Math 231E, Lecture 29.
Power Series

1 Definition of power series

Definition 1.1. A **power series** (at $x = 0$) is a series of the form

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

(Note that we typically start power series at $n = 0$.)

Example 1.2. Let $C_n = 1$ for all n, then we get the power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Definition 1.3. A **power series** (centered at $x = a$) is a series of the form

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots$$

2 Domain of convergence — examples

Remark 2.1. Notice that a power series is an infinite series that depends on the variable x. As such, it might converge for some x and diverge for other x!

Definition 2.2. Given a power series, the **domain of convergence** is the set of x that makes the series converge.

Example 2.3. Let us consider the case $C_n = 1/n^2, n \geq 1$ and center the power series at $a = 0$, so we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \cdots$$

Where does this series converge?
Let us first apply the Ratio Test. The nth term of this series is

$$a_n = C_n x^n = \frac{x^n}{n^2}.$$

We then have

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} = \left(\frac{n}{n+1}\right)^2 x.$$

Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = x.$$

By the Ratio Test, if $|x| > 1$, then the series diverges. If $|x| < 1$, then the series converges.

But what happens if $|x| = 1$?

Let us first consider $x = 1$. We then obtain the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges by the p-test.

If $x = -1$, then we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

and since this converges absolutely, it converges.

In summary, the domain of convergence of this series is $[-1, 1]$.

Example 2.4. Let us consider the case $C_n = n^3, n \geq 0$ and center the power series at $a = 0$, so we have

$$\sum_{n=0}^{\infty} n^3 x^n = x + 8x^2 + 27x^3 + \cdots$$

Where does this series converge?

Let us first apply the Ratio Test. The nth term of this series is

$$a_n = C_n x^n = n^3 x^n.$$

We then have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3 x^{n+1}}{n^3 x^n} = \left(\frac{n+1}{n}\right)^3 x.$$

Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 = x.$$

By the Ratio Test, if $|x| > 1$, then the series diverges. If $|x| < 1$, then the series converges.

But what happens if $|x| = 1$?

Let us first consider $x = 1$. We then obtain the series

$$\sum_{n=1}^{\infty} n^3$$

which diverges, and if $x = -1$, then we obtain

$$\sum_{n=1}^{\infty} (-1)^n n^3,$$

which diverges.

In summary, the domain of convergence of this series is $(-1, 1)$.

Example 2.5. Now we consider
$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$
Then we have
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}.$$
Then
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{x}{n+1} = 0$$
for all x. Therefore this series converges for every x, and the domain of convergence is $(-\infty, \infty)$.

Example 2.6. One more example! Now we consider
$$\sum_{n=0}^{\infty} n!x^n.$$
We have
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!x^{n+1}}{n!x^n} = (n+1)x.$$
Then
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} x(n+1) = \infty$$
for all $x \neq 0$. Therefore the domain of convergence is $\{0\}$.

Example 2.7. All of the power series we did above were centered at $a = 0$, but we can also do series centered elsewhere. Consider
$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{n4^n}.$$
Applying the Ratio Test, we consider
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}n4^n}{(x-3)^n(n+1)4^{n+1}}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right) \frac{|x-3|}{4} = \frac{|x-3|}{4}.$$
This series converges if
$$\frac{|x-3|}{4} < 1, \text{ or } |x-3| < 4.$$
The series diverges if
$$\frac{|x-3|}{4} > 1, \text{ or } |x-3| > 4. \tag{1}$$
Now, what happens at the two points where $|x-3| = 4$? These are the points $x = 7, x = -1$.
At $x = 7$, we have
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n},$$
which converges by the Alternating Series Test.
At $x = -1$, we have
$$\sum_{n=0}^{\infty} \frac{1}{n},$$
which diverges.
Therefore, the domain of convergence is $(-1, 7]$.

3 Domain of convergence — theorem

Theorem 3.1. For any power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n,$$

there are exactly three possibilities for the domain of convergence (DOC) and radius of convergence (ROC).

1. Converges only at $x = a$, or
   DOC = $\{a\}$, ROC = 0;
2. Converges for all x, or
   DOC = $(-\infty, \infty))$, ROC = ∞;
3. There is an R such that the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$,
   ROC = R.

Remark 3.2. In the case with a radius of convergence R with $0 < R < \infty$, we have to check the endpoints "by hand".

Proof. Let us assume that

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

exists. (The theorem is still true even if it doesn't, but proving this is a bit beyond the tools we have.)

Apply the Ratio test to the sum

$$\sum_{n=0}^{\infty} a_n, \tag{2}$$

where

$$a_n = c_n (x-a)^n.$$

Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| |x-a| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

We assume that the last limit exists, so let us call it L. Therefore, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x-a|.$$

Now, there are three possibilities:

1. $L = \infty$. In this case, the limit is ∞ for all $x \neq a$, and so the series converges only for $x = a$.
2. $L = 0$. In this case, the limit is 0 for all x, and therefore the series converges for all x.
3. $0 < L < \infty$. In this case, the condition for convergence is

   $$L|x-a| < 1, \text{ or } |x-a| < \frac{1}{L},$$

   and the condition for divergence is

   $$L|x-a| > 1, \text{ or } |x-a| > \frac{1}{L}.$$

   So we see that we have established that not only is the conclusion of the theorem true, but in fact it tells us that $R = 1/L$.

□