Math 415. Exam 2. October 26, 2017

Full Name: ____________________

Net ID: ________________________

Discussion Section: ___________

- There are 18 problems worth 5 points each.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 70 minute exam.
- Do not turn this page until instructed to.
- Fill in the answers on the scantron form provided. Also circle your answers on the exam itself.
- Hand in both the exam and the scantron.
- On the scantron make sure you bubble in **your name, your UIN and your NetID**.
- There are several different versions of this exam.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- Good luck!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**!
On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) Let $\mathcal{B} := \{\mathbf{b}_1, \mathbf{b}_2\}$ be an orthonormal basis of $\mathbb{R}^2$ such that $\mathbf{b}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and let c_1, c_2 be scalars such that $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$. What is c_2?

(A) $\frac{5}{\sqrt{2}}$

(B) ★ $\frac{3}{\sqrt{2}}$

(C) -$\frac{3}{\sqrt{2}}$

(D) -3

(E) There is not enough information to determine c_2

Solution. From page 4, Lecture 18, since the basis is orthonormal we obtain that $c_2 = \mathbf{v} \cdot \mathbf{b_2} = \frac{3}{\sqrt{2}}$.

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2. (5 points) Let H be a subspace of $\mathbb{R}^6$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis of H. What is the dimension of H?

(A) It cannot be determined from the given information

(B) ★ 4

(C) 6

(D) 3

(E) 2

Solution. By definition, the dimension of a vector space H is the number of elements in a basis of H. Thus, $\dim H = 4$.

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3. (5 points) Let A be an $m \times n$ matrix with rank r. Which of the following statements is always true?

The maximal number of linearly independent vectors orthogonal to the row space of A is equal to

(A) ★ the number of free variables of A.

(B) r.

(C) $m - r$.

(D) the dimension of the column space of A^T.

(E) the dimension of the left null space of A.

Solution. The row space of A is a subspace of $\mathbb{R}^n$ of dimension $r = \text{rank}(A)$. Its orthogonal complement has dimension $n - r$, which is the number of free variables of A. The orthogonal complement of the row space of A is the null space of A.

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4. (5 points) Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 4 \\ 1 & -1 & 2 \end{bmatrix}$. Which one of the following sets is a basis for $\mathrm{Col}(A)$?

(A) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}$

(B) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \right\}$

(C) None of the other answers.

(D) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \right\}$

(E) ★ $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

Solution. The third column is redundant since $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

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5. (5 points) Let $M_{2\times 2}$ be the vector space of 2×2 matrices. Consider the subspace $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = b = c \right\}$ of $M_{2\times 2}$. What is the dimension of W?

(A) 4

(B) None of the other answers

(C) 1

(D) ★ 2

(E) 3

Solution. A basis of W is given by $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

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6. (5 points) For which values of h is $\begin{bmatrix} 1 \\ 0 \\ h \end{bmatrix}$ in the column space of

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 10 \end{bmatrix}?$$

(A) For all values of h

(B) Only for $h = 10$

(C) For no values of h

(D) Only for $h = 0$

(E) ★ Only for $h = 2$

Solution. To answer this question, we have to check for which values of h the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ h \end{bmatrix}$ is consistent. So we form an augmented matrix and row reduce:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 10 & h \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 10 & h \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & h-2 \end{array}\right]$$

This system is only consistent if $h - 2 = 0$, i.e. if $h = 2$.

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7. (5 points) Consider the two matrices

$$A = \begin{bmatrix} -1 & -2 & 1 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Which of the following statements is correct?

(A) $\mathrm{Col}(A) \neq \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) = \mathrm{Col}(B^T)$

(B) $\mathrm{Col}(A) = \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) = \mathrm{Col}(B^T)$

(C) ★ $\mathrm{Col}(A) = \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) \neq \mathrm{Col}(B^T)$

(D) $\mathrm{Col}(A) \neq \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) \neq \mathrm{Col}(B^T)$

Solution. Since $\begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $\mathrm{Col}(A) = \mathrm{Col}(B)$. Since $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in $\mathrm{Col}(B^T)$, but not in $\mathrm{Col}(A^T)$, we get $\mathrm{Col}(A^T) \neq \mathrm{Col}(B^T)$.

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8. (5 points) Let A be an $m \times n$ matrix. Which of the following statements is always true?

(A) ★ If $\mathbf{x}$ is a non-zero vector in the null space of A, then $\mathbf{x}$ is not in the row space of A.

