Math 415. Exam 1. September 28, 2017

Full Name: ____________________

Net ID: ________________________

Discussion Section: ___________

- There are 18 problems worth 5 points each.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 70 minute exam.
- Do not turn this page until instructed to.
- Fill in the answers on the scantron form provided. Also circle your answers on the exam itself.
- Hand in both the exam and the scantron.
- On the scantron make sure you bubble in **your name, your UIN and your NetID**.
- There are several different versions of this exam.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- Good luck!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**!
On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) Consider the following multiplication of two 3×3-matrices, where the question marks represent unknown coefficients:

$$\begin{bmatrix} 9 & 8 & 6 \\ ? & 9 & ? \\ ? & 3 & 1 \end{bmatrix} \begin{bmatrix} 8 & 1 & 4 \\ 2 & ? & ? \\ 3 & 3 & ? \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}.$$

With the information given, only one coefficient in the matrix on the right hand side can be calculated. Which one is it?

(A) c_{11}

(B) c_{31}

(C) c_{21}

(D) c_{12}

(E) None of the other answers.

2. (5 points) Let $\mathbf{a}, \mathbf{b}$ be non-zero vectors in $\mathbb{R}^3$, where $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$. Which of the following is a description of the set $\operatorname{span}(\mathbf{a}, \mathbf{b})$?

(A) It is $\mathbb{R}^2$.

(B) It is a line in $\mathbb{R}^3$ through the origin.

(C) It is the union of two lines in $\mathbb{R}^3$.

(D) It is a plane in $\mathbb{R}^3$ through the origin.

(E) None of the other answers.

3. (5 points) Let $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{b}$ be three vectors in $\mathbb{R}^3$ and suppose $\mathbf{b}$ is in span$(\mathbf{a}_1, \mathbf{a}_2)$. Consider the matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{bmatrix}$, that is the matrix whose first column is $\mathbf{a}_1$, whose second column is $\mathbf{a}_2$ and whose third column is $\mathbf{b}$. Which of the following statements is true?

(A) A is row equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(B) None of the other answers.

(C) A has at most 2 pivots.

(D) A is invertible.

4. (5 points) For which values of b is the matrix $\begin{bmatrix} 1 & b \\ 2 & 2b \end{bmatrix}$ invertible?

(A) For any number b different from 1.

(B) For no number b.

(C) None of the other answers.

(D) For any number b.

(E) For any number b different from 0.

5. (5 points) Let A be a 3×3 matrix and $\mathbf{b} \in \mathbb{R}^{\mathbf{3}}$. Consider the following statements:

I. The row reduced echelon form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

II. If the linear system $A\mathbf{x} = \mathbf{b}$ has a solution, then it has a unique solution.

Which one of these statements is always true?

(A) Statement I only.

(B) Statement II only.

(C) Statement I and Statement II.

(D) Neither of Statements I or II.

6. (5 points) Let

$$\boldsymbol{w} = \begin{bmatrix} 1 \\ h \\ h \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

For which values of h is $\boldsymbol{w}$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?

(A) Only when $h = -1$.

(B) Only when $h = 1$.

(C) For no value of h.

(D) None of the other answers.

7. (5 points) The matrix $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ is reduced to the identity matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ using the following row operations (in the given order):

(1) $R_2 \leftrightarrow R4$.

(2) $R_2 \to R_2 + 2R_1$,

What is A^{-1}?

(A) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(D) None of the other answers.

(E) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

8. (5 points) Let $A = \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Which one of the following statements is true?

(A) There is $\mathbf{b} \in \mathbb{R}^{\mathbf{4}}$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(B) A is invertible.

(C) None of the other answers.

(D) $A\mathbf{x} = \mathbf{0}$ has exactly one solution.

(E) A does not have an LU decomposition.

9. (5 points) Let A be an $m \times n$ matrix. Which one of the following statements is true?

(A) $A^T A$ is an $n \times n$ matrix.

(B) $A^T A$ is an $m \times n$ matrix.

