This week's highlights

- Evaluate which proof technique(s) is appropriate for a given proposition
- Compare sets using one-to-one, onto, and invertible functions.
- Define cardinality using one-to-one, onto, and invertible functions.
- Differentiate between important sets of numbers
- Classify sets by cardinality into: Finite sets, countable sets, uncountable sets.
- Explain the central idea in Cantor's diagonalization argument.

Lecture videos

Week 8 Day 1 YouTube playlist
Week 8 Day 2 YouTube playlist
Week 8 Day 3 YouTube playlist

Monday February 22

$\mathbb{Z}$	The set of integers	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}^+$	The set of positive integers	$\{1, 2, \ldots\}$
$\mathbb{N}$	The set of nonnegative integers	$\{0, 1, 2, \ldots\}$
$\mathbb{Q}$	The set of rational numbers	$\left\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0\right\}$
$\mathbb{R}$	The set of real numbers	

$$\underline{\qquad} \subsetneq \underline{\qquad} \subsetneq \underline{\qquad} \subsetneq \underline{\qquad} \subsetneq \underline{\qquad}$$

The above sets are all **infinite**.

A **finite** set is one whose distinct elements can be counted by a natural number.

Examples of finite sets: $\emptyset$, $\{\sqrt{2}\}$

Motivating question: Are some of the above sets *bigger than* others?

Analogy: Musical chairs

People try to sit down when the music stops

Person☼ sits in Chair 1, Person☺ sits in Chair 2,

Person☹ is left standing!

What does this say about the number of chairs and the number of people?

Defining functions A function is defined by its (1) domain, (2) codomain, and (3) rule assigning each element in the domain exactly one element in the codomain. The domain and codomain are nonempty sets. The rule can be depicted as a table, formula, English description, etc.

(Rosen p139)

Example: $f_A : \mathbb{R}^+ \to \mathbb{Q}$ with $f_A(x) = x$ is **not** a well-defined function because

Example: $f_B : \mathbb{Q} \to \mathbb{Z}$ with $f_B\left(\frac{p}{q}\right) = p + q$ is **not** a well-defined function because

Example: $f_C : \mathbb{Z} \to \mathbb{R}$ with $f_C(x) = \frac{x}{|x|}$ is **not** a well-defined function because

Definition (Rosen p141): A function $f : D \to C$ is **one-to-one** (or injective) means for every a, b in the domain D, if $f(a) = f(b)$ then $a = b$.

Definition: For sets A, B, we say that **the cardinality of A is no bigger than the cardinality of B**, and write $|A| \leq |B|$, to mean there is a one-to-one function with domain A and codomain B.

In the analogy: The function $sitter : \{Chair1, Chair2\} \to \{Person\text{☼}, Person\text{☺}, Person\text{☹}\}$ given by $sitter(Chair1) = Person\text{☼}$, $sitter(Chair2) = Person\text{☺}$, is one-to-one and witnesses that

$$|\{Chair1, Chair2\}| \leq |\{Person\text{☼}, Person\text{☺}, Person\text{☹}\}|$$

Let S_2 be the set of RNA strands of length 2.

Statement	True/False , justification
$\lvert\{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\}\rvert \leq \lvert S_2\rvert$	
$\lvert\{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\} \times \{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\}\rvert \leq \lvert S_2\rvert$	

Definition (Rosen p143): A function $f : D \to C$ is **onto** (or surjective) means for every b in the codomain, there is an element a in the domain with $f(a) = b$.

Formally, $f : D \to C$ is onto means ________________________ .

Definition: For sets A, B, we say that **the cardinality of A is no smaller than the cardinality of B**, and write $|A| \geq |B|$, to mean there is an onto function with domain A and codomain B.

In the analogy: The function $triedToSit : \{Person\text{☼}, Person\text{☺}, Person\text{☹}\} \to \{Chair1, Chair2\}$ given by $triedToSit(Person\text{☼}) = Chair1$, $triedToSit(Person\text{☺}) = Chair2$, $triedToSit(Person\text{☹}) = Chair2$, is onto and witnesses that

$$|\{Person\text{☼}, Person\text{☺}, Person\text{☹}\}| \geq |\{Chair1, Chair2\}|$$

Let S_2 be the set of RNA strands of length 2.

