Partial fractions requires some facts about factoring polynomials that you may not remember, so let me remind you

Definition 0.1. The *order* of a polynomial is the degree of the highest (non-zero) power. A polynomial of n^{th} degree always has exactly n roots according to multiplicity.

We have to be a little bit careful with the counting here. We have to count by the number of times that $x - x_0$ divides the polynomial. For instance the cubic polynomial $P_3(x) = x^3 - x^2 - x + 1$ has three roots *counted according to multiplicity.* $x = -1$ *is a simple root (multiplicity 1) and* $x = 1$ *is a double root (multiplicity 2) since* $x^3 - x^2 - x + 1 = (x-1)^2(x+1)$.

Polynomials will generally have complex roots. However for a real polynomial the complex roots come in complex conjugate pairs. If x_0 *is a root then* $\bar{x}_0$ *is also a root. It is often convenient to factor off an* irreducible *quadratic factor.*

Fact *if a polynomial* $P(x)$ *has a complex conjugate pair of roots* $x = a \pm ib$ *then the quadratic* $(x-a)^2 + b^2$ *divides* $P(x)$

Throughout this section when integrating a rational function $\int \frac{P(x)}{Q(x)} dx$ *we assume that the degree of the numerator is less than the degree of the denominator, and that the denominator* $Q(x)$ *can be factored as a product of real linear and irreducible quadratic factors. Then we give the form of the partial fractions expansion for each factor.*

Type of Root	*Factor*	*Expansion*
Real, Simple	$(x - x_0)$	$\frac{A}{x-x_0}$
Real, Multiplicity k	$(x-x_0)^k$	$\frac{A_1}{x-x_0} + \frac{A_2}{(x-x_0)^2} + \ldots + \frac{A_k}{(x-x_0)^k}$
Complex, Simple	$((x-a)^2 + b^2)$	$\frac{Ax+B}{(x-a)^2+b^2}$
Complex, Multiplicity k	$((x-a)^2+b^2)^k$	$\frac{A_1x+B_1}{(x-a)^2+b^2} + \frac{A_2x+B_2}{((x-a)^2+b^2)^2} + \ldots + \frac{A_kx+B_k}{((x-a)^2+b^2)^k}$

Lets do a couple of examples:

Example 0.2. Find the partial fractions expansion of

$$f(x) = \frac{3x^2+4x+5}{x^4 - 2x^3 + 2x^2 - 2x + 1}$$

The denominator factors as $(x^4 - 2x^3 + 2x^2 - 2x + 1) = (x^2+1)(x-1)^2$, so consulting the chart above the form of the expansion ought to be

$$\frac{3x^2+4x+5}{x^4 - 2x^3 + 2x^2 - 2x + 1} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

If we multiply through by the denominator and expand we find that

$$3x^2+4x+5 = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1) = (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + (B-C+D)$$

collecting and equating powers of x we get four equations for the four unknowns A, B, C, D

$$\begin{aligned}
&x^3 \qquad && A + C = 0 \\
&x^2 && -2A + B - C + D = 3 \\
&x^1 && A - 2B + C = 4 \\
&x^0 && B - C + D = 5
\end{aligned}$$

If we solve these equations we find that

$$\begin{aligned}
A &= 1 \\
B &= -2 \\
C &= -1 \\
D &= 6.
\end{aligned}$$

Example 0.3. Give the form of the partial fractions expansion for $f(x) = \frac{(1+x+x^2)}{(x-1)^5(((x-1)^2+4)^3}$

The numerator is of lower degree than the denominator, so we don't have to divide. The form of the expansion would be

$$f(x) = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3} + \frac{A_4}{(x-1)^4} + \frac{A_5}{(x-1)^5} + \frac{B_1 x + C_1}{((x-1)^2+4)} + \frac{B_2 x + C_2}{((x-1)^2+4)^2} + \frac{B_3 x + C_4}{((x-1)^2+4)^3}$$

Notice that the denominator has degree 11, and that there are 11 coefficients to be determined.

The actual solution is

$$\begin{aligned}
A_1 &= \frac{3}{512} \\
A_2 &= -\frac{9}{256} \\
A_3 &= -\frac{5}{256} \\
A_4 &= \frac{3}{64} \\
A_5 &= \frac{3}{64} \\
B_1 &= -\frac{3}{512} \\
B_2 &= -\frac{1}{256} \\
B_3 &= \frac{1}{64} \\
C_1 &= \frac{21}{512} \\
C_2 &= \frac{25}{256} \\
C_3 &= \frac{11}{64}
\end{aligned}$$

but I cheated and used a computer algebra system to do this.