Math120 Spring 2016
Final Review

Solution by Yajie Zhang

Study Notes:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
F	T	T	F	T	T	F
T	F		F	T	F	F
F	F		F	F	T	T

1. Determine the truth values for each of the following statements.

   (a) The sum of 4 and 6 is prime if and only if either 4 or 6 is prime.
   T, since F $\leftrightarrow$ (F $\vee$ F) $\equiv$ F $\leftrightarrow$ F $\equiv$ T

   (b) If 3 divides 20, then 6 divides 20.
   T, since F $\to$ F $\equiv$ T

2. Show that the following statements are tautology using truth tables.

   (a) $p \wedge (p \to q) \to q$

p	q	$p \to q$	$p \wedge (p \to q)$	$p \wedge (p \to q) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

   (b) $((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$

p	q	r	$\neg p$	$p \vee q$	$\neg p \vee r$	$(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$
T	T	T	F	T	T	T	T	T
T	T	F	F	T	F	F	T	T
T	F	T	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	F	T	F	F	F	T
F	T	F	T	T	T	T	T	T
F	F	T	T	F	T	F	T	T
F	F	F	T	F	F	F	F	T

3. Show that the following statements are logically equivalent using truth tables.

(a) $p \vee (p \wedge q) \equiv p$

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

(b) $q \to (p \vee r) \equiv \neg p \to (q \to r)$

p	q	r	$p \vee r$	$q \to (p \vee r)$	$\neg p$	$q \to r$	$\neg p \to (q \to r)$
T	T	T	T	T	F	T	T
T	T	F	T	T	F	F	T
T	F	T	T	T	F	T	T
F	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T
F	T	F	F	F	T	F	F
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

Study Notes:

$p \wedge \text{T} \equiv p \vee \text{F} \equiv p \wedge p \equiv p \vee p \equiv \neg(\neg p) \equiv p$

$p \vee \text{T} \equiv p \vee \neg p \equiv \text{T}$

Commutative Law $\begin{cases} p \wedge q \equiv q \wedge p \\ p \vee q \equiv q \vee p \end{cases}$

$p \wedge \text{F} \equiv p \wedge \neg p \equiv \text{F}$

Associative Law $\begin{cases} (p \wedge q) \wedge r \equiv q \wedge (p \wedge r) \\ (p \vee q) \vee r \equiv q \vee (p \vee r) \end{cases}$

$p \to q \equiv \neg p \vee q$

Distributive Law $\begin{cases} p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \end{cases}$

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

De Morgan's Law $\begin{cases} \neg(p \wedge q) \equiv \neg p \vee \neg q \\ \neg(p \vee q) \equiv \neg p \wedge \neg q \end{cases}$

4. Re-do #2 and #3 using equivalent formulas.

(a) $(p \wedge (p \to q)) \to q$

$$\begin{aligned} &\equiv \neg(p \wedge (p \to q)) \vee q \\ &\equiv \neg p \vee \neg(p \to q) \vee q \\ &\equiv (\neg p \vee q) \vee \neg(p \to q) \\ &\equiv (p \to q) \vee \neg(p \to q) \\ &\equiv \text{T} \end{aligned}$$

(b) $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

$$\begin{aligned}
&\equiv \neg((p \vee q) \wedge (\neg p \vee r)) \vee (q \vee r) \\
&\equiv \neg(p \vee q) \vee \neg(\neg p \vee r) \vee q \vee r \\
&\equiv ((\neg p \wedge \neg q) \vee q) \vee ((p \wedge \neg r) \vee r) \\
&\equiv ((\neg p \vee q) \wedge (\neg q \vee q)) \vee ((p \vee r) \wedge (\neg r \vee r)) \\
&\equiv ((\neg p \vee q) \wedge \mathrm{T}) \vee ((p \vee r) \wedge \mathrm{T}) \\
&\equiv \neg p \vee q \vee p \vee r \\
&\equiv (\neg p \vee p) \vee (q \vee r) \\
&\equiv \mathrm{T} \vee (q \vee r) \\
&\equiv \mathrm{T}
\end{aligned}$$

(c) $p \vee (p \wedge q) \equiv p$
$(p \wedge \mathrm{T}) \vee (p \wedge q) \equiv p \wedge (\mathrm{T} \vee q) \equiv p \wedge \mathrm{T} \equiv p$

(d) $q \rightarrow (p \vee r) \equiv \neg p \rightarrow (q \rightarrow r)$

$$\begin{aligned}
LHS &\equiv \neg q \vee p \vee r \\
RHS &\equiv \neg(\neg p) \vee (\neg q \vee r) \\
&\equiv p \vee \neg q \vee r \\
&\equiv \neg q \vee p \vee r
\end{aligned}$$

5. For each of the following statements, determine its truth value and write a proof to support your conclusion. Assume the domain is the set of all real numbers.

(a) $\forall x \exists y\, (\sqrt{x+y} = x)$ F
Proof by negation: $\exists x \forall y\, (\sqrt{x+y} \neq x)$
Let $x = -1$, then $\forall y \sqrt{-1+y} \geq 0 > -1 \neq -1$

(b) $\exists x \forall y (x^2 + 2y^2 \geq 10)$ T
Proof: $\exists x = \sqrt{10}, \forall y (y^2 \geq 0) \Rightarrow 2y^2 \geq 0 \Rightarrow (\sqrt{10})^2 + 2y^2 \geq 10$

(c) $\exists x \exists y ((x < y) \rightarrow (x^3 > y^3))$ T
Proof: Let $x = 1$, $y = 1$, then $\mathrm{F} \rightarrow \mathrm{F} \equiv \mathrm{T}$

(d) $\forall x \forall y ((x^y = y^x) \rightarrow (x = y))$ F
Proof by negation: $\exists x \exists y (x^y = y^x \wedge x \neq y)$, let $x = 4, y = 2$, then $4^2 = 2^4 \wedge 4 \neq 2$

(e) $\exists x \forall y \exists z \left(\dfrac{1}{x+y+1} = \dfrac{1}{y+z+2} \right)$ F

Proof by negation:$\forall x \exists y \forall z (\dfrac{1}{x+y+1} \neq \dfrac{1}{y+z+2}$

