3.2

Consider an ODE defined by the following equation:

$$\dot{x}(t) = f(x(t)) \tag{4}$$

Consider, now, numerically approximating the solution to (4) with the following algorithm:

$$\hat{x}_{k+1} = \hat{x}_k + T(\theta f(\hat{x}_k) + (1+\theta) f(\hat{x}_{k+1})) \tag{5}$$

3.2.A Order of Local Error at $\theta = 0, 0.5, 1$

When $\theta = 0$, we have the Backwards Euler Method:

$$\hat{x}_{k+1} = \hat{x}_k + T f(\hat{x}_{k+1}) \tag{6}$$

To determine the Order of Local Error of the Backwards Euler Method, we will begin by representing the exact update algorithm with a Taylor Series Approximation:

$$x_{k+1} = x_k + T f(x_k) + \frac{T^2}{2} f'(x_k) + O(T^3) \tag{7}$$

Now we will represent our Backwards Euler update function in the same form:

$$\hat{x}_{k+1} = \hat{x}_k + T f(\hat{x}_{k+1}) \tag{8}$$

$$\hat{x}_{k+1} = \hat{x}_k + T(f(\hat{x}_k) + T f'(\hat{x}_k) + \frac{T^2}{2} f''(\hat{x}_k) + O(T^3)) \tag{9}$$

$$\hat{x}_{k+1} = \hat{x}_k + T f(\hat{x}_k) + T^2 f'(\hat{x}_k) + \frac{T^3}{2} f''(\hat{x}_k) + T O(T^3)) \tag{10}$$

Now, we subtract our exact x_{k+1} from our approximated $\hat{x}_{k+1}$ to get:

$$\hat{x}_{k+1} - x_{k+1} = T^2 f'(\hat{x}_k) - \frac{T^2}{2} f'(x_k) \tag{11}$$

Thus, we can see that our error occurs in the term of order 2, and so the order of local error in the Backwards Euler Method is 2.

When $\theta = 0.5$, we have the Trapezoidal Method:

$$\hat{x}_{k+1} = \hat{x}_k + \frac{T}{2}(f(\hat{x}_k) + f(\hat{x}_{k+1})) \tag{12}$$

To determine the Order of Local Error of the the Trapezoidal Method, we will begin once again by representing our exact update function in the form of a Taylor Series:

$$x_{k+1} = x_k + T f(x_k) + \frac{T^2}{2} f'(x_k) + \frac{T^3}{3} f''(x_k) + O(T^4) \tag{13}$$

Next, we will manipulate our Trapezoidal Approximation update function into a similar form:

$$\hat{x}_{k+1} = \hat{x}_k + \frac{T}{2}(f(\hat{x}_k) + f(\hat{x}_{k+1})) \tag{14}$$

$$\hat{x}_{k+1} = \hat{x}_k + \frac{T}{2}(f(\hat{x}_k) + f(\hat{x}_k) + Tf'(\hat{x}_k) + \frac{T^2}{2}f''(\hat{x}_k) + O(T^3)) \tag{15}$$

Simplifying gives us:

$$\hat{x}_{k+1} = \hat{x}_k + Tf(\hat{x}_k) + \frac{T^2}{2}f'(\hat{x}_k) + \frac{T^3}{4}f''(\hat{x}_k) + ... \tag{16}$$

Comparing with our exact representation (13), we can see the first discrepancy occurs in the 3rd order term, thus the order of local error of the Trapezoidal Method is 3.

When $\theta = 1$, we have the Forwards Euler Method:

$$\hat{x}_{k+1} = \hat{x}_k + Tf(\hat{x}_k) \tag{17}$$

If we recall the Taylor approximation of the exact equation (equation (7)) and subtract our Forwards Euler update equation from it, we will be left with a 2nd order term:

$$(7) - (17) = \frac{T^2}{2}f'(x_k) \tag{18}$$

Thus, the order of error of the Forwards Euler Method is 2.

3.2.B

Consider the test equation:

$$\dot{x}(t) = \lambda x(t) \tag{19}$$

where:

$$\lambda = \sigma + \omega j \tag{20}$$

We will determine the stability of (5) with the above test equation (19). Using this specific test equation will allow us to generalize the stability of (5) for real and complex ODEs. Our goal is to model how any error in $\hat{x}_k$ will propagate to $\hat{x}_{k+1}$. We begin by rewriting the update formula as follows, substituting $\hat{x}_k$ for $x_k + \delta x_k$, where $\hat{x}_k$ is the exact solution value at $t = kT$, and $\delta x_k = \hat{x}_k - x_k$

$$x_{k+1} = x_k + T(\theta f(x_k + \delta x_k) + (1-\theta)f(x_{k+1})) \tag{21}$$

Substituting (19) and rearranging the equation, intentionally isolating x_{k+1}, we get:

$$x_{k+1} = x_k\frac{1 - T\lambda\theta}{1 - T\lambda(1-\theta)} + \delta x_k\frac{1 + T\lambda\theta}{1 - T\lambda(1-\theta)} \tag{22}$$

The second term on the left side of (22) is δx_{k+1}, the error in x_{k+1} as a function of δx_k:

$$\delta x_{k+1} = \delta x_k\frac{1 + T\lambda\theta}{1 - T\lambda(1-\theta)} \tag{23}$$

As we learned in ROB310 Lecture, satisfying the following inequality will ensure stability:

$$\left|\frac{\delta x_{k+1}}{\delta x_k}\right| \leq 1 \tag{24}$$

So:

$$\left|\frac{\delta x_{k+1}}{\delta x_k}\right| = \left|\frac{1 + T\lambda\theta}{1 - T\lambda(1-\theta)}\right| \tag{25}$$

Simplifying with $k = T\lambda$, we get:

$$\left|\frac{\delta x_{k+1}}{\delta x_k}\right| = \left|\frac{1 + k\theta}{1 - k(1-\theta)}\right| \tag{26}$$

Thus, the stability is:

$$\left|\frac{1 + k\theta}{1 - k(1-\theta)}\right| \leq 1 \tag{27}$$