<u>Goals</u> : • learn how to add & subtract vectors and multiply a vector by a number
- geometrically
- algebraically

• discuss main properties of vectors

• introduce vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ and the notion "unit vector"

12.2 Vectors and the Geometry of Space

scalar : magnitude

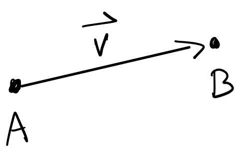
vector : magnitude + direction (e.g. force, velocity, ...)

"arrows where length and direction are important"

notation : $\vec{u}, \vec{v}, \vec{w}, \vec{a}, \vec{b}, \ldots$

$\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{a}, \mathbf{b}, \ldots$ bold (used in our book)

$\vec{u} = \langle 5, 3 \rangle$, $\vec{u} = 5\hat{\imath} + 3\hat{\jmath}$



$\vec{v}$ displacement vector from A to B

$\vec{v} = \overrightarrow{AB}$

A: initial point (tail)

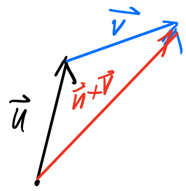
B: terminal point (tip/head)

Two vectors are equal if they are the same up to translation (starting point doesn't matter, only length and direction)

Combining vectors

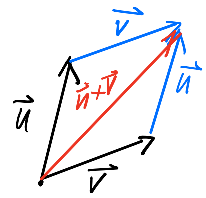
- **SUM** $\vec{u}+\vec{v}$

"tip-to-tail"
move one vector so that its tail lies on the tip of the first vector



Triangle Law

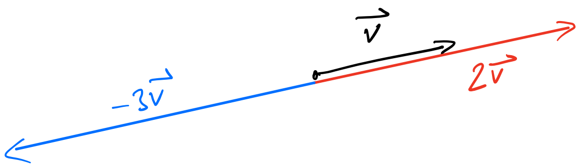
or



Parallelogram Law

$$\boxed{\vec{u}+\vec{v}=\vec{v}+\vec{u}}$$

- **scalar multiple** $k\cdot\vec{v}$



length of $k\vec{v} = |k|\cdot|\vec{v}|$
direction of $k\vec{v}$ = same as $\vec{v}$ if $k>0$
opposite to $\vec{v}$ if $k<0$

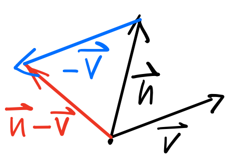
real numbers work like scaling factors
$\Longrightarrow$ thus the name 'scalars'

$-\vec{v} = (-1)\vec{v}$
the negative of $\vec{v}$

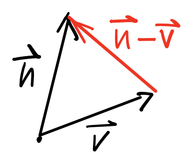
$0\cdot\vec{v} = \vec{0}$ the zero vector
(the only vector with no direction)

- **DIFFERENCE** $\vec{u}-\vec{v} = \vec{u}+(-1)\vec{v} = \vec{u}+(-\vec{v})$

tip-to-tail:

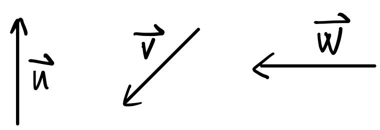


OR

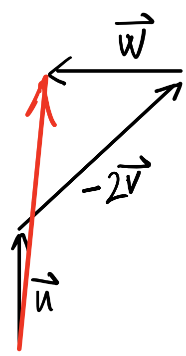


because:
$\vec{v}+(\vec{u}-\vec{v})=\vec{u}$
(Triangle law)

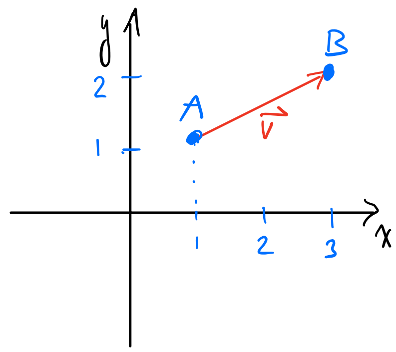
EX:



Sketch $\vec{u}-2\vec{v}+\vec{w}$



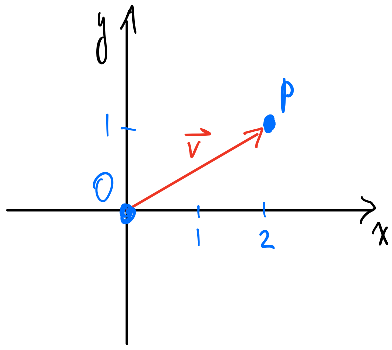
Vectors in $\mathbb{R}^2$



$\vec{v} = \vec{AB} = \langle 3-1, 2-1 \rangle = \langle 2, 1 \rangle$ — components of $\vec{v}$

