Math 231E, Lecture 9.
Implicit and Logarithmic Differentiation

1 Implicit Differentiation

In most of the cases we have already seen, we have y explicitly defined as a function of x, e.g.

$$y = x^2, \quad y = \sin(x)\cos(x), \ldots$$

What if, on the other hand, we have

$$x^2 + y^2 = 25,$$

how can we find dy/dt?

Of course, we could solve for y and then proceed:

$$y^2 = 25 - x^2, \quad y = \pm\sqrt{25 - x^2},$$

but then we do not have a function, and we have to take care of the fact that we are dealing with a multi-valued object.

And in any case, there are some functions of x and y which we could not possibly solve, say

$$x^2 y^2 \sin(xe^y) = 0.$$

Because of all of this, we need a way to differentiate when we only know the function y *implicitly*. For example:

$$\begin{aligned}
x^2 + y^2 &= 25\\
\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}25\\
\frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= \frac{d}{dx}25\\
2x + 2y\frac{dy}{dx} &= 0\\
2y\frac{dy}{dx} &= -2x\\
\frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y}.
\end{aligned}$$

We can even do the more difficult example without much more trouble:

$$
\begin{aligned}
x^2y^2\sin(xe^y) &= 0 \\
\frac{d}{dx}(x^2y^2\sin(xe^y)) &= 0 \\
2xy^2\sin(xe^y) + x^2\left(2y\frac{dy}{dx}\right) + x^2y^2\cos(xe^y)\left(xe^y\frac{dy}{dx}\right) &= 0 \\
\left(2x^2y + x^3y^2e^y\cos(xe^y)\right)\frac{dy}{dx} &= -2xy^2\sin(xe^y) \\
\frac{dy}{dx} &= \frac{-2xy^2\sin(xe^y)}{2x^2y + x^3y^2e^y\cos(xe^y)}
\end{aligned}
$$

One notices that we always seem to be able to solve for dy/dx in these equations. Will this always be true?

1.1 Inverse Trigonometric Functions

Let us compute the derivative of the inverse trig function

$$y = \sin^{-1}(x) = \arcsin(x).$$

(See Figure 1.)

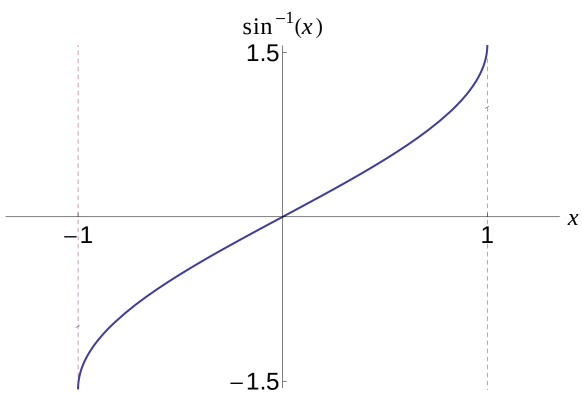

Figure 1: A plot of $y = \arcsin(x)$

Of course, simply using the definition we can write

$$x = \sin(y),$$

and now we differentiate this implicitly, as:

$$
\begin{aligned}
1 &= \cos(y)\frac{dy}{dx} \\
\frac{1}{\cos(y)} &= \frac{dy}{dx}.
\end{aligned}
$$

This statement is technically correct, but not entirely useful for us. We would like an expression that depends explicltly on x. But we have a formula, so we can write

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\cos(\sin^{-1}(x))},$$

and as long as we know what $\cos(\sin^{-1}(x))$ is, we are good. But what is that?

We have to use a triangle!

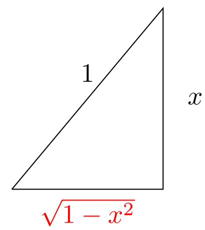

Figure 2: We choose a right triangle such that the interior angle θ has $\sin(\theta) = x$, and then Pythagoras gives us the third side!

We see from this triangle that $\sin(\theta)$ is x, and using the Pythagorean theorem, we have that $\cos(\theta) = \sqrt{1-x^2}$. Therefore

$$\sqrt{1-x^2} = \cos(\theta) = \cos(\sin^{-1}(x)),$$

so we have

$$y = \arcsin(x) \implies \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

We can also compute the derivative of $\arctan(x)$ similarly. If we write

$$y = \arctan(x), \quad x = \tan(y),$$

then differentiating gives

$$1 = \sec^2(y)\frac{dy}{dx}, \text{ or, } \frac{dy}{dx} = \cos^2(y) = \cos^2(\tan^{-1}(x)).$$

Using a different triangle (see Figure 3) gives

$$\cos^2(\tan^{-1}(x)) = \frac{1}{1+x^2}.$$

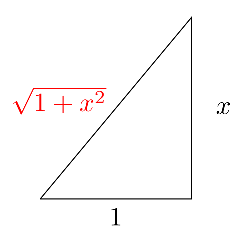

Figure 3: Another triangle!

