Math 231E, Lecture 12.
Maxima and Minima

From here on out, I will stand for an interval. It can be open, closed, or semi-open.

Definition 0.1. The **absolute maximum** of f on I is the largest value f takes on I. Specifically, we say that M is the absolute maximum of f if there is an x with $f(x) = M$, and, for any $y \in I$, $f(y) \leq M$.

Note! it must be a value of $f(x)$. The function $f(x) = 2x + 1$ on the interval $(3, 5)$ has no absolute maximum! Also, the function $f(x) = 1/x$ on $(0, \infty)$ also has no maximum.

Definition 0.2. A **local maximum** is a value $f(c)$ such that $f(c) \geq f(x)$ for all x near c.

(The same definitions work for absolute and local minimum in the obvious manner.)

Theorem 0.3 (Extreme Value Theorem)**.** If $f(x)$ is continuous on $[a, b]$, then f attains an absolute maximum and an absolute minimum in $[a, b]$.

Theorem 0.4 (Fermat's (not last) Theorem)**.** If $f(x)$ has a local maximum, or local minimum, at c, and $f'(c)$ exists, then $f'(c) = 0$.

Proof. Say $f(x)$ has a local maximum at c (the proof is similar for a minimum). For $h > 0$ small enough, we have

$$f(c + h) \leq f(c).$$

This means

$$\begin{aligned} f(c + h) - f(c) &\leq 0, \\ \frac{f(c + h) - f(c)}{h} &\leq 0, \\ \lim_{h \to 0+} \frac{f(c + h) - f(c)}{h} &\leq 0. \end{aligned}$$

We also have for $h < 0$ and small enough, that $f(c + h) \leq f(c)$, or

$$\begin{aligned} f(c + h) - f(c) &\leq 0, \\ \frac{f(c + h) - f(c)}{h} &\geq 0, \\ \lim_{h \to 0-} \frac{f(c + h) - f(c)}{h} &\geq 0. \end{aligned}$$

(**Note!** Why did the inequality flip??) But the limit only exists if both the left-hand and right-hand limits are the same, so this limit must be zero. □

Definition 0.5. A **critical point** is a number c such that $f'(c) = 0$ or $f'(c)$ does not exist.

Algorithm to find absolute max/min

- Find critical points of f,
- Evaluate f at all of these points,
- Evaluate f at the endpoints,
- the absolute max and min must be one of these.

Let us work a few examples:

Example 0.6. Given a fixed perimeter, how do we maximize and minimize the area of a rectangle?

Let us consider a fixed perimeter of $100\,\mathrm{m}$. We have $2x + 2y = 100\,\mathrm{m}$, and the area is $A = xy$. (See figure on next page) Let us solve for y and get $y = 50\,\mathrm{m} - x$, so

$$A(x) = x(50\,\mathrm{m} - x) = 50\,\mathrm{m}x - x^2.$$

The domain of validity for this function is clearly $0\,\mathrm{m} \le x \le 50\,\mathrm{m}$, since we cannot allow x or y to be negative. We then check for critical points: $A'(x) = 50\,\mathrm{m} - 2x$, so setting this equal to zero gives $x = 25\,\mathrm{m}$. Thus we have to check three points; the critical point and the two boundary points:

$$A(0\,\mathrm{m}) = 0\,\mathrm{m}^2, \quad A(25\,\mathrm{m}) = (25\,\mathrm{m})(25\,\mathrm{m}) = 625\,\mathrm{m}^2, \quad A(50\,\mathrm{m}) = 0\,\mathrm{m}^2.$$

Clearly the interior point is an absolute maximum, and the two boundary points are absolute minima. Thus we have proved that the optimal rectangle is a square; the pessimal rectangle is the degenerate one where we choose one side length or the other to be zero.

