Version 2024.1

Assignment 4

Mathematics for Robotics

C. Gummaluru, A. Shoelling *University of Toronto*

This assignment is part of the *Fall 2024* offering of *ROB310: Mathematics for Robotics* at the University of Toronto. Please review the information below carefully. Submissions that do not adhere to the guidelines below may lose marks.

Release Date: .... October 07, 2024 - 12:00
Due Date: .... October 14, 2024 - 11:59
Weight: .... 3.75% or 0%
Late Penalties: .... -10% per day and -100% after 3 days

Submission Information:

- Download this document as a PDF and fill in your information below.
- Answer the questions to the best of your ability.
- Typeset or **neatly hand-write** your answers, labelling the questions identically to this document. We recommend using LaTeX, but will accept the use of other typesetting software (such as Microsoft Word or Google Docs).
- Save each question as a separate PDF document. Do NOT include the questions.
- Submit your PDFs on Crowdmark titled `lastname-firstname-rob310f24-a3-q#.pdf`.

Problem 4.1

Find all of the critical points for each of the following functions, $f : \mathbb{R}^2 \to \mathbb{R}$. Determine whether any of these critical points are local/global minimizers. Show your work.

Problem 4.1.a

$f(\mathbf{x}) = x_1 \sin x_2$

Problem 4.1.b

$f(\mathbf{x}) = \dfrac{x_1}{1 + x_1^2 + x_2^2}$

Problem 4.2

Prove the following properties about convex functions. Make sure your proof is clear. Do not just use mathematical notation. Explain your steps.

Problem 4.2.a

Show that the sum of two convex functions is convex.

Problem 4.2.b

Suppose $\mathbf{A} \in \mathbb{R}^{n,n}$, $\mathbf{b} \in \mathbb{R}^n$ are constants and let

$$\|\mathbf{x}\|_2 = \sqrt{\sum_i x_i^2}$$

Show that $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2$ is convex in $\mathbf{x}$.

Problem 4.3

Let $\mathbf{C} \in \mathbb{R}^{n,n}$ be fixed and consider the following functions over $\mathbb{R}^n$:

$$f_{\text{square}}(\mathbf{x}) = \mathbf{x}^\top \mathbf{C}\mathbf{x}$$

and

$$f_{\text{hole}}(\mathbf{x}) = 1 - \exp\left(\mathbf{x}^\top \mathbf{C}\mathbf{x}\right).$$

For this problem, assume that $\mathbf{C}$ is a diagonal matrix, with the entry on the i^{th} diagonal element being

$$\mathbf{C}_{i,i} = c^{\frac{i-1}{n-1}},$$

where $c = 10$.

Problem 4.3.a

Compute ∇f_{square} and ∇f_{hole}.

Problem 4.3.b

Compute $\nabla^2 f_{\text{square}}$ and $\nabla^2 f_{\text{hole}}$.

Problem 4.3.c

Using `Python` with the `matplotlib` library, plot f_{square} and f_{hole} for $n = 2$ over $\mathbf{x} \in [-1, 1]^2$.

Problem 4.3.d

Implement a simple fixed-stepsize gradient descent algorithm, iterating $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha \nabla f_{\text{square}}(\mathbf{x}_k)$, with start point $\mathbf{x}_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$. Adjust α as needed. Submit the relevant sections of your code and the output of the algorithm in each step for both functions.

Problem 4.3.e

Implement Newton's method and compare your results to the ones you got in Part 4.3.d in terms of computation time per step, overall computation time, and number of steps required for the same accuracy of the result.