Final exam cheat sheet

May 1, 2024

A. If $\mathcal{A} = (\mathbf{a}_1, ..., \mathbf{a}_n)$ and $\mathcal{B} = (\mathbf{b}_1, ..., \mathbf{b}_n)$ are two bases of $\mathbb{R}^n$ then $I_{\mathcal{AB}}$ is the $n \times n$ matrix such that $\mathbf{v}_{\mathcal{A}} = I_{\mathcal{AB}}\mathbf{v}_{\mathcal{B}}$ for every $\mathbf{v} \in \mathbb{R}^n$. In other words, the linear transformation associated to $I_{\mathcal{AB}}$ "turns the $\mathcal{B}$-coordinates of $\mathbf{v}$ into the $\mathcal{A}$-coordinates of $\mathbf{v}$. Moreover $I_{\mathcal{BA}} = I_{\mathcal{AB}}^{-1}$.

B. If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are three (ordered) bases of $\mathbb{R}^n$ then $I_{\mathcal{AB}} = I_{\mathcal{AC}} I_{\mathcal{CB}}$.

C. if $\mathcal{E}$ is the standard basis of $\mathbb{R}^n$ and $\mathcal{B} = (\mathbf{b}_1, ..., \mathbf{b}_n)$ is a basis of $\mathbb{R}^n$ then $I_{\mathcal{EB}}$ is the matrix whose ith column is $\mathbf{b}_i$.

D. Let V and W be vector spaces and $T : V \to W$ a linear transformation. If $\mathcal{B} = (\mathbf{b}_1, ..., \mathbf{b}_n)$ is a basis of V and $\mathcal{C} = (\mathbf{c}_1, ..., \mathbf{c}_m)$ is a basis of W then the coordinate matrix $T_{\mathcal{CB}}$ is the $m \times n$ matrix whose ith column is $T(\mathbf{b}_i)_{\mathcal{C}}$, the coordinate vector of $T(\mathbf{b}_i)$ in the $\mathcal{C}$ basis.

E. Let A be an $m \times n$ matrix. Then Col(A)=Nul(A^T)$^{\perp}$.

F. An orthogonal matrix is a square matrix with orthonormal columns.

G. Gram-Schmidt process: suppose $\mathbf{v}_1, ..., \mathbf{v}_k$ is a basis of some subspace W in $\mathbb{R}^n$. The Gram Schmidt process produces an orthogonal basis $\mathbf{b}_1, ..., \mathbf{b}_k$ of W such that Span($\{\mathbf{b}_1, .., \mathbf{b}_j\}$) is equal to Span($\{\mathbf{v}_1, .., \mathbf{v}_j\}$) for every $j = 1, 2..., k$. Moreover, for each j there is a formula for $\mathbf{b}_j$: let $W_j = \text{Span}(\{\mathbf{v}_1, .., \mathbf{v}_j\})$. Then $\mathbf{b}_1 = \mathbf{v}_1$ and for $j > 1$

$$\mathbf{b}_j = \mathbf{v}_j - \text{proj}_{W_{j-1}} \mathbf{v}_j$$

H. Recall the quadratic formula says: the solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

I. Let A be an $n \times n$ matrix. Let C_{ij} be the determinant of the $(n-1) \times (n-1)$ matrix gotten by removing from A the ith row and jth column. The cofactor method for calculating the determinant says that: chose any r from 1 to n. The determinant of a matrix A is equal to

$$\sum_{j=1}^{n} (-1)^{r+j} C_{rj}$$

J. Let A be an $m \times n$ matrix of rank r. Recall that the singular value decomposition of A is a factorization

$$A = U \Sigma V^T$$

where U is an orthogonal matrix whose first r columns span Col(A), V is an orthogonal matrix whose columns are eigenvectors of $A^T A$, and Σ is an $m \times n$ matrix whose only nonzero entries are on the diagonal, whose main diagonal entries Σ_{ii} are in non-increasing order ($\Sigma_{11} \geq \Sigma_{22} \geq \cdots$), and whose nonzero entries are the *singular values* of A, i.e. the positive square roots of the nonzero eigenvalues of $A^T A$.

K. If $\mathbf{b}_1, ..., \mathbf{b}_k$ is an orthogonal basis of a subspace W in $\mathbb{R}^n$, and $\mathbf{v}$ is any vector in $\mathbb{R}^n$, then

$$\text{proj}_W(\mathbf{v}) = \sum_{j=1}^{k} \frac{\mathbf{v} \cdot \mathbf{b}_j}{\mathbf{b}_j \cdot \mathbf{b}_j} \mathbf{b}_j$$

L. If A is an $m \times r$ matrix with rank r, and $\mathbf{b}$ is any vector in $\mathbb{R}^m$, then

$$\text{proj}_{\text{Col}(A)}(\mathbf{b}) = A(A^T A)^{-1} A^T \mathbf{b}$$