Statement	True/False , justification
$\lvert S_2 \rvert \geq \lvert \{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\} \rvert$	
$\lvert S_2 \rvert \geq \lvert \{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\} \times \{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\} \rvert$	

Definition (Rosen p144): A function $f : D \to C$ is a **bijection** means that it is both one-to-one and onto. The **inverse** of a bijection $f : D \to C$ is the function $g : C \to D$ such that $g(b) = a$ iff $f(a) = b$.

For nonempty sets A, B we say

$|A| \leq |B|$ means there is a one-to-one function with domain A, codomain B

$|A| \geq |B|$ means there is an onto function with domain A, codomain B

$|A| = |B|$ means there is a bijection with domain A, codomain B

Properties of cardinality

$$\forall A \; (\; |A| = |A| \;)$$
$$\forall A \; \forall B \; (\; |A| = |B| \;\to\; |B| = |A| \;)$$
$$\forall A \; \forall B \; \forall C \; (\; (|A| = |B| \;\wedge\; |B| = |C|) \;\to\; |A| = |C| \;)$$

Extra practice with proofs: Use the definitions of bijections to prove these properties.

Cantor-Schroder-Bernstein Theorem: For all nonempty sets,

$$|A| = |B| \qquad \text{if and only if} \qquad (|A| \leq |B| \text{ and } |B| \leq |A|) \qquad \text{if and only if} \qquad (|A| \geq |B| \text{ and } |B| \geq$$

To prove $|A| = |B|$, we can do any **one** of the following

- Prove there exists a bijection $f : A \to B$;
- Prove there exists a bijection $f : B \to A$;
- Prove there exists two functions $f_1 : A \to B$, $f_2 : B \to A$ where each of f_1, f_2 is one-to-one.
- Prove there exists two functions $f_1 : A \to B$, $f_2 : B \to A$ where each of f_1, f_2 is onto.

Wednesday February 24

Countably infinite: A set A is **countably infinite** means it is the same size as $\mathbb{N}$.

Natural numbers $\mathbb{N}$ *List*: 0 1 2 3 4 5 6 7 8 9 10...

$identity : \mathbb{N} \to \mathbb{N}$ with $identity(n) = n$

Claim: *identity* is a bijection. *Proof*: Ex. **Corollary**: $|\mathbb{N}| = |\mathbb{N}|$

Positive integers $\mathbb{Z}^+$ *List*: 1 2 3 4 5 6 7 8 9 10 11...

$positives : \mathbb{N} \to \mathbb{Z}^+$ with $positives(n) = n + 1$

Claim: *positives* is a bijection. *Proof*: Ex. **Corollary**: $|\mathbb{N}| = |\mathbb{Z}^+|$

Negative integers $\mathbb{Z}^-$ *List*: $-1\ -2\ -3\ -4\ -5\ -6\ -7\ -8\ -9\ -10\ -11\ldots$

$negatives : \mathbb{N} \to \mathbb{Z}^-$ with $negatives(n) = -n - 1$

Claim: *positives* is a bijection. **Corollary**: $|\mathbb{N}| = |\mathbb{Z}^-|$

Proof: We need to show it is a well-defined function that is one-to-one and onto.

- Well-defined?

  Consider an arbitrary element of the domain, $n \in \mathbb{N}$. We need to show it maps to exactly one element of $\mathbb{Z}^-$.

- One-to-one?

  Consider arbitrary elements of the domain $a, b \in \mathbb{N}$. We need to show that

  $$(\ negatives(a) = negatives(b)\ ) \to (a = b)$$

- Onto?

  Consider arbitrary element of the codomain $b \in \mathbb{Z}^-$. We need witness in $\mathbb{N}$ that maps to b.

