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Assignment 2

Mathematics for Robotics

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This assignment is part of the *Fall 2024* offering of *ROB310: Mathematics for Robotics* at the University of Toronto. Please review the information below carefully. Submissions that do not adhere to the guidelines below may lose marks.

Release Date: .... September 23, 2024 - 00:00
Due Date: .... September 29, 2024 - 23:59
Weight: .... 3.75% or 0%
Late Penalties: .... -10% per day and -100% after 3 days

Submission Information:

- Download this document as a PDF and fill in your information below.
- Answer the questions to the best of your ability.
- Typeset or **neatly hand-write** your answers, labelling the questions identically to this document. We recommend using LaTeX, but will accept the use of other typesetting software (such as Microsoft Word or Google Docs).
- Save your answers as a PDF and combine it with the cover-sheet of this document. Do NOT include the question descriptions.
- Submit your PDF on Crowdmark titled `lastname-firstname-rob310f24-a2.pdf`.

Name (First, Last): .... Jaden, Stanshall

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Acknowledgement:

I hereby affirm that the work I am submitting for this assignment is entirely my own. I have not received any unauthorized assistance, and I understand that plagiarism or any other form of academic dishonesty is a violation of the University's academic integrity policy.

I agree to the above statement.......... ✔

Date (dd/mm/yyyy): 29/11/2024

2 Assignment 2

2.1 Centered Difference Derivative Approximation

Consider the following derivative approximation formula (the **Centered Difference Approach**):

$$\hat{f}'(x) = \frac{f(x+\Delta) - f(x-\Delta)}{2\Delta} \tag{1}$$

2.1.A

We can say this approximation is **consistent** if the following condition holds:

$$\lim_{\Delta \to 0} \hat{f}'(x) = f'(x) \tag{2}$$

Where $f'(x)$ is the actual derivative of $f(x)$. To show this approximation is consistent, we will take the limit of $\hat{f}'(x)$ as Δ approaches 0:

$$\lim_{\Delta \to 0} \hat{f}'(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x-\Delta)}{2\Delta} \tag{3}$$

Using L'Hôpital's Rule:

$$\lim_{\Delta \to 0} \hat{f}'(x) = \frac{f'(x+\Delta) + f'(x-\Delta)}{2} \tag{4}$$

And as $\Delta \to 0$:

$$\lim_{\Delta \to 0} \hat{f}'(x) = \frac{f'(x) + f'(x)}{2} \tag{5}$$

and so,

$$\lim_{\Delta \to 0} \hat{f}'(x) = f'(x) \tag{6}$$

Therefore, equation (2) holds and we have shown the centered difference approach is consistent.

2.1.B

The **order of accuracy** of the centered difference approach is 2. To show this, we will begin by representing the derivative of our function $f(x)$ as a Taylor Series approximation (where $\dot{x}(t_k) = f'(x)$):

$$x_{k+1} = x_k + T(\dot{x}(t_k)) + O(T^2) \tag{7}$$

We will express (1) in the same form (where $\dot{x}(t_k) = f'(x)$ as we have shown $\hat{f}'(x)$ is consistent):

$$\hat{x}_{k+1} = x_x + T(\dot{x}(t_k)) \tag{8}$$

Subtracting the actual function x_{k+1} from the approximation $\hat{x}_{k+1}$, we get:

$$\hat{x}_{k+1} - x_{k+1} = [x_x + T(\dot{x}(t_k))] - [x_k + T(\dot{x}(t_k)) + O(T^2)] \tag{9}$$

Simplifying, we get:

$$\hat{x}_{k+1} - x_{k+1} = O(T^2) \tag{10}$$

Thus, we have shown the centered difference approach has an order of accuracy equal to 2.

