Math 231E, Lecture 10.
Differentials, Exponential Decay & Related Rates

---

1 Differentials

A differential is an expression of the idea of the linear or tangent line approximation. Differentials are often useful when one has a function depending on a number of quantities that is constant – in other words an implicit relationship between the quantities. This is very common in (for instance) thermodynamics.

For instance the *adiabatic gas law* says that

$$PV^{\gamma} = \text{constant}$$

where P is the pressure, V is the volume, γ is a constant determined by the thermal properties of the material, and "adiabatic" means that changes to the volume/pressure are made very slowly so that the system remains close to equilibrium at all times.

Example 1.1. Suppose that air in a chamber satisfies the adiabatic gas law with $\gamma = 1.4$. Suppose that the volume is increased by 10%. How does the pressure change?

Taking logarithms we have

$$\ln|PV^{\gamma}| = ln|P| + \gamma \ln|V| = \ln|\text{constant}|$$

Taking the differential gives

$$\frac{dP}{P} + \gamma \frac{dV}{V} = 0$$

A relative change of +10% in the volume means that $\frac{dV}{V} = 0.1$, so $\frac{dP}{P} = -\gamma \frac{dV}{V} = -.14$, so the pressure decreases by 14%.

Example 1.2. The *ideal gas law* is a law that relates pressure, volume, and temperature of a gas, and is given by

$$PV = nRT,$$

where P is the pressure, V is the volume, T is the temperature, n is the number of moles in the gas, the R is the *ideal gas constant*

$$R \approx 8.314 \, \frac{\mathrm{J}}{\mathrm{K} \cdot \mathrm{mol}}.$$

Let us use differentials to solve the following problem: given a (relative) increase in pressure by 2% and a (relative) decrease in temperature by 3%, what the is approximate relative change in volume?

We have

$$\ln(PV) = \ln(nRT)$$
$$\ln(P) + \ln(V) = \ln(nR) + \ln(T)$$
$$\frac{dP}{P} + \frac{dV}{V} = \frac{dT}{T}.$$

Thus we have $(2\%) + dV/V = (-3\%)$, or $dV/V = -5\%$.

2 Exponential Decay

We will solve our first *differential equation* now. Consider the equation

$$\frac{dy}{dt} = ky, \tag{1}$$

where k is a constant. Notice that the variable that we are differentiating with respect to is t and not the usual x. No big deal, since it's just names ... but start thinking about changes in time.

We can solve (formally) as follows:

$$\begin{aligned} \frac{dy}{y} &= k\,dt \\ \ln|y| &= kt + C, \\ |y(t)| &= e^{kt+C} = \hat{C}e^{kt}. \end{aligned}$$

We wrote a $\hat{C}$ there instead of C to point out that the two constants are different, but since they're arbitrary we'll just write C from now on. Also notice that since the left-hand side of the equation is $|y(t)|$, this means that $y(t)$ could be positive or negative, so our solution is

$$y(t) = Ce^{kt}$$

where C is any arbitrary constant, and k is given by (1). Also notice that if we plug in $t = 0$ to both sides of the equation we obtain

$$y(0) = C,$$

so this constant has the meaning of the initial value of y at time $t = 0$. Thus we write

$$y(t) = y(0)e^{kt}.$$

A short way of saying all of this is that if the **rate of change of a quantity** is linear, then the quantity itself grows **exponentially**. (Or shrinks, since k can be negative!)

There are two places where these models are commonly used: modeling population growth and modeling exponential decay.

2.1 Population growth

Let us think of the model where we write $P(t)$ as the size of some population at time t, and k is the growth rate, so that

$$\frac{dP}{dt} = kP,$$

or

$$P(t) = P(0)e^{kt}.$$

Biological populations tend to grow, so here we think of $k > 0$.

Let us first define the **doubling time** of the population as the time it takes for the population to double, and denote this as τ_2. Specifically, we mean $P(\tau_2) = 2P(0)$. Then notice we have

$$\begin{aligned} 2P(0) &= P(0)e^{k\tau_2} \\ 2 &= e^{k\tau_2} \\ \ln(2) &= k\tau_2 \\ \tau_2 &= \frac{\ln(2)}{k}. \end{aligned}$$

Since $\ln(2) > 0$, as long as $k > 0$ this means that τ_2 is positive, which makes sense. Also note that τ_2 and k are inversely proportional: doubling the growth rate constant will halve the doubling time, etc.

Now, let us model the human population of Earth. Looking at some historical data, we have

Year	Population (in millions)
1500	458
1600	580
1700	682
1750	791
1800	1000
1850	1262
1900	1650
1950	2525
1975	4061
2000	6127
2015	7349

Let us consider just the second half of the twentieth century to calibrate k. We have

$$\frac{6127}{2525} = e^{k50\,yr},$$

which we can solve for

$$k = 0.0177\text{yr}^{-1} = 1.77\%\text{yr}^{-1}.$$

Let us first ask if this seems like a reasonable number. Under this growth rate, the doubling time is

$$\tau_2 = \frac{\ln(2)}{1.77\%/\text{yr}} \approx 39.1\,\text{yr}$$

This seems pretty plausible. If you assume that worldwide the average couple has four children and has them before age 40, then this is a reasonable doubling estimate.

So let us then ask the question: assume that humanity maintains an average growth of 1.77%, what happens next? In particular, how long would it take for us to consume the entire biosphere of Earth?

