Problem Set 0: Simplify

1. Find the difference between the sum of the first thousand positive even integers and the sum of the first thousand positive odd integers.

$2+4+6+8+10+\dots$
$1+3+5+7+\ 9\ +\dots$ $\Longleftarrow$ Start writing out the 1st few even 's, and the 1st few odd 's below it.

$$\begin{array}{r} 2+4+6+8+10+\dots \\ -\,(1+3+5+7+\ 9\ +\dots) \\ \hline \underbrace{1+1+1+1+1+\dots}_{\text{1000 columns}} \end{array}$$

$\Longleftarrow$ Focus on the columns in what we wrote. Notice that each # in the top row is 1 more than the # right below it. If we subtract 2 rows, it would look like this.

$1+1+1+1+1+\dots$ 1000 times $= \boxed{1000}$ $\Longleftarrow$ Every column has a difference of 1. How many columns are there? We wrote the list of 1000 positive integers, so there are 1000 columns.

2. Leo the Rabbit is climbing up a flight of 20 steps. Leo can only hop up 1 or 2 steps each time he hops. He never hops down, only up. How many different ways can Leo hop up the flight of 20 steps?

$$\begin{tikzcd}[column sep=small] & 20 & \\ 18 \arrow[ur] & & 19 \arrow[ul] \end{tikzcd}$$

$\Longleftarrow$ Let's work backwards, think about what happened right before Leo goes to the top, where did he come from?
He can only hop 1 or 2 steps, so he either came from step 19 or step 18.

$\#$ ways to get to 20
$= \#$ ways to get to 19 + # ways to get to 18 $\Longleftarrow$ So the # of ways he can get to 20th step is the sum of the # of ways he can get to 18th and 19th steps

$W_{20} = W_{19} + W_{18}$ $\Longleftarrow$ Let's introduce some notation to make it easier to write.
Let $W_x = \#$ of ways he can get to xth step.

$W_{19} = W_{18} + W_{17}$ $\Longleftarrow$ Keep working backwards, how many ways can he get to 19th step?
He must have come from 17th or 18th step.

$W_{18} = W_{17} + W_{16}$ $\Longleftarrow$ Similarly, and so on, we just add # of ways for the previous two steps.

$W_1 = 1$ $\Longleftarrow$ To get an actual #, think about the 1st step. Obviously, there is only 1 way to get step 1.

$W_2 = 2$ $\Longleftarrow$ For step 2, there are 2 ways to get there, either 2 hops of 1 step, or 1 hop of 2 steps.

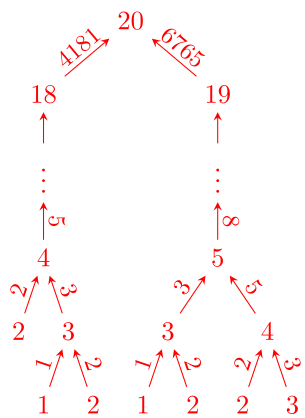
$$\begin{aligned} W_3 &= W_2 + W_1 \\ &= 2+1 \\ &= 3 \end{aligned}$$

$\Longleftarrow$ From there, we can calculate the rest by adding the previous 2

$$\begin{aligned} W_4 &= W_3 + W_2 \\ &= 3+2 \\ &= 5 \\ &\ \vdots \end{aligned}$$

$$\begin{aligned} W_{20} &= W_{19} + W_{18} \\ &= 6765 + 4181 \\ &= \boxed{10,946} \end{aligned}$$

$\Longleftarrow$ If you keep this up, you will eventually get $W_{20} = 10,946$



There is a formula to get this, but it's not easy to find and it's pretty complicated. We call these #'s Fibonacci numbers. You can look it up if you're interested.

3. Find the value of this difference:

$$987654320 \times 987654322 - 987654321 \times 987654321$$

Notice that the 1st 2 #'s are 1 less and 1 more than the last 2 #'s.

