Math 415. Exam 3. April 18, 2019

Full Name: ____________________

Net ID: ________________________

Discussion Section: ___________

- Do not turn this page until instructed to.
- There are 20 problems worth 5 points each.
- Each question has only one correct answer. You can choose up to two answers. If you choose just one answer, then you will get 5 points if the answer is correct, and 0 points otherwise. However, if you choose two answers, you will get 2.5 points if one of the answers is correct, and 0 points otherwise.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 75 minute exam. There are several different versions of this exam.
- No bathroom breaks.
- Fill in the answers on the scantron form provided, **and** circle your answers on the exam itself. Hand in both the exam and the scantron.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- If you have to erase something on the scantron, please make sure to do so thoroughly.
- Good luck!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**! On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) What is the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 2 \\ 2 & 0 & 3 & 0 \end{bmatrix}?$$

(A) -12

(B) ★ -24

(C) 20

(D) None of the other answers.

(E) 0

Solution. There are several possible ways to do the cofactor expansion. Choosing columns and rows with many zeros makes things easier. Expanding along the 2nd row and then along 3rd row yields

$$\det(A) = -3 \begin{vmatrix} 1 & 2 & 0 \\ 1 & 2 & 2 \\ 2 & 0 & 0 \end{vmatrix} = -6 \begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} = -24.$$

---

2. (5 points) Let A be an $m \times n$-matrix and $\mathbf{b} \in \mathbb{R}^m$. Consider the following statements:

I. Let $\mathbf{y}$ be in $\mathbb{R}^n$ such that $A^T A\mathbf{y} = A^T\mathbf{b}$, then $A\mathbf{y}$ is in $\mathrm{Col}(A)$.

II. The linear system $A\mathbf{x} = \mathbf{b}$ always has a least-squares solution.

Which one of these statements is always true?

(A) ★ Statement I and Statement II.

(B) Neither of Statements I or II.

(C) Statement II only.

(D) Statement I only.

Solution. By the definition, y is the least square solution of linear system $Ax = b$, and then it satisfies $Ay = \hat{b}$, where $\hat{b}$ is the orthogonal projection of b on $\mathrm{Col}(A)$. Therefore, $Ay = \hat{b}$ is in $\mathrm{Col}(A)$.

For Statement II, by the definition, the least square solution $\hat{x}$ is the solution of linear system $A\hat{x} = \hat{b}$, where $\hat{b}$ is the orthogonal projection of b on $\mathrm{Col}(A)$. Since $\hat{b}$ is in $\mathrm{Col}(A)$, the linear system $A\hat{x} = \hat{b}$ is always consistent.

---

3. (5 points) Let $W := \operatorname{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$. What is the projection matrix for the orthogonal projection onto W with respect to the standard basis?

(A) ★ $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$

(B) $\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{3}{2}\end{bmatrix}$

(C) $\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$

(D) None of the other answers.

(E) $\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\\frac{1}{2} & \frac{3}{2} & 0\\0 & 0 & 0\end{bmatrix}$

Solution. Let π_W be the projection onto W and P be the corresponding projection matrix. Then

$$\pi_W(\begin{bmatrix}1\\0\\0\end{bmatrix}) = \begin{bmatrix}1\\0\\0\end{bmatrix},$$

since $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is in W. Furthermore,

$$\pi_W(\begin{bmatrix}0\\1\\0\end{bmatrix}) = \begin{bmatrix}0\\1\\0\end{bmatrix},$$

becuause $\begin{bmatrix}0\\1\\0\end{bmatrix}$ is also in W. Since $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is orthogonal to W, we get that

$$\pi_W(\begin{bmatrix}0\\0\\1\end{bmatrix}) = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Thus

$$P = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}.$$

---

4. (5 points) Consider the following vectors

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

Which of the following is the ordered list of orthonormal vectors obtained from $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ by the Gram-Schmidt process **applied to the vectors in the given order**?

(A) $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$

(B) $\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{6}}\\\frac{\sqrt{2}}{\sqrt{3}}\\-\frac{1}{\sqrt{6}}\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}$

(C) None of the other answers.

(D) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

(E) ★ $\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$

Solution. The first vector in the orthonormal basis is

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

Next,

$$\mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \sqrt{2}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

So

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1}{\|\mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1\|} = \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Next

$$\mathbf{v}_3 - (\mathbf{u}_1 \cdot \mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v}_3)\mathbf{u}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix} - 0\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} - 0\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

So

$$\mathbf{u}_3 = \frac{\mathbf{v}_3 - (\mathbf{u}_1 \cdot \mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v}_3)\mathbf{u}_2}{\|\mathbf{v}_3 - (\mathbf{u}_1 \cdot \mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v}_3)\mathbf{u}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

---

5. (5 points) Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$, and let $\mathcal{E}$ be the standard basis for $\mathbb{R}^2$. Which of the following is the change of basis matrix $I_{\mathcal{B}\mathcal{E}}$?

