Math 231E, Lecture 20.
Improper Integrals

1 Examples of weirdness

Example 1.1. Let us consider the indefinite integral

$$\int \frac{dx}{(x-1)(x-2)}.$$

We will use partial fractions:

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}.$$

We obtain

$$1 = A(x-2) + B(x-1).$$

If we plug in $x = 1$ this gives $A = -1$, and if we plug in $x = 2$ this gives $B = 1$, so

$$\frac{1}{(x-1)(x-2)} = \frac{1}{x-2} - \frac{B}{x-1}.$$

Therefore

$$\int \frac{dx}{(x-1)(x-2)} = \ln|x-2| - \ln|x-1| + C = \ln\left|\frac{x-2}{x-1}\right| + C.$$

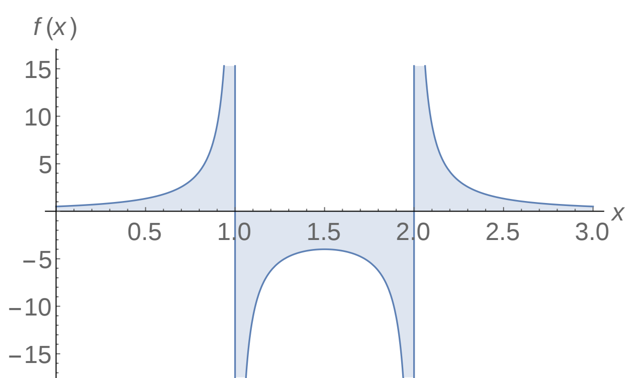

Figure 1: The function $1/((x-1)(x-2))$

Ok, great. So let us use this to compute

$$\int_0^3 \frac{dx}{(x-1)(x-2)} = \left(\ln\left|\frac{x-2}{x-1}\right|\right)\Bigg|_{x=0}^{x=3} = \ln(1/2) - \ln 2 = \ln(1/4) \approx -1.38.$$

However, let us look at the graph in Figure 1.

The number that popped out of this computation doesn't seem to be correct at all. But, maybe, just maybe, that infinite area below the axis cancels most of the infinite area above the axis, and we end up with a bit of negative excess??

Example 1.2. Let us now consider

$$\int_0^2 \frac{dx}{(x-1)^2} = \frac{-1}{x-1}\Bigg|_{x=0}^{x=2} = -1 - (1) = -2.$$

Ok, now this is just completely weird. This function is positive everywhere on the line (see Figure 2, how can it have a negative integral???

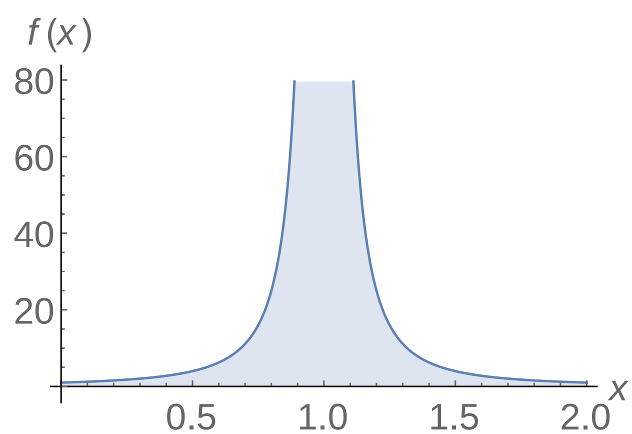

Figure 2: The function $1/(x-1)^2$

IS THE FUNDAMENTAL THEOREM OF CALCULUS WRONG WHAT IS THIS I DONT EVEN

2 Improper Integrals

The problem in the previous two examples is that the Fundamental Theorem of Calculus is not wrong because it does not apply to those situations. Recall that the FTC is only guaranteed to work for functions that are continuous on their domains, and these functions are not!

So to deal with the kinds of situations that we just saw, we need to talk about **improper integrals**. As it turns out, there are two types of improper integrals, and we refer to them differently.

1. **Type I** improper integrals are those we need to compute on an infinite interval.
2. **Type II** improper integrals are those for which the function is discontinuous on the integration interval.

The two cases we saw above are both of Type II.

3 Type I integrals

Let us consider the integral

$$\int_1^\infty \frac{dx}{x^2}.$$

See Figure 3.

