

Math 231E, Lecture 22.

More reasons to Integrate: Work, Average Value, Arclength, Surface Area

1 Work

We know from physics that work is force times distance. If we apply a constant force F over a distance d , then the total amount of work done is $W = F \cdot d$. But what if the force depends on position, i.e. we have $F = F(x)$? If we move the object over a very small distance Δx , then the force is roughly constant over that distance, so the amount of work done is $\Delta W = F(x)\Delta x$. Taking the limit as $\Delta x \rightarrow 0$, we get the infinitesimal form $dW = F(x) dx$. Then the total amount of work done is

$$W = \int dW = \int F(x) dx.$$

Example 1.1. Let us imagine that we have a spring with rest length 10 cm, and under a force of 40 N, it stretches to 15 cm. How much work does it take to stretch the spring from 15 cm to 25 cm?

Let us assume that the spring satisfies “Hooke’s Law”, i.e. that the force needed to stretch the spring is linear in the distance, i.e.

$$F(x) = k(x - x_0),$$

where x_0 is the rest length, and k is the spring constant. In this case, we have the equation

$$40 \text{ N} = k(15 \text{ cm} - 10 \text{ cm}),$$

so

$$k = \frac{40 \text{ N}}{5 \text{ cm}} = 8 \frac{\text{kg} \cdot \text{m}}{\text{cm} \cdot \text{s}^2} = 800 \text{ kg} \cdot \text{s}^{-2}.$$

So then $F(x) = [800 \text{ kg} \cdot \text{s}^{-2}](x - x_0)$. Therefore the total work to be done is

$$\begin{aligned} \int_{15 \text{ cm}}^{25 \text{ cm}} [800 \text{ kg} \cdot \text{s}^{-2}](x - x_0) dx &= [400 \text{ kg} \cdot \text{s}^{-2}](x - x_0)^2 \Big|_{15 \text{ cm}}^{25 \text{ cm}} \\ &= [400 \text{ kg} \cdot \text{s}^{-2}](225 - 25) \text{ cm}^2 \\ &= 80000 \frac{\text{kg} \cdot \text{cm}^2}{\text{s}^2} = 80000 \text{ J} \left(\frac{\text{m}}{100 \text{ cm}} \right)^2 = 8 \text{ J}. \end{aligned}$$

Note that the change in energy we computed above was the difference between two values of the same function. We can generalize this idea, as follows: Let $F(x)$ be a (one-dimensional) force field for a system, i.e. at each point x the system requires a force $-F(x)$ to hold it there. Then we define the **potential energy** of the system as

$$P(x) = - \int^x F(t) dt.$$

Notice that $P'(x) = -F(x)$ at all x , so that the infinitesimal rate of change of the energy is the force we need to apply to hold it where it is.

So, for example, for a spring following Hooke's Law, we have $P(x) = (k/2)(x - x_0)^2$. If we consider the spring above with $k = 800 \text{ kg/s}^2$, then $P(x) = 400 \text{ kg/s}^2 \cdot (x - x_0)^2$ for all x . In particular,

$$P(15 \text{ cm}) = 400 \text{ kg/s}^2 (5 \text{ cm})^2 = 10000 \frac{\text{kg} \cdot \text{cm}}{\text{s}^2} = 1 \text{ J},$$

$$P(25 \text{ cm}) = 400 \text{ kg/s}^2 (15 \text{ cm})^2 = 90000 \frac{\text{kg} \cdot \text{cm}}{\text{s}^2} = 9 \text{ J},$$

and we see that

$$P(25 \text{ cm}) - P(15 \text{ cm}) = 8 \text{ J},$$

the same energy that we computed above.

Example 1.2. Consider the same spring from above, and assume that we have a mass on the end of the spring of 5 kg. If we stretch the spring to 25 cm and let it go, how fast will the mass be moving when the spring has returned to its rest length of 10 cm?

A calculation similar to the one above shows that the potential energy to stretch it from equilibrium to 25 cm is $P(25 \text{ cm}) = 9 \text{ J}$. If this is converted completely into kinetic energy (which it must, since $P(10 \text{ cm}) = 0$, then we have

$$9 \text{ J} = \frac{1}{2}mv^2, \quad (1)$$

$$18 \text{ J} = 5 \text{ kg} \cdot v^2, \quad (2)$$

$$\frac{18}{5} \frac{\text{m}^2}{\text{s}^2} = v^2, \quad (3)$$

or $v = \sqrt{18/5} \text{ m/s}$.

2 Average Value

Definition 2.1. The average value of the function $f(x)$ on the interval $[a, b]$ is defined to be

$$\frac{1}{b-a} \int_a^b f(x) dx.$$

Example 2.2. For example, we could ask what the average force we have to apply to move the spring above from 15 cm to 25 cm. As we computed below, the potential energy difference is 8 J, so the average force is

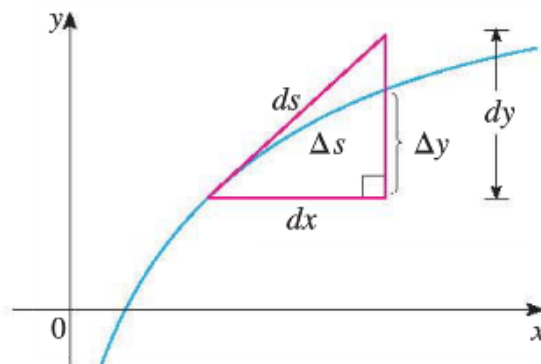
$$\frac{8 \text{ J}}{10 \text{ cm}} = 80 \frac{\text{kg} \cdot \text{m}}{\text{s}^2} = 80 \text{ N}.$$

This perhaps seems weird, since (as stated above) we only need 40 N to hold the spring at 15 cm, but note that we need 120 N to hold it at 25 cm. Therefore the average force is between 40 N and 120 N, but **note** it is not equal to the average of the two quantities!

3 Arclength

The main question here is: what is the length of a given curve?

As always: we break it up into pieces and approximate the length of each piece:



We have the computation $ds^2 = dx^2 + dy^2$, or $ds = \sqrt{dx^2 + dy^2}$. Writing

$$dy = \frac{dy}{dx} dx,$$

we have

$$ds = \sqrt{dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

If we are computing the length of the graph of a function $y = f(x)$, we obtain

$$ds = \sqrt{1 + (f'(x))^2} dx.$$

Example 3.1. Consider the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$. What is its length? We have

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2},$$

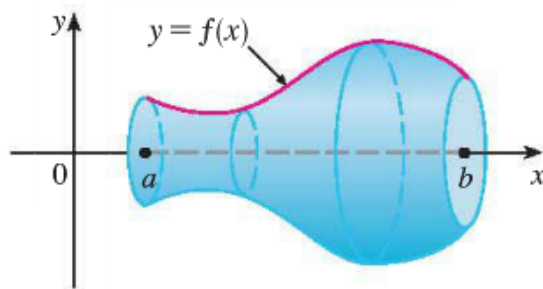
so we have

$$\int_1^2 \sqrt{1 + \frac{9}{4}x} dx = \frac{8}{27} \left(1 + \frac{9}{4}x\right)^{3/2} \Big|_{x=1}^{x=2} = \frac{8}{27} \left(\left(\frac{11}{2}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2} \right) \approx 2.08581$$

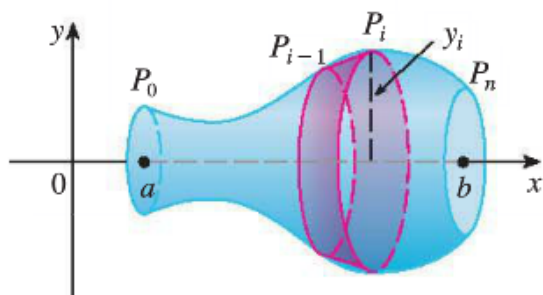
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4 Surface Area

Consider the following question: consider the curve given by the graph of $y = f(x)$. Rotate this curve around the x -axis, and consider the “shell” carved out by this curve. What is its surface area?



(a) Surface of revolution



(b) Approximating band

If we consider a small slice of this surface, it gives a cylinder with radius $f(x)$ and “height” $ds = \sqrt{1 + (f'(x))^2} dx$. Therefore we do the integral

$$\int 2\pi f(x) \sqrt{1 + (f'(x))^2} dx.$$

Example 4.1. Consider the curve $y = x^3$ with $x \in [1, 2]$. Rotate around the x -axis and compute its surface area. We have

$$\begin{aligned} \int_1^2 2\pi x^3 \sqrt{1 + (3x^2)^2} dx &= \int_1^2 2\pi x^3 \sqrt{1 + 9x^4} dx \\ &= \frac{\pi}{27} (1 + 9x^4)^{3/2} \Big|_{x=1}^{x=2} \\ &= \frac{\pi}{27} ((145)^{3/2} - (10)^{3/2}) \approx 199.48 \end{aligned}$$

Example 4.2 (Gabriel’s Horn). Consider the curve given by the graph of $y = x^{-p}$ for $x \in [1, \infty)$. Rotate this around the x -axis and compute its volume and surface area.

Volume. We have $V = \int A(x) dx$, where $A(x)$ is the area of a slice:

$$V = \int_1^\infty \pi (f(x))^2 dx = \int_1^\infty \frac{\pi}{x^{2p}} dx.$$

Surface Area. We have

$$SA = \int_1^\infty 2\pi f(x) \sqrt{1 + (f'(x))^2} dx = \int_1^\infty 2\pi \frac{\sqrt{1 + p^2 x^{-2(p+1)}}}{x^p} dx.$$

Now we might ask for which p these integrals converge. Note that the volume is finite iff $2p > 1$ or $p > 1/2$, which we get directly from the p -test.

Now, notice that since

$$\lim_{x \rightarrow \infty} \sqrt{1 + p^2 x^{-2(p+1)}} = 1,$$

we see that there must be an $M > 0$ such that for all $x > M$,

$$\frac{1}{2} < \sqrt{1 + p^2 x^{-2(p+1)}} < \frac{3}{2}.$$

This means, by a comparison, that the SA is finite iff $p > 1$.

Note then, this means that is $1/2 < p < 1$, then

the surface area is infinite, but the volume is finite!

This means, for example, that this horn cannot hold enough paint to paint its own surface!!