Review for Exam 1

1. Let p, q and r be the propositions

   p: Jim always studies hard.
   q: Jim doesn't get 100 on the final exam.
   r: Jim receives an A+ in this course.

   Express the following propositions in English.
   (a) $p \to r$ (b) $p \leftrightarrow (\neg q)$ (c) $(p \vee \neg q) \to r$

   (a) If Jim always studies hard, then he'll receive an A+ in this course.

   (b) Jim always studies hard, if and only if he get 100 on the final exam.

   (c) If Jim either always studies hard or get 100 on the final exam, then he'll receive an A+ in this course.

2. Determine the truth values of the following statements.
   (a) If 2 is odd or 3 is odd, then the sum of 2 and 3 is even.
   (b) If 5 is larger than 10 and 10 is less than 20, then 5 is larger than 20.
   (c) If 12 is a multiple of 3, then either 12 is odd or 3 is odd.
   (d) The product of 4 and 5 is 20 if and only if both 4 and 5 are even.

   (a) F, since $(\mathrm{F} \vee \mathrm{T}) \to \mathrm{F} \equiv \mathrm{T} \to \mathrm{F} \equiv \mathrm{F}$

   (b) T, since $\mathrm{F} \vee \mathrm{T} \to \mathrm{F} \equiv \mathrm{F} \to \mathrm{F} \equiv \mathrm{T}$

   (c) T, since $\mathrm{T} \to \mathrm{F} \vee \mathrm{T} \equiv \mathrm{T} \to \mathrm{T} \equiv \mathrm{T}$

   (d) F, since $\mathrm{T} \leftrightarrow \mathrm{T} \wedge \mathrm{F} \equiv \mathrm{T} \leftrightarrow \mathrm{F} \equiv \mathrm{F}$

3. Show that the following statements are tautology using truth tables.
   (a) $p \to (p \vee \neg q)$
   (b) $((p \vee q) \wedge (\neg p)) \to q$

(a)

p	q	$\neg q$	$p \vee \neg q$	$p \to (p \vee \neg q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

(b)

p	q	$\neg p$	$p \vee q$	$(p \vee q) \wedge (\neg p)$	$(p \vee q) \wedge (\neg p) \to q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

4. Show that the following statements are logically equivalent using truth tables.
   (a) $p \wedge (p \vee q) \equiv p$
   (b) $(p \wedge q) \to r \equiv (p \to q) \to (p \to r)$

(a)

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

(b)

p	q	r	$p \wedge q$	$(p \wedge q) \to r$	$p \to q$	$p \to r$	$(p \to q) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F
T	F	T	F	T	F	T	T
F	T	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	F	T	F	T	T	T	T
F	T	F	F	T	T	T	T
F	F	F	F	T	T	T	T

5. Re-do #3 and #4 using equivalent formulas.

(a) $p \to (p \vee \neg q) \equiv \neg p \vee (p \vee \neg q) \equiv (\neg p \vee p) \vee \neg q \equiv \mathrm{T} \vee \neg q \equiv \mathrm{T}$

(b) $$((p \vee q) \wedge (\neg p)) \to q \equiv \neg\,((p \vee q) \wedge (\neg p)) \vee q \equiv \neg(p \vee q) \vee \neg(\neg p) \vee q$$
$$\equiv \neg(p \vee q) \vee p \vee q \equiv \neg(p \vee q) \vee (p \vee q) \equiv \mathrm{T}$$

(c) $p \wedge (p \vee q) \equiv (p \vee \mathrm{F}\,) \wedge (p \vee q) \equiv p \vee (\,\mathrm{F} \wedge q) \equiv p \vee \mathrm{F} \equiv p$

(d) $(p \wedge q) \to r \equiv \neg(p \wedge q) \vee r$
$$(p \to q) \to (p \to r) \equiv \neg(p \to q) \vee (p \to r) \equiv \neg(\neg p \vee q) \vee (\neg p \vee r)$$
$$\equiv \neg(\neg p \vee q) \vee (\neg p \vee r) \equiv (p \wedge \neg q) \vee \neg p \vee r$$
$$\equiv [\,(p \wedge \neg q) \vee \neg p\,] \vee r \equiv [\,(p \vee \neg p) \wedge (\neg q \vee \neg p)\,] \vee r$$
$$\equiv [\,\mathrm{T} \wedge (\neg q \vee \neg p)\,] \vee r \equiv (\neg q \vee \neg p) \vee r \equiv \neg(q \wedge p) \vee r$$

6. For each of the following statements, determine its truth value and write a proof for your conclusion. Assume the domain is the set of all real numbers.

(a) $\exists x(x^2 = 2x - 1)$: T
Proof: Let $x = 1$, then $1^2 = 2 \cdot 1 - 1$

(b) $\forall x((x^2 > 1) \to (x > 1))$: F
Proof of Negation: $\exists x((x^2 > 1) \wedge (x \not> 1))$
Let $x = -2$, then $(-2)^2 > 1 \wedge (-2) \not> 1$

(c) $\forall x \forall y(x^2 + y^2 \geq 2xy)$: T
Proof: $\forall x \forall y, (x - y)^2 \geq 0 \Rightarrow x^2 - 2xy + y^2 \geq 0 \Rightarrow x^2 + y^2 \geq 2xy$

(d) $\forall x \exists y(2x + y > 0)$: T
Proof: There is a $y = -2x + 1$, such that $2x - 2x + 1 > 0$

(e) $\exists x \forall y(x + y^2 > 0)$: T
Proof: Let $x = 1$, since $y^2 \geq 0$, then $1 + y^2 > 0$

