§ Lecture 14 dkml5p

last time: change of coordinates $I_{\mathcal{A}\mathcal{B}}$ or $I_{\mathcal{B}\mathcal{A}}$, linear transformations (LT)

$$L(r\vec{x}+s\vec{y}) = rL(\vec{x}) + sL(\vec{y})$$

§ Matrix of LT

V vector space of dim $= n$ ordered basis $\mathcal{B} = (\vec{b}_1, \dots, \vec{b}_n)$

W ——— $= m$ ——— $\mathcal{C} = (\vec{c}_1, \dots, \vec{c}_m)$

LT $L: V \longrightarrow W$

$$\begin{tikzcd} \vec{v} \arrow[r, mapsto] & L(\vec{v}) \\ V \arrow[r, "L"] \arrow[d] & W \arrow[d] \\ \mathbb{R}^n \arrow[r, dashed] & \mathbb{R}^m \\ \vec{v}_{\mathcal{B}} \arrow[r, mapsto] & A\vec{v}_{\mathcal{B}} \end{tikzcd}$$

$\mathcal{B}$-coords $\qquad$ $\mathcal{C}$-coords

this LT is matrix-tuple product

$$L(\vec{v})_{\mathcal{C}} = A\,\vec{v}_{\mathcal{B}}$$

usually write $L_{\mathcal{C}\mathcal{B}}$ for matrix

matrix of LT

Fact: $L_{\mathcal{C}\mathcal{B}} = \begin{pmatrix} L(\vec{b}_1)_{\mathcal{C}} & \cdots & L(\vec{b}_n)_{\mathcal{C}} \end{pmatrix}_{m\times n}$

Example: LT identity function $V \longrightarrow V$ $\quad \vec{v} \longmapsto \vec{v}$ $\quad \dim(V) = n$

$\mathcal{B}, \mathcal{C}$ ordered bases on V

$$\begin{tikzcd} \vec{v} \arrow[r, mapsto] & \vec{v} \\ V \arrow[r, "\text{identity}"] \arrow[d] & V \arrow[d] \\ \mathbb{R}^n \arrow[r, dashed, "L_{\mathcal{C}\mathcal{B}}"] & \mathbb{R}^n \\ \vec{v}_{\mathcal{B}} \arrow[r, mapsto] & \vec{v}_{\mathcal{C}} = \boxed{L_{\mathcal{C}\mathcal{B}}}\,\vec{v}_{\mathcal{B}} \end{tikzcd}$$

$\mathcal{B}$-coords $\qquad$ $\mathcal{C}$-coords

$L_{\mathcal{C}\mathcal{B}}$ in this case is just $I_{\mathcal{C}\mathcal{B}}$

$$I_{\mathcal{C}\mathcal{B}} = \begin{pmatrix} (\vec{b}_1)_{\mathcal{C}} & \cdots & (\vec{b}_n)_{\mathcal{C}} \end{pmatrix}$$

get formula for change of coordinates matrix

Example: derivative $D: \mathbb{P}_2 \longrightarrow \mathbb{P}_1$ $\quad \mathcal{A} = (1, t, t^2)$ $\quad \mathcal{C} = (1, t)$

Compute: (1) $D_{\mathcal{C}\mathcal{A}}$

(2) $D(1+t+t^2)$ in two ways

$$\begin{tikzcd} 1+t+t^2 \arrow[r, mapsto, "D"] & \boxed{1+2t} \\ \mathbb{P}_2 \arrow[r] \arrow[d] & \mathbb{P}_1 \arrow[d] \\ \mathbb{R}^3 \arrow[r] & \mathbb{R}^2 \\ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{tikzcd}$$

$\mathcal{A}$-coords $\qquad$ $\mathcal{C}$-coords

$\vec{x} \longmapsto D_{\mathcal{C}\mathcal{A}}\vec{x}$

$1\vec{c}_1 + 2\vec{c}_2 = \boxed{1+2t}$

(1) $D_{\mathcal{C}\mathcal{A}} = \begin{pmatrix} D(\vec{a}_1)_{\mathcal{C}} & D(\vec{a}_2)_{\mathcal{C}} & D(\vec{a}_3)_{\mathcal{C}} \end{pmatrix}_{2\times 3}$

$= \begin{pmatrix} 0_{\mathcal{C}} & 1_{\mathcal{C}} & (2t)_{\mathcal{C}} \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$0 = \underline{0}\,1 + \underline{0}\,t$

$1 = \underline{1}\,1 + \underline{0}\,t$

$2t = \underline{0}\,1 + \underline{2}\,t$

Moral: coords make abstract concrete by translation into tuple-world

relative coords, each abstract LT is translated into matrix-tuple product

Example: $\mathcal{B} = (1, 1+t, 1+t+t^2)$ for $\mathbb{P}_2$
$\mathcal{D} = (1, 1+t)$ for $\mathbb{P}_1$ $\Big\} \Rightarrow D_{\mathcal{D}\mathcal{B}}$