(C) $A^T A$ is an $m \times m$ matrix.

(D) $A^T A$ is an $n \times m$ matrix.

(E) None of the other answers.

10. (5 points) Which of the following vectors does NOT belong to the set

$$\text{span}\left(\begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \begin{bmatrix}2\\4\\6\\0\end{bmatrix}, \begin{bmatrix}4\\8\\12\\0\end{bmatrix}\right)?$$

(A) $\begin{bmatrix}3\\6\\9\\0\end{bmatrix}$

(B) $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

(C) $\begin{bmatrix}11\\22\\33\\0\end{bmatrix}$

(D) $\begin{bmatrix}2\\4\\6\\0\end{bmatrix}$

(E) $\begin{bmatrix}5\\10\\20\\0\end{bmatrix}$

11. (5 points) Which one of the following statements is FALSE?

(A) If a system of linear equations has two different solutions, it must have infinitely many solutions.

(B) If A is invertible, the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

(C) Every matrix is row equivalent to a unique matrix in reduced row echelon form.

(D) Every system of n linear equations in n variables has exactly one solution.

12. (5 points) Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$ be a 3×2 matrix and suppose that

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{for } \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and } A\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{for } \mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then the following holds:

(A)

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\mathbf{a}_1 + 1\mathbf{a}_2, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\mathbf{a}_1 + 2\mathbf{a}_2$$

(B) None of the other answers.

(C)

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{a}_1 + \mathbf{a}_2, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{a}_1 + 2\mathbf{a}_2$$

(D)

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{a}_1 + 2\mathbf{a}_2, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{a}_1 + \mathbf{a}_2$$

(E)

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{a}_1 + 0\mathbf{a}_2 + 3\mathbf{a}_3, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\mathbf{a}_1 + 1\mathbf{a}_2 + 1\mathbf{a}_3$$

13. (5 points) Let P be the 4×4-permutation matrix that permutes the second row and the third row. Which of the following statements is true?

(A) $P^7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $(P^{-1})^9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(B) $P^7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $(P^{-1})^9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(C) $P^7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $(P^{-1})^9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(D) None of the other answers.

(E) $P^7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $(P^{-1})^9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

14. (5 points) Find an explicit description of the null space of

$$A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

that is, find a minimal set of vectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}$ such that $\operatorname{Nul}(A) = \operatorname{span}(\boldsymbol{v_1}, \ldots, \boldsymbol{v_p})$.

(A) None of the other answers.

(B) $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(E) $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$

15. (5 points) Consider the following subsets of $\mathbb{R}^2$:

$$W_1 = \{\begin{bmatrix} a \\ a \end{bmatrix} : a \in \mathbb{R}\},\ W_2 = \{\begin{bmatrix} a \\ b \end{bmatrix} : ab = 1\}.$$

Then:

(A) Only W_1 is a subspace of $\mathbb{R}^2$.

(B) Only W_2 is a subspace of $\mathbb{R}^2$.

(C) Neither W_1 nor W_2 is a subspace of $\mathbb{R}^2$.

(D) Both W_1 and W_2 are subspaces of $\mathbb{R}^2$.

16. (5 points) Consider the following matrix

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -2 & 2 & 0 \\ 2 & 2 & -2 & 0 \end{bmatrix}.$$

What is the number of pivot positions of this matrix?

(A) 3

(B) 0

(C) None of the other answers.

(D) 1

(E) 2

17. (5 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

Which of the following is the matrix L in an LU factorization of A?

(A) $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}.$

(B) $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}.$

(C) $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}.$

(D) $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}.$

(E) None of the other answers.

18. (5 points) Let $D = \{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \}$ denote the set of vectors in the unit disk of $\mathbb{R}^2$. It can be shown that D is NOT a subspace of $\mathbb{R}^2$. Which of the following tests does D **fail to satisfy**? (Select all that apply.)

I. contains the zero vector

II. closed under vector addition

III. closed under scalar multiplication

(A) III. only

(B) I., II., and III.

(C) II. only

(D) II. and III. only

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