CSE 102 Spring 2021
Homework Assignment 1

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1 HW1

1. (Problem 3.1-1) Let $f(n)$ and $g(n)$ asymptotically positive functions. Prove that $f(n)+g(n) = \Theta(max(f(n), g(n)))$.

Proof. Let $h(n) = max(f(n), g(n))$, so now we need to prove $f(n)+g(n) = \Theta(h(n))$
Note that $h(n) \leq f(n)+g(n)$ and $h(n) \leq f(n)+g(n)$. Hence, $f(n)+g(n) = \Omega(h(n))$.
Note that $f(n)+g(n) \leq 2h(n)$ and $f(n)+g(n) \leq 2h(n)$. Hence, $f(n)+g(n) = O(h(n))$
Hence we get $f(n)+g(n) = \Theta(max(f(n), g(n)))$. □

2. Prove or disprove: If $f(n) = \Theta(g(n))$, then $f(n)^2 = \Theta(g(n)^2)$.

Proof. The statement is true. $f(n) = \Theta(g(n))$ states that $\exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$. For convenience, we can let $c_3 = c_1^2, c_4 = c_2^2$

$$\begin{aligned}
c_1 g(n) \leq& f(n) \leq c_2 g(n) \\
(c_1 g(n))^2 \leq& f(n)^2 \leq (c_2 g(n))^2 \\
c_1^2 g(n)^2 \leq& f(n)^2 \leq c_2^2 g(n)^2 \\
c_3 g(n)^2 \leq& f(n)^2 \leq c_4 g(n)^2
\end{aligned}$$

We can easily observe that $c_3, c_4 > 0$. Hence $\exists c_3 > 0, \exists c_4 > 0, \exists n_0 > 0, \forall n \geq n_0$, $0 \leq c_3 g(n) \leq f(n) \leq c_4 g(n)$, and so $f(n)^2 = \Theta(g(n)^2)$. □

3. Prove or disprove: If $f(n) = \Theta(g(n))$, then $2^{f(n)} = \Theta(2^{g(n)})$.

Solution. [1]This statement is false. Let $f(n) = 2n$ and $g(n) = n$. Since $0 \leq g(n) \leq f(n) \leq 3g(n)$, then $f(n) = \Theta(g(n))$.
However, $2^{f(n)} = 2^{2n} = 4^n$ and $2^{g(n)} = 2^n$. And we can observe that:

$$\begin{aligned} \lim_{n\to\infty} \frac{4^n}{2^n} &= \lim_{n\to\infty} 2^n \\ &= \infty \end{aligned}$$

This means that 4^n increases faster than 2^n, so we cannot find a constant c satisfy $c2^n \geq 4^n$ when $n \to \infty$, and $2^{f(n)} = \omega(2^{g(n)})$. Thus, $2^{f(n)} \neq \Theta(2^{g(n)})$ if $f(n) = \Theta(g(n))$. □

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions, and assume that $\lim_{n\to\infty} g(n) = \infty$. Prove that if $f(n)$ = $\Theta(g(n))$,then $\ln(f(n)) = \Theta(\ln(g(n)))$.

Proof. $f(n) = \Theta(g(n))$ states that $\exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0, 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$.
For convenience, we can let $c_3 = c_1^2, c_4 = c_2^2$

$$\begin{aligned} c_1 g(n) \leq f(n) \leq c_2 g(n) \\ \ln(c_1 g(n)) \leq \ln(f(n)) \leq \ln(c_2 g(n)) \\ \ln(c_1) + \ln(g(n)) \leq \ln(f(n)) \leq \ln(c_2) + \ln(g(n)) \end{aligned}$$

Since c_1 and n_0 are constants, there must exist a constant c' such that:[3]

$$\begin{aligned} c' &\geq \frac{\ln c_1}{\ln g(n_0)} + 1 \\ c' - 1 &\geq \frac{\ln c_1}{\ln g(n_0)} \\ (c' - 1) \ln g(n_0) &\geq \ln c_1 \\ c' \ln g(n_0) &\geq \ln c_1 + \ln g(n_0) \end{aligned}$$

Therefore, we have $\exists c', n_0, \forall n > n_0$,

$$c' \ln g(n) \geq \ln c_1 + \ln g(n) \geq \ln f(n)$$

so $\ln f(n) = O(\ln g(n))$
Similarly, there must exist a constant c'' such that:

$$c'' \leq \frac{\ln c_2}{\ln g(n_0)} + 1$$

and do the same thing, we can get $c'' \ln g(n_0) \leq \ln c_2 + \ln g(n_0)$
Therefore, we have $\exists c', n_0, \forall n > n_0$,

$$c'' \ln g(n_0) \leq \ln c_2 + \ln g(n_0) \leq \ln f(n)$$

so $\ln f(n) = \Omega(\ln g(n))$.
Hence $\ln f(n) = \Theta(\ln g(n))$. □

5. (Problem 3.2-8) Show that if $f(n)\ln f(n) = \Theta(n)$, then $f(n) = \Theta(n/\ln n)$. Hint: use the result of the preceding problem.

