

# Mathematical Induction

**Invariant:** A property that is true about our algorithm no matter what.  
Rosen p375

**Theorem:** Statement that can be shown to be true, usually an important one.  
Rosen p81

Less important theorems can be called **proposition, fact, result**.

A less important theorem that is useful in proving a theorem is called a **lemma**.

A theorem that can be proved directly after another one has been proved is called a **corollary**

**Theorem:** A robot on an infinite 2-dimensional integer grid starts at  $(0,0)$  and at each step moves to diagonally adjacent grid point. This robot can / cannot (*circle one*) reach  $(1,0)$ .

**Definition** The set of positions the robot can visit  $P$  is defined by:

Basis Step:  $(0,0) \in P$

Recursive Step: If  $(x,y) \in P$ , then

**Lemma:**  $\forall(x,y) \in P((x+y \text{ is an even integer}) )$

Proof of theorem using lemma: To show is  $(1,0) \notin P$ . Rewriting the lemma to explicitly restrict the domain of the universal, we have  $\forall(x,y) ( (x,y) \in P \rightarrow (x+y \text{ is an even integer}) )$ . Since the universal is true,  $( (1,0) \in P \rightarrow (1+0 \text{ is an even integer}) )$  is a true statement. Evaluating the conclusion of this conditional statement: By definition of long division, since  $1 = 0 \cdot 2 + 1$  (where  $0 \in \mathbb{Z}$  and  $1 \in \mathbb{Z}$  and  $0 \leq 1 < 2$  mean that 0 is the quotient and 1 is the remainder),  $1 \bmod 2 = 1$  which is not 0 so the conclusion is false. A true conditional with a false conclusion must have a false hypothesis. Thus,  $(1,0) \notin P$ , QED.  $\square$

Proof of lemma by structural induction:

**Basis Step**

**Recursive Step.** Consider arbitrary  $(x,y) \in P$ . To show is:

$(x+y \text{ is an even integer}) \rightarrow (\text{sum of coordinates of next position is even integer})$

Assume **as the induction hypothesis, IH** that:

<p><b>“New”! Proof by Mathematical Induction</b> (Rosen 5.1 p329)</p> <p>To prove a universal quantification over the set of all integers greater than or equals some base integer <math>b</math>:</p> <p><b>Basis Step:</b> Show the statement holds for <math>b</math>.</p> <p><b>Recursive Step:</b> Consider an arbitrary integer <math>n</math> greater than or equal to <math>b</math>, assume (as the <b>induction hypothesis</b>) that the property holds for <math>n</math>, and use this and other facts to prove that the property holds for <math>n + 1</math>.</p>
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<p>Recall that the set of linked lists of natural numbers <math>L</math></p> <p>Basis Step: <math>[] \in L</math></p> <p>Recursive Step: If <math>l \in L</math> and <math>n \in \mathbb{N}</math> then <math>(n, l) \in L</math></p>	<p>Recall that length of a linked list of natural numbers <math>L</math>, <math>length : L \rightarrow \mathbb{N}</math> is defined by:</p> <p>Basis step: <math>length([]) = 0</math></p> <p>Recursive step: If <math>l \in L</math> and <math>n \in \mathbb{N}</math> then <math>length((n, l)) = 1 + length(l)</math></p>
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Prove or disprove:  $\forall n \in \mathbb{N} \exists l \in L ( length(l) = n )$

**Proof of  $\star$  by mathematical induction** ( $b = 8$ )

**Basis step:** WTS property is true about 8

**Recursive step:** Consider an arbitrary  $n \geq 8$ . Assume (as the IH) that there are nonnegative integers  $x, y$  such that  $n = 5x + 3y$ . WTS that there are nonnegative integers  $x', y'$  such that  $n + 1 = 5x' + 3y'$ . We consider two cases, depending on whether any 5 cent coins are used for  $n$ .

*Case 1:* Assume  $x \geq 1$ . Define  $x' = x - 1$  and  $y' = y + 2$  (both in  $\mathbb{N}$  by case assumption).

$$\begin{aligned} 5x' + 3y' &\stackrel{\text{by def}}{=} 5(x - 1) + 3(y + 2) = 5x - 5 + 3y + 6 \\ &\stackrel{\text{rearranging}}{=} (5x + 3y) - 5 + 6 \\ &\stackrel{\text{IH}}{=} n - 5 + 6 = n + 1 \end{aligned}$$

*Case 2:* Assume  $x = 0$ . Therefore  $n = 3y$ , so since  $n \geq 8$ ,  $y \geq 3$ . Define  $x' = 2$  and  $y' = y - 3$  (both in  $\mathbb{N}$  by

case assumption). Calculating:

$$\begin{aligned} 5x' + 3y' &\stackrel{\text{by def}}{=} 5(2) + 3(y - 3) = 10 + 3y - 9 \\ &\stackrel{\text{rearranging}}{=} 3y + 10 - 9 \\ &\stackrel{\text{IH and case}}{=} n + 10 - 9 = n + 1 \end{aligned}$$

<p><b>Proof of <math>\star</math> by strong induction</b>  <math>(b = 8 \text{ and } j = 2)</math></p>	<p><b>Recursive step:</b> Consider an arbitrary <math>n \geq 10</math>. Assume (as the IH) that the property is true about each of <math>8, 9, 10, \dots, n</math>. WTS that there are nonnegative integers <math>x', y'</math> such that <math>n + 1 = 5x' + 3y'</math>.</p>
<p><b>Basis step:</b> WTS property is true about <math>8, 9, 10</math></p>	