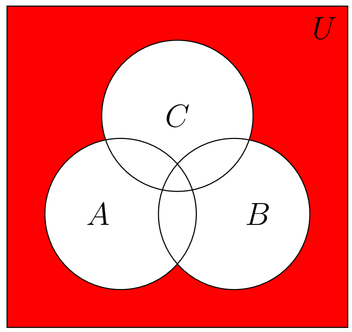
Let $y = -x-1$, $\forall x \forall z : \dfrac{1}{x-x-1+1} = \dfrac{1}{0}$ =undefined$\neq \dfrac{1}{y+z+2}$

(f) $\forall x \exists y \exists z \left(\sqrt{x-y}+x=\frac{1}{z}\right)$ T

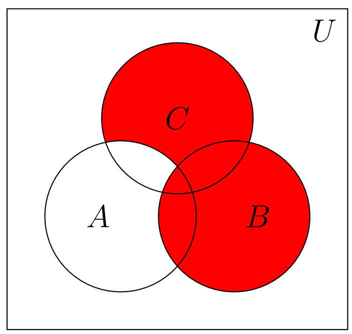
Proof: $\forall x \begin{cases} x\neq 0, \text{ let } y=x,\ z=\frac{1}{x} \Rightarrow \sqrt{x-x}+x=\frac{1}{\frac{1}{x}}=x \\ x=0, \text{ let } y=-1,\ z=1 \Rightarrow \sqrt{0-(-1)}+0=\frac{1}{1}=1 \end{cases}$

6. Let A, B and C be sets. For each of the following sets, draw a Venn diagram to represent the set.

(a) $\overline{A} \cap (\overline{B}-C)$



(b) $\overline{A-B} \cap (B \cup C)$



7. Suppose the universal set $\mathsf{U}=[0,9]$. Let $A=(0,4)$, $B=[3,9]$, and $C=\{1,6\}$. Find each of the following sets.

(a) $\overline{A} \cup B = \{0\} \cup [3,9]$
$\overline{A}=\{0\}\cup[4,9]$

(b) $A \cap (\overline{B}-C) = (0,1)\cup(1,3)$
$\overline{B}=[0,3) \Rightarrow \overline{B}-C=[0,1)\cup(1,3)$

(c) $(A\cup B)-C = (0,1)\cup(1,6)\cup(6,9]$
$A\cup B=(0,9]$

8. (a) Let function $f:[-4,\infty)\to R$ be defined as $f(x)=\sqrt{x+4}$. Find each of the following sets.

i. $f(\{0,5\})=\{2,3\}$

ii. $f((0,5))=(2,3)$

iii. $f([-4,0])=[0,2]$

iv. $f^{-1}([3,5])=[5,21]$
$\sqrt{x+4}=3\Rightarrow x+4=9\Rightarrow x=5$
$\sqrt{x+4}=5\Rightarrow x+4=25\Rightarrow x=21$

v. $f^{-1}((-4,4)) = [-4,12)$
$\sqrt{x+4} = 0 \Rightarrow x+4=0 \Rightarrow x=-4$
$\sqrt{x+4} = 4 \Rightarrow x+4=16 \Rightarrow x=12$

vi. $f^{-1}(\{-2,-1,0,1,2\}) = \{-4,-3,0\}$
$\sqrt{x+4} = 0 \Rightarrow x+4=0 \Rightarrow x=-4$
$\sqrt{x+4} = 1 \Rightarrow x+4=1 \Rightarrow x=-3$
$\sqrt{x+4} = 2 \Rightarrow x+4=4 \Rightarrow x=-0$

(b) Let function $f:\mathbb{R}\to\mathbb{R}$ be defined as $f(x) = \left\lfloor \frac{3x+1}{2} \right\rfloor$. Find each of the following sets.

i. $f((2,9)) = \{3,4,5,6,7,8,9,10,11,12,13\}$

ii. $f([2,9]) = \{3,4,5,6,7,8,9,10,11,12,13,14\}$

iii. $f(\{0,2,4,6,8\}) = \{0,3,6,9,24\}$

iv. $f^{-1}([3,5]) = \left[\frac{5}{3}, \frac{11}{3}\right)$
$3 \le \frac{3x+1}{2} < 6 \Rightarrow 6 \le 3x+1 < 12 \Rightarrow \frac{5}{3} \le x < \frac{11}{3}$

v. $f^{-1}((2,3)) = \varnothing$

vi. $f^{-1}((1.5,3.2)) = \left[1, \frac{7}{3}\right)$
$2 \le \frac{3x+1}{2} < 4 \Rightarrow 4 \le 3x+1 < 8 \Rightarrow 1 \le x < \frac{7}{3}$

Study Notes:

$$f: A \to B$$

1-1: $\forall a_1, a_2 \in A \quad (\, f(a_1) = f(a_2) \to a_1 = a_2 \,)$
Not 1-1: $\exists a_1, a_2 \in A \quad (\, f(a_1) = f(a_2)$ but $a_1 \neq a_2 \,)$

Onto: $\forall b \in B, \exists a \in A \quad (\, f(a) = b \,)$
Not Onto: $\exists b \in B, \forall a \in A \quad (\, f(a) \neq b \,)$

9. (a) For each of the following function $f: D \to \mathbb{R}$, where $D = \{x \in \mathbb{R} \mid f(x) \text{is defined}\}$, determine whether it's one-to-one and write a proof to support your conclusion.

i. $f(x) = \sqrt{1-2x}$ 1-1
Proof: $\forall a_1, a_2 \in D$, assume $f(a_1) = f(a_2)$
$\Rightarrow \sqrt{1-2a_1} = \sqrt{1-2a_2}$
$\Rightarrow 1-2a_1 = 1-2a_2$
$\Rightarrow -2a_1 = -2a_2$
$\Rightarrow a_1 = a_2$
Therefore, $f(x) = \sqrt{1-2x}$ is 1-1

ii. $f(x) = x^3 - x + 1$ Not 1-1

Proof: Let $a_1 = 0,\ a_2 = 1$

Then $\begin{cases} f(a_1) = 0^3 - 0 + 1 = 1 \\ f(a_2) = 1^3 - 1 + 1 = 1 \end{cases}$

