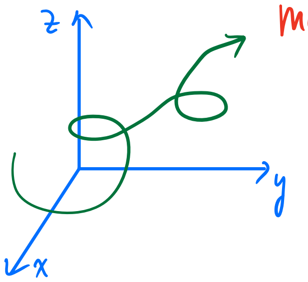
Chapter 13: Vector Functions



mosquito flying in space can be described using a vector fn

$t \longrightarrow \boxed{\text{vector fn}} \longrightarrow \langle f(t), g(t), h(t) \rangle$

Goals:
- Define vector functions and space curves
- Learn how to differentiate and integrate vector functions
  - Hint: component by component
- Find Arc Length and Curvature of curves
- Apply this knowledge to Velocity and Acceleration in space

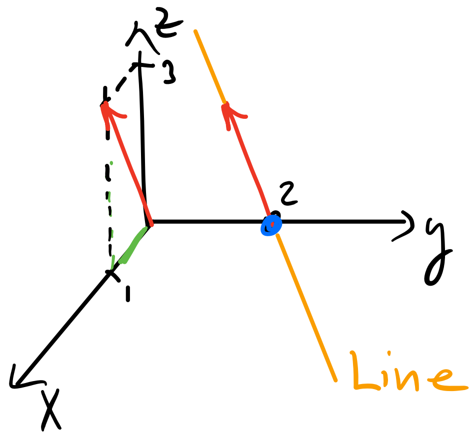
13.1 Vector Functions and Space Curves

Vector function:
(or vector-valued function)

scalar ($\mathbb{R}$) $\longrightarrow \boxed{\text{vector function}} \longrightarrow$ vector ($\mathbb{R}^3$)

Ex: $\vec{r}(t) = \underline{\langle 0, 2, 0 \rangle} + t\underline{\langle 1, 0, 3 \rangle}$

$= \langle t, 2, 3t \rangle$



In general:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

component fns
= real valued functions of 1 variable

DOMAIN: all possible choices of t for which $f(t)$, $g(t)$, and $h(t)$ are all defined

Limits

Def: $$\boxed{\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) \right\rangle}$$

Intuitively, $\boxed{\lim_{t\to a} \vec{r}(t) = \vec{L}}$ means that the length and direction of $\vec{r}(t)$ approach the length and direction of $\vec{L}$

Ex: $\vec{r}(t) = \left\langle 3 - t^2,\ \frac{\sin(t)}{t},\ t\cdot 2^{-t} \right\rangle$

$$\lim_{t\to 0} \vec{r}(t) = \langle 3,\ 1,\ 0 \rangle$$

Standard basis vector notation:

$$\vec{r}(t) = (3-t)\hat{\imath} + \frac{\sin(t)}{t}\hat{\jmath} + t\,2^{-t}\hat{k}$$

$$\lim_{t\to 0} \vec{r}(t) = 3\hat{\imath} + \hat{\jmath}$$

Continuity

Recall: $f(t)$ is cont. at a if $\lim_{x\to a} f(x) = f(a)$

Def: A vector fn $\vec{r}(t)$ is continuous at a if $\boxed{\lim_{t\to a} \vec{r}(t) = \vec{r}(a)}$,

Equivalently: $\vec{r}(t)$ is continuous at a whenever $f(t)$, $g(t)$, and $h(t)$ are continuous at a

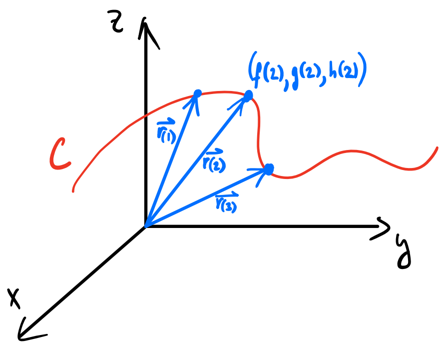
Ex: Is $\vec{r}(t) = \left\langle 3 - t^2,\ \frac{\sin(t)}{t},\ t\cdot 2^{-t} \right\rangle$ continuous at $t=0$?

NOT cont. at 0 (undefined at 0)

$\Longrightarrow$ $\vec{r}(t)$ is NOT continuous at 0.

Space Curves

Think of $\vec{r}(t)$ as representing a point in 3D space



1 point for each value of t

$(f(t), g(t), h(t))$

vector function:

$$\boxed{\vec{r}(t) = \langle f(t), g(t), h(t) \rangle} \iff \boxed{\begin{aligned} x &= f(t) \\ y &= g(t) \\ z &= h(t) \end{aligned}}$$

parametric equations:

C = curve traced out by the tip of the moving vector $\vec{r}(t)$

Important example

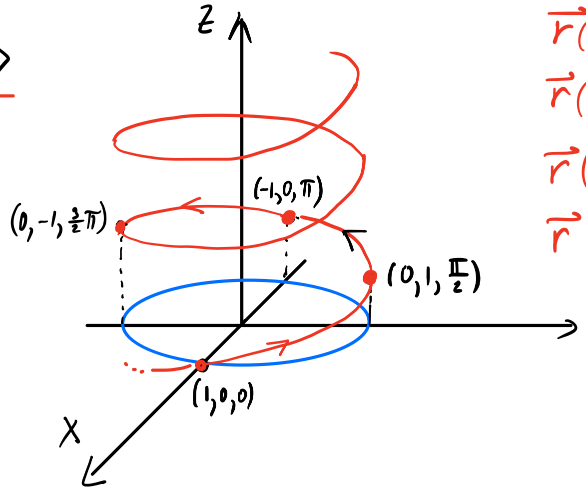
EX: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

parametric eqns:

$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\} \Longrightarrow x^2 + y^2 = 1$$

$$z = t$$

Helix



$\vec{r}(0) = \langle 1, 0, 0 \rangle$

$\vec{r}(\frac{\pi}{2}) = \langle 0, 1, \frac{\pi}{2} \rangle$

$\vec{r}(\pi) = \langle -1, 0, \pi \rangle$

$\vec{r}(\frac{3}{2}\pi) = \langle 0, -1, \frac{3}{2}\pi \rangle$

EX: $\vec{r}(t) = \langle 2t, 3\sin t, 3\cos t \rangle$

Q: Does this curve lie on the cylinder $y^2 + z^2 = 9$?

$$(3\sin t)^2 + (3\cos t)^2 = 9(\sin^2 t + \cos^2 t) = 9 \ \checkmark$$

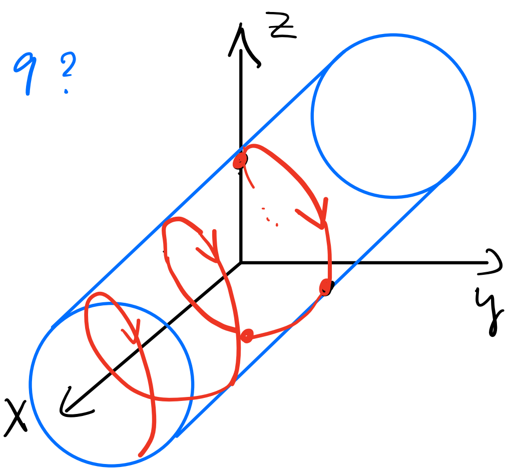
Yes.

$\vec{r}(0) = \langle 0, 0, 3 \rangle$

$\vec{r}(\frac{\pi}{2}) = \langle \pi, 3, 0 \rangle$

$\vec{r}(\pi) = \langle 2\pi, 0, -3 \rangle$

$\vdots$



Did you know? DNA molecule = two parallel helixes linked together (discovered in 1953 by James Watson and Francis Crick)

Intersection of two Surfaces

Ex: Find the curve of intersection of:

① $y - x^2 = 0$ parabolic cylinder

② $y + z = 2$ plane

pick $x = t \rightarrow$ ① becomes $y - t^2 = 0$

$y = t^2$

② becomes $t^2 + z = 2$

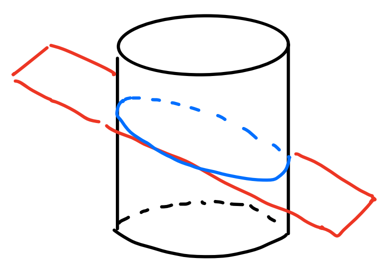
$z = 2 - t^2$

$$\vec{r}(t) = \langle t, t^2, 2 - t^2 \rangle, \quad t \in \mathbb{R}$$

Ex: Find the curve of intersection of:

① $x^2 + y^2 = 9$ circular cylinder

② $z = x + 5$ plane



TRICK: Projection of ① onto xy-plane: circle $x^2 + y^2 = 9$, $z = 0$.

Find parametric eqns of ① as this was a curve in a plane:

$$\left.\begin{aligned} x &= 3\cos t \\ y &= 3\sin t \end{aligned}\right\} \text{ so that } x^2 + y^2 = 9$$

Pick z so that C lies on ② (i.e., plug-in x and y into ②)

$z = 3\cos t + 5$ think: you are adding height to every point on the circle

$$\boxed{\vec{r}(t) = \langle 3\cos t, 3\sin t, 3\cos t + 5 \rangle}, \quad 0 \le t \le 2\pi$$

(once around)

Intersection of two Curves

EX: • $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$ ← twisted cubic

• $\vec{r}_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle$

(A) Do these two curves intersect?

Check if $\vec{r}_1(t) = \vec{r}_2(s)$ for some value of s and some value of t.

Solve the system of 3 eqns in 2 variables:

① $t = 1+2s$
② $t^2 = 1+6s$
③ $t^3 = 1+14s$

Substitute ① into ②: $(1+2s)^2 = 1+6s$

$$1+4s+4s^2 = 1+6s$$
$$-2s+4s^2 = 0$$
$$2s(-1+2s) = 0$$
$$s=0 \quad \text{or} \quad s=\tfrac{1}{2}$$

If $s=0$, then $t=1$ by ①. Check ③: $1^3 = 1+14\cdot 0$ ✓ $\boxed{\text{Point: } (1,1,1)}$

If $s=\frac{1}{2}$, then $t=2$ by ①. Check ③: $2^3 = 1+14\cdot\frac{1}{2}$ ✓ $\boxed{\text{Point: } (2,4,8)}$

(B) Do these two curves collide? i.e., intersect at the same time t.

Check if $\vec{r}_1(t) = \vec{r}_2(t)$ for some value t.

Solve the system of 3 eqns in 1 variable:

① $t = 1+2t \Rightarrow t=-1$
② $t^2 = 1+6t$
③ $t^3 = 1+14t$

Substitute $t=-1$ into ②: $(-1)^2 \neq 1+6\cdot(-1)$ NO SOLUTION

$\Rightarrow$ CURVES DO NOT COLLIDE

Intersection of Curve and Surface

Same method as for Intersection of Line and Plane

Curve

Ex: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$\Rightarrow \quad x = f(t)$

$y = g(t)$

$z = h(t)$

Surface

$x^2 - y + z^2 = 0$

substitute param. eqns into surface eqn.,

$\rightarrow$ solve for t.