Math 415. Exam 3. November 30, 2017

Full Name: ____________________

Net ID: ________________________

Discussion Section: ____________

- There are 17 problems worth 5 points each.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 70 minute exam.
- Do not turn this page until instructed to.
- Fill in the answers on the scantron form provided. Also circle your answers on the exam itself.
- Hand in both the exam and the scantron.
- On the scantron make sure you bubble in **your name, your UIN and your NetID**.
- There are several different versions of this exam.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- Good luck!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**! On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) Consider the following basis $\mathcal{B}$ of $\mathbb{R}^3$:

$$\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right).$$

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. What is the coordinate vector $\mathbf{x}_{\mathcal{B}}$ of $\mathbf{x}$ with respect to the basis $\mathcal{B}$?

(A) $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(B) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(C) ★ $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(D) $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(E) None of the other answers

Solution. The basis is orthonormal, so the matrix Q with these columns has inverse Q^T. Now $Q = I_{\mathcal{E}\mathcal{B}}$ and $Q^T = I_{\mathcal{B}\mathcal{E}}$, so $x_{\mathcal{B}} = I_{\mathcal{B}\mathcal{E}} x_E = Q^T x$.

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2. (5 points) Let A, B be 3×3 matrices such that $\det(A) = -1$ and $\det(B) = 2$. What is $\det(2A^2B^{-1})$?

(A) None of the other answers.

(B) 8

(C) 1

(D) ★ 4

(E) -4

Solution. Now $\det(2A^2B^{-1}) = \det(2I)\det(A)^2\det(B)^{-1} = 2^3(-1)^2\frac{1}{2} = 4$.

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3. (5 points) Let λ be an eigenvalue of an $n \times n$ **non-zero** matrix A. Which of the following statements is **not** true for all such A and λ?

(A) λ is an eigenvalue of A^T.

(B) λ^{-1} is an eigenvalue of A^{-1}, if A is invertible.

(C) ★ At least one eigenvalue of A is non-zero.

(D) 2λ is an eigenvalue of $2A$.

(E) $\lambda^2 + 1$ is an eigenvalue of $A^2 + I$, where I is the identity matrix.

Solution. The matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nonzero but has all eigenvalues equal to zero. To see that the other statements are true, see Worksheet 13 Problem 5.

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4. (5 points) Let A be a 3×3 matrix with columns $\mathbf{a}_1, \mathbf{a_2}, \mathbf{a_3}$. Which of the following statements is *not* true for all such matrices?

(A) If $\mathbf{a}_2 = \mathbf{0}$, then the determinant of A is zero.

(B) If B is obtained from A by adding the third row of A to the first row of A, then $\det(A) = \det(B)$.

(C) ★ $\det(\begin{bmatrix} \mathbf{a}_2 & (\mathbf{a_2} + 6\mathbf{a_1}) & \mathbf{a}_3 \end{bmatrix}) = -\det(A)$.

(D) If $\mathbf{a}_1 + 3\mathbf{a_2} + \mathbf{a_3} = \mathbf{0}$, then the determinant of A is zero.

(E) $\det(-A) = -\det(A)$.

Solution. If $\mathbf{a}_1 + 2\mathbf{a_2} + 3\mathbf{a_3} = \mathbf{0}$ holds, then the columns of A are linear dependent and hence A is not invertible. Thus $\det(A) = 0$.

Every matrix with zero column (or a zero row) has determinant 0.

Note that $\det(-A) = (-1)^3 \det(A)$, since A is 3×3. Thus $\det(A) = -\det(-A)$.

Since the determinant is preserved under row replacement, we get that if B is obtained from A by adding the second row of A to the first row of A, then $\det(A) = \det(B)$. Observe that we obtained $\begin{bmatrix} \mathbf{a}_2 & \mathbf{a_2} + 6\mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}$ by using the following two column operations: $C1 \leftrightarrow C2$, $C2 \to 6C2$ and finally $C2 \to C2 + C1$. Thus

$$\det(A) = -\det(\begin{bmatrix} \mathbf{a}_2 & \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}) = -\frac{1}{6}\det(\begin{bmatrix} \mathbf{a}_2 & 6\mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}) = -\frac{1}{6}\det(\begin{bmatrix} \mathbf{a}_2 & \mathbf{a_2} + 6\mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}).$$

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5. (5 points) Let Q be an orthogonal matrix. Consider the following statements:

(T1) The columns of Q are orthonormal.

