# CSE 102 Spring 2021
# Homework Assignment 2

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## 1 HW2

**1. Use induction to prove that $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$.**

*Proof.* We use induction to prove this, let $P(n)$ be the equation above.

1. **Base step:**
   $\sum_{k=1}^{1} k^4 = \frac{1(1+1)(6+9+1-1)}{30} = 1^4$, show that $P(1)$ is true.

2. **Induction step:**
   Let $n \geq 1$ and assume $P(n)$is true. That is, for this particular value of $n$, the equation holds. Then

$$
\begin{aligned}
\sum_{k=1}^{n+1} k^4 &= \sum_{k=1}^{n} k^4 + (n+1)^4 \\
&= \frac{n(n+1)(6n^3+9n^2+n-1)}{30} + (n+1)^4 \\
&= \frac{(n+1)(6n^4+9n^3+n^2-n+30(n+1)^3)}{30} \\
&= \frac{(n+1)(6n^4+9n^3+n^2-n+30n^3+90n^2+90n+30)}{30} \\
&= \frac{(n+1)(6n^4+39n^3+91n^2+89n+30)}{30} \\
&= \frac{(n+1)(n+2)(6n^3+27n^2+37n+15)}{30} \\
&= \frac{(n+1)((n+1)+1)(6(n+1)^3+9(n+1)^2+(n+1)-1)}{30}
\end{aligned}
$$

   showing that $P(n+1)$ is true.

   We conclude that $P(n)$ is true for all $n \geq 1$.

$\square$

**2. Let $T(n)$ be defined by the recurrence:**

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor \frac{n}{2} \rfloor) + n^2 & n \geq 2 \end{cases}$$

**Use substitution method to show that $\forall n \geq 1$: $T(n) \leq \frac{4}{3}n^2$, and hence $T(n) = O(n^2)$.**

*Proof.* We use induction to prove this, let $P(n)$ be the inequality above.

1. **Base step:**
   We can easily observe that $T(1) = 1 \leq \frac{4}{3}1^2$, show that $P(1)$ is true.

2. **Induction step:**
   Let $n > 1$ and assume for all $k$ in the range $1 \leq k < n$ that $P(k)$ is true. In particular when $k = \lfloor \frac{n}{2} \rfloor$, we have $T(\lfloor \frac{n}{2} \rfloor) \leq \frac{4}{3}(\lfloor \frac{n}{2} \rfloor)^2$. We must show $T(n) \leq \frac{4}{3}n^2$ is true.

$$\begin{aligned} T(n) &= T(\lfloor \frac{n}{2} \rfloor) + n^2 \\ &\leq \frac{4}{3}(\lfloor \frac{n}{2} \rfloor)^2 + n^2 \\ &\leq \frac{4}{3}(\frac{n}{2})^2 + n^2 \\ &= \frac{4}{3}n^2 \end{aligned}$$

Therefore, we conclude that $P(n)$ is true and $T(n) \leq \frac{4}{3}n^2$ for all $n \geq 1$. $\square$

**3.**

*Proof.* We use induction to prove this, let $P(n)$ be the equation above.

1. **Base step:**
   We can easily observe that $\sum_{k=1}^{1} kH_k = \frac{1}{2}1(1+1)H_1 - \frac{1}{4}1(1-1) = H_1$, show that $P(1)$ is true.

2. **Induction step:**
   Let $n \geq 1$ and assume $P(n)$is true. That is, for this particular value of $n$, the equation holds.

Then

$$\begin{aligned}
\sum_{k=1}^{n+1} kH_k &= \sum_{k=1}^{n} kH_k + (n+1)H_{n+1} \\
&= \frac{1}{2}n(n+1)H_n - \frac{1}{4}n(n-1) + (n+1)(H_n + \frac{1}{n+1}) \\
&= (\frac{1}{2}n+1)(n+1)H_n - \frac{1}{4}n(n-1) + 1 \\
&= \frac{1}{2}(n+2)(n+1)H_n - \frac{1}{4}n(n+1) + \frac{1}{2}(n-2) \\
&= \frac{1}{2}(n+2)(n+1)(H_n + \frac{1}{n+1}) - \frac{1}{4}n(n+1) \\
&= \frac{1}{2}(n+1)(n+2)H_{n+1} - \frac{1}{4}(n+1)n
\end{aligned}$$

showing that $P(n+1)$ is true.

We conclude that $P(n)$ is true for all $n \geq 1$.

□

**4.**

*Proof.*

$$\begin{aligned}
T(n) &= T(n-1) + n \\
&= T(n-2) + n + n - 1 \\
&= T(n-3) + n + n - 1 + n - 2 \\
&\vdots \\
&= T(n-k) + kn - (1 + 2 + \cdots + n - 1) \\
&= T(n-k) + kn - \frac{n(n-1)}{2}
\end{aligned}$$

This process must terminate when the recursion depth k reaches its maximum, i.e. when $n-k=1$.

$$\begin{aligned}
n - k &= 1 \\
k &= n - 1
\end{aligned}$$

Thus for the recursion depth $k = n-1$ we have $T(n-k) = T(1) = 1$, and hence the solution to the above recurrence is $T(n) = 1 + \frac{(n-1)n}{2}$.

