

## This week's highlights

- Evaluate the truth value of a compound proposition given truth values of its constituent variables.
- Prove propositional equivalences using truth tables
- Prove propositional equivalences using other known equivalences, e.g.
  - DeMorgan's laws
  - Double negation laws
  - Distributive laws, etc.
- Compute the CNF and DNF of a given compound proposition.
- Define a predicate over a finite domain using a table of values and as properties
- Determine the truth value of the proposition resulting from evaluating a predicate
- Describe the set of domain elements that make a predicate with one input evaluate to true.
- Evaluate universal and existential statements about finite domains (with no quantifier alternations).
- Counterexample and witness-based arguments for predicates with infinite domains
- Practice combinations of  $\wedge$ ,  $\rightarrow$  in conjunction with universal and existential quantifiers
- State and apply DeMorgan's law for quantified statements.
- Translate sentences from English to propositional logic using appropriate propositional variables and boolean operators.

- Decide and justify whether or not a collection of propositions is consistent.
- Use predicates with set of tuples as their domain to relate values to one another
- Evaluate nested quantifiers: both alternating and not.

## **Lecture videos**

Week 4 Day 1 YouTube playlist

Week 4 Day 2 YouTube playlist

Week 4 Day 3 YouTube playlist

## Monday January 25

|                            |   |
|----------------------------|---|
| <b>Logical equivalence</b> | Two compound propositions are <b>logically equivalent</b> means that they have the same truth values for all settings of truth values to their propositional variables. |
| <b>Tautology</b>           | A compound proposition that evaluates to true for all settings of truth values to its propositional variables; it is abbreviated $T$ .                                  |
| <b>Contradiction</b>       | A compound proposition that evaluates to false for all settings of truth values to its propositional variables; it is abbreviated $F$ .                                 |
| <b>Contingency</b>         | A compound proposition that is neither a tautology nor a contradiction.   |

*Can replace  $p$  and  $q$  with any compound proposition*

**Definition:** A collection of compound propositions is called **consistent** if there is an assignment of truth values to the propositional variables that makes each of the compound propositions true.

**Consistency:**

Whenever the system software is being upgraded, users cannot access the file system. If users can access the file system, then they can save new files. If users cannot save new files, then the system software is not being upgraded.

1. Translate to symbolic compound propositions
2. Look for some truth assignment to the propositional variables for which all the compound propositions output  $T$

Consider the following algorithm to introduce redundancy in a string of 0s and 1s.

Create redundancy by repeating each bit three times

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```

1 procedure redun3( $a_{k-1} \cdots a_0$ : a binary string)
2 for  $i := 0$  to  $k-1$ 
3    $c_{3i} := a_i$ 
4    $c_{3i+1} := a_i$ 
5    $c_{3i+2} := a_i$ 
6 return  $c_{3k-1} \cdots c_0$ 

```

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Decode sequence of bits using majority rule on consecutive three bit sequences

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```

1 procedure decode3( $c_{3k-1} \cdots c_0$ : a binary string whose length is an integer multiple of 3)
2 for  $i := 0$  to  $k-1$ 
3   if exactly two or three of  $c_{3i}, c_{3i+1}, c_{3i+2}$  are set to 1
4      $a_i := 1$ 
5   else
6      $a_i := 0$ 
7 return  $a_{k-1} \cdots a_0$ 

```

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Give a recursive definition of the set of outputs of the *redun3* procedure, *Out*,  
**Basis step:** \_\_\_\_\_

**Recursive step:** \_\_\_\_\_

Consider the message  $m = 0001$  so that the sender calculates  $\text{redun3}(m) = \text{redun3}(0001) = 000000000111$ .

Introduce \_\_\_\_ errors into the message so that the signal received by the receiver is \_\_\_\_\_ but the receiver is still able to decode the original message.

*Challenge: what is the biggest number of errors you can introduce?*

Building a circuit for line 3 in *decode* procedure: given three input bits, we need to determine whether the majority is a 0 or a 1.

| $c_{3i}$ | $c_{3i+1}$ | $c_{3i+2}$ | $a_i$ | Circuit |
|----------|------------|------------|-------|---------|
| 1        | 1          | 1          |       |         |
| 1        | 1          | 0          |       |         |
| 1        | 0          | 1          |       |         |
| 1        | 0          | 0          |       |         |
| 0        | 1          | 1          |       |         |
| 0        | 1          | 0          |       |         |
| 0        | 0          | 1          |       |         |
| 0        | 0          | 0          |       |         |

## Wednesday January 27

**Definition:** A **predicate** is a function from a given set (domain) to  $\{T, F\}$ .

A predicate can be applied, or **evaluated** at, an element of the domain.

