Math 415. Exam 1. February 14, 2019

Full Name: ____________________

Net ID: ________________________

Discussion Section: ___________

- There are 20 problems worth 5 points each.
- Each question has only one correct answer. You can choose up to two answers. If you choose just one answer, then you will get 5 points if the answer is correct, and 0 points otherwise. However, if you choose two answers, you will get 2.5 points if one of the answers is correct, and 0 points otherwise.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 75 minute exam.
- Do not turn this page until instructed to.
- Fill in the answers on the scantron form provided. Also circle your answers on the exam itself.
- Hand in both the exam and the scantron.
- There are several different versions of this exam.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- Good luck!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**! On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) Consider the following subsets of $\mathbb{R}^3$.

   I. All vectors of the form $\begin{bmatrix} a \\ a+2 \\ b \end{bmatrix}$, with a, b in $\mathbb{R}$.

   II. All vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying $a - b \leq 0$.

   III. All vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying $b = 0$ and $c = 0$.

Which of these are subspaces of $\mathbb{R}^3$?

(A) None of these

(B) I only

(C) I, II, and III

(D) II only

(E) ★ III only

Solution. III. This is a subspace, since it is the set of solutions of $A\mathbf{x} = \mathbf{0}$ where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Another way of seeing that this set is a subspace, is to observe that it is equal to the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and hence a line through the origin. Hence it is a subspace of $\mathbb{R}^3$.

I. Does not contain the zero vector.

II. Not closed under multiplication by -1.

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2. (5 points) Let

$$\mathbf{w} = \begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ h \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ h \\ 1 \end{bmatrix}.$$

For which values of h is $\mathbf{w}$ a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$?

(A) ★ Only when $h = -1$.

(B) For no value of h.

(C) Only when $h = 2$.

(D) None of the other answers.

(E) Only when $h \neq -1$.

Solution. We have to determine when the following system of equations is consistent:

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ h & h & -3 \\ -1 & 1 & -1 \end{array}\right] \xrightarrow{R2 \to R2 - hR1} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -2h & -3-5h \\ -1 & 1 & -1 \end{array}\right] \xrightarrow{R3 \to R3 + R1} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -2h & -3-5h \\ 0 & 4 & 4 \end{array}\right]$$

$$\xrightarrow{R3 \to R3/4} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -2h & -3-5h \\ 0 & 1 & 1 \end{array}\right]$$

The system is consistent if and only if $-2h = -3 - 5h$, i.e. $h = -1$.

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3. (5 points) Let A be an $m \times n$-matrix and $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be in $\mathbb{R}^n$. Consider the following statements:

I. If $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, then $A\mathbf{w}$ is a linear combination of $A\mathbf{u}$ and $A\mathbf{v}$.

II. If $A\mathbf{w}$ is a linear combination of $A\mathbf{u}$ and $A\mathbf{v}$, then $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Which of the two statements is always true?

(A) Both I and II are correct.

(B) ★ Only I is correct.

(C) Neither I nor II are correct.

(D) Only II is correct.

Solution. For I., suppose that $\mathbf{w} = c\mathbf{u} + d\mathbf{v}$. Then

$$A(\mathbf{w}) = A(c\mathbf{u} + d\mathbf{v}) = cA(\mathbf{u}) + dA(\mathbf{v}).$$

For II., let A be the zero matrix. Then $A\mathbf{w}$ is always the zero vector and hence a linear combination of $A\mathbf{u}$ and $A\mathbf{v}$. However, $\mathbf{w}$ does not have to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

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4. (5 points) Which columns of the matrix $\begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 2 & 3 & 1 \\ -1 & -2 & -3 & 1 \end{bmatrix}$ are pivot columns? Column 1 is the left-most column, column 2 is second from the left, etc.

(A) Columns 2 and 3

(B) ★ Columns 1 and 4

(C) Columns 1, 3, and 4

(D) Columns 1, 2, and 4

(E) None of the other choices

Solution. Bring the matrix to echelon form and then circle the pivots.

$$\begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 2 & 3 & 1 \\ -1 & -2 & -3 & 1 \end{bmatrix} \overset{R1\leftrightarrow R2}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

$$\overset{R3\rightarrow R3+R1}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \overset{R3\rightarrow R3-R2}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. (5 points) Let A be a 3×3 matrix. Consider the following statements:

I. If A has 2 pivots, then A is not invertible.

II. If a matrix B is the inverse of A, then B^T is also the inverse of A^T.

Which of these statements are always true?

(A) Statement I only.

(B) Statement II only.

