Math 231E, Lecture 17.
Trigonometric Integrals

In this lecture we will learn how to integrate expressions involving trigonometric functions, or integrals that can be converted into same.

1 Review of some trig derivatives

We have from before that

$$
\begin{aligned}
&\frac{d}{dx}\sin x = \cos x,\\
&\frac{d}{dx}\cos x = -\sin x,\\
&\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \sec^2 x,\\
&\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \csc^2 x,\\
&\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = (-1)\cos^{-2}(x)(-\sin x) = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \sec x\tan x,\\
&\frac{d}{dx}\csc x = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = (-1)\sin^{-2}(x)(\cos x) = -\frac{\cos x}{\sin x}\cdot\frac{1}{\sin x} = -\csc x\cot x.
\end{aligned}
$$

2 Antiderivatives of the Big Six

Ok, we see directly that

$$\int \sin x\,dx = -\cos x + C, \qquad \int \cos x\,dx = \sin x + C.$$

Now let us consider

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$$

If we make the substitution $u = \cos x$ and $du = -\sin x\,dx$, this gives

$$\int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C.$$

Similarly,

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx.$$

If we make the substitution $u = \sin x$ and $du = \cos x\,dx$, this gives

$$\int \frac{du}{u} = \ln|u| + C = \ln|\sin(x)| + C.$$

Again, notice the difference between the two is the flipping "co-" on or off and changing signs.

Finally, $\int \sec x \, dx$ is probably the trickiest. But if we notice that $(\sec x)'$ is $\sec x$ times $\tan x$, and $(\tan x)'$ is $\sec x$ times $\sec x$, then we might guess that we should do the following:

$$\begin{aligned} \int \sec x \, dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx, \end{aligned}$$

and with the substitution $u = \sec x + \tan x$, this gives

$$\int \frac{1}{u} \, du = \ln |u| + C = \ln |\sec x + \tan x| + C.$$

Similarly, we obtain

$$\int \csc x \, dx = -\ln |\csc x + \cot x| + C.$$

We will consider integrals of the form $\int \sin^k(x) \cos^l(x) dx$ as well as integrals of the form $\int \sec^k(x) \tan^l(x) dx$. All of these can be done in elementary terms, although some of them are a bit tedious.

We will begin with integrals involving sines and cosines

3 Tricks for odd powers

Example 3.1. Let us consider the integral

$$\int \sin^5(x) \cos^2(x) \, dx.$$

Recalling the identity

$$\sin^2(x) + \cos^2(x) = 1,$$

we can write $\sin^2(x) = 1 - \cos^2(x)$, and

$$\sin^4(x) = (\sin^2(x))^2 = (1 - \cos^2(x))^2 = 1 - 2\cos^2(x) + \cos^4(x).$$

Thus we can rewrite our integral

$$\begin{aligned} \int \sin^5(x) \cos^2(x) \, dx &= \int \sin(x) \sin^4(x) \cos^2(x) \, dx \\ &= \int \sin(x)(1 - 2\cos^2(x) + \cos^4(x)) \cos^2(x) \, dx \\ &= \int \sin(x)(\cos^2(x) - 2\cos^4(x) + \cos^6(x)) \, dx. \end{aligned}$$

Now, since we have a $\sin(x)$ in the integral, we can write

$$u = \cos(x), \quad du = -\sin(x) \, dx,$$

giving

$$-\int (u^2 - 2u^4 + u^6) \, du = -\frac{u^3}{3} + \frac{2}{5}u^5 - \frac{u^7}{7} + C,$$

or

$$\int \sin^5(x) \cos^2(x) \, dx = -\frac{\cos^3(x)}{3} + \frac{2\cos^5(x)}{5} - \frac{\cos^7(x)}{7} + C.$$

