Math 415. Exam 1. September 28, 2017

Full Name: ____________________

Net ID: ________________________

Discussion Section: ___________

- There are 18 problems worth 5 points each.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 70 minute exam.
- Do not turn this page until instructed to.
- Fill in the answers on the scantron form provided. Also circle your answers on the exam itself.
- Hand in both the exam and the scantron.
- On the scantron make sure you bubble in **your name, your UIN and your NetID**.
- There are several different versions of this exam.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- Good luck!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**!
On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) Consider the following multiplication of two 3×3-matrices, where the question marks represent unknown coefficients:

$$\begin{bmatrix} 9 & 8 & 6 \\ ? & 9 & ? \\ ? & 3 & 1 \end{bmatrix} \begin{bmatrix} 8 & 1 & 4 \\ 2 & ? & ? \\ 3 & 3 & ? \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}.$$

With the information given, only one coefficient in the matrix on the right hand side can be calculated. Which one is it?

(A) ★ c_{11}

(B) c_{31}

(C) c_{21}

(D) c_{12}

(E) None of the other answers.

Solution. Only the first row of the first matrix and the first column of the second matrix are fully determined. Thus we can only compute c_{11}, which is $9 \cdot 8 + 8 \cdot 2 + 6 \cdot 3$.

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2. (5 points) Let $\mathbf{a}, \mathbf{b}$ be non-zero vectors in $\mathbb{R}^3$, where $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$. Which of the following is a description of the set $\operatorname{span}(\mathbf{a}, \mathbf{b})$?

(A) It is $\mathbb{R}^2$.

(B) It is a line in $\mathbb{R}^3$ through the origin.

(C) It is the union of two lines in $\mathbb{R}^3$.

(D) ★ It is a plane in $\mathbb{R}^3$ through the origin.

(E) None of the other answers.

Solution. If the two vectors are not scalar multiples of each other, $\mathbf{b}$ is not in the line spanned by $\mathbf{a}$ in $\mathbb{R}^3$. So the two vectors span a subspace which is a plane (2-dimensional) and the origin is a point on that plane, because $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a scalar multiple of either vector. Note that while $\operatorname{span}(\mathbf{a}, \mathbf{b})$ is 2-dimensional, it is not $\mathbb{R}^2$. Because vectors in $\mathbb{R}^2$ have just two entries, but vectors in $\operatorname{span}(\mathbf{a}, \mathbf{b})$ have three entries.

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3. (5 points) Let $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{b}$ be three vectors in $\mathbb{R}^3$ and suppose $\mathbf{b}$ is in $\text{span}(\mathbf{a}_1, \mathbf{a}_2)$. Consider the matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{bmatrix}$, that is the matrix whose first column is $\mathbf{a}_1$, whose second column is $\mathbf{a}_2$ and whose third column is $\mathbf{b}$. Which of the following statements is true?

(A) A is row equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(B) None of the other answers.

(C) ★ A has at most 2 pivots.

(D) A is invertible.

Solution. In order for $\mathbf{b}$ to be in $\text{span}(\mathbf{a}_1, \mathbf{a}_2)$, the linear system with the augmented matrix $\left[\begin{array}{cc|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{array}\right]$ has to be consistent. Thus $\left[\begin{array}{cc|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{array}\right]$ can not have 3 pivots. Thus A can not have three pivots.

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4. (5 points) For which values of b is the matrix $\begin{bmatrix} 1 & b \\ 2 & 2b \end{bmatrix}$ invertible?

(A) For any number b different from 1.

(B) ★ For no number b.

(C) None of the other answers.

(D) For any number b.

(E) For any number b different from 0.

Solution. This matrix is not invertible for any choice of b, as $1 * 2b - b * 2 = 0$.

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5. (5 points) Let A be a 3×3 matrix and $\mathbf{b} \in \mathbb{R}^3$. Consider the following statements:

I. The row reduced echelon form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

II. If the linear system $A\mathbf{x} = \mathbf{b}$ has a solution, then it has a unique solution.

Which one of these statements is always true?

(A) Statement I only.

(B) Statement II only.

(C) Statement I and Statement II.

(D) ★ Neither of Statements I or II.

Solution. We do not know how many pivots the matrix A has. Both of these statements are true only when the matrix A has exactly 3 pivots.

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6. (5 points) Let

$$\boldsymbol{w} = \begin{bmatrix} 1 \\ h \\ h \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

For which values of h is $\boldsymbol{w}$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?

(A) Only when $h = -1$.

(B) ★ Only when $h = 1$.

(C) For no value of h.

(D) None of the other answers.

