Math 231E, Lecture 27.
Alternating Series

1 Definition

Definition 1.1. An **alternating sequence/series** is a sequence/series where the signs of the term are alternatively positive and negative.

Examples of alternating series include

$$1-2+3-4+5-6+7-\dots, \quad 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\dots$$

We continue the convention that a_n is the nth term in the sequence, and we will always write $b_n = |a_n|$. Then if a_n is an alternating sequence then either

$$a_n = (-1)^n b_n, \quad a_n = (-1)^{n+1} b_n.$$

2 Alternating Series Test

We have a nice theorem to test for convergence of an alternating series:

Theorem 2.1 (Alternating Series Test)**.** Let a_n be an alternating series, and $b_n = |a_n|$. If

1. $b_{n+1} \le b_n$
2. $\lim_{n\to\infty} b_n = 0$,

then $\sum_n a_n$ converges.

Note! Remember how we said you cannot deduce that a series converges just because its terms go to zero? If the series is decreasing and alternating, then you can!

Proof. Let us say that $a_1 \ge 0$ (the proof is similar if the first term is negative.)

We have

$$\begin{aligned} s_2 &= b_1 - b_2, \\ s_4 &= b_1 - b_2 + b_3 - b_4 = s_2 + (b_3 - b_4) \ge s_2. \end{aligned}$$

More generally,

$$s_{2n} = s_{2n-2} + b_{2n-1} - b_{2n} \ge s_{2n-2},$$

so s_{2n} is an increasing sequence.

Also notice that

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \dots$$

and every term in a pair of parentheses is positive, so $s_{2n} \leq b_1$. Therefore the sequence s_{2n} is increasing and bounded above, and by the Monotone Convergence Theorem, has a limit.

Let us write

$$\lim_{n\to\infty} s_{2n} = S.$$

We then compute

$$\begin{aligned}\lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\ &= S + 0 = S,\end{aligned}$$

so

$$\lim_{n\to\infty} s_{2n+1} = S.$$

From this we see that s_n converges to S, and we are done.

□

Remark 2.2. There is one way in which we can weaken the assumptions of the AST slightly. If it turns out that there exists an N such that for all $n > N$, $b_{n+1} \leq b_n$, then this is good enough to obtain convergence.

Example 2.3. Let us consider the Alternating Harmonic Series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Here $b_n = 1/n$ which is clearly decreasing, and $b_n \to 0$. So by the AST, this converges.

In fact, it converges to $\ln(2)$. (If you are interested to see how to compute this, see here.)

Example 2.4. Consider

$$\sum_{n=1}^{\infty} (-1)^n \frac{6n+7}{8n+9}.$$

This is an alternating series, but notice that

$$\lim_{n\to\infty} b_n = \frac{6}{8} = \frac{3}{4} \neq 0.$$

Since the limit of the b_n is not zero, then the series does not converge!

Example 2.5. What about

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2n^3+3}$$

Now, the limit of these terms is zero. But are the terms decreasing?

Let us consider the function

$$f(x) = \frac{x^2}{2x^3+3},$$

which is continuous on the whole real line. Now we compute

$$f'(x) = \frac{2x(2x^3+3) - x^2(6x)}{(2x^3+3)^2} = \frac{2x(3-x^3)}{(2x^3+3)^2}.$$

Now, notice that we do **not** have $f'(x) < 0$ for all x, and therefore x is not a decreasing function. However, if $x > 0$ and $3 - x^3 < 0$, then $f'(x) < 0$, so for $x > \sqrt[3]{3}$, we have $f'(x) < 0$. Therefore, for $n \geq 3$, the sequence a_n is decreasing. Also we can see that $a_n \to 0$, so by Remark 2.2 this series converges.

3 Something to be careful about!

As mentioned above, we need that the sequence b_n is decreasing, or at least **eventually** decreasing. Is that really necessary?

As it turns out, this is in fact necessary. Consider the (admittedly weird) example where

$$b_{2n} = \frac{1}{n}, \quad b_{2n+1} = \frac{1}{2^n},$$

and let $a_n = (-1)^n b_n$. Then this is an alternating sequence, and definitely

$$\lim_{n\to\infty} b_n = 0,$$

but just as clearly, this is not a decreasing function. For example, the first few terms of b_n are

$$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{8}, \frac{1}{5}, \frac{1}{16}, \frac{1}{7}, \dots$$

and we can see from this pattern that it is not decreasing. So the sum we are considering is

$$-\frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{5} - \frac{1}{16} + \frac{1}{7} + \dots$$

Let us compute some of the partial sums of this series:

$$s_2 = 0, \quad s_4 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \quad s_6 = s_4 + \frac{1}{5} - \frac{1}{8} = \frac{1}{12} + \frac{3}{40} = \frac{19}{120}.$$

More generally, let's just look at the even partial sums: we obtain

$$s_{2n} = s_{2(n-1)} + \frac{1}{n} - \frac{1}{2^{n-1}}.$$

It's not too hard to check that for $n \geq 5$,

$$\frac{1}{n} - \frac{1}{2^{n-1}} > \frac{1}{2n},$$

and therefore

$$s_{2n} > s_{2(n-1)} + \frac{1}{2n}$$

If we start this at $k = 5$, then this means that

$$s_{2n} > s_{10} + \sum_{k=5}^{n} \frac{1}{2k}.$$

Now take the limit $n \to \infty$. The right-hand side of this inequality goes to ∞, since the harmonic series diverges, and so then $s_{2n} \to \infty$. Therefore the limit of the partial sums cannot converge, and therefore the sum cannot converge.

tl;dr: The decreasing condition is actually quite important.

4 Estimating alternating sums

Theorem 4.1. Let us consider the convergent alternating sum

$$\sum_{n=1}^{\infty} a_n,$$

and call the sum S. Define the remainder $R_n = S - s_n$ as in last lecture, so

$$R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k.$$

Then

$$|R_n| \leq b_{n+1} = |a_{n+1}|.$$

Example 4.2. Let us consider the alternating series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

From the Alternating Series Test, this converges: notice that $1/n!$ is both decreasing and converges to zero.

Let's imagine that we want to compute this sum to within 2 decimal places, i.e. to have a remainder less than $1/100$.

Basically, we need to compute which term is less than $1/100$ and we can stop there. But since $5! > 100$ we can just go to five terms:

$$s_5 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} = \frac{44}{120} = 0.366666\ldots \tag{1}$$

So how close is this? In fact,

$$S = \frac{1}{e}.$$

(Why is this??) Then we can compute

$$S = 0.367879\cdots$$

and we got the first two digits correct!