2   Assignment 2

2.1   Centered Difference Derivative Approximation

Consider the following derivative approximation formula (the **Centered Difference Approach**):

$$\hat{f}'(x) = \frac{f(x + \Delta) - f(x - \Delta)}{2\Delta} \tag{1}$$

2.1.A

We can say this approximation is **consistent** if the following condition holds:

$$\lim_{\Delta \to 0} \hat{f}'(x) = f'(x) \tag{2}$$

Where $f'(x)$ is the actual derivative of $f(x)$. To show this approximation is consistent, we will take the limit of $\hat{f}'(x)$ as Δ approaches 0:

$$\lim_{\Delta \to 0} \hat{f}'(x) = \lim_{\Delta \to 0} \frac{f(x + \Delta) - f(x - \Delta)}{2\Delta} \tag{3}$$

Using L'Hôpital's Rule:

$$\lim_{\Delta \to 0} \hat{f}'(x) = \frac{f'(x + \Delta) + f'(x - \Delta)}{2} \tag{4}$$

And as $\Delta \to 0$:

$$\lim_{\Delta \to 0} \hat{f}'(x) = \frac{f'(x) + f'(x)}{2} \tag{5}$$

and so,

$$\lim_{\Delta \to 0} \hat{f}'(x) = f'(x) \tag{6}$$

Therefore, equation (2) holds and we have shown the centered difference approach is consistent.

2.1.B

The **order of accuracy** of the centered difference approach is 2. To show this, we will begin by representing the derivative of our function $f(x)$ as a Taylor Series approximation (where $\dot{x}(t_k) = f'(x)$):

$$x_{k+1} = x_k + T(\dot{x}(t_k)) + O(T^2) \tag{7}$$

We will express (1) in the same form (where $\dot{x}(t_k) = f'(x)$ as we have shown $\hat{f}'(x)$ is consistent):

$$\hat{x}_{k+1} = x_x + T(\dot{x}(t_k)) \tag{8}$$

Subtracting the actual function x_{k+1} from the approximation $\hat{x}_{k+1}$, we get:

$$\hat{x}_{k+1} - x_{k+1} = [x_x + T(\dot{x}(t_k))] - [x_k + T(\dot{x}(t_k)) + O(T^2)] \tag{9}$$

Simplifying, we get:

$$\hat{x}_{k+1} - x_{k+1} = O(T^2) \tag{10}$$

Thus, we have shown the centered difference approach has an order of accuracy equal to 2.

2.1.C

We would like to determine the conditions under which the centered difference approach is **stable**. To do this, we must determine the conditions under which the following is true:

$$\left\lVert \left. \frac{d\hat{f}'(x)}{dx} \right|_x \right\rVert \leq 1 \tag{11}$$

We begin by computing the derivative of (1) with respect to x:

$$\left\lVert \left. \frac{d}{dx}\frac{\hat{f}(x+\Delta) - \hat{f}(x-\Delta)}{2\Delta} \right|_x \right\rVert \leq 1 \tag{12}$$

$$\left\lVert \left. \frac{f'(x+\Delta) - f'(x-\Delta)}{2\Delta} \right|_x \right\rVert \leq 1 \tag{13}$$

By definition (1), we get:

$$\left\lVert \hat{f}''(x) \right\rVert \leq 1 \tag{14}$$

So, the second derivative of $f(x)$ must be less than or equal to 1 for the centered difference approximation to be stable. Additionally, the the second derivative of $f(x)$ must be strictly less than 1 for the centered difference approximation to converge.