Math 231E, Lecture 30.
Representing Functions as Power Series

1 Geometric Series

Recall the formula for a geometric series:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}.$$

Recall that this converges if and only if $|r| < 1$. If we choose $a = 1, r = x$, this gives the expression

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x},$$

and this converges on the set $|x| < 1$. In particular, the radius of convergence is 1.

Now consider the function

$$\frac{1}{1+x^2}.$$

Writing

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$$

and plugging in $y = -x^2$, we obtain

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

Moreover, the domain of convergence is $|y| < 1$, or $\left|-x^2\right| < 1$, or $|x| < 1$. Similarly, we can write

$$\frac{1}{1+x^6} = 1 - x^6 + x^{12} - x^{18} + \cdots$$

and again the domain of convergence is $|x| < 1$.

Another example:

$$\frac{1}{x+4} = \frac{1}{4} \cdot \frac{1}{1+x/4}.$$

So we can do the following:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \frac{1}{1-y} = \sum_{n=0}^{\infty} y^n.$$

Write $y = -x/4$ and obtain

$$\frac{1}{x+4} = \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{4^{n+1}}.$$

We might ask what the domain of convergence is for this series. We can approach it one of two ways: First apply the Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{4},$$

so this converges on $|x| < 4$ and thus has a radius of convergence of 4. Plugging in $x = \pm 4$ to the series gives a divergence series, so the domain of convergence is exactly $(-4, 4)$.

Conversely, if the original series in the y variables converges for $|y| < 1$, then this will converge for $|-x/4| < 1$ or $|x| < 4$. This should be, and is, the same answer.

2 Differentiation and Integration of Power Series

Theorem 2.1. Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series with radius of convergence R. Then, on the domain $(a-R, a+R)$, we can differentiate, or integrate, the power series term-by-term.

The two examples we are discussing here:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n \\ &= \sum_{n=0}^{\infty} c_n n(x-a)^{n-1} = \sum_{n=1}^{\infty} c_n n(x-a)^{n-1} \\ &= \sum_{n=0}^{\infty} c_{n+1}(n+1)(x-a)^n. \end{aligned}$$

Also:

$$\begin{aligned} \int f(x)\,dx &= C + \sum_{n=0}^{\infty} c_n \int (x-a)^n\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \\ &= C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} = C + \sum_{n=1}^{\infty} \frac{c_{n-1}}{n}(x-a)^n. \end{aligned}$$

So, if we summarize: When $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, then the nth coefficient of $f'(x)$ is

$$c_{n+1}(n+1),$$

while the nth coefficient of $\int f(x)\,dx$ is

$$\frac{c_{n-1}}{n}.$$

The theorem tells us that this always works, but only if we are strictly inside the interval! So, for example, if $f(x)$ is a power series that has a domain of convergence of $[-2, 2]$, then doing this term-by-term trick is only guaranteed to work on $(-2, 2)$.

Example 2.2. Let us write down the Taylor series for $1/(1-x)^2$. Note first that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x},$$

and this is valid for $x \in (-1, 1)$. Also note that

$$\frac{1}{1-x} = 1 + x + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

Then

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^3 + 4x^3 = \sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=1}^{\infty} nx^{n-1}.$$

(Either of the two are acceptable, since they are just renumberings of the series.)

So, for example, by plugging in $x = 1/3$, we have

$$\sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n-1} = \frac{1}{(2/3)^2} = \frac{9}{4}.$$

Plugging in $x = -3/4$ gives

$$\sum_{n=1}^{\infty} n \left(-\frac{3}{4}\right)^{n-1} = \frac{1}{(7/4)^2} = \frac{16}{49}.$$

Notice that we can also compute

$$\sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n} = \sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n-1} \left(\frac{1}{3}\right) = \frac{1}{3} \sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n-1} = \frac{3}{4}.$$

Also,

$$\sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n+12} = \sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n-1} \left(\frac{1}{3}\right)^{13} = \left(\frac{1}{3}\right)^{13} \sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n-1} = \frac{1}{4 \cdot 3^{11}}.$$

Note: This won't give the correct answer at the boundary points $-1, 1$. For example, plugging in -1 gives

$$\sum_{n=1}^{\infty} n(-1)^{n-1} = \frac{1}{4},$$

but clearly the equation

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots = \frac{1}{4}$$

doesn't make much sense.

Example 2.3. We can extend this to higher powers. Consider, for example, $1/(1-x)^4$. Note that

$$\frac{d^3}{dx^3} \frac{1}{1-x} = \frac{6}{(1-x)^4},$$

so we have

$$\frac{1}{(1-x)^4} = \frac{1}{6} \sum_{n=0}^{\infty} \frac{d^3}{dx^3}(x^n).$$

We note that

$$\frac{d^3}{dx^3}(x^n) = (x^n)''' = (nx^{n-1})'' = (n(n-1)x^{n-2})' = n(n-1)(n-2)x^{n-3}.$$

Therefore

$$\frac{1}{(1-x)^4} = \sum_{n=0}^{\infty} \frac{n(n-1)(n-2)}{6} x^{n-3}.$$

Notice that when $n = 0, 1, 2$, all of these terms are zero, so we can start at $n = 3$:

$$\frac{1}{(1-x)^4} = \sum_{n=3}^{\infty} \frac{n(n-1)(n-2)}{6} x^{n-3}.$$

Moreover, we can shift indices as well: if we write $m = n - 3$, or $n = m + 3$, then we also obtain

$$\frac{1}{(1-x)^4} = \sum_{m=0}^{\infty} \frac{(m+3)(m+2)(m+1)}{6} x^{m}.$$

(Of course, we can replace m with n everywhere in the last sum as well.) We can also write out several of the first few terms

$$\frac{1}{(1-x)^4} = 1 + 4x + 10x^2 + 20x^3 + \dots$$

Example 2.4. Let us now consider the power series for $\ln(1+x)$. Notice that we have

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

Integrating both sides of this equation gives

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{1}$$

We have to determine C, but if we plug in $x = 0$ we obtain

$$\ln 1 = C,$$

or $C = 0$. Therefore

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{2}$$

3 Approximating Integrals by Power Series

Let $f(x) = 1/(1+x^6)$, and let us imagine that we need to know the function

$$F(x) = \int_0^x f(t)\,dt = \int_0^x \frac{dt}{1+t^6}.$$

If we don't know how to do the integral, then we must approximate it somehow. One idea is to use power series. We have already computed

$$\frac{1}{1+x^6} = 1 - x^6 + x^{12} - x^{18} + \cdots$$

so

$$F(x) = x - \frac{x^7}{6} + \frac{x^{13}}{13} - \frac{x^{19}}{19} + \cdots$$

Note that $F(0) = 0$ by definition, so the constant $C = 0$.

Let us then assume that we will use the approximation

$$F(x) \approx G(x) := x - \frac{x^7}{6} + \frac{x^{13}}{13}$$

everywhere on the interval $x \in [-1/2, 1/2]$. We can ask what the error is, using this approximation. Note that the infinite series is an alternating series, and as such the error we obtain after taking these three terms is no larger than the first term we ignore. Specifically, if $E(x) = |F(x) - G(x)|$, then

$$\max_{x\in[-1/2,1/2]} E(x) \le \max_{x\in[-1/2,1/2]} \frac{|x|}{19} = \frac{2^{-19}}{19} = 1.004 \times 10^{-7}.$$