Proof. [4]Since $f(n)\ln f(n) = \Theta(n)$, we have:

$$\begin{aligned}
c_1 n \leq f(n)\ln f(n) &\leq c_2 n \\
\ln(c_1 n) \leq \ln(f(n)\ln f(n)) &\leq \ln(c_2 n) \\
\ln(c_1) + \ln(n) \leq \ln(f(n)) + \ln(\ln f(n)) &\leq \ln(c_1)\ln(n) \\
\frac{1}{2}\ln(n) \leq \ln(f(n)) + \ln(\ln f(n)) &\leq 2\ln(n)
\end{aligned}$$

Hence, $\ln(f(n)) + \ln(\ln f(n)) = \Theta(\ln(n))$.
Next, we can observe that:

$$\begin{aligned}
0 \leq \ln(f(n)) &\leq f(n) \\
0 \leq \ln(\ln f(n)) &\leq \ln(f(n)) \\
\ln(f(n)) \leq \ln(f(n)) + \ln(\ln f(n)) &\leq 2\ln(f(n))
\end{aligned}$$

Hence, we get $\ln(f(n)) + \ln(\ln f(n)) = \Theta(\ln(f(n)))$.
By reflexivity, we get $\ln(f(n)) = \Theta(\ln(f(n)) + \ln(\ln f(n)))$. Using transitivity, we get $\ln f(n) = \Theta(\ln n)$.

$$\begin{aligned}
f(n)\ln f(n) &= \Theta(n) \\
f(n)\ln f(n)/\ln f(n) &= \Theta(n)/\Theta(\ln n) \\
f(n) &= \Theta(n/\ln n)
\end{aligned}$$

□

6. Consider the statement: $f(cn) = \Theta(f(n))$.
a. Determine a function $f(n)$ and a constant $c > 0$ for which the statement is false.
b. Determine a function $f(n)$ for which the statement is true for all $c > 0$.

Solution of 6.a. Let $f(n) = 2^n$, $c = 2$, we have $f(2n) = 2^{2n} = 4^n$, by what have been proved in *problem*3, $4^n \neq \Theta(2^n)$. This case lead the statement false. □

Solution of 6.b. Let $f(n) = n$, then $f(cn) = cn$. $\forall c > 0$, there must exist $c_1 = c - 1$ and $c_2 = c + 1$ that makes $c_1 n \leq cn \leq c_2 n$. Hence, $f(cn) = \Theta(f(n))$. □

7. Determine the asymptotic order of the expression $\sum_{i=1}^{n} a^i$ where $a > 0$ is a constant, i.e. find a simple function $g(n)$ such that the expression is in the class $\Theta g(n)$. (Hint: consider the cases $a = 1$, $a > 1$, and $0 < a < 1$ separately.)

Proof. Case 1: when $a = 1$, we can get:

$$\begin{aligned}
f(n) &= \sum_{i=1}^{n} a^i \\
&= 1^1 + 1^2 + \cdots\cdots + 1^n \\
&= n
\end{aligned}$$

Let $g(n) = n$, we have the limit:

$$\begin{aligned}\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \lim_{n\to\infty}\frac{n}{n}\\ &= 1\end{aligned}$$

$\sum_{i=1}^{n} a^i$=$\Theta(n)$.

Case 2: when $a > 1$, we can get:

$$\begin{aligned}f(n) &= \sum_{i=1}^{n} a^i\\ &= a^1 + a^2 + \cdots\cdots + a^n\\ &= a(\frac{a^n-1}{a-1})\\ &= \frac{a^{n+1}-a}{a-1}\end{aligned}$$

Let $g(n) = a^n$, we have the limit:

$$\begin{aligned}\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \lim_{n\to\infty}\frac{\frac{a^{n+1}-a}{a-1}}{a^n}\\ &= \lim_{n\to\infty}\frac{a^{n+1}-a}{a^{n+1}-a^n}\\ &= \lim_{n\to\infty}\frac{(n+1)a^n}{(n+1)a^n-na^{n-1}}\\ &= \cdots\cdots \quad \text{(apply L.Hospital theorem n times)}\\ &= \lim_{n\to\infty}\frac{(n+1)!}{(n+1)!}\\ &= 1\end{aligned}$$

Hence, $\sum_{i=1}^{n} a^i = \Theta(a^n)$.

