

Math 231E Engineering Calculus: Notes on Complex numbers and complex arithmetic.

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1 Introduction

We are going to start off with something that is a little bit removed from the rest of the course content, but something that is important for you to be exposed to. This is the idea of complex numbers. Complex numbers are motivated by the problem of finding the roots of a polynomial. For instance you may be aware that a quadratic equation

$$ax^2 + bx + c = 0$$

always has two roots, a cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

always has three, etc. We have to be a bit careful how we count here, and for this to be true we need to consider not just real roots but *complex* roots. For instance the two roots of the quadratic equation

$$x^2 + 2x + 5 = 0$$

are given by

$$x = 1 \pm 2\sqrt{-1}.$$

Of course there is no *real* number a such that $a^2 = -1$, and so the quantity $\sqrt{-1}$ is usually denoted i (except in some branches of engineering where it is called j . I will use i) It turns out that allowing complex numbers makes many things in mathematics much much simpler. Over the complex numbers quadratic equations always have two roots, cubic equations always have three roots, etc. The goal of this submodule is to get you comfortable in working with complex numbers.

A bit of notation: if $a + ib$ is a complex number the a is called the real part and b is called the imaginary part. So for instance the real part of $-2 + 4i$ is -2 , while the imaginary part is 4 (we generally leave out the i)

2 Addition and Multiplication

2.1 Addition

Addition of complex numbers is easy: if we add two complex numbers then the real parts add and the imaginary parts add. For instance

$$\begin{aligned} (-3 + 4i) + (5 + 7i) &= (5 + -3) + (4 + 7)i \\ &= 2 + 11i. \end{aligned}$$

If you have had some exposure to vectors then this is the same as the parallelogram rule for adding vectors: vectors add component-wise. You can think of complex numbers as a two component vector: a complex number has a real component and an imaginary component. We'll talk more about this in a little while.

2.2 Multiplication

Multiplication is a little bit harder, but hopefully should be familiar. To multiply complex numbers we use our old friend FOIL (First, Outer, Inner, Last). For instance in order to multiply $4 - i$ by $3 + 2i$ we have

$$\begin{aligned}(4 - i)(3 + 2i) &= 4 \times 3 + 4 \times (2i) + (-i) \times 3 + (-i) \times (2i) \\ &= 12 + 8i - 3i - 2i^2 \\ &= 12 + 8i - 3i - 2(-1) \\ &= 14 + 5i.\end{aligned}$$

Notice that we had to use the fact that $i^2 = -1$, and also that the F and L pieces always end up being real, and the I and O pieces always end up being imaginary, so it is really easier to do FLIO, but whatever.

3 Complex conjugate, modulus (absolute value), and ratios.

Complex numbers are usually denoted by the letter z . The complex conjugate, which is denoted by \bar{z} or sometimes z^* , is the complex number with the same real part but with an imaginary part with the opposite sign. I will always use the overbar to denote complex conjugate. For instance if $z = 3 - 7i$ then

$$\begin{aligned}z &= 3 - 7i \\ \bar{z} &= \overline{3 - 7i} \\ &= 3 + 7i\end{aligned}$$

Similarly the complex conjugate of $4 + 11i$ is $4 - 11i$.

Notice that if z is a complex number then $z\bar{z}$ is always a real positive number.

$$\begin{aligned}z &= a + ib \\ \bar{z} &= a - ib \\ z\bar{z} &= (a + ib)(a - ib) \\ &= a^2 - iab + iab + b^2 \\ &= a^2 + b^2.\end{aligned}$$

Definition 3.1. If $z = a + ib$ is a complex number then the modulus or absolute value is defined to be

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$$

Notice that if we think of the real part a and the imaginary part b as being the components of a vector then the absolute value is, by the Pythagorean theorem, just the length of the vector.

Example 3.2. Find the modulus of the complex number $5 + 12i$

$$\begin{aligned}|5 + 12i| &= \sqrt{5^2 + 12^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} = 13\end{aligned}$$

3.1 Ratios/division

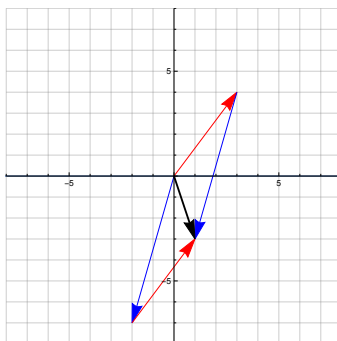
One divides complex numbers in a similar fashion to multiplication. The idea is to multiply the numerator and denominator of the fraction by the complex conjugate of the denominator so that one ends up with a real denominator. For instance

Example 3.3. Simplify $\frac{2+4i}{3-7i}$.

$$\begin{aligned}\frac{2+4i}{3-7i} &= \frac{2+4i}{3-7i} \frac{3+7i}{3+7i} \\ &= \frac{(2+4i)(3+7i)}{(3-7i)(3+7i)} \\ &= \frac{6+28i^2+12i+14i}{3^2-49i^2+21i-21i} \\ &= \frac{-22+26i}{58} \\ &= -\frac{11}{29} + \frac{13}{29}i\end{aligned}$$

4 The complex plane. Euler's theorem. DeMoivre's theorem.

In the same way that the number line provides a geometric way to picture the real numbers the plane provides a geometric way to picture complex numbers. We can associate the complex number $a + ib$ with the vector (a, b) . For instance here is an illustration of the addition $(3 + 4i) + (-2 - 7i) = 1 - 3i$. The red arrow represents the complex number $3 + 4i$ or the vector $(3, 4)$, the blue arrow represents the complex number $-2 - 7i$ or the vector $(-2, -7)$, while the black arrow represents the sum $1 - 3i$ or $(1, -3)$.



