Math 231E, Lecture 26.
Integral Test and Estimating Sums

1 Integral Test

The definition of determining whether the sum $\sum_{n=1}^{\infty} a_n$ converges is:

1. Compute the partial sums
$$s_n = \sum_{k=1}^{n} a_k,$$
2. Check that s_n is a convergent sequence, i.e.
$$\lim_{n\to\infty} s_n = L,$$
where L is finite.

However, this is typically quite difficult to do. The challenge tends to be part #1 of the above.
So we state the integral test:

Theorem 1.1. Suppose that $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$. Let $a_n = f(n)$. Then
$$\int_1^{\infty} f(x)\, dx$$
converges if and only if
$$\sum_{n=1}^{\infty} a_n$$
converges. In the case that both converge, we have
$$\left(\int_1^{\infty} f(x)\, dx\right) \le \sum_{n=1}^{\infty} a_n \le \left(a_1 + \int_1^{\infty} f(x)\, dx\right).$$

Note that this allows us to check the convergence of a series without having to compute it!

2 p-test for series

Example 2.1. Does the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

Let us use the integral test. Note that
$$\int_1^{\infty} \frac{1}{x}\, dx$$
is a divergent integral from the p-test for integrals, so therefore the harmonic series diverges.

Example 2.2. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

Let us use the integral test. Note that

$$\int_1^\infty \frac{1}{x^2}\,dx = 1,$$

so is a convergent integral from the p-test for integrals, so therefore the series converges.

You perhaps already see the pattern, and this gives the p-test for series:

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

This is just inherited from the p-test for integrals, and noticing that $1/x^p$ is continuous, positive, and decreasing on $[1, \infty)$.

Note! There is something to be careful about here. Let us consider the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We know it is convergent since

$$\int_1^\infty \frac{dx}{x^2} = 1.$$

However, what is **not true** is that the sum and the integral have the same value! For example, we might think that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1,$$

but this is **false**. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

which is a fact that is a bit beyond the scope of this class, but you will see in MATH 285/286.

In fact, let us compare formulas. We know that if $p > 1$, then

$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}, \tag{1}$$

which means that the sums converge. However, the sums don't have such a simple formula. In fact, it is known that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90},$$
$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945},$$
$$\sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450},$$

and so on. (Trust me on this!)

However, notice that I only put even powers up there. In fact, there is no known formula for any of the odd powers! So, for example, the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \tag{2}$$

is known to be finite and is a well-studied constant (known as Apéry's constant) but there is no known formula for it.

3 Estimating sums

Let's say that we have a convergent series

$$\sum_{n=1}^{\infty} a_n = S,$$

and we want to approximate its value. If we have a formula for it, great, but usually we do not have a formula for it. The best thing we can do is compute the partial sum s_n at some n, and then hopefully it is close to the actual sum. So let us try to estimate the error, or the remainder. If we write $R_n = S - s_n$, then we would like a formula for this.

Theorem 3.1. Let $f(x)$ be a continuous, positive function that is decreasing for all $x \geq n$. Then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

Example 3.2. Let us say that we want to compute

$$s = \sum_{n=1}^{\infty} \frac{1}{n^3} \tag{3}$$

with and error of no more than 10^{-4}, or 0.0001. How many terms do we need to compute?

From the theorem above, we have

$$R_n \leq \int_n^{\infty} \frac{dx}{x^3} = \frac{1}{2n^2}.$$

So as long as

$$\begin{aligned} \frac{1}{2n^2} &< 10^{-4} \\ n^2 &> 10^4 \\ n &> 100, \end{aligned}$$

then our error is less than 0.0001.

Of course, we should probably use a computer for this, but it's definitely something a computer can do. In fact, we can compute that

$$\sum_{n=1}^{100} \frac{1}{n^3} = 1.20201$$

whereas the actual sum is 1.20206, so we're only off by 5×10^{-5}.

4 Proof of Test

Proof of Integral Test.

$\boxed{\text{Converges.}}$ Let us first assume that the integral of $f(x)$ converges. Consider the quantity

$$\int_1^n f(x)\,dx,$$

the area under the curve from 1 to n.

Let us consider a Riemann sum approximation to this area, with $\Delta x = 1$. If we use left Riemann sums, we obtain

$$f(1) + f(2) + \cdots + f(n-1) = a_1 + a_2 + \cdots + a_{n-1},$$

and since f is decreasing, left sums are always **no smaller than** the integral, so we have

$$a_1 + a_2 + \cdots + a_{n-1} \geq \int_1^n f(x)\,dx. \tag{4}$$

If we now consider right Riemann sums, we obtain

$$f(2) + f(3) + \cdots + f(n) = a_2 + a_3 + \cdots + a_n.$$

Since f is decreasing, right sums are always **no larger than** the integral, so we have

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx. \tag{5}$$

Since the integral of $f(x)$ is convergent, this means that

$$\int_1^\infty f(x)\,dx = \lim_{n\to\infty} \int_1^n f(x)\,dx$$

exists and is finite. And, since $f(x) \geq 0$ and using (5), we know that

$$\sum_{k=2}^n a_k \leq \int_1^n f(x)\,dx \leq \int_1^\infty f(x)\,dx.$$

This means that the series of partial sums

$$s_n = \sum_{k=1}^n a_k$$

is bounded above (in fact by $a_1 + \int_1^\infty f(x)\,dx$).

Moreover, notice that

$$s_{n+1} - s_n = a_{n+1} \geq 0,$$

so s_n is increasing. Since s_n is increasing and bounded above, by the Monotone Convergence Theorem the limit

$$\lim_{n\to\infty} s_n$$

exists and is bounded (again by $a_1 + \int_1^\infty f(x)\,dx$).

$\boxed{\text{Diverges.}}$ If the integral of $f(x)$ diverges, since $f(x) \geq 0$ this means that

$$\lim_{n\to\infty} \int_1^n f(x)\,dx = \infty.$$

But from (4) we have

$$s_{n-1} \geq \int_1^n f(x)\,dx,$$

which means that

$$\lim_{n\to\infty} s_{n-1} \lim_{n\to\infty} \geq \int_1^n f(x)\,dx = \infty,$$

so the sum diverges.

□