Math 231E Engineering Calculus: Notes on Taylor Polynomials.

June 20, 2020

1 Motivating Problem:

The motivating problem here is the very practical question of being able to approximate transcendental functions such as $\sin(x), e^x, \log(x)$ to some given accuracy.

Taylor series/polynomials give us a way to approximate a given function $f(x)$ in a neighborhood of a given point a. These approximations will be good for points x near to a, though they will typically not be very good when x is far from a.

The simplest example is one that you have probably seen before, the tangent line approximation. In this approximation we approximate the function by the tangent line: the line with the same y value and the same slope at the point a.

Definition 1.1 (Tangent Line Approximation)**.** *Given a function $f(x)$ the tangent line is given by*

$$T_1(x) = f(a) + f'(a)(x - a)$$

Note that $T_1(a) = f(a)$ and $T_1'(a) = f'(a)$, so $T_1(X)$ and $f(x)$ have the same value and the same slope at $x = a$. The second and higher derivatives will generally **not** agree. They are zero for the tangent line, since it is linear.

Example 1.2. *Find the tangent line approximation to $f(x) = \sin(x)$ at the point $a = \frac{\pi}{4}$ and use it to approximate* $\sin(.75)$.

We have that $f(a) = f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ and $f'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ so that

$$T_1(x) = \frac{\sqrt{2}}{2}(1 + (x - \frac{\pi}{4}))$$

Plugging in gives $T_1(0.75) \approx 0.6820$, while $\sin(0.75) \approx 0.6816$. So the two differ in the fourth decimal place.

Note that the approximation gets less good as the point x gets further from $\frac{\pi}{4}$. For example $\sin(1) \approx 0.8415$ and $T_1(1) \approx 0.8589$ so the two differ in the second decimal place. Figure 1 gives a plot of $\sin(x)$ (in black) and the tangent line (in dashed red) for $x \in (0, \frac{\pi}{2})$.

2 Higher Order Approximation

The idea of the tangent line approximation is that we find a linear function that has the same value and the same slope as our given function $f(x)$ at a point a. This can be extended to higher order. We can find a quadratic function that has the same value, slope and second derivative as $f(x)$ at a point a. Similarly we could find a cubic polynomial with the same value, first derivative, second derivative and third derivative as $f(x)$ at the point a. This motivates the next definition

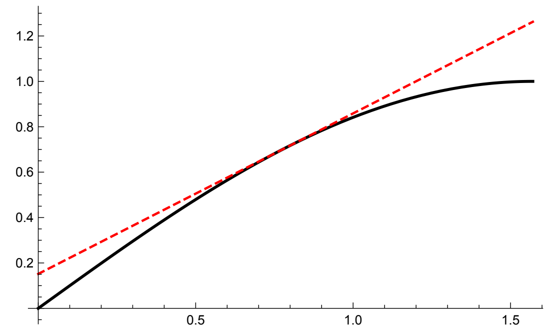

Figure 1: The function $f(x) = \sin(x)$ and the tangent line approximation $T_1(x) = \frac{\sqrt{2}}{2}(1 + (x - \frac{\pi}{4}))$ for $x \in (0, \frac{\pi}{2})$

Definition 2.1. *The nth degree or nth order Taylor polynomial approximation to $f(x)$ at the point a is given by*

$$T_n(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \ldots + f^n(a)\frac{(x-a)^n}{n!}$$

This polynomial has the following important properties:

a) T_n is a polynomial of degree at most n. Usually it is exactly n but sometimes $f^(n)(a) = 0$.

b) $\frac{d^k T_n}{dx^k}(a) = f^{(k)}(a)$ if $k \leq n$, so T_n has the same first n derivatives at $x = a$.

c) $\frac{d^k T_n}{dx^k}(a) = 0$ if $k > n$, since T_n is a polynomial of degree at most n.

So in this Taylor approximation business there are three things to keep track of:

- The function that you are approximating $f(x)$.
- The point at which you are approximating a.
- The order of approximation n.

Example 2.2. *Find the third order Taylor polynomial to $f(x) = e^x$ at the point $x = 1$.*
We need to make a small table of the derivatives of the function $f(x)$ at $x = 1$:

$$\begin{array}{ll} f(x) = e^x & f(1) = e^1 = e \\ f'(x) = e^x & f'(1) = e^1 = e \\ f''(x) = e^x & f''(1) = e^1 = e \\ f'''(x) = e^x & f'''(1) = e^1 = e \end{array}$$

so, if we pull out the common factor of e we find that

$$T_3(x) = e(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6})$$

Figure 2 shows the exponential function $f(x) = e^x$ (in solid black) together with the third order Taylor approximation $T_3(x)$ about the point $a = 1$ (in dashed red). You can see that the approximation is very good in the range $(0, 2)$– the maximum absolute error in this range is about $.14$, and the maximum relative error is about 0.04. We have also included the tangent line approximation as well (green dotted). Note that the third order Taylor approximation does a much better job than the tangent line in approximating the function.

