How Much Time Does Speeding Actually Save?

a Probabilistic Model

Jadon Jones

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Quick Hypothetical

Consider the following scenario:

- You are driving down the road when a driver speeds past you
- The next traffic light happens to be red when you reach it, but you notice that the same car from before is stopped right next to you at the light

Question: Did that car save any time by speeding?

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- No!

What exactly are we modeling?

- The situation described in the previous hypothetical is what we seek to model
- Suppose you have a car going the speed limit and a car speeding by a constant amount
- How much time does the speeding car save when there traffic lights are placed regularly on the road?

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- Assume the law abiding car is traveling at speed v and the speeding car is speeding by Δv.
- If the road is of length d, then the amount of time saved by the second car is difference in times it takes the two cars to make it to the end of the road
- Using basic kinematics, we get that the time difference Δt is given by

$$\Delta t = \frac{d}{v} - \frac{d}{v + \Delta v} = \frac{d\Delta v}{v(v + \Delta v)}$$

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- For example, suppose a driver on a highway is okay with speeding by 15 mph on a 30 mile stretch of uninterrupted highway. The graph of time saved vs speed limit is given below.

Consequences

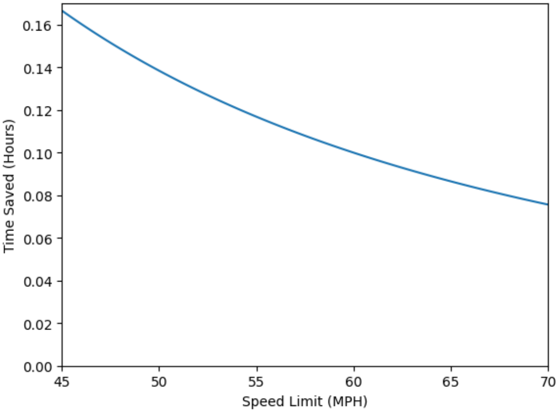

Figure: Speed Limit vs Time Saved

Model Assumptions pt. 1

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- Both cars have a fixed driving speed for the duration of the trip
- At all times cars are either stopped at a red light or moving at a constant speed on the road
- Traffic lights are evenly spaced and have identical light cycles with different phases

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- We choose units of time so that the traffic lights have a period of 1 (A.K.A normalization)

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- for example, we could represent the cycle as a function that assigns to each time either red or green based on the color of the light
- however a more useful concept for this model is to represent cycles as wait functions
- a wait function W is a function $W : \mathbb{R} \to [0, 1]$ with period 1, that given a time produces the amount of time a car would wait if the car arrived at the light at that time

Examples

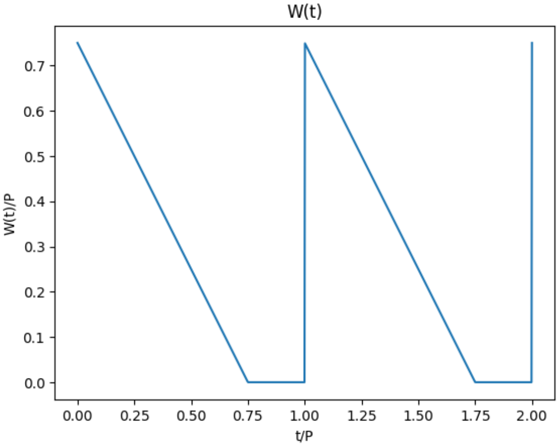
Example:
If the cycle consists of a red light for 0.75 of the cycle and a green/yellow light for 0.25, the wait function would be

$$W_{0.75}(t) = (.75 - \{t\})\chi_{[0,0.75]}(t)$$

Where the brackets refer to the fractional part of t and $\chi_{[0,0.75]}$ is the characteristic function of the set $[0, 0.75]$

Examples (cont.)