Case 3: when $a < 1$, let $a = \frac{1}{b}$, we can get:

$$\begin{aligned}f(n) &= \sum_{i=1}^{n} a^i\\ &= a^1 + a^2 + \cdots\cdots + a^n\\ &= \frac{1}{b} + \frac{1}{b^2} + \cdots\cdots + \frac{1}{b^n}\\ &= \frac{1}{b}(\frac{1-\frac{1}{b^n}}{1-\frac{1}{b}})\\ &= \frac{1-\frac{1}{b^n}}{b-1}\end{aligned}$$

Let $g(n) = 1$, we have the limit:

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{\frac{1-\frac{1}{b^n}}{b-1}}{1} \\
&= \lim_{n\to\infty} \frac{1-\frac{1}{b^n}}{b-1} \\
&= \lim_{n\to\infty} \frac{1-0}{b-1} \\
&= \frac{1}{b-1} \quad \text{(which is a constant)}
\end{aligned}$$

Hence, $\sum_{i=1}^{n} a^i = \Theta(1)$. □

8. Use limits to prove the following:
a. $n\ln(n) = o(n^2)$
b. $n^5 2^n = \omega(n^{10})$**.**
c. If $P(n)$ **is a polynomial of degree** $k \geq 0$**, then** $P(n) = \theta(n^k)$**. State any assumptions you need to make for the above statement to be true**

Proof of a.

$$\begin{aligned}
\lim_{n\to\infty} \frac{n\ln(n)}{n^2} &= \lim_{n\to\infty} \frac{\ln n + 1}{2n} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}}{2} \\
&= 0
\end{aligned}$$

Hence, $n\ln(n) = o(n^2)$. □

Proof of b.

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^5 2^n}{n^{10}} &= \lim_{n\to\infty} \frac{2^n}{n^5} \\
&= \lim_{n\to\infty} \frac{2^n \ln(2)}{5n^4} \\
&= \lim_{n\to\infty} \frac{2^n \ln^2(2)}{5\times 4n^3} \\
&= \cdots\cdots \quad \text{(apply L.Hospital theorem 3 more times)} \\
&= \lim_{n\to\infty} \frac{2^n \ln^5(2)}{5!} \\
&= \infty
\end{aligned}$$

Hence, $n^5 2^n = \omega(n^{10})$. □

Solution of c. Let's assume $P(n) = c_1 n^k + c_2 n^{k-1} + c_3 n^{k-2} + \cdots\cdots + c_k n$, with $c_1 > 0$ and $c_2, c_3, \cdots\cdots, c_k \in \mathbb{R}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{P(n)}{n^k} &= \lim_{n\to\infty} \frac{c_1 n^k + c_2 n^{k-1} + c_3 n^{k-2} + \cdots\cdots + c_k n}{n^k} \\
&= \lim_{n\to\infty} \frac{k c_1 n^{k-1} + c_2 (k-1) n^{k-2} + c_3 (k-2) n^{k-3} + \cdots\cdots + c_k}{k n^{k-1}} \\
&= \cdots\cdots \quad \text{(apply L.Hospital theorem k-1 more times)} \\
&= \lim_{n\to\infty} \frac{k! c_1}{k!} \\
&= c_1
\end{aligned}$$

Hence, $P(n) = \theta(n^k)$. □

9. Determine whether the first function is o, θ, or ω of the second function.
a. n^n **compared to** $2^{n \ln n}$
b. $\sqrt{\ln n}$ **compared to** $\ln(\ln n)$

Solution of a.

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^n}{2^{n \ln n}} &= \lim_{n\to\infty} \frac{n^n(\ln(n)+1)}{\ln(2) n^{\ln(2) n}(\ln(n)+1)} \\
&= \lim_{n\to\infty} \frac{1}{\ln(2) n^{\ln(2)}} \\
&= 0 \;\; (\textit{since} \;\ln(2) > 1)
\end{aligned}$$

Hence, $n^n = o(2^{n \ln n})$ □

Solution of b.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{\ln n}}{\ln(\ln n)} &= \lim_{n\to\infty} \frac{\frac{1}{2n\sqrt{\ln n}}}{\frac{1}{n \ln n}} \\
&= \lim_{n\to\infty} \frac{1}{2}\sqrt{\ln n} \\
&= \frac{1}{2} \lim_{n\to\infty} \sqrt{\ln n} \\
&= \frac{1}{2} \sqrt{\lim_{n\to\infty} \ln n} \\
&= \infty
\end{aligned}$$

Hence, $\sqrt{\ln n} = \omega(\ln(\ln n))$. □

References

[1] templatetypedef, https://stackoverflow.com/questions/2820211/if-fn-%CE%98gn-is-2fn-%CE%982gn

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[3] https://www3.cs.stonybrook.edu/ rob/teaching/cse373-fa15/sol2.pdf

[4] https://math.stackexchange.com/questions/179176/show-that-k-ln-k-in-thetan-implies-k-in-thetan-lnn