Math 231E, Lecture 28.
Absolute Convergence; Root and Ratio Tests

1 Absolute and Conditional Convergence

Definition 1.1. Let a_n be a sequence, and $\sum_{n=1}^{\infty} a_n$ the corresponding series. We say the series **converges absolutely** if

$$\sum_{n=1}^{\infty} |a_n|$$

converges. If a series is convergent but not absolutely convergent, we say it is **conditionally convergent**.

Example 1.2. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6}.$$

Here $a_n = (-1)^n/n^6$ and $|a_n| = 1/n^6$. Since $1/n^6$ is a convergent series (by the p-test), then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Example 1.3. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Here $a_n = (-1)^n/n$ and $|a_n| = 1/n$. First notice that $\sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test, but $\sum_{n=1}^{\infty} |a_n|$ diverges by the p-test. Therefore the $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Theorem 1.4. If a series is absolutely convergent, then it is convergent.

Proof. Consider the series $\sum_{n=1}^{\infty} a_n$, and assume it is absolutely convergent, i.e. $\sum_{n=1}^{\infty} |a_n|$ is convergent. Notice that

$$\begin{aligned} -|a_n| \leq a_n \leq |a_n|, \\ 0 \leq a_n + |a_n| \leq 2|a_n|. \end{aligned}$$

Note that $\sum_{n=1}^{\infty} 2|a_n|$ is convergent by assumption, and since $\sum_{n=1}^{\infty} a_n + |a_n|$ is positive, then by the Comparison Theorem we have that $\sum_{n=1}^{\infty} a_n + |a_n|$ is convergent. Then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n + |a_n| - \sum_{n=1}^{\infty} |a_n|$$

is the difference of two convergent series and is itself convergent. □

tl;dr: Minus signs can only help convergence.

Example 1.5. Now let us consider

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

Since the terms are not positive, we cannot use the Comparison Theorem directly. But notice that if

$$a_n = \frac{\sin n}{n^2} \tag{1}$$

then

$$|a_n| = \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}.$$

This means that

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$$

is convergent by comparison and the p-test, which means that

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

is absolutely convergent, and thus convergent.

2 Root Test

Theorem 2.1 (Root Test)**.** Let a_n be a sequence and $\sum_{n=1}^{\infty} a_n$ be the associated series. Let us define

$$b_n = \sqrt[n]{|a_n|} = |a_n|^{1/n},$$

and assume that $\lim_{n\to\infty} b_n = L$. Then

1. if $L < 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely;
2. if $L > 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges;
3. if $L = 1$, then no information is obtained.

Proof. Let us say that

$$\lim_{n\to\infty} |a_n|^{1/n} = L$$

and $L < 1$. Choose L' with $L < L' < 1$. Then there is an $N > 0$ such that for $n > N$,

$$|a_n|^{1/n} < L',$$

or

$$|a_n| < (L')^n$$

Since $L' < 1$, then

$$\sum_{n=N}^{\infty} |a_n|$$

converges, since it is a convergent geometric series. Therefore $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Now, assume that $L > 1$, and choose $1 < L' < L$. Similarly, there is an $N > 0$ such that for $n > N$,

$$a_n^{1/n} > L',$$

or

$$a_n > (L')^n$$

Since $L' > 1$, then

$$\lim_{n\to\infty} a_n \neq 0,$$

so $\sum_{n=1}^{\infty} a_n$ diverges.

□

Example 2.2. Consider the series

$$\sum_{n=1}^{\infty} \left(\frac{2n+5}{5n+7}\right)^n.$$

We have

$$|a_n|^{1/n} = \frac{2n+5}{5n+7},$$

and

$$\lim_{n\to\infty} |a_n|^{1/n} = \frac{2}{5} < 1,$$

so the series converges.

3 Ratio Test

Theorem 3.1. Let a_n be a sequence and $\sum_{n=1}^{\infty} a_n$ be the associated series. Let us assume that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L.$$

Then

1. if $L < 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely;
2. if $L > 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges;
3. if $L = 1$, then no information is obtained.

Proof. Similar to proof of Root Test. □

Example 3.2. Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^{64}}{3^n}.$$

We compute:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{64}}{3^{n+1}} \cdot \frac{3^n}{n^{64}} = \frac{1}{3}\left(\frac{n+1}{n}\right)^{64}.$$

Since

$$\lim_{n\to\infty} \frac{n+1}{n} = 1,$$

so

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{64} = 1^{64} = 1,$$

and therefore

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}.$$

Since $1/3 < 1$, this series converges absolutely.

Example 3.3. Now consider

$$\sum_{n=1}^{\infty} \frac{n^{6000}}{(1.1)^n}. \tag{2}$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{6000}}{(1.1)^{n+1}} \cdot \frac{(1.1)^n}{n^{6000}} = \frac{1}{1.1}\left(\frac{n+1}{n}\right)^{6000},$$

and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{1.1} < 1, \tag{3}$$

so the series converges absolutely.

Example 3.4. Now consider

$$\sum_{n=1}^{\infty} \frac{2^n}{n^6}. \tag{4}$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^6} \cdot \frac{n^6}{2^n} = 2\left(\frac{n}{n+1}\right)^6$$

Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2 > 1, \tag{5}$$

so the series diverges.

Example 3.5. Consider

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = (n+1)\frac{n^n}{(n+1)^n(n+1)} = \left(\frac{n}{n+1}\right)^n.$$

Now we have to compute

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n.$$

Note that as $n \to \infty$, the term on the inside goes to 1, but the power gets large. So it's not obvious where this limit goes. Let us write

$$b_n = \left(\frac{n}{n+1}\right)^n.$$

Then

$$\ln|b_n| = n\ln\left|\frac{n}{n+1}\right| = \frac{\ln\left|\frac{n}{n+1}\right|}{1/n}$$

We can use l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{\ln\left|\dfrac{n}{n+1}\right|}{1/n} = \lim_{n\to\infty} \frac{\dfrac{n+1}{n}\cdot\dfrac{1}{(n+1)^2}}{-\dfrac{1}{n^2}} = -1,$$

so

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} e^{\ln(b_n)} = e^{\lim_{n\to\infty}\ln(b_n)} = e^{-1}.$$

Since $e^{-1} < 1$, then

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

converges absolutely.