) Set up and evaluate the integral for the volume of the solid of revolution obtained by rotating the curve $y = \sin(\pi x)$ for $x \in (0, 1)$ about the y-axis.

A. ★ 2

B. 2π

C. π

D. 1

E. $\frac{1}{4}$

Solution. If we imagine infinitesimal shells, then above the point x we have a shell of height $\sin(\pi x)$ and radius x, with thickness dx, so the infinitesimal element is

$$2\pi(\text{radius})(\text{height})(\text{thickness}) = 2\pi x \sin(\pi x)\, dx.$$

This is an easy integration by parts, giving

$$\int_0^1 2\pi x \sin(\pi x) dx = (-2x \cos(\pi x) + \frac{2\sin(\pi x)}{\pi})|_0^1 = 2$$

11/1. (5 points) Determine the convergence of the following improper integral. If the integral converges compute its value.

$$I = \int_0^\infty \frac{1}{1+4x^2}\, dx$$

A. ★ Converges to $I = \frac{\pi}{4}$

B. Converges to $I = \frac{\pi}{2}$

C. Converges to $I = \pi$

D. Diverges

E. Converges to $I = \frac{1}{4}$.

Solution. Since $1 + \tan^2\theta = \sec^2\theta$, we have

$$I = \lim_{t\to\infty} \int_0^t \frac{1}{1+4x^2}\, dx \xrightarrow[dx=\frac{1}{2}\sec^2\theta\, d\theta]{x=\frac{1}{2}\tan\theta} \lim_{t\to\infty} \int_0^{\arctan(2t)} \frac{\frac{1}{2}\sec^2\theta}{\sec^2\theta}\, d\theta = \lim_{t\to\infty} \frac{1}{2}\arctan(2t) = \frac{\pi}{4}.$$

11/2. (5 points) Determine the convergence of the following improper integral. If the integral converges compute its value.

$$I = \int_0^\infty \frac{1}{1+9x^2} dx$$

A. ★ Converges to $I = \frac{\pi}{6}$

B. Converges to $I = \frac{\pi}{6}$

C. Converges to $I = \frac{2\pi}{3}$

D. Diverges

E. Converges to $I = \frac{1}{3}$.

Solution. Since $1 + \tan^2\theta = \sec^2\theta$, we have

$$I = \lim_{t\to\infty} \int_0^t \frac{1}{1+9x^2}\, dx \xrightarrow[dx=\frac{1}{3}\sec^2\theta\, d\theta]{x=\frac{1}{3}\tan\theta} \lim_{t\to\infty} \int_0^{\arctan(3t)} \frac{\frac{1}{3}\sec^2\theta}{\sec^2\theta}\, d\theta = \lim_{t\to\infty} \frac{1}{3}\arctan(3t) = \frac{\pi}{6}.$$

12/1. (5 points) Set up but do not evaluate the integral giving the surface area of the figure found by rotating the curve $y = \sqrt{x^2+1}$ around the x-axis for $x \in [0,1]$

A. ★ $2\pi \int_0^1 \sqrt{1+2x^2}dx$

B. $2\pi \int_0^1 x\frac{\sqrt{1+2x^2}}{1+x^2}dx$

C. $2\pi \int_0^1 \sqrt{1+x^2}dx$

D. $2\pi \int_0^1 x\frac{\sqrt{1-2x^2}}{1+x^2}dx$

E. $2\pi \int_0^1 x\sqrt{1+x^2}dx$

Solution. The formula for the surface area of a solid of rotation around the x axis is

$$\int 2\pi y ds = \int 2\pi y \sqrt{1+(\frac{dy}{dx})^2}dx$$

Given $y = \sqrt{x^2+1}$ we find $\frac{dy}{dx} = \frac{x}{\sqrt{x^2+1}}$ and thence $1 + (\frac{dy}{dx})^2 = 1 + \frac{x^2}{x^2+1} = \frac{1+x^2+x^2}{1+x^2}$ giving $2\pi \int_0^1 \sqrt{1+2x^2}dx$

13/1. (5 points) Evaluate convergence or divergence of the following integral. If the integral converges calculate the value to which it converges.

$$I = \int_0^\infty \frac{x}{\sqrt{x^2+1}} dx$$

A. ★ Integral Diverges

B. Integral Converges, $I = 2\sqrt{2}$

C. Integral Converges, $I = \sqrt{2}$

D. Integral Converges, $I = \sqrt{2} - 1$

E. Integral Converges $I = \log |1 + \sqrt{2}|$

Solution. Diverges by limit comparison test with $f(x) = \frac{1}{x}$

14/1. (5 points) Compute the volume of the solid found by rotating the area between the curves $y = 1 - x$ and $y = \cos(\frac{\pi x}{2})$ around the x-axis for $x \in (0, 1)$.

