Math 231E, Lecture 21.
Improper Integrals Lecture II – Comparison Theorems

1 Comparison Theorems

As I have said throughout this course the way that we define things in calculus is almost never the way that we want to compute them. This is true of improper integrals as well. If we want to evaluate the convergence using the definition then we have to

- Do the integral.
- Evaluate the limit.

Usually the second step is not so bad but the first one may be very hard. Many integrals are simply not expressible in terms of elementary functions. Even if they are that is a lot of work to do. I want to briefly state a couple of comparison theorems.

Theorem 1. Consider the improper integral

$$\int_a^b f(x)dx.$$

We are assuming that b is the (only) point of discontinuity of f, or $b = \infty$ or $b = -\infty$, so this covers both Type I and Type II integrals. Suppose that $0 \leq f(x) \leq g(x)$ for x sufficiently close to b. Then the following are true.

- If $\int_a^b g(x)dx$ converges then $\int_a^b f(x)dx$ converges
- If $\int_a^b f(x)dx$ diverges then $\int_a^b f(x)dx$ diverges

An example of this is given by

Example 1.1. Show that $\int_{12}^{\infty} \frac{1}{x^5+4x^4+27x^3+2x^2+x+1}dx$ converges. It is not hard to see that

$$x^5 + 4x^4 + 27x^3 + 2x^2 + x + 1 > x^5$$

for $x > 0$ since the lefthand side is a sum of positive terms. Taking the reciprocal gives

$$\frac{1}{x^5} > \frac{1}{x^5 + 4x^4 + 27x^3 + 2x^2 + x + 1}$$

. Since we have that

$$\int_{12}^{\infty} \frac{dx}{x^5}$$

converges by the p-test we know that $\int_{12}^{\infty} \frac{1}{x^5+4x^4+27x^3+2x^2+x+1}dx$ converges.

As a second example lets do one with a finite point of discontinuity

Example 1.2. Show that

$$\int_0^1 \frac{1+\sin^2(x)}{x} dx$$

diverges.

In order to show that this integral diverges we have to find a smaller integrand that also has a divergent integral. Obviously we have

$$\frac{1}{x} \leq \frac{1+\sin^2(x)}{x}$$

. This integral we can just do by hand:

$$\lim_{L\to 0^+} \int_L^1 \frac{1}{x} dx = \lim_{L\to 0^+} \ln(1) - \ln(L)$$

and that limit does not exist as a real number (it is ∞, but that is just a particular kind of does not exist) so the original integral is divergent.

This is all well and good, but it is a little bit of a pain to use in practice. One has to find and prove an inequality, and you kind of have to know the answer going it – the inequality goes one way if you are trying to prove divergence and the other way to prove convergence. This motivates the limit comparison theorem

Theorem 2. Consider the improper integral

$$\int_a^b f(x)dx.$$

We are assuming that b is the (only) point of discontinuity of f, or $b = \infty$ or $b = -\infty$, so this covers both Type I and Type II integrals. Suppose that $0 \leq f(x)$ and $0 \leq g(x)$ for x sufficiently close to b, and that

$$\lim_{x\to b} \frac{f(x)}{g(x)} = L$$

where L is a finite non-zero number. Then
$\int_a^b g(x)dx$ and $\int_a^b f(x)dx$ both converge or both diverge.

This theorem is usually easier to use since you don;t really have to know ahead of time whether the integral converges or diverges – you just need to find a simpler integral with the same behavior.

Example 1.3. Evaluate the convergence of the integral

$$\int_1^\infty (1-\cos(\frac{1}{x}))dx$$

The way to proceed here is to figure out what the integrand "looks like" when x is large, since that determines the convergence. When x is large we have that $\frac{1}{x}$ is small. So from the Taylor expansion

$$\cos(\frac{1}{x}) = 1 - \frac{1}{2x^2} + O(x^{-4})$$

when x is large. Therefore $1-\cos(\frac{1}{x}) \approx \frac{1}{2x^2}$ when x is large. So lets take our comparison function to be $f(x) = \frac{1}{2x^2}$

We have that $\lim_{x\to\infty} \frac{1-\cos(\frac{1}{x})}{2x^2} = 1$, so we know that $\int_1^\infty (1-\cos(\frac{1}{x}))dx$ and $\int_1^\infty \frac{1}{2x^2}dx$ both converge or both diverge. But $\int_1^\infty \frac{1}{2x^2}dx$ converges by the p-test.