(A) $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$

(B) None of the other answers

(C) $\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

(D) ★ $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

(E) $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

Solution. Since $\mathcal{E}$ is the standard basis, it is immediate that:

$$I_{\mathcal{E}\mathcal{B}} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}.$$

Next, we invert to get the desired matrix:

$$I_{\mathcal{B}\mathcal{E}} = I_{\mathcal{E}\mathcal{B}}^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

---

6. (5 points) Which of the following is NOT an eigenvector of the matrix $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$?

(A) $\begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix}$

(B) ★ $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(D) All of the others answers are eigenvectors of A.

(E) $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Solution. We can check whether each of the vectors v is an eigenvector by checking whether Av is a multiple of v. In particular, we find that

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

---

7. (5 points) Let W be a subspace of $\mathbb{R}^n$, P be the projection matrix (in the standard basis) of the orthogonal projection onto W and Q be the projection matrix (in the standard basis) of the orthogonal projection onto $W^\perp$. Consider the following statements:

T1. $P + Q = I_n$, where I_n is the identity matrix.

T2. $PQ = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix.

T3. $P^2 = P$.

Which of the statements are always correct?

(A) ★ All statements are correct.

(B) Only statement T3.

(C) Only statement T1.

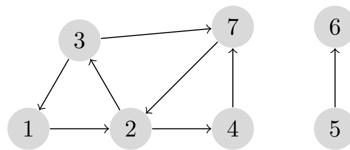
(D) Only statements T1 and T3.

(E) None of the statements.

Solution. From class we know that $Q = I_n - P$. Thus $P + Q = I_n$. Moreover, $P^2 = P$. Thus $PQ = P(I_n - P) = P^2 - P = \mathbf{0}$.

---

8. (5 points) Let A be the edge-node incidence matrix of the graph below.



Which of the following statements is true?

(A) ★ $\dim \mathrm{Nul}(A) = 2$ and $\dim \mathrm{Nul}(A^T) = 3$

(B) $\dim \mathrm{Nul}(A) = 2$ and $\dim \mathrm{Nul}(A^T) = 2$

(C) $\dim \mathrm{Nul}(A) = 3$ and $\dim \mathrm{Nul}(A^T) = 3$

(D) None of the other answers.

(E) $\dim \mathrm{Nul}(A) = 3$ and $\dim \mathrm{Nul}(A^T) = 2$

Solution. $\dim \mathrm{Nul}(A)$ in the number of components and $\dim \mathrm{Nul}(A^T)$ is the number of independent cycles

---

9. (5 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal basis of $\mathbb{R}^3$. Which of the following statements is always true about the set $\mathcal{C} = \{A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3\}$?

(A) ★ $\mathcal{C}$ is a basis of $\mathbb{R}^3$ but not necessarily orthogonal.

(B) $\mathcal{C}$ is an orthonormal basis of $\mathbb{R}^3$.

(C) None of the other answers.

(D) $\mathcal{C}$ is an orthogonal basis of $\mathbb{R}^3$ but not necessarily orthonormal.

(E) $\mathcal{C}$ is not a basis of $\mathbb{R}^3$.

Solution. The matrix has orthogonal columns, so it has full rank. This means that $\mathcal{C}$ is always a basis of $\mathbb{R}^3$. To see that $\mathcal{C}$ is not necessarily orthogonal, consider the example where

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$A\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, A\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

are not orthogonal.

---

10. (5 points) If $\lambda_1 \geq \lambda_2$ are the eigenvalues of the matrix $\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$, what is λ_1?

(A) 0

(B) ★ 1

(C) None of the other answers.

(D) -2

(E) 2

Solution. The eigenvalues of $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ are the roots of the characteristic polynomial

$$\det(xI - A) = (x-3)(x+3) + 8 = x^2 - 1 = (x-1)(x+1).$$

Thus, $\lambda_1 = 1$ and $\lambda_2 = -1$.

11. (5 points) Let A be an $m \times n$-matrix, let Q be an $m \times n$ matrix with orthonormal columns and let R be an upper triangular invertible $n \times n$-matrix such that $A = QR$. Consider the following statements:

(T1) The columns of Q form an orthonormal basis of the column space of A.

(T2) The column space of A is equal to the column space of R.

(T3) The null space of A is equal to the null space of R.

Which of these statements are always true?

(A) ★ Only (T1) and (T3).

(B) (T1), (T2) and (T3).

(C) Only (T1) and (T2).

(D) Only (T1)

(E) None of the other answers.