$(x-1)(x+1) - x \cdot x$ $\leftarrow$ we can generalize the pattern with variables

$= x^2 + x - x - 1 - x^2$

$= \boxed{-1}$ $\leftarrow$ doing some algebra, we get

Since x can represent any #, including 987654321, the answer for this problem and any other problem with #'s in the same pattern, will be -1.

4. Fine the value of this product:

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{25}\right)\cdots\left(1-\frac{1}{576}\right)\left(1-\frac{1}{625}\right)$$

You probably notice that denominators are all perfect squares, but more than that the entire fraction is a perfect square, $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$, note that 1 is also a perfect square, $1^2 = 1$.

So we have bunch of terms of the form $(a^2 - b^2)$, this factors in a special way, $a^2 - b^2 = (a-b)(a+b)$.

Let's factor the 1st few terms, then simplify.

$$\underbrace{\left(1-\frac{1}{4}\right)}_{(1-\frac{1}{2})(1+\frac{1}{2})}\underbrace{\left(1-\frac{1}{9}\right)}_{(1-\frac{1}{3})(1+\frac{1}{3})}\underbrace{\left(1-\frac{1}{16}\right)}_{(1-\frac{1}{4})(1+\frac{1}{4})}\underbrace{\left(1-\frac{1}{25}\right)}_{(1-\frac{1}{5})(1+\frac{1}{5})}\cdots\underbrace{\left(1-\frac{1}{576}\right)}_{(1-\frac{1}{24})(1+\frac{1}{24})}\underbrace{\left(1-\frac{1}{625}\right)}_{(1-\frac{1}{25})(1+\frac{1}{25})}$$

$$=\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{5}\right)\cdots\left(1-\frac{1}{25}\right)\left(1+\frac{1}{25}\right)$$

$$=\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{4}{5}\right)\left(\frac{6}{5}\right)\cdots\left(\frac{24}{25}\right)\left(\frac{26}{25}\right)$$

$$=\left(\frac{1}{2}\right)\underbrace{\cancel{\left(\frac{3}{2}\right)}\cancel{\left(\frac{2}{3}\right)}}_{=1}\underbrace{\cancel{\left(\frac{4}{3}\right)}\cancel{\left(\frac{3}{4}\right)}}_{=1}\cancel{\left(\frac{5}{4}\right)}\cancel{\left(\frac{4}{5}\right)}\cancel{\left(\frac{6}{5}\right)}\cdots\cancel{\left(\frac{24}{25}\right)}\left(\frac{26}{25}\right)$$ ← notice many of these new terms will cancel to 1, because they are reciprocals.

$$=\left(\frac{1}{2}\right)\left(\frac{26}{25}\right)$$ ← in fact, every term will cancel out except the first and last.

$$=\boxed{\frac{13}{25}}$$

5. Fine the value of this sum:

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\frac{1}{\sqrt{2498}+\sqrt{2499}}+\frac{1}{\sqrt{2499}+\sqrt{2500}}$$

Typically, when we have radicals in the denominator, we try to rationalize them. Since we have a radical to a radical, to rationalize we have to multiply by the conjugate.

Let's just look at the 1st 2 terms to start.

$$\frac{1}{\sqrt{1}+\sqrt{2}}\cdot\frac{\sqrt{1}-\sqrt{2}}{\sqrt{1}-\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}\cdot\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}}+\ldots$$

$$=\frac{\sqrt{1}-\sqrt{2}}{1-2}+\frac{\sqrt{2}-\sqrt{3}}{2-3}$$

$$=\frac{\sqrt{1}-\sqrt{2}}{-1}+\frac{\sqrt{2}-\sqrt{3}}{-1}$$

$$=-\sqrt{1}+\sqrt{2}-\sqrt{2}+\sqrt{3}$$ ← notice how the middle terms canceled out.

$$=-\sqrt{1}+\sqrt{3}$$

Let's look at the 1st 3 terms.

