CSE 102
Homework Assignment 1
Solutions

1. (Problem 3.1-1) Let $f(n)$ and $g(n)$ asymptotically positive functions. Prove that $f(n) + g(n) = \Theta(\max(f(n),\ g(n)))$.

   Proof:
   Since $f(n)$ and $g(n)$ are asymptotically positive, we know that there exists a positive constant n_0 such that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. For any such n we have

$$\begin{aligned} 0 &\leq \max\big(f(n), g(n)\big) \\ &\leq \min\big(f(n), g(n)\big) + \max\big(f(n), g(n)\big) \\ &\leq 2 \cdot \max\big(f(n), g(n)\big). \end{aligned}$$

   But $f(n) + g(n) = \min\big(f(n), g(n)\big) + \max\big(f(n), g(n)\big)$, and therefore

$$0 \leq 1 \cdot \max\big(f(n), g(n)\big) \leq f(n) + g(n) \leq 2 \cdot \max\big(f(n), g(n)\big)$$

   for all $n \geq n_0$, showing that $f(n) + g(n) = \Theta(\max\,(f(n), g(n)))$. ■

2. Prove or disprove: If $f(n) = \Theta(g(n))$, then $f(n)^2 = \Theta(g(n)^2)$.

   Proof:
   Since $f(n) = \Theta(g(n))$, there are positive constants c_1, c_2 and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

   for all $n \geq n_0$. Squaring this inequality gives $0 \leq c_1^2 g(n)^2 \leq f(n)^2 \leq c_2^2 g(n)^2$, showing that $f(n)^2 = \Theta(g(n)^2)$. ■

3. Prove or disprove: If $f(n) = \Theta(g(n))$, then $2^{f(n)} = \Theta(2^{g(n)})$.

   Counter Example:
   Let $f(n) = 2n$ and $g(n) = n$. Then $f(n) = \Theta(g(n))$, but $2^{2n} = 4^n = \omega(2^n)$, and therefore $2^{f(n)} = \omega(2^{g(n)})$, whence $2^{f(n)} \neq \Theta(2^{g(n)})$. ■

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions, and assume that $\lim\limits_{n\to\infty} g(n) = \infty$. Prove that if $f(n) = \Theta(g(n))$, then $\ln\big(f(n)\big) = \Theta\big(\ln\big(g(n)\big)\big)$.

   Proof:
   Assume $f(n) = \Theta(g(n))$. Then there exist positive constants c_1, c_2 and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0$. Since $\lim_{n\to\infty} g(n) = \infty$, the constant n_0 can be chosen large enough so that

$$1 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0$. Take ln() of all terms in the preceding inequality to get

$$0 \leq \ln(c_1) + \ln\big(g(n)\big) \leq \ln\big(f(n)\big) \leq \ln(c_2) + \ln\big(g(n)\big)$$

for all $n \geq n_0$. Since $\lim_{n\to\infty} g(n) = \infty$, the term $\ln\big(g(n)\big)$ dominates the constants $\ln(c_1)$ and $\ln(c_2)$, whence $\ln(c_1) + \ln\big(g(n)\big) = \Omega\Big(\ln\big(g(n)\big)\Big)$ and $\ln(c_2) + \ln\big(g(n)\big) = O\Big(\ln\big(g(n)\big)\Big)$. The desired result $\ln\big(f(n)\big) = \Theta\Big(\ln\big(g(n)\big)\Big)$ now follows by Exercise 4 on page 4 of the handout on asymptotic growth rates. ■

5. (Problem 3.2-8) Show that if $f(n)\ln f(n) = \Theta(n)$, then $f(n) = \Theta(n/\ln n)$. Hint: use the result of the preceding problem.

   Proof:
   Assume $f(n)\ln f(n) = \Theta(n)$. Since $\lim_{n\to\infty} n = \infty$, we can, by the result of problem (4), take ln () of both sides of this equation to get

$$\ln\big(f(n)\big) + \ln\Big(\ln\big(f(n)\big)\Big) = \Theta(\ln(n)).$$

   Since the term $\ln\Big(\ln\big(f(n)\big)\Big) = o(\ln\big(f(n)\big)$, we can, by Exercise 9g on page 8 of the handout on asymptotic growth rates, drop the lower order term and write this as $\ln\big(f(n)\big) = \Theta(\ln(n))$. Substituting this into our assumption gives $f(n)\Theta(\ln n) = \Theta(n)$, and therefore

$$f(n) = \Theta(n) \cdot \frac{1}{\Theta(\ln n)} = \Theta(n) \cdot \Theta\left(\frac{1}{\ln n}\right) = \Theta\left(\frac{n}{\ln n}\right)$$

   where we have used Exercise 11bc on page 8 of the handout on asymptotic growth rates. ■

6. Consider the statement: $f(cn) = \Theta(f(n))$.

   a. Determine a function $f(n)$ and a constant $c > 0$ for which the statement is false.

