Math 415. Practice Exam 2. October 26, 2017

Full Name: ____________________

Net ID: ______________________

Discussion Section: __________

- There are 20 problems worth 5 points each.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 70 minute exam.
- Do not turn this page until instructed to.
- Fill in the answers on the scantron form provided. Also circle your answers on the exam itself.
- Hand in both the exam and the scantron.
- On the scantron make sure you bubble in **your name, your UIN and your NetID**.
- There are several different versions of this exam.

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**!
On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Consider the following sets of polynomials in $\mathbb{P}_2$.

$$\begin{aligned}\mathcal{A} &= \{1, t, t+1\}\\ \mathcal{B} &= \{1, t, t^2\}\\ \mathcal{C} &= \{1, 2t, t^2, t^2 - t\}.\end{aligned}$$

Which of these sets is a spanning set of all of $\mathbb{P}^2$?

(A) None of them.

(B) $\mathcal{A}$ only.

(C) $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$.

(D) $\mathcal{B}$ only.

(E) ★ $\mathcal{B}$ and $\mathcal{C}$ only.

Solution. $\mathcal{B}$ is the standard basis of $\mathbb{P}_2$, hence spans $\mathbb{P}_2$. The span of $\mathcal{A}$ does not contain t^2. Finally,

$$at^2 + bt + c = at^2 + \frac{b}{2}2t + c.$$

Therefore $\mathcal{C}$ spans $\mathbb{P}_2$.

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2. (5 points) Let B be a $m \times m$-matrix such that $B^T = B$. Which of the following statements is false?

(A) The dimension of the column space of B is equal to the dimension of the row space of B.

(B) ★ The dimension of the null space of B is equal to the dimension of the column space of B.

(C) The dimension of the null space of B is equal to the dimension of the left null space of B.

(D) The null space of B is orthogonal to the column space of B.

Solution. The dimension of the column space of B is equal to the dimension of the row space of B (see Lecture 15 Theorem 2). The column space is orthogonal to left null space of B. Since $B^T = B$, the left null space of B is equal to the null space of B. Therefore the column space of B is orthogonal to the null space of B. Because B is square matrix, the dimension of the left null space is equal to the dimension of the null space (see Lecture 15 Theorem 2). Finally, consider $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This matrix satisfies $B^T = B$, but the $\dim \operatorname{Nul}(B) = 2 \neq 0 = \dim \operatorname{Col}(B)$.

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3. (5 points)

Let A be a 3×4-matrix. Which of the following statements is correct for all such matrices?

(A) Any three of columns of A form a basis of $\mathbb{R}^3$.

(B) The columns of A span $\mathbb{R}^3$.

(C) One of the columns is a multiple of one of the other columns.

(D) The first three columns of A are linearly independent.

(E) ★ The columns of A are linearly dependent.

Solution. Four vectors in $\mathbb{R}^3$ are always linearly dependent. For the other statements, consider $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then the columns of A do not span $\mathbb{R}^3$, hence no three of these columns are linearly independent and can not form a basis of $\mathbb{R}^3$. Also note that the none of the columns is a multiple of one of the other vectors.

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4. (5 points) Suppose $\mathbf{v}$ has coordinate vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the basis

$$\{\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}.$$

Then $\mathbf{v}$ is the vector:

(A) $\begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$

(B) ★ $\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$

(C) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solution. $1\mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3 = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$.

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5. (5 points) Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} \frac{-5}{\sqrt{42}} \\ \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{bmatrix}.$$

Which of the following vectors can be added as a third vector $\mathbf{v}_3$ such that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthonormal basis of $\mathbb{R}^3$?

(A) none of the vectors stated in the other answers.

(B) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(C) $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

(D) ★ $\begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \end{bmatrix}$.

Solution. The vector is $\begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \end{bmatrix}$ is a unit vector and orthogonal to $\mathbf{v}_1, \mathbf{v}_2$. Therefore

$$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{-5}{\sqrt{42}} \\ \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \end{bmatrix}$$

is an orthonormal basis of $\mathbb{R}^3$.

Note that $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ is also orthogonal to $\mathbf{v}_1, \mathbf{v}_2$, but not a unit vector. Therefore

$$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{-5}{\sqrt{42}} \\ \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

is only a orthogonal basis of $\mathbb{R}^3$, but not an orthonormal basis.

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6. (5 points)
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation with

$$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-1\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}.$$

Find the matrix A which represents T with respect to the following bases:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ of } \mathbb{R}^3, \text{ and } \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \text{ of } \mathbb{R}^2.$$

(A) $\begin{bmatrix}1.5 & -.5 & 1\\ .5 & -1.5 & 1\end{bmatrix}$.

