Math 415. Exam 3. November 30, 2017

Full Name: ____________________

Net ID: ________________________

Discussion Section: ___________

- There are 20 problems worth 5 points each.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 70 minute exam.
- Do not turn this page until instructed to.
- Fill in the answers on the scantron form provided. Also circle your answers on the exam itself.
- Hand in both the exam and the scantron.
- On the scantron make sure you bubble in **your name, your UIN and your NetID**.
- There are several different versions of this exam.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- Good luck!

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95. D
96. C

1. (5 points) Let V be a subspace of $\mathbb{R}^n$, $n > 0$ and let P be the projection matrix of the projection onto V. Which of the following statements is not always true?

(A) ★ $\mathrm{Col}(P^T) = V^\perp$.

(B) $\mathrm{Col}(P) = V$.

(C) $\mathrm{Nul}(P) = V^\perp$.

(D) $\mathrm{rank}(P) = \dim V$.

Solution. Since $\mathrm{Col}(P) = \{Px \; : \; x \in \mathbb{R}^n\}$ and the projection of any vector onto V is in V, we get that $\mathrm{Col}(P) = V$. It follows that $\mathrm{rank}(P) = \dim V$. Since $P\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x}$ is orthogonal to V, we have $\mathrm{Nul}(P) = V^\perp$. Therefore the statement $\mathrm{Col}(P^T) = V^\perp$ must be false.

For any projection matrix P, $P^T = P$ (Can you see why? Use that $P = A(A^TA)^{-1}A^T$ for some matrix A). So $\mathrm{Col}(P^T) = \mathrm{Col}(P) = V$ and this is never equal to $V^\perp$.

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2. (5 points) Let $\mathcal{E} = \left(\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ be the standard basis of $\mathbb{R}^2$ and let $\mathcal{B}$ be another basis: $\mathcal{B} = \left(\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)$. Let $I_{\mathcal{E},\mathcal{B}}$ be the change of basis matrix from the $\mathcal{B}$-basis to the standard basis, and let $I_{\mathcal{B},\mathcal{E}}$ be the change of basis matrix from the standard basis to the $\mathcal{B}$ basis. Then

(A)

$$I_{\mathcal{E},\mathcal{B}} = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}, \quad I_{\mathcal{B},\mathcal{E}} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$$

(B) ★

$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad I_{\mathcal{B},\mathcal{E}} = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

(C)

$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad I_{\mathcal{B},\mathcal{E}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(D)

$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad I_{\mathcal{B},\mathcal{E}} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

Solution.

$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad I_{\mathcal{B},\mathcal{E}} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}^{-1} = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

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3. (5 points) Consider two bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be such that the matrix $T_{\mathcal{B},\mathcal{B}}$ that represents T with respect to $\mathcal{B}$ is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Suppose that the change of basis matrices are given by

$$I_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } I_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

What is $T_{\mathcal{C},\mathcal{C}}$?

(A) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(B) ★ $\begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$.

(C) None of the other matrices.

(D) $\begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$.

Solution.

$$T_{\mathcal{C},\mathcal{C}} = I_{\mathcal{C},\mathcal{B}} T_{\mathcal{B},\mathcal{B}} I_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}.$$

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4. (5 points) Let Q be an orthogonal $n \times n$-matrix. Which of the following statements is not true for all such Q?

(A) Q^T is the inverse of Q.

(B) The columns of Q are orthonormal.

(C) ★ $\det(A) = 1$.

(D) If Q is upper triangular, then Q is diagonal.

Solution. If $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then Q is orthogonal, but the determinant is -1.

Q^T is the inverse of Q by Theorem 2 of Lecture 25.

An orthogonal square matrix is a matrix whose columns are orthonormal. See Definition in Lecture 25.

Let Q be orthogonal and upper triangular. Check that if a matrix is upper triangular, then its inverse is upper triangular and its transpose is lower triangular. However, the inverse of Q is Q^T. Thus Q^T is both upper and lower triangular. Hence Q^T is diagonal. Since Q^T is diagonal, so is Q.