      Example:
      Let $f(n) = 2^n$ and $c = 2$. Then the statement says $2^{2n} = \Theta(2^n)$, i.e. $4^n = \Theta(2^n)$. This is false since $4^n = \omega(2^n)$ by Exercise 4e on page 7 of the handout on Asymptotic Growth Rates, and since $\omega(2^n) \cap \Theta(2^n) = \emptyset$ by Exercise 6 on page 5 of the same handout. ■

   b. Determine a function $f(n)$ for which the statement is true for all $c > 0$.

      Example:
      Let $f(n) = n$. Then the statement says $cn = \Theta(n)$. This is true for any $c > 0$ by Exercise 2 on page 3 of the handout on Asymptotic Growth Rates. ■

7. Determine the asymptotic order of the expression $\sum_{i=1}^{n} a^i$ where $a > 0$ is a constant, i.e. find a simple function $g(n)$ such that the expression is in the class $\Theta\big(g(n)\big)$. (Hint: consider the cases $a = 1$, $a > 1$, and $0 < a < 1$ separately.)

   Solution: In the case $a = 1$ we have $\sum_{i=1}^{n} a^i = n = \Theta(n)$. Now assume $a \neq 1$ and use the formula for the sum of a geometric series:

$$\sum_{i=1}^{n} a^i = a\left(\sum_{i=0}^{n-1} a^i\right) = a\left(\frac{a^n - 1}{a - 1}\right) = \left(\frac{a}{a-1}\right) \cdot a^n - \left(\frac{a}{a-1}\right)$$

   If $a > 1$ then $a^n \to \infty$ as $n \to \infty$, so the first term dominates the second, and $\sum_{i=1}^{n} a^i = \Theta(a^n)$. If $0 < a < 1$ then $a^n \to 0$, so the constant term (which is positive in this case) dominates, and therefore $\sum_{i=1}^{n} a^i = \Theta(1)$. ■

⑧ ⓐ $\lim_{n\to\infty} \frac{n\ln(n)}{n^2} = \lim_{n\to\infty} \frac{\cancel{n}\cdot\ln(n)}{\cancel{n}\cdot n} = \lim_{n\to\infty} \frac{\ln(n)}{n}$

$=$ By l'hopital's rule $\lim_{n\to\infty} \frac{[\ln(n)]'}{[n]'} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1}$

$= \lim_{n\to\infty} \frac{1}{n} = 0$

By Lemma 1 of Handout, $n\ln(n) = o(n^2)$ ▨

ⓑ $\lim_{n\to\infty} \frac{n^5\cdot 2^n}{n^{10}} = \lim_{n\to\infty} = \frac{n^5\cdot 2^n}{n^5\cdot n^5} = \lim_{n\to\infty} \frac{2^n}{n^5}$

$=$ By l'hopital's rule $\lim_{n\to\infty} \frac{(2^n)'}{(n^5)'} = \lim_{n\to\infty} \frac{2^n\cdot(\ln 2)}{5n^4}$

$=$ 〃 $\lim_{n\to\infty} \frac{(2^n)(\ln 2)^2}{5\cdot 4\cdot n^3} = \cdots = \lim_{n\to\infty} = \frac{(2^n)(\ln 2)^5}{5!}$

$= \infty$. Since $L=\infty \Rightarrow n^5 2^n = \omega(n^{10})$

ⓒ $P(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k$ where $a_k \neq 0$

$\lim_{n\to\infty} \frac{(P(n))}{n^k} = \lim_{n\to\infty} \frac{a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k}{n^k}$

$= \lim_{n\to\infty} \left(\frac{a_0}{n^k} + \frac{a_1}{n^{k-1}} + \cdots + a_k\right)$

$= a_k$

By Exercise 6 of Handout, $P(n) = \Theta(n^k)$

$\boxed{\text{with the assumption that } a_k > 0}$

▨

(9) (a) compare n^n with $2^{n \ln n}$

$$f(n) = n^n \Rightarrow \ln f(n) = n \ln n \Rightarrow f(n) = e^{n \ln n}$$

$$g(n) = 2^{n \ln n}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{e^{n \ln n}}{2^{n \ln n}} = \lim_{n\to\infty} \left(\frac{e}{2}\right)^{n \ln n} = \infty$$

$$\therefore \quad n^n = \omega(2^{n \ln n}). \quad \boxtimes$$

(b) $\sqrt{\ln(n)}$ compared to $\ln(\ln n)$

$$\lim_{n\to\infty} \frac{\ln(\ln(n))}{\sqrt{\ln(n)}} = \text{By l'hopital's rule } \lim_{n\to\infty} \frac{[\ln(\ln(n))]'}{[\sqrt{\ln(n)}]'}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\ln(n)} \cdot [\ln(n)]'}{\frac{1}{2}[\ln(n)]^{-1/2} [\ln(n)]'} \quad \text{By composite rule of differentiation}$$

$$= \lim_{n\to\infty} 2 \cdot \frac{1}{[\ln(n)]^{1/2}} \frac{\cancel{\frac{1}{n}}}{\cancel{\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{\ln(n)}} = 0$$

By Lemma 1 of Handout, $\ln(\ln n) = o(\sqrt{\ln(n)})$ $\boxtimes$