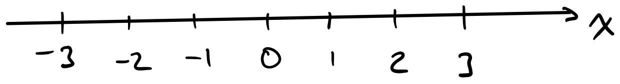
12.1 Three-Dimensional Coordinate Systems

- $\mathbb{R}$ = real numbers

line



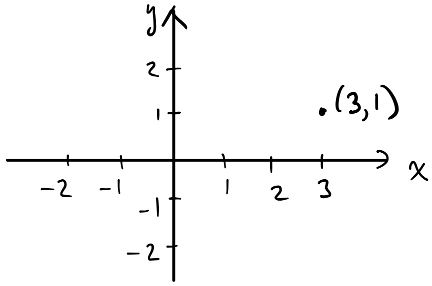
each real number specifies a location on line

- $\mathbb{R}^2$ = ordered pairs (x,y) of real numbers

$\mathbb{R} \times \mathbb{R} = \{(x,y) \mid x,y \in \mathbb{R}\}$

each pair specifies a location in plane

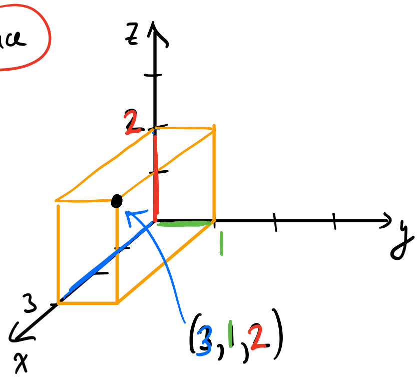
plane

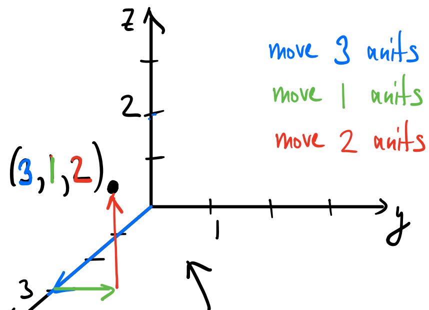


- $\mathbb{R}^3$ = ordered triples (x,y,z) of real numbers

$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$

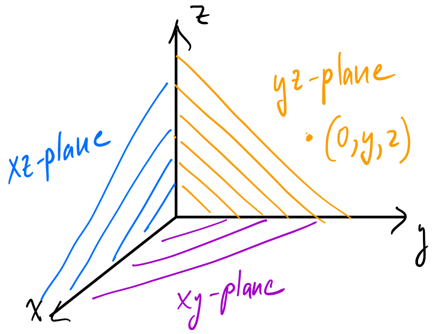
Space





move 3 units along the x-axis
move 1 units along the y-axis
move 2 units along the z-axis

Right-hand Rule: If you point fingers of your Right hand toward positive x-axis and curl them toward positive y-axis, your thumb should point to positive z-axis.



Ex. Which point is closest to the xz-plane?

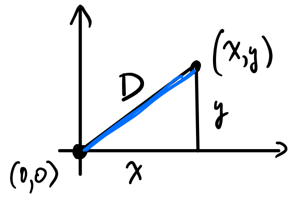
A $(-1, 2, 7)$
B $(0, -3, 2)$
C $(5, -1, 3)$

- $\mathbb{R}^n$ = n-tuples $(x_1, x_2, \dots, x_n)$ of real numbers

n-dim. space

Distance

$\mathbb{R}^2$:

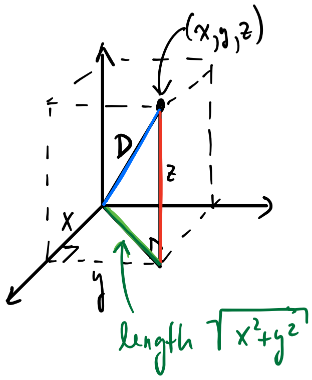


Distance D from $(0,0)$ to (x,y)

$$D^2 = x^2 + y^2$$

$$D = \sqrt{x^2+y^2}$$

$\mathbb{R}^3$:



Distance D from $(0,0,0)$ to (x,y,z)

$$D^2 = \left(\sqrt{x^2+y^2}\right)^2 + z^2$$

$$D^2 = x^2+y^2+z^2$$

$$D = \sqrt{x^2+y^2+z^2}$$

Q: Which object do you get if you take all points (x,y,z) of distance 5 from the origin?

