

## Strong Induction

For which nonnegative integers  $n$  can we make change for  $n$  with coins of value 5 cents and 3 cents?

Restating: We can make change for \_\_\_\_\_, we cannot make change for \_\_\_\_\_, and

\_\_\_\_\_★

### New! Proof by Strong Induction (Rosen 5.2 p337)

To prove that a universal quantification over the set of all integers greater than or equal to some base integer  $b$  holds, pick a fixed nonnegative integer  $j$  and then:

Basis Step: Show the statement holds for  $b, b + 1, \dots, b + j$ .

Recursive Step: Consider an arbitrary integer  $n$  greater than or equal to  $b + j$ , assume (as the **strong induction hypothesis**) that the property holds for **each of**  $b, b + 1, \dots, n$ , and use this and other facts to prove that the property holds for  $n + 1$ .

$\mathbb{N}$	The set of natural numbers	$\{0, 1, 2, 3, \dots\}$
$\mathbb{Z}^{\geq b}$	The set of integers greater than or equal a basis element $b$	$\{b, b + 1, b + 2, b + 3, \dots\}$

**Proof of  $\star$  by mathematical induction** ( $b = 8$ )

**Basis step:** WTS property is true about 8

**Recursive step:** Consider an arbitrary  $n \geq 8$ . Assume (as the IH) that there are nonnegative integers  $x, y$  such that  $n = 5x + 3y$ . WTS that there are nonnegative integers  $x', y'$  such that  $n + 1 = 5x' + 3y'$ . We consider two cases, depending on whether any 5 cent coins are used for  $n$ .

*Case 1:* Assume  $x \geq 1$ . Define  $x' = x - 1$  and  $y' = y + 2$  (both in  $\mathbb{N}$  by case assumption).

$$\begin{aligned} 5x' + 3y' &\stackrel{\text{by def}}{=} 5(x - 1) + 3(y + 2) = 5x - 5 + 3y + 6 \\ &\stackrel{\text{rearranging}}{=} (5x + 3y) - 5 + 6 \\ &\stackrel{\text{IH}}{=} n - 5 + 6 = n + 1 \end{aligned}$$

*Case 2:* Assume  $x = 0$ . Therefore  $n = 3y$ , so since  $n \geq 8$ ,  $y \geq 3$ . Define  $x' = 2$  and  $y' = y - 3$  (both in  $\mathbb{N}$  by

case assumption). Calculating:

$$\begin{aligned} 5x' + 3y' &\stackrel{\text{by def}}{=} 5(2) + 3(y - 3) = 10 + 3y - 9 \\ &\stackrel{\text{rearranging}}{=} 3y + 10 - 9 \\ &\stackrel{\text{IH and case}}{=} n + 10 - 9 = n + 1 \end{aligned}$$

<p><b>Proof of <math>\star</math> by strong induction</b>  <math>(b = 8 \text{ and } j = 2)</math></p>	<p><b>Recursive step:</b> Consider an arbitrary <math>n \geq 10</math>. Assume (as the IH) that the property is true about each of <math>8, 9, 10, \dots, n</math>. WTS that there are nonnegative integers <math>x', y'</math> such that <math>n + 1 = 5x' + 3y'</math>.</p>
<p><b>Basis step:</b> WTS property is true about <math>8, 9, 10</math></p>	

### Representing positive integers

**Theorem:** Every positive integer is a sum of (one or more) distinct powers of 2. *binary expansions exist!*

**Proof by strong induction,** with  $b = 1$  and  $j = 0$ .

**Basis step:** WTS property is true about 1.

**Recursive step:** Consider an arbitrary integer  $n \geq 1$ . Assume (as the IH) that the property is true about each of  $1, \dots, n$ . WTS that the property is true about  $n + 1$ .

**Definition** (Rosen p257): An integer  $p$  greater than 1 is called **prime** if the only positive factors of  $p$  are 1 and  $p$ . A positive integer that is greater than 1 and is not prime is called composite.

**Theorem** (Rosen p336): Every positive integer *greater than 1* is a product of (one or more) primes.

**Proof by strong induction,** with  $b = 2$  and  $j = 0$ .

**Basis step:** WTS property is true about 2.

**Recursive step:** Consider an arbitrary integer  $n \geq 2$ . Assume (as the IH) that the property is true about each of  $2, \dots, n$ . WTS that the property is true about  $n + 1$ .

**Case 1:**

**Case 2:**