Math 231E, Lecture 35.
Polar Calculus

1 Area inside of a polar curve

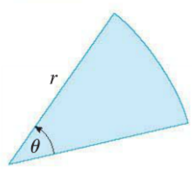


FIGURE 1

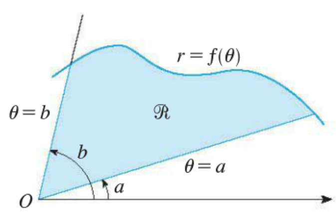


FIGURE 2

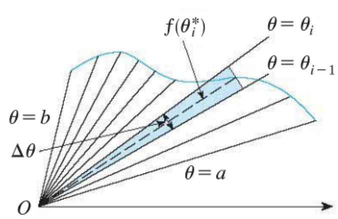


FIGURE 3

To the left we have a picture from the text book, giving a graphical argument for the formula for area. The idea here is to use the three standard principles: slice up the region, approximate it, then add it back together.

Let us consider a polar area given by the equation $r = f(\theta)$, where $\theta \in [a, b]$. Let us break up this angular domain by the points $a = \theta_0, \theta_1, \ldots, \theta_n = b$, where $\theta_{i+1} - \theta_i = \Delta\theta$ is small. Each of these slices will be approximately equal to a sector of a circle with radius $f(\theta_i)$ and subtended angle $\Delta\theta$. The formula for the area of such a sector is

$$\frac{1}{2}(f(\theta_i))^2\Delta\theta,$$

and thus the total area is

$$\frac{1}{2}\sum_{i=1}^{n}(f(\theta_i))^2\Delta\theta,$$

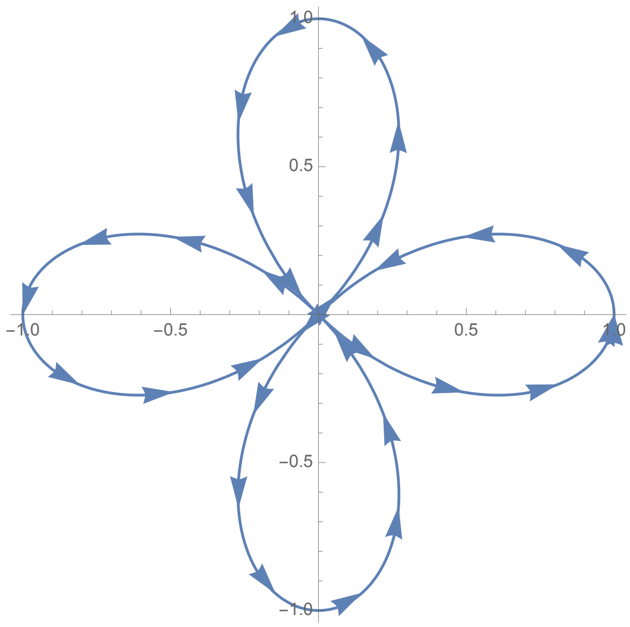
and as $n \to \infty$, this becomes the integral

$$\frac{1}{2}\int_a^b (f(\theta))^2\,d\theta. \tag{1}$$

Since $r = f(\theta)$, we also write this as

$$\int_a^b \frac{r^2}{2}\, d\theta.$$

Example 1.1. Let us compute the area in one leaf of the curve $r = \cos 2\theta$. Recall the picture from last time:



We need to figure out a θ domain to obtain one leaf. Recall from before that if we choose $\theta \in [-\pi/4, \pi/4]$ then $r \geq 0$ and this in fact gives the right leaf of the figure. Therefore our integral is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta.$$

Using our trig identities, we have

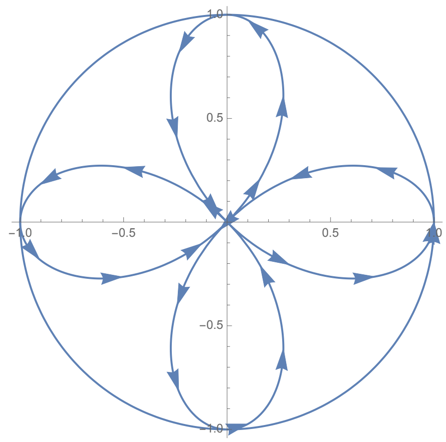
$$\cos^2 2\theta = \frac{1}{2}(1 + \cos(4\theta)),$$

so our area is

$$\frac{1}{4}\int_{-\pi/4}^{\pi/4} (1 + \cos(4\theta))\, d\theta = \frac{1}{4}\left(\theta + \frac{\sin 4\theta}{4}\right)\Bigg|_{\theta=-\pi/4}^{\theta=\pi/4} = \frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}.$$

From this, we get that the whole area of the four-leaf clover is $4 \cdot \pi/8 = \pi/2$.

