Math 231E Fall 2018 Final Exam.

- This exam has 14 questions.

- You must not communicate with other students during this test. No books, notes, **calculators**, or electronic devices allowed.

- Please fill out all of the information below. Make sure to fill out your Scantron form as directed in class; fill in name, UIN number, and NetID.

- You have three hours to complete this exam.

- Make your own luck.

1. Fill in your information:

Full Name: _____

UIN (Student Number): _____

NetID: _____

2. Fill out name, student number (UIN) and NetID on Scantron sheet. Then fill in the following answers on the Scantron form:

Zone 1

1/1. (3 points) If we have a point with polar coordinates $r = 1$, $\theta = \frac{5\pi}{6}$, then the Cartesian coordinates of the point are:

A. ★ $(-\sqrt{3}/2, 1/2)$

B. $(\sqrt{2}/2, \sqrt{2}/2)$

C. $(1/2, \sqrt{3}/2)$

D. $(-\sqrt{3}/2, -1/2)$

E. $(-1/2, -\sqrt{3}/2)$

Solution. Using the formulas

$$x = r\cos(\theta), \quad y = r\sin(\theta),$$

we have

$$x = 1 \cdot \cos(\tfrac{5}{6}\pi) = -\sqrt{3}/2, \quad y = 1 \cdot \sin(\tfrac{5}{6}\pi) = 1/2.$$

1/2. (3 points) If we have a point with polar coordinates $r = 1$, $\theta = \frac{7\pi}{6}$, then the Cartesian coordinates of the point are:

A. ★ $(-\sqrt{3}/2, -1/2)$

B. $(\sqrt{2}/2, \sqrt{2}/2)$

C. $(1/2, \sqrt{3}/2)$

D. $(\sqrt{3}/2, 1/2)$

E. $(-1/2, -\sqrt{3}/2)$

Solution. Using the formulas
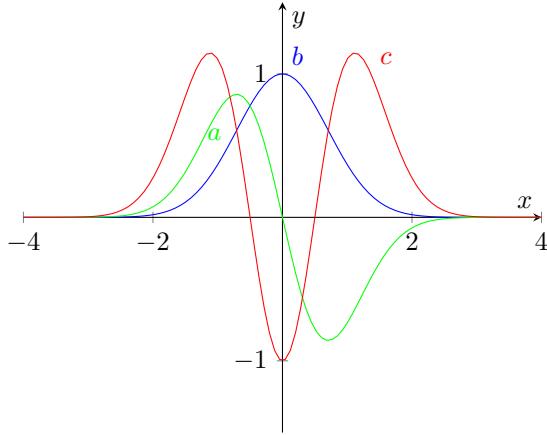
$$x = r\cos(\theta), \quad y = r\sin(\theta),$$

we have

$$x = 1 \cdot \cos(\tfrac{7}{6}\pi) = -\sqrt{3}/2, \quad y = 1 \cdot \sin(\tfrac{7}{6}\pi) = -1/2.$$

2/1. (3 points) This figure shows the graphs of f, f', and f''. Identify each curve.
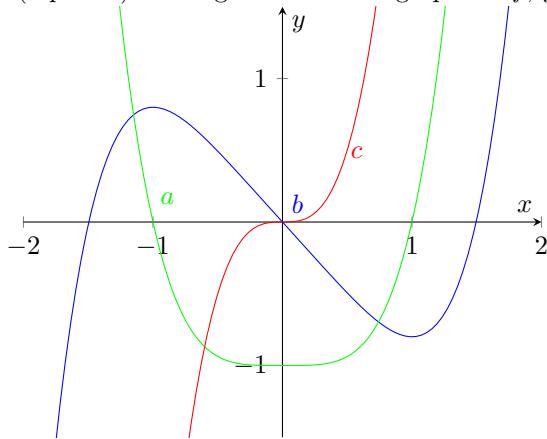
A. ★ f is (b), f' is (a), f'' is (c)

B. f is (a), f' is (b), f'' is (c)

C. f is (c), f' is (a), f'' is (b)

D. f is (c), f' is (b), f'' is (a)

E. f is (a), f' is (c), f'' is (b)

Solution. Matching zeros with local max/min shows that f is (b), f' is (a), f'' is (c).

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2/2. (3 points) This figure shows the graphs of f, f', and f''. Identify each curve.

A. ★ f is (b), f' is (a), f'' is (c)

B. f is (a), f' is (b), f'' is (c)

C. f is (c), f' is (a), f'' is (b)

D. f is (c), f' is (b), f'' is (a)

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E. f is (a), f' is (c), f'' is (b)

Solution. Matching zeros with local max/min shows that f is (b), f' is (a), f'' is (c).

3/1. (3 points) Recall the notation $\exp(A) = e^A$ for any expression A. Find the limit

$$\lim_{x \to 0} \frac{\sin(x^4) - x^4}{4\exp(x^{12}) - 4}$$

A. ★ $-\frac{1}{24}$

B. $-\frac{1}{3}$

C. $-\frac{1}{4}$

D. $\frac{1}{12}$

E. Does not exist

   Solution. Using Taylor expansions around the origin we have

$$\lim_{x \to 0} \frac{\sin(x^4) - x^4}{4\exp(x^{12}) - 4} = \lim_{x \to 0} \frac{\left(x^4 - \frac{x^{12}}{6} + \mathcal{O}(x^{20})\right) - x^4}{4\left(1 + x^{12} + \mathcal{O}(x^{24})\right) - 4} = \lim_{x \to 0} \frac{-\frac{x^{12}}{6} + \mathcal{O}(x^{20})}{4x^{12} + \mathcal{O}(x^{24})} = -\frac{1}{24}$$

