4 Assignment 4

4.1 Finding Critical Points and Local/Global Minimizers

4.1.A

Consider the following function $f : \mathbb{R}^n \to \mathbb{R}$:

$$f(\mathbf{x}) = x_1 \sin(x_2) \tag{1}$$

We would like to find the critical points of this function. To begin, we will compute the first partial derivatives of the function:

$$f_{x_1} = \frac{\partial f}{\partial x_1} = \sin(x_2) \tag{2}$$

$$f_{x_2} = \frac{\partial f}{\partial x_2} = x_1 \cos(x_2) \tag{3}$$

A critical point $\mathbf{x}^* \in \mathbb{R}^n$ of the function will exist where both partial derivatives are 0. Thus, we will need to find:

$$\mathbf{x}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \tag{4}$$

Such that:

$$f_{x_1}(x_1^*, x_2^*) = f_{x_2}(x_1^*, x_2^*) = 0 \tag{5}$$

Thus, we have the following system of equations which, once solved, will give us our critical point $\mathbf{x}^*$:

$$\sin(x_2^*) = 0 \tag{6}$$

$$x_1^* \cos(x_2^*) = 0 \tag{7}$$

Consider first the case where:

$$x_2^* = \frac{\pi}{2}(2k-1), \qquad k \in \mathbf{Z} \tag{8}$$

As this would satisfy (7). Subbing this into (6) we get:

$$\sin(x_2^*) = \sin(\frac{\pi}{2}(2k-1)) = \pm 1 \neq 0 \tag{9}$$

invalidating our initial guess for x_2^*. Consider instead $x_2^* = k\pi, k \in \mathbf{Z}$ which would satisfy (6). Subbing this into (7) we get:

$$x_1^* \cos(x_2^*) = x_1^* \cos(k\pi) = x_1^*(\pm 1) \tag{10}$$

And so we can set $x_1^* = 0$ to satisfy (7) with $x_2^* = k\pi$. This means our critical points are:

$$\mathbf{x}^* = \begin{bmatrix} 0 \\ k\pi \end{bmatrix}, \qquad k \in \mathbf{Z} \tag{11}$$

To check if any of these critical points are local or global minimizers, we will compute the Hessian of the function, compute its eigenvalues, and then substitute our critical points into the eigenvalues. If all eigenvalues are non-negative, the Hessian is positive semi-definite, meaning the critical point is a local minimum. The Hessian of $f(\mathbf{x}) = x_1 \sin(x_2)$ and its eigenvalues, computed with Wolfram Alpha, are as follows:

$$H = \begin{bmatrix} 0 & \cos(x_2) \\ \cos(x_2) & -x_1 \sin(x_2) \end{bmatrix} \tag{12}$$

$$\lambda_1 = \frac{-\sqrt{2}}{4}\sqrt{(x_1^2(-\cos(2x_2)) + x_1^2 + 4\cos(2x_2) + 4)} - 2x_1\sin(x_2) \tag{13}$$

$$\lambda_2 = \frac{\sqrt{2}}{4}\sqrt{(x_1^2(-\cos(2x_2)) + x_1^2 + 4\cos(2x_2) + 4)} - 2x_1\sin(x_2) \tag{14}$$

Substituting our $\mathbf{x}^*$ into λ_1, we get:

$$\lambda_1 = \frac{-\sqrt{2}}{4}\sqrt{0 + 0 + 4\cos(2k\pi + 4)} - 0 \tag{15}$$

And since the cosine of any even multiple of π is $+1$, the term under the root is positive, and λ_1 is negative. Thus, not all eigenvalues of the Hessian are non-negative and the Hessian is not positive semi-definite, thus our critical point $\mathbf{x}^*$ is not a local minimizer. And since $\mathbf{x}^*$ is not a local minimum, it certainly is not a global minimum.

4.1.B

Consider the following function $f : \mathbb{R}^n \to \mathbb{R}$:

$$f(\mathbf{x}) = \frac{x_1}{1 + x_1^2 + x_2^2} \tag{16}$$

We would like to find the critical points of this function. To begin, we will compute the first partial derivatives of the function:

$$f_{x_1} = \frac{\partial f}{\partial x_1} = \frac{-x_1^2 + x_2^2 + 1}{(1 + x_1^2 + x_2^2)^2} \tag{17}$$

$$f_{x_2} = \frac{\partial f}{\partial x_2} = \frac{-2x_1x_2}{(1 + x_1^2 + x_2^2)^2} \tag{18}$$

