JadeWatsonComputationalAssignment

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0.1 MTHE 472: Computational Assignment

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Define Parameters

[1]:
```
# B = fading channel in bad state and G = channel in good state
X = ['B','G']
# u = 0 not to use the channel and u = 1 to use channel
U = ['0','1']

# define transition function
def T(x_next,x,u):
    if x_next == 'G' and u == '1' and x == 'G':
        prob  = 0.3
        return prob
    elif x_next == 'G' and u == '0' and x == 'G':
        prob = 0.7
        return prob
    elif x_next == 'G' and u == '1' and x == 'B':
        prob = 0.5
        return prob
    elif x_next == 'G' and u == '0' and x == 'B':
        prob = 0.9
        return prob
    elif x_next == 'B' and u == '1' and x == 'G':
        prob = 0.7
        return prob
    elif x_next == 'B' and u == '0' and x == 'G':
        prob = 0.3
        return prob
    elif x_next == 'B' and u == '1' and x == 'B':
        prob = 0.5
        return prob
    elif x_next == 'B' and u == '0' and x == 'B':
        prob = 0.1
        return prob

# cost function
```

```
def C(x,u):
    if x == 'G' and u == '1':
        c = -1 + e*(float(u))
        return c
    else:
        c = e*(float(u))
        return c
```

Problem 1(i): Value Iteration

[2]:
```
def value_iteration(X,U,T,C):
    """
    :param list X: set of states
    :param list U: encoder use
    :param function P: transition function
    :param function C: cost function
    """
    # set value function for each state x to zero
    # initialize value function V as a dictionary to track the state as a key
    V = {x: 0 for x in X}
    optimal_policy = {x: 0 for x in X}

    while True:
        # store value functions at previous iteration
        prev_V = V.copy()
        for x in X:
            # Q(x,u) expected total cost for taking action u in state x
            Q = {}
            # calculate Q for each encoder u to update V(s)
            for u in U:
                Q[u] = C(x,u) + beta*sum(T(x_next,x,u)*prev_V[x_next] for x_next␣
↪in X)
            # V(x): minimum expected total cost starting from state x (value␣
↪function)
            V[x] = min(Q.values())
            optimal_policy[x] = min(Q, key=Q.get)
        # check if the value function is the same by comparing it to prev value
        if all(prev_V[x] == V[x] for x in X):
            break
    return V, optimal_policy
```

[3]:
```
eta = [2/3, 0.95, 0.05]
beta = 0.8
for e in eta:
    V, optimal_policy = value_iteration(X,U,T,C)
    print("The optimal solution with eta value ",e,"is: ",V)
    print("The corresponding policy is: ",optimal_policy)
```

```
The optimal solution with eta value  0.6666666666666666 is:  {'B':
-0.810810810810811, 'G': -1.0360360360360361}
The corresponding policy is:  {'B': '0', 'G': '1'}
The optimal solution with eta value  0.95 is:  {'B': -0.12162162162162174, 'G':
-0.15540540540540554}
The corresponding policy is:  {'B': '0', 'G': '1'}
The optimal solution with eta value  0.05 is:  {'B': -2.31081081081081, 'G':
-2.9527027027027017}
The corresponding policy is:  {'B': '0', 'G': '1'}
```

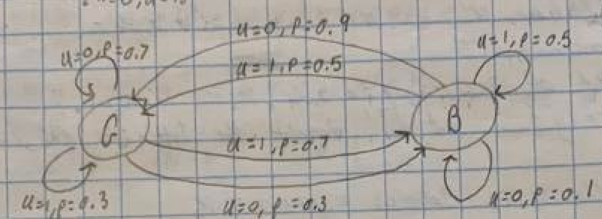
MTHE 472 - Computational Assignment

1\. i) $\mathbb{X} = \{B, G\}$ $\quad \mathbb{U} = \{0, 1\}$

G = channel in good state, $\quad 0$ = channel not used

B = channel in bad state, $\quad 1$ = channel used

$$C(x,u) = -1_{\{x=G,\,u=1\}} + \eta u \quad \text{for } \eta \in \mathbb{R}$$



Consider $\beta \in (0,1)$,

$$\inf_{\gamma} E_x^{\gamma}\left[\sum_{t=0}^{\infty} \beta^t c(x_t, u_t)\right]$$

Use value iteration (rate cost is bounded)

$$V(G) = \min_u \left\{ c(G,u) + \beta\left(V(G) P(x_{t+1} = G \mid x_t = G, u) + V(B) P(x_{t+1} = B \mid x_t = G, u)\right)\right\}$$

$$= \min_u \Big\{ c(G,0) + \beta\left(V(G)P(x_{t+1} = G \mid x_t = G, u=0) + V(B)P(x_{t+1} = B \mid x_t = G, u=0)\right),$$

$$c(G,1) + \beta\left(V(G)P(x_{t+1} = G \mid x_t = G, u=1) + V(B)P(x_{t+1} = B \mid x_t = G, u=1)\right)\Big\}$$

$$= \min_u \left\{ \beta\left(V(G)(0.7) + V(B)(0.3)\right),\ \eta - 1 + \beta\left(V(G)(0.3) + V(B)(0.7)\right)\right\}$$

