Math 231E, Fall 2016. Final Exam.

- This exam has 34 questions.
- You must not communicate with other students during this test. No books, notes, **calculators**, or electronic devices allowed.
- Please fill out all of the information below. Make sure to fill out your Scantron form as directed in class; fill in name, UIN number, and NetID.
- Draw a snowman somewhere on the test booklet for good luck.

1. Fill in your information:

Full Name: ______________________________

UIN (Student Number): ______________________________

NetID: ______________________________

2. Fill out name, student number (UIN) and NetID on Scantron sheet. Then fill in the following answers on the Scantron form:

Zone 1

1/1. (3 points) Which of the following is equivalent to

$$\frac{1}{2+i}?$$

A. ★ $\dfrac{2}{5} - \dfrac{i}{5}$

B. $\dfrac{2}{\sqrt{5}} - \dfrac{i}{\sqrt{5}}$

C. $\dfrac{1}{2} + i$

D. $2 - i$

E. $i + 2$

Solution. Using the formula $|z|^2 = z\overline{z}$, we have

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

In this case, this is

$$\frac{2-i}{(2^2+1^2)} = \frac{2-i}{5} = \frac{2}{5} - \frac{i}{5}.$$

1/2. (3 points) Which of the following is equivalent to

$$\frac{1}{3+i}?$$

A. ★ $\dfrac{3}{10} - \dfrac{i}{10}$

B. $\dfrac{3}{\sqrt{10}} - \dfrac{i}{\sqrt{10}}$

C. $\dfrac{1}{3} + i$

D. $3 - i$

E. $i + 3$

Solution. Using the formula $|z|^2 = z\overline{z}$, we have

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

In this case, this is

$$\frac{3-i}{(3^2+1^2)} = \frac{3-i}{10} = \frac{3}{10} - \frac{i}{10}.$$

2/1. (3 points) What is the second-order Taylor series for $\dfrac{1}{1-3x}$ at $a = 0$?

A. ★ $1 + 3x + 9x^2 + O(x^3)$

B. $1 + 2x + 3x^2 + O(x^3)$

C. $1 - \dfrac{x^2}{9} + O(x^3)$

D. $3x - 2x^2 + O(x^3)$

E. $1 + \dfrac{1}{3x} + \dfrac{1}{9x^2} + O(x^3)$

Solution. The definition of the second-order series of $f(x)$ at $a = 0$ is

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + O(x^3)$$

Noting that $f(x) = \dfrac{1}{1-3x}$, $f'(x) = \dfrac{3}{(1-3x)^2}$, $f''(x) = \dfrac{18}{(1-2x)^3}$, we have

$$1 + 3x + 9x^2 + O(x^3).$$

Alternately recall that $\dfrac{1}{1-x}$ is the geometric series $1 + x + x^2 + x^3 + \dots$ and plug in $3x$.

2/2. (3 points) What is the second-order Taylor series for $\dfrac{1}{1+3x}$ at $a = 0$?

A. ★ $1 - 3x + 9x^2 + O(x^3)$

B. $1 - 2x + 3x^2 + O(x^3)$

C. $1 + \dfrac{x^2}{9} + O(x^3)$

D. $3x + 2x^2 + O(x^3)$

E. $1 - \dfrac{1}{3x} + \dfrac{1}{9x^2} + O(x^3)$

Solution. The definition of the second-order series of $f(x)$ at $a = 0$ is

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + O(x^3)$$

Noting that $f(x) = \dfrac{1}{1+3x}$, $f'(x) = \dfrac{-3}{(1+3x)^2}$, $f''(x) = \dfrac{18}{(1+2x)^3}$, we have

$$1 - 3x + 9x^2 + O(x^3).$$

Alternately recall that $\dfrac{1}{1-x}$ is the geometric series $1 + x + x^2 + x^3 + \dots$ and plug in $-3x$.

2/3. (3 points) What is the second-order Taylor series for $\dfrac{1}{1-4x}$ at $a = 0$?

A. ★ $1 + 4x + 16x^2 + O(x^3)$

B. $1 + 4x + 4x^2 + O(x^3)$

C. $1 - \dfrac{x^2}{16} + O(x^3)$

D. $4x - 2x^2 + O(x^3)$

E. $1 + \dfrac{1}{4x} + \dfrac{1}{16x^2} + O(x^3)$

Solution. The definition of the second-order series of $f(x)$ at $a = 0$ is

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + O(x^3)$$

Noting that $f(x) = \dfrac{1}{1-4x}$, $f'(x) = \dfrac{4}{(1-4x)^2}$, $f''(x) = \dfrac{32}{(1-4x)^3}$, we have

$$1 + 4x + 16x^2 + O(x^3).$$

Alternately recall that $\dfrac{1}{1-x}$ is the geometric series $1 + x + x^2 + x^3 + \ldots$ and plug in $4x$.

3/1. (3 points) Evaluate $\lim_{x \to 0} \frac{\sin(2x)}{x}$.

A. ★ 2

B. 0

C. 1

D. $+\infty$

E. does not exist

Solution. We can use Taylor series here. Recall that the Taylor series for $\sin(x)$ is

$$x - \frac{x^3}{6} + O(x^5)$$

and plugging in $x \mapsto 2x$ gives

$$2x - \frac{4}{3}x^3 + O(x^5)$$

This means that

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = \lim_{x \to 0} \frac{2x - \frac{4}{3}x^3 + O(x^5)}{x} = \lim_{x \to 0} 2 - \frac{4}{3}x^2 + O(x^4) = 2.$$

3/2. (3 points) Evaluate $\lim_{x \to 0} \frac{\sin(3x)}{x}$.

A. ★ 3

B. 0

C. 1

D. $+\infty$

E. does not exist

Solution. We can use Taylor series here. Recall that the Taylor series for $\sin(x)$ is

$$x - \frac{x^3}{6} + O(x^5)$$

and plugging in $x \mapsto 3x$ gives

$$3x - \frac{9}{2}x^3 + O(x^5)$$

This means that

$$\lim_{x \to 0} \frac{\sin(3x)}{x} = \lim_{x \to 0} \frac{3x - \frac{9}{2}x^3 + O(x^5)}{x} = \lim_{x \to 0} 3 - \frac{9}{2}x^2 + O(x^4) = 3.$$

4/1. (3 points) Evaluate $\lim\limits_{x\to\infty} \dfrac{\sin(2x)}{x}$.

A. ★ 0

B. 2

C. 1

D. $+\infty$

E. does not exist

Solution. The definition of the limit at ∞ means

$$\lim_{x\to\infty} \frac{\sin(2x)}{x} = \lim_{x\to 0+} \frac{\sin(2/x)}{1/x} = \lim_{x\to 0+} x\sin(2/x).$$

We can use the Squeeze Theorem. Recall that sin takes values between -1 and 1 and so:

$$-1 < \sin(2/x) < 1,$$
$$-x < x\sin(2/x) < x,$$

and

$$\lim_{x\to 0} x = \lim_{x\to 0}(-x) = 0,$$

so the limit is 0.

4/2. (3 points) Evaluate $\lim\limits_{x\to\infty} \dfrac{\sin(3x)}{x}$.

A. ★ 0

B. 3

C. 1

D. $+\infty$

E. does not exist

Solution. The definition of the limit at ∞ means

$$\lim_{x\to\infty} \frac{\sin(3x)}{x} = \lim_{x\to 0+} \frac{\sin(3/x)}{1/x} = \lim_{x\to 0+} x\sin(3/x).$$

We can use the Squeeze Theorem. Recall that sin takes values between -1 and 1 and so:

$$-1 < \sin(3/x) < 1,$$
$$-x < x\sin(3/x) < x,$$

and

$$\lim_{x\to 0} x = \lim_{x\to 0}(-x) = 0,$$

so the limit is 0.

5/1. (3 points) Let $y = f(x)$ be defined implicitly by

$$y^2 - xy = 1.$$

Find $\dfrac{dy}{dx}$ in terms of y and x.