A critical point $\mathbf{x}^* \in \mathbb{R}^n$ of the function will exist where both partial derivatives are 0. Thus, we will need to find:

$$\mathbf{x}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \tag{19}$$

Such that:

$$f_{x_1}(x_1^*, x_2^*) = f_{x_2}(x_1^*, x_2^*) = 0 \tag{20}$$

We know the denominator of the function will always be non-negative, given that it is squared, so we will focus on setting the numerator of the partial derivatives to 0. Thus, we have the following system of equations which, once solved, will give us our critical point $\mathbf{x}^*$:

$$-x_1^{*2} + x_2^{*2} + 1 = 0 \tag{21}$$

$$-2x_1^*x_2^* = 0 \tag{22}$$

To solve, first consider the case where $x_1^* = 0$ which satisfies (22). Subbing this into (21), we get:

$$0 + x_2^{*2} + 1 = 0 \tag{23}$$

$$x_2^* = \sqrt{-1} \tag{24}$$

Since we define the inputs of f to be real, this is not a valid value for x_2^*. We try again with $x_2^{*2} = 0$ which satisfies (22). Subbing into (21), we get:

$$-x_1^* = \sqrt{1} = \pm 1 \tag{25}$$

And so, our critical points for f are:

$$\mathbf{x}_a^* = \begin{bmatrix} +1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_b^* = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{26}$$

To check if any of these critical points are local or global minimizers, we will compute the Hessian of the function, compute its eigenvalues, and then substitute our critical points into the eigenvalues. If all eigenvalues are non-negative, the Hessian is positive semi-definite, meaning the critical point is a local minimum. The Hessian of $f(\mathbf{x}) = x_1 \sin(x_2)$ and its eigenvalues, computed with Wolfram Alpha, are as follows:

$$H = \begin{bmatrix} \frac{8x^3}{(x^2+y^2+1)^3} - \frac{6x}{(x^2+y^2+1)^2} & \frac{8x^2y}{(x^2+y^2+1)^3} - \frac{2y}{(x^2+y^2+1)^2} \\ \frac{8x^2y}{(x^2+y^2+1)^3} - \frac{2y}{(x^2+y^2+1)^2} & \frac{8xy^2}{(x^2+y^2+1)^3} - \frac{2x}{(x^2+y^2+1)^2} \end{bmatrix} \tag{27}$$

$$\lambda_1 = \frac{2\left(-\sqrt{x_1^6 + 3x_1^4x_2^2 - 2x_1^4 + 3x_1^2x_2^4 + x_1^2 + x_2^6 + 2x_2^4 + x_2^2} - 2x_1\right)}{(x_1^2 + x_2^2 + 1)^3} \tag{28}$$

$$\lambda_2 = \frac{2\left(\sqrt{x_1^6 + 3x_1^4x_2^2 - 2x_1^4 + 3x_1^2x_2^4 + x_1^2 + x_2^6 + 2x_2^4 + x_2^2} - 2x_1\right)}{(x_1^2 + x_2^2 + 1)^3} \tag{29}$$

We would like to check, now, to see if any of our critical points are local or global minimizers. Substituting $\mathbf{x}_a^*$ into λ_1, we get:

$$\lambda_1 = \frac{2\left(-\sqrt{1+0-2+0+1+0+0+0} - 2\right)}{(1+0+1)^3} = \frac{2\,(-2)}{8} = -\frac{1}{2} \tag{30}$$

Since λ_1 is negative at $\mathbf{x}_a^*$, $\mathbf{x}_a^*$ is not a local minimum of f. Let's try substituting $\mathbf{x}_b^*$ into λ_1 and λ_2:

$$\lambda_1 = \frac{2\left(-\sqrt{1+0-2+0+1+0+0+0} + 2\right)}{(1+0+1)^3} = \frac{2\,(+2)}{8} = \frac{1}{2} \tag{31}$$

$$\lambda_2 = \frac{2\left(\sqrt{1+0-2+0+1+0+0+0} + 2\right)}{(1+0+1)^3} = \frac{2\,(+2)}{8} = \frac{1}{2} \tag{32}$$

And since both eigenvalues of the Hessian of f are positive at $\mathbf{x}_b^*$, the Hessian is positive semi-definite at that critical point which we know means the point $\mathbf{x}_b^*$ is a local minimum of f. Since the denominator of f grows faster than the numerator, the function is decreasing the further away we move from the origin, approaching 0 in every direction. This means there are no edge points of the domain of the function which could be a global minimum, and so the critical point and local minimum we found, $\mathbf{x}_b^*$ is the global minimum of f.