$$V(B) = \min_u \left\{ c(B,u) + \beta\left(V(G) P(x_{t+1} = G \mid x_t = B, u) + V(B) P(x_{t+1} = B \mid x_t = B, u)\right)\right\}$$

$$= \min_u \Big\{ c(B,0) + \beta\left(V(G)P(x_{t+1} = G \mid x_t = B, u=0) + V(B)P(x_{t+1} = B \mid x_t = B, u=0)\right),$$

$$c(B,1) + \beta\left(V(G)P(x_{t+1} = G \mid x_t = B, u=1) + V(B)P(x_{t+1} = B \mid x_t = B, u=1)\right)\Big\}$$

$$= \min_u \left\{ \beta\left(V(G)(0.9) + V(B)(0.1)\right),\ \eta + \beta\left(V(G)(0.5) + V(B)(0.5)\right)\right\}$$

Problem 1(ii): Policy Iteration

[4]:
```
#  in policy value, want to update the value function at a policy until they are␣
↪the same
def policy_value(policy,X):
    # initialize value function as a dictionary
    V = {x: 0 for x in X}
    while True:
        # stores previous value to be used for termination condition
        prev_V = V.copy()
        # update value for each state x
        for x in X:
            u = policy[x]
            V[x] = C(x,u) + beta*sum(T(x_next,x,u)*prev_V[x_next] for x_next in␣
↪X)
        # termination condition: if previous value is the same as current
        if all(prev_V[x] == V[x] for x in X):
            break
    return V
```

[5]:
```
# find the best policy for given value function
def best_policy(V,X,U):
    policy = {x: U[0] for x in X}

    for x in X:
        Q = {}
        for u in U:
            Q[u] = C(x,u) + beta*sum(T(x_next,x,u)*V[x_next] for x_next in X)

            policy[x] = min(Q,key=Q.get)
    return policy
```

[6]:
```
def policy_iteration(X,U):
    """
    :param list X: set of states
    :param list U: encoder use
    :param function P: transition function
    :param function C: cost function
    """
    # initialize policy function, policy, as a dictionary by setting its value␣
↪to the first element of encoder set U
    policy = {x: U[0] for x in X}

    while True:
        # store policy at previous iteration
        prev_policy = policy.copy()
        # calculate value function under current policy
        V = policy_value(policy,X)
```

```python
        # update the policy
        policy = best_policy(V,X,U)
        # termination condition: does the policy remain the same
        if all(prev_policy[x] == policy[x] for x in X):
            break
    return V,policy
```

[7]:
```python
eta = [2/3, 0.95, 0.05]
beta = 0.8
for e in eta:
    V,optimal_policy = policy_iteration(X,U)
    print("The optimal policy with eta value ", e,"is: ",optimal_policy)
    print("The optimal solution with eta value ",e,"is: ",V)
```

```
The optimal policy with eta value  0.6666666666666666 is:  {'B': '0', 'G': '1'}
The optimal solution with eta value  0.6666666666666666 is:  {'B':
-0.810810810810811, 'G': -1.0360360360360361}
The optimal policy with eta value  0.95 is:  {'B': '0', 'G': '1'}
The optimal solution with eta value  0.95 is:  {'B': -0.12162162162162174, 'G':
-0.15540540540540554}
The optimal policy with eta value  0.05 is:  {'B': '0', 'G': '1'}
The optimal solution with eta value  0.05 is:  {'B': -2.31081081081081, 'G':
-2.9527027027027017}
```

1. ii) let $\gamma_0 = \begin{array}{cc} 'G' & 'B' \\ \end{array} \begin{bmatrix} 0 & 0 \end{bmatrix}$

$$Z_0(x) = E_x^{\gamma_0}\left[\sum_{t=0}^{\infty} \beta^t c(x_t, \gamma_0(x_t))\right]$$

$Z_0(G) = Z_0(B) = 0 \quad \forall x \in \{G, B\}$

$$\pi(Z_0)(B) = \min_{u \in \{0,1\}} \{c(B,u) + \beta(Z_0(x=G)P(x_{t+1}=G|x_t=B, u_t=u) + Z_0(x=B)P(x_{t+1}=B|x_t=B, u_t=u))\}$$

$\hookrightarrow u=1$ achieves min @ $x=G$

$$\pi(Z_0)(G) = c(G,1) + \beta(Z_0(x=G)P(x_{t+1}=G|x_t=G, u_t=1) + Z_0(x=B)P(x_{t+1}=B|x_t=G, u_t=1))$$

$$= (\eta-1) + \beta(0 \cdot 0.3 + 0 \cdot 0.7)$$

$$= \eta - 1$$

$$\pi(Z_0)(B) = \min_{u \in \{0,1\}} \{c(B,u) + \beta(Z_0(x=G)P(x_{t+1}=G|x_t=B, u_t=u) + Z_0(x=B)P(x_{t+1}=B|x_t=B, u_t=u))\}$$