1.2 Inverse Functions in General

Whenever we have an equation defined using an inverse function, i.e.

$$y = f^{-1}(x),$$

then we proceed in the same manner. We write

$$\begin{aligned} x &= f(y), \\ 1 &= f'(y)\frac{dy}{dx}, \\ \frac{dy}{dx} &= \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x)))}, \end{aligned}$$

The only trick will be, in general, to figure out how to compute $f'(f^{-1}(x))$ explicitly, and this is usually done in a function-dependent manner.

2 Logarithmic Differentiation

We know that

$$\frac{d}{dx}\ln(x) = \frac{1}{x}. \tag{1}$$

Using the chain rule, this allows us to differentiate the logarithm composed with any function, e.g.

$$\frac{d}{dx}\ln(\sin(x)) = \frac{1}{\sin(x)} \cdot \cos(x) = \frac{\cos(x)}{\sin(x)} = \cot(x).$$

This rule is great, but it has the problem that it is not defined when $\sin(x) < 0$, which is a lot of x! If $x < 0$, then $\ln(x)$ is not defined. However, $\ln(-x)$ is defined, and

$$\frac{\ln(-x)}{=}\frac{1}{-x} \cdot (-1) = \frac{1}{x}. \tag{2}$$

We can combine (2, 1) together to form:

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

and this formula is defined for all $x \neq 0$. In general, we have that
This formula can be very useful for complicated derivatives quickly!

Example 1. Let

$$f(x) = \frac{(x-9)^6(x^2+2x-3)^7}{(x^2-1)^8}.$$

We could use the product and chain rules to compute this derivative exactly, but this would take a lot of work. We can also do the following:

$$\begin{aligned} \ln|f(x)| &= \ln\left|\frac{(x-9)^6(x^2+2x-3)^7}{(x^2-1)^8}\right|, \\ &= \ln|(x-9)^6| + \ln|(x^2+2x-3)^7| - \ln|(x^2-1)^8| \\ &= 6\ln|x-9| + 7\ln|x^2+2x-3| - 8\ln|x^2-1|. \end{aligned}$$

We then take derivatives of both sides, to obtain

$$\begin{aligned}
\frac{d}{dx}\ln|f(x)| &= \frac{d}{dx}\left(6\ln|x-9| + 7\ln|x^2+2x-3| - 8\ln|x^2-1|\right)\\
\frac{f'(x)}{f(x)} &= \frac{6}{x-9} + \frac{7(2x+2)}{x^2+2x-3} - \frac{8(2x)}{x^2-1}\\
&= \frac{6}{x-9} + \frac{14x+14}{x^2+2x-3} - \frac{16x}{x^2-1}.
\end{aligned}$$

Now, this is not the formula for $f'(x)$, but it is the formula for $f'(x)/f(x)$. To get $f'(x)$, we multiply both sides by $f(x)$ (which we know) to obtain:

$$\begin{aligned}
f'(x) &= f(x)\left(\frac{6}{x-9} + \frac{14x+14}{x^2+2x-3} - \frac{16x}{x^2-1}\right)\\
&= \frac{(x-9)^6(x^2+2x-3)^7}{(x^2-1)^8}\left(\frac{6}{x-9} + \frac{14x+14}{x^2+2x-3} - \frac{16x}{x^2-1}\right).
\end{aligned}$$

More on this in a bit.

3 Differentials

We know from Taylor series that a function $f(x)$ can be approximated near a by

$$f(x) = f(a) + f'(a)(x-a) + O(x-a)^2.$$

Now let us write $x = a + \epsilon$ and note that $x - a = \epsilon$, so

$$f(a+\epsilon) = f(a) + f'(a)\epsilon + O(\epsilon^2).$$

We write an ϵ to remind us that it is small!

The first-order Taylor polynomial in this expansion is commonly called the **linearization** (since we're keeping only the linear term). It is very useful when approximating.

Example 2. Let $f(x) = \sqrt{x}$ and choose $a = 9$. Note that $f'(x) = 1/2\sqrt{x}$. Then

$$f(a+\epsilon) \approx f(a) + f'(a)\epsilon = 3 + \frac{1}{6}\epsilon.$$

So, for example, if we want to approximate $\sqrt{9.03}$, we choose $a = 9, \epsilon = 0.03$, and we would guess

$$\sqrt{9.03} = 3 + \frac{1}{6}(0.03) = 3.005.$$

In fact, $\sqrt{9.03} = 3.00499584026\cdots$, so notice that this approximation is off only by 5×10^{-6}.