Solution. Statement (T1) is true. Let Qx be a vector in the column space of Q. Let $y := R^{-1}x$. Then $Qx = QRy = Ay$ is in the column space of A. Thus every vector in the column space of Q is in the column space of A. Thus the columns of Q are in the column space of A and thus form an orthonormal basis of the column space of A.

Statement (T2) is false. If $m \neq n$, the column space of A is a subspace of $\mathbb{R}^n$, while the column space of R is a subspace of $\mathbb{R}^m$.

Statement (T3) is true. Let $x \in \mathrm{Nul}(A)$. Then

$$Rx = Q^T QRx = Q^T Ax = Q^T 0 = 0.$$

Thus $x \in \mathrm{Nul}(R)$. On the other hand, if $x \in \mathrm{Nul}(R)$, then $Ax = QRx = Q0 = 0$. Thus $x \in \mathrm{Nul}(A)$.

---

12. (5 points) Consider the data points $(1, 2), (0, 1), (-1, 3)$. What is the least-squares line for the three data points given above?

(A) ★ $y = -\frac{1}{2}x + 2$

(B) $y = \frac{1}{2}x - 2$

(C) $y = -2x + \frac{1}{2}$

(D) $y = 2x - \frac{1}{2}$

(E) None of the other answers.

Solution. Assume the least square line is $y = mx + b$. Then we obtain the linear system

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

The normal equation corresponding to this linear system is

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}.$$

Therefore, we have $m = -\frac{1}{2}$, $b = 2$, and the least square line is $y = -\frac{1}{2}x + 2$.

---

13. (5 points) Let W be a subspace of $\mathbb{R}^n$ with $\dim W < n$. Let P be the projection matrix of the orthogonal projection onto W with respect to the standard basis. Select the statement about P which is FALSE, or if all statements about P are correct, select "None of the other options".

(A) None of the other options.

(B) ★ The columns of P form a basis of W.

(C) All eigenvalues of P are either 0 or 1.

(D) The nullspace of P is equal to $W^{\perp}$.

(E) The rank of P is equal to the dimension of W.

Solution. The columns of P span W, but may not be linearly independent. For example, take the orthogonal projection onto $W := \operatorname{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$. Then $P := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The columns of P are not linearly independent, but span W.

---

14. (5 points) Let A be a 3×3 matrix with rows $\mathbf{R}_1, \mathbf{R}_2$ and $\mathbf{R}_3$. Suppose the following row operations bring A to the identity matrix:

$$A \xrightarrow{R_1 \leftrightarrow R_2} A_1 \xrightarrow{R_3 \to \frac{1}{5} R_3} A_2 \xrightarrow{R_3 \to R_3 + 2R_1} I_3$$

where I_3 is the 3×3 identity matrix. Then,

(A) None of the other answers.

(B) $\det A = 5$

(C) ★ $\det A = -5$

(D) $\det A = -\frac{1}{5}$

(E) $\det A = \frac{1}{5}$

Solution. Of course, $\det(I_3) = 1$. Then also $\det(A_2) = 1$, since this row operation does not change determinants. Then $\det(A_1) = 5$, (row 3 in A_1 is 5 times row 3 in A_2) and finally $\det(A) = -5$ because of the row swap.

---

15. (5 points) Let a_1, a_2 and a_3 be three orthonormal vectors in $\mathbb{R}^n$. Which of the following matrices can be the matrix R in an QR-decomposition of

$$A = \begin{bmatrix} | & | & | \\ a_1 & 2a_2 - a_1 & a_3 + 3a_2 \\ | & | & | \end{bmatrix}?$$

(A) Not enough information to determine whether any of the matrices listed in the other answers can be the matrix R in an QR-decomposition of A.

(B) ★ $R = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

(C) None of the other answers.

(D) $R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

(E) $R = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$

Solution. The matrix $Q = \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}$ is the matrix we get from carrying out the Gram-Schmidt procedure. Thus

$$R = Q^T A = \begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot (2a_2 - a_1) & a_1 \cdot (a_3 + 3a_2) \\ a_2 \cdot a_1 & a_2 \cdot (2a_2 - a_1) & a_2 \cdot (a_3 + 3a_2) \\ a_3 \cdot a_1 & a_3 \cdot (2a_2 - a_1) & a_3 \cdot (a_3 + 3a_2) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

---

16. (5 points) Let A be an $n \times n$ matrix. Consider the following two statements:

(U1) The matrix $3A$ has the same eigenvectors as A.

(U2) The matrix $3A$ has the same eigenvalues as A.

Which of the two statements are always true?

(A) ★ Only Statement U1 is correct.

(B) Statement U1 and Statement U2 are correct.

(C) Neither Statement U1 nor Statement U2 is correct.

(D) Only Statement U2 is correct.