$\hookrightarrow u=0$ achieves min @ $x=B$

$\pi(Z_0)(B) = 0$

let $\gamma_1 = [1 \quad 0]$

$$Z_1(G) = c(G,1) + \beta(Z_1(G)P(x_{t+1}=G|x_t=G, u_t=1) + Z_1(B)P(x_{t+1}=B|x_t=G, u_t=1))$$

$$= (\eta-1) + \beta(Z_1(G)(0.3) + Z_1(B)(0.7))$$

$$Z_1(B) = c(B,0) + \beta(Z_1(G)P(x_{t+1}=G|x_t=B, u_t=0) + Z_1(B)P(x_{t+1}=B|x_t=B, u_t=0))$$

$$= 0 + \beta(Z_1(G)(0.9) + Z_1(B)(0.1))$$

Note $\gamma_1 = \gamma^*$ since thats where the policy iteration ends.

$\therefore, \gamma^* = \begin{bmatrix} 1 & 0 \end{bmatrix} \Rightarrow$
$\quad\quad\quad 'G' \; 'B'$

Problem 1 (iii): Q Learning

[8]:
```
import numpy as np
import random

beta = 0.8
eta = 2/3

# state is 0 <=> state is G
# state is 1 <=> state is B

# initializing Q matrix
Q = np.array([[1,1],[1,1]])

# initializing states and action space
T = 10 # shape of np array and total time
states = np.empty(T,int)
actions = np.empty(T,int)
# set all to -1
for i in range(T):
    states[i] = -1
    actions[i] = -1

# learning rate function where t is the time, x is the state, u is the action
def alpha(t,x,u):
    summation = 0
    for i in range(0,t):
        if states[i] == x and actions[i] == u:
            summation = summation + 1
    return 1/(1+summation)

# re-define cost function such that is uses numbers instead of strings
def C(x,u):
    indicator = 0
    if x == 0 and u == 1:
        indicator = 1
    return eta*u-indicator

# choose an action using the epsilon-greedy exploration strategy
# each time you need to choose an action, randomly generate a value␣
↪(probability). If the value has prob less than an epsilon, choose a random␣
↪action. Otherwise, take the best known action at the agents current state
def choose_action(x,u):
    transition_var = random.uniform(0,1) # generates random number between 0 and␣
↪1
    if x == 0 and u == 0:
        if transition_var < 0.7:
            # explore random action
```

```
            x_next = 0
            return x_next
        else:
            # select the action with max value (future reward)
            x_next = 1
            return x_next
    elif x == 0 and u == 1:
        if transition_var < 0.3:
            x_next = 0
            return x_next
        else:
            x_next = 1
            return x_next
    elif x ==1 and u ==0:
        if transition_var < 0.1:
            x_next = 1
            return x_next
        else:
            x_next = 0
            return x_next
    elif x == 1 and u == 1:
        if transition_var < 0.5:
            x_next = 1
            return x_next
        else:
            x_next = 0
            return x_next
    else:
        return "error"
```

[9]:

```
x_t = 0
u_t = 0
j = 0

states[0] = x_t
actions[0] = u_t

# Q-learning function
while j < T-1:
    # choose action u from X using epsilon-greedy policy derived from Q
    x_next = choose_action(x_t,u_t)
    minArg = 100.0
    if Q[x_next,0] > Q[x_next,1]:
        u_next = 1
        minArg = Q[x_next,1]
    else:
        u_next = 0
```

```
        minArg = Q[x_next,0]

    # Take action U, then observe cost C and next state X
    Q[states[x_t],actions[u_t]] = Q[states[x_t],actions[u_t]] +alpha(j,x_t,u_t)␣
↪* (C(x_t,u_t)+beta*minArg-Q[states[x_t],actions[u_t]])

    # repeat until time limit is reached and update values
    j = j+1
    x_t = x_next
    u_t = u_next
    states[j] = x_t
    actions[j] = u_t

# Print result of Q-table containing Q(X,U) pairs defining optimal policy
print("Q = ", Q)
print("states = ",states)
print("actions = ",actions)
```

```
Q =  [[0 1]
 [0 1]]
states =  [0 1 0 0 0 0 0 1 0 1]
actions =  [0 0 0 0 0 0 0 0 0 0]
```

Upon comparing the three methods above (value iteration, policy iteration, and Q learning), each method outputs a result that further proves the validity of the next method. I.e. all results coincide with each other. For example, value iteration with eta 2/3 outputs {'B': -0.810810810810811, 'G': -1.0360360360360361} The corresponding policy is: {'B': '0', 'G': '1'}. The result of the policy iteration matches these values exactly. Q-learning further emphasizes the optimal policy to be [0,1].