(B) If $\mathbf{x}$ is in the column space of A and $\mathbf{y}$ is in the row space of A, then $\mathbf{x}$ is orthogonal to $\mathbf{y}$.

(C) If $\mathbf{x} \in \mathrm{Nul}(A)$ and $\mathbf{y} \in \mathrm{Nul}(A^T)$, then $\mathbf{x} \cdot \mathbf{y} = 0$.

(D) If $\mathbf{x} \in \mathrm{Nul}(A)$ and $\mathbf{y} \in \mathrm{Col}(A)$, then $\mathbf{x} \cdot \mathbf{y} = 0$.

Solution. The null space of A and the row space of A are orthogonal complements of each other. Therefore, their intersection is the zero vector, and $\mathbf{x}$ is assumed to be a non-zero vector. Vectors in the null space of A are not generally in the same space as vectors in the column space of A. Similarly vectors in the null space of A and A^T, and for vectors in the row space and column space.

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9. (5 points) Which of the following sets of vectors is an **orthonormal basis** for $\mathbb{R}^4$?

(A) None of the other answers

(B) ★ $\left\{ \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \\ 0 \end{bmatrix} \right\}$

(C) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(D) $\left\{ \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

(E) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solution. We need 4 vectors (since we want a basis for $\mathbb{R}^4$) that all have length 1, and have dot product zero among distinct elements.

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10. (5 points) Let $W = \left\{ \begin{bmatrix} 0 \\ 2a \\ a \end{bmatrix} : a \in \mathbb{R} \right\}$. Which one of the following is a basis of $W^{\perp}$?

(A) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(B) None of the other answers.

(C) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$

(D) $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(E) ★ $\left\{ \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Solution. Notice that as $\dim(W) = 1$, $\dim(W^{\perp}) = 2$. The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$ are orthogonal to W and linearly independent. Thus $W^{\perp} = \text{span}(\left\{ \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\})$.

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11. (5 points) Let $\mathcal{B} = \{\mathbf{v}_1, ..., \mathbf{v}_d\}$ be a set of d vectors in the vector space V. If the vectors in $\mathcal{B}$ span V, then which of these statements is **false**?

(A) $\mathcal{B}$ is a basis of V if the dimension of V is d.

(B) $\dim(V) \leq d$.

(C) ★ $\dim(V) > d$.

(D) The vectors in $\mathcal{B}$ are linearly independent if the dimension of V is d.

(E) A subset of the vectors in $\mathcal{B}$ is a basis of a subspace of V.

Solution. If the dimension of V is d and we have a set of d vectors which span it, then this set is a basis, and its vectors are linearly independent. If they are not linearly independent, then a subset of $\mathcal{B}$ is linearly independent and spans V. This subset has $d' < d$ vectors, and the dimension of V is d' which is less than d in this case.

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12. (5 points) Let A be an $m \times n$ matrix with $m < n$. Consider the following two statements:

I. $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = n$.

II. $\dim \mathrm{Col}(A^T) + \dim \mathrm{Nul}(A^T) = n$.

Which one of these statements is always true?

(A) Statement I and Statement II.

(B) Statement II only.

(C) ★ Statement I only.

(D) Neither of Statements I or II.

Solution. For any matrix A, $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = r + (n - r) = n$ and $\dim \mathrm{Col}(A^T) + \dim \mathrm{Nul}(A^T) = r + (m - r) = m$, where r is the rank of matrix A. Since $m < n$, only statement I is true.

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13. (5 points) Consider the bases $\mathcal{A} = (\mathbf{a}_1, \mathbf{a}_2)$ of $\mathbb{R}^2$ and $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$ of $\mathbb{R}^4$. Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T(\mathbf{a}_1) = \mathbf{b}_3, \qquad T(\mathbf{a}_2) = -8\mathbf{b}_3 + 6\mathbf{b}_4.$$

Which of the following matrices is $T_{\mathcal{B}\mathcal{A}}$?

(A) $\begin{bmatrix} 1 & 0 \\ -8 & 6 \end{bmatrix}$

(B) ★ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -8 \\ 0 & 6 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -8 & 6 \end{bmatrix}$

(D) Not enough information to tell

(E) $\begin{bmatrix} 1 & -8 \\ 0 & 6 \end{bmatrix}$

Solution. The columns of $T_{\mathcal{B}\mathcal{A}}$ are given by $\begin{bmatrix} T(\mathbf{a}_1)_{\mathcal{B}} & T(\mathbf{a}_2)_{\mathcal{B}} \end{bmatrix}$. Note that $T(\mathbf{a}_1)_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, since $T(\mathbf{a}_1) = 0\mathbf{b}_1 + 0\mathbf{b}_2 + \mathbf{b}_3 + 0\mathbf{b}_4$, and that $T(\mathbf{a}_2)_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ -8 \\ 6 \end{bmatrix}$, since $T(\mathbf{a}_2) = 0\mathbf{b}_1 + 0\mathbf{b}_2 - 8\mathbf{b}_3 + 6\mathbf{b}_4$.

