Math 231E Engineering Calculus Module 2b: One sided Limits and Limit laws

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1 One-sided limits

We can also weaken the notion of limit, and only require that the limit work "on one side" of the point in question. This is why we define limits "from the left" and "from the right".

We say that

$$\boxed{\lim_{x\to a+} f(x) = L} \text{ or } \boxed{\lim_{x\searrow a} f(x) = L},$$

if,

$$\boxed{\text{for all } \epsilon > 0\text{, there is a } \delta > 0 \text{ such that if } a < x < a+\delta\text{, then } |f(x) - L| < \epsilon.}$$

This is called the "right-hand limit".

We can also consider the left-hand limit:

We say that

$$\boxed{\lim_{x\to a-} f(x) = L} \text{ or } \boxed{\lim_{x\nearrow a} f(x) = L},$$

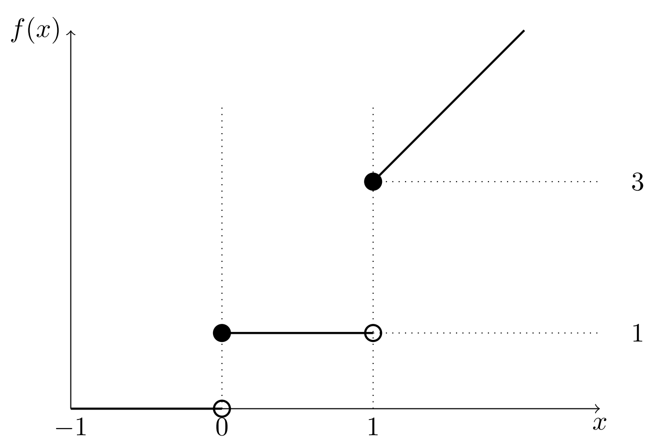
if,

$$\boxed{\text{for all } \epsilon > 0\text{, there is a } \delta > 0 \text{ such that if } a-\delta < x < a\text{, then } |f(x) - L| < \epsilon.}$$

Consider the function

$$f(x) = \begin{cases} 0, & x < 0, \\ 1, & 0 \le x < 1, \\ 2x+1, & 1 \le x, \end{cases}$$

which has a graph that looks like:



Let us now choose $a = 1$. Our intuition tells us that

$$\lim_{x \to 1-} f(x) = 1, \quad \lim_{x \to 1+} f(x) = 3.$$

Let us prove the latter. We want to show that for any $\epsilon > 0$, we can find a δ so that if $1 < x < x + \delta$, then $|f(x) - 3| < \epsilon$. Noting $f(x) = 2x + 1$, we have $f(x) - 3 = 2x - 2$ **if** $x > 1$. So we have

$$\begin{aligned} |f(x) - 3| &< \epsilon \\ |2x - 2| &< \epsilon \\ 2 - \epsilon < 2x &< 2 + \epsilon \\ 1 - \frac{\epsilon}{2} < x &< 1 + \frac{\epsilon}{2}. \end{aligned}$$

Thus we need to satisfy

$$x > 1, \text{ and } x < 1 + \frac{\epsilon}{2},$$

and it is clear that if we choose $\delta = \epsilon/2$, then we have satisfied the right-hand limit condition.

Here is the following important statement:

$$\boxed{\begin{gathered} \lim_{x \to a} f(x) = L \\ \text{if and only if} \\ \lim_{x \to a-} f(x) = L, \text{ and } \lim_{x \to a+} f(x) = L. \end{gathered}}$$

In short, if the left-hand limit exists, and the right-hand limit exists, and they are equal, then the limit exists.

2 Infinite limits and limits at infinity

We want to make sense of the notation

$$\boxed{\lim_{x \to a} f(x) = \infty},$$

and we define it as such:

$$\boxed{\text{for all } M > 0, \text{ there is a } \delta > 0 \text{ such that } f(x) > M \text{ whenever } 0 < |x - a| < \delta}$$

Example 2.1. We will prove that

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Choose any $M > 0$. We need to show find a δ such that $|x| < \delta$ implies

$$\frac{1}{x^2} > M,$$

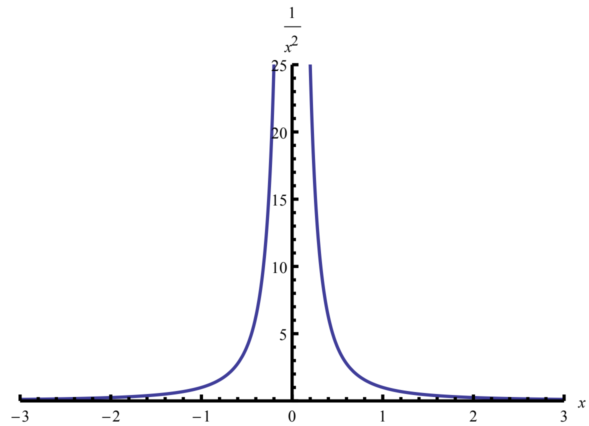


Figure 1: Graph of $f(x) = 1/x^2$. fig:xm2

and as usual we work backward.

$$\frac{1}{x^2} > M$$
$$\frac{1}{M} > x^2$$
$$\frac{1}{\sqrt{M}} > |x|,$$

and so we choose $\delta = 1/\sqrt{M}$. See Figure fig:xm2 1.

One can do the same thing with limits going to $-\infty$ (is it easy to see what that definition should be?) and with one-sided limits.

For example, we would say

$$\boxed{\lim_{x \to a+} f(x) = \infty}$$

whenever

$$\boxed{\text{for all } M > 0\text{, there is a } \delta > 0 \text{ such that } f(x) > M \text{ whenever } a < x < a + \delta,}$$

and similarly for other cases.

Question: Think about the function $f(x) = 1/x$. Does $\lim_{x \to 0} 1/x$ exist? What about the one-sided limits?

Limits at infinity are similar – these two cases are basically vertical and horizontal asymptotes that you probably learned about in high-school. We want to be precise about what it means for

$$\lim_{x \to \infty} f(x) = L$$

This is similar to the previous definition: the mathematical definition of this is that for every $\epsilon > 0$ there exists an $M > 0$ such that if $x > M$ then $|f(x) - L| < \epsilon$

In other words, being slightly informal, for x bigger than a certain value M then $f(x)$ is wihin ϵ of L

3 Limit Laws

We have the following **limit laws** (see §2.3 from the book). Let c be a number and assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

1. $$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$
2. $$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$
3. $$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$
4. $$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \text{ but only if } \lim_{x\to a} g(x) \neq 0!$$
5. $$\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x)$$
6. $$\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$$
7. $$\lim_{x\to a} c = c$$
8. $$\lim_{x\to a} x = a$$
9. $$\lim_{x\to a} x^n = a^n$$
10. $$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$$

Example 3.1.

$$\begin{aligned}
\lim_{x\to 3} 3x^2 - 4x + 2 &= \lim_{x\to 3} 3x^2 - \lim_{x\to 3} 4x + \lim_{x\to 3} 2 \\
&= 3\lim_{x\to 3} x^2 - 4\lim_{x\to 3} x + \lim_{x\to 3} 2 \\
&= 3\cdot 9 - 4\cdot 3 + 2 = 27 - 12 + 2 = 17.
\end{aligned}$$

(In short, just plug in!)

Example 3.2. Let us consider some rational functions:

1. $$\lim_{x\to 1}\frac{x^2-2x+7}{x+3} = \frac{\lim_{x\to 1} x^2-2x+7}{\lim_{x\to 1} x+3} = \frac{6}{4} = \frac{2}{3}.$$
2. $$\lim_{x\to 1}\frac{x^2-2x+1}{x+3} = \frac{\lim_{x\to 1} x^2-2x+7}{\lim_{x\to 1} x+3} = \frac{0}{4} = 0.$$
3. $$\lim_{x\to 1}\frac{x^2-2x+7}{x-1} = \frac{\lim_{x\to 1} x^2-2x+7}{\lim_{x\to 1} x-1} = \frac{6}{0}, \textbf{ Law \#4 fail}$$
4. $$\lim_{x\to 1}\frac{x^2-2x+1}{x-1} = \frac{\lim_{x\to 1} x^2-2x+1}{\lim_{x\to 1} x-1} = \frac{0}{0}, \textbf{ Law \#4 fail}$$

4 Piecewise limits

Example 4.1. What about $f(x) = |x|$ near 0? None of the Limit Laws work. We have to break the function up into pieces.

However, notice that

$$|x| = \begin{cases} x, & x > 0, \\ -x, & x < 0. \end{cases}$$

So, we can compute

$$\lim_{x \to 0+} |x| = \lim_{x \to 0+} x = 0,$$

and

$$\lim_{x \to 0-} |x| = \lim_{x \to 0-} (-x) = 0,$$

and since both left- and right-hand limits exist, and are the same, we can say that

$$\lim_{x \to 0} |x| = 0.$$

Question: What happens when we try

$$f(x) = \frac{|x|}{x}?$$