(B) $\begin{bmatrix}1 & 2 & 1\\ -1 & 2 & 1\end{bmatrix}$

(C) ★ $\begin{bmatrix}1 & -1 & 1\\ .5 & .5 & 0\end{bmatrix}$

(D) $\begin{bmatrix}1 & .5\\ -1 & .5\\ 1 & 0\end{bmatrix}$.

(E) None of the other answers.

Solution.

$$T(\begin{bmatrix}1\\0\\0\end{bmatrix}) = .5T(\begin{bmatrix}1\\1\\0\end{bmatrix}) + .5T(\begin{bmatrix}1\\-1\\0\end{bmatrix}) = .5\begin{bmatrix}1\\-1\end{bmatrix} + .5\begin{bmatrix}2\\2\end{bmatrix} = 1\begin{bmatrix}1\\1\end{bmatrix} + .5\begin{bmatrix}1\\-1\end{bmatrix}.$$

$$T(\begin{bmatrix}0\\1\\0\end{bmatrix}) = .5T(\begin{bmatrix}1\\1\\0\end{bmatrix}) - .5T(\begin{bmatrix}1\\-1\\0\end{bmatrix}) = .5\begin{bmatrix}1\\-1\end{bmatrix} - .5\begin{bmatrix}2\\2\end{bmatrix} = -1\begin{bmatrix}1\\1\end{bmatrix} + .5\begin{bmatrix}1\\-1\end{bmatrix}.$$

$$T(\begin{bmatrix}0\\0\\1\end{bmatrix}) = \begin{bmatrix}1\\1\end{bmatrix}.$$

Thus

$$A = \begin{bmatrix}1 & -1 & 1\\ .5 & .5 & 0\end{bmatrix}.$$

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7. (5 points) Let V be the following subspace of $\mathbb{R}^4$.

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 - 2x_2 + x_3 + x_4 = 0, \quad 2x_1 - 4x_2 + 2x_3 + x_4 = 0 \right\}$$

What is the dimension of V?

(A) 0.

(B) ★ 2.

(C) 1.

(D) 4.

(E) 3.

Solution. Note that $V = \mathrm{Nul}(A)$, where $A = \begin{bmatrix} 1 & -2 & 1 & 1 \\ 2 & -4 & 2 & 1 \end{bmatrix}$. An echelon form is of A is

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Because there are two free variables, $\dim \mathrm{Nul}(A) = 2$.

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8. (5 points) Consider the following two statements:

(T1) Every linearly independent set of vectors in a vector space V forms a basis of V.

(T2) If $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ are linearly independent vectors in a vector space V, then $\dim(V) \geq n$.

Then:

(A) Neither Statement T1 nor Statement T2 is correct.

(B) ★ Only Statement T2 is correct.

(C) Statement T1 and Statement T2 are correct.

(D) Only Statement T1 is correct.

Solution. (T1) fails. For example, $\{\begin{bmatrix}1\\0\end{bmatrix}\}$ is linear independent, but it is not a basis of $\mathbb{R}^2$. (T2) is correct, because every set of linearly independent vectors can be extended to basis of V.

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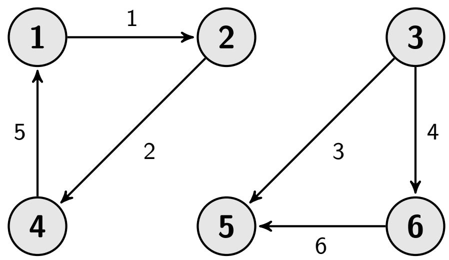
9. (5 points) Let A be the edge-node incidence matrix of the directed graph below.



What is the dimension of Nul(A)?

(A) 1.

(B) None of the above.

(C) ★ 2.

(D) 3.

(E) 6.

Solution. See Theorem 3 Lecture 20, this graph has two connected subgraphs.

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10. (5 points) Let $L : M_{2\times 2} \to M_{2\times 2}$ be the linear transformation given by $L(A) = A^T$. Consider the bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of $M_{2\times 2}$. What ia $L_{\mathcal{B}\mathcal{B}}$?

(A) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(B) ★ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(C) None of the other answers.

(D) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(E) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Solution.

$$L(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$L(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$L(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$L(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus

$$L_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

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11. (5 points) Let V be the vector space of 2×2 matrices and let W be the following subspace of V:

$$\mathrm{Span}\left(\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right).$$

The dimension of W is

(A) 2

(B) 0

(C) 3

(D) ★ 1

Solution. The matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are redundant, so the dimension is 1.

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12. (5 points) Let $B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The dimension of the left null space of B is

(A) ★ 3

(B) 2

(C) 0

(D) 4

(E) 1

Solution. See Lecture 15 Theorem 2.

