Math 415. Exam 3. November 30, 2017

Full Name: ____________________

Net ID: ________________________

Discussion Section: ___________

- There are 20 problems worth 5 points each.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 70 minute exam.
- Do not turn this page until instructed to.
- Fill in the answers on the scantron form provided. Also circle your answers on the exam itself.
- Hand in both the exam and the scantron.
- On the scantron make sure you bubble in **your name, your UIN and your NetID**.
- There are several different versions of this exam.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- Good luck!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**!
On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) Let V be a subspace of $\mathbb{R}^n$, $n > 0$ and let P be the projection matrix of the projection onto V. Which of the following statements is not always true?

(A) $\mathrm{Col}(P^T) = V^\perp$.

(B) $\mathrm{Col}(P) = V$.

(C) $\mathrm{Nul}(P) = V^\perp$.

(D) $\mathrm{rank}(P) = \dim V$.

2. (5 points) Let $\mathcal{E} = \left(\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ be the standard basis of $\mathbb{R}^2$ and let $\mathcal{B}$ be another basis: $\mathcal{B} = \left(\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)$. Let $I_{\mathcal{E},\mathcal{B}}$ be the change of basis matrix from the $\mathcal{B}$-basis to the standard basis, and let $I_{\mathcal{B},\mathcal{E}}$ be the change of basis matrix from the standard basis to the $\mathcal{B}$ basis. Then

(A)

$$I_{\mathcal{E},\mathcal{B}} = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}, \quad I_{\mathcal{B},\mathcal{E}} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$$

(B)

$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad I_{\mathcal{B},\mathcal{E}} = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

(C)

$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad I_{\mathcal{B},\mathcal{E}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(D)

$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad I_{\mathcal{B},\mathcal{E}} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

3. (5 points) Consider two bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be such that the matrix $T_{\mathcal{B},\mathcal{B}}$ that represents T with respect to $\mathcal{B}$ is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Suppose that the change of basis matrices are given by

$$I_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } I_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

What is $T_{\mathcal{C},\mathcal{C}}$?

(A) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(B) $\begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$.

(C) None of the other matrices.

(D) $\begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$.

4. (5 points) Let Q be an orthogonal $n \times n$-matrix. Which of the following statements is not true for all such Q?

(A) Q^T is the inverse of Q.

(B) The columns of Q are orthonormal.

(C) $\det(A) = 1$.

(D) If Q is upper triangular, then Q is diagonal.

5. (5 points) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Suppose that Q is a 3×2-matrix with orthonormal columns and R is an invertible upper triangular matrix such that $A = QR$. Which of the following matrices can be R?

(A) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

(B) $\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 2 \end{bmatrix}$

(C) None of the other answers.

(D) $\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}$

(E) $\begin{bmatrix} 2 & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}$

6. (5 points) Let C be a $n \times n$-matrix such that $C^T = C^{-1}$. Then

(A) $\det(C) = 1$ or $\det(C) = -1$,

(B) $\det(C) \geq 0$,

(C) $\det(C)$ can be any real number,

(D) $\det(C) \leq 0$.

7. (5 points) The least squares solution of

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is

(A) $\begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$.

(B) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(C) $\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$.

(D) $\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$.

(E) none of the other answers.

8. (5 points) Let A, B, C be invertible $n \times n$-matrices such that $AB = B^2C$, $\det(B) = 3$ and $\det(C) = 2$. Then:

(A) $A = BC$ and $\det(A) = 5$.

(B) $A = B^2CB^{-1}$ and $\det(A) = 6$.

(C) $A = B^2CB^{-1}$ and $\det(A) = 5$.

(D) $A = BC$ and $\det(A) = 6$.

(E) $A = B^3C$ and $\det(A) = 11$.

9. (5 points) Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ and $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Let $\boldsymbol{w}_1$ be the orthogonal projection of $\boldsymbol{v}_1$ onto W, and let $\boldsymbol{w}_2$ be the orthogonal projection of $\boldsymbol{v}_2$ onto W. Then:

(A) $\boldsymbol{w}_1 = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(B) $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(C) $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(D) $\boldsymbol{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(E) $\boldsymbol{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

10. (5 points) Which of the following statements is true? Let A, B be $n \times n$-matrices.

(A) if every row of A adds up to 0, then $\det(A) = 0$.

(B) if A is invertible and B is not invertible, then AB is invertible.

(C) the determinant of A is the product of the diagonal entries of A.

(D) if every row of A adds up to 1, then $\det(A) = 1$.

