Math 415. Practice Exam 1. September 28, 2017

Full Name: ____________________
Net ID: ____________________

- There are 20 problems worth 5 points each.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 70 minute exam.
- Do not turn this page until instructed to.
- Fill in the answers on the scantron form provided. Also circle your answers on the exam itself.
- Hand in both the exam and the scantron.
- On the scantron make sure you
- There are several different versions of this exam.

Fill in the following answers on the Scantron form:

On the first page of the scantron, bubble in **your name, your UIN and your NetID**!
On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) The matrix $A = \begin{bmatrix} 0 & 0 & 2 \\ 3 & 3 & 6 \\ 1 & 1 & 6 \end{bmatrix}$ does not have an $A = LU$ decomposition. However, after *interchanging the first and third rows* of A, we can find a $PA = LU$ decomposition, where P is the permutation matrix that interchanges the first and the third row. What are the appropriate matrices P, L, and U for such a decomposition?

(A) $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & \frac{1}{6} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 1 & 6 \\ 0 & 0 & -12 \\ 0 & 0 & 0 \end{bmatrix}$

(B) $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & 2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 3 & 3 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(C) ★ $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -\frac{1}{6} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 1 & 6 \\ 0 & 0 & -12 \\ 0 & 0 & 0 \end{bmatrix}$

(D) $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & -2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 3 & 3 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Solution. The permutation matrix for the interchange of the first and the third row is $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Thus $PA = \begin{bmatrix} 1 & 1 & 6 \\ 3 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$. Bring PA to echelon form:

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} \underset{R2 \to R2 - 3R1}{\longrightarrow} \begin{bmatrix} 1 & 1 & 6 \\ 0 & 0 & -12 \\ 0 & 0 & 2 \end{bmatrix} \underset{R3 \to R2 + \frac{1}{6}R3}{\longrightarrow} \begin{bmatrix} 1 & 1 & 6 \\ 0 & 0 & -12 \\ 0 & 0 & 0 \end{bmatrix} = U$$

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -\frac{1}{6} & 1 \end{bmatrix}$.

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2. (5 points) Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ k & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. For which value of k does the system $A\mathbf{x} = b$ have a unique solution?

(A) There is no such value for k.

(B) $k = 0$

(C) ★ $k = -1$

(D) $k = 1$

Solution.

$$[A|b] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ -1 & 1 & 0 \\ k & 2 & 1 \end{array}\right] \underset{R2\to R2+R1, R3\to R3-kR1}{\longrightarrow} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1-k \end{array}\right] \underset{R3\to R3-2R2}{\longrightarrow} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1-k \end{array}\right]$$

Therefore the system is consistent if and only if $-1 - k = 0$.

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3. (5 points) Consider the following subsets of $\mathbb{R}^3$:

$$W_1 = \{ \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \ : \ a+b=0\}, \ W_2 = \{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \ : \ a+b \geq 0\}.$$

Then:

(A) Both W_1 and W_2 are subspaces of $\mathbb{R}^3$.

(B) Only W_2 is a subspace of $\mathbb{R}^3$.

(C) ★ Neither W_1 nor W_2 is a subspace of $\mathbb{R}^3$.

(D) Only W_1 is a subspace of $\mathbb{R}^3$.

Solution. W_1 is not a subspace, because obviously $\mathbf{0}$ is not in W_1. The second subset W_2 is also not a subspace. Observe that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in W_2, but $(-5)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not. Therefore W_2 is not closed under scalar multiplication.

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4. (5 points) Which of the following matrices is the inverse of

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(A) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(C) ★ $\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution. Use Gauss Jordan algorithm do find:

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

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5. (5 points) Let $A = \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}$. Then

(A) $\mathrm{Nul}(A) = \mathbb{R}^2$.

(B) $\mathrm{Nul}(A) = \mathrm{Span}(\begin{bmatrix} 1 \\ 4 \end{bmatrix})$.

(C) $\mathrm{Nul}(A) = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$.

(D) $\mathrm{Nul}(A) = \mathrm{Span}(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 16 \end{bmatrix})$.

(E) ★ $\mathrm{Nul}(A) = \mathrm{Span}(\begin{bmatrix} 4 \\ -1 \end{bmatrix})$.

Solution. The reduced echelon form of A is $\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$. Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is in $\mathrm{Nul}(A)$ if and only if $x_1 = -4x_2$. Thus $\mathrm{Nul}(A) = \mathrm{Span}(\begin{bmatrix} -4 \\ 1 \end{bmatrix})$. The latter is equal to $\mathrm{Span}(\begin{bmatrix} 4 \\ -1 \end{bmatrix})$.

