1. Base step $n=0$

$T(0) = 0 \leq 0^2 = 0$

2. Induction step.

Assume $T(n) \leq n^2$, we need to prove $T(n+1) \leq (n+1)^2$

$$
\begin{aligned}
T(n+1) &= T(n) + n \\
&\leq n^2 + n \\
&\leq n^2 + 2n \\
&\leq n^2 + 2n + 1 \\
&= (n+1)^2
\end{aligned}
$$

Thus we get $T(n+1) \leq (n+1)^2$, by the induction. we prove $T(n) \leq n^2 \ \forall n \geq 0$

2.
$$
\begin{aligned}
T(n) &= T(n-1) + 1 \\
&= T(n-2) + 1 + 1 \\
&= \cdots \\
&= T(n-k) + k
\end{aligned}
$$

The formula reach the end when $n-k=0$. so we subsitute $k=n$.

$$
\begin{aligned}
T(n) &= T(0) + n \\
&= 2 + n
\end{aligned}
$$

3. "a" $= \sqrt{a}$ "b" $= a$ "$\log_b a$" $= \log_a \sqrt{a} = \frac{1}{2}$ "$f(n)$" $= n^{\frac{1}{2}}$

Since $\underbrace{f(n) = \theta(n^{\frac{1}{2}})}_{\text{which falls to case 2.}}$ then $T(n) = \theta(n^{\frac{1}{2}} \cdot \log(n))$.

4. $\sum_{i=1}^{n} (i a^i) = 1 \cdot a^1 + 2 \cdot a^2 + \cdots + n \cdot a^n$

$$
\begin{aligned}
&\geq 1 \cdot a^1 + 2a + \cdots + na \\
&= \frac{(1+n) \cdot n}{2} a \\
&= \frac{n^2+n}{2} a \\
&\geq \frac{1}{2} n^2 a
\end{aligned}
$$

So the inequality holds when $c = \frac{1}{2}$.