Math 231E, Lecture 7.
Limits using Taylor Series

1 L'Hôpital's rule

One important technique for computing limits is given by L'Hôpital's rule, which says the following:

Theorem 1.1. Suppose that we have a limit in the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where a can be a finite point or $\pm\infty$ and *the limit is in one of the indeterminate forms* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Note that while the Marquis de l'Hôpital discovered many things he did not, in fact, discover this rule – it is actually due to Jakob and Johann Bernoulli. Also the Marquis actually spelled his name l'Hospital: French spelling has changed in the intervening time.

Example 1.1. Compute the limit $\lim_{x \to 0} \frac{1-\cos(x)}{x^2}$.

Note first that this is in the required form $\frac{0}{0}$ since the numerator and the denominator each tend to zero as $x \to 0$. We need to apply the l'Hôpital rule twice:

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{\sin(x)}{2x} = \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2}$$

Example 1.2. Compute the limit $\lim_{x \to \infty} \frac{3x^2+2x+5}{4x^2-11x-7}$.

Note first that this is in the required form $\frac{\infty}{\infty}$ since the numerator and the denominator each tend to ∞ as $x \to \infty$. We need to apply the l'Hôpital rule twice:

$$\lim_{x \to \infty} \frac{3x^2 + 2x + 5}{4x^2 - 11x - 7} = \lim_{x \to \infty} \frac{6x + 2}{8x - 11} = \lim_{x \to \infty} \frac{6}{8} = \frac{3}{4}$$

note that you should be able to just eyeball this using the rules of thumb given in lecture: the largest term in the numerator for x large is $3x^2$, the largest in the denominator is $4x^2$, so the ratio tends to $\frac{3}{4}$.

l'Hôpital's rule is useful but in practice it is usually more convenient to use Taylor series. This is the main topic for this set of lecture notes.

2 Computing limits using Taylor series

Example 2.1. Let us now consider the limit

$$\lim_{x \to 0} \frac{\sin(x)}{x}.$$

We cannot use the Limit Law, since the denominator goes to zero. We know that one way to do this is l'Hôpital's Rule, but if we have Taylor series there is a better way to go.

Recall the Taylor series for $\sin(x)$:

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7),$$

so

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6).$$

It is easy to see if we take the limit as $x \to 0$, the right-hand side goes to one, so

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

In fact, we can use Taylor series to derive l'Hôpital's Rule, as follows: let us say that $\lim_{x\to a} f(a) = \lim_{x\to a} g(x) = 0$. Then we compute

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(a) + f'(a)(x-a) + O(x-a)^2}{g(a) + g'(a)(x-a) + O(x-a)^2}.$$

We know that $f(a) = g(a) = 0$, so

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a) + O(x-a)^2}{g'(a)(x-a) + O(x-a)^2} = \lim_{x\to a} \frac{f'(a) + O(x-a)}{g'(a) + O(x-a)},$$

where we got the last equality by dividing by $(x-a)$. But we can then use the Limit Law (as long as $g'(a) \neq 0$) and obtain

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example 2.2. We can do much more complicated examples using Taylor series. For example, say that we want to compute

$$\lim_{x\to 0} \frac{\cos(x^2) - e^{x^4}}{\sin(x^4)}.$$

Let us use Taylor series. We have

$$\begin{aligned}
\cos(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6),\\
\cos(x^2) &= 1 - \frac{x^4}{2} + \frac{x^8}{24} + O(x^{12}),\\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4),\\
e^{x^4} &= 1 + x^4 + \frac{x^8}{2} + \frac{x^{12}}{6} + O(x^{16}),\\
\sin(x) &= x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7),\\
\sin(x^4) &= x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28}).
\end{aligned}$$

So we have

$$\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = \frac{\left(1 - \frac{x^4}{2} + \frac{x^8}{24} + O(x^{12})\right) - \left(1 + x^4 + \frac{x^8}{2} + \frac{x^{12}}{6} + O(x^{16})\right)}{x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28})}.$$

Adding like terms in the numerator gives

$$\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = \frac{-\frac{3}{2}x^4 - \frac{11}{24}x^8 + O(x^{12})}{x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28})}.$$

We see that every term in the expression is divisible by x^4, so divide this out to obtain

$$\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = \frac{-\frac{3}{2} - \frac{11}{24}x^4 + O(x^8)}{1 - \frac{x^8}{6} + \frac{x^{16}}{120} + O(x^{24})},$$

and taking limits as $x \to 0$ on both sides gives

$$\lim_{x\to 0} \frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = -\frac{3}{2}.$$

Example 2.3. Here's a more complicated example. Let us compute:

$$\lim_{x\to 0} \frac{\cos(\sin(\sin(\sin(\sin(\sin(\sin(\sin(x)))))))) - 1}{\cos(x) - 1}.$$

(That's seven iterations of $\sin(\cdot)$ inside there, for those scoring at home.)