Two predicates over the same domain are **equivalent** means they evaluate to the same truth values for all possible assignments of domain elements to the input.

| Input | Output                     |                            |              |
|-------|----------------------------|----------------------------|--------------|
| $x$   | $P(x)$<br>$[x]_{2c,3} > 0$ | $N(x)$<br>$[x]_{2c,3} < 0$ | $Mystery(x)$ |
| 000   | $F$                        |                            | $T$          |
| 001   | $T$                        |                            | $T$          |
| 010   | $T$                        |                            | $T$          |
| 011   | $T$                        |                            | $F$          |
| 100   | $F$                        |                            | $F$          |
| 101   | $F$                        |                            | $T$          |
| 110   | $F$                        |                            | $F$          |
| 111   | $F$                        |                            | $T$          |

The domain for each of the predicates  $P(x)$ ,  $N(x)$ ,  $Mystery(x)$  is \_\_\_\_\_

**Definition:** The **truth set** of a predicate is the collection of all elements in its domain where the predicate evaluates to  $T$ .

The truth set for the predicate  $P(x)$  is \_\_\_\_\_.

The truth set for the predicate  $N(x)$  is \_\_\_\_\_.

The truth set for the predicate  $Mystery(x)$  is \_\_\_\_\_.

**Definition 15 (Universal quantification)** The **universal quantification** of  $P(x)$  is the statement “ $P(x)$  for all values of  $x$  in the domain” and is written  $\forall xP(x)$ . An element for which  $P(x) = F$  is called a **counterexample** of  $\forall xP(x)$ .

The **existential quantification** of  $P(x)$  is the statement “There exists an element  $x$  in the domain such that  $P(x)$ ” and is written  $\exists xP(x)$ . An element for which  $P(x) = T$  is called a **witness** of  $\exists xP(x)$ .

**Example:** \_\_\_\_\_ is a true existential quantification.

Statements involving predicates and quantifiers are **logically equivalent** means they have the same truth value no matter which predicates (domains and functions) are substituted in.

**Quantifier version of De Morgan’s laws:**  $\neg\forall xP(x) \equiv \exists x(\neg P(x))$

$$\neg\exists xQ(x) \equiv \forall x(\neg Q(x))$$

**Example:** \_\_\_\_\_ is a false universal quantification. It is logically equivalent to \_\_\_\_\_

Recall: Each RNA strand is a string whose symbols are elements of the set  $B = \{\mathbf{A}, \mathbf{C}, \mathbf{G}, \mathbf{U}\}$ . The **set of all RNA strands** is called  $S$ . The function *rlen* that computes the length of RNA strands in  $S$  is:

$$\begin{array}{lll} \text{Basis Step:} & \text{If } b \in B \text{ then} & \begin{array}{ll} rlen : S & \rightarrow \mathbb{Z}^+ \\ rlen(b) & = 1 \end{array} \\ \text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B, \text{ then} & rlen(sb) = 1 + rlen(s) \end{array}$$

### Example predicates on $S$

|  |  |
|--|--|
| $H(s) = T$   | Truth set of $H$ is _____  |
| $L_3(s) = \begin{cases} T & \text{if } rlen(s) = 3 \\ F & \text{otherwise} \end{cases}$  | Strand where $L_3$ evaluates to $T$ is e.g. _____<br>Strand where $L_3$ evaluates to $F$ is e.g. _____         |
| $F_A$ is defined recursively by:<br>Basis step: $F_A(\mathbf{A}) = T, F_A(\mathbf{C}) = F_A(\mathbf{G}) = F_A(\mathbf{U}) = F$<br>Recursive step: If $s \in S$ and $b \in B$ , then $F_A(sb) = F_A(s)$ | Strand where $F_A$ evaluates to $T$ is e.g. _____<br>Strand where $F_A$ evaluates to $F$ is e.g. _____         |
| $P_{AUC}$ is defined as the predicate whose truth set is the collection of RNA strands where the string <b>AUC</b> is a sub-string (appears inside $s$ , in order and consecutively)                   | Strand where $P_{AUC}$ evaluates to $T$ is e.g. _____<br>Strand where $P_{AUC}$ evaluates to $F$ is e.g. _____ |

