# Making change proof two ways

**Proof of $\star$ by mathematical induction** ($b = 8$)

**Basis step**: WTS property is true about 8

**Recursive step**: Consider an arbitrary $n \geq 8$. Assume (as the IH) that there are nonnegative integers $x, y$ such that $n = 5x + 3y$. WTS that there are nonnegative integers $x', y'$ such that $n + 1 = 5x' + 3y'$. We consider two cases, depending on whether any 5 cent coins are used for $n$.

*Case 1*: Assume $x \geq 1$. Define $x' = x - 1$ and $y' = y + 2$ (both in $\mathbb{N}$ by case assumption). Calculating:

$$\begin{aligned} 5x' + 3y' &\overset{\text{by def}}{=} 5(x-1) + 3(y+2) = 5x - 5 + 3y + 6 \\ &\overset{\text{rearranging}}{=} (5x + 3y) - 5 + 6 \\ &\overset{\text{IH}}{=} n - 5 + 6 = n + 1 \end{aligned}$$

*Case 2*: Assume $x = 0$. Therefore $n = 3y$, so since $n \geq 8$, $y \geq 3$. Define $x' = 2$ and $y' = y - 3$ (both in $\mathbb{N}$ by case assumption). Calculating:

$$\begin{aligned} 5x' + 3y' &\overset{\text{by def}}{=} 5(2) + 3(y-3) = 10 + 3y - 9 \\ &\overset{\text{rearranging}}{=} 3y + 10 - 9 \\ &\overset{\text{IH and case}}{=} n + 10 - 9 = n + 1 \end{aligned}$$

**Proof of $\star$ by strong induction** ($b = 8$ and $j = 2$)

**Basis step**: WTS property is true about $8, 9, 10$

**Recursive step**: Consider an arbitrary $n \geq 10$. Assume (as the IH) that the property is true about each of $8, 9, 10, \ldots, n$. WTS that there are nonnegative integers $x', y'$ such that $n + 1 = 5x' + 3y'$.

# Binary expansions exist proof

**Representing positive integers**

**Theorem**: Every positive integer is a sum of (one or more) distinct powers of 2. *binary expansions exist!*

**Proof by strong induction**, with $b = 1$ and $j = 0$.

**Basis step**: WTS property is true about 1.

**Recursive step**: Consider an arbitrary integer $n \geq 1$. Assume (as the IH) that the property is true about each of $1, \ldots, n$. WTS that the property is true about $n + 1$.

# Fundamental theorem proof

**Theorem**: Every positive integer *greater than 1* is a product of (one or more) primes.

**Proof by strong induction**, with $b = 2$ and $j = 0$.

**Basis step**: WTS property is true about 2.

**Recursive step**: Consider an arbitrary integer $n \geq 2$. Assume (as the IH) that the property is true about each of $2, \ldots, n$. WTS that the property is true about $n + 1$.

**Case 1**:

**Case 2**: