Math 231E Engineering Calculus: Notes on Taylor series

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1 Taylor Polynomials and Series

1.1 Polynomials

Last time we discussed Taylor polynomials. Recall that the Taylor polynomial for the function $f(x)$ of order (degree) n at the point a is defined to be

$$T_n(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \ldots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

For instance

Example 1.1. *Find the Taylor polynomial of order 5 for the function $f(x) = \frac{1}{x}$ at the point $a = 1$.*

$$\begin{aligned}
f(x) &= \frac{1}{x} \qquad f(1) = 1 \\
f'(x) &= -\frac{1}{x^2} \qquad f'(1) = -1 \\
f''(x) &= \frac{2}{x^3} \qquad f''(1) = 2 \\
f'''(x) &= -\frac{6}{x^4} \qquad f'''(1) = -6 \\
f^{(4)}(x) &= \frac{24}{x^5} \qquad f^{(4)}(1) = 24 \\
f^{(5)}(x) &= -\frac{120}{x^6} \qquad f^{(5)}(1) = -120
\end{aligned}$$

which gives

$$\begin{aligned}
T_5(x) &= 1 - 1(x-1) + \frac{2}{2!}(x-2)^2 - \frac{6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4 - \frac{120}{5!}(x-1)^5 \\
&= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5
\end{aligned}$$

1.2 Series

For many functions, and many choices of a and x it happens that as $n \to \infty$ we have that $T_n(x) \to f(x)$. One of the big goals of this semester is to understand what it means for such a series to converge: what it really means to add up an infinite set of numbers. For the time being we will just be thinking of a Taylor series as a Taylor polynomial of unspecified degree. We will tend to write these using our Big O notation.

Example 1.2. *Some Taylor series for some common functions (for various choices of a)*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + O(x^{n+1})$$
$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \ldots + (-1)^n(x-1)^n + O((x-1)^{n+1})$$
$$\sin(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) - \frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^2}{2!} - \frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^3}{3!} + \frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^4}{4!} + \ldots \pm \frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^n}{n!} + O((x-\frac{\pi}{4})^{n+1})$$

The sign pattern in the last example is a bit tricky: the signs have the pattern $++--++--\ldots$ so the nth term has a plus sign if n is divisible by four or has a remainder of 1 when divided by four, and has a minus sign otherwise. The first series is about $a=0$, the second about $a=1$ and the third about $a=\frac{\pi}{4}$. The first and third converge for all values of x, while the second converges for $x \in (0,2)$. You'll have no trouble showing these things by the end of the semester.

As kind of a preview let me do a numerical experiment with the function $f(x) = \sin(x)$

Example 1.3. *Find the general form of the Taylor series for $f(x) = \sin(x)$ about the point $a = 0$ and plot the Taylor polynomials of various orders against $\sin(x)$.*

The Taylor series takes the form

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots (-1)^k \frac{x^{2k+1}}{(2k+1)!} + O(x^{2k+3})$$

Since $sin(x)$ is an odd function the Taylor series about $a=0$ has only odd terms. Figure 1 shows the graph of $\sin(x)$ and $T_n(x)$ on $x \in (0, 2\pi)$ for (left to right, top to bottom) $n = 1, 3, 5, 7, 9, 11, 13, 15$.

It is clear that, as n gets larger, the Taylor polynomials approach the $\sin(x)$ curve. We will prove this later in the course.

1.3 Applications

As mentioned before Taylor polynomials and series are useful for many things. One thing that they are good for is estimating the value of a function:

Example 1.4. *Estimate the value of $\sin(.5)$ using a Taylor polynomial of degree five.*

$$T_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120} \quad T_5(0.5) \approx 0.479427 \quad \sin(0.5) = 0.479426$$

The difference between $\sin(x)$ and $T_5(x)$ on $(0, \pi/2)$ is at most .005, and you can always express the sin of any angle in terms of an angle in the first quadrant, so this would be a good way to numerically estimate the sin function. With a couple of more terms in the series you could reduce the maximum error even more.

Series are not so good for other things. For instance the fact that $\sin(x)$ and $\cos(x)$ are periodic is not at all obvious from the series. Obviously the Taylor polynomials themselves are generally not periodic.