Integers $\mathbb{Z}$ *List*: $0\ -1\ 1\ -2\ 2\ -3\ 3\ -4\ 4\ -5\ 5\ldots$

$$f : \mathbb{Z} \to \mathbb{N} \text{ with } f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$$

Claim: f is a bijection. *Proof*: Ex. **Corollary**: $|\mathbb{Z}| = |\mathbb{N}|$

More examples of countably infinite sets

Claim: S is countably infinite

Similarly: The set of all strings over a specific alphabet is countably infinite.

Bijection using alphabetical-ish (first order by length, then alphabetically among strings of same length) of strands

Claim: L is countably infinite

One-to-one function from $\mathbb{N}$ to L

One-to-one function from L to $\mathbb{N}$

Claim: $|\mathbb{Z}^+| = |\mathbb{Q}|$

One-to-one function from $\mathbb{Z}^+$ to $\mathbb{Q}$

One-to-one function from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{Z}$

One-to-one function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}^+ \times \mathbb{Z}^+$

One-to-one function from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to $\mathbb{Z}^+$

A set A is **finite** means it is empty or it is the same size as $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$.

A set A is **countably infinite** means it is the same size as $\mathbb{N}$.

A set A is **countable** means it is either finite or countably infinite.

A set A is **uncountable** means it is not countable.

All countably infinite sets are the same size as one another!

Useful Lemmas

If A and B are countable sets, then $A \cup B$ is countable. *Theorem 1, page 174*

If A and B are countable sets, then $A \times B$ is countable. *Generalize pairing ideas from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to $\mathbb{Z}^+$*

If A is a subset of B , to show that $|A| = |B|$, it's enough to give one-to-one function from B to A or an onto function from A to B. *Exercise 22, page 176*

If A is a subset of a countable set, then it's countable. *Exercise 16, page 176*

If A is a superset of an uncountable set, then it's uncountable. *Exercise 15, page 176*

Friday February 26

Countable sets:

A set A is **finite** means it is empty or it is the same size as $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$.

A set A is **countably infinite** means it is the same size as $\mathbb{N}$.

A set A is **countable** means it is either finite or countably infinite.

A set A is **uncountable** means it is not countable.

Key idea: For finite sets, the power set of a set has strictly greater size than the set itself.

Does this extend to infinite sets?

$\mathbb{N}$ **and its power set**

Example elements of $\mathbb{N}$

Example elements of $\mathcal{P}(\mathbb{N})$ *Recall*: For set A, its power set is $\mathcal{P}(A) = \{X \mid X \subseteq A\}$

Claim: $|\mathbb{N}| \leq |\mathcal{P}(\mathbb{N})|$

Claim: There is an uncountable set. Example: ________

Proof: By definition of countable, since ________ is not finite, **to show** is $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$.

Rewriting using the definition of cardinality, **to show** is

Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

To show: f is not a bijection. It's enough to show that f is not onto.

Rewriting using the definition of onto, **to show**:

$$\neg \forall B \in \mathcal{P}(\mathbb{N}) \; \exists a \in \mathbb{N} \; (\; f(a) = B \;)$$

. By logical equivalence, we can write this as an existential statement:

____

In search of a witness, define the following collection of nonnegative integers:

$$D_f = \{n \in \mathbb{N} \;\mid\; n \notin f(n)\}$$

. By definition of power set, since all elements of D_f are in $\mathbb{N}$, $D_f \in \mathcal{P}(\mathbb{N})$. It's enough to prove the following Lemma:

Lemma: $\forall a \in \mathbb{N} \; (\; f(a) \neq D_f \;)$.

Proof of lemma:

By the Lemma, we have proved that f is not onto, and since f was arbitrary, there are no onto functions from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$. QED

Where does D_f come from? The idea is to build a set that would "disagree" with each of the images of f about some element.

