Inequalities and Sums

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This document gives some rules for manipulating inequalities and sums.

1 Inequalities

Although stated for non-strict ($\leq$) inequalities, similar forms hold for strict inequalities. The rules are stated in the following form:

$$\frac{\begin{array}{c}\text{inequality 1}\\ \text{inequality 2}\end{array}}{\text{resulting inequality}}$$

with the meaning that inequalities 1 and 2 imply the resulting inequality. If additional constraints are needed, they will be mentioned before the inequalities.

1. Adding inequalites. This is the basic rule

$$\begin{array}{rcl} A & \leq & B \\ C & \leq & D \\ \hline A+C & \leq & B+D \end{array}$$

   Note the special case when $C = 0$ implies you can add something positive to the big side, and the case when $D = 0$ implies you can add something negative to the small side.

2. multiplying by a constant, if $c > 0$ then

$$\begin{array}{rcl} A & \leq & B \\ \hline cA & \leq & cB \end{array}$$

   Also, if $c < 0$, then the inequality flips after multiplying by c.

$$\begin{array}{rcl} A & \leq & B \\ \hline cB & \leq & cA \end{array}$$

3. multiplying positive inequalities

$$\begin{array}{rcl} 0 < A & \leq & B \\ 0 < C & \leq & D \\ \hline AC & \leq & BD \end{array}$$

4. Nondecreasing function: if $f(x)$ is a non-decreasing function (for example, if $f'(x)$ exists and is never negative), then

$$\begin{array}{rcl} A & \leq & B \\ \hline f(A) & \leq & f(B) \end{array}$$

   Note that the functions $\log_b(x)$ (for base $b > 1$) and x^c (for $c > 0$) are non-decreasing.

The flip side of this requires that function $f()$ be *strictly increasing* (i.e. $f'(x) > 0$). Then:

$$\frac{f(A) \quad \leq \quad f(B)}{A \quad \leq \quad B}$$

5. Staying above: for a constant a, if $f(a) \leq g(a)$ and $f'(x) \leq g'(x)$ for all $x \geq a$, then

$$f(x) \leq g(x) \quad \texttt{for all } x \geq a.$$

   This last is useful for proving big-O style relationships as it allows you to show that an infinite family of inequalities are true.

2 Sumations

The meaning of a sum is to define a bag (set with possible duplicates) of values and add up all the elements of the set. The sum of the empty bag (or set) is 0. Most commonly, the set is defined using an index of summation as follows:

$$\sum_{i=1}^{n} f(i)$$

where f is some function. This represents the bag $\{f(i) : 1 \leq i \leq n\}$. Note that the value of this sum is a function of n and the variable i is bound by the summation. Also i is implicitly typed as an integer, so

$$\sum_{i=1}^{1} f(i) = \sum_{i=1}^{1.9} = f(1).$$

This means that one must be somewhat careful when splitting summations. In general:

$$\sum_{i=1}^{n} f(i) \neq \sum_{i=1}^{n/2} f(i) + \sum_{i=n/2+1}^{n} f(i)$$

because when n is odd, $f((n+1)/2)$ is added in on the left, but not on the right.

Note that if the range doesn't contain an integer, the sum is empty. So

$$\sum_{i=1}^{0} f(i) = 0.$$

See appendix A of Cormen, Leiserson, Rivest, and Stein for several formulas and properties of summations.