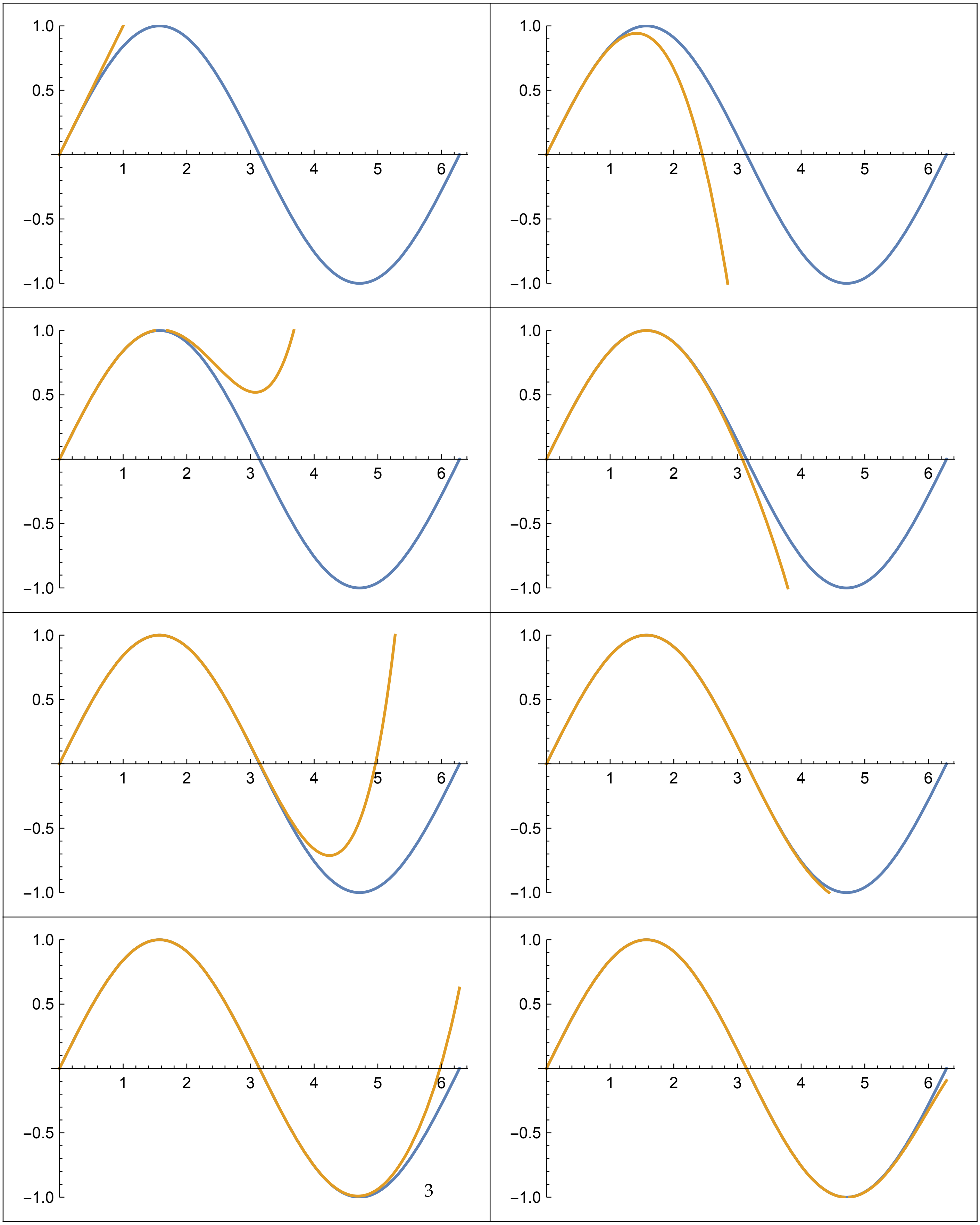

Figure 1: The Taylor polynomial approximants $T_n(x)$ for $\sin(x)$ for $n = 1, 3, 5, 7, 9, 11, 13, 15$.