Solution. We have to determine when the following system of equations is consistent:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & h \\ 1 & 3 & h \end{array}\right] \xrightarrow{R2 \to R2 - R1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & h-1 \\ 1 & 3 & h \end{array}\right] \xrightarrow{R3 \to R3 - R1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & h-1 \\ 0 & 2 & h-1 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - 2R2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & h-1 \\ 0 & 0 & -h+1 \end{array}\right]$$

The system is consistent if and only if $-h+1 = 0$, i.e., $h = 1$.

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7. (5 points) The matrix $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ is reduced to the identity matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ using the following row operations (in the given order):

(1) $R_2 \leftrightarrow R4$.

(2) $R_2 \rightarrow R_2 + 2R_1$,

What is A^{-1}?

(A) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}$

(C) ★ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(D) None of the other answers.

(E) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Solution. Let $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be the elementary matrix corresponding to $R2 \rightarrow R2{+}2R1$, and $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ be the elementary matrix corresponding to $R2 \leftrightarrow R4$.

Then $E_2E_1A = I$. Thus E_2E_1 is the inverse of A. Thus

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

8. (5 points) Let $A = \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Which one of the following statements is true?

(A) ★ There is $\mathbf{b} \in \mathbb{R}^{\mathbf{4}}$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(B) A is invertible.

(C) None of the other answers.

(D) $A\mathbf{x} = \mathbf{0}$ has exactly one solution.

(E) A does not have an LU decomposition.

Solution. Since an echelon form of A is $\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ there are free variables, thus the system $A\mathbf{x} = \mathbf{b}$ cannot have only one solution. The vector $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ makes the system inconsistent. Since the number of pivots is strictly less than 4, A is noninvertible. Since A is upper triangular $A = IA$ is a valid LU decomposition.

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9. (5 points) Let A be an $m \times n$ matrix. Which one of the following statements is true?

(A) ★ $A^T A$ is an $n \times n$ matrix.

(B) $A^T A$ is an $m \times n$ matrix.

(C) $A^T A$ is an $m \times m$ matrix.

(D) $A^T A$ is an $n \times m$ matrix.

(E) None of the other answers.

Solution. Since A^T is an $n \times m$ matrix and A is an $m \times n$ matrix , we know that $A^T A$ is an $n \times n$ matrix.

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10. (5 points) Which of the following vectors does NOT belong to the set

$$\operatorname{span}\left(\begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \begin{bmatrix}2\\4\\6\\0\end{bmatrix}, \begin{bmatrix}4\\8\\12\\0\end{bmatrix}\right)?$$

(A) $\begin{bmatrix}3\\6\\9\\0\end{bmatrix}$

(B) $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

(C) $\begin{bmatrix}11\\22\\33\\0\end{bmatrix}$

(D) $\begin{bmatrix}2\\4\\6\\0\end{bmatrix}$

(E) ★ $\begin{bmatrix}5\\10\\20\\0\end{bmatrix}$

Solution. $\operatorname{span}\left(\begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \begin{bmatrix}2\\4\\6\\0\end{bmatrix}, \begin{bmatrix}4\\8\\12\\0\end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix}1\\2\\3\\0\end{bmatrix}\right)$, so each vector in the set must be a scalar multiple of $\begin{bmatrix}1\\2\\3\\0\end{bmatrix}$. Of the choices, only $\begin{bmatrix}5\\10\\20\\0\end{bmatrix}$ is not.

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11. (5 points) Which one of the following statements is FALSE?

(A) If a system of linear equations has two different solutions, it must have infinitely many solutions.

(B) If A is invertible, the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

(C) Every matrix is row equivalent to a unique matrix in reduced row echelon form.

(D) ★ Every system of n linear equations in n variables has exactly one solution.

Solution. For a matrix A, its reduced echelon form is unique, so this option is true. There are $n \times n$ inconsistent linear systems, so this option is also false. The possibilities for a system of linear equations are "no solutions","exactly one solution" and "infinitely many solutions".Thus if a system has at least 2 solutions, then it must in fact have infinitely many solutions,so this option is true. If A is invertible, the solution is unique and given by $\mathbf{x} = A^{-1}b$, so this option is also true.

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12. (5 points) Let $A = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2\end{bmatrix}$ be a 3×2 matrix and suppose that

$$A\mathbf{x} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \text{for } \mathbf{x} = \begin{bmatrix}1\\1\end{bmatrix}, \quad \text{and } A\mathbf{y} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \text{for } \mathbf{y} = \begin{bmatrix}1\\2\end{bmatrix}$$

Then the following holds:

(A)

$$\begin{bmatrix}1\\1\end{bmatrix} = 1\mathbf{a}_1 + 1\mathbf{a}_2, \quad \begin{bmatrix}1\\2\end{bmatrix} = 1\mathbf{a}_1 + 2\mathbf{a}_2$$

(B) None of the other answers.