Again the modulus of the complex number is the length of the vector and the complex conjugate can be thought of as reflecting the vector across the x -axis.

Since we know how to add and multiply complex numbers we can compute any polynomial of a complex number. One might also be interested in computing more complicated functions of a complex number. One important result is Euler's¹ Theorem, which tells us how to exponentiate complex numbers.

Theorem 4.1 (Euler). *If $z = a + ib$ is a complex number then we have that*

$$e^z = e^{a+ib} = e^a(\cos(b) + i \sin(b))$$

Sometimes you will hear DeMoivre's theorem mentioned: this is just a special case of Euler's theorem

Theorem 4.2 (DeMoivre).

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

¹Pronounced "Oiler"

Anything you can do with DeMoivre you can do with Euler, but I thought that I ought to mention it.

Example 4.3. Compute $e^{i\frac{\pi}{4}}$.

$$\begin{aligned} e^{i\frac{\pi}{4}} &= e^0(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})) \\ &= 1(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) \\ &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \end{aligned}$$

Euler's theorem gives us a kind of "polar coordinates" for complex numbers.

Theorem 4.4. Every complex number $z = a + ib$ can be written in polar form $z = \rho e^{i\theta}$. The quantities a and b are related to ρ, θ via

$$\begin{aligned} a &= \rho \cos \theta \\ b &= \rho \sin \theta \end{aligned}$$

and conversely

$$\begin{aligned} \rho &= \sqrt{a^2 + b^2} \\ \theta &= \arctan \frac{b}{a} \end{aligned}$$

Note that if you are given ρ, θ it is easy to find a, b . If you are given a and b it is easy to find ρ (just the modulus/absolute value), but one has to be a little bit careful when finding θ to pick the right branch.

Example 4.5. Given the polar form $4e^{\frac{3\pi}{4}i}$ find the Cartesian form $a + ib$.

$$\begin{aligned} a + ib &= 4e^{\frac{3\pi}{4}i} \\ &= 4(\cos \frac{3\pi}{4} + i\sin \frac{3\pi}{4}) \\ &= 4(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) \\ &= -2\sqrt{2} + 2\sqrt{2}i \end{aligned}$$

Example 4.6. Given the Cartesian form $\sqrt{3} - i$ find the polar form $\rho e^{i\theta}$

First we find ρ the absolute value

$$\rho = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{3 + 1} = \sqrt{4} = 2.$$

Next we find the angle θ

$$\theta = \arctan\left(\frac{1}{-\sqrt{3}}\right) = \arctan\left(-\frac{\sqrt{3}}{3}\right)$$

Here is where we have to be a little bit careful. There are two possible values of $\theta = \arctan(-\frac{\sqrt{3}}{3})$: either $\theta = \frac{5\pi}{6}$ or $\theta = -\frac{\pi}{6}$.

To resolve this note that $\sqrt{3} - i$ has a positive x value and a negative y value – it lies in the fourth quadrant. An angle of $\theta = \frac{5\pi}{6}$ lies in the second quadrant and an angle of $\theta = -\frac{\pi}{6}$ lies in the fourth quadrant. So $\theta = -\frac{\pi}{6}$ is the right solution

$$\sqrt{3} - i = 2e^{-\frac{i\pi}{6}}$$

As a last example

Example 4.7. Compute $(\sqrt{2} + i\sqrt{2})^{2020}$.

Of course we could simply multiply $(\sqrt{2} + i\sqrt{2})$ by itself many, many times, but that could get dull. Instead let's use Euler's theorem and the polar form. Note that

$$\begin{aligned}(\sqrt{2} + i\sqrt{2}) &= 2e^{i\frac{\pi}{4}} \\(\sqrt{2} + i\sqrt{2})^{2020} &= 2^{2020}(e^{i\frac{\pi}{4}})^{2020} \\(\sqrt{2} + i\sqrt{2})^{2020} &= 2^{2020}(e^{i\frac{\pi}{4}2020}) \\(\sqrt{2} + i\sqrt{2})^{2020} &= 2^{2020}(e^{i505\pi}) \\(\sqrt{2} + i\sqrt{2})^{2020} &= 2^{2020}(\cos(505\pi) + i\sin(505\pi)) \\(\sqrt{2} + i\sqrt{2})^{2020} &= 2^{2020}(-1 + i0) = -2^{2020}\end{aligned}$$

Note that here we have used the polar form of $(\sqrt{2} + i\sqrt{2})$ (check it!) the laws of exponents $((e^a)^b = e^{ab})$ and the fact that if k is an integer then $\sin k\pi = 0$ and $\cos k\pi = -1$ if k is odd and $\cos k\pi = 1$ if k is even.