3/2. (3 points) Recall the notation $\exp(A) = e^A$ for any expression A. Find the limit

$$\lim_{x \to 0} \frac{\sin(x^3) - x^3}{4\exp(x^9) - 4}$$
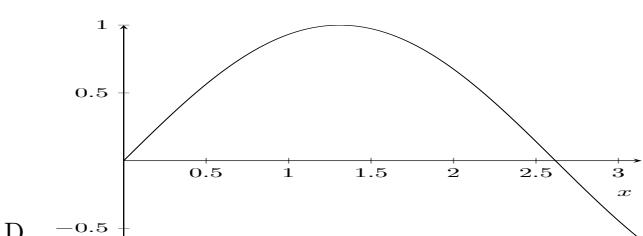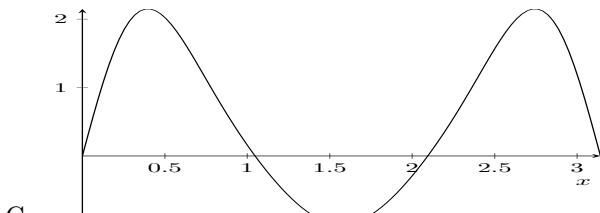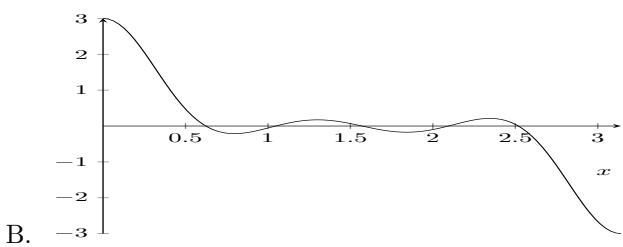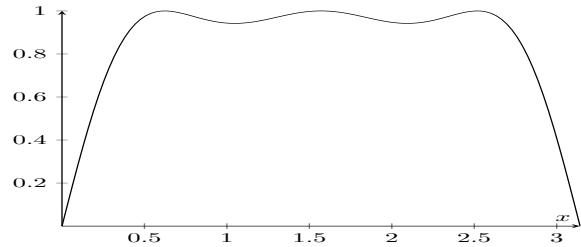
A. ★ $-\frac{1}{24}$

B. $-\frac{1}{3}$

C. $-\frac{1}{4}$

D. $\frac{1}{12}$

E. Does not exist

   Solution. Using Taylor expansions around the origin we have

$$\lim_{x \to 0} \frac{\sin(x^3) - x^3}{4\exp(x^9) - 4} = \lim_{x \to 0} \frac{\left(x^3 - \frac{x^9}{6} + \mathcal{O}(x^{15})\right) - x^3}{4\left(1 + x^9 + \mathcal{O}(x^{18})\right) - 4} = \lim_{x \to 0} \frac{-\frac{x^9}{6} + \mathcal{O}(x^{15})}{4x^9 + \mathcal{O}(x^{18})} = -\frac{1}{24}$$

4/1. (3 points) Identify the graph of $f(x) = \sin(x + \sin 2x)$

A. ★

B.

C.

D.

E. None of these graphs

5/1. (3 points) Find the derivative of the function

$$\int_{1-2x}^{1+2x} t \, \sin(t) \, dt$$

A. ★ $2(1+2x)\sin(1+2x) + 2(1-2x)\sin(1-2x)$

B. $\sin x + x \cos x$

C. $\sin(1+2x) + (1+2x)\cos(1+2x) - \sin(1-2x) - (1-2x)\cos(1-2x)$

D. $-4x \sin(4x)$

E. 0

Solution.

$$\frac{d}{dx} \int_{1-2x}^{1+2x} t \, \sin(t) \, dt = \frac{d}{dx} \left(\int_0^{1+2x} t \, \sin t \, dt - \int_0^{1-2x} t \, \sin t \, dt \right) = 2(1+2x)\sin(1+2x)+2(1-2x)\sin(1-2x)$$

5/2. (3 points) Find the derivative of the function

$$\int_{1-2x}^{1+2x} t \, \cos(t) \, dt$$

A. ★ $2(1+2x)\cos(1+2x) + 2(1-2x)\cos(1-2x)$

B. $\cos x - x \sin x$

C. $\cos(1+2x) - (1+2x)\sin(1+2x) - \cos(1-2x) + (1-2x)\sin(1-2x)$

D. $-4x \cos(4x)$

E. 0

Solution.

$$\frac{d}{dx} \int_{1-2x}^{1+2x} t \, \cos(t) \, dt = \frac{d}{dx} \left(\int_0^{1+2x} t \, \cos t \, dt - \int_0^{1-2x} t \, \cos t \, dt \right) = 2(1+2x)\cos(1+2x)+2(1-2x)\cos(1-2x)$$

6/1. (3 points) Find the indefinite integral

$$\int \sin^2 x \, dx$$

A. ★ $\frac{x}{2} - \frac{\sin 2x}{4} + C$

B. $\frac{x}{2} - \frac{\sin 2x}{2} + C$

C. $\frac{x}{4} - \frac{\cos 2x}{2} + C$

D. $\frac{x}{2} - \frac{\cos 2x}{4} + C$

E. $\frac{1}{3}\sin^3 x + C$

Solution.

$$\int \sin^2 x \quad dx \quad = \quad \int \left(\frac{1 - \cos 2x}{2} \right) \quad dx \quad = \quad \frac{1}{2}(x \quad - \quad \frac{\sin(2x)}{2}) \quad + \quad C$$

7/1. (3 points) Make an appropriate trigonometric substitution to convert this integral

$$\int \frac{dx}{x^2\sqrt{x^2+4}}$$

into a simpler form

A. ★ $\int \frac{\cos t}{4\sin^2 t}dt$

B. $\int \frac{\cos^2 t}{4\sin^2 t}dt$

C. $\int \frac{\cos^2 t}{16\sin t}dt$

D. $\int \frac{\cos t}{4\sin t}dt$

E. $\int \frac{4\cos t}{\sin^2 t}dt$

Solution.

$$\int \frac{dx}{x^2\sqrt{x^2+4}} \xrightarrow[dx=2\sec^2\theta\ d\theta]{x=2\tan\theta} \int \frac{2\sec^2\theta\ d\theta}{4\tan^2\theta(2\sec\theta)} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\ d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\ d\theta \xrightarrow[du=\cos\theta\ d\theta]{u=\sin\theta} \frac{1}{4}\int \frac{du}{u^2} = -\frac{1}{4u}+C = -\frac{1}{4\sin\theta}+C = -\frac{\sqrt{x^2+4}}{4x}+C$$

where in the last step we use a right triangle with $\tan\theta = \frac{x}{2}$.

7/2. (3 points) Make a trigonometric substitution to convert this integral

$$\int \frac{dx}{x^2\sqrt{x^2+9}}$$

into a simpler form

A. ★ $\int \frac{\cos t}{9\sin^2 t}dt$

B. $\int \frac{\cos^2 t}{9\sin^2 t}dt$

C. $\int \frac{\cos^2 t}{81\sin t}dt$

D. $\int \frac{\cos t}{9\sin t}dt$

E. $\int \frac{9\cos t}{\sin^2 t}dt$

Solution.

$$\int \frac{dx}{x^2\sqrt{x^2+9}} \xrightarrow[dx=3\sec^2\theta\ d\theta]{x=3\tan\theta} \int \frac{3\sec^2\theta\ d\theta}{9\tan^2\theta(3\sec\theta)} = \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\ d\theta$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\ d\theta \xrightarrow[du=\cos\theta\ d\theta]{u=\sin\theta} \frac{1}{9}\int \frac{du}{u^2} = -\frac{1}{9u}+C = -\frac{1}{9\sin\theta}+C = -\frac{\sqrt{x^2+9}}{9x}+C$$

where in the last step we use a right triangle with $\tan\theta = \frac{x}{3}$.

