Math 231E, Lecture 8.
Continuity, Differentiation, Chain Rule

1 Continuity

We say $f(x)$ **is continuous at** a if

$$\lim_{x \to a} f(x) = f(a).$$

- Every polynomial is continuous
- Every rational function $f(x) = P(x)/Q(x)$, where P, Q are polynomials, is continuous at all x such that $Q(x) \neq 0$
- The composition of continuous functions is continuous.

We have defined continuity at a point above, and we can now define it on an interval. There are three types of interval we consider:

- We say that $f(x)$ is continuous **everywhere** if it is continuous for every x with $-\infty < x < \infty$;
- We say that $f(x)$ is continuous **on the open interval** (a, b) if it is continuous for all x, $a < x < b$;
- We say that $f(x)$ is continuous **on the closed interval** $[a, b]$ if it is:
  - continuous for all x, $a < x < b$;
  - $\lim_{x \to a+} f(x) = f(a)$;
  - $\lim_{x \to b-} f(x) = f(b)$;

We can now define the very powerful

Theorem 1.1 (Intermediate Value Theorem)**.** Assume $f(x)$ is continuous on $[a, b]$ and $f(a) \neq f(b)$. Assume z is between $f(a)$ and $f(b)$. Then there is a c with $a < c < b$ and $f(c) = z$.

Example 1.2.

Question: Is there a real root of $x^3 - x^2 + 2x - 4 = 0$? Can we approximate it?

Let us take $f(x) = x^3 - x^2 + 2x - 4 = 0$. Then

$$f(1) = -2 < 0, \quad f(2) = 4 > 0,$$

so $f(x)$ has a root (and, in fact, it must be between 1 and 2!).

But, we can do better. Since we know that there is a root between 1 and 2, let us plug in the average, and we have

$$f(1.5) = 0.125 > 0,$$

so the root is between 1 and 1.5. So plug in that midpoint, and we obtain

$$f(1.25) = -1.10938 < 0$$

so the root is between 1.25 and 1.5. We can continue this for as long as we like, and get as close as we want to the root. This type of numerical method is called a em bisection method and is the basic method used to compute roots numerically. Notice that nowhere in this technique are we required to solve the equation! We only need to compute $f(x)$ multiple times. For those scoring at home, the actual root is

$$x = \frac{1}{3}\left(\sqrt[3]{46+3\sqrt{249}} - \frac{5}{\sqrt[3]{46+3\sqrt{249}}} + 1\right) \approx 1.47797.$$

Example 1.3.

Question: Does $\ln 2$ exist?

We define the natural logarithm as the inverse of the natural exponential, i.e. we define $\ln x$ to be the number that satisfies the equation

$$e^{\ln(x)} = x.$$

So, do we know that there is an x that makes $e^x = 2$?

We know $e^0 = 1$ and $e^1 = e$, and $e \approx 2.7182818 > 2$. Since e^x is continuous, there must be a z with $0 < z < 1$ such that $e^z = 2$, so the answer is **yes**.

1.1 Composition

Continuous functions act nicely under composition, i.e. we have the following:

Theorem 1.4. If $f(x)$ is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\boxed{\lim_{x\to a} f(g(x)) = f(b),}$$

or, said another way,

$$\boxed{\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).}$$

Example 1.5.

Question: How do we compute the limit below?

$$\lim_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^n$$

Let us define $f(n) = (1+\frac{\alpha}{n})^n$. Then

$$\ln(f(n)) = \ln\left(\left(1+\frac{\alpha}{n}\right)^n\right) = n\ln\left(1+\frac{\alpha}{n}\right).$$

If we write

$$\lim_{n\to\infty} n\ln\left(1+\frac{\alpha}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{\alpha}{n}\right)}{1/n} = \lim_{x\to 0}\frac{\ln(1+\alpha x)}{x},$$

and we can use l'Hôpital's Rule to get

$$\lim_{n\to\infty}\ln(f(n)) = \frac{\frac{\alpha}{1+\alpha x}}{1} = \alpha.$$

Then

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} e^{\ln(f(n))} = e^{\lim_{n\to\infty}\ln(f(n))} = e^{\alpha}.$$

2 Differentiation

Definition 2.1. Let $f(x)$ be a function, and we define the derivative of f at the point x, denoted either $f'(x)$ or $df/dx(x)$, as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

There are many differentiation rules: the power rule

$$\frac{d}{dx} x^n = nx^{n-1},$$

the product rule

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x),$$

and the quotient rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x)^2)}.$$

The derivative is also linear:

$$\frac{d}{dx}(af(x) + bg(x)) = af'(x) + bg'(x),$$

etc. Combining this with the power rule allows us to differentiate any polynomial.

Review differentiation rules from book.

Most of these are relatively easy, but we will stress the tricky one, the Chain Rule!

3 Chain Rule

The chain rule is the differentiation rule that allows us to differentiate the composition of two functions:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

The way to think of this is "the derivative of the outside times the derivative of the inside".

Basically this allows us to differentiate almost anything. For example, consider the function

$$f(x) = \cos^6(x)\sin^7(x).$$

We use the product rule first, then the chain rule and power rule on each half.

For example, consider the function $\cos^6(x)$. We can write this as $g(h(x))$, where $h(x) = \cos(x)$ and $g(x) = x^6$. So then

$$\frac{d}{dx}(g(h(x))) = g'(h(x)) \cdot h(x) = 6\cos^5(x) \cdot (-\sin(x)).$$

Similarly,

$$\frac{d}{dx}\sin^7(x) = 7\sin^6(x)\cos(x).$$

Therefore we have

$$\begin{aligned}\frac{d}{dx}(\cos^6(x)\sin^7(x)) &= \frac{d}{dx}(\cos^6(x)) \cdot \sin^7(x) + \cos^6(x) \cdot \frac{d}{dx}(\sin^7(x)) \\ &= 6\cos^5(x) \cdot (-\sin(x)) \cdot \sin^7(x) + \cos^6(x) \cdot (7\sin^6(x)\cos(x)) \\ &= -6\cos^5(x)\sin^8(x) + 7\cos^7(x)\sin^6(x).\end{aligned}$$