Similarly, if we want to compute $\sqrt{8.91}$, we choose $a = 9$ and $\epsilon = -0.09$, and we have

$$\sqrt{8.91} \approx 3 + \frac{1}{6}(-0.09) = 2.985,$$

where $\sqrt{8.91} \approx 2.98496231132\ldots$ and again we are only off by 4×10^{-5}.

The strength of differentials is to use this idea in functional form. If we write

$$f(a+\epsilon) \approx f(a) + f'(a)\epsilon$$

Rearrange this to obtain

$$f(a+\epsilon)-f(a)=f'(a)\epsilon.$$

Thinking of ϵ as "a small change in x" we now write it as Δx, and we think of $f(a+\epsilon)-f(a)$ as "the change in f" and write it as Δf, and we thus obtain the equation for a **differential**, namely

$$\Delta f = f'(a)\Delta x.$$

The way to interpret this equation is:

"Changes in output are (approximately) proportional to changes in input, and the constant of proportionality is $f'(a)$."

This is the way to think of the $1/6$ in the previous example: when we kick the input by a small amount ϵ, the function's value gets kicked by about $\epsilon/6$.

Example 3. Given that we have measured the radius of a sphere as $10\,\mathrm{cm}$ with a relative error of 5%, what is the error in measuring its volume?

We know that $V(r)=\dfrac{4}{3}\pi r^3$, so we use the differential approach, writing

$$dV = V'(r)\,dr = 4\pi r^2\,dr.$$

Therefore

$$\begin{aligned} dV &= 4\pi(10\,\mathrm{cm})^2(10\,\mathrm{cm}\cdot 5\%) \\ &= 4\pi(100\,\mathrm{cm}^2)(0.5\,\mathrm{cm}) \\ &= 200\pi\,\mathrm{cm}^3 \approx 628\,\mathrm{cm}^3. \end{aligned}$$

The represents the **absolute error** given by the approximation. The volume is

$$V(10\,\mathrm{cm}) = \frac{4}{3}\pi(10\,\mathrm{cm})^3 = \frac{4000\pi}{3}\,\mathrm{cm}^3,$$

so the **relative error** is

$$\frac{dV}{V} = \frac{200\pi\,\mathrm{cm}^3}{\frac{4000\pi}{3}\,\mathrm{cm}^3}\frac{3}{20} = 15\%.$$

In short, the relative error gets multiplied by about a factor of three (again, we say *about* since these calculations are approximations).

Again, to see this factor of three in action, we write

$$\frac{dV/V}{dr/r} = \frac{dV}{dr}\cdot\frac{r}{V} = 4\pi r^2\cdot\frac{r}{\frac{4}{3}\pi r^3} = 3.$$

Generalizing the pattern seen in the previous example, if we ever have a function $f(x)=Cx^p$, then $df = Cpx^{p-1}\,dx$, so

$$\frac{df}{f} = \frac{Cpx^{p-1}\,dx}{Cx^p} = \frac{p\,dx}{x} = p\frac{dx}{x}.$$

So we see that relative errors get multiplied by about a factor of p.

Another way to see this (remember logarithmic derivatives!!) is that

$$d(\ln f) = \frac{df}{f},$$

so if $f(x)=Cx^p$, then $\ln(f)=\ln(Cx^p)=p\ln(x)+C$, so

$$\frac{df}{f} = p\frac{dx}{x}.$$

Example 4. The *ideal gas law* is a law that relates pressure, volume, and temperature of a gas, and is given by

$$PV = nRT,$$

where P is the pressure, V is the volume, T is the temperature, n is the number of moles in the gas, the R is the *ideal gas constant*

$$R \approx 8.314 \, \frac{\mathrm{J}}{\mathrm{K} \cdot \mathrm{mol}}.$$

Let us use differentials to solve the following problem: given a (relative) increase in pressure by 2% and a (relative) decrease in temperature by 3%, what the is approximate relative change in volume?

We have

$$\begin{aligned}
\ln(PV) &= \ln(nRT) \\
\ln(P) + \ln(V) &= \ln(nR) + \ln(T) \\
\frac{dP}{P} + \frac{dV}{V} &= \frac{dT}{T}.
\end{aligned}$$

Thus we have $(2\%) + dV/V = (-3\%)$, or $dV/V = -5\%$.