Interesting fact! If we embed this clover inside a circle of radius one, we get the following picture:



Notice that we just proved that the area inside the clover is $\pi/2$. But also notice that the area of the circle is π, so that means that the area inside the circle, but outside the clover, is also $\pi/2$. So in fact each of the leaves has area $\pi/8$, and each of those "interleaf regions" also has area $\pi/8$.

Example 1.2. What is the area inside the cardioid? Similarly, we compute

$$\int_0^{2\pi} \frac{1}{2}(1+\sin\theta)^2\,d\theta$$

Using some algebra and another trig identity, we have

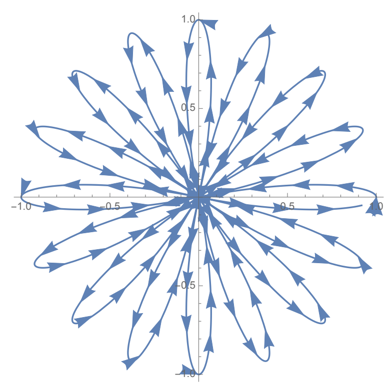
$$\frac{1}{2}(1+\sin\theta)^2 = \frac{1}{2}(1+2\sin\theta+\sin^2\theta) = \frac{1}{2}\left(1+2\sin\theta+\frac{1}{2}(1-\cos 2\theta)\right) = \frac{3}{4}+\sin\theta-\frac{1}{4}\cos 2\theta.$$

But notice that

$$\int_0^{2\pi}\sin\theta\,d\theta = \int_0^{2\pi}\cos 2\theta\,d\theta = 0,$$

so our integral is $2\pi\cdot 3/4 = 3\pi/2$.

Example 1.3. Let us compute the area inside the clover for $r = \cos n\theta$, where n is even. Recall from last time that this shape has $2n$ leaves, here we put a picture with $n = 8$:



Let us first compute one leaf. If we choose $\theta \in [-\pi/(2n), \pi/(2n)]$ then $r \geq 0$ and this in fact gives the rightmost leaf of the figure. Therefore our integral is

$$\frac{1}{2}\int_{-\pi/(2n)}^{\pi/(2n)}\cos^2 n\theta\,d\theta.$$

Using our trig identities, we have

$$\cos^2 n\theta = \frac{1}{2}(1+\cos(2n\theta)).$$

Again note that so our area is

$$\frac{1}{4}\int_{-\pi/(2n)}^{\pi/(2n)} (1+\cos(2n\theta))\,d\theta = \frac{1}{4}\left(\theta + \frac{\sin 2n\theta}{2n}\right)\Big|_{\theta=-\pi/(2n)}^{\theta=\pi/(2n)} = \frac{1}{4}\left(\frac{\pi}{n}\right) = \frac{\pi}{4n}.$$

From this, we get that the whole area of the four-leaf clover is $2n \cdot \pi/(4n) = \pi/2$. This is the same as the case where $n = 1$! The total area inside the clover is $\pi/2$, and again the total area outside the clover but inside the circle is also $\pi/2$.

2 Arclength in polar coordinates

To get a formula for arclength in polar coordinates is not so hard conceptually, but we need to do some complicated calculations.

If we have $r = f(\theta)$, then we can convert this to rectangular coordinates as

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta.$$

We then compute:

$$\begin{aligned}\frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta,\\ \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta.\end{aligned}$$

We can then compute:

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= (f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2\\ &= (f'(\theta))^2\cos^2\theta - 2f'(\theta)f(\theta)\cos\theta\sin\theta + (f(\theta))^2\sin^2\theta\\ &\quad + f'(\theta))^2\sin^2\theta + 2f'(\theta)f(\theta)\cos\theta\sin\theta + (f(\theta))^2\cos^2\theta\\ &= (f'(\theta))^2(\cos^2\theta + \sin^2\theta) + (f(\theta))^2(\cos^2\theta + \sin^2\theta)\\ &= (f(\theta))^2 + (f'(\theta))^2.\end{aligned}$$

We then have

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{(f(\theta))^2 + (f'(\theta))^2}, \tag{2}$$

or, recalling $r = f(\theta)$,

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}. \tag{3}$$

Therefore the formula for arclength of the curve specified by $r = f(\theta)$ for $\theta \in [a, b]$ is

$$\int_a^b \sqrt{(f(\theta))^2 + (f'(\theta))^2}\,d\theta.$$

Example 2.1. One simple example of the circle of radius a, $r = a$, so that $f(\theta) = a$. Then we have

$$\int_0^{2\pi} \sqrt{a^2 + 0^2}\,d\theta = 2\pi a.$$

Of course we knew that.

Example 2.2. Consider the polar curve $r = \cos(\theta)$ from last time. We go around the circleLet us compute the length of one leaf of the clover, specifically for $\theta \in [-\pi/4, \pi/4]$. We have

$$\int_{-\pi/4}^{\pi/4} \sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\, d\theta. \tag{4}$$

It turns out that this is a difficult integral to do, but this is the integral to do. We can compute this numerically as approximately 2.42211.