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5. (5 points) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Suppose that Q is a 3×2-matrix with orthonormal columns and R is an invertible upper triangular matrix such that $A = QR$. Which of the following matrices can be R?

(A) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

(B) $\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 2 \end{bmatrix}$

(C) None of the other answers.

(D) ★ $\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}$

(E) $\begin{bmatrix} 2 & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}$

Solution. We apply the Gram-Schmidt method to the columns of A: Then

$$\mathbf{q}_1 = \frac{1}{\left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\|} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then

$$\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}.$$

Then

$$\mathbf{q}_2 = \frac{1}{\left\| \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} \right\|} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}.$$

Thus

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

Then

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}$$

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6. (5 points) Let C be a $n \times n$-matrix such that $C^T = C^{-1}$. Then

(A) ★ $\det(C) = 1$ or $\det(C) = -1$,

(B) $\det(C) \geq 0$,

(C) $\det(C)$ can be any real number,

(D) $\det(C) \leq 0$.

Solution. Let $C^T = C^{-1}$

$$\det(C) = \det(C^T) = \det(C^{-1}) = \frac{1}{\det(C)}.$$

Thus $\det(C)^2 = 1$.

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7. (5 points) The least squares solution of

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is

(A) $\begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$.

(B) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(C) ★ $\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$.

(D) $\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$.

(E) none of the other answers.

Solution. We need to solve

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Multiplying out we get

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

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8. (5 points) Let A, B, C be invertible $n \times n$-matrices such that $AB = B^2C$, $\det(B) = 3$ and $\det(C) = 2$. Then:

(A) $A = BC$ and $\det(A) = 5$.

(B) ★ $A = B^2CB^{-1}$ and $\det(A) = 6$.

(C) $A = B^2CB^{-1}$ and $\det(A) = 5$.

(D) $A = BC$ and $\det(A) = 6$.

(E) $A = B^3C$ and $\det(A) = 11$.

Solution. Since B is invertible, we get $A = B^2CB^{-1}$. By the multiplication rule for determinants, we get

$$\det(A) = \det(B^2CB^{-1}) = \det(B)^2 \det(C) \det(B)^{-1} = \det(B) \det(C) = 6.$$

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9. (5 points) Let $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ and $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Let $\boldsymbol{w}_1$ be the orthogonal projection of $\boldsymbol{v}_1$ onto W, and let $\boldsymbol{w}_2$ be the orthogonal projection of $\boldsymbol{v}_2$ onto W. Then:

(A) $\boldsymbol{w}_1 = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(B) ★ $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(C) $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(D) $\boldsymbol{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(E) $\boldsymbol{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution. Note that

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in W$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in W^\perp$, the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the projection of v_1 onto W.

We note again that

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in W^\perp$, the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the projection of v_2 onto W.

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10. (5 points) Which of the following statements is true? Let A, B be $n \times n$-matrices.

(A) ★ if every row of A adds up to 0, then $\det(A) = 0$.

(B) if A is invertible and B is not invertible, then AB is invertible.

(C) the determinant of A is the product of the diagonal entries of A.

(D) if every row of A adds up to 1, then $\det(A) = 1$.

Solution. Let $C = [\mathbf{c}_1 \dots \mathbf{c}_n]$ be A^T. If every row of A add up to 0, that means every columns of C adds up to 0. Thus $\mathbf{c}_1 + \cdots + \mathbf{c_n} = \mathbf{0}$. Thus the columns of C are linearly dependent and thus $\det C = 0$. Since $A^T = C$ and $\det A = \det A^T$, we get that $\det A = 0$.

Let $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Observe that each row of A adds up to 1, but $\det(A) = 0$. Moreover, the determinant of A is not the product of the diagonal entries of A.

Let A be invertible and let B be the zero matrix. Then AB is not invertible.

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11. (5 points) Let $\mathcal{B} = \left(\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$. Let $\mathbf{x} = \begin{bmatrix} 4 \\ 8 \\ 12 \\ 8 \end{bmatrix}$. Let $W = \operatorname{Span}\{\mathbf{b_2}, \mathbf{b}_3\}$. Find the projection of $\mathbf{x}$ onto the orthogonal complement of W.