Problem 1(b)

[10]:
```
# use convex analytical method to get expected cost
beta = 0.8
eta = 2/3

# initialize starting values
u_G = 0
u_B = 0
min_cost = 0
optimalPolicy = [0,0]

# check handwritten notes for derivation of the numerator and denominator
for u_G in range(2):
    for u_B in range(2):
        numerator = eta*(0.5*u_G+0.4*(1-u_B)*u_G+0.7*u_B-0.4*(1-u_G)*u_B) - (0.
↪5*u_G+0.4*(1-u_B)*u_G)
        denominator = 1.2 - 0.4*(1-u_G)+0.4*(1-u_B)
        expected_cost = numerator / denominator
```

```python
        # check if we have reached minimum cost and take policy at such cost
        if expected_cost < min_cost:
            min_cost = expected_cost
            optimalPolicy[0] = u_G
            optimalPolicy[1] = u_B

print("The minimum expected cost is ", min_cost)
print("The optimal policy is ",optimalPolicy)
```

```
The minimum expected cost is  -0.18750000000000003
The optimal policy is  [1, 0]
```

1.b) Consider

$$\inf_{\gamma} \lim_{T\to\infty} \frac{1}{T} E_x^{\gamma}\left[\sum_{t=0}^{T-1} c(x_t, u_t)\right]$$

consider a deterministic policy $\gamma = [u^0 \; u^1]$

$$T = \begin{bmatrix} 0.3\,\mathbb{1}_{\{u^G=1\}} + 0.7\,\mathbb{1}_{\{u^G=0\}} & 0.7\,\mathbb{1}_{\{u^G=1\}} + 0.3\,\mathbb{1}_{\{u^G=0\}} \\ 0.5\,\mathbb{1}_{\{u^B=1\}} + 0.9\,\mathbb{1}_{\{u^B=0\}} & 0.5\,\mathbb{1}_{\{u^B=1\}} + 0.1\,\mathbb{1}_{\{u^B=0\}} \end{bmatrix}$$

$$= \begin{bmatrix} 0.3 + 0.4(1-u^G) & 0.7 - 0.4(1-u^G) \\ 0.5 + 0.4(1-u^B) & 0.5 - 0.4(1-u^B) \end{bmatrix}$$

$$[\pi_G \;\; \pi_B] = [\pi_G \;\; \pi_B] \cdot \begin{bmatrix} 0.3 + 0.4(1-u^G) & 0.7 - 0.4(1-u^G) \\ 0.5 + 0.4(1-u^B) & 0.5 - 0.4(1-u^B) \end{bmatrix}$$

$$\pi_G = \pi_G(0.3 + 0.4(1-u^G)) + \pi_B(0.5 + 0.4(1-u^B))$$

$$\pi_G - \pi_G(0.3 + 0.4(1-u^G)) = \pi_B(0.5 + 0.4(1-u^B))$$

$$\pi_G(1 - 0.3 - 0.4(1-u^G)) = \pi_B(0.5 + 0.4(1-u^B))$$

① $$\pi_G = \pi_B\left(\frac{0.5 + 0.4(1-u^B)}{0.7 - 0.4(1-u^G)}\right)$$

② $\pi_G + \pi_B = 1$

$$\pi_G = (1-\pi_G)\left(\frac{0.5 + 0.4(1-u^B)}{0.7 - 0.4(1-u^G)}\right)$$

$$\pi_G = \left(\frac{0.5 + 0.4(1-u^B)}{0.7 - 0.4(1-u^G)}\right) - \pi_G\left(\frac{0.5 + 0.4(1-u^B)}{0.7 - 0.4(1-u^G)}\right)$$

$$\pi_G\left(1 + \frac{0.5 + 0.4(1-u^B)}{0.7 - 0.4(1-u^G)}\right) = \frac{0.5 + 0.4(1-u^B)}{0.7 - 0.4(1-u^G)}$$

$$\pi_G\left(\frac{0.7 - 0.4(1-u^G) + 0.5 + 0.4(1-u^B)}{0.7 - 0.4(1-u^G)}\right) = \frac{0.5 + 0.4(1-u^B)}{0.7 - 0.4(1-u^G)}$$

$$\pi_G\left(\frac{1.2 - 0.4(1-u^G) + 0.4(1-u^B)}{0.7 - 0.4(1-u^G)}\right) = \frac{0.5 + 0.4(1-u^B)}{0.7 - 0.4(1-u^G)}$$

$$\pi_G = \left(\frac{0.5 + 0.4(1-u^B)}{0.7 - 0.4(1-u^G)}\right)\left(\frac{0.7 - 0.4(1-u^G)}{1.2 - 0.4(1-u^G) + 0.4(1-u^B)}\right)$$

$$\therefore,\; \pi_G = \frac{0.5 + 0.4(1-u^B)}{1.2 - 0.4(1-u^G) + 0.4(1-u^B)}$$

$$
\begin{aligned}
\pi_B &= 1 - \pi_G \\
&= 1 - \frac{0.5 + 0.4(1-u^B)}{1.2 - 0.4(1-u^G) + 0.4(1-u^B)} \\
&= \frac{1.2 - 0.4(1-u^G) + 0.4(1-u^B) - 0.5 - 0.4(1-u^B)}{1.2 - 0.4(1-u^G) + 0.4(1-u^B)}
\end{aligned}
$$

