Math 231E, Lecture 25.
Series

1 Infinite sums

When we say

$$\pi = 3.1415926535\cdots$$

what do we mean? What do the dots really mean here? How do we know this makes sense?

Definition 1.1. Given a sequence $\{a_k\}$, let us define the n**th partial sum**

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_{n-1} + a_n.$$

If the limit $\lim_{n\to\infty} s_n$ exists and is finite, then we say that the infinite sum of the a_k **converges**, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

Example 1.2. So, for example, for π, we write the sequence

$$a_0 = 3, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 1, \quad a_4 = 5, \ldots$$

and generally a_k is the kth digit after the decimal point in the decimal expansion for π. Define

$$b_k = \frac{a_k}{10^k}.$$

Then

$$\pi = \sum_{k=1}^{\infty} b_k.$$

(Of course, we still need to show that this makes sense!!)

2 Geometric Series

Definition 2.1. The **geometric series with ratio** r **and initial term** a is the sum

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \ldots$$

Theorem 2.2. The geometric series converges if and only if $|r| < 1$.

Proof. We compute

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} + ar^n,$$

and

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}.$$

Notice that if we subtract the second equation from the first, a lot of terms cancel, so we have

$$\begin{aligned} s_n - rs_n &= a - ar^{n+1} \\ (1-r)s_n &= a - ar^{n+1} \\ s_n &= \frac{a - ar^{n+1}}{1-r}, \qquad (r \neq 1). \end{aligned}$$

Then we have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{a - ar^{n+1}}{1-r} \right) = \frac{a}{1-r} \lim_{n\to\infty} \left(1 - r^{n+1}\right).$$

If $|r| > 1$, then this limit cannot be finite. If $r = -1$, then the sequence $(1 - r^{n+1})$ is

$$0, 2, 0, 2, 0, 2, \ldots$$

and the limit does not exist.

Now, if $|r| < 1$, then

$$\lim_{n\to\infty} \left| r^{n+1} \right| = \lim_{n\to\infty} |r|^{n+1} = 0,$$

so

$$\lim_{n\to\infty} \left(1 - r^{n+1}\right) = 1.$$

Therefore, if $|r| < 1$, then

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}.$$

Finally, we have to consider the case $r = 1$ (since we had to exclude it earlier). But if $r = 1$, then

$$s_n = \sum_{k=1}^{n} a = an, \tag{1}$$

and $\lim_{n\to\infty} s_n = \infty$.

□

Example 2.3. Consider the infinite sum

$$3 - 2 + \frac{4}{3} - \frac{8}{9} + \frac{16}{27} - \frac{32}{81} + \ldots$$

This is a geometric sum with ratio $r = 2/3$ and initial term $a = 3$. Since $|r| < 1$, this sum converges, and in fact it converges to

$$\frac{a}{1-r} = \frac{3}{1 - 2/3} = 9.$$

However, make sure you only apply the formula if you know the sum converges!

Example 2.4. Consider the sum

$$1 + 2 + 4 + 8 + 16 + 32 + \cdots$$

This is a geometric sum with ratio $r = 2$ and initial term $a = 1$. The formula then tells us that

$$1 + 2 + 4 + 8 + 16 + 32 + \cdots = \frac{1}{1-2} = -1.$$

But, no. Just, no.

3 Some other series

Example 3.1. Consider

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

We can write

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1},$$

Let us consider the partial sums

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Thus we have

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} s_n = 1.$$

One very important series is the **harmonic series**:

Example 3.2. Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

Note that we have

$$s_1 = 1, \quad s_2 = 1 + \frac{1}{2} = \frac{3}{2}.$$

Now note

$$a_3 + a_4 = \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Therefore $s_4 > 1 + 1/2 + 1/2 = 2$. Now see that

$$a_5 + a_6 + a_7 + a_8 = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 4\frac{1}{8} = \frac{1}{2}.$$

So $s_8 > 1 + 1/2 + 1/2 + 1/2 = 5/2$. Contining:

$$a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} = \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} > 8\frac{1}{16} = \frac{1}{2}.$$

So $s_{16} > 1 + 1/2 + 1/2 + 1/2 + 1/2 = 3$. More generally,

$$\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} > \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}} = (2^{n+1} - 2^n)\frac{1}{2^{n+1}} = 2^n \frac{1}{2^{n+1}} = 1/2.$$

Therefore

$$\sum_{k=1}^{2^{n+1}} > 1 + \frac{n}{2}.$$

Therefore

$$\lim_{n\to\infty} s_{2^n} > \lim_{n\to\infty} \left(1 + \frac{n-1}{2}\right) = \infty.$$

Therefore the harmonic series diverges.

4 Terms in a convergent series, and a warning

First of all, we have the following theorem:

Theorem 4.1. If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. Let us say that $\sum_{k=1}^{\infty} a_k = L$. Then we have

$$a_n = s_{n+1} - s_n,$$

so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_{n+1} - s_n) = \lim_{n\to\infty} s_{n+1} - \lim_{n\to\infty} s_n = L - L = 0.$$

$\square$

From this, we also see that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

So, for example, we know that

$$\sum_{k=1}^{\infty} \frac{2k^2 + 2}{k^2 + k + 1}$$

diverges, since the limit of each sum is 2, not 0.

Warning! If we try to reverse the statement in the above theorem, it does not work. We might like to say "If $a_n \to 0$, then $\sum_{k=1}^{\infty} a_k$ converges." But this is false! For example, consider the harmonic series. The terms go to zero, but the sum does not converge.

5 Properties

We have all the standard properties when series are **convergent**:

1. If $\sum a_n$ and $\sum b_n$ both converge, then

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

2. If $\sum a_n$ and $\sum b_n$ both converge, then

$$\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n.$$

3. If $\sum a_n$ converges, then for any constant c,

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$