Mathematical Induction

Invariant: A property that is true about our algorithm no matter what. Rosen p375

Theorem: Statement that can be shown to be true, usually an important one. Rosen p81

> Less important theorems can be called **proposition**, **fact**, **result**.
>
> A less important theorem that is useful in proving a theorem is called a **lemma**.
>
> A theorem that can be proved directly after another one has been proved is called a **corollary**

Theorem: A robot on an infinite 2-dimensional integer grid starts at $(0, 0)$ and at each step moves to diagonally adjacent grid point. This robot can / cannot (*circle one*) reach $(1, 0)$.

Definition The set of positions the robot can visit P is defined by:

Basis Step:	$(0, 0) \in P$
Recursive Step:	If $(x, y) \in P$, then

are

Lemma: $\forall (x, y) \in P((x + y \text{ is an even integer}))$

Proof of theorem using lemma: To show is $(1, 0) \notin P$. Rewriting the lemma to explicitly restrict the domain of the universal, we have $\forall (x, y)\ (\ (x, y) \in P \to (x + y \text{ is an even integer})\ )$. Since the universal is true, $(\ (1, 0) \in P \to (1 + 0 \text{ is an even integer})\ )$ is a true statement. Evaluating the conclusion of this conditional statement: By definition of long division, since $1 = 0 \cdot 2 + 1$ (where $0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$ and $0 \leq 1 < 2$ mean that 0 is the quotient and 1 is the remainder), $1 \textbf{ mod } 2 = 1$ which is not 0 so the conclusion is false. A true conditional with a false conclusion must have a false hypothesis. Thus, $(1, 0) \notin P$, QED. $\square$

Proof of lemma by structural induction:

Basis Step

Recursive Step. Consider arbitrary $(x, y) \in P$. To show is:

$$(x+y \text{ is an even integer}) \to (\text{sum of coordinates of next position is even integer})$$

Assume **as the induction hypothesis, IH** that:

“New”! Proof by Mathematical Induction (Rosen 5.1 p329)
To prove a universal quantification over the set of all integers greater than or equals some base integer b:

> **Basis Step**: Show the statement holds for b.
>
> **Recursive Step**: Consider an arbitrary integer n greater than or equal to b, assume (as the **induction hypothesis**) that the property holds for n, and use this and other facts to prove that the property holds for $n+1$.

Recall that the set of linked lists of natural numbers L

> Basis Step: $[] \in L$
> Recursive Step: If $l \in L$ and $n \in \mathbb{N}$ then $(n, l) \in L$

Recall that length of a linked list of natural n
bers L, $length : L \to \mathbb{N}$ is defined by:

> Basis step: $length([]) = 0$
> Recursive step: If $l \in L$ and $n \in \mathbb{N}$
> $length((n, l)) = 1 + length(l)$

Prove or disprove: $\forall n \in \mathbb{N}\ \exists l \in L\ (\ length(l) = n\ )$

Proof of $\star$ by mathematical induction $(b = 8)$

Basis step: WTS property is true about 8

Recursive step: Consider an arbitrary $n \geq 8$. Assume (as the IH) that there are nonnegative integers x, y such that $n = 5x + 3y$. WTS that there are nonnegative integers x', y' such that $n + 1 = 5x' + 3y'$. We consider two cases, depending on whether any 5 cent coins are used for n.

Case 1: Assume $x \geq 1$. Define $x' = x - 1$ and $y' = y + 2$ (both in $\mathbb{N}$ by case assumption). Calculating:

$$\begin{aligned} 5x' + 3y' &\overset{\text{by def}}{=} 5(x-1) + 3(y+2) = 5x - 5 + 3y + 6 \\ &\overset{\text{rearranging}}{=} (5x + 3y) - 5 + 6 \\ &\overset{\text{IH}}{=} n - 5 + 6 = n + 1 \end{aligned}$$

Case 2: Assume $x = 0$. Therefore $n = 3y$, so since $n \geq 8$, $y \geq 3$. Define $x' = 2$ and $y' = y - 3$ (both in $\mathbb{N}$ by case assumption). Calculating:

$$\begin{aligned} 5x' + 3y' &\overset{\text{by def}}{=} 5(2) + 3(y-3) = 10 + 3y - 9 \\ &\overset{\text{rearranging}}{=} 3y + 10 - 9 \\ &\overset{\text{IH and case}}{=} n + 10 - 9 = n + 1 \end{aligned}$$

Proof of $\star$ by strong induction ($b = 8$ and $j = 2$)

Basis step: WTS property is true about $8, 9, 10$

Recursive step: Consider an arbitrary $n \geq 10$. Assume (as the IH) that the property is true about each of $8, 9, 10, \ldots, n$. WTS that there are nonnegative integers x', y' such that $n + 1 = 5x' + 3y'$.