Hence, $f(a_1) = f(a_2)$, but $a_1 \neq a_2$

Therefore, $f(x) = x^3 - x + 1$ is not 1-1

iii. $f(x) = \dfrac{x+1}{x+2}$ 1-1

Proof: $\forall a_1, a_2 \in D$, assume $f(a_1) = f(a_2)$

$\Rightarrow \dfrac{a_1+1}{a_1+2} = \dfrac{a_2+1}{a_2+2}$

$\Rightarrow (a_1+1)(a_2+2) = (a_2+1)(a_1+2)$

$\Rightarrow \cancel{a_1 a_2} + a_2 + 2a_1 + \cancel{2} = \cancel{a_1 a_2} + a_1 + 2a_2 + \cancel{2}$

$\Rightarrow 2a_1 - a_1 = 2a_2 - a_2$

$\Rightarrow a_1 = a_2$

Therefore, $f(x) = \dfrac{x+1}{x+2}$ is 1-1

(b) For each of the following function $f : D \to \mathbb{R}$, where $D = \{x \in \mathbb{R} \mid f(x) \text{ is defined}\}$, determine whether it's onto and write a proof to support your conclusion.

i. $f(x) = \sqrt[3]{4x+3}$ Onto

Proof: $\forall b \in \mathbb{R}$, let $a = \dfrac{1}{4}(b^3 - 3)$

$$\begin{aligned} \text{then } f(a) &= \sqrt[3]{4a+3} \\ &= \sqrt[3]{\cancel{4}\frac{1}{\cancel{4}}(b^3-3)+3} \\ &= \sqrt[3]{b^3 - \cancel{3} + \cancel{3}} \\ &= b \end{aligned}$$

ii. $f(x) = 4 - 3^x$ Not Onto

Proof: Let $b = 5 \in \mathbb{R}$, then $\forall a \in D$

$3^a \geq 0$

$\Rightarrow -3^a \leq 0$

$\Rightarrow 4 - 3^a \leq 4$

$\Rightarrow f(a) \leq 4$

Since $b = 5$, then $f(a) \neq b$

iii. $f(x) = \dfrac{2x-3}{x}$ Not Onto

Proof: Let $b = 2 \in \mathbb{R}$, then $\forall a \in D$

$f(a) = \dfrac{2a-3}{a} = \dfrac{2\cancel{a}}{\cancel{a}} - \dfrac{3}{a} = 2 - \dfrac{3}{a}$

Since $\dfrac{3}{a} \neq 0$, then $2 - \dfrac{3}{a} \neq 2$

Hence, $f(a) \neq b$

Study Notes:
$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

10. Evaluate each of the following sums, and write the answers without sigma notation. (No need to simplify)

(a) $\displaystyle\sum_{n=1}^{60}(2n^3 + 3n^2 - 4n)$

$$= 2\sum_{n=1}^{60} n^3 + 3\sum_{n=1}^{60} n^2 - 4\sum_{n=1}^{60} n$$
$$= 2 \cdot \left[\frac{60 \cdot 61}{2}\right]^2 + 3 \cdot \frac{60 \cdot 61 \cdot (2 \cdot 60 + 1)}{6} - 4 \cdot \frac{60 \cdot 61}{2}$$
$$= 6697800 + 221430 - 7320$$
$$= 6911910$$

(b) $\displaystyle\sum_{k=100}^{150}(5k^3 - 1)$

$$= \sum_{k=1}^{150}(5k^3 - 1) - \sum_{k=1}^{99}(5k^3 - 1)$$
$$= 5\sum_{k=1}^{150} k^3 - 150 - \left(5\sum_{k=1}^{99} k^3 - 99\right)$$
$$= 5 \cdot \left[\frac{150 \cdot 151}{2}\right]^2 - 150 - 5 \cdot \left[\frac{99 \cdot 100}{2}\right]^2 + 99$$
$$= 641278125 - 150 - 122512500 + 90$$
$$= 518765565$$

(c) $\displaystyle\sum_{i=1}^{2n}\sum_{j=1}^{n+1}(8ij-1)$

$$
\begin{aligned}
&= \sum_{i=1}^{2n}\left(8i\sum_{j=1}^{n+1} j - (n+1)\right) \\
&= \sum_{i=1}^{2n}\left(8i \cdot \frac{(n+1)(n+2)}{2} - (n+1)\right) \\
&= \sum_{i=1}^{2n}\left(4i \cdot (n+1)(n+2) - (n+1)\right) \\
&= 4(n+1)(n+2)\sum_{i=1}^{2n} i - 2n(n+1) \\
&= 4(n+1)(n+2)\frac{\cancel{2}n(2n+1)}{\cancel{2}} - 2n(n+1) \\
&= 4n(n+1)(n+2)(2n+1) - 2n(n+1)
\end{aligned}
$$

(d) $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{i}(24j^2+12i)$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\left(24\sum_{j=1}^{i} j^2 + 12i^2\right) \\
&= \sum_{i=1}^{n}\left(24\frac{i(i+1)(2i+1)}{6} + 12i^2\right) \\
&= \sum_{i=1}^{n}(4i(i+1)(2i+1) + 12i^2) \\
&= \sum_{i=1}^{n}(4i(2i^2+2i+i+1) + 12i^2) \\
&= \sum_{i=1}^{n}(8i^3+12i^2+4i+12i^2) \\
&= \sum_{i=1}^{n}(8i^3+24i^2+4i) \\
&= 8\sum_{i=1}^{n} i^3 + 24\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i \\
&= 8\left[\frac{n(n+1)}{2}\right]^2 + 24\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} \\
&= 2n^2(n+1)^2 + 4n(n+1)(2n+1) + 2n(n+1)
\end{aligned}
$$

11. Let A, B, C and D be zero-one matrices, where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find each of the following matrices.

