Strong Induction

For which nonnegative integers n can we make change for n with coins of value 5 cents and 3 cents?

Restating: We can make change for __________, we cannot make change for _______________, and

____ $\star$

New! Proof by Strong Induction (Rosen 5.2 p337)
To prove that a universal quantification over the set of all integers greater than or equal to some base integer b holds, pick a fixed nonnegative integer j and then:

Basis Step:	Show the statement holds for b, $b+1$, $\ldots$, $b+j$.
Recursive Step:	Consider an arbitrary integer n greater than or equal to $b+j$, assume (as the **strong induction hypothesis**) that the property holds for **each of** b, $b+1$, $\ldots$, n, and use this and other facts to prove that the property holds for $n+1$.

$\mathbb{N}$	The set of natural numbers	$\{0, 1, 2, 3, \ldots\}$
$\mathbb{Z}^{\geq b}$	The set of integers greater than or equal a basis element b	$\{b, b+1, b+2, b+3, \ldots\}$

Proof of $\star$ by mathematical induction ($b = 8$)

Basis step: WTS property is true about 8

Recursive step: Consider an arbitrary $n \geq 8$. Assume (as the IH) that there are nonnegative integers x, y such that $n = 5x + 3y$. WTS that there are nonnegative integers x', y' such that $n + 1 = 5x' + 3y'$. We consider two cases, depending on whether any 5 cent coins are used for n.

Case 1: Assume $x \geq 1$. Define $x' = x - 1$ and $y' = y + 2$ (both in $\mathbb{N}$ by case assumption). Calculating:

$$\begin{aligned} 5x' + 3y' &\overset{\text{by def}}{=} 5(x-1) + 3(y+2) = 5x - 5 + 3y + 6 \\ &\overset{\text{rearranging}}{=} (5x + 3y) - 5 + 6 \\ &\overset{\text{IH}}{=} n - 5 + 6 = n + 1 \end{aligned}$$

Case 2: Assume $x = 0$. Therefore $n = 3y$, so since $n \geq 8$, $y \geq 3$. Define $x' = 2$ and $y' = y - 3$ (both in $\mathbb{N}$ by case assumption). Calculating:

$$\begin{aligned} 5x' + 3y' &\overset{\text{by def}}{=} 5(2) + 3(y-3) = 10 + 3y - 9 \\ &\overset{\text{rearranging}}{=} 3y + 10 - 9 \\ &\overset{\text{IH and case}}{=} n + 10 - 9 = n + 1 \end{aligned}$$

Proof of $\star$ by strong induction ($b = 8$ and $j = 2$)

Basis step: WTS property is true about $8, 9, 10$

Recursive step: Consider an arbitrary $n \geq 10$. Assume (as the IH) that the property is true about each of $8, 9, 10, \ldots, n$. WTS that there are nonnegative integers x', y' such that $n + 1 = 5x' + 3y'$.

Representing positive integers

Theorem: Every positive integer is a sum of (one or more) distinct powers of 2. *binary expansions exist!*

Proof by strong induction, with $b = 1$ and $j = 0$.

Basis step: WTS property is true about 1.

Recursive step: Consider an arbitrary integer $n \geq 1$. Assume (as the IH) that the property is true about each of $1, \ldots, n$. WTS that the property is true about $n + 1$.

Definition (Rosen p257): An integer p greater than 1 is called **prime** if the only positive factors of p are 1 and p. A positive integer that is greater than 1 and is not prime is called composite.

Theorem (Rosen p336): Every positive integer *greater than 1* is a product of (one or more) primes.

Proof by strong induction, with $b = 2$ and $j = 0$.

Basis step: WTS property is true about 2.

Recursive step: Consider an arbitrary integer $n \geq 2$. Assume (as the IH) that the property is true about each of $2, \ldots, n$. WTS that the property is true about $n + 1$.

Case 1:

Case 2: