Math 231E, Lecture 31.
Taylor Series, Revisited

1 Definition of Taylor Series

Recall our formula for the Taylor series of a function at $a = 0$:

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

We have a similar definition if we would like to expand the series at a different point $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots$$

Notice that Taylor series are always power series, and as such we can apply all of the theory of the last two lectures to them.

2 Domain of Convergence

Let us consider a few examples:

Example 2.1. Let us expand e^x at $a = 0$.

$$e^x = 1 + x + x^2/2 + x^3/6 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

What is the domain of convergence of this series? As before, we first apply the ratio test. Writing

$$a_n = \frac{x^n}{n!}, \quad a_{n+1} = \frac{x^{n+1}}{(n+1)!},$$

we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Therefore this converges for all x and has an infinite radius of convergence!

Example 2.2. We can also expand e^x at a different point, say $a = 2$. Note that if $f(x) = e^x$, then

$$f^{(n)}(x) = e^x, \quad f^{(n)}(2) = e^2,$$

so our Taylor series is

$$\sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n.$$

To compute the domain of convergence we proceed as above. Writing

$$a_n = \frac{e^2(x-2)^n}{n!}, \quad a_{n+1} = \frac{e^2(x-2)^{n+1}}{(n+1)!},$$

we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^2(x-2)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^2(x-2)^n}\right| = \lim_{n\to\infty}\frac{|x-2|}{n+1} = 0.$$

Therefore this also converges for all x and has an infinite radius of convergence.

Example 2.3. Just as power series can have a finite radius of convergence, so can Taylor series. For example, if we choose $f(x) = 1/(x-1)$, then

$$f'(x) = (x-1)^{-2}, \quad f''(x) = 2(x-1)^{-3}, \quad f'''(x) = 6(x-1)^{-4},$$

and in general,

$$f^{(n)}(x) = n!(x-1)^{-(n+1)}.$$

This gives

$$f^{(n)}(0) = n!,$$

and therefore the Taylor series is

$$\sum_{n=0}^{\infty}\frac{n!}{n!}x^n = \sum_{n=0}^{\infty}x^n,$$

which we already knew from before. Recall that this Taylor series has a finite radius of convergence and its domain of convegnece is $(-1,1)$. Note that the original function has a singularity at $x=1$ as well, so the domain of convergence of the power series is, in some sense, the largest that it could possibly be.

We see from earlier results that if we know something about how large the derivatives of a function are, we can say something about the radius of convergence:

Theorem 2.4. Let us assume that the derivatives of the function $f(x)$ at the point $x=a$ grow no faster than exponentially, i.e. there are $C>0$ and $0\le q<\infty$ such that

$$\left|f^{(n)}(a)\right| \le Cq^n.$$

Then the Taylor series for $f(x)$ at $x=a$ has an infinite radius of convergence.

Proof. Notice that we have

$$a_n = \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

and

$$\frac{\left|f^{(n)}(a)\right|}{n!} \le C\frac{q^n}{n!}.$$

Applying the Ratio Test to this will give

$$\frac{q\,|x-a|}{n+1}$$

and this has a limit of zero, so the series will converge for all x.

$\square$

Conversely, for a Taylor series to have a finite radius of convergence, the derivatives have to grow much more rapidly, e.g. factorially as in the case of $1/(1-x)$.

3 Approximation

Recall our definition of Taylor polynomials. Let $f(x)$ be a function, and we define the nth-order Taylor polynomial of $f(x)$ at $x = a$ as

$$T_n f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We now ask how much of an error we introduce by replacing a function with its Taylor polynomial. Let us define

$$R_n f(x) = f(x) - T_n f(x).$$

First, if

$$\lim_{n\to\infty} R_n f(x) = 0$$

for all x, then we say that "$f(x)$ is equal to its Taylor series".

For the record, this is true of any reasonable function we might encounter, although one can construct examples where it doesn't work (see book). The practical question is then, given a function, and given an approximating Taylor polynomial, how close are they? We have a theorem for this:

Theorem 3.1. Let $f(x)$ be a function with the property that

$$\left|f^{(n+1)}(x)\right| \le M,$$

for all $|x-a| \le L$. Then

$$|R_n f(x)| \le \frac{M}{(n+1)!}\,|x-a|^{n+1}$$

for all $|x-a| \le L$.

This theorem gives us a worst-case estimate of how bad our approximation is if we truncate a Taylor series at a certain point. It is called the *Taylor Remainder Theorem* or *Taylor Approximation Theorem*.

We consider a few examples

Example 3.2. Let us take $f(x) = e^x$, and we want to approximate it by its third-order Taylor polynomial at $a = 0$, namely

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Let us ask: how bad of an approximation is this on the domain $[-1/2, 1/2]$?

According to the Taylor Approximation Theorem, since $f^{(4)}(x) = e^x$ for all x, then we have

$$\left|f^{(4)}(x)\right| \le e^{1/2} \approx 1.64872.$$

We then choose $M = e^{1/2}$ and obtain

$$R_n(x) \le \frac{e^{1/2}}{4}!\,|x|^4 = \frac{e^{1/2}\,|x|^4}{24}.$$

The maximum error on the entire interval $[-1/2, 1/2]$ is obtained at $|x| = 1/2$, and the error is

$$\frac{\sqrt{e}}{24 2^4} = \frac{1.649}{384} \approx 0.00429.$$

This is a reasonable error. For the record, we have

$$e^{1/2} = 1.64872, \quad T_3(1/2) = \frac{79}{48} = 1.64583$$

so the error is about 0.00289 which is a bit better than the estimate, but comparable.

Example 3.3. What about the same question, except now we are interested in our approximation being on the interval $[-5, 5]$? Again we have $f^{(4)}(x) = e^x$, and the maximum of this function on $[-5, 5]$ is e^5. Now we have

$$R_n(x) \leq \frac{e^5}{4!} |x|^4 = \frac{e^5 |x|^4}{24}.$$

But now the maximal error is as high as

$$\max_{x\in[-5,5]} |R_n(x)| \leq \frac{e^5 5^4}{24} = \frac{e^5 625}{24} \approx 3864.93$$

and this is a very bad error bound. In point of fact, we have

$$e^5 \approx 148.41, \quad T_3(5) = \frac{118}{5} \approx 39.33.$$

so the actual error is quite a bit better than the estimate, but it's still not good. If we want to approximate e^5 by a third order Taylor series at $a = 0$ we're going to have a bad time.

Example 3.4. What if, however, we took more terms? Let us consider $T_{100}(x)$ instead, i.e.

$$T_{100}(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{100}}{100!}.$$

The Taylor Remainder Theorem tells us that

$$R_{100}(x) \leq \frac{e^5}{(101)!} |x|^{101}.$$

Note that by taking more terms, the e^5 term didn't get worse, but the denominator got way better. If we then used this to compute the maximal error on the interval

$$\max_{x\in[-5,5]} |R_n(x)| \leq \frac{e^5 5^{101}}{(101)!} \approx \frac{1.64872 \cdot 3.9443 \times 10^{70}}{9.42595 \times 10^{159}} = 6.8991 \times 10^{-90}.$$

This is a good estimate.