Practice problems for the Normal Distribution and the Central Limit Theorem

Problem 1. Given a standard normal distribution, find the area under the curve which lies

(a) to the left of $z = 1.43$;

$\Phi(1.43) = 0.9236$

(b) to the right of $z = -0.89$;

$\Phi(0.89) = 0.8133$

(c) between $z = -2.16$ and $z = -0.65$;

$\Phi(2.16) - \Phi(0.65) = 0.2424$

(d) to the left of $z = -1.39$;

$1 - \Phi(1.39) = 0.0823$

(e) to the right of $z = 1.96$;

$1 - \Phi(1.96) = 0.025$

(f) between $z = -0.48$ and $z = 1.74$;

$\Phi(1.74) - 1 + \Phi(0.48) = 0.6435$

Problem 2. Given a standard normal distribution, find the value of k such that

(a) $P(Z < k) = 0.0427$;

$k = \Phi^{-1}(0.0427)$. This value is not available in a table with only positive values. Thus, $\Phi(k) = 0.027$ implies $\Phi(-k) = 0.9573$. Then $k = -\Phi^{-1}(0.9573) = -1.72$.

(b) $P(Z > k) = 0.2946$;

$k = \Phi^{-1}(0.7054) = 0.54$

(c) $P(-0.93 < Z < k) = 0.2946$.

$k = -\Phi^{-1}(0.5292) = -0.073$

Problem 3. Given the normally distributed variable X with mean 18 and standard deviation 2.5, find

(a) $P(X < 15)$;

$Pr(Z < -1.2) = 0.1151$

(b) the value of k such that $P(X < k) = 0.2236$;

$(k-18)/2.5 = \Phi^{-1}(0.2236) = -\Phi^{-1}(1-0.2236) = -0.76 \implies k = 16.1$

(c) the value of k such that $P(X > k) = 0.1814$;

$(k-18)/2.5 = \Phi^{-1}(1-0.1814) = 0.91 \implies k = 20.275$

(d) $P(17 < X < 21)$.

0.5403

Problem 4. A soft-drink machine is regulated so that it discharges an average of 200 milliliters per cup. If the amount of drink is normally distributed with a standard deviation equal to 15 milliliters,

(a) What fraction of the cups will contain more than 224 milliliters?

$Pr(X > 224) = Pr(Z > (224-200)/15) = 0.0548$

(b) What is the probability that a cup contains between 191 and 209 milliliters?

$Pr(191 < X < 209) = 0.4514$

(c) How many cups are expected to overflow if 230 milliliter cups are used for the next 1000 drinks?

$Pr(X > 230) = 0.0227$. The expected number of a binomial with this probability of success and $n = 1000$ is $22.7 \approx 23$ cups.

(d) Below what value do we get the smallest 25% of the drinks?

$Pr(X < k) = 0.25, \implies (k-200)/15 = -\Phi^{-1}(0.75) = -0.675, \implies k = 189.875$.

Problem 5. The random variable X, representing the number of cherries in a cherry puff, has the following probability distribution

x	4	5	6	7
P(X=x)	0.2	0.4	0.3	0.1

(a) Find the mean μ and the variance σ^2 of X.

$EX = 5.3$, $\text{var}X = 0.81$

(b) Find the mean $\mu_{\overline{x}}$ and the variance $\sigma^2_{\overline{x}}$ of the mean $\overline{X}$ for random samples of 36 cherry puffs.

$\mu_{\overline{x}} = 5.3$, $\sigma^2_{\overline{x}} = 0.81/36 = 0.0225$.

(c) Approximate the probability that the average number of cherries in 36 cherry puffs will be less than 5.5.

$Pr(\overline{X} < 5.5) = Pr(Z < (5.5 - 5.3)/0.15) = 0.9082$.

Problem 6. The average life of a bread-making machine is 7 years, with a standard deviation of 1 year. Assuming that the lives of these machines follow approximately a normal distribution, find:

(a) The probability that the mean life of a random sample of 9 such machines falls between 6.4 and 7.2 years;

$Pr(6.4 < \overline{X} < 7.2) = 0.6898$.

(b) The value x that is such that 15% of the means computed from random samples of size 9 would fall above x.

$Pr(\overline{X} > x) = 0.15, \implies x = 7.34667$.

Problem 7. An astronomer is interested in measuring, in light years the distance to a distant star. The values of the measurements are i.i.d. with a common mean d, and a common variance of light 2 years square. How many measurements does the astronomer need to make in order to estimate the distance to within $\pm.5$ light years with probability 95%?

$$Pr(|\overline{X}_n - d| < .5) = 0.95$$

then

$$0.95 = Pr(-.5 < \overline{X}_n - d < .5) = Pr\left(\frac{-.5\sqrt{n}}{2} < Z < \frac{.5\sqrt{n}}{2}\right) = 2\Phi(\sqrt{(n)}/4) - 1$$

Then $n \approx 61.47$, thus $n \geq 62$.