(T2) $Q^TQ = I$, where I is the identity matrix.

(T3) $|\det(Q)| = 1$.

(T4) Q^T is the inverse of Q.

(T5) The rows of Q are orthonormal.

Which of the above statements are true?

(A) Only the statement (T1) is true.

(B) Only the statements (T1),(T2),(T3) and (T4) are true.

(C) Only the statements (T1) and (T2) are true.

(D) None of the statements are true.

(E) ★ All statements are true.

Solution. An orthogonal matrix is a square matrix with orthonormal columns. The columns of an orthogonal matrix are orthonormal by definition. Since $Q^TQ = I$, $Q^T = Q^{-1}$. Since $Q^{-1} = Q^T$ it follows that $QQ^T = I$ so the rows are also orthonormal. Also, since $\det(Q^T) = \det(Q)$, $\det(Q^TQ) = \det(Q)^2 = \det(I) = 1$ so the determinant of an orthogonal matrix is equal to ± 1.

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6. (5 points) You wish to find the parabola of best fit for the data points $(1,-1), (-3,1), (2,4)$, and $(-1,1)$. Given that a least squares solution for the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 4 \\ 1 \end{bmatrix}$$

is $\hat{\mathbf{x}} = \frac{1}{398}\begin{bmatrix} 157 \\ 277 \\ -22 \end{bmatrix}$, which of the following is the parabola of best fit?

(A) $y = \frac{157}{398} + \frac{277}{398}x - \frac{22}{398}x^2$

(B) ★ $y = \frac{157}{398}x^2 + \frac{277}{398}x - \frac{22}{398}$

(C) $y = -\frac{22}{398}x^2 + \frac{157}{398}x + \frac{277}{398}$

(D) None of the other answers

(E) $y = -\frac{22}{398} + \frac{157}{398}x + \frac{277}{398}x^2$

Solution. We are given a least-squares solution to the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 4 \\ 1 \end{bmatrix}.$$

Since $\hat{\mathbf{x}}$ represents $\begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix}$ where $\hat{a}$ is the coefficient for the quadratic term, $\hat{b}$ for the linear term, and $\hat{c}$ for the constant term, the parabola of best fit has the equation $y = \frac{157}{398}x^2 + \frac{277}{398}x - \frac{22}{398}$.

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7. (5 points) Suppose for some matrix A, you are given the QR-factorization

$$A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

What is the least-squares solution $\hat{\mathbf{x}}$ to the system $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$?

(A) $\hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$.

(B) $\hat{\mathbf{x}} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$.

(C) ★ $\hat{\mathbf{x}} = \begin{bmatrix} 0 \\ \frac{1}{8} \end{bmatrix}$.

(D) None of the other answers.

(E) $\hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{8} \\ 0 \end{bmatrix}$.

Solution. The normal equations reduce in this case to

$$R\hat{\mathbf{x}} = Q^T\mathbf{b} = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}.$$

Therefore the solution is $\hat{\mathbf{x}} = \begin{bmatrix} 0 \\ 1/8 \end{bmatrix}$.

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8. (5 points) Which one of the following vectors is an eigenvector with eigenvalue 2 for the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$?

(A) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(B) None of the other answers.

(C) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

(D) ★ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(E) $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

Solution. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

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9. (5 points) Let $\mathcal{B} := \{\boldsymbol{b_1}, \boldsymbol{b_2}\}$ and $\mathcal{C} := \{\boldsymbol{c_1}, \boldsymbol{c_2}\}$ be two bases of $\mathbb{R}^2$ such that

$$\boldsymbol{b_1} = \boldsymbol{c_1} - \boldsymbol{c_2} \text{ and } \boldsymbol{b_2} = 2\boldsymbol{c_1} - \boldsymbol{c_2}.$$

What is the change of basis matrix $I_{\mathcal{C},\mathcal{B}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{C}$?

(A) ★ $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

(B) None of the other answers.