13. (5 points) Consider the vector space $\mathbb{P}_2$ of polynomials of degree at most 2, and the basis $\mathcal{B} = (1, t+1, t^2+t)$ of $\mathbb{P}_2$. Let $p(t) = 1 + t + t^2$. What is the coordinate vector $p(t)_\mathcal{B}$ of $p(t)$ with respect to $\mathcal{B}$?

(A) ★ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(B) $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

(C) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(D) $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Solution. Note that

$$p(t) = 1 + t + t^2 = 1 \cdot 1 + 0 \cdot (t+1) + 1 \cdot (t^2 + t).$$

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14. (5 points)
Consider the following two statements:

(S1) There exists a subspace V of $\mathbb{R}^7$ such that $\dim V = \dim V^\perp$.

(S2) If V is a subspace of $\mathbb{R}^7$, then the zero vector is the only vector which is in V as well as in $V^\perp$.

Then:

(A) ★ Only Statement S2 is correct.

(B) Statement S1 and Statement S2 are correct.

(C) Neither Statement S1 nor Statement S2 is correct.

(D) Only Statement S1 is correct.

Solution. Statement (S1) is false, because $\dim V + \dim V^\perp = 7$, and there is no integer x such that $x + x = 7$. Statement (2) is correct, since the zero vector is the only vector orthogonal to itself.

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15. (5 points) Suppose

$$A = \begin{bmatrix} 1 & 0 & -5 & 1 \\ 1 & 1 & -5 & 3 \\ 1 & 2 & -5 & 5 \end{bmatrix}$$

and its reduced echelon form is

$$U = \begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Which of these is a basis for the row space $\mathrm{Col}(A^T)$?

(A) None of these form a basis for the row space.

(B) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -5 \\ 5 \end{bmatrix} \right\}$

(C) $\left\{ \begin{bmatrix} 1 & 0 & -5 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 2 \end{bmatrix} \right\}$

(D) Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

(E) ★ $\left\{ \begin{bmatrix} 1 \\ 0 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

Solution. The basis is given by the non zero rows columns of U, transposed. See Lecture 15 Example 1.

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16. (5 points) Let A be an $m \times n$-matrix with echelon form U. Which of the following statement is true for all such A?

(T1) $\mathrm{Col}(A) = \mathrm{Col}(U)$.

(T2) $\mathrm{Nul}(A) = \mathrm{Nul}(U)$.

(A) Neither (T1) nor (T2).

(B) ★ Only (T2).

(C) Both (T1) and (T2)

(D) Only (T1).

Solution. The solutions of $Ax = 0$ and $Ux = 0$ are the same, so (T2) holds. The column space is not preserved by elementary row operations in general so (T1) does not hold.

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17. (5 points)
Suppose

$$A = \begin{bmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

and its reduced echelon form is

$$U = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which of these is a basis for the column space Col(A)?

(A) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(B) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(C) ★ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} \right\}$

(D) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Solution. The basis is given by the pivot columns of A, columns 1 and 2.

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18. (5 points) Let $A = \begin{bmatrix} 0 & 1 & 3 & 2 & 3 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. The dimension of the column space of A is

(A) 5

(B) 3

(C) 6

(D) ★ 4

(E) 7

Solution. See Lecture 15 Theorem 2.

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19. (5 points) Let $\mathbb{P}_3$ be the vector space of all polynomials of degree up to 3, and let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the linear transformation defined by

$$T(p(t)) = 3p(t) + 2p'(t) + 4p''(t).$$

Which matrix A represents T with respect to the standard bases?
(Recall that the standard basis for $\mathbb{P}_3$ is given by $1, t, t^2, t^3$.)

(A) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 8 & 4 & 3 & 0 \\ 0 & 24 & 6 & 3 \end{bmatrix}$.

(B) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

(C) ★ $\begin{bmatrix} 3 & 2 & 8 & 0 \\ 0 & 3 & 4 & 24 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

(D) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

Solution. $T(t^3) = 3t^3 + 6t^2 + 24t$, $T(t^2) = 3t^2 + 4t + 8$, $T(t) = 3t + 2$ and $T(1) = 3$.
Therefore $A = \begin{bmatrix} 3 & 2 & 8 & 0 \\ 0 & 3 & 4 & 24 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

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20. (5 points) Determine a basis of the orthogonal complement of

$$V = \{ \begin{bmatrix} 0 \\ a \\ b \\ a+b \end{bmatrix} \ : \ a, b \text{ in } \mathbb{R} \}.$$

(A) ★ $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \}$.

(B) $\{ \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \}$

(C) $\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \}$.

(D) $\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \}$.

(E) None of the other answers.

Solution. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then $V = \mathrm{Col}(A)$. By Lecture 19 Theorem 2, $V^{\perp} = \mathrm{Nul}(A^T)$. The $A^T = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is already in reduced echelon form. Therefore $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ is a basis of $V^{\perp}$.
