Quiz 2

① Base case: $n=0$; $T(0) = 0 \le 0^2$. True — no points

Inductive Hypothesis; $T(n) \le n^2$ — 0 Point

To Prove : $T(n+1) \le (n+1)^2$

Proof:
$$\begin{aligned} T(n+1) &= T(n) + (n+1) \\ &\le n^2 + (n+1) && \text{By inductive Hypothesis} \quad 1 \text{ Point} \\ &\le n^2 + 2n + 1 && \text{because } n \le 2n \\ &= (n+1)^2 \end{aligned}$$
$\Big\}$ 1 Point

$\boxtimes$

variation: $T(n-1) \le (n-1)^2$ inductive Hypothesis is also ok

②
$$\begin{aligned} T(n) &= T(n-1) + 1 \\ &= T(n-2) + 1 + 1 \\ &= \cdots \\ &= T(n-k) + \underbrace{1+1+\cdots+1}_{k \text{ times}} \end{aligned}$$
$\longleftarrow$ 1 Point

choose $k=n$ then $= T(0) + n$
$= 2 + n$ $\boxtimes$ $\Big\}$ 1 Point

③ $T(n) = \sqrt{a}\, T\left(\frac{n}{a}\right) + \sqrt{n}$

$\log_b(a) = \log_a \sqrt{a} = \log_a a^{1/2} = \frac{1}{2}$ — 1 Point

$f(n) = \sqrt{n} = \Theta(n^{1/2}) \Rightarrow$ Case 2 $\Rightarrow T(n) = \Theta(\sqrt{n} \log n)$ — 1 Point

$\boxtimes$

④ $\sum_{i=1}^{n}(ia^i) = a + 2a^2 + 3a^3 + \dots + na^n$

$\geq a + 2a + 3a + \dots + na$ because $a^n > a$ for $a > 1$ ← 1 Point

$= a(1+2+\dots+n)$

$= a\,\dfrac{n(n+1)}{2}$

$= a\left(\dfrac{n^2+n}{2}\right)$ 1 Point

$\geq a\dfrac{n^2}{2}$

$\boxed{c = 1/2}$

A smaller value of c will also work so long $c > 0$

⑤ $\sum_{j=1}^{n}(ja^n) = \left(\sum_{j=1}^{n} j\right)a^n$

$= \dfrac{n(n+1)a^n}{2}$

$= \left(\dfrac{n^2+n}{2}\right)a^n$

Method 1: ignoring lower order terms, $= \dfrac{n^2a^n}{2} = \Theta(n^2a^n) \Rightarrow \boxed{k=2}$

Method 2: $\lim_{n\to\infty}\dfrac{\left(\frac{n^2+n}{2}\right)\cancel{a^n}}{n^k\cancel{a^n}} = \lim_{n\to\infty}\dfrac{n^2+n}{2(n^k)}$; $\boxed{\text{choose } k=2} \Rightarrow \lim_{n\to\infty}\dfrac{n^2+n}{2(n^2)} =$

Hence Θ.

<u>Any</u> solution that breaks the above series separately in erroneous examples below earn ZERO credit

$\sum_{i=1}^{n} ia^i = \left(\sum_{i=1}^{n} i\right)\cdot\left(\sum_{i=1}^{n} a^i\right)$ ← This is incorrect

$\sum_{j=1}^{n} ja^n = \left(\sum_{j=1}^{n} j\right)\left(\sum_{j=1}^{n} a^n\right)$ ← This is incorrect.

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$$T(n) = \begin{cases} 0 & \text{for } n=0 \\ 1 & \text{for } n=1 \\ T(n-2)+(n-2) & \text{for } n \geq 2 \end{cases}$$

a

$$\begin{aligned} T(n) &= T(n-2)+(n-2) \\ &= T(n-4)+(n-4)+(n-2) \\ &= \cdots \\ &= T(n-2k) + \underbrace{(n-2k)+\cdots+(n-2)}_{k \text{ terms}} \end{aligned}$$

1 Point
1 Point

b Let $n = 2k$;

$$\begin{aligned} T(n) &= T(0)+0+2+4\cdots+(2k-2) \\ &= 0+0+2(1+2+\cdots+(k-1)) \\ &= \cancel{2}\cdot\frac{(k-1)\cdot k}{\cancel{2}} \\ &= \frac{(k-1)\cdot k}{\cancel{2}} \\ &= \left(\frac{n}{2}-1\right)\left(\frac{n}{2}\right). \end{aligned}$$

1 Point
1 Point
1 Point