Therefore, $T(n) = \Theta(n^2)$ since $n$ did not increase as rapidly as $n^2$ does. □

**5.**

*Proof.*

$$
\begin{aligned}
T(n) &= T(\lfloor n/2 \rfloor) + 6 \\
&= T((\lfloor n/2 \rfloor)/2) + 12 \\
&= T(\lfloor n/2^2 \rfloor/2) + 18 \\
&\vdots \\
&= 6k + T(\lfloor n/2^k \rfloor)
\end{aligned}
$$

We seek the first integer k such that $\lfloor n/2^k \rfloor < 15$, which is equivalent to $n/2^k < 15$ .

$$
\begin{aligned}
&n/15 < 2^k \\
&k - 1 \leq \log_2(n/15) < k \\
&k - 1 = \lfloor \log_2(n/15) \rfloor \\
&\quad\; k = \lfloor \log_2(n/15) \rfloor + 1
\end{aligned}
$$

Thus, we have $T(n) = 6(\lfloor \log_2(n/15) \rfloor + 1) + 9$.
Hence $T(n) = \Theta(\log(n))$ since $\log(n)$ increase faster then constant does. □

**6.**

*Solution for a.* Observe that $a = 3$, $b = 3/2$, and $\log_b a = 2.70951 \cdots < 3$. Upon setting $\varepsilon = 3 - \log_{\frac{3}{2}} 3$ we have $\varepsilon > 0$, and therefore $f(n) = n^3 = \Omega(n^{\log_{\frac{3}{2}} 3 + \varepsilon})$. We can find $3f(\frac{n}{\frac{3}{2}}) \leq cf(n)$, since $\frac{3 \cdot 8 \cdot n^3}{27} \leq cn^3$, which is true when $\frac{24}{27} \leq c < 1$. Therefore, $T(n) = \Theta(n^3)$. □

*Solution for b.* Observe that $a = 2$, $b = 3$, and $\log_b a = 0.63092 \cdots > \frac{1}{2}$. Upon setting $\varepsilon = \log_3 2 - \frac{1}{2}$ we have $\varepsilon > 0$, and therefore $f(n) = n^{\frac{1}{2}} = O(n^{\log_3 2 - \varepsilon})$. Thus $T(n) = \Theta(n^{\log_3 2})$. □

*Solution for c.* Observe that $a = 5$, $b = 4$, and $\log_b a = 1.16096 \cdots < \lg \sqrt{5}$. Upon setting $\varepsilon = \lg \sqrt{5} - \log_4 5$ we have $\varepsilon > 0$, and therefore $f(n) = n^{\lg \sqrt{5}} = \Omega(n^{\log_4 5 + \varepsilon})$. We can find $5f(n/4) \leq cf(n)$ such c that $\frac{1}{5} \leq c < 1$. Thus $T(n) = \Theta(n^{\lg \sqrt{5}})$. □

*Solution for d.* Observe that $a = 3$, $b = \frac{5}{2}$, and $\log_b a = 1.19897\cdots > 1$. Upon setting $\varepsilon = 0.1$ we have $\varepsilon > 0$. Let $g(n) = n^{\log_{\frac{5}{2}} 3-\varepsilon}$, now we need to prove $f(n) = n \log n = O(g(n)) = O(n^{\log_{\frac{5}{2}} 3-0.1})$:

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \frac{n \log n}{n^{\log_{\frac{5}{2}} 3-0.1}} \\
&= \frac{\log n}{(\log_{\frac{5}{2}} 3 - 0.1)n^{\log_{\frac{5}{2}} 3-0.1-1}} \\
&= \frac{n^{-1}}{(\log_{\frac{5}{2}} 3 - 0.1) \cdot (\log_{\frac{5}{2}} 3 - 0.1 - 1)n^{\log_{\frac{5}{2}} 3-0.1-2}} \\
&= \frac{n^{-1-(\log_{\frac{5}{2}} 3-2.1)}}{(\log_{\frac{5}{2}} 3 - 0.1) \cdot (\log_{\frac{5}{2}} 3 - 0.1 - 1)} \\
&= \frac{n^{1.1-(\log_{\frac{5}{2}} 3-2.1)}}{(\log_{\frac{5}{2}} 3 - 0.1) \cdot (\log_{\frac{5}{2}} 3 - 0.1 - 1)} \\
&= 0 \quad \textbf{since } \log_{\frac{5}{2}} 3 \approx 1.19\textbf{, then the numerator is zero}
\end{aligned}
$$

Therefore $f(n) = n \log n = o(n^{\log_{\frac{5}{2}} 3-1})$. Since $o$ is a subset of $O$, then $f(n) = n \log n = O(n^{\log_{\frac{5}{2}} 3-1})$. Thus $T(n) = \Theta(n^{\log_{\frac{5}{2}} 3})$.

□

*Solution for e.*

- Case 1: $1 \le a < 16$
  We can observe that $\log_4 a < 2$, then setting $\varepsilon = 2 - \log_4 a$, we have $\varepsilon > 0$. Therefore $f(n) = n^2 = \Omega(n^{log_4 a+\varepsilon})$. Next we need to check $af(n/4) \le cf(n)$ for some $0 < c < 1$ and all sufficiently large $n$. This inequality says $a(n/4)^2 \le cn^2$, which is true as long as $\frac{a}{16} \le c < 1$. Therefore, $T(n) = \Theta(n^2)$.

- Case 2: $a = 16$
  We can observe that $\log_4 16 = 2$, then $f(n) = n^2 = \Theta(n^{\log_4 16}) = \Theta(n^2)$, then $T(n) = \Theta(n^2 \cdot \log(n))$

- Case 3: $a > 16$
  We can observe that $\log_4 a > 2$, then setting $\varepsilon = \log_4 a - 2$, we have $\varepsilon > 0$. Therefore $f(n) = n^2 = O(n^{\log_4 a-\varepsilon})$. Thus $T(n) = \Theta(n^{\log_4 a})$.

□

# References

[1] Suresh Lodha, MasterTheoremPracticeProblemsSKL.pdf

[2] https://web.csulb.edu/ tebert/teaching/lectures/528/recurrence/recurrence.pdf

[3] https://cs.pomona.edu/ dkauchak/classes/algorithms/handouts/hw2solution.pdf