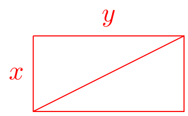
Problem Set 1: Chunks of Info

1. The area of a rectangle is 12.5 square meters. The perimeter of the same rectangle is 20 meters. Find the product of the diagonals of this rectangle.



diagonal $= \sqrt{x^2+y^2}$ $\leftarrow$ by Pythagorean Theorem

product of diagonals $= \left(\sqrt{x^2+y^2}\right)\left(\sqrt{x^2+y^2}\right) = x^2+y^2$ $\leftarrow$ here's what we want to find

Let's try to state the given info:

$A = xy = 12.5$

$P = \dfrac{2x+2y=20}{2}$

$x+y=10$ $\leftarrow$ here we have $x+y$, we want x^2+y^2. Let's try squaring this.

$(x+y)^2 = 10^2$

$x^2+2xy+y^2 = 10^2$

$x^2+y^2+2xy = 100$ $\leftarrow$ try to isolate what we want (x^2+y^2).

$x^2+y^2 = 100 - 2xy$ $\leftarrow$ recall that $xy = 12.5 =$ Area.

$x^2+y^2 = 100 - 2(12.5)$

$= 100 - 25$

$\boxed{x^2+y^2=75}$ $\leftarrow$ by focusing (x^2+y^2) as a chunk of info, we found the answer without having to solve x or y.

2. If $\sqrt{4+x}+\sqrt{10-x}=6$, find the value of $\sqrt{(4+x)(10-x)}$

Normally when we have radicals in an equation, we square both sides to try to get rid of radicals.

$$\left(\sqrt{4+x}+\sqrt{10-x}\right)^2 = 6^2$$

$\leftarrow$ Remember, we have to foil this:

$$(\sqrt{4+x}+\sqrt{10-x})\cdot(\sqrt{4+x}+\sqrt{10-x})$$

$$(4+x)+2\left(\sqrt{4+x}\right)\left(\sqrt{10-x}\right)+(10-x)=36$$

$$14+2\left(\sqrt{4+x}\right)\left(\sqrt{10-x}\right)=36$$

$\leftarrow (\sqrt{a})(\sqrt{b}) = \sqrt{(a)(b)}$

$$14+2\sqrt{(4+x)(10-x)}=36$$

$\leftarrow$ Look familiar? This is what we're try to find, so isolate it

$$\frac{2\sqrt{(4+x)(10-x)}}{2}=\frac{22}{2}$$

$$\boxed{\sqrt{(4+x)(10-x)}=11}$$

$\leftarrow$ answer

3. If $\dfrac{1}{x^3}-\dfrac{1}{x^2}-\dfrac{1}{x}-1=0$, find the value of x^3+x^2+x+1

Normally when we have variables in denominators, we multiply everything by LCD which gets rid of all the fractions, let's try that. $LCD = x^3$

$$x^3\left(\frac{1}{x^3}-\frac{1}{x^2}-\frac{1}{x}-1\right)=0(x^3)$$

$$1-x-x^2-x^3=0$$

$\leftarrow$ this looks similar to what we want, but it's negative, multiply through by (-1)

$$-1+x+x^2+x^3=0$$

$$x^3+x^2+x-1=0$$

$\leftarrow$ we want x^3+x^2+x+1, not x^3+x^2+x-1

$$+2 \quad +2$$

Can we just add 2?

$$\boxed{x^3+x^2+x+1=2}$$

$\leftarrow$ answer

Typeset @ Jae

4. If $x = \sqrt[3]{20+14\sqrt{2}} + \sqrt[3]{20-14\sqrt{2}}$, find the value of $x^3 - 6x$

This looks complicated, it might help to simplify the problem by introducing alternate notation.

Let $a = 20 + 14\sqrt{2}$
$b = 20 - 14\sqrt{2}$

Now we can rewrite x as:

$x = \sqrt[3]{a} + \sqrt[3]{b}$ $\leftarrow$ much easier to work with

$= a^{\frac{1}{3}} + b^{\frac{1}{3}}$ $\leftarrow$ rewrite in exponential form, it's easier to multiply stuff that way

We want to find $x^3 - 6x$, let's try cubing x, and see if we can eventually make it look like that.
Cubing means multiple by itself 3 times, we'll need to foil and then multiply again by the binomial.

$$\begin{aligned}
x^3 &= (a^{\frac{1}{3}} + b^{\frac{1}{3}})^3 \\
&= (a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{1}{3}} + b^{\frac{1}{3}}) \\
&= \underbrace{(a^{\frac{2}{3}} + 2a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})}_{\text{product of 1st 2 binomials}} \cdot \underbrace{(a^{\frac{1}{3}} + b^{\frac{1}{3}})}_{\text{3rd binomial}} && \leftarrow \text{now we have to multiple these two together} \\
&= a + 2a^{\frac{2}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} + a^{\frac{2}{3}}b^{\frac{1}{3}} + 2a^{\frac{1}{3}}b^{\frac{2}{3}} + b && \leftarrow \text{combine like terms} \\
&= a + b + 3a^{\frac{2}{3}}b^{\frac{1}{3}} + 3a^{\frac{1}{3}}b^{\frac{2}{3}} && \leftarrow \text{the last 2 terms has GCF(Great Common Factor), let's factor out} \\
&= a + b + 3a^{\frac{1}{3}}b^{\frac{1}{3}}\underline{(a^{\frac{1}{3}} + b^{\frac{1}{3}})} && \leftarrow \text{looks familiar? remember } x = a^{\frac{1}{3}} + b^{\frac{1}{3}}
\end{aligned}$$

$$\begin{aligned}
x^3 &= a + b + 3a^{\frac{1}{3}}b^{\frac{1}{3}}(x) \\
&= a + b + 3(ab)^{\frac{1}{3}}(x) && \leftarrow \text{if only we knew what } a+b \text{ and } ab \text{ were...}
\end{aligned}$$

$$\begin{aligned}
a + b &= 20 + 14\sqrt{2} + 20 - 14\sqrt{2} \\
&= 40
\end{aligned}$$

recall that $a = 20 + 14\sqrt{2}$, $b = 20 - 14\sqrt{2}$.
they are conjugates, conjugates have special sums and products.

$$\begin{aligned}
ab &= (20 + 14\sqrt{2})(20 - 14\sqrt{2}) \\
&= 20^2 - (14\sqrt{2})^2 \\
&= 400 - 392 \\
&= 8
\end{aligned}$$

so $a + b = 40$, $ab = 8$
substitute these value into $x^3 = a + b + 3(ab)^{\frac{1}{3}}(x)$

$x^3 = 40 + 3(8)^{\frac{1}{3}}(x)$

$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

$\frac{1}{3}$ exponent means cube root

$x^3 = 40 + 3(2)(x)$

$x^3 = 40 + 6x$ $\leftarrow$ whoa! this looks familiar, we're looking for $x^3 - 6x$

$-6x \qquad -6x$

$\boxed{x^3 - 6x = 40}$