Math 231E, Lecture 11.
Mean Value Theorem

Theorem 1 (Mean Value Theorem)**.** Let $f(x)$ be a function such that f is continuous on $[a, b]$ and f is differentiable on (a, b). Then there is a number c with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

This says that there is a point where the *instantaneous* rate of change is equal to the *average* rate of change. A corollary of this is Rolle's Theorem:

Theorem 2. Let $f(x)$ be a function such that f is continuous on $[a, b]$ and f is differentiable on (a, b), and $f(a) = f(b)$. Then there is a number c with $a < c < b$ such that $f'(c) = 0$.

Example 1. If a car goes 2 miles in one minute, then at some instant it had to be going exactly $120\,\mathrm{mph}$.

We have another corollary of the MVT, namely:

Theorem 3. If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x)$ is constant on (a, b).

Proof. Choose $x_1, x_2 \in (a, b)$. Then

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1),$$

for some $c \in (a, b)$. But $f'(c) = 0$ for all $c \in (a, b)$, so $f(x_2) - f(x_1) = 0$. □

From this, we get another nice result:

Theorem 4. If f and g are two functions defined on (a, b) such that $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + C$ for some constant C.

Proof. Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$, so $h(x)$ is constant. □

Example 2. Note! that the previous few statements need to be true for every point in the interval; just one point can mess it up. For example, consider $f(x) = |x|/x$ on the interval $(-1, 1)$. This function is definitely not constant on $(-1, 1)$, but its derivative is zero at every point but one!

Theorem 5. If $f'(x) < 0$ on (a, b), then $f(x)$ is decreasing on (a, b). If $f'(x) > 0$ on (a, b), then $f(x)$ is increasing on (a, b).

Proof. Let $f'(x) < 0$ for all $x \in (a, b)$. Let $x_1 < x_2$. Then

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) < 0.$$

□