Math 231E, Lecture 18.
Trigonometric Substitution

1 Quadratic Forms and their Square Roots

One of the main reasons these types of trig integrals are useful is that they allow us to deal with square roots of quadratic forms.

Example 1.

$$\int x^3\sqrt{1-x^2}\,dx.$$

If we make the substitution $x = \sin\theta$, $dx = \cos\theta\,d\theta$, then we have

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = |\cos(\theta)|.$$

As long as we take $\theta \in (-\pi/2, \pi/2)$, then $\cos(\theta) > 0$ and so $|\cos(\theta)| = \cos(\theta)$. We then have

$$\begin{aligned}\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3(\theta)\cdot\cos(\theta)\cdot\cos(\theta)\,d\theta\\ &= \int \sin^3(\theta)\cos^2(\theta)\,d\theta.\end{aligned}$$

Since we have an odd number of $\sin$ terms above, we "peel off" all but one of them and proceed as above:

$$\begin{aligned}\int \sin^3(\theta)\cos^2(\theta)\,d\theta &= \int \sin\theta\sin^2\theta\cos^2\theta\,d\theta\\ &= \int \sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta\\ &= \int(-du)(1-u^2)u^2 = \int u^4 - u^2\,du\\ &= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{(\cos\theta)^5}{5} - \frac{(\cos\theta)^3}{3} + C.\end{aligned}$$

Now we have to substitute back to get a function of x. Using the same triangle from Lecture 9, we have

$$\cos(\sin^{-1}(x)) = \sqrt{1-x^2},$$

so finally our answer is

$$\frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C.$$

2 Area of an Ellipse

We can, finally, use all of these integrals to compute the area of an ellipse.

We know that the formula for an ellipse whose two axes are a and b is given by the formula

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This means that the upper edge of the ellipse is given by the curve

$$y = \sqrt{b^2 - \frac{b^2}{a^2}x^2} = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Clearly, we integrate from $x = -a$ to $x = a$, so the top half of the area is

$$\frac{A}{2} = \int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\, dx.$$

We should use the substitution $x = a\sin\theta$ and $dx = a\cos\theta\, d\theta$, to obtain

$$\begin{aligned}
\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\, dx &= \int_{-\pi/2}^{\pi/2} \frac{b}{a}\sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} ab\sqrt{1 - \sin^2\theta}\cos\theta\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} ab\cos^2\theta\, d\theta \\
&= ab\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1 + \cos(2\theta))\, d\theta = \frac{ab\pi}{2}.
\end{aligned}$$

Thus

$$\frac{A}{2} = \frac{ab\pi}{2},$$

so $A = ab\pi$.

3 Saving time (and pain) for definite integrals

Let us consider the integral

$$\int_0^1 x^2\sqrt{1 - x^2}\, dx.$$

If we make the change $x = \sin\theta$, $dx = \cos\theta\, d\theta$, then we obtain

$$\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta.$$

Note that we made the change of integration domain already. This will save us a significant amount of labor later. We then have

$$\begin{aligned}
\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta &= \int_0^{\pi/2} \frac{1}{4}(1 - \cos 2\theta)(1 + \cos 2\theta)\, d\theta \\
&= \int_0^{\pi/2} \frac{1}{4}(1 - \cos^2 2\theta)\, d\theta = \int_0^{\pi/2} \frac{1}{4}(1 - \frac{1}{2}(1 + \cos 4\theta))\, d\theta \\
&= \int_0^{\pi/2} \frac{3}{8} - \frac{1}{8}\cos 4\theta\, d\theta = \frac{3\pi}{16}.
\end{aligned}$$

The reason this is much easier than doing the whole antiderivative is this: if we had computed the antiderivative in θ, we would have had a term of the form $\sin 4\theta$, which we then have to expand as

$$\sin 4\theta = 2\sin 2\theta\cos 2\theta = 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta),$$

etc, and then do the triangle, etc. This saves all that work!