## Making change proof two ways

**Proof** of  $\star$  by mathematical induction (b = 8)

Basis step: WTS property is true about 8

Recursive step: Consider an arbitrary  $n \geq 8$ . Assume (as the IH) that there are nonnegative integers x, y such that n = 5x + 3y. WTS that there are nonnegative integers x', y' such that n + 1 = 5x' + 3y'. We consider two cases, depending on whether any 5 cent coins are used for n.

Case 1: Assume  $x \ge 1$ . Define x' = x - 1 and y' = y + 2 (both in  $\mathbb N$  by case assumption). Calculating:

$$5x' + 3y' \stackrel{\text{by def}}{=} 5(x - 1) + 3(y + 2) = 5x - 5 + 3y + 6$$

$$\stackrel{\text{rearranging}}{=} (5x + 3y) - 5 + 6$$

$$\stackrel{\text{IH}}{=} n - 5 + 6 = n + 1$$

Case 2: Assume x = 0. Therefore n = 3y, so since  $n \ge 8$ ,  $y \ge 3$ . Define x' = 2 and y' = y - 3 (both in  $\mathbb N$  by case assumption). Calculating:

$$5x' + 3y' \stackrel{\text{by def}}{=} 5(2) + 3(y - 3) = 10 + 3y - 9$$

$$\stackrel{\text{rearranging}}{=} 3y + 10 - 9$$

$$\stackrel{\text{IH and case}}{=} n + 10 - 9 = n + 1$$

Proof of  $\star$  by strong induction (b = 8 and j = 2)

Basis step: WTS property is true about 8, 9, 10

**Recursive step**: Consider an arbitrary  $n \ge 10$ . Assume (as the IH) that the property is true about each of  $8, 9, 10, \ldots, n$ . WTS that there are nonnegative integers x', y' such that n + 1 = 5x' + 3y'.

## Binary expansions exist proof

## Representing positive integers

**Theorem**: Every positive integer is a sum of (one or more) distinct powers of 2. binary expansions exist!

**Proof by strong induction**, with b = 1 and j = 0.

Basis step: WTS property is true about 1.

**Recursive step**: Consider an arbitrary integer  $n \ge 1$ . Assume (as the IH) that the property is true about each of  $1, \ldots, n$ . WTS that the property is true about n + 1.

## Fundamental theorem proof

**Theorem**: Every positive integer greater than 1 is a product of (one or more) primes.

**Proof by strong induction**, with b = 2 and j = 0.

Basis step: WTS property is true about 2.

**Recursive step**: Consider an arbitrary integer  $n \ge 2$ . Assume (as the IH) that the property is true about each of  $2, \ldots, n$ . WTS that the property is true about n + 1.

Case 1:

Case 2: