§Lecture 18 1f8web

Last time: diagonalizability, Markov matrices

§Matrix Powers

Recall $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $A^2 = AA = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1^2 & 1^2 \\ 0 & 1^2 \end{pmatrix}$

Example: $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

$$D^2 = DD = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3^2 & 0 \\ 0 & (-1)^2 \end{pmatrix}$$

$$D^3 = D^2 D = \begin{pmatrix} 3^2 & 0 \\ 0 & (-1)^2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3^3 & 0 \\ 0 & (-1)^3 \end{pmatrix}$$

$$\vdots$$

$$D^k = \begin{pmatrix} 3^k & 0 \\ 0 & (-1)^k \end{pmatrix}$$

Moral: matrix power of diagonal matrix is easy to compute

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}_{n\times n} \qquad D^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}_{n\times n}$$

Example: diagonalizable matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = PDP^{-1}$ $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$A^2 = AA = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

$$A^3 = A^2 A = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1}$$

$$\vdots$$

$$A^k = PD^kP^{-1} = P\begin{pmatrix} (3)^k & 0 \\ 0 & (-1)^k \end{pmatrix}P^{-1}$$

Moral: matrix power of a diagonalizable matrix is straightforward once eigenvalues & an eigenbasis are known.

If $A = PDP^{-1}$ P invertible & $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ diagonal then

$$A^k = PD^kP^{-1} = P\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}P^{-1}$$

§ Matrix Exponential

Recall Taylor series: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{k\geq 0} \frac{1}{k!} x^k$ for any $x \in \mathbb{R}$.

let $A_{n\times n}$ square matrix,

Define matrix exponential of A:

$$e^A = \exp(A) := I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots = \sum_{k\geq 0} \frac{1}{k!} A^k \quad \text{where } A^0 := I$$

Example: $A = O_{n\times n}$ zero matrix

$$\exp(O) = I + O + \frac{1}{2}O^2 + \frac{1}{6}O^3 + \cdots = I_n$$

Example: diagonal matrix $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

$$\exp(D) = \sum_{k\geq 0} \frac{1}{k!} D^k = \sum_{k\geq 0} \frac{1}{k!} \begin{pmatrix} 3^k & 0 \\ 0 & (-1)^k \end{pmatrix} = \begin{pmatrix} \sum_{k\geq 0} \frac{1}{k!} 3^k & 0 \\ 0 & \sum_{k\geq 0} \frac{1}{k!} (-1)^k \end{pmatrix}$$

$$= \begin{pmatrix} e^3 & 0 \\ 0 & e^{-1} \end{pmatrix}$$

Moral: matrix exponential of a diagonal matrix is easy

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}_{n\times n} \qquad \exp(D) = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}_{n\times n}$$

Example: diagonalizable matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = PDP^{-1}$, invertible $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, diagonal $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\exp(A) = \sum_{k\geq 0} \frac{1}{k!} A^k = \sum_{k\geq 0} \frac{1}{k!} PD^kP^{-1} = P\left(\sum_{k\geq 0} \frac{1}{k!} D^k\right)P^{-1} = Pe^DP^{-1}$$

$$= P \begin{pmatrix} e^3 & 0 \\ 0 & e^{-1} \end{pmatrix} P^{-1}$$

Moral: matrix exponential of diagonalizable matrix is straightforward one eigenvalues & an eigenbasis are known.

$$A = PDP^{-1} \quad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \Rightarrow \exp(A) = Pe^DP^{-1} = P\begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}P^{-1}$$

Properties: A, B square $\overset{n\times n}{\vee}$ matrices

(1) $\exp(A)$ well-defined

(2) if $AB = BA$ then $e^A e^B = e^{A+B} = e^B e^A$

we say A, B commute

(3) $\exp(A)^T = \exp(A^T)$

Example: $A_{n\times n}$ square matrix

$$A(-A) = -A^2 = (-A)A \overset{(2)}{\Rightarrow} e^A e^{-A} = \underbrace{e^{A+(-A)}}_{e^0 = I} = e^{-A} e^A$$

$$\Rightarrow e^A e^{-A} = I = e^{-A} e^A \Rightarrow e^A \text{ is an invertible matrix with inverse } e^{-A}$$

Moral: $\exp(A)$ is always an invertible matrix

Example: $A_{n\times n}$ skew-symmetric, $A^T = -A$

$$AA^T = A(-A) = (-A)A = A^T A \overset{(2)}{\Rightarrow} e^A e^{A^T} = \underbrace{e^{A+A^T}}_{e^0 = I} = e^{A^T} e^A$$

$e^{A^T} = \exp(A^T)$

$= \exp(A)^T$

$$\exp(A)\exp(A)^T = I = \exp(A)^T \exp(A) \Rightarrow \exp(A) \text{ is an orthogonal matrix}$$

Moral: $A_{n\times n}$ skew-symmetric $\Rightarrow \exp(A)$ is orthogonal matrix

Fact: $A_{n\times n}$ square matrix, then $\det(e^A) = e^{tr(A)}$

Example: A diagonalizable matrix $A = PDP^{-1}$ $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}_{n\times n}$ λ_i: eigenvalues of A

$$\det(e^A) = \det\left(Pe^D P^{-1}\right) = \cancel{\det(P)} \det(e^D) \cancel{\det(P^{-1})} = \det\begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}$$

$$= e^{\lambda_1} \cdots e^{\lambda_n} = e^{\Sigma \lambda_i} = e^{tr(A)}$$