ML HW2

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1 KMeans Clustering and DBScan

1.1 Programming K-Means

2 EM Algorithm

2.1 Write the marginal probability of x, (p(x))

Sum Rule:

$$\text{p(x)} = \text{P}(\text{x}_n|z=0) * P(z=0) + P(x_n|z=1) * P(z=1) =$$

Given: $\text{P(z=0)} = \alpha, P(z=1) = 1-\alpha$

$$\text{P}(\text{x}_n|z=0) = N(6v+5\omega^2)$$
$$\text{P}(\text{x}_n|z=1) = N(3v+7\omega^2)$$

Therefore,

$$\text{p(x)} = \alpha(N(6v+5\omega^2)) + (1-\alpha)(N(3v+7\omega^2))$$

2.2 E-Step: Compute the posterior probabilities, $p(z_0|x), p(z_1|x)$

Bayes Rule:

$$\mathrm{P}(\mathrm{z}_0|x) = \frac{P(x|z_0)*P(z_0)}{P(x)} = T_0$$

$$\mathrm{P}(\mathrm{z}_1|x) = \frac{P(x|z_1)*P(z_1)}{P(x)} = T_1$$

$$\mathrm{T}_0 = \frac{\alpha*(N(6v+5\omega^2))}{\alpha(N(6v+5\omega^2))+(1-\alpha)(N(3v+7\omega^2))}$$

$$\mathrm{T}_1 = \frac{(1-\alpha)(N(3v+7\omega^2))}{\alpha(N(6v+5\omega^2))+(1-\alpha)(N(3v+7\omega^2))}$$

2.3 M-Step: Compute the updated value of ω^2

$$l(\theta|x) = \textstyle\sum^N \sum^Z p(z_k|x_n, \theta_{old}) ln[p(x_n, z_k|\theta)]$$

$$l(v,\ \omega^2, \alpha|x) = \textstyle\sum^N \sum^Z p(z_k|x_n, \theta_{old}) ln[p(x_n, z_k|\mu_k, \sigma_k^2, \alpha)]$$

$$=$$

$$\sum^N \sum^Z p(z_k|x_n, \theta_{old}) ln[p(z_k|\alpha) p(x_n|z_k, v, \omega^2]$$

$$\mathrm{p}(\mathrm{z}_0|\alpha) = \alpha, p(z_1|\alpha) = 1-\alpha$$

$$\mathrm{T}_0 = p(z_0|x_n, \theta_{old}) = \frac{\alpha*(N(6v+5\omega^2))}{\alpha(N(6v+5\omega^2))+(1-\alpha)(N(3v+7\omega^2))}$$

$$\mathrm{T}_1 = p(z_1|x_n, \theta_{old}) = \frac{(1-\alpha)(N(3v+7\omega^2))}{\alpha(N(6v+5\omega^2))+(1-\alpha)(N(3v+7\omega^2))}$$

p($x_n|z_0,\theta) = N(x_n|6v{+}5\omega^2), p(x_n|z_1,\theta) = N(x_n|3v{+}7\omega^2)$

$= \Sigma^N (T_0 * ln(\alpha * N(x_n|6v + 5\omega^2)) + T_1 * ln((1-\alpha) * N(x_n|3v + 7\omega^2)))$

first part of summation = $T_0 * ln(\alpha * N(x_n|6v + 5\omega^2))$

second part of summation = $T_1 * ln((1-\alpha) * N(x_n|3v + 7\omega^2))$

LN properties:
$\ln(a^b) = b * ln(a)$
ln(ab) = ln(a) + ln(b)
$\ln(e^x) = x$
ln(a/b) = ln(a) - ln(b)

simplification of first part of summation:

$$T_0 * ln(\alpha * (\frac{1}{\sqrt{2*\pi*\sigma^2}} * e^{\frac{(x_n-\mu)^2}{-2*\sigma^2}})$$

$$T_0 * ln(\alpha * (\frac{1}{\sqrt{2*\pi*5\omega^2}} * e^{\frac{(x_n-6v)^2}{-2*5\omega^2}})$$

$$T_0 * (ln(a) + ln(\frac{1}{\sqrt{2*\pi*5\omega^2}}) + ln(e^{\frac{(x_n-6v)^2}{-2*5\omega^2}}))$$

$$T_0 * (ln(a) + ln(\frac{1}{\sqrt{10\pi\omega^2}}) + \frac{(x_n-6v)^2}{-10\omega^2})$$

$$T_0 * (ln(a) + (-0.5)ln(10\pi\omega^2) - \frac{(x_n-6v)^2}{10\omega^2})$$

df/d$\omega^2 \rightarrow$ **differentiate w/ respect to** ω^2

$$\text{df/d}\omega^2(ln(\alpha)) = 0$$

$$\text{df/d}\omega^2((-0.5)ln(10\pi\omega^2)) = 0.5 * 10\pi * \frac{1}{10\pi\omega^2}$$

$$\text{df/d}\omega^2(\frac{(x_n-6v)^2}{-10\omega^2}) = \frac{(x_n-6v)^2}{10\omega^4}$$

