# APPM 4600 — Homework #3

Jaden Snell

## Problem 1

Consider the equation $2x - 1 = \sin(x)$.

(a) Find a closed interval $[a, b]$ on which the equation has a root $r$ and use the Intermediate Value Theorem to prove that $r$ exists.

The equation, $f(x) = 2x - 1 - sin(x)$, on the closed interval $[0.5, 1]$, has a root $r$. If $a = 0.5$ and $b = 1$ then. $f(a) = -\sin(0.5) < 0$ and $f(b) = 1 - \sin(1) > 0$. Therefore, through Intermediate Value Theorem, there exists at least one number $r$ such that $f(r) = 0$ as 0 is in between $f(a)$ and $f(b)$.

(b) Prove that $r$ from part (a) is the only root of the equation (on all of $\mathbb{R}$).

First, it's important to note that the function is continuous everywhere and differentiable. The derivative,

$$f'(x) = 2 - \cos(x) > 0, \ \forall x \in \mathbb{R}$$

meaning the the function is increasing thus unable to cross the x-axis more than the unique root $r$.

(c) Use the bisection code from class (or your own) to approximate $r$ to eight correct decimal places. Include the calling script, the resulting final approximation, and the total number of iterations used.

The final approximation was 0.8878622110933065 after 28 iterations. Code is pushed to the github in the Homework 3 folder contained in the `hw3bisection.py` file.

## Problem 2

The function $f(x) = (x-5)^9$ has a root (with multiplicity 9) at $x = 5$ and is monotonically increasing (decreasing) for $x > 5$ ($x < 5$). Use $a = 4.82$ and $b = 5.2$ with $tol = 1e - 4$ and use bisection method.

(a) $f(x) = (x-5)^9$

The result from using this condensed version of the function yields $x = 4.9999804687500005$. Code is pushed to the github in the Homework 3 folder contained in the `hw3bisection.py` file.

(b) The expanded version of $(x-5)^9$ is $f(x) = x^9 - 45x^8 + \cdots - 1953125$.

The result from using this expanded version of the function yields $x = 5.12875$. Code is pushed to the github in the Homework 3 folder contained in the `hw3bisection.py` file.

(c) Explain what is happening.

Due to the multiple subtraction operations that occur in the polynomial, as x approaches 5, cancellation errors increase substantially leading to an approximate root that is significantly wrong.

# Problem 3

(a) Use a theorem from class (Theorem 2.1 from text) to find an upper bound on the number of iterations in the bisection needed to approximate the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$ with an accuracy of $10^{-3}$.

Theorem 2.1 states that for an $f \in C[a, b]$ and $f(a)f(b) < 0$, the Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \leq \frac{b-a}{2^n}, \text{ when } n \geq 1.$$

Utilizing this theorem, the upper bound on the number of iterations in the bisection needed to approximate the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$ with an accuracy of $10^{-3}$ is

$$\begin{aligned} \frac{4-1}{2^n} &= 10^{-3} \\ 3 &= 2^n * 10^{-3} \\ 3 * 10^3 &= 2^n \\ n &= \log_2(3 * 10^3) \\ n &= 11.55. \end{aligned}$$

However $n \in \mathbb{Z}$, so the upper limit is 12 iterations.

(b) Find an approximation of the root using the bisection code from class to this degree of accuracy. How does the number of iterations compare with the upper bound you found in part (a)?

The approximate root calculated was 1.378662109375 and the number of iterations was 11 to find the root to the accuracy prescribed which makes since as the theorem only provides an upper limit. The code is pushed to the github in the Homework 3 folder contained in the `hw3bisection.py` file.

## Problem 4

Definition 1: Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to $p$ with $p_n \neq p$ for all $n$. If there exists positive constants $\lambda$ and $\alpha$ such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda,$$

then $\{p_n\}_{n=1}^{\infty}$ converges to $p$ with an order $\alpha$ and asymptotic error constant $\lambda$. Which of the following iterations will converge to the indicated fixed point $x^*$ (provided $x_0$ is sufficiently close to $x^*$)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.

Note: The limit must exist and be finite as that represents the error decreasing as $n$ increases implying that $p_n$ must converge to $p$. In addition, $\alpha$ and $\lambda$ must be positive as $\alpha$ represents the order of convergence and $\lambda$ represents the asymptotic error . For the different types of convergence given by $\alpha$, when $\alpha = 1$, the convergence is linear. If $1 < \alpha < 2$, the convergence is super-linear, and if $\alpha = 2$, then the convergence is quadratic.

(a) (10 points) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}, x_* = 2$

For $g(x) = -16 + 6x + \frac{12}{x}$, $g'(x) = 6 - \frac{12}{x^2}$. Plugging in $x_* = 2$ yields $g'(2) = 6 - \frac{12}{4} = 3 > 1$ which does not converge to the indicated fixed point.

