Solutions to practice problems for conditional distributions and functions of random variables.

Problem 1. The amount of kerosene, in thousands of liters, in a tank at the beginning of any day is a random amount Y from which a random amount X is sold during that day. Suppose that the tank is not resupplied during the day so that $x \leq y$, and assume that the joint density function of these variables is

$$f(x,y) = \begin{cases} 2, & 0 < x < y,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Determine if X and Y are independent.

(b) Find $P\left(\frac{1}{4} < X < \frac{1}{2} | Y = \frac{3}{4}\right)$.

Solution:

(a) $f_Y(y) = \int_0^y 2dx = 2y$, $0 < y < 1$
$g_X(x) = \int_x^1 2dy = 2(1-x)$, $0 < x < 1$
$f(x,y) \neq f_Y(y) g_X(x)$. They are dependent.

(b) $f_X(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2}{2y} = \frac{1}{y}$, $0 < x < y < 1$
$P\left(\frac{1}{4} < X < \frac{1}{2} | Y = \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{\frac{3}{4}} dx = \frac{1}{3}$.

Problem 2. Three cards are drawn without replacement from the 12 face cards (jacks, queens and kings) of an ordinary deck of 52 playing cards. Let X be the number of kings selected and Y the number of jacks. Find

(a) the joint probability distribution of X and Y;

(b) $P[(X,Y) \in A]$, where A is the region given by $\{(x,y) | x + y \geq 2\}$.

Solution:

(a)

y, x	0	1	2	3
0	$\frac{1}{55}$	$\frac{6}{55}$	$\frac{6}{55}$	$\frac{1}{55}$
1	$\frac{6}{55}$	$\frac{16}{55}$	$\frac{6}{55}$	
2	$\frac{6}{55}$	$\frac{6}{55}$		
3	$\frac{1}{55}$			

(b) A = {(x,y): (1,1);(2,0);(0,2);(1,2);(2,1);(3,0);(0,3)}
$P[(X,Y) \in A] = \frac{16}{55} + \frac{6}{55} + \frac{6}{55} + \frac{6}{55} + \frac{6}{55} + \frac{1}{55} + \frac{1}{55} = \frac{42}{55}$.

Problem 3. Given the joint density function

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find $f_X(x)$, $f_Y(y)$, $g_X(x|y)$, and evaluate $P\left(\frac{1}{4} < X < \frac{1}{2} | Y = \frac{1}{3}\right)$.

Solution:

$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\ dy = \int_0^1 \frac{x(1+3y^2)}{4}\ dy = \frac{x}{2}$, $0 < x < 2$

$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\ dx = \int_0^2 \frac{x(1+3y^2)}{4}\ dx = \frac{1+3y^2}{2}$, $0 < y < 1$

$g_X(x|y) = \frac{f(x,y)}{h(y)} = \frac{x(1+3y^2)/4}{1+3y^2)/2} = \frac{x}{2}$, $0 < x < 2, 0 < y < 1$,

$P\left(\frac{1}{4} < X < \frac{1}{2} | Y = \frac{1}{3}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{x}{2} dx = \frac{3}{64}$.

Problem 4. The joint pdf of random variables X and Y is

$$f(x,y) = \begin{cases} xe^{-x-y} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the pdf of $Z = X + Y$.

Solution: X and Y are independent random variables with $f_X(x) = xe^{-x}$ for $x > 0$ and $f_Y(y) = e^{-y}$ for $y > 0$. The pdf of the sum is given as

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-u) f_Y(u) du.$$

As $z - u > 0$ and $u > 0$, we have that $0 < u < z$. Thus

$$f_Z(z) = \int_0^z (z-u)e^{-z+u}e^{-u}du = e^{-z}\int_0^z (z-u)du = e^{-z}\frac{z^2}{2}, \quad z > 0$$

and 0 otherwise.

Problem 5. For a meeting at 9 am Jim and Laura will arrive at a time uniformly distributed between 8:55 and 9:10. Assuming that the meeting starts sharp on time, and that Jim and Laura's arrivals are independent, what is the probability that they will both be in the room for the beginning of the meeting?
Solution: Let J be the random variable denoting Jim's arrival time, and L be Laura's arrival time. Then, setting the origin at 8:55, $J, L \sim Unif(0, 15)$. If they are both on time that means that the latest of the two is on time or, letting $X = \max\{J, L\}$ then we want $Pr(X \leq 5) = Pr(J \leq 5)Pr(L \leq 5) = (5/15)^2 = 1/9$.

Problem 6. Let X be a random variable with probability distribution

$$f_X(x) = \begin{cases} \frac{1}{3}, & x = 1, 2, 3 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability distribution of the random variable $Y = 2X - 1$.
Solution: $y = 2x - 1 \ \Rightarrow x = \frac{y+1}{2}$
$f_Y(y) = f\left(\frac{y+1}{2}\right) = \frac{1}{3}, y = 1, 3, 5.$

Problem 7. The hospital period, in days, for patients following treatment for a certain type of kidney disorder is a random variable $Y = X + 4$, where X has the density function

$$f_X(x) = \begin{cases} \frac{32}{(x+4)^3}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the probability density function of the random variable Y.

(b) Using the density function of Y, find the probability that the hospital period for a patient following this treatment will exceed 8 days.

Solution:

(a) $y = x + 4 \Rightarrow x = y - 4$
$\frac{dx(y)}{dx} = 1$
$f_Y(y) = f\left[w(y)\right] \, \frac{dx(y)}{dx} = f\left[y - 4\right] \cdot 1 = \frac{32}{y^3}, y > 4$

(b) $P\left(Y > 8\right) = 1 - P\left(Y \leq 8\right) = 1 - \int_4^8 \frac{32}{y^3} dy = \frac{1}{4}$

Problem 8. The speed of a molecule in a uniform gas at equilibrium is a random variable V whose probability distribution is given by

$$f_V(v) = \begin{cases} kv^2 e^{-bv^2}, & v > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

where k is an appropriate constant and b depends on the absolute temperature and mass of the molecule. Find the probability distribution of the kinetic energy of the molecule W, where $W = mV^2/2$.
Solution:

$$f_V(v) = \begin{cases} kv^2 e^{-bv^2}, & v > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

$w = mv^2/2 \Rightarrow v = \sqrt{\frac{2w}{m}}$

$\frac{dv}{dw} = \frac{1}{\sqrt{2mw}}$

$f_W(w) = k\left(\sqrt{\frac{2w}{m}}\right)^2 e^{b\frac{2w}{m}} \ \frac{1}{\sqrt{2mw}} = k\sqrt{\frac{2w}{m^3}} e^{b\frac{2bw}{m}},\ w > 0$