8/1. (3 points) Use polynomial division to express the rational function

$$\frac{x^4 + x^3 - 3x^2 + 3x + 2}{(x-1)^2(x+1)}$$

in a simpler form

A. ★ $x + 2 + \frac{4x}{(x-1)^2(x+1)}$

B. $x^2 + x + \frac{4x}{(x-1)^2(x+1)}$

C. $x + 2 - \frac{4x}{(x-1)^2(x+1)}$

D. $x + 2 - \frac{(x-1)^2(x+1)}{4x}$

E. $4x - \frac{x+2}{(x-1)^2(x+1)}$

Solution. First we note $(x-1)^2(x+1) = x^3 - x^2 - x + 1$ and use long division to find

$$\frac{x^4 + x^3 - 3x^2 + 3x + 2}{(x-1)^2(x+1)} = x + 2 + \frac{4x}{(x-1)^2(x+1)}.$$

8/2. (3 points) Use polynomial division to express the rational function

$$\int \frac{2x^4 - x^3 - 3x^2 + 5x + 1}{(x-1)^2(x+1)} \, dx$$

in a simpler form

A. ★ $2x + 1 + \frac{4x}{(x-1)^2(x+1)}$

B. $2x + 1 + (x-1)^2(x+1)4x$

C. $4x + \frac{2x+1}{(x-1)^2(x+1)}$

D. $4x + \frac{(x-1)^2(x+1)}{2x+1}$

E. $2x + 1 - \frac{4x}{(x-1)^2(x+1)}$

Solution. First we note $(x-1)^2(x+1) = x^3 - x^2 - x + 1$ and use long division to find

$$\frac{2x^4 - x^3 - 3x^2 + 5x + 1}{(x-1)^2(x+1)} = 2x + 1 + \frac{4x}{(x-1)^2(x+1)}.$$

9/1. (3 points) Evaluate $\displaystyle\int_0^3 \frac{dx}{x-2}$

A. ★ Divergent

B. $-\log 2$

C. $\log(-2)$

D. 3

E. $\log 3$

Solution. This integral is improper of type II at $x = 2$, we have

$$\lim_{t \to 2-} \int_0^t \frac{dx}{x-2} = \lim_{t \to 2-} \log|t-2| - \log|-2| = -\infty$$

so the integral is divergent.

10/1. (3 points) Evaluate $\int (\ln x)^2 \, dx$
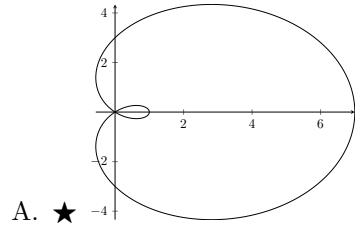
A. ★ $x(\ln x)^2 - 2x \ln x + 2x + C$

B. $x(\ln x)^2 - x \ln x + 2x + C$

C. $x(\ln x)^2 + 2x \ln x + 2x + C$

D. $x(\ln x)^2 - 2x \ln x + x + C$

E. $x(\ln x)^2 + x \ln x + x + C$

Solution. First we find the antiderivative
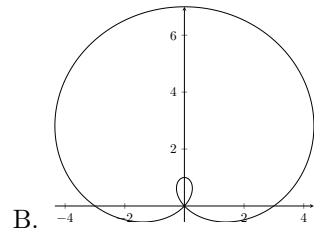
$$\int (\ln x)^2 \, dx \xrightarrow[du=(2 \ln x)/x \, dx, v=x]{u=(\ln x)^2, dv=dx} x(\ln x)^2 - 2 \int \ln x \, dx$$

$$\xrightarrow[du=1/x \, dx, v=x]{u=\ln x, dv=dx} x(\ln x)^2 - 2(x \ln x - \int dx) = x(\ln x)^2 - 2x \ln x + 2x + C,$$
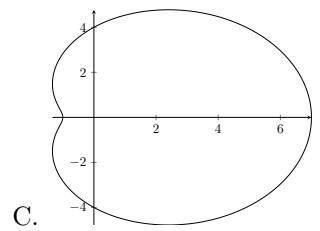
11/1. (3 points) Identify the graph of $r = 3 + 4\cos(t)$

A. ★

B.

C.

D.
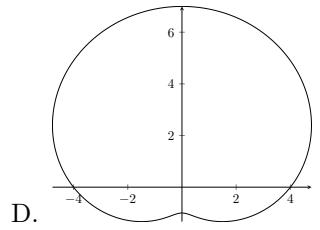
E. None of these graphs

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12/1. (3 points) Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1}$$

A. ★ diverges

B. converges

C. impossible to say

Solution. The nth term is larger than $\frac{1}{n}$

13/1. (3 points) Find the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

A. ★ $(-5, 1)$

B. the point $x = -2$

C. all x

D. $(-3, 3)$

E. $(-1, 5)$

Solution. The ratio a_{n+1}/a_n converges to $|x+2|/3$, so the series converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. At the end points the nth term does not go to zero, so the series can not converge.

13/2. (3 points) Find the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{n(x-2)^n}{3^{n+1}}$

A. ★ $(-1, 5)$

B. the point $x = 2$

C. all x

D. $(-3, 3)$

E. $(-5, 1)$

Solution. The ratio a_{n+1}/a_n converges to $|x-2|/3$, so the series converges if $|x-2| < 3$ and diverges if $|x-2| > 3$. At the end points the nth term does not go to zero, so the series can not converge.