We can see in the scenario above that the trick has nothing to do with the power of 5 on the $\sin x$, but it uses the fact that it is odd. So, for example, if we consider any integral of the form

$$\int \sin^{2k+1}(x)\cos^m(x)\,dx,$$

where k, m are integers, then we write

$$\begin{aligned}\int \sin^{2k+1}(x)\cos^m(x)\,dx &= \int \sin(x)\sin^{2k}(x)\cos^m(x)\,dx\\ &= \int \sin(x)(1-\cos^2(x))^k\cos^m(x)\,dx,\end{aligned}$$

and then we make the substitution $u = \cos(x)$, $du = -\sin(x)\,dx$, and we obtain

$$\boxed{\int \sin^{2k+1}(x)\cos^m(x)\,dx = \int -(1-u^2)^k u^m\,du.}$$

This is a polynomial that we can always integrate. Similarly, for any integral of the form $\int \cos^{2k+1}(x)\sin^m(x)\,dx$, we can peel off all but one of the cosines:

$$\begin{aligned}\int \cos^{2k+1}(x)\sin^m(x)\,dx &= \int \cos(x)\cos^2(x)\sin^m(x)\,dx\\ &= \int \cos(x)(1-\sin^{2k}(x))^k\sin^m(x)\,dx,\end{aligned}$$

and then we make the substitution $u = \sin(x)$, $du = \cos(x)\,dx$, and we obtain

$$\boxed{\int \cos^{2k+1}(x)\sin^m(x)\,dx = \int (1-u^2)^k u^m\,du.}$$

4 Tricks for even powers

Of course, this algorithm will only work when one (or both) of the powers on sines and cosines are odd. What about an integral of the form

$$\int \cos^2(x)\sin^4(x)\,dx?$$

Here we use the "half-angle formulas"

$$\sin^2(x) = \frac{1}{2}(1-\cos(2x)), \quad \cos^2(x) = \frac{1}{2}(1+\cos(2x)).$$

For example, we can compute

$$\begin{aligned}\int_0^{2\pi} \sin^2(x)\,dx &= \int_0^{2\pi} \frac{1}{2}(1-\cos(2x))\,dx\\ &= \int_0^{2\pi}\frac{1}{2}\,dx - \int_0^{2\pi}\frac{1}{2}\cos(2x)\,dx\\ &= \frac{x}{2} - \frac{1}{4}\sin(2x)\Big|_0^{2\pi} = \pi.\end{aligned}$$

Now, how do we deal with the original problem? We can write

$$\begin{aligned}\int \cos^2(x)\sin^4(x)\,dx &= \int \frac{1}{2}(1+\cos(2x))\left(\frac{1}{2}(1-\cos(2x))\right)^2\,dx\\ &= \frac{1}{8}\int (1+\cos(2x))(1-\cos(2x))^2\,dx\\ &= \frac{1}{8}\int (1-\cos(2x)-\cos^2(2x)+\cos^3(2x))\,dx\end{aligned}$$

Now, we can deal with these four integrals separately. The constant is easy, as is the simple cosine. The cube we deal with as we described above, i.e.

$$\begin{aligned}\int \cos^3(2x) &= \int \cos^2(2x)\cos(2x)\,dx\\ &= \int (1-\sin^2(2x))\cos(2x)\,dx,\end{aligned}$$

and make the substitution $u=\sin(2x)$, $du = 2\cos(2x)\,dx$, to obtain

$$\int \cos^3(2x) = \frac{1}{2}\int (1-u^2)\,du = \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + C.$$

We use the half-angle again to obtain

$$\int \cos^2(2x)\,dx = \int \frac{1}{2}(1+\cos(4x)) = \frac{x}{2} + \frac{1}{8}\sin(4x) + C.$$

Putting all of this together gives

$$\begin{aligned}\int \cos^2(x)\sin^4(x)\,dx &= \frac{x}{8} - \frac{1}{16}\sin(2x) - \frac{x}{16} - \frac{1}{64}\sin(4x) + \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + C\\ &= \frac{x}{16} + \frac{7}{16}\sin(2x) - \frac{1}{64}\sin(4x) - \frac{1}{6}\sin^3(2x) + C.\end{aligned}$$