## Friday January 29

**Definition** (Rosen p123): The **Cartesian product** of the sets  $A$  and  $B$ ,  $A \times B$ , is the set of all ordered pairs  $(a, b)$ , where  $a \in A$  and  $b \in B$ . That is:  $A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}$ . The Cartesian product of the sets  $A_1, A_2, \dots, A_n$ , denoted by  $A_1 \times A_2 \times \dots \times A_n$ , is the set of ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$ , where  $a_i$  belongs to  $A_i$  for  $i = 1, 2, \dots, n$ . That is,  $A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$

Recall: Each RNA strand is a string whose symbols are elements of the set  $B = \{A, C, G, U\}$ . The **set of all RNA strands** is called  $S$ . The function  $rnalen$  that computes the length of RNA strands in  $S$  is:

$$\begin{array}{lll} & & rnalen : S \rightarrow \mathbb{Z}^+ \\ \text{Basis Step:} & \text{If } b \in B \text{ then} & rnalen(b) = 1 \\ \text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B, \text{ then} & rnalen(sb) = 1 + rnalen(s) \end{array}$$

A function  $basecount$  that computes the number of a given base  $b$  appearing in a RNA strand  $s$  is:

$$\begin{array}{lll} & & basecount : S \times B \rightarrow \mathbb{N} \\ \text{Basis Step:} & \text{If } b_1 \in B, b_2 \in B & basecount(b_1, b_2) = \begin{cases} 1 & \text{when } b_1 = b_2 \\ 0 & \text{when } b_1 \neq b_2 \end{cases} \\ \text{Recursive Step:} & \text{If } s \in S, b_1 \in B, b_2 \in B & basecount(sb_1, b_2) = \begin{cases} 1 + basecount(s, b_2) & \text{when } b_1 = b_2 \\ basecount(s, b_2) & \text{when } b_1 \neq b_2 \end{cases} \end{array}$$

$L$  with domain  $S \times \mathbb{Z}^+$  is defined by, for  $s \in S$  and  $n \in \mathbb{Z}^+$ ,  $BC$  with domain \_\_\_\_\_ is defined   
  $S$  and  $b \in B$  and  $n \in \mathbb{N}$ ,

$$L(s, n) = \begin{cases} T & \text{if } rnalen(s) = n \\ F & \text{otherwise} \end{cases} \qquad BC(s, b, n) = \begin{cases} T & \text{if } basecount(s, b) = n \\ F & \text{otherwise} \end{cases}$$

Element where  $L$  evaluates to  $T$ : \_\_\_\_\_

Element where  $BC$  evaluates to  $T$ : \_\_\_\_\_

Element where  $L$  evaluates to  $F$ : \_\_\_\_\_

Element where  $BC$  evaluates to  $F$ : \_\_\_\_\_

**Notation:** for a predicate  $P$  with domain  $X_1 \times \dots \times X_n$  and a  $n$ -tuple  $(x_1, \dots, x_n)$  with each  $x_i \in X$ , we write  $P(x_1, \dots, x_n)$  to mean  $P((x_1, \dots, x_n))$ .

$\exists t BC(t)$       In English: \_\_\_\_\_  
 Witness that proves this existential quantification is true: \_\_\_\_\_  
 $\forall (s, b, n) ( BC(s, b, n) )$       In English: \_\_\_\_\_  
 Counterexample that proves this universal quantification is false: \_\_\_\_\_  
**New predicates from old**       $BC(s, b, n)$  means *basecount*( $s, b$ ) =  $n$ .

| Predicate  | Domain | Example domain element where predicate is $T$ |
|--|--------|---|
| $basecount(s, b) = 3$                            |        |   |
| $basecount(s, A) = n$                            |        |   |
| $\exists n \in \mathbb{N} (basecount(s, b) = n)$ |        |   |
| $\forall b \in B (basecount(s, b) = 1)$          |        |   |

### Alternating quantifiers

$$\forall s \exists n BC(s, A, n)$$

In English: \_\_\_\_\_

$$\exists n \forall s BC(s, U, n)$$

In English: \_\_\_\_\_

Evaluate each quantified statement as  $T$  or  $F$ .

|   |   |   |
|---|---|---|
| $\forall s \forall b \exists n BC(s, b, n)$ | $\forall s \forall n \exists b BC(s, b, n)$ | $\forall b \forall n \exists s BC(s, b, n)$ |
| $\exists s \forall b \exists n BC(s, b, n)$ | $\forall s \exists n \forall b BC(s, b, n)$ | $\exists b \exists n \forall s BC(s, b, n)$ |

*Extra example:* Write the negation of each of the statements above, and use De Morgan's law to find a logically equivalent version where the negation is applied only to the  $BC$  predicate (not next to a quantifier).