2.1.C

We would like to determine the conditions under which the centered difference approach is **stable**. To do this, we must determine the conditions under which the following is true:

$$\left|\left| \frac{d\hat{f}'(x)}{dx}\bigg|_x \right|\right| \leq 1 \tag{11}$$

We begin by computing the derivative of (1) with respect to x:

$$\left|\left| \frac{d}{dx}\frac{\hat{f}(x+\Delta)-\hat{f}(x-\Delta)}{2\Delta}\bigg|_x \right|\right| \leq 1 \tag{12}$$

$$\left|\left| \frac{f'(x+\Delta)-f'(x-\Delta)}{2\Delta}\bigg|_x \right|\right| \leq 1 \tag{13}$$

By definition (1), we get:

$$\left|\left| \hat{f}''(x) \right|\right| \leq 1 \tag{14}$$

So, the second derivative of $f(x)$ must be less than or equal to 1 for the centered difference approximation to be stable. Additionally, the the second derivative of $f(x)$ must be strictly less than 1 for the centered difference approximation to converge.

2.2 Root Finding

2.3 Newton's Method

Given a function $f(x)$ which is once continuously differentiable and its derivative $f'(x)$ is inexpensive to evaluate, we would use **Newton's Method** to numerically fund the root of $f(x)$.

2.3.A Fixed Point Method

Given a function $f(x)$ which is Lipschitz continuous with constant $0L < 1$, we would use the **Fixed Point Method** to numerically find the root of $f(x)$.

2.3.B Bisection Method

Given a function $f(x)$ which is once continuously differentiable and its derivative $f'(x)$ is expensive to evaluate, we would use the **Bisection Method** to numerically find its root as computing the derivative is not required for the Bisection Method to find the root of $f(x)$.

2.3.C Bisection Method

Given a function $f(x)$ which is continuous but not differentiable, we would use the **Bisection Method** to numerically find its root, as once again, computing the derivative of $f(x)$ is not required for the Bisection Method to find the root of $f(x)$.

2.4 Numerical Integration

Consider a function $f(x)$ where f is continuous.

2.4.A

We will use the **Trapezoidal Method** to integrate the following series of points $(x_i, f(x)_i))$ from $x = 0$ to $x = 0.2$:

$$(0, 0), (0.05, 0.15), (0.1, 0.3), (0.15, 0.45), (0.2, 0.6) \tag{15}$$

A plot of the series:

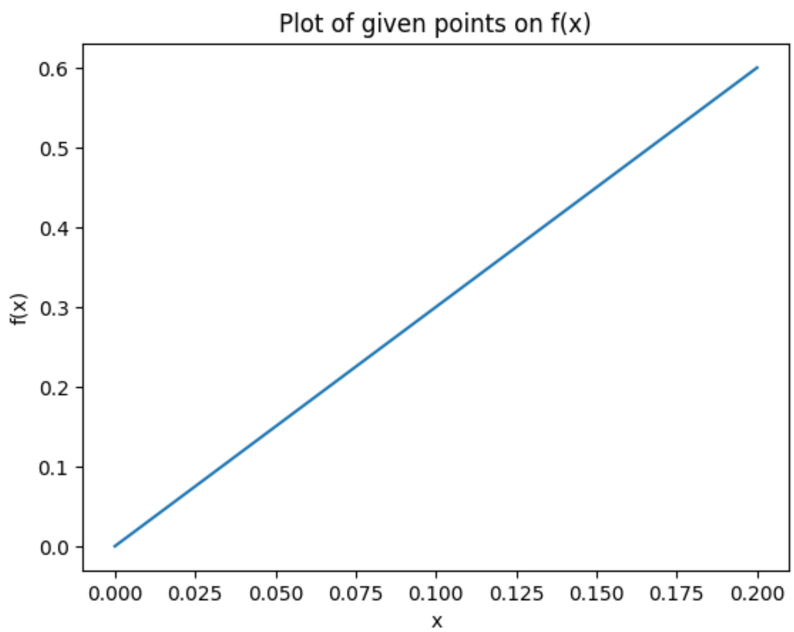

Figure 1:

The following is a Python implementation of the Trapezoidal Method:

```
# define points
x = np.array([0, 0.05, 0.1, 0.15, 0.2])
y = np.array([0, 0.15, 0.3, 0.45, 0.6])
```

```
# define function
def my_trapezoidal(x, y):
    sum = 0
    for i in range(1, len(x)):
        sum = sum + (x[i] - x[i - 1]) * (y[i] + y[i - 1]) /2
    return sum
```

Calling `my_trapezoidal()` on the above points will produce an output of 0.060 (within the margin of error of floating point calculations). This is equal to the analytical solution determined by treating the set of points as a triangle and computing its area.