A. ★ $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$

B. $\dfrac{dy}{dx} = -\dfrac{y}{2y + x}$

C. $\dfrac{dy}{dx} = \dfrac{1}{xy^3}$

D. $\dfrac{dy}{dx} = \dfrac{x}{x + y^2}$

E. $\dfrac{dy}{dx} = \arctan(x/y)$

Solution. The derivative is given by implicit differentiation

$$\begin{aligned} y^2 - xy &= 1, \\ 2y\frac{dy}{dx} - y - x\frac{dy}{dx} &= 0, \\ (2y - x)\frac{dy}{dx} &= y, \\ \frac{dy}{dx} &= \frac{y}{2y - x}. \end{aligned}$$

5/2. (3 points) Let $y = f(x)$ be defined implicitly by

$$y^2 + xy = 1.$$

Find $\dfrac{dy}{dx}$ in terms of y and x.

A. ★ $\dfrac{dy}{dx} = -\dfrac{y}{2y + x}$

B. $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$

C. $\dfrac{dy}{dx} = \dfrac{1}{xy^3}$

D. $\dfrac{dy}{dx} = \dfrac{x}{x + y^2}$

E. $\dfrac{dy}{dx} = \arctan(x/y)$

Solution. The derivative is given by implicit differentiation

$$\begin{aligned} y^2 + xy &= 1, \\ 2y\frac{dy}{dx} + y + x\frac{dy}{dx} &= 0, \\ (2y + x)\frac{dy}{dx} &= -y, \\ \frac{dy}{dx} &= \frac{-y}{2y + x}. \end{aligned}$$

6/1. (3 points) Compute the following limit: $\lim_{x \to 0^-} \frac{|2x|}{e^{2x} - 1}$

A. ★ -1

B. 1

C. 0

D. $+\infty$

E. does not exist

Solution. For $x < 0$ we have $|2x| = -2x$ and apply l'Hôpital. Alternately, after replacing $|2x| = -2x$, we can use the Taylor series $e^{2x} = 1 + 2x + O(x^2)$ to find

$$\lim_{x \to 0^-} \frac{|2x|}{e^{2x} - 1} = \lim_{x \to 0^-} \frac{-2x}{2x + O(x^2)} = -1.$$

6/2. (3 points) Compute the following limit: $\lim_{x \to 0^-} \frac{|3x|}{1 - e^{3x}}$

A. ★ 1

B. -1

C. 0

D. $+\infty$

E. does not exist

Solution. For $x < 0$ we have $|3x| = -3x$ and apply l'Hôpital. Alternately, after replacing $|3x| = -3x$, we can use the Taylor series $e^{3x} = 1 + 3x + O(x^2)$ to find

$$\lim_{x \to 0^-} \frac{|3x|}{1 - e^{3x}} = \lim_{x \to 0^-} \frac{-3x}{-3x + O(x^2)} = 1.$$

7/1. (3 points) Compute $f'(x)$, where

$$f(x) = x\cos(x^3).$$

A. ★ $\cos(x^3) - 3x^3\sin(x^3)$

B. $\cos(x^3) - 3x^2\sin(x^3)$

C. $x\cos(x^3) + 3x^3\sin(x^3)$

D. $-x\sin(x^3) + \cos(x^3)$

E. $x\sin(x^3) + 3\cos(x^3)$

Solution. We use the product and chain rules, so

$$\frac{d}{dx}(x\cos(x^3)) = \cos(x^3) + x\frac{d}{dx}(\cos(x^3)) = \cos(x^2) + x(-3x^2)(\sin(x^2))$$

7/2. (3 points) Compute $f'(x)$, where

$$f(x) = x\sin(x^3).$$

A. ★ $\sin(x^3) + 3x^3\cos(x^3)$

B. $-x\cos(x^3)$

C. $\sin(x^3) - 3x^3\cos(x^3)$

D. $-3x\cos(x^3) + \sin(x^3)$

E. $3x\cos(x^3) + \sin(x^3)$

Solution. We use the product and chain rules, so

$$\frac{d}{dx}(x\sin(x^3)) = \sin(x^3) + x\frac{d}{dx}(\sin(x^3)) = \sin(x^3) + x(3x^2)(\cos(x^3))$$

8/1. (3 points) Compute

$$L = \lim_{x \to 1-} \frac{d}{dx}(\sin^{-1}(x)).$$

A. ★ $L = \infty$

B. $L = 0$

C. $L = 1$

D. $L = -\infty$

E. limit does not exist

Solution. We first use implicit differentiation to compute the derivative. If $y = \sin^{-1}(x)$, then

$$x = \sin(y)$$

and differentiating both sides gives

$$1 = \cos(y)\frac{dy}{dx},$$

or $y'(x) = 1/\cos(y) = 1/\cos(\sin^{-1}(x))$. Drawing the applicable triangle gives

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Now, we see that the limit as $x \to 1-$ is a bit complicated, since we cannot plug in. Notice that this function has a vertical asymptote at $x = 1$ when approaching from the left, and is undefined to the right of $x = 1$. For x less than 1, but close to 1, we have a very small positive denominator, so the fraction is large and positive. From this we see that the limit from the left is $+\infty$.

8/2. (3 points) Compute

$$L = \lim_{x \to 1-} \frac{d}{dx}(\cos^{-1}(x)).$$

A. ★ $L = -\infty$

B. $L = 0$

C. $L = 1$

D. $L = \infty$

E. limit does not exist

Solution. We first use implicit differentiation to compute the derivative. If $y = \sin^{-1}(x)$, then

$$x = \cos(y)$$

and differentiating both sides gives

$$1 = -\sin(y)\frac{dy}{dx},$$

or $y'(x) = -1/\sin(y) = -1/\sin(\cos^{-1}(x))$. Drawing the applicable triangle gives

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

Now, we see that the limit as $x \to 1-$ is a bit complicated, since we cannot plug in. Notice that this function has a vertical asymptote at $x = 1$ when approaching from the left, and is undefined to the right of $x = 1$. For x less than 1, but close to 1, we have a very small positive denominator, so the fraction is large and negative. From this we see that the limit from the left is $-\infty$.

8/3. (3 points) Compute

$$L = \lim_{x \to 1-} \frac{d}{dx}(\cos^{-1}(x)).$$

A. ★ $L = -\infty$

B. $L = 0$

C. $L = 1$

D. $L = \infty$

E. limit does not exist

Solution. We first use implicit differentiation to compute the derivative. If $y = \sin^{-1}(x)$, then

$$x = \cos(y)$$

and differentiating both sides gives

$$1 = -\sin(y)\frac{dy}{dx},$$

or $y'(x) = -1/\sin(y) = -1/\sin(\cos^{-1}(x))$. Drawing the applicable triangle gives

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

Now, we see that the limit as $x \to 1-$ is a bit complicated, since we cannot plug in. Notice that this function has a vertical asymptote at $x = 1$ when approaching from the left, and is undefined to the right of $x = 1$. For x less than 1, but close to 1, we have a very small positive denominator, so the fraction is large and negative. From this we see that the limit from the left is $-\infty$.

9/1. (3 points) Compute the limit

$$L = \lim_{x\to 0} \sin\left(\frac{\pi \sin(2x)}{8x}\right)$$

if it exists.

A. ★ $L = \sqrt{2}/2$

B. $L = 0$

C. $L = 1/2$

D. $L = \infty$

E. $L = \sqrt{2}$

Solution. Let us first look at the "inside" part of the function, namely $\sin(2x)/8x$. Using L'Hôpital:

$$\lim_{x\to 0} \frac{\sin(2x)}{8x} = \lim_{x\to 0} \frac{2\cos(2x)}{8} = \frac{1}{4}.$$

Since $\sin(x)$ is continuous,

$$\lim_{x\to 0} \sin\left(\frac{\pi \sin(2x)}{8x}\right) = \sin\left(\lim_{x\to 0} \pi \frac{\sin(2x)}{8x}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

9/2. (3 points) Compute the limit

$$L = \lim_{x\to 0} \cos\left(\frac{\pi \sin(5x)}{20x}\right)$$

if it exists.