11. (5 points) Let $\mathcal{B} = \left(\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$. Let $\mathbf{x} = \begin{bmatrix} 4 \\ 8 \\ 12 \\ 8 \end{bmatrix}$. Let $W = \text{Span}\{\mathbf{b_2}, \mathbf{b}_3\}$. Find the projection of $\mathbf{x}$ onto the orthogonal complement of W.

(A) None of the other answers.

(B) $\begin{bmatrix} 8 \\ 8 \\ 10 \\ 6 \end{bmatrix}$

(C) $\begin{bmatrix} 8 \\ 8 \\ 8 \\ 8 \end{bmatrix}$.

(D) $\begin{bmatrix} -4 \\ 0 \\ 2 \\ 2 \end{bmatrix}$.

12. (5 points) Let $A = \begin{bmatrix} 6 & 5 & 3 \\ 2 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$.

(A) $\det(A) = 14$

(B) $\det(A) = 30$

(C) $\det(A) = -8$

(D) $\det(A) = -4$

(E) $\det(A) = 22$

13. (5 points) Which of the following vectors is an eigenvector for the matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & \frac{1}{2} \end{bmatrix}$$

with eigenvalue $\frac{1}{2}$?

(A) $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$.

(B) $\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}$.

(C) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

(D) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

14. (5 points) Let $A = \begin{bmatrix} 19 & -3 \\ -3 & 11 \end{bmatrix}$. This matrix has two eigenvalues: 10 and 20. Which of the following pairs of vectors contain a vector in each eigenspace of A?

(A) $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(D) None of these pairs contains a vector in each eigenspace of A.

(E) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

15. (5 points) Let $\mathcal{B} = \left(\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$. Let $\mathbf{x} = \begin{bmatrix} 4 \\ 8 \\ 12 \\ 8 \end{bmatrix}$. Let $W = \text{Span}\{\mathbf{b_1}, \mathbf{b}_3\}$. Find the coordinate vector $\mathbf{x}_{\mathcal{B}}$ of $\mathbf{x}$ with respect to the basis $\mathcal{B}$.

(A) $\begin{bmatrix} 8 \\ -2 \\ -2 \\ 2 \end{bmatrix}$

(B) $\begin{bmatrix} 4 \\ 8 \\ 12 \\ 8 \end{bmatrix}$

(C) $\begin{bmatrix} 6 \\ 0 \\ -2 \\ 6 \end{bmatrix}$

(D) $\begin{bmatrix} 32 \\ -8 \\ -4 \\ 4 \end{bmatrix}$

(E) None of the other answers

16. (5 points) Let A be an $n \times n$-matrix and let λ be an eigenvalue of A. Which of the following statements is incorrect?

(A) All the other statement are true.

(B) λ^2 is an eigenvalue of A^2,

(C) λ can not be 0.

(D) λ^{-1} is an eigenvalue of A^{-1} whenever A^{-1} exists,

(E) $\lambda + 1$ is an eigenvalue of $A + I$,

17. (5 points) Suppose you are given the data points $(-1, 1), (1, 1), (2, 3)$. In addition you are told that the least square solutions to

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

is $\begin{bmatrix} \frac{4}{7} \\ \\ \frac{9}{7} \end{bmatrix}$. What is the least squares line for the three data points given above?

(A) $y = \frac{4}{7} - \frac{9}{7}x$.

(B) $y = \frac{9}{7} + \frac{4}{7}x$.

(C) $y = \frac{4}{7} + \frac{9}{7}x$.

(D) $y = \frac{9}{7} - \frac{4}{7}x$.

18. (5 points) Let W be the subspace $\text{Span}\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\}$ and let P be the projection matrix for W. Then

(A)

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}^{-1}$$

(B)

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(C)

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(D)

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(E)

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

19. (5 points) Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ be 4 row vectors of length 4 (so that the transpose $\mathbf{a}_i^T$ belongs to $\mathbb{R}^4$). Let

$$A = \begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ - & \mathbf{a}_3 & - \\ - & \mathbf{a}_4 & - \end{bmatrix}$$

be the 4×4 matrix with these vectors as rows. Assume $\det(A) = 2$. What is the determinant of the matrix

$$B = \begin{bmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 - 2\mathbf{a}_2 \\ 2\mathbf{a}_3 + \mathbf{a}_2 \\ -\mathbf{a}_4 \end{bmatrix}$$

(A) $\det(A) = 4$.

(B) None of the other answers is correct.

(C) $\det(A) = -4$.

(D) $\det(A) = 2$.

(E) $\det(A) = -2$.

20. (5 points) Let $A = \begin{bmatrix} 14 & -2 \\ -2 & 11 \end{bmatrix}$. Find the eigenvalues of A.

(A) 1 and 5

(B) 10 and 15.

(C) 15 and 25

(D) None of the other answers

(E) -10 and 10