The graph of $W_{.75}$ is pictured below



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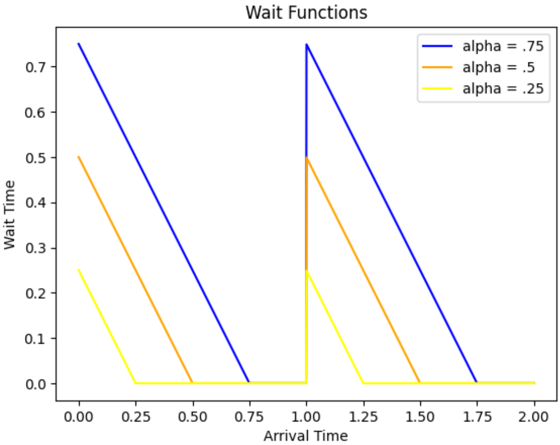
- These being the wait functions that are red for α units of time and green for $1-\alpha$ units of time.
- For $\alpha \in [0,1]$ we denote the corresponding wait function W_α

The graph of W_α for various values of α is pictured below



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  - The time saved right before the speeding car reaches the light
  - The amount of time the speeding car saves(or loses) at the light.

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- Suppose that the amount of time it takes the regular car to travel the distance between lights is t, then it takes the speeding car $t - \Delta t$ amount of time to travel the same distance

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- If the speeding car hits light k at time zero, then it hits the next light at time $t - \Delta t$. The regular car hits light $k - 1$ at T_{k-1}^{α} and so it hits light k at time $T_k^{\alpha} + t$

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- Thus the difference between these times is $T^{\alpha}_{k-1} + \Delta t$

Time saved/lost at the light

- The change in time saved due to the light is simply the amount of time the normal car waited at the light minus the time that the speeding car waited at that light.
- Since we choose the phases of the lights uniformly at random, we can assume that the speeding car hits the light at time X_k in it's cycle
- That means the regular car hits the light at time $X_k + T_{k-1}^{\alpha} + \Delta t$ in it's cycle
- Therefore the time saved/lost at the light is

$$W(X_k + T_{k-1}^{\alpha} + \Delta t) - W(X_k)$$

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- For $k > 0$ define T_k^α inductively by

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Time-Space Diagrams

- this model can be visualized using time-space diagrams
- these diagrams visualize the positions of both cars as functions of time as well as the cycle of the traffic lights
- from these graphs, you can see T_k^α for a particular run as the horizontal separation between the two cars' plots at the k^{th} light