A. ★ $L = \sqrt{2}/2$

B. $L = 0$

C. $L = 1/2$

D. $L = \infty$

E. $L = \sqrt{2}$

Solution. Let us first look at the "inside" part of the function, namely $\sin(3x)/12x$. Using L'Hôpital:

$$\lim_{x\to 0} \frac{\sin(5x)}{20x} = \lim_{x\to 0} \frac{5\cos(5x)}{20} = \frac{1}{4}.$$

Since $\sin(x)$ is continuous,

$$\lim_{x\to 0} \cos\left(\frac{\pi \sin(5x)}{20x}\right) = \cos\left(\lim_{x\to 0} \pi \frac{\sin(5x)}{20x}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

10/1. (3 points) Consider the expression

$$L = \lim_{x\to 0} \frac{\cos(\sin(x)) - q}{\cos^{-1}(\sin^{-1}(x^2)) - \pi/2}.$$

First determine the value of q so that the limit exists and is finite, and then compute L.

A. ★ $q = 1, \quad L = 1/2$

B. $q = 1, \quad L = 1$

C. $q = 0, \quad L = -1$

D. $q = \pi/2, \quad L = -1/2$

E. $q = \pi/2, \quad L = 2$

Solution. The answer is $q = 1$ and $L = 1/2$.

First let us see why q must be 1. Notice that as $x \to 0$, the denominator goes to 0: $\sin^{-1}(0) = 0$ and $\cos^{-1}(0) = \pi/2$. If the numerator does not go to zero, then the limit will not exist and be finite. Therefore we need the numerator to go to zero, and since $\cos(\sin(0)) = 1$, we need to choose $q = 1$.

Now, we compute the limit using Taylor series. First recall

$$\cos(x) = 1 - \frac{x^2}{2} + O(x^5),$$
$$\sin(x) = x - \frac{x^3}{6} + O(x^5),$$

so plugging the second into the first gives

$$\cos(\sin(x)) = 1 - \frac{1}{2}\left(x - \frac{x^3}{6} + O(x^4)\right)^2 = 1 - \frac{x^2}{2} + O(x^4).$$

We don't have the Taylor series for $\sin^{-1}(x)$ and $\cos^{-1}(x)$ memorized but we can derive them. Note that if $f(x) = \sin^{-1}(x)$, then

$$f'(x) = (1-x^2)^{-1/2}, \quad f''(x) = x(1-x^2)^{-3/2}, \quad f'''(x) = (2x^2+1)(1-x^2)^{-5/2},$$

and thus

$$f(0) = 0, \quad , f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = 1.$$

Therefore

$$\sin^{-1}(x) = x + \frac{x^3}{6} + O(x^4).$$

Similarly, if $g(x) = \cos^{-1}(x)$, then

$$g'(x) = -(1-x^2)^{-1/2}, \quad g''(x) = -x(1-x^2)^{-3/2}, \quad g'''(x) - (2x^2+1)(1-x^2)^{-5/2},$$

and thus

$$g(0) = \frac{\pi}{2}, \quad , f'(0) = -1, \quad f''(0) = 0, \quad f'''(0) = -1.$$

Therefore

$$\cos^{-1}(x) = \frac{\pi}{2} - x - \frac{x^3}{6} + O(x^4).$$

Finally, plugging in we have

$$\cos^{-1}(\sin^{-1}(x^2)) = \frac{\pi}{2} - \left(x^2 + \frac{x^6}{6} + O(x^8)\right) - \frac{1}{6}\left(x^2 + \frac{x^6}{6} + O(x^8)\right)^3$$

and so

$$\cos^{-1}(\sin^{-1}(x^2)) - \frac{\pi}{2} = -x^2 + O(x^4).$$

Then we have

$$\frac{\cos(\sin(x)) - 1}{\cos^{-1}(\sin^{-1}(x^2)) - \pi/2} = \frac{-x^2/2 + O(x^4)}{-x^2 + O(x^4)} = \frac{1}{2} + O(x^2),$$

so as $x \to 0$ we obtain $1/2$.

Zone 2

11/1. (3 points) Find the general solution to the equation

$$\frac{dy}{dt} = 3y.$$

A. ★ Ce^{3t}

B. $2e^{3t}$

C. Ce^{3x}

D. $\dfrac{t}{Ct^2+3}$

E. $x^3 + C$

Solution. We solve

$$\begin{aligned} \frac{dy}{dt} &= 3y \\ \frac{dy}{y} &= 3\,dt \\ \ln|y| &= 3t + C \\ y(t) = Ce^{3t}. \end{aligned}$$

11/2. (3 points) Find the general solution to the equation

$$\frac{dy}{dt} = 2y.$$

A. ★ Ce^{2t}

B. $3e^{2t}$

C. Ce^{2x}

D. $\dfrac{t}{Ct^2+2}$

E. $x^2 + C$

Solution. We solve

$$\begin{aligned} \frac{dy}{dt} &= 2y \\ \frac{dy}{y} &= 2\,dt \\ \ln|y| &= 2t + C \\ y(t) = Ce^{2t}. \end{aligned}$$

11/3. (3 points) Find the general solution to the equation

$$\frac{dy}{dt} = 4y.$$

A. ★ Ce^{4t}

B. $3e^{4t}$

C. Ce^{4x}

D. $\dfrac{t}{Ct^2+4}$

E. $x^4 + C$

Solution. We solve

$$\begin{aligned} \frac{dy}{dt} &= 4y \\ \frac{dy}{y} &= 4\,dt \\ \ln|y| &= 4t + C \\ y(t) = Ce^{4t}. \end{aligned}$$

12/1. (3 points) A 8 m long ladder is propped up against a wall. The ladder begins to slip. At time $t = 3\,\mathrm{s}$, the base of the ladder is 6 m from the wall and moving away from the wall at 7 m/s. How fast is the top of the ladder moving along the wall?

A. ★ $-3\sqrt{7}\,\mathrm{m/s}$

B. $-\sqrt{7}\,\mathrm{m/s}$

C. $-\dfrac{30}{8}\,\mathrm{s}$

D. $-24\sqrt{5}\,\mathrm{m/s}$

E. $-12\sqrt{5}\,\mathrm{m/s}$

Solution. Let us denote the height of the ladder's contact with the wall as h, and the distance from the wall to the base of the ladder as w. Then we know

$$w^2 + h^2 = 64\,\mathrm{m}^2$$

by the Pythagorean theorem. Differentiating gives

$$2w\frac{dw}{dt} + 2h\frac{dh}{dt} = 0.$$

We are given $w = 6\,\mathrm{m}$ and $dw/dt = 7\mathrm{m/s}$. We need h, but we can use the first equation to obtain

$$h = \sqrt{(8\,\mathrm{m})^2 - (6\,\mathrm{m})^2} = \sqrt{64\,\mathrm{m}^2 - 36\,\mathrm{m}^2} = \sqrt{28\,\mathrm{m}^2} = 2\sqrt{7}\ \mathrm{m}.$$

Thus we plug in, and obtain

$$(6\,\mathrm{m})(7\mathrm{m/s}) + (2\sqrt{7}\mathrm{m})\frac{dh}{dt} = 0,$$

or

$$\frac{dh}{dt} = -\frac{42\,\mathrm{m}^2/\mathrm{s}}{2\sqrt{7}\,\mathrm{m}} = -3\sqrt{7}\,\mathrm{m/s}.$$

12/2. (3 points) A 6 m long ladder is propped up against a wall. The ladder begins to slip. At time $t = 5\,\mathrm{s}$, the base of the ladder is 3 m from the wall and moving away from the wall at 3m/s. How fast is the top of the ladder moving down the wall?