(C) $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

(E) $\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

Solution. $I_{\mathcal{C},\mathcal{B}}$ is the matrix representing the identity transformation I in bases $\mathcal{B}$ (input) and $\mathcal{C}$ (output). Since

$$\boldsymbol{b_1} = 1\boldsymbol{c_1} - 1\boldsymbol{c_2} \text{ and } \boldsymbol{b_2} = 2\boldsymbol{c_1} - 1\boldsymbol{c_2},$$

we get that $I(\mathbf{b}_1)_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $I(\mathbf{b}_2)_{\mathcal{C}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Therefore we have that

$$I_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}.$$

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10. (5 points) What is the determinant of the matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 1 & 0 & 2 & 3 \end{bmatrix}$?

(A) None of the other answers.

(B) 1

(C) -1

(D) ★ 4

(E) -4

Solution.

$$\det(A) = (-1)\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ 1 & 2 & 3 \end{vmatrix} = -1\left(\begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}\right) = (-1)\left((6-8)+(4-6)\right) = 4.$$

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11. (5 points) Let $A = QR$ be the QR decomposition of A, and let $\mathbf{a_1}, ..., \mathbf{a_n}$ be the columns of A. Which of the following statements are true:

(S1) $A^T A = R^T R$.

(S2) If $\mathbf{a_1}, ..., \mathbf{a_n}$ are orthogonal, then $R = \begin{bmatrix} \|\mathbf{a_1}\| & 0 & \cdots & 0 \\ 0 & \|\mathbf{a_2}\| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|\mathbf{a_n}\| \end{bmatrix}$.

(A) Neither (S1) nor (S2) is true.

(B) Only (S2) is true.

(C) Only (S1) is true.

(D) ★ Both (S1) and (S2) are true.

Solution. Since Q is a matrix with orthonormal columns, we that $Q^T Q = I$. Therefore $A^T A = (QR)^T(QR) = R^T(Q^T Q)R = R^T R$. Therefore (S1) is true.

To see that (S2) is also true, note that if the columns of A are orthogonal, all we need to make them orthonormal is to normalize them. So in this case we have

$$Q = \left[\frac{\mathbf{a_1}}{||\mathbf{a_1}||}, \frac{\mathbf{a_2}}{||\mathbf{a_2}||}, \cdots, \frac{\mathbf{a_n}}{||\mathbf{a_n}||}\right]$$

and therefore

$$R = Q^T A = \begin{bmatrix} \frac{\mathbf{a_1}^T}{||\mathbf{a_1}||} \\ \vdots \\ \frac{\mathbf{a_n}^T}{||\mathbf{a_n}||} \end{bmatrix} [\mathbf{a_1} \ \cdots \ \mathbf{a_n}] = \begin{bmatrix} ||\mathbf{a_1}|| & 0 & \cdots & 0 \\ 0 & ||\mathbf{a_2}|| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ||\mathbf{a_n}|| \end{bmatrix}.$$

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12. (5 points) Let $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$. Let Q be a 2×2 matrix with orthonormal columns and let R be an invertible upper triangular matrix such that $A = QR$. Then Q is equal to which of the following matrices?

(A) $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-2}{\sqrt{3}} \end{bmatrix}$

(B) ★ $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

(C) $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(D) None of the other answers.

(E) $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Solution. We apply Grand-Schmidt to the columns of A. The first column has norm $||\mathbf{a_1}|| = \sqrt{2}$, so its normalized version is $\mathbf{u_1} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, which is the first column of Q. Now we need to subtract the projection of $\mathbf{a_2}$ onto $\mathbf{a_1}$ from $\mathbf{a_2}$. A quick computation shows that $\mathbf{a_2'} = \mathbf{a_2} - \frac{\mathbf{a_1}^T\mathbf{a_2}}{\mathbf{a_1}^2}\mathbf{a_1} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, which has norm $||\mathbf{a_2'}|| = \sqrt{8} = 2\sqrt{2}$. Therefore the second column of Q is the vector $\mathbf{u_2} = \frac{\mathbf{a_2'}}{||\mathbf{a_2'}||^2} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

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13. (5 points) Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$ of $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be such that the matrix $T_{\mathcal{B},\mathcal{B}}$ that represents T with respect to $\mathcal{B}$ is

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The change of basis matrix $I_{\mathcal{B},\mathcal{C}}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and the change of basis matrix $I_{\mathcal{C},\mathcal{B}}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$. What is $T_{\mathcal{C},\mathcal{C}}$?

(A) None of the other answers.