(a) $(A \vee D) \odot B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \odot B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(b) $C \odot (A \wedge D) = C \odot \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $B \odot C \odot D = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \odot D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(d) $(A \odot B \odot C)^{[2]} = \left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \odot C \right)^{[2]} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

12. Conversion.

(a) Convert $(3AC.8)_{14}$ to base 10.

$$\begin{aligned} (3AC.8)_{14} &= 3 \times 14^2 + 10 \times 14^1 + 12 \times 14^0 + 8 \times 14^{-1} \\ &= 588 + 140 + 12 + 0.\overline{571428} \\ &= (740.\overline{571428})_{10} \end{aligned}$$

(b) Convert $(782.75)_{10}$ to base 2.

$$\begin{array}{r} 391 \\ 2\overline{\smash{)}782} \\ \underline{600} \\ 182 \\ \underline{180} \\ 2 \\ \underline{2} \\ 0 \end{array} \quad \begin{array}{r} 195 \\ 2\overline{\smash{)}391} \\ \underline{200} \\ 191 \\ \underline{180} \\ 11 \\ \underline{10} \\ 1 \end{array} \quad \begin{array}{r} 97 \\ 2\overline{\smash{)}195} \\ \underline{180} \\ 15 \\ \underline{14} \\ 1 \end{array} \quad \begin{array}{r} 48 \\ 2\overline{\smash{)}97} \\ \underline{80} \\ 17 \\ \underline{16} \\ 1 \end{array} \quad \begin{array}{r} 24 \\ 2\overline{\smash{)}48} \\ \underline{40} \\ 8 \\ \underline{8} \\ 0 \end{array} \quad \begin{array}{r} 12 \\ 2\overline{\smash{)}24} \\ \underline{20} \\ 4 \\ \underline{4} \\ 0 \end{array} \quad \begin{array}{r} 6 \\ 2\overline{\smash{)}12} \\ \underline{12} \\ 0 \end{array} \quad \begin{array}{r} 3 \\ 2\overline{\smash{)}6} \\ \underline{6} \\ 0 \end{array}$$

$$\begin{array}{r} 1 \\ 2\overline{\smash{)}3} \\ \underline{2} \\ 1 \end{array} \quad \begin{array}{r} 0 \\ 2\overline{\smash{)}1} \end{array}$$

$$\begin{array}{r} 0.7\,5 \\ \times \quad\;\; 2 \\ \hline 1.5\,0 \end{array} \qquad \begin{array}{r} 0.5 \\ \times \quad 2 \\ \hline 1.0 \end{array}$$

$(782.75)_{10} = (11\ 0000\ 1110.11)_2$

(c) Convert $(B93.A2)_{12}$ to base 6.

$$\begin{aligned}(B93.A2)_{12} &= 11 \times 12^2 + 9 \times 12^1 + 3 \times 12^0 + 10 \times 12^{-1} + 2 \times 12^{-2} \\ &= 1584 + 108 + 3 + 10/12 + 2/12^2 \\ &= 1695 + \frac{10 \cdot 12 + 2}{12^2} \\ &= 1695 + 0.847\overline{2} \\ &= (1695.847\overline{2})_{10}\end{aligned}$$

$$\begin{array}{r} 282 \\ 6\,\overline{)\,1695} \\ \underline{1200} \\ 495 \\ \underline{480} \\ 15 \\ \underline{12} \\ 3 \end{array} \qquad \begin{array}{r} 47 \\ 6\,\overline{)\,282} \\ \underline{240} \\ 42 \\ \underline{42} \\ 0 \end{array} \qquad \begin{array}{r} 7 \\ 6\,\overline{)\,47} \\ \underline{42} \\ 5 \end{array} \qquad \begin{array}{r} 1 \\ 6\,\overline{)\,7} \\ \underline{6} \\ 1 \end{array} \qquad \begin{array}{r} 0 \\ 6\,\overline{)\,1} \end{array}$$

$$0.847\overline{2} \times 6 = 5.08\overline{3} \qquad 0.08\overline{3} \times 6 = 0.5 \qquad \begin{array}{r} 0.5 \\ \times \quad 6 \\ \hline 3.0 \end{array}$$

$(B93.A2)_{12} = (11503.503)_6$

(d) Convert $(11\ 1011\ 0101.1101)_2$ to base 8.
$(11\ 1011\ 0101.1101)_2 = (1665.64)_8$

(e) Convert $(5CF.B4)_{16}$ to base 2.
$(5CF.B4)_{16} = (101\ 1100\ 1111.1011\ 01)_2$

(f) Convert $(3673.52)_{16}$ to base 8.
$(3673.52)_{16} = (11\ 0110\ 0111\ 0011.0101\ 001)_2 = (33163.244)_8$

13. Perform indicated operations.

(a) $(892.7B)_{13} + (7A.9)_{13} = (940.3B)_{13}$

$$\begin{array}{rrrrrl} & \overset{1}{8} & \overset{1}{9} & \overset{1}{2}. & 7 & B \\ + & & 7 & A. & 9 & {}_{13} \\ \hline & 9 & 4 & 0. & 3 & B \end{array} \qquad \begin{array}{r} 1 \\ 13\,\overline{)\,16} \\ \underline{13} \\ 3 \end{array} \qquad \begin{array}{r} 1 \\ 13\,\overline{)\,13} \\ \underline{13} \\ 0 \end{array} \qquad \begin{array}{r} 1 \\ 13\,\overline{)\,17} \\ \underline{13} \\ 4 \end{array}$$

(b) $(541.03)_7 - (62.542)_7 = (445.155)_7$

$$\begin{array}{cccccccc} & \overset{1}{5} & \overset{1}{4} & \overset{1}{1}. & \overset{1}{0} & \overset{1}{3} & & \\ - & & 6 & 2. & 5 & 4 & 2 & {}_7 \\ \hline & 4 & 4 & 5. & 1 & 5 & 5 & \end{array}$$