$n \in \mathbb{N}$	$f(n) = X_n$	Is $0 \in X_n$?	Is $1 \in X_n$?	Is $2 \in X_n$?	Is $3 \in X_n$?	Is $4 \in X_n$?	...	Is $n \in$
0	$f(0) = X_0$	**Y / N**	Y / N	Y / N	Y / N	Y / N	...	**N /**
1	$f(1) = X_1$	Y / N	**Y / N**	Y / N	Y / N	Y / N	...	**N /**
2	$f(2) = X_2$	Y / N	Y / N	**Y / N**	Y / N	Y / N	...	**N /**
3	$f(3) = X_3$	Y / N	Y / N	Y / N	**Y / N**	Y / N	...	**N /**
4	$f(4) = X_4$	Y / N	Y / N	Y / N	Y / N	**Y / N**	...	**N /**
$\vdots$								

Additional examples and applications: countable vs. uncountable

Comparing $\mathbb{Q}$ and $\mathbb{R}$

No greatest element?				**Both sets**
No least element?				**Both sets**
$\forall x \forall y (x < y \to \exists z (x < z < y))$				**Both sets**
Least upper bound property?			**Only** $\mathbb{R}$	
Finite?	**Neither set**			
Countably infinite?		**Only** $\mathbb{Q}$		
Uncountable?			**Only** $\mathbb{R}$	

The set of real numbers

$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Order axioms (Rosen Appendix 1):

Reflexivity	$\forall a \in \mathbb{R} (a \leq a)$
Antisymmetry	$\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ (\ (a \leq b\ \land\ b \leq a) \to (a = b)\ )$
Transitivity	$\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \forall c \in \mathbb{R}\ (\ (a \leq b \land b \leq c)\ \to\ (a \leq c)\ )$
Trichotomy	$\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ (\ (a = b\ \lor\ b > a\ \lor\ a < b)$

Completeness axioms (Rosen Appendix 1):

Least upper bound	Every nonempty set of real numbers that is bounded above has a least upp
Nested intervals	For each sequence of intervals $[a_n, b_n]$ where, for each n, $a_n < a_{n+1} < b_{n+1} <$ is at least one real number x such that, for all n, $a_n \leq x \leq b_n$.

Each real number $r \in \mathbb{R}$ is described by a function to give better and better approximations

$$x_r : \mathbb{Z}^+ \to \{0, 1\} \qquad \text{where } x_r(n) = n^{th} \text{ bit in binary expansion of } r$$

r	Binary expansion	x_r
0.1	$0.00011001\ldots$	$x_{0.1}(n) = \begin{cases} 0 & \text{if } n > 1 \text{ and } (n \textbf{ mod } 4) = 2 \\ 0 & \text{if } n = 1 \text{ or if } n > 1 \text{ and } (n \textbf{ mod } 4) = \\ 1 & \text{if } n > 1 \text{ and } (n \textbf{ mod } 4) = 0 \\ 1 & \text{if } n > 1 \text{ and } (n \textbf{ mod } 4) = 1 \end{cases}$
$\sqrt{2} - 1 = 0.4142135\ldots$	$0.01101010\ldots$	Use linear approximations (tangent lines from calcul to get algorithm for bounding error of successive op ations. Define $x_{\sqrt{2}-1}(n)$ to be n^{th} bit in approximat that has error less than $2^{-(n+1)}$.

Claim: $\{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}$ is uncountable.

Approach 1: Mimic proof that $\mathcal{P}(\mathbb{Z}^+)$ is uncountable.

Proof: By definition of countable, since $\{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}$ is not finite, **to show** is $|\mathbb{N}| \neq |\{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}|$.

To show is $\forall f : \mathbb{Z}^+ \to \{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\} \cdot (f \text{ is not a bijection})$. Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{Z}^+ \to \{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}$. **To show**: f is not a bijection. It's enough to show that f is not onto. Rewriting using the definition of onto, **to show**:

$$\exists x \in \{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\} \; \forall a \in \mathbb{N} \; (\; f(a) \neq x \;)$$

In search of a witness, define the following real number by defining its binary expansion

$$d_f = 0.b_1 b_2 b_3 \cdots$$

where $b_i = 1 - b_{ii}$ where b_{jk} is the coefficient of 2^{-k} in the binary expansion of $f(j)$. Since[1] $d_f \neq f(a)$ for any positive integer a, f is not onto.

Approach 2: Nested closed interval property

To show $f : \mathbb{N} \to \{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}$ is not onto. **Strategy**: Build a sequence of nested closed intervals that each avoid some $f(n)$. Then the real number that is in all of the intervals can't be $f(n)$ for any n. Hence, f is not onto.