## Review quiz questions

1. **Monday** Real-life representations are often prone to corruptions. Biological codes, like RNA, may mutate naturally<sup>1</sup> and during measurement; cosmic radiation and other ambient noise can flip bits in computer storage<sup>2</sup>. One way to recover from corrupted data is to exploit redundancy. Consider the following algorithm to introduce redundancy in a string of 0s and 1s.

Create redundancy by repeating each bit three times

---

```
1 procedure redun3( $a_{k-1} \dots a_0$ : a binary string)
2 for  $i := 0$  to  $k-1$ 
3    $c_{3i} := a_i$ 
4    $c_{3i+1} := a_i$ 
5    $c_{3i+2} := a_i$ 
6 return  $c_{3k-1} \dots c_0$ 
```

---

Decode sequence of bits using majority rule on consecutive three bit sequences

---

```
1 procedure decode3( $c_{3k-1} \dots c_0$ : a binary string whose length is an integer multiple of 3)
2 for  $i := 0$  to  $k-1$ 
3   if exactly two or three of  $c_{3i}, c_{3i+1}, c_{3i+2}$  are set to 1
4      $a_i := 1$ 
5   else
6      $a_i := 0$ 
7 return  $a_{k-1} \dots a_0$ 
```

---

For each of the following, type in your answers precisely including all notational punctuation.

- (a) Give the output of *redun3*(100).
- (b) If the output of running *redun3* is 000000111000111, what was its input?
- (c) Give the output of *decode3*(100).
- (d) How many distinct possible inputs to *decode3* give the output 01?

2. **Wednesday** Consider the following predicates, each of which has as its domain the set of all bitstrings whose leftmost bit is 1

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<sup>1</sup>Mutations of specific RNA codons have been linked to many disorders and cancers.

<sup>2</sup>This RadioLab podcast episode goes into more detail on bit flips: <https://www.wnycstudios.org/story/bit-flip>

$E(x)$  is  $T$  exactly when  $(x)_2$  is even, and is  $F$  otherwise

$L(x)$  is  $T$  exactly when  $(x)_2 < 3$ , and is  $F$  otherwise

$M(x)$  is  $T$  exactly when  $(x)_2 > 256$  and is  $F$  otherwise.

(a) What is  $E(110)$ ?

(b) Why is  $L(00)$  undefined?

- i. Because the domain of  $L$  is infinite
- ii. Because  $00$  does not have 1 in the leftmost position
- iii. Because  $00$  has length 2, not length 3
- iv. Because  $(00)_{2,3} = 0$  which is less than 3

(c) Is there a bitstring of width (where width is the number of bits) 6 at which  $M(x)$  evaluates to  $T$ ?

3. **Friday** Recall that  $S$  is defined as the set of all RNA strands, strings made of the bases in  $B = \{\mathbf{A}, \mathbf{U}, \mathbf{G}, \mathbf{C}\}$ . Define the functions *mutation*, *insertion*, and *deletion* as described by the pseudocode below:

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```

1  procedure mutation( $b_1 \cdots b_n$ : a RNA strand,  $k$ : a positive integer,  $b$ : an element of  $B$ )
2  for  $i := 1$  to  $n$ 
3      if  $i = k$ 
4           $c_i := b$ 
5      else
6           $c_i := b_i$ 
7  return  $c_1 \cdots c_n$  {The return value is a RNA strand made of the  $c_i$  values}

```

---

```

1  procedure insertion( $b_1 \cdots b_n$ : a RNA strand,  $k$ : a positive integer,  $b$ : an element of  $B$ )
2  if  $k > n$ 
3      for  $i := 1$  to  $n$ 
4           $c_i := b_i$ 
5       $c_{n+1} := b$ 
6  else
7      for  $i := 1$  to  $k - 1$ 
8           $c_i := b_i$ 
9       $c_k := b$ 
10     for  $i := k + 1$  to  $n + 1$ 
11          $c_i := b_{i-1}$ 
12 return  $c_1 \cdots c_{n+1}$  {The return value is a RNA strand made of the  $c_i$  values}

```

---