(C) ★ Statement I and Statement II.

(D) Neither of Statements I or II.

Solution. For Statement I, if A has 2 pivots, then A is not row-equivalent to the identity matrix I_3, which indicates that A is not invertible.
For Statement II, if $AB = BA = I_3$, then we have $(AB)^T = (BA)^T = I_3$, i.e. $B^T A^T = A^T B^T = I_3$, which indicates that B^T is the inverse of A^T.

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6. (5 points) The matrix $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ is reduced to the identity matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ using the following row operations (in the given order):

(1) $R3 \to R3 - R4$,

(2) $R2 \leftrightarrow R4$

(3) $R2 \to R2 + R1$.

Which of the following matrices is A^{-1}?

(A) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(C) ★ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(D) None of the other answers.

(E) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution. Let $E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be the elementary matrix corresponding to $R2 \to R2 + R1$, and $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ be the elementary matrix corresponding to $R2 \leftrightarrow R4$, and let $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be the elementary matrix corresponding to $R3 \to R3 - R4$. Then $E_3E_2E_1A = I$. Thus $E_3E_2E_1$ is the inverse of A.

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7. (5 points) Let A, B be two $n \times n$-matrices. Consider the following statements:

(S1) If $A^2 - A$ is the zero matrix, then either A is the identity matrix or A is the zero matrix.

(S2) If $AB = BA$, then $(B + A)A = A^2 + AB$.

Which of the two statements is true for every possible choice of A and B?

(A) Neither statement S1 or S2.

(B) ★ Statement S2 only.

(C) Statement S1 and Statement S2.

(D) Statement S1 only.

Solution. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $A^2 = A$. So Statement S1 is false.

Since $AB = BA$, we have

$$(B + A)A = BA + A^2 = A^2 + BA = A^2 + AB.$$

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8. (5 points) Let A be an $m \times n$ matrix. Consider the following statements:

I. The linear system $A\mathbf{x} = \mathbf{0}$ is consistent if and only if the augmented matrix $[A|\mathbf{0}]$ has a pivot in every row.

II. If $\mathbf{b}$ is a vector in $\mathbb{R}^m$ and the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then $\mathbf{b}$ is a linear combination of the columns of A.

Which one of these statements is always true?

(A) Neither statement I or II.

(B) Statement I and Statement II.

(C) ★ Statement II only.

(D) Statement I only.

Solution. Statement I is false. The linear system $A\mathbf{x} = \mathbf{0}$ is consistent, because $\mathbf{0}$ is always a solution. However, the augmented matrix $[A|\mathbf{0}]$ doesn't have to have a pivot in every row. For example, take A to be the zero matrix.

Statement II is true. If $A\mathbf{x} = \mathbf{b}$ is consistent, there is a vector $\mathbf{z} \in \mathbb{R}^n$ such that $A\mathbf{z} = \mathbf{b}$. By definition of matrix-vector multiplication $A\mathbf{z}$ is a linear combination of the columns of A. Thus so is $\mathbf{b}$.

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9. (5 points) Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ be a 3×3-matrix such that the columns of A sum up to the zero vector (i.e. $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$), and let $B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Which of the following matrices could be the product matrix AB?

(I) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(II) $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

(III) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

(IV) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(A) The matrices in (I) and (II).

(B) All four matrices can be AB.

(C) None of the matrices can be AB.

(D) ★ Only the matrix in (I).

(E) The matrices in (I), (II) and (IV).

Solution. Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$ such that $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$. Then

$$AB = [0\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3 \quad \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \quad 2\mathbf{a}_1 + 2\mathbf{a}_2 + 2\mathbf{a}_3]$$

Since $2\mathbf{a}_1 + 2\mathbf{a}_2 + 2\mathbf{a}_3 = 2(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3) = \mathbf{0}$, the product AB has to be the zero matrix. Observe that

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus (I) is the only valid option.

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10. (5 points) Let A, B be two $n \times n$-matrices. Which one of the following statements is true for all such matrices A and B?

(A) If A is a permutation matrix, then A^n is the identity matrix.

(B) Each column of AB is a linear combination of the columns of B.

(C) If the $(1,1)$-entry of A is nonzero, then A has an LU-decomposition.

(D) If $A = LU$ with L lower triangular and U upper triangular, then U is invertible.

(E) ★ If A invertible, then B and AB have the same reduced row echelon form.

Solution.

- Each column of AB is a linear combination of the columns of B.

  FALSE. Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Then

  $$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

  It is easy to see that the columns of A are not linear combinations of the columns of B.