A. ★ $-\sqrt{3}\,\mathrm{m/s}$

B. $-3\sqrt{3}\,\mathrm{m/s}$

C. $-\dfrac{13}{3}\,\mathrm{s}$

D. $-2\sqrt{2}\,\mathrm{m/s}$

E. $-4\sqrt{2}\,\mathrm{m/s}$

Solution. Let us denote the height of the ladder's contact with the wall as h, and the distance from the wall to the base of the ladder as w. Then we know

$$w^2 + h^2 = 36\,\mathrm{m}^2$$

by the Pythagorean theorem. Differentiating gives

$$2w\frac{dw}{dt} + 2h\frac{dh}{dt} = 0.$$

We are given $w = 3\,\mathrm{m}$ and $dw/dt = 3\mathrm{m/s}$. We need h, but we can use the first equation to obtain

$$h = \sqrt{(6\,\mathrm{m})^2 - (3\,\mathrm{m})^2} = \sqrt{36\,\mathrm{m}^2 - 9\,\mathrm{m}^2} = \sqrt{27\,\mathrm{m}^2} = \sqrt{27}\ \mathrm{m} = 3\sqrt{3}\ \mathrm{m}.$$

Thus we plug in, and obtain

$$(3\,\mathrm{m})(3\mathrm{m/s}) + (3\sqrt{3}\ \mathrm{m})\frac{dh}{dt} = 0,$$

or

$$\frac{dh}{dt} = -\frac{9\,\mathrm{m}^2/\mathrm{s}}{3\sqrt{3}\,\mathrm{m}} = -\sqrt{3}\,\mathrm{m/s}.$$

13/1. (3 points) Suppose that the function $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$, and that $f(0)=1$ and $f(1)=0$. Which of the following statements are guaranteed to be true?

1. $f(x)$ is decreasing everywhere.
2. There is a point $b \in (0,1)$ such that $f(b)=1/2$.
3. There is a point $c \in (0,1)$ such that $f'(c)=-1$.
4. There is a point $d \in (0,1)$ such that $f'(d)=1/2$.
5. There is a point $e \in (0,1)$ such that $f(e)=-1$.

A. ★ Statements 2 and 3 must be true.

B. Statements 4 and 5 must be true.

C. Statements 1, 2, and 4 must be true.

D. Statements 3, 4, and 5 must be true.

E. Statements 2, 3, and 4 must be true.

Solution. By continuity and the Intermediate Value Theorem, [2] must be true since $1/2$ is between 1 and 0. By differentiability and the Mean Value Theorem, [3] must be true, since the average rate of change of the function over the interval is -1. However, none of the others need be true. For example, the function $f(x)=1-x$ makes [4] and [5] false, and the function $f(x)=x^2-2x+1$ makes [1] false.

13/2. (3 points) Suppose that the function $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$, and that $f(0)=3$ and $f(1)=5$. Which of the following statements are guaranteed to be true?

1. $f(x)$ is increasing everywhere.
2. There is a point $b \in (0,1)$ such that $f(b)=4$.
3. There is a point $c \in (0,1)$ such that $f'(c)=2$.
4. There is a point $d \in (0,1)$ such that $f'(d)=4$.
5. There is a point $e \in (0,1)$ such that $f(e)=2$.

A. ★ Statements 2 and 3 must be true.

B. Statements 4 and 5 must be true.

C. Statements 1, 2, and 4 must be true.

D. Statements 3, 4, and 5 must be true.

E. Statements 2, 3, and 4 must be true.

Solution. By continuity and the Intermediate Value Theorem, [2] must be true since 4 is between 3 and 5. By differentiability and the Mean Value Theorem, [3] must be true, since the average rate of change of the function over the interval is 2. However, none of the others need be true. For example, the function $f(x)=3+2x$ makes [4] and [5] false, and the function $f(x)=3x^2-x+3$ makes [1] false.

14/1. (3 points) Let $f(x)$ be the function

$$f(x) = \int_{-2}^{x} \sin(2t)\, dt.$$

Compute $f'(x)$.

A. ★ $\sin(2x)$

B. $-2\sin(2x)$

C. $\frac{1}{2}\cos(x) - \frac{1}{2}\cos(-2)$

D. $\frac{1}{2}\cos(2x) - \frac{1}{2}\cos(-4)$

E. $2\cos(2x)$

Solution. The fundamental theorem of calculus tells us that

$$f'(x) = \frac{d}{dx}\int_{-2}^{x} \sin(2t)\, dt = \sin(2x).$$

14/2. (3 points) Let $f(x)$ be the function

$$f(x) = \int_{-2}^{x} \sin(3t)\, dt.$$

Compute $f'(x)$.

A. ★ $\sin(3x)$

B. $-2\sin(3x)$

C. $\frac{1}{3}\cos(x) - \frac{1}{3}\cos(-2)$

D. $\frac{1}{3}\cos(3x) - \frac{1}{3}\cos(-6)$

E. $3\cos(3x)$

Solution. The fundamental theorem of calculus tells us that

$$f'(x) = \frac{d}{dx}\int_{-2}^{x} \sin(3t)\, dt = \sin(3x).$$

14/3. (3 points) Let $f(x)$ be the function

$$f(x) = \int_{-2}^{x} \sin(4t)\, dt.$$

Compute $f'(x)$.

A. ★ $\sin(4x)$

B. $-2\sin(4x)$

C. $\dfrac{1}{4}\cos(x) - \dfrac{1}{4}\cos(-2)$

D. $\dfrac{1}{4}\cos(4x) - \dfrac{1}{3}\cos(-8)$

E. $4\cos(4x)$

Solution. The fundamental theorem of calculus tells us that

$$f'(x) = \frac{d}{dx}\int_{-2}^{x} \sin(4t)\,dt = \sin(4x).$$

15/1. (3 points) Compute

$$\int_0^{\pi/2} e^{\sin(x)} \cos(x)\, dx.$$

A. ★ $e - 1$

B. $e - \frac{1}{e}$

C. $1 - e$

D. $e^{\sqrt{2}/2} - 1$

E. $e - \sqrt{2}/2$

Solution. We do a u-substitution of $u = \sin(x)$ and thus $du = \cos(x)\, dx$, giving

$$\int_0^{\pi/2} e^{\sin(x)} \cos(x)\, dx = \int_0^1 e^u\, du = e^u \Big|_{u=0}^{u=1} = e^1 - e^0 = e - 1.$$

15/2. (3 points) Compute

$$\int_{-\pi/2}^{\pi/2} e^{\sin(x)} \cos(x)\, dx.$$

A. ★ $e - \frac{1}{e}$

B. $e - 1$

C. $1 - e$

D. $e^{\sqrt{2}/2} + 1$

E. $e - \sqrt{2}/2$

Solution. We do a u-substitution of $u = \sin(x)$ and thus $du = \cos(x)\, dx$, giving

$$\int_{-\pi/2}^{\pi/2} e^{\sin(x)} \cos(x)\, dx = \int_{-1}^1 e^u\, du = e^u \Big|_{u=-1}^{u=1} = e^1 - e^{-1} = e - \frac{1}{e}.$$

16/1. (3 points) Compute

$$\int_0^{\pi/2} \sin^3(x)\cos^2(x)\,dx.$$

A. ★ $\frac{2}{15}$

B. 0

C. $\frac{7}{15}$

D. $\frac{\pi}{5}$

E. $\frac{2\pi}{5}$

Solution. Note that there is an odd power on sine, so there is a technique that is guaranteed to work. Write $\sin^3(x)\cos^2(x) = \sin x(1-\cos^2 x)(\cos^2 x)$ and then do a u-substitution $u = \cos x$. Thus we have

$$\int_0^{\pi/2} \sin^3(x)\cos^2(x)\,dx = \int_0^{\pi/2} \sin x(1-\cos^2 x)(\cos^2 x)\,dx$$

If $u = \cos(x)\,dx$, then $du = -\sin(x)\,dx$, and thus

$$\int_0^{\pi/2} \sin x(1-\cos^2 x)(\cos^2 x)\,dx = \int_1^0 (-du)(1-u^2)u^2 = \int_0^1 u^2 - u^4\,du = \frac{1}{3} - \frac{1}{5} = \frac{5}{15} - \frac{3}{15} = \frac{2}{15}.$$