$$\text{T}_0 * (-0.5 * 10\pi * \frac{1}{10\pi\omega^2} + \frac{(x_n-6v)^2}{10\omega^4})$$

$$\text{T}_0 * (\frac{-1}{2\omega^2} + \frac{(x_n-6v)^2}{10\omega^4})$$

simplification of second part of summation:

$$\text{T}_1 * ln((1-\alpha) * (\frac{1}{\sqrt{2*\pi*\sigma^2}} * e^{\frac{(x_n-\mu)^2}{-2*\sigma^2}})$$

$$\text{T}_1 * ln((1-\alpha) * (\frac{1}{\sqrt{2\pi*7\omega^2}} * e^{\frac{(x_n-3v)^2}{-2*7\omega^2}})$$

$$\text{T}_1 * (ln(1-a) + ln(\frac{1}{\sqrt{2\pi*7\omega^2}}) + ln(e^{\frac{(x_n-3v)^2}{-2*7\omega^2}}))$$

$$\text{T}_1 * (ln(1-a) + ln(\frac{1}{\sqrt{14\pi\omega^2}}) + \frac{(x_n-3v)^2}{-14\omega^2})$$

$$\text{T}_1 * (ln(1-a) + (-0.5)ln(14\pi\omega^2) - \frac{(x_n-3v)^2}{14\omega^2})$$

df/d$\omega^2 \rightarrow$ **differentiate w/ respect to** ω^2

$$\text{df/d}\omega^2(ln(1-\alpha)) = 0$$

$$\text{df/d}\omega^2((-0.5)ln(14\pi\omega^2)) = 0.5 * 14\pi * \frac{1}{14\pi\omega^2}$$

$$\text{df/d}\omega^2(\frac{(x_n-6v)^2}{-14\omega^2}) = \frac{(x_n-3v)^2}{14\omega^4}$$

$$\text{T}_1 * (-0.5 * 14\pi * \frac{1}{14\pi\omega^2} + \frac{(x_n-3v)^2}{14\omega^4})$$

$$\mathrm{T}_1 * (\frac{-1}{2\omega^2} + \frac{(x_n-3v)^2}{14\omega^4})$$

$$\partial l(\partial|x)/\partial\omega^2 =$$
$$\Sigma^N[T_0(\frac{-1}{2\omega^2} + \frac{(x_n-6v)^2}{10\omega^4}) + T_1 * (\frac{-1}{2\omega^2} + \frac{(x_n-3v)^2}{14\omega^4})] = 0$$

Solve For ω^2 :

$$\tau_0 = \alpha * N(6v + 5\omega^2),$$
$$\tau_1 = (1-\alpha) * N(3v + 7\omega^2)$$

$$\partial l(\partial|x)/\partial\omega^2 =$$

$$\Sigma^N[T_0(\frac{-1}{2\omega^2} + \frac{(x_n-6v)^2}{10\omega^4}) + T_1 * (\frac{-1}{2\omega^2} + \frac{(x_n-3v)^2}{14\omega^4})] = 0$$

$$\Sigma_{n=1}^N c = c + c + c... = cN$$

$$\Sigma^N(T_0(\frac{-1}{2\omega^2}) + T_1(\frac{-1}{2\omega^2})) + \Sigma^N(T_0(\frac{(x_n-6v)^2}{10\omega^4}) + T_1(\frac{(x_n-3v)^2}{14\omega^4})) = 0$$

$$\mathrm{N}^*\mathrm{T}_0 * \frac{-1}{2\omega^2} + N * T_1 * \frac{-1}{2\omega^2} + \Sigma^N(T_0(\frac{(x_n-6v)^2}{10\omega^4}) + T_1(\frac{(x_n-3v)^2}{14\omega^4}))$$

$$\mathrm{NT}_0 * \frac{-1}{2\omega^2} + NT_1 * \frac{-1}{2\omega^2} + \frac{T_0}{10\omega^4}\Sigma^N(x_n - 6v)^2 +$$
$$\frac{T_1}{14\omega^4}\Sigma^N(x_n - 3v)^2 = 0$$

multiply by ω^4

$$\text{-}(\omega^2)/2*(T_0+T_1) + \frac{T_0}{10}\Sigma^N(x_n-6v)^2 + \frac{T_1}{14}\Sigma^N(x_n-3v)^2 = 0$$

$$(\text{-}\omega^2)/2 * N * (T_0 + T_1) =$$
$$-(\frac{T_0}{10}\Sigma^N(x_n - 6v)^2 + \frac{T_1}{14}\Sigma^N(x_n - 3v)^2)$$

$$\omega^2 = \frac{1}{5}\Sigma^N(x_n - 6v)^2 + \frac{1}{7}\Sigma^N(x_n - 3v)^2)$$

$$\omega^2 = \frac{\tau_0}{5N} \Sigma^N (x_n - 6v)^2 + \frac{\tau_1}{7N} \Sigma^N (x_n - 3v)^2)$$

$$\omega = \sqrt{\frac{\tau_0}{5N} \Sigma^N (x_n - 6v)^2 + \frac{\tau_1}{7N} \Sigma^N (x_n - 3v)^2)}$$