Iterated Maps

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November 18, 2024

Overview

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- Now type cos(ans) and press enter as many times as you can.
- Your calculator should now show the number **0.7390851332**

Iterated Maps

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Given a function $f : A \to A$ where $A \subset \mathbb{R}$, Define the sequence of iterate functions $\{f^n\}_{n=1}^{\infty}$.

- For a given $x_0 \in A$ we call the sequence $\{x_n\}_{n=0}^{\infty}$ where $x_{n+1} = f(x_n)$ the orbit of x_0

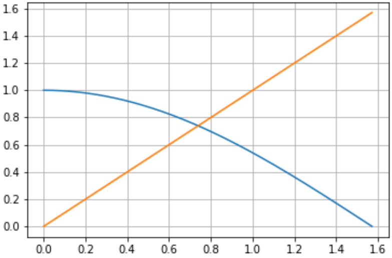
What is a Cobweb diagram?

- for simplicity lets assume our map is continuous
- With these definitions in place, we can now try to visualize iterated maps
- For a given map f, we can use a Cobweb diagram to graphically see the behavior of a given orbit.

Cobweb Diagram Example

- Step 1: graph your map and the line $y = x$

$$f(x) = \cos(x),\ y = x$$



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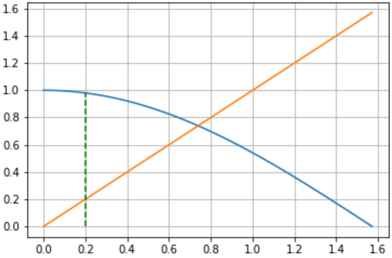

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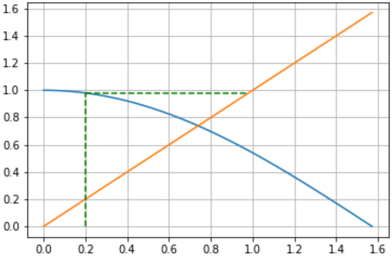

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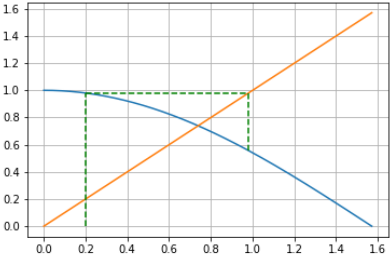

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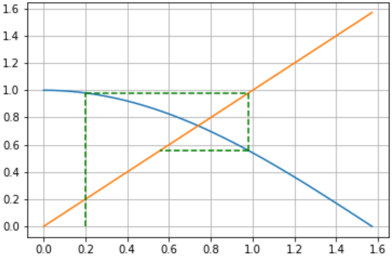

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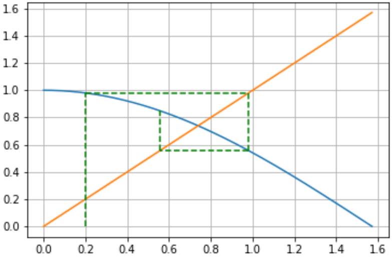

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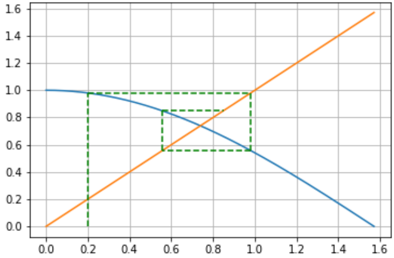

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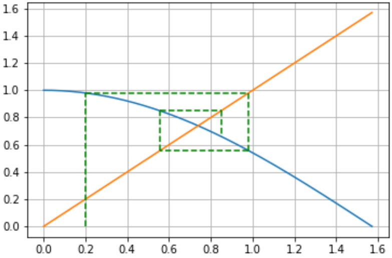

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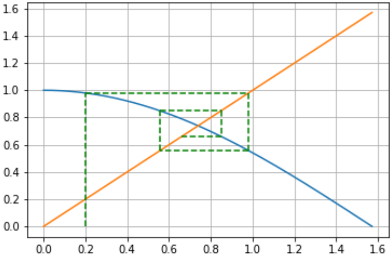

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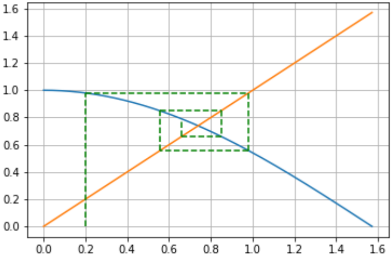

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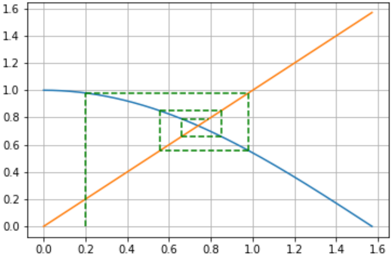

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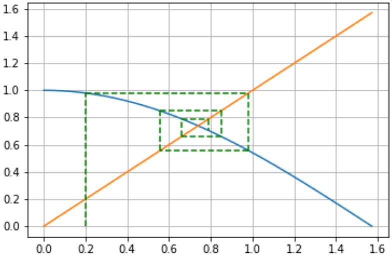

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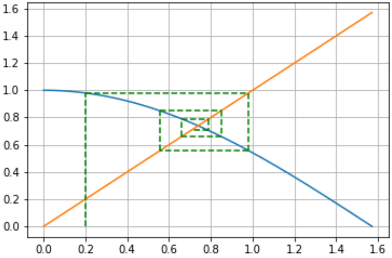

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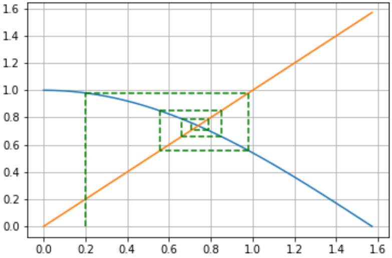

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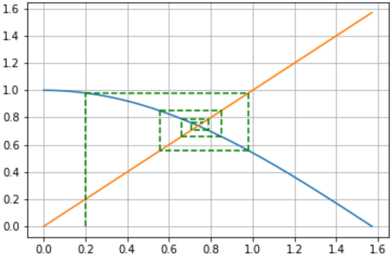

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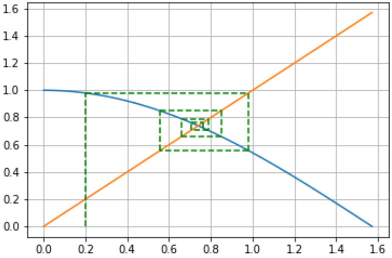

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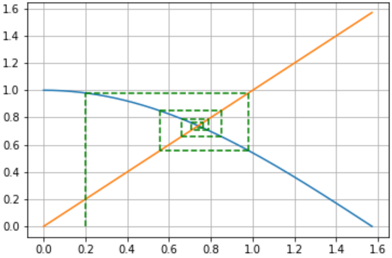

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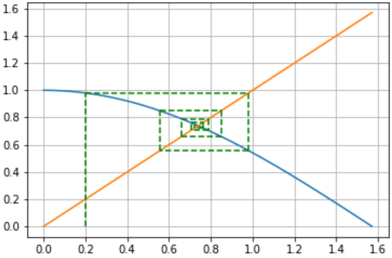

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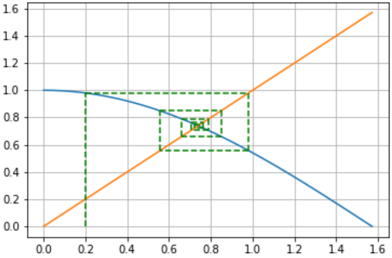

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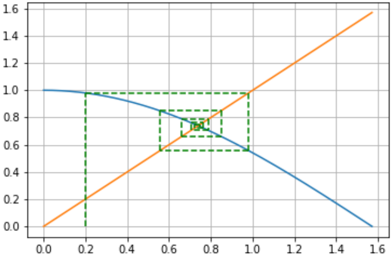

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- In the previous slide, the fixed point of our map is precisely the point which the orbits of the map converged to

Contractions

Definition of Contraction

A function $f : [a, b] \to [a, b]$ where $[a, b] \subset \mathbb{R}$ is a contraction if $|f(x) - f(y)| \leq K|x - y|$ for all points x, y in $[a, b]$ and for some $0 < K < 1$

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- In fact for contractions, the distance between $f^n(x)$ and $f^n(y)$ shrinks on the order of K^n
- In other words, $\lim_{n\to\infty} f^n(x)$ converges to a single point for every x in A

Banach Fixed Point Theorem

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- Well, what happens if we apply f to x^*?
- $f(x^*) = f(\lim_{n\to\infty} f^n(x)) = \lim_{n\to\infty} f(f^n(x)) = \lim_{n\to\infty} f^{n+1}(x) = x^*$
- That means x^* is a fixed point of f. This is the essence of the Banach Fixed Point Theorem.

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- Rearranging this gives $\frac{|f(x)-f(y)|}{|x-y|} \leq K < 1$.
- If we take the limit as y approaches x(supposing it exists), this just says that $|f'(x)| < 1$
- For continuously differentiable maps, if we have that the magnitude of the derivative of the map is less than one, we get that the map is a contraction and has a fixed point.

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- If x is a fixed point of f such that $|f'(x)| < 1$ then there is an interval around x on which f is a contraction
- we call this kind of point an attracting fixed point
- On the other hand, if $|f'(x)| > 1$, then near, x, f pushes points away from each other
- In this case, we call x a repelling fixed point

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- This equation has one solution at approximately $x = 0.7390851332$
- $|f'(0.7390851332)| = .6736... < 1$, so we have an attracting fixed point
- This(plus a few other reasons) is why repeatedly applying the cosine function to any number converges to one number

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- The Banach fixed point theorem can be used to prove may important results such as the existence and uniqueness of solutions to differential equations, the existence of steady state vectors in Markov chains and many other key results
- Iterated maps are also the basis for some more applied things such as the Newton-Raphson method for finding roots, the Lorenz map used to understand dynamical systems and many more numerical approximation algorithms

Thanks For Listening!