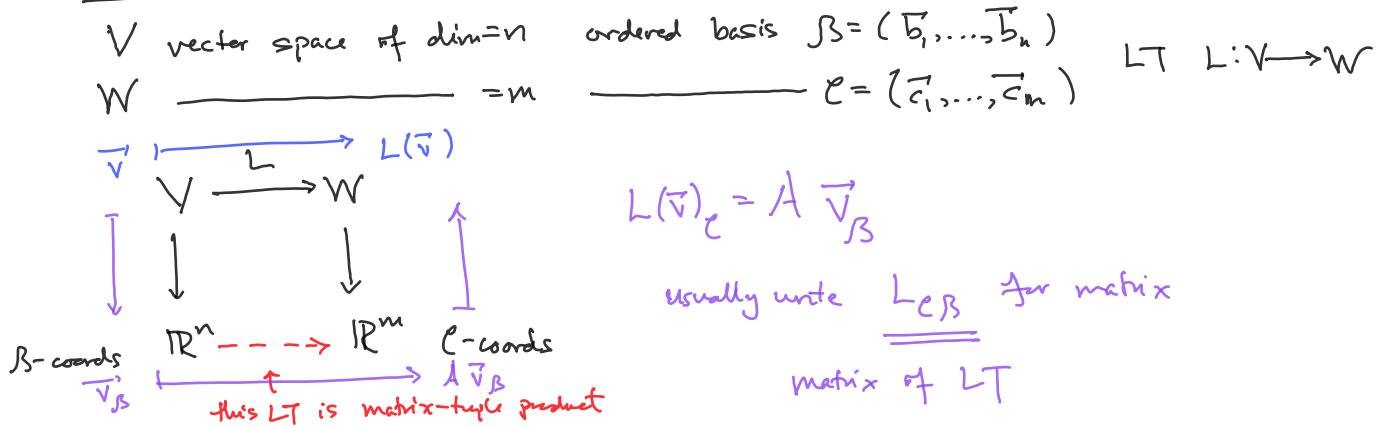


§ Lecture 14 dkml5p

last time: change of coordinates I_{AB} or I_{BA} , linear transformations (LT)

$$L(r\vec{x} + s\vec{y}) = rL(\vec{x}) + sL(\vec{y})$$

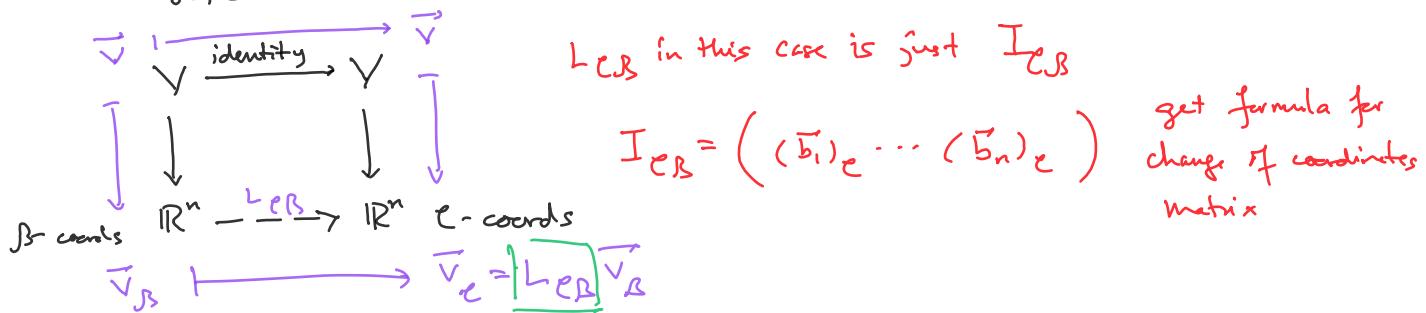
§ Matrix of LT



Fact: $L_{CB} = \begin{pmatrix} L(\vec{b}_1)_\gamma & \cdots & L(\vec{b}_n)_\gamma \end{pmatrix}_{m \times n}$

Example: LT identity function $V \rightarrow V$ $\vec{v} \mapsto \vec{v}$ $\dim(V) = n$

β, γ ordered bases on V



Example: derivative $D: P_2 \rightarrow P_1$, $A = (1, +, +^2)$ $\gamma = (1, t)$

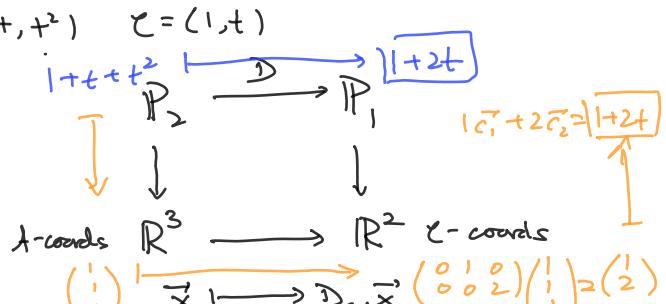
Compute: (1) D_{CA}

(2) $D(1+t+t^2)$ in two ways

$$(1) D_{CA} = \begin{pmatrix} D(\vec{a}_1)_\gamma & D(\vec{a}_2)_\gamma & D(\vec{a}_3)_\gamma \end{pmatrix}_{2 \times 3}$$

$$= \begin{pmatrix} 0_\gamma & 1_\gamma & (2t)_\gamma \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$