$2+7=9-4=5 \qquad 3+7=10-6=4$

(c) $(8D.A)_{15} \times (0.BC)_{15} = (70.24)_{15}$

$$\begin{array}{cccccccc} & & & \overset{10}{8} & \overset{8}{D}. & A & & \\ \times & & & \overset{10}{0}. & \overset{7}{B} & C & & {}_{15} \\ \hline & & 7 & 1 & E & 0 & & \\ + & 6 & 8 & 0 & 5 & & & \\ \hline & \overset{1}{7} & 0. & \overset{1}{2} & 4 & 0 & & \end{array}$$

$$\begin{array}{r} 8 \\ 15\overline{)\,120} \\ \underline{120} \\ 0 \end{array} \qquad \begin{array}{r} 10 \\ 15\overline{)\,164} \\ \underline{150} \\ 14 \end{array} \qquad \begin{array}{r} 7 \\ 15\overline{)\,106} \\ \underline{105} \\ 1 \end{array} \qquad \begin{array}{r} 7 \\ 15\overline{)\,110} \\ \underline{105} \\ 5 \end{array}$$

$$\begin{array}{r} 10 \\ 15\overline{)\,150} \\ \underline{150} \\ 0 \end{array} \qquad \begin{array}{r} 6 \\ 15\overline{)\,98} \\ \underline{90} \\ 8 \end{array} \qquad \begin{array}{r} 1 \\ 15\overline{)\,19} \\ \underline{15} \\ 4 \end{array} \qquad \begin{array}{r} 1 \\ 15\overline{)\,15} \\ \underline{15} \\ 0 \end{array}$$

(d) $(1\ 0110.101)_2 \times (11.0011\ 01)_2 = (100\ 1000.0111\ 1000\ 1)_2$

$$\begin{array}{cccccccccccccccccc} & & & & & & & & & 1 & 0 & 1 & 1 & 0. & 1 & 0 & 1 & \\ \times & & & & & & & & & 1 & 1. & 0 & 0 & 1 & 1 & 0 & 1 & {}_2 \\ \hline & & & & & & & & & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \\ & & & & & & & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & & \\ & & & & & & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & & & \\ & & & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & & & & & & \\ & & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & {}^1 & {}^1 & {}^1 & {}^1 & & & & \\ & & {}^1 & {}^1 & {}^2 & {}^2 & {}^2 & {}^2 & {}^2 & {}^1 & & & & & & & & \\ \hline & 1 & 0 & 0 & 1 & 0 & 0 & 0. & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & \end{array}$$

$$\begin{array}{r} 1 \\ 2\overline{)\,2} \\ \underline{2} \\ 0 \end{array} \qquad \begin{array}{r} 1 \\ 2\overline{)\,3} \\ \underline{2} \\ 1 \end{array}$$

$$\begin{array}{r} 2 \\ 2\overline{)\,5} \\ \underline{4} \\ 1 \end{array} \qquad \begin{array}{r} 2 \\ 2\overline{)\,4} \\ \underline{4} \\ 0 \end{array}$$

14. For each of the following integers a and b, use extended Euclidean algorithm to find integers s and t, such that $GCD(a,b) = s \cdot a + t \cdot b$.

(a) $a = 595, b = 48$

$$\begin{array}{|r|l} 595 & \\ 48 & \\ 19 & = 595 - 12(48) \\ 10 & = 48 - 2(19) \\ 9 & = 19 - 10 \\ 1 & = 10 - 9 \end{array}$$

$$\begin{array}{r} 12 \\ 48\overline{)\,595} \\ \underline{480} \\ 115 \\ \underline{96} \\ 19 \end{array} \qquad \begin{array}{r} 2 \\ 19\overline{)\,48} \\ \underline{38} \\ 10 \end{array} \qquad \begin{array}{r} 1 \\ 10\overline{)\,19} \\ \underline{10} \\ 9 \end{array} \qquad \begin{array}{r} 1 \\ 9\overline{)\,10} \\ \underline{9} \\ 1 \end{array}$$

$$
\begin{aligned}
1 &= 10 - 9 \\
&= 10 - 19 + 10 \\
&= 2(10) - 19 \\
&= 2(48 - 2(19)) - 19 \\
&= 2(48) - 5(19) \\
&= 2(48) - 5(595 - 12(48)) \\
&= 62(48) - 5(595)
\end{aligned}
$$

$GCD(595, 48) = -5 \cdot 595 + 62 \cdot 48$

(b) $a = 50209, b = 1426$

$$
\begin{array}{|r|l}
50209 & \\
1426 & \\
299 & = 50209 - 35(1426) \\
230 & = 1426 - 4(299) \\
69 & = 299 - 230 \\
23 & = 230 - 3(69)
\end{array}
$$

$$
\begin{array}{r}
35 \\
1426\,\overline{)\,50209} \\
\underline{42780} \\
7429 \\
\underline{7130} \\
299
\end{array}
\qquad
\begin{array}{r}
4 \\
299\,\overline{)\,1426} \\
\underline{1196} \\
230
\end{array}
\qquad
\begin{array}{r}
1 \\
230\,\overline{)\,299} \\
\underline{230} \\
69
\end{array}
\qquad
\begin{array}{r}
3 \\
69\,\overline{)\,230} \\
\underline{207} \\
23
\end{array}
$$

$$
\begin{aligned}
23 &= 230 - 3(69) \\
&= 230 - 3(299 - 230) \\
&= 4(230) - 3(299) \\
&= 4(1426 - 4(299)) - 3(299) \\
&= 4(1426) - 19(299) \\
&= 4(1426) - 19(50209 - 35(1426)) \\
&= 669(1426) - 19(50209)
\end{aligned}
$$

$GCD(50209, 1426) = -19 \cdot 50209 + 669 \cdot 1426$

15. Find the least non-negative residue.

(a) $x \equiv 22^{4999} \pmod{108}$
$22^1 \equiv 22 \pmod{108}$
$22^2 \equiv 484 \equiv 52 \pmod{108}$
$22^4 \equiv 52^2 \equiv 4 \pmod{108}$
$22^8 \equiv 4^2 \equiv 16 \pmod{108}$
$22^{16} \equiv 16^2 \equiv 40 \pmod{108}$
$22^{32} \equiv 40^2 \equiv 88 \pmod{108}$
$22^{64} \equiv 88^2 \equiv 76 \pmod{108}$
$22^{128} \equiv 76^2 \equiv 52 \pmod{108}$
$22^{256} \equiv 4 \pmod{108}$