```

1  procedure deletion( $b_1 \cdots b_n$ : a RNA strand,  $k$ : a positive integer)
2  if  $k > n$ 
3       $m := n$ 
4      for  $i := 1$  to  $n$ 
5           $c_i := b_i$ 
6  else
7       $m := n - 1$ 
8      for  $i := 1$  to  $k - 1$ 
9           $c_i := b_i$ 
10     for  $i := k$  to  $n - 1$ 
11          $c_i := b_{i+1}$ 
12 return  $c_1 \cdots c_m$  {The return value is a RNA strand made of the  $c_i$  values}

```

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For this question, we will use the following predicates.

$F_{\mathbf{A}}$  with domain  $S$  is defined recursively by:

Basis step:  $F_{\mathbf{A}}(\mathbf{A}) = T$ ,  $F_{\mathbf{A}}(\mathbf{C}) = F_{\mathbf{A}}(\mathbf{G}) = F_{\mathbf{A}}(\mathbf{U}) = F$

Recursive step: If  $s \in S$  and  $b \in B$ , then  $F_{\mathbf{A}}(sb) = F_{\mathbf{A}}(s)$

$P_{\text{AUC}}$  with domain  $S$  is defined as the predicate whose truth set is the collection of RNA strands where the string **AUC** is a substring (appears inside  $s$ , in order and consecutively)

$L$  with domain  $S \times \mathbb{Z}^+$  is defined by, for  $s \in S$  and  $n \in \mathbb{Z}^+$ ,

$$L(s, n) = \begin{cases} T & \text{if } rnalen(s) = n \\ F & \text{otherwise} \end{cases}$$

$Mut$  with domain  $S \times S$  is defined by, for  $s_1 \in S$  and  $s_2 \in S$ ,

$$Mut(s_1, s_2) = \exists k \in \mathbb{Z}^+ \exists b \in B( \text{mutation}(s_1, k, b) = s_2 )$$

$Ins$  with domain  $S \times S$  is defined by, for  $s_1 \in S$  and  $s_2 \in S$ ,

$$Ins(s_1, s_2) = \exists k \in \mathbb{Z}^+ \exists b \in B( \text{insertion}(s_1, k, b) = s_2 )$$

$Del$  with domain  $S \times S$  is defined by, for  $s_1 \in S$  and  $s_2 \in S$ ,

$$Del(s_1, s_2) = \exists k \in \mathbb{Z}^+ ( \text{deletion}(s_1, k) = s_2 )$$

(a) Which of the following is true? (Select all and only that apply.)

- i.  $F_A(\text{AA})$
- ii.  $F_A(\text{AC})$
- iii.  $F_A(\text{AG})$
- iv.  $F_A(\text{AU})$
- v.  $F_A(\text{CA})$
- vi.  $F_A(\text{CC})$
- vii.  $F_A(\text{CG})$
- viii.  $F_A(\text{CU})$

(b) Which of the following is true? (Select all and only that apply.)

- i.  $\exists s \in S \exists n \in \mathbb{Z}^+ (L(s, n))$
- ii.  $\exists s \in S \forall n \in \mathbb{Z}^+ (L(s, n))$
- iii.  $\forall n \in \mathbb{Z}^+ \exists s \in S (L(s, n))$
- iv.  $\forall s \in S \exists n \in \mathbb{Z}^+ (L(s, n))$
- v.  $\exists n \in \mathbb{Z}^+ \forall s \in S (L(s, n))$
- vi.  $\forall s \in S \forall n \in \mathbb{Z}^+ (L(s, n))$

(c) Which of the following is true? (Select all and only that apply.)

- i.  $\exists s \in S Mut(s, s)$

- ii.  $\forall s \in S \text{ } Mut(s, s)$
- iii.  $\exists s \in S \text{ } Ins(s, \mathbf{A})$
- iv.  $\exists s \in S \text{ } Ins(\mathbf{A}, s)$
- v.  $\exists s \in S \text{ } Del(s, \mathbf{A})$
- vi.  $\forall s \in S \text{ } Del(s, \mathbf{A})$