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14. (5 points) Let $\mathbb{P}_n$ be the vector space of all polynomials of degree at most n. Let $T : \mathbb{P}_2 \to \mathbb{P}_1$ be defined by

$$T(p(t)) = -2\frac{d}{dt}p(t)$$

and let $\mathcal{A} = (1+t, t^2, 1-t)$ and $\mathcal{B} = (1, t)$ be bases for $\mathbb{P}_2$ and $\mathbb{P}_1$ respectively. Which one of the following matrices is $T_{\mathcal{B}\mathcal{A}}$?

(A) None of the other answers.

(B) ★ $\begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} -2 & 0 \\ 0 & -4 \\ 2 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} -2 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

(E) $\begin{bmatrix} -2 & 0 \\ 2 & 0 \\ 0 & -4 \end{bmatrix}$

Solution. From Sections 3 and 4, Lecture 17, we obtain that

$$T_{\mathcal{B}\mathcal{A}} = \left[T(1+t)_{\mathcal{B}}\ T(t^2)_{\mathcal{B}}\ T(1-t)_{\mathcal{B}}\right] = [(-2)_{\mathcal{B}}\ (-4t)_{\mathcal{B}}\ 2_{\mathcal{B}}] = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \end{bmatrix}.$$

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15. (5 points) Let V be a vector space that is spanned by three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Which of the following vectors form a basis of V?

(A) None of the other answers.

(B) ★ $\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3$.

(C) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_1$.

(D) $\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_2 + \mathbf{v}_3$.

Solution. We know already from the statement of the question that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of V and so $\dim(V) = 3$. Thus, we are looking for a set of three linearly independent vectors spanning V. Among the given choices only $\{\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3\}$ satisfies these properties.

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16. (5 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -3 \end{bmatrix}.$$

What is the dimension of $\mathrm{Nul}(A^T)$?

(A) 3

(B) 0

(C) 4

(D) ★ 1

(E) 2

Solution.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow[R3 \to R3 - 2R2]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow[R4 \to R4 - R3]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $r = rank(A) = 3$ and $dim\,\mathrm{Nul}(A^T) = 4 - r = 4 - 3 = 1$.

17. (5 points) Consider the following 2×2-matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$ is a basis of $M_{2\times 2}$, the vector space of 2×2 matrices. Let M be the 2×2-matrix whose coordinate vector with respect to this basis $\mathcal{B}$ is $\begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

Which matrix is M?

(A) $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

(B) ★ $\begin{bmatrix} -2 & 2 \\ -2 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}$

(E) None of the other answers.

Solution. From the definition of coordinate vector, it follows that

$$2\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & -1 \end{bmatrix}.$$

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18. (5 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be such that

$$T(\mathbf{x}) = \mathbf{0}, \qquad \text{and there exists a vector } \mathbf{z} \in \mathbb{R}^n \text{ such that } T(\mathbf{z}) = \mathbf{y}.$$

Let A be the matrix representing T with respect to the standard basis of $\mathbb{R}^n$ (that is, $A = T_{\mathcal{E}\mathcal{E}}$). Consider the following statements.

I. $\mathbf{x} \in \mathrm{Nul}(A)$.

II. For every vector $\mathbf{v} \in \mathbb{R}^n$, $A\mathbf{v} = T(\mathbf{v})$.

III. $\mathbf{y} \in \mathrm{Col(A)}$.

Which of the statements are ALWAYS TRUE?

(A) I. and II. only

(B) I. only

(C) I. and III. only

(D) II. only

(E) ★ I., II., and III.

Solution. Because A is the matrix of T with respect to the standard basis, for any $\mathbf{v} \in \mathbb{R}^n$, $T(\mathbf{v}) = A\mathbf{v}$. In particular, if $T(\mathbf{x}) = \mathbf{0}$, so $A\mathbf{x} = \mathbf{0}$. Therefore $\mathbf{x} \in \mathrm{Nul}(A)$. Also, $A\mathbf{z} = T(\mathbf{z}) = \mathbf{y}$, so $\mathbf{y} \in \mathrm{Col}(A)$.