$$\frac{1}{\sqrt{1}+\sqrt{2}}\cdot\frac{\sqrt{1}-\sqrt{2}}{\sqrt{1}-\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}\cdot\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}\cdot\frac{\sqrt{3}-\sqrt{4}}{\sqrt{3}-\sqrt{4}}+\ldots$$

$$=\frac{\sqrt{1}-\sqrt{2}}{1-2}+\frac{\sqrt{2}-\sqrt{3}}{2-3}+\frac{\sqrt{3}-\sqrt{4}}{3-4}$$

$$=\frac{\sqrt{1}-\sqrt{2}}{-1}+\frac{\sqrt{2}-\sqrt{3}}{-1}+\frac{\sqrt{3}-\sqrt{4}}{-1}$$

$$=-\sqrt{1}+\sqrt{2}-\sqrt{2}+\sqrt{3}--\sqrt{3}+\sqrt{4}$$

$$=-\sqrt{1}+\sqrt{4}$$ ← again, middle term canceled, leaving just the 1st and the last.

This pattern will continue, so it will end up looking like:

$$
\begin{aligned}
& -\sqrt{1}+\cancel{\sqrt{4}}+\ldots-\cancel{\sqrt{2499}}+\sqrt{2500} \\
= & -\sqrt{1}+\sqrt{2500} \\
= & -1+50 \\
= & \boxed{49}
\end{aligned}
$$

6. Fine the value of this sum:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{999\times 1000}$$

If you look at sums of just the first few terms, a simple pattern should be easy to spot, but there is another way to solve the problem:

Notice every denominator is the product of 2 #'s, what is common thing we do in math where we multiple denominator by something? Finding common denominator.

What if these fractions came from finding common denominator of 2 separate fraction? What would these other fractions look like?

$$
\begin{aligned}
\frac{1}{1\times 2} &= \frac{?}{?}+\frac{?}{?} \\
&= \frac{?}{1}+\frac{?}{2} \quad \leftarrow \text{one fraction must have denominator equal to the 1st \# we're multiplying, and the other denominator must be the 2nd.} \\
&= \frac{x}{1}+\frac{y}{2} \quad \leftarrow \text{Let's put in some variables, and see if we can figure out what the numerators are.} \\
&= \frac{x}{1}\cdot\frac{2}{2}+\frac{y}{2}\cdot\frac{1}{1} \\
&= \frac{2x}{1\times 2}+\frac{y}{1\times 2}
\end{aligned}
$$

$$
\begin{aligned}
1 &= 2x+y \\
y &= 1-2x
\end{aligned}
$$

Let $x=1$, $y=1-2(1)=-1$ ← if we say that $x=1$ (let's choose a nice easy #), then y would be -1.

Let's check it.

$$
\begin{aligned}
\frac{1}{1\cdot 2} &\stackrel{?}{=} \frac{1}{1}-\frac{1}{2} \\
\frac{1}{2} &= \frac{1}{2} \quad \checkmark
\end{aligned}
$$

Let's also see if this pattern works to the other terms:

$$
\begin{aligned}
\frac{1}{2\cdot 3} &\stackrel{?}{=} \frac{1}{2}-\frac{1}{3} \\
\frac{1}{6} &= \frac{1}{2}\cdot\frac{3}{3}-\frac{1}{3}\cdot\frac{2}{2} \\
\frac{1}{6} &= \frac{3}{6}-\frac{2}{6} \\
\frac{1}{6} &= \frac{1}{6} \quad \checkmark
\end{aligned}
$$

Seems like it works, so we can rewrite original problem as:

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{999}-\frac{1}{1000}\right)$$

$$=\frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}+\ldots+\cancel{\frac{1}{999}}-\frac{1}{1000}$$ ← notice that all the middle terms will cancel out.

$$=\frac{1}{1}-\frac{1}{1000}$$ ← so we've left with only 1st and last term

$$=\boxed{\frac{999}{1000}}$$