2.4.B

Given the linearity of the set of points in Part 2.3.A, it may have been reasonable to use an integral approximation algorithm which simply treated the integral as a triangle with:

$$base = x[-1] - x[0], \qquad height = y[-1] - y[0] \tag{16}$$

However, the Trapezoidal Method would, in theory, be more applicable to a variety of series of points (such as the series in Part 2.3.B), so I decided to use the Trapezoidal Method to approximate the integral.

Aside from the given information that f is continuous, we have no information describing the behaviour of f between the provided discrete points, so any attempts to predict what those may be would further reduce the accuracy of our approximation; this is why I thought it best to simply connect the dots and compute the area underneath using the Trapezoidal Method.

2.4.C

Consider the following series of points $(x_i, f(x_i))$:

(0, 1), (0.13, 0.5198), (0.37, 0.6207), (0.49, 0.1728), (0.81, 1.259),
(1.06, 0.121), (1.19, 0.6467), (1.61, 0.6537), (1.94, 1.113), (2.06, 1.835)

A plot of the series:

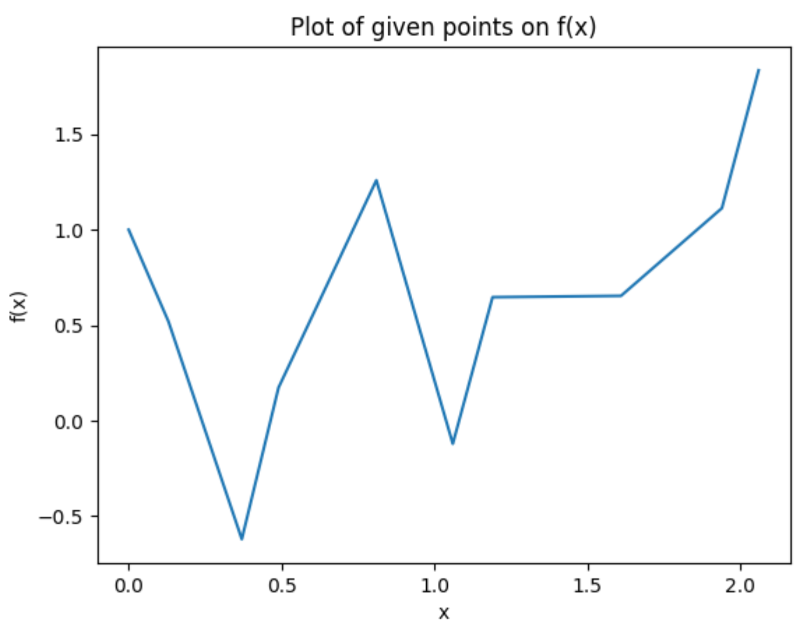

Figure 2:

Calling `my_trapezoidal()` on the above points (to integrate the series from $x = 0$ to $x = 2.06$) will produce a result of approximately 1.20678.

2.4.D

In terms of integrating the given discrete series', I am 100% confident that we have accurately determined the area under each set of points. In terms of approximating the integral of $f(x)$ using the above series', I am not very confident. The Trapezoidal Method's job is to connect the points and compute the area underneath the set of discrete points. As such, the Trapezoidal Method does include any estimation of the behaviour of $f(x)$ between the given points. This is good for us, however, since we have no information regarding the behaviour of $f(x)$ between the given points and any guesses would likely impact our approximated integral of $f(x)$ negatively. For this reason, the Trapezoidal Method will give the best approximation given the information in this scenario, however, I am not confident in even our best approximation.

In order to be more confident, it would be helpful to have more points of $f(x)$ on the given interval. This would disambiguate the behaviour of $f(x)$ and allow us to make a more informed (and likely more accurate) approximation. In theory, the more we know about $f(x)$, the more accurately (and confidently) we can approximate its integral, so the accuracy (and confidence) of our approximation should be proportional to the number of points of $f(x)$ we are given (assuming the given points are relatively evenly distributed across the given interval).