Consider the function $f : \mathbb{N} \to \{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}$ with $f(n) = \frac{1+\sin(n)}{2}$

n	$f(n)$	Interval that avoids $f(n)$
0	0.5	
1	$0.920735\ldots$	
2	$0.954649\ldots$	
3	$0.570560\ldots$	
4	$0.121599\ldots$	
$\vdots$		

Examples of uncountable sets

- $\mathcal{P}(\mathbb{N})$, $\mathcal{P}(\mathbb{Z}^+)$, $\mathcal{P}(\mathbb{Z})$, power set of any countably infinite set.
- The closed interval $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, any other nonempty closed interval of real numbers whose endpoints are unequal, as well as the related intervals that exclude one or both of the endpoints.

[1] There's a subtle imprecision in this part of the proof as presented, but it can be fixed.

- $\mathbb{R}$
- $\overline{\mathbb{Q}}$, the set of irrational numbers

Review quiz questions

1. **Monday** Consider the following input-output definition tables with $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4, 5\}$ and $Z = \{10, 20, 30\}$

Table 1

Input	Output
1	10
2	20
3	30

Table 2

Input	Output
a	1
b	4
c	5

Table 3

Input	Output
10	a
20	b
30	a

   (a) Select all and only the tables that define a well-defined function whose domain and codomain is each one of the sets X, Y, Z.

   (b) Select all and only the tables that define a well-defined function (with domain X or Y or Z and with codomain X or Y or Z) and that is one-to-one.

   (c) Select all and only the tables that define a well-defined function (with domain X or Y or Z and with codomain X or Y or Z) and that is onto.

2. **Monday** Consider the following functions:

$$f : \mathbb{Z} \to \mathbb{N} \qquad f(n) = \begin{cases} 0 & \text{when } n = 0 \\ (-2 \cdot n) - 1 & \text{when } n < 0 \\ 2 \cdot n & \text{when } n > 0 \end{cases}$$

$$g : \mathbb{Z} \to \mathbb{N} \qquad g(n) = \begin{cases} -1 \cdot n & \text{when } n < 0 \\ n & \text{when } n \geq 0 \end{cases}$$

$$h : \mathbb{N} \to \mathbb{Z} \qquad h(n) = \begin{cases} (-2 \cdot n) + 1 & \text{when } n \text{ is even} \\ 2 \cdot n & \text{when } n \text{ is odd} \end{cases}$$

$$q : \mathbb{N} \to \mathbb{Z} \qquad q(n) = \begin{cases} -1 \cdot ((n+1) \textbf{ div } 2) & \text{when } n \text{ is} \\ n \textbf{ div } 2 & \text{when } n \text{ is} \end{cases}$$

   (a) What is the result of $f(-3)$?

   (b) What is the result of $q(f(-4))$?

   For a review of function composition, see page 146, definition 10 in the textbook.

(c) What is the result of $f(h(4))$?

For a review of function composition, see page 146, definition 10 in the textbook.

(d) What is the result of $g(-4)$?

(e) What is the result of $g(4)$?

(f) Consider the following statements, and indicate if they are true for each of f, g, h, and q.

i. This function is one-to-one.

ii. This function is onto.

iii. This function is a bijection.

iv. This function could serve as a witness for $|\mathbb{Z}| \leq |\mathbb{N}|$.

v. This function could serve as a witness for $|\mathbb{Z}| \geq |\mathbb{N}|$.

vi. This function could serve as a witness for $|\mathbb{N}| \leq |\mathbb{Z}|$.

vii. This function could serve as a witness for $|\mathbb{N}| \geq |\mathbb{Z}|$.

3. **Wednesday** Consider the function $f : \mathbb{N} \to \mathbb{Z}$ given by $f(n) = \begin{cases} n \textbf{ div } 4 & \text{if } n \text{ is even} \\ -((n+1) \textbf{ div } 4) & \text{if } n \text{ is odd} \end{cases}$

Select all and only the true statements below.