Example 1

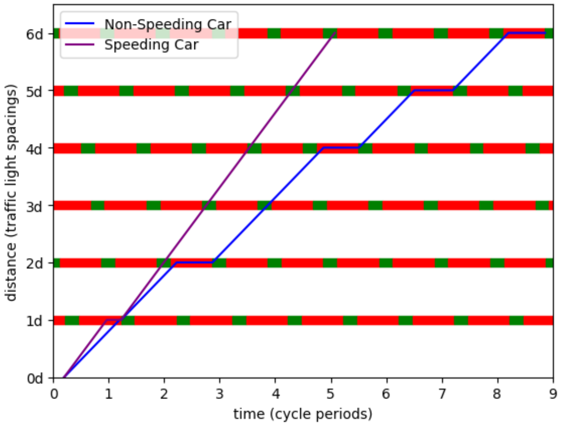

Figure: Time Space Diagram when T_6^{α} is not zero

Example 2

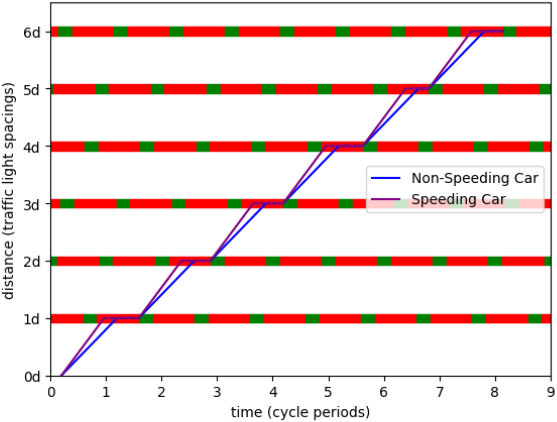

Figure: Time Space Diagram when T_6^α is zero

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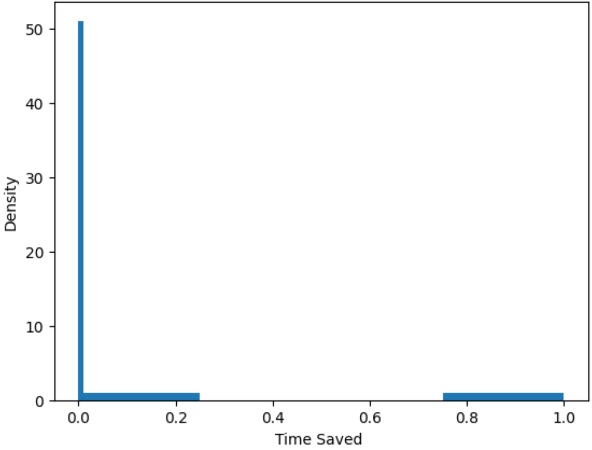
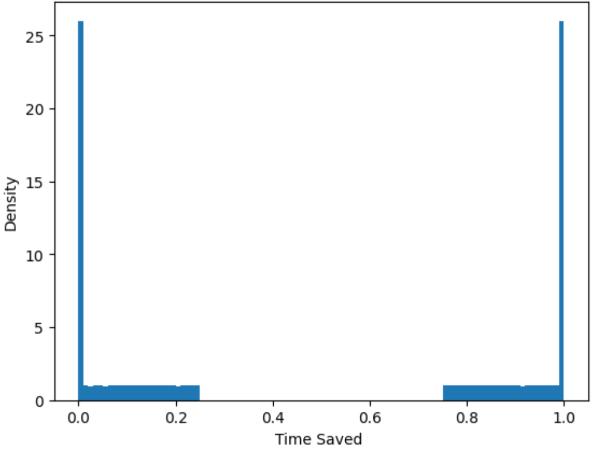
- From this definition, we can simulate specific values T_k^{α} by randomly generating numbers for the X_i's
- If we calculate a large number of such values and plot and create density plots, we obtain an approximation of the PDF of T_k^{α}
- In the following slides, the distributions of T_k^{α} have been simulated for various values of k, α, and Δt