Solution. Let $\mathbf{x}$ be an eigenvector of A with eigenvalue λ. Then $3A\mathbf{x} = 3(\lambda\mathbf{x}) = (3\lambda)\mathbf{x}$, therefore $3A$ has $\mathbf{x}$ as eigenvector with eigenvalue 3λ. Thus the matrices have the same eigenvectors, but not the same eigenvalues.

---

17. (5 points) Let A be an $n \times n$-matrix and let B be an $m \times n$-matrix. Consider the following statements.

(S1) If B has orthonormal columns, then B^T has orthonormal columns.

(S2) If A^T is an orthogonal matrix, then A is an orthogonal matrix.

Which of the two statements are always true?

(A) Neither (S1) nor (S2).

(B) Only (S1).

(C) Both (S1) and (S2).

(D) ★ Only (S2).

Solution.

(S1) This claim is false. The matrix below is a counterexample:

$$B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(S2) This claim is TRUE: A^T must have orthonormal columns, so A^T must be an orthogonal matrix. So $AA^T = A^T A = I$ and thus A is an orthogonal matrix.

---

18. (5 points) Let A be an $m \times n$-matrix and $\mathbf{b}$ be in $\mathbb{R}^m$.

(T1) If $\hat{\mathbf{x}}$ is a least-squares solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{y}$ is in $\mathrm{Nul}(A)$, then $\hat{\mathbf{x}} + \mathbf{y}$ is a least-squares solution to $A\mathbf{x} = \mathbf{b}$.

(T2) If $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2$ are least-squares solutions to $A\mathbf{x} = \mathbf{b}$, then $\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2$ is a least-squares solution to $A\mathbf{x} = \mathbf{b}$.

Which of the two statements is always true?

(A) Only (T2).

(B) Both (T1) and (T2).

(C) ★ Only (T1).

(D) Neither (T1) or (T2).

Solution. Statement (T1) is true. Suppose $\hat{\mathbf{x}}$ is a least-squares solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{y}$ is $\mathrm{Nul}(A)$. Then $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$. Thus

$$A^TA(\hat{\mathbf{x}} + \mathbf{y}) = A^TA\hat{\mathbf{x}} + A^TA\mathbf{y} = A^TA\hat{\mathbf{x}} + A^T\mathbf{0} = A^TA\hat{\mathbf{x}} = A^T\mathbf{b}.$$

Thus $\hat{\mathbf{x}} + \mathbf{y}$ is a least-squares solution to $A\mathbf{x} = \mathbf{b}$.

Statement (T2) is false. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then the projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus the least-squares solutions to $A\mathbf{x} = \mathbf{b}$ are the solutions to the equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are both solutions to this equation, however their sum $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not.

---

19. (5 points) Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -2 \end{bmatrix}$. Calculate $\mathbf{b}_W$, the orthogonal projection of $\mathbf{b}$ onto W.

(A) ★ $\begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix}$

(B) $\begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$

(D) $\begin{bmatrix} 2 \\ 0 \\ 2 \\ -2 \end{bmatrix}$

(E) None of the other answers

Solution. Then

$$b_W = \frac{\begin{bmatrix} 2 \\ 0 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 2 \\ 0 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 2 \\ 0 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix}$$

---

20. (5 points) If $\mathcal{A}$ and $\mathcal{B}$ are bases for $\mathbb{R}^n$, $\mathcal{E}$ is the standard basis for $\mathbb{R}^n$, and $\mathbf{v} \in \mathbb{R}^n$, then which of the following is always equal to $\mathbf{v}_{\mathcal{B}}$?

(A) $I_{\mathcal{B}\mathcal{E}}\mathbf{v}_{\mathcal{A}}$

(B) $I_{\mathcal{B}\mathcal{A}}^{-1}I_{\mathcal{B}\mathcal{E}}\mathbf{v}$

(C) ★ $I_{\mathcal{A}\mathcal{B}}^{-1}I_{\mathcal{A}\mathcal{E}}\mathbf{v}$

(D) $I_{\mathcal{A}\mathcal{B}}\mathbf{v}_{\mathcal{A}}$

(E) None of the other answers.

Solution. Using change of basis we obtain that

$$I_{\mathcal{A}\mathcal{B}}^{-1}I_{\mathcal{A}\mathcal{E}}\mathbf{v} = I_{\mathcal{A}\mathcal{B}}^{-1}I_{\mathcal{A}\mathcal{E}}\mathbf{v}_{\mathcal{E}} = I_{\mathcal{A}\mathcal{B}}^{-1}\mathbf{v}_{\mathcal{A}} = I_{\mathcal{B}\mathcal{A}}\mathbf{v}_{\mathcal{B}} = \mathbf{v}_{\mathcal{B}}.$$
