

## Netflix intro

What data should we encode about each Netflix account holder to help us make effective recommendations?

In machine learning, clustering can be used to group similar data for prediction and recommendation. For example, each Netflix user's viewing history can be represented as a  $n$ -tuple indicating their preferences about movies in the database, where  $n$  is the number of movies in the database. People with similar tastes in movies can then be clustered to provide recommendations of movies for one another. Mathematically, clustering is based on a notion of distance between pairs of  $n$ -tuples.

# Set operations

To define a set we can use the roster method, set builder notation, a recursive definition, and also we can apply a set operation to other sets.

## New! Cartesian product of sets and set-wise concatenation of sets of strings

**Definition:** Let  $X$  and  $Y$  be sets. The **Cartesian product** of  $X$  and  $Y$ , denoted  $X \times Y$ , is the set of all ordered pairs  $(x, y)$  where  $x \in X$  and  $y \in Y$

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$$

**Definition:** Let  $X$  and  $Y$  be sets of strings over the same alphabet. The **set-wise concatenation** of  $X$  and  $Y$ , denoted  $X \circ Y$ , is the set of all results of string concatenation  $xy$  where  $x \in X$  and  $y \in Y$

$$X \circ Y = \{xy \mid x \in X \text{ and } y \in Y\}$$

**Pro-tip:** the meaning of writing one element next to another like  $xy$  depends on the data-types of  $x$  and  $y$ . When  $x$  and  $y$  are strings, the convention is that  $xy$  is the result of string concatenation. When  $x$  and  $y$  are numbers, the convention is that  $xy$  is the result of multiplication. This is (one of the many reasons) why is it very important to declare the data-type of variables before we use them.

Fill in the missing entries in the table:

Set	Example elements in this set:			
$B$	A	C	G	U
	(A, C)		(U, U)	
$B \times \{-1, 0, 1\}$				
$\{-1, 0, 1\} \times B$				
	(0, 0, 0)			
$\{A, C, G, U\} \circ \{A, C, G, U\}$				
	GGGG			

# Defining functions

**New! Defining functions** A function is defined by its (1) domain, (2) codomain, and (3) rule assigning each element in the domain exactly one element in the codomain.

The domain and codomain are nonempty sets.

The rule can be depicted as a table, formula, or English description.

The notation is

“Let the function  $\text{FUNCTION-NAME}: \text{DOMAIN} \rightarrow \text{CODOMAIN}$  be given by  
 $\text{FUNCTION-NAME}(x) = \dots$  for every  $x \in \text{DOMAIN}$ ”.

or

“Consider the function  $\text{FUNCTION-NAME}: \text{DOMAIN} \rightarrow \text{CODOMAIN}$  given by  
 $\text{FUNCTION-NAME}(x) = \dots$  for every  $x \in \text{DOMAIN}$ ”.

Example: The absolute value function

**Domain**

**Codomain**

**Rule**

# Defining functions recursively

When the domain of a function is a *recursively defined set*, the rule assigning images to domain elements (outputs) can also be defined recursively.

Recall: The set of RNA strands  $S$  is defined (recursively) by:

$$\begin{array}{ll} \text{Basis Step:} & \mathbf{A} \in S, \mathbf{C} \in S, \mathbf{U} \in S, \mathbf{G} \in S \\ \text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B, \text{ then } sb \in S \end{array}$$

where  $sb$  is string concatenation.

**Definition** (Of a function, recursively) A function  $rnalen$  that computes the length of RNA strands in  $S$  is defined by:

$$\begin{array}{lll} & & rnalen : S \rightarrow \mathbb{Z}^+ \\ \text{Basis Step:} & \text{If } b \in B \text{ then} & rnalen(b) = 1 \\ \text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B, \text{ then} & rnalen(sb) = 1 + rnalen(s) \end{array}$$

The domain of  $rnalen$  is

The codomain of  $rnalen$  is

Example function application:

$$rnalen(\mathbf{ACU}) =$$

*Extra example:* A function  $basecount$  that computes the number of a given base  $b$  appearing in a RNA strand  $s$  is defined recursively: *fill in codomain and sample function applications*

$$\begin{array}{lll} & & basecount : S \times B \rightarrow \\ \text{Basis Step:} & \text{If } b_1 \in B, b_2 \in B & basecount(b_1, b_2) = \begin{cases} 1 & \text{when } b_1 = b_2 \\ 0 & \text{when } b_1 \neq b_2 \end{cases} \\ \text{Recursive Step:} & \text{If } s \in S, b_1 \in B, b_2 \in B & basecount(sb_1, b_2) = \begin{cases} 1 + basecount(s, b_2) & \text{when } b_1 = b_2 \\ basecount(s, b_2) & \text{when } b_1 \neq b_2 \end{cases} \end{array}$$

$$basecount(\mathbf{ACU}, \mathbf{A}) =$$

$$basecount(\mathbf{ACU}, \mathbf{G}) =$$

*Extra example:* The function which outputs  $2^n$  when given a nonnegative integer  $n$  can be defined recursively, because its domain is the set of nonnegative integers.

