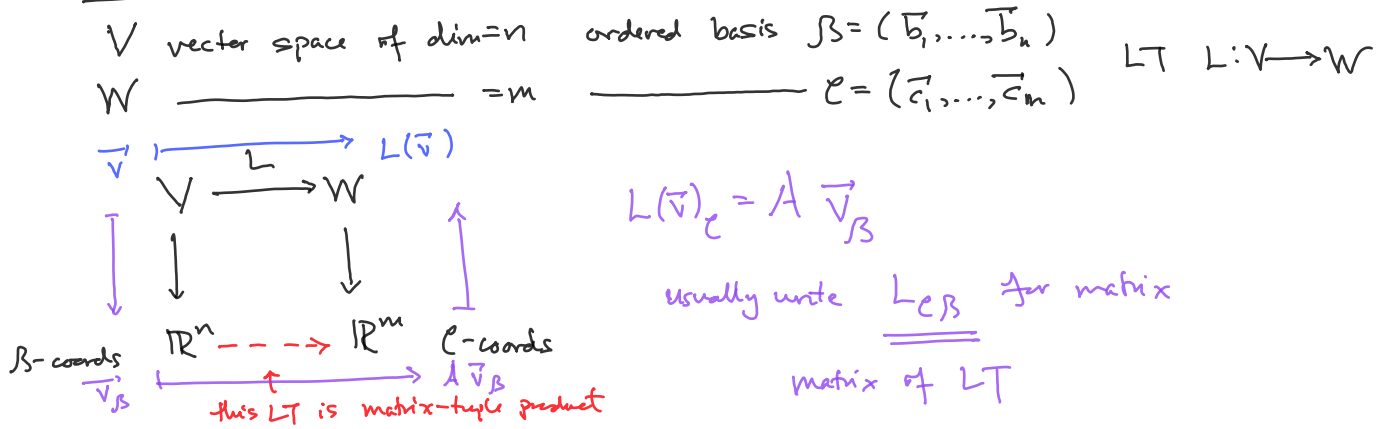


§ Lecture 14 dkm15p

last time: change of coordinates I_{AB} or I_{BA} , linear transformations (LT)

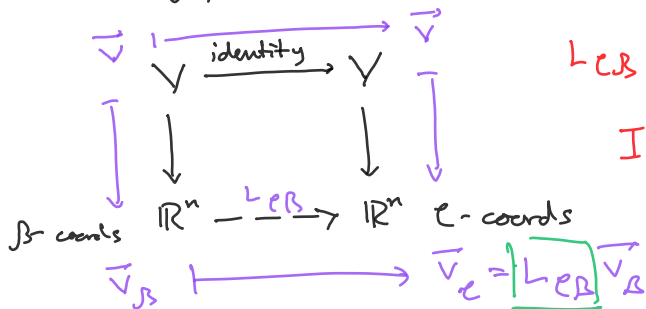
$$L(r\vec{x} + s\vec{y}) = rL(\vec{x}) + sL(\vec{y})$$

§ Matrix of LT



Fact: $L_{\mathcal{C}\beta} = \begin{pmatrix} L(\vec{b}_1)_{\mathcal{C}} & \dots & L(\vec{b}_n)_{\mathcal{C}} \end{pmatrix}_{m \times n}$

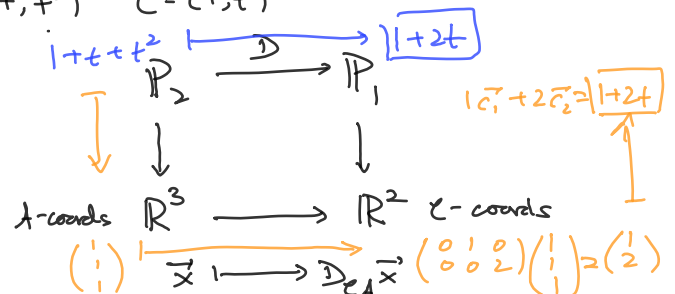
Example: LT identity function $V \rightarrow V$ $\vec{v} \mapsto \vec{v}$ $\dim(V) = n$
 β, \mathcal{C} ordered bases on V



Example: derivative $D: P_2 \rightarrow P_1$ $A = (1, t, t^2)$ $\mathcal{C} = (1, t)$