A. ★ $V = \frac{\pi}{6}$

B. $V = \frac{\pi}{2}$

C. $V = \frac{\pi}{3}$

D. $V = \pi$

E. $V = \frac{3\pi}{4}$

Solution. By washers the integral is given by

$$\int_0^1 \pi(y_1^2(x) - y_2^2(x))dx = \pi \int_0^1 \cos^2(\frac{\pi x}{2}) - (1-x)^2 dx$$

This is easily done with the trig identity $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$ to give

$$\pi \left(-\frac{x}{2} + x^2 - \frac{x^3}{3} + \frac{\sin(\pi x)}{2\pi} \right) |_0^1 = \frac{\pi}{6}$$

14/2. (5 points) Compute the volume of the solid found by rotating the area between the curves $y = x$ and $y = \sin(\frac{\pi x}{2})$ around the x-axis for $x \in (0, 1)$.

A. ★ $V = \frac{\pi}{6}$

B. $V = \frac{\pi}{2}$

C. $V = \frac{\pi}{3}$

D. $V = \pi$

E. $V = \frac{3\pi}{4}$

Solution. By washers the integral is given by

$$\int_0^1 \pi(y_1^2(x) - y_2^2(x))dx = \pi \int_0^1 \sin^2(\frac{\pi x}{2}) - (x)^2 dx$$

This is easily done with the trig identity $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$ to give

$$\pi \left(\frac{x}{2} - \frac{x^3}{3} - \frac{\sin(\pi x)}{2\pi} \right) |_0^1 = \frac{\pi}{6}$$

15/1. (5 points) Let $a_n = \dfrac{1}{n(\ln n)^2}$ and S be the series $S = \sum_{n=2}^{\infty} a_n$. Which of the following statements are true.

A. The sequence converges and the series diverges.

B. The series converges and the sequence diverges.

C. Both the series and sequence diverge.

D. ★ Both the series and sequence converge.

E. None of the other statements are true.

Solution. Since $a_n < 1/n$, we know $0 < a_n < 1/n$, and $\lim_{n\to\infty} 1/n = 0$, so by the Squeeze Theorem we know that $a_n \to 0$. Moreover, the function is positive continuous and decreasing, so we know that the sum converges if and only if the integral converges, and

$$\begin{aligned}\int_2^{\infty} \frac{dx}{x(\ln x)^2} &= \lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^2} \\ &= \lim_{t\to\infty} \int_{\ln(2)}^{\ln t} \frac{du}{u^2} = \lim_{t\to\infty} \frac{1}{\ln(2)} - \frac{1}{\ln(t)} = \frac{1}{\ln(2)}\end{aligned}$$

16/1. (4 points) Find the sum

$$7 + 7(\frac{5}{12}) + 7(\frac{5}{12})^2 + 7(\frac{5}{12})^3 + \dots$$

A. ★ 12

B. 8

C. $\frac{17}{5}$

D. 19

E. 21

Solution. This is a geometric series. Applying the geometric series formula

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad |r| < 1$$

with $a = 7$ and $r = \frac{5}{12}$ we find that the sum is 12

16/2. (4 points) Find the sum

$$12 + 12(\frac{7}{19}) + 12(\frac{7}{19})^2 + 12(\frac{7}{19})^3 + \dots$$

A. ★ 19

B. 8

C. $\frac{17}{5}$

D. 12

E. 21

Solution. This is a geometric series. Applying the geometric series formula

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad |r| < 1$$

with $a = 12$ and $r = \frac{7}{19}$ we find that the sum is 19

17/1. (4 points) Determine if the following improper integral converges. If it converges, determine its value.

$$\int_0^\infty \frac{2x^2}{1+x^6}\,dx$$

A. ★ Converges to $\dfrac{\pi}{3}$

B. Diverges

C. Converges to $\dfrac{\pi}{2}$

D. Converges to $\dfrac{\pi}{4}$

E. Converges to $\dfrac{\pi}{6}$

Solution. Applying the limit comparison test with $g(x) = 2/x^4$, the integral of which converges, we find that the original integral converges.

We first make the substitution $u = x^3$, $du = 3x^2\,dx$, and we obtain the integral

$$\begin{aligned}\frac{2}{3}\int_0^\infty \frac{du}{1+u^2} &= \frac{2}{3}\lim_{t\to\infty}\int_0^t \frac{du}{1+u^2}\\ &= \frac{2}{3}\lim_{t\to\infty}(\arctan(t) - \arctan(0)) = \frac{2}{3}\cdot\frac{\pi}{2} = \frac{\pi}{3}.\end{aligned}$$

18/1. (3 points) Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

A. ★ Converge

B. Diverge

C. Tertium non datur

Solution. Alternating series test. The coefficients are positive, monotone and tend to zero so the alternating series converges.

19/1. (3 points) Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{\sqrt{n^3+3}}$$

A. ★ Converge

B. Diverge

C. Bah, a pox on your Aristotelian logic.

Solution. The series is absolutely convergent, so it is convergent. Indeed, note that

$$0 \leq \frac{|\sin(n)|}{\sqrt{n^3+3}} \leq \frac{1}{\sqrt{n^3}}$$

and the series $\sum \frac{1}{n^{3/2}}$ converges by the p-test.