14/1. (3 points) Find the partial fraction expansion of $\frac{2x+6}{(x^2+1)(x-1)}$

A. ★ $\frac{-4x-2}{x^2+1} + \frac{4}{x-1}$

B. $\frac{-4x+2}{x^2+1} + \frac{4}{x-1}$

C. $\frac{-2x+4}{x^2+1} + \frac{4}{x-1}$

D. $\frac{-2x+4}{x^2+1} + \frac{2}{x-1}$

E. $\frac{-2x+2}{x^2+1} + \frac{2}{x-1}$

Solution. Take the partial fraction expansion $\frac{Ax+B}{x^2+1} + \frac{C}{x-1}$ Cross multiplying we get $(A+C)x^2 + (B-A)x + (C-A) = 2x+6$ Thus we have three equation $A+C = 0$, $-A+B = 6$ and $-B+C = 2$. Solving gives $A = -4, B = -2, C = 4$

14/2. (3 points) Find the partial fraction expansion of $\frac{6x+2}{(x^2+1)(x-1)}$

A. ★ $\frac{-4x+2}{x^2+1} + \frac{4}{x-1}$

B. $\frac{-4x-2}{x^2+1} + \frac{4}{x-1}$

C. $\frac{-2x+4}{x^2+1} + \frac{4}{x-1}$

D. $\frac{-2x+4}{x^2+1} + \frac{2}{x-1}$

E. $\frac{-2x+2}{x^2+1} + \frac{2}{x-1}$

Solution. Take the partial fraction expansion $\frac{Ax+B}{x^2+1} + \frac{C}{x-1}$ Cross multiplying we get $(A+C)x^2 + (B-A)x + (C-A) = 6x+2$ Thus we have three equation $A+C = 0$, $-A+B = 6$ and $-B+C = 2$. Solving gives $A = -4, B = 2, C = 4$

Zone 2

15/1. (3 points) Find the radius of convergence of the series $\sum_{n=0}^{\infty} \dfrac{(n!)^2}{(2n!)} x^n$

A. ★ 4

B. 2

C. 0

D. ∞

E. 8
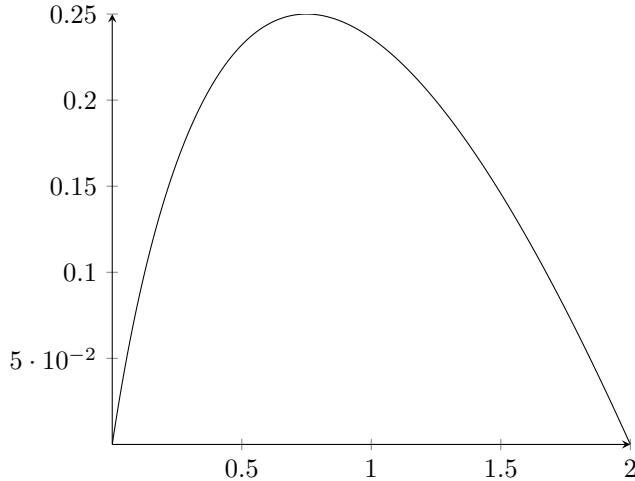
    Solution. The ratio $|a_{n+1}/a_n|$ is equal to

$$\frac{((n+1)!)^2}{(n!)^2} \frac{(2n!)}{(2(n+1))!}|x| = \frac{(n+1)^2}{(2n+1)(2n+2)}|x|$$

and so converges to $|x|/4$. It follows that the radius of convergence is 4.

16/1. (3 points) Find the area of the region enclosed by the x-axis and the curve $x = t + t^2$, $y = t - t^2$.
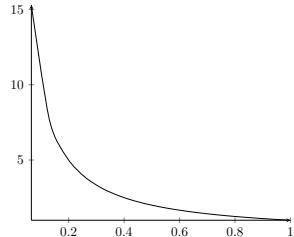
A. ★ $\frac{1}{3}$

B. $\frac{1}{4}$

C. $\frac{1}{2}$

D. 1

E. $\frac{1}{5}$

Solution.
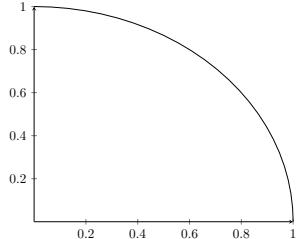
$$A = \int_0^1 y \, dx = \int_0^1 (t - t^2)(1 + 2t) \, dt = \int_0^1 (t + t^2 - 2t^3 \, dt$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{2} = \frac{1}{3}$$
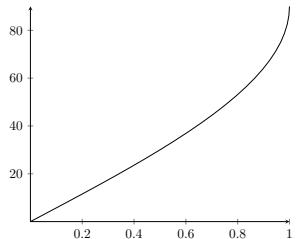
17/1. (3 points) Identify the graph of $x(t) = \sin t$, $y(t) = \csc t$, $0 < t < \frac{\pi}{2}$.

A. ★

B.

C.

D.
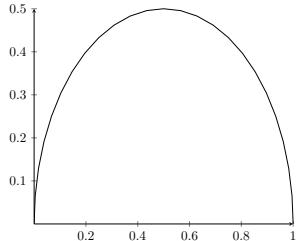
E. None of these graphs

Solution. The product of the two coordinates is one, so the graph is part of $y = \frac{1}{x}$.

18/1. (3 points) A group of engineers is building a parabolic satellite dish whose shape will be formed by rotating the curve $y = \frac{x^2}{2}$ between $x = 0$ and $x = 2$ about the y-axis. Find the surface area of the dish.

A. ★ $\frac{2\pi}{3}(5\sqrt{5} - 1)$

B. $\frac{\pi}{6}(17\sqrt{17} - 1)$

C. $\frac{26\pi}{3}$

D. $\frac{\pi}{24}(65\sqrt{65} - 1)$

E. $\frac{\pi}{8}(19\sqrt{19} - 1)$

Solution.

$$\int_0^2 2\pi x \; ds = 2\pi \int_0^2 x\sqrt{1 + 4\frac{1}{4}x^2} \; dx \xrightarrow[du=2xdx]{u=1+x^2} \pi \int_1^5 \sqrt{u} \; du = \pi\frac{2}{3}u^{3/2}\Big|_1^5 = \frac{2\pi}{3}(5\sqrt{5} - 1)$$

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19/1. (3 points) Compute the indefinite integral

$$\int e^{2\theta} \sin(3\theta) \, d\theta.$$

A. ★ $\frac{e^{2\theta}}{13}(2\sin(3\theta) - 3\cos(3\theta)) + C$

B. $\frac{e^{2\theta}}{6}\cos(3\theta) + C$

C. $\frac{e^{2\theta}}{6}(\sin(3\theta) - \cos(3\theta)) + C$

D. $\frac{3e^{2\theta}}{2}(\sin(3\theta) - \cos(3\theta)) + C$

E. $e^{2\theta}\sin(3\theta) + C$

Solution.

