CSE 102
Homework 2 Solutions

1. Use induction to prove that $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$ for all $n \geq 1$.

Proof:

I. $\sum_{k=1}^{1} k^4 = 1 = \frac{2\cdot 15}{30} = \frac{1\cdot(1+1)\cdot(6\cdot 1^3+9\cdot 1^2+1-1)}{30}$, and the base case is established.

II. Let $n \geq 1$. Assume for this *n* that

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$$

We must show that

$$\sum_{k=1}^{n+1} k^4 = \frac{(n+1)(n+2)(6(n+1)^3+9(n+1)^2+(n+1)-1)}{30}$$

We have

$$\sum_{k=1}^{n+1} k^4 = \left(\sum_{k=1}^{n} k^4\right) + (n+1)^4$$

$$= \frac{n(n+1)(6n^3+9n^2+n-1)}{30} + (n+1)^4 \qquad \text{(by the induction hypothesis)}$$

$$= \frac{(n+1)\left[n(6n^3+9n^2+n-1)+30(n+1)^3\right]}{30}$$

$$= \frac{(n+1)\left[6n^4+39n^3+91n^2+89n+30)\right]}{30}$$

$$= \frac{(n+1)(n+2)\left[6n^3+27n^2+37n+15)\right]}{30}$$

$$= \frac{(n+1)((n+1)+1)\left[6(n+1)^3+9(n+1)^2+(n+1)-1)\right]}{30}$$

Thus the formula holds for $n+1$ as well. The formula is valid for all *n* by induction. ■

2. Let $T(n)$ be defined by the recurrence

$$T(n) = \begin{cases} 1 & \text{if } \ n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & \text{if } \ n \geq 2 \end{cases}$$

Show that $\forall n \geq 1$: $T(n) \leq (4/3)n^2$, and hence $T(n) = O(n^2)$.

Proof:

I. For $n = 1$ we have $T(1) = 1 \leq 4/3 = (4/3) \cdot 1^2$, so the base case is satisfied.

II. Let $n > 1$ and assume for all k in the range $1 \leq k < n$ that $T(k) \leq (4/3)k^2$. In particular we have $T(\lfloor n/2 \rfloor) \leq (4/3)\lfloor n/2 \rfloor^2$. We must show that $T(n) \leq (4/3)n^2$. We have

$$
\begin{aligned}
T(n) &= T(\lfloor n/2 \rfloor) + n^2 \\
&\le (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis} \\
&\le (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \le x \text{ for any } x \\
&= (1/3)n^2 + n^2 \\
&= (4/3)n^2.
\end{aligned}
$$

By the Second Principle of Mathematical Induction, $T(n) \le (4/3)n^2$ for all $n \ge 1$. ■

3. Recall the n^{th} harmonic number was defined to be $H_n = \sum_{k=1}^{n} \left(\frac{1}{k}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$. Use induction to prove that

$$\sum_{k=1}^{n} kH_k = \frac{1}{2}n(n+1)H_n - \frac{1}{4}n(n-1)$$

for all $n \geq 1$. (Hint: Use the fact that $H_n = H_{n-1} + \frac{1}{n}$.)

Proof:

I. If $n = 1$ we have $\sum_{k=1}^{1} kH_k = 1 \cdot H_1 = 1$ and $\frac{1}{2} \cdot 1 \cdot (1+1)H_1 - \frac{1}{4} \cdot 1 \cdot (1-1) = 1$, so the base case is satisfied.

II. Let $n \geq 1$ and assume that $\sum_{k=1}^{n} kH_k = \frac{1}{2}n(n+1)H_n - \frac{1}{4}n(n-1)$ holds. We must show that $\sum_{k=1}^{n+1} kH_k = \frac{1}{2}(n+1)(n+2)H_{n+1} - \frac{1}{4}(n+1)n$ is also true.

