CSE 101
Algorithms and Abstract Data Types
Master Theorem Practice Problems

Master Theorem
Let $a \geq 1$, $b > 1$, $f(n)$ be asymptotically positive, and let $T(n)$ be defined by $T(n) = a\mathrm{T}(n/b) + f(n)$. Then we have three cases:

1. If $f(n) = O\left(n^{\log_b(a)-\varepsilon}\right)$ for some $\varepsilon > 0$, then $T(n) = \Theta\left(n^{\log_b(a)}\right)$.
2. If $f(n) = \Theta\left(n^{\log_b(a)}\right)$, then $T(n) = \Theta\left(n^{\log_b(a)} \cdot \log(n)\right)$.
3. If $f(n) = \Omega\left(n^{\log_b(a)+\varepsilon}\right)$ for some $\varepsilon > 0$, and if $af(n/b) \leq cf(n)$ for some c in the range $0 < c < 1$ and for all sufficiently large n, then $T(n) = \Theta\big(f(n)\big)$.

Practice Problems
For each of the following recurrences, if the Master Theorem can be applied, give a tight asymptotic bound on the solution $T(n)$. Otherwise, indicate that (and explain why) the Master Theorem does not apply.

1. $T(n) = 3T(n/2) + n^2$
2. $T(n) = 4T(n/2) + n^2$
3. $T(n) = T(n/2) + 2^n$
4. $T(n) = 2^n T(n/2) + n^n$
5. $T(n) = 16T(n/4) + n$
6. $T(n) = 2T(n/2) + n\log(n)$
7. $T(n) = 2T(n/2) + n/\log(n)$
8. $T(n) = 2T(n/4) + n^{0.51}$
9. $T(n) = (0.5)T(n/2) + 1/n$
10. $T(n) = 16T(n/4) + n!$
11. $T(n) = \sqrt{2}\,T(n/2) + \log(n)$
12. $T(n) = 3T(n/2) + n$
13. $T(n) = 3T(n/3) + \sqrt{n}$
14. $T(n) = 4T(n/2) + cn$
15. $T(n) = 3T(n/4) + n\log(n)$
16. $T(n) = 3T(n/3) + n/2$
17. $T(n) = 6T(n/3) + n^2\log(n)$
18. $T(n) = 4T(n/2) + n/\log(n)$
19. $T(n) = 64T(n/8) - n^2\log(n)$
20. $T(n) = 7T(n/3) + n^2$
21. $T(n) = 4T(n/2) + \log(n)$
22. $T(n) = T(n/2) + n(2 - \cos(n))$

Answers:

1. $T(n) = 3T(n/2) + n^2$	Case 3: $T(n) = \Theta(n^2)$
2. $T(n) = 4T(n/2) + n^2$	Case 2: $T(n) = \Theta(n^2 \log(n))$
3. $T(n) = T(n/2) + 2^n$	Case 3: $T(n) = \Theta(2^n)$
4. $T(n) = 2^n T(n/2) + n^n$	Does not apply since a is not constant
5. $T(n) = 16T(n/4) + n$	Case 1: $T(n) = \Theta(n^2)$
6. $T(n) = 2T(n/2) + n \log(n)$	Does not apply since theorem conditions not satisfued
7. $T(n) = 2T(n/2) + n/\log(n)$	Does not apply since theorem conditions not satisfied
8. $T(n) = 2T(n/4) + n^{0.51}$	Case 3: $T(n) = \Theta(n^{0.51})$
9. $T(n) = (0.5)T(n/2) + 1/n$	Does not apply since $a < 1$
10. $T(n) = 16T(n/4) + n!$	Case 3: $T(n) = \Theta(n!)$
11. $T(n) = \sqrt{2}\, T(n/2) + \log(n)$	Case 1: $T(n) = \Theta(\sqrt{n})$
12. $T(n) = 3T(n/2) + n$	Case 1: $T(n) = \Theta(n^{\log_2(3)})$
13. $T(n) = 3T(n/3) + \sqrt{n}$	Case 1: $T(n) = \Theta(n)$
14. $T(n) = 4T(n/2) + cn$	Case 1: $T(n) = \Theta(n^2)$
15. $T(n) = 3T(n/4) + n \log(n)$	Case 3: $T(n) = \Theta(n \log(n))$
16. $T(n) = 3T(n/3) + n/2$	Case 2: $T(n) = \Theta(n \log(n))$
17. $T(n) = 6T(n/3) + n^2 \log(n)$	Case 3: $T(n) = \Theta(n^2 \log(n))$
18. $T(n) = 4T(n/2) + n/\log(n)$	Case 1: $T(n) = \Theta(n^2)$
19. $T(n) = 64T(n/8) - n^2 \log(n)$	Does not apply since $f(n)$ is not asymptotically positive
20. $T(n) = 7T(n/3) + n^2$	Case 3: $T(n) = \Theta(n^2)$
21. $T(n) = 4T(n/2) + \log(n)$	Case 1: $T(n) = \Theta(n^2)$
22. $T(n) = T(n/2) + n(2 - \cos(n))$	Does not apply since case 3, but regularity condition violated