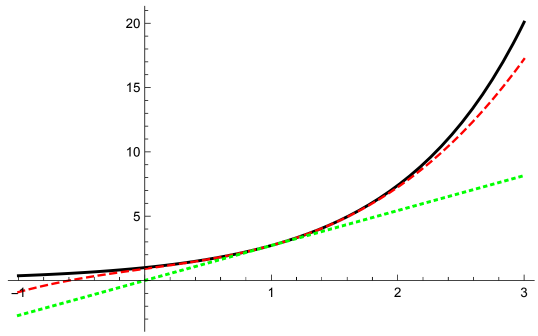

Figure 2: The function $f(x) = e^x$ (black) and the third order Taylor approximation at $a = 1$ $T_3(x) = e(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6})$ (red dashed) for $x \in (-1, 3)$

3 Order of Approximation/ Big O notation

As we saw in the previous example a good rule of thumb is that taking a higher degree Taylor polynomial will give a better approximation to the function. We will give error estimates later in the course that will make this more precise. For now we will just try to give some intuition:

If the quantity $(x - a)$ is small then

- $(x - a)^2$ will be smaller
- $(x - a)^3$ will be smaller still
- $(x - a)^4$ will be even smaller still

Often in engineering we aren't really interested in keeping too many digits of precision. Sometimes it is because it is just not worth the effort – we don't need to know the answer that precisely. Sometimes it is because we simply don't have the information: if our data is only good to one decimal place then it is usually not necessary or even meaningful to carry out our calculation to seven decimal places.

Big O notation is a mathematical way to discuss this idea of finite precision. When we say something like

$$f(x) = 1 + 2x + \frac{3}{17}x^2 + O(x^3)$$

we mean that when x is small we have that $f(x) - (1 + 2x + \frac{3}{17}x^2)$ is no larger than a constant times x^3. In other words we are neglecting terms of order x^3 as well as all higher orders $x^4, x^5, x^6 \ldots$.

We can do this about *a*ny point, not just zero. For instance

$$\sin(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) + O((x - \frac{\pi}{4})^2).$$

Example 3.1. *Find* $\cos(x)$ *to fourth order about the point* $a = 0$*.*

$$\begin{aligned}
&f(x) = \cos(x) \qquad f(0) = 1 \\
&f'(x) = -\sin(x) \qquad f'(0) = 0 \\
&f''(x) = -\cos(x) \qquad f''(0) = -1 \\
&f'''(x) = \sin(x) \qquad f'''(0) = 0 \\
&f''''(x) = \cos(x) \qquad f''''(0) = 1
\end{aligned}$$

So we have that

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^5)$$

Actually if we were to compute the fifth derivative we would find that it is equal to zero at 0*, so this is really good to order six.*

3.1 Adding and multiplying with errors

It is often the case that we want to add, multiply, etc series to a finite order. The basic rul is that one should not keep terms that are the same or lower order than your errors. For instance

Example 3.2. *Suppose that*

$$f(x) = 1 + 3x + O(x^2)$$

and that

$$g(x) = 2 - 2x + 4x^2 + O(x^3)$$

What are $f(x) + g(x)$ *and* $f(x)g(x)$*?*

First let's do the sum. It isn't hard to see that

$$f(x) + g(x) = 3 + x + O(x^2)$$

There is no point in keeping the $4x^2$ *term from* $g(x)$ *since there are* $O(x^2)$ *terms in* $f(x)$ *that we are not keeping.*

Multiplying out is similar. expaninf out all of the terms gives

$$\begin{aligned}(1 + 3x + O(x^2))(2 - 2x + 4x^2 + O(x^3)) &= 2 + 6x - 2x + 2O(x^2) - 2xO(x^2) + 4x^2O(x^2) + O(x^2)O(x^3) + O(x^3) + 3xO(x^3) \\ &= 2 + 4x + O(x^2)\end{aligned}$$

since all of those terms are smaller than the error, $O(x^2)$*. In general something that is* $O(x^k)$ *multiplied by something that is* $O(x^J)$ *is* $O(x^{j+k})$*. Also a constant times something* $O(x^k)$ *is still* $O(x^k)$*, since we are not keeping track of constants.*