(C) ★

$$\begin{bmatrix}1\\0\\1\end{bmatrix} = \mathbf{a}_1 + \mathbf{a}_2, \quad \begin{bmatrix}1\\1\\1\end{bmatrix} = \mathbf{a}_1 + 2\mathbf{a}_2$$

(D)

$$\begin{bmatrix}1\\0\\1\end{bmatrix} = \mathbf{a}_1 + 2\mathbf{a}_2, \quad \begin{bmatrix}1\\1\\1\end{bmatrix} = \mathbf{a}_1 + \mathbf{a}_2$$

(E)

$$\begin{bmatrix}1\\1\end{bmatrix} = \mathbf{a}_1 + 0\mathbf{a}_2 + 3\mathbf{a}_3, \quad \begin{bmatrix}1\\2\end{bmatrix} = 1\mathbf{a}_1 + 1\mathbf{a}_2 + 1\mathbf{a}_3$$

Solution. Matrix Multiplication is linear combination, so $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$ for a 3×2 matrix.

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13. (5 points) Let P be the 4×4-permutation matrix that permutes the second row and the third row. Which of the following statements is true?

(A) $P^7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $(P^{-1})^9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(B) $P^7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $(P^{-1})^9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(C) $P^7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $(P^{-1})^9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(D) None of the other answers.

(E) ★ $P^7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $(P^{-1})^9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution. From class we know that $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $P^{-1} = P$. Observe that $P^n = I_4$ whenever n is even, and $P^n = P$ whenever n is odd. Thus $P^7 = P$ and $(P^{-1})^9 = P^9 = P$.

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14. (5 points) Find an explicit description of the null space of

$$A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

that is, find a minimal set of vectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}$ such that $\mathrm{Nul}(A) = \mathrm{span}\left(\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}\right)$.

(A) None of the other answers.

(B) ★ $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(E) $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$

Solution. $\mathrm{Nul}(A)$ is the set of all solutions to the homogeneous equation

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We identify x_1, x_2 and x_4 as pivot variables and x_3 as the free variable. Thus we can write the general solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

where $x_3 \in \mathbb{R}$ is free. We conclude that

$$\mathrm{Nul}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}\right).$$

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15. (5 points) Consider the following subsets of $\mathbb{R}^2$:

$$W_1 = \{\begin{bmatrix} a \\ a \end{bmatrix} \ : \ a \in \mathbb{R}\}, \ W_2 = \{\begin{bmatrix} a \\ b \end{bmatrix} \ : \ ab = 1\}.$$

Then:

(A) ★ Only W_1 is a subspace of $\mathbb{R}^2$.

(B) Only W_2 is a subspace of $\mathbb{R}^2$.

(C) Neither W_1 nor W_2 is a subspace of $\mathbb{R}^2$.

(D) Both W_1 and W_2 are subspaces of $\mathbb{R}^2$.

Solution. W_1 is a subspace, because $W_1 = span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$. The second subset W_2 is not a subspace. Simply observe that the $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in W_2.

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16. (5 points) Consider the following matrix

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -2 & 2 & 0 \\ 2 & 2 & -2 & 0 \end{bmatrix}.$$

What is the number of pivot positions of this matrix?

(A) 3

(B) 0

(C) None of the other answers.

(D) ★ 1

(E) 2

Solution.

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -2 & 2 & 0 \\ 2 & 2 & -2 & 0 \end{bmatrix} \underset{R2 \to R2+2R1}{\longrightarrow} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 \end{bmatrix} \underset{R3 \to R3-2R1}{\longrightarrow} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

One pivot!

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17. (5 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

Which of the following is the matrix L in an LU factorization of A?

(A) ★ $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$.

(B) $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}$.

(C) $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$.

(D) $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$.

(E) None of the other answers.

Solution.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & -4 & -8 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & -7 \end{bmatrix}, \textit{ and } E_2E_1A = U$$

So

$$L = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

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18. (5 points) Let $D = \{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1\}$ denote the set of vectors in the unit disk of $\mathbb{R}^2$. It can be shown that D is NOT a subspace of $\mathbb{R}^2$. Which of the following tests does D **fail to satisfy**? (Select all that apply.)

I. contains the zero vector

II. closed under vector addition

III. closed under scalar multiplication

(A) III. only

(B) I., II., and III.

(C) II. only

(D) ★ II. and III. only

Solution. The zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ satisfies $0^2 + 0^2 \leq 1$. The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ each belong in D, but their sum $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does not, as $1^2 + 1^2 > 1$. Finally, if we scale $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by 2, then $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ does not belong in D since $2^2 + 0^2 > 1$.
