

Mathematical Induction

Invariant: A property that is true about our algorithm no matter what.
Rosen p375

Theorem: Statement that can be shown to be true, usually an important one.
Rosen p81

Less important theorems can be called **proposition, fact, result**.

A less important theorem that is useful in proving a theorem is called a **lemma**.

A theorem that can be proved directly after another one has been proved is called a **corollary**

Theorem: A robot on an infinite 2-dimensional integer grid starts at $(0,0)$ and at each step moves to diagonally adjacent grid point. This robot can / cannot (*circle one*) reach $(1,0)$.

Definition The set of positions the robot can visit P is defined by:

Basis Step: $(0,0) \in P$

Recursive Step: If $(x,y) \in P$, then

Lemma: $\forall(x,y) \in P((x+y \text{ is an even integer}))$

Proof of theorem using lemma: To show is $(1,0) \notin P$. Rewriting the lemma to explicitly restrict the domain of the universal, we have $\forall(x,y) ((x,y) \in P \rightarrow (x+y \text{ is an even integer}))$. Since the universal is true, $((1,0) \in P \rightarrow (1+0 \text{ is an even integer}))$ is a true statement. Evaluating the conclusion of this conditional statement: By definition of long division, since $1 = 0 \cdot 2 + 1$ (where $0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$ and $0 \leq 1 < 2$ mean that 0 is the quotient and 1 is the remainder), $1 \bmod 2 = 1$ which is not 0 so the conclusion is false. A true conditional with a false conclusion must have a false hypothesis. Thus, $(1,0) \notin P$, QED. \square

Proof of lemma by structural induction:

Basis Step

Recursive Step. Consider arbitrary $(x,y) \in P$. To show is:

$(x+y \text{ is an even integer}) \rightarrow (\text{sum of coordinates of next position is even integer})$

Assume **as the induction hypothesis, IH** that:

“New”! Proof by Mathematical Induction (Rosen 5.1 p329)
 To prove a universal quantification over the set of all integers greater than or equals some base integer b :

Basis Step: Show the statement holds for b .

Recursive Step: Consider an arbitrary integer n greater than or equal to b , assume (as the **induction hypothesis**) that the property holds for n , and use this and other facts to prove that the property holds for $n + 1$.

Recall that the set of linked lists of natural numbers L	Recall that length of a linked list of natural numbers L , $length : L \rightarrow \mathbb{N}$ is defined by:
Basis Step: $[] \in L$	Basis step: $length([]) = 0$
Recursive Step: If $l \in L$ and $n \in \mathbb{N}$ then $(n, l) \in L$	Recursive step: If $l \in L$ and $n \in \mathbb{N}$ then $length((n, l)) = 1 + length(l)$

Prove or disprove: $\forall n \in \mathbb{N} \exists l \in L (length(l) = n)$

Proof of \star by mathematical induction ($b = 8$)

Basis step: WTS property is true about 8

Recursive step: Consider an arbitrary $n \geq 8$. Assume (as the IH) that there are nonnegative integers x, y such that $n = 5x + 3y$. WTS that there are nonnegative integers x', y' such that $n + 1 = 5x' + 3y'$. We consider two cases, depending on whether any 5 cent coins are used for n .

Case 1: Assume $x \geq 1$. Define $x' = x - 1$ and $y' = y + 2$ (both in \mathbb{N} by case assumption).

$$\begin{aligned} 5x' + 3y' &\stackrel{\text{by def}}{=} 5(x - 1) + 3(y + 2) = 5x - 5 + 3y + 6 \\ &\stackrel{\text{rearranging}}{=} (5x + 3y) - 5 + 6 \\ &\stackrel{\text{IH}}{=} n - 5 + 6 = n + 1 \end{aligned}$$

Case 2: Assume $x = 0$. Therefore $n = 3y$, so since $n \geq 8$, $y \geq 3$. Define $x' = 2$ and $y' = y - 3$ (both in \mathbb{N} by

case assumption). Calculating:

$$\begin{aligned} 5x' + 3y' &\stackrel{\text{by def}}{=} 5(2) + 3(y - 3) = 10 + 3y - 9 \\ &\stackrel{\text{rearranging}}{=} 3y + 10 - 9 \\ &\stackrel{\text{IH and case}}{=} n + 10 - 9 = n + 1 \end{aligned}$$

<p>Proof of \star by strong induction $(b = 8 \text{ and } j = 2)$</p>	<p>Recursive step: Consider an arbitrary $n \geq 10$. Assume (as the IH) that the property is true about each of $8, 9, 10, \dots, n$. WTS that there are nonnegative integers x', y' such that $n + 1 = 5x' + 3y'$.</p>
<p>Basis step: WTS property is true about $8, 9, 10$</p>	