(f) $\forall x \exists y \left(\dfrac{y}{x+2} = 1\right)$: F

Proof of negation: $\exists x \forall y \left(\dfrac{y}{x+2} \neq 1\right)$

Let $x = -2$, such that $\dfrac{y}{-2+2}$ is undefined

(g) $\forall x \forall y \left((x > y > 0) \to (\frac{1}{x} < \frac{1}{y})\right)$: T

Proof: $x > y > 0 \Rightarrow x \cdot y > 0 \Rightarrow \dfrac{x}{xy} > \dfrac{y}{xy} \Rightarrow \dfrac{1}{x} < \dfrac{1}{y}$

(h) $\forall x \exists y \forall z (x + y = z)$: F

Proof of negation: $\exists x \forall y \exists z (x + y \neq z)$

Let $x = 1$ and $z = y$, then $1 + y \neq y$

(i) $\forall x \forall y \exists z (2x + z = 3y + 2z)$: T

Proof: There is a $z = 2x - 3y$

such that $2x + 2x - 3y = 3y + 2(2x - 3y) \Rightarrow 4x - 3y = 4x - 3y$

7. (a) Show that $\frac{1}{2}$ is not an integer.

(b) Let x and y be integers. Show that $2xy + 3y^2$ is even if and only if y is even.

(c) Show that if x is irrational, then $4x - 1$ is irrational.

(a) Proof by contradition:

Assume $\frac{1}{2}$ is an integer, then $\frac{1}{2} = x$, $x \in \mathbb{Z}$

$\frac{1}{2} = x \Rightarrow 1 = 2x \Rightarrow 1$ is even, contradict to 1 is odd

Hence, $\frac{1}{2}$ is not an integer.

(b) $p \to q$:

Assume $2xy + 3y^2$ is even, then $2xy$ and $3y^2$ must be either both odd or both even;

Since $2(xy)$ is even, then $3y^2$ must be even too;

Since $3y^2$ is even, then at least one of 3 and y^2 is even;

Since 3 is odd, then y^2 is even;

Hence, y must be even.

$q \to p$:

Assume y is even, then $y = 2n$, $n \in \mathbb{Z}$

$2xy + 3y^2 = 2x(2n) + 3(2n)^2 = 4xn + 12n^2 = 2(2xn + 6n^2)$

Since $x, n \in \mathbb{Z}$, then $2xn + 6n^2 \in \mathbb{Z}$

Hence, $2xy + 3y^2$ is even.

(c) Proof by contraposition:

Assume $4x - 1$ is rational, then $4x - 1 = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$

$$4x - 1 = \frac{a}{b} \Rightarrow 4x = \frac{a}{b} + \frac{b}{b} = \frac{a+b}{b} \Rightarrow \frac{1}{4}(4x) = \frac{1}{4} \cdot \frac{a+b}{b} \Rightarrow x = \frac{a+b}{4b}$$

Since $a, b \in \mathbb{Z}$, then $a+b \in \mathbb{Z}$ and $4b \in \mathbb{Z}$
Since $b \neq 0$, then $4b \neq 0$
Hence, x is rational.

8. Prove or disprove the following statements

(a) if $4x^2+1$ is odd, then $3x+1$ is odd.
F, Let $x=1$, then $4x^2+1=5$, which is odd, but $3x+1=4$, which is not odd

(b) if $2x+3y-1$ is irrational, then either x is irrational or y is irrational.
T, Proof by contraposition:
Assume both x and y is rational, then $2x+3y-1$ is rational

$$x=\frac{a}{b}, y=\frac{c}{d}, a, b, c, d \in \mathbb{Z}, b, d \neq 0$$

$$2 \cdot \frac{a}{b}+3 \cdot \frac{c}{d}-1=\frac{2ad+3bc-bd}{bd}$$

Since $a, b, c, d \in \mathbb{Z}$, then $2ad+3bc-bd \in \mathbb{Z}$ and $bd \in \mathbb{Z}$
Since $b, d \neq 0$, then $bd \neq 0$ Hence, $2x+3y-1$ is rational

(c) If both x and y are irrational, then $2x^2+y$ is irrational.
F, Let $x=\sqrt[4]{2}$ and $y=-2\sqrt{2}$, which are irrational, then $2x^2+y=0$, which is rational.

9. Show that the following statements are equivalent.
(i) $3x^2+7$ is even
(ii) $x+6$ is odd
(iii) x^3+4x+3 is even

(a) (i) $\rightarrow$ (ii):
$3x^2+7$ is even, then $3x^2$ and 7 must be either both odd or both even;
since 7 is odd, then $3x^2$ must be odd;
since $3x^2$ is odd, then both 3 and x^2 must be odd, then x must be odd.
Since x is odd and 6 is even, $x+6$ must be odd.

(b) (ii) $\rightarrow$ (iii):
$x+6$ is odd, then exactly one of x and 6 is odd, and the other one is even;
since 6 is even, then x must be odd;
since x is odd, then x^2 is odd, then x^3 is odd;
since 4 is even, $4x$ must be even;
since x^3 is odd, $4x$ is even, then x^3+4x must be odd;
since 3 is odd, x^3+4x is odd, then x^3+4x+3 is even.

(c) (iii) $\rightarrow$ (i):
x^3+4x+3 is even, then x^3+4x and 3 must be either both odd or both even;
since 3 is odd, then x^3+4x must be odd;
since $x^3+4x=x(x^2+4)$ is odd, then both x and x^2+4 are odd.
Since x is odd, then x^2 is odd;
since 3 is odd, then $3x^2$ must be odd;
Since 7 is odd, then $3x^2+7$ must be even.