$22^{512} \equiv 16 \pmod{108}$
$22^{1024} \equiv 40 \pmod{108}$
$22^{2048} \equiv 88 \pmod{108}$
$22^{4096} \equiv 76 \pmod{108}$

$$\begin{aligned} 4999 - 4096 &= 903 - 512 = 391 - 256 = 135 - 128 \\ &= 7 - 4 = 3 - 2 = 1 - 1 = 0 \end{aligned}$$

$$\begin{aligned} x \equiv 22^{4999} &\equiv 22^{4096} \cdot 22^{512} \cdot 22^{256} \cdot 22^{128} \cdot 22^{4} \cdot 22^{2} \cdot 22 \\ &\equiv 76 \cdot 16 \cdot 4 \cdot 52 \cdot 4 \cdot 52 \cdot 22 \equiv 1157398528 \equiv 4 \pmod{108} \end{aligned}$$

$LNR = 4$

(b) $x \equiv 21^{2345} \pmod{57}$
$x \equiv (21^5)^{469} \equiv 51^{469} \pmod{57}$

$51^1 \equiv 51 \pmod{57}$
$51^2 \equiv 2601 \equiv 36 \pmod{57}$
$51^4 \equiv 36^2 \equiv 42 \pmod{57}$
$51^8 \equiv 42^2 \equiv 54 \pmod{57}$
$51^{16} \equiv 54^2 \equiv 9 \pmod{57}$
$51^{32} \equiv 9^2 \equiv 24 \pmod{57}$
$51^{64} \equiv 24^2 \equiv 6 \pmod{57}$
$51^{128} \equiv 6^2 \equiv 36 \pmod{57}$
$51^{256} \equiv 42 \pmod{57}$

$469 - 256 = 213 - 128 = 85 - 64 = 21 - 16 = 5 - 4 = 1 - 1 = 0$

$$\begin{aligned} x \equiv 51^{469} &\equiv 51^{256} \cdot 51^{128} \cdot 51^{64} \cdot 51^{16} \cdot 51^{4} \cdot 51 \equiv 42 \cdot 36 \cdot 6 \cdot 9 \cdot 42 \cdot 51 \\ &\equiv 174890016 \equiv 51 \pmod{57} \end{aligned}$$

$LNR = 51$

16. Solve for x.

(a) $\begin{cases} x \equiv 1 \pmod{17} \\ x \equiv 5 \pmod{13} \\ x \equiv 2 \pmod{5} \end{cases}$

$x = 1 + 17t, t \in \mathbb{Z}$
$1 + 17t \equiv 5 \pmod{13} \Rightarrow 17t \equiv 5 - 1 \equiv 4 \pmod{13} \Rightarrow 4t \equiv 4 \pmod{13}$
$1 = 13 - 3(4)$
$4^{-1} \equiv -3 + 13 \equiv 10 \pmod{13}$
$t \equiv 40 \equiv 1 \pmod{13}$
$t = 1 + 13u, u \in \mathbb{Z}$
$x = 1 + 17(1 + 13u) = 1 + 17 + 221u = 18 + 221u$

$18 + 221u \equiv 2 \pmod{5} \Rightarrow 221u \equiv 2 - 18 \equiv -16 + 4(5) \equiv 4 \pmod{5}$
$\Rightarrow u \equiv 4 \pmod{5}$

$u = 4 + 5v, v \in \mathbb{Z}$
$x = 18 + 221(4 + 5v) = 18 + 884 + 1105v = 902 + 1105v, v \in \mathbb{Z}$

(b) $\begin{cases} x \equiv 3 \pmod{22} \\ x \equiv 7 \pmod{36} \\ x \equiv 1 \pmod{51} \end{cases}$

$x = 1 + 51t, t \in \mathbb{Z}$
$1 + 51t \equiv 7 \pmod{36} \Rightarrow 51t \equiv 6 \pmod{36} \Rightarrow 15t \equiv 6 \pmod{36}$
$5t \equiv 2 \pmod{12}$
$1 = 5(5) - 2(12)$
$5^{-1} \equiv 5 \pmod{12}$
$t \equiv 10 \pmod{12}$
$t = 10 + 12u, u \in \mathbb{Z}$
$x = 1 + 51(10 + 12u) = 511 + 612u$

$511 + 612u \equiv 3 \pmod{22} \Rightarrow 612u \equiv -508 + 24(22) \equiv 20 \pmod{22}$
$\Rightarrow 18u \equiv 20 \pmod{22}$
$9u \equiv 10 \pmod{11}$
$1 = 5(9) - 4(11)$
$9^{-1} \equiv 5 \pmod{11}$
$u \equiv 50 \equiv 6 \pmod{11}$
$u = 6 + 11v, v \in \mathbb{Z}$
$x = 511 + 612(6 + 11v) = 511 + 3672 + 6732v = 4183 + 6732v, v \in \mathbb{Z}$

17. Prove each of the following statements.

Study Notes:

Addition $\begin{cases} \text{Even + Even = Even} \\ \text{Odd + Odd = Even} \\ \text{Even + Odd = Odd} \end{cases}$ Subtraction $\begin{cases} \text{Even - Even = Even} \\ \text{Odd - Odd = Even} \\ \text{Even - Odd = Odd} \end{cases}$

Multiplication $\begin{cases} \text{Even} \times \text{Odd = Even} \\ \text{Even} \times \text{Even = Even} \\ \text{Odd} \times \text{Odd = Odd} \end{cases}$ x^2 rule $\begin{cases} x^2 = \text{Even} \leftrightarrow x = \text{Even} \\ x^2 = \text{Odd} \leftrightarrow x = \text{Odd} \end{cases}$

(a) Let x and y be integers. Show that $3x + y$ is odd if and only if $x^2 + 5y - 7$ is even.
Proof: $3x + y \rightarrow x^2 + 5y - 7$
Assume $3x + y$ is odd,
then exactly one of $3x$ and y is odd, and the other one is even.

Case 1: $3x$ is odd, and y is even
$3x$ is odd $\Rightarrow$ both 3 and x are odd
x is odd $\Rightarrow x^2$ is odd
y is even $\Rightarrow 5y$ is even
Hence $x^2 + 5y$ is odd

Since 7 is odd, then $x^2 + 5y - 7$ is even

Case 2: $3x$ is even, and y is odd
$3x$ is even $\Rightarrow$ at least one of 3 and x is even
Since 3 is odd, then x must be even
x is even $\Rightarrow x^2$ is even
y is odd $\Rightarrow 5y$ is odd
Hence $x^2 + 5y$ is odd
Since 7 is odd, then $x^2 + 5y - 7$ is even

Proof: $x^2 + 5y - 7 \to 3x + y$
Assume $x^2 + 5y - 7$ is even,
then x^2 and $5y - 7$ are either both odd or both even.

