Math 231E, Lecture 23.
Sequences

1 Definitions and examples

Definition 1.1. A **sequence** is a (typically infinite) list of numbers.

The best description is having a formula for the nth term in the sequence:

$$a_n = n, \text{ or } \quad 1, 2, 3, 4, 5, \dots,$$
$$a_n = \frac{1}{n}, \text{ or } \quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$$
$$a_n = \left(1 - \frac{1}{n}\right)^n, \text{ or } \quad 0, \frac{1}{4}, \frac{8}{27}, \frac{81}{256}, \frac{1024}{3125}, \dots$$

Unless stated otherwise, we typically start the sequence numbering at $n = 1$, but sometimes we start at $n = 0$.

We can also describe this by a "starting sequence":

$$1, 2, 3, 4, \dots, \quad 1, 3, 5, 7, \dots, \quad 1, 2, 4, 8, 16, \dots$$

But be careful! Sometimes patterns can be deceiving.

Example 1.2. Let r_n be the maximum number of regions that we can obtain when we put n points on the edge of a circle and draw chords between them. Then it turns out that this number is

$$1, 2, 4, 8, 16, 31, 57, 99, 163, \dots$$

The first five entries of this sequence are pretty misleading!

Example 1.3 (Fibonacci sequence)**.** An old problem. We are given a pair of newborn rabbits, which will give birth to another pair of rabbits after one month. Let us also assume that each new pair of rabbits can give birth in their second month, and give birth to one new breeding pair each month thereafter, and none of our rabbits die. How many rabbits do we have in one year?

— in *Liber Abaci*, 1202, by Leonardo Pisano Bigollo (Fibonacci)

Let a_n be the number of rabbits after n months, so $a_1 = 1, a_2 = 1$. The number of rabbits we have in month n will be the number we had last month, plus the number born this month. The number last month is a_{n-1}, and the number born now is the number of rabbits at least two months old, which is a_{n-2}. Therefore

$$a_n = a_{n-1} + a_{n-2}.$$

So we have

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$$

and the answer is $a_{12} = 144$ rabbits in one year! Going forward, how many after two years?

$$233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368.$$

We have $a_{24} = 46368$. A lot of rabbits!

(We will study this sequence quite a bit in a future worksheet.)

2 Limits

We are typically interesting in computing the limit of a sequence:

Definition 2.1. We say that the **limit of the sequence** a_n **is** L, denoted

$$\lim_{n\to\infty} a_n = L,$$

if for any $\epsilon > 0$, there is $N > 0$ such that for all $n > N$, we have

$$|a_n - L| < \epsilon.$$

If this happens, we say that the sequence **converges**.

Remark 2.2. There are two ways for a sequence to diverge: the limit could be ∞, or the limit could not exist. We will sometimes say it "diverges to infinity" in the former case for clarity.

Theorem 2.3. Let $a_n = f(n)$ for some function f. If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$.

Example 2.4. We want to compute the limit of the sequence $a_n = n/(n^2 + 4n + 1)$. We have

$$\lim_{x\to\infty} \frac{x}{x^2 + 4x + 1} = 0,$$

so

$$\lim_{n\to\infty} \frac{n}{n^2 + 4n + 1} = 0.$$

So from this, all of the old Limit Laws work as well. For example, if $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n,$$

etc. We also have the Squeeze Theorem:

Theorem 2.5. If $a_n \le b_n \le c_n$ for all $n > N$, and $\lim_{n\to\infty} a_n = L$, and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

A very useful theorem that comes from this is:

Theorem 2.6. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof. Assume that $\lim_{n\to\infty} |a_n| = 0$. Notice that $-|a_n| \le a_n \le |a_n|$. The left sequence limits to zero, and the right sequences limits to zero. Therefore the middle one does as well. □

Remark 2.7. This doesn't work with a number different than zero! For example, if $a_n = (-1)^n$, then $\lim_{n\to\infty} |a_n| = 1$ but $\lim_{n\to\infty} a_n$ doesn't exist.

Theorem 2.8. If $f(x)$ is continuous at L, and $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

More compactly, if f is continuous,

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right).$$

Example 2.9. Let us compute

$$\lim_{n\to\infty} \frac{n!}{n^n}.$$

First note that $n!/n^n \geq 0$. Now, let us write

$$\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n}{n \cdot n \cdot n \ldots n} = \frac{1}{n} \cdot \frac{2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n}{n \cdot n \ldots n} \leq \frac{1}{n}.$$

Then we have

$$0 \leq \frac{n!}{n^n} \leq \frac{1}{n},$$

and by Squeeze this limit is zero!

Example 2.10. Now consider $a_n = r^n$, where r is a real number. This is called a **geometric sequence**. What is $\lim_{n\to\infty} a_n$?

We do all of the cases. If $r = 1$, then $r^n = 1^n = 1$, so the limit is 1. If $r = 0$, then $a_n = 0^n = 0$ and the limit is zero. Now, let $0 < r < 1$. Then $\lim_{n\to\infty} r^n = 0$, which we can see a few ways. Let $f(x) = r^x$ with $r \in (0,1)$. Then we have $f(x) = e^{x \ln r}$ with $\ln r < 0$, and therefore $\lim_{x\to\infty} f(x) = 0$. Similarly, if $r > 1$ then we have $f(x) = e^{x \ln r}$ with $\ln r > 0$ and thus $\lim_{x\to\infty} f(x) = \infty$.

Now, let us assume that $-1 < r < 0$. Then $|r^n| \to 0$, so $r^n \to 0$. If $r = -1$, however, a_n does not converge. Thus we have

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & -1 < r \leq 1, \\ 1, & r = 1, \\ \text{diverges}, & r > 1 \text{ or } r \leq -1. \end{cases}$$

3 Monotone Convergence Theorem

Definition 3.1. A sequence is **increasing** if $a_{n+1} \geq a_n$ for all n.

A sequence is **decreasing** if $a_{n+1} \leq a_n$ for all n.

Definition 3.2. A sequence is **bounded below** if there is a C such that $a_n \geq C$ for all n.

A sequence is **bounded above** if there is a C such that $a_n \leq C$ for all n.

Theorem 3.3. If a sequence is increasing and bounded above, it is convergent.

If a sequence is decreasing and bounded below, it is convergent.

Example 3.4. Let us assume that the sequence a_n satisfies the recursion relation

$$a_{n+1} = \frac{1}{2}(a_n + 8), \quad a_1 = 0.$$

Compute $\lim_{n\to\infty} a_n$ if it exists.

We compute the first few terms:

$$0, 4, 6, 7, 7.5, 7.75, 7, 875 \cdots \to ?$$

It looks like it's heading toward 8.

First we need to show that the limit exists. But notice that if $a_n < 8$, then

$$a_{n+1} = \frac{1}{2}(a_n + 8),$$

is larger than a_n, since it is averaging a_n and 8. More specifically, we see that

$$\frac{1}{2}(a_n + 8) > \frac{1}{2}(a_n + a + n) = a_n.$$

Moreover, notice that if $a_n < 8$, then

$$a_{n+1} = \frac{1}{2}(a_n + 8) < \frac{1}{2}(8 + 8) = 8,$$

so $a_{n+1} < 8$.

Therefore the sequence a_n is increasing and bounded above. By the Convergence Theorem, this means that $\lim_{n\to\infty} a_n$ exists. Let us call this limit L, then we have

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 8) = \frac{1}{2}(L + 8),$$

so

$$2L = L + 8,$$

or $L = 8$.