CSE 102 Spring 2021
Advanced Homework Assignment 2

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1 AdvHW2

5. Consider the recurrence relation: $T(1) = a$ **and** $T(n) = nT(n-1) + bn$ **for** $n \geq 2$**. Prove, by induction, that for sufficiently large integer n, there exists two positive real constants** P **and** Q **such that** $Pn! \leq T(n) \leq Qn!$**. Solve the recurrrence in Problem 4. You are allowed a term of the form** $\sum_{i=1}^{n} \frac{1}{i!}$ **in your solution. What is the value of** $\lim\limits_{n\to\infty} T(n)/n!$ **as a function of a and b?**

Proof. First we assume $a, b > 0$. To prove $Pn! \leq T(n) \leq Qn!$, we can equaivalently prove $T(n) = \Theta(n!)$. First, we compute the recurrence:

$$
\begin{aligned}
T(n) &= nT(n-1) + bn \\
&= n \cdot ((n-1)T(n-2) + b(n-1)) + bn \\
&= n \cdot ((n-1)((n-2)T(n-3) + b(n-2)) + b(n-1)) + bn \\
&\vdots \\
&= \frac{n!}{(n-k)!}T(n-k) + b(n + n(n-1) + n(n-1)(n-2) + \cdots + \frac{n!}{(n-k)!}) \\
&= \frac{n!}{(n-k)!}T(n-k) + bn!(\sum_{i=1}^{k} \frac{1}{(n-i)!})
\end{aligned}
$$

The recurrence stop when $n-k=1$, then we can subsitute $k=n-1$, then we get:

$$\begin{aligned}
T(n) &= \frac{n!}{(n-k)!}T(n-k) + bn!(\sum_{i=1}^{k}\frac{1}{(n-i)!}) \\
&= n!T(1) + bn!(\sum_{i=1}^{n-1}\frac{1}{(n-i)!}) \\
&= an! + bn!(\sum_{i=1}^{n-1}\frac{1}{(n-i)!}) \\
&= an! + bn!(\sum_{i=1}^{n-1}\frac{1}{i!}) \quad \text{Since it just reverses the sum order}
\end{aligned}$$

Let $f(n) = n!$, we compute the limit $\lim_{n\to\infty}\frac{T(n)}{f(n)}$:

$$\begin{aligned}
\lim_{n\to\infty}\frac{T(n)}{f(n)} &= \lim_{n\to\infty}\frac{n!T(1) + b(n + n(n-1) + n(n-1)(n-2) + \cdots + n!)}{n!} \\
&= T(1) + \lim_{n\to\infty} b(\frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \frac{1}{(n-3)!} + \cdot + 1) \\
&= T(1) + b\lim_{n\to\infty}(\sum_{i=1}^{n-1}\frac{1}{i!}) \\
&= T(1) + b(e-1) \\
&= a + b(e-1)
\end{aligned}$$

Since $0 < a + b(e-1) < \infty$, then $T(n) = \Theta(n!)$. Thus, there must exist P and Q such that $Pn! \leq T(n) \leq Qn!$. $\square$

References