$$
\therefore \pi_B = \frac{0.7 - 0.4(1-u^G)}{1.2 - 0.4(1-u^G) + 0.4(1-u^B)}
$$

Now calculate the expected cost

$$
\begin{aligned}
&= \pi_G(c(G, u^G)) + \pi_B(c(B, u^B)) \\
&= \pi_G(\eta u^G - u^G) + \pi_B(\eta u^B) \\
&= \pi_G u^G(\eta - 1) + \pi_B \eta u^B \\
&= \left(\frac{0.5 + 0.4(1-u^B)}{1.2 - 0.4(1-u^G) + 0.4(1-u^B)}\right)(u^G)(\eta - 1) \\
&\quad + \left(\frac{0.7 - 0.4(1-u^G)}{1.2 - 0.4(1-u^G) + 0.4(1-u^B)}\right)(u^B)(\eta) \\
&= \frac{(0.5 + 0.4(1-u^B))(u^G)(\eta - 1) + (0.7 - 0.4(1-u^G))(u^B)(\eta)}{1.2 - 0.4(1-u^G) + 0.4(1-u^B)} \\
&= \frac{(\eta)(0.5u^G + 0.4(1-u^B)(u^G) + 0.7u^B - 0.4(1-u^G)(u^B)) - (0.5u^G + 0.4(1-u^B)u^G)}{1.2 - 0.4(1-u^G) + 0.4(1-u^B)}
\end{aligned}
$$

Recall $u \in \{0, 1\}$ $\therefore$ the only possible policy γ exist in $\{[0\ 0], [0\ 1], [1\ 0], [1\ 1]\}$

Thus, can evaluate the average cost under each gamma to find optimal γ^*, where

$$
\inf_{\gamma} \limsup_{T \to \infty} \frac{1}{T} E_x^{\gamma}\left[\sum_{t=0}^{T-1} c(x_t, u_t)\right] = \limsup_{T \to \infty} \frac{1}{T} E^{\gamma^*}\left[\sum_{t=0}^{T-1} c(x_t, u_t)\right]
$$

See python code for exact results.

$\therefore$ $\gamma^* = [1\ 0]$ and the expected minimum cost is -0.1875

Problem 2(a)

2.a) $x_{t+1} = Ax_t + w_t$ where $A = \begin{bmatrix} 0.75 & 1 & 0 \\ 0 & 0.75 & 1 \\ 0 & 0 & 0.75 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}$

$y_t = Cx_t + v_t$

where $\left.\begin{aligned} w_t &\sim N(0, I) \\ v_t &\sim N(0,1) \end{aligned}\right\}$ noise

By our lecture notes,

$$m_{t+1} = Am_t + A\Sigma_{t|t-1} C^T \left(C \Sigma_{t|t-1} C^T + V\right)^{-1} \left(y_t - Cm_t\right)$$

and

$$\Sigma_{t+1|t} = A \Sigma_{t|t-1} A^T + W - \left(A \Sigma_{t|t-1} C^T\right)\left(C\Sigma_{t|t-1} C^T + V\right)^{-1}\left(C\Sigma_{t|t-1} A^T\right)$$

where $m_t = E[X_t | y_{[0,t-1]}]$ is our belief variable and

$$\Sigma_{t|t-1} = E\left[(x_t - E[x_t | y_{[0,t-1]}])(x_t - E[x_t | y_{[0,t-1]}])^T\right]$$

is the estimation error covariance

$$\therefore \Sigma_{0|-1} = E[x_0 x_0^T] = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad m_0 = E[x_0]$$

Riccati Recursions:

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B \underbrace{(B^T P_{t+1} B + R)^{-1} B^T P A}_{\to 0}$$

In our case $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ => last term = 0

$$\therefore P_t = CC^T + A^T P_{t+1} A$$

Problem 2(b)

[11]:
```python
import numpy as np
from numpy.linalg import matrix_rank

# define given matrices and noise
T = 1000
A = np.matrix([[0.75,1,0],[0,0.75,1],[0,0,0.75]])
C = np.matrix([2,0,0])
I = np.matrix([[1,0,0],[0,1,0],[0,0,1]])
W = I # noise
V = 1 # noise

# initialize and set up array for inputs and outputs
sigma_t = I
x_t = np.matrix.transpose(np.matrix([1,0,0]))
m_t = x_t
results = np.zeros(T)
states = np.zeros((T,3,1))
belief_variables = np.zeros((T,3,1))
error_covariances = np.zeros((T,3,3))

# initialize states, beliefs, and error
x_states = np.zeros(T)
y_states = np.zeros(T)
z_states = np.zeros(T)

x_beliefs = np.zeros(T)
y_beliefs = np.zeros(T)
z_beliefs = np.zeros(T)

x_error = np.zeros(T)
y_error = np.zeros(T)
z_error = np.zeros(T)

# check observability
observability_matrix = [[2,0,0],[1.5,2,0],[9/8,3,2]]
rank = np.linalg.matrix_rank(observability_matrix)
print("Rank of observability matrix is: ",rank)
```

```
Rank of observability matrix is:  3
```

[12]:
```python
# Refer to handwritten results from a for defined equations

import random

t = 0

while t < T:
```