$$\sum_{k=1}^{n+1} kH_k = \sum_{k=1}^{n} kH_k + (n+1)H_{n+1}$$

$$= \frac{1}{2}n(n+1)H_n - \frac{1}{4}n(n-1) + (n+1)H_{n+1} \qquad \text{(by the induction hypothesis)}$$

$$= \frac{1}{2}n(n+1)\left\{H_{n+1} - \frac{1}{n+1}\right\} - \frac{1}{4}n(n-1) + (n+1)H_{n+1} \qquad \text{(using the hint)}$$

$$= \left(\frac{n}{2} + 1\right)(n+1)H_{n+1} - \frac{1}{2} \cdot n - \frac{1}{4} \cdot n(n-1)$$

$$= \left(\frac{n+2}{2}\right)(n+1)H_{n+1} - \left(\frac{n}{2} + \frac{n^2}{4} - \frac{n}{4}\right)$$

$$= \frac{1}{2} \cdot (n+1)(n+2)H_{n+1} - \frac{n^2+n}{4}$$

$$= \frac{1}{2} \cdot (n+1)(n+2)H_{n+1} - \frac{1}{4} \cdot (n+1)n,$$

as required. ■

4. Define $T(n)$ by the recurrence

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n-1) + n & \text{if } n \geq 2 \end{cases}$$

Use the iteration method to find the exact solution to this recurrence, then determine an asymptotic solution.

Solution:
Applying iteration to the above recurrence gives

$$\begin{aligned} T(n) &= n + T(n-1) \\ &= n + (n-1) + T(n-2) \\ &= n + (n-1) + (n-2) + T(n-3) \\ &\;\;\vdots \\ &= \sum_{i=0}^{k-1}(n-i) + T(n-k) \end{aligned}$$

The recurrence stops when the recursion depth k satisfies $n - k = 1$, so $k = n - 1$. Thus

$$\begin{aligned} T(n) &= \textstyle\sum_{i=0}^{n-2}(n-i) + 1 \\ &= \textstyle\sum_{i=0}^{n-2} n - \sum_{i=0}^{n-2} i + 1 \\ &= n(n-1) - \tfrac{1}{2}(n-1)(n-2) + 1 \\ &= \Theta(n^2)\ . \end{aligned}$$

■

5. Define $T(n)$ by the recurrence

$$T(n) = \begin{cases} 9 & \text{if } 1 \le n < 15 \\ T(\lfloor n/2 \rfloor) + 6 & \text{if } n \ge 15 \end{cases}$$

Use the iteration method to find the exact solution to this recurrence, then determine an asymptotic solution.

Solution:
Iteration yields

$$\begin{aligned} T(n) &= 6 + T(\lfloor n/2 \rfloor) \\ &= 6 + 6 + T\left(\left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor\right) = 6 \cdot 2 + T(\lfloor n/2^2 \rfloor) \\ &= 6 \cdot 3 + T(\lfloor n/2^3 \rfloor) \\ &\ \ \vdots \\ &= 6k + T(\lfloor n/2^k \rfloor). \end{aligned}$$

The process terminates when the recursion depth k first satisfies $1 \le \lfloor n/2^k \rfloor < 15$, which is equivalent to $1 \le n/2^k < 15$. Thus we seek the smallest k satisfying $2^k \le n < 15 \cdot 2^k$. Since k is to be minimized, we ignore the left hand inequality and concentrate on the right: $n < 15 \cdot 2^k \Rightarrow n/15 < 2^k \Rightarrow \lg(n/15) < k$. The smallest such k must satisfy $k - 1 \le \lg(n/15) < k$, whence $k - 1 = \lfloor \lg(n/15) \rfloor$, and $k = \lfloor \lg(n/15) \rfloor + 1$. For this k we have $T(\lfloor n/2^k \rfloor) = 9$, and therefore

$$\begin{aligned} T(n) &= 6(\lfloor \lg(n/15) \rfloor + 1) + 9 \\ &= 6\lfloor \lg(n/15) \rfloor + 15 \\ &= 6\lfloor \lg(n) - \lg(15) \rfloor + 15. \end{aligned}$$

Ignoring the constants, the floor function and the base of the log, we get $T(n) = \Theta(\log(n))$. ■