16/2. (3 points) Compute

$$\int_0^{\pi/2} \cos^3(x)\sin^2(x)\,dx.$$

A. ★ $\frac{2}{15}$

B. 0

C. $\frac{7}{15}$

D. $\frac{\pi}{5}$

E. $\frac{2\pi}{5}$

Solution. Note that there is an odd power on cosine, so there is a technique that is guaranteed to work. Write $\cos^3(x)\sin^2(x) = \cos x(1-\sin^2 x)(\sin^2 x)$ and then do a u-substitution $u = \sin x$. Thus we have

$$\int_0^{\pi/2} \cos^3(x)\sin^2(x)\,dx = \int_0^{\pi/2} \cos x(1-\sin^2 x)(\sin^2 x)\,dx$$

If $u = \sin(x)\,dx$, then $du = \cos(x)\,dx$, and thus

$$\int_0^{\pi/2} \cos x(1-\sin^2 x)(\sin^2 x)\,dx = \int_0^1 (du)(1-u^2)u^2 = \int_0^1 u^2 - u^4\,du = \frac{1}{3} - \frac{1}{5} = \frac{5}{15} - \frac{3}{15} = \frac{2}{15}.$$

17/1. (3 points) Compute

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx.$$

A. ★ $\pi/2$

B. $\pi/4$

C. π

D. 2π

E. $3\pi/4$

Solution. The best thing here is a trig sub. We write $x = \sin\theta$ and $dx = \cos\theta\, d\theta$ to obtain

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\cos\theta} = \int_0^{\pi/2} d\theta = \frac{\pi}{2}.$$

17/2. (3 points) Compute

$$\int_0^1 \sqrt{1-x^2}\,dx.$$

A. ★ $\pi/4$

B. $\pi/2$

C. π

D. 2π

E. $3\pi/4$

Solution. The obvious thing here is a trig sub. We write $x = \sin\theta$ and $dx = \cos\theta\, d\theta$ to obtain

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1}{2}(1+\cos(2\theta))\, d\theta = \frac{\pi}{4}.$$

An easier way to get this is to observe (as we did in lecture) that this function is the top half of a circle and this integral represents one quarter of the area of a circle of radius 1, i.e. $\pi/4$.

18/1. (3 points) Compute
$$\int_0^1 \frac{1}{(x+3)(x+5)}\,dx.$$

A. ★ $\ln(10/9)/2$

B. $\ln(9/10)/2$

C. $\ln(5/3)$

D. $\ln(24/15)/2$

E. $\ln(24/15)$

Solution. We use the method of partial fractions. Writing
$$\frac{1}{(x+3)(x+5)} = \frac{A}{x+3} + \frac{B}{x+5},$$
we can solve to obtain $A = 1/2, B = -1/2$, so that
$$\begin{aligned}\int_0^1 \frac{1}{(x+3)(x+5)}\,dx = \int_0^1 \left(\frac{1}{2}\frac{1}{x+3} - \frac{1}{2}\frac{1}{x+5}\right)dx &= \frac{1}{2}(\ln|x+3| - \ln|x+5|)\Big|_{x=0}^{x=1} \\ &= \frac{1}{2}(\ln 4 - \ln 6 - \ln 3 + \ln 5) \\ &= \frac{1}{2}\ln(20/18) = \frac{1}{2}\ln(10/9).\end{aligned}$$

18/2. (3 points) Compute
$$\int_0^1 \frac{1}{(x+5)(x+7)}\,dx.$$

A. ★ $\ln(21/20)/2$

B. $\ln(20/21)/2$

C. $\ln(7/5)$

D. $\ln(29/15)/2$

E. $\ln(29/15)$

Solution. We use the method of partial fractions. Writing
$$\frac{1}{(x+5)(x+7)} = \frac{A}{x+5} + \frac{B}{x+7},$$
we can solve to obtain $A = 1/2, B = -1/2$, so that
$$\begin{aligned}\int_0^1 \frac{1}{(x+5)(x+7)}\,dx = \int_0^1 \left(\frac{1}{2}\frac{1}{x+5} - \frac{1}{2}\frac{1}{x+7}\right)dx &= \frac{1}{2}(\ln|x+5| - \ln|x+7|)\Big|_{x=0}^{x=1} \\ &= \frac{1}{2}(\ln 6 - \ln 8 - \ln 5 + \ln 7) \\ &= \frac{1}{2}\ln(42/40) = \frac{1}{2}\ln(21/20).\end{aligned}$$

18/3. (3 points) Compute
$$\int_0^1 \frac{1}{(x+4)(x+6)}\,dx.$$

A. ★ $\ln(15/14)/2$

B. $\ln(14/15)/2$

C. $\ln(3/2)$

D. $\ln(25/12)/2$

E. $\ln(25/12)$

Solution. We use the method of partial fractions. Writing
$$\frac{1}{(x+4)(x+6)} = \frac{A}{x+4} + \frac{B}{x+6},$$
we can solve to obtain $A = 1/2, B = -1/2$, so that
$$\begin{aligned}\int_0^1 \frac{1}{(x+4)(x+6)}\,dx = \int_0^1 \left(\frac{1}{2}\frac{1}{x+4} - \frac{1}{2}\frac{1}{x+6}\right)dx &= \frac{1}{2}(\ln|x+4| - \ln|x+6|)\Big|_{x=0}^{x=1} \\ &= \frac{1}{2}(\ln 5 - \ln 7 - \ln 4 + \ln 6) \\ &= \frac{1}{2}\ln(30/28) = \frac{1}{2}\ln(15/14).\end{aligned}$$

19/1. (3 points) A farmer wants to fence in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. She has a length L of fence available. What is the maximum total area that she can fence in?

A. ★ $L^2/24$

B. $L^2/16$

C. $L^2/4$

D. $L^2/12$

E. $L^2/32$

Solution. Let us denote the east-west width of the field by x and the north-south height by y. Let us also imagine that the subdivision is a north-south fence. Then the total amount of fence used is $2x + 3y$ and this must be equal to L. (Clearly, if we use less than L, we can increase the area by increasing either dimension, the optimal solution must use all of the fence.)

The area is

$$A(x) = xy = x\left(\frac{L-2x}{3}\right) = \frac{L}{3}x - \frac{2}{3}x^2.$$

The domain of this function is $[0, L/2]$, and clearly $A(0) = A(L/2) = 0$ so these minimize area. The maximum must be at a critical point in the interior. We compute

$$A'(x) = \frac{L}{3} - \frac{4}{3}x,$$

which is zero when $x = L/4$. And we have $A(L/4) = (L/4)(L/6) = L^2/24$.

19/2. (3 points) A farmer wants to fence in a rectangular field and then divide it in three pieces with two parallel fences, both parallel to one of the sides of the rectangle. She has a length L of fence available. What is the maximum total area that she can fence in?

A. ★ $L^2/32$

B. $L^2/16$

C. $L^2/4$

D. $L^2/12$

E. $L^2/24$

Solution. Let us denote the east-west width of the field by x and the north-south height by y. Let us also imagine that the subdivisions are north-south fences. Then the total amount of fence used is $2x + 4y$ and this must be equal to L. (Clearly, if we use less than L, we can increase the area by increasing either dimension, the optimal solution must use all of the fence.)

The area is

$$A(x) = xy = x\left(\frac{L-2x}{4}\right) = \frac{L}{4}x - \frac{1}{2}x^2.$$

The domain of this function is $[0, L/2]$, and clearly $A(0) = A(L/2) = 0$ so these minimize area. The maximum must be at a critical point in the interior. We compute

$$A'(x) = \frac{L}{4} - x,$$

which is zero when $x = L/4$. And we have $A(L/4) = (L/4)(L/8) = L^2/32$.