Technique for evaluating integrals like $\int \sin^k x \cos^m x dx$

Case	Substitution/Identity
k odd	$u=\cos x$, $\sin^2 x = 1-\cos^2 x = 1-u^2$
m odd	$u=\sin x$, $\cos^2 x = 1-\sin^2 x = 1-u^2$
k,m even	$\cos^2(x) = \frac{1+\cos(2x)}{2}, \sin^2(x) = \frac{1-\cos(2x)}{2}$

5 Angle Addition Formulas

Similar to the half-angle formulas, there are the "angle-addition" formulas, namely:

$$\begin{aligned}\sin A\cos B &= \frac{1}{2}(\sin(A-B)+\sin(A+B)),\\ \sin A\sin B &= \frac{1}{2}(\cos(A-B)-\cos(A+B)),\\ \cos A\cos B &= \frac{1}{2}(\cos(A-B)+\cos(A+B)).\end{aligned}$$

In fact, you can see that these addition formulas actually give the half-angle formulas: Choose $A=B$ in either of the last two lines, and we recover the half-angle formulas.

This formula can be useful in integrals of the following form:

Example 5.1. Consider

$$\int \sin(3x)\cos(4x)\,dx.$$

Using the first of the two formulas gives

$$\sin(3x)\cos(4x) = \frac{1}{2}(\sin(3x-4x)+\sin(3x+4x)) = \frac{1}{2}(\sin(-x)+\sin(7x)) = \frac{1}{2}(\sin(7x)-\sin(x)).$$

Then we have

$$\int \sin(3x)\cos(4x)\,dx = \frac{1}{2}\int(\sin(7x)-\sin(x))\,dx = -\frac{1}{14}\cos(7x)+\frac{1}{2}\cos(x)+C.$$

6 Summary of Trig Identities

We have four useful classes of trig identities:

A1 $\sin^2 x + \cos^2 x = 1$

A2 $\tan^2 x + 1 = \sec^2 x$

B1 $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

B2 $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

B3 $\sin 2x = 2\sin x\cos x$

B4 $\cos 2x = \cos^2 x - \sin^2 x$

C1 $\sin A\cos B = \frac{1}{2}(\sin(A-B)+\sin(A+B))$

C2 $\sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B))$

C3 $\cos A\cos B = \frac{1}{2}(\cos(A-B)+\cos(A+B))$

D1 $\sin(A+B) = \sin A\cos B + \cos A\sin B$

D2 $\cos(A+B) = \cos A\cos B - \sin A\sin B$

The formulas B are called "half-angle" or "double-angle" formulas, those in C are "product" formulas, and D are "angle addition" formulas.

First note that everything follows from the two formulas in D. If we replace the B in D2 with a $-B$, we obtain:

$$\cos(A-B) = \cos A\cos(-B) - \sin A\sin(-B) = \cos A\cos B - \sin A(-\sin B) = \cos A\cos B + \sin A\sin B.$$

We can then add and subtract

$$\cos(A-B)+\cos(A+B) = 2\cos A\cos B,$$
$$\cos(A-B)-\cos(A+B) = 2\sin A\sin B,$$

recovering two out of three of the laws above. Doing similar manipulations on the $\sin$ equation gives the third.

Identities B1–4 correspond to replacing $A = B = x$ in identities C2, C3, D1, D2, respectively. Finally, B1 + B2 gives A1 and dividing A1 by $\cos^2 x$ gives B1. So it all follows from the formulas in D. But where do they come from??

7 Derivation of Angle Addition Formulas

We should not think of the half-angle formulas, or the angle addition formulas, as "something to memorize". This is because we know complex numbers and Euler's Formula! Recall Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

This means that we have

$$e^{i(A+B)} = \cos(A+B) + i\sin(A+B).$$

We also have, from the rules of multiplying exponents,

$$e^{i(A+B)} = e^{iA}e^{iB} = (\cos(A) + i\sin(A))(\cos(B) + i\sin(B)).$$

Multiplying the last two binomials out gives

$$\cos(A+B) + i\sin(A+B) = \cos A\cos B - \sin A\sin B + i(\cos A\sin B + \sin A\cos B).$$

If two complex expressions are equal, then their real parts, and their imaginary parts, must be equal, so we have

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B,\\ \sin(A+B) &= \cos A\sin B + \sin A\cos B.\end{aligned}$$

8 Integrals involving $\tan(x)$ and $\sec(x)$.

Integrals of the form $\int \sec^l(x)\tan^m(x)dx$ can be done similarly. As was the case with the integrals involving since and cosines there are three easy cases that can be done by a u-substitution and one difficult case that requires some tricks. We will begin with the easy cases.

