Math 415. Exam 2. March 14, 2019

Full Name: ____________________

Net ID: ________________________

Discussion Section: ___________

- Do not turn this page until instructed to.
- There are 20 problems worth 5 points each.
- Each question has only one correct answer. You can choose up to two answers. If you choose just one answer, then you will get 5 points if the answer is correct, and 0 points otherwise. However, if you choose two answers, you will get 2.5 points if one of the answers is correct, and 0 points otherwise.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 75 minute exam. There are several different versions of this exam.
- Fill in the answers on the scantron form provided, **and** circle your answers on the exam itself. Hand in both the exam and the scantron.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- If you have to erase something on the scantron, please make sure to do so thoroughly.
- Good luck! Have a nice Spring break!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**! On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) Consider the subspace $W := \{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \in \mathbb{R}\}$ of $\mathbb{R}^3$. Which of the following sets is a basis of W?

(A) None of the other answers.

(B) $\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$

(C) $\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \}$

(D) $\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \}$

(E) $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$

2. (5 points) Let $\mathcal{B} = \big(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\big)$ be an orthonormal basis of $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Suppose we know that,

$$(T(\mathbf{v}_1)) \cdot \mathbf{v}_1 = 1, \qquad (T(\mathbf{v}_1)) \cdot \mathbf{v}_2 = 2, \qquad (T(\mathbf{v}_1)) \cdot \mathbf{v}_3 = 3.$$

What is the first column of the matrix $T_{\mathcal{B}\mathcal{B}}$?

(A) None of the other answers.

(B) Not enough information to determine the first column.

(C) $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

(D) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(E) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

3. (5 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Which of the following is a basis of Nul(A)?

(A) $\left\{ \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$

(B) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

(C) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

(D) None of the other answers

(E) $\left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

4. (5 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

What are the dimensions of the null space and the row space of A?

(A) $\dim \mathrm{Nul}(A) = 3, \dim \mathrm{Col}(A^T) = 0$

(B) None of the other answers.

(C) $\dim \mathrm{Nul}(A) = 2, \dim \mathrm{Col}(A^T) = 0$

(D) $\dim \mathrm{Nul}(A) = 3, \dim \mathrm{Col}(A^T) = 2$

(E) $\dim \mathrm{Nul}(A) = 2, \dim \mathrm{Col}(A^T) = 3$

5. (5 points) Let A be a 6×6 matrix and let $\mathbf{x}$ be a vector in $\mathbb{R}^6$. Which of the following statements is true?

(A) The vectors $\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, A^4\mathbf{x}, A^5\mathbf{x}, A^6\mathbf{x}$ are linear independent.

(B) The linear independence/dependence of $\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, A^4\mathbf{x}, A^5\mathbf{x}, A^6\mathbf{x}$ can not be determined from the given data.

(C) The vectors $\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, A^4\mathbf{x}, A^5\mathbf{x}, A^6\mathbf{x}$ are linear dependent.

(D) None of the other answers.

6. (5 points) Let A be a 10×4 matrix with $\dim \mathrm{Nul}(A^T) = 6$. Which of the following statements is NOT true?

(A) The nullspace of A has dimension 6.

(B) The matrix A has rank 4.

(C) The equation $A\mathbf{x} = 0$ has a unique solution.

(D) The columns of A are linearly independent.

(E) The row space of A has dimension 4.

7. (5 points) Consider the polynomial $p(t) = 4t^2 - 6t - 1$. Which of the following ordered sets of vectors is an ordered basis $\mathcal{B}$ for $\mathbb{P}_2$ such that the coordinate vector for $p(t)$ with respect to that basis is $p(t)_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ -8 \end{bmatrix}$?

(A) $\left(2t^2, 6, 1\right)$

(B) None of the other answers.

(C) $\left(2t^2 + t, t, 1\right)$

(D) $\left(2t^2, 6, 0\right)$

(E) $\left(2t^2 + t, 1, t\right)$

8. (5 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ y \\ 2x - y \end{bmatrix}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be the ordered bases $\mathcal{A} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ and $\mathcal{B} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$ of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Compute the coordinate matrix $T_{\mathcal{B}\mathcal{A}}$.

(A) $\begin{bmatrix} 0 & -2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$

(E) None of the other answers

9. (5 points) Recall that $M_{2\times 2}$ is the vector space of 2×2-matrices. Let T be the linear transformation from $M_{2\times 2}$ to $M_{2\times 2}$ that performs the row operation $R2 \to 2R2$. That is, $T\left(\begin{bmatrix} R1 \\ R2 \end{bmatrix}\right) = \begin{bmatrix} R1 \\ 2R2 \end{bmatrix}$. Let $\mathcal{A}$ be the ordered basis $\mathcal{A} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$ of $M_{2\times 2}$. Which one of the following equals $T_{\mathcal{A}\mathcal{A}}$?