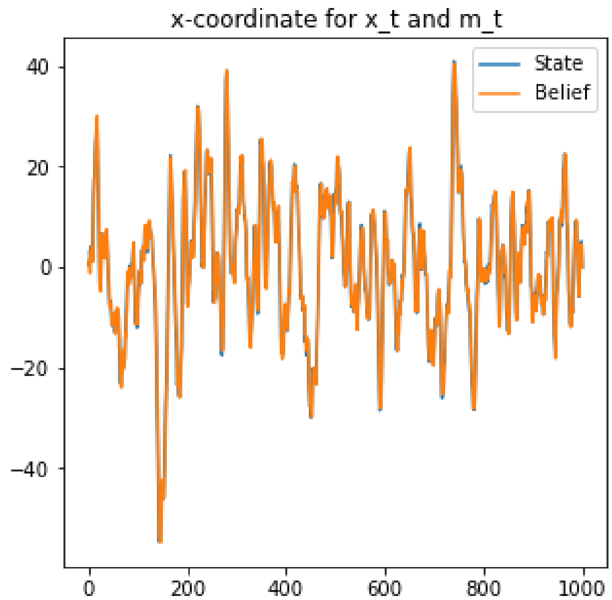
```
    # Noise initialization
    v_t = np.random.normal(0,1,1)
    w_t = np.matrix.transpose(np.matrix(np.random.normal(0,1,3)))
    # update system
    y_t = np.matmul(C,x_t) + np.random.normal(0,1,1)
    x_next = np.matmul(A,x_t) + w_t

    # update error covariance
    sigma_next = W + np.matmul(A,np.matmul(sigma_t,np.matrix.transpose(A))) -
 ↪(np.matmul(A,np.matmul(sigma_t,np.matrix.transpose(C))))*np.linalg.inv((np.
 ↪matmul(C,np.matmul(sigma_t,np.matrix.transpose(C)))+V))*(np.matmul(C,np.
 ↪matmul(sigma_t,np.matrix.transpose(A))))

    # update beliefs
    # m_tilda_t
    m_next = np.matmul(A,m_t)+np.matmul(np.matmul(sigma_next,np.matrix.
 ↪transpose(C)),(y_t-np.matmul(C,np.matmul(A,m_t))))*np.linalg.inv(np.matmul(np.
 ↪matmul(C,sigma_next),np.matrix.transpose(C))+v_t)

    # Update output matrices
    results[t] = y_t
    states[t,:,:] = x_t
    belief_variables[t,:,:] = m_t
    error_covariances[t,:,:] = sigma_t

    #Update Recursion
    sigma_t = sigma_next
    m_t = m_next
    x_t = x_next
    t = t+1
```

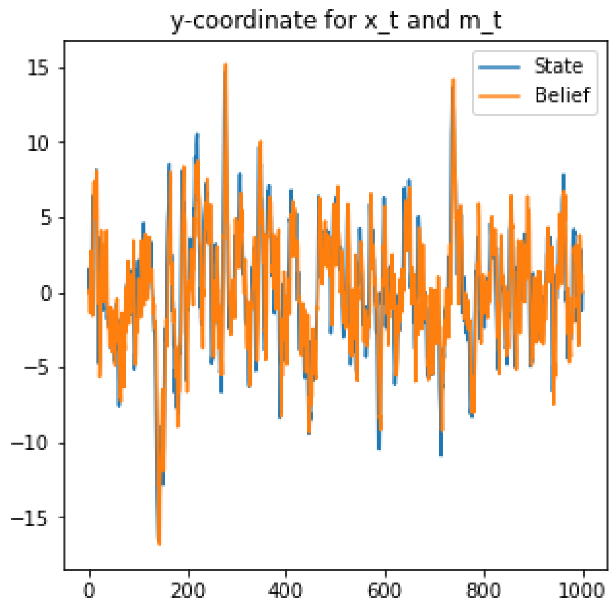
[13]:
```
# Update states, beliefs, and estimation error
for j in range(T-1):
    x_states[j] = states[j,0,0]
    y_states[j] = states[j,1,0]
    z_states[j] = states[j,2,0]

    x_beliefs[j] = belief_variables[j,0,0]
    y_beliefs[j] = belief_variables[j,1,0]
    z_beliefs[j] = belief_variables[j,2,0]

    x_error[j] = x_states[j] - x_beliefs[j]
    y_error[j] = y_states[j] - y_beliefs[j]
    z_error[j] = z_states[j] - z_beliefs[j]
```

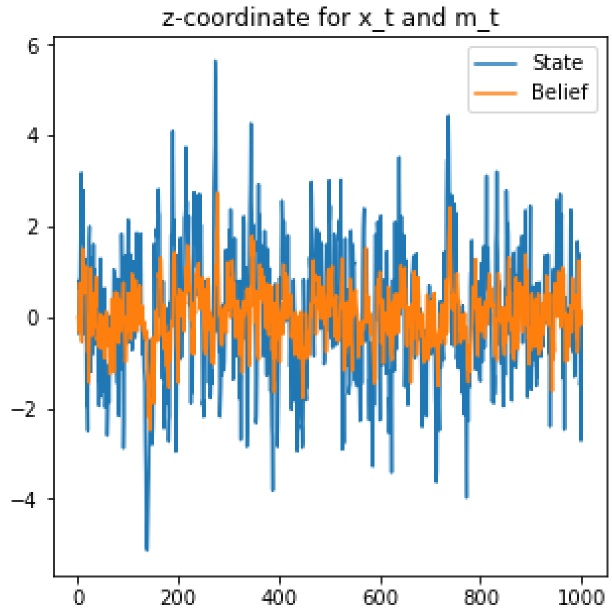
[14]:
```
from matplotlib import pyplot as plt
```