(A) None of the other answers.

(B) ★ $\begin{bmatrix} 8 \\ 8 \\ 10 \\ 6 \end{bmatrix}$

(C) $\begin{bmatrix} 8 \\ 8 \\ 8 \\ 8 \end{bmatrix}$.

(D) $\begin{bmatrix} -4 \\ 0 \\ 2 \\ 2 \end{bmatrix}$.

Solution. Observe that the orthogonal complement of W is $\operatorname{Span}\{\mathbf{b}_1, \mathbf{b}_4\}$. Thus the orthogonal projection is $\mathbf{x}_{W^\perp} = \frac{\mathbf{b}_1 \cdot \mathbf{x}}{\mathbf{b}_1 \cdot \mathbf{b}_1} \mathbf{b}_1 + \frac{\mathbf{b}_4 \cdot \mathbf{x}}{\mathbf{b}_4 \cdot \mathbf{b}_4} \mathbf{b}_4 = 8\mathbf{b}_1 + 2\mathbf{b}_4 = \begin{bmatrix} 8 \\ 8 \\ 10 \\ 6 \end{bmatrix}$.

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12. (5 points) Let $A = \begin{bmatrix} 6 & 5 & 3 \\ 2 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$.

(A) ★ $\det(A) = 14$

(B) $\det(A) = 30$

(C) $\det(A) = -8$

(D) $\det(A) = -4$

(E) $\det(A) = 22$

Solution.

$$\det(A) = 3 \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} + 4 \begin{vmatrix} 6 & 5 \\ 2 & 2 \end{vmatrix} = 3 \cdot 2 \cdot 1 + 4(12 - 10) = 6 + 8 = 14$$

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13. (5 points) Which of the following vectors is an eigenvector for the matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & \frac{1}{2} \end{bmatrix}$$

with eigenvalue $\frac{1}{2}$?

(A) ★ $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$.

(B) $\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}$.

(C) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

(D) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 1 & 2 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{1}{2} \end{bmatrix}$$

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14. (5 points) Let $A = \begin{bmatrix} 19 & -3 \\ -3 & 11 \end{bmatrix}$. This matrix has two eigenvalues: 10 and 20. Which of the following pairs of vectors contain a vector in each eigenspace of A?

(A) $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(C) ★ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(D) None of these pairs contains a vector in each eigenspace of A.

(E) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution. First consider the eigenvalue 10. Then

$$A - 10I = \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}.$$

The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}.$$

Thus $\mathrm{Nul}(A - 10I) = \mathrm{span}\left(\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right)$. Thus $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector of A to the eigenvalue 10.

First consider the eigenvalue 10. Then

$$A - 20I = \begin{bmatrix} -1 & -3 \\ -3 & -9 \end{bmatrix}.$$

The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}.$$

Thus $\mathrm{Nul}(A - 20I) = \mathrm{span}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$. Thus $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is an eigenvector of A to the eigenvalue 20.

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15. (5 points) Let $\mathcal{B} = \left(\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$. Let $\mathbf{x} = \begin{bmatrix} 4 \\ 8 \\ 12 \\ 8 \end{bmatrix}$. Let $W = \operatorname{Span}\{\mathbf{b_1}, \mathbf{b}_3\}$. Find the coordinate vector $\mathbf{x}_{\mathcal{B}}$ of $\mathbf{x}$ with respect to the basis $\mathcal{B}$.

(A) ★ $\begin{bmatrix} 8 \\ -2 \\ -2 \\ 2 \end{bmatrix}$

(B) $\begin{bmatrix} 4 \\ 8 \\ 12 \\ 8 \end{bmatrix}$

(C) $\begin{bmatrix} 6 \\ 0 \\ -2 \\ 6 \end{bmatrix}$

(D) $\begin{bmatrix} 32 \\ -8 \\ -4 \\ 4 \end{bmatrix}$

(E) None of the other answers

Solution.