(A) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(B) None of the other answers

(C) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(E) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

10. (5 points) Let V be a vector space, and let A, B be two $m \times n$-matrices. Consider the following two statements:

(I) If B is an echelon form of A, then the pivot columns of B form a basis of the column space of A.

(II) If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and S is a set of more than p vectors in V, then S is linearly dependent.

Which of the two statements is always true?

(A) Both statement (I) and statement (II) are true.

(B) Neither statement (I) nor statement (II) is true.

(C) Only statement (I) is true.

(D) Only statement (II) is true.

11. (5 points) What is the smallest possible dimension of $\text{Nul}(A)$ for a 9×14 matrix A?

(A) 14

(B) None of the other answers.

(C) 5

(D) 0

(E) 9

12. (5 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Which of the following subsets of $\mathbb{P}_2$ is linearly ***dependent***?

(A) $\{t^2 + t, 1 + t, 1\}$

(B) $\{1, t, t^2\}$

(C) $\{1 + t, 1 - t, t^2\}$

(D) None of the other answers.

(E) $\{t^2 + t, 1 + t, t^2 + 2t + 1\}$

13. (5 points) Let A, B be two $m \times n$-matrices that are row equivalent. Consider the following statements:

I. $\dim(\mathrm{Nul}(A)) = \dim(\mathrm{Nul}(B))$.

II. $\dim(\mathrm{Col}(A)) = \dim(\mathrm{Col}(B))$.

Which one of these statements is always true?

(A) Statement I and Statement II.

(B) Statement II only.

(C) Neither of Statements I or II.

(D) Statement I only.

14. (5 points) Consider the following ordered basis $\mathcal{B}$ of $\mathbb{R}^3$:

$$\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right).$$

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. What is the coordinate vector $\mathbf{x}_{\mathcal{B}}$ of $\mathbf{x}$ with respect to the basis $\mathcal{B}$?

(A) None of the other answers

(B) $\begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(D) $\begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix}$

15. (5 points) Consider the subspace $V = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^4$. Which of the following is a basis for the orthogonal complement $V^\perp$?

(A) $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(B) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(C) None of the other answers.

(D) $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

(E) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

16. (5 points) Recall that $M_{2\times 2}$ is the vector space of 2×2-matrices. Consider the following subspaces of $M_{2\times 2}$:

$$V = \{A \in M_{2\times 2} \mid A^T = -A\} \qquad W = \{B \in M_{2\times 2} \mid B \text{ is diagonal}\}.$$

What are the dimensions of V and W?

(A) $\dim V = 3$ and $\dim W = 2$

(B) $\dim V = 1$ and $\dim W = 2$

(C) $\dim V = 4$ and $\dim W = 4$

(D) $\dim V = 2$ and $\dim W = 2$

(E) None of the other answers

17. (5 points) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree up to 2. Let $T : \mathbb{R}^2 \to \mathbb{P}_2$ be a linear transformation such that $T\left(\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) = t - 1$ and $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = t^2 - 1$. What is $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$?

(A) None of the other answers

(B) $3t + 2$

(C) $3t^2 - t - 2$

(D) $-3t^2 + t - 2$

(E) $3t^2 + 2$

18. (5 points) Suppose A is a 3×6 matrix

$$A = \begin{bmatrix} 1 & * & * & 0 & * & * \\ 0 & * & * & 1 & * & * \\ 1 & * & * & -2 & * & * \end{bmatrix}$$

whose column space is

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

Which of the following is a basis for $\mathrm{Nul}(A^T)$?

(A) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

(B) There is not enough information to determine the answer.

(C) None of the other answers.

(D) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

(E) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

19. (5 points) Let V be a vector space of dimension at least 3. Let $\mathcal{B} = \big(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\big)$, $\mathcal{C} = \big(\mathbf{c}_1, \mathbf{c}_2, \ldots \mathbf{c}_n\big)$ be two different ordered bases of V. Which of the following statements are always true?

(I) If the second entry of $\mathbf{v}_{\mathcal{B}}$ is zero, then at least one entry of $\mathbf{v}_{\mathcal{C}}$ is zero.

(II) If $\mathbf{v}_{\mathcal{B}}$ is not the zero vector, then $\mathbf{v}_{\mathcal{C}}$ is not the zero vector.

(III) If $\mathbf{c}_1 = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, then $\{\mathbf{c_1}, \mathbf{b_2}, \mathbf{b_3}, \ldots, \mathbf{b_n}\}$ is a basis for V. That is, if $\mathbf{c}_1 = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, then substituting $\mathbf{c_1}$ for $\mathbf{b_1}$ in $\mathcal{B}$ creates a basis for V.

(A) All three statements.

(B) Only statements (I) and (II).

(C) Only statements (I) and (III).

(D) None of the other answers.

(E) Only statements (II) and (III).

20. (5 points) Let A be a 7×5 matrix and B be a 5×6 matrix such that the rank of AB is 5. What is the rank of A?

(A) None of the other answers.

(B) 5

(C) 7

(D) 8

(E) Not enough information to determine the rank of A.