20/1. (3 points) Compute

$$L = \lim_{x\to 0} \frac{\displaystyle\int_0^x e^{2-3t^2}\,dt}{\sin(2x)e^{\cos(x)}}$$

A. ★ $\dfrac{e}{2}$

B. e^2

C. $\dfrac{e^{-3}}{2}$

D. $2e$

E. 2

Solution. If we just try to plug in $x = 0$, we obtain the indeterminate $0/0$, so we need to do something else. But l'Hôpital's Rule can work for us.

We have

$$\lim_{x\to 0} \frac{\displaystyle\int_0^x e^{2-3t^2}\,dt}{\sin(2x)e^{\cos(x)}} = \lim_{x\to 0} \frac{\dfrac{d}{dx}\displaystyle\int_0^x e^{2-3t^2}\,dt}{\dfrac{d}{dx}\left(\sin(2x)e^{\cos(x)}\right)} = \lim_{x\to 0} \frac{e^{2-3x^2}}{2\cos(2x)e^{\cos(x)} - \sin(2x)\sin(x)e^{\cos(x)}}.$$

But we can just plug in $x = 0$ to numerator and denominator now, and obtain $e^2/(2e) = e/2$.

20/2. (3 points) Compute

$$L = \lim_{x\to 0} \frac{\displaystyle\int_0^x e^{3-4t^2}\,dt}{\sin(2x)e^{\cos(x)}}$$

A. ★ $\dfrac{e^2}{2}$

B. e^3

C. $\dfrac{e^{-4}}{2}$

D. $3e$

E. 3

Solution. If we just try to plug in $x = 0$, we obtain the indeterminate $0/0$, so we need to do something else. But l'Hôpital's Rule can work for us.

We have

$$\lim_{x\to 0} \frac{\displaystyle\int_0^x e^{3-4t^2}\,dt}{\sin(2x)e^{\cos(x)}} = \lim_{x\to 0} \frac{\dfrac{d}{dx}\displaystyle\int_0^x e^{3-4t^2}\,dt}{\dfrac{d}{dx}\left(\sin(2x)e^{\cos(x)}\right)} = \lim_{x\to 0} \frac{e^{3-4x^2}}{2\cos(2x)e^{\cos(x)} - \sin(2x)\sin(x)e^{\cos(x)}}.$$

But we can just plug in $x = 0$ to numerator and denominator now, and obtain $e^3/(2e) = e^2/2$.

Zone 3

21/1. (3 points) Compute

$$\int_1^\infty \frac{dx}{x^4}.$$

A. ★ 1/3

B. 1/4

C. 1/2

D. 1/5

E. diverges

Solution. We can say it converges by the p-test for integrals, but this wouldn't give us the value. We compute directly:

$$\lim_{t\to\infty}\int_1^t x^{-4}\,dx = \lim_{t\to\infty}\left(-\frac{1}{3}x^{-3}\right)\Big|_1^t = \lim_{t\to\infty}\frac{1}{3}\left(1-\frac{1}{t^3}\right) = \frac{1}{3}.$$

21/2. (3 points) Compute

$$\int_1^\infty \frac{dx}{x^5}.$$

A. ★ 1/4

B. 1/5

C. 1/3

D. 1/6

E. diverges

Solution. We can say it converges by the p-test for integrals, but this wouldn't give us the value. We compute directly:

$$\lim_{t\to\infty}\int_1^t x^{-5}\,dx = \lim_{t\to\infty}\left(-\frac{1}{4}x^{-4}\right)\Big|_1^t = \lim_{t\to\infty}\frac{1}{4}\left(1-\frac{1}{t^4}\right) = \frac{1}{4}.$$

21/3. (3 points) Compute

$$\int_1^\infty \frac{dx}{x^6}.$$

A. ★ 1/5

B. 1/6

C. 1/4

D. 1/7

E. diverges

Solution. We can say it converges by the p-test for integrals, but this wouldn't give us the value. We compute directly:

$$\lim_{t\to\infty} \int_1^t x^{-6}\, dx = \lim_{t\to\infty} \left(-\frac{1}{5}x^{-5}\right)\Bigg|_1^t = \lim_{t\to\infty} \frac{1}{5}\left(1 - \frac{1}{t^5}\right) = \frac{1}{5}.$$

22/1. (3 points) Compute

$$\int_1^\infty \frac{dx}{\sqrt[4]{x}}$$

A. ★ diverges

B. 1/4

C. 1/2

D. 1/5

E. 1/3

Solution. We can say it diverges by the p-test for integrals, since the power $p = 1/4 < 1$. We could also compute directly:

$$\lim_{t\to\infty} \int_1^t x^{-1/4}\,dx = \lim_{t\to\infty} \left(\frac{4}{3}x^{3/4}\right)\Big|_1^t = \infty.$$

22/2. (3 points) Compute

$$\int_1^\infty \frac{dx}{\sqrt[5]{x}}$$

A. ★ diverges

B. 1/5

C. 1/3

D. 1/6

E. 1/4

Solution. We can say it diverges by the p-test for integrals, since the power $p = 1/5 < 1$. We could also compute directly:

$$\lim_{t\to\infty} \int_1^t x^{-1/5}\,dx = \lim_{t\to\infty} \left(\frac{5}{4}x^{4/5}\right)\Big|_1^t = \infty.$$

22/3. (3 points) Compute

$$\int_1^\infty \frac{dx}{\sqrt[6]{x}}$$

A. ★ diverges

B. 1/6

C. 1/4

D. 1/7

E. 1/5

Solution. We can say it diverges by the p-test for integrals, since the power $p = 1/6 < 1$. We could also compute directly:

$$\lim_{t\to\infty} \int_1^t x^{-1/6}\, dx = \lim_{t\to\infty} \left(\frac{6}{5} x^{5/6} \right) \Big|_1^t = \infty.$$

23/1. (3 points) Compute

$$\int_0^1 \frac{dx}{\sqrt[4]{x}}$$

A. ★ 4/3

B. 1/4

C. 3/4

D. 1/3

E. diverges

Solution. We compute:

$$\lim_{t\to 0+} \int_t^1 x^{-1/4}\,dx = \lim_{t\to 0+} \left(\frac{4}{3}x^{3/4}\right)\Big|_t^1 = \lim_{t\to 0+} \frac{4}{3}(1-t^{3/4}) = \frac{4}{3}.$$

23/2. (3 points) Compute

$$\int_0^1 \frac{dx}{\sqrt[5]{x}}$$

A. ★ 5/4

B. 1/5

C. 4/5

D. 1/4

E. diverges

Solution. We compute:

$$\lim_{t\to 0+} \int_t^1 x^{-1/5}\,dx = \lim_{t\to 0+} \left(\frac{5}{4}x^{4/5}\right)\Big|_t^1 = \lim_{t\to 0+} \frac{5}{4}(1-t^{4/5}) = \frac{5}{4}.$$

23/3. (3 points) Compute

$$\int_0^1 \frac{dx}{\sqrt[6]{x}}$$

A. ★ 6/5

B. 1/6

C. 5/6

D. 1/5

E. diverges

Solution. We compute:

$$\lim_{t\to 0+}\int_t^1 x^{-1/6}\,dx = \lim_{t\to 0+}\left(\frac{6}{5}x^{5/6}\right)\Big|_t^1 = \lim_{t\to 0+}\frac{6}{5}(1-t^{5/6}) = \frac{6}{5}.$$

24/1. (3 points) Compute

$$\int_0^1 \frac{dx}{x^4}$$

A. ★ diverges

B. 4/3

C. 3/4

D. 1/5

E. 1/4

Solution. We compute directly:

$$\lim_{t\to 0+} \int_t^1 x^{-4}\,dx = \lim_{t\to 0+} \left(-\frac{1}{3}x^{-3}\right)\Big|_1^t = \lim_{t\to 0+} \frac{1}{3}\left(1-\frac{1}{t^3}\right) = -\infty,$$

so the integral diverges.