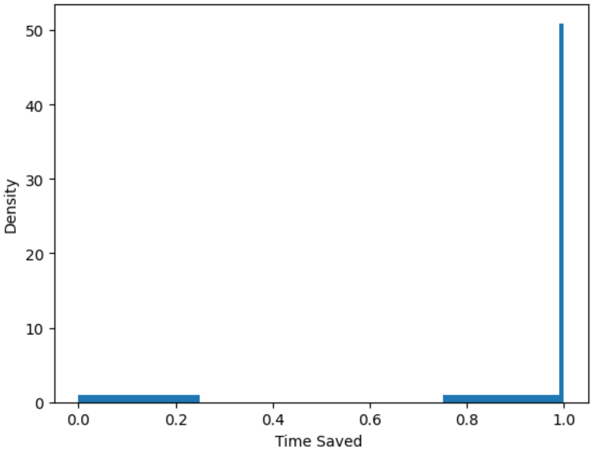
$\alpha = .75, \Delta\ t = .25$



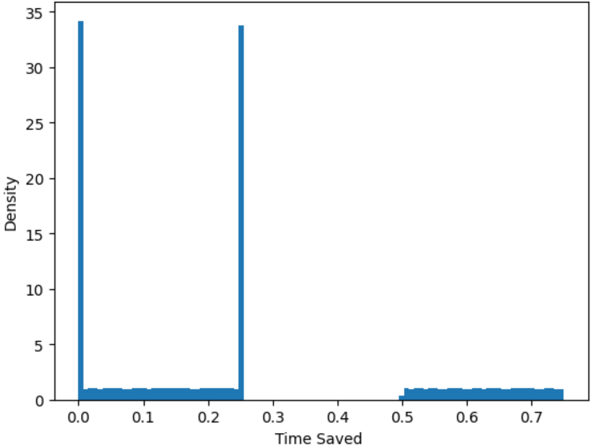
$\alpha = .75, \Delta\ t = .5$



$\alpha = .75, \Delta\ \mathrm{t} = .75$



$\alpha = .5, \Delta\ t = .25$



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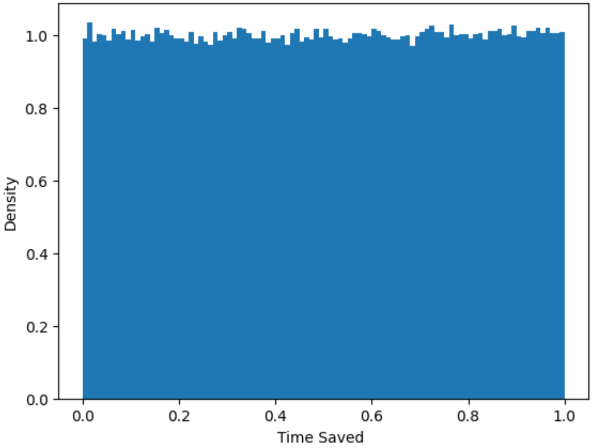

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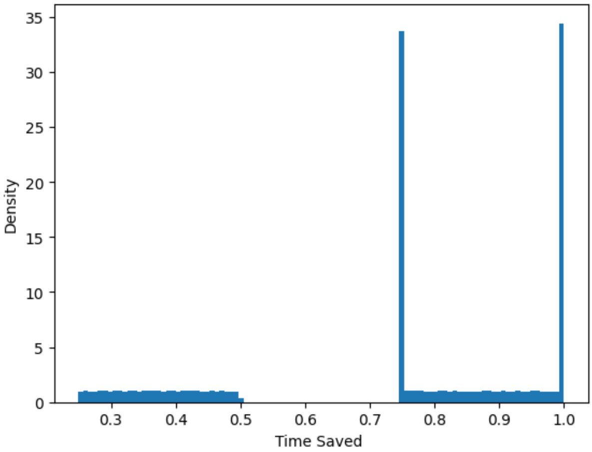

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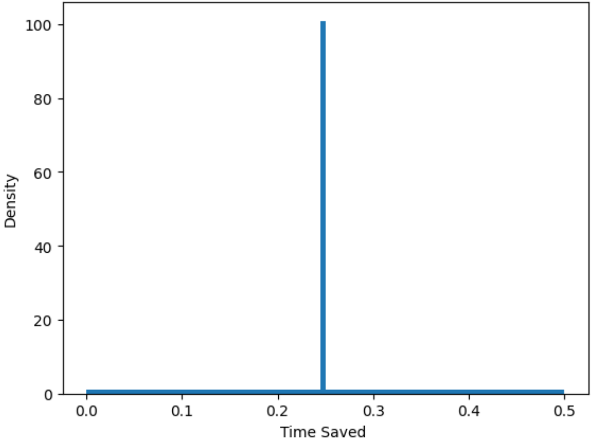

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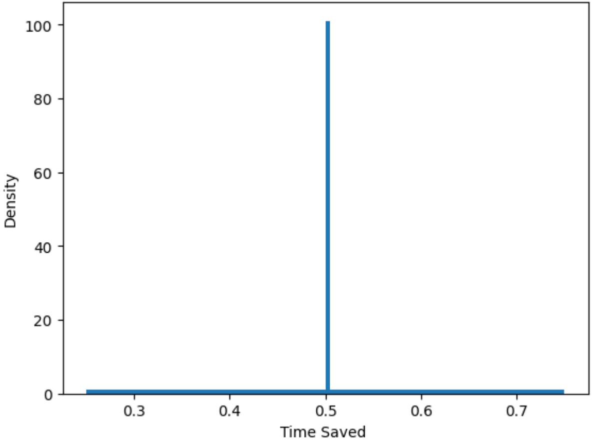