Case 1: x^2 and $5y - 7$ are both odd
x^2 is odd $\Rightarrow x$ is odd $\Rightarrow 3x$ is odd
$5y - 7$ is odd, then exactly one of $5y$ and 7 is odd, and the other is even
Since 7 is odd, then $5y$ must be even
then at least one of 5 amd y must be even
since 5 is odd, y must be even
since $3x$ is odd, y is even, then $3x + y$ must be odd

Case 2: x^2 and $5y - 7$ are both even
x^2 is even $\Rightarrow x$ is even $\Rightarrow 3x$ is even
$5y - 7$ is even, then $5y$ and 7 are both odd or both even
Since 7 is odd, then $5y$ must be odd
then both 5 amd y must be odd
since $3x$ is even, y is odd, then $3x + y$ must be odd

(b) Let A, B and C be non-empty sets. Show that if $A \times B = B \times C$, then $A = B = C$.
Proof: $A \times B = B \times C \to A = B = C$ ($A \subseteq B \subseteq C$ and $C \subseteq B \subseteq A$)
Let $x \in A$
Since B is non-empty, then $\exists b \in B$
Hence, $(x, b) \in A \times B$
Since $A \times B = B \times C$, then $(x, b) \in B \times C$
Hence, $x \in B$
Therefore, $A \subseteq B$

Let $x \in B$
Since A is non-empty, then $\exists a \in A$
Hence, $(a, x) \in A \times B$
Since $A \times B = B \times C$, then $(a, x) \in B \times C$
Hence, $x \in C$
Therefore, $B \subseteq C$

Let $x \in C$
Since B is non-empty, then $\exists b \in B$
Hence, $(b, x) \in B \times C$
Since $A \times B = B \times C$, then $(b, x) \in A \times B$
Hence, $x \in B$
Therefore, $C \subseteq B$

Let $x \in B$
Since C is non-empty, then $\exists c \in C$
Hence, $(x, c) \in B \times C$
Since $A \times B = B \times C$, then $(x, c) \in A \times B$
Hence, $x \in A$
Therefore, $B \subseteq A$

(c) Let f be a one-to-one function from A to B and let S and T be subsets of A. Show that if S and T are disjoint, then their images $f(S)$ and $f(T)$ must be disjoint.
Proof by contradiction:
Assume $f(S)$ and $f(T)$ are not disjoint
Then $\exists b \in f(S) \cap f(T)$
Hence, $b \in f(S)$ and $b \in f(T)$
$b \in f(S) \Rightarrow \exists a_1 \in S$, such that $f(a_1) = b$
$b \in f(T) \Rightarrow \exists a_2 \in T$, such that $f(a_2) = b$
Hence, $f(a_1) = f(a_2)$
Since f is 1-1, then $a_1 = a_2$
Since $a_1 \in S, a_2 \in T$ and $a_1 = a_2$, then $a_1 = a_2 \in S \cap T$
Hence, S and T are not disjoint. (contradiction)

(d) Show that if x is an odd integer, then $8 \mid (x^2 - 1)$
Proof: Assume x is odd, then $x = 2n + 1, n \in \mathbb{Z}$

$$\begin{aligned} x^2 - 1 &= (2n+1)^2 - 1 \\ &= 4n^2 + 4n + \cancel{1} - \cancel{1} \\ &= 4n(n+1) \end{aligned}$$

Case 1:	If n is even,	Case 2:	If n is odd,
	then $n = 2k, k \in \mathbb{Z}$		then $n+1$ is even
	Hence, $4n(n+1)$		Hence, $n + 1 = 2m, m \in \mathbb{Z}$
	$= 4(2k)(n+1)$		Hence, $4n(n+1)$
	$= 8k(n+1)$		$= 4n2m = 8nm$
	Hence, $8 \mid x^2 - 1$		Hence, $8 \mid x^2 - 1$

18. Prove or disprove each of the following statements.

(a) If both $1/x$ and $1/(x + y)$ are rational, then y is also rational. T
Proof: $\frac{1}{x}$ and $\frac{1}{x+y}$ are rational

$\Rightarrow \begin{cases} \dfrac{1}{x} = \dfrac{a}{b} \\ \dfrac{1}{x+y} = \dfrac{c}{d} \end{cases}$, where $a, b, c, d \in \mathbb{Z}$, and $b, d \neq 0$

$\dfrac{1}{x} = \dfrac{a}{b} \Rightarrow b = ax$, since $b \neq 0$, then $a \neq 0$

$\dfrac{1}{x+y} = \dfrac{c}{d} \Rightarrow d = c(x+y)$, since $d \neq 0$, then $c \neq 0$

$$\left.\begin{array}{l} b = ax \Rightarrow x = \dfrac{b}{a} \\ d = c(x+y) \Rightarrow x + y = \dfrac{d}{c} \end{array}\right\} \Rightarrow y = \frac{d}{c} - \frac{b}{a} = \frac{ad - bc}{ac}$$

Since $a, b, c, d \in \mathbb{Z}$, then $(ad - bc), (ac) \in \mathbb{Z}$

Since $a \neq 0, c \neq 0$, then $ac \neq 0$

Hence, y is rational.