(a) f is one-to-one

(b) f is onto

(c) f is a bijection

(d) f witnesses that $|\mathbb{N}| \leq |\mathbb{Z}|$

(e) f witnesses that $|\mathbb{N}| \geq |\mathbb{Z}|$

(f) f witnesses that $|\mathbb{N}| = |\mathbb{Z}|$

(g) There is a one-to-one function with domain $\mathbb{N}$ and codomain $\mathbb{Z}$

(h) There is an onto function with domain $\mathbb{N}$ and codomain $\mathbb{Z}$

(i) There is a bijection with domain $\mathbb{N}$ and codomain $\mathbb{Z}$

(j) $|\mathbb{N}| \leq |\mathbb{Z}|$

(k) $|\mathbb{N}| \geq |\mathbb{Z}|$

(l) $|\mathbb{N}| = |\mathbb{Z}|$

4. **Friday** The diagonalization argument constructs, for each function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, a set D_f defined as

$$D_f = \{x \in \mathbb{N} \mid x \notin f(x)\}$$

which has the property that, for all $n \in \mathbb{N}$, $f(n) \neq D_f$. Consider the following two functions with domain $\mathbb{N}$ and codomain $\mathcal{P}(\mathbb{N})$

$$f_1(x) = \{y \in \mathbb{N} \mid y \textbf{ mod } 3 = x \textbf{ mod } 3\}$$

$$f_2(x) = \{y \in \mathbb{N} \mid (y > 0) \wedge (x \textbf{ mod } y \neq 0)\}$$

Select all and only the true statements below.

(a) $0 \in D_{f_1}$

(b) D_{f_1} is infinite

(c) D_{f_1} is uncountable

(d) $1 \in D_{f_2}$

(e) D_{f_2} is empty

(f) D_{f_2} is countably infinite

5. **Friday** Recall the definitions from previous assignments and class: The bases of RNA are elements of the set $B = \{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{U}\}$. The set of RNA strands S is defined (recursively) by:

Basis Step:	$\mathtt{A} \in S, \mathtt{C} \in S, \mathtt{U} \in S, \mathtt{G} \in S$
Recursive Step:	If $s \in S$ and $b \in B$, then $sb \in S$

For b an integer greater than 1 and n a positive integer, the **base b expansion of n** is

$$(a_{k-1} \cdots a_1 a_0)_b$$

where k is a positive integer, $a_0, a_1, \ldots, a_{k-1}$ are nonnegative integers less than b, $a_{k-1} \neq 0$, and

$$n = a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$$

For b an integer greater than 1, w a positive integer, and n a nonnegative integer with $n < b^w$, the **base b fixed-width w expansion of n** is

$$(a_{w-1} \cdots a_1 a_0)_{b,w}$$

where $a_0, a_1, \ldots, a_{w-1}$ are nonnegative integers less than b and

$$n = a_{w-1}b^{w-1} + \cdots + a_1 b + a_0$$

For b an integer greater than 1, w a positive integer, w' a positive integer, and x a real number the **base b fixed-width expansion of x with integer part width w and fractional part width w'** is

$$(a_{w-1} \cdots a_1 a_0 . c_1 \cdots c_{w'})_{b,w,w'}$$

where $a_0, a_1, \ldots, a_{w-1}, c_1, \ldots, c_{w'}$ are nonnegative integers less than b and

$$x \geq a_{w-1}b^{w-1} + \cdots + a_1 b + a_0 + c_1 b^{-1} + \cdots + c_{w'} b^{-w'}$$

and

$$x < a_{w-1}b^{w-1} + \cdots + a_1 b + a_0 + c_1 b^{-1} + \cdots + (c_{w'} + 1) b^{-w'}$$

For each set below, determine if it is empty, nonempty and finite, countably infinite, or uncountable.

Challenge - not to hand in: how would you prove this?

(a) B

(b) S

(c) $\{x \in \mathbb{N} \mid x = (4102)_3\}$

(d) $\{x \in \mathbb{R} \mid x \text{ has a binary fixed-width 5 expansion}\}$

(e) $\{x \in \mathbb{R} \mid x = (0.10)_{(2,1,2)}\}$