We can generalize this to an arbitrary perimeter L. If we form a rectangle with width x and height y, then the perimeter is $L = 2x+2y$, while the area is $A = xy$. Since L is the constraint, let us write $y = L/2-x$, and then the area becomes

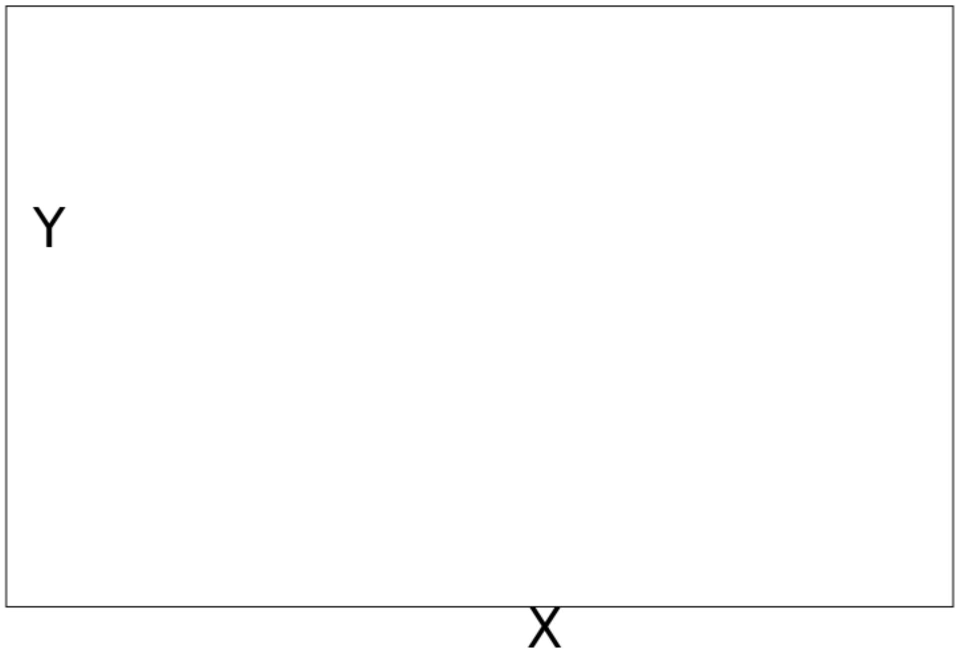
$$A(x) = x\left(\frac{L}{2} - x\right) = \frac{L}{2}x - x^2.$$

The domain of validity of this function is $0 \le x \le L/2$ (if $x > L/2$ then y would be negative!). Again we find

$$A'(x) = \frac{L}{2} - 2x,$$

and thus our critical point is $x = L/4$. Again, we check the boundary points, and we obtain

$$A(0) = 0, \quad A(L/4) = (L/4)(L/4) = \frac{L^2}{16}, \quad A(L/2) = 0.$$



$$2X + 2Y = 100$$

$$A = XY$$

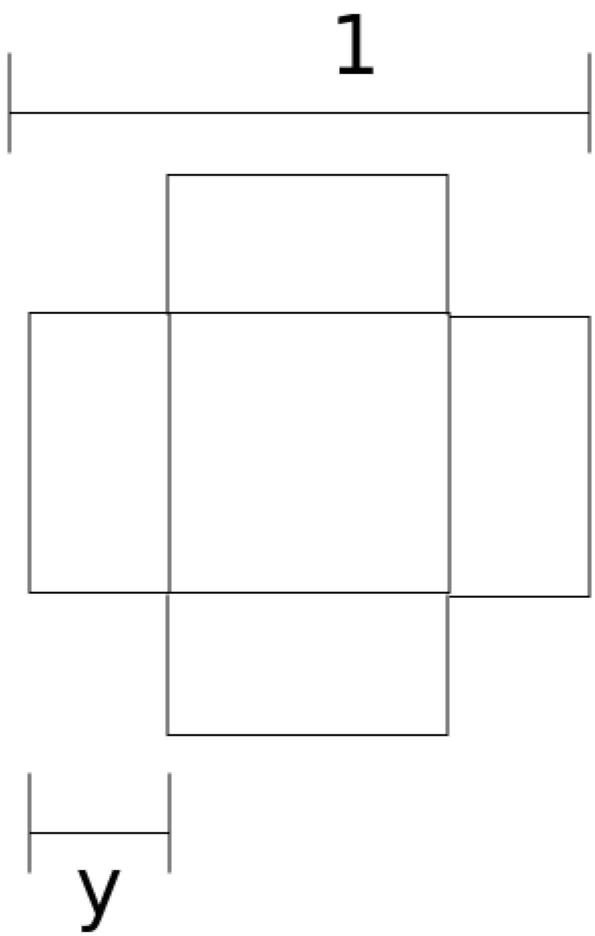
Example 0.7. Consider a $1 \text{ m} \times 1 \text{ m}$ sheet of cardboard. Suppose that an $y \times y$ square is cut from each corner, and the resulting shape is folded up into a (lidless) box. How should y be chosen in order to maximize the volume of the resulting box. Please see figure on the next page as a visual aid.