$$\begin{aligned} \mathbf{x} &= \frac{\mathbf{b}_1 \cdot \mathbf{x}}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \frac{\mathbf{b}_2 \cdot \mathbf{x}}{\mathbf{b}_2 \cdot \mathbf{b}_2}\mathbf{b}_2 + \frac{\mathbf{b}_3 \cdot \mathbf{x}}{\mathbf{b}_3 \cdot \mathbf{b}_3}\mathbf{b}_3 + \frac{\mathbf{b}_4 \cdot \mathbf{x}}{\mathbf{b}_4 \cdot \mathbf{b}_4}\mathbf{b}_4 = \\ &= \frac{32}{4}\mathbf{b}_1 + \frac{-8}{4}\mathbf{b}_2 + \frac{-4}{2}\mathbf{b}_3 + \frac{4}{2}\mathbf{b}_4 \\ &= 8\mathbf{b}_1 - 2\mathbf{b_2} - 2\mathbf{b}_3 + 2\mathbf{b}_4 \end{aligned}$$

so $\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} 8 \\ -2 \\ -2 \\ 2 \end{bmatrix}$.

16. (5 points) Let A be an $n \times n$-matrix and let λ be an eigenvalue of A. Which of the following statements is incorrect?

(A) All the other statement are true.

(B) λ^2 is an eigenvalue of A^2,

(C) ★ λ can not be 0.

(D) λ^{-1} is an eigenvalue of A^{-1} whenever A^{-1} exists,

(E) $\lambda + 1$ is an eigenvalue of $A + I$,

Solution. 0 is an eigenvalue of the zero matrix. See Worksheet 12 Problem 5.

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17. (5 points) Suppose you are given the data points $(-1,1),(1,1),(2,3)$. In addition you are told that the least square solutions to

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

is $\begin{bmatrix} \frac{4}{7} \\ \\ \frac{9}{7} \end{bmatrix}$. What is the least squares line for the three data points given above?

(A) $y = \frac{4}{7} - \frac{9}{7}x$.

(B) ★ $y = \frac{9}{7} + \frac{4}{7}x$.

(C) $y = \frac{4}{7} + \frac{9}{7}x$.

(D) $y = \frac{9}{7} - \frac{4}{7}x$.

Solution. We try to find $y = ax + b$, so that the equations for the coefficients a, b have the form

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

From the given information $\hat{a} = \frac{4}{7}$ and $\hat{b} = \frac{9}{7}$, so that the line is $y = \frac{4}{7}x + \frac{9}{7}$.

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18. (5 points) Let W be the subspace Span$\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\}$ and let P be the projection matrix for W. Then

(A)

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}^{-1}$$

(B)

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(C) ★

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(D)

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(E)

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

Solution. If $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ then $P = A\frac{1}{A^TA}A^T = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

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19. (5 points) Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ be 4 row vectors of length 4 (so that the transpose $\mathbf{a}_i^T$ belongs to $\mathbb{R}^4$). Let

$$A = \begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ - & \mathbf{a}_3 & - \\ - & \mathbf{a}_4 & - \end{bmatrix}$$

be the 4×4 matrix with these vectors as rows. Assume $\det(A) = 2$. What is the determinant of the matrix

$$B = \begin{bmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 - 2\mathbf{a}_2 \\ 2\mathbf{a}_3 + \mathbf{a}_2 \\ -\mathbf{a}_4 \end{bmatrix}$$

(A) ★ $\det(A) = 4$.

(B) None of the other answers is correct.

(C) $\det(A) = -4$.

(D) $\det(A) = 2$.

(E) $\det(A) = -2$.

Solution.

$$\det(B) = \det \begin{bmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ 2\mathbf{a}_3 \\ -\mathbf{a}_4 \end{bmatrix} = -2\det \begin{bmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{bmatrix} = 2\det(A) = 4$$

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20. (5 points) Let $A = \begin{bmatrix} 14 & -2 \\ -2 & 11 \end{bmatrix}$. Find the eigenvalues of A.

(A) 1 and 5

(B) ★ 10 and 15.

(C) 15 and 25

(D) None of the other answers

(E) -10 and 10

Solution. The characteristic polynomial is $p(\lambda) = \lambda^2 - 25\lambda + 150$. The roots of this polynomial are 10, 15.