24/2. (3 points) Compute

$$\int_0^1 \frac{dx}{x^5}$$

A. ★ diverges

B. 6/5

C. 5/6

D. 1/6

E. 1/5

Solution. We compute directly:

$$\lim_{t\to 0+} \int_t^1 x^{-5}\,dx = \lim_{t\to 0+} \left(-\frac{1}{4}x^{-4}\right)\Big|_1^t = \lim_{t\to 0+} \frac{1}{4}\left(1-\frac{1}{t^4}\right) = -\infty,$$

so the integral diverges.

24/3. (3 points) Compute

$$\int_0^1 \frac{dx}{x^6}$$

A. ★ diverges

B. 7/6

C. 6/7

D. 1/7

E. 1/6

Solution. We compute directly:

$$\lim_{t\to 0+}\int_t^1 x^{-6}\,dx = \lim_{t\to 0+}\left(-\frac{1}{5}x^{-5}\right)\Big|_1^t = \lim_{t\to 0+}\frac{1}{5}\left(1-\frac{1}{t^5}\right) = -\infty,$$

so the integral diverges.

25/1. (3 points) Consider the area lying under the curve $y = 3x - x^2$ and above the x-axis. Rotate this curve around the x-axis, and compute the volume of the resulting shape.

A. ★ $81\pi/10$

B. $16\pi/15$

C. $9\pi/2$

D. $3\pi/10$

E. $\pi/3$

Solution. Note that the function $f(x) = 3x - x^2$ intersects the x-axis in the points $x = 0, 3$. This will be the domain of integration. If we rotate the region around the x-axis, each slice will be a disc with radius $f(x)$, so will have area $\pi(f(x))^2$, so we compute

$$\int_0^3 \pi(3x - x^2)^2\,dx = \pi \int_0^3 9x^2 - 6x^3 + x^4\,dx = \pi\left(3x^3 - \frac{3x^4}{2} + \frac{x^5}{5}\right)\Bigg|_0^3 = \pi\left(81 - \frac{243}{2} + \frac{243}{5}\right) = \frac{81\pi}{10}.$$

25/2. (3 points) Consider the area lying under the curve $y = 2x - x^2$ and above the x-axis. Rotate this curve around the x-axis, and compute the volume of the resulting shape.

A. ★ $16\pi/15$

B. $81\pi/10$

C. $4\pi/3$

D. $3\pi/2$

E. $\pi/3$

Solution. Note that the function $f(x) = 2x - x^2$ intersects the x-axis in the points $x = 0, 2$. This will be the domain of integration. If we rotate the region around the x-axis, each slice will be a disc with radius $f(x)$, so will have area $\pi(f(x))^2$, so we compute

$$\int_0^2 \pi(2x - x^2)^2\,dx = \pi \int_0^2 4x^2 - 4x^3 + x^4\,dx = \pi\left(\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right)\Bigg|_0^2 = \pi\left(\frac{32}{3} - 16 + \frac{32}{5}\right) = \frac{16\pi}{15}.$$

26/1. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n}.$$

A. ★ The series converges conditionally.

B. The series converges absolutely.

C. The series diverges.

D. Not enough information is given to make a determination.

E. The series converges to $\pi^2/6$.

Solution. The answer is that the series converges conditionally. If we write $a_n = (-1)^n/n$, then first note that $|a_n| = 1/n$, and the sum of $1/n$ diverges by the p-test for series. Therefore it does not converge absolutely. However, since a_n is decreasing, and $a_n \to 0$ as $n \to \infty$, the series does converge conditionally.

26/2. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}.$$

A. ★ The series converges conditionally.

B. The series converges absolutely.

C. The series diverges.

D. Not enough information is given to make a determination.

E. The series converges to $\pi^2/6$.

Solution. The answer is that the series converges conditionally. If we write $a_n = (-1)^n/(2n)$, then first note that $|a_n| = 1/(2n)$, and the sum of $1/(2n)$ diverges by the p-test for series. Therefore it does not converge absolutely. However, since a_n is decreasing, and $a_n \to 0$ as $n \to \infty$, the series does converge conditionally.

26/3. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}.$$

A. ★ The series converges conditionally.

B. The series converges absolutely.

C. The series diverges.

D. Not enough information is given to make a determination.

E. The series converges to $\pi^2/6$.

Solution. The answer is that the series converges conditionally. If we write $a_n = (-1)^n/(3n)$, then first note that $|a_n| = 1/(3n)$, and the sum of $1/(3n)$ diverges by the p-test for series. Therefore it does not converge absolutely. However, since a_n is decreasing, and $a_n \to 0$ as $n \to \infty$, the series does converge conditionally.

27/1. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

A. ★ The series converges absolutely.

B. The series converges conditionally.

C. The series diverges.

D. Not enough information is given to make a determination.

E. The series converges to $\pi^2/6$.

Solution. The answer is that the series converges absolutely. If we write $a_n = (-1)^n/n^2$, then first note that $|a_n| = 1/n^2$, and the sum of $1/n^2$ converges by the p-test for series. However, it does not converge to $\pi^2/6$.

27/2. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}.$$

A. ★ The series converges absolutely.

B. The series converges conditionally.

C. The series diverges.

D. Not enough information is given to make a determination.

E. The series converges to $\pi^3/17$.

Solution. The answer is that the series converges absolutely. If we write $a_n = (-1)^n/n^3$, then first note that $|a_n| = 1/n^3$, and the sum of $1/n^3$ converges by the p-test for series.

27/3. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}.$$

A. ★ The series converges absolutely.

B. The series converges conditionally.

C. The series diverges.

D. Not enough information is given to make a determination.

E. The series converges to $\pi^4/90$.

Solution. The answer is that the series converges absolutely. If we write $a_n = (-1)^n/n^4$, then first note that $|a_n| = 1/n^4$, and the sum of $1/n^4$ converges by the p-test for series.

28/1. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^6}{2^n}.$$

A. ★ The series converges absolutely.

B. The series converges conditionally.

C. The series diverges.

D. Not enough information is given to make a determination.

E. The series converges to 1/5.

Solution. The answer is that the series converges absolutely. If we consider the ratio

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^6 2^n}{n^6 2^{n+1}} = \frac{1}{2}\frac{(n+1)^6}{n^6},$$

and as $n \to \infty$ this limit is 1/2.

28/2. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^5}{3^n}.$$

A. ★ The series converges absolutely.

B. The series converges conditionally.

C. The series diverges.

D. Not enough information is given to make a determination.

E. The series converges to 1/4.

Solution. The answer is that the series converges absolutely. If we consider the ratio

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^5 3^n}{n^5 3^{n+1}} = \frac{1}{3}\frac{(n+1)^5}{n^5},$$

and as $n \to \infty$ this limit is 1/3.

28/3. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{4^n}{n^7}.$$

A. ★ The series diverges.

B. The series converges conditionally.

C. The series converges absolutely.

D. Not enough information is given to make a determination.

E. The series converges to 1/5.

Solution. The answer is that the series diverges. If we consider the ratio

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^7 4^{n+1}}{(n+1)^7 4^n} = 4\frac{n^7}{(n+1)^7},$$

and as $n \to \infty$ this limit is 4. By the Ratio Test, this series diverges.

29/1. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{2n+4}{5n+7}\right)^n.$$

A. ★ The series converges absolutely.

B. The series converges conditionally.

C. The series diverges.

D. Not enough information is given to make a determination.

E. The series converges to 2/5.

Solution. The answer is that the series converges absolutely. If we consider the nth root

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{2n+4}{5n+7}=\frac{2}{5}<1.$$

This converges absolutely by the Root Test.

29/2. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{5n+1}{2n+9}\right)^n.$$

A. ★ The series diverges.

B. The series converges conditionally.

C. The series converges absolutely.

D. Not enough information is given to make a determination.

E. The series converges to 5/2.

Solution. The answer is that the series diverges. If we consider the nth root

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{5n+1}{2n+9}=\frac{5}{2}>1.$$

This diverges by the Root Test.