- If the $(1,1)$-entry of A is nonzero, then A has an LU-decomposition.

  FALSE. Take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. This matrix does not have an LU-decomposition.

- If A is a permutation matrix, then A^n is the identity matrix.

  FALSE. The permutation matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ satisfies $A^3 = A \neq I$.

- If A invertible, then B and AB have the same reduced row echelon form.

  TRUE. The matrix A is a product of elementary matrices, and so AB is obtained from B by performing row operations. Since B and AB are row equivalent, they have the same RREF.

- If $A = LU$ with L lower triangular and U upper triangular, then U is invertible.

  FALSE. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then

  $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

  is an LU-factorization of A in which U is not invertible.

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11. (5 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be non-zero vectors in $\mathbb{R}^3$ such that

$$2\mathbf{a} - \mathbf{b} = 0$$

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$

Which of the following describes the set $\operatorname{span}(\mathbf{a}, \mathbf{b}, \mathbf{c})$?

(A) It is empty.

(B) It is a plane in $\mathbb{R}^3$.

(C) ★ It is a line in $\mathbb{R}^3$.

(D) None of the other answers.

(E) It is $\mathbb{R}^3$.

Solution. The given relations let us rewrite any vector in the span

$$\mathbf{v} = d_1\mathbf{a} + d_2\mathbf{b} + d_3\mathbf{c}$$

as

$$\begin{aligned} \mathbf{v} &= d_1\mathbf{a} + d_2(2\mathbf{a}) + d_3(-\mathbf{a} - \mathbf{b}) \\ &= d_1\mathbf{a} + d_2(2\mathbf{a}) + d_3(-3\mathbf{a}) \\ &= (d_1 + 3d_2 - 3d_3)\mathbf{a} \end{aligned}$$

Hence the span is the set of all scalar multiples of the nonzero vector $\mathbf{a}$, i.e. a line.

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12. (5 points) For which of the following pairs of matrices is A row equivalent to B (that is A and B can be transformed to each other by a sequence of elementary row operations)?

Pair 1: $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

Pair 2: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(A) ★ A and B are row equivalent in Pair 1 only.

(B) In both Pair 1 and Pair 2, A and B are row equivalent.

(C) In neither Pair 1 nor Pair 2 are A and B row equivalent.

(D) A and B are row equivalent in Pair 2 only.

Solution. In Pair 2, both A and B are in reduced row echelon form but are not equal, and by a theorem from class a matrix can be row equivalent to only one reduced row echelon form matrix. In Pair 1, A can be transformed to B via the row operations $R1 \leftrightarrow R2$, $R2 \leftrightarrow R3$.

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13. (5 points) Let A and B be two $n \times n$ invertible matrices. Consider the following statements:

I. $A - B$ is invertible.

II. BA^2 is invertible.

Which one of the statements I. and II. is true for all possible choices of A and B?

(A) Both I and II are correct.

(B) Only I is correct.

(C) Neither I nor II are correct.

(D) ★ Only II is correct.

Solution. Let $A = B = I_2$. Both matrices are invertible, however $A - B = \mathbf{0}$ is noninvertible. By the property $(CD)^{-1} = D^{-1}C^{-1}$ we can conclude that the product of invertible matrices is invertible. By the same property, the power of an invertible matrix is also invertible, thus BA^2 is invertible.

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14. (5 points) Which of the following matrices, when multiplied on the left of a 3×3 matrix, performs the row operation $R1 \to R1 - 4R2$?

(A) $\begin{bmatrix} 0 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(C) ★ $\begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(E) None of the other answers

Solution.

$$\begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 - 4r_2 \\ r_2 \\ r_3 \end{bmatrix}$$

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15. (5 points) A system of linear equations with 5 variables and 3 equations must have:

(A) at most 2 free variables.

(B) no solution.

(C) infinitely many solutions.

(D) ★ at most 3 pivot variables.

(E) at least 3 pivot variables.

Solution. The number of pivot variables is capped by the number of equations you have (which is the number of rows in the coefficient matrix), so there are at most 3 pivot variables.

16. (5 points) Let $A = \begin{bmatrix} -1 & -3 & 2 \\ 1 & 2 & 3 \\ 2 & 5 & 3 \end{bmatrix}$. If $A = LU$ is an LU-decomposition with all diagonal entries of L equal to 1, what is the sum of the entries of L (including the entries on the diagonal)?

(A) 5

(B) ★ 1

(C) 2

(D) None of the other answers.

(E) This matrix does not have an LU-decomposition.