Compute: (1) $D_{\mathcal{C}A}$

(2) $D(1+t+t^2)$ in two ways



$$\begin{aligned} (1) D_{\mathcal{C}A} &= \begin{pmatrix} D(\vec{a}_1)_{\mathcal{C}} & D(\vec{a}_2)_{\mathcal{C}} & D(\vec{a}_3)_{\mathcal{C}} \end{pmatrix}_{2 \times 3} \\ &= \begin{pmatrix} 0_{\mathcal{C}} & 1_{\mathcal{C}} & (2t)_{\mathcal{C}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

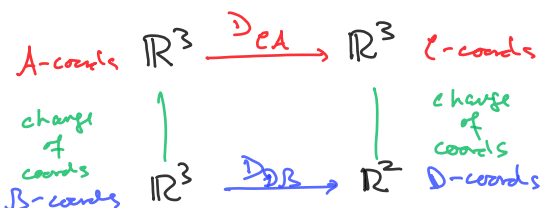
$$\begin{aligned} 0 &= 0 \cdot 1 + 0 \cdot t \\ 1 &= 1 \cdot 1 + 0 \cdot t \\ 2t &= 0 \cdot 1 + 2 \cdot t \end{aligned}$$

Moral: coords make abstract concrete by translation into tuple-words

relative coords, each abstract LT is translated into matrix-tuple product

Example: $\beta = (1, 1+t, 1+t+t^2)$ for $\mathbb{P}_2 \Rightarrow D_{DB}$
 $\beta = (1, 1+t)$ for \mathbb{P}_1

Q: how compare D_{CA} & D_{DB}



Fact: $D_{DB} = I_{DC} D_{CA} I_{AB}$

let's compute D_{DB} in two ways:

(1) direct computation

(2) leverage knowledge of D_{CA}

(1) direct computation

$$\begin{aligned} D_{DB} &= \begin{pmatrix} D(\vec{b}_1)_D & D(\vec{b}_2)_D & D(\vec{b}_3)_D \end{pmatrix} & \begin{aligned} 0 &= 0 \cdot 1 + 0 \cdot (1+t) \\ 1 &= 1 \cdot 1 + 0 \cdot (1+t) \\ 1+t &= -1 \cdot 1 + 2 \cdot (1+t) \end{aligned} \\ &= \begin{pmatrix} 0_D & 1_D & (1+2t)_D \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

(2) leverage knowledge of D_{CA}

$$D_{DB} = I_{DC} D_{CA} I_{AB}$$

let's find the change of coord. matrices

$$\begin{aligned} I_{AB} &= \begin{pmatrix} (\vec{b}_1)_A & (\vec{b}_2)_A & (\vec{b}_3)_A \end{pmatrix} & \begin{aligned} 1 &= 1 \cdot 1 + 0 \cdot t + 0 \cdot t^2 \\ 1+t &= 1 \cdot 1 + 1 \cdot t + 0 \cdot t^2 \\ 1+t+t^2 &= 1 \cdot 1 + 1 \cdot t + 1 \cdot t^2 \end{aligned} \\ &= \begin{pmatrix} 1_A & (1+t)_A & (1+t+t^2)_A \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} I_{DC} &= (I_{CD})^{-1} & I_{CD} &= \begin{pmatrix} (\vec{d}_1)_C & (\vec{d}_2)_C \end{pmatrix} & \begin{aligned} 1 &= 1 \cdot 1 + 0 \cdot t \\ 1+t &= 1 \cdot 1 + 1 \cdot t \end{aligned} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} & &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$D_{DB} = I_{DC} D_{CA} I_{AB} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

many:

$$\begin{array}{ccc} P_2 & \xrightarrow{D} & P_1 \\ | & & | \\ B\text{-cells} & R^3 \xrightarrow{D_{DB}} & R^2 \text{ D-cells} \\ \downarrow & & \downarrow \\ A\text{-cells} & R^3 \xrightarrow{D_{CA}} & R^2 \text{ C-cells} \end{array}$$

Motivation: recall for 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible $\iff \det(A) = ad - bc \neq 0$

One way to do this:

Elem row op

Effect on det

Swap

 $(-1) \times$

scalar ($c \neq 0$)

(c) x

shear ($i \neq j$)

 $(1) \times$
$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[\text{new } R_3]{\frac{1}{2} R_3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-R_3 + R_2 \text{ new } R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{new } R_1]{R_2 + R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2} R_2} \text{new } R_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\times \frac{1}{3}$$

$$I = \det(I) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \det(A) = \frac{1}{6} \det(A)$$

$$\Rightarrow \det(A) = 6$$