(b) (10 points) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}, x_* = 3^{1/3}$

For $g(x) = \frac{2}{3}x + \frac{1}{x^2}$, $g'(x) = \frac{2}{3} - \frac{2}{x^3}$. Plugging in $x_* = 3^{\frac{1}{3}}$ yields $g'(3^{\frac{1}{3}}) = \frac{2}{3} - \frac{2}{3} = 0$ meaning that the iteration will converge to the fixed point. Because the first derivative is 0, we utilize Taylor expansion about $x_*$ for $g(x)$.

$$g(x) = g(x_*) + g'(x_*)(x - x_*) + \frac{g"(x_*)}{2}(x - x_*)^2$$

We then substitute $x_n$ for x, $x_{n+1} = g(x_n)$, $g(x_*) = x_*$, and $g'(x_*) = 0$ which yields

$$x_{n+1} = x_* + \frac{g"(x_*)}{2}(x_n - x_*)^2$$

which is equivalent to

$$\frac{x_{n+1} - x_*}{(x_n - x_*)^2} = \frac{g"(x_*)}{2}$$

as $n \to \infty$,

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|^2} = \frac{|g"(x_*)|}{2}.$$

Comparing this equation to the one given in the problem, one can observe that $\alpha = 2$ thus showing the iteration converges quadratically.

(c) (10 points) $x_{n+1} = \frac{12}{1+x_n}, x_* = 3$

For $g(x) = \frac{12}{1+x}$, $g'(x) = -\frac{12}{(1+x)^2}$. Plugging in $x_* = 3$ yields $g'(3) = -\frac{12}{16} = -\frac{3}{4}$ meaning that the iteration will converge to the fixed point as $|g'(3)| < k < 1$. To find the order of convergence, we utilize a similar proof to the one given in part (b). First, use Taylor expansion about $x_*$ for $g(x)$.

$$g(x) = g(x_*) + g'(x_*)(x - x_*)$$

We then substitute $x_n$ for x, $x_{n+1} = g(x_n)$, and $g(x_*) = x_*$ which yields

$$x_{n+1} = x_* + g'(x_*)(x_n - x_*)$$

which is equivalent to

$$\frac{x_{n+1} - x_*}{(x_n - x_*)} = g'(x_*)$$

as $n \to \infty$,

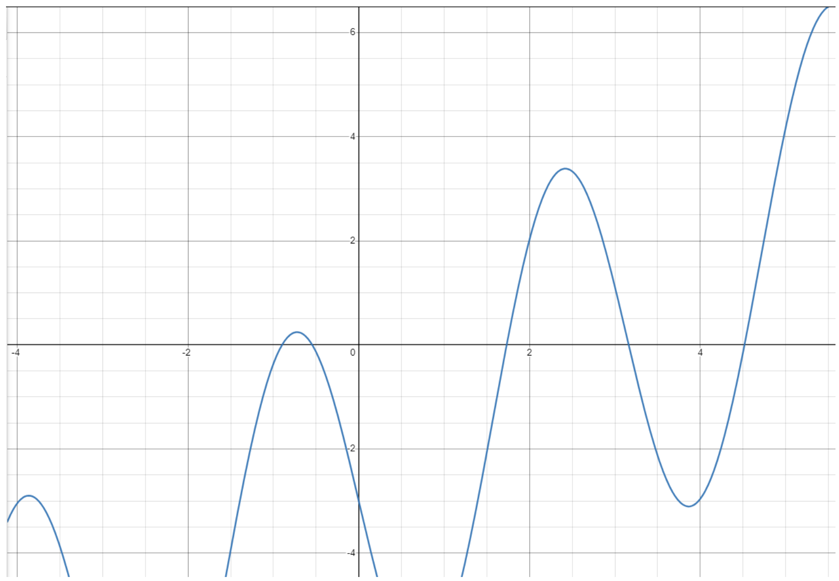
$$\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|} = |g'(x_*)|$$

. Comparing this equation to the one given in the problem, one can observe that $\alpha = 1$ thus showing the iteration converges linearly. The rate of convergence is

$$|g'(x_*)| = |g'(3)| = |-\frac{3}{4}| = \frac{3}{4}.$$

# Problem 5

(a) Plot $f(x) = x - 4\sin(2x) - 3$. Determine the number of zero crossings.




As shown on the graph, there are 5 zero crossings.

(b) Compute the roots using the fixed point iteration $x_{n+1} = -\sin(2x_n) + \frac{5}{4}x_n - \frac{3}{4}$ with a stopping criterion for ten correct digits. Find empirically which of the roots can be found with the above iteration and provide a theoretical explanation.

The only roots that could be found by the fixed point iteration listed above is $x_0 = -0.5444424007$ and $x_1 = 3.1618264865$. The code running the iteration is in the `hw3fixedpt.py` file of the Homework 3 folder in the github repository.

To empirically find which of the roots can be found with the above iteration, we take the derivative of $g(x) = -\sin(2*x) + \frac{5}{4}*x - \frac{3}{4}$ which is

$$g'(x) = -2\cos(2*x) + \frac{5}{4}$$

and find the sub-interval where $|g'(x)| < 1$ in the interval $-2 < x < 5$ (determined through inspection of the graph). This sub-interval is where $-\frac{\arccos(\frac{1}{8})}{2}+n\pi < x < \frac{\arccos(\frac{1}{8})}{2}+n\pi$, $n \in \{0, 1\}$. Both roots found by the above iteration fall within this interval which allows the iterations to converge to the root. This explains why the other roots couldn't be found using this specific iteration, the condition that $|g'(x)| < 1$ was not met for the intervals of $x$ containing the root.