Q: how compare $D_{\mathcal{C}\mathcal{A}}$ & $D_{\mathcal{D}\mathcal{B}}$

$$\begin{tikzcd}
\mathcal{A}\text{-coords } \mathbb{R}^3 \arrow[r, "D_{\mathcal{C}\mathcal{A}}"] & \mathbb{R}^3 \ \mathcal{C}\text{-coords} \\
\mathcal{B}\text{-coords } \mathbb{R}^3 \arrow[u, dash, "\text{change of coords}"] \arrow[r, "D_{\mathcal{D}\mathcal{B}}"'] & \mathbb{R}^2 \ \mathcal{D}\text{-coords} \arrow[u, dash, "\text{change of coords}"']
\end{tikzcd}$$

Fact: $\boxed{D_{\mathcal{D}\mathcal{B}} = I_{\mathcal{D}\mathcal{C}} D_{\mathcal{C}\mathcal{A}} I_{\mathcal{A}\mathcal{B}}}$

Let's compute $D_{\mathcal{D}\mathcal{B}}$ in two ways:

(1) direct computation

(2) leverage knowledge of $D_{\mathcal{C}\mathcal{A}}$

(1) direct computation

$$D_{\mathcal{D}\mathcal{B}} = \left(D(\vec{b}_1)_{\mathcal{D}} \quad D(\vec{b}_2)_{\mathcal{D}} \quad D(\vec{b}_3)_{\mathcal{D}} \right)$$

$$= \left(0_{\mathcal{D}} \quad 1_{\mathcal{D}} \quad (1+2t)_{\mathcal{D}} \right)$$

$$= \underline{\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}}$$

$0 = \underline{0}\, 1 + \underline{0}\,(1+t)$

$1 = \underline{1}\, 1 + \underline{0}\,(1+t)$

$1+2t = \underline{-1}\, 1 + \underline{2}\,(1+t)$

(2) leverage knowledge of $D_{\mathcal{C}\mathcal{A}}$

$$D_{\mathcal{D}\mathcal{B}} = I_{\mathcal{D}\mathcal{C}} D_{\mathcal{C}\mathcal{A}} I_{\mathcal{A}\mathcal{B}}$$

Let's find the change of coord. matrices

$$I_{\mathcal{A}\mathcal{B}} = \left((\vec{b}_1)_{\mathcal{A}} \quad (\vec{b}_2)_{\mathcal{A}} \quad (\vec{b}_3)_{\mathcal{A}} \right)$$

$$= \left(1_{\mathcal{A}} \quad (1+t)_{\mathcal{A}} \quad (1+t+t^2)_{\mathcal{A}} \right)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$1 = \underline{1}\, 1 + \underline{0}\, t + \underline{0}\, t^2$

$1+t = \underline{1}\, 1 + \underline{1}\, t + \underline{0}\, t^2$

$1+t+t^2 = \underline{1}\, 1 + \underline{1}\, t + \underline{1}\, t^2$

$$I_{\mathcal{D}\mathcal{C}} = (I_{\mathcal{C}\mathcal{D}})^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$I_{\mathcal{C}\mathcal{D}} = \left((\vec{d}_1)_{\mathcal{C}} \quad (\vec{d}_2)_{\mathcal{C}} \right) = \left(1_{\mathcal{C}} \quad (1+t)_{\mathcal{C}} \right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$1 = \underline{1}\, 1 + \underline{0}\, t$

$1+t = \underline{1}\, 1 + \underline{1}\, t$

$$D_{\mathcal{D}\mathcal{B}} = I_{\mathcal{D}\mathcal{C}} D_{\mathcal{C}\mathcal{A}} I_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \underline{\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}}$$

Summary:

$$\begin{tikzcd}
 & \mathbb{P}_2 \arrow[rr, "D"] \arrow[d, dash] \arrow[ddl] & & \mathbb{P}_1 \arrow[d, dash] \arrow[ddl] & \\
 & \mathbb{R}^3 \arrow[rr, "D_{DB}"] \arrow[dl, dash] & & \mathbb{R}^2 \arrow[dl, dash] & \\
\mathbb{R}^3 \arrow[rr, "D_{CA}"] & & \mathbb{R}^2 & &
\end{tikzcd}$$

$\mathcal{B}$-coords, $\mathcal{D}$-coords, $\mathcal{A}$-coords, $\mathcal{C}$-coords

§ Determinants

Motivation: recall for 2x2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible $\Longleftrightarrow$ $\det(A) = ad - bc \neq 0$

Goal: extend this idea for $n \times n$ matrices

One way to do this:

set $\det(I_n) = 1$

Elem row op	Effect on det
swap	$(-1)\times$
scalar $(c \neq 0)$	$(c)\times$
shear $(i \neq j)$	$(1)\times$

Example:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[\text{new } R_3]{\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-R_3 + R_2 \text{ new } R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{new } R_1]{R_3 + R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\times \frac{1}{2}$ $\quad \times 1$ $\quad \times 1$

$$\xrightarrow[\text{new } R_2]{\frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\times \frac{1}{3}$

$$1 = \det(I) = (\times\tfrac{1}{3})(\times 1)(\times 1)(\times \tfrac{1}{2}) \det(A) = \frac{1}{6}\det(A)$$

$$\Rightarrow \det(A) = 6$$