Solution.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

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17. (5 points) Consider the system of linear equations

$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 3 \\ 2x_1 + 8x_2 + 3x_3 &= 1 \end{aligned}$$

Which of the following describes its solution set?

(A) None of the other answers.

(B)

$$\begin{aligned} x_1 &= -7 - 4x_2 \\ x_2 &= \text{free} \\ x_3 &= 2 \end{aligned}$$

(C)

$$\begin{aligned} x_1 &= -7 - 4x_3 \\ x_2 &= 2 - x_3 \\ x_3 &= \text{free} \end{aligned}$$

(D) ★

$$\begin{aligned} x_1 &= -7 - 4x_2 \\ x_2 &= \text{free} \\ x_3 &= 5 \end{aligned}$$

(E) No solution

Solution. Work with the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 2 & 8 & 3 & 1 \end{array}\right] \underset{R2 \to R2 - 2R1}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 0 & -1 & -5 \end{array}\right] \underset{R2 \to -R2}{\longrightarrow}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] \underset{R1 \to R1 - 2R2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 4 & 0 & -7 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The pivot variables are x_1 and x_3 while x_2 is a free variable. Back in equation form, we have

$$\begin{aligned} x_1 + 4x_2 &= -7 \\ x_3 &= 5 \end{aligned}$$

Solving for the pivot variables gives the answer.

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18. (5 points) Which of the following statements are true for ALL 4×5 matrices A, B?

(S1) If A and B both have 0 pivots, then A and B have the same reduced row echelon form.

(S2) If A and B both have 2 pivots, then A and B have the same reduced row echelon form.

(S3) If A and B both have 4 pivots, then A and B have the same reduced row echelon form.

(A) Only S2 and S3 are true.

(B) Only S2 is true.

(C) ★ Only S1 is true.

(D) Only S1 and S3 are true.

(E) None are true.

Solution. S1 is true because the only matrix with 0 pivots is the zero matrix, so A and B must both be 0. S2 is false by considering the examples $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. S3 is false by considering the examples $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

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19. (5 points) Let A be a 4×4-matrix and let $\mathbf{b}, \mathbf{c}$ be two vectors in $\mathbb{R}^4$ such that the equation $A\mathbf{x} = \mathbf{b}$ has no solution. What can you say about the number of solutions of the equation $A\mathbf{x} = \mathbf{c}$?

(A) There is nothing further we can say about the number of solutions of $A\mathbf{x} = \mathbf{c}$.

(B) ★ The equation $A\mathbf{x} = \mathbf{c}$ either has no solution or infinitely many solutions.

(C) The equation $A\mathbf{x} = \mathbf{c}$ has no solution.

(D) The equation $A\mathbf{x} = \mathbf{c}$ either has no solution or exactly one solution.

(E) None of the other answers.

Solution. The equation $A\mathbf{x} = \mathbf{c}$ can not have a unique solution. Suppose it does. Then the equation can not have free variables. Thus the echelon form of A has a pivot in the every row. However, then echelon from of the augmented matrix $[A|\mathbf{b}]$ can not have a row of the form $[0\ 0\ 0\ 0 \mid z]$, where $z \neq 0$. Thus the equation $A\mathbf{x} = \mathbf{b}$ can not be inconsistent.

To see that $A\mathbf{x} = \mathbf{c}$ can have infinitely many solutions, consider $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Check that $A\mathbf{x} = \mathbf{c}$ has infinitely many solutions, while $A\mathbf{x} = \mathbf{b}$ has no solution.

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20. (5 points) Let A be a 2×1-matrix and B be a 1×3-matrix such that

$$AB = \begin{bmatrix} x & 1 & -2 \\ 6 & -2 & y \end{bmatrix}.$$

What can you say about x and y?

(A) Not enough information to determine x and y.

(B) ★ $x = -3$ and $y = 4$.

(C) None of the other answers.

(D) $x = -4$ and $y = -3$.

(E) $x = 3$ and $y = 4$.

Solution. Let $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} b_1 a_1 & b_2 a_1 & b_3 a_1 \\ b_1 a_2 & b_2 a_2 & b_3 a_2 \end{bmatrix} = \begin{bmatrix} x & 1 & -2 \\ 6 & -2 & y \end{bmatrix}.$$

Thus $b_2 a_1 = 1$ and $b_2 a_2 = -2$. Thus $a_2/a_1 = -2$. Therefore

$$x = b_1 a_1 = b_1 a_2/(-2) = 6/(-2) = -3,$$

and

$$y = b_3 a_2 = b_3 a_1 \cdot (-2) = -2 \cdot (-2) = 4.$$