# Why represent numbers

Positional representation.

Computers.

## Base expansion definition

**Definition** For  $b$  an integer greater than 1 and  $n$  a positive integer, the **base  $b$  expansion of  $n$**  is

$$(a_{k-1} \cdots a_1 a_0)_b$$

where  $k$  is a positive integer,  $a_0, a_1, \dots, a_{k-1}$  are nonnegative integers less than  $b$ ,  $a_{k-1} \neq 0$ , and

$$n = a_{k-1}b^{k-1} + \cdots + a_1b + a_0$$

Notice: *The base  $b$  expansion of a positive integer  $n$  is a string over the alphabet  $\{x \in \mathbb{N} \mid x < b\}$  whose leftmost character is nonzero.*

Base $b$	Collection of possible coefficients in base $b$ expansion of a positive integer
Binary ( $b = 2$ )	$\{0, 1\}$
Ternary ( $b = 3$ )	$\{0, 1, 2\}$
Octal ( $b = 8$ )	$\{0, 1, 2, 3, 4, 5, 6, 7\}$
Decimal ( $b = 10$ )	$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Hexadecimal ( $b = 16$ )	$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$ letter coefficient symbols represent numerical values $(A)_{16} = (10)_{10}$ $(B)_{16} = (11)_{10}$ $(C)_{16} = (12)_{10}$ $(D)_{16} = (13)_{10}$ $(E)_{16} = (14)_{10}$ $(F)_{16} = (15)_{10}$

## Base expansion examples

Binary $b = 2$	Octal $b = 8$	Decimal $b = 10$	Hexadecimal $b = 16$
$(1401)_2$			
	$(1401)_8$		
		$(1401)_{10}$	
			$(1401)_{16}$

## Algorithm definition

**New!** An algorithm is a finite sequence of precise instructions for solving a problem.

## Division algorithm

**The Division Algorithm** Let  $n$  be an integer and  $d$  a positive integer. There are unique integers  $q$  and  $r$ , with  $0 \leq r < d$ , such that  $n = dq + r$ . In this case,  $d$  is called the divisor,  $n$  is called the dividend,  $q$  is called the quotient, and  $r$  is called the remainder. We write  $q = n \text{ div } d$  and  $r = n \text{ mod } d$ .

*Extra example:* How do **div** and **mod** compare to  $/$  and  $\%$  in Java and python?

## Algorithm log

Algorithm for calculating integer part of log

```

1  procedure  $\log(n$ : a positive integer)
2     $r := 0$ 
3    while  $n > 1$ 
4       $r := r + 1$ 
5       $n := n \text{ div } 2$ 
6    return  $r$  { $r$  holds the result of the log operation}
```

$n$	$r$	$n > 1?$
6		

# Base expansion algorithms

## Two algorithms for constructing base $b$ expansion from decimal representation

**Algorithm 1:** Start with highest power of  $b$ , i.e. at left-most coefficient of expansion

Calculating integer part of  $\log_b$

```
1 procedure logb( $n, b$ : positive integers with  $b > 1$ )
2    $r := 0$ 
3   while  $n > 1$ 
4      $r := r + 1$ 
5      $n := n \text{ div } b$ 
6   return  $r$  { $r$  holds the result of the log operation}
```

Calculating base  $b$  expansion, from left

```
1 procedure baseb1( $n, b$ : positive integers with  $b > 1$ )
2    $v := n$ 
3    $k := \text{logb}(n, b) + 1$ 
4   for  $i := 1$  to  $k$ 
5      $a_{k-i} := 0$ 
6     while  $v \geq b^{k-i}$ 
7        $a_{k-i} := a_{k-i} + 1$ 
8        $v := v - b^{k-i}$ 
9   return  $(a_{k-1}, \dots, a_0)$  { $(a_{k-1} \dots a_0)_b$  is the base  $b$  expansion of  $n$ }
```

**Algorithm 2:** Start with right-most coefficient of expansion

$n$	$b$	$q$	$k$	$a_k$	$q \neq 0?$

Calculating base  $b$  expansion, from right

```
1 procedure baseb2( $n, b$ : positive integers with  $b > 1$ )
2    $q := n$ 
3    $k := 0$ 
4   while  $q \neq 0$ 
5      $a_k := q \text{ mod } b$ 
6      $q := q \text{ div } b$ 
7      $k := k + 1$ 
8   return  $(a_{k-1}, \dots, a_0)$  { $(a_{k-1} \dots a_0)_b$  is the base  $b$  expansion of  $n$ }
```

Idea: (when  $k > 1$ )  $n = a_{k-1}b^{k-1} + \dots + a_1b + a_0 = b(a_{k-1}b^{k-2} + \dots + a_1) + a_0$  so  $a_0 = n \bmod b$  and  $a_{k-1}b^{k-2} + \dots + a_1 = n \text{ div } b$ .

Algorithm 1 for the base 3 expansion of 17	Algorithm 2 for the base 3 expansion of 17