Example 8.1. Compute the integral

$$\int \sec^2(x)\tan^5(x)dx$$

We will use the fact that $(\tan x)' = \sec^2(x)$. If we make the substitution $u = \tan(x); du = \sec^2(x)dx$ the integrand becomes

$$\int \sec^2(x)\tan^5(x)dx = \int u^5 du = \frac{u^6}{6} + C = \frac{\tan^6(x)}{6} + C$$

As before there is nothing special about the powers 2 and 5. This same trick will work whenever we have an even power of $\sec(x)$ so that we can borrow a $\sec^2$ for the $\sec^2(x)dx$. If there are any remaining powers of $\sec(x)$ we can use the identity $\sec^2(x) = 1 + \tan^2(x)$ to express these in terms of $u = \tan(x)$: $\sec^2(x) = \tan^2(x) - 1 = u^2 - 1$ For instance

Example 8.2. Compute the integral

$$\int \sec^4(x)\tan^2(x)dx$$

Again making the substitution $u = \tan(x); du = \sec^2(x)dx$ the integrand becomes

$$\int \sec^2(x)\sec^2(x)\tan^2(x)dx = \int (u^2+1)u^2 du = \frac{u^5}{5} + \frac{u^3}{3} + C = \frac{\tan^5(x)}{5} + \frac{\tan^3(x)}{3} + C$$

There is a similar trick involving the substitution $u = \sec(x); du = \sec(x)\tan(x)dx$. If we have an integral of the form $\int \sec^l(x)\tan^m(x)dx$ then this can be done with the substitution $u = \sec(x); du = \sec(x)\tan(x)dx$ if $\tan(x)$ appears to an odd power.

Example 8.3. Compute the integral $\int \sec^2(x)\tan^3(x)dx$.

We can make the substitution $u = \sec(x); du = \sec(x)\tan(x)dx$ and borrow a factor of $\sec(x)\tan(x)dx$ to form the du. This leaves a factor of $\sec(x)\tan^2(x) = \sec(x)\left(\sec^2(x) - 1\right) = u(u^2 - 1)$so the integral becomes

$$\int \sec^2(x)\tan^3(x)dx = \int \sec(x)\tan^2(x)\sec(x)\tan(x)dx = \int u(u^2-1)du = \frac{u^4}{4} - \frac{u^2}{2} + C = \frac{\sec^4(x)}{4} - \frac{\sec^2(x)}{2} + C$$

The difficult case is when $\sec(x)$ appears to an odd power and $\tan(x)$ appears to an even power. In this case one can usually evaluate the integral by integration by parts and by knowing the following integral.

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$$

I would suggest memorizing this integral.

Example 8.4. Compute the integral $\int \sec(x)\tan^2(x)dx$

This can be done by parts. Letting $u = \tan(x); dv = \sec(x)\tan(x)dx$, which gives $v = \sec(x); du = \sec^2(x)dx$. Integration by parts gives

$$\int \sec(x)\tan^2(x)dx = \sec(x)\tan(x) - \int \sec^3(x)dx = \sec(x)\tan(x) - \int \sec(x)\left(1 + \tan^2(x)\right)dx$$

Note that the original integral appears on the right. Moving it over gives

$$2\int \sec(x)\tan^2(x)dx = \sec(x)\tan(x) - \int \sec(x)dx = \sec(x)\tan(x) - \ln|\sec(x) + \tan(x)| + C$$

or $\int \sec(x)\tan^2(x)dx = \frac{1}{2}\sec(x)\tan(x) - \frac{1}{2}\ln|\sec(x) + \tan(x)| + C$