A: a Sphere of radius 5.

Distance between $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$

$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}}$$

Ex: $P(4,-1,-3)$, $Q(8,0,-5)$

$$|PQ| = \sqrt{(8-4)^2+(0-(-1))^2+(-5-(-3))^2} = \sqrt{4^2+1^2+(-2)^2} = \sqrt{16+1+4} = \sqrt{21}$$

Sphere with center (x_0,y_0,z_0) and radius r has equation:

$$\boxed{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2}$$

Ex: Is this a sphere? If so, find its center and radius.

complete the square

$$x^2+y^2+z^2+2x-6y+7z-10=0$$

$$(x^2+2x) + (y^2-6y) + (z^2+7z) = 10$$

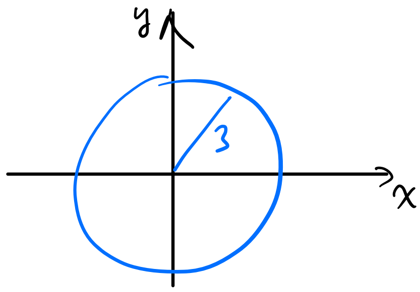
$$(x^2+2x+1) + (y^2-6y+9) + \left(z^2+7z+\left(\tfrac{7}{2}\right)^2\right) = 10+1+9+\left(\tfrac{7}{2}\right)^2$$

$$(x+1)^2 + (y-3)^2 + \left(z+\tfrac{7}{2}\right)^2 = 20+\tfrac{49}{4}$$

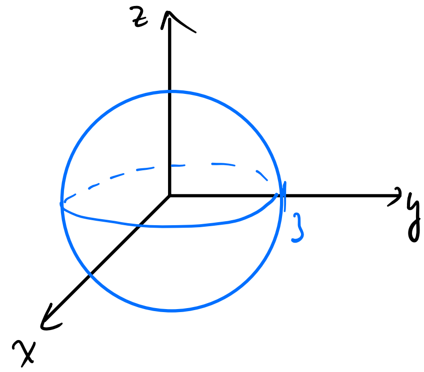
center: $\left(-1,3,-\frac{7}{2}\right)$ radius: $\sqrt{\dfrac{129}{4}} = \dfrac{\sqrt{129}}{2}$

Examples: Sketch the following:

- $x^2+y^2=9$ in $\mathbb{R}^2$
  - collection of all points (x,y) of distance 3 from origin



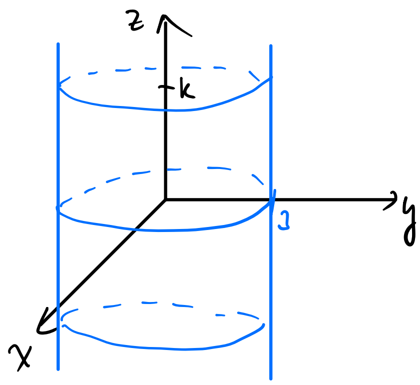
- $x^2+y^2+z^2=9$ in $\mathbb{R}^3$
  - collection of all points (x,y,z) of distance 3 from origin



- $x^2+y^2=9$ in $\mathbb{R}^3$

If $z=0$, then we get a circle of rad. 3 in the xy-plane

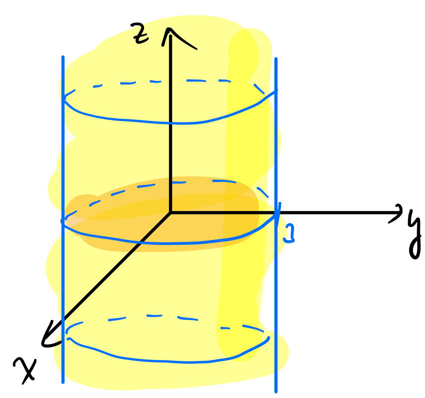
For any fixed $z=k$, we get a circle of rad. 3 in the horizontal plane $z=k$.