$$L = \int e^{2\theta} \sin(3\theta) \, d\theta \xrightarrow[du=3\cos(3\theta)d\theta, v=\frac{1}{2}e^{2\theta}]{u=\sin(3\theta), dv=e^{2\theta}d\theta} = \frac{1}{2}\sin(3\theta)e^{2\theta} - \frac{3}{2}\int e^{2\theta}\cos(3\theta)\, d\theta$$

$$\xrightarrow[du=-3\sin(3\theta)d\theta, v=\frac{1}{2}e^{2\theta}]{u=\cos(3\theta), dv=e^{2\theta}d\theta} \frac{1}{2}\sin(3\theta)e^{2\theta} - \frac{3}{2}\left(\frac{1}{2}\cos(3\theta)e^{2\theta} + \frac{3}{2}\int e^{2\theta}\sin(3\theta)\right)$$

$$= \frac{1}{2}\sin(3\theta)e^{2\theta} - \frac{3}{4}\cos(3\theta)e^{2\theta} - \frac{9}{4}L$$

and hence

$$L = \frac{4}{13}\left(\frac{1}{2}\sin(3\theta)e^{2\theta} - \frac{3}{4}\cos(3\theta)e^{2\theta}\right) = \frac{e^{2\theta}}{13}(2\sin(3\theta) - 3\cos(3\theta))$$

19/2. (3 points) Compute the indefinite integral

$$\int e^{3\theta} \sin(2\theta) \, d\theta.$$

A. ★ $\frac{e^{3\theta}}{13}(3\sin(2\theta) - 2\cos(2\theta)) + C$

B. $\frac{e^{3\theta}}{6}\cos(2\theta) + C$

C. $\frac{e^{3\theta}}{6}(\sin(2\theta) - \cos(2\theta)) + C$

D. $\frac{2e^{3\theta}}{3}(\sin(2\theta) - \cos(2\theta)) + C$

E. $e^{3\theta}\sin(2\theta) + C$

Solution.

$$L = \int e^{3\theta} \sin(2\theta) \, d\theta \xrightarrow[du=2\cos(2\theta)d\theta, v=\frac{1}{3}e^{3\theta}]{u=\sin(2\theta), dv=e^{3\theta}d\theta} = \frac{1}{3}\sin(2\theta)e^{3\theta} - \frac{2}{3}\int e^{3\theta}\cos(2\theta)\, d\theta$$

$$\xrightarrow[du=-2\sin(2\theta)d\theta, v=\frac{1}{3}e^{3\theta}]{u=\cos(2\theta), dv=e^{3\theta}d\theta} \frac{1}{3}\sin(2\theta)e^{3\theta} - \frac{2}{3}\left(\frac{1}{3}\cos(2\theta)e^{3\theta} + \frac{2}{3}\int e^{3\theta}\sin(2\theta)\right)$$

$$= \frac{1}{3}\sin(2\theta)e^{3\theta} - \frac{2}{9}\cos(3\theta)e^{2\theta} - \frac{4}{9}L$$

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and hence
$$L = \frac{9}{13}\left(\frac{1}{3}\sin(2\theta)e^{3\theta} - \frac{2}{9}\cos(2\theta)e^{3\theta}\right) = \frac{e^{3\theta}}{13}(3\sin(2\theta) - 2\cos(2\theta))$$

20/1. (3 points) The region $\mathcal{R}$ enclosed by the curves $y = 2x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

A. ★ $\frac{64\pi}{15}$

B. $\frac{2\pi}{15}$

C. $\frac{32\pi}{15}$

D. $\frac{16\pi}{15}$

E. $\frac{128\pi}{15}$

    Solution. The curves intersect at $(0,0)$ and $(2,4)$. The area of the cross section is

$$A(x) = \pi(2x)^2 - \pi(x^2)^2 = 4\pi x^2 - \pi x^4$$

and hence the volume is

$$V = \int_0^2 4\pi x^2 - \pi x^4 \, dx = 4\pi\left(\frac{2^3}{3}\right) - \pi\frac{2^5}{5} = \frac{64\pi}{15}$$

20/2. (3 points) The region $\mathcal{R}$ enclosed by the curves $y = 3x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

A. ★ $\frac{162\pi}{5}$

B. $\frac{243\pi}{5}$

C. $\frac{81\pi}{5}$

D. $\frac{161\pi}{15}$

E. $\frac{128\pi}{15}$

    Solution. The curves intersect at $(0,0)$ and $(3,9)$. The area of the cross section is

$$A(x) = \pi(3x)^2 - \pi(x^2)^2 = 9\pi x^2 - \pi x^4$$

and hence the volume is

$$V = \int_0^3 9\pi x^2 - \pi x^4 \, dx = 9\pi\left(\frac{3^3}{3}\right) - \pi\frac{3^5}{5} = \frac{162\pi}{5}$$

21/1. (3 points) Find L where
$$L = \lim_{x \to 0} \frac{\cos(\alpha x) - 1}{2x^2}$$

A. ★ $-\frac{\alpha^2}{4}$

B. $-\frac{\alpha}{2}$

C. $-\frac{\alpha^3}{6}$

D. $-\frac{\alpha^4}{6}$

E. Does not exist

Solution. We have
$$L = \lim x \to 0 \frac{\left(1 - \frac{1}{2}(\alpha x)^2 + \mathcal{O}(x^3)\right) - 1}{2x^2} = \lim x \to 0 \frac{-\frac{1}{2}\alpha^2 x^2 + \mathcal{O}(x^3)}{2x^2} = -\frac{\alpha^2}{4}.$$

21/2. (3 points) Find L where
$$L = \lim_{x \to 0} \frac{\cos(\alpha x) - 1}{3x^2}$$

A. ★ $-\frac{\alpha^2}{6}$

B. $-\frac{\alpha}{3}$

C. $-\frac{\alpha^3}{9}$

D. $-\frac{\alpha^4}{3}$

E. Does not exist

Solution. We have
$$L = \lim x \to 0 \frac{\left(1 - \frac{1}{2}(\alpha x)^2 + \mathcal{O}(x^3)\right) - 1}{3x^2} = \lim x \to 0 \frac{-\frac{1}{2}\alpha^2 x^2 + \mathcal{O}(x^3)}{3x^2} = -\frac{\alpha^2}{6}.$$

22/1. (3 points) At noon, ship A is 20 km west of ship B. Ship A is sailing south at 3 km/h and ship B is sailing north at 2 km/h. How fast is the distance between the ships changing at 3:00 pm?