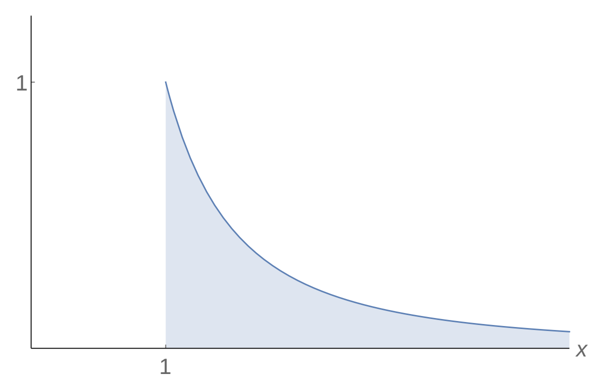

Figure 3: $1/x^2$

How should we deal with this integral? If we think of the integral as the area from 1 to infinity, let us try to compute the area from 1 to t, where t is a large number:

$$\int_1^t \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{x=1}^{x=t} = 1 - \frac{1}{t}.$$

Now notice that as t gets large, this last quantity behaves pretty well — in fact the t-dependence completely disappears. This seems like a good thing.

So let us make a definition:

Definition 3.1. We define the integral

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,$$

and

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_{t}^{b} f(x)\,dx,$$

if these limits exist.

If the limit exists, we say the integral **converges**, and if it does not exist we say it **diverges**.

Example 3.2. The integral

$$\int_1^\infty = \lim_{t\to\infty}\int_1^t \frac{dx}{x^2} = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1.$$

So we say the integral converges to 1.

We have to be a bit careful about how we define integrals on the whole real line. We do it as follows:

Definition 3.3. If both of the integrals

$$\int_{-\infty}^{a} f(x)\,dx, \quad \int_a^\infty f(x)\,dx$$

exist for some a, then we say that the integral

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^\infty f(x)\,dx,$$

and that the integral converges.

Question: For which p does the integral

$$\int_1^\infty \frac{dx}{x^p}$$

converge?

1. Let us first consider the case with $p=1$. Then

$$\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{t\to\infty}\ln|t| = \infty.$$

Therefore this diverges for $p=1$.

2. Now assume that $p\neq 1$. Then we have

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^p} = \lim_{t\to\infty}\left(\frac{x^{1-p}}{1-p}\right)\Bigg|_{x=1}^{x=t} = \lim_{t\to\infty}\frac{1}{1-p}\left(t^{1-p}-1\right).$$

Clearly the constant out front will not effect the finiteness of this limit. But notice that if we have a **positive** power on t, then the limit will be ∞, whereas if we have a **negative** power on t then the limit is finite. Therefore, we have

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & p>1, \\ \\ \text{diverges}, & p\le 1. \end{cases}$$

(This is sometimes called "the p-test for integrals". See more on this below.)

4 Type II Improper Integrals

These are integrals where the function has a discontinuity in the domain, especially if the function blows up to $\pm\infty$. So we have three cases:

1. If f is continuous on $[a, b)$ but discontinuous at b, then we define
$$\int_a^b f(x)\,dx = \lim_{t\to b-} \int_a^t f(x)\,dx.$$

2. If f is continuous on $(a, b]$ but discontinuous at a, then we define
$$\int_a^b f(x)\,dx = \lim_{t\to a+} \int_t^b f(x)\,dx.$$

3. If f has a discontinuity at c, with $a < c < b$, and both
$$\int_a^c f(x)\,dx, \quad \int_c^b f(x)\,dx$$
exist, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 4.1. Let us consider the integral
$$\int_{-1}^1 \frac{dx}{\sqrt{|x|}}.$$
This function clearly has a discontinuity at $x = 0$, so we need to consider the integrals
$$\int_{-1}^0 \frac{dx}{\sqrt{|x|}}, \quad \int_0^1 \frac{dx}{\sqrt{|x|}}$$
separately. However, we can see by the change of variables $x \mapsto -x$ that these two integrals are the same, so if we can compute either one then the target integral is just twice that.

So we compute
$$\int_0^1 x^{-1/2}\,dx = \lim_{t\to 0+} \int_t^1 x^{-1/2}\,dx = \lim_{t\to 0+} 2(1-\sqrt{t}) = 2.$$
Therefore the original integral is 4.

5 Comparison Principle

In this section we consider only functions that are continuous and positive (although we can expand these ideas much more generally).

Theorem 5.1 (Comparison Principle)**.** Let $g, h \geq 0$ be continuous, and assume $g(x) \leq h(x)$ for all x. Then

1. If $\int_a^\infty g(x)\,dx$ diverges, so does $\int_a^\infty h(x)\,dx$.
2. If $\int_a^\infty h(x)\,dx$ converges, so does $\int_a^\infty g(x)\,dx$.