Let us first consider the iterated applications of the sine function. First we note that

$$\sin(x) = x - \frac{x^3}{6} + O(x^5).$$

Now we want to compute the first two terms in the Taylor series for $\sin(\sin(x))$. Let us write $y = \sin(x)$, and then we have

$$\begin{aligned}\sin(y) &= y - \frac{y^3}{6} + O(y^5)\\ &= \sin(x) - \frac{1}{6}(\sin(x))^3 + O(\sin(x)^5).\end{aligned}$$

Now we plug in the Taylor series for sine. Note that since every term in the Taylor series is at least linear, then terms of $O(\sin(x)^5)$ will be $O(x^5)$, and we have

$$\begin{aligned}\sin(\sin(x)) &= \left(x - \frac{x^3}{6} + O(x^5)\right) - \frac{1}{6}\left(x - \frac{x^3}{6} + O(x^5)\right)^3 + O(x^5)\\ &= x - \frac{x^3}{6} - \frac{x^3}{6} + O(x^5)\\ &= x - \frac{x^3}{3} + O(x^5).\end{aligned}$$

The only thing that might not be obvious in the expansion above is how we got from the first to the second line, i.e. how do we know that every term that we neglected to write down is $O(x^5)$ or higher order? The real question is the expression

$$\frac{1}{6}\left(x - \frac{x^3}{6} + O(x^5)\right)^3,$$

and let's think about these terms. We can clearly get an x^3 term by taking x three times. Now notice that the next order term must be five: we take two x and one $-x^3/6$, and this will be $O(x^5)$. And of course everything else will be of even higher order.

But we have noticed a pattern here. If we are only interested in the **one term** Taylor expansion of $\sin(x)$, then we have

$$\sin(\sin(x)) = x + O(x^3),$$

which is the same as for $\sin(x)$! So then let us plug in the Taylor series for $\sin(x)$ into $\sin(\sin(x))$:

$$\begin{aligned}\sin(\sin(y)) &= y + O(y^3)\\ &= \left(x - \frac{x^3}{6} + O(x^5)\right) + O\left(\left(x - \frac{x^3}{6} + O(x^5)\right)^3\right)\end{aligned}$$

and we can see that $\sin(\sin(\sin(x))) = x + O(x^3)$ again.

We also have

$$\cos(x) - 1 = \frac{x^2}{2} + O(x^4)$$

and

$$\cos(x + O(x^3)) - 1 = \frac{1}{2}(x + O(x^3))^2 + O(x^4) = \frac{x^2}{2} + O(x^4)$$

so that the numerator and the denominator of the original fraction have the same first-order Taylor series. In short

$$\lim_{x\to 0} \frac{\cos(\sin(\sin(\sin(\sin(\sin(\sin(\sin(x)))))))) - 1}{\cos(x) - 1} = \frac{\frac{x^2}{2} + O(x^4)}{\frac{x^2}{2} + O(x^4)}$$

and as $x \to 0$ clearly the limit is 1.

3 Using Taylor Series for vertical asymptotes

Recall the calculation from last time:

$$\begin{aligned}\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} &= \frac{\left(1 - \frac{x^4}{2} + \frac{x^8}{24} + O(x^{12})\right) - \left(1 + x^4 + \frac{x^8}{2} + \frac{x^{12}}{6} + O(x^{16})\right)}{x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28})}\\ &= \frac{-\frac{3}{2}x^4 - \frac{11}{24}x^8 + O(x^{12})}{x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28})}\\ &= \frac{-\frac{3}{2} - \frac{11}{24}x^4 + O(x^8)}{1 - \frac{x^8}{6} + \frac{x^{16}}{120} + O(x^{24})},\end{aligned}$$

and so

$$\lim_{x\to 0} \frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = -\frac{3}{2}.$$

What worked out here was that the numerator and the denominator had the same leading order, so we were able to cancel terms and get a nice finite number.

What if this doesn't happen? Let us consider the case

$$\lim_{x\to 0} \frac{\cos(x)}{x^2} = \lim_{x\to 0} \frac{1 - \frac{x^2}{2} + O(x^4)}{x^2} = \frac{1}{x^2} + \frac{1}{2} + O(x^2).$$

Now, we know

$$\lim_{x\to 0}\frac{1}{x^2}=\infty,$$

so $\lim_{x\to 0}\frac{\cos(x)}{x^2}=\infty$ as well.

Moreover, this tells us more information that just that the limit is ∞, it also tells us how quickly we approach ∞;