Discrete joint distributions

Example

			Y		
	x/y	1	2	3	4
	1	0.1	0	0.1	0
X	2	0.3	0	0.1	0.2
	3	0	0.2	0	0

This table defines the joint p.f. (or p.m.f.) of (X, Y)

$$f_{X,Y}(x, y) = \Pr(X = x, Y = y)$$

- $f(1,4) = \Pr(X = 1, Y = 4) = 0$
- $\Pr(X \geq 2, Y = 4) = \Pr(X = 2, Y = 4) + \Pr(X = 3, Y = 4) = 0.2 + 0 = 0.2$
- $\Pr(X \geq 2, Y \geq 2) = 0 + 0.1 + 0.2 + 0.2 + 0 + 0 = 0.5$

Note that:

$$\Pr((X, Y) \in A) = \sum_{(x,y) \in A} f(x, y)$$

Properties:

i) $0 \leq f(x, y) \leq 1$ for all (x, y)

ii) $\sum_{i} \sum_{j} f(x, y) = 1$

Continuous Joint Distributions

The joint p.d.f /density of two cont. r.v.s X, Y is given by

$f_{X,Y}(x,y)$ such that

$$\Pr((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

Properties:

i) $f_{X,Y}(x,y) = f(x,y) \geq 0$ for all x, y

ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Example Let X, Y be r.v.s with joint pdf /density

$$f(x,y) = \begin{cases} c x^2 y & x^2 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

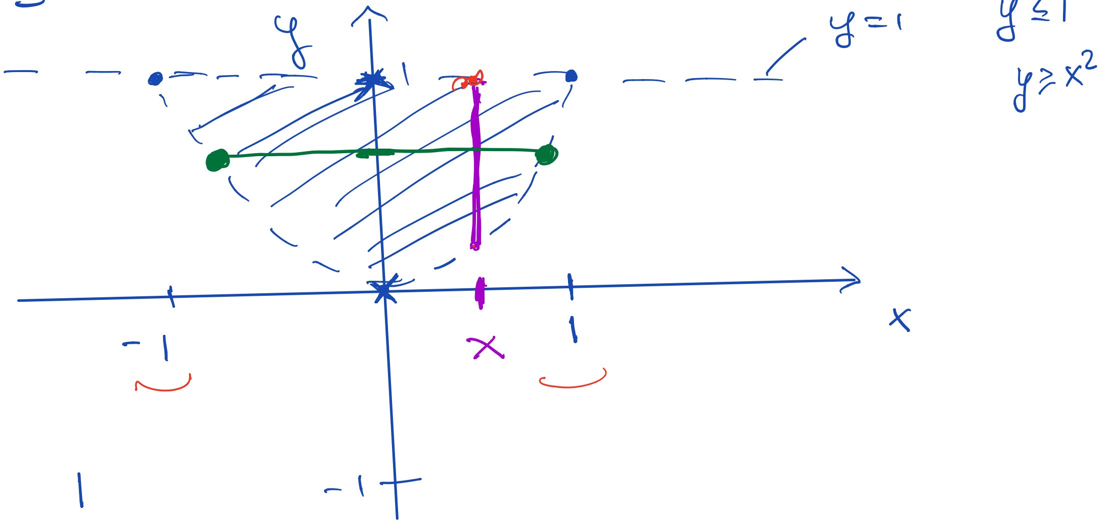
a) Find c

b) Find $\Pr(X \geq Y)$

a) We know that $f(x,y)$ has to be such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

Let's think about $x^2 \le y \le 1$



$$\int_{-1}^{1}\left(\int_{x^2}^{1} c x^2 y \; dy\right)dx = 1$$

$$1 = \int_{-1}^{1} \left(\int_{x^2}^{1} c x^2 y \, dy \right) dx =$$

$$= \int_{-1}^{1} c x^2 \left(\int_{x^2}^{1} y \, dy \right) dx =$$

$$= \int_{-1}^{1} c x^2 \left[\frac{y^2}{2} \right]_{x^2}^{1} dx =$$

$$= \int_{-1}^{1} c x^2 \left(\frac{1}{2} - \frac{x^4}{2} \right) dx =$$

$$= \frac{c}{2} \left[\int_{-1}^{1} x^2 \, dx - \int_{-1}^{1} x^6 \, dx \right] =$$

$$= \frac{c}{2} \left[\frac{x^3}{3} \Big|_{-1}^{1} - \frac{x^7}{7} \Big|_{-1}^{1} \right] = \frac{4c}{21}$$

$$\Rightarrow \quad \frac{4c}{21} = 1 \quad \Rightarrow$$

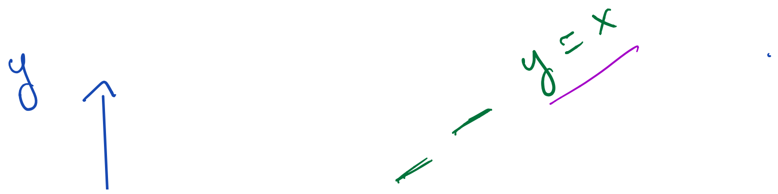
$$\boxed{c = \frac{21}{4}}$$

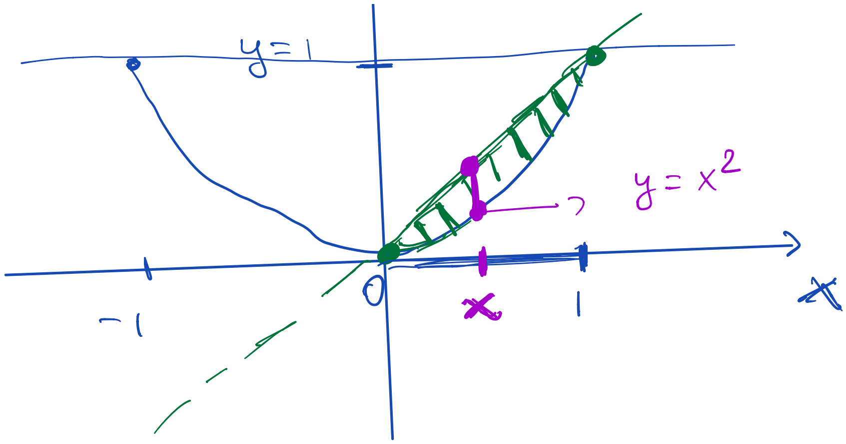
you could have solved

$$1 = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} c x^2 y \, dx \, dy$$

and the answer would be the same.

b) $Pr(\underline{X \geq Y}) = ?$





$y \leq x$

$$\Pr(X \geq Y) = \int_{0}^{1} \left(\int_{x^2}^{x} \frac{21}{4} x^2 y \, dy \right) dx = \frac{3}{20}$$

↑ exercise

Univariate Case:

$$\Pr(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx$$

Joint c.d.f./distribution function

$$F(x,y) = \underline{Pr(X \le x, Y \le y)}$$

↑
c.d.f. /distribution function

In the continuous case :

$$F(x,y) = Pr(X \le x, Y \le y) =$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\,dv\,du =$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f(u,v)\,du\,dv$$

and also

$$\frac{d^2F(x,y)}{dx\,dy} = \frac{d^2\overset{\text{distribution}}{F}(x,y)}{dy\,dx} = \overset{\text{density}}{f}(x,y)$$

Note that:

- $\lim_{y \to \infty} F(x, y) = F(x) = Pr(X \leq x)$
- $\lim_{x \to \infty} F(x, y) = F(y) = Pr(Y \leq y)$