Zone 3

20/1. (3 points) Given the infinite series

$$\sum_{n=1}^{\infty} a_n$$

and the sequence of partial sums

$$S_N = \sum_{n=1}^{N} a_n$$

Consider the following statements

1) $\lim_{N\to\infty} S_N = L$ implies that the series $\sum_{n=1}^{\infty} a_n$ converges.
2) $\lim_{N\to\infty} S_N = L$ implies that the series $\sum_{n=1}^{\infty} a_n$ diverges.
3) $\lim_{N\to\infty} S_N = L$ implies that $\lim_{n\to\infty} a_n = 0$.
4) $\lim_{n\to\infty} a_n = 0$ implies that $\lim_{N\to\infty} S_N = L$

Which of these statements are true.

A. ★ Statements 1) and 3) are true.

B. Statements 2) and 3) are true.

C. Statements 1) and 4) are true.

D. Statements 2) and 4) are true.

E. Statements 1), 3) and 4) are true.

Solution. Statement 1) is the definition of convergence. Statement 3) is the n^{th} term test or test for divergence. Statement 2) is always false. Statement 4) need not be true (for instance the harmonic series.)

21/1. (2 points) Define the sequence a_n by the following rule

$$a_{n+1} = 2a_n + 1$$

with $a_1 = 1$. Compute the limit

$$L = \lim_{n\to\infty} \frac{a_n}{2^n}$$

A. ★ $L = 1$

B. $L = \pi$

C. $L = 65535$

D. $L = 256$

E. Limit does not exist.

Solution. The first few terms of the series are $1, 3, 7, 15, 31$, so it is not so hard to guess that $a_n = 2^n - 1$ therefore the limit is $\lim_{n\to\infty} \frac{2^n-1}{2^n} = 1$. (If you do not want to guess, notice that $a_{n+1} + 1 = 2(a_n + 1)$ which shows that $a_n + 1 = 2^n$.)

22/1. (2 points) Evaluate the following indefinite integral

$$\int \frac{x^2}{(1-x^2)^{3/2}}\,dx$$

A. ★ $\dfrac{x}{\sqrt{1-x^2}} - \arcsin(x) + C$

B. $\dfrac{x}{\sqrt{1-x^2}} + \arcsin(x) + C$

C. $\dfrac{-x}{\sqrt{1-x^2}} - \arcsin(x) + C$

D. $\sqrt{1-x^2} + \arcsin(x) + C$

E. $\sqrt{1-x^2} + C$

Solution. We write $x = \sin(\theta)$, $dx = \cos(\theta)\,d\theta$, and obtain

$$\int \frac{\sin^2(\theta)}{\cos^3(\theta)} \cos(\theta)\,d\theta = \int \tan^2(\theta)\,d\theta = \int (\sec^2(\theta) - 1)\,d\theta = \tan\theta - \theta + C.$$

If we draw a triangle with opposite side x and hypothenuse 1, then we see $\tan\theta = x/\sqrt{1-x^2}$ and $\theta = \arcsin(x)$.

23/1. (2 points) Compute the arclength of the function $y = 1 + 2x^{3/2}$ between $x = 0$ and $x = 1$

A. ★ $\frac{2}{27}(10\sqrt{10} - 1)$

B. $\frac{2}{27}(10\sqrt{10} + 1)$

C. $\frac{14}{9}$

D. $\frac{10}{9}$

E. $\frac{2}{9}\sqrt{10}$

Solution. The element of arc length is $ds = \sqrt{dx^2 + dy^2}$, we have $y'(x) = 3x^{1/2}$ and hence

$$\int_0^1 \sqrt{1+9x}\, dx \xrightarrow[ds=9dx]{s=1+9x} \frac{1}{9}\int_1^{10} \sqrt{s}\, ds = \frac{1}{9}\frac{s^{3/2}}{\frac{3}{2}}\Bigg|_1^{10} = \frac{2}{27}(10\sqrt{10} - 1)$$

24/1. (2 points) Let P_n denote the n^{th} prime number (In other words, $P_1 = 2, P_2 = 3, P_3 = 5, P_4 = 7, P_5 = 11, P_6 = 13$.) Let a_n be the sequence defined by $a_n = \dfrac{1}{(P_n)^2}$. Let S be the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{P_n^2}$. Which of the following are true

A. ★ Both the series and sequence converge.

B. The sequence converges and the series diverges.

C. The series converges and the sequence diverges.

D. Both the series and sequence diverge.

E. None of the other statements are true.

Solution. Obviously the n^{th} prime number is larger than n.

$$P_n > n$$

Thus we have $0 \leq a_n \leq \frac{1}{n^2}$. Then the sequence converges by the squeeze theorem and the series converges by the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$
