CSE 102: Spring 2021

Quiz # 3: April 21 (26 points)

45 minutes (Quiz) + 10 minutes (uploading) = 55 minutes

If needed, you can use the following facts:

- $\log_2 7 = 2.8073$; $\log_3 20 = 2.7268$; $\log_4 49 = \log_2 7$; $\log_4 45 = 2.7459$.
- You must use the following definition of the Median, presented in the class: Median of a sorted array of odd size n is $\frac{n+1}{2}$-th element of the array. Median of a sorted array of even size n is the average of $\frac{n}{2}$ and $(\frac{n}{2}+1)$-th element of the array.
- Median of two sorted arrays $[a_1, a_2]$ and $[b_1, b_2]$ is $\frac{max(a_1,b_1)+min(a_2,b_2)}{2}$.

Problems

1. (3 points)
   (a) In the context of asymptotic growth of functions, what does *decimal war* mean?
   (b) Is Karatsuba's integer multiplication algorithm with asymptotic complexity $O(n^{\log_2 7})$ the most efficient algorithm to solve this problem?
   (c) What may be *one* of the main challenges with highly efficient algorithms?

2. (3 points) In an integer multiplication algorithm using regular divide-and-conquer algorithm, if we divide the integer into two equal parts, one needs four multiplications. For example, to multiply 2354 with 1265, four multiplications needed are: $23 * 12, 23 * 65, 54 * 12$ and $54 * 65$.

   What are the *three* multiplications needed using efficient multiplication algorithm? You may assume that the base case or the lowest level of multiplication needs *two* digit integers, instead of one digit integer, as described in the example above.

3. (5 points) Strassen's square matrix multiplication algorithm divides the matrices into 4 blocks of $n/2$ partitions. Instead of 8 multiplications needed, Strassen designed an algorithm that requires only 7 multiplications at each level thereby reducing the asymptotic complexity of algorithm from $O(n^3)$ to $O(n^{\log_2 7})$.

   If you were to design a divide-and-conquer algorithm for this problem where you will divide the matrices into $n/4$ partitions by rows and columns, then

   (a) How many blocks will be there?

   (b) How many multiplications per level will be needed in a regular case? Write down the recurrence relation in this case.

   (c) How many *maximum* number of multiplications per level will be needed to improve the asymptotic complexity of algorithm which is strictly better than $O(n^{\log_2 7})$? Write down the recurrence relation. Justify. [Although you do not need to use a calculator to solve this problem; facts provided above suffice; however, you may use a calculator.]

4. (5 points) A majority element of an array $A[1, .., n]$ is any element occurring in strictly more than $n/2$ positions. A student wants to design a divide-and-conquer algorithm by dividing the array into *three* sub-problems rather than two sub-problems.

   In the following problems, you can assume that $n = 3m$. Furthermore, if you need to provide a counter-example, use $n = 9$. If you need to provide a proof, prove formally by using the $<$ or $\leq$ or $>$ or $\geq$ symbols precisely in your proof.

   (a) Suppose each of the three sub-parts have a majority element. Then prove or disprove that the array must have a majority element.

   (b) Suppose none of the three sub-parts have a majority element. Then prove or disprove that the array has no majority element.

5. (6 points) Consider the problem of finding median of two *sorted* arrays of same size n. Recall the divide-and-conquer algorithm that subdivides the problem into sub-problems of *sorted* arrays of *smaller* sizes.

(a) What will be the successive smaller sizes of sub-problems if you started with two sorted arrays of size 24? Provide one sentence justification either in text or by a diagram. Be sure to terminate by specifying the size of arrays in the base case.

(b) Use divide-and-conquer algorithm to find the median of following two sorted arrays of size 5: $[4, 5, 9, 11, 12]$ and $[3, 6, 7, 8, 10]$. Be sure to write down the arrays after each step of divide-and-conquer.

[You may use HW 3 solution web link:
https://www.geeksforgeeks.org/median-of-two-sorted-arrays/
or a solution by you in your homework, assuming that it is correct.]

6. (4 points) We are going to use quicksort to sort the following integer array: $[4, 9, 6, 8]$.

   Choose the last element of the array as pivot.

   Demonstrate clearly step-by-step how the partition algorithm will work at the very first step. You may use diagrams similar to the diagram used in CLRS Section 7.1 on Quicksort. While re-drawing the array, draw the vertical lines clearly, longer, and darker, to distinguish the boundaries of the partitions.