29/3. (3 points) Determine which of the following is true about the series

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{6n+4}{15n+7}\right)^n.$$

A. ★ The series converges absolutely.

B. The series converges conditionally.

C. The series diverges.

D. Not enough information is given to make a determination.

E. The series converges to 2/5.

Solution. The answer is that the series converges absolutely. If we consider the nth root

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{6n+4}{15n+7} = \frac{6}{15} < 1.$$

This converges absolutely by the Root Test.

30/1. (3 points) Consider the sequences

$$a_n = \frac{2n+6}{3n+7},$$

and the series $\sum_{n=1}^{\infty} a_n$. Which of the following is true?

A. ★ The sequence converges and the series diverges.

B. The sequence diverges and the series converges.

C. Both the sequence and the series converge.

D. Both the sequence and the series diverge.

E. None of the others are true.

Solution. We can compute directly that

$$\lim_{n\to\infty} \frac{2n+6}{3n+7} = \frac{2}{3}$$

by looking at the leading-order terms. So the sequence converges. However, since the sequence converges to a nonzero number, this implies that the series diverges.

30/2. (3 points) Consider the sequences

$$a_n = \frac{4n^2+6}{3n^2+7},$$

and the series $\sum_{n=1}^{\infty} a_n$. Which of the following is true?

A. ★ The sequence converges and the series diverges.

B. The sequence diverges and the series converges.

C. Both the sequence and the series converge.

D. Both the sequence and the series diverge.

E. None of the others are true.

Solution. We can compute directly that

$$\lim_{n\to\infty} \frac{4n^2+6}{3n^2+7} = \frac{4}{3}$$

by looking at the leading-order terms. So the sequence converges. However, since the sequence converges to a nonzero number, this implies that the series diverges.

Zone 4

31/1. (3 points) Compute R, the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n n^6 (x-2)^n}{4^{n+5}}.$$

A. ★ $R = 4$

B. $R = \infty$

C. $R = 5$

D. $R = 2$

E. $R = 6$

Solution. If we write

$$a_n = \frac{(-1)^n n^6 (x-2)^n}{4^{n+5}},$$

then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{n}\right)^6 \frac{|x-2|}{4}.$$

Thus

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|}{4}.$$

By the Ratio Test, this converges if $|x-2| < 4$ and diverges if $|x-2| > 4$, so the radius of convergence is $R = 4$.

31/2. (3 points) Compute R, the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n n^8 (x-3)^n}{5^{n+7}}.$$

A. ★ $R = 5$

B. $R = \infty$

C. $R = 7$

D. $R = 3$

E. $R = 8$

Solution. If we write

$$a_n = \frac{(-1)^n n^8 (x-3)^n}{5^{n+7}},$$

then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{n}\right)^8 \frac{|x-3|}{5}.$$

Thus

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-3|}{5}.$$

By the Ratio Test, this converges if $|x-3| < 5$ and diverges if $|x-3| > 5$, so the radius of convergence is $R = 5$.

31/3. (3 points) Compute R, the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n n^2 (x-4)^n}{3^{n+9}}.$$

A. ★ $R = 3$

B. $R = \infty$

C. $R = 9$

D. $R = 4$

E. $R = 2$

Solution. If we write

$$a_n = \frac{(-1)^n n^2 (x-4)^n}{3^{n+9}},$$

then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left(\frac{n+1}{n} \right)^2 \frac{|x-4|}{3}.$$

Thus

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-4|}{3}.$$

By the Ratio Test, this converges if $|x-4| < 3$ and diverges if $|x-4| > 3$, so the radius of convergence is $R = 3$.

32/1. (3 points) Compute the domain of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n n^9 (x-1)^n}{n!}.$$

A. ★ $(-\infty, \infty)$

B. $(0, 2)$

C. $[0, 2]$

D. $(-8, 10)$

E. $(-8, 10]$

Solution. If we write

$$a_n = \frac{(-1)^n n^9 (x-1)^n}{n!},$$

then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{n}\right)^9 \frac{|x-1|}{n+1}.$$

Thus

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0,$$

and the sum converges for all x.

32/2. (3 points) Compute the domain of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n n^6 (x-2)^n}{n!}.$$

A. ★ $(-\infty, \infty)$

B. $(1, 3)$

C. $[1, 3]$

D. $(-4, 8)$

E. $(-4, 8]$

Solution. If we write

$$a_n = \frac{(-1)^n n^6 (x-2)^n}{n!},$$

then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{n}\right)^6 \frac{|x-2|}{n+1}.$$

Thus

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0,$$

and the sum converges for all x.

33/1. (2 points) Recall the cycloid generated by a circle of radius 2 has parametric description

$$x = 2(t - \sin t), \quad y = 2(1 - \cos t).$$

The "first leaf" of this cycloid is the curve traced out on the domain $t \in [0, 2\pi]$. Find the area underneath this leaf.

A. ★ 12π

B. 4π

C. 8π

D. 2π

E. 10π

Solution. Recall that we write the area under the curve as

$$\int_{x(0)}^{x(2\pi)} y\,dx = \int_0^{2\pi} 2(1 - \cos t) \cdot 2(1 - \cos t)\,dt = \int_0^{2\pi} 4(1 - \cos t)^2\,dt.$$

To do this integral, we use the trig identity

$$\cos^2 t = \frac{1}{2}(1 + \cos 2t),$$

and thus

$$\int_0^{2\pi} 4(1 - \cos t)^2\,dt = \int_0^{2\pi} (4 - 8\cos t + 2(1 + \cos 2t))\,dt = \int_0^{2\pi} (6 - 8\cos t + 2\cos 2t)\,dt.$$

Since $\cos t$ and $\cos 2t$ both have zero integrals over the interval $[0, 2\pi]$, this integral is 12π.

33/2. (2 points) Recall the cycloid generated by a circle of radius 3 has parametric description

$$x = 3(t - \sin t), \quad y = 3(1 - \cos t).$$

The "first leaf" of this cycloid is the curve traced out on the domain $t \in [0, 2\pi]$. Find the area underneath this leaf.

A. ★ 27π

B. 9π

C. 18π

D. 3π

E. 15π

Solution. Recall that we write the area under the curve as

$$\int_{x(0)}^{x(2\pi)} y\,dx = \int_0^{2\pi} 3(1-\cos t)\cdot 3(1-\cos t)\,dt = \int_0^{2\pi} 9(1-\cos t)^2\,dt.$$

To do this integral, we use the trig identity

$$\cos^2 t = \frac{1}{2}(1+\cos 2t),$$

and thus

$$\int_0^{2\pi} 9(1-\cos t)^2\,dt = \int_0^{2\pi} \left(9 - 8\cos t + \frac{9}{2}(1+\cos 2t)\right)\,dt = \int_0^{2\pi} \left(\frac{27}{2} - 8\cos t + 2\cos 2t\right)\,dt.$$

Since $\cos t$ and $\cos 2t$ both have zero integrals over the interval $[0, 2\pi]$, this integral is 27π.

34/1. (2 points) Recall the cardioid in polar coordinates can be written

$$r = 1 + \sin\theta, \quad \theta \in [0, 2\pi].$$

Which of the following integrals represents the arc length of this curve?

A. ★ $\int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$

B. $\int_0^{2\pi} \sqrt{1 + \sin\theta}\, d\theta$

C. $\int_0^{2\pi} \sqrt{2 - \cos\theta}\, d\theta$

D. $\int_0^{2\pi} \sqrt{1 + 2\cos\theta}\, d\theta$

E. $\int_0^{2\pi} \sqrt{2\theta - \cos\theta}\, d\theta$

Solution. The formula for arc length of a curve is

$$\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

In this case,

$$r = 1 + \sin\theta, \quad \frac{dr}{d\theta} = \cos\theta,$$

so

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 + \sin\theta)^2 + \cos^2\theta = 1 + 2\sin\theta + \sin^2\theta + \cos^2\theta = 2 + 2\sin\theta.$$

(Note that this example is also in the book on p. 668.)