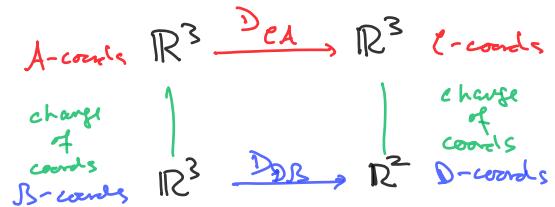
$$0 = \underline{0} 1 + \underline{0} t$$

$$1 = \underline{1} 1 + \underline{0} t$$

$$2t = \underline{0} 1 + \underline{2} t$$

Moral: coords make abstract concrete by translation into tuple-world
relative coords, each abstract LT is translated into matrix-tuple product

Example: $\beta = (1, 1+t, 1+t+t^2)$ for $\mathbb{P}_2 \Rightarrow D_{DB}$ Q: how compare D_{CA} & D_{DB}
 $\beta = (1, 1+t)$ for \mathbb{P}_1 .



Fact: $D_{DB} = I_{DC} D_{CA} I_{AB}$

Let's compute D_{DB} in two ways:

- (1) direct computation
- (2) leverage knowledge of D_{CA}

(1) direct computation

$$\begin{aligned} D_{DB} &= \left(D(\vec{b}_1)_D \quad D(\vec{b}_2)_D \quad D(\vec{b}_3)_D \right) & 0 &= \underline{0} 1 + \underline{0} (1+t) \\ &= \left(\begin{matrix} 0_D & 1_D & (1+2t)_D \end{matrix} \right) & 1 &= \underline{1} 1 + \underline{0} (1+t) \\ &= \left(\begin{matrix} 0 & 1 & -1 \\ 0 & 0 & 2 \end{matrix} \right) & 1+2t &= \underline{-1} 1 + \underline{2} (1+t) \end{aligned}$$

(2) leverage knowledge of D_{CA}

$$D_{DB} = I_{DC} D_{CA} I_{AB}$$

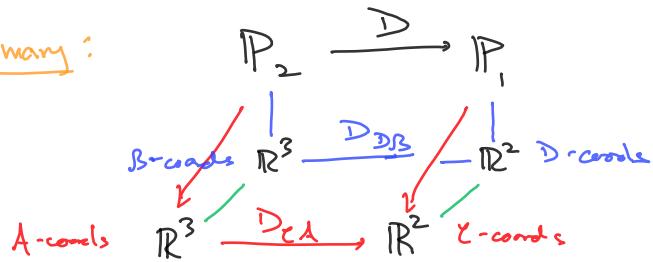
Let's find the change of coord. matrices

$$\begin{aligned} I_{AB} &= \left((\vec{b}_1)_A \quad (\vec{b}_2)_A \quad (\vec{b}_3)_A \right) & 1 &= \underline{1} 1 + \underline{0} t + \underline{0} t^2 \\ &= \left(\begin{matrix} 1_A & (1+t)_A & (1+t+t^2)_A \end{matrix} \right) & 1+t &= \underline{1} 1 + \underline{1} t + \underline{0} t^2 \\ &= \left(\begin{matrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \right) & 1+t+t^2 &= \underline{1} 1 + \underline{1} t + \underline{1} t^2 \end{aligned}$$

$$\begin{aligned} I_{DC} &= (I_{CD})^{-1} & I_{CD} &= \left((\vec{d}_1)_C \quad (\vec{d}_2)_C \right) & 1 &= \underline{1} 1 + \underline{0} t \\ &= \left(\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} \right)^{-1} & &= \left(1_C \quad (1+t)_C \right) & 1+t &= \underline{1} 1 + \underline{1} t \\ &= \left(\begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix} \right) & &= \left(\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} \right) \end{aligned}$$

$$D_{DB} = I_{DC} D_{CA} I_{AB} = \left(\begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix} \right) \left(\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix} \right) \left(\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \right) = \left(\begin{matrix} 0 & 1 & -2 \\ 0 & 0 & 2 \end{matrix} \right) \left(\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \right) = \boxed{\left(\begin{matrix} 0 & 1 & -1 \\ 0 & 0 & 2 \end{matrix} \right)}$$

Summary:



§ Determinants

Motivation: recall for 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible $\Leftrightarrow \det(A) = ad - bc \neq 0$

Goal: extend this idea for $n \times n$ matrices

One way to do this:

$$\text{set } \det(I_n) = 1$$

<u>Elem row op</u>	<u>Effect on det</u>
swap	$(-1) \times$
scalar ($c \neq 0$)	$(c) \times$
shear ($i \neq j$)	$(1) \times$

Example:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[\text{new } R_3]{\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[-R_3 + R_2]{\text{new } R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3 + R_1]{\text{new } R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\text{new } R_2]{\frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \times 1 \quad \times 1$$

$$\times \frac{1}{3}$$

$$| = \det(I) = (\times \frac{1}{3})(\times 1)(\times 1)(\times \frac{1}{2}) \quad \det(A) = \frac{1}{6} \det(I)$$

$$\Rightarrow \det(A) = 6$$