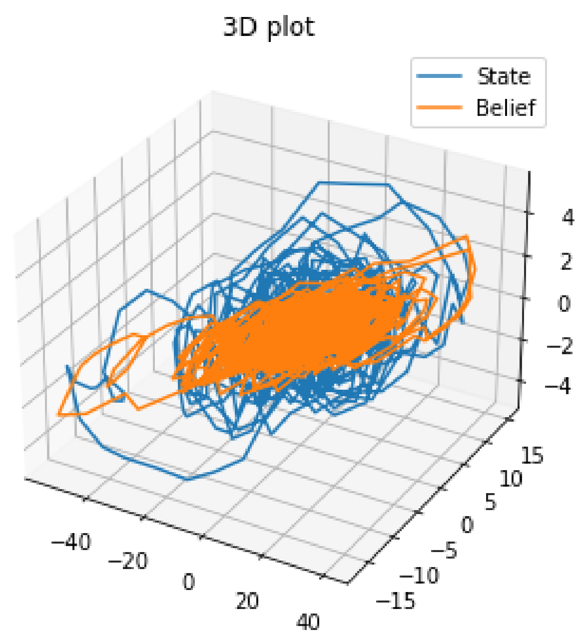
```
# plot x-coordinate of x_t, m_t
figure1 = plt.figure(figsize=(5,5))
x = np.linspace(1,1000,T)
ax1 = figure1.add_subplot(111)
ax1.plot(x,x_states)
ax1.plot(x,x_beliefs)
ax1.legend(['State','Belief'])
ax1.set_title("x-coordinate for x_t and m_t")

# plot y-coordinate of x_t, m_t
figure2 = plt.figure(figsize=(5,5))
y = np.linspace(1,1000,T)
ax2 = figure2.add_subplot(111)
ax2.plot(y,y_states)
ax2.plot(y,y_beliefs)
ax2.legend(['State','Belief'])
ax2.set_title("y-coordinate for x_t and m_t")

# plot z-coordinate of x_t, m_t
figure3 = plt.figure(figsize=(5,5))
z = np.linspace(1,1000,T)
ax3 = figure3.add_subplot(111)
ax3.plot(z,z_states)
ax3.plot(z,z_beliefs)
ax3.legend(['State','Belief'])
ax3.set_title("z-coordinate for x_t and m_t")

# 3D of all coordinates
figure4 = plt.figure(figsize=(5,5))
ax4 = figure4.add_subplot(111, projection='3d')
ax4.plot(x_states,y_states,z_states)
ax4.plot(x_beliefs,y_beliefs,z_beliefs)
ax4.legend(['State','Belief'])
ax4.set_title("3D plot")

# x-coordinate error
figure5 = plt.figure(figsize=(5,5))
q = np.linspace(0,1000,T)
ax5 = figure5.add_subplot(111)
ax5.plot(q,x_error)
ax5.set_title("x-coordinate error")

# y-coordinate error
figure6 = plt.figure(figsize=(5,5))
ax6 = figure6.add_subplot(111)
ax6.plot(q,y_error)
ax6.set_title("y-coordinate error")
```

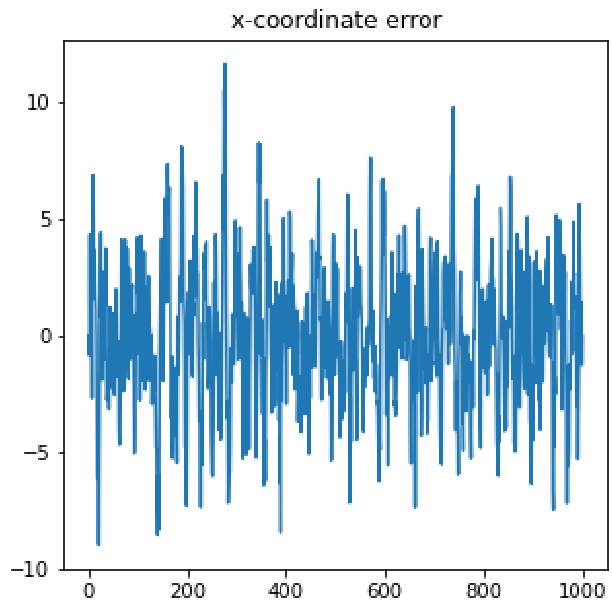
```
# z-coordinate error
figure7 = plt.figure(figsize=(5,5))
ax7 = figure7.add_subplot(111)
ax7.plot(q,z_error)
ax7.set_title("z-coordinate error")

plt.show()
```