It is not hard to see that, once the sides of the box are folded up the base will be a square that is $(1 - 2y) \times (1 - 2y)$ and is y units high. The volume is base x height or $V(y) = y(1 - 2y)^2$. We want to maximize this over y.

According to Fermat we have to check the boundaries and the critical points. The quantity y can lie between $y = 0$ and $y = \frac{1}{2}$. In each case this gives a volume of $V = 0$. This is probably not the maximum volume. So lets check the critical points.

$$\begin{aligned} V(y) &= 4y^3 - 4y^2 + y \\ V'(y) &= 12y^2 - 8y + 1 \end{aligned}$$

The critical points occur when $12y^2 - 8y + 1 = 0$. The roots are $y = \frac{1}{6}$ m and $y = \frac{1}{2}$ m. The first gives $V = \frac{2}{27}$ m^3 and the second $V = 0$ m^3. So we ought to choose $y = \frac{1}{6}$ m for a volume of $V = \frac{2}{27}$ m^3.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1 Tests for Maxima and Minima: The first and second derivative tests

I wanted to remind you of a pair of tests that are often used to decide if a critical point is a maximum or a minimum (or neither!). These are the first and the second derivative tests.

Theorem 1.1. Suppose that there exists a neighborhood around the critical point c such that $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$. Then c is a local minimum.

Similarly suppose that there exists a neighborhood around the critical point c such that $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$. Then c is a local maximum.

The proof here is pretty straightforward. If $f'(x) > 0$ for $x > c$ then the function is increasing for $x > c$. Similarly if $f'(x) < 0$ for $x < c$ then f is decreasing for $x < c$. Thus c must be a local minimum. This result is nice because it does not require the function to have a second derivative, or even to have a first derivative at c.

Example 1.2. Consider the function $f(x) = |x|$. The point $c = 0$ is a critical point. The function is not differentiable here, but $f'(x) = -1$ for $x < 0$ and $f'(x) = 1$ for $x > 0$. Thus 0 is a local minimum.

More well-known is the second derivative test.

Theorem 1.3. Suppose that the second derivative exists at the critical point c. If $f''(c) > 0$ then the critical point is a local minimum. If $f''(c) < 0$ then the critical point is a local maximum. If $f''(c) = 0$ then the test fails.

I won't give a full proof but this is a nice application of Taylor series. If we Taylor expand about the critical point then

$$f(x) = f(c) + \frac{f''(c)}{2}(x-c)^2 + O((x-c)^3)$$

As long as $f''(c)$ is non-zero then for x sufficiently close to c the term $\frac{f''(c)}{2}(x-c)^2$ is larger than the term $O((x-c)^3)$. Therefore the critical point is a local minimum of $f''(c) > 0$ and a local maximum if $f''(c) < 0$.

Finally I wanted to remind you that a critical point need not be either a local minimum or a local maximum.

Example 1.4. Consider the function $f(x) = x^3$. The point 0 is a critical point but it is neither a local maximum or a local minimum since x^3 is a monotone function.