A. ★ 3 km/h

B. 2 km/h

C. 1 km/h

D. 4 km/h

E. 5 km/h

Solution. Let us have ship A start at the origin and so ship B starts at $(20, 0)$. At t hours after noon, ship A is at $(0, -3t)$ and ship B is at $(20, 2t)$, so the distance between them is

$$d(t) = \sqrt{(20)^2 + (5t)^2} = 5\sqrt{16 + t^2}.$$

Thus

$$d'(t) = \frac{5t}{\sqrt{16 + t^2}}, \text{ and } d'(3) = \frac{15}{5} = 3km/h.$$

22/2. (3 points) At noon, ship A is 20 km west of ship B. Ship A is sailing south at 2 km/h and ship B is sailing north at 3 km/h. How fast is the distance between the ships changing at 3:00 pm?

A. ★ 3 km/h

B. 2 km/h

C. 1 km/h

D. 4 km/h

E. 5 km/h

Solution. Let us have ship A start at the origin and so ship B starts at $(20, 0)$. At t hours after noon, ship A is at $(0, -2t)$ and ship B is at $(20, 3t)$, so the distance between them is

$$d(t) = \sqrt{(20)^2 + (5t)^2} = 5\sqrt{16 + t^2}.$$

Thus

$$d'(t) = \frac{5t}{\sqrt{16 + t^2}}, \text{ and } d'(3) = \frac{15}{5} = 3km/h.$$

23/1. (3 points) Evaluate
$$L = \lim_{x \to 1} \frac{\sin(\pi(t^2 - 2t))}{(t-1)^2}$$

A. ★ $-\pi$

B. ∞

C. $\frac{1}{2}$

D. 0

E. π

Solution. Applying L'Hôpital

$$L = \lim_{x \to 1} \frac{\pi(2t-2)\cos(\pi(t^2 - 2t))}{2(t-1)} = \lim_{x \to 1} \frac{\pi\cos(\pi(t^2 - 2t))}{1} = \pi\cos(-\pi) = -\pi$$

23/2. (3 points) Evaluate
$$L = \lim_{x \to 1} \frac{\sin(\pi(2t - t^2))}{(t-1)^2}$$

A. ★ $-\pi$

B. ∞

C. $\frac{1}{2}$

D. 0

E. π

Solution. Applying L'Hôpital

$$L = \lim_{x \to 1} \frac{\pi(2 - 2t)\cos(\pi(t^2 - 2t))}{2(t-1)} = \lim_{x \to 1} \frac{-\pi\cos(\pi(t^2 - 2t))}{1} = -\pi\cos(-\pi) = \pi$$

24/1. (3 points) Which of the following parametrizes the astroid $x^{2/3} + y^{2/3} = 4$.

A. ★ $x(\theta) = 8\cos^3\theta$, $y(\theta) = 8\sin^3\theta$

B. $x(\theta) = 4\cos^{2/3}\theta$, $y(\theta) = 4\sin^{2/3}\theta$

C. $x(\theta) = \sqrt[3]{4}\cos^{1/3}\theta$, $y(\theta) = \sqrt[3]{4}\sin^{1/3}\theta$

D. $x(\theta) = 2\cos\theta$, $y(\theta) = 2\sin\theta$

E. Not possible.

Solution. We want to use $(2\cos\theta)^2 + (2\sin\theta)^2 = 4$, so we set $x = (2\cos\theta)^3$ and $y = (2\sin\theta)^3$.

24/2. (3 points) Which of the following parametrizes the astroid $x^{2/3} + y^{2/3} = 9$.

A. ★ $x(\theta) = 27\cos^3\theta$, $y(\theta) = 27\sin^3\theta$

B. $x(\theta) = 9\cos^{2/3}\theta$, $y(\theta) = 9\sin^{2/3}\theta$

C. $x(\theta) = \sqrt[3]{9}\cos^{1/3}\theta$, $y(\theta) = \sqrt[3]{9}\sin^{1/3}\theta$

D. $x(\theta) = 3\cos\theta$, $y(\theta) = 3\sin\theta$

E. Not possible.

Solution. We want to use $(3\cos\theta)^2 + (3\sin\theta)^2 = 9$, so we set $x = (3\cos\theta)^3$ and $y = (3\sin\theta)^3$.

25/1. (3 points) If $xy + e^y = 1$ find y'.

    A. ★ $y' = -\frac{y}{x+e^y}$

    B. $y' = \frac{1}{x+e^y}$

    C. $y' = \frac{1-y}{x+e^y}$

    D. $y' = \frac{x+e^y}{1-y}$

    E. $y' = \frac{1-e^y}{x}$

Solution.
By implicit differentiation we find

$$y + xy' + y'e^y = 0, \text{ so } y' = -\frac{y}{x + e^y}.$$

Zone 3

26/1. (3 points) Find the angles θ at which the curve $r = 1 + \cos\theta$ has tangent line either horizontal or vertical.

   A. ★ vertical tangents at $\theta = 0, 2\pi/3, 4\pi/3$, horizontal tangents at $\theta = \pi/3, \pi, 5\pi/3$

   B. vertical tangents at $\theta = \pi/3, \pi, 5\pi/3$, horizontal tangents at $\theta = 0, 2\pi/3, 4\pi/3$

   C. vertical tangents at $\theta = 0, 2\pi/3, \pi, 4\pi/3$, horizontal tangents at $\theta = \pi/3, 5\pi/3$

   D. vertical tangents at $\theta = \pi/3, 5\pi/3$, horizontal tangents at $\theta = 0, 2\pi/3, \pi, 4\pi/3$

   E. vertical tangents at $\theta = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$ horizontal tangents at $\theta = 0, \pi$

   Solution. We have

$$\begin{cases} x(\theta) = (1 + \cos\theta)\cos\theta \\ y(\theta) = (1 + \cos\theta)\sin\theta \end{cases} \implies \begin{cases} x'(\theta) = -\sin\theta\cos\theta - (1+\cos\theta)\sin\theta = -\sin\theta(1 + 2\cos\theta) \\ y'(\theta) = -\sin^2\theta + (1+\cos\theta)\cos\theta = (2\cos\theta - 1)(\cos\theta + 1) \end{cases}$$

and so

$$x'(\theta) = 0 \iff \sin\theta = 0 \text{ or } \cos\theta = -1/2 \iff \theta = 0, \pi, 2\pi/3, 4\pi/3$$
$$y'(\theta) = 0 \iff \cos\theta = 1/2 \text{ or } \cos\theta = -1 \iff \theta = \pi/3, \pi, 5\pi/3$$

Thus we have vertical tangents at $\theta = 0, 2\pi/3, 4\pi/3$ and horizontal tangents at $\theta = \pi/3, 5\pi/3$, closer inspection shows that $\theta = \pi$ is a horizontal tangent.