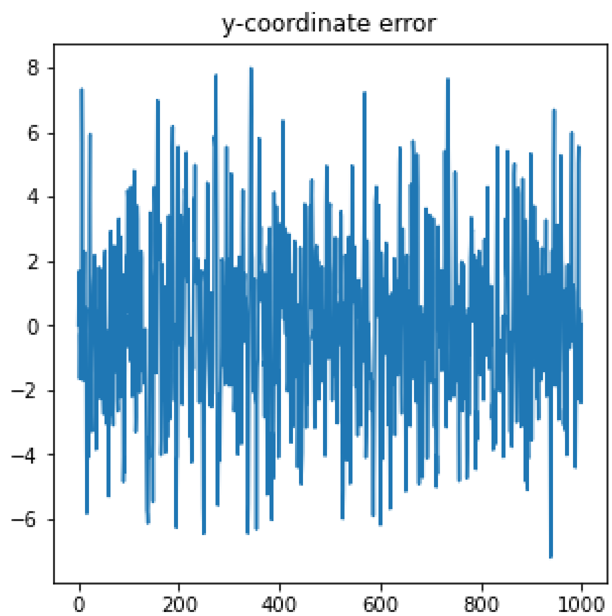


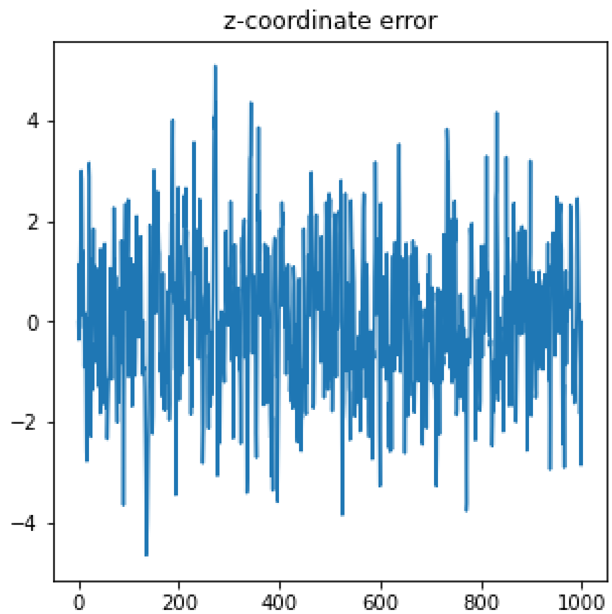



z-coordinate for x_t and m_t
State
Belief
6
4
2
0
−2
−4
0
200
400
600
800
1000


3D plot
State
Belief
4
2
0
−2
−4
15
10
5
0
−5
−10
−15
−40
−20
0
20
40


x-coordinate error
10
5
0
−5
−10
0
200
400
600
800
1000


y-coordinate error
8
6
4
2
0
−2
−4
−6
0
200
400
600
800
1000



Problem 2(c)

[15]:
```
from scipy.optimize import fsolve

def riccati(P):
    print(P.shape)
    print(A.shape)

    riccati_recursion = np.matmul(C,np.matrix.transpose(C)) + np.matmul(np.
 ↪matmul(np.matrix.transpose(A),P),A) - P
    return riccati_recursion

riccati_converge = fsolve(riccati, np.matrix([[1,1,1],[1,1,1],[1,1,1]]))
```

```
(9,)
(3, 3)
```

Does not stabilize around P because the Riccati conditions are not met as expected from the explanation in c.

[16]:
```
print("The 999th covariance matrix is ", error_covariances[999])
```

```
The 999th covariance matrix is  [[4.98145231 4.12780699 1.01482848]
 [4.12780699 6.8086493  2.11746902]
 [1.01482848 2.11746902 2.03260534]]
```

[17]:
```
print("The 60th covariance matrix is ",error_covariances[60])
```

```
The 60th covariance matrix is  [[4.98145231 4.12780699 1.01482848]
```

```
 [4.12780699 6.8086493  2.11746902]
 [1.01482848 2.11746902 2.03260534]]
```

[18]:
```python
print("The 123rd covariance matrix is ",error_covariances[123])
```

```
The 123rd covariance matrix is  [[4.98145231 4.12780699 1.01482848]
 [4.12780699 6.8086493  2.11746902]
 [1.01482848 2.11746902 2.03260534]]
```

Therefore, the sigma matrices recursion converge and is unique as seen in the matrices above. This is expected as explained in the handwritten solution c.

c) By thrm 5.2.2, if (A,B) is controllable, $\exists$ a solution the the Riccati eqns:

$$P = Q + A^T P A - A^T P B (B^T P B + R)^{-1} B^T P A$$

But $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow (A,B)$ not controllable

By thrm 6.2.2 if (A,C) observable and $V>0$, then the Σ matrices recursion converge, thus our Kalman filter solution exist. Since (A,B) not controllable, the fixed point solution is not necessarily unique

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1.5 & 2 & 0 \\ 9/8 & 3 & 2 \end{bmatrix} \quad \text{rank } 3 \Rightarrow \text{observable}$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} \vec{0} & \vec{0} & \vec{0} \end{bmatrix} \Rightarrow \text{not controllable}$$