Math 231E, Lecture 27.
Comparison Test(s)

1 Direct Comparison Test

Theorem 1.1. Let $0 \leq a_n \leq b_n$. Then

1. If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

Note: This looks just like the comparison tests we had for integrals, and for the same reason.

Example 1.2. Does the series

$$\sum_{n=1}^{\infty} \frac{2n}{n^4+1}$$

converge?

Note that we can write

$$n^4 + 1 > n^4,$$

so

$$\frac{2n}{n^4+1} < \frac{2n}{n^4} = \frac{2}{n^3},$$

and the latter series converges by the p-test. Therefore so does our series.

Example 1.3. What about

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}?$$

Notice that it is not strictly true that

$$\ln n > 1$$

for all $n = 1, 2, 3, \ldots$, since $\ln 1 = 0, \ln 2 < 1$. But $\ln n > 1$ for all $n \geq 3$. Therefore

$$\sum_{n=3}^{\infty} \frac{\ln n}{n}$$

diverges, since $\ln n/n > 1/n$ and the latter diverges. But if this sum diverges, then adding 0 and $\ln 2/2$ to it doesn't change that fact.

2 Limit Comparison Test

Theorem 2.1. Assume $a_n, b_n \geq 0$ and moreover assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C,$$

where $0 < C < \infty$. Thus C is a finite but nonzero number. Then the two sums

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} b_n,$$

either **both converge** or **both diverge**.

Proof. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C,$$

then there exists an $N > 0$ such that, for all $n > N$,

$$\frac{C}{2} \leq \frac{a_n}{b_n} \leq (C+1),$$

or

$$\frac{C}{2} b_n \leq a_n \leq (C+1) b_n.$$

Now, if $\sum b_n$ converges, then $\sum a_n$ converges by the previous comparison theorem. Conversely, if $\sum a_n$ converges, then $\sum b_n$ converges by the previous comparison theorem.

□

Example 2.2. Consider the series

$$\sum_{n=1}^{\infty} \frac{n^2+1}{3n^4+5}.$$

Looking at this term, we see that it should roughly be $1/n^2$ when n is large. So let us write

$$a_n = \frac{n^2+1}{3n^4+5}, \quad b_n = \frac{1}{n^2}, \tag{1}$$

and then we compute

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^2(n^2+1)}{3n^4+5} \\ &= \lim_{n\to\infty} \frac{n^4+n^2}{3n^4+5} = \frac{1}{3}. \end{aligned}$$

Since $0 < 1/3 < \infty$, and $\sum b_n$ converges, then $\sum a_n$ converges.

Example 2.3. Consider the series

$$\sum_{n=1}^{\infty} \frac{n^2+1}{2n^3+5}.$$

Looking at this term, we see that it should roughly be $1/n$ when n is large. So let us write

$$a_n = \frac{n^2+1}{2n^3+5}, \quad b_n = \frac{1}{n}, \tag{2}$$

and then we compute

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n(n^2+1)}{2n^3+5}$$
$$= \lim_{n\to\infty}\frac{n^3+n}{2n^3+5} = \frac{1}{2}.$$

Since $0 < 1/2 < \infty$, and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example 2.4. Let us test the series

$$\sum_{n=1}^{\infty}\frac{\sqrt[7]{n^3+4n+7}}{\sqrt[6]{n^{13}+n^5+1}}$$

for convergence.

At first, this looks kind of tricky, since there is a lot going on. But let's just look at the approximate size of this term for n large. The numerator looks like $n^{3/7}$ and the denominator looks like $n^{13/6}$, so we expect this quotient to be approximately

$$\frac{n^{3/7}}{n^{13/6}} = n^{\frac{3}{7}-\frac{13}{6}} = n^{-73/42}.$$

The good news is that $73/42 > 1$, and so $\sum_{n=1}^{\infty} n^{-73/42}$ converges.

To make this more precise, let us define

$$a_n = \frac{\sqrt[7]{n^3+4n+7}}{\sqrt[6]{n^{13}+n^5+1}}, \quad b_n = n^{-73/42}.$$

Let us compute:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^{73/42}\sqrt[7]{n^3+4n+7}}{\sqrt[6]{n^{13}+n^5+1}}$$
$$= \lim_{n\to\infty}\frac{n^{73/42}n^{3/7}\sqrt[7]{1+4/n^2+7/n^3}}{n^{13/6}\sqrt[6]{1+1/n^8+1/n^{13}}}$$
$$= \lim_{n\to\infty}\frac{\sqrt[7]{1+4/n^2+7/n^3}}{\sqrt[6]{1+1/n^8+1/n^{13}}} = 1.$$

Since we know $\sum_n b_n$ converges, then we also know that $\sum_n a_n$ converges.

Example 2.5. How about

$$\sum_{n=1}^{\infty}\frac{(\ln n)^6}{n^2}.$$

This is a case where we can't quite use the limit comparison test, but we can get an idea of how it goes anyway.

First note that

$$\lim_{n\to\infty}\frac{\ln n}{n^a} = 0$$

for any $a > 0$. To see this we can use l'Hôpital's Rule:

$$\lim_{n\to\infty}\frac{\ln n}{n^a} = \lim_{n\to\infty}\frac{1/n}{an^{a-1}} = \frac{1}{an^a} = 0.$$

From this, notice that

$$\lim_{n\to\infty}\frac{(\ln n)^6}{n^a} = \lim_{n\to\infty}\left(\frac{\ln n}{n^{a/6}}\right)^6 = 0^6 = 0$$

as well. So, choose $0 < a < 1$. Then

$$\frac{(\ln n)^6}{n^2} = \frac{(\ln n)^6}{n^a} \cdot \frac{1}{n^{2-a}}.$$

Since

$$\lim_{n\to\infty} \frac{(\ln n)^6}{n^a} = 0,$$

then there is an N such that for all $n > N$,

$$\frac{(\ln n)^6}{n^a} < 1.$$

Therefore, for all $n > N$, we have

$$\frac{(\ln n)^6}{n^a} < \frac{1}{n^{2-a}}.$$

Since we chose $a < 1$, this latter sum is convergent, so we have

$$\sum_{n=N+1}^{\infty} \frac{(\ln n)^6}{n^2} < \sum_{n=N+1}^{\infty} \frac{1}{n^{2-a}},$$

and this means that both of these terms are finite. This will mean that the whole sum converges, since

$$\sum_{n=1}^{\infty} \frac{(\ln n)^6}{n^2} = \sum_{n=1}^{N} \frac{(\ln n)^6}{n^2} + \sum_{n=N+1}^{\infty} \frac{(\ln n)^6}{n^2}$$

and the first term on the right-hand side, no matter how large it is, is still finite.