6. Use the Master Theorem to find tight asymptotic bounds on the following recurrences.

   a. $T(n) = 3T(2n/3) + n^3$
   Solution:
   Compare n^3 to $n^{\log_{3/2}(3)}$. Observe $3 = \frac{24}{8} < \frac{27}{8} = \left(\frac{3}{2}\right)^3 \Rightarrow \log_{3/2}(3) < 3$. Thus if we set $\epsilon = 3 - \log_{3/2}(3)$, then $\epsilon > 0$ and $n^3 = n^{\log_{3/2}(3)+\epsilon} = \Omega(n^{\log_{3/2}(3)+\epsilon})$. Select any c in the range $\frac{8}{9} \le c < 1$, so that $3(2n/3)^3 = (8/9)n^3 \le cn^3$, establishing the regularity condition. Case 3 now gives $T(n) = \Theta(n^3)$. ■

   b. $T(n) = 2T(n/3) + \sqrt{n}$
   Solution:
   We compare $n^{1/2}$ to $n^{\log_3(2)}$. Observe $3 < 4 \Rightarrow 1 < \log_3(4) = \log_3(2^2) = 2\log_3(2) \Rightarrow 1/2 < \log_3(2)$. Thus setting $\epsilon = \log_3(2) - (1/2)$, we have $\epsilon > 0$, and $1/2 = \log_3(2) - \epsilon$. Therefore $\sqrt{n} = n^{1/2} = O\left(n^{1/2}\right) = O\left(n^{\log_3(2)-\epsilon}\right)$. By case (1): $T(n) = \Theta(n^{\log_3(2)})$. ■

   c. $T(n) = 5T(n/4) + n^{\lg\sqrt{5}}$
   Solution:
   We compare $n^{\log_2\sqrt{5}}$ to $n^{\log_4(5)}$. Observe $5 = \sqrt{5}^2 = \sqrt{5}^{\log_2(4)} = 4^{\log_2(\sqrt{5})} \Rightarrow \log_4(5) = \log_2\left(\sqrt{5}\right)$. Therefore $n^{\log_2\sqrt{5}} = n^{\log_4(5)}$, and by case (2): $T(n) = \Theta\left(n^{\log_4(5)}\log(n)\right)$. ■

   d. $T(n) = 3T(2n/5) + n\log n$
   Solution:
   We compare $n\log(n)$ to $n^{\log_{5/2}(3)}$. Observe that $3 > 5/2 \Rightarrow \log_{5/2}(3) > 1$, so upon setting $\epsilon = \frac{1}{2}\left(\log_{5/2}(3) - 1\right)$ we have $\epsilon > 0$, and $2\epsilon = \log_{5/2}(3) - 1 \Rightarrow 1 + \epsilon = \log_{5/2}(3) - \epsilon$. Thus

   $$\lim_{n\to\infty}\frac{n\log(n)}{n^{\log_{5/2}(3)-\epsilon}} = \lim_{n\to\infty}\frac{n\log(n)}{n^{1+\epsilon}} = \lim_{n\to\infty}\frac{\log(n)}{n^{\epsilon}} = 0$$

   and therefore $n\log(n) = o\left(n^{\log_{5/2}(3)-\epsilon}\right) \subseteq O\left(n^{\log_{5/2}(3)-\epsilon}\right)$, so $T(n) = \Theta(n^{\log_{5/2}(3)})$ by case (1) of the Master Theorem. ■

   e. $S(n) = aS(n/4) + n^2$ (your answer will depend on the parameter a.)
   Solution:
   We compare n^2 to $n^{\log_4(a)}$. The answer depends on whether $\log_4(a)$ is greater than, equal to or less than 2. This is equivalent to asking whether a is greater than, equal to or less than 16.

   If $a > 16$ then $\log_4(a) > 2$. In this case set $\epsilon = \log_4(a) - 2$ so $\epsilon > 0$. Then $2 = \log_4(a) - \epsilon$ and $n^2 = O(n^{\log_4(a)-\epsilon})$. By case (1) we have $S(n) = \Theta(n^{\log_4(a)})$.

   If $a = 16$ then $\log_4(a) = 2$, and $n^2 = n^{\log_4(a)}$. Case (2) gives $S(n) = \Theta(n^2\log(n))$.

   If $1 \le a < 16$, then $0 \le \log_4(a) < 2$. Let $\epsilon = 2 - \log_4(a)$, so $\epsilon > 0$ and $2 = \log_4(a) + \epsilon$. Then $n^2 = \Omega(n^2) = \Omega(n^{\log_4(a)+\epsilon})$. Since $\frac{a}{16} < 1$ we can pick c satisfying $\frac{a}{16} \le c < 1$, and hence $a(n/4)^2 = (a/16)n^2 \le cn^2$ for all $n \ge 1$, establishing the regularity condition. Case (3) of the Master Theorem now yields $S(n) = \Theta(n^2)$. ■