Say the population now is 7 billion. Some data we need to use for this computation is that the average human weighs $62\,\text{kg}$, is 18% carbon, and the biosphere has 560 billion metric tones of carbon. We can compute that the mass of humanity is $7.8 \times 10^{10}\,\text{kg}$ of carbon, meaning that the ratio of biosphere to humanity (counting carbon) is just about 7170. So we're asking how long it would take humanity to grow by a factor of 7170 at a fixed growth rate of $1.77\%/\text{yr}$, which we can solve by

$$7170 = e^{.0177\,\text{yr}^{-1}t}$$
$$t = \frac{\ln(7170)}{.0177\,\text{yr}^{-1}} \approx 507\,\text{yr}.$$

Not so long!

Obviously, the takeaway from this shoudn't be that we will eat everything and die in five centuries; it is that the model is very much wrong.

More accurate modeling is given at `https://en.wikipedia.org/wiki/Population_growth#Human_population_growth_rate`

2.2 Radioactive decay

In radioactive decay, we typically think of $k < 0$, and then it makes sense to talk about the **half-life**:

$$\frac{1}{2} = e^{k\tau_{1/2}},$$

or

$$\tau_{1/2} = \frac{-\ln 2}{k}.$$

Note that since $k < 0$, this fraction is positive.

3 Related rates

We would like to solve problems where multiple variables are changing, and we would like to understand the relationship between the various rates of change.

We do some examples:

Example 3.1. Question: Assume that water is being pumped into a balloon at $50\,\mathrm{cm}^3/\mathrm{s}$. How fast is the radius increasing when the radius is $10\,\mathrm{cm}$?

We know that the relationship between volume and radius is

$$V = \frac{4}{3}\pi r^3.$$

Differentiating both sides of this equation gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt},$$

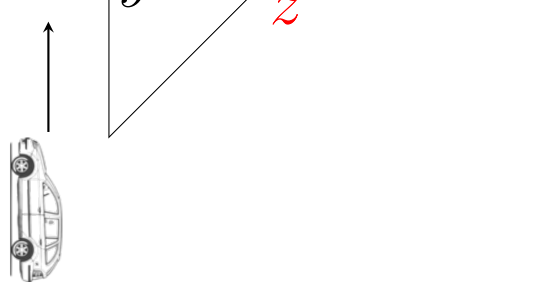
so

$$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} = \frac{50\,\mathrm{cm}^3/\mathrm{s}}{4\pi(10\,\mathrm{cm})^2} = \frac{1}{8\pi}\frac{\mathrm{cm}}{\mathrm{s}}.$$

Since $1/8\pi = .0397\cdots$, our answer is approximately $.397\,\mathrm{mm/s}$.

Example 3.2. Question: Two cars are approaching an intersection, one from the east and one from the south. Let's say that each car is traveling $60\,\mathrm{mph}$ towards the intersection, and they are currently each one mile from the intersection. How fast is the distance between the two decreasing?

We need to choose some coordinate system for these cars. Let us put the intersection at the origin, define the distance of the east-west car from the origin as x, and define the distance from the north-south car from the origin as y. We also define the distance between the cars as z.



We know from the Pythagorean Theorem that $z^2 = x^2 + y^2$. Differentiate both sides to obtain

$$\begin{aligned}
2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt}, \\
z\frac{dz}{dt} &= x\frac{dx}{dt} + y\frac{dy}{dt}, \\
&= (1\,\mathrm{mi})(-60\,\mathrm{mph}) + (1\,\mathrm{mi})(-60\,\mathrm{mph}) \\
&= -120\frac{\mathrm{mi}^2}{\mathrm{hr}}.
\end{aligned}$$

We also know that $z = \sqrt{2}\,\mathrm{mi}$, so

$$\frac{dz}{dt} = \frac{-120\,\mathrm{mi}^2/\mathrm{hr}}{\sqrt{2}\,\mathrm{mi}} = -\frac{120}{\sqrt{2}}\text{ mph.}$$

We can compute that $120/\sqrt{2} \approx 84.85$.

Example 3.3. Ladder problem. It wouldn't be Calc II with at least one ladder problem.

A four meter tall ladder is leaning against a wall, and the base of the ladder is one meter from the wall. The base of the ladder starts to slide away from the wall at $1\mathrm{m/s}$. How fast is the top of the ladder moving down the wall?

Drawing a picture it is clear that the ladder, the wall, and the ground form a right triangle with the ladder as the hypotenuese and the wall and the ground as the two sides. If we denote the distance from the ground to the top of the ladder by x and the distance from the wall to the base of the ladder by y then we have $x^2 + y^2 = 16\mathrm{m}^2$. Initially we have $y = 1\mathrm{m}$ so $x = \sqrt{15}\mathrm{m}$. Taking the relation between x and y and differentiating with respect to time gives

$$\begin{aligned}
x^2 + y^2 &= 16\mathrm{m}^2 \\
2x\tfrac{dx}{dt} + 2y\tfrac{dy}{dt} &= 0 \\
2x\tfrac{dx}{dt} &= -2y\tfrac{dy}{dt} \\
\tfrac{dx}{dt} &= -\tfrac{y}{x}\tfrac{dy}{dt}
\end{aligned}$$

Now we are set. We know that $y = 1\mathrm{m}$ ("The base of the ladder is initially $1\mathrm{m}$ from the wall"), $x = \sqrt{15}\mathrm{m}$ (using $x^2 + y^2 = 16\mathrm{m}^2$), and that $\frac{dy}{dt} = 1\mathrm{m/s}$ ("The base of the ladder starts to slide away from the wall at $1\mathrm{m/s}$" –The base is moving away from the wall, so distance should be increasing.) Therefore $\frac{dx}{dt} = -\frac{1}{\sqrt{15}}\mathrm{m/s} \approx 0.26\mathrm{m/s}$. Note the minus sign: if the base is moving away from the wall the height of the top of the ladder above the ground should decrease, which checks out.