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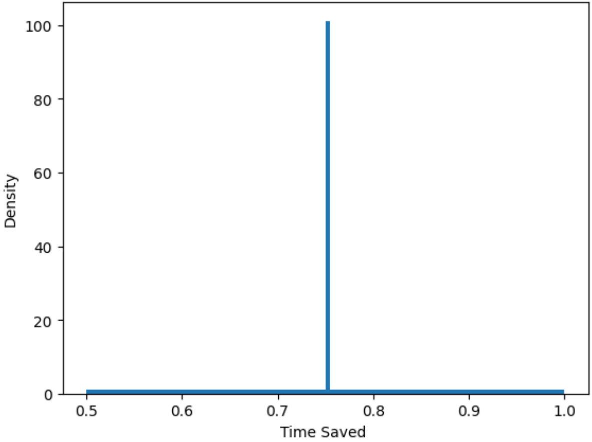

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- One very helpful tool is the Law of Total Expectation
- The Law of Total Expectation allows us to calculate the expected value of a random variable if we know information about it's conditional distribution
- The LoTE states that for a random variable X that can is conditioned on Y we have $\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X)$

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- This random variable is a function of only X_k
- We can use the fact that if X is uniform on $[0, 1)$ then $\mathbb{E}(f(X)) = \int_0^1 f(x)dx$

Expected Value of the Model

From this we can deduce that for $k > 0$

$$\begin{aligned}
\mathbb{E}(T_k^\alpha | T_{k-1}^\alpha) &= \int_0^1 (T_{k-1}^\alpha + \Delta t + W_\alpha(x + T_{k-1}^\alpha + \Delta t) - W_\alpha(x))dx \\
&= \int_0^1 (T_{k-1}^\alpha + \Delta t)dx + \int_0^1 W_\alpha(x + T_{k-1}^\alpha + \Delta t)dx \\
&\quad - \int_0^1 W_\alpha(x)dx \\
&= T_{k-1}^\alpha + \Delta t
\end{aligned}$$

Expected Value cont.

Using the LoTE, we can see

$$\begin{aligned}\mathbb{E}(T_k^\alpha) =& \mathbb{E}(\mathbb{E}(T_k^\alpha | T_{k-1}^\alpha)) \\ =& \mathbb{E}(T_{k-1}^\alpha + \Delta t) \\ =& \mathbb{E}(T_{k-1}^\alpha) + \Delta t\end{aligned}$$

Since $T_0^\alpha = 0$, by an inductive argument $\mathbb{E}(T_k^\alpha) = k\Delta t$

Backing Up

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- Think back to our Hypothetical Scenario. Was it just bad luck that the speeding car hit the light and saved no time?
- Yes, the expected value of the time saved is the same as if there were no stop lights at all.

So What?

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- If you had another model where there is either no time saved or twice the average time saved, with both options equally likely, it greatly changes how someone might make decisions
- One quantity of this model that helps to quantify this idea is variance

- Roughly, the variance tell you how far on average a random variable differs from it's mean
- When it exists, the variance is defined as

$$\mathbb{V}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$$

- Or using a more convenient form

$$\mathbb{V}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

Variance of T_k^α

- Using the LoTE we can begin to work with the variance of T_k^α
- Using the properties of Expectation we can write

$$\mathbb{E}((T_k^\alpha)^2 | T_{k-1}^\alpha) = (T_{k-1}^\alpha)^2 + 2T_{k-1}^\alpha \Delta t + \Delta t^2 + \frac{2}{3}\alpha^3 - 2\int_0^1 W_\alpha(T_{k-1}^\alpha + \Delta t + x) W_\alpha(x) dx$$

For convenience we define the integral term to be $I(T_{k-1}^\alpha)$

Variance of T_k^α Cont.

Using the LoTE we get

$$\begin{aligned}\mathbb{V}(T_k^\alpha) =& \mathbb{E}((T_k^\alpha)^2) - \mathbb{E}(T_k^\alpha)^2 \\ =& \mathbb{E}((T_{k-1}^\alpha)^2) - \mathbb{E}(T_{k-1}^\alpha)^2 + \frac{2}{3}\alpha^3 - 2\mathbb{E}(I(T_{k-1}^\alpha)) \\ =& \mathbb{V}(T_{k-1}^\alpha) + \frac{2}{3}\alpha^3 - 2\mathbb{E}(I(T_{k-1}^\alpha))\end{aligned}$$