26/2. (3 points) Find the angles θ at which the curve $r = 1 + \sin\theta$ has tangent line either horizontal or vertical.

   A. ★ vertical tangents at $\theta = \pi/6, 3\pi/2, 5\pi/6$, horizontal tangents at $\theta = \pi/2, 7\pi/6, 11\pi/6$

   B. vertical tangents at $\theta = \pi/2, 7\pi/6, 11\pi/6$, horizontal tangents at $\theta = \pi/6, 3\pi/2, 5\pi/6$

   C. vertical tangents at $\theta = \pi/6, 5\pi/6$, horizontal tangents at $\theta = \pi/2, 7\pi/6, 3\pi/2, 11\pi/6$

   D. vertical tangents at $\theta = \pi/2, 7\pi/6, 3\pi/2, 11\pi/6$, horizontal tangents at $\theta = \pi/6, 5\pi/6$

   E. vertical tangents at $\theta = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$ horizontal tangents at $\theta = \pi/2, 3\pi/2$

   Solution. We have

$$\begin{cases} x(\theta) = (1 + \sin\theta)\cos\theta \\ y(\theta) = (1 + \sin\theta)\sin\theta \end{cases} \implies \begin{cases} x'(\theta) = \cos^2\theta - (1+\sin\theta)\sin\theta = -(2\sin\theta - 1)(\sin\theta + 1) \\ y'(\theta) = \cos\theta\sin\theta + (1+\sin\theta)\cos\theta = \cos\theta(1 + 2\sin\theta) \end{cases}$$

and so

$$x'(\theta) = 0 \iff \sin\theta = 1/2 \text{ or } \sin\theta = -1 \iff \theta = \pi/6, 3\pi/2, 5\pi/6$$
$$y'(\theta) = 0 \iff \cos\theta = 0 \text{ or } \sin\theta = -1/2 \iff \theta = \pi/2, 7\pi/6, 3\pi/2, 11\pi/6$$

Thus we have vertical tangents at $\theta = \pi/6, 5\pi/6$, and horizontal tangents at $\theta = \pi/2, 7\pi/6, 11\pi/6$, closer inspection shows that $\theta = 3\pi/2$ is a vertical tangent.

27/1. (3 points) Is this series convergent or divergent? If convergent, what does it converge to?

$$\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}$$

A. ★ $\frac{3}{2}$

B. $\frac{2}{n-1}$

C. $\frac{1}{2}$

D. 2

E. Diverges

Solution. Using partial fractions we can write

$$\frac{2}{n^2 - 1} = \frac{1}{n - 1} - \frac{1}{n + 1}, \quad \sum_{n=2}^{\ell} \frac{2}{n^2 - 1} = 1 + \frac{1}{2} - \frac{1}{\ell} - \frac{1}{\ell + 1}$$

hence the series converges to $\frac{3}{2}$.

27/2. (3 points) Is this series convergent or divergent? If convergent, what does it converge to?

$$\sum_{n=2}^{\infty} \frac{3}{n^2 - 1}$$

A. ★ $\frac{9}{4}$

B. $\frac{2}{n-1}$

C. $\frac{3}{2}$

D. 1

E. Diverges

Solution. Using partial fractions we can write

$$\frac{3}{n^2 - 1} = \frac{3}{2}\left(\frac{1}{n - 1} - \frac{1}{n + 1}\right), \quad \sum_{n=2}^{\ell} \frac{2}{n^2 - 1} = \frac{3}{2}\left(1 + \frac{1}{2} - \frac{1}{\ell} - \frac{1}{\ell + 1}\right)$$

hence the series converges to $\frac{9}{4}$.

28/1. (3 points) Find the values of x for which the series converges and the sum of the series for those values of x.

$$\sum_{n=0}^{\infty} \frac{\sin^n(x)}{3^n}$$

A. ★ $\frac{3}{3-\sin(x)}$ for all values of x.

B. $\frac{\sin(x)}{3-\sin(x)}$ for x in $[-2\pi, 2\pi]$.

C. $\frac{\cos(x)}{3-\sin(x)}$ for x in $[-2\pi, 2\pi]$.

D. $\frac{3}{3-\cos(x)}$ for all values of x.

E. For no value of x

Solution. This is a geometric series with $r = \frac{1}{3}\sin(x)$, so the sum converges for all values of x to

$$\frac{1}{1 - \frac{\sin(x)}{3}} = \frac{3}{3 - \sin(x)}.$$

28/2. (3 points) Find the values of x for which the series converges and the sum of the series for those values of x.

$$\sum_{n=0}^{\infty} \frac{\cos^n(x)}{3^n}$$

A. ★ $\frac{3}{3-\cos(x)}$ for all values of x.

B. $\frac{\cos(x)}{3-\cos(x)}$ for x in $[-2\pi, 2\pi]$.

C. $\frac{\sin(x)}{3-\cos(x)}$ for x in $[-2\pi, 2\pi]$.

D. $\frac{3}{3-\sin(x)}$ for all values of x.

E. For no value of x

Solution. This is a geometric series with $r = \frac{1}{3}\cos(x)$, so the sum converges for all values of x to

$$\frac{1}{1 - \frac{\cos(x)}{3}} = \frac{3}{3 - \cos(x)}.$$

29/1. (3 points) Find the arclength of the curve $y = 2/3(x)^{\frac{3}{2}}$ between $x = 2$ and $x = 4$ ft.

A. ★ $\frac{2}{3}(5\sqrt{5} - 3\sqrt{3})$

B. $\frac{2}{3}(8 - 3\sqrt{3})$

C. $\frac{2}{3}(3\sqrt{3} - 2\sqrt{2})$

D. $\frac{2}{3}(6\sqrt{6} - 11)$

E. $\frac{2}{3}(8 - 2\sqrt{2})$

Solution. We are looking for the arc length. We can write $y = \frac{2}{3}(x)^{3/2}$, so that the arc length is

$$\int_2^4 ds = \int_0^2 \sqrt{1 + ((x)^{1/2})^2} \, dx = \int_2^4 \sqrt{1 + (x)} \, dx \xrightarrow[du=dx]{u=x+1} \int_3^5 \sqrt{u} \, du$$

$$= \frac{2}{3} u^{3/2} \Big|_3^5 = \frac{2}{3}(5\sqrt{5} - 3\sqrt{3})$$

29/2. (3 points) Find the arclength of the curve $y = 2/3(x)^{\frac{3}{2}}$ between $x = 1$ and $x = 2$ ft.

