4.2 Properties of Convex Functions

4.2.A Sum of 2 Convex Functions

We would like to show that the sum of two convex functions is also convex. We begin with the following convex functions:

$$f(\mathbf{x}), \qquad f : \mathbb{R}^n \to \mathbb{R} \tag{33}$$

$$g(\mathbf{x}), \qquad g : \mathbb{R}^n \to \mathbb{R} \tag{34}$$

Each defined by the definition of a convex function:

$$f(t\mathbf{x} + (1-t)\mathbf{x}') \leq tf(\mathbf{x}) + (1-t)f(\mathbf{x}') \tag{35}$$

$$g(t\mathbf{x} + (1-t)\mathbf{x}') \leq tg(\mathbf{x}) + (1-t)g(\mathbf{x}') \tag{36}$$

Where $t \in [0, 1]$ and x, x' are in the domain of f and g (they have the same domain, $\mathbb{R}^n$). We will also define the sum of f and g as:

$$h(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x}) \tag{37}$$

We would like to show that $h(\mathbf{x})$ is convex; ie. we would like to show:

$$h(t\mathbf{x} + (1-t)\mathbf{x}') \leq th(\mathbf{x}) + (1-t)h(\mathbf{x}') \tag{38}$$

We begin by writing the sum of f and g, by definition:

$$f(t\mathbf{x} + (1-t)\mathbf{x}') + g(t\mathbf{x} + (1-t)\mathbf{x}') \leq tf(\mathbf{x}) + (1-t)f(\mathbf{x}') + tg(\mathbf{x}) + (1-t)g(\mathbf{x}') \tag{39}$$

Factoring out the ts and $(1-t)$s on the left hand side and rearranging, we get:

$$f(t\mathbf{x} + (1-t)\mathbf{x}') + g(t\mathbf{x} + (1-t)\mathbf{x}') \leq t(f(\mathbf{x}) + g(\mathbf{x})) + (1-t)(f(\mathbf{x}') + g(\mathbf{x}')) \tag{40}$$

Substituting the definition of $h(\mathbf{x})$ (37) into (40), we get:

$$h(t\mathbf{x} + (1-t)\mathbf{x}') \leq th(\mathbf{x}) + (1-t)h(\mathbf{x}') \tag{41}$$

which is (38), the inequality which shows $h(\mathbf{x})$ is convex by the definition of convexity. Since f and g are convex and as we have shown h is convex directly using the definition of convexity, we have shown, in the general case, that the sum of two convex functions is convex.

4.2.B Norm of Convex Functions

Consider the function:

$$\mathbf{A}\mathbf{x} - \mathbf{b}, \qquad \mathbf{A} \in \mathbb{R}^{n,n}, \qquad \mathbf{b} \in \mathbb{R}^n \tag{42}$$

And consider the norm operator:

$$\|\mathbf{x}\|_2 = \sqrt{\sum_i x_i^2} \tag{43}$$

We would like to show that:

$$\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \tag{44}$$

is convex in $\mathbf{x}$. We begin by showing $\mathbf{A}\mathbf{x} - \mathbf{b}$ is convex itself to begin with. Then we can show that the norm operator (43) preserves convexity. And finally, we will show that the composition of two functions is also convex. First, define $g : \mathbb{R}^n \to \mathbb{R}^n$:

$$g(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b}, \qquad \mathbf{A} \in \mathbb{R}^{n,n}, \qquad \mathbf{b} \in \mathbb{R}^n \tag{45}$$

g is the image under an affine map, and as such, is convex if its domain is convex. The domain of g is the set $S = \mathbb{R}^n$. Since the set of reals, $\mathbb{R}^n$, is convex, the domain of g is convex, and so, by this, we know g is also be convex. Next, we will show the norm operator is a convex function. Consider next the function $f : \mathbb{R}^n \to \mathbb{R}$:

$$f(\mathbf{x}) = \|\mathbf{x}\|_2 = \sqrt{\sum_i x_i^2} \tag{46}$$

Consider also the $p-$norm operator, which we know is convex, defined by the following:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \tag{47}$$

We can see that f is simply a special case of the $p-$norm, where $p = 2$. And thus, we conclude f is also convex. Next, we will use the composition of functions to show that:

$$f(g(\mathbf{x})) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \tag{48}$$

is convex. In order for the composition of two functions to be convex, the following conditions must be satisfied:

a) The dimensionality of f and g must be consistent.

b) f is non-decreasing.

c) g is convex.

Since $g : \mathbb{R}^n \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$, the dimensions of f and g are consistent. We have already shown that f is convex, thus it is non-decreasing, and we have also shown that g is convex. Thus, we have shown $f(g(\mathbf{x})) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2$ is convex.