$\vec{v}$: vector represented by the directed line segment from $A=(1,1)$ to $B=(3,2)$

We can move any vector to "standard position" (translate so that the initial point is at the origin)



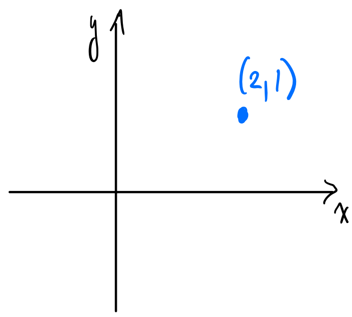
$\vec{v} = \vec{OP} = \langle 2-0, 1-0 \rangle = \langle 2,1 \rangle$ ← Component form

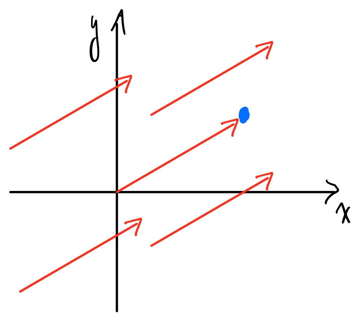
$\vec{v}$ = position vector of the point P

Points vs Vectors

- points and vectors can be both represented by a pair of numbers

BUT these are different concepts : absolute vs relative position





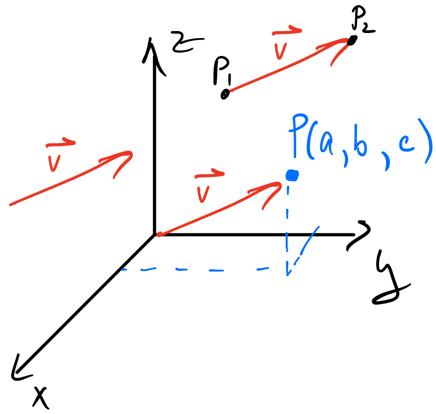
all these red vectors are representations of vector $\vec{v} = \langle 2,1 \rangle$

Point : (a, b) — coordinates

Vector $\langle a, b \rangle$ — components

The length of a vector $\vec{v} = \langle a, b \rangle$ is $|\vec{v}| = \sqrt{a^2+b^2}$

Vectors in $\mathbb{R}^3$



"think of as arrows in 3-space"

$$\vec{v} = \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \qquad P_1 = (x_1, y_1, z_1), \quad P_2 = (x_2, y_2, z_2)$$

$$\vec{v} = \overrightarrow{OP} = \langle a, b, c \rangle$$

$\vec{v}$ = position vector of the point P

Again, there are infinitely many representations of $\vec{v}$ in $\mathbb{R}^3$.

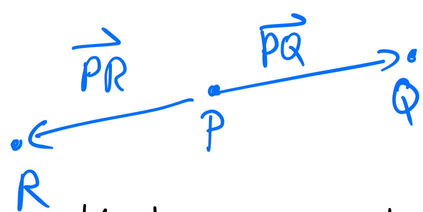
Point : (a, b, c) ← coordinates

Vector $\langle a, b, c \rangle$ ← components

The length of a vector $\vec{v} = \langle a, b, c \rangle$ is $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$

If $\vec{v} = \overrightarrow{P_1P_2}$, then $|\vec{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Ex: $P(4,-1,-3)$, $Q(8,0,-5)$, $R(0,-2,-1)$ } Do these points lie on the same line?



$$\overrightarrow{PQ} = \langle 8-4, 0-(-1), -5-(-3) \rangle = \langle 4, 1, -2 \rangle$$

$$\overrightarrow{PR} = \langle 0-4, -2-(-1), -1-(-3) \rangle = \langle -4, -1, 2 \rangle = -\overrightarrow{PQ}$$

$\Rightarrow$ Yes! On the same line.

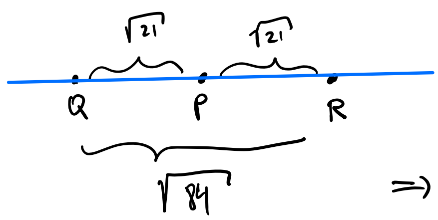
OR, using distances :

Is the longest distance is equal to the sum of the shortest two?