A. ★ $\frac{2}{3}(3\sqrt{3} - 2\sqrt{2})$

B. $\frac{2}{3}(8 - 3\sqrt{3})$

C. $\frac{2}{3}(5\sqrt{5} - 3\sqrt{3})$

D. $\frac{2}{3}(6\sqrt{6} - 11)$

E. $\frac{2}{3}(8 - 2\sqrt{2})$

Solution. We are looking for the arc length. We can write $y = \frac{2}{3}(x)^{3/2}$, so that the arc length is

$$\int_1^2 ds = \int_1^2 \sqrt{1 + ((x)^{1/2})^2} \, dx = \int_1^2 \sqrt{1 + x} \, dx \xrightarrow[du=dx]{u=x+1} \int_2^3 \sqrt{u} \, du$$

$$= \frac{2}{3} u^{3/2} \Big|_2^3 = \frac{2}{3}(3\sqrt{3} - 2\sqrt{2})$$

30/1. (3 points) Find numbers a and b such that $\lim\limits_{x \to 0} \dfrac{\sqrt{ax+b}-2}{x} = 4$

   A. ★ $a = 16$ and $b = 4$

   B. $a = 16$ and $b = 16$

   C. $a = 4$ and $b = 4$

   D. $a = 4$ and $b = 16$

   E. impossible

   Solution. Since the denominator is going to zero, we need the numerator to go to zero as well, hence $b = 4$. Applying L'Hôpital the limit is equal to

$$\lim_{x \to 0} \frac{\frac{a}{2\sqrt{ax+4}}}{1} = \frac{a}{4}$$

and so we need to set $a = 16$.

30/2. (3 points) Find numbers a and b such that $\lim\limits_{x \to 0} \dfrac{\sqrt{ax+b}-3}{x} = 4$

   A. ★ $a = 24$ and $b = 9$

   B. $a = 24$ and $b = 24$

   C. $a = 9$ and $b = 9$

   D. $a = 9$ and $b = 24$

   E. impossible

   Solution. Since the denominator is going to zero, we need the numerator to go to zero as well, hence $b = 9$. Applying L'Hôpital the limit is equal to

$$\lim_{x \to 0} \frac{\frac{a}{2\sqrt{ax+9}}}{1} = \frac{a}{6}$$

and so we need to set $a = 24$.

31/1. (3 points) If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, what is the smallest possible value of $f(4)$?

A. ★ 16

B. 10

C. 4

D. 18

E. Impossible to say

Solution. We have

$$f(4) = f(1) + \int_1^4 f'(x)\,dx \geq 10 + \int_1^4 2\,dx = 16.$$

31/2. (3 points) If $f(1) = 10$ and $f'(x) \geq 3$ for $1 \leq x \leq 4$, what is the smallest possible value of $f(4)$?

A. ★ 19

B. 10

C. 1

D. 13

E. Impossible to say

Solution. We have

$$f(4) = f(1) + \int_1^4 f'(x)\,dx \geq 10 + \int_1^4 3\,dx = 19.$$

32/1. (3 points) Find the area of the largest rectangle that can be inscribed in the ellipse

$$x^2/9 + y^2/25 = 1$$

A. ★ 30

B. 25

C. 9

D. 34

E. 225

Solution. For each point (x, y) on the ellipse in the first quadrant, there is an inscribed rectangle with this point as a vertex, and area $4xy$. Parametrize these points by

$$x(t) = 3\cos t, \ y(t) = 5\sin t, \quad 0 \le t \le \frac{\pi}{2}.$$

The optimal value of $A = 60\cos(t)\sin(t) = 30\sin(2t)$ is 30 achieved at $t = \frac{\pi}{4}$.

32/2. (3 points) Find the area of the largest rectangle that can be inscribed in the ellipse

$$x^2/4 + y^2/16 = 1$$

A. ★ 16

B. 4

C. 9

D. 32

E. 64

Solution. For each point (x, y) on the ellipse in the first quadrant, there is an inscribed rectangle with this point as a vertex, and area $4xy$. Parametrize these points by

$$x(t) = 2\cos t, \ y(t) = 4\sin t, \quad 0 \le t \le \frac{\pi}{2}.$$

The optimal value of $A = 32\cos(t)\sin(t) = 16\sin(2t)$ is 16 achieved at $t = \frac{\pi}{4}$.

Zone 4

33/1. (3 points) Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

If so indicate approximately how many terms are required to estimate the sum to within 10^{-4}

A. ★ Converges, $N = 10$

B. Converges, $N = 100$

C. Converges, $N = 5$

D. Converges, $N = 1,414$

E. Diverges

Solution.

33/2. (3 points) Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

If so indicate approximately how many terms are required to estimate the sum to within 10^{-4}

A. ★ Converges, $N = 100$

B. Converges, $N = 10$

C. Converges, $N = 5$

D. Converges, $N = 1,414$

E. Diverges

Solution.

34/1. (3 points) Compute the first two nonzero terms in the Taylor expansion at $a = 0$ of $f(x) = \sin(e^x - 1)$

A. ★ $x + \frac{x^2}{2}$

B. $1 - \frac{x^2}{2}$

C. $1 + x$

D. $x - \frac{x^3}{6}$

E. $x^2 - \frac{x^4}{24}$

Solution. The first few terms in the expansion of $e^x - 1$ and $\sin x$ are

$$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \mathcal{O}(x^4), \quad \sin x = x - \frac{x^3}{6} + \mathcal{O}(x^5)$$

hence

$$\sin(e^x - 1) = \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \mathcal{O}(x^4) \right) - \frac{\left(x + \frac{x^2}{2} + \mathcal{O}(x^3) \right)^3}{6} + \mathcal{O}(x^5) = x + \frac{x^2}{2} + \mathcal{O}(x^3).$$

34/2. (3 points) Compute the first two nonzero terms in the Taylor expansion at $a = 0$ of $f(x) = \cos(e^x - 1)$

A. ★ $1 - \frac{x^2}{2}$

B. $x + \frac{x^2}{2}$

C. $1 + x$

D. $x - \frac{x^3}{6}$

E. $x^2 - \frac{x^4}{24}$

Solution. The first few terms in the expansion of $e^x - 1$ and $\cos x$ are

$$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \mathcal{O}(x^4), \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \mathcal{O}(x^6)$$

hence

$$\cos(e^x - 1) = 1 - \frac{\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \mathcal{O}(x^4) \right)^2}{2} + \mathcal{O}(x^4) = 1 - \frac{x^2}{2} + \mathcal{O}(x^3).$$