Monday February 22

$\mathbb{Z}$	The set of integers	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}^+$	The set of positive integers	$\{1, 2, \ldots\}$
$\mathbb{N}$	The set of nonnegative integers	$\{0, 1, 2, \ldots\}$
$\mathbb{Q}$	The set of rational numbers	$\left\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0\right\}$
$\mathbb{R}$	The set of real numbers	

$$\_\_ \subsetneq \_\_ \subsetneq \_\_ \subsetneq \_\_ \subsetneq \_\_$$

The above sets are all **infinite**.

A **finite** set is one whose distinct elements can be counted by a natural number.

Examples of finite sets: $\emptyset$, $\{\sqrt{2}\}$

Motivating question: Are some of the above sets *bigger than* others? *Analogy*: Musical chairs

People try to sit down when the music stops

Person☼ sits in Chair 1, Person☺ sits in Chair 2,

Person☹ is left standing!

What does this say about the number of chairs and the number of people?

Defining functions A function is defined by its (1) domain, (2) codomain, and (3) rule assigning each element in the domain exactly one element in the codomain. The domain and codomain are nonempty sets. The rule can be depicted as a table, formula, English description, etc.

(Rosen p139)

Example: $f_A : \mathbb{R}^+ \to \mathbb{Q}$ with $f_A(x) = x$ is **not** a well-defined function because

Example: $f_B : \mathbb{Q} \to \mathbb{Z}$ with $f_B\left(\frac{p}{q}\right) = p + q$ is **not** a well-defined function because

Example: $f_C : \mathbb{Z} \to \mathbb{R}$ with $f_C(x) = \frac{x}{|x|}$ is **not** a well-defined function because

Definition (Rosen p141): A function $f : D \to C$ is **one-to-one** (or injective) means for every a, b in the domain D, if $f(a) = f(b)$ then $a = b$.

Definition: For sets A, B, we say that **the cardinality of A is no bigger than the cardinality of B**, and write $|A| \leq |B|$, to mean there is a one-to-one function with domain A and codomain B. *In the analogy*: The function $sitter : \{Chair1, Chair2\} \to \{Person☼, Person☺, Person☹\}$ given by $sitter(Chair1) = Person☼$, $sitter(Chair2) = Person☺$, is one-to-one and witnesses that

$$|\{Chair1, Chair2\}| \leq |\{Person☼, Person☺, Person☹\}|$$

Let S_2 be the set of RNA strands of length 2.

Statement	True/False , justification
$\lvert\{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\}\rvert \leq \lvert S_2\rvert$	
$\lvert\{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\} \times \{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\}\rvert \leq \lvert S_2\rvert$	

Definition (Rosen p143): A function $f : D \to C$ is **onto** (or surjective) means for every b in the codomain, there is an element a in the domain with $f(a) = b$.

Formally, $f : D \to C$ is onto means ____________________.

Definition: For sets A, B, we say that **the cardinality of A is no smaller than the cardinality of B**, and write $|A| \geq |B|$, to mean there is an onto function with domain A and codomain B. *In the analogy*: The function $triedToSit : \{Person☼, Person☺, Person☹\} \to \{Chair1, Chair2\}$ given by $triedToSit(Person☼) = Chair1$, $triedToSit(Person☺) = Chair2$, $triedToSit(Person☹) = Chair2$, is onto and witnesses that

$$|\{Person☼, Person☺, Person☹\}| \geq |\{Chair1, Chair2\}|$$

Let S_2 be the set of RNA strands of length 2.

Statement	True/False , justification
$\lvert S_2\rvert \geq \lvert\{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\}\rvert$	
$\lvert S_2\rvert \geq \lvert\{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\} \times \{\mathtt{A}, \mathtt{U}, \mathtt{G}, \mathtt{C}\}\rvert$	

Definition (Rosen p144): A function $f : D \to C$ is a **bijection** means that it is both one-to-one and onto. The **inverse** of a bijection $f : D \to C$ is the function $g : C \to D$ such that $g(b) = a$ iff $f(a) = b$.

For nonempty sets A, B we say

$|A| \leq |B|$ means there is a one-to-one function with domain A, codomain B

$|A| \geq |B|$ means there is an onto function with domain A, codomain B

$|A| = |B|$ means there is a bijection with domain A, codomain B

Properties of cardinality

$$\forall A \ (\ |A| = |A|\ )$$
$$\forall A\ \forall B\ (\ |A| = |B| \ \to\ |B| = |A|\ )$$
$$\forall A\ \forall B\ \forall C\ (\ (|A| = |B| \ \land\ |B| = |C|) \ \to\ |A| = |C|\ )$$

Extra practice with proofs: Use the definitions of bijections to prove these properties.

Cantor-Schroder-Bernstein Theorem: For all nonempty sets,

$$|A| = |B| \qquad \text{if and only if} \qquad (|A| \leq |B| \text{ and } |B| \leq |A|) \qquad \text{if and only if} \qquad (|A| \geq |B| \text{ and } |B| \geq |A|)$$

To prove $|A| = |B|$, we can do any **one** of the following

- Prove there exists a bijection $f : A \to B$;
- Prove there exists a bijection $f : B \to A$;
- Prove there exists two functions $f_1 : A \to B$, $f_2 : B \to A$ where each of f_1, f_2 is one-to-one.
- Prove there exists two functions $f_1 : A \to B$, $f_2 : B \to A$ where each of f_1, f_2 is onto.