$|PQ| = \sqrt{21}$

$|PR| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$

$|QR| = \sqrt{8^2 + 2^2 + 4^2} = \sqrt{84}$



$\sqrt{21} + \sqrt{21} = 2\sqrt{21} = \sqrt{4 \cdot 21} = \sqrt{84}$

Yes. $|PQ| + |PR| = |QR|$

$\Rightarrow$ P, Q, R lie on a straight line

$\boxed{\vec{u}+\vec{v}}$

To add (subtract) 2 vectors algebraically, add (subtract) their corresponding components

EX: $\vec{u} = \langle 2, 8, -2 \rangle$, $\vec{v} = \langle -1, 3, 0 \rangle$

$$\vec{u}+\vec{v} = \langle 2+(-1), 8+3, -2+0 \rangle = \langle 1, 11, -2 \rangle$$

$$\vec{u}-\vec{v} = \langle 2-(-1), 8-3, -2-0 \rangle = \langle 3, 5, -2 \rangle$$

$\boxed{k \cdot \vec{v}}$

To multiply a vector by a scalar, multiply each component by that scalar

$$\vec{v} = \langle -1, 3, 0 \rangle$$

$$5 \cdot \vec{v} = \langle 5 \cdot (-1), 5 \cdot 3, 5 \cdot 0 \rangle = \langle -5, 15, 0 \rangle$$

Properties of vectors — works in any #dimensions!

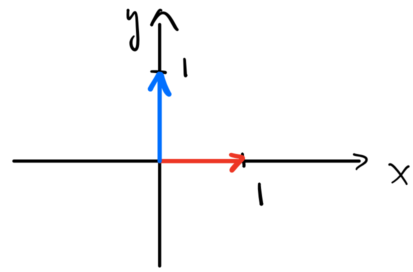
1. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (commutative)
2. $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ (associative)
3. $\vec{v}+\vec{0} = \vec{v}$
4. $\vec{v}+(-\vec{v}) = \vec{0}$
5. $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$
6. $(c+d)\cdot\vec{v} = c\vec{v}+d\vec{v}$
7. $(cd)\cdot\vec{v} = c(d\vec{v})$
8. $\underset{\substack{\uparrow \\ \text{scalar}}}{1}\vec{v} = \vec{v}$

think about each of these, do they make sense:
- geometrically?
- algebraically?

V_n = set of all n-dim. vectors $\langle v_1, v_2, \dots, v_n \rangle$

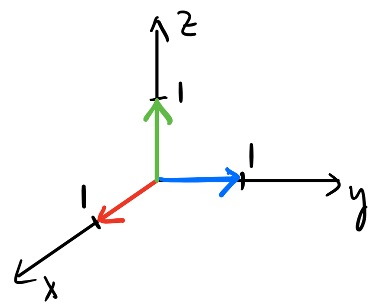
V_2 = set of all vectors $\langle v_1, v_2 \rangle$

standard basis vectors : $\boxed{\begin{aligned} \hat{\imath} &= \langle 1, 0 \rangle \\ \hat{\jmath} &= \langle 0, 1 \rangle \end{aligned}}$



V_3 = set of all vectors $\langle v_1, v_2, v_3 \rangle$

standard basis vectors : $\boxed{\begin{aligned} \hat{\imath} &= \langle 1, 0, 0 \rangle \\ \hat{\jmath} &= \langle 0, 1, 0 \rangle \\ \hat{k} &= \langle 0, 0, 1 \rangle \end{aligned}}$



Ex: Express the vector $\vec{a} = \langle 2, 3, 4 \rangle$ in terms of $\hat{\imath}, \hat{\jmath}, \hat{k}$

$$\vec{a} = \langle 2,0,0 \rangle + \langle 0,3,0 \rangle + \langle 0,0,4 \rangle = 2\langle 1,0,0 \rangle + 3\langle 0,1,0 \rangle + 4\langle 0,0,1 \rangle$$
$$= 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$$

A unit vector = a vector of length 1

$\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vectors

Ex: Find the unit vector that has the same direction as $\vec{b} = 4\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

$$|\vec{b}| = \sqrt{4^2 + (-2)^2 + 3^2} = \sqrt{16 + 4 + 9} = \sqrt{29}$$

unit vector : $$\frac{\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{29}}\left(4\hat{\imath} - 2\hat{\jmath} + 3\hat{k}\right) = \frac{4}{\sqrt{29}}\hat{\imath} - \frac{2}{\sqrt{29}}\hat{\jmath} + \frac{3}{\sqrt{29}}\hat{k}$$

$$= \frac{1}{\sqrt{29}}\langle 4, -2, 3 \rangle$$

EX: $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{a} + \vec{b} = 6\hat{i} + \hat{j} + 7\hat{k}$

On WebAssign, you cannot just type i, j, k. Instead, use CalcPad → Vectors to get bolded vectors **i**, **j**, **k**.

OR: use angle bracket notation $\langle 1, 2, 3 \rangle$

Multiplying vectors

DOT PRODUCT (Next time!)

SCALAR

$$\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = \underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3}_{}$$

CROSS PRODUCT (on Wednesday)

$$\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle \quad , \quad , \quad \rangle$$

VECTOR