CSE 102: Spring 2021

Advanced Homework # 2

Induction Proofs and Recurrence Relations

Attempt only **ONE** of the following problems. All questions do NOT carry equal weights.

1. (easy: 1 point) Prove by induction that the sum of the cubes of the first n positive integers is equal to the square of the sum of these integers.

2. (medium: 2 points) First, prove using induction, that, $n^3 < 2 \times (n-1)^3$ for $n \geq 5$. Then, use this result, to prove by induction that $n^3 < 2^n$ for $n \geq 10$. In both cases, clearly state the base case and the induction hypothesis.

3. (medium: 2.5 points) Consider the recurrence relation: $T(1) = 1$. $T(n) = 4T(n/2) + n^2 \log n$. Using iteration, prove that $T(n) = \theta(n^2 \log^2 n)$ *using iteration method*. You MUST write down the expression after k steps and then use the substitution $k = \log_2 n$ to simplify and derive the final result. [Stay away from long answers copied from internet without understanding.]

4. Consider the recurrence relation: $T(1) = a$ and $T(n) = nT(n-1) + bn$ for $n \geq 2$. Prove, by induction, that for sufficiently large integer n, there exists two positive real constants P and Q such that $Pn! \leq T(n) \leq Qn!$.

   [This problem shows that the time taken by direct use of the recursive definition to compute the determinant of an $n \times n$ matrix is proportional to $n!$, which is much worse than merely exponential. The determinant can be computed more efficiently by Gauss-Jordan elimination.]

5. [If you are not getting enough challenge in this course, try this:]

   Solve the above recurrence in Problem 4 exactly. You are allowed a term of the form $\sum_{i=1}^{n} 1/i!$ in your solution. What is the value of $lim_{n\to\infty} T(n)/n!$ as a function of a and b?

   [Hint: Use induction. Note that, $\sum_{i=1}^{\infty} 1/i! = e - 1$.]

6. Let n be a positive integer. Draw a circle and mark n points irregularly on the circle. Now, draw a chord inside the circle between each pair of points. In case $n = 1$, there are no pairs of points and thus no chords are drawn. Let $c(n)$ denote the number of sections thus carved inside the circle. You should find that $c(1) = 1$, $c(2) = 2$, $c(3) = 4$, $c(4) = 8$, $c(5) = 16$. Using induction, find a formula for $c(n)$.

   [Hint: This is Moser's circle problem. Do not underestimate!]