§Lecture 3 qwf17e

<u>last time:</u> row-échelon form (REF), RREF, utility for answering fundamental questions

§ Linear Combinations

$$\left(\begin{array}{ccc|c} \boxed{0 & 1 & -1 & 0} \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 4 & 1 \end{array}\right)$$

Q: can we read this vertically?

addition $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3} + \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 2+1 & 3-1 \\ 4+1 & 5+0 & 6+1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 5 & 5 & 7 \end{pmatrix}$

Scalar multiplication $(-2)\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1\cdot(-2) & 2\cdot(-2) & 3\cdot(-2) \\ 4\cdot(-2) & 5\cdot(-2) & 6\cdot(-2) \end{pmatrix} = \begin{pmatrix} -2 & -4 & -6 \\ -8 & -10 & -12 \end{pmatrix}$

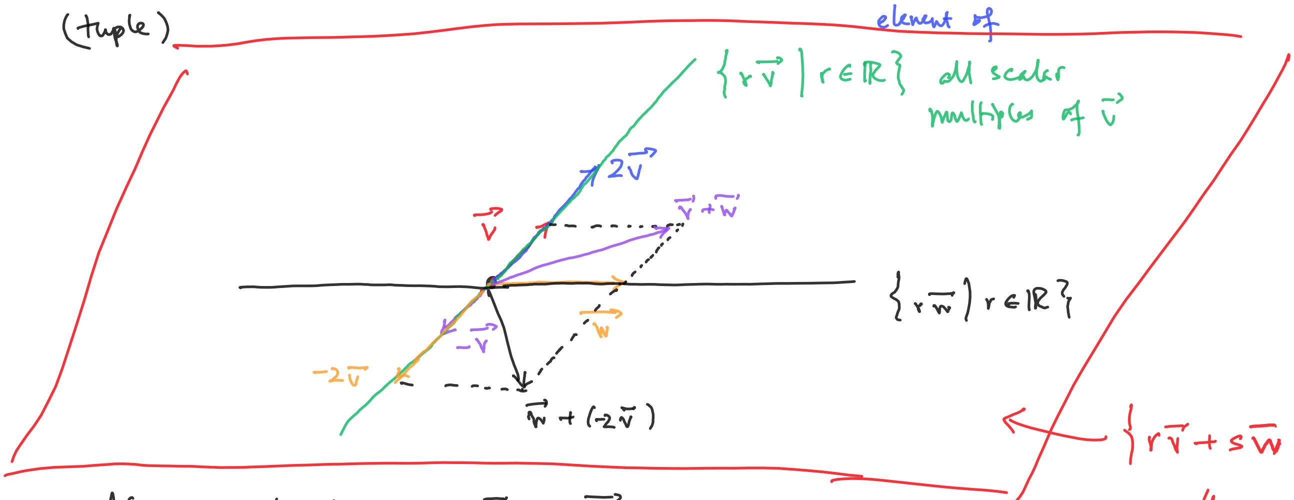
transpose $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

symmetric matrix $A^T = A$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ skew-symmetric matrix $A^T = -A$ $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$

column matrix (tuple) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\mathbb{R}^3 := \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \,\middle|\, a_1, a_2, a_3 \boxed{\in} \boxed{\mathbb{R}} \right\}$

set of real #s

element of



$\{r\vec{v} + s\vec{w} \mid r, s \in \mathbb{R}\}$

$= \operatorname{span}(\{\vec{v}, \vec{w}\})$

linear combination $r\vec{v} + s\vec{w}$

linear system

$$\begin{aligned} 0 &= x_2 - x_3 \\ 1 &= x_1 + 2x_2 + 3x_3 \\ 1 &= x_1 + x_2 + 4x_3 \end{aligned}$$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_2 - x_3 \\ x_1 + 2x_2 + 3x_3 \\ x_1 + x_2 + 4x_3 \end{pmatrix} = \underbrace{x_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}}_{\text{linear combination}}$$

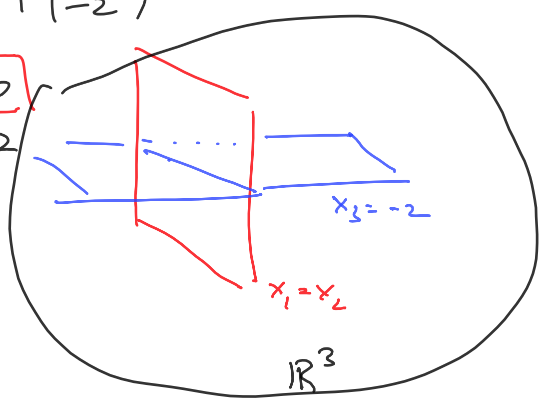
Restatement: Is $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \right\}$? this is existence question.

Moral: existence question for linear system is equivalent to a span membership question.

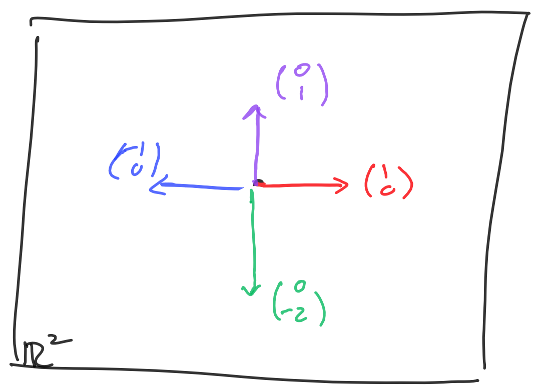
Example:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$\boxed{x_1 - x_2 = 0}$$
$$x_3 = -2$$



Is $\begin{pmatrix} 0 \\ -2 \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$?



§ Matrix-tuple product

$$\underset{2\times\boxed{3}}{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{\boxed{3}\times 1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

matrix-tuple product: make lin. comb. of col. of matrix using entries of tuple as coeff.

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right) \qquad x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{\vec{x}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{\vec{b}}{\begin{pmatrix} 0 \\ -2 \end{pmatrix}}$$

writes linear system as matrix equation $A\vec{x} = \vec{b}$

Given $A, \vec{b}$ find $\vec{x}$

Fact: (Linearity) $A(r\vec{x}+s\vec{y}) = r(A\vec{x}) + s(A\vec{y})$

Given matrix $A_{m\times n}$ get a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ via $\vec{x} \longmapsto A\vec{x}$

$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \vec{x} \in \mathbb{R}^3 \longmapsto A\vec{x} \in \mathbb{R}^2$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\text{input}} \longmapsto \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} x_1 - x_2 \\ x_3 \end{pmatrix}}_{\text{output}}$$

Moral: any matrix is a function in disguise!

solving linear system $A\vec{x} = \vec{b}$ $\quad f(\boxed{\vec{x}}) = \vec{b}$ given $\vec{b}$ want $\vec{x}$

pre-image question: given output want input yielding this desired output

Get perspectives on solving linear system

1. hyperplane intersection
2. span membership
3. pre-image question

§ Matrix Product

$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$JF = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{2\times 2}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{2\times 2} = \underbrace{\begin{pmatrix} \boxed{\begin{matrix} -1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} \end{pmatrix}}_{2\times 2}$$

1st column: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

2nd column: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Computed result column-by-column

Other ways

entry-by-entry: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & \boxed{0} \\ & \end{pmatrix}$ (1,2)-entry $\qquad 0\cdot 1 + (-1)\cdot 0 = 0$

Sum of "simple" matrices: see lecture slides.