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6. (5 points) If $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent for

(A) all $\mathbf{b} \in \mathbb{R}^2$.

(B) all $\mathbf{b}$ such that $\mathbf{b} = c \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(C) ★ all $\mathbf{b}$ such that $\mathbf{b} = c \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

(D) only for $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(E) no $\mathbf{b} \in \mathbb{R}^2$.

Solution. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be such that $Ax = b$. Then

$$b = A\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 8 \end{bmatrix} = (x_1 + 2x_2) \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Thus $\mathbf{b} = c \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ for some c if $A\mathbf{x} = b$ is consistent.

Suppose $\mathbf{b} = c \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Then $A \begin{bmatrix} c \\ 0 \end{bmatrix} = \mathbf{b}$.

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7. (5 points) Find an explicit description of the null space of

$$A = \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

that is, find a minimal set of vectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}$ such that $\mathrm{Nul}(A) = \mathrm{span}\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_p}\}$.

(A) ★ $\begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

(B) $\begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

Solution. $\mathrm{Nul}(A)$ is the set of all solutions to the homogeneous equation

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We identify x_1, x_2 and x_4 as pivot variables and x_3 as the free variable. Thus we can write the general solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

where $x_3 \in \mathbb{R}$ is free. We conclude that

$$\mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

8. (5 points) Let A be a 3×2-matrix and $\boldsymbol{b}$ in $\mathbb{R}^3$. Consider the following two statements:

(S1) If $\boldsymbol{b} \neq \mathbf{0}$, then the set of solutions of $A\boldsymbol{x} = \boldsymbol{b}$ can be a plane through the origin.

(S2) If $A\boldsymbol{x} = \boldsymbol{b}$ is consistent, the solution to $A\boldsymbol{x} = \boldsymbol{b}$ is unique if and only if the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

Then

(A) Statement S1 and Statement S2 are correct.

(B) Only Statement S1 is correct.

(C) ★ Only Statement S2 is correct.

(D) Neither Statement S1 nor Statement S2 is correct.

Solution. For statement (S1), note that $A\mathbf{0} = \mathbf{0} \neq \boldsymbol{b}$. So the set of solutions of $A\boldsymbol{x} = \boldsymbol{b}$ can not contain the origin. For statement (S2), let $\boldsymbol{y}$ in $\mathbb{R}^3$ such that $A\boldsymbol{y} = \boldsymbol{b}$. Then for every $\boldsymbol{z} \in \mathbb{R}^2$, we have

$$A(\boldsymbol{y} + \boldsymbol{z}) = A\boldsymbol{y} + A\boldsymbol{z}.$$

Thus $A(\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{b}$ if and only if $A\boldsymbol{z} = 0$. Therefore $\boldsymbol{y}$ is the only solution to $A\boldsymbol{x} = \boldsymbol{z}$ if and only if $\mathbf{0}$ is the only solution to $A\boldsymbol{x} = \mathbf{0}$.

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9. (5 points) The matrix $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ is reduced to the echelon matrix $U = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ using the following row operations (in the given order):

A. $R_2 \to R_2 + 2R_1$,

B. $R_3 \to R_3 - R_1$,

C. $R_3 \to R_3 + R_2$,

D. $R_4 \to R_4 + R_2$.

What is the matrix L in the $A = LU$ decomposition?

(A) ★ $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

(B) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(C) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

(D) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution. Since row operations correspond to multiplying with elementary matrices, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = U$$

Thus

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

10. (5 points) If A and B are $m \times n$-matrices with the same reduced row echelon form U, then there exists a sequence of elementary matrices $E_1, \ldots, E_k$, with each E_i of size $m \times m$, such that

$$E_k \cdots E_1 A = B.$$

(A) Always false

(B) Sometimes true and sometimes false

(C) ★ Always true

Solution. Note that there are elementary row operations that change A into U, and there are elementary row operations that change B into U. Since each elementary row is reversible, there are elementary row operations that change A into B. By Theorem 1 of Lecture 6 there is a sequence of elementary matrices $E_1, \ldots, E_k$ corresponding to these row operations such that

$$E_k \cdots E_1 A = B.$$

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11. (5 points) Let A, B be $n \times n$-matrices. Which of the following statements is false?

(A) if A is invertible and its rows are in reverse order in B, then B is invertible.

(B) if A and B are invertible, then BA is invertible.

(C) if A is invertible, A^T is invertible.

(D) ★ if A is invertible, then A can be factored into the product $A = LU$ of a lower triangular matrix L and an upper triangular matrix U.

Solution. See Worksheet 3 Problem 6.