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- **Conjecture:** $\mathbb{E}(I(T_{k-1}^\alpha))$ is constant for all $k > 1$
- If this is true then by an inductive argument we know $\mathbb{V}(T_k^\alpha) = C(k-1) + D$ for each α and Δt

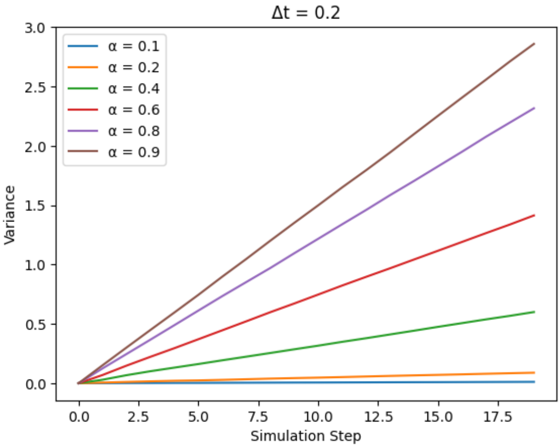
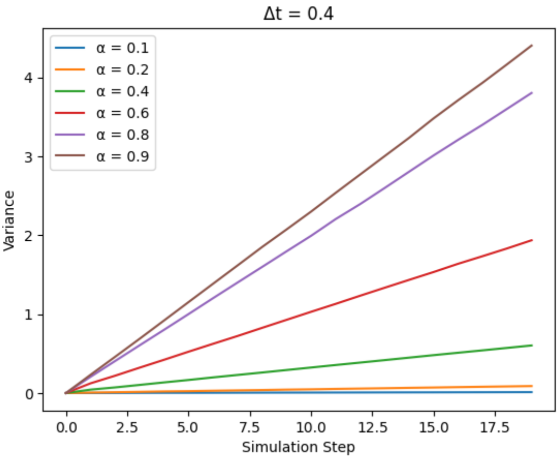

Δt = 0.2
α = 0.1
α = 0.2
α = 0.4
α = 0.6
α = 0.8
α = 0.9
Variance
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
Simulation Step

Figure: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α = 0.2
α = 0.4
α = 0.6
α = 0.8
α = 0.9
Variance
0
1
2
3
4
0.0
2.5
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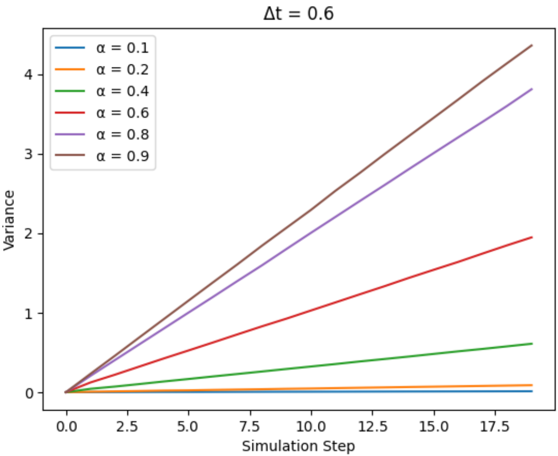
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Simulation Step

Figure:

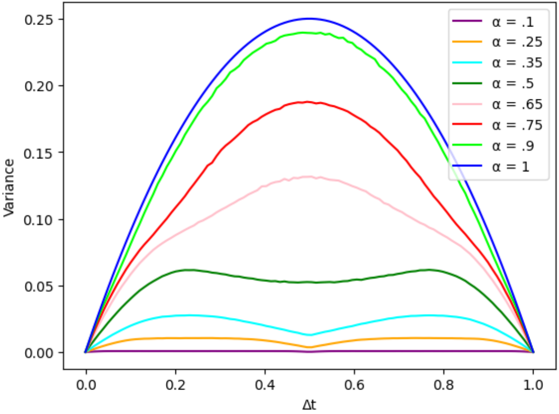

Figure: $\mathbb{V}(T_1^\alpha)$ vs Δt for various values of α

The End

Thanks for coming to my Honors Thesis Presentation