

## Making change proof two ways

**Proof of  $\star$  by mathematical induction** ( $b = 8$ )

**Basis step:** WTS property is true about 8

**Recursive step:** Consider an arbitrary  $n \geq 8$ . Assume (as the IH) that there are nonnegative integers  $x, y$  such that  $n = 5x + 3y$ . WTS that there are nonnegative integers  $x', y'$  such that  $n + 1 = 5x' + 3y'$ . We consider two cases, depending on whether any 5 cent coins are used for  $n$ .

*Case 1:* Assume  $x \geq 1$ . Define  $x' = x - 1$  and  $y' = y + 2$  (both in  $\mathbb{N}$  by case assumption). Calculating:

$$\begin{aligned} 5x' + 3y' &\stackrel{\text{by def}}{=} 5(x - 1) + 3(y + 2) = 5x - 5 + 3y + 6 \\ &\stackrel{\text{rearranging}}{=} (5x + 3y) - 5 + 6 \\ &\stackrel{\text{IH}}{=} n - 5 + 6 = n + 1 \end{aligned}$$

*Case 2:* Assume  $x = 0$ . Therefore  $n = 3y$ , so since  $n \geq 8$ ,  $y \geq 3$ . Define  $x' = 2$  and  $y' = y - 3$  (both in  $\mathbb{N}$  by case assumption). Calculating:

$$\begin{aligned} 5x' + 3y' &\stackrel{\text{by def}}{=} 5(2) + 3(y - 3) = 10 + 3y - 9 \\ &\stackrel{\text{rearranging}}{=} 3y + 10 - 9 \\ &\stackrel{\text{IH and case}}{=} n + 10 - 9 = n + 1 \end{aligned}$$

**Proof of  $\star$  by strong induction** ( $b = 8$  and  $j = 2$ )

**Basis step:** WTS property is true about 8, 9, 10

**Recursive step:** Consider an arbitrary  $n \geq 10$ . Assume (as the IH) that the property is true about each of  $8, 9, 10, \dots, n$ . WTS that there are nonnegative integers  $x', y'$  such that  $n + 1 = 5x' + 3y'$ .

# Binary expansions exist proof

## Representing positive integers

**Theorem:** Every positive integer is a sum of (one or more) distinct powers of 2. *binary expansions exist!*

**Proof by strong induction,** with  $b = 1$  and  $j = 0$ .

**Basis step:** WTS property is true about 1.

**Recursive step:** Consider an arbitrary integer  $n \geq 1$ . Assume (as the IH) that the property is true about each of  $1, \dots, n$ . WTS that the property is true about  $n + 1$ .

## Fundamental theorem proof

**Theorem:** Every positive integer *greater than 1* is a product of (one or more) primes.

**Proof by strong induction,** with  $b = 2$  and  $j = 0$ .

**Basis step:** WTS property is true about 2.

**Recursive step:** Consider an arbitrary integer  $n \geq 2$ . Assume (as the IH) that the property is true about each of  $2, \dots, n$ . WTS that the property is true about  $n + 1$ .

**Case 1:**

**Case 2:**