Math 231E, Lecture 15.
Indefinite Integral, and Computing Integrals by Substitution

1 Indefinite Integral

We first lay out the example of the *indefinite integral*.

Let us consider the integral

$$\int_0^4 e^x\,dx.$$

From the Fundamental Theorem of Calculus, we know that the antiderivative of e^x is itself, so

$$\int_0^4 e^x\,dx = e^x\Big|_{x=0}^{x=4} = e^4 - e^0 = e^4 - 1.$$

But we also have

$$\int_2^3 e^x\,dx = e^x\Big|_{x=2}^{x=3} = e^3 - e^2,$$

and

$$\int_{-2}^7 e^x\,dx = e^x\Big|_{x=-2}^{x=7} = e^7 - e^{-2}.$$

And we see that in all these cases, the formula into which we are plugging in is the same; we just use different values to plug in. Moreover, as we will see below, it is typical that the vast majority of the work that we need to do will involve finding this function, and the plugging in of the boundary values is the easy part.

As such, we would like to just refer as this antiderivative function, and when we need to do so we will talk about the indefinite integral, which is just another word for antiderivative.

The one thing to point out here is that an antiderivative is not unique. For example, we know that

$$\frac{d}{dx}(x^3) = \frac{d}{dx}(x^3+2) = \frac{d}{dx}(x^3-12) = \frac{d}{dx}(x^3+\pi^2 e^{17}\cos\sin(4)) = 3x^2.$$

In particular, this means that we can always add any constant we like to an antiderivative, since all of the functions

$$x^3, \quad x^3+2, \quad x^3-12, \quad x^3+\pi^2 e^{17}\cos\sin(4)$$

are all antiderivatives of $3x^2$. In general, we will write this as

$$x^3 + C,$$

where C is an arbitrary constant.

And, again, we stress that the choie of constant C will not matter when computing a definite integral, since

$$\int_{-2}^7 e^x\,dx = e^x + C\Big|_{x=-2}^{x=7} = (e^7+C) - (e^{-2}+C) = e^7 - e^{-2} + (C-C) = e^7 - e^{-2}.$$

2 Examples

So, the hard part is finding antiderivatives. Basically, all of the rules we come up with are just rules for differentiation run backwards! Examples:

1. $$\int x^n = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

2. $$\int e^x = e^x + C$$

3. $$\int \frac{1}{x} = \ln|x| + C$$

4. $$\int cf(x)\,dx = c\int f(x)\,dx$$

5. $$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Unfortunately, there isn't a rule like

$$\int f(x)g(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx.$$

There is a way to use the product rule in reverse, which we will see later — it is called "integration by parts".

Note! There is a simple enough pattern. Every "integration rule" is just a differentiation rule run backwards. We will see this theme over and over.

3 Integration by Substitution

However, we can use the Chain Rule is reverse pretty easily. Say that $F'(x) = f(x)$, and then

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x).$$

This means that

$$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C.$$

This gives rise to the technique of **u-substitution**.

Example 3.1. We want to compute

$$\int x\sqrt{3+x^2}\,dx.$$

Let us try $u = 3 + x^2$, which gives $du = 2x\,dx$, so that $x\,dx = du/2$. We then have

$$\int x\sqrt{3+x^2}\,dx = \int \sqrt{u}\,\frac{1}{2}du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}(3+x^2)^{3/2} + C.$$

A crucial feature that made this work is that there was an extra power of x in the integrand. For example, if we were given

$$\int \sqrt{3+x^2}\,dx,$$

and tried this trick, we would get

$$\int \sqrt{3+x^2}\,dx = \int \sqrt{u}\,\frac{du}{2x},$$

and now what do we do?

Example 3.2. Let's try the integral

$$\int e^{x+e^x}\,dx.$$

Now, we might first think that our sub should be $u = x + e^x$, which gives $du = (1+e^x)\,dx$, so we have

$$\int e^{x+e^x}\,dx = \int e^u\,\frac{du}{1+e^x},$$

which doesn't seem that helpful. But, let us go back to the original integral and multiply out, recalling that $e^{a+b} = e^a e^b$, so we have

$$\int e^{x+e^x}\,dx = \int e^x \cdot e^{e^x}\,dx,$$

and now we might see that it makes sense ot use $u = e^x$, $du = e^x\,dx$, and then we have

$$\int e^{x+e^x}\,dx = \int e^u\,du = e^u + C = e^{e^x} + C.$$

Example 3.3. Sometimes we might have to do a lot more work than we originally thought! Let's try

$$\int \frac{x}{1+\sqrt{1-x^2}-x^2}\,dx.$$

Let us try $u = 1 - x^2$, so $du = -2x\,dx$, and we have

$$\int \frac{x}{1+\sqrt{1-x^2}-x^2}\,dx = -\frac{1}{2}\int \frac{du}{u+\sqrt{u}}.$$

Since we have a $\sqrt{u}$ term, let us choose $v = \sqrt{u}$, and thus $dv = 1/(2\sqrt{u})\,du$, or $du = 2v\,dv$, and we have

$$\int \frac{x}{1+\sqrt{1-x^2}-x^2}\,dx = -\int \frac{v\,dv}{v^2+v} = -\int \frac{dv}{v+1}.$$

Finally, choose $w = v + 1$ and $dw = dv$, and we have

$$\int \frac{x}{1+\sqrt{1-x^2}-x^2}\,dx = -\int \frac{dw}{w} = -\ln|w| + C.$$

Unpacking all of the definitions, we have

$$\begin{aligned}\int \frac{x}{1+\sqrt{1-x^2}-x^2}\,dx &= -\ln|w| + C\\ &= -\ln|v+1| + C\\ &= -\ln|\sqrt{u}+1| + C\\ &= -\ln|\sqrt{1-x^2}+1| + C.\end{aligned}$$

4 u-sub for Definite Integrals

If we are doing definite integrals, then we can also use u-sub, but we should change the limits of integration as we go.

Let me say that again: If we are doing definite integrals, then we can also use u-sub, but we should change the limits of integration as we go.

Perhaps you didn't hear me.

If we are doing definite integrals, then we can also use u-sub, but we should change the limits of integration as we go.

Example 4.1. Let us consider the definite integral

$$\int_1^e \frac{\ln x}{x}\,dx.$$

It seems clear that we should make the substitution $u = \ln x$, which gives $du = dx/x$. But also notice that we should change the limits on the integral: when $x = 1$, we have $u = \ln 1 = 0$, and when $x = e$, we have $u = \ln e = 1$. So we have

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big|_{u=0}^{u=1} = \frac{1}{2}.$$

This, for example, will allow us to derive the *even–odd* relations. For example, we can show that if $f(x)$ is odd, then

$$\int_{-a}^a f(x)\,dx = 0$$

for any a. This is because

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx,$$

and if we make the substitution $u = -x$ in the first integral, this becomes

$$\int_{-a}^0 f(x)\,dx = \int_a^0 f(-u)(-du) = -\int_0^a (-f(u))(-du) = -\int_0^a f(u)\,du.$$

Similarly, if $f(x)$ is even, then

$$\int_{-a}^0 f(x)\,dx = \int_a^0 f(-u)(-du) = -\int_0^a (f(u))(-du) = \int_0^a f(u)\,du.$$

and so

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$