(b) Let A, B and C be sets. If $A - C \subseteq B - C$, then $A \subseteq B$ F

Counter example: $A = \{1\}$, $B = \{2\}$, $C = \{1, 2\}$

$A - C = \varnothing$, and $B - C = \varnothing$

Hence, $A - C \subseteq B - C$, but $A \nsubseteq B$

(c) Let A and B be sets. Then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$ T

Proof: $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$

$\Leftrightarrow S \in \mathcal{P}(A)$ and $S \in \mathcal{P}(B)$

$\Leftrightarrow S \subseteq A$ and $S \subseteq B$

$\Leftrightarrow S \subseteq A \cap B$

$\Leftrightarrow S \in \mathcal{P}(A \cap B)$

Hence, $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$ and $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$

Therefore, $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$

(d) Let x and m be integers with $m \geq 1$. If $x^2 \equiv 1 \pmod{m}$, then either $x \equiv 1 \pmod{m}$ or $x \equiv -1 \pmod{m}$ F

Let $x = 10$, $m = 99$

Then $x^2 \equiv 100 \equiv 1 \pmod{99}$

But $10 \not\equiv 1 \pmod{99}$ and $10 \not\equiv -1 \pmod{99}$

19. Prove each of the following statements using mathematical inductions.

(a) Show that $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \ldots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$ for all integers $n \geq 1$.

Basic Step: $n = 1$

$$LHS = \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2} = RHS$$

Inductive Step: Assume $n = k$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$$

Show $n = k + 1$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$$

$$\begin{aligned} LHS &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \\ &= 1 - \frac{2}{2^k \cdot 2} + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^{k+1}} \\ &= RHS \end{aligned}$$

(b) Show that $89 \mid (5^{3n} - 6^{2n}$ for all integer $n \geq 0$.

Basic Step: $n = 0$

$89 \mid 5^0 - 6^0 = 0$

Inductive Step: Assume $n = k$

$89 \mid 5^{3k} - 6^{2k}$

Show $n = k + 1$

$89 \mid 5^{3(k+1)} - 6^{2(k+1)}$

$$\begin{aligned} 5^{3(k+1)} - 6^{2(k+1)} &= 5^{3k+3} - 6^{2k+2} \\ &= 5^3 \cdot 5^{3k} - 6^2 \cdot 6^{2k} \\ &= 125 \cdot 5^{3k} - 36 \cdot 6^{2k} \\ &= (89 + 36) \cdot 5^{3k} - 36 \cdot 6^{2k} \\ &= 89 \cdot 5^{3k} + 36 \cdot 5^{3k} - 36 \cdot 6^{2k} \\ &= 89 \cdot 5^{3k} + 36(5^{3k} - 6^{2k}) \end{aligned}$$

Since $89 \mid 89 \cdot 5^{3k}$ and $89 \mid 36(5^{3k} - 6^{2k})$, then $89 \mid 5^{3(k+1)} - 6^{2(k+1)}$

Alternative Solution:

Basic Step: $n = 0$

$89 \mid 5^0 - 6^0 = 0$

Inductive Step: Assume $n = k$

$89 \mid 5^{3k} - 6^{2k} \Rightarrow 5^{3k} - 6^{2k} = 89m, m \in \mathbb{Z} \Rightarrow 5^{3k} = 89m + 6^{2k}$

Show $n = k + 1$

$89 \mid 5^{3(k+1)} - 6^{2(k+1)}$

$$
\begin{aligned}
5^{3(k+1)} - 6^{2(k+1)} &= 5^{3k+3} - 6^{2k+2} \\
&= 5^3 \cdot 5^{3k} - 6^2 \cdot 6^{2k} \\
&= 125(89m + 6^{2k}) - 36 \cdot 6^{2k} \\
&= 125 \cdot 89m + 125 \cdot 6^{2k} - 36 \cdot 6^{2k} \\
&= 125 \cdot 89m + 89 \cdot 6^{2k} \\
&= 89(125m + 6^{2k})
\end{aligned}
$$

Hence, $89 \mid 5^{3(k+1)} - 6^{2(k+1)}$

20. Consider all strings with 9 English letters.

(a) If repetitions are not allowed, how many strings contain the letter x?
$9 \times (25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19 \times 18)$

(b) If repetitions are not allowed, how many strings contain both letter x and y?
$9 \times 8 \times (24 \times 23 \times 22 \times 21 \times 20 \times 19 \times 18)$

(c) If repetitions are not allowed, how many strings contain both letter x and y in consecutive positions?
$(2 \times 8) \times (24 \times 23 \times 22 \times 21 \times 20 \times 19 \times 18)$

(d) If repetitions are allowed, how many strings contain the letter x?
$A =$ {Strings with x}
$\overline{A} =$ {Strings with no x}
$U =$ {All strings with 9 letters}

$|U| = 26^9, \quad |\overline{A}| = 25^9$
$|A| = |U| - |\overline{A}| = 26^9 - 25^9$

(e) If repetitions are allowed, how many strings contain either letter x or letter y?
$S =$ {All strings with either x or y}
$\overline{S} =$ {All strings with no x and y}
$U =$ {All strings with 9 letters}

$|U| = 26^9, \quad |\overline{S}| = 24^9$
$|S| = |U| - |\overline{S}| = 26^9 - 24^9$

(f) If repetitions are allowed, how many strings contain both letter x and y?
$A \cap B =$ {All strings with both x and y}
$A =$ {Strings with x}
$B =$ {Strings with y}
$A \cup B =$ {All strings with either x or y}

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cup B| = 26^9 - 24^9, \quad |A| = 26^9 - 25^9, \quad |B| = 26^9 - 25^9$

$$\begin{aligned} |A \cap B| = |A| + |B| - |A \cup B| &= (26^9 - 25^9) + (26^9 - 25^9) - (26^9 - 24^9) \\ &= 26^9 - 25^9 + \cancel{26^9} - 25^9 - \cancel{26^9} + 24^9 \\ &= 26^9 - 2(25^9) + 24^9 \end{aligned}$$

21. Consider a set of seven digits $\{0, 1, 3, 5, 6, 7, 9\}$, and a four-digit number is to be constructed by using the numbers in the set without repetitions.

(a) How many different numbers can be formed?
$7 \times 6 \times 5 \times 4 = 840$

(b) How many such numbers are odd?
$6 \times 5 \times 4 \times 5 = 600$

(c) How many such numbers are greater than 5000?
$4 \times 6 \times 5 \times 4 = 480$

(d) How many such numbers are greater than 5300?
Case 1: 1st digit is 5, then 2nd digit must be ≥ 3
$1 \times 5 \times 5 \times 4 = 100$
Case 2: 1st digit is > 5
$3 \times 6 \times 5 \times 4 = 360$
Total$= 100 + 360 = 460$