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12. (5 points) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. Then the inverse of A is

(A) $A^{-1} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

(B) ★ $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

(C) $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(D) $A^{-1} = \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$

(E) A^{-1} does not exist.

Solution. Use Lecture 8 Theorem 2.

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13. (5 points)

Let A be an $m \times n$-matrix and let B be an $n \times m$-matrix. Then which of the following statement is not true for all such matrices?

(A) BA is defined

(B) ★ the columns of AB are linear combinations of the columns of B

(C) AB is defined

(D) the columns of AB are linear combinations of the columns of A

Solution. Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

But $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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14. (5 points) Let $A = \begin{bmatrix} 1 & 7 \\ 2 & 8 \\ 3 & 9 \end{bmatrix}$. Then the *transpose* of A is

(A) ★ $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

(B) is only defined for $n \times n$-matrices.

(C) $A^T = \begin{bmatrix} 7 & 1 \\ 8 & 2 \\ 9 & 3 \end{bmatrix}$

(D) $A^T = \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}$

Solution. See Lecture 6 for a definition of the transpose.

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15. (5 points) Let A be an $\ell \times m$-matrix such that for every $\boldsymbol{b}$ in $\mathbb{R}^\ell$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution. What does this statement imply about the relative size of ℓ and m?

(A) ★ $\ell \leq m$

(B) $\ell \geq m$

(C) nothing (ℓ and m can be any positive integers)

(D) $\ell = m$

Solution. If for every $\boldsymbol{b}$ in $\mathbb{R}^\ell$ the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, then $\mathrm{Col}(A) = \mathbb{R}^\ell$. Therefore the columns of A span $\mathbb{R}^\ell$. Hence there must be at least ℓ columns.

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16. (5 points) Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$. Let $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Which of the following statements is true?

(A) ★ $\boldsymbol{v}$ is a linear combination of the columns of B but not of the columns of A.

(B) $\boldsymbol{v}$ is a linear combination of the columns of A and of the columns of B.

(C) $\boldsymbol{v}$ is a linear combination of the columns of A but not of the columns of B.

(D) $\boldsymbol{v}$ is neither a linear combination of the columns of A nor of the columns of B.

Solution. It is enough to check whether the two system $Ax = v$ and $Bx = v$. The matrix $[A|v]$ can be row reduced to $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$. Therefore $Ax = v$ has no solution. Thus v is not a linear combination of the columns of A. The matrix $[B|v]$ can be row reduced to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \end{array}\right]$. Therefore $Bx = v$ is consistent. Thus v is a linear combination of the columns of B.

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17. (5 points) Which of the following choices for a makes A invertible, where A is $\begin{bmatrix} 1 & a \\ 1 & a \end{bmatrix}$?

(A) ★ no real number.

(B) Any real number.

(C) any real number except for -1 and 1.

(D) Only for -1 and 1.

Solution. Because $a - a = 0$ for every a.

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18. (5 points) For what values of h is the system

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + x_3 &= h \\ x_1 - x_3 &= 1 \end{aligned}$$

consistent?

(A) It is consistent for $h = 0$.

(B) It is always inconsistent.

(C) It is consistent for $h = 1$.

(D) It is consistent for all values of h.

(E) ★ It is consistent for $h = -1$.

Solution. We bring the augmented matrix to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & h \\ 1 & 0 & -1 & 1 \end{array}\right] \underset{R3 \to R3 - R1}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & h \\ 0 & -1 & -1 & 1 \end{array}\right] \underset{R3 \to R3 + R2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & h \\ 0 & 0 & 0 & 1+h \end{array}\right].$$

Thus the system is consistent if and only if $h = -1$.

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19. (5 points) Consider the following four 3×3-matrices: $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Which matrix is not invertible?

(A) D

(B) ★ C

(C) B

(D) A

Solution. Bring all four matrices to reduced echelon form. Then realize that C is the only one whose reduced echelon form is not $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore it's the only one that is not invertible.

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20. (5 points) Let A be a 2×3-matrix and b a vector in $\mathbb{R}^2$.

Consider the following two statements:

(P1) A has at most two pivots,

(P2) Assuming $Ax = b$ has a solution, then it has infinitely many solutions.

Then:

(A) Only Statement P2 is correct.

(B) Neither Statement P1 nor Statement P2 is correct.

(C) Only Statement P1 is correct.

(D) ★ Statement P1 and Statement P2 are correct.

Solution. A has only two rows, so it can have at most two pivots (there is only one pivot per row). Since A has only two pivots, but three columns, there must be a free variable. So, whenever $Ax = b$ is consistent, there are infinitely many solutions.