Proof. Let us write

$$G(t) = \int_a^t g(x)\,dx, \quad H(t) = \int_a^t h(x)\,dx.$$

Since $g, h \geq 0$, it follows that $G, H \geq 0$ for all $t \geq a$. Moreover, we also know from FTC that $G', H' \geq 0$, so that G, H are nondecreasing. Also, we have that $H(t) \geq G(t)$ for all $t \geq a$.

Now, assume that

$$\lim_{t\to\infty} H(t) < \infty.$$

Then we have

$$G(t) \leq \lim_{t\to\infty} H(t) < \infty$$

for all t. Since G is nondecreasing and bounded above, its limit exists and is finite.

The proof in the other direction is similar. If $\lim_{t\to\infty} G(t) = \infty$, then for every M, there is some t for which $G(t) > M$. But then $H(t) \geq G(t) > M$, so this measn that we can make $H(t)$ as large as we would like. From this it follows that $\lim_{t\to\infty} H(t) = \infty$. □

Remark 5.2. An intuitive way to put the above theorem is like this. Say $G \leq H$. If $G = \infty$ then $H \geq \infty$ and therefore $H = \infty$. Conversely, if H is finite, and G is smaller, then G is also finite.

Note, however, that we cannot always use this theorem to get more information. For example, in this case, knowing that $\int_a^\infty h(x)\,dx$ diverges tells us nothing. The way to think about this is if $G \leq H$ and $H = \infty$ then all we know is that $G \leq \infty$, which is true but is actually content-free.

We can actually extend the Comparison Principle in a simple but useful manner:

Theorem 5.3. Let $f, g \geq 0$ be continuous, and assume there is an x_0 such that $f(x) \leq g(x)$ for all $x > x_0$. Then

1. If $\int_a^\infty f(x)\,dx$ diverges, so does $\int_a^\infty g(x)\,dx$.
2. If $\int_a^\infty g(x)\,dx$ converges, so does $\int_a^\infty f(x)\,dx$.

Remark 5.4. A short way to read this is that as long as g is bigger than f **eventually**, then we can still use the theorem.

Example 5.5. Can we determine whether or not

$$\int_0^\infty \frac{dx}{x^5 + 4x^4 + 3x^2 + 6x + 1}$$

converges or diverges?

One idea might be to try and do the integral, but that looks insane. Instead, let us note that for all $x > 0$, we have

$$x^5 + 4x^4 + 3x^2 + 6x + 1 > x^5,$$

and thus

$$\frac{1}{x^5 + 4x^4 + 3x^2 + 6x + 1} < \frac{1}{x^5}.$$

Therefore our integral converges if $\int_0^\infty x^{-5}\,dx$ does, but we know that this does from the p-test earlier.

Remark 5.6. A short way to read this is that as long as g is bigger than f **eventually**, then we can still use the theorem.

Example 5.7. Can we determine whether or not

$$\int_a^\infty \frac{dx}{x^5 - 4x^4 - 3x^2 - 6x + 1}$$

converges or diverges?

Let us try and mimic the idea of the last example. If the denominator is greater than x^5, then we are done by the comparison principle. However, it unfortunately is not, since we have some negative signs in there.

However, we ask the question: determine for which x can we guarantee that

$$x^5 - 4x^4 - 3x^2 - 6x + 1 > \frac{1}{2}x^5.$$

This simplifies to

$$\frac{1}{2}x^5 > 4x^4 + 3x^2 + 6x - 1.$$

There is clearly an x_0 such that for all $x > x_0$, this equation is correct. To see this, divide through by x^5 to obtain

$$\frac{1}{2} > \frac{4}{x} + \frac{3}{x^2} + \frac{6}{x^4} - \frac{1}{x^5}.$$

Since the limit of the right-hand side is 0 as $x \to \infty$, it must eventually become less than $1/2$ and stay there. Therefore our integral converges if $\int_0^\infty (x/2)^{-5}\,dx$ does, but just is just a constant times the p-test earlier.

Now, we didn't specify the lower limit a in the integral above: notice that for any of this to be correct, we have to assume that a is large enough that the denominator is not zero on $[a, \infty)$, so that we really have a Type II integral. If there were a discontinuity in the function because the denominator was zero, we would have to split up the domain, etc.

We can see the general principle from these examples should be as follows: given the integral

$$\int_a^\infty \frac{P(x)}{Q(x)}\,dx$$

where $Q(x) \neq 0$ on $[a, \infty)$ and $\deg Q - \deg P > 1$, then this integral converges.

How can we deal with

1. $\int_1^\infty \dfrac{dx}{x \ln x}$,
2. $\int_1^\infty \dfrac{dx}{x^p \ln x}$?