# Lecture: Integration Using Partial Fractions

## Motivation: When $u$-substitution fails

Consider the integral

$$I = \int \frac{1}{x^2 - 1}\, dx.$$

Note that it is not suitable for a substitution such as $u = x^2 - 1$. Since then $du = 2x\, dx$, but our numerator is 1, which is not a multiple of $2x$.

However, observe that the denominator can be *factored* and we can rewrite the fraction as a sum of simpler ones:

$$\frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}.$$

Multiplying both sides by $(x-1)(x+1)$ gives

$$1 = A(x+1) + B(x-1) = (A+B)x + (A-B).$$

Equating coefficients:

$$A + B = 0, \qquad A - B = 1.$$

Solving these equations gives $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.

Hence,

$$\frac{1}{x^2 - 1} = \frac{1}{2}\left(\frac{1}{x - 1} - \frac{1}{x + 1}\right).$$

Now the integral becomes

$$I = \frac{1}{2}\int \frac{1}{x - 1}\, dx - \frac{1}{2}\int \frac{1}{x + 1}\, dx = \frac{1}{2}\big(\ln|x - 1| - \ln|x + 1|\big) + C.$$

This example shows that factoring the denominator and decomposing the fraction leads to simpler terms that we can integrate using standard formulas.

## General Idea: Decomposing Rational Functions

Let

$$R(x) = \frac{P(x)}{Q(x)}$$

be a **proper rational function**, meaning that the degree of the numerator is smaller than that of the denominator ($\deg P < \deg Q$).

The goal of the *partial fraction decomposition* method is to express $R(x)$ as a sum of simpler fractions whose denominators are of lower degree and whose antiderivatives are already known (typically logarithmic or arctangent forms).

**Case 1: Distinct Real roots**

If $Q(x)$ factors completely into distinct real linear terms:

$$Q(x) = (x - r_1)(x - r_2)\cdots(x - r_n),$$

then we can write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \cdots + \frac{A_n}{x - r_n}.$$

Each term of the form $\frac{A}{x-r}$ integrates to a logarithm:

$$\int \frac{A}{x - r}\, dx = A \ln|x - r| + C.$$

**Example 1** Compute the following integral

$$\int \frac{2x + 5}{(x - 2)(x - 3)}\, dx.$$

**Step 1:** Decompose

$$\frac{2x + 5}{(x - 2)(x - 3)} = \frac{A}{x - 2} + \frac{B}{x - 3}.$$

Multiply through by the denominator:

$$2x + 5 = A(x - 3) + B(x - 2).$$

**Step 2:** Solve for $A, B$

$$\begin{cases} x = 2 \Rightarrow 9 = -A \Rightarrow A = -9, \\ x = 3 \Rightarrow 11 = B. \end{cases}$$

**Step 3:** Integrate

$$I = -9 \int \frac{dx}{x - 2} + 11 \int \frac{dx}{x - 3} = -9 \ln|x - 2| + 11 \ln|x - 3| + C.$$

**Example 2** Compute the following integral

$$\int \frac{2x + 3}{x^2 - x - 2}\, dx.$$

**Step 1:** Factor denominator

$$x^2 - x - 2 = (x - 2)(x + 1).$$

**Step 2:** Decompose

$$\frac{2x + 3}{(x - 2)(x + 1)} = \frac{A}{x - 2} + \frac{B}{x + 1}.$$

Multiply through:

$$2x + 3 = A(x + 1) + B(x - 2) = (A + B)x + (A - 2B).$$

**Step 3:** Solve
$$A + B = 2, \qquad A - 2B = 3 \quad \Rightarrow \quad A = \tfrac{7}{3},\ B = -\tfrac{1}{3}.$$

**Step 4:** Integrate
$$I = \tfrac{7}{3}\ln|x-2| - \tfrac{1}{3}\ln|x+1| + C.$$

**Practice Problems**: Compute each of the following:

1. $\int \frac{3x+1}{x^2-4}\,dx$
2. $\int \frac{2x+5}{x^2-3x+2}\,dx$
3. $\int \frac{4x+7}{x^2-x-6}\,dx$

**Case 2: Irreducible Quadratic Factors (Complex Roots)**

If $Q(x)$ has factors that cannot be further factored over the reals, they will appear as irreducible quadratic terms of the form
$$(x-a)^2 + b^2, \qquad (b \neq 0).$$

Before we discuss how to handle such factors in a general setting, recall the following result:
$$\int \frac{dx}{x^2+1} = \tan^{-1}(x) + C.$$

Next consider the integral,
$$I = \int \frac{dx}{(x-a)^2 + b^2}.$$

Then if you let
$$u = \frac{x-a}{b} \quad \Rightarrow \quad du = (1/b)dx,$$

Then after substitution we have,
$$I = \int \frac{b\,du}{(bu)^2 + b^2} = \int \frac{b\,du}{b^2(u^2+1)} = \frac{1}{b}\int \frac{du}{u^2+1}.$$

$$I = \frac{1}{b}\tan^{-1}(u) + C = \frac{1}{b}\tan^{-1}\left(\frac{x-a}{b}\right) + C.$$

$$\boxed{I = \frac{1}{b}\tan^{-1}\left(\frac{x-a}{b}\right) + C.}$$

In the next class we will see how to factorize a fraction that has such type of irreducible quadratic terms.