§ Lecture 15 20s01x

last time: matrix of LT, determinants

§ Determinants

Goal: determine whether square matrix is invertible.

$\det(I_n) = 1$

elem row op	effect on det
swap	$(-1) \times$
scalar ($c \neq 0$)	$(c) \times$
shear ($i \neq j$)	$(1) \times$

Fact: det of square triangular matrix is the product of main diagonal values

Example:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ \boxed{-1} & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow[\text{new } R_2]{R_1+R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \boxed{1} & 1 & 0 \end{pmatrix} \xrightarrow[\text{new } R_3]{-R_1+R_3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & \boxed{1} & 1 \end{pmatrix} \xrightarrow[\text{new } R_3]{-R_2+R_3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$\times 1$ $\times 1$ $\times 1$

row-échelon form

main diagonal

$\det(\text{REF}) = 1 \cdot 1 \cdot 2 = 2$

$2 = \det(\text{REF}) = \times 1 \ \times 1 \ \times 1 \ \det(A) = \det(A)$

$\Rightarrow \det(A) = 2$

Properties:

(1) $\det(AB) = \det(A)\det(B)$

(2) $\det(A^T) = \det(A)$

(3) $\det(cA) = c^n \det(A)$ $\quad A_{n\times n}$

(4) $\det(A) = 0 \iff A$ is not invertible

Another method for computing determinant: cofactor expansion

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $\quad$ $(1,3)$-cofactor $= (-1)^{1+3} \det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} = c_{13}$

$(3,2)$-cofactor $= (-1)^{3+2} \det\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = c_{32}$

expand along first row:

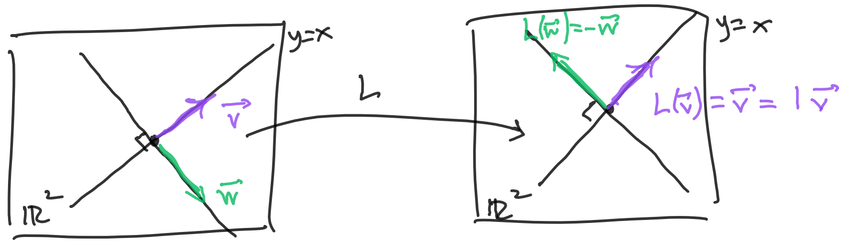
$$\det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = 1\,(-1)^{1+1}\det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} + 2\,(-1)^{1+2}\det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3\,(-1)^{1+3}\det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

expand along second column:

$$\det(A) = a_{12}c_{12} + a_{22}c_{22} + a_{32}c_{32} = 2\,(-1)^{1+2}\det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 5\,(-1)^{2+2}\det\begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} + 8\,(-1)^{3+2}\det\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$$

§ Eigenvalues & Eigenvectors

Motivation: LT $L: \mathbb{R}^2 \to \mathbb{R}^2$ reflection about line $y=x$



$L(\vec{v}) = 1\vec{v}$

$L(\vec{w}) = (-1)\vec{w}$

$\vec{v}, \vec{w}$ are examples of eigenvectors of LT L

Defn. LT $L: V \to V$ say non-zero vector $\vec{v} \neq \vec{0}$ is an eigenvector for L if $L(\vec{v})$ is a scalar multiple of $\vec{v}$, i.e. $L(\vec{v}) = \lambda \vec{v}$ say λ is the eigenvalue associated to $\vec{v}$.

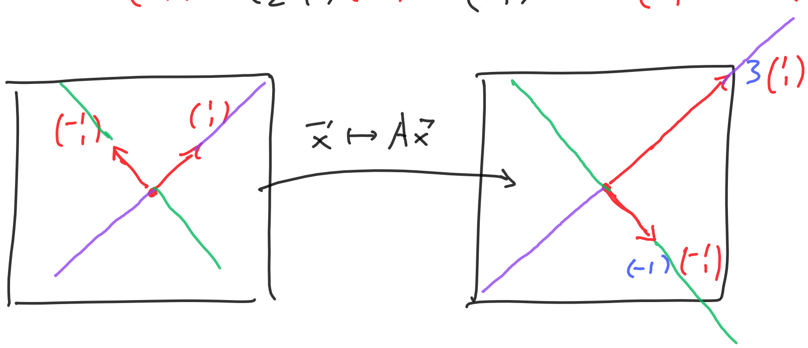
Example: In the case LT $L: \mathbb{R}^n \to \mathbb{R}^n$ $\vec{x} \mapsto A\vec{x}$ $A_{n \times n}$ matrix

we say matrix A has $\left\{ \begin{array}{l} \text{eigenvalue } \lambda \\ \text{eigenvector } \vec{v} \end{array} \right.$ when $A\vec{v} = \lambda \vec{v}$ for some $\vec{v} \neq \vec{0}$.

Example: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $L_A: \mathbb{R}^2 \to \mathbb{R}^2$ $\vec{x} \mapsto A\vec{x}$

Notice: $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector for A with eigenvalue 3

$A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)\begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ eigenvector for A with eigenvalue -1



$$A\, r\begin{pmatrix} 1 \\ 1 \end{pmatrix} = r A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = r\left(3\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = 3r\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Q: How compute eigenvalues & eigenvectors for matrix A?

Fact: eigenvalues of A are roots of the polynomial $p(\lambda) = \det(A - \lambda I)$ called characteristic polynomial

λ-eigenspace $E_\lambda := \mathrm{Nul}(A - \lambda I)$

Example: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

characteristic polynomial $p(\lambda) = \det(A - \lambda I) = \det\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$

$$= (1-\lambda)^2 - 2^2 = \lambda^2 - 2\lambda + 1 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$$

eigenvalues of A are roots of $p(\lambda)$: $\lambda = -1, 3$

Eigenspaces:

$\lambda = -1$ $\quad E_{-1} = \mathrm{Nul}(A-(-1)I) = \mathrm{Nul}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} = \mathrm{span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ $\quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ eigenvector for eigenvalue (-1)

$\lambda = 3$ $\quad E_3 = \mathrm{Nul}(A-3I) = \mathrm{Nul}\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ $\quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector for eigenvalue 3