(B) ★ $\begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Solution. $T_{\mathcal{C},\mathcal{C}} = I_{\mathcal{C},\mathcal{B}} T_{\mathcal{B},\mathcal{B}} I_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix}.$

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14. (5 points) Let $\hat{\mathbf{x}}$ be a least-squares solution of the system $A\mathbf{x} = \mathbf{b}$. Which of the following statements may be FALSE?

(A) If $\mathbf{b}$ is orthogonal to Col(A), then $\hat{\mathbf{x}} \in$ Nul(A).

(B) The error vector $A\hat{\mathbf{x}} - \mathbf{b}$ is orthogonal to Col(A).

(C) ★ $A\hat{\mathbf{x}} - \mathbf{b} = \mathbf{0}$.

(D) If Nul(A) $= \{\mathbf{0}\}$, then $\hat{\mathbf{x}}$ is the UNIQUE least-squares solution.

Solution. The error $\|A\hat{\mathbf{x}} - \mathbf{b}\|$ is minimized whenever $A\hat{\mathbf{x}} - \mathbf{b}$ is orthogonal to Col(A).

If $\mathbf{b}$ is orthogonal to Col(A), then $\hat{\mathbf{b}} = \mathbf{0}$. Since the least-squares solution must satisfy $A\hat{\mathbf{x}} = \hat{\mathbf{b}} = \mathbf{0}$, $\hat{\mathbf{x}} \in$ Nul(A).

Finally, recall that if A has trivial nullspace, then for each consistent system $A\mathbf{x} = \mathbf{y}$, there is a unique solution. In particular, $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$ has a unique solution.

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15. (5 points) Let $W = \text{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ and $\mathbf{b} = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$. Which one of the following is the orthogonal projection of $\mathbf{b}$ onto $W^{\perp}$?

(A) $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

(B) $\begin{bmatrix}1\\-1\\-1\end{bmatrix}$.

(C) None of the other answers.

(D) ★ $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

(E) $\begin{bmatrix}0\\-1\\-1\end{bmatrix}$.

Solution. We can check that $W^{\perp} = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$. Then, by definition of orthogonal projection, $\mathbf{b}_{W^{\perp}} = (\mathbf{b}\cdot\mathbf{e_1})\mathbf{e_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

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16. (5 points) What are the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 2 \end{bmatrix}?$$

(A) ★ $\lambda = -1, 0, 2$

(B) $\lambda = 0, 1, -1$

(C) $\lambda = 0, 2$

(D) $\lambda = 0, 1, 2$

(E) None of the other answers

Solution. Straightforward determinant calculation shows that the characteristic equation is $p(\lambda) = -\lambda^3 + \lambda^2 + 2\lambda = -\lambda(\lambda^2 - 4\lambda + 3) = -\lambda(-\lambda - 1)(2 - \lambda) = 0$ so that

$$\lambda_1 = -1, \quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = 0, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \lambda_3 = 2, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

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17. (5 points) Let P denote the projection matrix of the orthogonal projection onto the subspace $\mathrm{Col}(A)$, where A is an $n \times m$-matrix with linearly independent columns, and let $\mathbf{b}$ be in $\mathbb{R}^n$. Consider the following three statements:

(S1) $P^2 = P$.

(S2) $P\mathbf{b} = \mathbf{b}$ if and only if $\mathbf{b}$ is a linear combination of the columns of A.

(S3) $P\mathbf{b} = \mathbf{0}$ if and only if $\mathbf{b}$ is in $\mathrm{Nul}(A^T)$.

Which of the three statements are always true?

(A) ★ All statements.

(B) None of the statements.

(C) Statements (S2) and (S3) only.

(D) Statements (S1) and (S2) only.

(E) Statement (S1) only.

Solution. For (S1): $P^2 = P$, because one we have projected down onto $\mathrm{Col}(A)$, projecting onto $\mathrm{Col}(A)$ again does not change anything!
For (S2): If $P\mathbf{b} = \mathbf{b}$, then $\mathbf{b} \in \mathrm{Col}(A)$ and thus a linear combination of the columns of A. Also, when $\mathbf{b}$ is in the column space of A, then $P\mathbf{b} = \mathbf{b}$, because the projection of vector in the subspace is just the vector itself.
For (S3): note that $P\mathbf{b} = \mathbf{0}$ if and only if $b \in W^{\perp} = \mathrm{Nul}\left(A^T\right)$.