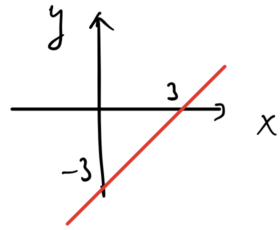
- $x^2+y^2\leq 9$ in $\mathbb{R}^3$

points inside the (infinitely tall) cyliner

(including the boundary)

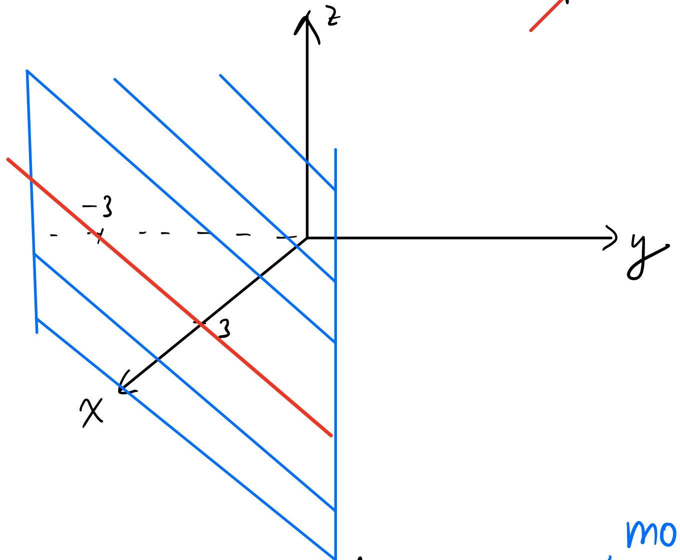


Ex: $x - y = 3$ in $\mathbb{R}^3$



In $\mathbb{R}^2$, this would be a line $y = x - 3$

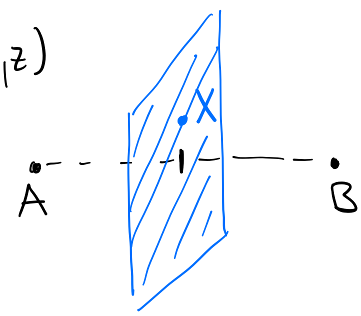
In $\mathbb{R}^3$, this is a plane:



Equation of a plane: $ax + by + cz + d = 0$ more details in section 12.5

Ex: Find the equation of the set of points equidistant from the points $A(1,4,2)$ and $B(3,3,3)$.

equidistant: X such that $|AX| = |BX|$, $X = (x, y, z)$



$$|AX| = \sqrt{(x-1)^2 + (y-4)^2 + (z-2)^2}$$

$$|BX| = \sqrt{(x-3)^2 + (y-3)^2 + (z-3)^2}$$

$$\sqrt{(x-1)^2 + (y-4)^2 + (z-2)^2} = \sqrt{(x-3)^2 + (y-3)^2 + (z-3)^2}$$

$$(x-1)^2 + (y-4)^2 + (z-2)^2 = (x-3)^2 + (y-3)^2 + (z-3)^2$$

$$(x^2 - 2x + 1) + (y^2 - 8y + 16) + (z^2 - 4z + 4) = (x^2 - 6x + 9) + (y^2 - 6y + 9) + (z^2 - 6z + 9)$$

$$-2x + 1 \quad - 8y + 16 \quad - 4z + 4 = -6x + 9 \quad - 6y + 9 \quad - 6z + 9$$

$$\boxed{4x - 2y + 2z = 6}$$

plane